Optimal encoding on discrete lattice with translational invariant constrains using statistical algorithms.

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Abstract

In this paper will be presented methodology of encoding information in valuations of discrete lattice with some translational invariant constrains in asymptotically optimal way. The method is based on finding statistical description of such valuations and changing it into statistical algorithm, which allows to construct deterministically valuation with given statistics. Optimal statistics allow to generate valuations with uniform distribution - we get maximum information capacity this way. It will be shown that we can reach the optimum for one-dimensional models using maximal entropy random walk and that for the general case we can practically get as close to the capacity of the model as we want (found numerically: lost $10^{-10}$ bit/node for Hard Square). There will be also presented simpler alternative to arithmetic coding method which can be used as cryptosystem and data correction method too.

1 Introduction

Consider all projections $\mathbb{Z}^2 \to \{0, 1\}$. In this way we can store 1 bit/node (point of the space). Now introduce some constrains, for example: there cannot be two neighboring ”1” (each node has 4 neighbors) - it’s so called Hard Square model(HS). It will occur that this restriction reduces the informational capacity to $H_{HS} \approx 0.5878911617753406$ bits/node.

The goal of this paper is to introduce methodology of encoding information in such models as near their capacity as required.

We will call a model such triplet - space ($\mathbb{Z}^2$), alphabet ($\{0, 1\}$) and some constrains. It’s elements are all projections fulfilling the constrains - we can think about them as valuations of nodes. Now the number of all such valuations over some finite
set of nodes \( A \) will asymptotically grow exponentially \( N \approx 2^{\#AH} \). Because in the possibility of choosing one of \( N \) choices can be stored \( \lg(N) \) bits, this \( H \) (Shannon’s entropy) is the maximal capacity in bits/node we can achieve.

We can really store \( \lg(N) \) bits in choosing one of \( N \) choices, only if all of them are equally probable only. So to get the whole available capacity, we have to make that all possible valuations are equally probable. Unfortunately the space of valuations over infinite space is usually quite complicated. But thanks of translational symmetry, elements should have the same local statistical behavior. If we find it and valuate the space accordingly, we should get near to the uniform distribution over all elements. The statistical algorithm have to encode some information in generating some specific valuation, fulfilling the optimal statistics of the space.

Statistical description \( (p) \) is a function, which for every finite set \( (shape) \) gives the probability distribution of valuations on it \( (patterns) \). Thanks of the translational invariance, we can for example write \( p(01) \) - the probability that while taking any two neighboring nodes, they give ‘01’ pattern. In one dimension we can find the optimal statistical description using pure combinatorics. In higher it’s much more complicated, but we can for example divide the space into short stripes, create new alphabet from their valuations and just use the one-dimensional method.

Having the statistical description, we can use it to construct the statistical algorithm. For example divide the space into straps and valuate them succeedingly. Now for succeeding nodes, depending on the valuations of already valuated neighbors, we get some probability from created previously table. According to this probability we valuate the node, encoding some information.

**Examples of usage**: We can think about for example hard disk, locally as valuating nodes (let say - magnetizing round dots) of two-dimensional lattice with 0 or 1 (without constrains).

![Figure 1: Rescaling the lattice without changing the size of magnetic dot.](image.png)

Now let us change the lattice constant, as on fig. 1 - we have \( \sqrt{2}^2 = 2 \) times more nodes, but we get some constrains - like in HS - so the capacity is now: \( 2 * 0.587 \approx 1.17 \) - we get 17% greater capacity. We’ve got it by more precise positioning of the head - it’s technically easier to achieve than shrinking the dot. We will see that going further, we can increase the capacity potentially to infinity.
1 INTRODUCTION

We can use statistical algorithm approach also to generate random states (e.g. spin alignment) in statistical physics. Usually we use Monte-Carlo methods - to generate ”good” alignment we have to make many runs (the more, the better).

But using for example many of such alignments, we can approximate its (local) statistical description with assumed ”goodness”. Now using statistical algorithm, we can generate so ”good” alignments in one run.

In the second section we will see how to solve analytically the one-dimensional model - find its optimal description and capacity. We will motivate that it should be Shannon’s entropy. To find the optimal description we will just average over all elements. In this case the statistical algorithm will be just Markov process - we will valuate node by node from left to right and the probability distribution for a node is found using only the valuation of the previous one. We get this way random walk on a graph (of symbols), which maximizes global entropy ([11]). This approach can be generalized to other than uniform distributions of sequences, by introducing some vertex/edge potentials.

In the third section there will be presented asymmetric numeral systems - a generalization of numeral systems, which are optimized for encoding sequences of equiprobable digits into which the probability distribution of digits is given. It’s natural way to encode data using given statistical algorithm. This algorithm can be alternative for widely used arithmetic coding method: in one table check it compress/decompress a few bits (a symbol) and have option that the output is encrypted, probably very well. It has also very nice data correction properties.

In the fourth section there will be introduced formality for general models. It will be shown that for ”reasonable” models: $X = \mathbb{Z}^n$, translative invariant constrains with finite range and which are ”simple” - valuation of some nodes cannot enforce valuation of distant enough ones - we can speak about their entropy, which is positive.

In the fifth section we will introduce methodology of statistical description. Now we will define the optimal description straightforward as the average over all elements. Unfortunately, in spite of giving many arguments, I couldn’t prove existence of such average - we will assume it and see that it’s equivalent to vanishing of long-range correlation. There will be also introduced alternative definition of optimality (LOC) - in a finite set of nodes, separated topologically from the rest of the space, all valuations are equally probable. Then there will be discussed how to generate elements for given statistical description (‘almost’ statistical algorithm): fix some order for nodes and get probability distribution for a node using valuation of the previous ones.

In the sixth section we will analyze some algorithms as above, but this time there can be infinite number of previous nodes. We will assume that the probability can be determined only by valuation of neighboring nodes. We will analyze two sim-
ple types of algorithms: valuate separately subsets inside which constrains doesn’t work or just take a random order. We will get near optimum this way and explain why we can’t get it in this way. There will be also described how to iteratively generate approximations of uniform distribution of elements on a finite set. The longer we will iterate the process, the nearer uniform probability we get. We can use this generator to find numerically approximation of the optimal description.

In the seventh section it will be finally shown how to get as close to the optimal algorithm as we want. We will do it by narrowing the space, so that it has only one infinite dimension and divide it into ”straps”, for which we can use one-dimensional analytical methods. There will be shown numerical results, showing that we are really tending quickly to the optimum in this way.

2 One-dimensional model

In this section we will look at the following transitively invariant model:

**Definition 1.** One-dimensional model will be called a triplet: \((X, \mathcal{A}, M)\):

- **space** \(X = \mathbb{Z}\)
- **alphabet** \(\mathcal{A}\) - finite set of symbols
- **constrains** \(M : \mathcal{A}^2 \rightarrow \{0, 1\}\)

Now elements of this model are \(V(X)\), where for \(A \subseteq X\):

\[
V(A) := \{v : A \rightarrow \mathcal{A} | \forall i \in A \cap (A - 1) M(v_i, v_{i+1}) = 1\}
\]  

(1)

2.1 Blocking symbols to reduce constrains to range one

I will shortly justify, that any general one-dimensional, transitively invariant model, can be easily reduced to above (with constrains of range one):

Let \(l\) be the largest range of constrains, for example \(v_k = a \Rightarrow v_{k+l} = b\). Take \(\mathcal{A}^l\) as the new alphabet, grouping \(l\) consecutive symbols.

Now we can construct matrix as above:

\[
M_{(v_i)_{i=1..l},(w_i)_{i=1..l}} = 1 \iff \forall i=2..l v_i = w_{i-1} \text{ and the sequence } v_1v_2...v_lw_l \text{ is consistent with the constrains}
\]

So we can restrict to the model defined above (\(l = 1\)).

Let’s visualize it to analyze some example:

**Definition 2.** \(k\)-model:

\(X = \mathbb{Z}\) \hspace{1cm} \(A = \{0, 1\}\) \hspace{1cm} constrain: after 1 follows at least \(k\) zeros.
Because constrains have range $k$, we should group $k$ symbols into one of new symbols "0", "1", ..."$k$":

- "0" : there were at least $k$ zeros before (there can be 1 now)
- "$i$" : $k - i + 1$ positions before was the last 1 (in $i$ positions there can be 1)

In states different than "0", we have to put 0.

So the whole algorithm (Markov process) is defined by the probability of putting 1 while in state "0" - denote it $q$ ($\tilde{q} := 1 - q$). Denote $p_i$ - the probability of state $i$.

Make one step ($p := p_0$): \[
\begin{align*}
  p_k &= pq \\
  p_i &= p_{i+1} & \text{for } i &\in \{1, ..., k - 1\} \\
  p &= p_1 + p\tilde{q}
\end{align*}
\]

So $p_1 = p_2 = ... = p_k = pq$

$p = 1 - p_1 - p_2 - ... - p_k = 1 - kpq \\
\frac{1}{1 + kq}$

We will explain later, that in a symbol with probability distribution $q/\tilde{q}$ is stored at average $h(q) := -q \log_2(q) - \tilde{q} \log_2(\tilde{q})$ bits. So the entropy of this model is the maximum over all algorithms:

$$H = \max_{q \in [0,1]} H_q = \max_{q \in [0,1]} \frac{h(q)}{1 + kq}. \quad (2)$$

In the introduction we’ve seen the example that this method can be used to store more data on two-dimensional plane. We’ve just found analytic expression for the one-dimensional case - we have constant length of "magnetic dot" - say $d$, but we don’t assume that potential positions cannot intersect (if we assume that - we can store 1bit/length $d$). We assume only that two 1 (magnetized dot) cannot intersect.

Let say that we can position the dot with precision $\frac{d}{k+1}$. That means exactly that after 1 there have to be $k$ zeros - analyzed model. We can now easily count that using $k + 1$ times more precise positioning, we get:

| $k$  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|------|---|---|---|---|---|---|---|---|---|---|----|----|----|
| benefit(%) | 0 | 39 | 65 | 86 | 103 | 117 | 129 | 141 | 151 | 160 | 168 | 176 | 183 |

more capacity of information. For larger $k$ we get fig. 2. It goes slowly to infinity

Figure 2: Informational capacity in bits/’old node’ for rescaled $k$ - models

with $k \to \infty$ - we could increase the capacity of information potentially as much as we need by more precise head positioning.
2.2 Maximal entropy random walk

Let’s think how much information can be stored in such sequence/path of length $k$? Denote $N_k = \#V(\overline{k})$ ($\overline{k} := \{0, 1, ..., k - 1\}$) the number of such sequences of length $k$. We can store some information by choosing one of them. To choose one of $2^n$ elements we need $n$ bits. So in sequence of length $k$ we can store about $\lg(N_k)$ bits. The average number of bits we can store in one node is:

$$H_k := \lim_{k \to \infty} H_k$$

(3)

Let’s introduce vector $c^k := (c_a^k)_{a \in A}$, where $c_a^k := \#\{\gamma \in V(\overline{k}) : \gamma_{k-1} = a\}$. Sequences of length $k + 1$ are obtained from sequences of length $k$:

$$c^{k+1} = c^k \cdot M$$

(4)

**Theorem 3** (Frobenius - Perron). Irreducible matrix with real, nonnegative coefficients have dominant real, nonnegative eigenvalue. Corresponding eigenvector has real, positive coefficients.

