Federated Averaging Langevin Dynamics: Toward a unified theory and new algorithms

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Abstract

This paper focuses on Bayesian inference in a federated learning context (FL). While several distributed MCMC algorithms have been proposed, few consider the specific limitations of FL such as communication bottlenecks and statistical heterogeneity. Recently, Federated Averaging Langevin Dynamics (FALD) was introduced, which extends the Federated Averaging algorithm to Bayesian inference. We obtain a novel tight non-asymptotic upper bound on the Wasserstein distance to the global posterior for FALD. This bound highlights the effects of statistical heterogeneity, which causes a drift in the local updates that negatively impacts convergence. We propose a new algorithm VR-FALD* that uses control variates to correct the client drift. We establish non-asymptotic bounds showing that VR-FALD* is not affected by statistical heterogeneity. Finally, we illustrate our results on several FL benchmarks for Bayesian inference.

1 Introduction

The paradigm of fully centralized machine learning is increasingly at odds with real-world use cases. Centralized machine learning leads to (a) data processing bottlenecks, (b) inefficient use of communication resources and (c) risks exposing individuals’ private data. As storage and computational capacity increases at the agent level, it becomes increasingly attractive to decentralize computational tasks whenever possible. The term federated learning (FL) was recently coined to capture some aspects of this grand challenge (McMahan et al., 2017; Kairouz et al., 2021; Yang et al., 2019; Alistarh et al., 2017; Horváth et al., 2022; Wang et al., 2021).

Reducing communication costs has been identified as one of the major challenges of FL (Kairouz et al., 2021). Two main approaches have been proposed to achieve this goal. In the former, agents perform multiple local optimization steps before sending a model update to the central node (McMahan et al., 2017). The latter consists in compressing the messages exchanged (Alistarh et al., 2017; Horváth et al., 2022). In this paper, we focus on the first approach which is widely used in practice. However, due to statistical heterogeneity, performing multiple steps can hinder convergence, as model updates target each agent’s local minimizer (Li et al., 2019; Ro et al., 2021). This results in a trade-off between communication cost and convergence (Wang et al., 2020) and a need for algorithms that mitigate client drift (Karimireddy et al., 2020).

Most of existing FL algorithms minimize a training loss. However, their results do not provide reliable uncertainty quantification, a strong requirement in safety-critical applications (Coglianese and Lehr, 2016; Fatima et al., 2017). We address this problem by considering the federated version of Bayesian inference (Welling and Teh, 2011; Yurochkin et al., 2019; Chen and Chao, 2021; Izmailov et al., 2021; Wilson et al., 2021). The objective is to compute the predictive distribution, highest posterior density regions (HPD). To this end, it is required to sample the posterior distribution \( \pi \propto \exp(-U) \) associated with the model at hand. This target posterior decomposes into the product of local posteriors \( \pi = \prod_{i \in [b]} \pi_i \). It is well known that sampling according to product distributions (Neiswanger et al., 2014; Hoffman et al., 2013; Minsker et al., 2014; Wang et al., 2015; Al-Shedivat et al., 2021; Dai et al., 2021) raises serious computational challenges even when sampling from each local posterior \( \pi_i \) is reasonably easy. We tackle this question in our contributions which can be summarized as follows.
Contributions.

- We study a random loop version of the FALD algorithm proposed in Deng et al. (2021), and we establish non-asymptotic upper bounds in Wasserstein distance for strongly convex potentials $U$. An analysis of FALD was conducted in (Deng et al., 2021, Theorem 5.7). However, the proof is plagued by an error; see Section 7.1.

- We give matching lower bounds to show that even with full batch gradients, FALD can be slower than Stochastic Gradient Langevin Dynamics (SGLD) due to client-drift.

- We propose a new method VR-FALD* that circumvents the shortcomings of FALD. This algorithm extends the Shifted Local-SVRG of Gorbunov et al. (2021) to the Bayesian context. It combines Stochastic Variance Reduced Gradient (SVRG) Langevin Dynamics (LD) (Dubey et al., 2016) and adapts the bias reduction techniques from SCAFFOLD (Karimireddy et al., 2020).

- We derive theoretical guarantees for VR-FALD* which highlight its gradient variance reduction effect and its ability to deal with data heterogeneity.

- The results are based on a general framework developed in the supplement, that encompasses a broad family of federated Bayes algorithms based on Langevin dynamics. This is the first unifying study among existing works on federated Bayesian inference.

- Finally, in Section 4 we illustrate our results using classical FL benchmarks and provide a thorough comparison with existing FL Bayesian methods.

Related works. Many distributed MCMC algorithms have been proposed in the last decade and it is difficult to credit all the references. The first significant contributions in this direction are the Consensus Monte Carlo (CMC) approach and “embarrassingly parallel” MCMC algorithms; see, e.g. Neiswanger et al. (2014); Wang and Dunson (2013); Scott et al. (2016). These methods require running separate MCMC chains on each client/computational node, with each chain targeting the local posterior $\pi^i$. In the final stage, the algorithms recombine the samples from these chains to generate samples from the desired global posterior $\pi$ (Minsker et al., 2014). The local posteriors may differ significantly from each other due to statistical heterogeneity, data imbalance, and / or inaccurate approximation. The effectiveness of the final combinations is either based on stringent assumptions on the local likelihoods (Liu and Ihler, 2014; Nemeth and Sherlock, 2018; Mesquita et al., 2020; Chittoor and Simeone, 2021) or on “fusion” algorithms that are exact but scale badly with the dimension; see, e.g. Dai et al. (2021); De Souza et al. (2022). Vono et al. (2020); Rendell et al. (2020); Plassier et al. (2021); Vono et al. (2022a) introduced hierarchical Bayesian models to simulate separate MCMC chains on each machine. Inspired by the alternating direction method of multipliers (Boyd et al., 2011), each client is assigned an auxiliary parameter that is conditionally independent given the server parameter. These authors developed MCMC schemes which alternate between sampling the clients parameters given the server parameter, and sampling the server parameter given the clients parameters. However, these approaches require tuning an additional hyperparameter to control the dispersion of the “local parameters”. This parameter characterizes the trade-off between computational tractability and closeness to the original target distribution.

A competing approach to Federated Averaging, the quantized-SGD scheme, has been proposed in (Alistarh et al., 2017) for non Bayesian FL. In this framework, the agents do not adapt parameters locally but a random subset of the agents compute at each iteration a new gradient estimator and transmit a compressed form—see Haddadpour et al. (2021), among many others, (Bernstein et al., 2018; Tang et al., 2021) for scalar quantization or (Shlezinger et al., 2020), for vector quantization. These approaches have been extended to the Bayesian inference context in Lee et al. (2020); Zhang et al. (2022); Vono et al. (2022b). Performance analysis is given in Vono et al. (2022b); Sun et al. (2022).

The Federated Gradient Stochastic Langevin Dynamics (FSGLD) algorithm introduced by El Mekkaoui et al. (2021) extends the distributed-SGLD (DSGLD) (Ahn et al., 2014) to the FL setting. Specifically, FSGLD operates passing a Markov chain between computing nodes and using only local data to estimate gradients at each step.

Methods with multiple local steps have been considered by several authors. Deng et al. (2021) designed FALD as a Bayesian version of FedAVG. Al-Shedivat et al. (2021) proposed FedPA as a generalization of FedAVG. This method performs several local steps to infer Gaussian approximations of the clients local parameters. These local parameters are then reweighted using the estimated local means and covariance matrices before being aggregated on the central server.

Notation and Convention. The Euclidean norm on $\mathbb{R}^d$ is denoted by $\| \cdot \|$, and we set $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. For $n \in \mathbb{N}^*$, we refer to $\{1, \ldots, n\}$ with the notation $[n]$. We denote by $P_2(\mathbb{R}^d)$ the set of probability measures on $\mathbb{R}^d$ with finite $2$-moment. For any random variable $\xi$ with values in $\mathbb{R}^d$, we define $\text{Var}(\xi) = \mathbb{E}[\|\xi - \mathbb{E}\xi\|^2]$. Let $\mu, \nu$ be in $P_2(\mathbb{R}^d)$, we define the Wasserstein distance of order 2 by $W_2(\mu, \nu) = \left(\inf_{\zeta \in \Pi(\mu, \nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - x'\|^2 \text{d}\zeta(x, x')\right)^{1/2}$, where $\Pi(\mu, \nu)$ is the set of transference plans of $\mu$ and $\nu$. 

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Algorithm derivation

We aim to sample a target probability density function $\pi$ defined for $x \in \mathbb{R}^d$ by

$$\pi(x) \propto \prod_{i=1}^b \pi^i(x), \quad \pi^i(x) \propto \exp(-U^i(x)), \quad (1)$$

where $b$ is the number of clients and the potential $U^i$ is a finite sum expressed by

$$U^i(x) = \omega^i U^0(x) + \sum_{j=1}^{N_i} U^{i,j}(x),$$

with $\{\omega^i\}_{i \in [b]} \in [0, 1]^b$ and $\sum_{i \in [b]} \omega^i = 1$. This setting encompasses the Bayesian federated learning as a particular case, in which $\pi$ stands for the global posterior distribution and $\{\pi^i\}_{i \in [b]}$ are referred to as local posteriors (Wu and Robert, 2017; Dai et al., 2021). In this case $U^0$ is the global negative log-prior, $N_i$ denotes the number of observations of client $i$, $U^{i,j}$ is the negative log-likelihood of the $j$-th data of client $i$, and $\omega^i U^0$ is the fraction of the negative log-prior allocated to this client (Rendell et al., 2020).

Federated Averaging Langevin Dynamics (FALD). FALD, proposed in Deng et al. (2021), is an extension to the Bayesian setting of FEDAVG (McMahan et al., 2017). The updates performed on the $i$th client define a sequence of local parameters $(X^i_k)_{k \in \mathbb{N}}$ which are transmitted according to some preset schedule (which is deterministic in Deng et al. (2021) and is random in this work) to a central server. The central server averages the local parameters to update the global parameter. This global parameter is finally transmitted back to each client, and is used as a starting point of a new round of local iterations. Hence, each iteration $k \geq 0$ of FALD can be decomposed into two steps:

(1) **Local iteration on each client.** Each client $i$ performs one step of the Langevin Monte Carlo algorithm (Grenander and Miller, 1994; Roberts and Tweedie, 1996) with a stochastic gradient associated with its local potential:

$$G^i_{k+1} = \nabla U^i_{k+1}(X^i_k),$$

$$X^i_{k+1} = X^i_k - \gamma G^i_{k+1} + \sqrt{2\gamma} Z^i_{k+1}, \quad (2)$$

where $\gamma > 0$ and for $x \in \mathbb{R}^d$, $\nabla U^i_{k+1}(x)$ is an unbiased estimator of $\nabla U^i(x)$ given by (see Welling and Teh (2011) – general updates are considered in the supplement)

$$\nabla U^i_{k+1} = \omega^i \nabla U^0 + (N_i/n_i) \sum_{j \in S^i_{k+1}} \nabla U_{i,j}^i, \quad (3)$$

where $S^i_k \in \mathbb{N}^*$ is a sequence of i.i.d uniform random subsets of $[N_i]$ of cardinal number $n_i$. Moreover, $(Z^i_k)_{k \in \mathbb{N}}$, $i \in [b]$ are sequence of i.i.d Gaussian random variables which might be correlated across the agents and the central server. More precisely, given independent sequences, $(Z^i_k)_{k \in \mathbb{N}^*}$, $i \in [b]$ and $(Z^i_k)_{k \in \mathbb{N}^*}$, of i.i.d. $d$-dimensional standard Gaussian random variables, for $\tau \in [0, 1]$ we set

$$Z^i_k = \sqrt{\tau} \tilde{Z}^i_k + \sqrt{1-\tau} \tilde{Z}^i_k. \quad (4)$$

(2) **A local update.** With probability $p_c \in (0, 1]$, the $i$th client communicates its parameter $X^i_{k+1}$, resulting from the first step, to the central server which in turns broadcasts the average $X^i_{k+1} = b^{-1} \sum_{i \in [b]} X^i_{k+1}$. Finally, each client updates its parameter as $X^i_{k+1} = \tilde{X}^i_{k+1}$. When no communication is performed, each client updates its parameter as $X^i_{k+1} = \tilde{X}^i_{k+1}$.

The local recursions defined by FALD can be written for $i \in [b]$ and $k \geq 0$ as

$$X^i_{k+1} = (1 - B_{k+1}) \tilde{X}^i_{k+1} + (B_{k+1}/b) \sum_{j \in [b]} \tilde{X}^j_{k+1}, \quad (5)$$

where $(B_k)_{k \in \mathbb{N}}$ is a sequence of i.i.d Bernoulli random variables with parameter $p_c$.

For $k \geq 1$, denote by $\mu^{(F)}_k$ the distribution of the average parameter

$$X_k = (1/b) \sum_{i \in [b]} X^i_k. \quad (6)$$

Non-asymptotic Wasserstein bounds between $\mu^{(F)}_k$ and the target distribution $\pi$ are established in Theorem 1 under the following assumptions.

**A1.** For any $i \in [b]$, $U^i$ is continuously differentiable. In addition, there exist $m, L > 0$ such that for any $i \in [b]$, the function $U^i$ is $L$-smooth and $m$-strongly convex, i.e., for any $x, x' \in \mathbb{R}^d$,

$$(m/2) \|x' - x\|^2 \leq U^i(x') - U^i(x) - \langle \nabla U^i(x), x' - x \rangle \leq (L/2) \|x' - x\|^2.$$  

**A2.** For any $i \in [b]$ , $(\{\tilde{U}^i_k\}_{k \in \mathbb{N}})_{i \in [b]}$ are i.i.d. unbiased estimates of $\{\nabla U^i\}_{i \in [b]}$. In addition, there exist $\hat{L} \geq 0$ such that for any $x, x' \in \mathbb{R}^d$ we have

$$\mathbb{E} \left[ \|\tilde{U}^i_k(x') - \tilde{U}^i_k(x)\|^2 \right] \leq \hat{L} \|x' - x\|^2.$$  

In the mini-batch scenario (3), A2 is satisfied if for $i \in [b]$, $j \in [N_i]$ there exists $L^i_j \geq 0$ such that for any $x, x' \in \mathbb{R}^d$, $\|\nabla U^{i,j}(x') - \nabla U^{i,j}(x)\| \leq L^i_j \|x' - x\|$.

Finally, we also consider the following optional smoothness condition on the potentials $\{U^i\}_{i \in [b]}$. This additional assumption, often satisfied in applications have been considered e.g. in Durmus and Moulines (2019); Dalalyan and Karagulyan (2019).

**HX1.** There exists $\hat{L} \geq 0$, such that for any $i \in [b]$, the function $U^i$ is three times continuously differentiable and for any $x, x' \in \mathbb{R}^d$, $\|\nabla^2 U^i(x) - \nabla^2 U^i(x')\| \leq \hat{L} \|x - x'\|$

We introduce some key quantities appearing in the theoretical derivations below. Denote by $x_*$ the minimizer of $\sum_{i \in [b]} U^i$ which exists and is unique under A1. We define

$$V_\pi = \int_{\mathbb{R}^d} \text{Var}\{b^{-1} \sum_{i \in [b]} \tilde{U}^i_k(x)\} \pi(dx),$$

$$V_* = \text{Var}\{b^{-1} \sum_{i \in [b]} \tilde{U}^i_k(x_*)\}. \quad (7)$$
the average of the stochastic gradient variance under the stationary distribution $\pi$ and at the minimum $x_\ast$, respectively. Finally, the statistical heterogeneity between the clients is quantified by (see, e.g., Stich et al. (2018))

$$H = b^{-1} \sum_{i \in [b]} \| \nabla U^i(x_\ast) \|^2 .$$

For ease of presentation, for two sequences $(a_k)_{k \in \mathbb{N}}$ and $(b_k)_{k \in \mathbb{N}}$ we write $a_k \lesssim b_k$ if there exists $C > 0$ only depending on the constants introduced in A1, A2 and HX1 such that $a_k \leq C b_k$, for any $k \in \mathbb{N}$.

**Theorem 1 (Simplified).** Assume A1, A2 and suppose for any $i \in [b]$, $X_0^i = X_0$. Then, there exist $\gamma > 0$, such that for any $\gamma \in (0, \gamma)$, $k \in \mathbb{N}$, $X_0 \sim \mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$, we have

$$W_2^2(\mu_k^{(F)}, \pi) \lesssim (1 - \gamma m/8)^k l(\mu_0) + \frac{\gamma c}{b} J + \gamma \nu_x + \frac{\gamma^2(1 - p_c)(J + p_c \nu_x + \frac{d}{p_c}) + \gamma(1 - \tau)(1 - b^{-1})d}{p_c},$$

where $J = d$, $e = 1$ and $l(\mu_0) < \infty$ is a function of the initial condition $\mu_0$. If HX1 holds, then $e = 2$ and $J = d(1 + d/b)$.

Elements of proof are provided in Section 3; a precise statement is given in Theorem 20 with detailed proofs. Note the step size upper bound $\gamma$ is proportional to $p_c$. In the single user case ($b = p_c = \tau = 1$), we recover up to numerical constants the results stated in Durmus and Moulines (2019); Dalalyan and Karagulyan (2019). Note that, under HX1 the leading term in the step size $\gamma$ is proportional to the stochastic gradient variance $\nu_x$, in accordance with the bounds obtained for SGLD by e.g., Dalalyan and Karagulyan (2019). More discussions on these bounds are postponed after the statement of Theorem 3.

**Lower bounding the effect of heterogeneity.** Similar to FedAvg, the convergence of FALD is impaired by data heterogeneity. Multiple local SGLD steps described in (2) cause $X_k^i$ to target the local posteriors $\pi^i \propto \exp(\nabla U^i)$.

We now provide lower bound on the Wasserstein distance between the distribution of the samples generated by FALD and the target distribution $\pi$ which is proportional to the heterogeneity $\gamma^2 H$.

**Proposition 2.** There exist $\gamma > 0$, potentials $\{U^i\}_{i \in [b]}$ on $\mathbb{R}$ satisfying A1, HX1 and an instance of FALD satisfying A2 such that for any $\gamma \in (0, \gamma)$, we have

$$\liminf_{k \to +\infty} W_2^2(\mu_k^{(F)}, \pi) \gtrsim \gamma^2 H .$$

This proposition extends Karimireddy et al. (2020, Theorem II) to the Bayesian context and underlines the same limitation as FedAvg. To circumvent this, various bias reduction techniques have been suggested in the stochastic optimization literature (Horváth et al., 2022; Gorbunov et al., 2021). In the next section, we adapt similar mechanisms to derive an alternative to FALD satisfying better finite bounds.

**FALD with control variates and bias reduction.** To mitigate the impact of local stochastic gradients, we adapt variance-reduction techniques (Wang et al., 2013; Kovalev et al., 2020) and bias-reduction techniques (Horváth et al., 2022; Gorbunov et al., 2021). This new approach introduces a different recursion rule in step (1) of FALD, while keeping step (2) unchanged. The local update rule is based on a reference point $Y_k \in \mathbb{R}^d$ common to all clients. This common point is updated with probability $q_e \in (0, 1]$ and allows the inclusion of a local shift $C_k$ to recenter the local gradients. This mechanism eliminates the “infamous non-stationarity of the local methods” (paraphrasing Gorbunov et al. (2021)) and therefore avoids extra bias. At each iteration $k$, the first step of the VR-FALD* algorithm is divided into two parts:

(1.1) **Update of the reference parameter and control variate.** The variance reduced gradient requires a sporadic computation of the full local gradient. Let $(B_k^Y)_{k \in \mathbb{N}}$ be a sequence of i.i.d. Bernoulli random variables with parameter $q_e \in (0, 1]$. If $B_k^Y = 1$, then the client reference point $Y_k$ is updated: the clients transmit their local parameter $\{X_k^i\}_{i \in [b]}$ to the central server which computes their average $Y_{k+1} = b^{-1} \sum_{i \in [b]} X_k^i$, which is sent back to the clients. The clients then compute the full gradients $\{\nabla U^i(Y_{k+1})\}_{i \in [b]}$ and transmit them to the central server which updates the shift $C_{k+1} = b^{-1} \sum_{i \in [b]} \nabla U^i(Y_{k+1})$.

To summarize, the reference point and the shift are updated according to

$$Y_{k+1} = (1 - B_k^Y)Y_k + (B_k^Y/b) \sum_{i \in [b]} X_k^i , \quad C_{k+1} = (1 - B_k^Y)C_k + (B_k^Y/b) \sum_{i \in [b]} \nabla U^i(Y_{k+1}) .$$

(1.2) **Local iteration on each client.** This step is similar to FALD, upon replacing the local updates (2) by the variance-reduced version

$$G_{k+1}^i = \nabla U^i_{k+1}(X_k^i) - \nabla U^i(Y_k) + C_k , \quad \tilde{X}_{k+1}^i = X_k^i - \gamma G_{k+1}^i + \sqrt{2\gamma Z_{k+1}^i} .$$

The VR-FALD* analysis relies on the following additional assumption.

**A3.** There exists $\omega \geq 0$ such that for any $i \in [b], k \in \mathbb{N}$ and $x, y \in \mathbb{R}^d$, the following inequality holds

$$\mathbb{E} \left[ \| \nabla U^i(x) - \nabla U^i(y) - \nabla U^i(x) + \nabla U^i(y) \|^2 \right] \leq \omega \| x - y \|^2 .$$

Under A1 and A2, A3 is satisfied with $\omega = 2L^2 + 2\tilde{L}^2$. However, using this result leads to some discrepancy in previous existing analysis, since $\omega = 0$ in the non-stochastic gradient case while $2L^2 + 2\tilde{L}^2 \neq 0$ in general. Finally, in the mini-batch scenario (3), if $\{\nabla U^i \}_{i \in [N]}$ are $L_i$-Lipschitz, then A3 holds with $\omega = \max_{i \in [b]} \{ N_i L_i^2 / n_i \}$; see Remark 15.
For $k \geq 0$, denote by $\mu_k^{(V_{\text{rs}})}$ the distribution of the average $X_k = b^{-1} \sum_{i \in [b]} X_i^k$, where $X_i^k$ is defined as in (5) with $X_i^0$ given in (10). With these notations, we obtain the following theoretical guarantee on VR-FALD*.

**Theorem 3** (Simplified). Assume $A1$, $A2$, $A3$ and suppose for $i \in [b]$, $X_0^i = Y_0 = X_0$. Then, there exist $\tilde{\gamma}_{VRs} > 0$, such that for any $q_c \leq p_c$, $\gamma \in \{0, \tilde{\gamma}_{VRs}\}$, $k \in \mathbb{N}$, $X_0 \sim \mu_0 \in P_2(\mathbb{R}^d)$, we have

$$W_2^2(\mu_k^{(V_{\text{rs}})}, \pi) \leq (1 - \gamma m/8)^k l_{\text{rs}}^*(\mu_0) + \frac{\gamma c}{b} J + \frac{\gamma^2 d}{bq_c^2}$$

$$+ \frac{\gamma(1 - \tau)(1 - b^{-1} d)}{p_c} + \frac{\gamma^2(1 - p_c)}{p_c^2} \left( \gamma V_s + \frac{d}{b} \right),$$

where $J = d$, $e = 1$, $V_s$ is defined in (7), $l_{\text{rs}}^*(\mu_0) < \infty$ is a function of the initial condition $\mu_0$. If $HX1$ holds, then $e = 2$ and $J = d(1 + d/b)$.

The proof is postponed to Section 7.2. Compared to Theorem 1, the client-drift term does not any longer appear, highlighting the advantage of VR-FALD* in dealing with data heterogeneity between agents.

Further, the variance of the stochastic gradients of VR-FALD* only appear in the factor $\gamma \omega$. This result agrees with Chatterji et al. (2018) for SVRG-LD, which might be seen as a particular instance of VR-FALD* with $b = 1$, $p_c = 1$. Nevertheless, a close inspection of the proof in Chatterji et al. (2018) reveals a gap—see Remark 31, which is corrected in the proof of Theorem 30.

**Complexity and Communication costs.** Now discuss the complexity and communication costs of FALD and VR-FALD*. We study two extreme cases: (A) the local computation cost is negligible and only the communication cost matters, which is typical in cross-device applications. (B) the communication cost is negligible and only the local computation cost (complexity) matters. More general scenarios are discussed in the supplement Section 9. In this discussion, it is assumed that $HX1$ is satisfied and $\tau = 1$. In both cases, for a target precision $\epsilon > 0$, we optimize the hyperparameters (number of iterations $K_\epsilon$, learning rate $\gamma_\epsilon$, probability of communication $p_{c, \epsilon}$) to ensure $W_2(\mu_k^{(V_{\text{rs}})}, \pi) \leq \epsilon$ (FALD) or $W_2(\mu_k^{(V_{\text{rs}})}, \pi) \leq \epsilon$ (VR-FALD*). The values of the parameters $d, m, \omega, H, J, V_s$ and $V_s$ are reported in Table 5.

(Scenario A) The objective is to minimize the number of communications $p_{c, \epsilon}K_\epsilon$. As $\gamma$ can be arbitrarily small, we set $K_\epsilon = \gamma^{-1} \lambda_\epsilon$, $p_{c, \epsilon} = \rho_\epsilon \gamma$, where $\lambda_\epsilon, \rho_\epsilon > 0$. Hence, the optimization problem becomes $\min_{\{\lambda_\epsilon, \rho_\epsilon\}} \exp(-\lambda_\epsilon m/8) + \rho_\epsilon^2 H + d/b) \leq c^2$. As $\epsilon \downarrow 0^+$, the minimum number of communications $p_{c, \epsilon}K_\epsilon$ scales as $O(\epsilon^{-1} \sqrt{H + b^{-1} d})$ for FALD and $O(\epsilon^{-1} \sqrt{b^{-1} d})$ for VR-FALD*.

(Scenario B) We take $p_{c, \epsilon} = 1$ and seek to minimize the total number of iterations $K_\epsilon$. As $\epsilon \downarrow 0^+$, $K_\epsilon$ scales as $O(1/\epsilon c \sqrt{b^{-1} J})$ for FALD and $O(1/\epsilon) (\text{Scenario A})$ Numerical results optimizing $p_{c, \epsilon}K_\epsilon$.

$O(\epsilon^{-1} \sqrt{b^{-1} J + b^{-1} \omega d})$ for VR-FALD*.

In Figures 1a-1b, we display the optimal number of communications $p_{c, \epsilon}K_\epsilon$ as a function of $\epsilon$ (left panels Figures 1a-1b). We also exhibit the physical time which corresponds to the time of the Langevin diffusion. The total physical times $- \lambda_\epsilon$ for (A) and $\gamma_\epsilon K_\epsilon$ for (B) are displayed in the middle panels Figures 1a-1b. Finally, the right panels Figures 1a-1b represent the average physical time between two consecutive communications $- \rho_\epsilon^{-1}$ for (A) and $\gamma_\epsilon/p_{c, \epsilon}$ for (B). Note that, the total physical time is (almost) the same for FALD, VR-FALD*, in scenarios (A) and (B). VR-FALD* significantly reduces the number of communications $p_{c, \epsilon}K_\epsilon$ in scenario (A) (top panel) and number of rounds $K_\epsilon$ (bottom panel) w.r.t. FALD.

Figures 1a-1b also illustrate that the “embarrassingly parallel” approach of (Neiswanger et al., 2014) is far from optimal. Indeed, our results show the importance of making multiple interactions (rather than a single consensus step) and using correlated noises between clients. In scenario (A), the optimal number of communications scales inversely proportional to $1/\epsilon$ which improve the bounds $O(1/\epsilon^2)$ derived in Deng et al. (2021, Section 5.3.1). For scenario (B), FALD has the same complexity as QLSD Vono et al. (2022b) under similar assumptions; see also Sun et al. (2022). VR-FALD* has the lowest complexity $(O(1/\epsilon))$ among the Bayesian Federated algorithms reported earlier. This bound matches the one obtained by Chatterji et al. (2018) for the fully centralized SVRG-LD (corresponding to $b = 1$).

### 3 Proofs outline

We briefly outline the main steps of the proof of Theorems 1 and 3. Details of the proofs can be found in the supple-
mentary paper, where we analyze the two algorithms under a common unifying framework. For both algorithms, the local parameters $(X_k^l)_{i \in [b]}$, $k \geq 0$, are given by (5), where $(\tilde{X}_k^l)_{i \in [b]}$ stands for local iterations, which are given in (2) for FALD and (9) for VR-FALD*.

Thus, we bound the Wasserstein distance between the target distribution $\pi$ and the distribution of $X_k = b^{−1} \sum_{i \in [b]} X^l_k$ which is denoted by $(\mu_k^γ)_{k \in \mathbb{N}}$. The Wasserstein distance is defined as the infimum over the coupling. We use below the synchronous coupling construction used in (Durmus and Moulines, 2019; Dalalyan and Karagulyan, 2019) for the analysis of Stochastic Gradient Langevin algorithms.

**Synchronous coupling.** We first construct a Brownian motion $(W_t)_{t \geq 0}$ by $W_t = \sqrt{t} W_t + \sqrt{(1−τ)/b} \sum_{i \in [b]} W_i^l$, starting from $b + 1$ independent $d$-dimensional standard Brownian motions $(W^l_t)_{t \geq 0}$, $i \in [b]$, and $(W_i)_i \geq 0$. Second, we define the following standard Gaussian random variables $Z^l_{k+1} = γ^{−1/2}(W_{(k+1)γ} − W_k^l)$, $k \in \mathbb{N}$, and $Z_k^l$ as in (4). For $k \in \mathbb{N}$, it holds that $\gamma \sum_{i \in [b]} Z^l_{k+1} = \sqrt{b}(W_{(k+1)γ} − W_k)$. Finally, we consider $(X_t)_{t \geq 0}$ the strong solution of the Langevin diffusion associated with $\pi$ and starting from $X_0 \sim \pi$ (see (1)) and driven by $(W_t)_{t \geq 0}$:

$$dX_t = −(1/b) \sum_{i \in [b]} \nabla U_i(X_t) dt + \sqrt{2/b} dW_t. \quad (11)$$

Under A1 and A2, $\pi$ is the unique stationary distribution for the Langevin diffusion, hence the distribution of $X_t$ is $\pi$ for all $t \geq 0$; see e.g. Roberts and Tweedie (1996). Hence, $(X_k, X_kγ)$ defines a coupling between $\mu_k^γ$ and $\pi$, thus for any $k \in \mathbb{N}$ we get

$$W_k^2(\mu_k^γ, \pi) \leq \mathbb{E}[\|X_k - X_kγ\|^2].$$

The rest of the proof then consists in bounding the right-hand side. It is worth noting that in contrast to most analysis on Langevin dynamics, we consider a Langevin diffusion (11) we scale the gradient term by $b^{−1}$ and the Brownian motion by $b^{−1/2}$. This scaling is adapted to the averaging procedure defining $(X_k)_{k \in \mathbb{N}}$.

**Decomposition of $\mathbb{E}[\|X_k - X_kγ\|^2]$.** Denote by $F_k$ the filtration generated by $X_0$, $(W_t)_{t \leq k}$, and $\{(X^l_i)_{i \leq k}\}_k$. Using the definition (6) of $(X_k)_{k \in \mathbb{N}}$ combined with A1, we show in Proposition 4 that for any $γ \geq 1$

$$\mathbb{E}^{F_k}[\|X_{(k+1)γ} - X_{k+1}\|^2] \lesssim (1−γ/m/2)\|X_{kγ} − X_k\|^2 + E_k + γ^2 S_k + V_k, \quad (12)$$

where $V_k = b^{−1} \sum_{i \in [b]} \|X^l_k - X_k\|^2$ and

$$S_k = \text{Var}^{F_k}(b^{−1} \sum_{i \in [b]} G_i^l), \quad E_k = γ^{−1}\mathbb{E}^{F_k}[\|I_k\|^2] + \mathbb{E}^{F_k}[\|I_k\|^2],$$

with $I_k = b^{−1} \sum_{i \in [b]} s_{kγ}^{(k+1)γ} \left(\nabla U_i(x) - \nabla U_i(x_kγ)\right) ds$.

