The quantum Hall effect: The symmetry and the velocity

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Abstract. The complete and the correct interpretation of all of the fractions found in the quantum Hall effect of AlGaAs/GaAs heterostructures is presented by use of a specific treatment of spin. Hence the flux quantization depends on spin and the Lande’s g value expression is replaced by a linear formula. The fractions occur in pairs of left and right helicities and the velocity of the particles is directed such that if one particle moves upstream then its conjugate moves downstream. The change in the sign of the spin is equivalent to the change in the sign of the velocity.

1. Introduction
The Hall resistivity is a linear function of magnetic field. In real systems it is found to have fractional values which require understanding. We explain the quantum Hall effect in GaAs heterostructures and in graphene. We find that the size of the devices is often only a few nm and temperatures of measurements are quite low such as mK. There has been considerable effort to understand the underlying quantum mechanics in terms of wave functions [1,2]. In this paper, we explain the general theory of the quantum Hall effect on the basis of angular momentum theory of quantum mechanics. We explain that electrons of positive as well as negative velocities occur and helicity is used to connect the spin with the velocity. The electrons occur as single particles which explains the “principal fractions”. There are resonances as well as “two-particle” states. A small part of the data requires that “electron clusters” occur which are subject to spin waves and rotations.

2. Theory of quantum Hall effect
We define the cyclotron frequency as,

\[ g\mu_B B = h\omega \]  

(1)

The Bohr magneton is,
\[ \mu_B = \frac{e\hbar}{2mc} \]  

(2)

Hence,

\[ \frac{g eB}{2 mc} = \omega_c. \]  

(3)

Since for \( L=0, g=2 \), it is a common practice to define the cyclotron frequency as,

\[ \omega_c = \frac{eB}{mc}. \]  

(4)

The electrons in a magnetic field behave like harmonic oscillators. Hence, the energy of a state is given by,

\[ E_n = \hbar \omega_c (n + \frac{1}{2}) = \hbar \frac{eB}{mc} (n + \frac{1}{2}) \]  

(5)

It is proper to include the factor of \( g/2 \) so that the above energy is modified to,

\[ E_n = \frac{1}{2} gh \frac{eB}{mc} (n + \frac{1}{2}). \]  

(6)

The same effect can be obtained by replacing \( e \) by \( e^* = (1/2) ge \), so that the energy can be written as,

\[ E_n = \hbar \frac{e^* B}{mc} (n + \frac{1}{2}). \]  

(7)

The transition can occur from \( E_n \) to \( E_n' \). The Hall effect resistivity is,

\[ \rho_{xy} = \frac{B}{ne} = \frac{B}{ne^* c}. \]  

(8)

Hence the effective charge can be measured. The flux quantization is given by,

\[ B.A = n \frac{hc}{e}. \]  

(9)

Substituting this field in the Hall effect formula gives,

\[ \rho_{xy} = \frac{n' \hbar c}{ne^* ceA} = \frac{n' \hbar}{nA ee^*} = n' \frac{\hbar}{\frac{1}{2} ge^2} \]  

(10)

which gives the quantum Hall effect. We do not use the Lande’s formula but suggest a formula linear in the angular momentum. This formula gives,

\[ g = \frac{2j + 1}{2l + 1}. \]  

(11)

Here, \( j = l \pm s \) is the total angular momentum quantum number. The effective charge is thus,

\[ \nu_\pm = \frac{1}{2} g_\pm e = \frac{l + \frac{1}{2} \pm s}{2l + 1}. \]  

(12)

For positive sign and \( s=1/2 \),
\[ v_+ = \frac{l+1}{2l+1} \]
and for the negative sign and \( s=1/2 \),
\[ v_- = \frac{l}{2l+1}. \]  
(14)

