COMPARISON RESULTS FOR SOLUTIONS OF POISSON EQUATIONS WITH ROBIN BOUNDARY ON COMPLETE RIEMANNIAN MANIFOLDS

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Abstract. In this paper, by using Schwarz rearrangement and isoperimetric inequalities, we prove comparison results for the solutions of Poisson equations on complete Riemannian manifolds with $\text{Ric} \geq (n-1)\kappa$, $\kappa \geq 0$, which extends the results in [2]. Furthermore, as applications of our comparison results, we obtain the Saint-Venant inequality and Bossel-Daners inequality for Robin Laplacian.

1. Introduction

Let $(M, g)$ be an $n$-dimensional complete Riemannian manifold with $\text{Ric}(g) \geq (n-1)\kappa$, where $\kappa \geq 0$. For $\kappa = 0$, we further assume that $M$ is noncompact with positive asymptotic volume ratio, i.e.

$$\text{AVR}(g) = \lim_{r \to \infty} \frac{|B_r(x_0)|}{\omega_n r^n} > 0,$$

where $B_r(x_0)$ denotes the geodesic ball in $M$ centered at $x_0$ with radius $r$, while $\omega_n$ denotes the volume of $n$-dimensional Euclidean unit ball. Let $\Omega$ be a bounded domain with smooth boundary $\partial \Omega$ on $M$. For a positive number $\beta$ and a nonnegative function $f$ in $L^2(\Omega)$, we consider the following problem

$$\begin{cases}
-\Delta_g u = f, & \text{in } \Omega, \\
\frac{\partial u}{\partial N} + \beta u = 0, & \text{on } \partial \Omega,
\end{cases}$$

where $N$ denotes the outer unit normal to $\partial \Omega$. For $\beta = \infty$, the equation (1.1) can be seen as the problem equipped with Dirichlet boundary condition.

According to isoperimetric inequalities in [8] for $\kappa = 0$ (See also [1, 6]) and in [20] for $\kappa = 1$, we define

$$\alpha_k = \begin{cases}
\text{AVR}(g), & \kappa = 0, \\
\frac{|M|}{|S^n|}, & \kappa = 1,
\end{cases}$$

where $|M|$, $|S^n|$ denote the volume of $M$ and $S^n$, respectively. Let $(\mathbb{R}^n(\kappa), g_\kappa)$ be Euclidean space when $\kappa = 0$ and be unit sphere when $\kappa = 1$ with canonical metrics. In this article, we intend to establish a comparison principle with the solution of the following problem

$$\begin{cases}
-\Delta_{g_\kappa} v = f^\sharp, & \text{in } \Omega^\sharp, \\
\frac{\partial v}{\partial N} + \beta v = 0, & \text{on } \partial \Omega^\sharp,
\end{cases}$$

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where \( \Omega^\sharp \) is a geodesic ball on \( \mathbb{R}^n(\kappa) \) satisfying \( |\Omega|_g = \alpha_\kappa |\Omega^\sharp|_{g^\sharp} \) and \( f^\sharp \) is the Schwarz rearrangement of \( f \).

For Poisson equations with Dirichlet boundary conditions in Euclidean spaces, Talenti [25] gave pointwise comparisons of \( u^\sharp \) and \( v \). Talenti’s comparison results were generalized to semilinear and nonlinear elliptic equations, for instance, in [26, 3, 15, 24]. Recently, Talenti’s comparison results were extended to solutions of Poisson equations on complete noncompact Riemannian manifolds with nonnegative Ricci curvature by the first two authors in [13]. We also refer the reader to excellent books [5, 21, 23] for related topics.

Recently, Alvino et al. [2] obtained the Talenti type comparison results of Poisson equations with Robin boundary conditions in Euclidean spaces. The results in [2] were generalized to the case of \( p \)-Laplace operator with Robin boundary conditions in [4] and to the torsion problem for the Hermite operator with Robin boundary conditions in [14]. Following the strategies in [2], we focus on Poisson equations with Robin boundary conditions on complete Riemannian manifolds and prove the following comparison results.

**Theorem 1.1.** Let \((M, g)\) be an \( n \)-dimensional complete Riemannian manifold with \( \text{Ric}(g) \geq (n - 1)\kappa \), where \( \kappa = 0 \) or \( 1 \). For \( \kappa = 0 \), we further assume that \( M \) is noncompact with \( \text{AVR}(g) > 0 \). Let \( \Omega \) be a bounded domain with smooth boundary on \( M \) and \( \Omega^\sharp \) be a geodesic ball on \( \mathbb{R}^n(\kappa) \) satisfying \( |\Omega|_g = \alpha_\kappa |\Omega^\sharp|_{g^\sharp} \), where \( \alpha_\kappa \) is defined in (1.2). Let \( u \) and \( v \) be solutions to (1.1) and (1.3) respectively, then

\[
\| u \|_{L^{p,1}(\Omega)} \leq \alpha^{\frac{1}{p}}_\kappa \| v \|_{L^{p,1}(\Omega^\sharp)} \quad \text{for } \kappa = 0, 1 \text{ and } 0 < p \leq \frac{n}{2n - 2},
\]

(1.4)

\[
\| u \|_{L^{2p,2}(\Omega)} \leq \alpha^{\frac{1}{p}}_\kappa \| v \|_{L^{2p,2}(\Omega^\sharp)} \quad \text{for } \kappa = 0 \text{ and } 0 < p \leq \frac{n}{3n - 4},
\]

(1.5)

\[
\| u \|_{L^{2p,2}(\Omega)} \leq \alpha^{\frac{1}{p}}_\kappa \| v \|_{L^{2p,2}(\Omega^\sharp)} \quad \text{for } \kappa = 1 \text{ and } 0 < p \leq \frac{n}{3n - 3},
\]

(1.6)

In particular, when \( n = 2, \kappa = 1 \) and \( 0 < p \leq 1 \),

\[
\| u \|_{L^{2p,2}(\Omega)} \leq \alpha^{\frac{1}{p}}_\kappa \| v \|_{L^{2p,2}(\Omega^\sharp)}.
\]

**Theorem 1.2.** Under the same assumptions as Theorem 1.1, letting \( u \) and \( v \) be solutions to (1.1) and (1.3) for \( f \equiv 1 \), respectively, we have

\[
\| u \|_{L^{p,1}(\Omega)} \leq \alpha^{\frac{1}{p}}_\kappa \| v \|_{L^{p,1}(\Omega^\sharp)} \quad \text{for } \kappa = 0, 1 \text{ and } p \leq \frac{n}{n - 2},
\]

(1.8)

and

\[
\| u \|_{L^{2p,2}(\Omega)} \leq \alpha^{\frac{1}{p}}_\kappa \| v \|_{L^{2p,2}(\Omega^\sharp)} \quad \text{for } \kappa = 0 \text{ and } 0 < p \leq \frac{n}{n - 2}.
\]

(1.9)

In particular, for \( n = 2 \) and \( \kappa = 0 \), we have a pointwise comparison result

\[
u^\sharp(x) \leq v(x) \quad \text{for all } x \in \Omega^\sharp.
\]
Let \((M, g)\) be a complete Riemannian manifold and let \(\Omega \subset M\) be a smoothly bounded domain in \(M\). Assume that \(u\) is a positive solution to the following torsion problem with Robin boundary condition

\[
\begin{cases}
-\Delta_g u = 1, & \text{in } \Omega, \\
\frac{\partial u}{\partial N} + \beta u = 0, & \text{on } \partial \Omega.
\end{cases}
\] (1.11)

The torsional rigidity \(T_\beta(\Omega)\) with Robin boundary condition of the domain \(\Omega\) is defined by

\[T_\beta(\Omega) = \int_\Omega u \, dV_g.\]

In 2015, by using free discontinuity techniques, Bucur et al. in [10] proved the Saint-Venant inequality for Robin Laplacian in Euclidean spaces. In 2019, Alvino et al. in [2] obtained the same Saint-Venant inequality for Robin Laplacian via a Talenti type comparison result in Euclidean spaces.

