Supersymmetry with Cadabra

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Abstract

This article provides a quick guide to the implementation of supersymmetry with Cadabra, a symbolic computer algebra system. Details are provided on the implementation of Grassmann variables, fermionic fields, supercharges, and superderivatives and the treatment of superfield expressions and supersymmetric Lagrangians. Finally, the automation of supersymmetric Lagrangian generation in the Cadabra programming framework is discussed.

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1 Introduction to Supersymmetry and Cadabra

Supersymmetry, strings and branes are conjectured to be key ingredients toward establishing a unique unified theory of physics. In addition to the extension of possible symmetry groups relating bosons to fermions, new structures within the theory seem to provide approaches to treating some conceptual problems in high-energy physics, including the hierarchy problem.

The underlying calculations in supersymmetric theories, especially in higher dimensions, are technically long, and using a symbolic computer algebra system for the implementation of such highly structured theories can facilitate calculations in the model under study. Cadabra is a new computer algebra system (CAS) that was developed specifically for solutions to problems that have emerged in field theory. It was first developed to study the supersymmetry of higher-derivative effective actions [1, 2]. The extensive capabilities of Cadabra, such as the ability to deal with any type of tensor, anticommuting variables, Clifford algebras and Fierz transformations, as well as its many ready-to-run simplification algorithms and fully programmable features, make it a powerful tool to efficiently study problems emerging in quantum field theory and string theory. In [3, 4, 5], Cadabra was introduced, and the various capabilities of this new computer algebra system, such as its comprehensive functionality for large-scale tensor computations, were discussed. In [6, 7, 8, 9, 10], Cadabra was extensively used to verify or derive results in the supergravity context.

The aim of this paper is to illustrate how Cadabra is used for supersymmetric computations and provide the source code for a variety of explicit calculations, paving the way for better and faster routes to study supersymmetric aspects of string theory and M-theory in more depth via Cadabra.
1.1 Indices, Properties and Algorithms in Cadabra

In Cadabra, all objects in the form of subscripts or superscripts are considered to be *indices*. Cadabra recognizes free and dummy indices, and it has an automatic index renaming feature. Cadabra uses LaTeX syntax.

\[ \text{ex1} := \Gamma^{i j r} \Gamma^{k l m n} \Gamma_{r}; \]

We can assign mathematical properties to symbols:

\[ \Gamma_{#}::\text{GammaMatrix}. \]
\[ \partial{#}::\text{PartialDerivative}. \]

With the above declaration, the properties of \text{GammaMatrix} and \text{PartialDerivative} are attached to the symbols \( \Gamma \) and \( \partial \), respectively.

After writing the mathematical expressions, computations can be executed on them by applying Cadabra’s algorithms. There are many built-in algorithms for substitution and variation, index manipulations, sorting and simplifications, as well as those that have been designed to act on spinors and much more. In the following sections, we demonstrate the use of a variety of Cadabra’s algorithms to perform different calculations.

1.2 Gamma Matrices and Fierz transformations

In this section, we show how to derive useful gamma matrices and Fierz identities within Cadabra. We define the indices and denote the underlying dimension and attach gamma matrix property to the \( \Gamma \) symbol, and we also associate a metric with it:

\[ \{i,j,k,l,m,n\}::\text{Indices(vector)}. \]
{i,j,k,l,m,n}::Integer(0..7).
\Gamma_{#}::GammaMatrix(metric=\delta).
\delta_{m n}::KroneckerDelta.

In Cadabra, simplification steps are contained in the function \texttt{post_process},
which is executed on every new input. Based on our needs, we update it to con-
sider \texttt{sort\_product}, \texttt{eliminate\_kronecker} and \texttt{canonicalise}:

\begin{verbatim}
def post_process(ex):
    sort_product(ex)
    eliminate_kronecker(ex)
    canonicalise(ex)
    collect_terms(ex)
\end{verbatim}

We aim to find an identity for the below expression:

\begin{verbatim}
ex1:= \Gamma^{i j m} \Gamma_{k l} \Gamma_{m};
\end{verbatim}

\begin{eqnarray}
\Gamma^{i j m} \Gamma^{k l} \Gamma_{m}
\end{eqnarray} \tag{1}

By executing \texttt{join\_gamma} on the expression, which joins two fully antisymmetrized
gamma matrix products, and distributing factors over sums by \texttt{distribute}, we arrive
at the below result:

\begin{verbatim}
join_gamma(_);
distribute(_);
\end{verbatim}

\begin{eqnarray}
\Gamma^{i j k l m} \Gamma_{m} + \Gamma^{i j l k} - \Gamma^{i j k l} + \Gamma^{i k m} \Gamma_{m} \delta_{j l} - \Gamma^{j k m} \Gamma_{m} \delta_{i l} + \Gamma^{i k} \Gamma^{j l} \Gamma_{m} \delta_{i k} \delta_{j l} - \Gamma^{i l} \Gamma^{j k} \delta_{i l} - \Gamma^{i j l k} - \Gamma^{j l} \Gamma^{i k} \\
\Gamma^{i l} \Gamma^{j k} \delta_{i l} - \Gamma^{i l} \Gamma^{j k} \delta_{k l} - \Gamma^{i k m} \Gamma_{m} \delta_{i j} \delta_{k l} + \Gamma^{i l} \Gamma^{j k} \delta_{i j} \delta_{k l} + \Gamma^{i l} \Gamma^{j k} \delta_{i j} - \Gamma^{j l} \Gamma^{i k} \delta_{i j}
\end{eqnarray} \tag{2}

