ENVELOPE OF HOLOMORPHY FOR BOUNDARY CROSS SETS

PETER PFLUG AND VIẾT-ANH NGUYỄN

Abstract. Let $D \subset \mathbb{C}^n$, $G \subset \mathbb{C}^m$ be open sets, let $A$ (resp. $B$) be a subset of the boundary $\partial D$ (resp. $\partial G$) and let $W$ be the 2-fold boundary cross $((D \cup A) \times B) \cup (A \times (B \cup G))$. An open subset $X \subset \mathbb{C}^{n+m}$ is said to be the “envelope of holomorphy” of $W$ if it is, in some sense, the maximal open set with the following property: Any function locally bounded on $W$ and separately holomorphic on $(A \times G) \cup (D \times B)$ “extends” to a holomorphic function defined on $X$ which admits the boundary values $f$ a.e. on $W$. In this work we will determine the envelope of holomorphy of some boundary crosses.

1. Introduction

In a series of articles [6, 7, 8] the authors establish various “boundary cross theorems”. These results deal with the continuation of holomorphic functions of several complex variables which are defined on some boundary crosses. The first theorem of this type was discovered and proved by Malgrange–Zerner [10].

However, the question naturally arises whether all these theorems are optimal. More precisely, are the extension domains in these theorems always maximal? In other words, are they always “envelopes of holomorphy”? In this work we investigate this question. We will show that under some conditions our boundary cross theorems are optimal.

1.1. Plurisubharmonic measures. Let $\Omega \subset \mathbb{C}^n$ be an open set. For any function $u$ defined on $\Omega$, let

$$
\hat{u}(z) := \begin{cases} 
u(z), & z \in \Omega, \\ \limsup_{\Omega \ni w \to z} u(w), & z \in \partial \Omega. 
\end{cases}
$$

For a set $A \subset \overline{\Omega}$ put

$$h_{A,\Omega} := \sup \{ u : u \in \mathcal{PSH}(\Omega), u \leq 1 \text{ on } \Omega, \hat{u} \leq 0 \text{ on } A \},$$

where $\mathcal{PSH}(\Omega)$ denotes the cone of all functions plurisubharmonic on $\Omega$.

The plurisubharmonic measure of $A$ relative to $\Omega$ is the function $\omega(\cdot, A, \Omega) \in \mathcal{PSH}(\Omega)$ defined by

$$\omega(z, A, \Omega) := h_{A,\Omega}^+(z), \quad z \in \Omega,$$

where $h^+$ denotes the upper semicontinuous regularization of the function $h$.

If $n = 1$ and $A \subset \partial \Omega$, then $\omega(\cdot, A, \Omega)$ is also called the harmonic measure of $A$ relative to $\Omega$. In this case, $\omega(\cdot, A, \Omega)$ is a harmonic function.

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1.2. **Cross, separate holomorphicity.** For open sets \( D \subset \mathbb{C}^n \), \( G \subset \mathbb{C}^m \) and subsets \( \emptyset \neq A \subset \overline{D}, \emptyset \neq B \subset \overline{G} \), we define the cross \( W \) and its interior \( W^o \) as
\[
W = \mathbb{X}(A, B; D, G) := ((D \cup A) \times B) \cup (A \times (G \cup B)),
\]
\[
W^o = \mathbb{X}^o(A, B; D, G) := (D \times B) \cup (A \times G).
\]
If \( A \subset \partial D \) and \( B \subset \partial G \) (resp. \( A \subset D \) and \( B \subset G \)), then \( W \) is called a boundary cross (resp. a classical cross).

For a cross \( W := \mathbb{X}(A, B; D, G) \) let
\[
\hat{W}^o = \hat{\mathbb{X}}^o(A, B; D, G) := \{(z, w) \in D \times G : \omega(z, A, D) + \omega(w, B, G) < 1 \}
\]
and
\[
\hat{W} = \hat{\mathbb{X}}(A, B; D, G) := W \cup \hat{W}^o.
\]

We say that a function \( f : W \rightarrow \mathbb{C} \) is separately holomorphic on \( W^o \) and write \( f \in \mathcal{O}_s(W^o) \), if for any \( a \in A \) the function \( f(a, \cdot)|_G \) is holomorphic on \( G \), and for any \( b \in B \) the function \( f(\cdot, b)|_D \) is holomorphic on \( D \).

We say that a function \( f : W \rightarrow \mathbb{C} \) is separately continuous and write \( f \in \mathcal{C}_s(W) \), if for any \( a \in A \) and for any \( b \in B \), the functions \( f(a, \cdot) \) and \( f(\cdot, b) \) are continuous.

For an open set \( \Omega \subset \mathbb{C}^n \), \( \mathcal{O}(\Omega) \) denotes the space of all holomorphic functions on \( \Omega \).

1.3. **Motivations for our work and envelope of holomorphy of a boundary cross.** We like to formulate the boundary cross theorems in one and higher dimensional contexts (see [6, 7, 8]).

A (Jordan) curve in \( \mathbb{C} \) is the image \( C := \{\gamma(t) : t \in [0, 1]\} \) of a continuous one-to-one map \( \gamma : [0, 1] \rightarrow \mathbb{C} \). The interior of the curve \( C \) given by \( \{\gamma(t) : t \in (0, 1)\} \) is said to be an open (Jordan) curve. A Jordan domain is the image \( \{\Gamma(t), t \in E\} \) of a one-to-one continuous map \( \Gamma : E \rightarrow \mathbb{C} \), where, in this work, \( E \) denotes the open unit disc in \( \mathbb{C} \). A closed (Jordan) curve is the boundary of a Jordan domain. An open set \( D \subset \mathbb{C} \) is said to be Jordan-curve-like at a point \( \zeta \in \partial D \) if there is a Jordan domain \( U \) such that \( \zeta \in U \) and \( U \cap \partial D \) is an open (Jordan) curve.