Choosing \(p = 1\) in (1.8), we can deduce the Saint-Venant inequality for Robin Laplacian on given manifolds.

**Corollary 1.3.** Under the same assumptions as Theorem 1.1, for the torsion problem (1.11), we obtain

\[T_\beta(\Omega) \leq \alpha_\kappa T_\beta(\Omega^\sharp).\] (1.12)

**Remark 1.4.** For the torsion problem with Dirichlet boundary condition on Riemannian manifold \((M, g)\) satisfying \(\text{Ric}(g) \geq (n - 1)\), the inequality (1.12) is due to [15] and [19]. In [13], the first two authors obtained sharp estimates for the \(L^1\)-moment spectrum and \(L^\infty\)-moment spectrum with Dirichlet boundary conditions on bounded domains in complete Riemannian manifolds with nonnegative Ricci curvature.

Let \(\Omega\) be a bounded domain with smooth boundary on \(M\) and \(\Omega^\sharp\) be a geodesic ball on \(\mathbb{R}^n(\kappa)\) satisfying \(|\Omega|_g = \alpha_\kappa |\Omega^\sharp|_g\). Let \(\lambda_{1,\beta}(\Omega)\) and \(\lambda_{1,\beta}(\Omega^\sharp)\) be the first eigenvalues to (1.13) and (1.14) respectively, i.e.

\[
\begin{cases}
-\Delta_g u = \lambda_{1,\beta}(\Omega) u, & \text{in } \Omega, \\
\frac{\partial u}{\partial N} + \beta u = 0, & \text{on } \partial \Omega,
\end{cases}
\] (1.13)

and

\[
\begin{cases}
-\Delta_g v = \lambda_{1,\beta}(\Omega^\sharp) v, & \text{in } \Omega^\sharp, \\
\frac{\partial v}{\partial N} + \beta v = 0, & \text{on } \partial \Omega^\sharp.
\end{cases}
\] (1.14)

As another application to Theorem 1.1, we use Talenti comparisons to deal with Bossel-Daners inequality for \(n = 2\). When \(n > 2\), however, \(p = 1\) no longer satisfies the conditions of (1.5) and (1.7), hence we fail to use Theorem 1.1 to give a direct proof.
Theorem 1.5. Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold with \(\text{Ric}(g) \geq (n - 1)\kappa\), where \(\kappa = 0\) or 1. For \(\kappa = 0\), we further assume that \(M\) is noncompact with \(\text{AVR}(g) > 0\). Let \(\Omega\) be a bounded domain with smooth boundary on \(M\) and \(\Omega^2\) be geodesic ball on \(\mathbb{R}^n(\kappa)\) satisfying \(|\Omega|_g = \alpha_{s}|\Omega^2|_{g_s}\). Then

\[
\lambda_{1, g}(\Omega) \geq \lambda_{1, g}(\Omega^2).
\]

Moreover, the inequality holds in (1.15) if and only if \((M, g)\) is isometric to \((\mathbb{R}^n(\kappa), g_s)\) and \(\Omega\) is isometric to a geodesic ball in \((\mathbb{R}^n(\kappa), g_s)\).

Remark 1.6. The inequality (1.15) is also called Faber-Krahn inequality for Robin Laplacian. When \((M, g)\) is Euclidean space, the result is due to Bossel in [7] for \(n = 2\) and Daners in [17] for all dimensions (See also [16, 9, 10]). For bounded domains in Riemannian manifolds with \(\text{Ric}(g) \geq n - 1\), the inequality (1.15) is due to Chen-Cheng-Li in [12].

The paper is organized as following. In Section 2, we recall the isoperimetric inequality of complete Riemannian manifolds with \(\text{Ric} \geq (n - 1)\kappa\), where \(\kappa \geq 0\), Schwarz rearrangement, Lorentz space and Gronwall’s inequality; In Section 3, we establish some integral inequalities for the solutions to (1.1) and (1.3) by using the isoperimetric inequality; In Section 4, we give the proofs of Theorem 1.1 and Theorem 1.2; In Section 5, we give the proof of Theorem 1.5.

2. Preliminaries

2.1. Isoperimetric inequalities. Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold with \(\text{Ric}(g) \geq (n - 1)\kappa\), where \(\kappa = 0\) or 1. For \(\kappa = 0\), we further assume that \(M\) is noncompact with \(\text{AVR}(g) > 0\). Let \(\Omega\) be a bounded domain with smooth boundary on \(M\) and \(\Omega^2\) be a geodesic ball on \(\mathbb{R}^n(\kappa)\) satisfying \(|\Omega|_g = \alpha_{s}|\Omega^2|_{g_s}\). According to Brendle [8] (See also [1, 6]) and Lévy-Gromov [20], we have the isoperimetric inequality

\[
|\partial\Omega|_g \geq \alpha_{s} |\partial\Omega^2|_{g_s}.
\]

The equality holds in (2.1) if and only if \((M, g)\) is isometric to \((\mathbb{R}^n(\kappa), g_s)\) and \(\Omega\) is isometric to a geodesic ball in \(\mathbb{R}^n(\kappa)\).

For a complete noncompact \(n\)-dimensional Riemannian manifold \((M, g)\) with nonnegative Ricci curvature and positive asymptotic volume growth, we also notice that the inequality (2.1) is proved by Agostiniani et al. in [1, Theorem 1.8] for \(n = 3\) and then extended to \(3 \leq n \leq 7\) by Fogagnolo and Mazzieri in [18]. Furthermore, (2.1) and its equality case still hold in \(\text{CD}(0, N)\) metric measure spaces based on the method of optimal mass transport by Balogh and Kristály in [6].

