Generalized Drinfeld-Sokolov Hierarchies, Quantum Rings, and W-Gravity

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Abstract

We investigate the algebraic structure of integrable hierarchies that, we propose, underlie models of W-gravity coupled to matter. More precisely, we concentrate on the dispersionless limit of the topological subclass of such theories, by making use of a correspondence between Drinfeld-Sokolov systems, principal $s\ell(2)$ embeddings and certain chiral rings. We find that the integrable hierarchies can be viewed as generalizations of the usual matrix Drinfeld-Sokolov systems to higher fundamental representations of $s\ell(n)$. Accordingly, there are additional commuting flows as compared to the usual generalized KdV hierarchy. These are associated with the enveloping algebra and account for degeneracies of physical operators. The underlying Heisenberg algebras are nothing but specifically perturbed chiral rings of certain Kazama-Suzuki models, and have an intimate connection with the quantum cohomology of grassmannians. Correspondingly, the Lax operators are directly given in terms of multi-field superpotentials of the associated topological LG theories. We view our construction as a prototype for a multi-variable system and suspect that it might be useful also for a class of related problems.
1. Introduction

There is considerable interest in the study of 2d conformal matter coupled to gravity. In particular, minimal models of type \((p, q)\) coupled to gravity can be described in terms of matrix models, whose dynamics are governed by the (generalized) KdV hierarchy. Recently, an infinite series of new theories based on \(W\)-matter coupled to \(W\)-gravity has been constructed. Since these theories appear to be on a footing similar to ordinary gravity, one might suspect that there exists an infinite sequence of new types of matrix models too, governed by certain integrable hierarchies.

The present paper is a first attempt to identify the algebraic structure of these integrable systems. We certainly would not be surprised if these “new” hierarchies ultimately turn out to be equivalent to some already known (say, multi-component) generalization of the KP-hierarchy. It seems that in particular the work in refs. is, in one way or another, related to ours. However, so far no connection of these integrable systems to concrete models of 2d-gravity was made, and such a connection will be one of the virtues of our construction.

Since we do not know the precise structure of the suspected new matrix models as yet, we cannot derive the hierarchies from first principles, i.e., in the way the KdV hierarchy can be extracted from the usual matrix models. Rather, we will follow an indirect route by making use of a connection to topological Landau-Ginzburg theory. For this, we will need to focus on a sub-class of theories, that is, on the topological models of type \((1, q)\), in the dispersionless limit (we also restrict to the “small phase space” of primary matter couplings, where this limit is exact). We reserve the extension to general \((p, q)\) for future work.

More specifically, recall that the ordinary \((1, q)\) Virasoro minimal models coupled to gravity are closely related to the twisted \(N = 2\) superconformal minimal models of type \(A_{k+1}\) (where \(k = q - 2\)). The point is that the Landau-Ginzburg potentials of these topological models, \(W(x, g)\), are very similar to the Lax operators \(L(D, g)\) of the \((1, q)\) type of KdV hierarchy. In fact, in the dispersionless limit, where \(D \to x\), \(L\) and \(W\) coincide and the KdV flow equations, which

\[\text{To prevent confusion at an early stage, note that we will not be talking about the well-known classical, Poisson bracket \(W\)-algebra of the generalized KdV hierarchy, which arises already in ordinary gravity. What we are about to discuss is an extension that sits on top of this structure. See section for an exposé of the general scheme.}\]
determine the dynamics of the \((1, q)\) matrix models, become equivalent to the Gauss-Manin equations \cite{19, 20}, which determine the correlation functions of the LG models \cite{14}.

Our strategy will be to first make an inspired guess about the structure of the integrable systems related to \(W\)-gravity, guided by the above-mentioned relationship between topological LG theory and KdV integrable systems. In particular, at the heart of our construction will be the generalization of the relationship between Landau-Ginzburg superpotential and dispersionless Lax operator to several variables.

We then would need to verify that the associated flow equations have the correct solutions. While we were not able to prove this in generality, we checked explicitly for a couple of examples related to \(W_3\) that the flow equations indeed reproduce at least the correct flat coordinates of the topological matter models. From the logic of the scheme it is fairly obvious that this feature should work in general. Note that this is, ultimately, more than just determining flat coordinates in topological matter systems: the integrable systems are supposed to describe not only the matter sector, but also the gravitational sector of a theory.

More specifically, after having recalled the relationship between \(W\)-matter-gravity systems and certain \(N = 2\) superconformal coset models, we will present in section 2 a useful characterization of the chiral rings of these coset models. It turns out that principal embeddings of \(s\ell(2)\) play a crucial rôle here, quite analogous to the well-known rôle they play in Drinfeld-Sokolov systems. In section 3, we will then first reformulate the relationship between the KdV hierarchy and the ordinary \(N = 2\) minimal models in a way that is most suitable for later generalization. We will in particular note a close correspondence between the KdV matrix Lax operators and the minimal model ring structure constants, and subsequently employ this correspondence for the generalization to \(W_3\). Section 3 will be concluded with an explicit example and with a brief discussion about some observations we made when we computed various examples. In particular, we will find that the generalized flow equations are just the integrability conditions for the existence of a “prepotential”. Finally, in section 4 we will review the philosophy of our construction and make as well some more speculative remarks.
2. Topological $W_n$ models

2.1. Equivariant cohomology of Kazama-Suzuki models and $W$-gravity

The physical theories in focus are tensor products $[21]$

\[ M_{p,q}^{(n)} \equiv \left[ M_{p,q}^{(n)} \otimes W_n \text{-Liouville} \otimes \text{ghosts} \right] \text{ with } p = 1 , \tag{2.1} \]

where $M_{p,q}^{(n)}$ denotes a $(p,q)$ type minimal model of the $W_n$ algebra $[22]$, and “$W_n$-Liouville” denotes an $sl(n)$ Toda theory that describes the coupling to $W_n$-gravity (ordinary gravity corresponds to $n = 2$). There is strong evidence $[6]$ that the sub-class of $(1,q)$ type of theories is equivalent to topological matter coupled to topological $W$-gravity $[23]$: \[ M_{1,q}^{(n)} \cong \left[ \text{CP}^\text{top}_{n-1,k} \otimes \text{topological } W_n \text{-gravity} \right] \tag{2.2} \]

Here, $\text{CP}^\text{top}_{n-1,k}$ denotes a minimal topological $W_n$ matter model $[24]$ at level $k$, which is nothing but the twisted version $[25]$ of a $N=2$ superconformal coset model $[26]$ based upon $SU(n)_k / U(n-1)$, with anomaly $c = 3k(n-1) / n + k$. The physical spectrum of (2.2) is given by a chiral ground ring $[27]$, which consists of two parts. The matter part is given by the primary chiral ring of the matter model $\text{CP}^\text{top}_{n-1,k}$, which is generated by fields $x_i, i = 1, \ldots , n - 1$ (with $U(1)$ charges $q(x_i) = i / (n+k)$) and which can be represented by

\[ \mathcal{R}_{n,k}^x = \left\{ \prod_{i=1}^{n-1} (x_i)^{m_i} \mid \sum_m \leq k \right\} . \tag{2.3} \]

The remaining part consists of the gravitational descendants and is generated by operators $\sigma_i, i = 1, \ldots , n - 1$ (with $U(1)$ charges $q(\sigma_i) = 1$). The complete ground ring thus has the form

\[ \mathcal{R}_{n,k}^{x,\sigma} = \mathcal{R}_{n,k}^x \otimes \left\{ \prod_{i=1}^{n-1} (\sigma_i)^{l_i} \mid l_i = 0, 1, 2, \ldots \right\} . \tag{2.4} \]

It is well-known that the matter models $\text{CP}^\text{top}_{n-1,k}$ have a Landau-Ginzburg realization, where the LG fields represent the ground ring generators $x_i$ above. The

\[ \text{Originally }[6], \text{ only a subset of these ring elements was considered, namely the subset that is associated with generators with charges } q(\sigma_i) = i. \]
superpotentials $W_{n,k}(x_i)$ were explicitly given in \cite{28,29} and can be compactly characterized by the following generating function:

$$-\log \left[ \sum_{i=1}^{n-1} (-t)^i x_i \right] = \sum_{k=-n+1}^{\infty} t^{n+k} W_{n,k}(x) \quad (2.5)$$

These superpotentials obey the following recursive identity that will be important in the following:

$$W_{n,k}(x) = \frac{1}{n+k} \nabla_x W_{n,k+1}(x),$$

$$\nabla_x \equiv \sum_{i=1}^{n-1} (n-i) x_{i-1} \frac{\partial}{\partial x_i} \quad (2.6)$$

This identity can easily be proven by observing that the differential operators $\nabla_x$ and

$$\nabla_t = t^2 \frac{\partial}{\partial t} + (n-1) t$$

give the same result when acting on the LHS of (2.5).

It is also known \cite{6} that by changing the cohomological definition of topological matter models $\text{CP}^\text{top}_{n-1,k}$, i.e., by requiring equivariant cohomology \cite{30}, the gravitational sector can be represented entirely in terms of the matter sector:

$$\left[ \text{CP}^\text{top}_{n-1,k} \otimes \text{topological } W_n \text{-gravity} \right] \cong \text{CP}^\text{top}_{n-1,k} \bigg|_{\text{equivariant \ cohomology}} \quad (2.7)$$

The gravitational sector is thus implicitly contained in the topological matter model, and for the LG representation this means that (in equivariant cohomology) one can represent the gravitational descendants in terms of LG fields as well. In particular, for ordinary minimal models $\text{CP}^\text{top}_{1,k} \sim A_{k+1}$, one has the following LG representatives \cite{31,32}

$$\sigma_l(x_i) \equiv (\sigma_1)^l(x_1)^i \cong x_1^{i+l(k+2)},$$

where $x_1^{(l+1)(k+2)-1} \equiv (\sigma_1)^l \partial_x W_{2,k}$ are BRST trivial. The equivariant LG spectrum for $k = 2$ is depicted in Fig.1.
Fig.1: Gravitational chiral ring associated with the model \( \text{CP}^\text{top}_{1,2} \sim A_3 \). The open dots describe null fields, and the repetitions of the matter subring \( \{1, x, x^2\} \) correspond to the gravitational excitations. Mathematically, the picture represents the highest weight representations of \( \hat{s}(2) \) at level 2; the matter subring corresponds to the subset of integrable representations. It also represents the spectrum of type \((1, k+2)\) KdV flows.

The situation is analogous for the models \( \text{CP}^\text{top}_{2,k} \) associated with \( W_3 \)-gravity, on which we will primarily focus in the following. It is convenient choose a specific ring basis in these models, namely the eigenbasis corresponding to the underlying \( W_3 \)-symmetry [33]. This basis is given by the LG polynomials

\[
\Phi^{m_1,m_2}(x_1, x_2) = x_2^{m_2} \left( \frac{\partial}{\partial x_1} W_{3,m_1-2}(x_1, x_2) \right), \quad m_{1,2} \geq 0,
\]

so that the matter chiral ring (2.3) can be represented as

\[
\mathcal{R}_{x,3,k} = \left\{ \Phi^{m_1,m_2}(x_1, x_2), \quad m_1 + m_2 \leq k \right\}.
\]

By writing \( \lambda^{m_1,m_2} = m_1 \lambda_1 + m_2 \lambda_2 \) (where \( \lambda_{1,2} \) are the fundamental weights of \( s\ell(3) \)), the ring elements (2.3) can be associated with the integrable highest weights of \( \hat{s}(3) \) at level \( k \). If one requires equivariant cohomology (in order to incorporate the \( W \)-gravitational descendants), one finds [33,34] that the LG polynomials \( \Phi^{m_1,m_2} \) corresponding to all (in general not integrable) highest weights become physical. Recalling the structure of the affine highest weights, one can thus represent the complete spectrum in terms of LG variables as follows:

\[
\mathcal{R}_{x,3,k}^{x,\sigma} = \left\{ \Phi^{m_1,m_2}(x_1, x_2) : \ (\lambda^{m_1,m_2} + \rho) \cdot \alpha_i \neq 0 \text{ mod } (k+3) \right\}
\]

(here, \( \rho \) and \( \alpha_i \) denote Weyl vector and roots of \( s\ell(3) \)). We depicted the spectrum for \( k = 2 \) in Fig.2.
Fig. 2: Landau-Ginzburg representation $\Phi^{m_1,m_2}(x_1,x_2)$ of the gravitational chiral ring associated with the model $\text{CP}^{\text{top}}_2$. The open dots describe null fields, and the repetitions of the matter “triangle” correspond to the $W_3$-gravitational excitations. The picture represents the highest weight representations of $\hat{s}l(3)$ at level 2.

