Sharp Z-Eigenvalue Inclusion Set-Based Method for Testing the Positive Definiteness of Multivariate Homogeneous Forms

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Abstract. In this paper, we establish a sharp $Z$-eigenvalue inclusion set for even-order real tensors by $Z$-identity tensor and prove that new $Z$-eigenvalue inclusion set is sharper than existing results. We propose some sufficient conditions for testing the positive definiteness of multivariate homogeneous forms via new $Z$-eigenvalue inclusion set. Further, we establish upper bounds on the $Z$-spectral radius of weakly symmetric nonnegative tensors and estimate the convergence rate of the greedy rank-one algorithms. The given numerical experiments show the validity of our results.

1. Introduction

Consider the following multivariate homogeneous forms with spherical constraint:

$$f_A(x) = A x^m = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \ldots i_m} x_{i_1} x_{i_2} \ldots x_{i_m}$$

s.t. $x^\top x = 1$, (1)

where $x \in \mathbb{R}^n$, $m, n \geq 2$, $f_A(x)$ is a multivariate homogeneous form of degree $m$ with $n$ variables, and $A \in \mathbb{R}^{[m,n]}$ is an $m$-order $n$-dimensional real tensor with entries $[12, 14]$

$$a_{i_1 \ldots i_m} \in \mathbb{R}, i_j \in N = \{1, \ldots, n\}, j = 1, \ldots, m.$$ 

Clearly, the critical points of (1) satisfy the following equations for some $\lambda \in \mathbb{R}$:

$$\mathcal{A} x^{m-1} = \lambda x \text{ and } x^\top x = 1,$$

(2)

where $(\mathcal{A} x^{m-1})_i = \sum_{i_1, \ldots, i_m \in N} a_{i_1 \ldots i_m} x_{i_1} \ldots x_{i_m}$. The real number $\lambda$ and the real vector $x$ satisfying with (2) are called $Z$-eigenvalue and $Z$-eigenvector, respectively.

The multivariate homogeneous form $f_A(x)$ is positive definite, which plays important roles in signal processing [15] and the stability study of nonlinear autonomous systems via Lyapunov’s direct method in
Each even tensor in the following is a $Z$-identity tensor with parameters by $Z$-identity tensor, which can identify the positive definiteness of an even-order real symmetric tensor. Very recently, Li et al. [10] proposed a sharp Brauer-type inclusion set with parameters by $Z$-identity tensor, which can identify the positive definiteness of an even-order real symmetric tensor. It is remarkable that Brauer-type inclusion set is tighter than Gershgorin-type inclusion set [20]. As a continuation of the article [20], we shall establish the positive definiteness of an even-order real symmetric tensor. To this end, we introduce a Z-identity tensor in [8, 10] and important results proposed in [10].

**Definition 1.1.** Assume that $m$ is even. We call $I_Z \in \mathbb{R}^{[m,n]}$ a Z-identity tensor if

$$I_Z x^{m-1} = x, \quad x^T x = 1, \quad \forall x \in \mathbb{R}^n.$$  

It is worth noting that the even-order $n$ dimension Z-identity tensor is not unique in general. For instance, each even tensor in the following is a Z-identity tensor:

Case I: $(I_Z)_{ii,i\ldots,i} = 1, \forall k \in N$ and $m = 2k$;

Case II (Property 2.4 of [8]): $(I_Z)_{ii,i\ldots,i} = \frac{1}{m!} \sum_{p \in \Pi_n} \delta_{p_1i} \delta_{p_{2}i} \ldots \delta_{p_{m-1}i} \delta_{p_{m}i}$, where $\delta$ is the standard Kronecker, i.e.,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

