FROM SCATTERING AMPLITUDES TO CROSS SECTIONS IN QCD

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Abstract. I describe how to calculate cross sections for hard-scattering processes in high energy collisions at next to leading order in QCD. I consider infrared-safe quantities and I assume that the scattering amplitudes are known in analytic form up to next-to-leading order. The main topic is the description of the algorithm for the analytic cancellation of the soft and collinear singularities in the loop and bremsstrahlung contributions. The method is systematic and general. It allows the construction of an analytic expression for finite next-to-leading order hard scattering cross sections suitable for numerical evaluation.

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1 Introduction

The application of perturbative QCD is not straightforward even for reactions with large momentum transfer $Q$. In higher order corrections soft and collinear momentum regions may lead to large contributions of order $(\alpha_s \log(Q/m_{\ell}))^n$ where $m_{\ell}$ denotes a light quark mass. These terms give $\mathcal{O}(1)$ corrections which destroy the validity of the perturbative treatment. The applications of perturbative QCD are limited to phenomena where such terms are either cancelled or can be controlled with improved treatment (resummation). Fortunately, in perturbative QCD the soft and collinear structure is relatively well understood. The main features are summarized by fundamental cancellation and factorization theorems valid in all orders in perturbation theory.

This lecture will show explicitly how the cancellation and factorization theorems “work” in next-to-leading order applications. I describe how to combine virtual and real next-to-leading order infrared singular amplitudes into finite physical cross sections. The method of calculating amplitudes in leading and next-to-leading order have been described in great detail by Lance Dixon. He also discussed the asymptotic properties of amplitudes in the soft and collinear regions as well as the soft and collinear singular terms appearing in one-loop amplitudes. I describe the singularities as they appear in various cross section contributions (loop contributions and bremsstrahlung-contributions). I consider their universal features and show explicitly that they cancel for infrared safe quantities. The examples will always be chosen from the physics of jet-production. In order to build a numerical program for efficiently calculating hard scattering cross sections in next-to-leading order (NLO) accuracy one should use a well defined algorithm for the analytic cancellation of the soft and collinear singularities. Several methods are used in the literature. I shall focus on the subtraction method.

2 Cancellation and factorization theorems

2.1 KLN cancellation theorem

In simple inclusive reactions, such as the total cross section of $e^+e^-$ annihilation into quarks and gluons, the soft and collinear contributions cancel. That is a consequence of the KLN theorem. In the simplest examples, only one high momentum transfer scale is relevant, the effective coupling becomes small and the cross section can reliably be calculated in power series of the
effective coupling up to small power corrections

\[ R = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} = (1 + \frac{\alpha_s}{\pi} + \ldots) \sum \frac{e_q^2}{2} \]

(1)

where

\[ \tilde{R} = 1 + \frac{\alpha_s(\mu)}{\pi} + \left(\frac{\alpha_s(\mu)}{\pi}\right)^2 \left[ \pi b_0 \ln \frac{\mu^2}{s} + B_2 \right] + \ldots \]

(2)

\( \alpha_S \) is the running coupling constant, \( \mu \) is the renormalization scale, \( B_2 \) is a known constant given by the NNLO calculation\(^5\) and \( b_0 \) is the first coefficient in the beta-function

\[ b_0 = \frac{11 - \frac{2}{3}n_f}{6\pi} \]

(3)

where \( n_f \) denotes the number of quark flavours. The truncated series is \( \mu \)-dependent but the \( \mu \) dependence is \( O(\alpha_S^2) \) if the cross section is calculated to \( O(\alpha_S^2) \).

The KLN theorem remains valid also for integrating over final states in a limited phase space region, as is the case of jet production. The Sterman-Weinberg two-jet cross section\(^6\) is defined by requiring that all the final state partons are within a back-to-back cone of size \( \delta \) provided their energy is less than \( \epsilon \sqrt{s} \). At NLO

\[ \sigma_{2\text{jet}} = \sigma_{\text{SW}}(s, \epsilon, \delta) \]

\[ = \sigma_{\text{tot}} - \sigma_{q\bar{q}g}^{(1)}(\text{all } E > \epsilon \sqrt{s}, \text{all } \theta_{ij} > \delta) \]

\[ = \sigma_0 \left[ 1 - \frac{4\alpha_s}{3\pi} \left(4 \ln 2 \epsilon \ln \delta + 3 \ln \delta - 5/2 + \frac{\pi^2}{3}\right) \right] \]

(4)

where \( \sigma_0 = 4\pi\alpha^2/3s \). The cancellation theorem here is hidden in the calculation of the total annihilation cross section.

2.2 Factorization theorem

The initial state collinear singularities, which in general do not cancel, are universal and process-independent in all orders in perturbation theory\(^7\). Therefore they can be cancelled by universal collinear counter terms generated by the ‘renormalization’ of the incoming parton densities. The rule for defining the finite part of this counter term is fixed by the factorization scheme. As in the case of ultraviolet renormalization\(^7\), the physics is unchanged under a change of the factorization scheme, provided the parton densities are also changed
suitably. This feature is expressed by the Altarelli-Parisi evolution equation of parton densities. The collinear subtraction terms define the kernels of the evolution equations.

The differential cross section for hadron collisions can be written as

\[ d\sigma_{AB}(p_A, p_B) = \sum_{ab} \int dx_1 dx_2 f_{a/A}(x_A) f_{b/B}(x_B) d\hat{\sigma}_{ab}(x_A p_A, x_B p_B), \quad (5) \]

where \( A \) and \( B \) are the incoming hadrons, \( p_A \) and \( p_B \) their momentum, and the sum runs over all the parton flavours which give a non-trivial contribution. The quantities \( d\hat{\sigma}_{ab} \) are the \textit{subtracted} partonic cross sections, in which the singularities due to collinear emission of massless partons from the incoming partons have been cancelled by some suitable counter terms.

