NONLINEAR SCHRÖDINGER EQUATIONS ON A FINITE INTERVAL WITH POINT DISSIPATION

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Abstract. The paper considers the initial value problem of a general type of nonlinear Schrödinger equations

\[ iu_t + u_{xx} + f(u) = 0, \quad u(x, 0) = w_0(x) \]

posed on a finite domain \( x \in [0, L] \) with an \( L^2 \)-stabilizing feedback control law

\[ u(0, t) = \beta u(L, t), \quad \beta u_x(0, t) - u_x(L, t) = i\alpha u(0, t), \]

where \( L > 0, \alpha, \beta \) are real constants with \( \alpha \beta < 0 \) and \( \beta \neq \pm 1 \), and \( f(u) \) is a smooth function from \( \mathbb{C} \) to \( \mathbb{C} \) satisfying some growth conditions. It is shown that for \( s \in \left( \frac{1}{2}, 1 \right] \) and \( w_0(x) \in H^s(0, L) \) with the boundary conditions described above, the problem is locally well-posed for \( u \in C([0, T]; H^s(0, L)) \). Moreover, the solution with small initial condition exists globally and approaches to 0 as \( t \to +\infty \).

1. Introduction. We consider the solutions of nonlinear Schrödinger (NLS) equation

\[ iu_t + u_{xx} + c|u|^2u = 0 \quad (1.1) \]

or more generally,

\[ iu_t + u_{xx} + f(u) = 0 \quad (1.2) \]

on a finite interval \( 0 \leq x \leq L \) with \( t \geq 0 \), where \( u(x, t) \) is a complex-valued function, \( c \) is a nonzero real constant, and \( f(u) \) satisfies certain conditions described later.

The NLS equation (1.1) has many applications and was derived as a model for a considerable range of physical problems, which include propagation of light in fiber optical cables, certain types of shallow and deep surface water waves, and Langmuir waves in a hot plasma or in general forms of Bose-Einstein condensate theory. Recently, the NLS equation has been used as a popular model in attempting to explain the formation of rogue waves observed in the seas or oceans [11, 29]. Here, we are only interested in the well-posedness and stabilization problems for (1.1) or (1.2) under certain boundary conditions.

Mathematical study of (1.1) or (1.2) can be traced back to several decades ago, when Zakharov and his collaborators [37, 38] considered the initial value problem (IVP) of (1.1) on \( \mathbb{R} \) using inverse scattering method. The rigorous theory on the well-posedness of the IVP of (1.1) in the classical Sobolev spaces was extensively

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studied afterwards and mainly focused on the pure IVP posed on the entire real line \( \mathbb{R} \) or the periodic initial-value problem posed on one-dimensional torus \( \mathbb{T} \) (for example, see [2, 3, 9, 10, 15, 16, 22, 23, 36] and a monograph by Cazenave [8]). Moreover, the boundary value problem of (1.1) posed in a finite interval or half line with boundary conditions has been discussed for solutions in Sobolev spaces (see [1, 4, 5, 6, 7, 18, 24, 34], and the references therein). For the control and stabilization problems of NLS equations, Illner, et al. [20, 21] applied an internal forcing to show the controllability of NLS equations posed on a finite interval with periodic boundary conditions, while Lange and Teismann [26] considered the internal controllability of NLS equations in a finite interval with Dirichlet boundary conditions. Rosier and Zhang [30] studied the exact controllability and stabilizability of (1.1) posed on a finite interval using both internal and boundary controls and showed that problems with those controls are locally exactly controllable in the Sobolev space \( H^s \) for any \( s \geq 0 \). It is also shown that the problem with an internal stabilizing forcing is locally exponentially stabilizable.

The main concern of this paper is the local and global well-posedness of (1.1) (or more generally (1.2)) and the asymptotic behavior of small solutions as \( t \to +\infty \) using a closed-loop point dissipation process (general discussions on such problems can be found in [17, 31]). This type of control problems for the KdV equation was first discussed by Russell and Zhang [32, 33] using dissipative point boundary condition, which, in control theory, is called a closed-loop control process that generally refers to control synthesis via some kind of state feedback and is mainly concerned with achieving asymptotic stability of an equilibrium or invariant set. Similar problems using dissipative point boundary conditions were studied for the KdV equation in a singular case [35] and other KdV type of equations [14]. To design a dissipation mechanism for (1.1) or (1.2), we first state some conditions on \( f(u) \) in (1.2).

**Assumption 1.1.** \( f(0) = 0 \) and \( f(u) \neq 0 \) with \( f(u) \in C^1(\mathbb{C}, \mathbb{C}) \) and \( f(u) \bar{u} \in \mathbb{R} \) for any \( u \in \mathbb{C} \);

**Assumption 1.2.** \[ |f(u) - f(v)| \leq f_0(u,v)|u - v| \] where \( f_0(u,v) \leq C_0|u|^{p_1-1} + |v|^{p_1-1} + C_1(|u|^{p_2-1} + |v|^{p_2-1}) \) for any \( u, v \in \mathbb{C} \) with \( 1 < p_1 < p_2 < \infty \).

From Assumptions 1.1 and 1.2, it is easy to see that \( |f(u)| \leq C_0|u|^{p_1} + C_1|u|^{p_2} \). Obviously, \( f(u) = \epsilon |u|^2u \) in (1.1) satisfies Assumptions 1.1 and 1.2.

Now, we multiply both sides of (1.2) by \( \bar{u}(x,t) \) and integrate it from zero to \( L \). Then, take the complex conjugate of the equations and subtract each other to obtain

\[
\frac{d}{dt} \int_0^L |u(x,t)|^2 dx = i \left( u_x(x,t) \bar{u}(x,t) - u(x,t) \bar{u}_x(x,t) \right) \bigg|_0^L = i \left( \bar{u}(L,t) u_x(L,t) - u_x(0,t) \beta \right) + i u(L,t) [\beta \bar{u}_x(0,t) - \bar{u}_x(L,t)] .
\]

(1.3)

To make the \( L^2 \)-norm of \( u \) dissipative, we need the right side of (1.3) non-positive. If we let the boundary values of \( u \) satisfy the conditions

\[ u(0,t) = \beta u(L,t), \quad \beta u_x(0,t) - u_x(L,t) = i \alpha u(0,t), \]

then, it can be shown that

\[
\frac{d}{dt} \int_0^L |u(x,t)|^2 dx = 2 \alpha \beta |u(L,t)|^2 \leq 0 ,
\]

(1.5)
where \( \alpha, \beta \) are any real constants satisfying \( \alpha \beta < 0 \). Thus, we make the following assumptions on \( \alpha, \beta \).

**Assumption 1.3.** \( \alpha, \beta \) are any real numbers satisfying \( \alpha \beta < 0 \) and \( \beta \neq \pm 1 \)

The second condition of Assumption 1.3 on \( \beta \) is technical. If \( \beta = 1 \) or \(-1\), the problem is singular (see also [35]). From (1.5), the boundary conditions (1.4) are considered as dissipative for (1.2) and it is reasonable to expect that the solution \( u(x,t) \) of (1.2) with (1.4) goes to zero as \( t \to \infty \).

The main result of this paper is stated as follows. Let an operator \( Au = iu_{xx} \) with domain \( \mathcal{D}(A) = \{ u \in H^2(0,L) \mid u \text{ satisfies } (1.4) \} \) and \( A^* \) be its adjoint operator in \( L^2 = L^2(0,L) \). Then, the following results are obtained.

**Theorem 1.4.** The operator \( A \) is dissipative and generates a \( C^\infty \)-semigroup \( S(t) \) in \( L^2 \) with \( t \geq 0 \) that decays exponentially as \( t \to +\infty \). Moreover, \( A \) and \( A^* \) have complete eigenfunctions \( \{ \phi_k(x) \mid k = 0, \pm 1, \pm 2, \ldots \} \) and \( \{ \psi_k(x) \mid k = 0, \pm 1, \pm 2, \ldots \} \) in \( L^2 \), respectively. The eigenfunctions are normalized with \( \psi_j^* \phi_k = \delta_{kj} \) and form dual Riesz bases in \( L^2 \).

From Theorem 1.4, the following function space can be defined. For any \( s \geq 0 \), let

\[
H^s_{\alpha, \beta} = \left\{ w = \sum_{k=-\infty}^{\infty} c_k \phi_k \mid \sum_{k=-\infty}^{\infty} (1 + |k|^{2s}) |c_k|^2 < \infty \right\}
\]

with the norm \( \|w\|_{H^s_{\alpha, \beta}}^2 = \|w\|_s^2 = \sum_{k=-\infty}^{\infty} (1 + |k|^{2s}) |c_k|^2 \). Then, the solutions of (1.2) with (1.4) and initial condition \( u(x,0) = w_0(x) \) can be found in \( H^s_{\alpha, \beta} \).

**Theorem 1.5.** Suppose that Assumptions 1.1-1.3 hold, \( s \in \left( \frac{1}{2}, 1 \right] \) and \( w_0 \in H^s_{\alpha, \beta} \).

(i) There exists a \( T = T(\|w_0\|_s) > 0 \) such that a unique solution \( u(x,t) \) of (1.2) with (1.4) exists and satisfies \( u \in C([0,T]; H^s_{\alpha, \beta}) \). The corresponding solution mapping \( N \) from \( H^s_{\alpha, \beta} \) to \( C([0,T']; H^s_{\alpha, \beta}) \) is Lipschitz continuous for any \( T' < T \).

(ii) There is a \( \delta > 0 \) such that if \( \|w_0\|_s < \delta \), then the unique solution \( u \) of (1.2) with (1.4) is in \( H^s_{\alpha, \beta} \) for any \( t \in [0, \infty) \) and decays exponentially to zero as \( t \to \infty \), i.e.,

\[
\|u(\cdot, t)\|_{L^2} \leq ce^{-ct} \|w_0\|_{L^2}, \quad t \geq 0, \tag{1.6}
\]

where \( c > 0 \) and \( c_0 > 0 \) are independent of \( w_0 \) and \( t \).

Note that (i) is a local well-posedness result and (ii) gives the global existence and stability results for small initial data.

The idea for the proof of Theorems 1.4 and 1.5 basically follows from the method introduced in the papers by Russell and Zhang [32, 33], which discuss a similar problem for the KdV equation. However, it is known that the solutions of the linearized KdV equation posed in a bounded interval have smoothing property, while the solutions of the linear Schrödinger equation have no smoothing property. Therefore, the estimates of the linear Schrödinger operator together with the estimates of the corresponding semigroup are more delicate. In this paper, it is first shown that the operator \( A \) is dissipative and generates a strongly \( C_0 \)-semigroup \( S(t) \). After deriving some estimates of the resolvent operator for \( A \), it is proved that the eigenvalues
of $A$ lie in the left half of complex plane and the semigroup $S(t)$ is $C^\infty$ and decays exponentially as $t \to \infty$. Then, under Assumption 1.3 and using the properties of the eigenvalues of $A$, it is shown that $A$ and $A^*$ are discrete spectral operators whose eigenfunctions form dual Riesz bases in $L^2$. Furthermore, the asymptotic forms of the eigenvalues are derived, from which it can be seen that the solutions of linear problem do not have any smoothing property. The spectral properties of $A$ are essential in obtaining the estimates of $S(t)$ in $H^s_{\alpha, \beta}$. The existence and uniqueness results for the nonlinear problem are deduced from those estimates using contraction mapping principle. The asymptotic behavior of the solutions of (1.2) with small initial data is then derived by use of Lyapunov techniques based upon the linear operator $A$ and its spectral properties.

The paper is organized as follows. Section 2 discusses the properties of the operator $A$ with its resolvent operator, the semi-group $S(t)$ generated by the operator $A$, and the exponential decay property of $S(t)$. The spectral properties of $A$ are provided and the function spaces are defined in Section 3. In Section 4, the various estimates of solutions for the linear problem in these function spaces are derived. Section 5 proves the local well-posedness of (1.2), while the global well-posedness and decay of the solutions of (1.2) with small initial data are presented in Section 6.

2. Exponential decay for the linear equation with boundary dissipation.

To study the solutions of (1.2) with (1.4), we may write (1.2) as $u_t = iu_{xx} - if(u)$ and then define a linear operator $A$ by

$$ Au = iu''(x) \quad (2.1) $$

with the domain

$$ \mathcal{D}(A) = \{ u \in H^2(0, L) \mid u(0) = \beta u(L), \; \beta u'(0) - u'(L) = i\alpha u(0) \} . \quad (2.2) $$

The following properties of $A$ hold.

**Lemma 2.1.** The operator $A$ is dissipative. For any $f$ in the range of $\lambda I - A$ with $\lambda > 0$, the pre-image of $f$, denoted by $(\lambda I - A)^{-1}f$, is unique with its $L^2$-norm bounded by $\lambda^{-1}\|f\|$.

