What makes a multi-complex exact?

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Abstract

In this paper, we give a sufficient condition which makes the total complex of a multi-complex exact. This can be regarded as a variant of the Buchsbaum-Eisenbud theorem which gives a characterization of what makes a complex of finitely generated free modules exact in terms of the grade of the Fitting ideals of boundary maps of the complex.

Introduction

In the celebrated paper [BE73], Buchsbaum and Eisenbud gave a necessary and sufficient condition for a complex \( x \) of finitely generated free modules to be a resolution of the 0-th homology group \( H_0 x \) of \( x \) in terms of the grade of the Fitting ideals of boundary maps of the complex \( x \). (See Theorem 5.17 and also [BE73 1.4.2].) The main goal of this paper is to give a variant of this theorem for multi-complexes. We consider a special class of multi-complexes which we call cubes.

Let \( S \) be a finite set. We denote the cardinality of the set \( S \) by \( \# S \). We write \( \mathcal{P}(S) \) for the power set of \( S \). An \( S \)-cube in a category \( \mathcal{D} \) is a contravariant functor from \( \mathcal{P}(S) \) to \( \mathcal{D} \). Let \( x \) be an \( S \)-cube in \( \mathcal{D} \). For any \( T \in \mathcal{P}(S) \), we denote \( x(T) \) by \( x_T \) and call it the vertex of \( x \) (at \( T \)). For any \( k \in T \), we also write \( d^c_{T,k} \) or shortly \( d^c_k \) for \( x(T \setminus \{k\} \to T) \) and call it the \((k)\)boundary morphism of \( x \) (at \( T \)). (See Definition 2.5 and Example 2.6.) Let \( F^k \) and \( B^k \) be the order preserving maps from \( \mathcal{P}(S \setminus \{k\}) \) to \( \mathcal{P}(S) \) defined by sending a subset \( U \) of \( S \setminus \{k\} \) to \( U \) and \( U \cup \{k\} \) respectively. For any \( S \)-cube \( x \) in a category \( \mathcal{D} \) and any \( k \in S \), the compositions \( xB^k \) and \( xF^k \) are called the backside \( k\)-face of \( x \) and the frontside \( k\)-face of \( x \) respectively. By a face of \( x \), we mean any backside or frontside \( k\)-face of \( x \). (See Example 3.2.) Let \( S \) be a non-empty finite set and \( x \) an \( S \)-cube in an additive category \( \mathcal{B} \). If we regard \( x \) as a multi-complex where \( x_0 \) is in degree \((0, \cdots, 0)\), we will take its total complex \( \text{Tot} x \). (See Notations 3.5.)

Example 0.1 (See Notations 5.1 and Example 5.4). Let \( A \) be a commutative ring with unit and \( \mathcal{C} \) an abelian category enriched over the category of \( A \)-modules. Namely for any pair of objects \( x \) and \( y \) in \( \mathcal{C} \), the set of morphisms from \( x \) to \( y \), \( \text{Hom}_\mathcal{C}(x, y) \) has a structure of \( A \)-module and the composition of morphisms \( \text{Hom}_\mathcal{C}(x, y) \times \text{Hom}_\mathcal{C}(y, z) \to \text{Hom}_\mathcal{C}(x, z) \) is an \( A \)-bilinear homomorphism for any objects \( x, y \) and \( z \) in \( \mathcal{C} \). In particular, it induces a homomorphism of \( A \)-modules \( \text{Hom}_\mathcal{C}(x, y) \otimes_A \text{Hom}_\mathcal{C}(y, z) \to \text{Hom}_\mathcal{C}(x, z) \). For any object \( x \) in \( \mathcal{C} \) and any element \( a \) in \( A \), we write \( a_x \) for the morphism \( a \text{id}_x : x \to x \). A typical example of \( \mathcal{C} \) is the category of \( A \)-modules. Let \( f_S = \{ f_s \}_{s \in S} \) a family of elements in \( A \) and \( x \) an object in \( \mathcal{C} \). We define the \( S \)-cube \( \text{Typ}(f_S; x) \) in \( \mathcal{C} \), called the typical cubes associated with \( f_S \) and \( x \), as follows. For any \( T \in \mathcal{P}(S) \) and any element \( t \) in \( T \), we put \( \text{Typ}(f_S; x)_T = x \) and \( d^c_{T,g} \text{Typ}(f_S; x) := (f_t)_{x} \). In particular, if \( \mathcal{C} \) is the category of \( A \)-modules, then \( \text{Tot} \text{Typ}(f_S; A) \) is isomorphic to the usual Koszul complex associated with the family of elements \( f_S \).
In particular, we study a specific class of cubes in an abelian category which is a categorical variant of the notion about regular sequences in commutative rings. Let \( S \) be a finite set and \( A \) an abelian category. Let us fix an \( S \)-cube \( x \) in \( A \). For each \( k \in S \), the \( k \)-th homology of \( x \) is the \( k \)-cube \( \theta_k(x) \) in \( A \) defined by \( \theta_k(x)_T := \text{Coker } d_k^T \). When \( \#S = 1 \), we say that \( x \) is admissible if its unique boundary morphism is a monomorphism. For \( \#S > 1 \), we define the notion of an admissible cube inductively by saying that \( x \) is admissible if its boundary morphisms are monomorphisms and if for every \( k \) in \( S \), \( \theta_k(x) \) is admissible. (See Definition 3.13.)

The relationship between admissibility of cubes and the classical notion of regular sequences are summed up with the following way. For any elements \( f_1, \ldots, f_q \) in \( A \) and an object \( x \) in \( C \), we simply write \( x/(f_1, \ldots, f_q) \) for \( x/(f_1)_x, \ldots, (f_q)_x \). Let us fix an object \( x \) in \( C \). A sequence of elements \( f_1, \ldots, f_q \) in \( A \) is an \( x \)-regular sequence if every \( f_i \) is a non-unit in \( A \), if \( (f_1)_x \) is a monomorphism in \( C \) and if \( (f_{i+1})_x/(f_1, \ldots, f_i)_x \) is a monomorphism for any \( 1 \leq i < q - 1 \). A finite family \( \{ f_s \}_{s \in S} \) of elements in \( A \) is an \( x \)-sequence if \( \{ f_s \}_{s \in S} \) forms an \( x \)-regular sequence with respect to every ordering of the members of \( \{ f_s \}_{s \in S} \). Let \( f_S := \{ f_s \}_{s \in S} \) be a family of elements in \( A \). Then the \( S \)-cube \( \text{Typ}(f_S; x) \) is admissible if and only if the family \( f_S \) is an \( x \)-sequence. (See Notations 5.5 and Lemma 5.6.)

We give several characterizations of admissibility of cubes in an abelian category. The following theorem shows that the admissibility of a cube \( x \) gives a sufficient condition that the cube \( x \) is a resolution of the 0-th homology group \( \text{H}_0(x) \) of \( x \).

**Theorem 0.2 (A part of Theorem 3.15).** Let \( x \) be an \( S \)-cube in an abelian category \( A \). Then the following conditions are equivalent.

1. The \( S \)-cube \( x \) is admissible.
2. All faces of the \( S \)-cube \( x \) are admissible and \( \text{H}_k(\text{Tot} x) = 0 \) for any \( k > 0 \).

In our main theorem, we give a sufficient condition of admissibility of cubes. To state the main theorem, we introduce some terminology about a categorical variant of adjectives of matrices. The condition about the grade of the Fitting ideals of boundary morphisms of complexes in the Buchsbaum-Eisenbud theorem in [BE73] is replaced with the existence of regular adjectives of cubes. (See Proposition 5.13.) Let \( S \) be a finite set and \( C \) be an abelian category as in Example 0.1. An **adjugate of an \( S \)-cube** \( x \) in \( C \) is a pair \( (a, b^*) \) consisting of a family of elements \( a = \{ a_s \}_{s \in S} \) in \( A \) and a family of morphisms \( b^* = \{ d_T^s : x_T \to x_{T \setminus \{ i \}} \}_{T \in \mathcal{P}(S), i \in T} \) in \( C \) which satisfies the following two conditions.

1. We have the equalities \( d_T^s d_T^s = (a_t)_{T \setminus \{ i \}} \) and \( d_T^s d_T^s = (a_t)_{T \setminus \{ i \}} \) for any \( T \in \mathcal{P}(S) \) and \( i \in T \).
2. For any \( T \in \mathcal{P}(S) \) and any distinct elements \( a \) and \( b \) in \( T \), we have the equality \( d_T^s d_T^s = \frac{d_T^s d_T^s}{d_T^s d_T^s} \) for any \( T \in \mathcal{P}(S) \). (See Definition 5.7.) Example 5.3 shows how to relate the notion about adjectives of cubes and the classical notion about adjectives of matrices. The following theorem is the main theorem in this paper.

**Theorem 0.3 (A part of Theorem 5.13).** Let \( C \) be an abelian category as in Example 0.1 and \( x \) be an \( S \)-cube in \( C \). If \( x \) admits a regular adjective, then \( x \) is admissible.

Theorem 5.13 is a consequence of Theorem 0.4, which is a purely general theorem in category theory. For any natural number \( n \), let \( [n]^S \) be the partially ordered set of maps from \( S \) to the totally ordered sets of integers \( 0 \leq k \leq n \). Let \( 2 : [1]^S \to [2]^S \) and \( e_T : [1]^S \to [2]^S \) be order preserving maps defined by sending a map \( f : S \to [1] \) to \( 2f : S \to [2] \) and \( f + \chi_T : S \to [2] \) respectively where \( \chi_T \) is the characteristic function of \( T \) in \( S \). (Compare Example 2.3.) A double \( S \)-cube \( x \) in a category \( D \) is a contravariant functor from \( [2]^S \) to \( D \). (See Example 2.6.)
Theorem 0.4 (Double cube theorem). Let $x$ be a double $S$-cube in an abelian category $\mathcal{A}$. We assume that the following conditions hold.

1. The $S$-cube $x^2$ is admissible.
2. For any ordering pair $j < j'$ in $[2]^S$, $x(j < j')$ is a monomorphism in $\mathcal{A}$.
3. If $\#S \geq 3$, all faces of the $S$-cube $x \in T$ are admissible for any proper subset $T$ of $S$. Then the $S$-cube $x_{\in T}$ is also an admissible $S$-cube.

The proof of Theorem 0.4 will be given at 4.11. We explain the structure of this paper. In section 1, we introduce and study the notion about universally admissible families in a lattice which is a lattice theoretic variant of regular sequences in commutative ring theory. In section 2, we introduce and study the notions of (co)cubes and fibered cubes. In section 3, we review and establish the foundation of admissible cubes in an abelian category from [Moc13]. In section 4, we develop an abstract version of the main theorem. In section 5, we state and prove the main theorem. The standard results in this paper will be frequently utilized in the authors’ subsequent works about studying the weight of Adams operations on topological filtrations of $K$-theory of commutative regular local rings.

Conventions.

0.1 General assumptions

Throughout this paper, we use the letters $A$, $D$, $\mathcal{A}$, $S$ and $P$ to denote a commutative ring with unit, a category, an abelian category, a set and a partially ordered set respectively.

0.2 Set theory

0.2.1. We denote the cardinality of a set $S$ by $\#S$.

0.2.2. For any pair of disjoint sets $S$ and $T$, we write $S \sqcup T$ for the union set $S \cup T$ of $S$ and $T$.

0.2.3. For any pair of sets $U$ and $V$, we put $U \setminus V := \{x \in U; x \notin V\}$ and $U \ominus V := (U \cup V) \setminus (U \cap V)$.

0.3 Partially ordered sets

0.3.1. For two elements $a, b$ in a partially ordered set $P$, we write $[a, b]$ for the set of all elements $u$ in $P$ satisfying $a \leq u \leq b$. We regard $[a, b]$ as a partially ordered subset of $P$ if $a \leq b$ and $[a, b] = \emptyset$ if otherwise. We often use this notation when $P = \mathbb{Z}$ is the partially ordered set of integers.

0.3.2. For a non-negative integer $n$ and a positive integer $m$, we denote $[0, n]$ and $[1, m]$ by $[n]$ and $(m)$ respectively.

0.3.3. The trivial ordering $\leq$ on a set $S$ is defined by $x \leq y$ if and only if $x = y$.

0.3.4. An element $x$ in a partially ordered set $P$ is said to be maximal (resp. minimal) if for any element $a$ in $P$, the inequality $x \leq a$ (resp. $a \leq x$) implies the equality $x = a$. An element $x$ in a partially ordered set $P$ is maximum (resp. minimum) if the inequality $a \leq x$ (resp. $x \leq a$) holds for any elements $a$ in $P$. 
0.4 Category theory

0.4.1. For a category $\mathcal{X}$, we denote the class of objects in $\mathcal{X}$ by $\text{Ob}\mathcal{X}$ and for any objects $x$ and $y$ in $\mathcal{X}$, we write $\text{Hom}_\mathcal{X}(x, y)$ (or shortly $\text{Hom}(x, y)$) for the class of the morphisms from $x$ to $y$. We say a category $\mathcal{X}$ is locally small (resp. small) if for any objects $x$ and $y$, $\text{Hom}_\mathcal{X}(x, y)$ forms a set (resp. if $\mathcal{X}$ is locally small and $\text{Ob}\mathcal{X}$ forms a set).

0.4.2. For two categories $\mathcal{X}$ and $\mathcal{Y}$, we denote the (large) category of functors from $\mathcal{X}$ to $\mathcal{Y}$ by $\mathcal{Y}^{\mathcal{X}}$. Here the morphisms between functors from $\mathcal{X}$ to $\mathcal{Y}$ are just natural transformations.

0.4.3. We regard a partially ordered set $P$ as a category in a natural way. Namely, $P$ is a small category whose set of objects is $P$ and for any elements $x$ and $y$ in $P$, $\text{Hom}_P(x, y)$ is the singleton $\{(x, y)\}$ if $x \leq y$ and is the emptyset $\emptyset$ if otherwise. In particular, we regard any set $S$ as a category by the trivial ordering on $S$.

0.4.4. For any category $\mathcal{X}$, we denote the dual category of $\mathcal{X}$ by $\mathcal{X}^{\text{op}}$. Namely $\text{Ob}\mathcal{X}^{\text{op}} = \text{Ob}\mathcal{X}$ and for any objects $x$ and $y$ in $\mathcal{X}$, $\text{Hom}_{\mathcal{X}^{\text{op}}}(x, y) := \text{Hom}_\mathcal{X}(y, x)$.

0.5 Chain complexes

For a chain complex, we use the homological notation. Namely a boundary morphisms are of degree $-1$.

1 Universally admissible families in lattices

In this section, we study the notion about (universally) admissible families in lattices. Let us start by recalling some basic concepts about lattices.

**Definition 1.1 (Lattice).** A lattice $L$ is a partially ordered set such that for any elements $a$ and $b$ in $L$, their supremum $a \lor b$ and their infimum $a \land b$ exist. We call $a \lor b$ (resp. $a \land b$) the join (resp. the meet) of $a$ and $b$.

**Notations 1.2.** Let $S$ be a non-empty finite set and $\mathcal{r} = \{x_s\}_{s \in S}$ a family of elements in a lattice $L$.

1) For any subset $T$ of $S$, we denote the subfamily $\{x_t\}_{t \in T}$ by $\mathcal{r}_T$.