According to the factorization theorem, the subtracted cross section is obtained by adding the collinear counter terms to the unsubtracted cross section. The latter quantity can be directly calculated in perturbative QCD. Due to universality, eq. (5) applies also when the incoming hadrons are formally substituted for partons. In this case, we are also able to evaluate the partonic densities, which at NLO read

\[ f_{a/d}(x) = \delta_{ad}(1 - x) - \frac{\alpha_s}{2\pi} \left( \frac{1}{\epsilon} P_{a/d}(x, 0) - K_{a/d}(x) \right) + \mathcal{O}(\alpha_s^2), \quad (6) \]

where \( P_{a/d}(x, 0) \) are the Altarelli-Parisi kernels in four dimensions (since we will usually work in \( 4 - 2\epsilon \) dimensions, the 0 in the argument of \( P_{a/d} \) stands for \( \epsilon = 0 \)) and the functions \( K_{a/d} \) depend upon the subtraction scheme in which the calculation is carried out. For \( \overline{\text{MS}} \), \( K_{a/d} \equiv 0 \). Writing the perturbative expansion of the unsubtracted and subtracted partonic cross sections at next-to-leading order as

\[ d\sigma_{ab} = d\sigma_{ab}^{(0)} + d\sigma_{ab}^{(1)}, \quad d\hat{\sigma}_{ab} = d\sigma_{ab}^{(0)} + d\hat{\sigma}_{ab}^{(1)}, \quad (7) \]

where the superscript 0 (1) denotes the leading (next-to-leading) order contribution, we have

\begin{align*}
\hat{d}\sigma_{ab}^{(0)}(p_1, p_2) &= d\sigma_{ab}^{(0)}(p_1, p_2) \quad (8) \\
\hat{d}\sigma_{ab}^{(1)}(p_1, p_2) &= d\sigma_{ab}^{(1)}(p_1, p_2) + d\sigma_{ab}^{\text{count}}(p_1, p_2) \quad (9)
\end{align*}
where
\[
d\sigma_{\text{count}}(p_1, p_2) = \frac{\alpha_S}{2\pi} \sum_d \int dx \left( \frac{1}{\epsilon} P_{d/a}(x, 0) - K_{d/a}(x) \right) d\sigma^{(0)}_{db}(xp_1, p_2) \\
+ \frac{\alpha_S}{2\pi} \sum_d \int dx \left( \frac{1}{\epsilon} P_{d/b}(x, 0) - K_{d/b}(x) \right) d\sigma^{(0)}_{ad}(p_1, xp_2) .
\]  

Eq. (10) defines the collinear counter terms for any finite hard scattering cross section for processes with quarks and/or gluons in the initial state. Notice that in this equation the Born terms \(d\sigma^{(0)}\) are evaluated in 4 \(-\) 2\(\epsilon\) dimensions.

The KLN theorem and the factorization theorems, constitute the theoretical basis of the description of scattering processes of hadrons in perturbative QCD. Those physical quantities for which these theorems remain valid are called infrared safe\(^a\). These theorems constitute the necessary consistency condition for the validity of the fundamental assumption of the QCD improved parton model. This assumption is that for the case of infrared safe quantities the perturbative QCD predictions given in terms of partons are a good approximation to the same quantities measured in terms of hadrons (up to power corrections which are small at high momentum scales).

Provided the higher order corrections at a given order are larger than the power corrections, one can systematically improve the accuracy of the predictions by calculating terms of higher and higher order. Indeed the analysis of the experimental results required the inclusion of higher order radiative corrections for a large number of measured quantities.

3 Jet cross sections at next-to-leading order

We consider the cross sections for three jet production in \(e^+e^-\) annihilation and two-jet production in hadron-hadron collisions. These cross sections are proportional to at least one power of \(\alpha_S\) and are studied experimentally with high precision.

3.1 Three-jet production in \(e^+e^-\) annihilation

Let us consider the process
\[
e^-(k_-) + e^+(k_+) = a_1(p_1) + ... + a_n(p_n) \tag{11}
\]
\(^a\) In other words a measurable is infrared safe if it is insensitive to collinear splittings of partons and/or emission of soft gluons.
where $a_i$ denote quarks or gluons and $n = 3, 4$. The amplitudes of these processes $A^{(n,i)}$ are known in the tree approximation ($i = 0$) for $n = 3, 4, 5$ partons and in the one-loop approximation ($i = 1$) for $(n = 3)$ partons. Lance Dixon explained how to calculate these amplitudes quickly with modern techniques.

It is convenient to consider the squared amplitude divided by the flux and the spin averaging factor $8s$ ($s = 2k_+ k_-$):

$$
\psi^{(n,0)}_{e^+ e^-}(\{a_l\}_{1,n}; \{p_l\}_{1,n}) = \frac{1}{8s} \sum_{\text{colour}} \sum_{\text{spin}} \left| A^{(n,0)}_{e^+ e^-} \right| \tag{12}
$$

and

$$
\psi^{(3,1)}_{e^+ e^-}(\{a_l\}_{1,3}; \{p_l\}_{1,3}) = \frac{1}{8s} \sum_{\text{colour}} \sum_{\text{spin}} \left( A^{(3,0)}_{e^+ e^-} A^{(3,1)*}_{e^+ e^-} + A^{(3,0)*}_{e^+ e^-} A^{(3,1)}_{e^+ e^-} \right) \tag{13}
$$

where $\{v\}_{m,m+n}$ is a short-hand notation for the list of variables $v_m, v_{m+1}, \ldots, v_{m+n}$

and we indicated the flavour and momentum dependence only for the $\psi$ functions. The one-loop corrections to the production of three partons are given by $\psi^{(3,1)}$. They were calculated first by R.K. Ellis, Ross and Terrano (ERT). Recently, a new derivation using the helicity method, where the orientation with respect to the beam direction is not averaged, was given by Giele and Glover.

The physical cross sections are obtained by integrating the product of the $\psi$ functions and some “measurement functions” $S_X$ over the corresponding phase space volume

$$
d\sigma^{\text{nlo}} = d\sigma^{\text{Born}} + d\sigma^{\text{virt}} + d\sigma^{\text{real}}, \tag{14}
$$

where

$$
d\sigma^{\text{Born}}(s, X) = \sum_{\{a_l\}_{1,3}} \psi^{(3,0)}_{e^+ e^-} S_X, 3(\{p_l\}_{1,3}; X) d\phi_3(\{p_l\}_{1,3}) \tag{15}
$$

$$
d\sigma^{\text{virt}}(s, X) = \sum_{\{a_l\}_{1,3}} \psi^{(3,1)}_{e^+ e^-} S_X, 3(\{p_l\}_{1,3}; X) d\phi_3(\{p_l\}_{1,3}) \tag{16}
$$

$$
d\sigma^{\text{real}}(s, X) = \sum_{\{a_l\}_{1,4}} \psi^{(4,0)}_{e^+ e^-} S_X, 4(\{p_l\}_{1,4}; X) d\phi_4(\{p_l\}_{1,3}) \tag{17}
$$
where $X$ stands for the measured physical quantity and

$$d\phi_n = \frac{1}{n!} \int \prod_{i=1}^{n} \frac{d^{d-1}p_i}{(2\pi)^{d-1}} (2\pi)^d \delta^{(d)}(k_+ + k_- - \sum_{i=1}^{n} p_i)$$

(18)

with $d = 4 - 2\epsilon$. As a result of complete flavour sum the final particles behave as though they were identical; this explains the identical particle factor of $n!$.

