Galois descent for higher Brauer groups

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Abstract

For $X$ a smooth projective variety over a field $k$, we consider the problem of Galois descent for higher Brauer groups. More precisely, we extend a finiteness result of Colliot-Thélène and Skorobogatov [5] to higher Brauer groups.

For $X$ a smooth projective variety over a field $k$, the Brauer group $\text{Br}(X) = H^2_{\text{ét}}(X, \mathbb{G}_m)$ is a fundamental invariant in arithmetic geometry. Of particular interest is its role in the Tate conjecture in codimension 1. Recall that the Tate conjecture in codimension $m$, which we denote by $\text{T}C^m(X)_{\mathbb{Q}_\ell}$, states that when $k$ is a finitely generated field and $\overline{k}$ is its separable closure the cycle class map

$$\text{CH}^m(X) \otimes \mathbb{Q}_\ell \to H^2_{\text{ét}}(X \times_k \overline{k}, \mathbb{Q}_\ell(m))$$  

surjects onto the subspace of Tate classes:

$$\bigcup_U H^2_{\text{ét}}(X \times_k \overline{k}, \mathbb{Q}_\ell(m))^{\text{Gal}(\overline{k}/k)}$$

where $U$ ranges over all open subgroups of $\text{Gal}(\overline{k}/k)$. When $k$ is a finite field, Tate showed in [17] that the Tate conjecture $\text{T}C^1(X)_{\mathbb{Q}_\ell}$ holds $\Leftrightarrow$ the $\ell$-primary torsion in $\text{Br}(X)$ is finite (for $\ell \neq \text{char } k$). For arbitrary fields, the Tate conjecture for divisors is equivalent to the finiteness of the $\ell$-primary torsion in $\text{Br}(X \times_k \overline{k})^{\text{Gal}(\overline{k}/k)}$ (see, for instance, [3] Prop. 2.1.1).

In higher codimension, the $m^{th}$ higher Brauer groups $\text{Br}^m(X)$ are defined by $H^2_{\text{ét}}(X, \mathbb{Z}(m))$, where $H^2_{\text{ét}}(X, \mathbb{Z}(m))$ denote the étale motivic cohomology groups of [15]. These latter are (étale) hyper-cohomology groups of the étale sheafification of Bloch’s cycle complexes [2], denoted by $\mathbb{Z}(m)$. When $m = 1$, the complex $\mathbb{Z}(1)$ is quasi-isomorphic to $\mathbb{G}_m[-1]$, which recovers the usual Brauer group. For finite fields, Rosenschon and Srinivas prove an analogue of Tate’s theorem for higher Brauer groups; namely, that when $k$ is a finite field $\text{T}C^m(X)_{\mathbb{Q}_\ell}$ holds if and only if the $\ell$-primary torsion in $\text{Br}^m(X)$ (for $\ell \neq \text{char } k$); see [15] Theorem 1.4. As motivation for the utility of higher Brauer groups, we begin with the following result:

**Proposition 0.1.** Let $k$ be a finitely generated field of characteristic 0. Then, $\text{T}C^m(X)_{\mathbb{Q}_\ell} \Leftrightarrow$ the $\ell$-primary torsion subgroup $\text{Br}^m(X \times_k \overline{k})^{\text{Gal}(\overline{k}/k)}[\ell^\infty]$ is finite.
The finiteness of $Br^m(X)^{Gal(\overline{k}/k)}$ in particular would imply that the cokernel of the restriction map:

$$Br^m(X)[\ell^{\infty}] \rightarrow Br^m(X)^{Gal(\overline{k}/k)}[\ell^{\infty}]$$

is at worst finite. Thus, if one believes the Tate conjecture, we should expect that the failure of Galois descent for (higher) Brauer classes is at worst finite. An unconditional result in this direction was proved by Colliot-Thélène and Skorobogatov for fields of characteristic 0 when $m = 1$; i.e., for the usual Brauer group. Our main result is to extend this Galois descent property for higher Brauer groups:

**Theorem 0.1.** Let $p$ be the exponential characteristic of $k$. Suppose that one of the following holds:

(a) $k$ has characteristic 0;

(b) $X$ satisfies the standard conjectures of Grothendieck [11].

Then, the cokernel of the map

$$Br^m(X)[\frac{1}{p}] \rightarrow Br^m(X)^{Gal(\overline{k}/k)}[\frac{1}{p}]$$

is finite.