**Proof.** By definition, $A$ is dissipative if and only if $\text{Re}(u, Au) \leq 0$ for every $u \in \mathcal{D}(A)$. It is straightforward to check that

$$ 2\text{Re}(u, Au) = (u, Au) + \overline{(u, Au)} = -i \int_0^L \left( uu'' - u'u' \right) dx $$

$$ = -i \left[ u\bar{u}'|_0^L - \int_0^L u'\bar{u}'dx - \bar{u}u'|_0^L + \int_0^L \bar{u}'u'dx \right] $$

$$ = i(u'\bar{u}' - \bar{u}u')|_0^L = 2\alpha\beta|u(L)|^2 \leq 0 , \quad (2.3) $$

i.e., $A$ is dissipative. From (2.3), it is obtained that

$$ \|(\lambda I - A)u\|^2 = \int_0^L |\lambda u - u''|^2 dx = \int_0^L \lambda^2|u|^2 + i\lambda(u\bar{u}'' - \bar{u}u'') + |u''|^2 dx $$

$$ = \lambda^2\|u\|^2 + \|u''\|^2 - 2\lambda\alpha\beta|u(L)|^2 \geq \lambda^2\|u\|^2 , \quad (2.4) $$

which implies that for any $f$ in the range of $\lambda I - A$ and $\lambda > 0$, $(\lambda I - A)^{-1}f$ is unique and $\|(\lambda I - A)^{-1}f\| \leq \frac{\|f\|}{\lambda}$.

\[\square\]
Theorem 2.2. The operator $A$ generates a strongly continuous semigroup $S(t)$ for $t \geq 0$ on $L^2(0,L)$.

Proof. By Lumer-Phillips theorem \cite{27} and Lemma 2.1, $A$ generates a strongly continuous semigroup on $L^2(0,L)$ if the range of $(\lambda I - A)$ is all of $L^2(0,L)$. Thus, we need to prove that there exists $u \in D(\lambda I - A)$, the domain of $\lambda I - A$, such that $(\lambda I - A)u = f$ for any $f \in L^2(0,L)$, i.e., we need to find $u \in D(\lambda I - A)$ satisfying $u'' + i\lambda u = if$. For $\lambda \neq 0$, we denote two square roots of $-i\lambda$ by $\mu_0$ and $\mu_1$, respectively. Denote $w' = z$ and rewrite $u'' + i\lambda u = if$ to a system of first-order differential equations,
\[
\begin{cases}
u' = z \\
z' = u'' = -i\lambda u + if
\end{cases}
\]
which is equivalent to
\[
\vec{u}' = F(\lambda) \vec{u} + \vec{\phi},
\]
with
\[
F(\lambda) = \begin{pmatrix} \mu_0 & 1 \\ -i\lambda & 0 \end{pmatrix},
\]
$\vec{u} = [u, z]^T$, and $\vec{\phi} = [0, if]^T$. We diagonalize (2.5) by a transformation using $\mu = (\mu_0, \mu_1)$,
\[
\vec{u} = \begin{pmatrix} 1 & \mu_0 \\ \mu_1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = M(\mu) \vec{v},
\]
and plug (2.6) into (2.5) to obtain
\[
\vec{v}' = \Omega(\mu) \vec{v} + \vec{\psi}
\]
where
\[
\Omega(\mu) = M(\mu)^{-1} F(\lambda) M(\mu) = \begin{pmatrix} \mu_0 & 0 \\ 0 & \mu_1 \end{pmatrix}, \quad \vec{\psi} = M(\mu)^{-1} \vec{\phi}.
\]
The solution of (2.7) can be written by
\[
\vec{v}(x) = e^{x\Omega(\mu)} \vec{v}(0) + \int_0^x e^{(x-s)\Omega(\mu)} \vec{\psi}(s) ds.
\]
Then, the boundary conditions (1.4) are changed to
\[
\vec{u}'(L) = \begin{pmatrix} \frac{1}{\beta} & 0 \\ -i\alpha & \beta \end{pmatrix} \begin{pmatrix} u(0) \\ z(0) \end{pmatrix} = B(\alpha, \beta) \vec{u}(0),
\]
or
\[
\vec{v}(L) = M(\mu)^{-1} B(\alpha, \beta) M(\mu) \vec{v}(0).
\]
Substituting (2.9) into (2.10) at $x = L$ yields
\[
\left[ B(\alpha, \beta) M(\mu) - M(\mu) e^{L\Omega(\mu)} \right] \vec{v}(0) = M(\mu) \int_0^L e^{(L-s)\Omega(\mu)} \vec{\psi}(s) ds.
\]
To show no positive $\lambda$ satisfying $\det (B(\alpha, \beta) M(\mu) - M(\mu) e^{L\Omega(\mu)}) = 0$, we let $\Phi(\mu, \alpha, \beta) = B(\alpha, \beta) M(\mu) - M(\mu) e^{L\Omega(\mu)}$ and assume that there exists $\lambda$ with $\det(\Phi(\mu, \alpha, \beta)) = 0$. Then, a vector $\vec{v}(0)$ satisfying $\Phi(\mu, \alpha, \beta) \vec{v}(0) = 0$ with $\vec{v}(0) \neq 0$ provides an eigenfunction
\[
\vec{v}(\mu, \alpha, \beta, x) = e^{x\Omega(\mu)} \vec{v}(0)
\]
for $A$, corresponding to the eigenvalue $\lambda$ associated with $\mu$ by $-i\lambda = \mu_j^2$, $j = 0, 1$. For this $\vec{v}(\mu, \alpha, \beta, x)$, there is a nonzero solution $u$ of $u'' + i\lambda u = 0$ (i.e. $\| (\lambda I - A) u \| = 0$), which contradicts to (2.4). Hence, $\Phi(\mu, \alpha, \beta)$ is invertible for any $\lambda > 0$ and

$$
\vec{v}(0) = \Phi(\mu, \alpha, \beta)^{-1} M(\mu) \int_0^L e^{(L-s)\Omega(\mu)} \vec{\psi}(s) ds.
$$

From (2.9), we have

$$
\vec{v}(x) = e^{x\Omega(\mu)} \Phi(\mu, \alpha, \beta)^{-1} M(\mu) \int_0^L e^{(L-x)\Omega(\mu)} \vec{\psi}(x) dx + \int_0^x e^{(x-s)\Omega(\mu)} \vec{\psi}(s) ds.
$$

(2.11)

Therefore, for any $f \in L^2(0, L)$, we can find $u = (\lambda I - A)^{-1} f$, which means that the range of $\lambda I - A$ is all the functions in $L^2(0, L)$ for any $\lambda > 0$. Thus, $A$ generates a strongly continuous semigroup on $L^2(0, L)$.

Now, we study the resolvent of $A$ for $\lambda$ on the imaginary axis.

**Lemma 2.3.** For any $\lambda$ on the imaginary axis, $R(\lambda, A) = (\lambda I - A)^{-1}$ exists on $L^2(0, L)$.

**Proof.** Let $\lambda = i\omega$ and assume that there is a $\tilde{u}(i\omega, \alpha, \beta, x) = \tilde{u} \in D(A)$ satisfying $(A - \lambda I) \tilde{u} = (A - i\omega I) \tilde{u} = 0$. By the identity

$$
(\tilde{u}, A \tilde{u}) + (\tilde{u}, A \tilde{u}) = (\tilde{u}, A \tilde{u}) + (A \tilde{u}, \tilde{u}) = (\tilde{u}, \lambda \tilde{u}) + (\lambda \tilde{u}, \tilde{u}) = i\omega \tilde{u} \tilde{u} - i\omega \tilde{u} \tilde{u} = 0,
$$

$\tilde{u}(L) = 0$ follows from (2.3). Thus, by the boundary conditions (1.4), $\tilde{u} \in D(A)$ satisfies

$$
\tilde{u}'' = i\omega \tilde{u}, \quad \text{with } \tilde{u}(0) = \tilde{u}(L) = 0, \quad \beta \tilde{u}_x(0) - \tilde{u}_x(L) = 0.
$$

(2.12)

If $\omega \neq 0$, since $\mu_0$ and $\mu_1$ are two different square roots of $-i\lambda = \omega$,

$$
\tilde{u} = c_0 e^{\mu_0 x} + c_1 e^{\mu_1 x}, \quad \tilde{u}_x = c_0 \mu_0 e^{\mu_0 x} + c_1 \mu_1 e^{\mu_1 x}.
$$

Applying the boundary conditions (2.12) yields

$$
\tilde{u}(0) = c_0 + c_1 = 0 \Rightarrow c_1 = -c_0,
$$

$$
\tilde{u}(L) = c_0 e^{\mu_0 L} + c_1 e^{\mu_1 L} = c_0 (e^{\mu_0 L} - e^{\mu_1 L}) = 0 \Rightarrow c_0 = 0 \Rightarrow c_1 = 0,
$$

which implies that $\tilde{u} = 0$ and $(A - \lambda I) \tilde{u} = 0$ has only trivial solution for $\omega \neq 0$. Similarly, the case for $\omega = 0$ also implies $\tilde{u} = 0$. Since $A$ has only discrete spectrum, we conclude that for any $\lambda$ on the imaginary axis, $R(\lambda, A)$ exists.

The following is the resolvent estimate for large $\lambda$ on the imaginary axis.

**Lemma 2.4.** The operator norm of the resolvent $R(i\omega, A)$ satisfies $\| R(i\omega, A) \| = O(\omega^{-\frac{1}{2}})$ when $|\omega| \to \infty$.

**Proof.** First, we find the solution $u$ of $(\lambda I - A) u = f$ with boundary conditions (1.4) using Green’s function. If we define $G(\lambda, x, \zeta)$ that satisfies

$$
\lambda \Gamma(\lambda, x, \zeta) - iG''(\lambda, x, \zeta) = \delta(x - \zeta),
$$

$$
G(\lambda, 0, \zeta) = \beta G(\lambda, L, \zeta),
$$

(2.13)

$$
\beta G'(\lambda, 0, \zeta) - G'(\lambda, L, \zeta) = i\alpha G(\lambda, 0, \zeta),
$$

(2.14)

then the solution $u$ is given by

$$
u(\lambda, \alpha, \beta, x) = \int_0^L G(\lambda, x, \zeta) f(\zeta) d\zeta.
$$

(2.15)
The Green’s function $G$ can be found as follows. From the homogeneous equation $(\lambda I - A)u = 0$, it is obtained that

$$G(\lambda, x, \zeta) = \begin{cases} G_1(\lambda, x, \zeta), & x > \zeta, \\ G_2(\lambda, x, \zeta), & x \leq \zeta, \end{cases}$$

$$= \begin{cases} (c_0 + \hat{c}_0)e^{\mu_0(x-\zeta)} + (c_1 + \hat{c}_1)e^{\mu_1(x-\zeta)}, & x > \zeta, \\ c_0e^{\mu_0(x-\zeta)} + c_1e^{\mu_1(x-\zeta)}, & x \leq \zeta. \end{cases}$$ (2.16)

By the conditions at $x = \zeta$,

$$G_1(\lambda, \zeta, \zeta) - G_2(\lambda, \zeta, \zeta) = 0, \quad G'(\lambda, \zeta, \zeta) - G''(\lambda, \zeta, \zeta) = i,$$

it is deduced that

$$\hat{c}_0 = \frac{i}{\mu_0 - \mu_1}, \quad \hat{c}_1 = \frac{i}{\mu_1 - \mu_0}. \tag{2.17}$$

The boundary conditions (2.13) and (2.14) imply that

$$\begin{align*}
\hat{c}_0 e^{-\mu_0 \zeta} \left( \frac{1}{\beta} - e^{\mu_0 L} \right) + c_1 e^{-\mu_1 \zeta} \left( \frac{1}{\beta} - e^{\mu_1 L} \right) &= \hat{c}_0 e^{\mu_0(L-\zeta)} + \hat{c}_1 e^{\mu_1(L-\zeta)}, \tag{2.18} \\
\hat{c}_0 e^{-\mu_0 \zeta} \left( \beta \mu_0 - \mu_0 e^{\mu_0 L} - i\alpha \right) + c_1 e^{-\mu_1 \zeta} \left( \beta \mu_1 - \mu_1 e^{\mu_1 L} - i\alpha \right) &= \hat{c}_1 e^{\mu_1(L-\zeta)} + \hat{c}_0 e^{\mu_0(L-\zeta)}. \tag{2.19}
\end{align*}$$

From the definitions in the proof of Theorem 2.2, we have that if $c = (c_0, c_1)^T$ and $\hat{c} = (\hat{c}_0, \hat{c}_1)^T$, then

$$\Phi(\mu, \alpha, \beta) = B(\alpha, \beta)M(\mu)M^2(\mu) - M(\mu)Pe^{i\Omega(\mu)} = \begin{pmatrix} \frac{1}{\beta} - e^{\mu_0 L} & \frac{1}{\beta} - e^{\mu_1 L} \\ -i\alpha + \beta \mu_0 - \mu_0 e^{\mu_0 L} & -i\alpha + \beta \mu_1 - \mu_1 e^{\mu_1 L} \end{pmatrix},$$

and

$$\Phi(\mu, \alpha, \beta) e^{-\mu_0 \zeta} c = \begin{pmatrix} \hat{c}_0 \left( \frac{1}{\beta} - e^{\mu_0 L} \right) e^{-\mu_0 \zeta} + c_1 \left( \frac{1}{\beta} - e^{\mu_1 L} \right) e^{-\mu_1 \zeta} \\ c_0 \left( -i\alpha + \beta \mu_0 - e^{\mu_0 L} \right) e^{-\mu_0 \zeta} + c_1 \left( -i\alpha + \beta \mu_1 - e^{\mu_1 L} \right) e^{-\mu_1 \zeta} \end{pmatrix},$$

with

$$M(\mu)Pe^{i\Omega(L-\zeta)} \hat{c} = \begin{pmatrix} \hat{c}_0 e^{i\Omega(L-\zeta)} e^{\mu_0(L-\zeta)} & \hat{c}_1 e^{i\Omega(L-\zeta)} e^{\mu_1(L-\zeta)} \\ \hat{c}_0 e^{i\Omega(L-\zeta)} e^{\mu_0(L-\zeta)} & \hat{c}_1 e^{i\Omega(L-\zeta)} e^{\mu_1(L-\zeta)} \end{pmatrix}.$$

and $\Omega(\mu)$ defined in (2.8). Thus, if

$$\overline{q}(\mu, \zeta) = e^{-\Omega(\mu) \zeta} c \quad \text{and} \quad \overline{p}(\mu, \zeta) = M(\mu)e^{i\Omega(L-\zeta)} \hat{c},$$

we can rewrite (2.18) and (2.19) by

$$\Phi(\mu, \alpha, \beta) \overline{q}(\mu, \zeta) = \overline{p}(\mu, \zeta).$$

From the Cramer’s rule, $\overline{q}(\mu, \zeta)$ can be solved as follows,

$$\overline{q}(\mu, \zeta) = \frac{\overline{p}(\mu, \zeta)}{\det(\Phi(\mu, \alpha, \beta))},$$

where $\overline{q}(\mu, \zeta) = (r_0, r_1)^T$, and $r_0$ (or $r_1$) is the determinant of the matrix obtained from $\Phi(\mu, \alpha, \beta)$ by replacing the first (or second) column of $\Phi$ by $\overline{p}(\mu, \zeta)$.  

