A CHARACTERIZATION OF SOME PRIME IDEALS IN CERTAIN
F-ALGEBRAS OF HOLOMORPHIC FUNCTIONS

ROMEO MEŠTROVIĆ

Abstract. The class $M^p (1 < p < \infty)$ consists of all holomorphic functions $f$ on the
open unit disk $D$ for which
$$\int_0^{2\pi} (\log^+ Mf(\theta))^p \frac{d\theta}{2\pi} < \infty,$$
where $Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|$. The class $M^p$ equipped with the topology given
by the metric $\rho_p$ defined by
$$\rho_p(f, g) = \|f - g\|_p = \left( \int_0^{2\pi} \log^+ (1 + M(f - g)(\theta)) \frac{d\theta}{2\pi} \right)^{1/p}$$
becomes an F-algebra.

In this paper, we consider the ideal structure of the classes $M^p (1 < p < \infty)$. Our main result gives a complete characterization of prime ideals in $M^p$ which are
not dense subsets of $M^p$. As a consequence, we obtain a related Mochizuki’s result
concerning the Privalov classes $N^p (1 < p < \infty)$.

1. Introduction and Preliminaries

Let $D$ denote the open unit disk in the complex plane and let $T$ denote the boundary
of $D$. Let $L^q(T) (0 < q \leq \infty)$ be the familiar Lebesgue space on the unit circle $T$.

Following Kim [1, 2], the class $M$ consists of all holomorphic functions $f$ on $D$ for which
$$\int_0^{2\pi} \log^+ Mf(\theta) \frac{d\theta}{2\pi} < \infty,$$
where $\log^+ |a| = \max\{\log |a|, 0\}$ and
$$Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|$$
is the maximal radial function of $f$.

The study of the class $M$ has been well established in [1, 2, 3] and M. Nawrocky [4].
Kim [2, Theorems 3.1 and 6.1] showed that the space $M$ with the topology given by the
metric $\rho$ defined by
$$\rho(f, g) = \int_0^{2\pi} \log(1 + M(f - g)(\theta)) \frac{d\theta}{2\pi}, \quad f, g \in M,$$
becomes an F-algebra.

Recall that the Smirnov class $N^+$ is the set of all functions $f$ holomorphic on $D$ such that
$$\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} < +\infty,$$
where $f^*$ is the boundary function of $f$ on $T$; that is,
$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta})$$

2010 Mathematics Subject Classification: 46E10, 46J15, 46J20, 30H50, 30H15.
Keywords and Phrases: class $M^p (1 < p < \infty)$, Privalov class $N^p (1 < p < \infty)$, F-algebra, prime
ideal, principal ideal, maximal ideal.
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is the radial limit of $f$ which exists for almost every $e^{i\theta} \in \mathbb{T}$.

The classical Hardy space $H^q$ ($0 < q \leq \infty$) consists of all functions $f$ holomorphic on $\mathbb{D}$, which satisfy

$$\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi} < \infty$$

(6)

if $0 < q < \infty$, and which are bounded when $q = \infty$:

$$\sup_{z \in \mathbb{D}} |f(z)| < \infty.$$  

(7)

Although the class $M$ is essentially smaller than the class $N^+$, it was showed in [4] that the class $M$ and the Smirnov class $N^+$ have the same corresponding locally convex structure which was already established for $N^+$ by N. Yanagihara for the Smirnov class in [5] and [6] (also see [7]).

The Privalov class $N^p$ ($1 < p < \infty$) is defined as the set of all holomorphic functions $f$ on $\mathbb{D}$ such that

$$\sup_{0 \leq r < 1} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p \frac{d\theta}{2\pi} < +\infty.$$  

(9)

holds. These classes were firstly considered by Privalov in [8, p. 93], where $N^p$ is denoted as $A^q$.

Notice that for $p = 1$, the condition (9) defines the Nevanlinna class $N$ of holomorphic functions in $\mathbb{D}$.

It is known (see [9, 10, 11, 12]) that the following inclusion relations hold:

$$N^r \subset N^p \quad (r > p), \quad \bigcup_{q > 0} H^q \subset \bigcap_{p > 1} N^p, \quad \bigcup_{p > 1} N^p \subset M \subset N^+ \subset N,$$  

(10)

where the above containment relations are proper.

The study of the spaces $N^p$ ($1 < p < \infty$) was continued in 1977 by M. Stoll [13] (with the notation $(\log^+ H)^{\alpha}$ in [13]). Further, the topological and functional properties of these spaces have been studied by several authors (see [9]–[26]).