2) We write $\mathcal{r} \lor_{S}$ (resp. $\mathcal{r} \land_{S}$ or $\bigwedge_{s \in S} x_s$) for $\sup\{x_s; s \in S\}$ (resp. $\inf\{x_s; s \in S\}$) and call $\mathcal{r} \lor_{S}$ (resp. $\mathcal{r} \land_{S}$) the join (resp. meet) of a family $\mathcal{r}$. For any non-empty subset $T$ of $S$, we write $\mathcal{r} \lor_{T}$ and $\mathcal{r} \land_{T}$ for $(\mathcal{r}_T) \lor_{T}$ and $(\mathcal{r}_T) \land_{T}$ respectively. If the lattice $L$ has the maximum element 1, we use the notation $\mathcal{r} \land_{S}$ or $\bigwedge_{s \in S} x_s$ for $S = \emptyset$, which stands for the element 1.

3) We write $\mathcal{r} \lor y$ (resp. $\mathcal{r} \land y$) for a family $\{x_s \lor y\}_{s \in S}$ (resp. $\{x_s \land y\}_{s \in S}$). We have the following inequalities.

\[(\mathcal{r} \land y)^{\lor_{S}} \leq \mathcal{r}^{\lor_{S}} \land y.\]  \hspace{1cm} (1)

\[(\mathcal{r} \lor y)^{\land_{S}} \geq \mathcal{r}^{\land_{S}} \lor y.\]  \hspace{1cm} (2)

**Definition 1.3 (Ideals).** A subset $I$ of a partially ordered set $P$ is an ideal if for any pair of elements $x \leq y$ of $P$, $x \in I$ implies $y \in I$. We write $\text{Ideal}(P)$ for the set of all ideals in $P$. 

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Remark 1.4 (Semi-modular law). Let \( L \) be a lattice and \( a, b \) and \( c \) elements in \( L \) such that \( a \leq c \). Then we have the following inequality called the **semi-modular law**.

\[
a \lor (b \land c) \leq (a \lor b) \land c.
\]  

Lemma-Definition 1.5 (Modular lattice). A lattice \( L \) is **modular** if the following equivalent conditions hold.

1. For any elements \( a, b \) and \( c \) in \( L \) such that \( a \leq c \), the following equality called the **modular law** holds.

\[
a \lor (b \land c) = (a \lor b) \land c.
\]

The modular law is equivalent to the inequality

\[
a \lor (b \land c) \geq (a \lor b) \land c
\]

by the inequality (3) in Remark 1.4.

2. For any elements \( a, b \) and \( c \) in \( L \) such that \( a \leq b \), the equalities \( a \lor c = b \lor c \) and \( a \land c = b \land c \) imply the equality \( a = b \).

Proof. First we assume that condition (1) holds. Then for any elements \( a, b \) and \( c \) in \( L \) such that \( a \leq b \), \( a \lor c = b \lor c \) and \( a \land c = b \land c \), we have the equalities

\[
a = a \lor (a \land c) = a \lor (b \land c) = (a \lor c) \land b = (b \lor c) \land b = b.
\]

Next we assume that condition (2) holds. Then for any elements \( a, b \) and \( c \) in \( L \) such that \( a \leq c \), we put \( x = a \lor (b \land c) \) and \( y = (a \lor b) \land c \). Then we have \( x \leq y \), \( x \lor b = y \lor b \) and \( x \land b = y \land b \). Hence we have the equalities

\[
a \lor (b \land c) = x = y = (a \lor b) \land c.
\]

\[ \Box \]

Example 1.6 (Well-powered abelian category). An abelian category \( A \) is **well-powered** if for any object \( x \) in \( A \), the isomorphism class of subobjects of \( x \) which is written by \( \mathcal{P}(x) \) forms a set. For example, it is well-known that if \( A \) is the category of \( A \)-modules, then \( A \) is well-powered. We claim that for any object \( x \) in a well-powered abelian category \( A \), the set \( \mathcal{P}(x) \) is a modular lattice with respect to the ordering given by the inclusion. For any abelian category \( A \), for any object \( x \) in \( A \) and for any family of subobjects \( \mathfrak{x} = \{ x_s \to x \} \) in \( A \) indexed by a set \( S \), we write \( \mathcal{P}(\mathfrak{x}) \) for the sublattice of \( \mathcal{P}(x) \) generated by \( \mathfrak{x} \). Then \( \mathcal{P}(\mathfrak{x}) \) is a modular lattice.

Proof of the claim in Example 1.6. In general, for any subobjects \( a \subset b \) and \( c \subset d \) of \( x \), we have the short exact sequence

\[
(b \land d)/(a \land c) \to b/a + d/c \to (b \lor d)/(a \lor c).
\]

By putting \( c = d \) in the short exact sequence (6) above, it turns out that the equalities \( a \land c = b \land c \) and \( a \lor c = b \lor c \) imply the equality \( a = b \). Hence \( \mathcal{P}(x) \) is modular by Lemma-Definition 1.5.

\[ \Box \]

Definition 1.7 (Distributive, regular and (universally) admissible sequences). Let \( L \) be a lattice, \( r \) a positive integer and \( \mathfrak{x} = \{ x_s \}_{s \in S} \) a non-empty family of elements in \( L \) and \( y \) an element in \( L \).

1. We say that a pair \( (\mathfrak{x}, y) \) is **distributive** if we have an equality \( \mathfrak{x}^\lor \land y = (\mathfrak{x} \land y)^\lor \). It is equivalent to the condition that \( \mathfrak{x}^\lor \land y \leq (\mathfrak{x} \land y)^\lor \) by the inequality (1) in Notation 1.2.

2. We say that the family \( \mathfrak{x} \) is **strictly distributive** (resp. **admissible**) if \( \# S \leq 1 \) or if \( \# S \geq 2 \)
and if for any element \( t \in S \) (resp. for any non-empty proper subset \( T \) of \( S \)) and for any element \( t \) in \( S \setminus T \) a pair \((x_{S \setminus \{t\}}, x_t)\) (resp. \((x_T, x_t)\)) is distributive.

(3) We say that a sequence \( z_1, \ldots, z_r \) of elements in \( L \) is regular if \( r = 1 \) or if \( r \geq 2 \) and for any integer \( i \in [2, r] \), a pair \((x_{[i-1]} \leq i \leq z_i)\) is distributive.

(4) We say that the family \( \tau \) is universally admissible if for any two non-empty subsets \( U \) and \( V \) of \( S \) with \( U \cap V = \emptyset \), the pair \((\tau_U, \tau^AV)\) is distributive, or equivalently, if for any disjoint decomposition \( S = U \cup V \) such that \( U \neq \emptyset \), a family \( \tau_U \cup \tau^AV \) is admissible.

Remark 1.8 (Universally admissible sequences). Let \( S \) be a finite set, \( T \) a non-empty subset of \( S \) and \( \tau = \{x_s\}_{s \in S} \) a family of elements in a lattice \( L \). Then

- (1) If a family \( \tau \) is admissible (resp. universally admissible), then a family \( \tau_T \) is also admissible (resp. universally admissible).
- (2) If \( |S| \leq 2 \), then a family \( \tau \) is universally admissible.
- (3) If \( |S| \geq 3 \), a family \( \tau \) is universally admissible if and only if \( \tau \) satisfies the following two conditions.
  - (i) \( \tau \) is admissible.
  - (ii) \( \tau_{S \setminus \{s\}} \cup x_s \) is universally admissible for any \( s \in S \).
- (4) In particular if \( |S| = 3 \), then a family \( \tau \) is universally admissible if and only if \( \tau \) is admissible.

Proof. Assertion (1) for the admissible case is trivial. Let us assume that a family \( \tau \) is universally admissible. For any disjoint decomposition \( T = U \cup V \) such that \( |U| \geq 3 \), a family \( \tau_{(S \setminus T) \cup U \cup \tau^AV} \) is admissible by the assumption. Therefore a family \( \tau_U \cup \tau^AV \) is also admissible by the assertion for the admissible case. Hence a family \( \tau_T \) is universally admissible. Next we prove assertion (3). Let us assume that a family \( \tau \) satisfies conditions (i) and (ii) and let us fix a pair of disjoint subsets \( U \) and \( V \) of \( S \) such that \( S = U \cup V \) and \( |U| \geq 3 \). If \( V = \emptyset \), then \( \tau_U \cup \tau^AV = \tau \) is admissible by condition (i). If there exists an element \( s \in V \), then \( \tau_U \cup \tau^AV = (\tau_{S \setminus \{s\}} \cup x_s)_L \cup (\tau_{S \setminus \{s\}} \cup x_s)^AV \setminus \{s\} \) is admissible by condition (ii). Hence \( \tau \) is universally admissible. Assertion of the other direction is trivial. Assertions (2) and (4) are easy.

Example 1.9 (Regular sequences). Let \( A \) be a commutative ring with unit, \( M \) an \( A \)-module, \( r \) an integer such that \( r \geq 3 \) and \( f_1, \ldots, f_r \) a sequence of non-unit elements in \( A \). Let us recall that we say a sequence \( f_1, \ldots, f_r \) is \( M \)-regular if the multiplication by \( f_i, M \rightarrow M \) is injective and for any \( i \in [r-1] \), the multiplication by \( f_{i+1}, M/(f_1, \ldots, f_i)M \rightarrow M/(f_1, \ldots, f_i)M \) is injective. Now assume that a sequence \( f_1, f_j \) is a \( M \)-regular sequence for any \( 1 \leq i, j \leq r \) with \( i \neq j \). Then a sequence \( f_1, f_2, \ldots, f_r \) in \( P(M) \) is a regular sequence if and only if the sequence \( f_1, \ldots, f_r \) is \( M \)-regular. (See also Lemma 5.6).

Example 1.10 (Distributive lattices). We say that a lattice \( L \) is distributive if for any finite subset \( \tau = \{x_s\}_{s \in S} \) of \( L \) indexed by a non-empty finite subset \( S \) with \( |S| \geq 2 \), a pair \((\tau_{S \setminus \{s\}}, x_s)\) is distributive for any \( s \in S \). For any non-empty finite subset \( \tau = \{x_s\}_{s \in S} \) of \( L \), we consider the following three assertions.

- (1) The sublattice of \( L \) generated by \( \tau \) is distributive.
- (2) The map \( \operatorname{Ideal}(P(S)) \rightarrow L \) defined by sending an ideal \( I \) to an element \( \bigvee V I \tau^AV \) in \( L \) preserves the meet operation.
- (3) The set \( \{V \in \tau^AV; V \subset S\} \) is admissible.

Then assertions (1) and (2) are equivalent and assertion (2) implies assertion (3). Moreover if \( \tau \) satisfies assertion (3), then \( \tau \) is universally admissible.

Proposition 1.11. Let \( L \) be a lattice, \( S \) a non-empty finite set, \( \tau = \{x_i\}_{i \in S} \) a family of elements in \( L \) and \( y \) an element in \( L \). Assume that the following two conditions hold.

- (1) The family \( \tau \) is admissible (resp. universally admissible).
- (2) If \( |S| \geq 2 \), then for any element \( s \in S \) and any non-empty subset \( U \) of \( S \setminus \{s\} \), a pair
\((I_U \land x_i, y)\) is distributive. (Resp. For any pair of non-empty disjoint subsets \(U\) and \(V\) of \(S\) such that \(\#V \geq 2\) and any element \(v \in V\), a pair \((I_U \land x_{AV}, I^AV \land (v) \land y)\) is distributive.)

Then a family \(I \land y\) is also admissible (resp. universally admissible).

**Proof.** We first prove the assertion for the admissible case. What we need to prove is that the family \(I_U \land y\) is strictly distributive for any non-empty subset \(U\) of \(S\). We shall assume that \(k := \#U \geq 3\) and we put \(U = \{i_1, \cdots, i_k\}\). Then without loss of generality, we just need to check the following (in)equalities.

\[
\bigwedge_{j=1}^{k-1} (x_{ij} \land x_{ik} \land y) = \left\{ \bigwedge_{j=1}^{k-1} (x_{ij} \land x_{ik}) \right\} \land y \geq \left\{ \bigwedge_{j=1}^{k-1} x_{ij} \right\} \land (x_{ik} \land y) \geq \left\{ \bigwedge_{j=1}^{k-1} (x_{ij} \land y) \right\} \land (x_{ik} \land y)
\]

where the equality I follows from assumption (2) and the equality II follows from assumption (1). Next we prove the assertion for the universally admissible case. What we need to prove is that for any disjoint decomposition \(U \sqcup V = S\) such that \(U \neq \emptyset\), \(I_U \land x^AV \land y\) is admissible. To prove the assertion above, we apply this Proposition to the family \(I_U \land x^AV\) and the element \(x^AV \land y\). What we need to check is the following two conditions.

(i) The family \(I_U \land x^AV\) is admissible.

(ii) For any \(u \in U\) and any non-empty subset \(W\) of \(U \setminus \{u\}\), a pair \((I_W \land x^AV \setminus \{u\}, x^AV \land y)\) is distributive.

Condition (i) is a consequence of assumption (1) and condition (ii) is just assumption (2). Hence we get the desired result. 

**Corollary 1.12.** Let \(L\) be a lattice, \(S\) a non-empty finite set and \(a = \{a_s\}_{s \in S}, b = \{b_s\}_{s \in S}\) families of elements indexed by \(S\) in \(L\) such that \(a_s \geq b_s\) for any \(s \in S\). Assume that the following two conditions hold.

(1) The family \(b\) is admissible (resp. universally admissible).

(2) If \(\#S \geq 2\), then for any proper subset \(W\) of \(S\), any non-empty subset \(U\) of \(S\) such that \(\#U \geq 2\), any elements \(u \in U\) and \(s \in S \setminus W\), a pair \((b_{U \setminus \{u\}} \land a_{AV}, a_s)\) is distributive. (Resp. If \(\#S \geq 2\), then for any proper subset \(W\) of \(S\), any pair of disjoint non-empty subsets \(U\) and \(V\) of \(S\) such that \(\#V \geq 2\) and any elements \(s \in S \setminus W\) and \(v \in V\), a pair \((b_{AV} \land a^AV, b_{AV} \land a^AV \setminus \{v\} \land a^AV \land \{s\})\) is distributive.)

Then a family \(b \land a^S\) is also admissible (resp. universally admissible).

**Proof.** We set \(r := \#S\). We may assume without loss of generality that \(S = \{r\}\) and \(r \geq 2\).

For any integers \(k \in [-1, r - 1]\) and \(s \in S\), we set \(c^{(k)}_s = b_s \land a^\land \{r-k, r\}\) and \(c^{(k)} := \{c^{(k)}_s\}_{s \in S}\). Notice that \(c^{(-1)} = b\) and \(c^{(r-1)} = b \land a^S\).

**Claim.** For any integer \(k \in [-1, r - 1]\), the family \(c^{(k)}\) is admissible (resp. universally admissible).

We prove the claim by induction on \(k\). For \(k = -1\), the assertion is nothing but assumption (1). Let us assume that the assertion is true for some integer \(k \in [-1, r - 2]\). Notice that we have the equality \(c^{(k+1)}_s = c^{(k)}_s \land a_{r-k+1}\) for any \(s \in \{r\}\). We apply Proposition to the family \(c^{(k)}\) and the element \(a_{r-k+1}\). What we need to check is the following two conditions.

(a) The family \(c^{(k)}\) is admissible (resp. universally admissible).

(b) For any element \(s \in S\) and any non-empty subset \(U\) of \(S \setminus \{s\}\), a pair \((c^{(k)} U_{AV}, a_{r-k+1})\) is distributive. (Resp. For any pair of non-empty subsets \(U\) and \(V\) of \(S\) and any element \(v \in V\), a pair \((c^{(k)} \land a_{r-k+1}) \land c^{(k)} \land a_{r-k+1} \land c^{(k)} \land a_{r-k+1} \land c^{(k)} \land a_{r-k+1})\) is distributive.)

Condition (a) is just an inductive hypothesis and condition (b) follows from assumption (2). Hence the family \(c^{(k+1)} = c^{(k)} \land a_{r-k+1}\) is admissible (resp. universally admissible), which completes the proof of the claim. Since \(c^{(r-1)} = b \land a^\land \{r\}\), we obtain the desired result. 

