TOTAL VARIATION APPROXIMATION OF RANDOM ORTHOGONAL MATRICES BY GAUSSIAN MATRICES

KATHRYN LOCKWOOD†

Abstract. The topic of this paper is the asymptotic distribution of random orthogonal matrices distributed according to Haar measure. We examine the total variation distance between the joint distribution of the entries of $Z_n$, the $p_n \times q_n$ upper-left block of a Haar-distributed matrix, and that of $p_n q_n$ independent standard Gaussian random variables. We show that the total variation distance converges to zero when $p_n q_n = o(n)$.

1. Introduction

Let $U_n$ be a random orthogonal matrix which is distributed according to Haar measure on the orthogonal group $O(n)$. The asymptotic distribution of the individual entries of such a Haar-distributed matrix is classical. Borel showed in 1906 [2] that a single coordinate of a randomly chosen point on the sphere is asymptotically Gaussian. That is, if $X = (X_1, \cdots, X_n)$ is a uniform random vector in $S^{n-1} \subset \mathbb{R}^n$ then

$$P[\sqrt{n}X_1 \leq t] \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2} dx.$$  

It follows by one of the standard constructions of Haar measure that if for each $n$, $U_n$ is a random orthogonal matrix, then the sequence $\{\sqrt{n}|U_n|_{1,1}\}$ converges weakly to the standard Gaussian distribution as $n \to \infty$. By symmetry, this means that all of the individual entries of a random orthogonal matrix are approximately Gaussian, for large matrices.

Let $||\mu - \nu||_{TV}$ denote the total variation distance between two probability measures $\mu$ and $\nu$. Diaconis and Freedman [3] gave a substantial strengthening of Borel’s result, showing that the joint distribution of the first $k$ coordinates of a uniform random point on the sphere is close in total variation distance to $k$ independent identically distributed Gaussian random variables if $k = o(n)$ in the following theorem:

**Theorem 1.1** (Diaconis-Freedman). Let $X$ be a uniform random point on $\sqrt{n}S^{n-1}$, for $n \geq 5$, and let $1 \leq k \leq n-4$. Let $Z$ be a standard Gaussian random vector in $\mathbb{R}^k$. Then the total variation distance between the distribution of the first $k$ coordinates of $X$ and the distribution of $Z$ is

$$||\left(X_1, \cdots, X_n\right) - Z||_{TV} \leq \frac{2(k+3)}{n-k-3}.$$  

The theorem implies that for $k = o(n)$, one can approximate any $k$ entries from the same row or column of a uniform random orthogonal matrix $U$ by independent Gaussian random variables. This led Diaconis to consider the question of how many entries of $U$ can be simultaneously approximated

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by independent normal random variables. This question was first answered by Diaconis, Eaton, and Lauritzen \cite{Diaconis-Eaton-Lauritzen} with the following:

**Theorem 1.2** (Diaconis-Eaton-Lauritzen). For each \( n \geq 1 \), let \( Z_n \) be the \( p_n \times q_n \) upper left block of a random matrix \( U_n \) which is uniformly distributed on the group \( O(n) \). Let also \( \delta_n \) be the total variation distance between the distribution of the \( p_n q_n \) entries of \( Z_n \) and the joint distribution of \( p_n q_n \) independent standard Gaussian random variables. Then \( \delta_n \to 0 \) if \( p_n = o(n^\alpha) \) and \( q_n = o(n^\alpha) \) for \( \alpha = 1/3 \).

There was much speculation on the maximum value of \( \alpha \) to make the total variation distance go to zero following the result of Diaconis, Eaton, and Lauritzen. The maximum such \( \alpha \) was found to be 1/2 by Tiefeng Jiang \cite{Jiang}.

**Theorem 1.3** (Jiang). For each \( n \geq 1 \), suppose that \( Z_n \) is the \( p_n \times q_n \) upper left block of a random matrix \( U_n \) which is uniformly distributed on the orthogonal group \( O(n) \). Let \( G_n \) be the joint distribution of \( p_n q_n \) independent standard Gaussian random variables and let \( \mathcal{L}(\sqrt{n}Z_n) \) represent the joint probability distribution of the \( p_n q_n \) random entries of \( \sqrt{n}Z_n \). If \( p_n = o(\sqrt{n}) \) and \( q_n = o(\sqrt{n}) \) as \( n \to \infty \), then

\[
\lim_{n \to \infty} ||\mathcal{L}(\sqrt{n}Z_n) - G_n||_{TV} = 0.
\]

Jiang further showed that the theorem was sharp in the sense that if \( x > 0 \) and \( y > 0 \) are two numbers and \( p_n \sim x\sqrt{n} \) and \( q_n \sim y\sqrt{n} \) then

\[
\liminf_{n \to \infty} ||\mathcal{L}(\sqrt{n}Z_n) - G_n||_{TV} \geq \phi(x, y) > 0
\]

where \( \phi(x, y) := \mathbb{E}|\exp(-\frac{x^2y^2}{8} + \frac{x^2}{4}\xi) - 1| \in (0, 1) \) and \( \xi \) is a standard normal.

Jiang also showed in \cite{Jiang} that relaxing the sense if which the entries of the random matrix should be simultaneously approximable by independent identically distributed Gaussian variables allows a larger collection of entries to be approximated.

**Theorem 1.4** (Jiang). For each \( n \geq 1 \), let \( Y_n = [y_{ij}]_{i,j=1}^n \) be an \( n \times n \) matrix of independent standard normals. Let \( U_n = [u_{ij}]_{i,j=1}^n \) be the orthogonal matrix obtained from performing the Gram-Schmidt procedure on the columns of \( Y_n \). Define \( \epsilon_n(m) = \max_{1 \leq i \leq n, 1 \leq j \leq m} |\sqrt{n}u_{ij} - y_{ij}| \). Let \( \{m_n < n : n \geq 1\} \) be a sequence of positive integers. Then:

(i) \( \epsilon_n(m_n) \to 0 \) in probability, provided \( m_n = o(n/\log n) \) and \( n \to \infty \);

(ii) for any \( \alpha > 0 \), \( \epsilon_n([n\alpha/\log n]) \to 2\sqrt{\alpha} \) in probability as \( n \to \infty \).

Thus when \( U_n \) is an orthogonal matrix obtained from performing the Gram-Schmidt procedure on a matrix whose elements are independent standard normals, the maximum order of \( m_n \) such that \( \epsilon_n(m_n) \to 0 \) in probability is \( m_n = o(n/\log n) \).

Let \( |f|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x, y)} \) be the Lipschitz constant of \( f \). Define the \( L_1 \) Kantorovich distance between two probability measures \( \mu \) and \( \nu \) to be

\[
W_1(\mu, \nu) = \sup_{|f|_L \leq 1} \left| \int f d\mu - \int f d\nu \right|.
\]

Chatterjee and Meckes \cite{Chatterjee-Meckes} have shown that relaxing the sense in which the entries of the random orthogonal matrix must be simultaneously approximated by independent standard Gaussian random variables in a different way (using the \( L_1 \) Kantorovich metric) allows for a coordinate free approach.
Theorem 1.5 (Chatterjee-Meckes). Let $A_1, \cdots, A_k$ be $n \times n$ matrices over $\mathbb{R}$ satisfying $\text{Tr}(A_i A_j^T) = n \delta_{ij}$; that is, $\{\frac{1}{\sqrt{n}} A_i\}_{1 \leq i \leq k}$ is orthonormal with respect to the Hilbert-Schmidt inner product. Let $U$ be a random orthogonal matrix, and consider the random vector

$$X = (\text{Tr}(A_1 U), \text{Tr}(A_2 U), \cdots, \text{Tr}(A_k U))$$

in $\mathbb{R}^k$. Let $Z = (Z_1, \cdots, Z_k)$ be a random vector whose components are independent standard normal random variables. Then for $n \geq 2$,

$$W_1(X, Z) \leq \frac{\sqrt{2k}}{n - 1}.$$

The preceding theorem implies that any collection of $o(n)$ entries of $U$ can be simultaneously approximated (in $W_1$) by independent identically distributed standard normal random variables, whereas Jiang’s result treats the square case and Diaconis-Freedman treat the opposite extreme of all entries being drawn from the same row or column. The Chatterjee-Meckes result together with the results of Jiang and Diaconis-Freedman suggest that one should be able to approximate the top left $p \times q$ block by independent identically distributed Gaussian random variables, in total variation distance, as long as $p n q n = o(n)$. The following theorem verifies this conjecture.

Theorem 1.6. For each $n \geq 1$, suppose that $Z_n$ is the $p n \times q n$ upper left block of a random matrix $U_n$ which is uniformly distributed on the orthogonal group $O(n)$. Let $G_n$ be the joint distribution of $p n q n$ independent standard Gaussian random variables. Let $L(\sqrt{n}Z_n)$ be the joint probability distribution of the $p n q n$ entries of $\sqrt{n}Z_n$. If $p n q n = o(n)$ as $n \to \infty$, then

$$\lim_{n \to \infty} ||L(\sqrt{n}Z_n) - G_n||_{TV} = 0.$$

This result recovers not only Jiang’s result, but also the Diaconis-Freedman result, and intermediate cases. The approach is the same as in Jiang’s paper, however the analysis is much more delicate if the only assumption is that $p n q n = o(n)$. In particular, the proof requires sharp asymptotics for the covariances of traces of powers of Wishart matrices, for powers growing with the size of the matrix. Bai [1] has developed asymptotics for the expected value of traces of powers of Wishart matrices. We give an extension of Bai’s result, as well as providing sharp asymptotics for the covariances, which may be of independent interest.

The contents of this paper are as follows. In section 2, we give the proof of the main theorem, making use of new estimates on the asymptotic means and covariances of traces of powers of Wishart matrices. Section 3 contains the proofs of these asymptotics; some asymptotics for the Gamma function used in section 2 are relegated to the appendix.

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where $\mathcal{B}$ is the Borel $\sigma$-algebra. If $\mu$ and $\nu$ have densities $f(x)$ and $g(x)$ with respect to Lebesgue measure, then

$$||\mu - \nu||_{TV} = \int_{\mathbb{R}^n} |f(x) - g(x)|dx_1dx_2 \cdots dx_n.$$ 

Let $f_n(z)$ be the joint density function of $\sqrt{n}Z_n$, the $p_n \times q_n$ upper left block of the random orthogonal matrix $\sqrt{n}U_n$. We will assume throughout that $g_n \leq p_n$. Let $g_n(z)$ be the joint density function of $p_nq_n$ independent standard Gaussian random variables, and $X_n$ be a matrix of $p_nq_n$ independent standard Gaussian random variables with density $g_n$. Then the total variation distance between the entries of $\sqrt{n}Z_n$ and $p_nq_n$ independent standard Gaussian random variables is

$$||\mathcal{L}(\sqrt{n}Z_n) - \mathcal{L}(X_n)|| = \int_{\mathbb{R}^n} |f_n(z) - g_n(z)|dz$$

$$= \int_{\mathbb{R}^n} \left| \frac{f_n(z)}{g_n(z)} - 1 \right| g_n(z)dz$$

$$= \mathbb{E} \left| \frac{f_n(X_n)}{g_n(X_n)} - 1 \right|.$$ 

The following formula for the joint density function $f_n(z)$ of the entries of $Z_n$ is due to Eaton [3].

