VOLUMES OF DEGENERATING POLYHEDRA – ON A CONJECTURE OF J. W. MILNOR

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Abstract. In his paper [4] J. Milnor conjectured that the volume $V_n$ of $n$-dimensional hyperbolic and spherical simplices, as a function of the dihedral angles, extends continuously to the closure $\overline{A}$ of the space $A$ of allowable angles (“The continuity conjecture”), and furthermore, $V_n(a \in \partial A) = 0$ if and only if $a$ lies in the closure of the space of angles of Euclidean tetrahedra (“the Vanishing Conjecture”). A proof of the Continuity Conjecture was given by F. Luo (3) – Luo’s argument uses Kneser’s formula [2] together with some delicate geometric estimates). In this paper we give a simple proof of both parts of Milnor’s conjecture, prove much sharper regularity results, and then extend the method to apply to all convex polytopes. We also give a precise description of the boundary of the space of angles of convex polyhedra in $\mathbb{H}^3$, and sharp estimates on the diameter of a polyhedron in terms of the length of the shortest polar geodesic.

1. Introduction

Consider the set of simplices in $\mathbb{H}^n$ or $S^n$. It is well-known that this set is parametrized by the (ordered) collection of dihedral angles, and we may call the set of assignements of dihedral angles of geometric simplices in $\mathbb{H}^n$ as a subset $\Omega_{\mathbb{H}^n} \subset \mathbb{R}^{n(n+1)/2}$ and similarly, the set of dihedral angle assignments of of geometric simplices in $S^n$ as a subset $\Omega_{S^n} \subset \mathbb{R}^{n(n+1)/2}$. These sets are open, since they are defined by collections of strict inequalities (which are polynomial in the cosines of the dihedral angles). One may then view the volume $V$ of a simplex as a function $V$ on $\Omega_{X^n}$. J. Milnor (4) conjectured:
**Conjecture 1.** The volume function $V$ admits a continuous extension to $\overline{\Omega}$. Furthermore, the points on $\partial \Omega$ where $V$ vanishes are precisely those which also lie in the closure of the set of angle assignments of Euclidean simplices.

Some comments are in order regarding Conjecture 1. Firstly, it falls into two parts: the “Continuity Conjecture” and the “Vanishing Conjecture.” The Vanishing Conjecture cannot be stated without knowing that the answer to the Continuity Conjecture is affirmative.

The Continuity Conjecture was first shown by F. Luo (in [3]), and then a sharper version was shown by me in a predecessor ([7]) of the current paper.

Milnor does not attribute the conjecture to himself, and his paper (which was written in the late seventies or early eighties) seems to imply that the conjecture precedes the paper.

The contents of this paper are as follows. First, we give a simple argument to show a sharp version of Milnor’s Continuity Conjecture for all hyperbolic polytopes of dimension greater than 3, and also all spherical polytopes. It should be noted that in many cases it is not known whether hyperbolic or spherical polytopes are determined by their dihedral angles and how to characterize the possible assignments of dihedral angles\(^\dagger\), it makes more sense to use polar metrics introduced in [6,8]. The argument shows that the extension is, in fact, Lipschitz.

Next we give an argument to show the Continuity Conjecture for three-dimensional hyperbolic tetrahedra, which is conceptually related to the higher-dimensional argument (via the Schl"afli differential formula), but is a little more delicate. The argument requires a version of Sobolev’s Embedding Theorem, but as a consequence, a sharp regularity result is obtained (this time the extension is shown to be in the class $C^{0,1}$.)

We then go on to arbitrary convex polyhedra in $\mathbb{H}^3$ (and polytopes in $\mathbb{H}^n$) and prove the same sharp version of the Continuity Conjecture for those. These results use (at least philosophically) the results of [6,8]. It should be noted that the estimates proved in this section work just as well for higher-dimensional convex polytopes (although they are not necessary for the regularity result). The results here are of independent interest, and can be summarized as follows:

**Theorem 1.** Let $P$ be a polyhedron with $N$ vertices in $\mathbb{H}^3$ of diameter $\rho \gg 1$. Let $M^{\ast}$ be the polar metric of $P$ (as in [8,6]). The $M^{\ast}$ lies within

\(^\dagger\)Simplices are a notable exception, and an excellent exposition is given in Milnor’s paper [4]
$c_1(N) \exp(-c_2(N)\rho)$ of the boundary of the space of admissible polar metrics, where $c_1, c_2$ are strictly positive functions of $N$.

