Template iterations with non-definable ccc forcing notions

Diego Alejandro Mejía*

Graduate School of System Informatics, Kobe University, Kobe, Japan.
damejiag@kurt.scitec.kobe-u.ac.jp

Abstract

We present a version with non-definable forcing notions of Shelah’s theory of iterated forcing along a template. Our main result, as an application of this, is to prove that, if \( \kappa \) is a measurable cardinal and \( \theta < \kappa < \mu < \lambda \) are uncountable regular cardinals, then there is a ccc poset forcing \( s = \theta < b = \mu < a = \lambda \). Another application is to get models with large continuum where the groupwise-density number \( g \) takes an arbitrary regular value.

1 Introduction

The technique of template iterations was first introduced by Shelah in [23] to prove the consistency of \( d < a \) where \( d \) is the dominating number and \( a \) is the almost disjointness number. There are two approaches for the construction of the models. Shelah first observed that, given a measurable cardinal \( \kappa \) witnessed by a \( \kappa \)-complete ultrafilter \( D \) and a ccc poset \( P \), forcing with the ultrapower \( P^\kappa/D \) destroys the maximality of any almost disjoint family of the P-extension of size \( \geq \kappa \), while it preserves all scales of length different from \( \kappa \) (see Lemma 8.2). Therefore, taking \( P \) as the finite support iteration (fsi) of length \( \mu > \kappa \) of Hechler forcing, with \( \mu \) regular, then, by taking ultrapowers \( \lambda \) times for some \( \lambda > \mu \) regular with \( \lambda^\omega = \lambda \) (with special care in the limit steps), the obtained poset forces \( b = d = \mu < a = c = \lambda \), where \( b \) is the unbounding number and \( c = 2^\omega \) is the size of the continuum. Although these ultrapowers can be represented by iterations along a template, it is not necessary to look into the template structure of the ultrapowers generated in the proof, but it is enough to understand its forcing equivalence with a fsi to get the consistency statement. This approach can be used to get the consistency of \( u < a \) modulo a measurable, where \( u \) is the ultrafilter number, by starting with a fsi of Laver forcing with an ultrafilter, but, as this forcing notion is not definable enough, it is not known whether this construction can be represented by a template iteration (for this last part, see also [10]).

The second approach consists in defining a template iteration where the ultrapower argument to increase \( a \) is replaced by an isomorphism-of-names argument, so the consistency result can be obtained modulo ZFC alone. Concretely, if \( \aleph_1 < \mu < \lambda \) are regular cardinals and \( \lambda^\omega = \lambda \), the statement \( b = d = \mu < a = c = \lambda \) is consistent by this method. However, this approach only works for definable ccc (Suslin) forcing notions, so it is still not known whether such a construction can be done to get the consistency of \( u < a \) on the basis of ZFC, which is still an open problem. All details of this discussion and more applications of the template technique can be also found in [9], [14] and [15].

In this paper, we investigate to what extent it is possible to obtain, with the techniques discussed above, models where the values of \( b, a \) and the splitting number \( s \) can be separated. The simplest of these results is the consistency of \( s < b = c \) that was proved in [1] by a fsi of Hechler forcing, even more, using techniques from [8] (see also [20] Sect. 3) a model of \( s = \theta \leq b = c = \mu \) can be obtained, where \( \theta \leq \mu \) are regular uncountable cardinals (fix these cardinals for this paragraph). Shelah [22] proved, by countable support iteration techniques, the consistency of \( b = \aleph_1 < a = s = \aleph_2 \) and the consistency of \( b = a = \aleph_1 < s = \aleph_2 \). Extensions of these results are the consistency of \( b = \mu < a = \mu^+ \) obtained by Brendle [12] with fsi techniques and, in [16], using matrix iterations, Brendle and Fischer proved the consistency of \( b = a = \theta \leq s = \mu \) with ZFC and the consistency of \( \kappa < b = \mu < a = s = \lambda \) where \( \kappa \) is measurable in the ground model and \( \mu < \lambda \) are regular uncountable cardinals (here, the ultrapower technique explained above is also used). In Shelah’s model for the consistency of \( u < a \) mentioned above, it is also true that \( \kappa < b = s = u = \mu < a = \lambda \) where \( \kappa \) is measurable in the ground model and the other

*Supported by the Monbukagakusho (Ministry of Education, Culture, Sports, Science and Technology) Scholarship, Japan.
two cardinals are regular. The consistency of \( b = s = \aleph_1 < a = \aleph_2 \) with ZFC is still and open problem (see [17]).

Concerning models where \( s, b \) and \( a \) are different, these are the possibilities.

**Problem 1.1.** Let \( \theta < \kappa < \lambda \) be uncountable regular cardinals. Is it consistent that

1. \( b = \theta < a = \mu < s = \lambda \)?
2. \( b = \theta < s = \mu < a = \lambda \)?
3. \( s = \theta < b = \mu < a = \lambda \)?

As models for \( b < s \) and \( b < a \) are hard to get, many difficulties arise to answer each question of this problem. In the case of (1) and (2) it seems required to construct a poset by starting with a matrix iteration and turning it into a three-dimensional iteration, but in such a construction it is not known how to guarantee embeddability between the intermediate stages. A way to think about the answer of (3) is to use the known techniques for obtaining posets that force \( b < a \) with large continuum and guarantee that these preserves splitting families of the ground model. This is not viable for the techniques of [12], so we are left with the elaborated technique of iterations along a template to attack (3).

In the models constructed in both approaches explained at the beginning of this introduction, \( s \) is preserved to be equal to \( \aleph_1 \) (see Remark 4.14 for details), so the consistency of (3) with ZFC is true for \( \theta = \aleph_1 \).

We obtain a partial answer to (3) for larger \( \theta \), which is the main result of this text. By using a forcing construction as in the first approach above, given a measurable cardinal \( \kappa \), we prove the consistency of \( s = \theta < b = \mu < a = \lambda \) for regular cardinals \( \theta < \kappa < \mu < \lambda \). The idea of the proof is to start with \( V \) a model of ZFC that satisfies \( s = \mathfrak{c} = \theta \) and \( \theta^< \theta = \theta \) and, with the measurable \( \kappa \), perform a forcing construction with iterations and ultrapowers. It is needed that the resulting poset preserves \( s \leq \theta \) and, moreover, we need to use posets with small filter bases (of size \( \theta \)), like Mathias or Laver forcing with a filter base, along the iteration to ensure that \( s \geq \theta \) in the final extension. Although this construction can be done without using the template structure of the iterations, it seems that knowledge about the template is necessary to get an easy proof of the preservation of \( s \leq \theta \) in the final extension.

As these posets with small filter bases are non-definable, we need to expand Shelah’s theory of iterated forcing along a template by explaining how to include certain non-definable posets in the template framework. This is the main technical achievement of this paper and it is presented in such generality that it can be used for other purposes. In particular, we use this to obtain models where the groupwise-density number \( g \) can assume an arbitrary regular value, even in models obtained by well-known fsi techniques.

Concerning this it is known, from results in [4], how to force \( g = \aleph_1 \) by a fsi of Suslin ccc forcing that adds new reals at many intermediate stages. Our application is an extension of this argument to force \( g \) to be an arbitrary regular uncountable cardinal by some template iteration constructions.

Throughout this text, we refer as a real to any member of a fixed Polish space (e.g. the Baire space \( \omega^\omega \) or the Cantor space \( 2^\omega \)). Our notation is quite standard. Given a measurable cardinal \( \kappa \) witnessed by a \( \kappa \)-complete ultrafilter \( D \), we denote the ultrapower of an object \( X \) by \( X^\kappa / D \). Say also that a property \( \varphi(\alpha) \) holds for \( D \)-many \( \alpha \) iff \( \{ \alpha < \kappa / \varphi(\alpha) \} \in D \). \( A \) represents the amoeba poset, \( B \) the random poset, \( C \) the Cohen poset, \( D \) is Hechler forcing, \( E \) is the eventually different reals forcing and \( I \) denotes the trivial poset \( \{ 0 \} \). Those posets are Suslin ccc forcing notions. See [2, Chapter 3, Section 7.4B] for definitions and properties. Basic notation and knowledge about forcing can be found in [13] and [19].

First, we fix some notation. A family \( A \subseteq [\omega]^{\omega} \) is said to be **almost disjoint** (a.d.) if the intersection of any two different members of \( A \) is finite. A maximal family of this kind is called **maximal almost disjoint (mad)**. \( a \), the **almost disjointness number**, is defined as the least size of an infinite mad family. For \( A \) and \( B \) subsets of \( \omega \), \( A \subseteq^* B \) denotes that \( A \setminus B \) is finite. A family \( F \) is contained in \( [\omega]^{\omega} \) is a **filter base** if the intersection of any finite subfamily of \( F \) is infinite and \( \{ X \in [\omega]^{\omega} / \exists F \subseteq [\omega]^{\omega} \cap X \subseteq^* X \} \) is the filter that generates. \( X \in [\omega]^{\omega} \) is said to be a **pseudo-intersection** of \( F \) if \( X \subseteq^* A \) for any \( A \in F \). The cardinal invariant \( p \), the **pseudo-intersection number**, is defined as the least size of a filter base that does not have a pseudo-intersection, and the cardinal invariant \( u \), the **ultrafilter number**, is the least size of a filter base that generates a non-principal ultrafilter on \( \omega \). \( G \subseteq [\omega]^{\omega} \) is groupwise-dense if \( G \) is downward closed under \( \subseteq^* \) and, for any interval partition \( \langle I_n \rangle_{n<\omega} \) of \( \omega \), there exists an \( A \in [\omega]^{\omega} \) such that \( \bigcup_{n \in A} I_n \in G \). The **groupwise-density number** \( g \) is defined as the least size of a family of groupwise-dense sets whose intersection is empty. In this text, without loss of generality, we only consider filter bases that contain the cofinite subsets of \( \omega \) and that are closed under finite intersections (e.g., the value of \( p \) and \( u \) does not change with this additional condition). For a filter base \( F \), \( M_F \) denotes **Mathias forcing with** \( F \), which is
a \sigma\text{-centered forcing notion that adds a pseudo-intersection of } \mathcal{F}. For definitions, properties and proofs, see [2], [5] and [6].

For a Polish space with a Lebesgue measure, let \( \mathcal{M} \) be the \sigma\text{-ideal of meager sets and } \mathcal{N} \text{ is the } \sigma\text{-ideal of null sets (from the context, it is clear which Polish space corresponds to such an ideal). For } \mathcal{I} \text{ being } \mathcal{M} \text{ or } \mathcal{N}, \text{ the following cardinal invariants are defined, whose value does not depend on the space used to define it:}

- \text{add}(\mathcal{I}) \text{ the least size of a family } \mathcal{F} \subseteq \mathcal{I} \text{ whose union is not in } \mathcal{I},
- \text{cov}(\mathcal{I}) \text{ the least size of a family } \mathcal{F} \subseteq \mathcal{I} \text{ whose union covers all the reals},
- \text{non}(\mathcal{I}) \text{ the least size of a set of reals not in } \mathcal{I}, \text{ and}
- \text{cof}(\mathcal{I}) \text{ the least size of a cofinal subfamily of } \langle \mathcal{I}, \subseteq \rangle.

The cardinal invariants \( b, d, s \) and \( r \) (the reaping number) are defined in Section 4. Recall the typical inequalities between these cardinal invariants that are true in ZFC. Clearly, they are between \( \aleph_1 \) and \( c \). Cichon’s diagram (figure 1) illustrates some provable inequalities in ZFC, where vertical lines from bottom to top and horizontal lines from left to right represent \( \leq \). Also, the dotted lines mean \( \text{add}(\mathcal{M}) = \min\{b, \text{cov}(\mathcal{M})\} \) and \( \text{cof}(\mathcal{M}) = \max\{d, \text{non}(\mathcal{M})\} \). We also know that \( p \leq \text{add}(\mathcal{M}), p \leq s, p \leq g, s \leq d, g \leq d, b \leq a, b \leq r, s \leq \text{non}(\mathcal{I}), \text{cov}(\mathcal{I}) \leq r \) (where \( \mathcal{I} \) is \( \mathcal{M} \) or \( \mathcal{N} \)) and \( r \leq u \). See [2, Chapter 2], [3] and [5] for details.

This paper is structured as follows. We introduce, in Section 2, the basic definitions of, and results about, the templates that are used as supports for the iterations in this paper. In Section 3 we present a version of Shelah’s theory of iterated forcing along a template for non-definable forcing notions, plus some basic results about ccc-ness, complete embeddability and equivalence for posets that come from a template iteration. Most of the concepts and results of these two sections are due to Shelah and many proofs of the extended results are not that different from the original proofs, which can be found in [9] and [14].

In Section 4 we extend some preservation theorems of [8] and [20] to some cases of template iterations with ccc posets. Our applications are included in Sections 5 and 6, and the latter presents the proof of our main result, Theorem 6.1. Section 7 contains questions and discussions about the material of this text.

2 Templates

We introduce Shelah’s notion of a template (in a simpler way than in the original work [23]), which represents the index set of a forcing iteration as defined in Section 3. The definitions and the criteria of construction of templates discussed here are relevant for the proof of many of the results concerning template iterations in Section 6 and for the construction of the model of our main result in Section 6. Except for Lemmas 2.5 and 2.8, all definitions and results are, in essence, due to Shelah [23], but for proofs we refer to [9].

For a linear order \( L := \langle L, \leq_L \rangle \) and \( x \in L \), denote \( L_x := \{ z \in L \mid z < x \} \).

**Definition 2.1 (Indexed template).** An indexed template is a pair \( \langle L, \mathcal{I} := \{ I_x \mid x \in L \} \rangle \) such that \( L \) is a linear order, \( I_x \subseteq \mathcal{P}(L_x) \) for all \( x \in L \) and

![Figure 1: Cichon’s diagram](image)
(1) \( \emptyset \in \mathcal{I}_x \).

(2) \( \mathcal{I}_x \) is closed under finite unions and intersections.

(3) if \( z < x \) then there is some \( A \in \mathcal{I}_x \) such that \( z \in A \).

(4) \( \mathcal{I}_x \subseteq \mathcal{I}_y \) if \( x < y \), and

(5) \( \mathcal{I} := \bigcup_{x \in L} \mathcal{I}_x \cup \{L\} \cup \{X \cup \{m\} \mid X \in \mathcal{I}_m \text{ and } m = \text{max}(L)\} \) (the last set of this union is only considered when such a maximum exists) is well-founded with the subset relation. Let \( \mathcal{D}_p := \mathcal{D}_{pI} : \mathcal{I} \rightarrow \text{On} \) be the rank function for this relation.

\( L \) is meant to be the index set of an iteration as defined in Section 3. The well-foundedness of \( \mathcal{I} \) allows to define, by recursion, such an iteration for each support \( A \in \mathcal{I} \). Note that properties (2) and (4) imply that \( \mathcal{I} \) is closed under finite unions and intersections.

If \( A \subseteq L \) and \( x \in A \), define \( \mathcal{I}_x | A := \{A \cap X \mid X \in \mathcal{I}_x\} \) the trace of \( \mathcal{I}_x \) on \( A \). Then, \( \langle A, \mathcal{I}|A := \langle \mathcal{I}_x | A \rangle \mid x \in A \rangle \) is also an indexed template.

For \( x \in L \), define \( \mathcal{I}_x := \{B \subseteq L_x \mid B \in \mathcal{I}_x (B \cup \{x\})\} \). This family is important at the time of the construction of an iteration because the generic object added at stage \( x \) is generic over all the intermediate extensions that come from any support in \( \mathcal{I}_x \) (see comment after Theorem 3.3). Note that \( B \in \mathcal{I}_x \) if and only if \( B \subseteq H \) for some \( H \in \mathcal{I}_x \). Also, (1), (2) and (3) imply that any finite subset of \( \mathcal{I}_x \) is in \( \mathcal{I}_x \).

Example 2.2. (1) Given a linear order \( L, \mathcal{I}_x = [L_x]^{<\omega} \) for \( x \in L \) form an indexed template on \( L \). Note that \( \mathcal{I}_x = \mathcal{I}_x \).

(2) (Template for a fsi) Let \( \delta \) be an ordinal number. Then, \( \mathcal{I}_x := \{\xi \mid \xi \leq \alpha\} \) for \( \alpha < \delta \) form an indexed template on \( \delta \). This is the template structure that corresponds to a fsi of length \( \delta \). Note that \( \mathcal{I}_x = \mathcal{P}(\alpha) \).

Definition 2.3 (Innocuous extension). Let \( \langle L, \mathcal{I} \rangle \) be an indexed template and \( \theta \) an uncountable cardinal.