**Theorem 2.1** (Eaton). Let $U_n$ be an $n \times n$ random orthogonal matrix, and let $Z_{p,q}$ denote the upper-left $p \times q$ subblock of $U_n$. For $q \leq p$ and $p + q \leq n$, with probability one the random matrix $Z_{p,q}$ lies in the set $\mathcal{X}$ of $p \times q$ matrices $X$ over $\mathbb{R}$ with the property that all of the eigenvalues of $X^TX$ lie in $(0,1)$, and the density of $Z_{p,q}$ with respect to Lebesgue measure on $\mathcal{X}$ is given by

$$f(z) = C_1 \det \left( I_q - \frac{z^Tz}{n} \right)^{\frac{n-p-q-1}{2}},$$

where the constant $C_1$ is

$$C_1 = \left(2\pi\right)^{-pq} \frac{\omega(n-p,q)}{\omega(n,q)},$$

with $\omega(\cdot, \cdot)$ denoting the Wishart constant defined by

$$\frac{1}{\omega(r,s)} = \pi^{\frac{s-r-1}{4}} 2^{\frac{s-r}{2}} \prod_{j=1}^{s-r} \Gamma \left( \frac{r-j+1}{2} \right).$$

Here $s$ is a positive integer and $r$ is a real number, $r > s - 1$.

If follows that the density function of $\sqrt{n}Z_n$ is

$$f_n(z) = \left(\frac{2\pi}{n}\right)^{-pq} \frac{\omega(n-p,q)}{\omega(n,q)} \left[ \det \left( I_q - \frac{z^Tz}{n} \right)^{\frac{n-p-q-1}{2}} \right] I_0 \left( \frac{z^Tz}{n} \right),$$

where $I_0 \left( \frac{z^Tz}{n} \right)$ is the indicator that all the eigenvalues of $\frac{z^Tz}{n}$ lie in $(0,1)$. The joint density function of $pq$ independent standard Gaussian random variables is

$$g_n(z) = \left(\frac{2\pi}{n}\right)^{-pq} \exp \left( -\frac{tr(z^Tz)}{2} \right),$$
where $z$ is a $p$ by $q$ matrix. Let $\lambda_1, \ldots, \lambda_q$ be the eigenvalues of $X^T_n X_n$. Then the ratio $\frac{f_n(z)}{g_n(z)}$ can be written as a product of a constant part $K_n$ and a random part $L_n$, where

$$K_n = \left(\frac{2}{n}\right)^{\frac{q}{2}} \prod_{j=1}^{q} \frac{\Gamma((n-j+1)/2)}{\Gamma((n-p-j+1)/2)},$$

$$L_n = \left[\prod_{i=1}^{q} \left(1 - \frac{\lambda_i}{n}\right)\right]^{\frac{a}{2} - 1} \exp\left(\frac{1}{2} \sum_{i=1}^{q} \lambda_i\right)$$

if all the $\lambda_i$ are in $(0, 1)$ and $L_n$ is zero otherwise. Then

$$||\mathcal{L}(\sqrt{n}Z_n) - \mathcal{L}(X_n)||_{TV} = E\left|\frac{f_n(X_n)}{g_n(X_n)} - 1\right| = E|K_n \cdot L_n - 1|.$$

To prove the theorem, it therefore suffices to show that $E|K_n \cdot L_n - 1| \to 0$ as $n \to \infty$. Note that $K_n L_n \geq 0$ and $E|K_n L_n| = \int_{\mathbb{R}^p} f_n(x)dx = 1$, so that $\{K_n L_n\}$ are uniformly integrable; it thus suffices to show that $K_n L_n \to^p 1$ as $n \to \infty$.

Define a function $F(x)$ by $F(x) = \frac{x}{2} + \frac{n - p - q - 1}{2} \log(1 - \frac{x}{n})$ if $0 \leq x < n$ and $F(x) = -\infty$ otherwise. Then $L_n = \exp(\sum_{i=1}^{q} F(\lambda_i))$, and showing that $K_n L_n \to^p 1$ as $n \to \infty$ is equivalent to showing that

$$\log(K_n) + \log(L_n) = \log(K_n) + \left\{\sum_{i=1}^{q} F(\lambda_i)\right\} \to^p 0.$$

If $p/q \to \eta \in (0, \infty)$, choose $l = 2$. Otherwise, let $l = \frac{\log p}{\log(n/pq)}$. Then using Taylor’s Theorem to expand $\log(1 - \frac{x}{n})$ up to order $l$,

$$\log(L_n) = \sum_{i=1}^{q} F(\lambda_i)$$

$$= \sum_{i=1}^{q} \left[\frac{\lambda_i}{2} + \frac{n - p - q - 1}{2} \log\left(1 - \frac{\lambda_i}{n}\right)\right]$$

$$= \text{tr}(X^T_n X_n) \left[\frac{p + 1}{2n} - \frac{(n - p - q - 1)}{2n^2}\right] - \frac{(n - p - q - 1)}{6n^3} \text{tr}(X^T_n X_n)^3$$

$$= \frac{1}{n} \left[\frac{p + 1}{2} \text{tr}(X^T_n X_n) - \frac{1}{4} \text{tr}(X^T_n X_n)^2\right] + \frac{1}{n^2} \left[\frac{p + 1}{4} \text{tr}(X^T_n X_n)^2 - \frac{1}{6} \text{tr}(X^T_n X_n)^3\right]$$

$$+ \frac{1}{n^3} \left[\frac{p + 1}{6} \text{tr}(X^T_n X_n)^3 - \frac{1}{8} \text{tr}(X^T_n X_n)^4\right] + \cdots + \frac{p + 1}{2l n^{2l}} \text{tr}(X^T_n X_n)^l + \frac{g_n \text{tr}(X^T_n X_n)^{l+1}}{n^{l+1}},$$

where $g_n = \sum_{i=1}^{q} g_n(\lambda_i)$ and $g_n(x) = \frac{-n^{(n-p-q-1)}}{2^{(i+1)q-n-p-1}}$ for some $\xi \in (0, x)$. Define

$$h_i = \frac{1}{n^i} \left[\frac{p + 1}{2i} \text{tr}(X^T_n X_n)^i - \frac{1}{2(i+1)} \text{tr}(X^T_n X_n)^{i+1}\right]$$
when \( i < l \) and \( h_i = \frac{1}{n^i} \left[ \frac{p^i q^{i+1}}{2^i} \operatorname{tr}(X^T X)^i \right] \). Let \( E_i = \mathbb{E}[h_i] \) and \( R_i = h_i - E_i \). Finally, let \( g = \frac{g_n \operatorname{tr}(X^T X)^{i+1}}{n^i} \), so \( \sum_{i=1}^q F(\lambda_i) = \sum_{i=1}^l R_i + \sum_{i=1}^l E_i + g \); the goal is to show that

\[
\left( \log(K_n) + \sum_{i=1}^l R_i + \sum_{i=1}^l E_i + g \right) \xrightarrow{p} 0.
\]

By lemmas 3.4 and 3.5 in the appendix, \( \log(K_n) + \sum_{i=1}^l E_i = o(1) \) for \( n \) sufficiently large. To show \( \log(K_n) + \sum_{i=1}^q F(\lambda_i) \xrightarrow{p} 0 \), it thus suffices to show that \( \sum_{i=1}^l R_i + g \rightarrow 0 \) in probability as \( n \rightarrow \infty \).

First consider the case where \( q/p \rightarrow 0 \) as \( n \rightarrow \infty \). Lemma 3.3 provides an explicit formula for \( \operatorname{Var}[R_i] = \operatorname{Var}[h_i] \). Recall that \( q \leq p \). Then,

\[
\begin{align*}
\operatorname{Var} \left[ \frac{p+q+1}{2i n^i} \operatorname{tr}(X^T X)^i - \frac{1}{2(i+1)n^i} \operatorname{tr}(X^T X)^{i+1} \right] \\
= \frac{(p+q+1)^2}{4i^2 n^{2i}} \operatorname{Var} \left[ \operatorname{tr}(X^T X)^i \right] + \frac{1}{4(i+1)^2 n^{2i}} \operatorname{Var} \left[ \operatorname{tr}(X^T X)^{i+1} \right] \\
- \frac{2(p+q+1)}{4i(i+1)n^{2i}} \operatorname{Cov} \left( \operatorname{tr}(X^T X)^i, \operatorname{tr}(X^T X)^{i+1} \right) \\
= \frac{(p+q+1)^2}{4i^2 n^{2i}} \left( 2i^2 (p^{2i-1}q + pq^{2i-1}) + \delta(i, i) \right) \\
+ \frac{1}{4(i+1)^2 n^{2i}} \left( 2(i+1)^2 (p^{2i+1}q + pq^{2i+1}) + \delta(i+1, i+1) \right) \\
- \frac{2(p+q+1)}{4i(i+1)n^{2i}} \left( 2i(i+1)(p^{2i}q + pq^{2i}) + \delta(i, i+1) \right) \\
\leq \frac{2p^{2i-1}q^3 + p^{3} q^{2i-1}}{n^{2i}} + 4p^{2i-1}q^2 + p^2 q^{2i-1} + 2p^{2i-1}q + pq^{2i-1} + \frac{C_p q^2 p^{2i} 2i}{(i+1)^{3/2} n^{2i}}.
\end{align*}
\]

Fix \( \epsilon > 0 \) and recall that in this case \( l = \frac{\log p}{\log(2pq)} \). Define \( \epsilon_i \) to be \( \frac{\epsilon}{f(i) Z_i} \), where \( f(i) = \left( \frac{n}{2pq} \right)^{i-1} \) and \( Z_i = \sum_{j=1}^l \frac{1}{f(j)} \), so that \( \sum_{i=1}^l \epsilon_i = \epsilon \). Note that \( \lim_{n \rightarrow \infty} Z_i = 1 \). It follows from Chebychev’s Inequality that

\[
P \left[ \sum_{i=1}^l R_i \geq \frac{\epsilon}{2} \right] \leq \sum_{i=1}^l \mathbb{P} \left[ R_i \geq \frac{\epsilon}{2} \right] \leq \sum_{i=1}^l \frac{\operatorname{Var}[R_i]}{\epsilon_i^2} = \sum_{i=1}^l \frac{\operatorname{Var}[h_i]}{\epsilon_i^2}.
\]

We show in lemma 3.1 that \( \sum_{i=1}^l \frac{p^{2i-1}q^3}{n^{2i}} \frac{1}{\epsilon_i^2} \rightarrow 0 \) and \( \sum_{i=1}^l \frac{2^i p^{2i} q^2}{(i+1)^{3/2} n^{2i}} \frac{1}{\epsilon_i^2} \rightarrow 0 \), and so \( \sum_{i=1}^l \frac{\operatorname{Var}[h_i]}{\epsilon_i^2} \rightarrow 0 \).

