HOMOLOGICAL PROPERTIES OF CONTRACTIBLE TRANSFORMATIONS OF GRAPHS

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Abstract. In [1, 2], A. Ivashchenko shows the family of contractible graphs, constructed from $K(1)$ by contractible transformations, and he proves that such transformations do not change the homology groups of graphs. In this paper, we show that a contractible graph is actually a collapsible graph (in the simplicial sense), from which the invariance of the homology follows. In addition, we extend the result in [2] to a filtration of graphs, and we prove that the persistent homology is preserved with respect to contractible transformations. We apply this property as an algorithm to preprocess a data cloud and reduce the computation of the persistent homology for the filtered Vietoris-Rips complex.

1. Introduction

In graph theory, several reductions have been studied that leave the homology invariant. In [1, 2], A. Ivashchenko shows a family of graphs constructed from $K(1)$ by contractible transformations (as in Definition 1), and he proves that such transformations do not change the homology groups of graphs. He started the study of these transformations because these are used in the theory of molecular spaces, digital topology. Modern references are [3, 4].

In [5], ws-dismantling and the collapse of graph are studied; see Remark 3. A graph is essentially collapsible if its complete complex is collapsible; see Section 3. Neither change the homology groups of graphs ([5]).

In Section 2, we introduce the graph homology as given in [1] and the contractible transformations as were presented in [2]. We conclude the section with Lemma 2 by showing the existence of an induced homomorphism between the graph homology groups of two graphs, given by the image of contractible transformations. In Section 3, we start with some terminology and notation about simplicial complexes, and we give the definition of collapsible graph in terms of the collapsibility of the clique complex. Furthermore, we prove that any contractible graph (in the sense of Ivashchenko [2]) is also a collapsible graph in Theorem 4. As a consequence, the homology groups of any contractible graphs are trivial, as was proven in [1].

We conjecture than any collapsible graph is also a contractible one. We show computational evidence that supports the conjecture through some algorithms. The scripts were written in C/C++, they are free/open and available on-line in [6]. However, we can not follow a similar arguments than used in Theorem 4. We start Section 4 with an example to show that Theorem 4 is not trivially reversible.

We conclude Section 5 with an application of this work to the computation of the persistent homology of the filtered Vietoris-Rips complex. This is a topic of interest in topological data analysis.

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2. Graph homology and contractible transformations

Let $\mathcal{G}$ be the set of all finite undirected graphs without multiple edges, and let $G = (V(G), E(G)) \in \mathcal{G}$ be a graph. To define the homology of a graph, we follow the notation of [1, Sec. 3], which is equivalent to considering a graph as a simplicial complex, for each simplex given by any clique.

On other words, the complete graph on $n + 1$ vertices $v_0, v_1, \ldots, v_n$, correspond to the $n$-simplex $\sigma^n = [v_0 v_1 \ldots v_n]$. An orientation for every complete subgraph is defined by restricting an arbitrary order for the vertices up to even permutations.

The oriented boundary $\partial \sigma^n$ of $\sigma^n$ is defined as the formal linear combination of its complete subgraphs on $n$ vertices:

$$\partial \sigma^n = \partial [v_0 v_1 \ldots v_n] = \sum_{k=0}^{n} (-1)^k [v_0 v_1 \ldots \hat{v}_k \ldots v_n].$$

The hat over the vertex $v_k$ means that such vertex must be omitted.

Let $A$ be an abelian group; usually $A$ is taken as the integers group or a finite field. An $n$-chain in the subgroup $\mathcal{G}$ is defined to be a formal linear combination of distinct complete subgraphs $\sigma^n$ of the graph:

$$c_n = \sum_k \alpha_k \sigma^n_k$$

for $\alpha_k \in A$. The addition of $n$-chains is defined in the obvious way. We have then the chain group $C_n(G)$ of all $n$-chains in $G$. Define the boundary operator $\partial : C_n(G) \rightarrow C_{n-1}(G)$ by linear extension.