Let \( D \subset \mathbb{C}, G \subset \mathbb{C} \) be two open sets and \( A \) (resp. \( B \)) a subset of \( \partial D \) (resp. \( \partial G \)) such that \( D \) (resp. \( G \)) is Jordan-curve-like at every point of \( A \) (resp. \( B \)), and let \( f : W \rightarrow \mathbb{C} \) be a function. We can define as in Subsections 2.1–2.3 of [7] various notions and terminology: Jordan-measurable sets, sets of positive length, sets of zero length, Jordan-measurable functions, the angular limit, the set of all locally regular points \( A^* \) (resp. \( B^* \)) relative to \( A \) (resp. \( B \)), almost everywhere (a.e.) etc.

**Theorem A.** In [7] may be restated, in a simple form, as follows:

**Theorem 1.** We keep the hypotheses and notation of the previous paragraph. Suppose in addition that \( A \) and \( B \) are of positive length and that \( f \) verifies the following properties:

(i) \( f \) is locally bounded on \( W \) and \( f \in \mathcal{O}_s(W^o) \);
(ii) \( f|_{A \times B} \) is Jordan-measurable;
(iii) for any \( a \in A \) (resp. \( b \in B \)), the holomorphic function \( f(a, \cdot)|_G \) (resp. \( f(\cdot, b)|_D \)) has the angular limit \( f_1(a, b) \) at \( b \) for a.e. \( b \in B \) (resp. \( f_2(a, b) \) at \( a \) for a.e. \( a \in A \)) and \( f_1 = f_2 = f \) a.e. on \( A \times B \).
Then there exists a unique function \( \hat{f} \in \mathcal{O}(\hat{W}^o) \) with the following property: There are subsets \( \tilde{A} \subset A \cap A^* \) and \( \tilde{B} \subset B \cap B^* \) such that
1a) the sets \( A \setminus \tilde{A} \) and \( B \setminus \tilde{B} \) are of zero length;
1b) \( \hat{f} \) admits the angular limit \( f(\zeta, \eta) \) at every point \( (\zeta, \eta) \in (\tilde{A} \times G) \cup (D \times \tilde{B}) \).

In fact, this theorem was formulated in [7] in a more general context: \( D \) and \( G \) are open sets of arbitrary complex manifolds of dimension 1 countable at infinity.

For the higher dimensional case we recall the following terminology from Section 2 in [8]. Let \( D \subset \mathbb{C}^n \) be a nonempty open set, and \( A \) a nonempty relatively open subset of \( \partial D \). Then \( A \) is said to be a topological hypersurface (in \( \mathbb{C}^n \equiv \mathbb{R}^{2n} \)) if, for every \( a \in A \) there exist an open neighborhood \( V \) of \( a \), an open subset \( U \subset \mathbb{R}^{2n-1} \), a continuous function \( h : U \to \mathbb{R} \) and an integer \( j : 1 \leq j \leq 2n \) such that

\[
V \cap A = \{ z = (x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n} : x_j = h(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{2n}), (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{2n}) \in U \}.
\]

The Main Theorem in [6, 8] may be restated, in a simple form, as follows:

**Theorem 2.** Let \( D \subset \mathbb{C}^n, G \subset \mathbb{C}^m \) be two nonempty open sets, let \( A \) (resp. \( B \)) be a nonempty relatively open subset of \( \partial D \) (resp. \( \partial G \)). Suppose in addition that \( A \) and \( B \) are topological hypersurfaces. Let \( f : W \to \mathbb{C} \) be such that:

(i) \( f \in \mathcal{C}_s(W) \cap \mathcal{O}_s(W^o) \);
(ii) \( f \) is locally bounded on \( W \);
(iii) \( f|_{A \times B} \) is continuous.

Then there exists a unique function \( \hat{f} \in \mathcal{O}(\hat{W}^o) \) such that

\[
\lim_{(z,w) \to (\zeta,\eta), (z,w) \in \hat{W}^o} \hat{f}(z, w) = f(\zeta, \eta), \quad (\zeta, \eta) \in W.
\]

In fact, this theorem was formulated in [8] in its full generality: \( D \) and \( G \) are open sets of arbitrary complex manifolds.

These results lead to the following concept.

**Definition 1.** Let \( D, G, A, B \) be as in the hypothesis of Theorem 1 (resp. Theorem 2) and \( W := X(A, B; D, G) \). We say that \( \hat{W}^o \) is the **envelope of holomorphy** of the boundary cross \( W \) if there do not exist nonempty open sets \( U_1, U_2 \subset \mathbb{C}^2 \) (resp. \( \mathbb{C}^{n+m} \)) with \( U_2 \) connected, \( U_2 \not\subset \hat{W}^o \), \( U_1 \subset U_2 \cap \hat{W}^o \), such that for every \( f : W \to \mathbb{C} \) which satisfies (i)-(iii) of Theorem 1 (resp. Theorem 2), there is a function \( h \in \mathcal{O}(U_2) \) such that \( h = \hat{f} \) on \( U_1 \), where \( \hat{f} \in \mathcal{O}(\hat{W}^o) \) is the unique function given by Theorem 1 (resp. Theorem 2).

