The Positive Occupation Time of Brownian Motion with Two-Valued Drift and Asymptotic Dynamics of Sliding Motion with Noise.

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Abstract

We derive the probability density function of the positive occupation time of one-dimensional Brownian motion with two-valued drift. Long time asymptotics of the density are also computed. We use the result to describe the transitional probability density function of a general \( N \)-dimensional system of stochastic differential equations representing stochastically perturbed sliding motion of a discontinuous, piecewise-smooth vector field on short time frames. A description of the density at larger times is obtained via an asymptotic expansion of the Fokker-Planck equation.

1 Introduction

Filippov systems are vector fields that are discontinuous on codimension-one surfaces, termed switching manifolds, and are used to model a wide variety of physical systems involving a discontinuity or abrupt change, \([1, 2]\). Whenever a forward orbit of a Filippov system reaches a section of a switching manifold for which the vector field points toward the manifold from both sides, subsequent forward evolution is constrained to the manifold for some time. Such evolution is called sliding motion. In applications, sliding motion often has an important physical interpretation. For instance, oscillators subject to dry friction exhibit nonsmooth dynamics when there
is both slipping and sticking behaviour [3, 4, 5, 6]. Here sliding motion corresponds to the sticking phase of the dynamics. Models of population dynamics involving assumptions that species make different selections between habitats or food sources at different times are often nonsmooth [7, 8, 9, 10]. In [7], for example, sliding motion is interpreted as the scenario that predators hesitate between two different food sources. Also, relay control systems undergo switching when a controlling signal reaches a threshold value [11, 12, 13]. Consequently relay control systems are often well-modelled by piecewise-smooth systems and sliding motion corresponds to the idealized limit of repeated switching events occurring instantaneously [14, 15, 16]. In this context sliding motion has been directly exploited to provide enhanced performance [17, 18].

Naturally it is important to understand the effects that noise or uncertainty may have on the dynamics of a Filippov system. With the addition of small noise in the form of additive Brownian motion, roughly speaking, sliding motion along a switching manifold becomes a nearby random motion repeatedly intersecting the switching manifold [19]. In general, expressions for the transitional PDF (probability density function) of a Filippov system with noise are unavailable because the discontinuity in the drift is an extreme nonlinearity that inhibits exact calculations. In this paper we derive two explicit asymptotic expressions for the transitional PDF of stochastically perturbed sliding motion. One expression is derived in the context of occupation times and is valid for short time frames; the other expression is derived via an asymptotic expansion and is valid for time frames that are long relative to the size of the noise.

As illustrated below in §4.1, over short time frames we find that the distance covered by a sample solution in a direction parallel to a switching manifold, is almost completely determined by the occupation time of the solution on each side of the switching manifold. Furthermore, over short time frames dynamics in the direction orthogonal to the switching manifold may be well-approximated by Brownian motion with two-valued drift:

\[
dx(t) = \begin{cases} 
a_L, & x(t) < 0 \\
-a_R, & x(t) > 0 \end{cases} \, dt + dW(t), \quad x(0) = x_0.
\]

(1.1)

Here \(a_L, a_R \in \mathbb{R}\) and \(W(t)\) is a standard Brownian motion. For any \(t > 0\), let

\[
\tau = \int_0^t \chi_{[0, \infty)}(x(s)) \, ds = \text{meas } \{ s \in [0, t] \mid x(s) \geq 0 \},
\]

denote the positive occupation time of \(x(t)\). In §4.1 we discuss the approximations more carefully and use the PDF of \(\tau\) to describe the transitional PDF for stochastically perturbed sliding motion in directions parallel to the switching manifold, over short time frames.

In this paper we derive the PDF of \(\tau\), call it \(p\), i.e.,

\[
\text{Prob}(\tau \in \Sigma) = \int_{\Sigma} p(\hat{\tau}; t; x_0, a_L, a_R) \, d\hat{\tau},
\]

(1.3)

for any measurable subset, \(\Sigma \subset [0, t]\). PDFs of occupation times of simple stochastic processes have found applications in control problems and mathematical finance [20]. Equation (1.1) has also arisen in a stochastic control problem [21], and general theoretical settings [22, 23]. In the context of sliding motion we require, \(a_L, a_R > 0\), such that solutions to (1.1) rarely stray far from the origin, but for generality we study (1.1) without this restriction. An explicit expression for the PDF of \(x(t)\) was first derived by Karatzas and Shreve in [24].

When \(x_0 = 0\), \(p\) is given by the following theorem.
Theorem 1.
For \( x_0 = 0 \) and any \( t > 0 \), the PDF of the positive occupation time (1.2) of Brownian motion with two-valued drift (1.1), is

\[
p(\tau; t; 0, a_L, a_R) = e^{-\frac{a_L^2(t-\tau)}{2}} - \frac{a_R e^{-\frac{a_R^2(t-\tau)}{2}} \text{erfc} \left( \frac{a_R \sqrt{\tau}}{\sqrt{2}} \right)}{\sqrt{2\pi(t-\tau)}} - \frac{a_L e^{-\frac{a_L^2(t-\tau)}{2}} \text{erfc} \left( \frac{a_L \sqrt{\tau}}{\sqrt{2}} \right)}{\sqrt{2\pi(t-\tau)}} + \mathcal{F}(\tau; t; a_L, a_R) + \mathcal{F}(t-\tau; t; a_R, a_L),
\]

(1.4)

where

\[
\mathcal{F}(\tau; t; a_L, a_R) = \frac{a_L(2a_L + a_R)}{2\sqrt{\pi}} \int_0^{t-\tau} \frac{e^{-\frac{a_L^2(z+\tau)}{2}} - e^{-\frac{a_R^2(z+\tau)}{2}}}{\sqrt{2(z+\tau)}} \text{erfc} \left( \frac{-(a_L + a_R)\sqrt{z+\tau}}{\sqrt{2(z+\tau)}} \right) dz.
\]

(1.5)

In §2 we prove Theorem 1 by applying the Feynman-Kac formula and taking inverse Laplace transforms. In §3 we discuss \( p \) when \( x_0 \neq 0 \), and in §3.1 we discuss the special cases, \( a_L = a_R = 0 \) and \( a_L = -a_R \). The long time asymptotics of (1.4) are described in §3.2 with a focus on the case \( a_L, a_R > 0 \).

In §4 we introduce an \( N \)-dimensional system of stochastic differential equations that describe stochastically perturbed sliding motion. For short time frames, in §4.1 we approximate the dynamics by a one-dimensional stochastic differential equation of the form, (1.1), and an algebraic equation involving occupation times for the remaining \( N-1 \) components of the system, and apply Theorem 1. An entirely different methodology is required for long time frames. In §4.2 we use the associated Fokker-Planck equation to derive the leading order term of an asymptotic expansion of the \( N \)-dimensional transitional PDF for stochastically perturbed sliding motion. This requires matching asymptotics to the third level in the expansion and imposing a consistency condition at the switching manifold. The resulting expression is a useful approximation to the PDF when the noise amplitude is small and at times that are long relative to the magnitude of the noise. In §4.3 we use Monte-Carlo simulations to illustrate the utility of both the short time and long time approximations. Finally §5 contains concluding remarks.