It is well known (see, e.g., [27, p. 26]) that a function $f \in N$ belongs to $N^+$ if and only if

$$f(z) = e^{i\gamma} B(z) S(z) F(z), \quad z \in \mathbb{D},$$  

(11)

where $\gamma$ is a real constant, $B$ is a Blaschke product with respect to zeros of $f(z)$, $S$ is a singular inner function and $F$ is an outer function for the class $N$, i.e.,

$$B(z) = z^m \prod_n \frac{|a_n|}{a_n} \cdot \frac{a_n - z}{1 - a_n \bar{z}}, \quad z \in \mathbb{D},$$  

(12)

where $m$ is a nonnegative integer and $\sum_n (1 - |a_n|) < \infty$,

$$S(z) = \exp \left( - \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu_k(t) \right), \quad z \in \mathbb{D},$$  

(13)

with positive singular measure $d\mu$, and

$$F(z) = \omega \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |f^*(e^{i\theta})| dt \right), \quad z \in \mathbb{D},$$  

(14)

where $\log |F^*| \in L^1(\mathbb{T})$.  

Recall that a function $I$ of the form

$$I = e^{i\gamma} BS$$  

(15)
is called an inner function. It is well known (see, e.g., [27, p.24]) that $I$ is a bounded holomorphic function on $\mathbb{D}$ such that $|I(z)| \leq 1$ for each $z \in \mathbb{D}$, $|I^*(e^{i\theta})| = 1$ and $|B^*(e^{i\theta})| = 1$ for almost every $e^{i\theta} \in \mathbb{T}$.

Privalov [8] showed the following canonical factorization theorem for the classes $N^p$ ($1 < p < \infty$).

**Theorem 1** ([8 pp. 98–100]; also see [15]). Let $p > 1$ be a fixed real number. A function $f \in N^p$ ($f \neq 0$) has a unique factorization of the form

$$f(z) = B(z)S(z)F(z),$$

where $B$ is the Blaschke product with respect to zeros of $f(z)$, $S(z)$ is a singular inner function and $F(z)$ is an outer function such that $\log^+ |F^*| \in L^p(\mathbb{T})$. Conversely, every such product $BSF$ belongs to $N^p$.

M. Stoll proved the following result.

**Theorem 2** ([13 Theorem 4.2]). For each $p > 1$ the Privalov space $N^p$ with the topology given by the metric $d_p$ defined by

$$d_p(f, g) = \left( \int_0^{2\pi} (\log^+ (1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|)^p \frac{d\theta}{2\pi}) \right)^{1/p}, \quad f, g \in N^p,$$

becomes an $F$-algebra, that is, $N^p$ is an $F$-space in which multiplication is continuous.

Recall that the function $d_1 = d$ defined on the Smirnov class $N^+$ by (17) with $p = 1$ induces the metric topology on $N^+$. N. Yanagihara [6] showed that under this topology, $N^+$ is an $F$-space.

Motivated by the mentioned investigations of the classes $M$ and $N^+$, and the fact that the classes $N^p$ ($1 < p < \infty$) are generalizations of the Smirnov class $N^+$, in [28, 29] and the author of this paper investigated the classes $M^p$ ($1 < p < \infty$) as generalizations of the class $M$. Accordingly, the class $M^p$ ($1 < p < \infty$) consists of all holomorphic functions $f$ on $\mathbb{D}$ for which

$$\int_0^{2\pi} (\log^+ Mf(\theta))^p \frac{d\theta}{2\pi} < \infty.$$

(18)

Obviously, $\bigcup_{p>1} M^p \subset M$ holds. Following [22], by analogy with the space $M$, the space $M^p$ can be equipped with the topology induced by the metric $\rho_p$ defined as

$$\rho_p(f, g) = \left( \int_0^{2\pi} \log^+ (1 + M(f - g)(\theta)) \frac{d\theta}{2\pi} \right)^{1/p},$$

with $f, g \in M^p$.

For our purposes, we will also need the following two results.

**Theorem 3** ([22 Corollary 18]). For each $p > 1$, $M^p$ is an $F$-algebra with respect to the metric topology induced by the metric $\rho_p$ defined by (19).

**Theorem 4** ([22 Theorem 16]). For each $p > 1$, the classes $M^p$ and $N^p$ coincide, and the metric spaces $(M^p, \rho_p)$ and $(N^p, d_p)$ have the same topological structure.

