General contact mechanics theory for randomly rough surfaces with application to rubber friction

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We generalize the Persson contact mechanics and rubber friction theory to the case where both surfaces have surface roughness. This theory includes elastic coupling and can be used for arbitrary large normal forces, e.g., as in tire dynamics. We generalize the Persson contact mechanics and rubber friction theory to the case where both solids have surface roughness, and arbitrary viscoelastic properties, e.g., a viscoelastic solid in contact with an elastic solid.

We consider the frictionless contact between elastic solids with nominally flat surfaces in normal (quasi-static) approach. Assume that the solids have roughness described by the height profiles $h_1(x)$ and $h_2(x)$, where $x = (x, y)$ is a two-dimensional position vector in the surface $xy$-plane, and different elastic properties (Young modulus $E_1$ and $E_2$, and Poisson ratio $\nu_1$ and $\nu_2$, respectively). If the root-mean-square slope of the roughness on both surfaces is small one can show that this contact problem can be mapped on a simpler problem with one elastic half space with a perfectly flat surface, and another rigid solid with the combined roughness (see Fig. ??)[26]:

$$h(x) = h_2(x) - h_1(x).$$

The effective elastic module $E$ and Poisson ratio $\nu$ of the elastic solid must satisfy

$$\frac{1 - \nu_2^2}{E} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2}.$$

We will refer to this transformation as surface-mapping. For viscoelastic solids both the Young’s modulus and the Poisson ratio depends on the frequency $\omega$ (and hence, during sliding (velocity $v$), on the length scale $v/\omega$). In this case the surface-mapping is no longer valid, and in this paper we will show, within the Persson approach, how to treat the more general problem of viscoelastic contact mechanics when both solids have surface roughness.

2 Qualitative discussion

Let us consider a rubber block sliding on a rigid solid substrate. We assume first that the rubber block has a perfectly flat surface and the rigid substrate a ran-
domly rough surface [see Fig. 2(a)]. In this case during sliding the asperities of the rigid solid will induce time-dependent deformations of the rubber, which results in the conversion of translation energy into heat via the internal damping in the rubber. Thus the viscoelastic deformations of the rubber will contribute to the kinetic friction coefficient.

Let us now consider the opposite case where the rigid solid has a perfectly smooth surface and the rubber block a rough surface [see Fig. 2(b)]. In this case during sliding the deformations of the rubber does not change in time and no viscoelastic energy dissipation occurs. Thus in this case there is no contribution to the friction from viscoelastic deformations of the rubber. As roughness occurring on both surfaces the situation is more complex but already the argument given here indicates that the energy dissipation is dominated by the roughness in the harder solid.

Another important difference between the two cases in Fig. 2 relates to the area of real contact. In Fig. 2(a) the contact area depends on the viscoelastic modulus of the rubber at finite frequencies. That is, during sliding at the speed $v$ a surface roughness component with wavelength $\lambda$ will result in oscillating deformations of the rubber surface with a frequency $\omega \approx v/\lambda$. Since the elastic modulus of rubber-like materials may increase by a factor of $\sim 1000$ when the perturbing frequency increases from the rubbery region to the glassy region, it is clear that a very strong dependency of the contact area is expected: An increase in the elastic modulus by a factor of $\sim 1000$ can result in a decrease in the contact area by a factor of $\sim 1000$. However, when the surface roughness occurs on the rubber surface while the hard counter-surface is perfectly flat, there are no time-dependent viscoelastic deformations of the rubber and the contact area is determined by the low-frequency viscoelastic modulus.

Let us now compare the contact between a randomly rough hard solid (the substrate) and (a) a rubber block with a flat surface [see Fig. 2(a)], and (b) with a rubber block with a simple periodic “roughness” as indicated in Fig. 2(b). We assume that the longest wavelength roughness component of the substrate surface is smaller than the linear size of the (nominal) contact regions in 2(b). A uniform stress or pressure $\sigma_0$ is assumed to act on the upper surface of the rubber block. In case (a) a similar pressure is assumed to act at the interface between the block and the substrate. According to the Persson contact mechanics theory, if frictional heating can be neglected, the viscoelastic contribution to the friction for small applied load is proportional to the normal load, i.e. the friction coefficient is independent of the load. Since the load is the same in both Fig. 2(a) and (b) it follows that the friction coefficient is the same too. This is only true as long as adhesion does not manifest itself macroscopically as a finite pull-off force, and as long as the contact area is small compared to the nominal contact area. It is clear that if the nominal contact area in case (b) is very small, the latter assumption will no longer hold. But if all the assumptions made above holds, introducing a periodic “roughness” on the rubber surface will have no influence on the friction coefficient and the contact area. We will show in the next section that random roughness on the rubber surface does influence the friction, but in most cases only to a very small extent.

