Spectrum of the Andreev Billiard and Giant Fluctuations of the Ehrenfest Time

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The density of states in the semiclassical Andreev billiard is theoretically studied and shown to be determined by the fluctuations of the classical Lyapunov exponent λ. The rare trajectories with a small value of λ give rise to an anomalous increase of the Ehrenfest time τE ≈ | ln h/λ| and, consequently, to the appearance of Andreev levels with small excitation energy. The gap in spectrum is obtained and fluctuations of the value of the gap due to different positions of superconducting lead are considered.

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Introduction. — The density of states in a metallic island coupled to a superconductor is modified due to the proximity effect [1]. The changes are most pronounced in the vicinity of Fermi energy, where there opens a gap in the spectrum of excitations. A ballistic chaotic normal region coupled to a superconductor via the small constriction (NS interface) is called the Andreev billiard [2]. The spectrum of such billiards was calculated a decade ago [3], assuming the random matrix description of the quantum dynamics. This approximation is valid if the number of channels N supported by the NS interface is small. In spite of large efforts [4, 5, 6, 7, 11, 13], there is currently no reliable calculation of the spectrum in the most interesting case of the semiclassical Andreev billiard, when the number of open superconducting channels scales with the Planck constant as N ~ 1/h.

The properties of Andreev billiards are governed by the Andreev reflection, when an electron trajectory is retraced by the hole that is produced upon absorption of a Cooper pair at the NS interface. At the Fermi energy EF, the classical dynamics of the hole is the time reverse of the electron dynamics, so that the motion is strictly periodic. This periodicity, however, comes in conflict with a quantum-mechanical evolution, since each Andreev reflection adds an extra phase π/2 to the electron-hole wave function. This phase is compensated by the difference of classical actions of electron (EF + ε) and hole (EF − ε) along the trajectory, where ε is the excitation energy. The longer the interval t between Andreev reflections, the smaller energy suffices to produce the missing phase ε = hπ/2t. On the other hand, the probability of particle trajectory to not touch the NS-interface for a long time becomes exponentially small ~ e−tD/τD, with τD being a dwell time. This leads to the prediction of an exponential suppression of the density of states [9, 12]

\[ \rho(\varepsilon) \approx N(\hbar \pi/\varepsilon^2 t_D) \exp(-\hbar \pi/2\varepsilon t_D). \]  

However, the density of states at small excitation energies, corresponding to large times, is not captured by this formula. A new time scale responsible for the spectrum at low energies [4] is the Ehrenfest time τE ~ | ln h/λ|, where λ is a Lyapunov exponent in the normal billiard. For longer times the initial quantum wave packet \[ \Delta x \Delta p \sim \hbar \] acquires a macroscopic size due to exponential ~ e^M divergence of trajectories. This invalidates the trajectory based derivation that led to [1]. It is believed that, below certain energy \[ \varepsilon_{\text{gap}} \sim \hbar/\tau_E, \] the density of Andreev states vanishes exactly, \( \rho(\varepsilon < \varepsilon_{\text{gap}}) = 0 \), but the magnitude of the gap and the mechanism of its formation remained a subject of controversial discussion [4, 7, 10].

At finite times, the value of the Lyapunov exponent λ depends on the specific trajectory [13], leading to fluctuations of the Ehrenfest time. Both the Andreev spectrum [3, 5, 6, 7, 8, 9, 10, 11, 17] and the quantum to classical crossover in ballistic transport [14, 15, 16, 17, 18] attracted a great deal of interest recently, but the role of fluctuations of the Ehrenfest time was never investigated. In this Letter we show how the low energy density of states is determined by the large Ehrenfest time fluctuations and solve the long standing problem of the Andreev gap.

The distribution of finite time Lyapunov exponents is parameterized as (here \( \tau_0 \) is, e.g., an averaged time between bounces at the walls of normal billiard)

\[ P(\lambda, t) = \tau_0 \exp(t F(\lambda)). \]  

