Dissipative Quantum Systems and the Heat Capacity Enigma

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We present a detailed study of the quantum dissipative dynamics of a charged particle in a magnetic field. Our focus of attention is the effect of dissipation on the low- and high-temperature behavior of the specific heat at constant volume. After providing a brief overview of two distinct approaches to the statistical mechanics of dissipative quantum systems, viz., the ensemble approach of Gibbs and the quantum Brownian motion approach due to Einstein, we present exact analyses of the specific heat. While the low-temperature expressions for the specific heat, based on the two approaches, are in conformity with power-law temperature-dependence, predicted by the third law of thermodynamics, and the high-temperature expressions are in agreement with the classical equipartition theorem, there are surprising differences between the dependencies of the specific heat on different parameters in the theory, when calculations are done from these two distinct methods.

In particular, we find puzzling influences of boundary-confinement and the bath-induced spectral cutoff frequency. Further, when it comes to the issue of approach to equilibrium, based on the Einstein method, the way the asymptotic limit \( t \to \infty \) is taken, seems to assume significance.

I. INTRODUCTION

Recent years have seen great strides in the statistical mechanics of dissipative quantum systems \([1]\). Dissipation arises when the quantum degrees of freedom of a heat bath, which is strongly coupled to a subsystem of interest, are projected (or integrated) out of the Hilbert space of the total system. Two different approaches, detailed below in Sec.II, have been used in this context: (i) the usual Gibbs approach that focuses on the partition function \([2]\) and (ii) the Einstein approach that hinges on a quantum Langevin equation for the subsystem \([3]\). Lately it has been argued that the presence of quantum dissipation yields a satisfactory behavior of the fundamental thermodynamic attribute, viz., the heat capacity, as far as the low-temperature properties are concerned \([1]\). Here we will point out that there are some puzzling issues even for the high temperature limit of the heat capacity, apart from the intriguing low-temperature attributes. Before we address this question, it is important to review the kind of subsystem we have in mind and the foundational basis of statistical mechanics, which we do below. While our present discussion as well as that in Sec.II are set within the domain of classical statistical mechanics, extension to quantum mechanics can be easily carried out, as indicated in Sec.III. But we want to first concentrate on some preliminaries about the subject of statistical mechanics itself.

Statistical Mechanics provides the microscopic basis of the macroscopic properties of a system described by the subject of thermodynamics. Though the power of statistical mechanics comes to the fore in its full glory for an interacting many body system, such as in the exact formulation of second order phase transitions by means of the two-dimensional Ising model \([3]\), many of the intricacies can be elucidated for just a single entity, albeit in contact with a heat bath comprising an infinitely large number of (invisible) degrees of freedom. It is this simplified approach to statistical mechanics in the context of a single particle embedded in a heat bath that we shall adopt in this paper.

The dynamics of a particle of mass \( m \) is described by the system Hamiltonian defined by

\[
\mathcal{H}_S = \frac{\vec{p}^2}{2m} + V(\vec{q}) ,
\]

where \( \vec{p} \) is the canonical momentum vector of the particle moving under an arbitrary potential \( V(\vec{q}) \) which is a function of the generalized coordinate vector \( \vec{q} \). We shall discuss three distinct cases in the sequel:

(a) Free particle:

\[
V(\vec{q}) = 0 ,
\]

(b) Harmonic oscillator:

\[
V(\vec{q}) = \frac{1}{2} m \omega_0^2 \vec{q}^2 ,
\]

\( \omega_0 \) being the frequency of the oscillator, and

(c) Charged oscillator in a magnetic field, that is described by a momentum and coordinate-dependent potential:

\[
V(\vec{q}, \vec{p}) = -\frac{e}{2mc} (\vec{p} \cdot \vec{A}(\vec{q}) + \vec{A}(\vec{q}) \cdot \vec{p}) + \frac{e^2}{2mc^2} \vec{A}^2(\vec{q}) + \frac{1}{2} m \omega_0^2 \vec{q}^2 ,
\]

\( \vec{A}(\vec{q}) \) being the vector potential, the curl of which yields the magnetic field \( \vec{B} \):

\[
\vec{B} = \nabla \times \vec{A}(\vec{q}) .
\]
It is evident that for zero vector potential, case (c) reduces to (b). If additionally, \( \omega_0 \) is also zero, case (a) is obtained. In what way are these limiting situations arrived at, for a quantum dissipative system, will indeed be the focus of our discussion below.

It should be mentioned here that the problem of a charged oscillator in a magnetic field is relevant in the context of Landau diamagnetism [8] which has had a deep impact on modern condensed matter physics through phenomena such as the quantum Hall effect [9]. Landau diamagnetism, which is purely quantum in origin, is characterized by strong boundary effects that can be mimicked by the oscillator potential [8]. The presence of a quantum bath, comprising of, say, bosonic excitations like phonons, lends additional richness to the problem as it allows us to study the effect of dissipation on Landau diamagnetism [8]. In this article however our focus of attention is not diamagnetism but the thermodynamic property of the heat capacity.

The microstate of the particle at a given time is specified by a point in the 6-dimensional (three for coordinates and three for momenta) phase space. As the time evolves the phase point curves out a phase trajectory. While in classical mechanics the trajectory is uniquely deterministic, once the initial values of \( \vec{q} \) and \( \vec{p} \) are given, the point of statistical mechanics is that the phase trajectory randomly changes from one ‘realization’ of the system to another. The meaning of ‘realization’ becomes clear if one considers how experiments are performed. A realization corresponds to a given experiment when one watches the trajectory evolve in time. Of course, the whole statistical basis of data collection is to repeat the experiment, this time tracking a different trajectory, even though the initial values of \( \{\vec{q}, \vec{p}\} \) are the same. It is this multitude of trajectories corresponding to multiple realizations of the system that yields the concept of ‘ensemble’ in statistical mechanics — an ensemble means a collection of possible realizations of the system. Thermal equilibrium is said to be reached when experiments are repeated so many times that all possible trajectories (realizations) in the phase space are explored—this yields the notion of ‘mixing’ [10].

With these preliminaries the outline of the paper is as follows. In Sec.II, we review the Gibbs and Einstein approaches to statistical mechanics. Although our treatments are couched in classical terms similar results hold for quantum phenomena as well. With these approaches in the background we summarize in Sec.III, the newly developed subject of dissipative quantum systems. In Sec.IV we analyze the results for the heat capacity for the three problems (a-c) and point out certain surprises when we consider the various limits of case (c). In Sec.V, we summarize the results.

II. GIBBS AND EINSTEIN APPROACHES TO STATISTICAL MECHANICS

The remarkable thesis of Gibbs is that for a system in thermal equilibrium the observed properties of the system can be computed from a weighted average of the values of the relevant observable at all possible phase points that lie on a constant time-slice. This approach is quite different from how experimental data are processed—by taking a time average of the ‘values’ of the observable at different times, over a very long time. The equivalence of this time-average to the Gibbsian ensemble average follows from the fascinating attribute called ‘ergodicity’, a property that is the consequence of mixing [10]. The ensemble average of an observable \( X(\vec{q}, \vec{p}) \) in equilibrium (indicated by the subscript ‘eq’ below) is defined by

\[
\langle X(\vec{q}, \vec{p}) \rangle_{eq} = \text{Tr} (\rho(\vec{q}, \vec{p}) X(\vec{q}, \vec{p})) ,
\]

where \( \text{Tr} \) (‘trace’) implies an integration over the entire phase space in classical statistical mechanics, whereas it is a sum over possible eigenstates of the full \( \mathcal{H}_S \) in Eq.(1) in quantum statistical mechanics. The Gibbs-Boltzmann weight function \( \rho(\vec{q}, \vec{p}) \) is what is called a density matrix, given by

\[
\rho(\vec{q}, \vec{p}) = \frac{\exp(-\beta \mathcal{H}_S(\vec{q}, \vec{p}))}{Z_S} ,
\]

where \( \beta = (k_B T)^{-1} \) is the inverse temperature, \( k_B \) being the Boltzmann constant. The normalization factor \( Z_S \), referred to as the partition function,

\[
Z_S = \text{Tr} (\exp(-\beta \mathcal{H}_S(\vec{q}, \vec{p}))) ,
\]

provides the critical link between statistical mechanics and thermodynamics as it leads to the Helmholtz free energy \( F \) through the relation:

\[
F_S = -\frac{1}{\beta} \ln Z_S .
\]

From \( F_S \) all thermodynamic properties can be derived.

