Alternating-sum statistics for certain sets of integers

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Abstract. We introduce a class of set families that includes the collection of primitive sets, pairwise coprime sets, and product-free sets. If \( \mathcal{F} \) is a set family in our class, we let \( F_{n,k} \) be the number of elements in \( \mathcal{F} \cap \mathcal{P} \) with cardinality exactly \( k \) and show that \( \sum_{k=0}^{n} (-1)^k F_{n,k} = K \mathcal{F} \), where \( K \mathcal{F} \) is a constant depending on the family \( \mathcal{F} \) but not on \( n \). This constant equals \(-1\) in the case of primitive sets and 0 in the case of pairwise coprime sets as well as the case of product-free sets. We generalise primitive sets by saying that a set is \( s \)-multiple if it does not contain more than \( s \) multiples of any integer. We show that if \( P_{s,n,k} \) is the number of \( s \)-multiple subsets of \( \{1,2,\ldots,n\} \) with cardinality \( k \), then
\[
\sum_{k=0}^{n} (-1)^k P_{s,n,k} = (-1)^s \binom{n-2}{s-1},
\]
generalising the identity above in the special case of primitive sets.

Keywords. Primitive sets, product-free sets, M"obius function, lattices.

1. Introduction

Prime numbers form the multiplicative building blocks of the ring of integers. The set \( \{2,3,5,7,11,\ldots\} \) of primes possesses many pleasant properties that may be generalised to other sets of integers. For instance, a set \( S \) of integers is said to be

i) \textit{primitive} if for any two distinct \( i,j \in S \), \( i \) does not divide \( j \);

ii) \textit{pairwise coprime} if any two distinct \( i,j \in S \) have \( \gcd(i,j) = 1 \); and

iii) \textit{product-free} if for any \( i,j \in S \) not necessarily distinct, \( ij \notin S \).

It is easily seen that the set of prime numbers satisfies the requirements for all three of these definitions, and of course, so does any finite subset of the primes. Denote \( \{1,2,\ldots,n\} \) by \([n]\) for short and \( P_n \) be the number of primitive subsets of \([n]\). Likewise, let \( Q_n \) be the number of pairwise coprime subsets of \([n]\) and let \( R_n \) be the number of product-free subsets of \([n]\). These quantities have been the subject of interest of several papers over the past thirty years. All of them were treated by a 1990 paper of P. J. Cameron and P. Erdős. They showed that

\[ c_1^n \leq P_n \leq c_2^n \]
where \( c_1 = 1.44967\ldots \) and \( c_2 = 1.59\ldots \),

\[ 2\pi(n)e^{(1/2+o(1))\sqrt{n}} \leq Q_n \leq 2\pi(n)e^{(2+o(1))\sqrt{n}} \]
where \( \pi \) is the prime-counting function. This result was subsequently improved to

\[
Q(n) = 2^{\pi(n)} e^{\sqrt{n}(1+O(\log \log x / \log x))}
\]

by N. J. Calkin and A. Granville [2]. The paper of Cameron and Erdős also gives the lower bound \( R_n \geq 2^{n-\sqrt{n}} \), and the authors note that any product-free sequence has upper density less than 1, but there are product-free sequences with density more than \( 1 - \epsilon \) for any \( \epsilon > 0 \).

Regarding primitive sets, Cameron and Erdős conjectured that the limit \( \lim_{n \to \infty} P(n)^{1/n} \) exists; this was proven by R. Angelo in 2017 [1]. A separate proof was given by N. McNew in 2021 [10], which also improves \( c_1 \) to \( 1.572939 \) and \( c_2 \) to \( 1.574445 \) in \( (\frac{3}{4}) \). Around the same time, an independent approach of H. Liu, P. P. Pach, and R. Palincza gave roughly the same upper bound and the slightly weaker base of \( 1.571068 \) in the lower bound [9].

A great deal of attention has been given to infinite primitive sets. In 1993, P. Erdős and Z. Zhang [4] proved that for any primitive set \( A \),

\[
\sum_{a \in A} \frac{1}{a \log a} \leq 1.84.
\]

The sum \( \sum_p 1/(p \log p) \) equals \( 1.6366 \ldots \) when \( p \) ranges over all primes, and Erdős conjectured that the bound he and Zhang gave could be improved to this value. This was proven very recently, in a 2022 preprint of J. D. Lichtman [8].

