The Chow groups and the motive of the Hilbert scheme of points on a surface

Mark Andrea A. de Cataldo* and Luca Migliorini†

October 23, 2018

Abstract

We compute the Chow motive and the Chow groups with rational coefficients of the Hilbert scheme of points on a smooth algebraic surface.

1 Introduction

In this paper we compute the Chow motive and the Chow groups with rational coefficients of the Hilbert scheme $X^{[n]}$ of $n$ points on a nonsingular algebraic surface $X$. The result holds over an arbitrary field.

The space $X^{[n]}$ is a remarkably rich object which finds itself, for reasons that are perhaps still mysterious, at the crossroads of geometry, representation theory and mathematical physics. In the context of testing the $S$-duality conjecture, using Göttsche’s calculation of the Betti numbers of $X^{[n]}$, Vafa and Witten suggested, for reasons stemming from orbifold cohomology, that the singular cohomology groups of these Hilbert schemes should be naturally linked to infinite dimensional graded Lie algebras. This fact was firmly established by Nakajima and Grojnowski, independently. Some of the contributors to this circle of ideas are: Fogarty [6], Briancon [1], Iarrobino [11], Ellingsrud-Stromme [4], Göttsche [8], Göttsche-Sorgel [9], Cheah [2], Vafa-Witten [15], Nakajima [13], Grojnowski [10]. The reader is referred to the beautiful lectures [14] and to our paper [3] for background, further results and references to the literature.

In our paper [3] we proved, directly, a precise form of the decomposition theorem for the so-called Hilbert-Chow map. We detected, via the action of the Lie algebra, certain subvarieties of $X^{[n]}$ carrying the topological information necessary to understand the additive structure of singular cohomology. Subsequently, we looked directly at certain related correspondences. In this paper we show how these correspondences identify the Chow groups with rational coefficients of $X^{[n]}$ with the Chow groups with rational coefficients of a certain collection of products of symmetric products of the surface $X$. Using this result we determine the Chow motive with rational coefficients of $X^{[n]}$.

Voevodsky’s theories of motivic cohomology incorporate the formalisms of Tate twists and shifts of complexes. Once the Chow motive of $X^{[n]}$ is computed, it is easy to compute the motive in these more general theories. Our result, read in these theories, harmonizes

---

*Partially supported by N.S.F. Grant DMS 9701779.
†member of GNSAGA, supported by MURST funds, and by SFB237
the shifts present in the decomposition theorem mentioned above and the Tate twists in the mixed Hodge structure of $X^{[n]}$. There is also a generating function for the motives in question.

The main tool used in our calculation is the Gysin formalism of Fulton-MacPherson. None of the tools used in the various derivations of the structure of the singular cohomology of $X^{[n]}$ in the literature seems applicable in the context of cycles. In fact, this cycle-theoretic approach can be easily supplemented to give another proof of Göttsche’s formula for the Betti numbers. The fact, due to Ellingsrud and Stromme, that punctual Hilbert schemes admit affine cellular decompositions is essential to our approach.

The outline of the paper is as follows. §2 is devoted to fixing the notation and to introducing the varieties naturally associated with $X^{[n]}$ and its natural stratification given by partitions. §3 reviews the basic results from intersection theory that we need. §4 reviews well-known facts concerning the Gysin formalism of correspondences in a “non-complete” situation. §4.1 discusses correspondences in this “refined” formalism. §4.2 discusses the composition of correspondences in the context of the refined formalism. §4.3 overviews the situation for quotient varieties. In §4.4 we define the natural map $\hat{\Gamma}$ via the correspondences $\hat{\Gamma}$. The main result of the paper is Theorem 5.4.1, which states that $\hat{\Gamma}$ is an isomorphism. The injectivity statement is Corollary 5.4.2. The surjectivity statement, Corollary 5.4.3, is the heart of the present paper. Corollary 5.4.2 and Remark 5.4.3 are standard consequences for the Grothendieck groups also in the equivariant context. §5 is devoted to the identification, Theorem 6.2.1, of the motive of the Hilbert scheme $X^{[n]}$ with a collection of motives of the Gysin formalism of correspondences in this “refined” formalism. §6 is devoted to the identification, Theorem 6.2.1, of the motive of the Hilbert scheme $X^{[n]}$ with a collection of motives of the surface $X$. Theorem 6.2.4 is the translation of Theorem 6.2.1 into Voevodsky’s categories. We give a “generating function” for this motivic structure at the end of §6.1.3.

Acknowledgments. Some ideas of N. Fakhruddin have influenced our approach to this question. We have asked him to co-author the present paper, but he has declined. The first author would like to thank Hong Kong University for its very generous hospitality while portions of this work have been carried out. We wish to thank M. Kumar, J.I. Manin and A. Vistoli for useful conversations. The commutative diagrams are produced by Paul Taylor’s diagrams macro package, available in CTAN in TeX/macros/generic/diagrams/taylor.

2 Some special varieties and fibrations

In these section we review a few definitions and introduce the correspondences which will play a major role in the paper. Proofs or references for all the results stated here can be found in our previous paper [3]. Let $X$ be an irreducible quasiprojective nonsingular surface defined over an algebraically closed field, $X^{(n)}$ be its $n$-th symmetric product, $X^{(n)}$ be the Hilbert scheme of 0-dimensional subschemes of $X$ of length $n$, $\pi : X^{[n]} \to X^{(n)}$ be the Hilbert-Chow morphism. It is well known that $X^{[n]}$ is nonsingular. $\mathfrak{P}(n)$ denotes the set of partitions of $n$ and $p(n)$ its cardinality. If $\nu \in \mathfrak{P}(n)$, we denote by $l(\nu)$ its length, and define $X^{(n)}_\nu$ to be the locally closed subset of points in $X^{(n)}$ of the type $\nu_1 x_1 + \cdots + \nu_l(\nu)x_l(\nu)$, with $x_h \in X$ and $x_i \neq x_j$ for every $i \neq j$. Similarly, $X^{(n)}_\nu := (\pi^{-1}(X^{(n)}_\nu))_{\text{red}}$. Let $X^{(n)}_\nu$ denote the closure of the stratum. It can be proved that $X^{(n)}_\nu = (\pi^{-1}(X^{(n)}_\nu))_{\text{red}}$. If $\nu = 1^{a_1} \cdots n^{a_n}$, then the finite group $\Sigma(\nu) := \Sigma_{a_1} \times \cdots \times \Sigma_{a_n}$ acts naturally on $X^{(\nu)}$. The quotient $X^{(\nu)}$ is isomorphic to $X^{(a_1)} \times \cdots \times X^{(a_n)}$. For the sake of uniformity of notation we shall denote $X^{(\nu)}$ by $X^{\nu}$. There
is a natural \( \Sigma_{\nu} \)-invariant map \( \nu : X^\nu \to X^{(n)} \) whose image is \( \overline{X}_\nu^{(n)} \). This map descends to a
map which we denote by the same symbol \( \nu : X^\nu \to X^{(n)} \). This map is the normalization
map of \( \overline{X}_\nu^{(n)} \). We define correspondences \( \Gamma^\nu \) and \( \hat{\Gamma}^\nu \) as follows:

\[
\Gamma^\nu = \{ (x_1, \ldots, x_{\ell(\nu)}, J) \in X^\nu \times X^{[n]} : \pi(J) = \nu x_1 + \cdots + \nu_{\ell(\nu)} x_{\ell(\nu)} \} \cong (X^\nu \times X^{(n)} X^{[n]})_{\text{red}}.
\]

Denote by \( X^\nu_{\text{reg}} \) (resp. \( X^{(n)}_{\text{reg}} \)) the set of points of \( X^\nu \) (resp. \( X^{(n)} \)) strictly of type \( \nu \).

**Remark 2.0.1** The restriction \( \Gamma^\nu_{\text{reg}} \) of \( \Gamma^\nu \) to \( X^\nu_{\text{reg}} \) is a Zariski locally trivial fibration with
irreducible fibers and it is open and dense in \( \Gamma^\nu \) (see [3] Lemma 3.6.1 and Remark 5.2.5). It follows that \( \Gamma^\nu \) is irreducible of dimension \( n + l(\nu) \).

The projection \( p_1 \) will be denoted by \( p_\nu \) in order to emphasize the dependence on \( \nu \). The
correspondence \( \Gamma^\nu \) is invariant under the action of \( \Sigma_{\nu} \) on the first factor of the product. We
can therefore define \( \hat{\Gamma}^\nu := \Gamma^\nu / \Sigma_{\nu} \). We summarize these constructions with the diagram:

![Diagram](image)

where \( q \) and \( q' \) are the quotient maps by the action of \( \Sigma_{\nu} \). The composition \( P \circ q' \) will still
be denoted by \( P \).

**Remark 2.0.2** The restriction of \( P \) to \( \hat{\Gamma}^\nu_{\text{reg}} = \Gamma^\nu_{\text{reg}} / \Sigma_{\nu} \) identifies \( \hat{\Gamma}^\nu_{\text{reg}} \) with the locally closed
stratum \( X^\nu_{\nu} \).

We introduce \( \mathcal{X} = \bigsqcup_{\nu \in \mathfrak{X}(n)} X^\nu \) and \( \hat{\mathcal{X}} = \bigsqcup_{\nu \in \mathfrak{X}(n)} X^{(\nu)} \), \( \Gamma = \bigsqcup_{\nu \in \mathfrak{X}(n)} \Gamma^\nu \), \( \hat{\Gamma} = \bigsqcup_{\nu \in \mathfrak{X}(n)} \hat{\Gamma}^\nu \) and
define \( p : \Gamma \to \mathcal{X} \) (resp. \( \hat{p} : \hat{\Gamma} \to \hat{\mathcal{X}} \)) to be the map induced by all the maps \( p_\nu \) (resp. \( \hat{p}_\nu \)).
The set of partitions has a natural partial order, which reflects the incidence relations of the
strata:

**Definition 2.0.3** Let \( \nu, \mu \in \mathfrak{X}(n) \). We say that \( \nu \geq \mu \) if there exists a decomposition
\( \{I_1, \ldots, I_{l(\mu)}\} \) of the set \( \{1, \ldots, l(\nu)\} \) such that \( \mu_1 = \sum_{i \in I_1} \nu_i, \ldots, \mu_{l(\mu)} = \sum_{i \in I_{l(\mu)}} \nu_i \).

