On the Performance of Downlink MIMO-ISAC

Chongjun Ouyang, Yuanwei Liu, and Hongwen Yang

Abstract—This letter analyzes the performance of sensing and communications (S&C) achieved by a multiple-input multiple-output downlink integrated S&C (ISAC) system. Three typical ISAC scenarios are studied, including the sensing-centric design, communications-centric design, and Pareto optimal design. For each scenario, diversity orders and high signal-to-noise ratio slopes of the sensing rate and communication rate are derived to gain further insights. It is found that ISAC can provide more degrees of freedom and a broader rate region than existing frequency-division S&C (FDSAC) techniques.

Index Terms—Communications-sensing tradeoff, integrated sensing and communications (ISAC), performance analysis.

I. INTRODUCTION

Integrated sensing and communications (ISAC) is a promising enabler for the development of the six-generation (6G) and beyond wireless networks [1]. The main feature of ISAC is to allow communications and sensing to share the same time-frequency-power-hardware resources. Compared with existing frequency-division-sensing and communications (FDSAC) techniques, in which sensing and communications (S&C) require isolated frequency bands as well as hardware infrastructures, ISAC is envisioned to be more spectrum-, energy-, and hardware-efficient [1], [2]. Due to these attractive characteristics, ISAC has received considerable attention from both industry and the research community [1]–[6].

From the perspective of information, the S&C performance of ISAC systems is evaluated by two metrics including the sensing rate (SR) and the communication rate (CR) [2]. Specifically, the SR measures how much environmental information can be extracted from the sensing echoes, whereas the CR measures how much data information can be recovered from the received symbols [2]–[4]. These two metrics evaluate the fundamental information-theoretic limits of the S&C performance achieved by ISAC. With this in mind, the authors in [3] optimized the weighted sum of the CR and the SR in a downlink multiple-input multiple-output (MIMO) ISAC system with a single communication user terminal (UT). This work focused on the dual-functional S&C (DFSAC) precoding design and neglected the discussion of basic system insights. As an advance, the authors in [4] extended the work in [3] to a multiuser case and discussed the high signal-to-noise ratio (SNR) slopes of the CR and SR as well as the achievable SR-CR region. Though these two works have made great progress in understanding the SR and CR in ISAC systems, there are still many important unsolved problems.

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Fig. 1: Illustration of a downlink MIMO-ISAC system. For example, the Pareto boundary of the SR-CR region has not been characterized. For another example, a rigorous comparison between the SR-CR regions achieved by ISAC and FDSAC is still missing.

In light of the above facts and in order to develop a deeper understanding of the S&C performance in ISAC, this letter considers a downlink MIMO-ISAC system with multiple UTs. Particularly, we provide DFSAC precoding design for three scenarios, including the sensing-centric (S-C) design (maximizing the SR only), the communications-centric (C-C) design (maximizing the CR only), and the Pareto optimal design (characterizing the Pareto boundary of the SR-CR region). For each scenario, the diversity orders and high-SNR slopes of the CR and SR are derived to unveil important system insights. Moreover, we provide a rigorous comparison between the SR-CR regions achieved by ISAC and FDSAC. It is found that ISAC is capable of achieving a broader rate region as well as providing more degrees of freedom than FDSAC.

II. SYSTEM MODEL

Consider a downlink MIMO-ISAC system as shown in Fig. 1 where a DFSAC base station (BS) is communicating with a set of $M$ UTs, while simultaneously sensing the targets in its surrounding environment. The BS has $M$ transmit antennas and $N$ receive antennas and each UT $m \in \mathcal{M} = \{1, \ldots, M\}$ has $K$ receive antennas. Let $\mathbf{X} = [\mathbf{x}_1 \ldots \mathbf{x}_L] \in \mathbb{C}^{M \times L}$ be a DFSAC signal matrix, with $L$ being the length of the communication frame/radar pulse. From a sensing perspective, $\mathbf{x}_l \in \mathbb{C}^{M \times 1}$ for $l \in \mathcal{L} = \{1, \ldots, L\}$ represents the radar snapshot transmitted at the $l$th time slot. For communications, $\mathbf{x}_l$ is the $l$th data symbol vector. Under the framework of MIMO-ISAC, the signal matrix $\mathbf{X}$ can be written as