2 Proof of Theorem 1

For all \( t > 0 \), the stochastic differential equation (1.1) has a unique strong solution [25, 26, 27, 28] and therefore the expectation

\[
u(x_0, t, \lambda) = \mathbb{E}_{x_0} \left( e^{-\lambda \int_0^t \chi_{[0,\infty)}(x(s)) \, ds} \right),
\]

(2.1)
where $\lambda > 0$, is well-defined. Equation (1.4) may be obtained from $u$ because we have

$$u(0, t, \lambda) = \int_0^t e^{-\lambda \tau} p(\tau; t; 0, a_L, a_R) d\tau . \quad (2.2)$$

Following a method applied to similar problems, see for instance [24, 29, 30], by the Feynman-Kac formula [31, 32], $u$ is the unique, bounded, continuous solution to the initial value problem

$$u_t = \begin{cases} 
    a_L u_x + \frac{1}{2} u_{xx}, & x_0 < 0 \\
    -a_R u_x + \frac{1}{2} u_{xx} - \lambda u , & x_0 > 0 
\end{cases}, \quad u(x_0, 0, \lambda) = 1 . \quad (2.3)$$

Integration of the PDE with respect to $x_0$ over an arbitrarily small neighbourhood of zero reveals that $u_{x_0}$ is also continuous.

To solve (2.3), we take the Laplace transform

$$U(x_0, \omega, \lambda) = \int_0^\infty e^{-\omega t} u(x_0, t, \lambda) dt , \quad (2.4)$$

to produce the piecewise-linear ordinary differential equation

$$- 1 = \begin{cases} 
    \frac{1}{2} U_{x_0 x_0} + a_L U_{x_0} - \omega U , & x_0 < 0 \\
    \frac{1}{2} U_{x_0 x_0} - a_R U_{x_0} - (\omega + \lambda)U , & x_0 > 0 
\end{cases} . \quad (2.5)$$

Via standard ODE methods, we can obtain an explicit expression for the unique bounded solution to (2.5) for which $u$ and $u_{x_0}$ are continuous at $x_0 = 0$, and from this arrive at

$$U(0, \omega, \lambda) = \frac{\omega \left( \sqrt{2(\omega + \lambda) + a_R^2} - a_R \right) + (\omega + \lambda) \left( \sqrt{2\omega + a_L^2} - a_R \right)}{\omega(\omega + \lambda) \left( \sqrt{2(\omega + \lambda) + a_R^2} - a_R + \sqrt{2\omega + a_L^2} - a_R \right)} . \quad (2.6)$$

Our goal is to obtain $p$, which by (2.2) and (2.4) is related to $U(0, \omega, \lambda)$ by

$$U(0, \omega, \lambda) = \int_0^\infty \int_0^t e^{-\omega \tau} e^{-\lambda \tau} p(\tau; t; 0, a_L, a_R) d\tau dt . \quad (2.7)$$

Equation (2.7) is now used to derive $p$ constructively. To make the task simply the evaluation of two inverse Laplace transforms, we first reverse the order of integration in (2.7), and let $v = t - \tau$ and $\mu = \omega + \lambda$, to obtain

$$U(0, \omega, \mu - \omega) = \int_0^\infty \int_0^\infty e^{-\omega v} e^{-\mu \tau} p(\tau, v + \tau; 0, a_L, a_R) dv d\tau . \quad (2.8)$$

Simplification is provided by noting from (2.6) that we can write

$$U(0, \omega, \mu - \omega) = \Xi(\mu, \omega; a_L, a_R) + \Xi(\omega, \mu; a_R, a_L) , \quad (2.9)$$

where

$$\Xi(\mu, \omega; a_L, a_R) = \frac{\sqrt{2\mu + a_R^2} - a_R}{\mu \left( \sqrt{2\mu + a_R^2} - a_R + \sqrt{2\omega + a_L^2} - a_L \right)} . \quad (2.10)$$
By (2.8), with a little care it follows that

\[ p(\tau; t; 0, a_L, a_R) = \xi(\tau, t - \tau; a_L, a_R) + \xi(t - \tau, 0; a_R, a_L), \tag{2.11} \]

where

\[ \Xi(\mu, \omega; a_L, a_R) = \int_0^\infty \int_0^\infty e^{-\omega^2} e^{-\mu t} \xi(\tau, \nu; a_L, a_R) \, d\nu \, d\tau. \tag{2.12} \]

A determination of \( \xi \) from (2.12) using (2.10) is achieved by inverting the double Laplace transform. The details of this calculation are given in Appendix A. The result for \( \xi \) combined with (2.11) then produces (1.4) with (1.5) as required. \( \square \)

3 The behaviour and asymptotics of \( p \)

Theorem \( \square \) gives the PDF, \( p \), of the positive occupation time, \( \tau \), when \( x_0 = 0 \) (where \( x_0 = x(0) \)). The result for \( x_0 \neq 0 \) may be written in terms of the result for \( x_0 = 0 \) by applying the strong Markov property of (1.1). Specifically, let

\[ h(s; x_0, a) = \frac{|x_0|}{\sqrt{2\pi s^3}} e^{-(x_0 + as)^2/2s}, \tag{3.1} \]

denote the PDF for the first passage time, \( s \), of \( dx(t) = a \, dt + dW(t) \), from \( x(0) = x_0 \) to \( x(t) = 0 \). Then

\[ p(\tau; t; x_0, a_L, a_R) = \begin{cases} \int_0^t h(s; x_0, a_L) p(\tau; t - s; 0, a_L, a_R) \, ds + \delta(\tau) \int_t^\infty h(s, x_0, a_L) \, ds, & x_0 < 0 \\ \int_0^t h(s; x_0, -a_R) p(\tau - s; t - s; 0, a_L, a_R) \, ds + \delta(\tau - t) \int_t^\infty h(s, x_0, -a_R) \, ds, & x_0 > 0 \end{cases} \tag{3.2} \]

where \( \delta \) is the Dirac-delta function and we set \( p(\tau; t; 0, a_L, a_R) = 0 \) whenever \( \tau \notin [0, t] \). For both \( x_0 < 0 \) and \( x_0 > 0 \), the first term of (3.2) corresponds to the case that \( x(t) \) first reaches zero at a time \( s \leq t \), and the second term corresponds to \( x(t) \) not reaching zero by the time \( t \).

3.1 Two special cases

In two special cases, (1.4) reduces to previously described results. First, if \( a_L = a_R = 0 \), then \( x(t) \) is simply regular Brownian motion. As first shown by Levy \[33\], in this case the occupation time is distributed by the arc-sine distribution \[31, 34\]. Indeed, in this case only the first term of (1.4) is nonzero, and

\[ p(\tau; t; 0, 0, 0) = \frac{1}{\pi \sqrt{t - \tau}}. \tag{3.3} \]

Equation (3.3) provides a reasonable approximation to \( p(\tau; t; 0, a_L, a_R) \) for small \( t \), see Fig. \( \square \) because over short time intervals the diffusion of a stochastic quantity dominates its drift.

Second, if \( a_L = -a_R \), then \( x(t) \) is Brownian motion with constant drift. In this case (1.4) simplifies to

\[ p(\tau; t; 0, -a, a) = \left( \frac{1}{\sqrt{\pi t}} e^{-\frac{a^2}{t}} - a \sqrt{\frac{t}{\pi}} \text{erfc} \left( \frac{a}{\sqrt{2}} \frac{\sqrt{t}}{\sqrt{t}} \right) \right) \left( \frac{1}{\sqrt{\pi (t - \tau)}} e^{-\frac{a^2}{t - \tau}} + a \sqrt{\frac{t - \tau}{\pi}} \text{erfc} \left( \frac{-a}{\sqrt{2}} \frac{\sqrt{t - \tau}}{\sqrt{t - \tau}} \right) \right), \tag{3.4} \]
Figure 1: The PDF, $p$, when $a_L = 2$ and $a_R = 1$, computed by numerically evaluating the integrals in (1.4)-(1.5). Note that the axes are scaled by $t$ such that the domain is the unit interval at all times. For small $t$, $p$ is well-approximated by the density of the arc-sine distribution, (3.3). For large $t$, $p$ is approximately the Gaussian density of (3.5), since $a_L, a_R > 0$. When $t = 10$, on the given axes the Gaussian approximation has mean $\frac{a_L}{a_L + a_R} = \frac{2}{3}$ and standard deviation $\frac{1}{(a_L + a_R)\sqrt{t}} = \frac{1}{3\sqrt{10}} \approx 0.11$, as shown.

which was first derived by Akahori [29]. The simplification is not straight-forward but may be demonstrated by using different expressions for Owen’s T-function [35].

