TODA P-BRANE BLACK HOLES AND POLYNOMIALS RELATED TO LIE ALGEBRAS

V. D. Ivashchuk\textsuperscript{1} and V. N. Melnikov\textsuperscript{2},

Center for Gravitation and Fundamental Metrology, VNIIMS, 46 Ozyornaya ul.,
Moscow 119361, Russia

Institute of Gravitation and Cosmology, Peoples’ Friendship University of Russia, 6
Miklukho-Maklaya ul., Moscow 117198, Russia

Abstract

Black hole generalized $p$-brane solutions for a wide class of intersection rules are obtained. The solutions are defined on a manifold that contains a product of $n - 1$ Ricci-flat internal spaces. They are defined up to a set of functions $H_s$ obeying non-linear differential equations equivalent to Toda-type equations with certain boundary conditions imposed. A conjecture on polynomial structure of governing functions $H_s$ for intersections related to semisimple Lie algebras is suggested. This conjecture is proved for Lie algebras: $A_m$, $C_{m+1}$, $m \geq 1$. For simple Lie algebras the powers of polynomials coincide with the components of twice the dual Weyl vector in the basis of simple coroots. The coefficients of polynomials depend upon the extremality parameter $\mu > 0$. In the extremal case $\mu = 0$ such polynomials were considered previously by H. Lü, J. Maharana, S. Mukherji and C.N. Pope. Explicit formulas for $A_2$-solution are obtained. Two examples of $A_2$-dyon solutions, i.e. dyon in $D = 11$ supergravity with $M2$ and $M5$ branes intersecting at a point and Kaluza-Klein dyon, are considered.

*PACS numbers: 04.20.Jb, 04.50.+h, 04.70.Bw, 02.20.Sv, 02.30.Hq.

\textsuperscript{1}e-mail: ivashchuk@mail.ru
\textsuperscript{2}e-mail: melnikov@phys.msu.ru
1 Introduction

At present there exists an interest to the so-called $M$-theory (see, for example, [1]-[2]). This theory is “supermembrane” analogue of superstring models [3] in $D = 11$. The low-energy limit of $M$-theory after a dimensional reduction leads to models governed by a Lagrangian containing a metric, fields of forms and scalar fields. These models contain a large variety of the so-called $p$-brane solutions (see [4]-[49] and references therein).

In [30] it was shown that after the dimensional reduction on the manifold $M_0 \times M_1 \times \ldots \times M_n$ when the composite $p$-brane ansatz for fields of forms is considered the problem is reduced to the gravitating self-interacting $\sigma$-model with certain constraints imposed. (For electric $p$-branes see also [22, 23, 31].) This representation may be considered as a tool for obtaining different solutions with intersecting $p$-branes. In [30, 31, 41, 42, 43] the Majumdar-Papapetrou type solutions (see [52]) were obtained (for non-composite case see [22, 23]). These solutions corresponding to Ricci-flat factor-spaces $(M_i, g^i)$, $(g^i)$ is metric on $M_i$) $i = 1, \ldots, n$, were also generalized to the case of Einstein internal spaces [30]. Earlier some special classes of these solutions were considered in [16, 12, 14, 25, 26, 27]. The obtained solutions take place, when certain (block-)orthogonality relations (on couplings parameters, dimensions of "branes", total dimension) are imposed. In this situation a class of cosmological and spherically-symmetric solutions was obtained [36, 44, 46]. Special cases were also considered in [18, 32, 34, 35]. The solutions with the horizon were studied in details in [15, 28, 29, 36, 38, 41, 44, 47].

In models under consideration there exists a large variety of Toda-chain solutions, when certain intersection rules are satisfied [36]. Cosmological and spherically symmetric solutions with $p$-branes and $n$ internal spaces related to $A_m$ Toda chains were previously considered in [18, 19] and [45, 48]. Recently in [49] a family of $p$-brane solutions depending on one harmonic function with nearly arbitrary (up to some restrictions) intersection rules were obtained. These solutions are defined up to solutions of Laplace and Toda-type equations and correspond to null-geodesics of the sigma-model target-space metric.

Here we consider a family of spherically-symmetric and cosmological type solutions from [49] (see Sect. 2) and single out a new subclass of black-hole configurations related to Toda-type equations with certain asymptotical conditions imposed (Sect. 3). These black hole solutions are governed by functions $H_s(z) > 0$ defined on the interval $(0, (2\mu^{-1})(\mu > 0))$ and obeying a set of second order non-linear differential equations

$$\frac{d}{dz} \left( \frac{1 - 2\mu z}{H_s} \frac{d}{dz}H_s \right) = \hat{B}_s \prod_{s' \in S} H_{s'}^{-A_{ss'}}, \quad (1.1)$$

with the following boundary conditions imposed:

(i) $H_s((2\mu)^{-1} - 0) = H_{s0} \in (0, +\infty)$; \hspace{1cm} (1.2)

(ii) $H_s(+0) = 1$, \hspace{1cm} (1.3)

$s \in S$. In (1.1) $\hat{B}_s > 0$, $s \in S$, and $(A_{ss'})$ is a "quasi-Cartan" matrix ($A_{ss} = 2$, $s \in S$) coinciding with the Cartan one when intersections are related to Lie algebras. Equations (1.1) are equivalent to Toda-type equations (see Sect. 2).
For positively defined scalar field metric \((h_{\alpha\beta})\) all \(p\)-branes in this solution should contain a time manifold (see Proposition 1 from Sect. 3). This agrees with Theorem 3 from Ref. [44] (for orthogonal case, see also [38]).

In Sect. 4 we suggest a hypothesis: the functions \(H_s\) are polynomials when intersection rules correspond to semisimple Lie algebras. This hypothesis (Conjecture) is proved for Lie algebras: \(A_m, C_{m+1}, m = 1, 2, \ldots\). It is also confirmed by special black-hole “block orthogonal” solutions considered earlier in [39, 41, 47, 46]. An analogue of this conjecture for extremal black holes was considered earlier in [33]. In Sect. 5 explicit formulas for the solution corresponding to the algebra \(A_2\) are obtained. These formulas are illustrated by two examples of \(A_2\)-dyon solutions: a dyon in \(D = 11\) supergravity (with \(M2\) and \(M5\) branes intersecting at a point) and Kaluza-Klein dyon.

2 The model and Toda-type solutions

We consider a model governed by the action [30]

\[
S = \int d^Dx \sqrt{|g|} \left\{ R[g] - h_{\alpha\beta} g^{MN} \partial_M \varphi^\alpha \partial_N \varphi^\beta - \sum_{a \in \triangle} \frac{\theta_a}{n_a!} \exp[2\lambda_a(\varphi)] (F^a)^2 \right\} \tag{2.1}
\]

where \(g = g_{MN}(x)dx^M \otimes dx^N\) is a metric, \(\varphi = (\varphi^\alpha) \in \mathbb{R}^l\) is a vector of scalar fields, \((h_{\alpha\beta})\) is a constant symmetric non-degenerate \(l \times l\) matrix \((l \in \mathbb{N})\), \(\theta_a = \pm 1\),

\[
F^a = dA^a = \frac{1}{n_a!} F_{M_1...M_{na}}^a dz^{M_1} \wedge \ldots \wedge dz^{M_{na}} \tag{2.2}
\]

is a \(n_a\)-form \((n_a \geq 1)\), \(\lambda_a\) is a 1-form on \(\mathbb{R}^l\): \(\lambda_a(\varphi) = \lambda_{aa} \varphi^\alpha, a \in \triangle, \alpha = 1, \ldots, l\). In (2.1) we denote \(|g| = |\det(g_{MN})|\),

\[
(F^a)^2 = F_{M_1...M_{na}}^a F_{N_1...N_{na}}^a g^{M_1 N_1} \ldots g^{M_{na} N_{na}}, \tag{2.3}
\]

\(a \in \triangle\). Here \(\triangle\) is some finite set.

Let us consider a family of one-variable sector solutions to field equations corresponding to the action (2.1) and depending upon one variable \(u\) [49]. These solutions are defined on the manifold

\[
M = (u_-, u_+) \times M_1 \times M_2 \times \ldots \times M_n, \tag{2.4}
\]

where \((u_-, u_+)\) is an interval belonging to \(\mathbb{R}\). The solutions read [49]

\[
g = \left( \prod_{s \in S} [f_s(u)]^{2d^2_1 + (D-2)} \right) \left\{ [f_1(u)]^{2d_1/(1-d_1)} \exp(2\varepsilon^1 u + 2\bar{\varepsilon}^1) \right. \tag{2.5}
\]

\[
\times [wdu \otimes du + f_1^2(u)g^1] + \sum_{i=2}^{n} \left( \prod_{s \in S} [f_s(u)]^{-2h_s s_i} \right) \exp(2\varepsilon^i u + 2\bar{\varepsilon}^i) g^i \right\},
\]

\[
\exp(\varphi^\alpha) = \left( \prod_{s \in S} f_s^{h_s x^s \lambda_s^\alpha} \right) \exp(c^\alpha u + \bar{c}^\alpha), \tag{2.6}
\]

\[
F^a = \sum_{s \in S} \delta^a_{as} F^s, \tag{2.7}
\]
\[\alpha = 1, \ldots, l. \text{ In (2.5) } w = \pm 1, \ g^i = g^i_{m_n_l}(y_i)dy^m_i \otimes dy^n_i \text{ is a Ricci-flat metric on } M_i, \ i = 2, \ldots, n, \text{ the space } (g^1, M_1) \text{ is an Einstein space of non-zero curvature:}\]

\[R_{mn}[g^1] = \xi^1 g^1, \quad (2.8)\]

\[\xi^1 \neq 0, \text{ and}\]

\[\delta_{iI} = \sum_{j \in I} \delta_{ij} \quad (2.9)\]

is the indicator of \( i \) belonging to \( I: \delta_{iI} = 1 \) for \( i \in I \) and \( \delta_{iI} = 0 \) otherwise.

