Magnetised Electron-Positron Plasmas

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Abstract. Electrostatic oscillations in cold electron-positron plasmas can be coupled to a propagating electromagnetic mode if the background magnetic field is inhomogeneous. Previous work considered this coupling in the quasi-linear regime, successfully simulating the electromagnetic mode. Here we present a stability analysis of the non-linear problem, perturbed from dynamical equilibrium, in order to gain some insight into the modes present in the system.

1. Introduction

Pulsar magnetospheres are composed of magnetised electron-positron plasmas. Conventional modelling uses the fact that the dominant Lorentz force produces a host of relativistic charged particles, each of which radiates strongly and stochastically, producing γ-ray photons. Such single-particle models have been explored as possible radiation sources (da Costa & Kahn 1997), but have not been able to recover the highest energy radiation observed.

However this strict single particle approach may be challenged as there are physical processes which develop in the rest frame of the plasma, and are strongly amplified when translated in the laboratory frame of the pulsar. In cold non-relativistic plasma theory non-linear electrostatic oscillations in electron-positron plasmas develop a density instability in which the density of both species grows sharply at the edges of the oscillation site (da Costa et al. 2001). Folding thermodynamics into the system provides a possible mechanism for avoiding the onset of the instability, since pressure effects would oppose the density build up. However, a more advantageous mechanism would be coupling the oscillation to an electromagnetic mode via an inhomogeneous background magnetic field. This would allow energy to be radiated away from the oscillation, quenching the density instability and giving a source of radiation in the pulsar magnetosphere. Previous work has studied this mechanism and successfully simulated the coupling between the oscillation and the electromagnetic mode in the quasi-linear regime (Diver et al. 2002). Therefore the study of these processes is of the greatest importance for the dynamics of the magnetosphere, and the radiation mechanisms of pulsars.

This paper addresses the basic mathematical formulation of the electron-positron cold magnetoplasma and the mode coupling mechanism in Section 2. In Section 3 linear analysis of the equations perturbed from dynamical equilibrium is performed to gain an understanding of the modes present in the plasma and their stability.

2. Model equations

In cylindrical polar coordinates (r, θ, z) with an axial magnetic field \( B = 2B_z \) and an electric field in the r, θ plane \( E_{r,\theta} \) the full non-linear model equations for a cold electron-positron plasma are

\[
\begin{align*}
    r\dot{n}_+ + (rn_+u_r)' &= 0 \\
    r\dot{n}_- + (rn_-v_r)' &= 0 \\
    \dot{u}_r + u_r u'_r - u_0^2/r &= (e/m)(E_r + u_0 B_z) \\
    \dot{u}_\theta + u_r u'_r + u_r u_\theta /r &= (e/m)(E_\theta - u_0 B_z) \\
    \dot{v}_r + v_r v'_r - v_0^2/r &= -(e/m)(E_r + v_0 B_z) \\
    \dot{v}_\theta + v_r v'_r + v_r v_\theta /r &= -(e/m)(E_\theta - v_0 B_z) \\
    (rE_r)' &= (e/e_0)c(n_+ - n_-) \\
    0 &= -\dot{E}_r/c^2 - \mu_0 e(n_+u_r - n_- v_r) \\
    (rE_\theta)' &= -r\dot{B}_z \\
    B'_z &= -\dot{E}_\theta/c^2 - \mu_0 e(n_+u_\theta - n_- v_\theta)
\end{align*}
\]

where \( n_+, n_- \) are the positron and electron number densities; \( u, v \) are the positron and electron velocities; \(' \) denotes \( \partial/\partial t; \) \( ' \) denotes \( \partial/\partial r; \) \( (\Box) \) is Poisson’s equation; \( (\mathbf{B}) \) is the single z component of the induction equation; and \( (\nabla \times \mathbf{B}) \) is the \( \theta \) and \( r \) components of the \( \nabla \times \mathbf{B} \) equation. The equations can be recast into a form that highlights the symmetry of the electron-positron plasma, this can be...
where \( L, \beta \) are characteristic length, time scales and \( \beta_0, \beta \) are the equilibrium and perturbed magnetic field respectively. The system of governing equations then becomes

\[
\begin{align*}
\dot{\Sigma} &= -\left(\Sigma\sigma + \Delta\delta\right)'
\end{align*}
\]