The purpose of this article is to investigate the question whether \( \hat{W}^o \) in Theorem 1 and 2 is always the envelope of holomorphy. This problem is motivated by the work of Alehyané-Zeriahi [11], where the envelope of holomorphy of a classical cross (i.e. \( A \subset D, B \subset G \)), \( D, G \) are subdomains of Stein manifolds, has been identified. See also [8] for further generalizations.

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2. Statement of the results

Let $D \subset \mathbb{C}$ be an open set which is Jordan-curve-like at a point $\zeta \in \partial D$. Then $\zeta$ is said to be of type 1 if there is a neighborhood $V$ of $\zeta$ such that $V \cap D$ is a Jordan domain. Otherwise, $\zeta$ is said to be of type 2. We easily see that if $\zeta$ is of type 2, then there are an open neighborhood $V$ of $\zeta$ and two Jordan domains $V_1$, $V_2$ such that $V \cap D = V_1 \cup V_2$. A (Jordan) curve or an open (Jordan) curve or a closed curve $C \subset \partial D$ is said to be of type 1 (resp. type 2) if all points of $C$ are of type 1 (resp. type 2).

The following simple example (see Subsection 2.1 in [7]) may clarify the above definitions.

**Example 1.** Let $H$ be the open square in $\mathbb{C}$ whose four vertices are $1 + i$, $-1 + i$, $-1 - i$, and $1 - i$. Define the domain $D := H \setminus \left[-\frac{1}{2}, \frac{1}{2}\right]$. Then $D$ is Jordan-curve-like on $\partial H \cup \left(-\frac{1}{2}, \frac{1}{2}\right)$. Every point of $\partial H$ is of type 1 and every point of $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is of type 2. In other words, $\partial G$ is a closed curve of type 1 and $\left(-\frac{1}{2}, \frac{1}{2}\right)$ is an open curve of type 2.

We continue with another example showing that in Theorem 1 above $\hat{W}^o$ is, in general, not the envelope of holomorphy of $W$.

**Example 2.** Let $D$ be as in Example 1, let $A := \left(-\frac{1}{4}, \frac{1}{4}\right)$, and $G := E$, $B := \partial G$. Then a direct computation shows that $\hat{W}^o = \hat{X}^o(A, B; D, G) = D \times G$.

Let $f$ be an arbitrary function satisfying the hypothesis of Theorem 1, and let $\hat{f} \in \mathcal{O}(\hat{W}^o)$ and $\tilde{A} \subset A$ be as in the conclusion this theorem. Therefore, using the Lindelöf Theorem, we have

$$\lim_{y \to 0^+} \hat{f}(x + iy) = f(x) = \lim_{y \to 0^-} \hat{f}(x + iy), \quad x \in \tilde{A}.$$ 

On the other hand, using the fact that the one dimensional Hausdorff measure of $A \setminus \tilde{A}$ is zero, the hypothesis that $f$ is locally bounded on $W$ and applying Two-Constant Theorem, one can prove that for every $(\zeta, \eta) \in A \times G$, there are open neighborhoods $U$ of $\zeta$ in $D \cup A$ and $V$ of $\eta$ in $G$ such that

$$\sup_{(U \setminus A) \times V} |\hat{f}(z, w)| < \infty.$$ 

Consequently, by Morera’s Theorem, $\hat{f}$ extends holomorphically through all points of $A \times G$. Therefore, one can take $U_1 := D \times G$ and $U_2 := (D \cup A) \times G$ in Definition 1. Hence, $\hat{W}^o$ is not the envelope of holomorphy of $W$.

This discussion leads us to the following

**Definition 2.** Let $D \subset \mathbb{C}$ be an open set and $A \subset \partial D$ be such that $D$ is Jordan-curve-like at every point of $A$. A point $a \in A$ is said to be an extendible point of $A$ if there exists an open (Jordan) curve $C := \{\gamma(t) : t \in (0, 1)\} \subset \partial D$ such that

- $a \in C$,
- the open curve $C$ is of type 2,
Now we are able to state the first result.

**Theorem A.** Let $D, G, A, B$ be as in the hypothesis of Theorem 1 and $W := \mathbb{X}(A, B; D, G)$. Suppose in addition that $D$ (resp. $G$) is Jordan-curve-like at all points of $\overline{A}$ (resp. $\overline{B}$) and that $A$ and $B$ do not possess any extendible points. Then $\hat{W}^o$ is the envelope of holomorphy of $W$.

For the higher dimensional situation we need to introduce a new concept.

**Definition 3.** A pair $(A, D)$ of an open set $D \subset \mathbb{C}^n$ and a relatively open set $A$ of $\partial D$ is said to satisfy hypothesis ($\mathcal{H}$) if there exists a sequence of pseudoconvex open sets $(D_k)_{k=1}^{\infty} \subset \mathbb{C}^n$ such that $D_k \cap \overline{D} = D \cup A$ and $\bigcap_{k=1}^{\infty} D_k = D \cup A$.

Now we are ready to state the second result.

**Theorem B.** Let $D, G, A, B$ be as in the hypothesis of Theorem 2 and $W := \mathbb{X}(A, B; D, G)$. Suppose in addition that the pairs $(A, D)$ and $(B, G)$ satisfy hypothesis ($\mathcal{H}$). Then $\hat{W}^o$ is the envelope of holomorphy of $W$.

Here is a simple sufficient condition.

