FACET VOLUMES OF POLYTOPES

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Abstract. In this paper, motivated by the work of Edelman and Strang, we show that for fixed integers $d \geq 2$ and $n \geq d + 1$ the configuration space of all facet volume vectors of all $d$-polytopes in $\mathbb{R}^d$ with $n$ facets is a full dimensional cone in $\mathbb{R}^n$. In particular, for tetrahedra ($d = 3$ and $n = 4$) this is a cone over a regular octahedron. Our proof is based on a novel configuration space / test map scheme which uses topological methods for finding solutions of a problem, and tools of differential geometry to identify solutions with the desired properties. Furthermore, our results open a possibility for the study of realization spaces of all $d$-polytopes in $\mathbb{R}^d$ with $n$ facets by the methods of algebraic topology.

1. Introduction and the statement of main results

Already in elementary school we learn that each triangle in the plane is defined, up to a plane isometry, by the length of its edges. These lengths fulfill three triangle inequalities. On the other hand, every triple of positive real numbers, satisfying all three triangle inequalities, gives rise to a unique triangle, up to a plane isometry, with edge lengths coinciding with the given numbers.

What would be a high dimensional analogue of this basic school fact? Let $d \geq 1$ be an integer, and let $(x_1, \ldots, x_{d+1})$ be a collection of $d + 1$ affinely independent points in $\mathbb{R}^d$. The convex hull of the collection

$$\Delta_d(x_1, \ldots, x_{d+1}) := \text{conv}\{x_1, \ldots, x_{d+1}\} \subseteq \mathbb{R}^d$$

is a $d$-dimensional simplex in $\mathbb{R}^d$. The simplex $\Delta_d(x_1, \ldots, x_{d+1})$ has $d + 1$ facets given by

$$F_i(x_1, \ldots, x_{d+1}) := \text{conv}\left(\{x_1, \ldots, x_{d+1}\} \setminus \{x_i\}\right),$$

where $1 \leq i \leq d + 1$. Hence, $F_i(x_1, \ldots, x_{d+1})$ is the facet opposite to the vertex $x_i$. Let us now consider the $(d - 1)$-dimensional volume of the facets, denoted by

$$v_i(x_1, \ldots, x_{d+1}) := \text{vol}_{d-1}(F_i(x_1, \ldots, x_{d+1})).$$

In this way we have defined a map $\nu_d: \Lambda(\mathbb{R}^d, d + 1) \to \mathbb{R}^{d+1}$ given by

$$\nu_d(x_1, \ldots, x_{d+1}) = (v_1(x_1, \ldots, x_{d+1}), \ldots, v_{d+1}(x_1, \ldots, x_{d+1})),

$$

where $\nu_d(x_1, \ldots, x_{d+1})$ is called the facet volume vector of the simplex $\Delta_d(x_1, \ldots, x_{d+1})$. Here $\Lambda(\mathbb{R}^d, d + 1)$ denotes the space of all collections of $d + 1$ affinely independent points in $\mathbb{R}^d$, or in other words,

$$\Lambda(\mathbb{R}^d, d + 1) := \{ (x_1, \ldots, x_{d+1}) \in (\mathbb{R}^d)^{d+1} : \left(\forall (\lambda_1, \ldots, \lambda_{d+1}) \in W_{d+1}\setminus\{0\}\right) \sum_{1 \leq i \leq d+1} \lambda_i x_i \neq 0\},$$

where $W_{d+1} := \{ (\lambda_1, \ldots, \lambda_{d+1}) \in \mathbb{R}^{d+1} : \sum_{1 \leq i \leq d+1} \lambda_i = 0 \}$. The space $\Lambda(\mathbb{R}^d, d + 1)$ is also known as the realization space of the $d$-dimensional simplex in $\mathbb{R}^d$. Note that the definition of space $\Lambda(\mathbb{R}^d, d + 1)$ can be extended to all collections of $r$ points where $1 \leq r \leq d + 1$. For example, $\Lambda(\mathbb{R}^d, 1) = \mathbb{R}^d$, and $\Lambda(\mathbb{R}^d, 2)$ coincides with the classical configuration space of two pairwise distinct points in $\mathbb{R}^d$. Furthermore, the first facet volume can be expressed as $v_1(x_1, \ldots, x_{d+1}) = \frac{1}{(d-1)!} \sqrt{\det(A^T A)}$, where $A$ is the matrix $[x_3 - x_2 \ldots \ x_{d+1} - x_2] \in \mathbb{R}^{d \times (d-1)}$. Analogous formulas hold for the other volumes. This shows that $\nu_d$ is a smooth map, when $\Lambda(\mathbb{R}^d, d + 1)$ is endowed with the subspace topology from $(\mathbb{R}^d)^{d+1}$.

Now, our elementary school knowledge tells us that

$$\text{im}(\nu_1) = \{(0, 0)\} \quad \text{and} \quad \text{im}(\nu_2) = \{(\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}_{>0})^3 : \alpha_1 + \alpha_2 < \alpha_3, \alpha_2 + \alpha_3 < \alpha_1, \alpha_3 + \alpha_1 < \alpha_2\}.$$

Here $\mathbb{R}_{>0} = (0, +\infty) \subseteq \mathbb{R}$ denotes the subset of the positive real numbers. Hence, the general analogous question we want to answer is: What is the image of the map $\nu_d$ for every $d \geq 1$, or in other words, what facet volume vectors can one get in an arbitrary dimension?
Theorem 1.1. Let \( d \geq 2 \) be an integer, and let \( \mathfrak{S}_{d+1} \) denote the symmetric group on \( d+1 \) letters. Then the space of all facet volume vectors is the open cone:

\[
\text{VOL}_{d,d+1} := \text{im}(\nu_d) = \{ (\alpha_1, \ldots, \alpha_{d+1}) \in (\mathbb{R}_0, d+1) : \alpha_{\pi(1)} + \cdots + \alpha_{\pi(d)} \in \mathfrak{S}_{d+1} \}. \tag{1}
\]

The statement of the theorem uses the natural (left) action of the symmetric group \( \mathfrak{S}_{d+1} \) on \( \mathbb{R}^{d+1} \) given by the permutation of the coordinates. For this reason, the defining inequalities in the right hand side of (1) repeat many times, but on the other hand it is clear that \( \text{VOL}_{d,d+1} \) is an \( \mathfrak{S}_{d+1} \)-invariant subspace of \( \mathbb{R}^{d+1} \). Furthermore, the map \( \nu_d \) is \( \mathfrak{S}_{d+1} \)-equivariant if the action on \( \Lambda(\mathbb{R}, d+1) \) is given by the permutation of the points \( \rightarrow \) the vertices of a simplex.

The points on the diagonal \( \{ (\alpha_1, \ldots, \alpha_{d+1}) \in (\mathbb{R}_0, d+1) : \alpha_1 = \cdots = \alpha_{d+1} \} \) which belong to \( \text{VOL}_{d,d+1} \) correspond to the simplices whose facet volumes all coincide. In the plane these are equilateral triangles.

In their recent publication Alan Edelman and Gilbert Strang [3] studied the shape of \( \text{VOL}_{2,3} = \text{im}(\nu_2) \). For example, they showed that when the edge lengths \( (\alpha_1, \alpha_2, \alpha_3) \) of a triangle are normalised to the unit sphere, that is \( \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1 \), and the point on the unit sphere \( e = \frac{1}{\sqrt{3}}(1, 1, 1) \) is considered as the north pole, then latitudinal circles correspond to the triangles of equal area.