$$\mathbf{X} = \mathbf{W}\mathbf{S},$$

(1)

where $\mathbf{W} = [\mathbf{w}_1 \ldots \mathbf{w}_M] \in \mathbb{C}^{M \times M}$ is the precoding matrix subject to the power budget $\text{tr}(\mathbf{W}\mathbf{W}^H) \leq p$ and $\mathbf{S} = [\mathbf{s}_1 \ldots \mathbf{s}_M]^H \in \mathbb{C}^{M \times L}$ contains $M$ unit-power data streams intended for the $M$ UTs. Here, $\mathbf{s}_m^H \in \mathbb{C}^{1 \times L}$ and $\mathbf{w}_m \in \mathbb{C}^{M \times 1}$ denote the data stream dedicated for UT $m$ and the associated precoding vector, respectively. The data streams are assumed to be independent with each other so that $L^{-1}\mathbf{SS}^H \approx \mathbf{I}_M$ [5].
A. Communication Model

By transmitting $X$ to the UTs, the received signal matrix at UT $m$ is given by

$$Y_{c,m} = H_m w_m s_m^H + \sum_{m' \neq m} H_{m'} w_{m'} s_{m'}^H + N_{c,m},$$

(2)

where $H_m \in \mathbb{C}^{K \times M}$ is the communication channel matrix and $N_{c,m} \in \mathbb{C}^{K \times L}$ is the additive white Gaussian noise (AWGN) matrix. It is assumed that all the communication links shown in Fig. 1 suffer Rayleigh fading. In this case, $N_{c,m}$ and $H_m$ are mutually independent complex Gaussian matrices, whose elements are independent and identically distributed (i.i.d.) with zero mean and unit variance. After receiving $Y_{c,m}$, UT $m$ adopts a normalized equalizer $v_m \in \mathbb{C}^{K \times 1}$ to eliminate the inter-user interference. By assuming $K \geq M$, we have $v_m = \frac{p_m}{\|p_m\|}$, where $p_m$ denotes the $m$th column of matrix $P_m = (P^H_m P_m)^{-1}$ with $P_m = H_m W \in \mathbb{C}^{K \times M}$. Accordingly, the received SNR at UT $m$ is given as $\gamma_m = \|v_m^H H_m w_m\|^2$. In order to reduce system overhead caused by acquiring channel state information (CSI) at the BS, it is assumed that the BS does not have the global CSI and only $v_m^H H_m$ is fed back to the BS for power controlling. The sum CR is given by $R_c = \sum_{m=1}^{M} \log_2 (1 + \gamma_m)$.

B. Sensing Model

By transmitting $X$ to sense the targets, the BS observes the following reflected echo signal matrix at its receiver: [4] - [6]

$$Y_s = G X + N_s,$$

(3)

where $N_s \in \mathbb{C}^{N \times L}$ is the AWGN matrix with each entry having zero mean and unit variance, and $G = [g_1 \ldots g_N]^T \in \mathbb{C}^{N \times M}$ represents the target response matrix with $g_n \in \mathbb{C}^{M \times 1}$ for $n \in \mathcal{N} = \{1, \ldots, N\}$ representing the target response from the transmit antenna array to the $n$th receive antenna. The target response matrix is modeled as [3 - 6]

$$G = \sum_{i} \beta_i a(\theta_i) b^H(\theta_i),$$

(4)

where $\beta_i \sim \mathcal{CN}(0, \sigma_i^2)$ is the complex amplitude of the $i$th target with $\sigma_i^2$ representing the average strength, $a(\theta_i) \in \mathbb{C}^{N \times 1}$ and $b(\theta_i) \in \mathbb{C}^{M \times 1}$ are the associated receive and transmit array steering vectors, respectively, and $\theta_i$ is its direction of arrival. By considering that the receive antennas at the BS are widely separated, we have $g_n \sim \mathcal{CN}(0, R)$ for $n \in \mathcal{N}$ and $\mathbb{E}(g_n g_{n'}^H) = 0$ for $n \neq n'$. Moreover, the BS is assumed to know the correlation matrix $R \in \mathbb{C}^{M \times M}$.