Without loss of generality, let \( \omega > 0 \) and \( \omega = \rho^2 \) with \( \rho > 0 \). Thus, the square roots of \(-i\lambda = -i(\omega) = \omega = \rho^2\) are \( \rho \) and \(-\rho\) and \( \mu_0 = \rho, \mu_1 = -\rho \). Hence, as \( \rho \to \infty \),

\[
r_0(\mu, \zeta) = \det \begin{pmatrix}
\hat{c}_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 e^{(L-\zeta)\mu_1} & 1 - \rho e^{\mu_1 L} \\
\hat{c}_0\mu_0 e^{(L-\zeta)\mu_0} + \hat{c}_1\mu_1 e^{(L-\zeta)\mu_1} & -i\alpha + \beta\mu_1 - \mu_1 e^{\mu_1 L}
\end{pmatrix}
\]

\[
= \hat{c}_0 e^{(L-\zeta)\rho} \left[ -i\alpha - \rho \left( \beta + \frac{1}{\beta} \right) + O(\rho e^{-L\rho}) \right] + \hat{c}_1 e^{-(L-\zeta)\rho} \left[ -i\alpha - \rho \left( \beta - \frac{1}{\beta} \right) + O(\rho e^{-L\rho}) \right],
\]

\[
r_1(\mu, \zeta) = \det \begin{pmatrix}
\frac{1}{\beta} - \rho e_{\mu_0 L} & \hat{c}_0 e^{(L-\zeta)\mu_0} + \hat{c}_1 e^{(L-\zeta)\mu_1} \\
-i\alpha + \beta\mu_0 - \mu_0 e^{\mu_0 L} & \hat{c}_0\mu_0 e^{(L-\zeta)\mu_0} + \hat{c}_1\mu_1 e^{(L-\zeta)\mu_1}
\end{pmatrix}
\]

\[
= \hat{c}_0 e^{(L-\zeta)\rho} \left[ i\alpha + \left( \frac{1}{\beta} - \beta \right) \rho + O(\rho e^{-L\rho}) \right] + \hat{c}_1 e^{-(L-\zeta)\rho} \left[ i\alpha - \left( \frac{1}{\beta} + \beta \right) \rho + O(\rho e^{-L\rho}) \right],
\]

\[
\det(\Phi(\mu, \alpha, \beta)) = \left( \frac{1}{\beta} - \rho e_{\mu_0 L} \right) \left( -i\alpha + \beta\mu_1 - \mu_1 e^{\mu_1 L} \right)
\]

\[
- \left( \frac{1}{\beta} - e^{\mu_1 L} \right) \left( -i\alpha + \beta\mu_0 - \mu_0 e^{\mu_0 L} \right)
\]

\[
= \frac{1}{\beta} \left( -2\beta\rho + \rho e^{\mu L} \right) + \rho e^L + \beta\rho e^{\mu L} - 2\rho + O(\rho e^{-L\rho})
\]

\[
= e^{\mu L} \left( \frac{\rho}{\beta} + \rho\beta + i\alpha + O(\rho e^{-L\rho}) \right),
\]

which imply that

\[
\begin{aligned}
\overrightarrow{q}(\mu, \zeta) &= \frac{\overrightarrow{q}(\mu, \zeta)}{\det(\Phi(\mu, \alpha, \beta))} \\
&\approx \hat{c}_0 e^{(L-\zeta)\rho} \left( \frac{-i\alpha - \rho(\beta + \frac{1}{\beta})}{e^{x(\mu_0 - \beta + \frac{1}{\beta})} + \rho e^L} \right) + \hat{c}_1 e^{-(L-\zeta)\rho} \left( \frac{-i\alpha - \rho(\beta - \frac{1}{\beta})}{e^{x(\mu_0 - \beta + \frac{1}{\beta})} + \rho e^L} \right),
\end{aligned}
\]

Let \( \epsilon = (1, 1)^T \) and \( \epsilon^* = (1, 1) \). Then,

\[
G(\lambda, x, \zeta) = \epsilon^* \left[ e^{\mu L} \overrightarrow{q}(\mu, \zeta) + H(x - \zeta) e^{\mu(x - \zeta)} \hat{c}(\mu) \right].
\]

For \( x \leq \zeta \),

\[
G(\lambda, x, \zeta) = \epsilon^* \left[ e^{\mu L} \overrightarrow{q}(\mu, \zeta) \right]
\]

\[
\approx (1 \ 1) \begin{pmatrix}
\epsilon^x & 0 \\
0 & e^{-\rho x}
\end{pmatrix} \begin{pmatrix}
\frac{-i\alpha - \rho(\beta + \frac{1}{\beta})}{e^{x(\mu_0 - \beta + \frac{1}{\beta})} + \rho e^L} & \hat{c}_0 e^{(L-\zeta)\rho} \\
\frac{-i\alpha - \rho(\beta - \frac{1}{\beta})}{e^{x(\mu_0 - \beta + \frac{1}{\beta})} + \rho e^L} & \hat{c}_1 e^{-(L-\zeta)\rho}
\end{pmatrix}.
\]
\[= \hat{c}_0 e^{\rho(x-\zeta)} \frac{-i\alpha - \rho(\beta + \beta^{-1})}{\rho \beta^{-1} + \rho \beta + i\alpha} + \hat{c}_0 \frac{i\alpha + (\beta^{-1} - \beta)\rho}{e^{\rho(x-\zeta)}(\rho \beta^{-1} + \rho \beta + i\alpha)} + \hat{c}_1 e^{\rho(x+\zeta-2L)} \frac{-i\alpha - \rho(\beta - \beta^{-1})}{\rho \beta^{-1} + \rho \beta + i\alpha} + \hat{c}_1 \frac{i\alpha - (\beta^{-1} + \beta)\rho}{e^{\rho(x-\zeta+2L)}(\rho \beta^{-1} + \rho \beta + i\alpha)}\]

\[= \hat{c}_0 \left( O\left(e^{\rho(x-\zeta)}\right) + O\left(e^{\rho(-x-\zeta)}\right) \right) + \hat{c}_1 \left( O\left(e^{\rho(x+\zeta-2L)}\right) + O\left(e^{\rho(-x+\zeta-2L)}\right) \right),\]

as \(\rho \to \infty\), where

\[e^{\rho(x-\zeta)}, \ e^{\rho(-x-\zeta)}, \ e^{\rho(x+\zeta-2L)} \text{ and } e^{\rho(-x+\zeta-2L)}\]

are uniformly bounded for \(0 \leq x \leq \zeta \leq L\). From (2.17), \(|\hat{c}_0| \approx \rho^{-1}, |\hat{c}_1| \approx \rho^{-1}\) as \(\rho \to \infty\). Thus, we conclude that if \(r > 0\) is given, there exists a constant \(\xi_r\) independent of \(\rho\) such that \(|G(\lambda, x, \zeta)| \leq \xi_r \rho^{-1}\) if \(x < \zeta\) and \(\rho > r\). If \(x > \zeta\), by the above calculations, it is deduced that

\[G(\lambda, x, \zeta) = e^{\rho(x-\zeta)} \left[ e^{\rho(-x-\zeta)} \hat{c}(\mu) + e^{\rho(x-\zeta)} \hat{c}(\mu) \right]\]

\[= \hat{c}_0 e^{\rho(x-\zeta)-i\alpha - \rho(\beta + \beta^{-1}) + O(\rho e^{-\rho L})} + \hat{c}_0 \left( e^{\rho(-x-\zeta)} \right) + \hat{c}_1 \left( O\left(e^{\rho(x+\zeta-2L)}\right) + O\left(e^{\rho(-x+\zeta-2L)}\right) \right) + \hat{c}_0 e^{\rho(x-\zeta)} + \hat{c}_1 e^{-\rho(x-\zeta)}\]

\[= \hat{c}_0 \frac{e^{\rho(x-\zeta)-i\alpha - \rho(\beta + \beta^{-1}) + O(\rho e^{-\rho L})} + \hat{c}_0 \left( e^{\rho(-x-\zeta)} \right) + \hat{c}_1 \left( O\left(e^{\rho(x+\zeta-2L)}\right) + O\left(e^{\rho(-x+\zeta-2L)}\right) \right),\]

as \(\rho \to \infty\). Similar to the case of \(x \leq \zeta\), we obtain that \(e^{-\rho(x-\zeta)}, e^{\rho(x-\zeta-L)}\) and other similar terms are uniformly bounded for \(L \geq x \geq \zeta \geq 0\) and (2.17) implies \(|\hat{c}_0|, |\hat{c}_1| \approx \rho^{-1}\) as \(\rho \to \infty\). Therefore, it is obtained that if \(r > 0\) is given, there exists a constant \(\zeta_r\) independent of \(\rho\) such that \(|G(\lambda, x, \zeta)| \leq \zeta_r \rho^{-1}\) if \(x > \zeta\) and \(\rho > r\).

From above discussion, it is found that \(|G(\lambda, x, \zeta)| \leq c_r \rho^{-1}\) for \(\rho > r\). Because of \(\omega = \rho^2\), (2.15) yields \(\|R(\omega, A)\| = O(\omega^{-\frac{1}{2}})\), as \(|\omega| \to \infty\). \(\square\)

Lemma 2.4 implies the following Corollary using Corollary 4.10 in Chapter 2 of [28].

**Corollary 2.5.** The semigroup \(S(t)\) generated by the operator \(A\) is a \(C^\infty\) semigroup on \(L^2(0, L)\).

The next theorem gives the exponential decay of the semi-group for the linear Schrödinger operator (2.1) with boundary conditions given by (2.2).

**Theorem 2.6.** There exist positive constants \(\xi\) and \(\eta\) such that

\[\|S(t)\| \leq \xi e^{-\eta t}, \quad t \geq 0.\]  

**Proof.** By Lemmas 2.3 and 2.4, it is obtained that for all \(\lambda\) on the imaginary axis, \(R(\lambda, A)\) exists and is uniformly bounded for large \(\lambda\). Thus, to derive the uniform exponential decay property of \(S(t)\) using the result by Huang [19], we only need to prove that \(R(\lambda, A)\) is bounded as \(\lambda \to 0\). From (2.6), (2.8) and (2.11), it is deduced
In this section, we will discuss the spectral properties of \( A \) with its semi-group \( \lambda \) formly bounded in a small neighborhood of \( \lambda \). Therefore, by the continuity of (2.21) with respect to \( \lambda \) and \( \omega \), it is straightforward to find the adjoint operator \( A^\ast \) of \( A \) in \( L^2 = L^2(0, L) \) as

\[
(A^\ast v)(x) = -i\nu''(x)
\]

with domain

\[
\mathcal{D}(A^\ast) = \left\{ v \in H^2(0, L) \mid v(0) = \beta v(L), \beta v'(0) - v'(L) = -i\alpha v(0) \right\}.
\]

**Proposition 3.1.** The operator \( A \) is a discrete spectral operator. All of its eigenvalues \( \lambda \) except for first few ones correspond to one-dimensional projections \( E(\lambda; T) \).