In the case of chaos the limit \( F_{\lambda \rightarrow \lambda_0}(\lambda) = F(\lambda) \) exists, with \( F(\lambda) \) specific for the dynamical model [20]. The function \( F(\lambda) \) has a maximum at \( \lambda = \lambda_0 \), which is the conventional self-averaging Lyapunov exponent \( F(\lambda_0) = 0 \). Since all small values of \( \lambda > 0 \) are present in the distribution [2], one may always find (rare) trajectories with any large value of the Ehrenfest time. To build a semiclassical eigenfunction of 2-dimensional billiard, however, one needs a family of trajectories, all having the same interval t between the Andreev reflections. The explicit construction of discrete Andreev levels from the tube of trajectories (whose transverse Poincaré section is quantized via the Bohr-Sommerfeld rule) is presented in Ref. [4]. The gap in the spectrum is determined by the largest time for which the number of trajectories is enough to form at least one eigenstate. Quantitative counting of the number of trajectories is done with the use of the concept of transmission band [13, 14]. This lead us to the expressions for the gap and the density of states [13 and 16] below depending on \( F(\lambda), t_D, \) and a number \( N \) of open channels in the NS interface.
Dwell time corresponding, respectively, to the 6th and 7th iterations of the standard map averaged over 200 positions of the NS interface. Dwell time \( t_D = 10 \); kicking strength \( K = 10 \). The dashed lines show the theoretical prediction (9), with the function \( F \) found numerically for 5 and 6 iterations. For \( M = 10^7 \), the value \( \lambda = 0 \) would correspond to \( N_b \approx 10^5 \) in Eq. (8).

**Stroboscopic model.**— All of the essential features of a generic Andreev billiard are captured by the stroboscopic model of the Andreev billiard, which was developed in Ref. [6] for the generic Andreev billiard. Here we present a semiclassical solution of this model. First, the quantum kicked rotator, a counterpart of the classical standard map (22),

\[
p_{n+1} = p_n + KI_0 \sin \theta_n/\tau_0 , \quad \theta_{n+1} = \theta_n + \tau_0 p_{n+1}/I_0 ,
\]

is defined by the Floquet operator

\[
U = \exp \left( \frac{i \hbar \tau_0}{2I_0} \frac{d^2}{d\theta^2} \right) \exp \left( i \frac{K I_0 \cos \theta}{\hbar \tau_0} \right).
\]

Here \( I_0, \tau_0, \) and \( K \) are the moment of inertia, the inter-


val between kicks, and the kicking strength, respectively. Next, introduce the dimensionless Planck constant \( \hbar_{\text{eff}} = \hbar/\tau_0 I_0 \). If \( \hbar_{\text{eff}} = 2\pi I/M \) with integer \( I \), the coordinate and the momentum \( \tilde{p} = -i\hbar_{\text{eff}} d/d\theta \) take discrete values \( \theta(k) = 2\pi k/M, p_n = 2\pi m/M, k, m = 1, 2, \ldots, M \). The Floquet operator now becomes \( M \times M \) matrix.

The electron and the hole components of the wave function of Andreev kicked rotator span over the doubled \( 2M \)-sites Hilbert space with their evolution given by the normal (\( U \)) or conjugated (\( U^* \)) Floquet operators. The electron is converted into the hole by reflection at the \( N \)-channel superconducting lead, attached at \( \theta_1 < \theta < \theta_2 \) \( (\theta_2 - \theta_1 = \hbar N) \). This is done with the help of projection matrix \( Q \) those \( N \) nonzero elements are \( Q_{k,k} = 1 \), for \( \theta_1 < \hbar k < \theta_2 \). Andreev levels are found from

\[
U \psi = e^{iK_0 \tau_0/\hbar} \psi , \quad U = \begin{pmatrix} 1 - Q & -i Q U^* \\ -i Q U & (1 - Q) U^* \end{pmatrix}.
\]

The classical limit corresponds to \( M, N \to \infty \), while the dwell time \( t_D = \tau_0 M/N \) is fixed. A classical particle at any time \( t = n\tau_0 \) has a definite position either inside the normal region or at the interface. A semiclassical quantization of the map (11) requires a construction of the quantum states having a similar property. A formal description of the wave packet \( \phi \), which is injected from the superconductor, stays inside the billiard for \( n - 1 \) kicks, and then hits the NS interface is given by \( 0 < m < n \)

\[
Q \phi = \phi , \quad QU_m \phi = 0 , \quad QU^n \phi = U^n \phi.
\]

Provided such a solution is found, one easily builds \( 2n \) eigenfunctions with the eigenvalues \( (r = 0, \ldots, n - 1) \)

\[
\varepsilon_{nr} = \pm \hbar \pi + \frac{2r + 1}{2n \tau_0} , \quad \psi = \left( \sum_{k=0}^{n-1} U_k e^{-i k \theta n/\hbar} \phi \right) - \sum_{k=n} \sum_{l=1} U_k e^{i k \theta n/\hbar} \phi \right).
\]

\( \varepsilon_{n0} \) with the largest \( n \) constitutes a gap. Below we always consider only the levels with \( r = 0 \). Equations (9) are the analog of adiabatic quantization, developed in Ref. [1] for the generic Andreev billiard.

