Fermion Loops, Linear Magnetoresistance, Linear In Temperature Resistance, and Bad Metals

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Bad metals including the high $T_c$ superconductors display an exotic resistance that is linear in both temperature and magnetic field. This hallmark of strong correlations is poorly understood. We show that Fourier transforming the magnetoconductance with respect to magnetic field obtains a curve describing the area distribution of loops traced by fermions within the sample. Analysis of this area distribution reveals that linear resistance is caused by scattering and quantum interference, but with more large loops than occur in ordinary 2-D and 3-D materials where scattering destroys quantum coherence and limits loop size. This limit is absent in linear resistance materials, resulting in larger loops limited only by thermal decoherence. Linear resistance signals that quantum coherence is maintained in the presence of scattering.

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In atomic units $m_e = e = \hbar = k_B = 1$ the magnetic field $B$ has dimensions of inverse area, which is key to understanding the electrical conductance’s dependence on $B$. Large fields probe areas at the scale of the unit cell, and small changes in field probe much larger areas. At large fields one finds Landau levels and Schubnikov-de Haas (SdH) oscillations, which are visible in the longitudinal conductance $G_{xx}(B)$ as peaks at characteristic field strengths determined by the Fermi energy $E_F$. The Fermi surface’s cross section, an inverse area, can be read off from the peak positions. Landau levels and SdH oscillations are signals of ballistic physics, i.e. weak scattering, and are extinguished once scattering becomes too frequent.

At small fields one finds weak (anti) localization (WL/WAL), where scattering and quantum interference cooperate to cause $G_{xx}$ to decrease or increase with field depending on whether the spin relaxation length $l_s$ is small or large. WL/WAL contrasts strongly against Landau levels and SdH oscillations. It is much more sensitive to magnetic field, is visible at much smaller field strengths, and varies smoothly with field rather than exhibiting oscillations. Moreover WL/WAL requires scattering and is weakly sensitive to length scales shorter than the scattering length, including the atomic unit cell and the Fermi wavelength.

Recent years have exposed mysteries which lie well beyond traditional SdH and WL/WAL physics. Many experiments report systems where the longitudinal resistance $R_{xx}$ increases linearly with $B$ and does not saturate.\cite{28} This contrasts both with SdH oscillations which oscillate periodically in $1/B$ and with WL/WAL which is logarithmic. Explanations have been given for the cases of ballistic conduction when the Fermi surface has a cusp, of a 3-D Dirac cone in the presence of a strong magnetic field, of a density gradient across the sample, and of classical transport with strong sample inhomogeneities.\cite{29,30} However this kind of case by case treatment is not entirely satisfactory given the wide range of experimental realizations.

Another long-standing mystery is that high $T_c$ superconductors (cuprates and pnictides) at temperatures above the superconducting phase display a resistance which increases linearly with temperature and does not saturate.\cite{30} This is inconsistent both with the WAL signal which increases much more slowly with $T$ (logarithmically in 2-D and as a square root in 3-D) and with phonon based scattering which should saturate at a maximum determined by the atomic spacing. Materials whose resistance is linear in temperature are called bad metals and are understood to be strongly correlated, but the details of the conduction process responsible for linear resistance are not understood. It is widely believed that linear-in-temperature resistance is correlated with high-$T_c$ superconductivity. Very recently the linear dependence on temperature has been linked to linear magnetoresistance found in the same bad metal regime.\cite{31,32}

This paper, inspired by the magnetic field’s units, develops a geometrical analysis of magnetotransport in terms of the loops traced by electrons and holes as they move through real space, magnetic fluxes through those loops, and the Aharonov-Bohm effect. Charge conservation, as we discuss below, requires electrons and holes to move in loops. These loops mean that either an electron returns to its starting point or it merges with a hole which has opposite charge, thereby conserving charge. The requirement to move in loops applies to the individual electrons and holes regardless of whether their collective many-body behavior forms a Fermi liquid with quasiparticles, and applies independently of magnetic fields, scattering, and interactions. Analysis focused on fermion loops is therefore a very powerful way of understanding the mechanisms responsible for electronic transport.

Experimental data on a sample’s magnetic field dependence gives direct access to the areas of the electron and hole loops within the sample. Each fermion loop couples to field via a phase which is proportional to the magnetic flux through the loop, and this phase is equal to the product of the loop area and the magnetic field strength. There is also a Zeeman term, which we neglect for the
moment and will return to later. Within the phase factor magnetic field has a conjugate relationship to area, with a proportionality constant that is determined by fundamental constants and can not be renormalized. Therefore measurements of a sample’s magnetic field dependence allow rigorous conclusions to be made about the areas of the loops traced by electrons within the sample, and give a window onto the physical processes and length scales at work in transport. The methodology of using magnetic fields to measure loops, which we explain further below, does not assume quasiparticle or Fermi liquid physics. It is a simple consequence of charge conservation.

In this article we apply this methodology to electrical conduction and obtain rigorous results about the electron and hole loops responsible for Landau levels, SdH oscillations, WL/WAL, and linear magneto-resistance. Our rigor here is founded on the firm link between experimental observables and loop areas, and does not concern details of the material’s Hamiltonian, interactions, Fermi surfaces, or disorder. In fact these points are discussed very little here. Instead we focus on reasoning about the fermion loops that are responsible for conduction, and on drawing conclusions about which physical mechanisms are responsible for producing them.

First we focus on scattering. The electron loops responsible for Landau levels and SdH oscillations, which are phenomena that do not rely on scattering, have weights that oscillate periodically with loop area, and the period of oscillation is determined by the Fermi surface. In contrast, the weights of the loops responsible for the WL/WAL signal decrease gradually and smoothly with area, and the decay extends to loop areas which are far larger than that determined by the Fermi surface. The very smooth distribution of loop areas and the very large area scales are caused by stochastic motion of the charge carriers, and are unmistakeable hallmarks of scattering.