**Proof.** The proposition is a direct consequence of Theorem 8 in Section 4 of Chapter XIX on p.2334 of Dunford and Schwartz’s book [12], if we can check that the hypotheses of the theorem are satisfied. By the notations introduced there, we let

\[
B_1(u) = u(0) - \beta u(L), \quad B_2(u) = \beta u'(0) - u'(L) - i\alpha u(0).
\]

Also, \( p = m_1 + m_2 = 0 + 1 = 1 \) and \( n = 2 \) with \( n = 2\nu \) and \( \nu = 1 \). \( \omega_0 = 1, \omega_\nu = \omega_1 = -1 \) and \( \sigma_k(x, \mu) = e^{i\mu kx} \) with \( k = 0, 1 \). Then, \( \sigma_k(0, \mu) = i\mu e^{i\mu x}, \sigma_k(1, \mu) = -i\mu e^{-i\mu x} \). Moreover,

\[
M_{ik}(\mu) = B_i(\sigma_k(\mu)), \quad i = 1, 2, \quad k = 0, 1
\]
\[ N(\mu) = \det \begin{pmatrix} B_1(\sigma_0) & B_1(\sigma_1) \\ B_2(\sigma_0) & B_2(\sigma_1) \end{pmatrix} = \det \begin{pmatrix} 1 - \beta e^{\mu L} & 1 - \beta e^{-\mu L} \\ i\mu \beta - i\mu e^{\mu L} - i\alpha & -i\mu \beta + i\mu e^{-\mu L} - i\alpha \end{pmatrix} = e^{\mu L} \left[ i\mu (\beta^2 + 1) + i\alpha \beta \right] + e^{-\mu L} \left[ i\mu (\beta^2 + 1) - i\alpha \beta \right] - 4i\mu \beta. \]

Thus, the coefficients are
\[ \Pi_1(\mu) = i\mu (\beta^2 + 1) + i\alpha \beta, \quad \Pi_2(\mu) = i\mu (\beta^2 + 1) - i\alpha \beta, \quad \Pi_3(\mu) = -4i\mu \beta, \]
with the leading order parts in terms of \( \mu \) as
\[ a_p = i(\beta^2 + 1), \quad b_p = i(\beta^2 + 1), \quad c_p = -4i \beta. \]
It is obtained that \( \kappa = a_p e^{i\alpha} = -b_p e^{-i\alpha}. \text{If } e^{i\alpha} = s, \text{ then } s = \pm i \text{ and } \kappa = \mp (\beta^2 + 1), \]
which implies that
\[ \theta = \frac{c_p}{2i\kappa} = \frac{-4i\beta}{\mp 2(\beta^2 + 1)} = \pm \frac{2\beta}{\beta^2 + 1} \neq \pm 1. \]
Hence, the hypotheses of Theorem 8 in Section 4 of Chapter XIX of Dunford and Schwartz’s book [12] on p.2334 are satisfied.

In the following, for \( \psi \in L^2 \), we denote \( \psi^* \) as the corresponding adjoint vector of \( \psi \) in the Hilbert space \( L^2 \); i.e., for any \( \phi \in L^2 \),
\[ \psi^* \phi = (\phi, \psi)_{L^2}. \quad (3.3) \]

**Proposition 3.2.** The operators \( A, A^* \) with their corresponding domains have compact resolvents and possess complete sets of eigenvectors,
\[ \{ \phi_k \mid -\infty < k < +\infty \}, \quad \{ \psi_k \mid -\infty < k < +\infty \}, \]
respectively. The eigenvectors form dual Riesz bases in \( L^2(0, L) \) and satisfy (with \( \delta_{kj} \), the Kronecker delta) \( \psi_j^* \phi_k = \delta_{kj} \), while the eigenvalues \( \lambda \) of \( A \) satisfy \( \text{Re} \lambda \leq -\gamma < 0 \) and have the asymptotic form
\[ \lambda_k = \frac{-4\pi \tau}{L^2} - i \frac{(2k\pi + O(1))^2}{L^2} \quad \text{as} \quad k \to \infty \quad (3.4) \]
with
\[ \tau = \frac{-\alpha \beta L}{2\pi(\beta^2 + 1)} > 0. \quad (3.5) \]

**Proof.** By Lemmas 2.1 and 2.3, it is obvious that any \( \lambda \) with \( \text{Re} \lambda \geq 0 \) is not an eigenvalue of operator \( A \) or \( A^* \). Thus, the eigenvalues must satisfy \( \text{Re} \lambda < 0 \). For \( \text{Im} \lambda > 0 \), the eigenfunction \( \phi \) satisfies
\[ \phi'' + i\lambda \phi = 0, \quad \phi(0) = \beta \phi(L), \quad \beta \phi'(0) - \phi'(L) = i\alpha \phi(0). \quad (3.6) \]
The general solution of (3.6) is
\[ \phi(x) = c_0 e^{\mu_0 x} + c_1 e^{\mu_1 x}. \quad (3.7) \]
Substituting (3.7) into the boundary conditions in (3.6), we have
\[ c_0 + c_1 = \beta c_0 + c_1, \quad \beta \mu_0 c_0 + \beta c_1 e^{\mu_1 L}, \quad \beta \mu_0 c_0 + \beta \mu_1 c_1 - (\mu_0 c_0 e^{\mu_0 L} + \mu_1 c_1 e^{\mu_1 L}) = i\alpha (c_0 + c_1), \quad (3.8) \]
which give a system of equations
\[
\begin{pmatrix}
1 - \beta e^{\mu_0 L} & 1 - \beta e^{\mu_1 L} \\
\beta \mu_0 - \mu_0 e^{\mu_0 L} - i \alpha & \beta \mu_1 - \mu_1 e^{\mu_1 L} - i \alpha
\end{pmatrix}
\begin{pmatrix}
c_0 \\
c_1
\end{pmatrix} = 0.
\]
Setting the determinant of the coefficient matrix equal to zero yields
\[
(1 - \beta e^{\mu_0 L}) (\beta \mu_1 - \mu_1 e^{\mu_1 L} - i \alpha) - (1 - \beta e^{\mu_1 L}) (\beta \mu_0 - \mu_0 e^{\mu_0 L} - i \alpha)
= -4\beta \mu_0 + e^{\mu_0 L} (\beta^2 \mu_0 + i \alpha \beta + \mu_0) + e^{-\mu_0 L} (\beta^2 \mu_0 + \mu_0 - i \alpha \beta) = 0.
\]
(3.9)

If the real part of \(\mu_0 \to \infty\), (3.9) implies \((\beta^2 + 1) = 0\), which does not hold. Thus, the real part of \(\mu_0\) must be bounded. Dividing (3.9) by \(\mu_0\) gives
\[
0 = \frac{-4\beta \mu_0}{\mu_0} + e^{\mu_0 L} \frac{\beta^2 \mu_0 + i \alpha \beta + \mu_0}{\mu_0} + e^{-\mu_0 L} \frac{\beta^2 \mu_0 + \mu_0 - i \alpha \beta}{\mu_0}
= -4\beta + e^{\mu_0 L} (\beta^2 + 1) + e^{-\mu_0 L} (\beta^2 + 1) + O(1/\mu_0), \quad \text{as } |\mu_0| \to \infty.
\]

Hence,
\[
4\beta = (e^{\mu_0 L} + e^{-\mu_0 L})(\beta^2 + 1) + O(1/\mu_0) \Rightarrow \cosh(\mu_0 L) = \frac{2\beta}{(\beta^2 + 1)} + O(1/\mu_0).
\]
If \(\mu_0 = a + bi\), then
\[
\cosh(aL) \cos(bL) = \frac{2\beta}{\beta^2 + 1} + O(1/\mu_0), \quad \sinh(aL) \sin(bL) = O(1/\mu_0).
\]

Since \(\cosh x \geq 1\) and \(|2\beta/(\beta^2 + 1)| < 1\) for \(\beta \neq \pm 1\), \(|\cos(bL)|\) cannot approach to one for large \(b\) or \(\sin(bL)\) cannot go to zero. Thus, \(\sin(aL) = O(1/\mu_0)\), \(\cosh(aL) = O(1)\), and
\[
\cos(bL) = \frac{2\beta}{\beta^2 + 1} + O(1/\mu_0), \quad \sin(bL) = \pm \frac{\beta^2 - 1}{\beta^2 + 1} + O(1/\mu_0).
\]

If \(\beta > 0\), let \(\theta = \sin^{-1} \frac{\beta^2 - 1}{\beta^2 + 1}\), and if \(\beta < 0\), let \(\theta = \pi - \sin^{-1} \frac{\beta^2 - 1}{\beta^2 + 1}\). Therefore, \(bL = 2k\pi + \theta + O(1/\mu_0)\) and \(\mu_0 L = i(2k\pi + \theta + O(1/\mu_0)) = i(2k\pi + \theta + \varepsilon_k)\). (3.9) gives
\[
0 = -4\beta i(2k\pi + \theta + \varepsilon_k)/L
+ (\beta^2 + 1) \left( e^{i(2k\pi + \theta + \varepsilon_k)} + e^{-i(2k\pi + \theta + \varepsilon_k)} \right) i(2k\pi + \theta + \varepsilon_k)/L
+ i\alpha \beta \left( e^{i(2k\pi + \theta + \varepsilon_k)} - e^{-i(2k\pi + \theta + \varepsilon_k)} \right)
-i(2k\pi + \theta + \varepsilon_k)/L \left( -4\beta + 2(\beta^2 + 1) \cos(\theta + \varepsilon_k) \right) - 2\alpha \beta \sin(\theta + \varepsilon_k).
\]
Since
\[
\cos(\theta + \varepsilon_k) = \cos \theta \cos \varepsilon_k - \sin \theta \sin \varepsilon_k = \cos \theta - \varepsilon_k \sin \theta + O(\varepsilon_k^2)
\]
and \(\sin(\theta + \varepsilon_k) = \sin \theta + O(\varepsilon_k)\), it is obtained that
\[
2(\beta^2 + 1) \varepsilon_k \sin \theta = -(\alpha \beta L/i k \pi) \sin \theta + O(1/k^2)
\]
where \(\cos \theta = 2\beta/(\beta^2 + 1)\) has been used. Thus,
\[
\varepsilon_k = -((\alpha \beta L)/(2i k \pi (\beta^2 + 1))) + O(1/k^2) \text{ or } \mu_0 L = i(2k\pi + \theta) + (\tau/k) + O(1/k^2),
\]
where \( \tau = (-\alpha \beta L)/(2(\beta^2 + 1)\pi) > 0 \). Hence,
\[
\lambda_k = i \mu_k^0 = i \left( \frac{T}{\kappa} + i(2k\pi + \theta) \right)^2 L^{-2} + O(1/k)
\]
\[
= i \left( 4\tau \pi - (2k\pi + \theta)^2 \right) L^{-2} + O(1/k)
\]
\[
= - \frac{4\tau \pi}{L^2} \frac{i(2k\pi + \theta)^2}{L^2} + O \left( \frac{1}{k} \right).
\]

Then, Rouché’s theorem yields a one-to-one correspondence between the eigenvalues \( \lambda_k \) and the indices \( k, k = 0, \pm 1, \pm 2, \ldots \). Therefore, there is a \( \gamma > 0 \) such that \( \Re \lambda_k \leq -\gamma < 0 \). A similar argument gives that \( \overline{\lambda_k} \), the complex conjugate of \( \lambda_k \), is the eigenvalue of adjoint operator \( A^* \). Hence, the eigenfunction of \( A \) for the eigenvalue \( \lambda_k \) is
\[
\phi_k(x) = c_{0,k} e^{\mu_0,kx} + c_{1,k} e^{\mu_1,kx}.
\]

(3.8) implies that
\[
c_{1,k} = -\frac{1 - \beta e^{\mu_0,kL}}{1 - \beta e^{\mu_1,kL}} c_{0,k}
\]
and \( c_{1,k} \) is uniformly bounded relative to \( c_{0,k} \) as \( |k| \to \infty \). The eigenfunctions \( \psi_k(x) \) of adjoint operator \( A^* \) take the form
\[
\psi_k(x) = \overline{\phi_k(x)}, \quad -\infty < k < \infty.
\]

Next, to proof \( \psi_k^* \phi_j \equiv (\phi_j, \psi_k)_{L^2} = \delta_{kj} \) for \( -\infty < k, j < \infty \), note that
\[
(\psi_k, \phi_j) = \int_0^L \phi_j \overline{\psi}_k \, dx = \int_0^L \phi_k \phi_j \, dx.
\]

By the boundary conditions (2.2), it is obtained that
\[
\int_0^L \phi_k^* \phi_j \, dx = \int_0^L \phi_j \overline{\phi}_k \, dx = \phi_j \overline{\phi}_k \bigg|_0^L - \int_0^L \phi_j^* \phi_k \, dx
\]
\[
= \phi_j \overline{\phi}_k \bigg|_0^L - \int_0^L \phi_j^* \phi_k \, dx = \phi_j \overline{\phi}_k \bigg|_0^L - \phi_j^* \phi_k \bigg|_0^L + \int_0^L \phi_j^* \phi_k \, dx = \int_0^L \phi_j^* \phi_k \, dx.
\]

Hence,
\[
\int_0^L \phi_k \phi_j \, dx = \frac{i}{\lambda_k} \int_0^L \phi_k^* \phi_j \, dx = \frac{i}{\lambda_k} \int_0^L \phi_j \overline{\phi}_k \, dx
\]
\[
= \frac{i}{\lambda_k} \int_0^L \phi_k \lambda_j \phi_j \, dx = \frac{\lambda_j}{\lambda_k} \int_0^L \phi_k \phi_j \, dx,
\]
which implies
\[
\left( 1 - \frac{\lambda_j}{\lambda_k} \right) \int_0^L \phi_k \phi_j \, dx = 0.
\]

When \( j \neq k \) that implies \( \lambda_k \neq \lambda_j \), we have that \( \int_0^L \phi_k \phi_j \, dx = 0 \) and \( (\psi_k, \phi_j) = 0 \). When \( j = k \), an appropriate choice of the coefficients \( c_0 \) and \( c_1 \) makes \( (\psi_k, \phi_j) = ||\psi_k||_{L^2} = ||\phi_k||_{L^2} = 1 \).