**Proposition C.** Let $D \subset \mathbb{C}^n$ be a pseudoconvex domain and $A \subset \partial D$ a relatively open subset. Suppose in addition that $A$ can be written in the form $A = \bigcup_{j \in J} A_j$, where

(i) $A_j$ is a connected relatively open and bounded subset of $\partial D$ for $j \in J$;
(ii) $A_j \cap \bigcup_{k \in J \setminus \{j\}} A_k = \emptyset$ for $j \in J$;
(iii) $\bigcup_{j \in J} A_j$ is contained in the set of $C^2$ smooth, strongly pseudoconvex points of $\partial D$.

Then the pair $(A, D)$ satisfies hypothesis ($\mathcal{H}$).

It seems to be of interest to find weaker conditions than hypothesis ($\mathcal{H}$) for Theorem B to be true. One may also seek to determine the envelope of holomorphy of boundary cross sets where some singularities are allowed or in the manifold context. The problem of determining the envelope of holomorphy of classical cross sets with singularities has been studied in many works (see [2, 3, 4, 5] and the references therein).

### 3. The proofs

The main idea of the proofs is contained in the following lemma.

**Lemma 3.1.** Let $D, G, A, B$ be as in the hypothesis of Theorem A (resp. Theorem B) and $W := \mathbb{X}(A, B; D, G)$. Assume that for every nonempty open connected set $U \subset \mathbb{C}^2$ (resp. $\mathbb{C}^{n+m}$) such that $U \not\subset \hat{W}^o$ and $U \cap \hat{W}^o \neq \emptyset$, there is a pseudoconvex open set $\Omega$ in $\mathbb{C}^2$ (resp. $\mathbb{C}^{n+m}$) such that $\hat{W} \subset \Omega$ and $U \cap \partial \Omega \neq \emptyset$. Then $\hat{W}^o$ is the envelope of holomorphy of $W$. 
Proof. Assume that \( \widetilde{W}^o \) is not the envelope of holomorphy of \( W \). Then there are non empty open connected sets \( U_1 \subset U_2 \), \( U_1 \subset \widetilde{W}^o \) but \( U_2 \not\subset \widetilde{W}^o \) such that for every \( f : W \rightarrow \mathbb{C} \) which satisfies (i)-(iii) of Theorem 1 (resp. Theorem 2), there is a function \( h \in \mathcal{O}(U_2) \) such that \( h = \hat{f} \) on \( U_1 \), where \( \hat{f} \in \mathcal{O}(\widetilde{W}^o) \) is the unique function given by Theorem 1 (resp. Theorem 2).

By the hypothesis, one may find a pseudoconvex open set \( \Omega \) such that \( \widetilde{W} \subset \Omega \) and \( U_2 \cap \partial \Omega \neq \emptyset \). Let \( U_3 \) be the connected component of \( U_2 \cap \Omega \) which contains \( U_1 \). It is clear that \( \partial U_3 \cap U_2 \cap \partial \Omega \neq \emptyset \). Therefore, by Cartan–Thullen Theorem, there are a function \( f \in \mathcal{O}(\Omega) \), a sequence of points \( ((z_j, w_j))_{j=1}^{\infty} \subset U_3 \) and a point \( (z_0, w_0) \in \partial U_3 \cap U_2 \cap \partial \Omega \) with \( |f(z_j, w_j)| \to \infty \) and \( (z_j, w_j) \to (z_0, w_0) \) as \( j \to \infty \). Then \( f \) restricted to \( W \) extends to a holomorphic function \( \hat{f} \) on \( \widetilde{W}^o \) and \( f = \hat{f} \) on \( \widetilde{W}^o \). By the above assumption there is \( h \in \mathcal{O}(U_2) \) such that \( h = \hat{f} \) on \( U_1 \). This implies that \( f = h \) on \( U_3 \). But the latter identity contradicts the fact that
\[
\lim_{j \to \infty} |f(z_j, w_j)| = \infty \quad \text{and} \quad \lim_{j \to \infty} |h(z_j, w_j)| = |h(z_0, w_0)|.
\]

Hence, the proof is finished. \( \square \)

Before we present the proof of Theorem A, we have to introduce some notation and terminology (see also Subsections 2.1–2.3 of [7])

For two sets \( T \subset S \), the characteristic function \( 1_{T,S} : S \rightarrow \{0, 1\} \) is given by:
\[
1_{T,S} = 1 \text{ on } T \quad \text{and} \quad 1_{T,S} = 0 \text{ on } S \setminus T.
\]

Let us come back to the beginning of Subsection 1.3. For a curve \( \mathcal{C} := \{\gamma(t) : t \in [0, 1]\} \) (resp. a closed curve which is the boundary of a Jordan domain \( \Gamma : E \rightarrow \mathbb{C} \)), \( \gamma : [0, 1] \rightarrow \mathbb{C} \) (resp. \( \Gamma_{|\partial E} \)) is said to be a parametrization. Moreover, two curves with corresponding parameterizations \( \gamma_1, \gamma_2 \) are said to have the same end-points if \( \gamma_1(0) = \gamma_2(0) \) and \( \gamma_1(1) = \gamma_2(1) \).