(1) An indexed template \( \langle L, \mathcal{J} \rangle \) is a \( \theta \)-innocuous extension of \( \langle L, \mathcal{I} \rangle \) if

(1) for every \( x \in L \), \( \mathcal{I}_x \subseteq \mathcal{J}_x \), and

(2) for any \( x \in L \), \( A \in \mathcal{J}_x \) and \( X \subseteq A \) of size \( < \theta \), there exists a \( C \in \mathcal{I}_x \) containing \( X \).

If in (2) we can even find \( C \subseteq A \), say that \( \langle L, \mathcal{J} \rangle \) is a strongly \( \theta \)-innocuous extension of \( \langle L, \mathcal{I} \rangle \).

(II) Let \( \langle L', \mathcal{J}' \rangle \) be an indexed template such that \( L' \) is a linear order extending \( L \). \( \langle L', \mathcal{J}' \rangle \) is a (strongly) \( \theta \)-innocuous extension of \( \langle L, \mathcal{J} \rangle \) if

(1) for every \( x \in L \), \( \mathcal{J}_x \mid L \subseteq \mathcal{J}_x \) and

(2) \( \langle L, \mathcal{J}' \rangle \mid L \) is a (strongly) \( \theta \)-innocuous extension of \( \langle L, \mathcal{J} \rangle \).

The main point of this definition is that, when two iterations are defined along templates where one is an innocuous extension of the other and where some “coherence” is ensured in the construction of both iterations, we can get complete embeddability or equivalence between the resulting posets. The results that express this are Corollaries 3.11 and Lemma 3.12.

Lemma 2.4 (III Lemma 1.3] Adding small sets to a template). Let \( \langle L, \mathcal{I} \rangle \) be an indexed template, \( L_0 \subseteq L \). For \( x \in L \), define \( \mathcal{J}_x := \{A \cup (B \cap L_0) \mid A, B \in \mathcal{I}_x\} \). Then, \( \langle L, \mathcal{J} \rangle \) is an indexed template which is a \( \theta \)-innocuous extension of \( \langle L, \mathcal{I} \rangle \) and a strongly \( \theta \)-innocuous extension of \( \langle L_0, \mathcal{I}_0 \rangle \) for any \( \theta \). Moreover, \( \mathcal{J}_x \mid L_0 = \mathcal{I}_x \mid L_0 \).

Fix a measurable cardinal \( \kappa \) with a non-principal \( \kappa \)-complete ultrafilter \( \mathcal{D} \) and let \( \langle L, \mathcal{I} \rangle \) be an indexed template. Put \( L^* := L^* / \mathcal{D} \), which is a linear order. For \( \vec{x} = (x_\alpha)_{\alpha < \kappa} / \mathcal{D} \in L^* \), let \( \mathcal{I}_x^* \) be the family of sets of the form \( A := \{A_\alpha \mid x_\alpha \in A_\alpha / \mathcal{D} \} \), where \( \{A_\alpha \mid x_\alpha \in A_\alpha / \mathcal{D} \} \) is a sequence of subsets of \( L \) such that \( A_\alpha \in \mathcal{I}_x \), for \( \mathcal{D} \)-many \( \alpha \). Identifying the members of \( L \) with constant functions in \( L^* \), \( L^* \) extends the linear order \( L \) and \( \mathcal{I}_x \subseteq \mathcal{I}_x^* \subseteq \mathcal{I}_x^* \mid L \) for all \( x \in L \). For \( \vec{x} \in L^* \), let \( \mathcal{I}_x^\kappa := \{A \cup (B \cap L) / A, B \in \mathcal{I}_x\} \). Notice that \( \mathcal{I}_x^\kappa = \mathcal{I}_x \mid L \subseteq \mathcal{I}_x^\kappa \) for all \( x \in L \). From Lemma 2.2 we get

Lemma 2.5 (IV Lemma 2.1) Ultrapowers of templates. (a) \( \langle L^*, \mathcal{I}^* \rangle \) is an indexed template.

(b) \( \langle L, \mathcal{I}^* \rangle \) is an indexed template which is a strongly \( \kappa \)-innocuous extension of \( \langle L, \mathcal{I} \rangle \).
Proof. Let $X = \{x^\xi / \xi < \nu\}$ for some $\nu < \theta$. For $\alpha < \kappa$ let $X_\alpha := \{x^\xi_\alpha / \xi < \nu\}$. Then, $\bar{X} = [X_\alpha]_{\alpha < \kappa}$, so any $Z \in \bar{I}^\theta|X$ comes from two objects of the form $Y = [Y_\alpha]_{\alpha < \kappa}$ where $Y_\alpha \in \bar{I}|X_\alpha$ for $\mathcal{D}$-many $\alpha$. But, as $\theta < \kappa$ and each $|Z|X_\alpha < \theta$, there exists $\nu' < \theta$ such that $|Z|X_\alpha = \nu'$ for $\mathcal{D}$-many $\alpha$. Therefore, $|\bar{I}^\theta|X| \leq (\nu')^2 < \theta$. \hfill \Box

Now we deal with the context of the construction of a “limit” of templates, which is relevant for the construction of the templates corresponding to the limit step in the proof of Theorem 6.1. Fix an uncountable cardinal $\theta$ and consider a chain of indexed templates $\{\bar{L}^\alpha, \bar{I}^\alpha\}_{\alpha < \delta}$ such that, for $\alpha < \beta < \delta$, $\langle \bar{L}^\alpha, \bar{I}^\alpha \rangle$ is a strongly $\theta$-innocuous extension of $\langle \bar{L}^\beta, \bar{I}^\beta \rangle$. Moreover, assume that there is an ordinal $\mu \subseteq L^0$ such that, for all $\alpha < \delta$,

(i) $\mu$ is cofinal in $L^\alpha$ and $0 = \min(L^\alpha)$,

(ii) $L^\alpha_\xi \subseteq I^\alpha_\xi$ for all $\xi \in \mu$.

Define $L^\delta := \bigcup_{\alpha < \delta} L^\alpha$ and, for $x \in L^\delta$, let $I^\delta_x := \bigcup_{\alpha \in \langle \alpha, \delta \rangle} I^\alpha_x$ where $\alpha_x$ is the least $\alpha$ such that $x \in L^\alpha$. Also, put $J^\delta_x := \{L^\delta_\xi \cup A \in \mu, \xi \leq x \text{ and } A \in I^\delta_x\}$.

Lemma 2.7 (M Lemma 1.8) Chains of templates. (a) $\langle L^\delta, \bar{I} \rangle$ is an indexed template which is a strongly $\theta$-innocuous extension of $\langle L^\alpha, \bar{I}^\alpha \rangle$ for all $\alpha < \delta$.

(b) $\langle L^\delta, \bar{J} \rangle$ is an indexed template which is a strongly $\theta$-innocuous extension of $\langle L^\alpha, \bar{I}^\alpha \rangle$ for all $\alpha < \delta$.

Note that properties (i) and (ii) also hold for the template $\langle L^\delta, \bar{J} \rangle$, but (ii) may not hold for $\langle L^\delta, \bar{I} \rangle$. Although, in many cases, both templates lead to the same template iteration construction when $\text{cf}(\delta) \geq \theta$, $\bar{J}$ is preferred over $\bar{I}$ because of property (ii).

As Lemma 2.6 the following result states that, in the resulting template, it is preserved the property of having small templates when restricting to a small set. This is used for the application of Theorem 5.13 which deals with the preservation properties of Section 4.

Lemma 2.8. Assume that $\nu \leq \theta$ is a regular cardinal and that, for each $\alpha < \delta$ and $X \in [L^\alpha]^{<\nu}$, $|T^\alpha|X| < \nu$. Then, $|I|X < \nu$ and $|J|X < \nu$ for any $X \in [L^\delta]^{<\nu}$.

Proof. If $\text{cf}(\delta) < \nu$, choose an increasing cofinal sequence $\{\alpha_\mu\}_{\mu < \text{cf}(\delta)}$ for $\delta$ and note that $\bar{I}|X = \bigcup_{\mu < \text{cf}(\delta)} \bar{I}^\alpha|X(\cap L^\alpha)$ for any $X \subseteq L^\delta$, so it has size $< \nu$ when $X$ does. In the case that $\text{cf}(\delta) \geq \nu$, if $X \in [L^\delta]^{<\nu}$, there exists an $\alpha < \delta$ such that $X \subseteq L^\alpha$. We claim that $\bar{I}|X = \bar{I}^\alpha|X$. If $Z \in \bar{I}|X$, then $Z = X \cap |T^\alpha|$ for some $H \in \bar{I}^\alpha$ with $\xi \in \mu$ and $\alpha < \beta < \delta$. As $|Z| < \nu$, by strong $\theta$-innocuity, we can find a $C \in I^\alpha$ such that $Z \subseteq C \subseteq H$, so $Z = C \cap X \in \bar{I}^\alpha|X$.

For the case of $\bar{J}$, note that $\{L^\delta_\xi \cap X / \xi \leq \mu\}$ has size $\leq |X|$. As, for any $X \subseteq L^\delta$, $\bar{J}|X = \{(L^\delta_\xi \cap X) / \xi \leq \mu\}$ and $Z \in \bar{I}|X$, then it has size $< \nu$ when $X$ does. \hfill \Box

(c) $\langle L^\delta, \bar{I} \rangle$ is an indexed template which is a $\theta$-innocuous extension of $\langle L^\delta, \bar{I} \rangle$ and a strongly $\theta$-innocuous extension of $\langle L^\delta, \bar{I} \rangle$ for any $\theta$.

(d) $\langle L^\delta, \bar{I} \rangle$ is a strongly $\kappa$-innocuous extension of $\langle L, \bar{I} \rangle$.
3 Iterations along templates

We present the theory of template iterations for non-definable posets. Although this approach is general, the proofs of the criteria of construction of the iterations and ccc-ness are not different from those in [13].

Definition 3.1 (Correct system of embeddings). Let $P_i$ be a poset for each $i \in I$ and let $e_{i,j} : P_i \to P_j$ be complete embeddings for $i < j$ in $I$ such that $e_{0,v} \circ e_{\wedge,0} = e_{1,v} \circ e_{\wedge,1}$. This system of embeddings is correct if, for each $p \in P_0$ and $q \in P_1$, if both have compatible reductions in $P_\wedge$, then $e_{0,v}(p)$ and $e_{1,v}(q)$ are compatible in $P_v$. An equivalent statement is that, for each $p \in P_0$ and for every reduction $r \in P_\wedge$ of $p$, $e_{\wedge,1}(r)$ is a reduction of $e_{0,v}(p)$.

When $e_{i,j}$ is the identity embedding for any $i < j$ in $I$ (which corresponds to all the cases in this text), we say that $(P_\wedge, P_0, P_1, P_j)$ is a correct system.

Recall that a partial order $\langle I, \leq \rangle$ is directed iff any two elements of $I$ have an upper bound in $I$. A sequence of posets $(P_i)_{i \in I}$ is a directed system of posets if, for any $i \leq j$ in $I$, $P_i$ is a complete suborder of $P_j$. In this case, the direct limit of $(P_i)_{i \in I}$ is defined as the partial order $\limdir_{i \in I} P_i := \bigcup_{i \in I} P_i$. It is clear that, for any $i \in I$, $P_i$ is a complete suborder of this direct limit.

Lemma 3.2 (Embeddability of direct limits [11], see also [13] Lemma 1.2). Let $I$ be a directed set, $(P_i)_{i \in I}$ and $(Q_i)_{i \in I}$ directed systems of posets such that

1. For each $i \in I$, $P_i$ is a complete suborder of $Q_i$, and

2. Whenever $i \leq j$, $(P_i, P_j, Q_i, Q_j)$ is a correct system.

Then, $\limdir_{i \in I} P_i$ is a complete suborder of $\limdir_{i \in I} Q_i$.

Theorem 3.3 (Iteration along a template). Given a template $(L, \bar{\bar{L}})$, a partial order $P|A$ can be defined by induction on $A \in L$ given the following conditions.

1. For $x \in L$ and $B \in L_x$, $\bar{\bar{L}}_x^B$ is a $P|B$-name of a poset given by reals. The following conditions should hold.
   (i) If $E \subseteq B$, then $\Vdash_{P|B} \bar{\bar{L}}_x^E \subseteq \Vdash_{P|E} \bar{\bar{L}}_x^B$.
   (ii) If $E \subseteq L_x$ and $q$ is a $P|B \cap E$-name for a real such that $\Vdash_{P|E} q \in \bar{\bar{L}}_x^E$ and $\Vdash_{P|B} q \in \bar{\bar{L}}_x^B$, then $\Vdash_{P|B \cap E} q \in \bar{\bar{L}}_x^{E \cap B}$.
   (iii) If $B', D \subseteq B$ and $(P \upharpoonright (B' \cap D), P \upharpoonright B', P \upharpoonright D, P \upharpoonright B)$ is a correct system, then the system $(P|B' \cap D) * \bar{\bar{L}}_x^{B' \cap D}, P|B' * \bar{\bar{L}}_x^{B'}, P|D * \bar{\bar{L}}_x^D, P|B * \bar{\bar{L}}_x^B)$ is correct.

2. The partial order $P|A$ is given by:
   (i) $P|A$ consists of all finite partial functions $p$ with domain contained in $A$ such that $p(0) = 0$, if $|p| > 0$ and $x = \max(\dom p)$, then there exists a $B \in L_x[A]$ such that $p[L_x \in P|B] \text{ and } p(x)$ is a $P|B$-name for a condition in $Q_x^B$.
   (ii) The ordering on $P|A$ is given by: $q \preceq_A p$ if $\dom q \subseteq \dom p$ and either $p = 0$ or, when $p \neq 0$ and $x = \max(\dom q)$, there is a $B \in L_x[A]$ such that $q[L_x \in P|B]$, and, either $x \notin \dom p$, $p \in P|B$ and $q[L_x \leq_B p]$, or $x \in \dom p$, $p[L_x \in P|B]$, $q[L_x \leq_B p][L_x]$ and $p(x), q(x)$ are $P|B$-names for conditions in $Q_x^B$ such that $q[L_x \upharpoonright P|B] q(x) \leq p(x)$.

Within this induction, the following properties are proved.

(a) For $D \subseteq A$, $P|D \subseteq P|A$ and, for $p, q \in P|D$, $q \leq_D p$ iff $q \leq_A p$.
(b) $P|A$ is a poset.

\footnote{In a more general way, we can think of a directed system with complete embeddings $e_{i,j} : P_i \to P_j$ for $i < j$ in $I$ such that, if $i < j < k$, then $e_{j,k} \circ e_{i,j} = e_{i,k}$. This allows to define a direct limit of the system as well.}
(c) \( P|A \) is obtained from \( P|B \) where \( B \subseteq A \) belongs to \( \mathcal{I} \) in the following way:

(i) If \( x = \max(A) \) exists and \( A' := A \cap L_x \in \check{\mathcal{I}}_x \), then \( P|A = P|A' \ast \check{Q}^A_x \).

(ii) If \( x = \max(A) \) but \( A' \notin \check{\mathcal{I}}_x \), then \( P|A \) is the direct limit of the \( P|B \) where \( B \subseteq A \) and \( B \cap L_x \in \check{\mathcal{I}}_x |A \).

(iii) If \( A \) does not have a maximum element, then \( P|A \) is the direct limit of the \( P|B \) where \( B \in \check{\mathcal{I}}_x |A \) for some \( x \in A \).

(d) If \( D \subseteq A \), then \( P|D \) is a complete suborder of \( P|A \).

(e) If \( D \subseteq L \) then \( P|D \) is \( \check{P}|A \cap P|D \) (this is proved by induction on \( \alpha \) for \( D \) and \( A \) such that \( D|\check{\mathcal{I}}_x(D)|D|P(A) \leq \alpha \)).

(f) If \( D \subseteq L \) then \( P|D \) is a complete suborder of \( P|A \) for some \( x \in A \).

Proof. By just changing certain notation, the proof follows the same lines as [14, Thm. 2.2].

Actually, considering the previous result proved by induction for all the templates of the form \( (X, \check{\mathcal{I}}|X) \) for any \( X \subseteq L \), \( P|X \) can be defined and Theorem 3.3 becomes valid for any subset of \( L \). In the same way, results proved throughout this text “by induction on \( \alpha \) in \( \mathcal{I} \)” become valid for any subset of \( L \).

Condition (1), particularly item (i), implies that, when we step into the generic extension of \( P|L \), the generic object added at stage \( x \) is generic over the intermediate extension by \( P|B \) for any \( B \in \check{\mathcal{I}}_x \).

