ON THE DIFFERENCE EQUATION OF THE
POINCARÉ TYPE

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Dedicated to the memory of Professor A.O. Gelfond.

Abstract. In the paper are proved theorems, which amplify the results of my paper "On the difference equation of Poincaré type (Part 3)", Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2004, 09, 1 – 34.

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§0. Foreword.

In [77]: was proved the following

Theorem 7. Let \( s \in \mathbb{N} - 1, n \in \mathbb{N} \),

\[
a_i^{-} \in \mathbb{C}, \quad a_i(\nu) \in \mathbb{C},
\]

(1) \( a_n(\nu) = 1, \quad a_i(\nu) - a_i^{-} = O(1/(\nu + 1)) \)

for \( \nu \in \mathbb{N} - 1 \) and \( i = 0, \ldots, n \). Let us consider the following difference equation

(2) \( \sum_{k=0}^{n} a_k(\nu)y(\nu + k) = 0, \)

where \( \nu \in \mathbb{N} - 1 \).
For \( m \in \mathbb{N} \) let \( V_m \) denotes the linear over \( \mathbb{C} \) space of solutions \( y = y(\nu) \) of the equation

\[
\sum_{k=0}^{n} a_k(\nu)y(\nu + k) = 0,
\]

where \( \nu \in m + \mathbb{N} - 1 \). Let the absolute values of all the roots of the characteristic polynomial

\[
T(z) = \sum_{k=0}^{n} a_k^{-1} z^k
\]

are among the numbers \( \{\rho_i: 1 \leq i \leq 1 + s\} \) such that \( \rho_{s+1} = 0 \) and \( \rho_j < \rho_i \) for \( 1 \leq i < j \leq s + 1 \). Let \( e_i \) and \( k_i \) denote respectively the sum and the maximum of the multiplicities of those roots, whose absolute value is equal to the number \( \rho_i \), where \( i = 1, \ldots, s + 1 \), and let \( k^* = k_{s+1} \). We suppose that, if \( s > 0 \), then

\[
e_i > 0
\]

for \( i = 1 \ldots, s \). For given \( y = y(\nu) \) in \( \mathbb{C}^{m-1+N} \), let

\[
\omega_{n,y}(\nu) = \max(|y(\nu)|, \ldots, |y(\nu + n - 1)|).
\]

Then there exist \( A > 0 \), \( m \in \mathbb{N} \), \( \alpha(\nu) > 0 \) with \( \nu \in m + \mathbb{N} - 1 \) and the subspaces \( V_{m,1}^\vee, \ldots, V_{m,s+1}^\vee \) such that

\[
\lim_{\nu \to \infty} \alpha(\nu) = 0,
\]

\[
V_m = V_{m,1}^\vee \oplus \ldots \oplus V_{m,s+1}^\vee, \quad \dim_{\mathbb{C}}(V_{m,i}^\vee) = e_i, \quad 1 \leq i \leq s + 1,
\]

and, if \( y \in V_{m,\theta}^\vee \) for some \( \theta \in \{1, \ldots, s\} \), then

\[
\exp(-A(\ln(\nu) + \nu^{1-1/k_\theta}))(\rho_\theta)^\nu \omega_n(y)(m) \leq \omega_{n,y}(\nu)
\]

for \( \nu \in m + \mathbb{N} - 1 \); moreover, the spaces

\[
V_{m,j}^\wedge = V_{m,j}^\vee \oplus \ldots \oplus V_{m,s+1}^\vee,
\]

where \( j = 1 \ldots, s + 1 \), and, if \( s \geq 1 \) natural projections \( \pi_j \),

\[
V_{m,j}^\wedge \mapsto V_{m,j}^\vee,
\]

where \( j = 1 \ldots, s \), have the following properties:

if \( y \in V_{m,\theta}^\wedge \) for some \( \theta \in \{1, \ldots, s\} \), then

\[
\omega_{n,y}(\nu) \leq \exp(A(\ln(\nu) + \nu^{1-1/k_\theta}))((\rho_\theta)^\nu \omega_{n,y}(m),
\]
\[(\omega_{n\theta(y)}(m) - \alpha(\nu)\omega_{n,y}(m))(\rho_\theta)^\nu
\exp(-A(\ln(\nu) + \nu^{1-1/k_\theta})) \leq \omega_{n,y}(\nu),\]

where \(\nu \in m + \mathbb{N} - 1;\) if
\[(9) \quad k^* > 0,\]
and \(y \in V_{m,s+1}^\land (= V_{m,s+1}^\lor),\) then
\[(10) \quad |y(\nu)| \leq (A/\nu)^{\nu/k^*}\omega_{n,y}(m),\]

where \(\nu \in m + \mathbb{N} - 1.\)

Remark 1. It follows from the Theorem 7 that the space \(V_{m,\theta}^\land,\)
where \(\theta = 1, \ldots, s + 1,\) does not depend from the construction and
is defined uniquelly by means of the equality
\[
V_{m,\theta}^\land = \{y \in V_m: \limsup |y(\nu)|^{1/\nu} \leq \rho_\theta\}.
\]

The presence of unkown \(\alpha(\nu)\) (even tending to zero) in (8) constrict the
possibilities of the application of this Theorem. Of course, in view of (6),
in the case \(e_{s+1} = 0, \theta = s\) this \(\alpha(\nu)\) cannot play devil with us, because it
vanishes then, but such happy case (see, for example, [75]-[76]) is rather an
exepction from the rule. One may attempt to estimate the specified value but
it would be better to get rid from it at all. With this goal I prove here the
following

**Theorem 10.** Let are fulfilled all the conditions of the Theorem 7. Let
further \(V\) be an arbitrary linear subspace of \(V_m\) such that
\[
V \cap V_{m,\theta+1} = \{0\},
\]

where \(\theta \in \{1 \ldots, s\}.\)

Then for this \(V\) there exists a constant \(A^* = A^*(V) > 0\) such that
\[(11) \quad \exp(-A^*(\ln(\nu) + \nu^{1-1/k}))\omega_n(y)(m) \leq \omega_{n,y}(\nu),\]

where \(y \in V, k = \max(k_1, \ldots, k_s)\) and \(\nu \in m + \mathbb{N} - 1.\)

First I prove the following

**Theorem 8.** Let are fulfilled all the conditions of the Theorem 7. Let
further
\[
V_{m,j}^* = V_{m,1}^\lor \oplus \ldots \oplus V_{m,j}^\lor,
\]
where \(j = 1 \ldots, s + 1,\) (and \(V_{m,1}^* = V_{m,1}^\lor,\)) Then for \(V = V_{m,\theta}^*\) with \(\theta \in \{1,\ldots, s\},\) holds the assertion of the Theorem 10.

Then I prove

**Theorem 9.** Let for some \(\theta \in \{1 \ldots, s\}\) is given a linear map \(\xi_\theta\) of the
space \(V_{m,\theta}^*\) into \(V_{m,\theta+1}^\land.\) Let \(I_\theta^*\) is the identity map \(V_{m,\theta}^* \to V_{m,\theta}^*\) Then for
\[
V = (I_\theta^* + \xi_\theta)(V_{m,\theta}^*)
\]
holds the assertion of the Theorem 10.
In the section 4 I prove the Theorem 10.
And in the section 5 I discuss, what will take place, if instead (1) the following conditions hold:
\[
\lim_{\nu \to \infty} a_i(\nu) = a_i^\sim,
\]
where \(i = 0, \ldots, n\),
\[
a_n(\nu) = 1,
\]
where \(\nu \in \mathbb{N} - 1\).

§1. Some preparatory results.

**Lemma 1.** Let \(a \in \mathbb{N}, b \in \mathbb{N} - 1 + a, C > 0\) Then
\[
0 < \sum_{\kappa = a}^{b} \ln(1 + C/\kappa) \leq \ln(1 + C/a) + b \ln(1 + C/b) - a \ln(1 + C/a) + C \ln((b + C)/(a + C)).
\]

**Proof.** See the proof of the Lemma 1 in [77].

**Corollary.** If \(a \in \mathbb{N}, b \in \mathbb{N} + a - 1, b < 2a, C > 0\), then
\[
\sum_{\kappa = a}^{b} \ln(1 + C/\kappa) \leq 3C.
\]

**Proof.** See the Proof of the Corollary of the Lemma 1 in [77].

**Lemma 2.** ([51], Lemma 2, [44], Lemma 2, [72], Lemma 8.) Let \(A \in \text{Mat}_n(\mathbb{C})\) an let \(k\) is a maximal order of its Jordan blocks. Then there exists a constant \(\gamma^*(A) > 0\) with the following properties:

for any \(\varepsilon > 0\) there exists a norm \(p_{A,\varepsilon}\) on \(\mathbb{C}^n\) such that
\[
p_{A,\varepsilon} \leq \gamma^*(A) \left(\max(1, 1/\varepsilon)\right)^{k-1}h,
\]
\[
h \leq \gamma^*(A) \left(\max(1, \varepsilon)\right)^{k-1}p_{A,\varepsilon},
\]
\[
(p_{A,\varepsilon})^\sim \leq \gamma^*(A)^2 \left(\max(1, 1/\varepsilon)\right)^{k-1}h^\sim,
\]
\[
h^\sim \leq \gamma^*(A)^2 \left(\max(1, \varepsilon)\right)^{k-1}(p_{A,\varepsilon})^\sim,
\]
\[
\|A\|_{sp} \leq (p_{A,\varepsilon})^\sim \leq \|A\|_{sp} + (\text{sign}(k - 1))\varepsilon,
\]
where \(\|A\|_{sp}\) denotes the maximum of the absolute values of eigenvalues of the matrix \(A\). If, moreover,
\[
det(A) \neq 0, \quad \|A^{-1}\|_{sp}^{-1} > (\text{sign}(k - 1))\varepsilon,
\]
then

\[(p_{A,\epsilon})^<=(A^{-1}) \leq \left(\|A^{-1}\|_{sp}^{-1} - (\text{sign}(k-1))\epsilon\right)^{-1} =
\]

\[|\text{sign}(k-1)| \|A^{-1}\|_{sp} + \epsilon \|A^{-1}\|_{sp} \left(\|A^{-1}\|_{sp}^{-1} - (\text{sign}(k-1))\epsilon\right)^{-1}.
\]

**Proof.** See the proof of the Lemma 8 in [72]. ■

**Corollary.** If all the eigenvalues of the matrix \(A\) are simple, then

\[(p_{A,\epsilon})^//=\|A\|_{sp}.
\]

If, moreover,

\[(\text{det}(A) \neq 0),
\]

then

\[(p_{A,\epsilon})^<=(A^{-1}) = \left(\|A^{-1}\|_{sp}^{-1}\right)^{-1}.
\]

**Proof.** See the proof of the Corollary of the Lemma 8 in [72]. ■

**Lemma 3.** Let all the conditions of the Theorem 7 are fulfilled, and let

\[k = \max(k_1, \ldots, k_s).
\]

Then there exist \(A > 0, m \in \mathbb{N}\) such that

\[\omega_{n,y}(\nu) \leq \exp(A(ln(\nu) + \nu^{1-1/k}))(\rho_1)^\nu\omega_n(y)(m)
\]

for any \(y \in V_m\) and \(\nu \in m + \mathbb{N} - 1\).