**Lemma 1.2.** (Theorem 2 of [10]) Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with $m$ being even. Let $\sigma_Z(\mathcal{A})$ be the set of all Z-eigenvalues of $\mathcal{A}$. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$, then

$$\sigma_Z(\mathcal{A}) \subseteq \mathcal{G}(\mathcal{A}, \alpha) = \bigcup_{i \in N} \mathcal{G}_i(\mathcal{A}, \alpha) := \{z \in \mathbb{R} : |z - \alpha_j| \leq R_i(\mathcal{A}, \alpha_i)\},$$

where $R_i(\mathcal{A}, \alpha_i) = \sum_{i_2\ldots,i_m \in N} |a_{i_2\ldots,i_m} - \alpha_i (I_Z)_{i_2\ldots,i_m}|$. Furthermore, $\sigma_Z(\mathcal{A}) \subseteq \bigcap_{\alpha \in \mathbb{R}^n} \mathcal{G}(\mathcal{A}, \alpha)$.

2. A sharp Z-eigenvalue inclusion set for even-order real tensors

In this section, we establish new Z-eigenvalue inclusion set for even-order tensors. To this end, we define

$$\Theta_i = \{(i_2, i_3, \ldots, i_m) : i_k = j \text{ for some } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in N\},$$

$$\overline{\Theta}_i = \{(i_2, i_3, \ldots, i_m) : i_k \neq j \text{ all any } k \in \{2, \ldots, m\}, \text{where } j, i_2, \ldots, i_m \in N\},$$

$$r^\Theta_i(\mathcal{A}, \alpha_i) = \sum_{\Theta_i \in \Theta} |a_{i_2\ldots,i_m} - \alpha_i (I_Z)_{i_2\ldots,i_m}|, \quad r^\overline{\Theta}_i(\mathcal{A}, \alpha_i) = \sum_{\overline{\Theta}_i \in \overline{\Theta}} |a_{i_2\ldots,i_m} - \alpha_i (I_Z)_{i_2\ldots,i_m}|.$$  

Obviously, $R_i(\mathcal{A}, \alpha_i) = r^\Theta_i(\mathcal{A}, \alpha_i) + r^\overline{\Theta}_i(\mathcal{A}, \alpha_i)$.

**Theorem 2.1.** Let $\mathcal{A} = (a_{i_1i_2\ldots i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with $m$ being even. For any real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T \in \mathbb{R}^n$, then

$$\sigma_Z(\mathcal{A}) \subseteq \bigcup_{i \in N} \bigcap_{j \in N, \neq i} \mathcal{U}_{ij}(\mathcal{A}, \alpha),$$

where $\mathcal{U}_{ij}(\mathcal{A}, \alpha) = \{z \in \mathbb{R} : |(\lambda - \alpha_j) - r^\overline{\Theta}_i(\mathcal{A}, \alpha_i)\lambda - \alpha_j| \leq r^\Theta_i(\mathcal{A}, \alpha_i) R_j(\mathcal{A}, \alpha_j)\}$. Furthermore, $\sigma_Z(\mathcal{A}) \subseteq \bigcap_{\alpha \in \mathbb{R}^n} \mathcal{U}(\mathcal{A}, \alpha)$.  

automatic control [3, 4, 13]. Note that $f_a(x)$ is positive definite if and only if tensor $\mathcal{A}$ is positive definite, and that an even-order real symmetric tensor is positive definite if and only if all of its Z-eigenvalues are positive [14]. Some effective algorithms for finding Z-eigenvalue and the corresponding eigenvector have been implemented [5–9, 11, 16, 18, 21–26], but it is difficult to compute all the Z-eigenvalues and judge the positive definiteness of an even-order real symmetric tensor. Very recently, Li et al. [10] proposed a sharp Brauer-type inclusion set with parameters by Z-identity tensor, which can identify the positive definiteness of an even-order real symmetric tensor. It is remarkable that Brauer-type inclusion set is tighter than Gershgorin-type inclusion set [20]. As a continuation of the article [20], we shall establish a sharp Brauer-type Z-eigenvalue localization set and propose some sufficient conditions for the positive definiteness of multivariate homogeneous forms.
Proof. Let \((\lambda, x)\) be a \(Z\)-eigenpair of \(A\) and \(I_z \in \mathbb{R}^{[m,n]}\) be a \(Z\)-identity tensor, i.e.,
\[
A x^m = \lambda x = A I_z x^m, \quad x^\top x = 1.
\]  
(3)
Assume \(|x_i| = \max\{|x_i|\}\), then \(0 < |x_i|^m \leq |x_i| \leq 1\).