Motivated by the idea of normalization, but now with respect to the affine hyperplane \( H_{d+1} \) given by the equality \( \sum_{1 \leq i \leq d+1} \alpha_i = 1 \), instead of the unit sphere, we get the following description of the cone \( \text{VOL}_{d,d+1} \). For this we take \( e_1, \ldots, e_{d+1} \) to be the standard basis of \( \mathbb{R}^{d+1} \), set \( c := \sum_{1 \leq i \leq d+1} e_i \), and denote by \( \Delta_d \) the \( d \)-dimensional simplex \( \text{conv} \{ e_1 - \frac{1}{\sqrt{d+1}} e, \ldots, e_{d+1} - \frac{1}{\sqrt{d+1}} e \} \subseteq W_{d+1} = H_{d+1} - \frac{1}{\sqrt{d+1}} e \). In the following, the standard scalar product of the Euclidean space \( \mathbb{R}^{d+1} \) will be denoted by \( \langle \cdot, \cdot \rangle \).

Corollary 1.2. Let \( d \geq 2 \) be an integer, \( P_d := \text{conv} \{ \frac{d+2}{d+1} \Delta_d \cup (d+1)(-\Delta_d) \} \subseteq W_{d+1} \), and let \( P_d^* \) be its polar in \( W_{d+1} \). Then,

\[
\text{VOL}_{d,d+1} \cap H_{d+1} = \text{relint}(P_d^*) + \frac{1}{\sqrt{d+1}} e \quad \text{and} \quad \text{VOL}_{d,d+1} = \text{cone}(\text{relint}(P_d^*) + \frac{1}{\sqrt{d+1}} e) \setminus \{0\}. \tag{2}
\]

Proof. From Theorem 1.1 we know that \( \text{VOL}_{d,d+1} \) is the open cone

\[
\{ \alpha = (\alpha_1, \ldots, \alpha_{d+1}) \in (\mathbb{R}_0, d+1) : \alpha_1 + \cdots + \alpha_{i-1} - \alpha_i + \alpha_{i+1} + \cdots + \alpha_{d+1} > 0, 1 \leq i \leq d+1 \},
\]

which, using the scalar product, can also be presented by

\[
\{ \alpha \in \mathbb{R}^{d+1} : \langle \alpha, e_i \rangle > 0, \langle \alpha, e - 2e_i \rangle > 0, 1 \leq i \leq d+1 \}.
\]

On the other hand, the hyperplane \( H_{d+1} \) can be presented by \( \{ \alpha \in \mathbb{R}^{d+1} : \langle \alpha, e \rangle = 1 \} \). Therefore,

\[
\text{VOL}_{d,d+1} \cap H_{d+1} = \{ \alpha \in \mathbb{R}^{d+1} : \langle \alpha, e \rangle = 1, 0 < \langle \alpha, e_i \rangle < \frac{1}{\sqrt{d+1}}, 1 \leq i \leq d+1 \}
\]

\[
= \{ \alpha \in H_{d+1} : -\frac{1}{\sqrt{d+1}} < \langle \alpha - \frac{d+2}{d+1} e, e_i - \frac{1}{\sqrt{d+1}} e \rangle < \frac{d+1}{2(d+2)}, 1 \leq i \leq d+1 \}
\]

\[
= \{ \alpha \in H_{d+1} : \langle \alpha - \frac{1}{\sqrt{d+1}} e, \frac{2d+2}{d+1} (e_i - \frac{1}{\sqrt{d+1}} e) \rangle < 1,
\]

\[
\langle \alpha - \frac{1}{\sqrt{d+1}} e, (d+1)(-e_i - \frac{1}{\sqrt{d+1}} e) \rangle < 1, 1 \leq i \leq d+1 \}
\]

\[
= \text{relint}(P_d^*) + \frac{1}{\sqrt{d+1}} e\text{,}
\]

as we have claimed. \( \square \)
Edelman and Strang [3] showed that in case of triangles, \( \mathrm{VOL}_{2,3} = \mathrm{im}(\nu_2) \) is a cone over the relative interior of an equilateral triangle where the apex, the origin, is deleted. This can also be deduced from Corollary 1.2 by observing the interesting phenomena that \(-3\Delta_2 \subseteq 6\Delta_2\). In the case of dimension \( d = 3 \), the polytope \( P_3 = \text{conv}(4\Delta_3 \cup 4(-\Delta_3)) \) is a cube and consequently the polar \( P_3^* \) is a regular octahedron.

Another interesting geometric observation can be made. Let \( P \) be a \( d \)-dimensional polytope with facets \( F_1, \ldots, F_n \). The \textit{dihedral angle} between the facets \( F_i \) and \( F_j \) of the polytope \( P \) is defined to be the angle \( \theta_{i,j} := \arccos(-u_i \cdot u_j) \) between the corresponding (unit) outer unit normals \( u_i \) and \( u_j \) to the facets \( F_i \) and \( F_j \). In his paper from 2003, Leng [8] proved that any \( d \)-dimensional simplex \( \Delta \) is a cone over the relative interior of an equilateral triangle where the apex, the origin, is deleted. This can also be seen in the right picture, which shows obtuse tetrahedra in blue. Merging the two pictures would give a single convex body of all tetrahedra, obtuse and acute. The pictures were created using \texttt{Plots.jl} [1].

The relationship between obtuse dihedral angles of a simplex and its facet volumes motivates yet another parametrization of simplices by, now, squared facet volumes. We consider the set

\[
\mathrm{VOL}^2_{d,d+1} := \{ (\alpha_1^2, \ldots, \alpha_{d+1}^2) \in \mathbb{R}^{d+1} : (\alpha_1, \ldots, \alpha_{d+1}) \in \mathrm{VOL}_{d,d+1} \},
\]

and the subspace \( A \subset \mathrm{VOL}^2_{d,d+1} \) of all acute simplices; these are the simplices whose facet volumes satisfy the inequalities \( \alpha_i^2 \geq \sum_{1 \leq j \leq d+1, j \neq i} \alpha_j^2 \) for all \( 1 \leq i \leq d+1 \). Then with \( P_d \) and \( H_{d+1} \), as defined in Corollary 1.2, we can show the following fact.

**Corollary 1.3.** \( A \cap H_{d+1} = \text{relint}(P_d^*) + \frac{1}{d+1}e \).

In the case of a plane this says that acute triangles form an equilateral triangle, while in the case \( d = 3 \) we have that acute tetrahedra form a regular octahedron.

After complete determination of the space of facet volume vectors of \( d \)-dimensional simplices in \( \mathbb{R}^d \) a natural question arrises: \textit{What about facet volume vectors of other polytopes?} More precisely, for given integers \( d \geq 2 \) and \( n \geq d+1 \), what is the space \( \mathrm{VOL}_{d,n} \) of all vectors \( (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}_{>0})^n \) such that there exists a \( d \)-dimensional polytope \( P \) in \( \mathbb{R}^d \) with \( n \) facets \( F_1, \ldots, F_n \) with the property that \( \alpha_i = \mathrm{vol}_{d-1}(F_i) \)
for every $1 \leq i \leq n$. As in the case of simplices ($n = d + 1$), the set $\VOL_{d,n}$ can also be seen as an image of the realization space of all $d$-polytopes with $n$ facets along the appropriate map.

The set of all volume vectors $\VOL_{d,n}$ can be described in general, and we show the following generalization of Theorem 1.1.