If, moreover, \(k^* = 0\), then there exist \(A > 0, m \in \mathbb{N}\) such that

\[\exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_s)^\nu\omega_{n,y}(m) \leq \omega_{n,y}(\nu)
\]

for any \(y \in V_m\) and \(\nu \in m + \mathbb{N} - 1\).

**Proof.** Since \(V_m = V_{m_1}^h\), it follows that the inequality (7) holds with 
\(\theta = 1\) for any \(y \in V_m\). For the full proof of the Lemma let us make some not large changes in the proof of the Lemma 2 in [77].

The condition \(k^* = 0\) implies the inequality

\[a_0^* = T(0) \neq 0,
\]

and for \(p\) in Theorem 6 of the paper [72] the equality \(p = n\). Let

\[A(\nu) = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ldots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-a_0(\nu) & -a_1(\nu) & -a_2(\nu) & \ldots & -a_{n-1}(\nu)
\end{pmatrix},
\]
where $\nu \in \mathbb{N} - 1$, and let

\[
A^- = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-a_0^- & -a_1^- & -a_2^- & \ldots & -a_{n-1}^-.
\end{pmatrix}
\] (29)

Let $\lambda_j$, where $j = 1, \ldots, r$, is the sequence of all the mutually distinct roots of the polynomial (4) and $k^*_j$ is the multiplicity of the root $\lambda_j$. Then

\[ k = \sup \{ k^*_j : j = 1, \ldots, r \}. \]

Clearly,

\[ \sum_{j=0}^{r} k^*_j = n, \]

\[ \rho_s \leq |\lambda_j| \leq \rho_1 = \|A\|_{sp} < R = h^- (A^-) + 1, \]

where $j = 1, \ldots, r$. In view of (1), there exists $C_1 > 0$ such that

\[ h^- (A(\nu) - A^-) \leq C_1 / (\nu + 1). \] (30)

Therefore, according to the Lemma 2, for any $\varepsilon > 0$ the following inequalities holds

\[ p_{A^-, \varepsilon}(A(\nu) - A^-) \leq \]

\[ (\gamma^* (A^-))^2 (\max(\varepsilon, 1/\varepsilon))^{k-1} h^- (A(\nu) - A^-) \leq \]

\[ (\gamma^* (A^-))^2 (\max(\varepsilon, 1/\varepsilon))^{k-1} C_1 / (\nu + 1), \] (31)

\[ p_{A^-, \varepsilon}(A(\nu)) \leq \rho_1 + \text{sign}(k - 1) \varepsilon + \]

\[ (\gamma^* (A^-))^2 (\max(\varepsilon, 1/\varepsilon))^{k-1} C_1 / (\nu + 1) \leq \]

\[ (\rho_1 + (\text{sign}(k - 1)) \varepsilon) \left( 1 + (\gamma^* (A^-))^2 (\max(\varepsilon, 1/\varepsilon))^{k-1} \frac{C_1}{\rho_1 (\nu + 1)} \right), \] (32)

\[ \ln(p_{A^-, \varepsilon}(A(\nu))) \leq \ln(\rho_1) + \ln(1 + (\text{sign}(k - 1))C_{1,2}) + \]

\[ \ln \left( 1 + (\max(\varepsilon, 1/\varepsilon))^{k-1} C_{1,2} / (\nu + 1) \right), \] (33)

where

\[ C_{1,1} = 1 / \rho_1, \quad C_{1,2} = (\gamma^* (A^-))^2 C_1 / \rho_1 \]

and $\nu \in \mathbb{N}$.

We consider first the case $k > 1$. For given $\nu \in \mathbb{N} + 1$ we take $d \in \mathbb{N}$ and $a_i \in [1, \nu] \cap \mathbb{N}$, where $i = 0, \ldots, d$, in such a way that

\[ a_0 = 1, \quad a_d = \nu, \quad a_{i-1} < a_i \leq 2a_{i-1}, \] (34)
where \( i = 1, \ldots , d \), and

\[
(35) \quad d \leq \frac{\ln(\kappa)}{\ln(2)} + 1.
\]

According to the Corollary of the Lemma 1 and (34) – (35),

\[
(36) \quad \sum_{\kappa=a_{i-1}}^{a_i-1} \ln(p_{A_{a_i,\varepsilon}}(A(\kappa))) \leq (a_i - a_{i-1}) \ln(\rho_1) + (a_i - a_{i-1}) \ln(1 + C_{1,1}) + \\
\sum_{\kappa=a_{i-1}}^{a_i-1} \ln \left( 1 + C_{1,2}(\max(\varepsilon, 1/\varepsilon))^{k-1}/(\kappa + 1) \right) \leq (a_i - a_{i-1}) \ln(\rho_1) + (a_i - a_{i-1}) \ln(1 + \varepsilon/\rho_1) + 3C_{1,2}(\max(\varepsilon, 1/\varepsilon))^{k-1},
\]

where \( i = 1, \ldots , d \). We take now in (36) \( \varepsilon = \varepsilon_i = (a_{i-1})^{-1/k} \). Then we obtain the inequality

\[
(37) \quad \ln(p_{A_{a_i,\varepsilon}}(A(\kappa))) \leq (a_i - a_{i-1}) \ln(\rho_1) + O(2^{i-1}(1-1/k))
\]

and, in view of (18),

\[
(38) \quad \ln(h_{a_i-a_{i-1}}(A(a_i - \kappa))) \leq \\
\ln((\gamma(A)^2)(\max(\varepsilon, 1/\varepsilon))^{k-1}) + (a_i - a_{i-1}) \ln(\rho_1) + O(2^{i-1}(1-1/k)) \leq 2 \ln(\gamma(A)^2) + (i-1)(1-1/k) \ln(2) + (a_i - a_{i-1}) \ln(\rho_1) + O(2^{i-1}(1-1/k)) = (i-1)(1-1/k) \ln(2) + (a_i - a_{i-1}) \ln(\rho_1) + O(2^{i-1}(1-1/k)),
\]

where \( i = 1, \ldots , d \). Consequently,

\[
(39) \quad \ln \left( h_{a_i-a_{i-1}}(A(\nu - \kappa)) \right) = \\
\ln \left( h_{a_i-a_{i-1}} \left( \prod_{\kappa=1}^{a_i-a_{i-1}} A(a_i - \kappa) \right) \right) = \\
\nu \ln(\rho_1) + O(\nu^{(1-1/k)}),
\]
where \( \nu \in \mathbb{N} \).

If \( k = 1 \), then, according to (33),
\[
\ln(p_{\tilde{A}^\sim,1}(A(\nu))) \leq \ln(\rho_1) + C_{1.2}/(\nu + 1),
\]
\[
\ln \left( h^\sim \left( \prod_{\kappa=1}^{\nu-1} A(\nu-\kappa) \right) \right) \leq \ln \left( (\gamma^*(\tilde{A}^\sim))^2 p_{\tilde{A}^\sim,1} \left( \prod_{\kappa=1}^{\nu-1} A(\nu-\kappa) \right) \right) \leq \nu \ln(\rho_1) + O(\ln(e\nu)),
\]
where \( \nu \in \mathbb{N} \). In view of (39) – (40),
\[
\ln \left( h^\sim \left( \prod_{\kappa=1}^{\nu-1} A(\nu-\kappa) \right) \right) = \nu \ln(\rho_1) + O((\nu + 1)^{(1-1/k)}) + O(\ln(\nu)),
\]
where \( \nu \in \mathbb{N} \).

As in Section 3 of [72], let \( K \) denotes one of the fields \( \mathbb{R} \) or \( \mathbb{C} \) and \( L \) denotes a linear normed space over \( K \) with norm \( p = p(x) \). If \( L = K^n \), we fix as \( p = p(x) \), where \( x \in K^n \), the maximum of the absolute values of coordinates of the element \( x \) in the standard basis, i.e.
\[
p(x) = h(x) = \sup(\{|x_1|, \ldots, |x_n|\}),
\]
where
\[
x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}.
\]

If \( L \) is a Banach space with the norm \( p \), then \( K \)-algebra of all the linear continuous operators acting in \( L \) will be denoted by \( \mathfrak{M}(L) \), and the norm on \( \mathfrak{M}(L) \), associated with the norm \( p \) will be denoted by \( p^\sim \). So,
\[
p^\sim(A) = \sup(\{p(AX) : X \in L, p(X) \leq 1\}).
\]
It is well known that the associated with \( h \) norm on \( Mat_n(\mathbb{C}) \) is defined as follows
\[
h^\sim(A) = \sup \left( \left\{ \sum_{k=1}^{n} |a_{i,k}| : i = 1, \ldots, n \right\} \right),
\]
where \( A = (a_{i,k}) \in Mat_n(\mathbb{C}) \). The norms \( h \) and \( h^\sim \) coincide respectively with the norms \( q_{\infty} \) and \( q^\sim_{\infty} \) considered in section 6 of the paper [68].

Let \( m \in \mathbb{N} \), and let \( E_m(L) \) be the set \( L^{m-1+N} \) of all the maps of the set \( m-1+N \) into \( L \). The set \( E_m(L) \) is a linear space over \( K \), where the multiplication of the elements by the number from \( K \) and addition of the elements are defined coordinate-wise. The subspace of \( E_m(L) \) composed by
all the constant maps is isomorphic to the space \( L \), and we identify this subspace with \( L \).

As in Section 4 of [72], for any \( y \in E_m(\mathbb{C}) \) and \( n \in \mathbb{N} \) let \( Y_{n,y} \) and \( Y^\#_{n,y} \) denote the elements in the space \( E_m(\mathbb{C}^n) \), which are determined respectively by means the following equalities:

\[
Y_{n,y}(\nu) = \begin{pmatrix} y(\nu) \\ \vdots \\ y(\nu + n - 1) \end{pmatrix},
\]

\[
Y^\#_{n,y}(\nu) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y(\nu) \end{pmatrix},
\]

where \( \nu \in m - 1 + \mathbb{N} \). Let is fixed \( m \in \mathbb{N} \). If \( y = y(\nu) \) is a solution of the equation (3) for \( \nu \in \mathbb{N} + m - 1 \), then

\[
Y_{n,y}(\nu) = \left( \prod_{\kappa=1}^{\nu-m} A(\nu - \kappa) \right) Y_{n,y}(m),
\]

where \( \nu \in \mathbb{N} \), and, in view of (41),

\[
\omega_{n,y}(\nu) = h(Y_{n,y}(\nu)) \leq \exp \left( O(1) \left( \nu^{1-1/k} + \ln(\nu) \right) \right) \rho_1^\nu \omega_{n,y}(m),
\]

where \( \nu \in \mathbb{N} + m - 1 \). So, with \( m = 1 \) the asserted by the Lemma the upper estimate (25) of the value \( \omega_{n,y}(\nu) \rho_1^{-\nu} = \omega_{n,y}(\nu)(\rho_1)^{-\nu} \) is obtained. We shall take now

\[
\varepsilon \in (0, \rho_s/2).
\]