Moving on to linear magneto-resistance, in this case the loop areas again decrease smoothly up to very large areas. These results are mathematically rigorous, and are conclusive evidence that scattering is key to linear magneto-resistance.

The weights governing loop areas for standard WL/WAL and for linear magneto-resistance show two additional similarities: both do not show any concentration near any particular area scale (i.e. both are scale-free), and both have a single sign. In our view the root cause of linear magneto-resistance is the same interplay of scattering and quantum coherence that causes WL/WAL in standard Fermi liquids. In standard WL/WAL scattering is caused by static impurities, and in bad metals scattering is caused by electronic interactions, but in both cases the effects of scattering on electronic conduction are broadly the same. First of all, at lengths exceeding the scattering length the electronic wave-function $|\psi\rangle$'s phase is randomized, rendering single wave-functions unable to mediate conduction. Therefore conduction over distances longer than $l$ is mediated only by pairing between $|\psi\rangle$ and its complex conjugate $\langle\psi|$ whose random phases cancel. This pairing produces the single-particle density matrix $\rho(x)$, which is the key mediator of conduction at distances longer than the scattering length. This result is well known from the extensive work on sigma models and bosonization techniques in disordered systems.

$\rho$'s evolution is sensitive only to processes where $|\psi\rangle$ and $\langle\psi|$ undergo the same scattering events and follow the same path between scatterings, because only these processes satisfy the requirement of phase cancellation. There are two such processes, the diffusion and the Cooperon. In the diffusion $|\psi\rangle$ and $\langle\psi|$ follow the same sequence in the same order, producing purely classical electronic motion. In the Cooperon $|\psi\rangle$ and $\langle\psi|$ follow the same sequence but in reversed order, and their phases seem to be randomized with respect to each other. It is not until their trajectories close a loop and come back to their origin that $|\psi\rangle$ and $\langle\psi|$'s phases suddenly cancel and $\rho(x)$ undergoes a revival. The Cooperon is a quantum interference process, distinct from classical motion, and is strongly dependent on even very weak magnetic fields. It is the Cooperon’s contribution to conduction which is responsible for WL/WAL, and which we assert is responsible for linear magneto-resistance. This should not be read as an assertion that bad metals are governed by impurities, elastic scattering instead of interaction-based scattering, Fermi liquid physics, or quasiparticles. We are simply stating that scattering and stochastic motion are key, and that both WL/WAL and linear magneto-resistance are caused by pairs of matching loops, one for $|\psi\rangle$ and one for $\langle\psi|$ following identical trajectories but in opposite directions. These paired loops are the Cooperon, whose distribution of loop areas is scale free and smooth and extends to very large areas, as seen both in standard WL/WAL and linear magneto-resistance.

Comparison of the loops responsible for standard WL/WAL and for linear magneto-resistance does reveal one important difference between the two: there are many more large loops in linear magneto-resistance than in WL/WAL. This distinction is mathematically rigorous: the probability distribution of WL/WAL’s loop areas decays like $1/A$ ($A$ is the area), while the area distribution for linear magneto-resistance decays like $\ln A$; i.e. it has a long tail. We argue that the long tail is caused by quantum coherence, which results in quantum mechanical harmonics. One example of harmonics is the sequence of Landau levels, where the lowest Landau level is accompanied by many siblings whose areas are multiples of the area in real space of the lowest level. In standard WL/WAL scattering randomizes momentum, and therefore each Cooperon loop is not accompanied by harmonics at multiples of its area. This leads directly to the standard WL/WAL signal. If however each Cooperon loop is accompanied by repeats at multiples of its area, then one finds a long tail in the loop area distribution and can obtain linear magneto-resistance. We therefore interpret linear magneto-resistance as unambiguous evidence that the charge carriers maintain their coherence.
and manifest many harmonics, despite undergoing scattering. In other words, linear magnetoresistance is caused by a combination of quantum coherence and scattering that is not present in ordinary materials. This deduction is based on the logarithmic tail in the loop areas that are responsible for linear magnetoresistance, which is a rigorous result. It concerns physics at distances much larger than the scattering length, and therefore is both agnostic about and independent of microscopic details of the Hamiltonian, interactions, and how quantum coherence is preserved.

Turning now to the linear-in-temperature resistance seen in bad metals, in our view it is nothing other than the zero-field manifestation of linear-in-field resistance. Quantum mechanical harmonics are regulated only by temperature and scattering. In linear magnetoresistance systems the higher harmonics responsible for large loops are resilient against scattering, but not against temperature, and increasing temperature reduces the number of harmonics. The precise mapping from linear in field resistance to linear in temperature resistance is required by a very simple and weighty fact: when the field is reduced to zero, temperature supplies the only area scale comparable to the areas of the very large loops in play in linear magnetoresistance.

Our explanation contrasts with most other work on bad metals, which typically neglects WL/WAL and therefore deduces that the scattering time scale is the inverse of temperature, opening up a range of questions about correlation effects at short time scales. We claim that quantum interference, not effects at short time scales, is key to understanding bad metals. In particular, linear magnetoresistance has been observed in bad metals at temperatures of 1000 K, and we understand this as clear evidence that quantum interference and quantum coherence continue up to the same temperatures, albeit with a number of harmonics that decreases steadily as temperature increases.

We have made one assumption throughout this discussion: that the phase factor associated with flux through fermion loops, and not the Zeeman coupling between field and spin, is responsible for linear resistance. In relativistic systems the Zeeman coupling is a seamless part of the flux phase factor, but in the condensed matter systems of interest here the Zeeman coupling takes on a separate role which we have neglected. As we have discussed earlier, disordered transport is mediated by the single particle density matrix \( \rho(x) \). The main effect of spin on disordered transport is that \( \rho(x) \) contains both a spin singlet (charge) and a spin triplet (spin polarization). In materials with strong spin-orbit coupling or strong spin-dependent scattering the spin polarization relaxes very quickly and only the spin singlet contributes to electronic conduction. In other words, when the spin relaxation length is short conduction is mediated by charges which effectively have zero spin. In this case the Zeeman coupling’s effect on electronic transport is negligible.

