INTRINSIC RECTIFIABILITY VIA FLAT CONES IN THE HEISENBERG GROUP

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Abstract. We give a geometric criterion for a topological surface in the first Heisenberg group to be an intrinsic Lipschitz graph, using planar cones instead of the usual open cones.

Introduction

We identify the first Heisenberg group $\mathbb{H}$ with the manifold $\mathbb{R}^3$ endowed with the group operation

$$(x, y, z)(x', y', z') = \left(x + x', y + y', z + z' + \frac{xy' - x'y}{2}\right).$$

The inverse of $(x, y, z)$ is $(-x, -y, -z)$. If $E \subset \mathbb{H}$ and $p \in \mathbb{H}$, we write $pE$ for the set $\{pe : e \in E\}$. The one-parameter family of group automorphisms $\delta_\lambda : (x, y, z) \mapsto (\lambda x, \lambda y, \lambda^2 z)$ are called dilations.

In recent years, intrinsic Lipschitz graphs in $\mathbb{H}$ have gained more attention, see [6, 7, 9] and [5, 2, 4, 3]. Indeed, these graphs yield a robust notion of rectifiable set in the Heisenberg group, and more generally in Carnot groups. The definition of intrinsic Lipschitz graphs is inspired by the euclidean characterization of Lipschitz graphs by cones. Here, a cone is a set $E \subset \mathbb{H}$ with $\delta_\lambda(E) = E$ for all $\lambda > 0$. A set is an intrinsic Lipschitz graph if it has the following full cone property.

Definition 1 (Full cone property). A set $S \subset \mathbb{H}$ has the full cone property if there is an open cone $C$ with $-C = C$, $C \cap \{z = 0\} \neq \emptyset$ and, for all $p \in S$,

$$C \cap p^{-1}S = \emptyset.$$ 

An example of open cone, for some $\alpha > 0$, is the $\alpha$-full cone:

$$C(\alpha) := \{(x, y, z) : |y| < \alpha |x|, \ |z| < \alpha x^2/2\}.$$ 

In this paper, we consider flat cones

$$fC(\alpha) := \{(x, y, 0) : |y| < \alpha |x|\}.$$ 

Our aim is to compare the full cone property with a flat cone property defined as follows.

Definition 2 (Flat cone property). A set $S \subset \mathbb{H}$ has the $\alpha$-flat cone property if for all $p \in S$

$$fC(\alpha) \cap p^{-1}S = \emptyset.$$ 

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Note that, if $C$ is an open cone with $-C = C$ and $C \cap \{z = 0\} \neq \emptyset$, then there exists $\alpha > 0$ such that $C(\alpha) \subset C$, up to a rotation around the $z$ axis. Recall that rotations around the $z$ axis are isomorphisms of the homogeneous structure of $\mathbb{H}$ given by left-translations and dilations. Noting that $fC(\alpha) = C(\alpha) \cap \{z = 0\}$, it is then clear that an intrinsic Lipschitz graphs also have the flat cone property, up to a rotation around the $z$-axis. We will prove the inverse implication for topological surfaces.

**Theorem.** If $S \subset \mathbb{H}$ is a topological surface with the flat cone property, then it has locally the full cone property. Quantitatively, the $\alpha$-flat cone property implies locally the $\alpha/4$-full cone property.

**Remark 3.** If $S$ is a topological surface with the flat cone property, then it is an intrinsic graph and characteristic lines on $S$ are Lipschitz with uniformly bounded slope. The Theorem is thus a geometric version of the result of [1] in the first Heisenberg group.

**Remark 4.** The hypothesis in the Theorem that $S$ be a topological surface is crucial. We register two examples where our strategy fails and that we consider prototypical. The first one has the flat cone property, but not the full cone property at $0$ in the direction of the $x$ axis:

$$\{(x, 0, x^2), x \in [0, 1]\}.$$ 

Another example is the following set, which has the flat cone property, but is not an intrinsic Lipschitz graph:

$$\{(0, 0, 0) \cup \{(x, y, z) : x = 1\} \setminus \{(1, s, 0) : |s| < 1\}.$$ 

However, both sets are clearly intrinsic 1-codimensional-rectifiable in the sense of [6, Definition 3.16]. We do not know if the flat cone property implies intrinsic rectifiability.