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The $n$-dimensional homology group of the graph $G$, with coefficients in $A$, is defined to be a formal linear combination of distinct complete subgraphs $\sigma^n$ of the graph:

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The boundary of a $0$-chain is defined as zero. It can be proven directly that $\partial^2 = 0$, then we have a chain complex

$$\cdots \xrightarrow{\partial} C_n(G) \xrightarrow{\partial} C_{n-1}(G) \xrightarrow{\partial} \cdots \xrightarrow{\partial} C_1(G) \xrightarrow{\partial} C_0(G) \xrightarrow{\partial} 0.$$

The $n$-dimensional homology group of the graph $G$, with coefficients in $A$, is defined by

$$H_n(G; A) = \frac{\ker (\partial : C_n(G) \rightarrow C_{n-1}(G))}{\text{Im} (\partial : C_{n+1}(G) \rightarrow C_n(G))}.$$

The $n$-chains in the subgroup $Z_n(G) := \ker (\partial : C_n(G) \rightarrow C_{n-1}(G))$ are called $n$-cycles, and the $n$-chains in the subgroup $B_n(G) := \text{Im} (\partial : C_{n+1}(G) \rightarrow C_n(G))$ are called $n$-boundaries. In the following, we will omit the coefficients group.

2.1. Contractible transformations. Let $G = (V(G), E(G)) \in \mathcal{G}$ be a graph, and let $v \in V(G)$ be a vertex. We denote $N_G(v) = \{u \in V(G) : \{u, v\} \in E(G)\}$ and $N_G(v, w) := N_G(v) \cap N_G(w)$. In addition, by abuse of notation we identify the graph with its set of vertices.

In [1] the next family of graphs was defined, and its elements are called contractible graphs.

**Definition 1.** Let $\mathcal{I} \subset \mathcal{G}$ be the family of graphs defined by

1. The trivial graph $K(1)$ is in $\mathcal{I}$.
2. Any graph of $\mathcal{I}$ can be obtained from $K(1)$ by the following transformations.
   1. Deleting of a vertex $v$. A vertex $v$ of a graph $G$ can be deleted if $N_G(v) \in \mathcal{I}$.
   2. Gluing of a vertex $v$. If a subgraph $G_1$ of the graph $G$ is in $\mathcal{I}$, then the vertex $v$ can be glued to the graph $G$ in such way that $N_G(v) = G_1$.
   3. Deleting of an edge $\{v_1, v_2\}$. The edge $\{v_1, v_2\}$ of a graph $G$ can be deleted if $N_G(v_1, v_2) \in \mathcal{I}$.
   4. Gluing of an edge $\{v_1, v_2\}$. Let two vertices $v_1$ and $v_2$ of a graph $G$ be nonadjacent. The edge $\{v_1, v_2\}$ can be glued if $N_G(v_1, v_2) \in \mathcal{I}$.

If $G$ belongs to $\mathcal{I}$, then $G$ is called a contractible graph.
The transformations \((II)-(I_4)\) were referred to in [2] as contractible transformations. The contractible transformations are used in molecular spaces, see [2], for more explanation. In addition, in [1] it was proved that contractible transformations do not change the homology groups of a graph, for any commutative group of coefficients \(A\), so the elements of \(\mathcal{F}\) have trivial groups of \(A\)-homology.

**Remark 1.** It was proved in [2] Th. 3.8] that the family \(\mathcal{F}\) can be constructed from \(K(1)\) by only the iteration of the transformation \((12)\).

**Notation 1.** We write \(I(G; v) = G - v\) when \(N_G(v)\) is in the family \(\mathcal{F}\) for the vertex \(v \in G\). We define \(I(G; (v_1, v_2, \ldots, v_k)) := I(I(G; v_1); (v_2, \ldots, v_k))\).

In [1] Th. 4.9], it was proved that the contractible transformation \(I : \mathcal{G} \to \mathcal{G}, G \mapsto I(G; v)\) does not change the homology groups of the graph \(G\) when \(N_G(v) \in \mathcal{F}\).