**Theorem 1.4.** Let $d \geq 2$ and $n \geq d + 1$ be integers. Then,

$$\VOL_{d,n} = \{(\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}_{>0})^n : \alpha_\pi(n) \leq \alpha_\pi(1) + \cdots + \alpha_\pi(n-1), \pi \in \mathfrak{S}_n\}.$$

In the statement of Theorem 1.4 we decide to neglect the description of the cone $\VOL_{d,n}$ as the image of a realization space of all $d$-dimensional polytopes in $\mathbb{R}^d$ with $n$ facets. Hence, we do not elaborate on the different presentations of the realization space which can be found in the literature. For more details on realization spaces consult for example the recent publication of Rastanawi, Sinn and Ziegler [10].

Theorem 1.1 is a special case of Theorem 1.4. Hence, in Section 2, we prove only Theorem 1.4 by combining classical results about polytopes with methods from differential geometry and topology. In fact, the case $d = 2$ was also proven by Manecke and Sanyal in [9, Proposition 3.8]. For completeness, we include this case in our proof.

The normalization, in the general case of a $d$ polytope with $n \geq d + 1$ facets, with respect to the hyperplane $H_n := \{(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n : \sum_{1 \leq i \leq n} \alpha_i = 1\}$, yields the following corollary.

**Corollary 1.5.** Let $d \geq 2$ and $n \geq d + 1$ be an integer, $P_{n-1} := \text{conv}\{(2n\Delta_{n-1} - n(-\Delta_{n-1})) \subseteq W_n,$ and let $\alpha_{n-1}'$ be its polar in $W_n.$ Then,

$$\VOL_{d,n} \cap H_n = \text{relint}(P_{n-1}^\ast) + \frac{1}{n} e \quad \text{and} \quad \VOL_{d,n} = \text{cone}(\text{relint}(P_{n-1}^\ast) + \frac{1}{n} e) \setminus \{0\},$$

(3)

where $e := \sum_{1 \leq i \leq n} e_i,$ and $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n.$

The statement can be verified along the lines of the proof of Corollary 1.2 where instead of Theorem 1.1 we use rather its generalization, Theorem 1.4.

**What comes next?** The fact that facet volumes of polytopes form a cone introduces a geometrically motivated stratification of the realization space of all $d$-dimensional polytopes in $\mathbb{R}^d$ with $n$ facets. A strata of a realization space is defined to be a fiber of the cone $\VOL_{d,n}$ as the image of an affine isometry class of triangles in the plane and the cone $\VOL_{2,3},$ that is $A(\mathbb{R}^2, 3)/(O(2) \times \mathbb{R}^2) \cong \VOL_{2,3}.$ Here $O(2) \times \mathbb{R}^2$ denotes the group of the affine Euclidean isometries in $\mathbb{R}^2.$

Consider a facet volume map, from $\mathcal{R}_{d,n},$ the realization space of all $d$-dimensional polytopes in $\mathbb{R}^d$ with $n$ facets, onto the cone $\VOL_{d,n},$ and its induced maps

$$\mathcal{R}_{d,n} \twoheadrightarrow \mathcal{R}_{d,n}/(O(d) \times \mathbb{R}^d) \twoheadrightarrow \VOL_{d,n},$$

and

$$\mathcal{R}_{d,n} \twoheadrightarrow \mathcal{R}_{d,n}/(O(d) \times \mathbb{R}^d) \twoheadrightarrow \VOL_{d,d+1} \twoheadrightarrow \VOL_{d,n}/\mathbb{S}_n.$$

The questions we should ask are: What are the fibers of all, or any of, the maps in the compositions? More concretely, what is the topology of strata of the realization space $\mathcal{R}_{d,n}$ or its orbit space $\mathcal{R}_{d,n}/(O(d) \times \mathbb{R}^d)?$ A pioneered work, related to the proposed questions, for the case of arbitrary, not necessarily convex, polygons, considered up to a positive similarity, was done in 1998 by Kapovich and Millson [5].

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2. **Facet Volume Vectors of Polytopes: Proof of Theorem 1.4**

Let $d \geq 2$ and $n \geq d + 1$ be integers. As before, let $e_1, \ldots, e_n$ denote the the standard basis of $\mathbb{R}^n$ and set $e := \sum_{1 \leq i \leq n} e_i.$ As before, $(\langle \cdot, \cdot \rangle)$ stands for the standard scalar product of the Euclidean space $\mathbb{R}^n.$

In this section we prove Theorem 1.4 by showing that a vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}_{>0})^n$ satisfies the system of inequalities

$$\langle \alpha, e - 2e_1 \rangle > 0, \ldots, \langle \alpha, e - 2e_n \rangle > 0$$

if and only if there exists a $d$-dimensional polytope $P$ in $\mathbb{R}^d$ with $n$ facets $F_1, \ldots, F_n$ such that

$$\alpha_1 = \text{vol}_{d-1}(F_1), \ldots, \alpha_n = \text{vol}_{d-1}(F_n).$$

The proof of this equivalence has two natural parts. Sufficiency is considerably simpler to prove, while necessity turns out to be more challenging.
Proposition 2.1. There exists a collection of pairwise distinct unit vectors several steps which follow. Before going into the proof let us note that the case of a triangle, that is $n = 3$ and $d = 2$. Thus, we completed the sufficiency part of the proof.

2.2. Necessity. Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}_{>0})^n$ be a given vector which satisfies the inequalities

$$\langle \alpha, e - 2e_1 \rangle > 0, \ldots, \langle \alpha, e - 2e_n \rangle > 0.$$

We prove that there exists a $d$-dimensional polytope $P$ in $\mathbb{R}^d$ with $n$ facets $F_1, \ldots, F_n$ having the property that $\alpha_i = \text{vol}_{d-1}(F_i)$ for every $1 \leq i \leq n$. For this it suffices to show that there exists a collection of pairwise distinct unit vectors $(u_1, \ldots, u_n)$ which linearly span $\mathbb{R}^d$ and satisfy the linear relation $\sum_{1 \leq i \leq n} \alpha_i u_i = 0$. Indeed, according to the Minkowski’s existence theorem [11, Theorem 8.2.1], for any collection of pairwise distinct unit vectors $(u_1, \ldots, u_n)$ which linearly span $\mathbb{R}^d$ and a collection of positive real numbers $(\alpha_1, \ldots, \alpha_n)$ satisfying the relation $\sum_{1 \leq i \leq n} \alpha_i u_i = 0$ and (6) there exists a $d$-dimensional polytope $P$ with $n$ facets $F_1, \ldots, F_n$ such that $u_i$ is the outer unit normal of the facet $F_i$ and $\alpha_i = \text{vol}_{d-1}(F_i)$, for every $1 \leq i \leq n$.

Thus, we complete the proof of the necessity part by proving the following proposition. Recall that the vector $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{R}_{>0})^n$ is already fixed and satisfies the system of inequalities (6).

**Proposition 2.1.** There exists a collection of pairwise distinct unit vectors $(u_1, \ldots, u_n) \in (\mathbb{R}^d)^n$ with

$$\text{span}\{u_1, \ldots, u_n\} = \mathbb{R}^d \quad \text{and} \quad \sum_{1 \leq i \leq n} \alpha_i u_i = 0.$$
Next, we consider the variety of all singular matrices $M'$ in $M$, that is

$$M' := \{ A \in M : \text{rank}(A) \leq d - 1 \}. \quad (8)$$

The variety $M'$ can be partitioned into a family of smooth manifolds which are given by fixing the rank, these are manifolds

$$M_r := \{ A \in M : \text{rank}(A) = r \}$$

for $0 \leq r \leq d - 1$. Hence, $M' = \coprod_{0 \leq r \leq d - 1} M_r$ is a stratification of $M'$ into smooth manifolds. Each $M_r$ is a smooth manifold of dimension $(d + n - 1 - r)r$ embedded in $M$. In addition, we denote by $M''$ the singular part of the variety $M'$, that is

$$M'' := \{ A \in M : \text{rank}(A) \leq d - 2 \} = \coprod_{0 \leq r \leq d - 2} M_r. \quad (10)$$

For more details on the manifolds of fixed rank consult for example [13, Section 2.1] or [12, Sec. 2.2].