In general, as \( L_x \) may not belong to \( \check{\mathcal{I}}_x \) (that is, in \( \check{\mathcal{I}}_x \)), this object added at stage \( x \) need not be generic over the intermediate extension by \( P|L_x \) or over the extension for any subset of \( L_x \) that is not in \( \check{\mathcal{I}}_x \).

The following examples present the types of template iterations that are used in our applications.

Definition 3.4 (Correctness-preserving Suslin ccc notion). A Suslin ccc notion \( S \) (with parameters in the ground model) is correctness-preserving if, for any correct system \((P, i) \in I_L \) with complete embeddings \( e_{i,j} \) for \( i < j \in L_4 \), \((P_i \ast \check{S}_i) \in L_4 \) is a correct system (with the obvious resulting embeddings), where \( \check{S}_i \) is a \( P_i \)-name for \( S \).

For example, the partial orders \( B, C, D, E \) are correctness-preserving forcing notions, see [14, 15] and also [13, Lemma 1.3]. So far, there is no known example of a Suslin ccc notion that is not correctness-preserving.

The following examples present the types of template iterations that are used in our applications.

Example 3.5 (Fsi in terms of a template iteration). Let \( \delta \) be an ordinal and consider the template \( \check{\mathcal{I}} \) defined in Example 2.2. An iteration along \((\delta, \check{\mathcal{I}})\) defined as in Theorem 3.3 is equivalent to the fsi \((P|\alpha, \check{Q}^\alpha_{\gamma(i)})_{\alpha < \delta}\) defined as in Example 3.3. Unlike a generic fsi, this iteration has the feature that it can be restricted to any subset of \( \delta \). To be more precise, if \( X \subseteq \delta \), then \( P|X \) is equivalent to the fsi \((P|X \cap \alpha, \check{Q}^X_{\gamma(i)})_{\alpha \in X} \) that is a complete suborder of \( P|\delta \). Recall that, for any \( \alpha < \delta \), \( \check{\mathcal{I}}_\alpha = \mathcal{P}(\alpha) \), so the generic object added at stage \( \alpha \) is generic over the intermediate extension by \( P|X \) for any \( X \subseteq \alpha \).

Of course, the proof of Theorem 3.3 is much simpler for this template, for it is enough to have the conditions in (1) and prove, by induction on \( \alpha \leq \delta \), that \( P|X \) is defined for any \( X \subseteq \alpha \) and that properties (a)-(f) hold.

Example 3.6. Let \( L = L_S \cup L_C \) be a disjoint union. For \( x \in L \) define the orders \( \check{Q}_x^B \) for \( B \in \check{\mathcal{I}}_x \) according to one of the following cases.

(i) If \( x \in L_S \), \( \check{Q}_x^B \) is a \( P|B \)-name for \( S_x^{x:1|B} \), where \( S_x \) is a fixed Suslin correctness-preserving ccc poset coded in the ground model.

(ii) If \( x \in L_C \), for a fixed \( C_x \in \check{\mathcal{I}}_x \) and a \( P|C_x \)-name \( \check{Q}_x \) for a poset given by reals,

\[
\check{Q}_x^B = \begin{cases} 
\check{Q}_x & \text{if } C_x \subseteq B \\
\check{1} & \text{otherwise.}
\end{cases}
\]

It is a straightforward calculation to see that the properties stated in (1) of Theorem 3.3 hold, so the template iteration can be defined as stated in that Theorem.

The following result is about complete embeddability between two template iterations. Although it is stated in a general way, Corollary 5.11 presents a particular case corresponding to what we need for our applications.
Theorem 3.7 (Complete embeddability of template iterations). Let $L$ be a linear order, $\overline{I}$ and $\overline{J}$ templates on $L$ such that $\overline{I}_x \subseteq \overline{J}_x$ for all $x \in L$. Consider two template iterations $\mathcal{P} \langle L, \overline{I} \rangle$ and $\mathcal{P} \langle L, \overline{J} \rangle$ such that the following conditions hold.

(1) For $x \in L$ and $B \in \overline{I}_x$, if $\mathcal{P} \upharpoonright B$ is a complete suborder of $\mathcal{P} \upharpoonright B$, then $\models \mathcal{P} \upharpoonright B \dot{\mathcal{Q}}^B_x \subseteq_{\forall \mathcal{P} \upharpoonright B} \dot{\mathcal{Q}}^B_x$.

(2) Whenever $B \in \overline{I}_x$, $A \subseteq B$ and $(\mathcal{P} \upharpoonright A, \dot{\mathcal{P}} \upharpoonright A, \mathcal{P} \upharpoonright B, \dot{\mathcal{P}} \upharpoonright B)$ is a correct system, then the system $(\mathcal{P} \upharpoonright A \ast \dot{\mathcal{Q}}^A_x, \dot{\mathcal{P}} \upharpoonright A \ast \dot{\mathcal{Q}}^A_x, \mathcal{P} \upharpoonright B \ast \dot{\mathcal{Q}}^B_x, \dot{\mathcal{P}} \upharpoonright B \ast \dot{\mathcal{Q}}^B_x)$ is correct.

(3) For $B \subseteq L$, $x \in B$, if $C \in \overline{J}_B \upharpoonright B$ and $p \in \mathcal{P} \upharpoonright C$, then there exists an $A \in \overline{I} \upharpoonright B$ such that $p \in \mathcal{P} \upharpoonright A$.

(4) For $B \subseteq L$, $x \in B$, if $C \in \overline{J}_B \upharpoonright B$ and $\mathcal{P} \upharpoonright C$-name for a condition in $\dot{\mathcal{Q}}^C_x$, then there exists an $A \in \overline{I} \upharpoonright B$ such that $A$ is a $\mathcal{P} \upharpoonright A$-name for a condition in $\dot{\mathcal{Q}}^A_x$.

Then, the following hold for each $B \in \overline{I}$.

(a) $\mathcal{P} \upharpoonright B$ is a complete suborder of $\mathcal{P} \upharpoonright B$.

(b) If $A \subseteq B$, then $(\mathcal{P} \upharpoonright A, \dot{\mathcal{P}} \upharpoonright A, \mathcal{P} \upharpoonright B, \dot{\mathcal{P}} \upharpoonright B)$ is a correct system.

Proof. Proceed by induction on $B \in \overline{I}$. The non-trivial case is when $B \neq \emptyset$. According to Theorem 3.3 consider the following cases.

(i) Case $x = \max(B)$ and $B_x = B \cap L_x \subseteq \overline{I}_x$. Then, $\mathcal{P} \upharpoonright B = \mathcal{P} \upharpoonright B_x \ast \dot{\mathcal{Q}}^B_x$ and $\dot{\mathcal{P}} \upharpoonright B = \dot{\mathcal{P}} \upharpoonright B_x \ast \dot{\mathcal{Q}}^B_x$. Then, by induction hypothesis and (1), $\mathcal{P} \upharpoonright B$ is a complete suborder of $\mathcal{P} \upharpoonright B$. This gives (a).

For (b), if $x \in A$, note that $\mathcal{P} \upharpoonright B_A \upharpoonright L_x \subseteq A_x$. By inductive hypothesis, $(\mathcal{P} \upharpoonright A_x, \dot{\mathcal{P}} \upharpoonright A_x, \mathcal{P} \upharpoonright B_x, \dot{\mathcal{P}} \upharpoonright B_x)$ is a correct system, so $(\mathcal{P} \upharpoonright A, \dot{\mathcal{P}} \upharpoonright A, \mathcal{P} \upharpoonright B, \dot{\mathcal{P}} \upharpoonright B)$ is a correct system by (2). The conclusion is simplier when $x \notin A$.

(ii) Case $x = \max(B)$ and $B_x \notin \overline{I}_x$. Then, with $B' : = \{B' \subseteq B \mid \exists x \in L \cap \overline{I}_x, B \subseteq \mathcal{P} \upharpoonright B' \}$, $\mathcal{P} \upharpoonright B = \lim \mathcal{P} \upharpoonright B' \upharpoonright B'$. By induction hypothesis and Lemma 3.2 it is enough to prove that $\mathcal{P} \upharpoonright B' = \lim \mathcal{P} \upharpoonright B' \upharpoonright B' \mathcal{P} \upharpoonright B'$ to see that $\mathcal{P} \upharpoonright B$ is a complete suborder of $\mathcal{P} \upharpoonright B$. If $p \in \mathcal{P} \upharpoonright B$, then, in the case that $x = \max(\text{dom}(p))$, there exists an $A' \in \overline{J}_B \upharpoonright B$ such that $p \mathcal{P} \upharpoonright A'$ and $p(x)$ is a $\mathcal{P} \upharpoonright A'$-name for a condition in $\dot{\mathcal{Q}}_{x'}^{A'}$. By (3) and (4), we can find $C \subseteq \mathcal{P} \upharpoonright B$ such that $p \mathcal{P} \upharpoonright C$ and $p(x)$ is a $\mathcal{P} \upharpoonright C$-name for a condition in $\dot{\mathcal{Q}}_{x'}^{C_x}$, so $p \mathcal{P} \upharpoonright (C \cup \{x\})$ with $C \cup \{x\} \subseteq B$. The case $\max(\text{dom}(p)) < x$ is treated in a similar way.

For (b), let $A \subseteq B$ and $p \in \mathcal{P} \upharpoonright A$ which is a reduction of $q \in \mathcal{P} \upharpoonright B$ and prove that $p$ is a reduction of $q$ with respect to the posets $\mathcal{P} \upharpoonright A$ and $\mathcal{P} \upharpoonright B$. Find $B' \subseteq B$ such that $p, q \in \mathcal{P} \upharpoonright B'$. Put $A' = A \cap B'$, so $p \in \mathcal{P} \upharpoonright A'$. It is easy to notice that $p$ is a reduction of $q$ with respect to the posets $\mathcal{P} \upharpoonright A'$ and $\mathcal{P} \upharpoonright B'$ so, by induction hypothesis, $p$ is a reduction of $q$ with respect to the posets $\mathcal{P} \upharpoonright A'$ and $\mathcal{P} \upharpoonright B'$. As $(\mathcal{P} \upharpoonright A', \mathcal{P} \upharpoonright A', \mathcal{P} \upharpoonright B', \mathcal{P} \upharpoonright B')$ is a correct system, our claim is proved.

(iii) Case $B$ does not have a maximum element. Then, $\mathcal{P} \upharpoonright B = \lim \mathcal{P} \upharpoonright B' \upharpoonright B'$ where $B' : = \{B' \subseteq B \mid \exists x \in B, B' \subseteq \mathcal{P} \upharpoonright B \}$. Like in the previous case, (3) and (4) imply that $\mathcal{P} \upharpoonright B = \lim \mathcal{P} \upharpoonright B' \upharpoonright B'$. Then, by Lemma 3.2 $\mathcal{P} \upharpoonright B$ is a complete suborder of $\mathcal{P} \upharpoonright B$. The argument for (b) is very similar to the one of the previous case.

Four our applications, we are interested in template iterations that produce ccc posets. The following result presents some conditions for this. Recall that a poset $\mathcal{P}$ has the Knaster condition if, for any sequence $\{p_n\}_{n<\omega}$ of conditions in $\mathcal{P}$, there exists an $E \subseteq \omega_1$ uncountable such that all the members of $\{p_n \mid n \in E\}$ are pairwise compatible.

Lemma 3.8 (Ccc-ness of template iterations). Consider an indexed template $(L, \overline{I})$ and $\mathcal{P} \upharpoonright L$ a corresponding template iteration such that the following conditions hold.

(i) For any $x \in L$ and $B \in \overline{I}_x$ there are $\mathcal{P} \upharpoonright B$-names $\langle \dot{\mathcal{Q}}^B_{x,n} \rangle_{n<\omega}$ witnessing that $\dot{\mathcal{Q}}^B_x$ is $\sigma$-linked and

(ii) if $D \subseteq B$ then $\mathcal{P} \upharpoonright B \dot{\mathcal{Q}}^D_{x,n} \subseteq \dot{\mathcal{Q}}^B_{x,n}$ for all $n < \omega$.

Then, for any $A \in \overline{I}$, $\mathcal{P} \upharpoonright A$ has the Knaster condition.

Proof. Same proof as [14] Lemma 2.3].
For this Lemma, if the template \((L, \mathcal{I})\) is as in Example 6.2, to obtain that \(P \upharpoonright L\) has the ccc conditions (i) and (ii) can be replaced by \(\mathcal{P}^{P,B}_{\mathcal{J}} \mathcal{Q}^{L}_{\mathcal{J}}\) has the ccc” for any \(x \in L\) and \(B \in \mathcal{I}_x\). The reason of this, as explained in Example 6.8 is that \(P \upharpoonright X\) is a fsi for any \(X \subseteq L\).

Recall from [14] that a forcing notion \(S\) is Suslin \(\sigma\)-linked if it is Suslin ccc and \(S = \bigcup_{n<\omega} S_n\) where all \(S_n\) are linked and “\(x \in \mathcal{S}_n\)” is a \(\Sigma^1_{1}\) -statement. This implies that “\(\mathcal{S}_n\) is linked” is \(\Pi^1_1\) and, thus, absolute. In a similar way, Suslin \(\sigma\)-centered forcing notion is defined. In particular, \(B\) is a Suslin \(\sigma\)-linked poset and \(C, D\) and \(E\) are Suslin \(\sigma\)-centered posets.

**Corollary 3.9 ([14] Lemmas 2.3, 2.4).** Any template iteration defined as in Example 3.6 where \(L_C = \emptyset\) and where only Suslin \(\sigma\)-linked correctness-preserving posets are involved satisfies the Knaster condition. Moreover, any condition and any name of a real for this template iteration poset has a support of countable size, that is, if \(p \in P \upharpoonright L\) and \(\dot{x}\) is a \(P \upharpoonright L\)-name for a real, then there exists \(C \in [L]^{<\omega}\) such that \(p \in P \upharpoonright C\) and \(\dot{x}\) is a \(P \upharpoonright C\)-name.

The last assertion of the preceding Corollary follows from the next result. In contrast with this, when we consider iterations as in Example 3.6 with \(L_C \neq \emptyset\), it is not possible to guarantee that the supports of a condition or a name for a real have countable size.

**Lemma 3.10** (Small support for ccc template iterations). Fix \(\theta\) a cardinal with uncountable cofinality. Consider a template iteration defined as in Example 5.6 where

- for \(x \in L_S\), \(S_x\) is a Suslin \(\sigma\)-linked correctness-preserving forcing notion and
- for \(x \in L_C\), \(Q_x\) is a \(P \upharpoonright C\)-name for a \(\sigma\)-linked poset of reals such that each linked component contains the trivial condition, and \(|C_x| < \theta\).

Then, for each \(A \in \mathcal{I}\), \(P \upharpoonright A\) has the Knaster condition and each condition and name of a real for this poset has a support of size \(< \theta\).

**Proof.** The Knaster condition follows from Lemma 6.8. The proof of the statement about the support follows the same lines of the proof of [14] Lemma 2.4 except for a further detail. Proceed by induction on \(A \in \mathcal{I}\) and let \(p \in P \upharpoonright A\) be such that \(x = \max(\text{dom}(p)) \in L_C\). Then, there exists a \(B \in \mathcal{I}_x \cup \mathcal{A}\) such that \(p \upharpoonright L_x \in P \upharpoonright B\) and \(p(x)\) is a \(P \upharpoonright B\)-name for a condition in \(\mathcal{Q}^{L}_{\mathcal{J}}\). By induction hypothesis, there exists \(D_0 \subseteq B\) of size \(< \theta\) such that \(p \in P \upharpoonright D_0\). If \(C_x \subseteq B\) then \(p(x)\) will be the trivial condition, so that \(p \in P \upharpoonright (D_0 \cup \{x\})\). Else, if \(C_x \subseteq B\), by induction hypothesis find \(D_1 \subseteq B\) of size \(< \theta\) such that \(p(x)\) is a \(P \upharpoonright D_1\)-name for a real. Without loss of generality, we may assume \(C_x \subseteq D_1\), so \(p(x)\) is a \(P \upharpoonright D_1\)-name for a condition in \(\mathcal{Q}^{L}_{\mathcal{J}}\). Then, \(p \in P \upharpoonright (D_0 \cup D_1 \cup \{x\})\). The argument when \(x \in L_S\) is similar.

Now, if \(\dot{x}\) is a \(P \upharpoonright A\)-name for a real, note that it can be determined by countably many conditions \(\langle r_n \rangle_{n<\omega}\) in \(P \upharpoonright A\). As each \(r_n\) has a support of size \(< \theta\) and \(\theta\) has uncountable cofinality, we can find \(X \subseteq A\) of size \(< \theta\) such that \(r_n \in P \upharpoonright X\) for all \(n < \omega\). This implies that \(\dot{x}\) is a \(P \upharpoonright X\)-name.