Since the space $M^p$ ($1 < p < \infty$) is an algebra, it can be also considered as a ring with respect to the usual ring’s operations addition and multiplication. Notice that these two operations are continuous in $M^p$ because $M^p$ is an $F$-algebra.

Motivated by several results on the ideal structure of some spaces of holomorphic functions given in [1, 9, 18] and [30–36], related investigations for the spaces $N^p$ ($1 < p < \infty$) and their Fréchet envelopes were given in [9, 37, 18, 38, 24] and [39]. Note that a survey of these results was given in [40]. A complete characterization
of principal ideals in $N^p$ which are dense in $N^p$ was given by Mochizuki [9, Theorem 3]. This result was used in [38] to obtain the $N^p$-analogue of the famous Beurling’s theorem for the Hardy spaces $H^q$ ($0 < q < \infty$) [30]. Moreover, it was proved in [37, Theorem B] that $N^p$ ($1 < p < \infty$) is a ring of Nevanlinna–Smirnov type in the sense of Mortini [34]. The structure of closed weakly dense ideals in $N^p$ was established in [24]. The structure of maximal ideals in $N^p$ was studied in [39], where related results are similar to those obtained by Roberts and Stoll [31] for the Smirnov class $N^+$. Let $R$ be a commutative ring. An ideal $I$ in $R$ is called principal if there is an element $a$ of $R$ such that $I = aR := \{ab : b \in R\}$. An ideal $I$ in $R$ is called maximal if $I \neq R$ and no proper ideal of $R$ properly contains $I$. An ideal $I$ in $R$ is called a prime ideal if for any $a, b \in R$ if $a, b \in I$ then either $a \in I$ or $b \in I$. Our goal in this paper is to characterize prime ideals in the algebras $M^p$ ($1 < p < \infty$) which are not dense subsets of $M^p$ (Theorem 5 in the next section). As an application of this result and Theorem 4, we obtain a related result of Mochizuki [9] concerning a characterization of closed prime ideals in the algebras $N^p$ ($1 < p < \infty$) which are not dense in $N^p$ (Corollary 6).

2. The main result and its proof

Let $p > 1$ be any fixed real number. For $\lambda \in \mathbb{D}$, we define

$$\mathcal{M}_\lambda = \{f \in M^p : f(\lambda) = 0\}. \quad (20)$$

It was proved in [26, Proposition 2.2] that for each $\lambda \in \mathbb{D}$, a set $\mathcal{M}_\lambda$, defined by (20) with $N^p$ instead of $M^p$, is a closed maximal ideal in $N^p$ and that $\mathcal{M}_\lambda = (z - \lambda) \quad (20)$ Proof of Theorem 2.3).

The main result of this paper (Theorem 5) is the $M^p$-analogue of Theorem 4 in [11] concerning a characterization of prime ideals in $M$ which are not dense in $M$.

**Theorem 5.** Let $\mathcal{M}$ be a nonzero prime ideal in $M^p$ ($1 < p < \infty$) which is not a dense subset of $M^p$. Then $\mathcal{M} = \mathcal{M}_\lambda$, for some $\lambda \in \mathbb{D}$, where $\mathcal{M}_\lambda$ is defined by (20).

As an immediate consequence of Theorems 4 and 5, we immediately obtain the following $N^p$-analogue of Theorem 5 established by N. Mochizuki [9].

**Corollary 6** [9, Theorem 35]]. Let $\mathcal{M}$ be a nonzero prime ideal in $N^p$ ($1 < p < \infty$) which is not a dense subset of $N^p$. Then $\mathcal{M} = \mathcal{M}_\lambda$, for some $\lambda \in \mathbb{D}$, where $\mathcal{M}_\lambda$ is defined by (20) with $N^p$ instead of $M^p$.

Notice that the $N^+$-analogue of Theorem 5 was proved by J. Roberts and M. Stoll [31, Theorem 1].

Proof of Theorem 5 is similar to those of Theorem 4 in [11], and it is based on the following five lemmas.

**Lemma 7.** Let $\mathcal{M}$ be a nonzero ideal od $M^p$. Then $\mathcal{M}$ contains a bounded holomorphic function which is not identically zero.