\(\square\)
Remark 1.13. (1) In the situation Corollary 1.12 condition (2), we shall assume $s \neq u$ for the admissible case and $s \neq v$ for the universally admissible case.

(2) Moreover if we assume that $L$ is modular, then we shall assume that $W \cup U \neq S$ and that $s$ is not in $U$ for the universally admissible case.

Proof. (1) For the admissible case (resp. the universally admissible case), if we assume $u = s$ (resp. $v = s$), then we have the (in)equality

\[ b_k \land b_u \land a^W \leq a_s \quad (\text{resp. } b_k \land b^V \land a^W \leq b^V \setminus \{v\} \land a^{W \cup \{s\}}). \]

Therefore we have the equality

\[
\begin{aligned}
\left( \bigvee_{k \in U \setminus \{u\}} (b_k \land b_u \land a^W) \right) \land a_s &= \bigvee_{k \in U \setminus \{u\}} (b_k \land b_u \land a^W) \\
(\text{resp. } \left( \bigvee_{k \in U} (b_u \land b^V \land a^W) \right) \land (a^{W \cup \{s\}} \land b^V \setminus \{v\}) &= \bigvee_{k \in U} (b_u \land b^V \land a^W)).
\end{aligned}
\]

Therefore a pair $(b_{U \setminus \{u\}} \land b_u \land a^W, a_s)$ (resp. $(b_U \land b^V \land a^W, b^V \setminus \{v\} \land a^{W \cup \{s\}})$) is distributive.

(2) Let us assume that $s$ is in $U$. Then since we have the inequality

\[ b_s \land b^V \land a^W \leq b^V \setminus \{v\} \land a^{W \cup \{s\}}, \]

we have the equalities

\[
\begin{aligned}
\left( \bigvee_{k \in U} (b_u \land b^V \land a^W) \right) \land (a^{W \cup \{s\}} \land b^V \setminus \{v\}) &= (b_k \land b_s \land a^W) \lor \left( \bigvee_{k \in U \setminus \{s\}} (b_u \land b^V \land a^W) \right) \land (a^{W \cup \{s\}} \land b^V \setminus \{v\}) \\
&= (b_k \land b_s \land a^W) \lor \left( \bigvee_{k \in U \setminus \{s\}} (b_u \land b^V \land a^W) \right) \land (a^{W \cup \{s\}} \land b^V \setminus \{v\})
\end{aligned}
\]

by the modularity of $L$. Hence we shall assume that $s$ is in $U$ by replacing $U \setminus \{s\}$ with $U$. □

2 Cubes

In this section, we introduce the notions of cubes. Let $S$ be a set, $P$ a partially ordered set and $D$ a category.

Definition 2.1 (Successor, Predecessor). Let $x$ be an element in a partially ordered set $P$. A successor (resp. predecessor) of $x$ in $P$ is an element $t$ in $P$ such that $x < t$ (resp. $x > t$) and there exists no element $u$ in $P$ such that $x < u < t$ (resp. $x > u > t$). If $P$ is a totally ordered set, then the successor (resp. predecessor) of $x$ is uniquely determined if it exists and we denote it by $\text{Suc}(x)$ (resp. $\text{Pre}(x)$).
Notations 2.2. The set of maps from $S$ to $P$ is denoted by $P^S$. We define the ordering $\leq$ on $P^S$ by $f \leq g$ if and only if $f(s) \leq g(s)$ for any element $s$ in $S$. Then $P^S$ is a partially ordered set. If $P$ is a lattice, then $P^S$ is also a lattice. Here for two elements $f, g$ in $P^S$, the maps $f \lor g, f \land g : S \to P$ send $s$ to $f(s) \lor g(s)$ and $f(s) \land g(s)$ respectively, for each element $s$ in $S$. (Compare with Conventions (4) (iii).) Notice that we have the equality as partially ordered sets
\[(P^S)^{op} = (P^{op})^S.\] (7)

Example 2.3 (Double Power sets). (1) For any subset $T$ of $S$, we denote the characteristic function (of $T$ on $S$) by $\chi_T : S \to \{1\}$. Namely $\chi_T(s) = 1$ if $s$ is in $T$ and otherwise $\chi_T(s) = 0$. We write $\mathcal{P}(S)$ for the power set of $S$. Namely $\mathcal{P}(S)$ is the set of all subsets of $S$. We regard $\mathcal{P}(S)$ as a partially ordered set ordered by set inclusion, a fortiori, a category. We also write $\mathcal{P}'(S)$ for the set $\mathcal{P}(S) \setminus \{\emptyset\}$. We have the canonical isomorphism of partially ordered sets
$$\mathcal{P}(S) \xrightarrow{\sim} [1]^S$$ (8)
which is defined by sending a subset $T$ of $S$ to the characteristic function $\chi_T$ of $T$ on $S$. If we regard $[1]$ as the Sierpinski space, namely the topological space whose class of open sets is $\{\emptyset, \{1\}, \{0, 1\}\}$, and $S$ as a discrete topological space, then $\mathcal{P}(S) \xrightarrow{\sim} [1]^S$ inherits the compact-open topology from $[1]$ and $S$. The class of open sets of $\mathcal{P}(S)$ is the set of all ideals $\text{Ideal}(\mathcal{P}(S))$.

(2) For any ordered pair of disjoint subsets $(U, V)$ of $S$, we define the characteristic function (of $(U, V)$ on $S$) $\chi_{U, V} : S \to [2]$ as follows. For any element $s$ in $S$, $\chi_{U, V}(s) = 0$ if $s$ is in $S \setminus (U \cup V)$, is $1$ if $s$ is in $U$ and is $2$ if $s$ is in $V$. We denote the set of all ordered pair of disjoint subsets of $S$ by $\mathcal{D}(S)$. Namely
$$\mathcal{D}(S) := \{(U, V) \in \mathcal{P}(S) \times \mathcal{P}(S); U \cap V = \emptyset\}.$$ We define the ordering $\leq$ on $\mathcal{D}(S)$ by declaring to be $(U, V) \leq (U', V')$ if and only if $V \subset V'$ and $U \subset U' \cup V'$. Then $\mathcal{D}(S)$ is a partially ordered set. We have the canonical isomorphism of partially ordered sets
$$\mathcal{D}(S) \xrightarrow{\sim} [2]^S$$ (9)
which is defined by sending an ordered pair of subsets $(U, V)$ of $S$ to the characteristic function $\chi_{U, V}$ of $(U, V)$ on $S$.

(3) For any pair of maps $(f, g)$ from $S$ to $[1]$, we define the map $f + g : S \to [2]$ by sending an element $s$ in $S$ to $f(s) + g(s)$. Then we have the map
$$+: [1]^S \times [1]^S \to [2]^S.$$ By the virtue of isomorphisms (8) and (9), we also have the map
$$+: \mathcal{P}(S) \times \mathcal{P}(S) \to \mathcal{D}(S), \ (U, V) \mapsto U + V := (U \cup V, U \cap V).$$ For any element $T$ in $\mathcal{P}(S)$, we write $e_T$ for the map from $\mathcal{P}(S)$ to $\mathcal{D}(S)$ which is sending a subset $U$ of $S$ to the element $U + T$ in $\mathcal{D}(S)$. For any pair of disjoint subsets $(U, V)$ in $\mathcal{D}(S)$ and for any disjoint decomposition $U = A \cup B$, we have the equality
$$e_{U,V}((A \cup B) \cup V).$$ (10)

Notations 2.4. Let $U$ be an element in $P^S$ and $s$ an element in $S$. We write $s \in \text{Supp} U$ for the condition that $U(s)$ is not a minimal element in $P$. Now let us assume that $P$ is a totally ordered set, $s \in U$ and there exists the element $\text{Pre}_U(s)$. Then we define the map $U \setminus \{s\} : S \to P$ by putting that $U \setminus \{s\}(t)$ is $\text{Pre}_U(t)$ if $s = t$ and, is $U(t)$ if $s \neq t$. Then obviously we have the inequality $U \setminus \{s\} < U$. These notations are compatible with the usual ones when $P = [1]$. 

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**Definition 2.5 (Cubes).** An \((S, P)\)-cube (resp. \((S, P)\)-cocube) in a category \(D\) is a contravariant (resp. covariant) functor from \(P^S\) to \(D\). We denote the category of \((S, P)\)-cubes (resp. \((S, P)\)-cocubes) in \(D\) by \(\text{Cub}^{(S, P)} D\) (resp. \(\text{CoCub}^{(S, P)} D\)). Here the morphisms between \((S, P)\)-(co)cubes are just natural transformations. The associations \(D \mapsto \text{Cub}^{(S, P)} D\) and \(D \mapsto \text{CoCub}^{(S, P)} D\) give endofunctors on the category of (small) categories. For any \((S, P)\)-(co)cube \(x\) in \(D\), any element \(T\) in \(P^S\), we write \(x_T\) for \(x(T)\) and call it the **vertex of** \(x\) (at \(T\)). We say that an \((S, P)\)-cocube \((S, P)\)-cube \(x\) in a category \(D\) is **monic** if for any pair of elements \(U\) and \(V\) in \(P^S\) such that \(U \leq V\) (resp. \(V \leq U\)), \(x(U) \leq x(V)\) is a monomorphism in \(D\). Now assume that \(P\) is a totally ordered set and let \(s\) be an element in \(S\) such that \(s\) \(\in U\) and there exists the element \(\text{Pre} U(s)\) in \(P\). We write \(d_U^s\) or shortly \(d_U^s\) for \(x(U \setminus \{s\} < U)\) and call it the **(s-)boundary morphism of** \(x\) (at \(T\)).

**Example 2.6.** (1) If \(P\) is a singleton \(P = \{\ast\}\), then an \((S, P)\)-(co)cube \(x\) in a category \(D\) is just a family \(\{x_s\}_{s \in S}\) of objects in \(D\) indexed by \(S\).
(2) If \(P = [1]\), then an \((S, P)\)-cube (resp. \((S, P)\)-cocube) \(x\) in a category \(D\) is regarded as a contravariant (resp. covariant) functor from \(\mathcal{P}(S)\) to \(D\) by the isomorphism \([8]\). We simply call \((S, [1])\)-(co)cubes \(S\)-(co)cubes and we write \(\text{Cub}^S D\) (resp. \(\text{CoCub}^S D\)) for \(\text{Cub}^{(S,[1])} D\) (resp. \(\text{CoCub}^{(S,[1])} D\)).
(3) If \(P = [2]\), then an \((S, P)\)-cube (resp. \((S, P)\)-cocube) \(x\) in a category \(D\) is regarded as a contravariant (resp. covariant) functor from \(\mathcal{D}(P)\) to \(D\) by the isomorphism \([9]\). We simply call \((S, [2])\)-(co)cubes double \(S\)-(co)cubes and we write \(\text{DCub}^S D\) (resp. \(\text{CoDCub}^S D\)) for \(\text{Cub}^{(S,[2])} D\) (resp. \(\text{CoCub}^{(S,[2])} D\)). For any double \(S\)-(co)cube \(x\), any pair of disjoint subsets \(U, V\) of \(S\) and any element \(s \in U \lor V\), we write \(x_{U,V}\) and \(d_{U,V}^s\) (or shortly \(d_{U,V}^s\)) for \(x(U,V)\) and \(d_{U,V}^s\).

**Definition 2.7 (Pull-back of cubes).** Let \(S\) and \(T\) be sets, \(P\) and \(Q\) partially ordered sets and \(f : P^S \to Q^T\) an order-preserving map. Then composition with \(f\) induces the canonical natural transformations \(f^* : \text{Cub}^{(T,Q)} \to \text{Cub}^{(S,P)}\) and \(f^* : \text{CoCub}^{(T,Q)} \to \text{CoCub}^{(S,P)}\). For any \((T, Q)\)-(co)cube \(x\) in a category \(D\), we call \(f^* x\) the **pull-back of** \(x\) (along \(f\)).

**Definition 2.8 (Attachment of objects to cubes).** Let \(S\) be a non-empty set, \(P\) a partially ordered set with the minimum element \(m\). We denote the minimum element in \(P^S\) by \(\emptyset\). Moreover let \(x\) be an \((S, P)\)-cube in a category \(D\), \(f : x_0 \to y\) a morphism in \(D\). We define an \((S, P)\)-cube \(x_{f,y} : (P^S)^{op} \to D\) as follows. \(x_{f,y}\) is equal to \(x\) on \(P^S \setminus \{\emptyset\}\) and we put \((x_{f,y})_{\emptyset} := y\) and \(x_{f,y}(\leq U) := f \circ x(\leq U)\) for any element \(U\) in \(P^S \setminus \{\emptyset\}\). We call \(x_{f,y}\) the **attachment of** \(y\) to \(x\) (by \(f\)).

**Notations 2.9 (Dual (co)cube).** Let \(T\) be a finite totally ordered set. Then we have a unique isomorphism \(D_T : T^{op} \cong T\) of partially ordered sets and the map \(D_T\) induces an isomorphism of partially ordered sets
\[
(T^S)^{op} = (T^{op})^S \cong T^S.
\]
Let \(x\) be an \((S, T)\)-cube (resp. \((S, T)\)-cocube) in a category \(D\). We define \(\hat{x}\) the **dual** \((S, T)\)-cocube (resp. **dual** \((S, T)\)-cube) (of \(x\)) by \(\hat{x} := (x(D_T)^S)^{-1}\) (resp. \(\hat{x} := x(D_T)^S\)).

**Example 2.10 (Dual of \(S\)-(co)cubes).** Let \(x\) be an \(S\)-cube in a category \(D\). Then we have the equalities \(\hat{x}_T = x_{S \setminus T}\) and \(d_T^{x_S} := d_{(S \setminus T) \cup (T)}\) for any \(T \in \mathcal{P}(S)\) and any \(t \in T\).

The following lemma is sometimes useful for studying morphisms of cubes.

**Lemma 2.11.** Let \(S\) be a set and \(P\) a totally ordered set and \(x\) and \(y\) \((S, P)\)-cubes in a category \(D\). Then
(1) Let $U$ and $V$ be elements in $P^S$ such that $U \leq V$ and the set $[U, V]$ is a finite set. Then the morphism $x(U \leq V)$ is described as compositions of boundary morphisms.

(2) Assume that $S$ and $P$ are finite sets. Let $f = \{f_U : x_U \to y_U\}_{U \in P^S}$ be a family of morphisms in $D$. Then $f : x \to y$ is a morphism of $(S, P)$-morphisms in $D$ if and only if for any $U \in P^S$, and $s \in U$, we have the equality $d^U_{x, s} f_U = f_{U \setminus \{s\}} d^U_{y, s}$.

**Notations 2.12.** Let $D$ be a category closed under finite limits, $\{x_s : x \to y\}_{s \in S}$ and $\{y_s : y \to z\}_{s \in S}$ families of morphisms to objects $x$ and $y$ in $D$ respectively indexed by a non-empty finite set $S = \{s_1, \ldots, s_r\}$, $f : x \to y$ a morphism in $D$ and $\{f_s : x_s \to y_s\}_{s \in S}$ a family of morphisms in $D$ indexed by $S$ such that $f_s y = f s x$ for any element $s$ in $S$. Then we write $\prod_{s \in S} x_s$ and $\prod_{s \in S} f_s$ for $s_1 x_s x_{s_2} \cdots x_{s_r}$ and $f_s f_{s_2} \cdots f s_{s_r}$ respectively.

**Definition 2.13 (Coverings, associated cubes of coverings).** Let $P$ be a partially ordered set with the minimum element $m$. $S$ a non-empty finite set and $x$ an object in a category $D$.