It is easily seen that \( \mu \geq \nu \) if and only if \( X^\mu_{\nu} \subseteq X^\mu_{\nu} \). Fix a total order \( \succeq \) on \( \mathfrak{X}(n) \) which is
compatible with \( \geq \). For any \( \mu \in \mathfrak{X}(n) \) we have the open subsets \( X^\mu_{\geq \mu} := \bigsqcup_{\nu \geq \mu} X^\nu_{\nu} \subseteq X^{(n)} \).
Similarly for \( X^\mu_{> \mu} \). Correspondingly, we have open sets \( \mathcal{X}_{\geq \mu}, \mathcal{X}_{> \mu}, \Gamma_{\geq \mu}, \Gamma_{> \mu}, X_{\geq \mu}^{[n]} \) and \( X_{> \mu}^{[n]} \)
obtained by base change followed by reduction. We have the corresponding quotients \( \mathcal{X}_{\geq \mu}, \mathcal{X}_{> \mu}, \Gamma_{\geq \mu}, \Gamma_{> \mu}, X_{\geq \mu}^{[n]} \) and \( X_{> \mu}^{[n]} \). Similarly, with the symbol \( \geq \) replaced by \( \succeq \).
The imbedding \( X^\mu_{\nu} \to X^\mu_{\nu} \) is closed. We have corresponding closed imbeddings \( \mathcal{X}_\mu \to \mathcal{X}_{\geq \mu} \)
and \( \Gamma_\mu \to \Gamma_{\geq \mu} \) obtained by base change followed by reduction. Note that \( \Gamma_\mu \not= \Gamma^\mu \). We thus
have a diagram:

\[
\begin{array}{ccc}
\Gamma_{\mu} & \xrightarrow{P} & X_{\mu} \\
\downarrow & & \downarrow \pi \\
X_{\geq \mu} & \xrightarrow{P} & X_{\geq \mu}
\end{array}
\]

3 Review of Intersection Theory

Our unique reference for this section is [7]. In what follows, the Chow groups \( A_*(X) \) of an algebraic scheme \( X \) over a field are always taken with rational coefficients, even though many results hold with integer coefficients. We recall that given a regular imbedding \( i : X \to Y \) of codimension \( d \) with normal bundle \( N_XY \) and a morphism \( f : Y' \to Y \) from a pure \( l \)-dimensional variety \( Y' \), there are refined Gysin homomorphisms \( i^! : A_*(Y') \to A_{*-d}(X') \) where \( X' = X \times_Y Y' \). The construction of \( i^!([Y']) \) goes as follows: the map \( X' \to Y' \) is a closed imbedding and the normal cone \( C_{X,Y'} \) is a pure \( l \)-dimensional subscheme of the pullback of \( N_XY \) to \( X' \); its cycle class is therefore equivalent to the flat pullback of a unique cycle class \( i^!([Y']) \in A_{l-d}(X') \). To define \( i^!(\alpha) \) for \( \alpha \in A_*(Y') \) one replaces \( Y' \) with the support of \( \alpha \) and maps the resulting class to \( A_*(Y') \).

Let \( f : X \to Y \) be a morphism from a scheme \( X \) to a nonsingular variety \( Y \). The graph morphism \( \gamma_f : X \to X \times Y \) is a regular imbedding. Given maps \( X' \to X \) and \( Y' \to Y \) there is a refined Gysin morphism \( \gamma^*_f : A_k(Y') \otimes A_l(X') \to A_{k+l-\dim Y}(X' \times_Y Y') \). If \( X' = X \), \( Y' = Y \) and both maps are the identity map, then we will denote \( \gamma^*_f(\alpha \otimes \beta) \), the image of \( \alpha \otimes \beta \) via the morphism \( A_k(Y) \otimes A_l(X) \to A_{k+l-\dim Y}(X) \), with the more suggestive piece of notation \( f^*(\alpha) \cap \beta \). If \( X' = X \), then we will use the notation \( f^!(\alpha) \cap \beta \) for the refined intersection.

**Remark 3.0.4** Let \( \alpha \in Z_k(Y') \), \( \beta \in Z_l(X') \) be irreducible cycles. If the irreducible components \( \xi_i \) of \( |\alpha| \times_Y |\beta| \) have the expected dimension \( k + l - \dim Y \), then \( \gamma_f^!(\alpha \otimes \beta) \) is a linear combination with strictly positive coefficients of the cycles \( \xi_i \); see [7], §7.1. In particular, if \( Y' \) is a closed subvariety of \( Y \) and \( f^{-1}(Y') \) is irreducible of dimension \( \dim Y' + \dim X - \dim Y \), then \( f^!(|Y'|) \cap [X] \) is a positive multiple of \( [f^{-1}(Y')] \).

The following well-known lemma will be used in the sequel of the paper. We omit the simple proof which follows easily from the case of a trivial fibration (see [7], Ex. 1.10.2) and by noetherian induction using the basic properties of the refined Gysin formalism.
Lemma 3.0.5 Let \( X \) be a nonsingular irreducible variety and \( p : \Gamma \to X \) a Zariski locally trivial fibration with fiber \( F \) admitting a cellular decomposition. Suppose \( \{\alpha_i\}_{i \in I} \) is a set of classes in \( A_*(\Gamma) \) whose restrictions generate \( A_*(p^{-1}(x)) \) for every \( x \in X \). Then \( \{\alpha_i\}_{i \in I} \) is a set of generators of the \( A_*(X) \)-module \( A_*(\Gamma) \). In other words, the map \( \Phi : A_*(X) \oplus I \to A_*(\Gamma) \), defined by \( \Phi(\{\beta_i\}) = \sum_i p_1^*(\beta_i) \cap \alpha_i \), is surjective.

4 Correspondences

4.1 Correspondences via refined Gysin maps

The standard reference is [7], Remark 16.1, §6 and §8. Consider a diagram

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{p_2} & X_2 \\
\downarrow^{p_1} & & \downarrow \\
X_1
\end{array}
\]

with \( X_1 \) nonsingular and \( p_2 \) proper. In section 3 we define a map

\[
A_k(X_1) \otimes A_l(\Gamma) \xrightarrow{p_1^*(-) \cap (-)} A_{k+l-\dim X_1}(\Gamma) \xrightarrow{p_{2*}} A_{k+l-\dim X_1}(X_2).
\]

The case we will mostly use is

\[
p_{2*}(p_1^*(-) \cap [\Gamma]) : A_k(X_1) \to A_{k+\dim \Gamma-\dim X_1}(X_2),
\]

where \([\Gamma]\) is some fixed cycle.

Most important for us will be the case of relative correspondences of the type

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{p_2} & X_2 \\
\downarrow^{p_1} & & \downarrow \\
X_1 & \xrightarrow{f} & Y
\end{array}
\]

where \( X_1 \) is a nonsingular variety, \( f \) is a proper morphism and \( \Gamma = X_1 \times_Y X_2 \). If \( V \to Y \) is a morphism, then we have a similar diagram by pulling back to \( V \). Set, for brevity, \( X_{1,V} = X_1 \times_Y V \) and similarly \( X_{2,V} = X_2 \times_Y V \) and \( \Gamma_V = \Gamma \times_Y V \). Note that, by the transitivity property of the fiber product, \( \Gamma_V = X_{1,V} \times_V X_{2,V} \). Fix a cycle \([\Gamma]\). The refined intersection product gives a map \( p_{1*}^*(-) \cap [\Gamma] : A_*(X_{1,V}) \to A_*(\Gamma_V) \).

Remark 4.1.1 It is important that we do not assume that \( X_1 \) is complete, i.e. that we do not assume that \( p_2 : X_1 \times X_2 \to X_2 \) is proper and that we do not factorize through \( A_*(X_1 \times X_2) \). Otherwise, we would lose too much information: e.g. the diagonal in \( A^1 \times A^1 \) is zero in \( A_1(A^1 \times A^1) \), but it induces the identity map via the formalism of refined intersections.

Let \( i : Z \to Y \) be a closed imbedding and \( j : U := Y \setminus Z \to Y \) be the resulting open imbedding.
Lemma 4.1.2 Let $[\Gamma]$ be any cycle in $A_*(\Gamma)$. The diagram

$$
\begin{array}{ccc}
A_*(X_{1,Z}) & \xrightarrow{p_1^1(-) \cap [\Gamma]} & A_*(\Gamma_Z) \\
A_*(X_1) & \xrightarrow{j^*} & A_*(\Gamma) \\
A_*(X_{1,U}) & \xrightarrow{j^*} & A_*(\Gamma_U) \\
0 & & 0
\end{array}
$$

is commutative and the columns are exact.

Proof. The exactness of the columns stems from [7], Proposition 1.8. The commutativity of the first column of diagrams stems from [7], Proposition 8.1.1.(c) and Theorem 6.2. The commutativity of the second column is obvious. \qed

Remark 4.1.3 Note that, even if \( \Gamma \) is reduced, \( \Gamma_Z \) may fail to be so. However, this causes no trouble in view of the obvious canonical isomorphism \( A_*(-_{red}) \to A_*(-) \). In the sequel we shall often take fibre products. We shall always assume that we are taking the reduced structure. When we write that a diagram is “cartesian modulo nilpotents”, we mean that the fibre product is to be taken with the reduced structure. We shall always take care to define the cycle $[\Gamma]$.