The following is a consequence of Theorem 3.7 that fits for the purposes of our applications. Although this type of results was considered originally to get only forcing equivalence, we need to extend to cases where we can get complete embeddability, fact that is needed in order to deal with the limit steps of small cofinality in the proof of Theorem 6.1.

**Corollary 3.11** (Complete embeddability of template iterations, particular case). Let \(\theta\) be a cardinal with uncountable cofinality, \(L\) a linear order, \(\mathcal{I}\) and \(\mathcal{J}\) templates on \(L\) such that \((L, \mathcal{J})\) is a \(\theta\)-innocuous extension of \((L, \mathcal{I})\). Consider two template iterations \(P \upharpoonright (L, \mathcal{I})\) and \(P \upharpoonright (L, \mathcal{J})\) defined with the conditions of Lemma 3.10 such that

\((0')\) The same \(L_S\) and \(L_C\) are considered for both iterations.
\((1')\) For \(x \in L_S\), the same Suslin forcing \(S_x\) is considered for both template iterations.
\((2')\) For \(x \in L_C\) either \(C_x = C_x\) and \(\dot{Q}_x = \dot{Q}_x\), or \(C_x = \emptyset\) and \(\dot{Q}_x\) is the trivial forcing.

Then, the following hold for each \(B \in \mathcal{I}\).

\((a)\) \(P \upharpoonright B\) is a complete suborder of \(P \upharpoonright B\).

\((b)\) If \(A \subseteq B\), then \(\langle P \upharpoonright A, \mathcal{P} \upharpoonright A, P \upharpoonright B, \mathcal{P} \upharpoonright B \rangle\) is a correct system.

**Proof.** It is enough to prove conditions (1)-(4) of Theorem 3.7.
(1) Straightforward from (0'), (1') and (2').

(2) For \( x \in L_G \), the result follows because \( S_x \) is a correctness-preserving Suslin ccc notion. For \( x \in L_C \), it is straightforward from (2').

(3) Let \( B \subseteq L, x \in B, C \in \mathcal{J}_x \mid B \) and \( p \in \dot{P} \mid C \). By Lemma 3.10, there exists \( K \subseteq C \) such that \( p \in \dot{P} \mid K \) and \( |K| < \theta \). Then, by \( \theta \)-innocuity, there exists \( H \in \mathcal{I}_x \) such that \( K \subseteq H \), so \( K \subseteq A \) and \( p \in \dot{P} \mid A \), where \( A := B \cap H \in \mathcal{I}_x \mid B \).

(4) Let \( B \subseteq L, x \in B, C \in \mathcal{J}_x \mid B \) and \( \dot{q} \) a \( \dot{P} \mid C \)-name for a condition in \( \dot{Q} \). A similar argument as before works with Lemma 3.10. It is clear for \( x \in L_S \), so assume \( x \in L_C \). If \( \dot{C}_x \subseteq C \), find \( K \subseteq C \) such that \( \dot{q} \) is a \( \dot{P} \mid K \)-name for a real, \( |K| < \theta \) and \( \dot{C}_x \subseteq K \). Then, \( \dot{q} \) is a \( \dot{P} \mid K \)-name for a condition in \( \dot{Q}_x \) so, by \( \theta \)-innocuity, find an \( A \in \mathcal{I}_x \mid B \) containing \( K \), so that \( \dot{q} \) is a \( \dot{P} \mid A \)-name for a condition in \( \dot{Q}_x \). The case \( \dot{C}_x \not\subseteq C \) is simpler because \( \dot{q} \) is a \( \dot{P} \mid C \)-name for the trivial condition.

We conclude this section with a version of a known result of forcing equivalence for the template iterations of Lemma 3.10.

**Lemma 3.12 (Forcing equivalence between template iterations, analog to [9, Lemma 1.7]).** Assume that \( \langle L, \mathcal{J} \rangle \) is a \( \theta \)-innocuous extension of \( \langle L, \mathcal{I} \rangle \). Consider \( \dot{P} \mid \langle L, \mathcal{I} \rangle \) and \( \dot{P} \mid \langle L, \mathcal{J} \rangle \) template iterations satisfying the hypothesis in Corollary 3.11, but in (2') always assume that \( \dot{C}_x = C_x \) and \( \dot{Q}_x = Q_x \). Then, there exists a dense embedding \( F: \dot{P} \mid L \to \dot{P} \mid L \).

**Proof.** Proceed like in the proof of [9, Lemma 1.7]. By recursion on \( B \in \mathcal{J} \), construct \( F_B: \dot{P} \mid B \to \dot{P} \mid B \) such that

1. \( F_B \) is a dense embedding and
2. \( F_B \subseteq F_{B'} \) whenever \( B, B' \in \mathcal{J} \) and \( B \subseteq B' \).

Let \( p \in \dot{P} \mid B \). If \( p = \emptyset \), put \( F_B(\emptyset) = \emptyset \), so assume that \( p \neq \emptyset \). Let \( x := \max(\text{dom} \ p) \) and find \( B \in \mathcal{J}_x \mid B \) such that \( p|L_x \in \dot{P} \mid B \) and \( p(x) \) is a \( \dot{P} \mid B \)-name for a condition in \( \dot{Q}_x \). Consider the following cases.

(i) \( x \in L_S \). By hypothesis, there exists an \( A \subseteq B \) of size \( < \theta \) such that \( p|L_x \in \dot{P} \mid A \) and \( p(x) \) is a \( \dot{P} \mid A \)-name for a condition in \( \dot{S} \). By innocuity, there exists a \( C \in \mathcal{I}_x \mid B \subseteq \mathcal{J}_x \mid B \) containing \( A \), so \( p|L_x \in \dot{P} \mid C \) and \( p(x) \) is a \( \dot{P} \mid C \)-name for a condition in \( \dot{S} \). As \( C \in \mathcal{J} \) and has rank less than \( B \), the embedding \( F_C \) has already been defined. So let \( F_B(p) := F_C(p|L_x) \cup \{ \langle x, p_0(x) \rangle \} \) where \( p_0(x) \) is the \( \dot{P} \mid C \)-name associated to \( p(x) \) with respect to the embedding \( F_C \). Notice that, because of (2), \( F_B(p) \) does not depend on the choice of \( C \).

(ii) \( x \in L_C \) and \( C_x \subseteq B \), so \( \dot{Q}_x = Q_x \). Proceed like before, but take \( A \) such that \( C_x \subseteq A \).

(iii) \( x \in L_C \) but \( C_x \not\subseteq B \), so \( \dot{Q}_x = Q_x \), that is, \( p(x) \) is forced to be the trivial condition. Proceed as in (i).

\( \square \)

4 Preservation theorems for iterations along templates

The main goal of this section is to prove preservation results for template iterations associated to some cardinal invariants. The preservation properties involved use the same notation as in [20, Sect. 2] and [21, Sect. 2].

We are going to use the notion of quotients between posets in the proofs of many results of this section. For \( P, Q \) posets and a complete embedding \( i: P \to Q \), define the quotient poset (with respect to \( i \)) \( Q/P := \{ q \in Q / \exists p \in C (p \text{ is a reduction of } q \text{ (with respect to } i)) \} \), which is a \( P \)-name of a poset which inherits the same order as \( Q \), where \( \dot{G} \) is the \( P \)-name for the \( P \)-generic set. Note that \( p \in P \) is a reduction of \( q \in Q \) iff \( p \Vdash_P q \in Q/P \). It is well known that \( Q \) is forcing equivalent to \( P * (Q/P) \). In the proofs of our results, \( i \) will be the identity embedding.

---

2Here, \( F_B(p) := F_B(p|L_x) \) would be ok, but proceeding as in (i) guarantees that \( \text{dom} F_B(p) = \text{dom} p \).
Lemma 4.1. Let \((P, P', Q, Q')\) be a correct system. Then, \(P'\) forces that \(Q/P \subseteq _{\forall \forall} Q'/P'\).

Proof. Correctness implies directly that \(\Vdash_{P'} Q/P \subseteq Q'/P'\). We prove first that \(P'\) forces that any pair of incompatible conditions in \(Q/P\) are incompatible in \(Q'/P'\). Let \(p' \neq q' \in Q'/P'\) be such that \(p' \Vdash_{P'} q'\). Then, \(p' \neq q'\) in \(P'\). As \(p' \Vdash_{P'} q'\), \(p'\) is a reduction of \(q'\). Let \(p \in Q\) such that \(q \leq p, q_p\) is a reduction of \(q\), \(p\) is a reduction of \(p'\) and \(q\) is a reduction of \(q'\). Indeed, \(p_0 \in P\) a reduction of \(p'\). Then, \(p_0 \in P\) is also a reduction of \(q'\), there exists a \(q' \in Q\) such that \(q' \leq q, p_0\). Now, \(p \leq p_0 \in P\) such that \(q\) is a reduction of \(q'\). Clearly, \(p\) and \(q\) are as desired. Now, \(p \Vdash_{P'} q \in Q/P\) and, as it is a reduction of \(p'\), \(p'\) is also a reduction of \(q'\). Then, \(p'\) is a reduction of \(q'\) such that \(q \leq q_0, q_1\). Now, \(p \leq p_0 \in P\) such that \(q\) is a reduction of \(q'\). Clearly, \(p\) is a reduction of \(q'\), so \(q'\) is a reduction of \(q'\), and \(q'\) is a reduction of \(q'\). Hence, \(q\) is a reduction of \(q'\), and \(\vdash_{P'\forall \forall} q' \subseteq Q'/P'\).

Context 4.2. Fix an increasing sequence \((\subseteq_n)\) of 2-place relations in \(\omega^\omega\) such that

- Each \(\subseteq_n (n < \omega)\) is a closed relation (in the arithmetical sense) and
- for all \(n < \omega\) and \(g \in \omega^\omega\), \((\subseteq_n)^g = \{f \in \omega^\omega \mid f \subseteq_n g\}\) is (closed) n.w.d.

Put \(\subseteq = \bigcup_{n<\omega} \subseteq_n\). Therefore, for every \(g \in \omega^\omega\), \((\subseteq)^g\) is an \(F_\sigma\) meager set.

\(F \subseteq \omega^\omega\) is a \(\omega\)-unbounded family if, for every \(g \in \omega^\omega\), there exists an \(f \in F\) such that \(f \not\subseteq g\). Define the cardinal \(b_\omega\) as the least size of a \(\omega\)-unbounded family. Besides, \(D \subseteq \omega^\omega\) is a \(\omega\)-dominating family if, for every \(x \in \omega^\omega\), there exists an \(f \in D\) such that \(x \subseteq f\). Likewise, define the cardinal \(b_\omega\) as the least size of a \(\omega\)-dominating family.

Given a set \(Y\), say that a real \(f \in \omega^\omega\) is \(\omega\)-unbounded over \(Y\) if \(f \not\subseteq Y\) for every \(g \in Y \cap \omega^\omega\).

Although we define Context 4.2 for \(\omega^\omega\), we can use, in general, the same notion by changing the space for the domain or the range of \(\subseteq\) to another uncountable Polish space, like \(2^\omega\) or other spaces whose members can be coded by reals in \(\omega^\omega\).

Definition 4.3. For a forcing notion \(P\), the property \((\forall P, \subseteq)\) holds if, for every \(P\)-name \(\dot{h}\) of a real in \(\omega^\omega\), there exists a set \(Y \subseteq \omega^\omega\) such that \(|Y| < \theta\) and, for every \(f \in \omega^\omega\), if \(f\) is \(\subseteq\)-unbounded over \(Y\), then \(\Vdash f \not\subseteq h\). When \(\theta = \aleph_1\), we just write \((\forall P, \subseteq)\).

\((\forall P, \subseteq)\) is a standard property associated to the preservation of \(b_\omega\) \(\leq \theta\) and the preservation of \(b_\omega\)-large through forcing extensions of \(P\). To explain this, first say that \(F \subseteq \omega^\omega\) is \(\omega\)-unbounded if, for any \(X \subseteq \omega^\omega\) of size \(\theta\), there exists an \(f \in F\) which is \(\omega\)-unbounded over \(X\). In practice, \(F\) has size \(\theta\)

Example 4.5 (Preserving unbounded families). For \(f, g \in \omega^\omega\), define \(f <^*_g g \iff \forall k \geq n (f(k) < g(k))\), so \(f <^*_g g \iff \forall k \geq n f(k) < g(k)\). The unbounding number is defined as \(b := b_{\omega^\omega}\) and \(d := d_{\omega^\omega}\) is the dominating number.

Example 4.6 (Preserving splitting families). For \(A, B \subseteq \omega^\omega\), define \(A \preceq_B B \iff (B \setminus A \subseteq A \setminus B \subseteq \omega \setminus A)\), so \(A \preceq_B B \iff B \subseteq_A A \setminus B \subseteq \omega \setminus A\). Note also that \(A \preceq B\) iff \(A\) splits \(B\), that is, \(A \cap B \) and \(B \setminus A\) are infinite. It is clear from the standard definitions that the splitting number is \(s = b_\omega\) and the reaping number is \(r = d_\omega\).
Lemma 4.7 (Baumgartner and Dordal [1], see also [7] Main Lemma 3.8). \(+_{D,\alpha}\) holds.

Example 4.8 (Preserving null-covering families). Fix, from now on, \(\langle I_n\rangle_{n<\omega}\) an interval partition of \(\omega\) such that \(\forall n<\omega(|I_n| = 2^{n+1})\). For \(f, g \in 2^\omega\) define \(f \triangleleft g \iff \forall k \geq n(I_k \neq g|I_k)\), so \(f \triangleleft g \iff \forall n<\omega(f|I_k \neq g|I_k)\). Clearly, \((\triangleleft)\) is a co-null \(F_\alpha\)-meager set.

Lemma 4.9 ([8] Lemma 1*]). Given \(\nu < \theta\) an infinite cardinal, every \(\nu\)-centered forcing notion satisfies \((+^\nu_\theta)\).

The following result shows why \(\triangleleft\) is useful to deal with preserving \(\text{cov}(\mathcal{N})\) small and non\(\mathcal{N}\) large.

Lemma 4.10 ([20] Lemma 7]). \(\text{cov}(\mathcal{N}) \leq b_\theta \leq \text{non}(\mathcal{M})\) and \(\text{cov}(\mathcal{N}) \leq d_\theta \leq \text{non}(\mathcal{N})\).

Example 4.11 (Preserving new reals). For \(f, g \in 2^\omega\) define \(f =_n^* g\) as \(\forall k \geq n(f(n) = g(n))\), so \(f =_n^* g \iff \forall n<\omega(f(k) = g(k))\). Note that, if \(M\) is a model of ZFC and \(c\) is a real, then \(c =_n^*\)-unbounded over \(M\) iff \(c \notin M\). It is also easy to see that \(b_\omega = 2\) and \(d_\omega = c\). For this relation, we are not interested in the cardinal invariants but in the “preservation” of new reals that are added at certain stage of an iteration and that cannot be added at other different stages. Concretely, we use this relation to prove Theorem 4.16.

Lemma 4.12. If \(P\) is a \(\theta\)-cc poset, then \((+^\theta_{\mathcal{P},\omega})\). In particular, \(\mathcal{P}\) posets satisfy \((+_{\mathcal{P},\omega})\).

Proof. Let \(h\) be a \(P\)-name for a real. Find a maximal antichain \(A \subseteq P\) such that, for \(p \in A\), either \(p \vDash \exists^* \dot{h} \notin V\) or there is a real \(f_p\) such that \(p \vDash \dot{h} = f_p\). Clearly, \(Y := \{f_p / p \in A\}\) (we include only those that exist) has size \(< \theta\) and it is a witness of \((+^\theta_{\mathcal{P},\omega})\) for \(\dot{h}\).

Theorem 4.13 (First preservation theorem for template iterations). Consider \(\langle L, \mathcal{T}\rangle\) an indexed template and \(P|L\) a corresponding template iteration such that it is \(\mathcal{C}\) and \(\nu \leq \theta\) is an uncountable cardinal such that

(i) for all \(B \in [L]<\nu\), \(\mathcal{T}|B\) has size \(< \nu\),

(ii) for all \(A \in \mathcal{T}\), every condition and name for a real in \(P|A\) has a support of size \(< \nu\) and

(iii) for all \(x \in L\) and \(B \in \mathcal{T}_x\), \(P_B \vDash (+^\theta_{Q^\mathcal{T}_x})\) holds.

Then, \((+^\theta_{P|L,\omega})\) holds. Moreover, if \(L'\) is an initial segment of \(L\) such that \(\forall x \in L\setminus L'(L' \in \mathcal{T}_x)\), then \(P|L'\) forces \((+^\theta_{P|L,P|L',\omega})\).