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By repeating the above procedure once more:

```math
\begin{align}
2\Gamma^{ijkl} + 4\Gamma^{il}\delta^{jk} - 4\Gamma^{jl}\delta^{ik} - 4\Gamma^{ik}\delta^{jl} + 4\Gamma^{jk}\delta^{il} - 6\delta^{ik}\delta^{jl} + 6\delta^{il}\delta^{jk}
\end{align}
```

Now, we perform a substitution for the term $\Gamma^{ijkl}$:

```math
\begin{align}
\text{substitute}(_, \Gamma^{ijkl} \rightarrow \Gamma^{kl}\Gamma^{ij} + \Gamma^{jk}\delta^{il} - \Gamma^{ik}\delta^{jl} - \Gamma^{jl}\delta^{ik} + \Gamma^{il}\delta^{jk} - \delta^{il}\delta^{jk} + \delta^{ik}\delta^{jl})
\end{align}
```

Finally, one can arrive at the below useful identity:

```math
\begin{align}
\Gamma^{ijm}\Gamma^{kl}\Gamma_m = 2\Gamma^{kl}\Gamma^{ij} + 6\Gamma^{jk}\delta^{il} - 6\Gamma^{ik}\delta^{jl} - 6\Gamma^{jl}\delta^{ik} + 6\Gamma^{il}\delta^{jk} + 4\delta^{il}\delta^{jk} - 4\delta^{ik}\delta^{jl}
\end{align}
```

The above substitution is obtained using the relation from the below manipulation:

```math
\begin{align}
\text{ex2:= } \Gamma^{kl}\Gamma^{ij}
\end{align}
```

On a product of four spinors, one can apply a Fierz transformation. Here, in eleven-dimensional Clifford algebra, we define our spinors as follows:
Consider the following expression of the spinor product:

\[ \bar{\epsilon} \Gamma_{m} \psi \bar{\chi} \Gamma^{m} \lambda \]  

Now, we apply a Fierz transformation, with the desired order as follows:

\[ \text{fierz}(\_, \epsilon, \lambda, \chi, \psi); \]

\[ - \frac{1}{32} \bar{\epsilon} \Gamma_{m} \Gamma^{m} \lambda \bar{\chi} \psi - \frac{1}{32} \bar{\epsilon} \Gamma_{m} \Gamma^{m} \lambda \bar{\chi} \Gamma^{m} \psi - \frac{1}{64} \bar{\epsilon} \Gamma_{m} \Gamma^{m} \lambda \bar{\chi} \Gamma^{m} \psi \\
- \frac{1}{192} \bar{\epsilon} \Gamma_{m} \Gamma^{m} \lambda \bar{\chi} \Gamma^{m} \psi - \frac{1}{768} \bar{\epsilon} \Gamma_{m} \Gamma^{m} \lambda \bar{\chi} \Gamma^{m} \psi - \frac{1}{3840} \bar{\epsilon} \Gamma_{m} \Gamma^{m} \lambda \bar{\chi} \Gamma^{m} \psi \]  

(8)

2 Superalgebra and Superfield Implementations

Here, we first look at super-Poincaré algebra. We establish the vector and spinorial indices, and then, by declaring the antisymmetric and noncommuting features available in super-Poincaré generators, we simplify a commutation relation in this regard. Additionally, in the following subsection, by introducing the superspace formalism and supercharges in Cadabra, we shed light on the implementation of superfields,
including scalar, chiral and vector superfields specifically for $N = 1$ and $D = 4$, to
derive supersymmetric transformations of component fields.

First, we define the vector and spinor indices for the superalgebra generators as
follows:

$$\{a,b,c\}::\text{Indices(spinor)};$$

$$\{\mu,\nu,\rho,\sigma,\lambda,\kappa,\alpha,\beta,\gamma,\xi\}::\text{Indices(vector)};$$

$$\{\mu,\nu,\rho,\sigma,\lambda,\kappa,\alpha,\beta,\gamma,\xi\}::\text{Integer(0..3)};$$

$$\delta{}^{#}::\text{KroneckerDelta};$$

$$\epsilon_{\mu\nu\lambda\rho}::\text{EpsilonTensor(delta=\delta);}$$

At this stage, we declare the antisymmetric and noncommuting nature of our gen-
erators by attaching the properties $\text{AntiSymmetric, SelfNonCommuting}$ and $\text{NonCommuting}$
to them.

$$\{J_{\mu\nu},\sigma^{4}_{\mu\nu}\}::\text{AntiSymmetric};$$

$$J_{\mu\nu}::\text{SelfNonCommuting};$$

$$\{J_{\mu\nu}, P_{\mu}, W_{\mu}\}::\text{NonCommuting};$$

$$\{J_{\mu\nu}, W_{\mu}, Q_{a}\}::\text{NonCommuting};$$

$$\{J_{\mu\nu}, P_{\mu}, W_{\mu}, Q_{a}\}::\text{Depends(\text{commutator}#)};$$
As mentioned in the previous section, simplification treatments are contained in the `post_process` function, which is executed on every new input. Here, we update it to consider `unwrap(ex)`, `eliminate_kronecker(ex)`, `canonicalise(ex)` and `rename_dummies(ex)`:

```python
def post_process(ex):
    unwrap(ex)
    eliminate_kronecker(ex)
    canonicalise(ex)
    rename_dummies(ex)
    collect_terms(ex)
```

Now, we establish the commutator relations for $J_{\mu\nu}, Q_a$ and $P_\mu$ generators.

```plaintext
superpoincare:= { \commutator{J_{\mu\nu}}{Q_{a}} -> -\sigma^{4}_{\mu \nu} Q_{b}, \commutator{P_{\mu}}{Q_{a}} -> 0 }; 
```
\[
[[J_{\mu\nu}, Q_a] \rightarrow -\sigma^4_{\mu\nu} Q_b, \ [P_{\mu}, Q_a] \rightarrow 0]
\]  