Let \( P_{n,k} \) be the number of primitive subsets of \( [n] \) of cardinality exactly \( k \), so that \( P_n = \sum_{k=0}^{n} P_{n,k} \). Define \( Q_{n,k} \) and \( R_{n,k} \) analogously. None of these quantities have any known formula, as far as the authors are aware. In this paper, we will show that

\[
\sum_{k=0}^{n} (-1)^k P_{n,k} = -1
\]

and

\[
\sum_{k=0}^{n} (-1)^k Q_{n,k} = \sum_{k=0}^{n} (-1)^k R_{n,k} = 0.
\]

We do this by first defining a class of families that encompasses these three examples. We then prove a more general theorem and show that the constant works out as stated in these three cases.

**Partition-intersecting families.** Let \( \mathcal{F} \) be a family of finite sets of positive integers. Let \( F_{n,k} \) be the number of elements of cardinality \( k \) in \( \mathcal{F} \cap 2^{[n]} \). In this paper we shall study the alternating sum \( \sum_{k=0}^{n} (-1)^k F_{n,k} \). Under certain conditions, satisfied by the three examples above, we are able to show that it equals a constant that depends on the family but not on \( n \). We shall say that \( \mathcal{F} \) is \( m \)-partition-intersecting if for all \( n \geq 2 \), the following three conditions are satisfied (condition (ii) does not depend on \( n \) but (i) and (iii) do). Fixing some \( n \), let \( \mathcal{F}_n = \mathcal{F} \cap 2^{[n]} \) for brevity and let \( M_n \) be the set of maximal elements in \( \mathcal{F}_n \). These are the elements \( S \in \mathcal{F}_n \) such that for all \( x \in [n] \setminus S \), \( S \cup \{x\} \notin \mathcal{F}_n \).
i) \([n] \notin F_n\).

ii) \(F\) is downward closed; that is, for every \(S \in F\), every subset \(T\) of \(S\) is also in \(F\).

iii) The set \(M_n \subseteq F_n\) of maximal elements can be partitioned into \(m\) disjoint nonempty classes 
\(M_n = C_1 \sqcup \cdots \sqcup C_m\) such that for all \(i\), the intersection 
\(\bigcap_{S \in C_i} S\) is nonempty, and for all \(i \neq j\) and any \(S \in C_i\) and \(T \in C_j\), 
\(S \cap T = \emptyset\).

We now state the main theorem.

**Theorem 1.** Let \(F\) be \(m\)-partition-intersecting and let \(n \geq 2\). For \(0 \leq k < n\), 
let \(F_{n,k}\) be the number of \(S \in F_n = F \cap 2^{[n]}\) with \(|S| = k\). Then

\[
\sum_{k=0}^{n-1} (-1)^k F_{n,k} = 1 - m. \quad (7)
\]

We prove the theorem in the following section using some results from the theory of partially ordered sets. In the third section, we shall apply it to the three aforementioned families of sets. In the fourth and final section, we introduce a generalisation of primitive sets; namely, we say that a set is \(s\)-multiple if it does not contain more than \(s\) multiples of any integer. These families are not partition-intersecting for any \(s \geq 2\), but we extend the techniques used to prove Theorem 1 to show that the corresponding alternating sum for each of these families of sets equals \((-1)^s \binom{n-2}{s-1}\).

2. Proof of the main theorem

The proof of Theorem 1 is relatively straightforward, but relies on a number of constructions and known results that we must set up beforehand. For this section, we carry over notation established in the introduction, and fix an integer \(n \geq 2\) as well as a family \(F\) of finite subsets of integers. Consider the partially ordered set (poset) obtained by taking \(F_n\) as a ground set and ordering its elements by inclusion. This poset has bottom element \(\emptyset\), but since \([n] \notin F_n\) and \(n \geq 2\), it does not have a top element, so we shall adjoin an artificial one, denoted \(\hat{1}\), to obtain a lattice \(L_n = F_n \cup \{\hat{1}\}\).