It is of course outside the realm of Gibbsian statistical mechanics to address the issue of how equilibrium is reached. That question has to be posed in terms of models of nonequilibrium statistical mechanics, which are however not as robust and time-tested as the formulation of equilibrium statistical mechanics encapsulated by Eqs.(7)-(9). One model that stands out in this regard is based on the idea of Brownian motion [11]. In the latter one imagines the particle (much like the pollen particle of Brown [12]), the Hamiltonian of which is given by Eq.(1), is in contact with a heat bath that drives stochastic (noisy) fluctuations into the system. The idea of Brownian motion is very physical in that if one tags the particle by taking camera snapshots at different times, its dynamics would indeed appear to be random, when the particle is out of equilibrium, and even when it is in equilibrium! The stochastic dynamics is captured
by the time-dependant distribution function $P(\vec{q}, \vec{p}, t)$ in phase space that obeys the Fokker-Planck-Smoluchowski-Kramers equation \cite{13}:

$$\frac{\partial}{\partial t} P(\vec{q}, \vec{p}, t) = \left\{ \frac{\vec{p}}{m} \hat{\nabla}_q + \hat{\nabla}_p (\vec{V}(\vec{q}) + \gamma \vec{p}) \right\} P(\vec{q}, \vec{p}, t) + m\gamma k_B T \nabla^2 P(\vec{q}, \vec{p}, t) \, ,$$  

(10)

where $\gamma$ is the friction constant. The quantity $P$ plays the same role in non-equilibrium as $\rho$ does in equilibrium. Thus the averaged time-evolution of the dynamical variable $X(\vec{q}, \vec{p})$ is given by

$$\bar{X}(t) = \int d\vec{q}d\vec{p} X(\vec{q}, \vec{p}) P(\vec{q}, \vec{p}, t) \, .$$  

(11)

With the temperature-dependant prefactor in front of $\nabla^2$, it is ensured that the stationary state is indeed the thermal equilibrium state, described by $\rho$ in Eq.(7). This is consistent with the fluctuation-dissipation theorem.

Although the fluctuation-dissipation relation is a necessary condition for guaranteeing that the system transits to the thermal equilibrium distribution, as $t \rightarrow \infty$, the Brownian motion model is far from being a unique description for the approach to equilibrium. More significantly, even within the Brownian motion model, there may be different routes to approach equilibrium. For instance, we can ask: does $\lim_{t \rightarrow \infty} \bar{X}(t)$ agree with $\langle X \rangle$, as defined in Eq.(6)? The resolution to this question helps our understanding of how to relate experimentally measured quantities to their theoretically calculated values in equilibrium, as prescribed by Eq.(6), for instance (cf., comments in the last but paragraph one in Sec.II).

It is pertinent to mention here that the time-dependent approach, as formulated through Eq.(10), is based on what is called the ‘Schrödinger picture’. An equivalent description obtains through the ‘Heisenberg picture’ in which one directly considers the dynamical equations of motion:

$$\frac{\partial \vec{q}}{\partial t} = \frac{\vec{p}}{m} \, ,$$

$$\frac{\partial \vec{p}}{\partial t} = -m\omega_0^2 \vec{q} - \frac{e}{c}(\vec{q} \times \vec{B}) - \gamma \vec{p}(t) + \vec{f}(t) \, .$$  

(12)

The set of equations (12) is called the Langevin equation in which the force $\vec{f}(t)$ is a stochastic noise, defined on an ensemble for which the distribution function is given by $P(\vec{q}, \vec{p}, t)$. A particular realization of $\vec{f}(t)$ corresponds to a given trajectory, and ensemble averages are obtained by imposing the following constraints on the spectral properties of $\vec{f}(t)$:

$$\langle \vec{f}(t) \rangle = 0$$

$$\langle f_\mu(t) f_\nu(t') \rangle = 2m\gamma k_B T \delta(t - t') \delta_{\mu\nu} , \, \mu, \nu = x, y, z \, .$$  

(13)

### III. DISSIPATIVE QUANTUM SYSTEMS

In this section we move from the classical to the quantum domain and consider the case in which the quantum subsystem is put into contact with a heat bath that is also quantum mechanical. Before we indicate the steps necessary for Brownian motion in terms of what is referred to as the quantum Langevin equations \cite{12}, it is useful to backtrack and indicate how the classical Langevin equations (12) themselves are derived from a system-plus-bath method. Here we start from a treatment of Zwanzig \cite{12} in which the Hamiltonian in Eq. (1) is extended as

$$\mathcal{H} = \mathcal{H}_S + \sum_j \left[ \frac{\vec{p}_j^2}{2m_j} + \frac{1}{2} m_j \omega_j^2 (\vec{q}_j - \frac{C_j \vec{q}}{m_j \omega_j^2})^2 \right] \, .$$  

(14)

Upon expanding the square over the round brackets it is evident that the Hamiltonian contains a linear coupling between the coordinate $\vec{q}$ of the subsystem and the coordinate $\vec{q}_j$ of the harmonic bath with $C_j$ being a coupling constant.

From Eq. (14) it is easy to write down Hamilton’s equations of motion, solve for the bath coordinates and momenta, put the solutions back in the equations of motion for the subsystem variables and derive for the momentum the generalized Langevin equation \cite{12, 15}:

$$m\ddot{q} = -m\omega_0^2 q - \frac{e}{c}(\dot{\vec{q}} \times \vec{B}) - m \int_0^t dt' \gamma(t - t') + \vec{f}(t) \, ,$$  

(15)

where the “friction“ $\gamma(t)$, that appears as a memory function, depends quadratically on $C_j$ and the noise $\vec{f}(t)$ depends explicitly on initial coordinates and the momenta of the bath oscillators:

$$\gamma(t) = \sum_j \frac{C_j^2}{m_j \omega_j^2} \cos(\omega_j t) \, ,$$

(16)

$$\vec{f}(t) = \sum_j \left[ C_j \vec{q}_j(0) - \frac{C_j \vec{q}(0)}{m_j \omega_j^2} \cos(\omega_j t) + \frac{C_j \vec{p}_j(0)}{m_j \omega_j} \sin(\omega_j t) \right] \, .$$  

(17)

Suffice it to note that Eq. (15) is exact and devoid of any assumption except that we have decided to integrate the equation of motion in the forward direction of time, thereby giving a sense to the ‘arrow of time’. The next step however is a crucial one of introducing irreversibility by considering an initial ensemble of states, a’ la Gibbs, in which the bath variables are drawn at random from a canonical distribution (Eq.(7)), yielding

$$\langle f_\mu(t) f_\nu(t') \rangle = \delta_{\mu\nu} 2m k_B T \gamma(t - t') \, .$$  

(18)

The final step is to go to the limit of an infinitely large system in order to endow the harmonic oscillator system
the attribute of a heat bath. Thus

$$\frac{1}{N} \sum_j C_j^2 \ldots \to \int d\omega g(\omega), \quad m_j = m, \quad C_j = \frac{C}{\sqrt{N}}, \quad (19)$$

where $g(\omega)$ is the ‘spectral density’. Equation (16) then yields

$$\gamma(t) = \frac{C^2}{m} \int_0^\infty d\omega \frac{g(\omega)}{\omega^2} \cos(\omega t). \quad (20)$$

A commonly assumed form of $g(\omega)$ is the one which yields what is called Ohmic dissipation, and is given by

$$g(\omega) = \begin{cases} \frac{\omega^2}{\bar{\omega}}, & \omega < \bar{\omega} \\ 0, & \omega > \bar{\omega}, \end{cases} \quad (21)$$

$\bar{\omega}$ being a high-frequency cut-off. Employing Eq.(21) we derive Eq.(12), implying that Ohmic dissipation corresponds to constant friction $\gamma$ because the generalized friction coefficient reduces to $\gamma \delta(t - t')$, wherein $\gamma$ equals $\frac{3\pi k^2}{2m\omega_0^2}$.