### 3.2 Long time asymptotics of $p$

Here we provide a statement of the long time asymptotics of (1.4), then discuss its derivation.

For large $t$,

$$
p(t; \tau; 0, a_L, a_R) \sim \begin{cases} \frac{a_L + a_R}{\sqrt{2\pi t}} e^{-\frac{(a_L + a_R)\tau - a_L}{a_L + a_R}^2}, & a_L > 0, a_R > 0 \\
G(t - \tau; a_L, a_R), & a_L < 0, a_R > 0 \\
G(t - \tau; a_R, a_L), & a_L > 0, a_R < 0 \\
G(t; a_L, a_R) + G(t - \tau; a_R, a_L), & a_L < 0, a_R < 0
\end{cases}
$$

(3.5)

where,

$$
G(t; a_L, a_R) = -\frac{\sqrt{2}a_L}{\sqrt{\pi t}} e^{-\frac{a_R}{\sqrt{2}}} - a_L(2a_L + a_R) e^{2a_L(a_R + a_L)\tau} \text{erfc} \left( \frac{-(2a_L + a_R)\sqrt{\tau}}{\sqrt{2}} \right).
$$

(3.6)
To obtain (3.5), note that it may be shown directly from (1.4), that for any fixed $\tau > 0$,

$$p(\tau; t; 0, a_L, a_R) \sim \begin{cases} G(\tau; a_L, a_R), & a_L < 0 \\ 0, & a_L > 0 \end{cases},$$

(3.7)

for large $t$. For brevity we omit a derivation of this statement which may be demonstrated by taking $t \to \infty$ in (1.5) and simplifying the result via integral transformations. Moreover, for $a_L < 0$,

$$\int_0^\infty G(\tau; a_L, a_R) \, d\tau = \begin{cases} \frac{-a_L}{a_L+a_R}, & a_R \leq 0 \\ \frac{a_R}{a_L+a_R}, & a_R \geq 0 \end{cases},$$

(3.8)

which ensures (3.5) has unit area in the limit $t \to \infty$. The term $G(t-\tau; a_R, a_L)$ appears in (3.5) by symmetry.

When $a_L, a_R > 0$, for large $t$ the first three terms of (1.4) may be neglected because their $L_1$ norms rapidly approach zero as $t \to \infty$. The fourth term of (1.4) approaches $\frac{4(a_L+a_R)}{\sqrt{2\pi t}} e^{-\frac{(a_L+a_R)^2}{2t}}$, because $\text{erfc}(\cdot) \to 2$. To simplify (1.5), we consider the two terms in the integrand of (1.5) separately. The first term may be neglected for large $t$ when $a_L, a_R > 0$. To deal with the second term we let

$$\rho = t^\eta \left( \tau - \frac{aLt}{aL+aR} \right),$$

(3.9)

be a scaled difference of $\tau$ from the mean, $\frac{aLt}{aL+aR}$, for some $\eta \in \mathbb{R}$. Then, substituting $u = \frac{aLz-aR\tau}{t}$ into (1.5) produces

$$\mathcal{F}(\tau; t; a_L, a_R) \sim \frac{a_L^2 (2a_L + a_R) \sqrt{t}}{2\sqrt{2\pi}} \int_{-\frac{aL}{aL+aR}}^{\frac{aR}{aL+aR}} \frac{u}{\sqrt{t}} e^{\frac{-u^2}{2}} \left( a_L + \frac{(a_L+a_R)\rho}{\sqrt{t+1}} + u \right)^2 e^{\frac{-aL}{aL+aR}u^2} \text{erfc}(\cdot) \, du,$$

(3.10)

where the argument of the complement error function is omitted for brevity. Equation (3.10) may be evaluated asymptotically (3.6). When $\rho \leq 0$, this is achieved by expanding the integrand in a Taylor series about $u = 0$; when $\rho > 0$, we expand about the upper limit of the integral. For instance, when $\rho \leq 0$ (the other case is similar and produces the same result), after integration we have

$$\mathcal{F}(\tau; t; a_L, a_R) \sim -\frac{\sqrt{aL^2 (2a_L + a_R)}}{\sqrt{2\pi \left( a_L + \frac{(a_L+a_R)\rho}{\sqrt{t+1}} \right) t}} e^{\frac{-aL(aL+aR)^2}{2t(aL+aR)^2 + (aL+aR)^2 \rho^2} + O \left( \frac{1}{t} \right)}.$$

(3.11)

Only when $\eta = -\frac{1}{2}$ is the leading order term of (3.11) non-constant in the limit $t \to \infty$, and in this case, after also substituting back $\tau$ using (3.9),

$$\mathcal{F}(\tau; t; a_L, a_R) \sim -\frac{2aL + a_R}{\sqrt{2\pi t}} e^{\frac{-(a_L+aR)^2}{2t}} + O \left( \frac{1}{t} \right).$$

(3.12)

In view of (3.12), summing the asymptotic limits of the fourth, fifth and sixth terms of (1.4) yields (3.5) for $a_L, a_R > 0$. 

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When $a_L, a_R > 0$, (3.3) is the PDF of a Gaussian with mean $\frac{a_L L}{a_L + a_R}$, and standard deviation $\frac{\sqrt{\pi}}{a_L + a_R}$; see Fig. 11 In this case the direction of the drift is toward $x = 0$ for both positive and negative $x$. The fraction of time spent in $[0, \infty)$, approaches $\frac{a_L}{a_L + a_R}$ as $t \rightarrow \infty$. This is consistent with the fact that the area under the steady-state density of $x(t)$ over $[0, \infty)$ is $\frac{a_L}{a_L + a_R}$ [19].

4 An application to stochastically perturbed sliding motion

Our interest is in the dynamics near a switching manifold of a general N-dimensional Filippov system. Assuming the manifold is smooth, we may choose our coordinate system such that the manifold coincides with points in $\mathbb{R}^N$ that are zero in their first coordinate [2, 37]. We let $x \in \mathbb{R}$ denote the first coordinate and $y \in \mathbb{R}^{N-1}$ denote the remaining coordinates. If we ignore any other switching manifolds, and add constant noise to the system in the form of additive Brownian motion, then the system may be written as

$$\begin{bmatrix}
\frac{dx(t)}{dy(t)} \\
\end{bmatrix} = \begin{cases}
\begin{bmatrix}
\phi^{(L)}(x(t), y(t)) \\
\psi^{(L)}(x(t), y(t)) \\
\phi^{(R)}(x(t), y(t)) \\
\psi^{(R)}(x(t), y(t)) \\
\end{bmatrix}, & x(t) < 0 \\
\begin{bmatrix}
\phi^{(L)}(x(t), y(t)) \\
\psi^{(L)}(x(t), y(t)) \\
\phi^{(R)}(x(t), y(t)) \\
\psi^{(R)}(x(t), y(t)) \\
\end{bmatrix}, & x(t) > 0
\end{cases}
\begin{aligned}
\frac{dx(t)}{dy(t)} &= dt + \sqrt{\varepsilon}DdW(t), \\
x(0) &= \begin{bmatrix} 0 \\
y_0 \end{bmatrix},
\end{aligned}
$$

