Boundedness of solutions for the reversible system with low regularity in time

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Abstract
In the present paper, it is proved that all solutions are bounded for the reversible system
\[\ddot{x} + \sum_{l=0}^{[\frac{n}{2}]-1} b_l(t)x^{2l+1} + \sum_{i=0}^{n-1} a_i(t)x^{2i+1} = 0, \quad 0 \leq l \leq [\frac{n}{2}] - 1, \quad t \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}, \quad \text{where} \quad a_i(t) \in C^1(\mathbb{T}^1) (0 \leq i \leq [\frac{n}{2}]-1), \quad b_l(t) \in C^1(\mathbb{T}^1) (0 \leq l \leq [\frac{n-1}{2}]).\]

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1. Introduction

The boundedness of all solutions for the differential equation
\[\ddot{x} + f(x, t)\dot{x} + g(x, t) = 0, \quad x \in \mathbb{R} \tag{1.1}
\]
has been widely and deeply investigated by many authors since 1940’s. The boundedness of solutions depends heavily on the structure of (1.1).

(i) When \(f(x, t) \equiv 0\), (1.1) is a Hamiltonian system. Let \(g(x, t) = x^{2n+1} + \sum_{j=0}^{2n} P_j(t)x^{j} \), where \(P_j(t)\)’s are of period 1. It has been proved by Dieckerhoff-Zehnder in [3] that all solutions of (1.1) are bounded in \(t \in \mathbb{R}\) if \(P_j(t) \in C^\infty\). The smoothness of \(P_j(t)\)’s has been recently reduced to \(C^\gamma\) with \(0 < \gamma < 1 - \frac{1}{n}\) in [17]. See [3, 5, 14–17] for more details.

(ii) When \(f(x, t) \neq 0\), (1.1) is dissipative with more appropriate conditions. A compact absorbing domain in the phase space can be constructed such that all solutions of (1.1) always go into this domain for \(t \geq t_0\). See [4, 6, 11], for example.

(iii) When \(f(x, t)\) and \(g(x, t)\) are odd in \(x\) and even in \(t\), (1.1) is neither Hamiltonian nor dissipative. In this case, (1.1) is actually in the class of so-called reversible systems. After the Kolmogorov-Arnold-Moser (KAM) theory was established, Arnold [1, 2] and Moser [9] among others proposed the study of the existence of invariant tori for the reversible systems by KAM technique, i.e. to establish KAM theory for the reversible system. See [13] for more details.

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Let \( f(x, t) = \sum_{i=0}^{l} b_i(t)x^{2i+1}, 0 \leq l \leq \lfloor \frac{d}{2} \rfloor - 1 \), and \( g(x, t) = x^{2r+1} + \sum_{i=0}^{n-1} a_i(t)x^{2i+1} \), where \( a_i(t) \)'s and \( b_i(t) \)'s are even and of period 1. Then (1.1) is a simple reversible system, and it has non-trivial dynamical behaviors. The KAM theory for reversible systems \([2, 3]\) deals with some integrable systems with small reversible perturbations. When \( a_i(t) \equiv b_i(t) \equiv 0 \), (1.1) is indeed integrable. However, (1.1) can not be regarded as an integrable system with small reversible perturbation when \( a_i(t) \not\equiv 0, b_i(t) \not\equiv 0 \). Following \([3]\), (1.1) can be transformed into a new system which consists of an integrable system with a small reversible one around the infinity by a number of so-called involution transforms. In that direction, Liu in \([7]\) proved that all solutions are bounded for \( \dot{x} + b \ddot{x} + c x^{2r+1} = p(t) \), where \( b, c \) are positive constants and \( p(t) \) is a continuous 1-period function. This result was generalized to the more general case where \( f(x, t) = \sum_{i=0}^{l} b_i(t)x^{2i+1}, 0 \leq l \leq \lfloor \frac{d}{2} \rfloor - 1 \), and \( g(x, t) = x^{2r+1} + \sum_{i=0}^{n-1} a_i(t)x^{2i+1} \), where \( a_i(t) \in C^2 \) and \( b_i(t) \in C^2 \). Later, the smoothness of \( a_i(t) \) and \( b_i(t) \) in \([10, 12]\) were furthermore relaxed to \( a_i(t), b_i(t) \in C^{1+\text{Lip}} \). In the present paper, we relax the smoothness to \( a_i(t), b_i(t) \in C^1 \). More exactly, we have the following theorem.