(1) A $P$-covering (of $x$ indexed by a non-empty set $S$) is a family of contravariant functors $x := \{x_s : P^{op} \to D\}_{s \in S}$ such that $x_s(m) = x$ for any element $s$ in $S$.

(2) Let us assume that $S$ is a finite set and $D$ is closed under finite limits. Then for any $P$-covering $x := \{x^S : P^{op} \to D\}_{s \in S}$ of an object $x$ in $D$ indexed by $S$, we can associate $x$ with the $(P, S)$-cube $\text{Fib}_x$ in $D$ as follows. For any elements $U$ and $V$ in $P^S$ such that $U \leq V$, we put $(\text{Fib}_x)_U := \prod_{s \in S} x^S(U(s))$ and $(\text{Fib}_x)(U \leq V) := \prod_{s \in S} x^S(U(s))$. We call $\text{Fib}_x$ the $P$-covering $x$ associated with the $P$-covering $x$.

**Definition 2.14 (Pull-back of coverings).** Let $P$ and $Q$ be partially ordered sets with the minimum elements, $f : P \to Q$ an order preserving map preserving the minimum element, $x$ an object in a category $D$ and $x := \{x^Q : Q^{op} \to D\}_{s \in Q}$ $Q$-covering of $x$ indexed by a non-empty set $S$. Then we put $f^* x := \{x^P = f^Q P \to D\}_{s \in P}$ and call it the pull-back of $x$ along $f$.

**Example 2.15.** For any positive integer $m$, the map $m : [1] \to [m]$ which sends 0 to 0 and 1 to $m$ is order preserving mapping preserving the minimum element. Therefore for any $[m]$-covering $x$ of an object $x$ in a category, we can define the $[1]$-covering $m^* x$ of $x$.

**Notations 2.16.** Let $P$ and $Q$ be partially ordered sets and $f : P \to Q$ an order preserving map. For any set $S$, composition with $f$ induces the order preserving map $f^S : P^S \to Q^S$. If $P$ and $Q$ possess the minimum elements and $f$ preserves the minimum element, then $P^S$ and $Q^S$ also possess the minimum elements and the ordered map $f^S$ also preserves the minimum element. The map $f^S$ is sometimes abbreviated $f$.

**Definition 2.17 (Attachment of morphisms to coverings).** Let $P$ be a partially ordered set with the minimum element $m$, $x := \{x^P : P^{op} \to D\}_{s \in P}$ a $P$-covering of an object $x$ in a category $D$ indexed by a non-empty set $S$ and $f : x \to y$ a morphism in $D$.

(1) We define a partially ordered set $P^* := P \cup \{-\infty\}$ where $-\infty$ is a symbol and $p > -\infty$ for any element $p$ in $P$.

(2) We define $i_P : P \to P^*$ to be an order preserving map by sending an element $p$ to $p$ if $p \neq m$ and $m$ to $-\infty$.

(3) We define the $P^*$-covering $i_P x^P := \{x^P : P^{op} \to D\}_{s \in S}$ of $y$ indexed by $S$ as follows. For any element $s$ in $S$, $x^P_{s} (m \leq p)$ is equal to $x^P_{s}$ on $P$ and $x^P_{s} (-\infty) = y$ and $x^P_{s} (-\infty < p) := f \circ x^P(m \leq p)$ for any element $p$ in $P$.

**Example 2.18.** Let $P$ be a partially ordered set with the minimum element $m$, $x := \{x^P : P^{op} \to D\}_{s \in S}$ a $P$-covering of an object $x$ in a category $D$ indexed by a non-empty set $S$ and $f : x \to y$ a morphism in $D$. Let us assume that $D$ is closed under taking finite limits. Then we have the canonical isomorphism $i_P^* \text{Fib}(x_{f, y}) \cong (\text{Fib} x)_{f, y}$. (11)
Example 2.19. Let $P$ be a partially ordered set with the minimum element $m$, $S$ a non-empty finite set and $x$ an $(S, P)$-cube in a category $D$.

(1) Recall the definition of the map $\emptyset : S \to P$ from Definition 2.8. It is the minimum element in $P^{\text{op}}$. Namely, it sends any element in $S$ to $m$.

(2) For any elements $p$ in $P$ and $s$ in $S$, we define a map $\delta_{s,p} : S \to P$ by sending an element $t$ in $S$ to $p$ if $s = t$ and to $m$ if $s \neq t$. Obviously for any elements $p \leq p'$, we have an inequality $\delta_{s,p} \leq \delta_{s,p'}$.

(3) We associate with $x$ a $P$-covering $\mathcal{U} x := \{ x^s : P^{\text{op}} \to D \}_{s \in S}$ of $x_\emptyset$ indexed by $S$ as follows. For any element $s$ in $S$, we define $x^s : P^{\text{op}} \to D$ to be the functor which sends an element $p$ in $P$ to $x_{s,p}$ and any pair $p \leq p'$ in $P$ to $x(\delta_{s,p})$.

(4) If $D$ is closed under finite limits and $S$ is a non-empty finite set, then we have the canonical morphism of $(S, P)$-cubes $C(x) : x \to \text{Fib}\, \mathcal{U} x$ which is induced from the identity morphisms on $x_{\delta_{s,p}}$ for any elements $s$ in $S$ and $p$ in $P$ by the universal property of fiber products.

Definition 2.20 (Fibered cubes). Let $P$ be a partially ordered set with the minimum element $m$, $S$ a non-empty finite set, $D$ a category closed under finite limits. An $(S, P)$-cube $x$ in $D$ is fibered if the canonical morphism of $(S, P)$-cubes $C(x) : x \to \text{Fib}\, \mathcal{U} x$ is an isomorphism.

Lemma 2.21 (Compatibility of pull-backs). Let $P$ and $Q$ be partially ordered sets with the minimum elements, $f : P \to Q$ an order-preserving map which preserves the minimum element, $S$ a non-empty set, $D$ a category closed under finite limits.

(1) For a $Q$-covering indexed by $S$, $\mathcal{U} z := \{ x^s : Q^{\text{op}} \to D \}_{s \in S}$ of an object $z$ in $D$, we have the canonical isomorphism of $(P, S)$-cubes

$$\text{Fib}(f^* \mathcal{U} z) \cong f^* \text{Fib}\, \mathcal{U} z.$$

(2) For a $(Q, S)$-cube $x$ in $D$, we have the canonical equality of $P$-coverings

$$f^* \mathcal{U} x = \mathcal{U} f^* x.$$

(3) For a fibered $(Q, S)$-cube $y$ in $D$, the pull-back $f^* y$ of $y$ along $f$ is a fibered $(P, S)$-cube.

Proof. For any elements $s$ in $S$ and $p$ in $P$ and any map $U$ from $S$ to $P$, we have the canonical equalities

$$\text{Fib}(f^* x)_U = \prod_{s \in S} x^s(f U(s)) = (f^* \text{Fib} x)_U$$

and

$$f \delta_{s,p} = \delta_{s,f(p)}.$$

Assertions (1) and (2) follow from the equalities above respectively. For assertion (3), we have the commutative diagram of $(P, S)$-cubes.

$$\begin{array}{ccc}
\text{Fib}(f^* y) & \xrightarrow{C(f^* y)} & \text{Fib}\, \mathcal{U} f^* y \\
\text{Fib} f^* y & \xrightarrow{f^* \text{Fib}} & \text{Fib} f^* \mathcal{U} y.
\end{array}$$

Since the morphisms $f^* C(y)$, $\text{I}$ and $\text{II}$ are isomorphisms by assumption and assertions (1) and (2), $C(f^* y)$ is also isomorphism. Therefore $f^* y$ is also fibered. □

Definition 2.22 (Power sets of cubes and coverings). Assume that $P$ has the minimum element $m$.

(1) Let $x$ be a monic $P^S$-cube in an abelian category $\mathcal{A}$. Then we may regard all vertices of $x$ as subobjects of $x_\emptyset$. We write $P(x)$ for the sublattice of $P(x_\emptyset)$ generated by all vertices of $x$.

(2) Let $z$ be an object in $\mathcal{A}$ and $\mathcal{A}$ is a $P$-covering of $z$. Assume that Fib $\mathcal{A}$ is monic. Then we write $P(z)$ for $P(\text{Fib} \mathcal{A})$.  

12
Lemma 2.23 (Characterization of fibered cubes). Let \( S \) be a finite set such that \( \# S \geq 2 \), then the following conditions are equivalent for any \( S \)-cube \( x \) in a category \( D \) closed under finite limits.

1. The \( S \)-cube \( x \) is fibered.
2. For any pair \((s, t)\) of distinct elements in \( S \) and for any subset \( T \subset S \setminus \{s, t\} \), the commutative diagram

\[
\begin{array}{ccc}
X_{T \cup \{s, t\}} & \xrightarrow{d^{x}_{T \cup \{s, t\}}} & X_{T \cup \{s\}} \\
\downarrow d^{x}_{T \cup \{s, t\}} & & \downarrow d^{x}_{T \cup \{s\}} \\
X_{T \cup \{t\}} & \xrightarrow{d^{x}_{T \cup \{t\}}} & X_{T}
\end{array}
\]

is a Cartesian square.

**Proof.** Obviously assertion (1) implies assertion (2). We prove the converse implication. Namely we prove that \( C(x)_{T} \) is an isomorphism for any \( T \) by induction on the cardinality of \( T \).

If \( \# T \leq 1 \), then \( C(x)_{T} \) is the identity morphism. For any subset \( T \) of \( S \) such that \( \# T \geq 2 \), we fix a pair of distinct elements \( s \) and \( t \) in \( T \). By hypothesis, the square

\[
\begin{array}{ccc}
X_{T} & \xrightarrow{d^{x}_{T}} & X_{T \setminus \{s\}} \\
\downarrow d^{x}_{T} & & \downarrow d^{x}_{T \setminus \{s\}} \\
X_{T \setminus \{t\}} & \xrightarrow{d^{x}_{T \setminus \{t\}}} & X_{T \setminus \{s, t\}}
\end{array}
\]

is a Cartesian square. Since \( C(x)_{U} \) is an isomorphism for any proper subset \( U \) of \( T \) by the inductive hypothesis, \( C(x)_{T} = C(x)_{T \setminus \{t\}} \times C(x)_{T \setminus \{s, t\}} \) is also an isomorphism by the universal property of the fiber product \((\text{Fib} \cup x)_{T}\). Hence we get the desired assertion. \( \square \)

Example 2.24. Let \( D \) be a category closed under finite limits, \( S \) a non-empty finite set, \( x \) an object in \( D \), \( x = \{x^{s} : [2]^{\text{op}} \rightarrow D\}_{s \in S} \) a \([2]\)-covering of \( x \) indexed by \( S \) and \( T \) a subset of \( S \). We set \( x|_{T} := \sqcup e^{x}_{T} \text{Fib} x \), which is a \([1]\)-covering of \((\text{Fib} x)_{T, \emptyset}\) indexed by \( S \), where \( e^{x} \) is as in Example 2.23(3). Then we have the canonical isomorphisms

\[
e^{x}_{x}(\text{Fib} x) \iso \text{Fib}(x|_{T}) \tag{12}
\]

\[
2^{*}(\text{Fib} x) \iso \text{Fib}(2^{*} x) \tag{13}
\]

by Lemma 2.21(1) and (3). Here \( 2 \) is as in Example 2.15.

Lemma 2.25 (Characterization of fibered double cubes). Let \( x \) be a double \( S \)-cube in a category \( D \) closed under finite limits.

1. For any subset \( T \) of \( S \), we have the commutative diagrams

\[
\begin{array}{ccc}
e^{z}_{x} & \xrightarrow{e^{z}_{x} C(x)} & e^{z}_{x}(\text{Fib} \cup x) \\
\downarrow C(e^{z}_{x}) & \downarrow i & \downarrow C(2^{*} x) \\
\text{Fib}(\sqcup(e^{z}_{x} x)) & \iso & \text{Fib}(\sqcup(2^{*} x))
\end{array}
\]

\[
\begin{array}{ccc}
2^{*} x & \xrightarrow{2^{*} C(x)} & 2^{*}(\text{Fib} \cup x) \\
\downarrow C(2^{*} x) & \downarrow i & \downarrow C(2^{*} x) \\
\text{Fib}(\sqcup x) & \iso & \text{Fib}(2^{*}(\text{Fib} x)).
\end{array}
\]
(2) If \( \#S \geq 2 \), then the following conditions are equivalent:
(i) The \( S \)-cubes \( e^*_T \) \( x \) is fibered \( S \)-cubes for any subset \( T \) of \( S \).
(ii) The \( S \)-cubes \( 2^* \) \( x \) and \( e^*_T \) \( x \) are fibered \( S \)-cubes for any proper subset \( T \) of \( S \).
(iii) The double \( S \)-cube \( x \) is fibered.
(iv) The canonical morphism \( e^*_T C(x) \) is an isomorphism of \( S \)-cubes for any subset \( T \) of \( S \).
(v) The canonical morphisms \( 2^* C(x) \) and \( e^*_T C(x) \) are isomorphisms of \( S \)-cubes for any proper subset of \( S \).

Proof. Assertion (1) is straightforward. In assertion (2), conditions (iii), (iv) and (v) are obviously equivalent. By the virtue of the assertion (1) and Lemma 2.23 condition (i) (resp. (ii)) is equivalent to condition (iv) (resp. (v)). Hence we get assertion (2). \( \square \)

Corollary 2.26. Let \( x \) be a monic double \( S \)-cube in an abelian category \( A \). We put \( a \) := \( \{ x_{(s, \{ \})} \} \) \( s \in S \) and \( b := \{ x_{(s, \{ \})} \} \) \( s \in S \). Moreover let us assume that \( 2^* x \) and \( e^*_T x \) are fibered for any proper subset \( T \) of \( S \). Then we have the canonical isomorphism \( x_{U,V} \sim a^\land U \land b^\land V \) as subobjects of \( x_{\emptyset, \emptyset} \) for any \( (U, V) \) in \( \mathcal{D} P(S) \). Here the symbol \( \land \) means the meet in the lattice \( \mathcal{P}(x) \).

Proof. We shall only remark that in this case, fiber products of vertices of \( x \) over \( x_{\emptyset, \emptyset} \) is just the wedge products in \( \mathcal{P}(x) \). The assertions follows from Lemma 2.25. \( \square \)

3 Admissible cubes

In this section, we review some notations of admissible cubes introduced in [Moc13]. Let \( S \) be a finite set and \( A \) an abelian category.

