Novel coupling scheme to control dynamics of coupled discrete systems

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Abstract

A new coupling scheme to control the dynamics of coupled discrete systems on 1-d and 2-d lattices and random networks is presented. The scheme involves coupling with an external lattice or network of damped systems. When the system network and external network are set in a feedback loop, the system network can be controlled to a homogeneous steady state with suppression of the chaotic dynamics of the individual units. The control scheme has the advantage that its design does not require any prior information about the system dynamics or its parameters and works effectively for a range of parameters of the control network. The stability analysis of the controlled steady state or amplitude death state of lattices is carried out using the theory of circulant matrices and Routh-Hurwitz’s criterion for discrete systems and this helps to isolate regions of effective control in relevant parameter planes. The conditions thus obtained are found to agree well with those obtained from direct numerical simulations in the specific context of lattices with logistic map and Henon map as on-site system dynamics. The control scheme is then extended to regulate dynamics on random networks. The master stability function method is used to analyze the stability of amplitude death state for various topologies and coupling strengths.

Keywords: Control of dynamics, amplitude death, coupled map lattice, random network

PACS: 05.45.Xt, 05.45.Gg, 05.45.Ra

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1. Introduction

The control of chaos in nonlinear systems has been an active field of research due to its potential applications in many practical situations where chaotic behaviour is not desirable. Since the time of Grebogi-Ott-Yorke [1], several methods were proposed to control chaotic dynamics to desired periodic states or to stabilize unstable fixed points [2] of the system. Recently these techniques have been extended to control spatiotemporal chaos in spatially extended systems. They are useful in many applications to control dynamics in plasma devices and chemical reactions [3, 4] and to reduce intensity fluctuations in laser systems [5, 6]. We note that control is important for emergence of regulated and sustainable phenomena in biological systems, for example to insure stability of signal-off state in cell signaling networks which is desirable to prevent autoactivation [7]. Also in general, chaotic oscillations can degrade the performance of engineered systems and hence effective and simple control strategies have immense relevance in such cases.

The dynamics of spatially extended systems can be modelled in a very simple but effective manner by using coupled map lattices (CML) introduced by Kaneko [8, 9]. This approach forms an efficient method to coarse grain the local dynamics in such systems and has been used to understand complex natural phenomena. The dynamical states in such systems are extremely rich and varied including spatiotemporal chaos, regular and irregular patterns, travelling waves, spiral waves etc. In case of coupled map lattices and other dynamical systems, several methods for control and synchronization of dynamics have been reported like nonlinear feedback control [10], constant and feedback pinnings [11, 12], random or constant delays in couplings [13, 14], disorder [15], asymmetric couplings [16], spatiotemporal perturbations etc. We note that most of the control schemes are often derived from the system and as such require information about the dynamics and parameters of the system for its design. However for many applications, it is important that the practical implementation of control should have minimum interference with the system.

In the present paper, we introduce a new coupling scheme which can be applied externally to the system and does not require a priori information about the system. As such it is very general and effective and can be implemented easily for control of spatiotemporal dynamics on a coupled map
lattices. This is made possible by designing an external lattice of damped discrete systems such that without feedback from the system, this control lattice stabilizes to a global fixed point. When both lattices are put in a feedback loop, the mutual dynamics works to control their dynamics to a homogeneous steady state or periodic state. We illustrate this using logistic map and Henon map as site dynamics.

In the case of continuous systems, one of the methods introduced to induce amplitude death in coupled systems is coupling to an external system referred to as environmental coupling [17, 18]. The reported method uses a single external system to control the dynamics of coupled chaotic systems. The present paper extends this particular method in two ways: the method proposed is for controlling dynamics in discrete systems and the external system is now a spatially extended system. The spatial extension has its own advantages as a method for suppressing dynamics. Moreover such an extended external system can model an external medium controlling the dynamics of real world systems.

The analysis is carried out by considering a single unit of the interacting system and the control, which is then extended to connected rings of systems, interacting 2-dimensional lattices and random networks. This ‘bottom up approach’ is chosen not only because it gives clarity in describing the mechanism but also because the cases of even single system or rings are not studied or reported so far for discrete systems. We discuss two cases for single units using logistic map and Henon map where control to periodic state as well as fixed point state is achieved due to coupling with an external map. We also report results from detailed numerical simulations in these cases that are found to agree with that from the stability analysis. In the context of coupled map lattices, we note that the amplitude death induced due to time delay in coupling has been reported previously [19]. So also, even in the presence of random delays, coupled chaotic maps can stabilize to a homogeneous steady state [13]. Using multiple delay feedback control unstable steady states are shown to stabilize in chaotic electronic circuits [14]. Here we report a similar phenomenon but induced due to coupling with an external network.