2.2. Schwarz rearrangement. Let \(u\) and \(v\) be solutions to (1.1) and (1.3) respectively. For \(t \geq 0\) we denote by

\[
U_t = \{x \in \Omega : u(x) > t\}, \quad U^i_t = \partial U_t \cap \Omega, \quad \partial U^c_t = \partial U_t \cap \partial \Omega, \quad \mu(t) = |U_t|_g,
\]

\[
V_t = \{x \in \Omega^2 : v(x) > t\}, \quad \phi(t) = |V_t|_{g_s}.
\]
Denoting by \( u_m \) and \( v_m \) the minimum of \( u \) and \( v \) respectively, thanks to the positiveness of \( \beta \) and Robin boundary conditions, we have \( u_m \geq 0 \) and \( v_m \geq 0 \). Since \( v \) is radial, positive and decreasing along the radius, we have \( V_t \) coincides with \( \Omega^\sharp \) when \( 0 \leq t < v_m \).

**Definition 2.1.** Letting \( h : \Omega \to \mathbb{R} \) be a measurable function, the distribution function of \( h \) is the function \( \mu_h : [0, +\infty) \to [0, +\infty) \) defined by

\[
\mu_h(t) = |\{ x \in \Omega : |h(x)| > t \}|_g.
\]

**Definition 2.2.** The decreasing rearrangement \( h^* : [0, |\Omega|_g] \to \mathbb{R} \) is defined based on the distribution function of \( h \), that is

\[
h^*(s) = \begin{cases} 
\text{esssup}_\Omega h, & s = 0, \\
\inf \{ t : \mu_h(t) \leq s \}, & s > 0.
\end{cases}
\]

**Definition 2.3.** The Schwarz rearrangement \( h^\# : \Omega^\# \to \mathbb{R} \) is defined based on the decreasing rearrangement of \( h \), that is

\[
h^\#(x) = h^*(\alpha_\kappa |B^\#_R|_{g_\kappa}),
\]

where \( B^\#_R \) is the geodesic ball in \( \mathbb{R}^n(\kappa) \) with radius \( R \).

The distribution function of \( h \) and \( h^\# \) satisfies

\[
\mu_h(t) = \alpha_\kappa \mu_{h^\#}(t).
\]

By definition, \( h, h^* \) and \( h^\# \) are equi-distributed in the sense that

\[
\|h\|_{L^p(\Omega)} = \|h^*\|_{L^p(\Omega^{\#})} = \alpha_\kappa^{\frac{1}{p}} \|h^\#\|_{L^p(\Omega^{\#})}.
\]

Given measurable functions \( h_1, h_2 \) on \( \Omega \), the Hardy-Littlewood inequality holds,

\[
\int_\Omega |h_1(x)h_2(x)|dV_g \leq \int_0^{\Omega|_g} h^*_1(s)h^*_2(s)ds.
\]

Choosing \( h_2 = \chi_{|u|>t} \) in (2.6), one has

\[
\int_{|u|>t} |h_1(x)|dV_g \leq \int_0^{\Omega|_g} h^*_1(s)ds.
\]

By strong maximum principle, both solutions \( u \) and \( v \) to (1.1) and (1.3) achieve their minimum on boundaries. Hence \( u \) and \( v \) are strictly positive in the interior of domains. Moreover, by isoperimetric inequality (2.1), we have

\[
\alpha_\kappa v_m|\partial \Omega^\sharp|_{g_\kappa} = \alpha_\kappa \int_{\partial \Omega^\sharp} v(x) d\mu_{g_\kappa} = \frac{\alpha_\kappa}{\beta} \int_{\Omega^\sharp} f^\#dV_{g_\kappa} = \frac{1}{\beta} \int_{\Omega} f dV_g
\]

\[
= \int_{\partial \Omega} u(x) d\mu_g
\]

\[
\geq u_m|\partial \Omega|_g
\]

\[
\geq u_m \alpha_\kappa |\partial \Omega^\sharp|_{g_\kappa},
\]
which implies that
\[(2.9) \quad \min_{\Omega} u = u_m \leq v_m = \min_{\Omega^g} v.\]

An important consequence of (2.9) is that
\[(2.10) \quad \mu(t) \leq |\Omega|_g = \alpha_\kappa \phi(t) \quad \text{for} \quad 0 \leq t < v_m.\]

2.3. Lorentz space.

**Definition 2.4.** Let \(0 < p < +\infty\) and \(0 < q \leq +\infty\). The Lorentz space \(L^{p,q}(\Omega)\) is the space of functions such that the quantity
\[
||h||_{L^{p,q}} = \begin{cases} 
\left( \int_0^\infty t^{\frac{p}{q}} \mu_h(t)^\frac{q}{p} \frac{dt}{t} \right)^\frac{1}{\frac{q}{p}}, & \text{if } 0 < q < \infty, \\
\sup_{t>0} (t^p \mu_h(t)), & \text{if } q = \infty,
\end{cases}
\]
is finite.

It’s well known that Lorentz space coincides with \(L^p\) space when \(p = q\) (See [27] for more details).

2.4. Gronwall’s inequality. Let \(\xi\) be a continuously differentiable function satisfying
\[
\tau \xi'(\tau) \leq \xi(\tau) + C
\]
for all \(\tau \geq \tau_0 > 0\), where \(C\) is a non-negative constant. For all \(\tau \geq \tau_0\), we have
\[
(2.11) \quad \xi'(\tau) \leq \frac{\xi(\tau_0) + C}{\tau_0}.
\]

3. Some lemmas

Let \((M, g)\) be an \(n\)-dimensional complete Riemannian manifold with \(\text{Ric}(g) \geq (n - 1)\kappa\), where \(\kappa = 0\) or 1. Moreover, for \(\kappa = 0\), assume \(M\) is non-compact with \(\text{AVR}(g) > 0\). Let \(\Omega\) be a bounded domain with smooth boundary on \(M\) and \(\Omega^2\) be a geodesic ball on \(\mathbb{R}^n(\kappa)\) satisfying \(|\Omega|_{g_\kappa} = \alpha_\kappa |\Omega^2|_{g_\kappa}\). Define a function \(G_\kappa(l)\),
\[
(3.1) \quad G_\kappa(l) = \frac{dI_\kappa}{dr} \circ I_\kappa^{-1}(l),
\]
where
\[
I_\kappa(r) = n \omega_n \alpha_\kappa \int_0^r \text{sn}_\kappa^{n-1}(s)ds,
\]
and
\[
\text{sn}_\kappa(s) = \begin{cases} 
s, & \kappa = 0, \\
\sin s, & \kappa = 1.
\end{cases}
\]

We then have the following lemma:
Lemma 3.1. Let \( u \) and \( v \) be solutions to (1.1) and (1.3) respectively. For almost every \( t > 0 \), we have

\[
G_\kappa(\mu(t))^2 \leq \int_0^{\mu(t)} f^*(s) ds \cdot \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial U^c_t} \frac{1}{u} d\mu_g \right),
\]

\[
\tilde{G}_\kappa(\phi(t))^2 = \alpha_\kappa^{-1} \int_0^{\alpha_\kappa \phi(t)} f^*(s) ds \cdot \left( -\phi'(t) + \frac{1}{\beta} \int_{\partial V^c_i \cap \partial \Omega} \frac{1}{v} d\mu_g \right),
\]

where \( \tilde{G}_\kappa(r) = \alpha_\kappa^{-1} G_\kappa(\alpha_\kappa r) \).