Definition 3.1 (Restriction of cubes). Let \( U, V \) and \( W \) be a triple of subsets of \( S \) such that \( U \cap V = \emptyset \) and \( U \cup V \subset W \). We define \( i^V_U : \mathcal{P}(U) \rightarrow \mathcal{P}(W) \) to be the functor which sends a set \( A \in \mathcal{P}(U) \) to the disjoint union set \( A \cup V \) of \( A \) and \( V \). Composition with \( i^V_U \) induces the natural transformation \( (i^V_U)^* : \text{Cub}^S \rightarrow \text{Cub}^W \). (See Definition 2.7.) For any \( S \)-cube \( x \) in a category \( D \), we write \( x^V_W \) for \( (i^V_U)^* x \) and call it the restriction of \( x \) to \( U \) along \( V \). For any pair of disjoint subsets \( U \) and \( V \) of \( S \) and subsets \( A \subset U \) and \( B \subset V \), we have the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{P}(S \setminus (U \cup V)) & \xrightarrow{i^A_{S \setminus (U \cup V), s \setminus U}} & \mathcal{P}(S \setminus U) \\
\downarrow{i^B_{S \setminus (U \cup V), s \setminus V}} & & \downarrow{i^A_{S \setminus U, s \setminus V}} \\
\mathcal{P}(S \setminus V) & \xrightarrow{i^B_{S \setminus V, s \setminus V}} & \mathcal{P}(S).
\end{array}
\]

In particular, for any \( S \)-cube \( x \) in \( C \), we have the equalities

\[(x|_{S \setminus (U \cup V)}|_{S \setminus (U \cup V)}) = (x|_{S \setminus U})|_{S \setminus (U \cup V)}.
\]

Example 3.2 (Faces of cubes). For any \( S \)-cube \( x \) in a category \( D \) and any \( k \in S \), \( x_{S \setminus \{ k \}}^{(k)} \) and \( x_{S \setminus \{ k \}}^{(k)} \) are called the backside \( k \)-face of \( x \) and the frontside \( k \)-face of \( x \) respectively. By a face of \( x \), we mean any backside or frontside \( k \)-face of \( x \). We write \( d^k_x : x_{S \setminus \{ k \}}^{(k)} \rightarrow x_{S \setminus \{ k \}}^{(k)} \)
for the natural transformation from \(x|_{S \setminus \{k\}}\) to \(x|_{S \setminus \{k\}}\) induced by the boundary morphisms 
\[d_{T∪\{k\}}^{d,x}: x_{T∪\{k\}} \to x_{T}\] for any \(T \in \mathcal{P}(S \setminus \{k\})\).

**Example 3.3.** Let \(\mathcal{D}\) be a category closed under finite limits and \(x\) an object in \(\mathcal{D}\) and \(x = \{x_\ell, d^x_\ell\}_{\ell \in \mathcal{S}}\) a family of morphisms to \(x\). For any element \(s \in S\), we obviously have the canonical isomorphism
\[\text{Fib}(\mathcal{D})|_{S \setminus \{s\}} \cong \text{Fib}(\mathcal{D})|_{S \setminus \{s\}}\]. Moreover if all \(d^x_\ell\) are monomorphisms, then we also have the canonical isomorphism
\[\text{Fib}(\mathcal{D})|_{S \setminus \{s\}} \cong \text{Fib}(\mathcal{D})|_{S \setminus \{s\} \land s}\]
where the symbol \(\land\) means the meet in \(\mathcal{P}(x)\).

**Definition 3.4 (Composition of cubes).** Let \(x\) and \(y\) be \(S\)-cubes in a category \(\mathcal{D}\), \(s\) an element in \(\mathcal{S}\) and \(\alpha \in x|_{S \setminus \{s\} \land s}\) a morphism of \(S \setminus \{s\}\)-cubes. We define \(x \circ_s \alpha \circ y\) to be an \(S\)-cube as follows. Let \(T\) be a subset of \(S\) and \(t\) an element in \(T \setminus \{s\}\). \((x \circ_s \alpha \circ y)_T = x_T\) if \(t\) is \(s\) is in \(T\), and \(y_T\) if \(s\) is not in \(T\). The morphism \(d_T^{x \circ_s \alpha \circ y}: (x \circ_s \alpha \circ y)_T \to (x \circ_s \alpha \circ y)_{T \setminus \{t\}}\) is \(d_T^x\) if \(t\) is in \(T \setminus \{s\}\), \(d_T^y\) if \(s\) is not in \(T\), and we put \(\lambda_T^{x \circ_s \alpha \circ y} := d_T^x \circ d_T^y\). We call \(x \circ_s \alpha \circ y\) the composition of \(S\)-cubes \(x\) and \(y\) (along \(s\)-direction by \(\alpha\)).

**Notations 3.5 (Total complexes).** Let \(S\) be a non-empty finite set such that \(\#S = n\) and \(x\) an \(S\)-cube in an additive category \(\mathcal{B}\). Let us fix a bijection \(\alpha\) from \(S\) to \(\{n\}\) and we will identify \(S\) with the set \(\{n\}\) via \(\alpha\). We associate an \(S\)-cube \(x\) with total complex \(\text{Tot}_x (x) = \text{Tot}_x\) as follows. \(\text{Tot}_x\) is a chain complex in \(\mathcal{B}\) concentrated in degrees \(0, \ldots, n\) whose component at degree \(k\) is given by
\[(\text{Tot}_x)_k := \bigoplus_{T \in \mathcal{P}(S), \#T = k} x_T\]
and whose boundary morphism \(d_k^{\text{Tot}_x}: (\text{Tot}_x)_k \to (\text{Tot}_x)_{k-1}\) are defined by
\[(-1)^{t+1} \chi_T(t) \cdot d_T^x: x_T \to x_{T \setminus \{j\}}\]
on its \(x_T\) component to \(x_{T \setminus \{j\}}\) component. Here \(\chi_T\) is the characteristic function of \(T\). (See Example 2.3)

**Example 3.6 (Mapping cone).** Let \(\mathcal{B}\) be an additive category. For a chain morphism between chain complexes \(f: a \to b\) in \(\mathcal{B}\), we denote the canonical mapping cone of \(f\) by \(\text{Cone}_f\). Namely \(\text{Cone}_f\) is a complex in \(\mathcal{B}\) and whose component at degree \(n\) is given by \((\text{Cone}_f)_n = a_n \oplus b_n\) and whose boundary morphism \(d_n^{\text{Cone}_f}: (\text{Cone}_f)_n \to (\text{Cone}_f)_{n-1}\) are defined by
\[d_n^{\text{Cone}_f} = \begin{pmatrix} d_n^a & 0 \\ -f_{n-1} & d_n^b \end{pmatrix}\]. Let \(x\) be an \(S\)-cube in \(\mathcal{B}\). Then for any \(s \in S\), we have the canonical isomorphism
\[\text{Cone}(\text{Tot} d^n x): \text{Tot}(x|_{S \setminus \{s\}}) \to \text{Tot}(x|_{S \setminus \{s\}}) \cong \text{Tot}\]
In particular, if \(\mathcal{B}\) is an abelian category, then we have the long exact sequence
\[\cdots \to H_{p+1} \text{Tot} x \to H_p \text{Tot}(x|_{S \setminus \{s\}}) \to H_p \text{Tot}(x|_{S \setminus \{s\}}) \to \cdots\]
(18)

**Definition 3.7 (Spherical complexes, spherical cubes).** Let \(n\) be an integer. We say that a complex \(y\) in an abelian category \(\mathcal{A}\) is \(n\)-spherical if \(H_k(y) = 0\) for any \(k \neq n\). We say that an \(S\)-cube \(x\) in an abelian category \(\mathcal{A}\) is \(n\)-spherical if the complex \(\text{Tot}\) is \(n\)-spherical.
By the long exact sequence \[ \text{in Example 3.6} \] we can easily get the following result.

**Lemma 3.8 (Homology groups of Total complexes).** Let \( x \) be an \( S \)-cube in an abelian category \( \mathcal{A} \) and let us assume that \( x|_{S \setminus \{s\}}^0 \) is 0-spherical, then we have the canonical isomorphisms

\[
H_p \text{Tot} x \cong \begin{cases} 
   \text{Coker} \; H_0 \text{Tot} d^{s,x} & \text{if } p = 0 \\
   \text{Ker} \; H_0 \text{Tot} d^{s,x} & \text{if } p = 1 \\
   H_{p-1} \text{Tot}(x|_{S \setminus \{s\}}^1) & \text{if } p \geq 2.
\end{cases}
\]

(19)

Here \( d^{s,x} \) is as in Example 3.2.

**Example 3.9.** Let \( x \) be an \( S \)-cube in \( \mathcal{A} \) such that all boundary morphisms are monomorphisms. Then for any \( s \in S \), we have the isomorphism

\[
\text{Ker}(H_0 \text{Tot} d^{s,x} : H_0 \text{Tot} x|_{S \setminus \{s\}}^1 \rightarrow H_0 \text{Tot} x|_{S \setminus \{s\}}^0) \cong \left( \bigvee_{t \in S \setminus \{s\}} \text{Im} d^t_x \right) \cap \text{Im} d^s_x
\]

(20)

where the symbols \( \bigvee \) and \( \cap \) are the join and the meet in \( \mathcal{P}(x) \) respectively. Therefore the morphism \( H_0 \text{Tot} d^{s,x} \) is a monomorphism if and only if a pair of (a family of) subobjects \( (\text{Im} d^t_x)_{t \in S \setminus \{s\}}, \text{Im} d^s_x) \) is distributive in \( \mathcal{P}(x) \).

**Example 3.10.** Let \( x \) and \( y \) be \( S \)-cubes in an additive category \( \mathcal{B} \), \( s \) an element in \( S \) and \( \alpha : x|_{S \setminus \{s\}}^0 \rightarrow y|_{S \setminus \{s\}}^1 \) an isomorphism of \( S \)-cubes. Then we have a sequence of complexes of \( S \setminus \{s\} \)-cubes.

\[
\begin{bmatrix}
   x|_{S \setminus \{s\}}^1 \\
   \downarrow d^{s,x} \\
   x|_{S \setminus \{s\}}^0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
   y|_{S \setminus \{s\}}^0 \\
   \downarrow d^{s,y} \alpha d^{s,x} \\
   y|_{S \setminus \{s\}}^1
\end{bmatrix}
\]

(21)

We regard sequence above as

\[
x \rightarrow x \circ_{s,\alpha} y \rightarrow y
\]

and it is a distinguished triangle in the triangulated category of the homotopy category of chain complexes of \( S \setminus \{s\} \)-cubes by the octahedron axiom. In particular if \( \mathcal{B} \) is an abelian category, we have a long exact sequence

\[
\cdots \rightarrow H_p+1 \text{Tot}(y) \rightarrow H_p \text{Tot}(x) \rightarrow H_p \text{Tot}(x \circ_{s,\alpha} y) \rightarrow H_p \text{Tot}(y) \rightarrow H_{p-1} \text{Tot}(x) \rightarrow H_{p-1} \text{Tot}(x \circ_{s,\alpha} y) \rightarrow \cdots
\]

(22)

In particular, let \( n \) be a non-negative integer and if \( x \) and \( y \) are \( n \)-spherical, then \( x \circ_{s,\alpha} y \) is also \( n \)-spherical.

**Definition 3.11 (Homology of cubes).** Let us fix an \( S \)-cube \( x \) in \( \mathcal{A} \). For each \( k \in S \), the \( k \)-direction 0-th homology of \( x \) is the \( S \setminus \{k\} \)-cube \( H^0_k(x) \) in \( \mathcal{A} \) defined by \( H^0_k(x)_T := \text{Coker} \; d^{k,x}_{T \cup \{k\}} \). For any \( T \in \mathcal{P}(S) \) and \( k \in S \setminus T \), we denote the canonical projection morphism \( x_T \rightarrow H^0_k(x)_T \) by \( \pi^k_T \) or simply \( \pi^k_T \).
Example 3.12 (Motivational example). Let $f_S = \{f_s\}_{s \in S}$ be a family of elements in $A$. The typical cube associated with $f_S = \{f_s\}_{s \in S}$ is an $S$-cube in the category of $A$-modules denoted by $\text{Typ}_A(f_S)$ and defined by $\text{Typ}_A(f_S)_T = A$ and $d_{ij}^{\text{Typ}_A(f_S)} = f_i$ for any $T \in \mathcal{P}(S)$ and $i < j$. The complex $\text{Tot} \text{Typ}_A(f_S)$ is the usual Koszul complex associated with a family $f_S$. If the family $f_S$ forms a regular sequence with respect to every ordering of the members of $f_S$, then for any $k \in (#S]$ and any distinct elements $s_1, \ldots, s_k$ in $S$, boundary maps of $H_*^S(\cdot \cdot \cdot (H_*^{s_k}(\text{Typ}_A(f_S))) \cdot \cdot \cdot )$ are injections.

Definition 3.13 (Admissible cubes). Let us fix an $S$-cube $x$ in $A$. When $#S = 1$, we say that $x$ is admissible if $x$ is monic, namely if its unique boundary morphism is a monomorphism. For $#S > 1$, we define the notion of an admissible cube inductively by saying that $x$ is admissible if $x$ is monic and if for every $k \in S$, $H_0^S(x)$ is admissible. If $x$ is admissible, then for any distinct elements $i_1, \ldots, i_k$ in $S$ and for any automorphism $\sigma$ of $S$, the identity morphism on $x|_{S \setminus \{i_1, \ldots, i_k\}}$ induces an isomorphism

$$H_n^S(\cdot \cdot \cdot (H_*^{i_k}(x)) \cdot \cdot \cdot ) \simeq H_n^S(\cdot \cdot \cdot (H_*^{\sigma(i_k)}(x)) \cdot \cdot \cdot )$$

(cf. [Moc13] 3.11). For an admissible $S$-cube $x$ and a subset $T = \{i_1, \ldots, i_k\} \subset S$, we set $H_0^S(T) := H_n^S(\cdot \cdot \cdot (H_*^{i_k}(x)) \cdot \cdot \cdot )$ and $H_0^S(T) = x$. By virtue of the isomorphism (23), the definition of $H_0^S(x)$ does not depend upon an ordering of the sequence $i_1, \ldots, i_k$, up to isomorphisms. Notice that $H_0^S(x)$ is an $S \setminus T$-cube for any $T \in \mathcal{P}(S)$. We have the isomorphisms

$$H_p(\text{Tot}(x)) \simeq \begin{cases} H_0^S(x) & \text{for } p = 0 \\ 0 & \text{otherwise.} \end{cases}$$

(See [Moc13] 3.13.) We denote the full subcategory of $\text{Cub}_A^S$ consisting of those admissible cubes by $\text{Cub}_{adm}^S(A)$. The category $\text{Cub}_{adm}^S(A)$ is closed under extensions in $\text{Cub}_A^S$ by [Moc13] 3.20 and therefore it naturally becomes an exact category.

Example 3.14 (Admissible squares). Let $x$ be a $(1, 2)$-cube in an abelian category such that the boundary morphisms $d_{(1,2)}^{1, x}$ and $d_{(2)}^{2, x}$ are monomorphisms. Then we can easily prove that the following conditions are equivalent.

1. The cube $x$ is $0$-spherical.
2. The cube $x$ is admissible.
3. The diagram

$$\begin{array}{ccc} x_{(1,2)} & \xrightarrow{d_{(1,2)}^{1, x}} & x_{(1)} \\ \downarrow{d_{(1,2)}^{1, x}} & & \downarrow{d_{(1)}^{1, x}} \\ x_{(2)} & \xrightarrow{d_{(2)}^{2, x}} & x_0 \end{array}$$

is a Cartesian square.

Theorem 3.15 (Characterization of admissibility). (cf. [Moc13] 3.15). Let $x$ be an $S$-cube in an abelian category $A$. Then the following conditions are equivalent.

1. The $S$-cube $x$ is admissible.
2. All faces of the $S$-cube $x$ are admissible and the $S$-cube $x$ is $0$-spherical.
3. All frontside faces of the $S$-cube $x$ are admissible and the $S$-cube $x$ is $0$-spherical.
4. The cube $x|_T^0$ is admissible for any $T \in \mathcal{P}(S)$.

