HOMOTOPY CLASSES OF NEWTONIAN SPACES

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Abstract. We study notions of homotopy in the Newtonian space $N^{1,p}(X;Y)$ of Sobolev type maps between metric spaces. After studying the properties and relations of two different notions we prove a compactness result for sequences in homotopy classes with controlled homotopies.

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1. Introduction

Interest in homotopy classes of mappings and energy minimizers arises naturally both in the theory of PDE’s – where certain energy minimizers in homotopy classes provide natural examples of non-uniqueness of some (systems) of partial differential equations (2) – and in the study of the geometry of manifolds. Minimizing some energy in a given homotopy class provides one with a well-behaved representative of that class. Topological conclusions from the study of harmonic maps in given homotopy classes were drawn, for instance, by R. Schoen and S.T. Yau in [38, 39, 37]. For $p$-harmonic maps, connections to higher homotopy groups, as well as to homotopy classes of maps arise, see e.g. [43, 46, 42].

From early on in the work of various authors, such as Eells and Sampson [9], it has been noted that certain methods of obtaining existence results for harmonic

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maps in homotopy classes are restricted to the setting of non-positively curved target manifolds (see the survey article [8] for further discussion). This paper is a continuation of that line of research. Of course some results, such as in the papers [5, 44] of Burstall and White, have been obtained for the existence, regularity (and, more rarely, uniqueness) of harmonic and $p$-harmonic maps between general Riemannian manifolds, with varying assumptions. More recently $p$-harmonic maps between general Riemannian manifolds in have been studied in [36, 21, 42] to mention but a few.

Towards a nonsmooth theory the assumption of (some sort of) nonpositive curvature on the target space seems to become compulsory. Starting with Gromov’s and Schoen’s work [10], continued in [31, 7] a theory of harmonic maps from a Riemannian manifold (or polyhedron) to a nonpositively curved metric space (in the sense of Alexandrov, see Section 1.2 below) was built. Jost, in a series of papers [23, 24, 25] studied harmonic maps from metric spaces with a doubling measure and a Poincaré inequality to metric spaces of nonpositive curvature. This setting is closest to ours; with basically the same assumptions we proceed to define and study homotopy classes using tools coming from analysis in metric spaces (more of which in Section 1.1 below).

The main goal in this paper has been to try and prove the stability of $p$-homotopy classes under $L^p$-convergence, in the spirit of [44]. A particular source of inspiration (as well as the source of a few definitions) have been the papers [3, 15]. Unfortunately the methods – and for the most part the definitions – in all of the above mentioned papers seem to be specific to the manifold setting. Therefore our approach to the stability problem is necessarily quite different from [44, 3, 15].

The strategy adopted here to reach the desired conclusion has been to look at a given homotopy class, $[v]$ and construct a covering space $H^v$ together with a covering map $p : H^v \to [v]$. Under the appropriate technical assumptions the stability result would follow from the fact that $H^v$ is a proper metric space (proven in section 3), $[v]$ is known to be precompact (the Rellich Kondrakov theorem) and $p$ is a covering map. However, I have been unable to prove this last part, and this inability comes from a lack of knowledge concerning the metric geometry of the space $N^{1,p}(X;Y) \supset [v]$. The numerous details of this (ultimately failed) attempt are presented in Section 3.

It nevertheless seemed reasonable to communicate the partial results obtained along the way, in hope of encouraging future research for a better understanding of the metric properties of Newtonian classes of maps.

The outline of the paper is as follows. In the first and second subsections of the introduction, relevant facts and notions concerning analysis on metric spaces are presented, including the concept of upper gradients, Poincaré inequalities and the Newtonian and Dirichlet classes of mappings. The third subsection serves as a brief review of the basics of nonpositively curved spaces. Both the definition of Alexandrov and that of Busemann are presented and briefly discussed.

In the second section we focus on two different notions of homotopy. Some properties of each are exhibited and the relationship between the different notions is studied.

The third and final section is devoted to "lifts" of homotopies. Using this point of view on homotopy between maps we construct a kind of (rectifiable) covering space over a given homotopy class, and use it to prove a weak compactness result.
of homotopy classes of maps between compact spaces. We also construct a group acting freely and properly on this covering and discuss the connection between the resulting quotient space and the (restricted) $p$-homotopy class $[v]_{p,M}$.

The paper is closed by a description of some open problems in this setup and a discussion of possible future research directions.

Notation and convention. Throughout this paper, the notation

$$f_A = \int_A f \, d\mu := \frac{1}{\mu(A)} \int_A f \, d\mu$$

will be used for the average of a locally $\mu$-integrable function $f$ over the $\mu$-measurable set $A$, with positive measure. The centered maximal function is denoted by

$$M_R f(x) := \sup_{0 < r < R \setminus B(x,r)} \int f \, d\mu.$$ 

For a number $\sigma > 0$, the dilated ball $\sigma B$ of a (open or closed) ball $B = B(x,r)$ is

$$\sigma B = B(x, \sigma r).$$

The length of a path $\gamma$ joining two points $x, y \in Z$ in a metric space is the following:

$$\ell(\gamma) = \sup \{ \sum_{k=1}^n d_Z(\gamma(a_k), \gamma(a_{k-1})) : a = a_0 < a_1 < \ldots < a_n = b \}.$$ 

In general, this quantity may be infinite. Paths $\gamma$ for which $\ell(\gamma) < \infty$ are called rectifiable. A rectifiable path $\gamma$ can always be affinely reparametrized so that $\gamma : [0,1] \to Y$ and $d(\gamma(t), \gamma(s)) \leq \ell(\gamma|[t,s]) = \ell(\gamma)|t-s|$ for all $t, s \in [0,1]$, $t < s$; see [35, Proposition 2.2.9]. We will call this the constant speed parametrization of a rectifiable path $\gamma$.

If not otherwise stated, we will always regard rectifiable curves $\gamma$ in a metric space $Z$ as being maps $\gamma : [0,1] \to Z$. Depending on the situation we may or may not assume that rectifiable curves are constant speed parametrized.

1.1. Upper gradients and Poincaré inequalities. A standing assumption on the domain space in this paper is that they are complete, doubling metric measure spaces supporting a weak $(1, p)$-Poincaré inequality.

A metric measure space is a locally compact metric space $(X, d)$ equipped with a Borel regular measure $\mu$ with the property that $0 < \mu(B) < \infty$ for all open balls $B \subset X$.

We say that the metric measure space is doubling if the measure is doubling, i.e. there is a constant $0 < C < \infty$ such that

$$\mu(B(x, 2r)) \leq C \mu(B(x, r))$$

for all balls $B(x,r) \subset X$ with $r < \text{diam } X$. Note the difference to saying that a metric space is doubling, which means that for some fixed number $N$, any ball can be covered with at most $N$ balls of half the radius.

To define what we mean by saying that a space supports a Poincaré inequality we need the concept of upper gradients. Let $(X, \mu, d)$ be a metric measure space, $(Y, d_Y)$ any metric space and $u : X \to Y$ a locally integrable map. This means that $x \mapsto d_Y(u(x), y)$ is measurable for every $y \in Y$ and for every compact $K \subset X$.
there is some \( y_0 \in Y \) with \( \int_K d_Y(u, y_0) d\mu < \infty \). See also [20] for the definition of measurability for maps with metric-space target.

A non-negative Borel function \( g : X \to [0, \infty] \) is said to be an upper gradient of \( u \) if, for every rectifiable curve \( \gamma \) with endpoints \( x \) and \( y \) we have the inequality

\[
(1.1) \quad d_Y(u(x), u(y)) \leq \int_\gamma g ds.
\]

Here \( \int_\gamma g ds \) denotes the path-integral of \( g \) along the path \( \gamma \). If \( g \) does not satisfy (1.1) for all curves, but the exceptional curve family \( \Gamma \) has \( p \)-modulus \( \text{Mod}_p(\Gamma) = 0 \) we say that \( g \) is a \( p \)-weak upper gradient of \( u \). See [12, 20, 17] for the definitions of path-integrals and \( p \)-modulus.

Upper gradients, then, are objects that control the behaviour of maps along paths, much like the norm of the gradient of a \( C^1 \)-function. (In fact, the term Newtonian space, to be defined shortly, arises from the analogy between (1.1) and Newtons’ fundamental theorem of calculus.)

An analytic way of imposing a condition that ties the (geo)metric properties of \( X \) and the behaviour of the measure \( \mu \) is to require that upper gradients also control the behaviour of maps in some integral average sense. This is done by the Poincaré inequality.

We say that a metric measure space \((X, d, \mu)\) supports a weak \((1, p)\)-Poincaré inequality if, whenever \( u : X \to \mathbb{R} \) is locally integrable and \( g : X \to [0, \infty] \) is a locally integrable upper gradient of \( u \) the inequality

\[
(1.2) \quad \int_B |u - u_B| d\mu \leq C \text{diam}(B) \left( \int_{\sigma B} g^p d\mu \right)^{1/p}
\]

is satisfied with constants \( C, \sigma \) independent of \( u, g \) and \( B \). The constants in the Poincaré inequality and the doubling constant of the measure will be referred to as the data of the space \( X \).

By now doubling metric measure space supporting a weak \((1, p)\)-Poincaré inequality are known to enjoy many geometric as well as analytic properties. We will only mention some of these that are relevant to this paper. There are numerous sources on the subject, and the interested reader is referred to [18, 33, 6, 20, 40, 13, 17, 29, 12, 19, 1] to name a few.

We record the following useful theorem from [20, Theorem 4.3].

**Theorem 1.1.** Suppose \((X, d, \mu)\) is a complete doubling metric measure space. Then \( X \) supports a weak \((1, p)\)-Poincaré inequality for \( p > 1 \) if and only if it supports a weak \((1, p)\)-Poincaré inequality for \( V \)-valued maps, for any Banach space \( V \), i. e. if there are constants \( C', \sigma' \in [1, \infty) \) such that for every locally integrable map \( u : X \to V \) and every upper gradient \( g \) of \( u \) the inequality

\[
\int_B \|u - u_B\|_V d\mu \leq C' \left( \int_{\sigma' B} g^p d\mu \right)^{1/p}
\]

holds. The constants \( C' \) and \( \sigma' \) then depend only on \( p \) and the data of \( X \).

An important result due to Zhong and Keith [28] is the so called self-improving property of Poincaré inequalities.
**Theorem 1.2.** Let \((X, d, \mu)\) be a doubling metric measure space such that \(X\) is complete. If \(X\) supports a weak \((1, p)\)-Poincaré inequality for some \(p > 1\) then it supports a weak \((1, q)\)-Poincaré inequality for some \(q < p\) with constants depending only on the data.

The assumption of completeness, as well as \(p > 1\), is crucial. There are examples \([32]\) of noncomplete spaces supporting a \((1, p)\)-Poincaré inequality but not a \((1, q)\)-Poincaré inequality for any \(1 \leq q < p\).

**\(p\)-Capacity.** Upper gradients and their \(p\)-weak counterparts enable us to define a concept of \(p\)-capacity of subsets of \(X\), analogously with the classical \(p\)-capacities. Let \((X, d, \mu)\) be a metric measure space, \(E \subset X\) a subset and \(p \geq 1\). The Sobolev \(p\)-capacity of the set \(E\) is defined by

\[
\text{Cap}_p(E) = \inf \left\{ \int_X (|u|^p + g^p) \, d\mu : g \text{ an upper gradient for } u \text{ s.t. } u \geq 1 \text{ on } E \right\}.
\]

The \(p\)-capacity of a condenser, that is a pair of subsets \((E, \Omega)\) where \(E \subset \Omega\) and \(\Omega\) is open is obtained by

\[
\text{Cap}_p(E; \Omega) = \inf \left\{ \int_\Omega g^p \, d\mu : u \geq 1 \text{ on } E, \ u = 0 \text{ on } X \setminus \Omega \right\},
\]

\(g\) an upper gradient of \(u\). As we shall see this concept will play an important role for us. More information on \(p\)-capacities, equivalent notions and variants, can be found for instance in \([30, 1]\).

One result we shall need is the following stronger version of \([1, \text{ Proposition 1.48}]\).

**Lemma 1.3.** Let \(E_n \subset X\) be a sequence of open sets with \(\varepsilon_n := \text{Cap}_p(E_n)\) converging to zero. Denote

\[
\Gamma_\infty = \{ \gamma : \gamma^{-1}(E_n) \neq \emptyset \forall n \}.
\]

Then \(\text{Mod}_p(\Gamma_\infty) = 0\).

**Proof.** Let \(u_m\) be such that \(u_m|_{E_m} = 1\), \(u_m \geq 0\) and \(g_m\) an upper gradient of \(u_m\) with

\[
\int_X (|u_m|^p + g_m^p) \leq 2\varepsilon_m.
\]

Since \(\varepsilon_m \to 0\) we have \(u_m \to 0\) in \(N^{1,p}(X)\) as \(m \to \infty\). Consequently we may pass to a subsequence \(u_{m_k}\) converging to zero outside a set \(F\) of \(p\)-capacity zero and satisfying

\[
\sum_{k=1}^{\infty} \varepsilon_k^{1/p} < \infty.
\]

For \(l \geq 1\), set

\[
\rho_l = \sum_{k \geq l} g_{m_k}.
\]

Then

\[
\left( \int_X \rho_l^p \, d\mu \right)^{1/p} \leq \sum_{k \geq l} \left( \int_X g_{m_k}^p \, d\mu \right)^{1/p} \leq 2 \sum_{k \geq l} \varepsilon_k^{1/p}.
\]
Denote by $\Gamma_F$ the family of paths $\gamma$ with the property that $\gamma^{-1}(F) \neq \emptyset$. Then for $\gamma \in \Gamma_{\infty} \setminus \Gamma_F$ we have that for each $k$ there exists $t_k \in [0, 1]$ with $\gamma(t_k) \in E_{m_k}$ while $\gamma(0) \notin F$. Given $l \geq 1$ we have, for all $k \geq l$ the estimate

$$|u_{m_k}(\gamma(0)) - 1| = |u_{m_k}(\gamma(0)) - u_{m_k}(\gamma(t_k))| \leq \int_{\gamma} g_{m_k} \leq \int_{\gamma} \rho_l.$$

Taking $k \to \infty$ we have $u_{m_k}(\gamma(0)) \to 0$, whence

$$\int_{\gamma} \rho_l \geq 1.$$

This shows that

$$\text{Mod}_p(\Gamma_{\infty} \setminus \Gamma_F) \leq \int_X \rho_l^p \, d\mu \leq 2^p \left( \sum_{k \geq l} 2^{l/p} \right)^{p} \to 0$$

as $l \to \infty$. Since by [1, Proposition 1.48] we have $\text{Mod}_p(\Gamma_F) = 0$ it follows that

$$\text{Mod}_p(\Gamma_{\infty}) \leq \text{Mod}_p(\Gamma_{\infty} \setminus \Gamma_F) + \text{Mod}_p(\Gamma_F) = 0$$

and the proof is complete. \qed

The following proposition, though not difficult to establish, does not seem to appear in the literature. A proof is therefore included.