In our case reducibility would mean that starting from proper subset of alphabet we have to stay in it - we could decompose it into irreducible cases.

Assuming irreducibility, we can say:

$$H = \lg\lambda$$

(5)

where $\lambda$ is the dominant eigenvalue of $M$.

Look at the normalized corresponding eigenvector: $\phi^T M = \lambda \phi^T$, $\sum_{a \in A} \phi_a = 1$. We could think that it is probability distribution of symbols in elements...

It’s not true: it’s the distribution of symbols on the end of a sequence which is unbounded in one direction.

We can use this probability distribution to initiate algorithm we will find.

Basing on above derivation we will find the probability distribution inside such sequences - unbounded in both directions. Now we have to expand in both sides.

For some $(v_i)_{i=0..m-1}$, $k \in \mathbb{N}$, $a, b \in A$, consider all paths from $a$ to $b$ of length $2k + m$, which has $v$ in the middle:

$$\Gamma_{v}^{k,a,b} := \{\gamma : m + 2k \to A, \gamma_k \gamma_{k+1}... \gamma_{k+m-1} = v, \gamma_0 = a, \gamma_{2k+m-1} = b\}$$

We will call $v$ a pattern, its domain ($\overline{m}$) - its shape and the extremal nodes of the domain - its boundary: $\hat{v} := v_0$, $\breve{v} := v_{m-1}$.

We want to count allowed paths among them. Define matrix $C_v^k$:

$$(C_v^k)_{a,b} := \#\{\gamma \in \Gamma_{v}^{k,a,b} \cap V(\overline{m+2k})\} = \sum_{\gamma \in \Gamma_{v}^{k,a,b}} M_{\gamma_0 \gamma_1} M_{\gamma_1 \gamma_2}...M_{\gamma_{m+2k-2} \gamma_{m+2k-1}}$$

(6)
The second form uses that such multiplication is equal 1 for allowed sequences and 0 otherwise. This form suggests generalization to other matrices and will be considered later. It also suggests $k \to k + 1$ iteration:

$$C_v^{k+1} = M \cdot C_v^k \cdot M$$

For the dominant eigenvalue we have left/right eigenvectors:

$$\phi^T M = \lambda \phi^T \quad M \psi = \lambda \psi$$

(7)

This time we take normalization $\phi^T \psi = 1$.

Usually we will consider real symmetric matrices, like for undirected graphs, for which $\phi = \psi$. Generally if $\phi$ is not real, we should use $\overline{\phi^T}$ instead of $\phi^T$.

$M$ has unique dominant eigenvalue, so for large $k$ we can assume:

$$M^k \simeq \lambda^k \psi \phi^T$$

(8)

This asymptotic behavior is the key point: it allows us to average over infinitely long sequences. Presented approach can be generalized to degenerated dominant eigenvalue by projecting into its eigenspace, but only if all dominant eigenvalues: with the same absolute value has also the same phase. In other case the final value would depend on $k$.

Substituting we get

$$C_v^k \simeq \lambda^{2k} \psi \phi^T C_v^0 \psi \phi^T = \lambda^{2k} (\psi \phi^T) \phi \psi$$

(9)

because $(C_v^0)_{a,b} = 1$ if $a = \check{v}$, $b = \check{v}$ and 0 otherwise.

Let’s look at the probability distribution of allowed patterns on the given shape $(\bar{m})$ inside such infinite sequences ($k \to \infty$).

$$p_v = \lim_{k \to \infty} \frac{\sum_{a,b \in A} (C_v^k)_{a,b}}{\sum_{w \in V(\bar{m})} \sum_{a,b \in A} (C_w^k)_{a,b}} = \frac{\phi_v \psi_v}{\sum_{w \in V(\bar{m})} \phi_w \psi_w}$$

(10)

Notice that we can forget about the normalization assumption in this equation.

We get

1. Patterns on the same shape and with equal boundary values are equally probable. In other words - after fixing values on the boundary of some set, probability distribution of valuations inside is uniform. We will see later that it can be generalized into higher dimensions.

2. For $m = 1$ we get the probability distribution of symbols:

$$p_a = \frac{\phi_a \psi_a}{\sum_{b \in A} \phi_b \psi_b} = \frac{\phi_a \psi_a}{\phi^T \psi}$$

(11)
3. For $m = 2$ we get

$$p_{ab} = \phi_a M_{ab} \psi_b = \frac{\phi_a \psi_b M_{ab}}{\phi T M \psi} = \frac{\phi_a \psi_b M_{ab}}{\lambda \phi T \psi} \quad (12)$$

4. The probability that from vertex $a$ we will jump to vertex $b$ is

$$S_{ab} = \frac{p_{ab}}{p_a} = \frac{M_{ab} \psi_b}{\lambda \psi_a} \quad (13)$$

The Markov process defined this way ($P(a \rightarrow b) = S_{ab}$) reproduces original statistics of the space of infinite allowed sequences with uniform probability distribution. In fact it gives uniform probability among finite paths also: for any two points, each allowed path of given length($k$) between them has the same probability

$$P(\text{path } \gamma_0 \gamma_1 \ldots \gamma_k) = \frac{1}{\lambda^k \psi_{\gamma_k} \psi_{\gamma_0}} \quad (14)$$

Taking $S^k$ or multiplying above by the combinatorial factor, we get:

$$(S^k)_{ab} = \frac{(M^k)_{ab} \psi_b}{\lambda^k \psi_a} \quad (15)$$

It suggests the statistical algorithm: after symbol $a$ choose the next one with $(S_{ab})_b$ probability distribution.

In this way we get uniform distribution among sequences - get maximal entropy.

We will also calculate, that it gives maximal possible entropy $\lg(\lambda)$. 

Firstly look at the problem: we have a sequence of 0 and 1 in which the probability of 1 is fixed to some $p \in (0, 1)$. Let’s calculate how much information corresponds asymptotically to one digit. Denote $\bar{p} := 1 - p$

$$\binom{n}{pm} = \frac{n!}{(pm)! \bar{p}^{nm}} \approx \frac{n^{n-1/2} e^{n}}{(pm)^{pm+1/2} \bar{p}^{pm+1/2} e^{n}} = (2\pi n \bar{p})^{-1/2} \bar{p}^{-pm} \bar{p}^{pm} = (2\pi n \bar{p})^{-1/2} 2^{-n(p \lg p - \bar{p} \lg \bar{p})}$$

$$H_p = \lim_{n \to \infty} \frac{\lg \binom{n}{pm}}{n} = -p \lg p - \bar{p} \lg \bar{p} =: h(p) \quad (16)$$

where we’ve used the Stirling’s formula: $\lim_{n \to \infty} \frac{n!}{\sqrt{2\pi n} \left(\frac{n}{e}\right)^n} = 1$

That means that when we choose from all sequences, which number grows like $2^n$, these with given asymptotical probability of 1 ($p$), we get $2^{nh(p)}$ sequences. If we restrict to sequences with uncorrelated bits with $p = 1/2$, we would get every sequence with the same probability - the uniform distribution of sequences. While storing information in this distribution we get the maximum capacity: 1bit/digit.
Inserting some correlations, favoring some symbols, or part of the space would create redundancy.

Now when we have larger alphabet $A$ and a probability distribution $p : \sum_{a \in A} p_a = 1$, the mean number of information per symbol is (by induction)

$$H_p = -\sum_{a \in A} p_a \log p_a$$  \hspace{1cm} (17)$$

it’s the Shannon’s entropy - in a symbol with given probability distribution $p$ we can store $H_p$ bits.

In the case of found above Markov process, $S = (S_{ab})_{a,b \in A}$:

1. $\forall_{a,b \in A} M_{ab} \geq S_{ab} \geq 0$
2. $\forall_{a \in A} \sum_{b \in A} S_{ab} = 1$
3. $pS = p$, $\sum_{a \in A} p_a = 1$

Generating path for a Markov process is a sequence of independent choices - the entropy of choosing one of sequences is the sum of entropies for single choices. So the average amount of information per symbol is:

$$H = -\sum_a p_a \sum_b S_{ab} \log S_{ab} =$$

$$= -\sum_a \phi_a \psi_a \sum_b \frac{M_{ab} \psi_b}{\lambda} \frac{1}{\psi_a} \log \frac{1}{\lambda} \psi_b = -\frac{1}{\lambda \phi^T \psi} \sum_{a,b} \phi_a M_{ab} \psi_b \log \frac{1}{\lambda} \psi_b =$$

$$= \frac{\phi^T M \psi \log \lambda}{\lambda \phi^T \psi} + \frac{1}{\lambda \phi^T \psi} \sum_{a,b} (\phi_a (\log \psi_a) M_{ab} \psi_b - \phi_a M_{ab} \psi_b (\log \psi_b)) =$$

$$= \log \lambda + \frac{1}{\lambda \phi^T \psi} \sum_{a,b} (\phi_a (\log \psi_a) \lambda \psi_b - \phi_a \lambda \psi_b (\log \psi_b)) = \log \lambda$$

We know that this is the limit for all allowed sequences, so among stochastic processes consistent with the graph ($S_{ab} \leq M_{ab}$) we cannot get higher entropy - encoding information this way gives the maximum capacity.
2.3 Generalization to other distributions of sequences

We’ve focused on 0/1 matrices, but above derivations are more general. Analogously we can enforce that paths are not equally probable, but their probability is proportional to the sum (integral) of some potential along the way:

\[ P(\text{path } \gamma_0 \gamma_1 \ldots \gamma_k) \text{ is proportional to } e^{-(\frac{1}{2} V(\gamma_0) + V(\gamma_0, \gamma_1) + V(\gamma_1) + \ldots + V(\gamma_{k-1}, \gamma_k) + \frac{1}{2} V(\gamma_k))} \]

where \( V, V' \) are some freely chosen real vertex/edge potentials.

In discretized euclidean path integrals, the potential says about the probability that the particle will decay in given place (for example by going to an absorbing vertex). In the presented approach, the particle doesn’t decay. The potential says about the probability that the particle will use given vertex/edge.

