Construction of Liouville Brownian motion via Dirichlet form theory

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Abstract. The Liouville Brownian motion which was introduced in [3] is a natural diffusion process associated with a random metric in two dimensional Liouville quantum gravity. In this paper we construct the Liouville Brownian motion via Dirichlet form theory. By showing that the Liouville measure is smooth in the strict sense, the positive continuous additive functional $(F_t)_{t \geq 0}$ of the Liouville measure in the strict sense w.r.t. the planar Brownian motion $(B_t)_{t \geq 0}$ is obtained. Then the Liouville Brownian motion can be defined as a time changed process of the planar Brownian motion $B_{F^{-1}}$.

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1 Introduction

In [3] C. Garban, R. Rhodes, V. Vargas constructed the Liouville Brownian motion (hereafter LBM) which is a canonical diffusion process in planar Liouville quantum gravity. By classical theory of Gaussian multiplicative chaos (cf. [5]), the Liouville measure $M_\gamma$, $\gamma \in (0, 2)$ (see Section 2.2) is formally defined as

$$M_\gamma(dz) = \exp \left( \gamma X(z) - \frac{\gamma^2}{2} E[X(z)^2] \right) dz,$$

where $X$ is a massive Gaussian free field on $\mathbb{R}^2$ on a probability space $(\Omega, \mathcal{F}, P)$ and $dz$ is the Lebesgue measure on $\mathbb{R}^2$. By constructing the positive continuous additive functional $(A_t)_{t \geq 0}$ of a Brownian motion $(W_t)_{t \geq 0}$ w.r.t. the Liouville measure $M_\gamma$, the Liouville Brownian motion is defined as $W_{A^{-1}}$. Subsequently, the heat kernel estimates and Dirichlet forms associated with the LBM were investigated in [4] and [1].

In this short paper, we are concerned with constructing the Liouville Brownian motion via Dirichlet form theory. It is known from [4] and [2] that the LBM is associated with the Liouville Dirichlet form. Therefore it is a natural question if one can construct the Liouville Brownian motion in view of Dirichlet form theory [2] (cf. [4] Section 1.4). In general there is no theory of Dirichlet forms which enables to get rid of the polar set and construct a Hunt process starting from all points of $\mathbb{R}^2$. Recently, using elliptic regularity results only n-regularized Liouville Brownian motion was constructed via Dirichlet form theory (see [7]). This approach, however, can not be applied to construct the LBM since the massive Gaussian free field is not a function. In principle our construction of LBM is based on the observation of power law decay of the size of balls of the Liouville measure and Proposition 3.4. By Lemma 3.1 and Proposition 3.4 we can show that $M_\gamma \in S_1$ (see Section 2.2 for the definition of $S_1$). Therefore there exists a positive continuous additive functional $(F_t)_{t \geq 0}$ in the strict sense of the planar Brownian motion $(B_t)_{t \geq 0}$.
are equipped with \( C \) w.r.t. \( \mu \) functions on \( \mathbb{R}^2 \). We denote the set of all Borel measurable functions and the set of all bounded Borel measurable \( \mathcal{A}_0 \) and \( \mathcal{B}_0(\mathbb{R}^2) \), respectively. The usual \( L^q \)-spaces \( L^q(\mathbb{R}^2, dx) \), \( q \in [1, \infty] \) are equipped with \( L^q \)-norm \( \| \cdot \|_{L^q(\mathbb{R}^2, dx)} \) with respect to the Lebesgue measure \( dx \) on \( \mathbb{R}^2 \) and \( \mathcal{A}_0 \) for \( \mathcal{A} \subset L^q(\mathbb{R}^2, dx) \). The inner product on \( L^2(\mathbb{R}^2, dx) \) is denoted by \( (\cdot, \cdot)_2 \). Let \( \mathcal{E}(f,g) := \int_{\mathbb{R}^2} (\nabla f, \nabla g) \, dx \), where \( f, g \in D(\mathcal{E}) = \{ f \in L^2(\mathbb{R}^2, dx) \mid \nabla f \in L^2(\mathbb{R}^2, dx) \} \). Let \( ((\mathcal{B}_0)_{x \in \mathbb{R}^2}, (\mathbb{P}_x)_{x \in \mathbb{R}^2}) \) be the planar Brownian motion associated to the Dirichlet form \( (\mathcal{E}, D(\mathcal{E})) \). A positive Radon measure \( \mu \) on \( \mathbb{R}^2 \) is said to be of finite energy integral if