Strictly speaking, Eqs. (9) for any \( n \) have no solutions. However, for \( \hbar \to 0 \) where exists \( N \) linearly independent wave packets satisfying (9) with practically any desired accuracy, \( \phi^I QU^n \phi \sim e^{-1/\hbar} \). Finding the number of such solutions reduces to the calculation of certain phase-space areas called transmission bands (17).

**Transmission bands.**— We call the transmission band a simply connected part of the phase-space area of the NS interface, \( \theta_1 < \theta < \theta_2, 0 < p_{\text{eff}} < 2\pi, \) each point \( \theta, \rho \) of which visits the interface at the \( n \)-th iteration of the map (and do not visit it earlier). The image of the stripe \( \theta_1 < \theta < \theta_2 \) after \( n \) iterations is another long and narrow (curved) stripe of a width \( \sim e^{-\lambda t} N/M \) (see examples in Refs. [16] (17)). Phase-space overlaps of the NS interface with its image, the transmission bands, are the areas with approximately the shape of parallelogram whose long and short sides have a length \( \sim N/M \) and \( \sim e^{-\lambda t} N/M \). The number \( N_b \) of (families of) Andreev levels supported by a single transmission band is calculated as its area divided by \( 2\pi \hbar_{\text{eff}} \).

\[
N_b \approx (N^2/M) e^{-\lambda t}.
\]

\( \lambda t \) the total number of levels composed from trajectories having the time \( t = n\tau_0 \) between hitting the NS interface, whose Lyapunov exponent falls in the interval \( \Delta \lambda \), is (2)

\[
dN_b = N P(\lambda, t) e^{-1/\hbar_{\text{eff}}} (\tau_0/t_D) \lambda d\lambda.
\]

These levels originate from the many transmission bands of various size: \( N_t = \sum N_b \). We may use (3) to express \( \lambda \) through \( N_b \) and to find the distribution of levels over the sizes of the bands

\[
\frac{dN_t}{d\ln N_b} = N^2/\lambda \exp \left\{ t F_t \left( \frac{1}{t} \ln \frac{N^2}{MN_b} - \frac{t}{t_D} \right) \right\}.
\]

The distribution \( dN_t/d\ln N_b \) found numerically for the model (5), is shown by the histogram in Fig. 1 for times \( t = 6\tau_0 \) and \( t = 7\tau_0 \). The choice of \( \hbar_{\text{eff}} = 2\pi \times 10^{-7} \) introduces a quantum-mechanical scale in classical area counting. Direct quantum-mechanical calculation of energy levels in such an Andreev billiard would require diagonalization of the matrix of the size \( M = 10^7 \), which is beyond the reach for existing computers.
exponentially small \( \bar{\epsilon} \). These two conflicting effects allow one to find the value of the Lyapunov exponent leading to the largest, for a given time, transmission band. Equating via \( \lambda \) the total number of expected levels with \( \lambda < \lambda_c \) to the area of a single band \( \bar{\epsilon} \) with \( \lambda = \lambda_c \), we find
\[
F(\lambda_c) = -\lambda_c + t_D^{-1}.
\]

The value of the gap now is
\[
\bar{\epsilon}_{\text{gap}} = \frac{\lambda_c}{2\ln(N\tau_0/t_D)}.
\]

The value of \( \lambda_c \) depends on details of the specific model. It was shown in Ref. [24] that the derivative \( F'(\lambda) \) has a maximum at \( \lambda = 0 \) and that \( F'(0) \leq 1 \). Since the only maximum of the function \( F \) is \( F(\lambda_0) = 0 \) we obtain
\[
\lambda_c < \lambda_0/2.
\]

Thus we found that a value of the Andreev gap is at least twice smaller than predicted previously [4, 5, 10].