We present a detailed proof in the subsequent sections. However, the expert reader could be satisfied with the following sketch.

Suppose $p = 0 \in S$: we want to find $\beta > 0$ and $0 \in U \subset \mathbb{H}$ open such that $C(\beta) \cap (S \cap U) = \emptyset$. We start by considering the intersection $S \cap H$, where $H = \{z = 0\}$ is the so called “horizontal plane”: for every $y_0$ close to $0$ there is $x \in [-y_0/\alpha, y_0/\alpha]$ such that $(x, y_0, 0) \in S$. Now, we notice a crucial fact: all the left-translations $\{(x, y_0, 0)H\}_{x \in \mathbb{R}}$, with $y_0$ fixed, are affine planes all containing a common line, which turns out to be inside the plane $\{y = 0\}$. Therefore,

$$\bigcap_{x \in [-y_0/\alpha, y_0/\alpha]} (x, y_0, 0)\mathbb{C}(\alpha)$$

is the union of half lines in the plane $\{y = 0\}$. By construction, these two half lines do not intersect $S$.

If we then consider all $y_0 \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$ small enough, we get a family of half lines in $\{y = 0\}$ that do not intersect $S$: the union of such a family yields a “vertical set” which, in a neighbourhood of $0$, coincides with $C(\alpha/2) \cap \{y = 0\}$. Although promising, this is still not enough to get a full cone. However, the same argument can be carried out in another system of coordinates $(x', y', z')$ for $\mathbb{H}$ in which the $x'$-axis is slightly tilted with respect to the $x$-axis; in other words, we consider $x' = x$, $y' = y + tx$, $z' = z$, where $t \in \mathbb{R}$. Indeed, for $t$ small, $S$ will still have the $(\alpha/2)$-flat cone property in the new coordinates and thus we obtain a vertical set in $\{y' = 0\}$ that does not intersect $S$ and coincides in a neighbourhood of $0$ with $C(\alpha/4) \cap \{y' = 0\}$.

The union of all these vertical sets coincides with $C(\alpha/4)$ near $0$, as desired. This is proved in Section 1. Of course, the arguments should be carried out in a uniform
way for $p$ close to 0 in $S$, this is the scope of Section 2. The full proof of the Theorem is in Section 3.

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1. Flat cones build up full cones

In this section, we show how we get truncated full cones from flat cones. It is worth mentioning [8], where another strategy to pass from flat cones to full open cones is discussed.

For $r, \beta > 0$, define the **truncated full cone**

$$C(\beta, r) := \{(x, y, z) : |y| < \beta|x|, |z| < \beta x^2/2, |x| < r\}$$

and the **vertical set**

$$vC(\beta, r) = \{(x, 0, ux^2/2) : |u| < \beta, |x| < r\}.$$

**Remark 5.** Note that if a set $S$ satisfies $C(\beta, r) \cap p^{-1}S = \emptyset$, for all $p \in S$, then, for each $o \in S$, the set $S \cap o\{|x| < r/2\}$ has the $\beta$-full cone property.

**Lemma 6.** Fix $\epsilon, \beta > 0$, then

$$vC\left(\frac{\beta}{2}, \frac{2\epsilon}{\beta}\right) \subseteq \bigcup_{|u| < \epsilon} \bigcap_{|s| < \beta^{-1}} (su, \eta, 0) C(\beta).$$