Finally, we check the convergence in probability of the error term \( g = \frac{g_n \operatorname{tr}(X^T X)^{i+1}}{n^i} \). By lemma 3.3

\[
P \left[ g \geq \frac{\epsilon}{4} \right] = \mathbb{P} \left[ g \geq \frac{\epsilon}{4} \right] \geq \frac{\epsilon}{4}.
\]
which tends to zero when

\[ \frac{p}{q} \to \eta \in (0, 1), \]

l is equal to 2 and the approach is simply a repackaging of Jiang’s result in [7]. Since it is relatively brief, we include it here for completeness. In this case, by lemma 3.2,

\[
\text{Var}(\text{tr}(X^T X)) = 2pq(1 + O(1/p^3)),
\]

\[
\text{Var}(\text{tr}(X^T X)^2) = (8p^3q + 8pq^3 + 17p^2q^2)(1 + O(2^6/p^3)),
\]

and

\[
\text{Cov}(\text{tr}(X^T X), \text{tr}(X^T X)^2) = (4p^3q + 4pq^3)(1 + O(2^6/p^3)).
\]

Then,

\[
P[h_1 \geq \frac{\epsilon}{3}] \leq \frac{9}{\epsilon^2} \text{Var}[h_1]
\]

\[
= \frac{9}{\epsilon^2} \text{Var} \left[ \frac{p + q + 1}{2n} \text{tr}(X^T X) - \frac{1}{4n} \text{tr}(X^T X)^2 \right]
\]

\[
= \frac{9}{\epsilon^2} \left( \frac{(p + q + 1)^2}{4n^2} \text{Var}(\text{tr}(X^T X)) + \frac{1}{16n^2} \text{Var}(\text{tr}(X^T X)^2) - \frac{2(p + q + 1)}{8n^2} \text{Cov}(\text{tr}(X^T X), \text{tr}(X^T X)^2) \right)
\]

\[
= \frac{9}{\epsilon^2} \left( \frac{p^2 + 2pq + 2p + q^2 + 2q + 1}{4n^2} (2pq) + \frac{8p^3q + 17p^2q^2 + 8pq^3}{16n^2} - \frac{2(p + q + 1)}{8n^2} (4p^2q + 4pq^2) \right)
\]

\[
= \frac{9}{\epsilon^2} \frac{p^2q^2 + 8pq}{16n^2},
\]

which tends to zero when \( p = q = o(\sqrt{n}) \). Similarly,

\[
P[\frac{p + q + 1}{4n^2} \text{tr}(X^T X)^2 \geq \frac{\epsilon}{3}] \leq \frac{9}{\epsilon^2} \text{Var} \left[ \frac{p + q + 1}{4n^2} \text{tr}(X^T X)^2 \right]
\]

\[
= \frac{9}{\epsilon^2} \frac{(p + q + 1)^2}{16n^4} \text{Var}(\text{tr}(X^T X)^2)
\]

\[
= \frac{9}{\epsilon^2} \frac{(p + q + 1)^2}{16n^4} (8p^3q + 8pq^3 + 17p^2q^2)
\]
where the sum is taken over all S-graphs \((G_i)\) which converges to 0 as \(n \to \infty\). Then, from Lemma 3.1.

\[ \text{Proof.} \]

Write \(\sigma_n(X^T X)^3\) in terms of \(\sigma_n(X^T X)^3 - \mathbb{E}[\sigma_n(X^T X)^3]\). By lemma 3.2,

\[ \text{Var}[\sigma_n(X^T X)^3] = 18(p^5q + pq^5) + 110(p^4q^2 + p^2q^4) + 202p^3q^3. \]

Then,

\[ \mathbb{P} \left[ \frac{\sigma_n(X^T X)^3 - \mathbb{E}[\sigma_n(X^T X)^3]}{\sqrt{\text{Var}[\sigma_n(X^T X)^3]}} > \epsilon \right] \leq \frac{\text{Var}[\sigma_n(X^T X)^3]}{\epsilon^2 p^8} \]

which converges to 0 as \(n \to \infty\) since \(\eta\) is bounded and \(p \to \infty\). By lemma 3.1.

\[ \mathbb{E}[\sigma_n(X^T X)^3] = \sum_{r=0}^{2} p^{3-r}q^{r+1} \frac{1}{r+1} \binom{3}{r} \binom{2}{r} \]

so that \(\frac{\sigma_n(X^T X)^3 - \mathbb{E}[\sigma_n(X^T X)^3]}{p^3q + 3pq^2 + pq^3}\) → 0 and \(\frac{\mathbb{E}[\sigma_n(X^T X)^3]}{p^3q + 3pq^2 + pq^3}\) → 0 in probability as \(n \to \infty\), and it follows that \(\mathbb{P} \left[ \frac{\sigma_n(X^T X)^3}{p^3q + 3pq^2 + pq^3} \geq \epsilon \right] \to 0\).

3. Combinatorics of Wishart Matrices

The following result is a slight extension of a result in [1] on means of Wishart matrices. The majority of the proof is the same as the one in [1]. However, somewhat more careful estimates of the error are needed to complete the proof of the main theorem.

**Lemma 3.1.** Let \(\{p_n : n \geq 1\}\) and \(\{q_n : n \geq 1\}\) be two sequences of positive integers such that \(p_n \to \infty\) and \(q_n \leq p_n\). For each \(n\), let \(X_n = (x_{ij})\) be a \(p_n \times q_n\) matrix where \(x_{ij}\) are independent standard Gaussian random variables. Then for each integer \(h \geq 1\),

\[ \mathbb{E}[\sigma_n(X_n)^h] = \left( \sum_{r=0}^{h-1} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h-1}{r} \binom{h-1}{r} \right) \left( 1 + O\left(\frac{h^2}{p}\right) \right). \]

**Proof.** Write \(\sigma_n(X_n)^h\) as

\[ \sum_{1 \leq i_1 \ldots i_h \leq p} \sum_{1 \leq j_1 \ldots j_h \leq q} x_{i_1j_1} x_{i_1j_2} x_{i_2j_2} \cdots x_{i_{h-1}j_h} x_{i_hj_h} x_{i_hj_1} = \sum_G x_G, \]

where \(G\) is a bipartite graph with the \(i_k\) on a top line and the \(j_k\) on a bottom line, with \(h\) up-edges from \(j_k\) to \(i_k\) and \(h\) down-edges from \(i_k\) to \(j_{k+1}\). We refer to such a graph as an S-graph. An edge (\(i, j\)) in the S-graph corresponds to the variable \(x_{ij}\). Now,

\[ \mathbb{E}[\sigma_n(X_n)^h] = \sum_G \mathbb{E}[x_G], \]

where the sum is taken over all S-graphs \(G\). If \(G\) contains any edges of odd multiplicity then \(\mathbb{E}[x_G] = 0\) so that the proof reduces to the case where \(G\) contains only edges of even multiplicity.
Each S-graph $G$ contains $2h$ edges, hence at most $h$ distinct edges and $h + 1$ distinct vertices. First consider the case when $G$ contains exactly $h$ distinct edges. For each $r = 0, \ldots, h - 1$ the calculation reduces to counting the number of graphs which have no single edges, $r + 1$ non-coincident $j$-vertices and $h - r$ non-coincident $i$-vertices.

Consider two S-graphs to be isomorphic if one can be converted to the other by permuting $\{1, \ldots, p\}$ on the top line and $\{1, \ldots, q\}$ on the bottom line. To compute the number of isomorphism classes define $u_l = -1$ if the graph leaves a bottom vertex for the final time after the $l$th up edge and $u_l = 0$ otherwise. Define $d_l = 1$ if the $l$th down edge leads to a new bottom vertex and $d_l = 0$ otherwise. The graph must return to the initial bottom vertex so $u_1 = 0$. Because the number of vertices seen for the final time cannot exceed the number of new vertices, we have $d_1 + \cdots + d_{L-1} + u_1 + \cdots + u_l \geq 0$ for every $l$.

There are $\binom{h}{r}$ ways to arrange $r$ ones into the $h$ positions of down edges that could lead to a new bottom vertex. There are $\binom{h-1}{r-1}$ ways to arrange $r$ minus ones into the $h - 1$ positions of up edges that leave a bottom vertex for the last time. ($h - 1$ since the first vertex can never be left for the last time.) Thus we have $\binom{h}{r}\binom{h-1}{r-1}$ ways to arrange our $d$-sequence and $u$-sequence.

However, not all of these $\binom{h}{r}\binom{h-1}{r-1}$ graphs are a proper S-graph. It is an improper graph if at some point $d_1 + \cdots + d_{L-1} + u_1 + \cdots + u_l < 0$. Let $L$ be the first integer at which this happens. Then we must have $d_{L-1} = 0$ and $u_L = -1$. That is, we have just returned to a vertex we have seen before and left it for the last time. To fix it we must instead see a new vertex that we will return to again later. To do this, change $d_{L-1}$ to 1 and $u_L$ to 0. The initial bad sequences contained $r$ ones and $r$ minus ones. The fixed sequences now contain $r + 1$ ones and $r - 1$ minus ones. Therefore we have $\binom{h}{r+1}\binom{h-1}{r-1}$ bad sequences. Thus the number of isomorphism classes is $\binom{h}{r}\binom{h-1}{r-1} - \binom{h}{r+1}\binom{h-1}{r-1} = \binom{h}{r}\binom{h-1}{r-1}$. The number of graphs in each isomorphism class is $p(p-1) \cdots (p-h+r-1)q(q-1) \cdots (q-r) \leq p^h q^{r+1}$. Thus there are $\sum_{r=0}^{h-1} p^h q^{r+1} \frac{1}{r+1} \binom{h}{r}\binom{h-1}{r-1}$ S-graphs with exactly $h$ distinct edges. This is the main term in the expectation. We will next show that all other terms are of smaller order.

Suppose that $G$ has $m < h$ distinct edges. Let $r = 0, 1, \ldots, m - 1$ and choose $r + 1$ bottom vertices and $m - r$ top vertices. There are $\frac{1}{r+1} \binom{m}{r}\binom{m-1}{r}$ isomorphism classes and $p^m q^{r+1}$ graphs per class. Now, $G$ contains $m$ distinct edges. So there are $h - m$ (double) edges left to place within the graph. Each of these edges can overlap with any of the $m$ distinct edges already in the graph. So there are $m^{h-m}$ ways to arrange the edges of multiplicity more than two, and there are thus $m^{h-m} \sum_{r=0}^{m-1} p^m q^{r+1} \frac{1}{r+1} \binom{m}{r}\binom{m-1}{r}$ such S-graphs with $m$ distinct edges.

Comparing the contribution of the $m$ edge case to that of the $h$ edge case,

$$\frac{m^{h-m} \sum_{r=0}^{m-1} p^m q^{r+1} \frac{1}{r+1} \binom{m}{r}\binom{m-1}{r}}{\sum_{r=0}^{h-1} p^h q^{r+1} \frac{1}{r+1} \binom{h}{r}\binom{h-1}{r}} \leq \frac{m^{h-m} \sum_{r=0}^{h-1} p^m q^{r+1} \frac{1}{r+1} \binom{h}{r}\binom{h-1}{r}}{\sum_{r=0}^{h-1} p^h q^{r+1} \frac{1}{r+1} \binom{h}{r}\binom{h-1}{r}}$$
\[ = \frac{m^{h-m}p^m}{p^h} \]
\[ \leq \left( \frac{h}{p} \right)^{h-m}. \]

Summing over all possibilities for \( m \),
\[ \sum_{m=1}^{h-1} \left( \frac{h}{p} \right)^{h-m} = \frac{h}{p} \sum_{m=1}^{h-1} \left( \frac{p}{h} \right)^m \leq h \left( \frac{h}{p} \right) \left( \frac{p}{h} \right)^{h-1} = \frac{h^2}{p}. \]

Then
\[ \mathbb{E}[\text{tr}(X_n^T X_n)^h] = \left( \sum_{r=0}^{h-1} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h-1}{r} \binom{k-1}{s} \left( \frac{k}{s} \right) \right) \left( 1 + O \left( \frac{k^6}{p^6} \right) \right) \quad \Box. \]

Careful asymptotics for the covariances of traces of powers of Wishart matrices are given below. The proof uses the same techniques as the proof above. The following lemma holds for any \( p_n, q_n = o(n) \) and in particular can be used when \( p_n = q_n = o(\sqrt{n}) \) to recover Jiang’s results.