The Möbius function \(\mu_X\) of a poset \(X\) is defined on pairs of elements \((x,y) \in X^2\) with \(x \leq y\), and is given by the recursive formula \(\mu_X(x,x) = 1\) for all \(x \in X\) and

\[
\mu_X(x,y) = - \sum_{x \leq z < y} \mu_X(x,z). \quad (8)
\]

This function generalises the counting method used to derive the inclusion-exclusion formula; indeed, letting \(B_n\) denote the lattice of all subsets of \([n]\), ordered by inclusion, one checks that \(\mu_{B_n}(\emptyset,S) = (-1)^{|S|}\) for any \(S \in B_n\). This fact is used to prove the following easy lemma.
Lemma 2. Let \( F_n \) be any set of subsets of \([n]\) that does not contain \([n]\) itself and is downward closed. Let \( L_n \) be the lattice obtained by adjoining a top element to this set system and let \( F_{n,k} \) be the number of elements of cardinality \( k \) in \( F_n \). Then
\[
\sum_{k=1}^{n-1} (-1)^k F_{n,k} = -\mu_{L_n}(\emptyset, \hat{1}).
\]

Proof. Since \( F_n \) is downward closed, the induced subposet
\[
\{ z \in L_n : 0 \leq x \leq z \}
\]
is isomorphic to the Boolean lattice \( B_{|x|} \) and we have \( \mu_{L_n}(0, x) = (-1)^{|x|} \). Then since \([n] \notin F_n\), we have
\[
\sum_{k=1}^{n-1} (-1)^k F_{n,k} = \sum_{x \in L_n \setminus \{\hat{1}\}} \mu_{L_n}(\emptyset, x) = -\mu_{L_n}(\emptyset, \hat{1}).
\]

Cross-cuts. A chain in a poset \( X \) is a set \( \{x_1, x_2, \ldots, x_k\} \subseteq X \) such that \( x_1 < x_2 < \cdots < x_k \). A chain is maximal if adding any new element to the chain causes the chain property to be violated. Let \( L \) be a lattice with top element \( \hat{1} \) and bottom element \( \hat{0} \). A cross-cut \( C \) is a subset of \( L \setminus \{\hat{1}, \hat{0}\} \) such that no two elements of \( C \) are comparable and every chain in \( L \) contains some element of \( C \).

A subset \( S \subseteq L \) is said to be spanning if the join of all its elements is \( \hat{1} \) and the meet of all its elements is \( \hat{0} \). We derive a simplicial complex \( \Delta(C) \) from any cross-cut \( C \) by letting the vertices be \( C \) itself and adding a face for any subset of \( C \) that is not spanning. (For us, a simplicial complex is a set of subsets of a finite set that is closed under intersection.) This topological space is germane to our study of the M"obius function because of the following theorem of J. Folkman [5], which relates its homological data to the M"obius function. From here on out, when we speak of homology of a space \( X \) and write \( \tilde{H}_k(X) \) for some \( k \geq 0 \), we mean reduced homology over \( \mathbb{Z} \).

Theorem A (Folkman, 1966). Let \( L \) be a lattice with bottom element \( \hat{0} \) and top element \( \hat{1} \). For any cross-cut \( C \) of \( L \),
\[
\mu_L(\hat{0}, \hat{1}) = \sum_{k=0}^{n} (-1)^k \text{rank } \tilde{H}_k(\Delta(C)),
\]
where \( \tilde{H}_k(\Delta(C), \mathbb{Z}) \) is the \( k \)th reduced homology group of \( \Delta(C) \).

An element \( y \) is said to cover an element \( x \neq y \) if \( x \leq y \) and \( x \leq z \leq y \) implies that \( z = x \) or \( z = y \). An element covered by \( \hat{1} \) is called a coatom. The set of coatoms in a lattice is a cross-cut; this is a direct consequence of definitions.

We are now able to prove Theorem 1. The proof is similar to one used by M. K. Goh, J. Hamdan, and J. Saks [6] to show that the M"obius function
of the poset of all arithmetic progressions contained in \([n]\) equals the number-theoretic Möbius function evaluated at \(n - 1\). Recall that \(\mathcal{F}\) is assumed to be \(m\)-partition-intersecting in this proof.

**Proof of Theorem 1.** Let \(M_n\) be the set of maximal elements of \(\mathcal{F}_n\); these are exactly the coatoms of \(L_n\). In view of Lemma 2 and Theorem A, we are done if we can show that

\[
\sum_{k=0}^{n} (-1)^k \text{rank } \tilde{H}_k(\Delta(M_n), \mathbb{Z}) = m - 1. \tag{12}
\]

By hypothesis, \(M_n = C_1 \sqcup C_2 \sqcup \cdots \sqcup C_m\) where each \(C_i\) is nonempty, and for all \(i\) and all subsets \(C'\) of \(C_i\), the intersection \(\bigcap_{S \in C'} S\) is nonempty. This tells us that for any \(i\), any subset of \(C_i\) is not spanning and is thus a face in \(\Delta(M_n)\). In other words, the simplicial complex \(\Delta(M_n)\) is always the union of \(m\) simplices, which has \(\tilde{H}_0(\Delta(M_n), \mathbb{Z}) = \mathbb{Z}^{m-1}\) and \(\tilde{H}_k(\Delta(M_n), \mathbb{Z}) = \{0\}\) for all \(k > 0\). This completes the proof.