Our plan will be as follows. We first give a proof of Proposition [22] which is fairly routine, given existing results in the literature. The proof of Theorem [0.1] on the other hand, will exploit some classical techniques of Deligne from [6] and [7]. These allow us to prove some basic degeneracy results for the Hochschild-Serre spectral sequence:

$$E^{p,q}_2 = H^p(k, H^q_{et}(X, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})) \Rightarrow H^{p+q}_{et}(X, \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell})$$

This will involve the introduction of a certain Serre localization which we call the isogeny category. This is the natural context in which to state a degeneracy result for this spectral sequence, as it will turn out. Then, using other techniques, we obtain the required statement about the higher Brauer groups. We do not obtain estimates for the size of the cokernel of

$$Br^m(X) \rightarrow Br^m(X)^{Gal(\overline{k}/k)}$$

as the authors of [3] do. This is mostly because any optimal estimate would require that we assume the standard conjectures (even when the assumption of the standard conjectures is not used.)

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Notation

Throughout this note, we will let $X$ be a smooth projective variety of dimension $d$ over a field $k$, which we will assume to be of characteristic 0 when necessary. We also let $\overline{k}$ be the separable closure of $k$ and $\overline{X} := X \times_k \overline{k}$. We also let $G_k$ denote the absolute Galois group of $k$.

1 Preliminaries

1.1 The isogeny category

Definition 1.1. Given an additive category $\mathcal{A}$, we define the associated isogeny category of $\mathcal{A}$ to be the Serre localization $\mathcal{A}_Q$ of $\mathcal{A}$ along the isogenies; more precisely, $\mathcal{A}_Q$ is the category whose objects are the same as those of $\mathcal{A}$ and whose morphisms are given by:

$$\text{Hom}_{\mathcal{A}_Q}(A, B) := \text{Hom}_\mathcal{A}(A, B) \otimes \mathbb{Q}$$

As a matter of convenience, we have the following straightforward lemma:

Lemma 1.1. Suppose that $\mathcal{A}$ is an additive category and that $\phi \in \text{Hom}_\mathcal{A}(A, B)$ is split-injective (resp., split-surjective) in the isogeny category $\mathcal{A}_Q$. Then, the kernel (resp., cokernel) of $\phi$ is of finite-exponent.

There is also the following analogue one of Deligne’s decomposition theorems set in the isogeny category:

Lemma 1.2. Let $\mathcal{A}$ be an Abelian category and let $D(\mathcal{A})$ be the derived category of bounded complexes in $\mathcal{A}$. Suppose that there exists $A^* \in D(\mathcal{A})$ and $\phi \in \text{Hom}_{D(\mathcal{A})}(A^*, A^*[2])$ for which the map on cohomology

$$H^{n-i}(\phi^i) : H^{n-i}(A^*) \to H^{n+i}(A^*)$$

is an isogeny for all $i > 0$. Then, there exists a non-canonical decomposition in $D(\mathcal{A}_Q)$:

$$A^* \cong \bigoplus_i H^i(A^*)[-i]$$

Proof. This is nothing more than [6] Theorem 1.5 after one realizes that $D(\mathcal{A}_Q)$ is the isogeny category of $D(\mathcal{A})$.

Corollary 1.1. Retain the notation and assumptions of the previous lemma and let $\Gamma : \mathcal{A} \to \text{Ab}$ be a left-exact functor to the category of Abelian groups. Then, for all $i \geq 0$ the edge map

$$H^i(\Gamma A^*) \to \Gamma H^i(A^*)$$

is split-surjective in the isogeny category of Abelian groups; in particular, the cokernel is of finite exponent.

Proof. The first statement follows from Lemma 1.2 and [6] Prop. 1.2. The second statement follows from Lemma 1.1.
1.2 Some Lefschetz-type Theorems

**Lemma 1.3.** Suppose that \( h \in \text{Pic}(X) \) is the class of an ample divisor and \( \ell \) is a prime different from the characteristic of \( k \).

(a) For all \( m \geq 0 \), \( H^{d-m}_{\text{ét}}(X, \mathbb{Z}_\ell) \xrightarrow{\cup h^m} H^{d+m}_{\text{ét}}(X, \mathbb{Z}_\ell(m)) \) is an isogeny in the category of \( G_k \)-modules.