All quantities are calculated in $d = 4 - 2\epsilon$ dimensions. The singular terms appear as single or double poles in $\epsilon$. The singularities, however, cancel in the sum (14).

**Infrared safe measurement function.** The cancellation of the soft and collinear singularities of the virtual corrections against the singular part of the real contribution is independent of the form of the measurement functions provided they are insensitive to collinear splitting and soft emission. This means that one obtains the same measured result whether or not a parton splits into two collinear partons and whether or not one parton emits another parton that carries infinitesimal momentum. A physical quantity that is designed to look at short distance physics should have this property, otherwise it will be sensitive to the details of parton shower development and hadronization. The mathematical requirements for $S_X,3$ and $S_X,4$ are that $S_X,4$ should reduce to $S_X,3$ when two of the outgoing partons become collinear:

$$S_{X,4}(p_1^\mu, p_2^\mu, (1 - \lambda)p_3^\mu, \lambda p_4^\mu) = S_{X,3}(p_1^\mu, p_2^\mu, p_3^\mu)$$

(19)

for $0 \leq \lambda \leq 1$ plus similar conditions where the $\lambda$ and $1 - \lambda$ factors are inserted to all possible pairs of the momenta in $S_{X,4}(p_1^\mu, p_2^\mu, p_3^\mu, p_4^\mu)$.

**Example of an infrared safe observable.** For the sake of illustration I give the definition of the measurement function for the shape variable thrust $T$

$$S_{T,n} = \delta(T - \tau_n(p_1^\mu, p_2^\mu, ..., p_n^\mu)), \quad \text{where} \quad \tau_n = \max_{\bar{u}} \frac{\sum_{i=1}^{n} |\vec{p}_i \bar{u}|}{\sum_{i=1}^{n} |\vec{p}_i|}.$$  

(20)

Thrust is well defined for arbitrary number of final-state particles. It is easy to check that it satisfies the conditions of infrared safety formulated with the help of eq. (19).

Thrust measures the sum of the lengths of the longitudinal momenta of the final particles relative to the thrust axis $\bar{u}$ chosen to maximize the sum. For two-particle final states its value is 1. Its allowed range changes with the particle number, therefore its differential distribution is only well defined after
Table 1: Range of thrust for various numbers of partons

| Number of Particles | $T$ Ranges          | Description                   |
|--------------------|---------------------|-------------------------------|
| 2                  | $T = 1$             | Discontinuous in particle     |
| 3                  | $1 > T > 2/3$       |                               |
| 4                  | $1 > T > 1/\sqrt{3}$|                               |
| $\infty$           | $1 > T > 1/2$       |                               |

some smearing\(^6\). Carrying out the phase-space integral for the Born term one gets

$$\frac{1}{\sigma_0} \frac{d\sigma^{\text{Born}}}{dT} = \frac{\alpha_s}{2\pi} \left[ \frac{2(3T^2 - 3T + 2)}{(1 - T)T} \ln \left( \frac{2T}{1 - T} \right) - \frac{3(3T - 2)(2 - T)}{(1 - T)} \right]$$

(21)

In the classic paper of ERT the singular pieces have been evaluated analytically, and it was demonstrated that indeed for infrared safe quantities they cancel as is required by the KLN theorem. The remaining finite next-to-leading order cross section is suitable for numerical evaluation. ERT calculated the distribution for the shape variable $C$. But their analytic result can be used to calculate any infrared safe three-jet-like quantity in next-to-leading order.

It is convenient to give the one-dimensional distributions of various three jet measures generically denoted as $X = t, C, E_J$ in a form that satisfies the renormalization group equation

$$\frac{1}{\sigma_0} \frac{d\sigma}{dX} = \frac{\alpha_s(\mu)}{2\pi} A_X(X) + \left( \frac{\alpha_s(\mu)}{\pi} \right)^2 \left[ A_X(X) 2\pi b_0 \log(\mu^2 / S) + B_X(X) \right]$$

(22)

$A(x)$ and $B(x)$ are scale-independent functions. Their values are tabulated for many quantities in ref.\(^1\). The next-to-leading order expression is scale independent up to $\mathcal{O}(\alpha_s^3)$

$$\frac{d}{d\mu^2} \left( \frac{d\sigma}{dX} \right) = \mathcal{O}(\alpha_s^3).$$

(23)

The size and sign of the corrections is rather different for the various jet measures. In many cases the corrections are substantial ($\approx 30\%$) even at LEP energy. Thanks to the technical development described by Lance Dixon, the NLO calculation will soon be available also for four-jet production ($\psi(4,1)$).

\(^6\)Smearing is required also by hadronization effects. The typical width of a Gaussian smearing is $\Delta T = m_h / \sqrt{S} \approx 0.02$. 

8
3.2 Jet cross section in hadron-hadron collisions

At the Tevatron, multijet cross sections are observed up to six jets. The analysis of the data requires the evaluation of the amplitudes of the parton processes

\[ a_1(p_1) + a_2(p_2) \rightarrow a_3(p_3) + ... + a_n(p_n) \]  

where \( a_i \) denotes parton flavour labels, \( p_i \) are their four-momenta and \( n \) is the number of the participating partons. The data are rather precise for two- and three-jet like quantities therefore the comparison with the theory has to be done at next-to-leading order. For this purpose the tree amplitudes have to be known for \( n = 4, 5, 6 \) while in the case of \( n = 4, 5 \) we have to know also the one-loop amplitudes. It is convenient again to introduce \( \psi \) functions giving the squared amplitude divided by the flux and spin averaging factors

\[ \psi^{(n,0)}(\{a_l\}_{1,n}; \{p_l\}_{1,n}) = \frac{1}{2s \omega(a_1)\omega(a_2)} \sum_{\text{colour, spin}} \left| A^{(n,0)} \right| \]  

\[ \psi^{(n,1)}(\{a_l\}_{1,n}; \{p_l\}_{1,n}) = \frac{1}{2s \omega(a_1)\omega(a_2)} \times \sum_{\text{colour, spin}} \left( A^{(n,0)} A^{(n,1)*} + A^{(n,0)*} A^{(n,1)} \right) \]  

where \( s = 2p_1p_2 \), \( A^{(n,i)} \) denote the tree- \( (i = 0) \) and one-loop \( (i = 1) \) amplitudes of the process (24) (helicity, flavour and momentum labels are all suppressed), and \( \omega(a) \) is the number of colour and spin degrees of freedom for the flavour \( a \), in 4 \(- \) 2\( \epsilon \) dimensions

\[ \omega(q) = 2N_c, \quad \omega(g) = 2(1-\epsilon)(N_c^2-1) \]  