(4.1)

where $W(t)$ is an $N$-dimensional standard Brownian motion, $\varepsilon > 0$ governs the overall strength of the noise, and the $N \times N$ matrix, $D$, specifies the relative strength of the noise in different directions. For simplicity we have set $x(0) = 0$, so that the solution is initially located on the switching manifold. We assume, at least locally, that the functions $\phi^{(L)}$, $\phi^{(R)}$, $\psi^{(L)}$, and $\psi^{(R)}$ are $C^2$ on the closure of their respective half spaces. The purpose of this assumption is to allow us to study their Taylor series centred at $x = 0$. We write

$$\begin{align*}
\phi^{(L)}(x, y) &= a_L(y) + c_L(y)x + O(x^2), \\
\psi^{(L)}(x, y) &= b_L(y) + d_L(y)x + O(x^2), \\
\phi^{(R)}(x, y) &= -a_R(y) + c_R(y)x + O(x^2), \\
\psi^{(R)}(x, y) &= b_R(y) + d_R(y)x + O(x^2),
\end{align*}
$$

(4.2)

where $a_L, a_R, b_L$ and $b_R$ are $C^2$ functions of $y$, and $c_L, c_R, d_L$ and $d_R$ are $C^1$ functions of $y$. Note that $a_L, a_R, c_L$ and $c_R$ are scalar functions, whereas $b_L, b_R, d_L$ and $d_R$ take values in $\mathbb{R}^{N-1}$.

In view of the minus sign in (4.2), subsets of $x = 0$ for which

$$a_L(y) > 0, \quad a_R(y) > 0,$$

(4.3)

are known as stable sliding regions, because when $\varepsilon = 0$, (4.1) is a vector field that points toward these regions from both sides. When $\varepsilon = 0$, forward evolution of a point on a stable sliding region is not defined in the classical sense. For this reason, as is usual, we employ Filippov’s method to define a sliding vector field on $x = 0$, $\dot{y}_S = \Omega(y_S)$, as the unique convex combination of the limiting left and right vector fields that is tangent to $x = 0$ [1, 2, 38]. Specifically, for (4.1) with
\[ \varepsilon = 0, \text{ on } x = 0 \text{ we write} \]
\[
\begin{bmatrix}
\dot{x}_S \\
\dot{y}_S
\end{bmatrix} = (1 - \kappa(y_S)) \begin{bmatrix}
a_L(y_S) \\
b_L(y_S)
\end{bmatrix} + \kappa(y_S) \begin{bmatrix}
a_R(y_S) \\
b_R(y_S)
\end{bmatrix},
\]
(4.4)
where \( \kappa \) is given by the requirement \( \dot{x}_S \equiv 0 \), thus \( \kappa = \frac{a_L}{a_L + a_R} \). The sliding vector field is therefore
\[
\begin{bmatrix}
\dot{y}_S
\end{bmatrix} = \frac{a_L(y_S)b_R(y_S) + a_R(y_S)b_L(y_S)}{a_L(y_S) + a_R(y_S)} = \Omega(y_S).
\]
(4.5)
We assume that forward orbits of (4.1) with \( \varepsilon = 0 \) evolve on \( x = 0 \), as governed by (4.5), until (4.3) ceases to be true and the orbits escape \( x = 0 \).

When \( \varepsilon > 0 \), an additional definition is not required to define solutions to (4.1). If \( \phi^{(L)}, \phi^{(R)}, \psi^{(L)}, \psi^{(R)} \) are bounded, then for any \( \varepsilon > 0 \), (4.1) has a unique strong stochastic solution \([25, 26, 27, 28]\).

Our interest is in the transitional PDF of (4.1) (i.e. the PDF of the point \((x(t), y(t)) \in \mathbb{R}^N\)), call it \( q(x, y, t) \). We assume that at all times in \([0, t]\), the deterministic solution lies in a stable sliding region, Therefore \( y_S(t) \) is determined by (4.5), and \( a_L(y_S(s)) > 0 \) and \( a_R(y_S(s)) > 0 \) for all \( s \in [0, t] \). Below we derive two new explicit limiting expressions for \( q(x, y, t) \). First we apply Theorem 1 to describe marginals of \( q \) in the limit \( t \to 0 \). Second we perform an asymptotic expansion to describe \( q \) in the limit \( \varepsilon \to 0 \). This result applies for times, \( t \gg \varepsilon \).

### 4.1 Asymptotics for small \( t \)

To obtain explicit results for small \( t \), we apply two simplifications to (4.1). First, we expect \( x(t) \) and \( y(t) \) to not vary greatly over short time frames, in which case it is reasonable to approximate the drift terms in (4.1) by their values at \( t = 0 \). This gives the approximation

\[
\begin{bmatrix}
\frac{dx(t)}{dt} \\
\frac{dy(t)}{dt}
\end{bmatrix} = \begin{cases}
\begin{bmatrix}
a_L \\
b_L
\end{bmatrix}, & x(t) < 0 \\
\begin{bmatrix}
-a_R \\
b_R
\end{bmatrix}, & x(t) > 0
\end{cases}dt + \sqrt{\varepsilon}DdW(t),
\]
(4.6)
where \( a_L, a_R, b_L \) and \( b_R \) are evaluated at \( y(0) = y_0 \). Second, we assume that the noise in \( x \) is independent to the noise in \( y \). Given these assumptions, in this section we derive explicit expressions for the following marginal densities of \( q \):

\[
q_{\text{orthogonal}}(x, t) = \int_{\mathbb{R}^{N-1}} q(x, y, t) dy, \quad q_{\text{parallel}}(y, t) = \int_{\mathbb{R}} q(x, y, t) dx.
\]
(4.7)
Difficulties in obtaining more general results are discussed at the end of \([15]\). The nomenclature of (4.7) was chosen because the \( x \)-axis is orthogonal to the switching manifold, whereas \( y \) comprises of all directions parallel to the switching manifold.

The crucial benefit obtained by the first simplification is that the right hand side of (4.6) is independent of \( y \). Hence we can decouple (4.6) and first write

\[
dx(t) = \begin{cases}
a_L, & x < 0 \\
-a_R, & x > 0
\end{cases} dt + \sqrt{\varepsilon}dW(t), \quad x(0) = 0,
\]
(4.8)
where we have contracted the vector noise term into an equivalent scalar noise term by letting \( \alpha = (DD^T)_{11} \). The transitional PDF of (4.8) is

\[
q_{\text{orthogonal}}(x, t) = \begin{cases} 
\frac{2}{\varepsilon^3 \alpha^2} e^{\frac{2 \alpha x^2}{\varepsilon^2 \alpha^2}} \int_0^\infty \int_0^t h\left(\frac{t-s}{\varepsilon \alpha}, \frac{b_R-a_L}{\varepsilon \alpha}\right) ds \ db, & x \leq 0 \\
\frac{2}{\varepsilon^3 \alpha^2} e^{-2 \alpha R^2} \int_0^\infty \int_0^t h\left(\frac{t-s}{\varepsilon \alpha}, \frac{b_R-a_L}{\varepsilon \alpha}\right) ds \ db, & x \geq 0
\end{cases},
\tag{4.9}
\]

where \( h \) is the first passage time PDF, (3.1). For a derivation of (4.9), refer to [24, 31].