Proof. Multiplying (1.1) by \( \phi \in H^1(\Omega) \) and integrating by parts, we have

\[
\int_{\Omega} \nabla g u \cdot \nabla g \phi dV_g + \beta \int_{\partial \Omega} u \phi d\mu_g = \int_{\Omega} \nabla g u \cdot \nabla g \phi dV_g + \beta \int_{\partial \Omega} (\phi \cdot \text{div}(\nabla g u)) dV_g
\]

\[
= \int_{\Omega} \left[ \nabla g u \cdot \nabla g \phi - \text{div}(\nabla g u) \right] dV_g
\]

\[
= \int_{\Omega} \phi \Delta_g u dV_g
\]

\[
= \int_{\Omega} f \phi dV_g.
\]

Define a test function, for \( h > 0 \),

\[
\phi_h = \begin{cases} 0, & 0 < u \leq t, \\ h, & u > t + h, \\ u - t, & t < u \leq t + h. \end{cases}
\]

Choosing \( \phi = \phi_h \) in (3.4), one has

\[
\int_{U_t} |\nabla g u|_g^2 dV_g + \beta \int_{\partial U^c_t} u h d\mu_g + \beta \int_{\partial U^c_t \cap \partial U^c_{t+h}} u(u - t) d\mu_g
\]

\[
= \int_{U_t \setminus U_{t+h}} f(u - t) dV_g + \int_{U_{t+h}} f h dV_g.
\]

Dividing by \( h \) and letting \( h \to 0^+ \), we have

\[
-\frac{d}{dt} \left( \int_{U_t} |\nabla g u|_g^2 dV_g \right) + \beta \int_{\partial U^c_t} u d\mu_g = \int_{U_t} f dV_g.
\]

Applying co-area formula, we then obtain

\[
\int_{\partial U^c_t} |\nabla g u|_g d\mu_g + \int_{\partial U^c_t} \beta u d\mu_g = \int_{U_t} f dV_g.
\]

Define

\[
A = \left\{ \frac{\nabla g u}{|\nabla g u|_g}, \frac{\partial U^c_t}{\beta u}, \partial U^c_t \right\}.
\]

Then (3.7) becomes

\[
\int_{\partial U^c_t} A d\mu_g = \int_{U_t} f dV_g.
\]
By (2.1), (2.7), (3.1), (3.8) and co-area formula, we have

\[ G_\kappa(\mu(t))^2 \leq |\partial U_t|^2 \]

\[
\leq \int_{\partial U_t} A d\mu_g \cdot \int_{\partial U_t} A^{-1} d\mu_g
\]

\[
= \int_{U_t} f dV_g \cdot \left( \int_{\partial U_t} \frac{1}{\nabla g} d\mu_g + \int_{\partial U_t} \frac{1}{\beta u} d\mu_g \right)
\]

\[
= \int_{U_t} f dV_g \cdot \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial U_t} \frac{1}{u} d\mu_g \right)
\]

\[
\leq \int_0^{\mu(t)} f^*(s) ds \cdot \left( -\mu'(t) + \frac{1}{\beta} \int_{\partial U_t} \frac{1}{u} d\mu_g \right).
\]

Similarly, by \( \check{G}_\kappa(\phi(t)) = |\partial V_t|^\kappa \) and \( \int_{U_t} f^\tau dV_g = \alpha_k^{-1} \int_0^{\alpha_k \phi(t)} f^*(s) ds \), we prove (3.3). \( \square \)

**Lemma 3.2.** Let \( u \) and \( v \) be solutions to (1.1) and (1.3) respectively. For all \( t > v_m \), we have

\[
\int_0^t \tau \left( \int_{\partial U_t} \frac{1}{u} d\mu_g \right) d\tau \leq \frac{1}{2\beta} \int_0^{\partial\Omega_g} f^*(s) ds,
\]

(3.10)

\[
\int_0^t \tau \left( \int_{\partial V_t \cap \partial \Omega} \frac{1}{v} d\mu_g \right) d\tau = \frac{\alpha_k^{-1}}{2\beta} \int_0^{\partial\Omega_g} f^*(s) ds.
\]

(3.11)

**Proof.** By Fubini’s theorem, (1.1) and (2.5), we have

\[
\int_0^{+\infty} \tau \left( \int_{\partial U_t} \frac{1}{u} d\mu_g \right) d\tau = \int_0^{+\infty} d\tau \left( \int_{\partial \Omega \cap \{u \geq \tau\}} \frac{\tau}{u} d\mu_g \right)
\]

\[
= \int_0^{+\infty} d\tau \left( \int_{\partial \Omega \cap \{u \geq \tau\}} \frac{\tau}{u} d\mu_g \right)
\]

\[
= \int_{\partial \Omega} \left( \int_0^\tau d\tau \frac{1}{u} d\mu_g \right)
\]

\[
= \frac{1}{2} \int_{\partial \Omega} u d\mu_g
\]

\[
= \frac{1}{2\beta} \int_{\Omega} f dV_g
\]

\[
= \frac{1}{2\beta} \int_0^{\partial\Omega_g} f^*(s) ds.
\]

(3.12)

Thus, for all \( t > v_m \), we obtain

\[
\int_0^t \tau \left( \int_{\partial U_t} \frac{1}{u} d\mu_g \right) d\tau \leq \int_0^{+\infty} \tau \left( \int_{\partial U_t} \frac{1}{u} d\mu_g \right) d\tau
\]

\[
= \frac{1}{2\beta} \int_0^{\partial\Omega_g} f^*(s) ds.
\]

(3.13)
With the fact that \( \int_{\Omega^k} f^k \, dV_{g_k} = \alpha_k^{-1} \int_{\Omega^k} f^*(s) \, ds \) and \( \partial V_t \cap \partial \Omega^k = \emptyset \) when \( t > \nu_m \), we deduce

\[
\int_0^\tau \left( \int_{\partial V_t \cap \partial \Omega^k} \frac{1}{v} d\mu_{g_k} \right) \, d\tau = \int_0^{\infty} \tau \left( \int_{\partial V_t \cap \partial \Omega^k} \frac{1}{v} d\mu_{g_k} \right) \, d\tau = \frac{\alpha_k^{-1}}{2\beta} \int_0^\tau f^*(s) \, ds.
\]

(3.14)

\[\square\]

4. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we give the proof of main Theorem 1.1 and Theorem 1.2 by using the lemmas in previous section.