Proof. The equivalence of conditions (1) and (2) is proven in [Moc13] 3.15. Obviously condition (2) implies condition (3). We prove the converse implication by induction on the cardinality
of $S$. If $\# S \leq 1$, the assertion is trivial. If $\# S = 2$, the assertion follows from Example 3.14.  Next let us assume that $\# S \geq 3$. Then for any distinct elements $s$ and $t \in S$, we have the equality

$$(x|_{S \setminus \{t\}}^{(t)})|_{S \setminus \{s, t\}} = (x|_{S \setminus \{s\}}^{(t)})|_{S \setminus \{s, t\}}$$

by the equality \[14\] in 3.1 and $(x|_{S \setminus \{s\}}^{(t)})|_{S \setminus \{s, t\}}$ is admissible by the assumption and the equivalence of conditions (1) and (2). Therefore all frontside faces of $x|_{S \setminus \{t\}}^{(t)}$ are admissible. On the other hand, since $x$ and $x|_{S \setminus \{t\}}^{(t)}$ are 0-spherical, $x|_{S \setminus \{t\}}^{(t)}$ is also 0-spherical by Lemma 3.8. Hence $x|_{S \setminus \{t\}}^{(t)}$ is admissible for any $t \in S$ by inductive hypothesis. The equivalence of conditions (3) and (4) follows from Example 3.14 and induction on the cardinality of $S$.

\[\square\]

**Corollary 3.16.** Let $x$ and $y$ be $S$-cubes in an abelian category $\mathcal{A}$, $S$ an element in $S$ and $\alpha : x|_{S \setminus \{s\}}^{0} \to y|_{S \setminus \{s\}}^{1}$ an isomorphism of $S \setminus \{s\}$-cubes. If $x$ and $y$ are admissible, then $x \circ_{s, \alpha} y$ is also admissible.

**Proof.** We proceed by induction on the cardinality of $S$. If $\# S \leq 1$, the assertion is trivial. We will check the condition \[3\] in Theorem 3.15. Since $x$ and $y$ are 0-spherical, $x \circ_{s, \alpha} y$ is also 0-spherical by Example 3.10. Let us assume $\# S \geq 2$. We can easily check that we have the equality

$$(x \circ_{s, \alpha} y)|_{T}^{0} = \begin{cases} y|_{T}^{0} & \text{if } s \notin T \\ x|_{T}^{0} \circ_{s, \alpha|_{T}} y|_{T}^{0} & \text{if } s \in T \end{cases}$$

for any $T \in \mathcal{P}(S)$ where $\alpha|_{T} : (x|_{T \setminus \{s\}}^{0})|_{S \setminus \{s\}}^{0} \to (y|_{T \setminus \{s\}}^{0})|_{S \setminus \{s\}}^{0}$ is a restriction of $\alpha$. Therefore $(x \circ_{s, \alpha} y)|_{T}^{0}$ is admissible by the assumption and the inductive hypothesis. Hence $x \circ_{s, \alpha} y$ is admissible.

\[\square\]

**Corollary 3.17.** Any admissible $S$-cube in an abelian category $\mathcal{A}$ is fibered.

**Proof.** The assertion follows from Lemma 2.23 and Example 3.14 and Theorem 3.15.

\[\square\]

**Corollary 3.18.** Let $x$ be an admissible $S$-cube and $f : x_{\emptyset} \to y$ a monomorphism in $\mathcal{A}$. Then $x_{f, y}$ is also admissible.

**Proof.** We prove the condition in Theorem 3.15 (3) to $x_{f, y}$ by induction on the cardinality of $S$. For $\# S = 1$, the assertion is trivial and for $\# S = 2$, the assertion follows from Example 3.14 and the standard result in the category theory Lemma 3.19 below. For $\# S \geq 3$, notice that we have the following equalities

$$H_{k} \text{ Tot } x_{f, y} = H_{k} \text{ Tot } x = 0 \quad \text{for } k > 0 \quad \text{and}$$

$$x_{f, y}|_{S \setminus \{s\}}^{0} = (x|_{S \setminus \{s\}}^{0})_{f, y} \quad \text{for any } s \in S.$$

Hence we get the desired result by inductive hypothesis.

\[\square\]

**Lemma 3.19.** For the commutative diagram below in a category,

\[\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
\downarrow & b \downarrow & \downarrow \text{id} \\
\bullet & \longrightarrow & \bullet \\
\end{array}\]

\[\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
I & \downarrow & H \\
\bullet & \longrightarrow & \bullet \\
\end{array}\]

\[\begin{array}{ccc}
\bullet & \longrightarrow & \bullet \\
\downarrow & a \downarrow & \downarrow \text{ab} \\
\bullet & \longrightarrow & \bullet \\
\end{array}\]

\[18\]
if the square \( I \) is Cartesian and the morphism \( a \) is a monomorphism, then the square \( II \) is also Cartesian.

**Proposition 3.20.** Let \( \tau = \{ x_s \xrightarrow{d_s} x \}_{s \in S} \) be a family of subobjects in \( A \). Then \( \text{Fib} \tau \) is an admissible \( S \)-cube if and only if a family \( \tau \) is universally admissible in \( P(\chi) \). Here \( P(\chi) \) is as in Example 1.6.

**Proof.** We proceed by induction on the cardinality of \( S \). If \( \#S \leq 1 \), the assertion is trivial. If \( \#S = 2 \), then the assertion follows from Example 3.14. Let us assume \( \#S \geq 3 \). First notice that \( \text{Fib} \tau \) is admissible if and only if the following three conditions hold by Theorem 3.15 and the equalities (15) and (16) in Example 3.3.

1. The \( S \setminus \{ s \} \)-cube \( \text{Fib}(\chi_{S \setminus \{ s \}}) \) is admissible for any \( s \in S \).
2. The \( S \setminus \{ s \} \)-cube \( \text{Fib}(\chi_{S \setminus \{ s \}} \land x_s) \) is admissible for any \( s \in S \).
3. The \( S \)-cube \( \text{Fib} \chi \) is 0-spherical.

Under conditions (1) and (2), condition (3) is equivalent to the condition that \( H_0 \text{Tot} d^S_x \) is a monomorphism for any \( s \in S \) by Lemma 3.8 and it is equivalent to the condition that a family \( \chi \) is admissible in \( P(\chi) \) by Example 3.9. Therefore by Remark 1.8, a family \( \chi \) is universally admissible in \( P(\chi) \).

4 Double cubes

In this section, we develop an abstract version of the main theorem in Theorem 4.9.

**Definition 4.1 (Disjoint systems).** Let \( S \) be a set and \( T \) a subset of \( S \). A system of subsets \((A, B, C, D)\) of \( S \) is a disjoint system of \( S \) with respect to \( T \) if the following three conditions hold.

1. The sets \( A, B, C \) and \( D \) are disjoint in each other.
2. The sets \( A \) and \( B \) are contained in \( T \).
3. The sets \( C \) and \( D \) are contained in \( S \setminus T \).

**Notations 4.2.** Let \( S \) be a set.

1. For any ordered pair of disjoint subsets \((A, B)\) of \( S \), we set

\[
\mathcal{D} \mathcal{P}(A, B)(S) := \{ (U, V) \in \mathcal{D} \mathcal{P}(S); A \subset U \cup V, B \cap V = \emptyset \}
\]

and regard it as a partially ordered subset of \( \mathcal{D} \mathcal{P}(S) \). Notice that \( \mathcal{D} \mathcal{P}(\emptyset, \emptyset)(S) = \mathcal{D} \mathcal{P}(S) \).

2. Let \( T \) be a subset of \( S \) and an ordered system \((A, B, C, D)\) a disjoint system of \( S \) with respect to \( T \). We define \( i_{(\mathcal{C}, \mathcal{D})}(T \subset S)_{(\mathcal{A}, \mathcal{B})}: \mathcal{D} \mathcal{P}(A, B)(T) \rightarrow \mathcal{D} \mathcal{P}(S) \) to be an order preserving map which sends an ordered pair \((U, V)\) in \( \mathcal{D} \mathcal{P}(A, B)(T) \) to an ordered pair \((U \cup C, V \cup D)\) in \( \mathcal{D} \mathcal{P}(S) \).

**Definition 4.3 (Total functor).** For any disjoint pair of subsets \((A, B)\) of \( S \), we define the Total functor \((\mathcal{A}, \mathcal{B})\) \( \text{Tot}(\mathcal{A}, \mathcal{B}) : \mathcal{P}(S) \rightarrow \mathcal{D} \mathcal{P}(A, B)(S) \) by sending a subset \( T \) of \( S \) to an ordered pair \((\{A \setminus T\} \cup (B \setminus T), T \setminus B)\). For any subset \( A \) of \( S \), we shortly write \( \text{Tot}(\mathcal{A}) = \text{Tot}(\mathcal{A}, \mathcal{A}) \) for \( \text{Tot}(\mathcal{A}, \mathcal{A}, \mathcal{A}) \). We also write \( e_{(\emptyset, \emptyset)}^S = e_{\emptyset, \emptyset} \) for \( \text{Tot}(\mathcal{A}, \emptyset) \).

**Lemma 4.4.** For any set \( S \) and any subset \( A \) of \( S \), we have an isomorphism of partially ordered sets

\[
\text{Tot}_{\mathcal{A}} : \mathcal{P}(S) \xrightarrow{\sim} \mathcal{D} \mathcal{P}_{\mathcal{A}}(S)
\]

which sends a subset \( T \) to an ordered pair of disjoint subsets \((T \cap A, T \cap A)\) of \( S \).
Proof. The inverse map of $\text{Tot}_A$ is given by sending an ordered pair $(U, V)$ in $\mathcal{D}P_A(S)$ to a subset $(U \setminus A) \cup V$ of $S$. \hfill \Box

Remark 4.5. Let $T$ be a subset of $S$. Recall the definition of $e_T$ from Example 4.3 (3). Then the equality $e_T = i_{(T,S \setminus T)}^T \text{Tot}_{(T,S \setminus T)}$.

In the rest of this section, let $S$ be a finite set and $A$ an abelian category.

Notations 4.6 (Restriction of double cubes). Let $T$ be a subset, $(A, B, C, D)$ a disjoint system of $S$ with respect to $T$ and $x$ a double $S$-cube. We define the restriction of $x$ to $(A, B)$ along $(C, D)$ by composition of the functors $x_{|T,(A,B)}^{(C,D)} := x_{|T \subseteq S,(A,B)}^{(C,D)} \text{op}$.

If $T = S$, we shortly write $x_{|T,(A,B)}^{(C,D)}$ for $x_{|(T,S),(A,B)}^{(C,D)} \text{op}$.

Lemma-Definition 4.7 (Patching of cubes). (1) A family $\mathfrak{r} = \{x_T\}_{T \in \mathcal{P}(S)}$ of $S$-cubes in a category $\mathcal{D}$ indexed by the subsets of $S$ is a patching family if it satisfies the following patching condition:

\begin{equation}
    x_T |_{S \setminus \{t\}} = x_T |_{S \setminus \{t\}} \quad \text{(25)}
\end{equation}

for any subset $T$ of $S$ and any element $t \in T$.

(2) Then there exists a unique double $S$-cube $\text{Pat} \mathfrak{r}$ in $\mathcal{D}$ such that

\begin{equation}
    e_T^* \text{Pat} \mathfrak{r} = x_T \quad \text{(26)}
\end{equation}

for any subset $T$ of $S$.

Proof. For any $(U, V)$ in $\mathcal{D}P(S)$, there exists a pair of subsets $T$ and $W$ of $S$ such that

\begin{equation*}
    (U, V) = e_T(W)
\end{equation*}

by the equality (10) in Notations 4.2. We put $(\text{Pat} \mathfrak{r})_{U,V} := x_T^W$. We need to check that this definition does not depend upon the choice of subsets $T$ and $W$ of $S$. By virtue of the equality (10) again, we shall assume that $W \neq S$ and $T \neq \emptyset$ and we just need to verify the equality $x_T^W = x_{W \cup \{t\}}^W$ for any $t \in T$. Since $W \subset S \setminus \{t\}$, we have the equality $x_T^W = (x_T^W |_{S \setminus \{t\}})_W = (x_T^W |_{S \setminus \{t\}} |_{S \setminus \{t\}})_W = x_{W \cup \{t\}}^W$ by the equality (25). Hence we get the well-definedness of $(\text{Pat} \mathfrak{r})_{U,V}$. The definition of boundary morphisms of $\text{Pat} \mathfrak{r}$ are similar. \hfill \Box

Example 4.8 (Reconstruction of double cubes). Let $x$ be a double $S$-cube in a category $\mathcal{D}$. Then the family $\mathfrak{r} = \{e_T^* x\}_{T \in \mathcal{P}(S)}$ is a patching family and $\text{Pat} \mathfrak{r} = x$.

Theorem 4.9. Let $x$ be a double $S$-cube in $\mathcal{A}$. We assume that the following four conditions hold.

(1) The $S$-cube $2^x x$ is admissible.
(2) The double $S$-cube $x$ is monic.
(3) The $S$-cube $e^*_S x$ is fibered for any proper subset $T$ of $S$.
(4) For any subset $W$ of $S$, any pair of non-empty disjoint subsets $U$ and $V$ of $S$ such that $W \cup U \neq S$ and any element $s \in S \setminus (W \cup U)$ and $v \in V$ such that $s \neq v$, we put $V' := V \setminus \{v\}$ and a family of morphisms to $x_{W \setminus V', V'}$

$$x := \{x_{W \setminus (V \cup \{k\}), V \cup \{k\}}\}_{k \in U} \cup \{x_{(W \cup (s)) \setminus V', V'} \}$$

where $d_k^x = x((W \setminus V', V')) \leq (W \setminus (V \cup \{k\}), V \cup \{k\}))$ for any $k \in U$ and $d_k^x = d_{(W \cup (s)) \setminus V', V'}$. Then the morphism

$$H_0 \text{Tot} \mathcal{F}ib \cdot: H_0 \mathcal{F}ib \{s\} \to H_0 \mathcal{F}ib \{s\}$$

is a monomorphism.

Then the $S$-cube $e^*_S x$ is also an admissible $S$-cube.

To prove the theorem, we utilize the following lemma which is a standard result in the category theory.

**Lemma 4.10.** Let $D$ be a category. Then

1. For a pair of composable morphisms $f \to g \to \cdot$ in $D$, if $g f$ is a monomorphism, then $f$ is also a monomorphism.
2. In the commutative diagram in $D$ below,

$$\begin{array}{ccc}
\bullet & \to & \bullet \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
\bullet & \to & \bullet \\
\downarrow & & \downarrow \\
\bullet & \to & \bullet \\
\end{array}$$

if the big square $\text{I} \to \text{II} \to \text{III} \to \text{IV}$ is a Cartesian square and if the morphisms $a$ and $b$ in the diagram above are monomorphisms, then the squares $\text{I} \to \text{II}$, $\text{I} \to \text{III}$ and $\text{I}$ are Cartesian. 

**Proof of Theorem 4.9.** If $\# S \leq 2$, the assertion follows from Lemma 4.10. We assume that $\# S \geq 3$. We put $a = \{x_{\{s\}, 0}\}_{s \in S}$ and $b = \{x_{\emptyset, \{s\}}\}_{s \in S}$. We have the canonical isomorphism $x_{UV} \cong a^U \land b^V$ as subobjects of $x_{\emptyset, 0}$ by Corollary 2.26. Then conditions (1) and (4) are equivalent to the following conditions (1)' and (4)' respectively by Proposition 3.20 and Example 3.9 respectively.

(1)' The family $b$ of subobjects in $x_{\emptyset, 0}$ is universally admissible in a lattice $P(x)$.

(4)' For any subset $W$ of $S$, any pair of non-empty disjoint subsets $U$ and $V$ of $S$ such that $W \cup U \neq S$ and any elements $s \in S \setminus (W \cup U)$ and $v \in V$ such that $s \neq v$, a pair $(b_U \land b^V \land a^U \land v) \land a^W \cup \{s\})$ is distributive.

The family of subobjects $\mathcal{F}ib \cdot := \{x_{S \setminus T, T} \to x_{\emptyset, T} \}_{T \in P(S)}$ of $x_{\emptyset, 0}$ is universally admissible in $P(\mathcal{F}ib \cdot)$ by Example 1.6 Corollary 1.12 and Remark 1.13. Hence $e^*_S x \cong \mathcal{F}ib \mathcal{F}ib$ is admissible by Proposition 3.20 again.