We analyze the stability of the coupled system and control lattices using the theory of circulant matrices and Routh-Hurwitz’s criterion for discrete systems. Thus we obtain regions in relevant parameter planes that correspond to effective control. In particular cases we observe control even when the units in the system lattice are uncoupled. Moreover by tuning the param-
eters of the control lattice, we observe control to regular periodic patterns on the system lattice.

We show that this method is effective in controlling the dynamics of discrete time systems on a random network. Using the framework of master stability analysis, the stability of the controlled state of the system network is studied.

2. Control scheme for 1-d coupled map lattice

In this section we introduce our scheme for controlling a 1-d CML of size $N$ by coupling with an equivalent lattice of damped systems. The dynamics at the $i$'th node of the system lattice constructed using the discrete dynamical system, $x(n+1) = f(x(n))$, with time index $n$, is given by:

$$x_i(n+1) = f(x_i(n)) + D\zeta(f(x_{i-1}(n)) + f(x_{i+1}(n)) - 2f(x_i(n)))$$ (1)

Here $x \in \mathbb{R}^m$ is the state vector of the discrete system, $f : \mathbb{R}^m \to \mathbb{R}^m$ is the nonlinear differentiable function. Also, $D$ represents the strength of diffusive coupling and $\zeta$ is a $m \times m$ matrix, entries of which decide the variables of the system to be used in the coupling.

The dynamics of a damped discrete map $z(n)$ in the external lattice is represented as:

$$z(n+1) = kz(n)$$ (2)

where $k$ is a real number with $|k| < 1$. Thus for positive values of $k$, this map is analogous to over-damped oscillator while for negative values of $k$ it is analogous to damped oscillator. The dynamics at the $i$'th node of the 1-d CML of size $N$ of these maps with diffusive coupling can be represented by the following equation:

$$z_i(n+1) = g(z_i(n)) + D_e(g(z_{i-1}(n)) + g(z_{i+1}(n)) - 2g(z_i(n)))$$ (3)

Here, $g(z) = kz$ and $D_e$ represents the strength of diffusive coupling among the nodes of the external lattice.

We consider periodic boundary conditions for both the lattices. When the system lattice is coupled to the lattice of external systems, node to node,
with feedback type of coupling, then the resulting system can be represented as:

\[
x_i(n + 1) = f(x_i(n)) + D\zeta(f(x_{i-1}(n)) + f(x_{i+1}(n)) - 2f(x_i(n))) + \varepsilon_1 \xi z_i(n)
\]

\[
z_i(n + 1) = g(z_i(n)) + D\varepsilon(g(z_{i-1}(n)) + g(z_{i+1}(n)) - 2g(z_i(n)))
+ \varepsilon_2 \xi^T x_i(n)
\]  

(4)

Here, \(\xi\) is the \(m \times 1\) matrix which determines the components of the state vector \(x\) which take part in the coupling with external system.

In the next section we start with the analysis of control for one unit of this coupled system. Then we study the case of 1-d CML and 2-d CML and extend the scheme to random networks. In all these cases, for direct numerical simulations, the typical discrete systems considered are the logistic map and the Henon map.

3. Analysis of control in a single unit of the model

The dynamics of a single unit of the model introduced in eq.(4) is given by a discrete system coupled to an external system in a feedback loop as:

\[
x(n + 1) = f(x(n)) + \varepsilon_1 \xi z(n)
\]

\[
z(n + 1) = g(z(n)) + \varepsilon_2 \xi^T x(n)
\]  

(5)

To proceed further, we consider logistic map as system dynamics to get:

\[
x(n + 1) = rx(n)(1 - x(n)) + \varepsilon_1 z(n)
\]

\[
z(n + 1) = k z(n) + \varepsilon_2 x(n)
\]  

(6)

Here \(r\) is the parameter whose value determines the intrinsic nature of dynamics of logistic map. We choose \(r = 3.6\) so that the map is in the chaotic regime. Time series \(x(n)\) obtained from eq.(6) numerically is plotted in Fig.1 for two different values of coupling strength with \(\varepsilon_1 = -\varepsilon_2\). The control is switched on at \(n = 50\) and we find that depending on the coupling strength, the chaotic dynamics can be controlled to periodic or steady state.
We find that there are 2 fixed points for the system in eq. (6) given by the following equations:

\[ x^* = 0, \ z^* = 0 \]  \hspace{1cm} (7)

and

\[ x^* = \left( \frac{1}{r} \right) \left( -1 + r + \frac{\varepsilon_1 \varepsilon_2}{1-k} \right); \]
\[ z^* = \frac{\varepsilon_2}{1-k} \left( \frac{1}{r} \right) \left( -1 + r + \frac{\varepsilon_1 \varepsilon_2}{1-k} \right) \]  \hspace{1cm} (8)