(b) Suppose further that \( k \) has characteristic 0 or that \( X \) satisfies the standard conjectures. Also set

\[
H^*(\overline{X}, \hat{\mathbb{Z}}') := \prod_{\ell \neq \text{char } k} H^*(\overline{X}, \mathbb{Z}_\ell)
\]

Then, for all \( m \geq 0 \), \( H^{d-m}_{\text{ét}}(X, \hat{\mathbb{Z}}') \xrightarrow{\cup h^m} H^{d+m}_{\text{ét}}(X, \hat{\mathbb{Z}}'(m)) \) is an isogeny in the category of \( G_k \)-modules.

**Proof.** Since \( X \) is defined over a finitely generated field, we can assume (by invariance of étale cohomology under separably closed extensions, [13] VI Cor. 2.6) that \( k \) is some finitely generated field. When \( k \) has characteristic 0, one thus reduces to the case that \( k \subset \mathbb{C} \). In this case, the classical hard Lefschetz theorem gives an isomorphism of singular cohomology:

\[
H^{d-m}(X_C, \mathbb{Q}) \xrightarrow{\cup h^m} H^{d+m}(X_C, \mathbb{Q}(m))
\]

which means that the corresponding map with integral coefficients is an isogeny of Abelian groups. Both statements (a) and (b) then follow from the comparison isomorphism between singular and étale cohomology. Now suppose that \( k \) has positive characteristic. Then, to prove statements (a) and (b), note that by the main result of [7] we have an isomorphism:

\[
H^{d-m}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) \xrightarrow{\cup h^m} H^{d+m}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(m))
\]

Since \( H^*_\text{ét}(\overline{X}, \mathbb{Z}_\ell) \) is a finitely generated \( \mathbb{Z}_\ell \) module, it follows that the corresponding map with \( \mathbb{Z}_\ell \) coefficients is an isogeny of \( G_k \)-modules. To obtain the corresponding statement for \( H^*_\text{ét}(\overline{X}, \hat{\mathbb{Z}}') \), we use the Lefschetz standard conjecture to obtain a correspondence

\[
\Lambda_m \in CH^{d-m}(X \times X)
\]

for which there exists some integer \( M \) satisfying

\[
(\cup h^m) \circ \Lambda_m = M \cdot \text{id}_{H^{d+m}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(m))}, \quad \Lambda_m \circ (\cup h^m) = M \cdot \text{id}_{H^{d-m}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)}
\]

for all \( \ell \). Thus,

\[
\cup h^m : H^{d-m}_{\text{ét}}(\overline{X}, \hat{\mathbb{Z}}') \rightarrow H^{d+m}_{\text{ét}}(\overline{X}, \hat{\mathbb{Z}}'(m))
\]

is an isogeny of \( G_k \)-modules, which gives statement (b) in positive characteristic. \( \square \)
Remark 1.1. The reader should note that the role of the Lefschetz standard conjecture in characteristic \( p > 0 \) is to ensure that the degree of the isogeny
\[
\cup h^m : H^d_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) \to H^{d+m}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(m))
\]
does not depend on \( \ell \). The author is not sure how to prove this in the absence of the Lefschetz standard conjecture.

Corollary 1.2. With the notation of Lemma 1.3,

(a) For all \( m \geq 0 \), \( H^d_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) \cup h^m \to H^{d+m}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(m)) \) is an isogeny in the category of \( G_k \)-modules.

(b) Suppose further that \( k \) has characteristic 0 or that \( X \) satisfies the standard conjectures and set
\[
H^*_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}') := \prod_{\ell \neq \text{char} k} H^*_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)
\]
Then, for all \( m \geq 0 \) \( H^d_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}') \cup h^m \to H^{d+m}_{\text{ét}}(\overline{X}, \mathbb{Q}/\mathbb{Z}'(m)) \) is an isogeny in the category of \( G_k \)-modules.