The discussion in the quantum case proceeds along similar lines in which one has to however keep track of the fact that $\hat{q}$ and $\hat{p}$ are non-commuting operators, and consequently, the noise $\hat{f}$ in Eq.(17) is also a quantum operator. Additionally, because the bath oscillators are to be treated quantum mechanically, the noise correlations are not ‘white’, as in Eq.(13), but are characterized by both a symmetric combination and a commutator structure, respectively given by

$$\langle [f_\mu(t), f_\nu(t')] \rangle = \delta_{\mu\nu} \frac{2}{\pi} \int_0^\infty d\omega \mathcal{R}[\hat{f}(\omega + i0^+)] \omega \coth\left(\frac{\beta \omega}{2}\right) \times \cos[\omega(t - t')]. \quad (22)$$

$$\langle [f_\mu(t), f_\nu(t')] \rangle = \delta_{\mu\nu} \frac{2}{i\pi} \int_0^\infty d\omega \mathcal{R}[\hat{f}(\omega + i0^+)] \times \omega \sin[\omega(t - t')]. \quad (23)$$

At this point it is pertinent to ask: which system is $\beta$ (as in Eq.(22)) the inverse temperature of? In the Einstein approach, discussed so far in this section, it is clear that $\beta$ represents the harmonic oscillator bath which the subsystem of interest, described by $\mathcal{H}_S$ in Eq.(1), is expected to eventually come to equilibrium with. However, because the interaction between the subsystem and the bath is treated exactly there is no reason for not thinking of the entire system, represented by the Hamiltonian $\mathcal{H}$ in Eq.(14), as one composite many body entity, which is further embedded in yet another external bath, the inverse temperature of which is also given by $\beta$! This then summarizes the Gibbsian approach in which one writes the full partition function by replacing $\mathcal{H}_S$ in Eq.(8) by Eq.(14):

$$Z = Tr \left( e^{\cdot\cdot\cdot[\beta \mathcal{H}]} \right). \quad (24)$$

It is customary to rewrite $Z$ as a functional integral:

$$Z = \int D[q, p, \hat{q}, \hat{p}] \exp\left(-\frac{1}{\hbar} A_e[q, p, \hat{q}, \hat{p}]\right), \quad (25)$$

where $\hbar$ is the Planck constant and $A_e$ is the so-called Euclidean action, defined by

$$A_e = \int_0^{\hbar \beta} d\tau \mathcal{L}(\tau), \quad (26)$$

$\mathcal{L}(\tau)$ being the Lagrangian written in terms of the ‘imaginary time’ $\tau = i\hbar \beta$. We illustrate in Sec.IV below the application of Gibbs and Einstein approaches to the calculation of the heat capacity for the charged oscillator in a magnetic field.

### IV. HEAT CAPACITY

The heat capacity or the specific heat at constant volume is the most basic thermodynamic property. It is defined by

$$C = -k_B \beta^2 \left( \frac{\partial U}{\partial \beta} \right)_V \quad , \quad (27)$$

where $U$ is the internal energy. From a statistical mechanical point of view $C$ is also related to the mean squared energy fluctuations given by

$$C = k_B \beta^2 \left( \langle \mathcal{H}^2 \rangle - \langle \mathcal{H} \rangle^2 \right). \quad (28)$$

While in the Gibbs approach $C$ can be directly computed from Eq.(25), employing the definition in either Eq.(27) or Eq.(28), the quantities $\langle \mathcal{H}^2 \rangle$ and $\langle \mathcal{H} \rangle^2$ are functions of the time $t$, in the Einstein approach. Correspondingly, $C$ will also be a function of $t$, and the question we address is under what circumstances do we have the following equality:

$$\lim_{t \to \infty} C(t)^{\text{Einstein}} = C^{\text{Gibbs}}? \quad (29)$$

#### A. Gibbs Approach ($\omega_0 \neq 0$)

Before we discuss the calculation of $C^{\text{Gibbs}}$ for the dissipative charged oscillator in a magnetic field it is useful to indicate the steps for the simpler problem without dissipative coupling, viz; that described by $\mathcal{H}_S$ alone (Eqs.(1) and (4)) [19]. The corresponding Lagrangian for the two-dimensional motion in the plane normal to the field is given by

$$\mathcal{L} = \frac{1}{2} m(x^2 + y^2) - \frac{1}{2} m\omega_0^2 (x^2 + y^2) - \frac{e}{c} (x A_x + y A_y). \quad (30)$$

It is customary to work in the so-called "symmetric gauge" in which

$$A_x = \frac{1}{2} y B, \quad A_y = \frac{1}{2} x B. \quad (31)$$
The Euclidean action can be written as
\[ A_e[x, y] = \frac{m}{2} \int_0^{\beta} dt [ (\dot{x}(\tau)^2 + \dot{y}(\tau)^2) + \omega_c^2 (x(\tau)^2 + y(\tau)^2) ] - i \omega_c (x(\tau)\dot{y}(\tau) - y(\tau)\dot{x}(\tau)) \] (32)
where, \( \omega_c \) being the "cyclotron frequency" given by
\[ \omega_c = \frac{eB}{mc} . \] (33)

Introducing
\[ x(\tau) = \sum_j \tilde{x}(\nu_j) \exp(-i\nu_j \tau) , \] (34)
where \( \nu_j \)'s are the so called Matsubara frequencies, defined by
\[ \nu_j = \frac{2\pi j}{\hbar \beta} \quad j = 0, \pm 1, \pm 2, \ldots , \] (35)
we find
\[ A_e[z_+, z_-] = \frac{1}{2} m \hbar \beta \sum_{j=-\infty}^{\infty} \left[ (\nu_j^2 + \omega_0^2 + i\omega_c \nu_j) \tilde{z}_+(\nu_j) \tilde{z}_+(\nu_j) + (\nu_j^2 + \omega_0^2 - i\omega_c \nu_j) \tilde{z}_-(\nu_j) \tilde{z}_-(\nu_j) \right] , \] (36)
where
\[ \tilde{z}_\pm(\nu_j) = \frac{1}{\sqrt{2 \pi}} (\tilde{x}(\nu_j) \pm i \tilde{y}(\nu_j)) . \] (37)

As shown in Ref.\[19\] the partition function \( Z_S \) in equation (8) can be written as (cf., also Eq.(25))
\[ Z = \prod_{j=1}^{\infty} Z_j^+ Z_j^- , \] (38)
where,
\[ Z_j^+ = \frac{1}{\sqrt{2\pi \hbar^2 \beta}} m \int_{-\infty}^{\infty} dz_+(0) \exp \left[ -\frac{m\beta \omega_0^2}{2} |z_+(0)|^2 \right] \times \prod_{j=1}^{\infty} \int_{-\infty}^{\infty} \frac{d \text{Re}z_+ d \text{Im}z_+}{\pi/(m \beta \nu_j^2)} \times \exp \left[ -m\beta (\nu_j^2 + \omega_0^2 - i\omega_c \nu_j) (\text{Re}z_{+j}^2 + \text{Im}z_{+j}^2) \right] , \] (39)
and
\[ Z_j^- = (Z_j^+)^* . \] (40)

Carrying out the Gaussian integrals we find
\[ Z_j^+ = \frac{1}{\beta \hbar \omega_0} \frac{\nu_j^2}{(\nu_j^2 + \omega_0^2 - i\omega_c \nu_j)} . \] (41)

Hence,
\[ Z_S = \left( \frac{1}{\beta \hbar \omega_0} \right)^2 \prod_{j=1}^{\infty} \frac{\nu_j^4}{(\nu_j^2 + \omega_0^2)^2 + \omega_c^2 \nu_j^2} . \] (42)

Turning now to the dissipative system described by the full many body Hamiltonian in Eq.(14) we can similarly derive \[19\]
\[ Z(\omega) = \frac{1}{(\hbar \beta \omega_0)^2} \prod_{j=1}^{\infty} \frac{\nu_j^4}{(\nu_j^2 + \omega_0^2 + \nu_j^2 \gamma(\nu_j))^2 + \omega_c^2 \nu_j^2} , \] (43)
where \( \gamma(\nu_j) \) is the frequency (ie., \( \nu_j \)) dependent friction coefficient. The Ohmic dissipation model, discussed earlier in Eq.(21) that yields constant friction, is not suitable for calculating \( Z \) as it leads to a singularity. In order to regularize the latter it is convenient to introduce a ‘Drude cut-off’ by writing the spectral density as (cf., Eq.(21))
\[ g(\omega) = \frac{2\sqrt{\pi} \omega^2}{\pi e^2} \frac{\omega^2}{1 + \omega^2/\omega_D^2} . \] (44)

Correspondingly (cf., Eq.(21)),
\[ \tilde{\gamma}(\nu_j) = \frac{\omega_D}{(\nu_j + \omega_D)} \quad \nu_j = 2\pi j / \hbar \beta . \] (45)

All our results in the sequel are restricted to Ohmic-Drude spectral density (Eqs.(21) and (44)), though it is known that other forms of frequency-dependence of the spectral density yield diverse forms of power-law dependence of the specific heat at low-temperatures \[20\].