To analyze the stability of these fixed points, we consider Jacobian of the system in (6) evaluated at the fixed point.

\[ J = \begin{pmatrix} r - 2rx^* & \varepsilon_1 \\ \varepsilon_2 & k \end{pmatrix} \]  \hspace{1cm} (9)
The characteristic equation for this matrix is given by:

\[ \lambda^2 + c_1 \lambda + c_2 = 0 \quad (10) \]

where

\[ c_1 = -(r - 2rx^* + k) \]
\[ c_2 = (r - 2rx^*)k - \varepsilon_1 \varepsilon_2 \quad (11) \]

The corresponding fixed point will be stable if the absolute values of all the eigenvalues given by eq.\((10)\) are less than 1. Using Routh-Hurwitz’s conditions for discrete systems, this would happen when the following conditions are satisfied \[20\]:

\[ b_0 = 1 + c_1 + c_2 > 0 \]
\[ b_1 = 1 - c_1 + c_2 > 0 \]
\[ \triangle = 1 - c_2 > 0 \quad (12) \]

Using these conditions, we find that the fixed point \((x^*, z^*) = (0, 0)\) is unstable. The other fixed point given by eq.\((8)\) is stable for a range of parameter values and these ranges can be obtained using the analysis described above. The numerical and analytical results thus obtained are shown in Fig. 2. From Fig. 2(a) it is clear that the steady state region (red) is symmetric about the line \(\varepsilon_1 = \varepsilon_2\) and the line \(\varepsilon_1 = -\varepsilon_2\). Fig. 2(b) shows the region of steady state behavior in \(\varepsilon-k\) plane where \(\varepsilon_1 = -\varepsilon_2 = \varepsilon\). In both these plots, black boundary is obtained analytically and green region, the region in which system is oscillatory, is obtained numerically. It is clear that curves in black on the boundaries of the region that correspond to steady state, obtained analytically using eq.\((12)\), agree well with the results of numerical analysis.
Figure 2: (Color online) Parameter planes for a chaotic logistic map coupled to an external map: (a) $\varepsilon_1 - \varepsilon_2$ plane for $k = 0.3$, (b) $\varepsilon$-$k$ plane where $\varepsilon_1 = -\varepsilon_2 = \varepsilon$. Here, the region where the fixed point of coupled system is stable is shown in red (dark gray), in the green (light gray) region system is oscillatory while in the white region system is unstable. The black boundary surrounding the red region, in both the plots, is obtained analytically.

The transition to steady state behaviour for this system, for a given value of $k$ as $\varepsilon$ is varied, happens through reverse doubling bifurcations, as shown in Fig. 3. This would mean that by tuning $\varepsilon$, it is possible to control the system to any periodic cycle or fixed point.
Figure 3: Transition to the steady state for a coupled system of logistic map and external map as $\varepsilon$ is increased. Here $k = 0.5$ and $r = 3.6$.

As a second example we consider the Henon map as the site dynamics. It is a 2-d map given by:

\[
\begin{align*}
x(n+1) &= 1 + y(n) - ax(n)^2 \\
y(n+1) &= bx(n)
\end{align*}
\] (13)

Then the dynamics of the coupled system is given by:

\[
\begin{align*}
x(n+1) &= 1 + y(n) - ax(n)^2 + \varepsilon_1 z(n) \\
y(n+1) &= bx(n) \\
z(n+1) &= kz(n) + \varepsilon_2 x(n)
\end{align*}
\] (14)

The intrinsic dynamics of Henon map is chaotic with parameters $a = 1.4$ and $b = 0.3$. It is clear from time series given in Fig. 4 that the coupled system in eq.(14), can be controlled to periodic state or fixed point state by varying the parameters $\varepsilon_1$, $\varepsilon_2$ and $k$. 
Figure 4: Time series, $x(n)$ of the system in eq.(14) for 2 different values of coupling strength with $\varepsilon_1 = -\varepsilon_2$ and $k = 0.5$. Initially Henon map is in the chaotic regime and the coupling with external system is switched on at time 50. (a) $\varepsilon_1 = 0.7$, chaotic dynamics is controlled to periodic dynamics of period two, (b) $\varepsilon_1 = 0.8$, the system is controlled to a fixed point.