3 Theory: special case

In the following we will extend the contact mechanics and rubber friction theory developed in Ref. to the
3.1 Contact area

In this section we study the contact mechanics for the most important case where solid 1 is viscoelastic (say rubber) and solid 2 rigid, and both surfaces have surface roughness which are uncorrelated so that \( \langle h_1(x)h_2(x) \rangle = \langle h_1(x) \rangle \langle h_2(x) \rangle \) where \( \langle .. \rangle \) stands for ensemble averaging. In this case, in order for solid 2 to make perfect contact with solid 1, one needs to deform the surface of solid 1 so that the normal surface displacement \( u_z(x,t) \) equals the gap function:

\[
u_z(x,t) = h_2(x,t) - h_1(x). \tag{1}
\]

Let us define the Fourier transform

\[
u_z(q,\omega) = (2\pi)^{-3} \int d^2x \int dt \, u_z(x,t)e^{-i(q\cdot x-\omega t)}.
\]

If the upper solid moves at a constant velocity \( v = (v_x, v_y) \) parallel to the bottom solid, we have \( h_2(x,t) = h_2(x-vt) \) and

\[
u_2(q,\omega) = h_2(q)\delta(\omega - q\cdot v)
\]

and \( \nu_1 \) takes the form

\[
u_1(q,\omega) = h_2(q)\delta(\omega - q\cdot v) - h_1(q). \tag{2}
\]

If a normal stress or pressure \( \sigma_z(x,t) \) acts on the surface of solid 1, from the theory of viscoelasticity,\(^3\)

\[
u_z(q,\omega) = M(q,\omega)\sigma_z(q,\omega). \tag{3}
\]

In the Persson contact mechanics theory the mean-square fluctuation of the (normal) interfacial stress for complete contact (see Appendix A in Ref. \( \text{[2]} \)) is needed to quantify the diffusion function:

\[
\langle \sigma_z^2(x,t) \rangle = \frac{(2\pi)^3}{v_0A_0} \int d^2q \int d\omega \quad |M(q,\omega)|^{-2} \times \langle u_z(q,\omega)u_z(-q,-\omega) \rangle,
\]

where \( A_0 \) is the surface area and \( v_0 \) the time over which we perform averaging. Using \( \nu_1 \) we get

\[
\langle u_z(q,\omega)u_z(-q,-\omega) \rangle = \langle h_1(q)h_1(-q)\rangle|\delta(\omega)|^2 + \langle h_2(q)h_2(-q)\rangle|\delta(\omega - q\cdot v)|^2
\]

where we have used that \( h_1(q) \) and \( h_2(q) \) are (assumed to be) uncorrelated. Next, using that

\[
\langle h_1(q)h_1(-q) \rangle = \frac{A_0}{(2\pi)^2}C_1(q) \tag{5}
\]

\[
\langle h_2(q)h_2(-q) \rangle = \frac{A_0}{(2\pi)^2}C_2(q)
\]

where \( C_1(q) \) and \( C_2(q) \) are the power spectra of the surfaces of solids 1 and 2, respectively, and that

\[
|\delta(\omega)|^2 = \frac{\delta(\omega) t_0}{2\pi}
\]

we get

\[
\langle u_z(q,\omega)u_z(-q,-\omega) \rangle = \frac{A_0 t_0}{(2\pi)^3} [C_1(q)|\delta(\omega)| + C_2(q)|\delta(\omega - q\cdot v)|]. \tag{6}
\]

Substituting this result in \( \langle \sigma_z^2(x,t) \rangle \) gives

\[
\langle \sigma_z^2(x,t) \rangle = \int d^2q d\omega \quad |M(q,\omega)|^{-2} \times [C_1(q)|\delta(\omega)| + C_2(q)|\delta(\omega - q\cdot v)|] \int d^2q \quad [|M(q,\omega\cdot v)|^{-2}C_1(q) + |M(q,0)|^{-2}C_1(q)]. \tag{7}
\]

For a homogeneous viscoelastic solid

\[
M^{-1}(q,\omega) = -\frac{1}{2} \frac{E(\omega)}{1 - \nu^2(\omega)}q. \tag{8}
\]

Substituting this in \( \langle \sigma_z^2(x,t) \rangle \) gives

\[
\langle \sigma_z^2(x,t) \rangle = \frac{1}{4} \int d^2q \quad q^2 \times \left(\frac{E(q\cdot v)}{1 - \nu^2(q\cdot v)}\right)^2 C_2(q) + \left|\frac{E(0)}{1 - \nu^2(0)}\right|^2 C_1(q). \tag{9}
\]