The \( p \)-brane set \( S \) is by definition

\[S = S_e \cup S_m, \quad S_v = \cup_{a \in \triangle} \{a\} \times \{v\} \times \Omega_{a,v}, \quad (2.10)\]

\( v = e, m \) and \( \Omega_{a,e}, \Omega_{a,m} \subset \Omega, \) where \( \Omega = \Omega(n) \) is the set of all non-empty subsets of \( \{2, \ldots, n\} \). Hence all \( p \)-branes do not “live” in \( M_1 \).

Any \( p \)-brane index \( s \in S \) has the form

\[s = (a_s, v_s, I_s), \quad (2.11)\]

where \( a_s \in \triangle, \ v_s = e, m \) and \( I_s \in \Omega_{a,v}. \) The sets \( S_e \) and \( S_m \) define electric and magnetic \( p \)-branes correspondingly. In (2.6)

\[\chi_s = +1, -1 \quad (2.12)\]

for \( s \in S_e, S_m \) respectively. In (2.7) forms

\[\mathcal{F}^s = Q_s \left( \prod_{s' \in S} f_{s'}^{-A_{s'}} \right) du \wedge \tau(I_s), \quad (2.13)\]

\( s \in S_e \), correspond to electric \( p \)-branes and forms

\[\mathcal{F}^s = Q_s \tau(\bar{I}_s), \quad (2.14)\]

correspond to magnetic \( p \)-branes; \( Q_s \neq 0, \ s \in S. \) In (2.14) and in what follows

\[\bar{I} \equiv \{1, \ldots, n\} \setminus I. \quad (2.15)\]

All the manifolds \( M_i, \ i > 1, \) are assumed to be oriented and connected and the volume \( d_i \)-forms

\[\tau_i \equiv \sqrt{|g^i(y_i)|} \ dy^1_i \wedge \ldots \wedge dy^d_i, \quad (2.16)\]

are well-defined for all \( i = 1, \ldots, n. \) Here \( d_i = \dim M_i, \ i = 1, \ldots, n \) (in spherically symmetric case \( M_1 = S^{d_1} ), \ d_1 > 1, \ D = 1 + \sum_{i=1}^n d_i, \) and for any \( I = \{i_1, \ldots, i_k\} \in \Omega, \ i_1 < \ldots < i_k, \) we denote

\[\tau(I) \equiv \tau_{i_1} \wedge \ldots \wedge \tau_{i_k}, \quad (2.17)\]

\[M_I \equiv M_{i_1} \times \ldots \times M_{i_k}, \quad (2.18)\]

\[d(I) \equiv \dim M_I = \sum_{i \in I} d_i. \quad (2.19)\]
The parameters $h_s$ appearing in the solution satisfy the relations
\[ h_s = K_s^{-1}, \quad K_s = B_{ss}, \]  \hspace{1cm} (2.20)
where
\[ B_{ss'} \equiv d(I_s \cap I_{s'}) + \frac{d(I_s)d(I_{s'})}{2 - D} + \chi_s\chi_{s'}\lambda_{\alpha\alpha_s}\lambda_{\beta\beta_{s'}}h^{\alpha\beta}, \]  \hspace{1cm} (2.21)
$s, s' \in S$, with $(h^{\alpha\beta}) = (h_{\alpha\beta})^{-1}$. Here we assume that
\[(i) \quad B_{ss} \neq 0, \]  \hspace{1cm} (2.22)
for all $s \in S$, and
\[(ii) \quad \det(B_{ss'}) \neq 0, \]  \hspace{1cm} (2.23)
i.e. the matrix $(B_{ss'})$ is a non-degenerate one. In (2.13) another non-degenerate matrix (“a quasi-Cartan” matrix) appears
\[ (A_{ss'}) = (2B_{ss'}/B_{s's'}). \]  \hspace{1cm} (2.24)
Here some ordering in $S$ is assumed.

This matrix also appears in the relations for
\[ f_s = \exp(-q^s), \]  \hspace{1cm} (2.25)
where $(q^s) = (q^s(u))$ is a solution to Toda-type equations
\[ \ddot{q}^s = -B_s\exp\left(\sum_{s' \in S} A_{ss'}q^{s'}\right), \]  \hspace{1cm} (2.26)
with
\[ B_s = 2K_sA_s, \quad A_s = \frac{1}{2}\varepsilon_sQ_s^2, \]  \hspace{1cm} (2.27)
$s \in S$. Here
\[ \varepsilon_s = (-\varepsilon[g])^{(1-\chi_s)/2}\varepsilon(I_s)\theta_{\alpha_s}, \]  \hspace{1cm} (2.28)
satisfy $\varepsilon[g] \equiv \text{sign} \det(g_{MN})$. More explicitly (2.28) reads: $\varepsilon_s = \varepsilon(I_s)\theta_{\alpha_s}$ for $v_s = e$ and $\varepsilon_s = -\varepsilon[g]\varepsilon(I_s)\theta_{\alpha_s}$, for $v_s = m$.

In (2.5)
\[ f_1(u) = R\sinh(\sqrt{C_1}u), \quad C_1 > 0, \quad \xi_1w > 0; \]  \hspace{1cm} (2.29)
\[ R\sin(\sqrt{|C_1|}u), \quad C_1 < 0, \quad \xi_1w > 0; \]  \hspace{1cm} (2.30)
\[ R\cosh(\sqrt{C_1}u), \quad C_1 > 0, \quad \xi_1w < 0; \]  \hspace{1cm} (2.31)
\[ |\xi_1(d_1 - 1)|^{1/2}u, \quad C_1 = 0, \quad \xi_1w > 0, \]  \hspace{1cm} (2.32)
where $C_1$ is constant and $R = |\xi_1(d_1 - 1)/C_1|^{1/2}$. 5
Vectors $c = (c^A) = (c^i, c^\alpha)$ and $\tilde{c} = (\tilde{c}^A)$ satisfy the linear constraints

$$U^s(c) = \sum_{i \in I_s} d_i c^i - \chi_s \lambda_{a,\alpha} c^\alpha = 0,$$

$$U^s(\tilde{c}) = \sum_{i \in I_s} d_i \tilde{c}^i - \chi_s \lambda_{a,\alpha} \tilde{c}^\alpha = 0,$$

$s \in S$,

$$U^1(c) = -c^1 + \sum_{j=1}^n d_j c^j = 0,$$

$$U^1(\tilde{c}) = -\tilde{c}^1 + \sum_{j=1}^n d_j \tilde{c}^j = 0,$$

and

$$C_1 \frac{d_1}{d_1 - 1} = 2E_{TL} + h_{\alpha \beta} c^\alpha c^\beta + \sum_{i=2}^n d_i (c^i)^2 + \frac{1}{d_1 - 1} \left( \sum_{i=2}^n d_i c^i \right)^2,$$

where

$$E_{TL} = \frac{1}{4} \sum_{s,s' \in S} h_s A_{ss'} q^s q^{s'} + \sum_{s \in S} A_s \exp \left( \sum_{s' \in S} A_{ss'} q^{s'} \right).$$

is an integration constant (energy) for the solutions from (2.26).

We note that the eqs. (2.26) correspond to the Toda-type Lagrangian

$$L_{TL} = \frac{1}{4} \sum_{s,s' \in S} h_s A_{ss'} q^s q^{s'} - \sum_{s \in S} A_s \exp \left( \sum_{s' \in S} A_{ss'} q^{s'} \right).$$

**Remark 1.** Here we identify notations for $g^i$ and $\hat{g}^j$, where $\hat{g}^j = p^*_i g^i$ is the pullback of the metric $g^i$ to the manifold $M$ by the canonical projection: $p_i : M \to M_i$, $i = 1, \ldots, n$. An analogous agreement will be also kept for volume forms etc.

Due to (2.13) and (2.14), the dimension of $p$-brane worldsheet $d(I_s)$ is defined by

$$d(I_s) = n_{a_s} - 1,$$

for $s \in S_e, S_m$, respectively. For a $p$-brane: $p = p_s = d(I_s) - 1$.

The solutions are valid if the following restrictions on the sets $\Omega_{a,v}$ are imposed. These restrictions guarantee the block-diagonal structure of the stress-energy tensor, like for the metric, and the existence of $\sigma$-model representation [23] (see also [25]). We denote $w_1 \equiv \{ i | i \in \{ 2, \ldots, n \}, \ d_i = 1 \}$, and $n_1 = |w_1|$ (i.e. $n_1$ is the number of 1-dimensional spaces among $M_i$, $i = 1, \ldots, n$).

**Restriction 1.** Let 1a) $n_1 \leq 1$ or 1b) $n_1 \geq 2$ and for any $a \in \Delta$, $v \in \{ e, m \}$, $i, j \in w_1$, $i < j$, there are no $I, J \in \Omega_{a,v}$ such that $i \in I$, $j \in J$ and $I \setminus \{ i \} = J \setminus \{ j \}$.

**Restriction 2.** Let 2a) $n_1 = 0$ or 2b) $n_1 \geq 1$ and for any $a \in \Delta$, $i \in w_1$ there are no $I \in \Omega_{a,m}$, $J \in \Omega_{a,e}$ such that $\tilde{I} = \{ i \} \cup J$.