Looking at (6) we see that to achieve this potential, we should choose:

\[ M_{ij} = e^{-(\frac{1}{2} V(i) + V'(i,j) + \frac{1}{2} V(j))} \]  

(18)

So if we want to use only vertex (edge) potential, we can set \( V' = 0 \) (\( V = 0 \)). If \( V' \) is symmetric, \( \psi = \phi \). To forbid some vertices/edges (like before), we can set their potential to infinity. We can also choose nonzero diagonal elements to allow self-loops.

Now for a connected weighted graph Frobenius - Perron theorem still works: we get the dominant eigenvalue (\( \lambda \)) and corresponding normalized right eigenvector \( \psi > 0 \), \( \sum_i \psi_i^2 = 1 \) and as before:

\[ P(\text{path } \gamma_0 \gamma_1 \ldots \gamma_k) = \frac{e^{-(\frac{1}{2} V(\gamma_0) + V(\gamma_0, \gamma_1) + V(\gamma_1) + \ldots + V(\gamma_{k-1}, \gamma_k) + \frac{1}{2} V(\gamma_k))}}{\lambda^k} \frac{\psi_{\gamma_k}}{\psi_{\gamma_0}} \]  

(19)

\[ p_i = \psi_i^2 \quad (S^k)_{ab} = \frac{(M^k)_{ab} \psi_b}{\psi_a}. \]  

(20)

2.4 Infinitesimal limit

To the end of this section we will informally derive infinitesimal limit with some time independent vertex potential density. We will do it by covering the space, let say \( \mathbb{R}^d \), with lattice and decrease its width to 0.

This time we would like that (informally):

\[ P(\text{path } \gamma) \text{ is proportional to } e^{-\int V(\gamma(t))dt} \]

The problem is that such Brownian-like path has infinite length - this probability distribution has to be thought for example as a limit of made for some finite length approximations of paths.
For simplicity we will take diagonal terms of $M$ equal 0, what corresponds to moving with constant speed. But we will see later that adding some constant on the diagonal doesn’t change the results.

This time $V$ is some (smooth) function $\mathbb{R}^d \to \mathbb{R}$. Let’s choose some time step $\epsilon > 0$ and cover the space with lattice of some width $\delta > 0$. We will treat it as a graph, in which each vertex is connected with $2d$ neighbors.

We assume that the potential around a node is constant and equal to the original potential $V$ in this node. One edge corresponds to time $\epsilon$, so such discretized potential should be chosen as $\epsilon$ times the original one.

This discretized model is symmetric, so $\phi = \psi$.

The eigenvalue relations ($M\psi = \lambda \psi$) for one-dimension ($d = 1$) are:

$$\text{for all } i, \quad \psi_{i-1} e^{-\frac{\epsilon}{2} (V_{i-1} + V_i)} + \psi_{i+1} e^{-\frac{\epsilon}{2} (V_i + V_{i+1})} = \lambda_i \psi_i$$

where $\psi_i := V(i\delta)$, $2dV_i := V(i\delta)$ (to simplify the final equation).

Now because $\epsilon \to 0$, we can use $e^{-\epsilon} \approx 1 - \epsilon$:

$$\left(1 - \frac{\epsilon}{2} (V_{i-1} + V_i)\right) \psi_{i-1} + \left(1 - \frac{\epsilon}{2} (V_i + V_{i+1})\right) \psi_{i+1} \approx \lambda_i \psi_i$$

$$\frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{\epsilon} - \frac{1}{2} ((V_{i-1} + V_i)\psi_{i-1} + (V_i + V_{i+1})\psi_{i+1}) + \frac{2}{\epsilon} \psi_i \approx \frac{\lambda_i}{\epsilon} \psi_i$$

It looks like there is a problem with $\frac{2}{\epsilon} \psi_i$ term - it goes to infinity. But it adds only a constant to the operator - it doesn’t change its eigenfunctions. The formula for the stochastic matrix also won’t be affected - this time the powers will become exponents and $e^{(M+C)\epsilon} = e^{M\epsilon} e^{C\epsilon}$. The $e^{i\lambda}$ term realizes the normalization of probability distribution. So we will be able to ignore this constant term of $\hat{M}$ later.

We have to connect $\delta$ and $\epsilon$. We see that to obtain 2nd derivative of $\psi$, we should take $\epsilon \sim \delta^2$, what is characteristic for Brownian motion. To simplify the final equation, let’s choose $\epsilon = 2\delta^2$ time scale.

Let’s multiply above equation by -1 and assume some smoothness of $\psi$ and $V$ to write it for $\epsilon \to 0$:

$$\frac{\psi_{i-1} - 2\psi_i + \psi_{i+1}}{2\delta^2} + 2V_i \psi_i \approx \frac{2 - \lambda_i}{\epsilon} \psi_i$$

We can sum such equations for all dimensions and take the limit $\epsilon \to 0$, getting the Schrödinger equation:

$$\hat{H}\psi = -\frac{1}{2} \Delta \psi + V \psi = E_0 \psi$$

where $\Delta := \sum_i \partial_{ii}$, $\hat{H} := -\frac{1}{2} \Delta + V$.

We’ve already explained that we can ignore the constant term for $\hat{M}$, so we can take $\hat{M} = -\hat{H}$. 
This time $E_0 = \lim_{\epsilon \to 0} \frac{2d - \lambda_0}{\epsilon}$ is some new dominant eigenvalue of differential operator $\hat{H}$, called Hamiltonian. Now the largest $\lambda$ corresponds to the smallest $E$. Corresponding eigenfunction $\psi$ should be obtained as the limit of succeeding approximations - can be chosen as real nonnegative - it’s the ground state.

The equation for probability distribution becomes

$$p(x) = \psi^2(x)$$

where $\int_{\mathbb{R}^d} \psi^2(x) dx = 1$ \hspace{1cm} (21)

which looks similarly to known from the quantum mechanics, but this time $\psi$ is real and nonnegative function. Sometimes $\psi$ cannot be normalized, but it still can be used to calculate the propagator.

The powers of $M$ matrix becomes the exponent of the differential operator $\hat{M}$. The stochastic matrix becomes propagator, which gives the probability density of finding a particle from $x$ after time $t$:

$$K(x, y, t) = \frac{\langle x | e^{-t\hat{H}} | y \rangle \psi(y)}{e^{-tE_0} \psi(x)}$$ \hspace{1cm} (22)

It’s easy to check, that as we would expected:

$$\int_{\mathbb{R}^d} K(x, y, t) dy = 1, \quad \int_{\mathbb{R}^d} K(x, y, t) K(y, z, s) dy = K(x, z, t + s),$$

$$\int_{\mathbb{R}^d} p(x) K(x, y, t) dx = p(y).$$

As it was already mentioned, $e^{-tE_0}$ term is for normalization. The $\psi$ division term corresponds to time discretization, I think. Because $\psi$ should be continuous, for small times the particle moves corresponding to the local potential only. For larger times, this term starts to be important - now the particle doesn’t just move straightforward between these points, but choose statistically some trajectory - this term corresponds to the global structure of potential/topology of the space.

### 3 Asymmetric Numeral Systems (ANS)

We will now show how to use found Markov process (or generally - statistical algorithm) to deterministically encode some information. Using the data, we have to generate succeeding symbols with given probability distribution $(q_s)_{s=0, \ldots, n-1}$. To do it we could use any entropy coder, but in reversed order: encoding into Markov’s process correspond to decompression, decoding to compression.

In practice there are used two approaches for entropy coding nowadays: building binary tree (Huffman coding) and arithmetic coding. The first one approximates
probabilities of symbols with powers of 2 - isn’t precise. Arithmetic coding is 
precise. It encodes symbol in choosing one of large ranges of length proportional to 
assumed probability distribution \((q)\). Intuitively, by analogue to standard numeral 
systems - the symbol is encoded on the most important position. To define the 
actual range, we need to use two numbers (states).

We will construct precise encoding that uses only one state. It will be obtained 
by distributing symbols uniformly instead of in ranges - intuitively: place infor-
mation on the least important position. Standard numeral systems are optimal 
for encoding streams of equiprobable digits. Asymmetric numeral systems is nat-
ural generalization into other, freely chosen probability distributions. If we choose 
uniform probability, with proper initialization we get standard numeral system.

### 3.1 General concept

We would like to encode an uncorrelated sequence of symbols of known probability 
distribution into as short as possible sequence of bits. For simplicity we will 
assume that the probability distribution is constant, but it can be naturally 
generalized for various distributions. The encoder will receive succeeding symbols 
and transform them into succeeding bits.

An symbol(event) of probability \(p\) contains \(\lg(1/p)\) bits of information - it 
doesn’t have to be a natural number. If we just assign to each symbol a sequence 
of bits like in Huffman coding, we approximate probabilities with powers of 
2. If we want to get closer to the optimal compression rates, we have to be 
more precise. We see that to do it, the encoder have to be more complicated 
- use not only the current symbol, but also relate to the previous ones. The 
encoder has to have some state in which is remembered unnatural number of bits of 
information. This state in arithmetic coder are two numbers describing actual range.

The state of presented encoder will be one natural number: \(x \in \mathbb{N}\). For this 
subsection we will forget about sending bits to output and focus on encoding sym-
bols. So the state \(x\) in given moment is a natural number which encodes all already 
processed symbols. We could just encode it as a binary number after processing the 
whole sequence, but because of it’s size it’s completely impractical. In 3.4 it will be 
shown that we can transfer the youngest bits of \(x\) to assure that it stays in the fixed 
range during the whole process. For now we are looking for a rule of changing the 
state while processing a symbol \(s\):

\[
\begin{align*}
\text{encoding} \\
(s, x) & \quad \rightarrow \quad x' \\
\text{decoding}
\end{align*}
\]
So our encoder starts with for example 0 and uses above rule on succeeding symbols. These rules are bijective, so that we can uniquely reverse whole process - decode the final state back into initial sequence of symbols in reversed order.

In given moment in $x$ is stored some unnatural number of bits of information. While writing it in binary system, we would round this value up. To avoid such approximations, we will use convention that $x$ is the possibility of choosing one of $\{0, 1, \ldots, x-1\}$ numbers, so $x$ contains exactly $\lg(x)$ bits of information.

For assumed probability distribution of $n$ symbols, we will somehow split the set $\{0, 1, \ldots, x-1\}$ into $n$ separate subsets - of sizes $x_0, \ldots, x_{n-1} \in \mathbb{N}$, such that $\sum_{n=0}^{n-1} x_s = x$. We can treat the possibility of choosing one of $x$ numbers as the possibility of choosing the number of subset$(s)$ and then choosing one of $x_s$ numbers. So with probability $q_s = \frac{x_s}{x}$ we would choose $s$-th subset. We can enumerate elements of $s$-th subset from 0 to $x_s-1$ in the same order as in the original enumeration of $\{0, 1, \ldots, x-1\}$.