Inserting this form of the friction coefficient in Eq.(43) the internal energy \( U \) can be calculated as
\[ U(\omega_0) = -\frac{2}{\beta} - \frac{1}{\beta} \sum_{j=1}^{3} \left[ \frac{\lambda_j}{\nu} \psi(\frac{\lambda_j}{\nu}) + \frac{\lambda_j^\prime}{\nu} \psi(\frac{\lambda_j^\prime}{\nu}) \right] \]
\[ + \frac{2}{\beta} \omega_D^2 \psi(\frac{\omega_D}{\nu}) , \] (46)
where \( \psi(z) \) is the digamma function and the arguments are:
\[ \lambda_1 + \lambda_2 + \lambda_3 = \omega_D + i\omega_c , \]
\[ \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = \omega_0^2 + \gamma \omega_D + i\omega_c \omega_D , \]
\[ \lambda_1 \lambda_2 \lambda_3 = \omega_0^2 \omega_D . \] (47)

The corresponding primed \( \lambda \)'s are obtained from the complex conjugate of Eq.(47). Finally, it is easy to derive for the heat capacity the expression (cf., Eq.(27)) \[19\]
\[ C_{\text{Gibbs}}(\omega_0 \neq 0) = -2k_B + k_B \sum_{k=1}^{3} \left( \frac{\lambda_k}{\nu} \right)^2 \psi(\frac{\lambda_k}{\nu}) \]
\[ + \left( \frac{\lambda_k^\prime}{\nu} \right)^2 \psi(\frac{\lambda_k^\prime}{\nu}) \]
\[ - 2k_B \left( \frac{\omega_D}{\nu} \right)^2 \psi(\frac{\omega_D}{\nu}) . \] (48)

We are now ready to discuss the low and high-temperature limits of the heat capacity.
(a) Low-\(T\) limit

\[
C^{\text{Gibbs}}_{(\omega_0 \neq 0)} = \frac{2\pi \gamma k_B^2 T}{3 \omega_0^2 \hbar} + \alpha_1^G T^3 + O(T^5) \tag{49}
\]

where

\[
\alpha_1^G = \frac{8\pi^3}{15} \frac{\omega_0^3}{(\hbar \omega_0)^3} \left\{ \frac{3(\omega_c^2 + \omega_0^2)}{\omega_0^2} \right\} - \left( \frac{\gamma}{\omega_0} \right)^2 - \frac{3\omega_0^2}{\omega_0^2} \frac{\gamma}{\omega_0} \frac{\omega_0}{\omega_0}.
\]

Curiously, to leading order, the presence of the magnetic field through the cyclotron frequency disappears from \(C^{\text{Gibbs}}_{(\omega_0 \neq 0)}\), the expression of which matches with that of a two-dimensional quantum oscillator (Einstein oscillator). The result in Eq.(49) has been much in discussion in recent times, in the context of the third law of thermodynamics as it provides a satisfactory power-law behavior in temperature [4].

(b) High-\(T\) limit

At high temperatures (\(\hbar \omega_c, \hbar \omega_0, \hbar \gamma, \hbar \omega_D \ll k_B T\)) our quantum system is expected to be described by classical statistical mechanics. We find

\[
C^{\text{Gibbs}}_{(\omega_0 = 0)} = 2k_B - \frac{\alpha_2^G}{T^2}. \tag{50}
\]

where

\[
\alpha_2^G = \frac{\hbar^2}{12k_B} (\omega_c^2 + 2\omega_0^2 + 2\gamma \omega_D)
\]

In the limit of infinite temperature, therefore, we recover the expected ‘equipartition’ result:

\[
C^{\text{Gibbs}}_{(\omega_0 \neq 0)} = 2k_B, \tag{51}
\]

where the factor of 2 comes from two dimensions, each of which contributes \(k_B\) to the specific heat, \(k_B\) arising from the kinetic energy while the other half from the potential energy.

B. Gibbs Approach (\(\omega_0 = 0\))

While studying dissipative Landau diamagnetism we have learnt that taking \(\omega_0 = 0\) at the outset yields puzzlingly different result from keeping \(\omega_0\) fixed, evaluating the partition function, calculating its derivatives and then setting \(\omega_0 = 0\) [1]. It is already evident from the low-temperature specific heat (Eq.(49)) that it is not meaningful to take the limit of \(\omega_0 = 0\) without ‘fixing’ the coupling with the heat bath characterized by the friction coefficient \(\gamma\)! It is therefore of interest to take a relook at the heat capacity calculation by investigating afresh the partition function for a charge in a magnetic field (without the oscillator potential). In this case only two roots \(\lambda_1\) and \(\lambda_2\) (cf., Eqs.(46)) matter [19] and we find

\[
Z(\omega_0 = 0) = \frac{N m \beta}{8\pi^3} (\gamma^2 + \omega_c^2)^2 \frac{\Pi_{k=1}^2 \Gamma(\frac{\lambda_k}{\nu}) \Gamma\left(\frac{\lambda_k}{\nu}\right)}{(\gamma(\omega_0^2))^\nu}. \tag{52}
\]

The heat capacity becomes

\[
C^{\text{Gibbs}}_{(\omega_0 = 0)} = -k_B + k_B \sum_{k=1}^2 \left\{ \left( \frac{\lambda_k}{\nu} \right)^2 \psi' \left( \frac{\lambda_k}{\nu} \right) \right\} - 2k_B \left( \frac{\omega_D}{\nu} \right)^2 \psi' \left( \frac{\omega_D}{\nu} \right). \tag{53}
\]

We now discuss the low and high temperature limits of Eq.(53).

(a) Low-\(T\) limit

Using asymptotic expansions as before, we find

\[
C^{\text{Gibbs}}_{(\omega_0 = 0)} = \frac{2\pi \gamma (1 - \frac{\gamma}{\omega_c}) k_B^2 T}{3 \hbar (\gamma^2 + \omega_c^2)} - (\alpha_3^G - \alpha_4^G) T^3 + O(T^5). \tag{54}
\]

where

\[
\alpha_2^G = \frac{8\pi^3}{15} \frac{k_B^4}{h^3} \left\{ \frac{(\omega_c^2 - 3\gamma \omega_c^2)}{\sqrt{(\gamma^2 + \omega_c^2)^3}} (1 - \frac{3\gamma}{\omega_D}) \right\} + \frac{(\omega_c^2 - 3\gamma \omega_c^2)}{\sqrt{(\gamma^2 + \omega_c^2)^3}} \left( \frac{\omega_c}{\omega_D} \right)^3 + 3\gamma \frac{\omega_c}{\omega_D} \right\} \tag{55}
\]

\[
\alpha_4^G = \frac{8\pi^3}{15} \frac{k_B^4}{h (\hbar \omega_D)^3}
\]

While Eq.(54) is in conformity with the third law of thermodynamics with identical linear temperature dependence as in the case of \(\omega_0 \neq 0\), but, is free from the singularity issue in Eq.(49) (for \(\omega_0 = 0\)). It leads, in the limit of \(\omega_D = \infty\) (infinite Drude cut-off) to the result:

\[
C^{\text{Gibbs}}_{(\omega_0 = 0)} = \frac{2\pi \gamma k_B^2 T}{3h} \frac{\gamma}{\gamma^2 + \omega_c^2}. \tag{55}
\]

Further, for very strong magnetic fields (\(\gamma \ll \omega_c\)),

\[
C^{\text{Gibbs}}_{(\omega_0 = 0)} = \frac{2\pi \gamma k_B^2 T}{3 \omega_c^2 h}, \tag{56}
\]

a harmonic oscillator like result with the cyclotron frequency \(\omega_c\) replacing \(\omega_0\). On the other hand, for weak magnetic fields (\(\gamma \gg \omega_c\)),

\[
C^{\text{Gibbs}}_{(\omega_0 = 0)} = \frac{2\pi k_B^2 T}{3 h} \frac{1}{\gamma}, \tag{57}
\]
the free particle result in which the friction coefficient \( \gamma \) appears in the denominator, in agreement to the corresponding result given in [21], after a proper counting of the degree of freedom.

(b) Hight \(-T\) limit

We find

\[
C_{(\omega_0=0)}^{\text{Gibbs}} = k_B - \frac{\hbar^2}{12k_B T^2} (\omega_c^2 + 2\gamma \omega_D) \tag{58}
\]

Again, equipartition theorem for a free particle (in 2 dimensions) prevails at \( T = \infty \).