Semiclassical methods may be applied for a description of eigenstates constructed from the trajectories with the Lyapunov exponents
\[
\lambda < \lambda_{\text{max}} = t^{-1}\ln(N\tau_0/t_D).
\]

For \( \lambda > \lambda_{\text{max}} \), the number of levels per transmission band became less than 1. The number of Andreev levels associated with the time \( t \) may now be found by integration of Eq. (10) over the interval \( \lambda_c < \lambda < \lambda_{\text{max}} \). If \( \lambda_{\text{max}} > \lambda_0 \) this allows one to recover the known result Eq. (1), which is now valid for \( \epsilon \ln(N\tau_0/t_D) > h\pi\lambda_0 \). The smaller energies may exist only due to \( \lambda < \lambda_{\text{max}} < \lambda_0 \), for which we found a novel form of the density
\[
\rho(\epsilon) \approx \frac{N\tau_0}{\epsilon t_D F'(\kappa)} \exp \left[ \frac{h\pi}{2\epsilon} F(\kappa) - \frac{h\pi}{2\epsilon t_D} \right],
\]
where \( \kappa = (\epsilon/h\pi) \ln(N\tau_0/t_D/\hbar\pi < \lambda_0) \). Equation (16) is valid for \( \lambda_c < \epsilon \ln(N\tau_0/t_D)/h\pi < \lambda_0 \).

Formulas (16) and (17) are the main results of this Letter. They describe not only the stroboscopic model, but any Andreev billiard with chaotic dynamics coupled to a superconductor through the \( N \)-channel lead. (In this case, \( \tau_0 \) may be replaced by the averaged time between bouncing of the billiard walls. The precise value of \( \tau_0 \) is not important since \( N \sim h^{-1} \gg t_D/\tau_0 \)). For the model \( \lambda \) the semiclassical density consists of a series of \( \delta \)-function peaks at \( \epsilon = \epsilon_{\text{nr}}(7) \), and Eq. (16) describes the smoothed envelope of this distribution. Such peaks in the Andreev spectrum were seen in simulations of Refs. [11].

In Eq. (13), we found the averaged value of the superconducting gap. Fluctuations of \( \bar{\epsilon}_{\text{gap}} \) are caused by the variations in the position of NS interface. These fluctuations result from the fluctuations of the area of largest transmission band at a given time, which is shown in

\[\text{FIG. 2: The number of Andreev levels vs logarithm of the area of transmission band } dN_t/d\ln N_b \text{ for the stroboscopic model with } K = 11, \ h_{\text{eff}} = 2\pi \times 10^{-7}. \text{ The superconducting lead is attached at } \pi/10 < \theta < 3\pi/10. \text{ Histograms for different number of kicks } n = 2 - 16 \text{ are offset vertically and multiplied by } 2^n. \text{ For } n \geq 6, \text{ theoretical distribution (10) is also shown (dashed line). The gap amounts to } \bar{\epsilon}_{\text{gap}} = h\pi/30\tau_0 \text{ instead of } \epsilon_{\text{gap}} = h\pi/14\tau_0 \text{ expected from Refs. [4, 5, 10].}
\]
levels, area should also decrease with time. No such increased number of the bands exponentially increases, we expected Fig. 3. Since the area of (even the largest) transmission

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\[ In\ the\ case\ of\ mixed\ phase\ space,\ when\ there\ may\ exist\ stable\ periodic\ orbits\ not\ touching\ the\ superconducting\ lead,\ the\ Andreev\ gap\ disappears.\]

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\[ To\ describe\ the\ transmission\ bands\ formed\ by\ the\ trajectories\ with\ \lambda < \lambda_c,\ one\ has\ to\ consider\ dynamical\ folding\ of\ the\ phase\ space\ 24.\ The\ area\ of\ these\ bands\ is\ much\ smaller\ than\ predicted\ by\ Eq. (3).\ Therefore,\ they\ do\ not\ appear\ in\ Fig. 1.\]

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\[ Finding\ the\ distribution\ of\ Lyapunov\ exponents,\ and\ consequently\ F_n(\lambda),\ is\ much\ easier,\ as\ is\ seen\ in\ Figs. 1 and 2.\]

\[ Semiclassical\ quantization\ based\ on\ the\ transmission\ bands\ picture\ was\ confirmed\ by\ Refs. 6, 11, 15.\]

\[ However,\ the\ value\ of\ the\ Planck\ constant\ h_{\text{eff}}\ available\]
in these simulations was too small to observe the suppression of the gap. [13]