Second, integration of (4.6) yields

\[
y(t) = y_0 + b_L t + (b_R - b_L) \int_0^t \chi_{[0,\infty)}(x(s)) \, ds + \sqrt{\varepsilon} \tilde{D} \mathbf{W}(t),
\tag{4.10}
\]

where we use \( \tilde{D} \) to denote the lower right \((N-1) \times (N-1)\) block of \( D \). Since the scaling, \( x \mapsto \frac{x}{\varepsilon \alpha}, \quad t \mapsto \frac{t}{\varepsilon \alpha} \), transforms (4.8) to (4.11) (with \( x_0 = 0 \)), here the PDF of \( \tilde{\tau} = \int_0^t \chi_{[0,\infty)}(x(s)) \, ds \) is

\[
p_{\text{scaled}}(\tilde{\tau}) = \frac{1}{\varepsilon \alpha} p\left(\frac{\tau}{\varepsilon \alpha}; 0, a_L, a_R\right).
\tag{4.11}
\]

Via straight-forward geometric arguments, it follows that the PDF of \( \tilde{y}(t) = y_0 + b_L t + (b_R - b_L) \int_0^t \chi_{[0,\infty)}(x(s)) \, ds \), is given by

\[
\tilde{q}_{\text{parallel}}(\tilde{y}, t) = \delta \left( \left\| \tilde{y} - y_0 - b_L t - (b_R - b_L) \frac{(\tilde{y} - y_0 - b_L t)^T (b_R - b_L)}{\|b_R - b_L\|^2} \right\| \right)
\times \frac{1}{\|b_R - b_L\|^2} p_{\text{scaled}} \left( \frac{(\tilde{y} - y_0 - b_L t)^T (b_R - b_L)}{\|b_R - b_L\|^2} \right),
\tag{4.12}
\]

where \( \delta \) is the Dirac-delta function. Finally, the PDF of \( y(t) \), is the convolution of \( \tilde{q}_{\text{parallel}}(\tilde{y}, t) \) and the PDF of \( \sqrt{\varepsilon} \tilde{D} \mathbf{W}(t) \):

\[
q_{\text{parallel}}(y, t) = \int_{\mathbb{R}^{N-1}} \tilde{q}_{\text{parallel}}(y - w, t) \frac{1}{(2\pi \varepsilon^2 t)^{\frac{N-1}{2}}} \sqrt{\det(\gamma)} e^{-\frac{1}{2\varepsilon^2} w^T \gamma^{-1} w} \, dw,
\tag{4.13}
\]

where \( \gamma = \tilde{D} \tilde{D}^T \) [4.13].

### 4.2 Asymptotics for small \( \varepsilon \)

The Fokker-Planck equation for (4.1) is

\[
q_t = \begin{cases} 
- (\phi^{(L)} q)_x - \left( \psi_i^{(L)} q \right) \gamma_i & x < 0 \\
- (\phi^{(R)} q)_x - \left( \psi_i^{(R)} q \right) \gamma_i & x > 0
\end{cases},
\tag{4.14}
\]

where

\[
DD^T = \begin{bmatrix} \alpha & \beta^T \\ \beta & \gamma \end{bmatrix},
\tag{4.15}
\]
and $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^{N-1}$ and $\gamma$ is an $(N-1) \times (N-1)$ matrix. In (4.14), and throughout this section, we use index notation to abbreviate summations in $i$ and $j$, which always range from 1 to $N-1$. The initial and boundary conditions for $q$ are

$$q(x,y,t) \to 0, \text{ as } |x|, ||y|| \to \infty,$$

$$q(x,y,0) = \delta(x)\delta(y-y_0),$$

(4.16)

(4.17)

together with a consistency condition at the switching manifold, $x = 0$. This consistency condition is given by the requirement that no probability is gained or lost at $x = 0$:

$$e_1^T \left( \lim_{x \to 0^-} J - \lim_{x \to 0^+} J \right) = 0,$$

(4.18)

where $J$ denotes the probability current of (4.1) [39, 40]. In the $x$-direction, $J$ is given by

$$e_1^T J = \begin{cases} \phi(L)q - \frac{\alpha}{2} q_x - \frac{\epsilon}{2} \sum \beta_i q_{y_i}, & x < 0 \\ \phi(R)q - \frac{\alpha}{2} q_x - \frac{\epsilon}{2} \sum \beta_i q_{y_i}, & x > 0 \end{cases}.$$  

(4.19)

To determine $q$ from the above boundary value problem when $\epsilon$ is small, a preliminary analysis indicates that a change of variables corresponding to tracking the system near the deterministic sliding solution is useful. Specifically we define

$$X = \frac{x}{\varepsilon}, \quad Y = \frac{y - y_S(t)}{\sqrt{\varepsilon}},$$

(4.20)

where $y_S(t)$ is the deterministic sliding solution determined by (4.5). The motivation for this scaling is discussed in greater detail in earlier work [19]. Under this change of variables, one can obtain an asymptotic approximation from the resulting Fokker-Planck equation for $Q(X,Y,t) = q(x,y,t)$, through a regular series expansion

$$Q = Q^{(0)} + \sqrt{\varepsilon}Q^{(1)} + \varepsilon Q^{(2)} + \cdots.$$  

(4.21)

Here we summarize the steps leading to the asymptotic approximation to $q$ with details given in Appendix B. We note that while the calculations for $X < 0$ and $X > 0$ may be treated separately, the equations for $X > 0$ are the same as those for $X < 0$ but with $-a_R$ in place of $a_L$, and $R$’s in place of the remaining $L$’s.

By (4.12) and (4.20), for $X < 0$ the leading order component of the Fokker-Planck equation (4.14) is

$$\frac{\alpha}{2} Q^{(0)}_{XX} - a_L(y_S(t))Q^{(0)}_X = 0.$$  

(4.22)

Noting that $Q \to 0$ as $X \to -\infty$, the leading order contribution to $Q$ takes the form

$$Q^{(0)}(X,Y,t) = f^{(0)}(Y,t) e^{\frac{2a_L(y_S(t))X}{\alpha}},$$  

(4.23)

for some function $f^{(0)}$. By also using the analogous form for $X > 0$: $Q^{(0)}(X,Y,t) = f^{(0)}(Y,t) e^{-\frac{2a_R(y_S(t))X}{\alpha}}$, we find that the consistency condition for $Q$ at $X = 0$, (4.18), is automatically satisfied for any function $f^{(0)}(Y,t)$. Therefore it is necessary to calculate higher
order contributions in order to determine \( f^{(0)} \), and hence completely determine the leading order term, \( Q^{(0)} \).