Proof. Let \(V\) be the ground model and let \(V'\) be a \(P|L'\)-generic extension of \(V\). In \(V'\), prove, by induction on \(A \in \mathcal{T}\), that if \(L' \subseteq A\) then \((+^\theta_{P|A/P|L',\omega})\) holds (the first claim is the particular case \(L' = \emptyset\)). Assume that \(L' \subseteq A\).

Case 1 \(x = \max(A)\) exists and \(A_0 := A \cap L_x \in \mathcal{T}x\). Clearly, \(P|A/P|L'\) is forcing equivalent to \((P|(A_0)/P|L') \ast Q^\mathcal{T}_x\) so, by (iii) and the induction hypothesis, \((+^\theta_{P|A/P|L',\omega})\) holds.

Case 2 \(x = \max(A)\) exists, but \(A_0 \notin \mathcal{T}_x\). First, note that \(P|A/P|L' = \text{limdir} \{P|B|P|L'/B \in A\}\), where \(A := \{B \subseteq A / L' \subseteq B\}\) and \(B \cap L_x \in I(A)\). Let \(h \in V\) be a \(P|A\)-name for a real. If there is some \(B \in A\) such that \(h\) is a \(P|B\)-name, then, in \(V'\), by induction hypothesis, there exists a witness of \((+^\theta_{P|B/P|L',\omega})\) for \(\dot{h}\) (which can be seen as a \(P|B/P|L'-\text{name}\)) and we are done.

Assume that \(h\) is not a \(P|B\)-name for any \(B \in A\). By (ii), there exists a \(C \in [A \setminus L']^<\nu\) such that \(h\) is a \(P|L'\cup C\)-name and \(x \in C\). In \(V'\), we can clearly think of \(h\) as a \(P|L'\cup C)/P|L'-\text{name}\). As \(L' \in \mathcal{T}_x\), note that

\[C := \{D \subseteq L' \cup C / L' \subseteq D\} \cap \{D \cap L_x \in \mathcal{T}_x[L' \cup C]\} = \{B \cap (L' \cup C) / B \in A\} = \{L' \cup E / E \subseteq C\}\] 

As \(\mu := |C| < \nu\) by (i), this equation implies that \(|C| \leq \mu\), so enumerate \(C := \{B_\alpha / \alpha < \mu\}\) where each \(D_\alpha = B_\alpha \cap (L' \cup C)\) and \(B_\alpha \in A\). Note also that \((L' \cup C)\cap L_x \notin \mathcal{C}\) (if so, there exists a \(B \in A\) such that \(L' \cup C \subseteq B\), so \(h\) would be a \(P|B\)-name, which is false), so \(P|L' \cup C|P|L'| = \text{limdir} \{P|D_\alpha / \alpha < \mu\}\) and, in consequence, \(P|L' \cup C|P|L'| = \text{limdir} \{P|D_\alpha/P|L' / \alpha < \mu\}\).

For each \(\alpha < \mu\), choose a \(P|D_\alpha/P|L'-\text{name}\) \(\dot{h}_\alpha\) for a real and \(\langle \dot{p}_{\alpha,h} \rangle_{k<\omega}\) \(P|D_\alpha/P|L'-\text{names}\) for
a decreasing sequence of conditions in \((P|(L' \cup C)/P|L')/(P|D_0/P|L')\) such that it is forced by \(P|D_0/P|L'\) that \(p_{\alpha,k}\) forces \(h[k] = h_\alpha[k]\). Choose \(Y_\alpha \in V'\) to be a witness of \((p^\theta|_{P|L'|}/\mathcal{C})\) for \(h_\alpha\).

Put \(Y := \bigcup_{\alpha \in \omega} Y_\alpha\) and prove that this is a witness of \((p^\theta|_{P|L'|}/\mathcal{C})\) for \(h\). Let \(f \in \mathcal{C}\) be unbounded real over \(Y\) and fix \(m < \omega\) and \(p \in P|(L' \cup C)/P|L'\). There exists an \(\alpha < \mu\) such that \(p \in P|D_0/P|L'\).

Case 3 There is no \(\max(A)\), so \(P|A = \limdir_{B \in P} B\) where \(B := \{B \in \mathcal{I}_x[A / x \in A \text{ and } L' \subseteq B]\} \rightarrow \mathcal{I}_{<\omega}\).

Let \(\dot{h}\) a \(P|A\)-name for a real, so there exists \(C \subseteq A\) of size \(\nu\) such that \(h\) is a \(P|(L' \cup C)-\)name and, without loss of generality, assume that \(C\) doesn’t have a maximum. Proceed like in case 2 to find a witness of \((+p^\theta|_{P|L'|}/\mathcal{C})\) for \(\dot{h}\).

Remark 4.14. Shelah’s model \((\mathfrak{p}a, \mathfrak{p}b)\) for the consistency of \(\varnothing < a\) with ZFC uses a template iteration like in Ex. 4.14 where \(L_C = \varnothing\) and \(S_x = D\) for every \(x \in L_S = L\). To use the isomorphism-of-names argument, the iteration is done under the continuum hypothesis so, by Lemma 4.13, the conditions of Theorem 4.13 with \(\theta = \mathfrak{p}_1\) and \(C = \alpha\) hold for that template iteration and, thus, \(s = \mathfrak{p}_1\) in the generic extension. Therefore, if \(\mathfrak{p}_1 < \mu < \lambda\) are regular cardinals and \(\lambda^+ = \lambda\), there is a model of ZFC such that \(s = \mathfrak{p}_1 < b = \varnothing = \mu < a = c = \lambda\). Moreover, the same model satisfies \(\cov(\mathcal{N}) = \mathfrak{p}_1\), \(\add(\mathcal{M}) = \cof(\mathcal{N}) = \mu\), and \(\non(\mathcal{N}) = \lambda\). See details in [9].

We introduce a preservation result of the same property for template iterations but with different conditions.

Theorem 4.15 (Second preservation theorem for template iterations). Consider \(\langle L, \mathcal{I} \rangle\) an indexed template and \(P|L\) a corresponding template iteration such that it is ccc,

(i) whenever \(A \in \mathcal{I}\) has a maximum \(x\) and \(A \cap L_x \notin \mathcal{I}_x\), if \(\dot{h}\) is a \(P|A\)-name for a real, then there exists an increasing sequence \(\langle B_n \rangle_{n < \omega}\) in \(B_A := \{B \subseteq A / \exists B \cap L_x \in \mathcal{I}_x[A]\}\) such that \(\dot{h}\) is a \(P|C\)-name for a real, where \(C := \bigcup_{n < \omega} B_n\), and \(P|C = \limdir_{n < \omega} P|B_n\),

(ii) whenever \(A \in \mathcal{I}\) does not have a maximum and \(\dot{h}\) is a \(P|A\)-name for a real, then there exists an increasing sequence \(\langle B_n \rangle_{n < \omega}\) in \(B_A := \{B \subseteq A / \exists x \in A(B \in \mathcal{I}_x[A])\}\) like in (i), and

(iii) for all \(x \in L\) and \(B \in \mathcal{I}_x\), \(\Vdash P|B (p^\theta|_{B|\mathcal{C}})\) holds.

Then, \((+p^\theta|_{P|L'|})\) holds.

Proof. Prove, by induction on \(A \in \mathcal{I}\), that \((+p^\theta|_{A|\mathcal{C}})\) holds. Assume \(A \neq \varnothing\) and consider the same cases as in the proof of Theorem 4.13.

Case 1 Like in the latter proof.

Case 2 Let \(\dot{h}\) a \(P|A\)-name for a real. Choose \(\langle B_n \rangle_{n < \omega}\) and \(C\) as in (i). Now, for each \(n < \omega\), choose a \(P|B_n\)-name \(\dot{h}_n\) for a real and \(\langle \dot{p}_{n,k} \rangle_{k < \omega}\) \(P|B_n\)-names for a decreasing sequence of conditions in \(P|C/P|B_n\) such that it is forced with \(P|B_n\) that \(\dot{p}_{n,k} \Vdash P|C/P|B_n h[k] = h_\alpha[k].\) Choose \(Y_\alpha\) to be a witness of \((+p^\theta|_{B_n|\mathcal{C}})\) for \(h_\alpha\).

Put \(Y := \bigcup_{n < \omega} Y_\alpha\). As in the proof of Theorem 4.13 this is a witness of \((+p^\theta|_{P|L'|})\) for \(\dot{h}\) (first note this for \(P|C\).

Case 3 Similar argument as in the previous case.
It is easy to note that any iteration as in Example 3.5, where all the involved posets have the ccc, satisfies the conditions of the previous theorem, moreover, any \( A \in \mathcal{I} \) that has a maximum \( x \) satisfies \( A \cap L_x \in \mathcal{I}_x \), so condition (i) becomes irrelevant in this case.

To finish this section, we prove the following result about new reals added in an intermediate extension of a template iteration.

**Theorem 4.16** (New reals not added at other stages of a template iteration). In a (ground) model \( V \) of ZFC, let \( P \upharpoonright (L, \mathcal{I}) \) be a template iteration as in Example 3.3, \( x \in L \) such that \( L_x \in \mathcal{I}_x \) and let \( f \) be a \( P\upharpoonright (L_x \cup \{ x \})\)-name of a real such that \( \Vdash_P L_x \cup \{ x \} \not\in V^{P}_{L_x} \). Then, \( P \upharpoonright L \) forces that \( f \not\in V^{P\upharpoonright (L_x \cup \{ x \})} \).

This result is a direct consequence of Theorem 4.15 which is a more general result about the preservation of \( \square \)-unbounded reals. Fix \( \square \) a relation as in Context \( \mathcal{C} \), \( M \subseteq N \) transitive models of ZFC, \( P \subseteq M \) and \( Q \subseteq N \) posets such that \( P \subseteq M Q \) (recall this notation from the first paragraph of Section 4) and let \( c \in \omega^\omega \cap N \) be a \( \square \)-unbounded real over \( M \). Recall the following property from [20] Sect. 4.

\[(\star, P, Q, M, N, \square, c) : \text{ For every } P\text{-name } h \in M \text{ for a real in } \omega^\omega, \Vdash_{Q, N} c \nsubseteq h.\]

This means that \( c \) is \( \square \)-unbounded over \( M[G \cap \mathbb{P}] \) for every \( Q \)-generic \( G \) over \( N \). Such a property was introduced for the first time in [18] and generalized in [19] and [20]. Recall

**Lemma 4.17.** (a) ([20] Thm. 7) Let \( S \) be a Suslin ccc poset coded in \( M \) such that \( (+_{S, \square}) \) is true in \( M \). Then, \( (\star, S^M, \mathcal{S}^N, M, N, \square, c) \) holds.

(b) ([18] Lemma 11) \( (\star, P, \mathcal{P}, M, N, \square, c) \) holds.

**Theorem 4.18** (Preservation of \( \square \)-unbounded reals in a template iteration). In a (ground) model \( V \) of ZFC, let \( P \upharpoonright (L, \mathcal{I}) \) be a template iteration as in Example 3.3 such that, for every \( x \in L_S \) and \( B \in \mathcal{I}_x \), \( P\upharpoonright B \) forces that \( (+_{Q, \square}) \) holds. Let \( x \in L \) such that \( L_x \in \mathcal{I}_x \), let \( c \) be a \( P\upharpoonright (L_x \cup \{ x \})\)-name for a real and assume that \( (L_x \cup \{ x \}) \) forces that \( c \) is \( \square \)-unbounded over \( V^{P\upharpoonright L_x} \). Then, \( P \upharpoonright L \) forces that \( c \) is \( \square \)-unbounded over \( V^{P\upharpoonright (L_x \cup \{ x \})} \).

**Proof.** We prove, by induction on \( A \in \mathcal{I} \), that if \( L_x \cup \{ x \} \subseteq A \) then \( P \upharpoonright (A \setminus \{ x \}) \) forces that \( c \) is \( \square \)-unbounded over \( V^{P\upharpoonright (A \setminus \{ x \})} \).

**Case 1** \( y = \max(A) \) exists and \( A_0 := A \cap L_y \in \mathcal{I}_y \). If \( y = x \) then \( A = L_x \cup \{ x \} \) and the conclusion is clear. Assume that \( x < y \). Clearly, \( A_0 \setminus \{ x \} \in \mathcal{I}_y \). Let \( G \) be \( P \upharpoonright A_0 \)-generic over \( V \), \( M := V[G \cap P\upharpoonright (A_0 \setminus \{ x \})] \) and \( N := V[G] \). By induction hypothesis, \( c \in N \) is \( \square \)-unbounded over \( M \) and, by cases when \( y \in L_S \) or \( y \in L_C \), Lemma 4.11 implies that, in \( N \), \( Q^A_\varphi \) forces that \( c \) is \( \square \)-unbounded over \( M_{Q^A_\varphi}^{\omega^\omega} \).

**Case 2** \( y = \max(A) \) exists and \( A_0 := A \cap L_y \notin \mathcal{I}_y \). The proof of this case follows the same idea as the argument of [18] Lemma 12. Clearly, \( x \leq y \) and, by (3) of Definition 3.3, \( A_0 \setminus \{ x \} \notin \mathcal{I}_y \). Therefore, \( P\upharpoonright A = \limdir_{B \in \mathcal{B}} P\upharpoonright B \) and \( P\upharpoonright (A \setminus \{ x \}) = \limdir_{B \in \mathcal{B}} P\upharpoonright B \) where \( B := \{ B \subseteq A / L_x \cup \{ x \} \subseteq B \) and \( B \cap L_y \in \mathcal{I}_y \cap \mathcal{A} \) and \( B' := \{ B \subseteq A \setminus \{ x \} / L_x \subseteq B \) and \( B \cap L_y \in \mathcal{I}_y \cap \mathcal{A} \}). By contradiction, assume that there are \( p \in P\upharpoonright A \), \( m < \omega \) and \( \hat{h} \) a \( P\upharpoonright (A \setminus \{ x \})\)-name of a real such that \( p \Vdash_{P\upharpoonright A} c \subseteq m \hat{h} \). Then, there exists a \( B \in \mathcal{B} \) such that \( p \in P\upharpoonright B \). Clearly, \( B \setminus \{ x \} \in B' \).

Let \( G \) be \( P\upharpoonright B\)-generic over \( V \) with \( p \in G \). By Lemma 4.11, \( P\upharpoonright (A \setminus \{ x \}) / P\upharpoonright (B \setminus \{ x \}) \subseteq V[G \upharpoonright (B \setminus \{ h \})] \).

**Case 3** \( A \) does not have a maximum. Same argument as in the previous case.

**Proof of Theorem 4.16** Apply Lemma 4.12 and Theorem 4.13 for the relation defined in Example 4.11
5 The groupwise-density number and fsi

With the fsi techniques of [3], the author constructed in [20 Sect. 3] and [21 Thm. 4.1-4.4] models with large continuum where the cardinal invariants defined in the introduction, with the exception of \( g \) and \( a_\alpha \), can take many different values. But, because the iterations used there can be defined as a template iteration as explained in Example [3] and Theorem [4.15] for preservation can be used, then we can also get a value of \( g \). We show how to do this in this section. First, recall the following result.

**Lemma 5.1** ([4 Thm. 2]). Let \( \theta \) be an uncountable regular cardinal, \((V_\alpha)_{\alpha \leq \theta}\) an increasing sequence of transitive models of \( \text{ZFC} \) such that
\[
\begin{align*}
(i) & \quad [\omega]^2 \cap (V_{\alpha+1} \setminus V_\alpha) \neq \emptyset, \\
(ii) & \quad [\omega]^\omega \cap (V_\alpha)_{\alpha < \theta} \in V_\theta \quad \text{and} \\
(iii) & \quad [\omega]^\omega \cap V_\theta = \bigcup_{\alpha < \theta} [\omega]^\omega \cap V_\alpha.
\end{align*}
\]
Then, in \( V_\theta \), \( g \leq \theta \).

For any infinite cardinal \( \lambda \), we use the notation

\[
\text{GCH}_\lambda. \quad \text{For any infinite cardinal } \mu,
\]

\[
2^\mu = \begin{cases} 
\lambda & \text{if } \mu < \text{cf} (\lambda), \\
\lambda^+ & \text{if } \text{cf} (\lambda) \leq \mu < \lambda, \\
\mu^+ & \text{if } \lambda \leq \mu.
\end{cases}
\]

Fix uncountable regular cardinals \( \mu_1 \leq \mu_2 \leq \mu_3 \leq \kappa \) and a cardinal \( \lambda \geq \kappa \).

**Theorem 5.2.** If \( \text{cf}(\lambda) \geq \mu_3 \), it is consistent with \( \text{ZFC} \) that \( \text{GCH}_\lambda \), \( \text{add}(N) = \mu_1 \), \( \text{cov}(N) = \mu_2 \), \( p = s = g = \mu_3 \), \( \text{non}(N) = \varepsilon = \zeta = \lambda \) and that one of the following statements hold.