**Lemma 3.2.** Let \( \{p_n : n \geq 1\} \) and \( \{q_n : n \geq 1\} \) be two sequences of positive integers such that \( p_n \to \infty \) and \( q_n \leq p_n \). For each \( n \), let \( X_n = (x_{ij}) \) be a \( p_n \times q_n \) matrix where \( x_{ij} \) are independent standard Gaussian random variables. Then for integers \( h \geq 1 \) and \( k \geq h, \)

\[
\text{Cov}(\text{tr}(X_n^T X_n)^h, \text{tr}(X_n^T X_n)^k) = \left( 2h \sum_{r=0}^{h-1} \sum_{s=0}^{k-1} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h-1}{r} \binom{k-1}{s} \left( \frac{k}{s} \right) \right) \left( 1 + O \left( \frac{k^6}{p^6} \right) \right) \]
\[ + \left( 2(k-h) p^{h-k} q^{h-k} \sum_{r=0}^{k-h} p^{k-h-r} q^{r+1} \frac{1}{r+1} \binom{k-h-1}{r} \right) \left( 1 + O \left( \frac{k^4}{p^4} \right) \right) \]
\[ + \left( 2(h-l) (k-l) p^{l} q^{l} \sum_{r=0}^{l} p^{l-r} q^{r+1} \frac{1}{r+1} \binom{h-l}{r} \binom{k-l}{s} \left( \frac{k}{s} \right) \right) \left( 1 + O \left( \frac{k^6}{p^6} \right) \right) \]

**Proof.** Write \( (\text{tr}(X_n^T X_n)^h) \) as
\[ \sum_{G} X_G, \]
where \( G \) is a bipartite graph with the \( i_k \) on a top line and the \( j_k \) on a bottom line, with \( h \) up-edges from \( j_k \) to \( i_k \) and \( h \) down-edges from \( i_k \) to \( j_{k+1} \). We refer to such a graph as an S-graph. An edge \( (i, j) \) in the S-graph corresponds to the variable \( x_{ij} \). Now,
\[
\text{Cov}(\text{tr}(X^T X)^h, \text{tr}(X^T X)^k) = \mathbb{E}[\text{tr}(X^T X)^h \text{tr}(X^T X)^k] - \mathbb{E}[\text{tr}(X^T X)^h] \mathbb{E}[\text{tr}(X^T X)^k] \]
\[ = \sum_{G,K} (\mathbb{E}[X_G X_K] - \mathbb{E}[X_G] \mathbb{E}[X_K]), \]
where $G, K$ are both S-graphs. If $G \cup K$ contains a single edge, or an edge of odd multiplicity, then either $G$ or $K$ also contains a single edge. In either case, $\mathbb{E}[X_G X_K]$ and $\mathbb{E}[X_G] \mathbb{E}[X_K]$ are both zero. If $G$ and $K$ do not have a coincident edge, then $\mathbb{E}[X_G X_K] = \mathbb{E}[X_G] \mathbb{E}[X_K]$ and the difference is zero. It thus suffices to consider the case where there are no edges of odd multiplicity and at least one coincident edge. There are three main cases:

1. $\mathbb{E}[X_G] = \mathbb{E}[X_K] = 1$ and $G \cup K$ contains an edge of multiplicity four. There are

$$\left( hk \sum_{r=0}^{h-1} \sum_{s=0}^{k-1} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h}{r} \binom{h-1}{r} p^{k-1-s} q^s \frac{1}{s+1} \binom{k}{s} \binom{k-1}{s} \right) \left( 1 + O \left( \frac{k^6}{p^3} \right) \right)$$

such pairs $(G, K)$ of graphs.

One can first build $G$ as in the lemma on the mean. In this case, there are

$$\left( \sum_{r=0}^{h-1} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h}{r} \binom{h-1}{r} \right) \left( 1 + O \left( \frac{h^2}{p} \right) \right)$$

such graphs.

Next, build the graph of $K$. There are $h$ possible choices of edges in the graph of $G$ with which one of the edges of $K$ can coincide and $k$ possible times in the construction of $K$ at which a coincident edge may be added. $K$ has an edge in common with $G$ and therefore $K$ can have at most $k-1$ new vertices. Let $s = 0, \cdots, k-1$ and choose $s$ bottom vertices and $k-s-1$ top vertices.

As with $G$, the number of isomorphism classes is

$$\binom{h}{s} \binom{k-1}{s} - \binom{k-1}{s+1} = \frac{1}{s+1} \binom{k}{s} \binom{k-1}{s}.$$ 

There will be at most $p^{k-1-s} q^s$ graphs in each isomorphism class.

Thus there are $hk \sum_{r=0}^{h-1} \sum_{s=0}^{k-1} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h}{r} \binom{h-1}{r} p^{k-1-s} q^s \frac{1}{s+1} \binom{k}{s} \binom{k-1}{s}$ possible graphs with exactly $h+1$ distinct vertices in $G$ and $k-1$ distinct vertices in $K$. $G$ and $K$ both contain only edges of multiplicity two. Thus $\mathbb{E}[X_G X_K] = 3$ and $\mathbb{E}[X_G] = \mathbb{E}[X_K] = 1$. This gives the factor of 2 in the first term of the expression in the lemma.

Any graphs with fewer distinct edges will be of smaller order than the term found above. The error is computed by counting the number of possible graphs. Therefore, suppose now that there is at least one less distinct edge in $G \cup K$. So that either $G$ contains at most $h-1$ distinct edges or $K$ contains at most $k-2$ distinct vertices. Without loss of generality, we assume that $G$ is the graph with one less distinct edge, then $G$ has at most $h-1$ distinct edges and $K$ has at most $k-1$ distinct vertices. Let $G$ have $m < h$ non-coincident edges, hence $m+1$ non-coincident vertices, and $K$ have $n \leq k-1$ non-coincident vertices. Taking $r = 0, \cdots, m-1, r+1$ bottom vertices, and $m-r$ top vertices, there are $\frac{1}{r+1} \binom{m}{r} \binom{m-1}{r}$ isomorphism classes and $p^{m-r} q^{r+1}$ graphs per class.

Now, $G$ contains $m$ distinct edges. So there are $h-m$ (double) edges left to place within the graph. Each of these (double) edges can come after any of the $m$ distinct edges. So there are $m^{h-m}$ ways to arrange the edges of multiplicity greater than two. In the same way, there are $n^{k-n}$ ways to arrange the edges of multiplicity greater than two in the construction of $K$. Let $s = 0, \cdots, n$, and choose $s$ bottom vertices and $n-s$ top vertices in the graph of $K$, then there are $\frac{1}{s+1} \binom{n+1}{s} \binom{n}{s}$ isomorphism classes and $p^{n-s} q^s$ graphs per class. ($n+1$ for the $n$ new vertices in $K$ and the one overlapping vertex). The number of such graphs is
\[ m^{m-n} n^k \sum_{r=0}^{m-1} p^{m-r} q^{r+1} \left( \frac{m}{r} \right) \sum_{s=0}^{n} p^{n-s} q^s \left( \frac{n}{s} \right). \]

Lastly, there exists a bound on \( E[X_G X_K] - E[X_G]E[X_K] \). This is less than or equal to \( E[X_G X_K] \), and the expected value will be largest when all but one of the edges in \( G \cup K \) have multiplicity two. That is, when all of the \( h-m \) double edges in \( G \) are coincident, all of the \( k-n \) double edges in \( K \) are coincident, and these are coincident with each other. This one edge will then have multiplicity \( 2(h-m+1) + 2(k-n+1) = 2h + 2k - 2m - 2n + 4 \). Thus, by Sterling’s Formula,

\[
E[X_G X_K] = (2h + 2k - 2m - 2n + 3)! \]

\[
= \frac{(2h + 2k - 2m - 2n + 4)!}{2^{h+m-k-n-2} (h+k-m-n+2)!} \sim \sqrt{2} \left( \frac{2h + 2k - 2m - 2n + 4}{e} \right)^{h+k-m-n+2}.
\]

Comparing this term to the number of graphs with exactly \( h+1 \) and \( k-1 \) distinct vertices that was calculated earlier,

\[
\sqrt{2} \left( \frac{2h+2k-2m-2n+4}{e} \right)^{h+k-m-n+2} m^{h-m} n^k \sum_{r=0}^{m-1} p^{m-r} q^{r+1} \left( \frac{m}{r} \right) \sum_{s=0}^{n} p^{n-s} q^s \left( \frac{n}{s} \right) \]

\[
= \frac{2hk}{\sqrt{2}} \left( \frac{2h+2k-2m-2n+4}{e} \right)^{h+k-m-n+2} m^{h-m} n^k \sum_{r=0}^{m-1} p^{m-r} q^{r+1} \left( \frac{m}{r} \right) \sum_{s=0}^{n} p^{n-s} q^s \left( \frac{n}{s} \right) \]

\[
\leq \frac{2hkp}{\sqrt{2}} \left( \frac{2h+2k-2m-2n+4}{e} \right)^{h+k-m-n+2} m^{h-m} n^k \sum_{r=0}^{m-1} p^{m-r} q^{r+1} \left( \frac{m}{r} \right) \sum_{s=0}^{n} p^{n-s} q^s \left( \frac{n}{s} \right) \]

\[
= \frac{1}{\sqrt{2}} \left( \frac{(2h+2k)h}{ep} \right)^{h-m-1} \left( \frac{(2h+2k)k}{ep} \right)^{k-n-1} \left( \frac{2h+2k}{e} \right)^3 \left( \frac{2h+2k}{e} \right).\]

Summing over all possible combinations of \( m \) and \( n \) gives all the graphs that can be constructed under the assumption of one or more additional coincident edges as follows,
\[
\sum_{m=1}^{h-1} \sum_{n=1}^{k-1} \frac{1}{\sqrt{e}} \left( \frac{(2h+2k)h}{ep} \right)^{k-m-1} \left( \frac{(2h+2k)k}{ep} \right)^{k-n-1} \left( \frac{(2h+2k)}{e} \right)^3 \left( \frac{2h+2k}{e} \right)^n
\]
\[
= \frac{1}{\sqrt{2}} \left( \frac{2h+2k}{ep} \right)^3 \left( \frac{2h+2k}{e} \right)^h \left( \frac{k(2h+2k)}{ep} \right)^{k-1} \sum_{m=1}^{h-1} \left( \frac{ep}{h(2h+2k)} \right)^m \sum_{n=1}^{k-1} \left( \frac{ep}{k(2h+2k)} \right)^n
\]
\[
\leq \frac{hk}{\sqrt{2}} \left( \frac{2h+2k}{ep} \right)^3 \left( \frac{2h+2k}{e} \right)^h \left( \frac{k(2h+2k)}{ep} \right)^{k-1} \frac{1}{\sqrt{2}} \frac{4^4 k^6}{e^2 p^3}
\]
\[
\leq \frac{4^4}{\sqrt{2} e^2} \left( \frac{2}{p} \right)^3,
\]
which tends to zero by choice of \( l \).

(2) Assume that \( \mathbb{E}[X_G] = \mathbb{E}[X_K] = 0 \), \( \mathbb{E}[X_GX_K] \) does not equal 0, and \( 2 < h < k \) where \( G \) is a graph consisting of \( 2h \) single edges. Then \( K \) must contain a subgraph that exactly overlaps the single edges of \( G \) in order that \( \mathbb{E}[X_GX_K] \neq 0 \). There are \( 2p^h q^h \) ways to construct \( G \) since there are two possible orientations and \( p^h q^h \) labels. Now \( K \) will have \( 2k - 2h \) vertices not contained in the subgraph of single edges, hence at most \( k - h \) distinct edges, and at most \( k - h \) distinct vertices. First consider the case with exactly \( k - h \) distinct vertices. As in the previous case, this will be the dominating term for case (2). The graph of \( G \) can be attached onto \( K \) in \( k - h \) different places. Let \( r = 0, \ldots, k - h \). We again consider all possible ways of breaking the \( k - h \) vertices into top and bottom vertices by choosing \( r \) bottom vertices and \( k - h - r \) top vertices.