For \( X < 0 \), by taking \( O(\sqrt{\varepsilon}) \) terms and \( O(\varepsilon) \) terms respectively, we find that the equations for \( Q^{(1)} \) and \( Q^{(2)} \) take the form

\[
\frac{\alpha}{2} Q^{(1)}_{XX} - a_L(y_S(t))Q^{(1)}_X = \Phi_1(Y, t)e^{\frac{2a_L(y_S(t))X}{\alpha}}, \tag{4.24}
\]

\[
\frac{\alpha}{2} Q^{(2)}_{XX} - a_L(y_S(t))Q^{(2)}_X = (\Phi_2(Y, t) + \Phi_3(Y, t)X)e^{\frac{2a_L(y_S(t))X}{\alpha}}, \tag{4.25}
\]

where the \( \Phi_i \) depend on \( f^{(0)} \). From (4.24) (and the analogous equation for \( X > 0 \)) we find that the consistency condition is automatically satisfied at \( O(\sqrt{\varepsilon}) \). It is only by using (4.25) to apply the consistency condition at \( O(\varepsilon) \) that we obtain an equation for \( f^{(0)} \), see Appendix B. The substitution

\[
g^{(0)}(Y, t) = \frac{a_L + a_R}{a_L a_R} \bigg|_{y=y_S(t)} f^{(0)}(Y, t), \tag{4.26}
\]

then produces

\[
g^{(0)}_i = -\Omega_{i,j}(Y_S y^{(0)})(Y_i) + \frac{1}{2} \left( \frac{r_{Li}r_{Lj}a^2}{a_L^2} - \frac{r_{Li}\beta_i}{a_L} - \frac{r_{Lj}\beta_j}{a_L} + \gamma_{ij} \right) \bigg|_{y=y_S(t)} g^{(0)}_{Y_i Y_j}, \tag{4.27}
\]

where we have introduced

\[
r_L = \frac{a_L(b_L - b_R)}{a_L + a_R}, \quad r_R = -\frac{a_R(b_L - b_R)}{a_L + a_R}. \tag{4.28}
\]

Equation (4.27) is the Fokker-Planck equation of the stochastic differential equation

\[
dY_{\text{approx}}(t) = (D_y \Omega)(y_S(t))Y_{\text{approx}}(t) dt + M(y_S(t)) dW(t), \quad Y_{\text{approx}}(0) = 0, \tag{4.29}
\]

where \( \Omega \) is given by (4.5) and

\[
M(y) = \left[ \begin{array}{c} -\frac{b_L(y) - b_R(y)}{a_L(y) + a_R(y)} \end{array} \right] I, \tag{4.30}
\]

where \( I \) is the \((N-1) \times (N-1)\) identity matrix. Equation (4.29) also arises when stochastic averaging arguments are applied to (4.1) with (4.20), [11]. Since (4.29) is a time-dependent Ornstein-Uhlenbeck process [39, 42], the PDF of \( Y_{\text{approx}}(t) \) is a zero-mean Gaussian with covariance matrix:

\[
\Theta(t) = \int_0^t e^{\int_s^t (D_y \Omega)(y_S(u)) du} M(y_S(s)) M(y_S(s))^T e^{\int_s^t (D_y \Omega)(y_S(u))^T du} ds. \tag{4.31}
\]

Since \( Q \to 0 \) as \( ||y|| \to \infty \), we therefore have

\[
g^{(0)}(Y, t) = \frac{K(t)}{\sqrt{\det(\Theta(t))}} e^{-\frac{1}{2} Y^T \Theta(t)^{-1} Y}, \tag{4.32}
\]
where $K(t)$ is determined by the requirement that $q$ is normalized. The combination of (4.23) and (4.26) and (4.32), provides the following expression for the transitional PDF of (4.1):

$$q(x, y, t) = \frac{1}{(2\pi)^{N/2} \sqrt{\det(\Theta(t))}} e^{-\frac{1}{2} Y^T \Theta(t)^{-1} Y} \frac{2a_L a_R}{\alpha(\alpha + a_R)} \left\{ \begin{array}{ll} e^{\frac{2\alpha - \alpha}{\alpha + a_R} X}, & X < 0 \\ e^{-\frac{2\alpha - \alpha}{\alpha + a_R} X}, & X > 0 \end{array} \right\} \bigg|_{y = y_S(t)} + O\left( \frac{1}{\epsilon^2} \right),$$

(4.33)

where $X$ and $Y$ are given by (4.20).

Now we consider the approximation obtained by omitting the error term in (4.33). Since $t$ is implicitly assumed to be independent of $\epsilon$, this approximation applies at times, $t \gg \epsilon$. The approximation is piecewise-exponential in $x$ and Gaussian in $y$. Moreover, $x$ and $y$ are independent in the sense that the value of $x(t)$ does not give us any information about the value of $y(t)$, and vice-versa. The mean value of $y(t)$ is Filippov’s sliding solution, $y_S(t)$. The covariance of $y$ limits to zero as $\epsilon \to 0$, which is consistent with the result of (4.3) that tells us that in the limit $\epsilon \to 0$, solutions to (4.1) are Filippov solutions. Finally, deviations of $y$ from the mean are $O(\sqrt{\epsilon})$, whereas deviations of $x$ from the mean are $O(\epsilon)$. Therefore, there is significantly more variability in the value of $y(t)$ than in the value of $x(t)$.

### 4.3 An illustration of the results for a simple example

To illustrate the formulas, (4.13) and (4.33), consider

$$\begin{bmatrix} dx(t) \\ dy(t) \end{bmatrix} = \begin{cases} \begin{bmatrix} -x(t) + y(t) + 1 \\ -x(t) + 1 \end{bmatrix}, & x(t) < 0 \\ \begin{bmatrix} -x(t) + y(t) - 3 \\ -x(t) - 2 \end{bmatrix}, & x(t) > 0 \end{cases} dt + \sqrt{\epsilon} \begin{bmatrix} 1 & 0 \\ 0 & 1/3 \end{bmatrix} dW(t), \quad \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

(4.34)

which is a two-dimensional example fitting the general form (4.1). Here, $a_L(y) = 1 + y$ and $a_R(y) = 3 - y$, thus, by (4.3), the interval $(-1, 3)$ of the $y$-axis is a stable sliding region. From (4.3) we find that the deterministic sliding solution is

$$y_S(t) = \frac{5}{3} e^{-\frac{1}{3} t} + \frac{1}{3}.$$  

(4.35)

Fig. 2 shows histograms of the value of $y(t)$ as determined from $10^5$ sample solutions to (4.34) that were computed numerically using the Euler-Maruyama method of step size $\Delta t = 10^{-5}$. We have the scaled the histograms to unit area so that they represent the PDF of $y(t)$. The solid curves show the short time approximation, (4.13). The dashed curves show the marginal density in $y$ of the long time approximation, (4.33).

As expected, the accuracy of the short time approximation, (4.13), decreases with increasing $t$, because the piecewise-constant approximation to (4.34) formed by evaluating the two pieces of the drift at $(x, y) = (0, 2)$, as in (4.6), worsens as $(x(t), y(t))$ becomes distant from $(0, 2)$. In contrast, the long time approximation, (4.33), is more accurate at larger values of $t$. This is because (4.33) may be thought of as a quasi-steady-state density that is achieved once sufficient time has passed that sample solutions to (4.1) have spent a significant amount of time on each side of $x = 0$, with high probability. In view of the scaling (4.20), this required length of time is of order $\epsilon$.  

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5 Discussion

This paper gives for the first time the PDF of the positive occupation time of Brownian motion with two-valued drift. The PDF, $p$, is given by the complicated expression (1.4). For large $t$, $p$ is asymptotic to the simpler form (3.5). In the case that the piecewise drift points toward the origin on both sides, (3.5) is Gaussian.

Figure 2: The PDF of $y(t)$ for (4.34) with $\varepsilon = 0.1$ at several different values of $t$. The histograms were computed from numerically computed sample solutions to (4.34). The solid curves show the small time approximation, (4.13). The dashed curves show the long time approximation, (4.33).
In §4 we used the result to improve our understanding of the influence of additive noise on sliding motion. When the noise amplitude, $\varepsilon$, is zero, the solution to the general $N$-dimensional system, (4.1), slides along $x = 0$ for some time. For $\varepsilon > 0$, we let $q(x, y, t)$ denote the transitional PDF of (4.1). To approximate $q$ for small $t$, we constructed a system with piecewise-constant drift, (4.6), by evaluating the drift functions at the initial point. With this approximation, $x(t)$ is Brownian motion with two-valued drift, (4.8), and $y(t)$ is a function of the positive occupation time of $x(t)$, (4.10). In the case that the noise applied to $x$ is independent to the noise applied to $y$, it follows immediately from (4.10) that we can write the PDF of $y(t)$ as a convolution involving $p$, (4.13). The accuracy of the PDF (4.13) for the general system (4.1) correlates with the suitability of approximating the drift terms by their initial values. For this reason (4.13) is less accurate at larger times, and indeed this is evident for the example considered in Fig. 2.