**Proof of Theorem A.** Since \( D \) is Jordan-curve-like on the set \( A \) of positive length, we may find a sequence \( (A_k)_{k=1}^{\infty} \) of relatively open subsets of \( \partial D \) such that \( D \) is Jordan-curve-like on \( A_k, k \geq 1 \),
\[
(3.1) \quad A \subset A_k, \quad A_k \searrow A_0 \text{ as } k \to \infty, \quad \text{and} \quad A_0 \setminus A \text{ is of zero length}.
\]

Using (3.1) and applying Theorem 4.6 in [7] yields that
\[
(3.2) \quad \omega(z, A, D) = \omega(z, A_0, D), \quad z \in D.
\]

Let \( \mathcal{P}_D \) be the Poisson projection of \( D \) (see [9] Subsection 4.3)). By Theorem 4.3.3 in [9], we have
\[
\omega(\cdot, A_k, D) = \mathcal{P}_D[1_{\partial D \setminus A_k, \partial D}], \quad k \geq 0.
\]

On the other hand, using (3.1) and applying the monotone convergence theorem yields that
\[
\lim_{k \to \infty} \mathcal{P}_D[1_{\partial D \setminus A_k, \partial D}] = \mathcal{P}_D[1_{\partial D \setminus A_0, \partial D}].
\]

This, combined with (3.2), implies that
\[
(3.3) \quad \omega(z, A_k, D) \nearrow \omega(z, A, D) \quad \text{as } k \to \infty, \quad z \in D.
\]
Now we use the hypothesis that $D$ (resp. $G$) is Jordan-curve-like at all points of $\overline{A}$ (resp. $\overline{B}$). After a possible change of $A_k$ (resp. $B_k$) we may assume that for every fixed $k \geq 1$, $(A_{kl})_{l \in I_k}$ (resp. $(B_{km})_{m \in J_k}$) are pairwisely disjoint, they are either curves or closed curves and their types are either 1 or 2. Moreover, using (if necessary) a conformal transformation: $z \mapsto \frac{z-z_0}{1-z_0}$ where $z_0$ is an arbitrary point in $D$, we may assume, without loss of generality that $\partial D$ and $\partial G$ are bounded (hence, compact) sets.

For every open (Jordan) curve $A_{kl}$ (resp. $B_{km}$) of type 1 we fix a parametrization $a_{kl} : [0, 1] \rightarrow \overline{A_{kl}}$ (resp. $b_{km} : [0, 1] \rightarrow \overline{B_{km}}$) and find, using a geometric argument which is based on the compactness and the connectedness of $\overline{A_{kl}}$ (resp. $\overline{B_{km}}$), an open curve $\Gamma_{kl}$ (resp. $\Lambda_{km}$) with a parametrization $\gamma_{kl} : [0, 1] \rightarrow \Gamma_{kl}$ (resp. $\lambda_{km} : [0, 1] \rightarrow \Lambda_{km}$) satisfying the following properties:

(a) $\Gamma_{kl}$ has the same end-points as those of $A_{kl}$ and $\max_{t \in [0, 1]} |a_{kl}(t) - \gamma_{kl}(t)| < \frac{1}{k}$ (resp. $\Lambda_{km}$ has the same end-points as those of $B_{km}$ and $\max_{t \in [0, 1]} |b_{km}(t) - \lambda_{km}(t)| < \frac{1}{k}$);
(b) $\overline{A_{kl}} \cup \overline{\Gamma_{kl}}$ is the boundary of a Jordan domain $\Delta_{kl}$ (resp. $\overline{B_{km}} \cup \overline{\Lambda_{km}}$ is the boundary of a Jordan domain $\Phi_{km}$);
(c) $\Delta_{kl} \cap D = \emptyset$ (resp. $\Phi_{km} \cap G = \emptyset$).

For every closed (Jordan) curve $\overline{A_{kl}}$ (resp. $\overline{B_{kr}}$) of type 1 we fix a parametrization $a_{kp} : \partial E \rightarrow A_{kp}$ (resp. $b_{kr} : \partial E \rightarrow B_{kr}$) and find, using above a geometric argument, a closed curve $\Gamma_{kp}$ (resp. $\Lambda_{kr}$) with a parametrization $\gamma_{kp} : \partial E \rightarrow \Gamma_{kp}$ (resp. $\lambda_{kr} : \partial E \rightarrow \Lambda_{kr}$) satisfying the following properties:

(d) $A_{kp} \cap \Gamma_{kp} = \emptyset$ and $\max_{t \in \partial E} |a_{kp}(t) - \gamma_{kp}(t)| < \frac{1}{k}$ (resp. $B_{kr} \cap \Lambda_{kr} = \emptyset$ and $\max_{t \in \partial E} |b_{kr}(t) - \lambda_{kr}(t)| < \frac{1}{k}$). Moreover, $A_{kp} \cup \Gamma_{kp}$ is the boundary of an open annulus $\Delta_{kp}$ (resp. $B_{kr} \cup \Lambda_{kr}$ is the boundary of an open annulus $\Phi_{kr}$) such that $\Delta_{kp} \cap D = \emptyset$ (resp. $\Phi_{kr} \cap G = \emptyset$).

For every curve or closed curve $\overline{A_{kl}}$ (resp. $\overline{B_{km}}$) of type 2 let

(e) $\Gamma_{kl} := \emptyset$, $\Delta_{kl} := \emptyset$ (resp. $\Lambda_{km} := \emptyset$, $\Phi_{km} := \emptyset$).