**Proof.** Let $f \in \mathcal{M}$ and suppose that $f \neq 0$. By Theorems 1 and 4, $f$ can be factorized as

$$f(z) = B(z)S(z)F(z), \quad z \in \mathbb{D}, \quad (21)$$

where $B$ is the Blaschke product with respect to zeros of $f$, $S$ is the singular inner function and

$$F(z) = \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |f^*(e^{it})| \, dt \right), \quad z \in \mathbb{D}, \quad (22)$$
is the outer function. Define the function \( g \) as
\[
g(z) = \exp \left( -\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log^+ |f^*(e^{it})| \, dt \right), \quad z \in \mathbb{D},
\] (23)
Then \( g \) is a bounded holomorphic function on \( \mathbb{D} \), and thus, \( g \in N^p \). Since \( \mathcal{M} \) is an ideal of \( M^p \), it follows that \( fg \in \mathcal{M} \). On the other hand, we have
\[
f(z)g(z) = B(z)S(z) \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log^- |f^*(e^{it})| \, dt \right), \quad z \in \mathbb{D},
\] (24)
where \( \log^- |a| = \log^+ |a| - \log |a| \). From the factorization (24) it follows that \( |f(z)g(z)| < 1 \) for each \( z \in \mathbb{D} \), and therefore, \( fg \in H^\infty \). Thus, the function \( fg \) has the desired property. \( \square \)

**Lemma 8.** Let \( F \in M^p \) be a function such that \( F(z) \neq 0 \) for each \( z \in \mathbb{D} \). Then there exists a sequence \( \{F_n\}_{n=1}^\infty \) of functions in \( M^p \) such that \( (F_n)^n = F \) for all positive integers \( n \) and \( F_n \to 1 \) in the space \( M^p \) as \( n \to \infty \).

**Proof.** Since by the assumption, \( F(z) \neq 0 \) for each \( z \in \mathbb{D} \), there exist a real-valued function \( \rho(z) \) on \( \mathbb{D} \) and a positive continuous function \( \phi(z) \) on \( \mathbb{D} \) for which
\[
F(z) = \rho(z)e^{i\phi(z)}, \quad z \in \mathbb{D}.
\] (25)
Now define the sequence \( \{F_n\}_{n=1}^\infty \) of functions as
\[
F_n(z) = \rho(z)^{1/n}e^{i \phi(z)/(n)}, \quad z \in \mathbb{D}.
\] (26)
Then \( F_n \) is a holomorphic function on \( \mathbb{D} \) and \( (F_n)^n = F \) for all positive integers \( n \). Since \( F \in M^p \subset N^+ \) and \( F(z) \neq 0 \) for each \( z \in \mathbb{D} \), there exists \( F^*(e^{it}) \) for almost every \( e^{it} \in \mathbb{T} \) and \( F^*(e^{it}) \neq 0 \) for almost every \( e^{it} \in \mathbb{T} \). For such a fixed \( e^{it} \), \( \rho(re^{it}) \) is a positive continuous function of the variable \( r \) on the segment \([0, 1] \). Therefore, there are positive real numbers \( l_\theta \) and \( L_\theta \) depending on \( \theta \) such that
\[
0 < l_\theta \leq \rho(re^{it}) \leq L_\theta < \infty, \quad \text{for each } r \in [0, 1].
\] (27)
Moreover, \( \phi(re^{it}) \) is also a continuous function of the variable \( r \) on the segment \([0, 1] \), and hence, it is bounded on \([0, 1] \). This shows that
\[
F_n(re^{it}) \to 1 \quad \text{as } n \to \infty,
\] (28)
uniformly on \( r \in [0, 1] \). This together with the fact that
\[
Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|,
\] (29)
yields
\[
M(F_n - 1)(\theta) \to 0 \quad \text{as } n \to \infty,
\] (30)
for almost every \( e^{it} \in \mathbb{T} \). Since \( (F_n)^n = F \), we find that
\[
\log^+ MF_n(\theta) \leq \frac{1}{n} \log^+ MF(\theta) \leq \log^+ MF(\theta), \quad n = 1, 2, \ldots,
\] (31)
Applying the elementary inequalities \( M(F_n - 1)(\theta) \leq 1 + MF_n(\theta) \), \( \log(2 + a) \leq 2 \log 2 + \log^+ a \) (\( a > 0 \)), \( (a + b)p \leq 2^{p-1}(a^p + b^p) \) (\( a, b \geq 0, p > 1 \)) and the inequality (31), we find that
\[
(\log(1 + M(F_n - 1)(\theta)))^p \leq (\log(2 + MF_n(\theta)))^p
\]
\[
\leq (2 \log 2 + \log^+ MF_n(\theta))^p
\]
\[
\leq 2^{p-1} ((2 \log 2)^p + (\log^+ MF_n(\theta))^p)
\]
\[
\leq 2^{p-1} ((2 \log 2)^p + (\log^+ MF(\theta))^p), \quad n = 1, 2, \ldots.
\] (32)
By the inequality (32) and the fact that $F \in M^p$, we conclude that $F_n \in M^p$ for every positive integer $n$. Finally, by the inequality (32) and (30), we can apply the Lebesgue dominated convergence theorem to obtain