Linear magnetoresistance has been seen chiefly in topological systems and in bad metals. In the first case spin-orbit interactions are very strong, while in the second case it is well known that spin fluctuations are strong; in both cases the spin relaxation length is short. The fact that resistance increases rather than decreases with field is also strong evidence that the spin triplet is negligible. The singlet contribution to WL/WAL decreases the resistance and therefore when magnetic field disrupts WL/WAL the resistance increases; this is called weak antilocalization. In contrast, the spin triplet contribution has the opposite sign causing weak localization, so if the spin relaxation length is long magnetic field causes the resistance to decrease. Since experimental observations of linear magnetoresistance universally find that resistance increases rather than decreases with field, we conclude that the spin relaxation length is short, and that the Zeeman effect is not relevant to conduction.

Section IV will explain how to start with experimental measurements of magnetic field dependence and then obtain rigorous information about the areas of fermion loops. Next section III applies this methodology to compare the areas of fermion loops in Landau levels, SdH oscillations, WL/WAL, linear magnetoresistance, and Levy flights. Section III shows how quantum coherence causes resistance to be linear in field and temperature and discusses coefficients that can be measured experimentally, and section IV wraps up with a few final thoughts.

## I. GEOMETRIC ANALYSIS OF FERMION LOOPS

We found our geometric analysis of magnetotransport on a basic result from relativistic quantum field theory: the unique way to build a theory of fermions which conserves charge is to first add a phase degree of freedom \( \phi(x) \) (a \( U(1) \) phase for electromagnetism) and then require that the theory be invariant under multiplications by \( \exp(i\phi) \). After rigorous development one obtains two requirements which constrain every condensed matter theory:

- The effect of an external magnetic field \( \vec{B} \) on electrons must be mediated by the minimal coupling: the momentum operator \( \vec{p} \) must be replaced by \( \vec{p} - e\vec{R} \), where \( \vec{R} \) is the gauge field determining the magnetic field by \( \vec{B} = \nabla \times \vec{R} \).
- All physical observables must be invariant under gauge transformations of \( \vec{R} \rightarrow \vec{R} + \nabla \phi \), no matter what value of \( \phi(x) \) is chosen.

In the non-relativistic limit one also obtains the Zeeman term, which we omit from the following discussion because in the systems of interest the spin relaxation length is short and spin therefore has little effect on transport. These two requirements have very important consequences for the structure of any physical observable. The only way to build a gauge invariant observable \( O \) from
the minimal coupling is by requiring each electron contributing to the observable to return to its starting position or merge with a hole, forming closed loops. The actual shape and size of the loops, and also the number of loops, is completely unrestricted by gauge invariance; the emphasis is on their closed nature.

It is important to be clear that the loops under discussion here are traced by bare electrons and holes, not by quasiparticles. Their existence is an immediate consequence of charge conservation and gauge invariance, and does not depend in any way on the existence of quasiparticles like those seen in a Fermi liquid, or on weak interactions. Within quantum field theory, the loops become explicit after calculating the expectation value of an observable, because all possible non-gauge-invariant terms average to zero while calculating the expectation value. This is seen in both strongly interacting lattice gauge theory where loops traverse the lattice and in perturbation theory where fermions always contribute via loops of one or more concatenated Green’s functions.

It is also important to be clear that the loops discussed here move through real space, and occur even when disorder is very strong. Therefore they are not the same loops used in the analysis of Berry phases and topological invariants, which are usually calculated in momentum space and assume a well defined band structure. Moreover the loops here concern the bare electron and hole carriers within a material; the U(1) symmetry which supports them comes from electromagnetism, not from effective interactions that occur in spin liquids and other interacting systems.

Because each electron is required to trace a loop, the only way that a magnetic field can affect any observable is by introducing a multiplicative factor, the Wilson loop $W_{\Gamma} = \exp(i \oint_{\Gamma} d\vec{x} \cdot \vec{B})$, where $\Gamma$ specifies the paths traced by a set of one or more electron and hole loops. By Stokes’ theorem the phase in the Wilson loop is equal to the magnetic flux through the one or more loops specified by $\Gamma$, i.e. $W_{\Gamma} = \exp(i \Phi_{\Gamma})$, where $\Phi_{\Gamma} = \int d\vec{S} \cdot \vec{B}$ and $e$ is the sum of the surfaces bounded by those loops. This is the fundamental reason why magnetic fields have units of inverse area.

We formalize these ideas using the loop representation of observables. The previous discussion can be summarized by the statement that every physical observable $O_i(\vec{B})$ must be written as

$$O_i(\vec{B}) = \sum_{\Gamma} (O_i | \Gamma) \exp(i \Phi_{\Gamma})$$

where $\sum_{\Gamma}$ is a sum over all the Wilson loops which contribute to the observable $O_i$, and $(O_i | \Gamma)$ is the weight with which a particular Wilson loop $\Gamma$ contributes to $O_i$. This weight is determined by both the observable and by the kinematics and interactions of the electrons tracing the loops in $\Gamma$.