(22)

\[
\dot{\Delta} = -\left(\Delta\sigma + \Sigma\delta\right)'
\]

(23)

\[
\dot{\sigma} = -\frac{1}{2} \left(\sigma^2 + \delta^2\right)' + \left(\chi^2 + \zeta^2\right)/\xi + \zeta (\beta_0 + \beta)
\]

(24)

\[
\dot{\delta} = -\left(\sigma\delta\right)' + 2\chi\zeta/\xi + \rho + \chi (\beta_0 + \beta)
\]

(25)

\[
\dot{\chi} = -\chi' (\sigma - \zeta\delta) - (\chi\sigma - \zeta\delta)/\xi - \delta (\beta_0 + \beta)
\]

(26)

\[
\dot{\zeta} = -a\zeta' + \delta' - \sigma' - (\sigma\zeta + \delta\chi)/\xi + \theta - \sigma (\beta_0 + \beta)
\]

(27)

\[
\dot{\rho} = -\frac{2}{\xi} (\Delta\sigma + \Sigma\delta)
\]

(28)

\[
\dot{\theta} = -p (\beta_0 + \beta) - \frac{2}{\xi} (\Sigma\zeta + \Delta\chi)
\]

(29)

\[
\dot{\beta} = -\theta' - \theta/\xi
\]

(30)

where \( p = c^2/(\omega_0^2 L^2) \).

### 2.1. The coupling mechanism

Consider a homogeneous background magnetic field. When the density of the plasma is perturbed, resulting in a charge imbalance, a radial electric field is created that accelerates the electrons and positrons in radially opposite directions. The plasma collectively is responding to the presence of the electric field and is trying to restore equilibrium. The magnetic field causes the particle trajectories of both species to participate in partial Larmor orbits with the same azimuthal velocity. The particles overshoot their initial positions, due to their acquired kinetic energy, and produce a new charge imbalance that they again try to correct. This is one half period of an electrostatic oscillation.

Introducing an inhomogeneous magnetic field induces a \( B \times \nabla B \) drift which causes a net current density in the azimuthal motion of the plasma during the electrostatic oscillation. The resulting current density induces axial magnetic field fluctuations which propagate away from the electrostatic oscillation site. This mechanism causes the electrostatic oscillations to couple to a propagating electromagnetic mode in the plasma.

This effect has been fully investigated in the quasi-linear regime, in which the variables were perturbed from a uniform equilibrium. However having an inhomogeneous magnetic field permeating the plasma requires the equilibrium to be non-uniform for the system to be self-consistent.

### 2.2. Dynamical equilibrium

In the equilibrium situation \( \partial/\partial \tau = 0 \) and \( \Delta = \rho = \theta = \beta = 0 \). The resulting dynamical equilibrium initial condition is described by

\[
\begin{align*}
\dot{\Sigma}_0 &= -\frac{\xi_0}{\zeta_0} - \beta_0
\end{align*}
\]

(31)

\[
\begin{align*}
\dot{\beta}_0 &= -\frac{2\kappa_0}{\kappa_1} \sqrt{\kappa_1^2 - \zeta_0^2}
\end{align*}
\]

(32)

\[
\begin{align*}
\sigma_0 &= \sqrt{\kappa_1^2 - \zeta_0^2}
\end{align*}
\]

(33)

\[
\begin{align*}
\Sigma_0 &= \kappa_0/\sigma_0
\end{align*}
\]

(34)

where 0 subscripts denote equilibrium value and \( \kappa_0, \kappa_1 \) are constants. The equilibrium equations describe the self-consistent response of the plasma to a prescribed background inhomogeneous magnetic field, \( \beta_0 \). Conversely they describe how the plasma must behave in order to generate the same magnetic field. Equations (32) defines the kinetic energy conservation of the equilibrium flow; equation (33) defines the conservation of the total number density flux; and equations (32), (31) describe the magnetic field generation via the \( \zeta_0, \beta_0 \) coupling.