### 3. Primitive, pairwise coprime, and product-free sets

Let \(n \geq 2\) be an integer and for \(0 \leq k < n\), recall that we let \(P_{n,k}\) be the number of primitive subsets of \([n]\) of cardinality \(k\), \(Q_{n,k}\) be the number of pairwise coprime subsets of \([n]\) of cardinality \(k\), and \(R_{n,k}\) be the number of product-free subsets of \([n]\) of cardinality \(k\). The following proposition lists some simple facts about these quantities in special cases.

**Proposition 3.** For \(n \geq 2\) we have

\begin{enumerate}
  \item \(P_{n,1} = Q_{n,1} = n\) and \(R_{n,1} = n - 1\);
  \item \(P_{n,2} = \sum_{i=2}^{n} (i - \text{d}(i))\), where \(\text{d}(i)\) counts the number of divisors of \(i\);
  \item \(Q_{n,2} = \sum_{i=2}^{n} \phi(i)\), where \(\phi\) is Euler's totient function;
  \item \(R_{n,2} = \binom{n}{2} - n - \lfloor \sqrt{n} \rfloor + 2\); and
  \item for any \(p\) prime and \(k \geq 3\), \(F(p, k) = F(p - 1, k) + F(p - 1, k - 1)\), where \(F\) can be any of \(P\), \(Q\), or \(R\).
\end{enumerate}

**Proof.** Assertion (i) is obvious, as are assertions (ii) and (iii) once one notes that in both cases the sum runs over the larger element in each 2-element set. To prove (iv), we start with all 2-element subsets of \([n]\), then remove the \(n - 1\) subsets containing 1 as well as the \(\lfloor \sqrt{n} \rfloor - 1\) subsets that contain a square and its square root.

For the final assertion, note that every primitive \(k\)-subset of \([p]\) either contains \(p\) or it does not. In the first case, it is counted by \(P_{p-1,k}\), and in the second case, it is counted by \(P_{p-1,k-1}\), since we can add \(p\) to any primitive \((k-1)\)-subset of \([p-1]\) without violating primitivity (for this we need \(k \geq 3\), otherwise the \((k-1)\)-subset could be \(\{1\}\)). The same argument works in the case of relatively coprime and product-free sets, since \(p\) is coprime to all smaller integers.
Values of $P_{n,k}$, $Q_{n,k}$, and $R_{n,k}$ for small $n$ and $k$ are presented in the appendix. We now state and prove the corollary to Theorem 1, as promised in the introduction.

**Corollary 4.** For $n \geq 2$ and $2 \leq k < n$ we have

i) $\sum_{k=0}^{n} (-1)^k P_{n,k} = -1$;

ii) $\sum_{k=0}^{n} (-1)^k Q_{n,k} = 0$; and

iii) $\sum_{k=0}^{n} (-1)^k R_{n,k} = 0$.

**Proof.** The interval $[n]$ is neither primitive, nor pairwise coprime, nor product-free. Furthermore, these three properties are all closed under taking subsets, so the families are downward closed. Thus we only need to show that condition (iii) in the definition of $m$-partition-intersecting holds with $m = 2$ in the case of primitive sets, and $m = 1$ in the other two cases. From here Theorem 1 does the trick. Fix $n \geq 2$ and let $p$ be the largest prime less than or equal to $n$. By the famous 1852 theorem of P. Chebyshev [3], we have $2^p > n$. This will be important in two of the cases.

We deal with primitive sets first. In this case, the collection $C$ of maximal primitive sets contains the singleton $\{1\}$, but no other element of $C$ contains 1, since 1 divides every other positive integer. So we set $C_1 = \{\{1\}\}$ and $C_2 = C \setminus \{1\}$. To any subset of $\{2,3,\ldots,n\}$, we may add the element $p$ without violating primitivity, since no element divides $p$ and $2^p > n$. Thus every element of $C_2$ contains $p$ and we see that $F$ is 2-partition-intersecting in this case.

For the family of pairwise coprime sets, every maximal set contains 1, since $\gcd(1,i) = 1$ for all $1 \leq i \leq n$. So we have $C_1 = C$ and $F$ is 1-partition-intersecting.