Essentially, the aim of sensing is to extract environmental information contained in $G$, e.g., the direction and reflection coefficient of each target, from the reflected echo signal $Y_s$ [3 - 6]. Particularly, the mutual information (MI) between $Y_s$ and $G$ conditioned on the DFSAC signal $X$ characterizes the information-theoretic limits on how much environmental information can be extracted, which is also referred to as the sensing MI [6]. On this basis, from an information-theoretic perspective, we adopt the SR as the performance metric of sensing, which is defined as the sensing MI per unit time [2 - 4]. Assuming that each DFSAC symbol lasts 1 unit time, we write the SR as $R_s = L^{-1} I(Y_s; G|X)$, where $I(X; Y|Z)$ denotes the MI between $X$ and $Y$ conditioned on $Z$.

Given the MIMO-ISAC framework, we intend to analyze its S&C performance by investigating the CR $R_c$ and SR $R_s$. Note that both $R_c$ and $R_s$ are influenced by the precoding matrix $W$. Yet, it is challenging to find a $W$ that can maximize $R_s$ and $R_c$ simultaneously. As a compromise, we consider three typical scenarios to unveil further system insights. The first scenario is the S-C design that aims to maximize the SR, the second scenario is the C-C design that aims to maximize the CR, and the third scenario is the Pareto optimal design that aims to find the Pareto boundary of the rate region.

III. PERFORMANCE OF ISAC

A. Sensing-Centric Design

1) Performance of Sensing: Under the S-C design, the precoding matrix $W$ is set to maximize $R_s$. To proceed, we characterize the SR as follows.

**Lemma 1.** For a given $W$, the SR can be calculated as $R_s = L^{-1} \log_2 (I_M + LW^H RW)$.  

**Proof:** Please refer to Appendix [A] for more details.  

Under the S-C design, the precoding matrix satisfies $W_s = \arg \max_{\|W\|_F^2 \leq p} \log_2 (I_M + LW^H RW)$. (5)

For analytical tractability, we assume that $R > 0$. Then, the following theorem provides an exact expression for the SR as well as its high-SNR approximation.

**Theorem 1.** In the S-C design, the maximum SR is given by

$$R_s^* = NL^{-1} \sum_{m=1}^{M} \log_2 (1 + \lambda_m s_m^*),$$

(6)

where $\{\lambda_m > 0\}_{m=1}^{M}$ are the eigenvalues of matrix $R$ and $s_m^* = \max \left\{ 0, \frac{1}{\nu} - \frac{1}{\lambda_m} \right\}$ with $\sum_{m=1}^{M} \max \left\{ 0, \frac{1}{\nu} - \frac{1}{\lambda_m} \right\} = p$. The maximum SR is attained when $W_s W_s^H = U \Delta U^H$, where $U \text{diag} \{\lambda_1, \ldots, \lambda_M\} U^H$ denotes the eigendecomposition (ED) of $R$ and $\Delta = \text{diag} \{s_1, \ldots, s_M\}$. When $p \rightarrow \infty$, the SR satisfies

$$R_s^* \approx \frac{NM}{L} \left( \log_2 p + \frac{1}{M} \sum_{m=1}^{M} \log_2 \left( \frac{\lambda_m}{M} \right) \right).$$

(7)

**Proof:** Please refer to Appendix [B] for more details.

**Remark 1.** The results in (7) suggest that the high-SNR slope of the SR under the S-C design is given by $\frac{NM}{L}$.

2) Performance of Communications: Turn now to the communication performance. Particularly, the CR of UT $m$ is given by $R_c = \log_2 (1 + s_m \rho_m)$ with $\rho_m = \|v_m^H h_{m}\|^2$ and $h_m \in \mathbb{C}^{K \times 1}$ being the $m$th column of matrix $H_m \in \mathbb{C}^{K \times M}$. Specifically, we utilize the outage probability (OP) and ergodic CR (ECR) to evaluate the communication performance. The following theorem provides an exact expression for the sum ECR $R_c^*$ with $R_c^* = \sum_{m=1}^{M} R_c$ as well as its high-SNR approximation.