The 3-d dynamical system in (14) has a pair of fixed points given by:

\[
x^* = \frac{1}{2a} \left[ -c \pm \sqrt{c^2 + 4a} \right] \\
y^* = \frac{b}{2a} \left[ -c \pm \sqrt{c^2 + 4a} \right] \\
z^* = \frac{\varepsilon}{2a(1-k)} \left[ -c \pm \sqrt{c^2 + 4a} \right]
\]  

(15)

where,

\[
c = 1 - b - \frac{\varepsilon_1\varepsilon_2}{1-k} \\
\]  

(16)

The stability of these fixed points is decided by the Jacobian matrix $J$ evaluated at that fixed point:

\[
J = \begin{pmatrix}
-2ax^* & 1 & \varepsilon_1 \\
b & 0 & 0 \\
\varepsilon_2 & 0 & k
\end{pmatrix}
\]  

(17)
The characteristic polynomial equation for the eigenvalue $\lambda$, in this case, is given by:

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0$$ (18)

where

$$
\begin{align*}
c_1 &= 2ax^* - k \\
c_2 &= -(b + 2akx^* + \epsilon_1\epsilon_2) \\
c_3 &= bk
\end{align*}
$$ (19)

Similar to the case of logistic map, the absolute values of all eigenvalues given by eq.(18) are less than 1 if the following conditions are satisfied [20]:

$$
\begin{align*}
b_0 &= 1 + c_1 + c_2 + c_3 > 0 \\
b_1 &= 3 + c_1 - c_2 - 3c_3 > 0 \\
b_2 &= 3 - c_1 - c_2 + 3c_3 > 0 \\
b_3 &= 1 - c_1 + c_2 - c_3 > 0
\end{align*}
$$ (20)

Here also, one of the solutions in Eq.(15) (with negative sign) is unstable while the other is stable for regions in parameter planes $\epsilon_1$-$\epsilon_2$ and $\epsilon$-$k$ as shown in Fig. 5. The color codes and details are as given in Fig. 2. For any particular value of parameter $k$, the nature of transition in this case is through a sequence of reverse period doubling bifurcations similar to the case of logistic map.
4. Control of dynamics in a ring of discrete systems

In this section, we consider control of dynamics on a ring of discrete systems. For this, as mentioned in section II, we construct a ring of external maps which are coupled to each other diffusively and then couple it to the ring of discrete systems in one-to-one fashion with feedback type of coupling. We start our analysis with logistic map as the nodal dynamics. With logistic map as on-site dynamics, the dynamics at the $i$'th site of the coupled system is given by:

\[
x_i(n+1) = f(x_i(n)) + D(f(x_{i-1}(n)) + f(x_{i+1}(n)) - 2f(x_i(n))) + \varepsilon_1 z_i(n)
\]
\[
z_i(n+1) = g(z_i(n)) + D_e(g(z_{i-1}(n)) + g(z_{i+1}(n)) - 2g(z_i(n))) + \varepsilon_2 x_i(n)
\]

where

\[
f(x) = rx(1-x)
\]
Numerically, we find that the system in eq.(21) can be controlled to a periodic state and a state of fixed point or amplitude death by adjusting the coupling strengths $\varepsilon_1$ and $\varepsilon_2$. Fig. 6 shows two such cases, where in (a) lattice is controlled to a temporal 2-cycle and in (b) temporal fixed point state.

Figure 6: (Color online) Space-time plots for a ring of logistic maps coupled to an external ring. The coupling is switched on at time 60. Parameter values of the system are: $r = 3.6$, $D = 0.1$, $D_e = 0.1$, $k = 0.3$. (a) For $\varepsilon_1 = -0.3; \varepsilon_2 = 0.3$, the dynamics is controlled to a 2-cycle state and (b) for $\varepsilon_1 = -0.5; \varepsilon_2 = 0.5$, the dynamics is controlled to a fixed point state. The color code is as per the value of $x(n)$

We see that, in the amplitude death state, the dynamics on the ring is controlled to a synchronized fixed point. To analyze the stability of this state, we consider the system in eq.(21) as a 1-d lattice of single units considered in section III coupled diffusively through $x$ and $z$. Because of the synchronized nature of the fixed point and periodic boundary conditions, Jacobian of this system is block circulant matrix as given below.

$$ J = \begin{pmatrix} a_0 & a_1 & 0 & \ldots & 0 & a_1 \\ a_1 & a_0 & a_1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 & 0 & \ldots & a_1 & a_0 \end{pmatrix} $$

(23)

For the case of logistic maps, $a_0$ and $a_1$ are $2 \times 2$ matrices and 0 denotes $2 \times 2$ zero matrix. In explicit form, these matrices are given by:
\[ a_0 = \begin{bmatrix} (1 - 2D)r(1 - 2x^*) & \varepsilon_1 \\ \varepsilon_2 & (1 - 2D_e)k \end{bmatrix} \]  
\hspace{1cm} (24)

and

\[ a_1 = \begin{bmatrix} Dr(1 - 2x^*) & 0 \\ 0 & D_e k \end{bmatrix} \]  
\hspace{1cm} (25)