Linearise the full set of governing equations and look at large values of \( \xi \). In this regime the magnetic field is asymptotically tending to a constant value, \( \beta_0 = constant \), corresponding to no motion of the plasma in the azimuthal direction \( \zeta_0 = \chi_0 = 0 \), saying that radial flow dominates. This requires \( \sigma_0 = constant \) to be consistent with equation (33) and implies that in this parameter set that there is a net motion of the plasma in the radial direction with no net current, \( \delta_0 = 0 \). The governing equa-
tions then become,
\[ \dot{\chi} = -\sigma_0 \chi - \sigma_0 \chi - \delta \beta_0 \]
\[ \dot{\xi} = -2 \sigma_0 \xi + \sigma_0 \xi - \sigma_0 \beta - \beta_0 \sigma + \theta \]
\[ \dot{\rho} = -2 \xi (\Delta \sigma_0 + \Sigma \sigma_0) \]
\[ \dot{\theta} = -\rho \beta + 2 \sigma_0 \xi / \xi \]

3. Stability analysis

Upon inspection of the linearised governing equations it is evident that they can be split into two independent, self-consistent sets namely equations (36), (38), (39), (41) forming one set and equations (35), (37), (40), (42) forming the other. This simplification was exploited to obtain the following solutions.

3.1. Electrostatic solution

The electrostatic solution is characterised by \( \beta = \beta_0, \theta = 0 \) and \( \xi = 0 \). Substituting these conditions into the governing equations yields,
\[ \dot{\Delta} = -\sigma_0 \Delta + 3 \sigma_0 \]
\[ \dot{\delta} = -\sigma_0 \delta + \rho + \beta_0 \chi \]
\[ \dot{\chi} = -\sigma_0 \chi - \sigma_0 \chi - \delta \beta_0 \]
\[ \dot{\xi} = -2 \sigma_0 \xi - \sigma_0 \beta - \beta_0 \sigma + \theta \]

Combing equations (45) and (46) to eliminate \( \chi \) and (47) to eliminate \( \Delta \) produces two differential equations both involving \( \delta \) and \( \rho \). Substituting one differential expression into the other to eliminate \( \rho \) yields the partial differential equation
\[ \ddot{\sigma}_0 \sigma'' + \sigma_0^2 \dot{\sigma} + \sigma_0 \dot{\sigma} / \xi + \sigma_0 \dot{\sigma} / \xi + (\beta_0^2 + 2 \sigma_0 / \xi) \delta - 2 \sigma_0 C_1 / \xi = 0 \]

where \( C_1 \) is a constant. Setting \( C_1 = 0 \) and setting \( \delta = y(\xi) \exp(-i\omega \tau) \) yields
\[ \sigma_0 \dot{\sigma} + (\sigma_0 / \xi - 2i\omega \sigma_0) \dot{\sigma} + (\beta_0^2 + 2 \sigma_0 / \xi - \omega^2) - i\omega \sigma_0 / \xi y = 0 \]

This has complete solution
\[ \delta(\xi, \tau) = C_2 e^{-1/2 \xi \sigma_0 - \tau} / \sigma_0 (\xi \sigma_0 - \tau) M(0, 2i \beta_0 \xi / \sigma_0) \]
\[ C_2 W(0, 2i \beta_0 \xi / \sigma_0) \]
evolution of $\Sigma$. Therefore,