Now we are able to define the two sequences of open sets $(D_k)_{k=1}^\infty \subset \mathbb{C}$ and $(G_k)_{k=1}^\infty \subset \mathbb{C}$. Indeed, for every $k \geq 1$ let

\begin{align}
D_k &= D \cup \left( \bigcup_{l \in I_k} (\Delta_{kl} \cup A_{kl}) \right) \cup \left( \bigcup_{p \in I_k''} (\Delta_{kp} \cup A_{kp}) \right), \\
G_k &= G \cup \left( \bigcup_{m \in J_k'} (\Phi_{km} \cup B_{km}) \right) \cup \left( \bigcup_{r \in J_k''} (\Phi_{kr} \cup B_{kr}) \right).
\end{align}
where $I'_k$ (resp. $J'_k$) is the set of all $l \in I_k$ (resp. $m \in J_k$) such that $A_{kl}$ (resp. $B_{km}$) are open curves, and $I''_k$ (resp. $J''_k$) is the set of all $p \in I_k$ (resp. $r \in J_k$) such that $\overline{A_{kp}}$ (resp. $\overline{B_{kr}}$) are closed curves.

As a consequence of the construction in (a)--(e) and (3.4)-(3.5) we are in position to show that

\begin{equation}
\omega(z, A_k, D_k) = \omega(z, A_k, D), \quad z \in D;
\end{equation}

\begin{equation}
\omega(w, B_k, G_k) = \omega(w, B_k, G), \quad w \in G, \ k \geq 1.
\end{equation}

We only need to prove the first identity in (3.6), the other one can be proved similarly. In fact, it suffices to show that

\begin{equation}
\omega(z, A_k, D) \leq \omega(z, A_k, D_k), \quad z \in D,
\end{equation}

since the converse inequality is evident as $D \subset D_k$, $A_k \subset D_k$ (see (3.5)). To this end we define the function $u : D_k \rightarrow [0, 1]$ as

\[
u(z) := \begin{cases} 
\omega(z, A_k, D), & z \in D, \\
0, & z \in D_k \setminus D.
\end{cases}
\]

Using [9] we see that

\[
\lim_{z \rightarrow \zeta} \omega(z, A_k, D) = 0, \quad \zeta \in A_k.
\]

Consequently, $u$ is subharmonic on $D_k$, and $u \leq \omega(z, A_k, D_k)$, which implies (3.7).

Next, we like to check the hypothesis of Lemma 3.1 in the present context. Therefore, fix a nonempty open connected set $U \subset \mathbb{C}^2$ such that $U \varsubsetneq \hat{W}^o$ and $U \cap \hat{W}^o \neq \emptyset$. We have to show that there is a pseudoconvex open set $\Omega$ in $\mathbb{C}^2$ such that $\hat{W} \subset \Omega$ and $U \cap \partial \Omega \neq \emptyset$. In fact, we will choose $\Omega$ as either

\begin{equation}
\Omega = \Omega_k := \hat{X}^o(A_k, B_k; D_k, G_k) \quad \text{or} \quad \Omega = D_k \times G_k
\end{equation}

for some $k \geq 1$. In virtue of (3.3), (3.5), and (3.6)-(3.8), we obtain that $\Omega_k$ is pseudoconvex and $\hat{W} \subset \Omega_k$. Therefore, we only have to check that $U \cap \partial \Omega \neq \emptyset$. Several cases are to be considered.

**Case I:** \((U \cap \partial \hat{W}^o) \cap (D \times G) \neq \emptyset\).

Fix a point \((z_0, w_0) \in \left(U \cap \partial \hat{W}^o\right) \cap (D \times G)\). Using the continuity of the harmonic measure, we see that

\[
\partial \hat{W}^o \cap (D \times G) = \{(z, w) \in D \times G : \omega(z, A, D) + \omega(w, B, G) = 1\}.
\]

In particular, $\omega(z_0, A, D) + \omega(w_0, B, G) = 1$.

Next, we fix two points \((z_1, w_1) \in U \cap \hat{W}^o \cap (D \times G)\) and \((z_2, w_2) \in (U \cap (D \times G)) \setminus \hat{W}^o\) close to \((z_0, w_0)\) such that

\[
1 < \omega(z_2, A, D) + \omega(w_2, B, G),
\]

and

\[
\gamma := \{t(z_1, w_1) + (1 - t)(z_2, w_2) : t \in [0, 1]\} \subset U \cap (D \times G).
\]

Hence,

\[
\omega(z_1, A, D) + \omega(w_1, B, G) < 1 < \omega(z_2, A, D) + \omega(w_2, B, G).
\]

In virtue of (3.3) and (3.6), the monotonically decreasing sequence $(\omega(\cdot, A_k, D_k))_{k=1}^\infty$ (resp. $(\omega(\cdot, B_k, G_k))_{k=1}^\infty$) of continuous functions converges uniformly to $\omega(\cdot, A, D)$.
Therefore, \( z \in \gamma \) such that \( \omega(z_3, A_k, D_k) + \omega(w_2, B_k, G_k) = 1 \). Hence, \( U \cap \partial \Omega \neq \emptyset \). So case I is complete.

**Case II:** \( (U \cap \partial \hat{W}^o) \cap (D \times G) = \emptyset \).

First we show that

\[
\partial \hat{W}^o \cap \left( \left( \partial D \setminus \overline{A} \right) \times \left( \partial G \setminus \overline{B} \right) \right) = \emptyset.
\]

Indeed, suppose the contrary in order to get a contradiction and let \((z_0, w_0)\) be an arbitrary point in the left hand side of (3.9). Since \((z_0, w_0) \in (\partial D \setminus \overline{A}) \times (\partial G \setminus \overline{B})\), we have (see [12])

\[
\lim_{z \to z_0} \omega(z, A, D) = 1, \quad \lim_{w \to w_0} \omega(w, B, G) = 1,
\]

which proves that \((z_0, w_0) \notin \overline{W}^o\). Hence, we obtain the desired contradiction, and the proof of (3.9) is complete.