\[
\int_{\mathbb{R}^2} |f(x)| \mu(dx) \leq c \sqrt{\mathcal{E}(f,f)}, \quad f \in D(\mathcal{E}) \cap C_0(\mathbb{R}^2),
\]

where \( c \) is some constant independent of \( f \) and \( \mathcal{E}_1(\cdot, \cdot) := \mathcal{E}(\cdot, \cdot) + (\cdot, \cdot)_2 \). A positive Radon measure \( \mu \) on \( \mathbb{R}^2 \) is of finite energy integral if and only if there exists a unique function \( U_1 \mu \in D(\mathcal{E}) \) such that

\[
\mathcal{E}_1(U_1 \mu, f) = \int_{\mathbb{R}^2} f(x) \mu(dx),
\]

for all \( f \in D(\mathcal{E}) \cap C_0(\mathbb{R}^2) \). \( U_1 \mu \) is called 1-potential of \( \mu \). For any set \( A \subset \mathbb{R}^2 \) the capacity of \( A \) is defined as

\[
\text{Cap}(A) = \inf_{A \subset B} \inf_{\gamma \in E(\mathcal{E})} \mathcal{E}_1(f,f).
\]

We denote by \( p_t(x,y) \) the transition kernel density of \( (B_t)_{t \geq 0} \). Taking the Laplace transform of \( p_t(x,y) \), we see that there exists a \( \mathcal{B}(\mathbb{R}^2) \times \mathcal{B}(\mathbb{R}^2) \) measurable non-negative map \( r_t(x,y) \) such that

\[
R_t f(x) := \int_{\mathbb{R}^2} r_t(x,y) f(y) \, dy \quad x \in \mathbb{R}^2, \quad f \in \mathcal{B}_b(\mathbb{R}^2).
\]
The density \( r_1(x, y) \) is called the resolvent kernel density. For a signed Radon measure \( \mu \) on \( \mathbb{R}^2 \), let us define
\[
R_1 \mu (x) = \int_{\mathbb{R}^2} r_1(x, y) \mu(dy), \quad x \in \mathbb{R}^2,
\]
whenever this makes sense. In particular, \( R_1 \mu \) is a version of \( U_1 \mu \) (see e.g. [2 Exercise 4.2.2]).

The family of the measures of finite energy integral is denoted by \( S_0 \). We further define \( S_{00} := \{ \mu \in S_0 \mid \mu(\mathbb{R}^2) < \infty, \| U_1 \mu \|_{L^\infty(\mathbb{R}^2, dx)} < \infty \} \). A positive Borel measure \( \mu \) on \( \mathbb{R}^2 \) is said to be smooth in the strict sense if there exists a sequence \((E_k)_{k \geq 1}\) of Borel sets increasing to \( \mathbb{R}^2 \) such that \( 1_{E_k} \cdot \mu \in S_{00} \) for each \( k \) and
\[
\mathbb{P}_x(\lim_{k \to \infty} \sigma_{x^1(E_k)} \geq \zeta) = 1, \quad \forall x \in \mathbb{R}^2,
\]
where \( \zeta \) is the lifetime of \((B_t)_{t \geq 0}\). The totality of the smooth measures in the strict sense is denoted by \( S_1 \) (see [2]). If \( \mu \in S_1 \), then there exists a unique \( A \in A^*_c \) with \( \mu = \mu_A \), i.e. \( \mu \) is the Revuz measure of \( A \) (see [2 Theorem 5.1.7]), where \( A^*_c \) denotes the family of the positive continuous additive functionals on \( \mathbb{R}^2 \) in the strict sense.