$$\int_0^{2\pi} (\log(1 + M(F_n - 1)(\theta)))^p \frac{d\theta}{2\pi} \to 0 \quad \text{as} \quad n \to \infty,$$

(33)
or equivalently, $\rho_p(F_n, 1) \to 0$ as $n \to \infty$. This completes the proof. □

The following result can be considered as the Hardy-Littlewood maximal theorem for holomorphic functions.

**Lemma 9** (Hardy-Littlewood; see [27, Theorem 1.9, p. 12]). Let $f \in H^p$ with some $p$ such that $1 < p < \infty$, and let

$$Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|,$$

(34)

Then $Mf \in L^p(\mathbb{T})$ and

$$\int_0^{2\pi} (Mf(\theta))^p \frac{d\theta}{2\pi} \leq C_p \int_0^{2\pi} |f(e^{i\theta})|^p \frac{d\theta}{2\pi},$$

(35)

where $C_p$ is a constant depending only on $p$.

In order to prove Lemma 11, we will need the following known result.

**Lemma 10** (see [41, pp. 65–66, Proof of Lemma]). Let $B$ be an infinite Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$ and let $\{B_n\}_{n=1}^{\infty}$ be a sequence of functions defined as

$$B_n(z) = \prod_{k=1}^{n} \frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \bar{a}_k z}, \quad z \in \mathbb{D}.$$  

(36)

Then $B_n \to B$ in the space $H^2$, or equivalently,

$$\int_0^{2\pi} |B(e^{i\theta}) - B_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} \to 0 \quad \text{as} \quad n \to \infty.$$

(37)

**Lemma 11.** Let $B$ be an infinite Blaschke product with zeros $\{a_k\}_{k=1}^{\infty}$. For each positive integer $n$, take $B(z) = B_n(z)g_n(z)$ ($z \in \mathbb{D}$), where

$$B_n(z) = \prod_{k=1}^{n} \frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \bar{a}_k z}, \quad z \in \mathbb{D}$$

and

$$g_n(z) = \prod_{k=n+1}^{\infty} \frac{|a_k|}{a_k} \cdot \frac{a_k - z}{1 - \bar{a}_k z}, \quad z \in \mathbb{D}.$$  

(38)

Then for each $p > 1$, $g_n \to 1$ in the space $M^p$ as $n \to \infty$.

**Proof.** Using the inequality $\log(1 + a) \leq pa^{1/p}$ for $a > 0$ and the Cauchy-Schwarz integral inequality, we find that

$$\rho_p(g_n, 1) = \left( \int_0^{2\pi} (\log(1 + M(g_n - 1)(\theta)))^p \frac{d\theta}{2\pi} \right)^{1/p} \leq p \left( \int_0^{2\pi} M(g_n - 1)(\theta) \frac{d\theta}{2\pi} \right)^{1/p} \leq p \left( \int_0^{2\pi} (M(g_n - 1)(\theta))^2 \frac{d\theta}{2\pi} \right)^{1/2p}. \quad (40)$$
Then using the inequality (35) of Lemma 9, the well known fact that $|B_n^*(e^{i\theta})| = 1$ for almost every $e^{i\theta} \in T$ and the inequality (40), we find that

\[
\rho_p(g_n, 1) \leq p(C_2)^{1/2p} \left( \int_0^{2\pi} |g_n(e^{i\theta}) - 1|^2 \frac{d\theta}{2\pi} \right)^{1/2p} = p(C_2)^{1/2p} \left( \int_0^{2\pi} |B(e^{i\theta}) - B_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} \right)^{1/2p}. \tag{41}
\]

Finally, (37) of Lemma 10 and the inequality (41) yield $\rho_p(g_n, 1) \to 0$, or equivalently, $g_n \to 1$ in the space $M^p$ as $n \to \infty$.