Then, in view of (21),

\[
1/\rho_s \leq (p_{A,\varepsilon})^\sim (A^{-1}) \leq 1/\rho_s + 2\text{sign}(k-1)\varepsilon/\rho_s^2 \leq 2/\rho_s.
\]

Let is fixed

\[
m \in \max([2/\rho_s]^k], [2C_1h^\sim ((A^\sim)^{-1})]) + \mathbb{N}.
\]

Then

\[
h^\sim ((A^\sim)^{-1}(A(\nu) - A^\sim)) \leq h^\sim ((A^\sim)^{-1}) C_1/(\nu + 1) \leq h^\sim ((A^\sim)^{-1}) C_1/([2C_1h^\sim ((A^\sim)^{-1})] + 1) \leq 1/2,
\]
if \( \nu \in m - 1 + \mathbb{N} \), the matrix \( E + ((A^\sim)^{-1}(A(\nu) - A^\sim)) \) is invertible,
\[
h^\sim \left( (E + ((A^\sim)^{-1}(A(\nu) - A^\sim)))^{-1} \right) \leq 2,
\]
if \( \nu \in m - 1 + \mathbb{N} \), there exists the matrix
\[
(A(\nu))^{-1} = (E + ((A^\sim)^{-1}(A(\nu) - A^\sim)))^{-1} (A^\sim)^{-1},
\]
if \( \nu \in m - 1 + \mathbb{N} \), moreover,
\[
h^\sim ((A(\nu))^{-1}) \leq h^\sim \left( (E + ((A^\sim)^{-1}(A(\nu) - A^\sim)))^{-1} \right) h^\sim ((A^\sim)^{-1}) \leq 2h^\sim ((A^\sim)^{-1})
\]
and, finally,
\[
h^\sim ((A(\nu))^{-1} - (A^\sim)^{-1}) = h^\sim ((A(\nu))^{-1} (A^\sim - A(\nu)) (A^\sim)^{-1}) \leq h^\sim ((A(\nu))^{-1}) h^\sim ((A^\sim - A(\nu))) h^\sim ((A^\sim)^{-1}) \leq C_2/(\nu + 1),
\]
where
\[
C_2 = (h^\sim ((A^\sim)^{-1}))^2 C_1
\]
and \( \nu \in m - 1 + \mathbb{N} \). Therefore
\[
(51) \quad p^\sim_{A^\sim, \varepsilon}((A(\nu))^{-1} - (A^\sim)^{-1}) \leq (\gamma^* (A^\sim))^2 (\max(\varepsilon, 1/\varepsilon))^{k-1} h^\sim ((A(\nu))^{-1} - (A^\sim)^{-1}) \leq (\gamma^* (A^\sim))^2 (\max(\varepsilon, 1/\varepsilon))^{k-1} C_2/(\nu + 1),
\]
where \( \nu \in m - 1 + \mathbb{N} \). In view of (51) and (21),
\[
(52) \quad p^\sim_{A^\sim, \varepsilon}((A(\nu))^{-1}) \leq 1/\rho_s + 2(\text{sign}(k-1)) \varepsilon / \rho_s^2 \leq 2/\rho_s + (\gamma^* (A^\sim))^2 (\max(\varepsilon, 1/\varepsilon))^{k-1} C_2/(\nu + 1) \leq (1/\rho_s + 2(\text{sign}(k-1)) \varepsilon / \rho_s^2) (1 + (\gamma^* (A^\sim))^2 (\max(\varepsilon, 1/\varepsilon))^{k-1} \rho_s C_2 \rho_s / (\nu + 1)),
\]
\[
(53) \quad \ln(p^\sim_{A^\sim, \varepsilon}((A(\nu))^{-1})) \leq \ln(1/\rho_s) + \ln(1 + (\text{sign}(k-1)) C_{2,1} \varepsilon) + \ln \left( 1 + (\max(\varepsilon, 1/\varepsilon))^{k-1} C_{2,2} / (\nu + 1) \right),
\]
where
\[
C_{2,1} = 2/\rho_s, \quad C_{2,2} = (\gamma^* (A^\sim))^2 C_2 \rho_s
\]
and \( \nu \in \mathbb{N} - 1 + m \). We take \( \nu \in m - 1 + \mathbb{N} \).

We consider first the case \( k > 1 \) again now. For given \( \nu \in \mathbb{N} + m \) we take \( d \in \mathbb{N} \) and \( a_i \in [m, \nu] \cap \mathbb{N} \), where \( i = 0, \ldots, d \), in such a way that
\[
(54) \quad a_0 = m, \quad a_d = \nu, \quad a_{i-1} < a_i \leq 2a_{i-1} \leq m2^i,
\]
where \( i = 1, \ldots, d \), and
\[
(55) \quad d \leq \frac{\ln(\nu)}{\ln(2)} + 1.
\]
According to the Corollary of the Lemma 1 and (54) – (55),

\[
\sum_{\kappa=a_{i-1}}^{a_i-1} \ln(p_{A^*,A}^{\kappa-1}(A(\kappa)^{-1})) \leq \\
(a_i - a_{i-1}) \ln(1/\rho_s) + (a_i - a_{i-1}) \ln(1 + C_{2,1}\varepsilon) + \\
\sum_{\kappa=a_{i-1}}^{a_i-1} \ln \left( 1 + \left( \max(\varepsilon, 1/\varepsilon) \right)^{k-1} C_{2,2}/(\kappa + 1) \right) \leq \\
(a_i - a_{i-1}) \ln(1/\rho_s) + \ln(1 + C_{2,1}\varepsilon) + 3(\gamma^*(A^*))^2(\max(\varepsilon, 1/\varepsilon))^{k-1} C_{2,2},
\]

where \( i = 1, \ldots, d \). We take now in (56) \( \varepsilon = \varepsilon_i = (a_{i-1})^{-1/k} \). Then, in view of (49),

\[ a_{i-1} \geq m > (2/\rho_s)^{1/k}, (a_{i-1})^{1/k} > 2/\rho_s, \varepsilon_i < \rho_s/2, \]

where \( i = 1, \ldots, d \), and therefore (47) and (56) hold. Consequently,

\[
\sum_{\kappa=a_{i-1}}^{a_i-1} \ln(p_{A^*,A}^{\kappa-1}(A(\kappa)^{-1})) \leq \\
(a_i - a_{i-1}) \ln(1/\rho_s) + (C_{2,1} + 3C_{2,2})(a_{i-1})^{1-1/k} = \\
(a_i - a_{i-1}) \ln(1/\rho_s) + O(2^{(i-1)(1-1/k)}),
\]

where \( i = 1, \ldots, d \). Therefore

\[
\ln(p_{A^*,A}^{\kappa-1}
\left( \prod_{\kappa=a_{i-1}}^{a_i-1} (A(\kappa)^{-1}) \right) \leq \\
(a_i - a_{i-1}) \ln(\rho_1) + O(2^{(i-1)(1-1/k)})
\]

and, in view of (18),

\[
\ln(h^{\sim}
\left( \prod_{\kappa=1}^{a_{i-1}-a_i} (A(\kappa)^{-1}) \right) \leq \\
\ln((\gamma^*(A^*))^2(\max(\varepsilon_i, 1/\varepsilon_i))^{k-1}) + (a_i - a_{i-1}) \ln(\rho_1) + O(2^{(i-1)(1-1/k)}) \leq \\
2 \ln(\gamma^*(A^*)) + (i - 1)(1 - 1/k) \ln(2) + (a_i - a_{i-1}) \ln(\rho_1) + O(2^{(i-1)(1-1/k)}) = \\
(i - 1)(1 - 1/k) \ln(2) + (a_i - a_{i-1}) \ln(\rho_1) + O(2^{(i-1)(1-1/k)})
\]

where \( i = 1, \ldots, d \). Consequently,

\[
\ln \left( h^{\sim}
\left( \prod_{\kappa=1}^{a_{i-1}-a_i} (A(\kappa)^{-1}) \right) \right) = \\
\ln \left( h^{\sim}
\left( \prod_{i=1}^{d} \prod_{\kappa=1}^{a_i-a_{i-1}} (A(\kappa)^{-1}) \right) \right) = \\
\nu \ln(1/\rho_s) + O(\nu^{(1-1/k)}),
\]
where $\nu \in \mathbb{N} + m$.

If $k = 1$, then, according to (53),

$$\ln(p_{n-1}(A(\nu))) \leq \ln(1/\rho_s) + C_{2,2}/(\nu + 1),$$

(61)

$$\ln \left( h^\sim \left( \prod_{\kappa=m}^{\nu-1} (A(\kappa))^{-1} \right) \right) \leq$$

$$\ln \left( (\gamma^*(A^\sim))^2 p_{n-1}^\sim \left( \prod_{\kappa=m}^{\nu-1} (A(\kappa))^{-1} \right) \right) \leq$$

$$\nu \ln(1/\rho_s) + O(\ln(\nu)).$$

where $\nu \in \mathbb{N}$. In view of (60) – (61),

(62)

$$\ln \left( h^\sim \left( \prod_{\kappa=m}^{\nu-1} A(\nu - \kappa) \right) \right) =$$

$$= \nu \ln(1/\rho_s) + O(\nu^{1-1/k}) + O(\ln(\nu)),$$

where $\nu \in \mathbb{N} + m$. Since

$$Y_{n,y}(m) = \left( \prod_{\kappa=1}^{\nu-m} A(\nu - \kappa) \right)^{-1} Y_{n,y}(\nu) =$$

$$\left( \prod_{\kappa=m}^{\nu-1} (A(\kappa))^{-1} \right) Y_{n,y}(\nu),$$

it follows that

(63)

$$\omega_{n,y}(m) = h \left( Y_{n,y}(m) \right) \leq$$

$$h^\sim \left( \prod_{\kappa=m}^{\nu-1} (A(\kappa))^{-1} \right) h \left( Y_{n,y}(\nu) \right) =$$

$$\exp \left( O(1) \left( \nu^{1-1/k} + \ln(\nu) \right) \right) (\rho_1)^{-\nu} \omega_{n,y}(\nu),$$

where $\nu \in \mathbb{N} + m$. So, with $m$ from (49) the asserted by the Lemma the lower estimate of the value $\omega_{n,y}(\nu) (\rho_s)^{-\nu}$ is obtained.

**Remark 1.** In the case $s = 1, k^* = 0$ the assertions of the Lemma and Theorem 7 coincide.

**Lemma 4 (Perron’s decomposition Lemma, [22], Hilfsatz 3).** Let the characteristic polynomial (4) is represented as product

(64)

$$T(z) = T_1(z)T_2(z),$$

where

(65)

$$T_1(z) = \sum_{\alpha=0}^{p} b^\sim_{\alpha} z^\alpha, T_2(z) = \sum_{\beta=0}^{q} u^\sim_{\beta} z^\beta,$$
with $b_p = u_q = a_q = 1$ and absolute value of each root of $T_1(z)$ is greater than the absolute value of each root of $T_2(z)$.