(a) \( \text{non}(M) = \mu_3 \) and \( \text{cov}(M) = \lambda \).

(b) \( \text{add}(M) = \text{col}(M) = \kappa \).

(c) \( b = \mu_3 \), \( \text{non}(M) = \text{cov}(M) = \kappa \) and \( d = \lambda \).

**Proof.** (a) In the proof of [20 Thm. 2] (see also [21 Thm. 4.1]) we constructed a model \( V^3 \), which is a generic extension by a ccc poset, that satisfies \( \text{add}(N) = \mu_1 \), \( \text{cov}(N) = \mu_2 \), \( \text{add}(M) = \varepsilon = \mu_3 \) and \( \text{GCH}_{\mu_3} \). Consider the template \( (\lambda, \vec{T}) \) corresponding to a fsi of length \( \lambda \) (Example [2.2.2]). For each \( \alpha < \lambda \), enumerate \( [\alpha]^{\mu_3} := \{C_{\alpha, \beta}\}_{\beta < \lambda} \). Fix a bijection \( g : \lambda \to \lambda^3 \) such that \( g^{-1}(\alpha, \beta, \gamma) \geq \alpha, \beta, \gamma < \lambda \). Consider a template iteration \( P \downarrow (\lambda, \vec{T}) \) as in Example 4.3 such that \( L_S = \{\xi < \lambda / \exists \bar{3} (\xi = 4\delta)\} \), \( S_\xi = C \) for \( \xi \in L_S \) and, for each \( \xi \in L_C \), if \( \xi = 4\delta + \varepsilon \) with \( 0 < \varepsilon < 4 \) and \( g(\delta_\varepsilon) = (\alpha, \beta, \gamma) \), then

- \( C_\xi := C_{\alpha, \beta} \).
- \( \{\bar{1}\}_{\eta < \lambda} \) is an enumeration of the \( P\downarrow C_{\alpha, \beta} \)-names for all the subalgebras of \( A_\eta \) of size \( < \mu_1 \).
- \( \{\bar{3}\}_{\eta < \lambda} \) is an enumeration of the \( P\downarrow C_{\alpha, \beta} \)-names for all the subalgebras of \( B_\eta \) of size \( < \mu_2 \).
- \( \{\bar{2}\}_{\eta < \lambda} \) is an enumeration of the \( P\downarrow C_{\alpha, \beta} \)-names for all the filter bases of size \( < \mu_3 \).
- If \( r_\varepsilon = 1 \), then \( Q_\varepsilon = \bar{A}_{\alpha, \beta, \gamma} \).
- If \( r_\varepsilon = 2 \), then \( Q_\varepsilon = \bar{B}_{\alpha, \beta, \gamma} \).
- If \( r_\varepsilon = 3 \), then \( Q_\varepsilon = \bar{M}_{\alpha, \beta, \gamma} \).

By Lemma 5.1, \( P\downarrow \lambda \) is ccc and each condition and name for a real has a support of size \( < \mu_3 \). Let \( V^3_\lambda \) be a generic extension by \( P\downarrow \lambda \). The same argument as in the proof of [20 Thm. 2], by the use of Theorem 4.15 yields \( \text{GCH}_\lambda \), \( \text{add}(N) = \mu_1 \), \( \text{cov}(N) = \mu_2 \), \( \text{non}(M) \leq \mu_3 \) and \( \text{cov}(M) = \varepsilon = \lambda \) in \( V^3_\lambda \). To get \( p \geq \mu_3 \) note that, if \( F \) is a filter base of size \( < \mu_3 \), then there is an \( \alpha < \lambda \) such that \( F \) is in the intermediate extension by \( P\downarrow \alpha \). Thus, there exists \( \beta < \lambda \) such that \( F \) is in the intermediate extension by \( P\downarrow C_{\alpha, \beta} \) and \( F = \bar{F}_{\alpha, \beta, \gamma} \) for some \( \gamma < \lambda \), so the Mathias real added at the coordinate \( \xi = 4\delta + 3 \) with \( g(\delta) = (\alpha, \beta, \gamma) \) is a pseudo-intersection of \( F \).

We are left with \( g \leq \mu_3 \). In \( V^3 \), let \( \{A_\varepsilon\}_{\varepsilon < \mu_3} \) be a partition of \( \lambda \) into sets of size \( \lambda \) such that \( A_\varepsilon \cap L_S \neq \emptyset \) for each \( \varepsilon < \mu_3 \). If \( G \) is \( P\downarrow \lambda \)-generic over \( V^3 \), apply Lemma 5.1 to \( V^3_\rho := V^3_\rho \downarrow [\bigcup_{\varepsilon < \rho} A_\varepsilon] \) for \( \rho \leq \mu_3 \). Also note that Theorem 4.16 is needed here to prove that condition (i) of Lemma 5.1 holds.
(b) This is a modification of the proof of [20, Thm. 5] (see also [21, Thm. 4.2]). Work in $W = V_\kappa$ the model obtained in (a). Consider a template $\mathcal{J}$ defined as in Example (2.2) for the ordinal $\lambda \kappa$. Fix a bijection $h : \lambda \to \lambda \times \lambda \times 3$ and, for each $\alpha < \kappa$, enumerate $|\lambda\alpha|^{<\mu_3} := \{D_{\alpha,\beta}\}_{\beta<\lambda}$. Perform a template iteration $\mathbb{P}(\langle \lambda \kappa, \mathcal{J}\rangle)$ such that $L_{S} = \{\lambda \alpha / \alpha < \kappa\}$, $S_{S} = \mathcal{D}$ for each $\xi \in L_{S}$ and, for each $\xi \in L_{C}$, if $\xi = \lambda \alpha + \eta$ for some $\alpha < \kappa$, $0 < \eta < \lambda$ and $h(\eta) = (\beta, \gamma, r)$, then
\begin{itemize}
  \item $C_{\xi} := D_{\alpha,\beta}$
  \item $\hat{A}_{\alpha,\beta,\eta}$ is an enumeration of the $\mathbb{P}(D_{\alpha,\beta})$-names for all the subalgebras of $\mathcal{A}$ of size $< \mu_1$.
  \item $\hat{B}_{\alpha,\beta,\eta}$ is an enumeration of the $\mathbb{P}(D_{\alpha,\beta})$-names for all the subalgebras of $\mathcal{B}$ of size $< \mu_2$.
  \item $\hat{F}_{\alpha,\beta,\eta}$ is an enumeration of the $\mathbb{P}(D_{\alpha,\beta})$-names for all the filter bases of size $< \mu_3$.
  \item If $r = 0$, then $\hat{Q}_{\xi} = \hat{A}_{\alpha,\beta,\gamma}$.
  \item If $r = 1$, then $\hat{Q}_{\xi} = \hat{B}_{\alpha,\beta,\gamma}$.
  \item If $r = 2$, then $\hat{Q}_{\xi} = \mathcal{M}_{\hat{F}_{\alpha,\beta,\gamma}}$.
\end{itemize}
Arguments as in the proof of (a) and [20, Thm. 5] give the result.

(c) Modifying the construction in the proof of [20, Thm. 3] (see also [21, Thm. 4.3]) as it is done in (b) gives the result.

The following results are modifications of proofs in [20, Sect. 3] as done in the proof of Theorem 5.2. We do not show the proofs but refer to the modified result instead.

**Theorem 5.3** ([20, Thm. 4], see also [21, Thm. 4.4]). Assume $\text{cf}(\lambda) \geq \mu_2$. It is consistent with ZFC that $\text{add}(\mathcal{N}) = \mu_1$, $p = b = s = g = \mu_2$, $\text{cov}(\mathcal{N}) = \text{non}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \text{non}(\mathcal{N}) = \kappa$, $\mathcal{D} = \mathcal{V} = \mathcal{E} = \lambda$ and $\text{GCH}_\kappa$.

**Theorem 5.4** ([20, Thm. 6]). Assume $\text{cf}(\lambda) \geq \mu_1$. It is consistent that $\text{GCH}_\lambda$, $\text{add}(\mathcal{N}) = p = g = \mu_1$, $\text{cov}(\mathcal{N}) = \text{add}(\mathcal{M}) = \text{cof}(\mathcal{M}) = \text{non}(\mathcal{N}) = \kappa$ and $\text{cof}(\mathcal{N}) = \mathcal{E} = \lambda$.

In this last result, we cannot say much about $s$, $r$ and $u$ because the corresponding iteration uses random and Hechler forcing for the $L_3$-coordinates (as explained in Example 3.6) and it is not known how to preserve splitting families when these two forcing notions are involved in this way.

6 Proof of the main result

**Theorem 6.1** (Main result). Let $\kappa$ be a measurable cardinal, $\theta < \kappa < \mu < \lambda$ all regular uncountable cardinals. Assuming $\text{GCH}$, there exists a ccc poset forcing that $s = \theta < b = \mu < a = \mathcal{E} = \lambda$. Moreover, this poset forces $p = g = \theta$ and $\text{GCH}_\lambda$.

The following result will be relevant at the end of the proof of this theorem.

**Lemma 6.2** (Destruction of mad families. Shelah [23], see also [8, Lemma 0.3]). Let $\mathbb{P}$ be a ccc poset and $\mathcal{A}$ a $\mathbb{P}$-name for an a.d. family of size $\geq \kappa$. Then, $\mathbb{P}_c / \mathcal{D}$ forces that $\mathcal{A}$ is not maximal.

As forcing notions of size $< \kappa$ preserve the measurable cardinal, by Theorem 5.2(a) with $\mu_1 = \mu_2 = \mu_3 = \lambda = \theta$, we work in a ZFC model $V$ that satisfies $\text{add}(\mathcal{N}) = s = \theta = \mathcal{E} = \lambda$ and $\text{GCH}_\theta$. Note that $Q := [\omega]^{\geq \theta}$ is a $\theta\mathcal{E}$-unbounded family.

Fix $\mathcal{D}$ a non-principal $\kappa$-complete ultrafilter on $\kappa$.

**Definition 6.3** (Appropriate template iteration). A template iteration $\mathbb{P} / \langle L, \mathcal{I}\rangle$ is appropriate (for the proof of Theorem 6.1) if the following conditions hold.

1. $\lambda \mu \subseteq L$, $|L| = \lambda$ and $\theta = \text{min}(L)$.
2. Every $x \in L$ has an immediate successor and, for $\xi \in \lambda \mu$, $\xi + 1$ is the immediate successor of $\xi$.
3. If $\gamma \in \lambda \mu$ is a limit ordinal of cofinality $\neq \kappa$, then $\gamma = \text{sup}_{L} \{\alpha \in \lambda \mu / \alpha < \gamma\}$.

\footnote{Much less is necessary to get such a model. For example, start with a model of ZFC + $\text{GCH}$ and perform a $\mathbb{P}\mathcal{E}$ of length $\theta$ alternating between amoeba forcing and Mathias forcing with an ultrafilter on $\kappa$.}
(IV) $L$ is partitioned into three disjoint sets $L_H$, $L_F$ and $L_T$.

(V) $L_H \cap \lambda \mu$ is unbounded in $\lambda \mu$.

(VI) For each $\alpha \in \lambda \mu$, $L_\alpha \in \mathcal{I}_\alpha$.

(VII) If $X \in [L]^\omega_\theta$, then $|\mathcal{I}[X]| < \theta$.

(VIII) For $x \in L_H$ and $B \in \mathcal{I}_x$, $\dot{Q}_x^B$ is a $P\upharpoonright B$-name for $\mathbb{D}^{V\upharpoonright H}$.

(IX) For $x \in L_F$ there is a fixed $C_x \in \mathcal{I}_x$ of size $< \theta$ and $\dot{F}_x$ a $P\upharpoonright C_x$-name for a filter base on $\omega$ of size $< \theta$ such that, for every $B \in \mathcal{I}_x$,

\[
\dot{Q}_x^B = \begin{cases} 
\mathbb{M}_{\dot{F}_x} & \text{if } C_x \subseteq B, \\
\text{otherwise.} & 
\end{cases}
\]

(X) For $x \in L_T$ and $B \in \mathcal{I}_x$, $\dot{Q}_x^B$ is the trivial forcing.

(XI) Given $\dot{F}$ a $P \upharpoonright L$-name for a filter base on $\omega$ of size $< \theta$, there exists an $x \in L_F$ such that $\mathbb{M}_{\dot{F} \upharpoonright L} \dot{F} = \dot{F}_x$.

Notice that an appropriate template iteration $P\upharpoonright \langle L, \mathcal{I} \rangle$ satisfies the hypothesis of Lemma 6.10 so it has the Knaster condition and the support of each condition and of each name of a real has size $< \theta$. Therefore, all conditions of Theorem 4.13 are satisfied for $\alpha$ (see Lemmas 4.3 and 4.7), so $(\text{V}P\downharpoonright L, \alpha)$ holds and, thus, $P\upharpoonright \langle L, \mathcal{I} \rangle$ forces $\delta \leq \theta$, moreover, equality is forced because $\mathbb{P} \geq \theta$ by (XI) and a similar argument as in the proof of Theorem 5.2(a) (notice that, for each $x \in L_F$, $C_x \in \mathcal{I}_x$ and $P(C_x \cup \{x\}) = P(\mathbb{C}_x \star M_{\dot{F}_x})$).

Also, by (V), (VI) and (VIII), $P\upharpoonright \mathcal{I}[L, \mathcal{I}]$ forces $\delta = 0 = \mu$. As $|L| = \lambda$, it is also clear that $\mathbb{P}\upharpoonright \langle L, \mathcal{I} \rangle$ forces $\epsilon \leq \lambda$, $\mathbb{g} \leq \theta$ is also forced: let $\{A_x\}_{\alpha < \mu_3}$ be a partition of $L$ into sets of size $\lambda$ such that $A_x \cap L_H \neq \emptyset$ for each $\alpha < \mu_3$ and, by using Lemma 5.1 and Theorem 4.10, proceed like at the end of the proof of Theorem 5.2(a).

Therefore, to prove Theorem 6.1 it is enough to construct an appropriate template iteration that forces $\alpha \geq \lambda$. This will be done by constructing a chain of appropriate template iterations of length $\lambda$ such that the inductive step is done by taking ultrapowers (so we can use Lemma 6.2 to force $\alpha$ to be large).

Before proceeding with this construction, we explain how we deal with the inductive and limit steps for the construction of that chain.

Fix an appropriate template iteration $P\upharpoonright \langle L, \mathcal{I} \rangle$. Recall from the context of Lemma 2.8 the templates $\mathcal{T}^*$ and $\mathcal{T}^!$ associated to the ultrapower $L^*$ of the linear order $L$. We show how to construct, in a canonical way, an appropriate template iteration $P\upharpoonright \langle L^*, \mathcal{T}^* \rangle$ that is forcing equivalent to the ultrapower of $P\upharpoonright L$.

As $c(\mathcal{I}[\alpha]) = \mu > \kappa$, it is easy to note that $\lambda \mu$ is still cofinal in $L^*$. By standard arguments with ultrapowers, conditions (I)-(III) of Definition 6.3 are satisfied by $L^*$. Let $L^*_H := L_H^*/D$, $L^*_F$ and $L^*_T$ defined likewise. (IV)-(VII) for $\langle L^*, \mathcal{T}^* \rangle$ and $\langle L^!, \mathcal{T}^! \rangle$ are clear, the last one by Lemma 2.6. Notice that $L^*_H \cap L = L_H$, $L^*_F \cap L = L_F$ and $L^*_T \cap L = L_T$.

**Lemma 6.4** (Ultrapower of a template iteration). There is a template iteration $P^* \upharpoonright \langle L^*, \mathcal{T}^* \rangle$ such that (VIII)-(X) hold and, for any $A \in \mathcal{T}^*$, there is an onto embedding $F_A : \bigsqcup_{\alpha < \kappa} P\upharpoonright A_\alpha/D \to P^*\upharpoonright A$ such that, for any $D = [\bigsqcup_{\alpha < \kappa}] \subseteq A$, $F_A[D] \subseteq F_A$.

**Proof.** To define the desired template iteration $P^* \upharpoonright \langle L^*, \mathcal{T}^* \rangle$, it is enough to show how $C_x$ and $\dot{F}_x$ are defined for (IX). This is done in parallel with the construction, by recursion on $A \in \mathcal{T}^*$, of the desired onto embeddings.