These restrictions are satisfied in the non-composite case [22, 23]: $|\Omega_{a,e}| + |\Omega_{a,m}| = 1$, (i.e. when there are no two $p$-branes with the same color index $a$, $a \in \Delta$.) Restriction 1 and 2 forbid certain intersections of two $p$-branes with the same color index for $n_1 \geq 2$. 


and \( n_1 \geq 1 \) respectively. Restriction 2 is satisfied identically if all \( p \)-branes contain a common manifold \( M_j \) (say, time manifold).

This solution describes a set of charged (by forms) overlapping \( p \)-branes \((p_s = d(I_s) - 1, \ s \in S)\) “living” on submanifolds of \( M_2 \times \ldots \times M_n \).

### 2.1 \( U^s \)-vectors and scalar products

Here we consider a minisuperspace covariant form of constraints and corresponding scalar products that will be used in the next section. The linear constraints (2.33)-(2.36) may be written in the following form

\[
U^r(c) = U^r_A c^A = 0, \quad U^r(\bar{c}) = U^r_A \bar{c}^A = 0, \quad r = s, 1,
\]

where

\[
(U^s_A) = (d_0 \delta_{ii_s} - \chi_s \lambda_{aa_s}), \quad (U^1_A) = (-\delta^i_1 + d_i, 0),
\]

\( s = (a_s, v_s, I_s) \in S \), and

\( A = (i, \alpha) \).

The quadratic constraint (2.37) reads

\[
E = E_1 + E_{TL} + \frac{1}{2} \hat{G}_{AB} c^A c^B = 0,
\]

where \( C_1 = 2E_1(U^1, U^1) \),

\[
(U^1, U^1) = 1/d_1 - 1,
\]

\( (d_1 > 1) \) and

\[
(\hat{G}_{AB}) = \begin{pmatrix} G_{ij} & 0 \\ 0 & h_{\alpha\beta} \end{pmatrix},
\]

is the target space metric with

\[
G_{ij} = d_i \delta_{ij} - d_i d_j,
\]

\( i, j = 1, \ldots, n \). In (2.45) a scalar product appears

\[
(U, U') = \hat{G}^{AB} U_A U_B',
\]

where \( U = U_A z^A \), \( U' = U'_A z^A \) are linear functions on \( \mathbb{R}^{n+l} \), and \((\hat{G}^{AB}) = (\hat{G}_{AB})^{-1}\).

The scalar products (2.48) for co-vectors \( U^s \) from (2.42) were calculated in [30]

\[
(U^s, U^{s'}) = B_{ss'},
\]
s, s' ∈ S (see (2.21)). It follows from (2.23) and (2.49) that the vectors \( U^s, s ∈ S \), are linearly independent. Hence, the number of the vectors \( U^s \) should not exceed the dimension of the dual space \((\mathbb{R}^{n+l})^*\), i.e.

\[ |S| ≤ n + l. \quad (2.50) \]

We also get [30]

\[ (U^s, U^1) = 0, \quad (2.51) \]

for all \( s ∈ S \). This relation takes place, since all \( p \)-branes do not live in \( M_1: I_s ∈ \{2, \ldots, n\} \).

**Intersection rules.** From (2.20), (2.21) and (2.24) we get the intersection rules corresponding to the quasi-Cartan matrix \((A_{ss'})\) [36]

\[ d(I_s ∩ I_{s'}) = \frac{d(I_s)d(I_{s'})}{D - 2} - \chi_sχ_{s'}λ_{as}λ_{as'} + \frac{1}{2}K_{ss'}A_{ss'}, \quad (2.52) \]

where \( λ_{as}λ_{as'} = λ_{as}λ_{as'}h^{αβ}, s, s' ∈ S \).

The contravariant components \( U^rA = \hat{G}^{AB}U_B \) reads [30, 36]

\[ U^{si} = G^{ij}U^s_j = δ_{is} - \frac{d(I_s)}{D - 2}, \quad U^{sα} = -\chi_sλ^α_{as}, \quad (2.53) \]

\[ U^{1i} = -\frac{δ^i_1}{d_1}, \quad U^{1α} = 0, \quad (2.54) \]

\( s ∈ S \). Here (as in [53])

\[ G^{ij} = \frac{δ^{ij}}{d_i} + \frac{1}{2 - D}, \quad (2.55) \]

\( i, j = 1, \ldots, n \), are the components of the matrix inverse to \((G_{ij})\) from (2.47). The contravariant components (2.53) and (2.54) occur as powers in relations for the metric and scalar fields in (2.5) and (2.6).

We note that the solution under consideration for the special case of the \( A_m \) Toda chain was obtained earlier in [45]. Special \( A_1 ⊕ \ldots ⊕ A_1 \) Toda case, when vectors \( U^s \) are mutually orthogonal, was considered earlier in [36] (for non-composite case see also [32, 34, 38]). For a (general) block-orthogonal set of vectors \( U^s \) special solutions were considered in [39, 46].

### 3 Black holes solutions

#### 3.1 The choice of parameters

Here we consider the spherically symmetric case:

\[ w = 1, \quad M_1 = S^{d_1}, \quad g^1 = dΩ^2_{d_1}, \quad (3.1) \]

where \( dΩ^2_{d_1} \) is the canonical metric on a unit sphere \( S^{d_1}, d_1 ≥ 2 \). In this case \( ξ^1 = d_1 - 1 \). We also assume that

\[ M_2 = \mathbb{R}, \quad g^2 = -dt ⊗ dt, \quad (3.2) \]
i.e. $M_2$ is a time manifold.

We put $C_1 \geq 0$. In this case relations (2.29)-(2.32) read

$$f_1(u) = \frac{d \text{sh}(\sqrt{C_1} u)}{\sqrt{C_1}}, \quad C_1 > 0, \tag{3.3}$$

$$\bar{d} = d_1 - 1. \tag{3.5}$$

Here and in what follows

$$C_1 \geq 0.$$

Let us consider the null-geodesic equations for the light “moving” in the radial direction (following from $ds^2 = 0$):

$$\frac{dt}{du} = \pm \Phi, \tag{3.6}$$

$$\Phi = \int_1^{d_1/(1-d_1)} e^{(c^1-c^2)u + \bar{c}^1 - \bar{c}^2} \prod_{s \in S} f_s^{h_s \delta_{2s}}, \tag{3.7}$$

equivalent to

$$t - t_0 = \pm \int_{u_0}^u d\bar{u} \Phi(\bar{u}), \tag{3.8}$$

where $t_0, u_0$ are constants.

Let us consider solutions (defined on some interval $[u_0, +\infty)$) with a horizon at $u = +\infty$ satisfying

$$\int_{u_0}^{+\infty} du \Phi(u) = +\infty. \tag{3.9}$$

Here we restrict ourselves to solutions with $C_1 > 0$ and linear asymptotics at infinity

$$q^s = -\beta^s u + \bar{\beta}^s + o(1), \tag{3.10}$$

$u \to +\infty$, where $\beta^s, \bar{\beta}^s$ are constants, $s \in S$. This relation gives us an asymptotical solution to Toda type eqs. (2.26) if

$$\sum_{s' \in S} A_{ss'} \beta^{s'} > 0, \tag{3.11}$$

for all $s \in S$. In this case the energy (2.38) reads

$$E_{TL} = \frac{1}{4} \sum_{s, s' \in S} h_s A_{ss'} \beta^s \beta^{s'}. \tag{3.12}$$

**Remark 2.** For positive-definite matrices $(h_s A_{ss'})$ and $(h_{\alpha\beta})$ we get from (2.37) and (3.12): $E_{TL} \geq 0$, $C_1 \geq 0$. (For the extremal case $E_{TL} = C_1 = 0$ see Sect. 7.) According to Lemma 2 from [44] black hole solutions can only exist for $C_1 \geq 0$ and the horizon is then at $u = \infty$.

For the function (3.7) we get

$$\Phi(u) \sim \Phi_0 e^{\beta u}, \quad u \to +\infty, \tag{3.13}$$
where $\Phi_0 \neq 0$ is constant,
\begin{equation}
\beta = c^1 - c^2 + \sqrt{C_1}h_1 + \sum_{s \in S} \beta_s b_s \delta_{2I_s},
\end{equation}
and
\begin{equation}
h_1 = (U^1, U^1)^{-1} = \frac{d_1}{1 - d_1}.
\end{equation}
Horizon at $u = +\infty$ takes place if and only if
\begin{equation}
\beta \geq 0.
\end{equation}
Let us introduce dimensionless parameters
\begin{equation}
b^s = \beta^s / \sqrt{C_1}, \quad b^A = c^A / \sqrt{C_1},
\end{equation}
where $s \in S$, $A = (i, \alpha)$, $C_1 > 0$.
Thus, a horizon at $u = +\infty$ corresponds to a point $b = (b^s, b^A) \in \mathbb{R}^{|S|+n+l}$ satisfying the relations following from (2.41), (2.44), (3.11), (3.12) and (3.14)-(3.17):
\begin{equation}
U^r_A b^A = 0, \quad r = s, 1; \ s \in S,
\end{equation}
\begin{equation}
\frac{1}{2} \sum_{s,s' \in S} h_s A_{ss'} b^s b^{s'} + \hat{G}_{AB} b^A b^B = |h_1|,
\end{equation}
\begin{equation}
\sum_{s \in S} A_{ss'} b^{s'} > 0,
\end{equation}
\begin{equation}
f(b) \equiv b^1 - b^2 + \sum_{s \in S} b_s h_s \delta_{2I_s} \geq |h_1|.
\end{equation}

**Proposition 1.** Let matrix $(h_{\alpha\beta})$ be positively defined. Then the point $b = (b^s, b^A)$ satisfying relations (3.18)-(3.21) exists only if
\begin{equation}
2 \in I_s, \quad \forall s \in S,
\end{equation}
(i. e. all $p$-branes have a common time direction $t$) and is unique: $b = b_0$, where
\begin{equation}
b^A_0 = -\delta^A_2 + h_1 U^{1A} + \sum_{s \in S} h_s b^s_0 U^{sA},
\end{equation}
\begin{equation}
b^s_0 = 2 \sum_{s' \in S} A^{ss'},
\end{equation}
where $s \in S$, $A = (i, \alpha)$, and the matrix $(A^{ss'})$ is inverse to the matrix $(A_{ss'}) = (2(U^s, U^{s'})/(U^s, U^{s'}))$.