(10)

Below, we aim to compute the commutator between the square of the Pauli–Lubanski vector and supercharge \( Q_a \):

\[
W_{sq} := \text{\textbackslash{commutator}{W_\mu W_\mu}{Q_a}};
\]

\[
\left[ W_\mu W_\mu, Q_a \right]
\]  

(11)

We substitute the explicit definition of the Pauli–Lubanski vector into the commutator:

\[
\text{substitute(_, } W_\mu \text{ } \rightarrow \text{ } 1/2 \epsilon_{\mu\nu\lambda\rho} J_{\nu\lambda} P_{\rho})
\]

\[
1/4 \epsilon_{\alpha\mu\nu\kappa} \epsilon_{\alpha\lambda\rho\sigma} [J_{\mu\nu} P_\lambda J_{\rho\sigma} P_\kappa, Q_a]
\]  

(12)

One can replace the product of two epsilon tensors with a generalized delta by the below command:

\[
\text{epsilon_to_delta(_);} \]

\[
3/2 \delta_{\mu\rho\sigma\kappa} [J_{\mu\nu} P_\lambda J_{\rho\sigma} P_\kappa, Q_a]
\]  

(13)

Now, we expand the generalized delta into standard two-index Kronecker deltas:

\[
\text{expand_delta(_);} \]

\[
8
\]
\[
\frac{3}{2} \left( \frac{1}{6} \delta_{\lambda \kappa} \delta_{\mu \rho} \delta_{\nu \sigma} - \frac{1}{6} \delta_{\lambda \nu} \delta_{\kappa \sigma} \delta_{\mu \rho} - \frac{1}{6} \delta_{\lambda \rho} \delta_{\kappa \sigma} \delta_{\mu \nu} + \frac{1}{6} \delta_{\lambda \sigma} \delta_{\kappa \mu} \delta_{\rho \nu} + \frac{1}{6} \delta_{\lambda \nu} \delta_{\kappa \rho} \delta_{\mu \sigma} - \frac{1}{6} \delta_{\lambda \mu} \delta_{\kappa \rho} \delta_{\sigma \nu} \right)
\]

\[ [J_{\mu \nu} P_{\lambda} P_{\kappa}, Q_a] \]  

(14)

By applying \texttt{product_rule}, we can simplify the commutator in the above:

\texttt{product_rule(_);} 

\[
\frac{3}{2} \left( \frac{1}{6} \delta_{\lambda \kappa} \delta_{\mu \rho} \delta_{\nu \sigma} - \frac{1}{6} \delta_{\lambda \nu} \delta_{\kappa \sigma} \delta_{\mu \rho} - \frac{1}{6} \delta_{\lambda \rho} \delta_{\kappa \sigma} \delta_{\mu \nu} + \frac{1}{6} \delta_{\lambda \sigma} \delta_{\kappa \mu} \delta_{\rho \nu} + \frac{1}{6} \delta_{\lambda \nu} \delta_{\kappa \rho} \delta_{\mu \sigma} - \frac{1}{6} \delta_{\lambda \mu} \delta_{\kappa \rho} \delta_{\sigma \nu} \right)
\]

\[
\left( [J_{\mu \nu}, Q_a] P_{\lambda} P_{\rho} P_{\kappa} + J_{\mu \nu} [P_{\kappa}, Q_a] J_{\rho \sigma} P_{\lambda} + J_{\mu \nu} P_{\lambda} [J_{\rho \sigma}, Q_a] P_{\kappa} + J_{\mu \nu} P_{\kappa} J_{\rho \sigma} [P_{\lambda}, Q_a] \right)
\]  

(15)

At this level, one can insert the super-Poincaré commutators defined previously into the expression and obtain:

\texttt{substitute(_, superpoincare);} 

\[
\frac{3}{2} \left( \frac{1}{6} \delta_{\lambda \kappa} \delta_{\mu \rho} \delta_{\nu \sigma} - \frac{1}{6} \delta_{\lambda \nu} \delta_{\kappa \sigma} \delta_{\mu \rho} - \frac{1}{6} \delta_{\lambda \rho} \delta_{\kappa \sigma} \delta_{\mu \nu} + \frac{1}{6} \delta_{\lambda \sigma} \delta_{\kappa \mu} \delta_{\rho \nu} + \frac{1}{6} \delta_{\lambda \nu} \delta_{\kappa \rho} \delta_{\mu \sigma} - \frac{1}{6} \delta_{\lambda \mu} \delta_{\kappa \rho} \delta_{\sigma \nu} \right)
\]

\[
( -\sigma^4_{\rho \sigma} Q_b P_{\lambda} J_{\mu \nu} P_{\kappa} - J_{\rho \sigma} P_{\lambda} \sigma^4_{\mu \nu} Q_b P_{\kappa} )
\]  

(16)

Finally, we can distribute factors over sums and obtain a non-vanishing result:

\texttt{distribute(_);} 

\[
-\frac{1}{2} \sigma^4_{\mu \nu} Q_b P_{\rho} J_{\mu \nu} P_{\rho} - \frac{1}{2} J_{\mu \nu} P_{\rho} \sigma^4_{\mu \nu} Q_b P_{\rho} - \sigma^4_{\rho \mu} Q_b P_{\rho} J_{\mu \nu} P_{\nu} - J_{\mu \nu} P_{\mu} \sigma^4_{\nu \rho} Q_b P_{\rho}
\]  

(17)


2.1 Scalar, Chiral and Vector Superfield Implementations

In this section, we first define a general Lorentz scalar superfield in terms of its power series expansion. Then, by defining the derivative and antiderivative operators (for Grassmann variables) and establishing the related anticommuting properties, we aim to derive supersymmetric transformations of component fields. We achieve this by first defining the infinitesimal supersymmetry variation and applying it to the superfield with substitute and further simplifications via distribute, product_rule and unwrap. At the final stage, we sort the coefficients based on the same powers of $\theta$ in the superfield expansion to read the SUSY transformation of the component field. We carry this out via hard-coded substitutions using take_match and replace_match.