**Bounding $E_k$.** The term $E_k$ accounts for the difference between the diffusion and its discretization; the bound is the same for FALD and VR-FALD*. By adapting Durmus and Moulines (2019, Lemma 21), we establish in Lemma 7 that

$$\mathbb{E}[E_k] \lesssim γ^2 d/b. \quad (13)$$

Under HX1, for $γ \lesssim 1$ the bound can be sharpened in

$$\mathbb{E}[E_k] \lesssim (γ^3d/b)(1 + d/b). \quad (14)$$

The right-hand side of (13) has a higher order with respect to the step size $γ$ in comparison to (14). This step is the reason why we consider the more restrictive assumption HX1, which leads to different guarantees depending on whether this condition is met or not.

**Bounding $S_k$.** $S_k$ is the conditional variance of the stochastic gradient. This is the main difference between the two algorithms. For FALD, we show in Lemma 19 that

$$\mathbb{E}[S_k] \lesssim \mathbb{E}[\|X_k - X_kγ\|^2] + γωd/bq_{c} + \omega \mathbb{E}[V_k] + \frac{γωd}{bq_{c}} \quad (15)$$

On the other hand, under A3, we establish in Lemma 27 that for VR-FALD*, it holds that

$$\mathbb{E}[S_k] \lesssim ω\mathbb{E}[\|X_k - X_kγ\|^2] + ω\mathbb{E}[V_k] + \frac{γωd}{bq_{c}} + \omega q_{c} \sum_{l=0}^{k-1} (1−q_{c})^{k-l-1} \mathbb{E}[\|X_{lγ} − X_{l}\|^2].$$

Compared to the inequality (15), which holds for FALD, the variance term $V_k$ for VR-FALD* is replaced by $γωd/bq_{c}$, which can be made arbitrarily small with $γ → 0$. Note that this term is inversely proportional to the update probability $q_{c}$ of the control variate. Interestingly, the term $S_k$ vanishes when $ω = 0$, i.e., when each client uses its full local gradient at each iteration.

**Bounding $V_k$.** We show in Lemma 18 (FALD) and Lemma 26 (VR-FALD*), there exist $a_0, a_1 \geq 0$ satisfying

$$\mathbb{E}[V_k] \leq (1−γ/m/8)^k a_0 + a_1. \quad (16)$$

To establish this result, we consider the sequence $(f_k)_{k \in \mathbb{N}}$ with general term given by

$$f_k = V_k + α_d d_k^2 + α_{σ} σ_k^2,$$

where $α_d, α_σ \geq 0$ are given in (97); $d_k = \|X_k − x_k\|$ denotes the distance between the average parameter $X_k$ and the minimizer $x_k$ of the global potential $U$: $σ_k = 0$ for FALD and $σ_k^2 = b^{−1} \sum_{i \in [b]} [\nabla U^l_k(x_kγ) − \nabla U^l_k(x_kγ)]^2$ for VR-FALD* with $Y_k$ defined in (8). The weights $α_d, α_{σ}$ are tailored to prove a contraction; more precisely, we show the existence of $a_2 > 0$ whose expression is given in Lemma 12, such that

$$f_{k+1} \leq (1−γ/m/4) f_k + γ^2 a_2 + 2γd(1−γ)(1−b^{−1}).$$
An immediate induction combines with $V_k \leq f_k$ yields a first bound for $E[V_k]$ of the form (16) with $a_1$ of order $\gamma$. In a final step Lemma 10, we refine this bound to obtain a term $a_1$ of order $\gamma^2$.

Gathering all the bounds. The proof is concluded by plugging the upper bounds derived for $E_k$, $S_k$, $V_k$ into (12).

4 Numerical experiments

To illustrate our findings, we perform three numerical experiments on both synthetic toy-examples and real datasets. We compare FALD, VR-FALD* with Bayesian federated learning benchmarks: DG-LMC (Plassier et al., 2021), the Federated Stochastic Langevin Dynamics FSGLD (El Mekkaoui et al., 2021), the Quantized Langevin Stochastic Dynamic QLSD and its variance-reduced version QLS-DPP (Vono et al., 2022b). We also include in our benchmark state of the art (centralized MCMC) algorithms: HMC (Brooks et al., 2011), the Stochastic Gradient Langevin (El Mekkaoui et al., 2021), the Quantized Langevin Stochastic Variance {\textit{⋆}}predictive distribution obtained for FALD is isotropic Gaussian prior with a mean of zero and variance $\sigma^2$. For different values of the hyperparameters $(p_c, \gamma, \tau)$, we run 100 chains with $k_1 = 10^5$ iterations $(X_k)_{k=1}^{k_1}$ and discard 10% of the samples (more details are reported in Section 10.1). For each chain, we estimate the posterior variance $\sigma^2 = \int \|x - x_x\|^2d\pi(x)$ using FALD and VR-FALD*, where $\pi \propto \exp(-\sum_{i \in \{b\}} U^i)$ and $x_x = \arg \max_{x \in \mathbb{R}^d} \pi(x)$. We compute a Monte-Carlo estimates (over $10^2$ independent replications) of the Mean Squared Error (MSE) given by \((k_1 - k_0)^{-1} \sum_{k=k_0+1}^{k_1} \|X_k - x_x\|^2 - \sigma^2)^2\) where $k_1$ is the total number of samples and $k_0$ is the burn-in period. The values of the hyperparameters are reported in Section 10.1. From Table 1, VR-FALD* always outperforms FALD for any choices of $p_c, \gamma$. This illustrates the impact of the heterogeneity and supports the theoretical findings given in Theorems 1 and 3. Furthermore, the asymptotic bias for VR-FALD* improves when $\tau = 1$ as derived in the theoretical analysis.

Bayesian Logistic Regression. We assess the performance of FALD and VR-FALD* using calibration metrics—the expected calibration error (ECE), the Brier score (BS), and the negative log likelihood (nNLL); see Guo et al. (2017)— and predictive accuracy. We consider Bayesian logistic regression applied to the Titanic dataset, which consists of $p = 2$ classes with $N = 2201$ samples in dimension $d = 4$. This dataset is allocated between $b = 10$ clients in a very heterogeneous manner, as displayed in Figure 3. We use an isotropic Gaussian prior with a mean of zero and variance 1. We also report the total variation distance between the predictive distribution obtained for FALD and VR-FALD* to the predictive distribution approximated by 100 long runs of Langevin Stochastic Dynamics (LSD). These metrics are evaluated on a test data sets of 441 samples, and the mean and standard deviation are reported in Table 2. Moreover, we illustrate the quality improvement of VR-FALD* over FALD in Figure 4. We compared the Wasserstein distance using POT (Flamary et al., 2021) between the empirical distributions generated by FALD, VR-FALD* to the estimated target distribution. Based on the same samples, we compute the relative highest posterior density (HPD) error; see Section 10.2 for details.
Bayesian Neural Network: MNIST. To illustrate the behavior of FALD and VR-FALD* in a non-convex setting, we perform Bayesian Neural Network (BNN) inference on the MNIST dataset (Deng, 2012). To this end, we distribute the dataset to \( b = 20 \) clients as follows: 80% of the data labeled \( y \in \{0, \ldots, 9\} \) are equally allocated to clients \( i = y + 1 \) and \( i = y + 10 \); the remaining data are evenly distributed among the \( b \) clients. The likelihood of the observations is computed using LeNet5 neural network (LeCun et al., 1998) with an isotropic Gaussian prior. Finally, we implement FALD and its variants with \( p_c = 1/b \) and \( q_c = \frac{N_b}{N_d} \), where \( N_b \) is the batch size used in the experiments and \( N_d \) is the total number of data. All standard deviations and the values of the other parameters are reported in Section 10.3.

In Table 3 we can observe that the best results are obtained by VR-FALD*: it achieves similar performance to the (fully centralized) SGLD and pSGLD. Alleviating client drift using control variates is still effective even in the highly non-convex BNN setting.

Bayesian Neural Network: CIFAR10. We consider the CIFAR10 dataset (Krizhevsky, 2009) and the ResNet-20 model (He et al., 2016). We split the data across 20 clients, similar to the previous example. Denote by \( Y = \{y_1, \ldots, y_{10}\} \) the set of labels. Then 80% of the data associated with a label \( y_j \in Y, j \in [10] \), is distributed among clients \( j \) and \( j + 10 \), while the rest of the data is evenly distributed among clients. We assess the performance of FALD and VR-FALD* against HMC, Deep Ensemble, and SGLD. We follow Izmailov et al. (2021) by computing the accuracy, agreement, and total deviation distance between the predictive distribution. All of these quantities are defined in the Appendix; see Section 10.4. We also report the calibration results and all resulting scores in Table 7; the results for HMC and SGLD are from Izmailov et al. (2021, Table 6). Details on the implementation and choice of hyperparameters can be found in Section 10.4. We can see that VR-FALD* gives very similar results to SGLD and performs favorably in terms of agreement. Finally, FALD and VR-FALD* outperform Deep Ensembles.

5 Conclusion

In this work, we propose VR-FALD* which extends the FALD Deng et al. (2021) algorithm by introducing control variates to mitigate client drift and reducing stochastic gradient variance. We develop a unifying framework for Bayesian FL combining ideas from Langevin Monte Carlo and Federated Averaging schemes. The theory covers a wide range of local stochastic gradient algorithms; connections can even be made with the global consensus Monte Carlo method (Rendell et al., 2020; Vono et al., 2022a). Using this theoretical framework, we develop non-asymptotic bounds for the algorithms FALD and VR-FALD*, and discuss the choice of hyperparameters (learning rate, communication probability, control variate update probability) to obtain optimal trade-offs. Our analysis allows to correct some errors in the results obtained previously for FALD. The results we obtain on both toy examples and applications to BNNs clearly show the importance of variance reduction and heterogeneity, even when the potential is non-convex.

| Method       | Accuracy | Agreement | 10×TV | 10×ECE | 10×BS | 10×nNLL |
|--------------|----------|-----------|-------|--------|-------|---------|
| LSD          | 72.4 ± 0.1 | 99.9 ± 0.1 | 5.53 ± 2.00 | 1.20 ± 0.01 | 3.44 ± 0.00 | 5.30 ± 0.00 |
| FALD         | 77.0 ± 0.8 | 91.3 ± 0.9 | 533.32 ± 8.13 | 1.05 ± 0.09 | 3.37 ± 0.01 | 5.19 ± 0.00 |
| VR-FALD*     | 74.9 ± 0.1 | 93.6 ± 0.1 | 287.81 ± 2.04 | 1.00 ± 0.05 | 3.51 ± 0.00 | 5.35 ± 0.00 |

Table 4: Performance of Bayesian FL algo. on CIFAR10.
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Notation and convention. The Euclidean norm and the scalar product on $\mathbb{R}^d$ are denoted by $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ respectively. We set $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ and denote by $N(m, \Sigma)$ the Gaussian distribution with mean vector $m$ and covariance matrix $\Sigma$. Finally, for any $f : \mathbb{R}^d \to \mathbb{R}$ twice continuously differentiable, we define the Laplacian $\Delta f$, which for all $x \in \mathbb{R}^d$ is given by $\Delta f(x) = \left\{ \sum_{j=1}^d \left( \frac{\partial^2 f_j}{\partial x_j^2} \right) x_j^2 \right\}_{j=1}^d$.

Theoretical road map. The derivations leading to Theorem 1 and Theorem 3 are split in two sections:

- Section 6 consists of general results under mild assumptions. In this section, we derive an upper bound on $V_k$ – see Section 6.3, and provide a Wasserstein upper bound holding for general federated averaging Langevin schemes in Theorem 8.

- Section 7 is subdivided between the results on FALD (Section 7.1) and VR-FALD* (Section 7.2). In each subsection, we prove intermediate results showing that results of Section 6.3 hold, and finally we apply Theorem 8 to derive the final theoretical guarantees on FALD and VR-FALD*.
6 General scheme and technical results

Problem statement. We consider a general recursion that includes both FALD and VR-FALD*. This general scheme is based on i.i.d. random variables \( \{ \xi_k : k \in \mathbb{N} \} \) taking values in a measurable space \((\mathcal{E}, \mathcal{C})\) and whose joint distribution is denoted by \( \nu_\xi \). Moreover, we introduce a family of measurable functions \( \{ \gamma^i : \mathbb{R}^d \times \mathbb{Y}_2 \times \mathbb{C}^2 \times \mathcal{E} \to \mathbb{R}^d, \gamma^j : \mathbb{R}^d \times \mathbb{Y} \times \mathbb{C}^2 \times \mathcal{E} \to \mathbb{C} \}_{i,j=1}^b \), where \((\mathbb{Y}, \mathcal{Y})\) and \((\mathbb{C}, \mathcal{C})\) are measurable spaces. For each \( i = \{0, [b]\} \), the functions \( (\gamma^i, \gamma^j, \gamma^k) \) correspond to the update of the local parameter and control variate by the \( i\)th agent. To define the global control variate update, we consider the function \( \mathcal{F} : \mathbb{Y} \times \mathbb{C}^{b+1} \times (\mathbb{R}^d)^{b+1} \times \mathcal{E} \to \mathbb{Y} \times \mathcal{C} \). Starting from \( \{ G_i^0 \}_{i=1}^b, \{ X_i^0 \}_{i=1}^b \in (\mathbb{R}^d)^b, \{ C_i^0 \}_{i=1}^b \in \mathbb{C}^{b+1}, \{ Y_i^0 \}_{i=1}^b \in \mathbb{Y}^{b+1} \) and set \( X_0 = b^{-1} \sum_{i=1}^b X_i^0 \). For each \( k \in \mathbb{N} \) the random variables are updated according to

\[
G_{k+1}^i = \gamma^i \left( X_k^i, Y_k^i, \xi_k, \tilde{X}_k^i, \tilde{Y}_k^i, \xi_k+1 \right),
\]

\[
\tilde{X}_{k+1}^i = X_k^i - \gamma G_{k+1}^i + \sqrt{2\mathcal{Y}} \left( \sqrt{\tau/b} \tilde{Z}_{k+1} + \sqrt{1-\mathcal{Y}} Z_{k+1}^i \right),
\]

\[
Y_{k+1}^i = \gamma^i \left( X_k^i, Y_k^i, \xi_k+1 \right),
\]

\[
C_{k+1}^i = \gamma^i \left( X_k^i, Y_k^i, C_k^i, \xi_k+1 \right),
\]

\[
X_{k+1}^i = B_{k+1} \sum_{j=1}^b \tilde{X}_j^i + (1 - B_{k+1}) \tilde{X}_{k+1}^i,
\]

\[
(Y_{k+1}, C_{k+1}) = \mathcal{F}(X_{k+1}, C_{k+1}, \{ C_{k+1}^i \}_{i=1}^b, \{ X_{k+1}^i \}_{i=1}^b, \xi_{k+1}),
\]

where \( \tau \in [0, 1]; \mathcal{Y} \in (0, \bar{\mathcal{Y}}) \) is the stepsize; \( \{ B_k, \xi_k, \tilde{Z}_k, Z^i_k, \ldots, Z_b^i_k \} : k \in \mathbb{N}^* \) is a set of independent sequences of i.i.d. random variables such that for any \( k \in \mathbb{N} \), \( B_k \) is a Bernoulli random variable with parameter \( p_k \in (0, 1] \); and \( (\tilde{Z}_k, Z^1_k, \ldots, Z^b_k) \) are \( d \)-dimensional standard Gaussian random variables. Recall that \((\xi_k)_{k \geq 1}\) is a set of i.i.d. random variables distributed according to \( \nu_\xi \) such that \( H1 \) holds to ensure that the combination of functions \( \{ \gamma^i \}_{i \in [b]} \) provides an unbiased estimate of \( \nabla U \).

In iteration \( k \geq 0 \), the local parameter of the \( i\)th client is denoted by \( X_k^i \), and \( G_k^i \) stands for its local gradient. If \( B_k = 1 \) (communication round), the local parameter \( X_k^i \) is set to the value of the global server parameter \( X_k \). If \( B_k = 0 \), \( X_k^i \) is set to the local update \( \tilde{X}_k^i \). Moreover, we write \( Y_k^i \) the reference point used to compute the control variate \( C_k^i \). The first step \((17)\) corresponds to the computation of a stochastic estimate of \( \nabla U^i \) by the \( i\)th client. Then, the client updates the reference point \( Y_k^i \) \((18)\) at which the local control variate is computed. The client also updates its own local control variate \( C_k^i \) \((19)\). If \( B_{k+1} = 1 \), then the server averages the parameter of each client, and broadcasts this average. If \( B_{k+1} = 0 \), then each client keeps \( \tilde{X}_{k+1}^i \) as its new local parameter. Finally, the server updates the reference point \( \tilde{Y}_k \) and the global control variate \( C_k \) according to \( (21) \). Denote the filtration \( \{ \mathcal{F}_k \}_{k \in \mathbb{N}} \) defined for any \( k \geq 0 \), by

\[
\mathcal{F}_k = \sigma \left( X_0, \left( B_i, C_i, Y_i, \tilde{Z}_i, \xi_i, \left( C_i^i, G_i^i, X_i^i, \tilde{X}_i^i, Y_i^i, Z^i_k \right) \right)_{i=1}^{n_k} \right)
\]

and consider the conditional expectation and variance denoted by \( E^{\mathcal{F}_k} \), \( \text{Var}^{\mathcal{F}_k} (\cdot) = E^{\mathcal{F}_k} [\| \cdot - E^{\mathcal{F}_k} [\cdot] \|^2] \) respectively. For \( k \in \mathbb{N} \), we introduce \( X_k \) the average of the local parameters given by

\[
X_k = \frac{1}{b} \sum_{i=1}^b X_k^i
\]

and we set

\[
V_k = \frac{1}{b} \sum_{i=1}^b \| X_k^i - X_k \|^2.
\]

Finally, to control the distance between the average parameter \( X_k \) and the minimizer \( x_\ast = \arg \min U \), we consider the parameter \( d_k \), which for \( k \geq 0 \) is given by

\[
d_k = \| X_k - x_\ast \|.
\]

For each \( k \in \mathbb{N} \) and \( \gamma \in (0, \bar{\mathcal{Y}}) \), we denote by \( \mu_k^{(\gamma)} \) the distribution of \( X_k \) defined by \((23)\). To ensure the quality of the samples generated by Algorithm 1, we control the Wasserstein distance \( W_2(\pi, \mu_k^{(\gamma)}) \). Recall that the Wasserstein distance
Algorithm 1 Stochastic Averaging Langevin Dynamics - FALD and its variants

**Input:** initial vectors \((X^i_0)_{i \in [b]}\), noise parameter \(\tau \in [0, 1]\), number of communication rounds \(K\), probability \(p_c \in (0, 1]\) of communication, probability \(q_c \in [0, 1]\) to update the control variates, and step-size \(\gamma\)

**Initialize:** \(Y_0 = (1/b) \sum_{i=1}^b X^i_0\) and \(C_0 = (1/b) \nabla U(Y_0)\)

**for** \(k = 0\) to \(K - 1\) **do**

- **Draw** \(B_{k+1} \sim B(p_c)\), \(\tilde{Z}_{k+1} \sim N(0, I_d)\) // On every client
- **for** \(i = 1\) to \(b\) **do** // In parallel on the \(b\) clients
  - **Draw** \(\xi^i_{k+1} \sim \nu^i_{k+1}\), \(\tilde{Z}^i_{k+1} \sim N(0, I_d)\)
  - Compute \(G^i_k\) following (17)
  - **Set** \(\tilde{X}^i_{k+1} = X^i_k - \gamma G^i_k + \sqrt{2\gamma}(\sqrt{\tau/b} \tilde{Z}^i_{k+1} + \sqrt{1-\tau} \tilde{Z}^i_{k+1})\)
  - **if** \(B_{k+1} = 1\) **then** // Communication round
    - Broadcast \(\tilde{X}^i_{k+1}\) to the server
  - **else** // Local step
    - **Update** \(X^i_{k+1} \leftarrow \tilde{X}^i_{k+1}\)
    - **else** // Control variate update round
      - Broadcast the necessary information to the server in order to update \((Y^i_k, C^i_k, Y_k, C_k)\)
    - **else** // No update
      - **Set** \((Y^i_{k+1}, C^i_{k+1}, Y_{k+1}, C_{k+1}) \leftarrow (Y^i_k, C^i_k, Y_k, C_k)\)

- **if** \(B_{k+1} = 1\) **then** // During communication round
  - **Update** then broadcast \(X_{k+1} \leftarrow (1/b) \sum_{i=1}^b \tilde{X}^i_{k+1}\) // On the central server
  - **Update** the local parameter \(X_{k+1} \leftarrow X^i_{k+1}\) // On every client

- **if** \(\tilde{B}_{k+1} = 1\) **then** // During control variate update round
  - If needed, update then broadcast \(Y_{k+1} \leftarrow (1/b) \sum_{i=1}^b X^i_k\) // On the central server
  - **Update** \((Y^i_k, C^i_k)\) using the parameters \((X^i_k, Y^i_k, Y_k, Y_{k+1}, C_k)\) // On every client
  - **Update** then broadcast \(C_{k+1} \leftarrow (1/b) \sum_{i=1}^b C^i_{k+1}\) // On the central server

**Output:** samples \(\{X_\ell\}_{\ell \in [K]: B_\ell = 1}\).
is the infimum of $E[||X_{k}\gamma - X_{k}||^2]$ over all couplings $(X_{k}\gamma, X_k)$ such that $X_{k}\gamma$ is distributed according to $\pi$. Thus, to study the convergence of $(\mu_{k}(\gamma))_{k\in\mathbb{N}}$, we introduce a synchronous coupling $(X_{k}\gamma, X_k)_{k\geq 0}$ with values in $(\mathbb{R}^d)^2$ between $\pi$ and $\mu_{k}(\gamma)$, starting from the couple $(X_0, X_0)$ distributed according to $\zeta \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$, i.e., $\zeta((\mathbb{R}^d, \cdot) = \mu_0(\gamma) \in \mathcal{P}_2(\mathbb{R}^d)$ and $\zeta(\cdot, \mathbb{R}^d) = \pi$. Since log $\pi$ is supposed $m$-strongly concave by A1, note that $\pi$ belongs in $\mathcal{P}_2(\mathbb{R}^d)$. Based on independent $d$-dimensional standard Brownian motions $\{\tilde{W}_t, \{W^i_t\}_{i=1}^{b}\}_{t\geq 0}$, we define $W_k = \sqrt{\tau}W_t + \sqrt{(1-\tau)/b}\sum_{i=1}^{b} W^i_t$. For $k \in \mathbb{N}^*$, we introduce $\tilde{Z}_k = \gamma^{-1/2}(W_k - \tilde{W}_{(k-1)\gamma})$, and for $r \in [b]$, we consider $\tilde{Z}^r_k = \gamma^{-1/2}(W^r_k - \tilde{W}^r_{(k-1)\gamma})$. Therefore, for all $k \in \mathbb{N}^*$ we can verify that $W_{k\gamma} - W_{(k-1)\gamma} = \sqrt{\tau}\tilde{Z}_k + \sqrt{(1-\tau)/b}\sum_{i=1}^{b} \tilde{Z}^i_k$. Moreover, consider $(X_t)_{t\geq 0}$ the strong solution of the Langevin stochastic differential equation (SDE) given by

$$dX_t = -\frac{1}{b}\nabla U(X_t)\, dt + \frac{\sqrt{2}}{b}\, dW_t.$$  \hspace{1cm} (26)

The Langevin diffusion defines a Markov semigroup $(\tilde{P}_t)_{t\geq 0}$ satisfying $\pi\tilde{P}_t = \pi$ for any $t \geq 0$, see for example Roberts and Tweedie (1996, Theorem 2.1). Note that $X_t$ and $X_k$ are distributed according to $\pi$ and $\mu_{k}(\gamma)$, respectively. From the definition of the Wasserstein distance of order 2 it follows that

$$W_2(\pi, \mu_{k}(\gamma)) \leq E[||X_{k\gamma} - X_k||^2]^{1/2}.$$  

So the proof consists mainly of upper bounding the squared norm $||X_{k\gamma} - X_k||$, from which we derive an explicit bound on the Wasserstein distance by the previous inequality.

**First upper bound on $E^{F_k}[||X_{(k+1)\gamma} - X_{k+1}||^2]$.** Under mild assumptions, we derive a first bound in Proposition 4 to control $||X_{(k+1)\gamma} - X_{k+1}||^2$ based on $||X_{k\gamma} - X_k||^2$, $(1/b)\sum_{i=1}^{b} G_i^k$ and $V_k$. This decomposition highlights the different approximations brought by the discretization of the Langevin diffusion (26) between the averaged parameter $(X_k)_{k\in\mathbb{N}}$ defined in (23) and $\{X_{k\gamma}\}_{k\in\mathbb{N}}$. Recall that $x_* = \arg \min U$ and for all $k \in \mathbb{N}$, consider $I_k$ the approximation error defined by

$$I_k = \int_{k\gamma}^{(k+1)\gamma} (\nabla U(x_*) - \nabla U(X_k))\, ds.$$  \hspace{1cm} (27)

For $\gamma > 0$ small enough and $k \in \mathbb{N}$, for all $\gamma \in (0, \gamma]$ and under the following assumption H1 we control the distance between the target distribution $\pi$ and $\mu_{k}(\gamma)$.

**H1.** For any $\{(x^i, y^i, c^i)\}_{i=1}^{b} \in \mathbb{R}^{3d}$, we have

$$\sum_{i=1}^{b} \int_{\mathcal{E}} q^{i}\left(\{(x^i, y^i, c^i)\}_{j=1}^{b}, \xi^i\right)\, d\nu_\gamma(\xi^i) = \sum_{i=1}^{b} \nabla U^i(x^i).$$

**Proposition 4.** Assume A1, H1 hold and let $\gamma \leq 2(3L)^{-1}$. Then, for any $k \in \mathbb{N}$, we have

$$E^{F_k}[||X_{(k+1)\gamma} - X_{k+1}||^2] \leq \left[1 - \gamma m(1 - 3\gamma L)\right]||X_{k\gamma} - X_k||^2 + \gamma \left(\frac{2L^2}{m} + 3\gamma L^2\right) V_k + \left(\frac{2}{\gamma m}\right) E^{F_k}[I_k]^2 + 3E^{F_k}[||I_k||^2] + \var\gamma^2 Var^{F_k}\left(\frac{1}{b}\sum_{i=1}^{b} G_i^k\right),$$

where $V_k, F_k, d_k$ are defined in (24), (22) and (25).

**Proof.** Let $k$ be in $\mathbb{N}$ and $\gamma$ in $(0, (3L)^{-1}]$. Recall the stochastic processes $X_{k+1}, X_{(k+1)\gamma}$ are defined in (23) and (26) by

$$\left\{\begin{array}{l}
X_{(k+1)\gamma} = X_{k\gamma} - \gamma \nabla U(X_{k\gamma}) - I_k + \sqrt{2/b}\left(W_{(k+1)\gamma} - W_{k\gamma}\right), \\
X_{k+1} = \frac{1}{b}\sum_{i=1}^{b} \left[X_{k} - \gamma G_i^k + \sqrt{2/b}\tilde{Z}_{k+1} + \sqrt{(1-\gamma)}\tilde{Z}_{k+1}^i\right],
\end{array}\right.$$  

with $I_k$ defined in (27). Substracting the two above equations gives

$$X_{(k+1)\gamma} - X_{k+1} = (X_{k\gamma} - X_k) - \left(\int_{k\gamma}^{(k+1)\gamma} \nabla U(X_s)\, ds - \frac{\gamma}{b}\sum_{i=1}^{b} G_i^k\right).$$
Taking the conditional expectation of the above equation and developing the squared norm, we obtain
\[ \mathbb{E}^{F_k} \left[ \|X_{k+1} - X_{k+1}\|^2 \right] = \mathbb{E}^{F_k} \left[ \|X_{k\gamma} - X_{k}\|^2 \right] - 2\gamma \langle X_{k\gamma} - X_{k}, \nabla \bar{U}(X_{k\gamma}) - \nabla \bar{U}(X_k) \rangle \\
- 2 \left\langle X_{k\gamma} - X_k, \mathbb{E}^{F_k} [I_k] + \gamma \nabla \bar{U}(X_k) - \gamma b \sum_{i=1}^{b} \mathbb{E}^{F_k} [G^i_k] \right\rangle + \mathbb{E}^{F_k} \left[ \left\| I_k + \gamma \nabla \bar{U}(X_{k\gamma}) - \gamma \frac{b}{b} \sum_{i=1}^{b} G^i_k \right\|^2 \right]. \tag{28} \]

Using that for all \( \alpha > 0 \), \( (a, b) \in \mathbb{R}^2 \), \( 2 \langle a, b \rangle \leq \alpha \|a\|^2 + (1/\alpha) \|b\|^2 \) combined with \( \text{H1} \), for any \( \epsilon > 0 \) we have
\[ - 2 \left\langle X_{k\gamma} - X_k, \mathbb{E}^{F_k} [I_k] + \gamma \nabla \bar{U}(X_k) - \gamma b \sum_{i=1}^{b} \mathbb{E}^{F_k} [G^i_k] \right\rangle \leq \epsilon \|X_{k\gamma} - X_k\|^2 + \frac{2}{\epsilon} \|\mathbb{E}^{F_k} [I_k]\|^2 \\
+ \frac{2\gamma^2}{\epsilon} \left\| \nabla \bar{U}(X_k) - \frac{1}{b} \sum_{i=1}^{b} \nabla U_i(X_k) \right\|^2. \tag{29} \]

In addition, the unbiased property \( \text{H1} \) implies that
\[ \mathbb{E}^{F_k} \left[ \left\| I_k + \gamma \nabla \bar{U}(X_{k\gamma}) - \gamma \frac{b}{b} \sum_{i=1}^{b} G^i_k \right\|^2 \right] = \gamma^2 \text{Var}^{F_k} \left( \frac{1}{b} \sum_{i=1}^{b} G^i_k \right) \\
\quad + \mathbb{E}^{F_k} \left[ \| \nabla \bar{U}(X_{k\gamma}) - \nabla \bar{U}(X_k) \| + I_k + \gamma \nabla \bar{U}(X_k) - \gamma \frac{b}{b} \sum_{i=1}^{b} \nabla U_i(X_k) \| \right]^2. \tag{30} \]

The Young inequality shows that
\[ \mathbb{E}^{F_k} \left[ \| \nabla \bar{U}(X_{k\gamma}) - \nabla \bar{U}(X_k) \|^2 + 3\text{Var}^{F_k} [I_k]^2 \right] + 3\gamma^2 \left\| \nabla \bar{U}(X_k) - \frac{1}{b} \sum_{i=1}^{b} \nabla U_i(X_k) \right\|^2. \]

By \( \text{A1} \) we know that \( \bar{U} \) is \( L \)-smooth and convex which imply the co-coercivity of \( \bar{U} \) (Nesterov, 2003, Theorem 2.1.5), that is for all \( x, y \in \mathbb{R}^d \), \( \| \nabla \bar{U}(y) - \nabla \bar{U}(x) \|^2 \leq L \langle \nabla \bar{U}(y) - \nabla \bar{U}(x), y - x \rangle \). Hence, we deduce that
\[ \| \nabla \bar{U}(X_{k\gamma}) - \nabla \bar{U}(X_k) \|^2 \leq L \langle X_{k\gamma} - X_k, \nabla \bar{U}(X_{k\gamma}) - \nabla \bar{U}(X_k) \rangle. \tag{31} \]

Setting \( \epsilon = \gamma m \), we have \( 0 < \epsilon \leq 1 \) and \( 1 + 1/\epsilon = 2(\gamma m)^{-1} \). Therefore, (29), (30) and (31) associated with (28) show that
\[ \mathbb{E}^{F_k} \left[ \|X_{k+1} - X_{k+1}\|^2 \right] \leq (1 + \gamma m) \|X_{k\gamma} - X_k\|^2 + \left( \frac{2}{\gamma m} \|\mathbb{E}^{F_k} [I_k]\|^2 + 3\text{Var}^{F_k} [I_k]^2 \right) \\
- \gamma \langle X_{k\gamma} - X_k, \nabla \bar{U}(X_{k\gamma}) - \nabla \bar{U}(X_k) \rangle \\
+ \gamma^2 \left( 3 + \frac{2}{\gamma m} \right) \left\| \nabla \bar{U}(X_k) - \frac{1}{b} \sum_{i=1}^{b} \nabla U_i(X_k) \right\|^2 + \gamma^2 \text{Var}^{F_k} \left( \frac{1}{b} \sum_{i=1}^{b} G^i_k \right). \tag{32} \]

For any \( i \in [b] \), by \( \text{A1} \), the \( m \)-convexity of \( \bar{U} \) gives that
\[ \langle X_{k\gamma} - X_k, \nabla \bar{U}(X_{k\gamma}) - \nabla \bar{U}(X_k) \rangle \geq m \|X_{k\gamma} - X_k\|^2 \tag{33} \]

In addition, under \( \text{A1} \) the Jensen inequality implies
\[ \left\| \nabla \bar{U}(X_k) - \frac{1}{b} \sum_{i=1}^{b} \nabla U_i(X_k) \right\|^2 \leq L^2 V_k, \tag{34} \]

where \( V_k \) is defined in (24). Therefore, using the assumption on \( \gamma \) and plugging (33) and (34) in (32) yields the expected inequality. \( \square \)
6.1 General supporting lemmas

In this subsection, we consider the stochastic processes \((X_k)_{k \in \mathbb{N}}, (X_{k\gamma})_{k \in \mathbb{N}}\) defined in (23) and (26). We derive several lemmas which allow us to derive a recursion on \(\mathbb{E}[\|X_{k\gamma} - X_k\|^2]\).