Lastly, we note that no product-free set contains 1, since $1 \cdot 1 = 1$, and every maximal product-free subset of $[n]$ must also contain $p$, since $p$ is not the product of any two elements of $\{2,3,\ldots,n\}$ and $2^p > n$. Hence again we have $C_1 = C$ and $F$ is 1-partition-intersecting in this case as well.

We have given only three examples, but the definition of an $m$-partition-intersecting family is fairly permissive, and we anticipate our result to be applicable to a broad class of set systems. Note that if $m$ is a constant not depending on $n$, then $m$ must either be 1 or 2, since there are at most 2 maximal elements of $F_2$. One could imagine extending the definition to let $m$ vary as a function of $n$, which may allow even more interesting set families to be treated.

4. A generalisation of primitive sets

A primitive set is a set that does not contain more than one multiple of any integer. In this section we generalise the condition to forbid having more than $s$ multiples of an integer. We will refer to sets satisfying this condition as $s$-multiple sets, and when $s = 1$ we recover the definition of a primitive set. In the more general case, we still have the pleasing property that any maximal $s$-multiple set intersects $\{1,p\}$. The following theorem gives an alternating-sum identity for $s$-multiple subsets of $[n]$. 
**Theorem 5.** Let \( s \geq 2 \) be an integer, let \( n > s \), and let \( P_{s,n,k} \) denote the number of \( s \)-multiple subsets of \([n]\) with cardinality exactly \( k \). We have

\[
\sum_{k=0}^{n} (-1)^k P_{s,n,k} = (-1)^s \binom{n-2}{s-1}.
\]

We shall prove this theorem after proving a slightly technical lemma. For the rest of the section, fix \( s \geq 2 \) and \( n > s \), let \( \mathcal{F}_{s,n} \) denote the set of \( s \)-multiple subsets of \([n]\), and let \( L_{s,n} \) be this set with a top element adjoined so as to make it a lattice under inclusion. Let \( \Delta \) be the simplicial complex obtained from the cross-cut of coatoms in \( L_{s,n} \). The vertices in this complex are maximal \( s \)-multiple subsets of \([n]\), with a face for every subset of the vertices that have nonempty intersection. Note that unlike in the primitive \((s=1)\) case, there are now vertices in the complex containing 1 and \( i \) for any \( 1 < i \leq n \), which shows that \( \Delta \) is connected.

We make use of the Mayer-Vietoris sequence, which for reduced homology says that

\[
\cdots \to \tilde{H}_s(A) \oplus \tilde{H}_s(B) \to \tilde{H}_s(X) \to \tilde{H}_{s-1}(A \cap B) \to \tilde{H}_{s-1}(A) \oplus \tilde{H}_{s-1}(B) \to \cdots \to \tilde{H}_0(A \cap B) \to \tilde{H}_0(A) \oplus \tilde{H}_0(B) \to \tilde{H}_0(X) \to 0,
\]

whenever \( A \) and \( B \) are topological spaces whose union is \( X \) and whose intersection is nonempty (see, e.g., p. 150 of [7]).

We will apply it in the case \( X = \Delta \), so we need to make a choice of \( A \) and \( B \). Let \( A = \{ v \in \Delta : 1 \in v \} \) be the simplex of all vertices containing 1. Let \( B \) be the smallest simplicial subcomplex of \( \Delta \) such that \( A \cup B = \Delta \).

**Lemma 6.** For all \( t \geq 0 \), we have \( \tilde{H}_t(A) \oplus \tilde{H}_t(B) \cong \{0\} \).

**Proof.** Being a simplex, it is clear that \( A \) has all homology groups zero. Next we show that \( \tilde{H}_{s-1}(B) \cong \{0\} \). Let \( S \) be a set of vertices in \( B \) that forms an \((s-1)\)-cycle and suppose further that \( v \in A \cap B \). We claim there is some \( w \neq v \) in \( S \cap A \) as well. In other words, there are no \((s-1)\)-cycles in \( \Delta \) consisting of exactly one element \( A \), except for those cycles that are the boundary of a simplex. Let

\[
v = \{1, x_1, x_2, \ldots, x_{s-1}\}, b_1, b_2, \ldots, b_s
\]

be elements of \( B \), where for all \( 1 \leq i \leq t \), \( b_i \) does not contain 1 (so it contains \( p \)). Suppose these \( s+1 \) vertices form an \((s-1)\)-cycle, so that if we let