Now, we show that \( \{\psi_k\} \) and \( \{\phi_k\} \) form Riesz bases. It was shown that both \( A \) and \( A^* \) are discrete spectral operators. Then, we apply the following slightly modified Ingham-Komornik result in [25]: Let \( \{\lambda_k\}_{k \geq 1} \) be a sequence of complex numbers with \( \Re \lambda_k \) uniformly bounded, \( \{b_k\}_{k \geq 1} \) a sequence of complex numbers satisfying
From (3.15) and (3.16), it is shown that the sequences \( \{b_k\}_{k \geq 1} \) are two Riesz bases in \( L^2 \). Then for any square-summable sequence \( \{x_k\}_{k \geq 1} \) of complex numbers,

\[
\int_\mathbb{R} \chi(x) \left\| \sum_{k \geq 1} x_k b_k e^{\lambda_k x} \right\|^2 \, dx \leq C(\chi) \sum_{k \geq 1} |x_k|^2
\]

where \( \chi(x) \) is an infinitely differentiable function of compactly supported in \( \mathbb{R} \) and \( C(\chi) \) is a constant dependent upon \( \chi \).

Thus, by the forms of \( c_{0,k}, c_{1,k}, \mu_{0,k}, \mu_{1,k} \) derived above, it is straightforward to check that the conditions in the Ingham-Komornik result with \( \tilde{\lambda}_k = \mu_{0,k} (\text{or } \mu_{1,k}) \) and \( b_k = c_{0,k} (\text{or } c_{1,k}) \) are satisfied. Thus, if \( \chi(x) = 1 \) for \( x \in [0, L] \), it is obtained that

\[
\left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 = \left\| \sum_{k=-\infty}^{\infty} f_k \left( c_{0,k} e^{\mu_{0,k} x} + c_{1,k} e^{\mu_{1,k} x} \right) \right\|_{L^2}^2 \\
\leq \left( \left\| \sum_{k=-\infty}^{\infty} f_k c_{0,k} e^{\mu_{0,k} x} \right\|_{L^2} + \left\| \sum_{k=-\infty}^{\infty} f_k c_{1,k} e^{\mu_{1,k} x} \right\|_{L^2} \right)^2 \leq D^2 \sum_{k=-\infty}^{\infty} |f_k|^2.
\]

A similar argument gives

\[
\left\| \sum_{j=-\infty}^{\infty} g_j \psi_j \right\|_{L^2}^2 \leq D^2 \sum_{j=-\infty}^{\infty} |g_j|^2.
\]

Hence, \( \phi_k \) and \( \psi_k \), \( k = 0, \pm 1, \pm 2, \ldots \), have the uniform \( l^2 \)-convergence property. Since \( \langle \psi_k, \phi_j \rangle_{L^2} = \delta_{kj} \),

\[
\left( \sum_{k=-\infty}^{\infty} |f_k|^2 \right)^2 = \left( \sum_{k=-\infty}^{\infty} f_k \phi_k, \sum_{j=-\infty}^{\infty} f_j \psi_j \right)_{L^2}^2 \\
\leq \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 \left\| \sum_{j=-\infty}^{\infty} f_j \psi_j \right\|_{L^2}^2 \leq \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2 D^2 \sum_{j=-\infty}^{\infty} |f_j|^2 \\
\Rightarrow D^{-2} \sum_{k=-\infty}^{\infty} |f_k|^2 \leq \left\| \sum_{k=-\infty}^{\infty} f_k \phi_k \right\|_{L^2}^2.
\]

A similar argument gives

\[
D^{-2} \sum_{j=-\infty}^{\infty} |g_j|^2 \leq \left\| \sum_{j=-\infty}^{\infty} g_j \psi_j \right\|_{L^2}^2.
\]

From (3.15) and (3.16), it is shown that the sequences \( \{\phi_k\} \) and \( \{\psi_k\} \) also have the uniform \( l^2 \)-independent property. Thus, from Proposition 3.1, we showed that both of \( \{\phi_k\} \) and \( \{\psi_k\} \) are complete in \( L^2(0, L) \), which yields that \( \{\phi_k\} \) and \( \{\psi_k\} \) are two Riesz bases in \( L^2 \). \(\square\)
Next, we derive the relations between Sobolev norms and the norms obtained from those Riesz bases.

**Proposition 3.3.**

\[
\left\| \sum_{k=-\infty}^{\infty} f_k \phi^{(n)}_k \right\|_{L^2}^2 = \left\| \sum_{k=-\infty}^{\infty} (|k| + 1)^n f_k \phi^{(n)}_k \right\|_{L^2}^2 \leq D_n^2 \sum_{k=-\infty}^{\infty} (|k| + 1)^n f_k^2,
\]

for any complex sequence \( \{f_k\} \in l^2_n \), where

\[
l^2_n = \left\{ \{a_k\} : \sum_{k=-\infty}^{\infty} |k^n a_k|^2 < \infty \right\}.
\]

**Proof.** The proof of this proposition is similar to the proof of uniform \( l^2 \)-convergence property of \( \{\phi_k\} \) and \( \{\psi_k\} \) in Proposition 3.2. By the Ingham-Komornik result stated in Proposition 3.2 (see the proof of (3.13)), it is deduced that

\[
\left\| \sum_{k=-\infty}^{\infty} f_k \phi^{(n)}_k \right\|_{L^2}^2 = \left\| \sum_{k=-\infty}^{\infty} f_k \left( c_{0,0,k} \mu_{n,k} + c_{1,1,k} \right) \right\|_{L^2}^2 \leq \left( \left\| \sum_{k=-\infty}^{\infty} f_k c_{0,0,k} \mu_{n,k} \right\|_{L^2} + \left\| \sum_{k=-\infty}^{\infty} c_{1,1,k} \right\|_{L^2} \right)^2 \leq D_n^2 \left( \sum_{k=-\infty}^{\infty} \mu_{n,k}^2 + \sum_{k=-\infty}^{\infty} \mu_{1,1,k}^2 \right) \leq D_n^2 \sum_{k=-\infty}^{\infty} (|k| + 1)^n f_k^2.
\]

A similar proof works for

\[
\left\| \sum_{j=-\infty}^{\infty} g_j \psi^{(n)}_j \right\|_{L^2}^2 = \left\| \sum_{j=-\infty}^{\infty} (|j| + 1)^n g_j \psi^{(n)}_j \right\|_{L^2}^2 \leq D_n^2 \sum_{j=-\infty}^{\infty} (|j| + 1)^n g_j^2,
\]

for any \( \{g_j\} \in l^2_n \). Thus, \( \left\{ \phi^{(n)}_k \right\}_{k \neq 0} \) and \( \left\{ \psi^{(n)}_k \right\}_{k = 0} \) are uniformly \( l^2_n \)-convergent in \( L^2(0, L) \).

**Proposition 3.4.** \( \left\{ \phi^{(n)}_k \right\}_{k \neq 0} \) is uniform \( l^2_n \)-independent in \( L^2 \) for \( n \geq 1 \), i.e. there exists a positive \( D_n^2 \) such that for any sequence of complex numbers \( \{f_k\} \in l^2_n \),

\[
\left\| \sum_{k=-\infty}^{\infty} k^n f_k \phi^{(n)}_k \right\|_{L^2}^2 + \left\| \sum_{k=-\infty}^{\infty} f_k \phi^{(n)}_k \right\|_{L^2}^2 \geq D_n^2 \sum_{k=-\infty}^{\infty} |k^n f_k|^2.
\]

**Proof.** The case \( n = 0 \) was proved in Proposition 3.2. For \( n = 1 \), from the boundary conditions (3.2), we have an identity,

\[
-(\phi'_k, \psi'_j) = \int_0^L \phi'_k(x) \psi'_j(x) dx = -i \alpha \phi_k(0) \bar{\psi}_j(L) + i \lambda_k \delta_{k,j},
\]
Hence, by the Sobolev embedding theorem and Proposition 3.3, it is obtained that

\[
\sum_{k=-\infty}^{\infty} |k f_k|^2 = \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} k j f_k \bar{f}_j \delta_{k,j}
\]

\[
= \sum_{k=-\infty, k \neq 0}^{\infty} \sum_{j=-\infty, j \neq 0}^{\infty} k j f_k \bar{f}_j \left[ i \frac{j^2}{\lambda_j} \left( \frac{\phi_k'}{k}, \frac{\psi_j'}{j} \right) + \alpha \frac{j^2}{\lambda_j} \frac{\phi_k(0)}{k} \frac{\bar{\psi}_j(L)}{j} \right]
\]

\[
\leq \left\{ \sum_{k=-\infty}^{\infty} f_k \phi_k' \right\}^2 \left\{ \sum_{j=-\infty}^{\infty} j^2 \psi_j' \right\}^2 \lambda_j^2
\]

which implies

\[
\frac{1}{D_l^2} \sum_{k=-\infty}^{\infty} |k f_k|^2
\]

\[
\leq \left( \left\{ \sum_{k=-\infty}^{\infty} f_k \phi_k' \right\}^2 + 4 |\alpha| \left\{ \sum_{k=-\infty}^{\infty} f_k \phi_k \right\}^2 \right)^2
\]

\[
\leq 2 \left\{ \sum_{k=-\infty}^{\infty} f_k \phi_k' \right\}^2 + 4 |\alpha| \left\{ \sum_{k=-\infty}^{\infty} f_k \phi_k \right\}^2
\]
The cases for $E_n > 0$, which may depend on $\alpha$, $\beta$, such that for $w \in H_{\alpha,\beta}^n$, 
\[
E_n^2 \sum_{k=-\infty}^{\infty} |c_k|^2 \leq \|w\|_{H_{\alpha,\beta}^n}^2 \leq E_n^2 \sum_{k=-\infty}^{\infty} (|c_k|^2 + |k^n c_k|^2).
\]
Proof. Since \(\|f^{(m)}\|^2_{L^2} \leq c(\epsilon)\|f\|^2_{L^2} + \epsilon\|f^{(n)}\|^2_{L^2}\) for \(m = 1, 2, \ldots, n-1\), Propositions 3.2 and 3.3 imply that

\[
\|w\|^2_{H^{n,\beta}_{\alpha,\beta}} = \sum_{j=0}^{n} \|w^{(j)}\|^2_{L^2} = \sum_{j=0}^{n} \left\| \sum_{k=-\infty}^{\infty} c_k \phi_k^{(j)} \right\|^2_{L^2} \\
\leq c_1 \left\| \sum_{k=-\infty}^{\infty} c_k \phi_k \right\|^2_{L^2} + c_2 \left\| \sum_{k=-\infty}^{\infty} c_k^{(n)} \phi_k \right\|^2_{L^2} \\
\leq E_n^2 \sum_{k=-\infty}^{\infty} |c_k|^2 + E_n^2 \sum_{k=-\infty}^{\infty} |k^n c_k|^2.
\]

From Propositions 3.2 and 3.4, we have

\[
2 \left\| \sum_{k=-\infty}^{\infty} c_k \phi_k^{(n)} \right\|^2_{L^2} + 2 \left\| \sum_{k=-\infty}^{\infty} c_k \phi_k \right\|^2_{L^2} \geq \tilde{D}_n^2 \sum_{k=-\infty}^{\infty} |k^n c_k|^2 + D_0^{-2} \sum_{k=-\infty}^{\infty} |c_k|^2 \\
\geq c_n^2 \sum_{k=-\infty}^{\infty} (|k^n c_k|^2 + |c_k|^2).
\]

The proof is completed. \(\square\)

By Corollary 3.6, we define a class of Banach spaces \(H^{s,p}_{\alpha,\beta}\). If \(\{\phi_k(x)\}\) is the Riesz basis of \(L^2(0, L)\) given in Proposition 3.2, then for any \(s \geq 0\) and \(p \geq 1\), define

\[
H^{s,p}_{\alpha,\beta} = \left\{ w = \sum_{k=-\infty}^{\infty} c_k \phi_k \mid \|w\|^p_{H^{s,p}_{\alpha,\beta}} = \sum_{k=-\infty}^{\infty} (1 + |k|^p s) |c_k|^p < \infty \right\}.
\]

By \(\ell^p \hookrightarrow \ell^q\) for any \(q > p \geq 2\), it is easy to see that

\[
H^{s,p}_{\alpha,\beta} \hookrightarrow H^{s',p'}_{\alpha,\beta}, \quad s' < s, \\
H^{s,p}_{\alpha,\beta} \hookrightarrow H^{s,q}_{\alpha,\beta}, \quad q > p \geq 2.
\]

For \(s = n\) an integer and \(p = 2\), the space \(H^{s,p}_{\alpha,\beta}\) is same as \(H^n_{\alpha,\beta}\) in Definition 3.5. Also, for \(s\) not an integer, \(H^{s}_{\alpha,\beta} = H^2_{\alpha,\beta}\) is a subspace of \(H^s\) as well for all \(s \geq 0\). Sometimes, we write \(\| \cdot \|_s\) as the norm of \(H^{s}_{\alpha,\beta}\) and \(\| \cdot \|_{H^s}\) as the regular Sobolev norm of \(H^s\).