**Proof.** Let $\mathcal{E}$ be a manifold described by relations (3.18)-(3.19). This manifold is an ellipsoid. Indeed, due to positively definiteness of $(h_{\alpha\beta})$ the matrix $\hat{G}_{AB}$ has a signature $(-, +, \ldots, +)$, since the matrix $(G_{ij})$ from (2.47) has a signature $(-, +, \ldots, +)$ [53]. Due to relations $(U^1, U^1) < 0$, $(U^1, U^s) = 0$, $(U^s, U^s) \neq 0$ for all $s \in S$, and (2.23) the matrices $(B_{ss'})$ and $(A_{ss'})$ are positively defined and all $h_s > 0$, $s \in S$. Then, the quadratic form in (3.19) has a pseudo-Euclidean signature. Due to $(U^1, U^1) < 0$ the
intersection of the hyperboloid (3.19) with the (multidimensional) plane \( U_A^1 z^A = 0 \) gives us an ellipsoid. Its intersection with the planes \( U_A^s z^A = 0, \ s \in S \), gives us to an ellipsoid, coinciding with \( \mathcal{E} \).

Let us consider a function \( f : \mathcal{E} \to \mathbb{R} \) that is a restriction of the linear function (3.21) on \( \mathcal{E} \). Let \( b_* \in \mathcal{E} \) be a point of maximum of \( f \). Using the conditional extremum method and the fact that \( \mathcal{E} \) is ellipsoid we prove that

\[
\begin{align*}
\frac{d}{dA} &= 0, \\
\sum_{s \in S} h_s b_*^s U^s A &= 0, \\
\sum_{s \in S} h_s b_*^s U^s &= 0.
\end{align*}
\]

Let us consider a function \( f|_{E} \) that is a restriction of the linear function (3.21) on \( E \). Let \( b_* \in E \) be a point of maximum of \( f|_{E} \). Using the conditional extremum method and the fact that \( E \) is ellipsoid we prove that

\[
\begin{align*}
b_*^A &= -\delta_2 A + h_1 U^1 A + \sum_{s \in S} h_s b_*^s U^s A, \\
b_*^s &= 2 \sum_{s' \in S} A_{ss'} \delta_{2I_s},
\end{align*}
\]

\( s \in S, \ A = (i, \alpha) \). Let us consider the function

\[
\tilde{f}(b, \lambda) \equiv f(b) - \lambda_1 U^1 A b^A - \sum_{s \in S} \lambda_s U^s A b^A - \lambda_0 \left( \sum_{s, s' \in S} \frac{h_s}{2} A_{ss'} b^s b^{s'} + \hat{G}_{AB} b^A b^B + h_1 \right),
\]

where \( \lambda = (\lambda_0, \lambda_1, \lambda_s) \) is a vector of Lagrange multipliers. The points of extremum for the function \( \tilde{f} \) from (3.27) have the form \( (\lambda_0 b_*, \lambda) \) with \( b_* \) from (3.25) and

\[
\begin{align*}
\lambda_0 &= \pm 1, \\
\lambda_1 &= 1/(d_1 - 1), \\
\lambda_s &= -2 \sum_{s' \in S} h_s A_{ss'} \delta_{2I_s}.
\end{align*}
\]

\( s \in S \). Then, the points \( b_* \) and \( -b_* \) are the points of maximum and minimum, respectively, for the function \( f|_{E} \) defined on the ellipsoid \( \mathcal{E} \). Since \( f(b_*) = |h_1| \), the only point satisfying the restriction \( f(b) \geq |h_1| \) is \( b = b_* \). From (3.20) we get

\[
\sum_{s' \in S} A_{ss'} b^{s'} = 2 \delta_{2I_s} > 0 \iff 2 \in I_s,
\]

for all \( s \in S \). The proposition is proved.

We introduce a new radial variable \( R = R(u) \) by relations

\[
\exp(-2\bar{\mu} u) = 1 - \frac{2\mu}{R^d} = F, \quad \bar{\mu} = \sqrt{C_1}, \quad \mu = \frac{\bar{\mu}}{\bar{d}} > 0,
\]

\( u > 0, \ R^d > 2\mu \ (\bar{d} = d_1 - 1) \). We put

\[
\begin{align*}
\hat{e}^A &= 0, \\
q^s(0) &= 0.
\end{align*}
\]

\( A = (i, \alpha), \ s \in S \). These relations guarantee the asymptotical flatness (for \( R \to +\infty \)) of the \((2 + d_1)\)-dimensional section of the metric.

Let us denote

\[
H_s = f_s e^{-\bar{\mu} b^s_0 u},
\]
s \in S$. Then, solutions (2.5)-(2.7) may be written as follows

\[ g = \left( \prod_{s \in S} H_s^{2h_s/d(I_s)/(D-2)} \right) \left\{ F^{-1} dR \otimes dR + R^2 dQ^2_{d_1} \right\} - \left( \prod_{s \in S} H_s^{-2h_s} \right) F dR \otimes dR + \sum_{i=3}^{n} \left( \prod_{s \in S} H_s^{-2h_s A_i s} \right) g^i, \]

\[ \exp(\varphi^a) = \prod_{s \in S} H_s^{h_s s} \chi^a s, \]

\[ F^a = \sum_{s \in S} \delta^a_{a_s} \mathcal{F}^s, \]

where

\[ \mathcal{F}^s = - \frac{Q_s}{\bar{R}^d_{d_1}} \left( \prod_{s' \in S} H_s'^{-A_{ss'}} \right) dR \wedge \tau(I_s), \]

\[ \mathcal{F}^s = Q_s \tau(I_s), \]

\[ s \in S_e, \]

\[ s \in S_m. \] Here \( Q_s \neq 0, \ h_s = K_s^{-1}; \) parameters \( K_s \neq 0 \) and the non-degenerate matrix \( (A_{ss'}) \) are defined by relations (2.52) and \( (A_{ss}) = 2, \ s \in S. \)

Functions \( H_s > 0 \) obey the equations

\[ R^{d_1} \frac{d}{dR} \left( R^{d_1} \frac{d H_s}{dR} \right) = B_s \prod_{s' \in S} H_{s'}^{-A_{ss'}}, \]

where \( B_s \neq 0 \) are defined in (2.27) and (2.28). These equations follow from Toda-type equations (2.26) and the definition (3.30) and (3.33).

It follows from (3.10), (3.17) and (3.33) that there exist finite limits

\[ H_s \rightarrow H_{s0} \neq 0, \]

for \( R^d \rightarrow 2\mu, \ s \in S. \) We note, that in this case the metric (3.34) does really have a horizon at \( R^d = 2\mu. \)

From (3.32) we get.

\[ H_s(R = +\infty) = 1, \]

\[ s \in S. \]

The metric (3.34) has a regular horizon at \( R^d = 2\mu. \) The Hawking temperature corresponding to the solution is (see also [29, 38] for orthogonal case) found to be

\[ T_H = \frac{\bar{d}}{4\pi(2\mu)^{1/d}} \prod_{s \in S} H_{s0}^{-h_s}, \]

where \( H_{s0} \) are defined in (3.40).

The boundary conditions (3.40) and (3.41) play a crucial role here, since they single out, generally speaking, only few solutions to eqs. (3.39).
Moreover for some values of parameters $\mu = \bar{\mu}/\bar{d}$, $\varepsilon_s$ and $Q_s^2$ the solutions to eqs. (3.39)-(3.41) do not exist. Indeed, from (2.27), (2.38), (3.12), (3.17), (3.24), (3.30) and (3.32) we get

$$E_{TL} = \bar{\mu}^2 \sum_{s,s' \in S} h_s A_{ss'} = \frac{1}{4} \sum_{s,s' \in S} h_s A_{ss'} \dot{q}^s(0) \dot{q}^s(0) + \sum_{s \in S} \frac{1}{2} \varepsilon_s Q_s^2. \quad (3.43)$$

Let the matrix $(h_s A_{ss'})$ be positive-definite (in this case matrix $(B_{ss'})$ is positive-definite too and all $h_s > 0$). Then $E_{TL} > 0$ and

$$\bar{\mu}^2 \sum_{s,s' \in S} h_s A_{ss'} \geq \sum_{s \in S} \frac{1}{2} \varepsilon_s Q_s^2. \quad (3.44)$$

If the parameters obey the relation

$$0 < \bar{\mu}^2 \sum_{s,s' \in S} h_s A_{ss'} < \sum_{s \in S} \frac{1}{2} \varepsilon_s Q_s^2, \quad (3.45)$$

e.g. for $\varepsilon_s = +1$ and big enough $Q_s^2$, the solution under consideration does not exist.