Proof. For statement [a] there is a commutative with rows exact:
\[
\begin{array}{cccccc}
H^d_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^d_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \longrightarrow & H^d_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\infty] \\
\cup h^d & & \cup h^d & & \cup h^d & \\
H^{d+m}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell & \longrightarrow & H^{d+m}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) & \longrightarrow & H^{d+m}_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)[\ell^\infty]
\end{array}
\]
(suppressing weights). Since \( H^d_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell) \) is a finitely generated \( \mathbb{Z}_\ell \)-module, both the rightmost terms are finite. So, to prove that the middle vertical arrow is an isogeny, it suffices to prove that the left vertical arrow is an isogeny. But this latter follows directly from Lemma 1.3 [a]. To prove the statement of [b] note that there is an identical diagram with coefficients in
\[
\mathbb{Q}/\mathbb{Z}' = \bigoplus_{\ell \neq \text{char} k} \mathbb{Q}_\ell/\mathbb{Z}_\ell
\]
and with the rightmost terms \( H^{d+m+1}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)_{\text{tors}}[\frac{1}{p}] \) and \( H^{d+m+1}_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell)_{\text{tors}}[\frac{1}{p}] \), where \( p \) is the exponential characteristic of \( k \). These latter groups are finite by the main result of [8]. The argument from statement [a] works mutatis mutandis (using Lemma 1.3 [b] this time).

Corollary 1.3. Retain the assumptions of Lemma 1.3.

(a) For all \( r \) and \( m \), the cokernel of the natural map
\[
H^m_{\text{ét}}(X, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r)) \to H^m_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(r))^{G_k}
\]
is finite.
(b) Suppose further that \( k \) has characteristic 0 or that \( X \) satisfies the standard conjectures. Then, for all \( r \) and \( m \), the cokernel of the natural map
\[
H^m_{\text{et}}(X, \mathbb{Q}/\mathbb{Z}'(r)) \to H^m_{\text{et}}(\overline{X}, \mathbb{Q}/\mathbb{Z}'(r))^{G_k}
\]
is finite.

**Proof.** For both statements, we first show that the cokernel is of finite exponent. This follows for (2) (resp., for (3)) by Corollary 1.2 (a) and Corollary 1.1 (resp., by Corollary 1.2 (b) and Corollary 1.1). The desired statement then follows from the lemma below, which is a straightforward group-theoretic fact (cf, [5] §1.2).

**Lemma 1.4.** Suppose that \( A \) is an Abelian group of the form \((\mathbb{Q}_\ell/\mathbb{Z}_\ell)^n \oplus F\) (or \((\mathbb{Q}/\mathbb{Z})^n \oplus F\)), where \( n \geq 0 \) and \( F \) is finite. Then, any finite-exponent subquotient of \( A \) is finite.

**Remark 1.2.** We remark that Corollary 1.3 is only interesting for \( m = 2r, 2r−1 \) since for all others, the right hand groups are already finite by the results of [4].

### 1.3 Étale motivic cohomology

For \( m \geq 0 \) an integer, let \( \tau \) denote either the Zariski or (small) étale topology and let \( \Delta^p \subset \mathbb{A}^{p+1}_k \) denote the algebraic simplex defined by \( x_0 + \ldots + x_p = 1 \) and let \( \partial \Delta^p \) denote its boundary. As in [15] one has sheaves of Abelian groups for the \( \tau \) topology on \( X \) for \( i \leq 0, U \mapsto z^m(U, i) \), where \( z^m(U, i) \) denotes the free Abelian group of codimension \( m \) cycles on \( U \times \Delta^i \) that intersect \( U \times \partial \Delta^i \) properly. These fit into a bounded above complex whose boundary maps are defined in the usual way. Then, for any Abelian group \( A \) consider the corresponding *cycle complex*:

\[
A_X(m)_{\tau} := (z^m(-, +)_{\tau} \otimes A)[-2m]
\]

and then define the corresponding hypercohomology groups, known as the motivic and étale motivic cohomology groups, respectively:

\[
H^p_{\text{M}}(X, A(m)) = H^p_{\text{Zar}}(X, A_X(m)_{\text{Zar}})
\]
\[
H^p_{\text{L}}(X, A(m)) = H^p_{\text{et}}(X, A_X(m)_{\text{et}})
\]

For the Zariski motivic cohomology groups, there are isomorphisms with Bloch’s higher Chow groups, \( CH^m(X, 2m - p) \cong H^p_{\text{M}}(X, \mathbb{Z}(m)) \) (see [15] p. 515). As per convention, we denote the Lichtenbaum Chow group by \( CH^p_{\text{L}}(X) = H^m_{\text{L}}(X, \mathbb{Z}(m)) \). We also observe that \( H^p_{\text{M}}(X, \mathbb{Q}(p)) \cong H^p_{\text{L}}(X, \mathbb{Q}(p)) \) by [12] Théorème 2.6.