On one hand, taking the \(t\)-th equation from (3), for any \(j \in \mathbb{N}, j \neq t\), we have
\[
\sum_{i_2,\ldots, i_m \in \mathbb{N}} \lambda(I_z)_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m} = \sum_{i_2,\ldots, i_m \in \mathbb{N}} a_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m}.
\]  
(4)
Hence, for any real number \(a_t\), it follows that
\[
(\lambda - a_t)x_t = \sum_{i_2,\ldots, i_m \in \mathbb{N}} (\lambda - a_t)(I_z)_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m} = \sum_{i_2,\ldots, i_m \in \mathbb{N}} (a_{ii_2\ldots i_m} - a_t(I_z)_{ii_2\ldots i_m}) x_{i_2} \cdots x_{i_m}
= \sum_{i_2,\ldots, i_m \in \mathbb{E}_j} (a_{ii_2\ldots i_m} - a_t(I_z)_{ii_2\ldots i_m}) x_{i_2} \cdots x_{i_m} + \sum_{i_2,\ldots, i_m \in \mathbb{B}_j} (a_{ii_2\ldots i_m} - a_t(I_z)_{ii_2\ldots i_m}) x_{i_2} \cdots x_{i_m}
\]  
(5)
Taking modulus in (5) and using the triangle inequality give
\[
|\lambda - a_t||x| \leq \sum_{i_2,\ldots, i_m \in \mathbb{E}_j} |a_{ii_2\ldots i_m} - a_t(I_z)_{ii_2\ldots i_m}| |x_{i_2}| \cdots |x_{i_m}| + \sum_{i_2,\ldots, i_m \in \mathbb{B}_j} |a_{ii_2\ldots i_m} - a_t(I_z)_{ii_2\ldots i_m}| |x_{i_2}| \cdots |x_{i_m}|
\leq r_j^\Theta (A, a_t)|x| + r_j^\Omega (A, a_t)|x|,
\]  
(6)
i.e.,
\[
(|\lambda - a_t| - r_j^\Omega (A, a_t))|x| \leq r_j^\Theta (A, a_t)|x|.
\]  
(7)
On the other hand, for \(t \neq j \in \mathbb{N}\), taking the \(j\)-th equation from (3), we obtain
\[
(\lambda - a_j)x_j = \sum_{i_2,\ldots, i_m \in \mathbb{N}} (\lambda - a_j)(I_z)_{ii_2\ldots i_m} x_{i_2} \cdots x_{i_m} = \sum_{i_2,\ldots, i_m \in \mathbb{N}} (a_{ij_2\ldots i_m} - a_j(I_z)_{ij_2\ldots i_m}) x_{i_2} \cdots x_{i_m},
\]  
(8)
Taking modulus in (8) and using the triangle inequality, one has
\[
|\lambda - a_j||x| \leq R_j(A, a_j)|x|.
\]  
(9)
If \(|x| = 0\), by (7), we obtain
\[
|\lambda - a_t| \leq r_j^\Theta (A, a_t).
\]
Thus, \(\lambda \in \mathcal{U}_{ij}(A, a) \subset \mathcal{U}(A, a)\).

Otherwise, \(|x| > 0\). Multiplying (9) with (8) yields
\[
(|\lambda - a_t| - r_j^\Omega (A, a_t))|\lambda - a_j||x| \leq r_j^\Theta (A, a_j) R_j(A, a_j)|x||x|,
\]
equivalently,
\[
(|\lambda - a_t| - r_j^\Omega (A, a_t))|\lambda - a_j| \leq r_j^\Theta (A, a_j) R_j(A, a_j),
\]
which implies \(\lambda \in \mathcal{U}_{ij}(A, a)\). From the arbitrariness of \(j\), we have \(\lambda \in \bigcup_{i \in \mathbb{N}} \bigcap_{j \in \mathbb{N}, j \neq i} \mathcal{U}_{ij}(A, a)\). Further, \(\sigma_Z(A) \subseteq \bigcap_{a \in \mathbb{R}^n} \mathcal{U}(A, a)\) by the arbitrariness of \(a\). \(\square\)