**Theorem 1.1.** Consider

\[
\dot{x} + \sum_{i=0}^{l} b_i(t)x^{2i+1} \dot{x} + x^{2r+1} + \sum_{i=0}^{n-1} a_i(t)x^{2i+1} = 0, \ t \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}, \tag{1.2}
\]

where \( 0 \leq l \leq \lfloor \frac{d}{2} \rfloor - 1 \), and

- \( b_i(t) \in C^1(\mathbb{T}^1), b_i(-t) = b_i(t), 0 \leq i \leq l \),
- \( a_i(t) \in C^1(\mathbb{T}^1), \lfloor \frac{n}{2} \rfloor + 1 \leq i \leq n - 1; a_i(t) \in L^1(\mathbb{T}^1), 0 \leq i \leq \lfloor \frac{n}{2} \rfloor; a_i(-t) = a_i(t), 0 \leq i \leq n - 1 \).

Then all solutions of (1.2) are bounded, i.e. the solution \((x(t), \dot{x}(t))\) with initial values \((x(0), \dot{x}(0))\) exists for all \( t \in \mathbb{R} \) and

\[
\sup_{t \in \mathbb{R}}(|x(t)| + |\dot{x}(t)|) \leq C_{(x(0), \dot{x}(0))},
\]

where \( C_{(x(0), \dot{x}(0))} > 0 \) is a constant depending on the initial values \((x(0), \dot{x}(0))\).

**Remark 1.** It is still open whether the smoothness of \( a_i(t) \) and \( b_i(t) \) can be relaxed to \( C^\kappa (0 < \kappa < 1) \) as in \([12]\).

### 2. Action-Angle variables

Consider (1.2). First, rescale \( x \to Ax \), where \( A \) is a large constant. Then (1.2) can be written as a system:

\[
\dot{x} = A^n y, \quad \dot{y} = -A^n x^{2r+1} - \sum_{i=0}^{n-1} A^{2i-n} a_i(t) x^{2i+1} - \sum_{i=0}^{l} A^{2i+1} b_i(t) x^{2i+1} y. \tag{2.1}
\]

First of all, we consider an unperturbed Hamiltonian system

\[
\frac{dx}{dt} = y = \frac{\partial H_0}{\partial y}, \quad \frac{dy}{dt} = -x^{2r+1} = -\frac{\partial H_0}{\partial x}. \tag{2.2}
\]

where \( H_0(x, y) = \frac{1}{2} y^2 + \frac{1}{2r+2} x^{2r+2} \). Assume \((S(t), C(t))\) is the solution of (2.2) with the initial condition \((S(0), C(0)) = (0, 1)\). Clearly, this solution is periodic. Let \( T_0 \) be its minimal positive period. It follows from (2.2) that \( S(t) \) and \( C(t) \) satisfy the following properties:
Thus \( \psi \) and \( C \) and \( f \) and \( ff \) and \( S(\omega) \).

Following [3], we define a di-

\[ f(\rho, \theta, t) = A^{-1} f(\rho, \theta, t) \]

\[ \frac{df}{dt} = \rho \frac{df}{dt} + g_1(\rho, \theta, t) + g_2(\rho, \theta, t) \]

where \( \beta = \frac{\alpha}{\rho} \). By a simple calculation, we have \( \det \frac{d(\rho, \theta)}{\rho, \theta} \) = 1.

Thus \( \psi \) is symplectic. So by \( \psi_0 \), (2.1) is changed into

\[ \frac{df}{dt} = f_1(\rho, \theta, t) + f_2(\rho, \theta, t), \]

(2.3)

where \( \beta = \beta e^{2\theta} \), and

\[ f_1(\rho, \theta, t) = -\sum_{i=0}^{n-1} A^{2^i}a_i(\rho, \theta, t)C(\Theta_0)S^{2^i}(\Theta_0) \]

\[ = -\sum_{i=0}^{n-1} A^{2^i}(t)C(\Theta_0)S^{2^i}(\Theta_0) \]

(2.4)

\[ f_2(\rho, \theta, t) = -\sum_{i=0}^{n-1} A^{2^i}a_i(\rho, \theta, t)C(\Theta_0)S^{2^i}(\Theta_0) \]

(2.5)

\[ g_1(\rho, \theta, t) = \alpha \sum_{i=0}^{n-1} A^{2^i}a_i(\rho, \theta, t)C(\Theta_0)S^{2^i}(\Theta_0) \]

