Inflation in asymptotically safe $f(R)$ theory

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Abstract. We discuss the existence of inflationary solutions in a class of renormalization group improved polynomial $f(R)$ theories, which have been studied recently in the context of the asymptotic safety scenario for quantum gravity. These theories seem to possess a nontrivial ultraviolet fixed point, where the dimensionful couplings scale according to their canonical dimensionality. Assuming that the cutoff is proportional to the Hubble parameter, we obtain modified Friedmann equations which admit both power law and exponential solutions. We establish that for sufficiently high order polynomial the solutions are reliable, in the sense that considering still higher order polynomials is very unlikely to change the solution.

1. Introduction

The idea that a perturbatively nonrenormalizable theory could be consistently defined in the UV limit at a nontrivial fixed point (FP), often called “asymptotic safety”, is theoretically very attractive, especially when applied to quantum gravity [1]. Much progress in this direction has come from the direct application of Renormalization Group (RG) techniques to gravity. A particularly useful tool has been the non-perturbative Functional Renormalization Group Equation (FRGE) [2], defining an RG flow on a theory space which consists of all diffeomorphism invariant functionals of the metric $g_{\mu\nu}$. It defines a one parameter family of effective field theories with actions $\Gamma_k(g_{\mu\nu})$ depending on a coarse graining scale (or “cutoff”) $k$, and interpolating between a “bare” action (for $k \to \infty$) and the ordinary effective action (for $k \to 0$).

The application of such concepts to a cosmological framework is based to the following logic [3]. If we want to study the quantum evolution of the cosmic scale factor, we should in principle use the full effective action $\Gamma(g_{\mu\nu})$, which, as we mentioned above, coincides with $\Gamma_k(g_{\mu\nu})$ in the limit $k \to 0$. However, our knowledge of this functional is rather poor. One way of gaining some traction on this issue is to observe that the Hubble parameter appears as a mass in propagators. Thus, the contributions of quantum fluctuations with wavelengths greater than $H^{-1}$ are suppressed. As a result, the functional $\Gamma_k$ at $k$ comparable to $H$ should be a reasonable approximation for the same functional at $k = 0$. We do not know $\Gamma_k$ much better than the full effective action, but we can easily calculate the dependence of some terms in $\Gamma_k$ on $k$. By doing so we effectively take into account nonlocal terms that would be very hard to calculate otherwise.

Previous investigations along these lines, in particular [4], have been based on the Einstein-Hilbert (EH) truncation. It is important to establish that the results obtained there persist when further operators are included in the truncation. We know that at the FP the coefficients of these terms are not very small, but their presence does not seem to affect the values of the

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cosmological constant and Newton’s constant too much. In other words, the FP that is is seen
in the EH truncation seems to be robust. The question then is to see if this stability of the FP
against the inclusion of new terms is reflected in the stability of the corresponding solutions.
This question is important because the values of the couplings at the FP are fixed and as a
consequence there are no free parameters to be varied. We will see that the (power-law or
exponential) inflationary solutions are indeed stable against the inclusion of new terms, but
establishing this fact requires including a rather large number of terms.

2. Asymptotic Safety

It is well known that general relativity can be treated as an effective field theory. This means
that it is possible to compute quantum effects due to graviton loops, as long as the momenta
of the particles in the loops are cut off at some scale. The results are independent of the structure
of any ultraviolet completion, and therefore constitute genuine low energy predictions of any
quantum theory of gravity.

Given for a field \( \phi \) an effective action \( \Gamma(\phi) \), it can always be written as a sum

\[
\Gamma(\phi) = \sum g_i \mathcal{O}_i(\phi)
\]

that runs over the infinitely many operators consistent with the underlying symmetries of the
theory. Given an energy scale \( k \), one can define dimensionless couplings \( \tilde{g}_i = g_i k^{-d_i} \), where
\( [g_i] = [k]^{d_i} \). The resulting picture is an infinite dimensional space parametrized by the \( \tilde{g}_i \)'s,
where a Renormalization Group flow can be seen as a vector field that shifts any action along a
trajectory

\[
\Gamma(\phi) \to \Gamma_k(\phi) = \sum \tilde{g}_i(k) k^{d_i} \mathcal{O}_i(\phi) .
\]

The particular realization of RG that was used in this work is the one \( à la \) Wilson, based on
the idea that the effective action describing physical phenomena at a momentum scale \( k \) can
be thought of as the result of having integrated out all fluctuations of the field with momenta
larger than \( k \). This can be achieved adding to the bare action \( S \) a term

\[
\Delta S_k = \int dx \phi(x) R_k \phi(x)
\]

where \( R_k \) is called the cutoff kernel (see [5] for details). This term acts as a mass for the
fluctuations with momenta smaller than \( k \). Following the usual procedure of background field
method, we define a \( k \)-dependent generating functional of connected Green functions by

\[
e^{-W_k(J)} = \int D\phi e^{-S(\phi)+\Delta S_k(\phi)} - \int dx J \phi
\]

and the effective action as its Legendre transform

\[
\Gamma_k(\phi) = W_k(J) - \int dx J \phi - \Delta S_k(\phi) .
\]