The constants in the statement of the Theorem above are completely explicit, and can be sharpened by taking into consideration finer invariants of the combinatorics of $P$ than the number of vertices.

In Section 6 we give the proof of the Vanishing Conjecture for simplices (that is, Milnor’s original conjecture) and then use our description (as given in Section 5) of the boundary of the set of polar metrics of convex polytopes to show the Vanishing Conjecture for arbitrary convex polytopes.

The main result of Section 5 is as follows:

**Theorem 2.** Let $P$ lie on the boundary of the space of polar metrics of compact convex polytopes in $\mathbb{H}^n$. Then either $P$ has a combinatorial closed geodesic of length $2\pi$, or $P$ is a metric suspension.

In the above, a combinatorial geodesic is one which is contained in the 1-dimensional skeleton of the cell-decomposition of $P$ coming from a family of polar metrics of degenerating polytopes.

2. A simple proof for simplices (among other things)

In dimension 2, the result follows immediately from Gauss’ formula, which states that area is a linear function of the angles, so we will only discuss dimensions 3 or above.

The simple proof relies on the Schlafli differential equality (see [4], which states that in a space of constant curvature $K$ and dimension $n$ the volumes of a smooth family of polyhedra $P$ satisfy the differential equation:

$$KdV(P) = \frac{1}{n-1} \sum_F V_{n-2}(F)d\theta_F,$$

where the sum is over all codimension-2 faces, $V_{n-2}$ is the $n-2$ dimensional volume of $F$, and $\theta_F$ is the dihedral angle at $F$.

Another way of writing the Schlafli formula is:

$$K \frac{dV(P)}{d\theta_F} = V_{n-2}(F).$$

This is the form we will use.

The first observation is that $V_{n-2}(F)$ is bounded by a constant (dimensional for $S^n$, depending on the number of vertices of $F$ in $\mathbb{H}^n$ for $n \geq 4$.)
This immediately shows the continuity of volume for all $S^n$, and for $\mathbb{H}^n$, whenever $n \geq 4$.

We are left with dimension 3. All we really need is the result that the partial derivatives of $V$ with respect to the dihedral angles develop at worst logarithmic singularities as we approach the frontier of $\Omega_{\mathbb{H}^3}$ – this result suffices by the following form of the Sobolev Embedding Theorem (this is [1, Theorem 7.26]):

**Theorem 3.** Let $\Omega$ be a $C^{0,1}$ domain in $\mathbb{R}^n$. Then,

- (i) If $kp < n$, the space $W^{k,p}(\Omega)$ is continuously imbedded in $L^{p^*}(\Omega)$, where $p^* = np/(n - kp)$, and compactly imbedded in $L^q(\Omega)$ for any $q < p^*$.
- (ii) If $0 \leq m < k - \frac{n}{p} < m + 1$, the space $W^{k,p}$ is continuously embedded in $C^{m,\alpha}(\overline{\Omega})$, $\alpha = k - \frac{n}{p} - m$, and compactly embedded in $C^{m,\beta}(\overline{\Omega})$ for any $\beta < \alpha$.

Here, the Sobolev space $W^{k,p}$ is the space of functions whose first $k$ (distributional) derivatives are in $L^p$.

In our case, we know that the domain $\Omega$ is bounded, convex “curvilinear polyhedral” (hence $C^{0,1}$) domain, volume is a bounded function, and we assume that the gradient grows logarithmically as we approach the boundary. This implies that $V$ is in $W^{1,p}$ for all $p > 0$, so we get the following corollary:

**Corollary 1.** Volume is in $C^{0,\alpha}(\overline{\Omega})$ for any $\alpha < 1$.

The logarithmic growth of diameter of the simplex as a function of the distance to $\partial \Omega$ can be shown in a completely elementary way using Eq. (2) and elementary reasoning about Gram matrices, as follows:

Let $G$ be “angle Gram matrix” of a simplex $\Delta$, that is, $G_{ij} = -\cos \theta_{ij}$, where $\theta_{ij}$ is the angle between the $i$-th and the $j$-th face. Let $S$ be the matrix whose columns are the normals to the faces of $\Delta$ (all the computations take place in Minkowski space, and we use the hyperboloid model of $\mathbb{H}^n$. It is immediate that $G = S^t S$.