By the definition of \(H^{s}_{\alpha,\beta}\), it is straightforward to see that \(H^{s}_{\alpha,\beta}\) is a subspace of \(H^s\). Similar to the classical Sobolev spaces \(H^s = H^s_0\) for \(s \in [0, 1/2]\), since in this case the boundary values of functions in \(H^s = H^s(0, L)\) are undefined and the boundary conditions are not necessary, we have that for \(0 \leq s < 1/2\), \(H^s_{\alpha,\beta} = H^s\). However, for \(1/2 < s < 3/2\), the boundary values of functions in \(H^s\) are well-defined by Sobolev imbedding theorem, which implies that one boundary condition \(w(0) = \beta w(L)\) is needed for an \(H^s\)-function \(w\) to be in \(H^s_{\alpha,\beta}\) (since \(w_x(0), w_x(L)\) are not defined for this case, the other boundary condition involving the derivative of \(w\) is not necessary).

4. Properties of semi-groups generated by linear Schrödinger operators.

From the discussion of Section 2, if

\[
P_k = \phi_k \psi_k^*: L^2 \rightarrow L^2, \quad -\infty < k < \infty,
\]
then we can obtain that the resolution of the identity associated with the operator $A$ is $I = \sum_{k=-\infty}^{\infty} P_k$, which is strongly convergent in $L(L^2, L^2)$. The corresponding strongly convergent semigroup generated by $A$ is

$$S(t) = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} \phi_k \psi_k^* = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} P_k.$$ 

Hence, the solution of the nonhomogeneous problem

$$\begin{cases} u_t - iu_{xx} = f, & x \in (0, L), \ t \geq 0, \\ u(x, 0) = w_0(x), \\ u(0, t) = \beta u(L, t), \beta u_x(0, t) - u_x(L, t) = i \alpha u(0, t), & L > 0, \end{cases} \tag{4.1}$$

is given by

$$u(t) = S(t)w_0(x) + \int_0^t S(t-\tau)f(\cdot, \tau)d\tau.$$ 

First, we derive the estimates for $S(t)w_0(x)$.

**Proposition 4.1.** For any given $s \geq 0$ and $T > 0$ and $w_0 \in H^s_{\alpha, \beta}$,

$$\|S(t)w_0\|_s \leq e^{-\gamma t}\|w_0\|_s, \quad t \geq 0, \tag{4.2}$$

$$\int_0^T \|S(t)w_0\|_s^2 dt \leq (2\gamma)^{-1}\|w_0\|_s^2. \tag{4.3}$$

**Proof.** If $w_0 = \sum_{k=-\infty}^{\infty} c_k \phi_k$, then $S(t)w_0 = \sum_{k=-\infty}^{\infty} e^{\lambda_k t} c_k \phi_k$. Since $\text{Re} \lambda_k \leq -\gamma < 0$, the definition of $H^s_{\alpha, \beta}$ gives

$$\|w_0\|_s^2 = \sum_{k=-\infty}^{\infty} (|c_k|^2 + |k^s c_k|^2) = \sum_{k=-\infty}^{\infty} |c_k|^2(1 + |k|^{2s}),$$

and

$$\|S(t)w_0\|_s^2 = \sum_{k=-\infty}^{\infty} (|e^{\lambda_k t} c_k|^2 + |k^s e^{\lambda_k t} c_k|^2)$$

$$= \sum_{k=-\infty}^{\infty} |c_k|^2(1 + |k|^{2s})e^{2\text{Re} \lambda_k t}$$

$$\leq e^{-2\gamma t} \sum_{k=-\infty}^{\infty} |c_k|^2(1 + |k|^{2s}) = e^{-2\gamma t}\|w_0\|_s^2.$$ 

(4.3) follows from (4.2). \hfill \Box

The following propositions will be used for the estimates of solutions corresponding to the nonhomogeneous terms.

**Proposition 4.2.** For any given $s \geq 0$, $T > 0$, and $f \in L^2(0, T; H^s_{\alpha, \beta})$,

$$\sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau)f(x, \tau)d\tau \right\|_s \leq (2\gamma)^{-1/2} \left( \int_0^T \|f(x, \tau)\|_s^2 d\tau \right)^{1/2}. \tag{4.4}$$
Proof. Since \( f(x, t) = \sum_{k=-\infty}^{\infty} f_k(t)\phi_k(x) \), we find that

\[
\sup_{0 \leq t \leq T} \left\| \int_0^t S(t - \tau)f(x, \tau)d\tau \right\|_s = \sup_{0 \leq t \leq T} \left\| \int_0^t \sum_{k=-\infty}^{\infty} e^{\lambda_k(t-\tau)} f_k(\tau)\phi_k(x)d\tau \right\|_s 
\]

\[
\leq \sup_{0 \leq t \leq T} \sum_{k=-\infty}^{\infty} (1 + |k|^{2s}) \left( \int_0^t \left| e^{\lambda_k(t-\tau)} \right| |f_k(\tau)| \, d\tau \right)^2 \leq \sum_{k=-\infty}^{\infty} \sup_{0 \leq t \leq T} \left( \int_0^t \left| e^{\lambda_k(t-\tau)} \right|^2 \, d\tau \int_0^t |f_k(\tau)|^2 \, d\tau \right) (1 + |k|^{2s}) \leq \sum_{k=-\infty}^{\infty} \sup_{0 \leq t \leq T} \left( \int_0^t \left| e^{2\Re\lambda_k(t-\tau)} \right| \, d\tau \int_0^t |f_k(\tau)|^2 \, d\tau \right) (1 + |k|^{2s}) \leq \sum_{k=-\infty}^{\infty} \sup_{0 \leq t \leq T} \frac{1}{2\gamma} \int_0^t |f_k(\tau)|^2 \, d\tau (1 + |k|^{2s}) \leq \frac{1}{2\gamma} \int_0^T \sum_{k=-\infty}^{\infty} |f_k(\tau)|^2 (1 + |k|^{2s}) \, d\tau = \frac{1}{2\gamma} \int_0^T \|f(x, \tau)\|^2_2 \, d\tau.
\]

Proposition 4.3. For any given \( s \geq 0 \), there is a \( B > 0 \) such that if \( f \in L^\infty(0, \infty; H^s_{1,2}) \), then

\[
\sup_{0 \leq t < \infty} \left\| \int_0^t S(t - \tau)f(x, \tau)d\tau \right\|_s \leq B \sup_{0 \leq t < \infty} \|f(x, t)\|_s. \tag{4.5}
\]

Proof. Define \( \hat{t} = \max\{t - 1, 0\} \). Then,

\[
\int_0^t S(t - \tau)f(\tau)d\tau = \int_0^\hat{t} S(t - \tau)f(\tau)d\tau + \int_\hat{t}^t S(t - \tau)f(\tau)d\tau = \hat{h}(\cdot, t) + h(\cdot, t).
\]

By Proposition 4.2 and \( t - \hat{t} \leq 1 \), there is a \( B_0 > 0 \) such that

\[
\|\hat{h}(\cdot, t)\|^2_s \leq B_0^2 \int_0^\hat{t} \|f(\cdot, \tau)\|^2_s \, d\tau \leq B_0^2 \sup_{0 \leq t < \infty} \|f(\cdot, t)\|_s^2 \int_0^t 1 \, d\tau \leq B_0^2 \sup_{0 \leq t < \infty} \|f(\cdot, t)\|^2_s.
\]

If \( t \leq 1 \), \( \hat{t} = 0 \) and \( h(\cdot, t) = 0 \). For \( t > 1 \), write

\[
f(x, \tau) = \sum_{j=-\infty}^{\infty} f_j(\tau)\phi_j(x)
\]

with
sider the local well-posedness of the IVP for the nonlinear Schrödinger equation

Local well-posedness of the nonlinear problem.

5. \( \frac{\partial u}{\partial t} + u_{xx} + f(u) = 0, \quad 0 < x < L, t \geq 0 \)

\[ u(x, 0) = u_0(x), \]

\[ u(0, t) = \beta u(L, t), \]

\[ \beta u_x(0, t) - u_x(L, t) = i\alpha u(0, t), \]

with \( \alpha, \beta \) real numbers satisfying \( \alpha \beta < 0 \) and \( \beta \neq \pm 1 \) and \( f(u) \) satisfying the assumptions (H1) and (H2) stated in Introduction. We can rewrite the equation in
(5.1) as \( u_t - iu_{xx} = if(u) \). By comparing to (4.1), the inhomogeneous term \( f \) is \( if(u) \) and the solution of (5.1) is
\[
u(x,t) = S(t)w_0 + \int_0^t S(t - \tau)(if(u))(x,\tau)d\tau. \tag{5.2}
\]
We will study the solution of (5.2) using the fixed-point theorem for the mapping \( F : v(x,t) \to u(x,t) \) defined by
\[
u(x,t) = (Fv)(x,t) := S(t)w_0 + \int_0^t S(t - \tau)(if(v))(x,\tau)d\tau. \tag{5.3}
\]
The following lemma gives that the Sobolev norm in \( H^s \) for \( s > 1/2 \) is an algebra and the proof is well-known.

**Lemma 5.1.** If \( s > \frac{1}{2} \), there exists a constant \( c_s \) such that \( \|fg\|_{H^s} \leq c_s \|f\|_{H^s}\|g\|_{H^s} \) for any functions \( f \) and \( g \) in \( H^s \).

Since \( H^s_{\alpha,\beta} \) is a subspace of \( H^s \), Lemma 5.1 is applicable if all of \( f, g \) and \( fg \) are in \( H^s_{\alpha,\beta} \). Now, we prove that the IVP of (5.1) is well-posed in the space \( H^s_{\alpha,\beta} \) for \( \frac{1}{2} < s \leq 1 \).

**Theorem 5.2.** Suppose that \( s \in (\frac{1}{2}, 1] \).

(i) If \( w_0 \in H^s_{\alpha,\beta} \) is given, there is a \( T = T(||w_0||_s) > 0 \) such that the IVP (5.1) has a unique solution
\[
v \in X_T := C(0,T; H^s_{\alpha,\beta})
\]
and \( T \to \infty \) as \( ||w_0||_s \to 0 \);

(ii) If \( 0 < T' < T \) is given, there is a neighborhood \( U \) of \( w_0 \) in \( H^s_{\alpha,\beta} \) and the map \( K : w_0 \to v(x,t) \) from \( U \) to \( X_{T'} \) is Lipschitz continuous.

**Proof.** Let
\[
S_{T,b} = \left\{ v \in X_T \mid \sup_{0 \leq t \leq T} ||v(x,t)||_s \leq b \right\}
\]
where \( b > 0 \) and \( T > 0 \) will be determined later. To find the desired solution of the IVP (5.1) using a fixed point theorem, some appropriate \( b \) and \( T \) must be chosen such that the map \( F \) in (5.3) is a contraction from \( S_{T,b} \) to \( S_{T,b} \).

First, we note that for \( v \in C(0,T; H^s_{\alpha,\beta}) \) and \( t \) fixed, it is not necessary that \( f(v(.,t)) \in H^s_{\alpha,\beta} \). However, for \( 1/2 < s \leq 1 \), since \( v(0,t), v(L,t) \) are well-defined (note that \( v_\alpha(0,t), v_\alpha(L,t) \) are not defined so that the boundary condition with derivatives in the definition of \( H^s_{\alpha,\beta} \) is not needed), we can let a function \( \tilde{f}(t) = [f(v(0,t)) - \beta f(v(L,t))]/(1 - \beta) \) such that \( g(v) = f(v(.,t)) - \tilde{f}(t) \in H^s_{\alpha,\beta} \) and the nonlinear part in (5.3) can be rewritten as \( f(v(.,t)) = g(v) + \tilde{f}(t) \) with
\[
(Fv)(x,t) = S(t)w_0 + \int_0^t S(t - \tau)(i\tilde{f}(v))(x,\tau)d\tau + \int_0^t S(t - \tau)(i\tilde{f})(\tau)d\tau = S(t)w_0 + F_1(v) + F_2(v).
\]
Since \( g(v) \in H^s_{\alpha,\beta} \), the propositions in Section 4 are applicable to \( g(v) \). Thus, the estimates of \( F_1(v) \) in the following can be obtained using the results in Section 4.