If the rubber surface is smooth \( C_1(q) = 0 \), \( \langle \sigma_z^2(x,t) \rangle \) reduces to the expression derived in Ref. \( \text{[2]} \). For an elastic solid \( E(\omega) = E(0) \) and \( \nu(\omega) = \nu(0) \) are frequency independent so in that case

\[
\langle \sigma_z^2(x,t) \rangle = \frac{1}{4} \int d^2q \quad q^2 \left[\frac{E}{1 - \nu^2}\right]^2 [C_2(q) + C_1(q)].
\]

This result is the expected one because for a rigid solid in contact with an elastic solid the surface-mapping effective modulus \( E/2(1-\nu^2) \) is just that of the elastic solid, while the effective power spectrum is that for the combined roughness given by

\[
C(q) = \frac{(2\pi)^2}{A_0} \langle [h_1(q) + h_2(q)] [h_1(-q) + h_2(-q)] \rangle,
\]

i.e. \( C(q) = C_1(q) + C_2(q) \), where we again have assumed
uncorrelated roughness profiles. Thus, for elastic solids the results above could be obtained directly using the surface-mapping.

In the Persson contact mechanics theory the relative contact area is given by

\[
\frac{A}{A_0} = \text{erf} \left( \frac{1}{2\sqrt{G}} \right),
\]

where erf is the error function, with the integral representation

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x dx \exp(-x^2),
\]

and where

\[
G = \frac{1}{2\sigma_0^2} \int d^2 q \left[ |M(q, q \cdot v)|^2 C_2(q) + |M(0, 0)|^2 C_1(q) \right],
\]

where \( M(q, q \cdot v) \) is given by \( E_1(q, q \cdot v) \). Note that \( G \) depends on the range of surface roughness included in the analysis. Thus, if \( q_0 \) correspond to the smallest surface roughness wavevector, and if we include all the roughness \( q_0 < q < q_1 \) then the function \( G(q_1) \) will mainly depend on the cut-off wavevector \( q_1 \). In general, as \( q_1 \) increases, \( G(q_1) \) increases and the contact area \( A(q_1) \) decreases.

For the rubber friction to be discussed next we need the relative contact area \( P(q) = A(q)/A_0 \) for all \( q \) between \( q_0 \) and \( q_1 \).

Using that for \( x < < 1 \) we have \( \text{erf}(x) \approx 2x/\sqrt{\pi} \) we get for large \( G \) that \( A/A_0 \approx (\pi G)^{-1/2} \). Note that \( G \to \infty \) as \( \sigma_0 \to 0 \) so for small nominal contact pressures

\[
\frac{A}{A_0} = \frac{\sigma_0}{E_1(0)} \frac{(8/\pi)^{1/2}}{\int d^2 q q^2 \left[ \left( \frac{E_1(q, q \cdot v)}{E_1(0)} \right)^2 C_2(q) + C_1(q) \right]^{-1/2}},
\]

It is clear that the surface roughness on the solid 1 will reduce the contact area.

3.2 Rubber friction

We now study the contribution to rubber friction from the viscoelastic deformations of the rubber surface induced by the surface roughness of the hard counter surface. Following Ref. [3] we first focus on the case of complete contact. In this case the dissipated energy is given by Eq. ?? in Ref. [3]:

\[
\Delta E = \int d^2 x dt \langle \dot{u}_z(x, t) \sigma_z(x, t) \rangle
\]

Next using ?? this equation takes the form

\[
\Delta E = A_0 t_0 \int d^2 q d\omega \left[ i\omega M^{-1}(q, \omega) \right] \langle \gamma \rangle
\]

\[
= A_0 t_0 \int d^2 q \left( i q \cdot v \right) M^{-1}(q, q \cdot v) C_2(q).
\]

As expected, for complete contact the dissipated energy does not depend on the surface roughness on the rubber, but only on the roughness on the hard counter-surface.

As a result the rubber friction coefficient within the Persson approach is given by the same expression as derived in Refs. [3, 19], except now enters the contact area function \( P(q) \) given by ?? and ???, rather than the expression derived in Ref. [3] for a perfectly smooth rubber surface:

\[
\mu = \frac{1}{2} \int_{q_0}^{q_1} dq q^3 C_2(q) P(q) S(q)
\]

\[
\times \int_0^{2\pi} d\phi \cos \phi \text{ Im} \frac{E(\omega)}{\sigma_0(1 - \nu^2)}
\]

where \( P(q) = A(q)/A_0 \) is given by ?? and where the correction factor [19]

\[
S(q) = \gamma + (1 - \gamma) P^2(q)
\]

with \( \gamma \approx 1/2 \). Note that \( P(q) \) and \( S(q) \) depend on the applied (or nominal) pressure \( \sigma_0 \) and sometimes, when necessary, we write \( P(q) = P(q, \sigma_0) \).