(IX) If $x \in L_F^*$, let $C_x^* := [(C_{x\alpha})_{\alpha < \kappa}] \in \mathcal{T}_x^*$. The $P^*\upharpoonright C_x^*$-name $\dot{F}_x^* := \langle \dot{F}_{x\alpha} \rangle_{\alpha < \kappa}/D$ is defined in the following way. By cc-ness find, for $D$-many $\alpha$, cardinals $\nu_{x\alpha} < \theta$ such that $\dot{F}_{x\alpha}$ is forced by $P\upharpoonright C_{x\alpha}$ to have size $< \nu_{x\alpha}$, but, as $\theta < \kappa$, there exists a $\nu < \theta$ such that $\nu_{x\alpha} = \nu$ for $D$-many $\alpha$. For those $\alpha$ put $\hat{F}_{x\alpha} := \langle \hat{U}_{x\alpha, \xi} / \xi < \nu \rangle$. Let $\hat{U}_x^* := (\hat{U}_{x, \xi})_{\alpha < \kappa}/D$, which is a $\bigsqcup_{\alpha < \kappa} P\upharpoonright C_{x\alpha}/D$-name (so a $P^*\upharpoonright C_x^*$-name) for an infinite subset of $\omega$. Let $\hat{F}_x^*$ be a $P^*\upharpoonright C_x^*$-name for $\langle \hat{U}_x^* / \xi < \nu \rangle$. By standard arguments with ultrapowers, it is easy to see that $\hat{F}_x^*$ is a $P^*\upharpoonright C_x^*$-name for a filter base. Notice that, if $x \in L_F^* \cap L = L_F$, then $C_x^* = C_x$ (as the size of $C_x$ is $< \theta$, its ultrapower is the same set) and, thus, $\hat{F}_x^* = \hat{F}_x$.
As mentioned, the onto embedding is constructed by induction on \( \bar{A} \in \mathcal{I}^\ast \). Let \( \bar{p} \in \prod_{\alpha<\kappa} P|A_\alpha/D \), that is, \( p_\alpha \in P|A_\alpha \) for \( \mathcal{D}\)-many \( \alpha \). Let \( x_\alpha := \max(\text{dom}(p_\alpha)) \), so there exists a \( B_\alpha \in \mathcal{I}_x|A_x \) such that \( p_\alpha|L_{x_\alpha} \in P|B_\alpha \), and \( p_\alpha(x_\alpha) \) is a \( P|B_\alpha \)-name for a condition in \( Q^B_\beta \). Let \( \bar{r} := (p_\alpha|L_{x_\alpha})_{\alpha<\kappa}/D \) and \( p(\bar{x}) := \langle p_\alpha(x) \rangle_{\alpha<\kappa}/D \) which is a \( P^\ast|B \)-name for a real (by inductive hypothesis), where \( B := \{ [B_\alpha]_{\alpha<\kappa} \} \in \mathcal{I}_x^\ast |A \). By considering cases on \((\text{VIII}), (\text{IX})\) and \((\text{X})\), \( p(\bar{x}) \) is actually a \( P^\ast|B \)-name for a condition in \( Q^B_\beta \), so by definition \( F_{\bar{A}}(\bar{p}) = F_B(\bar{r})\gamma(p(\bar{x})). \) Note that this definition does not depend on \( B \).

A template iteration \( P^1|\langle L^\ast, \bar{I}^\ast \rangle \) can be defined in a similar way as in the previous proof, so that \( P^1|A \) is forcing equivalent to \( \prod_{\alpha<\kappa} P|A_\alpha/D \) for any \( A = \{ A_\alpha \}_{\alpha<\kappa} \) given by subsets of \( L \). Notice that \( \langle L^\ast, \bar{I}^\ast \rangle \) is a \( \theta \)-innocuous extension of \( \langle L^\ast, \bar{I}^\ast \rangle \) (Lemma 2.3), so by Lemma 3.12 \( P^1|A \) is forcing equivalent to \( P^\ast|A \).

**Lemma 6.5.** \( P^\ast|\langle L^\ast, \bar{I}^\ast \rangle \) and \( P^1|\langle L^\ast, \bar{I}^\ast \rangle \) are appropriate template iterations. Moreover, \( P|A \) is forcing equivalent to \( P^1|A \) and \( P^1|A \) for any \( A \in \mathcal{I} \).

**Proof.** It remains to prove condition (XI) for both iterations. As every set in \( \mathcal{I}_x^\ast \) is contained in some set in \( \mathcal{I}_x^\ast \) for any \( \bar{x} \in L^\ast \), it is enough to consider only the case for \( \bar{I}^\ast \). Indeed, let \( \bar{F} \) be a \( P^\ast|L^\ast \)-name for a filter base on \( \omega \) of size \( < \theta \). By ccc-ness, find \( \nu < \theta \) such that \( \bar{F} = \{ \bar{U}_\epsilon \mid \epsilon < \nu \} \). Each \( \bar{U}_\epsilon \) is of the form \( \langle U_{\alpha, \epsilon} \rangle_{\alpha<\kappa}/D \) where each \( U_{\alpha, \epsilon} \) is a \( P|L \)-name for an infinite subset of \( \omega \). As \( \nu < \theta \), \( \bar{F}_\alpha := \{ U_{\alpha, \epsilon} \mid \epsilon < \nu \} \) is a \( P|L \)-name for a filter for \( \mathcal{D}\)-many \( \alpha \), so, by (XI), there exists an \( x_\alpha \in L_\alpha \) such that \( \Vdash_{\alpha \mathcal{L}} \bar{F}_{\alpha} = \bar{F}_{Fact \alpha} \). Then \( \Vdash_{\mathcal{P}_1 \mathcal{L}} \bar{F} = \bar{F}^\ast \).

The second part of the proof follows from Lemma 3.12 because, for \( x \in L, \mathcal{I}_x^\ast = \mathcal{I}_x \mathcal{L} \) and \( (\bar{L}, \bar{I}^\ast |L) \) is a strongly \( \theta \)-innocuous extension of \( (L, \bar{I}) \).

Now, we explain how we deal, in general, with the limit step. Let \( \delta \) be a limit ordinal and consider a chain \( \{ \langle L^\alpha, \bar{I}^\alpha \rangle \}_{\alpha<\delta} \) of templates and appropriate template iterations \( P^\alpha|\langle L^\alpha, \bar{I}^\alpha \rangle \) with the following properties for all \( \alpha < \beta < \delta \).

1. \( \langle L^\beta, \bar{I}^\beta \rangle \) is a strongly \( \theta \)-innocuous extension of \( \langle L^\alpha, \bar{I}^\alpha \rangle \).
2. For \( x \in L^\alpha \), its immediate successor in \( L^\alpha \) is the same as in \( L^\beta \).
3. \( L^\alpha_H = L^\beta_H \cap L^\alpha \) and \( L^\alpha_F \subseteq L^\beta_F \).
4. If \( x \in L^\beta_F \), then \( \bar{C}^\alpha_x = \bar{C}^\beta_x \) and \( \Vdash_{P^\delta|C^\beta_x} \bar{F}^\alpha_x = \bar{F}^\beta_x \).

Note that Corollary 3.11 implies that \( P^\alpha|X \) is a complete suborder of \( P^\delta|X \) for any \( X \subseteq L^\alpha \). Consider \( L^\delta \) and the templates \( \bar{I}^\delta \) as in the context of Lemma 2.7. Let \( L^\delta_H = \bigcup_{\alpha<\delta} L^\alpha_H \) and \( L^\delta_F = \bigcup_{\alpha<\delta} L^\alpha_F \). Properties (I)-(V) are straightforward for \( L^\delta \), moreover, properties (1)-(3) hold for any \( \alpha < \delta \) by replacing \( \beta \) by \( \delta \) and for both templates \( \bar{I}^\delta \). (VII) also holds for both templates because of Lemma 2.8. Nevertheless, (VI) holds for \( \bar{I} \) but it need not hold for \( \bar{I}^\delta \).

We show how to define template iterations \( P^\delta_0|\langle L^\delta, \bar{I}^\delta \rangle \) and \( P^\delta_1|\langle L^\delta, \bar{J}^\delta \rangle \) such that they are close to be appropriate and have nice agreement with the template iterations \( P^\alpha|\langle L^\alpha, \bar{I}^\alpha \rangle \) for \( \alpha < \delta \). We just need to be specific about (IX) in order to define the iterations. In the case of \( \bar{I} \), for \( x \in L^\beta_F \), if there is some \( \alpha < \delta \) such that \( x \in L^\alpha_F \), let \( C^\alpha_x := C^\beta_x \) and \( \bar{F}^\alpha_x = \bar{F}^\beta_x \). Otherwise, choose \( C^\alpha_x \) and \( \bar{F}^\alpha_x \) freely. For this to be defined, it is necessary to proceed inductively and guarantee that, for each \( \alpha < \delta \), \( P^\alpha|X \) is a complete suborder of \( P^\delta_0|X \) for any \( X \subseteq L^\alpha \), but this can be done along the way using Corollary 3.11.

Notice that (4) holds in this case by replacing \( \beta \) by \( \delta \).

\( P^\delta_1|\langle L^\delta, \bar{J}^\delta \rangle \) is defined in the same way by just ensuring to make the same choices of \( C^\alpha_x \) and \( \bar{F}^\alpha_x \) as for \( \bar{I} \). The same conclusions as in the previous case hold in the same way. However, it is not always the case that property (XI) holds, moreover, it will depend on the particular “free” choices of \( C^\alpha_x \) and \( \bar{F}^\alpha_x \). There is one case in which (XI) holds for both template iterations.

**Lemma 6.6.** (Direct limit of a chain of template iterations). Assume that \( c(\delta) \geq \theta \) and that \( L^\beta_F = \bigcup_{\alpha<\delta} L^\alpha_F \). Then, both template iterations \( P^\delta_0|\langle L^\delta, \bar{I}^\delta \rangle \) and \( P^\delta_1|\langle L^\delta, \bar{J}^\delta \rangle \) are forcing equivalent and satisfy (XI). Moreover, \( P^\delta_0|L^\delta = \text{limdir}_{\alpha<\delta} P^\alpha|L^\alpha \) and \( P^\delta_1|L^\delta \) is appropriate.
Proof. By Lemmas 2.27 and 3.12 both template iterations are equivalent, so it is enough to prove (XI) for the iteration along $\mathcal{I}$.

We claim that $\mathbb{P}_0^{\mathfrak{a}} \restriction A = \text{limdir}_{\alpha < \delta} \mathbb{P}_0^{\mathfrak{a}} \restriction A$ for any $A \in \mathcal{I}^\delta$. Proceed by induction. Let $p \in \mathbb{P}_0^{\mathfrak{a}} \restriction A$ and $x = \max(\text{dom}(p))$, so there exists a $B \in \mathcal{I}_x \restriction A$ such that $p \upharpoonright L^x_\delta \in \mathbb{P}_0^{\mathfrak{a}} \restriction B$ and $p(x)$ is a $\mathbb{P}_0^{\mathfrak{a}} \restriction A$-name for a real in $\check{Q}^{\mathfrak{a},B}_0$. By induction hypothesis and cc+ness, find $\alpha < \delta$ such that $x \in L^\alpha$, $B \in \mathcal{I}_x^\alpha$. Then, $p(L^\alpha_\delta) = p(L^\delta_\delta) \in \mathbb{P}_0^{\mathfrak{a}} \restriction B$ and $p(x)$ is a $\mathbb{P}_0^{\mathfrak{a}} \restriction B$-name for a real. In the case that $x \in L^\alpha_0$, then $x \in L^\alpha _0$ and so $p(x)$ is a $\mathbb{P}_0^{\mathfrak{a}} \restriction B$-name for a condition in Hechler forcing; in the case that $x \in L^\delta_0$, increasing $\alpha$ if necessary, $x \in L^\delta_0$, so clearly $p(x)$ is a $\mathbb{P}_0^{\mathfrak{a}} \restriction B$-name for a condition in $\mathbb{M}_x$, if $x \in L^\delta_0$, then $x \in L^\delta_0$ and so $p(x)$ is clearly a name for the trivial condition. Then, in any case, $p \in \mathbb{P}_0^{\mathfrak{a}} \restriction A$.

Let $\check{F}$ be a $\mathbb{P}_0^{\mathfrak{a}} \restriction A$-name for a filter base on $\omega$ of size $< \theta$. Then, as $\text{cf}(\delta) \geq \theta$ and the previous claim, find $\alpha < \delta$ such that $\check{F}$ is a $\mathbb{P}_0^{\mathfrak{a}} \restriction A$-name, so there exists an $x \in L^\alpha_0 \subseteq L^\delta_0$ such that $\check{F}$ is forced to be equal to $\check{F}^\alpha_x = \check{F}^\alpha_x$.

The use of the template $\mathcal{I}$ is to prove the preceding result, but for the construction of the model of the main result, $\mathcal{J}$ is the one used for the limit step.

Proof of Theorem 7.7. Fix a bijective enumeration $\gamma \downharpoonright \{0\} = \{\delta_{\alpha, \beta} / \alpha, \beta < \lambda\}$, a bijection $g : \lambda \rightarrow \lambda \times \theta$ and an increasing enumeration $\langle \delta_{\alpha, \beta} \rangle_{\alpha < \lambda}$ of $\theta$ and all the limit ordinals below $\lambda$ that have cofinality $< \theta$. For an ordered pair $z = (x, y)$, denote $z_0 := x$ and $z_1 := y$.

By recursion on $\gamma \leq \lambda$, define a chain of templates $\{\langle L^\gamma, \check{\mathcal{I}}^\gamma \rangle\}_{\gamma < \lambda}$. Clearly, conditions (I)-(VII) hold for $\langle L^\gamma, \check{\mathcal{I}}^\gamma \rangle$.

Given $\langle L^\gamma, \check{\mathcal{I}}^\gamma \rangle$, let $\langle L^\gamma + 1, \check{\mathcal{I}}^\gamma + 1 \rangle := (\langle L^\gamma, \check{\mathcal{I}}^\gamma \rangle)^+$ in the previous discussion of ultrapowers.

For $\langle L^\gamma, \check{\mathcal{I}}^\gamma \rangle$, recall that $\langle L^\gamma + 1, \check{\mathcal{I}}^\gamma + 1 \rangle$ is a strongly $\theta$-approximation of $\langle L^\gamma, \check{\mathcal{I}}^\gamma \rangle$, so it is non-trivial to prove that it is also a strongly $\theta$-approximation of $\langle L^\delta, \check{\mathcal{I}}^\delta \rangle$ for each $\beta < \gamma$. Indeed, the non-trivial part is to see that, for $x \in L^\delta$, $\check{\mathcal{I}}^\beta \upharpoonright L^\delta \subseteq \check{\mathcal{I}}^\beta$. If $A \in \check{\mathcal{I}}^\alpha$, then, as $\check{\mathcal{I}}^\alpha \subseteq L^\gamma$, there exists $H \in (\check{\mathcal{I}}^\gamma)^\delta$ such that $A = H \cap L^\delta$. Then, $A = \langle \langle H, \alpha \cap L^\delta \rangle \rangle \in (\check{\mathcal{I}}^\gamma)$ because $\check{\mathcal{I}}^\gamma \subseteq L^\gamma$.

If $\delta$ is a limit ordinal, define $L^\delta = \bigcup_{\beta < \delta} L^\beta$ and $\check{\mathcal{I}}^\delta := \mathcal{J}$ according to the previous discussion about chains of templates. Hence, it is only needed to be specific about how to define $L^\alpha_0$. If $\text{cf}(\delta) \geq \theta$, put $L^\alpha_0 := \bigcup_{\beta < \alpha} L^\beta_0$, but otherwise, let $L^\alpha_0 := L^\alpha_0 \cup \{\lambda \in \check{\mathcal{I}}^\alpha \cap L^\delta / \xi < \mu$ and $\beta < \lambda\}$ where $\alpha < \lambda$ is such that $\delta = \delta_\alpha$. Clearly, (I)-(VII) and (1)-(3) are satisfied.

Again, by recursion on $\gamma \leq \lambda$, define appropriate template iterations $\mathbb{P}^\gamma(\langle L^\gamma, \check{\mathcal{I}}^\gamma \rangle)$ such that (4) is satisfied for them.

Looking at the template $\langle L^0, \check{\mathcal{I}}^0 \rangle$, for each $\xi < \mu$ enumerate $[\lambda \xi]^{< \theta} = \{C^0_{\xi, \alpha} / \alpha < \lambda\}$. In an inductive way, define the iteration $\mathbb{P}^0(\langle L^0, \check{\mathcal{I}}^0 \rangle)$ in the following way.