### 2.2 Massive Gaussian free field and Liouville measure

The massive Gaussian free field on \( \mathbb{R}^2 \) is a centered Gaussian random distribution (in the sense of Schwartz) on a probability space \((\Omega, \mathcal{A}, P)\) with covariance function given by the Green function \( G^{(m)} \) of the operator \( m^2 - \Delta, m > 0 \), i.e.
\[
(m^2 - \Delta)G^{(m)}(x, \cdot) = 2\pi \delta_x, \quad x \in \mathbb{R}^2,
\]
where \( \delta_x \) stands for the Dirac mass at \( x \). The massive Green function with the operator \( (m^2 - \Delta) \) can be written as
\[
G^{(m)}(x, y) = \int_0^\infty e^{-\frac{m^2}{2} s} \frac{m^2 s - \| x - y \|^2}{2s} ds = \int_1^\infty \frac{k_m(s(x - y))}{s} ds, \quad x, y \in \mathbb{R}^2,
\]
where
\[
k_m(z) = \frac{1}{2} \int_0^z e^{-\frac{m^2}{2} s} s^{\frac{d-2}{2}} ds.
\]

Note that this massive Green function is a kernel of \( \sigma \)-positive type in the sense of Kahane [5] since we integrate a continuous function of positive type w.r.t. a positive measure. Let \((c_k)_{k \geq 1}\) be an unbounded strictly increasing sequence such that \( c_1 = 1 \) and \((Y_k)_{k \geq 1}\) be a family of independent centered continuous Gaussian fields on \( \mathbb{R}^2 \) on the probability space \((\Omega, \mathcal{A}, P)\) with covariance kernel given by
\[
E[Y_k(x) Y_\ell(y)] = \int_{|x-y|} \frac{k_m(s(x - y))}{s} ds.
\]

The massive Gaussian free field is the Gaussian distribution defined by
\[
X(x) = \sum_{k \geq 1} Y_k(x).
\]
We define \( n \)-regularized field by
\[
X_n(x) = \sum_{k=1}^n Y_k(x), \quad n \geq 1.
\]
and the associated \( n \)-regularized Liouville measure by

\[
M_n(dz) = \exp \left( \gamma X_n(z) - \frac{\gamma^2}{2} \mathbb{E}[X_n(z)^2] \right) dz, \quad \gamma \in (0, 2).
\]

By the classical theory of Gaussian multiplicative chaos (see [5]), \( P \)-a.s. the family \((M_n)_{n \geq 1}\) weakly converges to the measure \( M_\gamma \), which is called Liouville measure. It is known from [5] that \( M_\gamma \) is a Radon measure on \( \mathbb{R}^2 \) and has full support.

### 3 Liouville Brownian motion via Dirichlet form theory

In this section we show that the LBM can be constructed in the framework of [2]. The following lemma is a direct consequence of [3, Proposition 2.3]:

**Lemma 3.1.** Almost surely in \( X \), the mapping

\[
x \mapsto -\int_{\mathbb{R}^2} \log \left( \frac{1}{|x-y|} \right) M_\gamma(dy)
\]

is continuous on \( \mathbb{R}^2 \) where \( \log_a := \max\{\log a, 0\} \).

The estimates of resolvent kernel density of the planar Brownian motion \((B_t)_{t \geq 0}\) are well known:

**Lemma 3.2.** For any \( x, y \in \mathbb{R}^2 \)

\[
r_1(x, y) \leq c_1 \log \frac{1}{|x-y|} + c_2,
\]

where \( c_1, c_2 > 0 \) are some constants.