Actually, the induced isomorphism \(I_* : H_n(G) \to H_n(I(G; v))\) can be given explicitly as follows. Let \(c_n \in C_n(G)\) be an \(n\)-chain of the graph \(G\); we then have that 
\[
c_n = a_n + v \star b_n - a_n \in I(G; v)
\]
and \(b_{n-1} \in N_G(v)\). If \(c_n \in Z_n(G)\), then \(\partial c_n = \partial a_n + b_{n-1} - v \star \partial b_{n-1} = 0\) implies \(\partial a_n + b_{n-1} = 0\) and \(\partial b_{n-1} = 0\). From the last equality and the contractibility of \(N_G(v)\), there exists an \(n\)-chain \(b_n \in C_n(N_G(v))\) such that \(\partial b_n = b_{n-1}\). We then have
\[
c_n = a_n + v \star \partial b_n \quad \text{and} \quad \partial(a_n + b_n) = 0.
\]

Since \(\partial(v \star b_n) = v_n - v \star \partial b_n\), the above equation can be written as
\[
c_n = (a_n + b_n) - \partial(v \star b_n).
\]

The isomorphism \(I_* : H_n(G) \to H_n(I(G; v))\) is induced by the homomorphism
\[
I_*(v) : Z_n(G) \to Z_n(I(G; v)), \quad (a_n + b_n) - \partial(v \star b_n) \mapsto a_n + b_n.
\]

For iterated contractible transformations, we use the following notation:
\[
I_#(v_1, v_2, \ldots, v_k) : Z_* (G) \xrightarrow{I_#(v_1)} Z_* (G - \{v_1\}) \xrightarrow{I_#(v_2)} \cdots \xrightarrow{I_#(v_k)} Z_* (G - \{v_1, v_2, \ldots, v_k\}).
\]

In short, write \(I_#(S)\) for the ordered set \(S = (v_1, v_2, \ldots, v_k)\). Define an analogous notation for the induced homomorphism in homology.

**Theorem 2.** Let \(G_1\) be a finite graph and let \(G_0 \subset G_1\) be a subgraph. There then exists an homomorphism \(i_* : H_* (I(G_0; S_0)) \to H_* (I(G_1; S_1))\) such that the following diagram commutes:
\[
\begin{array}{ccc}
H_* (G_0) & \xrightarrow{i_*} & H_* (G_1) \\
\downarrow{I_*} & \cong & \downarrow{I_*} \\
H_* (I(G_0; S_0)) & \xrightarrow{i_#} & H_* (I(G_1; S_1)).
\end{array}
\]

**Proof.** The iterated contractible contractions over \(G_0\) and \(G_0\), given by the sets \(S_0\) and \(S_1\), respectively, induces the following diagram:
\[
\begin{array}{ccc}
Z_* (G_0) & \xrightarrow{I_#(v_1)} & Z_* (I(G_0; v_1)) & \xrightarrow{I_#(v_2)} & \cdots & \xrightarrow{I_#(v_m)} & Z_* (I(G_0; S_0)) \\
\downarrow{i_#} & & \downarrow{i_#} & & \cdots & \downarrow{i_#} & \\
Z_* (G_1) & \xrightarrow{I_#(w_1)} & Z_* (I(G_1; w_1)) & \xrightarrow{I_#(w_2)} & \cdots & \xrightarrow{I_#(w_n)} & Z_* (I(G_1; S_1)).
\end{array}
\]

The vertical arrow is induced by the inclusion \(i : G_0 \hookrightarrow G_1\). From [1], each horizontal arrow in the above diagram is an isomorphism, then \(I_#(S_0) := I_#(v_m) \circ \cdots \circ I_#(v_1)\) and \(I_#(S_1) := I_#(w_n) \circ \cdots \circ I_#(w_1)\) are also isomorphisms. Let \(z : Z_*(I(G_0; S_0)) \to Z_*(I(G_1; S_1))\) be defined by \(z \mapsto I_#(S_1) \circ i_# \circ I_#(S_0)^{-1}(z)\). Thus, we have the following two commutative diagrams:
The first diagram from left to right is commutative by construction, and the second one is commutative by restriction to the boundaries’ subgroup. Taking the induced morphisms in the quotient groups (homology groups), we obtain the commutative diagram \( \square \).

### 3. Collapsible graphs

The property of a graph to be collapsible is established in the category of simplicial complexes. Let us remember some terminology.