**Proof.** Pick $(x, 0, ux^2/2) \in vC(\beta/2, 2\epsilon/\beta)$, and notice that $|−ux| < \epsilon$. We will show that $(x, 0, ux^2/2) \in (−sux, −ux, 0) C(\beta)$, i.e. that $(sux, ux, 0)(x, 0, ux^2/2) \in C(\beta)$, for all $s \in (−\beta^{-1}, \beta^{-1})$. We have

$$(sux, ux, 0)(x, 0, ux^2/2) = (x + sux, ux, 0),$$

so we only need to prove that $|ux| < \beta|x + sux|$, which is clear as $|u| < \beta/2$, $|su| < 1/2$ and thus

$$\beta|x + sux| \geq \beta|x|(1 − |su|) \geq \frac{\beta}{2} |x| \geq |ux|.$$

For $t \in \mathbb{R}$, define the map $M_t(x, y, z) := (x, y + tx, z)$, which is a group automorphism $\mathbb{H} \to \mathbb{H}$, with inverse $M_{t^{-1}} = M_{−t}$. An immediate consequence of the above definitions is that, for all $r, \beta > 0$,

$$C(\beta, r) = \bigcup_{|t| \leq \beta} M_t(vC(\beta, r)).$$

2. Intrinsic graphs

Given $\Omega \subset \mathbb{R}^2$ and $\phi : \Omega \to \mathbb{R}$, the **intrinsic graph** of $\phi$ is the subset

$$\Gamma_\phi := \{(0, \eta, \tau)(\phi(\eta, \tau), 0, 0) : (\eta, \tau) \in \Omega\} = \{(\phi(\eta, \tau), \eta, \tau − \eta\phi(\eta, \tau)/2) : (\eta, \tau) \in \Omega\}.$$

Define the projection $\pi : \mathbb{H} \to \mathbb{R}^2$, $(x, y, z) \mapsto (y, z + xy/2)$. Notice that if $(\eta_0, \tau_0) = \pi(x_0, y_0, z_0)$, then $(x_0, y_0, z_0) = (0, \eta_0, \tau_0)(x_0, 0, 0)$.

**Proposition 7.** If $S$ is a topological surface with the flat cone property, then it is the intrinsic graph of a continuous function defined on an open subset of $\mathbb{R}^2$.

**Proof.** Since $\{(\xi, 0, 0) : \xi \in \mathbb{R}\} \subset C(\alpha)$, the map $\pi|S : S \to \mathbb{R}^2$ is injective. Thus, $S = \Gamma_\phi$ for some $\phi : \pi(S) \to \mathbb{R}$. As $S$ is a topological surface and by the Invariance of Domain Theorem, $\pi(S)$ is open and $\phi$ is continuous. □
As we explained in the introduction, in order to apply Lemma 6, we will use the fact that for every $y_0$ close to 0 there is $x \in [-y_0/\alpha,y_0/\alpha]$ such that $(x,y_0,0) \in S$:

**Proposition 8.** Fix $\eta_0, \tau_0, \beta > 0$ and let $\psi : [-\eta_0,\eta_0] \times [-\tau_0,\tau_0] \to \mathbb{R}$ be continuous, such that $\psi(0) = 0$ and $\Gamma_\psi$ has the $\beta$-flat cone property. Then, letting $\epsilon := \min\{\eta_0, \sqrt{\tau_0}\}$, for all $\eta \in [-\epsilon, \epsilon]$, there exists $s \in [-\beta^{-1}, \beta^{-1}]$ with $(s\eta, \eta, 0) \in \Gamma_\psi$.

**Proof.** Consider the function $\zeta(\eta, \tau) := \tau - \frac{\eta^2}{2\beta^2}$ (yielding the third coordinate of the point $(0, \eta, \tau)(\psi(\eta, \tau), 0)$). Notice that, since $\mathcal{C}(\beta) \cap \Gamma_\psi = \emptyset$, we have

$$\{(\eta, \tau) : \zeta(\eta, \tau) = 0\} \subset \mathbb{R}^2 \setminus \pi(\mathcal{C}(\beta)) = \{(\eta, \tau) : \eta^2 \geq 2\beta^2|\tau|\}.$$  