Then there exist $m \in \mathbb{N}$, $a_\alpha(\nu) \in \mathbb{C}$, $\alpha = 0, \ldots, p, \nu \in \mathbb{N} + m - 1$, and $a_\beta(\nu) \in \mathbb{C}$, $\beta = 0, \ldots, q, \nu \in \mathbb{N} + m - 1$, such that

\begin{equation}
\lim_{\nu \to \infty} b_\alpha(\nu) = b_\alpha^\sim, \alpha = 0, \ldots, p, b_p(\nu) = 1, b_0(\nu) \neq 0,
\end{equation}

\begin{equation}
\lim_{\nu \to \infty} u_\beta(\nu) = u_\beta^\sim, \beta = 0, \ldots, q, u_q(\nu) = 1,
\end{equation}

where $\nu \in \mathbb{N} + m - 1$, and, moreover, the equation (3) is equivalent to the equation

\begin{equation}
\sum_{\alpha=0}^{p} b_\alpha(\nu)y(\nu + \alpha) = r(\nu),
\end{equation}

where $\nu \in \mathbb{N} - 1 + m$ and $r(\nu)$ satisfies to the equation

\begin{equation}
\sum_{\beta=0}^{q} u_\beta(\nu)r(\nu + \beta) = 0
\end{equation}

with $\nu \in \mathbb{N} - 1 + m$.

**Lemma 5.** Let all the conditions of the Perron’s decomposition Lemma are fulfilled and $a_k(\nu) - a_k^\sim = O(1/(\nu + 1)), k = 0, \ldots, n$, when $\nu \to \infty$.

Then for $b_\alpha(\nu)$ with $\alpha = 0, \ldots, p$, and $u_\beta(\nu)$ with $\beta = 0, \ldots, q$, from the assertion of the Perron’s decomposition Lemma the following conditions are fulfilled (cf. with (66) and (67)):

\begin{equation}
b_\alpha(\nu) - b_\alpha^\sim = O(1/\nu), \alpha = 0, \ldots, p, b_0(\nu) = 1, b_q(\nu) \neq 0,
\end{equation}

\begin{equation}u_\beta(\nu) - u_\beta^\sim = O(1/\nu), \beta = 0, \ldots, q, u_0(\nu) = 1,
\end{equation}

where $\nu \in \mathbb{N} - 1 + m$.

**Proof.** The Lemma is direct corollary of the Theorem 5 in [68]. □

**§2. Proof of the theorem 8.**

We use below the notations of the section 3 in [77]. Let $K$ be one of the fields $\mathbb{R}$ or $\mathbb{C}$. Let $L$ be a linear normed space over $K$ with a norm $p(x)$. In the case $L = K^n$ we fix as $p(x)$, wehre $x \in K^n$, the maximum of the absolute values of coordinates of $x$ in the standard basis, i.e.

\begin{equation}
p(x) = h(x) = \sup(|x_1|, \ldots, |x_n|),
\end{equation}

where

\[ x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}. \]

If $L$ is a Banach space with the norm $p$, then $K$–algebra of all the linear continuous operators acting in $L$ will be denoted by $\mathfrak{M}^\wedge(L)$, and the norm on $\mathfrak{M}^\wedge(L)$, associated with the norm $p$ will be denoted by $p^\sim$. So,

\[ p^\sim(A) = \sup\{|p(AX)| : X \in L, p(X) \leq 1\}. \]
It is well known that the associated with $h$ norm on $Mat_n(\mathbb{C})$ is defined as follows

$$h^\sim(A) = \sup \left( \left\{ \sum_{k=1}^{n} |a_{i,j}| : i = 1, \ldots, n \right\} \right),$$

where $A = (a_{i,k}) \in Mat_n(\mathbb{C})$. The norms $h$ and $h^\sim$ coincide respectively with the norms $q_\infty$ and $q^\sim_\infty$ considered in [68], section 6. Let $m \in \mathbb{N} - 1$, and we denote by $E_m(L)$ the set $L^{m-1+N}$ of all the maps of the set $m - 1 + N$ into $L$. The set $E_m(L)$ is a linear space over $K$, where the multiplication of the elements by the number from $K$ and addition of the elements is defined coordinate-wise. The subspace of $E_m(L)$ composed by all the constant maps is isomorphic to $L$, and we identify this subspace with $L$.

We denote by $\mathcal{M}^\vee(L)$ the space of all the $K-$linear maps of the space $L$ in $L$. If $\phi \in \mathcal{M}^\vee(L)$ and $\psi \in \mathcal{M}^\vee(L)$, then $\phi \circ \psi$ denotes the composition of operators $\phi$ and $\psi$, so that $(\phi \circ \psi)f = \phi((\psi f))$ for each $f \in L$. For $x \in E_m(L)$ let

$$p_{m,\infty}(x) = \sup\{p(x(\nu)) : \nu \in m - 1 + \mathbb{N}\}.$$

Let further

$$E_{m,\infty}(L) = \{x \in E_m(L) : p_{m,\infty}(x) \neq \infty\},$$
$$E_{m,0}(L) = \{x \in E_m(L) : \lim_{\nu \to \infty} p(x(\nu)) = 0\},$$
$$E_m^\sim(L) = L + E_{m,0}(L).$$

Clearly, the space $E_m^\sim(L)$ consists of all the $y \in E_m(L)$, for which there exists

$$\lim(y) = \lim_{\nu \to \infty}(y(\nu)).$$

Let $m \in \mathbb{N} - 1$, $\mu \in m - 1 + \mathbb{N}$ and let $r_{m,\mu}$ be the operator of restriction of the elements $y \in E_m(L)$ on the set $m - 1 + \mathbb{N}$. Clearly, the map $r_{m,\mu}$ is an epimorphism of the space $E_m(L)$ onto the space $E_\mu(L)$. If $L$ is a $K$-algebra, then $E_m(L)$ is a $K$-algebra, where the multiplication and addition of the elements is defined coordinate-wise; so, in this case $r_{m,\mu}$ is an epimorphism of $K$-algebra $E_m(L)$ onto $K$-algebra $E_\mu(L)$. If $L$ be an algebra with unity, let $L^*$ denotes the group of all its invertible elements. Then

$$(L^*)^{m-1+N} \subset L^{m-1+N};$$

we denote below $(L^*)^{m-1+N}$ by $E_m(L^*)$. Clearly,

$$E_m(L^*) = (E_m(L))^*.$$
the space $E_m(L)$ acts also $K$-linear operator $\nabla \in \mathfrak{M}^\vee (L)$, which turns any element $y \in E_m(L)$ in the $\nabla y \in E_m(L)$ such that

$$(\nabla y)(\nu) = y(\nu + 1)$$

for any $\nu \in m - 1 + \mathbb{N}$. Let us consider the subring $\mathfrak{A}_m(L)$ of the ring $\mathfrak{M}^\vee (L)$ generated by the operator $\nabla$ and by all the operators $\mu_a$, where $a \in E_m(L)$. Clearly,

$$(70) \quad \mu_a \circ \nabla^r \circ \mu_b \circ \nabla^s = \mu_{a \nabla^r b} \circ \nabla^{r+s},$$

where $\{r, s\} \subset \mathbb{N} - 1$, $\{a, b\} \subset E_m(L)$. For each $\alpha \in \mathfrak{A}_m(L) \setminus \{0_{\mathfrak{A}_m(L)}\}$ are uniquely defined the number $\text{deg}(\alpha)$ and representation of $\alpha$ in the form

$$(71) \quad \alpha = \sum_{k=0}^{\text{deg}(\alpha)} \mu_{a_k} \circ \nabla^k,$$

where $a_k \in E_m(L)$ for $k = 0, \ldots, \text{deg}(\alpha)$ and $a_{\text{deg}(\alpha)} \neq 0_{E_m(L)}$. Clearly, (71) may be rewritten in the form

$$(72) \quad \alpha = \sum_{k=0}^{\infty} \mu_{a_k} \circ \nabla^k,$$

where $a_k = 0_{E_m(L)}$ for $k \in \text{deg}(\alpha) + \mathbb{N}$. It follows from (70) that $\mathfrak{A}_m(L)$ is a graduated algebra, and if

$$(73) \quad \beta = \sum_{r=0}^{p} \mu_{b_r} \circ \nabla^r \in \mathfrak{A}_m(L),$$

$$(74) \quad \gamma = \sum_{s=0}^{q} \mu_{c_s} \circ \nabla^s \in \mathfrak{A}_m(L),$$

then

$$(75) \quad \beta \gamma = \sum_{k=0}^{p+q} \sum_{\begin{array}{c} 0 \leq r \leq p \\ 0 \leq s \leq q \\ r+s=k \end{array}} \mu_{b_r \nabla^r c_s} \circ \nabla^{r+s};$$

clearly, $\text{deg}(\beta \gamma) = \text{deg}(\beta) + \text{deg}(\gamma)$, if $b_p(\nu)^r c_q(\nu + p)$ is different from 0 at least for one $\nu \in m - 1 + \mathbb{N}$. Let $\mathfrak{A}_m^\rightarrow (L)$ be the ring generated by the operator $\nabla$ and by all the operators $\mu_a$, where $a \in E_m^\rightarrow (L)$. Since $\nabla a \in E_m^\rightarrow (L)$, if $a \in E_m^\rightarrow (L)$, it follows, in view of (70), that $\mathfrak{A}_m^\rightarrow (L)$ is a graduated subalgebra $\mathfrak{A}_m^\rightarrow (L)$ of the algebra $\mathfrak{A}_m(L)$, each $\alpha \in \mathfrak{A}_m(L) \setminus \{0_{\mathfrak{A}_m(L)}\}$ admits a representation in the form (71) with $a_k \in E_m^\rightarrow (L)$ for $k = 0, \ldots, \text{deg}(\alpha)$ and $a_{\text{deg}(\alpha)} \neq 0_{E_m(L)}$; to each such $\alpha$ corresponds the limit operator

$$(76) \quad \lim(\alpha) = \sum_{k=0}^{\text{deg}(\alpha)} \mu_{\lim(a_k)} \circ \nabla^k;$$
and polynomial

\[ P(\alpha, z) = \sum_{k=0}^{\text{deg}(\alpha)} \lim(a_k)z^k \in L[z]. \]  

If \( \alpha = 0_{A_m(L)} \), then we put

\[ \lim(\alpha) = 0_{A_m(L)}, P(\alpha, z) = 0_{L[z]}. \]

The equality (70) shows that the map

\[ \alpha \rightarrow P(\alpha, z) \]

is an epimorphism of the algebra \( A_m(L) \) on the algebra \( L[z] \) (if the algebra \( L \) is noncommutative, then we can treat the algebra \( L[z] \) as a semigroup ring of the semigroup \((\mathbb{N} - 1, +)\) over the algebra \( L \)). We note that, if \( \alpha \in \mathfrak{A}_m(\mathbb{C}) \), then, clearly, \( \text{Ker}(\alpha) \) coincides with the linear space of all the solutions of the equation (3), and, moreover, if \( \alpha \in \mathfrak{A}_m(\mathbb{C}) \), then the corresponding to \( \alpha \) equation (3) is an equation of the Poincaré type and \( P(\alpha, z) \) is its characteristic polynomial.