Here, 
\[ v_\pm = (1/2)g \]
(15)
gives the correct fractional charge as, for \( L=1 \), \( v_- = 1/3 \) and \( v_+ = 2/3 \), etc. These values agree with experimental data. We call these as “principal fractions”. Instead of the principal values, we can also generate the resonances, \( v_1 - v_2 \). This process produces some more fractions not already present in the “principal fractions”, except in the case when we consider the energy level difference \( E_1 - E_2 \) with \( E_2 = 0 \). Indeed, there is a zero energy state for \( l=0 \), with negative sign, \( s=1/2 \), \( v_- = 0 \). Many resonances are in fact present in the experimental data and indeed predicted from the linear theory. At low temperatures, the excitation populations are small so that interactions are minimized. Hence we predict the “resonances”. We are thus able to produce a large number of principal fractions and resonances so that “two-particle states” occur. For these particles we have \( v_1 + v_2 \) so that these processes produce more fractions than are found in resonances. Hence we have (i) principal fractions, (ii) resonances and the (iii) two-particle states. The real material is often having electron clusters so that the spin need not be \( 1/2 \). The spin becomes
\[ N_S = (N_\uparrow - N_\downarrow)S. \]  
(16)
Hence the spin may be \( 3/2, 5/2 \), etc. For example for \( l=0 \)
\[ \ell = 0 \]  
(17)
the fractional charge can occur at,
\[ e^* = \frac{1}{2} \pm s. \]  
(18)
For finite \( L \),
\[ e^* / e = \frac{l+1}{2l+1} \pm s \]
(19)
For \( S=0 \) which can arise in pairs
\[ N_\uparrow - N_\downarrow = 0 \]  
(20)
so that the effective charge becomes,
\[ e^* / e = \frac{1}{2l+1} + \frac{1}{2}. \]  
(21)
which has even denominator. For \( S=1 \),
\[ e^* / e = \frac{l+1}{2l+1} \pm \frac{1}{2} \]
(22)
which has even denominators and occurs in pairs.
3. Spin deviation

For a large cluster, there is a small spin deviation so that the spin ceases to be an integer. For example, \( S - \delta S = 2 - 0.02 = 1.98 \)

which has 1% spin deviation. All of the data is correctly explained on the basis of this theory and we have written down the interpretation of 101 fractions satisfactorily. The fractions of charge which are measured from the plateaus in the Hall resistivity

\[ \rho_{xy} \]

are usually the same as those measured from the minima in \( \rho_{xx} \). In the case of electron clusters, the fractions derived from xx direction are slightly different from those found from xy values. The effective charge of the electron becomes anisotropic due to the spin wave propagation in the micro-cluster of electrons. The wave vector of the spin waves appears in the effective value of the spin. There is an explicit dependence of the charge on the spin which modifies the condition of the flux quantization and leads to an anisotropic charge. In the Hall effect, the resistivity is a linear function of magnetic field. When field is quantized there occur plateaus in the \( \rho_{xy} \) and minima in \( \rho_{xx} \). The plateaus at the integer multiples of \( \hbar e/2 \) were first detected by von Klitzing et al [3]. They found that the resistivity is quantized at \( \rho_{xy} = \hbar / ie^2 \) where \( i \) is an integer.

Later on Tsui et al [4] working at still lower temperatures and higher magnetic fields discovered that \( i \) need not be an integer as it becomes a fraction such as 1/3 or 2/3. At this stage, Laughlin wrote a wave function [1] by using the first principles and suggested that the charge may become fractional such as \( e/3 \) or \( e/5 \) due to electron correlations. Many efforts have been made to suggest composite fermions, parafermions and non-Abelian fermion pairs to explain the fractionally charged quasiparticles. Instead of starting from a wave function, we looked at the factors which multiply the Bohr magneton which change the effective charge of the electron. We found that there is a possibility of a spin-charge coupling which changes the charge of a particle [2]. The factor of \( g = (2j + 1)/(2l + 1) \) multiplies the Bohr magneton and hence changes the charge of the electron to \( e^* = (1/2)ge \) which is \( e \) for \( g = 2 \). In actual samples which have a layer of GaAs over Al doped GaAs, electron clusters occur. In the clusters the spin need not be \( 1/2 \). For example, it can be \( 3/2 \) if three electrons are aligned parallel to each other as in a ferromagnet. Hence the spin becomes large, such as NS. It is also possible that the electrons are aligned in such a way that there are \( N_+ \) electrons with spin up and there are \( N_- \) electrons with spin down so that the spin of the electrons in the cluster becomes NS with \( N = N_+ - N_- \). Hence the idea of “clusters” of electrons explains the data. In this way, all of the 101 fractions can be explained. Instead of detecting one fraction at a time, some times a full series of fractions can be written down. Such series can be generated by using higher than \( s=1/2 \) and by varying the orbital angular momentum quantum number \( l \). These series of fractions are observed in graphene and correctly predicted [2]. Recently, Kumar et al [6] have found that a fraction which shows one value when measured from the plateau in the \( \rho_{xy} \) need not show the same value when measured from the minimum in \( \rho_{xx} \). We show that there are spin waves in small clusters of electrons which create a finite “spin deviation” giving rise to anisotropy in the fractional charge. The spin deviation depends on the spin wave vector and makes the spin value anisotropic which due to spin-charge coupling makes the charge anisotropic and hence the effective charge along xy plane is not equal to that in the xx direction. We find that a small difference occurs in the value of the fraction in going from xx to xy value. The Hamiltonian of the ferromagnetic spin waves with exchange interaction is given by,