Thus the classical limit of the Landau problem, as far as the heat capacity is concerned, is that of free particle whereas an additional (parabolic) constraining potential yields harmonic oscillator behavior.

C. Einstein approach \((\omega_0 \neq 0)\)

We will now focus on the Einstein approach based on the Langevin equation (15) which can be recast into the following convenient form [9]:

\[
\ddot{z} + \int_0^t dt' \dot{\gamma}(t-t') \dot{z}(t') + \omega_0^2 z = \frac{F(t)}{m}, \tag{59}
\]

where

\[
z = x + iy, \quad F = f_x = i f_y, \text{ and } \dot{\gamma}(t) = \gamma(t) + i \omega_c. \tag{60}
\]

In order to find the time-dependent specific heat we need the internal energy which is the statistical average of the Hamiltonian given by

\[
\mathcal{H} = \frac{1}{2} m \dot{z} \dot{z}^\dagger + \frac{i}{2} \hbar \omega_c + \frac{1}{2} m \omega_0^2 z z^\dagger. \tag{61}
\]

We therefore need the equal-time correlation functions:

\[
\zeta_1(t) = \langle z(t) z^\dagger(t) \rangle \tag{62a},
\]

\[
\zeta_2(t) = \langle \dot{z}(t) \dot{z}^\dagger(t) \rangle \tag{62b}.
\]

The correlation functions in Eq.(62) can be found from the analytic continuation to \( t' = t \) of the unequal time correlation functions, eg.,

\[
\zeta_1(t, t') = \langle z(t) z^\dagger(t') \rangle \tag{63},
\]

where \( z(t) \) can be further expressed in terms of the response function \( \chi(t) \) as

\[
z(t) = \int_0^t dt' \chi(t-t') \frac{F(t)}{m}. \tag{64}
\]

The former is the inverse Fourier transform of \( \chi(\omega) \) that can be easily written from Eq.(59) as

\[
\chi(\omega) = \frac{1}{2\pi} \frac{1}{(-\omega^2 - i\omega \gamma + \omega_0^2)}, \tag{65}
\]

with

\[
\tilde{\gamma}(\omega) = i \omega_c + \gamma(\omega) = i \omega_c + \gamma \frac{\omega_D}{\omega_D - i \omega}. \tag{66}
\]

From Eq.(63),

\[
\zeta_1(t, t') = \int_0^t dt' \int_0^{t'} dt'' \chi(t-t) \chi^*(t'-t') \times \frac{\langle F(\tau) F^\dagger(\tau') \rangle}{m}, \tag{67}
\]

where [9],

\[
\langle F(\tau) F^\dagger(\tau') \rangle = \int_{-\infty}^{+\infty} d\tilde{\omega} f(\tilde{\omega}) e^{-i\omega(\tau-\tau')} , \tag{68}
\]

with

\[
f(\tilde{\omega}) = \frac{m}{\pi} \frac{\gamma \omega_D^2}{(\omega_D^2 + \tilde{\omega}^2)} \hbar \omega \coth\left( \frac{\hbar \tilde{\omega}}{2k_B T} \right) - 1. \tag{69}
\]

Our strategy is to first calculate \( \zeta_1(t, t') \) and \( \zeta_2(t, t') \) (for details, see the Appendix A), then set \( t = t' \) and finally, in order to extract the thermal equilibrium internal energy \( E \), take the limit \( t = \infty \). We find

\[
E = \langle \mathcal{H} \rangle = \frac{1}{2} \hbar \omega_c + \frac{1}{2} m \omega_0^2 \lim_{t \to \infty} \zeta_1(t) + \frac{1}{2} m \lim_{t \to \infty} \zeta_2(t)
\]

\[
= 2k_B T + \frac{\hbar}{2\pi} \sum_{j=1}^{3} \left\{ \psi(1 + \frac{\lambda_j}{\nu}) [2 \omega_0^2 q_j + p_j] + \psi(1 + \frac{\lambda'_j}{\nu}) [2 \omega_0^2 q'_j + p'_j] \right\}, \tag{70}
\]

where

\[
q_j = \frac{(\lambda_j - \omega_D)}{\prod'_{j'}(\lambda_j - \lambda'_{j'})}, \tag{71a}
\]

\[
p_j = \frac{\lambda_j [\gamma \omega_D - i \omega_c (\lambda_j - \omega_D)]}{\prod'_{j'}(\lambda_j - \lambda'_{j'})}. \tag{71b}
\]

In the denominators of Eqs.(71), the notation \( \prod'_{j'} \) implies that the \( j = j' \) terms are excluded from the product. The quantities \( q_j \) and \( p_j \) are obtained by priming the \( \lambda \)'s, the latter having been already defined in Eq.(47).

Finally, the equilibrium specific heat is given by

\[
C_{(\omega_0 \neq 0)}^{\text{Einstein}} = \frac{\partial E}{\partial T} = -2k_B - k_B \beta \frac{\hbar}{2\pi} \sum_{j=1}^{3} \left\{ \frac{\lambda_j}{\nu} \psi'(\frac{\lambda_j}{\nu}) [2 \omega_0^2 q_j + p_j] + \frac{\lambda'_j}{\nu} \psi'(\frac{\lambda'_j}{\nu}) [2 \omega_0^2 q'_j + p'_j] \right\}, \tag{72}
\]

where \( \psi' \) are the trigamma functions [19].

We may now discuss the low and the high temperature limits of Eq.(72).
Employing the asymptotic expansion of the digamma function:

\[ \psi'(z) = \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} - \frac{1}{30z^5} - \ldots, \]  

we find

\[ C_{\omega \neq 0}^{\text{Einstein}} = \frac{2\pi}{3} \frac{\gamma k_B^2 T}{\hbar} + \alpha_1^E T^3 + O(T^5). \]  

where

\[ \alpha_1^E = \frac{8\pi^3}{15} \frac{k_B^2}{\omega_0 (\hbar \omega_0)^3} \left\{ \frac{3(\omega_c^2 + \omega_0^2)}{\omega_0^2} \right\} \]

\[ - \left( \frac{\gamma}{\omega_0} - \frac{\omega_c^2}{\omega_D} \frac{\gamma}{\omega_0} - \frac{2\gamma}{\omega_0} - \frac{\omega_0}{\omega_D} \right) \]

As required by the third law of thermodynamics the specific heat does vanish as a power law as \( T \to 0 \), exactly in the same manner as in the corresponding Gibbs expression (cf., Eq.(49)), but interestingly the coefficient of the next higher order term (\( \propto T^3 \)) differs from the Gibbs result.

(b) High-T limit

At high temperatures, \( C_{\omega \neq 0}^{\text{Einstein}} = 2k_B - \frac{\alpha_2^E}{T^2} \).  

where

\[ \alpha_2^E = \frac{k_B^2}{12}\left( \omega_c^2 + 2\omega_0^2 + \gamma \omega_D \right) \]

At infinite temperatures the classical equipartition result is restored. But again, in the next higher order term (in \( \frac{1}{T^2} \)), the Einstein result differs from the Gibbs result by a cut-off-dependent term:

\[ C_{\omega \neq 0}^{\text{Einstein}} = C_{\omega \neq 0}^{\text{Gibbs}} + \frac{\hbar^2 \gamma \omega_D}{12k_B T^2}. \]  

D. Einstein approach (\( \omega_0 = 0 \))

We now return to discuss the Einstein result for the specific heat due to the presence of the magnetic field alone, i.e., in the absence of the parabolic well. The relevant Hamiltonian is

\[ \mathcal{H} = -\frac{1}{2} \hbar \omega_c + \frac{1}{2} m \dot{z} \beta^+, \]

and hence

\[ E = -\frac{1}{2} \hbar \omega_c + \frac{1}{2} m \lim_{t \to \infty} \langle \mathcal{Z}_c(t) \rangle_{\omega_0 = 0}. \]  

As discussed in Ref.[19], one of the three roots, viz. \( \lambda_1 \) vanishes for \( \omega_0 = 0 \). Consequently (see Appendix B, for details),

\[ E(\omega_0 = 0) = k_B T \]

\[ + \frac{\hbar}{2\pi} \left\{ p_2^2 \psi(1 + \frac{\lambda_2}{\nu}) + p_3^2 \psi(1 + \frac{\lambda_3}{\nu}) \right\}. \]  

As before, the derivative of \( E \) with respect to temperature yields an expression for the specific heat in terms of the digamma functions, which can be further analyzed in the low- and high-temperature limits.