The hard-scattering cross section is decomposed into four contributions

\[ d\sigma^{nlo} = d\sigma^{\text{Born}} + d\sigma^{\text{virt}} + d\sigma^{\text{real}} + d\sigma^{\text{count}}, \]  

where

\[ d\sigma^{\text{Born}}_{a_1a_2}(p_1, p_2; X) = \sum_{\{a_l\}_{3,n}} \psi^{(n,0)} S_{X,n-2}(\{p_l\}_{3,n}; X) \times d\phi_{n-2}(p_1, p_2; \{p_l\}_{3,n}) \]  

\[ d\sigma^{\text{virt}}_{a_1a_2}(p_1, p_2; X) = \sum_{\{a_l\}_{3,n}} \psi^{(n,1)} S_{X,n-2}(\{p_l\}_{3,n}; X) \times \]
\[ d\sigma_{a_1 a_2}^{\text{real}}(p_1, p_2; X) = \sum_{\{a\}_{1,n+1}} \psi^{(n+1,0)}(n+1,n+1) S_{X,n-1}(\{p_l\}_{3,n+1}; X) \times \]
\[ \frac{d\phi_{n-2}(p_1, p_2; \{p_l\}_{3,n})}{d\phi_{n-2}(p_1, p_2; \{p_l\}_{3,n})} \] (30)

where \( d\phi_{n-2}(p_1, p_2; \{p_l\}_{3,n}) \) is the phase-space volume for \( n-2 \) final particles with total energy defined by the incoming four-momenta \( p_1 \) and \( p_2 \), \( X \) is a generic notation of the measured physical quantity, \( S_{X,n} \) denotes the measurement functions of \( X \) defined in terms of \( n \) partons and the counter-term cross sections \( d\sigma_{ab}^{(\text{count})}(p_1, p_2; X) \) are given by eq. (10). The loop corrections \( \psi^{(4,1)} \) have been obtained for the spin-independent case by Ellis and Sexton [12]; the spin-dependent one-loop corrections have been obtained by the helicity method [13] and very recently the one-loop corrections to the five-parton processes \( \psi^{(5,1)} \) have also been calculated (see the lecture of Lance Dixon). All four terms contributing to the hard scattering cross section (28) have to be calculated in \( 4-2\epsilon \) dimensions. The individual terms have singular \( 1/\epsilon \) and \( 1/\epsilon^2 \) contributions. The singularities cancel in the sum, provided the measurement functions satisfy the conditions of infrared safety formulated in subsection 3.1.c.

The hard scattering cross section given by eq. (28) is finite and, provided the cancellation of the singular terms is achieved analytically, it is suitable for the numerical evaluation of physical cross sections with the use of eq. (5).

4 Methods of analytic cancellation of the singularities

Although eqs. (14), (28) define finite cross-sections, they cannot be used directly for numerical evaluation since the singular terms in the real contributions are obtained by integrating over the soft and collinear kinematical range. Since the phase space is large and its boundary due to the presence of arbitrary measurement functions is complicated, analytic evaluation is impossible. Fortunately, in the soft and collinear regions the cross-sections and the measurement functions have a simple universal behaviour such that the integration relevant for the calculation of the singular terms becomes feasible analytically. This feature can be implemented in two basically equivalent methods.

In the first case (phase space slicing method) one excludes (slices) from the numerical integration domain the singular regions such that the numerical integrations becomes well defined. In the excluded regions the \( \psi \) functions, \( S \) functions and the relevant phase-space factors can be replaced with their

\[ ^c \text{In the case of hadron-hadron collisions the measurement functions should also fulfil the condition that } S_{X,n-1} \text{ should reduce to } S_{X,n-2} \text{ when one of the partons becomes parallel to one of the beam momenta.} \]
limiting values and the integrals in these regions are performed analytically. The boundary of the excluded regions are defined with some small parameters (invariant mass parameter or some small angle and energy-fraction parameter as in the case of the Sterman-Weinberg jet cross-section). Let us consider as illustration a one-dimensional problem with integration domain $0 \leq x \leq 1$ and an integrand which has a simple pole at $x = 0$. One slices the integration region into two pieces, $0 < x < \delta$ and $\delta < x < 1$. We choose $\delta \ll 1$, thus allowing us to use the simple approximation $F(x) \to F(0)$ for $0 < x < \delta$. This gives

$$I \sim \lim_{\epsilon \to 0} \left\{ F(0) \int_{0}^{\delta} \frac{dx}{x} x^\epsilon + \int_{\delta}^{1} \frac{dx}{x} x^\epsilon F(x) - \frac{1}{\epsilon} F(0) \right\}$$

$$= F(0) \ln(\delta) + \int_{\delta}^{1} \frac{dx}{x} F(x).$$

(32)

Now the second integral can be performed by normal Monte Carlo integration. As long as $\delta$ is small, the sum of the two terms will be independent of $\delta$. About the first use of this method, see refs. The method was further developed using systematically the universal features of the soft and collinear limits by Giele, Glover and Kosower. The actual implementation is not completely straightforward. The numerical integration over the real contribution has to be done with a certain accuracy. Since the integrand near the boundaries of the singular regions increases steeply the parameters defining the boundaries must not be very small. Furthermore, the result of the analytic integration is approximate since the integrand is replaced with its limiting value. The result is more and more accurate as the parameters defining the boundary of the singular regions become smaller and smaller. Fortunately, the measured cross sections have finite accuracy, which sets the required precision for the theoretical evaluation. In practical applications one compromises over these conflicting requirements to achieve the best efficiency.

The other method is called the subtraction method. This method takes account of the fact that the singular behaviour of the real contribution is just some simple pole with well known simple residue. After subtraction the numerical integration over the subtracted integrand becomes convergent. The quantity subtracted should be added back on. The subtraction terms, however, are simple, the dimensionally regulated singular integrations can be carried out analytically. This can be illustrated again with a simple one-dimensional integral. We write

$$I = \lim_{\epsilon \to 0} \left\{ \int_{0}^{1} \frac{dx}{x} x^\epsilon [F(x) - F(0)] + F(0) \int_{0}^{1} \frac{dx}{x} x^\epsilon \right\}$$

11
\[ \int_0^1 \frac{dx}{x} \left[ F(x) - F(0) \right] + \frac{1}{\epsilon} F(0). \]  

The integral can now be performed by Monte Carlo integration.