We specialize to the case of a uniform magnetic field $\vec{B}$ originating externally to the sample, which allows us to assign to each Wilson loop a vector-valued cross-section $\vec{A}_\Gamma = \int d\vec{S}$, a vector generalization of the Wilson loop’s surface area. The magnetic flux through $\Gamma$ is then $\Phi_{\Gamma} = \vec{A}_\Gamma \cdot \vec{B}$, leading to our central result:

$$O_i(\vec{B}) = \int d\vec{A} \, O_i(\vec{A}) \exp(i \vec{A} \cdot \vec{B}),$$

$$O_i(\vec{A}) = \sum_{\Gamma} (O_i | \Gamma) \delta(\vec{A}_\Gamma - \vec{A})$$

In other words, every physical observable can be resolved into contributions corresponding to specific areas $\vec{A}$, each with weight $O_i(\vec{A})$:

$$O_i(\vec{A}) = \int \frac{d\vec{B}}{(2\pi)^3} \, O_i(\vec{B}) \exp(-i \vec{A} \cdot \vec{B})$$

In summary, equations 2 and 3 state that measurements of an observable $O_i(\vec{B})$’s magnetic field dependence give direct and rigorous information about the areas traced out by electron loops within a sample, simply by performing a Fourier transform. The resulting loop area distribution $O_i(\vec{A})$ completely describes how different loops contribute. In particular, the external field $\vec{B}$ is conjugate to a specific area $\vec{A}$, $A_i = B^{-1}$, and there is no possibility that any process could renormalize this relation.

From a theorist’s point of view, the problem of how to perform a regularized sum over loops, i.e. how to control both long distance and short distance contributions to the sum, remains in general unsolved. This makes evaluation of the sum over loops $\sum_{\Gamma}$ in equations 1 and 2 a difficult task. It is therefore both remarkable and encouraging that we can measure the sum experimentally and directly using equation 3 experimental measurements of the loop area distribution $O_i(\vec{A})$ can lead the way in guiding theorists to correct procedures for performing the sum over loops.

II. COMPARISON OF LOOP AREA DISTRIBUTIONS

A. Landau levels and SdH Oscillations

To illustrate the transformation from experimental observables to loop area distributions, consider the case where the dominant energy scale is the Fermi level $E_F$, in which case a delta function - the first Landau level - will be found in $G_{xx}(B)$ at $B^{-1} = \frac{1}{2} A_F$. Here $2\pi/A_F \propto 2\pi E_F$ is the cross-section of the Fermi surface. The loop area distribution is therefore a cosine $G_{xx}(A) \propto \cos(A/2 A_F)$, which means that loops with every area contribute to the conductance and that there is ringing corresponding to the characteristic Fermi area $A_F$.

The most interesting aspect of this result is that quantum coherence causes the first Landau level to be repeated exactly at characteristic areas $(N + 1/2) A_F$, with
We now apply the geometric interpretation to 2-D weak WL/WAL, which is mediated by the Cooperon. At magnetic fields larger than the inverse cutoff $q_c B > 1/L_{max}^2$ the conductance $G_{xx}^{WL}(B)$ is a logarithm $G_{xx}^{WL}(B) \propto -\ln B$, and at smaller fields it quickly transitions to zero. The transition is quadratic in $B$ because time reversal symmetry requires that $G_{xx}$ be even under $B \rightarrow -B$. These results are well known and strongly attested by many sources. Aside from minor numerical changes, the log-quadratic combination is universal for both Dirac and $p^2/2m$ dispersions, for both lattice and continuum models, for both short and long range scattering, and for a wide variety of quasi-2-D geometries.\(^{28–41}\)

These results for the Cooperon conductance are usually obtained by starting with a microscopic model and then deriving from that another model which describes Cooperon movement at distances longer than the scattering length. It is invariably found, without any loss of information or generality, that at long enough distance scales the trajectories of Cooperons are simply random walks. This is a rigorous result, since momentum is randomized at distances longer than the scattering length, and since spin polarization disappears at the spin decay length. The WL/WAL contribution to the conductance is determined by assuming that disorder causes Cooperons to move diffusively (random walks), introducing an infrared cutoff $A_{max} = L_{max}^2$, and then calculating the probability that the Cooperon will return to its origin. Quantum coherent repeats of loops are not allowed because Cooperon momentum is random and therefore even when a Cooperon returns to its origin it does not have the same momentum that it started with. This procedure always produces the standard 2-D conductance profile, which is quadratic at small $B$ and logarithmic at large $B$.

We demonstrate here that the Fourier transform of the WL/WAL conductance $G_{xx}(B) \propto \ln B$ varies inversely with area, i.e. that the loop area distribution obeys $G_{xx}(A) \propto A^{-1}$. The main difficulty is controlling the ultraviolet and infrared cutoffs, so we perform the demonstration three ways. First, we use purely dimensional analysis: $G_{xx}(A) = (2\pi)^{-1} \int dB \exp(-iAB) \ln B$ and therefore must carry units of inverse area. Second, we use specific functional forms for $G_{xx}(A)$ and carry out analytically the inverse Fourier transform from $G_{xx}(A)$ to $G_{xx}(B)$. For instance, if we use hard cutoffs at $A_0$ and $A_{max}$ then we arrive at $G_{xx}^{WL}(B) \approx R_0 - \ln[q_c B] + (q_c B A_0)^{3/4} / 4$ for $1/A_{max} < q_c B < 1/A_0$, where $R_0 = -\gamma E - \ln |A_0|$ and $\gamma E$ is the Euler-Mascheroni constant. Here $R_0$ should be adjusted to zero, as a regularization compensating for the hard cutoffs, in order to make the conductance be always positive. As shown in the appendix analytical results can be obtained for other cutoffs and invariably produce logarithmic forms for $G_{xx}(A)$. Lastly, Figure 2a summarizes the results of extensive numerical Monte Carlo simulations of random

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**FIG. 1:** (Color online.) Quantum coherence and the hierarchy of Landau levels. Panel (a) shows the magnetococonductance $G_{xx}(B)$ of the $N = 1, 2, 3$ Landau levels and panel (b) shows their loop area distributions $G_{xx}(A)$, which have been vertically shifted for clarity. Quantum coherence is responsible for producing higher harmonics of the lowest Landau level, causing higher Landau levels at multiples $(N + \frac{1}{2})A_F$ of the characteristic area $A_F = E_F^{-1}$. Decoherence, i.e. loss of quantum coherence, is caused by either the scattering energy scale $\Gamma$ or the temperature $T$. Decoherence widens peaks in $G_{xx}(B)$, suppresses $G_{xx}(A)$ at large $A$, and completely suppresses oscillations when the field $B$ is as small as $\Gamma, T, E_F = 1, \Gamma = 0, T = 0.05$.