A finite (abstract) simplicial complex is a finite set \( V \) together with a collection \( \Delta \subseteq 2^V \) closed under contention, i.e. if \( \tau \in \Delta \) and \( \sigma \subseteq \tau \), then \( \sigma \in \Delta \). An element \( \sigma \in \Delta \) is called a simplex; if additionally \( \sigma \subseteq A \), then \( \sigma \) is called a vertex of \( \Delta \). The dimension of the simplex \( \sigma \) is defined as \( \dim(\sigma) := |\sigma| - 1 \). Therefore, the 0-simplices are just the vertices of \( \Delta \). We denote by \( \Delta^n \) the complete simplicial structure associated with the set \( \{0,1,\ldots,n\} \).

Given a simplicial complex \( \Delta \), a couple of simplices \( \sigma, \tau \in \Delta \) are called a free face if the following two conditions are satisfied: (1) \( \sigma \subseteq \tau \) and, (2) \( \tau \) is a maximal face of \( \Delta \) with respect to inclusion, and no other maximal face of \( \Delta \) contains \( \sigma \).

A simplicial collapse of \( \Delta \) in \( \tilde{\Delta} \) is obtained by removal of all simplices \( \gamma \in \Delta \) such that \( \sigma \subseteq \gamma \subseteq \tau \), provided that \( (\sigma, \tau) \) is a free face, we will write \( \Delta \backslash \gamma \Delta \). Additionally, if \( \dim(\tau) = \dim(\sigma) + 1 \), then \( \Delta \backslash \gamma \Delta \) is called an elementary simplicial collapse. It is not hard to see that any simplicial collapse can be realized by elementary ones (cf. [1] Sec. III.4)

If there are \( \Delta_1, \Delta_2, \ldots, \Delta_n \) simplicial complexes such that \( \Delta_1 \backslash \Delta_2 \backslash \cdots \backslash \Delta_n \), we say that \( \Delta_1 \) is collapsible to \( \Delta_n \), with \( n = 1 \) inclusive. We denote this also by \( \Delta_1 \backslash \Delta_n \). In particular, if \( \Delta \) is collapsible to \( \Delta^0 \) we just say that \( \Delta \) is collapsible. In the same way as Notation [1] we write \( C(\Delta; (\sigma, \tau)) = \Delta - (\sigma, \tau) \) when \( (\sigma, \tau) \) is a free face of the simplicial complex \( \Delta \). We write \( C(\Delta; ((\sigma_1, \tau_1), (\sigma_2, \tau_2), \ldots, (\sigma_k, \tau_k))) = C(C(\Delta; (\sigma_1, \tau_1)); ((\sigma_2, \tau_2), \ldots, (\sigma_k, \tau_k))) \) for a sequence of free faces collapses.

Let \( G = (V(G), E(G)) \) be a finite undirected, without multiple edges. The clique complex \( \Delta(G) \) of the graph \( G \) is the (finite) simplicial complex with all complete subgraphs of \( G \) as simplices. The 1-skeleton of \( \Delta(G) \) can be identified with \( G \) itself. A simplicial complex \( \Delta \) is called a flag complex if there exists a graph \( G \) such that \( \Delta = \Delta(G) \).

Some barycentric division (not necessarily the first one) of any simplicial complex is a clique complex of some graph: for example the dunce hat needs the second barycentric division.

**Definition 3.** Let \( \mathfrak{C} \) denote the family of graphs \( G \in \mathfrak{G} \) such that \( \Delta(G) \) is collapsible. A graph \( G \in \mathfrak{C} \) is called a collapsible graph.

For example, any complete graph \( K(n) \) is collapsible: \( \Delta(K(n)) \backslash \Delta^0 \).

In [8] it was proved that any dismantleable graph is contraible. In [5] this family was extended to \( s \)-dismantlables graphs. Furthermore, the \( s \)-moves were defined, and it was proved that \( s \)-moves do not change the homology of graphs. Particularly it was proved that the graph \( s \)-dismantlables have trivial homology.

In [9] there is a survey about the relation of \( k \)-behavior to homology. In [10] a study was performed on the triviality of the homology groups of random clique complexes.

The next theorem proves that every contractible graph is also a collapsible graph.

**Theorem 4.** If \( G \) is a contractible graph, then \( \Delta(G) \) is a collapsible simplicial complex.
Proof. Let $n = \dim(\Delta(G))$ be the dimension of the clique complex $\Delta(G)$, and let $k = |V(\Delta(G))|$ denote its number of vertices.