Geisser and Levine (in [9] and [10]) establish quasi-isomorphisms in the (bounded below) derived category of étale sheaves over \( X \):

\[
(\mathbb{Z}/\ell^r)_{X(m)_{\text{et}}} \xrightarrow{\sim} \mathbb{Z}/\ell^r(m)
\]
for $\ell$ a prime and $r \geq 1$. In particular, the corresponding map on hypercohomology is an isomorphism:

\[
H^p_L(X, \mathbb{Z}/\ell^r(m)) \cong H^p_{\text{ét}}(X, \mathbb{Z}/\ell^r(m))
\]  

(5)

**Definition 1.2.** We denote by $N^m(X)_\ell$ (or when there is no confusion, $N^m(X)$) the image of the $\ell$-adic cycle class map:

\[
\lim_{\leftarrow r \geq 0} c^m_\ell : CH^m_L(X) \rightarrow H^{2m}_{\text{ét}}(X, \mathbb{Z}_\ell(m))
\]

Also, define the $m^{th}$ higher Brauer group $Br^m(X) := H^{2m+1}_L(X, \mathbb{Z}(m))$.

**Theorem 1.1.** With the above notation,

(a) There exists a natural short exact sequence:

\[
0 \rightarrow H^p_L(X, \mathbb{Z}(m)) \otimes \mathbb{Q}/\mathbb{Z} \rightarrow H^p_{\text{ét}}(X, \mathbb{Q}/\mathbb{Z}(m)) \rightarrow H^p_L(X, \mathbb{Z}(m))_{\text{tors}} \rightarrow 0
\]

where the first non-zero arrow is the cycle class map.

(b) $H^p_L(X, \mathbb{Z}(m))$ is torsion for $p > 2m$.

(c) When $k$ is algebraically closed of characteristic 0, $N^m(X) \otimes \mathbb{Q} \subset A^m(X)$, the $\mathbb{Q}$-vector space of absolute Hodge cycles in $H^{2m}_{\text{ét}}(X, \mathbb{Q}(m))$.

### 1.4 A splitting result

For the applications to follow, we will need the following auxiliary results.

**Lemma 1.5.** $N^m(\overline{X})$ is a finitely generated Abelian group.

**Proof.** This follows from the fact that

\[
CH^n(X) \otimes \mathbb{Q} \cong CH^n_L(X) \otimes \mathbb{Q}
\]

and the fact that the image of the cycle class map

\[
CH^m(\overline{X}) \otimes \mathbb{Q} \rightarrow H^{2m}_{\text{ét}}(\overline{X}, \mathbb{Q}(m))
\]

is a finitely generated vector space. \qed

Now, we adopt the notation:

\[
\hat{\mathbb{Z}}' := \prod_{\ell \neq \text{char } k} \mathbb{Z}_\ell, \hat{\mathbb{Q}}' := \prod_{\ell \neq \text{char } k} \mathbb{Q}_\ell
\]

**Lemma 1.6.** Suppose that $k$ has characteristic 0. Then, there exists a $G_k$-invariant pairing:

\[
H^{2m}_{\text{ét}}(\overline{X}, \hat{\mathbb{Z}}'(m)) \otimes H^{2m}_{\text{ét}}(\overline{X}, \hat{\mathbb{Z}}'(m)) \xrightarrow{(\cdot, \cdot)} \hat{\mathbb{Z}}'
\]

and a discrete subgroup $N^m(\overline{X})' \subset H^{2m}_{\text{ét}}(\overline{X}, \hat{\mathbb{Z}}'(m))$ for which:
\[(a) \ N^m(\overline{X}) \subset N^m(\overline{X})'\]
\[(b) \ N^m(\overline{X})' \text{ is } G_k\text{-stable}\]
\[(c) \ (, ) \text{ restricted to } N^m(\overline{X})' \text{ is integral and non-degenerate.}\]