Let $0 \leq r \leq d - 1$ be an integer, and let $X \in M_r$ be a fixed matrix. We describe the tangent space $T_X M_r$ as it was done in [13, Section 2.1]. The singular value decomposition of a rank $r$ matrix gives a presentation of $X$ in the form $U \cdot D \cdot V^t$ where

(i) $U \in \text{Mat}_{d \times r}(\mathbb{R})$ is a matrix of rank $r$ with the property that $U^t \cdot U = 1_r$,

(ii) $D = \text{diag}(d_1, \ldots, d_r) \in \text{Mat}_{r \times r}(\mathbb{R})$ is a diagonal matrix with diagonal entries $d_1 \geq \cdots \geq d_r > 0$, and

(iii) $V \in \text{Mat}_{(n-1) \times r}(\mathbb{R})$ is also a matrix of rank $r$ with the property that $V^t \cdot V = 1_r$.

Here $1_r$ denotes the $r \times r$ unit matrix. The manifold $M_r$ can be described as follows:

$M_r = \{ U \cdot D \cdot V^t : U \in \text{Mat}_{d \times r}(\mathbb{R}), \text{rank}(U) = r, U^t \cdot U = 1_r, V \in \text{Mat}_{(n-1) \times r}(\mathbb{R}), \text{rank}(V) = r, V^t \cdot V = 1_r, D = \text{diag}(d_1, \ldots, d_r) \in \text{Mat}_{r \times r}(\mathbb{R}), d_1 \geq d_2 \geq \cdots \geq d_r > 0 \}.

Then, for $X = U \cdot D \cdot V^t \in M_r$, as explained in [13, Proposition 2.1], we have that

$$\tau_X M_r = \{ U \cdot Z \cdot V^t + U_p \cdot V^t + U \cdot V_p^t : Z \in \text{Mat}_{r \times r}(\mathbb{R}), U_p \in \text{Mat}_{d \times r}(\mathbb{R}), U_p^t \cdot U = 0, V_p \in \text{Mat}_{(n-1) \times r}(\mathbb{R}), V_p^t \cdot V = 0 \}. \quad (11)$$

In the first auxiliary lemma we prove that the submanifolds $S$ and $M_r$, $0 \leq r \leq d - 1$, intersect transversally, in symbols $S \pitchfork M_r$. This means that for every point in the intersection $X \in S \cap M_r$ the sum of vector spaces $\tau_X S + \tau_X M_r$, where $X$ is now considered as the origin, coincides with the tangent vector spaces $\tau_X M$ to the ambient manifold $M = (\mathbb{R}^d)^{n-1}$ at the point $X$. For more details of transversal intersections see for example [7, Chapter 6, p. 143].

**Lemma 2.2.** $S \pitchfork M_r$, for all $1 \leq r \leq d - 1$.

**Proof.** Let $X \in S \cap M_r$, with $X = U \cdot D \cdot V^t$ for some $U \in \text{Mat}_{d \times r}(\mathbb{R})$, $V \in \text{Mat}_{(n-1) \times r}(\mathbb{R})$ and $D = \text{diag}(d_1, \ldots, d_r) \in \text{Mat}_{r \times r}(\mathbb{R})$, where $\text{rank}(U) = r$, $\text{rank}(V) = r$, $U^t \cdot U = 1_r$, $V^t \cdot V = 1_r$, and $d_1 \geq d_2 \geq \cdots \geq d_r > 0$. We have seen in (7) and (11) than the tangent spaces to $S$ and $M_r$ at the point $X$ can be described by:

$$\tau_X S = \{ Y \in M : (X^t \cdot Y)_{i,i} = 0, 1 \leq i \leq n - 1 \},$$

and

$$\tau_X M_r = \{ U \cdot Z \cdot V^t + U_p \cdot V^t + U \cdot V_p^t : Z \in \text{Mat}_{r \times r}(\mathbb{R}), U_p \in \text{Mat}_{d \times r}(\mathbb{R}), U_p^t \cdot U = 0, V_p \in \text{Mat}_{(n-1) \times r}(\mathbb{R}), V_p^t \cdot V = 0 \}. \quad (12)$$

Thus, in order to prove that $S \pitchfork M_r$ we need to check that $\tau_X M = \tau_X S + \tau_X M_r$. Here $\tau_X M$ is a $d(n - 1)$ dimensional real vector space, since $M$ is also a $d(n - 1)$ dimensional real vector space. Recall, $\dim \tau_X S = (d - 1)(n - 1)$ and $\dim \tau_X M = (d + n - 1 - r)r$.

Further on, let us denote by $E_i \in \text{Mat}_{(n-1) \times (n-1)}(\mathbb{R})$ the matrix with all entries zero except the entry $(i, i)$ which is assumed to be 1. We show that $X \cdot E_i \in \tau_X M_r$ for every $1 \leq i \leq n - 1$. For that we note that $P_V := V \cdot V^t$, considered as the linear map $\mathbb{R}^{n-1} \rightarrow \mathbb{R}^{n-1}, x \mapsto P_V \cdot x$, is a projection onto the subspace $C$ spanned by the columns of the matrix $V$. Indeed, $P_V \cdot V = (V \cdot V^t)\cdot V = V \cdot (V^t \cdot V) = V \cdot 1_r = V$, meaning
that \( P_V(x) = x \) for \( x \in C \), and for \( x \in C^\perp \) \( V^t \cdot x = 0 \) we have that \( P_V(x) = (V \cdot V^t) \cdot x = V \cdot (V^t \cdot x) = 0 \). We set:

\[
Z := D \cdot V^t \cdot E_1 \cdot V \in \text{Mat}_{r \times r}(\mathbb{R}),
\]
\[
U_p := 0 \in \text{Mat}_{d \times r}(\mathbb{R}), \text{ and }
\]
\[
V_p^t := D \cdot V^t \cdot E_1 \cdot (1_{n-1} - P_V) \in \text{Mat}_{r \times (n-1)}(\mathbb{R}).
\]

Then, \( U_p \cdot U = 0 \), and also \( V_p^t \cdot V = D \cdot V^t \cdot E_1 \cdot (1_{n-1} - P_V) \cdot V = D \cdot V^t \cdot E_1 \cdot (V - V) = 0 \). Furthermore,

\[
U \cdot Z \cdot V^t + U_p \cdot V^t = U \cdot (Z \cdot V^t + V^t) = U \cdot (D \cdot V^t \cdot E_1 \cdot V + D \cdot V^t \cdot E_1 \cdot (1_{n-1} - P_V)) = U \cdot D \cdot V^t \cdot E_1 \cdot (P_V + 1_{n-1} - P_V) = (U \cdot D \cdot V^t) \cdot E_1 = X \cdot E_1.
\]

Hence, \( X \cdot E_1 \in \tau_X M_r \).