Following the analysis given in [21] for eigenvalues of the general block circulant matrix as given below:

\[
\begin{pmatrix}
  a_0 & a_1 & a_2 & \ldots & a_{N-2} & a_{N-1} \\
  a_{N-1} & a_0 & a_1 & \ldots & a_{N-3} & a_{N-2} \\
  a_{N-2} & a_{N-1} & a_0 & \ldots & a_{N-4} & a_{N-3} \\
  \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  a_1 & \ldots & \ldots & \ldots & a_{N-1} & a_0
\end{pmatrix}
\hspace{1cm} (26)
\]

we construct \( N \) blocks each of order \( 2 \times 2 \) as follows:

\[ H_j = a_0 + a_1 \rho_j + a_2 \rho_j^2 + \ldots + a_{N-1} \rho_j^{N-1} \]  
\hspace{1cm} (27)

where \( \rho \) is a complex \( N \)'th root of unity:

\[ \rho_j = \exp \left( \frac{2\pi j}{N} \right) i \]  
\hspace{1cm} (28)

where \( i = \sqrt{-1} \)

For our problem, \( a_1 = a_{N-1} \) and \( a_2, a_3, \ldots, a_{N-2} = 0 \). Thus we get,

\[ H_j = a_0 + a_1 (\rho_j + \rho_j^{N-1}) = a_0 + 2\theta_j a_1 \]  
\hspace{1cm} (29)

where

\[ \theta_j = \cos \left( \frac{2\pi j}{N} \right) \]

Thus, in this case, \( H_j \) is given by:

14
\[ H_j = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]  

(30)

where,

\[
\begin{align*}
a_{11} &= (1 - 2D + 2D\theta_j)r(1 - 2x^*) \\
a_{12} &= \varepsilon_1 \\
a_{21} &= \varepsilon_2 \\
a_{22} &= (1 - 2D_e + 2D_e\theta_j)k
\end{align*}
\]

(31)

Now we make use of the fact that eigenvalues of Jacobian when evaluated at the synchronized fixed point are the same as eigenvalues of these \( H_j \) matrices since the Jacobian is block circulant matrix \([21]\). This means that, to check the stability of the fixed point of the two coupled rings, instead of directly calculating eigenvalues of Jacobian matrix of size \( 2N \times 2N \), we can calculate eigenvalues of all \( H_j \) matrices, each of which is \( 2 \times 2 \) matrix, which is a much simpler task.

The characteristic equation for \( H_j \) is given by:

\[ \lambda^2 + c_1\lambda + c_2 = 0 \]

(32)

where

\[
\begin{align*}
c_1 &= -(a_{11} + a_{22}) \\
c_2 &= a_{11}a_{22} - a_{12}a_{21}
\end{align*}
\]

(33)

(34)

In this case, the corresponding fixed point with coordinates given by eq.(7) or eq.(8) is stable when the conditions given in eq.(12) are satisfied for each \( H_j \). These conditions then give us the regions in parameter space where the fixed point state of the whole system is stable i.e. the regions where the control of the original dynamics to the steady state is possible. Similar to the case of single unit, the fixed point with coordinates given by eq.(7) turns out to be unstable for all values of control parameters. The other fixed point is stable for a range of parameter values. These amplitude death regions obtained numerically in \( \varepsilon-k \) and \( D-D_e \) planes for this system of coupled rings are shown in Fig. 7. In each of these plots, the amplitude death region is shown with red color while the black boundary around that
region is obtained analytically. From Fig. 7(b), it is clear that the amplitude death can happen even when the logistic maps are not coupled to each other directly (corresponding to $D = 0$) or when external maps are not coupled to each other (corresponding to $D_e = 0$).

As our next example, we illustrate the control scheme for a ring of Henon maps. The coupled system in this case can be represented as follows:

$$
\begin{align*}
x_i(n+1) &= f(x_i(n), y_i(n)) + D(f(x_{i-1}(n), y_{i-1}(n)) + f(x_{i+1}(n), y_{i+1}(n)) \\
&\quad - 2f(x_i(n), y_i(n))) + \varepsilon_1 z_i(n) \\
y_i(n+1) &= bx_i(n) \\
z_i(n+1) &= g(z_i(n)) + D_e(g(z_{i-1}(n)) + g(z_{i+1}(n)) - 2g(z_i(n))) + \varepsilon_2 x_i(n)
\end{align*}
$$

where

$$f(x, y) = 1 + y - ax^2$$

Here also the system can be controlled to a synchronized fixed point and because of the periodic boundary conditions and because of the synchronized nature of fixed point, Jacobian becomes block circulant as given in eq. (23).
In this case, $a_0$ and $a_1$ are $3 \times 3$ matrices and 0 represents $3 \times 3$ zero matrix. In explicit form, $a_0$ and $a_1$ are given by:

$$
a_0 = \begin{pmatrix}
(1 - 2D)(-2ax^*) & (1 - 2D) & \varepsilon_1 \\
\varepsilon_2 & 0 & (1 - 2D)e k
\end{pmatrix}
$$

and

$$
a_1 = \begin{pmatrix}
D(-2ax^*) & D & 0 \\
0 & 0 & 0 \\
0 & 0 & D e k
\end{pmatrix}
$$