**Proposition 1.4.** Let $(X, d_X, \mu)$ and $(Y, d_Y, \nu)$ be complete doubling metric measure spaces supporting $(1, p)$–Poincaré inequalities. Then the product space $(X \times Y, d_{\text{max}}, \mu \times \nu)$ is a complete doubling metric measure space and also supports a $(1, p)$–Poincaré inequality.

**Proof.** We will use the maximum metric $d_{\text{max}}$ on $X \times Y$ for technical convenience, and abbreviate notation by dropping the subscript. Completeness of $X \times Y$ is elementary, and the doubling condition on $\mu \times \nu$ is also easy to establish:

$$\mu \times \nu(B((x, y); 2r)) = \mu(B_X(x, 2r))\nu(B_Y(y, 2r)) \leq C_{\mu} C_{\nu} \mu(B_X(x, r))\nu(B_Y(y, r)) = C_{\mu} C_{\nu} \mu \times \nu(B((x, y); r)).$$

It remains to be shown that $X \times Y$ supports a $(1, p)$–Poincaré inequality. By [26, Theorem 2] it suffices to consider Lipschitz functions and their pointwise upper Lipschitz constants as upper gradients (we do this mainly for expository purposes: the argument can be made to work without restricting to Lipschitz functions).

Let $u : X \times Y \to \mathbb{R}$ be a Lipschitz function and fix a ball $B_0 = B_X(x, r) \times B_Y(y, r)$. For fixed $z \in X$ denote by $u^z : Y \to \mathbb{R}$ the function $u^z(w) = u(z, w)$, and $v : X \to \mathbb{R}$ the function $z \mapsto (u^z)_{B_Y(y, r)}$. Note that both are again Lipschitz, with

$$\text{Lip } u^z(w) \leq \text{Lip } u(z, w) \text{ and } \text{Lip } v(z) \leq \int_{B_Y(y, r)} \text{Lip } u^z(w) \, d\nu(w).$$

Note also that

$$u_{B_0} = \int_{B_X(x, r)} \int_{B_Y(y, r)} u(z, w) \, d\nu(w) \, d\mu(z) = \int_{B_X(x, r)} v \, d\mu.$$


With these observations we may estimate
\[\int_{B_0} |u(z, w) - u_{\sigma B_0}| d(\mu \times \nu)(z, w) \leq \int_{B_X(x, r)} \int_{B_Y(y, r)} |u^\sigma(w) - (u^\sigma)_{B_Y(y, r)}| d\nu(w) d\mu(z)\]
\[+ \int_{B_X(x, r)} \int_{B_Y(y, r)} |(u^\sigma)_{B_Y(y, r)} - (u^\sigma)_{B_Y(y, r)}| d\nu(w) d\mu(z).\]

The first summand on the right-hand side of this inequality may be estimated, using the Poincaré inequality of \(X\), to be at most
\[Cr \left( \int_{B_X(x, r)} (\text{Lip } u^\sigma)^p d\nu \right)^{1/p} \leq Cr \left( \int_{B_X(x, r)} (\text{Lip } u)^p d\nu d\mu \right)^{1/p},\]
while the second is at most
\[Cr \left( \int_{B_X(x, r)} (\text{Lip } u)^p d\mu \right)^{1/p} \leq Cr \left( \int_{B_X(x, r)} (\text{Lip } u)^p d\nu d\mu \right)^{1/p}\]
by the Poincaré inequality of \(X\). Thus we arrive at
\[\int_{B_0} |u - u_{\sigma B_0}| d(\mu \times \nu) \leq Cr \left( \int_{\sigma B_0} (\text{Lip } u)^p d(\mu \times \nu) \right)^{1/p},\]
and are done. \(\square\)

**Remark 1.5.** The proof yields some quantitative information on the constants: the doubling constant of the measure \(\mu \times \nu\) is at most the product of the doubling constants of \(\mu\) and \(\nu\). Furthermore, the constant in the Poincaré inequality is at most \(C_X + C_Y\) where \(C_X\) and \(C_Y\) are the constants in the Poincaré inequalities of \(X\) and \(Y\), respectively. Provided the dilation constant \(\sigma\) (appearing in the dilated balls in the right-hand side of the Poincaré inequality) is such that it works for both the Poincaré inequalities of \(X\) and \(Y\).

1.2. Maps with (locally) \(p\)-integrable upper gradients. To study maps between metric spaces we adopt the framework used in [20]. Let \((X, d, \mu)\) be a metric space and \(V\) a Banach space with the Lipschitz extension property. This property states for any metric space \(Z\) and any \(L\)-Lipschitz map \(f : A \to V\) from an arbitrary subset \(A \subset Z\) may be extended to an \(CL\)-Lipschitz map \(\overline{f} : X \to V\), with constant \(C\) independent of \(Z, A\) and \(f\). Examples of such spaces are \(V = \mathbb{R}\) and \(V = \ell^\infty\). See [20 Chapter 2] for the definitions of measurability, integrability of a map \(u : X \to V\) as well as being essentially separably valued.

The Dirichlet class \(D^{1,p}(X; V)\) consists of measurable maps \(u : X \to V\) which have a \(p\)-integrable \(p\)-weak upper gradient \(g^u\).

Since for any \(p\)-weak upper gradient \(g\) of \(u\) (not necessarily \(p\)-integrable) there is a sequence of \(g_k\) of upper gradients such that \(\|g_k - g\|_{L^p(X)} \to 0\), [11 Lemma 1.46], it follows that the requirement of \(u\) having a \(p\)-integrable \(p\)-weak upper gradient is equivalent to requiring that \(u\) has a \(p\)-integrable upper gradient.

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1It is implicitly understood that maps \(u, v\) which agree outside a set of \(p\)-capacity zero are identified, similarly to the usual \(L^p\)-theory.
Measurability and local $p$-integrability. Note that in the definition of $D^{1,p}(X; V)$ no local integrability assumption is made. However, if $v^* \in V^*$ then the function $u_{v^*} := v^*(u)$ has upper gradient $g$ whenever $g$ is an upper gradient for $u$:

$$|u_{v^*}(\gamma(1)) - u_{v^*}(\gamma(0))| \leq ||u(\gamma(1)) - u(\gamma(0))||_V \leq \int \gamma \cdot g.$$  

This remark remains true without any measurability assumptions on $u$. From [22, Theorem 1.1] we have the following.

**Theorem 1.6.** If $(X, d, \mu)$ supports a weak $(1, p)$-Poincaré inequality and $f : X \to [-\infty, \infty]$ is a function that has a $p$-integrable upper gradient, then $f$ is measurable and locally $p$-integrable.

Applying this to $f = u_{v^*}$ (with arbitrary $v^* \in V^*$) we see that in fact $u$ is weakly measurable (see [20]) and locally $p$-integrable. From the Pettis measurability theorem [20, Theorem 2.1] it follows that if $u$ is essentially separably valued, the existence of a $p$-integrable (p-weak) upper gradient implies both measurability and local $p$-integrability of $u$.

**Minimal $p$-weak upper gradients.** As in [1, Section 2.2] (or [19, Chapter 6 and 7]) it can be seen that the set

$$G_u = \{ g \in L^p(X) : g \text{ is a $p$-weak upper gradient for } u \}$$

is a closed and convex lattice, if $p > 1$. It follows that there is a unique minimal element $g_u$ in the sense that for all $g \in G_u$, one has $g_u \leq g$ almost everywhere. We arrive at the following [1, Theorem 2.5].

**Theorem 1.7.** For $p > 1$, every map $u \in D^{1,p}(X; V)$ has a unique minimal $p$-integrable $p$-weak upper gradient, denoted $g_u$.

**Remark 1.8.** Regarding the minimal $p$-weak upper gradient of a locally lipschitz map $u : X \to V$, with $(X, d, \mu)$ a doubling metric measure space supporting a weak $(1, p)$-Poincaré inequality, we note that there is a constant $C$ depending only on the data of $X$ such that

$$\text{Lip } u \leq C g_u$$

almost everywhere. To see this we simply note that Keith’s proof of his result [27, Proposition 4.3.3],

$$\text{Lip } u(x) \leq C \limsup_{r \to 0} \frac{1}{r} \int_{B(x, r)} |u - u_B| d\mu \text{ almost everywhere}$$

applies to Lipschitz maps with a Banach space target. From this our claim follows by a straightforward application of the Poincaré inequality:

$$\limsup_{r \to 0} \frac{1}{r} \int_{B(x, r)} |u - u_B| d\mu \leq C \limsup_{r \to 0} \left( \int_{B(x, r)} g_{u_B}^p d\mu \right)^{1/p} = C g_u(x)$$

almost everywhere. When the target is a locally compact CAT(0) space $Y$ (which is locally geodesically complete, see section 1.2 below) it is proven in [34, Corollary 5.10] that $\text{Lip } u = g_u$ for locally lipschitz maps $u : X \to Y$. 
(local) Newtonian classes. We say that a map $u : X \to V$ belongs to the \textit{local Newtonian class}, $N^{1,p}_{\text{loc}}(X;V)$, if $u$ is locally $p$-integrable and possesses a ($p$-weak) upper gradient $g \in L^p_\text{loc}(X)$, while the \textit{Newtonian class}, $N^{1,p}(X;V)$, consists of maps $u \in L^p(X;V)$ with a ($p$-weak) upper gradient $g \in L^p(X)$. The inclusions $N^{1,p}(X;V) \subset D^{1,p}(X;V) \subset N^{1,p}_{\text{loc}}(X;V)$ are valid by definition and the previous discussion.

$p$-\textit{quasicontinuity}. It follows from the properties of the $p$-modulus $\text{Mod}_p$ and the definition of $p$-weak upper gradients that, given $u \in D^{1,p}(X;V)$, there exists a curve family $\Gamma$ with $\text{Mod}_p(\Gamma) = 0$ so that if $\gamma \notin \Gamma$ then

$$
\|u(\gamma(b)) - u(\gamma(a))\|_V \leq \int_{\gamma([a,b])} g_u, \ a, b \in [0,1].
$$

In particular $u$ is absolutely continuous along $p$-almost every curve $\gamma$.

For us a crucial continuity property is the following concept of $p$-quasicontinuity: a map $u : X \to V$ is said to be $p$-quasicontinuous if for every $\varepsilon > 0$ there exists an open set $E \subset X$ with $\text{Cap}_p(E) < \varepsilon$ such that $u|_{X \setminus E} : X \setminus E \to V$ is continuous.

\textbf{Lemma 1.9.} Suppose $(X,d,\mu)$ is a proper metric measure space supporting a weak $(1,p)$-Poincaré inequality. Then every map $u \in N^{1,p}_{\text{loc}}(X;V)$ is $p$-quasicontinuous.

\textbf{Proof.} From \cite{17} Corollary 6.8 we have the claim for maps $v \in N^{1,p}(X;V)$. Since $u \in N^{1,p}_{\text{loc}}(X;V)$ we have $u_j := \varphi_j u \in N^{1,p}(X;V)$ for all $j \geq 1$, where

$$
\varphi_j(x) = \begin{cases} 
1, & x \in B(x_0,j) \\
 j - d(x_0,x), & x \in B(x_0,j+1) \setminus B(x_0,j) \\
 0, & x \in X \setminus B(x_0,j+1).
\end{cases}
$$

(If $g$ is an upper gradient of $u$ then $g_j = \varphi_j g + \|u\|_V \chi_{B(x_0,j+1) \setminus B(x_0,j-1)}$ is an upper gradient for $u_j$, see \cite{1} Section 2.3.) Given $\varepsilon > 0$ we choose open sets $E^j \subset X$ with $\text{Cap}_p(E^j) < 2^{-j}\varepsilon$ and $u_j|_{X \setminus E^j}$ continuous. Set

$$
E = \bigcup_j E^j.
$$

Then $E$ is open in $X$, $\text{Cap}_p(E) < \varepsilon$ and each $u_j$ restricted to $X \setminus E$ is continuous. Moreover $u_j \to u$ locally uniformly in $X \setminus E$, whence $u$ restricted to $X \setminus E$ is continuous. \hfill \Box

\textbf{Remark 1.10.} We will often use the following equivalent formulation of $p$-quasicontinuity: there is a decreasing sequence $E_n \supset E_{n+1}$ of open sets in $X$ with $\text{Cap}_p(E_n) < 2^{-n}$ such that $u|_{X \setminus E_n}$ is continuous. Indeed, using $p$-quasicontinuity to select open sets $E^k$ with $\text{Cap}_p(E^k) < 2^{-k}$ such that $u|_{X \setminus E^k}$ is continuous, the sets $E_n = \bigcup_{k=n} E^k$ satisfy the conditions of this alternative formulation.

If $(u_k)_{k \in D}$ is a countable collection of maps in $N^{1,p}_{\text{loc}}(X;V)$ we may, by a similar procedure, produce a decreasing sequence $E_n \supset E_{n+1}$ of open sets so that $\text{Cap}_p(E_n) < 2^{-n}$ and $u_k|_{X \setminus E_n}$ is continuous for all $k \in D, n \in \mathbb{N}$. 

Maps with metric space target. Let \( Y \) be a complete metric space. Recall the Kuratowski embedding \( Y \to \ell^\infty(Y) \) where we send a point \( y \in Y \) to the function \( d_y - d_e \). Here \( e \in Y \) is a fixed point and \( d_y(x) := d(x, y) \). We define the classes

\[
N^1_{loc}(X; Y) = \{ u \in N^1_{loc}(X; \ell^\infty(Y)) : u(x) \in Y \text{ for } p\text{-quasievery } x \in X \}
\]

\[
D^1_p(X; Y) = \{ u \in D^1_p(X; \ell^\infty(Y)) : u(x) \in Y \text{ for } p\text{-quasievery } x \in X \}
\]

\[
N^1_p(X; Y) = \{ u \in N^1_p(X; \ell^\infty(Y)) : u(x) \in Y \text{ for } p\text{-quasievery } x \in X \}.
\]

We will mainly concern ourselves with \( D^1_p(X; Y) \).

Metrics on \( D^1_p(X; Y) \). Let us note that the family of seminorms

\[
\|u\|_{D^1_p(\Omega; V)} := \int_\Omega \|u\|^p d\mu + \int_X g_u^p d\mu,
\]

for compact \( \Omega \subset X \) gives rise to a metric on \( D^1_p(X; V) \) so that \( u_j \to u \) as \( j \to \infty \) if and only if

\[
\|u - u_j\|_{D^1_p(\Omega; V)} \to 0 \text{ as } j \to \infty
\]

for all compact \( \Omega \subset X \).