Let \( L \) be an algebra with unity. The set of all \( \alpha \in \mathfrak{A}_m(L) \setminus \{0_{A_m(L)}\} \), which have the representation (71) with \( a_{\text{deg}(\alpha)} \in E_m(L^*) \) will be denoted further by \( \mathfrak{A}_m(L)^\circ \). The set of all the the elements \( \alpha \in \mathfrak{A}_m(L) \setminus \{0_{A_m(L)}\} \), which have the representation (71) with \( a_{\text{deg}(\alpha)} = 1_{E_m(L)} \) will be denoted further by \( \mathfrak{A}_m(L)^\wedge \). The set of all the \( \alpha \in \mathfrak{A}_m(L) \setminus \{0_{A_m(L)}\} \), which have the representation (71) with \( a_0 \in E_m(L^*) \), will be denoted further by \( \mathfrak{A}_m(L)^\land \).

Let further

\[ \mathfrak{A}_m(L)^\land = \mathfrak{A}_m(L)^\circ \cap \mathfrak{A}_m(L)^\wedge, \quad \mathfrak{A}_m(L)^\land = \mathfrak{A}_m(L)^\circ \cap \mathfrak{A}_m(L)^\wedge, \]

\[ \mathfrak{A}_m(L)^\land = \mathfrak{A}_m(L)^\circ \cap \mathfrak{A}_m(L)^\wedge, \quad \mathfrak{A}_m(L)^\land = \mathfrak{A}_m(L)^\circ \cap \mathfrak{A}_m(L)^\wedge, \]

Clearly, \( \mathfrak{A}_m(L)^\circ \) consists of epimorphisms of the space \( E_m(L) \) onto \( E_m(L) \). The above map \( r_{m,\mu} \) induces epimorphism \( r_{m,\mu}^\circ \) of the algebra \( \mathfrak{A}_m(L) \) on the algebra \( \mathfrak{A}_m(L) \) defined as follows:

if \( \alpha \in \mathfrak{A}_m(L) \),

\[ \alpha = \sum_{k=0}^{n} \mu_{a_k} \circ \nabla^k, \]

then

\[ r_{m,\mu}^\circ (\alpha) = \sum_{k=0}^{n} \mu_{r_{m,\mu}(a_k)} \circ \nabla^k, \]

where the operator \( \nabla \) in (79) acts in \( E_m(L) \) and the operator \( \nabla \) in (80) acts in \( E_m(L) \). Clearly, \( r_{m,\mu}^\circ \) surjectively maps

\[ \mathfrak{A}_m(L)^\circ \text{ onto } \mathfrak{A}_m(L)^\circ, \mathfrak{A}_m(L)^\wedge \text{ onto } \mathfrak{A}_m(L)^\wedge, \]

\[ \mathfrak{A}_m(L)^\land \text{ onto } \mathfrak{A}_m(L)^\land, \mathfrak{A}_m(L)^\land \text{ onto } \mathfrak{A}_m(L)^\land, \]

\[ \mathfrak{A}_m(L)^\wedge \text{ onto } \mathfrak{A}_m(L)^\wedge, \mathfrak{A}_m(L)^\land \text{ onto } \mathfrak{A}_m(L)^\land, \]

\[ \mathfrak{A}_m(L)^\land \text{ onto } \mathfrak{A}_m(L)^\land, \mathfrak{A}_m(L)^\land \text{ onto } \mathfrak{A}_m(L)^\land, \]

\[ \mathfrak{A}_m(L)^\land \text{ onto } \mathfrak{A}_m(L)^\land, \mathfrak{A}_m(L)^\land \text{ onto } \mathfrak{A}_m(L)^\land, \]

\[ \mathfrak{A}_m(L)^\land \text{ onto } \mathfrak{A}_m(L)^\land, \mathfrak{A}_m(L)^\land \text{ onto } \mathfrak{A}_m(L)^\land, \]
\( \mathbb{A}_m(L)^\wedge \) onto \( \mathbb{A}_\mu(L)^\wedge \), \( \mathbb{A}_m(L)^\rightarrow \) onto \( \mathbb{A}_\mu(L)^\rightarrow \), \\
\( \mathbb{A}_m(L)^\rightarrow \) onto \( \mathbb{A}_\mu(L)^\rightarrow \), \( \mathbb{A}_m(L)^\wedge \) onto \( \mathbb{A}_\mu(L)^\wedge \),\n\( \mathbb{A}_m(L)^\wedge \) onto \( \mathbb{A}_\mu(L)^\wedge \). Since the diagram

\[
\begin{array}{ccc}
E_m(L) & \xrightarrow{r_{m,\mu}} & E_\mu(L) \\
\alpha \downarrow & & \downarrow r_{m,\mu}(\alpha) \\
E_m(L) & \xrightarrow{r_{m,\mu}} & E_\mu(L)
\end{array}
\]

is commutative and therefore

\[(81)\]

\[r_{m,\mu}\alpha = r_{m,\mu}(\alpha)r_{m,\mu},\]

it follows that \( r_{m,\mu} \) surjectively maps \( \text{Ker}(r_{m,\mu}\alpha) \) onto

\[\text{Ker}(r_{m,\mu}(\alpha)) \supset r_{m,\mu}\text{Ker}(\alpha).\]

**Lemma 6.** If \( \mu \in m - 1 + \mathbb{N} \) and \( \alpha \in \mathbb{A}_m(L)^\wedge \), then the operator \( \alpha \)
bijectively maps \( \text{Ker}(r_{m,\mu}) \) onto \( \text{Ker}(r_{m,\mu}) \).

**Proof.** Proof is given in [72], Lemma 3.

**Corollary 1.** Let \( \mu \in m - 1 + \mathbb{N} \) and let \( \alpha \in \mathbb{A}_m(L)^\wedge \). If

\[g \in E_m(L), \ x \in E_\mu(L), \ m \leq \mu, \ \alpha \in \mathbb{A}_m(L)^\wedge,\]

\[r_{m,\mu}(g) = (r_{m,\mu}(\alpha))(x),\]

then there exists a unique \( y \in E_m(L) \) such that

\[\alpha(y) = g, \ r_{m,\mu}(y) = x;\]

**Proof.** Proof is given in [72], Corollary 1 to the Lemma 3.

**Corollary 2.** Let \( \mu \in m - 1 + \mathbb{N} \) and \( \alpha \in \mathbb{A}_m(L)^\wedge \). Then and \( r_{m,\mu} \)
bijectively maps \( \text{Ker}(\alpha) \) onto \( \text{Ker}(r_{0,\mu}(\alpha)) = r_{m,\mu}(\text{Ker}(\alpha)) \).

**Proof.** Proof is given in [72], Corollary 2 to the Lemma 3.

If for the equation (2) are fulfilled the conditions (1) then

\[(82)\]

\[a_k = (a_k(0), a_k(1), \ldots, a_k(\nu), \ldots) \in E_0^\rightarrow(\mathbb{C}),\]

where \( k = 0, \ldots, n \). Moreover, \( a_n = 1_{E_0(\mathbb{C})} \), for \( \alpha \) in (79) \( \text{Ker}(\alpha) \) coincides with the linear over \( \mathbb{C} \) space of all the solutions of the equation (2), polynomial (4) is equal to the polynomial \( P(\alpha, z) = P(r_{0,m}(\alpha), z) \), where \( m \in \mathbb{N} \), and the set \( \text{Ker}(r_{0,m}(\alpha)) \) coincides with linear over \( \mathbb{C} \) space \( V_m \) of all the solutions of the equation (3).

Let \( \nu \) be the element in \( E_{0,0} \), for which

\[\nu(\nu) = \frac{1}{\nu + 1},\]

where \( \nu \in \mathbb{N} - 1 \). Clearly, \( r_{0,m}(\nu)E_{m,\infty}(\mathbb{C}) \subset E_{m,0}(\mathbb{C}) \) for any \( m \in \mathbb{N} - 1 \). Let

\[E_{m,0}(L) = r_{0,m}(\nu)E_{m,\infty}(L), \ E^\nu_m(L) = L + E^\nu_m(L).\]
Let us consider the ring $\mathfrak{A}_m(\mathbb{L})$ generated by the operator $\nabla$ and by all the operators $\mu_a$, where $a \in E_m(\mathbb{L})^\triangleright$. The Lemma 5 may be reformulated as follows:

**Lemma 7.** Let $\alpha \in \mathfrak{A}_0^\triangleright(\mathbb{C}) \cap \mathfrak{A}_0^\blacktriangleleft(\mathbb{C})$, and $P(\alpha, z)$ coincides with the polynomial $T(z)$ in (4) and (64).

Then there exist $m \in \mathbb{N}$ and representation of the operator $r_{0,m}^\triangleright(\alpha)$ in the form

$$r_{0,m}^\triangleright(\alpha) = \eta \beta$$

such that

$$\eta \in \mathfrak{A}_m^\triangleright(\mathbb{C}) \cap \mathfrak{A}_m^\blacktriangleleft(\mathbb{C}), \ deg(\eta) = q$$

$$\beta \in \mathfrak{A}_m^\triangleright(\mathbb{C}) \cap \mathfrak{A}_m^\blacktriangleleft(\mathbb{C}), \ deg(\beta) = p = n - q,$$

and the polynomials $P(\beta, z), P(\eta, z)$ coincide respectively with the polynomials $T_1(z), T_2(z)$ in (65).

**Lemma 8.** Let are fulfilled all the conditions of the Theorem 7. Let $\alpha \in \mathfrak{A}_0^\triangleright(\mathbb{C}) \cap \mathfrak{A}_0^\blacktriangleleft(\mathbb{C})$ corresponds to the equation (2), i.e. (with $m = 0$) (79) holds

$$\alpha = \sum_{k=0}^{n} \mu_{a_k} \circ \nabla^k,$$

where $a_n = 1_{E(\mathbb{C})}$ and

$$a_k = (a_k(0), a_k(1), \ldots, a_k(\nu), \ldots) \in E_0^\triangleright(\mathbb{C}),$$

for $k = 0, \ldots, n$.

Let the characteristic polynomial (4) is represented as product

$$P(\alpha, z) = T(z) = \prod_{i=1}^{s+1} T_i(z),$$

where

$$T_i(z) = \sum_{\alpha=0}^{e_i} b_{i,\alpha}^\sim z^\alpha,$$

with $b_{i,e_i} = a_n^\sim = 1$ and absolute value of each root of the polynomial $T_i(z)$ is equal to $\rho_i$.

Then there exist $m \in \mathbb{N}$ and representation of the operator $r_{0,m}^\triangleright(\alpha)$ in the form

$$r_{0,m}^\triangleright(\alpha) = \prod_{i=0}^{s} \beta_{s+1-i}$$

such that

$$\beta_i \in \mathfrak{A}_m^\triangleright(\mathbb{C}) \cap \mathfrak{A}_m^\blacktriangleleft(\mathbb{C}), \ deg(\beta_i) = e_i,$$
where \( i = 1, \ldots, s + 1 \) and

\[
P(\beta_i, z) = T_i(z),
\]

where \( i = 1, \ldots, s \).

**Proof.** The assertion of the Lemma may be obtained by means of the sequentially applying of the Lemma 7. ■

Let

\[
(\beta_1^\ast, \theta) = \prod_{i=0}^{\theta-1} \beta_{\theta-i}, \beta_\theta^\wedge = \prod_{i=s+1-\theta}^{s} \beta_{s+1-i},
\]

where \( \theta = 1, \ldots s + 1 \). In view of (refeq:2cdj),

\[
(94) \quad r_{0,m}^\wedge (\alpha) = \beta_\theta^\wedge \beta_\theta^\ast,
\]

\( \theta = 1, \ldots s \).