One can show that it obeys the so called functional renormalization group equation (FRGE)

\[
k \frac{d\Gamma_k}{dk} = \frac{1}{2} \text{Tr} \left( \Gamma_k^{(2)} + R_k \right)^{-1} k \frac{dR_k}{dk}
\]

where \( \Gamma_k^{(2)} \) is the inverse propagator of the graviton. From the r.h.s. of (6) one can then extract
the \( \beta \)-functions of the couplings, as it is done in [5], and study the RG flow of the theory towards
the UV limit \( k \to \infty \).

A theory is then said to be asymptotically safe if
• for any coupling $\tilde{g}_i$ there exists a finite UV limit $\tilde{g}_i^*$ that is called a fixed point

• the stability matrix $M_{ij} = \left. \frac{\partial \beta_{i*}}{\partial \tilde{g}_j} \right|_{\tilde{g}_i^*}$ evaluated in the fixed point has a finite number of negative eigenvalues $m_i$ (in the following, it will come useful to define critical exponents $\theta_i = -m_i$).

If both the requests are satisfied one finds that the critical surface, the locus of the points that are attracted towards the FP, is finite dimensional. This means that only a finite number of experiments is needed to univocally identify a trajectory, and the theory is thereby predictive.

In [5] this procedure was applied to a (truncated) gravitational action consisting of a sum of powers of the curvature scalar $R$

$$\Gamma(g) = \int d^4x \sqrt{|g|} \sum_{i=0}^{n} g_i R^i$$

with $n = 2, \ldots, 8$, and it was further extended to $n = 9, 10$ in [6]. For every $n$ a fixed point was found, with only three positive critical exponents. To understand if a finite truncation is a reliable approximation of the action (1), one can study the stability of the position of the fixed point $\tilde{g}_i^*$ and of the values of the critical exponents $\theta_i$: these values should stay almost constant with respect to the inclusion of new operators in the action. In figure (2) it is clearly visible that this seems to be the case, so that our finite sum is a reliable truncation.

**Figure 1.** Position of the fixed point and values of the critical exponents as functions of the truncation order $n$. 
3. Asymptotically safe cosmology

It is now time to apply the above described formalism to a cosmological background. There are two main issues that are worth stressing, the first concerning the possibility to obtain any inflationary solution, and the second about the resulting stability of such solutions with respect to the inclusion of new terms in the action. We will address them after having defined a key ingredient of our analysis: the identification of the RG scale $k$ with a characteristic scale of the described system.

As it was said before, in the Wilsonian approach the cutoff scale is the momentum scale at which the process is taking place. This means that, in a Friedmann-Robertson-Walker metric, it should be set to the order of magnitude of the Hubble rate $H(t)$. Another argument can be found in the comparison with the framework of quantum field theory in curved spaces, where the Hubble rate appears as a mass term in the propagator of the field. We then set $k = \xi H$, where $\xi = \mathcal{O}(1)$ is a free parameter that in principle can be tuned to offset the scheme dependence carried by the cutoff scale.

3.1. Inflationary cosmology

We have now all the ingredients to extract the desired inflationary solutions. From the action

$$\Gamma(g) = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} (f(R) - \Lambda)$$

where $f(R) = \sum f_i R^i$, we derive the modified Einstein equations that, after having imposed the FRW symmetry, appear as (modified) Friedmann equations

$$A(H) = 8\pi G\rho + \Lambda$$
$$B(H) = 8\pi G\rho(1 - 3w) + 4\Lambda$$

where $w$ is the equation of state of the ordinary matter and $A(H)$, $B(H)$ are two functions of the Hubble rate whose shape depends on the chosen $f(R)$, and can be rearranged in the form

$$B(H) = (1 - 3w)A(H) + 3(1 + w)\Lambda$$
$$\rho = \frac{1}{8\pi G} (A(H) - \Lambda).$$

Before solving the equations, we have to perform the above mentioned cutoff identification: the couplings are rewritten as

$$\Lambda = \tilde{\Lambda}k^2; \quad G = \tilde{G}/k^2; \quad f_i = \tilde{f}_i k^{2-2i}$$

and, because we are interested in describing the dynamics of the very early universe, are assumed to lie in the UV fixed point. With these substitutions, we look for solutions of the (11) in the form of a de Sitter ($H(t) = \bar{H}$ and $a(t) \propto e^{Ht}$) and a power law ($H(t) = \frac{p}{t}$ and $a(t) \propto t^p$). It comes out that the system admits the former only for discrete values $\xi_{dS}$ of the free parameter, while we found continuous solutions of the latter type, with $p = p(\tilde{\Lambda}^*, \tilde{G}^*, \tilde{f}^*_i, w, \xi)$. As a final remark, the functions $p(\xi)$ show vertical asymptotes, located exactly at the values $\xi_{dS}$: de Sitter is then recovered as a limit of power law.