(a) Low-T limit

Again, using the asymptotic expansion of the digamma function (cf., Eq.(73)), we find

\[ C_{\omega = 0}^{\text{Einstein}} = \frac{2\pi}{3} \frac{\gamma}{\hbar} \frac{1}{\gamma^2 + \omega_c^2} k_B^2 T - \alpha_3^E T^3 + O(T^5). \]  

where

\[ \alpha_3^E = \frac{8\pi^3}{15} \frac{k_B^2}{\hbar^3} \left( \frac{3(\gamma^3 - 3\gamma^2 \omega_c^2)}{\sqrt{(\gamma^2 + \omega_c^2)^3}} \right) \]

\[ \times \left\{ (1 - \frac{2\gamma}{\omega_D}) \left( \frac{\omega_c^2}{\omega_D} \right) \right\} \]

\[ + 10(\frac{\omega_c^2}{\omega_D})^2 \frac{\gamma}{\sqrt{(\gamma^2 + \omega_c^2)^3}} \}

While the expression in Eq.(80) is in conformity with the third law of thermodynamics, as expected, it differs from the corresponding Gibbsian result of Eq.(54) in terms of different dependencies on the Drude cut-off \( \omega_D \)!

Apart from this issue the strong and weak magnetic field cases follow the behavior discussed earlier, below Eq.(54).

(b) High-T limit

\[ C_{\omega = 0}^{\text{Einstein}} = k_B - \frac{\hbar^2}{12k_B T^2}(\omega_c^2 + \gamma \omega_D). \]  

Finally, in the high-temperature limit, equipartition result obtains, but once again, there is a correction term over and above the Gibbs result that is cut-off dependent, as we found earlier in the \( \omega_0 \neq 0 \) case in Eq.(76):

\[ C_{\omega_0 = 0}^{\text{Einstein}} = C_{\omega_0 = 0}^{\text{Gibbs}} + \frac{\hbar^2 \gamma \omega_D}{12k_B T^2}, \]  

where \( C_{\omega_0 = 0}^{\text{Gibbs}} \) is given by the high-T expression in Eq.(58).
\[ \omega_0 \neq 0, \omega_0 \neq 0 \quad \omega_0 \neq 0, \omega_0 = 0 \quad \omega_0 = 0, \omega_0 = 0 \]

| Low Temperature | High Temperature |
|-----------------|------------------|
| $\frac{2\pi}{3} \frac{k_B T}{\hbar} - \frac{1}{\omega_0^2} T^3 + O(T^5)$ | $2k_B - \frac{\alpha G}{T^2}$ |
| $\frac{2\pi}{3} \frac{k_B T}{\hbar} - \frac{1}{\omega_0^2} T^3 + O(T^5)$ | $2k_B - \frac{\alpha G}{T^2}$ |
| $\frac{2\pi}{3} \frac{k_B T}{\hbar} (1 - \omega_T) - (\alpha_3 G - \alpha_4 E) T^3 + O(T^5)$ | $2k_B - \frac{\alpha G}{T^2}$ |
| $\frac{2\pi}{3} \frac{k_B T}{\hbar} (1 - \omega_T) - (\alpha_3 G - \alpha_4 E) T^3 + O(T^5)$ | $2k_B - \frac{\alpha G}{T^2}$ |

TABLE I: Comparison of Specific Heat in the Gibbs Approach and the Einstein Approach in different limits.

| Specific Heat | Magnetization |
|---------------|---------------|
| Low Temperature | High Temperature |
| $\omega_0 \to 0$, $t \to \infty$ | $2k_B - \frac{\alpha G}{T^2}$ |
| $t \to \infty$, $\omega_0 \to 0$ | $2k_B - \frac{\alpha E}{T^2}$ |
| Singularity | $2k_B - \frac{\alpha E}{T^2}$ |

TABLE II: Specific Heat and Magnetization in the limit of vanishing confinement frequency in two sequences.

V. SUMMARY

Summarising, we study the various limiting behavior of the specific heat of a dissipative charged harmonic oscillator in a uniform magnetic field, obtained from the partition function approach (Gibbs’ method) and from the steady state of corresponding quantum Langevin equation (Einstein’s approach). The specific heat obtained from both these methods shows linear $T$ dependence at low temperatures, which is in agreement with the third law of thermodynamics. At high temperatures the specific heat approaches a constant value depending on the number of degrees of freedom of the system. Although, both the Gibbs and Einstein approaches are in conformity with the third law of thermodynamics and the equipartition theorem, at low and high temperatures respectively, they differ from each other in detail, beyond the leading order. In the limit of vanishing confinement frequency ($\omega_0 \to 0$), the specific heat of the oscillator becomes singular at low-temperatures and manifests extra degrees of freedom counting at high temperatures. The specific heat of the free particle cannot be obtained from the equilibrium value ($t \to \infty$) of the specific heat of the oscillator just by taking the $\omega_0 \to 0$ limit. It is evident that the order in which one takes the $t = \infty$ and $\omega_0 = 0$ limits yield qualitatively different answers for the specific heat. While in the Einstein approach, the free particle-like specific heat emerges by taking the $\omega_0 = 0$ limit first before considering the $t = \infty$ limit, the Gibbs approach is plagued by a singularity issue, for $\omega_0 = 0$, in the low-temperature limit (cf., Eq.(49)).

In Table I, we summarise our results for the Specific Heat in different limits for both the Gibbs and Einstein approaches. In the limit of $\omega_0 \to \infty$, both the Gibbs and Einstein approaches give the same thermodynamic results. However, for a finite cutoff frequency $\omega_0$, the results differ in next to the leading order at both high and low temperatures. The results summarized in Table I lead to the following conclusions:

1. At low temperatures the specific heat is linear in temperature and hence the dissipative environment restores the third law of thermodynamics.

2. In the presence of the oscillator potential, the low temperature behavior of the specific heat goes as $1/\omega_0^2$ and is therefore singular in the limit of $\omega_0 \to 0$. Thus the results of the unconfined particle cannot be recovered in this limit.

3. The high temperature specific heat approaches a constant value independent of the confinement potential and depends only on the number of degrees
of freedom in agreement with the equipartition law. Again, the results of the unconfined system cannot be recovered in the limit of vanishing confinement frequency $\omega_0$.

While the issue of recovering the results of the unconfined particle, starting from the confined system and taking the limit of vanishing confinement frequency $\omega_0$ cannot be resolved at the equilibrium level, the Einstein approach has the intrinsic advantage of obtaining the results in the process of equilibration. The equilibrium results can be arrived at by taking the limit of $t \to \infty$. Hence, one could in principle ask the question, what would happen if the confinement frequency $\omega_0$ is taken to zero, before the limit $t \to \infty$ is taken. A similar result was obtained for the case of a particle in a harmonic oscillator potential [22]. The results for the two different sequences of taking the limits is summarised in Table 11. It is clear from the table that, if the limit of $\omega_0 \to 0$ is taken before the limit of $t \to \infty$, one can actually recover the results of the unconfined system for the specific heat and magnetization. It is curious to note that the result for magnetization obtained from this sequence of taking the limits is inconsistent with the Landau results, whereas when the limits are taken in the other way round, the Landau result is recovered. This is, however, due to the fact that the Landau result for magnetization can only be recovered in the presence of a confinement potential.

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APPENDIX A: EINSTEIN APPROACH ($\omega_0 \neq 0$)

With the help of the Drude cut-off frequency we can write $\chi(\omega)$ as

$$\chi(\omega) = \frac{(\omega_D - i \omega)}{\pi [\omega^2 - \omega_D^2 - i \omega \gamma \omega_D]} + \frac{\omega_D^2 \omega_D}{\pi [\omega^2 - \omega_D^2]}. \tag{A1}$$

Alternatively,

$$\chi(\omega) = \frac{(\omega + i \omega_D)}{(\omega + i \lambda_1)(\omega + i \lambda_2)(\omega + i \lambda_3)}. \tag{A2}$$

where $\lambda_j$s and $\lambda'_j$s are given by the Vieta equations (Eq.(47)). We can write Eq.(67) as

$$\zeta_1(t, t') = \langle \chi(t) \chi(t') \rangle = \frac{1}{4\pi^2m^2} \int_{-\infty}^{+\infty} d\tilde{\omega} f(\tilde{\omega})$$

$$\times \int_{-\infty}^{+\infty} d\omega \chi(\omega) \frac{(e^{i\tilde{\omega}t} - e^{-i\tilde{\omega}t})}{i(\tilde{\omega} - \omega)}$$

$$\times \int_{-\infty}^{+\infty} d\omega' \chi^*(\omega') \frac{(e^{i\omega't} - e^{-i\omega't})}{-i(\omega' - \omega)}. \tag{A3}$$