4.2 Composition of correspondences

This formalism of correspondences via refined Gysin maps extends easily to the case of composition of correspondences. The reference is [7], Remark 16.1, Proposition 16.1.2, §6 and §8.

Let $X_i$, $i = 1, 2, 3$ be nonsingular varieties. Let $p_{ij} : X_1 \times X_2 \times X_3 \to X_i \times X_j$ be the obvious projections. Let $\Phi \subseteq X_1 \times X_2$ and $\Gamma \subseteq X_2 \times X_3$ be two irreducible cycles such that all the following maps are proper: $[\Phi] \to X_2$, $\Gamma \to X_3$, $p_{12}^1([\Phi]) \cap p_{23}^{-1}([\Gamma]) \to X_1 \times X_3$, $p_{13}(p_{12}^{-1}([\Phi]) \cap p_{23}^{-1}([\Gamma])) \to X_3$. In addition, assume that $p_{13}(p_{12}^{-1}([\Phi]) \cap p_{23}^{-1}([\Gamma]))$ has the expected dimension.

We can therefore define a refined cycle $\Gamma \circ \Phi$ in $Z_*(p_{13}(p_{12}^{-1}([\Phi]) \cap p_{23}^{-1}([\Gamma])))$ and maps: $\Phi_*$, $\Gamma_*$ and $(\Gamma \circ \Phi)_*$. We have $\Gamma_* \circ \Phi_* = (\Gamma \circ \Phi)_*$. 
4.3 Quotient varieties

The formalism recalled so far extends to the case of quotient varieties. See [7], Ex. 16.1.13. In this case rational coefficients are necessary. The basic set-up we need is as follows.

\( Z \) is a nonsingular integral variety; \( Z' \) is an integral variety; \( p_Z : Z' \to Z \) is a surjective morphism; \( G \) is a finite group acting on \( Z \) and \( Z' \), compatibly with \( p_Z \). There is a commutative diagram:

\[
\begin{array}{ccc}
Z' & \xrightarrow{p'} & Z' \\
p_Z \downarrow & & \downarrow p_{Z_1} \\
Z & \xrightarrow{p} & Z_1
\end{array}
\]

where \( Z_1 = Z/G \), \( p \) is a quotient Galois morphism of degree \( |G| \), \( Z'_1 = Z'/G \) and \( p' \) is a quotient Galois morphism of degree \( |G| \). Note that it follows that \( p \) and \( p' \) are separable.

Define \( [Z] \) and \( [Z_1] \) to be the corresponding fundamental classes.

We have that \( p^*[Z'_1] = [Z] \). We can define a Gysin-type map as follows:

\[
\Phi_{[Z'_1]} : A_*(Z_1) \to A_*(Z'_1), \quad z_1 \mapsto \frac{1}{|G|} p'_*(p_Z^*(p^*z_1) \cap [Z'_1]).
\]

We need the following elementary fact which can be checked directly using the definition of \( p^* \) for quotient maps; see [7], Ex. 1.7.6.

**Lemma 4.3.1** Let \( R \) be a variety endowed with an action of a finite group \( G \) on it. Denote the quotient \( p : R \to S \). Let \( \sigma : A_*(R) \to A_*(R), z \to \sum_{g \in G} \gamma_*z \), be the so-called symmetrization operator. Then \( \sigma = p^*p_* \).

**Lemma 4.3.2** We have the following commutative diagram:

\[
\begin{array}{ccc}
A_*(Z') & \xrightarrow{p'^*} & A_*(Z'_1) \\
\Phi_{[Z'_1]} \downarrow & & \downarrow \Phi_{[Z'_1]} \\
A_*(Z) & \xrightarrow{p^*} & A_*(Z_1)
\end{array}
\]

**Proof.** For ease of notation we denote a class \( p^*_Z(a) \cap p'^*_g z'_1 \), by \( a \cdot p'^*_g z'_1 \).

Let \( z \in A_*(Z) \) and \( z' \in A_*(Z') \). Define \( \sigma : z \to \sum_{g \in G} \gamma_*z \) and \( \sigma' : z' \to \sum_{g \in G} \gamma_*z' \). By virtue of Lemma 4.3.1, we have that \( \sigma = p^*p_* \) and that \( \sigma' = p'^*p'_* \).

It follows that \( p'^*p'_*(z \cdot p'^*_g z'_1) = \sigma'(z \cdot p'^*_g z'_1) = \sum_{g \in G} \gamma_*z \cdot p'^*_g z'_1 = (\text{by the proper push-forward property}) = \sum_{g \in G} (\gamma_*z \cdot \gamma_*p'^*_g z'_1) = (\text{since } p'^*_g z'_1 \text{ is } G\text{-invariant}) = \sum_{g \in G} (\gamma_*z \cdot z'_1) = (p^*p_* z) \cdot p'^*_g z'_1 = (\text{since } p^*p_* z \text{ is } G\text{-invariant}) = \frac{1}{|G|} \Phi_{[Z'_1]}(p^*p_* z) \cdot p'^*_g z'_1 = \frac{1}{|G|} \Phi_{[Z'_1]}(p'^*p'_*(p^*p_* z \cdot z'_1)). \)

Since \( p'^* \) is injective, we have that \( p'^*_g(z \cdot p'^*_g z'_1) = \frac{1}{|G|} p'_*(p^*p_* z \cdot p'^*_g z'_1) \) which is what we wanted to prove. \( \square \)
4.4 The morphisms $\Gamma_*$ and $\hat{\Gamma}_*$

Let $X$ be an irreducible nonsingular quasi projective surface defined over an algebraically closed field.

Recalling the notation introduced in §2, we summarize the results of this section by stating that the following diagram is commutative:

$$
\begin{array}{ccc}
A_*^\Gamma & \longrightarrow & A_*^{\hat{\Gamma}} \\
\downarrow \Gamma_* & & \downarrow \hat{\Gamma}_* \\
A_*(X^{[n]}) & \longrightarrow & A_*(\hat{X})
\end{array}
$$

By abuse of notation, the corresponding maps into $A_*^\Gamma(X^{[n]})$ will be denoted with the same symbol, e.g. $\Gamma_* : A_*(X) \rightarrow A_*(X^{[n]})$ and $\hat{\Gamma}_* : A_*(\hat{X}) \rightarrow A_*(X^{[n]})$.

5 The Chow groups of $X^{[n]}$

Let $X$ be an irreducible nonsingular quasi projective surface defined over an algebraically closed field. The reader should keep in mind §2.

5.1 The injectivity of $\hat{\Gamma}_*$

We shall freely use the formalism developed for correspondences over quotient varieties in §4.3.

Lemma 5.1.1 Let $Z$ be an irreducible algebraic scheme of dimension $n$, quotient of a nonsingular algebraic scheme via a finite group. Let $V$ and $W$ be pure-dimensional cycles on $Z$ of dimensions $k$ and $l$. Let $f : Z \rightarrow Y$ be a proper morphism of algebraic schemes. If $\dim f(|V| \cap |W|) < k + l - n$, then $f_*(V \cdot W) = 0$.

Proof. There is a well-defined refinement of the product in $A_k^{k+l}(V \cap W)$ which must map to zero. See [8], Ex. 8.3.12. ☐

The following lemma is essentially a reformulation of [5]. Before stating it, we introduce some notation: Let $\nu$ be a partition of $n$. For $x \in X_{\text{reg}}^{[\nu]}$, we set $F_\nu = (\pi^{-1}(x))_{\text{red}}$, which we identify via $P$ with $p_{\nu}^{-1}(x)$ (cf. 2.0.2). We set $m_\nu := (-1)^{n-l(\nu)} \prod_{j=1}^{l(\nu)} \nu_j$. We denote by $[F_\nu] \cdot X^{[n]}_{\geq \nu} \in A_0(F_\nu)$ the refined intersection defined by the closed imbeddings of $F_\nu$ and $X^{[n]}_{\geq \nu}$ in $X^{[n]}$.

Lemma 5.1.2 $\deg ([F_\nu] \cdot X^{[n]}_{\geq \nu}) = m_\nu$.

Proof. The case $\nu = n^1$ is precisely the main result in [5]. The general case follows from the K"unneth formula. ☐

In the following two propositions we compute the compositions $\iota^* \hat{\Gamma}_* \circ \hat{\Gamma}_*$. Here $\iota^*$ denotes, as usual, the transposed correspondence; see [8], §16.1.
Proposition 5.1.3 Let \( \mu \neq \nu \) be two distinct partitions of \( n \). Then \( \hat{\Gamma}^\nu \circ \hat{\Gamma}^\mu = 0 \) in \( A_{l(\mu) + l(\nu)}(X(\mu) \times X(\nu)) \).

Proof. Consider \( X(\mu) \times X[n] \times X(\nu) \) together with the natural projections \( p, \pi \) and \( q \) to \( X(\mu) \times X[n] \), \( X(\mu) \times X(\nu) \) and \( X[n] \times X(\nu) \), respectively. By virtue of Lemma 5.1.1 it is enough to show that \( \dim \pi(p^{-1}\hat{\Gamma}^\mu \cap q^{-1} \hat{\Gamma}^\nu) < l(\mu) + l(\nu) \).

One checks directly that \( \dim \pi(p^{-1}\hat{\Gamma}^\mu \cap q^{-1} \hat{\Gamma}^\nu) = \dim (\overline{X}_\mu(n) \cap \overline{X}_\mu(n)) \). Since \( \mu \neq \nu \), this last dimension is strictly less than \( 2 \min (l(\mu), l(\nu)) \).

\( \square \)

Proposition 5.1.4 \( \hat{\Gamma}^\nu \circ \hat{\Gamma}^\nu = m_\nu \Delta_{X(\nu)} \) in \( A_{2l(\nu)}(X(\nu) \times X(\nu)) \).