**Lemma 5.** Assume A1 holds. Then, for any \(k \in \mathbb{N}\) and \(\gamma > 0\) we have

\[
\mathbb{E}[\|I_k\|^2] \leq \frac{d^3\gamma L^2}{b} \left(1 + \frac{\gamma L^2}{2m} + \frac{\gamma^2 L^2}{12}\right).
\]

**Proof.** Let \(k\) be in \(\mathbb{N}\). Using the Jensen inequality, we have

\[
\mathbb{E}[\|I_k\|^2] = \mathbb{E} \left[\left\|\int_{k\gamma}^{(k+1)\gamma} (\nabla U(X_s) - \nabla U(X_{k\gamma})) ds\right\|^2\right]
\leq \gamma \int_{k\gamma}^{(k+1)\gamma} \mathbb{E}[\|\nabla U(X_s) - \nabla U(X_{k\gamma})\|^2] ds
\leq L^2 \gamma \int_{k\gamma}^{(k+1)\gamma} \mathbb{E}[\|X_s - X_{k\gamma}\|^2] ds.
\]

Further, for any \(s \in \mathbb{R}_+\), using Durmus and Moulines (2019, Lemma 21) applied to \((X_{bt})_{t \in \mathbb{R}_+}\) we obtain

\[
\mathbb{E}_{\mathbb{P}^{X_{bt}}} [\|X_s - X_{k\gamma}\|^2] \leq \frac{d(s - k\gamma)}{b} \left(2 + (s - k\gamma)^2 \frac{L^2}{3}\right) + \frac{3}{2} (s - k\gamma)^2 L^2 \|X_{k\gamma} - x_*\|^2.
\]

Integrating the previous inequality on \([k\gamma, (k+1)\gamma]\), it implies

\[
\int_{k\gamma}^{(k+1)\gamma} \mathbb{E}[\|X_s - X_{k\gamma}\|^2] ds \leq \frac{\gamma^2}{b} \left(d + \frac{b L^2 \gamma}{2} \mathbb{E}[\|X_{k\gamma} - x_*\|^2] + \frac{d L^2 \gamma^2}{12}\right).
\]

Plugging (36) in (35) gives

\[
\mathbb{E}[\|I_k\|^2] \leq \frac{L^2 \gamma^3}{b} \left(d + \frac{b L^2 \gamma}{2} \mathbb{E}[\|X_{k\gamma} - x_*\|^2] + \frac{d L^2 \gamma^2}{12}\right).
\]

Applying Durmus and Moulines (2019, Proposition 1) to \((X_{bt})_{t \in \mathbb{R}_+}\), we get

\[
\mathbb{E}[\|X_{k\gamma} - x_*\|^2] \leq \frac{d}{b m}.
\]

Thus, combining (37) with (38) completes the proof.

**Lemma 6.** Assume A1 and HX1 hold. Then, for any \(k \in \mathbb{N}\) and \(\gamma > 0\) we have

\[
\mathbb{E} \left[\|\mathbb{E}^{X_k}[I_k]\|^2\right] \leq \frac{2 \gamma^4 d}{3b} \left(L^3 + \frac{d L^2}{b}\right),
\]

where \(I_k\) is defined in (27).

**Proof.** Denote \(\Delta\) the Laplacian defined, for all \(x \in \mathbb{R}^d\), by \(\Delta U(x) = \{\sum_{j=1}^d (\partial^2 U_j)(x)/\partial x^2\}_{j=1}^d\), moreover let \(k \in \mathbb{N}\) be a fixed integer and \(\gamma > 0\). Using the Itô formula, we have for \(s \in [k\gamma, (k+1)\gamma]\)

\[
\nabla U(X_s) - \nabla U(X_{k\gamma}) = \int_{k\gamma}^{s} \frac{1}{b} \Delta(\nabla U)(X_u) - \nabla^2 U(X_u) \nabla U(X_u) du + \sqrt{\frac{\gamma}{b}} \int_{k\gamma}^{s} \nabla^2 \tilde{U}(X_u) dB_u.
\]

We will upper bound separately the three terms of the previous equality. First, the \(L\)-Lipschitz property of \(\nabla \tilde{U}\) given by A1 implies for any \(u \in \mathbb{R}_+\) that

\[
\|\nabla^2 \tilde{U}(X_u)\| \leq L \|\nabla \tilde{U}(X_u) - \nabla \tilde{U}(x_*)\|.
\]
In addition, since for \( u \in \mathbb{R}_+ \), the random variable \( X_u \) is distributed according to the stationary distribution \( \pi \propto \exp(-U) \), we know from Dalalyan (2017, Lemma 2) that

\[
\mathbb{E} \left[ \| \nabla \bar{U}(X_u) - \nabla \bar{U}(x_*) \|^2 \right] \leq \frac{dL}{b}.
\]  

(41)

Therefore, we deduce from (40) and (41) the following bound

\[
\mathbb{E} \left[ \| \nabla^2 \bar{U}(X_u) \nabla \bar{U}(X_u) \|^2 \right] \leq \frac{dL^3}{b}.
\]  

(42)

Denote \( (e_i)_{i=1}^d \) the canonical basis of \( \mathbb{R}^d \); using that \( U \) is three times continuously differentiable we can apply the Schwarz’s theorem which combined with HX1, immediately yield that

\[
\| \Delta(\nabla \bar{U})(x) \|^2 = \sum_{j=1}^d \left( \sum_{j=1}^d \sum_{i=1}^d \partial_i \partial_j \bar{U}(x) \right)^2 \leq d \sum_{i=1}^d \sum_{j=1}^d |\partial_i \partial_j \bar{U}(x)|^2
\]

\[
= d \sum_{i=1}^d \lim_{\epsilon \to 0} \left\{ e^{-2} \sum_{j=1}^d |\partial_j \bar{U}(x + \epsilon \cdot e_i) - \partial_j \bar{U}(x)|^2 \right\}
\]

\[
\leq d \sum_{i=1}^d \lim_{\epsilon \to 0} \left\{ e^{-2} \left( \bar{L} \| (x + \epsilon \cdot e_i) - x \|^{-1} \right)^2 \right\} \leq \left( dL \right)^2 .
\]  

(43)

Lastly, we upper bound the third term derived in (39). Since the potentials \( \{ U^i \}_{i \in [b]} \) are supposed \( L \)-smooth and \( \bar{U} \) twice continuously differentiable, for \( s \in [k\gamma, (k+1)\gamma] \) we know that \( \int_{k\gamma}^s \nabla^2 \bar{U}(X_u) dB_u \) is a \( \mathcal{F}_s \)-martingale. Thus, for \( k \geq 0 \) we deduce that

\[
\mathbb{E} \int_{k\gamma}^{(k+1)\gamma} \nabla^2 \bar{U}(X_u) \, du = 0.
\]  

(44)

Eventually, combining (39), (42), (43) and (44) with the Jensen and Young inequalities give

\[
\frac{1}{\gamma} \mathbb{E} \left[ \| \mathbb{E} \mathcal{F}_k [I_k] \|^2 \right] = \frac{1}{\gamma} \mathbb{E} \left[ \left\| \int_{k\gamma}^{(k+1)\gamma} \mathbb{E} \mathcal{F}_k \left[ \nabla \bar{U}(X_s) - \nabla \bar{U}(X_{k\gamma}) \right] \, ds \right\|^2 \right]
\]

\[
\leq \int_{k\gamma}^{(k+1)\gamma} \mathbb{E} \left[ \left\| \int_{k\gamma}^{s} \mathbb{E} \mathcal{F}_k \left[ \nabla \bar{U}(X_s) - \nabla \bar{U}(X_{k\gamma}) \right] \, ds \right\|^2 \right] \, ds
\]

\[
= \int_{k\gamma}^{(k+1)\gamma} \mathbb{E} \left[ \left\| \mathbb{E} \mathcal{F}_k \left[ \int_{k\gamma}^{s} \frac{1}{\gamma} \Delta(\nabla \bar{U})(X_u) \, du \right] \right\|^2 \right] \, ds
\]

\[
\leq 2 \int_{k\gamma}^{(k+1)\gamma} (s - k\gamma) \int_{k\gamma}^{s} \mathbb{E} \left[ \frac{1}{\gamma^2} \left\| \int_{k\gamma}^{s} \Delta(\nabla \bar{U})(X_u) \, du \right\|^2 + \left\| \nabla^2 \bar{U}(X_u) \, du \right\|^2 \right] \, ds
\]

\[
\leq 2 \int_{k\gamma}^{(k+1)\gamma} (s - k\gamma)^2 \left( \frac{dL^3}{b} + \frac{(d\bar{L})^3}{b^2} \right) \, ds = \frac{2\gamma^3 dL^3}{3b} \left( L^3 + \frac{d\bar{L}^3}{b} \right).
\]

Multiplying this last inequality by \( \gamma > 0 \) proves the expected result. \( \square \)

**Lemma 7.** Assume \( A1 \) hold. Then, for any \( k \in \mathbb{N} \) and \( \gamma \in (0, (3m)^{-1}] \) we have

\[
\frac{2}{\gamma m} \mathbb{E} \left[ \| \mathbb{E} \mathcal{F}_k [I_k] \|^2 \right] + 3 \mathbb{E} \left[ \| I_k \|^2 \right] \leq \begin{cases} \frac{3^3 dL^2}{bm} \left( 1 + \frac{19\gamma L^2}{36m} \right) & \text{if HX1 holds and } \gamma \leq L^{-1}. \\
\frac{3^3 dL^2}{bm} \left( 5L^3 + \frac{4 d\bar{L}^2}{3b} \right) & \text{if HX1 holds and } \gamma \leq L^{-1}. \end{cases}
\]

**Proof.** Let \( k \in \mathbb{N} \) and \( \gamma \in (0, (3m)^{-1}] \), using Lemma 5 we have

\[
\mathbb{E} \left[ \| I_k \|^2 \right] \leq \frac{\gamma^3 dL^2}{b} \left( 1 + \frac{\gamma L^2}{2m} + \frac{\gamma^2 L^2}{12} \right).
\]
Therefore, we deduce
\[
\frac{2}{\gamma m} E \left[ \left\| \mathbb{E}^F_k [I_k] \right\|^2 \right] + 3 E \left[ \left\| I_k \right\|^2 \right] \leq \frac{3 \gamma^2 d L^2}{b m} \left( 1 + \frac{\gamma L^2}{2m} + \frac{\gamma^2 L^2}{12} \right).
\]

Moreover, if we additionally suppose the regularity of the Hessian of the potentials \((U^i)^b_{i=1}\) as stated in \textbf{HX1}, we sharpen the upper bound on \(E[\left\| \mathbb{E}^F_k [I_k] \right\|^2]\). Indeed, we show in Lemma 6 that
\[
\frac{2}{\gamma m} E \left[ \left\| \mathbb{E}^F_k [I_k] \right\|^2 \right] + 3 E \left[ \left\| I_k \right\|^2 \right] \leq \frac{4 \gamma^3 d}{3bm} \left( L^3 + \frac{d L^2}{b} \right).
\]

Hence, we deduce that
\[
\frac{2}{\gamma m} E \left[ \left\| \mathbb{E}^F_k [I_k] \right\|^2 \right] + 3 E \left[ \left\| I_k \right\|^2 \right] \leq \frac{\gamma^3 d L^3}{bm} \left( 3 + \frac{4}{3} \right) + \frac{4 \gamma^3 d^2 L^2}{3b^2 m}.
\]

\[\square\]

6.2 Derivation of the central theorem

**H2.** There exist \( \alpha_v \in (0, 1) \) and \((v_1, v_2) \in (\mathbb{R}^+)^2\) such that for any \( k \in \mathbb{N} \), \( V_k \) satisfies
\[
E [V_k] \leq v_1 \alpha_v^k + v_2,
\]
where \( V_k \) is defined in (24).

**HX2.** There exist \( \alpha_q \in (0, 1) \) and \( \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}_+ \) satisfying \( (1 - \alpha_q)(1 + \alpha_0 + \sqrt{(\alpha_0 - 1)^2 + 4 \alpha_1}) < 2 \) such that for \( k \geq 0 \) the following inequality holds
\[
(1 - \alpha_q)^{-1} E [\left\| X_{(k+1)\gamma} - X_{k+1} \right\|^2] \leq \alpha_0 E \left[ \left\| X_{k\gamma} - X_k \right\|^2 \right] + \alpha_1 \sum_{l=0}^{k-1} (1 - \alpha_q)^{k-l} E \left[ \left\| X_{l\gamma} - X_l \right\|^2 \right] + \alpha_2 E [V_k] + \alpha_3 \sum_{l=0}^{k-1} (1 - \alpha_q)^{k-l} E [V_l] + \alpha_4.
\]

With the notation introduced in \textbf{HX2}, consider
\[
\delta = \frac{-1 - \alpha_0 + \sqrt{(\alpha_0 - 1)^2 + 4 \alpha_1}}{2}.
\]

At iteration \( k \geq 0 \), recall that \( \mu_k^{(\gamma)} \) denotes the distribution of the average parameter \( X_k \) (23). The next result controls the Wasserstein distance between \( \mu_k^{(\gamma)} \) and the posterior distribution \( \pi \).

**Theorem 8.** Assume \textbf{HX2} and \textbf{H2} hold. Then, for any probability measure \( \mu_0^{(\gamma)} \in \mathcal{P}_2(\mathbb{R}^d) \), \( k \in \mathbb{N} \), we have
\[
W_2^2 \left( \mu_k^{(\gamma)}, \pi \right) \leq (1 + \alpha_0 + \delta)^k (1 - \alpha_q)^k W_2^2 \left( \mu_0^{(\gamma)}, \pi \right) + (1 - \alpha_q) v_1 \left( \alpha_2 + \alpha_3 \alpha_0 + \delta \right) \left( \alpha_0 - (1 + \alpha_0 + \delta) \left( 1 - \alpha_q \right) \right) + \frac{1 - \alpha_q}{\alpha_0 - (1 - \alpha_q)(\alpha_0 + \delta)} \left( \alpha_2 + \alpha_3 \alpha_0 + \delta \right) v_2 + \alpha_4.
\]

**Proof.** For any \( n \in \mathbb{N} \), define
\[
u_n = (1 - \alpha_q)^{-n} E \left[ \left\| X_{n\gamma} - X_n \right\|^2 \right], \quad S_n = \sum_{l=0}^{n} u_l, \quad u_n = (1 - \alpha_q)^{-n} E \left[ \left\| X_{n\gamma} - X_n \right\|^2 \right], \quad S_n = \sum_{l=0}^{n} u_l,
\]
and
\[
u_n = (1 - \alpha_q)^{-n} \left( \alpha_2 E [V_n] + \alpha_4 \right) + \alpha_3 \sum_{l=0}^{n-1} (1 - \alpha_q)^{-l} E [V_l].
\]
With the above notations, \( HX2 \) becomes
\[
u_{k+1} \leq \alpha_0 u_k + \alpha_1 \sum_{l=0}^{k-1} u_l + v_k,
\]
which can be rewritten as
\[
S_{k+1} - S_k \leq \alpha_0 (S_k - S_{k-1}) + \alpha_1 S_{k-1} + v_k.
\]
Since \( \delta \) is solution of \( \delta(1 + \alpha_0 + \delta) + \alpha_0 - \alpha_1 = 0 \), adding \( (1 + \delta) S_k \) in (47) gives that
\[
S_{k+1} + \delta S_k \leq (1 + \alpha_0 + \delta) \left( S_k - \frac{\alpha_0 - \alpha_1}{1 + \alpha_0 + \delta} S_{k-1} \right) + v_k
\]
\[
= (1 + \alpha_0 + \delta) (S_k + \delta S_{k-1}) + v_k.
\]
Using the fact that \( \alpha_0 \leq 1 + \sqrt{(\alpha_0 - 1)^2 + 4\alpha_1} \), we obtain \( 2(1 + \delta) = 1 - \alpha_0 + \sqrt{(\alpha_0 - 1)^2 + 4\alpha_1} \geq 0 \). Hence \( 1 + \delta > 0 \), which leads to the following upper bound
\[
u_{k+1} \leq \nu_{k+1} + (1 + \delta) \sum_{l=0}^{k} u_l = S_{k+1} + \delta S_k.
\]
Thus, we obtain that
\[
u_k \leq S_k + \delta S_{k-1} \leq (1 + \alpha_0 + \delta)^{k-1} (u_1 + (1 + \delta) u_k) + \sum_{l=1}^{k-1} (1 + \alpha_0 + \delta)^{k-l-1} v_l.
\]
Plugging the definition (46) of \( u_k \) and \( v_l \) inside the previous inequality, we get
\[
(1 - q_e) - k \mathbb{E} \left[ \|X_{k+1} - X_k\|^2 \right] \leq (1 + \alpha_0 + \delta)^{k-1} \left( (1 - q_e)^{-1} \mathbb{E} \left[ \|X_0 - X_1\|^2 \right] + (1 + \delta) \mathbb{E} \left[ \|X_0 - X_0\|^2 \right] \right)
\]
\[
+ \sum_{l=1}^{k-1} (1 + \alpha_0 + \delta)^{k-l-1} \left( (1 - q_e)^{-l} (\alpha_2 \mathbb{E} [V_l] + \alpha_4) + \alpha_3 \sum_{j=0}^{l-1} (1 - q_e)^{-j} \mathbb{E} [V_j] \right).
\]
Moreover, using \( HX2 \) we obtain that
\[
\mathbb{E} \left[ \|X_0 - X_1\|^2 \right] \leq (1 - q_e) \alpha_0 \mathbb{E} \left[ \|X_0 - X_0\|^2 \right] + (1 - q_e) \alpha_2 \mathbb{E} [V_0] + \alpha_4,
\]
combining (48) with (49) yield
\[
\mathbb{E} \left[ \|X_{k+1} - X_k\|^2 \right] \leq (1 + \alpha_0 + \delta)^{k-1} (1 - q_e)^{k} \mathbb{E} \left[ \|X_0 - X_0\|^2 \right] + \alpha_2 \sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^{k-l-1} (1 - q_e)^{k-l} \mathbb{E} [V_l]
\]
\[
+ \alpha_3 \sum_{j=0}^{k-2} (1 - q_e)^{-j} \mathbb{E} [V_j] \sum_{l=j+1}^{k-1} (1 + \alpha_0 + \delta)^{k-l-1} + (1 - q_e) \alpha_4 \sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^l (1 - q_e)^l.
\]
Consider the function \( f : \alpha \in \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(\alpha) = \alpha(1 + \alpha_0 + \alpha) - \alpha_0 - \alpha_1 \). Using the definition (45) of \( \delta \) combined with the increasing property of \( f \), we deduce from \( f(\delta) = 0 > f(-\alpha_0) = -\alpha_1 \) that \( \delta > -\alpha_0 \), and thus we get \( 1 + \alpha_0 + \delta > 1 \) which implies that
\[
\sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^{k-l-1} \leq \sum_{l=0}^{k-2} (1 + \alpha_0 + \delta)^{k-j-l-2}
\]
\[
\leq \frac{(1 + \alpha_0 + \delta)^{k-1}}{\alpha_0 + \delta}.
\]
Therefore, plugging (51) in (50) gives
\[
\sum_{j=0}^{k-2} (1 - q_e)^{-j} \mathbb{E} [V_j] \sum_{l=j+1}^{k-1} (1 + \alpha_0 + \delta)^{k-l-1} \leq \sum_{l=0}^{k-2} (1 - q_e)^{-l} (1 + \alpha_0 + \delta)^{k-l-1} \mathbb{E} [V_l].
\]
In addition, since $HX2$ ensures that $(1 - q_c)(1 + \alpha_0 + \delta) < 1$, we have

$$\sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^l (1 - q_c)^l \leq \frac{1}{q_c - (1 - q_c)(\alpha_0 + \delta)}.$$  \hfill (53)

The last inequality combined with (50) and (52) show that

$$\mathbb{E} [||X_k\gamma - X_k||^2] \leq (1 + \alpha_0 + \delta)^k (1 - q_c)^k \mathbb{E} [||X_0 - X_0||^2]
+ \left( \alpha_2 + \frac{\alpha_3}{\alpha_0 + \delta} \right) \sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^k (1 - q_c)^k k^{-l-1} \mathbb{E} [V_l] + \frac{(1 - q_c)\alpha_4}{q_c - (1 - q_c)(\alpha_0 + \delta)}.$$  \hfill (54)

Further, since we assume $H2$, we have

$$\sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^k (1 - q_c)^k \mathbb{E} [V_l] \leq \nu_1 \sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^k (1 - q_c)^k \alpha_v^l + \nu_2 \sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^k (1 - q_c)^k.$$  \hfill (55)

A calculation gives that

$$\sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^k (1 - q_c)^k \alpha_v^l \leq (1 - q_c)\frac{\alpha_v^k}{\alpha_v - (1 + \alpha_0 + \delta)(1 - q_c)}.$$  \hfill (56)

and combining (53), (55) with (56), we find that

$$\sum_{l=0}^{k-1} (1 + \alpha_0 + \delta)^k (1 - q_c)^k \mathbb{E} [V_l] \leq (1 - q_c)\nu_1 \frac{\alpha_v^k}{\alpha_v - (1 + \alpha_0 + \delta)(1 - q_c)} + \frac{(1 - q_c)\nu_2}{q_c - (1 - q_c)(\alpha_0 + \delta)}.$$  \hfill (57)

Therefore, plugging (57) inside (54) shows that

$$\mathbb{E} [||X_k\gamma - X_k||^2] \leq (1 + \alpha_0 + \delta)^k (1 - q_c)^k \mathbb{E} [||X_0 - X_0||^2]
+ (1 - q_c)\nu_1 \left( \alpha_2 + \frac{\alpha_3}{\alpha_0 + \delta} \right) \frac{\alpha_v^k}{\alpha_v - (1 + \alpha_0 + \delta)(1 - q_c)}
+ \frac{1 - q_c}{q_c - (1 - q_c)(\alpha_0 + \delta)} \left[ \left( \alpha_2 + \frac{\alpha_3}{\alpha_0 + \delta} \right) \nu_2 + \nu_4 \right].$$  \hfill (58)

Eventually, since the Wasserstein distance $W_2(\pi, \mu_k^{(\gamma)})$ is the infimum over all couplings, we obtain that $W_2^2(\pi, \mu_k^{(\gamma)}) \leq \mathbb{E} [||X_k\gamma - X_k||^2]$. Moreover, it follows from the strongly convex assumption $A1$ that $\pi \in \mathcal{P}_2(\mathbb{R}^d)$. Thus, we can apply Villani (2009, Theorem 4.1) to prove the existence of an optimal coupling $\zeta$ such that taking $(X_0, X_0)$ distributed according to $\zeta$ implies that $\mathbb{E} [||X_0 - X_0||^2]^{1/2} = W_2(\pi, \mu_0^{(\gamma)})$. Substituting these results into (58) completes the proof. \hfill $\square$

### 6.3 Upper bound on $V_k$

The goal of this subsection is to prove the upper bound derived in Lemma 11 for $(\mathbb{E} [V_k])_{k \in \mathbb{N}}$ to ensure that $H2$ holds. Recall that for $k \geq 0$, $V_k$ is defined in (24), $d_k$ in (25), $G_k^\pi$ in (17) and we introduce $G_k^{\ast} = \mathbb{E}^2 [G_k]$ to prove the central lemma of this subsection, we also consider the assumptions $HX3$ and $HX4$ given below.

**HX3.** There exist $A_d, A_\sigma \in (0, 1), B_d, B_\sigma, C_d, C_\sigma, D_d, D_\sigma \in \mathbb{R}_+$, such that for any $k \in \mathbb{N}$, we have

$$\mathbb{E} [d_{k+1}^2] \leq (1 - A_d) \mathbb{E} [d_k^2] + B_d \mathbb{E} [\sigma_k^2] + C_d \mathbb{E} [V_k] + D_d,$$

$$\mathbb{E} [\sigma_{k+1}^2] \leq (1 - A_\sigma) \mathbb{E} [\sigma_k^2] + B_\sigma \mathbb{E} [d_k^2] + C_\sigma \mathbb{E} [V_k] + D_\sigma.$$
**HX4.** There exist $A, \tilde{A}, B, \tilde{B}, C, \tilde{C}, D, \tilde{D} \geq 0$ such that for any $i \in [b], k \in \mathbb{N}$, we have

$$
\frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \|G_k^i\|^2 \right] \leq \tilde{A} \mathbb{E} [V_k] + \tilde{B} \mathbb{E} [d_k^2] + \tilde{C} \mathbb{E} [\sigma_k^2] + \tilde{D},
$$

$$
\frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \|G_k^i - \bar{G}_k\|^2 \right] \leq \tilde{A} \mathbb{E} [V_k] + \tilde{B} \mathbb{E} [d_k^2] + \tilde{C} \mathbb{E} [\sigma_k^2] + \tilde{D}.
$$

With the notation considered in **HX3** and **HX4**, for any $\gamma > 0$ we also introduce the following quantities:

$$
C_\gamma = \frac{4(1 - p_c)\gamma^2}{\gamma^2},
$$

$$
C_\gamma = \frac{4(1 - p_c)\gamma^2}{\gamma^2} \left[ \frac{2 + p_c}{p_c} \bar{B} + \frac{B_\sigma}{A_\sigma - A_d} \left( \frac{2 + p_c}{p_c} \bar{C} \right) \right],
$$

$$
C_\gamma = \frac{g(1 - p_c)\gamma^2}{\gamma^2} \left[ \frac{C + 2 + p_c}{p_c} \bar{C} + 3C \gamma \left( C_d + \frac{B_dC_\sigma}{A_\sigma - A_d} \right) \right],
$$

$$
C_\gamma = \frac{4(1 - p_c)\gamma^2}{\gamma^2} \left[ \frac{C + 2 + p_c}{p_c} \bar{C} + C^\gamma B_d \left( \frac{2 + p_c}{p_c} \bar{C} \right) + 4(1 - p_c)^\gamma \gamma^2 \left( D + \frac{2 + p_c}{p_c} \bar{D} \right) \right],
$$

$$
+ \frac{C_\gamma}{A_d} \left( 1 + \frac{2B_dB_{\sigma}}{A_d(A_\sigma - A_d)} \right) \left( D_d + \frac{B_dD_\sigma}{A_\sigma} \right) + \frac{8(1 - \tau)(b - 1)\gamma}{b p_c}.
$$

If $A_d \leq A_\sigma/2$ and $A_d A_\sigma \geq 8B_dB_{\sigma}$, we also introduce a convergence rate (proved later in Lemma 10) defined by

$$
\alpha = A_d - \frac{2(A_\sigma - A_d)^{-1}B_dB_{\sigma}}{1 + \sqrt{1 + 4(1 - A_d)^{-1}(A_\sigma - A_d)^{-1}B_dB_{\sigma}}}. \tag{60}
$$

**Lemma 9.** Assume **HX3** and also that $A_d \leq A_\sigma/2, A_d A_\sigma \geq 8B_dB_{\sigma}$ hold. Then, we have

$$
A_d/2 < \alpha \leq A_d.
$$

**Proof.** First, introduce $\delta_\alpha \in \mathbb{R}_+$ the unique non-negative solution of

$$
\delta_\alpha^2 + \delta_\alpha = \frac{B_dB_{\sigma}}{(A_\sigma - A_d)}.
$$

Since we suppose $A_d \leq A_\sigma/2$, thus we have $A_d \leq 1/2$ which implies that $(1 - A_d)(A_\sigma^2/4 + A_d/2) \geq A_d/4$. In addition, using $A_d A_\sigma \geq 8B_dB_{\sigma}$, we get that

$$
(1 - A_d) \left( \frac{A_d^2}{4} + \frac{A_d}{2} \right) \geq \frac{A_d}{4} \geq \frac{B_dB_{\sigma}}{A_\sigma} \geq (1 - A_d) \left( \delta_\alpha^2 + \delta_\alpha \right).
$$

Hence, the increasing property of the function $x \in \mathbb{R}_+ \mapsto x^2 + x$ combined with the fact that $\delta_\alpha \geq 0$ prove that $A_d \geq 2\delta_\alpha$. Moreover, a calculation shows that $\alpha$ satisfies $\alpha = 1 - (1 - A_d)(1 + \delta_\alpha)$. Thus, using $0 \leq 2\delta_\alpha \leq A_d$ implies that $\alpha \in (A_d/2, A_d]$. \hfill \Box

The random variable $V_k$ given in (24) measures the averaged distance between the global parameter $X_k$ and the local ones $(X_k^i)_{i \in [b]}$. The first lines of the proof of the next lemma are based on Gorbunov et al. (2021, Lemma E.3), however their purpose was to upper bound $\sum_i w_i \mathbb{E} V_k$ for some weights $w_i > 0$, while we prefer to control $\mathbb{E} V_k$ to combine this bound with that of Proposition 4. Moreover, the assumptions considered in this work are different, so the proof requires the development of other techniques.