There are \( \binom{k-h}{r} \) way to arrange \( r \) ones into the \( k - h \) positions of down edges that could lead to new bottom vertices and \( \binom{k-h-1}{r} \) ways to arrange \( r \) minus ones into the \( k - h - 1 \) positions of up edges that leave a bottom vertex for the last time. As before, not all of these \( \binom{k-h}{r} \binom{k-h-1}{r} \) arrangements are proper S-graphs. There are \( \binom{k-h}{r+k-h} \binom{k-h-1}{r-h+1} \) bad sequences. Then there are \( \binom{k-h}{r} \binom{k-h-1}{r} - \binom{k-h}{r+k-h} \binom{k-h-1}{r-h+1} = \frac{1}{r+1} \binom{k-h}{r} \binom{k-h-1}{r} \) possible isomorphism classes. Thus when \( k > h \), there are
\[
2(k-h)p^h q^h \sum_{r=0}^{k-h} p^{k-h-r} q^r \frac{1}{r+1} \binom{k-h}{r} \binom{k-h-1}{r}
\]
graphs.

Now, to compute the error, suppose there is at least one less distinct edge in \( K \). Assume there are \( n < k - h \) distinct vertices. As before, the graph of \( G \) can be attached onto \( K \) in \( n \) places. There are \( k-h-n \) double edges left to place, each of which can coincide with any of the \( n \) distinct edges. So there are \( n^{k-h-n} \) ways to arrange the non-distinct edges. Take \( r \) bottom vertices and \( n - r \) top vertices where \( r = 0, \ldots, n \). Then there are \( \binom{n-r}{r} \) possible isomorphism classes of
graphs and at most $p^{n-r}q^r$ graphs per class. Such graphs will have maximal expectation when all of the edges have multiplicity two, except for one edge of multiplicity $2(k-h-n+1)$. By Sterling’s formula, $E \sim \sqrt{2} \left( \frac{2k-2h-2n+2}{e} \right)^{k-h-n+1} n^{k-h-n+1} p^k q^h \sum_{r=0}^{n} p^{n-r} q^r \binom{n}{r}^{(n-1)}$. Comparing to the case with exactly $k-h$ distinct vertices,

\[
2 \sqrt{2} \left( \frac{2k-2h-2n+2}{e} \right)^{k-h-n+1} n^{k-h-n+1} p^k q^h \sum_{r=0}^{k-h} p^{k-h-r} q^r \binom{k-h-1}{r} \binom{k-h-1}{r} \leq \frac{2(k-h) p^{k-h}}{2k-h-n} \leq \frac{2(p^{k-h-n})}{2(p^{k-h-n})} = 1 \sqrt{2} \left( \frac{2k-2h-2n+2}{e} \right)^{k-h-n} \frac{2k-2h-2n+2}{e} \leq \frac{1}{\sqrt{2}} \left( \frac{2k^2}{e} \right)^{k-h-n} \left( \frac{2k}{e} \right).
\]

Summing over all possible constructions,

\[
\sum_{n=0}^{k-h-1} \frac{1}{\sqrt{2} \left( \frac{2k^2}{e} \right)^{k-h-n}} \left( \frac{2k}{e} \right) = \frac{1}{\sqrt{2} \left( \frac{2k^2}{e} \right)^{k-h-n}} \left( \frac{2k}{e} \right) \left( \sum_{n=0}^{k-h} \left( \frac{2k}{e} \right) \right) = \frac{2k}{\sqrt{2} \left( \frac{2k^2}{e} \right)^{k-h-n}} \left( \frac{2k}{e} \right) \left( \sum_{n=0}^{k-h} \left( \frac{2k}{e} \right) \right) = \frac{4k^4}{\sqrt{2}e^2 p}.
\]

Then there are

\[
\left( 2(k-h) p^{k-h} q^h \sum_{r=0}^{k-h} p^{k-h-r} q^r \frac{1}{r+1} \binom{k-h-1}{r} \binom{k-h-1}{r} \right) \left( 1 + O \left( \frac{k^4}{p} \right) \right)
\]

possible graphs from case (2).

(3) Assume that $\mathbb{E}[X_G] = \mathbb{E}[X_K] \neq 0$, $\mathbb{E}[X_G X_K]$ does not equal 0. Then for each $2 \leq l \leq h-1$, we will count the number of ways that both $G$ and $K$ could contain proper subgraphs consisting of $2l$ single edges that overlap exactly. In this case, for each $l$, there are

\[
2(h-l)(k-l) p^{h-l} q^l \sum_{r=0}^{h-l} s=0 \sum_{r=0}^{h-l} p^{h-l-r} q^r \frac{1}{r+1} \binom{h-l-1}{r} \binom{h-l-1}{r} p^{k-l-s} q^s \frac{1}{s+1} \binom{k-l-1}{s} \binom{k-l-1}{s}
\]
possible graphs.

This can be seen by first building up the closed cycle with $2l$ vertices. There are two orientations for the cycle and $p^l q^l$ choices of vertices. The rest of $G$ will have at most $h - l$ remaining distinct edges not within this cycle. The rest of $K$ will have at most $k - l$ distinct edges. First, consider the case when $G$ has exactly $h - l$ distinct edges and $K$ has exactly $k - l$ distinct edges. This will be the leading term for case (3). The subgraph of $2l$ single edges can be inserted into the construction of $G$ before any one of these edges. $G$ has $2h - 2l$ vertices not in the cycle. Each edge must have multiplicity two and the cycle must attach to $G$ at one vertex. So $G$ has exactly $h - l$ distinct vertices outside of the cycle. Let $r = 0, \cdots , h - l$ and take $r$ bottom vertices and $h - l - r$ top vertices. As before, there are $\frac{1}{r + 1}(\alpha r)_{r+1} \alpha r$ isomorphism classes and $p^{h - l - r}q^r$ choices of labels. Then build up $K$ in the same way by choosing $s = 0, \cdots , k - l$ bottom vertices and $k - l - s$ top vertices and inserting the cycle after any of the $k - l$ edges. Then there are $(k - l) \sum_{s=0}^{k-l} \frac{1}{s+1}(\beta s)_{s+1} \beta s q^s$ non-isomorphic graphs.

To compute the error, assume for some $l$ that $G \cup K$ contains at least one less distinct edge. First, we will assume that all of the edges with multiplicity more than two all lie within the cycle consisting of $2l$ distinct edges. The cycle part of $G$ will then contain $2l$ distinct edges. Let the $i$th edge have multiplicity $m_i$. Then $m_i \geq 1$, each $m_i$ is odd by assumption, and $2l = \sum_{i=1}^{2l} m_i := m \leq 2h$. Similarly, the cycle part of $K$ consists of the overlapping $2l$ edges each with multiplicity $n_i \geq 1$, odd, and $2l = \sum_{i=1}^{2l} n_i := n \leq 2k$. Where, without loss of generality, it has been assumed that at least one edge in the cycle piece of $K$ has multiplicity at least $3$. There are two orientations for the subgraph and $p^l q^l$ choices of vertices.

Next, build the rest of $G$. There are $\alpha := \binom{2h - m}{2}$ (double) edges left to place outside of the cycle. Note that $0 \leq \alpha \leq h - l$. Let $r = 0, \cdots , \alpha$ and choose $r$ bottom vertices and $\alpha - r$ top vertices. There are $\frac{1}{r + 1}(\beta r)_{r+1}$ ways to arrange the edges in $G$ outside of the cycle and $p^\alpha q^r$ ways to label the edges. Furthermore, there are $\alpha$ places in the construction of $G$ where the cycle can be inserted. Similarly, build up the rest of $K$ by taking $\beta := \binom{2k - m}{2}$ (double) edges outside of the cycle with $0 \leq \beta < h - l$.

The largest expectation will occur when all of the edges in the combined cycles are double edges except for one with multiplicity $m + n - (4l - 2)$. The corresponding expectation for such a graph is

$$(m + n - 4l + 1)! = \frac{(m + n - 4l + 2)!}{2 \alpha + \beta + 2l + 1} \alpha \beta \sum_{r=0}^\alpha \sum_{s=0}^\beta p^{\alpha - r} q^r \frac{1}{r+1}(\alpha r) \frac{1}{s+1}(\beta s) \sim \sqrt{2} \binom{m + n - 4l + 2}{e} \frac{1}{4 + 2l + 1} \alpha \beta \sum_{r=0}^\alpha \sum_{s=0}^\beta p^{\alpha - r} q^r \frac{1}{r+1}(\alpha r) \frac{1}{s+1}(\beta s)$$

by Sterling’s Formula. Comparing this to the term for the same cycle of $2l$ edges but where each edge has exact multiplicity two,

$$\sqrt{2} \binom{m + n - 4l + 2}{e} \frac{1}{4 + 2l + 1} p^l q^l \sum_{r=0}^{h - l} \sum_{s=0}^{h - l - r} p^{h - l - r} q^r \frac{1}{r+1}(h - l - r) \frac{1}{s+1}(h - l - s)$$

by Sterling’s Formula. Comparing this to the term for the same cycle of $2l$ edges but where each edge has exact multiplicity two,
\[
\frac{\sqrt{2} \left( \frac{m+n-4l+2}{e} \right) \frac{h+k-2l-a-\beta}{h+k-2l-a-\beta} \left( \frac{m+n-4l+2}{e} \right) \left( \frac{2h+2k-4l+2}{ep} \right)^{h+k-2l-a-\beta} \left( \frac{2h+2k-4l+2}{e} \right)^{h+k-2l-a-\beta} \sum_{\alpha=0}^{h-1} \sum_{\beta=0}^{k-l-1} \frac{ep}{2h+2k-4l+2}^{h+k-2l-a-\beta}}{2(h-l)(k-l) \frac{q^l}{h+k-2l+1} \frac{p^{r-1}}{e}} \leq \sqrt{2} \left( \frac{16k^4}{e^2p} \right) \]
case,

\[
E_{\alpha\beta} a^{\alpha-a+1} b^{\beta-b+1} 2^{p} q^{l} \sum_{r=0}^{a} p^{a-r} q^{r} \binom{a}{r} \sum_{s=0}^{b} p^{b-s} q^{s} \binom{b}{s} \sqrt{2}\left(\frac{2a+2\beta-2a-2b+4}{e}\right)^{\alpha+\beta-a-b+2} (h-k)^{\alpha+\beta-a-b} p^{a+b}
\]

\[
\leq \sqrt{2}\left(\frac{4k^{2}}{ep}\right)^{\alpha+\beta-a-b} \left(\frac{4k^{2}}{e}\right)^{2}.
\]

Summing over all possible \(a\), \(b\) and \(l\),

\[
\sum_{l=2}^{h-1} \sum_{a=0}^{h-1-\alpha-\beta} \sum_{b=0}^{h-1-\alpha-\beta} \sqrt{2}\left(\frac{4k^{2}}{ep}\right)^{\alpha+\beta-a-b} \left(\frac{4k^{2}}{e}\right)^{2} = \sqrt{2}\sum_{l=2}^{h-1} \left(\frac{4k^{2}}{ep}\right)^{\alpha+\beta} \left(\frac{4k^{2}}{e}\right)^{2} \sum_{a=1}^{h-1} \sum_{b=1}^{h-1} \left(\frac{ep}{4k^{2}}\right)^{\alpha+\beta-1}
\]

\[
\leq \sqrt{2}\sum_{l=2}^{h-1} \alpha\beta \left(\frac{4k^{2}}{ep}\right)^{\alpha+\beta} \left(\frac{4k^{2}}{e}\right)^{2} \left(\frac{ep}{4k^{2}}\right)^{\alpha+\beta-1}
\]

\[
\leq \sum_{l=2}^{h-1} \sqrt{2}k^{2} \left(\frac{64k^{6}}{e^{3}p}\right)
\]

\[
\leq \sqrt{2}\left(\frac{64hk^{8}}{e^{3}p}\right).
\]

This completes the proof. \(\square\)

When \(q_{n}/p_{n} \to 0\), the formula for the covariance can be simplified further.