Let us present \( X \) as the collection of its column vectors, say \( X = [x_1 \ldots x_{n-1}] \). Hence, \( X \cdot E_1 \) is the matrix with all columns zero except the \( i \)th column which is \( x_i \), that means \( X \cdot E_i = [0 0 \ldots 0 x_i 0 \ldots 0] \). Each vector \( x_i \) belongs to the sphere \( S_i \), and so cannot be zero implying that

\[
\dim \left( \text{span}\{X \cdot E_1,\ldots,X \cdot E_{n-1}\} \right) = n - 1.
\]

The subspaces \( \tau_X S \) and \( \text{span}\{X \cdot E_1,\ldots,X \cdot E_{n-1}\} \subseteq \tau_X M_r \) are orthogonal with respect to the scalar product defined on \( \tau_X M \) by \( \langle (A, B) \rangle := \text{trace}(A^t \cdot B) \). Therefore,

\[
\tau_X M \supseteq \tau_X S + \tau_X M_r \supseteq \tau_X S + \text{span}\{X \cdot E_1,\ldots,X \cdot E_{n-1}\} \cong \tau_X S \oplus \text{span}\{X \cdot E_1,\ldots,X \cdot E_{n-1}\}.
\]

Since,

\[
\dim \left( \tau_X S \oplus \text{span}\{X \cdot E_1,\ldots,X \cdot E_{n-1}\} \right) = \dim(\tau_X S) + \dim \left( \text{span}\{X \cdot E_1,\ldots,X \cdot E_{n-1}\} \right)
= (d - 1)(n - 1) + (n - 1)
= d(n - 1) = \dim(\tau_X M),
\]

we proved that \( \tau_X M = \tau_X S + \tau_X M_r \), and consequently \( S \cap M_r \).

As a direct consequence of the previous lemma we determine the codimension of the each intersection \( S \cap M_r \) in \( S \).

**Corollary 2.3.** \( \text{codim}_M(S \cap M_r) = (d - r)(n - r - 1) \), for all \( 1 \leq r \leq d - 1 \).

**Proof.** According to Lemma 2.2 we have seen that \( S \cap M_r \), and so by [7, Theorem 6.30] follows that \( \text{codim}_M(S \cap M_r) = \text{codim}_M(S) + \text{codim}_M(M_r) = (d - r)(n - r - 1) \). \( \square \)

The complement \( S \setminus M' \) is the space of all collections of \( n - 1 \) vectors \( (x_1,\ldots,x_{n-1}) \) which span the ambient vector space \( \mathbb{R}^d \) and each vector is of the prescribed norm \( \|x_i\| = \alpha_i \) for \( 1 \leq i \leq n - 1 \). In other words, these are the candidates, up to a scaling, for the first \( n - 1 \) vectors, out of \( n \), whose existence is claimed by Proposition 2.1. Hence, we call \( S \setminus M' \) the space of solution candidates.

2.2.2. The test map. Let \( \varphi : S \to \mathbb{R}^d \) be the restriction of the linear map

\[
(\mathbb{R}^d)^{n-1} \to \mathbb{R}^d, \quad (x_1,\ldots,x_{n-1}) \mapsto \sum_{1 \leq i \leq n-1} x_i,
\]

onto the product of spheres \( S = S_1 \times \cdots \times S_{n-1} \). The first property of the map \( \varphi \) we show is that its image \( \varphi(S) \) intersects the sphere \( S_n \subseteq \mathbb{R}^d \).

**Lemma 2.4.** \( \varphi(S) \cap S_n \neq \emptyset \).

**Proof.** Let \( v \in S^{d-1} = S_n \) be an arbitrary unit vector in \( \mathbb{R}^d \). Without loss of generality we can assume that the values \( \alpha_i \) are ordered in the non-decreasing order \( 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \). Consider the vectors

\[
w_0 := \varphi(\alpha_1 v,\ldots,\alpha_{n-1} v) = (\alpha_1 + \cdots + \alpha_{n-1}) v,
\]

and

\[
w_1 := \varphi(-\alpha_1 v,\alpha_2 v,\ldots,(-1)^{n-2}\alpha_{n-2} v,(-1)^{n-1}\alpha_{n-1} v)
= (-\alpha_1 + \alpha_2 - \cdots + (-1)^{n-2}\alpha_{n-2} + (-1)^{n-1}\alpha_{n-1}) v.
\]

From the assumption (6) we can estimate the norm of \( w_0 \) by:

\[
\|w_0\| > \alpha_1 + \cdots + \alpha_{n-1} > \alpha_n.
\]
In the case of the vector $w_1$ we have that
\[ \|w_1\| = | -\alpha_1 + \alpha_2 - \cdots + (-1)^{n-2} \alpha_{n-2} + (-1)^{n-1} \alpha_{n-1} | < \alpha_n. \]
Indeed, if $n$ is odd then, from the assumption on the ordering of values of $\alpha_i$’s, we see that
\[ \|w_1\| = | -\alpha_1 + \alpha_2 - \cdots - \alpha_{n-2} + \alpha_{n-1} | = (\alpha_2 - \alpha_1) + \cdots + (\alpha_{n-1} - \alpha_{n-2}). \]
Consider the differences $\alpha_2 - \alpha_1, \ldots, \alpha_{n-1} - \alpha_{n-2}$ as lengths of the intervals $[\alpha_1, \alpha_2], \ldots, [\alpha_{n-2}, \alpha_{n-1}]$, all contained in the interval $[0, \alpha_n]$. Since these intervals are either disjoint or intersect in a boundary point, and in addition $\alpha_0 > 0$ we see that $\|w_1\| < \alpha_n$. On the other hand, if $n$ is even then,
\[ \|w_1\| = | -\alpha_1 + \alpha_2 - \cdots + (-1)^{n-2} \alpha_{n-2} + (-1)^{n-1} \alpha_{n-1} | = |(\alpha_2 - \alpha_1) + \cdots + (\alpha_{n-2} - \alpha_{n-3}) - \alpha_{n-1}|. \]
Using the same reasoning as in the case when $n$ is odd, now for the sequence of values $\alpha_1, \ldots, \alpha_{n-1}$, we have that $0 < (\alpha_2 - \alpha_1) + \cdots + (\alpha_{n-2} - \alpha_{n-3}) < 1$. Consequently, $\|w_1\| < \alpha_n$.

The complement $\mathbb{R}^d \setminus S_n$ is a disconnected space with two connected components
\[ C_0 := \{ w \in \mathbb{R}^d : \|w\| > \alpha_n \} \quad \text{and} \quad C_1 := \{ w \in \mathbb{R}^d : \|w\| < \alpha_n \}, \]
which are also its path-connected components. We have seen that $w_0 \in C_0$ and $w_1 \in C_1$. Thus, $\varphi(S) \cap C_0 \neq \emptyset$ and $\varphi(S) \cap C_1 \neq \emptyset$. The map $\varphi$ is continuous, $S$ is path-connected, and therefore its image $\varphi(S)$ is also path-connected. Consequently, $\varphi(S)$ cannot be contained in $\mathbb{R}^d \setminus S_n$ and intersect non-trivially both connected components. Hence, $\varphi(S) \cap S_n \neq \emptyset$. \(\square\)

As a direct consequence of the previous lemma we get the following corollary which would already give us Proposition 2.1 if we did not require the vectors spanning the ambient $\mathbb{R}^d$ to be pairwise distinct. For this reason we call the map $\varphi$, as well as its restrictions, a test map for our problem.