Although the approximate system, (4.6), is in general only applicable for small $t$, it is instructive to apply the long time asymptotic result, (3.5). For large $t$, the occupation time, $\int_0^t \chi_{[0,\infty]}(x(s)) \, ds$, is asymptotic to a Gaussian random variable of mean, $\frac{a_L t}{a_L + a_R}$, and variance, $\varepsilon t$. By substituting this into (4.10), we obtain

$$E(y(t)) \sim y_0 + \Omega(y_0) t, \quad \text{Cov}(y(t)) \sim \left(\frac{\alpha(b_R - b_L)(b_R - b_L)^T}{(a_L + a_R)^2} + \gamma\right) \varepsilon t,$$

(5.1)

where $\Omega$ is given by (4.5). Thus the mean of $y(t)$ is asymptotic to the deterministic sliding solution of the approximate system, (4.6), and the covariance of $y(t)$ recovers a result of [19] that was obtained directly from the PDF of (1.1).

For large $t$, the distribution of $y(t)$ for the original system, (4.1), is approximately Gaussian if the noise amplitude, $\varepsilon$, is small. Although the approximation, (4.13), is asymptotically Gaussian for large $t$, it gives incorrect values for the mean and covariance because the derivation of (4.13) assumes $t$ is small and, specifically, does not account for $O(1)$ global changes in the drift as the system evolves. We derived the small $\varepsilon$ approximation, (4.33), which is Gaussian with a mean and covariance obtained by appropriately tracking the deterministic sliding solution. For the example shown in Fig. 2 this approximation shows good agreement to Monte-Carlo simulations for relatively large values of $t$.

Since $x$ and $\tau$ are strongly correlated, the product $q_{\text{orthogonal}}(x, t)q_{\text{parallel}}(y, t)$ is not a sensible approximation to $q(x, y, t)$. It remains to compute the joint PDF of $x$, (1.1), and $\tau$, (1.2), from which we could obtain an accurate expression of $q(x, y, t)$ for small $t$. Similarly it remains to compute the joint PDF of $W(t)$ and $\tau$, which we could use for the general case that the noise applied to $x$ is not independent to the noise applied to $y$.

A A calculation of the double inverse Laplace transform

In this section we invert the double Laplace transform (2.12) to obtain $\xi$. Using the inversion formula, $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{\pi + k}}\right) = \frac{1}{\sqrt{\pi t}} - ke^{k^2 t} \text{erfc}\left(k\sqrt{t}\right)$, together with the standard shifting and scaling
rules for Laplace transforms, the inverse Laplace transform of (2.10) with respect to \( \omega \) is

\[
\int_0^\infty e^{-\mu \tau} \xi(\tau, v; a_L, a_R) d\tau = \frac{\sqrt{2 \mu + a_R^2 - a_R}}{2\mu} e^{-\frac{a_R^2}{2\mu}}\left( \frac{\sqrt{2}}{\sqrt{\pi v}} - \left( \sqrt{2 \mu + a_R^2 - a_R - a_L} \right) \right) 
\times e^{\left( \sqrt{2 \mu + a_R^2 - a_R - a_L} \right) / \sqrt{2}} \text{erfc} \left( \left( \sqrt{2 \mu + a_R^2 - a_R - a_L} \right) \frac{\sqrt{v}}{\sqrt{2}} \right). \tag{A.1}
\]

To perform the daunting inverse Laplace transform of (A.1), we rewrite it as

\[
\int_0^\infty e^{-\mu \tau} \xi(\tau, v; a_L, a_R) d\tau = e^{-\frac{a_R^2}{2}} \left( \frac{1}{\sqrt{\pi v}} H_1(\mu; a_R) - H_2(\mu; v, a_L, a_R) \right) + \frac{a_L + 2a_R}{\sqrt{2}} \frac{H_1(\mu; a_R) H_2(\mu, v; a_L, a_R)}{\sqrt{2}}, \tag{A.2}
\]

where

\[
H_1(\mu; a_R) = \frac{\sqrt{2 \mu + a_R^2 - a_R}}{\sqrt{2\mu}}, \tag{A.3}
\]

\[
H_2(\mu, v; a_L, a_R) = e^{\left( \sqrt{2 \mu + a_R^2 - a_R} \right) / \sqrt{2}} \text{erfc} \left( \left( \sqrt{2 \mu + a_R^2 - a_R - a_L} \right) \frac{\sqrt{v}}{\sqrt{2}} \right), \tag{A.4}
\]

so that

\[
\xi(\tau, v; a_L, a_R) = e^{-\frac{a_R^2}{2}} \left( \frac{1}{\sqrt{\pi v}} h_1(\tau; a_R) - h_2(\tau, v; a_L, a_R) + \frac{a_L + 2a_R}{\sqrt{2}} h_1(\tau; a_R) * h_2(\tau, v; a_L, a_R) \right), \tag{A.5}
\]

where \( h_1 \) and \( h_2 \) denote the inverse Laplace transforms of \( H_1 \) and \( H_2 \) respectively, and \(*\) denotes convolution with respect to \( \tau \). By writing \( H_1(\mu; a_R) = -\frac{a_R}{\sqrt{2\mu}} + \frac{\sqrt{3}}{\sqrt{2\mu + a_R^2}} + \frac{a_R^2}{\sqrt{2\mu \sqrt{2\mu + a_R^2}}} \), and using, in particular, \( \mathcal{L}^{-1} \left( \frac{1}{\sqrt{s + k^2}} \right) = \frac{1}{\sqrt{k}} \left( 1 - \text{erf} \left( \sqrt{kt} \right) \right) \), we obtain

\[
h_1(\tau; a_R) = e^{-\frac{a_R^2}{2\tau}} - \frac{a_R}{\sqrt{\tau}} \text{erfc} \left( \frac{a_R\sqrt{\tau}}{\sqrt{2}} \right). \tag{A.6}
\]

Also by using \( \mathcal{L}^{-1} \left( e^{(\sqrt{s+k)}^2} \text{erfc} \left( \sqrt{s + k} \right) \right) = \frac{1}{2\pi^2} \int_0^\infty ze^{-\frac{z^2}{2}}(1+\frac{1}{4})e^{-kz} dz \) (obtained by combining formulas in \[44\]), we obtain

\[
h_2(\tau, v; a_L, a_R) = e^{-\frac{a_R^2}{2\tau}} \left( \sqrt{\frac{v}{\pi(\tau + v)}} + \frac{a_L + a_R}{\sqrt{2(\tau + v)}} e^{-\frac{(a_L+a_R)^2}{2(\tau + v)}} \text{erfc} \left( \frac{(a_L + a_R)\sqrt{v}}{\sqrt{2(\tau + v)}} \right) \right). \tag{A.7}
\]

After an involved algebraic simplification that we omit for brevity, the combination of (2.11), (A.5), (A.6) and (A.7) yields (1.4) with (1.5) as required.
B Details of the asymptotic expansion