**Theorem 2.** In the S-C design, the sum ECR is given by

$$R_c^* = \sum_{m=1}^{M} \left[ \begin{array}{c} (K' - \mu) \left( -1/s_m^* \right)^{K' - \mu} e^{-\frac{1}{s_m^*}} \text{Ei} \left( \frac{1}{s_m^*} \right) \\ + \sum_{i=1}^{K' - \mu} (i - 1)! \left( -1/s_m^* \right)^{i-1} \end{array} \right],$$

(8)
where $K' = K - M$ and $E_1(x) = -\int_0^\infty e^{-t}t^{-1}dt$ denotes the exponential integral function [9] Eq. (8.211.1)]. When $p \to \infty$, the ECR satisfies

$$R_c^C \approx M(\log_2 p - \log_2 M + \psi(K' + 1) / \ln 2),$$  \hfill (9)$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the Digamma function [12] Eq. (6.461)] and $\Gamma(x) = \int_0^\infty t^{-x}e^{-t}dt$ is the gamma function [12] Eq. (6.1.1)].

Proof: Please refer to Appendix C for more details.

Remark 2. The results in (9) suggest that the high-SNR slope of the sum ECR under the S-C design is given by $M$.

It is challenging to derive a closed-form expression for the OP, i.e., $P_c^C = \Pr(R_c^C < R_0)$ with $R_0$ denoting the target sum CR. Thus, we focus more on its high-SNR properties.

Theorem 3. As $p \to \infty$, the OP of the sum CR achieved by the S-C design satisfies $P_c^C \approx O(p^{-M(K+M-1)})$. The notation $f(x) = O(g(x))$ means that $\limsup_{x \to \infty} \frac{|f(x)|}{g(x)} < \infty$.

Proof: Please refer to Appendix D for more details.

Remark 3. The above results suggest a diversity of $M(K - M + 1)$ is achievable for the OP under the S-C design.

B. Communications-Centric Design

Having investigated the S&C performance under the S-C design, we now move to the C-C design.

1) Performance of Communications: It is worth noting that $H_m$ has the same statistical properties as $H_mU$. In light of this fact as well as the conclusion drawn in Theorem 1, we design the C-C precoding matrix as $W = U\Delta_1^{1/2}$ with $\Delta_c = \text{diag}\{c_1, \ldots, c_M\}$, where $\sum_{m=1}^M c_m \leq p$ and $c_m \geq 0$ for $m \in M$. The resulting precoding matrix satisfies

$$W_c = \arg\max_{W = U\Delta_1^{1/2}} \sum_{m=1}^M \log_2 (1 + c_m \rho_m),$$

where $\log_2 (1 + c_m \rho_m)$ calculates the CR of UT m. Particularly, the following lemma provides an expression for the maximum sum CR achieved by $W_c$.

Lemma 2. Under the C-C design, the maximum sum CR is

$$R_c^C = \sum_{m=1}^M \log_2 (1 + c_m \rho_m),$$

where $c_m^* = \max\left\{0, \frac{1}{\rho_m} - \frac{1}{\rho_m}\right\}$ with $\sum_{m=1}^M c_m^* = p$. The maximum sum CR is attained when $c_m = c_m^*$ for $m \in M$.

Proof: This lemma can be directly proved by using the water-filling procedure [8].

We note that deriving a closed-form expression of the sum ECR $R_c^C = \mathbb{E}(R_c^C)$ is a hard job. To unveil more insights, we characterize its high-SNR behaviour in Theorem 4.

Theorem 4. For a sufficiently large value of $p$, the sum ECR achieved by the C-C design satisfies

$$R_c^C \approx M\log_2 p - \log_2 M + \psi(K' + 1) / \ln 2,$$  \hfill (12)$$

where $\psi(x) = \frac{d}{dx} \ln \Gamma(x)$ is the Digamma function [12] Eq. (6.461)] and $\Gamma(x) = \int_0^\infty t^{-x}e^{-t}dt$ is the gamma function [12] Eq. (6.1.1)].