Proof. The cycle \( \hat{\Gamma}^\nu \circ \hat{\Gamma}^\nu \) is supported on the diagonal of \( X(\nu) \times X(\nu) \) which is irreducible of the expected dimension, therefore \( \hat{\Gamma}^\nu \circ \hat{\Gamma}^\nu = c \Delta_{X(\nu)} \), with \( c \in \mathbb{Q} \). Let \( x \in X_{\nu_reg} \). Since the map \( \hat{\Gamma}_{\nu_reg} \rightarrow X_{\nu_reg} \) is flat, we have that \( \hat{\Gamma}_x^\nu([\underline{x}]) = [F_{\nu}] \). By virtue of Lemma 5.1.2 and of the projection formula we have that

\[
c = \deg (\hat{\Gamma}_X^\nu \circ \hat{\Gamma}_X^\nu ([\underline{x}])) = \deg (\hat{\Gamma}_X^\nu ([F_{\nu}])) = \deg ([F_{\nu}] \cdot X_{\nu}^-[n]) = m_\nu.
\]

\( \square \)

Corollary 5.1.5 The natural map \( \hat{\Gamma}_X = \bigoplus_{\nu \in \Psi(n)} \hat{\Gamma}_X^\nu : A_*(\hat{X}) \rightarrow \bigoplus_{\nu \in \Psi(n)} A_*(X(\nu)) \rightarrow A_*(X[n]) \) is injective.

Proof. Propositions 5.1.4 and 5.1.3 imply that \( \hat{\Gamma}_X \circ \hat{\Gamma}_X \) is a non zero multiple of the identity being the map induced by a non zero multiple of the diagonal.

\( \square \)

Remark 5.1.6 An argument identical to the one given above shows that Corollary 5.1.3 holds if we replace all spaces by the ones obtained by base change with respect to any open immersion \( U \rightarrow X(n) \).

5.2 A surjectivity statement

The following surjectivity statement is essential to proving Proposition 5.3.1 which, in turn, constitutes the surjectivity part of the main result of this paper, Theorem 5.4.1. The proof will be given at the end of this section, after a series of preparatory results.

Proposition 5.2.1 Let \( \mu \in \Psi(n) \): the map \( P_*(P_{\geq \mu}(-) \cap [\Gamma_{\geq \mu}]) : A_*(X_\mu) \rightarrow A_*(X_{\mu}[n]) \) is surjective.

In order to prove Proposition 5.2.1 we need to make explicit the combinatorics involved. Let \( l \) be an integer, and \( \mathfrak{R} \) be the set of decompositions \( \rho = \{I_1, \cdots, I_r\} \) of the set \( \{1, \cdots, l\} \) into disjoint subsets, i.e. \( \{1, \cdots, l\} = I_1 \sqcup \cdots \sqcup I_r \). For \( \rho \in \mathfrak{R} \) define \( D_\rho = \{(x_1, \cdots, x_l) \in X^l : x_i = x_j \Leftrightarrow i, j \in I_k \text{ for some } k\} \). For \( \rho = \{1, \cdots, l\} \) we have \( D_\rho = X_{\nu_reg} = X^l \setminus \text{Diagonals} \). For \( \rho = \{1, \cdots, l\} \), we have that \( D_\rho \) is the small diagonal.
Let \( \mathfrak{P}_{\leq \nu}(n) = \{ \mu \in \mathfrak{P}(n) \text{ such that } \mu \preceq \nu \} \). Define \( Q_\nu : \mathfrak{R}_l(\nu) \to \mathfrak{P}_{\leq \nu}(n) \) by setting \( Q_\nu(\{I_1, \ldots, I_r\}) = (\sum_{i \in I_1} \nu_i, \ldots, \sum_{i \in I_r} \nu_i) \). Note that this partition is not ordered, i.e. \( \sum_{i \in I_1} \nu_i \geq \sum_{i \in I_2} \nu_i \cdots \geq \sum_{i \in I_r} \nu_i \), but see Remark 5.2.4.

Let \( \nu : X^\nu \to X^{(n)}_\nu \) be the already mentioned map defined by \( \nu(x_1, \ldots, x_{l(\nu)}) = \nu_1 x_1 + \cdots + \nu_{l(\nu)} x_{l(\nu)} \). Note that \( \nu^{-1}(X^{(n)}_\nu) = X^{(n)}_{\text{reg}} \). Note that \( l(Q_\nu(\{I_1, \ldots, I_r\})) = r \). The proof of the following lemma is immediate:

**Lemma 5.2.2**

\[
X^\nu \times X^{(n)}_\mu \neq \emptyset \iff \nu \succeq \mu \text{ and } (X^\nu \times X^{(n)}_\mu)_{\text{red}} = \coprod_{\rho \in Q_\nu^{-1}(\mu)} \mathcal{D}_\rho.
\]

Let \( \Gamma^\nu \) be the correspondences introduced in \( \S 2 \).

**Remark 5.2.3** The group \( \Sigma_\nu \) acts on \( \mathfrak{R}_l(\nu) \) and \( Q_\nu \) is \( \Sigma_\nu \)-invariant. In particular, \( \Sigma_\nu \) acts on \( Q_\nu^{-1}(\mu) \).

**Remark 5.2.4** For \( \nu \in \mathfrak{P}(n) \) we can fix a total order on the set \( \mathfrak{R}_l(\nu) \) so that we always have \( \sum_{i \in I_1} \nu_i \geq \sum_{i \in I_2} \nu_i \cdots \geq \sum_{i \in I_r} \nu_i \).

Let \( \rho \in \mathfrak{R}(\nu) \) and \( \mu := Q_\nu(\rho) \). We can identify \( \mathcal{D}_\rho \) with \( X^\mu_{\text{reg}} \) via the map \( \delta_\rho : X^\mu_{\text{reg}} \to X^\nu \) sending the point \( (y_1, \ldots, y_r) \in X^\mu_{\text{reg}} \) to \( (x_1, \ldots, x_{l(\nu)}) \) defined by \( x_j = y_i \) precisely if \( j \in I_i \).

In this case \( \nu \circ \delta_\rho = \mu |_{X^\mu_{\text{reg}}} : X^\mu_{\text{reg}} \to X^{(n)} \), hence \( (\Gamma^\nu \times X^\nu \mathcal{D}_\rho)_{\text{red}} = \Gamma^\mu_{|X^\mu_{\text{reg}}} =: \Gamma^\mu_{\text{reg}} \). These identifications will always be tacitly made in the sequel.

**Remark 5.2.5** \( \rho_\mu : \Gamma^\mu_{\text{reg}} \to X^\mu_{\text{reg}} \) is a Zariski locally trivial fibration. The fiber is isomorphic to the product of punctual Hilbert schemes \( \mathcal{H}_{\mu_1} \times \cdots \times \mathcal{H}_{\mu_i} \) and it admits a cellular decomposition. See \( \S 2 \), Lemma 2.1.4. and \( \S 4 \).

Let \( (\nu, \rho) \) be such that \( Q_\nu(\rho) = \mu \). The pair \( (\nu, \rho) \) defines a set of partitions \( \nu^i \in \mathfrak{P}(\mu_i) \) as follows: if \( \rho = \{I_1, \ldots, I_{l(\mu)}\} \), then \( \mu_i = \sum_{j \in I_i} \nu_j \) and therefore \( \nu^i = \{\nu_j\}_{j \in I_i} \) is a partition of \( \mu_i \).

**Definition 5.2.6** Define the open sets

\[
U_\rho = \{(x_1, \ldots, x_{l(\nu)}) \in X^\nu, \text{ such that the subsets } \{x_j\}_{j \in I_i} \text{ are pairwise disjoint}\}.
\]

In other words we allow only points belonging to the same \( I_j \) to collapse: in particular \( U_\rho \supseteq \mathcal{D}_\rho \).

**Lemma 5.2.7** Let \( (\nu, \rho) \) be such that \( Q_\nu(\rho) = \mu \), and \( \nu^i \in \mathfrak{P}(\mu_i) \) be the corresponding set of partitions. There is a canonical isomorphism

\[
\Gamma^\nu_{|U_\rho} = \prod \Gamma^{\nu^i}_{|U_\rho}.
\]
Proof. Let \((x_1, \ldots, x_{|\mu|}) \in \Gamma^\nu_{|U^\rho|}\). From the definition of \(U^\rho\) it follows that \(U\) is the product of ideals \(\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_r\) of lengths \(\mu_1, \mu_2, \ldots, \mu_r\) supported on \(\{x_j\}_{j \in I_1}, \{x_j\}_{j \in I_2}, \ldots, \{x_j\}_{j \in I_r}\), whence the obviously bijective map from \(\Gamma^\nu_{|U^\rho|}\) to \(\prod \Gamma^\nu_{|U^\rho|}\).

We have the following diagram

\[
\begin{array}{ccc}
\Gamma^\nu_{\text{reg}} & \longrightarrow & \Gamma^\nu_{|U^\rho|} \\
p^\mu & & p^\nu \\
\mathcal{D}_\rho = X^\mu_{\text{reg}} & \delta^\rho & U^\rho,
\end{array}
\]

which is cartesian modulo nilpotents (see Remark \[\ref{rem:4.1.3}\]) and a map \(\gamma_{\nu, \rho} : A_*(X^\mu_{\text{reg}}) \to A_*(\Gamma^\nu_{\text{reg}})\) defined by \(\gamma_{\nu, \rho}(\beta) = p^\nu_{i^\nu}(\beta) \cap \Gamma^\nu\) (cf. \[\ref{lem:5.2.4}\]).