4 Numerical results
As on the road surface. If one instead use the actual surface roughness power spectrum shown in Fig. 5. The calculated contact area (a) and friction coefficient (b) for a rubber tread block sliding on a road surface with the surface roughness power spectrum shown in Fig. ?? (red curve). The red curve is assuming the rubber surface as rough as the asphalt surface i.e. it has a power spectrum given by the log-log curve in Fig. ?? C. Calculations are without including the effect of frictional heating.

In this section we will illustrate the theory presented above with an example. Fig. ?? shows the surface roughness power spectra of an asphalt road surface and a rubber tread block surface after use. Note that, as expected, the road surface exhibits much larger roughness than the tread block.

By using the theory above one finds that the friction (and the contact area) between the asphalt road surface and the rubber block surface after use. Note that, as expected, the road surface exhibits much larger roughness than the tread block. Figure 5. The calculated contact area (a) and friction coefficient (b). The red curves in the same figure are the results when the rubber surface is completely flat (no surface roughness). It is remarkable how small influence the surface roughness on the rubber block would have on the contact area and the friction even in this extreme case. Note in particular that for high sliding speeds the roughness on the rubber surface has negligible influence on the calculated friction. This is easy to understand from ?? and ??.

5 Theory: general case

We now consider the general case where both solids have surface roughness and arbitrary viscoelastic properties. Using the Persson contact mechanics theory, in Appendix A we derive expressions for the contact area, the viscoelastic contribution to the friction coefficient and contact area becomes so similar to the case of the flat rubber surface that it overlap the red curves. For the nominal contact pressure $\sigma_0 = 0.15$ MPa and the temperature $T = 20^\circ$C. Calculations are without including the effect of frictional heating.

For small sliding speeds, $\omega = q \cdot v$ is very small in a large part of the $q$-plane, and the $\omega = q \cdot v$ and $\omega = 0$ terms in ?? are of nearly the same magnitude in a large part of the $q$-integration domain. However, for high sliding speeds $\omega = q \cdot v$ will be very large in a large part of the $q$-plane, and since $E(\omega)$ increases strongly as the frequency increases from the rubbery region (small $\omega$) to the glassy region (large $\omega$), it is clear that for large sliding speeds the $\omega = q \cdot v$ term in ?? will be much larger than the $\omega = 0$ term in a large region of the $q$-plane. This explain why the two curves in Fig. ?? (a) and (b) approach each other as the velocity $v$ increases.

\[
G(q_1) = \frac{1}{2\sigma_0^2} \int_{q_0}^{q_1} d^2q \sum_{j=1}^2 \frac{C_j(q)}{|M(q, q \cdot v_j)|^2}, \tag{16}
\]

where

\[
M(q, \omega) = M_1(q, \omega - q \cdot v_1) + M_2(q, \omega - q \cdot v_2). \tag{17}
\]

The viscoelastic contribution to the friction coefficient is given by:

\[
\mu = \int d^2q \left| S(q) \right| P(q) \sum_{j=1}^2 \frac{\tilde{M}(q, q \cdot v_j) C_j(q)}{\sigma_0 |M(q, q \cdot v_j)|^2}, \tag{18}
\]

where

\[
\tilde{M}(q, \omega) = \tilde{M}_1(q, \omega - q \cdot v_1) + \tilde{M}_2(q, \omega - q \cdot v_2)
\]
The average interfacial separation is given by:

\[ \bar{\sigma} = \sum_{j=1}^{2} C_j(\mathbf{q}) M(\mathbf{q}, q \cdot \mathbf{v}_j)^{-1} \]

The average interfacial separation depends on \( \bar{u}(\sigma_0) \) as expected.

Finally, the average interfacial separation depends on [see ??]:

\[ \sum_{j=1}^{2} \frac{C_j(\mathbf{q})}{M(\mathbf{q}, q \cdot \mathbf{v}_j)^2} = \frac{C_1(\mathbf{q})}{M_1(\mathbf{q}, 0)^2} + \frac{C_2(\mathbf{q})}{M_1(\mathbf{q}, q \cdot \mathbf{v}_1)^2}. \]  

(21)

As discussed in Sec. 4, we note that, at least for high sliding speed, the power spectrum of the compliant solid usually plays a negligible role in the generation of the contact area. Similar, the average interface separation \( \bar{u} \) is determined mainly by the power spectrum of the rigid solid, but note the difference in the potency of the \( M_1 \)-term (two for the contact area and one for the average separation).