We note that the solution to eqs. (3.39)-(3.41) may not be unique. The simplest example occurs in the case of one $p$-brane, when $h_s > 0$, $\varepsilon_s = +1$ and $\bar{\mu}^2 h_s > Q_s^2$. In this case we have two solutions to (3.39)-(3.41) corresponding to two possible values of $\dot{q}^s(0)$.

**Hypothesis.** For positive-definite matrix $(h_s A_{ss'})$ and $\varepsilon_s = -1$, $s \in S$, the solution to (3.39)-(3.41) is uniquely defined.

This hypothesis will be a subject of a future investigation. It implies a ”no-hair theorem” for black hole solutions under consideration.

Thus, we obtained a family of black hole solutions up to solutions of radial equations (3.39) with the boundary conditions (3.40) and (3.41). In the next sections we consider several exact solutions to eqs. (3.39)-(3.41).

**Remark 3.** Let $M_i = \mathbb{R}$ and $g^i = -d\bar{t} \otimes d\bar{t}$ for some $i \geq 3$. Then the metric (3.34) has no a horizon with respect to the “second time” $\bar{t}$ for $R^d \to 2\mu$. Thus, we a led to a “single-time” theorem from [44]. Relation (3.22) from Proposition 1 coincides with the “no-hair” theorem from [44].

## 4 Polynomial structure of $H_s$ for Lie algebras

### 4.1 Conjecture on polynomial structure

Now we deal with solutions to second order non-linear differential equations (3.39) that may be rewritten as follows

$$\frac{d}{dz} \left( \frac{F}{H_s} \frac{d}{dz} H_s \right) = \bar{B}_s \prod_{s' \in S} H_s^{-A_{ss'}}. \quad (4.1)$$
where \( H_s(z) > 0 \), \( F = 1 - 2\mu z \), \( \mu > 0 \), \( z = R^{-d} \), \( \bar{B}_s = B_s/\bar{d}^2 \neq 0 \). Eqs. (3.41) and (3.40) read
\[
H_s((2\mu)^{-1} - 0) = H_{s0} \in (0, +\infty), \quad (4.2)
\]
\[
H_s(+0) = 1, \quad (4.3)
\]
\( s \in S \). (Here we repeat equations (1.1)-(1.3)) for convenience.)

It seems rather difficult to find the solutions to a set of eqs. (4.1)-(4.3) for arbitrary values of parameters \( \mu \), \( \bar{B}_s \), \( s \in S \) and quasi-Cartan matrices \( A = (A_{ss'}) \). But we may expect a drastically simplification of the problem under consideration for certain class of parameters and/or \( A \)-matrices.

In general we may try to seek solutions of (4.1) in a class of functions analytical in a disc \(|z| < L\) and continuous in semi-interval \( 0 < z \leq (2\mu)^{-1} \). For \(|z| < L\) we get
\[
H_s(z) = 1 + \sum_{k=1}^{\infty} P_s^{(k)} z^k, \quad (4.4)
\]
where \( P_s^{(k)} \) are constants, \( s \in S \). Substitution of (4.4) into (4.1) gives us an infinite chain of relations on parameters \( P_s^{(k)} \) and \( \bar{B}_s \). In general case it seems to be impossible to solve this chain of equations.

Meanwhile there exist solutions to eqs. (4.1)-(4.3) of polynomial type. The simplest example occurs in orthogonal case [15, 28, 29, 36, 38], when
\[
(U_s, U_{s'}) = B_{ss'} = 0, \quad (4.5)
\]
for \( s \neq s' \), \( s, s' \in S \). In this case \( (A_{ss'}) = \text{diag}(2, \ldots, 2) \) is a Cartan matrix for semisimple Lie algebra \( A_1 \oplus \cdots \oplus A_1 \) and
\[
H_s(z) = 1 + P_s z, \quad (4.6)
\]
with \( P_s \neq 0 \), satisfying
\[
P_s(P_s + 2\mu) = -\bar{B}_s, \quad (4.7)
\]
\( s \in S \).

In [39, 46, 47] this solution was generalized to a block orthogonal case:
\[
S = S_1 \cup \ldots \cup S_k, \quad S_i \cap S_j = \emptyset, \quad i \neq j, \quad (4.8)
\]
\( S_i \neq \emptyset \), i.e. the set \( S \) is a union of \( k \) non-intersecting (non-empty) subsets \( S_1, \ldots, S_k \), and
\[
(U_s, U_{s'}) = 0 \quad (4.9)
\]
for all \( s \in S_i \), \( s' \in S_j \), \( i \neq j \); \( i, j = 1, \ldots, k \). In this case (4.6) is modified as follows
\[
H_s(z) = (1 + P_s z)^{b_0^s}, \quad (4.10)
\]
where \( b_0^s \) are defined in (3.24) and parameters \( P_s \) are coinciding inside blocks, i.e. \( P_s = P_{s'} \) for \( s, s' \in S_i \), \( i = 1, \ldots, k \). Parameters \( P_s \neq 0 \) satisfy the relations
\[ P_s(P_s + 2\mu) = -B_s/b^*_0, \]

\[ b^*_0 \neq 0, \] and parameters \( B_s/b^*_0 \) are also coinciding inside blocks, i.e. \( B_s/b^*_0 = B_{s'}/b^*_0 \)

for \( s, s' \in S_i, \ i = 1, \ldots, k \). In this case \( H_s \) are analytical in \( |z| < L \), where \( L = \min(|P_s|^{-1}, s \in S) \).

Let \( (A_{ss'}) \) be a Cartan matrix for a finite-dimensional semisimple Lie algebra \( G \). In this case all powers in (3.24) are natural numbers [43]

\[ b^*_0 = 2 \sum_{s' \in S} A^{ss'} = n_s \in \mathbb{N}, \quad (4.11) \]

and hence, all functions \( H_s \) are polynomials, \( s \in S \).

Integers \( n_s \) coincide with the components of twice the dual Weyl vector in the basis of simple coroots (see Sect. 13.7 in [50]).

**Conjecture.** Let \( (A_{ss'}) \) be a Cartan matrix for a semisimple finite-dimensional Lie algebra \( G \). Then the solution to eqs. (4.1)-(4.3) (if exists) is a polynomial

\[ H_s(z) = 1 + \sum_{k=1}^{n_s} P_s^{(k)} z^k, \quad (4.12) \]

where \( P_s^{(k)} \) are constants, \( k = 1, \ldots, n_s \), integers \( n_s = b^*_0 \) are defined in (4.11) and \( P_s^{(n_s)} \neq 0, \ s \in S \).

In extremal case (\( \mu = +0 \)) an analogue of this conjecture was suggested previously in [33].

### 4.2 Proof of Conjecture for \( A_m \) and \( C_{m+1} \)

First, we prove the **Conjecture** for simple Lie algebras \( A_m = sl(m+1), \ m \geq 1 \). Let us consider exact solutions to equations of motion of a Toda-chain corresponding to the Lie algebra \( A_m \) [54, 55],

\[ \dot{q}^s = -B_s \exp \left( \sum_{s'=1}^{m} A_{ss'} q^{s'} \right), \quad (4.13) \]

where

\[
(A_{ss'}) = \begin{pmatrix}
 2 & -1 & 0 & \ldots & 0 & 0 \\
 -1 & 2 & -1 & \ldots & 0 & 0 \\
 0 & -1 & 2 & \ldots & 0 & 0 \\
 \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
 0 & 0 & 0 & \ldots & 2 & -1 \\
 0 & 0 & 0 & \ldots & -1 & 2 \\
\end{pmatrix}
\]

is the Cartan matrix of the Lie algebra \( A_m \) and \( B_s > 0, \ s, s' = 1, \ldots, m \). Here we put \( S = \{1, \ldots, m\} \).

The equations of motion (4.13) correspond to the Lagrangian

\[ L_T = \frac{1}{2} \sum_{s,s'=1}^{m} A_{ss'} q^s \dot{q}^{s'} - \sum_{s=1}^{m} B_s \exp \left( \sum_{s'=1}^{m} A_{ss'} q^{s'} \right). \quad (4.15) \]
This Lagrangian may be obtained from the standard one [54] by separating a coordinate describing the motion of the center of mass.

Using the result of A. Anderson [55] we present the solution to eqs. (4.13) in the following form

\[
C_s \exp(-q^s(u)) = \sum_{r_1<\ldots<r_s} v_{r_1} \cdots v_{r_s} \Delta^2(w_{r_1}, \ldots, w_{r_s}) \exp[(w_{r_1} + \ldots + w_{r_s})u],
\]

\(s = 1, \ldots, m,\)

where

\[
\Delta(w_{r_1}, \ldots, w_{r_s}) = \prod_{i<j} (w_{r_i} - w_{r_j}); \quad \Delta(w_{r_1}) \equiv 1,
\]

denotes the Vandermonde determinant. The real constants \(v_r\) and \(w_r, r = 1, \ldots, m+1,\) obey the relations

\[
\prod_{r=1}^{m+1} v_r = \Delta^{-2}(w_1, \ldots, w_{m+1}), \quad \sum_{r=1}^{m+1} w_r = 0.
\]

In (4.16)

\[
C_s = \prod_{s'=1}^{m} B_{s's'}^{-A^{ss'}},
\]

where

\[
A^{ss'} = \frac{1}{m+1} \min(s, s')[m+1 - \max(s, s')],
\]

\(s, s' = 1, \ldots, m,\) are components of a matrix inverse to the Cartan one, i.e. \((A^{ss'}) = (A_{ss'})^{-1}\) (see Sect. 7.5 in [50]).