**Lemma 10.** Assume **HX3, HX4** hold with $A_d < \min(A_\sigma/2, p_c/4), A_d A_\sigma \geq 8B_dB_{\sigma}$ and consider $\gamma \leq p_c^{1/2}(2 - p_c)^{-1/2}[A + (1 + 2/p_c)\bar{A}]^{-1/2}$. Then, for any $k \in \mathbb{N}$, we have

$$
\mathbb{E} [V_k] \leq (1 - \alpha)^k \left( C_\gamma \mathbb{E} [V_0] + C_\gamma^2 \mathbb{E} [d_0^2] + C_\gamma \mathbb{E} [\sigma_0^2] + 2D_d + C_\gamma \sum_{i=0}^{k-2} (1 - \alpha)^{k-i-1} \mathbb{E} [V_i] + C_\gamma^2 \right),
$$

where $V_k$ is defined in (24).
Proof. Let $k \in \mathbb{N}^*$, using for $i \in [b]$ the definitions (20), (23) of $X_i^k$ and $X_k$

\[ X_{k+1}^i = X_k^i - \gamma G_k^i + \sqrt{2\gamma} \left( \frac{\sqrt{\tau}}{b} \tilde{Z}_{k+1} + \sqrt{1-\tau} \tilde{Z}_{k+1} \right), \]

\[ X_{k+1} = X_k - \gamma b \sum_{j=1}^{b} G_k^j + \sqrt{\frac{2\gamma\sqrt{\tau}}{b}} \tilde{Z}_{k+1} + \sqrt{\frac{2(1-\tau)\gamma}{b}} \sum_{i=1}^{b} Z_{k+1}^i. \]

**First upper bound on $\mathbb{E}[V_k]$.** Substracting the two above equations combined with the Jensen inequality give

\[
\mathbb{E}[V_{k+1}] = \frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \|X_{k+1}^i - X_{k+1}\|^2 \right] = \frac{1-p_c}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \left( X_k^i - X_k - \gamma (G_k^i - G^k) + \sqrt{2(1-\tau)\gamma} Z_{k+1}^i - \sqrt{\frac{2(1-\tau)\gamma}{b}} \sum_{j=1}^{b} Z_{k+1}^j \right)^2 \right]
\]

\[
= \frac{1-p_c}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \left( X_k^i - X_k - \gamma (G_k^i - G^k) \right)^2 \right] + \frac{(1-p_c)\gamma^2}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \|G_k^i - G_k^i\|^2 \right] + 2(1-\tau)(1-1/b)\gamma d.
\]

Hence, we get

\[
\mathbb{E}[V_{k+1}] \leq \frac{1-p_c}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \left( X_k^i - X_k \right)^2 \right] + \frac{(1-p_c)\gamma^2}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \|G_k^i\|^2 \right] + 2(1-\tau)(1-1/b)\gamma d.
\]

Using $(1-p_c)(1+p_c/2) \leq 1 - p_c/2$, we finally obtain

\[
\mathbb{E}[V_{k+1}] \leq (1-p_c/2) \mathbb{E}[V_k] + \frac{(1-p_c)(2+p_c)}{p_c b} \sum_{i=1}^{b} \mathbb{E} \left[ \|G_k^i\|^2 \right] + (1-p_c)\gamma^2 \sum_{i=1}^{b} \mathbb{E} \left[ \|G_k^i\|^2 \right] + 2(1-\tau)(1-1/b)\gamma d.
\]

Combining the last inequality with $HX4$, it shows

\[
\mathbb{E}[V_{k+1}] \leq \left( 1 - \frac{p_c}{2} + (1-p_c)\gamma^2 \left[ A + \frac{2+p_c}{p_c} \tilde{A} \right] \right) \mathbb{E}[V_k] + (1-p_c)\gamma^2 \left( D + \frac{2+p_c}{p_c} \tilde{D} \right)
\]

\[
+ (1-p_c)\gamma^2 \left( B + \frac{2+p_c}{p_c} \tilde{B} \right) \mathbb{E} [\sigma_k^2] + (1-p_c)\gamma^2 \left( C + \frac{2+p_c}{p_c} \tilde{C} \right) \mathbb{E} [\sigma_k^2] + 2(1-\tau)(1-1/b)\gamma d.
\]

Since $\gamma \leq \frac{p_c^{1/2}}{2(1-p_c)^{1/2}[A+(1+2/p_c)\tilde{A}]^{1/2}}$, the above inequality implies that

\[
\mathbb{E}[V_{k+1}] \leq \left( 1 - \frac{p_c}{4} \right) \mathbb{E}[V_k] + (1-p_c)\gamma^2 \left( D + \frac{2+p_c}{p_c} \tilde{D} \right) + 2(1-\tau)(1-1/b)\gamma d.
\]
Thus, using that
\[ \sum \] Hence, by plugging (63) in (61) we obtain that
\[ \text{and unrolling the recursion gives that} \]
\[ \text{Moreover, for any } l \in \mathbb{N}^* \text{ the assumption HX3 implies that} \]
\[ \text{and unrolling the recursion gives that} \]
\[ \text{Similarly, we also have} \]
\[ \text{Hence, by plugging (63) in (61) we obtain that} \]
\[ \text{In addition, interchanging the summations gives} \]
\[ \text{Thus, using that } \sum_{l=0}^{k-1-i-2} (1 - p_c/4)^{k-l-2} (1 - A_\sigma)^{l-1} \leq 4 (1 - A_d)^{k-l-1} (p_c - 4A_d)^{-1}, \text{ we can simplify the upper bound of } \mathbb{E} [V_k] \text{ derived in (64). Indeed, we can write} \]
The above inequality leads to the next inequality

\[
\mathbb{E}[V_k] \leq \left(1 - \frac{p_c}{4}\right)^k \mathbb{E}[V_0] + \frac{4(1 - p_c)\gamma^2}{p_c} \left(1 - Ad\right)^k \left(C + \frac{2 + p_c}{p_c} \sigma^2\right) + \frac{4(1 - p_c)\gamma^2}{p_c} \left(D + \frac{2 + p_c}{p_c} \bar{D}\right) + \frac{8(1 - \tau)(b - 1)\gamma d}{\bar{b}p_c} + \frac{4(1 - p_c)\gamma^2 D_{\sigma}}{A_{\sigma}(p_c - 4A_d)} \left(C + \frac{2 + p_c}{p_c} \sigma^2\right) \right]
\]

Similarly, we can also get that

\[
\mathbb{E}[V_{k+1}] \leq \left(1 - \frac{p_c}{4}\right)^{k+1} \mathbb{E}[V_0] + \frac{4(1 - p_c)\gamma^2}{p_c} \left(1 - Ad\right)^{k+1} \left(C + \frac{2 + p_c}{p_c} \sigma^2\right) + \frac{4(1 - p_c)\gamma^2}{p_c} \left(D + \frac{2 + p_c}{p_c} \bar{D}\right) + \frac{8(1 - \tau)(b - 1)\gamma d}{\bar{b}p_c} + \frac{4(1 - p_c)\gamma^2 D_{\sigma}}{A_{\sigma}(p_c - 4A_d)} \left(C + \frac{2 + p_c}{p_c} \sigma^2\right) \right]
\]

Upper bound on \(\mathbb{E}[d_i^2]\). For \(l \geq 1\), plugging (63) into (62) yields the following upper bound

\[
\mathbb{E}[d_i^2] \leq \left(1 - Ad\right)^l \mathbb{E}[d_0^2] + C_d \sum_{j=1}^{l} \left(1 - Ad\right)^{l-j} \mathbb{E}[V_{j-1}] + \frac{D_d}{A_d} \left(1 - Ad\right)^l \mathbb{E}[V_l] + \sum_{j=1}^{l} \left(1 - Ad\right)^{l-j} \left(1 - Ad\right)^{l-j} \mathbb{E}[d_{i-1}^2] + \frac{B_d}{A_d} \left(1 - Ad\right)^{l-j} \mathbb{E}[d_l^2]
\]

The above inequality leads to the next inequality

\[
\mathbb{E}[d_i^2] \leq \left(1 - Ad\right)^l \mathbb{E}[d_0^2] + B_d \sum_{j=1}^{l} \left(1 - Ad\right)^{l-j} \mathbb{E}[V_{j-1}] + C_d \sum_{j=1}^{l} \left(1 - Ad\right)^{l-j} \mathbb{E}[d_{i-1}^2] + \frac{B_d}{A_d} \sum_{j=1}^{l} \left(1 - Ad\right)^{l-j} \mathbb{E}[d_l^2]
\]

By interchanging the double summations in (66), we obtain

\[
\sum_{j=1}^{l} \sum_{i=1}^{j-1} \left(1 - Ad\right)^{l-j} \left(1 - Ad\right)^{j-i-1} \mathbb{E}[d_{i-1}^2] = \sum_{i=1}^{l-1} \left[ \sum_{j=i+1}^{l} \left(1 - Ad\right)^{l-j} \left(1 - Ad\right)^{j-i-1} \mathbb{E}[d_{i-1}^2] \right]
\]

Similarly, we can also get that

\[
\sum_{j=1}^{l} \sum_{i=1}^{j-1} \left(1 - Ad\right)^{l-j} \left(1 - Ad\right)^{j-i-1} \mathbb{E}[V_{j-1}] \leq \frac{1}{A_\sigma - A_d} \sum_{i=0}^{l-2} \left(1 - Ad\right)^{l-i-1} \mathbb{E}[d_i^2]
\]

Plugging back (67) and (68) in (66) shows

\[
\mathbb{E}[d_i^2] \leq \left(1 - Ad\right)^l \mathbb{E}[d_0^2] + \frac{B_d}{A_\sigma - A_d} \sum_{i=0}^{l-2} \left(1 - Ad\right)^{l-i-1} \mathbb{E}[d_i^2] + \sum_{j=1}^{l} \left(1 - Ad\right)^{l-j} \mathbb{E}[V_{j-1}] \leq \frac{1}{A_\sigma - A_d} \sum_{i=0}^{l-2} \left(1 - Ad\right)^{l-i-1} \mathbb{E}[d_i^2]
\]
Now, we want to control \( \sum_{l=0}^{k-1} (1 - A_d)^{k-l-1} \mathbb{E} [d_i^2] \). For this, for any \( l \in \mathbb{N} \) define

\[
U_l = \mathbb{E} [d_0^2] + \frac{B_d}{A_d - A_{\sigma}} \mathbb{E} [\sigma_0^2] + \frac{D_d (1 - A_d)^{-l}}{A_d} + \frac{B_d D_{\sigma} (1 - A_d)^{-l}}{A_d A_{\sigma}} + C_d \sum_{i=0}^{l-1} (1 - A_d)^{-i-1} \mathbb{E} [V_i] + \frac{B_d C_{\sigma}}{A_d - A_{\sigma}} \sum_{i=0}^{l-2} (1 - A_d)^{-i-1} \mathbb{E} [V_i] \tag{70}
\]

and consider

\[
S_l = \sum_{i=0}^{l} (1 - A_d)^{-i} \mathbb{E} [d_i^2].
\]

With the above notation, (69) can be rewritten as

\[
S_l - S_{l-1} \leq \frac{B_d B_{\sigma}}{(1 - A_d) (A_{\sigma} - A_d)} S_{l-2} + U_l. \tag{71}
\]

For \( l \geq 2 \), using the upper bound derived in (71) gives

\[
\mathbb{E} [d_l^2] = (1 - A_d)^l (S_l - S_{l-1}) \leq \frac{B_d B_{\sigma} (1 - A_d)^{l-1} S_{l-2}}{(A_{\sigma} - A_d)} + (1 - A_d)^l U_l. \tag{72}
\]

Finally, we define

\[
\delta_\alpha = \frac{-1 + \sqrt{1 + 4(1 - A_d)^{-1} (A_{\sigma} - A_d)^{-1} B_d B_{\sigma}}}{2}
\]

such that \( \delta_\alpha \) is solution of the equation

\[
\delta_\alpha^2 + \delta_\alpha = \frac{B_d B_{\sigma}}{(1 - A_d) (A_{\sigma} - A_d)} \tag{73}
\]

Thus for \( l \geq 2 \), the definition of \( \delta_\alpha \) combined with (71) show

\[
S_l + \delta_\alpha S_{l-1} \leq (1 + \delta_\alpha) (S_{l-1} + \delta_\alpha S_{l-2}) + U_l.
\]

Unrolling this recursion gives

\[
S_k + \delta_\alpha S_{k-1} \leq (1 + \delta_\alpha)^k (S_1 + \delta_\alpha S_0) + \sum_{l=2}^{k} (1 + \delta_\alpha)^{k-l} U_l. \tag{74}
\]

**Upper bound on** \( \sum_{l=0}^{k-1} (1 - \tilde{\alpha})^{l-j-1} \mathbb{E} [d_j^2] \). Let consider a fixed \( \tilde{\alpha} \in \{ p_c / 4, A_{\sigma} \} \), by assumption we have \( A_d < \tilde{\alpha} < 1 \). Since we want to control \( \sum_{l=0}^{k-1} (1 - p_c / 4)^{k-l-1} \mathbb{E} [d_l^2] \) and \( \sum_{l=0}^{k-1} (1 - p_c / 4)^{k-l-1} \sum_{j=0}^{l} (1 - A_d)^{l-j-1} \mathbb{E} [d_j^2] \) involved in the inequality (65), we first study \( \sum_{l=0}^{k-1} (1 - \tilde{\alpha})^{k-l-1} \mathbb{E} [d_l^2] \). From (72), we deduce that

\[
\sum_{l=0}^{k-1} (1 - \tilde{\alpha})^{k-l-1} \mathbb{E} [d_l^2] \leq \frac{B_d B_{\sigma}}{(1 - A_d) (A_{\sigma} - A_d)} \sum_{l=0}^{k-1} (1 - A_d)^{l} (1 - \tilde{\alpha})^{k-l-1} S_{l-2} + \sum_{l=0}^{k-1} (1 - A_d)^{l} (1 - \tilde{\alpha})^{k-l-1} U_l. \tag{75}
\]

Since we suppose \( HX^3 \) and \( A_d \leq A_{\sigma} / 2, A_d A_{\sigma} \geq 8 B_d B_{\sigma} \) we can apply Lemma 9 which shows that \( 1 - \alpha = (1 - A_d)(1 + \delta_\alpha) \in (0, 1 - \tilde{\alpha}) \) and leads to

\[
\sum_{l=0}^{k-1} (1 - A_d)^l (1 - \tilde{\alpha})^{k-l-1} (1 + \delta_\alpha)^{l-3} \leq (1 + \delta_\alpha)^{-3} \sum_{l=0}^{k-1} (1 - \alpha)^l (1 - \tilde{\alpha})^{k-l-1} \leq \frac{(1 - \alpha)^k}{(\tilde{\alpha} - \alpha)(1 + \delta_\alpha)^3}. \tag{76}
\]
Moreover, for $l \geq 2$ applying the result given by (74), we have

$$S_{l-2} \leq (1 + \delta_\alpha)^{l-3} (S_1 + \delta_\alpha S_0) + \sum_{j=2}^{l-2} (1 + \delta_\alpha)^{l-j-2} U_j.$$  \hspace{1cm} (77)

Using the definition of $U_l$ given by (70), we can write the following equality

$$\begin{align*}
\sum_{l=0}^{k-1} (1 - A_d)^l (1 - \tilde{\alpha})^{k-l-1} \sum_{j=2}^{l-2} (1 + \delta_\alpha)^{l-j-2} U_j \\
= \left( \mathbb{E} [d_0^2] + \frac{B_d}{A_d} \mathbb{E} [\sigma_0^2] \right) \sum_{l=0}^{l-2} (1 - A_d)^l (1 - \tilde{\alpha})^{k-l-1} (1 + \delta_\alpha)^{l-j-2} \\
+ \left( \frac{D_d}{A_d} + \frac{B_d D_\sigma}{A_d A_\sigma} \right) \sum_{l=0}^{l-2} (1 - A_d)^l (1 - \tilde{\alpha})^{k-l-1} (1 + \delta_\alpha)^{l-j-2} \\
+ \left( C_d + \frac{B_d C_\sigma}{A_d} \right) \sum_{l=0}^{l-2} (1 - A_d)^l (1 - \tilde{\alpha})^{k-l-1} (1 + \delta_\alpha)^{l-j-2} \sum_{i=0}^{l-1} (1 - A_d)^{-i-1} \mathbb{E} [V_i] \hspace{1cm} (78)
\end{align*}$$

We now upper bound each quantity separately. Regarding the first double sum, since $(1 - A_d)(1 + \delta_\alpha) = 1 - \alpha$ we get

$$\sum_{l=0}^{l-2} (1 - A_d)^l (1 - \tilde{\alpha})^{k-l-1} (1 + \delta_\alpha)^{l-j-2} \leq \frac{1}{\tilde{\alpha} - \alpha} \sum_{j=4}^{k-1} (1 - A_d)^j (1 - \alpha)^{k-j} \leq \frac{(1 - A_d)^4 (1 - \alpha)^{k-3}}{(A_d - \alpha)(\tilde{\alpha} - \alpha)}. \hspace{1cm} (79)$$

Using $(1 - A_d)(1 + \delta_\alpha) = 1 - \alpha$ combined with $\sum_{i=0}^{k-1} (1 - \alpha)^{l-j-2} (1 - \tilde{\alpha})^{k-l-1} \leq (\tilde{\alpha} - \alpha)^{-1}(1 - \alpha)^{k-j-2}$ give

$$\begin{align*}
\sum_{l=0}^{l-2} (1 - A_d)^l (1 - \tilde{\alpha})^{k-l-1} (1 + \delta_\alpha)^{l-j-2} \\
= (1 - A_d)^2 \sum_{l=0}^{l-2} (1 - \alpha)^{l-j-2} (1 - \tilde{\alpha})^{k-l-1} \\
= (1 - A_d)^2 \sum_{j=2}^{k-3} \sum_{l=j+2}^{k-1} (1 - \alpha)^{l-j-2} (1 - \tilde{\alpha})^{k-l-1} \\
\leq \frac{(1 - A_d)^2}{\tilde{\alpha} - \alpha} \sum_{j=2}^{k-3} (1 - \alpha)^{k-j-2} \leq \frac{(1 - \alpha) (1 - A_d)^2}{\alpha(\tilde{\alpha} - \alpha)}. \hspace{1cm} (80)
\end{align*}$$

The same arguments show that

$$\begin{align*}
\sum_{l=0}^{l-2} (1 - A_d)^l (1 - \tilde{\alpha})^{k-l-1} (1 + \delta_\alpha)^{l-j-2} \sum_{i=0}^{l-1} (1 - A_d)^{-i-1} \mathbb{E} [V_i] \\
\leq \sum_{i=0}^{k-4} \sum_{j=i+1}^{k-3} \sum_{l=j+2}^{k-1} (1 - A_d)^l (1 - \tilde{\alpha})^{k-l-1} (1 + \delta_\alpha)^{l-j-2} (1 - A_d)^{-i-1} \mathbb{E} [V_i] \\
\leq \sum_{i=0}^{k-4} \sum_{j=i+1}^{k-3} \sum_{l=j+2}^{k-1} \mathbb{E} [V_i] (1 - A_d)^{j-i+1} \sum_{l=j+2}^{k-1} (1 - \tilde{\alpha})^{k-l-1} (1 - \alpha)^{l-j-2}
\end{align*}$$
Therefore, plugging (79), (80), (81) inside (78) implies

\[
\sum_{i=0}^{k-4} \sum_{j=i+1}^{k-3} (1 - A_d)^{j-i+1} (1 - \tilde{\alpha})^{k-j-2} \leq \frac{1}{\alpha - \tilde{\alpha}} \sum_{i=0}^{k-3} (1 - A_d)^{j-i+1} (1 - \alpha)^{k-j-3}
\]

\[
= \frac{(1 - \alpha)(1 - A_d)^2}{\alpha - \tilde{\alpha}} \sum_{i=0}^{k-4} \sum_{j=i+1}^{k-3} (1 - A_d)^{j-i+1} (1 - \alpha)^{k-j-3}
\]

\[
\leq \frac{(1 - \alpha)^{-1}(1 - A_d)^2}{(A_d - \alpha)(\tilde{\alpha} - \alpha)} \sum_{i=0}^{k-4} (1 - \alpha)^{k-i-1} \mathbb{E}[V_i].
\]

Therefore, plugging (79), (80), (81) inside (78) implies

\[
\sum_{i=0}^{k-4} \sum_{j=i+1}^{k-3} (1 - A_d)^{j-i+1} (1 + \delta_\alpha)^{k-j-2} U_j \leq \frac{(1 - \alpha)(1 - A_d)^2}{\alpha(\tilde{\alpha} - \alpha)} \left( D_d + B_d D_\sigma \right) \sum_{i=0}^{k-4} (1 - \alpha)^{k-i-1}
\]

\[
+ \frac{(1 - A_d)^3 (1 - \alpha)^{k-3}}{(A_d - \alpha)(\tilde{\alpha} - \alpha)} \left( \mathbb{E}[d_0^2] + \frac{B_d}{A_d - A_d} \mathbb{E}[\sigma_0^2] \right) \sum_{i=0}^{k-4} (1 - A_d)^{i} (1 - \tilde{\alpha})^{k-i-1}
\]

\[
+ \left( C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \right) \sum_{i=0}^{k-4} (1 - A_d)^{i} (1 - \alpha)^{k-i-1} \sum_{i=0}^{k-4} (1 - A_d)^{-i-1} \mathbb{E}[V_i].
\]

In addition, by definition of \( U_l \) provides in (70) we have

\[
\sum_{i=0}^{k-4} (1 - A_d)^{i} (1 - \tilde{\alpha})^{k-i-1} U_l = \frac{D_d}{A_d} + \frac{B_d D_\sigma}{A_d A_\sigma} \sum_{i=0}^{k-4} (1 - \tilde{\alpha})^{k-i-1}
\]

\[
+ \left( \mathbb{E}[d_0^2] + \frac{B_d}{A_d - A_d} \mathbb{E}[\sigma_0^2] \right) \sum_{i=0}^{k-4} (1 - A_d)^{i} (1 - \tilde{\alpha})^{k-i-1}
\]

\[
+ \left( C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \right) \sum_{i=0}^{k-4} (1 - A_d)^{i} (1 - \alpha)^{k-i-1} \sum_{i=0}^{k-4} (1 - A_d)^{-i-1} \mathbb{E}[V_i].
\]

Thus, a calculation yields that

\[
\sum_{i=0}^{k-4} (1 - A_d)^{i} (1 - \tilde{\alpha})^{k-i-1} U_l \leq \frac{(1 - A_d)^k}{\alpha - A_d} \left( \mathbb{E}[d_0^2] + \frac{B_d}{A_d - A_d} \mathbb{E}[\sigma_0^2] \right)
\]

\[
+ \frac{1}{\alpha} \left( D_d + \frac{B_d D_\sigma}{A_d A_\sigma} \right) + \frac{1}{\alpha - A_d} \left( C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \right) \sum_{i=0}^{k-4} (1 - A_d)^{k-i-1} \mathbb{E}[V_i].
\]

Plugging (77) in (75) shows

\[
\sum_{i=0}^{k-4} (1 - \tilde{\alpha})^{k-i-1} \mathbb{E}[d_0^2] \leq \frac{B_d B_\sigma (S_1 + \delta_\alpha S_0)}{(1 - A_d)(A_\sigma - A_d)} \sum_{i=0}^{k-4} (1 - A_d)^{i} (1 - \tilde{\alpha})^{k-i-1} (1 + \delta_\alpha)^{l-3}
\]

\[
+ \frac{B_d B_\sigma}{(1 - A_d)(A_\sigma - A_d)} \sum_{i=0}^{k-4} \sum_{j=0}^{k-3} (1 - A_d)^{j-i+1} (1 + \delta_\alpha)^{j-2} U_j
\]

\[
+ \sum_{i=0}^{k-4} (1 - A_d)^{i} (1 - \tilde{\alpha})^{k-i-1} U_l.
\]

Hence, by combining (76), (82), (83) and (84) we obtain for \( A_d > \alpha \), that

\[
\sum_{i=0}^{k-4} (1 - \tilde{\alpha})^{k-i-1} \mathbb{E}[d_0^2] \leq \frac{B_d B_\sigma (S_1 + \delta_\alpha S_0)(1 - \alpha)^k}{(1 - A_d)(A_\sigma - A_d)(\tilde{\alpha} - \alpha)(1 + \delta_\alpha)^3}
\]
Therefore, permuting the summations implies

\[ + \left( \frac{(1 - A_d)^k}{\tilde{\alpha} - A_d} + \frac{B_d B_\sigma (1 - \alpha)^k}{(A_\sigma - A_d)(A_d - \alpha)(\tilde{\alpha} - \alpha)} \right) \left( \mathbb{E} \left[ d_0^2 \right] + \frac{B_d}{A_\sigma - A_d} \mathbb{E} \left[ \sigma_0^2 \right] \right) \]

\[ + \left( \frac{1}{\tilde{\alpha} + \alpha(\tilde{\alpha} - \alpha)(A_\sigma - A_d)} \right) \left( D_d + \frac{B_d D_\sigma}{A_d A_\sigma} \right) \]

\[ + \left( C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \right) \sum_{i=0}^{k-2} \left( \frac{(1 - A_d)^{k-i-1}}{\tilde{\alpha} - A_d} + \frac{B_d B_\sigma (1 - \alpha)^{k-i-1}}{(A_\sigma - A_d)(A_d - \alpha)(\tilde{\alpha} - \alpha)} \right) \mathbb{E} \left[ V_i \right]. \quad (85) \]

In addition, the above bound holds even if \( A_d = \alpha \) by considering that \( (A_d - \alpha)^{-1} B_d B_\sigma = 0 \).

**Upper bound on** \( \sum_{i=0}^{k-1} (1 - p_c/4)^k \sum_{j=0}^{l-1} (1 - A_d)^l (1 - \alpha)^j \sum_{i=0}^{j-1} \mathbb{E} \left[ d_i^2 \right] \). Applying (85) with \( \tilde{\alpha} = p_c/4 \) gives

\[ \sum_{i=0}^{k-1} \left( 1 - \frac{p_c}{4} \right)^{k-1-i} \mathbb{E} \left[ d_i^2 \right] \leq \]

\[ + \frac{4 (1 - A_d)^k}{p_c - 4 A_d} + \frac{4 B_d B_\sigma (1 - \alpha)^k}{(A_\sigma - A_d)(A_d - \alpha)(p_c - 4 \alpha)(1 + \delta \alpha)^3} \]

\[ + \left( \frac{4}{p_c + \alpha(p_c - 4 \alpha)(A_\sigma - A_d)} \right) \left( \mathbb{E} \left[ d_0^2 \right] + \frac{B_d}{A_\sigma - A_d} \mathbb{E} \left[ \sigma_0^2 \right] \right) \]

\[ + \left( C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \right) \sum_{i=0}^{k-2} \left( \frac{(1 - A_d)^{k-i-1}}{p_c - 4 A_d} + \frac{B_d B_\sigma (1 - \alpha)^{k-i-1}}{(A_\sigma - A_d)(A_d - \alpha)(p_c - 4 \alpha)} \right) \mathbb{E} \left[ V_i \right]. \quad (86) \]

**Upper bound on** \( \sum_{i=0}^{k-1} (1 - p_c/4)^k \sum_{j=0}^{l-1} (1 - A_d)^l (1 - \alpha)^j \sum_{i=0}^{j-1} \mathbb{E} \left[ d_i^2 \right] \). Recall that we consider that \( (A_d - \alpha)^{-1} B_d B_\sigma = 0 \) in the specific case where \( A_d = \alpha \). This time, setting \( \tilde{\alpha} = A_d \) in (85) shows that

\[ \sum_{j=0}^{l-1} (1 - A_d)^l (1 - \alpha)^j \sum_{i=0}^{j-1} \mathbb{E} \left[ d_i^2 \right] \leq \]

\[ + \left( \frac{1}{A_\sigma - A_d} + \frac{B_d B_\sigma (1 - \alpha)^j}{(A_\sigma - A_d)(A_d - \alpha)} \right) \left( \mathbb{E} \left[ d_0^2 \right] + \frac{B_d}{A_\sigma - A_d} \mathbb{E} \left[ \sigma_0^2 \right] \right) \]

\[ + \left( C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \right) \sum_{i=0}^{l-2} \left( \frac{(1 - A_d)^{l-i-1}}{A_\sigma - A_d} + \frac{B_d B_\sigma (1 - \alpha)^{l-i-1}}{(A_\sigma - A_d)(A_d - \alpha)(A_\sigma - \alpha)} \right) \mathbb{E} \left[ V_i \right]. \quad (87) \]

Moreover, we have the two following bounds

\[ \sum_{i=0}^{k-1} \left( 1 - \frac{p_c}{4} \right)^k (1 - A_d)^l \leq \frac{4 (1 - A_d)^k}{p_c - 4 A_d}, \]

\[ \sum_{i=0}^{k-1} \left( 1 - \frac{p_c}{4} \right)^k (1 - \alpha)^l \leq \frac{4 (1 - \alpha)^k}{p_c - 4 \alpha}. \quad (88) \]

Therefore, permuting the summations implies

\[ \sum_{i=0}^{k-1} \left( 1 - \frac{p_c}{4} \right)^k \sum_{i=0}^{l-2} (1 - A_d)^l (1 - \alpha)^j \sum_{i=0}^{j-1} \mathbb{E} \left[ V_i \right] \leq \sum_{i=0}^{k-3} \mathbb{E} \left[ V_i \right] \sum_{i=0}^{k-3} \left( 1 - \frac{p_c}{4} \right)^k (1 - A_d)^l \]

\[ \leq \frac{4}{p_c - 4 A_d} \sum_{i=0}^{k-3} (1 - A_d)^{k-i-1} \mathbb{E} \left[ V_i \right]. \quad (89) \]
In a similar way, we obtain

\[
\sum_{i=0}^{k-1} \left(1 - \frac{p_c}{4}\right)^{k-i-1} \sum_{i=0}^{l-2} (1 - \alpha)^{i-1} \mathbb{E}[V_i] \leq \frac{4}{p_c - 4\alpha} \sum_{i=0}^{k-1} (1 - \alpha)^{k-i-1} \mathbb{E}[V_i]. \tag{90}
\]

Hence, the combination of (87) with (88), (89), (90) yields

\[
\sum_{i=0}^{k-1} \left(1 - \frac{p_c}{4}\right)^{k-i-1} \sum_{j=0}^{l-1} (1 - \alpha^j)^{l-j-1} \mathbb{E}[d_j^2] \leq \frac{4B_d B_\sigma (S_1 + \delta_\alpha S_0) (1 - \alpha)^k}{(p_c - 4\alpha)(1 - A_d)(A_\sigma - A_d)(A_\sigma - \alpha)(1 + \delta_\alpha)^3} \]

\[+ \frac{4}{A_\sigma - A_d} \left(1 - \frac{B_d B_\sigma (1 - \alpha)^k}{p_c - 4A_d} + \frac{B_d B_\sigma (1 - \alpha)^k}{(p_c - 4\alpha)(A_\sigma - A_d)(A_\sigma - \alpha)} \mathbb{E}[d_0^2]\right) \]

\[+ \frac{4}{A_\sigma - A_d} \left(\frac{1}{A_\sigma} + \frac{B_d B_\sigma}{(A_\sigma - \alpha)(A_\sigma - A_d)} \frac{D_d}{A_d} + \frac{B_d D_\sigma}{A_d A_\sigma}\right) \]

\[+ \frac{4}{A_\sigma - A_d} \left(C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \sum_{i=0}^{k-3} \left(1 - \frac{A_d}{p_c - 4A_d} + \frac{B_d B_\sigma (1 - \alpha)^k}{(A_\sigma - A_d)(A_\sigma - \alpha)(p_c - 4\alpha)} \mathbb{E}[V_i]\right) \]

\[4(1 - p_c) \gamma^2 B_\sigma \left(C + \frac{2 + p_c}{p_c} \frac{B_d B_\sigma (1 - \alpha)^k}{(p_c - 4\alpha)(1 - A_d)(A_\sigma - A_d)(A_\sigma - \alpha)(1 + \delta_\alpha)^3} \mathbb{E}[d_0^2]\right) \]

\[+ \frac{1}{A_\sigma - A_d} \left(1 - \frac{A_d}{p_c - 4A_d} + \frac{B_d B_\sigma (1 - \alpha)^k}{(p_c - 4\alpha)(A_\sigma - A_d)(A_\sigma - \alpha)} \mathbb{E}[d_0^2]\right) \]

\[+ \frac{1}{A_\sigma - A_d} \left(\frac{1}{A_\sigma} + \frac{B_d B_\sigma}{(A_\sigma - \alpha)(A_\sigma - A_d)} \frac{D_d}{A_d} + \frac{B_d D_\sigma}{A_d A_\sigma}\right) \]

\[+ \frac{1}{A_\sigma - A_d} \left(C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \sum_{i=0}^{k-3} \left(1 - \frac{A_d}{p_c - 4A_d} + \frac{B_d B_\sigma (1 - \alpha)^k}{(p_c - 4\alpha)(A_\sigma - A_d)(A_\sigma - \alpha)} \mathbb{E}[V_i]\right) \]

\[+ \frac{4}{p_c} \gamma^2 B_\sigma \left(C + \frac{2 + p_c}{p_c} \sum_{i=0}^{k-2} (1 - A_d)^{k-i-1} \mathbb{E}[V_i]\right). \tag{91}
\]

**Upper bound on** \(\mathbb{E}[V_k].\) **Plugging** (86) and (91) in (65), we obtain

\[
\mathbb{E}[V_k] \leq \left(1 - \frac{p_c}{4}\right)^k \mathbb{E}[V_0] + \frac{4(1 - p_c) \gamma^2 (1 - A_d)^k}{p_c - 4A_d} \left(C + \frac{2 + p_c}{p_c} \mathbb{E}[d_0^2]\right) \]

\[+ \frac{1}{A_\sigma - A_d} \left(1 - \frac{A_d}{p_c - 4A_d} + \frac{B_d B_\sigma (1 - \alpha)^k}{(p_c - 4\alpha)(A_\sigma - A_d)(A_\sigma - \alpha)} \mathbb{E}[d_0^2]\right) \]

\[+ \frac{1}{A_\sigma - A_d} \left(\frac{1}{A_\sigma} + \frac{B_d B_\sigma}{(A_\sigma - \alpha)(A_\sigma - A_d)} \frac{D_d}{A_d} + \frac{B_d D_\sigma}{A_d A_\sigma}\right) \]

\[+ \frac{1}{A_\sigma - A_d} \left(C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \sum_{i=0}^{k-3} \left(1 - \frac{A_d}{p_c - 4A_d} + \frac{B_d B_\sigma (1 - \alpha)^k}{(p_c - 4\alpha)(A_\sigma - A_d)(A_\sigma - \alpha)} \mathbb{E}[V_i]\right) \]

\[+ \frac{4}{p_c} \gamma^2 B_\sigma \left(C + \frac{2 + p_c}{p_c} \sum_{i=0}^{k-2} (1 - A_d)^{k-i-1} \mathbb{E}[V_i]\right). \tag{92}
\]