Moreover, by the choice of $\epsilon$, for $\eta \in [-\epsilon, \epsilon]$, there holds $\eta^2 < 2\beta \tau_0$ and therefore $\zeta(\eta, \pm\tau_0) \neq 0$. Since $\zeta$ is continuous, and $\zeta(0, \tau_0) = \tau_0 > 0 > -\tau_0 = \zeta(0, -\tau_0)$, we have for all $\eta \in [-\epsilon, \epsilon]$:

$$\zeta(\eta, \tau_0) > 0 > \zeta(\eta, -\tau_0).$$

Using once again the continuity of $\zeta$, it follows that for every $\eta \in [-\epsilon, \epsilon]$ there is $\tau \in [-\tau_0, \tau_0]$ with $\zeta(\eta, \tau) = 0$ and thus a point $(s\eta, \eta, 0) \in \Gamma_\psi$. By the $\beta$-flat cone property of $\Gamma_\psi$, either $\eta = 0$ (and we can take $s = 0$) or there holds $|s| \leq \beta^{-1}$.

We want to apply Proposition 8 and Lemma 6, not only to $S$ but also to $M_t(p^{-1}S)$ for $p$ in a neighborhood of 0 and $t$ in a compact interval. Lemma 10 allows us to do this.

**Remark 9.** Notice that, for $t \in [-\alpha/2, \alpha/2]$, there holds $\mathcal{C}(\alpha/2) \subset M_t \mathcal{C}(\alpha)$. In particular, if $S$ has the $\alpha$-flat cone property, then for such $t$, $M_t(p^{-1}S)$ has the $\alpha/2$-flat cone property, for every $t \in [-\alpha/2, \alpha/2]$ and $p \in \mathbb{R}$.

**Lemma 10.** Let $S \subset \mathbb{R}$ be a topological surface with the $\alpha$-cone property and $0 \in S$. Then there are $\eta_0, \tau_0 > 0$ such that, defining $V_\alpha := \pi^{-1}([-\eta_0, \eta_0] \times [-\tau_0, \tau_0])$, for all $t \in [-\alpha/2, \alpha/2]$ and $p \in S \cap V_\alpha$, there exists a continuous function $\phi_{t,p} : [-\eta_0, \eta_0] \times [-\tau_0, \tau_0] \to \mathbb{R}$ such that

$$\Gamma_{\phi_{t,p}} = M_t(p^{-1}S) \cap V_\alpha.$$

**Proof.** By Remark 9 and Proposition 7, for all $p \in S$ and $t \in [-\alpha/2, \alpha/2]$, $M_t(p^{-1}S)$ is an intrinsic graph over $\pi(M_t(p^{-1}S))$.

Denote by $D_r$ the disk of radius $r$ in $\mathbb{R}^2$. Let $\Phi : D_1 \to S$ be a local chart with $\Phi(0,0) = (0,0,0)$. For $p \in \Phi(D_1)$ and $t \in [-\alpha/2, \alpha/2]$, define

$$f_{t,p} : D_1 \to \mathbb{R}^2, \quad f_{t,p}(v) = \pi(M_t(p^{-1}\Phi(v))).$$

Notice that, since $M_t(p^{-1}S)$ is an intrinsic graph, the $f_{t,p}$ are homeomorphisms onto their images. Define

$$\Omega := \bigcap_{p \in \Phi(D_{1/2})} \bigcap_{|t| \leq \alpha/2} f_{t,p}(D_1).$$

We claim that $\Omega$ is a neighborhood of $(0,0)$. We argue by contradiction. Suppose that there exists a sequence $g_n \to 0$ in $\mathbb{R}^2$, along with $p_n \in \Phi(D_{1/2})$ and $t_n \in [-\alpha/2, \alpha/2)$, with $g_n \notin f_{t_n,p_n}(D_1)$. By compactness, we can suppose that $p_n \to p_\infty \in \Phi(D_{1/2})$ and that $t_n \to t_\infty \in [-\alpha/2, \alpha/2]$. Since the $f_{t,p}$ are homeomorphisms onto their images, there is $r > 0$ such that $D_{2r} \subset f_{t_\infty,p_\infty}(D_1)$. Since $f_{t_n,p_n} \to f_{t_\infty,p_\infty}$ uniformly, there holds $f_{t_n,p_n}(\partial D_1) \cap D_r = \emptyset$ for $n$ large enough. In particular, as $0 \in f_{t_n,p_n}(D_1)$, we have $D_r \subset f_{t_n,p_n}(D_{\tau_0})$, however, as $g_n \to 0$, for $n$ large enough there also holds $g_n \in \partial D_r$, a contradiction. This proves the claim.