**Proof of Theorem 5.** We follow the proof of Theorem 4 in [1]. Suppose that $M \neq M_\lambda$ for every $\lambda \in \mathbb{D}$. By Lemma 2, $M$ contains a bounded holomorphic function $f$ on the disk $\mathbb{D}$. Then by Theorems 1 and 4, $f$ can be factorized as a product $f = BF$, where $B$ is the Blaschke product whose zeros are the same as those of $f$, and $F$ is a bounded holomorphic function on $\mathbb{D}$ such that $F(z) \neq 0$ for all $z \in \mathbb{D}$. Since $M$ is a prime ideal, we conclude that either $F \in M$ or $B \in M$. Let $F \in M$ and let $\{F_n\}_{n=1}^\infty$ be a sequence of functions defined as in Lemma 8. Since $(F_n)^n = f$ for all $n = 1, 2, \ldots$, and taking into account that $M$ is a prime ideal, it follows that $F_n \in M$ for all $n = 1, 2, \ldots$. This together with the fact that by Lemma 8, $F_n \to 1$ in $M^p$ as $n \to \infty$, shows that $1 \in cl(M)$ (cl($M$) is the closure of $M$ in the space $M^p$). Hence, since $M$ is also an ideal, we conclude that cl($M$) = $M^p$, which is a contradiction in view of the assumption that $M$ is not dense in $M^p$. This shows that it must be $B \in M$. Then we consider the cases when $B$ is a finite Blaschke product and when $B$ is an infinite Blaschke product.

In the first case, set

\[
B(z) = z^n \prod_{k=1}^n \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \bar{a}_k z}. \tag{42}
\]

Since $B \in M$ and $M$ is a prime ideal, it follows that either $z \in M$ or $(a_k - z)/(1 - \bar{a}_k z) \in M$ for some $k \in \{1, 2, \ldots\}$. If $z \in M$, then

\[
M_0 = z M^p \subset M. \tag{43}
\]

Since by Theorem 4, $M^p = N^p$, it follows from [26 Proposition 2.2] that $M_0$ is a maximal ideal in $M^p$. Hence, from (43) we conclude that $M = M_0$. A contradiction.

If $(a_k - z)/(1 - \bar{a}_k z) \in M$ for some $k \in \{1, 2, \ldots\}$, then for such a $k$ it must be

\[
M_{a_k} = \frac{a_k - z}{1 - \bar{a}_k z} M^p \subset M. \tag{44}
\]

Notice that in view of the fact that $M^p = N^p$, by [26 Proposition 2.2], the set $(a_k - z)M^p := \{(a_k - z)f : f \in M^p\}$ is a maximal ideal in $M^p$. This together with the fact that $1/(1 - \bar{a}_k z) \in H^\infty \subset M^p$ implies that $M_{a_k}$ is a maximal ideal in $M^p$. Therefore, the inclusion (44) yields $M = M_{a_k}$. A contradiction.

It remains to consider the case when $B$ is an infinite Blaschke product, i.e.,

\[
B(z) = z^n \prod_{k=1}^\infty \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \bar{a}_k z}. \tag{45}
\]

Define the sequence $\{g_n\}_{n=1}^\infty$ of functions on $\mathbb{D}$ as

\[
g_n(z) = \prod_{k=n+1}^\infty \frac{|a_k|}{a_k} \frac{a_k - z}{1 - \bar{a}_k z}, \quad z \in \mathbb{D}, \quad n = 1, 2, \ldots. \tag{46}
\]
Then \( g_n \in \mathcal{M} \) for all \( n = 1, 2, \ldots \) because of the facts that \( \mathcal{M} \) is a prime ideal in \( M^p \) and \( B \in M^p \). By Lemma 11, \( g_n \to 1 \) in the space \( M^p \) as \( n \to \infty \). Therefore, it must be \( 1 \in \text{cl}(\mathcal{M}) \). This contradiction implies that \( \mathcal{M} = \mathcal{M}_\lambda \) for some \( \lambda \in \mathbb{D} \). This concludes proof of Theorem 5. \( \square \)



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MARITIME FACULTY KOTOR, UNIVERSITY OF MONTENEGRO, DOBROTA 36, 85330 KOTOR, MONTENEGRO
E-mail address: romeo@ucg.ac.me

MARITIME FACULTY KOTOR, UNIVERSITY OF MONTENEGRO, DOBROTA 36, 85330 KOTOR, MONTENEGRO
E-mail address: romeo@ucg.ac.me