\[ H = -J \sum_{j<\delta} S_j^x S_\delta^x - g \mu_B \sum_j S_j^z \]

(24)
where $J$ is the exchange interaction and $H$ is the magnetic field. $S_j$ are the spin operators at the $j$th site and the summation can be carried out to nearest neighbors, $S_{j+r,s}$. The number of nearest neighbors is $z$ so that the Fourier transforms of spin operators requires,

$$\gamma_k = \frac{1}{z} \sum_S \exp(ik\cdot\delta)$$

(25)

Leaving out the magnetic field dependent term, the unperturbed frequency of a magnon is,

$$\hbar\omega_k = 2JzS(1-\gamma_k)$$

(26)

Hence, the spin has been changed from $S$ to $S(1-\gamma_k)$ in going from the site variables to spin wave variables. This is called the spin deviation and it amounts to a few per cent in real materials. For $k\cdot\delta \ll 1$, $\exp(ik\cdot\delta)$ has only the sine term and the cos term is zero so that for small wave vectors,

$$z(1-\gamma_k) \approx \frac{1}{2} \sum_S (k\cdot\delta)^2$$

(27)

so that

$$\hbar\omega_k \approx JS\sum_S (k\cdot\delta)^2$$

(28)

which for cubic lattices is $2JS(k.a)^2$ where a is the lattice constant. Hence, the spin is changed to $S(k.a)^2$. In the case of two-sublattice antiferromagnets, $\hbar\omega_k \approx 4\sqrt{3}Jsk_a$. In a cluster, in the a-b plane, the frequency of a magnon is given by the expression $2JS[(k.a)^2 + (k.b)^2]$ whereas along the x direction it appears as $2JS(k.a)^2$. Hence, the spin value becomes anisotropic.

The Hall effect resistivity is,

$$\rho_{sy} = \frac{h}{2e^2} = \frac{h}{l + \frac{1}{2} + \frac{s}{2(l+1)}}$$

(29)

In the case of an electron cluster, the value of $S$ along xx direction is $NS$ which due to spin waves becomes $S(k.a)^2N$ whereas along the xy direction it will be $\frac{1}{2}S[(k.a)^2 + (k.b)^2]N$. Hence the value of $(1/2)g$ depends of the direction. The value of the filling factor $\nu = (1/2)g$ also depends on the direction. Hence $\nu_{xx}$ need not be equal to $\nu_{sy}$ where $\nu_{xx}$ is deduced from $\rho_{xx}$ and $\nu_{sy}$ from $\rho_{sy}$. Kumar et al [6] have found that in the xx direction $\nu = 2.463\pm0.002$ whereas in the xy direction $\nu = 2.461$. Hence, the value deduced from the xy is slightly different from that deduced from the xx which we assign to the value of the spin due to spin waves. For $L=0$, the filling fraction is

$$\frac{1}{2} \pm s = \frac{2463}{1000}.$$ 

Hence $\pm s = 1.963$ which shows 1.8 % spin deviation compared with $S=2$. The Hall effect experiment is performed on a polycrystalline heterostructure film so that the crystallographic directions are not the same as those used in the Hall effect.