The restriction of this metric to \( D^1_p(X; Y) \) is referred to as the standard metric on \( D^1_p(X; Y) \). This way \( D^1_p(X; Y) \) becomes a closed subspace of \( D^1_p(X; \ell^\infty(Y)) \), hence a complete metric space.

There is also a different metric we may put on \( D^1_p(X; Y) \): the one induced by the family of pseudometrics

\[
d_\Omega(u, v) = \int_\Omega dy(u, v) d\mu + \left( \int_X |g_u - g_v|^p d\mu \right)^{1/p},
\]

with compact \( \Omega \subset X \). We say that \( u_j \to u \) in this metric if \( d_\Omega(u, u_j) \to 0 \) as \( j \to \infty \) for each compact \( \Omega \subset X \). The metric defined above is based on the notion used by Ohta in [34]. Since \( |g_u - g_v| \leq |g_u|_p \) almost everywhere we see that the standard metric dominates the one defined through \( d_\Omega \). We therefore refer to this as the weak metric on \( D^1_p(X; Y) \).

Compactness results for local Newtonian classes. We state two results we shall need.

**Theorem 1.11** (Rellich-Kondrakov). Let \( X \) be a doubling metric measure space supporting a weak \((1, p)\)– Poincaré inequality, and \( Y \) a proper metric space. If \( u_j \) is a sequence in \( N^1_{loc}(X; Y) \), and \( v \in N^1_{loc}(X; Y) \) with

\[
\sup_j \left[ \int_B d^p_v(u_j) d\mu + \int_{5\sigma B} g^p_{u_j} d\mu \right] < \infty,
\]

for a given ball \( B \subset X \), then there is a subsequence (denoted by the same indices) and \( u \in L^p(B; Y) \) so that

\[
\|u_j - u\|_{L^p(B; Y)} \to 0
\]

as \( j \to \infty \) and, moreover,

\[
\int_B g^p_{u_j} d\mu \leq \liminf_{j \to \infty} \int_B g^p_{u_j} d\mu.
\]
Proof. Note that the assumptions imply for \( q \in Y \),

\[
\sup_j \left[ \int_B d_Y^p(u_j, q) d\mu + \int_{5\varepsilon B} g_{u_j}^p d\mu \right]
\sup_j \left[ 2^{p-1} \int_B d_Y^p(v, u_j) d\mu + 2^{p-1} \int_B d_Y^p(v, q) d\mu + \int_{5\varepsilon B} g_{u_j}^p d\mu \right] < \infty.
\]

We have the scalar valued case of the claim by [13, Theorem 8.3]. Using the argument presented in the proof of [31, Theorem 1.3] we may reduce the claim to the scalar valued case, and hence we are done. \( \square \)

Lemma 1.12. Suppose \( f_n \) is a sequence in \( D^{1,p}(X; V) \) and \( f_n \to f \) in \( L^p_{\text{loc}}(X; V) \). If \( g_n \) is a sequence of \( p \)-weak upper gradients of \( f_n \) and \( g_n \to g \) weakly in \( L^p(X) \). Then \( g \) is a (\( p \)-integrable) \( p \)-weak upper gradient for \( f \).

Proof. By Mazur’s lemma [1, Lemma 6.1] a sequence of convex combinations of the \( g_n \)’s converge to \( g \) in norm. (In particular we may choose the convex combinations so that the \( j \)th element is a convex combination of \( g_j, g_{j+1}, g_{j+2}, \ldots \) The corresponding sequence of convex combinations of the \( f_n \)’s converges to \( f \) in \( L^p_{\text{loc}}(X; V) \) and therefore, by the proof of [1, Proposition 2.3] \( g \) is a \( p \)-weak upper gradient for \( f \). The \( p \)-integrability is obvious. \( \square \)

1.3. Spaces of nonpositive curvature: Busemann and Alexandrov. Let us mention to start with that of the two notions of nonpositive curvature, Busemann’s and Alexandrov’s, the more widely used is the notion given by Alexandrov. However, we shall use Busemann’s definition of nonpositive curvature for the simple reason that the nature of the methods used in this paper corresponds quite naturally to the notions used in Busemann’s definition.

A central theme in the theory of spaces of nonpositive curvature, both Busemann’s and Alexandrov’s, is convexity.

Recall that a geodesic \( \gamma \) joining two points \( x, y \in Y \) in a metric space is a path satisfying \( \ell(\gamma) = d(x, y) \). A geodesic \( \gamma \) can always be constant speed parametrized so that \( \gamma : [0, 1] \to Y \) and \( d(\gamma(t), \gamma(s)) = \ell(\gamma)|t - s| \) for all \( t, s \in [0, 1] \); see [35, Proposition 2.2.9].

We call a (path connected) metric space \((Y, d)\) locally complete and geodesic if each point has a closed neighbourhood that is a complete geodesic space.

Definition 1.13. A function \( f : Y \to \mathbb{R} \) from a geodesic space \( Y \) is said to be convex if, for every affinely reparametrized geodesic \( \gamma : [0, 1] \to Y \) the composition

\[
\gamma \circ f : [0, 1] \to \mathbb{R}
\]

is convex in the usual sense.

Definition 1.14. (a) A metric space \( Y \) is called a Busemann space if it is complete, geodesic and for every pair of affinely reparametrized geodesics \( \gamma, \sigma : [0, 1] \to Y \) the distance map

\[
t \mapsto d(\gamma(t), \sigma(t)) : [0, 1] \to \mathbb{R}
\]

is convex.

(b) A metric space \( Y \) is locally convex if each point has a neighbourhood that is a Busemann space with the induced metric. Such neighbourhoods are called convex neighbourhoods.
Note that many authors define (local) convexity by considering geodesics with common starting point (see, for instance [4, Chapter II.4]). However, this seemingly weaker notion of (local) convexity is easily seen to be equivalent to the definition presented here.

To speak about Alexandrov’s notion of nonpositive curvature we need to introduce the concept of geodesic triangles and comparison triangles. Let $Y$ be a locally complete and geodesic space. A geodesic triangle $\Delta \subset Y$ consists of three points $x, y, z \in Y$ and affinely reparametrized geodesics $\gamma_{xy}, \gamma_{xz}, \gamma_{yz}$ connecting $x$ with $y$, $x$ with $z$ and $y$ with $z$, respectively. A comparison triangle $\overline{\Delta} \subset \mathbb{R}^2$ is a euclidean triangle with vertices $\overline{x}, \overline{y}, \overline{z}$ such that the side lengths agree, i.e.

$$|\overline{x} - \overline{y}| = d(x, y), \quad |\overline{x} - \overline{z}| = d(x, z), \quad |\overline{y} - \overline{z}| = d(y, z).$$

For any geodesic triangle a comparison triangle always exists, [4, Lemma 1.2.14]. Given this, the notion of a comparison point to $w \in \Delta$ is self-explanatory.

**Definition 1.15.**

(a) A complete geodesic space $Y$ is said to be of \textit{global nonpositive curvature} if, for all geodesic triangles $\Delta$ with comparison triangle $\overline{\Delta}$ and any two points $a, b \in \Delta$, the comparison points $\overline{a}, \overline{b} \in \overline{\Delta}$ satisfy

$$d(a, b) \leq |\overline{a} - \overline{b}|.$$

(b) A locally complete and geodesic space is said to be of \textit{nonpositive curvature} (an \textit{NPC} space for short) if each point has a closed neighbourhood that is a space of global nonpositive curvature when equipped with the inherited metric.

In the literature one often encounters the name CAT(0) space for spaces of global nonpositive curvature.

In intuitive terms a globally nonpositively curved space is one where geodesic triangles are “thinner” than their corresponding Euclidean comparison triangles. One sees from the two definitions that that of Alexandrov does not directly pertain to convexity whereas Busemann’s definition does. It is however true that a nonpositively curved space is locally convex (and similarly for the global notions). The converse fails to hold and so the class of locally convex spaces is strictly larger than that of nonpositively curved ones. In fact a Banach space is of global nonpositive curvature if and only if its norm comes from an inner product. In contrast, a Banach space is a Busemann space if and only if its unit ball is strictly convex.

For a good account of convexity in normed spaces see [35, Chapters 7 and 8], and also [4, Chapter 4]. The difference between the two notions is that in Alexandrov’s definition the points $a, b \in \Delta$ are allowed to be arbitrary while Busemann’s definition only allows one to compare certain pairs of points (ones that are of the form $a = \gamma(t), b = \sigma(t)$ for some $\gamma, \sigma \subset \Delta$ and the same $t$ for both).

Let us mention the following result, due to Alexandrov, from which the convexity of a globally nonpositively curved space follows, [21, Cor. 2.5]

**Proposition 1.16.** Let $Y$ be a space of global nonpositive curvature and let $\gamma, \sigma : [0, 1] \to Y$ be two affinely reparametrized geodesics. Then for all $t \in [0, 1]$ the inequality

$$d^2(\gamma(t), \sigma(t)) \leq td^2(\gamma(1), \sigma(1)) + (1 - t)d^2(\gamma(0), \sigma(0)) - t(1 - t)[d(\gamma(1), \gamma(0)) - d(\sigma(1), \sigma(0))]^2$$

(1.3)
holds. In particular the metric \(d: Y \times Y \to \mathbb{R}\) is convex whence \(Y\) is a Busemann space.

**Corollary 1.17.** A nonpositively curved space \(Y\) is also locally convex.

The main reason for interest in locally convex spaces is the validity of a strong "local to global" principle. Below is a very general notion of this, see [4 The Cartan-Hadamard Theorem 4.1, p.193], but the punchline of the principle is that a simply connected, locally convex metric space is globally convex, i.e. a Busemann space (and similarly for the nonpositive curvature case).

**Theorem 1.18.** Let \(Y\) be a locally convex space. Then \(Y\) admits a universal covering \(\tilde{Y}\) with a unique metric with the properties that the covering map \(\pi: \tilde{Y} \to Y\) is a local isometry and \(\tilde{Y}\) is a Busemann space.

If \(Y\) is of nonpositive curvature (in the sense of Alexandrov) then the universal cover is a CAT(0) space.

In fact the universal covering space may be constructed as follows. Take any \(q \in Y\) and consider the set \(\tilde{Y}_q\) consisting of all constant speed parametrized local geodesics \(\gamma: [0, 1] \to Y\) starting at \(q\) (i.e. \(\gamma(0) = q\)). The map \(p_q: \tilde{Y}_q \to Y\) given by \(p(\gamma) = \gamma(1)\) is a covering map and the metric of \(Y\) pulls back under \(p_q\) to produce a length metric \(d_q\) on \(\tilde{Y}_q\) such that the claims of Theorem 1.18 are valid. For details, see [4, Chapter II.4].

The proof of Theorem 1.18 relies in large part on the following lemma which (along with its refinement 1.20) will be of independent use for us. For a reference see [4, Lemma 4.3, p. 194].

**Lemma 1.19.** Suppose \(Y\) is a locally convex space. Let \(x' \in B(x, \varepsilon), y' \in B(y, \varepsilon)\), \(\gamma: [0, 1] \to Y\) an affinely reparametrized local geodesic joining \(x\) and \(y\), with \(\varepsilon > 0\) such that \(B(\gamma(t), 2\varepsilon)\) is convex for all \(t\). Then there exists a unique affinely reparametrized local geodesic \(\alpha: [0, 1] \to Y\) so that

\[
    t \mapsto d(\gamma(t), \alpha(t))
\]

is a convex function.

Moreover, \(\alpha\) satisfies

\[
    \ell(\alpha) \leq \ell(\gamma) + d(x', x) + d(y', y).
\]

In particular \(d(\gamma(tL), \alpha(tL')) < \varepsilon\) for each \(t \in [0, 1]\).

We shall require a sharpening of Lemma 1.19 a proof of which can be found in [35, Theorem 9.2.4].

**Lemma 1.20** (refinement of Lemma 1.19). Suppose \(\varepsilon > 0\) and \(\gamma\) are as in the previous lemma; \(u, v \in B(x, \varepsilon)\) and \(u', v' \in B(y, \varepsilon)\). If \(\alpha\) and \(\beta\) are the unique local geodesics provided by Lemma 1.19 connecting \(u\) with \(u'\) and \(v\) with \(v'\), respectively. Then the unique local geodesic connecting \(v\) and \(v'\) with respect to \(\alpha\), provided by Lemma 1.19 is also \(\beta\).

A particular consequence of Theorem 1.18 is a uniqueness property for homotopy classes of paths in locally convex spaces. Here we give a formulation identical to [35, Corollary 9.3.3] apart from the local compactness –assumption made there. For non-locally compact spaces the proof of this proposition is included in the proof of [4, Corollary 4.7, p. 197].
Proposition 1.21. Let $Y$ be a complete locally convex space. Then each path $\gamma$ in $Y$ is (endpoint-preserving –) homotopic to a local geodesic $\sigma$, unique up to reparametrization.

2. Homotopies

Now we introduce a notion of homotopy for $p$-quasicontinuous maps, generalizing the classical one. This so called $p$-homotopy is then compared to a different notion, appearing in [3, 14, 15]. The notion $p$-homotopy utilizes the geometric structure of the target space whereas the second notion, path homotopy, which is stated for maps in the Dirichlet class, in fact relies on the topology of that class.

2.1. $p$-homotopy and path homotopy. Throughout this section $X$ stands for a complete doubling metric measure space $(X, d, \mu)$ supporting a weak $(1,p)$-Poincaré inequality for some $p > 1$ which will be fixed for the rest of the paper. In the definitions below $Y$ stands for a complete separable metric space. Completeness ensures that $D^{1,p}(X;Y)$ is closed and separability that maps in $D^{1,p}(X;Y)$ are essentially separably valued. By the remark after Theorem 1.6 it is therefore always enough to find a $p$-integrable ($p$-weak) upper gradient for a map in order to show it is in $D^{1,p}(X;Y)$. We shall gradually add assumptions on the target space $Y$.

Definition 2.1. Let $u, v : X \to Y$ be $p$-quasicontinuous. We say that $u$ and $v$ are $p$-homotopic if there exists a map $H : X \times [0,1] \to Y$ with the following property.

For every $\varepsilon > 0$ there exists an open set $E \subset X$ with $\text{Cap}_p(E) < \varepsilon$ such that $H|_{X \setminus E \times [0,1]}$ is a usual homotopy between $u|_{X \setminus E}$ and $v|_{X \setminus E}$.

The notion of $p$-homotopy is in the spirit of [31], where homotopy of maps into nonpositively curved spaces is studied. Explicit emphasis is given to the topology (structure) of the target space instead of the topology of the Sobolev (or in our case Newtonian) space.