This method was first used for QCD calculations by R.K. Ellis, Ross, and Terrano and it was further developed with systematic use of the simplicity of the soft and collinear limit in ref. The disadvantage of this method is that outside the singular region the values of the physical parameters at a given phase-space point will change if we take the limit corresponding to the soft or collinear configurations of the subtraction terms. In the case of Monte Carlo numerical evaluation this can lead to relatively large fluctuations in the binned distributions. This problem can be avoided by using Gaussian smearing over the bin size.

5 Virtual contributions

In the following I shall consider only the squared matrix elements of process. The virtual contributions to the cross sections are given by eq. (30). We are interested in the form of the singular terms of the functions \( \psi_{ij}^{(n,1)} \). It turns out that they have a very simple general structure:

\[
\psi_{ij}^{(n,1)}(\{a_i\}_{1,n}, \{p_i\}_{1,n}) = \frac{\alpha S}{2\pi} \left( \frac{4\pi \mu^2}{Q^2} \right)^\epsilon c_T \left\{ -\frac{1}{\epsilon^2} \sum_{i=1}^n C(a_i) - \frac{1}{\epsilon} \sum_{i=1}^n \gamma(a_i) \right\} \psi_{ij}^{(n,0)}(\{a_i\}_{1,n}, \{p_i\}_{1,n}) \\
+ \frac{1}{2\epsilon} \sum_{i=1}^n \ln \left( \frac{2 p_i \cdot p_i}{Q^2} \right) \psi_{ij}^{(n,0)}(\{a_i\}_{1,n}, \{p_i\}_{1,n}) \\
+ \psi_{NS}^{(n,1)}(\{a_i\}_{1,n}, \{p_i\}_{1,n}) \right\}
\]  

(34)

where we introduced the short-hand notation

\[ c_T = \frac{\Gamma^2(1-\epsilon)\Gamma(1+\epsilon)}{\Gamma(1-2\epsilon)}, \]  

(35)

\( Q^2 \) is an auxiliary variable which cancels in the full expression, \( \mu \) is the scale introduced by dimensional regularization, \( \psi_{ij}^{(n,0)} \) denotes the colour-connected Born squared matrix elements, \( C(a_i) \) is the colour charge of parton \( a_i \) and the constant \( \gamma(a_i) \) gives the size of the virtual contributions to the diagonal.
Altarelli-Parisi kernel $P_{a/a}(\xi)$

\[ C(g) = N_c; \quad \gamma(g) = \frac{11N_c - 2N_f}{6} \text{ for gluons} \quad (36) \]

\[ C(q) = \frac{N_c^2 - 1}{2N_c}; \quad \gamma(q) = \frac{3(N_c^2 - 1)}{4N} \text{ for quarks}, \quad (37) \]

finally $\psi^{(3,1)}_{NS}$ represents the remaining finite terms.

The derivation of this result is rather simple. First we observe that in axial gauge the collinear singularities come from the self energy corrections to the external lines. For each helicity and colour sub-amplitudes they are proportional to the Born term since the Altarelli-Parisi functions, $P_{a/a}(z)$, for diagonal splitting preserve helicity in the $z \to 1$ limit. Therefore, the collinear singularities of the one-loop amplitudes have the form

\[ A_{\text{loop}}^{\text{col}} = - \left(\frac{g}{4\pi}\right)^2 \sum_a \gamma(a) \epsilon A^{\text{tree}}. \quad (38) \]

There is a contribution for every external leg and the full contribution to $\psi^{(n,1)}$ is easily obtained using eq. $(30)$.

The structure of multiple soft emission from hard processes in QED was investigated by Grammer and Yennie. They have shown that the energetic electrons participating in a hard process receive an eikonal phase factor. In quantum chromodynamics, the situation is very similar except, that the eikonal factor is a matrix equal to the path ordered product of the matrix-valued gluon field.

For one soft gluon, the main result is very simple: it states that the singular contributions come from configurations where the soft gluons are attached to the external legs of the graphs. Therefore, the soft contribution can easily be calculated in terms of the Born amplitude. The insertion of a soft gluon that connects the external legs $i$ and $j$ has a twofold effect. First, after carrying out the loop integral and dropping singular terms corresponding to collinear configurations, we pick up the same eikonal factor as in QED.

\[ E_{ij} = - \left(\frac{g}{4\pi}\right)^2 \frac{1}{c^2} \left( - \frac{\mu^2}{s_{ij}} \right) \epsilon. \quad (39) \]

Secondly, the remaining part of the amplitude is the same as the Born amplitude except that it gets rotated in the colour space by the insertion of the

\[ ^{d}\text{Using unitarity this form of the soft factor can be confirmed by integrating the gluon momenta over the bremsstrahlung eikonal factor, as we shall see in the next section.} \]
colour matrices appearing in the two vertices of the soft line connecting the hard lines $i$ and $j$, therefore we have the replacement

$$A^{(n,0)}_{e_1e_2...e_n} \rightarrow E_{i,j} \sum_{a,c_i,c_j'} t_{c_i,c_i'}^{a} t_{c_j,c_j'}^{a} A^{(n,0)}_{c_1...c_i'...c_j'...c_n} \quad (40)$$

where $c_i$'s denote colour indices for the external partons and $t_{c,c_i}^{a}$ is the SU(3) generator matrix for the colour representation of line $i$, that is $t_{c,c_i}^{a} = (1/2)\lambda_{c,i}^{a}$ for an outgoing quark, $-(1/2)\lambda_{c,i}^{a}$ for an outgoing antiquark, and $i f_{aij}$ for an outgoing gluon. For an incoming parton, the same formula can be used as long as we use the conjugate colour representations, $-(1/2)\lambda_{c,i}^{a}$ for a quark, $(1/2)\lambda_{c,i}^{a}$ for an antiquark, and $i f_{aij}$ for a gluon. Finally, using the definition of $\psi^{(n,1)}$ we obtain

$$\psi^{(n,0)}_{ij}(\{a_l\}_{1,n},\{p_l\}_{1,n}) = \frac{1}{4p_i \cdot p_j \omega(a_1)\omega(a_2)} \sum_{\text{spin}} \sum_{c_i,c_i',c_j,c_j'} t_{c_i,c_i'}^{a} t_{c_j,c_j'}^{a} A^{(n,0)}_{c_1...c_i'...c_j'...c_n} A^{(n,0)*}_{c_1...c_i'...c_j'...c_n} \quad (41)$$