Proof: Please refer to Appendix D for more details.

Remark 4. The results in (12) suggest that the high-SNR slope of the sum ECR under the C-C design is given by $M$.

Remark 5. By comparing (9) and (12), we observe that the CR achieved by the C-C design yields the same asymptotic behaviour as that achieved by the S-C design.

Turn to the OP $P_c^C = \Pr(R_c^C < R_0)$, whose asymptotic behaviour is characterized as follows.

Theorem 5. As $p \to \infty$, the OP of the sum CR achieved by the C-C design satisfies $P_c^C \approx \mathcal{O}(p^{-M(K+M-1)})$.

Proof: Please refer to Appendix E for more details.

Remark 6. The above results suggest that a diversity order of $M(K - M + 1)$ is achievable for the OP under the C-C design, which is the same as that achieved by the S-C design.

2) Performance of Sensing: When $W_c$ is utilized, the SR can be written as

$$R_s^C = NL^{-1} \sum_{m=1}^M \log_2 (1 + L\lambda_m c_m^*).$$

Due to the statistics of $c_m^*$, we define the average SR as $R_s^C = \mathbb{E}(R_s^C)$, which can be evaluated numerically. Besides, Theorem 6 describes the high-SNR behaviour of $R_s^C$.

Theorem 6. As $p \to \infty$, the average SR achieved by the C-C design satisfies

$$R_s^C \approx \frac{NM}{L} \left(\log_2 p + \frac{1}{M} \sum_{m=1}^M \log_2 \left(\frac{L\lambda_m}{M}\right)\right).$$

Proof: Similar to the proof of Theorem 4.

Remark 7. The SR achieved by the C-C design achieves the same asymptotic behaviour as that achieved by the S-C design. This means that a high-SNR slope of $\frac{\lambda}{M}$ is achievable for the SR under the C-C design.

Taken the conclusions in Remarks 5 and 7 together, we find that the C-C design degrades to the S-C design in the high-SNR regime. Moreover, by considering the equal power allocation (EPA), i.e., $W = \sqrt{p/M}U \triangleq W_c$, we derive the following corollary.

Corollary 1. In the high-SNR regime, the SR and the CR achieved by $W_c$ have the same high-SNR slopes and asymptotic behaviours as $R_s^C$ and $R_c^C$, respectively.

Proof: Similar to the proofs of Theorems 4 and 5.

Remark 8. The EPA-based precoding can achieve asymptotically maximum SR and CR in the high-SNR regime.

C. Pareto Optimal Design

In practice, the precoding matrix $W$ can be designed to satisfy different qualities of services, which results in a communication-sensing performance tradeoff. To evaluate this tradeoff, we resort to the Pareto boundary of the SR-CR region. Particularly, we design the precoding matrix as $W_p = U\text{diag}\{p_1, \ldots, p_M\}U^H$ with $\sum_{m=1}^M p_m = p$ and $p_m \geq 0, \forall m \in M$. Accordingly, any rate-tuple on the Pareto boundary of the rate region can be obtained via solving the following optimization problem [9]:

$$\max_{p, R} R, \text{ s.t. } R_s \geq \alpha R, R_c \geq \beta R, 1^T p \leq p, p_m \geq 0, \alpha, \beta \in [0, 1].$$

where $\alpha \in [0, 1]$ is a particular rate-profile parameter, $\alpha = 1 - \alpha$, and $p = [p_1, \ldots, p_M]^T$. We find that problem (13) is convex and it can be solved via standard convex problem solvers such as CVX. For a given $\alpha$, let $R_c^C$ and $R_s^C$ denote the sum ECR and average SR achieved by the corresponding optimal precoding matrix, respectively. It follows that $R_c^C \in [R_s^C, R_c^C]$ and $R_s^C \in [R_s^C, R_c^C]$ with $R_c^C = R_s^C$ and $R_s^C = R_c^C$. On this basis, we conclude the following corollaries.
Corollary 2. For a sufficiently larger SNR, \( \mathcal{R}_s^\alpha \approx \frac{N}{M} \left( \log_2 p + \frac{1}{M} \sum_{m=1}^{M} \log_2 \left( \frac{L_m}{M} \right) \right) \) and \( \mathcal{R}_c^\alpha \approx M \log_2 p - \log_2 M + \psi (K' + 1) / \ln 2 \).