It is noteworthy that for $p > Q = \log_2 C_\mu$ the $p$-capacity $\text{Cap}_p$ becomes trivial in the sense that $\text{Cap}_p(A) = 0$ if and only if $A = \emptyset$. Therefore for these values of $p$, the notions of $p$-homotopy and usual homotopy agree. This is natural in view of the Sobolev embedding theorem [20 Theorem 6.2], which states that Newtonian maps, for $p > Q$, are in fact $(1 - p/Q)$ – Hölder continuous.

To talk about path homotopy one needs to specify the metric used in $D^{1,p}(X;Y)$. If not otherwise stated, $D^{1,p}(X;Y)$ will be equipped with the standard metric.

Definition 2.2. We say that $u, v \in D^{1,p}(X;Y)$ are path-homotopic if there exists a continuous path $h \in C([0,1]; D^{1,p}(X;Y))$ connecting $u$ and $v$.

This definition appears in [3] where much more concerning it can be found. (See also [2, 14, 15] and the references therein.) The study of path homotopy classes is equivalent to the study of path components of the Newtonian space $N^{1,p}(X;Y)$.

We shall see that a path-homotopy satisfying certain rectifiability assumptions can always be modified to become a $p$-homotopy. Conversely, a locally geodesic $p$-homotopy between maps with locally convex target defines a path in the Dirichlet class between the endpoint maps, but continuity must be taken with respect to the weak metric. (Such a $p$-homotopy even satisfies some rectifiability assumptions, but again in a weaker sense than even the weak metric on the Dirichlet class.)
A first result reflects the geometric structure of the target space $Y$ in the $p$-homotopies between maps.

**Theorem 2.3.** Suppose $Y$ is a complete, locally convex space and let $u, v : X \to Y$ be $p$-quasicontinuous and $p$-homotopic. Given a $p$-homotopy $H : u \simeq v$, there exists a $p$-homotopy $\tilde{H} : u \simeq v$, unique in the following sense. For $p$-quasievery $x \in X$ the path $t \mapsto \tilde{H}_t(x)$ is the unique local geodesic between $u(x)$ and $v(x)$ belonging to the homotopy class of $t \mapsto H_t(x)$.

**Remark 2.4.**

a) Such a $p$-homotopy is called locally geodesic. Sometimes, for brevity, the word “locally” is omitted. (This does not mean that the paths $H(x, \cdot)$ are geodesic.)

b) We use the notation $H : u \simeq v$ to signify that $H$ is a $p$-homotopy between the maps $u$ and $v$.

**Proof.** To prove the claim let $\tilde{H} : u \simeq v$ be a $p$-homotopy. For $p$ a.e. $x$ set

$$H(x, t) = \gamma^x(t)$$

where $\gamma^x$ is the unique constant speed parametrized local geodesic in the homotopy class of $\alpha^x$, $\alpha^x(t) = H(x, t)$, given by Proposition 1.21.

It suffices to prove that $H$ is a $p$-homotopy. To this end let $\varepsilon > 0$ be arbitrary and let $E$ be an open set such that $\text{Cap}_p(E) < \varepsilon$ and $H|_{X \setminus E \times [0, 1]}$ is a usual homotopy $u|_{X \setminus E} \simeq v|_{X \setminus E}$. We shall show that $H|_{X \setminus E \times [0, 1]}$ is also a usual homotopy. If $x \in X \setminus E$ and $\delta > 0$ are given, let $\delta_0 \leq \delta$ be such that $B(\tilde{H}_t(x), \delta_0)$ and $B(\gamma^x(t), 2\delta_0)$ are convex, for all $t \in [0, 1]$. By the continuity of $\tilde{H}|_{X \setminus E \times [0, 1]}$ we may find $r > 0$ such that $\tilde{H}_t(B(x, r) \setminus E) \subset B(\tilde{H}_t(x), \delta_0)$ for all $t \in (0, 1]$.

For $y \in B(x, r) \setminus E$ let $\gamma'$ be the unique local geodesic, guaranteed by Lemmata 1.19 and 1.20, such that

$$t \mapsto d(\gamma^x(t), \gamma'(t))$$

is convex. Then $\gamma'$ is necessarily homotopic to $\alpha^y$:

$$\gamma' \simeq \beta^y_u \cdot \gamma^x \cdot \beta^y_v \simeq \beta^y_u \cdot \alpha^x \cdot \beta^y_v \simeq \alpha^y.$$

Here $\beta^y_u$ is the geodesic from $u(x)$ to $u(y)$ and $\beta^y_v$ the geodesic from $v(y)$ to $v(x)$. The last homotopy follows since for all $t$ the points $\alpha^y(t)$ belong to the convex ball $B(\alpha^y(t), \delta)$ by the choices of $y$ and $\delta$.

This shows that in fact $\gamma' = \gamma^y$ (by uniqueness) and from the estimates in Lemma 1.19 we have, for $y, z \in B(x, r) \setminus E$

$$d_Y(H_t(y), H_t(z)) \leq td_Y(v(y), v(z)) + (1 - t)d_Y(u(y), u(z))$$

and

$$d_Y(H_t(z), H_s(z)) \leq |t - s|d_Y(v(x), v(z)).$$

These estimates prove the continuity of $H|_{X \setminus E}$.  

**Theorem 2.5.** Suppose $Y$ is a separable, complete locally convex space. If $H : u \simeq v$ is a locally geodesic $p$-homotopy between two $p$-quasicontinuous maps $u, v : X \to Y$ then it satisfies the following convexity estimate: whenever $x \in X$ and
$\varepsilon > 0$ is such that $B(H_t(x), 2\varepsilon)$ is a convex ball for all $t \in [0, 1]$, $y, z \in X$ satisfy 
\[ \max_{0 \leq t \leq 1} d_Y(H_t(x), H_t(y)) < \varepsilon \text{ and } \max_{0 \leq t \leq 1} d_Y(H_t(x), H_t(z)) < \varepsilon, \]
we have
\[ d_Y(H_t(y), H_t(z)) \leq td_Y(v(y), v(z)) + (1-t)d_Y(u(y), u(z)). \]  
(2.1) \[ |\ell(H^y) - \ell(H^z)| \leq d_Y(u(y), u(z)) + d_Y(v(y), v(z)). \]
(2.2) 
Here $H^w$ denotes the local geodesic $t \mapsto H_t(w), w \in X$.

**Proof.** The paths $\gamma_1 = t \mapsto H_t(y)$ and $\gamma_2 = t \mapsto H_t(z)$ are local geodesics. For each $t_0 \in [0, 1]$ there is a neighbourhood $U \ni t_0$ such that $\gamma_1|U$ and $\gamma_2|U$ are geodesics in the Busemann space $B(H_{t_0}(x), 2\varepsilon)$ and thus the function $d_Y(H_t(y), H_t(z)) = d_Y(\gamma_1(t), \gamma_1(t))$ is convex in $U$. Therefore it is convex in $[0, 1]$, proving (2.1).

To prove (2.2) we may assume, without loss of generality, that $\ell(H^y) \geq \ell(H^z)$. Let $\gamma'$ be the local geodesic from $u(y)$ to $v(z)$ guaranteed by Lemmata 1.19 and 1.20. Then by the convexity of $d_Y(H^y, \gamma')$ and the local geodesic property we have, for small $t > 0$
\[ t\ell(H^y) = d_Y(u(y), H_t(y)) = d_Y(\gamma'(0), H_t(y)) \leq d_Y(\gamma'(0), \gamma'(t)) + d_Y(\gamma'(t), H_t(y)) \]
\[ \leq t\ell(\gamma') + td_Y(v(z), v(y)). \]

The same argument for the inverse paths $(\gamma')^{-1}$ and $(H^z)^{-1}$ yields
\[ t\ell((\gamma')^{-1}) \leq t\ell((H^z)^{-1}) + td_Y(u(z), u(y)). \]
Cancelling out $t$ and moving $\ell(H^z) = \ell((H^z)^{-1})$ to the other side we obtain (2.2).

Let us introduce some notation. Given a $p$-homotopy $H : u \simeq v$ we denote by $\langle H \rangle : u \simeq v$ the locally geodesic $p$-homotopy associated to $H$, given by Theorem 2.23. It is evident that, given two $p$-homotopies $H_1 : u \simeq v$ and $H_2 : v \simeq w$ the conjunction $H_2H_1 : u \simeq w$ is a $p$-homotopy, and we may consider the locally geodesic representative $\langle H_2H_1 \rangle$. We call this the product of $H_2$ and $H_1$. The inverse $H^{-1}$ of a $p$-homotopy $H : u \simeq v$ is simply the $p$-homotopy $H^{-1} : v \simeq u$ given by
\[ H^{-1}(x, t) = H(x, 1-t). \]

### 2.2. $p$-homotopies as paths in the Dirichlet class.

We may view a $p$-homotopy $H : u \simeq v$ as gliding the map $u$ to $v$ through the path $t \mapsto H_t$ in a somehow quasi-continuous manner. Our aim in this subsection is to promote this view and study $p$-homotopies as paths in the Dirichlet class $D^{1,p}(X; Y)$. To pass from pointwise information to paths in the Dirichlet class we need sufficiently good geometric behaviour from the target space. We continue assuming that $(X, d, \mu)$ is a complete doubling metric measure space supporting a weak $(1,p)$-Poincaré inequality and $Y$ is a (complete and separable) locally convex space (see section 1.2).

From now on, instead of using the cumbersome notation $\ell(H^x)$ we will denote by $l_H$ the function $x \mapsto \ell(H^x)$, for a given $p$-homotopy $H$.

**Theorem 2.6.** Let $u, v \in D^{1,p}(X; Y)$ and $H : u \simeq v$ be a locally geodesic $p$-homotopy. Then for $t \in [0, 1]$ we have
\[ g_{H_t} \leq tg_v + (1-t)g_u \]
almost everywhere. In particular $H_t \in D^{1,p}(X; Y)$ for all $t$. 

Proof. Let $E_m \supset E_{m+1}$ be a sequence of open sets in $X$, with $\text{Cap}_p(E_m) < 2^{-m}$ and $\{H|_{X \setminus E_m} \times [0,1]\}$ continuous. By Lemma 1.3 and the fact that $u$ and $v$ are absolutely continuous continuous on $p$-almost every curve, there is a curve family $\Gamma$ with $\text{Mod}_p(\Gamma) = 0$ such that each $\gamma \notin \Gamma$ satisfies

1. there exists $m_0$ so that $\gamma^{-1}(E_{m_0}) = \emptyset$ (and consequently $\gamma^{-1}(E_{m}) = \emptyset$ for every $m \geq m_0$)
2. the inequalities

$$d_Y(u(\gamma(b)), u(\gamma(a))) \leq \int_{\gamma[a,b]} g_u$$

$$d_Y(v(\gamma(b)), v(\gamma(a))) \leq \int_{\gamma[a,b]} g_v$$

hold for $a, b \in [0,1]$.

Fix such a $\gamma$ and a compact set $K \subset X \setminus E_{m_0}$ with $|\gamma| \subset K$. Since $H|_{K \times [0,1]}$ is uniformly continuous there exists $\varepsilon > 0$ so that $B(H_t(\gamma(s)), 2\varepsilon)$ is convex for all $t, s \in [0,1]$. Furthermore the uniform continuity implies the existence of $\delta > 0$ so that $\max_{0 \leq t \leq 1} d_Y(H_t(\gamma(b)), H_t(\gamma(a))) < \varepsilon$ whenever $|a - b| < \delta$. By the estimate (2.1) we therefore have

$$d_Y(H_t(\gamma(1)), H_t(\gamma(0))) \leq \sum_k d_Y(H_t(\gamma(a_k)), H_t(\gamma(a_{k-1}))) \leq \int_{\gamma} (t g_v + (1 - t) g_u).$$

Partitioning $[0,1]$ into subintervals of length $< \delta$ and applying the estimate above yields

$$d_Y(H_t(\gamma(1)), H_t(\gamma(0))) \leq \int_{\gamma} (t g_v + (1 - t) g_u).$$

This proves that $tg_v + (1 - t) g_u$ is a $p$-weak upper gradient for $H_t$, and the claim follows.

Lemma 2.7. Suppose $u, v \in D^{1,p}(X; Y)$ and $H : u \simeq v$ is a locally geodesic $p$-homotopy. Then $l_H \in D^{1,p}(X)$, and

$$g_{l_H} \leq g_u + g_v.$$

Proof. It suffices to prove the inequality. For this let $\Gamma$ be as in the previous proof and fix $\gamma \notin \Gamma$. By the reasoning in the previous proof we have the existence of $\delta > 0$ so that $|l_H(\gamma(b)) - l_H(\gamma(a))| \leq d_Y(u(\gamma(b)), u(\gamma(a))) + d_Y(v(\gamma(b)), u(\gamma(a)))$ whenever $|a - b| < \delta, a, b \in [0,1]$. By the same partitioning argument we arrive at

$$|l_H(\gamma(1)) - l_H(\gamma(0))| \leq \int_{\gamma} (g_u + g_v)$$

and this proves the claim.

Corollary 2.8. In the situation of Lemma 2.7 we have, for each compact $K \subset X$, the inequality

$$\int_K d_{pV}(H_t, H_s) d\mu \leq |t - s|^p \int_K l_{pH} d\mu.$$

Consequently $H_s \rightarrow H_t$ in $L^p_{\text{loc}}(X; V)$ as $s \rightarrow t$. 
Proof. The inequality follows directly from the fact that
\[ d_Y(H_t(x), H_s(x)) \leq |t-s|d(H^{p}) = |t-s|l_H(x) \]
for \( p \)-quasievery \( x \in X \). The second claim is immediate from the first and the fact that \( l_H \in L^p_{loc}(X) \).

\[ \square \]

**Theorem 2.9.** Suppose \( H : u \approx v \) is a locally geodesic \( p \)-homotopy. Then the map
\[ t \mapsto H_t : [0,1] \to D^{1,p}(X;Y) \]
is a continuous path when \( D^{1,p}(X;Y) \) is equipped with the weak metric.

Proof. That \( H_s \to H_t \) in \( L^p_{loc}(X;Y) \) as \( s \to t \) follows from Corollary [2,8]. Therefore we only need to focus on the convergence of the \( p \)-weak upper gradients. Let \( t \in [0,1] \). We will show that the one-sided limits exist and agree:
\[ \lim_{s \to t^+} g_{H_s} = g_{H_t} = \lim_{s \to t^-} g_{H_s} \]
in the \( L^p \)-sense. We will make use of the following well known fact about uniformly convex Banach spaces:

**Fact 2.10.** Let \( V \) be a uniformly convex Banach space, \( x_k \rightarrow x \) as \( k \to \infty \) weakly and further \( ||x_k|| \to ||x|| \). Then \( x_k \to x \) in norm.