This result of eq. (35) is scheme-dependent\[3\]. In conventional dimensional regularization $\psi^{(3,0)}$ and $\psi^{(n,0)}_{mn}$ denote the Born and colour-connected Born cross section in $4-2\epsilon$ dimensions and $\psi^{(3,1)}_{NS}$ denotes the remaining final terms in this scheme. Finally, we note that to obtain the form of eq. (34) one should expand the eikonal factor in $\epsilon$ and apply the soft-colour identity\[18\]

$$\sum_{j=1}^{n} \psi^{(n,0)}_{ij}(\{a_l\}_{1,n},\{p_l\}_{1,n}) = 2 C(a_i) \psi^{(n,0)}_{ij}(\{a_l\}_{1,n},\{p_l\}_{1,n}). \quad (42)$$

6 Real contributions

In this section I consider the limiting behaviour of the real contribution of the process (24) in the soft and collinear limit. I also give the local subtraction terms which render the real contributions integrable over the whole phase-space region. The discussion will be detailed on the soft contributions. In the case of the collinear limits I only summarise their most salient features.
6.1 Kinematics

The four-momenta of the reaction (24) can be parameterised, for example, in terms of transverse momenta, rapidities and azimuthal angles

\[ p_1^\mu = \frac{\sqrt{s}}{2}(x_1, 0, 0, x_1), \quad p_2^\mu = \frac{\sqrt{s}}{2}(0, 0, -x_2) \]

\[ p_i^\mu = p_{\perp i} (\cos \phi_i, \sin \phi_i, \sin \theta_i), \quad i \in \{3, \ldots, n\}. \] (43)

From energy and longitudinal momentum conservation we obtain for the momentum fractions

\[ x_1 = \frac{1}{\sqrt{s}} \sum_{i=3}^{n} p_i e^{y_i}, \quad x_2 = \frac{1}{\sqrt{s}} \sum_{i=3}^{n} p_i e^{-y_i}. \] (44)

Considering singular limits we shall always assume the use of some suitable set of independent variables appropriate for the phase-space integration after imposition of four-momentum conservation. For the definition of measurable quantities, rapidity and transverse momentum variables are particularly convenient since they are boost-invariant. In the evaluation of the soft and collinear limit it appears more convenient, however, to use energy-angle variables

\[ p_i^\mu = (E_i, \sin \phi_i \sin \theta_i, \cos \phi_i, \sin \theta_i). \] (45)

6.2 Soft subtraction terms and soft contributions

The cross section of the real contribution (see eq. (31)) is constructed from products of three factors: function \( \psi^{(n+1,0)} \), the measurement function \( S_{X,n-1} \) and the phase-space factor \( d\phi_{n-1} \). Considering their soft limits let us assume that parton \( a_k \) is soft (3 \( \leq k \leq n+1 \)). Its energy is denoted by \( E_k \) and its angular correlations are controlled by the four-vector \( n_k^\mu \) defined by the relation

\[ k^\mu = E_k n_k^\mu = E_k (1, \tilde{n}_k). \] (46)

The method that has been used in the previous section to calculate the soft limit of the virtual contribution \( \psi^{(n+1)} \) also applies for the real emission, and we obtain

\[ \lim_{E_k \to 0} \psi^{(n+1,0)}(\{a_i\}_{1,n+1};\{p_i\}_{1,n+1}) = \delta_{gak} \frac{4\pi\alpha_s}{E_k^2} \sum_{i,j;i < j} c_{ij} (p_i, p_j, n_k) \times \psi^{(n,0)}_ij(\{a_l\}_{1,n+1};\{p_l\}_{1,n}) \] (47)
where the list \(\{v_i\}_{m,m+n}\) denotes the same list as \(\{v_i\}_{m,m+n}\) but \(v_k\ (m \leq k \leq m+n)\) is left out, \(\psi^{(n,0)}_{ij}\) is the colour-correlated Born contribution defined by eq. (41) in the previous section and \(e(p_i, p_j, n_k)\) is the eikonal factor for real emissions:

\[
e(p_i, p_j, n_k) = \frac{p_i \cdot p_j}{p_i \cdot n_k p_j \cdot n_k},
\]

(48)

We note that \(e(p_i, p_j, n_k)\) is independent of \(E_k\) but is dependent on the angular variables of the soft line. The colour-correlated Born terms \(\psi_{ij}\), however, are completely independent of the soft momenta. The soft limit of the measurement function is given by the requirement of infrared safety

\[
\lim_{E_k \rightarrow 0} S_{X,n-1}(\{p_i\}_{3,n+1}; X) = S_{X,n-2}(\{p_i\}_{3,n+1}; X)
\]

(49)

and the phase-space factor behaves as

\[
\lim_{E_k \rightarrow 0} d\phi_{n-1}(p_1, p_2 \rightarrow \{p_i\}_{3,n+1}) = \frac{1}{n-1} d\phi_1[k] d\phi_{n-2}(p_1, p_2 \rightarrow \{p_i\}_{3,n+1})
\]

(50)

where

\[
d\phi_1([k]) = \mu^{2\epsilon} \int \frac{d^{d-1}p_k}{(2\pi)^{d-1} 2p_k^0} = \mu^{2\epsilon} \int \frac{E_k^{1-2\epsilon}}{2(2\pi)^{3-2\epsilon}} dE_k \int d\Omega_{3-2\epsilon}
\]

(51)

is the phase-space integral over parton \(a_k\) and the decomposition into energy and angular integrals is also shown.\(^e\)

The local soft subtraction term for subtracting the soft singular behaviour in the \(E_k \rightarrow 0\) limit is given by the integrand of the expression below

\[
d\sigma^{(\text{soft, sub})}_{a_1a_2,k}(p_1, p_2; X) = -\frac{\delta_{a_2a_3}}{n-1} \frac{\alpha_S}{2\pi} \sum_{[k]} \sum_{i,j; i<j} \left[ 8\pi^2 e(p_i, p_j, n_k) \frac{1}{E_k} \Theta(E_k \geq E_a) d\phi_1[k] \right] \psi^{(n,0)}_{ij} \{a_i\}_{1,n+1} \{p_i\}_{1,3,n+1} S_{X,n-2}(\{p_i\}_{3,n+1}; X) \times \frac{1}{(n-2)!} d\phi_{n-2}(p_1, p_2 \rightarrow \{p_i\}_{3,n+1})
\]