Let now $W$ be the matrix whose columns are the (possibly scaled) vertices of $\Delta$. $W$ satisfies the equation $S^t W = I$, and to get the vertices to lie on the hyperboloid $\langle x, x \rangle = -1$ we must rescale in such a way that the squared norms of the columns of $W$ become $-1$. Call the scaled matrix $W_s$. Since the usual “length” Gram matrix $G^*$ of $\Delta S$ can be written as $W_s^t W_s$, and $G^*_{ij} = -\cosh(d(v_i, v_j))$, a simple computation using Cramer’s rule gives:
\[
\cosh d(v_i, v_j) = \frac{c_{ij}}{\sqrt{c_{ii}c_{jj}}},
\]
where \(c_{ij}\) is the \(ij\)-th cofactor of \(G\). (see \([5]\) for many related results).

It follows that the distances between the vertices (which are the lengths of the edges, which are the faces of codimension 2.) behave as \(|\log c_{ii}|\). Since the cofactors are polynomial in the cosines of the angles, we are done.

It should be noted that this argument works mutatis mutandis for hyperideal simplices, or simplices with some finite and some hyper-infinite vertices.

3. Convex polytopes

For arbitrary convex polytopes in dimension \(n > 3\) (and convex spherical polytopes in all dimensions) the proof given in Section 2 goes through without change, with the one proviso that it is not currently known whether the volume of a polytope is determined up to congruence by its dihedral angles. Such a uniqueness result is conjectured (indeed, it is conjectured that a polytope is determined up to congruence by the dihedral angles), and is easy to prove for simple polytopes – those with simplicial links of vertices – this follows in arbitrary dimension from the corresponding result in 3 dimensions (\([8, 6]\)). The uniqueness issue can be finessed (in dimension 3, at least) by using the results of \([8, 6]\):

**Theorem 4** \(([8, 6])\). A metric space \((M, g)\) homeomorphic to \(S^2\) can arise as the Gaussian image \(G(P)\) of a compact convex polyhedron \(P\) in \(H^3\) if and only if the following conditions hold:

- (a) The metric \(g\) has constant curvature 1 away from a finite collection of cone points.
- (b) The cone angle at each \(c_i\) is greater than 2\(\pi\).
- (c) The lengths of closed geodesics of \((M, g)\) are all strictly greater than 2\(\pi\).

The space of admissible metrics \(\Omega_P\) (as per Theorem 4) is parametrized by the exterior dihedral angles (the cell decomposition dual to that of \(P\) gives a triangulation of the Gaussian image, and the (exterior) dihedral angles are the lengths of edges of the triangulation.) Theorems 7,8 immediately imply the following:

**Theorem 5.** There exists a constant \(L_0\), such that the maximal length \(\ell_P\) of an edge of \(P\) is bounded as follows:

\[
\ell_P \leq \max(L_0, -2N \log(d(P, \partial \Omega_P)/12N)),
\]
where $N$ is the number of vertices of $P$.

**Proof.** Assume the contrary. Then, there exists a sequence of polyhedra $P_1, \ldots, P_n, \ldots$ with diameter $\rho(P_i) \geq \ell P_i$ going to infinity, which are farther than $12N \exp(-\rho/2N)$. By choosing a subsequence, we may assume that there is a fixed cycle of faces $F_1, \ldots, F_k$ of $P$, such that the sum of dihedral angles along the edges $e_i = F_i \cap F_{i+1}$ is smaller than $2\pi + 12N \exp(-\rho/2N)$, (by Theorem 7) and which are a quasigeodesic (by Theorem 8). Since the limit point of the $P_i$ is not in $\Omega_P$ (by Theorem 4), the result follow.

The following corollary is immediate (by Schl"afli, see Section 2):

**Corollary 2.** The volume is in $W^{1,p}(\Omega_P)$ for all $p > 0$.

We now have almost enough to show that volume extends to $\Omega_P$, except for the slight matter of not having the required (by Theorem 5) regularity result for $\partial \Omega_P$. Such a result seems quite non-trivial, since the length of the shortest closed geodesic is a rather badly behaved quantity, but the results of Section 5 show that things are well enough behaved.