Let

$$
1 - 2D + 2D\theta_j = \eta_1
$$

$$
1 - 2D_e + 2D_e\theta_j = \eta_2
$$

Then,

$$
H_j = \begin{pmatrix}
\eta_1(-2ax^*) & \eta_1 & \varepsilon_1 \\
b & 0 & 0 \\
\varepsilon_2 & 0 & \eta_2 e k
\end{pmatrix}
$$

The characteristic equation for $H_j$ is given by:

$$
\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0
$$

where

$$
c_1 = 2ax^*\eta_1 - \eta_2 k
$$

$$
c_2 = -(b\eta_1 + 2ax^*k\eta_1\eta_2 + \varepsilon_1\varepsilon_2)
$$

$$
c_3 = bke\eta_1\eta_2
$$
Using these parameters $c_1, c_2, c_3$, we define $b_0, b_1, b_2, b_3$ and $\triangle$ as in the case of single unit in eq. (20). This allows us to check stability for each of the $H_j$ matrices which in turn gives us regions in parameter space where the control of the dynamics is possible for a ring of Henon maps. In Fig. 8 we show these regions in $\varepsilon$-$k$ parameter plane, for $D_e = 0$ and for $D = 0$.

Figure 8: (Color online) $\varepsilon$-$k$ parameter planes for a ring of Henon maps coupled to an external ring: (a) when $D = 0.3$, $D_e = 0$, (b) when $D = 0$, $D_e = 0.3$. Color code is same as that for Fig. 2

5. Control of dynamics on a 2d-Lattice

Now we consider controlling the dynamics on 2-d lattice of discrete systems by coupling to an external lattice as follows:

$$\begin{align*}
\mathbf{x}_{i,j}(n+1) &= f(\mathbf{x}_{i,j}(n)) + D\zeta(f(\mathbf{x}_{i-1,j}(n)) + f(\mathbf{x}_{i+1,j}(n)) + f(\mathbf{x}_{i,j-1}(n)) \\
&+ f(\mathbf{x}_{i,j+1}(n)) - 4f(\mathbf{x}_{i,j}(n)) + \varepsilon_1\xi z_{i,j}(n) \quad (42) \\
\mathbf{z}_{i,j}(n+1) &= g(\mathbf{z}_{i,j}(n)) + D_e(g(\mathbf{z}_{i-1,j}(n)) + g(\mathbf{z}_{i+1,j}(n)) + g(\mathbf{z}_{i,j-1}(n)) \\
&+ g(\mathbf{z}_{i,j+1}(n)) - 4g(\mathbf{z}_{i,j}(n))) + \varepsilon_2\xi^T\mathbf{x}_{i,j}(n)
\end{align*}$$
Here, $\xi$ is the $m \times 1$ matrix which determines the components of state vector $x$ which take part in the coupling with an external system.

In Fig. 9 we show the numerically obtained time series from a typical node of a $100 \times 100$ lattice of logistic maps coupled to an external lattice for 3 different coupling strengths $\varepsilon$. In all three cases, coupling with an external lattice is switched on at time 50.

![Time series plots](image)

Figure 9: The time series $x(n)$ of a typical node from a $100 \times 100$ lattice of logistic maps coupled to an external lattice when $\varepsilon_1 = -\varepsilon_2$. Parameter values are: $D = 0.1$, $D_e = 0.1$ and $r = 3.6$. In all the cases coupling with external lattice is switched on at time 50. (a) For $\varepsilon_2 = 0.2$, the chaotic dynamics is controlled to 4-cycle state. (b) For $\varepsilon_2 = 0.3$, it is a 2-cycle state. (c) For $\varepsilon_2 = 0.5$, the dynamics is quenched to fixed point state or amplitude death state.

In Fig. 10 we show space-time plots of 2-d lattice of logistic maps with and without coupling with an external lattice. Here the time series from nodes along one of the main diagonals of the lattice are plotted on the y-axis and time on the x-axis. The coupling strength with the external lattice is adjusted so that lattice of logistic maps is controlled to a 2-cycle state temporally.
Figure 10: (a) Space-time plot of patterns formed on 2-d lattice of logistic maps when the individual maps are in chaotic regime, \( r = 3.6 \) and \( D = 0.1 \). (b) When this lattice is coupled to the external lattice with \( \varepsilon_1 = -0.3 \), \( \varepsilon_2 = 0.3 \), \( D_e = 0.1 \), the dynamics is controlled to a 2-cycle state temporally. Also, spatially pattern on the lattice becomes regular. Color coding in the plot is according the values of \( x(n) \).