Summarizing: we’ve exchanged the possibility of choosing one of $x$ numbers into the possibility of choosing a pair: a symbol$(s)$ with known probability distribution $(q_s)$ and the possibility of choosing one of $x_s$ numbers. This $(x \Leftarrow (s, x_s))$ will be the bijective coding we are looking for.

We will now describe how to split the range. In arithmetic coding approach (Range Coding), we would divide $\{0, \ldots, x-1\}$ into ranges. In ANS we will distribute these subsets uniformly.

We can describe this split using distribution function $D_1 : \mathbb{N} \rightarrow \{0, \ldots, n-1\}$:

$$\{0, \ldots, x-1\} = \bigcup_{s=0}^{n-1} \{y \in \{0, \ldots, x-1\} : D_1(y) = s\}$$

We can now enumerate numbers in these subsets by counting how many are there smaller of them in the same subset:

$$x_s := \#\{y \in \{0, 1, \ldots, x-1\}, D_1(y) = s\} \quad D_2(x) := x_{D_1(x)}$$

getting bijective decoding function $(D)$ and it’s inverse coding function $(C)$:

$$D(x) := (D_1(x), D_2(x)) = (s, x_s) \quad C(s, x_s) := x.$$ 

Assume that our sequence consist of $n \in \mathbb{N}$ symbols with given probability distribution $(q_s)_{s=0}^{n-1}$ ($\forall_{s=0}^{n-1} q_s > 0$). We have to construct a distribution function and coding/decoding function for this distribution: such that

$$\forall_{s,x} \quad x_s \text{ is approximately } x \cdot q_s.$$  

We will now show informally how essential above condition is. In 3.3 and 3.5 will be shown two ways of making such construction.
Statistically in a symbol is encoded $H(q) := - \sum_s q_s \lg q_s$ bits. ANS uses $\lg(x) - \lg(x_s) = \lg(x/x_s)$ bits of information to encode a symbol $s$ from $x_s$ state. Using second Taylor’s expansion of logarithm (around $q_s$), we can estimate that our encoder needs at average:

$$- \sum_s q_s \lg \left( \frac{x_s}{x} \right) \approx - \sum_s q_s \left( \lg(q_s) + \frac{x_s/x - q_s}{q_s \ln(2)} - \frac{(x_s/x - q_s)^2}{2q_s^2 \ln(2)} \right) =$$

$$= H(q) + \frac{1 - 1}{q_s \ln(2)} + \sum_s \frac{(x_s/x - q_s)^2}{2q_s \ln(2)} \text{ bits/symbol.}$$

We could average

$$\frac{1}{2 \ln(2)} \sum_s q_s x_s^2 (x_s/q_s - x) = \frac{1}{\ln(4)} \sum_s q_s x_s^2 (x_s/q_s - C(s,x_s))^2 \quad (26)$$

over all possible $x_s$ to estimate how many bits/symbols we are wasting.

### 3.2 Asymmetric Binary System (ABS)

It occurs that in the binary case, we can find simple explicit formula for coding/decoding functions.

We have now two symbols: ”0” and ”1”. Denote $q := q_1$, so $\bar{q} := 1 - q = q_0$.

To get $x_s \approx x \cdot q_s$, we can for example take

$$x_1 := \lfloor xq \rfloor \quad \text{(or alternatively } x_1 := \lfloor xq \rfloor \rfloor) \quad (27)$$

$$x_0 = x - x_1 = x - \lfloor xq \rfloor \quad \text{(or } x_0 = x - \lfloor xq \rfloor \rfloor) \quad (28)$$

Now using [24]: $D_1(x) = 1 \iff$ there is a jump of $\lfloor xq \rfloor$ after it:

$$s := \lfloor (x+1)q \rfloor - \lfloor xq \rfloor \quad \text{(or } s := \lfloor (x+1)q \rfloor - \lfloor xq \rfloor \rfloor) \quad (29)$$

We’ve just defined **decoding** function: $D(x) = (s,x_s)$.

For example for $q = 0.3$:

| $x$   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
|-------|---|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|
| $x_0$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |
| $x_1$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 |

We will find coding function now: we have $s$ and $x_s$ and want to find $x$. Denote $r := \lfloor xq \rfloor - xq \in [0,1)$

$$s := \lfloor (x+1)q \rfloor - \lfloor xq \rfloor = \lfloor (x+1)q - \lfloor xq \rfloor \rfloor = \lfloor (x+1)q - r - xq \rfloor = \lfloor q - r \rfloor$$

$s = 1 \iff r < q$
3 ASYMMETRIC NUMERAL SYSTEMS (ANS)

- $s = 1$: $x_1 = \lceil xq \rceil = xq + r$
  
  $x = \frac{x_1 - r}{q} = \left\lfloor \frac{x_1}{q} \right\rfloor$ because it's natural number and $0 \leq r < q$.

- $s = 0$: $q \leq r < 1$ so $\tilde{q} \geq 1 - r > 0$
  
  $x_0 = x - \lfloor xq \rfloor = x - xq - r = x\tilde{q} - r$
  
  $x = \frac{x_0 + r}{\tilde{q}} = \frac{x_0 + 1}{\tilde{q}} - \frac{1 - r}{\tilde{q}} = \left\lfloor \frac{x_0 + 1}{\tilde{q}} \right\rfloor - 1$

Finally coding:

$$C(s, x) = \begin{cases} 
\left\lceil \frac{x + 1}{1 - q} \right\rceil - 1 & \text{if } s = 0 \\
\left\lfloor \frac{x}{q} \right\rfloor & \text{if } s = 1
\end{cases} \left( \text{or } \begin{cases} 
\left\lfloor \frac{x}{1 - q} \right\rfloor & \text{if } s = 0 \\
\left\lceil \frac{x + 1}{q} \right\rceil - 1 & \text{if } s = 1
\end{cases} \right)$$ (30)

For $q = 1/2$ it’s usual binary system (with switched digits).

### 3.3 Stream coding/decoding

Now we can encode into large natural numbers ($x$). We would like to use ABS/ANS to encode data stream - into potentially infinite sequence of digits(bits) with expected uniform distribution. To do it we can sometimes transfer a part of information from $x$ into a digit from a standard numeral system to enforce $x$ to stay in some fixed range ($I$).

Let the data stream be encoded as $\{0, \ldots, b - 1\}$ digits - in standard numeral system of base $b \geq 2$. In practice we use binary system ($b = 2$), but thanks of this general approach, we can for example use $b = 2^8$ to transfer whole byte at once. Symbols contain correspondingly $\log(1/q_s)$ bits of information. When they cumulate into $\log b$ bits, we will transfer full digit to/from output, moving $x$ back to $I$.

Observe that taking interval in form ($l \in \mathbb{N}$):

$$I := \{l, l + 1, \ldots, bl - 1\}$$ (31)

for any $x \in \mathbb{N}$ we have exactly one of three cases:

- $x \in I$ or
- $x > bl - 1$, then $\exists! k \in \mathbb{N}$ $\lfloor x/b^k \rfloor \in I$ or
- $x < l$, then $\forall_{(d_i) \in \{0, \ldots, b - 1\}^n} \exists! k \in \mathbb{N}$ $xb^k + d_1 b^{k - 1} + \ldots + d_k \in I$. 
We will call such intervals \(b\)-absorbing: starting from any natural number \(x\), after eventual a few reductions \((x \to \lfloor x/b \rfloor)\) or placing a few youngest digits in \(x\) \((x \to xb + d_1)\) we would finally get into \(I\) in unique way.

For some interval \((I)\), define

\[
I_s = \{ x : C(s, x) \in I \}, \quad \text{so that } I = \bigcup_s C(s, I_s). \tag{32}
\]

Define stream decoding:

\[
\{(s, x) = D(x); \quad \text{use } s; \quad \text{(for example to generate symbol)} \]

\[\text{while} (x \notin I) \quad x = xb + \text{‘digit from input’} \}

Stream coding(s):

\[
\{ \text{while} (x \notin I_s) \}
\]

\{ \text{put } \text{mod}(x, b) \text{ to output; } x = \lfloor x/b \rfloor \}

\[x = C(s, x) \}

We need that above functions are ambiguous reverses of each other. Observe that we would have it iff \(I_s\) for \(s = 0, \ldots, n - 1\) and \(I\) are \(b\)-absorbing:

\[
I = \{ l, \ldots, lb - 1 \}, \quad I_s = \{ l_s, \ldots, lb_s - 1 \} \tag{33}
\]

for some \(l, l_s \in \mathbb{N}\).
We have: \( \sum_s l_s(b - 1) = \sum_s #I_s = #I = l(b - 1) \). Remembering that \( C(s, x) \approx x/q_s \), we finally have:

\[
l_s \approx l q_s \quad \sum_s l_s = l. \tag{34}
\]

We will look at the behavior of \( \lg x \) while stream coding \( s \) now:

\[
\lg x \rightarrow \sim \lg x + \lg(1/q_s) \pmod{\lg b} \tag{35}
\]

We have three possible sources of random behavior of \( x \):

- we choose one of symbol (behavior) in statistical(random) way,
- usually \( \frac{\lg q_s}{\lg b} \) are irrational,
- \( C(s, x) \) is near but not exactly \( x/q_s \).

It suggests that \( \lg x \) should cover uniformly the possible space, what agrees with statistical simulations. That means that the probability that we are visiting given state \( x \) should be approximately proportional to \( 1/x \).

Thanks of this observation we can for example estimate the number of wasted bits/symbol \( (26) \). Mean value of \( (x/q_s - C(s, x))^2 \) depends on used distribution function - for precise functions like in ABS, it’s smaller than 1. More important is the mean value of \( \frac{1}{x^2} \) - it should be about \((\text{by integrating}) \frac{1}{x^2} \frac{b^q - 1}{b} \ln b\). We can manipulate \( l \) and \( b \) parameters to achieve wanted compromise between speed and precision of our encoder.

We will now focus on the **stream version of ABS**.

For practical reason we can take:

\[
l = 2^R \quad b = 2^w \quad I = \{2^R, ..., 2^{R+w} - 1\} \tag{36}
\]

We have to check when \( I_0, I_1 \) are \( 2^w \)-absorbing.

Observe that \( D_2(bl) \in \{bl - [blq], [blq]\} \) have to be the smallest number above correspondingly \( I_0 \) or \( I_1 \) - have to be equal \( bl_0 \) or \( bl_1 \).

In both cases \( I_0, I_1 \) are \( 2^w \)-absorbing iff

\[
[blq] = [2^{R+w}q] \quad \text{is a multiplicity of} \ 2^w \tag{37}
\]

So if we want to use formulas explicitly, the precision of \( q \) should be at most \( R \) bits.

In implementation of data compressor using ABS, we can:

- calculate formulas for every symbol while processing data - it is more precise and we can transfer a few bits at once, but it can be a bit slower, and we need to be careful with \( (37) \), or
• store the coding tables in memory - smaller precision, needs memory and time for initialization, but should be faster and we have large freedom in choosing coding/decoding functions. For example by changing a few last symbols we can repair \[37\] for more precise \(q\).