\[
S_i = v \cap b_1 \cap \cdots b_{i-1} \cap b_{i+1} \cap \cdots \cap b_s,
\]

then \( S_i \neq \emptyset \) for all \( 1 \leq i \leq s \). The claim is that this \((s-1)\)-cycle is the boundary of an \( s \)-simplex. Note that none of these \( S_i \) can contain 1, so each \( S_i \) must contain some element of \( \{x_1, \ldots, x_{s-1}\} \). But there are \( s \) different sets \( S_i \) and
only \( s - 1 \) choices for \( x_j \), so by the pigeonhole principle there must be \( i \neq j \) such that \( S_i \cap S_j = \emptyset \). We have shown that

\[
v \cap b_1 \cap \cdots \cap b_s \neq \emptyset,
\]

so that \( \{v, b_1, \ldots, b_s\} \) is a face in \( B \).

Thus the only \((s - 1)\)-cycles in \( \Delta \) that contain vertices from \( B \) contain either no vertices of \( A \), or at least two. In the first case, it is the boundary of an \( s \)-simplex, since every vertex of \( \Delta \setminus A \) contains \( p \). In the second case, denote the vertices of the cycle that also belong to \( A \) by \( v_1, \ldots, v_r \), so that \( r \geq 2 \). The face \( \{v_1, \ldots, v_r\} \) belongs to \( A \) already, since all these vertices contain \( 1 \). Hence if this face were also in \( B \), then it and all its sub-faces of positive dimension can be removed from \( B \), contradicting minimality of \( B \). Hence this is not an \((s - 1)\)-cycle in \( B \). We conclude that \( B \) does not contain an \((s - 1)\)-cycle that is not the boundary of an \( s \)-simplex and consequently \( H_{s-1}(B) \cong \{0\} \).

For \( t > s - 1 \), the same reasoning can be used to show that every \( t \)-cycle that contains exactly one vertex in \( A \cap B \) must be the boundary of an \((t + 1)\)-simplex, and thus \( H_t(B) \cong \{0\} \) as well.

Now for \( t < s - 1 \), it is possible for a \( t \)-cycle to have exactly one vertex in \( A \), but we show that this also causes no issue. Let

\[
v, b_1, b_2, \ldots, b_{t+1}
\]

be the vertices of the \( t \)-cycle in question; so none of the \( b_i \) contain \( 1 \). If \( v \cap b_1 \cap \cdots \cap b_{t+1} \) is nonempty, then this is the boundary of a simplex and we are done. If not, then for all \( 1 \leq i \leq t + 1 \), we can pick

\[
y_i \in v \cap b_1 \cap \cdots \cap b_{i-1} \cap b_{i+1} \cap \cdots \cap b_{t+1}
\]

such that \( y_i \notin b_i \). The set \( \{y_1, \ldots, y_{t+1}\} \) does not contain \( 1 \) and it has at most \( t + 1 \leq s - 1 \) multiples of any element, just by virtue of its cardinality, so it is a subset of some maximal element \( b \in B \) that is not equal to any of the \( b_i \) by construction. Then the set \( \{v, b_1, b_2, \ldots, b_{t+1}, b\} \) forms a \((t + 2)\)-simplex, missing its interior as well as one of its \((t + 1)\)-faces. The \( t \)th homology of such an object is the same as the \( t \)th homology of the \((t + 2)\)-simplex, namely zero, therefore it does not contribute anything to the \( t \)th homology group of \( B \).

Proof of Theorem 5. The condition on \( n \) ensures that \([n]\) is not \( s \)-multiple, and any subset of an \( s \)-multiple set is also \( s \)-multiple. Thus the set \( \mathcal{F}_{s,n} \) of all \( s \)-multiple subsets of \([n]\) is downward closed and does not contain \([n]\), and in light of Lemma 2 and Theorem A, we need only show that

\[
\sum_{k=0}^{n} (-1)^k \text{rank } H_k(\Delta, \mathbb{Z}) = (-1)^{s-1} \binom{n-2}{s-1},
\]

(19)
where $\Delta$ is the cross-cut consisting of the coatoms of $L_n$ (the maximal elements in $\mathcal{F}_n$).