**4.11. Proof of Theorem 0.4.** If $\# S \leq 2$, then the assertion follows from Lemma 4.10. If $\# S \geq 3$, then we will prove condition (3) implies conditions (3) and (4) in Theorem 4.9. Condition
(3) follows from Corollary 3.17. Inspection shows that $\text{Fib}_F$ in condition (3) in Theorem 4.9 is written by compositions of restrictions of faces of $c_T F$ for some proper subsets $T$ of $S$. More precisely, we put $U_1 := \{s, v\} \cup U$ and $U_2 := U \cup \{s\}$ and

\[ f_1 := \{x((W \setminus V) \cup \{k\}, V') \mapsto x(W \setminus V', V')\}_{k \in U_1}, \]

\[ f_2 := \{x(W \setminus V, V) \mapsto x((W \setminus V') \cup \{s\}, V') \} \sqcup \{x((W \setminus V') \cup \{k, v\}, V') \mapsto x((W \setminus V') \cup \{v\}, V')\}_{k \in U_2}. \]

Then $\text{Fib}_x$ can be written by the composition of

\[ \text{Fib}_{f_1} = \text{Tot}_{(\emptyset, U_1)} x^{((W \setminus V, V')}, \]

and

\[ \text{Fib}_{f_2} = \text{Tot}_{(\{v\}, U_2)} x^{((W \setminus V, V')}, \]

and $\text{Tot}_{(\emptyset, \{s\})} x^{((W \setminus V, V')$. Therefore $\text{Fib}_F$ is admissible by Corollary 3.16. Hence we obtain the desired result.

\[ \square \]

5 Regular adjugates of cubes

In this section, let $A$ be a commutative ring with unit and we study the notion about (regular) adjugates of cubes in an $A$-linear abelian category.

**Notations 5.1.** An $A$-linear category $\mathcal{C}$ is a category enriched over the category of $A$-modules. Namely for any pair of objects $x$ and $y$ in $\mathcal{C}$, the set of morphisms from $x$ to $y$, $\text{Hom}_\mathcal{C}(x, y)$ has a structure of $A$-module and the composition of morphisms $\text{Hom}_\mathcal{C}(x, y) \otimes_A \text{Hom}_\mathcal{C}(y, z) \to \text{Hom}_\mathcal{C}(x, z)$ is a homomorphism of $A$-modules for any objects $x$, $y$ and $z$ in $\mathcal{C}$. We fix an $A$-linear category $\mathcal{C}$. For an element $a$ in $A$ and an object $x$ in $\mathcal{C}$, we write $a_x$ for the morphism $a \text{id}_x : x \to x$. In particular, we notice that we have the equality $f a_x = a_y f$ for any morphism $f : x \to y$ in $\mathcal{C}$ and an element $a$ in $A$. The category of $A$-modules is a typical example of $A$-linear category.

**Lemma-Definition 5.2 (Adjugate).** Let $f : x \to y$ and $\phi : y' \to y$ be morphisms in $\mathcal{C}$ and let us assume that there exists an element $a$ in $A$ and a morphism $f^* : y \to x$ such that we have the equalities $f^* f = a_x$ and $f^* f = a_y$. We call a pair $(f^*, a)$ or simply $f^*$ an **adjugate of $f$**. Moreover let us assume that the diagram below is the pull back of $f$ along $\phi$.

\[
\begin{array}{ccc}
x' & \xrightarrow{f'} & y' \\
\phi' \downarrow & & \downarrow \phi \\
x & \xrightarrow{f} & y.
\end{array}
\]

Then there exists a unique morphism $f^{**} : y' \to x'$ which satisfies the following equalities.

1. $f^{**} f' = a_{y'}$ and $f' f^{**} = a_{y'}$.
2. $\phi^* f^{**} = f^* \phi$. Namely the following diagram is commutative.

\[
\begin{array}{ccc}
y' & \xrightarrow{f^{**}} & x' \\
\phi \downarrow & & \downarrow \phi' \\
y & \xrightarrow{f^*} & x.
\end{array}
\]
We call the morphism $f^{**}$ the **adjugate of $f'$ induced from $f^*$ (along $\phi$).**

**Proof.** Since we have the equality $ff^* \phi = a_\phi \phi \circ \phi a_{\phi'}$, there exists, by the universal property of fiber product, a unique morphism $f^{**} : y' \to x'$ with which we can fill in the dotted arrow in the following commutative diagram:

$$
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow{f} & & \downarrow{f'} \\
x' & \xrightarrow{f'} & y'
\end{array}
\quad
\begin{array}{ccc}
x & \xrightarrow{f} & y \\
\downarrow{f} & & \downarrow{f'} \\
x' & \xrightarrow{f^{**}} & y'
\end{array}
$$

Applying the two morphisms $a_{x'} : x' \to x'$ and $f^{**} f' : x' \to x'$ to the universal property of $x'$ again, we acquire the equality $f^{**} f' = a_{x'}$. \qed

**Example 5.3** (Adjugate of matrices). Let $X$ be an $n \times n$ matrix whose coefficients are in $A$. We regard $X$ as a homomorphism of free $A$-modules $X : A^{\oplus n} \to A^{\oplus n}$. We denote the **adjugate** of $X$ by $\text{adj} X$. Namely the matrix $\text{adj} X$ is an $n \times n$ matrix whose $(i, j)$-entry is given by $(-1)^{i+j} \det X_{j,i}$ where $X_{j,i}$ is the $(j, i)$-cofactor of $X$ and $\det X_{j,i}$ means the **determinant** of $X_{j,i}$. It is well-known that we have the equality $(\text{adj} X)X = X \text{adj} X = (\det X)E_n$ where $E_n$ is the $n$-th unit matrix. Then a pair $(\text{adj} X, \det X)$ is an adjugate of $X$.

**Example 5.4** (Typical cubes). Let $f_S = \{f_s\}_{s \in S}$ a family of elements in $A$ and $x$ an object in $C$. We define $\text{Typ}(f_S; x)$ an $S$-cube in $C$ called the **typical cubes associated with** $f_S$ and $x$ as follows. For any $T \in P(S)$ and any element $t$ in $T$, we put $\text{Typ}(f_S; x)_T = x$ and $d^T_{i,T} : \text{Typ}(f_S; x)_T := (f_t)_x$. For example, if $C$ is the category of $A$-modules and $x = A$, then $\text{Typ}(f_S; A)$ is just the typical cube associated with $f_S$ as in Example 3.12.

**Notations 5.5** ($x$-regular sequences and $x$-sequences). Let us assume that $C$ is an additive category and moreover for any finite family of morphisms $\{\phi_i : x_i \to y\}_{1 \leq i \leq q}$ in $C$, there exists the cokernel $x/(\phi_1, \ldots, \phi_q) := \text{Coker}(\oplus_i \phi_i \otimes y)$ in $C$. For any elements $f_1, \ldots, f_q$ in $A$ and an object $x$ in $C$, we simply write $x/(f_1, \ldots, f_q)$ for $x/((f_1)_x, \ldots, (f_q)_x)$. Let us fix an object $x$ in $C$.

1. A sequence of elements $f_1, \ldots, f_q$ in $A$ is an **$x$-regular sequence** if every $f_i$ is a non-unit in $A$, if $(f_1)_x$ is a monomorphism in $C$ and if $(f_{i+1})_{x/(f_1, \ldots, f_i)}$ is a monomorphism for any $1 \leq i \leq q - 1$.
2. A finite family $\{f_s\}_{s \in S}$ of elements in $A$ is an **$x$-sequence** if $\{f_s\}_{s \in S}$ forms an $x$-regular sequence with respect to every ordering of the members of $\{f_s\}_{s \in S}$.

**Lemma 5.6.** Let $f_S = \{f_s\}_{s \in S}$ be a family of elements in $A$ and $x$ an object in $C$ and we put $f_{x,S} := \{((f_s)_x : x \to x)_{s \in S}\}$ a family of morphisms in $C$. Then the following conditions are equivalent.

1. The $S$-cube $\text{Typ}(f_S; x)$ is admissible.
2. The family $f_S$ is an $x$-sequence.
3. The morphism $(f_s)_x : x \to x$ is a monomorphism for any $s \in S$ and the family $f_{x,S}$ is admissible in $P(f_{x,S})$.
4. The morphism $(f_s)_x : x \to x$ is a monomorphism for any $s \in S$ and the family $f_{x,S}$ is universally admissible in $P(f_{x,S})$. 23
Proof. We prove that condition (1) implies condition (4). If $\text{Typ}(f_S; x)$ is admissible, then $\text{Typ}(f_S; x) \to \text{Fib}(f_x, y)$ by Corollary 3.17. Therefore a family $f_{x,S}$ is universally admissible by Proposition 3.20. Obviously condition (4) implies condition (3). Since for any non-empty subset $T$ of $S$ and any element $s \in S \setminus T$, we have the isomorphism

$$\text{Ker}((f_s)_{x/} : x/ \to x/ f_s) \cong f^T \land (f_s)_x$$

by Example 3.9 where $(f_s)_x$ means the subobject $(f_s)_x : x \to x$ of $x$. We can easily notice that condition (3) implies condition (2). We prove condition (2) implies condition (1) by induction on the cardinality of $S$. If $\# S = 1$, the assertion is trivial. For $\# S \geq 2$, we first notice that we have the equalities

$$\text{Typ}(f_S; x)|_{S \setminus \{s\}} = \text{Typ}(f_S; x)|_{S \setminus \{s\}} = \text{Typ}(f_S; x)$$

(27) for any $s \in S$. Therefore all faces of $\text{Typ}(f_S; x)$ is admissible by the inductive hypothesis and Example 3.14. Moreover by Lemma 3.8, we have the isomorphisms

$$H_p(\text{Tot Typ}(f_S; x)) \cong \begin{cases} x/ f_S & \text{if } p = 0 \\ f^{x \setminus \{s\}} \land (f_s)_x & = 0 \\ (f_S \setminus \{s\}) \land (f_s)_x & = 0 \\ H_{p-1}(\text{Tot Typ}(f_S \setminus \{s\}; x)) & = 0 \text{ if } p \geq 2. \end{cases}$$

Therefore $\text{Typ}(f_S; x)$ is 0-spherical and hence $\text{Typ}(f_S; x)$ is admissible by Example 3.9 and Theorem 3.15.

Definition 5.7 (Adjugate of cubes). (1) An adjugate of an $S$-cube $x$ in $C$ is a pair $\mathfrak{A} = (a, a^*)$ consisting of a family of elements $a = \{a_t\}_{t \in S}$ in $A$ and a family of morphisms $a^* = \{d_T^a : x_{T \setminus \{t\}} \to x_T\}_{T \in \mathcal{P}(S), t \in T}$ in $C$ which satisfies the following two conditions.

(i) We have the equalities $d_T^a d_T^{a*} = (a_t)_{x_{T \setminus \{t\}}}$ and $d_T^{a*} d_T^a = (a_t)_{x_T}$ for any $T \in \mathcal{P}(S)$ and $t \in T$.

(ii) For any $T \in \mathcal{P}(S)$ and any distinct elements $a$ and $b \in T$, we have the equality $d_T^a d_T^{a*} = d_T^b d_T^{b*}$. Namely, the following diagram is commutative.

\[
\begin{array}{ccc}
 x_{T \setminus \{a\}} & \xrightarrow{d_T^a} & x_T \\
 \downarrow d_{T \setminus \{a\}} & & \downarrow d_T^a \\
 x_{T \setminus \{a, b\}} & \xrightarrow{d_T^{a*}} & x_{T \setminus \{b\}}.
\end{array}
\]

(2) An adjugate of an $S$-cube $\mathfrak{A} = (a, a^*)$ is regular if $a$ forms an $x_T$-sequence for any $T \in \mathcal{P}(S)$.

Remark 5.8. Let $x$ be an $S$-cube in $C$ and $\mathfrak{A} = (a = \{a_t\}_{t \in S}, a^* = \{d_T^a : x_{T \setminus \{t\}} \to x_T\}_{T \in \mathcal{P}(S), t \in T}$) an adjugate of $x$. For any subset $T$ of $S$ such that $\# T \geq 2$ and any pair of distinct elements $s$ and $t$ in $T$, if $d_T^{a*}$ or $d_T^a$ is a monomorphism, then the following diagram is commutative.

\[
\begin{array}{ccc}
 x_{T \setminus \{s, t\}} & \xrightarrow{d_T^{a*}} & x_{T \setminus \{s\}} \\
 \downarrow d_T^{a*} & & \downarrow d_T^{a*} \\
 x_{T \setminus \{t\}} & \xrightarrow{d_T^a} & x_T.
\end{array}
\]


In particular, if \( x \) is monic, then we can define the \( S \)-cubecube \( x^* \) by \( x^* = x_T \) and \( d^{x*}_T := d^x_T \).

We call \( x^* \) the **adjugate** \( S \)-cubecube (associated with an adjugate \( \mathfrak{A} = (a, \mathfrak{d}^* ) \)). In this case, notice that a pair \( \mathfrak{d}^* := (a = \{a_s\}_{s \in S}, \mathfrak{d} := d^{x*}_T : x_T \to x_{-\{t\}}(U) \in \mathcal{P}(S), t \in T) \) is an adjugate of an \( S \)-cube \( \hat{x} \). We call \( \mathfrak{d}^* \) the **dual adjugate** of \( \mathfrak{A} \). Obviously we have the equality \( (\hat{x}^*)^* = x \).

**Example 5.9** (Patching families associated to adjugates). Let \( x \) be a monic \( S \)-cube in \( \mathcal{C} \) and \( \mathfrak{A} = (a = \{a_s\}_{s \in S}, \mathfrak{d}^* := \{d^{x*}_T : x_T \to x_{\mathfrak{d}^*}(U) \in \mathcal{P}(S), t \in T) \) an adjugate of \( x \). We construct the **patching family** \( \mathfrak{P}_\mathfrak{A} := \{x^{\mathfrak{d}, \mathfrak{A}}_T \}_{T \in \mathcal{P}(S)} \) associated to an adjugate \( \mathfrak{A} \) as follows. For \( T \in \mathcal{P}(S) \), we define the \( S \)-cube \( x^{\mathfrak{d}, \mathfrak{A}} \) by setting \( d^{\mathfrak{d}, \mathfrak{A}}_U : x_T \to x_{-\{t\}}(U) \in \mathcal{P}(S), t \in T) \) for any \( U \in \mathcal{P}(S) \) and for any \( u \in U \), \( d^{\mathfrak{d}, \mathfrak{A}}_U : d^{\mathfrak{d}}_{T \cap U} := d^{\mathfrak{d}}_{U \cap T} \) if \( u \in U \setminus T \) and if \( d^{\mathfrak{d}}_{U \cap T} = d^{\mathfrak{d}}_{U \cap T} \cup \{u\} \) if \( u \in U \cap T \). As in Remark 5.8, we can easily check that \( x^{\mathfrak{d}, \mathfrak{A}} \) is an \( S \)-cube for any \( T \in \mathcal{P}(S) \). For any subsets \( U \) and \( T \) of \( S \) and any element \( u \) in \( U \), we define a morphism \( d^{\mathfrak{d}, \mathfrak{A}}_U : x^{\mathfrak{d}, \mathfrak{A}}_U : x^{\mathfrak{d}, \mathfrak{A}}_U \to x^{\mathfrak{d}, \mathfrak{A}}_U \) by \( d^{\mathfrak{d}, \mathfrak{A}}_U = d^{\mathfrak{d}}_{U \cap T} \) if \( u \in U \setminus T \) and \( d^{\mathfrak{d}, \mathfrak{A}}_U = d^{\mathfrak{d}}_{U \cap T} \cup \{u\} \) if \( u \in U \cap T \). Then a pair \( \mathfrak{A}^T = (a = \{a_s\}_{s \in S}, \mathfrak{d}^{\mathfrak{d}, \mathfrak{A}} := \{d^{\mathfrak{d}, \mathfrak{A}}_U : x^{\mathfrak{d}, \mathfrak{A}}_U \to x^{\mathfrak{d}, \mathfrak{A}}_U \}_{U \in \mathcal{P}(S), u \in U} \) is an adjugate of \( x^{\mathfrak{d}, \mathfrak{A}} \). Notice that \( x^{\mathfrak{d}, \mathfrak{A}} = x \) and \( \mathfrak{A}^0 = \mathfrak{d} \) and \( x \mathfrak{A}^0 = x \mathfrak{d} \) and \( \mathfrak{d}^0 = \mathfrak{d} \). We check the patching condition for \( \mathfrak{P}_\mathfrak{A} \). For any \( T \in \mathcal{P}(S) \) and any \( t \), we need to check the equality \( x^{\mathfrak{d}, \mathfrak{A}} \mid S \setminus \{t\} = x^{\mathfrak{d}, \mathfrak{A}} \mid S \setminus \{t\} \). We fix a subset \( W \) of \( S \setminus \{t\} \) and an element \( w \) in \( W \). Then we have the equalities.

\[
(x^{\mathfrak{d}, \mathfrak{A}} \mid S \setminus \{t\} \mid W) = x\mathfrak{d} \mathfrak{A} \mid S \setminus \{t\} = (x^{\mathfrak{d}, \mathfrak{A}} \mid S \setminus \{t\} \mid W) \text{ and }
\]

\[
d^{\mathfrak{d}, \mathfrak{A}}_U = \begin{cases} d^{\mathfrak{d}}_U & \text{if } w \in W \setminus T \\ d^{\mathfrak{d}}_{U \cup \{w\}} & \text{if } w \in W \cap T \end{cases}
\]

Hence \( \mathfrak{P}_\mathfrak{A} \) is a patching family of \( S \)-cubes in \( \mathcal{C} \).