In fact it suffices to prove that \( g_{H_s} \to g_u \) as \( s \to 0 \). This is because of the following: the restriction \( H|_{X \times [t,1]} \) is a \( p \)-homotopy between \( H_t \) and \( v \), so by rescaling the parameter side we obtain a locally geodesic \( p \)-homotopy \( \tilde{H} : H_t \approx v \), \( \tilde{H}(x,s) = H(x,t+s(1-t)) \) so that
\[ \lim_{s \to t^+} g_{H_s} = \lim_{s \to t^-} g_{H_s} ; \]
to study \( \lim_{s \to t^-} g_{H_s} \) we simply replace \( \tilde{H} \) by \( \tilde{H} : H_t \approx u \), \( \tilde{H}(x,s) = H(x,t(1-s)) \).

Now, to prove that \( \lim_{s \to 0} g_{H_s} = g_u \) in \( L^p(X) \), take any sequence \( s_k \to 0 \). Since
\[ \| g_{H_{s_k}} \|_{L^p(X)}^p \leq s_k \| g_v \|_{L^p(X)}^p + (1 - s_k) \| g_u \|_{L^p(X)}^p \leq \| g_v \|_{L^p(X)}^p + \| g_u \|_{L^p(X)}^p \]
for all \( k \), the reflexivity of \( L^p(X) \) implies that there is a subsequence (denoted by the same indices) converging to some \( g \in L^p(X) \), whence
\[ \| g \|_{L^p(X)}^p \leq \liminf_{k \to \infty} \| g_{H_{s_k}} \|_{L^p(X)}^p. \]
On the other hand, since \( H_{s_k} \to u \) in \( L^p_{loc}(X;Y) \) (Corollary [2,8]) it follows by Lemma [1,12] that \( g \) is a \( p \)-weak upper gradient for \( u \). This and the convexity estimate together imply
\[ \lim_{k \to \infty} \| g_{H_{s_k}} \|_{L^p(X)}^p \leq \limsup_{k \to \infty} \| s_k \|_{L^p(X)}^p + (1 - s_k) \| g_u \|_{L^p(X)}^p \]
\[ \leq \| g_u \|_{L^p(X)}^p \leq \| g \|_{L^p(X)}^p. \]
Consequently, \( g_{H_{s_k}} \to g \) in \( L^p(X) \), therefore \( g_{H_{s_k}} \to g \) in norm.

Let us still prove that \( g = g_u \). From the fact that \( g \) is a \( p \)-weak upper gradient for \( u \) it follows that \( g_u \leq g \) almost everywhere, so it suffices to prove \( \| g \|_{L^p(X)}^p \leq \| g_u \|_{L^p(X)}^p \). This, however follows immediately from the convexity estimate:
\[ \| g \|_{L^p(X)}^p = \lim_{k \to \infty} \| g_{H_{s_k}} \|_{L^p(X)}^p \leq \lim_{k \to \infty} \| s_k \|_{L^p(X)}^p + (1 - s_k) \| g_u \|_{L^p(X)}^p = \| g_u \|_{L^p(X)}^p. \]
Altogether we have shown that for every \( s_k \to 0 \) we have \( g_{H_{s_k}} \to g_u \) in \( L^p(X) \) up to a subsequence. Therefore \( g_{H_{s_k}} \to g_u \) in \( L^p(X) \) and the proof is complete. \( \square \)

2.3. Pointwise properties of path homotopies. In this subsection the assumptions on the domain space \( X \) remain the same but the target \( Y \) may be an arbitrary complete and separable metric space.

**Theorem 2.11.** Let \( u, v \in D^{1,p}(X;Y) \) and \( h : [0,1] \to D^{1,p}(X;Y) \) be a map joining \( u \) and \( v \) (i.e. \( h_0 = u, h_1 = v \)). Suppose that there exists a constant \( C \) and, for every compact \( K \subset X \) some \( C_K \in (0, \infty) \) so that

\[
\|d_Y(h_t, h_s)\|_{L^p(X)} \leq C|t - s| \quad \text{and} \quad \int_K d_Y(h_t, h_s)d\mu \leq C_K|t - s|
\]

for all \( t, s \in [0,1] \). Then \( u \) and \( v \) are \( p \)-homotopic.

**Remarks.**

(1) We pose no control on the \( L^p \)-norms of \( d_Y(h_t, h_s) \).

(2) The conditions of the theorem are a sort of rectifiability requirement for the path \( h \). Since \( g_{d_Y(h_t, h_s)} \leq g_{u-v} \) almost everywhere the condition is implied if \( h \) is a rectifiable path \( h : [0,1] \to D^{1,p}(X;Y) \) when \( D^{1,p}(X;Y) \) is equipped with the standard metric.

(3) They are however weaker than rectifiability in the standard metric, since \( g_{d_Y(u,v)} \) may vanish without the same being true of \( g_{u-v} \) (think of maps \( u \equiv 0 \in \mathbb{R}^n \) and \( v \) taking values in \( S^{n-1} \)).

(4) The relation between \( g_{d_Y(u,v)} \) and \( |g_u - g_v| \) is not clear; when \( Y = \mathbb{R} \) we have \( g_{d_Y(u,v)} = g_{[u-v]} = g_{u-v} \geq |g_u - g_v| \) but the previous example shows that \( g_{d_Y(u,v)} \) may vanish without \( |g_u - g_v| \) vanishing. This question is related to the open question [1, 2.13].

**Proof.** Let \( D_n = \{k/2^n : k = 0, \ldots, 2^n\} \) and \( D = \bigcup_{n=1}^\infty D_n \) (the dyadic rationals on the interval \([0,1]\)). We may find a sequence \( E_m \supset E_{m+1} \) of open subsets of \( X \) with \( \text{Cap}_p(E_m) < 2^{-m} \) and \( h_s|_{X \setminus E_m} \) continuous for all \( m \in \mathbb{N} \) and \( s \in D \). For \( x \notin E := \bigcap_mE_m \) define

\[
L_n(x) = 2^{n(1-1/p)} \left( \sum_{k=1}^{2^n} d_Y^p(h_{k/2^n}(x), h_{(k-1)/2^n}(x)) \right)^{1/p}, \quad \text{and}
\]

\[
L(x) = \sup_n L_n(x).
\]

Note that, if we set \( p = 1 \) in the definition of \( L_n \), we are in fact measuring the “length” of the “path” \( D \ni s \mapsto h_s(x) \). The finiteness of this “length” however only guarantees that \( s \mapsto h_s(x) \) is a sort of BV-map. The significance of \( L \) (as defined above) is shown by the next lemma.

**Lemma 2.12.** If \( L(x) < \infty \) then the map \( D \ni s \mapsto h_s(x) \) extends to a \((1 - 1/p)\)-Hölder continuous path, denoted \( h^x : [0,1] \to Y \) (joining the points \( u(x) \) and \( v(x) \)) with Hölder constant \( L(x) \).
Proof of Lemma 2.12. Let us define \( h^x(s) \) for \( s \in D \) by \( h^x(s) = h_*(x) \). For \( n \in \mathbb{N} \) and \( j = 1, \ldots, 2^n \) we have
\[
\sum_{k<j} d_j \leq 2^{-n(p-1)} L_n(x)^p \leq 2^{-n(p-1) L(x)^p}.
\]

For \( k, l \in \{0, \ldots, 2^n \} \), \( l > k \), the triangle inequality implies
\[
\sum_{j=k+1}^l d_Y(h_x(j/2^n), h_x((j-1)/2^n)) \leq \sum_{j=k+1}^l d_Y(h_x(j/2^n), h_x((j-1)/2^n)) := \sum_{k<j \leq l} d_j.
\]

We may use the Hölder inequality as follows:
\[
\sum_{k<j \leq l} d_j = \sum_{k<j \leq l} 1 \cdot d_j \leq \left( \sum_{k<j \leq l} 1^{p/(p-1)} \right)^{1-1/p} \left( \sum_{k<j \leq l} d_j^p \right)^{1/p} \leq (l-k)^{1-1/p} 2^{-n(1-1/p)2^{n(1-1/p)}} \left( \sum_{k<j \leq l} d_j^p \right)^{1/p} \leq \left( \frac{l-k}{2^n} \right)^{1-1/p} L_n(x).
\]

If \( s, t \in D \) we may write \( s = k/2^n \) and \( t = l'/2^m \). Assuming, without loss of generality that \( n \geq m \) we have \( t = 2^n-m'/2^n, s = k/2^n \) and putting the above estimates together yields
\[
d_Y(h_x(t), h_x(s)) \leq |t-s|^{1-1/p} L_n(x) \leq |t-s|^{1-1/p} L(x).
\]

This proves the claim. \( \square \)

The result is very much in the spirit of the Sobolev embeddings; by this analogy the need for \( p > 1 \) in the definition of \( L_n \) becomes apparent.

The rest of the proof is devoted to obtaining pointwise control over \( L \). We start with the following lemma.

Lemma 2.13. We have \( L_n \to L \) in \( L^p_{\text{loc}}(X) \) and the functions
\[
g_n := 2^{n(1-1/p)} \left( \sum_{k=1}^{2^n} g_{d_Y(h_{(k/2^n)}, (h_{(k-1)/2^n)}^p) \right)^{1/p}
\]
are \( p \)-weak upper gradients for \( L_n \), satisfying
\[
\sup_n \|g_n\|_{L^p(X)} \leq C
\]
where \( C \) is the constant in the claim.

Proof of Lemma 2.13. We note that \( (L_n)_n \) is a pointwise increasing sequence, so by the monotone convergence theorem
\[
\lim_{n \to \infty} \int_K |L_n - L|^p \, d\mu = \lim_{n \to \infty} \int_K (L - L_n)^p \, d\mu = \int_K \lim_{n \to \infty} (L - L_n)^p \, d\mu,
\]
provided \( L \in L^p_{\text{loc}}(X) \). But again,
\[
\int_K L^p \, d\mu = \lim_{n \to \infty} \int_K L_n^p \, d\mu
\]
by monotone convergence.
The Poincaré inequality implies, for balls $B \subset X$ that
\[
\left(\int_B d^p_Y(h_t, h_s)\,d\mu\right)^{1/p} \leq \int_B d_Y(h_t, h_s)\,d\mu + Cr \left(\int_B g^p_{dY}(h_t, h_s)\,d\mu\right)^{1/p} \\
\leq C_B/\mu(B)|t-s| + Cr/\mu(B)^{1/p}|t-s| = C'_B|t-s|.
\]

Therefore
\[
\int_B L_n^p\,d\mu = \int_B 2^{n(p-1)} \sum_{k=1}^{2^n} d^p_Y(h_{k/2^n}, h_{(k-1)/2^n})\,d\mu \\
= 2^{n(p-1)} \sum_{k=1}^{2^n} \int_B d^p_Y(h_{k/2^n}, h_{(k-1)/2^n})\,d\mu \leq 2^{n(p-1)} \sum_{k=1}^{2^n} \mu(B)(C'_B)^p2^{-np} \\
= \mu(B)(C'_B)^p
\]
for all $n \in \mathbb{N}$ and we have the first claim of the lemma. (Incidentally, this implies that $L(x) < \infty$ almost everywhere.)

For the second claim fix a family of curves $\Gamma$ with $\text{Mod}_p(\Gamma) = 0$ so that whenever $\gamma \notin \Gamma$ the upper gradient inequality
\[
|d_Y(h_{k/2^n}, h_{(k-1)/2^n})(x) - d_Y(h_{k/2^n}, h_{(k-1)/2^n})(y)| \leq \int_\gamma g_{dY}(h_{k/2^n}, h_{(k-1)/2^n})
\]
is satisfied. For these curves we may estimate
\[
|L_n(x) - L_n(y)| = \\
2^{n(1-1/p)} \left| \sum_{k=1}^{2^n} d^p_Y(h_{k/2^n}, h_{(k-1)/2^n})(x) - \left(\sum_{k=1}^{2^n} d^p_Y(h_{k/2^n}, h_{(k-1)/2^n})(y)\right)^{1/p} \right| \\
\leq 2^{n(1-1/p)} \left(\sum_{k=1}^{2^n} |d_Y(h_{k/2^n}, h_{(k-1)/2^n})(x) - d_Y(h_{k/2^n}, h_{(k-1)/2^n})(y)|p\right)^{1/p} \\
\leq 2^{n(1-1/p)} \left(\sum_{k=1}^{2^n} \left(\int_\gamma g_{dY}(h_{k/2^n}, h_{(k-1)/2^n})\right)p\right)^{1/p} \\
= 2^{n(1-1/p)} \int_\gamma \left(\sum_{k=1}^{2^n} g_{dY}(h_{k/2^n}, h_{(k-1)/2^n})\right)^{1/p}.
\]
The rightmost term may be estimated using the Minkowski inequality in integral form [10 Theorem 2.12, p. 148] by
\[
2^{n(1-1/p)} \int_\gamma \left(\sum_{k=1}^{2^n} g_{dY}(h_{k/2^n}, h_{(k-1)/2^n})\right)^{1/p}
\]
We arrive at
\[
|L_n(x) - L_n(y)| \leq \int_\gamma g_n.
\]
To see the last part use the condition in the statement of the theorem to compute
\[
\int_X g^p_n\,d\mu = 2^{n(p-1)} \sum_{k=1}^{2^n} \int_X g^p_{dY}(h_{k/2^n}, h_{(k-1)/2^n})\,d\mu \leq 2^{n(p-1)} \sum_{k=1}^{2^n} C^p2^{-np} = C^p.
\]
This completes the proof of Lemma 2.13. \qed
Since \((g_n)\) is bounded in \(L^p(X)\) there is a subsequence converging weakly to some \(g \in L^p(X)\). By Lemma 1.12 \(g\) is a \(p\)-integrable \(p\)-weak upper gradient for \(L\).

We conclude that \(L \in D^{1,p}(X)\). In particular \(L\) is \(p\)-quasiconstant and finite \(p\)-quasieverywhere.

Define \(H(x,t) = h^x(t)\) for every \(x \in X\) for which \(L(x) < \infty\), \(h^x\) being the path from \(u(x)\) to \(v(x)\) given by Lemma 2.12. Let us prove that \(H\) is a \(p\)-homotopy.

To this end let \(F_m \supset F_{m+1}\) be a sequence of open sets in \(X\) with \(\text{Cap}_p(F_m) < 2^{-m}\) and \(L|X \setminus F_m\) continuous, for \(m \in \mathbb{N}\). Set \(U_m = E_m \cup F_m\). We claim that \(H|X \setminus U_m\times [0,1]\) is a continuous homotopy between \(u|X \setminus U_m\) and \(v|X \setminus U_m\), for all \(m\).