For any negative number \(j < 0,\) using the convention that \(\sum_{i=0}^j = 0\) and simplifying the calculations provided by (92), we find that

\[
\mathbb{E}[V_k] \leq \left(1 - \frac{p_c}{4}\right)^k \mathbb{E}[V_0] + \frac{4(1 - p_c) \gamma^2 (1 - A_d)^k}{p_c - 4A_d} \left(C + \frac{2 + p_c}{p_c} \mathbb{E}[d_0^2]\right)
\]
As explained in (73), recall that

\[ \delta_\alpha = \frac{B_d B_\sigma}{(1 - A_d)(A_\sigma - A_d)}, \quad \alpha = A_d - \delta_\alpha(1 - A_d). \]

Thus, when \( B_d B_\sigma \neq 0 \) then \( \delta_\alpha \neq 0 \), which implies that \( A_d \neq \alpha \) and gives

\[ \frac{B_d B_\sigma}{(1 - A_d)(A_\sigma - A_d)} = 1 + \delta_\alpha. \]

In addition, in the proof of Lemma 9 we saw that \( 2\delta_\alpha \leq A_d \leq 1/2 \) and also that \( A_d/2 \leq \alpha \leq A_d \). Therefore, we can regroup several terms in (93) and write

\[
\begin{align*}
\mathbb{E}[V_k] &\leq \left(1 - \frac{p_c}{4}\right)^k \mathbb{E}[V_0] + \frac{4(1 - p_c)\gamma^2(1 - A_d)^k}{p_c - 4A_d} \left[ C + \frac{2 + p_c}{p_c} \bar{C} \right] \mathbb{E}[\sigma^2] \\
&+ \frac{4(1 - p_c)\gamma^2\delta_\alpha(S_1 + \delta_\alpha S_0)(1 - \alpha)^k}{p_c - 4A_d}(1 + \delta_\alpha)^2 \left[ B + \frac{2 + p_c}{p_c} \bar{B} + \frac{B_\sigma}{A_\sigma - A_d} \left( C + \frac{2 + p_c}{p_c} \bar{C} \right) \right] \mathbb{E}[\sigma^2] \\
&+ \frac{9\gamma^2(1 - p_c)(1 - \alpha)^k}{p_c - 4A_d} \left[ B + \frac{2 + p_c}{p_c} \bar{B} + \frac{B_\sigma}{A_\sigma - A_d} \left( C + \frac{2 + p_c}{p_c} \bar{C} \right) \right] \mathbb{E}[\sigma^2] + \frac{B_d B_\sigma}{A_\sigma - A_d} \mathbb{E}[\sigma^2] \\
&+ \frac{4(1 - p_c)\gamma^2 D_\sigma}{A_\sigma(p_c - 4A_d)} \left[ C + \frac{2 + p_c}{p_c} \bar{C} \right] + \frac{4(1 - p_c)\gamma^2(1 - A_d)^k}{p_c - 4A_d} \left[ D + \frac{2 + p_c}{p_c} \bar{D} \right] + \frac{B_\sigma}{A_\sigma - A_d} \mathbb{E}[\sigma^2] \\
&+ \frac{B_\sigma}{p_c} \left[ \frac{1}{A_\sigma} + \frac{2B_d B_\sigma}{A_d(A_\sigma - A_d)^2} \right] \left( C + \frac{2 + p_c}{p_c} \bar{C} \right) \mathbb{E}[\sigma^2] \\
&+ \frac{B_\sigma}{p_c} \left[ \frac{1}{A_\sigma} + \frac{2B_d B_\sigma}{A_d(A_\sigma - A_d)^2} \right] \left( C + \frac{2 + p_c}{p_c} \bar{C} \right) \mathbb{E}[\sigma^2] \\
&+ \frac{9\gamma^2(1 - p_c)}{p_c - 4A_d} \left[ C_\sigma \left( C + \frac{2 + p_c}{p_c} \bar{C} \right) + \left( C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \right) \left( B + \frac{2 + p_c}{p_c} \bar{B} + \frac{B_\sigma}{A_\sigma - A_d} \left( C + \frac{2 + p_c}{p_c} \bar{C} \right) \right) \right].
\end{align*}
\]

Therefore, we have
\[
\sum_{i=0}^{k-2} (1 - \alpha)^{k-i-1} \mathbb{E}[V_i]. \tag{94}
\]

Recall that we defined \( C^\gamma \) in (59) by
\[
C^\gamma = \frac{4(1 - p_c)\gamma^2}{p_c - 4A_d} \left[ B + \frac{2 + p_c}{p_c} D + \frac{B_{\sigma}}{A\sigma - A_d} \left( C + \frac{2 + p_c}{p_c} C \right) \right].
\]

Hence, using (94) we get that
\[
\mathbb{E}[V_k] \leq \left( 1 - \frac{p_c}{4} \right)^k \mathbb{E}[V_0] + \frac{4(1 - p_c)\gamma^2}{p_c - 4A_d} \left( C + \frac{2 + p_c}{p_c} C \right) + \frac{4(1 - p_c)\gamma^2}{p_c} \left( D + \frac{2 + p_c}{p_c} D \right)
+ \frac{C^\gamma}{A_d} \left( \frac{1}{b} + \frac{2B_dB_\sigma}{A_d(A\sigma - A_d)} \right) \left( D_d + B_dD_\sigma \right) + \frac{8(1 - \tau)(b - 1)\gamma d}{b p_c}
+ \left( \frac{4(1 - p_c)\gamma^2}{p_c - 4A_d} \right) \left( C + \frac{2 + p_c}{p_c} C \right) + \frac{9C^\gamma(1 - \alpha)^k}{4(A\sigma - A_d)} \mathbb{E}[\sigma_0^2]
+ \left( \frac{9r^2(1 - p_c)C_d}{p_c - 4A_d} \right) \left( C + \frac{2 + p_c}{p_c} C \right) + 3C^\gamma \left( C_d + \frac{B_dC_\sigma}{A\sigma - A_d} \right) \sum_{i=0}^{k-2} (1 - \alpha)^{k-i-1} \mathbb{E}[V_i].
\]

Finally, we conclude the proof remarking that
\[
C^\gamma (1 - \alpha)^{k-2} (A_d - \alpha) [(1 - A_d)S_1 + (A_d - \alpha)S_0]
\leq C^\gamma (1 - \alpha)^{k-2} (A_d - \alpha) [(2 - A_d - \alpha)\mathbb{E}[d_0^2] + B_d\mathbb{E}[\sigma_0^2] + C_d\mathbb{E}[V_0] + D_d]
\leq C^\gamma (1 - \alpha)^k \left( 4\mathbb{E}[d_0^2] + 2B_d\mathbb{E}[\sigma_0^2] + 2C_d\mathbb{E}[V_0] + 2D_d \right).
\]

In order to ease notation, with the definitions used in \textbf{HX4} and (59), consider for any \( \gamma \in \mathbb{R}_+ \) the variable \( C^\gamma_\epsilon \in \mathbb{R}_+ \) defined by
\[
C^\gamma_\epsilon = C^\gamma_\epsilon \mathbb{E}[V_0] + C^\gamma_\epsilon \mathbb{E}[d_0^2] + C^\gamma_\epsilon \mathbb{E}[\sigma_0^2] + 2D_d
\tag{95}
\]
In addition, with the previous notations consider
\[
\delta = \frac{2(1 - A_d/2)^{-1} C^\gamma_\epsilon}{1 + \sqrt{1 + 4(1 - A_d/2)^{-1} C^\gamma_\epsilon}}.
\]
and define
\[
\gamma_\nu = \frac{p_c^{1/2}}{(2 - 2p_c)^{1/2} \left[ \epsilon + (1 + 2/p_c)\bar{A} \right]^{1/2}}.
\]

\textbf{Lemma 11.} \textit{Assume \textbf{HX3}, \textbf{HX4} hold with \( 4C^\gamma_\epsilon \leq A_d < \min(A_\sigma/2, p_c/4) \), \( A_dA_\sigma \geq 8B_dB_\sigma \) and let \( \gamma \in (0, \gamma_\nu] \). Then, for any \( k \geq 1 \), we have}
\[
\mathbb{E}[V_k] \leq \left( 1 - \frac{A_d}{4} \right)^k \left( 2C^\gamma_\epsilon + \frac{4C^\gamma_\epsilon C^\gamma_\epsilon}{A_d} \right) + C^\gamma_\epsilon,
\]
\textit{where \( V_k \) is defined in (24), \( C^\gamma_\epsilon, C^\gamma, C^\gamma_\delta \) in (59) and (95).}

\textit{Proof.} Let \( k \in \mathbb{N} \) be fixed. Since the assumptions of Lemma 10 are satisfied, we know that
\[
\mathbb{E}[V_k] \leq (1 - \alpha)^k C^\gamma_\epsilon + C^\gamma_\epsilon \sum_{i=0}^{k-2} (1 - \alpha)^{k-i-1} \mathbb{E}[V_i] + C^\gamma_\epsilon,
\]
where $\alpha$ is defined in (60). In addition, Lemma 9 shows that $A_d/2 \leq \alpha$. Hence, multiplying the last inequality by the weight $\omega_k$ defined for any $l \in \mathbb{N}$, by

$$\omega_l = (1 - A_d/2)^{-l},$$

we obtain the following inequality

$$\omega_k \mathbb{E}[V_k] \leq C^*_\gamma + \frac{C^*_r}{1 - A_d/2} \sum_{l=0}^{k-2} \omega_l \mathbb{E}[V_l] + C^*_\delta \omega_k.$$

Applying the sharp Gronwall inequality (Holte, 2009), we get

$$\omega_k \mathbb{E}[V_k] \leq C^*_\gamma + \omega_k C^*_\gamma + \frac{C^*_r}{1 - A_d/2} \sum_{l=0}^{k-1} (C^*_\gamma + \omega_l C^*_r) \left(1 + \frac{C^*_r}{1 - A_d/2}\right)^{k-l-1}.$$

Therefore, a calculation shows that

$$\omega_k \mathbb{E}[V_k] \leq C^*_\gamma + \omega_k C^*_\gamma + C^*_r \left(1 + \frac{C^*_r}{1 - A_d/2}\right)^k + \frac{C^*_r}{A_d/2} \sum_{l=0}^{k-1} \left(1 - \frac{A_d}{2} + C^*_r\right)^{k-l-1},$$

and simplifying the previous inequality gives the following upper bound:

$$\mathbb{E}[V_k] \leq C^*_\gamma + \omega_k^{-1} C^*_\gamma + C^*_r \left(1 - \frac{A_d}{2} + C^*_r\right)^k + \frac{C^*_r C^*_\gamma}{A_d/2 - C^*_r} \left(1 - \frac{A_d}{2} + C^*_r\right)^k. \quad (96)$$

In addition, using $4C^*_r < A_d < p_c/4$ implies $0 < 1 - A_d/2 + C^*_r < 1$ which combined with (96) gives

$$\mathbb{E}[V_k] \leq C^*_\gamma + \omega_k^{-1} C^*_\gamma + C^*_r \left(1 - \frac{A_d}{2} + C^*_r\right)^k + \frac{C^*_r C^*_\gamma}{A_d/2 - C^*_r} \left(1 - \frac{A_d}{2} + C^*_r\right)^k.$$

Eventually, combining the last inequality with the assumption $4C^*_r < A_d$ completes the proof. \qed

With the notation of the assumptions HX3 and HX4, we define

$$\alpha_d = \frac{4\gamma^2}{p_c A_d} \max \left\{ p_c B + 3B, \frac{4B_\sigma A_\sigma}{p_c}, \left( p_c C + 3C \right) \right\}, \quad \alpha_\sigma = \frac{4\gamma^2 \left( p_c C + 3C \right)}{p_c A_\sigma}. \quad (97)$$

The following lemma is used in the convergence proof of VR-FALD* (see Lemma 26).

**Lemma 12.** Assume HX3, HX4 hold with

$$A_d \leq \min \left( A_\sigma, \frac{p_c}{4} \right), \quad \alpha_d C_d + \alpha_\sigma C_\sigma \leq \frac{p_c}{8}, \quad \alpha_d B_d + \gamma^2 \left( C + \frac{3}{p_c} \tilde{C} \right) \leq \frac{\alpha_\sigma A_\sigma}{2},$$

and consider $\gamma \leq p_c^{-1/2} \left( 2 - 2p_e \right)^{-1/2} \left( A + (1 + 2/p_c) \tilde{A} \right)^{-1/2}$. Then, for any $k \in \mathbb{N}$, we have

$$\mathbb{E}[V_k] + \alpha_d \mathbb{E}[d_k^2] + \alpha_\sigma \mathbb{E}\left[ \sigma_k^2 \right] \leq \left( 1 - \frac{A_d}{2} \right)^k \left( \mathbb{E}[V_0] + \alpha_d \mathbb{E}[d_0^2] + \alpha_\sigma \mathbb{E}\left[ \sigma_0^2 \right] \right) + \frac{2(1 - p_c)\gamma^2}{A_d} \left( D + \frac{2 + p_c}{p_c} \tilde{D} \right) + \frac{2\alpha_d D_d + 2\alpha_\sigma D_\sigma}{A_d} + \frac{4 (1 - \tau) (b - 1) \gamma d}{b A_d},$$

where $V_k$ is defined in (24).

**Proof.** Let $k \in \mathbb{N}^*$, using for $i \in [b]$ the definitions (20), (23) of $X_i^k$ and $X_k$

$$X_{k+1}^i = X_k^i - \gamma G_k^i + \sqrt{2\gamma} \left( \sqrt{\tau / b} \tilde{Z}_{k+1} + \sqrt{1 - \tau} \tilde{Z}_{k+1}^i \right),$$
\[ X_{k+1} = X_k - \frac{\gamma}{b} \sum_{j=1}^{b} G_j^i + \sqrt{\frac{2\gamma}{b} Z_{k+1}} + \frac{\sqrt{2(1-\tau)\gamma}}{b} \sum_{i=1}^{b} Z_{k+1}. \]

Subtracting the two above equations combined with the Jensen inequality give

\[
\mathbb{E} [V_{k+1}] = \frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| X_{k+1}^i - X_k \|^2 \right]
= \frac{1-p_c}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| (X_{k+1}^i - X_k) - \gamma (G_k^i - G_k^i) \|^2 \right] + \frac{1-p_c}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| (G_k^i - G_k^i) - (G_k^i - G_k^i) \|^2 \right]
+ 2(1-\tau)\gamma \mathbb{E} \left[ \left\| Z_{k+1}^i - \frac{1}{b} \sum_{j=1}^{b} Z_{k+1}^j \right\|^2 \right].
\]

Hence, we get

\[
\mathbb{E} [V_{k+1}] \leq \frac{1-p_c}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| (X_{k+1}^i - X_k) - \gamma (G_k^i - G_k^i) \|^2 \right] + \frac{(1-p_c)^2}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| G_k^i - G_k^i \|^2 \right]
+ 2(1-\tau)(1-1/b)\gamma c.
\]

We finally obtain

\[
\mathbb{E} [V_{k+1}] \leq \frac{(1-p_c)(2+p_c)}{p_c} \mathbb{E} [V_k] + \frac{(1-p_c)(2+p_c)}{p_c} \mathbb{E} [G_k^i] \mathbb{E} [V_k] + (1-p_c)^2 \mathbb{E} [G_k^i] + (1-p_c) \mathbb{E} [G_k^i] + 2(1-\tau)(1-1/b)\gamma c.
\]

Combining the last inequality with \( \textbf{HX4} \) shows

\[
\mathbb{E} [V_{k+1}] \leq \left( 1 - \frac{p_c}{2} + (1-p_c) \gamma A \left[ A + \frac{2 + p_c}{p_c} \bar{A} \right] \right) \mathbb{E} [V_k] + (1-p_c) \gamma D \left( D + \frac{2 + p_c}{p_c} D \right)
+ (1-p_c) \gamma C \left( C + \frac{2 + p_c}{p_c} \bar{C} \right) \mathbb{E} [\sigma_k^2] + 2(1-\tau)(1-1/b)\gamma c.
\]

Since \( \gamma \geq \frac{p_c}{2(1-p_c)^{1/2}[A+(1+2/p_c)\bar{A}]^{1/2}} \), the above inequality implies that

\[
\mathbb{E} [V_{k+1}] \leq \left( 1 - \frac{p_c}{4} \right) \mathbb{E} [V_k] + (1-p_c) \gamma D \left( D + \frac{2 + p_c}{p_c} D \right) + 2(1-\tau)(1-1/b)\gamma c
+ (1-p_c) \gamma C \left( C + \frac{2 + p_c}{p_c} \bar{C} \right) \mathbb{E} [\sigma_k^2].
\]

The previous bound combined with \( \textbf{HX3} \) gives that
we denote $A_4$. The assumption $A$ is equivalent to

$$E[V_{k+1}] + \alpha_d E[d^2_{k+1}] + \alpha_E E[\sigma^2_{k+1}] \leq \left(1 - \frac{p_c}{4}\right) + \alpha_d C_d + \alpha_E C_E E[V_k]$$

\[
\begin{align*}
&+ \left[\alpha_d(1 - A_d) + \alpha_L B_d + (1 - p_c)\gamma^2 \left(B + \frac{2 + p_c}{p_c} \bar{B}\right)\right] E[d^2_k] \\
&+ \left[\alpha_L(1 - A_d) + \alpha_d B_d + (1 - p_c)\gamma^2 \left(C + \frac{2 + p_c}{p_c} \bar{C}\right)\right] E[\sigma^2_k] \\
&+ (1 - p_c)\gamma^2 \left(D + \frac{2 + p_c}{p_c} \bar{D}\right) + 2 \left(1 - \frac{b}{b} - \gamma d + \alpha_d D_d + \alpha_E D_E. \right)
\end{align*}
\]

By assumption, we have

$$\alpha_d C_d + \alpha_E C_E \leq \frac{p_c}{8},$$

$$\alpha_d B_d + \gamma^2 \left(C + \frac{3}{p_c} \bar{C}\right) \leq \frac{\alpha_d A_d}{2},$$

and by definition of $\alpha_d, \alpha_L$ given in (97), we know that $\alpha_d B_d + \gamma^2 (B + 3 \bar{B} + p_c) \leq \alpha_d A_d/2$. In addition, since we suppose that $A_d \leq \min(p_c/4, A_d)$, the last inequalities combined with (99) imply

$$1 - \frac{p_c}{4} + \alpha_d C_d + \alpha_E C_E \leq 1 - \frac{A_d}{2}$$

$$1 - A_d + \alpha_d B_d + \frac{(1 - p_c)\gamma^2}{\alpha_d} \left(B + \frac{2 + p_c}{p_c} \bar{B}\right) \leq 1 - \frac{A_d}{2}$$

$$1 - A_d + \frac{\alpha_d}{\alpha_L} B_D + \frac{(1 - p_c)\gamma^2}{\alpha_L} \left(C + \frac{2 + p_c}{p_c} \bar{C}\right) \leq 1 - \frac{A_d}{2}.$$

Thus, by taking up (98) and using (100), we get

$$E[V_{k+1}] + \alpha_d E[d^2_{k+1}] + \alpha_E E[\sigma^2_{k+1}] \leq \left(1 - \frac{A_d}{2}\right) \left(E[V_k] + \alpha_d E[d^2_k] + \alpha_E E[\sigma^2_k]\right)$$

\[
+ (1 - p_c)\gamma^2 \left(D + \frac{2 + p_c}{p_c} \bar{D}\right) + 2 \left(1 - \frac{b}{b} - \gamma d + \alpha_d D_d + \alpha_E D_E. \right)
\]

Finally, the stated result follows by induction.

7 Main results

Section 7 is divided into four subsections in which we prove theoretical results for the FALD and VR-FALD* algorithms. These analyses are presented in Theorem 20 and Theorem 28. The proofs are based on Lemma 11 proved in Section 6.3 to ensure that the local parameters $\{X_i\}_{i \in [b]}$ do not deviate too much from $X_k$, then we apply the general result given in Section 6 to obtain explicit upper bounds for $W_2^2(\pi, \mu_k^{(\gamma)})$.

Until the end of the paper, we consider a family of independent random variables $(\xi^i, x, y, \nu^i, \mu_k^{(\gamma)})$, and we denote $(H^i)_{i \in [b]}$ a family of functions defined on $\mathbb{R}^d \times \mathbb{E} \to \mathbb{R}^d$ such that for each $i \in [b], x \in \mathbb{R}^d$, $H^i(x, \xi^i, \nu^i, \mu_k^{(\gamma)})$ is measurable on $(\mathbb{E}, \mathcal{E})$ and satisfies the following condition:

A4. Assume there exists $\hat{L} \geq 0$, such that for any $i \in [b], x, y \in \mathbb{R}^d$, we have

$$E[H^i(x, \xi^i)] = \nabla U^i(x),$$

$$E \left[\left\|H^i(y, \xi^i) - H^i(x, \xi^i)\right\|^2\right] \leq \hat{L}^2 \left\|y - x\right\|^2.$$

The assumption A4 is equivalent to A2 written in the main paper, though for clarity we prefer to replace the stochastic gradient $\nabla U_k^i$ by $H^i(\cdot, \xi^i)$. To simplify the notation, in what follows we consider the random variable $\xi = (\xi^1, \ldots, \xi^b)$, and we denote

$$H : \left(\mathbb{R}^d \times \mathbb{E}^b \to \mathbb{R}^d \right)$$

$$\left(x, z\right) \mapsto \sum_{i=1}^b H^i(x, z^i).$$
Thus, for each $x \in \mathbb{R}^d$, with this notation we have $H(x, \xi) = \sum_{i=1}^{b} H^i(x, \xi_i)$. We also introduce the averaged versions $\bar{U}, \bar{H}$ of the local potentials $\{U^i\}_{i \in [b]}$ and the stochastic gradients $\{H^i\}_{i \in [b]}$ defined by

$$
\bar{U}(x) = \frac{1}{b} \sum_{i=1}^{b} U^i(x), \quad \bar{H}(x, z) = \frac{1}{b} \sum_{i=1}^{b} H^i(x, z^i).
$$

**Remark 13.** In the mini-batch scenario without replacement, the $i$th client draws a mini-batch $J_i$ subset $[N_i]$ of size $n_i = |J_i| \in [N_i]$ among $N_i$ data and computes its stochastic gradient, which for $x \in \mathbb{R}^d$ is given by $H^i(x, \xi_i) = \sum_{j \in J_i} \nabla U^{i,j}(x)$. Using the result provided in Vono et al. (2022b, Lemma S4), we know that

$$
\mathbb{E}\left[\left\|H^i(y, \xi_i) - H^i(x, \xi_i)\right\|^2\right] = \left\|\nabla U^i(y) - \nabla U^i(x)\right\|^2 + \text{Var}(H^i(y, \xi_i) - H^i(x, \xi_i)) \leq \left(1 + \frac{n_i(N_i - n_i) \max_{j=1}^{N_i} L_j}{N_i(N_i - 1)L}\right)L^2 \|y - x\|^2.
$$

Therefore, A4 is satisfied for a choice of $\hat{L} > 0$ such that

$$
\hat{L} \leq L \sqrt{1 + \max_{i=1}^{b} \left\{n_i(N_i - n_i)[N_i(N_i - 1)]^{-1}(\max_{j=1}^{N_i} L_j)^{-1}\right\}}.
$$

**A5.** For $i \in [b], j \in [N_i]$, assume that $U^{i,j}$ is continuously differentiable, convex and there exists $L_{ij} > 0$ such that for any $x, y \in \mathbb{R}^d$,

$$
U^{i,j}(y) \leq U^{i,j}(x) + \langle \nabla U^{i,j}(x), y - x \rangle + \frac{L_{ij}}{2} \|y - x\|^2.
$$

**A6.** Assume there exists $\tilde{\omega} > 0$ such that for any $x \in \mathbb{R}^d$,

$$
\mathbb{E}\left[\left\|H(x, \xi) - H(x_*, \xi) - \nabla U(x)\right\|^2\right] \leq \tilde{\omega} b^2 \|x - x_*\|^2.
$$

A1 combined with A4 implies A6 with $\tilde{\omega} = 2L^2 + 2\hat{L}^2$. However, this new assumption A6 is interesting because without stochastic gradient we obtain $\tilde{\omega} = 0$, which allows us to recover the classical Langevin bounds.

**Remark 14.** Consider the same scenario as detailed in Remark 13 and define

$$
\tilde{\omega} = \left(\sum_{i=1}^{b} \frac{n_i(N_i - n_i) \max_{j=1}^{N_i} L_j}{b^2N_i(N_i - 1)}\right)L.
$$

Applying Vono et al. (2022b, Lemma S4) we have the following lines

$$
\mathbb{E}\left[\left\|H(x, \xi) - H(x_*, \xi) - \nabla \bar{U}(x)\right\|^2\right] = \text{Var}(H(x, \xi) - H(x_*, \xi)) = \frac{1}{b^2} \sum_{i=1}^{b} \text{Var}(H^i(x, \xi_i) - H^i(x_*, \xi_i)) \leq \tilde{\omega} \|x - x_*\|^2.
$$

Therefore, A6 is satisfied and in the deterministic case where all data are used to calculate the gradient, we have $\tilde{\omega} = 0$.

To deal with variance reduction based algorithms, we consider the following assumption A7, which is also implied by A1-A4, however the constant $\omega$ vanishes with exact gradient computation.

**A7.** Assume there exists $\omega \geq 0$ such that for any $i \in [b]$ and $x, y \in \mathbb{R}^d$,

$$
\mathbb{E}\left[\left\|H^i(x, \xi_i) - H^i(y, \xi_i) - \nabla U^i(x) + \nabla U^i(y)\right\|^2\right] \leq \omega \|x - y\|^2.
$$

**Remark 15.** In the mini-batch scenario without replacement detailed in Remark 13, the use of Vono et al. (2022b, Lemma S4) implies that

$$
\mathbb{E}\left[\left\|H^i(x, \xi_i) - H^i(y, \xi_i) - \nabla U^i(x) + \nabla U^i(y)\right\|^2\right] = \text{Var}(H^i(x, \xi_i) - H^i(y, \xi_i))
$$
Thus, \( A7 \) is satisfied by setting

\[
\omega = \max_{i=1}^{N} \left\{ \frac{n_i(N_i - n_i)}{N_i(N_i - 1)} \right\} L \frac{n_i}{\max_j L_j} \left\| x - y \right\|^2.
\]

In the deterministic case, we obtain \( \omega = 0 \). Similarly, in the mini-batch scenario with replacement it is sufficient to set

\[
\omega = \frac{N_i - n_i}{n_i} \sum_{j=1}^{N_i} \left( L_j^i \right)^2
\]

to ensure that \( A7 \) holds.

### 7.1 Study of FALD

#### 7.1.1 Remark on the theoretical analysis of Deng et al. (2021)

FALD has been proposed in Deng et al. (2021), the authors develop an MCMC algorithm targeting the distribution proportional to \( \exp\left(-b^{-1} \sum_{i=1}^{b} U^i \right) \) and also establish non-asymptotic bounds. They introduce (Deng et al., 2021, Lemma B.2) the stochastic processes \( \{ \theta_t^i \}_{t \geq 0} \) satisfying the Langevin stochastic differential equations for \( t \geq 0 \),

\[
d\theta_t^i = -\nabla U^i(\theta_t^i) + \sqrt{2b} dW_t^i \quad \text{where} \quad \{ (W_t^i)_{t \geq 0} \}_{i \in [b]} \text{ are independent } d \text{-dimensional standard Brownian motion and define } \theta_t = b^{-1} \sum_{i=1}^{b} \theta_t^i.
\]

Then, it is asserted (Deng et al., 2021, Lemma B.5) that \( \theta_t \) is solution of the Langevin stochastic differential equation \( d\theta_t = -b^{-1} \sum_{i=1}^{b} \nabla U^i(\theta_t) + \sqrt{2} dW_t \), where \( W_t = b^{-1/2} \sum_{i=1}^{b} W_t^i \). However, this statement cannot hold in all generalities, and we give a counter-example. For instance, consider the Gaussian potentials \( \{ U^i : x \in \mathbb{R}^d \to \Sigma_i^{-1}(x - m^i) \}_{i \in [b]} \) where \( \{ (m^i, \Sigma_i) \}_{i \in [b]} \) are the mean and the covariance parameters; if for \( i \in [b] \), \( \theta_0^i \) is distributed according to \( \exp(-U^i) \), then \( b^{-1} \sum_{i=1}^{b} \theta_t^i \) follows \( N(b^{-1} \sum_{i=1}^{b} m^i, b^{-2} \sum_{i=1}^{b} \Sigma_i) \) whereas \( \exp(-b^{-1} \sum_{i=1}^{b} U^i) \) corresponds to the density of the Gaussian \( N(\sum_{i=1}^{b} \Sigma_i^{-1} m^i, b \Sigma) \) where \( \Sigma = (\sum_{i=1}^{b} \Sigma_i)^{-1} \). Therefore, for any \( t \geq 0 \), in this case \( \theta_t \) is distributed according to \( N(b^{-1} \sum_{i=1}^{b} m^i, b^{-2} \sum_{i=1}^{b} \Sigma_i) \) and thus cannot be distributed according to \( \exp(-b^{-1} \sum_{i=1}^{b} U^i) \) as crucially used in the proof of Deng et al. (2021, Lemma B.5).