Let \( C > 0, n \in \mathbb{N} \),

\[
(95) \quad w_{C,n}(\nu) = \left( \frac{C}{\nu} \right)^{\nu/n},
\]

where \( \nu \in \mathbb{N} \); let further \( \rho > 0 \) and

\[
(96) \quad v_{C,n,\rho}(\nu) = \rho^\nu \exp \left( C \left( (\nu)^{1-1/n} + \ln(\nu) \right) \right),
\]

where \( \nu \in \mathbb{N} \).

Let \( \theta_1 \in \{1, \ldots s\} \). Replacing \( m \) by some bigger \( m \) and applying to \( \beta_\theta^\wedge \) in (93) the Theorem 7, we see that there exist \( A^\wedge > 0, m \in \mathbb{N}, \alpha^\wedge(\nu) > 0 \) with \( \nu \in m + \mathbb{N} - 1 \) and the subspaces \( R_{m, \theta_1, \theta_1}^\wedge, \ldots, R_{m, \theta_1, s+1}^\wedge \) of the space \( R_m = \text{Ker}(\beta_\theta^\wedge) \) such that

\[
\lim_{\nu \to \infty} \alpha^\wedge(\nu) = 0,
\]

\[
R_m = R_{m, \theta_1, \theta_1}^\wedge \oplus \ldots \oplus R_{m, \theta_1, s+1}^\wedge, \quad \dim_C(R_{m, \theta_1,i}^\wedge) = e_i, \quad \theta_1, \leq i \leq s + 1,
\]

if

\[
(97) \quad r \in R_{m, \theta_1, \theta_1}^\wedge,
\]

for some \( \theta \in \{\theta_1, \ldots, s\} \), then

\[
(98) \quad \exp(-A^\wedge(\ln(\nu) + \nu^{1-1/k_\rho}))\omega_{q,r}(m)(\rho_\theta)^{\nu} \leq \omega_{q,r}(\nu),
\]

where \( \nu \in \mathbb{N} + m - 1 \); moreover, the spaces

\[
R_{m, \theta_1,j}^\wedge = R_{m, \theta_1,j}^\wedge \oplus \ldots \oplus R_{m, \theta_1,s+1}^\wedge,
\]

where \( j = \theta_1, \ldots, s + 1 \) and natural projections \( \pi_{\theta_1,j}^\wedge \) of the space \( R_{m, \theta_1,j}^\wedge \) onto the space \( R_{m, \theta_1,j}^\wedge \), where \( j = \theta_1, \ldots, s \), have the following properties:

\[
(99) \quad r \in R_{m, \theta_1, \theta_1}^\wedge
\]
for some \( \theta \in \{\theta_1, \ldots, s\} \), then

\begin{align}
(100) \quad \omega_{q,r}(\nu) & \leq (\rho_0)^{\nu} \exp(A^\nu (\ln(\nu) + \nu^{1-k_0})\omega_{q,r}(m), \\
(101) \quad \omega_{q,r}(\nu) & \geq (\omega_{q,\pi_{q,s}(r)}(m) - \alpha^\nu(\nu)\omega_{q,r}(m)) \times \\
& \quad (\rho_0)^{\nu} \exp(-A^\nu (\ln(\nu) + \nu^{1-k_0})),
\end{align}

where \( \nu \in \mathbb{N} + m - 1 \); if \( e_{s+1} > 0 \), and

\begin{equation}
(102) \quad r \in R^\nu_{m,\theta_1,s+1} (= R^\nu_{m,\theta_1,s+1}),
\end{equation}

then

\begin{equation}
(103) \quad |r(\nu)| \leq (A^\nu/\nu)^{r/\kappa^s} \omega_{q,r}(m),
\end{equation}

where \( \nu \in \mathbb{N} + m - 1 \).

Let

\begin{equation}
(104) \quad \beta = \beta_1^* = \beta_1, \quad \eta = \beta_2^*.
\end{equation}

Then, in view of (94), (83) holds.

**Lemma 9.** Let all the conditions of the Theorem 7 are fulfilled. For sufficient big \( m \in \mathbb{N} \) there exists the splitting monomorphism

\[ \psi_2^\wedge: \text{Ker}(\eta) \to \text{Ker}(\eta \beta) \]

with following properties:

(a) \[ \text{Ker}(\beta) = V^\wedge_{m,1}, \psi_2^\wedge(\text{Ker}(\eta)) = V^\wedge_{m,2}, \]

where \( V^\wedge_{m,1} \) and \( V^\wedge_{m,2} \) are defined in the assertion of the Theorem 7;

(b) the monomorphism \( \psi_2^\wedge \) maps isomorphically the space \( R^\wedge_{m,2,j} \), where \( j = 2, \ldots, s + 1 \), onto the space \( V^\wedge_{m,j} \);

(c) the map \( \beta_1 \psi_2^\wedge \text{edge}_2 \) coincides with identity map \( \text{Ker}(\eta) \to \text{Ker}(\eta) \);

(d) the restriction of the map \( \psi_2^\wedge \text{edge}_2 \beta_1 \) on the space \( V^\wedge_{m,2} \) coincides with the identity map \( V^\wedge_{m,2} \to V^\wedge_{m,2} \);

(e) let \( I \) be the identiti map \( V_m \to V_m \); then the natural projection \( \pi_1 \) of the space \( V^\wedge_{m,1} = V_m \) onto \( V^\wedge_{m,1} \) coincides with restriction of the map \( I - \psi_2^\wedge \beta_1 \) on the space \( V^\wedge_{m,1} \);

(f) if \( j \in [2, s] \cap \mathbb{N} \), then natural projection \( \pi_j \) of the space \( V^\wedge_{m,j} \) onto the space \( V^\wedge_{m,j} \) coincides with the restriction of the map \( \psi_2^\wedge \pi_2^\wedge \beta_1 \) on the space \( V^\wedge_{m,j} \) and the projection \( \pi_2^\wedge \beta_1 \) coincides with the restriction of the map \( \beta_1 \pi_j, \psi_2^\wedge \) on the space \( R^\wedge_{2,j} \).

**Proof.** See the proof of the Theorem 7 in [77], especially the section 2 and section 3.

**Remark 2.** I use this opportunity to make a corrections in [77]. On the page 13, third line from the bottom must stand \( \psi_i \pi_j ^\wedge \phi_m \) instead \( \psi_i \pi_j ^\wedge \); on the second line from the bottom must stand \( \pi_j ^\wedge = \phi_m \pi_j \psi_m \) instead \( \pi_j ^\wedge = \phi_m \pi_j \).

**Lemma 10.** Let all the conditions of the Lemma 9 are fulfilled. Then (for sufficient big \( m \))

\begin{equation}
(105) \quad V^\wedge_{m,\theta} = \text{Ker}(\beta_1^*),
\end{equation}
where \( \theta = 1, \ldots, s \).

**Proof.** We use induction on \( s \) For \( s = 1 \) or \( \theta = 1 \) the assertion of the Lemma directly follows from the Lemma 9.

Let \( s > 1 \) and assertion of the lemma is true for \( s - 1 \). Let \( \theta > 1 \). Since the monomorphism \( \psi^\theta_2 \) maps isomorphically the space \( R^\theta_{m,2,j} \) where \( j = 2, \ldots, \theta \), onto the space \( V^\theta_{m,j} \) and the map \( \beta_1 \psi^\theta_2 \) coincides with identity map \( \text{Ker}(\eta) \rightarrow \text{Ker}(\eta) \), it follows that the restriction of the map \( \beta_1 \) on the space \( V^\theta_{m,2} \oplus \ldots \oplus V^\theta_{m,\theta} \) is an isomorphism of \( V^\theta_{m,2} \oplus \ldots \oplus V^\theta_{m,\theta} \) onto \( R^\theta_{m,2,\theta} \).

According to the inductive hypothesis,

\[
R^\theta_{m,2,\theta} = \text{Ker}(\beta^\theta_{2,\theta}).
\]

Therefore, if \( y \in V^\theta_{m,2} \oplus \ldots \oplus V^\theta_{m,\theta} \) then \( \beta^\theta_{1,\theta}(y) = \beta^\theta_{2,\theta}(\beta_1(y)) = 0 \); moreover, since also \( V^\theta_{m,1} = \text{Ker}(\beta_1) \), it follows that \( V^\theta_{m,\theta} \subset \text{Ker}(\beta^\theta_{1,\theta}) \). Since, according to the Theorem 7,

\[
(106) \quad \dim_C(V_{m,\theta}) = \sum_{i=1}^{\theta} e_i = \dim_C(\text{Ker}(\beta^\theta_{1,\theta})) = n_\theta := \deg(\beta^\theta_{1,\theta}),
\]

it follows that (105) holds. \( \blacksquare \)

**Proof of the Theorem 8.** In view of (105), (106), we apply to \( \beta^\theta_{1,\theta} \) the Lemma 3. Since \( \theta \) plays the role of \( s \) in the lemma 3 now and

\[
\max(k_1, \ldots, k_\theta) \leq k = \max(k_1, \ldots, k_s),
\]

it follows that there exist \( A = A_\theta > 0, m \in \mathbb{N} \) such that

\[
(107) \quad \exp(-A(ln(\nu) + \nu^{1-1/k}))(\rho_s)\omega_{n_\theta,y}(m) \leq \omega_{n_\theta,y}(\nu)
\]

for any \( y \in V^\theta_{m,\theta} \). Since both the functions \( y \rightarrow \omega_{n_\theta,y}(m), y \in V^\theta_{m,\theta} \), and \( y \rightarrow \omega_{n,y}(m), y \in V^*_{m,\theta} \) are two norms on the \( n_\theta \)-dimensional space \( V^*_{m,\theta} \) there exists \( B = B_\theta > 0 \) such that

\[
\omega_{n,y}(m) \exp(-B) \leq \omega_{n_\theta,y}(m)
\]

for any \( y \in V^*_{m,\theta} \). Then

\[
(108) \quad \exp(-A(ln(\nu) + \nu^{1-1/k} - B))(\rho_\theta)\omega_{n,y}(m) \leq \omega_{n,y}(\nu)
\]

for any \( y \in V^*_{m,\theta} \). \( \blacksquare \)