**Proof of Theorem 1.1.** Multiplying (3.2) by \( t \mu(t)^{\frac{1}{p}} G_k(\mu(t))^{-2} \) and integrating from 0 to \( \tau > \nu_m \), we have

\[
\int_0^\tau t \mu(t)^{\frac{1}{p}} \, dt \leq \int_0^\tau -t \mu'(t) \mu(t)^{\frac{1}{p}} G_k(\mu(t))^{-2} \int_0^{\mu(t)} f^*(s) \, ds \, dt
\]

(4.1)

\[+ \int_0^\tau \left[ \frac{t}{\beta} \int_{\partial \Omega^k} \frac{1}{u} d\mu_k \right] \mu(t)^{\frac{1}{p}} G_k(\mu(t))^{-2} \int_0^{\mu(t)} f^*(s) \, ds \, dt.
\]

From (3.1), it’s obvious that \( t^{\frac{1}{p}} G_k(l)^{-2} = n^{-2}(\omega n \alpha_0)^{-\frac{1}{p}} l^{2-\frac{2}{p}} \) is non-decreasing when \( 0 < p \leq \frac{n}{2n-2} \). Besides, \( t^{\frac{1}{p}} G_1(l)^{-2} \) is also non-decreasing when \( 0 < p \leq \frac{n}{2n-2} \). By definition of \( G_1 \) in (3.1), we have

\[
\frac{d}{dl} \left( t^{\frac{1}{p}} G_1(l)^{-2} \right) = \frac{1}{p} l^{\frac{1}{p}-1} \cdot \frac{G_1(l) - 2pI_1G_1'(l)}{G_1(l)^3}
\]

(4.2)

\[= \frac{p^{-1} l^{\frac{1}{p}-1}}{(I_1 \circ I_1^{-1}(l))^4} \cdot (I_1^2 - 2pI_1I_1''(l) \circ I_1^{-1}(l)).
\]

It is sufficient to show \( k(r) := I_1^2(r)^2 - 2pI_1(r)I_1''(r) \geq 0 \) on \([0, \pi]\). Noticing that

\[\cos r \cdot \int_0^r \sin^{n-1}s \, ds \cdot \sin^n r \leq \frac{1}{n}\]

on \([0, \pi]\), one has

\[
k(r) = \left( n\omega n \alpha_1 \sin^{n-1} r \right)^2 \left[ 1 - 2p(n-1) \frac{\cos r \cdot \int_0^r \sin^{n-1}s \, ds}{\sin^n r} \right] \geq 0
\]

when \( 0 < p \leq \frac{n}{2n-1} \). By monotonicity of \( t^{\frac{1}{p}} G_k(l)^{-2} \) when \( 0 < p \leq \frac{n}{2n-2} \) and (3.10), it follows

\[
\int_0^\tau \left[ \frac{t}{\beta} \int_{\partial \Omega^k} \frac{1}{u} d\mu_k \right] \mu(t)^{\frac{1}{p}} G_k(\mu(t))^{-2} \int_0^{\mu(t)} f^*(s) \, ds \, dt
\]

(4.4)

\[\leq \frac{\Omega^k}{2\beta^2} \int_0^\tau f^*(s) \, ds \cdot \int_0^\tau \frac{t}{\beta} \int_{\partial \Omega^k} \frac{1}{u} d\mu_k \, dt
\]

\[\leq \frac{\Omega^k}{2\beta^2} \left( \int_0^\tau f^*(s) \, ds \right)^2.
\]
By (4.1) and (4.4), we have
\[
\int_0^\tau t \mu(t)^\frac{1}{2} dt \leq - \int_0^\tau t \left( \mu'(t) \mu(t)^\frac{1}{2} G_\alpha(\mu(t))^{-2} \int_0^\mu(t) f^*(s) ds \right) dt \\
\quad + \frac{[\Omega_\gamma]^2 G_\alpha([\Omega_\gamma])^{-2}}{2\beta^2} \left( \int_0^{[\Omega_\gamma]} f^*(s) ds \right)^2
\]
(4.5)
\[
= - \int_0^\tau t dF_\alpha(\mu(t)) + \frac{[\Omega_\gamma]^2 G_\alpha([\Omega_\gamma])^{-2}}{2\beta^2} \left( \int_0^{[\Omega_\gamma]} f^*(s) ds \right)^2,
\]
where \( F_\alpha(l) = \int_0^l \tilde{w}^* G_\alpha(w)^{-2} \int_0^w f^*(s) ds dw \). Integrating by parts and applying (2.11), we obtain
\[
\int_0^\tau \mu(t)^\frac{1}{2} dt + F_\alpha(\mu(\tau)) \leq \frac{1}{V_m} \left[ \int_0^{V_m} dt \int_0^{\mu(t)} \mu(r)^\frac{1}{2} \int_0^{\mu(t)} f^*(s) ds dt \\
\quad + \frac{[\Omega_\gamma]^2 G_\alpha([\Omega_\gamma])^{-2}}{2\beta^2} \left( \int_0^{[\Omega_\gamma]} f^*(s) ds \right)^2 \right].
\]
(4.6)
Making similar computations and using (3.3) and (3.11), we have
\[
\int_0^\tau \phi(t)^\frac{1}{2} dt = - \int_0^\tau t \left( \alpha_k^{-1} \phi'(t) \phi(t)^\frac{1}{2} \tilde{G}_\alpha(\phi(t))^{-2} \int_0^{\alpha_k \phi(t)} f^*(s) ds \right) dt \\
\quad + \alpha_k^{-1} \int_0^\tau \left[ \frac{t}{\beta} \int_{\partial V_m \cap \partial \gamma} \frac{1}{V} d\mu_\gamma \right] \phi(t)^\frac{1}{2} \tilde{G}_\alpha(\phi(t))^{-2} \int_0^{\alpha_k \phi(t)} f^*(s) ds dt \\
\quad = - \int_0^\tau t d\left( \alpha_k^{-\frac{1}{2}} F_\alpha(\alpha_k \phi(t)) \right) + \alpha_k^{-\frac{1}{2}} \frac{[\Omega_\gamma]^2 G_\alpha([\Omega_\gamma])^{-2}}{2\beta^2} \left( \int_0^{[\Omega_\gamma]} f^*(s) ds \right)^2.
\]
(4.7)
Thus,
\[
\int_0^\tau (\alpha_k \phi(t))^{\frac{1}{2}} dt + F_\alpha(\alpha_k \phi(\tau)) = \frac{1}{V_m} \left[ \int_0^{V_m} dt \int_0^{\alpha_k \phi(r)} \alpha_k \phi(r)^\frac{1}{2} \int_0^{\alpha_k \phi(r)} f^*(s) ds dr \\
\quad + \frac{[\Omega_\gamma]^2 G_\alpha([\Omega_\gamma])^{-2}}{2\beta^2} \left( \int_0^{[\Omega_\gamma]} f^*(s) ds \right)^2 \right].
\]
(4.8)
Since (2.10) holds, by (4.6) and (4.8), we conclude
\[
\int_0^\tau \mu(t)^\frac{1}{2} dt + F_\alpha(\mu(\tau)) \leq \int_0^\tau (\alpha_k \phi(t))^{\frac{1}{2}} dt + F_\alpha(\alpha_k \phi(\tau)).
\]
(4.9)
Letting \( \tau \to +\infty \), we prove (1.4).