Practical problem is that decoded and encoded sequences of symbols are in reverse order - to use probability prediction methods, we have to make predictions to the end, than encode in backward order. Now decompression is straightforward. In Matt Mahoney’s implementations (fpaqb, fpaqc on [9]) the data is divided into compressed separately segments, for which we store \(q\) from the prediction process.

### 3.4 Asymmetric Numeral Systems(ANS)

In the general case: encoding a sequence of symbols with probability distribution \(q_0, \ldots, q_{n-1}\) for some \(n > 2\), we could divide the selection of symbol into a few binary choices and just use ABS. We will see that we can also encode such symbols straightforward. Unfortunately I couldn’t find practical explicit formulas for \(n > 2\), but we can calculate coding/decoding functions while the initialization, making processing of the data stream extremely fast. The problem is that we rather cannot table all possible probability distributions - we have to initialize for a few and eventually reinitialize sometimes.

Fix some \(l, b, (q_s)\) and choose some \(l_s \in \mathbb{N}: l_s \approx l q_s, \sum_s l_s = l\).
We have to choose the distribution function \((D_1)\) for \(x\) from \(l\) to \(bl - 1\) - distribute \((b-1)l_0\) appearances of symbol ‘0’, \((b-1)l_1\) of ‘1’, \ldots, \((b-1)l_{n-1}\) of ‘\(n-1\)’.

We could do it using that \(D_2(x)\) is about \(x q_s\) as previously to choose the most appropriate symbol for succeeding \(x\). It would require priority queue for symbols.

In this section we will focus on a bit less precise but faster statistical method: fill a table of size \((b-1)l\) with proper number of appearances of symbols and for succeeding \(x\) take symbol of random number from this table, reducing the table.

Another advantage of this approach is that after fixing \((l_s)\), we still have huge (exponential in \(\#I\)) number of possible coding functions - we can choose one using some key, additionally encrypting the data.

**Initialization:** choose some \(l_s \in \mathbb{N}: l_s \approx l q_s, \sum_s l_s = l;\)
\(m=(b-1)l; \) symbols \((0, 0, \ldots, 0, 1, 1, \ldots, 1, \ldots, n-1, \ldots, n-1);\)
For \(x=1\) to \(b*l-1\)
{\(i=\)random natural number from 1 to \(m;\)
\(s=\)symbols\([i]\); symbols\([i]=\)symbols\([m]\); \(m--;\)
\(D[x]=(s, l_s) \) or \(C[s, l_s]=x\)
\(l_s++\)}
Where we can use practically any deterministic pseudorandom number generator, like Mersenne Twister ([10]) and use eventual key for its initialization. Practically without any cost we can increase the preciseness of this algorithm as much as we want by dividing $I$ into subsegments initialized separately.

Modern random number generators are practically unpredictable, so the ANS initialization would be. It chooses for each $x \in I$ different random local behavior, making the state practically unpredictable hidden variable.

Encryption based on ANS instead of making calculation while taking succeeding blocks as standard ciphers, makes all calculations while initialization - processing of the data is much faster: just using the tables.

Another advantage of preinitialized cryptosystem is that it’s more resistant to brute force attacks - while taking a new key to try we cannot just start decoding as usual, but we have to make whole initialization earlier, what can take as much time as the user wanted.

We can use this approach as an error correction method also. For example by introducing a new symbol - the forbidden one and rescaling the probability of the allowed ones. Now if this symbol occurs, we know that there was an error nearby. Knowing the statistical error distribution model, we can create a long list of possible errors, sorted by the probability. Now we can try to correct as it would be this case for succeeding points of this list and verify by trying to encode the following message. In this way we use that the most of blocks are correct, so we can transfer surpluses of unused redundancy to help with pessimistic cases.

We can also use huge freedom of choice for ANS to make the correction easier and faster - for example by enforcing that two allowed symbols has Hamming distance at least some number. We can for example get Hamming code this way as degenerated case - each allowed symbol appears once, so the intermediate state is just 1 - we don’t connect redundancy of blocks. In other cases we can use the connection between their redundancy to cope with badly damaged fragments.

4 Information capacity of general models

In this section we will give formalism and methodology for general models and prove existence of entropy for for large class of them.

4.1 Basic definitions

We will operate on $f : A \to \mathcal{A}$ ($A \subset X$) functions. They will be called as before patterns and their domain, denoted $\mathcal{D}(f)$ - their shape.

**Definition 4.** Model will be called a triple $(X, \mathcal{A}, C)$
space $X \subset \mathbb{Z}^n$

alphabet $\mathcal{A}$ - finite set of symbols

constraints (by forbidden valuations): $C \subset \{c : A \to \mathcal{A} : A \subset X\}$

then elements of model are $V(X)$, where for $A \subset X$ its valuations are:

$$V(A) := \{v : A \to \mathcal{A} | \forall c \in C : x \in D(c) \exists z \neq v(x)\}$$

$$V := \bigcup_{A \in X : |A| < \infty} V(A)$$

Digression: it is a very general definition. Instead of defining by forbidden states, we could do it by allowed ones (for example - take large enough subset of $X$ and take all allowed valuations).

We can write for example tiling problems in that formalism (denote each tile with a separate symbol and put their shapes in "allowed constrains"). We know that there can happen very different situations - even enforcing nonperiodic tiling.

To control our model we will have to limit this class.

The main example we will use is the Hard Square model.

**Definition 5.** Hard-square model (HS):

$$(X = \mathbb{Z}^2, \mathcal{A} = \{0, 1\}, C = \{((i, j), 1), ((k, l), 1)) : i, j, k, l \in \mathbb{Z}, |i - k| + |j - l| = 1\})$$

In each node of two-dimensional lattice we put 0 or 1 such that there are no neighboring ‘1’. It is one of the simplest models which haven’t been solved analytically. In [5] we can find exact formula for entropy of similar Hard Hexagon Model (we add upper left and lower right to the neighborhood), but this methodology creates some unsolvable singularities for HS. In [6] 43 digits of entropy of HS are found - I will use it to evaluate algorithms.

Unfortunately this methodology cannot be used to find statistics.

Define:

**Definition 6.**

Neighborhood of $x \in X$: $N_x := \{y \in X : \exists c \in C \in y \in D(c)\}$

Range of constrains: $L := \max\{|y_i - x_i| : x \in X, y \in N_x, i \in \mathbb{N}\}$

Interior of $A \subset X$: $A^- := \{x \in A : N_x \subset A\}$

Boundary of $A \subset X$: $A^0 := A \setminus A^-$

Thickening of $A \subset X$: $A^+ := \bigcup_{x \in A} N_x$

We will call set $A \subset X$ connected, if

$$\exists_{x, y \in A} \exists_{n \in \mathbb{N}, z^0, \ldots, z^n} \in X \quad z^0 = x, z^n = y, \forall i \sum_{j} |z^i_j - z^{i+1}_j| = 1.$$
For translative invariant models, we need to know only the neighborhood around one point, e.g. for HS: $N_x = x + \{(0,0), (1,0), (0,1), (-1,0), (0,-1)\}$.

Digression: neighborhood is always symmetric set ($x \in N_y \iff y \in N_x$), so

$$\rho(x, y) = \min \exists (x_0, x_1, \ldots, x_n = y) \forall i x_{i+1} \in N_{x_i}$$

is a natural metric for a given model.

HS model has many symmetries: generated by translations, axial symmetry and rotations by $\pi/2$.

**Definition 7.** A bijection $S : X \hookrightarrow X$ will be called *symmetry* if

$$c \in \mathcal{C} \iff c \circ S \in \mathcal{C}$$

(38)

### 4.2 Existence of entropy

We will now reduce the family of models we will focus on and prove the existence of positive average entropy for them.

**For the rest of this paper we assume, that:**

- $X = \mathbb{Z}^m$
- has finite range of constrains
- is translational invariant - every translation is its symmetry($t_x(y) := y + x$)
- is simple(#):

**Definition 8.** We call model *simple(#)*, if $\#V(X) > 1$ and there exists natural numbers $N > n \geq 0$, such that

$$\forall_{\text{connected} \, A \subset X} \forall_{u \in V((A^n)^o)} \forall_{v \in V(A^n)} \exists_{w \in V((A^n)^- \setminus A^n)} \; u \cup w \cup v \in V(A^N)$$

where $A^0 := A$, $A^{i+1} := (A^i)^+$.

In other words: for sets "nice" enough (of the form $A^n$), exists thickening ($N - n$ times) that for any valuation of its boundary, every valuation of $A^n$ is still allowed.

For example models with neutral symbol - which isn’t in any constrain (we can always use it: 0 in HS) are simple(#): $n = 0, \; N = 2$ - we fill $A^+ \setminus A$ with this symbol.

To the end of this paper we use notation:

$$N^A_u \equiv N(A, u) := \#{w \in V(A) : w|_{D(u)} = u}, \; N(A) \equiv N(A, \{\}) .$$

(39)
Lemma 9. Denote $k$-block: $\beta_k := \{0, 1, \ldots, k - 1\}^m$, now:
$$\exists n' \geq \max(L, 1) \forall v \in V(\beta_n^+ \setminus \beta_n) N(\beta_n^+, v) > 1.$$ 

Simply speaking: for every valuation of the neighborhood of large enough block we can valuate it in more than one way.

Proof: Because $\#V(X) > 1$, we can choose $k$: $\#V(\beta^k) > 1$.

$\beta^k$ can be placed in some $n'$ - block ($n' \geq L$). $\square$

Theorem 10. For models as above, there exists entropy $(H)$ and is positive:

There exists increasing sequence of sets $(A_i)$: $A_i \subset X$, $\#A_i < \infty$, $\bigcup_i A_i = X$, such that there exists a limit:

$$0 < H := \lim_{i \to \infty} \frac{\log(N(A_i))}{\#A_i} \leq \log \#A$$

Proof:

Take $n'$ like in the lemma. We will operate on blocks: $B \equiv \beta_n'$ placed in nodes of lattice $n'\mathbb{Z}^m$.

Because $n' \geq L$, valuations of a block can be restricted only by valuations of adjoined blocks: $B + n'(-1,0,1)^m \setminus \{0\}$.

![Figure 5](image)

Figure 5: Block division for $A_1, C'_2, A_2, C'_3, A_3$. Black - ”essence”. The rest (filler) we valuate such that all blocks denoted with the same color has the same pattern.

First of all we will find the filling pattern - periodic valuation of the space.

Take the lattice $Y := 2n'\mathbb{Z}^m$ and numerate anyhow $\{0, 1\}^m = \{x^i\}_{i=1,\ldots,2^m}$.

Now every block from $Y + B + x^i$ can be valuated independently from the other ($n' \geq$ range of constrains). Choose any $w_1 \in V(B)$.