Here

\[
v_r \neq 0, \quad w_r \neq w_{r'}, \quad r \neq r',
\]

\(r, r' = 1, \ldots, m+1.\) We note that the solution with \(B_s > 0\) may be obtained from the solution with \(B_s = 1\) (see [55]) by a certain shift \(q^s \mapsto q^s + \delta^s.\)

The energy reads [55]

\[
E_T = \frac{1}{2} \sum_{s,s'=1}^{m} A_{ss'} q^s q^{s'} + \sum_{s=1}^{m} B_s \exp \left( \sum_{s'=1}^{m} A_{ss'} q^{s'} \right) = \frac{1}{2} \sum_{r=1}^{m+1} w_r^2.
\]

If \(B_s > 0, s \in S,\) then all \(w_r, v_r\) are real and, moreover, all \(v_r > 0, r = 1, \ldots, m+1.\) In a general case \(B_s \neq 0, s \in S,\) relations (4.16)-(4.19) also describe real solutions to eqs. (4.13) for suitably chosen complex parameters \(v_r\) and \(w_r.\) These parameters are either real or belong to pairs of complex conjugate (non-equal) numbers, i.e., for example, \(w_1 = \bar{w}_2, v_1 = \bar{v}_2.\) When some of \(B_s\) are negative, there are also some special (degenerate) solutions to eqs. (4.13) that are not described by relations (4.16)-(4.19), but may be obtained from the latter by certain limits of parameters \(w_r.\)
For the energy (2.38) we get
\[
E_{TL} = \frac{1}{2K}E_T = \frac{h}{4} \sum_{r=1}^{m+1} w_r^2. \tag{4.23}
\]
Here
\[
K_s = K, \quad h_s = h = K^{-1}, \tag{4.24}
\]
s \in S. Thus, in the \(A_m\) Toda chain case eqs. (4.16)-(4.24) should be substituted into relations (2.25) and (2.37).

Now we consider \(A_m\)-solutions with asymptotics (3.10). In this case all \(w_1, \ldots, w_{m+1}\) are real and without loss of generality \(w_1 < \ldots < w_{m+1}\). For \(b_0^s = n_s\) from (3.24) we get
\[
n_s = b_0^s = s(m - s + 1), \tag{4.25}
\]
s = 1, \ldots, m, or explicitly
\[
b_1 = m, \quad b_0^2 = 2(m - 1), \ldots, b_0^m = m. \tag{4.26}
\]
From (3.10), (3.17), \(\bar{\mu} = \sqrt{C_1}\) and (4.16) we get (\(w_1 < \ldots < w_{m+1}\))
\[
\bar{\mu}b_1^1 = \bar{\mu}m = w_{m+1}, \tag{4.27}
\]
\[
\bar{\mu}b_0^2 = 2\bar{\mu}(m - 1) = w_m + w_{m+1}, \tag{4.28}
\]
\[
\ldots
\]
\[
\bar{\mu}b_0^m = \bar{\mu}m = w_2 + \ldots + w_{m+1}. \tag{4.29}
\]
These relations imply
\[
w_{m+1} = \bar{\mu}m, \quad w_m = \bar{\mu}(m - 2), \ldots, w_1 = -\bar{\mu}m, \tag{4.30}
\]
or,
\[
w_j = (2j - m - 2)\bar{\mu}, \tag{4.31}
\]
j = 1, \ldots, m + 1. From (4.16) and (4.30) we get
\[
f_s = e^{-q^s} = \alpha_s^{(0)} e^{n_s\bar{\mu}m} + \alpha_s^{(1)} e^{(n_s-2)\bar{\mu}m} + \ldots + \alpha_s^{(n_s)} e^{-n_s\bar{\mu}m}, \tag{4.32}
\]
where \(\alpha_s^{(k)}\) are constants, \(k = 1, \ldots, n_s\), \(\alpha_s^{(n_s)} \neq 0\). Hence, due to (3.30), (3.33) we obtain the relations
\[
H_s = e^{-q^s - n_s\bar{\mu}m} = \alpha_s^{(0)} + \alpha_s^{(1)} F + \ldots + \alpha_s^{(n_s)} F^{n_s}, \tag{4.33}
\]
equivalent to (4.12) (\(\alpha_s^{(0)} + \alpha_s^{(1)} + \ldots + \alpha_s^{(n_s)} = 1\)) with \(\alpha_s^{(n_s)} = P_s^{(n_s)} \neq 0\), s = 1, \ldots, m. Thus, the Conjecture is proved for the Lie algebras \(G = A_m\), \(m \geq 1\).

Now we prove the Conjecture for simple Lie algebras \(C_{m+1} = sp(m + 1), m \geq 1\). (Remind that for \(m = 1\): \(C_2 = B_2 = so(5)\)). The Cartan matrix for the Lie algebra \(C_{m+1} (m \geq 1)\) reads
\[
(A_{ss'}) = \begin{pmatrix}
2 & -2 & 0 & \ldots & 0 & 0 \\
-1 & 2 & -1 & \ldots & 0 & 0 \\
0 & -1 & 2 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 2 & -1 \\
0 & 0 & 0 & \ldots & -1 & 2
\end{pmatrix} \tag{4.34}
\]
s, s' = 0, ..., m. The set of equations (4.1) with the Cartan matrix (4.34) and s = 0, ..., m, may be embedded into a set of equations (4.13) corresponding to the Cartan matrix of the Lie algebra $A_{2m+1}$ (see (4.14)) with $s = -m, ..., 0, ..., m$, if the following identifications: $\bar{B}_{-k} = \bar{B}_k$ and $H_{-k} = H_k$, $k = 1, ..., m$, are adopted. This proves the Conjecture for $C_{m+1}$, since it was proved for $A_{2m+1}$.

5 Some examples

5.1 Solution for $A_2$

Here we consider some examples of solutions related to the Lie algebra $A_2 = sl(3)$. According to the results of previous section we seek the solutions to eqs. (4.1)-(4.3) in the following form (see (4.12); here $n_1 = n_2 = 2$):

$$H_s = 1 + P_s z + P_s^{(2)} z^2,$$

where $P_s = P_s^{(1)}$ and $P_s^{(2)} \neq 0$ are constants, $s = 1, 2$.

The substitution of (5.1) into equations (4.1) and decomposition in powers of $z$ lead us to the relations

$$-P_s(P_s + 2\mu) + 2P_s^{(2)} = \bar{B}_s,$$  

(5.2)

$$-2P_s^{(2)}(P_s + 4\mu) = P_{s+1}\bar{B}_s,$$  

(5.3)

$$-2P_s^{(2)}(\mu P_s + P_s^{(2)}) = P_{s+1}^{(2)}\bar{B}_s,$$  

(5.4)

corresponding to powers $z^0, z^1, z^2$ respectively, $s = 1, 2$. Here we denote $s + 1 = 2, 1$ for $s = 1, 2$ respectively. For $P_1 + P_2 + 4\mu \neq 0$ the solutions of (5.2)-(5.4) read

$$P_s^{(2)} = \frac{P_s P_{s+1}(P_s + 2\mu)}{2(P_1 + P_2 + 4\mu)},$$  

(5.5)

$$\bar{B}_s = -\frac{P_s(P_s + 2\mu)(P_s + 4\mu)}{P_1 + P_2 + 4\mu},$$  

(5.6)

$s = 1, 2$. For $P_1 + P_2 + 4\mu = 0$ there exist also a special solution with

$$P_1 = P_2 = -2\mu, \quad 2P_s^{(2)} = \bar{B}_s > 0, \quad \bar{B}_1 + \bar{B}_2 = 4\mu^2.$$  

(5.7)

Thus, in the $A_2$-case the solution is described by relations (3.34)-(3.38) with $S = \{s_1, s_2\}$, intersection rules (2.52), or, equivalently,

$$d(I_{s_1} \cap I_{s_2}) = \frac{d(I_{s_1})d(I_{s_2})}{D - 2} - \chi_{s_1}\chi_{s_2} \lambda_{a_{s_1}} \cdot \lambda_{a_{s_2}} - \frac{1}{2} K,$$  

(5.8)

$$d(I_{s_i}) - \left(\frac{d(I_{s_i})}{D - 2}\right)^2 + \lambda_{a_{s_i}} \cdot \lambda_{a_{s_i}} = K,$$  

(5.9)

where $K = K_{s_i} \neq 0$, and functions $H_{s_i} = H_i$ are defined by relations (5.1) and (5.5)-(5.7) with $z = R^{-d}$, $i = 1, 2$. 

18
5.2 *A_2*-dyon in *D* = 11 supergravity

Consider the “truncated” bosonic sector of *D* = 11 supergravity (“truncated” means without Chern-Simons term). The action (2.1) in this case reads [51]

\[ S_{tr} = \int_M d^{11}z \sqrt{|g|} \left\{ R[g] - \frac{1}{4!} F^2 \right\}. \]  

(5.10)

where rank*F* = 4. In this particular case, we consider a dyonic black-hole solutions with electric 2-brane and magnetic 5-brane defined on the manifold

\[ M = (2\mu, +\infty) \times (M_1 = S^2) \times (M_2 = \mathbb{R}) \times M_3 \times M_4, \]  

(5.11)

where dim*M* = 2 and dim*M* = 5.