**Lemma 5.2.8** Let \(\alpha_{\nu, \rho} := \gamma_{\nu, \rho}([X^\mu_{\text{reg}}]) \in A_*(\Gamma^\nu_{\text{reg}})\) and \(\beta \in A_*(X^\mu_{\text{reg}})\). Then

\[
\gamma_{\nu, \rho}(\beta) = p^\nu_{i^\nu}(\beta) \cap \alpha_{\nu, \rho}.
\]

**Proof.** It follows from the associativity property of refined products, \[\ref{lem:8.1.1.a}\]. \(\square\)

What has been done so far can be summarized by the following:

**Lemma 5.2.9** Let \(\mu\) be a partition, denote by \(\tilde{q} : \Gamma^\nu_{\text{reg}} \to X^{|n|}_\mu\) the quotient map by the action of \(\Sigma_\mu\). Let \(W \subseteq A_*(\Gamma^\nu_{\text{reg}})\) be the submodule generated by the classes \(\{\alpha_{\nu, \rho}\}_{\rho \in Q^\nu_{1^\nu}(\mu)}\). The surjectivity of the map \(P_*(p^\nu_{\geq \mu}(-) \cap [\Gamma_{\geq \mu}]) : A_*(X^\mu_\mu) \to A_*(X^{|n|}_\mu)\) is equivalent to the surjectivity of the restriction of \(\tilde{q}_* : A_*(\Gamma^\nu_{\text{reg}}) \to A_*(X^{|n|}_\mu)\) to \(W\).

**Proof.** By virtue of Lemma \[\ref{lem:5.2.2}\],

\[
X^\mu_\mu = \prod_{\rho \in Q^\nu_{1^\nu}(\mu)} \mathcal{D}_\rho = \prod_{\rho \in Q^\nu_{1^\nu}(\mu)} X^\mu_{\text{reg}},
\]

this last identification is made using the map described in \[\ref{lem:5.2.4}\]. The following diagram, where the equality \(\sum \gamma_{\nu, \rho} = \sum p^\nu_{i^\nu}(-) \cap \alpha_{\nu, \rho}\) is a consequence of \[\ref{lem:5.2.8}\], commutes:

\[
\begin{array}{ccc}
A_*(X^\mu_\mu) & \longrightarrow & A_*(X^{|n|}_\mu) \\
& \downarrow & \downarrow q_* \\
\oplus_{\rho \in Q^\nu_{1^\nu}(\mu)} A_*(X^\mu_{\text{reg}}) & \sum \gamma_{\nu, \rho} = \sum p^\nu_{i^\nu}(-) \cap \alpha_{\nu, \rho} & A_*(\Gamma^\nu_{\text{reg}}),
\end{array}
\]

whence the statement. \(\square\)

We now study a special case which is crucial to the proof of Proposition \[\ref{prop:5.2.1}\]. We consider \(\mu = n^1\). For every partition \(\nu\) there is only one \(\delta = \delta : X \to X^{|n|}\) and \(p := p_{n^1} : \Gamma^n \to X\) is a Zariski locally trivial fibration, whose fibre is isomorphic to the length \(n\) punctual Hilbert scheme \(\mathcal{H}_n\). For each \(\nu \in \Psi(n)\) we have \(\alpha_{\nu} = p^\nu_{|\nu|}([X]) \cap [\Gamma^{n^1}]\) and the map \(P : A_*(X)^{\oplus_{\nu}(n)} \to A_*([\Gamma^{n^1}])\) sending the collection \(\{\beta_{\nu}\}\) to the class \(\sum_{\nu} p^\nu_{i^\nu}([\beta_{\nu}]) \cap \alpha_{\nu}\).
**Lemma 5.2.10** The map $P$ is surjective

**Proof.** By virtue of Lemma [3.0.5], it is enough to prove that the restrictions $\hat{\alpha}_\nu$ of the classes $\alpha_\nu$ to any fibre generate $A_*(\mathcal{H}_n)$. We may assume, without loss of generality, that $\hat{\alpha}_\nu$ is the cycle $\hat{\gamma}_n$. Consider the pairing

$$A_*(X[n]) \times A_*(X[n]) \to \mathbb{Q}$$

given by taking $(a, b) \to \deg a \cdot b = \int_{X[n]} a \cdot b$, where the last product is the product in the Chow ring. Note that this pairing is almost never perfect. It descends to algebraic equivalence.

CLAIM. $\deg(\hat{\beta}_\nu \cdot [\underline{X}_\mu^{[n]}]) = 0$ if and only if $\mu \neq \nu$.

Proof of the CLAIM. Consider $\Gamma^\nu$ and $X^\nu \times \underline{X}_\nu^{[n]}$ as a pair of cycles in $X^\nu \times X[n]$. They define two families of cycles on $X[n]$: $\Gamma^\nu_\tau$ and $\{X^\nu \times \underline{X}_\nu^{[n]}\}_\tau$, $\tau \in X^\nu$. See [7], §10. In particular, we have $\Gamma^\nu_y = i^!_y([\Gamma^\nu])$. If $y \in X^\nu_{\text{reg}}$, then $\Gamma^\nu_y = [(p^{-1}(y))_{\text{red}}]$. Recalling that we have the canonical $\delta_\nu : X \to X^\nu$, we see that $\hat{\beta}_\nu = \Gamma^\nu_y$. In other words: $\Gamma^\nu_y$ and $\hat{\beta}_\nu$ belong to the same family of cycles. By construction, $\hat{\beta}_\nu$ can be represented, modulo algebraic equivalence, by a cycle supported at $(p^{-1}(y))_{\text{red}}$. Note that $\hat{\beta}_\nu \cdot [\underline{X}_\mu^{[n]}] \in A_{l(\nu)-l(\mu)}(X[n])$.

If $l(\nu) < l(\mu)$, or if $l(\nu) > l(\mu)$, then the degree is zero for trivial reasons. If $l(\nu) = l(\mu)$, but $\mu \neq \nu$, then the degree is still zero. In fact, using refined intersection products in the context of algebraic equivalence, we can represent $\hat{\beta}_\nu$ as a cycle supported on a fiber $\pi^{-1}(y)$. This fiber does not meet $\underline{X}_\mu^{[n]}$.

Finally, if $\nu = \mu$, then, by virtue of [7], Corollary 10.1 and Proposition 10.2, we have that

$$\deg(\hat{\beta}_\nu \cdot [\underline{X}_\mu^{[n]}]) = \deg(\Gamma^\nu \cdot [\underline{X}_\mu^{[n]}]) = \deg([p^{-1}(y)] \cdot [\underline{X}_\mu^{[n]}]) = m_\nu.$$

The CLAIM follows easily. By virtue of Corollary 5.1.5 (see also [8]), the classes $[\underline{X}_\mu^{[n]}]$ are independent. Jointly with the CLAIM just proved, this shows that the classes $\hat{\beta}_\nu$ are independent. It follows that the classes $\hat{\alpha}_\nu$ are independent. Their number is $p(n)$, which is equal to the dimension of $A_*(\mathcal{H}_n)$ (cf. [8], for example). This proves the lemma. \hfill $\square$

Let us go back to the situation dealt with in Lemma 5.2.7: the diagram after that lemma and 5.2.10 imply immediately:

**Lemma 5.2.11** The restriction of the cycle $\alpha_{\mu, \rho} := p_\rho^!([\mathcal{D}_\mu]) \cap [\Gamma^\nu] \in A_*(\Gamma^\mu)$ to a fibre of $p_\mu$ is the cycle $\hat{\alpha}_\mu_1 \times \cdots \times \hat{\alpha}_\nu \in A_*(\mathcal{H}_\mu \times \cdots \times \mathcal{H}_\mu)$.

Let $\eta = (\eta^1, \ldots, \eta^r)$ be a multipartition of $\mu$. By this we mean that $\eta^i \in \mathfrak{P}(\mu_i)$. Set $l_i = l(\eta^i)$ and define the open sets $U_{\eta} \subseteq \prod X^{\eta^i}$ by

$$U_{\eta} = \{(x_1^{(1)}, \ldots, x_{l_1}^{(1)}, \ldots, x_1^{(r)}, \ldots, x_{l_r}^{(r)}) \text{ such that } x_k^{(i)} \neq x_l^{(j)} \text{ if } i \neq j\}.$$

In other words we allow collisions only among points of the same group. $X^\mu_{\text{reg}}$ can be identified with the closed subset of points in $U_{\eta}$ such that $x_k^{(i)} = x_l^{(j)}$ if and only if $i = j$. According to [7], 8.1.4, the diagram

12
Lemma 5.2.12 The set \( \{ \alpha_\nu \} \) generates \( A_*(\Gamma^\nu_{\text{reg}}) \) as a \( A_*(X^\nu_{\text{reg}}) \)-module.

Proof. Since \( \mathcal{H}_{\mu_1} \times \cdots \times \mathcal{H}_{\mu_r} \) has a cellular decomposition, \( A_*(\mathcal{H}_{\mu_1} \times \cdots \times \mathcal{H}_{\mu_r}) = A_*(\mathcal{H}_{\mu_1}) \otimes \cdots \otimes A_*(\mathcal{H}_{\mu_r}) \) has dimension \( \prod p(\mu_i) \). The set in question is therefore a basis of this space, and the statement is a direct consequence of Lemma 5.2.7 and Lemma 5.2.11.

Remark 5.2.13 We have already observed that a couple \( (\nu, \rho) \) with \( Q_\nu(\rho) = \mu \) determines a multipartition \( \varphi_\rho \) of \( \mu \); Lemma 5.2.7 implies that \( \alpha_{\omega_\rho} = \alpha_{\varphi_\rho} \).

Remark 5.2.14 The set \( \{ \alpha_{\nu, \rho} \} \) is only a subset of \( \{ \alpha_\nu \} \) as shown by the following example.