Now using lemma 9 after valuating these blocks we can find some $w_2 \in V(B)$ for $Y + B + x^2$ not colliding with all $w_1$. And so on we get the universal periodic filling:

$$V(X) \ni w : \forall \ y \in Y, x \in B, i \in \{1, \ldots, 2^m\} \quad w(y + x + x^i) := w_i(x).$$

Now we can go to the main construction.

As the sequence we are looking for, let’s take

$$A_i := B + n'\{0, \ldots, 2^i\}^m, \quad H_i := \frac{\log(N(A_i))}{\#A_i}$$
Notice that $H_i \leq \#A$ - we would get this entropy without constrains. To prove that $H_i$ has a limit, we will construct increasing sequence $H'_i$, that $\forall i, H'_i \leq H_i \leq \#A$ and show that $\lim_{i \to \infty} H'_i - H_i = 0$. Monotone and bounded sequence has a limit, so $H_i$ will have the same.

We will make some construction to ensure the monotonicity:

$$C_i := B + n'\{x \in \{0, ..., 2^i\}^m : \exists x_i \in \{0, 2^i\}\}$$

external blocks of $A_i$ which will be filled with $w$.
We will need intermediate step:

$$C'_{i+1} := B + n'\{x \in \{0, ..., 2^i, 2^{i+1}\}^m : \exists x_i \in \{0, 2^i, 2^{i+1}\}\}$$

$$H'_i = \frac{\lg(N(A_i, w|C_i))}{\#A_i} \quad H''_{i+1} = \frac{\lg(N(A_{i+1}, w|C'_{i+1}))}{\#A_{i+1}}$$

Of course $H''_i \leq H'_i \leq H_i$.
Now the "essence" of $A_{i+1} \setminus C''_{i+1}$ is made exactly of $2^m$ "essences" from the previous step, with the same valuation of surrounding blocks.
So $N(A_{i+1}, w|C'_{i+1}) = 2^m N(A_i, w|C_i), \#A_{i+1} = \left(\frac{2^{i+1}+1}{2^i+1}\right)^m \#A_i < 2^m \#A_i$
We get: $H'_i < H''_{i+1}$,

$$H'_i < H''_{i+1} \leq H_{i+1}' < ... \leq \#A.$$

There left only to show, that $\lim_{i \to \infty} H_i - H'_i = 0$
Look at $D_i := (A_i \setminus C_i)^{N_i-1} - \ldots - + \ldots +, (A^+ \subset A)$, so from (#):
we can freely valuate $D_i$, independently to valuation on $C_i$.
Because $\beta_k \supset \beta_{k-2L} + (l, l, \ldots, l)$, so $\beta_{2i-2NL} + (NL, \ldots, NL) \subset D_i$, for large $i$:

$$\#D_i \sim (2^i)^m \quad (A_i \setminus D_i) \sim (2^i)^{m-1}$$

So $\lim_{i \to \infty} \frac{\#(A_i \setminus D_i)}{\#D_i} = 0$.

We have $N(D_i) \leq N(A_i, w|C_i) \leq N(A_i)$
But $\frac{N(A_i)}{N(D_i)} \leq (\#A)^{\#(A \setminus D_i)}$ - equality would be without constrains.
So $\frac{\lg N(A_i) - \lg N(D_i)}{\#A_i} \leq \frac{(\#(A_i \setminus D_i)) \lg(\#A)}{\#A_i} \to 0$.

From three sequences: $\lim_{i \to \infty} H_i - H'_i = 0$ \hfill $\square$.
For large class of models we can speak about their entropy.

For the rest of this work we add assumption of irreductibility:
5 Statistical approach

Now we will want to find optimal statistical description, by averaging over all valuations, like in one-dimensional case.

5.1 Statistical description

Definition 12. (Statistical) description is a function:

\[ p : \{(A,f) : A \subset X, \#A < \infty, f : A \to A\} \to [0,1] \]

such that

\[ \forall_{A \subset X, \#A < \infty} \forall_{x \in X \setminus A} \forall_{f : A \to A} \sum_{a \in A} p_{f \cup \{(x,a)\}} = p_f \tag{41} \]

\[ p(\emptyset) = 1 \]

where \( p_f \equiv p(\mathcal{D}(f), f) \).

It gives for each shape \( A \), the probability distribution of valuations on this shape. Because of translational invariance, it will be shown that \( f \) doesn’t depend on its position - for example \( p(01) \) denotes the probability that when choosing a node, it and its right neighbor are valued correspondingly to 01.

The conditions (41) ensure normalization to 1. (e.g. \( p(01) + p(00) = p(0) \)).
Now we would like to take an average over elements, but we can count only valuations on finite sets - we have to choose some sequence of finite sets tending to the whole space.

**Definition 13.**
We call series of finite sets *normal sequence of sets* \((A_i)_{i \in \mathbb{N}}\) if

\[
A_0 \subset A_1 \subset A_2 \subset \ldots 
\]

\[
\bigcup_{i \in \mathbb{N}} A_i = X.
\]

For \(f, v \in V, A \subset X : D(v) \cap D(f) = \emptyset, D(f) \subset A, D(v) \subset A\) we will call its \((A, v)\) approximation of optimal description:

\[
p_{f}^{A,v} := \frac{N(A, v \cup f)}{N(A, v)}.
\]  

(42)

Where \(N(A, u) = \# \{ w \in V(A) : w|D(u) = u \}\). Denote: \(p_{f}^{A} := p_{f}^{A,\emptyset}\).

Now for some \(B \supset A, v \in V(B^o), D_f \subset A^-, f \in V(A)\):

\[
p_{f}^{B,v} = \frac{N(B, v \cup f)}{N(B, v)} = \frac{\sum_{u \in V(A^o \setminus D_v)} N(B \setminus A^-, u \cup v) N(A, u \cup f)}{N(B, v)} = 
\]

\[
= \sum_{u \in V(A^o \setminus D_v)} p_{f}^{A,u} \frac{N(B \setminus A^-, u \cup v) N(A, u)}{N(B, v)} = \sum_{u \in V(A^o \setminus D_v)} p_{f}^{A,u} p_{u}^{B,v}
\]  

(43)

Where we’ve divided the sum over all valuations into the sum within and outside topologically separating set \(A^o\).

We want to find the optimal description \(p_{f}^{o}\) as the limit of succeeding approximations \(p_{f}^{A}\) for sets from a normal sequence. We’ve just shown that we get approximation for the succeeding set, as weighted average \(\left(\sum_{u \in V(A^o \setminus D_v)} p_{u}^{B,v} = 1\right)\) of approximations from the previous one for the same pattern \(f\).

So going to the next set won’t give worse approximation:

\[
A \subset B \Rightarrow \hat{p}_{f}^{A} \geq \hat{p}_{f}^{B} \geq \check{p}_{f}^{B} \geq \check{p}_{f}^{A}
\]

where

\[
\hat{p}_{f}^{A} := \min_{v \in V(A^o)} p_{f}^{A,v} \quad \check{p}_{f}^{A} := \max_{v \in V(A^o)} p_{f}^{A,v} \quad d_{f}^{A} := \check{p}_{f}^{A} - \hat{p}_{f}^{A}
\]

We’ve explained that \(d_{f}^{A}\) isn’t growing in normal sequence. If we would prove that for some normal sequence \((A_i)\), \(d_{f}^{A}\) is decreasing to 0, than taking any other normal sequence \((B_i)\), because \(\forall i \exists j A_i \subset B_j, \lim_{i \to \infty} p_{f}^{A_i} = \lim_{i \to \infty} p_{f}^{B_j}\).

So this limit would be the only reasonable optimal description.

Unfortunately I cannot prove formally its existence. I have to assume it:
Assumption 14 (*). For any pattern \( f \), \( d_f^A_i \to 0 \) for some normal sequence \( (A_i) \).

We can now define:

**Definition 15.** Optimal description, is \( p_f^S := \lim_{i \to \infty} p_f^A_i \)

where \( (A_i) \) - any normal sequence.

### 5.2 Optimal description fulfills general Markov’s property

Standard Markov’s property can be thought that knowing only both ending symbols of a sequence, we know the probability distribution of its interior. In the second section we’d shown that one-dimensional optimal description fulfills Markov’s property. Now we will see that in the general case analogous property is fulfilled - knowing symbols on the boundary of some set, we know the probability distribution in its interior. As previously, it will be uniform distribution among possible valuations.

Let’s show before, that optimal description preserves symmetries

\( S \) - a symmetry of model

\[ S' : V(X) \to V(S(X)) = V(X), S'(v)(x) = v(S^{-1}(x)) - \text{bijection on } V(X) \]

Now for a normal sequence \( (A_i) \) and some pattern \( f \), take (using freedom of choice) normal sequence \( (B_i) : B_i = S(A_i) \) and pattern \( S'(f) \):

\[
p_f^S = \lim_{i \to \infty} \frac{|\{w \in V(A_i) : w|_{D(f)} = f\}|}{|\{w \in V(A_i)\}|} = \lim_{i \to \infty} \frac{|\{S'(w) \in V(B_i) : S'(w)|_{D(S'(f))} = S(D(f)) = S'(f)\}|}{|\{S'(w) \in V(B_i)\}|} = p_{S'(f)}^{S'(f)}
\]

**Observation 16.** For any symmetry of model \( S \) and pattern \( f \)

\[
p_f^S = p_{S'(f)}^{S'(f)}.
\]

We will now discuss local optimality condition (LOC).

While having finite space, we have finite \( N \) number of elements - we can store there the largest number of information \( \lg(N) \), if they have uniform probabilistic distribution - we don’t favor any. We have analog to this condition in infinite space: when we valuate boundary of some finite set, all available valuations of its interior are equally probable - its LOC.