Finally, if $\eta_0, \tau_0 > 0$ are such that $[-\eta_0, \eta_0] \times [-\tau_0, \tau_0] \subset \Omega$, then we have all the properties stated in the lemma. □
3. Proof of the Theorem

Let \( \alpha > 0 \) and \( S \subset \mathbb{H} \) be a topological surface with the \( \alpha \)-flat cone property. We will show that \( S \) has locally the \( (\alpha/4) \)-full cone property, that is, that for every \( o \in S \) there is an open neighborhood \( U \subset \mathbb{H} \) of \( o \) such that, for all \( p \in S \cap U \)
\[
\mathcal{C}(\alpha/4) \cap p^{-1}(S \cap U) = \emptyset.
\]
It suffices to work for \( o = 0 \).

By Lemma 10, there are \( \eta_0, \tau_0 > 0 \) such that, defining \( V_0 := \pi^{-1}(\lbrack -\eta_0, \eta_0 \rbrack \times \lbrack -\tau_0, \tau_0 \rbrack) \), for all \( t \in [-\alpha/2, \alpha/2] \) and \( p \in S \cap V_0 \), there exists a continuous function \( \phi_{t,p} : [-\eta_0, \eta_0] \times [-\tau_0, \tau_0] \to \mathbb{R} \) such that
\[
\Gamma_{\phi_{t,p}} = M_t(p^{-1}S) \cap V_0.
\]
Fix \( p \in S \cap V_0 \), \( t \in [-\alpha/2, \alpha/2] \) and set \( \epsilon := \min\{\eta_0, \sqrt{\tau_0 \alpha/2}\} \). Applying Proposition 8 to \( \phi_{t,p} \), we obtain for each \( \eta \in [-\epsilon, \epsilon] \) a number \( s \in [-2\alpha/\alpha, 2/\alpha] \) such that \( (s\eta, \eta, 0) \in \Gamma_{\phi_{t,p}} \). Therefore, as \( \Gamma_{\phi_{t,p}} \subset M_t(p^{-1}S) \) and since \( M_t(p^{-1}S) \) has the \( (\alpha/2) \)-flat cone property by Remark 9, there holds
\[
M_t(p^{-1}S) \cap \bigcap_{s \in [-2\alpha/\alpha, 2/\alpha]} (s\eta, \eta, 0) \in \mathcal{C}(\alpha/2) = 0, \quad \forall \eta \in [-\epsilon, \epsilon].
\]
Taking the union over \( \eta \in [-\epsilon, \epsilon] \) and applying Lemma 6 yields
\[
M_t(p^{-1}S) \cap \nu\mathcal{C}(\alpha/4, 4\epsilon/\alpha) = 0,
\]
which holds for all \( p \in S \cap V_0 \) and \( t \in [-\alpha/2, \alpha/2] \). Apply \( M_{-t} \) and the left translation by \( p \) to (5), then take a union over \( t \in [-\alpha/2, \alpha/2] \) and use (2) to get
\[
S \cap p\mathcal{C}(\alpha/4, 4\epsilon/\alpha) = \emptyset,
\]
which holds for all \( p \in S \cap V_0 \). Remark 5 implies that \( S \cap V_0 \) has the \( (\alpha/4) \)-cone property in a neighborhood of \( 0 \). Thus, \( S \) has locally the \( (\alpha/4) \)-cone property. \( \square \)

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