It is clear that Fig. 1 represents simultaneously the spectrum of type $(1, k + 2)$ KdV flows, which describe the dynamics of the ordinary matter-gravity systems $\text{CP}^{\text{top}}_1$.

Each black dot corresponds to a flow Hamiltonian, and each open dot to a trivial flow. In the following, we attempt to find integrable systems that pertain in an analogous way to spectra of $W$-gravity systems, like to the spectrum in Fig. 2.

2.2. Chiral rings and principal embeddings of $SU(2)$

We like to characterize in this section the algebraic structure of the chiral rings $\mathcal{R}^{x}_{n,k}$, which will turn out to be closely related to the structure of the generalized KdV flow Hamiltonians.

There is a well-known duality symmetry [26,35,36] among some of the $N = 2$ superconformal coset models, and in particular there is an isomorphism between the coset models based on $rac{SU(m+l)_k}{SU(m) \times SU(n) \times U(1)}$ for all permutations of $m, l, k$. This allows to associate the $N = 2$ $W_n$ minimal models $\text{CP}^{\text{top}}_{n-1,k}$ (based on $G/k/H = \frac{SU(n)_k}{U(n-1)}$) also with the Grassmannians $G_1/H = \frac{SU(n+k-1)}{SU(n-1) \times SU(k) \times U(1)}$. The point is that for any such model where $G/H$ is a hermitian symmetric space, $G$ is simply laced and the level
of $G$ is equal to one, there exist powerful theorems about the structure of the chiral rings.

Specifically, it was noticed \[28,37\] that in such models, the chiral, primary fields are in one-to-one correspondence with the Lie-algebra cohomology, and the primary chiral ring of a given such model based on $G/H$ is isomorphic to the Dolbeault cohomology ring of $G/H$. In particular,

$$R^\tau_{n,k} \cong H^*_\overline{G}(SU(n-1) \times SU(k) \times U(1), \mathbb{R}) \ .$$

That is, a chiral, primary field with $N = 2 U(1)$ charge $\frac{p_g}{q+1}$ can be thought of as an element of $H^p_{\overline{G}}(G/H, \mathbb{R})$ ($g$ denotes the Coxeter number of $G$; for $G = SU(m)$, $g = m$). A characterization of these chiral rings in terms of Chern classes of vector bundles over $G/H$ was given in \[38,29\].

In addition, it was shown that the chiral ring of any such theory has a close relationship to a particular fundamental representation, $\Xi$, of $G$. The primary, chiral fields are one-to-one to the weights of $\Xi$ and their $U(1)$ charges are given, up to a uniform shift, by the dot product of the corresponding weight with the Weyl vector $\rho_G \equiv \frac{1}{2} \sum_{\text{positive roots}} \alpha$. The highest weight of $\Xi$ (also denoted by $\Xi$) is defined by the fundamental weight of $G$ corresponding to the node of the $G$-Dynkin diagram that defines the embedding of the $U(1)$ factor in $G$. The possible choices of $U(1)$ factors correspond to the Dynkin nodes which have Kac weight equal to one, i.e., the allowed representations $\Xi$ are the level one representations of affine-$G$. For the specific models under consideration, $\mathbb{C}P_{n-1,k}^{\text{top}}$, $\Xi$ is given by the $(n-1)$-th fundamental representation of $G = SU(n + k - 1)$, with highest weight $\Xi = (0, \ldots, 0, 1, 0, \ldots, 0)$ (where "1" appears at the $(n-1)$-th entry).

This leads to a further characterization of the chiral rings that is most useful in the present context. Crucial to this is the principal $SU(2)$ subgroup $S \subset G$ generated by

$$I_+ = \sum_{\text{simple roots } \alpha} a^{(1)}_\alpha E_\alpha$$

$$I_- = \sum_{\text{simple roots } \alpha} a^{(-1)}_\alpha E_{-\alpha}$$

$$I_0 = \rho_G \cdot H \ ,$$

where $E_{\pm \alpha}$ are, as usual, the generators of $G$ in the Cartan-Weyl basis and $a^{(\pm 1)}_\alpha$ are coefficients such that $[I_+, I_-] = I_0$. One can always take $a^{(1)}_\alpha \equiv 1$, and this is what we
will assume henceforth. The particular choice of $I_0$ induces the “principal” gradation of the generators of $G$: the $I_0$ charge of a generator $E_\alpha$ is given by $p = \rho_G \cdot \alpha$. It is a well-known mathematical fact [39] that the possible values of $|p|$ are just given by the exponents $m_i, i = 1, \ldots, \ell$, of $G$. One can group the generators into sets of equal $I_0$ grade, and build the following linear combinations:

$$ \Lambda_p = \sum_{\{\alpha: \rho_G \cdot \alpha = p\}} a^{(p)}_{\alpha} E_\alpha, \text{ for each } p \in \{\pm m_1, \pm m_2, \ldots, \pm m_\ell\}, \quad (2.13) $$

(whence $\Lambda_1 = I_+, \Lambda_{-1} = I_-$). The coefficients $a^{(p)}_{\alpha}$ (for $|p| > 1$) are determined from (2.12) by requiring:

$$ \left[ \Lambda_p, \Lambda_q \right] = 0 \quad \text{for} \quad \begin{cases} p > 0, q > 0 \\ p < 0, q < 0 \end{cases} \quad (2.14) $$

For $G = SU(m)$, one can take $\Lambda_p = \sum_{i=1}^{m-p} E_{e_i-e_i+p}$ ($p = 1, 2, \ldots, m-1$).

A useful theorem [40] can now be formulated as follows:

*In the representation $\Xi$ of $G$ defined above, the matrices $\Lambda_p$ (with $p > 0$) generate an algebra that is isomorphic to the cohomology ring $H^*_\partial(G/H, \mathbb{R})$.*

It follows that the matrices $\Lambda_p, p > 0$ represent the Landau-Ginzburg fields $x_p$ and that they are identical to the (unperturbed) ring structure constants:

$$ (C_p)_j^k(0) = (\Lambda_p)_j^k \quad (i, k = 1, \ldots, \dim \Xi). \quad (2.15) $$

Ring multiplication is represented by simple matrix multiplication. Generic ring elements are given by polynomials in $\Lambda_p$, which do not necessarily belong to the algebra of $G$; by definition, they belong to the enveloping algebra of $G$. The $U(1)$ charge of a ring element is equal to its $I_0$ grade in units of $1/(g + 1)$. Obviously, $\Lambda_1 \equiv I_+$ is always a generator of the ring, and this corresponds to the fact that in each coset model, there is a unique Landau-Ginzburg field of lowest $U(1)$ charge, $q(x_1) = \frac{1}{g+1}$.

In a sense, the powers of $\Lambda_1$ always describe the chiral ring of an ordinary $N=2$ minimal model, but if $\Xi$ is larger than the defining representation of $G$, there exist additional, independent commuting matrices (proportional to $\Lambda_{p \neq 1}$) that represent additional ring elements. It is precisely this point of view that we will take later to describe the integrable hierarchies: if one considers representations higher than

\[\text{† One can equally well restrict to } p < 0.\]
the defining representation, then there will exist extra, commuting flows besides the ordinary KdV flows.

In general, various $\Lambda_p$ can be expressed in terms of powers of other $\Lambda_q$ so that they are not independent generators; which $\Lambda_p$ are independent for a given group $G$ depends on the representation $\Xi$, i.e., on the choice of $H$. We will denote in the following the independent generators by $\Lambda_i$, $i = 1, \ldots, M$. For the grassmannians associated with $W_n$ gravity, there are $M \equiv (n-1)$ independent generators with degrees $1, \ldots, (n-1)$, in accordance with (2.3). The independent generators represent the independent Landau-Ginzburg fields $x_i$, and satisfy certain relations. There is always a subset of relations that generate all other ones; it is non-trivial [28] that these generating vanishing relations are always of the form $\partial x, W(x) = 0$ for some $W$ and thus can be interpreted as the equations of motion of a Landau-Ginzburg theory with superpotential $W$.

The fact that $\Lambda_i$ represents the Landau-Ginzburg field $x_i$, when acting on the space of primary chiral fields, can be expressed by the following matrix system:

$$\begin{bmatrix} x_i \mathbf{1} - \Lambda_i \end{bmatrix} \cdot \Psi = 0, \quad \Psi_j \in \mathcal{R}^x .$$

The components of the solution vector $\Psi$ represent the weights of $\Xi$ and can be recursively solved for. The solution $\Psi_j(x_i)$ in terms of polynomials in the Landau-Ginzburg fields thus gives the precise relationship between the weights of $\Xi$ and the Landau-Ginzburg fields. When all unknown $\Psi_j$ are eliminated in favor of the first component $\Psi_0 \equiv 1$, the system (2.16) is reduced to a set of vanishing relations for the Landau-Ginzburg fields $x_i$. One can thus regard (2.16) as fundamental equations characterizing the chiral ring. As we will see, $\Lambda_i$ are closely related to Lax operators of integrable systems.

To give some examples, consider first the models $\text{CP}^{\text{top}}_{1,k}$, which are the same as the ordinary twisted $N=2$ minimal models of type $A_{k+1}$. These models can be associated to cosets $SU(k+1)_1/U(k)$, and $\Xi$ is the defining representation of $SU(k+1)$. There is just one independent generator of the chiral ring, which can be represented by the familiar, ubiquitous $(k+1) \times (k+1)$ dimensional matrix

$$\Lambda_1 \equiv I_+ = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix} ,$$

(2.17)
in terms of which the other $\Lambda_p$ are given by

$$\Lambda_p = (\Lambda_1)^p, \quad p = 1, 2, \ldots, k.$$  \hspace{1cm} (2.18)

The vanishing relation is

$$(\Lambda_1)^{k+1} = 0,$$  \hspace{1cm} (2.19)

and this corresponds to the vanishing relation in $H^*(\mathbb{C}P_k)$, and to the equation of motion of a LG model with superpotential $W_{2,k}(x,0) = \frac{1}{k+2}x^{k+2}$. The matrix equation $[x1 - \Lambda_1] \Psi = 0$ has as solution vector

$$\Psi = (1, x, x^2, \ldots, x^k)^t, \quad \Psi_j \in \mathcal{R}^{x}_{2,k},$$  \hspace{1cm} (2.20)

and leads to

$$x^{k+1} \Psi_0 = 0.$$  

This is just the characteristic equation

$$0 = \det[x1 - \Lambda_1] = x^{k+1},$$  \hspace{1cm} (2.21)

and thus is the same as the relation (2.19) that the matrix $\Lambda_1$ itself satisfies.