**Lemma 3.3.** Let \(\{p_{n} : n \geq 1\}\) and \(\{q_{n} : n \geq 1\}\) be two sequences of positive integers such that \(p_{n} \to \infty\) and \(q_{n}/p_{n} \to 0\). For each \(n\), let \(X_{n} = (x_{ij})\) be a \(p_{n} \times q_{n}\) matrix where \(x_{ij}\) are independent standard Gaussian random variables. Then for integers \(h \geq 1\) and \(k \geq h\),

\[
\text{Cov}(tr(X_{n}^{h} X_{n}^{h}), tr(X_{n}^{k} X_{n}^{k})) = 2hk(p^{h+k-1}q + pq^{h+k-1}) + \delta(h, k),
\]

where \(\delta(h, k) \leq C\frac{e^{h+k}2^{h+k+1}}{\sqrt{h}}\).

**Proof.** When \(q_{n}/p_{n} \to 0\) the \(l = 2\) term in the last sum of the covariance is of larger order than all the terms with larger \(l\). Comparing any of the \(l \neq 2\) terms to the \(l = 2\) term,

\[
2(h-l)(k-l)p^{h-l}q^{l} \sum_{r=0}^{h-l} \sum_{s=0}^{k-l} p^{h-l-r} q^{l-r} \binom{h-l}{r} \binom{k-l}{s} p^{k-l-s} q^{l-s} \binom{k-l-1}{s} \leq \frac{(h-l)(k-l)p^{h+k-2l}}{(h-2)(k-2)p^{h+k-4}} \leq \frac{2^{h+k-2l}}{h-k+2l}
\]
\[ \left( \frac{q}{p} \right)^{l-2} \]

Summing over all \( l \):

\[ \sum_{l=3}^{h-1} \left( \frac{q}{p} \right)^{l-2} \leq h \left( \frac{q}{p} \right)^{h-2} \rightarrow 0, \text{ since } q/p \rightarrow 0 \text{ as } n \rightarrow \infty. \]

When \( q/p \rightarrow 0 \) the first term in the covariance that came from case (1) is of larger order than each of the remaining two terms from cases (2) and (3). First, the ratio between the second and third terms will be shown to converge to 0. Then the ratio of the third to first term will be shown to also converge to 0. Comparing the term from (2) to the term from (3),

\[
\begin{align*}
2(k-h)p^h q^h \sum_{r=0}^{k-h} p^{r-h} q^{r+1} \frac{1}{r+1} (k-h)^{r} (k-h-1)^{r-1} \\
2(h-2)(k-2)p^2 q^2 \sum_{r=0}^{h-2} \sum_{s=0}^{k-h} p^{h-2-s} q^{s+1} \frac{1}{s+1} (h-2)^{r} (h-3)^{r-1} p^{s} q^{s} \frac{1}{s+1} (k-s)^{r} (k-s-1)^{r-1} \\
= \frac{2(k-h)p^h q^h}{2(h-2)(k-2)p^2 q^2} \sum_{r=0}^{h-2} \sum_{s=0}^{k-h} p^{h-2-s} q^{s+1} \frac{1}{s+1} (h-2)^{r} (h-3)^{r-1} p^{s} q^{s} \frac{1}{s+1} (k-s)^{r} (k-s-1)^{r-1} \\
\leq \frac{p^h q^h}{(h-2)p^{h-2} q^2} \\
\leq \frac{q^{h-2}}{(h-2)p^{h-2}}
\end{align*}
\]

which tends to zero as \( n \rightarrow \infty \). Comparing the term from case (3) to the term from (1),

\[
\begin{align*}
2(h-2)(k-2)p^2 q^2 \sum_{r=0}^{h-2} \sum_{s=0}^{k-h} p^{h-2-s} q^{s+1} \frac{1}{s+1} (h-2)^{r} (h-3)^{r-1} p^{s} q^{s} \frac{1}{s+1} (k-s)^{r} (k-s-1)^{r-1} \\
2hk \sum_{r=0}^{h-1} \sum_{s=0}^{h-1} p^{h-r} q^{r+1} \frac{1}{r+1} (h)^{r} (h-1)^{r-1} p^{s} q^{s} \frac{1}{s+1} (k)^{r} (k-1)^{r-1} \\
\leq \frac{q}{p}
\end{align*}
\]

The expression for the covariance may be further simplified by finding bounds on the only remaining sum. First, bound the individual sums that appear in the covariance.

\[
\begin{align*}
\sum_{r=1}^{h-2} p^{r-h} q^{r+1} \frac{1}{r+1} (h)^{r} (h-1)^{r-1} \\
= p^h q \sum_{r=1}^{h-2} \frac{q}{p} \frac{1}{r+1} (h)^{r} (h-1)^{r-1} \\
= p^{h-1} q^2 \sum_{r=1}^{h-2} \frac{1}{r+1} \frac{h}{r+1} (h-1)! (h-1-r)! \\
\leq p^{h-1} q^2 \sum_{r=1}^{h-2} \frac{1}{r+1} r!(h-r)! (r+1)! (h-1-r)! \\
= p^{h-1} q^2 \sum_{r=1}^{h-2} \frac{h}{r+1} (h-1)! \\
\end{align*}
\]
\[\begin{align*}
&= p^{h-1}q^{1/2} \sum_{r=1}^{h-2} \binom{h}{h-r} \binom{h}{r+1} \\
&\leq p^{h-1}q^{1/2} \binom{2h}{h+1} \\
&\sim \frac{p^{h-1}q^{1/2}}{h} \left(\frac{\sqrt{4\pi}he^{-2h}(2h)^{2h}}{\sqrt{2\pi(h+1)e^{-(h+1)}}} \right) \\
&= \frac{p^{h-1}q^{1/2}2^{2h}}{h^{1/2}},
\end{align*}\]

where Sterling’s formula was used in the second to last line. Similarly,

\[
\sum_{s=1}^{k-2} p^{k-s-1} q^s \frac{1}{s+1} \binom{k}{s} \binom{k-1}{s} \sim \left( p^{h-2} q^{k+2k}\right). 
\]

Then the covariance is bounded as follows,

\[
2hk \sum_{r=0}^{h-1} \sum_{s=0}^{k-2} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h-1}{r} \binom{h-1}{r} p^{k-s-1} q^s \frac{1}{s+1} \binom{k}{s} \binom{k-1}{s} \\
= 2hk \left( p^h q + pq^h + \sum_{r=1}^{h-2} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h-1}{r} \binom{h-1}{r} \right) \left( p^{k-1} + q^{k-1} + \sum_{s=1}^{k-2} p^{k-s-1} q^s \frac{1}{s+1} \binom{k}{s} \binom{k-1}{s} \right) \\
= 2hk \left( p^{h+k-1} q + p^h q^k + pq^h + \sum_{s=1}^{k-2} p^{k-s-1} q^s \frac{1}{s+1} \binom{k}{s} \binom{k-1}{s} \right) \\
+ q^{k-1} \sum_{r=1}^{h-2} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h-1}{r} \binom{h-1}{r} + \sum_{s=1}^{k-2} p^{k-s-1} q^s \frac{1}{s+1} \binom{k}{s} \binom{k-1}{s} \sum_{r=1}^{h-2} p^{h-r} q^{r+1} \frac{1}{r+1} \binom{h-1}{r} \binom{h-1}{r} \\
= 2hk \left( p^{h+k-1} q + pq^{h+k-1} + p^k q^h + p^h q^k \right) \\
+ \left( \frac{p^{h+k-2} q^{2k+1} \sqrt{2^{2h+1}h}}{\sqrt{k}} \frac{p^{h-k-2} q^{2k+1} \sqrt{h}}{\sqrt{k}} \frac{p^{h-k-2} q^{2k+1} \sqrt{h}}{\sqrt{k}} \frac{p^{h-k-2} q^{2k+1} \sqrt{h}}{\sqrt{k}} \frac{p^{h-k-2} q^{2k+1} \sqrt{h}}{\sqrt{k}} \right) \\
:= 2hk(p^{h+k-1} q + pq^{h+k-1}) + \delta(h,k) \\
\leq 2hk(p^{h+k-1} q + pq^{h+k-1}) + C\left( \frac{2^{2h+1}k}{\sqrt{k}} \right). 
\]

4. Appendix

**Lemma 4.1.** Let \( pq = o(n) \) and \( l = \frac{\log p}{\log n/pq} \). Then \( \sum_{j=1}^{l} \left( \frac{Cpq}{n} \right)^{1/j} \) converges to 0 as \( n \to \infty \) for any constant \( C \).
Proof.

\[
\sum_{j=1}^{l} \left( \frac{Cpq}{n} \right)^j \frac{1}{j} = \sum_{k=0}^{l-1} \left( \frac{Cpq}{n} \right)^k \frac{1}{k+1}
\]

\[
= \sum_{k=0}^{l-1} \int \left( \frac{Cpq}{n} \right)^k \, d \left( \frac{Cpq}{n} \right)
\]

\[
= \int \sum_{k=0}^{l-1} \left( \frac{Cpq}{n} \right)^k \, d \left( \frac{Cpq}{n} \right)
\]

\[
\leq \int \frac{1 - \left( \frac{Cpq}{n} \right)^i}{1 - \left( \frac{Cpq}{n} \right)} \, d \left( \frac{Cpq}{n} \right)
\]

\[
= \log(1 - \frac{Cpq}{n}) \rightarrow 0
\]

So that \(\sum_{j=1}^{l} \left( \frac{Cpq}{n} \right)^j \frac{1}{j}\) converges to 0 as \(n \to \infty\) for any constant \(C\). □

Lemma 4.2. Let \(f(u, v)\) be a real-valued function. Suppose the three second-order derivatives of \(f\) exist, bounded below and above by \(-M\) and \(M\), respectively, over \([a, b] \times [c, d]\). Then

\[
\frac{1}{n^2} \sum_{j=j_1}^{j_2} \sum_{i=i_1}^{i_2} f \left( \frac{j+1}{n}, \frac{i}{n} \right) = \int_{j_1/n}^{(j_2+1)/n} \int_{i_1/n}^{(i_2+1)/n} f(x, y) \, dx \, dy - \frac{1}{2n^2} \sum_{j=j_1}^{j_2} \sum_{i=i_1}^{i_2} f' \left( \frac{j}{n}, \frac{i}{n} \right) - \frac{1}{2n^2} \sum_{j=j_1}^{j_2} \sum_{i=i_1}^{i_2} f'' \left( \frac{j}{n}, \frac{i}{n} \right) + \varepsilon,
\]

where \(|\varepsilon| \leq (i_2 - i_1)(j_2 - j_1)M/n^4\) for any \(i_1, i_2, j_1,\) and \(j_2\) such that \(na \leq i_1 < i_2 \leq nb - 1\) and \(nc \leq j_1 < j_2 \leq nd - 1\).

For a proof see [7] Lemma 2.2.