In a broader design, one can develop sorting rules to simplify general expressions containing Grassmann variables. A similar approach can be applied for the SUSY transformation of chiral and vector superfields.

Here, we start by defining a general scalar superfield $\Phi$:

\[
\text{Superfield} := \Phi \rightarrow (f(x) + \theta^\beta \phi_{\beta} + \bar{\theta}_{\dot{\beta}} \bar{\chi}^\dot{\beta} + \text{indexbracket}{\theta\theta} m(x) + \text{indexbracket}{\bar{\theta}\bar{\theta}} n(x) + \theta^\beta \sigma_{\beta \dot{\beta}}^\nu \bar{\theta}^\dot{\beta} V_{\nu} + \text{indexbracket}{\theta\theta} \bar{\theta}_{\dot{\beta}} \bar{\lambda}_{\dot{\beta}} + \text{indexbracket}{\bar{\theta}\bar{\theta}} \theta^\beta \psi_{\beta})
\]

10
\phi \rightarrow f(x) + \theta^\beta \phi_\beta + \bar{\theta} \bar{\chi} + (\theta \theta)m(x) + (\bar{\theta} \bar{\theta})n(x) + \theta^\beta \sigma_{\beta \dot{\nu}} V_{\nu} + (\theta \theta)\bar{\theta} \bar{\lambda} \bar{\psi}_\beta + (\bar{\theta} \bar{\theta})\theta \psi_\beta
+ (\theta \theta)(\bar{\theta} \bar{\theta})d(x) \\
(18)

\\parbar{#}::\LaTeXForm("\bar{\partial}").
\\partialmu{#}::\LaTeXForm("\hat{\partial}").
\\epsbar{#}::\LaTeXForm("\bar{\epsilon}").
superderivative:={\partial_{\alpha}{\theta^\beta}\to\delta_{\alpha}^\beta, 
\parbar^\aldot{\tbar_{\betdot}}\to\delta^\aldot_{\betdot}, 
\parbar^\aldot{\tbar^\betdot}\to-e^\aldot\betdot, 
\partial_{\alpha}{\indexbracket{\theta\theta}}\to2\theta_{\alpha}, 
\parbar^\aldot{\indexbracket{\tbar\tbar}}\to2\tbar^\aldot};

Property PartialDerivative attached to \[\partial\#, \barpartial\#, \hatpartial\#\].

Property AntiCommuting attached to \[[\theta\#, \bartheta\#, \epsilon\#, \barepsilon\#]\].

Property SelfAntiCommuting attached to \[[\theta^\alpha, \bartheta^\dotalpha]\].

Property ImplicitIndex attached to \[\partial(\#)\].

Property ImplicitIndex attached to \[\barpartial(\#)\].

\[e^\alpha\partial_\alpha(\Phi) + \barepsilon_\dotalpha\barpartial^\dotalpha(\Phi) + (i\theta^\mu\barepsilon)\barpartial_\mu(\Phi) - (i\epsilon^\mu\bartheta)\barpartial_\mu(\Phi)
\]
\[[\partial_\alpha(\theta^\beta)\to\delta_{\alpha}^\beta, \barpartial^{\dotalpha}(\bartheta_{\dotbeta})\to\delta^{\dotalpha}_{\dotbeta}, \barpartial^{\dotalpha}(\bartheta^{\dotbeta})\to-e^{\dotalpha\dotbeta}, \partial_\alpha((\theta\theta))\to2\theta_{\alpha}, \barpartial^{\dotalpha}((\bartheta\bartheta))\to2\bartheta^{\dotalpha}\]

(19)

substitute(susy, Superfield);
distribute(_);
product_rule(_)

{{\theta^{\#},\indexbracket{\theta\theta}} 
::Depends(\partial\#);
{\tbar_{\#},\indexbracket{\tbar\tbar}} 
::Depends(\parbar\#);
{f(x),m(x),n(x),\phi_{\alpha},\barcbar^{\aldot},V_{\nu},\lambar^{\betdot},
\psi_{\beta}}::Depends(\partialmu\#);}
unwrap(_);

\text{Property Depends attached to } [\theta^#, (\theta \theta)].

\text{Property Depends attached to } [\bar{\theta}^#, (\bar{\theta} \bar{\theta})].

\text{Property Depends attached to } [f(x), m(x), n(x), \phi_\alpha, \bar{\chi}^\alpha, V_\nu, \bar{\lambda}^\beta, \psi_\beta].

\text{take}_\text{match}(_\text{, } ][i \theta \sigma^\mu \bar{\epsilon}_\beta] [\text{mu}] [\text{epsbar}]

\text{take}_\text{match}(_\text{, } ][\theta \theta \text{theta} ] [Q??\$]

\text{substitute}(_\text{, } ][A??+ B?? \rightarrow 0\$)

\text{replace}_\text{match}(\_)

\text{take}_\text{match}(_\text{, } ][i\epsilon \sigma^\mu \bar{\epsilon}_\beta] [\text{mu}] [\text{tbar}]

\text{take}_\text{match}(_\text{, } ][\epsilon \sigma^\mu \bar{\epsilon}_\beta] [\text{mu}] [\text{tbar} \text{tbar} ] [Q??\$]

\text{substitute}(_\text{, } ][A??+ B?? \rightarrow 0\$)