Corollary 2.2. Let \(\mathcal{A} = (a_{ij_2\ldots i_m}) \in \mathbb{R}^{[m,n]}\) with \(m\) being even. For any real vector \(a = (a_1, \ldots, a_n)^\top \in \mathbb{R}^n\), then
\[
\mathcal{U}(A, a) \subseteq \mathcal{G}(A, a).
\]
Proof. For any \( \lambda \in \mathcal{U}(\mathcal{A}, a) \), without loss of generality, there exists \( t \in N \) such that \( \lambda \in \mathcal{U}_{t,a}(\mathcal{A}) \), that is,

\[
(10)
\]

Next, the following argument is divided into two cases.

Case I: \( r^\Theta_t(\mathcal{A}, a_t)R_s(\mathcal{A}, a_s) = 0 \). Since \(|\lambda - a_i| \geq 0\), from (10), we deduce \(|\lambda - a_i| - r^\Xi_t(\mathcal{A}, a_t) \leq 0\). Further, it holds that

\[
|\lambda - a_i| \leq r^\Xi_t(\mathcal{A}, a_t) \leq R_s(\mathcal{A}, a_s),
\]

i.e., \( \lambda \in \mathcal{G}_s(\mathcal{A}, a) \). So, we have \( \mathcal{U}_{t,a}(\mathcal{A}, a) \subseteq \mathcal{G}_s(\mathcal{A}, a) \).

Case II: \( r^\Theta_t(\mathcal{A}, a_t)R_s(\mathcal{A}, a_s) > 0 \). Then dividing both sides by \( r^\Theta_t(\mathcal{A}, a_t)R_s(\mathcal{A}, a_s) \) in (10), we obtain

\[
(11)
\]

which implies

\[
(12)
\]

or

\[
(13)
\]

If (12) holds, then we have \(|\lambda - a_i| - r^\Xi_t(\mathcal{A}, a_t) \leq r^\Theta_t(\mathcal{A}, a_t)\), i.e.,

\[
|\lambda - a_i| \leq r^\Xi_t(\mathcal{A}, a_t) + r^\Theta_t(\mathcal{A}, a_t) = R_s(\mathcal{A}, a_s).
\]

So, \( \lambda \in \mathcal{G}_s(\mathcal{A}, a) \). Otherwise, (13) holds, we can verify \( \lambda \in \mathcal{G}_s(\mathcal{A}, a) \).

From the above two cases, we can get \( \mathcal{U}_{t,a}(\mathcal{A}, a) \subseteq \mathcal{G}_s(\mathcal{A}, a) \cup \mathcal{G}_s(\mathcal{A}, a) \). Thus, \( \mathcal{U}(\mathcal{A}, a) \subseteq \mathcal{G}(\mathcal{A}, a) \) for a given parameter \( a \).

Next, we give a numerical comparison between Theorem 2.1 and Theorem 2 of [10].

**Example 2.3.** Consider \( \mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{4^2} \) defined by

\[
a_{ijkl} = \begin{cases} 
a_{1111} = 10; a_{1122} = 9; a_{1211} = a_{1221} = -1; 
a_{2222} = 5; a_{2211} = 6; a_{2122} = a_{2212} = -1; 
                  & a_{ijkl} = 0, \text{ otherwise.}
\end{cases}
\]

All Z-eigenvalues of \( \mathcal{A} \) are 5.0000 and 10.0000. We choose different parameters \( a_1 = [3, 8]^T, a_2 = [10, 7]^T, a_3 = [9, 5]^T \) and \( a_4 = [9, 5, 5]^T \), respectively. Set \( a_1 = [3, 8]^T \) and \( I_Z = (l_{ijkl}) \) as Case I of Definition 1.1

\[
l_{ijkl} = \begin{cases} 
l_{1111} = l_{1122} = l_{2211} = l_{2222} = 1; 
                  & 0, \text{ otherwise.}
\end{cases}
\]