To study \( F_2(v) \), by the explicit form of \( S(t) \), it is deduced that
\[
F_2(v) = i \int_0^t S(t - \tau)\tilde{f}(\tau)d\tau = \sum_{k=1}^{+\infty} i \int_0^t e^{\lambda_k(t-\tau)}\tilde{f}(\tau)d\tau \phi_k(x) \int_0^L \overline{\phi_k(x)}dx.
\]
By the properties of $\phi_k$, it is shown that $\int_0^L \phi_k(x)dx \leq C|k|^{-1}$, which gives

$$\|F_2(v)\|_{H^1_{\alpha, \beta}}^2 \leq C \left( \sum_{k=-\infty}^{+\infty} \left( \int_0^t e^{\lambda_k(t-\tau)} \tilde{f}(\tau) d\tau \right)^2 + \sum_{k=-\infty}^{+\infty} \left( \int_0^L \phi_k(x)dx \right)^2 \right) \leq C \sum_{k=-\infty}^{+\infty} \left( \int_0^t e^{\lambda_k(t-\tau)} \tilde{f}(t-\tau) d\tau \right)^2.$$  

(5.4)

Now, we state the Selberg’s inequality, a generalization of Bessel’s inequality (see Eq. 4 in [13]): Let $H$ be a Hilbert space with its inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$, and $z_j, j = 1, 2, \ldots$ be in $H$. Then, for any $x \in H$,

$$\sum_{k} \frac{|(x,z_k)|^2}{\sum_{j}|(z_k,z_j)|} \leq \|x\|^2.$$

To apply the Selberg’s inequality for $F_2(v)$, we let $H = L^2(0,t)$ for fixed $t > 0$ with functions $f(\tau)$ and $z_k = e^{\lambda_k \tau}$. Using the asymptotic form of $\lambda_k$ in (3.10), a straightforward calculation shows that $\sum_{j} |(z_k,z_j)| \leq C_0$ where $C_0 > 0$ is a constant independent of $k$. From (5.4), Selberg’s inequality, and Assumptions 1.1 and 1.2, it is obtained that

$$\|F_2(v)\|_{H^1_{\alpha, \beta}}^2 \leq C \sum_{k=-\infty}^{+\infty} \left( \int_0^t e^{\lambda_k \tau} \tilde{f}(t-\tau) d\tau \right)^2 \leq C \sum_{k=-\infty}^{+\infty} \left( \int_0^t e^{\lambda_k \tau} \tilde{f}(t-\tau) d\tau \right)^2 \leq C \left( \sup_{0 \leq t \leq T} \|v(x,t)\|_{p_2}^2 + \sup_{0 \leq t \leq T} \|v(x,t)\|_{p_2}^2 \right)^2.$$

Here, $C$ may be independent of $t$ if $\int_0^t \|\tilde{f}(\tau)\|_2 d\tau$ is uniformly bounded for any $t \geq 0$. Thus, for any $s \in (\frac{1}{2}, 1]$, $F_2(v)$ is in $C(0,T; H^1_{\alpha, \beta})$ and the corresponding norm is bounded by $\sup_{0 \leq t \leq T} \|v(x,t)\|_s$. Therefore, in the following, for the sake of simplicity, we will always implicitly use this procedure to deal with the boundary conditions of $f(v)$ when we apply the propositions in Section 4 directly to $f(v)$, instead of $g(v)$, by assuming that $f(v)$ satisfies the boundary conditions for $1/2 < s \leq 1$.

Now, applying Propositions 4.1 and 4.2 to (5.3) yields

$$\sup_{0 \leq t \leq T} \|Fv\|_s \leq C \left( \sup_{0 \leq t \leq T} \left\|S(t)w_0 + \int_0^t S(t-\tau)(if(v))(x,\tau)d\tau \right\|_s \right) \leq C \left( \sup_{0 \leq t \leq T} \|S(t)w_0\|_s + \sup_{0 \leq t \leq T} \left\|S(t-\tau)(if(v))(x,\tau)d\tau \right\|_s \right) \leq c \|w_0\|_s + B_s \left( \int_0^T \|if(v)\|_s^2 d\tau \right)^{\frac{1}{2}}.$$
\[
\begin{align*}
&\leq c \|w_0\|_s + B_s T^{\frac{1}{2}} \left( \sup_{0 \leq t \leq T} \|f(v)\|_s^2 \right)^{\frac{1}{2}} \\
&= c \|w_0\|_s + B_s T^{\frac{1}{2}} \sup_{0 \leq t \leq T} \|f(v)\|_{H^s} \\
&\leq c \|w_0\|_s + B_s' T^{\frac{1}{2}} \sup_{0 \leq t \leq T} (\|v\|^p_{H^s} + \|v\|^q_{H^s}) \\
&= c \|w_0\|_s + B_s' T^{\frac{1}{2}} \sup_{0 \leq t \leq T} (\|v\|^p_s + \|v\|^q_s),
\end{align*}
\]

where the assumptions (H1) and (H2) on \(f(u)\) have been used. Here, we have used the facts that the norms in \(H^s\) and \(H^s_{\alpha, \beta}\) are equivalent if a function is in \(H^s_{\alpha, \beta}\), and \(f(v)\) is assumed to be in \(H^s_{\alpha, \beta}\). From now on, we will not emphasize this detail again.

Choose \(b = 2\tilde{c} \|w_0\|_s\) where \(\tilde{c} = \max\{B'_s, c\}\), and \(T > 0\) such that

\[
\frac{3}{2} \tilde{c} T^{\frac{1}{2}} (b^p + b^q) \leq b
\]

which implies that

\[
3\tilde{c} T^{\frac{1}{2}} (b^{p-1} + b^{q-1}) \leq \frac{1}{2}.
\]

Then by (5.5) and the definition of \(S_{T,b}\),

\[
\sup_{0 \leq t \leq T} \|Fv\|_s \leq c \|w_0\|_s + B_s' T^{\frac{1}{2}} \sup_{0 \leq t \leq T} (\|v\|^p_s + \|v\|^q_s)
\]

\[
\leq \tilde{c} \|w_0\|_s + \tilde{c} T^{\frac{1}{2}} (b^p + b^q) \leq \frac{b}{2} + \frac{b}{2} = b.
\]

Hence, \(F\) maps \(S_{T,b}\) to \(S_{T,b}\). Next, we show that \(F\) is a contraction on \(S_{T,b}\). For any \(v_1, v_2 \in S_{T,b}\), let \(\tilde{v} = v_1 - v_2\). Then

\[
\sup_{0 \leq t \leq T} \|Fv_1 - Fv_2\|_s
\]

\[
= \sup_{0 \leq t \leq T} \left\| S(t)w_0 + \int_0^t S(t - \tau)(if(v_1))(x, \tau)d\tau \\ - \left( S(t)w_0 + \int_0^t S(t - \tau)(if(v_2))(x, \tau)d\tau \right) \right\|_s
\]

\[
= \sup_{0 \leq t \leq T} \left\| \int_0^t S(t - \tau)i(f(v_1) - f(v_2))(x, \tau)d\tau \right\|_s
\]

\[
\leq B_s \left( \int_0^T \|f(v_1) - f(v_2)\|_s^2 d\tau \right)^{\frac{1}{2}}
\]

\[
\leq T^{\frac{1}{2}} B_s \left( \sup_{0 \leq t \leq T} \|f(v_1) - f(v_2)\|_s^2 \right)^{\frac{1}{2}} = T^{\frac{1}{2}} B_s \sup_{0 \leq t \leq T} \|f_0(v_1, v_2)\|_s
\]

\[
\leq T^{\frac{1}{2}} B_s' \left[ \left( \sup_{0 \leq t \leq T} \|v_1\|_s \right)^{p_1-1} + \left( \sup_{0 \leq t \leq T} \|v_1\|_s \right)^{p_2-1} \\
+ \left( \sup_{0 \leq t \leq T} \|v_2\|_s \right)^{p_1-1} + \left( \sup_{0 \leq t \leq T} \|v_2\|_s \right)^{p_2-1} \right] \sup_{0 \leq t \leq T} \|\tilde{v}\|_s
\]
Then, Lemma 5.3.

Thus \( u \) 4.1 gives the regularity of solutions for the linear case. For the nonlinear case, the

\[ \frac{3}{2} \epsilon T^2 (b^{p_1} + b^{p_2}) \leq \frac{b}{4} \] which implies

\[ T \leq \left( \frac{6 \epsilon (b^{p_1} + b^{p_2})}{(2 \epsilon \|w_0\|_s)^{p_1 - 1} + (2 \epsilon \|w_0\|_s)^{p_2 - 1}} \right)^{-2} \]

Choose \( T = \left[ \frac{6 \epsilon (2 \epsilon \|w_0\|_s)^{p_1 - 1} + (2 \epsilon \|w_0\|_s)^{p_2 - 1}}{2} \right]^{-2} \), which gives that \( T \to \infty \) as \( \|w_0\|_s \to 0 \).

To prove the second part of the theorem, it is straightforward to see that for any \( T' \leq T \), there exists a neighborhood \( U \) of \( w_0 \) in \( H^s_{\alpha, \beta} \) and the map \( K \) is well-defined from \( U \) to \( X_T \). If \( w_1, w_2 \in U \), define \( u_1 = K w_1 \), \( u_2 = K w_2 \) and \( u = u_1 - u_2 \). Then, Proposition 4.1 and the contraction property of \( F \) yield

\[
\sup_{0 \leq t \leq T'} \|u\|_s = \sup_{0 \leq t \leq T'} \left\| S(t)(w_1 - w_2) + \int_0^t S(t - \tau)i(f(u_1) - f(u_2))(x, \tau) d\tau \right\|_s \\
\leq \sup_{0 \leq t \leq T'} \left\| S(t)(w_1 - w_2) \right\|_s \\
+ \sup_{0 \leq t \leq T'} \left\| \int_0^t S(t - \tau)i(f(u_1) - f(u_2))(x, \tau) d\tau \right\|_s \\
\leq c \left\| w_1 - w_2 \right\|_s + \rho \sup_{0 \leq t \leq T'} \|u_1 - u_2\|_s,
\]

where \( \rho \leq 1/2 \). By \( T' < T \),

\[
\sup_{0 \leq t \leq T'} \|u_1 - u_2\|_s \leq c \left\| w_1 - w_2 \right\|_s + \rho \sup_{0 \leq t \leq T'} \|u_1 - u_2\|_s \quad \text{or}
\sup_{0 \leq t \leq T'} \|u_1 - u_2\|_s \leq \frac{c}{1 - \rho} \left\| w_1 - w_2 \right\|_s.
\]

Thus \( K \) is Lipschitz continuous from \( U \) to \( X_{T'} \). \( \Box \)

Next, we consider the regularity of solutions of (5.1), i.e., for any given \( n > 1 \), the solution \( u(\cdot, t) \in H^s_{\alpha, \beta} \) if its initial state \( w_0 \in H^s_{\alpha, \beta} \). Note that Proposition 4.1 gives the regularity of solutions for the linear case. For the nonlinear case, the regularity is more complicated since \( u \) is defined in a special space \( H^s_{\alpha, \beta} \) which is a Hilbert space inherited from Sobolev space \( H^n \) with boundary conditions (1.4). Thus,

\[ u \in H^n_{\alpha, \beta} \neq u_x \in H^{n-1}_{\alpha, \beta} \quad n > 1. \]

However, note that \( \partial_x u \) and \( \partial_x^2 u \) are in the same space \( H^n \) and \( \partial_x u \) satisfies the boundary conditions in (5.1). Hence, we should be able to prove the regularity of the solution after proving the regularity of \( \partial_x u \).

To this end, we first give the estimates for \( \tilde{f}(\cdot, t) \equiv \partial_x f(\cdot, t) \).