Next, we come to prove (1.5), (1.6) and (1.7). Letting \( \tau \to +\infty \) in (4.5) and (4.7), we have
\[
\int_0^{+\infty} t \mu(t)^\frac{1}{2} dt \leq \int_0^{+\infty} F_\alpha(\mu(t)) dt + \frac{[\Omega_\gamma]^2 G_\alpha([\Omega_\gamma])^{-2}}{2\beta^2} \left( \int_0^{[\Omega_\gamma]} f^*(s) ds \right)^2,
\]
(4.10)
\[
\int_0^{+\infty} t (\alpha_k \phi(t))^{\frac{1}{2}} dt = \int_0^{+\infty} F_\alpha(\alpha_k \phi(t)) dt + \frac{[\Omega_\gamma]^2 G_\alpha([\Omega_\gamma])^{-2}}{2\beta^2} \left( \int_0^{[\Omega_\gamma]} f^*(s) ds \right)^2.
\]
(4.11)
We then need to prove
\[ \int_0^{+\infty} F(x)\mu(t)dt \leq \int_0^{+\infty} F(x)\phi(t)dt. \]
Multiplying (3.2) by \( tF(x)\mu(t)G_s\mu(t)^{-2} \) and integrating from 0 to \( \tau > \nu_1 \), we get
\[ \int_0^\tau tF(x)\mu(t)dt \leq -\int_0^\tau t\left( \mu'(t)F(x)\mu(t)G_s\mu(t)^{-2} \right) \int_0^{\mu(t)} f^*(s)ds \, dt 
+ \int_0^\tau \frac{t}{\beta} \int_{\beta_0\tau}^\infty \frac{1}{\mu} F(x)\mu(t)G_s\mu(t)^{-2} \int_0^{\mu(t)} f^*(s)ds \, dt. \]

(4.12)

It’s not hard to verify that \( F_0(l)G_0(l)^{-2} \) is non-decreasing when \( 0 < p \leq \frac{n}{2n-3} \). By definitions of \( G_1 \) and \( F_1 \), when \( 0 < p \leq \frac{n}{2n-3} \), one has
\[ G''_1(l) = \frac{d}{dl} \frac{n-1}{\tan(I^{-1}_1(l))} \leq 0, \quad G_1(l) - 3plG'_1(l) \geq 0. \]

Hence we have
\[ \frac{d}{dl} \left( F'_1(l)G_1(l) - 2F_1(l)G'_1(l) \right) = F''_1(l)G_1(l) - F'_1(l)G'_1(l) - 2F_1(l)G''_1(l) \]
\[ \geq F''_1(l)G_1(l) - F'_1(l)G'_1(l) \]
\[ = \frac{1}{p} l^{\frac{1}{2}} G_1(l)^{-2} \left( G_1(l) - 3plG'_1(l) \right) \int_0^q f^*(t)dt + l^{\frac{1}{2}} G_1(l)^{-1}f^*(l) \]
\[ \geq 0. \]

Noticing that \( F'_1(0)G_1(0) - 2F_1(0)G'_1(0) = 0 \), we have
\[ F'_1(l)G_1(l) - 2F_1(l)G'_1(l) \geq 0. \]

Thus, \( F_1(l)G_1(l)^{-2} \) is non-decreasing since
\[ (F_1G_1^{-2})'(l) = G_1(l)^{-3}(F'_1(l)G_1(l) - 2F_1(l)G'_1(l)) \geq 0. \]

When \( n = 2, \ 0 < p \leq 1 \), we have
\[ (G_1G'_1)'(l) = 2\pi \alpha_1 \frac{d}{dl} \left( \frac{\cos(I^{-1}_1(l))}{\cos((I^{-1}_1(1)^{-2}) \right) \leq 0, \quad G_1(l) - 2plG'_1(l) \geq 0. \]

Hence we deduce
\[ \frac{d}{dl} \left( F'_1(l)G_1(l)^2 - 2F_1(l)G_1(l)G'_1(l) \right) = F''_1(l)G_1(l)^2 - 2F_1(l)(G_1G'_1)'(l) \]
\[ \geq F''_1(l)G_1(l)^2 \]
\[ = \frac{1}{p} l^{\frac{1}{2}} G_1(l)^{-1} \left( G_1(l) - 2plG'_1(l) \right) \int_0^q f^*(t)dt + l^{\frac{1}{2}} f^*(l) \]
\[ \geq 0. \]

Together with \( F'_1(0)G_1(0)^2 - 2F_1(0)G_1(0)G'_1(0) = 0 \), we have
\[ F'_1(l)G_1(l)^2 - 2F_1(l)G_1(l)G'_1(l) \geq 0. \]

Then \( F_1(l)G_1(l)^{-2} \) is non-decreasing by
\[ (F_1G_1^{-2})'(l) = G_1(l)^{-4}(F'_1(l)G_1(l)^2 - 2F_1(l)G_1(l)G'_1(l)) \geq 0. \]
Since \( F_\kappa(l)G_\kappa(l)^{-2} \) is non-decreasing, by (3.10) we have
\[
\int_0^\tau tF_\kappa(\mu(t))dt \leq -\int_0^\tau tdH_\kappa(\mu(t)) + \frac{F_\kappa(|\Omega|^g_\kappa)G_\kappa(|\Omega|^g_\kappa)^{-2}}{2\beta^2} \left( \int_0^{|\Omega|^g_\kappa} f^+(s)ds \right)^2,
\]
where \( H_\kappa(l) = \int_0^l F_\kappa(w)G_\kappa(w)^{-2} \int_0^w f^+(s)dsdw. \) Integrating by parts and applying (2.11), one gets
\[
\int_0^\tau F_\kappa(\mu(t))dt + H_\kappa(\mu(\tau)) \leq \frac{1}{v_m} \left[ \int_0^{\nu_m} dt \int_0^\tau F_\kappa(\mu(r))dr + \int_0^{\nu_m} H_\kappa(\mu(t))dt \right.
\]
\[
+ \frac{F_\kappa(|\Omega|^g_\kappa)G_\kappa(|\Omega|^g_\kappa)^{-2}}{2\beta^2} \left( \int_0^{|\Omega|^g_\kappa} f^+(s)ds \right)^2
\]
(4.14)

Analogously, by (3.3) and (3.11), we obtain
\[
\int_0^\tau tF_\kappa(\alpha_\kappa \phi(t))dt = -\int_0^\tau t \left( \alpha_\kappa^{-1} \phi' \left( t F_\kappa(\alpha_\kappa \phi(t)) \tilde{G}_\kappa(\phi(t)) \right)^{-2} \int_0^{\alpha_\kappa \phi(t)} f^+(s)ds \right) dt
\]
\[
+ \alpha_\kappa^{-1} \int_0^\tau \left[ \frac{t}{\beta} \int_{\partial V_\tau \cap \partial \Omega^2} \frac{1}{\nu} d\mu_k \right] F_\kappa(\alpha_\kappa \phi(t)) \tilde{G}_\kappa(\phi(t))^{-2} \int_0^{\alpha_\kappa \phi(t)} f^+(s)ds dt
\]
\[
= -\int_0^\tau tdH_\kappa(\alpha_\kappa \phi(t)) + \frac{F_\kappa(|\Omega|^g_\kappa)G_\kappa(|\Omega|^g_\kappa)^{-2}}{2\beta^2} \left( \int_0^{|\Omega|^g_\kappa} f^+(s)ds \right)^2.
\]
(4.15)