As above, we split $\Delta = A \cup B$, where $A = \{v \in \Delta : 1 \in v\}$ and $B$ be the smallest simplicial subcomplex of $\Delta$ such that $A \cup B = \Delta$. The complex $A \cap B$ has a vertex $1 \cup S$ for every $S \subseteq \{2, \ldots, n\}$ of size $s - 1$, with a face for every subset of these vertices whose intersection contains an element other than $1$. This is isomorphic to the simplicial complex $Y$ whose vertices are elements of $[n-1]^{(s-1)}$, with a face for every collection of vertices with nonempty intersection. For every $1 \leq i < n$, it is easy to see that the set

$$ f_i = \{S \in [n-1]^{(s-1)} : i \in S\} $$

is a maximal face, and in fact, these are the only maximal faces. Any set of $s - 1$ distinct maximal faces $f_{i_1}, \ldots, f_{i_{s-1}}$ has nonempty intersection (it contains, in particular, the element $\{i_1, \ldots, i_{s-1}\}$), But any $s$ distinct maximal faces cannot have any intersection. Contracting each of these faces down to a point does not change the homology, and the reasoning above shows that we are left with $n - 1$ points, every $s - 1$ of which belong to a maximal face. This is the $(s-2)$-skeleton of an $(n-2)$-simplex, which we shall call $X$. We have $\tilde{H}_t(A \cap B) = \tilde{H}_t(X)$ for all $t$, and it is a well-known fact that

$$ \text{rank } \tilde{H}_t(X) = \begin{cases} \binom{n-2}{s-1}, & \text{if } t = s - 1; \\ 0, & \text{otherwise.} \end{cases} \quad (20) $$

(See, for example, exercise 16 on p. 156 of [7].)

For all $t \geq 1$, the relevant section of the Mayer-Vietoris sequence is

$$ \cdots \to \tilde{H}_t(A) \oplus \tilde{H}_t(B) \to \tilde{H}_t(\Delta) \to \tilde{H}_{t-1}(A \cap B) \to \tilde{H}_{t-1}(A) \oplus \tilde{H}_{t-1}(B) \to \cdots, \quad (21) $$

and by Lemma 6, we can fill this sequence in to get

$$ \cdots \to 0 \to \tilde{H}_t(\Delta) \to \tilde{H}_{t-1}(A \cap B) \to 0 \to \cdots \quad (22) $$

for all $t \geq 1$. Thus we have an isomorphism $\tilde{H}_t(\Delta) \cong \tilde{H}_{t-1}(A \cap B)$ for all $t \geq 1$. This, along with the fact that $\Delta$ is connected, shows that

$$ \text{rank } \tilde{H}_t(\Delta) = \begin{cases} \binom{n-2}{s-1}, & \text{if } t = s - 1; \\ 0, & \text{otherwise.} \end{cases} \quad (23) $$

Hence the alternating sum in (19) works out to $(-1)^{s-1} \binom{n-2}{s-1}$, which is what we needed.

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A. Numerical tables

This appendix contains values of $P_{n,k}$, $Q_{n,k}$, and $R_{n,k}$ for small values of $n$ and $k$. The row sums of $P_{n,k}$, which we called $P_n$ in the introduction, appears as A051026 in the On-line Encyclopedia of Integer Sequences (OEIS). The row sums of $Q_{n,k}$ and $R_{n,k}$ were referred to as $Q_n$ and $R_n$ in the introduction and appear as A084422 and A326489, respectively. The triangular numbers $P_{n,k}$, $Q_{n,k}$, and $R_{n,k}$ are A355145, A355146, and A355147 in the OEIS, respectively.
**Table 1**
THE NUMBER $P_{n,k}$ OF PRIMITIVE $k$-SUBSETS OF $[n]$

| $n$ | $P_{n,0}$ | $P_{n,1}$ | $P_{n,2}$ | $P_{n,3}$ | $P_{n,4}$ | $P_{n,5}$ | $P_{n,6}$ | $P_{n,7}$ | $P_{n,8}$ | $P_{n,9}$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1   | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 2   | 1         | 2         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 3   | 1         | 3         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 4   | 1         | 4         | 2         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 5   | 1         | 5         | 5         | 2         | 0         | 0         | 0         | 0         | 0         | 0         |
| 6   | 1         | 6         | 7         | 3         | 0         | 0         | 0         | 0         | 0         | 0         |
| 7   | 1         | 7         | 12        | 10        | 3         | 0         | 0         | 0         | 0         | 0         |
| 8   | 1         | 8         | 16        | 15        | 5         | 0         | 0         | 0         | 0         | 0         |
| 9   | 1         | 9         | 22        | 26        | 13        | 2         | 0         | 0         | 0         | 0         |
| 10  | 1         | 10        | 28        | 38        | 22        | 4         | 0         | 0         | 0         | 0         |
| 11  | 1         | 11        | 37        | 66        | 60        | 26        | 4         | 0         | 0         | 0         |
| 12  | 1         | 12        | 43        | 80        | 76        | 35        | 6         | 0         | 0         | 0         |
| 13  | 1         | 13        | 54        | 123       | 156       | 111       | 41        | 60        | 0         | 0         |
| 14  | 1         | 14        | 64        | 161       | 227       | 180       | 74        | 12        | 0         | 0         |
| 15  | 1         | 15        | 75        | 206       | 323       | 299       | 161       | 47        | 6         | 0         |
| 16  | 1         | 16        | 86        | 253       | 425       | 421       | 242       | 75        | 10        | 0         |
| 17  | 1         | 17        | 101       | 339       | 678       | 846       | 663       | 317       | 85        | 10        |