The only limit to these repetitions (harmonics) is decoherence caused by the scattering energy scale $\Gamma$ or temperature $T$, which causes a power law decay in the loop area distribution.\(^{22}\) Decoherence broadens the Landau levels into peaks with width $\Gamma, T$, which eventually merge into SdH oscillations. These oscillations have equal height instead of the height proportional to $N$ seen in Landau levels. They decay exponentially when the value of $B = N^{-1}A_F^{-1}$ descends to $\Gamma, T$, and their loop area distribution also decays exponentially at areas larger than $\Gamma^{-1}, T^{-1}$.\(^{24}\)
walks, all of which verify that the loop area distribution of random walks scales with $A^{-1}$. Since standard WL/WAL calculations start with random walks and arrive at $G_{xx}(A) \propto \ln B$, the Monte Carlo results shown here again verify that the Fourier transform of $\ln B$ is $G_{xx}(A) \propto A^{-1}$.

### C. Linear magnetoresistance

This result can be reversed and applied to the case of linear magnetoresistance, where $G_{xx}(B) \propto B^{-1}$. The loop area distribution is therefore logarithmic, $G_{xx}(A) \propto -\ln A$. This profile, whose broad features are general for any roughly linear resistance, is remarkably different from the $A^{-1}$ distribution that governs loops generated by random walks. It has fewer small loops and many more large loops; a small head and a very long tail. We emphasize that the logarithmic loop area distribution is a rigorous result and can be inferred whenever a linear magnetoresistance is seen, no matter what physical mechanism is responsible for producing the loops.

Both the logarithmic area distribution $-\ln A$ associated with linear magnetoresistance and the standard WAL result $1/A$ are monotonically decreasing functions of $A$, without any hint of oscillations. This contrasts very strongly with the area profiles of Landau levels and SdH oscillations. It is very compelling evidence that linear magnetoresistance is caused by Cooperons, i.e. paired particles and holes scattering together, rather than by any species of single-particle motion. The phase of single electrons and holes oscillates vary rapidly at the Fermi wavelength, which is the ultimate reason why the loop area distribution of Landau levels exhibits an oscillating sign. In contrast the Cooperon phase varies at the much longer localization length, resulting in a very smooth single-sign loop area distribution.

### D. Levy Flights and 3-D

The $1/A$ decay seen in 2-D random walks seems to be the slowest decay that can be produced by purely stochastic random walks. Increasing the dimensionality produces a steeper decay, $A^{-3/2}$ in 3-D. If instead diffusion is replaced by Levy flights where step lengths follow a distribution including both short and very long steps, the result is again a steeper decay than $A^{-1}$, because longer steps tend to decrease the probability that the walker’s path will complete a loop. Figure 2b shows that $G_{xx}(A) \propto A^{-2+\alpha/2}$, where $\alpha \leq 2$ is the Levy distribution stability parameter which controls the step lengths. Therefore neither changing the step distribution nor changing the dimensionality can reproduce the $1/A$ tail seen in linear magnetoresistance systems.

### III. WEAK ANTILocalIZATION WITH QUANTUM COHERENCE

Since linear magnetoresistance can not be explained by random walks or by Levy flights, we propose instead that the Cooperons responsible for linear magnetoresistance do follow random walks, but with the special feature that they maintain quantum coherence, allowing them to repeat their loops many times, in the same way that Landau levels are repeated in a hierarchy at $(N + \frac{1}{2})A_F$. This depletes the loop distribution’s head, because if a loop area distribution has a UV cutoff at $A_0$, then its $N = 2$ first repetition will have a higher UV cutoff at $2A_0$, its second repetition will have its cutoff at $3A_0$, etc. Therefore between $A_0$ and $2A_0$ the total loop area distribution will have a contribution from only the base $N = 1$ loop distribution, while at larger areas higher and higher $N$ will contribute. For example, consider the case of 2-D random walks with hard cutoffs at $A_0$ and $A_{\text{max}}$ and $G_{xx}(B,A_0, A_{\text{max}}) = \int_{A_0}^{A_{\text{max}}} dA \cos(qCBA)/A$. Allowing up to $N$ repetitions with inverse weighting and regularizing with $\gamma_E$ obtains $G_{xx}(B) = \gamma_E + \sum_{n=1}^{N} n^{-1} G_{xx}^{1}(B,NA_0, A_{\text{max}})$. Figure 3a compares this loop area distribution to the diffusive
returns to its starting point it does so with a momentum that is different from its original momentum, and therefore it is unable to repeat its first loop. There are notable exceptions, most notably in one dimension where both particle and hole are constrained to follow the same trajectory. The resulting Cooperon coherence causes 1-D systems to be almost universally localized. Moreover, it is well known that higher harmonics of the Cooperon occur in 2-D cylindrical geometries, resulting in Altshuler-Aronov-Spivak oscillations which deviate strongly from a simple cosine signal. However these exceptions are not relevant to linear magnetoresistance, which is generally observed in non-cylindrical (simply connected) 3-D or 2-D geometries.

In the 2-D and 3-D systems of interest here at least two avenues are available for protecting Cooperon coherence against scattering. In certain materials transport may be locally one-dimensional, with carriers constrained along certain race-track like trajectories. This is seen in snake states in graphene, in edge states in the quantum Hall effect, and might also be realized in $C_4$ symmetry-broken states in underdoped cuprates. When particle motion is locked to a locally one dimensional track, at any particular point the momentum is limited to only two values differing only in sign, and therefore the Cooperon’s position and momentum are locked to each other, up to the same sign. Therefore Cooperon phase coherence is protected in materials where carriers are constrained to move along locally one dimensional tracks. In conjunction with the fact that linear resistance is a kind of WAL not WL, this scenario suggests that topological physics could play an important role in linear resistance.