Accordingly to Theorem 2.1, we obtain

\[
\mathcal{U}(\mathcal{A}, a_1 = (3, 8)) = [-7.5917, 16.5498] \cup [-3.8102, 15.7178] = [-7.5917, 16.5498];
\]

Similarly, we can obtain the following table:

| \( a \)  | [3, 8] | [10, 7] | [9, 5] | [9, 5, 5] |
|----------|--------|--------|--------|----------|
| \( \mathcal{U}(\mathcal{A}, a) \) | [-7.5917, 16.5498] | [3.5949, 12.6533] | [3.6277, 11] | [3.6088, 10.6225] |
| \( \mathcal{G}(\mathcal{A}, a) \) | [-12, 18] | [2, 13] | [2, 12] | [2.5, 12] |

Numerical results show that the bound of Theorem 2.1 is tighter than that of Theorem 2 of [10] and the suitable parameter \( a \) has a great influence on the numerical effect.
3. Positive definiteness of multivariate homogeneous forms

In this section, based on the inclusion set $\mathcal{U}(\mathcal{A}, \alpha)$ in Theorem 2.1, we propose a sufficient condition for the positive definiteness of even-order tensors. Before proceeding further, we introduce the results of \cite{1, 10}.

**Definition 3.1.** (i) We say that $\mathcal{A}$ is symmetric if

$$a_{i_1...i_m} = a_{i_{m-i}}^{-1} a_{i_{m-1}}^{-1} \forall \pi \in \Gamma_m,$$

where $\Gamma_m$ is the permutation group of $m$ indices.

(ii) We say that $\mathcal{A}$ is weakly symmetric if the associated homogeneous polynomial $f_\mathcal{A}(x)$ satisfies

$$\nabla f_\mathcal{A}(x) = m \mathcal{A} x^{m-1}.$$  

Obviously, if tensor $\mathcal{A}$ is symmetric, then $\mathcal{A}$ weakly symmetric. However, the converse result may not hold.

**Lemma 3.2.** (Theorem 3 of \cite{10}) Let $\lambda$ be a Z-eigenvalue of $\mathcal{A} = (a_{i_1...i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with $m$ being even. If there exists a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ such that

$$\alpha_i > R((\mathcal{A}, \alpha_i)), \forall i \in N,$$

then $\lambda > 0$. Further, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_\mathcal{A}(x)$ defined in (1) is positive definite.

**Theorem 3.3.** Let $\lambda$ be a Z-eigenvalue of $\mathcal{A} = (a_{i_1...i_m}) \in \mathbb{R}^{[m,n]}$ and $I_Z \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor with $m$ being even. For $i \in N$, if there exist a positive real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^T$ and $j \neq i$ such that

$$\lambda - r_j^\Theta(\mathcal{A}, \alpha_i) \alpha_j > r_j^\Theta(\mathcal{A}, \alpha_i) R_j(\mathcal{A}, \alpha_i),$$

then $\lambda > 0$. Further, if $\mathcal{A}$ is symmetric, then $\mathcal{A}$ is positive definite and $f_\mathcal{A}(x)$ defined in (1) is positive definite.

**Proof.** Suppose on the contrary that $\lambda = 0$. From Theorem 2.1, there exists $t \in N$ with $\lambda \in \mathcal{U}_t(\mathcal{A}, \alpha_i)$, i.e.,

$$|\lambda - a_i| > r_i^\Theta(\mathcal{A}, \alpha_i) R_j(\mathcal{A}, \alpha_i), \forall j \neq t.$$