### §3. Proof of the theorem 9.

**Theorem 9.** Let for some \( \theta \in \{1, \ldots, s\} \) is given a linear map \( \xi_\theta \) of the space \( V^*_{m,\theta} \) into \( V^\wedge_{m,\theta+1} \). Let \( I^*_\theta \) is the identity map \( V^*_{m,\theta} \rightarrow V^*_{m,\theta} \) Then for

\[
V = (I^*_\theta + \xi_\theta)(V^*_{m,\theta})
\]

holds the assertion of the Theorem 10.
Proof of the theorem 9. Since $V^*_{m,\theta}$ is finite-dimensional linear space over $\mathbb{C}$, it follows that the map $\xi_\theta$ is continuous. Therefore there exists $C > 0$ such that

\begin{equation}
\omega_{n,\xi_\theta(y)}(m) \leq \exp(C)\omega_{n,\xi_\theta(y)}(m)
\end{equation}

for any $y \in V^*_{m,\theta}$. According to the Theorem 7 and Theorem 8, there exist numbers $A > 0$, $m \in \mathbb{N}$, such that, if $y \in V^*_{m,\theta}$, $\nu \in m - 1 + \mathbb{N}$, then

\begin{equation}
\omega_{n,\xi_\theta(y)}(\nu) \leq \exp(A(\ln(\nu) + \nu^{1-1/k_0})) (\rho_\theta)^{\nu} \omega_{n,\xi_\theta(y)}(m)
\end{equation}

for $\theta < s$,

\begin{equation}
\omega_{n,\xi_\theta(y)}(m) = 0
\end{equation}

for $\theta = s$ and $k^* = k_{s+1} = 0$,

\begin{equation}
|\xi_\theta(y)(\nu)| \leq (A/\nu)^{\nu/k^*} \omega_{n,\xi_\theta(y)}(m)
\end{equation}

for $\theta = s$ and $k^* = k_{s+1} > 0$,

\begin{equation}
\exp(-A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} \omega_{n,y}(m) \leq \omega_{n,y}(\nu).
\end{equation}

Therefore

\begin{equation}
\omega_{n,y+\xi_\theta(y)}(\nu) \geq \omega_{n,y}(\nu) - \omega_{n,\xi_\theta(y)}(\nu) \geq \exp(-A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} \omega_{n,y}(m) -
\end{equation}

\begin{equation}
|\xi_\theta(y)(\nu)| \leq (A/\nu)^{\nu/k^*} \exp(C)\omega_{n,y}(m) = \exp(-A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} \omega_{n,y}(m) 
\end{equation}

\begin{equation}
(1 + \exp(A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} (A/\nu)^{\nu/k^*} \exp(C))
\end{equation}

for $\theta = s$ and $k^* = k_{s+1} > 0$,

\begin{equation}
\omega_{n,y+\xi_\theta(y)}(\nu) \geq \omega_{n,y}(\nu) - \omega_{n,\xi_\theta(y)}(\nu) \geq \exp(-A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} \omega_{n,y}(m) -
\end{equation}

\begin{equation}
(A/\nu)^{\nu/k^*} \exp(C)\omega_{n,y}(m) = \exp(-A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} \omega_{n,y}(m) 
\end{equation}

\begin{equation}
(1 - \exp(A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} (A/\nu)^{\nu/k^*} \exp(C))
\end{equation}

for $\theta = s$ and $k^* = k_{s+1} > 0$,

\begin{equation}
\omega_{n,y+\xi_\theta(y)}(\nu) \geq \omega_{n,y}(\nu) - \omega_{n,\xi_\theta(y)}(\nu) \geq \exp(-A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} \omega_{n,y}(m) -
\end{equation}

\begin{equation}
(A/\nu)^{\nu/k^*} \exp(C)\omega_{n,y}(m) = \exp(-A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} \omega_{n,y}(m) 
\end{equation}

\begin{equation}
(1 - \exp(A(\ln(\nu) + \nu^{1-1/k})) (\rho_\theta)^{\nu} (A/\nu)^{\nu/k^*} \exp(C))
\end{equation}

for $\theta = s$ and $k^* = k_{s+1} > 0$,
it follows from (118) that 

\[ \pi \text{ isomorphism to the isomorphism } \]

the space \( V \) with some \( \theta \).

Since (118), and let \( I \) be the identity map \( V \rightarrow V \). Then

\[ \exp(A(\ln(\nu) + \nu^{1-1/k}))(\rho) \exp(C)\omega_{n,y}(m) = \]

\[ \exp(-A(\ln(\nu) + \nu^{1-1/k}))(\rho) \omega_{n,y}(m) \times \]

\[ (1 - \exp(2A(\ln(\nu) + \nu^{1-1/k}))(\rho \nu + /\rho) \exp(C)) \]

for \( \theta < s \).

\textbf{Remark 3.} The value of \( \omega_{n,y}(m) \) in (8) may be much bigger than the value of \( \omega_{n\pi\omega_y}(m) \). Therefore we cannot (as show a simple examples) for fixed value of \( A \) in the inequality (11) to replace the linear space \( V \) of the Theorem 10 by the set \( V_m \setminus V_{m,\theta+1}^\wedge \). But in (116) – (117) \( \omega_{n,y}(m) \) is carried out the brackets, and the value in the brackets tends to 1, when \( \nu \rightarrow +\infty \), being greater than \( 1/2 \) for \( \nu \in m - 1 + N \), with sufficient big \( m \in N \), which depends only from the equation.

\section{4. Proof of the theorem 10.}

\textbf{Proof of the theorem 10.} Let \( \pi^* \) and \( \pi^\wedge \) are restrictions on \( V \) of the natural endomorphisms of the space \( V_m \) onto respectively \( V^*_{m,\theta} \) and \( V^\wedge_{m,\theta+1} \) and let \( I_0 \) be the identity map \( V \rightarrow V \). Then

\[ (118) \quad I_0 = \pi^* + \pi^\wedge \]

Since \( \text{Ker}(\pi^*) \subset V \cap V^\wedge_{m,\theta+1} = \{0\} \), it follows that \( \pi^* \) is an isomorphism of the space \( V \) onto linear subspace \( V' \) of the space \( V^*_{m,\theta} \). Let \( \tau \) be the inverse isomorphism to the isomorphism \( \pi^* \). Then \( \pi^*\tau \) is identity map \( V' \rightarrow V' \) and it follows from (118) that

\[ (119) \quad \tau = \pi^*\tau + \pi^\wedge\tau. \]

Clearly, the linear map \( \pi^\wedge\tau : V' \rightarrow V^\wedge_{m,\theta+1} \) have an extension

\[ (120) \quad \xi_\theta : V^*_{m,\theta} \rightarrow V^\wedge_{m,\theta+1}. \]

It follows from (119) that

\[ V = \tau(V') \subset (I + \xi_\theta)V^*_{m,\theta}, \]

where \( I \) is identity map \( V_{m,\theta} \rightarrow V_{m,\theta} \) and \( \xi_\theta \) is a linear map in (120).

So, the Theorem 10 is Corollary of the Theorem 9.

\section{5. The case of the general difference equation of the Poincaré type.}

If (12) and (13) hold instead of (1), then making use of the above arguments we obtain the following changes in the Theorem 7 and theorem 10. For any \( \varepsilon > 0 \) there exists a constant \( A^\wedge = A^\wedge(\varepsilon) > 0 \) such that, if \( y \in V^\wedge_{m,\theta} \) with some \( \theta \in \{1, ..., s\} \), then (instead of (7) the following inequality holds)

\[ (121) \quad \omega_{n,y}(\nu) \leq \exp(A^\wedge(\rho \exp(\varepsilon))\nu \omega_{n,y}(m), \]

where \( \nu \in m + N - 1 \); if \( y \in V^\wedge_{m,s+1} (= V^\wedge_{m,s+1}) \), then (instead of (10) the following inequality holds)

\[ (122) \quad \omega_{n,y}(\nu) \leq \exp(A^\wedge(\exp(-\varepsilon))\nu \omega_{n,y}(m), \]
where $\nu \in m + \mathbb{N} - 1$.

Let further $V$ be an arbitrary linear subspace of $V_m$ such that

$$V \cap V_{m,\theta+1} = \{0\},$$

where $\theta \in \{1, \ldots, s\}$. Then for this subspace $V$ and any $\varepsilon > 0$ there exists a constant $A^\nu = A^\nu(V, \varepsilon) > 0$ such that (instead of (11) the following inequality holds)

$$\exp(-A^\nu)(\rho_{\theta}\exp(-\varepsilon))^\nu \omega_n(y)(m) \leq \omega_{n,y}(\nu)$$

where $y \in V$ and $\nu \in m + \mathbb{N} - 1$.

**Corollary.** (See [51], Theorem 3 and [75], Lemma 16). Let as before (12) and (13) hold instead of (1). Let $V$ be a $r$-dimensional subspace of $V_m$, let

$$V \cap V_{m,s+1}^\nu = \{0\}$$

and let $\{y_1(\nu), \ldots, y_r(\nu)\}$ be a basis of the space $V$. Let

$$k_3(V) = \max\{k \in \mathbb{Z}: 1 \leq k \leq s, V \subset V_{m,k}^\nu\},$$

and

$$k_4(V) = \min\{k \in \mathbb{Z}: 1 \leq k \leq s, V \cap V_{m,k+1}^\nu = \{0\}\}.$$

For $X = (x_1, \ldots, x_r)$, $X \in \mathbb{C}^r$, let

$$h(X) = \max\{|x_1|, \ldots, |x_r|\},$$

$$y = y^\nu(X, \nu) = x_1y_1^\nu(\nu) + \ldots + x_r y_r^\nu(\nu).$$

Then for every $\varepsilon \in (0, 1)$ there exist $C_3(\varepsilon) > 0$ and $C_4(\varepsilon) > 0$ such that

$$C_3(\varepsilon)(\rho_{k_4}(1-\varepsilon))^\nu h(X) \leq \omega_{n,y}(\nu) \leq C_3(\varepsilon)(\rho_{k_4} + \varepsilon)^\nu h(X).$$

**Proof** The functions $h(X)$ and the restriction of $\omega_{n,y}(m)$ on $V$ are two norms on the $r$-dimensional over $\mathbb{C}$ linear space $V$. Therefore there exists a constant $C_5 > 0$ such that $h(X) \leq C_5 \omega_{n,y}(m)$ and $\omega_{n,y}(m) \leq C_5 h(X)$ The assertion of the Corollary directly follows from (121) and (123) now. ■

**References.**

[1] R.Apéry, Interpolation des fractions continues
et irrationalité de certaines constantes,
Bulletin de la section des sciences du C.T.H., 1981, No 3, 37 – 53;
[2] F.Beukers, A note on the irrationality of $\zeta(2)$ and $\zeta(3)$,
Bull. London Math. Soc., 1979, 11, 268 – 272;
[3] A.van der Porten, A proof that Euler missed...Apéry’s proof of the irrationality of $\zeta(3)$,
Math Intellegencer, 1979, 1, 195 – 203;
[4] W. Maier, Potenzreihen irrationalen Grenzwertes,
J.reine angew. Math., 156, 1927, 93 – 148;
[5] E.M. Nikišin, On irrationality of the values of the functions $F(x,s)$ (in Russian),
Mat.Sb. 109 (1979), 410 – 417;
English transl. in Math. USSR Sb. 37 (1980), 381 – 388;
[6] G.V. Chudnovsky, Pade approximations to the generalized hyper-geometric functions
I,J.Math.Pures Appl., 58, 1979, 445 – 476;
[7] ______________, Transcendental numbers, Number Theory,Carbondale,
L.A. Gutnik, ON THE DIFFERENCE EQUATION OF THE POINCARÉ TYPE

[8] __________, Approximations rationnelles des logarithmes de nombres rationnels, C.R. Acad. Sci. Paris, Série A, 1979, 288, 607 – 609;