\text{replace}_\text{match}(\_)

\text{substitute}(_\text{, } ][\text{superderivative})

13
eliminate_kronecker(_);

\[ e^\beta \phi_\beta + 2 \epsilon^x \theta_a m(x) + e^\beta \sigma_{\beta \bar{\beta}} \nu \bar{\theta}^\beta V_\nu + 2 \epsilon^x \theta_a \bar{\theta}^\beta \lambda^\beta + e^\beta (\bar{\theta} \theta) \psi_\beta + 2 \epsilon^x \theta_a (\bar{\theta} \theta) d(x) + \bar{\epsilon}^{\alpha} \bar{\chi}^\beta \\
+ 2 \bar{\epsilon}^{\alpha} n(x) + \bar{\epsilon}^{\alpha} \theta^\beta \sigma_{\beta \bar{\beta}} \nu e^\alpha \bar{\theta}^\beta V_\nu + \bar{\epsilon}^{\alpha} (\theta \theta) \lambda^\beta + 2 \bar{\epsilon}^{\alpha} \bar{\theta}^\alpha \theta^3 \psi_\beta + 2 \bar{\epsilon}^{\alpha} (\theta \theta) \bar{\theta}^\alpha d(x) \\
+ (i \theta \sigma^\nu \bar{\epsilon}) \partial_{\mu} f(x) + (i \theta \sigma^\nu \bar{\epsilon}) \theta^\beta \bar{\partial}_{\mu} \phi_\beta + (i \theta \sigma^\nu \bar{\epsilon}) \bar{\theta}^\beta \bar{\partial}_{\mu} \bar{\chi}^\beta + (i \theta \sigma^\nu \bar{\epsilon}) (\bar{\theta} \theta) \bar{\partial}_{\mu} n(x) \\
+ (i \theta \sigma^\nu \bar{\epsilon}) \sigma_{\beta \bar{\beta}} \nu \bar{\theta}^\beta \bar{\partial}_{\mu} V_\nu + (i \theta \sigma^\nu \bar{\epsilon}) (\theta \theta) \theta^\beta \bar{\partial}_{\mu} \psi_\beta - (i \epsilon \sigma^\mu \bar{\theta}) \bar{\partial}_{\mu} f(x) \\
- (i \epsilon \sigma^\mu \bar{\theta}) \theta^\beta \bar{\partial}_{\mu} \phi_\beta - (i \epsilon \sigma^\mu \bar{\theta}) \bar{\theta}^\beta \bar{\partial}_{\mu} \bar{\chi}^\beta - (i \epsilon \sigma^\mu \bar{\theta}) (\theta \theta) \bar{\partial}_{\mu} m(x) \\
- (i \epsilon \sigma^\mu \bar{\theta}) \sigma_{\beta \bar{\beta}} \nu \bar{\theta}^\beta \bar{\partial}_{\mu} V_\nu - (i \epsilon \sigma^\mu \bar{\theta}) (\theta \theta) \bar{\theta}^\beta \bar{\partial}_{\mu} \bar{\lambda}^\beta \\
\] (22)

take_match(_, \$\theta^x\{\beta\}\partial_{\mu}\{\mu\}\{\phi\}_{\{\beta\}} Q??$

take_match(_, \$\indexbracket{i}\theta^x\{\sigma\}_{\{\mu\}}\bar{\epsilon} Q??$

substitute(_,\$A?? B?? C??->\indexbracket{\theta\theta}$

\indexbracket{-i/2 \partial_{\nu}\{\phi\}_{\sigma\{\nu\}}\bar{\epsilon}$

replace_match(_)

replace_match(_)


take_match(_, \$\bar{t}^x_{\{\bar{\beta\dot{\beta}}\}}\partial_{\mu}\{\bar{\mu}\}\{\bar{\cbar}\}_{\{\bar{\dot{\beta}}\}} Q??$

take_match(_, \$\indexbracket{i}\epsilon^x\{\sigma\}_{\{\mu\}}\bar{\theta} Q??$

substitute(_,\$A?? B?? C??-> -\indexbracket{\theta\theta}$

\indexbracket{i/2 \epsilon^x\{\nu\}\partial_{\nu}\{\mu\}\{\sigma\}_{\{\cbar\}}}$

replace_match(_)

replace_match(_)


take_match(_, \$\indexbracket{i}\theta^x\{\sigma\}_{\{\mu\}}\bar{\epsilon} Q??$

\[14\]
\partial_{\mu}(V_{\nu} Q)\\
substitute(_, A \rightarrow i/2 \index{\theta\theta}\index{\bar{\epsilon}}\partial_{\mu}(V_{\mu} Q)\\
replace_match(_,)\\
take_match(_, \partial_{\mu}(m(x)) Q)\\
substitute(_, \index{i\epsilon}\index{\sigma}\index{\bar{\epsilon}}\index{\bar{\theta}\theta} \rightarrow -i \index{\theta\theta}\bar{\epsilon}_{\bar{\beta}}^\dot{\beta} \epsilon^{\mu} e)\\
replace_match(_,)\\
take_match(_, \partial_{\mu}(V_{\nu}) Q)\\
substitute(_, K \rightarrow i/2 \index{\theta\theta}\index{\bar{\epsilon}}\partial_{\nu}(\psi_{\beta}) Q)\\
replace_match(_,)\\
take_match(_, \partial_{\mu}(\lambda_{\dot{\beta}}) Q)\\
substitute(_, K \rightarrow -i/2 \index{\theta\theta}\index{\bar{\epsilon}}}
\epsilon\sigma^\nu_\partial\mu_\nu^{\lambda\bar{\rho}}$

replace_match(_)