We now turn to the more interesting models $\mathbb{C}P_{2,k}^{\text{top}}$, on which we will primarily focus in the paper. These topological minimal $W_3$ models are associated with the Grassmannians $G_1/H = \frac{SU(k+2)}{SU(k) \times SU(2) \times U(1)}$, where $\Xi$ is the $\mathbb{B}$-representation $(0, 1, 0, \ldots, 0)$, with $\dim \Xi = \frac{1}{2}(k+1)(k+2) = \dim \mathcal{R}^{x}_{3,k}$. The independent ring generators are $x_1 \simeq \Lambda_1$ and $x_2 \simeq \Lambda_2$, with $\Lambda_2 \neq (\Lambda_1)^2$ (for $k \geq 2$).

For example, for $k = 3$ one has in the ten dimensional representation of $SU(5)$:

$$
\Lambda_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
\Lambda_2 = \begin{pmatrix}
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$  \hspace{1cm} (2.22)
where $\Lambda_3 = \frac{1}{2} \Lambda_1^3 + \frac{3}{2} \Lambda_1 \Lambda_2$, $\Lambda_4 = -\frac{1}{2} \Lambda_1^4 - \frac{5}{2} \Lambda_1^2 \Lambda_2$ are not independent; the generators satisfy the relations $\Lambda_3 \Lambda_2 + \Lambda_1^2 = \Lambda_2^2 - \Lambda_4^2 - 4 \Lambda_1^2 \Lambda_2 = 0$.

For general $k$, the ring generators satisfy certain relations that are the analogs of the characteristic equation (2.21). These relations are generated by solving the matrix system

$$[x_1 \mathbf{1} - \Lambda_1] \cdot \Psi = [x_2 \mathbf{1} - \Lambda_2] \cdot \Psi = 0,$$

and integrate exactly to the superpotentials given in (2.25), up to reparametrizations. More precisely, we find that the superpotentials associated with the ring structure constants $\Lambda_{1,2}$ are given by $W_{3,k}(x_1, \frac{1}{2} x_2 + \frac{1}{2} x_1^2)$, and we will consider (implicitly) only this parametrization of the superpotentials in the following. In this parametrization, the relevant recursion relation (2.6) looks

$$W_{3,k}(x,0) = \frac{1}{k+3} [ \partial_{x_1} - x_1 \partial_{x_2} ] W_{3,k+1}(x,0).$$

Specifically, the first few such potentials for the models $\text{CP}^{\text{top}}_{2,k}$ are

$$W_{3,0}(x) = \frac{1}{6} x_1^3 + \frac{1}{2} x_1 x_2 \quad \text{(trivial, } c = 0)$$
$$W_{3,1}(x) = \frac{1}{4} x_1^4 - \frac{1}{4} x_2^2 + \frac{1}{2} x_1^2 x_2 \quad \text{(A}_3, \ c = \frac{3}{2})$$
$$W_{3,2}(x) = \frac{1}{6} x_1^5 - x_1 x_2^2 \quad \text{(D}_6, \ c = \frac{12}{5})$$
$$W_{3,3}(x) = \frac{1}{3} x_2^3 - x_1^4 x_2 - 2 x_1^2 x_2^2 \quad \text{(J}_{10}, \ c = 3),$$

where we also indicated the singularity types as well as the central charges of the corresponding $N = 2$ theories. For $k = 1$, the $N = 2$ $W_3$ algebra truncates to the ordinary $N = 2$ superconformal algebra, so that the $D_6$ model represents the simplest non-trivial $N = 2$ $W_3$ minimal model with a non-null chiral spin-3 current.

The components of the solution vectors to (2.23) are given, up to reparametrization and vanishing relations, precisely by the $W_3$-eigenpolynomials:

$$\{ \Psi_i \} = \{ \Phi^{m_1, m_2}(x_1, \frac{1}{2} x_2 + \frac{1}{2} x_1^2), m_1 + m_2 \leq k \} = \mathcal{R}_{3,k}^x$$

$$= \{ 1, x_1, \frac{1}{2} (x_1^2 + x_2), \frac{1}{2} (x_1^2 - x_2), x_1 x_2, x_1^3 + x_1 x_2, \ldots \}.$$
3. Matrix formulation of integrable hierarchies

3.1. Dispersionless KdV hierarchy

In order to present further below a matrix formulation of the ordinary KdV hierarchy [41,9,42] (pertaining to the LG models $A_{k+1} \cong CP^{top}_{1,k}$), we will first review some facts about the quasi-classical limit of the scalar Lax formulation.

The scalar Lax operator is given by

$$L(D,g) = \frac{1}{k+2}D^{k+2} - \sum_{i=0}^{k} g_{i+2}(t) D^{k-i}, \quad (3.1)$$

where $D \equiv \partial_{\xi}$, and where $g_i$ are certain functions of the KdV times $t_i$ and the cosmological constant $\xi$. (We will restrict ourselves to the “small phase space”, i.e., the non-vanishing couplings correspond to the topological primary fields: $t_i = 0$ for $i \leq 1$. A subscript will always denote the scaling degree.) The functions $g(t,\xi)$ are determined by the flow equations

$$\partial_{t_{i+2}} L(D,g) = [\Omega_{k+1-i}, L](D,g), \quad (3.2)$$

$i = 0,\ldots,k$, where

$$\Omega_i(D,g) = \frac{1}{i}((k+2)L)^{\frac{1}{i+2}}(D,g) \quad (3.3)$$

are the hamiltonians. The subscript “+” denotes, as usual, the truncation to non-negative powers of $D$. The associated linear system is [41]

$$[\partial_{t_{i+2}} - \Omega_{k+1-i}] \cdot \Psi_0 = 0, \quad (3.4)$$

together with

$$L(D,g) \Psi_0 = \frac{1}{k+2} z \Psi_0, \quad (3.5)$$

where $\Psi_0$ is the Baker-Akhiezer function and $z$ the spectral parameter.

Boundary conditions are imposed via the string equation $[L,D] = 1$, that is,

$$D g_i = \delta_{i,k+2} \quad \rightarrow \quad \xi \equiv t_{k+2}. \quad (3.6)$$

† With “KdV hierarchy” we will always mean the type $(1,k+2)$ generalized KdV hierarchy.
Since for the topological models $D$ never acts on $g_i$ ($i \neq k + 2$), one may regard $D$ as a $c$-number variable, and call it $x$. The Lax operator turns then into the LG superpotential of type $A_{k+1}$:

$$W_{2,k}(x, g) = L(D \rightarrow x, g). \quad (3.7)$$

(for convenience, we will write $W(x, g) \equiv (k+2) W_{2,k}$ in the following). This amounts to going to the dispersionless limit of the KdV hierarchy \[18\], where one replaces the conjugate variables $(\xi, D)$ by $(\xi, x)$ and the commutator in (3.2) by a Poisson bracket, $\{\Omega, W\} \equiv \partial_x \Omega \partial_\xi W - \partial_\xi \Omega \partial_x W$. Note that this limit is exact for the topological models \[3\], as long as one restricts to the small phase space. Since $\partial_\xi \Omega = 0$ on the small phase space hamiltonians, the flow equations become

$$- \partial_{t_{i+2}} W(x, g(t)) = \partial_x \Omega_{k+1-i}(x, g(t)). \quad (3.8)$$

These equations determine the dependence of the couplings $g(t)$ on the KdV times $t_i$, which now have the interpretation as flat coordinates on the LG deformation space. The flat fields, i.e., the perturbed ring elements in a flat basis, are \[17\]

$$\phi_i(x, t) = \partial_x \Omega_{i+1}(x, g(t)), \quad i = 0, \ldots, k. \quad (3.9)$$

Note that one can define hamiltonians $\Omega_i$ for arbitrary integers $i \geq 1$; these describe flows associated with the gravitational descendants \[32\]:

$$\sigma_l(\phi_i)(x, t) = \partial_x \Omega_{i+(k+2)l+1}(x, g(t)). \quad (3.10)$$

The above operators are flat in the sense that the associated Gauss-Manin connection vanishes \[19,20,32\].

Note that one can write

$$W(x, t) = \partial_x V(x, t), \quad (3.11)$$

so that the flow equations are equivalent to

$$\Omega_{k+1-i}(x, t) = - \partial_{t_{i+2}} V(x, t). \quad (3.12)$$

Thus, one may regard the prepotential $V$ as a more fundamental object, and indeed, $V$ is precisely the potential of the Kontsevich model \[43\], which is the matrix model that underlies the topological matter-gravity system.
3.2. Matrix formulation

In the spirit of the KP hierarchy, one may define \[42\] a “pseudo-polynomial”,
\[
K = x + \sum c_l x^{-l},
\]
by
\[
K^{k+2}(x, x^{-1}, g) = W(x, g).
\]
(3.13)
Accordingly, the LG field, viewed as a multi-valued functional, can be expanded in an infinite series:
\[
x(W(g)) = K + \mathcal{O}(K^{-1}).
\]
(3.14)
In terms of \(K\), the hamiltonians (3.3) are given by
\[
\Omega_i(x, g) = \frac{1}{i} (K^i(x, x^{-1}, g))_+ =: \frac{1}{i} K^i + \sum_{l=1}^\infty b^i_l(g) K^{-l},
\]
(3.15)
where “+” denotes the truncation to non-negative powers of \(x\) in the expansion
\[
K^i(x, x^{-1}, g) =: x^i + \sum_{l=2}^\infty c^i_l(g) x^{i-l}.
\]
(3.16)
In terms of these quantities, the flow equations (3.8) are equivalent to
\[
\partial_t x(g(t)) = \partial_\xi \Omega_{k+1-i}(g(t)) ,
\]
(3.17)
which is actually part of the zero curvature system
\[
[\partial_{t_{i+2}} - \Omega_{k+1-i}, \partial_{t_{j+2}} - \Omega_{k+1-j}] = 0
\]
(3.18)
\((\Omega_1 \equiv x, \xi \equiv t_{k+2}, i, j = 0, \ldots, k)\). Note that the hamiltonians commute identically.

The above can given a matrix representation in the following way. One first needs to find a \((k+2) \times (k+2)\) dimensional matrix Lax operator such that the first order Drinfeld-Sokolov system
\[
\left[ D1 - \mathcal{L}_1 \right] \cdot \Psi = 0
\]
(3.19)
reproduces, upon recursively solving for the components of \(\Psi\), the spectral equation (3.5). It is well-known \[41\] that this operator has the form
\[
\mathcal{L}_1 = \mathcal{L}_1^{(z)} + Q_1(g),
\]
(3.20)
\[\dagger\] We will use the term “Lax operator” for the derivative-free piece \(\mathcal{L}_1\).
where $\Lambda_1^{(z)}$ can be taken as
\[
\Lambda_1^{(z)} = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
z & 0 & 0 & \ldots & 0
\end{pmatrix},
\tag{3.21}
\]
and where $Q_1$ is usually taken to be a lower triangular matrix that is determined only up to gauge transformations belonging to the nilpotent subgroup $N^-$. In the dispersionless limit, the spectral equation (3.3) becomes $[W(x,g) - z] \Psi_0 = 0$, which is the equation that is supposed to be obtained by recursively solving for the components of $\Psi$ in $[x_1 1 - L_1] \Psi = 0$. Since this is precisely the characteristic equation of $L_1$, which captures the gauge invariant content of (3.19), it follows that the matrix itself must satisfy
\[
W(L_1, g) = z 1.
\tag{3.22}
\]
This “superpotential spectral equation” will prove crucial for our purposes and can be taken as the definition of $L_1$ in terms of the Landau-Ginzburg superpotential; it (non-uniquely) determines $Q_1(g)$. The gauge freedom can be fixed by going to any particular gauge. The choice that is most appropriate for us is however not given by taking $Q_1$, as usual, to be a lower triangular matrix, but by taking $Q_1$ to belong to the Heisenberg subalgebra generated by $\Lambda_1^{(z)}$ ($Q_1$ is then lower triangular only up to $O(1/z)$). That is, we have an infinite expansion
\[
L_1(g) = \Lambda_1^{(z)} + \sum_{l=1}^{\infty} q_l(g)(\Lambda_1^{(z)})^{-l},
\tag{3.23}
\]
whose coefficients can be computed from (3.22) in a recursive way.