We will prove the claim by induction on $n$ and $k$. The claim is true for $n = 0$ or if $k = 1$, in which case $\Delta(G) = \Delta^0$.

Now suppose that the claim is true for all $m < n$ and for $j < k$ if $m = n$.

Let $v \in G$ be a vertex such that $N_G(v) \in \mathcal{J}$, then it follows from the induction hypothesis over $n$ that

$$\Delta(N_G(v)) = \text{Lk}(v; \Delta(G)) \searrow \Delta^0.$$ 

Therefore, there exists a sequence of elementary collapses and their corresponding free faces $(\sigma_i, \tau_i)$, $\dim(\tau_i) = \dim(\sigma_i) + 1$, in $\text{Lk}(v; \Delta(G))$ such that

$$\Delta^0 = (\cdots ((\Delta(N_G(v)) - (\sigma_1, \tau_1)) - (\sigma_2, \tau_2)) \cdots).$$

On the other hand, each couple $(v \ast \sigma_i, v \ast \tau_i)$ is also a free face of the simplicial subcomplex $v \ast \text{Lk}(v; \Delta(G))$ for each $i$. In fact, they are free faces in the clique complex $\Delta(G)$. Thus,

$$v \ast \text{Lk}(v; \Delta(G)) \searrow v \ast \Delta^0 \searrow \Delta^0$$

and consequently, $\Delta(G) \searrow \Delta(G - v)$. Now $|V(\Delta(G - v))| = k - 1$ and by the induction hypothesis over $k$, we have $\Delta(G - v) \searrow \Delta^0$, which induces the desired simplicial collapses $\Delta(G) \searrow \Delta(G - v) \searrow \Delta^0$.

\[ \Box \]

Remark 2. Theorem 4 lets us prove the claim in [1, Th. 4.9], i.e. that the contractible deleting of a vertex does not change the homology group of a graph. More precisely, if $G$ is a graph and $v$ is a vertex in $G$ such that $N_G(v) \in \mathcal{J}$, then

$$H_\ast(G) \cong H_\ast^\Delta(\Delta(G)) \cong H_\ast^\Delta(\Delta(G - v)) \cong H_\ast(I(G; v)).$$

We write $H_\ast^\Delta(-)$ to denote the functor of simplicial homology to differentiate from the graph homology.

4. Computational approach in the study of the contractible graphs

To verify that $\mathcal{J} \subseteq \mathcal{C}$, we can use as a guide the sequence of contractible transformations to make the collapses, as shown in Theorem 4. In the other way, however, it is not possible. We now examine the example below.

Example 5. Let $G$ be the graph to the left in Figure 1. The clique complex $\Delta(G)$ is given at the right in the picture.
Clearly $G \in \mathcal{I}$, and $\Delta(G)$ is collapsible, i.e. $G \in \mathcal{C}$. In order to reverse the arguments in Theorem 4, any removing sequence of free faces should induce a sequence of contractible transformations. The collapses, however, cannot induce a contractible transformation:

$$
\Delta_0 := \Delta(G) \setminus \Delta_1 := C(\Delta_0; \{(A, B, E), \{A, B, E, F\}\}) \\
\Delta_2 := C(\Delta_1; \{(A, E, F), \{A, D, E, F\}\}) \\
\Delta_3 := C(\Delta_2; \{(D, E, F), \{C, D, E, F\}\}) \\
\Delta_4 := C(\Delta_3; \{(C, E, F), \{B, C, E, F\}\}) \\
\Delta_5 := C(\Delta_4; \{E, F\}, \{B, E, F\}).
$$

In fact, the 1-skeleton of $\Delta_5$ is not a contractible graph, and $N_G(\{E, F\}) \notin \mathcal{I}$.

**Remark 3.** In [5], the author presents the concept of ws-dismantlable; they say that a vertex $v$ in a graph $G$ is ws-dismantlable if $N(v)$ is dismantlable, in this case they write $WS(G; v) = G - v$. An edge $\{u, v\}$ in the graph $G$ is called ws-dismantlable if $N(u) \cap N(v)$ is dismantlable, in this case we write $WS(G; \{u, v\}) = G - \{u, v\}$.