take_match(_, \$t\bar{\rho}_\beta^\dot{\gamma}\lambda^{\dot{\gamma}}\bar{Q}\$)

substitute(_, $K?? L?? M?? N?? \rightarrow$

1/2 \indexbracket{\theta\sigma^\mu_\nu\lambda\bar{\rho}}

\indexbracket{\epsilon\sigma_{\mu}\lambda\bar{\rho}}$

replace_match(_)

take_match(_, \$\theta^\beta_\psi_\beta\bar{Q}\$)

substitute(_, $K?? L?? M?? N?? \rightarrow$

1/2 \indexbracket{\theta\sigma^\mu_\nu\lambda\bar{\rho}}

\indexbracket{\psi\sigma_{\mu}\epsilon\bar{\rho}}$

replace_match(_)

take_match(_, \$\partial_\mu_\nu^{\phi_\beta_\gamma}\bar{Q}\$)

substitute(_, $K?? L?? M?? \rightarrow$

-i/2 \indexbracket{\theta\sigma^\nu_\partial\mu_\nu^{\phi_\beta_\gamma}}

\indexbracket{\epsilon\partial_\nu_\partial\mu_\psi_{\phi_\gamma}}$

replace_match(_)

take_match(_, \$\bar{t}_\beta^\dot{\gamma}\partial_\mu_\nu^{\phi_\beta_\gamma}\bar{Q}\$)

substitute(_, $K?? L?? M?? \rightarrow$

-i/2 \indexbracket{\theta\sigma^\nu_\partial\mu_\nu^{\phi_\beta_\gamma}}

\indexbracket{\partial_\nu_\partial^\nu_\phi_{\phi_\gamma}\epsilon\bar{\rho}}$

replace_match(_);
\[ \begin{align*}
e^\beta \phi_\beta + 2e^\alpha \theta_\alpha m(x) + e^\beta \sigma^{\beta \gamma} \phi^\gamma V_\nu + e^\beta (\bar{\theta} \theta) \phi_\beta + 2e^\alpha \theta_\alpha (\bar{\theta} \theta) d(x) + \bar{\epsilon}_\beta \chi^\beta + 2\bar{\epsilon}_\alpha \tilde{\theta}^\alpha n(x) \\
+ \bar{\epsilon}_\alpha \theta^\alpha \sigma_\alpha \epsilon^\alpha \phi^\beta V_\nu + \bar{\epsilon}_\beta (\theta \theta) \bar{\lambda}^\beta + 2\bar{\epsilon}_\alpha (\theta \theta) \tilde{\theta}^\alpha d(x) + (i \theta \sigma^\nu \bar{\epsilon}) \partial_\mu f(x) + (i \theta \sigma^\nu \bar{\epsilon})(\bar{\theta} \theta) \partial_\mu \tilde{n}(x) \\
- (i \epsilon^\nu \bar{\theta}) \partial_\mu f(x) + (\theta \theta)(-\frac{1}{2} i \partial_\nu \phi \sigma^\nu \epsilon) + (\bar{\theta} \theta)(\frac{1}{2} i \epsilon \sigma^\nu \partial_\nu \chi) + \frac{1}{2} i (\theta \theta)(\bar{\theta} \epsilon) \partial^\mu V_\mu \\
+i(\theta \theta) \bar{\theta} (\epsilon \sigma^\nu \epsilon) \partial_\mu m(x) - \frac{1}{2} i (\bar{\theta} \theta)(\theta \epsilon) \partial^\mu V_\mu + \frac{1}{2} i (\theta \theta)(\bar{\theta} \epsilon) \partial_\nu \psi \sigma^\nu \bar{\epsilon} \\
+ \frac{1}{2} i (\theta \theta)(\bar{\theta} \epsilon) \sigma^\nu \partial_\nu \chi + (\theta \sigma^\nu \bar{\theta})(i \epsilon \sigma^\nu \bar{\lambda}) + (\theta \sigma^\nu \bar{\theta})(\psi \sigma^\nu \epsilon) \\
+ \frac{1}{2} i (\theta \sigma^\nu \bar{\theta})(i \epsilon \sigma^\nu \bar{\epsilon}) - \frac{1}{2} i (\theta \sigma^\nu \bar{\theta})(i \epsilon \sigma^\nu \bar{\epsilon}) \\
\end{align*} \]

Now, one can easily read off the supersymmetry transformation of the component fields from the above expression.

One may find that the various hard-coded substitution routines at specific terms that we used to achieve the canonical form for \( \theta \) expansions make the computation procedure more difficult. As mentioned previously, a broader algorithm would be more feasible to sort these Grassmann expressions at least to a better stopping point.

For example, in the explicit indices, we can have:

\begin{verbatim}
{\bar{t} ^ { # }, \theta ^ { # } }::SelfAntiCommuting;
{\bar{t} ^ { # }, \epsilon ^ { # } }::AntiCommuting;
{\bar{t} ^ { # }, \theta ^ { # } }::SortOrder;
\partial ^ { # }::PartialDerivative;
ex:=\epsilon^{\alpha} M^{\mu}_{\alpha \aldot} \bar{t}^{\aldot} \theta^{\beta} M^{\nu}_{\beta \betdot} \bar{t}^{\betdot} \partial_{\mu} V_{\nu} + \theta^{\beta} M^{\mu}_{\beta \aldot} \epsilon^{\aldot} \bar{t}^{\betdot} \theta^{\alpha} \partial_{\mu} \psi_{\alpha};
\end{verbatim}

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Property SelfAntiCommuting attached to $[\bar{\theta}^#, \theta^#]$.

Property AntiCommuting attached to $[\bar{\theta}^#, \epsilon^#, \theta^#, \psi^#]$.

Property SortOrder attached to $[\bar{\theta}^#, \theta^#, \epsilon^#]$.