Comparing with (3.14), we see that $L_1$ is a matrix representation of the LG field $x$ and that $\Lambda_1^{(z)}$ is a matrix representation of $K$. The defining equation (3.13) of $K$ is trivially satisfied:
\[
K^{k+2} \equiv (\Lambda_1^{(z)})^{k+2} = z 1 = W(L_1, g).
\tag{3.24}
\]
We can thus interpret the foregoing equations as matrix equations, and it is this form of the hierarchy that is most useful for our generalization. In particular, the flow equations are
\[
\partial_{t_{i+2}} \Omega_{k+1-j}(g(t)) = \partial_{t_{j+2}} \Omega_{k+1-i}(g(t)) ,
\Omega_i(L_1(g), g) \equiv \frac{1}{i} (\Lambda_1^{(z)}(L_1(g), L_1^{-1}(g), g))^i_+ , \quad \Omega_1 \equiv L_1 .
\tag{3.25}
\]
Obviously, the flows associated with $\Omega_{l(k+2)} = \frac{1}{l(k+2)} z^l 1$ are trivial, and correspond to perturbations by the null operators $\sigma_l(\phi_{k+1})$. 

\[ - 15 - \]
3.3. Heisenberg algebras and “quantum” chiral rings

We notice that the matrix $\Lambda_1^{(z)} \ (3.21)$ figuring in the $(1, k+2)$-type KdV hierarchy has the same form as the structure constant matrix $\Lambda_1 \ (2.17)$ of the $A_{k+1}$-type chiral rings (apart from the spectral parameter $z$). In both cases, the matrix represents a principally embedded $s\ell(2)$ step generator $I_+$. Note, however, that despite of this similarity, the hamiltonians $\Omega_i(t) = (\Lambda_1^{(z)})^i + O(t)$ that figure in the KdV equations are $(k+2) \times (k+2)$-dimensional matrices, whereas the ring structure constants $C_i(t) = (\Lambda_1)^i + O(t)$ are only $(k+1) \times (k+1)$ dimensional. The point is that the KdV system does not only describe the matter sector, but also the null state and gravitational sector of the theory (which is non-trivial even for trivial matter where $k = 0$). More precisely, the perturbed version of (2.16) is

$$
\left[x_1 - C_1(t)\right] \cdot \Psi = 0 \, , \quad (3.26)
$$

and the solution vector components are the flat fields, $\Psi_i \equiv \phi_i(x, t), i = 0, \ldots, k$. One the other hand, the KdV linear system (3.19), (3.23) in the dispersionless limit is:

$$
\left[x_1 - \Omega_1(t, z)\right] \cdot \Psi = 0 \, . \quad (3.27)
$$

Here one obtains in a recursive manner the flat fields $\phi_i, i = 0, \ldots, (k + 1)$, plus all their gravitational descendants, organized by $z$-series expansion:

$$
\Psi_i(x, t, z) = \sum_{l=0}^{\infty} z^{-l} \psi_{i,l}(x, t, z) \quad (3.28)
$$

where

$$
\psi_{i,l}(x, t, 0) = -\partial_x \psi_{i,l+1}(x, t, z) = \frac{1}{(i+(k+2))l+1} \partial_x \left(W(x, t)\right)^{i+l+1} \equiv \sigma_l(\phi_i)(x, t) \, . \quad (3.29)
$$

We thus see that the LG field $x$ can be represented either by the ring structure constant $C_1$ or (formally) by the KdV Lax operator $L_1 \equiv \Omega_1$, depending on whether the space it acts upon is the finite dimensional Hilbert space of the topological matter model, or the infinite dimensional space of “flat” polynomials $\sigma_l(\phi_i)$ of the matter-gravity system. To obtain the Hilbert space of the matter-gravity system, one needs to mod out the space of polynomials $\{\sigma_l(\phi_i)\}$ by the null fields, $\sigma_l(\phi_{k+1}) \equiv (\partial_x W)^l$; it is
therefore precisely the null fields that account for the difference in the dimensions of
the matrices $C_1$ and $\Omega_1$.

Our plan is to make use of this correspondence between chiral ring structure
constants $C_i$ and hamiltonians $\Omega_i$ for the generalization to $W$-gravity, by relating the
hamiltonians to certain other, higher dimensional chiral rings.

The relevant underlying algebraic structure of the hamiltonians, $\Omega_i \equiv \Lambda^{(z)}_i$, that gives the complete characterization of the KdV flows (including the trivial ones), is given by

$$\tilde{H}^+ \equiv \{(\Lambda^{(z)}_1)^m, m \in \mathbb{Z}_+\}.$$ (3.30)

This is the positive part of the infinite dimensional, maximally commuting principal
Heisenberg subalgebra $\mathcal{H} \subset \hat{sl}(k+2)$ generated by

$$\Lambda^{(z)}_1 \equiv (\sum_{\text{simple roots } \alpha} E_\alpha) + z E_{-\psi},$$ (3.31)

where $\psi$ denotes the highest root. Such an extension of a Lie algebra to an affine
algebra, via addition of $(-\psi)$ to the set of simple roots, is well-known to play an
important rôle in the context of integrable perturbations. Indeed, $\Lambda^{(z)}_1$ is identical to
the chiral ring structure constant $C_1(z)$ associated with the following perturbed LG
potential “at one level higher”:

$$W_{2,k+1}(x, t_{k+2} = z, t_l = 0) =$$

$$\frac{1}{k+3} x^{k+3} - z x.$$ (3.32)

That is, the underlying algebraic structure, $\mathcal{H}^+$, of the $A_{k+1}$ matter-gravity integrable
system is that of a specifically deformed chiral ring pertaining to the LG theory $A_{k+2}$:

$$\mathcal{H}^+ \cong \mathcal{R}_{2,k+1}(t_{k+2} = z, t_l = 0).$$ (3.33)

Mathematically, the deformation by the spectral parameter $z$ is precisely what deforms
the cohomology ring (2.11) $H^*$ into the quantum cohomology ring $QH^*$ [47], whence

$$\mathcal{H}^+ \cong QH^*_g(CP_{k+1}, \mathbb{R}).$$ (3.34)

(The word “quantum” indicates that the deformation of the classical cohomology ring
by the spectral parameter $z$ is precisely the effect of the instanton corrections in a

---

† This perturbation is known to be quantum integrable and can be described in terms of affine
Toda theory [45, 38, 46].
supersymmetric CP$_{k+1}$ σ-model. From this viewpoint, the gravitational descendant sectors correspond to the non-trivial instanton sectors of the σ-model).

Most importantly, the spectral equation (3.3) in the dispersionless limit is precisely the vanishing relation of the LG theory (3.32) at one level higher,

$$W_{2,k}(x,0) - \frac{1}{k+2} z = \frac{1}{k+2} \partial_x W_{2,k+1}(x,z) ,$$

and this is what is at the heart of the matrix spectral equation (3.22), $W_{2,k}(\Lambda_1^{(z)}) \equiv \frac{1}{k+2} (\Lambda_1^{(z)})^{k+2} = \frac{1}{k+2} z 1$.

We will take these facts, which are rather trivial here for one LG variable, as starting points for our generalization.

### 3.4. Generalization to $W_3$

We are now in the position to formulate, tentatively, a generalization of the dispersionless KdV hierarchy to several LG variables. As we have seen in section 2.2, the chiral rings $\mathcal{R}_{n,k}^x$ for $n > 2$ are characterized by taking for $\Xi$ not the defining representation, but the $(n-1)$th fundamental representation of $SU(n+k-1)$. Thus, the most direct extension of the KdV system would be to just consider matrix Lax operators in the corresponding higher fundamental representations of $SU(n+k)$. As we will see, this amounts to describe the integrable systems in terms of perturbed chiral rings at one level higher, $\mathcal{R}_{n,k+1}^x$, just like for $n=2$.

Specifically, focusing on $W_3$ at level $k$ associated with chiral rings $\mathcal{R}_{3,k}^x$, we consider the perturbed $SU(2)$ generator

$$\Lambda_1^{(z)} \equiv \Lambda_1 + z \Lambda_{-(k+2)}$$

now in the $\frac{1}{2}(k+2)(k+3)$ dimensional representation of $SU(k+3)$. From section 2.2 we know that in this representation there exists an additional, independent ring generator at grade two; requiring it to commute with $\Lambda_1^{(z)}$, we find

$$\Lambda_2^{(z)} = \Lambda_2 + z \Lambda_{-(k+1)} .$$

\[\Diamond\text{ Heuristically, one may view this shift } k \rightarrow k + 1 \text{ as due to the well-known shift of the } \hat{\mathfrak{sl}}(n) \text{ integrable weights by the Weyl vector } \rho_G.\]
The generators $\Lambda_1^{(z)}, \Lambda_2^{(z)}$ are identical to the ring structure constants corresponding to the following LG potential with (integrable) perturbation: $W_{3,k+1}(x, z) \equiv W_{2,k+1}(x, t_{k+3} = z, t_i = 0) - \alpha z x_1$ ($\alpha$ is some numerical factor that we will neglect in the following). Using (2.24) we can thus write:

$$W_{3,k}(x_1, x_2, 0) - \frac{1}{k+3} z = \frac{1}{k+3} [\partial x_1 - x_1 \partial x_2] W_{3,k+1}(x_1, x_2, z). \quad (3.38)$$

This gives us the rationale for considering this construction, since what (3.38) means is that we have found matrices that satisfy a generalized super potential spectral equation

$$W_{3,k}(\Lambda_1^{(z)}, \Lambda_2^{(z)}, 0) = \frac{1}{k+3} z 1. \quad (3.39)$$

This is indeed precisely what we have been looking for: $\Lambda_1^{(z)}, \Lambda_2^{(z)}$ play the rôle of $K$ that represents the $(k+2)$-th root of the ordinary, dispersionless Lax operator (cf., (3.24)). In fact, writing down such a matrix equation is non-trivial, since for large $k$ the superpotential $W_{3,k}(\Lambda_1^{(z)}, \Lambda_2^{(z)}, 0)$ has an arbitrary number of terms whose relative coefficients are fixed and correspond to the correct, specific point in the LG moduli space. Note also that by virtue of (2.6), equation (3.39) can be immediately generalized to all $W_n$.

We now introduce LG couplings $g(t)$ according to

$$W_{3,k}(x, g) = W_{3,k}(x_1, x_2, 0) - \sum_{m_1 + m_2 \leq k} g_{m_1, m_2}(t) \Phi_{m_1, m_2}(x_1, x_2), \quad (3.40)$$

where $\Phi_{m_1, m_2}$ denotes the unperturbed ring elements (2.26). Writing $W \equiv (k+3)W_{3,k}$, the spectral equation that generalizes (3.22) then looks

$$W(L_1, L_2, g) = z 1, \quad (3.41)$$

which involves perturbed Lax operators of the form

$$L_1(g, z) = \Lambda_1^{(z)} + Q_1(g) \equiv \Lambda_1^{(z)} + \sum_{l,m} q_{l,m}(g) (\Lambda_1^{(z)})^{-l}(\Lambda_2^{(z)})^{-m}$$
$$L_2(g, z) = \Lambda_2^{(z)} + Q_2(g) \equiv \Lambda_2^{(z)} + \sum_{l,m} q_{l,m}(g) (\Lambda_1^{(z)})^{-l}(\Lambda_2^{(z)})^{-m}. \quad (3.42)$$

The superpotential spectral equation (3.41) corresponds to the multi-variable analog of the condition $W(L_1, g)_- \equiv (\Lambda_1^{(z)}(L_1, L_1^{-1}))^{k+2}_- = 0$, which implements the reduction from the KP to the generalized KdV hierarchy.
In order to obtain flow equations that will ultimately determine the couplings \( g(t) \) as functions of the flat coordinates, we need first to construct appropriate hamiltonians. As we will see, the structure of these hamiltonians is considerably more complicated as before (c.f., (3.25)).