**Proof.** Select an ample divisor \(h \in Pic(X)\). Then, for \(2m \leq d\), we define the pairing as:
\[
(, ) : H^{2m}_{\text{ét}}(X, \hat{Z}'(m)) \otimes H^{2m}_{\text{ét}}(X, \hat{Z}'(m)) \to \hat{Z}', \ (\alpha, \beta) = \alpha \cup \beta \cup h^{d-2m}
\]
This is certainly \(G_k\)-invariant. Now, if \(d < 2m\) we define the pairing as follows. By Lemma 1.3 there is an isogeny \(\Lambda_m : H^{2m}_{\text{ét}}(X, \hat{Z}'(m)) \to H^{2d-2m}_{\text{ét}}(X, \hat{Z}'(d-m))\) which is inverse to \(\cup h^{2m-d}\). In fact, since \(k\) has characteristic 0, this isogeny is defined for singular cohomology with integral coefficients. Thus, we can define the pairing as
\[
(, ) : H^{2m}_{\text{ét}}(X, \hat{Z}'(m)) \otimes H^{2m}_{\text{ét}}(X, \hat{Z}'(m)) \to \hat{Z}', \ (\alpha, \beta) = \Lambda_m(\alpha) \cup \beta
\]
Again, this is a \(G_k\)-invariant pairing since \(\Lambda_m\) is the inverse isogeny of \(\cup h^{2m-d}\). Now, we set \(N^m(\overline{X})'\) to be a lattice in the \(\mathbb{Q}\)-vector space \(A^m_{\text{ét}}(X) \subset H^{2m}_{\text{ét}}(X, \hat{Q}'(m))\) of absolute Hodge classes of degree \(2m\). Note that \(A^m_{\text{ét}}(X)\) does not depend on \(\ell\), contains \(N^m(\overline{X})\) (Theorem 1.1) and is \(G_k\)-stable. Thus, upon refining this lattice, we can assume that it contains \(N^m(\overline{X})\) and is \(G_k\)-stable. That the pairing \((, )\) is non-degenerate on \(A^m_{\text{ét}}(X)\) is well-known (see, for instance, [1] Prop. 3.3).

**Lemma 1.7.** Suppose that \(\hat{V}\) is a continuous \(\hat{Z}'[G_k]\)-module possessing a \(G_k\)-invariant pairing:
\[
(, ) : \hat{V} \otimes \hat{V} \to \hat{Z}'
\]
and \(M \subset \hat{V}\) is a finitely generated Abelian group which is \(G_k\)-stable, discrete and for which \((, )\) restricted to \(M\) is integral and non-degenerate. Then, the inclusion
\[
M \otimes \hat{Z}' \hookrightarrow \hat{V}
\]
is split-injective in the isogeny category of \(G_k\)-modules.

**Proof.** As a first reduction, we may freely pass to a finite extension of \(k\) whenever necessary. To see why, let \(k' / k\) be a finite Galois extension and observe that if a splitting \(\rho\) exists which is \(G_{k'}\)-invariant, then one can set
\[
\rho' := \sum_{h \in H} h^* \rho
\]
where \(H = G_k / G_{k'}\) to obtain a \(G_k\)-invariant splitting.

Now, there is the composition of \(G_k\)-modules:
\[
\phi : M \otimes \hat{Z}' \hookrightarrow \hat{V} \to \hat{V}^\vee = \text{Hom}_{\hat{Z}'}(\hat{V}, \hat{Z}') \xrightarrow{res_{M \otimes \hat{Z}'}} \text{Hom}_{\hat{Z}'}(M \otimes \hat{Z}', \hat{Z}')
\]
defined by
\[\alpha \mapsto (\alpha, -), \alpha \in M\]
Here, the middle arrow is induced by the pairing \((, )\). It suffices to show that \(\phi\) is split-injective. By the non-degeneracy of \((, )\) on \(M\), \(\phi\) is injective. Upon passing to a finite extension of \(k\), we can assume that \(M\) is a trivial \(G_k\)-module (since \(M\) is discrete by assumption). Certainly, \(\phi\) is split-injective in the isogeny category of Abelian groups, from which it follows that \(\phi\) is split-injective in the isogeny category of \(G_k\)-modules, as desired.

\[\square\]

**Corollary 1.4.** Suppose \(k\) has characteristic 0 or that \(X\) satisfies the standard conjectures. Then, the natural inclusion
\[N^m(X) \otimes \hat{\mathbb{Z}}' \hookrightarrow H_2^{2m}(\overline{X}, \hat{\mathbb{Z}}'(m))\]
is split-injective in the isogeny category of \(G_k\)-modules.