$$\dot{\Sigma} = -\sigma_0 \Sigma' + C_5$$

$$\dot{\Delta} = -\sigma_0 \Delta'$$

$$\dot{\chi} = -\sigma_0 (\xi^\prime)/\xi$$

$$\dot{\rho} = -\sigma_0 (\rho^\prime)/\xi$$

where $C_5$ is a constant. These have the general solution,

$$\Sigma(\xi, \tau) = f(\tau - \xi/\sigma_0) - \xi C_5/\sigma_0$$

$$\Delta(\xi, \tau) = f(\tau - \xi/\sigma_0)$$

$$G(\xi, \tau) = \frac{1}{\xi} f(\tau - \xi/\sigma_0)$$

where $G = \rho, \chi$ and $f$ is an arbitrary functions. Here $\Sigma, G$ and $\Delta$ are being convected at the streaming velocity of the plasma $\sigma_0$. Assuming $G, \Delta, \Sigma \sim \exp(-i\omega\tau)$ gives the particular solutions $\propto \exp[-i\omega(\tau - \xi/\sigma_0)]$ describing a perturbation in the plasma variables propagating through the plasma at the streaming velocity $\sigma_0$.

### 3.3. General solution

If we prescribe $\theta = f(\gamma)/\xi$ where $\gamma = k\xi - (\omega \pm k\sigma_0)\tau = k\xi - 2\sigma_0\tau$, looking at (12) this implies $\beta = \eta f(\gamma)/\xi$, where $\eta$ is a constant. Substituting this expression into (12) yields

$$\xi = \frac{(\Omega - pk\eta) f}{2\Sigma_0} + \frac{p\eta f}{2\Sigma_0\xi}$$

(70)

Substituting (70) into (57) and further prescribing $\sigma = \lambda f(\gamma)/\xi$, where $\eta$ is a constant, requires

$$\left(\sigma_0 k - \Omega\right)\lambda/\xi = \frac{(\Omega - pk\eta)\beta_0}{2\Sigma_0}$$

$$-\sigma_0 \lambda/\xi = p\beta_0\eta/2\Sigma_0$$

(71)

(72)

for $\sigma$, $\xi$, $\theta$ and $\beta$ to be consistent. Substituting the expressions for $\sigma$, $\xi$, $\theta$ and $\beta$ into equation (10) defines the function $f$ by

$$A (\sigma_0 k - \Omega) f_{\gamma\gamma} + [A\sigma_0 + B (\sigma_0 k - \Omega)] \frac{f_{\gamma}}{\xi}$$

$$- (1 - \sigma_0 \eta - \beta_0 \lambda) \frac{f}{\xi} = 0$$

(73)

where

$$A = (\Omega - pk\eta)/2\Sigma_0$$

$$B = p\eta/2\Sigma_0$$

(74)

(75)

The solution of which is a Bessel function of non-integer order which shows presence of Doppler effect as electromagnetic wave propagates in the plasma that is streaming in the radial direction with a velocity $\sigma_0$.

### 4. Discussion and Further Developments

The linear perturbation analysis presented here has shown some of the dynamical responses of the magnetised electron-positron streaming plasma. These offer guidance in respect of possible wave modes and plasma stability. In the linear regime the electrostatic, convective and general solutions describe the simplified streaming system perturbed from equilibrium. The non-linear system of equations does not have a closed-form analytical solution so the problem has to be solved numerically, work on which is currently still in progress. The linear analysis presented here will help in the development of the numerical simulations. Future considerations include extending the cold plasma treatment to a kinetic one. In this context the possibility exists of coupling Bernstein modes (Laing & Diver 2005) (electrostatic waves) to electromagnetic modes.

These results imply a change in the study of pulsar radiation mechanisms. In pulsar magnetospheres the electromagnetic field distribution of the star is the superposition of the underlying dipolar electromagnetic field of the star, plus the self-field of the flowing plasma (da Costa & Kahn 1982). Collective processes in the pulsar rest frame depend very strongly on the local plasma and field conditions. Earlier work (Diver et al. 2002) explored quasilinear coupling processes in a stationary inhomogeneous plasma. This article presents the preliminary analysis of the stability of the full non-linear mechanism in a streaming plasma prior to a full numerical simulation.

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