**Case II**: A rigid solid 2 with smooth surface sliding on a viscoelastic solid 1 with rough surface.

With \( M_2 = 0 \) and \( h_2 = 0 \), we have

\[ M(\mathbf{q}, \omega) = M_1(\mathbf{q}, \omega - \mathbf{q} \cdot \mathbf{v}_1) \]

\[ \bar{M}(\mathbf{q}, \omega) = i(\omega - \mathbf{q} \cdot \mathbf{v}_1) M(\mathbf{q}, \omega). \]

The contact area depends on:

\[ \sum_{j=1}^{2} \frac{C_j(\mathbf{q})}{M(\mathbf{q}, q \cdot \mathbf{v}_j)^2} = \frac{C_1(\mathbf{q})}{M_1(\mathbf{q}, 0)^2}. \]  

(22)

The viscoelastic contribution to the friction depends on:

\[ \sum_{j=1}^{2} \frac{\bar{M}(\mathbf{q}, q \cdot \mathbf{v}_j) C_j(\mathbf{q})}{M(\mathbf{q}, q \cdot \mathbf{v}_j)^2} = 0. \]  

(23)

Hence, as expected, no hysteresis friction is generated during the sliding interaction. This is in accordance with the qualitative discussion presented in Sec. 2.

The average interfacial separation depends on:

\[ \sum_{j=1}^{2} \frac{C_j(\mathbf{q})}{M(\mathbf{q}, q \cdot \mathbf{v}_j)^2} = \frac{C_1(\mathbf{q})}{M_1(\mathbf{q}, 0)}. \]  

(24)

Note that both the contact area and the average interfacial separation depends on the viscoelastic modulus at vanishing frequency (i.e., in the rubbery regime). This is of course expected as there are no time-dependent deformations of the rubber surface.
**Case III:** An elastic solid 2 with a smooth surface sliding on a viscoelastic solid 1 with rough surface.

With $M_2(q, \omega) = M_2(q)$ and $h_2 = 0$, we have

\[
\dot{M}(q, \omega) = \dot{M}_1(q, \omega - q \cdot v_1) + \dot{M}_2(q, \omega - q \cdot v_2)
\]

\[
M(q, \omega) = M_1(q, \omega - q \cdot v_1) + M_2(q).
\]

The contact area depends on:

\[
\sum_{j=1}^{2} \frac{C_j(q)}{|M(q, q \cdot v_j)|^2} = \frac{C_1(q)}{|M_1(q, 0) + M_2(q)|^2}.
\]

(25)

The viscoelastic contribution to the friction depends on:

\[
\sum_{j=1}^{2} \frac{\dot{M}(q, q \cdot v_j)C_j(q)}{|M(q, q \cdot v_j)|^2} = \frac{-i(q \cdot v)M_2(q)C_1(q)}{|M_1(q, 0) + M_2(q)|^2}.
\]

(26)

The average interfacial separation depends on:

\[
\sum_{j=1}^{2} \frac{C_j(q)}{M(q, q \cdot v_j)} = \frac{C_1(q)}{M_1(q, 0) + M_2(q)}.
\]

(27)

We note first in ?? and ?? that the equivalent interfacial compliance, related to the contact area as well as to the average interfacial separation, is determined by a simple summation rule of each solid compliance, evaluated under static interaction. Moreover, even if ?? is non-zero, $(q \cdot v)M_2(q)$ is an odd function of $q$, resulting into a zero micro rolling friction [see ??].

**Case IV:** A viscoelastic solid 2 with a smooth surface sliding on a rigid solid 1 with rough surface.

With $M_1 = 0$ and $h_2 = 0$ we have

\[
M(q, \omega) = M_2(q, \omega - q \cdot v_2)
\]

\[
\dot{M}(q, \omega) = i(\omega - q \cdot v_2)M(q, \omega).
\]

The contact area depends on:

\[
\sum_{j=1}^{2} \frac{C_j(q)}{|M(q, q \cdot v_j)|^2} = \frac{C_1(q)}{|M_2(q, -q \cdot v)|^2}.
\]

(28)

The viscoelastic contribution to the friction depends on:

\[
\sum_{j=1}^{2} \frac{\dot{M}(q, q \cdot v_j)C_j(q)}{|M(q, q \cdot v_j)|^2} = \frac{-i(q \cdot v)C_1(q)}{M_2(-q, q \cdot v)}.
\]

(29)

Assuming $v = v_2 - v_1$ is aligned along the x-axis, using ?? in ?? we obtain

\[
\mu = \int d^2q S(q)P(q) \frac{iq_x}{\sigma_0M_2(-q, q_x v)}C_1(q),
\]

which for a viscoelastic half space reads (removing subscripts for simplicity) [19]

\[
\mu(\sigma_0) = \frac{1}{2\sigma_0} \int d^2q q_x q S(q)P(q)C_1(q)\text{Im}[E_r(q_x v)]
\]

which corresponds to ??.