The stability of the synchronized steady state of coupled 2-d lattices can be analyzed in exactly similar fashion to the coupled rings discussed in detail in previous section by re-indexing lattice site \((i, j)\) with a running index \( l \) defined as \( l = i + (j - 1)N \). Then Jacobian for the whole system becomes \( N^2 \times N^2 \) block circulant matrix with each block’s order being \((m + 1) \times (m + 1)\). The intersection of regions satisfying the stability conditions for all the \( H_l \) matrices will then give region of amplitude death in the corresponding parameter planes. The results thus obtained are qualitatively similar to the case of ring and therefore not included here.

6. Control scheme for random networks

Now we generalize the control scheme given above to control the dynamics on random networks of discrete systems. Given the random network of discrete systems, we construct external network with the same topology as this network and then we couple these networks in one to one fashion with feedback type of coupling. This coupled system of two networks, when the coupling between the nodes of the same network is of diffusive type, can be represented by:
\[ x_i(n+1) = f(x_i(n)) + D \sum_{j=1}^{N} A_{ij} \zeta(f(x_j(n)) - f(x_i(n))) + \varepsilon_1 \zeta z_i(n) \]

\[ z_i(n+1) = g(z_i(n)) + D_\varepsilon \sum_{j=1}^{N} A_{ij} (g(z_j(n)) - g(z_i(n))) + \varepsilon_2 \xi^T x_i(n) \] (43)

In this equation, intrinsic \( m \)-dimensional dynamics on the nodes of the network is given by \( x_i(n+1) = f(x_i(n)) \) and the same function \( f(x) \) is used to couple nodes in the network diffusively. Also, \( A_{ij} \) is the \((i, j)\)’th entry of the adjacency matrix \( A \) of the network, \( \zeta \) is the \( m \times m \) matrix which determines the components of state vector \( x \) to be used in the coupling with the other nodes and \( \xi \) is the \( m \times 1 \) matrix which determines the components of state vector \( x \) to be used in the coupling with nodes of the external network.

We perform numerical simulations for a network of 50 nodes for 5 different random topologies which include 3 realizations of Erdős-Rényi network with average degree equal to 4 and 2 realizations of Scale-Free network. The dynamics on the nodes is of logistic map in all these simulations with \( \varepsilon_1 = -\varepsilon_2 = \varepsilon \). We find that for all topologies, as the coupling strength \( \varepsilon \) is increased both networks reach synchronized fixed point of the single unit given in eq.(6). To analyze the stability of this fixed point, we invoke the master stability analysis for this fixed point [22].

Since the external network has exactly the same topology as the original network and since nodes of two networks are coupled in one to one fashion, this system of two coupled networks can be considered as a single network of the single units considered in section III. Then if \( \varepsilon_{\mu}^i(n) \) represents \( \mu \)'th component of perturbation to the synchronized fixed point of the network at node \( i \), we expand this perturbation as:

\[ \varepsilon_{\mu}^i(n) = \sum_r c_{\mu}^r(n) v_r^i \] (44)

where \( v_r^i \) is \( i \)'th component of \( r \)'th eigenvector of Laplacian matrix of the network. Following the analysis given in [23] we then get:

\[ c^r(n+1) = [\alpha + \lambda_r \beta] c^r(n) \] (45)

for logistic map as nodal dynamics and for diffusive coupling among the nodes, matrices \( \alpha \) and \( \beta \) are given by:
\[ \alpha = \begin{pmatrix} r(1 - 2x^*) & \varepsilon_1 \\ \varepsilon_2 & k \end{pmatrix} \] (46)

and

\[ \beta = \begin{pmatrix} -rD(1 - 2x^*) & 0 \\ 0 & -D_e k \end{pmatrix} \] (47)

This gives,

\[ \alpha + \lambda \beta = \begin{pmatrix} r(1 - \lambda_r D)(1 - 2x^*) & \varepsilon_1 \\ \varepsilon_2 & k(1 - \lambda_r D_e) \end{pmatrix} \] (48)

Then for the synchronized fixed point state of the system of coupled networks to be stable, for every eigenvalue \( \lambda_r \) of the Laplacian matrix of the network, absolute values of all the eigenvalues of the matrix \( (\alpha + \lambda \beta) \) should be less than 1. We now consider \( \lambda_r \) in \( (\alpha + \lambda \beta) \) as a parameter \( \lambda \) and for a range of \( \lambda \), find eigenvalue with largest absolute value and use this eigenvalue as master stability function. The results are shown in Fig. [11] where for 3 different values of \( \varepsilon \), the master stability curves are shown. We see that all the curves are continuous over the whole range of \( \lambda \). The region of the curve for which \(-1 \leq MSF \leq 1\) is the stable region for that curve. If all the eigenvalues of Laplacian matrix for a given topology fall inside this region, then the state of synchronized fixed point is stable. Since the smallest eigenvalue of Laplacian matrix for unweighted and undirected network is always 0, we need to worry about the largest eigenvalue only. Hence if both \( MSF(\lambda = 0) \) and \( MSF(\lambda = \lambda_{\text{max}}) \) are inside this region then fixed point for the network is stable.