The hamiltonians for a given integrable system are usually directly given in terms of the underlying Heisenberg algebra: \( \Omega = (H_+)^+ \). However, in the present context, where we consider representations larger than the defining representation, there exist in general more commuting matrices at a given grade than there are elements of the Heisenberg algebra at this grade (e.g., \( (\Lambda_1^{(z)})^2 \) does not belong to \( H_+ \)). A priori, any commuting matrix may give rise to a valid flow hamiltonian. Therefore, the appropriate algebraic structure to look at is the enveloping algebra \( \tilde{H}_+ \) of the principal Heisenberg algebra \( H_+ \) that consists of the complete set of commuting matrices. It is given by the perturbed chiral ring corresponding to the potential \( W_{3,k+1}(x_1, x_2, z) \) (3.38)

\[
\tilde{H}_+ \equiv \left\{ (\Lambda_1^{(z)})^{l_1}(\Lambda_2^{(z)})^{l_2}, \ l_i \geq 0 \right\} \\
\cong \mathcal{R}_{k+3}(t_{k+3} = z, t_l = 0). \tag{3.43}
\]

To characterize it mathematically, note that in precise analogy to (3.34), this ring is isomorphic to the “quantum” cohomology ring [47] of the underlying grassmannian:

\[
\tilde{H}_+ \cong QH^*_g\left( SU(k+3) / SU(k+1) \times SU(2) \times U(1) \right). \tag{3.44}
\]

The structure of \( \tilde{H}_+ \) (3.43) is more complicated than what the notation in (3.43) might suggest, in that \( \Lambda_1^{(z)}, \Lambda_2^{(z)} \) satisfy non-trivial relations. These relations, among which is the superpotential spectral equation \( W(\Lambda_1^{(z)}, \Lambda_2^{(z)}, 0) = z1 \), truncate the Fock space generated by \( \Lambda_1^{(z)}, \Lambda_2^{(z)} \) such that there is only a finite number of different Heisenberg algebra elements at any given grade. Accordingly, we can write

\[
\tilde{H}_+ = \bigoplus_{i=1,\ldots,k+3} \widetilde{H}_{i;l}, \\
\tilde{H}_{i;l} = \left\{ z^l \otimes \left[ \oplus_{j=1}^{b_i} \Lambda_{i;j}^{(z)} \right] \right\}, \\
b_i \equiv \dim \widetilde{H}_{i;\ast}, \quad \sum b_i = \dim \Xi, \tag{3.45}
\]

‡ This corresponds to “type II hierarchies” in the nomenclature of ref. [4].
where $\{\Lambda^{(z)}_{i,j}\}$ denotes the set of Heisenberg algebra elements at grade $i$. The fact that $b_i$ may be larger than one reflects the possibility of having several degenerate flows at a given grade. (In the following, we will drop the redundant label $l$ and take $i \in \mathbb{Z}_+$.)

Candidate commuting Hamiltonians are given by

$$
\Omega_{i,j}(\mathcal{L}_1(g), \mathcal{L}_2(g), g) = (\Lambda^{(z)}_{i,j}(\mathcal{L}_1, \mathcal{L}_1^{-1}, \mathcal{L}_2, \mathcal{L}_2^{-1}, g))_+, \quad \Lambda^{(z)}_{i,j} \in \tilde{\mathcal{H}}^+,
$$

(3.46)

where the subscript ”+” indicates the truncation to positive powers of both $\mathcal{L}_1$ and $\mathcal{L}_2$ in the Laurent expansion of $\Lambda^{(z)}_{i,j}$. The flow equations can then be written in the zero curvature form

$$
[D_{i,j}, D_{i', j'}] = 0,
$$

$$
D_{i,j} \equiv \frac{\partial}{\partial t_{i,j}} - \sum_{m=1}^{b_k+i-4} Z^{(m)}_{i,j} \Omega_{k+4-i, m}(g(t)),
$$

(3.47)

where $Z$ are normalization constants (we can always put $Z^{(m)}_{1,j} = \delta_{j,1} \delta_{m,1}$). Whether these equations really are meaningful or not is a question that we cannot decide at this point. However, as we will explain in the next section, the equations have indeed precisely the correct, unique solutions $g(t)$ for the examples that we checked explicitly.

Note that the degeneracy of the Hamiltonians is equal to or larger than the degeneracy of the ring elements at the corresponding grade, so that the equations are consistent; this can be deduced from (3.43). If the degeneracy of the Hamiltonians $\Omega$ at a given grade is larger than the degeneracy of the corresponding ring elements, then there should be non-trivial relations between the coefficients $Z$. This means that the normalization constants $Z$ should be non-trivially determined by the flow equations (modulo trivial rescalings of the $t_{i,j}$), and, as a consequence, that in general not all Heisenberg algebra elements will generate independent non-trivial flows. The situation is actually more involved than that, as we will see momentarily.

The extra complication comes from the fact that the single equation (3.41) cannot fully determine all the coefficients $q(g)$ of the matrices $\mathcal{L}_{1,2}$ simultaneously; for this, one actually needs to impose still another relation between $\mathcal{L}_1$ and $\mathcal{L}_2$. To understand the correct choice of the extra relation, recall what the rôles of the superpotential
spectral equation \((3.41)\) is in terms of the hamiltonian flows. The point is that the superpotential represents a constant hamiltonian

\[
\left( \Lambda_{k+3,1}^{(z)} \right)_+ = W(L_1, L_2, g(t))_+ = z1 = \text{const},
\]

which leads to a trivial flow. This trivial flow is just the flow that is associated with the null field \(\partial_{x_1} W\). It is then clear that the extra relation we need to impose should correspond to a constant hamiltonian that is related to the other null field, \(\partial_{x_2} W\). This means that the correct extra relation should have the form

\[
\tilde{W}(L_1, L_2, \tilde{g}(t)) = \text{const} \in \tilde{\mathcal{H}}_{k+2;0},
\]

where \(\tilde{W}\) has degree \(k + 2\) and whose explicit form is a priori undetermined and is supposed to be determined from the flow equations \((3.47)\).

The other vanishing relation, \(\partial_{x_2} W_{3,k+1}(\Lambda^{(z)}) = 0\), that the matrices \(\Lambda_1^{(z)}, \Lambda_2^{(z)}\) satisfy, implies that

\[
(\partial_{x_2} W_{3,k+1})(L_1(g), L_2(g)) \bigg|_{g=0} = 0,
\]

and this means that the expansion of constant Heisenberg algebra elements \((3.46)\), which yields the hamiltonians, is in general not unique, but rather is determined only up to terms proportional to \(\partial_{x_2} W_{3,k+1}\). For example, the following hamiltonians are a priori defined only up to certain additional free parameters \(\alpha\):

\[
\Omega_{k+2,j}(L_1, L_2, g, \tilde{g}, \alpha) = \left( \Lambda_{k+2,j}^{(z)}(L_1, L_1^{-1} L_2, L_2^{-1}, g, \tilde{g}) \right)_+ + \alpha_{k+2,j} (\partial_{x_2} W_{3,k+1})(L_1, L_2).
\]

It turns out that, in general, some of the extra parameters \(\alpha\) will be fixed by the flow equations, and the dependence of the hamiltonians on the surviving free parameters will just reflect the freedom of adding terms proportional to vanishing relations to the ring elements. That is, if \(\phi(x, t)\) represents a perturbed chiral ring element in a flat basis, then \(\phi(x, t) + \sum p_a(x)(\partial_{x_a} W(x, t))\) (for some polynomial \(p\) of appropriate degree) is a flat ring element as well.

The fact that the hamiltonians are not a priori completely defined in terms of the Heisenberg algebra \(\tilde{\mathcal{H}}^+\), but initially depend on some free parameters \(Z, \alpha\), is in contrast to the ordinary KdV system, where all hamiltonians \((3.10)\) are completely
determined in terms of powers of $\Lambda_1^{(2)}$ (up to normalization); this is precisely because there is just one candidate Hamiltonian at any given grade.

As a consequence of these ambiguities, the Hamiltonians (3.46) associated with the gravitational descendants are not completely determined in terms of the small phase space flow equations. Thus, in order to construct the gravitational descendants, one needs to consider the corresponding flow equations to fix these parameters. Since these equations involve couplings to the gravitational descendants, one would need to step beyond the small phase space and the dispersionless limit. This is beyond the scope of the present work, and thus we cannot gain insight into the precise structure of the $W$-gravitational descendants at this point.

3.5. An example

We like to demonstrate our ideas by presenting as an example the model $\mathbb{CP}^2_{\kappa}$ for $\kappa = 2$. We actually also fully computed the model with $\kappa = 1$, which is however not that much interesting, and obtained many terms of the Lax operators of the model with $\kappa = 3$ as well. The latter model, though more interesting, turns out to be technically too involved for us to be solved completely, and we decided to refrain from writing down the correct, but partial results we got. However, as far as we can see, the treatment of this $\kappa = 3$ model is indeed completely parallel to $\kappa = 2$, except for the appearance of a dimensionless modulus $t_0$, which is at the origin of the technical complications.

The reader might object that since the model $\mathbb{CP}^2_{2,2}$ is identical to the twisted $N=2$ minimal model of type $D_6$, the example will not be very meaningful, and also that it has been already discussed in the literature [14,48,49]. However, this model really is a true $N=2$ $W_3$ minimal model with a non-zero spin-3 current, and exhibits non-trivial degenerate, extra flows precisely according to the scheme that we have in mind. Moreover, our treatment of this model is quite different from the treatment in the literature, where the variable $x_2$ is eliminated at the expense of introducing terms proportional to $(x_1)^{-1}$; this is reflected in the appearance of inverse powers of $D$ in the differential scalar Lax operator [14,48,49]. In our approach we do not eliminate

\[ [x_1 - \Lambda_p - Q_p(g)] \Psi = 0, \quad p > 1. \]

Such inverse powers in scalar Lax operators derived from $[x_1 - \Lambda_1 - Q_1(g)] \Psi = 0$ occur whenever the representation $\Xi$ is reducible under the principal $SU(2)$ subgroup $S$ [50]. In our approach, negative powers do not arise due to the additional equations $[x_p - \Lambda_p - Q_p(g)] \Psi = 0, \quad p > 1$. In our approach we do not eliminate
We have $b_2 = 2$, and this is precisely how our treatment of the model $\text{CP}^{\text{top}}_{2,2}$ captures the essence of our construction, which is supposed to work for all $k$ in an analogous way.