Further, it follows from $\alpha_i > 0$ and $\lambda = 0$ that

$$|\lambda - a_i| > r_i^\Theta(\mathcal{A}, \alpha_i) R_j(\mathcal{A}, \alpha_i), \forall j \neq t,$$

which contradicts (14). Thus, $\lambda > 0$. When $\mathcal{A}$ is a symmetric tensor and all Z-eigenvalues are positive, $\mathcal{A}$ is positive definite and $f_\mathcal{A}(x)$ defined in (1) is positive definite. \qed

**Example 3.4.** Consider $f_\mathcal{A}(x) = \mathcal{A} x^m$ deduced by symmetric tensor $\mathcal{A} = (a_{i,j}) \in \mathbb{R}^{[4,4]}$ as follows

$$a_{1111} = 1.4; a_{2222} = 3.2; a_{3333} = 2.6; a_{1112} = a_{1121} = a_{2121} = a_{2112} = 0.8;$$
$$a_{1133} = a_{1313} = a_{1331} = a_{3113} = a_{3131} = a_{3311} = 1.1;$$
$$a_{1233} = a_{1323} = a_{2133} = a_{2313} = a_{2331} = 0.1;$$
$$a_{3123} = a_{3213} = a_{3231} = a_{3312} = a_{3321} = 0.1;$$
$$a_{2223} = a_{2323} = a_{3223} = 0.1;$$
$$a_{2233} = a_{2332} = a_{3233} = a_{3322} = 1.0; a_{j,kl} = 0, \text{otherwise.}$$
Taking $I_2$ as Case II (Case I) of Definition 1.1, by simple computations, we cannot find positive real number $\alpha_1$ such that

$$\alpha_1 > R_1(\mathcal{A}, \alpha_1),$$

which shows that Theorem 3 of [10] cannot check the positive definiteness of $\mathcal{A}$ and $f_{\mathcal{A}}(x)$.

Set $a = (2.85, 3.0, 2.7)$ and let $I_2 = (i_{jkl})$ be Case II of Definition 1.1

$$I_{jkl} = \begin{cases} 
I_{1111} = I_{2222} = I_{3333} = 1; \\
I_{1112} = I_{1212} = I_{1122} = I_{1221} = I_{2211} = I_{2121} = I_{1313} = I_{2323} = I_{2232} = I_{3223} = I_{3322} = \frac{1}{3}; \\
I_{3113} = I_{3311} = I_{3313} = I_{3223} = I_{1333} = I_{2322} = I_{2332} = \frac{2}{3}; \\
0, \text{ otherwise.}
\end{cases}$$

From Theorem 3.3, we can calculate the following corresponding values

| $i,j$ | $(a_i - r^\circ(\mathcal{A}, a_i))a_j$ | $r^\circ(\mathcal{A}, a_i)R_i(\mathcal{A}, a_i)$ |
|-------|----------------------------------|----------------------------------|
| $1,2$ | $2.85$                           | $1.575$                          |
| $1,3$ | $1.755$                          | $1.275$                          |
| $2,1$ | $4.56$                           | $2.065$                          |
| $2,3$ | $6.21$                           | $2.55$                           |
| $3,1$ | $6.27$                           | $3.54$                           |
| $3,2$ | $6$                              | $1.5$                            |

From the above table, we verify

$$(a_i - r^\circ(\mathcal{A}, a_i))a_j > r^\circ(\mathcal{A}, a_i)R_i(\mathcal{A}, a_i), \forall i \neq j \in N,$$

which implies that $\mathcal{A}$ is positive definite and $f_{\mathcal{A}}(x)$ is positive definite.

4. Estimations of $Z$-spectral radius and convergence rate on the greedy rank-one algorithms

As we know, the best rank-one approximation which has numerous applications in wireless communication systems, image processing, data analysis [7, 15–17, 21]. The best rank-one approximation of $\mathcal{A} = (a_{i_{1}i_{2}...i_{m}})$ is to find a rank-one tensor $x x^m = (x_{i_1} x_{i_2} ... x_{i_m})$ such that

$$\min_{x \in \mathbb{R}^n} \|\mathcal{A} - x x^m\|_F : x^T x = 1,$$

where $\|\mathcal{A}\|_F := \sqrt{\sum_{i_{1}i_{2}...i_{m} \in N} a_{i_{1}i_{2}...i_{m}}^2}$. When $\mathcal{A}$ is nonnegative and weakly symmetric, $\rho(\mathcal{A})x_0^m$ is a best rank-one approximation of $\mathcal{A}$, i.e.,

$$\min_{x \in \mathbb{R}^n} \|\mathcal{A} - x x^m\|_F = \|\mathcal{A} - \rho(\mathcal{A})x_0^m\|_F = \sqrt{\|\mathcal{A}\|_F^2 - \rho(\mathcal{A})^2}.$$