The average interfacial separation depends on:

\[
\sum_{j=1}^{2} \frac{C_j(q)}{M(q, q \cdot v_j)} = \frac{C_1(q)}{M_2(q, -q \cdot v)}.
\]

(30)

6 Summary and conclusion

We have studied the contact mechanics and sliding friction between a viscoelastic solid and a rigid solid where both objects have random uncorrelated roughness. We have shown that for a rubber-like material, where the Young’s modulus may increase by a factor $\sim 1000$ when going from the rubbery region to the glassy region, roughness on the rubber surface will in most cases have a rather small influence on the sliding (velocity $v$) contact. However, for static contact, obtained in the limit $v \to 0$, the roughness on both surfaces is equally important. As a numerical application we considered the contact between a rubber tread block and a road surface, and showed that during sliding the surface roughness on the rubber surface has in most cases only a very small influence on the contact area and the viscoelastic sliding friction coefficient.

We have also studied the general case where both solids have random roughness and arbitrary (linear) rheology properties. For this case, within the Persson contact mechanics approach, we have derived expressions for the contact area, the viscoelastic contribution to the friction force, and for the average surface separation. We have studied 4 different limiting cases of the theory.

The theory presented in this paper can be extended to correlated surface roughness, and to include frictional heating and adhesion.
Appendix A: General contact mechanics theory

Here we derive the general contact mechanics theory for steady sliding rough solids (to which a moving reference \( \xi_i \) is associated, where \( i = 1 \) for the lower solid, see Fig. ??), in adhesive- and friction-less interaction. We define

\[
u_x(q, \omega) = (2\pi)^{-3} \int dt \int d^2x \ u_x(x, \omega) e^{-i(q \cdot x - \omega t)}, \tag{A1}\]

where \( u_x(x, \omega) \) is the generic surface out-of-plane displacement, and

\[
\sigma(q, \omega) = (2\pi)^{-3} \int dt \int d^2x \ \sigma(x, \omega) e^{-i(q \cdot x - \omega t)}, \tag{A2}\]

where \( \sigma(x, \omega) = \sigma(x, \omega) \) is the contact stress, both observed in the fixed reference \( \xi \). Hence, given that \( \xi_j = \xi - \nu_j t \) [resulting into \( \sigma_{\xi}(q, \omega) = \sigma_{\xi_j}(q, \omega) \), and similarly for \( u_x \)], we get

\[
\sigma(q, \omega) = M_j^{-1}(q, \omega - q \cdot \nu_j) \ u_{x,j}(q, \omega), \tag{A3}\]

whereas in term of the displacement time derivatives [note: \( \frac{d}{d\xi_j}[u_{x,j}(q, \omega)] = \frac{d}{d\xi}[u_{x,j}(q, \omega)] + u_{x,j}(q, \omega) i q \cdot \nu_j \), resulting into \( i (q \cdot \nu_j) u_{x,j}(q, \omega) = \dot{u}_{x,j}(q, \omega) \)]

\[
\sigma(q, \omega) = M_j^{-1}(q, \omega - q \cdot \nu_j) \dot{w}_j(q, \omega), \tag{A4}\]

where simply

\[
M_j(q, \omega) = i \omega M_j(q, \omega). \tag{A5}\]

Hence, by equating the stress in ?? for both top and bottom solid we get

\[
u_{x,2}(q, \omega) = M_2(q, \omega - q \cdot v_2) M_1^{-1}(q, \omega - q \cdot v_1) u_{x,1}(q, \omega) \tag{A6}\]

so that

\[
u_x(q, \omega) = u_{x,1}(q, \omega) + u_{x,2}(q, \omega) = \left[ 1 + M_2(q, \omega - q \cdot v_2) M_1^{-1}(q, \omega - q \cdot v_1) \right] u_{x,1}(q, \omega). \tag{A7}\]

We can determine the equivalent interface compliance \( M(q, \omega) \) from

\[
\sigma(q, \omega) = M^{-1}(q, \omega) \ u_x(q, \omega), \]

where

\[
M(q, \omega) = M_1(q, \omega - q \cdot v_1) + M_2(q, \omega - q \cdot v_2). \tag{A8}\]