**Corollary 2.5.** There exists a collection of unit vectors $(u_1, \ldots, u_n) \in (\mathbb{R}^d)^n$ such that
\[ \sum_{1 \leq i \leq n} \alpha_i u_i = 0. \]

**Proof.** As we have seen in Lemma 2.4 the intersection $\varphi(S) \cap S_n$ is non-empty. This implies that there exists $(x_1, \ldots, x_{n-1}) \in S$ with $\|\varphi(x_1, \ldots, x_{n-1})\| = \alpha_n$. Taking
\[ u_1 := \frac{x_1}{\|x_1\|}, \ldots, u_{n-1} := \frac{x_{n-1}}{\|x_{n-1}\|}, \quad u_n := -\frac{\varphi(x_1, \ldots, x_{n-1})}{\|\varphi(x_1, \ldots, x_{n-1})\|} \]
we get that $\sum_{1 \leq i \leq n} \alpha_i u_i = 0$. \(\square\)

In order, find the collection of vectors $(u_1, \ldots, u_n)$ which satisfy the desired properties we study further the map $\varphi$ and its restrictions. Recall from (8) and (10) that we denote by $M'$ the variety of $d \times (n-1)$ matrices of rank at most $d-1$ and by $M''$ the variety of matrices of rank at most $d-2$. Let $X$ be the open $d(n-1)$-dimensional manifold given by
\[ X := \begin{cases} S \setminus M'', & d \geq 3, \quad n = d+1, \\ S \setminus M', & n \geq d+2, \end{cases} \]
and let us denote by $\psi : X \to \mathbb{R}^d$ the restriction of $\varphi$ to $X$, that is $\psi := \varphi|_X$.

**Lemma 2.6.** Every point contained in the image of $\psi$ is a regular value of $\psi$.

**Proof.** Consider first the case of a simplex, that is $d \geq 3$ and $n = d+1$. Pick $w \in \text{im}(\psi) = \varphi(S \setminus M'')$ and $x = (x_1, \ldots, x_{n-1}) \in \psi^{-1}(\{w\}) \subseteq X$. From Lemma 2.2 we have that $\dim(S \cap M') < \dim(S)$ because $S \cap M_r$ for every $1 \leq r \leq n-1$. Hence,
\[ \tau_x X = \tau_{x}(S \setminus M'') = \tau_{x}(S) = \{ y = (y_1, \ldots, y_{n-1}) \in (\mathbb{R}^d)^{n-1} : \langle x_i, y_i \rangle = 0, \ 1 \leq i \leq n-1 \}. \quad (13) \]
Then, since $\psi$ is a restriction of a linear map we have that the differential $(D\psi)_x : \tau_x X \to \tau_{w\mathbb{R}^d}$ is given by $y = (y_1, \ldots, y_{n-1}) \mapsto y_1 + \cdots + y_{n-1}$.

Now, we show that $(D\psi)_x$ is surjective; for relevant definitions see for example [7]. For this we denote by $H_i := (\text{span}(x_i))^{\perp}, 1 \leq i \leq n-1$, the orthogonal complement hyperplane of the vector $x_i$ in $\mathbb{R}^d$. Then,
\[ \text{im}(D\psi)_x = H_1 + \cdots + H_{n-1} \subseteq \tau_{w\mathbb{R}^d}. \]
Since $x \in \psi^{-1}(\{w\}) \subseteq X = S \setminus M''$ and $d \geq 3$ we have that $\text{rank}(x) = \text{rank}[x_1 \cdots x_{n-1}] \geq d-1 \geq 2$, implying that not all of the hyperplanes $H_1, \ldots, H_{n-1}$ coincide. Consequently, $\dim(H_1 + \cdots + H_{n-1}) = d$.
and so \( \text{im}(D\psi)_x = H_1 + \cdots + H_{n-1} = \tau_w \mathbb{R}^d \), implying that \((D\psi)_x\) is surjective. Since \( x \) was chosen arbitrary in \( \psi^{-1}([w]) \) we have proved that \( w \) is a regular value of \( \psi \).

In the case \( n \geq d + 2 \) we proceed in the same way as in the case of a simplex with the only difference that now \( x \in \psi^{-1}([w]) \subseteq X = S \setminus M' \) implies that \( \text{rank}(x) = d \geq 2 \). Thus, once more we have that \( w \) is a regular value of the map \( \psi \). \qedhere

Furthermore, an analogue of Lemma 2.4 for the map \( \psi \) holds.

**Lemma 2.7.** \( \psi(X) \cap S_n \neq \emptyset \).

**Proof.** From Lemma 2.2 follows that both, \( M'' \) in the case \( d \geq 3 \), \( n = d + 1 \), and \( M' \) in the case \( d \geq 2 \), \( n \geq d + 2 \), are of codimension at least 2 in \( S \). Consequently, the complement of \( X \) in \( S \) is path-connected.

Let \( a := (\alpha_1, \ldots, \alpha_{n-1}) \) and \( b := (\alpha_1 v, \alpha_2 v, \ldots, (-1)^{n-2} \alpha_{n-2} v, (1)^{n-1} \alpha_{n-1} v) \) be points in \( S \) considered in the proof of Lemma 2.4. Then there exists a path \( \gamma : [0, 1] \rightarrow S \) from \( a = \gamma(0) \) to \( b = \gamma(1) \) with the property that \( \gamma((0, 1)) \subseteq X \). As in the proof of Lemma 2.4, we set \( C_0 := \{ w \in \mathbb{R}^d : \| w \| > \alpha_n \} \) and \( C_1 := \{ w \in \mathbb{R}^d : \| w \| < \alpha_n \} \). Since \( w_0 = \varphi(a) \in C_0 \) and \( w_1 = \varphi(b) \in C_1 \), and \( C_0 \) and \( C_1 \) are the path-connected components of the complement \( \mathbb{R}^d \setminus S_n \), we can find \( t \in (0, 1) \) with the property that \( \varphi(\gamma(t)) \in S_n \). Hence, \( \psi(X) \cap S_n \neq \emptyset \). \( \Box \)

### 2.2.3. **Proof of Proposition 2.1.** We are going to prove the existence of a collection of pairwise distinct unit vectors \((u_1, \ldots, u_n) \in (\mathbb{R}^d)^n \) with the property that

\[
\text{span}\{u_1, \ldots, u_n\} = \mathbb{R}^d \quad \text{and} \quad \sum_{1 \leq i \leq n} \alpha_i u_i = 0.
\]

For that we show the existence of another collection of vectors \( x = (x_1, \ldots, x_{n-1}) \in S \) with the property that

\[
x \notin M' \quad \text{and} \quad \varphi(x) \in S_n \quad \text{and} \quad \left( \frac{x_1}{\| x_1 \|}, \ldots, \frac{x_{n-1}}{\| x_{n-1} \|}, -\frac{\varphi(x)}{\| \varphi(x) \|} \right) \in F(S^{d-1}, n),
\]

(14)

where \( F(S^{d-1}, n) \subseteq (S^{d-1})^n \) denotes the ordered configuration space of \( n \) pairwise distinct points on the sphere \( S^{d-1} = \{ x \in \mathbb{R}^d : \| x \| = 1 \} \). If \( x \in S \) does not satisfy conditions (14) then, either

(i) \( x \in M' \), or

(ii) \( \varphi(x) \notin S_n \), or

(iii) \( \frac{x_j}{\| x_j \|} = \frac{x_i}{\| x_i \|} \Rightarrow x_i = x_j \) for some \( 1 \leq i < j \leq n-1 \), or

(iv) \( \frac{x_i}{\| x_i \|} = -\frac{\varphi(x)}{\| \varphi(x) \|} \) for some \( 1 \leq i \leq n-1 \).

We will find a point \( x \in S \) satisfying none of the properties (i)–(iv). In the following, as before, for the collection of vectors \( x \) we also use the matrix notation \( X \) when convenient.