Property PartialDerivative attached to $\partial^#$.

\[
e^\alpha M^\mu_{\alpha\alpha} \bar{\theta}^\alpha \theta^3 M^\nu_{\beta\beta} \bar{\theta}^\beta \partial^\nu + \theta^3 M^\mu_{\beta\beta} \bar{\epsilon}^\alpha \bar{\theta}^\beta \theta^\alpha \psi_\alpha
\] (24)

Here, $M^\mu$ is the Pauli matrices and the anticommutativity of our spinors is indicated using the \texttt{AntiCommuting} property. We also define our preferred order using \texttt{SortOrder}. Now, we apply \texttt{sort_product} to sort factors in our Grassmann expression defined above:

\[
\texttt{sort\_product(\_)};
\]

\[
M^\mu_{\alpha\alpha} M^\nu_{\beta\beta} \bar{\theta}^\alpha \theta^3 \bar{\theta}^\beta \partial^\nu + M^\mu_{\beta\alpha} \bar{\epsilon}^\alpha \partial^\mu \psi_\alpha \bar{\theta}^\beta \theta^\alpha \theta^\beta
\] (25)

After this stage, one can use the relation among the Grassmann variables for further simplifications. Finally, working with $\theta$-expansions can be cumbersome. One can handle things in a covariant way, using projection techniques to find component field expansions of supersymmetric actions. The implementation of projection operators via supercovariant derivatives in the current stage of Cadabra is a little tricky, but it is still possible to perform computations on them.

At the end, chiral and vector superfields can be implemented in Cadabra in the same way as general scalar superfields. Here, we quickly review chiral and vector superfields. We use these multiplets in the next section for the automation of supersymmetric Lagrangians.

A superfield $\Phi$ satisfying the constraint $\bar{D} \Phi = 0$ is called a left chiral superfield, where $\bar{D}$ is a covariant derivative for the superfield. In terms of new variable $y^\mu =$
\[ x^\mu + i \theta \sigma^\mu \bar{\theta}, \] the superfield has the following power series expansion in \( \theta \):

\[ \Phi(y^\mu, \theta^\alpha) = A(y) + \sqrt{2} \theta \psi(y) + \theta \theta F(y) \]  

(26)

where \( A(y) \) and \( F(y) \) are complex scalar fields, and \( \psi(y) \) is a left-handed Weyl spinor.

By expanding the component fields in terms of variables \( x, \theta, \bar{\theta} \), we have

\[
\Phi(x^\mu, \theta^\alpha, \bar{\theta}^\dot{\alpha}) = \phi(x) + \sqrt{2} \theta \psi(x) + \theta \theta F(x) + i \theta \sigma^\mu \bar{\theta} \partial^\mu \phi(x) \\
- \frac{i}{\sqrt{2}} (\theta \bar{\theta}) \partial^\mu \psi(x) \sigma^\mu \bar{\theta} - \frac{1}{4} (\theta \theta) (\bar{\partial} \bar{\partial}) \partial^\mu \phi(x)
\]  

(27)

Under supersymmetry transformation, we have

\[ \delta \Phi = i (\varepsilon Q + \bar{\varepsilon} \bar{Q}) \Phi \]  

(28)

As we defined the differential operator representations of these supercharges, one can easily compute the SUSY transformation for the chiral field using the same approach as we used for a general scalar superfield.

A vector superfield \( V(x^\mu, \theta^\alpha, \bar{\theta}^\dot{\alpha}) \) is defined by the reality condition \( V = V^\dagger \). In the Wess–Zumino gauge, a vector superfield is expressed as:

\[ V_{W.Z} = (\theta \sigma^\mu \bar{\theta}) V_\mu + (\theta \bar{\theta}) \bar{\partial} \lambda(x) + (\bar{\theta} \bar{\partial}) \partial \lambda(x) + (\theta \theta)(\bar{\partial} \bar{\partial}) D(x) \]  

(29)

where \( V_\mu(x) \) is a real vector field, \( \lambda(x) \) is a complex spinor field and \( D(x) \) is a real scalar field.

In section 3, we choose a vector superfield to define the invariant kinetic part of the superspace Lagrangian density and to show how one can achieve simple automation for SUSY Lagrangian generation.
3 Supersymmetric Lagrangian Implementations

In an automated way, one can construct supersymmetric Lagrangians and actions and verify the various properties and structures of the related SUSY model directly within Cadabra.

Here, we shed light on the potential capabilities of Cadabra in the automatic generation of supersymmetric Lagrangians, which can speed up verification tasks and facilitate the study of SUSY theories that are analytically computation-intensive. We do not address all of the details of Cadabra implementations and only elaborate on areas in which one can develop Cadabra’s algorithms and functions to handle the automation.

Part of the supersymmetric Lagrangian describing the dynamics of different component fields in chiral and vector supermultiplets can be automated. The most general supersymmetry Lagrangian describing the dynamics between various multiplets can be constructed based on invariant kinetic and superpotential parts and gauge interactions:

\[ L = \Phi_i^\dagger e^{\hat{q}}_{\hat{V}^i} \Phi_i \bigg|_{\theta \bar{\theta} \theta \bar{\theta}} + W[\Phi] \bigg|_{\theta \bar{\theta}} + \bar{W}[\Phi^\dagger] \bigg|_{\theta \bar{\theta}} + \frac{1}{16} W^\alpha_A W^A_\alpha \bigg|_{\theta \bar{\theta}} + \frac{1}{16} \bar{W}^A_\dot{\alpha} \bar{W}^\dot{\alpha}_A \bigg|_{\theta \bar{\theta}} \] (30)

where \( W[\Phi] \) is a polynomial of the superfield \( \Phi \) and is called the superpotential. \( W^\alpha_A \) and \( \bar{W}^A_\dot{\alpha} \) are our spinorial superfields. Based on the above structure, Lagrangian density generation can be automated by defining three main functions in Cadabra as the extractor of the \( \theta \theta \bar{\theta} \bar{\theta} \), \( \theta \theta \) and \( \bar{\theta} \bar{\theta} \) components.