Obviously $\{E, F\}$ is not ws-dismantlable in $\Delta_4$. Thus, a collapse of a free face is not, in general, either a contractible transformation nor ws-dismantling. Compare this remark to the proof of Lemma 4.4 of [5].

Despite Example 5, we conjecture that the reverse inclusion in Lemma 4 is also true: that any contractible graph is also a collapsible one. In this section, we show some algorithms we had used to verify the inclusion $\mathcal{C} \subset \mathcal{I}$ to several graphs. These algorithms were written in C/C++ and are available in the repository [6].

To support the conjecture ($\mathcal{C} \subset \mathcal{I}$), a subset of graphs of these families will be calculated, restricting the number of vertices to $n \leq 8$.

Since connectivity is a common characteristic of these families, the strategy is first to obtain a collection of related graphs, then for each graph, two algorithms will be applied to determine if $G \in \mathcal{I}$ and if $G \in \mathcal{C}$. The codes and the results will be found in the aforementioned repository.

4.1. **Connected graphs.** To generate the collection of connected graphs, we start with a connected graph, making the next modifications to obtain other connected graphs.

- Add vertex $v_{n+1}$ and edge $\{v_i, v_{n+1}\}$ for some $i$ from $\{1, 2, \ldots, n\}$,
- Add edge $\{v_i, v_j\}$ to $G$ if $\{v_i, v_j\} \notin E$ with $i \neq j$.

To add a graphic candidate to the collection, it is verified that it is not isomorphic to any already added through the canonical labeling technique.

The process starts with $K(1)$, and a generated graph is used to generate new connected graphs. The maximum number of vertices is determined by the computing capacity and storage for the generated graphs.

We are using a personal desktop computer, with ten vertices, which begins to require secondary memory instead of RAM for both calculations and storage. For this reason, the work is limited to nine vertices.

In the repository, codes in C/C++ are shared, with also files containing adjacency matrices as well as geometric representations of some the graphics generated. We are even sharing the code for the canonical labeling necessary for solving the graph isomorphism problem.

4.2. **Iterated construction of contractible graphs.** A recursive algorithm is then shown to determine if a family belongs to the family $\mathcal{I}$. In other words, if a graph can be reduced to a point using only the elimination operation of vertices, then the order in which you choose to delete vertices does not matter. The only graph assumed to be known as part of the family $\mathcal{I}$ is $K(1)$. 

The proposed algorithm assumes that there are functions or procedures to calculate the open neighborhood of a vertex $N_G(v_i)$, and to remove a vertex from the graph $G - v_i$.

**Algorithm 1: contractible.graph**

| Input | A graph $G$ and the cardinality $n$ of the vertex set. |
|-------|-------------------------------------------------------|
| Output| The logical TRUE if $G \in \mathcal{I}$, or FALSE in otherwise. |

1. if $n = 0$ then
   2. return FALSE;
3. else
   4. if $n = 1$ then
     5. return TRUE;
   6. else
   7. for $i \leftarrow 1$ to $n$ do
     8. if contractible.graph($N_G(v_i), k$) = TRUE then
       9. return contractible.graph($G - v_i, n - 1$)
     10. end
   11. return FALSE;
 12. end
13. end
14. end

The following lemma proves that Algorithm 1 is consistent.

**Lemma 6.** Let $G$ be a finite graph. The algorithm contractible.graph then returns the logical TRUE if and only if $G$ is a contractible graph.

*Proof.* This lemma is a consequence of the results in [2, 1].

4.3. The contractible reduction algorithm. Many graphs are not in the family $\mathcal{I}$; however, it is desirable to determine how far it is possible to eliminate vertices from the graph $G$. The next algorithm uses Algorithm 1 to eliminate vertices when possible.