The Heisenberg algebra $\mathcal{H}$ that is relevant for the model $\text{CP}^{\text{top}}_{2,2}$ coupled to topological $W$-gravity (with potential $W_{3,2}(x,z,0) = \frac{1}{10}x^5 - \frac{1}{2}x_1 x_2^2 - \frac{4}{5} z$) is given by the perturbed chiral ring $\mathcal{R}_{3,3}^2(t_1 = \frac{8}{5} z)$ of $\text{CP}^{\text{top}}_{2,2}$, with potential $W_{3,3}(x,z,0) = \frac{1}{3} x_2^3 - x_1^4 x_2 - 2 x_1^2 x_2^2 - \frac{8}{5} z x$. This ring is associated with the ten dimensional representation of $SU(5)$ and is generated by

$$
\Lambda^{(z)}_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
\quad
\Lambda^{(z)}_2 = \begin{pmatrix}
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -z & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -z & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

We have $b_i = 2$, and a basis for the enveloping algebra $\mathcal{H}$ is

$$
\Lambda^{(z)}_{i,1} = \Lambda^{(z)}_1,
\Lambda^{(z)}_{i,2} = \frac{1}{z} (\Lambda^{(z)}_1)^2 (\Lambda^{(z)}_2)^2,
\Lambda^{(z)}_{2,1} = \Lambda^{(z)}_2
\Lambda^{(z)}_{2,2} = (\Lambda^{(z)}_1)^2
\Lambda^{(z)}_{3,1} = \frac{1}{7} [ (\Lambda^{(z)}_1)^3 + 3 \Lambda^{(z)}_1 \Lambda^{(z)}_2 ]
\Lambda^{(z)}_{3,2} = \frac{1}{7} [ (\Lambda^{(z)}_1)^3 - \Lambda^{(z)}_1 \Lambda^{(z)}_2 ]
\Lambda^{(z)}_{4,1} = -\frac{1}{7} [ (\Lambda^{(z)}_1)^2 \Lambda^{(z)}_2 + (\Lambda^{(z)}_2)^2 ]
\Lambda^{(z)}_{4,2} = -\frac{1}{7} [ (\Lambda^{(z)}_1)^2 \Lambda^{(z)}_2 - (\Lambda^{(z)}_2)^2 ]
\Lambda^{(z)}_{5,1} = \frac{1}{8} [ (\Lambda^{(z)}_1)^5 - 5 \Lambda^{(z)}_1 (\Lambda^{(z)}_2)^2 ] = z \mathbf{1}
\Lambda^{(z)}_{5,2} = \frac{1}{7} [ (\Lambda^{(z)}_1)^3 \Lambda^{(z)}_2 - 3 \Lambda^{(z)}_1 (\Lambda^{(z)}_2)^2 ]
\quad (3.52)
$$
which is complete up to arbitrary powers of the spectral parameter $z$ (we use the
convention that $\Lambda_i^{(z)} \in \mathcal{H} \subset \hat{\mathfrak{sl}}(5)$ for $i = 1, \ldots, 4$). Accordingly, the Lax operators can be expanded as

$$
\mathcal{L}_1(g, \hat{g}, z) = \Lambda_1^{(z)} + \sum_{i=1}^5 \sum_{j=1}^2 q_{i,j}^1(g, \hat{g}, z) \Lambda_{i,j}^{(z)} \in \tilde{\mathcal{H}}
$$

(3.53)

$$
\mathcal{L}_2(g, \hat{g}, z) = \Lambda_2^{(z)} + \sum_{i=1}^5 \sum_{j=1}^2 q_{i,j}^2(g, \hat{g}, z) \Lambda_{i,j}^{(z)} \in \tilde{\mathcal{H}}.
$$

The coefficients $q$ depend on the coupling constants $g, \hat{g}$ that figure in the superpotential

$$
W(x_1, x_2, g(t)) = \frac{1}{10} x_1^5 - \frac{1}{2} x_1 x_2^2 - g_1(t) x_1^4 - \ldots - g_5(t) x_2 - g_5(t),
$$

(3.54)

and in the extra relation

$$
\hat{W}(x_1, x_2, \hat{g}) = \hat{g}_0^{(1)} x_1^4 + \hat{g}_0^{(2)} x_1 x_2^2 + \hat{g}_0^{(3)} x_2^2 + \hat{g}_1(t) x_1^3 - \ldots - \hat{g}_4(t)
$$

(3.55)

(we choose here as ring basis not the one given in (2.26), but the basis used in [14, 48, 49]). It turns out that writing the Lax operators and other hamiltonians in terms of a large number of unknowns takes an enormous amount of space. Therefore, we will write all these quantities directly in terms of the solutions $g(t)$. We will in addition suppress the dependence on the parameter $t_1$, which is not important for demonstrating the existence of consistent, non-trivial degenerate flows. We emphasize, however, that we did make the most general ansatz and indeed found flat coordinates as *unique* solution of the flow equations.

Written in terms of the flat coordinates, the superpotential spectral equation is

$$
W(\mathcal{L}_1, \mathcal{L}_2, t) = \frac{1}{10} \mathcal{L}_1^5 - \frac{1}{2} \mathcal{L}_1 \mathcal{L}_2^2 - t_2 \mathcal{L}_1^3 - t_3 \mathcal{L}_1^2 + \frac{1}{2} \left(3t_2^2 - 2t_4\right) \mathcal{L}_1 - t_5 \mathcal{L}_2 + t_2 t_3 - t_5
$$

(3.56)

$$
= \frac{4}{5} \mathcal{L}_1 + \mathcal{L}_2.
$$

The extra relation, as determined by the flow equations, turns out to be

$$
\hat{W}(\mathcal{L}_1, \mathcal{L}_2, t) = \frac{3}{10} \mathcal{L}_1^4 - \frac{3}{8} \mathcal{L}_1^2 \mathcal{L}_2 - \frac{1}{16} \mathcal{L}_2^2 - \frac{1}{8} t_2 \mathcal{L}_1^2 + \frac{1}{8} t_2 \mathcal{L}_2 - t_5 \mathcal{L}_1 + \frac{3}{16} t_2^2 - \frac{t_4}{2}
$$

(3.57)

$$
= \Lambda_{4,1}^{(z)} + \frac{5}{4} \Lambda_{4,2}^{(z)}.
$$

† Unique up to the freedom of adding trivial vanishing relations to the flat ring elements.
These equations imply for the coefficients $q(t, z)$ in (3.53):

\[
\begin{align*}
g_{1,1}^1 & = \frac{1}{2z} (4t_2t_3 - 2t_5 + 13t_2t_*) + O(z^{-2}) \\
g_{1,2}^1 & = -\frac{1}{4z} (9t_2t_3 - 7t_5 + 43t_2t_*) + O(z^{-2}) \\
g_{2,1}^1 & = \frac{1}{8z} (26t_2^2 + 9t_4) + O(z^{-2}) \\
g_{2,2}^1 & = \frac{1}{8z} (8t_2^2 + 3t_4) + O(z^{-2}) \\
g_{3,1}^1 & = -\frac{1}{4z} (2t_3 - t_*) + O(z^{-2}) \\
g_{3,2}^1 & = \frac{1}{4z} (t_3 + O(z^{-2}) \\
g_{4,1}^1 & = \frac{1}{2z} (709t_2^2t_3 + 27t_3t_4 - 112t_2t_5 + 1011t_2^2t_* + 99t_4t_*) + O(z^{-3}) \\
g_{4,2}^1 & = \frac{1}{16z} (486t_2^2t_3 + 21t_3t_4 + 2019t_2^2t_* + 151t_4t_*) + O(z^{-3}) \\
g_{5,1}^1 & = -\frac{1}{64z} (2360t_2^3 + 21t_3^2 + 488t_2t_4 - 278t_3t_* + 17t_*^2) + O(z^{-3}) \\
g_{5,2}^1 & = -\frac{1}{64z} (1096t_2^3 + 19t_3^2 + 232t_2t_4 - 130t_3t_* + 7t_*^2) + O(z^{-3})
\end{align*}
\]

(3.58)

and

\[
\begin{align*}
g_{1,1}^2 & = \frac{1}{12} (240t_2^2 + 3t_3^2 + 56t_2t_4 - 23t_3t_* + t_*^2) + O(z^{-2}) \\
g_{1,2}^2 & = -\frac{1}{12} (1480t_2^2 + 3t_3^2 + 336t_2t_4 + 142t_3t_* + 7t_*^2) + O(z^{-2}) \\
g_{2,1}^2 & = -\frac{1}{4z} (20t_2t_3 - 9t_5 + 48t_2t_*) + O(z^{-2}) \\
g_{2,2}^2 & = \frac{3}{4z} (t_5 - 4t_2t_*) + O(z^{-2}) \\
g_{3,1}^2 & = -\frac{1}{4z} (25t_2^2 + 4t_4) + O(z^{-2}) \\
g_{3,2}^2 & = \frac{1}{4z} (9t_2^2 + 2t_4) + O(z^{-2}) \\
g_{4,1}^2 & = -\frac{1}{2} t_* + O(z^{-2}) \\
g_{4,2}^2 & = \frac{1}{2} (3t_3 - t_*) + O(z^{-2}) \\
g_{5,1}^2 & = \frac{1}{16z} (1469t_2^2t_3 + 87t_3t_4 + 244t_2t_5 + 2019t_2^2t_* + 151t_4t_*) + O(z^{-3}) \\
g_{5,2}^2 & = -\frac{1}{2} t_2 + \frac{1}{16z} (589t_2^2t_3 + 33t_3t_4 - 116t_2t_5 + 919t_2^2t_* + 69t_4t_*) + O(z^{-3})
\end{align*}
\]

(3.59)

The flow equations (3.47) can be written as ($I \equiv (2, \ldots, 5, *)$, $\Omega_{6-\ast} \equiv \Omega_*$):

\[
\partial_{t_1} \Omega_{6-\ast}(\mathcal{L}_1(t), \mathcal{L}_2(t), t) = \partial_{t_2} \Omega_{6-\ast}(\mathcal{L}_1(t), \mathcal{L}_2(t), t),
\]

(3.60)
and are satisfied by the matrix-valued hamiltonians $\Omega_I \equiv \Omega_I(\mathcal{L}_1(t), \mathcal{L}_2(t), t)$:

$$\Omega_1 = (\Lambda_{1,1}^{(z)})_+ = \mathcal{L}_1$$
$$\Omega_2 = (\frac{1}{2} \Lambda_{2,1}^{(z)})_+ = \frac{1}{2} \mathcal{L}_2$$
$$\Omega_3 = (\frac{1}{6} \Lambda_{3,1}^{(z)} + \frac{1}{2} \Lambda_{3,2}^{(z)})_+ = \frac{1}{3} \mathcal{L}_1^3 - \mathcal{L}_1 t_2 - \frac{5}{8} t_3 - \frac{1}{8} t_*$$
$$\Omega_* = (\frac{1}{6} \Lambda_{3,1}^{(z)} - \frac{3}{2} \Lambda_{3,2}^{(z)})_+ = \mathcal{L}_1 \mathcal{L}_2 - \frac{2}{3} \mathcal{L}_1^3 + \frac{15}{8} t_* - \frac{1}{8} t_3$$
$$\Omega_4 = (\frac{1}{2} \Lambda_{4,1}^{(z)} + \frac{3}{4} \Lambda_{4,2}^{(z)})_+ + \alpha_1 (\Lambda_{4,1}^{(z)} + \frac{5}{4} \Lambda_{4,2}^{(z)})$$
$$= \frac{1}{3} \mathcal{L}_1^4 + \frac{1}{8} \mathcal{L}_1^2 \mathcal{L}_2 - \frac{1}{16} \mathcal{L}_2^2 + \frac{31}{16} t_2^2 - \frac{9}{8} t_2 \mathcal{L}_1^2 - \frac{3}{8} t_2 \mathcal{L}_2 - t_3 \mathcal{L}_1$$
$$+ \alpha_1 \hat{W}(\mathcal{L}_1, \mathcal{L}_2, t).$$

Note that $\Omega_3, \Omega_*$ represent two degenerate flows, as advertised, and that $\alpha_1$ is a free parameter which corresponds to the freedom of adding a constant matrix to $\Omega_4$.