**Definition 5.10** (Restriction of adjugates). Let \( x \) be an \( S \)-cube in \( \mathcal{C} \) and \( \mathfrak{A} = (a = \{a_s\}_{s \in S}, \mathfrak{d}^* := \{d^{x*}_T : x_T \to x_{\mathfrak{d}^*}(U) \in \mathcal{P}(S), t \in T) \) an adjugate of \( x \). For any disjoint pair of subsets \( U \) and \( V \) of \( S \), we define an adjugate \( \mathfrak{A}^V \mid U : (a \mid U \mathfrak{d}^* \mid V) \) as follows. We put \( \mathfrak{d}^* \mid U \mathfrak{d}^* \mid V := \{a_{s \in U} \cup V \} \) and \( \mathfrak{d}^* \mid U \mathfrak{d}^* \mid V := \{d^{x*}_U : x_{U \cup V} \mid T \to x_{U \cup V} \mid U \}_{U \in \mathcal{P}(U), t \in T} \). We call \( \mathfrak{A}^V \mid U \) the **restriction** of \( \mathfrak{A} \) (to \( U \) along \( V \)).

**Lemma 5.11** (Compatibility of restriction and patching families). Let \( x \) be a monic \( S \)-cube in \( \mathcal{C} \). Then for any disjoint pair of subsets \( U \) and \( V \) of \( S \) and any subset \( T \) of \( S \), if we suppose either the condition (1) or the condition (2) below, then we have the equality

\[
(x^{\mathfrak{d}, \mathfrak{A}} \mid U \mathfrak{d}^* \mid V) \mid T \cap U = (x^{\mathfrak{d}, \mathfrak{A}} \mid U \mathfrak{d}^* \mid V) \mid T \cap U.
\]

(1) \( T \subset U \).

(2) \( V \subset T \) and \( V = S \setminus U \).

**Proof.** We suppose either condition (1) or condition (2). For any subset \( W \) of \( S \) and any element \( w \) of \( W \), we have the equalities

\[
\{(x^{\mathfrak{d}, \mathfrak{A}} \mid U \mathfrak{d}^* \mid V) \mid W = x^{\mathfrak{d}, \mathfrak{A}} \mid U \mathfrak{d}^* \mid V \mid W = x^{\mathfrak{d}, \mathfrak{A}} \mid U \mathfrak{d}^* \mid V \mid W \}
\]
\[d_{W}^{w,x}:T \to V = d_{W}^{w,x}:T \cup V = \begin{cases} d_{(W \cup V) \cap T}^{w,x} & \text{if } w \in (W \cup V) \setminus T \\ d_{(W \cup V) \cap T \cup \{w\}}^{w,x} & \text{if } w \in (W \cup V) \cap T \end{cases} \]

Hence we have the equality \([28]\).

The following result is an easy corollary of Lemma-Definition 5.2.

**Corollary 5.12.** Let \(r = \{x_{i} \to x_{j}\}_{i \in S}\) be a family of subobjects in an \(A\)-linear abelian category. If all \(d_{x}^{*}\) admit non-trivial adjugates \((d_{x}^{*} : x \to x_{a}, a)\), then \(\text{Fib} r\) also admits the adjugate \((\{a_{s}\}_{s \in S}, \{d_{T}^{a_{s}} : x_{T} \setminus \{t\} \to x_{T} \cap (S) \}_{t \in T})\) such that \(d_{0}^{*} = d_{x}^{*}\) for any \(s \in S\) and \(d_{T}^{*}\) is the adjugate of \(d_{T}^{a_{i}}\) induced from \(d_{T}^{a_{i}}\) along \(d_{T}^{a_{i}}\) for any subset \(T\) of \(S\) with \(#T \geq 2\) and for any distinct pair of elements \(s\) and \(t\) in \(T\).

**Proof.** Let \(\{d_{T}^{a_{s}} : x \to x_{a}, a\}_{s \in S}\) be a family of adjugates of \(r\). Namely \(\{a_{s}\}_{s \in S}\) is a family of elements in \(A\) and we have equalities \(d_{s}^{a_{s}} = d_{s}^{a_{s}}\) and \(d_{T}^{a_{s}a_{t}} = d_{T}^{a_{s}a_{t}}\) for any \(s \in S\). We inductively define a morphism \(d_{T}^{a_{i}} : x_{T} \setminus \{t\} \to x_{T}\) for any non-empty subset \(T\) of \(S\) and any element \(t\) in \(T\). For \(#T \geq 2\), we fix an element \(s\) in \(T\) and let \(d_{T}^{a_{i}}\) be the adjugate of \(d_{T}^{a_{i}}\) induced from \(d_{T}^{a_{i}}\) along \(d_{T}^{a_{i}}\). If \(#T \geq 3\), for any element \(u \in T \setminus \{s, t\}\), we consider the following diagram.

\[
\begin{array}{ccc}
x_{T \setminus \{s, t\}} & \overset{d_{T}^{a_{s}}}{\rightarrow} & x_{T \setminus \{s\}} \\
| \downarrow d_{T}^{a_{i}} \downarrow & & \downarrow d_{T}^{a_{i}} \\
x_{T \setminus \{t\}} & \overset{d_{T}^{a_{t}}}{\rightarrow} & x_{T} \\
| \downarrow d_{T}^{a_{i}} \downarrow & & \downarrow d_{T}^{a_{t}} \\
x_{T \setminus \{s, t, u\}} \end{array}
\]

Since \(\text{Fib} r\) is an \(S\)-cube and by the inductive hypothesis, all squares except \(\star\) in the diagram above are commutative. Notice that the morphism \(d_{T}^{a_{i}} \downarrow \{u\}\) is a monomorphism and hence the square \(\star\) is also commutative. Namely we have the equality \(d_{T}^{a_{i}} d_{T}^{a_{i}} = d_{T}^{a_{t}}\). This equality means that the definition of \(d_{T}^{a_{i}}\) does not depend on the choice of \(s\) and the pair of families \(\{a_{s}\}_{s \in S}, \{d_{T}^{a_{s}} : x_{T} \setminus \{t\} \to x_{T} \cap (S) \}_{t \in T}\) forms an adjugate of \(\text{Fib} r\).

**Theorem 5.13.** Let \(C\) be an \(A\)-linear abelian category and \(x\) an \(S\)-cube in \(C\) which admits a regular adjugate \(d_{T}^{a_{i}}\). Then

1. The \(S\)-cube \(x\) is monic. In particular, we can define the patching family \(\{x_{T}^{a_{i}}\}_{T \in (S)}\) associated to the adjugate \(d_{T}^{a_{i}}\) as in Example 5.9.
2. The \(S\)-cube \(x_{T}^{a_{i}}\) is admissible for any subset \(T\) of \(S\). In particular, the \(S\)-cube \(x_{\{a_{i}\}}^{a_{i}}\) is admissible.
Proof. (1) We apply Lemma [4.10](1) to pairs of composable morphisms \( x_T \xrightarrow{d^1_T} x_{T \setminus \{t\}} \xrightarrow{d^0_T} x_T \) and \( x_T \xrightarrow{d^{2}_T} x_{T \setminus \{t\}} \xrightarrow{d^0_T} x_{T \setminus \{t\}} \) for any subset \( T \) of \( S \) and any element \( t \) of \( T \), and we notice that the morphisms \( d^1_T \) and \( d^0_T \) are monomorphisms. Hence we get the desired result.

(2) By replacing \( x^A.T \) and \( A^T \) with \( x \) and \( A \) respectively, we may assume without loss of generality that \( T = \emptyset \). We check the assertion for \( x^A.\emptyset \). If \( S \geq 1 \), then \( x \) is admissible by assertion (1). We apply Theorem [0.4] to the double \( S \)-cube \( \hat{\mathcal{P}} \mathcal{A} \). Then we obtain the desired result by the equality \( e_\hat{S} \hat{\mathcal{P}} \hat{\mathcal{A}} = (\hat{x}^*_{\hat{T}})^{A^*.T} = (\hat{x}^*)^2 = x \). What we need to check is the following conditions:

(A) \( 2^*(\hat{\mathcal{P}} \hat{\mathcal{A}}) \) is admissible.
(B) All boundary morphisms are monomorphisms.
(C) If \( \#S \geq 3 \), then all the faces of \( e_\hat{T}^* \hat{\mathcal{P}} \hat{\mathcal{A}} \) are admissible for any proper subset \( T \) of \( S \). Condition (B) follows from assertion (1). Since we have the equality \( 2^*(\hat{\mathcal{P}} \hat{\mathcal{A}}) = \mathsf{Typ}(a; x_S) \) and \( a \) is \( x_S \)-sequence, \( 2^*(\hat{\mathcal{P}} \hat{\mathcal{A}}) \) is admissible by Lemma [5.6]. Therefore we get the desired result for \( \#S = 2 \). For \( \#S \geq 3 \), we need to check that all the faces of \( e_\hat{T}^* \hat{\mathcal{P}} \hat{\mathcal{A}} = (\hat{x}^*_{\hat{T}})^{A^*.T} \) are admissible for any proper subset \( T \) of \( S \). By replacing \( (x^*)^{A^*.T} \) with \( x \) and \( A \) respectively, we shall just check that all the faces of \( x^A.T \) are admissible for any proper subset \( T \) of \( S \). We have the equality

\[
(x^A.T)_{V_{S \setminus \{k\}}} = \begin{cases} 
(x|_{S \setminus \{k\}})_{V_{S \setminus \{k\}}} & \text{if } V = \{k\} \\
(x|_{S \setminus \{k\}})_{V_{S \setminus \{k\}}} & \text{if } V = \emptyset \text{ and } k \notin T \\
(x|_{S \setminus \{k\}})_{V_{S \setminus \{k\}}} & \text{if } V = \emptyset \text{ and } k \in T
\end{cases}
\]

for any element \( k \) of \( S \) and any subset \( V \) of \( \{k\} \) by Lemma [5.11] and the patching conditions for \( \hat{\mathcal{P}} \hat{\mathcal{A}} \). Therefore the \( S \setminus \{k\} \)-cube \( (x^A.T)_{V_{S \setminus \{k\}}} \) is admissible by induction of the cardinality of \( S \).

The main theorem has the following application.

Corollary 5.14. Let \( \hat{f}_S = \{f_s\}_{s \in S} \) and \( \hat{g}_S = \{g_s\}_{s \in S} \) be families of elements in \( A \). We set \( h_s = f_s g_s \) for any \( s \in S \) and \( \hat{h}_S := \{h_s\}_{s \in S} \). Assume that \( \hat{h}_S \) is an \( A \)-sequence and \( f_s \) is not a unit element in \( A \) for any \( s \in S \). Then \( \hat{f}_S \) is also an \( A \)-sequence.

Proof. We put \( d^0_{f_s} = g_t \) for any \( T \in \mathcal{P}(S) \) and \( t \in T \) and \( \hat{\sigma} := \{d^0_{f_s}\}_{T \in \mathcal{P}(S), t \in T} \). Then a family \( (\hat{h}, \hat{\sigma}) \) is a regular adjugate of an \( S \)-cube \( \mathsf{Typ}(\hat{f}_S; A) \). Therefore \( \mathsf{Typ}(\hat{f}_S; A) \) is admissible by Theorem [5.13] and a family \( \hat{f}_S \) is an \( A \)-sequence by Lemma [5.6].

We give an explanation about the relationship between Theorem [5.13] and theorem of Buchsbaum and Eisenbud [BE73].

Definition 5.15 (Fitting ideal). Let \( U \) be an \( m \times n \) matrix over \( A \) where \( m, n \) are positive integers. For \( t \) in \( \{\min(m, n)\} \) we then denote by \( I_t(U) \) the ideal generated by the \( t \)-minors of \( U \), that is, the determinant of \( t \times t \) sub-matrices of \( U \).

For an \( A \)-module homomorphism \( \phi : M \to N \) between free \( A \)-modules of finite rank, let us choose a matrix representation \( U \) with respect to bases of \( M \) and \( N \). One can easily prove that the ideal \( I_t(U) \) only depends on \( \phi \). Therefore we put \( I_t(\phi) := I_t(U) \) and call it the \( t \)-th Fitting ideal associated with \( \phi \).
Notations 5.16 (Grade). For an ideal $I$ in $A$, we put

$$S_I := \{ n; \text{There are } f_1, \cdots, f_n \in I \text{ which forms an } A\text{-regular sequence.} \}, \text{ and}$$

$$\text{grade } I := \begin{cases} 0 & \text{if } S_I = \emptyset \\ \max S_I & \text{if } S_I \text{ is a non-empty finite set} \\ +\infty & \text{if } S_I \text{ is an infinite set.} \end{cases}$$

Theorem 5.17 (Buchsbaum-Eisenbud [BE73]). Assume that $A$ is noetherian. For a complex of free $A$-modules of finite rank.

$$F_* : 0 \to F_s \xrightarrow{\phi_s} F_{s-1} \xrightarrow{\phi_{s-1}} \cdots \to F_1 \xrightarrow{\phi_1} F_0 \to 0,$$

set $r_i = \sum_{j=i}^{s} (-1)^{j-i} \text{rank } F_j$. Then the following are equivalent:

1. $F_*$ is $0$-spherical.
2. $\text{grade } I_{r_i}(\phi_i) \geq i$ for any $i$ in $[s]$.

The following proposition is essentially proven in the proof of [Moc13, 4.15].

Proposition 5.18. Let $x$ be an $S$-cube of free $A$-modules of finite ranks. Assume that all vertices of $x$ have same rank and $x$ admits a regular adjugate, then $\text{Tot } x$ satisfies the condition (2) in Theorem 5.17. \qed

References

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