**Observation 17.** Optimality of statistical description \( p \) is equivalent to LOC(Local Optimality Condition):

for any \( B \subset X \), \( \#B < \infty \), \( A \subset B^{-} \), \( f \in V(B) \):

\[
p(f) = \frac{p(f|_{B \setminus A})}{N(B, f|_{B \setminus A})}
\]

**Proof:**
1. Take \( p^p = p^0 \), \((A_i)\) - normal sequence, \( A, B, f \) like above

\[
N(A_i, f|_{B\setminus A}) = \#\{u \in V(A) : u \cup f|_{B\setminus A} \in V(B)\} \quad N(A_i, f) = N(B, f|_{B\setminus A}) N(A_i, f)
\]

\[
p_f^0 = \lim_{i \to \infty} \frac{N(A_i, f)}{N(A_i)} = \lim_{i \to \infty} \frac{N(A_i, f|_{B\setminus A})}{N(B, f|_{B\setminus A})} \frac{p(f|_{B\setminus A})}{N(B, f|_{B\setminus A})}
\]

2. assume now LOC for \( p : B \subset X : \#B < \infty, \mathcal{D}(f) \subset A = B^- \)

\[
p(f) = \sum_{v \in V(B^o)} p(f \cup v) = \sum_{v \in V(B^o)} N(B, v \cup f) \frac{p(v)}{N(B, v)} = \sum_{v \in V(B^o)} p(v) p_f^{B,v} \quad (44)
\]

we get the first equality using normalization (41), the second from LOC (we can fill \( A \setminus \mathcal{D}(f) \) in \( N(B, v \cup f) \) ways)

Taking as \( B \) succeeding elements of some normal sequence, we get thesis. \( \square \)

Remarks

1. For a pattern \( f, A \subset X : \mathcal{D}(f) \subset A^- \) we have(44):

\[
p_f^0 = \sum_{v \in V(A^o)} p_f^0 p_f^{A,v}
\]

Now take for example sequence \( A_0 := 0, A_{i+1} := A_i^+ \), from (*): \( p_f^{A_i,v} \to p_f^0 \) - dependence of probability distribution of distant nodes decrease:

Assumption (*) is equivalent vanishing of long distance correlations.

2. LOC is equivalent pLOC - point local optimality condition:

\[
\forall_{B : N_0^0 \subset B \subset X \setminus \{0\}, \#B < \infty} \forall_{v \in V(B)} \forall_{a,b \in A} p_0 v \cup \{(0, a)\} = p_0 v \cup \{(0, b)\}
\]

where \( N_0^o = N_0 \setminus \{0\}, N_0 \) - neighborhood of 0.

It gives smaller set of conditions for optimality - we have only to check LOC for \( \#A = 1 \). We will use it later to generate approximations of the uniform distribution over elements. It says for example for HS that for any finite pattern:

\[
p \begin{pmatrix} x & 0 & x \\ 0 & 0 & 0 \\ x & 0 & x \end{pmatrix} = p \begin{pmatrix} x & 0 & x \\ 0 & 1 & 0 \\ x & 0 & x \end{pmatrix}
\]

Where ”\( x \)” denotes some valuations outside \( N_0 \).

formally:

\[
\forall_{A \subset X \setminus N_0} \forall_{v \in V(A)} p_f^0 v \cup v = p_f v \cup v
\]
where $f^* := 0|_{N_0 \setminus \{0\}}$, $f_a^* := f^* \cup \{(0, a)\}$

Other valuations of $N_0^0$ enforce 0 in the middle.

**Proof:** by induction over $\#A$: for $\#A = 1$ - pLOC

assume we’ve proved LOC for $\#A = k - 1$

Take some $A, B, v : \#A = k$, $B \supset A^+$, $v \in V(B \setminus A)$

We want to show that for any $f, g \in V(A)$

$$p_{v \cup f} = p_{v \cup g}$$

If there exists $x \in A$ such, that $f(x) = g(x)$, than we move $x$ to $B$ and use induction assumption. If not, we make intermediate step to $f' = f \setminus \{(x, f(x))\} \cup \{(x, g(x))\}$ for some $x \in A$. □

3. Take a sequence $A_0 := \{0\}$, $A_{i+1} := A_i^+$

then:

$$\lim_{i \to \infty} \frac{\lg(N(A_i))}{-\sum_{v \in V(A_i)} p_v^o \log p_v^o} = 1$$

**Proof:** set $i \in \mathbb{Z}$

Take any valuation of $A_{i+1}$, we have uniform distribution of available valuations on $A_i$. Using (#) $(n | N)$, we can freely valuate $A_{i-N+n}$, so we have more valuations than $N(A_{i-N+n})$:

$$-\sum_{v \in V(A_i)} p_v^o \log p_v^o \geq \lg N(A_{i-N+n})$$

Now we repeat discussion from the end of proof of theorem [10] ($\lim_{i \to \infty} \frac{\#A_{i-N+n}}{\#A_i} = 1$) and get the thesis. □

So we’ve justified that **optimal description has the same average capacity as the model.**

### 5.3 Sequential statistical description

In statistical description we have some excessive information because of the normality conditions. We can get rid of them in e.g. two ways:

1. We can limit to patterns without fixed symbol $(a)$. E.g. for HS, $s : A \rightarrow p_0|_A$

**Proof:** We want to get probability of some $f$ with $k + 1$ appearances of $a$, e.g.: $f(x) = a$, now:

$$p_f = p_f \setminus \{(x, a)\} - \sum_{b \in A \setminus \{a\}} p_f \setminus \{(x, a)\} \cup \{(x, b)\}.$$
2. Sequential description:
Take any numeration of nodes of the space: \( \{x^i\}_{i=0,1,...,\infty} = X \)
and fix some \( a \in A \)
\[
\forall v = (v_0, v_1, ..., v_{k-1}) \in A^k \quad \forall b \in A \setminus \{a\} \quad q_b(v) := \frac{p_{f_v \cup \{(x_k, b)\}}}{p_{f_v}}
\]
where \( D(f_v) := \{x^i\}_{i=0,...,k-1}, f_v(x^i) := v_i \).
We are taking successively \( x^i \) and using valuation of previous nodes we get its probability distribution of valuations.

Proof: We want to find some \( p_f \),
\[
\exists_k \{x_0, ..., x_k\} \supset D(f), A = \{x_0, ..., x_k\}
\]
\[
\forall u \in V(A) \quad P_u = \prod_{i=0}^{k} q_{u(x_i)}(u(x_0), ..., u(x_{i-1})) \left( q_a(w) = 1 - \sum_{b \in A \setminus \{a\}} q_b(w) \right)
\]
\[
p_f = \sum_{u \in V(A \setminus D(f)) : u \cup f \in V(A)} p_{f \cup u}.
\]

In the next section we will forget about assumption that there are only finite number of previous nodes.

So we are able to generate elements with given statistical description - visit successively \( x^i \) and generate its valuation with appropriate distribution.

Digression: assume now that we have some element \( f : X \rightarrow A \) generated from uniform distribution (e.g. using optimal algorithm). For a given shape \( A \), the value of \( f|_A \) should statistically correspond to optimal statistical description. So assuming vanishing of long-range correlation, we should get optimal description just by "averaging" \( f|_{A+x} \) over \( x \in nX \) where \( n \) is some large number. The same would be for any translations of \( nX \), so:
We could get the optimal description from any random element: by taking "average" of \( f|_{A+x} \) over all points of the space: \( x \in X \).

6 Statistical algorithms

Let’s say we have a statistical description, the nearer to the optimal, the better.
Now we want to construct an element using this description. If it would be optimal - we get uniform distribution of elements this way - we can store the same amount of information as model’s capacity.
We can use sequential approach like in the previous section, but it would favor some points (e.g. the first). We would like to use transitional invariance of the space - we cannot assume that there were only finite number of points before.

We are still assuming that long range correlations vanishes, so as an approximation of the optimal algorithm, we can assume that probability distribution for a given point depends only on valuations of neighboring ones.

**Definition 18.** *Statistical algorithm* is a pair \((<, q)\):

- \("<" \subset X^2\) - linear order \(X\)
- \(q_a : \{(x,v) : v \in V(x_<)\} \rightarrow [0,1]\) \((v \cup \{(x,a)\} \notin V \Rightarrow q_a(x,v) = 0)\)

such that \(\sum_{a \in A} q_a(x,v) = 1\),

where \(x_< := \{y \in X : y < x\}\).

Usually \(q\) will be translational invariant and depend only on neighboring nodes. Algorithm have to be consistent with the model - some \(q\) are enforced to 0.

In practice we cannot just "start" algorithm with infinite number of previous nodes - we usually need some initialization - it will be discussed on the end of the next section.

While generating an element, we will get some average entropy per choice(node) - we will count it on examples. By *optimality of algorithm* we will think: how distant is the capacity given by the algorithm to the real capacity (model’s entropy). For multidimensional models we usually won’t be able to construct optimal algorithm, only approximate it.

We will analyze now two simple algorithms for Hard Square model.

### 6.1 Algorithm I: Filling over independent sets

Divide the space into separate subsets \(Y_i\), inside which constrains doesn’t work (each valuation is allowed).

For example generally for given range of constrains \(L\), take

\[ Y = LZ^m, \ I = \{1, ..., L^m\}, \ \{x^i\}_{i \in I} = \{0, ..., L - 1\}^m, \ \text{now} \ Y_i := Y + x^i \]

For HS we can take \(Y_i = \{(x,y) : \text{mod}(x + y, 2) = i\}\) - nodes with even/odd sum of coordinates.

Algorithm: fill \(Y_0\) using some simple probability distribution (e.g.: with probability \(q\) put 1). Now do the same with \(Y_1\) ( with probability \(q'\)). This time only some of nodes can be valued to 1. This second valuation doesn’t influence any more nodes - we can store here as much information as possible - take \(q' = 1/2\).
So our algorithm is described by one number: $q$.

Let’s calculate the average entropy. Because in one half of nodes we get entropy $h(q)$ and in the rest, 1 bit/node if it’s possible - all neighbors have zeros (probability $(1 - q^4)$):

$$H_q = \frac{1}{2} h(q) + \frac{1}{2} (1 - q)^4$$

The best capacity we can achieve this way is $\max_q H_q \approx H - 0.0217$

So we lose about 4% of capacity: $\Delta H = 0.0217$ bit/node.

Why couldn’t we get the optimum?
We can take $q$ to fulfill relation:

$$(p_* :=) p \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = p \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ but then e.g. } p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} > p \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Explanation: If the middle node is in $Y_1$ - we have equality in both cases ($q' = 1/2$).
The strong inequality is got, if the middle is in $Y_0$. Probability that we value the middle to 1 is $q$ (probability on the right side). If we value it to 0, we’d have to additionally value its neighbor to 0, which is more probable when we fix 1 in the corner.

Remind that we have countable number of equalities to fulfill (pLOC), so usually finite number of parameters won’t be enough - we rather cannot get practical optimal algorithm, only its approximation.

### 6.2 Element generator using thermalization

To study the next algorithm, we will need to generate valuations on a finite set ($A$) with probability distribution close to uniform. We could do it using approximations of statistical algorithm. In presented here method, we will do it straightforward - by allowing random evolution. This process will approach to kind of thermal equilibrium - we can get to the uniform distribution as close as we want. The disadvantage comparing to the statistical algorithm, is that this time we have to visit every node many times - using this method for coding is highly unpractice. But we can use this generator for example to find the optimal statistics, but it occurs that the approaching is slow.

It is based on pLOC - we will equalize iteratively probabilities, by enforcing some random fluctuations:

For HS after any initialization from $V(A)$ (e.g. using some statistical algorithm)

**Repeat many times:** choose randomly a point in $A$. If all of its 4 neighbors has 0 value, then change its value ($0 \leftrightarrow 1$).
Let’s explain why we will tend to the uniform distribution. We are making some Markov process on $V(A)$. We can go from $f \in V(A)$ to $g \in V(A)$ in one step only if they differ on exactly one node - the probability of this transition is $\frac{1}{\#A}$ for both directions.