**Algorithm 2: contractible.reduction**

| Input | A finite graph $G$. |
|-------|----------------------|
| Output| A reduced graph $I(G; S)$ and a ordered maximal set of vertices $S$. |

1. reduced $\leftarrow$ FALSE;
2. $S \leftarrow \emptyset$;
3. while reduced = FALSE do
   4. reduced $\leftarrow$ TRUE ;
   5. for $v \in V(G)$ do
     6. if contractible.graph($N_G(v), k$) = TRUE then
       7. reduced $\leftarrow$ FALSE;
       8. update graph $G \leftarrow G - v$;
       9. $S \leftarrow S \cup \{v\}$;
      10. break for
     11. end
   12. end
13. end
14. return $(G, S)$
In [6] can be found the C/C++ code for the Algorithm 2. The algorithm contractible.reduction preserve the homology groups; moreover, as we explain in the next section, the algorithm contractible.reduction is compatible with persistent homology, i.e. the persistent homology of a family of graphs can be recovered from the sequence of the homology groups of the reduced graphs.

The algorithm contractible.reduction returns a maximally reduced graph with respect to deleting vertices. Reducing the graph $I(G; S)$, however, by using a contractible deleting edge is also possible as shown in Figure 2.

![Figure 2. Contractible-reduced graph with a contractible edge.](image)

For the graph in Figure 2 we have $I(G; S) = G$, i.e. $S = \emptyset$. The edge $\{v_1, v_2\}$, however, can be deleted by the contractible transformation ($I_3$), even when it is not possible to delete any vertex of $G$ through a contractible transformation.

**Remark 4.** The families $\mathcal{I}$ and $\mathcal{C}$ are infinite; however, subfamilies can be calculated to make the comparison. In the repository [6] are the C/C++ codes to calculate one subfamily limited to eight vertices, also the adjacent matrix including some geometrical representations. To compare these subfamilies, we need to calculate also. The algorithm to obtain the subfamily $\mathcal{C}$ is by simple inspection; however, the code is provided too.

5. Application: Persistent homology of the Vietoris-Rips complex

In some applications of algebraic topology, as topological data analysis (TDA), the shape of a data cloud is studied through the persistent homology of a filtered simplicial structure (cf. [11, 12]). The appropriated simplicial structure and the parameter or variable to construct the filtration are key steps in the topological analysis. We can find several simplicial structures usually used in TDA. A very standard simplicial structure with which define a geometry over a data cloud is known as the Vietoris-Rips complex.

Let $N$ be a finite point cloud in some metric space $(M, d)$, and let $\varepsilon$ be a nonnegative real number. The Vietoris-Rips complex $\text{VR}(N; \varepsilon)$ is the abstract simplicial complex with a set of vertices that is $N$, and $\sigma \subset N$ is a simplex if and only if the distance between any two points in $\sigma$ is less than or equal to $\varepsilon$. Clearly, $\text{VR}(N; \varepsilon) \subset \text{VR}(N; \varepsilon')$ for all $\varepsilon \leq \varepsilon'$. In addition, taking all values $d(u,v)$ for any $u,v \in N$, say $\{0 = \varepsilon_0 < \varepsilon_1 < \cdots < \varepsilon_m\}$, we have the Vietoris-Rips filtration

$$\text{VR}(N; \varepsilon_0) \subset \text{VR}(N; \varepsilon_1) \subset \cdots \subset \text{VR}(N; \varepsilon_m).$$

We can denote $\text{VR}(N; \varepsilon_i)$ simply by $\text{VR}_i$ when there is no risk of confusion about the point cloud or the filtration. In [13] several algorithms can be found for the construction of this simplicial structure.

The Vietoris-Rips filtration is a very useful tool in the applications of the algebraic topology for the data analysis. The way to study a data cloud with the Vietoris-Rips complex (or any other filtered simplicial structure) is through its persistent homology.
Given a finite sequence of simplicial complexes $\Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_m$, for $i, j \in \{0, 1, \ldots, m\}$ such that $i \leq j$, the $(i, j)$-persistent $p$-homology group $H_p^{i,j}$ of the filtration is defined as $\text{Im}(H_\Delta^p(\Delta_i) \to H_\Delta^p(\Delta_j))$, where $H_\Delta^\Delta(-)$ denotes the functor of simplicial homology. See [14] for a deeper analysis.