#### 7.1.2 Theoretical analysis

In this section, we prove the first theoretical guarantee on FALD stated in Theorem 1. Similar to McMahan et al. (2017), the clients update their local parameters \( \{ X_k^i \}_{i \in [b]} \) several times before transmitting them to the server with probability \( p_c \in (0, 1] \). Then, the server aggregates the local parameters to update its own parameter \( X_k \) as in (23). For all \( i \in [b], k \in \mathbb{N} \), consider the stochastic gradients defined by

\[
G_k^i = H^i(X_k^i, \xi_{k+1}^i),
\]

\[
\overline{G}_k^i = \nabla U^i(X_k^i).
\]

**Lemma 16.** Assume \( A1, A4 \) and \( A6 \) hold. Then for any \( k \in \mathbb{N} \), we have

\[
\frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| \bar{G}_k^i \right\|^2 \right] \leq 3L^2 \mathbb{E} \left[ V_k \right] + 3L^2 \mathbb{E} \left[ d_k^2 \right] + \frac{3}{b} \sum_{i=1}^{b} \left\| \nabla U^i(x_*) \right\|^2,
\]

\[
\frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| G_k^i - \bar{G}_k^i \right\|^2 \right] \leq 3L^2 \mathbb{E} \left[ V_k \right] + 3\xi \mathbb{E} \left[ d_k^2 \right] + 3\mathbb{E} \left[ \left\| \tilde{H}(x_*, \xi) \right\|^2 \right].
\]

For any \( i \in [b], k \in \mathbb{N} \), recall the stochastic gradients \( G_k^i, \overline{G}_k^i \) are defined in (101) and (102), respectively.
Algorithm 2 Stochastic Averaging Langevin Dynamics - FALD

**Input:** initial vectors \((X^i_0)_{i \in [b]}\), noise parameter \(\tau \in [0, 1]\), number of communication rounds \(K\), probability \(p_e\) of communication, step-size \(\gamma\).

for \(k = 0\) to \(K - 1\) do
  // On each client
  Draw \(B_{k+1} \sim B(p_e)\), \(\tilde{Z}_{k+1} \sim N(0_d, I_d)\)
  // In parallel on the \(b\) clients
  for \(i = 1\) to \(b\) do
    Draw \(\xi^i_{k+1} \sim \nu_c\) and \(\tilde{Z}^i_{k+1} \sim N(0_d, I_d)\)
    Compute \(G^i_k = H^i(X^i_k, \xi^i_{k+1})\)
    Set \(\bar{X}^i_{k+1} = X^i_k - \gamma G^i_k + \sqrt{2\gamma}(\sqrt{\tau/b} \tilde{Z}_{k+1} + \sqrt{1 - \tau} \tilde{Z}^i_{k+1})\)
    if \(B_{k+1} = 1\) then
      Update the local parameter \(X^i_{k+1} \leftarrow \bar{X}^i_{k+1}\)
    else
      Update \(X^i_{k+1} \leftarrow \bar{X}^i_{k+1}\)
  end for
if \(B_{k+1} = 1\) then
  // On the central server
  Update then broadcast the global parameter \(X_{k+1} = \frac{1}{b} \sum_{i=1}^b \bar{X}^i_{k+1}\)
  // On each client
  Update the local parameter \(X^i_{k+1} \leftarrow X_{k+1}\)

**Output:** samples \(\{X^i\}_{i \in [K] : B_i = 1}\).

**Proof.** Using the Young inequality combined with the Lipschitz property A.1 of the gradients \((U^i)^b_i\), for \(k \geq 0\) we get

\[
\frac{1}{b} \sum_{i=1}^b \mathbb{E} \left[ \|G^i_k\|^2 \right] = \frac{1}{b} \sum_{i=1}^b \mathbb{E} \left[ \|\nabla U^i(X^i_k) - \nabla U^i(X_k) + \nabla U^i(X_k) - \nabla U^i(x_*) + \nabla U^i(x_*)\|^2 \right] \leq 3L^2 \mathbb{E} [V_k] + 3L^2 \mathbb{E} [d^2_k] + \frac{3}{b} \sum_{i=1}^b \|\nabla U^i(x_*)\|^2.
\]

In addition, since the random variables \((G^i_k - \bar{G}^i_k)_{i=1}^b\) are centered and independent, the Young and the Jensen inequality imply that

\[
\frac{1}{b} \sum_{i=1}^b \mathbb{E} \left[ \|G^i_k - \bar{G}^i_k\|^2 \right] = \mathbb{E} \left[ \frac{1}{b} \sum_{i=1}^b \left( G^i_k - \bar{G}^i_k \right) \right]^2 = \mathbb{E} \left[ \frac{1}{b} \sum_{i=1}^b H^i(X^i_k, \xi^i_{k+1}) - \bar{H}(X_k, \xi_{k+1}) + \bar{H}(X_k, \xi_{k+1}) - \bar{H}(x_*, \xi_{k+1}) + \bar{H}(x_*, \xi_{k+1}) - \nabla \bar{U}(X_k) + \nabla \bar{U}(X_k) - \frac{1}{b} \sum_{i=1}^b \nabla U^i(X^i_k) \right]^2 \leq 3 \mathbb{E} \left[ \left( \frac{1}{b} \sum_{i=1}^b H^i(X^i_k, \xi^i_{k+1}) - \bar{H}(X_k, \xi_{k+1}) \right)^2 \right] + 3 \mathbb{E} \left[ \|\bar{H}(x_*, \xi)\|^2 \right] + 3 \mathbb{E} \left[ \|\bar{H}(x_*, \xi)\|^2 \right] \leq 3L^2 \mathbb{E} [V_k] + 3 \mathbb{E} [d^2_k] + 3 \mathbb{E} \left[ \|\bar{H}(x_*, \xi)\|^2 \right].
\]

\[\Box\]

**Lemma 17.** Assume A1 and A4 hold. Then, for any \(\gamma \in (0, m(6L^2)^{-1})\), we have

\[
\mathbb{E} [d^2_{k+1}] \leq \left(1 - \frac{\gamma m}{2}\right) \mathbb{E} [d^2_k] + \frac{2\gamma L^2}{m} \mathbb{E} [V_k] + 3\gamma^2 \mathbb{E} \left[ \|\bar{H}(x_*, \xi)\|^2 \right] + \frac{2\gamma d}{b},
\]
where $V_k, d_k$ are defined in (24) and (25).

**Proof.** Let $k$ be in $\mathbb{N}$. Rewriting the expression of $X_{k+1}$ defined in (23), we obtain

\[
E[d^2_{k+1}] = E\left[\|X_{k+1} - x_+\|^2\right] = E\left[\|X_k - x_+ - \frac{1}{b}\sum_{i=1}^{b} H^i(X^i_k, \xi^i_{k+1}) + \sqrt{2\gamma} \left(\sqrt{\frac{\tau}{b}} \bar{Z}_{k+1} + \frac{\sqrt{1 - \tau}}{b} \sum_{i=1}^{b} Z^i_{k+1}\right)\|^2\right]
\]

\[
= E\left[\|X_k - x_+\|^2\right] - 2\gamma E\left[\left\langle X_k - x_+, \frac{1}{b}\sum_{i=1}^{b} H^i(X^i_k, \xi^i_{k+1}) \right\rangle\right] + \gamma^2 E\left[\left\|\frac{1}{b}\sum_{i=1}^{b} H^i(X^i_k, \xi^i_{k+1})\right\|^2\right] + \frac{2\gamma d}{b}.
\]  

(103)

Further, the Young inequality combined with A4 give

\[
E\left[\left\|\frac{1}{b}\sum_{i=1}^{b} H^i(X^i_k, \xi^i_{k+1})\right\|^2\right] \leq \frac{3}{b} \sum_{i=1}^{b} E\left[\|H^i(X^i_k, \xi^i_{k+1}) - H^i(X_k, \xi^i_{k+1})\|^2\right] + 3E \|\bar{H}(x_+, \xi)\|^2 + 3E \|\bar{H}(X_k, \xi) - \bar{H}(x_+, \xi)\|^2
\]

\[
\leq 3\bar{L}^2 E[V_k] + 3\bar{L}^2 E[d_k^2] + 3E \|\bar{H}(x_+, \xi)\|^2.
\]

(104)

In addition, using the fact that for any vectors $a, b \in \mathbb{R}^d$, $2 \langle a, b \rangle \leq m\|a\|^2 + \|b\|^2 / m$ we can upper bound the inner product derived in (103) as follows

\[
-E\left[\left\langle X_k - x_+, \frac{1}{b}\sum_{i=1}^{b} H^i(X^i_k, \xi^i_{k+1}) \right\rangle\right] = -E \left[\left\langle X_k - x_+, \nabla \bar{U}(X_k) \right\rangle\right]
\]

\[
+ E\left[\left\langle X_k - x_+, \frac{1}{b}\sum_{i=1}^{b} [H^i(X^i_k, \xi^i_{k+1}) - H^i(X_k, \xi^i_{k+1})] \right\rangle\right]
\]

\[
\leq -mE \left[\left\langle X_k - x_+, \nabla \bar{U}(X_k) \right\rangle\right] + mE \left[\frac{d_k^2}{2} + \frac{L^2}{m} E[V_k] / (2m)\right]
\]

\[
\leq -mE \left[\frac{d_k^2}{2} + \frac{L^2}{m} E[V_k] / (2m)\right].
\]

(105)

Therefore, plugging (104) and (105) in (103) shows

\[
E[d_{k+1}^2] \leq \left(1 - \gamma \left[m - 3\gamma \bar{L}^2\right]\right) E[d_k^2] + \gamma \left(3\gamma \bar{L}^2 + \frac{L^2}{m}\right) E[V_k] + \gamma^2 E \|\bar{H}(x_+, \xi)\|^2 + \frac{2\gamma d}{b}.
\]

Eventually, the assumption $\gamma \leq m(6\bar{L}^2)^{-1}$ completes the proof. \qed

For any $\gamma \in (0, m(6\bar{L}^2)^{-1})$, under A1, A4 and A6 using Lemma 16 and Lemma 17 we have shown that HX3 and HX4 hold with the following quantities

\[
A = 3\bar{L}^2, \quad B = 3\tilde{\omega}, \quad C = 0, \quad D = 3E \left[\|\bar{H}(x_+, \xi)\|^2\right],
\]

\[
\bar{A} = 3\bar{L}^2, \quad \bar{B} = 3L^2, \quad \bar{C} = 0, \quad \bar{D} = (3/b) \sum_{i=1}^{b} \|\nabla U(x_+)^i\|^2,
\]

\[
A_d = \gamma m / 2, \quad B_d = 0, \quad C_d = 2\gamma L^2 / m, \quad D_d = 3\gamma^2 E \left[\|\bar{H}(x_+, \xi)\|^2\right] + 2\gamma d / b, 
\]

\[
A_\sigma = 1, \quad B_\sigma = 0, \quad C_\sigma = 0, \quad D_\sigma = 0.
\]

(106)

For any $\gamma > 0$, consider the following variables

\[
C^\gamma = \frac{4(1 - pc)c^2}{pc - 4A_d} \left( B + \frac{2 + pc}{pc} \bar{B} \right), \quad C^\gamma = 3C^\gamma c_d, \quad C^\gamma_d = 1 + 2C_d c^\gamma,
\]

\[
C^\gamma_0 = C^\gamma_0 E[V_0] + 7C^\gamma E[d_0^2] + 2D_d, \quad C^\gamma_1 = \frac{4(1 - pc)c^2}{pc} \left( D + \frac{2 + pc}{pc} D \right) + C^\gamma D_d A_d + \frac{r(1 - \tau)(b - 1)c^d}{bp_c}.
\]

(107)
We also introduce $\gamma_1$ and $I_\gamma$, which are defined for any $\gamma > 0$ by

$$
\gamma_1 = \frac{p_c^{1/2}}{(2 - 2p_c)^{1/2} [A + (1 + 2/p_c)A]^{1/2}} \wedge \frac{m}{6L^2} \wedge \frac{p_c}{2m} \wedge \frac{q_c}{m},
$$

$$
I_\gamma = \{ \gamma \in (0, \gamma_1) : \gamma m \geq 8C_7^5 \}.
$$

Based on Lemma 11, we derive the following result.

**Lemma 18.** Assume $A_1, A_4$ and $A_6$ hold. Then, for any $\gamma \in I_\gamma$ and $k \geq 1$, we have

$$
\mathbb{E}[V_k] \leq \left(1 - \frac{A_d}{4}\right)^k \left(2C_7 + \frac{4C_5^7 C_5^7}{A_d}\right) + C_7^5.
$$

where $V_k$ is defined in (24) and $C_7^5, C_5^7, C_6^5$ in (107).

**Proof.** For any $\gamma \in I_\gamma$, we have $4C_7^5 \leq A_d$ and moreover it is easy to check that $A_d < \min(A_c/2, p_c/4)$, $A_d A_c \geq 8B_2B_\sigma = 0$. In addition, since $A_1, A_4$ and $A_6$ are satisfied we can apply Lemma 16 and Lemma 17 which show that $Hx_4$ hold with the variables introduced in (106). Therefore, we can use Lemma 11 to complete the proof.

Based on the results presented in this section, we can rewrite the upper bound on $(\mathbb{E}[V_k])_{k \in \mathbb{N}}$ given in Lemma 18 into the format of $H_2$. We consider for $\gamma > 0$,

$$
v_1 = 2C_7 + \frac{4C_5^7 C_5^7}{A_d}, \quad v_2 = C_7^5.
$$

**Lemma 19.** Assume $A_1, H_1, A_4$ hold and let $\gamma \leq 2(3L)^{-1}$. Then for any $k \in \mathbb{N}$, we have

$$
\mathbb{E} \left[ \|X_{(k+1)\gamma} - X_{k+1}\|^2 \right] \leq \left[1 - \gamma m (1 - 3\gamma L) + \gamma \left(\frac{2L^2}{m} + 3\gamma L^2 + \bar{L}^2\right)\right] \mathbb{E}[V_k] + \left(\frac{2}{\gamma m}\right) \mathbb{E} \left[ \|\mathcal{F}_k [I_k]\|^2 \right] + 3 \mathbb{E} \left[ \|I_k\|^2 \right] + \gamma^2 \mathbb{E} \left[ \left\| \mathcal{F}_k \left( \frac{1}{b} \sum_{i=1}^b C_k^i \right) \right\|^2 \right] + \frac{3\gamma^2}{b^2} \int_{\mathbb{R}^d} \mathbb{E} \left[ \mathcal{F}_k \left( H(x, \xi) \right) \pi(dx) \right].
$$

**Proof.** For any $k \in \mathbb{N}$, recall that $\mathcal{F}_k$ is defined in (22) and using Proposition 4 we obtain

$$
\mathbb{E} \mathcal{F}_k \left[ \|X_{(k+1)\gamma} - X_{k+1}\|^2 \right] \leq \left[1 - \gamma m (1 - 3\gamma L) \right] \|X_{k\gamma} - X_k\|^2 + \gamma \left(\frac{2L^2}{m} + 3\gamma L^2\right) V_k + \left(\frac{2}{\gamma m}\right) \mathbb{E} \mathcal{F}_k \left[ \|I_k\|^2 \right] + 3 \mathbb{E} \mathcal{F}_k \left[ \|I_k\|^2 \right] + \gamma^2 \mathbb{E} \mathcal{F}_k \left( \frac{1}{b} \sum_{i=1}^b C_k^i \right). \quad (109)
$$

Since the stochastic gradients $(H^i(\cdot, \xi_{k+1}))_{i=1}^b$ are unbiased, $A_4$ with the Young inequality imply that

$$
\mathbb{E} \mathcal{F}_k \left( \frac{1}{b} \sum_{i=1}^b C_k^i \right) = \mathbb{E} \mathcal{F}_k \left[ \left\| \frac{1}{b} \sum_{i=1}^b \left[ H^i(X_k, \xi_{k+1}) - \nabla U^i(X_k) \right] \right\|^2 \right]
$$

$$
= \mathbb{E} \mathcal{F}_k \left[ \left\| \frac{1}{b} \sum_{i=1}^b \left( H^i(X_k, \xi_{k+1}) - H(X_k, \xi_{k+1}) - \frac{1}{b} \sum_{i=1}^b \nabla U^i(X_k) + \nabla U(X_k) + H(X_{k\gamma}, \xi_{k+1}) - H(X_{k\gamma}, \xi_{k+1}) - \nabla U(X_{k\gamma}) \right) \right\|^2 \right]
$$

$$
\leq 3\bar{L}^2 V_k + 3\bar{L}^2 \|X_k - X_{k\gamma}\|^2 + 3 \mathbb{E} \mathcal{F}_k \left( H(X_{k\gamma}, \xi_{k+1}) \right).
$$

Taking the expectation and using that $X_{k\gamma}$ has distribution $\pi$ combined with (109) complete the proof. \qed
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For notational convenience, we also introduce the time step-size \( \gamma_2 \) defined by
\[
\gamma_2 = \frac{p_c}{4m} \wedge \frac{1}{6(L + L^2 / m)} \wedge \frac{p_c m}{38(1 - p_c)^{1/2} (p_c \tilde{\omega} + 3L^2)^{1/2} L}.
\]

**Theorem 20.** Assume A1, A4 and A6 hold and let \( \gamma \in (0, \gamma_1 \wedge \gamma_2) \). Then, for any initial probability measure \( \mu_0^{(F)} \in \mathcal{P}_2(\mathbb{R}^d) \), \( k \in \mathbb{N} \), we have
\[
W_2^2 (\mu_k^{(F)}, \pi) \leq \left( 1 - \frac{\gamma m \gamma}{2} \right)^k W_2^2 (\mu \gamma_0^{(F)}, \pi) + \frac{8L^2}{m^2} v_1 \left( 1 - \frac{\gamma m \gamma}{8} \right) + \frac{6L^2}{m^2} v_2 \frac{6d \gamma}{b m^2} \kappa_f + \frac{6 \gamma}{b^2 m} \int_{\mathbb{R}^d} \text{Var}^{\gamma_0} (H(x, \xi_1)) \pi(dx).
\]

where \( v_1, v_2 \) are defined in (108) and \( \kappa_f = L^2(1 + \gamma L^2 / m) \). If in addition we suppose HX1, set \( \kappa_f = 2\gamma (L^3 + dL^2 / b) \).

**Proof.** We know that H1 is satisfied since for any \( i \in [b], x \in \mathbb{R}^d \) the stochastic gradient \( H^i(x, \xi^i) \) is unbiased. The constraint \( \gamma \leq \gamma_1 \) combined with Lemma 17 implies HX3 and plugging the expression of \( A_d, A_\sigma, B_d, C, C_\gamma \) provided in (106) into \( C_r \) defined in (107) gives that
\[
C_r = \frac{72 \gamma^3 (1 - p_c) L^2 (\tilde{\omega} + (1 + 2/p_c) L^2)}{(p_c - 2\gamma m) m}.
\]
For any \( \gamma \in (0, \gamma_2) \), we have \( (p_c - 2\gamma m) m^2 \geq 576(1 - p_c) \gamma^2 L^2 (\tilde{\omega} + (1 + 2/p_c) L^2) \) which shows that \( \gamma \in I_\gamma \). Thus, we can apply Lemma 18 which proves that H2 holds with \( q_c = \gamma m \) and \( \alpha_v = 1 - A_d / 4 \) and \( v_1, v_2 \) defined in (108). Since the assumptions of Lemma 19 are satisfied, HX2 holds, and therefore we can apply Theorem 8 with
\[
(1 - q_c) \alpha_0 = 1 - \gamma (1 - 3\gamma L) + 3\gamma^2 \hat{L}, \quad \alpha_1 = 0, \quad (1 - q_c) \alpha_2 = \gamma \left( \frac{2L^2}{m} + 3\gamma (L + \hat{L}) \right), \quad \alpha_3 = 0,
\]
\[
(1 - q_c) \alpha_4 = \gamma \left( \frac{2}{\gamma m} \mathbb{E} \left[ \left\| \mathbb{E}^{\mathbb{F}_k} \left[ I_k \right] \right\|^2 \right] + 3 \mathbb{E} \left[ \left\| I_k \right\|^2 \right] \right) + \frac{3 \gamma^2}{b^2} \int_{\mathbb{R}^d} \text{Var}^{\gamma_0} (H(x, \xi_1)) \pi(dx).
\]
Furthermore, using Lemma 7 we have
\[
\frac{2}{\gamma m} \mathbb{E} \left[ \left\| \mathbb{E}^{\mathbb{F}_k} \left[ I_k \right] \right\|^2 \right] + 3 \mathbb{E} \left[ \left\| I_k \right\|^2 \right] \leq \frac{3 \gamma^2 (L^2)}{b m} \left( 1 + \frac{19 \gamma L^2}{36 m} \right).
\]
Moreover, if we suppose HX1, we obtain
\[
\frac{2}{\gamma m} \mathbb{E} \left[ \left\| \mathbb{E}^{\mathbb{F}_k} \left[ I_k \right] \right\|^2 \right] + 3 \mathbb{E} \left[ \left\| I_k \right\|^2 \right] \leq \frac{\gamma^2 L^2}{b m} \left( 5 L^2 + \frac{4 \hat{L} L^2}{3 b} \right).
\]
Finally, with the notation of Theorem 8 we obtain \( 1 + \delta = 0 \), and using \( \gamma \leq (6(L + m^{-1} L^2))^{-1} \) combined with (110) or (111) if we suppose HX1 give the expected result.

Now, consider the time stepsizes \( \gamma_3 \) and \( \gamma_* \) defined by
\[
\gamma_3 = \frac{p_c m}{3L^2 + p_c \tilde{\omega}}, \quad \gamma_* = \gamma_1 \wedge \gamma_2 \wedge \gamma_3.
\]
From the previous result, the next corollary controls the asymptotic bias obtained by Algorithm 2.

**Corollary 21.** Assume A1, A4 and A6 hold and let \( \gamma \in (0, \gamma_*), \tau = 1 \). Then, for any initial probability measure \( \mu_0^{(F)} \in \mathcal{P}_2(\mathbb{R}^d) \), \( k \in \mathbb{N} \), we have
\[
\frac{6^{-4}}{\gamma d} \limsup_{k \to \infty} W_2^2 (\mu_k^{(F)}, \pi) \leq \frac{\int_{\mathbb{R}^d} \text{Var}^{\gamma_0} (H(x, \xi_1)) \pi(dx)}{b d m} + \frac{\tilde{\kappa}_f}{m^2} + \frac{(1 - p_c) \gamma L^2}{p_c m^2} \left( \frac{1}{b} \sum_{i=1}^{b} \left\| \nabla U^i (x) \right\|^2 + \frac{p_c}{b d} \mathbb{E} \left[ \left\| H(x, \xi) \right\|^2 \right] + \frac{L^2 + p_c \tilde{\omega}}{m} \right).
\]
where \( \tilde{\kappa}_f = L^2 \) and if we suppose HX1, \( \tilde{\kappa}_f = \gamma (L^3 + dL^2 / b) \).
For any $i$, we also show how this new global control variate mitigates the effect of heterogeneity in the convergence rate. To do so, we consider the stochastic gradients defined for any $i$ and denote $\bar{X}$. Upper bound on the Wasserstein distance between the distribution of the server parameter $X$ and the proxy $H$ is provided in (106) and $C^2_\delta$ is defined in (107) by

$$C^2_\delta = \frac{4(1-p_c)\gamma^2}{pc} \left( D + \frac{2 + p_c}{p_c} \tilde{D} \right) + \frac{8(1-\tau)(b-1)\gamma d}{bp_c} A_d + \frac{12(1-p_c)\gamma^2}{pc} \left[ 1 + \frac{12\gamma}{m} \left( \tilde{\omega} + \frac{3L^2}{m} \right) \right] E \left[ \|H(x_*,\xi)\|^2 \right] + \frac{8(1-\tau)(b-1)\gamma d}{bp_c} \left( \tilde{\omega} + \frac{3L^2}{m} \right) \sum_{i=1}^b \|\nabla U^i(x_*)\|^2 + \frac{36(1-p_c)\gamma^2}{pc} \sum_{i=1}^b \|\nabla U^i(x_*)\|^2$$

Finally, setting $\tau = 1$ combined with (112) and (113) show that

$$\limsup_{k \to \infty} W_2^2 \left( \mu_k^{(F)}, \pi \right) \leq \frac{6\gamma}{b^2m} \int_{S^d} \text{Var}_0 \left( H(x,\xi_1) \right) \pi(dx) + \frac{6\gamma d}{bm^2} \kappa_I + \frac{L^2}{m} \tilde{\omega} \left( \tilde{\omega} + \frac{3L^2}{m} \right).$$

\[\-boxed{}\]

### 7.2 Study of VR-FALD$^*$

In this alternative of FALD derived in Section 7.1, we introduce control variates to cope with both heterogeneity and variance in local gradients. Instead of using $H_i^i(X_k^i)$ to update the local parameter $X_k^i$, this time the $i$th client uses the proxy $H_i^i(X_k^i,\xi_{k+1}^i) - H_i^i(Y_k,\xi_{k+1}^i) + \nabla U^i(Y_k)$ based on an analog of the SVRG algorithm (Johnson and Zhang, 2013; Karimireddy et al., 2020) and where $Y_k$ is a global reference point updated with probability $q_c \in (0,1]$. We derive an explicit upper bound on the Wasserstein distance between the distribution of the server parameter $X_{k,\gamma}$ and the target distribution $\pi$. We also show how this new global control variate mitigates the effect of heterogeneity in the convergence rate. To do so, we consider the stochastic gradients defined for any $i \in [b], k \in \mathbb{N}$, by

$$G_k^i = H_i^i(X_k^i,\xi_{k+1}^i) - H_i^i(Y_k,\xi_{k+1}^i) + C_k$$

and denote

$$\sigma_k = \left( \frac{1}{b} \sum_{i=1}^b \mathbb{E}[F_k] \left[ \|H_i^i(Y_k,\xi_{k+1}^i) - H_i^i(x_*,\xi_{k+1}^i)\|^2 \right] \right)^{1/2}$$

#### Lemma 22. Assume A1, A4 and A6 hold. Then for any $k \in \mathbb{N}$, we have

$$\frac{1}{b} \sum_{i=1}^b \mathbb{E}[G_k^i] \leq 3L^2 \mathbb{E}[V_k] + 3L^2 \mathbb{E}[d_k^2] + 3\mathbb{E}[\sigma_k^2],$$

$$\frac{1}{b} \sum_{i=1}^b \mathbb{E}[\|G_k^i - \bar{G}_k^i\|^2] \leq 3L^2 \mathbb{E}[V_k] + 3\tilde{\omega} \mathbb{E}[d_k^2] + 3\mathbb{E}[\sigma_k^2].$$

For any $i \in [b], k \in \mathbb{N}$, recall the stochastic gradients $G_k^i, \bar{G}_k^i$ are defined in (114) and (115), respectively.
Algorithm 3 VR-FALD*  

Input: initial vectors $(X_0^i)_{i \in [b]}$, noise parameter $\tau \in [0, 1]$, number of communication rounds $K$, probability $p_c$ of communication, probability $q_c$ to update the control variates, step-size $\gamma$ and batch size $r$.

Initialize $Y_0 = (1/b) \sum_{i=1}^{b} X_0^i$ and $C_0 = (1/b) \nabla U(Y_0)$

for $k = 0$ to $K - 1$ do
  // On each client
  Draw $B_{k+1} \sim B(p_c), \tilde{Z}_{k+1} \sim N(0_d, I_d)$
  // In parallel on the $b$ clients
  for $i = 1$ to $b$ do
    Draw $\xi_{k+1}^i \sim \nu, \tilde{Z}_{k+1}^i \sim N(0_d, I_d)$
    Compute $G_{k+1}^i = H^i(X_k^i, \xi_{k+1}^i) - H^i(Y_k, \xi_{k+1}^i) + C_k$
    Set $\tilde{X}_{k+1}^i = X_k^i - \gamma G_{k+1}^i + \sqrt{2\gamma} (\sqrt{\tau/b} \tilde{Z}_{k+1} + \sqrt{1-\tau} \tilde{Z}_{k+1})$
    if $B_{k+1} = 1$ then
      Broadcast $\tilde{X}_{k+1}^i$ to the server
    else
      Update $X_{k+1}^i \leftarrow \tilde{X}_{k+1}^i$
    if $\tilde{B}_{k+1} = 1$ then
      Broadcast $X_{k+1}^i$ to the server
    else
      Update $Y_{k+1} \leftarrow Y_k$ and $C_{k+1} \leftarrow C_k$
  if $B_{k+1} = 1$ then
    // On the central server
    Update then broadcast the global parameter $X_{k+1} \leftarrow (1/b) \sum_{i=1}^{b} \tilde{X}_{k+1}^i$
  // On each client
  Update the local parameter $X_{k+1}^i \leftarrow X_{k+1}^i$
  if $\tilde{B}_{k+1} = 1$ then
    // On the central server
    Update then broadcast $Y_{k+1} \leftarrow (1/b) \sum_{i=1}^{b} X_{k}^i$
  // On each client
  Compute and broadcast $\nabla U(Y_{k+1})$
  // On the central server
  Update then broadcast $C_{k+1} \leftarrow (1/b) \nabla U(Y_{k+1})$

Output: samples $\{X_t\}_{t \in [K]} : B_t = 1$.  


Proof. For $k \geq 0$, Lipschitz property of $\{\nabla U^i\}_{i \in [b]}$ supposed in $A1$ gives that
\[
\frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| \bar{G}^i_k \|^2 \right] = \frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| \nabla U^i(X_k^i) - \nabla U^i(Y_k) + \nabla \hat{U}(Y_k) \|^2 \right] \\
\leq \frac{3}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| \nabla U^i(X_k^i) - \nabla U^i(Y_k) \|^2 \right] + \frac{3}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| \nabla U^i(Y_k) \|^2 \right] \\
+ \frac{3}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \| \nabla U^i(X_k^i) - \nabla U^i(x_*) \|^2 \right] \\
\leq 3L^2 \mathbb{E} [V_k] + 3L^2 \mathbb{E} [d_k^2] + 3 \mathbb{E} [\sigma_k^2]
\]
and the proof is concluded by noting that $A4$ gives
\[
\frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \| G^i_k - \bar{G}^i_k \|^2 = \mathbb{E} \left[ \text{Var}^F \left( \frac{1}{b} \sum_{i=1}^{b} G^i_k \right) \right] \\
\leq \mathbb{E} \left[ \left\| \frac{1}{b} \sum_{i=1}^{b} H^i(X_k^i, \xi_{k+1}^i) - \hat{H}(X_k, \xi_{k+1}) \right\|^2 \right] \\
\leq 3 \mathbb{E} \left[ \left\| \frac{1}{b} \sum_{i=1}^{b} H^i(X_k^i, \xi_{k+1}^i) \right\|^2 \right] + 3 \mathbb{E} \left[ \left\| \nabla \hat{H}(X_k, \xi_{k+1}) - \nabla \hat{H}(x_*, \xi_{k+1}) \right\|^2 \right] \\
+ 3 \mathbb{E} \left[ \left\| \nabla \hat{H}(X_k, \xi_{k+1}) \right\|^2 \right].
\]

**Lemma 23.** Assume $A1$ and $A4$ hold. Then, for any $\gamma \in (0, m(6L^2)^{-1})$, we have
\[
\mathbb{E} [d_{k+1}^2] \leq \left( 1 - \frac{\gamma m}{2} \right) \mathbb{E} [d_k^2] + \frac{2\gamma L^2}{m} \mathbb{E} [V_k] + 4\gamma^2 \mathbb{E} [\sigma_k^2] + 10\gamma^2 \mathbb{E} [\| \nabla \hat{H}(x_*, \xi) \|^2] + \frac{2\gamma d}{b}.
\]
where $V_k, d_k, \sigma_k$ are defined in (24), (25) and (116).

**Proof.** Let $k$ be in $\mathbb{N}$. Writing the expression of $X_{k+1}$ defined in (23) and developing the expectation of the squared norm give
\[
\mathbb{E} [d_{k+1}^2] = \mathbb{E} \left[ \| X_{k+1} - x_* \|^2 \right] \\
= \mathbb{E} \left[ \| X_k - x_* - \frac{\gamma}{b} \sum_{i=1}^{b} H^i(X_k^i, \xi_{k+1}^i) + \gamma \hat{H}(Y_k, \xi_{k+1}) - \gamma \nabla \hat{U}(Y_k) + \sqrt{2\gamma} \left( \sqrt{\frac{\tau}{b}} \tilde{z}_{k+1} + \frac{\sqrt{1 - \tau}}{b} \sum_{i=1}^{b} Z_k^i \right) \|^2 \right] \\
= \mathbb{E} \left[ \| X_k - x_* \|^2 \right] - 2\gamma \mathbb{E} \left[ \left( X_k - x_* \right) \left( \frac{1}{b} \sum_{i=1}^{b} H^i(X_k^i, \xi_{k+1}^i) \right) \right] + \gamma^2 \mathbb{E} \left[ \left\| \frac{1}{b} \sum_{i=1}^{b} H^i(X_k^i, \xi_{k+1}^i) \right\|^2 \right] \\
- 2\gamma^2 \mathbb{E} \left[ \left( \frac{1}{b} \sum_{i=1}^{b} H^i(X_k^i, \xi_{k+1}^i), \hat{H}(Y_k, \xi_{k+1}) \right) - \gamma \nabla \hat{U}(Y_k) \right] + \gamma^2 \mathbb{E} \left[ \left\| \hat{H}(Y_k, \xi_{k+1}) - \nabla \hat{U}(Y_k) \right\|^2 \right] + \frac{2\gamma d}{b} \\
= \mathbb{E} [d_k^2] - 2\gamma \mathbb{E} \left[ \left( X_k - x_* \right) \left( \frac{1}{b} \sum_{i=1}^{b} \nabla U^i(X_k^i) \right) \right] + 2\gamma^2 \mathbb{E} \left[ \left\| \frac{1}{b} \sum_{i=1}^{b} H^i(X_k^i, \xi_{k+1}^i) \right\|^2 \right] \\
+ 2\gamma^2 \mathbb{E} \left[ \left\| \hat{H}(Y_k, \xi_{k+1}) - \nabla \hat{U}(Y_k) \right\|^2 \right] + \frac{2\gamma d}{b}. \tag{117}
\]
Using the Young inequality combined with $A4$ show
\[
\mathbb{E} \left[ \left\| \frac{1}{b} \sum_{i=1}^{b} H^i(X_k^i, \xi_{k+1}^i) \right\|^2 \right] \leq \frac{3}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| H^i(X_k^i, \xi_{k+1}^i) \right\|^2 \right]
\]
We also have that
\[ \gamma \]
Which shows the expected result.