(2.6)

\[ g_2(\rho, \theta, t) = \alpha \sum_{i=0}^{n-1} A^{2^i}a_i(\rho, \theta, t)C(\Theta_0)S^{2^i}(\Theta_0) \]

(2.7)

Recall \( C(-t) = C(t) \), \( S(-t) = -S(t) \), and \( a_i(-t) = a_i(t) \) and \( b_i(-t) = b_i(t) \). So we have

\[ f_1(\rho, \theta, t) = -f_1(\rho, \theta, t), f_2(\rho, \theta, t) = -f_2(\rho, \theta, t), f_3(\rho, \theta, t) = -f_3(\rho, \theta, t) \]

(2.8)

\[ g_1(\rho, \theta, t) = g_1(\rho, \theta, t), g_1(\rho, \theta, t) = g_1(\rho, \theta, t), g_2(\rho, \theta, t) = g_2(\rho, \theta, t) \]

(2.9)

In addition, by (2.4)-(2.7), we have \( f_1 = O_1(A^{n-1}), f_2 = O(A^{n-1}), g_1 = O_1(A^{n-1}), g_2 = O(A^{n-1}) \), where \( f(\rho, \theta, t, A) = O_1(A^{n-1}) \) means

\[ \sup_{(\rho, \theta, t) \in D_{A^n} \times \mathbb{T}^1} \left| \sum_{k=q}^{n} \frac{\partial f}{\partial \rho} \right| \leq C_k A^\Gamma, \quad A \gg 1, \quad k \in \mathbb{Z}_+ \]

(2.10)

with constant \( C_k \) depending on \( k \) and

\[ D_s = \{(\rho, \theta) \in \mathbb{R} \times \mathbb{T}^1 : 1 \leq \rho \leq s, \theta \in \mathbb{T}^1 \} \]

(2.11)

and \( f(\rho, \theta, t, A) = O(A^\Gamma) \) means

\[ \sup_{(\rho, \theta, t) \in D_{A^n} \times \mathbb{T}^1} \left| \sum_{k=q}^{n} \frac{\partial f}{\partial \rho} \right| \leq C_k A^\Gamma, \quad A \gg 1, \quad k \in \mathbb{Z}_+ \]

(2.12)
3. Coordinate changes

Lemma 3.1. There exists a diffeomorphism \( \Psi^1 : \rho = \mu + U_1(\mu, \phi, \theta), \theta = \phi \) such that \( \Psi^1(D_4 - CA^{a_1}) \subset D_4 \) with a constant \( C_0 > 0 \) and \( (2.3) \) is changed into \( \frac{d\rho}{dt} = f_1^{(1)}(\rho, \theta, t) + f_2^{(1)}(\rho, \theta, t), \frac{d\theta}{dt} = dA^\mu e^{2\mu - 1} + g_1^{(1)}(\rho, \theta, t) + g_2^{(1)}(\rho, \theta, t), \) where \( f_1^{(1)}, f_2^{(1)} \) and \( g_1^{(1)}, g_2^{(1)} \) satisfy (2.8) and (2.9), respectively, and

\[
\begin{align*}
\dot{f}_1^{(1)} &= O_1(A^{n-2}), \quad \dot{f}_2^{(1)} = O(A^{-1}), \quad \dot{g}_1^{(1)} = O_1(A^{n-1}), \quad \dot{g}_2^{(1)} = O(A^{-1}), \quad (\mu, \phi) \in D_4 - CA^{a_1}. (3.1)
\end{align*}
\]

Proof. Set \( \Phi^1 : \mu = \rho + V_1(\rho, \theta, t), \theta = \phi. \) Under \( \Phi^1, \) we have \( \frac{d\mu}{dt} = \frac{d\rho}{dt} + \frac{\partial V_1}{\partial \rho} \frac{d\rho}{dt} + \frac{\partial V_1}{\partial \theta} \frac{d\theta}{dt} + \frac{\partial V_1}{\partial \phi} \frac{d\phi}{dt} \). By (2.3), \( \frac{d\theta}{dt} = f_1 + f_2 + \frac{\partial V_1}{\partial \rho}(f_1 + f_2) + \frac{\partial V_1}{\partial \theta} (d\rho + g_1 + g_2) + \frac{\partial V_1}{\partial \phi} \frac{d\phi}{dt} \). Since \( f_1(\rho, -\theta, t) = -f_1(\rho, \theta, t), \) \( [f_1] = \int_{T_1} f_1(\rho, \theta, t) d\theta = 0. \) So by setting \( dA^\mu e^{2\mu - 1} + f_1(\rho, \theta, t) = 0, \) we can get \( V_1(\rho, \theta, t) = -\int_0^t \frac{\partial V_1}{dA^\mu} ds \). Recall \( f_1(\rho, -\theta, t) = -f_1(\rho, \theta, t) \) and \( f_1 = O(A^{-1}). \) We have