Proof: This corollary can be proved by using the results in Theorems 2 and 4 as well as the Sandwich theorem.

Corollary 3. Let \( \mathcal{R}_c^\alpha \) denote the sum CR for a given \( \alpha \). Then, it has \( \lim_{p \to \infty} \Pr (\mathcal{R}_c^\alpha < \mathcal{R}_0) \approx O \left( p^{-M(1/M-1)} \right) \).

Proof: Similar to the proof of Corollary 2.

Remark 9. Any CR-SR pair on the Pareto boundary has the same asymptotic behaviour in the high-SNR regime.

D. Baseline Scheme

We consider FDSAC as a baseline scenario, where the total bandwidth is partitioned into two sub-bands, one for sensing only and the other for communications. Besides, the total power is also partitioned into two parts for sensing and communications, respectively. Specifically, we assume \( \kappa \) fraction of the total bandwidth as well as \( \mu \) fraction of the total power is used for communications with the \( M \) UTs. In this case, the sum ECR and the SR are given by \( \mathcal{R}_c^\alpha = \mathbb{E} \{ \max_{\sum_{m=1}^{M} b_m \leq (1-\mu)p} \sum_{m=1}^{M} \kappa \log_2 (1 + \frac{a_m}{\mu p_m}) \} \) and \( \mathcal{R}_s^\alpha = \frac{N}{M} \max_{\sum_{m=1}^{M} b_m \leq (1-\mu)p} \sum_{m=1}^{M} \log_2 \left( 1 + \frac{a_m}{\mu p_m} \right) \), respectively. Accordingly, we derive the following corollary.

Corollary 4. The high-SNR slopes of \( \mathcal{R}_s^\alpha \) and \( \mathcal{R}_c^\alpha \) are given by \( \kappa \mathcal{M} \) and \( (1 - \kappa) \frac{N}{M} \mathcal{E} \), respectively.

Proof: Similar to the proofs of Theorems 1 and 4.

Remark 10. Comparing Corollary 2 with Corollary 4, we note that ISAC yields larger high-SNR slopes than FDSAC. In other words, ISAC provides more degrees of freedom than FDSAC in terms of both communications and sensing.

IV. RATE REGION CHARACTERIZATION

We now characterize the CR-SR region of ISAC and FDSAC systems. In particular, let \( \mathcal{R}_s^\alpha \) and \( \mathcal{R}_c^\alpha \) denote the achievable SR and CR, respectively. Then, the rate regions achieved by ISAC and FDSAC are, respectively, given by

\[
\mathcal{C}_1 = \left\{ (\mathcal{R}_s^\alpha, \mathcal{R}_c^\alpha) \mid \mathcal{R}_s^\alpha \in [0, \mathcal{R}_s^\alpha], \mathcal{R}_c^\alpha \in [0, \mathcal{R}_c^\alpha], \alpha \in [0, 1] \right\}, \quad (16)
\]

\[
\mathcal{C}_f = \left\{ (\mathcal{R}_s^\alpha, \mathcal{R}_c^\alpha) \mid \mathcal{R}_s^\alpha \in [0, \mathcal{R}_s^\alpha], \mathcal{R}_c^\alpha \in [0, \mathcal{R}_c^\alpha], \kappa \in [0, 1], \mu \in [0, 1] \right\}. \quad (17)
\]

Theorem 7. The achievable CR-SR regions of ISAC and FDSAC satisfy \( \mathcal{C}_f \subseteq \mathcal{C}_1 \).

Proof: Please refer to Appendix C for more details.

Remark 11. The results in Theorem 7 suggest that the rate region achieved by FDSAC is entirely covered by that achieved by ISAC. This superiority mainly originates from ISAC’s integrated utilization of spectrum and power resources.