In addition, using the fact that for any vectors \( a, b \in \mathbb{R}^d \), \( 2 \| (a, b) \| \leq m \| a \|^2 + \| b \|^2 / m \), we can upper bound the inner product derived in (117) as follows
\[
-\mathbb{E} \left[ \left( X_k - x_*, \frac{1}{b} \sum_{i=1}^{b} \nabla U^i(X_k^i) \right) \right] = -\mathbb{E} \left[ \langle X_k - x_*, \nabla U(X_k) \rangle \right] \\
+ \mathbb{E} \left[ \left( X_k - x_*, \frac{1}{b} \sum_{i=1}^{b} \left[ H^i(X_k, \xi_{k+1}^i) - H^i(X_k, \xi_{k+1}^i) \right] \right) \right] \\
\leq -\mathbb{E} \left[ \langle X_k - x_*, \nabla U(X_k) \rangle \right] + m \mathbb{E} \left[ d_k^2 \right] / 2 + L^2 \mathbb{E} \left[ V_k \right] / (2m) \\
\leq -m \mathbb{E} \left[ d_k^2 \right] / 2 + L^2 \mathbb{E} \left[ V_k \right] / (2m). \\
(120)

Hence, combining (117), (118), (119) and (120) implies that
\[
\mathbb{E} \left[ d_{k+1}^2 \right] \leq \left( 1 - \gamma m + 6 \gamma^2 L^2 \right) \mathbb{E} \left[ d_k^2 \right] + \left( \frac{\gamma L^2}{m} + 6 \gamma^2 L^2 \right) \mathbb{E} \left[ V_k \right] + 4 \gamma^2 \mathbb{E} \left[ \sigma_k^2 \right] + 10 \gamma^2 \mathbb{E} \left[ \| H(x_*, \xi) \|^2 \right] + \frac{2 \gamma d}{b}.
\]

Using the assumption on \( \gamma \) completes the proof.

\[ \square \]

**Lemma 24.** Assume the \( L \)-smoothness of the potentials \( \{ U^i \}_{i \in [b]} \) and A4 hold. Then, for any \( k \in \mathbb{N} \), we have
\[
\mathbb{E} \left[ \sigma_{k+1}^2 \right] \leq (1 - q_\epsilon) \mathbb{E} \left[ \sigma_k^2 \right] + 2q L^2 \mathbb{E} \left[ d_k^2 \right] + 2q L^2 \mathbb{E} \left[ V_k \right],
\]
where \( V_k, d_k, \sigma_k \) are defined in (24), (25) and (116).

**Proof.** Let’s consider \( k \geq 0 \), using A4 implies that
\[
\mathbb{E} \left[ \sigma_{k+1}^2 \right] = \frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| H^i(Y_{k+1}^i, \xi_{k+1}^i) - H^i(x_*, \xi_{k+1}^i) \right\|^2 \right] \\
= \frac{1 - q_\epsilon}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| H^i(Y_{k+1}^i, \xi_{k+1}^i) - H^i(x_*, \xi_{k+1}^i) \right\|^2 \right] + \frac{q_\epsilon}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \left\| H^i(X_k, \xi_{k+1}^i) - H^i(x_*, \xi_{k+1}^i) \right\|^2 \right] \\
= (1 - q_\epsilon) \mathbb{E} \left[ \sigma_k^2 \right] + 2q L^2 \mathbb{E} \left[ d_k^2 \right] + 2q L^2 \mathbb{E} \left[ V_k \right].
\]

Which shows the expected result. \[ \square \]

For any \( \gamma \in (0, m(6L^2)^{-1}) \), under A1, A4 and A6 we have shown that Lemma 22 and Lemma 23 imply HX3 and HX4 with
\[
A = c_V = 3L^2, \quad B = c_d = 3\bar{\omega}, \quad C = c_\sigma = 3, \quad D = c = 0, \\
\bar{A} = 3L^2, \quad \bar{B} = 3L^2, \quad \bar{C} = 3, \quad \bar{D} = 0, \\
A_d = \gamma m / 2, \quad B_d = 4\gamma^2, \quad C_d = 2\gamma L^2 / m, \quad D_d = 10\gamma^2 \mathbb{E} \left[ \left\| H(x_*, \xi) \right\|^2 \right] + 2\gamma d / b, \\
A_\sigma = q, \quad B_\sigma = 2q L^2, \quad C_\sigma = 2q L^2, \quad D_\sigma = 0.
\]
\[ (121) \]
For any $\gamma > 0$, consider the following variables

$$\alpha_d = \frac{4\gamma^2}{p_c A_d} \max \left\{ p_c B + 3B, \frac{4B^2}{A_d} \left( p_c C + 3\overline{C} \right) \right\}, \quad \alpha_d = \frac{4\gamma^2}{p_c A_d} \left( p_c C + 3\overline{C} \right).$$

(122)

**Lemma 25.** Assume $A_1, A_4$ and $A_6$ hold with

$$A_d \leq \min \left( A_d, \frac{p_c}{4} \right), \quad \alpha_d C_d + \alpha_d C_{\sigma} \leq \frac{p_c}{8}, \quad \alpha_d B_d + \gamma^2 \left( C + \frac{3}{p_c} \overline{C} \right) \leq \frac{\alpha_\sigma A_\sigma}{2},$$

and consider $\gamma \leq m(6\hat{L}^2)^{-1} \land p_c^{1/2}(2 - 2p_c)^{-1/2}[A + (1 + 2/p_c)\hat{A}]^{-1/2}$. Then, for any $k \in \mathbb{N}$, we have

$$E[V_k] \leq \left( 1 - \frac{A_d}{2} \right)^k \left( E[V_0] + \alpha_d E[d^2_0] + \alpha_d E[\sigma_0^2] \right) + \frac{2\alpha_d D_d}{A_d} + \frac{4(1 - \tau)(b - 1) \gamma d}{bA_d},$$

where $V_k$ is defined in (24).

**Proof.** Applying Lemma 12 with the variables provided in (121) gives the result. \( \Box \)

Let's introduce $\gamma_1 > 0$ such that

$$\gamma_1 \leq \frac{m}{128\hat{L}^2} \land 8 \max \left( \frac{m}{3L^2 + p_c \overline{\omega}, 24\hat{L}^2} \right) \land \frac{2q}{m} \land \frac{p_c}{2m} \land \frac{p_c}{2(1 - p_c)(p_c A + 3\hat{A})^{1/2}} \land \frac{p_c}{8 \left( \frac{L^2}{m^2} \max \left( 3L^2 + p_c \overline{\omega}, 24\hat{L}^2 \right) \right) + \frac{2}{q_c}}^{1/2}. $$

Under $A_1, A_4$ and $A_6$, for all $\gamma \in (0, \gamma_1]$ the assumptions of Lemma 25 are satisfied. The upper bound on $(E[V_k])_{k \in \mathbb{N}}$ derived in Lemma 25 can be rewritten into the format of $H2$ by considering

$$\tilde{v}_1 = E[V_0] + \alpha_d E[d^2_0] + \alpha_d E[\sigma_0^2], \quad \tilde{v}_2 = \frac{2\alpha_d D_d}{A_d} + \frac{4(1 - \tau)(b - 1) \gamma d}{bA_d}. $$

(123)

In addition, for any $\gamma > 0$, consider the following variables

$$C^* = \frac{4(1 - p_c)\gamma^2}{p_c - 4A_d} \left[ B + \frac{2 + p_c}{p_c} B + \frac{B_{\sigma}}{A_\sigma - A_d} \left( C + \frac{2 + p_c}{p_c} \overline{C} \right) \right],$$

$$C^*_\sigma = \frac{9\gamma^2 (1 - p_c) C_{\sigma}}{p_c - 4A_d} \left( C + \frac{2 + p_c}{p_c} \overline{C} \right) + 3C^* \left( C + \frac{B_d C_{\sigma}}{A_\sigma - A_d} \right),$$

$$C^*_d = \frac{4(1 - p_c)\gamma^2}{p_c - 4A_d} \left( C + \frac{2 + p_c}{p_c} \overline{C} \right) + C^* B_d \left( 2 + \frac{3}{A_\sigma - A_d} \right),$$

$$C^*_d = 7C^*, \quad C^*_{\sigma} = 1 + 2C^* C_d,$$

$$C^*_d = \frac{C^* D_d}{A_d} \left( 1 + \frac{2B_d B_{\sigma}}{A_d(A_\sigma - A_d)} \right) + \frac{8(1 - \tau)(b - 1) \gamma d}{b p_c},$$

$$C^*_d = C^*_d E[V_0] + C^*_d E[d^2_0] + C^*_d E[\sigma_0^2] + 2D_d.$$

Based on Lemma 10, we derive the following result.

**Lemma 26.** Assume $A_1, A_4$ and $A_6$ hold and consider $\gamma \in (0, \gamma_1]$. Then, for any $k \in \mathbb{N}$, we have

$$E[V_k] \leq \left( 1 - \frac{A_d}{4} \right)^k \left( C^* + \frac{4C^*_d \tilde{v}_1}{A_d} \right) + \frac{2C^*_d \tilde{v}_2}{A_d} + C^*_d,$$

where $V_k$ is defined in (24) and $C^*_d, C^*_\sigma, C^*_d$ in (124).
Proof. Since we suppose \( A_1, A_4 \) and \( A_6 \) hold with \( \gamma \leq \gamma_1 \), the assumptions of Lemma 25 are satisfied. Therefore, for any \( l \in \mathbb{N} \), we obtain
\[
\mathbb{E} [V_l] \leq \left(1 - \frac{A_d}{2}\right)^l \hat{v}_1 + \hat{v}_2.
\] (125)
Moreover, the condition \( \gamma \leq m/128L^2 \) ensures that \( A_d A_\sigma = q\gamma m/2 \geq 8B_d B_\sigma = 64q\gamma^2 L^2 \), hence we can apply Lemma 10. Then, plugging (125) in the bound derived in Lemma 10 gives
\[
\mathbb{E} [V_k] \leq (1 - \alpha)^k C_\gamma^\gamma + C_\gamma^\gamma \sum_{i=0}^{k-2} (1 - \alpha)^{k-i-1} \mathbb{E} [V_i] + C_\gamma^\gamma,
\] (126)
where \( \alpha \) is defined in (60) by
\[
\alpha = A_d - \frac{2(A_\sigma - A_d)^{-1} B_d B_\sigma}{1 + \sqrt{1 + 4(1 - A_d)^{-1}(A_\sigma - A_d)^{-1} B_d B_\sigma}}.
\]
Using Lemma 9, we know that \( A_d/2 < \alpha \leq A_d \) and combining this bound with (125) and (126) leads to
\[
\mathbb{E} [V_k] \leq \left(1 - \frac{A_d}{4}\right)^k \left(C_\gamma^\gamma + \frac{4C_\gamma^\gamma \hat{v}_1}{A_d}\right) + \frac{2C_\gamma^\gamma \hat{v}_2}{A_d} + C_\gamma^\gamma.
\]
\[\square\]
In order to rewrite the upper bound on \( \mathbb{E} [V_k] \) given in Lemma 26 in the format of \( \mathbf{H}_2 \), we consider for \( \gamma > 0 \)
\[
v_1 = C_\gamma^\gamma + \frac{4C_\gamma^\gamma \hat{v}_1}{A_d}, \quad v_2 = \frac{2C_\gamma^\gamma \hat{v}_2}{A_d} + C_\gamma^\gamma.
\] (127)

Lemma 27. Assume \( A_1, A_7, H_1 \) and hold and let \( \gamma \leq (6L)^{-1} \). Using the convention that \( \sum_{i=0}^{-1} = 0 \), then for any \( k \in \mathbb{N} \), we have
\[
\mathbb{E} \left[\|X_{(k+1)\gamma} - X_{k+1}\|^2\right] \leq [1 - \gamma m + \gamma^2 (3mL + 4\omega)] \mathbb{E} \left[\|X_{k\gamma} - X_k\|^2\right] + 4\gamma^2 \omega q \sum_{l=0}^{k-1} (1 - q_c)k-l-1 \mathbb{E} \left[\|X_{l\gamma} - X_l\|^2\right] + \gamma \left(\frac{2L^2}{m} + 3\gamma L^2 + 4\gamma \omega\right) \mathbb{E} [V_k] + \left(\frac{2}{\gamma m} \mathbb{E} \left[\|X_{k}\|^2\right] + 3\mathbb{E} \left[\|I_k\|^2\right]\right) + \frac{16\gamma^3 \omega d}{bq_c} \left(1 + \frac{\gamma L}{q_c}\right).
\]

Proof. For \( k \in \mathbb{N} \), using the independence of \( (\xi^i_{k+1})_{i \in [b]} \) combined with \( H_1 \) and \( A_7 \), we obtain
\[
\text{Var}^F_k \left(\frac{1}{b} \sum_{i=1}^b G^i_k\right) = \mathbb{E}^F_k \left[\left(\frac{1}{b} \sum_{i=1}^b \left\|\nabla U^i(X_k^i) - \nabla U^i(Y_k) - H^i(X_k^i, \xi^i_{k+1}) + H^i(Y_k, \xi^i_{k+1})\right\|^2\right)\right]
\]
\[
= \frac{1}{b} \sum_{i=1}^b \mathbb{E}^F_k \left[\left\|\nabla U^i(X_k^i) - \nabla U^i(Y_k) - H^i(X_k^i, \xi^i_{k+1}) + H^i(Y_k, \xi^i_{k+1})\right\|^2\right]
\]
\[
\leq \frac{\omega^i}{b} \sum_{i=1}^b \|X_k^i - Y_k\|^2.
\] (128)
Denote \( t_k \in \mathbb{N} \) the time when the reference point of the control variate is updated, therefore we have
\[
t_k = \begin{cases} 0, & \text{if } k = 0 \\ \max \left\{ l \in \{0, \ldots, k - 1\} : Y_k = b^{-1} \sum_{i=1}^b X_k^i \right\}, & \text{if } k \geq 1 \end{cases}.
\] (129)
Thus, setting $x_k = (X_k - X_k) + (X_k - X_{k,t}) + (X_{k,t} - X_k)$.

Thus for $k \geq 0$, combining the previous line with Young’s inequality, it yields that

$$\frac{1}{b} \sum_{i=1}^{b} \mathbb{E} \left[ \|X_k - Y_k\|^2 \right] \leq 4 \mathbb{E} [V_k] + 4 \mathbb{E} \left[ \|X_k - X_{k,t}\|^2 \right] + 4 \mathbb{E} \left[ \|X_{k,t} - X_k\|^2 \right] + 4 \mathbb{E} \left[ \|X_{k,t} - Y_k\|^2 \right].$$

(130)

For $k \geq 1$, by definition of $t_k$, we have

$$\mathbb{E} [V_k] = \sum_{l=0}^{k-1} \mathbb{P} (t_k = l) \mathbb{E} [V_l] = q \sum_{l=0}^{k-1} (1 - q_c)^{k-l-1} \mathbb{E} [V_l].$$

Moreover, for $k \geq 1$ we get

$$\mathbb{E} \left[ \|X_{k,t} - X_t\|^2 \right] = \sum_{l=0}^{k-1} \mathbb{P} (t_k = l) \mathbb{E} \left[ \|X_{k,t} - X_t\|^2 \right]$$

$$= q \sum_{l=0}^{k-1} (1 - q_c)^{k-l-1} \mathbb{E} \left[ \left\| \int_{t_{l-1}}^{t_l} \nabla \overline{U}(X_s) \, ds + \sqrt{\frac{2}{b}} (W_{k_{l-1}} - W_{l_{l}}) \right\|^2 \right]$$

$$\leq 2 \gamma q \sum_{l=0}^{k-1} (k-l)(1 - q_c)^{k-l-1} \left( \int_{t_{l-1}}^{t_l} \mathbb{E} \left[ \|\nabla \overline{U}(X_s)\|^2 \right] \, ds + \frac{2d}{b} \right).$$

(131)

Using Dalalyan (2017, Lemma 2) with $s \in \mathbb{R}_+$, we obtain

$$\mathbb{E} \left[ \|\nabla \overline{U}(X_s)\|^2 \right] \leq dL/b.$$ Using by convention that $\sum_{t=1}^{0} = 0$, for any $k \in \mathbb{N}$ and $x \neq 1$ we have

$$\sum_{t=1}^{k} t^2 x^{l-1} = (1 - x)^{-3} \left( 1 + x - x^k \left[ 2x + kx(1 - x) + (k + 1)(1 + k(1 - x))(1 - x) \right] \right).$$

Thus, setting $x = 1 - q$ inside the last shows that

$$\sum_{t=1}^{k} t^2 (1 - q_c)^{l-1} \leq 2/q_c^3.$$

Hence, the above line combined with $\sum_{t=1}^{k} l(1 - q_c)^{l-1} = q^{-2} \left[ 1 - (1 + kq)(1 - q_c)^k \right]$ and (131) yield the following upper bound

$$\mathbb{E} \left[ \|X_{k,t} - X_t\|^2 \right] \leq \frac{2 \gamma dq}{b} \sum_{l=0}^{k-1} [(k-l)(1 - q_c)^{k-l-1} (2 + (k-l) \gamma L)]$$

$$\leq \frac{2 \gamma dq}{b} \sum_{l=0}^{k-1} [(k-l)(1 - q_c)^{k-l-1} (2 + (k-l) \gamma L)]$$

$$\leq \frac{4 \gamma d}{b q_c} \left( 1 + \frac{\gamma L}{q_c} \right).$$

(132)

In addition, by definition (129) of $t_k$, we immediately get for any $k \geq 1$, that

$$\mathbb{E} \left[ \|X_{k,t} - X_t\|^2 \right] = \sum_{l=0}^{k-1} \mathbb{P} (t_k = l) \mathbb{E} \left[ \|X_{t_{l-1}} - X_t\|^2 \right]$$
Theorem 28. Assume $A1, A4, A6, A7$ and let $\gamma \in (0, \gamma_1 \land \gamma_2)$. Then, for any initial probability measure $\mu_0^{(\text{Vr}^*)} \in \mathcal{P}_2(\mathbb{R}^d)$, $k \in \mathbb{N}$, we have

$$W_2^2 \left( \mu_k^{(\text{Vr}^*)}, \pi \right) \leq \left( 1 - \frac{\gamma m}{2} \right)^k W_2^2 \left( \mu_0^{(\text{Vr}^*)}, \pi \right) + \left( 1 - \frac{\gamma m}{8} \right)^k \frac{3L^2}{m^2 v_1} + \frac{6L^2}{m^2} v_2 + \frac{6\gamma d}{bm^2 \kappa_I} + \frac{32\gamma qd}{bm} ,$$

where $v_1, v_2$ are defined in (127) and $\kappa_I = L^2(1 + \gamma L^2/m)$. If in addition we suppose $HX1$, set $\kappa_I = 2\gamma(L^3 + dL^2/b)$.

Proof. We know that $H1$ is satisfied since for any $i \in [b], x \in \mathbb{R}^d$ the stochastic gradient $H'(x, \xi')$ is unbiased. Lemma 26 proves that $H2$ holds with $\alpha_v = 1 - A_4/4$ and $v_1, v_2$ defined in (127). Lemma 27 implies that $HX2$ holds with the choice of $(\alpha_i)_{i=0}^4$ detailed in (134). Finally, since $HX2$ and $H2$ hold, we can apply Theorem 8 to show that

$$W_2^2 \left( \mu_k^{(\text{Vr}^*)}, \pi \right) \leq (1 + \alpha_0 + \delta)^k (1 - q_c)^k W_2^2 \left( \mu_0^{(\text{Vr}^*)}, \pi \right) + (1 - q_c)v_1 \left( \alpha_2 + \frac{\alpha_3}{\alpha_0 + \delta} \right) \left( \frac{\alpha_2 - (1 + \alpha_0 + \delta)^k (1 - q_c)^k}{\alpha_v - (1 + \alpha_0 + \delta)(1 - q_c)} + \frac{1 - q_c}{q_c - (1 - q_c)(\alpha_0 + \delta)} v_2 + \frac{\alpha_2 + \alpha_3}{\alpha_0 + \delta} v_2 + \alpha_4 \right) .$$

Combining (128), (130) with (132), for any $k \geq 1$ we obtain

$$\mathbb{E} \left[ \text{Var}^{\text{K}} \left( \frac{1}{b} \sum_{i=1}^b G_i^k \right) \right] \leq 4\omega \mathbb{E} \left[ \| X_k - X_k \|^2 \right] + 4\omega q \frac{1}{b} \sum_{i=1}^b (1 - q_c)^{k-1} \mathbb{E} \left[ \| X_{i\gamma} - X_i \|^2 \right] + 4\omega \mathbb{E} \left[ V_k \right] + \frac{16\gamma qd}{b q c} \left( 1 + \frac{\gamma L}{q c} \right) . \quad (133)$$

Since $Y_0 = b^{-1} \sum_{i=1}^b X_i^0$, we have $\text{Var}^{\text{K}}(b^{-1} \sum_{i=1}^b G_i^k) \leq \mathcal{W} \mathcal{V} k$ and therefore the above inequality also holds for $k = 0$. Lastly, using Proposition 4 gives

$$\mathbb{E}^{\text{K}} \left[ \| X_{(k+1)\gamma} - X_{k+1} \|^2 \right] \leq \left( 1 - \gamma m (1 - 3\gamma L) \right) \| X_{k\gamma} - X_k \|^2 + \gamma \left( \frac{2L^2}{m} + 4\gamma L^2 \right) V_k + \left( \frac{2}{\gamma m} \mathbb{E}^{\text{K}} \left[ \| I_k \|^2 \right] + 3 \mathbb{E}^{\text{K}} \left[ \| I \|^2 \right] \right) + \gamma^2 \text{Var}^{\text{K}} \left( \frac{1}{b} \sum_{i=1}^b G_i^k \right) .$$

Hence, plugging (133) in the above inequality yields the expected result. \hfill \Box
where $\delta = 2^{-1}(\sqrt{(a_0 - 1)^2 + 4a_1} - a_0)$ is defined in (45). Using for any $a > 0, b \geq 0$, that $\sqrt{a} + b \leq \sqrt{a+b}/(2\sqrt{a})$, we obtain

$$
\alpha_0 + \sqrt{(a_0 - 1)^2 + 4a_1} = 1 + (a_0 - 1) \left(1 + \sqrt{1 + \frac{4a_1}{(a_0 - 1)^2}}\right) \leq 1 + 2(a_0 - 1) \left(1 + \frac{a_1}{(a_0 - 1)^2}\right) = 2a_0 - 1 + \frac{2a_1}{a_0 - 1}.
$$

Thus, we deduce that

This upper bound gives that

$$
(1 - q_c)(\alpha_0 + \delta) = (1 - q_c)(1 + \alpha_0 + \delta) + q - 1 \leq q - \gamma m/2.
$$

Thus, we deduce that

$$
\frac{1}{q_c - (1 - q_c)(\alpha_0 + \delta)} \leq \frac{2}{\gamma m}. \tag{137}
$$

Further, using $\gamma \leq \gamma_2$ combined with the definitions of $\alpha_0, \alpha_2, \alpha_3, \alpha_v$ and $\delta$ show that

$$
\frac{\alpha^k_v - (1 + \alpha_0 + \delta)^k (1 - q_c)^k}{\alpha_v - (1 + \alpha_0 + \delta)(1 - q_c)} \leq \frac{8}{3\gamma m} \left(1 - \frac{\gamma m}{8}\right)^k, \tag{138}
$$

$$
\alpha_2 + \frac{\alpha_3}{\alpha_0 + \delta} = \frac{\gamma}{1 - q_c} \left(\frac{2L^2}{m} + 3\gamma L^2 + 4\gamma \omega\right) \leq \frac{3\gamma L^2}{(1 - q_c)m}.
$$

Lastly, plugging (136), (137) and (138) in (135) yields

$$
W_2^\delta \left(p_k^{(v*, \pi)}, \pi\right) \leq \left(1 - \frac{\gamma m}{2}\right)^k W_2^\delta \left(p_0^{(v*, \pi)}, \pi\right) + \left(1 - \frac{\gamma m}{8}\right)^k \frac{3L^2}{m^2} v_1 + \frac{6L^2}{m^2} v_2 + \frac{2(1 - q_c)\alpha_4}{\gamma m}. \tag{139}
$$

In addition, following the lines provided in the proof of Theorem 20, we deduce

$$
\frac{2(1 - q_c)\alpha_4}{\gamma m} \leq \frac{6\gamma dL^2}{bm^2} \left(1 + \frac{19\gamma L^2}{36m}\right) + \frac{32\gamma^2 \omega d}{bmq}. \tag{140}
$$

If in addition we suppose $HX1$, then we obtain

$$
\frac{2(1 - q_c)\alpha_4}{\gamma m} \leq \gamma mL^2 \left(1 + \frac{\gamma L^2}{2m} + \frac{\gamma^2 L^2}{12}\right) + \frac{4\gamma}{9} \left(L^3 + \frac{dL^2}{b}\right) + \frac{32\gamma^2 \omega d}{bmq}. \tag{141}
$$

Finally, plugging (140) or (141) if $HX1$ holds inside (139) combined with $\gamma \leq qL^{-1}$ lead to the expected result. \qed

Now, consider the time stepsizes $\gamma_3$ and $\gamma_*$ defined by

$$
\gamma_3 = \frac{p_c m}{3L^2 + 16L^2 + p_c \omega}, \quad \gamma_* = \gamma_1 \wedge \gamma_2 \wedge \gamma_3.
$$

From the previous result, the next corollary controls the asymptotic bias obtained by Algorithm 3.
Corollary 29. Assume A1, A4, A6, A7 and let $\gamma \in (0, \gamma_*)$ with $\tau = 1$. Then, for any initial probability measure $\mu_0^{(Vr*)} \in \mathcal{P}_2(\mathbb{R}^d)$, $k \in \mathbb{N}$, we have

$$
\frac{9^{-9}b}{\gamma_d} \limsup_{k \to \infty} \mathbb{W}_2^2(\mu_k^{(Vr*)}, \pi) \leq \frac{\kappa_I + \gamma \omega}{m^2} + \frac{\gamma \omega}{mq} + \frac{(1 - p_c)\gamma L^2}{p_c^2m^3} \left( L^2 + L^2 + p_c \bar{\omega} \right) \left( 1 + \frac{\gamma}{bd} \mathbb{E} \left[ \|H(x_*, \xi)\|^2 \right] \right) \left( L^2 + \frac{q_c}{p_c} \bar{L}^2 \right),
$$

where $\bar{\kappa}_I = L^2 (1 + \gamma L^2 m^{-1})$ and if we suppose HXL, $\bar{\kappa}_I = \gamma (L^3 + d \bar{L}^2 b^{-1})$.

Proof. Applying Theorem 28 with $\gamma \in (0, \gamma_1 \land \gamma_2)$ shows that

$$
\limsup_{k \to \infty} \mathbb{W}_2^2(\mu_k^{(Vr*)}, \pi) \leq \frac{6L^2}{m^2} \left( \frac{2 + 8 (1 - \tau) (b - 1) \gamma}{b \sigma} \right) \left( \frac{4B}{d} \right) \left( \frac{3L^2 + p_c \bar{\omega}, 8(p_c + 3) \bar{L}^2}{m^2} \right) \leq \frac{768 \gamma}{p_c m} \left( L^2 + L^2 + p_c \bar{\omega} \right).
$$

Further, recall that $A_d, B, \bar{B}, D, \bar{D}, D_d$ are provided in (121) and $\alpha_d$ is defined in (122) by

$$
\alpha_d = \frac{4\gamma^2}{p_c A_d} \max \left\{ p_c B + 3 \bar{B}, \frac{4B}{A_d} \right\} = \frac{24 \gamma^2}{p_c m} \left( 3L^2 + p_c \bar{\omega}, 8(p_c + 3) \bar{L}^2 \right) \leq \frac{768 \gamma}{p_c m} \left( L^2 + L^2 + p_c \bar{\omega} \right).
$$

Moreover, $C_\gamma^D, C_\gamma^\alpha$ are defined in (124) by

$$
C_\gamma^D = \frac{C_\gamma D_d}{A_d} \left( 1 + \frac{2B_d B_\sigma}{A_d (A_\sigma - A_d)} \right) + \frac{8 (1 - \tau) (b - 1) \gamma d}{bp_c} = \frac{10C_\gamma}{m} \left( 1 + \frac{64 \gamma q L^2}{(2 - \gamma - \gamma)m} \right) \left( 5 \gamma \mathbb{E} \left[ \|H(x_*, \xi)\|^2 \right] \right) \left( 9 \gamma \mathbb{E} \left[ \|H(x_*, \xi)\|^2 \right] \right) + \frac{8 (1 - \tau) (b - 1) \gamma d}{bp_c} \leq \frac{360 (1 - p_c) \gamma^2}{mp_c^2} \left( 3L^2 + 11 \bar{L}^2 + p_c \bar{\omega} \right) \left( 5 \gamma \mathbb{E} \left[ \|H(x_*, \xi)\|^2 \right] \right) \left( 9 \gamma \mathbb{E} \left[ \|H(x_*, \xi)\|^2 \right] \right) + \frac{8 (1 - \tau) (b - 1) \gamma d}{bp_c},
$$

$$
C_\gamma^\alpha = \frac{9 \gamma^2 (1 - p_c)}{p_c - 4A_d} \left( C + \frac{2 + p_c \bar{C}}{bp_c} \right) + 3C_\gamma \left( C_d + \frac{B_d C_\sigma}{A_\sigma - A_d} \right) \leq \frac{144 \gamma^2 (1 - p_c)}{p_c^2} \left[ 3q \bar{L}^2 + \gamma \left( \frac{L^2}{m} + 8 \gamma \bar{L}^2 \right) \right] \left( p_c \bar{\omega} + 3L^2 + 16 \bar{L}^2 \right) \leq \frac{432 \gamma^2 (1 - p_c)}{p_c^2} \left( p_c L^2 + q \bar{L}^2 \right).
$$

Eventually, for the specific choice $\tau = 1$ combined with (142) and (143), it yields that

$$
\limsup_{k \to \infty} \mathbb{W}_2^2(\mu_k^{(Vr*)}, \pi) \leq \frac{6L^2}{m^2} \left( \frac{2 + 8 (1 - \tau) (b - 1) \gamma}{b \sigma} \right) \left( \frac{4B}{d} \right) \left( \frac{3L^2 + p_c \bar{\omega}, 8(p_c + 3) \bar{L}^2}{m^2} \right) \leq \frac{768 \gamma}{p_c m} \left( L^2 + L^2 + p_c \bar{\omega} \right).
$$

Therefore, using (143) and (144) we can finally conclude that

$$
9^g \limsup_{k \to \infty} \mathbb{W}_2^2(\mu_k^{(Vr*)}, \pi) \leq \frac{\gamma d}{bm^2} \kappa_I + \frac{\gamma \omega}{mq} \left( \frac{1 - p_c)\gamma L^2}{p_c^2m^3} \left( L^2 + L^2 + p_c \bar{\omega} \right) \left( \gamma \mathbb{E} \left[ \|H(x_*, \xi)\|^2 \right] \right) \left( 1 + \frac{d}{b} \right) \left( L^2 + \frac{q_c}{p_c} \bar{L}^2 \right). \quad \Box
$$
Theorem 30. Assume $A_l, A_4, A_6, A_7$ and let $\gamma \in (0, \gamma_1 \wedge \gamma_2)$. Then, for any initial probability measure $\mu_0^{(V_{\gamma})} \in \mathcal{P}(\mathbb{R}^d)$, $k \in \mathbb{N}$, we have

$$W_2^2\left(\mu_k^{(V_{\gamma})}, \pi\right) \leq \left(1 - \frac{\gamma m}{2}\right)^k W_2^2\left(\mu_0^{(V_{\gamma})}, \pi\right) + \frac{6\gamma d}{b m^2} \kappa I + \frac{32\gamma^2 w d}{n q},$$

where $\kappa I = L^2(1 + \gamma L^2/2m + \gamma^2 L^2/12)$. If in addition we suppose $HX$, set $\kappa I = 3\gamma(L^3 + d L^2/b)$.

Remark 31.

- The constants obtained in this result can be refined by directly using that $\mathbb{E}[V_k] = 0$ in the proof of Lemma 27 and by simplifying the calculations detailed in Theorem 28.