4. Helicity and velocity

In single particles, the helicity is defined as the projection of its spin along the direction of motion. It is impossible to change the helicity of a particle because it would then become its antiparticle. In the solid state it is possible to consider a cluster of particles which permits the observation of a
phenomenon not found in single particles. Similarly, in molecules it is possible to see some phenomenon which is not seen in single particles. The wave function of a particle may be made of a linear combination of two particles which differ in helicity. The helicity of one particle may be positive while that of the other may be negative and we can make a third particle which is a combination of the first and the second particle. A particle of mixed helicity may mean that it is made of particles of different helicities. It does not mean that a particle of positive helicity can change itself into that of a negative helicity. This means that “reversed helicity” does not occur. Under the rotation it is possible to change the direction of the velocity and hence the helicity of a particle in a molecule or in a solid. In the quantum Hall effect, the resistivity is quantized at the plateau. The usual Hall effect occurs with the resistivity linearly proportional to the magnetic field, \( \rho_{xy} = B / ne c \) where \( B \) is the magnetic induction, \( e \) is the electron charge, \( c \) is the velocity of light and \( n \) is the electron concentration, the number of electrons per unit area for a slab of a metal. The flux quantization leads to plateaus at which \( \rho_{xy} = h / ie^2 \) where \( h \) is the Planck’s constant and \( i \) is an integer. When angular momentum is considered, \( i \) becomes a fraction. Due to particle-antiparticle symmetry the fractions occur in pairs, one for the positive spin and the other for the negative spin. For single particles, the resistivity at the plateau is given by,

\[
\rho_{xy} = \frac{h}{2g_\pm e^2}
\]

where \( g = (2j+1)/(2l+1) \) and \( j = l \pm s \). The \( \pm \) sign in the expression for \( j \) introduces the particle-antiparticle symmetry. Since the signs occur in the spin, the values of the resistivity correspond to two helicities. The values of \( (1/2)g_\pm \) for the two helicities give two different plateaus in the resistivity. The spin as well as the helicities thus enter in the problem of Hall resistivity. The plateaus which occur for a single value of \( l \) and \( s \) are arising from the “principal fractions”. The two-particle states occur for \( \nu_1+\nu_2 \) and resonances occur at \( \nu_1-\nu_2 \) where \( \nu = (1/2)g_\pm \) and the effective charge of a particle becomes \( e^\# = (1/2)e \). In Al\textsubscript{1-x}Ga\textsubscript{x}As/GaAs heterostructures, clusters of electrons are often formed in which case the spin is different from 1/2. The large values of spin are required to understand the plateaus which depend on the sample and show spin wave reduction in the value of the spin. The spin larger than 1/2 occurs in clusters which are subject to rotation. Hence, “roton” type effect is seen in the quantum Hall effect of clusters of electrons. The single-particle states with spin 1/2 occur at \( (1/2)g_\pm \). The positive sign corresponds to + helicity (right handed) and the negative sign corresponds to - helicity (left handed). For \( l=0 \) the above expression reduces to \( (1/2)\pm s \) so that the effective charge for \( s=1/2 \) becomes 1 and corresponds to + helicity and negative sign gives zero charge and corresponds to - helicity. For \( l = 1 \), the denominator is 2 \( l+1 = 3 \) and the numerator gives \( (3/2)\pm s \) which is 2 for the + sign and 1 for the negative sign. Hence 1/3 belongs to negative helicity and 2/3 belongs to positive helicity. These are single-particle states which can not rotate. There is no doubt that all of the predicted series of fractions \( l/(2l+1) \) as well as \( (l+1)/(2l+1) \) are the same as in the experimental data. The two particle state of charge 1/3 occurs at \( (1/3)+(1/3) = 2/3 \) with negative helicity. The one-particle state at 2/3 has positive helicity. Hence the two particle state at 2/3 and the one-particle state also at 2/3 are degenerate. Hence a degenerate state of mixed helicities is predicted at 2/3. The two-particle state at 2/3 can rotate so that the direction of the velocity changes. The state at 1/3 has negative helicity. The two-particle state at \( (1/3)+1/3 = 2/3 \) also has negative helicity. Hence 2/3 corresponds to mixed helicity. The sign of the charge is that of the electron charge in both cases. When a particle has energy proportional to 2/3, then two-particle state occurs at \( (2/3)+(2/3) = 4/3 \). Now there is a particle at 4/3 with positive helicity and it is made of two-particles of charge 2/3 each. The particle of charge 1/3 is of negative helicity. The three-particle state occurs at \( (4/3)+(1/3) = 5/3 \). It is made of particles of positive as well as negative helicities. By rotation, the direction of velocity changes. One 2/3 becomes
1/3 and one 1/3 becomes 2/3. Hence, \((2/3)+(2/3)+(1/3) = (2/3)+(1/3)+(2/3) = 5/3\) so that charge is conserved and the helicities of some of the charges are reversed except that one of the values should not rotate. We started with the odd number of particles and reversed the helicities of only two values. It is possible to start with even number of values in which we reverse the helicities of even number of charges so that we reverse the helicities without loosing the charge conservation. The state 7/3 is made of \((2/3)+(2/3)+(1/3)+(2/3) = 7/3\). This is a cluster of 4 electrons. Upon rotation 1/3 changes into 2/3 and 2/3 changes to 1/3 so that 7/3 is an example of mixed helicities as well as “reversed helicities”.