Thus,
\[
\int_0^\tau F_\kappa(\alpha_\kappa \phi(t))dt + H_\kappa(\alpha_\kappa \phi(\tau)) = \frac{1}{v_m} \left[ \int_0^{\nu_m} dt \int_0^\tau F_\kappa(\alpha_\kappa \phi(r))dr + \int_0^{\nu_m} H_\kappa(\alpha_\kappa \phi(t))dt \right.
\]
\[
+ \frac{F_\kappa(|\Omega|^g_\kappa)G_\kappa(|\Omega|^g_\kappa)^{-2}}{2\beta^2} \left( \int_0^{|\Omega|^g_\kappa} f^+(s)ds \right)^2
\]
(4.16)

By (2.10), (4.14) and (4.16), we come up with
\[
\int_0^\tau F_\kappa(\mu(t))dt + H_\kappa(\mu(\tau)) \leq \int_0^\tau F_\kappa(\alpha_\kappa \phi(t))dt + H_\kappa(\alpha_\kappa \phi(\tau)).
\]
(4.17)

Letting \( \tau \to +\infty \) and using (4.10), (4.11), we prove (1.5), (1.6) and (1.7).

\[ \square \]

**Proof of Theorem 1.2.** If \( f \equiv 1 \), then (4.1) becomes
\[
\int_0^\tau t\mu(t)^{\frac{1}{p}} dt \leq \int_0^\tau -\mu'(t)\mu(t)^{\frac{1}{p}+1} G_\kappa(\mu(t))^{-2} dt
\]
\[
+ \int_0^\tau \left[ \frac{t}{\beta} \int_{\partial V_\tau \cap \partial \Omega^2} \frac{1}{\nu} d\mu_k \right] \mu(t)^{\frac{1}{p}+1} G_\kappa(\mu(t))^{-2} dt.
\]
(4.18)

It’s easy to show that \( l^{\frac{1}{p}+1} G_\kappa(l)^{-2} \) is non-decreasing when \( \kappa = 0, 1 \) and \( 0 < p \leq \frac{n}{n-2} \), hence
\[
\int_0^\tau t\mu(t)^{\frac{1}{p}} dt \leq -\int_0^\tau tdF_\kappa(\mu(t)) + \frac{|\Omega|^g_\kappa^{\frac{1}{p}+2} G_\kappa(|\Omega|^g_\kappa)^{-2}}{2\beta^2}.
\]
(4.19)

Following the similar arguments in proof of (1.4), we are able to prove (1.8).

Noticing that \( lF_0(0)G_0(l)^{-2} \) is still non-decreasing when \( 0 < p \leq \frac{n}{n-2} \), we adopt same as the proof of (1.5), (1.6) and (1.7) and finish the proof of (1.9).
Finally, we give the proof of the pointwise comparison result (1.10). For \( n = 2, \kappa = 0 \) and \( f \equiv 1 \), (3.2) becomes
\[
4\pi\alpha_0 \leq -\mu'(t) + \frac{1}{\beta} \int_{\partial U^r} \frac{1}{v} d\mu_g,
\]
while (3.3) yields
\[
4\pi = -\phi'(t) + \frac{1}{\beta} \int_{\partial V \cap \partial \Omega} \frac{1}{v} d\mu_g.
\]
Multiplying (4.20) by \( t \), integrating from 0 to \( \tau > v_m \) and applying (3.10), we obtain
\[
2\pi\alpha_0 \tau^2 \leq \int_0^\tau -t\mu' dt + \frac{|\Omega|}{2\beta^2}.
\]
Meanwhile, by (4.21), one has
\[
2\pi\alpha_0 \tau^2 = \int_0^\tau -t(\alpha_0\phi(t))' dt + \frac{|\Omega|}{2\beta^2}.
\]
Thus,
\[
\int_0^\tau -t\mu' dt \geq \int_0^\tau -t(\alpha_0\phi(t))' dt.
\]
Integrating by parts and using (2.11), for all \( \tau > v_m \), we have
\[
\mu(\tau) - \alpha_0\phi(\tau) \leq \frac{1}{v_m} \int_0^{v_m} \mu(t) - \alpha_0\phi(t) dt.
\]
Together with (2.10), we find out that for all \( \tau \geq 0 \),
\[
\mu(\tau) \leq \alpha_0\phi(\tau).
\]
By definition of Schwarz rearrangement, we complete the proof of (1.10). \( \square \)

5. Proof of Theorem 1.5

For \( n = 2 \), it is independently interesting to prove Bossel-Daners inequality by using our Theorem 1.1. For \( n \geq 2 \), the methods in [7] and [17] can be adopted to prove the Bossel-Daners inequality. For bounded domains in Riemannian manifolds with \( \text{Ric}(g) \geq (n - 1) \), the inequality (1.15) has already been proven in [12]. Here we only give the proof of (1.15) for complete noncompact Riemannian manifolds with nonnegative Ricci curvature and positive asymptotic volume growth.

Proof of Theorem 1.5. Following the idea in [22], we can prove the Bossel-Daners inequality for the first eigenvalue of Robin Laplacian in dimension 2. Let \( u \) and \( z \) be solutions to (1.13) and (1.14) respectively. When \( n = 2 \), by choosing \( p = 1 \) in (1.5) and (1.7), we have
\[
\int_{\Omega^r} (u^#)^2 dV_g = \alpha_1^{-1} \int_{\Omega} u^2 dV_g \leq \int_{\Omega^r} z^2 dV_g.
\]
By Cauchy-Schwarz inequality, one has
\[
\int_{\Omega^r} u^# z dV_g \leq \int_{\Omega^r} z^2 dV_g.
\]
By Rayleigh quotient for eigenvalues, we obtain

$$\lambda_{1,\beta}(\Omega) = \frac{\int_{\Omega} |\nabla_{g}\zeta|_{g}^{2} dV_{g} + \beta \int_{\partial \Omega} \zeta^{2} d\mu_{g}}{\int_{\Omega} u^{2} dV_{g}}$$

(5.3)

$$\geq \frac{\int_{\Omega} |\nabla_{g}\zeta|_{g}^{2} dV_{g} + \beta \int_{\partial \Omega} \zeta^{2} d\mu_{g}}{\int_{\Omega} \zeta^{2} dV_{g}} \geq \lambda_{1,\beta}(\Omega^{\sharp}).$$

Now we are in position to give the Bossel-Daners inequality for \( n \geq 2 \) and \( \kappa = 0 \). Let \( B_{R}^{0} \) be the geodesic ball of radius \( R \) in Euclidean space \((\mathbb{R}^{n}, 0)\) such that \( |\Omega|_{g} = \alpha_{0} |B_{R}^{0}|_{g_{0}} \). Let \( u_{0} \) be the eigenfunction associated to \( \lambda_{1,\beta}(B_{R}^{0}) \) on geodesic ball \( B_{R}^{0} \), i.e.