This simplicial complex has too many simplices, however, and an appropriated reduction or simplification can be useful. Some work in this direction can be found in [15]. On the other hand, there are many libraries to compute the persistent homology; see [16] for an extensive list. One of these libraries is the software Ripser (cf. [17]), a very efficient software for computing the persistent homology of the Vietoris-Rips complex. The next lemma provides an alternative method for reaching the persistent homology of such a simplicial structure through contractible reductions.

**Theorem 7.** Let $VR_0 \subset VR_1 \subset \cdots \subset VR_m$ be the Vietoris-Rips filtration of any finite point cloud, and let $H_p^{i,j}$ be the $(i, j)$-persistent $p$-homology group of the filtration. Define the graph $G_i := VR_i^{(1)}$, i.e. the 1-skeleton of the simplicial complex $VR_i$, then

$$H_p^{i,j} \cong \text{Im}(\iota_* : H_p(I(G_i; S_i)) \to H_p(I(G_j; S_j))).$$

**Proof.** By definition $H_\Delta^\Delta(\Delta_i) \cong H_p(G_i)$ and from Theorem 2 we have the commutative diagram

$$
\begin{array}{ccc}
H_\Delta^\Delta(\Delta_i) & \cong & H_p(G_i) \\
\downarrow I_* & & \downarrow I_* \\
H_\Delta^\Delta(I(G_i; S_i)) & \cong & H_p(I(G_i; S_i)) \\
\iota_* & & \iota_* \\
& & H_\Delta^\Delta(I(G_{i+1}; S_{i+1})) \cong H_p(I(G_{i+1}; S_{i+1})).
\end{array}
$$

The isomorphism (2) is a consequence of the functoriality of the homology theory and the Persistence Equivalence Theorem [7, Sec. VII.2].

Another way to compute the persistent homology is through reductions coming from discrete Morse theory (cf. [18]). This approach is not trivial because the free face removed must preserve some filtration structure and the persistent homology; see [19] for details. In [20], the Perseus software can be accessed, which performs certain homology-preserving Morse theoretic reductions on several structures including the simplicial.

Our approach based on contractible reductions does not care about filtration-preserving reductions; however, the induced homomorphisms are actually persistent homology-preserving as Theorem 7 establishes.

We conclude this section with an example of the persistent homology of the Vietoris-Rips complex of a point cloud in the plane.

**Example 8.** Let $N = \{v_1, \ldots, v_6\} \subset \mathbb{R}^2$ be a point cloud, and let

$VR(N; \varepsilon_1 = 0) \subset VR(N; \varepsilon_2 = 1.5) \subset VR(N; \varepsilon_3 = 2.1) \subset VR(N; \varepsilon_4 = 2.6) \subset VR(N; \varepsilon_5 = 2.7)$

be the Vietoris-Rips filtration, as shown in the next picture.

The following picture show the sequence of graphs given by the contractible reduction algorithm over each simplicial complex, $G_i := I(VR(N; \varepsilon_i))$.  

![Diagram showing the sequence of graphs](attachment:image.png)
Clearly, the sequence $\{G_i\}$ does not have a filtration structure. The following table shows the generators of the $p$-homology groups ($p = 0, 1$) of each reduced graph. On the right is the barcode for the persistent homology.

| $p = 0$ | $H_p(G_1)$ | $H_p(G_2)$ | $H_p(G_3)$ | $H_p(G_4)$ | $H_p(G_5)$ |
|---------|-------------|-------------|-------------|-------------|-------------|
| $v_1$   |             |             | $v_3$       |             |             |
| $v_2$   |             |             |             | $v_1$       |             |
| $v_3$   | $v_6$       |             |             |             |             |
| $v_4$   |             |             |             |             |             |
| $v_5$   |             |             |             |             |             |
| $v_6$   |             |             |             |             |             |

$p = 1$

| $0$ | $0$ | $\alpha$ | $\beta$ | $0$ |

we are denoting $\alpha = [v_1v_2] + [v_2v_3] + [v_3v_4] + [v_4v_5] + [v_5v_6] + [v_1v_6]$ and $\beta = [v_1v_3] + [v_3v_4] + [v_4v_6] + [v_1v_6]$. 

At the repository in [6], some visual material about the construction of the Vietoris-Rips complex (as a graph) can be found, next to the corresponding contractible reduced structure. The experiments show a behavior similar to the above example.

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