V. NUMERICAL RESULTS

Simulation results will be presented to evaluate the S&C performance of ISAC systems and also verify the accuracy of the developed analytical results. The parameters used for simulation are listed as follows: \( M = 4, N = 5, K = 4, L = 30 \), and the eigenvalues of \( \mathbf{R} \) are \{1, 0.1, 0.05, 0.01\}.

Fig. 2(a) and Fig. 2(b) plot the OP and ECR versus the SNR \( p \), respectively. As Fig. 2(a) shows, C-C ISAC achieves the lowest OP while FDSAC achieves the highest OP. In the high-SNR regime, the OP curves for all the presented four cases (S-C ISAC, C-C ISAC, EPA-based ISAC, and FDSAC) are parallel to the one representing \( p^{-M(1/M-1)} \), which suggests that the achievable diversity order obtained in the previous section is tight. This also suggests that ISAC yields the same diversity order as FDSAC. Let us now turn to Fig. 2(b). As expected, C-C ISAC attains the best ECR performance among the four presented cases. Besides, the asymptotic results track the provided simulation results accurately in the high-SNR regime. Particularly, it can be seen from Fig. 2(b) that ISAC achieves a high-SNR slope than FDSAC and the ECRs achieved by C-C ISAC, S-C ISAC, and EPA-based ISAC have the same high-SNR behaviour, which is consistent with the results shown in Remark 5, Remark 10, and Corollary 1.

In Fig. 3 the SR is shown as a function of the SNR \( p \). It can be seen from this graph that S-C ISAC is capable of achieving the best SNR performance among the presented four cases. Moreover, as Fig. 3 shows, the asymptotic results track the provided simulation results accurately in the high-SNR regime. From the data in Fig. 3 it is apparent that ISAC achieves a high-SNR slope than FDSAC and S-C ISAC, C-C ISAC, and EPA-based ISAC yield the same high-SNR slope, which agrees with the conclusions drawn in Remark 7, Remark 10, and Corollary 1. Taking the results in Fig. 2(b) and Fig. 3 together, we find that EPA-based ISAC achieves asymptotically optimal SR and CR in the high-SNR regime, which supports the discussion in Remark 8.

Fig. 4 compares the SR-CR regions achieved by the ISAC system (presented in (16)) and the baseline FDSAC system (presented in (17)). Particularly, the point \( \mathcal{P}_s \) and the point \( \mathcal{P}_c \) are achieved by the S-C design and the C-C design, respectively. As shown in Fig. 4, the curve segment connecting \( \mathcal{P}_s \) and \( \mathcal{P}_c \) represents the Pareto boundary of ISAC’s rate region, which is obtained by solving (15) for \( \alpha \) changing from 1 to 0. What stands out in Fig. 4 is that the rate region achieved...
by FDSAC is completely included in that achieved by ISAC, which, hence, verifies the correctness of Theorem 7.

VI. CONCLUSION

In this letter, we have analyzed the S&C performance of ISAC systems under three typical DFSAC designs. The high-SNR slopes and diversity orders achieved by ISAC have been derived to highlight its superiority. Theoretical analyses have demonstrated that ISAC can provide more degrees of freedom and achieve a broader rate region than DFSAC.

APPENDIX

A. Proof of Lemma 7

Following similar steps as listed in [4] Appendix C), we get $I(Y; G|X) = N \log_2 \det(I_M + H^H RX)$, which together with the Sylvester’s identity yields $I(Y; G|X) = N \log_2 \det(I_M + RXX^H)$. It follows from the fact of $XX^H = LW^W$ as well as the Sylvester’s identity that $\mathcal{R}_s = L^{-1}N \log_2 \det(I_M + LW^W RW)$. 