It is clear that \(H_0 = u\) and \(H_1 = v\) \(p\)-quasieverywhere so only the continuity remains to be proven. Let \(x_k \in X \setminus U_m\), \(t_k \in [0,1]\), \((x_k, t_k) \to (x, t)\). There is a compact set \(K \subset X \setminus U_m\) containing all \(x_k\)'s, and \(L(z) < \infty\). Therefore the \(z \in K\) paths \(h^{x_k}\) are equicontinuous and pointwise bounded (since \(h^{x_k}(s) = h_s(x_k) \to h_s(x) = h^x(s), s \in D\)). By the Arzela-Ascoli theorem \(h^{x_k}\) converges uniformly up to a subsequence to a path \(\gamma\). But since \(h^{x_k} \to h^x\) pointwise in a dense set \(D\) it follows that \(\gamma = h^x\). This argument shows that any subsequence of \(h^{x_k}\) has a further subsequence converging uniformly to \(h^x\). From this it follows that \(h^{x_k} \to h^x\) uniformly. In particular \(h^{x_k}(t_k) \to h^x(t)\), as \(k \to \infty\). The proof of Theorem 2.11 is now complete.

\[\tag*{\square}\]

Remark 2.14. Suppose \(Y\) is a locally convex space and \(H : u \simeq v\) a locally geodesic \(p\)-homotopy. It is not difficult to see, using the argument in the proofs of Theorem 2.6 and Lemma 2.7 that \(g_{d_Y(H, H_\simeq)} \leq |t-s|g_\mu\). This, together with Corollary 2.8 implies that a locally geodesic \(p\)-homotopy satisfies the conditions of Theorem 2.11.

We may conclude, in the case where \(Y\) is a locally convex space, that two maps \(u, v \in D^{1,p}(X; Y)\) being \(p\)-homotopic is equivalent to the existence of a path joining \(u\) and \(v\), satisfying the conditions of Theorem 2.11.

3. "Lifting" \(p\)-Homotopies

Besides thinking of \(p\)-homotopies as paths in \(D^{1,p}(X; Y)\), there is another way of looking at them. In this section we concentrate on this view, which is reminiscent of lifting paths in covering space theory.

The aim is to view a (locally geodesic) \(p\)-homotopy \(H : u \simeq v\) between two maps \(u, v \in D^{1,p}(X; Y)\) as a single Newtonian map, with target space \(\hat{Y}\) a certain covering space of \(Y \times Y\).

We start by contructing the space \(\hat{Y}\) and recalling some useful facts. Throughout this section \((X, d, \mu)\) stands for a complete doubling metric measure space supporting a weak \((1, p)\)-Poincaré inequality. After Definition 3.5 \(X\) will be assumed to be compact.

Let \(Y\) be a locally convex space. Equip \(Y^2\) with the metric
\[d_{Y^2}((x_1, y_1); (x_2, y_2))^2 = d_Y^2(x_1, x_2) + d_Y^2(y_1, y_2)\]
The product space \(Y^2\) remains a locally convex space – and nonpositively curved in case \(Y\) is nonpositively curved. Set
\[\hat{Y} = \{\gamma : [0, 1] \to Y : \gamma\ \text{a constant speed local geodesic}\}.\]
With metric $d_{\infty}(\alpha, \beta) = \max_{0 \leq t \leq 1} d_Y(\alpha(t), \beta(t))$ the map

$$p : \hat{Y} \rightarrow Y^2, \ p(\gamma) = (\gamma(0), \gamma(1))$$

is a local bilipschitz map: given $\gamma \in \hat{Y}$ and $\varepsilon > 0$ satisfying the conditions of Lemma 1.19 any $\alpha, \beta \in B_{\infty}(\gamma, \varepsilon) = \{ \sigma \in \hat{Y} : d_{\infty}(\gamma, \sigma) < \varepsilon \}$ the distance function $t \mapsto d_Y(\alpha(t), \beta(t))$ is convex, implying

$$d_{\infty}(\alpha, \beta) \leq \max\{d_Y(\alpha(0), \beta(0)); d_Y(\alpha(1), \beta(1))\} \leq d_{Y^2}(p(\alpha), p(\beta))$$

while the estimate

$$d_{Y^2}(p(\alpha), p(\beta)) \leq \sqrt{2} d_{\infty}(\alpha, \beta)$$

holds always. Therefore $p$ restricted to $B_{\infty}(\gamma, \varepsilon)$ is a $\sqrt{2}$-bilipschitz map $B_{\infty}(\gamma, \varepsilon) \rightarrow p(B_{\infty}(\gamma, \varepsilon))$.

We may pull back the length metric from $Y^2$ to obtain a unique length metric $d_{\hat{Y}}$ on $\hat{Y}$ such that $p : (\hat{Y}, d_{\hat{Y}}) \rightarrow (Y^2, d_{Y^2})$ is a local isometry. (This metric is given by $d_{\hat{Y}}(\alpha, \beta) := \inf \ell(p \circ h)$ where the infimum is taken over all the paths $h$ in $\hat{Y}$ joining $\alpha$ and $\beta$.)

In particular, in case $\sigma \in \hat{Y}$ and $\varepsilon > 0$ is such that $B_{\hat{Y}}(\sigma(t), 2\varepsilon)$ is convex for all $t \in [0, 1]$, we have that $p : B_{\hat{Y}}(\sigma, \varepsilon) \rightarrow B_{Y^2}(p(\sigma), \varepsilon)$ is a surjective isometry.

Since $Y^2$ is locally convex (nonpositively curved) it follows that $\hat{Y}$ is locally convex (nonpositively curved) and, by Proposition I.3.28 $p$ is a covering map.

If $Y$ is locally compact it follows from the Hopf-Rinow theorem that $\hat{Y}$ is a complete, proper geodesic space. By the general theory $p$ is a 1-Lipschitz map:

$$d_{Y^2}(p(\alpha), p(\beta)) \leq d_{\hat{Y}}(\alpha, \beta).$$

In the event that $\alpha, \beta \in \hat{Y}_q$ (see the discussion after Theorem I.18) we have

$$d_{\hat{Y}}(\alpha, \beta) \leq d_q(\alpha, \beta).$$

Indeed the identity map $\iota : (\hat{Y}_q, d_{\hat{Y}}) \rightarrow (\hat{Y}_q, d_{\hat{Y}})$ is a local isometry: for every $\alpha \in \hat{Y}_q$ the restriction $\iota|_{B_q(\alpha, \varepsilon)}$ is a surjective isometry whenever $\varepsilon > 0$ is such that $B_Y(\alpha(t), \varepsilon)$ is a convex neighbourhood for all $t \in [0, 1]$.

A fact we shall use is that, for $\alpha, \beta \in \hat{Y}_q$ the distance in the $d_q$ metric is given by

$$d_q(\alpha, \beta) = \ell((\alpha \beta^{-1})),$$

where $(\alpha \beta^{-1})$ denotes the unique local geodesic homotopic to $\alpha \beta^{-1}$. To see this let $h$ be the lift (to $\hat{Y}_q$) of the local geodesic $\gamma = (\alpha \beta^{-1})$, starting at $h_0 = \beta$. As a lift of a local geodesic, $h$ itself is a local geodesic and since $\hat{Y}_q$ is a Busemann space, $h$ is actually a geodesic. Furthermore by the homotopy lifting property $h$ is homotopic to $h'$, the lift (to $\hat{Y}_q$) of the path $\alpha \beta^{-1}$ starting at $h'_0 = \beta$. We see that this lift is given by

$$h'_s(t) = \begin{cases} \beta((1 - 2s)t) & 0 \leq s \leq 1/2, \\
\alpha((2s - 1)t) & 1/2 \leq s \leq 1 
\end{cases}$$
because this clearly defines a lift of $\alpha \beta^{-1}$ starting at $\beta$ and such a lift is necessarily unique. Since $h'_1 = \alpha$ it follows that $h_1 = \alpha$ and therefore $h$ is a geodesic joining $\beta$ and $\alpha$. Consequently

$$d_\gamma(\alpha, \beta) = \ell(h) = \ell(\gamma),$$

as desired.

Let us now turn to “lifting” $p$-homotopies.

**Definition 3.1.** Let $H : u \simeq v$ be a locally geodesic $p$-homotopy between two maps $u, v \in D^{1,p}(X; Y)$. The lift $\hat{H}$ of $H$ is the map $\hat{H} : X \to \hat{Y}$ given by mapping $x \in X$ to the local geodesic path $t \mapsto H_t(x) \in \hat{Y}$.

The covering map $p : \hat{Y} \to Y^2$ also induces a map $p : D^{1,p}(X; \hat{Y}) \to D^{1,p}(X; Y)^2$,

$$pF(x) = (F_0(x), F_1(x)).$$

The fact that each component $F_0, F_1 \in D^{1,p}(X; Y)$ follows from the fact that $p$ is a Lipschitz map. Note that, if $H : u \simeq v$ is a locally geodesic $p$-homotopy and $\hat{H}$ its lift, the identity $p \circ \hat{H} = (u, v)$ holds.

**Proposition 3.2.** Let $H : u \simeq v$ be as in Definition 3.1. Then $\hat{H} \in D^{1,p}(X; \hat{Y})$ with

$$1/2 \ (g_u + g_v) \leq g_{\hat{H}} \leq g_u + g_v.$$ 

**Proof.** As in the proof of Theorem 2.6 let us take a sequence $E_m \supset E_{m+1}$ of open sets in $X$ with $\operatorname{Cap}_p(E_m) \leq 2^{-m}$ and $H|_{X \setminus E_m \times [0, 1]}$ continuous homotopy between $u|_{X \setminus E_m}$ and $v|_{X \setminus E_m}$, and a path family $\Gamma$ with $\operatorname{Mod}_p(\Gamma) = 0$ so that whenever $\gamma \notin \Gamma$,

1. there exists $m_0$ so that $\gamma^{-1}(E_{m_0}) = \emptyset$ (and consequently $\gamma^{-1}(E_m) = \emptyset$ for every $m \geq m_0$)

2. the inequalities

$$d_Y(u(\gamma(b)), u(\gamma(a))) \leq \int_{\gamma|_{[a,b]}} g_u$$

$$d_Y(v(\gamma(b)), v(\gamma(a))) \leq \int_{\gamma|_{[a,b]}} g_v$$

hold for $a, b \in [0, 1]$.

Let $\gamma \notin \Gamma$ and let $K \subset X \setminus E_{m_0}$ be a compact set containing the image of $\gamma$. Since $H|_{K \times [0, 1]}$ is uniformly continuous there is some $\varepsilon > 0$ so that $B(z, 2\varepsilon) \subset Y$ is a convex ball for all $z \in H(K \times [0, 1])$ (the image being a compact set). By the uniform continuity of $H|_{K \times [0, 1]}$ there is $\delta > 0$ so that whenever $a, b \in [0, 1]$ are such that $|a - b| < \delta$ we have $d_Y(H_t(\gamma(b)), H_t(\gamma(a))) < \varepsilon$ for all $t \in [0, 1]$. It follows that

$$d_Y(\hat{H}(\gamma(b)), \hat{H}(\gamma(a))) = d_Y(p \circ \hat{H}(\gamma(b)), p \circ \hat{H}(\gamma(a)))$$

for $a, b \in [0, 1]$ with $|a - b| < \delta$. On the other hand

$$\max\{d_Y(u(\gamma(b)), u(\gamma(a))), d_Y(v(\gamma(b)), v(\gamma(a)))\}$$

$$\leq d_Y(p \circ \hat{H}(\gamma(b)), p \circ \hat{H}(\gamma(b)))$$

$$\leq d_Y(u(\gamma(b)), u(\gamma(a))) + d_Y(v(\gamma(b)), v(\gamma(a))).$$
From this we see, as in the proof of Theorem 2.6, that $g_{\tilde{H}} \leq g_u + g_v$. To arrive at the other inequality note that by the leftmost inequality above, any $p$-weak upper gradient for $\tilde{H}$ is also a $p$-weak upper gradient for both $u$ and $v$. Thus $g_u \leq g_{\tilde{H}}$ and $g_v \leq g_{\tilde{H}}$ almost everywhere, from which we have

$$1/2 \ (g_u + g_v) \leq g_{\tilde{H}}.$$ 

\[\square\]

Using the map $p$ introduced after Definition 3.1 we also have a converse result.

**Proposition 3.3.** Suppose $F \in D^{1,p}(X;\hat{Y})$ and $p \circ F = (u,v)$. Then $H(x,t) = F_t(x)$ defines a locally geodesic $p$-homotopy $H : u \simeq v$.

**Proof.** By definition for $p$-quasievery $x \in X$ the path $t \mapsto H_t(x) = F_t(x)$ is a local geodesic. Suppose $\varepsilon > 0$ is given, and let $E \subset X$ be an open set with $\text{Cap}_p(E) < \varepsilon$ so that $F|_{X \setminus E}$ is continuous. We claim that $H|_{X \setminus E \times [0,1]}$ is a continuous homotopy between $u|_{X \setminus E}$ and $v|_{X \setminus E}$.

From the fact that $p \circ F(x) = (u(x),v(x))$ it is clear that $H|_{X \setminus E \times [0,1]}$ connects $u|_{X \setminus E}$ and $v|_{X \setminus E}$. To see continuity let $(x,t) \in X \setminus E \times [0,1]$ and $\delta > 0$ be arbitrary. Choose $\delta_0 < \delta$ so that $p : B(F(x), \delta_0) \to B(p \circ F(x), \delta_0)$ is an isometry and, moreover, $B_Y(F_1(x), 2\delta_0)$ is a convex ball in $Y$ for every $s \in [0,1]$. By the continuity of $F|_{X \setminus E}$ we find $r > 0$ so that $d_Y(F(x), F(y)) < \delta_0$ whenever $y \in B(x,r) \setminus E$. These choices ensure that the distance function

$$t \mapsto d_Y(F_t(x), F_t(y))$$

is convex, in particular

$$d_{\infty}(F(x), F(y)) \leq d_{\infty}(p \circ F(x), p \circ F(y)).$$

Let us use this to estimate

$$d_Y(H(x,t), H(y,s)) \leq d_Y(H(x,t), H(x,s)) + d_Y(H(x,s), H(y,s))$$

$$\leq |t-s|\ell(F(x)) + d_{\infty}(F(x), F(y))$$

$$\leq |t-s|\ell(F(x)) + d_{\infty}(p \circ F(x), p \circ F(y))$$

$$= |t-s|\ell(F(x)) + \delta < |t-s|\ell(F(x)) + \delta.$$ 

Therefore whenever $(y,s) \in B(x,r) \setminus E \times B(t, \delta/\ell(F(x)))$ we have

$$d_Y(H(x,t), H(y,s)) < 2\delta.$$ 

Since $\delta > 0$ was arbitrary we have the desired continuity. \[\square\]

Propositions 3.2 and 3.3 demonstrate a one-to-one correspondence between locally geodesic $p$-homotopies between maps that are in $D^{1,p}(X;Y)$, and elements in $D^{1,p}(X;\hat{Y})$; any locally geodesic $p$-homotopy $H$ lifts to a map $\tilde{H} \in D^{1,p}(X;\hat{Y})$ and, conversely, any map $F \in D^{1,p}(X;\hat{Y})$ yields a locally geodesic $p$-homotopy.