- For each $\xi < \mu$ and $\alpha < \lambda$, let $\langle \check{F}^0_{\xi, \alpha, \eta} \rangle_{\eta < \theta}$ be an enumeration of all the $\mathbb{P}^0 \restriction C^0_{\xi, \alpha}$-names of filter bases on $\omega$ of size $< \theta$. This can be done because $[\mathbb{P}^0]^\theta \restriction X \leq \theta$ when $|X| < \theta$. Indeed, as $|\mathbb{P}^0| |X| < \theta$ and $\epsilon = 0$, by induction on $A \in \mathbb{V}^X$ and Theorem 3.3(6) it is not difficult to see that $[\mathbb{P}^0]^\theta |A| \leq \theta$.

- For $x \in L^0$ and $B \in \check{\mathcal{I}}_x$, $\check{Q}^0_x \check{B}$ is defined as indicated in (VIII)-(X). For (IX), if $x \in L^0$ then $x = \lambda \xi + \tau_{0, \beta}$ for some $\beta < \lambda$, so put $C^0_x := C^0_{\xi, g(\beta)}$ and $\check{F}^0_\xi := \check{F}^0_{\xi, g(\beta)}$.

To see that $\mathbb{P}^0(\langle L^0, \check{\mathcal{I}}^0 \rangle)$ is appropriate, it remains to prove (XI). Indeed, if $\check{F}$ is a $\mathbb{P}^0 \restriction A$-name for a filter for size $< \theta$ then, as the support of every name of a real has size $< \theta$, find $C \subseteq [L^0]^{< \theta}$ such that $\check{F}$ is a $\mathbb{P}^0 \restriction C$-name. Clearly, there exist $\xi < \mu$ and $\alpha < \lambda$ such that $C = C_{\xi, \alpha}$, so $\check{F}$ is forced to be equal to $\check{F}^0_{\xi, \alpha, \eta} = \check{F}^0_{\xi, \alpha, \eta}$ for some $\eta < \theta$.

The iteration $\mathbb{P}^{\gamma + 1}(\langle L^\gamma + 1, \check{\mathcal{I}}^\gamma + 1 \rangle)$ is defined from $\mathbb{P}^\gamma(\langle L^\gamma, \check{\mathcal{I}}^\gamma \rangle)$ as explained in the previous discussion of ultrapowers. From the proof of Lemma 6.3(4) (4) is satisfied.

For the limit step for $\delta$ limit consider two cases. When $\text{cf}(\delta) \geq \theta$, define $\mathbb{P}^\delta \restriction \langle L^\delta, \check{\mathcal{I}}^\delta \rangle$ as in Lemma 6.3(4) so assume that $\text{cf}(\delta) < \theta$, that is, $\delta = \delta_\epsilon$ for some $\epsilon < \lambda$. For each $\xi < \mu$, enumerate $[\lambda \xi]^{< \theta} := \{C^\delta_{\xi, \alpha} / \alpha < \lambda\}$. As it was done for $\mathbb{P}^0 \restriction L^0$, define the iteration corresponding to $\delta$ inductively in the following way.
• For each $\xi < \mu$ and $\alpha < \lambda$, let $\langle \check{F}_\alpha^\xi, \eta < \theta \rangle$ be an enumeration of all the $P|^\xi \check{F}_\alpha^\xi$-names of filter bases on $\omega$ of size $< \theta$.

• For $x \in L^\delta$ and $B \in \check{I}_x$, $\check{Q}^0_B$ is defined as indicated in (VIII)-(X). For (IX), if $x \in \bigcup_{\beta < \delta} L^\beta$, let $C^\beta_x := C^\beta \omega$ and $\check{F}^\beta_x := F^\beta$ for some $\beta < \delta$ such that $x \in L^\beta$ (this does not depend on the chosen $\beta$ by (4)); if $x \in L^\beta = \bigcup_{\beta < \delta} L^\beta$, then $x = \lambda \xi + \tau, \beta$ for some (unique) $\beta < \lambda$, so put $C^\beta_x := C^\beta_{\xi, \eta < \theta (\beta)}$ and $\check{F}^\beta_x := F^\beta_{\xi, \eta < \theta (\beta)}$.

According to the previous discussion with chains of templates, we only need to prove condition (XI) for $P|^\delta L^\delta$, but its proof follows the same lines as in the case of $P^0|L^0$. From the discussion following Definition 6.8, it is enough to prove that $P\lambda|L^\lambda$ forces that $a = \lambda$. Indeed, let $A$ be a $P\lambda|L^\lambda$-name for an a.d. family of size $\nu < \lambda$ with $\nu \geq \kappa$ (we don’t need to consider a.d. families of size $< \kappa$ because $b$ is forced to be equal to $\mu > \kappa$ and $b \leq a$ is true in ZFC). By Lemma 6.9 $P\lambda|L^\lambda = \text{limdir}_{\nu < \lambda} P^\nu|L^\nu$, so there exists an $\alpha < \lambda$ such that $A$ is a $P^\alpha|L^\alpha$-name. As $P^{\alpha + 1}|L^{\alpha + 1}$ is forcing equivalent to the ultrapower of $P^\alpha|L^\alpha$ (Lemma 6.3), by Lemma 6.2 this poset forces that $A$ is not mad, and so does $P\lambda|L^\lambda$.

With this same type of construction, it is possible to use small suborders of $A$ to get consistency results with some other cardinal invariants.

**Theorem 6.7.** It is consistent with ZFC and the existence of a measurable cardinal $\kappa$ that, for $\theta < \kappa < \mu < \lambda$ all regular uncountable cardinals, $add(\mathcal{N}) = cov(\mathcal{N}) = p = s = g = \theta < add(\mathcal{M}) = cof(\mathcal{M}) = \mu < non(\mathcal{N}) = a = \tau = c = \lambda$.

**Proof.** We imitate the preceding construction, but in this case we include suborders of $A$ of size $< \theta$ along all the iterations. Redefine $L^\delta_{\eta^\beta} := \{\lambda \eta + \xi < \mu \mid \eta < \lambda \}$ and consider the enumeration $\{\tau_{\alpha, \beta} < \alpha, \beta < \lambda \}$ for all the odd ordinals $< \lambda$. For the chain of template iterations, use the ordinals congruent to 1 modulo 4 to force with Mathias forcing with a filter base of size $< \theta$ (like in (IX)) and use the ordinals congruent to 3 modulo 4 to force with suborders of $A$ of size $< \theta$. For $P^0|L^0$, we ensure (XII) for any sequence $\{N_\eta\}_{\eta < \gamma}$ of $P^0|L_0$-names for Borel-null subsets of $2^\omega$ and $\gamma < \theta$, there exists an ordinal $\beta < \lambda$ such that $\tau_{\beta, \theta} \equiv 3$ mod 4 and the forcing at coordinate $\tau_{\beta, \theta}$ adds a Borel-null set that covers $\{N_\eta\}_{\eta < \gamma}$.

We imitate the construction of the chains of templates of length $\lambda$ in such a way that, for each template iteration, an condition like (XII) is satisfied. The same argument as in the proof of Theorem 6.9 implies that $P\lambda|L^\lambda$ forces $add(\mathcal{N}) \geq \theta$, $p = s = g = \theta < add(\mathcal{M}) = cof(\mathcal{M}) = \mu < non(\mathcal{N}) = a = \tau = c = \lambda$ and GCH$\lambda$. Also, Theorem 1.13 implies that $(+^{P\lambda|L^\lambda})$ holds, so we get that $cov(\mathcal{N}) \leq \theta$ is preserved in its generic extension.

The redefinition of $L^\delta_{\eta^\beta}$ is done in order to get $\lambda$ cofinally many Cohen reals in some intermediate extension of the iteration $P\lambda|L^\lambda, \check{I}_x$ and to be able to extend the argument of the proof of [33, Prop 4.3] to get $non(\mathcal{N}) \geq \lambda$ and $\tau \geq \lambda$. Fix $L := L^\lambda, \check{I}_x := \check{I}_x$, the iteration $P|L, \check{I}_x := P\lambda|L, \check{I}_x$ and $L' := L_\lambda = \{x \in L \mid x < \lambda\}$. Note that the even ordinals $< \lambda$ are cofinal in $L'$.

Let $V' = V_0^{|L'}$ and, for $\eta < \lambda$ even, let $c_\eta$ be a Cohen real over the intermediate extension $V_0^{|L'}$ added at coordinate $\eta$ (this exists because Hechler forcing adds Cohen reals).

**Claim 6.8.** Let $\bar{\eta}$ be a relation as in Context 1.12 such that, for any $x \in L$ and $B \in \check{I}_x$, $P|L \models (\forall^{P|L} \bar{\eta})^\check{I}_x$. Then, $P|L$ forces $\exists \bar{\eta}$, $\forall \bar{\eta}$.

**Proof.** Let $\{\check{z}_\xi\}_{\xi < \nu}$, with $\nu < \lambda$, be a sequence of $P|L$-names of reals. In $V'$, by Theorem 1.13, for each $\xi < \nu$ there exists a set of reals $Y_\xi \in V'$ of size $< \theta$ such that, for any real $y \in V'$, if $y$ is $\check{I}_x$-unbounded over $Y_\xi$, then $\exists y \in V'|L, y \cap \check{z}_\xi$. Let $Y := \bigcup_{\xi < \nu} Y_\xi$, which clearly has size $< \lambda$. Then, as $P|L = \text{limdir}_{\nu < \lambda} P|^\nu|L_\nu$, there exists an $\eta < \lambda$ such that $Y \in V_\eta|L_\eta$, so, as $(\bar{\eta})^\eta$ is meager for any real $y$, $c_\eta$ is $\check{I}_x$-unbounded over $Y$. Therefore, $\exists y \in V_\eta|L_\eta, c_\eta \cap \check{z}_\xi$ for any $\xi < \nu$.

As the hypotheses of the Claim are satisfied for the relations $\bar{\alpha}$ and $\bar{\eta}$, $P|L$ forces that $\tau = \exists \bar{\eta} \geq \lambda$ and $non(\mathcal{N}) \geq \eta_\eta \geq \lambda$.

**Remark 6.9.** Shelah’s model discussed in Remark 1.12 satisfies $cov(\mathcal{N}) = s = g = \aleph_1 < add(\mathcal{M}) = cof(\mathcal{M}) = \mu < non(\mathcal{N}) = a = \tau = c = \lambda$ by the same arguments as in this section for the values of $g$, $cov(\mathcal{N})$, $non(\mathcal{N})$ and $\tau$. 

20
7 Questions

Question 7.1. Can we solve Problem 1.1(3) with respect to ZFC, that is, if \( \theta < \mu < \lambda \) are regular cardinals, is it consistent with ZFC that \( s = \theta < \mu = b < a = \lambda \)?

As Theorem 6.1 was an extension of Shelah’s argument for the consistency of \( d < a \) modulo a measurable, we can think about extending the isomorphism-of-names argument to our context in order to obtain a proof without the measurable. This argument works for Shelah’s proof of the consistency of \( \mathfrak{d} < a \) with ZFC alone because the template iteration for this, where only Hechler forcing is involved, has enough uniformity. But in the case of our question, we need to include Mathias forcing with a filter base of size \( < \theta \) in many coordinates of the template (as done in Section 6), but there, the iteration is not uniform enough to do an isomorphism-of-names argument.

Within the results in Section 6, we ask the following about Theorem 6.7.

Question 7.2. Is it consistent with ZFC and the existence of a measurable cardinal \( \kappa \) that add(\( N \)) = \( \theta_0 < \text{cov}(N) = \theta_1 < s = p = g = \theta < \kappa < b = \mathfrak{d} = \mu < a = r = \mathfrak{c} = \lambda \) with all these cardinals uncountable and regular?

Working in a model of ZFC + GCH + add(\( N \)) = \( \theta_0 < \text{cov}(N) = \theta_1 < s = p = g = \theta < \kappa < b = \mathfrak{d} = \mu < a = r = \mathfrak{c} = \lambda \) (this can be obtained by Theorem 5.2(a)), but techniques for an easier construction of this model can be found in [8], [20, Sect. 3] and [24 Thm. 4.1]), the idea of this would be to include suborders of \( \mathcal{A} \) of size \( < \theta_0 \) and suborders of \( \mathcal{B} \) of size \( < \theta_1 \) in the construction of the template iteration of Theorem 6.1. The only problem is that Theorem 4.13 does not work anymore to prove that add(\( N \)) \( \leq \theta_0 \) and cov(\( N \)) \( \leq \theta_1 \) are preserved in the final forcing extension. This is because a name of a real may not have a support of size \( < \theta_1 \) in any of the iterations.

A way to solve this would be to prove by induction on \( \alpha \leq \lambda \) that the hypothesis of Theorem 4.15 hold for the \( \alpha \)-th template iteration in the chain. This can be done for the basic and the successor steps, but the limit step for \( \delta \) with \( \text{cf}(\delta) < \theta \) is problematic.

Shelah’s proof of the consistency of \( \mathfrak{d} < a \) and \( u < a \) modulo a measurable involves an easier construction that does not appeal to templates but to iterations (forcing) equivalent to a fsi ([23], see also [10]). In this way, the construction of the chain of iterations of the results of Section 5 can be simplified, but we do not know if preservation results as in Section 4 can be proved in this simplification.

Question 7.3. Can we simplify the construction of the chain of iterations of Section 6 like in [24] and [10] and have a preservation result for this like the one for templates in Section 4?

Acknowledgements. I am very thankful with professor J. Brendle for all the guidance and support provided during the research that precedes this paper. He kindly taught me Shelah’s theory of iterated forcing along a template and offered a lot of his time for discussions that concluded in the results that are presented in this text. Professor Brendle also provided help with revisions, proof reading and language corrections of the material of this paper.

References

[1] J. Baumgartner, P. Dordal, Adjoining dominating functions, J. Symbolic Logic 50, no. 1 (1985) 94-101.
[2] T. Bartoszynski, H. Judah, Set theory: on the structure of the real line, A. K. Peters, Massachusetts, 1995.
[3] T. Bartoszynski, Invariants of measure and category, in: A. Kanamori, M. Foreman (Eds.), Handbook of Set-Theory, Springer, Heidelberg, 2010, pp. 491-555.
[4] A. Blass, Applications of superperfect forcing and its relatives, in: J. Steprans, S. Watson (Eds.), Set Theory and Its Applications, in: Lecture Notes in Math. 1401, Springer-Verlag, Berlin, 1989, pp. 18-40.
[5] A. Blass, Combinatorial cardinal characteristics of the continuum, in: A. Kanamori, M. Foreman (Eds.), Handbook of Set-Theory, Springer, Heidelberg, 2010, pp. 395-490.
[6] A. Blass, S. Shelah, Ultrafilters with small generating sets, Israel J. Math. 65 (1984) 259-271.
[7] J. Brendle, Forcing and the structure of the real line: the Bogotá lectures, lecture notes, 2009.
[8] J. Brendle, Larger cardinals in Cichon’s diagram, J. Symb. Logic 56, no. 3 (1991) 795-810.
[9] J. Brendle, *Mad families and iteration theory*, in: Logic and Algebra, Y. Zhang (Ed.), Contemp. Math. 302, Amer. Math. Soc., Providence, RI, 2002, pp. 1-31.

[10] J. Brendle, *Mad families and ultrafilters*, AUC 49 (2007) 19-35.

[11] J. Brendle, *Measure, category and forcing theory*, preprint.

[12] J. Brendle, *Mob families and mad families*, Arch. Math. Logic 37 (1998) 183-197.

[13] J. Brendle, *Shattered iterations*, in preparation.

[14] J. Brendle, *Templates and iterations, Luminy 2002 lecture notes*, Kyōto daigaku sūrikaiseki kenkyūsho kōkyūroku (2005) 1-12.

[15] J. Brendle, *The almost disjointness number may have countable cofinality*, Trans. Amer. Math. Soc. 355 (2003) 2633-2649.

[16] J. Brendle, V. Fischer, *Mad families, splitting families and large continuum*, J. Symb. Logic 76, no. 1 (2011) 198-208.

[17] J. Brendle, D. Raghavan, *Bounding, splitting and almost disjointness*, preprint.

[18] T. Jech, *Set theory*, 3rd millenium edition, Springer, Heidelberg, 2002.

[19] K. Kunen, *Set theory: an introduction to independence proofs*, North-Holland, Amsterdam, 1980.

[20] D. A. Mejía, *Matrix iterations and Cichon’s diagram*, Arch. Math. Logic 52 (2012) 261-278.

[21] D. A. Mejía, *Models of some cardinal invariants with large continuum*, Kyōto daigaku sūrikaiseki kenkyūsho kōkyūroku, to appear.

[22] S. Shelah, *On cardinal invariants of the continuum*, Contemp. Math. 31 (1984) 184-207.

[23] S. Shelah, *Two cardinal invariants of the continuum (θ < a) and FS linearly ordered iterated forcing*, Acta Math. 192 (2004) 187-223 (publication number 700).