\[
V_1(\rho, -\theta, t) = V_1(\rho, \theta, t) = V_1(\rho, \theta, t), \quad V_1(\rho, \theta, t) = O(A^{-1}), \quad (\rho, \theta, t) \in D_4 \times \mathbb{T}^1. (3.2)
\]

By the implicit function Theorem, we have that there exists the inverse of \( \Phi^1, \) say \( \Psi^1, \) which can be written as \( \Psi^1 = (\Phi^1)^{-1} : \rho = \mu + U_1(\mu, \phi, \theta), \theta = \phi, \) where

\[
U_1(\mu, \theta, t) = U_1(\mu, \phi, \theta) = U_1(\mu, \phi, \theta), U_1 = O(A^{-1}), \quad (\mu, \phi, \theta) \in D_4 - CA^{a_1} \times \mathbb{T}^1. (3.3)
\]

Let \( f_1^{(1)} = \frac{\partial V_1(\mu + U_1(\mu, \phi, \theta), \theta)}{\partial \rho} f_1(\mu + U_1(\mu, \phi, \theta), \rho, \theta) + \frac{\partial V_1(\mu + U_1(\mu, \phi, \theta), \theta)}{\partial \theta} g_1(\mu + U_1(\mu, \phi, \theta), \rho, \theta), f_2^{(1)} = f_2(\mu + U_1(\mu, \phi, \theta), \rho, \theta) + \frac{\partial V_1(\mu + U_1(\mu, \phi, \theta), \theta)}{\partial \rho} f_1 + \frac{\partial V_1(\mu + U_1(\mu, \phi, \theta), \theta)}{\partial \theta} g_2(\mu + U_1(\mu, \phi, \theta), \rho, \theta), \)

\( g_1^{(1)} = g_1(\mu + U_1(\mu, \phi, \theta), \rho, \theta) - dA^\mu e^{2\mu - 1} + dA^\mu(\mu + U_1(\mu, \phi, \theta)) e^{2\mu - 1} + g_2^{(1)} = g_2(\mu + U_1(\mu, \phi, \theta), \rho, \theta). \)

Using (2.8), (2.9), (3.2) and (3.3), we have that \( f_1^{(1)}, f_2^{(1)} \) and \( g_1^{(1)}, g_2^{(1)} \) satisfy (2.8) and (2.9), respectively. Using (3.2) and (3.3), we have \( f_1^{(1)} = O(A^{n-2}), f_2^{(1)} = O(A^{-1}), g_1^{(1)} = O(A^{n-1}), g_2^{(1)} = O(A^{-1}), (\mu, \phi) \in D_4 - CA^{a_1}. \) This completes the proof of Lemma 3.1.

Repeating Lemma 3.1 \( n \) times, we have a new equation (still by \( (\rho, \theta) \) denoting the variables, for brevity):

\[
\dot{\rho} = f_1^{(n)}(\rho, \theta, t) + f_2^{(n)}(\rho, \theta, t), \quad \dot{\theta} = dA^\mu e^{2\mu - 1} + g_1^{(n)}(\rho, \theta, t) + g_2^{(n)}(\rho, \theta, t), (\rho, \theta, t) \in D_4 - CA^{a_1} \times \mathbb{T}^1. (3.4)
\]

where \( f_1^{(n)}, f_2^{(n)} \) and \( g_1^{(n)}, g_2^{(n)} \) satisfy (2.8) and (2.9), respectively, and

\[
\begin{align*}
\dot{f}_1^{(n)} &= f_2^{(n)} = O(A^{-1}), \quad \dot{g}_1^{(n)} = O_1(A^{n-1}), \quad \dot{g}_2^{(n)} = O(A^{-1}). (3.5)
\end{align*}
\]