A second route to Cooperon coherence is to explicitly invoke strong correlations. For example, there has been much speculation that cuprates and pnictides above the superconducting transition host preformed pairs, precursors of superconducting Cooper pairs which at temperatures above $T_c$ maintain phase coherence up to a length scale $\lambda_\phi$, and which at lower temperatures unite into the superconducting condensate. The phase coherence of these preformed pairs may play a role in protecting the Cooperon’s phase coherence at scales similar to $\lambda_\phi$.

Linear magnetoresistance does not require the IR cutoff $A_{\text{max}}$, i.e. the area scale where phase coherence is suppressed, to be very large. For instance, a coherence area of $(100 \text{ Å})^2$ would allow linear magnetoresistance to extend down to $\approx 0.3 \text{T}$.

### B. Linear in temperature resistance

We turn to the linear-in-temperature resistance observed in bad metals. As we have seen, the loop area distribution has a smooth logarithmic form with only two characteristic area scales: the UV cutoff $A_0$, and the infrared cutoff $A_{\text{max}}$ which regulates large loops. The ultraviolet cutoff $A_0$ may be controlled by many length scales: the spin relaxation length with its crossover from...
WAL to WL, the scattering length, the lattice spacing, the scale of the Fermi surfaces, etc. These mechanisms depend weakly on temperature.

The main source of temperature dependence comes instead from the infrared cutoff $A_{\text{max}}$ which regulates large loops. The tail of the ln $A$ loop area distribution reflects quantum harmonics where Cooperons repeat the same loop many times, and these harmonics have areas far larger than those caused by pure diffusion. Regardless of which short distance physics controls the loop area distribution of the lowest Cooperon harmonic (i.e. without coherence-based repetitions), this physics has no control over the number of higher harmonics, and therefore cannot determine the infrared cutoff $A_{\text{max}}$ regulating large loops. The only process available to supply the cut off $A_{\text{max}}$ on the loop area distribution’s tail is quantum decoherence, which is controlled by temperature. In atomic units inverse temperature has units of area, and therefore the decoherence-based cutoff $A_{\text{max}} \propto T^{-1}$ scales inversely with temperature. We have already noted this behavior in the specific case of SnIH oscillations. This $A_{\text{max}} \propto T^{-1}$ scaling is however very robust, both because decoherence is the only process available to regulate the tail of ln $A$ distribution, and also because the loops in the tail are so large that $T^{-1}$ is the only area scale large enough match them.

It is then a simple matter to show that at $B = 0$ the resistance must be linear in temperature. Most simply, the linear magnetoresistance is $R_{xx}(B) = \pi q_e B A_0$. When $B = 0$ the inverse area $B^{-1}$ diverges. The only scale available to take the place of $B^{-1}$ is $T^{-1}$, and plugging this into the magnetoresistance formula immediately gives linear in temperature resistance $R_{xx}(B) = \pi q_e T A_0$.

Going into more detail, the logarithmic loop area distribution is $G_{xx}(A) \approx -(\pi A_0)^{-1} \ln |A|$, which should be regularized to give positive values. One possible regularization is to add $(\pi A_0)^{-1} \ln |A_{\text{max}}|$. Setting $B = 0$ and integrating with respect to area gives $R_{xx}(B = 0) \approx \pi A_0 / A_{\text{max}} \propto \pi A_0 T$ for $A_0 \ll A_{\text{max}} \propto T^{-1}$, plus regularization-dependent terms which are sensitive to the cutoffs. When $A_{\text{max}} \propto T^{-1}$ is comparable to $A_0$ the linear dependence on $T$ collapses. At smaller temperatures the linear in temperature resistance is robust because $B^{-1}$ and $T^{-1}$ both act at large length scales and are therefore roughly interchangeable.

C. Coefficients of the linear resistivity

We have shown that quantum coherence and loop repetitions lead to a resistance which scales linearly with both magnetic field and temperature as long as the quantum decoherence area $A_{\text{max}}$ is more than both the inverse field $(q_e B)^{-1}$ and the ultraviolet cutoff $A_0$. Within our theory the coefficient $\alpha \mu_B T$ of linear temperature dependence, the coefficient $\beta \mu_B B \approx \beta \mu_B A_{\text{max}}$ of linear field dependence, and their ratio $\gamma = \alpha / \beta$ are

$$\alpha = \pi A_0 / (A_{\text{max}} T)$$

$$\beta = \pi q_e \mu_B A_0$$

$$\gamma = q_e \mu_B / (A_{\text{max}} T)$$

where $\mu_B = 1/2$ and $q_e = 2$ are the Bohr magneton and the Cooperon charge.

Hayes et al measured $\gamma$ in the high-$T_c$ pnictide superconductor BaFe$_2$(As$_{1-x}$P$_x$)$_2$ at dopings ranging between 0.31 and 0.41, and found that $\gamma$ was identical to one in atomic units, within their experimental error bar of 7%. Further studies of the cuprate La$_{2-x}$Sr$_x$CuO$_4$ and of Yb$_{1-x}$La$_x$Rh$_2$Si$_2$ at various dopings near optimal doping have found constants between 0.7 and 2.3. We conclude that $\gamma$ does depend on the host material, but only mildly, and that it is of order 1.