Example 5.2.15 Let \( n = 4 \), and \( \mu = 2^2 \). There are 4 multipartitions: \( (2, 2) \), \( (2, 1^2) \), \( (1^2, 2) \), \( (1^2, 1^2) \) and 4 corresponding cycles which restrict to a basis for \( A_*(H_2 \times H_2) \). The couples \( (\nu, \rho) \) such that \( Q_\nu(\rho) = 2^2 \) are: if \( \nu = 1^4 \), then there are 3 different \( \rho \)'s which are conjugate by \( \Sigma_\nu \) and give the same (0-dimensional) cycle corresponding to the multipartition \( (1^2, 1^2) \); if \( \nu = 2^2 \), then there is only one \( \rho \) which gives the 2-dimensional cycle corresponding to \( (2, 2) \); if \( \nu = 2 \cdot 1^2 \), then the only \( \rho \) is \( \{1\} \), \( \{2\} \), \( \{3\} \) which induces the multipartition \( (2, 1^2) \).

This is consistent with the fact that \( X^{2 \cdot 1^2} \times_{X^{(4)}} X^{(4)}_{2^2} \) contains only one component \( D_\rho \) given by points of type \( x_2 = x_3 \). Note also that the map \( D_\rho \to X^{(4)}_{2^2} \) is \( 2 : 1 \), the quotient map by \( \Sigma_{2^2} \).

This can be easily explained: note first that the group \( \Sigma_\mu \) acts on the set of multipartitions of \( \mu \). Given a multipartition \( \eta = (\eta^1, \ldots, \eta^r) \) of \( \mu \), let \( \eta^i = \eta^i_{l_i} \geq \cdots \geq \eta^i_1 \). The sequence of the \( \eta^i_j \) is a partition \( \nu \in \mathfrak{P}(n) \), let \( l = \sum l_i \) be its length. A permutation \( \sigma \in \Sigma_l \), reordering the sequence in a non-increasing one, is identified up to left multiplication with \( \Sigma_\nu \). Define the subsets \( I_i := \{ \sigma((\sum_{k=0}^{i-1} l_k) + 1), \ldots, \sigma((\sum_{k=0}^i l_k)) \} \). We get \( \{ I_i \} \in \mathfrak{S}_l \), and \( \sum_{k \in I_i} \nu_k = \mu_i \). This associates with \( \eta \) a couple \( (\nu, \rho) \) such that \( \rho \in Q^{-1}_\nu(\mu) \). A different choice of the permutation \( \sigma \) gives the same partition \( \nu \) whereas \( \rho \) is changed by the action of \( \Sigma_\nu \) (cf. Remark 5.2.3).

Starting from the couple \( (\nu, \rho) \), instead, gives, as we have already observed, a multipartition \( \tilde{\eta} \) of \( \mu \). Note that this multipartition depends on the way the \( I_i \)'s were ordered. We thus have:

Lemma 5.2.16 \( \tilde{\eta} \) and \( \eta \) are in the same \( \Sigma_\mu \)-orbit.
Finally, let \( q : D_\rho = X_{\rho, \text{reg}}^\mu \rightarrow X^\mu_{\rho} \) and \( \tilde{q} : \Gamma_{\rho, \text{reg}}^\mu \rightarrow X^\mu_{\rho} \) be the quotient maps by \( \Sigma_\mu \). It follows from \( \square \), Ex. 1.7.6, that \( q^* \) (resp. \( \tilde{q}^* \)) identify \( A_\ast(X^\mu_{\rho, \text{reg}}) \) (resp. \( A_\ast(\Gamma_{\rho, \text{reg}}^\mu) \)) with \( A_\ast(D_\rho)^\Sigma_\mu \), (resp. \( A_\ast(\Gamma_{\rho, \text{reg}}^\mu)^\Sigma_\mu \)). By virtue of lemma \( \sqcup \), if \( \alpha \in A_\ast(D_\rho) \), then \( q^* q_\ast \alpha = \sum_{\sigma \in \Sigma_\mu} \sigma \cdot \alpha \). Similarly, if \( \alpha \in A_\ast(\Gamma_{\rho, \text{reg}}^\mu) \), then \( \tilde{q}^* \tilde{q}_\ast \alpha = \sum_{\sigma \in \Sigma_\mu} \sigma \cdot \alpha \).

**Lemma 5.2.17** The images under the map \( \tilde{q}^* \tilde{q}_\ast \) of the \( A_\ast(X_{\rho, \text{reg}}^\mu) \)-submodules generated by \( \{\alpha_\ast\} \) and \( \{\alpha_{\rho, \ast}\} \) coincide.

**Proof.** We fix our attention on a single \( \Sigma_\mu \)-orbit. Set \( \alpha_{\rho, \ast} = \alpha_\ast \). Let \( H \) denote the stabilizer of \( \alpha_0 \) in \( \Sigma_\mu \), and choose a set \( g_1 = e, g_2, \ldots, g_r \) of representatives for \( \Sigma_\mu / H \), so that the orbit of \( \alpha_0 \) is \( \alpha_0, g_2 \alpha_0, \ldots, g_r \alpha_0 \). Given a cycle \( \alpha = \sum_{i=1}^r p^* (\beta_i) g_i \alpha_0 \), let \( \beta = (\sum_{i=1}^r g_i^{-1} p^* (\beta_i)) \alpha_0 \). Then \( \tilde{q}^* \tilde{q}_\ast \alpha = \tilde{q}^* \tilde{q}_\ast \beta \). In fact, \( \tilde{q}^* \tilde{q}_\ast \alpha = \sum_{g, j} g p^* (\beta_j) g g_j \alpha_0 = \sum_i g_i (\sum_{g \in H} h g_j^{-1} p^* (\beta_j)) g_i \alpha_0 \), while \( \tilde{q}^* \tilde{q}_\ast \beta = \sum_g g (\sum_{g_j} g_j^{-1} p^* (\beta_j)) g \alpha_0 = \sum_i (\sum_{g \in H} h g_j^{-1} p^* (\beta_j)) g_i \alpha_0 \). \( \square \)

We are now in the position to prove Proposition 5.2.1. Lemma 5.2.12 and Lemma 5.2.17 imply that the restriction of the push-forward map \( \tilde{q}_\ast \) to the \( A_\ast(X_{\rho, \text{reg}}^\mu) \)-submodule of \( A_\ast(\Gamma_{\rho, \text{reg}}^\mu) \) generated by the classes \( \{\alpha_{\rho, \ast}\} \) is surjective. This implies Proposition 5.2.1 by virtue of Lemma 5.2.4.

### 5.3 The surjectivity of \( \hat{\Gamma}_\ast \)

Proposition 5.2.1 implies easily the following surjectivity result:

**Proposition 5.3.1** Let \( [\Gamma] \) be the fundamental cycle of \( \Gamma \). The map

\[
P_\ast(p^*(-) \cap [\Gamma]) = \Gamma_\ast : A_\ast(X) = \bigoplus_{\nu \in \mathfrak{P}(n)} A_\ast(X^{\nu}) \rightarrow A_\ast(X^{[n]})
\]

is surjective.

**Proof.** By virtue of Lemma 4.1.2 we have, for every \( \mu \in \mathfrak{P}(n) \), a commutative diagram with exact columns:

\[
\begin{array}{cccc}
A_\ast(X_\mu) & \xrightarrow{p_{\geq \mu}^*(-) \cap [\Gamma_{\geq \mu}]} & A_\ast(\Gamma_{\mu}) & \xrightarrow{P_\ast} & A_\ast(X^{[n]}) \\
\downarrow{i_\ast} & & & & \downarrow{=} \\
A_\ast(X_{\geq \mu}) & \xrightarrow{p_{\geq \mu}^*(-) \cap [\Gamma_{\geq \mu}]} & A_\ast(\Gamma_{\geq \mu}) & \xrightarrow{P_\ast} & A_\ast(X^{[n]}) \\
\downarrow{j_\ast} & & & & \downarrow{=} \\
A_\ast(X_{> \mu}) & \xrightarrow{p_{\geq \mu}^*(-) \cap [\Gamma_{> \mu}]} & A_\ast(\Gamma_{> \mu}) & \xrightarrow{P_\ast} & A_\ast(X^{[n]}) \\
\end{array}
\]

0 0 0.
We proceed by decreasing induction on \( \mu \), proving that \( P_\ast(p^\ast(-) \cap [\Gamma_{\geq \mu}]) : A_\ast(X_{\geq \mu}) \to A_\ast(X^{[n]}_{\geq \mu}) \) is surjective. The statement in the case \( \mu = 1 \) is clearly true since \( X^{[n]}_1 = X^{(n)}_1 = X_1/\Sigma_n \). Suppose now that the surjectivity of \( P_\ast(p^\ast(-) \cap [\Gamma_{>\mu}]) : A_\ast(X_{>\mu}) \to A_\ast(X^{[n]}_{>\mu}) \) has been established. By virtue of Proposition 5.2.1, the map in the first line of the diagram is surjective, whence the surjectivity of the second line and the statement. \( \square \)

**Corollary 5.3.2** The natural map

\[ \hat{\Gamma}_\ast : A_\ast(\hat{X}) = \bigoplus_{\nu \in \Psi(n)} A_\ast(X^{(\nu)}) \longrightarrow A_\ast(X^{[n]}) \]

is surjective.

**Proof.** Immediate from Proposition 5.3.1 and §4.4.

**Remark 5.3.3** Corollary 5.3.2 holds if we replace all spaces by the ones obtained by base change with respect to any open immersion \( U \to X^{(n)} \). In fact, it is sufficient to use the corollary together with the standard exact sequence \([7]\), Proposition 1.8.

### 5.4 The isomorphism of Chow groups

The main result of this paper now follows easily. It does not hold with \( \mathbb{Z} \) coefficients.