$$\begin{cases}
u'' - \frac{n-1}{r} \nu' + \lambda_{1,\beta}(B_{R}^{0}) \nu = 0, & \text{on } B_{R}^{0}, \\
u(0) = 0, & u_{0}(R) + \beta u_{0}(R) = 0.
\end{cases}$$

(5.4)

It is known that the function \( u_{0}(r) \) is positive for \( r \in [0, R] \). Define \( \nu(r) = (\ln u_{0})'(r) \). According to Lemma 2.1 and Proposition 2.2 in [12] (See also [17, 9]), the function \( \nu(r) = (\ln u_{0})'(r) \) is strictly decreasing and then \( 0 \leq -\nu(r) < \beta \) for \( r \in [0, R] \) and the first eigenvalue \( \lambda_{1,\beta}(B_{R}^{0}) \) of Robin Laplacian is strictly decreasing with respect to radius \( R \). Let \( u \) be the first eigenfunction of Laplacian equation on \( \Omega \). We can choose \( u > 0 \) and normalize it such that

$$\max_{x \in \Omega} u(x) = 1, \quad \min_{x \in \Omega} u(x) = m.$$

For \( t \in (m, 1) \), set

$$U_t = \{ x \in \Omega : u(x) > t \}, \quad \partial U_t^j = \{ x \in \Omega : u(x) = t \}, \quad \partial U_t^c = \partial U_t \cap \partial \Omega = \{ x \in \partial \Omega : u(x) \geq t \}.$$

Define the admissible subset \( M_{\beta}(\Omega) \) of \( C(\Omega) \) by

$$M_{\beta}(\Omega) = \left\{ \varphi \in C(\Omega) : \varphi(x) \geq 0, \limsup_{x \to z} \varphi(x) \leq \beta, \text{ for all } z \in \partial \Omega \right\}.$$

For \( \varphi \in M_{\beta}(\Omega) \), we can define the functional \( H_{\Omega}(U_t, \varphi) \) as in [7] and [17]

$$H_{\Omega}(U_t, \varphi) = \frac{1}{|U_t|_{g}} \left( \beta |\partial U_t^c|_{g} + \int_{\partial U_t^j} \varphi d\mu_{g} - \int_{U_t} \varphi^{2} dV_{g} \right).$$

(5.7)

For the first eigenfunction \( u \), one can infer that \( \frac{|\nabla_{g}u|_{g}}{u} \in M_{\beta}(\Omega) \) and

$$\lambda_{1,\beta}(\Omega) = H_{\Omega} \left( U, \frac{|\nabla_{g}u|_{g}}{u} \right), \quad \text{for almost all } t \in (m, 1).$$

(5.8)

For \( \varphi \in M_{\beta}(\Omega) \), set

$$w = \varphi - \frac{|\nabla_{g}u|_{g}}{u}, \quad \text{and} \quad F(t) = \int_{U_t} w^2 dV_{g}.$$

From Lemma 3.1 in [12] (See also [9, 17]), the functional \( H_{\Omega}(U_t, \varphi) \) satisfies

$$H_{\Omega}(U_t, \varphi) = \lambda_{1,\beta}(\Omega) - \frac{1}{|U_t|_{g}} \left( \frac{1}{t} \frac{d}{dt} (t^2 F(t)) + \int_{U_t} w^2 dV_{g} \right).$$

(5.9)
for almost all \( t \in (m, 1) \). Furthermore, suppose that \( \varphi \neq \frac{|\nabla u|}{u} \), then there exits a set \( S \subset (m, 1) \) with \(|S| > 0\) such that

\[
\lambda_{1, \varphi}(\Omega) > H_{\Omega}(U, \varphi), \quad \text{for all} \quad t \in S.
\]

(5.10)

For \( t \in (m, 1) \), we define the ball \( B^{0}_{r(t)} \) with radius \( r(t) \) such that \(|U|_{g} = \alpha_{0} |B^{0}_{r(t)}|_{g_0} \). For \( x \in \partial U_{t}^{i} \) and \( t \in (m, 1) \), we define

\[
\varphi(x) = -v(r(t)) = \frac{-u_{0}^{\prime}(r(t))}{u_{0}(r(t))}.
\]

(5.11)

It is easy to check that \( \varphi : \Omega \to (0, \infty) \) is a measurable function and

\[
U_{i} = \{ x \in \Omega : u(x) > t \} = \{ x \in \Omega : \varphi(x) < -v(r(t)) \}
\]

is open in \( \Omega \) for \( t \in (m, 1) \). From the construction of \( \varphi \) in (5.11) and the monotonic decreasing property of \( v(r) = (\ln u_{0})^{\prime} (r) \), we notice that

\[
0 \leq \varphi(x) < \beta, \quad \text{for} \quad x \in \Omega,
\]

which implies that \( \varphi \in M_{\beta}(\Omega) \). Since \(|U|_{g} = \alpha_{0} |B^{0}_{r(t)}|_{g_0} \), we have

\[
\int_{U_{i}} \varphi^{2} dV_{g} = \alpha_{0} \int_{B^{0}_{r(t)}|_{g_0}} (-v(r(t)))^{2} dV_{g_0}, \quad \text{for} \quad t \in (m, 1).
\]

(5.13)

Moreover, by isoperimetric inequality (2.1) and \( \varphi = -v(r(t)) \leq \beta \), we obtain that

\[
\alpha_{0} |\partial B^{0}_{r(t)}|_{g_0} (-v(r(t))) \leq (-v(r(t))) |\partial U_{i}^{g}| \leq \int_{\partial U_{t}^{i}} \varphi(x) d\mu_{g} + \beta |\partial U_{i}^{g}|_{g}, \quad \text{for} \quad t \in (m, 1),
\]

(5.14)

From (5.11) and (5.14), we conclude

\[
\lambda_{1, \varphi}(\Omega) = H_{B_{R}}(B^{0}_{r(t)}, -v(r(t)))
\]

\[
= \frac{1}{\alpha_{0} |B^{0}_{r(t)}|_{g_0}} \left( \alpha_{0} \int_{\partial B^{0}_{r(t)}} (-v(r(t))) d\mu_{g_0} - \alpha_{0} \int_{B^{0}_{r(t)}} (-v(r(t)))^{2} dV_{g_0} \right)
\]

\[
\leq \frac{1}{|U|_{g}} \left( \beta |\partial U_{i}^{g}|_{g} + \int_{\partial U_{t}^{i}} \varphi(x) d\mu_{g} - \int_{U_{i}} \varphi^{2} dV_{g} \right)
\]

\[
\leq \lambda_{1, \varphi}(\Omega),
\]

which completes the proof of the inequality (1.15). When equality occurs in (1.15), the above inequalities become equalities. Therefore, according to [8], the equality holds in (1.15) if and only if \((M, g)\) is isometric to \((\mathbb{R}^{n}(0), g_0)\) and \(\Omega\) is isometric to a ball \(B^{0}_{R}\), where \(g_0\) is the canonical metric of Euclidean space.

\(\square\)
References

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