B. Proof of Theorem 7

It is noteworthy that maximizing $\mathcal{R}_s$ is equivalent to maximizing the MI of a virtual MIMO channel $\hat{y} = R^{1/2} \hat{x} + \tilde{n}$ with $E[\hat{x}\hat{x}^H] = W^W$ and $\tilde{n} \sim CN(0, L^{-1}I_M)$. Thus, when $\mathcal{R}_s$ is maximized, the eigenvalues of $W^W$ should equal the left eigenvectors of $R^{1/2}$, with the eigenvalues chosen by the water-filling procedure, which yields $\mathcal{R}_s = \frac{M}{N} \sum_{m=1}^{M} \log_2 (1 + \lambda_m s_m^* w)$ with $s_m^* = \max(0, 1/\nu - 1/|\lambda_m|)$ and $\sum_{m=1}^{M} |s_m^*| = p$. When $p \to \infty$, we have $\nu \to 0$ and $\sum_{m=1}^{M} s_m^* = M - \sum_{m=1}^{M} \frac{1}{\lambda_m} = p$. It follows that $\lim_{p \to \infty} s_m^* \approx \frac{p}{L M} - \frac{1}{\lambda_m} + \frac{1}{\sum_{m=1}^{M} \frac{1}{\lambda_m}}$, which together with the fact of $\lim_{x \to \infty} \log_2 (1 + x) \approx \log_2 x$ yields $\lim_{p \to \infty} \mathcal{R}_s = \frac{N M}{L} \log_2 p + \frac{N}{L} \sum_{m=1}^{M} \log_2 \left( \frac{L}{\lambda_m} \right)$.

C. Proof of Theorem 2

We note that $s_m^*$ is a constant and the probability density function (PDF) of $\rho_m = |y_m|^2 h_m^2$ is given by $f_m(x) = \frac{x^{M-1} e^{-x}}{K^M}$, where $K = \int_0^\infty e^{-x} x^{M-1} dx$. With this in mind, we can get $\mathcal{R}_m$ by [7] Eq. (4.337.5)]. As mentioned earlier, $\lim_{p \to \infty} s_m^* \approx \frac{p}{L M} - \frac{1}{\lambda_m} + \frac{1}{\sum_{m=1}^{M} \frac{1}{\lambda_m}}$. This together with the fact of $\lim_{x \to \infty} \log_2 (1 + x) = \log_2 x$ and $\mathcal{R}_m$ by [7] Eq. (4.352.1)] leads to $\mathcal{R}_m$.

D. Proof of Theorem 3

As stated before, $\lim_{p \to \infty} \mathcal{R}_m \approx \frac{p}{L M} - \frac{1}{\lambda_m} + \frac{1}{\sum_{m=1}^{M} \frac{1}{\lambda_m}}$, which together with the fact of $\lim_{x \to \infty} \log_2 (a + x) \approx \log_2 x$ yields $\lim_{p \to \infty} \mathcal{R}_m \approx \log_2 (\frac{p}{L M} + \frac{1}{\sum_{m=1}^{M} \frac{1}{\lambda_m}})$. Therefore, the OP satisfies $\lim_{p \to \infty} \mathcal{R}_m \approx \Pr(\prod_{m=1}^{M} \rho_m < 2^{\mathcal{R}_m})$. This basis, we can obtain $\lim_{p \to \infty} \mathcal{R}_m \approx O(p^{-M(K'+1)})$ by exploiting the approach in deriving [10] Eq. (39).

E. Proof of Theorem 2

Clearly, as $p \to \infty$, we have $\nu \to 0$ and thus $c_m^* \approx \frac{p}{L M} - \frac{1}{\lambda_m} + \frac{1}{\sum_{m=1}^{M} \frac{1}{\lambda_m}}$. It follows from the fact of $\lim_{x \to \infty} \log_2 (a + x) \approx \log_2 x$ that the ECR of UT $m$ satisfies $\lim_{p \to \infty} E[\log_2 (1 + \rho_m c_m^*)] \approx \log_2 \left( \frac{p}{L M} + \frac{1}{\sum_{m=1}^{M} \frac{1}{\lambda_m}} \right)$ and the OP satisfies $\lim_{p \to \infty} \mathcal{R}_c \approx \Pr(\prod_{m=1}^{M} \rho_m < 2^{\mathcal{R}_c})$. Exploiting the approach in deriving [10] Eq. (39), we can get $\lim_{p \to \infty} \mathcal{R}_c \approx O(p^{-M(K'+1)})$.

F. Proof of Theorem 5

The fact of $\lim_{p \to \infty} \mathcal{R}_c \approx O(p^{-M(K'+1)})$ is equivalent to $

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