The solution reads,

\[ g = H_1^{1/3} H_2^{2/3} \left\{ \frac{dR \otimes dR}{1 - 2\mu/R} + R^2 d\Omega_2^2 \right\} - H_1^{-1} H_2^{-1} \left( 1 - \frac{2\mu}{R} \right) dt \otimes dt + H_1^{-1} g^3 + H_2^{-1} g^4 \}, \]

\[ F = -\frac{Q_1}{R^2} H_1^{-2} H_2 dR \wedge dt \wedge \tau_3 + Q_2 \tau_1 \wedge \tau_3, \]  

(5.13)

where metrics *g*² and *g*³ are Ricci-flat metrics of Euclidean signature, and *H*ₙ are defined as follows

\[ H_s = 1 + \frac{P_s}{R} + \frac{P_s^{(2)}}{R^2}, \]  

(5.14)

where parameters *P*ₙ, *µ* > 0 and *P*ₙ⁻², *B*ₙ = *B*ₙ = −2*Q*ₙ², *n* = 1, 2, satisfy relations (5.5) and (5.6).

The solution describes *A_2*-dyon consisting of electric 2-brane with world sheet isomorphic to (*M_2 = \mathbb{R}) × *M*³ and magnetic 5-brane with worldsheet isomorphic to (*M_2 = \mathbb{R}) × *M*₄. The “branes” are intersecting on the time manifold *M_2 = \mathbb{R}*. Here \(K_s = (U^s, U^s) = 2, \varepsilon_s = -1\) for all *s* ∈ *S*. The *A_2* intersection rule reads (see (2.52))

\[ 2 \cap 5 = 1 \]  

(5.15)

Here and in what follows \((p_1 \cap p_2 = d) \Leftrightarrow (d(I) = p_1 + 1, d(J) = p_2 + 1, d(I \cap J) = d)\).

The solution (5.12), (5.13) satisfies not only equations of motion for the truncated model, but also the equations of motion for *D* = 11 supergravity with the bosonic sector action

\[ S = S_{tr} + c \int_M A \wedge F \wedge F \]  

(5.16)

\((c = \text{const}, F = dA)\), since the only modification related to “Maxwells” equations

\[ d \ast F = \text{const} \ F \wedge F, \]  

(5.17)

is trivial due to \(F \wedge F = 0\) (since \(\tau_i \wedge \tau_i = 0\)).

This solution in a special case \(H_1 = H_2 = H^2 \ (P_1 = P_2, Q_1^2 = Q_2^2)\) was considered in [47]. The 4-dimensional section of the metric (5.12) in this special case coincides with the Reissner-Nordström metric. For the extremal case, *µ* → +0, and multi-black-hole generalization see also [41].
5.3 $A_2$-dyon in Kaluza-Klein model

Let us consider 4-dimensional model

$$S = \int_M d^4z \sqrt{|g|} \left\{ R[g] - g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2!} \exp[2\lambda \varphi] F^2 \right\}$$  \hspace{1cm} (5.18)

with scalar field $\varphi$, two-form $F = dA$ and

$$\lambda = -\sqrt{3/2}.$$  \hspace{1cm} (5.19)

This model originates after Kaluza-Klein reduction of 5-dimensional gravity. The 5-dimensional metric in this case reads

$$g^{(5)} = \phi g_{\mu\nu} dx^\mu \otimes dx^\nu + \phi^{-2}(dy + A) \otimes (dy + A),$$  \hspace{1cm} (5.20)

where

$$\begin{align*}
A &= \sqrt{2} A = \sqrt{2} A_\mu dx^\mu, \\
\phi &= \exp(2\varphi/\sqrt{6}).
\end{align*}$$  \hspace{1cm} (5.21)

We consider the dyonic black-hole solution carrying electric charge $Q_1$ and magnetic charge $Q_2$, defined on the manifold

$$M = (2\mu, +\infty) \times (M_1 = S^2) \times (M_2 = \mathbb{R}).$$  \hspace{1cm} (5.22)

This solution reads

$$g = (H_1 H_2)^{1/2} \left\{ \frac{dR \otimes dR}{1 - 2\mu/R} + R^2d\Omega^2 - H_1^{-1} H_2^{-1} \left( 1 - \frac{2\mu}{R} \right) dt \otimes dt \right\},$$  \hspace{1cm} (5.23)

$$\exp(\varphi) = H_1^{\lambda/2} H_2^{-\lambda/2},$$  \hspace{1cm} (5.24)

$$F = dA = -\frac{Q_1}{R^2} H_1^{-2} H_2 dR \wedge dt + Q_2 \tau_1,$$  \hspace{1cm} (5.25)

where functions $H_s$ are defined by relations (5.1), (5.5) and (5.6) with $\bar{B}_s = -2Q_s^2$, $z = R^{-1}$, $s = 1, 2$; where $\tau_1$ is volume form on $S^2$.

For 5-metric we obtain from (5.20)-(5.24)

$$g^{(5)} = H_2 \left\{ \frac{dR \otimes dR}{1 - 2\mu/R} + R^2d\Omega^2 - H_1^{-1} H_2^{-1} \left( 1 - \frac{2\mu}{R} \right) dt \otimes dt \right\} + H_1 H_2^{-1}(dy + A) \otimes (dy + A),$$  \hspace{1cm} (5.26)

$$dA = \sqrt{2}F.$$  \hspace{1cm} (5.26)

For $Q_2 \to 0$ we get the black hole version of Dobiash-Maison solution from [56] and for $Q_1 \to 0$ we are led to the black hole version of Gross-Perry-Sorkin monopole solution from [57, 58], see [61]. The solution coincides with Gibbons-Wiltshire dyon solution [60]. Our notations are related to those from ref. [60], as following : $H_1 R^2 = B$, $H_2 R^2 = A$, $R^2 - 2\mu R = \Delta$, $Q_1 = \sqrt{2}q$, $Q_2 = -\sqrt{2}p$, $R - \mu = r - m$, $\mu^2 = m^2 + d^2 - p^2 - q^2$, $(P_2 - P_1)/2(P_2 + 1) = d/(d - \sqrt{3}m)$. (For general spherically symmetric configurations see also ref. [59].)

We note that, quite recently, in [62] the KK dyon solution [60] was used for constructing the dyon solution in $D = 11$ supergravity (5.12)-(5.13) for flat $g^3$ and $g^4$ and its rotating version.
6 Conclusions

Thus here we obtained a family of black hole (BH) solutions with intersecting $p$-branes with nearly arbitrary intersection rules, see relations (2.22), (2.23) and Restriction 1. (Restriction 2 is satisfied, since all $p$-branes have a common time manifold.) These BH solutions are given by relations (3.34)-(3.41). The metric of solutions contains $n - 1$ Ricci-flat “internal” space metrics. The solutions are defined up to a set of ("moduli") functions $H_s$ obeying a set of equations (3.39) with boundary conditions (3.40)-(3.41) (or, equivalently, eqs. (1.1)-(1.3)).

These solutions are new and generalize a lot of special classes of BH solutions considered earlier in the literature. It is not necessary in future investigations to consider special models and setups, find spherically symmetric solutions and single out BH ones. All this program is fulfilled in this paper (with the use of results of ref. [42]). What we only need is to find explicit relations for moduli functions $H_s$, when matrix $A$ is fixed, i.e. to solve equations (1.1) with boundary conditions (1.2)-(1.3) imposed. The problem (1.1)-(1.3) seems to be rather difficult and may be of interest from the pure mathematical point of view, regardless to possible physical applications.

Here we suggested a conjecture on polynomial structure of $H_s$ for intersections related to semisimple Lie algebras and proved it for $A_m$ and $C_{m+1}$ algebras, $m \geq 1$. This result may be interesting, since any appearance of polynomials in mathematical physics, especially related to Lie algebras, is always a rather attractive for mathematicians (and physicists, as well).

Here we also obtained explicit relations for the solutions in the $A_2$-case and considered two examples of $A_2$-dyon solutions: one in $D = 11$ supergravity (with $M2$- and $M5$-branes intersecting at a point ) and another in 5-dimensional Kaluza-Klein theory (Gibbons-Wiltshire solution). Explicit relations for $H_s$ corresponding to other examples of Lie algebras (e.g. $A_3$, $B_2$ etc) will be considered in future publications.

Acknowledgments

This work was supported in part by the Russian Ministry for Science and Technology, Russian Foundation for Basic Research, and project SEE.

References

[1] E. Witten, Nucl. Phys. B 443, 85 (1995); hep-th/9503124;  
   P. Townsend, Phys. Lett. B 350, 184 (1995); hep-th/9612121;  
   C. Hull and P. Townsend, Nucl. Phys. B 438, 109 (1995); hep-th/9610167;  
   P. Horava and E. Witten, Nucl. Phys. B 460, 506 (1996); hep-th/9510209.

[2] J.M. Schwarz, Lectures on Superstring and M-theory Dualities, hep-th/9607201;  
   M.J. Duff, M-theory (the Theory Formerly Known as Strings), hep-th/9608117.

[3] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, vol. 1, 2, Cambridge, 1987.

[4] K.S. Stelle, Lectures on Supergravity p-branes, hep-th/9701088.
[5] M.J. Duff, R.R. Khuri and J.X. Lu, *Phys. Rep.* **259**, 213 (1995).

[6] A. Dabholkar, G. Gibbons, J.A. Harvey and F. Ruiz Ruiz, *Nucl. Phys.* **B 340**, 33 (1990).

[7] G.T. Horowitz and A. Strominger, *Nucl. Phys.* **B 360**, 197 (1990).

[8] M.J. Duff and K.S. Stelle, *Phys. Lett.* **B 253**, 113 (1991).