- The proof given in Chatterji et al. (2018, Theorem 4.2-Option 2) on the convergence of SVRG-LD seems to have some gaps since the authors use Grönwall’s inequality (Clark, 1987) as if $\rho = \tau^2 (8d^2 + 4M\delta^2 d + 42M\Omega_1)$ were constant, which is not the case because $\Omega_1 = \langle \nabla f(y_k) - \nabla f(x_k), y_k - x_k \rangle$ depends on the iteration $k$. If we denote $\rho_k$ instead of $\rho$ and adopt their other notation (we also correct a typography in the right-hand term), we obtain

$$\mathbb{E}\left[\|x_k - \hat{x}\|^2\right] \leq \rho_k + \sum_{j=0}^{k-1} \mathbb{E}\left[\|x_j - \hat{x}\|^2\right].$$

(145)

Then, it is claimed in the proof of Chatterji et al. (2018, Theorem 4.2-Option 2) that (145) implies $\mathbb{E}[\|x_k - X_k\|^2] \leq \rho_k \exp(\tau \rho)$. But this inequality cannot hold in all generalities, for example if we consider $\tau s = 0$, for $j \leq k$, $\rho_j = 1$, $x_j = \hat{x} + \sqrt{2/d} \cdot 1$ and $\rho_k = 0$, $x_k = \hat{x} + 1/\sqrt{d}$, then (145) holds for $j \notin [k]$ but $\mathbb{E}[\|x_k - X_k\|^2] = 1$ whereas $\rho_k \exp(\tau \rho) = 0$.

8 Lower bound on the heterogeneity in a Gaussian case

In this section, we want to illustrate the heterogeneity problem by lower bounding the Wasserstein distance $W_2$ in a simple case. To simplify the calculations, we assume that each client performs 2 local iterations following the FALD update before communicating its local parameter to the central server. More specifically, take $(\mu_1, \mu_2, \sigma_1, \sigma_2) \in \mathbb{R}^2 \times (\mathbb{R}^2)^2$ and define the potentials $U^1 : x \in \mathbb{R}^d \mapsto \sigma_1^{-2}(x - \mu_1)^2, U^2 : x \in \mathbb{R}^d \mapsto \sigma_2^{-2}(x - \mu_2)^2$. Thus, the global posterior distribution $\pi$ is Gaussian with mean $\bar{m}$ and variance $\bar{\sigma}^2$ given by

$$\bar{m} = \frac{\mu_1 \sigma_2^2 + \mu_2 \sigma_1^2}{\sigma_1^2 + \sigma_2^2}, \quad \bar{\sigma} = \left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)^{-1/2}.$$

(146)

The objective is to illustrate the problem of heterogeneity in the basic version of FALD. To do so, we first show that this algorithm generates samples targeting a distribution $\pi_{\gamma} \in \mathcal{P}(\mathbb{R}^d)$ where the distance $W_2(\pi, \pi_{\gamma})$ is lower bounded by a heterogeneity term. To this end, we introduce the Markov kernel, which for each $\gamma > 0$, $B \in \mathcal{B}(\mathbb{R}^d)$ is given by

$$P_{\gamma}(x, B) = \int_B \exp \left( -\frac{\|x' - \left(1 - \frac{\gamma}{\bar{\sigma}^2} + \frac{\gamma}{2\bar{\sigma}}\left(\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2}\right)\right) x - \frac{\gamma}{\bar{\sigma}} \frac{\mu_1}{\sigma_1^2} + \frac{\gamma}{2\bar{\sigma}} \frac{\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2} - \sqrt{\gamma} \left(1 - \frac{\gamma}{2\bar{\sigma}^2}\right) Z_{k+1} + Z_{k+2} \right) \frac{dx'}{(2\pi)^{d/2}},$$

and we define the stochastic processes $(A_k, \tilde{A}_k)_{k \geq 0}$ on $\mathbb{R}^d \times \mathbb{R}^d$ starting from $(X_0, X_0) = (x, \hat{x})$ and following the recursion for $k \geq 0$,

$$A_{k+1} = A_k - \frac{\gamma}{\sigma_2^2} (A_k - \bar{m}) + \frac{\gamma}{2} \left( \frac{A_k - \mu_1}{\sigma_1^2} + \frac{A_k - \mu_2}{\sigma_2^2} \right) + \sqrt{\gamma} \left(1 - \frac{\gamma}{2\bar{\sigma}^2}\right) Z_{k+1} + Z_{k+2},$$

$$\tilde{A}_{k+1} = \tilde{A}_k - \frac{\gamma}{\sigma_2^2} (\tilde{A}_k - \bar{m}) + \frac{\gamma}{2} \left( \frac{\tilde{A}_k - \mu_1}{\sigma_1^2} + \frac{\tilde{A}_k - \mu_2}{\sigma_2^2} \right) + \sqrt{\gamma} \left(1 - \frac{\gamma}{2\bar{\sigma}^2}\right) Z_{k+1} + Z_{k+2}.\quad (147)$$

It is possible to verify that $(A_k, \tilde{A}_k)$ is distributed according to $(\delta_x P^k, \delta_x P^k_{\gamma})$. 
Lemma 32. Let \( \gamma \in (0, 2(\sigma_1 \sigma_2)^4[\sigma^2(\sigma_1^4 + \sigma_2^4)]^{-1}) \). Then, there exists \( \pi_\gamma \in \mathcal{P}_2(\mathbb{R}^d) \) such that for any distribution \( \pi^0 \in \mathcal{P}_2(\mathbb{R}^d) \), the sequence \( (\pi^0 P^k_\gamma)_{k \in \mathbb{N}} \) converges to \( \pi_\gamma \) in \( \mathcal{P}_2(\mathbb{R}^d) \).

\[
\text{Proof.} \text{ Let } k \in \mathbb{N} \text{ and consider the stochastic processes } (A_t, \tilde{A}_t)_{t \in \mathbb{N}} \text{ defined in } (147), \text{ subtracting the two recursions we obtain}
\]
\[
A_{k+1} - \tilde{A}_{k+1} = \left(1 - \frac{\gamma}{\sigma^2} + \frac{\gamma^2}{2} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \right) (A_k - \tilde{A}_k).
\]

Since \( 0 < \gamma < 2(\sigma_1 \sigma_2)^4[\sigma^2(\sigma_1^4 + \sigma_2^4)]^{-1} \), taking the norm in the previous inequality implies that
\[
\|A_{k+1} - \tilde{A}_{k+1}\| \leq \left(1 - \frac{\gamma}{\sigma^2} + \frac{\gamma^2}{2} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \right) \|A_k - \tilde{A}_k\|.
\]

Finally, combining (148) with Douc et al. (2018, Lemma 20.3.2), we deduce that the \( c \)-Dobrushin coefficient of \( P_\gamma \) is upper bounded by \( 1 - \gamma/\sigma^2 + \gamma^2/2 \left(1/\sigma_1^4 + 1/\sigma_2^4\right) \). Hence, applying Douc et al. (2018, Theorem 20.3.4) we deduce the existence and uniqueness of a stationary distribution \( \pi_\gamma \in \mathcal{P}_2(\mathbb{R}^d) \) for the Markov Kernel \( P_\gamma \) such that \( W_2(\pi^0 P^k_\gamma, \pi) \leq \left(1 - \gamma/\sigma^2 + \gamma^2/2 \left(1/\sigma_1^4 + 1/\sigma_2^4\right)\right)^k W_2(\pi^0, \pi_\gamma) \).

Lemma 32 shows the existence of an invariant distribution \( \pi_\gamma \in \mathcal{P}_2(\mathbb{R}^d) \) for \( P_\gamma \) and the next lemma specifies this distribution of \( \pi_\gamma \).

Lemma 33. Assume \( \gamma \in (0, 2(\sigma_1 \sigma_2)^4[\sigma^2(\sigma_1^4 + \sigma_2^4)]^{-1}) \). Then, the stationarity distribution \( \pi_\gamma \) is Gaussian with parameters given by

\[
m(\gamma) = \frac{\bar{m} - \gamma \bar{\sigma}^2/2 \left( \frac{\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2} \right)}{1 - \gamma \bar{\sigma}^2/2 \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)}, \quad \sigma(\gamma)^2 = \frac{\bar{\sigma}^2 - \gamma^2/2 \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)}{1 - \gamma \bar{\sigma}^2/2 \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)}.
\]

\[
\text{Proof.} \text{ First, let } k \in \mathbb{N} \text{ be fixed and introduce}
\]
\[
\alpha = 1 - \frac{\gamma}{\bar{\sigma}^2} + \frac{\gamma^2}{2} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right), \quad \beta = \frac{\gamma \bar{m}}{\bar{\sigma}^2} - \frac{\gamma^2}{2} \left( \frac{\mu_1}{\sigma_1^2} + \frac{\mu_2}{\sigma_2^2} \right),
\]
\[
\tilde{Z}_k = \left(1 - \frac{\gamma}{2\bar{\sigma}^2}\right) Z_{2k-1} + Z_{2k}.
\]

Moreover, consider \( (A_t)_{t \in \mathbb{N}} \) the stochastic process following (147) and initialized at \( \pi_\gamma \). By induction, we know that
\[
A_k = \alpha^k A_0 + \beta \sum_{l=0}^{k-1} \alpha^l + \sqrt{\gamma} \sum_{l=0}^{k-1} \alpha^{k-l-1} \tilde{Z}_l.
\]

Since \( A_k \) is distributed according to \( \pi_\gamma P^k_\gamma \), we have that \( A_k \) follows \( \pi_\gamma \). Denote \( \nu^k_\gamma \) the distribution of \( \sqrt{\gamma} \sum_{l=0}^{k-1} \alpha^{k-l-1} \tilde{Z}_l - \beta \sum_{l=0}^{k-1} \alpha^l \), combining (149) with the definition of the Wasserstein, we have
\[
W_2^2 \left( \pi_\gamma, \nu^k_\gamma \right) \leq \mathbb{E} \left[ \left\| A_k - \sqrt{\gamma} \sum_{l=0}^{k-1} \alpha^{k-l-1} \tilde{Z}_l - \beta \sum_{l=0}^{k-1} \alpha^l \right\|^2 \right] = \alpha^{2k} \mathbb{E} \left[ \|A_0\|^2 \right].
\]

Since \( A_0 \) is distributed according to \( \pi_\gamma \) belonging to \( \mathcal{P}_2(\mathbb{R}^d) \), we deduce that \( \mathbb{E} \|A_0\|^2 < \infty \). Consequently, (150) implies that \( (\nu^k_\gamma)_{k \in \mathbb{N}} \) converges to \( \pi_\gamma \), but using the fact that \( (\nu^k_\gamma)_{k \in \mathbb{N}} \) converges to a Gaussian distribution, we obtain by uniqueness of the limit in metric space \( \mathcal{P}_2(\mathbb{R}^d) \), \( W_2 \) that \( \pi_\gamma \) is a Gaussian distribution. Recalling that \( m(\gamma) \) denotes the expectation of the random variable distributed according to \( \pi_\gamma \), using (147) at stationarity yields
\[
m(\gamma) = m(\gamma) - \frac{\gamma}{\bar{\sigma}^2} \left( m(\gamma) - \bar{m} \right) + \frac{\gamma^2}{2} \left( \frac{m(\gamma) - \mu_1}{\sigma_1^2} - \frac{m(\gamma) - \mu_2}{\sigma_2^2} \right).
\]

Thus, we deduce that
\[
m(\gamma) = \frac{\bar{m} - (\gamma \bar{\sigma}^2/2) \left( \mu_1/\sigma_1^2 + \mu_2/\sigma_2^2 \right)}{1 - (\gamma \bar{\sigma}^2/2) \left( 1/\sigma_1^4 + 1/\sigma_2^4 \right)}.
\]
In addition, we can obtain the standard deviation \( \sigma_{c(\gamma)} \) of \( \pi_\gamma \) since we have

\[
\text{Var} \left( \beta \sum_{l=0}^{k-1} \alpha^l + \sqrt{\gamma} \sum_{l=0}^{k-1} \alpha^{k-l-1} \tilde{Z}_l \right) = \gamma \text{Var} \left( \sum_{l=0}^{k-1} \alpha^{k-l-1} \tilde{Z}_l \right) = \frac{\gamma(1 - \alpha^{2k})}{1 - \alpha^2} \text{Var}(\tilde{Z}_0)
\]

\[
\overset{k \to \infty}{\longrightarrow} \frac{\gamma \text{Var}(\tilde{Z}_0)}{1 - \alpha^2} = \frac{\gamma \left( 2 - \frac{\gamma}{\sigma^2} + \frac{\gamma^2}{\sigma^4} \right)}{1 - \left( 1 - \frac{2}{\sigma^2} + \frac{\gamma^2}{\sigma^4} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \right)^2} = \frac{1}{\bar{\sigma}^2 - \frac{\gamma}{\bar{\sigma}} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) - \frac{\gamma^2}{\sigma_1^2} \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right)^2}.
\]

\[\square\]

**Theorem 34.** Assume \( \gamma \in (0, 2(\sigma_1, \sigma_2)^4[\sigma^2(\sigma_1^2 + \sigma_2^2)]^{-1}) \). Then, the Wasserstein distance between the stationary distribution \( \pi_\gamma \) and the target \( \pi \) of FALD is lower bounded as

\[
W_2(\pi_\gamma, \pi) \geq \frac{\gamma}{2} |\mu_1 - \mu_2| \left| \frac{\bar{\sigma}^2}{\sigma_1^2} - \frac{\bar{\sigma}^2}{\sigma_2^2} \right|.
\]

**Proof.** Based on Lemma 33, we know that \( \pi_\gamma \) is Gaussian with parameters \((m_\gamma, \sigma^2_{\gamma})\) and using that \( \pi \) is Gaussian too with parameters \((\bar{m}, \bar{\sigma}^2)\) given in (146), we have that

\[
W_2^2(\pi_\gamma, \pi) = (m_\gamma - \bar{m})^2 + (\sigma_\gamma - \bar{\sigma})^2 \geq \gamma^2 \bar{\sigma}^4 \left( \frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2} \right) \left| \bar{m} - \frac{\mu_1}{\sigma_1^2} - \frac{\mu_2}{\sigma_2^2} \right|^2 = \frac{\gamma^2 \bar{\sigma}^4}{4} \left( \frac{1}{\sigma_1^2} - \frac{1}{\sigma_2^2} \right)^2.
\]

\[\square\]

### 9 Analysis of the complexity and communication cost

In this section, we study the optimal choices of \( k, \gamma \) when \( p_c \) is fixed. For \( c_0, c_1, c_2 \geq 0 \) fixed, we consider the following optimization problem:

\[
\left\{ \begin{array}{l}
\min_{k \in \mathbb{N}^+, \gamma > 0} \{ k \} \\
\text{Subject to} \ \left\{ c_0 \exp(-8k\gamma/m) + c_1\gamma + c_2\gamma^2 \leq \epsilon \right\}.
\end{array} \right.
\]

Using that the constraint must be saturated at the optimum (which can be proved), we can write \( k \) as a function of \( \gamma \). Hence, the problem becomes

\[
\left\{ \begin{array}{l}
\min_{k, \gamma} \left\{ \frac{8}{\gamma m} \log \left( \frac{c_0}{\epsilon - c_1\gamma - c_2\gamma^2} \right) \right\} \\
\text{Subject to} \ 0 < \gamma \text{ and } \epsilon^2 - c_1\gamma - c_2\gamma^2 > 0.
\end{array} \right.
\]

Let us introduce \( x \in \mathbb{R}^*_+ \), defined by \( x = \epsilon^{-2}\gamma \) and let \( \tilde{c}_2 = \epsilon^2 c_2 \). We can rewrite (151) as

\[
\left\{ \begin{array}{l}
\min_{k, x} \left\{ \frac{8}{\gamma mx} \log \left( \frac{c_0}{\epsilon(1-c_1x-\tilde{c}_2x^2)} \right) \right\} \\
\text{Subject to} \ 0 < x \text{ and } 1 - c_1x - \tilde{c}_2x^2 > 0.
\end{array} \right.
\]

Consider \( \mu = -c_1/(2\tilde{c}_2), \sigma = \sqrt{c_1^2/(4\tilde{c}_2^2) + 1/\tilde{c}_2} \), and denote \( z = (x-\mu)/\sigma \). Since \( x = \mu + z\sigma \), we can verify that \( 1 - c_1x - \tilde{c}_2x^2 = \tilde{c}_2\sigma^2(1 - z^2) \). Hence, (153) is equivalent to

\[
\left\{ \begin{array}{l}
\min_{k, \gamma} \left\{ \frac{8}{\epsilon m(\mu + z\sigma)} \log \left( \epsilon(1-c_1x-\tilde{c}_2x^2) \right) \right\} \\
\text{Subject to} \ -\mu/\sigma < z < 1.
\end{array} \right.
\]
According to the intermediate value theorem, we have the existence of $z_\varepsilon$ (not necessarily unique, but we can consider one of the solutions) such that

$$z_\varepsilon = \arg \max_{-\mu/\sigma < z < 1} \left\{ \frac{\log(1 - z^2)}{\mu + z\sigma} \right\}.$$  

Thus, the solution is

$$\gamma_\varepsilon = c_2 \times z_\varepsilon^2 + (4\varepsilon^2 c_2)^{-1}(z_\varepsilon^2 - 1)c_1^2,$$

$$K_\varepsilon = \frac{8(c_1/2 + z_\varepsilon \sqrt{4^{-1} c_1^2 + \varepsilon^2 c_2})}{\varepsilon^2 m (z_\varepsilon^2 + (4\varepsilon^2 c_2)^{-1}(z_\varepsilon^2 - 1)c_1^2)} \log \left( \frac{c_0}{\varepsilon^2(c_1^2/4 + \varepsilon^2 c_2)^{1/2}(1 - z_\varepsilon^2)} \right).$$

**FALD.** According to Theorem 1, we have

$$\begin{cases}  
c_0 = 1(\mu_0) 
c_1 = V_\pi + (1 - 1_{H\mathbf{X}1}) J/b + (1 - \tau)(1 - b^{-1})d/p_c. 
c_2 = 1_{H\mathbf{X}1} J/b + (1 - p_c) \{H + p_c V_\varepsilon + d/b \}/p_c^2. 
\end{cases}$$

If $c_1 > 0$, define $w = \varepsilon^2 c_2/c_1^2$. For $\varepsilon \in (0, c_1/\sqrt{2c_2}]$, we have $0 < w \leq 1/2$. Consider $z = 1 - w$, we get that

$$\left( \frac{\mu}{\sigma} \right)^2 = \frac{1 + 4\varepsilon^2 c_2/c_1^2}{1 + 2w} \leq 1 - w \leq 1 - 2w + w^2 = z^2 < 1.$$  

Hence, the previous inequalities show that $-\mu/\sigma < z < 1$, and for this choice

$$\frac{c_1/2 + z_\varepsilon \sqrt{4^{-1} c_1^2 + \varepsilon^2 c_2}}{z^2 + (4\varepsilon^2 c_2)^{-1}(z^2 - 1)c_1^2} \leq \frac{c_1 + \varepsilon(1 - w)\sqrt{c_2}}{7/8 + (w - 2 + 1/64)w}.$$  

Thus, for any $\varepsilon \in (0, c_1(2\sqrt{c_2}^{-1})^{-1}]$, we deduce that $w < 1/4$. Therefore, we have shown that $K_\varepsilon = \tilde{O}((\varepsilon^2 m)^{-1}(c_1 + \varepsilon\sqrt{c_2}))$. Moreover, this result is immediately valid when $c_1 = 0$ since $z_\varepsilon = \arg \max_{0 < z < 1} \{z^{-1}\log(1 - z^2)\}$. Furthermore, when $p_{c,\varepsilon} \downarrow 0^+$, $p_{c,\varepsilon} K_\varepsilon = \tilde{O}((\varepsilon m)^{-1}\sqrt{b^{-1}})$ as stressed in the main paper.

**VR-FALD.** Using Theorem 3, we obtain

$$\begin{cases}  
c_0 = 1^{V_\varepsilon\varepsilon}(\mu_0) 
c_1 = (1 - 1_{H\mathbf{X}1}) J/b + (1 - \tau)(1 - b^{-1})d/p_c. 
c_2 = 1_{H\mathbf{X}1} J/b + (1 - p_c) \{p_c V_\varepsilon + d/b \}/p_c^2. 
\end{cases}$$

When assuming $H\mathbf{X}1$ and $\tau = 1$, we have $c_1 = 0$. Hence, $z_\varepsilon = \arg \max_{0 < z < 1} \{z^{-1}\log(1 - z^2)\}$ and therefore

$$K_\varepsilon = \frac{8\sqrt{c_2}}{\varepsilon m z_\varepsilon} \log \left( \frac{c_0}{\varepsilon^3 \sqrt{c_2}(1 - z_\varepsilon^2)} \right).$$

When $p_{c,\varepsilon} \downarrow 0^+$, the minimum number of communications becomes $p_{c,\varepsilon} K_\varepsilon = \tilde{O}(\varepsilon^{-1}\sqrt{b^{-1}}d)$. Finally, setting $p_{c,\varepsilon} = 1$ gives $K_\varepsilon = \tilde{O}(\varepsilon^{-1}\sqrt{b^{-1}}J + b^{-1}\omega d)$.

| Parameter | $d$ | $m$ | $\omega$ | $H$ | $J$ | $V_\pi$ | $V_*$ |
|-----------|-----|-----|----------|-----|-----|--------|-------|
| Value     | 10  | 1   | 10       | 100 | 20  | 10     | 30    |
10 Numerical experiments

10.1 Gaussian example

In this first experiment, we consider $b = 100$ clients associated with potentials: $\forall i \in [b], U^i : x \in \mathbb{R}^d \mapsto (1/2)(x - \mu_i)^\top \Sigma_i^{-1}(x - \mu_i)$ in dimension $d = 20$. In this particular case, we know, that the posterior distribution $\pi \propto \exp(-\sum_{i=1}^b U^i)$ is Gaussian with mean $x_* = \sum_{i=1}^b (\Sigma_i^{-1} \mu_i)$ and covariance $\Sigma_* = (\sum_{i=1}^b \Sigma_i^{-1})^{-1}$. Also, we have a close formula to calculate $\int f(x - x_*, \Sigma_*) \, dx_*$, since this quantity is equal to $\text{Tr}(\Sigma_*)$. To speed up the calculations, we initialize all chains at $x_*$, we discard the first 10% of the samples and keep all others. Moreover, we consider the step size $\gamma = 2[\lambda_{\min}(\Sigma_*) + \lambda_{\max}(\Sigma_*)]^{-1}$ for Langevin Monte Carlo (Dalalyan and Karagulyan, 2019; Durmus and Moulines, 2019), and we run the algorithms for the step sizes $\gamma \in \{0.5, \frac{1}{20}, \frac{1}{200}\}$ associated with $p_c = \{\frac{1}{5}, \frac{1}{10}, \frac{1}{20}\}$. We set the probability of updating the control variates $q_c = p_c$ so as not to increase the communication cost too much. We also consider the two extreme values of the parameter $\tau \in \{0, 1\}$ to determine whether it is preferable to have independent Gaussian noise on each client or if it is better to have a common one.

10.2 Bayesian Logistic Regression

The second experiment is performed on the Titanic dataset, which is in the public domain and licensed under the Commons Public Domain Dedication License (PDDL-1.0). We distribute this dataset heterogeneously across $b = 10$ clients by drawing a Dirichlet random variable for each label on the standard $b - 1$ simplex. Since the sum of the coordinates of these random variables equals 1, each coordinate indicates the fraction of labels to be distributed to each client. To have access to ground truth, we also implement Langevin Stochastic Dynamics (LSD). We compute $K = 250000$ iterations, each time considering a burn-in period of length 10% initialized with a warm start provided by SGD. The $i$th client uses its local dataset $\{(z_{ij}, o_{ij}) \in \mathbb{R}^d \times \{0, 1\} : j \in [N_i]\}$ to calculate the local potential $U^i = \sum_{j=1}^{N_i} o_{ij} \log(1 + \exp(z_{ij}^\top x)) + (1 - o_{ij}) \log(1 + \exp(-z_{ij}^\top x)) + \lambda \|x\|^2$, where $\lambda = 1$ is associated with the Gaussian prior. Denote $Z_{\text{train}}$ the matrix whose lines are the covariates $z_{ij}^\top$, and write $\Sigma = Z_{\text{train}}^\top Z_{\text{train}}$. We run the algorithms with mini-batches of size $n_i = 1$; a step size $\gamma = 2[\lambda_{\min}(\Sigma) + \lambda_{\max}(\Sigma)]^{-1}$ for FALD, VR-FALD* and equal to $\gamma/b$ for LSD with thinning inversely proportional to the step size. Moreover, we consider a communication probability of $p_c = 1/20$ and clients update their control variables with probability $q_c = p_c$. Finally, to evaluate the obtained results, we consider the accuracy, agreement, and total variation, as well as the calibration results such as ECE, BS, and NLL, which are described below.

Accuracy. Based on samples from the approximate posterior distribution, we compute the minimum mean squared estimator (i.e., which corresponds to the posterior mean) and use it to make predictions for the test dataset. The Accuracy metric corresponds to the percentage of well-predicted labels.

Agreement. Let $p_{\text{ref}}$ and $p$ denote the predictive densities associated with HMC and an approximate simulation-based algorithm, respectively. Similar to Izmailov et al. (2021), we define the agreement between $p_{\text{ref}}$ and $p$ as the proportion of test data points for which the top-1 predictions of $p_{\text{ref}}$ and $p$, i.e.

$$\text{agreement}(p_{\text{ref}}, p) = \frac{1}{|D_{\text{test}}|} \sum_{x \in D_{\text{test}}} 1 \left\{ \arg\max_{y'} p_{\text{ref}}(y' \mid x) = \arg\max_{y'} p(y' \mid x) \right\} .$$

Total variation (TV). By denoting $\mathcal{Y}$ as the set of possible labels, we consider the total variation metric between $p_{\text{ref}}$ and $p$, i.e.

$$\text{TV}(p_{\text{ref}}, p) = \frac{1}{2|D_{\text{test}}|} \sum_{x \in D_{\text{test}}} \sum_{y' \in \mathcal{Y}} |p_{\text{ref}}(y' \mid x) - p(y' \mid x)| .$$

Expected Calibration Error (ECE). To measure the difference between the accuracy and confidence of the predictions, we group the data into $M \geq 1$ buckets defined for each $m \in [M]$ by $B_m = \{(x, y) \in D_{\text{test}} : p(y_{\text{pred}}(x) \mid x) \in$
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\[ |(m - 1)/M, m/M| \}. As in the previous work of Ovadia et al. (2019), we denote the model accuracy on \( B_m \) by

\[
\text{acc} (B_m) = \frac{1}{|B_m|} \sum_{(x,y) \in B_m} 1_{y_{\text{pred}}(x) = y}
\]

and define the confidence on \( B_m \) by

\[
\text{conf} (B_m) = \frac{1}{|B_m|} \sum_{(x,y) \in B_m} p(y_{\text{pred}}(x)|x).
\]

As emphasized in Guo et al. (2017), for any \( m \in [M] \) the accuracy \( \text{acc} (B_m) \) is an unbiased and consistent estimator of \( \mathbb{P} (y_{\text{pred}}(x) = y \mid (m - 1)/M < p(y_{\text{pred}}(x)|x) \leq m/M) \). Therefore, the ECE is defined by

\[
\text{ECE} = \sum_{m=1}^{M} \frac{|B_m|}{|D_{\text{test}}|} |\text{acc} (B_m) - \text{conf} (B_m)|
\]

and is an estimator of

\[
\mathbb{E}_{(x,y)} \left[ |\mathbb{P} (y_{\text{pred}}(x) = y \mid p(y_{\text{pred}}(x)|x)) - p(y_{\text{pred}}(x)|x)| \right].
\]

Thus, the ECE measures the absolute difference between the confidence level of a prediction and its accuracy.

Brier Score (BS). The BS is a proper scoring rule (see for example Dawid and Musio (2014)) that can only evaluate random variables taking a finite number of values. Denote by \( \mathcal{Y} \) the finite set of possible labels, the BS measures the confidence of the model in its predictions and is defined by

\[
\text{BS} = \frac{1}{|D_{\text{test}}|} \sum_{(x,y) \in D_{\text{test}}} \sum_{c \in \mathcal{Y}} (p(y = c|x) - 1_{y=c})^2.
\]

Normalized Negative Log Likelihood (nNLL). This classical score defined by

\[
n\text{NLL} = -\frac{1}{|D_{\text{test}}|} \sum_{(x,y) \in D_{\text{test}}} \log p(y|x)
\]

measures the ability of the model to predict good labels with high probability.

Highest posterior density (HPD). Under the Bayesian paradigm, we are interested in quantifying uncertainty by estimating the regions of high probability. For all \( \alpha \in (0, 1) \), we run each algorithm to estimate \( \eta_{\alpha}^{\text{algo}} > 0 \) such that \( \int_{R_\alpha} \pi(x)dx = 1 - \alpha \), where \( R_\alpha = \{ x \in \mathbb{R}^d : \pi(x) \geq \exp(-\eta_{\alpha}^{\text{algo}}) \} \). Then we define the relative HPD error as

\[
|\eta_{\alpha}^{\text{algo}} / \eta_{\alpha}^{\text{LSD}} - 1|,
\]

where \( \eta_{\alpha}^{\text{LSD}} \) is estimated based on the samples drawn with the Langevin Stochastic Dynamics method.

10.3 Bayesian Neural Network: MNIST

To investigate the behavior of the proposed algorithms in a highly non-convex setting, we perform a first Deep Learning experiment on the MNIST dataset (Deng, 2012), which can be publicly downloaded using the torchvision package and is available under the Creative Commons Attribution-Share Alike 3.0 license. To this end, we distribute the entire dataset across \( b = 20 \) clients in a highly heterogeneous manner to train the LeNet5 neural network (LeCun et al., 1998). The MNIST real-world dataset consists of 70000 grayscale images of size \( 28 \times 28 \) associated with the 10 digits. This dataset is divided into two subsets: the training set, which contains 60000 images, and the test set, which consists of the remaining 10000 images. We report the median of the scores with their associated hyperparameters in Table 6. The burn-in corresponds to the number of steps performed before we start storing the samples, and the thinning is the frequency with which we keep the samples. We also consider a Gaussian prior which corresponds to a squared norm regularizer with weight decay. We initialized FSGLD (El Mekkaoui et al., 2021) with a global SGD warm start combined with local SWAG (Maddox et al., 2019) to learn Gaussian conducive gradients.
expected calibration error (the one associated with the HMC sampler. We also give some classical calibration scores (Guo et al., 2017), such as the agreement, i.e., the percentage of time the top-1 prediction of an algorithm matches that given by the HMC, and the total variation (TV) between the predictive distribution given by an algorithm with the one associated with the HMC sampler. We also give some classical calibration scores (Guo et al., 2017), such as the expected calibration error (ECE), the Brier score (BS), and the negative log-likelihood (nNLL).

In this last experiment, we consider the more challenging dataset CIFAR10 (Krizhevsky, 2009), which is available under license MIT and contains images of size (3, 32, 32). We used different approaches to sample the weights for the ResNet-20 model (He et al., 2016), which is publicly available in the pytorchcv library. We initialized the algorithms with 10 different parameters using SGD (400 epochs) trained with a OneCycleLR scheduler (Smith and Topin, 2019), and we also use data augmentation with a mini-batch of size 128 and a learning rate of 2e-7. Based on these initializations, we ran 10 chains in parallel for SGLD, FALD, and VR-FALD* with step sizes of 1e-7, 2e-8, 1e-8. We considered 1e4 iterations with only one stored sample every 1e3 iterations (we did not keep the initial weights obtained by SGD to make the predictions). For each chain, we can see that Bayesian model averaging increases the accuracy. To compare the behavior of the mentioned algorithms, we compute the accuracy, the agreement, i.e., the percentage of time the top-1 prediction of an algorithm matches that given by the HMC, and the total variation (TV) between the predictive distribution given by an algorithm with the one associated with the HMC sampler. We also give some classical calibration scores (Guo et al., 2017), such as the expected calibration error (ECE), the Brier score (BS), and the negative log-likelihood (nNLL).