### 3.1. $p$-Homotopy Classes of Maps

Given a map $v \in D^{1,p}(X;Y)$ we want to study the $p$-homotopy class of $v$, denoted $[v]_p$.

A first observation is that

$$[v]_p = \{F_1 : F \in D^{1,p}(X;\hat{Y}), \ F_0 = v\}.$$
This is easy to see using the one-to-one correspondence of $p$-homotopies and maps in $D^{1,p}(X; \hat{Y})$ presented above. Let us set

$$H^v = \{F \in D^{1,p}(X; \hat{Y}) : F_0 = v\}.$$ 

Abusing notation slightly we denote by $p : H^v \to [v]_p$ the map

$$F \mapsto F_1$$

induced by the covering map $p : \hat{Y} \to Y^2$ (since for $F \in H^v$ the first projection $F_0 = v$ always holds we may disregard it).

Let $G_v$ denote the set of locally geodesic $p$-homotopies $H : v \simeq v$. Let us define multiplication in $G_v$. Since for $H \in G_v$, the path $H^x$ is a geodesic loop based at $v(x)$ for $p$-quasievery $x \in X$ we may concatenate two elements $H, F \in G_v$ to obtain a path $(HF)^x = H^x F^x$ for $p$-quasievery $x$. This defines a $p$-homotopy $HF : v \simeq v$. By \[2.\] we have the unique locally geodesic $p$-homotopy, denoted here $\langle HF \rangle : v \simeq v$ so that for $p$-quasievery $x \in X$ the paths $H^x F^x$ and $\langle HF \rangle^x$ are homotopic.

Thus $G_v$ becomes a group with multiplication given by (quasieverywhere) pointwise concatenation (and taking locally geodesic representatives from the homotopy class of the concatenated loops).

Furthermore the group acts on $H^v$ (from the right): given elements $\sigma \in G_v$ and $F \in H^v$ we set $F.\sigma = \langle F \sigma \rangle$, the unique locally geodesic $p$-homotopy given by Theorem \[2.\] associated with $F \sigma : v \simeq F_1$. Indeed, the map

$$(F, \sigma) \mapsto F.\sigma : H^v \times G_v \to H^v$$

defines a right group action on $H^v$. This is easily seen: $(F.1)^x = F^x$ for all $F \in H^v$ and $(F.(\sigma_2 \sigma_1))^x = \langle F(\sigma_2 \sigma_1) \rangle^x = \langle F^x \sigma_2^x \sigma_1^x \rangle = \langle (F \sigma_2) \sigma_1 \rangle^x = ((F.\sigma_2).\sigma_1)^x$ for $p$-quasievery $x \in X$.

Pointwise, this is actually the action of $\pi_1(Y, v(x))$ on the universal covering space $\hat{Y}_{v(x)}$ (for $p$-quasievery $x \in X$).

**Remark 3.4.** The group $G_v$ acts on $H^v$ by “deck transformations”, i.e.

$$p \circ (F \sigma) = p \circ F$$

for $F \in H^v, \sigma \in G_v$. This is directly seen from the definitions.

Next we demonstrate that the action of $G_v$ on $H^v$ is in fact both free and proper (in the sense of \[4.\] Chapter I.8, Definition 8.2)). The following definition and lemma will prove useful.

**Definition 3.5.** We say that a set $U \subset X$ is $p$-quasiopen ($p$-quasiclosed), or quasiopen (quasiclosed) for short, if, for every $\varepsilon > 0$ there exists an open set $E \subset X$ with $\text{Cap}_p(E) < \varepsilon$ so that $U \setminus E$ is open (closed) in $X \setminus E$.

**Lemma 3.6.** Suppose $X$ is compact, $f \in N^{1,p}(X)$ and the set $\{f = 0\}$ both quasiclosed and quasiopen. Then either

$$\text{Cap}_p(\{f = 0\}) = 0$$

or

$$\text{Cap}_p(X \setminus \{f = 0\}) = 0.$$
Proof. Set $A = \{f = 0\}$. Let $F_n \supseteq F_{n+1}$ be a decreasing sequence of open sets in $X$ such that $\text{Cap}_p(F_n) < 2^{-n}$, $f|_{X \setminus F_n}$ is continuous and $A \setminus F_n$ is both closed and open in $X \setminus F_n$. We further denote by $F$ the intersection of all $F_n$’s.

Suppose $\text{Cap}_p(A) > 0$. Then also $\text{Cap}_p(A \setminus F) > 0$. First we will show that $\mu(X \setminus A) = 0$. If $\mu(X \setminus A) > 0$ then also $\mu(X \setminus (F \setminus A)) > 0$. Since, for given $n \in \mathbb{N}$ the set $A \setminus F_n$ is both closed and open in $X \setminus F_n$ the same is true of $X \setminus (A \cup F_n) = (X \setminus A) \setminus F_n$. Therefore the sets $A \setminus F_n$ and $X \setminus (A \cup F_n)$ form a separation of $X \setminus F_n$, for all $n$.

Take $x \in A \setminus F$ and $y \in X \setminus (A \cup F)$ with $\text{Cap}_p(B(x, r) \setminus F) > 0$ and $\mu(B(y, r) \setminus (A \cup F)) > 0$ for all $r > 0$. The condition is automatic for $x$ since by [1, Theorem 6.7 (xii)] $\text{Cap}_p(B(x, r)) = \text{Cap}_p(B(x, r) \setminus F)$ and it is true for $y$ provided we choose $y$ to be a density point of $X \setminus (A \cup F)$.

Take $0 < r < d(x, y)/2$ whence $\overline{B}(x, r) \cap \overline{B}(y, r) = \emptyset$. Furthermore the sets $B_n = (A \cap \overline{B}(x, r)) \setminus F_n$, $B'_n = \overline{B}(y, r) \setminus (A \cup F_n)$ are disjoint and compact. Now [L7, Theorem 7.33] implies (for compact $X$!)

$$\text{Mod}_p(\Gamma_n) = \text{Cap}_p(B_n, B'_n) = \inf \left\{ \int_X g_u^p d\mu; u \in L^p(X), u|_{B_n} \equiv 0, u|_{B'_n} \geq 1 \right\}$$

where $\Gamma_n = \Gamma_{B_n, B'_n}$. Choose a ball $B_0 \subset X$ so that $B_n \cup B'_n \subset \overline{B}(x, r) \cup \overline{B}(y, r) \subset B_0$ and estimate, for any $u$ as in the above infimum by [1, Theorem 5.53]

$$\frac{\mu(B'_n)}{\mu(2B_0)} \leq \int_{2B_0} |u|^p d\mu \leq \frac{C}{\text{Cap}_p(B_0 \cap \{ u = 0 \})} \int_{2\sigma B_0} g_u^p d\mu \leq \frac{C/\mu(2B_0)}{\text{Cap}_p(B_0)} \int_X g_u^p d\mu$$

from which we get, taking infimum over $u$,

$$\text{Mod}_p(\Gamma_n) \geq 1/C \mu(B'_n) \text{Cap}_p(B_0).$$

Since $(A \cap \overline{B}(x, r)) \setminus F = \bigcup_n B_n$, $\overline{B}(y, r) \setminus (A \cup F) = \bigcup_n B'_n$ we have

$$\lim_{n \to \infty} \mu(B_n) \text{Cap}_p(B_n) = \mu(\overline{B}(y, r) \setminus (A \cup F)) \text{Cap}_p((A \cap \overline{B}(x, r)) \setminus F)$$

and thus

$$\text{Mod}_p(\Gamma_0) \geq \limsup_{n \to \infty} \text{Mod}_p(\Gamma_n) \geq \alpha > 0,$$

where

$$\Gamma_0 = \Gamma_{\overline{B}(y, r) \setminus (A \cup F), (A \cap \overline{B}(x, r)) \setminus F}.$$ Let $\Gamma_\infty = \{ \gamma : \gamma^{-1}(F_n) \neq \emptyset \forall n \}$ whence by Lemma [15,3] $\text{Mod}_p(\Gamma_\infty) = 0$. From the fact that

$$\text{Mod}_p(\Gamma_0 \setminus \Gamma_\infty) \geq \text{Mod}_p(\Gamma_0) - \text{Mod}_p(\Gamma_\infty) \geq \alpha > 0$$

we conclude that there exists a curve $\gamma \in \Gamma_0 \setminus \Gamma_\infty$. In other words there exists an index $n_0$ and a curve $\gamma \in \Gamma_0$ with $|\gamma| \subset X \setminus F_{n_0}$. Such a curve joins the sets $A \setminus F_{n_0}$ and $X \setminus (A \cup F_{n_0})$ in $X \setminus F_{n_0}$. This, however should be impossible since these two sets separate $X \setminus F_{n_0}$.

We conclude that $\mu(X \setminus A) = 0$, that is, $f = 0$ almost everywhere. Since $f$ is $p$-quasicontinuous it follows [1, Proposition 1.59] that $f = 0$ $p$-quasieverywhere, i.e $\text{Cap}_p(X \setminus A) = 0$. The proof is now complete. □

This lemma will be used to prove that the projection $p : H^0 \to [v]_p$ is a discrete map. Namely we have
Proposition 3.7. Let \( u \in [v]_p \). Then the set
\[
p^{-1}(u) = \{ F \in H^v : F_1 = u \}
\]
is discrete with respect to the metric
\[
\bar{d}(F,H) := \left( \int_X d_{\bar{Y}}^p(F,H)d\mu \right)^{1/p}.
\]

Proof. Suppose \( H, F \in p^{-1}(u) \) are distinct and let \( \sigma = \langle HF^{-1} \rangle \) be the locally geodesic \( p \)-homotopy \( u \simeq u \) in the q.e pointwise homotopy class of \( HF^{-1} : u \simeq u \).

Consider the map \( l_{\sigma} \in N^{1,p}(X) \). Let \( \varepsilon_Y \) be half the injectivity radius of \( Y \), i.e. the largest number \( r \) with the property that every ball \( B(y,2r) \), \( y \in Y \), is a Busemann space. It follows that if \( l_{\sigma}(x) < \varepsilon_Y \) then the loop \( \sigma(x) \) is contractible. Therefore we have
\[
\{ x \in X : l_{\sigma}(x) < \varepsilon_Y \} = \{ x \in X : l_{\sigma}(x) = 0 \} =: U.
\]
Since \( l_{\sigma} \) is \( p \)-quasicontinuous it follows that \( U \) is both \( p \)-quasiclosed and \( p \)-quasiopen, whence by Lemma 3.6 either \( \text{Cap}_p(U) = 0 \) or \( \text{Cap}_p(X \setminus U) = 0 \).

Note further that
\[
U = \{ x \in X : d_{\bar{Y}}(F(x),H(x)) < \varepsilon_Y \}.
\]
This is because for any \( q \in Y \) the map inclusion map \( \iota_q : (\bar{Y},d_{\bar{Y}}) \to (\hat{Y},d_{\bar{Y}}) \) is a local isometry with every restriction \( \iota_{B(\alpha,\varepsilon_Y)} \), \( \alpha \in \hat{Y} \), an isometry. This in turn implies
\[
d_{\bar{Y}}(F(x),H(x)) = d_{\bar{Y}}(F(x),H(x)) = l_{\sigma}(x)
\]
whenever \( l_{\sigma}(x) < \varepsilon_Y \) (or equivalently \( d_{\bar{Y}}(F(x),H(x)) < \varepsilon_Y \)), yielding the desired identity.

Now suppose that \( \bar{d}(F,H) := \varepsilon < \varepsilon_Y \mu(X)^{1/p} \). Then we have
\[
\mu(\{ x \in X : d_{\bar{Y}}(F(x),H(x)) \geq \varepsilon_Y \}) \leq \left( \frac{\varepsilon}{\varepsilon_Y} \right)^p < \mu(X),
\]
implying
\[
\mu(U) = \mu(X) - \mu(\{ x \in X : d_{\bar{Y}}(F(x),H(x)) \geq \varepsilon_Y \}) > 0.
\]
By Lemma 3.6 we therefore have \( \text{Cap}_p(X \setminus U) = 0 \), in other words \( l_{\sigma} = 0 \) \( p \)-quasi-everywhere which implies \( \bar{d}(F,H) = 0 \).

This, however is not possible since \( F \) and \( H \) are distinct and therefore we conclude that any two distinct \( F, H \in p^{-1}(u) \) must satisfy
\[
\bar{d}(F,H) \geq \varepsilon_Y \mu(X)^{1/p}.
\]

We now introduce two minor alterations to the discussion above. The first one is a change of metric; for us it is convenient to use the metric
\[
\bar{d}(F,H)^p = \int_X d_{\bar{Y}(x)}^p(F(x),H(x))d\mu(x)
\]
on \( H^v \) instead of \( \bar{d} \). This way we ensure that \( G_{v} \) acts on \( H^v \) by isometries. Indeed, since for \( p \)-quasi-every \( x \in X \) we have
\[
d_{\bar{Y}}(\langle F(x)\sigma(x) \rangle, \langle H(x)\sigma(x) \rangle) = d_{\bar{Y}}(H(x), F(x)),
\]
From the elementary inequality
\[ d_\gamma(\alpha, \beta) \leq d_q(\alpha, \beta), \quad \alpha, \beta \in \hat{Y}_q, \]
it follows that
\[ \hat{d}(F\sigma, H\sigma) = \hat{d}(F, H). \]
In particular, the claim of Proposition 3.7 remains true if the metric \( \hat{d} \) is replaced by \( \hat{d} \).

The second alteration is on the space \( H^v \). We introduce a parameter \( M \in (0, \infty) \) and denote by \( H^v_M \) the set
\[ H^v_M = \{ F \in H^v : \|g_{p\circ F}\|_{L^p} \leq M \}. \]
In other words, we restrict our attention to those maps in the \( p \)-homotopy class of \( v \) which have controlled \( L^p \)-norm of their minimal upper gradients. By Theorem 2.6 if \( u \in H^v_M, M \geq \|g_v\|_{L^p} \), and \( H: v \approx u \) is a locally geodesic \( p \)-homotopy then \( H_\ast \in H^v_M \) for every \( t \). Clearly the claim of Proposition 3.7 remains true if, in addition to the change of metric, the space \( H^v \) is replaced by \( H^v_M \). We use the notation \([v]_{p,M}\) for the image set \( p(H^v_M) \subset [v]_p \).