[9] __________, Formules d’Hermite pour les approximants de Padé de logarithmes et de fonctions binômes, et mesures d’irrationalité, C.R. Acad. Sci. Paris, Série A, 1979, t.288, 965 – 967;

[10] __________, Un système explicite d’approximants de Padé pour les fonctions hypégéométriques généralisées, avec applications à l’arithmétique, C.R. Acad. Sci. Paris, Série A, 1979, t.288, 1001 – 1004;

[11] __________, Recurrences defining Rational Approximations to the irrational numbers, Proceedings of the Japan Academie, Ser. A, 1982, 58, 129 – 133;

[12] __________, On the method of Thue-Siegel, Annals of Mathematics, 117 (1983), 325 – 382;

[13] K. Alladi and M. Robinson, Legendre polynomials and irrationality, J. Reine Angew. Math., 1980, 318, 137 – 155;

[14] A. Dubitskas, An approximation of logarithms of some numbers, Diophantine approximations II, Moscow, 1986, 20 – 34;

[15] __________, On approximation of $\pi/\sqrt{3}$ by rational fractions, Vestnik MGU, series 1, 1987, 6, 73 – 76;

[16] S. Eckmann, Über die lineare Unabhangigkeit der Werte gewisser Reihen, Results in Mathematics, 11, 1987, 7 – 43;

[17] M. Hata, Legendre type polynomials and irrationality measures, J. Reine Angew. Math., 1990, 407, 99 – 125;

[18] A.O. Gelfond, Transcendental and algebraic numbers (in Russian), GIT-TL, Moscow, 1952;

[19] H. Bateman and A. Erdélyi, Higher transcendental functions, 1953, New-York – Toronto – London, Mc. Grow-Hill Book Company, Inc.;

[20] E.C. Titchmarsh, The Theory of Functions, 1939, Oxford University Press;

[21] E.T. Whittaker and G.N. Watson, A course of modern analysis, 1927, Cambridge University Press;

[22] O. Perron, Über die Poincaresche Differenzengleichung, Journal für die reine und angewandte mathematik, 1910, 137, 6 – 64;

[23] A.O. Gelfond, Differenzenrechnung (in Russian), 1967, Nauka, Moscow.

[24] A.O. Gelfond and I.M. Kubenskaya, On the theorem of Perron in the theory of difference equations (in Russian), IAN USSR, math. ser., 1953, 17, 2, 83 – 86.

[25] M.A. Evgrafov, New proof of the theorem of Perron (in Russian), IAN USSR, math. ser., 1953, 17, 2, 77 – 82;

[26] G.A. Frejman, On theorems of Poincaré and Perron (in Russian), UMN, 1957, 12, 3 (75), 243 – 245;

[27] N.E. Nörlund, Differenzenrechnung, Berlin, Springer Verlag, 1924;

[28] I.M. Vinogradov, Foundations of the Number Theory, (in Russian), 1952, GIT-TL;

[29] __________, J. Dieudonné, Foundations of modern analysis, Institut des Hautes Études Scientifiques, Paris, Academic Press, New York and London, 1960;

[30] CH.-J. de la Vallée Poussin, Course d’analyse infinitésimale, Russian translation by G.M. Fikhtengolts, GT-TI, 1933;

[31] H. Weyl, Algebraic theory of numbers, 1940, Russian translation by L.I. Kopejkina;

[32] G. Rhin and C. Viola, On a permutation group related to $\zeta(2)$, Acta Arithmetica 77(1996), 25 – 56;

[33] G. Rhin and C. Viola, The group structure for $\zeta(3)$, Acta Arithmetica 97(2001), 269 – 293;
[34] M. Hata. A new irrationality measure for $\zeta(3)$, Acta Arithmetica 92(2000), 47 – 57;

[35] T. Rivual. La fonction zêta de Riemann prend une infinimité valeurs irrationnelles aux entiers impairs, C.R. Acad. Sci. Paris, série 1, p. 267 – 270, 2000.

[36] K. Boll, T. Rivual. Irrationalité d’une infinité valeurs la fonction zêta aux entiers impairs, Invent. math., 146, 193 – 207 (2001);

[37] W. Zudilin. One from the numbers $\zeta(5), \zeta(7), \zeta(9), \zeta(11)$ is irrational. (in Russian) UMN 56(4) (2001), 149 – 150;

[38] W. Zudilin. Apéry’s Theorem and problems for the values of $\zeta$-function of Riemann and their $q$-analogues. (in Russian) Moscow state university, doctoral thesis, Moscow 2004;

[39] L. A. Gutnik, On the decomposition of the difference operators of Poincaré type (in Russian), VINITI, Moscow, 1992, 2468 – 92, 1 – 55;

[40] ————. On the decomposition of the difference operators of Poincaré type in Banach algebras (in Russian), VINITI, Moscow, 1992, 3443 – 92, 1 – 36;

[41] ————. On the difference equations of Poincaré type (in Russian), VINITI, Moscow 1993, 443 – B93, 1 – 41;

[42] ————. On the difference equations of Poincaré type in normed algebras (in Russian), VINITI, Moscow, 1994, 668 – B94, 1 – 44;

[43] ————. On the decomposition of the difference equations of Poincaré type (in Russian), VINITI, Moscow, 1997, 2062 – B97, 1 – 41;

[44] ————. The difference equations of Poincaré type with characteristic polynomial having roots equal to zero (in Russian), VINITI, Moscow, 1997, 2418 – 97, 1 – 20;

[45] ————. On the behavior of solutions of difference equations of Poincaré type (in Russian), VINITI, Moscow, 1997, 3384 – B97, 1 – 41;

[46] ————. On the variability of solutions of difference equations of Poincaré type (in Russian), VINITI, Moscow, 1999, 361 – B99, 1 – 9;

[47] ————. To the question of the variability of solutions of difference equations of Poincaré type (in Russian), VINITI, Moscow, 2000, 2416 – B00, 1 – 22;

[48] ————. On linear forms with coefficients in $\mathbb{N}\zeta(1 + \mathbb{N})$, Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2000, 3, 1 – 13;

[49] ————. On the Irrationality of Some Quantities Containing $\zeta(3)$ (in Russian), Uspekhi Mat. Nauk, 1979, 34, 3(207), 190;

[50] ————. On the Irrationality of Some Quantities Containing $\zeta(3)$, Eleven papers translated from the Russian, American Mathematical Society, 1988, 140, 45 - 56;

[51] ————. Linear independence over $\mathbb{Q}$ of dilogarithms at rational points (in Russian), UMN, 37 (1982), 179-180; english transl. in Russ. Math. surveys 37 (1982), 176-177;

[52] ————. On a measure of the irrationality of dilogarithms at rational points (in Russian), VINITI, 1984, 4345-84, 1 – 74;

[53] ————. To the question of the smallness of some linear forms (in Russian), VINITI, 1993, 2413-B93, 1 – 94;

[54] ————. About linear forms, whose coefficients are logarithms of algebraic numbers (in Russian), VINITI, 1995, 135-B95, 1 – 149;

[55] ————. About systems of vectors, whose coordinates are linear combinations of logarithms of algebraic numbers with algebraic coefficients (in Russian), VINITI, 1994, 3122-B94, 1 – 158;
coefficients are $A$- linear combinations
of logarithms of $A$- numbers,
VINITI, 1996, 1617-B96, pp. 1 – 23.

[56] On systems of linear forms, whose
coefficients are $A$- linear combinations
of logarithms of $A$- numbers,
VINITI, 1996, 2663-B96, pp. 1 – 18.

[57] About linear forms, whose coefficients
are $\mathbb{Q}$-proportional to the number log 2, and the values
of $\zeta(s)$ for integer $s$ (in Russian),
VINITI, 1996, 3258-B96, 1 – 70;

[58] The lower estimate for some linear forms,
coefficients of which are proportional to the values
of $\zeta(s)$ for integer $s$ (in Russian),
VINITI, 1997, 3072-B97, 1 – 77;

[59] On linear forms with coefficients in $\mathbb{N}\zeta(1 + N)$
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2000, 3, 1 – 13;

[60] On linear forms with coefficients in $\mathbb{N}\zeta(1 + N)$
(the detailed version, part 1), Max-Plank-Institut für Mathematik,
Bonn, Preprint Series, 2001, 15, 1 – 20;

[61] On linear forms with coefficients in $\mathbb{N}\zeta(1 + N)$
(the detailed version, part 2), Max-Plank-Institut für Mathematik,
Bonn, Preprint Series, 2001, 104, 1 – 36;

[62] On linear forms with coefficients in $\mathbb{N}\zeta(1 + N)$
(the detailed version, part 3), Max-Plank-Institut für Mathematik,
Bonn, Preprint Series, 2002, 57, 1 – 33;

[63] On the rank over $\mathbb{Q}$ of some real matrices (in Russian),
VINITI, 1984, 5736-84; 1 – 29;

[64] On the rank over $\mathbb{Q}$ of some real matrices,
Max-Plank-Institut für Mathematik,
Bonn, Preprint Series, 2002, 27, 1 – 32;

[65] On linear forms with coefficients in $\mathbb{N}\zeta(1 + N)$
(the detailed version, part 4), Max-Plank-Institut für Mathematik,
Bonn, Preprint Series, 2002, 142, 1 – 27;

[66] On the dimension of some linear spaces
over finite extension of $\mathbb{Q}$ (part 2),
Max-Plank-Institut für Mathematik, Bonn, Preprint Series,
2002, 107, 1 – 37;

[67] On the dimension of some linear spaces over $\mathbb{Q}$ (part 3),
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 16, 1 – 45.

[68] On the difference equation of Poincaré type (Part 1).
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 52, 1 – 44.

[69] On the dimension of some linear spaces over $\mathbb{Q}$ (part 4)
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 73, 1 – 38.

[70] On linear forms with coefficients in $\mathbb{N}\zeta(1 + N)$
(the detailed version, part 5),
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 83, 1 – 13.

[71] On linear forms with coefficients in $\mathbb{N}\zeta(1 + N)$
(the detailed version, part 6),
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 99, 1 – 33.

[72] On the difference equation of Poincaré type (Part 2).
Max-Plank-Institut für Mathematik, Bonn,
Preprint Series, 2003, 107, 1 – 25.
[73] On the asymptotic behavior of solutions of difference equation (in English). Chebyshevskij sbornik, 2003, v.4, issue 2, 142 – 153.

[74] On linear combinations of logarithms of algebraic numbers with algebraic coefficients. International conference "Diophantine analysis, uniform distributions and applications" August 25-30, 2003, Minsk, Belarus Abstracts pp. 16 – 17.

[75] On linear forms with coefficients in $\mathbb{N}\zeta(1 + N)$, Bonner Mathematishe Schriften Nr. 360, Bonn, 2003, 360.

[76] On linear forms with coefficients in $\mathbb{N}\zeta(1 + N)$ (the detailed version, part 7), Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2004, 88, 1 – 27.

[77] On the difference equation of Poincaré type (Part 3). Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2004, 09, 1 – 34.

[78] On the dimension of some linear spaces over $\mathbb{Q}$, (part 5) Max-Plank-Institut für Mathematik, Bonn, Preprint Series, 2004, 1 – 42.

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