**Proof.** First assume that \(k\) has characteristic 0. From Lemma 1.7 it follows that the natural inclusion
\[N^m(\overline{X})' \otimes \hat{\mathbb{Z}}' \hookrightarrow H_2^{2m}(\overline{X}, \hat{\mathbb{Z}}'(m))\]
is split-injective, where \(N^m(\overline{X})'\) is the discrete subgroup from Lemma 1.3 Again, we can assume (upon passing to a finite extension of \(k\)) that \(N^m(X)'\) is a trivial \(G_k\)-module, from which it follows that the injection of (trivial) \(G_k\)-modules
\[N^m(\overline{X}) \hookrightarrow N^m(\overline{X})'\]
is also split. It follows that that the composite injection:
\[N^m(\overline{X}) \otimes \hat{\mathbb{Z}}' \hookrightarrow N^m(\overline{X})' \otimes \hat{\mathbb{Z}}' \hookrightarrow H_2^{2m}(\overline{X}, \hat{\mathbb{Z}}'(m))\]
is also split-injective, as desired.

On the other hand, if \(X\) satisfies the standard conjectures, then homological equivalence coincides with numerical equivalence, which means that the pairing \((, )\) is non-degenerate on \(N^m(X)\). The above proof then works mutatis mutandis with \(N^m(\overline{X}) = N^m(\overline{X})'\). \(\square\)

2 Proof of Proposition 0.1

(Proof of \(\Leftarrow\)) Consider the short exact sequence of \(G_k\)-modules:
\[0 \to N^m(\overline{X}) \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \to H_2^{2m}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(m)) \to Br^m(\overline{X})[\ell^\infty] \to 0 \quad (6)\]
Applying the \(G_k\)-invariants functor to (6), we obtain an exact sequence:
\[0 \to N^m(\overline{X})^{G_k} \otimes \mathbb{Q}_\ell/\mathbb{Z}_\ell \xrightarrow{\phi} H_2^{2m}(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell(m))^{G_k} \to Br^m(\overline{X})[\ell^\infty]^{G_k} \]
Suppose that $Br^m(\overline{X})[\ell^\infty]^{G_k}$ is finite, so that $\text{coker } \phi$ is also. Then, there is a commutative diagram with rows exact:

$$
\begin{array}{cccc}
N^m(\overline{X})^{G_k} \otimes \mathbb{Z}_\ell & \rightarrow & N^m(\overline{X})^{G_k} \otimes \mathbb{Q}_\ell & \rightarrow & N^m(\overline{X})^{G_k} \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \\
\downarrow & & \downarrow & & \downarrow \\
H^m_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(m))^{G_k}_{/\text{tors}} & \rightarrow & H^m_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(m))^{G_k} & \rightarrow & H^m_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell / \mathbb{Z}_\ell(m))^{G_k}
\end{array}
$$

A diagram chase then shows that there is an inclusion

$$C \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell \hookrightarrow \text{coker } \phi$$

where

$$C := \text{coker } \{ N^m(\overline{X})^{G_k} \rightarrow H^m_{\text{ét}}(\overline{X}, \mathbb{Z}_\ell(m))^{G_k} \}$$

It follows that $C \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell$ is finite. Since $C$ is a finitely generated $\mathbb{Z}_\ell$-module, it then follows that $C \otimes \mathbb{Q}_\ell / \mathbb{Z}_\ell = 0$ and hence that $C$ is torsion; in particular, $TC^m(X)_{\mathbb{Q}_\ell}$ holds. (Proof of $\Rightarrow$) Suppose conversely that the Tate conjecture $TC^m(X)_{\mathbb{Q}_\ell}$ holds. Then, the proof proceeds as in [16] Prop. 2.5. Indeed, let

$$T_\ell(\text{Br}^m(\overline{X})) := \text{Hom}(\mathbb{Q}_\ell / \mathbb{Z}_\ell, \text{Br}^m(\overline{X})) = \lim_{\substack{\text{r}\rightarrow \infty}} \text{Br}^m(\overline{X})[\ell^r]$$

Then, by Corollary [14], there is a splitting of $G$-modules

$$H^m_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(m)) \cong N^m(\overline{X}) \otimes \mathbb{Q}_\ell \oplus T_\ell(\text{Br}^m(\overline{X})) \otimes \mathbb{Q}_\ell$$

(7)