4. Degeneration estimates

The results of this section are a quantitative version of the results of the compactness results of [8, 6]. First, some key lemmas. The general setup will be as follows: $L$ is a geodesic in $\mathbb{H}^3$, $t$ is a real number (generally large) and $P, P^-, P^+$ are three planes, all orthogonal to $L$, and such that $d(P, P^-) = d(P, P^+) = t$, and $d(P^-, P^+) = 2t$. We denote $x_0 = L \cap P$.

In the sequel, we use the hyperboloid model of $\mathbb{H}^3$, where $\mathbb{H}^3$ is represented by the set $\langle x, x \rangle = -1; x_0 > 0$, in the $\mathbb{R}^4$ equipped with the scalar product $\langle x, y \rangle = -x_1y_1 + \sum_{i=2}^4 x_iy_i$. The reader is referred to [9] (as well as [8]) for the details (which will be used below).

Returning back to our setup, we can assume, without loss of generality, that

$$x_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

that

$$P^\perp = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix},$$
and hence, that $P^+ = \phi(t)P$, while $P^- = \phi(-t)P$, where

$$
\phi(r) = \begin{pmatrix}
\cosh(r) & \sinh(r) & 0 & 0 \\
\sinh(r) & \cosh(r) & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
$$

Since $\phi(r)$ is symmetric, it follows that

$$
P^{+\perp} = \phi(t)P^{\perp} = \begin{pmatrix}
\cosh(t) \\
\sinh(t) \\
0 \\
0
\end{pmatrix},
$$

while

$$
P^{-\perp} = \phi(t)P^{\perp} = \begin{pmatrix}
\cosh(t) \\
-\sinh(t) \\
0 \\
0
\end{pmatrix},
$$

Lemma 1. Let $Q$ be a plane in $\mathbb{H}^3$ which intersects both $P^-$ and $P^+$. Then, there exists $t_0$, such that $Q$ intersects $P$, and the cosine of the angle $\alpha$ of intersection satisfies $|\cos(\alpha)| < 3e^{-t}$, as long as $t > t_0$. The number $t_0$ can be picked independently of $Q$.

Proof. Let the unit normal $Q^{\perp}$ to $Q$ be $Q^{\perp} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$. Since two planes intersect if and only if the scalar product of their unit normals is less than 1 in absolute value, we have, from the hypotheses of the lemma and the description of the unit normals to $P^-$ and $P^+$ above that:

(3) $|a \cosh(t) + b \sinh(t)| < 1$

(4) $|a \cosh(t) - b \sinh(t)| < 1$.

Squaring the two inequalities, and adding them together we obtain:

$$
a^2 \cosh^2(t) + b^2 \sinh^2(t) < 1.
$$

Since, under the hypotheses of the lemma, $\min(\cosh(t), \sinh(t)) > e^t/3$, it follows that

$$
a^2 + b^2 < 3/e^t,
$$

and so $\max(a, b) < 3e^{-t}$. Now, the cosine of the angle between $Q$ and $P$ equals $\langle Q^{\perp}, P^{\perp} \rangle = b$, so the result follows.

Remark 6. The constant 3 is far from sharp (especially for larger $t$).
Lemma 2. There exists a $t_0$, such that if $M$ is a line in $\mathbb{H}^3$ which intersects both $P^-$ and $P^+$, then $M$ intersects $P$, and $\cosh(d(P \cap M, x_0)) < 4e^{-2t} + 1$, as long as $t > t_0$.

Proof. Assume that $M \cap P^+ = \phi(t)p_1$, and $M \cap P^- = \phi(-t)p_1$, where $p_{1,2} \in P$. (This is always possible, since $P^+ = \phi(t)P$, $P^- = \phi(-t)$.) The intersection of $M$ with $P$ is then given by

$$M \cap P = \frac{x(M \cap P^+) + y(M \cap P^-)}{\|x(M \cap P^+) + y(M \cap P^-)\|},$$

where $x$ and $y$ are chosen so that the linear combination is actually in $P$, or, in other words, the second coordinate of the linear combination vanishes. We abuse notation above by writing $\|Z\| = \sqrt{-\langle Z, Z \rangle}$.