(52)

\(^e\) We note that in the soft limit the condition imposed by the delta function of momentum conservation is different from the original one; therefore our notation is not completely precise. In the soft limit some of the components of the momenta \(\{p_i\}_{3,n+1}\) will be different from the original ones and instead \(\{p_i\}_{3,n+1}\) we should write \(\lim_{E_k \rightarrow 0} \{p_i\}_{3,n+1}\). We tacitly assume that in the soft limit \(p_1\) denotes its limiting value. It is uniquely defined after choosing the independent set of variables in the phase space-integral \(d\phi_{n-1}\).
In eqs. (31) and (52), we can use the same independent integration variables, therefore, by adding eq. (52) to the integrand of (31) we subtract its singular regions defined by the $E_k \to 0$ limit. Again, to ensure that the subtraction does not change the value of the original expression what is subtracted has to be added back on but in an integrated form where the singular terms are calculated analytically. This is achieved by carrying out the phase-space integral $d\phi_1[k]$. The expression in the square bracket in eq.(52) defines the soft integral $I_{\text{soft}}(z_{ij}, \mu/E_c, \epsilon) = 8\pi^2 \mu \int d\phi d\rightarrow \int d\phi \int d\rightarrow \int d\phi \int d\phi \int d\phi \int d\phi = \Theta(E_k \geq E_c) \tag{53}$ where $E_c$ is a cut-off value on the energy of the particle which is allowed to be soft. We also indicated the remaining dependences of the integral; the angular variable $z_{ab}$ is defined as

$$z_{ab} = \cos \theta_{ab} = \frac{\vec{p}_a \cdot \vec{p}_b}{|\vec{p}_a| |\vec{p}_b|}$$

| Definition | Answer |
|------------|--------|
| $J(\epsilon) = \frac{\Omega_{1-\epsilon} d - \Omega_{1-\epsilon} d}{1 - \epsilon} d \cos \theta \int_0^\pi d\phi \frac{(\sin \theta \sin \phi)^{-2\epsilon}}{1 - \cos \theta}$ | $-\frac{1}{\epsilon} + 2 \ln 2$ |
| $I_\phi(z_{ij}, z_{ik}) = \int_0^\pi d\phi_k \frac{(1 - z_{ij}^2)^{-\epsilon}}{1 - z_{ij}^2}$ | $+\epsilon(\text{Li}_2(1) - 2 \ln^2 2)$ |
| $J^{(0)}(z_{ij}) = \frac{1}{\pi} \int_1^1 d\rightarrow \int_0^\pi d\phi_k \ln \sin^2 \phi i_{ij}^R$ | $\ln \frac{1}{1 - z_{ij}^2}$ |
| $J^{(z)}(z_{ij}) = \frac{1}{\pi} \int_1^1 d\rightarrow \int_0^\pi d\phi_k \ln \sin^2 \phi i_{ij}^R$ | $- \ln 2(1 + z_{ij}) \ln \frac{1}{1 - z_{ij}^2}$ |
| $J^{(z)}(z_{ij}) = \frac{1}{\pi} \int_1^1 d\rightarrow \int_0^\pi d\phi_k \ln \sin^2 \phi i_{ij}^R$ | $\frac{1}{2} \ln^2 2(1 - z_{ij}) - 2 \ln^2 2 + \text{Li}_2(1) - \text{Li}_2(\frac{1}{2})$ |

Table 2: List of angular integrals

It is convenient to decompose the eikonal factor into a term which has one
collinear singularity and terms which are finite

\[
\frac{p_i \cdot p_j}{(p_i \cdot p_k)(p_j \cdot p_k)} = \frac{1}{E_k^2} \left[ l(z_{ij}, z_{ik}, z_{jk}) + (i \leftrightarrow j) \right]
\]

(54)

where

\[
l(z_{ij}, z_{ik}, z_{jk}) = \frac{1}{1 - z_{ik}} + l^R(z_{ij}, z_{ik}, z_{jk})
\]

(55)

and

\[
l^R(z_{ij}, z_{ik}, z_{jk}) = \frac{1}{2} \left[ \frac{1 - z_{ij}}{(1 - z_{ik})(1 - z_{jk})} - \frac{1}{1 - z_{ik}} - \frac{1}{1 - z_{jk}} \right].
\]

(56)

The integral over the energy is trivial and the angular integrals have to be calculated only up to order \(\epsilon\) so one obtains

\[
I_{ij}^\text{soft} = c_\Gamma \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \left[ J(\epsilon) + J^{(0)}(z_{ij})(1 - \epsilon \ln 4) - \epsilon J^{(z)}(z_{ij}) + J^{(\phi)}(z_{ij}) \right]
\]

(57)

where \(c_\Gamma\) was defined in eq. (35). The integrals \(J^{(0)}, J^{(\phi)}\) and \(J^{(z)}\) are listed in Table (2). Inserting their values into eq. (57) and expanding in \(\epsilon\) we get both the singular and finite parts

\[
I_{ij} = I_{ij}^\text{soft,sing} + I_{ij}^\text{soft,fin}
\]

(58)

\[
I_{ij}^\text{soft,sing} = c_\Gamma \left( \frac{4\pi\mu^2}{Q^2} \right)^\epsilon \left[ \frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{2p_i p_j}{Q^2} + \frac{1}{\epsilon} \ln \frac{E_i E_j}{E^2} \right].
\]

(59)

The explicit form of the finite part is not interesting for us here; it can be found in ref. [21]. We can use completely covariant notation by replacing the energies with the covariant expressions

\[
E_i = (\mathcal{P} p_i)/\sqrt{s}, \quad \text{where} \quad \mathcal{P}^\mu = p_A^\mu + p_B^\mu = (\sqrt{s}, 0, 0, 0)
\]

(60)

where \(p_A\) and \(p_B\) are the four momenta of the incoming hadrons.