The value of exactly one for $\gamma$ is easily explained by combining the Heisenberg uncertainty relation with a $p^2/2m$ dispersion: $B^{-1} = A = (\langle \Delta x \rangle^2)^{1/2}$, where the momentum scale is determined by $(p^2/2m_c) = T$ and the Cooperon mass is $m_c = 2m_e$. This produces $A_{\text{max}} \approx (4T)^{-1}$ and $A_{\text{max}} T = 1/4$. The Heisenberg relation used here is a quantum mechanical upper bound on the decoherence scale which can be obtained at a given temperature. These arguments indicate that $\gamma$’s physical meaning is simply the effective mass of the charge carriers, i.e. $\gamma = m/m_e$. The experimental observation of $\gamma = 1$, i.e. a $p^2/2m_e$ dispersion, implies that in the compounds studied by Refs. the electrons and holes which contribute to the Cooperon are itinerant carriers with bare electron mass $m_e$, and are insensitive to the ionic potential of their host material. When $\gamma = 1$ the coefficients $\alpha$ and $\beta$ of both the linear-in-temperature resistance and the linear magnetoresistance are direct measures of the UV cutoff $A_0$, with $R_{xx}(B = 0) = \alpha T \approx \pi A_0 / A_{\text{max}}$ and $R_{xx}(B) = \beta \mu_B B \approx 4\pi B A_0$. Where the carrier mass $m$ differs from the bare electron mass $m_e$, $R_{xx}(B = 0)$ will be multiplied by $m/m_e$.

As a case in point we determine the UV cutoff $A_0$’s scaling in several cuprates analyzed by Refs., which determined the linear coefficient $\alpha$ in La$_{2-x}$Sr$_x$CuO$_4$ (LSCO), YBa$_2$Cu$_3$O$_{6+\delta}$ (YBCO), Tl$_2$Ba$_2$CuO$_{6+\delta}$ (Tl2201), and HgBa$_2$CuO$_{4+\delta}$ (Hg21201). Ref. found that $\alpha$ is the same in all four compounds if one uses the sheet resistance per CuO$_2$ plane, not per unit cell. They also found that $\alpha$ is inversely proportional to the doping $p$ for $p \leq 0.20$, where $p$ is the number of holes per unit cell, not per CuO$_2$ plane. After conversion to atomic units one finds that $\alpha = \pi \times 64 A_0^2 \times p^{-1}$. We surmise on dimensional grounds that $\alpha$ should not depend on the doping, but instead on the 2-D carrier density $\rho_{2D}$. Using $p = \rho_{2D} A$, where $A \approx 53a_0^2$ is the cross-section of the unit cell in the cuprates’ copper oxide plane, we arrive at $R_{xx}(B = 0) = 0.30 \times \pi \times (\rho_{2D} A_{\text{max}})^{-1}$ and $R_{xx}(B) = 0.30 \times 4\pi \mu_B B \rho_{2D}^{-1} B$, where $A_{\text{max}} = (4T)^{-1}$. This suggests that in these compounds the UV cutoff $A_0$ of the loop area distribution is the inverse of the car-
carrier density, i.e. $A_0 = \rho_{2D}^{-1}$, and that the resistance includes a dimensionless normalization factor $\mathcal{N} = 0.30 \times \pi$. This normalization factor $\mathcal{N}$ may reflect details of the Cooperon coherence mechanism and the UV and IR cutoffs. It also reflects the loop area distribution prior to coherence-induced repetitions, where we chose the most obvious normalization: 1 in atomic units, resulting in $\mathcal{N} = \pi$. Rounding the experimentally determined normalization factor $\mathcal{N}$ from 0.30 to 1, with a 6% difference that is close to Ref.\textsuperscript{58}’s experimental error, obtains $R_{xx}(T) = (A_{max}(\rho_{2D})^{-1}$ and $R_{xx}(B) = 4\mu_B \times \rho_{2D}B$, where $A_{max} = (4T)^{-1/6}$. These formulas offer the possibility of determining the charge carrier density directly from either the linear magnetoresistance or the linear in temperature resistance, without any speculation about the compound’s chemistry or band structure.

D. The weak-field and small-temperature regimes

These two regimes are distinct. Time reversal symmetry, combined with the presence of a fairly sharp (faster than power law) infrared cutoff $A_{max}$, requires a $B^2$ behavior at small fields, i.e. $\rho = \rho_0 + (B/B_0)^2$. This was found in Ref.\textsuperscript{55} and confirmed by Ref.\textsuperscript{50}. In particular, Ref.\textsuperscript{55} verified that $\rho = \rho_0 + (\mu_B B/T)^2/2$ in BaFe\textsubscript{2}(As\textsubscript{1−x}P\textsubscript{x}) and found a smooth form which interpolates between small and large fields: $\rho_0 + \sqrt{\mu_B^2 B^2 + T^2}$. This form does not give a general description of the small T limit because finite $B$ does not impose a sharp cutoff on the integral $G_{xx}(B) = \int dA \exp(iAB)G_{xx}(A)$. Therefore the small T behavior may be sensitive to the tail of the loop area distribution $G_{xx}(A)$. In point of fact Ref.\textsuperscript{55} saw a quadratic form in BaFe\textsubscript{2}(As\textsubscript{1−x}P\textsubscript{x}) which indicates that in this material $G_{xx}(A)$ decays slowly compared to the experimental values of $B^{-1}$, while Ref.\textsuperscript{56} saw a linear form in La\textsubscript{2−x}Sr\textsubscript{x}CuO\textsubscript{4} indicating a sharper cutoff.

Systematic studies of the tail of $G_{xx}(A)$ throughout the high-$T_C$ phase diagram are likely to be very illuminating. Such studies require only an increased attention to the magnetoconductance’s sensitivity to small changes in $B$. Any sharp cutoff (faster than power law) at $A_{max}$ will manifest in the magnetoconductance as fast ripples which will be visible even at large fields. The ripples will be superimposed on the linear signal, with a profile qualitatively similar to that seen in Ref.\textsuperscript{57}. Using $A_{max} = (4T)^{-1}$, at $T = 0.1K$ the ripple period $2\pi/A_{max}$ will be 1.87 Tesla.