The two integrals, defined by

$$I_1 = \int_{-\infty}^{+\infty} d\omega \chi(\omega) \frac{(e^{i\tilde{\omega}t} - e^{-i\tilde{\omega}t})}{i(\tilde{\omega} - \omega)}, \tag{A4}$$

$$I_2 = \int_{-\infty}^{+\infty} d\omega' \chi^*(\omega') \frac{(e^{i\omega't} - e^{-i\omega't})}{-i(\omega' - \omega)}, \tag{A5}$$

can be expressed as

$$I_1 = \frac{2\pi}{iA} \left( \frac{\lambda_1 - \lambda_2 - \lambda_3}{\omega + i \lambda_1} \right) (\tilde{\omega} + i \lambda_1)$$

$$+ \frac{(\lambda_2 - \lambda_D)(\lambda_3 - \lambda_1)}{(\omega + i \lambda_2)} (\tilde{\omega} + i \lambda_2)$$

$$+ \frac{(\lambda_3 - \lambda_D)(\lambda_1 - \lambda_2)}{(\omega + i \lambda_3)} (\tilde{\omega} + i \lambda_3), \tag{A6}$$

$$I_2 = -\frac{2\pi}{iA} \left( \frac{\lambda'_1 - \lambda'_2 - \lambda'_3}{\omega + i \lambda'_1} \right) (\omega + i \lambda'_1)$$

$$+ \frac{(\lambda'_2 - \lambda'_D)(\lambda'_3 - \lambda'_1)}{(\omega + i \lambda'_2)} (\omega + i \lambda'_2)$$

$$+ \frac{(\lambda'_3 - \lambda'_D)(\lambda'_1 - \lambda'_2)}{(\omega + i \lambda'_3)} (\omega + i \lambda'_3). \tag{A7}$$

where

$$A = (\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_3), \tag{A8}$$

$$A' = (\lambda'_1 - \lambda'_2)(\lambda'_1 - \lambda'_3)(\lambda'_2 - \lambda'_3). \tag{A9}$$

Eq.(A3) then yields

$$\zeta_1(t, t') = \frac{1}{4\pi^2 m^2} \int_{-\infty}^{+\infty} d\tilde{\omega} f(\tilde{\omega}) I_1 I_2$$

$$= \frac{1}{4\pi^2 m^2} \int_{-\infty}^{+\infty} d\omega \frac{m}{\pi} \frac{\gamma \omega_D^2}{(\omega^2 + \tilde{\omega}^2)} h\tilde{\omega}$$

$$\times \text{coth} \left( \frac{h\tilde{\omega}}{2kT} \right) I_1 I_2$$

$$- \frac{1}{4\pi^2 m^2} \int_{-\infty}^{+\infty} d\omega' \frac{m}{\pi} \frac{\gamma \omega'_D^2}{(\omega'_2 + \tilde{\omega}^2)} h\tilde{\omega} I_1 I_2. \tag{A10}$$

The second integral vanishes for symmetry reasons, so that only the integral containing cotangent hyperbolic contributes. In order to find out the equal time correlation function $\zeta_1(t)$, we set $t = t'$. In that case the coefficients of $e^{-i\omega t}$ and $e^{i\omega t}$ matter, because in the product,
these are the only time independent parts. Now substituting Eqs.(A6) and (A7) in (A10), we can easily separate the mean squared average into two parts, one that is completely time independent and the other which is an exponentially decaying (time dependent) one. In the limit of \( t \to \infty \), the time dependent parts vanish and we are left with the equilibrium value. Finally,

\[
\zeta_1(t) = \frac{\hbar}{4\pi^2m} \int_{-\infty}^{+\infty} d\tilde{\omega} \frac{\text{coth}(\frac{\hbar\tilde{\omega}}{2k_BT})}{(\omega_D - \tilde{\omega})^2} + \omega_D^2 \tilde{\omega} 
\]

\[
\times \text{coth}\left(\frac{\hbar\tilde{\omega}}{2k_BT}\right) I'_1 I'_2 \, . \quad (A11)
\]

where

\[
I'_1 = \frac{2\pi}{i} \frac{\omega_D - i\tilde{\omega} e^{-i\tilde{\omega}t}}{(\tilde{\omega} + i\lambda_1)(\tilde{\omega} + i\lambda_2)(\tilde{\omega} + i\lambda_3)} \, , \quad (A12)
\]

\[
I'_2 = -\frac{2\pi}{i} \frac{\omega_D + i\tilde{\omega}}{(\tilde{\omega} - i\lambda'_1)(\tilde{\omega} - i\lambda'_2)(\tilde{\omega} - i\lambda'_3)} \, . \quad (A13)
\]

We can write

\[
\zeta_1(t) = \langle z(t)z^\dagger(t) \rangle = Q_1 - Q_2 \, . \quad (A14)
\]

where

\[
Q_1 = -\frac{\hbar}{2\pi m} \int_{-\infty}^{+\infty} d\tilde{\omega} \frac{\text{coth}(\frac{\hbar\tilde{\omega}}{2k_BT})}{(\omega_D - i\tilde{\omega})} \times \frac{(\tilde{\omega} + i\lambda_1)(\tilde{\omega} + i\lambda_2)(\tilde{\omega} + i\lambda_3)}{\tilde{\omega}},
\]

\[
Q_2 = \frac{h}{2\pi m} \int_{-\infty}^{+\infty} d\tilde{\omega} \frac{\text{coth}(\frac{\hbar\tilde{\omega}}{2k_BT})}{(\omega_D + i\tilde{\omega})} \times \frac{(\tilde{\omega} - i\lambda'_1)(\tilde{\omega} - i\lambda'_2)(\tilde{\omega} - i\lambda'_3)}{\tilde{\omega}} \, . \quad (A15)
\]

Assuming that the time is long enough compared to the relaxation time, we can ignore the integrals containing \( \lambda_1, \lambda_2, \lambda_3 \). After simplifications

\[
\zeta_1(t) = \langle z(t)z^\dagger(t) \rangle = \frac{2kT}{m\omega_0^2} + \frac{\hbar}{m\pi} \sum_{j=1}^{3} \left\{ q_j \psi(1 + \frac{\lambda_j}{\nu}) + q'_j \psi(1 + \frac{\lambda'_j}{\nu}) \right\} \, . \quad (A16)
\]

where \( \psi(1 + z) \) is a digamma function, \( \nu = 2\pi kT \), and the \( q_j \) and the \( q'_j \) are defined in Eq.(71a). We can observe from Eq.(A16), since \( \langle z^2 \rangle = \langle z(t)z^\dagger(t) \rangle \) that the equipartition theorem is satisfied for this two-dimensional problem.

We will calculate \( \zeta_2(t, t') \), which is defined as

\[
\zeta_2(t, t') = \langle z(t)z^\dagger(t') \rangle = \frac{\hbar}{m\pi} \int_{-\infty}^{+\infty} d\omega \omega^2 \zeta'' \text{coth}(\frac{\hbar\omega}{2kT}) \]

\[
- \frac{\hbar}{m\pi} \int_{-\infty}^{+\infty} d\omega \omega^2 \zeta'' \text{coth}(\frac{\hbar\omega}{2kT}) \quad (A17)
\]

\[
= \frac{2kT}{m} + \frac{\hbar\omega_o^2}{m\pi} \sum_{j=1}^{3} \left\{ q_j \psi(1 + \frac{\lambda_j}{\nu}) + q'_j \psi(1 + \frac{\lambda'_j}{\nu}) \right\} \]

\[
- \frac{\hbar}{m\pi} \left\{ \sum_{j=1}^{3} p_j \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda_j}{\nu}} + \sum_{j=1}^{3} p'_j \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda'_j}{\nu}} \right\} \]

\[
+ \frac{\hbar\omega_e}{m} \, . \quad (A18)
\]

where \( q_j \) and the \( q'_j \) are defined in Eq.(71a), and \( p_j \) and \( p'_j \) are given by (71b). We now use a transformation \( p_j = p_j + i\omega_j \), such that \( \sum_{j=1}^{3} p_j = 0 \) since \( \sum_{j=1}^{3} p_j = -i\omega_e \). Therefore

\[
\sum_{j=1}^{3} p_j \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda_j}{\nu}} = \sum_{j=1}^{3} P_j \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda_j}{\nu}} \]

\[
- \sum_{j=1}^{3} \frac{\omega_e}{3} \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda_j}{\nu}} \]

\[
= - \sum_{j=1}^{3} P_j \psi(1 + \frac{\lambda_j}{\nu}) \]

\[
- \sum_{j=1}^{3} \frac{\omega_e}{3} \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda'_j}{\nu}} \, . \quad (A19)
\]