**Theorem 5.4.1** Let \( X \) be an irreducible nonsingular algebraic surface defined over an algebraically closed field. The natural map

\[ \hat{\Gamma}_\ast = \bigoplus_{\nu \in \Psi(n)} \hat{\Gamma}_\ast^\nu : A_\ast(\hat{X}) = \bigoplus_{\nu \in \Psi(n)} A_\ast(X^{(\nu)}) \longrightarrow A_\ast(X^{[n]}) \]

is an isomorphism.

**Proof.** Injectivity and surjectivity are proved in Corollary 5.1.3 and 5.3.2, respectively. \( \square \)

The following is an immediate consequence of Theorem 5.4.1. Let \( Y \) be a scheme. We denote by \( K_o(Y) \) the Grothendieck group of coherent sheaves on \( Y \) and we define \( K_o(Y)_\mathbb{Q} := K_o(Y) \otimes_{\mathbb{Z}} \mathbb{Q} \).

**Corollary 5.4.2** There is a natural isomorphism of Grothendieck groups with \( \mathbb{Q} \)-coefficients

\[ \hat{\Gamma}_\ast = \bigoplus_{\nu \in \Psi(n)} \hat{\Gamma}_\ast^\nu : K_o(\hat{X})_\mathbb{Q} = \bigoplus_{\nu \in \Psi(n)} K_o(X^{(\nu)})_\mathbb{Q} \longrightarrow K_o(X^{[n]})_\mathbb{Q}. \]

**Proof.** It follows immediately from Theorem 5.4.1 and \([7]\), Corollary 18.3.2. \( \square \)

**Remark 5.4.3** Corollary 5.4.2 and the localization theorem for equivariant \( K \)-theory \([16]\), imply that one has an isomorphism \( K_o^{\Sigma_n}(X^n)_\mathbb{Q} \simeq K_o(X^{[n]})_\mathbb{Q} \). In a paper in preparation, we construct this isomorphism directly, i.e. without the localization theorem. We expect similar statements to hold for the higher \( K \)-theory and for the equivariant derived category.
Remark 5.4.4 Theorem 5.4.1 and Corollary 5.4.2 hold if we replace $X$ and $X^{[n]}$ by the corresponding open sets obtained by base change with respect to any open immersion $U \to X^{(n)}$. See Remark 5.1.6 and 5.3.3.

Remark 5.4.5 Note that, by virtue of [7], Ex. 1.7.6, $A^\ast(X^{(\nu)}) = A^\ast(X^{(\nu)}) \otimes \mathbb{Q}$. However, unless $X$ itself has a cellular decomposition, the natural map $A^\ast(X) \otimes \mathbb{Q} \to A^\ast(X^{(\nu)})$ is not necessarily surjective. This contrasts singular cohomology, where one has Künneth Formula.

Remark 5.4.6 With minor modifications, which we leave to the interested reader, Theorem 5.4.1 holds: 1) for a geometrically irreducible smooth surface $X$ defined over any field, and 2) in the analytic context where $X$ is a smooth complex analytic surface (see [7], Ex. 19.2.5).

6 The motive of $X^{[n]}$

Let $X$ be an irreducible nonsingular quasi projective surface defined over an algebraically closed field.

6.1 The correspondences $\Delta_{\nu}$

We freely use refined products over quotient varieties and the formalism of correspondences between quotient varieties together with their standard properties. See §4. Let $\nu \in \mathfrak{P}(n)$ and consider $(X^{[n]}_{\nu} \times X^{(n)}_{\nu})_{\text{red}}$. Being the quotient of a Zariski locally trivial fibration with irreducible fiber by the action of a finite group, this space is a $2n$-dimensional irreducible locally closed subset of $X^{[n]} \times X^{[n]}$. Let $D^\nu \subseteq X^{[n]} \times X^{[n]}$ be its closure:

$$D^\nu := \{(a, b) \in X^{[n]} \times X^{[n]} \mid \pi(a) = \pi(b) \in X^{(n)}_{\nu}\} \in Z_{2n}(X^{[n]} \times X^{[n]}).$$

Note that the image of $D^\nu$ under either projection is the closed stratum $X^{[n]}_{\nu}$.

Remark 6.1.1 $D^\nu$ is an irreducible component of $(X^{[n]}_{\nu} \times X^{(n)}_{\nu})_{\text{red}}$. The latter contains all the $D^\mu$ such that $\mu \leq \nu$ precisely as its irreducible components.

Let $\nu \in \mathfrak{P}(n)$. The components of $p_{1,2}^{-1}(|\hat{\Gamma}^\nu|) \cap p_{2,3}^{-1}(|\hat{\Gamma}^\nu|)$ in $X^{[n]} \times X^{(\nu)} \times X^{[n]}$ are all of the expected dimension. The refined formalism mentioned above, Remark 3.0.4 and Remark 6.1.1 show that the following is a well-defined $2n$-dimensional cycle with rational coefficients

$$\Delta_{\nu} := \frac{1}{m_{\nu}} \hat{\Gamma}^\nu \circ \hat{\Gamma}^\nu \in Z_{2n}(X^{[n]} \times X^{[n]}) \otimes \mathbb{Q},$$

with the property that $\Delta_{\nu}$ is supported precisely on $(X^{[n]}_{\nu} \times X^{(n)}_{\nu})_{\text{red}}$ and that

$$\Delta_{\nu} = \sum_{\nu \geq \nu'} \epsilon_{\nu}' D^\nu',$$

where the numbers $\epsilon_{\nu}'$ are nonzero rational numbers with the same sign as $m_{\nu}$.
Lemma 6.1.2 Let $\nu$ and $\mu$ be partitions of $n$. Then
\[ \Delta_{\nu} \circ \Delta_{\mu} = \delta_{\mu\nu} \Delta_{\nu} \in \mathbb{Z}_{2n}(X^{[n]} \times X^{[n]}) \otimes \mathbb{Q}. \]
where $\delta_{\mu\nu}$ is the usual Kronecker function.

In particular, the $\Delta_{\nu}$ are mutually orthogonal projectors and:
\[ \Delta_{\nu \ast} : A_{*}(X^{[n]}) \to A_{*}(X^{[n]}), \quad \Delta_{\nu \ast} \circ \Delta_{\mu \ast} = \delta_{\mu\nu} \Delta_{\nu \ast}. \]

Proof. By the associativity of the composition of correspondences we have
\[ \Delta_{\nu \ast} \circ \Delta_{\mu} = \frac{\hat{\Gamma}_{\nu} \circ \hat{T}_{\nu}}{m_{\nu}} \circ \frac{\hat{\Gamma}_{\mu} \circ \hat{T}_{\mu}}{m_{\mu}} = \frac{\hat{\Gamma}_{\nu}}{m_{\nu}} \circ \left( \frac{\hat{T}_{\nu}}{m_{\nu}} \circ \hat{T}_{\mu} \right) \circ \hat{T}_{\mu}. \]
We conclude by Proposition 5.1.3 and Proposition 5.1.4. \hfill \Box

Recall that $F_{\mu}$ denotes the reduced fiber $(\pi^{-1}(x))_{red}$, where $x \in X^{(n)}_\ast$.

Lemma 6.1.3 Let $\nu$ and $\mu$ be partitions of $n$.
(i) If $\nu \not\geq \mu$, then $\Delta_{\nu \ast}([F_{\mu}]) = 0$. If $\nu = \mu$, then $\Delta_{\nu \ast}([F_{\mu}]) = [F_{\mu}]$.
(ii) If $\nu \not\geq \mu$, then $D_{\nu \ast}([F_{\mu}]) = 0$. If $\nu = \mu$, then $\Delta_{\nu \ast}([F_{\mu}]) = c_{\nu}[F_{\nu}]$, where $\mathbb{Q} \ni c_{\nu} \neq 0$.

Proof. We compute $\Delta_{\nu \ast}$ using refined intersections so that the classes $[F_{\mu}]$, which are non zero in $A_{*}(F_{\mu})$, may be zero in $A_{*}(X^{[n]})$. Of course this does not happen if, for example, $X$ is projective.

If $\nu$ does not refine $\mu$, then $p^{-1}(F_{\mu}) \cap \Delta_{\nu} = \emptyset$, and the first parts of (i) and (ii) follow at once. Note that since we have proved the first parts of (i) and (ii), (i) implies (ii) by consideration of supports. We now prove the second part of (i). Let $\mu = \nu$. Note that $\Delta_{\nu \ast}([F_{\nu}]) = \frac{1}{m_{\nu}} \hat{\Gamma}_{\nu} \circ \hat{T}_{\nu}([F_{\nu}])$. The zero cycle $\hat{T}_{\nu}([F_{\nu}])$ is supported at the point $\nu^{-1}(x) \in X^{(\nu)}$. By virtue of the projection formula, its degree equals $\deg([F_{\nu}] \cdot [X^{[n]}_{\nu}]) = m_{\nu}$; see Lemma 5.1.2. In other words $\hat{T}_{\nu}([F_{\nu}]) = m_{\nu} \nu^{-1}(x)$. We apply $\hat{T}_{\nu}$ and we find the conclusion. \hfill \Box

Note that the diagonal $\Delta = D_{1}^{n}$. The main reason for introducing the correspondences $\Delta_{\nu}$ is the following

Proposition 6.1.4 The map $(\sum_{\nu \in \Psi(n)} \Delta_{\nu} - \Delta)_{\ast}$ is the zero map. In particular, $\sum_{\nu} \Delta_{\nu \ast} = \Delta_{\ast} = Id_{A_{*}(X^{[n]})}$.

Proof. This follows immediately from Theorem 5.4.1 which implies that $\hat{T}_{\ast}$ is the inverse of $\hat{T}_{\ast}$ and from the fact that $\hat{T}_{\ast} \circ \hat{T}_{\ast} = \sum_{\nu \in \Psi(n)} \Delta_{\nu \ast}$.