**Table 2**
THE NUMBER $Q_{n,k}$ OF PAIRWISE COPRIME $k$-SUBSETS OF $[n]$

| $n$ | $Q_{n,0}$ | $Q_{n,1}$ | $Q_{n,2}$ | $Q_{n,3}$ | $Q_{n,4}$ | $Q_{n,5}$ | $Q_{n,6}$ | $Q_{n,7}$ | $Q_{n,8}$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1   | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 2   | 1         | 2         | 1         | 0         | 0         | 0         | 0         | 0         | 0         |
| 3   | 1         | 3         | 3         | 1         | 0         | 0         | 0         | 0         | 0         |
| 4   | 1         | 4         | 5         | 2         | 0         | 0         | 0         | 0         | 0         |
| 5   | 1         | 5         | 9         | 7         | 2         | 0         | 0         | 0         | 0         |
| 6   | 1         | 6         | 11        | 8         | 2         | 0         | 0         | 0         | 0         |
| 7   | 1         | 7         | 17        | 19        | 10        | 2         | 0         | 0         | 0         |
| 8   | 1         | 8         | 21        | 25        | 14        | 3         | 0         | 0         | 0         |
| 9   | 1         | 9         | 27        | 37        | 24        | 6         | 0         | 0         | 0         |
| 10  | 1         | 10        | 31        | 42        | 26        | 6         | 0         | 0         | 0         |
| 11  | 1         | 11        | 41        | 73        | 68        | 32        | 6         | 0         | 0         |
| 12  | 1         | 12        | 45        | 79        | 72        | 33        | 6         | 0         | 0         |
| 13  | 1         | 13        | 57        | 124       | 151       | 105       | 39        | 6         | 0         |
| 14  | 1         | 14        | 63        | 138       | 167       | 114       | 41        | 6         | 0         |
| 15  | 1         | 15        | 71        | 159       | 192       | 128       | 44        | 6         | 0         |
| 16  | 1         | 16        | 79        | 183       | 228       | 157       | 56        | 8         | 0         |
| 17  | 1         | 17        | 95        | 262       | 411       | 385       | 213       | 64        | 8         |
Table 3
THE NUMBER $R_{n,k}$ OF PRODUCT-FREE $k$-SUBSETS OF $[n]$

| $n$ | $R_{n,0}$ | $R_{n,1}$ | $R_{n,2}$ | $R_{n,3}$ | $R_{n,4}$ | $R_{n,5}$ | $R_{n,6}$ | $R_{n,7}$ | $R_{n,8}$ | $R_{n,9}$ |
|-----|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 1   | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 2   | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 3   | 1         | 2         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 4   | 1         | 3         | 2         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| 5   | 1         | 4         | 5         | 2         | 0         | 0         | 0         | 0         | 0         | 0         |
| 6   | 1         | 5         | 9         | 6         | 1         | 0         | 0         | 0         | 0         | 0         |
| 7   | 1         | 6         | 14        | 15        | 7         | 1         | 0         | 0         | 0         | 0         |
| 8   | 1         | 7         | 20        | 29        | 22        | 8         | 1         | 0         | 0         | 0         |
| 9   | 1         | 8         | 26        | 43        | 38        | 17        | 3         | 0         | 0         | 0         |
| 10  | 1         | 9         | 34        | 68        | 76        | 47        | 15        | 2         | 0         | 0         |
| 11  | 1         | 10        | 43        | 102       | 144       | 123       | 62        | 17        | 2         | 0         |
| 12  | 1         | 11        | 53        | 143       | 234       | 238       | 149       | 55        | 11        | 1         |