In a similar fashion we can use a transformation \( P'_j = p'_j - i\omega'_j \), in such a way that \( \sum_{j=1}^{3} P'_j = 0 \) since \( \sum_{j=1}^{3} p'_j = i\omega_e \), hence

\[
\sum_{j=1}^{3} p'_j \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda'_j}{\nu}} = \sum_{j=1}^{3} P'_j \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda'_j}{\nu}} \]

\[
+ \sum_{j=1}^{3} \frac{\omega_e}{3} \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda'_j}{\nu}} \]

\[
= - \sum_{j=1}^{3} P'_j \psi(1 + \frac{\lambda'_j}{\nu}) \]

\[
+ \sum_{j=1}^{3} \frac{\omega_e}{3} \sum_{n=1}^{\infty} \frac{1}{n + \frac{\lambda'_j}{\nu}} \, . \quad (A20)
\]

Substituting Eqs.(A19) and (A20) in Eq.(A18) and using
three important properties of the digamma functions, we obtain

\[ \psi(x) - \psi(y) = \frac{(x - y)}{xy} + \sum_{n=1}^{\infty} \left[ \frac{1}{n + y} - \frac{1}{n + x} \right], \]

\[ \psi(1 + z) = \psi(z) + \frac{1}{z}, \]

\[ \sum_{j=1}^{N} a_j \sum_{n=1}^{\infty} \frac{1}{n + z_j} = - \sum_{j=1}^{N} a_j \psi(1 + z_j) \left[ \sum_{j=1}^{N} a_j = 0 \right], \tag{A21} \]

we obtain

\[ \zeta_2(t) = \langle \dot{z}(t) \dot{z}^\dagger(t) \rangle = \frac{2kT}{m} + \frac{\hbar \omega_c^2}{2\pi m} \sum_{j=1}^{3} \left\{ q_j \psi(1 + \frac{\lambda_j}{\nu}) + q_j^\dagger \psi(1 + \frac{\lambda_j^*}{\nu}) \right\} \]

\[ + \frac{\hbar}{2\pi m} \sum_{j=1}^{3} \left\{ p_j \psi(1 + \frac{\lambda_j}{\nu}) + p_j^\dagger \psi(1 + \frac{\lambda_j^*}{\nu}) \right\} \]

\[ + \frac{\hbar \omega_c}{m}. \tag{A22} \]

From Eq.(A22), we can calculate the mean squared average of the kinematic momentum of the particle in a magnetic field, given by

\[ \langle (\vec{P} - \frac{e}{c} \vec{A})^2 \rangle = m^2 \langle \dot{z}(t) \dot{z}^\dagger(t) \rangle - m \hbar \omega_c \]

\[ = 2mkT + \frac{m \hbar \omega_c^2}{\pi} \sum_{j=1}^{3} \left\{ q_j \psi(1 + \frac{\lambda_j}{\nu}) + q_j^\dagger \psi(1 + \frac{\lambda_j^*}{\nu}) \right\} \]

\[ + \frac{m \hbar}{\pi} \sum_{j=1}^{3} \left\{ p_j \psi(1 + \frac{\lambda_j}{\nu}) + p_j^\dagger \psi(1 + \frac{\lambda_j^*}{\nu}) \right\} . \tag{A23} \]

In the limit of a vanishing magnetic field, the two average values which we calculate are similar to the result obtained for a damped harmonic oscillator, as given by Weiss, of course with a different degree of freedom.

The internal energy can be obtained as

\[ E(\omega_0) = \langle H \rangle = \frac{1}{2} m \langle \dot{z} \dot{z}^\dagger \rangle - \frac{1}{2} \hbar \omega_c - \frac{1}{2} m \omega_0^2 \langle z z^\dagger \rangle . \tag{A24} \]

Taking the derivative with respect to temperature, we find

\[ C_{\omega_0 \neq 0}^{\text{Einstein}} = 2k_B \]

\[ - k_B \beta \frac{\hbar \omega^2}{2\pi} \sum_{j=1}^{3} \left\{ q_j \frac{\lambda_j}{\nu} \psi'(1 + \frac{\lambda_j}{\nu}) \right\} \]

\[ + q_j^\dagger \frac{\lambda_j^*}{\nu} \psi'(1 + \frac{\lambda_j^*}{\nu}) \}

\[ - k_B \beta \frac{\hbar}{2\pi} \sum_{j=1}^{3} \left\{ p_j \frac{\lambda_j}{\nu} \psi'(1 + \frac{\lambda_j}{\nu}) \right\} \]

\[ + p_j^\dagger \frac{\lambda_j^*}{\nu} \psi'(1 + \frac{\lambda_j^*}{\nu}) \}, \tag{A25} \]

where \( \psi'(z) \) are the trigamma functions and \( k_B \) is the Boltzmann constant. Finally employing the recurrence formula for trigamma functions leads to

\[ \psi'(1 + z) = \psi'(z) - \frac{1}{z^2}, \]

and also

\[ \sum_{j=1}^{3} \left\{ \frac{p_j}{\lambda_j} + \frac{p_j^\dagger}{\lambda_j^*} \right\} = 0 , \]

\[ \sum_{j=1}^{3} \left\{ \frac{q_j}{\lambda_j} + \frac{q_j^\dagger}{\lambda_j^*} \right\} = - \frac{1}{\omega_0^2} , \tag{A26} \]

from which we obtain Eq.(72).

APPENDIX B: EINSTEIN APPROACH \((\omega_0 = 0)\)

Here we provide details of the calculations for the case of \( \omega_0 = 0 \). Here, one of the three roots, viz., \( \lambda_1 \) vanishes and we are left with just two roots. From the Vieta equations given in Eq.(47), we can write the new equations for this particular case as \( \lambda_2 + \lambda_3 = \omega_D + i \omega_c \), \( \lambda_2 \lambda_3 = \omega_D (\gamma + i \omega_c) \). In the limit of vanishing harmonic oscillator frequency, the energy is obtained as Eq.(78)

\[ E = - \frac{1}{2} \hbar \omega_c + \frac{1}{2} m \lim_{t \to \infty} \langle \zeta_2(t) \rangle_{\omega_0 = 0} . \tag{B1} \]

We can write \( \zeta_2(t) = \langle \dot{z} \dot{z}^\dagger \rangle \) as

\[ \lim_{t \to \infty} \langle \zeta_2(t) \rangle_{\omega_0 = 0} = \frac{2k_B T}{m} + \frac{\hbar \omega_c}{m} \]

\[ + \frac{\h}{2\pi m} \left\{ p_2 \psi(1 + \frac{\lambda_2}{\nu}) + p_3 \psi(1 + \frac{\lambda_3}{\nu}) \right\} + p_2^\dagger \psi(1 + \frac{\lambda_2^*}{\nu}) + p_3^\dagger \psi(1 + \frac{\lambda_3^*}{\nu}) \right\} . \tag{B2} \]
where

\[ p_2 = \frac{[\gamma \omega_D - i \omega_c (\lambda_2 - \omega_D)]}{(\lambda_2 - \lambda_3)} \]
\[ p_3 = -\frac{[\gamma \omega_D - i \omega_c (\lambda_3 - \omega_D)]}{(\lambda_2 - \lambda_3)} . \]  (B3)

The primed roots are calculated from complex conjugates. Hence, the internal energy is

\[ E(\omega_0 = 0) = -k_B T \]
\[ + \frac{\hbar}{2\pi} \left\{ p_2 \psi(1 + \frac{\lambda_2}{\nu}) + p_3 \psi(1 + \frac{\lambda_3}{\nu}) \right\} . \]  (B4)

Correspondingly, the specific heat becomes

\[ C_{\omega_0=0}^{\text{Einstein}} = -k_B \]
\[ - k_B \beta \frac{\hbar}{2\pi} \left\{ p_2 \frac{\lambda_2}{\nu} \psi'(\frac{\lambda_2}{\nu}) + p_3 \frac{\lambda_3}{\nu} \psi'(\frac{\lambda_3}{\nu}) \right\} . \]  (B5)

This form of the specific heat has been used in the text as the basis of our discussions of the low and high temperature limits, via Eqs.(80) and (81).

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