Let us now compute. Set (for $i = 1, 2$)

$$p_i = \begin{pmatrix} a_i \\ 0 \\ c_i \\ d_i \end{pmatrix}.$$ 

It follows that

$$M \cap P^+ = \begin{pmatrix} a_1 \cosh(t) \\ a_1 \sinh(t) \\ c_1 \\ d_1 \end{pmatrix},$$

while

$$M \cap P^- = \begin{pmatrix} a_2 \cosh(t) \\ -a_2 \sinh(t) \\ c_2 \\ d_2 \end{pmatrix}.$$

It follows that we can choose $x = 1/(2a_1)$, $y = 1/(2a_2)$, so that

$$m = xM \cap P^+ + yM \cap P^- = \begin{pmatrix} \cosh(t) \\ 0 \\ \frac{1}{2}(c_1/a_1 + c_2/a_2) \\ \frac{1}{2}(d_1/a_1 + d_2/a_2) \end{pmatrix}.$$

It follows that

$$- \cosh(d(M \cap P, x_0)) = \begin{pmatrix} m \\ \|m\|, x_0 \end{pmatrix} = -\frac{\cosh(t)}{\sqrt{\cosh^2(t) - 1/4 ((c_1/a_1 + c_2/a_2)^2 + (d_1/a_1 + d_2/a_2)^2)}}.$$
Since \( c_i^2 + d_i^2 + 1 = a_i^2 \), for \( i = 1, 2 \) it follows that \(|c_i/a_i| < 1\), and similarly \(|d_i/a_i| < 1\), so that

\[
cosh^2(t) \geq \cosh^2(t) - 1/4 \left( (c_1/a_1 + c_2/a_2)^2 + (d_1/a_1 + d_2/a_2)^2 \right) > \cosh^2(t) - 2.
\]

It follows that

\[
\cosh(d(M \cap P, x_0)) \leq \frac{1}{\sqrt{1 - 2/\cosh^2(t)}}
\]

and the assertion of the lemma follows by elementary calculus. □

**Lemma 3.** Let \( T \) be a spherical triangle with sides \( A, B, C \) and (opposite) angles \( \alpha, \beta, \gamma \). Suppose that \( |\cos(\beta)| < \epsilon \ll 1 \), \( |\cos(\gamma)| < \epsilon \ll 1 \). Then \( |\alpha - A| < 2\epsilon \).

**Proof.** The spherical Law of Cosines states that:

\[
\cos(A) = \frac{\cos(\alpha) + \cos(\beta) \cos(\gamma)}{\sin(\beta) \sin(\gamma)}.
\]

It follows that

\[
\cos(A) - 2\epsilon^2 \leq \cos(A)(1 - \epsilon^2) - \epsilon^2 \leq \cos(\alpha) \leq \cos(A) + \epsilon^2.
\]

The assertion of the lemma follows immediately. □

**Corollary 3.** Let \( F_1 \) and \( F_2 \) be two planes intersecting at a dihedral angle \( \alpha \), with both \( F_1 \) and \( F_2 \) intersecting a third plane \( P \), at angles whose cosines are smaller than \( \epsilon \). Let \( A \) be the angle between \( F_1 \cap P \) and \( F_2 \cap P \). Then \( |\alpha - A| < 2\epsilon \).

**Proof.** Apply Lemma 3 to the link of the point \( F_1 \cap F_2 \cap P \). □

**Lemma 4.** Let \( V \) be a convex polygon in the hyperbolic plane \( \mathbb{H}^2 \), such that all the vertices of \( V \) lie within a distance \( r \) of a certain point \( O \). Then, the sum of the exterior angles of \( V \) is smaller than \( 2\pi \cosh(r) \).

**Proof.** The area of a disk of radius \( r \) in \( \mathbb{H}^2 \) equals \( 4\pi \sinh^2(r/2) = 2\pi(\cosh(r) - 1) \) (see [10]). Since \( V \) is contained in such a disk, its area is at most \( 2\pi(\cosh(r) - 1) \), and since the area of \( V \) equals the difference between the sum of the exterior angles and \( 2\pi \), the statement of the lemma follows. □

Now we are ready to show the following:

**Theorem 7.** Let \( X \) be a convex polyhedron with \( N \) vertices in \( \mathbb{H}^3 \) of diameter \( \rho \gg 1 \). Then, there exists a cyclic sequence of faces \( F_1, \ldots, F_k = F_1 \), with \( F_i \) sharing an edge \( e_i \) with \( F_{i+1} \) (indices taken \( \text{mod} \ k \)) so that the sum of exterior dihedral angles at \( e_1, \ldots, e_k \) is smaller than \( 2\pi + 12N \exp(-\rho/2N) \).
Proof. Take a diameter \( D \) of \( X \) of length \( \rho \), place points \( p_1, \ldots, p_N \) equally spaced on \( D \). By the pigeonhole principle, one of the segments \( p_ip_{i+1} \) contains no vertices of \( X \). Let \( x_0 \) be the midpoint of the segment \( p_ip_{i+1} \). Construct planes orthogonal to \( D \) at \( x_0 (P^-) \) and \( p_i (P^+) \). Let \( t = \rho/(2N) \). The portion of \( X \) contained between \( P^- \) and \( P^+ \) is a polyhedral cylinder, consisting of faces \( F_1, \ldots, F_k \). By Lemma 2, the intersection of \( X \) with \( P \) is a polygon \( P \), whose sum of exterior angles is at most \( 2\pi(4 \exp(-2t) + 1) \), and so by Corollary 3, combined with Lemma 1, the sum of the dihedral angles corresponding to pairs \( F_iF_{i+1} \) is at most \( 2\pi(4 \exp(-2t) + 1) + 6k \exp(-t) \). Since \( k \) is no greater than the number of faces of \( X \), which, in turn, is at most \( 2N - 4 \). □

Theorem 8. With notation as in Theorem 7, the faces \( F_1, \ldots, F_k \) form a curve in the Gaussian image of \( X \) with geodesic curvature not exceeding \( 3k \exp(-\rho/N) \).

Remark 9. The reader is referred to [6, 8] for a more thorough discussion of geodesics on spherical cone manifold, but suffice it to say that the contribution of the face \( F_i \) to the geodesic curvature is 0 if the two edges are (hyper)parallel, and equal to the angle of intersection if they intersect.

Proof. Let \( e_1 \) and \( e_2 \) be the two edges of \( F \). If \( e_1 \) and \( e_2 \) do not intersect, there is nothing to prove (by the remark above. If they do intersect at a point \( C \), note that \( C \) is at a distance at least \( \rho/2N \) from \( x_0 \), while the intersections \( A \) and \( B \) of \( e_1 \) and \( e_2 \) with \( P \) are at most \( \text{arccosh}(4 \exp(-\rho/2N) + 1) \approx \sqrt{8} \exp(-\rho/2N) \) away from \( x_0 \), and so at most (for large \( \rho \)) \( 6 \exp(-\rho/2N) \) away from each other. We will only use the (much cruder) estimate \( \cosh(AB) \leq 2 \). Now, apply the hyperbolic law of cosines to the triangle \( ABC \), to get:

\[
1 \geq \cos(\gamma) = \frac{-\cosh(AB) + \cosh(AC) \cosh(BC)}{\sinh(AC) \sinh(BC)} \geq 1 - \frac{2}{\sinh(AC) \sinh(BC)} \geq 1 - 8 \exp(-\rho/2N).
\]

The estimate now follows. □

Remark 10. The argument above is easily modified to show that the curve dual to \( F_1, \ldots, F_k \) has small geodesic curvature viewed as a curve in \( S^3 \), and not just in \( X^* \).
5. The boundary of the space of polar metrics of convex polytopes.

Consider a sequence of degenerating polytopes. We have two possibilities: the diameter stays bounded or it does not. If the diameter does not stay bounded, then the results of Section 4 indicate that one can pick a subsequence in such a way that the length of a (quasi)-geodesic in the dual 1-skeleton converges to $2\pi$, while the quasi-geodesic itself converges to a dual 1-skeleton geodesic. The other possibility is that the polytopes degenerate while the diameter is bounded. In this case there are the following possibilities:

First, the diameter goes to 0. In this case, it is clear that the polar is a round sphere.

Secondly, the diameter stays bounded away from 0, but the limit is 1-dimensional. In this case the polar metric is still a round sphere.

Thirdly, the limit may be 2-dimensional (a doubled polygon). In this case the polar is a metric suspension with two cone points with curvature equal to the area of the (doubled) polygon.

In higher dimensions the analysis is the same, though the number of suspension possibilities increases.