**Lemma 3.3.** Almost surely in \( X \), for any relatively compact open set \( G \), \( 1_G \cdot M_\gamma \in S_0 \).

**Proof.** Note that by Lemma [3.1] and Lemma [3.2]

\[
\int_G \int_G r_1(x, y) M_\gamma(dy) M_\gamma(dx) < \infty.
\]

Hence \( 1_G \cdot M_\gamma \in S_0 \) by [2, Example 4.2.2] \( \square \)

Now we restate [8, Proposition 2.13] in our setting, which is a main ingredient to construct the LBM:

**Proposition 3.4.** Let \( \mu \) be a positive Radon measure on \( \mathbb{R}^2 \). Suppose that for some relatively compact open set \( G \subset \mathbb{R}^2 \), \( 1_G \cdot \mu \in S_0 \) and \( R_t(1_G \cdot \mu) \) is bounded \( dx \)-a.e. by a continuous function \( r_1^G \in C(\mathbb{R}^2) \). Then \( 1_G \cdot \mu \in S_0 \). In particular, if this holds for any relatively compact open set \( G \), then \( \mu \in S_1 \).

**Theorem 3.5.** Almost surely in \( X \), the Liouville measure \( M_\gamma \in S_1 \).
Proof. Let \((E_k)_{k \geq 1}\) be an increasing sequence of relatively compact open sets with \(\bigcup_{k \geq 1} E_k = \mathbb{R}^2\). Since \(M_r\) is a Radon measure, \(1_{E_k} \cdot M_r(\mathbb{R}^2) < \infty\). For any \(x \in \mathbb{R}^2\)

\[
R_1(1_{E_k} \cdot M_r)(x) = \int_{E_k} r_1(x, y) M_r(dy) \leq \int_{E_k} \left( c_1 \log_+ \frac{1}{|x-y|} + c_2 \right) M_r(dy)
\]

\[
\leq c_1 \int_{E_k} \log_+ \frac{1}{|x-y|} M_r(dy) + c_2,
\]

where \(c_i = c_2 M_r(E_k)\). By Lemma 3.1, the right hand side of (3.1) is continuous on \(\mathbb{R}^2\). Therefore by Proposition 3.4, \(M_r \in S_1\).

Therefore by [2, Theorem 5.1.7] there exists a positive continuous additive functional \((F_t)_{t \geq 0}\) of the Brownian motion \((B_t)_{t \geq 0}\) in the strict sense w.r.t. \(M_r\). Finally the LBM can be defined as

\[
\mathcal{B}_t = B_{F^{-1}_t}, \quad t \geq 0,
\]

where \(F^{-1}_t := \inf\{s > 0 \mid F_s > t\}\). As a by-product of our approach, we can directly obtain that the Liouville measure \(M_r\) charges no set of zero capacity (see [4, Lemma 1.5]).

Corollary 3.6. Let \((E_k)_{k \geq 1}\) be an increasing sequence of relatively compact open sets with \(\bigcup_{k \geq 1} E_k = \mathbb{R}^2\). Suppose \(\mu \in S_1\) with respect to the sequence of the sets \((E_k)_{k \geq 1}\). Then \(\mu\) does not charge capacity zero sets. In particular, \(M_r\) charges no set of zero capacity.

Proof. Note that by [2, Lemma 2.2.3], \(1_{E_k} \cdot \mu\) charges no set of zero capacity (see Lemma 3.3). Suppose \(N \subset \mathbb{R}^2\) be an open set such that \(\text{Cap}(N) = 0\). Then the statement follows from

\[
\mu(N) \leq \sum_{k \geq 1} \mu(1_{E_k} \cap N) = 0.
\]

Remark 3.7. (i) Note that the support and quasi support of \(M_r\) is the whole space \(\mathbb{R}^2\) (see [4]). Since \(M_r\) charges no set of zero capacity, the LBM is associated to the Liouville Dirichlet form

\[
\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^2} \langle \nabla f, \nabla g \rangle \, dx, \quad f, g \in \mathcal{F},
\]

where \(\mathcal{F} = \{ f \in L^2(\mathbb{R}^2, M_r) \cap H^1_{\text{loc}}(\mathbb{R}^2, dx) \mid \nabla f \in L^2(\mathbb{R}^2, dx)\} \) (see [2]).

(ii) The LBM at criticality (i.e. \(\gamma = 2\)) was introduced in [6]. In this case our approach can not be applied to construct the LBM at criticality since the continuous mapping as in Lemma 3.1 for the Liouville measure at criticality can not be obtained.

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