**Proposition 3.8.** The action of \( G_v \) on \( H^v_M \) is proper and free. Moreover, if \( F \in H^v_M \) and \( u = p \circ F \in [v]_{p,M} \) then \( F.G_v = p^{-1}_M(u) \). Here \( p_M = p|_{H^v_M} \).

**Proof.** Let us first show that the action is free. If \( F\sigma = H\sigma \) then for \( p \)-quasievery \( x \in X \) one has \( \langle F(x)\sigma(x) \rangle = \langle H(x)\sigma(x) \rangle \). Since the action of \( \pi_1(Y, v(x)) \) on \( \hat{Y}_v(x) \) is free this implies that \( \sigma(x) \) the neutral element of \( \pi_1(Y, v(x)) \), i.e. the constant path \( v(x) \). Since \( \sigma(x) \) is the path \( t \mapsto v(x) \) for \( p \)-quasievery \( x \in X \) we have that \( \sigma \) is the trivial \( p \)-homotopy \( v \approx v \), i.e. \( \sigma_t = v \) for all \( t \in [0, 1] \).

Now suppose \( H \in B(F, \varepsilon) \cap B(F\sigma, \varepsilon) \). Then \( \hat{d}(F, F\sigma) \leq 2\varepsilon \). By Remark 3.3 \( p \circ F = p \circ (F\sigma) \), and thus Proposition 3.7 implies that if \( 2\varepsilon < \varepsilon_Y \mu(X)^{1/p} =: \varepsilon_0 \) then \( F = F\sigma \), i.e. \( \sigma \) is the trivial \( p \)-homotopy \( v \approx v \), the neutral element of the group \( G_v \). This shows that for \( \varepsilon < \varepsilon_0/2 \) the collection of \( \sigma \in G_v \) for which \( B(F, \varepsilon) \cap B(F\sigma, \varepsilon) \neq \emptyset \) consists only of the neutral element.

Finally let \( F \in H^v_M \) and \( u = p \circ F \in [v]_{p,M} \). Obviously \( F.G_v \subset p^{-1}_M(u) \) since for all \( \sigma \in G_v \) it holds that \( p \circ F\sigma = p \circ F \). But if \( H \in p^{-1}_M(u) \), let \( \sigma = (F^{-1}H) \in G_v \) and calculate \( F\sigma = \langle FF^{-1}H \rangle = H \) so that \( H \in F.G_v \).

**3.2. A weak compactness result and further discussion.** Unfortunately I have been unable to prove that the (restricted) \( p \)-homotopy class \([v]_{p,M}\) is compact with respect to the \( L^p \)-metric \( \hat{d} \).

To look for weaker results we shall look at the metric properties of the spaces \((H^v_M, \hat{d})\) and \((H^v_M, \hat{d})\). A little care is needed when considering \( H^v_M \) as a metric space with either of the metrics \( \hat{d} \) or \( \hat{d} \). Instead of speaking of the equivalence classes of maps agreeing \( p \)-quasieverywhere (as in the case of \( D^{1,p}(X; \hat{Y}) \)) we identify a map \( F \) as belonging to \( H^v_M \) if it agrees almost everywhere with a map \( F' \in D^{1,p}(X; \hat{Y}) \) which satisfies \( (F')_0 = v \) and \( \|g_{p\circ F'}\|_{L^p} \leq M \). When speaking of an element \( F \in H^v_M \) we always pick the “good” representative given by a Dirichlet map agreeing with \( F \) almost everywhere.
From now on we assume that $X$ and $Y$ are both compact. In particular $\tilde{Y}$ is then proper.

A direct application of the Rellich Kondrakov compactness theorem yields that the space $H^1_M$ equipped with metric $\tilde{d}$ is proper. The situation with $\hat{d}$ is similar.

**Lemma 3.9.** The set $H^1_M$ equipped with the metric $\hat{d}$ is a proper metric space.

**Proof.** Take a sequence

$$F_n \in \hat{B}(H, L) =: \{F \in H^1_M : \hat{d}(H, F) \leq L\}.$$  

Each $F_n$ is the lift of the locally geodesic $p$-homotopy $t \mapsto (F_n)_t : v \simeq p \circ F_n$, so using Proposition 3.2 we may estimate

$$\hat{d}(H, F_n) + \|g_{F_n}\|_{L^p} \leq L + \|g_v\|_{L^p} + \|g_{p_0F_n}\|_{L^p} \leq \|g_v\|_{L^p} + L + M$$  

for all $n$ and therefore the Rellich Kondrakov theorem implies that a subsequence denoted $F_n$ converges to some $F \in N^{1,p}(X; \tilde{Y})$ in the metric $\hat{d}$. By passing to a further subsequence we may assume that $F_n \to F$ pointwise almost everywhere. (In particular $d_{v(x)}(F_n(x), F(x)) \to 0$ as $n \to \infty$ for almost every $x \in X$.)

From the fact that $d_q(\alpha, \beta) = \ell(\beta\alpha^{-1})$ for paths $\alpha, \beta \in \tilde{Y}_q$ (see discussion before Definition 3.1) we observe that $l_{F_n} = d_{v(x)}(F_n(x), \hat{v})$, where $\hat{v}$ denotes the lift of the trivial $p$-homotopy $v \simeq \hat{v}$. Using this and Lemma 2.7 we may estimate

$$\|l_{F_n}\|_{L^p} + \|g_{l_{F_n}}\|_{L^p} = \hat{d}(F_n, \hat{v}) + \|g_{F_n}\|_{L^p}$$  

$$\leq \hat{d}(\hat{v}, H) + \hat{d}(H, F_n) + \|g_{p_0F_n}\|_{L^p} + \|g_v\|_{L^p}$$  

$$\leq \hat{d}(\hat{v}, H) + L + M + \|g_v\|_{L^p}$$

for all $n$, so for a still further subsequence the function $l_{F_n}$ converges to some $f \in N^{1,p}(X)$ in $L^p$-norm and pointwise almost everywhere. We shall use the following General Lebesgue Dominated Convergence Theorem.

**Lemma 3.10.** Let $f_n$ be a sequence of measurable functions on a measure space $(\Omega, \nu)$ that converges $\nu$-almost everywhere to $f$. Suppose there is a sequence $g_n$ of $\nu$-integrable functions that converge pointwise $\nu$-almost everywhere to a $\nu$-integrable function $g$, such that $|f_n| \leq g_n$ for each $n$, and

$$\lim_{n \to \infty} \int_\Omega g_n d\nu = \int_\Omega g d\nu.$$  

Then

$$\lim_{n \to \infty} \int_\Omega f_n d\nu = \int_\Omega f d\nu.$$  

By the inequality

$$d_v^p(F_n, F) \leq 2^{p-1}d_v^p(F_n, \hat{v}) + 2^{p-1}d_v^p(\hat{v}, F) = 2^{p-1}l_{F_n}^p + 2^{p-1}l_F^p$$

we may take $g_n = 2^{p-1}l_{F_n}^p + 2^{p-1}l_F^p$ and $g = 2^{p-1}l_F^p$, and use the above theorem to conclude

$$\lim_{n \to \infty} \int_X d_v^p(F_n, F) d\mu = \int_X \lim_{n \to \infty} d_v^p(F_n, F) d\mu = 0.$$  

Having established $\hat{d}(F_n, F) \to 0$ as $n \to \infty$ it is evident that $F \in \hat{B}(H, L)$ and therefore we have shown the compactness of $\hat{B}(H, L)$.  

An immediate corollary is the following weak compactness result.
Corollary 3.11. Suppose \( v \in N^{1,p}(X;Y) \) and \( u_n \) is a sequence in \([v]_p\) with
\[
\sup_n \|g_{u_n}\|_{L^p} < \infty
\]
converging to \( u \) in \( L^p(X;Y) \). If the maps \( u_n \) can be connected to \( v \) by \( p \)-homotopies \( H_n : v \simeq u_n \) satisfying
\[
(3.1) \quad \sup_n \int_X l_{H_n} d\mu < \infty,
\]
then \( u \in [v]_p \).

Proof. Let \( M_0 = \sup_n \|g_{u_n}\|_{L^p} \) and \( M_1 = \sup_n \int_X l_{H_n} d\mu \). Using the Poincaré inequality we estimate
\[
\tilde{d}(\hat{v}, \hat{H}_n) = \left( \int_X l^p_{H_n} d\mu \right)^{1/p} \leq \int_X l_{H_n} d\mu + C \text{diam}(X) \left( \int_X g^p_{H_n} d\mu \right)^{1/p} \leq \mu(X)^{-1} M_1 + C \text{diam}(X) M_0
\]
(We use the notation \( \hat{v} \) for the lift of the trivial \( p \)-homotopy \( v \simeq v \) again.) Therefore \( \hat{H}_n \in \hat{H}_v M_0 \cap \tilde{B}(\hat{v}, L) \), where \( L = \mu(X)^{1/p - 1} M_1 + C \text{diam}(X) M_0 \). By the previous lemma a subsequence \( \hat{H}_n \) converges to some \( \hat{H} \in \hat{H}^p_v \) in the metric \( \tilde{d} \).

Furthermore we have
\[
\int_X d^p_Y(p \circ H, u) d\mu = \lim_{n \to \infty} \int_X d^p_Y(p \circ H, p \circ H_n) d\mu \leq \lim_{n \to \infty} \tilde{d}(H, H_n) = 0
\]
so that \( u = p \circ H \). Therefore \( u \in [v]_p \). \( \square \)

This is an unsatisfactory result because of the extra assumption (3.1) of having to control the lengths of the homotopies \( H_n \). The result is basically a restatement of the fact that, given \( M > 0 \) the space \( H^p_v \) equipped with metric \( \tilde{d} \) is a proper (which in turn followed easily from the Rellich Kondrakov compactness theorem [1.11]).

Removing the extra assumption (3.1) on the homotopies \( H_n \) in Corollary 3.11 amounts to ensuring that the space \( H^p_v / G_v \), arising from the action of \( G_v \) on \( H^p_v \) in the previous subsection, equipped with the metric
\[
\overline{d}(F.G_v, H.G_v) := \text{dist}_{\tilde{d}}(F.G_v, H.G_v)
\]
has finite diameter; notice that the action gives rise to a covering map
\[
\pi : H^p_v \to H^p_v / G_v
\]
and the metric \( \overline{d} \) makes \( \pi \) into a local isometry, see [4] Chapter I.8, Proposition 8.5(3)].

To see the claim about the finite diameter take two elements \( F.G_v, H.G_v \in H^p_v / G_v \) and let \( u = p \circ F, w = p \circ H \). Note that
\[
\overline{d}(F.G_v, H.G_v) = \inf_{\sigma \in G_v} \overline{d}(F, (H\sigma)) = \inf_{\sigma \in G_v} \left( \int_X l^p_{H\sigma F^{-1}} d\mu \right)^{1/p}
\]
The rightmost infimum is equal to the infimum over all locally geodesic \( p \)-homotopies \( H : u \simeq w \) of the quantity
\[
\left( \int_X l^p_H d\mu \right)^{1/p},
\]
since for each \( \sigma \in G_v \), \( (H \sigma F^{-1}) : u \simeq w \) is a locally geodesic \( p \)-homotopy. Conversely, given any locally geodesic \( p \)-homotopy \( H' : u \simeq w \) we may write it as \( H' = (H(H^{-1}H')F^{-1}) \), where \( H^{-1}H'F \in G_v \).

We obtain

\[
(3.2) \quad \bar{d}(p^{-1}(u), p^{-1}(w)) = \inf_{H: u \simeq w} \left( \int_X l_p \mu \right)^{1/p}.
\]

With this in hand it is easy to see that if \( H^v_M / G_v \) has finite diameter then \( (3.1) \) is automatically satisfied.

On the other hand, requiring that for every sequence \( u_n \in [v]_M \) condition \( (3.1) \), rewritten

\[
\sup_n \inf_{H: u \simeq u_n} \int_X l_p \mu < \infty,
\]

is satisfied, is equivalent to requiring that there is some \( C < \infty \) so that

\[
\sup_{u \in [v]_M} \inf_{H: u \simeq u} \int_X l_p \mu \leq C
\]

(if such a constant did not exist we would have a sequence \( u_n \) contradicting the condition). Thus we see that \( (3.1) \) is automatically satisfied if and only if \( H^v_M / G_v \) has finite diameter.

Observe that since \( (H^v_M, \bar{d}) \) is proper the same is true of \( H^v_M / G_v \) and therefore it has finite diameter if and only if it is compact.

What, then, can we say about the quotient space \( H^v_M / G_v \)?

We may define a map \( \overline{p}_M : H^v_M / G_v \to [v]_{p,M} \) by

\[
\overline{p}_M(F,G_v) = p \circ F.
\]

This is well-defined by Remark 3.3. By the last assertion in Proposition 3.8 we see that \( \overline{p}_M \) is bijective.

With the continuous bijection \( \overline{p}_M : H^v_M / G_v \to [v]_{p,M} \) at hand it is immediate that compactness of \([v]_{p,M}\) is implied by the compactness of \( H^v_M / G_v \); indeed assuming this, the map \( \overline{p}_M \) is a homeomorphism (by elementary topological considerations). In this event, furthermore, we see that \( p_M = \overline{p}_M \circ \pi : H^v_M \to [v]_{p,M} \) is a covering map. (Conversely, assuming that \( p_M \) is a covering map we have that \( \overline{p}_M \) is also a covering map, and therefore a homeomorphism.)

Another way of interpreting the identity \( (3.2) \) is to identify \( H^v_M / G_v \) with \([v]_{p,M}\) (through the map \( \overline{p}_M \)) and \( \bar{d} \) with a metric induced by a certain length structure on \([v]_{p,M}\) (see 11, Definition 1.3, p.3)). Indeed, the length structure is given by the family of paths \( H \) that are (locally geodesic) \( p \)-homotopies between the endpoint maps, and the length functional is simply

\[
\ell(H) = \left( \int_X l_p \mu \right)^{1/p}.
\]

This point of view emphasizes the (geo)metric structure of \([v]_{p,M}\) or \( N^{1,p}(X;Y) \) and in particular the question of compactness of \([v]_{p,M}\) is reduced to asking does
the length structure $\overline{d}$ give rise to the same topology on $[v]_{p,M}$ as does the original $L^p$-metric

$$d(u, w) = \left( \int_X d^p_{Y}(u, w) d\mu \right)^{1/p}.$$ 

This question remains open, encouraging the study of geometry of the Newtonian spaces $N^{1,p}(X; Y)$

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