Lemma 4.3. Let \(\Gamma(x), x > 0\) be the standard Gamma function. Then for all \(n \geq 1,\)

\[
(i) \quad 1 - \frac{1}{6n} < \frac{\Gamma(n + (1/2))}{\sqrt{n}\Gamma(n)} < 1
\]

\[
(ii) \quad \left| \frac{\Gamma((n+1)/2)}{\sqrt{n/2}\Gamma(n/2)} - 1 \right| < \frac{3}{5n}.
\]

For a proof see [7] Lemma 2.1.

Lemma 2.6 in [7] showed that under the assumption \(p, q = o(\sqrt{n})\), the constant part of the density ratio, \(K_n\) was equal to \(\exp \left[ -\frac{p^2 q + p q^2}{4n} + o(1) \right]\) for sufficiently large \(n\). When \(p\) and \(q\) are no longer assumed to be individually \(o(\sqrt{n})\) and instead \(pq = o(n)\) an extended result is necessary. The approach follows that of Jiang but involves more technical calculations.
Lemma 4.4. Let \( pq = o(n) \). Set

\[
K_n = \left( \frac{2}{n} \right)^{pq/2} \prod_{j=1}^{q} \frac{\Gamma((n-j+1)/2)}{\Gamma((n-p-j+1)/2)}.
\]

Then

\[
\log(K_n) = -\sum_{j=1}^{q} \frac{pq^{j+1} + pq^{j+1}}{2(j+1)n^j} + o(1),
\]

for sufficiently large \( n \).

**Proof.** First consider the case when \( p = 2k \) is even. Using the property that \( \Gamma(x+1) = x\Gamma(x) \),

\[
K_n = \left( \frac{2}{n} \right)^{pq/2} \prod_{j=1}^{q} \frac{\Gamma((n-j+1)/2)}{\Gamma((n-p-j+1)/2)}
= \prod_{j=0}^{q-1} \prod_{i=1}^{k} \left( 1 - \frac{2i+j}{n} \right),
= e^{B_n},
\]

where \( B_n := \sum_{j=0}^{q-1} \sum_{i=1}^{k} \log(1 - \frac{2i+j}{n}) \). Let \( f(s,t) = \log(1 - 2s - t) \) with \( 2s + t < 1 \). Now, if \( s \in [0,p/n] \) and \( t \in [0,q/n] \) then \( 2s + t \leq \frac{2pn+q}{n} \). Consider the partial derivatives of \( f \).

\[
\frac{\partial f}{\partial s} = \frac{2}{(1-2s-t)^2}, \quad \frac{\partial f}{\partial t} = \frac{-2}{(1-2s-t)^2},
\]

and

\[
\left| \frac{\partial^2 f}{\partial s^2} \right| = \left| \frac{-4}{(1-2s-t)^2} \right| \leq 5, \quad \left| \frac{\partial^2 f}{\partial t^2} \right| = \left| \frac{2}{(1-2s-t)^2} \right| \leq 5
\]

for all \((s,t) \in [0,p/n] \times [0,q/n]\), since \( n \) is sufficiently large. Then lemma 4.2 gives

\[
\frac{1}{n^2} B_n = \frac{1}{n^2} \sum_{j=0}^{q-1} \sum_{i=1}^{k} \log \left( 1 - \frac{2i+j}{n} \right)
= \int_0^{q/n} \int_{1/n}^{(k+1)/n} \log(1 - 2s - t)dsdt - \frac{1}{2n^3} \sum_{j=0}^{q-1} \sum_{i=1}^{k} -2n \sum_{j=0}^{q-1} \sum_{i=1}^{k} n - 2i - j + \epsilon
= \int_0^{q/n} \int_{1/n}^{(k+1)/n} \log(1 - 2s - t)dsdt + \frac{3kq}{2n^3} + \epsilon
\]

where \( |\epsilon| \leq \frac{5(k-1)(q-1)}{n^4} \). Multiplying through by \( n^2 \),

\[
B_n = n^2 \int_0^{q/n} \int_{1/n}^{(k+1)/n} \log(1 - 2s - t)dsdt + \frac{3kq}{2n^3} + n^2 \epsilon = n^2 \int_0^{q/n} \int_{1/n}^{(k+1)/n} \log(1 - 2s - t)dsdt + o(1).
\]

Taking \( b = -2s, c = -t \),

\[
B_n = n^2 \int_0^{q/n} \int_{-2/n}^{-(k+1)/n} \log(1 + b + c) \frac{1}{2} dbdc + o(1)
\]
\[
\frac{n^2}{2} \int_0^v \int_0^u \log(1 + s + t) \, ds \, dt - \frac{n^2}{2} \int_0^v \int_0^{-2/n} \log(1 + s + t) \, ds \, dt + o(1)
\]

since \( n \) is sufficiently large, where \( u = \frac{(p+2)}{n} \) and \( v = \frac{2}{n} \). By Taylor’s Theorem, \( \log(1 + s + t) \leq \sum_{j=1}^{l} (-1)^{j+1} \frac{(s+t)^j}{j} \). Then the first integral is bounded as follows,

\[
\frac{n^2}{2} \int_0^v \int_0^u \log(1 + s + t) \, ds \, dt \leq \frac{n^2}{2} \sum_{j=1}^{l} \int_0^v \int_0^u \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \, ds \, dt
\]

\[
= \frac{n^2}{2} \sum_{j=1}^{l} \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \left[ (u + v)^{j+2} - u^{j+2} - v^{j+2} \right]
\]

\[
= \frac{n^2}{2} \sum_{j=1}^{l} \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \left( \frac{j+2}{r} \right) u^{j+2-r} v^r
\]

\[
= \frac{n^2}{2} \sum_{j=1}^{l} \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \left( \frac{j+2}{r} \right) \left( \frac{-(p+2)}{n} \right)^{j+2-r} \left( \frac{-q}{n} \right)^r
\]

\[
= \frac{n^2}{2} \sum_{j=1}^{l} \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \left( \frac{j+2}{r} \right) \left( \frac{-(p+2)}{n} \right)^{j+2-r} \left( \frac{-q}{n} \right)^r
\]

\[
= \frac{n^2}{2} \sum_{j=1}^{l} \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \left( \frac{j+2}{r} \right) \left( \frac{-(p+2)}{n} \right)^{j+2-r} \left( \frac{-q}{n} \right)^r
\]

where for the third equality the fact that \( \int_0^v \int_0^u (s+t)^j \, ds \, dt = \frac{1}{(j+1)(j+2)} \left[ (u + v)^{j+2} - u^{j+2} - v^{j+2} \right] \) was used. Similarly, for the second integral,

\[
\frac{n^2}{2} \int_0^v \int_0^{-2/n} \log(1 + s + t) \, ds \, dt \leq \frac{n^2}{2} \sum_{j=1}^{l} \int_0^v \int_0^{-2/n} \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \, ds \, dt
\]

\[
= \frac{n^2}{2} \sum_{j=1}^{l} \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \left[ (-\frac{2}{n} + v)^{j+2} - (-\frac{2}{n})^{j+2} - v^{j+2} \right]
\]

\[
= \frac{n^2}{2} \sum_{j=1}^{l} \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \left( \frac{j+2}{r} \right) \left( \frac{-2}{n} \right)^{j+2-r} \left( \frac{-q}{n} \right)^r
\]

\[
= \frac{1}{2} \sum_{j=1}^{l} \frac{(-1)^{j+1}}{j} \frac{1}{(j+1)(j+2)} \frac{t^j}{n} \left( \frac{j+2}{r} \right) 2^{j+2-r} q^r.
\]
Therefore,

\[ B_n = \frac{n^2}{2} \int_0^\infty \int_0^u \log(1 + s + t) \, ds \, dt - \frac{n^2}{2} \int_0^\infty \int_0^{-2/n} \log(1 + s + t) \, ds \, dt + o(1) \]

\[ \leq -\frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)(j+2)n^j} \sum_{r=1}^{j+1} \binom{j+2}{r} \left( p + 2 \right)^{j+2-r} q^r + \frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)(j+2)n^j} \sum_{r=1}^{j+1} \binom{j+2}{r} 2^{j+2-r} q^r + o(1) \]

\[ = -\frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)(j+2)n^j} \sum_{r=1}^{j+1} \sum_{s=0}^{j+1-r} \binom{j+2}{r} \left( j + 2 - r \right) \left( s \right) p^{j+2-r}s^s q^r + \frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)(j+2)n^j} \sum_{r=1}^{j+1} \binom{j+2}{r} 2^{j+2-r} q^r + o(1) \]

\[ = -\frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)(j+2)n^j} \left[ \left( j + 2 \right) pq^{j+1} + \left( j + 2 \right) \sum_{s=0}^{j+1} \binom{j+1}{s} p^{j+1-s} q \right.
\]

\[ + \sum_{r=2}^{j+1} \sum_{s=0}^{j+1-r} \binom{j+2}{r} \left( j + 2 - r \right) \left( s \right) p^{j+2-r}s^s q^r \left( 1 \right) \] + o(1) \]

\[ = -\frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)(j+2)n^j} \left[ \left( j + 2 \right) pq^{j+1} + \left( j + 2 \right) p^{j+1} q + \left( j + 2 \right) \sum_{s=1}^{j+1} \binom{j+1}{s} p^{j+1-s} q \right.
\]

\[ + \sum_{r=2}^{j+1} \sum_{s=0}^{j+1-r} \binom{j+2}{r} \left( j + 2 - r \right) \left( s \right) p^{j+2-r}s^s q^r \left( 1 \right) \] + o(1) \]

\[ = -\frac{1}{2} \sum_{j=1}^l \frac{p^{j+1} q + pq^{j+1}}{j(j+1)n^j} + o(1), \]

where the remaining sums were bounded as follows,

\[ \frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)n^j} \sum_{s=1}^{j+1} \binom{j+1}{s} p^{j+1-s} q^s = \frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)n^j} \sum_{s=1}^{j+1} \binom{j+1}{s} p^{-s} 2^s \]

\[ = \frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)n^j} \sum_{s=1}^{j+1} \binom{j+1}{s} \left( \frac{2}{p} \right)^{s-1} \]

\[ \leq \frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)n^j} \sum_{s=1}^{j+1} \binom{j+1}{s} \left( \frac{1}{s} \right) \]

\[ \leq \frac{1}{2} \sum_{j=1}^l \frac{1}{j(j+1)n^j} \sum_{s=1}^{j+1} \binom{j+1}{s} \left( \frac{1}{s} \right) \]
\[ \frac{1}{2} \sum_{j=1}^{l} \frac{1}{j(j+1)n} 2p^j q^{2j+1} \leq 2 \sum_{j=1}^{l} \frac{1}{j(j+1)} \left( \frac{2pq}{n} \right)^j, \]

which tends to zero by lemma 4.1. Similarly,

\[ \sum_{r=2}^{j} \sum_{s=0}^{j+1-r} \binom{j+2}{r} \binom{j+2-r}{s} p^{j+2-r-s} q^r \to 0. \]

Next, suppose that \( p = 2k - 1 \) is odd. Let

\[ C_n = \prod_{j=1}^{q} \frac{\Gamma((n-j+p+1)/2)}{\Gamma((n-j+p)/2) \sqrt{(n-j+p)/2}} \]

Then by lemma 4.3 the jth term of the product \( C_{n,j} \) has the following property:

\[ 1 - \frac{1}{n+p-q} \leq C_{n,j} \leq 1 + \frac{1}{n+p-q} \]

for all \( j = 1, \ldots, q \) as long as \( p + q \leq n - 3 \). Therefore,

\[ \left( 1 - \frac{1}{n+p-q} \right)^q \leq C_n \leq \left( 1 + \frac{1}{n+p-q} \right)^q. \]