Further, Qi [17] defined the quotient on the residual of the best rank-one approximation of tensor $\mathcal{A}$ as follows:

$$\omega = \frac{\|\mathcal{A} - \rho(\mathcal{A})x_0^m\|_F}{\|\mathcal{A}\|_F} = \sqrt{1 - \frac{\rho(\mathcal{A})^2}{\|\mathcal{A}\|_F^2}},$$

which can estimate the convergence rate of the greedy rank-one algorithm [2, 17, 18, 25]. Hence, we shall devote to finding sharp upper bounds of the $Z$-spectral radius of weakly symmetric nonnegative tensors to estimate the convergence rate of the greedy rank-one algorithms. We recall some fundamental results of nonnegative tensors [1].
Lemma 4.1. (Theorem 3.11 of [1]) Assume $\mathcal{A}$ is a weakly symmetric nonnegative tensor. Then, $\rho(\mathcal{A}) = \lambda^*$, where $\lambda^*$ denotes the largest Z-eigenvalue.

Lemma 4.2. (Corollary 4.10 of [1]) Assume $\mathcal{A}$ is a weakly symmetric nonnegative tensor. Then,

$$\rho(\mathcal{A}) \geq \max_{i \in \mathbb{N}} a_{i,i,i}.$$  

Theorem 4.3. Let $\mathcal{A} = (a_{i_0i_1\ldots i_m}) \in \mathbb{R}^{[m,n]}$ be a weakly symmetric nonnegative tensor and $I_{Z} \in \mathbb{R}^{[m,n]}$ be a Z-identity tensor (Case I or Case II) with $m$ being even. For real vector $\alpha = (\alpha_1, \ldots, \alpha_n)^{\top} \in \mathbb{R}^{n}$ with $\alpha_i \leq \max_{i \in \mathbb{N}} a_{i,i,i}$, then

$$\rho(\mathcal{A}) \leq \max_{i \in \mathbb{N}} \min_{j \in \mathbb{N}, j \neq i} \frac{1}{2} (\alpha_i + \alpha_j + \bar{r}^\Theta_i (\mathcal{A}, \alpha_i) + \Lambda_{i,j}^\Theta (\mathcal{A}, \alpha_i)),$$

where $\Lambda_{i,j}(\mathcal{A}) = (\alpha_i - \alpha_j + \bar{r}^\Theta_i (\mathcal{A}, \alpha_i))^2 + 4r^\Theta_j (\mathcal{A}, \alpha_j)i\sigma_j(\mathcal{A}, \alpha_j)$. 

Proof. From Lemma 4.1, we assume that $\rho(\mathcal{A}) = \lambda^*$ is the largest Z-eigenvalue. It follows from Theorem 2.1 that there exists $t \in \mathbb{N}$ such that

$$(|\rho(\mathcal{A}) - \alpha_i| - \bar{r}^\Theta_i (\mathcal{A}, \alpha_i))|\rho(\mathcal{A}) - \alpha_i| \leq r^\Theta_j (\mathcal{A}, \alpha_j)i\sigma_j(\mathcal{A}, \alpha_j), \forall j \neq t. \tag{15}$$

Since $\mathcal{A}$ is nonnegative and Lemma 4.2 holds, for $\alpha_i \leq \max_{i \in \mathbb{N}} a_{i,i,i}$, we have

$$\rho(\mathcal{A}) \geq \alpha_i \text{ and } \rho(\mathcal{A}) \geq \alpha_j.$$