Let \( F^{(n)} = f_1^{(n)}(\rho, \theta, t) + f_2^{(n)}(\rho, \theta, t). \) Then \( F(\rho, -\theta, t) = -F(\rho, \theta, t), (\rho, \theta) \in D_4 - CA^{a_1}, t \in \mathbb{T}^1, F = O(A^{-1}). \) Rewrite (3.4) as follows

\[
\begin{align*}
\dot{\rho} &= F(\rho, \theta, t), \quad \dot{\theta} = dA^\mu e^{2\mu - 1} + c_0 h(\rho, t) + g_1(\rho, \theta, t) + g_2(\rho, \theta, t),
\end{align*}
\]

where \( (\rho, \theta) \in D_4 - CA^{a_1}, t \in \mathbb{T}^1, \) and

\[
\begin{align*}
F(\rho, -\theta, t) &= -F(\rho, \theta, t), \quad F = O(A^{-1}), \quad h(\rho, t) = h(\rho, t), \quad h = O_1(A^{-1}),
\end{align*}
\]

\[
\begin{align*}
g_1(\rho, -\theta, t) &= g_1(\rho, \theta, t), \quad g_1(\rho, \theta, -t) = g_1(\rho, \theta, t), \quad g_2(\rho, -\theta, t) = g_2(\rho, \theta, t),
\end{align*}
\]

\[
\begin{align*}
g_1 = O_1(A^{a_1}), \quad g_2 = O(A^{-1}), \quad c_0 = 0.
\end{align*}
\]
Lemma 3.2. There exists a diffeomorphism $\psi_2 : \rho = \mu, \phi = \theta + U_2(\mu, \phi, t), (\mu, \phi, t) \in D_{2-C_0 \text{A}^1 \times \mathbb{T}^1}$ such that $\psi_2(D_{2-C_0 \text{A}^1 \times \mathbb{T}^1}) \subset D_{2-C_0 \text{A}^1 \times \mathbb{T}^1}$, $(C_1 \geq C_0)$, and (3.6) is transformed into

$$
\dot{\rho} = F^{(1)}(\rho, \theta, t), \quad \dot{t} = dA^n \rho^{2b-1} + h^{(1)}(\rho, t) + g_1^{(1)}(\rho, \theta, t) + s_2^{(1)}(\rho, \theta, t),
$$

(3.10)

where $F^{(1)}$, $h^{(1)}$ satisfy (3.7), $g_1^{(1)}, s_2^{(1)}$ satisfy (3.8) and $g_1^{(1)} = O_1(A^{n-2})$, $s_2^{(1)} = O(A^{-1})$.

Proof. Define a transformation $\Phi_2 : \mu = \rho, \phi = \theta + V_2(\rho, \theta, t)$, where

$$
V_2(\rho, \theta, t) = - \int_0^\rho \frac{g_1(\rho, \theta, t)}{1 + dA^n \rho^{2b-1}} d\rho, [g_1] = \int_0^\rho g_1(\rho, s, t) d\rho.
$$

(3.11)

By (3.7) - (3.9), we have

$$
V_2(\rho, -\theta, t) = - V_2(\rho, \theta, t), \quad V_2(\rho, 0, t) = 0, \quad V_2(\rho, -\theta, t) = - V_2(\rho, \theta, t).
$$

(3.12)

Moreover, doing as in the proof of Lemma 1, we have that there exists $U_2 = U_2(\mu, \phi, t)$ satisfying

$$
U_2(\mu, \theta, t) = - U_2(\mu, \phi, t), \quad U_2(\mu, \phi, t) = U_2(\mu, \phi, t), \quad U_2 = O_1(A^{-1}),
$$

(3.13)

and $\psi_2 = \Phi_2^{-1} : \rho = \mu, \phi = \theta + U_2(\mu, \phi, t)$ such that $\psi_2(D_{2-C_0 \text{A}^1 \times \mathbb{T}^1}) \subset D_{2-C_0 \text{A}^1 \times \mathbb{T}^1}$, $C_3 > C_2$.

Then (3.10) is changed into $\mu = F(\mu, \phi, t), \phi = dA^n \rho^{2b-1} + h^{(1)}(\mu, t) + g_1^{(1)}(\mu, \phi, t) + s_2^{(1)}(\mu, \phi, t)$, where $F^{(1)}(\mu, \phi, t) = F(\mu, \phi, t) + U_2(\mu, \phi, t), h^{(1)}(\mu, t) = c_0 h(\mu, t) + [g_1](\mu, t) + g_1^{(1)}(\mu, \phi, t) + g_2^{(1)}(\mu, \phi, t) = F(\mu, \phi, t) + U_2(\mu, \phi, t), t^{(2)}(\mu, \phi, t) = (\partial_t + \mu^{-1} \partial_{\phi}) + s_2^{(1)}(\mu, \phi, t)$.