IV. FINAL COMMENTS

If, as we have suggested, WAL caused by interaction-induced scattering is responsible for the bad metals’ linear resistance, then translational symmetry is broken in these materials. As a consequence a diffuse (non-momentum-conserving) component should be seen in the single particle density of states and in excitation spectra such as ARPES, when analyzed as functions of momentum. This is in fact true in the cuprates and pnictides. In principle the scattering could be caused by anything that fluctuates at a time scale longer than the time $\tau = 4A$ required to go around a loop, which is three picoseconds for loops of area $A = (100 \text{ Å})^2$.

If the scattering source were truly static, then Universal Conductance Fluctuations should be seen in the magnetoconductance, and possibly individual phase coherent loops would be visible to STM experiments. We expect instead that the scattering is caused by interactions, so that UCFs will average to zero. In fact UCFs have not been reported in bad metals, although very few studies of the bad metals’ magnetoconductance have been performed. If UCFs cannot be found in bad metals, this would indicate that scattering is indeed caused by time dependent fluctuations, like the fluctuating and glassy signals that have been observed in Refs.\textsuperscript{51,52} using muon spin relaxation.

It is also worthwhile to point out that, if it is true that the linear magnetoresistance is caused by Cooperons and that the scattering source fluctuates with time, then magnetoresistance experiments on small nanorings fabricated from linear magnetoresistance materials and bad metals should observe that the carriers have charge $q_C = 2q_e$. In other words, Aharonov-Bohm oscillations will average to zero at the same time scale at which the scattering source fluctuates, while Altshuler-Aronov-Spivak oscillations will remain because they are robust against scattering.

Lastly we point out that, in our view, one of the most important aspects of the present work is its methodology, which focuses on geometric analysis of electron and hole loops, and especially on the loop area distribution that can be obtained from Fourier transforms of magnetoconductance data. We have presented a non-perturbative framework for understanding fermion behavior that is completely independent of any assumptions about Fermi liquid physics and gives additional physical insight. We anticipate experiments focusing on the loop area distribution, with special attention to accessing large areas using carefully controlled small increments of the field, to removing leads effects, and to performing careful Fourier transforms. We also expect increased use of vector magnets and multi-dimensional Fourier transforms.

Appendix A: Alternate Regularizations.

We demonstrate in this supporting information that reasonable regularizations of the Fourier transform of $x^{-1}$ produce a logarithm, and vice versa. In our first example
we perform the Fourier transform of $x^{-1}$.

$$A(\gamma, x_0, k) = \int_0^\infty dx \cos(\gamma x) \frac{1}{x + x_0}$$

$$= \text{Re}(\exp(x_0(\gamma + ik))\Gamma(0, x_0(\gamma + ik)))(A1)$$

$A(\gamma = 0, x_0 = 1, k)$ is approximately logarithmic in the range $x = [0, 0.1] \times 2\pi$, as can be verified by plotting $A(\gamma = 0, x_0 = 1, k)/\log(k)$.

Our second example again performs the Fourier transform of $x^{-1}$, but with a different UV cutoff.

$$B(\gamma, x_0, k) = \int_0^\infty dx \cos(\gamma x) \frac{1}{\sqrt{1 + x^2/x_0^2}}$$

$B(\gamma, x_0, k)$ can be integrated exactly, and includes Bessel, logarithmic, Struve, and hypergeometric functions. $B(\gamma = 0, x_0 = 1, k)$ is approximately logarithmic in the range $x = [0, 0.1] \times 2\pi$.

Our third example performs the Fourier transform of $\log|x|$.

$$C(\gamma, x_0, k, \nu) = \int_0^\infty dx \cos(\gamma x) \exp(\nu x) \log[1 + (x/\lambda)^2\Delta k]$$

This integral can be performed analytically for many values of $\nu = 1, 3/2, 5/3, 7/4, 19/10, 2$, and is always proportional to $k^{-1}$ plus corrections of order $k$ at small $k$ in the range $[0, 0.2]$. For $\nu = 3/2, 5/3, 7/4, 19/10$ it is written in terms of Meijer functions. The value of $k \times C(\gamma = 0, x_0 = 1, k, \nu) \Delta k$ at $k = 0$ is generally a rational fraction times $\pi$. For instance,

$$C(\gamma, x_0, k, \nu = 1) = \text{Re}(\omega^{-1} \exp(\omega x_0)\Gamma(0, \omega x_0))$$

$$\omega = -ik + \gamma. \quad (A4)$$

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In the systems of interest here the spin relaxation length is short and therefore the spin triplet component of the single particle density does not contribute to transport. When this is not the case it will be useful to extend the geometric analysis advocated here to a special relativistic formulation where electron/hole loops extend both in time and position (4 dimensions), loop areas $A_{\mu

53}$ have six components (three rotations and three boosts), the conjugate to area is the electromagnetic tensor $F^\mu\nu$ which contains both $\vec{B}$ and $\vec{E}$, and the physical observables are $O(\langle F^\mu\nu \rangle)$ and $O(\langle A_{\mu\nu} \rangle)$. This extension allows seamless treatment of the spin degree of freedom, and its $(2+1)$ dimensional version is natural for analysis of Hall bar experiments.

A slightly different result $\cos(A/N, A^\parallel)$ obtains for the $N$-th Landau level of Dirac fermions.$^{54}$

Up to the multiplicative constant, this formula $R \propto T/p$ is the same as Ref. 63's RVB result. See also Ref. 58 for discussion of the resistance per CuO$_2$ plane.

Ref. 59 has argued that the linear coefficient of the resistivity $dR/dT$ is proportional to $\lambda_T^2$ across a range of cuprate, pnictide, and heavy fermion materials, where $\lambda_T$ is the London penetration depth. Since both Ref. 60 and Ref. 61 analyze the same data on LSCO, this suggests a scaling relationship between $\lambda_T^2$ and the inverse carrier density $\rho_\parallel^{-1}$. 