5. Noise power and helicity

The concept of holes is obtained from the absence of electrons from the valence band. Hence, the holes occur in the band theory of solids. Since the electrons are negatively charged, the holes are positively charged and there is a one to one correspondence due to the conservation of energy. When light shines on the solid which has a periodic potential, a hole with positive charge is created in the valence band along with an electron in the conduction band. In the periodic potential, the 1/3 charge is replaced by \(\pm 1/3\) and 2/3 is replaced by \(\pm 2/3\) and so on and so forth. For electronic states of fractional charges such as 2/3, 3/5, 8/3, 5/2 etc. there are hole states of opposite charge. Unfortunately, the experimentalist neither identify the spin nor the symmetries of the fractions. So they are unable to identify the conjugate pairs. The noise measurements \([7]\) indicate that there are neutral modes which accompany the charged modes. The noise power at a point is of the form,

\[
S = 2g_Q \int_0^\infty f(1-f) dE
\]

where \(f(1-f)\) is a fermion factor which arises from the creation of an electron at one site accompanied with the destruction of an electron at another site and \(g_Q\) is the conductance of the material at the quantum contact point. The thermal excess noise power is of the form,

\[
S_{\text{excess}} = 2\Delta E_g g_Q t(1-t) [\coth(\Delta E_g /2k_B T) + 2k_B T/\Delta E_g]
\]

where \(\Delta E_g\) is the change in the Fermi energy, \(t(1-t)\) is the transmission probability and \(T\) is the temperature. The conductance at a point at 1/3 is , for example,

\[
g_Q = (1/3) e^2/h.
\]

For \(t = 0, s = +1/2, (1/2)g_+ = (1/2)-s=0\) which is electrically neutral or a neutral particle. Two particles of charge 1/3 each and one hole of charge -2/3 create a zero charge. Similarly, two holes of charge -1/3 each and one particle of charge 2/3 can produce charge neutrality. In the experimental data \([8]\) of AlGaAs/GaAs also there is a zero-energy state with \(g = 0\). The excess noise power has the factor \(g_Q = (1/3)e^2/h\) accompanied with \((2/3)e^2/h\) which differ in helicity. Hence, the noise power changes upon change in the helicity. Some times the experimentalists \([9,10]\) use the idea of chirality which is similar to that of the helicity defined in the present work as the sign of \(p\). The sign of spin \(s\) becomes the sign of the velocity. So that the particles of + helicity do not move in the same direction as those of - helicity. By applying a heater on the sample we can see the electrons move with fractional charges and the electrons of opposite helicities move in opposite directions. The particles which are clusters of electrons with mixed helicity rotate with velocity in two directions.

6. Conclusions

We find that every fraction at which a plateau occurs in the Hall resistivity is subject to helicity. There are both right as well as left handed fractions in the quantum Hall effect. We find that mixed helicity as well as “reversed helicity” occurs in various fractions in the quantum Hall effect. The activation
energy is found to become double valued. The doubling occurs in the conductivity so that the noise power changes upon change in the helicity. All of the experimental data[11,12] is consistent with the angular momentum theory [13,14].

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