Using (3.9), the obvious inclusion \((U \cap \partial \hat{W}^o) \subset \overline{D \times G}\) and the assumption of Case II, we see that there are two subcases to consider.

**Subcase** \((U \cap \partial \hat{W}^o) \cap (\overline{A} \times \overline{G}) \neq \emptyset\). Let \((z_0, w_0)\) be a point in this intersection. Since \(D\) is Jordan-curve-like at all points of \(\overline{A}\), we may choose a sufficiently large \(k_0\) such that

\[
\left\{ (z, w_0) : |z - z_0| < \frac{4}{k_0} \right\} \subseteq U,
\]

and that all points of \(T := \left\{ z \in \partial D : |z - z_0| < \frac{4}{k_0} \right\} \) are either of the same type 1 or of the same type 2. There are two subsubcases to consider.

**Subsubcase** The points of \(T\) are of type 1. Since \(z_0 \in \overline{A}\), there exist \(z_1 \in A\) and \(l \in I_{k_0}\) such that \(|z_0 - z_1| < \frac{1}{k_0}\) and \(z_1 \in A_{k_0}\). Using (a) above we can choose \(t_1 \in (0, 1)\) such that \(z_1 = a_{k_0}(t_1)\). Now setting \(z_2 := \Gamma_{k_0}(t_1)\), we see that \(z_2 \in \Gamma_{k_0}\) and \(|z_2 - z_1| < \frac{1}{k_0}\). This, coupled with (3.10), gives that \((z_2, w_0) \in U\).

Now we choose \(\Omega := D_k \times G_k\) in (3.8). It remains to show that \((z_2, w_0) \in \partial \Omega\). Since \(\Gamma_{k_0}\) is of type 1, it follows from (3.5) that \(\Gamma_{k_0} \subset \partial D_{k_0}\). Hence, \(z_2 \in \partial D_{k_0}\). Therefore, \((z_2, w_0) \in \partial \Omega\).

In summary, we have shown that \((z_2, w_0) \in U \cap \partial \Omega\). Hence, this subsubcase is completed.

**Subsubcase** The points of \(T\) are of type 2.

Since \(z_0 \in \overline{A}\), there exists \(z_1 \in A\) such that \(|z_0 - z_1| < \frac{1}{k_0}\). For every \(k \geq 1\), let \(l_k\) be the unique index in \(I_k\) such that \(z_1 \in A_{k_l}\). Since \((A_k)_{k=1}^{\infty}\) is a decreasing sequence of relatively open subsets of \(\partial D\), so is the sequence \((A_{kl})_{k=1}^{\infty}\). Put \(H := \bigcap_{k=1}^{\infty} A_{kl}\). We like to show that

\[
H = \{z_1\}.
\]
Indeed, suppose in order to reach a contradiction that $H \neq \{z_1\}$. Then the interior of $H$ (in the relative topology of $\partial D$) contains an open (Jordan) curve $C$. On the other hand, we know from (3.11) that $A_k \setminus A_0$ and $A_0 \setminus A$ is of zero length, and it is easy to see that $H \subset A_0$. Consequently, $C \setminus A$ is of zero length. Hence, by Definition 2 all points of $A \cap C$ are extendible points of $A$, this contradicts the hypotheses of Theorem A. Hence, (3.11) has been proved.

In virtue of (3.11) we may find a sufficiently large $k$ such that $\sup_{x,y \in A_{klk}} |x - y| < \frac{1}{k_0}$ and $A_{klk}$ is an open (Jordan) curve. Let $z_2$ be an end-point of $A_{klk}$. So by the choice of $k$, we have $|z_1 - z_2| < \frac{2}{k_0}$. This, coupled with the previous estimate $|z_0 - z_1| < \frac{1}{k_0}$ and (3.10), gives that $(z_2, w_0) \in U$.

On the other hand, since $z_2$ is an end-point of $A_{klk}$, it follows from (3.3) that $z_2 \in \partial D_k$. Now we choose $\Omega := D_k \times G_k$ in (3.8). Consequently, $(z_2, w_0) \in \partial \Omega$.

Summarizing, we obtain $(z_2, w_0) \in U \cap \partial \Omega$. Hence, this subcase is completed.

Subcase $(U \cap \partial \hat{W}^o) \cap (\hat{D} \times \hat{B}) \neq \emptyset$. It is similar to the previous subcase.

Hence, the proof of the theorem is complete. \qed

Proof of Theorem B. Using this and hypothesis ($\mathcal{H}$) and applying Proposition 3.7 in [8], we see that, for every $k \geq 1$,

\begin{equation}
\omega(\zeta, A, D_k) = \lim_{z \to \zeta, z \in D_k} \omega(z, A, D_k) = \lim_{z \to \zeta, z \in D} \omega(z, A, D) = 0, \quad \zeta \in A,
\end{equation}

\begin{equation}
\omega(\eta, B, G_k) = \lim_{w \to \eta, w \in G_k} \omega(w, B, G_k) = \lim_{w \to \eta, w \in G} \omega(w, B, G) = 0, \quad \eta \in B.
\end{equation}

Consequently, arguing as in the proof (3.6) one can show that
\begin{equation}
\omega(z, A, D) = \omega(z, A, D_k), \quad z \in D, \quad \omega(w, B, G) = \omega(w, B, G_k), \quad w \in G.
\end{equation}