6. The Vanishing Conjecture

We will first need the following observation:

**Lemma 5.** The set of (hyper)planes intersecting a fixed ball in $\mathbb{H}^n$ is compact.

*Proof.* There are a number of arguments, the simplest of which would appear to be that the set of planes going through a fixed point in $\mathbb{H}^n$ is compact (being in one-to-one correspondence with the unit sphere $S^{n-1}$) and then identifying the set of planes intersecting a ball $B$ with a quotient of $S^{n-1} \times B$. □

We will actually need the following:

**Corollary 4.** A sequence of polytopes all faces of which intersect a fixed ball contains a convergent subsequence.

*Proof.* Immediate by compactness. □

In order to deal with the vanishing conjecture for simplices, we now make the following:

**Observation 11.** There exists a universal constant $K$ such that for any triangle $T \subset \mathbb{H}^2$, there exists a disk of radius $K$ intersecting all of the sides of $T$. 
The observation can be rephrased as saying that the hyperbolic plane is Gromov-hyperbolic. The constant $K$ can be chosen to be $\log 2/2$.

**Proof.** Since every triangle is contained in an ideal triangle, it is enough to show the result for the ideal triangle. There, the result follows by construction. \qed

**Corollary 5.** There exists a universal constant $K$ such that for any simplex $T \subset \mathbb{H}^n$, there exists a ball of radius $K$ intersecting all of the faces of $T$.

**Proof.** By induction on dimension. Pick any face $F$ of the simplex $T \in \mathbb{H}^n$. By induction, there is an $n-1$-dimensional ball of radius $K$ which intersects all of the faces of $F$, and thus all of the faces of $T$. \qed

Observation 11 shows that any sequence of simplices contains a convergent subsequence, and hence the volume of a sequence of simplices with degenerating dihedral angles is the volume of an actual simplex $T_\infty$ in $\mathbb{H}^n$. The only way that volume could be equal to 0 is if $T_\infty$ is degenerate (that is, lower dimensional). It is easy to see that the dihedral angles of $T_\infty$ then lie in the closure of the set of angles of Euclidean simplices.

To show the Vanishing Conjecture for an arbitrary sequence of polytopes, we consider two possibilities. The first is that all the faces of the polytopes of the (sub)sequence intersect a fixed ball. This case is the same as the case of the simplex consider above, and there is nothing left to prove.

For the other possibility, we will first need the following:

**Lemma 6.** Let $T$ be a simplex in $\mathbb{H}^n$, and $B$ a ball intersecting all the faces of $T$. Let $P$ be a plane which does not intersect $B$. Then at least 2 vertices of $T$ lie on the same side of $P$ as $B$.

**Proof.** Suppose not. Then at least $n$ vertices of $T$ are separated from $B$ by $P$, and hence so is their convex hull, which is then a face of $T$ not intersecting $B$, contradicting the hypothesis. \qed

**Corollary 6.** Let $T_1$ and $T_2$ be two simplices in $\mathbb{H}^n$, let $B_1$ and $B_2$ be balls intersecting all the faces of $T_1$ and $T_2$, respectively, and let $P$ be a hyperplane such that $B_1$ and $B_2$ are on different sides of $P$, and which does not contain $T_1 \cap T_2$. Then there are at least 2 vertices of $T_1$ on one side of $P$ and at least 2 vertices of $T_2$ on the other side.

**Proof.** Follows immediately from Lemma 6. \qed

Let us now assume that there is no ball which all the faces intersect. Let us assume, for convenience, that all the faces of the polytopes in
the sequence are simplicial (if not, we can always triangulate them, with the additional dihedral angles equal to \( \pi \). For each face \( F_i \) we have the ball \( B_i \) which intersects all of its faces and there must be a pair of adjacent faces \( F_i, F_j \) such that the \( B_i \) and \( B_j \) are far apart. Let \( E_{ij} \) be \( F_i \cap F_j \), and there must be a cycle of faces \( f_1 = F_i, f_2 = F_j, f_3, \ldots, f_n = f_1 \) which give a dual quasi-geodesic of length close to \( 2\pi \) and a corresponding plane \( P \) (as in Section 4). By the lemma, the set of vertices of our polytope is separated by \( P \) into two sets, the cardinality of each of which is at least 2, and the limiting object is the disjoint union of two limits, one on each side of \( P \), and the limiting volume is the sum of the two volumes. One can then induct on the number of vertices to show that both halves are degenerate, and hence so is the limit.

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