### 3.6. Prepotential and scalar Lax system

So far we considered the flow equations in matrix form, cf., (3.25), (3.47). There is also a scalar version of these equations, which is obtained by replacing $\mathcal{L}_i$ by the LG fields $x_i$. Because of the dependence of $\mathcal{L}_i$ on the coordinates $t$, the form of the scalar equations is different from their matrix versions.

We find for the examples for $W_3$-gravity that we computed explicitly a structure that is very similar to the one of ordinary gravity. For instance, for the $k = 2$ model discussed in the previous section (and similarly for the other examples), we find that the scalar Lax system can be written as:

$$-\partial_t W(x_1, x_2, t) = \{ \Omega_{6-I}(x_1, x_2, t), W \}$$
$$-\partial_t \hat{W}(x_1, x_2, t) = \{ \Omega_{6-I}(x_1, x_2, t), \hat{W} \},$$

with the Poisson bracket

$$\{ \Omega, L \} \equiv \left[ (\partial_t \Omega)(\partial_x L) - (\partial_x \Omega)(\partial_t L) \right]$$
$$+ 2 \left[ (\partial_{t_4} \Omega)(\partial_{x_2} L) - (\partial_{x_2} \Omega)(\partial_{t_4} L) \right].$$

This corresponds to two pairs of conjugate variables, $(t_5, x_1)$ and $(t_4, \frac{1}{2} x_2)$. The equations (3.62) can be viewed as the dispersionless limit of a differential scalar system of the form

$$-\partial_t L(D_1, D_2, t) = \left[ \Omega_{6-I}(D_1, D_2, t), L \right]$$
$$-\partial_t \hat{L}(D_1, D_2, t) = \left[ \Omega_{6-I}(D_1, D_2, t), \hat{L} \right],$$

$$- 27 -$$
if one imposes as string equations
\[
D_1 t_5 = 1 \\
D_2 t_4 = 2 .
\] (3.65)

Presumably, the system (3.64) gives the complete characterization of the \( W_3 \)-matter-gravity model beyond the small phase space.

The terms in (3.62) that are proportional to \( \partial_{x_i} W \) precisely cancel out the dependence on the free parameter \( \alpha_1 \) in the hamiltonian \( \Omega_4 \) (3.61). If one puts this parameter to zero, one can write the scalar equations in a simpler form:
\[
-\partial_t W(x_1, x_2, t) = (\partial_{x_1} + 2x_1 \partial_{x_2}) \Omega_{6-I}(x_1, x_2, t) \\
-\partial_t \hat{W}(x_1, x_2, t) = \partial_{x_2} \Omega_{6-I}(x_1, x_2, t)
\] (3.66)

which is the two-variable generalization of (3.8). The first equation gives the flat fields \( \phi_5 - I \equiv -\partial_t W \) directly in terms of the hamiltonians.

Note that the equations (3.66) are precisely the integrability conditions for the existence of a prepotential \( V \) with
\[
W(x, t) = (\partial_{x_1} + 2x_1 \partial_{x_2}) V(x, t) \\
\hat{W}(x, t) = \partial_{x_2} V(x, t) \\
\Omega_I(x, t) = -\partial_{6-I} V(x, t)
\] (3.67)

which is given by:
\[
V(x, t) = \frac{3}{10} x_1^4 x_2 - \frac{11}{240} x_1^6 - \frac{3}{16} x_1^2 x_2^2 - \frac{1}{16} x_2^3 - \frac{3}{16} t_2 x_1^4 - \frac{1}{8} t_2 x_1^2 x_2 \\
+ \frac{9}{16} t_2^2 x_1^2 + \frac{1}{16} (3t_2^2 - 8t_4) x_2 + (t_2 t_3 - t_5) x_1 - \frac{1}{3} (t_3 - 2t_s) x_1^3 \\
- t_s x_1 x_2 + \frac{1}{16} t_2 x_2^2 + \frac{5}{16} t_3^2 - \frac{31}{48} t_2^3 + \frac{1}{8} t_3 t_s - \frac{15}{16} t_s^2 .
\] (3.68)

The fact that
\[
-\partial_{t_5} V = x_1 \\
-\partial_{t_4} V = \frac{1}{2} x_2
\] (3.69)

just expresses that \((t_5, x_1)\) and \((t_4, \frac{1}{2} x_2)\) are conjugate pairs. Moreover, note that the scalar spectral equations can be thought of as variations of the prepotential:
\[
\delta_r V(x, t) = 0 ,
\] (3.70)

\[ - 28 - \]
where
\[
\begin{align*}
\delta_1 &= \partial_{x_1} + 2x_1\partial_{x_2} \\
\delta_2 &= \partial_{x_2}
\end{align*}
\] (3.71)
are nothing but Killing vectors of the “\(W_3\)-plane” \([52]\). This hints at an interesting underlying geometrical structure.

It is fascinating to speculate that, in analogy to ordinary gravity \((3.11)\), \(V\) might be the potential of an appropriate underlying, generalized two-matrix Kontsevich \([43]\) (or Kazakov-Migdal \([53]\)) model.

4. Discussion and Outlook

We proposed in this paper a construction of integrable systems that extends the dispersionless, \((1, k + 2)\) type generalized KdV hierarchy to several variables \(x_i\). We focused mainly on two variables, which corresponds to topological \(W_3\)-gravity coupled to matter, but it is pretty obvious how to generalize this to \(M\) variables.

A key point was to represent the superpotential spectral equations as matrix relations:
\[
W(\mathcal{L}_1, \mathcal{L}_2, \ldots, g) = z \ 1
\] (4.1)
(these equations are the multi-variable analogs of the constraint that implements the reduction from the KP to the KdV hierarchy). That these matrix relations happen to be identical to chiral ring vanishing relations of certain other LG theories was very convenient and allowed us to realize them in terms of known chiral ring structure constants. Accordingly, the Hamiltonians can be constructed in terms of the chiral rings of these other LG theories, which are “at one level higher”. In practice, one considers Drinfeld-Sokolov matrix systems associated with the \((n-1)\)-th fundamental representation of \(SU(n+k)\) (for models \(\mathcal{M}_{1,n+k}^{(n)}\)). In such representations, there exist in general more commuting matrices at a given grade than there are powers of the \(s\ell(2)\) step generator, \(\Lambda_1\), and this leads to the possibility of having more commuting flows as compared to the ordinary generalized KdV hierarchy.

In mathematical terms, the underlying algebraic structure of the relevant Heisenberg algebras is that of “quantum” cohomology rings of grassmannians, \(\tilde{\mathcal{H}}^{+} \cong \overline{QH}^{*}_{\mathbb{G}_m}(SU(n+k)/SU(n-1)\times SU(k+1)\times U(1), \mathbb{R})\), and the superpotential spectral equations (4.1) are specific relations in these rings.
So far, $W$-algebras were most often discussed \[54\] in the context of ordinary gravity coupled to matter and not in the context of higher $W_n$-gravities. For ordinary topological gravity coupled to matter, which is described by the model $\text{CP}_{1,k}^{\text{top}}$, the basic underlying structure is the one of Fig.1, which is essentially the positive affine weight space of $\hat{s}l(2)$. The picture may simultaneously represent the chiral ring $\mathcal{R}_{2,k}^{x,\sigma}$ of the topological LG model, the spectrum of KdV hamiltonians $L^{(i+1)/(k+2)}_+$, the $W_{k+2}$-constraints of the matrix model, and also the classical $W_{k+2}$ Poisson bracket algebra of the KdV hierarchy (note that these classical $W$-algebras have nothing to do with $W$-gravity.) The important point is that all these objects are not truly independent, but live in the enveloping algebra of a single generator (which is, morally speaking, given by an $sl(2)$ step generator).

The generalization to $W_n$-gravity coupled to matter discussed in the present paper proceeds in an orthogonal direction and is essentially a generalization to $M$ independent generators, where $M \equiv n-1$ counts the number of independent generators of the $W_n$ algebra. The situation for $n=3$ can be schematically depicted as in Fig.2. That is, $M$ also counts the number of independent LG fields (which generate the chiral rings $\mathcal{R}_{n,k}^x$), the number of gravitational descendant generators $\sigma_i$, the number of independent $N=2$ $W_n$ supercurrents that generate the chiral algebra of the coset models $\text{CP}_{M,k}^{\text{top}}$, the number of Heisenberg algebra generators $\Lambda_i^{(z)}$ and the number of independent derivatives in the differential scalar Lax operators $L(D_i)$ (cf., (3.64)). Moreover, we conjecture that there exist generalized Kontsevich \[43\], or Kazakov-Migdal \[53\] type matrix models that give an equivalent description of the $W_n$-matter-gravity theories, where $M$ is the number of matrices. These models are supposedly defined in terms of prepotentials $V$ like (3.68). If this were true, one would have a matrix model description for at least some models that are directly relevant for string compactification (in particular, the models $\text{CP}_{4,15}^{\text{top}}, \text{CP}_{5,9}^{\text{top}}$ and $\text{CP}_{6,7}^{\text{top}}$ have $c=9$).

What these $M$ generators have in common is that they are graded like the exponents $(1, 2, 3, ..., M)$ (sometimes shifted by one or two units), and this is ultimately inherited from the principal embedding $sl(2) \hookrightarrow sl(n)$. We believe that our findings can be generalized to other embeddings of $sl(2)$ into Lie algebras, which would give rise more general kinds of gradations.

A generalization of Drinfeld-Sokolov systems with different than principal gradations was considered before by refs. \[9,55\], and one might ask about the relationship of this to our work. Indeed one may view the generators $\Lambda_i$, when taken in some higher fundamental representation, also as generators in the fundamental representation of
some larger group with appropriately chosen gradation \( s \) (where \( I_0 = \sum s_i \lambda_i \cdot H \)). For example, the matrices in (2.22), which represent \( \Lambda_1 \) and \( \Lambda_2 \) in the 10 dimensional representation of \( SU(5) \) with principal gradation \( s = (1,1,1,1) \), can also be interpreted as matrices in the fundamental representation of \( SU(10) \) with gradation \( s = (1,1,0,1,0,1,0,1,1) \). However, the total grade,

\[
d_s = N_s (z \partial_z) + \sum s_i \lambda_i \cdot H ,
\]

involves in addition the grade of the spectral parameter \( z \), which is given by \( N_s = 1 + \sum s_i \) (for \( SU(m) \)). Obviously, the grade of the spectral parameter in our construction (in the example, \( [z] = 5 \)) is different from the corresponding grade of [9] (\( [z] = 7 \)). This means that the matrices \( \Lambda^{(z)}_i \) cannot be the same in the two approaches (although they agree for \( z = 0 \)), and that the Heisenberg algebras differ. It would be interesting to see how the work of [9] can be generalized such as to include the construction discussed in the present paper.

We approached the problem of integrable systems pertaining to \( W \)-gravity from a very specific viewpoint, namely by relating dispersionless Lax operators with certain chiral rings. It is quite clear that we just barely scratched the surface, and many important questions are left unanswered. Among these open problems is a proof that our method works in general and indeed describes topological \( W \)-gravity including gravitational descendants, and this would probably require to first develop an appropriate kind of intersection theory of the correlation functions.