Now we are able to check the hypothesis of Lemma 3.1 in the present context. To this end, fix a nonempty connected open set $U \subset C^{n+m}$ such that $U \not\subset \hat{W}^o$ and $U \cap \hat{W}^o \neq \emptyset$. We will choose the pseudoconvex open set $\Omega$ as
\begin{equation}
\text{either } \Omega = \Omega_k := \hat{\mathcal{K}}^o(A, B; D_k, G_k) \text{ or } \Omega = D_k \times G_k,
\end{equation}

and need to verify that $\hat{W} \subset \Omega$ and $U \cap \partial \Omega \neq \emptyset$. Observe that $\Omega$ is pseudoconvex which follows from the definition of $\hat{\mathcal{K}}^o(A, B; D_k, G_k)$ and the fact that $D_k, G_k$ are pseudoconvex. On the other hand, using (3.12)–(3.14) and the fact that $A \subset D_k, B \subset G_k, D \subset D_k, G \subset G_k$, we easily see that $\hat{W} \subset \Omega$. Therefore, it remains to show that $U \cap \partial \Omega \neq \emptyset$. To do this let $(z_0, w_0)$ be an arbitrary point in $U \cap \partial \hat{W}^o$. There are several cases to consider.

Case I: $(z_0, w_0) \in D \times G$.

Then there is a sequence $((z_j, w_j))_{j=1}^\infty \subset \hat{W}^o$ such that
\[
\lim_{j \to \infty} (z_j, w_j) = (z_0, w_0), \quad \omega(z_0, A, D) + \omega(w_0, B, G) \geq 1.
\]

This, combined with (3.13)–(3.14), implies that $(z_0, w_0) \in \partial \Omega_k$ for arbitrary $k \geq 1$. Hence, choosing $\Omega := \Omega_k$ for any $k \geq 1$ in (3.14) case I is completed.

Case II: $(z_0, w_0) \in \partial D \times \hat{C}$.

Two subcases are to be considered.
Subcase \((z_0, w_0) \in A \times \overline{G}\). Recall from Definition 3 that \(D_k \cap \overline{D} = D \cup A\) and \(\bigcap_{k=1}^\infty D_k = D \cup A\). Consequently, there are an integer \(k\) and a point \(z_1 \in \partial D_k\) such that \((z_1, w_0) \in U\). Now put \(\Omega := D_k \times G_k\). Then we have that \((z_1, w_0) \in U \cap \partial \Omega\).

Subcase \((z_0, w_0) \in (\partial D \setminus A) \times \overline{G}\). Since \(z_0 \in \partial D \setminus A\), it follows from Definition 3 that \(z_0 \in \partial D_k\) for \(k \geq 1\). Now choosing \(\Omega := D_1 \times G_1\), we see that \((z_0, w_0) \in U \cap \partial \Omega\).

This completes case II.

Case III: \((z_0, w_0) \in \overline{D} \times \partial G\). It is similar to case II.

Hence, the proof of Theorem B is finished. \(\square\)

Proof of Proposition C. First we consider the case \(|J| = 1\). Using (i) and (iii) we may find an open neighborhood \(U\) of \(A\) in \(\mathbb{C}^n\) which is relatively compact and a \(C^2\) smooth strictly plurisubharmonic defining function \(\rho\) on \(U\) such that \(D \cap U = \{z \in U : \rho (z) < 0\}\). For every \(z \in U \cap \partial D\), let \(v_z\) be the outward normal vector \(v_z\) of \(D\) at \(z\). Using the smoothness in (iii) one may find a sufficiently small number \(t_0 > 0\) such that the map \(\Theta : (U \cap \partial D) \times [0, t_0) \rightarrow \mathbb{C}^n\), given by

\[
\Theta(z, t) := z + v_z,
\]

is diffeomorphic onto the set \(V \subset \mathbb{C}^n\). Geometrically, \(V\) is a tube with the base \(U \cap \partial D\) and with the height \(t_0\).

Since \(\Theta\left((U \cap \partial D) \setminus A\right) \times [0, t_0)\) is relatively closed in \(V\), there is a smooth function \(\lambda\) defined on \(V\) such that \(0 \leq \lambda \leq 1\) on \(V\) and

\[
\{z \in V : \lambda(z) = 0\} = \Theta\left(((U \cap \partial D) \setminus A) \times [0, t_0)\right).
\]

For every \(k \geq 1\), we define an open set

\[
D_k := D \cup \left\{z \in U : \rho(z) - \frac{1}{k}\lambda(z) < 0\right\}.
\]

Then using the above properties of \(\rho\), \(\Theta\) and \(\lambda\), it can be checked that there exists a sufficiently large \(N > 0\) such that \((D_{N+k})_{k=1}^\infty\) satisfies Definition 3. Hence, the pair \((A, D)\) satisfies hypothesis \((\mathcal{H})\).

The general case (i.e. \(|J|\) is at most countable) may be done in the same way using (ii) and the fact that an increasing union of pseudoconvex open sets is again pseudoconvex. \(\square\)

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Peter Pflug, Carl von Ossietzky Universität Oldenburg, Fachbereich Mathematik, Postfach 2503, D–26111, Oldenburg, Germany
E-mail address: pflug@mathematik.uni-oldenburg.de

Việt-Anh Nguyên, Mathematics Section, The Abdus Salam international centre for theoretical physics, Strada costiera, 11, 34014 Trieste, Italy
E-mail address: vnguyen0@ictp.trieste.it