Moreover since \( \dot{u}_{x,1}(q, \omega) / \dot{M}_1(q, \omega - q \cdot v_1) = u_x(q, \omega) / M(q, \omega) \) we can determine the equivalent interface compliance derivative \( \dot{M}(q, \omega) \) from

\[
\sigma(q, \omega) = M^{-1}(q, \omega) \dot{u}_x(q, \omega), \]

where

\[
\dot{M}(q, \omega) = \dot{M}_1(q, \omega - q \cdot v_1) + \dot{M}_2(q, \omega - q \cdot v_2). \tag{A9}\]

Observe finally that from ?? and ?? it follows

\[
\dot{M}^{-1}(q, \omega) \dot{M}^{-1}(-q, -\omega) = \left| \dot{M}(q, \omega) \right|^2
\]

\[
M^{-1}(q, \omega) M^{-1}(-q, -\omega) = \left| M(q, \omega) \right|^2
\]

i.e. \( [M^{-1}(-q, -\omega)] = [M^{-1}(q, \omega)]^* \) [and similarly for \( \dot{M}^{-1}(q, \omega) \)].
We calculate first the stress auto-correlation for full contact. In such a case

\[ u_z(q, \omega) = \sum_{j=1}^{2} h_j(q) \delta(\omega - q \cdot v_j) \]

i.e.

\[ \sigma(q, \omega) = M^{-1}(q, \omega) \sum_{j=1}^{2} h_j(q) \delta(\omega - q \cdot v_j). \]

Hence, considering that \( C_j(q) \delta(q + \bar{q}) = \langle h_j(q) h_j(\bar{q}) \rangle \) and assuming the surface roughness are uncorrelated

\[
\langle \sigma^2(x, t) \rangle = \int d^2q d\omega \int d^2q d\bar{\omega} \langle \sigma(q, \omega) \sigma(\bar{q}, \bar{\omega}) \rangle e^{i(q \cdot x - \omega t)} e^{i(\bar{q} \cdot x - \bar{\omega} t)} = \int d^2q \sum_{j=1}^{2} C_j(q) M(q, q \cdot v_j)^2.
\]

(A10)

With similar considerations we can calculate the correlation function \( \langle \sigma(x, t) \dot{u}_z(x, t) \rangle \)

\[
\langle \sigma(x, t) \dot{u}_z(x, t) \rangle = \int d^2q d\omega \int d^2q d\bar{\omega} \langle \sigma(q, \omega) \dot{u}_z(\bar{q}, \bar{\omega}) \rangle e^{i(q \cdot x - \omega t)} e^{i(\bar{q} \cdot x - \bar{\omega} t)}
\]

\[ = \int d^2q S(q) P(q) \sum_{j=1}^{2} C_j(q) M(q, q \cdot v_j)^2. \]

(A11)

Furthermore, the correlation function \( \langle \sigma(x, t) u_z(x, t) \rangle \) reads

\[
\langle \sigma(x, t) u_z(x, t) \rangle = \int d^2q d\omega \int d^2q d\bar{\omega} \langle \sigma(q, \omega) u_z(\bar{q}, \bar{\omega}) \rangle e^{i(q \cdot x - \omega t)} e^{i(\bar{q} \cdot x - \bar{\omega} t)}
\]

\[ = \int d^2q S(q) P(q) \sum_{j=1}^{2} C_j(q) \text{Re}[M(q, q \cdot v_j)]. \]

(A12)

In \ref{eq:corr1} and \ref{eq:corr2} we have added the \( S(q) P(q) \) term to take into account the partial contact in the evaluation of \( \langle \sigma(x, t) \dot{u}_z(x, t) \rangle \) and \( \langle \sigma(x, t) u_z(x, t) \rangle \), where \( S(q) \) has been introduced before. \( M_i(q, \omega) = 2 [q E_i, r(\omega) \bar{S}(qd)] \) for a generic viscoelastic solid of finite thickness, where the real function \( \bar{S}(qd) \) is a correction factor related to the adopted boundary conditions [e.g. for a slab of thickness \( d \) on a rigid substrate we have

\[
\bar{S}(qd) = \frac{(3 - 4\nu) \cosh(2qd) + 2(qd)^2 - 4\nu (3 - 2\nu) + 5}{(3 - 4\nu) \sinh(2qd) - 2qd},
\]

where \( \bar{S}(qd) \to 1 \) for \( d \to \infty \)].