Proposition 6.1.5 $\sum_{\nu} \Delta_{\nu} - \Delta = 0$ in $\mathbb{Z}_{2n}(X^{[n]} \times X^{[n]}) \otimes \mathbb{Q}$.

Proof. Without loss of generality, we may assume that $X$ is projective. Note that, for every $\nu \neq 1^{n}$, $D_{\nu}$ is identically zero on $A_{0}(X^{[n]})$. It follows, by virtue of Lemma 5.1.3, that we can write $\sum_{\nu \in \Psi(n)} \Delta_{\nu} - \Delta = \sum_{\nu \neq 1^{n}} p_{\nu} D_{\nu}$, where $p_{\nu}$ are suitable rational numbers. Seeking a contradiction, assume that there is at least one partition $\mu$ for which $p_{\mu} \neq 0$. Choose a partition $\theta$ which is maximal, with respect to the partial ordering on partitions, among all partitions $\mu$ for which $p_{\mu} \neq 0$. We have, again by virtue of Lemma 5.1.3, that
\[ 0 = \left( \sum_{\nu \neq 1} p_{\nu} D_{\nu} \right)_{\ast} [F_{\theta}] = p_{\theta} D_{\theta}^{\theta}([F_{\theta}]) \neq 0. \]

The contradiction stems from the projectivity assumption, for then no effective cycle can be trivial. \hfill \Box
6.2 The structure and the generating function of motives

Let $X$ be an irreducible nonsingular projective surface defined over an algebraically closed field. The correspondence $\hat{\Gamma}^\nu$ defines a morphism, which by abuse of notation we denote by the same symbol, of effective Chow motives with rational coefficients:

$$\hat{\Gamma}^\nu : (X^{(\nu)}, \Delta_{X^{(\nu)}})(n - l(\nu)) \rightarrow (X^{[n]}, \Delta_{X^{[n]}}).$$

See [12], page 459. The Tate-type shift is in accordance with the usual convention $(\mathbb{P}^1, \Delta_{\mathbb{P}^1})(1) = (\mathbb{P}^1 \times \{0\})$. This morphism admits, in the category of Chow motives with rational coefficients, a right inverse. This is given, again by abuse of notation, by $t\hat{\Gamma}^\nu$. By virtue of [12], page 453, we can split-off a direct summand:

$$\hat{\Gamma}^{\nu} : (X^{(\nu)}, \Delta_{X^{(\nu)}})(n - l(\nu)) \simeq (X^{[n]}, \Delta_{\nu}).$$

By virtue of Lemma 6.1.2, the projectors $\Delta_{\nu}$ are mutually orthogonal so that we can split-off a direct summand for each partition:

$$\hat{\Gamma} : \bigoplus_{\nu \in \Psi(n)} \hat{\Gamma}^{\nu} : \bigoplus_{\nu \in \Psi(n)} (X^{(\nu)}, \Delta_{X^{(\nu)}})(n - l(\nu)) \simeq \bigoplus_{\nu \in \Psi(n)} (X^{[n]}, \Delta_{\nu}) = (X^{[n]}, \sum_{\nu \in \Psi(n)} \Delta_{\nu}).$$

The following is the main theorem of this section and follows immediately from what above and Proposition 6.1.5.

**Theorem 6.2.1** Let $X$ be an irreducible nonsingular projective surface defined over an algebraically closed field. There is a natural isomorphism of effective Chow motives with rational coefficients

$$(X^{[n]}, \Delta_{X^{[n]}}) \simeq \bigoplus_{\nu \in \Psi(n)} (X^{(\nu)}, \Delta_{X^{(\nu)}})(n - l(\nu)).$$

**Remark 6.2.2** Theorem 6.2.1 and Theorem 6.2.4 below hold over any field. We leave the necessary but easy modifications to the reader.

**Remark 6.2.3** Over the complex numbers, Theorem 6.2.1 gives immediately the structure of the singular cohomology $H^*(X^{[n]}, \mathbb{Q})$, together with its Hodge structure. The resulting isomorphisms coincide with the ones obtained in [3].

Voevodsky [17] has defined several motivic categories over a field $k$: $DM_{eff}^g(k)$ (cf. [17], 2.1.1), $DM_{eff}^g(k)$ (cf. [17], 3.1.12), $DM_{eff}^g(k)$ (cf. [17], 3.3), $DM_{h}(k)_\mathbb{Q}$ (cf. [17] and [18]). These are, essentially, categories of bounded complexes of pre-sheaves on certain sites. There is an obvious formalism of shifts of complexes and a formalism of Tate-type shifts. In addition, there are natural transformations between the category of effective Chow motives with rational coefficients and these other categories (see [17], 2.1.4):

$$Chow_{eff}^g(k)_\mathbb{Q} \rightarrow DM_{g}^g(k) \rightarrow DM_{-et}^g(k) \rightarrow DM_{-et}^g(k) \rightarrow DM(h)_\mathbb{Q}.$$

The formalism holds for quotient varieties.

Let us denote by $M(Z)$ the object in any of the above categories corresponding to a variety $Z$. Theorem 6.2.1 and the natural transformations mentioned above imply the following
**Theorem 6.2.4** Let $X$ be an irreducible nonsingular projective surface defined over an algebraically closed field. Then

$$M(X^{[n]}) \simeq \bigoplus_{\nu \in \mathfrak{P}(n)} M(X^{(\nu)})(n - l(\nu))[2n - 2l(\nu)].$$

If we use the dual Tate twist, then we can re-write the conclusion of the theorem as follows:

$$M(X^{[n]})(2n) \simeq \bigoplus_{\nu \in \mathfrak{P}(n)} M(X^{(\nu)})(l(\nu))[2l(\nu)].$$

This is formally reminiscent of 1) the decomposition theorem for the Douady-Barlet morphism \[3\], where shifts for complexes of sheaves appear and 2) the Hodge structure of $X^{[n]}$ (see \[3\], for example), where Tate shifts appear. It is by staring at those two formulas that we became convinced that the statement of Theorem 6.2.4 should hold. Theorem 5.4.1 and Theorem 6.2.1 were the means to realize this isomorphism.

It is amusing to note that there is a generating function for this picture, the one that generates partitions. In fact, skipping the bookkeeping details, the formula is as follows:

$$\sum_{n=0}^{\infty} X^{[n]}(n)[2n] q^n \equiv \prod_{m=1}^{\infty} \frac{1}{(1 - X_m(1)[2] q^m)},$$

where it is understood that in the r.h.s. we identify monomials of type $X_1^{a_1} \cdots X_t^{a_t}$ with $X^{(\nu)}$, where $\nu = a_1 \cdots a_t \in \mathfrak{P}(t)$. Note that this suggests a series of identities in the Grothendieck ring of varieties, among cohomology groups, mixed Hodge structures, intersection cohomology, equivariant $K$-theories, Chow groups, motives, etc., all of which are true, except, possibly, for the first one.

**References**

[1] J. Briançon, “Description de $\text{Hilb}^n C\{x, y\}$,” Invent. Math. 41 (1977), 45-89.

[2] J. Cheah, “The Hodge numbers of the Hilbert scheme of points of a smooth projective surface,” J. Algebraic Geometry 5 (1996), 479-511.

[3] M. de Cataldo, L. Migliorini, “The Douady space of a complex surface,” to appear in Adv. in Math.

[4] G. Ellingsrud, S.A. Stromme, “On a cell decomposition of the Hilbert scheme of points in the plane,” Invent. Math. 91 (1988) 365-370.

[5] G. Ellingsrud, S.A. Stromme, “An intersection number for the punctual Hilbert scheme of a surface,” Trans. Amer. Math. Soc. 350 (1998), no.6, 2547-2552.

[6] J. Fogarty, “Algebraic families on an algebraic surface,” American Journal of Math. 90 (1968), 511-521.

[7] W. Fulton, *Intersection Theory*, Ergebnisse der Mathematik, 3.folge. Band 2, Springer-Verlag, Berlin Heidelberg 1984.
[8] L. Göttsche, *Hilbert schemes of zero-dimensional subschemes of smooth varieties*, LNM 1572, Springer-Verlag Berlin 1994.

[9] L. Göttsche, W. Soergel, “Perverse sheaves and the cohomology of the Hilbert schemes of smooth algebraic surfaces,” Math. Ann. 296 (1993), 235-245.

[10] L. Grojnowski, “Instantons and affine algebras, I. The Hilbert scheme and vertex operators,” Math. Res. Letters 3 (1996), no. 2, 275-291.

[11] A. Iarrobino, *Punctual Hilbert schemes*, Mem. Amer. Math. Soc. 188, 1977.

[12] Ju. I. Manin, “Correspondences, motifs and monoidal transformations,” Math. USSR-Sb. 6 (1968), 439–470.

[13] H. Nakajima, “Heisenberg algebra and Hilbert schemes of points on projective surfaces,” Ann. of Math. (2) 145 (1997), no.2, 379-388.

[14] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*. University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999.

[15] Vafa, E. Witten, “A strong coupling test for S-duality,” Nucl. Phys. 431 (1994), 3-77.

[16] A. Vistoli, “Higher equivariant K-theory for finite group actions,” Duke Math. J. 63 (1991), no.2, 399–419.

[17] V. Voevodsky, “Triangulated categories of motives over a field,” preprint, k-theory arch. n.74.

[18] V. Voevodsky, “Homology of schemes, I,” preprint, k-theory arch. n.31.

Authors’s addressees:
Mark Andrea A. de Cataldo, Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794, USA. e-mail: mde@math.sunysb.edu
Luca Migliorini, Dipartimento di Matematica , Università di Trento, Via Sommarive, 14, 38050 Povo (Tn), ITALY. e-mail: luca@alpha.science.unitn.it