[9] R. Güven, *Phys. Lett.* **B 276**, 49 (1992); *Phys. Lett.* **B 212**, 277 (1988).

[10] A. Strominger, *Phys. Lett.* **B 383**, 44 (1996); hep-th/9512059.

[11] P.K. Townsend, *Phys. Lett.* **B 373**, 68 (1996); hep-th/9512062.

[12] G. Papadopoulos and P.K. Townsend, *Phys. Lett.* **B 380**, 273 (1996); hep-th/9603087.

[13] A.A. Tseytlin, *Nucl. Phys.* **B 475**, 149 (1996); hep-th/9604035.

[14] J.P. Gauntlett, D.A. Kastor, and J. Traschen, *Nucl. Phys.* **B 478**, 544 (1996); hep-th/9604179.

[15] M. Cvetic and A.A. Tseytlin, Nucl. Phys. B 478, 181 (1996).

[16] A.A. Tseytlin, *Nucl. Phys.* **B 487**, 141 (1997); hep-th/9609212.

[17] H. Lü, C.N. Pope, SL(N+1,R) Toda Solitons in Supergravities, *Int. J. Mod. Phys.* **A 12**, 2061 (1997); hep-th/9607027.

[18] H. Lü, C.N. Pope, and K.W. Xu, Liouville and Toda Solitons in M-theory, *Mod. Phys. Lett.* **A 11**, 1785 (1996); hep-th/9604058.

[19] H. Lü, S. Mukherji, C.N. Pope and K.-W. Xu, Cosmological Solutions in String Theories, *Phys. Rev.* **D 55**, 7926 (1997); hep-th/9610107.

[20] A. Volovich, *Nucl. Phys.* **B 487** (11), 141 (1997); hep-th/9608095.

[21] I.Ya. Aref'eva and A.I. Volovich, *Class. Quantum Grav.* **B 14**, 29901 (1997); hep-th/9611026.

[22] V.D. Ivashchuk and V.N. Melnikov, Intersecting p-brane Solutions in Multidimensional Gravity and M-theory, hep-th/9612089; *Grav. and Cosmol.* **2**, No 4, 204 (1996).

[23] V.D. Ivashchuk and V.N. Melnikov, *Phys. Lett.* **B 403**, 23 (1997).

[24] E. Bergshoeff, M. de Roo, E. Eyras, B. Janssen and J.P. van der Schaar, *Class. Quantum Grav.* **14**, 2757 (1997); hep-th/9612095.

[25] I.Ya. Aref’eva and O.A. Rytchkov, Incidence Matrix Description of Intersecting p-brane Solutions, *Preprint* SMI-25-96, hep-th/9612236.

[26] R. Argurio, F. Englert and L. Hourant, *Phys. Lett.* **B 398**, 2991 (1997); hep-th/9701042.

[27] I.Ya. Aref’eva, M.G. Ivanov and O.A. Rytchkov, Properties of Intersecting p-branes in Various Dimensions, *Preprint* SMI-05-97, hep-th/9702077.

[28] I.Ya. Aref’eva, M.G. Ivanov and I.V. Volovich, Non-Extremal Intersecting p-Branes in Various Dimensions, hep-th/9702079; *Phys. Lett.* **B 406**, 44 (1997).
[29] N. Ohta, Intersection rules for non-extreme p-branes, hep-th/9702164; Phys. Lett. B 403, 218-224 (1997).

[30] V.D. Ivashchuk and V.N. Melnikov, Sigma-model for the Generalized Composite p-branes, hep-th/9705036; Class. Quantum Grav. 14, 3001 (1997); Corrigenda ibid. 15 (12), 3941 (1998).

[31] V.D. Ivashchuk, V.N. Melnikov and M. Rainer, Multidimensional \(\sigma\)-models with Composite Electric p-branes, gr-qc/9705005; Grav. and Cosmol. 4, No 1 (13), (1998).

[32] K.A. Bronnikov, M.A. Grebeniuk, V.D. Ivashchuk and V.N. Melnikov, Integrable Multidimensional Cosmology for Intersecting p-branes, Grav. and Cosmol. 3, No 2(10), 105 (1997).

[33] H. Lü, J. Maharana, S. Mukherji and C.N. Pope, Cosmological Solutions, p-branes and the Wheeler De Witt Equation, Phys. Rev. D 57, 2219 (1998); hep-th/9707182.

[34] M.A. Grebeniuk, V.D. Ivashchuk and V.N. Melnikov, Integrable Multidimensional Quantum Cosmology for Intersecting p-Branes, Grav. and Cosmol. 3, No 3 (11), 243 (1997), gr-qc/9708031.

[35] K.A. Bronnikov, U. Kasper and M. Rainer, Interesecting electric and magnetic p-branes: spherically-symmetric solutions, gr-qc/9708058; GRG, 31, No 11, 1681 (1999).

[36] V.D. Ivashchuk and V.N. Melnikov, Multidimensional Classical and Quantum Cosmology with Intersecting p-branes, J. Math. Phys., 39, 2866 (1998); hep-th/9708157.

[37] D. Youm, Phys. Rept., 316 (1999) 1-232; hep-th/9710046. (this review on black holes in string theories is rather complete although it does not contain some important citations, e.g. [25], [28], [39] etc.)

[38] K.A. Bronnikov, V.D. Ivashchuk and V.N. Melnikov, The Reissner-Nordström Problem for Intersecting Electric and Magnetic p-branes, gr-qc/9710054; Grav. and Cosmol., 3, No 3(11), 203 (1997).

[39] K.A. Bronnikov, Block-orthogonal Brane systems, Black Holes and Wormholes, hep-th/9710207; Grav. and Cosmol. 4, No 1 (13), 49 (1998).

[40] D.V. Gal’tsov and O.A. Rytchkov, Generating Branes via Sigma models, hep-th/9801180.

[41] V.D. Ivashchuk and V.N. Melnikov, Madjundar-Papapetrou Type Solutions in Sigma-model and Intersecting p-branes, Class. Quantum Grav. 16, 849 (1999); hep-th/9802121.

[42] V.D.Ivashchuk, S.-W.Kim and V.N.Melnikov, Hyperbolic Kac-Moody Algebra from Intersecting p-branes, J. Math. Phys. 40, 4072 (1999); hep-th/9803006.

[43] M.A. Grebeniuk and V.D. Ivashchuk, Sigma-model Solutions and Intersecting p-branes Related to Lie Algebras, Phys. Lett. B 442, 125 (1998); hep-th/9805113.

[44] K.A. Bronnikov, Gravitating Brane Systems: Some General Theorems, gr-qc/9806102; J. Math. Phys. 40, 924 (1999).
[45] V.R. Gavrilov and V.N. Melnikov, Toda Chains with Type $A_m$ Lie Algebra for Multidimensional Classical Cosmology with Intersecting p-branes, In: Proceedings of the International seminar ”Current topics in mathematical cosmology”, (Potsdam, Germany, 30 March - 4 April 1998), Eds. M. Rainer and H.-J. Schmidt, World Scientific, 1998, p. 310; hep-th/9807004.

[46] V.D. Ivashchuk and V.N. Melnikov, Multidimensional Cosmological and Spherically Symmetric Solutions with Intersecting p-branes, gr-qc/9901001; Cosmological and Spherically Symmetric Solutions with Intersecting p-branes, J. Math. Phys. 40, No 10 (1999), 6558.

[47] S. Cotsakis, V.D. Ivashchuk and V.N. Melnikov, P-branes Black Holes and Post-Newtonian Approximation, Grav. and Cosmol. 5, No 1 (17), 52 (1999); gr-qc/9902148.

[48] V.R. Gavrilov and V.N. Melnikov, Toda Chains Associated with Lie Algebras $A_m$ in Multidimensional Gravitation and Cosmology with Intersecting p-branes, Theor. Math. Phys. 123, No 3, 374-394 (2000) (in Russian).

[49] V.D. Ivashchuk and S.-W. Kim, Solutions with intersecting p-branes related to Toda chains, J. Math. Phys. 41 (1), 444-460 (2000); hep-th/9907019.

[50] J. Fuchs and C. Schweigert, Symmetries, Lie algebras and Representations. A graduate course for physicists (Cambridge University Press, Cambridge, 1997).

[51] E. Cremmer, B. Julia, J. Scherk. Phys. Lett. B 76, 409 (1978).

[52] S.D. Majumdar, Phys. Rev. 72, 930 (1947); A. Papapetrou, Proc. R. Irish Acad. A51, 191 (1947).

[53] V.D. Ivashchuk, V.N. Melnikov and A.I. Zhuk, Nuovo Cimento B 104, 575 (1989).

[54] M. Toda, Progr. Theor. Phys. 45, 174 (1970).

[55] A. Anderson, J. Math. Phys. 37, 1349 (1996); hep-th/9507092.

[56] P. Dobiash and D. Maison, Gen. Rel. Grav. 14, 231 (1982).

[57] D.J. Gross and M.J. Perry, Nucl. Phys. B 226, 29 (1983).

[58] R.D. Sorkin, Phys. Rev. Lett. 51, 87 (1983).

[59] S.-C. Lee, Phys. Lett. 149, 98 (1984).

[60] G. Gibbons and D. Wiltshire, Ann. Phys. 167, 201 (1986); Erratum: ibid 176, 393 (1987).

[61] C.-M. Chen, D. V. Gal’tsov, K. Maeda and S. Sharakin, Phys. Lett. B 453, 7 (1999).

[62] C.-M. Chen, D. V. Gal’tsov and S. Sharakin, Einstein Gravity — Supergravity Correspondence, hep-th/9912127.