According to the Lemma 2.6 and Lemma 2.7 there exists a regular value \( w \) of the map \( \psi \) which belongs to the sphere \( S_n \), that is \( w \in \mathbb{S}_n \). We proceed by applying different strategies in the case of simplices, and then in the case of arbitrary polytope. Recall from (9) that \( M_d \) denotes the manifold of \( d \times (n-1) \) matrices of rank equal to \( r \).

(1) We start with the case of simplices. For this, let \( d \geq 3 \) and \( n = d + 1 \) be integers. Take an arbitrary \( x \in \psi^{-1}([w]) \), then condition (ii) does not hold. If \( \text{rank}(x) = d = n - 1 \), or equivalently \( x \in X \setminus M' \), condition (i) is not satisfied, and conditions (iii) and (iv) do not hold because they contradict the assumption that vectors \( x_1, \ldots, x_{n-1} \) are linearly independent. Thus, if \( \text{rank}(x) = d \) the proof is complete, otherwise we assume that \( x \in X \setminus M_d = M_{d-1} \) and show that there exists another point in \( \psi^{-1}([w]) \) whose rank is maximal.

We start with \( x \in X \setminus M_d = S \cap M_{d-1} \), and assume the contrary, that \( \psi^{-1}([w]) \subseteq M_{d-1} \). Since \( w \) is a regular value of \( \psi \) the preimage \( \psi^{-1}([w]) \) is a smooth submanifold of \( X \) of codimension \( d \). Then, the tangent space of \( \psi^{-1}([w]) \) at the point \( x \) is \( \tau_x \psi^{-1}([w]) = \ker(D\psi)_x \), see for example [2, Theorem A.9]. Note that the assumption \( \psi^{-1}([w]) \subseteq M_{d-1} \) implies that \( \tau_x \psi^{-1}([w]) \subseteq \tau_x M_{d-1} \). Hence, we consider next the tangent space \( \tau_x M_{d-1} \) of \( M_{d-1} \) at \( x \). We now describe it using its normal (bundle) subspace with respect to the ambient \( M = (\mathbb{R}^d)^{n-1} \) and the scalar product we already used, \( \langle A, B \rangle := \text{trace}(A^\top B) \) for \( A, B \in M \). That is,

\[
\nu_x M_{d-1} = \{ Y \in M : (\forall Z \in \tau_x M_{d-1}) \langle Y, Z \rangle = 0 \}.
\]

Alternatively, the normal space can be described as follows. Let us consider the point \( x \) as a matrix \( X \in M_{d-1} \) and present it as a product \( X = U \cdot V^\top \) where \( U \in \text{Mat}_{d \times (d-1)}(\mathbb{R}) \), \( V \in \text{Mat}_{d \times (d-1)}(\mathbb{R}) \) and \( \text{rank}(U) = \text{rank}(V) = d - 1 \). Furthermore, denote by \( U^\perp \subseteq \mathbb{R}^d \) and \( V^\perp \subseteq \mathbb{R}^d \) the subspaces orthogonal
to the column spans of the matrices $U$ and $V$, respectively. Note that $\dim(U^\perp) = \dim(V^\perp) = 1$. We claim that

$$\nu_x M_{d-1} = \text{span}\{u \cdot v^\perp : u \in U^\perp, v \in V^\perp\}. \quad (15)$$

We use the description of $\tau_x M_{d-1}$ in (12) to see that the right-hand side in (15) is indeed perpendicular to $\tau_x M_{d-1}$. This shows that the right-hand side of (15) is included in $\nu_x M_{d-1}$. Equality follows, because $M_{d-1}$ is of codimension 1 in $\text{Mat}_{d \times d}(\mathbb{R})$ and so both sides are of dimension 1.

To complete the proof we will find a vector $Y$ which belongs to $\tau_x \psi^{-1}(\{w\})$ but is not in $\tau_x M_{d-1}$, contradicting the assumption that $\tau_x \psi^{-1}(\{w\}) \subseteq \tau_x M_{d-1}$. For that we fix two unit vectors $u_0 \in U^\perp$ and $v_0 \in V^\perp$, and set $\varepsilon := (1,1,\ldots,1) \in \mathbb{R}^{n-1}$. First, we observe that

$$X^t \cdot u_0 = (U \cdot V^t)^t \cdot u_0 = V \cdot (U^t \cdot u_0) = 0$$

which in particular implies that $x_i^t \cdot u_0 = 0$ for all $1 \leq i \leq d$. Next, we consider the vector $Y_\lambda := (\lambda_1 u_0, \ldots, \lambda_d u_0) \in U' \mathbb{R}$ for an arbitrary choice of the vector $\lambda := (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$. According to (13), we have that $Y_\lambda \in \tau_x \mathcal{X} = \tau_x (S' M^n)$ because $(x_i, \lambda_i u_0) = x_i^t \cdot (\lambda_i u_0) = \lambda_i (x_i^t \cdot u_0) = 0$. On the other hand, if the coordinates of the vector $\lambda$ sum to zero, then $Y_\lambda \in \ker(D\psi)_x = \tau_x \psi^{-1}(\{w\})$ because

$$(D\psi)_x(Y_\lambda) = (D\psi)_x(\lambda_1 u_0, \ldots, \lambda_d u_0) = (\lambda_1 + \cdots + \lambda_d) u_0 = 0$$

Next, $\psi(x) = X \cdot \varepsilon \neq 0$ because $\psi(x) \in S_n$, and $X \cdot v_0 = (U \cdot V^t)^t \cdot v_0 = 0$, implying the linear independence of $e^t$ and $v_0$. Therefore, we can find a vector $\lambda_0 \in \mathbb{R}^d$ with the property that $\langle \lambda_0, \varepsilon \rangle = \lambda_0^t \cdot \varepsilon = 0$ and $\langle \lambda_0, v_0 \rangle = \lambda_0^t \cdot v_0 = 0$ since the orthogonal complements of $\varepsilon$ and $v_0$ do not coincide. Then we have that

$$\langle \langle Y_{\lambda_0}, u_0 \cdot v_0^t \rangle \rangle = \text{trace} \left( Y_{\lambda_0} \cdot (u_0 \cdot v_0^t) \right) = \text{trace} \left( (u_0 \cdot v_0^t) \cdot \lambda_0 \right) = \lambda_0^t \cdot v_0 \neq 0,$$

because $u_0$ is chosen to be a unit vector. Here we use the identity $\text{trace}(A^t \cdot B) = \text{trace}(A \cdot B^t)$. Thus, the fact $\langle \langle Y_{\lambda_0}, u_0 \cdot v_0^t \rangle \rangle \neq 0$ implies that $Y_{\lambda_0} \notin \tau_x M_{d-1}$ but $Y_{\lambda_0} \in \tau_x \psi^{-1}(\{w\})$, and the contradiction we announced is reached. This completes the proof of Proposition 2.1 in the case $d \geq 3$ and $n = d + 1$.

We prove that the relation (17) cannot hold.

(2) Now we consider the case of polytopes. Let $d \geq 2$ and $n \geq d + 2$ be integers. For every point $x \in \mathcal{X}$ we have that $\text{rank}(x) = d$ so that conditions (i) and (ii) do not hold. Hence, to prove the proposition, we need to find a point in $\psi^{-1}(\{w\})$ which does not satisfy either condition (iii) or condition (iv).