For $X < 0$, the Fokker-Planck equation for $Q(X, Y, t)$ is

\[
Q_t = -\frac{1}{\varepsilon} a_L Q_X - \frac{1}{\sqrt{\varepsilon}} a_{L,i} Y_i Q_X - \frac{1}{2} a_{L,ij} Y_i Y_j Q_X - c_L (XQ)_X - \frac{1}{\sqrt{\varepsilon}} b_{Li} Q_{Y_i} - b_{Li,j} (Y_j Q)_{Y_i} + \frac{\alpha}{2\varepsilon} Q_{XX} + \frac{1}{\sqrt{\varepsilon}} \beta_i Q_{XY_i} + \frac{1}{2} \gamma_{ij} Q_{Y_i Y_j} + \frac{1}{\sqrt{\varepsilon}} \Omega_i Q_Y + O(\varepsilon), \tag{B.1}
\]

where for the remainder of this section, unless otherwise specified, the coefficients, $a_J, \ldots, d_J$, and their derivatives, are evaluated at $y = y_S(t)$, and we have abbreviated derivatives with respect to components of $y$ with indices in subscripts following commas:

\[
a_{L,i} \equiv \frac{\partial a_L}{\partial y_i}, \quad a_{L,ij} \equiv \frac{\partial^2 a_L}{\partial y_i \partial y_j}, \quad b_{Li,j} \equiv \frac{\partial b_{Li}}{\partial y_j}. \tag{B.2}
\]

By multiplying (B.1) (and the analogous expression for $X > 0$) through by $\varepsilon$ and collecting terms of the same order, we arrive at

\[
\frac{\alpha}{2} Q_{XX} - a_L Q_X = \begin{cases} \sqrt{\varepsilon} (a_{L,i} Y_i Q_X + r_{Li} Q_{Y_i} - \beta_i Q_{XY_i}) + \varepsilon \left( \frac{1}{2} a_{L,ij} Y_i Y_j Q_X + c_L (XQ)_X \right) + b_{Li,j} (Y_j Q)_{Y_i} + Q_t + O \left( \varepsilon^{\frac{3}{2}} \right), & X < 0 \\ \sqrt{\varepsilon} (-a_{R,i} Y_i Q_X + r_{Ri} Q_{Y_i} - \beta_i Q_{XY_i}) + \varepsilon \left( -\frac{1}{2} a_{R,ij} Y_i Y_j Q_X + c_R (XQ)_X \right) + b_{Ri,j} (Y_j Q)_{Y_i} + Q_t + O \left( \varepsilon^{\frac{3}{2}} \right), & X > 0 \end{cases} \tag{B.3}
\]

where we have substituted (1.28). In terms of the scaled variables, the consistency condition, (1.15), may be written as

\[
\lim_{X \to 0^-} Q_X(X, Y, t) - \lim_{X \to 0^+} Q_X(X, Y, t) = \frac{2}{\alpha} \left( a_L + a_R + \sqrt{\varepsilon} (a_{L,i} + a_{R,i}) Y_i \right. + \varepsilon \left. \left( a_{L,ij} + a_{R,ij} \right) Y_i Y_j + O \left( \varepsilon^{\frac{3}{2}} \right) \right) Q(0, Y, t) \tag{B.4}
\]

The leading order term, $Q^{(0)}$, of the asymptotic expansion of $Q$ is given by (1.28). The equation for the $O(\sqrt{\varepsilon})$ correction to $Q^{(0)}$ is given by

\[
\frac{\alpha}{2} Q_{XX}^{(1)} - a_L Q_X^{(1)} = \Phi_1(Y, t) e^{\frac{2a_L X}{\alpha}}, \tag{B.5}
\]

where

\[
\Phi_1(Y, t) = \frac{2a_L}{\alpha} a_{L,i} Y_i f^{(0)} + r_{Li} f^{(0)} - \frac{2a_L}{\alpha} \beta_i f_Y^{(0)}. \tag{B.6}
\]

Consequently,

\[
Q^{(1)}(X, Y, t) = \left( f^{(1)}(Y, t) + \frac{1}{a_L} \Phi_1(Y, t) X \right) e^{\frac{2a_L X}{\alpha}}, \tag{B.7}
\]

for some function $f^{(1)}$. Since the consistency condition (B.4) is automatically satisfied to $O(1)$ and $O(\sqrt{\varepsilon})$, $f^{(0)}$ and $f^{(1)}$ are undetermined. The equation for $Q^{(2)}$ is

\[
\frac{\alpha}{2} Q_{XX}^{(2)} - a_L Q_X^{(2)} = \left( \Phi_2(Y, t) + \Phi_3(Y, t) X \right) e^{\frac{2a_L X}{\alpha}}, \tag{B.8}
\]

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where
\[
\Phi_2(Y, t) = \frac{2a_L}{\alpha} a_{L,i} Y_i f^{(1)} + \frac{2}{\alpha} a_{L,i} a_{L,j} Y_j f^{(0)} + \frac{1}{a_L} a_{L,i} r_{L,j} Y_j f^{(0)}_j \\
- \frac{2}{\alpha} a_{L,i} \beta_j Y_i f^{(1)}_j + r_{L,i} f^{(1)}_i - \frac{2a_L}{\alpha} \beta_i f^{(1)}_i - \frac{2}{\alpha} a_{L,j} \beta_i (Y_j f^{(0)}) y_i \\
- \frac{1}{a_L} r_{L,j} \beta_i f^{(0)}_i y_j + \frac{2}{\alpha} \beta_i j f^{(0)}_i y_j + \frac{a_L}{\alpha} a_{L,i} y_i f^{(0)} + c_L f^{(0)} \\
+ b_{L,i} (Y_j f^{(0)}) y_i - \frac{1}{2} \gamma_{ij} f^{(0)}_i y_j + f^{(0)}_t,
\]
(B.9)
\[
\Phi_3(Y, t) = \frac{4a_L}{\alpha^2} a_{L,i} a_{L,j} Y_j f^{(0)} + \frac{2}{\alpha} a_{L,i} r_{L,j} Y_i f^{(0)} - \frac{4a_L}{\alpha^2} a_{L,i} \beta_j Y_i f^{(0)}_j \\
+ \frac{2}{\alpha} a_{L,i} r_{L,i} (Y_j f^{(0)}) y_i + \frac{1}{a_L} r_{L,i} r_{L,j} f^{(0)}_i y_j - \frac{2}{\alpha} r_{L,i} \beta_j f^{(0)}_i y_j \\
- \frac{4a_L^2}{\alpha^2} a_{L,i} \beta_i (Y_j f^{(0)}) y_i - \frac{2}{\alpha} r_{L,i} \beta_j f^{(0)}_i y_j + \frac{4a_L}{\alpha^2} \beta_i j f^{(0)}_i y_j \\
+ \frac{2a_L c_L}{\alpha} f^{(0)} + \frac{2}{\alpha} a_{L,i} \Omega_i f^{(0)}.
\]
(B.10)

Consequently
\[
Q^{(2)}(X, Y, t) = \left( f^{(2)}(Y, t) + \left( \frac{1}{a_L} \Phi_2(Y, t) - \frac{\alpha}{2a_L^2} \Phi_3(Y, t) \right) X + \frac{1}{2a_L} \Phi_3(Y, t) X^2 \right) e^{\frac{2a_L X}{\alpha}},
\]
(B.11)

for some function \(f^{(2)}\). By applying the consistency condition at \(O(\varepsilon)\), after an involved algebraic reduction we arrive at (4.27). The observation that (4.27) is the Fokker-Planck equation for the stochastic differential equation, (4.29), leads to the desired result (4.33) as described in the main text.

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