Thus, (15) is equivalent to

$$(\rho(\mathcal{A}) - \alpha_i - \bar{r}^\Theta_i (\mathcal{A}, \alpha_i))(\rho(\mathcal{A}) - \alpha_j) \leq r^\Theta_j (\mathcal{A}, \alpha_j)i\sigma_j(\mathcal{A}, \alpha_j), \forall j \neq t. \tag{16}$$

Solving for (16), we obtain

$$\rho(\mathcal{A}) \leq \frac{1}{2} (\alpha_i + \alpha_j + \bar{r}^\Theta_i (\mathcal{A}, \alpha_i) + \Lambda_{i,j}^\Theta (\mathcal{A}, \alpha_i)),$$

where $\Lambda_{i,j}(\mathcal{A}) = (\alpha_i - \alpha_j + \bar{r}^\Theta_i (\mathcal{A}, \alpha_i))^2 + 4r^\Theta_j (\mathcal{A}, \alpha_j)i\sigma_j(\mathcal{A}, \alpha_j)$. Since $j \in \mathbb{N}$ and $\alpha$ are chosen arbitrarily, it holds

$$\rho(\mathcal{A}) \leq \min_{i \in \mathbb{N}, j \neq i} \frac{1}{2} (\alpha_i + \alpha_j + \bar{r}^\Theta_i (\mathcal{A}, \alpha_i) + \Lambda_{i,j}^\Theta (\mathcal{A}, \alpha_i)).$$

Consequently,

$$\rho(\mathcal{A}) \leq \max_{i \in \mathbb{N}} \min_{j \in \mathbb{N}, j \neq i} \frac{1}{2} (\alpha_i + \alpha_j + \bar{r}^\Theta_i (\mathcal{A}, \alpha_i) + \Lambda_{i,j}^\Theta (\mathcal{A}, \alpha_i)).$$

Thus, the conclusion holds. \( \square \)

The following numerical experiment shows validity of Theorem 4.3 and gives an estimation for the convergence rate of the greedy rank-one algorithms.

Example 4.4. Consider tensor $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ defined by

$$a_{ijkl} = \begin{cases} a_{1111} = 1; & a_{2222} = 3; a_{1122} = a_{1212} = a_{2112} = a_{2121} = a_{2212} = a_{2211} = 1; \\ a_{1112} = a_{1121} = a_{1211} = a_{2111} = 1; \end{cases}$$

By simple computation, we obtain $(\rho(\mathcal{A}), x) = (3, (0, 1))$ and $\|\mathcal{A}\|_F = 3.3166$. For this tensor, set $\alpha = (1, 1)$ and let $I_{Z} = (I_{ijkl})$ be Case II of Definition 1.1. The bounds via different estimations given in the literature are shown in the following table:
From the table above, it is easy to see that only the upper bound obtained by Theorem 4.1 is smaller than $\|A\|_F$. Consequently, we have

$$\min_{x\in\mathbb{R}^n, x^T x = 1} \|A - xx^T\|_F = \sqrt{\|A\|_F^2 - \rho(A)^2} \geq 1.3559.$$ 

Further, we obtain that the quotient on the residual of the best rank-one approximation of $A$ is

$$\omega = \frac{\|A - \rho(A)x^*_0\|_F}{\|A\|_F} = \sqrt{1 - \frac{\rho(A)^2}{\|A\|_F^2}} \geq 0.3511,$n

which implies the convergence rate of the greedy rank-one algorithm [2, 17, 18, 24, 25].

5. Conclusions

In this paper, we established a Brauer-type $Z$-eigenvalue inclusion set for even-order real tensors by $Z$-identity tensor and proposed some sufficient conditions for the positive definiteness of multivariate homogeneous forms. Note that the suitable parameter $\alpha$ has a great influence on the numerical effects and positive definiteness of $f_A(\alpha)$. Therefore, how to select the suitable parameter $\alpha$ is our further research.

Competing Interests
The authors declare that they have no competing interests.

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