Let $x = (x_1, \ldots, x_{n-1}) \in X \in \psi^{-1}(\{w\})$ be fixed and let $x$ satisfy one of the conditions (iii) or (iv). Consequently, there is a linear dependence between vectors $x_1, \ldots, x_{n-1}$ which can be encoded by $X \cdot \ell = 0$ for some concrete non-zero vector $\ell \in \mathbb{R}^{n-1}$. Consider the linear subspace of $M$:

$$\mathcal{L} := \{Y \in M : Y \cdot \ell = 0\} = \{Y \in M : \langle\langle Y, e_i \cdot \ell^t \rangle\rangle = 0, \ 1 \leq i \leq d\}.$$ 

Here $e_1, \ldots, e_d$ denotes the standard basis of $\mathbb{R}^d$, and $f_1, \ldots, f_{n-1}$ is the standard basis of $\mathbb{R}^{n-1}$. Then, $X \in \mathcal{L}$ and $\tau_x \mathcal{L} = \mathcal{L}$. According to Lemma 2.2 we have that $\dim(S \cap M') < \dim(S)$, and therefore $\tau_x \mathcal{X} = \tau_x (S' M^n) = \tau_x \mathcal{S}$. From the equality $\ker(D\psi)_x = \tau_x \psi^{-1}(\{w\})$ we get that

$$\tau_x \psi^{-1}(\{w\}) = \{y = (y_1, \ldots, y_n) \in Y \in M : \langle\langle Y, x_j \cdot f_j^t \rangle\rangle = 0, \ 1 \leq j \leq n, \ y_1 + \cdots + y_{n-1} = 0\}$$

$$= \{Y \in M : \langle\langle Y, x_j \cdot f_j^t \rangle\rangle = 0, \ 1 \leq j \leq n-1, \ \langle\langle Y, e_i \cdot f_i^t \rangle\rangle = 0, \ 1 \leq i \leq d\}.$$

where $f := \sum_{1 \leq j \leq n-1} f_j$. Let $a := (a_1, \ldots, a_d) \neq 0$ be a non-zero vector in $\mathbb{R}^{d}$ which is not a multiple of any of the vectors $x_1, \ldots, x_{n-1}, \psi(x)$. The vector $\sum_{1 \leq j \leq d} a_j e_i \cdot \ell^t$ belongs to the orthogonal complement of $\mathcal{L}$. If $\tau_x \psi^{-1}(\{w\}) \subseteq \mathcal{L}$, then on the level of orthogonal complements the inclusion changes direction, that is $\mathcal{L}^\perp \subseteq (\tau_x \psi^{-1}(\{w\}))^\perp$. Consequently,

$$\sum_{1 \leq i \leq d} a_i e_i \cdot \ell^t = \sum_{1 \leq i \leq d} b_i e_i \cdot f_i^t + \sum_{1 \leq j \leq n-1} c_j x_j \cdot f_j^t$$

for some vectors $b := (b_1, \ldots, b_d)$ and $c := (c_1, \ldots, c_{n-1})$. The relation (16) simplifies into:

$$a \cdot \ell^t = b \cdot f^t + X \cdot \text{diag}(c). \quad (17)$$

We prove that the relation (17) cannot hold.
Assume the opposite, that equality (17) holds. In the first step, we show that the vectors $a$ and $b$ are linearly independent. If $b = \lambda a$ for some $\lambda \in \mathbb{R}$, it implies that $a \cdot (\ell^t - \lambda \cdot f^t) = X \cdot \text{diag}(c)$. Since

$$
\ell = \alpha_p f_q - \alpha_q f_p, \quad \text{for some } 1 \leq p < q \leq n - 1, \text{ or }
\ell = \alpha_p f + \alpha_n f_p, \quad \text{for some } 1 \leq p \leq n - 1,
$$

we have that $\ell \notin \text{span}\{f\}$. Then there exists a coordinate of $\ell$ different from $\lambda$, that is $\ell_r \neq \lambda$ for some $1 \leq r \leq n - 1$. Now the $r$th coordinate of the equality $a \cdot (\ell^t - \lambda \cdot f^t) = X \cdot \text{diag}(c)$ implies that $a = \frac{\ell_r - X}{\ell_r - X} f_j$, a contradiction with the choice of the vector $a$.

We conclude that $a$ and $b$ have to be linearly independent. Moreover, since $\text{rank}(X) = d$ then, $a \cdot \ell^t - b \cdot f^t = X \cdot \text{diag}(c)$ has rank 2 implying that only two entries of $\text{diag}(c)$ are non-zero. Without loss of generality we can assume that $c = c_1 f_1 + c_2 f_2$, where $c_1 \neq 0$ and $c_2 \neq 0$. Hence,

$$
a \cdot \ell^t - b \cdot f^t = X \cdot \text{diag}(c) = c_1 (x_1 \cdot f_1^t) + c_2 (x_2 \cdot f_2^t).
$$

Recall that $n \geq d + 2 \geq 4$ which implies $n - 1 \geq 3$. Therefore, the third coordinate of equation (17) implies $\ell_3 a = b + 0$, contradicting the linear independence of $a$ and $b$.

Consequently, the equality (17) cannot hold, which implies that $\mathcal{L}^\perp \not\subseteq (\tau_X \psi^{-1}([w]))^\perp$, or dually $\tau_X \psi^{-1}([w]) \not\subseteq \mathcal{L}$. Hence, there exists a non-zero tangent vector in $\tau_X \psi^{-1}([w])$ which does not belong to any of the linear spaces defined by conditions (iii) and (iv) for a fixed matrix $X$. Therefore, there is a curve in $\psi^{-1}([w])$ passing through $X$ that contains points which do not satisfy (iii) or (iv) as well. Any such point satisfies conditions (14), and the proof of the proposition is complete. 

\section*{References}

[1] Tom Breloff and other contributors, JuliaPlots/Plots.jl.
[2] Peter Bürgisser and Felipe Cucker, Condition, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 349, Springer, Heidelberg, 2013, The geometry of numerical algorithms.
[3] Alan Edelman and Gilbert Strang, Random triangle theory with geometry and applications, Foundations of Computational Mathematics 15 (2015), no. 3, 681–713.
[4] David Eppstein, John M. Sullivan, and Alper Üngür, Tiling space and slabs with acute tetrahedra, Computational Geometry 27 (2004), no. 3, 237–255.
[5] Michael Kapovich and John Millson, On the moduli space of polygons in the Euclidean plane, Journal of Differential Geometry 42 (1998).
[6] Michal Křížek, There is no face-to-face partition of $\mathbb{R}^3$ into acute simplices, Discrete & Computational Geometry 36 (2006), no. 2, 381–390.
[7] John M. Lee, Introduction to smooth manifolds, second ed., Graduate Texts in Mathematics, vol. 218, Springer, New York, 2013.
[8] Gangsong Leng, The minimum number of acute dihedral angles of a simplex, Proceedings of the American Mathematical Society 131 (2003), no. 10.
[9] Sebastian Manecke and Raman Sanyal, Inscribable fans I: Inscribed cones, virtual polytopes, and routed particle trajectories, Preprint, 40 pages, December 2020; arXiv:2012.07724.
[10] Laith Rastanawi, Rainer Sinn, and Günter M. Ziegler, On the dimensions of the realizations spaces of polytopes, Mathematika 67 (2021), no. 2, 342–365.
[11] Rolf Schneider, Convex bodies: The Brunn-Minkowski theory, expanded ed., Encyclopedia of Mathematics and its Applications, vol. 151, Cambridge University Press, Cambridge, 2014.
[12] André Uschmajew and Bart Vandereycken, Geometric methods on low-rank matrix and tensor manifolds, Handbook of variational methods for nonlinear geometric data, Springer, Cham, 2020, pp. 261–313.
[13] Bart Vandereycken, Low-rank matrix completion by Riemannian optimization, SIAM J. Optim. 23 (2013), no. 2, 1214–1226.

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