Note that since \( pq = o(n) \),

\[ \left( 1 + \frac{1}{n+p-q} \right)^q = \exp \left\{ q \log \left( 1 + \frac{1}{n+p-q} \right) \right\} = 1 + O \left( \frac{q}{n+p-q} \right). \]

Similarly, \( (1 - \frac{1}{n+p-q})^q = 1 + O \left( \frac{q}{n+p-q} \right) \). Thus \( C_n = 1 + O \left( \frac{q}{n+p-q} \right) \). So that in this case,

\[ K_n = \frac{1}{C_n} \left( 2 \right)^{pq} \prod_{j=1}^{q} \frac{\Gamma((n-j+1)/2)}{\Gamma((n-j+2k+1)/2) \sqrt{(n-j+2k+1)/2}} \]

\[ = \left[ \prod_{j=1}^{q} \prod_{i=1}^{k} n - 2i - j + 1 \right] \left[ \prod_{j=1}^{q} n - j - 2k + 1 \right]^{-1/2} \left( 1 + O \left( \frac{q}{n+p-q} \right) \right) \]

\[ := K'_n K''_n, \]

where

\[ \log(K''_n) = -\frac{1}{2} \sum_{j=1}^{q} \log \left( 1 - \frac{j+2k-1}{n} \right) \]

\[ = -\frac{1}{2} \sum_{j=1}^{q} \frac{j+2k-1}{n} + O \left( \frac{(j+2k-1)^2}{n^2} \right). \]
\[
- \frac{1}{2} \sum_{j=1}^{q} \frac{-j + 2k - 1}{n} + O\left(\frac{(p + q)^2}{n^2}\right)
\]

\[
= \frac{1}{2n} \sum_{j=1}^{q} j + 2k - 1 + o(1)
\]

\[
\leq \frac{q(q + p)}{2n} + o(1)
\]

\[
= o(1).
\]

In notation \( K'_n \) is the same as \( K_n \) from the previous case, now with \( k = (p + 1)/2 \). The conclusion of the lemma holds for \( K'_n \). This can be seen by defining \( B'_n \) so that

\[
e^{B'_n} = K'_n = \prod_{j=0}^{q-1} \left( 1 - \frac{2i + j}{n} \right).
\]

\( B'_n \) is almost identical to \( B_n \), so that

\[
B'_n = \frac{n^2}{2} \int_0^u \int_0^v \log(1 + s + t)dsdt - \frac{n^2}{2} \int_0^v \int_0^{-2/n} \log(1 + s + t)dsdt + o(1),
\]

where \( v = -q/n \) as before, but \( u = -(p + 4)/2 \). Using Taylor’s expansion for the logarithm,

\[
B'_n \leq -\frac{1}{2} \sum_{j=1}^{j} \frac{1}{j(j+1)(j+2)n^3} \sum_{r=1}^{j+1} \binom{j+2}{r} (p + 4)^{j+2-r} q^r
\]

\[
+ \frac{1}{2} \sum_{j=1}^{j} \frac{1}{j(j+1)(j+2)n^3} \sum_{r=1}^{j+1} \binom{j+2}{r} 2^{j+2-r} q^r + o(1)
\]

\[
= -\frac{1}{2} \sum_{j=1}^{j} \frac{1}{j(j+1)(j+2)n^3} \sum_{r=1}^{j+1} \binom{j+2}{r} (j+2-r) \binom{j+2-r}{s} p^{j+2-r-s} 4^s q^r
\]

\[
+ \frac{1}{2} \sum_{j=1}^{j} \frac{1}{j(j+1)(j+2)n^3} \sum_{r=1}^{j+1} \binom{j+2}{r} 2^{j+2-r} q^r + o(1)
\]

\[
= -\frac{1}{2} \sum_{j=1}^{j} \frac{1}{j(j+1)(j+2)n^3} \left[ (j+2) p^{j+1} + (j+2) \sum_{s=1}^{j+1} \binom{j+1}{s} p^{j+1-s} 4^s q^r + (j+2) p q^{j+1}
\]

\[
+ (j+2) 4q^{j+1} + \sum_{r=2}^{j+1} \sum_{s=0}^{j+1-r} \binom{j+2}{r} \binom{j+2-r}{s} p^{j+2-r-s} 4^s q^r + \sum_{r=2}^{j} \binom{j+2}{r} 4^{j+2-r} q^r
\]

\[
+ \frac{1}{2} \sum_{j=1}^{j} \frac{1}{j(j+1)(j+2)n^3} \left[ (j+2) + \sum_{r=1}^{j} \binom{j+2}{r} 2^{j+2-r} q^r \right] + o(1).
\]
Note that \((j + 2) \sum_{s=1}^{j+1} \binom{j+1}{s} p^{j+1-s} q^s \to 0\) and \(\sum_{r=2}^{j+1} \frac{\sum_{s=0}^{j+1-r} \binom{j+2}{r} (j+2-r) p^{j+2-r} q^r}{(j+1)(j+2)} \to 0\) as before, and

\[
\sum_{j=1}^{l} \frac{4q^{j+1}}{j(j+1)n^j} = \frac{4q^2}{n} \sum_{j=0}^{l-1} \left( \frac{q}{n} \right)^j \frac{1}{(j+1)(j+2)} \leq \frac{4q^2}{n} \sum_{j=0}^{\infty} \left( \frac{q}{n} \right)^j = \frac{4q^2}{n} \frac{1}{1-q/n}
\]
tends to 0 as well. Finally,

\[
\sum_{j=1}^{l} \frac{1}{j(j+1)(j+2)n^j} \sum_{r=2}^{j} \binom{j+2}{r} p^{j+2-r} q^r \leq \sum_{j=1}^{l} \frac{16}{jn^j} \sum_{r=2}^{j} \binom{j}{r} 4^{j-r} q^r \leq \sum_{j=1}^{l} \frac{16(4+q)^j}{jn^j}
\]
which tends to 0 since \(\sum_{j=1}^{l} \frac{q^j}{jn^j} \to 0\). Similarly, \(\sum_{j=1}^{l} \frac{1}{j(j+1)(j+2)n^j} \sum_{r=1}^{j} \binom{j+2}{r} 2^{j+2-r} q^r \to 0\). Therefore, \(\log(K_n') = -\sum_{j=1}^{l} \frac{p^{j+1} + q^{j+1}}{2j(j+1)n^j} + o(1) = \log(K_n)\).

**Lemma 4.5.** Let \(\{p_n : n \geq 1\}\) and \(\{q_n : n \geq 1\}\) be two sequences of positive integers such that \(p_n \to \infty\) and \(q_n \leq p_n\). For each \(n\), let \(X_n = (x_{ij})\) be a \(p_n \times q_n\) matrix where \(x_{ij}\) are independent standard Gaussian random variables. Let

\[
E_j = \mathbb{E} \left[ \frac{1}{n^j} \left( \frac{p + q + 1}{2j} tr(X^T X)^j - \frac{1}{2(j+1)} tr(X^T X)^{j+1} \right) \right].
\]

Then

\[
\sum_{j=1}^{l} E_j = \left( \sum_{j=1}^{l} \frac{p^{j+1} + q^{j+1}}{2j(j+1)n^j} + o(1) \right) \left( 1 + O \left( \frac{j^2}{p} \right) \right).
\]

**Proof.**

\[
E_j = \mathbb{E} \left[ \frac{1}{n^j} \left( \frac{p + q + 1}{2j} tr(X^T X)^j - \frac{1}{2(j+1)} tr(X^T X)^{j+1} \right) \right]
= \frac{p + q + 1}{2jn^j} \mathbb{E}[tr(X^T X)^j] - \frac{1}{2(j+1)n^j} \mathbb{E}[tr(X^T X)^{j+1}]
= \left( \frac{p + q + 1}{2jn^j} p^j q^j \sum_{s=0}^{j-1} \frac{1}{s+1} \left( \frac{q}{p} \right)^s \binom{j}{s} \binom{j-1}{s} \right) (1 + O (j^2/p))
- \left( \frac{1}{2(j+1)n^j} p^{j+1} q^j \sum_{s=0}^{j} \frac{1}{s+1} \left( \frac{q}{p} \right)^s \binom{j+1}{s} \binom{j}{s} \right) (1 + O (j+1)^2/p))
= \frac{p + q + 1}{2jn^j} \left( p^j q^j + p^j q^{j+1} \sum_{s=1}^{j-2} p^{j-s} q^{s+1} \frac{1}{s+1} \binom{j}{s} \binom{j-1}{s} \right) (1 + O (j^2/p))
- \frac{1}{2(j+1)n^j} \left( p^{j+1} q + q^{j+1} + \sum_{s=1}^{j-1} p^{j+1-s} q^{s+1} \frac{1}{s+1} \binom{j+1}{s} \binom{j}{s} \right) (1 + O (j^2/p))
= \left( p^{j+1} q + p^2 q^j + p^2 q^j + p^{j+1} q + q^{j+1} + \sum_{s=1}^{j-1} p^{j+1-s} q^{s+1} \frac{1}{s+1} \binom{j+1}{s} \binom{j}{s} \right) (1 + O (j^2/p))
- \frac{p + q + 1}{2jn^j} \left( p^j q^j + p^j q^{j+1} \sum_{s=1}^{j-2} \frac{q}{p} \frac{1}{s+1} \binom{j}{s} \binom{j-1}{s} \right)
\]
\[-\frac{1}{2(j+1)n^j} p^{j+1} q \sum_{s=1}^{j-1} \left( \frac{q}{p} \right)^s \frac{1}{s+1} \binom{j+1}{s} \binom{j}{s} \left( 1 + O\left(j^2/p\right) \right) \]
\begin{align*}
&= \left( \frac{pq^{j+1} + pj^{j+1} q}{2(j+1)n^j} \right) + \frac{p+q+1}{2jn^j} p^j q \sum_{s=1}^{j-2} \left( \frac{q}{p} \right)^s \frac{1}{s+1} \binom{j}{s} \binom{j-1}{s} \\
&\quad - \frac{1}{2(j+1)n^j} p^{j+1} q \sum_{s=1}^{j-1} \left( \frac{q}{p} \right)^s \frac{1}{s+1} \binom{j+1}{s} \binom{j}{s} \left( 1 + O\left(j^2/p\right) \right) \\
&= \left( \frac{pq^{j+1} + pj^{j+1} q}{2(j+1)n^j} + o(1) \right) \left( 1 + O\left(j^2/p\right) \right),
\end{align*}

where the last line follows by bounding the sums using Sterling’s formula as follows,

\[
p^j q \sum_{s=1}^{j-2} \left( \frac{q}{p} \right)^s \frac{1}{s+1} \binom{j}{s} \binom{j-1}{s} \leq \frac{p^{j-1} q^2}{\sqrt{\pi j^{3/2}}} 2^{2j} \left( 1 + O\left(1/(j-1)\right) \right).
\]

Lemma 4.1 then implies convergence to 0 even when summed over all \( j \). Similarly,

\[
p^{j+1} q \sum_{s=1}^{j-1} \left( \frac{q}{p} \right)^s \frac{1}{s+1} \binom{j+1}{s} \binom{j}{s} \leq \frac{p^{j+2} q^2}{\sqrt{\pi (j+1)^{3/2}}} 2^{2j+2} \left( 1 + O\left(1/j\right) \right).
\]

\[\square\]

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E-mail address: kathryn.lockwood@case.edu

(Kathryn Lockwood) DEPARTMENT OF MATHEMATICS, CASE WESTERN RESERVE UNIVERSITY 231 YOST HALL, CLEVELAND, OH 44106 USA