If the sum over flavour is carried out the result becomes independent of the label of the soft line. Therefore every leg in the final state gives the same contributions and the factor \(1/(n - 1)\) in eq. (52) gets cancelled. As a result, the soft contribution of the real leading order process with \(n + 1\) partons can be written in the form of the virtual contribution of the corresponding \(n\)-parton process. It can be obtained form the right hand side of eq. (52) by multiplying
it with $n - 1$, changing its sign and inserting the integrated value of $\mathcal{I}_{ij}^{\text{soft}}$ (eq. 59):

$$d\sigma_{a_1a_2}^{(n+1,\text{soft})}(p_1,p_2;X) = \sum_{\{a_i\}_{3,n}} \psi^{(n+1,\text{soft})}_{X,n-2}(\{p_i\}_{3,n};X) \times d\phi_{n-2}(p_1,p_2;\{p_i\}_{3,n})$$

(61)

where

$$\psi^{(n+1,\text{soft})}(\{a_i\}_{1,n},\{p_i\}_{1,n}) = \frac{\alpha_s}{2\pi} \left( \frac{4\pi\mu^2}{Q^2} \right)^c e \Gamma \left\{ \sum_{i=1}^n \frac{C(a_i)}{2} + 2 \frac{C(a_i)}{\epsilon} \ln \frac{E_i}{E_c} \right\} \psi^{(n,0)}(\{a_i\}_{1,n},\{p_i\}_{1,n}) + \frac{1}{2\epsilon} \sum_{i,j=1}^n \ln \left( \frac{2 p_i \cdot p_j}{Q^2} \right) \psi^{(n,0)}_{ij}(\{a_i\}_{1,n},\{p_i\}_{1,n}) + \psi^{(n+1,\text{soft})}_{\text{NS}}(\{a_i\}_{1,n},\{p_i\}_{1,n})$$

(62)

The first term is the soft-collinear singularity and it cancels the soft-singular terms appearing in the virtual corrections. The third term is the soft contribution proportional to the colour correlated Born-term and again it cancels the corresponding terms in the virtual contributions. The second term comes from the collinear singularities of the eikonal factors and is cancelled by the direct singular collinear contributions. The last term is finite and can be evaluated in four dimensions.

6.3 Collinear subtraction terms and collinear contributions

In the previous section we constructed the local subtraction term for the soft singular region. We also demonstrated that the soft singularities of the virtual corrections are cancelled by the soft contributions of the real corrections after the integrals over the energy and angular variables of the soft line are carried out.

The subtraction and addition procedure can also be applied to the singular collinear regions. Similarly to the soft case, one finds again simple limiting behaviours for the $\psi$-functions, the measurement functions and the phase space. We shall discuss only the collinear limit of the $\psi$ function. The reader can find further details in refs. 17, 18, 21, 19.
From the simple behaviour of the helicity amplitudes in the collinear limit we obtain for the \( \psi \) function the limiting behaviour

\[
\psi^{(n+1,0)}(p_1, p_2; \ldots, p_i, \ldots, p_j, \ldots) \xrightarrow{\text{collinear}} \frac{4\pi \alpha_s}{p_i \cdot p_j} \left[ \frac{p_{S(a_i, a_j)}^c(z)\psi^{(n,0)}(p_1, p_2; \ldots, p_i, \ldots, p_j, \ldots)}{p_i \cdot p_j} + \frac{4\pi \alpha_s Q_{a_i, S(a_i, a_j)}^c(z)\tilde{\psi}^{(n,0)}(p_1, 2; \ldots, p_i, \ldots, p_j, \ldots)}{p_i \cdot p_j} \right],
\]

where \( z \) denotes the momentum fraction defined through the equations

\[
p_i = z p_F, \quad k_j = (1 - z) p_F,
\]

\( P_{ab}(z) \) denotes the standard Altarelli-Parisi splitting functions and \( Q_{ab*}(z) \)'s are some new universal functions which control the azimuthal angle behaviour of the collinear limit. Assuming that the collinear particles belong to the final state

\[
Q_{gg*}(z) = -4C_A z(1 - z),
\]

\[
Q_{gq*}(z) = 4T_F z(1 - z),
\]

\[
Q_{qq*}(z) = 0,
\]

\[
Q_{gq*}(z) = 0.
\]

In these equations, the \( * \) symbol over the flavour of the particle that eventually splits reminds that this particle is off-shell. In principle, this notation should be extended also to the Altarelli-Parisi splitting kernels, but at the leading order \( P_{ab*} = P_{a*b} \), and therefore there is no need to keep track of the off-shell particle; the \( \tilde{\psi} \) function is constructed from the helicity amplitudes of \( n \)-parton processes but with some linear dependence on the azimuthal angle of the quasi-collinear configuration. The \( \tilde{\psi} \) functions therefore are as simple as the Born-terms. They are important in constructing a correct local subtraction term. However, upon integrating over the azimuthal angle of the collinear momenta their contributions vanish and they do not appear in the integrated collinear contributions. It is an interesting simplifying feature of the five parton processes that the \( \tilde{\psi} \) functions vanish identically.

Using crossing properties of the splitting functions and of the \( \psi \) functions similar relations remain valid also for initial collinear singularities.

The remaining procedure, in principle, is the same what we used in the soft case although it is somewhat more tedious. The properties of the \( S \)-functions and the phase-space again ensure that we can easily construct the
local collinear counter terms and in the collinear contributions we can obtain the singular terms analytically. For further details the reader should consult the original literature\textsuperscript{17,18}.

**Exercise:** Calculate the inclusive one-jet cross section $d\sigma/dE_Jd\cos\Theta_J$ for $e^+e^-$ annihilation following the general algorithm explained above.

7 Conclusion

In this lecture I described the methods and techniques to get differential cross-section formulae from NLO singular scattering amplitudes. These are free from singular terms well defined in all integration regions and therefore suitable for numerical evaluation. The ingredients of the methods are the collinear counter terms, the local subtractions terms, the virtual, the soft and the collinear contributions. Their sum defines the *finite hard scattering cross-section*

$$d\hat{\sigma}^{\text{hard}}_{a_1a_2} = d\sigma^{\text{born}}_{a_1a_2} + d\sigma^{\text{virt}}_{a_1a_2} + d\sigma^{\text{soft}}_{a_1a_2} + d\sigma^{(\text{coll,initial})}_{a_1a_2} + d\sigma^{(\text{coll,final})}_{a_1a_2} + d\sigma^{(\text{coll,counter})}_{a_1a_2} + d\sigma^{(\text{real,subtracted})}_{a_1a_2}$$

(69)

where the last term has the kinematics of an $n + 1$ parton process while all the other terms have the kinematics of an $n$ parton process. In addition, the evaluation of some physical quantity requires the explicit construction of the corresponding measurement functions. We note that in the case of jet-production this is a non-trivial exercise (see refs.\textsuperscript{27,28,29}).

We have seen that although the methods are conceptually simple, their implementation is non-trivial. Recently, several papers attempted to give a comprehensive documentation\textsuperscript{17,18,21,30}, which are recommended for further reading.

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