In figure (3.1) the power law exponents $p$ obtained for $n = 8$ are plotted as functions of $\xi$. There is in particular one growing solution that is greater than 1 (i.e. $\ddot{a}/a > 0$) for most values of the free parameter.
Figure 2. Power law exponents as functions of the parameter \( \xi \). The asymptote is clearly visible for \( \xi \approx 3.9 \).

3.2. Reliability analysis

As it was said before, the UV fixed point of the gravitational action (7) tends to stabilize as one widens the truncation. The issue is then whether the values of the solutions will show the same behaviour. We thus define a reliability criterion based on the following logic: taken the dimensionless curvature scalar \( \tilde{R} = Rk^{-2} \), we write the dimensionless action with truncation \( n \) evaluated at the fixed point \( \tilde{f}(\tilde{R}) \bigg|_n \) and we compare it with the same object in \( n-1 \). A truncation is then said to be reliable if

\[
\left| \frac{\tilde{f}(\tilde{R})\bigg|_n - \tilde{f}(\tilde{R})\bigg|_{n-1}}{\tilde{f}(\tilde{R})\bigg|_{n-1}} \right| < 5\% \tag{14}
\]

and in general this is only true for a limited range of values of \( \tilde{R} \). One can schematically rewrite this as \( \tilde{R} < c \), where

| \( n \) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| \( c \) | 0.40 | 0.24 | 0.45 | 1.07 | 1.21 | 0.90 | 0.79 | 0.92 | 1.09 | 1.09 |

The table shows that the reliable range tends to increase, so that a larger truncation is more likely to correctly approximate the infinite truncation \( n \to \infty \).

For a power law solution, the constraint on the dimensionless curvature scalar can be translated into a constraint on the values of the exponents, namely

\[
\alpha \xi^2 \gtrsim 6 \left( 2 + \frac{\dot{H}}{H^2} \right) = 6 \left( 2 - 1/p \right) \tag{15}
\]

that can be plotted together with the exponents themselves, as it is shown in figure (3.2) on the left. From the plot it comes clear that only a small portion of the growing solution is to be considered reliable, so we decided to extend the truncation to \( n = 10 \). The result is plotted in figure (3.2) on the right, that shows a great improvement in the reliability of the solutions.
Figure 3. Power law exponents as functions of $\xi$ for $n = 8$ (left) and $n = 10$ (right). Areas to the right of the dashed lines are the reliability regions, while the dotted line indicates the $n = 1$ solution described in [4].

4. Conclusions

The results obtained here can be summarized saying that, if we identify the cutoff with a multiple of the Hubble parameter, inflationary power law solutions (i.e. with exponent $p > 1$) exist for some range of values of $\xi$. The dependence on such parameter is strong, with the exponents diverging at some values $\xi_{dS}$. Exactly at those points, the theory admits also de Sitter solutions. Provided that $\xi$ lies in the above range, that the starting point is close enough to the FP and that $p > 1$, it should always be possible to have a sufficient number of $e$-foldings.

Moreover, it has been shown that the increasing stability of the fixed point with respect to the widening of the truncation seems to be reflected in an increasing reliability of the cosmological solutions obtained using the truncated action. The overall conclusion that one can draw is that we found a strong hint of the convergence of the series in our fixed point action, so that any result coming from low order truncations should not be too misleading.

References

[1] S. Weinberg, “Ultraviolet divergences in quantum theories of gravitation”, in “General Relativity, an Einstein Centenary Survey” ed. S. W. Hawking and W. Israel, Cambridge University Press, 1979.
[2] C. Wetterich, Phys. Lett. B301 (1993) 90.
[3] M. Reuter and H. Weyer, Phys. Rev. D69 (2004) 104022 and hep-th/0311196
[4] A. Bonanno and M. Reuter, JCAP 08 (2007) 024 and arxiv:hep-th/07060174.
[5] A. Codello, R. Percacci and C. Rahmede, Int. J. Mod. Phys. A 23 (2008) 143, arXiv:hep-th/07051709.
[6] A. Bonanno, A. Contillo and R. Percacci arXiv:gr-qc/10060192.