Moreover, from the Tate conjecture there exists some finite Galois extension $k'/k$ for which

$$N^m(\overline{X})^{\text{Gal}(\overline{K}/k')} \otimes \mathbb{Q}_\ell \cong H^m_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(m))^{\text{Gal}(\overline{K}/k')}$$

and hence that $T_\ell(\text{Br}^m(\overline{X}))^{\text{Gal}(\overline{K}/k')}$ is finite. Finally, since $\mathbb{Q}_\ell / \mathbb{Z}_\ell$ is a trivial $G$-module, it follows that

$$T_\ell(\text{Br}^m(\overline{X}))^{\text{Gal}(\overline{K}/k')} = \text{Hom}(\mathbb{Q}_\ell / \mathbb{Z}_\ell, \text{Br}^m(\overline{X})^{\text{Gal}(\overline{K}/k')})$$

and since this is finite, we deduce that so is $\text{Br}^m(\overline{X})^{\text{Gal}(\overline{K}/k')}$. 

**Remark 2.1.** The main difficulty in extending this result to characteristic $p > 0$ is that there is no notion of absolute Hodge classes in this case and so even with the Tate conjecture for $X$, it is not clear that the splitting of $G_k$-modules (7) holds. If one instead assumes the Tate conjecture for all products $X^n$, then the results of [14] (for instance) show that $H^m_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell(m))$ is semi-simple as a $G_k$-module. This would then imply (7).
3 Proof of Theorem 0.1

Note that there is a commutative diagram with rows exact:

\[
\begin{array}{cccccc}
N^m(X)_{Q/Z'} & \longrightarrow & H^2_{\text{et}}(X, Q/Z'(m)) & \longrightarrow & Br^m(X)_{[1/p]} & \longrightarrow 0 \\
\downarrow & & \downarrow & & \downarrow \phi & \\
0 & \longrightarrow & N^m(X)^{G_k}_{Q/Z'} & \longrightarrow & H^2_{\text{et}}(X, Q/Z'(m))^{G_k} & \longrightarrow Br^m(X)^{G_k}_{[1/p]} & \longrightarrow K
\end{array}
\]

where \(A_{Q/Z'} = A \otimes Q/Z'\) for any Abelian group \(A\) and where

\[K := \ker \{H^1(k, N^m(X)_{Q/Z'}) \to H^1(k, H^2_{\text{et}}(X, Q/Z'(m)))\}\]

Since the middle vertical arrow has finite cokernel by Corollary 1.3, what remains is to show that \(K\) is finite. By Lemma 1.4, it suffices to prove that \(K\) has finite exponent. To this end, observe that \(\alpha\) factors as

\[H^1(k, N^m(X)_{Q/Z}) \xrightarrow{\beta} H^1(k, H^2_{\text{et}}(X, \hat{Z}'(m))_{Q/Z}) \xrightarrow{\gamma} H^1(k, H^2_{\text{et}}(X, Q/Z'(m)))\]

From Corollary 1.4

\[N^m(X) \xrightarrow{\beta} H^2_{\text{et}}(X, \hat{Z}'(1))\]

is split-injective in the isogeny category of \(G_k\)-modules, from which it follows that \(\beta\) is split-injective in the isogeny category of Abelian groups. From Lemma 1.4 it follows that the kernel of \(\beta\) is of finite exponent. On the other hand, there is a short exact sequence of \(G_k\)-modules:

\[0 \longrightarrow H^2_{\text{et}}(X, \hat{Z}'(m))_{Q/Z'} \longrightarrow H^2_{\text{et}}(X, Q/Z'(m)) \longrightarrow H^2_{\text{et}}(X, \hat{Z}'(m))_{\text{tors}} \longrightarrow 0\]

By the comparison isomorphism with singular cohomology, \(H^2_{\text{et}}(X, \hat{Z}'(m))_{\text{tors}}\) is finite (since singular cohomology with integral coefficients is finitely generated). Applying \(H^*(k, -)\) to this exact sequence, one obtains an exact sequence:

\[H^2_{\text{et}}(X, \hat{Z}'(m))^{G_k}_{\text{tors}} \xrightarrow{\gamma} H^1(k, H^2_{\text{et}}(X, \hat{Z}'(m))_{Q/Z'}) \xrightarrow{\gamma} H^1(k, H^2_{\text{et}}(X, Q/Z'(m)))\]

from which it follows that the kernel of \(\gamma\) has finite exponent. We deduce that \(K\) has finite exponent, as desired.

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