Here, for educational purposes, we show how to achieve simple automation for the gauge covariant kinetic term (\( \theta \theta \bar{\theta} \bar{\theta} \) component extractor). Similarly, one can develop
$$\theta\theta$$ and $$\bar{\theta}\bar{\theta}$$ component extractors out of scalar superfields of $$W_\alpha^A W_\alpha^A$$ and $$\bar{W}_\alpha^A \bar{W}_\alpha^A$$ and from superpotentials $$W[\Phi]$$ and $$W[\Phi^\dagger]$$.

We define a simple toy function called T2TBar2 to derive the invariant kinetic part of the SUSY Lagrangian from a list of chiral superfields in the Wess–Zumino gauge. Here, for simplicity of the algorithm, we do not search for the component of the vector superfield and use preconfigured components of $$V_\mu(x)$$, $$\lambda(x)$$ and $$D(x)$$.

```python
def T2TBar2(SF):
    F_terms:=0;
    Scalar_terms:=0;
    Spinor_lambda:=0;
    Spinors_psi:=0;
    Total_Terms:=0;

    for element in SF["F"]: 
        F_term:= |\indexbracket{@(element)}|^2.
        F_terms += F_term

    for element in SF["A"]: 
        indexa = element.indices().__next__()
        Scalar_term:= q_{ @(indexa)} D(x) |\indexbracket{@(element)}|^2 + (\partial_{\mu}{ @(element)}\partial^{\mu}{ @(element)^{\star}} + 1/4 q_{ @(indexa)}^2 V_{\mu}(x) V^{\mu}(x) |\indexbracket{@(element)}|^2.
        Scalar_terms += Scalar_term
```

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for psi in SF["\psi"]:
    indexpsi = psi.indices().__next__()
    if str(indexpsi)==str(indexa):
        Spinor_term1:= -1/sqrt(2} q_{@indexa}
        \indexbracket{\lambda(x) @(psi)} @(element)^{\star}.
        Spinor_term2:= -1/sqrt(2} q_{@indexa}\bar{\lambda(x)}
        \bar{@(psi)} @(element).
    else:
        Spinor_term1:=0 ;
        Spinor_term2:=0 ;
        Spinor_lambda +=Spinor_term1+Spinor_term2

for psi in SF["\psi"]:
    indexpsi = psi.indices().__next__()
    Spinor:= i \partial_{\mu}{\bar{@(psi)}}\bar{\sigma}^{\mu}
    @(psi)-1/2 q_{@indexpsi} V(x)_{\mu}\bar{@(psi)}
    \bar{\sigma}^{\mu} @(psi).
    Spinors_psi += Spinor

Total_Terms=  F_terms + Scalar_terms + Spinor_lambda + Spinors_psi
return Total_Terms

The input is a simple list of field contents. For example, consider the case of two scalar superfields (chiral):
\[ SF := [A_1(x), \psi_1(x), F_1(x), q_1, A_2(x), \psi_2(x), F_2(x), q_2, V_\mu(x), \lambda(x), D(x)]; \]

\[ [A_1(x), \psi_1(x), F_1(x), q_1, A_2(x), \psi_2(x), F_2(x), q_2, V_\mu(x), \lambda(x), D(x)] \quad (31) \]

T2TBar2(SF);

\[ |(F_1(x))^2 + (F_2(x))^2 + q_1 D(x) | (A_1(x))^2 + \partial_\mu [A_1(x)] \partial^\mu [A_1(x)^*]| + \]
\[ \frac{1}{4} q_1^2 V_\mu(x) V^\mu(x) | (A_1(x))^2 + q_2 D(x) | (A_2(x))^2 + \partial_\mu [A_2(x)] \partial^\mu [A_2(x)^*]| + \]
\[ \frac{1}{4} q_2^2 V_\mu(x) V^\mu(x) | (A_2(x))^2 - 2^{-\frac{1}{2}} q_1 (\lambda(x) [\psi_1(x)]) [A_1(x)^*] - 2^{-\frac{1}{2}} q_1 \lambda(x) [\psi_1(x)] [A_1(x)] - \]
\[ 2^{-\frac{1}{2}} q_2 (\lambda(x) [\psi_2(x)]) [A_2(x)^*] - 2^{-\frac{1}{2}} q_2 \lambda(x) [\psi_2(x)] [A_2(x)] + i \partial_\mu (\psi_1(x)) \bar{\sigma}^\mu [\psi_1(x)] - \]
\[ \frac{1}{2} q_1 V(x) \mu [\psi_1(x)] \bar{\sigma}^\mu [\psi_1(x)] + i \partial_\mu (\psi_2(x)) \bar{\sigma}^\mu [\psi_2(x)] - \frac{1}{2} q_2 V(x) \mu [\psi_2(x)] \bar{\sigma}^\mu [\psi_2(x)] \quad (32) \]

The above example can provide the reader with simple guidance on how to initiate programming in Cadabra specifically for supersymmetry use cases. Obviously, the real value of such automation or other types of computations will be much more noticeable when dealing with supersymmetric models in higher dimensions. For example, many fields emerge in supergravity theories in ten dimensions, usually restricting us to checking supersymmetry only with respect to a subset of the transformation rules. With the help of Cadabra, heavy computations, especially in the context of field theory, can be addressed more easily.

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