The red curve (thick continuous) in the Fig. [11] is for \( \varepsilon = 0.41 \) which is the minimum coupling strength required for any topology for fixed point of the whole network to be stable. For any coupling strength less than this, absolute value of \( MSF(\lambda = 0) \) is greater than 1 and hence fixed point of the network cannot be stable. To illustrate this, we have plotted master stability curve for \( \lambda = 0.3 \) shown in figure as dotted blue curve. The third curve (thin continuous) is for \( \varepsilon = 0.5 \). In the figure we have marked largest eigenvalues of Laplacian matrices of 5 different networks for which numerical simulations are performed.
Figure 11: (Color online) Master stability functions for a network of logistic maps coupled to an external network for three values of coupling strengths $\varepsilon$. The red curve (thick curve) is for critical value of $\varepsilon = 0.41$ which is the minimum coupling strength necessary for the fixed point state of the network to be stable when all other parameters are kept constant. The blue curve (dotted) is for $\varepsilon = 0.3$ and it is clear from the figure that for all topologies, fixed point state is unstable at this coupling strength. The black curve (thin continuous) is for $\varepsilon = 0.5$. For a network of 50 nodes, with different random topologies (3 Erdős-Rényi topologies and 2 Scale-free topologies), the largest eigenvalue is marked on the graph.

To support the master stability analysis, we present numerical results for one particular realization of Erdős-Rényi network. We calculate an index $\langle A \rangle$ to characterize amplitude death by taking the difference between global maximum and global minimum of variable $x$ in time at each node of the network after neglecting the transients and averaging this quantity over all the nodes. For parameter values corresponding to amplitude death, this index has value 0 \cite{17}. In Fig. 12 we show variation of $\langle A \rangle$ with respect to $\varepsilon$ and it is clear that $\langle A \rangle$ goes to 0 at $\varepsilon = 0.41$ indicating that amplitude death happens as predicted by master stability analysis.
Figure 12: The index $\langle A \rangle$ obtained numerically as a function of coupling strength $\varepsilon$ for a system of logistic network coupled to an external network. The topology of both networks in this case is of Erdős-Rényi type. At approximately $\varepsilon = 0.41$, amplitude death happens.

7. Conclusion

The suppression of unwanted or undesirable excitations or oscillations in systems is necessary procedure in a variety of fields and therefore mechanisms for achieving this suppression or amplitude death in connected systems is of great relevance. In the specific context of neurons, it is known that exaggerated oscillatory neuronal synchronization can lead to pathological situations and hence methods for their suppression are very important. Methods to avoid oscillations by design of system controls such as power system stabilizers are important in engineering too.

In this paper we present a novel scheme where suppression of dynamics can be induced by coupling the system with an external system. Our study is mostly on extended systems like rings and lattices in interaction with similar systems with different dynamics. However the method is quite general and is applicable to other such situations also where control and stabilization to steady states are desirable. Even in the case of single systems, as illustrated
here for Henon and logistic maps, this method can be thought of as a control mechanism for targeting the system to steady state.

We develop the general stability analysis for the coupled systems using the theory of maps and circulant matrices and identify the regions where suppression is possible in the different cases considered. The results from direct numerical simulations carried out using two standard discrete systems, logistic amp and Henon map, agree well with that from the stability analysis. Our method will work even with a single external system as control, however, extended or coupled external system enhances the region of effective suppression in the parameter planes. To the best of our knowledge, this is the first time an extended control system is being considered. This has relevance in understanding and modeling controlled dynamics in real systems due to interaction with external medium.

Our study indicates that by tuning the parameters of the external control system, we can regulate the dynamics of the original system to desired periodic behavior with consequent spatial order. We also report how the present coupling scheme can be extended to suppress dynamics on random networks by connecting the system network with an external similar network in a feedback loop. In this context the stability analysis using Master stability function, gives the critical strength of feedback coupling required for suppression.

Our analysis leads to the interesting result that amplitude death occurs even when the systems are not coupled among themselves but coupled individually to the connected external systems. So also, individual external systems, which are not connected among themselves but each one connected to one node in the system, can suppress the dynamics of the connected nodes in the system. This facilitates design of control through connected or isolated external systems depending on the requirement.

8. Acknowledgements

One of the authors (S.M.S.) would like to thank University Grants Commission, New Delhi, India for financial support.
9. References

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