So the stochastic matrix describing this process is symmetric - bistochastic - $(1,1,\ldots,1)$ is the dominant eigenvector corresponding to eigenvalue 1 (matrix is irreducible) - iterating this process we will tend to the uniform distribution.

Generally we see that any bistochastic process would be appropriate.

Practically I’ve used about 2-5 $\#A$ iteration to get the first valuation and than to get next uncorrelated about $\#A$ iterations.

We can now analyze ”intuitively optimal” algorithm - with random order.

### 6.3 Algorithm II: Random seed

For every node take with uniform distribution a random real number from $[0,1]$, 

$$t : X \rightarrow [0,1]$$

It defines our order: $x < y \equiv t(x) < t(y)$

So in the following step of the algorithm we will take random, unvisited node:
- if it has a neighbor valuated to 1 - we valuate it to 0
- else - we valuate it to 0 or 1 with given probability.

We could chose this probability as a constant, but we will be more sophisticated. When we are in a point with some $t$, that means that statistically $t$ of nodes were already visited. So we can take some function:

charging profile - $q : [0,1] \rightarrow [0,1]$ - if we are in a node $t$, with probability $q(t)$ value it to 1, if we can.

To calculate entropy, we need to find for a given $q$ a second function:

$a : [0,1] \rightarrow [0,1]$ - probability that a node with given $t$ can still be valuated to 1.

Assume that $a, q$ are continuous.

Now entropy will be:

$$H_q = \int_0^1 a(t)h(q(t))dt \quad (45)$$

Notice that for optimality we would need:

- $a(0) = 1$ - we have no 1 yet
- $q(1) = \frac{1}{2}$ - these 1 are not blocking anything
7 APPROACHING OPTIMUM

- \( a(1) = 2p_0^* \) - they are neighbored by 4 zeros
- \( q(0) = p_0^* \) - this elements will be 1 for sure \((a = 1)\)

Unfortunately finding \( a \) from \( q \) seems very difficult, so finding optimal \( q \) seems even worse.

We can approximate it numerically, using Monte-Carlo type method:

On some finite set (say: a square), generate different random orders and valuations with uniform distribution (using found generator) and take the averages to find \( q \) and \( a \). On fig.6 are shown results \((A = \{1, ..., 300\}^2, 4000 \) measures\).

![Figure 6: Numerically found \( q \) i \( a \).](image)

This graphs are very blurred. It’s because we’ve simplified: \( q \) parameter should depends on the order of neighboring nodes. This algorithm looks translative invariant, but in fact only choosing the order is so.

After fitting 4th order polynomials and integrating \((45)\), I’ve got entropy larger then model’s. It’s the lesson that found \( a \) doesn’t corresponds to \( q \) now.

We can do it exactly by generating valuations using found \( q \): I’ve got about \( \Delta H \approx 0.01 - 0.02 \) bit/node.

7 Approaching optimum

In this section will be shown practical method which (if (*) is true) can gives us description as near to the optimal one as needed. We can use it to encode information with practically real model’s capacity. We will show numerical results for HS - there is very good tendency to the optimum.

7.1 Method

The idea is to approximate the model to be able to use one-dimensional solution:
1. Approximation of model - all dimensions but one (we will call it essential) are shrunk to a finite width \( n \),

2. New alphabet - all allowed valuations of cross-section orthogonal to the essential direction of width: range of constrains \( L \),

3. Transfer matrix: introduce (transfer) matrix of allowed succeeding (in essential direction) new symbols,

4. Solution to one-dimensional model - as in section 2,

5. Find algorithm - using found description. It will fill lines successively, treating every node in the same way,

It’s good to make one more step:

6. Algorithm evaluation - use it to reconstruct the real statistical description used and calculate entropy.

We will go through these steps for HS:

1. Fix \((1,0)\) as the essential direction on \( \mathbb{Z}^2 \). Fix width \( n \).

\[
Y = \mathbb{Z}(1,0) + (1,1) \overline{n},
\]

where \( \overline{n} = \{0, \ldots, n - 1\} \).

We have to chose some boundary conditions. We can do it e.g. in 2 ways:

(a) cyclic: \( \forall i, v(i + n, n) = v(i, 0) \),

(b) zero: \( \forall i, v(i, -1) = v(i, n) = 0 \).

We can think about cyclic conditions as additional constrains, so we get smaller entropy than original.

Zero conditions - the space is split into straps - we have all constrains instead those between straps - lines \( ni \) and \( ni + 1 \) \( (i \in \mathbb{Z}) \) - we reduce number of constrains - increase entropy.

So choosing proper boundary conditions we can bound entropy from below or above.

In this paper we are not interested in calculating entropy, but in statistical description - we want alphabet to be as small as possible. So the best will be cyclic conditions - thanks of symmetries, we will be able to identify many states.
2. We are interested in valuations of \((1, 1)\pi\) - we don’t have constrains inside - new alphabet has \(2^n\) symbols: \(A' := (1, 1)\pi \to \{0, 1\}\) but we can identify \(v, w \in A'\), such that:

(a) cyclic translation: \(\exists k \forall i \ v(i) = w(i + k \mod n)\)
(b) symmetry: \(\forall i \ v(i) = w(n - 1 - i)\)
(c) unimportant zero: \(v(0) = v(1) = v(2) = 1 \land w(1) = 0 \land \forall i \neq 1 \ v_i = w_i\)

We have first two conditions from the cycle symmetry.

The third says that two 1 blocks neighbor of node between them - its value is unimportant. This condition is the advantage of taking diagonal - the number of symbols behave like \(1.5^n\) and for vertical straps: \(\varphi^n \approx 1.618^n\).

Using this identifications for example for \(n = 21\) we get 3442 symbols instead of \(2^{21} - 609\) times less.

3. \(M_{vw} = 1 \iff u \in V\)
where \(D(u) := \{(0, 0), (1, 0)\} + (1, 1)\pi, \ u(i,i) := v(i), \ u(i+1,i) := w(i)\).

4. For \(M\) find the dominant eigenvector (e.g. using power method) and the method from the second section gives the probability distribution of two succeeding straps.

5. **Algorithm III**: straps of precision \((a,b)\)
Order: \((i,j) < (k,l) \iff (i - j < k - l) \lor (i - j = k - l \land i < k)\)
\(q : \{0, 1\}^a \times \{0, 1\}^b \to [0, 1]\)
\(q(u,v) = \) if there are no neighboring 1 and \(u\) - valuation of previous \(a\) nodes,
\(v\) - valuation of \(b\) nodes which will influence to following nodes,
then with this \((q)\) probability valuate this node (?) to 1.

We get \(q\) using two-straps probability distribution - look fig. 7.

**Figure 7**: Finding \(q\) for \(a = b = 2\).

Here are some algorithms as an example, found this way \((n = 24)\):

(a) \(a = b = 0\) \quad \(q = 0.3602994\)
(b) \( a = b = 1 \quad q = \begin{pmatrix} 0.3365899 & 0.2946553 \\ 0.4406678 & 0.3999231 \end{pmatrix} \)

(c) \( a = b = 2 \quad q = \begin{pmatrix} 0.3350090 & 0.2918920 & 0.3265314 & 0.2910261 \\ 0.3507593 & 0.3075943 & 0.3423356 & 0.3067152 \\ 0.4419816 & 0.4001011 & 0.4342912 & 0.3992940 \\ 0.4421921 & 0.4004149 & 0.4345211 & 0.3996084 \end{pmatrix} \)

Where the number of line denotes \( u (00?, 10?, 01?, 11?) \), column - \( v \) analogically.

6. Algorithm evaluation.

We need to find statistical the description that is really archived using given algorithm. It should be "similar" to used to find the algorithm (we will use it as the starting point), but while constructing the algorithm, we’ve lost some information. The description we are looking for, should be fixed point of iteration below (fig. 8):

![Figure 8: Iteration transforming algorithm into description](image)

Where the probability distribution for \( d \) is given by statistical algorithm.

There is a problem with \( c \) - we cannot find it straightforward. Although we can find its distribution, assuming that we’ve already found good approximation of probability distribution of strap valuation.

Algorithm:

(a) as starting description take used to find algorithm

(b) while in following iteration we get description more distant than some fixed boundary:

i. using actual description find probability distribution for \( (1, 1)^k \)

ii. using algorithm and this distribution make some iterations from the fig. 8

While having statistical description we can count capacity we get this way. Then we can compare it to the result from [6] - 43 digits of model’s entropy.
8 Conclusions

Here are results: $-\log_{10}(H - H_q)$ for different $a$, $b$:

| $a \setminus b$ | 0   | 1   | 2   | 3   | 4   | 5   |
|-----------------|-----|-----|-----|-----|-----|-----|
| 0               | 2.06| 2.113| 2.1153| 2.1153| 2.1153| 2.1153|
| 1               | 3.32| 3.82| 3.99| 4.001| 4.002| 4.003|
| 2               | 3.50| 4.37| 5.19| 5.44| 5.466| 5.436|
| 3               | 3.50| 4.42| 5.55| 6.43| 6.74| 6.78|
| 4               | 3.50| 4.42| 5.58| 6.71| 7.60| 7.96|
| 5               | 3.50| 4.42| 5.58| 6.74| 7.83| 8.72|

7.2 Initialization

To use found algorithm in practice, we still need to initiate it. We could valuate the first strap anyhow and in a few straps we would tend to assumed statistical description - we loose only some information on the boundary.

But assume we would like to use the whole space optimally.

On the first look - to valuate the first strap we should use the probability distribution for one strap. But on one side of this strap there will be not constrains - we should be able to store here a bit more of information.

So for the first strap, we should use the statistical description for straps following strap filled with zeros (or suitable boundary conditions): $p(v) = S_{0v}$.

Second: Straps (first($v$),second($u$)) should have probability distribution: $p(v, u) = S_{0v}S_{vu}$ - we can find statistical algorithm for second line from this distribution.

And so on. Of course we are tending to original algorithm this way.

8 Conclusions

- For one-dimensional codings, we can analytically find the optimal statistical description - we can encode it with full capacity,

- We have simpler alternative for arithmetic coding, which can be used to quickly compress, encrypt and add redundancy for correction in the same time,

- We have criteria to ensure that given model has average informational capacity,

- For models in which long range correlations vanishes, we can speak about optimal statistical description, which gives us uniform distribution over elements,

- If we don’t need exact optimality, we can use some simple algorithms, like filling over independent sets or over random sequence,

- We can generate approximations of uniform distribution on finite set,
Sometimes we can find algorithm as close to optimal as needed - there are results for Hard Square model in this paper.

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