**Contact area**

The contact area \( A/A_0 = P(q_0) \) can be calculated from \ref{eq:corr1}, with \( G(\bar{q}) = \langle \sigma^2(x, t) \rangle_{\bar{q}}/\langle \sigma_0^2 \rangle \) given by \ref{eq:corr2}

\[
G(\bar{q}) = \frac{1}{2\sigma_0^2} \int_{q_0}^{\bar{q}} d^2q \sum_{j=1}^{2} \frac{C_j(q)}{|M(q, q \cdot v_j)|^2}.
\]

(A13)

**Stored and dissipated interfacial energy**

In the process of determining the average interfacial separation \( \bar{u} \) as well as the micro-rolling friction, we assume the normal approach to occur quasi-statically with respect to the sliding kinematics. Within this assumption (i.e. \( t_0 \gg t_1 \), where \( t_0 \propto \dot{u}^{-1} \) and \( t_1 \propto v^{-1} \) are, respectively, the time scale of the normal and sliding motion), the time-derivative of the displacement field \( \dot{u}_z(x, t) \) can be expanded into two uncorrelated sources \( \dot{u}_0(x, t) + \dot{w}_1(x, t) \) (\( \dot{w}_1 \) is random in
time), where $\dot{w}_0 (x, t) \propto t_0^{-1}$ and $\dot{w}_1 (x, t) \propto t_1^{-1}$. Here we calculate the average external work over a $t_1$ time span as

$$W_1 = \int_{t_1} dt \bar{W} = \int_{t_1} dt \left[ -\sigma_0 A_0 \dot{u} + \tau_0 A_0 v \right].$$

Since $t_1/t_0 \ll 1$ the term $\sigma_0 A_0 \dot{u}$ can be assumed constant and given that $\int_{t_1} dt \tau_0 = \langle \tau \rangle$, we get $W_1 = -t_1 \sigma_0 A_0 \dot{u} + t_1 A_0 v \langle \tau \rangle$. However $\dot{u} \propto t_0^{-1}$ and $v \propto t_1^{-1}$, resulting into

$$W_1 \approx t_1 A_0 v \langle \tau \rangle.$$

Now the average external work over a $t_0$ time span reads

$$W = \int_{t_0} dt \bar{W} = \int_{t_0} dt \left[ -\sigma_0 A_0 \dot{u} \right] + \frac{t_0}{t_1} W_1. \tag{A14}$$

However the energy conservation requires that $W = A_0 \langle \sigma (x, t) \dot{w} (x, t) \rangle t_0 = A_0 \langle \sigma_0 (x, t) \dot{w}_0 (x, t) \rangle t_0 + A_0 \langle \sigma_1 (x, t) \dot{w}_1 (x, t) \rangle t_0$. However since $\langle \sigma_0 (x, t) \dot{w}_0 (x, t) \rangle \propto t_0^{-1}$ and $\langle \sigma_1 (x, t) \dot{w}_1 (x, t) \rangle \propto t_1^{-1}$ we can separate the different energy contribution in ?? as

$$\langle \sigma_0 (x, t) \dot{w}_0 (x, t) \rangle t_0 = \int_{t_0} dt \left[ -\sigma_0 \dot{u} \right]$$

$$A_0 \langle \sigma_1 (x, t) \dot{w}_1 (x, t) \rangle t_1 = W_1. \tag{A16}$$

Manipulating the LHS of ?? we have

$$\langle \sigma_0 (x, t) \dot{w}_0 (x, t) \rangle t_0 = \langle \sigma_0 (x, t) \Delta w_0 (x, t) \rangle = \frac{1}{2} \Delta \langle \sigma_0 (x, t) w_0 (x, t) \rangle,$$

resulting into

$$\frac{1}{2} \Delta \langle \sigma_0 (x, t) w_0 (x, t) \rangle = -\sigma_0 \Delta \dot{u}, \tag{A17}$$

whereas ?? leads to

$$\langle \sigma_1 (x, t) \dot{w}_1 (x, t) \rangle = (v_2 - v_1) \cdot \langle \tau \rangle. \tag{A18}$$

Hence, by using ?? in ?? we get the sliding-induced micro-rolling shear stress

$$(v_2 - v_1) \cdot \langle \tau \rangle (p) = \int d^2 q \sum_{j=1}^2 \frac{M(q, q \cdot v_j) C_j(q)}{|M(q, q \cdot v_j)|^2}, \tag{A19}$$

whereas the average interfacial separation can be calculated by substituting ?? into ??,

$$\dot{u} (p) = \int d^2 q \sum_{j=1}^2 C_j(q) \text{Re} \left[ M(q, q \cdot v_j) \right] \int_0^\infty d\sigma_0 \frac{\partial [S(q, \sigma_0) P(q, \sigma_0)]}{\partial \sigma_0}. \tag{A20}$$

For an elastic solid in contact with a hard randomly rough surface, this equation reduces to the equation derived in Ref. 28.

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with rubber surface roughness

with smooth rubber surface

\[ A/A_0 \]

\[ \log_{10} v \text{ (m/s)} \]
Macroscopic contact geometry

REV

Surface (2)

Surface (1)