By (3.7) - (3.9) and (3.13), we have that

$$
F^{(1)}(\mu, \phi, t), g_1^{(1)}, s_2^{(1)}
$$

satisfy (3.7), (3.8) and $g_1^{(1)} = O_1(A^{n-2}), g_2^{(1)} = O(A^{-1})$.

This completes the proof of Lemma 2.

Note that if $g(\rho, -\theta, -t) = g(\rho, \theta, t)$, we have $g(\rho, -\theta, -t) = g(\rho, \theta, t)$.

Repeating Lemma 2 $n$ times, we have that (3.6) is changed into

$$
\dot{\rho} = F(\rho, \theta, t), \quad \dot{t} = dA^n \rho^{2b-1} + H(\rho, t) + G(\rho, \phi, t),
$$

(3.12)

where $F(\rho, -\theta, -t) = - F(\rho, \theta, t), F = O(A^{-1}), H(\rho, -\theta, -t) = H(\rho, t), H = O_1(A^{n-1}), G(\rho, -\theta, -t) = G(\rho, \theta, t), G = O(A^{-1})$.

(3.13)
Introduce the notation $\Omega_{\gamma,C} = \{ \omega \in dA^m\Omega \mid \omega \text{ is of type } M_m(dA^m\gamma_C, 1) \}$. Then for each $\varepsilon > 0$ there exists $\delta > 0$, depending only on $\varepsilon$, $D$, and $C$ but not on $\gamma$, such that if on $D |f'| < \gamma \delta$ and $|x^2| < \gamma \delta$ then for each $\omega \in \Omega_{\gamma,C}$ the mappings $A$ and $G$ have a common invariant $(m + \kappa)$-dimensional manifold

$$x = \varphi + \Phi^1_\omega(\varphi, \chi), \quad y = \gamma^{-1}\omega + \Phi^2_\omega(\varphi, \chi), \quad \eta = \chi + \Phi^3_\omega(\varphi, \chi),$$

(3.14)

where $\Phi^i_\omega$ are normal in $[\varphi \in C^\infty, \text{Im} \varphi < i \delta] \times \{ \chi \in C^\infty, |x_1 - b_i| < R + i \delta \}$ functions, such that diffeomorphisms of the manifold (3.13) induced by the mappings $A$ and $G$ are $(\varphi, \chi) \mapsto (\varphi + \omega, \chi)$ and $(\varphi, \chi) \mapsto (-\varphi, \chi)$ respectively (so that (3.14) is foliated into invariant under $A$ and $G$ spaces $\chi = \text{const}$) and the following inequality holds $\Phi^1_\omega < \varepsilon$. Moreover, for every two $\omega^1$ and $\omega^2$ in $\Omega_{\gamma,C}$ the following estimate holds $|\Phi^1_\omega - \Phi^1_\omega^2| < \gamma^{-1}|\omega^1 - \omega^2|\varepsilon$.

The present theorem is Theorem 1.1 of [13] when $dA^n = 1$. When $dA^n \neq 1$, the proof is similar to that of Theorem 1.1 in [13] and so is omitted. See [13] for the details.

**Proof of Theorem 1.1** Let $G : (\rho, \theta) \mapsto (\rho, -\theta)$ in Lemma 3.3 and let $\eta$ vanish. By Lemma 3.3 and 3.13, $\mathcal{P}^\omega$ has an invariant curve $\mathcal{T}$ in the annulus $[2, 3] \times \mathbb{T}^1$. Since $A \gg 1$, it follows that the time-1 map of the original system has an invariant curve $\mathcal{T}_A$ in $[2A + C, 3A - C] \times \mathbb{T}^1$ with $C$ being a constant independent of $A$. Choosing $A = A_k \to \infty$ as $k \to \infty$, we have that there are countable many invariant curves $\mathcal{T}_{A_k}$, clustering at $\infty$. Then any solution of the original system is bounded. Incidentally, we can obtained that there are many infinite number of quasi-periodic solutions around the infinity in the $(x, \dot{x})$ plane.

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