Furthermore, one would also like to go beyond the dispersionless limit, in order to describe more general, non-topological models of type \((p,q)\). As indicated in section 3.6, this would amount to introducing an independent, commuting derivative for each LG field, such that the Kazama-Suzuki superpotential turns into a differential Lax operator: \( W(x_1, x_2, \ldots, x_M, g) \to L(D_1, D_2, \ldots, D_M, g) \). This scalar differential operator would be associated to a matrix system of the form

\[
\begin{align*}
[ D_1 \mathbf{1} - \mathcal{L}_1(g) ] \cdot \Psi &= 0 \\
[ D_2 \mathbf{1} - \mathcal{L}_2(g) ] \cdot \Psi &= 0 \\
& \vdots \\
[ D_M \mathbf{1} - \mathcal{L}_M(g) ] \cdot \Psi &= 0 ,
\end{align*}
\]

(4.2)
with $\mathcal{L}_p(g) = \Lambda_p^{(z)} + Q_p(g)$, in generalization of (2.16) and (3.19). The appearance of several kinds of derivatives would be a direct manifestation of $W$-geometry, via association with coordinates in a “$W$-superspace”, i.e., $D_1 \rightarrow \partial z_1 = L_{-1}, D_2 \rightarrow \partial z_2 = (W_3)_{-2}$, and so on.

We believe that the ideas presented in this paper can be applied to a variety of related problems, presumably to all problems that are based on the principal subgroup $SU(2)$, signalled by the appearance of the familiar matrix (2.17). For example, we expect that the construction of singular vectors of the Virasoro algebra given in [56] can be adapted to $W$-algebras by making use of equations similar to (4.2).

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References

[1] D.J. Gross and A.A. Migdal, Phys. Rev. Lett. **64** (1990) 717; M. Douglas and S. Shenker, Nucl. Phys. **B235** (1990) 635; E. Brezin and V. Kazakov, Phys. Lett. **236B** (1990) 144.

[2] M. Douglas, Phys. Lett. **238B** (1990) 176.

[3] R. Dijkgraaf and E. Witten, Nucl. Phys. **B342** (1990) 486; R. Dijkgraaf and E. and H. Verlinde, Nucl. Phys. **B348** (1991) 435; For a review, see: R. Dijkgraaf, Intersection theory, integrable hierarchies and topological field theory, preprint IASSNS-HEP-91/91.

[4] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov and A. Zabrodin, Nucl. Phys. **B380** (1991) 181; C. Itzksen and J. B. Zuber, Int. Journ. Mod. Phys. A7 (1992) 5661; M. Fukuma, H. Kawai and R. Nakayama, Int. Journ. Mod. Phys. A6 (1992) 1385; S. Kharchev, A. Marshakov, A. Mironov and A. Morozov, Mod. Phys. Let. **A8** (1993) 1047; A. Marshakov, Integrable structures in matrix models and physics of 2D-gravity, preprint NORDITA-93/21 P.

[5] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Phys. Lett. **B292** (1992) 35, E. Bergshoeff, A. Sevrin and X. Shen, Phys. Lett. **B296** (1992) 95, J. de Boer and J. Goeree, Nucl. Phys. **B405** (1993) 669.

[6] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Nucl. Phys. **B401** (1993) 304.

[7] See e.g., V. Kac and J. van de Leur, The n-component KP hierarchy and representation theory, MIT preprint 1993.

[8] J. Lacki, Int. J. Mod. Phys. A7 (1992) 4871.

[9] M. de Groot, T. Hollowood and J. Miramontes, Comm. Math. Phys. **145** (1992) 57; N. Burroughs, M. De Groot, T. Hollowood and J. Miramontes, Phys. Lett. **B277** (1992) 89; T. Hollowood, J. Miramontes and J. Guillen, Generalized integrability and two-dimensional gravitation, preprint CERN-TH-6678-92.
[10] J. de Boer, *Extended conformal symmetry in non-critical string theory*, Ph.D. thesis, 1993.

[11] L. Bonora, Q. Liu and C. Xiong, *The integrable hierarchy constructed from a pair of higher KdV hierarchies and its associated W algebra*, preprint BONN-TH-94-17.

[12] C. Vafa, Mod. Phys. Lett. **A6** (1991) 337.

[13] K. Li, Nucl. Phys. **B354** (1991) 711; Nucl. Phys. **B354** (1991) 725.

[14] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. **B352** (1991) 59.

[15] B. Gato-Rivera and A.M. Semikhatov, Phys. Lett. **B293** (1992) 72, vrule width0pt hep-th/9207004.

[16] E. Witten, Comm. Math. Phys. **117** (1988) 353; Comm. Math. Phys. **118** (1988) 411; Nucl. Phys. **B340** (1990) 281.

[17] T. Eguchi and S. Yang, Mod. Phys. Lett. **A4** (1990) 1693.

[18] I. Krichever, Comm. Math. Phys. **143** (1992) 415; B. Dubrovin, Nucl. Phys. **B379** (1992) 627, Comm. Math. Phys. **145** (1992) 195, Comm. Math. Phys. **152** (1993) 539.

[19] K. Saito, J. Fac. Sci. Univ. Tokyo Sec. IA.28 (1982) 775; M. Noumi, Tokyo. J. Math. 7 (1984) 1; B. Blok and A. Varchenko, *Topological conformal field theories and the flat coordinates*, preprint IASSNS-HEP-91/5.

[20] S. Cecotti and C. Vafa, Nucl. Phys. **B367** (1991) 359; W. Lerche, D. Smit and N. Warner, Nucl. Phys. **B372** (1992) 87.

[21] A. Bilal and J. Gervais, Nucl. Phys. **B326** (1989) 222; P. Mansfield and B. Spence, Nucl. Phys. **B362** (1991) 294; M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, Phys. Lett. **B292** (1992) 35.

[22] For reviews on W-algebras, see: P. Bouwknegt and K. Schoutens, Phys. Rep. 223 (1993) 183; J. de Boer, *Extended conformal symmetry in non-critical string theory*, Ph.D. thesis, 1993; J. Goeree, *Higher spin extensions of two-dimensional gravity*, P.D. thesis, 1993; T. Tjin, *Finite and infinite W-algebras*, P.D. thesis, 1993.

[23] K. Li, Phys. Lett. **B251** (1990) 54, Nucl. Phys. **B346** (1990) 329; H. Lu, C.N. Pope and X. Shen, Nucl. Phys. **B366** (1991) 95; S. Hosono, *Algebraic definition*
of topological W-gravity, preprint UT-588; H. Kunitomo, Prog. Theor. Phys. 86 (1991) 745.

[24] K. Ito, Phys. Lett. **B259** (1991) 73; Nucl. Phys. **B370** (1992) 123; D. Nemeschansky and S. Yankielowicz, N=2 W-algebras, Kazama-Suzuki models and Drinfeld-Sokolov reduction, preprint USC-91-005A; L.J. Romans, Nucl. Phys. **B369** (1992) 403; W. Lerche, D. Nemeschansky and N.P. Warner, unpublished.

[25] W. Lerche, Phys. Lett. **B259** (1990) 349; T. Eguchi, S. Hosono and S.K. Yang, Comm. Math. Phys. **140** (1991) 159; T. Eguchi, T. Kawai, S. Mizoguchi and S.K. Yang, Rev. Math. Phys. 4 (1992) 329.

[26] Y. Kazama and H. Suzuki, Nucl. Phys. **B321** (1989) 232.

[27] E. Witten, Nucl. Phys. **B373** (1992) 187.

[28] W. Lerche, C. Vafa and N.P. Warner, Nucl. Phys. **B324** (1989) 427.

[29] D. Gepner, Comm. Math. Phys. **141** (1991) 381.

[30] E. Witten, Nucl. Phys. **B340** (1990) 281; E. and H. Verlinde, Nucl. Phys. **B348** (1991) 457.

[31] A. Lossev, Descendants constructed from matter field and K. Saito higher residue pairing in Landau-Ginzburg theories coupled to topological gravity, preprint TPI-MINN-92-40-T, 

[32] T. Eguchi, H. Kanno, Y. Yamada and S.-K. Yang, Phys. Lett. **B305** (1993) 235, 

[33] W. Lerche and A. Sevrin, On the Landau-Ginzburg Realization of Topological Gravities, preprint CERN-TH.7210/94, 

[34] W. Lerche and N. Warner, On the Algebraic Structure of Gravitational Descendants in CP(n–1) Coset Models, preprint CERN-TH.7442/94, 

[35] D. Gepner, Comm. Math. Phys. **142** (1991) 433.

[36] J. Fuchs and C. Schweigert, Level-rank duality of WZW theories and isomorphisms of N=2 coset models, preprint NIKHEF-H-93-16.
[37] D. Gepner, *A comment on the chiral algebras of quotient superconformal field theories*, preprint PUPT-1130; S. Hosono and A. Tsuchiya, Comm. Math. Phys. 136 (1991) 451.

[38] P. Fendley, W. Lerche, S. Mathur and N.P. Warner, Nucl. Phys. B348 (1991) 66; W. Lerche and N.P. Warner, Nucl. Phys. B358 (1991) 571.

[39] B. Kostant, Am. J. Math. 81 (1959) 973.

[40] Theorem by Dale Petersen, private communication by B. Kostant.

[41] Drinfel’d and V. G. Sokolov, Jour. Sov. Math. 30 (1985) 1975.

[42] I. Krichever, *Topological minimal models and soliton equations*, Landau Institute preprint.

[43] M. Kontsevich, Comm. Math. Phys. 147 (1992) 1.

[44] V. Kac and D. Petersen, *112 constructions of the basic representation of the loop group of E8*, in: Symp. on Anomalies, geometry and Topology, eds, W. Bardeen and A. White, World Scientific, Singapore 1985.

[45] P. Fendley, S. Mathur C. Vafa and N.P. Warner, Nucl. Phys. B348 (1991) 66.

[46] S. Cecotti and C. Vafa, Nucl. Phys. B367 (1991) 359.

[47] E. Witten, Comm. Math. Phys. 118 (1988) 411, Nucl. Phys. B340 (1990) 281; K. Intriligator, Mod. Phys. Let. A6 (1991) 3543; C. Vafa, *Topological mirrors and quantum rings*, in: Essays in Mirror Symmetry, ed. S.T. Yau, 1992; V. Sadov, *On the equivalence of Floer’s and quantum cohomology*, preprint HUTP-93/A027; E. Witten, *The Verlinde algebra and the cohomology of the grassmannian*, preprint IASSNS-HEP-93/41.

[48] T. Eguchi, Y. Yamada and S. Yang, Mod. Phys. Let. A8 (1993) 1627.

[49] K. Takasaki, *Integrable hierarchies underlying topological Landau-Ginzburg models of D-type*, preprint KUCP-0061/93.

[50] J. Balog, L. Feher, P. Forgacs, L. O’Raifeartaigh and A. Wipf, Phys. Lett. B244 (1990) 435.

[51] P. Di Francesco and D. Kutasov, Nucl. Phys. B342 (1990) 589.

[52] H. Ooguri, K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Comm. Math. Phys. 145 (1992) 515.

[53] V. Kazakov and A. Migdal, Nucl. Phys. B397 (1993) 214.
[54] M. Fukuma, H. Kawai and R. Nakayama, Int. J. Mod. Phys. A6 (1991) 1385; R. Dijkgraaf and E. and H. Verlinde, Nucl. Phys. B356 (1991) 574; A. Marshakov, A. Mironov and A. Morozov, Phys. Lett. 274 (1992) 280.

[55] L. Feher, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Ann. Phys. 213 (1992) 1; L. Feher, J. Harnad and I. Marshall, Comm. Math. Phys. 154 (1993) 181.

[56] M. Bauer, P. Di Francesco, C. Itzykson and J. Zuber, Phys. Lett. B260 (1991) 323, Nucl. Phys. B362 (1991) 515.