**Lemma 5.3.** For \( s \geq 0 \), assume that \( f \in C[0, T; H^s_{\alpha, \beta}] \) and \( \partial_x f \in L^2[0, T; H^s_{\alpha, \beta}] \). Then,

\[
\partial_x \left( \int_0^t S(t - \tau)f(x, \tau) d\tau \right) = S(t)f(x, 0) + \int_0^t S(t - \tau)\tilde{f}(x, \tau) d\tau \quad (5.6)
\]
satisfies
\[ \sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t} \int_0^t S(t - \tau) f(x, \tau) d\tau \right\|_s \leq c \| f(x, 0) \|_s + c \left( \int_0^T \left\| \dot{f}(x, \tau) \right\|^2_s d\tau \right)^{\frac{1}{2}} \]
where \( c > 0 \) is independent of \( T \). Moreover,
\[ \sup_{0 \leq t < \infty} \left\| \frac{\partial}{\partial t} \int_0^t S(t - \tau) f(x, \tau) d\tau \right\|_s \leq c \| f(x, 0) \|_s + c \sup_{0 \leq t < \infty} \left\| \dot{f}(x, \tau) \right\|_s. \] (5.7)

Proof. Consider the initial boundary value problem
\[
\begin{cases}
  u_t - iu_{xx} = f, & 0 < x < L, t \geq 0 \\
  u(x, 0) = 0, \\
  u(0, t) = \beta u(L, t), \\
  \beta u_x(0, t) - u_x(L, t) = i\alpha u(0, t),
\end{cases}
\]
which has a solution
\[ u = \int_0^t S(t - \tau) f(x, \tau) d\tau. \] (5.8)

Let \( v = \partial_t u \). Then
\[ v(x, t) = \frac{\partial}{\partial t} \int_0^t S(t - \tau) f(x, \tau) d\tau = f(x, t) + \int_0^t \frac{\partial}{\partial t} S(t - \tau) f(x, \tau) d\tau, \]
which implies \( v(x, 0) = f(x, 0) \). Also,
\[
\begin{cases}
  \partial_t v - iv_{xx} = \dot{f}, & 0 < x < L, t \geq 0, \\
  v(x, 0) = f(x, 0), \\
  v(0, t) = \beta v(L, t), \\
  \beta v_x(0, t) - v_x(L, t) = i\alpha v(0, t),
\end{cases}
\]
has a solution
\[ v(x, t) = S(t) f(x, 0) + \int_0^t S(t - \tau) \dot{f}(x, \tau) d\tau. \] (5.9)

From (5.8) and (5.9), we have
\[ \frac{\partial}{\partial t} u = v \quad \Rightarrow \quad \frac{\partial}{\partial t} \int_0^t S(t - \tau) f(x, \tau) d\tau = S(t) f(x, 0) + \int_0^t S(t - \tau) \dot{f}(x, \tau) d\tau. \]

Then by Propositions 4.1 and 4.2,
\[
\sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t} \int_0^t S(t - \tau) f(x, \tau) d\tau \right\|_s = \sup_{0 \leq t \leq T} \left\| S(t) f(x, 0) + \int_0^t S(t - \tau) \dot{f}(x, \tau) d\tau \right\|_s \\
\leq \sup_{0 \leq t \leq T} \| S(t) f(x, 0) \|_s + \sup_{0 \leq t \leq T} \left\| \int_0^t S(t - \tau) \dot{f}(x, \tau) d\tau \right\|_s \\
\leq c \| f(x, 0) \|_s + c \left( \int_0^T \left\| \dot{f}(x, \tau) \right\|_s^2 d\tau \right)^{\frac{1}{2}},
\]
and by Propositions 4.1 and 4.3,
\[
\sup_{0 \leq t < \infty} \left\| \frac{\partial}{\partial t} \int_0^t S(t - \tau) f(x, \tau) d\tau \right\|_s = \sup_{0 \leq t < \infty} \left\| S(t) f(x, 0) + \int_0^t S(t - \tau) \dot{f}(x, \tau) d\tau \right\|_s.
\]
Define

\[ K(i) \]

If \( v \) satisfies Assumptions 1.1 and 1.2 with \( p \),

\[ \text{Assume that} \]

Theorem 5.4.

with the norm \( \| \cdot \|_s \).

\[ \text{(ii) If} \quad T > b \quad \text{where} \]

is well defined on \( S \) for \( s \leq \frac{T}{2} \).

The proof of Lemma 5.3 is completed.

Now we discuss the regularity of solutions for (5.1) with the initial value

\[ u_0 \in X = \{ \phi \in H^{s+2} \cap H^2_{\alpha,\beta} : i\phi_{xx} + |\phi|^2 \phi \in H^s_{\alpha,\beta} \} \]

for \( s \in \left( \frac{1}{2}, 1 \right) \). If \( T > 0 \) is given, define a Banach space \( Y_T \) by

\[ Y_T = \left\{ v \in C^1(0,T; H^s_{\alpha,\beta}) : \sup_{0 \leq t \leq T} \| v \|_s < \infty, \sup_{0 \leq t \leq T} \| \dot{v}(t) \|_s < \infty \right\}, \]

with the norm

\[ \| v \|_{Y_T} := \left( \sup_{0 \leq t \leq T} \| v \|_s^2 + \sup_{0 \leq t \leq T} \| \dot{v}(t) \|_s^2 \right)^{\frac{1}{2}} \approx \sup_{0 \leq t \leq T} \| v \|_s + \sup_{0 \leq t \leq T} \| \dot{v} \|_s. \]

If we can show that for any \( u_0 \in X \) there is a unique solution \( v \in Y_T \), then

\[ \partial_t v \in C([0,T]; H^s_{\alpha,\beta}) \quad \text{and} \quad v \in C([0,T]; H^{s+2} \cap H^2_{\alpha,\beta}). \]

**Theorem 5.4.** Assume that \( f(u) \) satisfies Assumptions 1.1 and 1.2, while \( f'(u) \) satisfies Assumptions 1.1 and 1.2 with \( p_1, p_2 \) replaced by \( p_1 - 1, p_2 - 1 \).

(i) If \( u_0 \in X \) is given, then there is a \( T = T(\| u_0 \|_X) > 0 \) and (5.1) has a unique solution \( v \in Y_T \).

(ii) If \( 0 < T' < T \) is given, then there is a neighborhood \( U \) of \( u_0 \) in \( X \) and the map \( K : u_0 \to v \) from \( U \) to \( Y_T \) is Lipschitz continuous.

**Proof.** Define

\[ S_{T,b} = \{ v \in Y_T : \| v \|_{Y_T} \leq b, v(x,0) = u_0(x) \} \]

where \( T > 0 \) and \( b > 0 \) will be determined later. First, we want to prove the map

\[ Fv = S(t)u_0 + \int_0^t S(t - \tau)(i f(v))(x, \tau) d\tau \]

is well defined on \( S_{T,b} \), i.e. for any \( v \in S_{T,b} \),

\[ \| Fv \|_{Y_T} \leq \sup_{0 \leq t \leq T} \| Fv \|_s + \sup_{0 \leq t \leq T} \| \partial_t Fv \|_s \leq b. \]

It is deduced that for any \( v \in S_{T,b} \),

\[ \sup_{0 \leq t \leq T} \| Fv \|_s = \sup_{0 \leq t \leq T} \left\| S(t)u_0 + \int_0^t S(t - \tau)(i f(v))(x, \tau) d\tau \right\|_s \leq c \| u_0 \|_s + cT^\frac{3}{2} \sup_{0 \leq t \leq T} (\| v \|_{H^1}^2 + \| v \|_{H^2}^2), \]

\[ \sup_{0 \leq t \leq T} \| \partial_t Fv \|_s = \sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t} S(t)u_0 + \frac{\partial}{\partial t} \int_0^t S(t - \tau)(i f(v))(x, \tau) d\tau \right\|_s. \]
To find $\frac{\partial}{\partial t} S(t)w_0$, we know that $u = S(t)w_0$ is the solution of
\[
\begin{cases}
    u_t - iu_{xx} = 0, \\
    u(x, 0) = w_0(x), \\
    u(0, t) = \beta u(L, t), \\
    \beta u_x(0, t) - u_x(L, t) = i\alpha u(0, t).
\end{cases}
\]

Let $z = \partial_t u$, which implies that $z = iu_{xx}$ and $z(0, x) = iu_{xx}(0, x)$. Thus, $z$ is the solution of
\[
\begin{cases}
    z_t - iz_{xx} = 0, \\
    z(x, 0) = iu_{xx}(x, 0), \\
    z(0, t) = \beta z(L, t), \\
    \beta z_x(0, t) - z_x(L, t) = i\alpha z(0, t).
\end{cases}
\]

Therefore, $\frac{\partial}{\partial t} S(t)w_0 = \frac{\partial}{\partial t} u = z = S(t)iu_{xx}(x, 0) = S(t)iw_0''(x)$. By Lemma 5.3 and $v(x, 0) = w_0(x)$,
\[
\frac{\partial}{\partial t} \int_0^t S(t-\tau) (if(v))(x, \tau) d\tau = S(t)iw_0'' + \int_0^t S(t-\tau) \frac{\partial}{\partial t} (if(v)) d\tau. \tag{5.10}
\]

Propositions 4.1 and 4.2 give
\[
\sup_{0 \leq t \leq T} \|\partial_t Fv\|_s
\]
\[
= \sup_{0 \leq t \leq T} \left\| S(t)iw_{xx}(x, 0) + \int_0^t S(t-\tau) \frac{\partial}{\partial t} (if(v)) d\tau \right\|_s
\]
\[
\leq \sup_{0 \leq t \leq T} \|S(t)iw_0'' + f(w_0)\|_s + \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-\tau) \frac{\partial}{\partial t} (if(v)) d\tau \right\|_s
\]
\[
\leq c(\|w_0\|_{s_1}^p + \|w_0\|_{s_2}^p + \|w_0\|_{s+2}) + c \left( \int_0^T \left\| \frac{\partial}{\partial t} f(v) \right\|_s^2 d\tau \right)^{1/2}
\]
\[
\leq c(\|w_0\|_{s_1}^p + \|w_0\|_{s_2}^p + \|w_0\|_{s+2}) + cT^{1/2} \sup_{0 \leq t \leq T} \|\partial_t (if(v))\|_s
\]
\[
= c(\|w_0\|_{s_1}^p + \|w_0\|_{s_2}^p + \|w_0\|_{s+2}) + c' T^{1/2} \sup_{0 \leq t \leq T} (\|v\|_{s_1}^{p_1} + \|v\|_{s_2}^{p_2}) \|\dot{v}\|_s.
\]

Thus,
\[
\|Fv\|_{Y_T} \leq B (\|w_0\|_s + \|w_0\|_{s_1}^p + \|w_0\|_{s_2}^p + \|w_0\|_{s+2})
\]
\[
+ BT^{1/2} \left( \|v\|_{s_1}^{p_1} + \|v\|_{s_2}^{p_2} \right) \left( \|\dot{v}\|_s + \|\dot{v}\|_s \right)
\]
\[
\leq B (\|w_0\|_s + \|w_0\|_{s_1}^p + \|w_0\|_{s_2}^p + \|w_0\|_{s+2}) + 2BT^{1/2} (b^{p_1} + b^{p_2}),
\]
where $B = \max(c, c')$. If we let $B (\|w_0\|_s + \|w_0\|_{s_1}^p + \|w_0\|_{s_2}^p + \|w_0\|_{s+2}) = \frac{b}{2}$ and choose $T > 0$ small such that $2BT^{1/2} (b^{p_1} + b^{p_2}) \leq \frac{b}{2}$, then $\|Fv\|_{Y_T} \leq b$.

Define $\dot{v} = v_1 - v_2$. It is deduced that
\[
Fv_1 - Fv_2 = \int_0^t S(t-\tau) i(f(v_1) - f(v_2))(x, \tau) d\tau.
\]
Similar to the proof of Theorem 5.2, it is obtained that
\[
\sup_{0 \leq t \leq T} \| Fv_1 - Fv_2 \|_{s} \leq c T^{\frac{1}{2}} \left( \sup_{0 \leq t \leq T} \| v_1 \|_{s}^{p_1 - 1} + \sup_{0 \leq t \leq T} \| v_1 \|_{s}^{p_2 - 1} \right.
+ \sup_{0 \leq t \leq T} \| v_2 \|_{s}^{p_1 - 1} + \sup_{0 \leq t \leq T} \| v_2 \|_{s}^{p_2 - 1} \left. \right) \sup_{0 \leq t \leq T} \| \hat{v} \|_{s}.
\]
From (5.10) and \( v_1(x,0) = v_2(x,0) \), we have
\[
\sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t} (Fv_1 - Fv_2) \right\|_{s}
= \sup_{0 \leq t \leq T} \left\| \int_0^t S(t-t') \frac{\partial}{\partial t} \left(i f(v_1) - i f(v_2)\right) dt \right\|_{s}
\leq c \left( \int_0^T \left\| \frac{\partial}{\partial t} \left(i f(v_1) - i f(v_2)\right) \right\|_{s} dt \right)^{\frac{1}{2}}
\leq c T^{\frac{1}{2}} \sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t} \left(i f(v_1) - i f(v_2)\right) \right\|_{s}
\leq c T^{\frac{1}{2}} \sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t} \left( \tilde{v} \int_0^1 f'(w \tilde{v} + v_2) dw \right) \right\|_{s}
\leq c T^{\frac{1}{2}} \sup_{0 \leq t \leq T} \left( \left( \| v_1 \|_{s}^{p_1 - 1} + \| v_1 \|_{s}^{p_2 - 1} + \| v_2 \|_{s}^{p_1 - 1} + \| v_2 \|_{s}^{p_2 - 1} \right) \| \hat{v} \|_{s}
+ \left( \| v_1 \|_{s}^{p_1 - 2} + \| v_1 \|_{s}^{p_2 - 2} + \| v_2 \|_{s}^{p_1 - 2} + \| v_2 \|_{s}^{p_2 - 2} \right) \| \hat{v} \|_{s} \right).
\]
Hence,
\[
\| Fv_1 - Fv_2 \|_{Y_T}
\leq c' T^{\frac{1}{2}} \sup_{0 \leq t \leq T} \left( \left( \| v_1 \|_{s}^{p_1 - 1} + \| v_1 \|_{s}^{p_2 - 1} + \| v_2 \|_{s}^{p_1 - 1} + \| v_2 \|_{s}^{p_2 - 1} \right) \| \hat{v} \|_{s} + \| \hat{v} \|_{s} \right)
+ \left( \| v_1 \|_{s}^{p_1 - 2} + \| v_1 \|_{s}^{p_2 - 2} + \| v_2 \|_{s}^{p_1 - 2} + \| v_2 \|_{s}^{p_2 - 2} \right) \| \hat{v} \|_{s} \hat{v} \|_{s} \right)
\leq 8 c' T^{\frac{1}{2}} (b^{p_1 - 1} + b^{p_2 - 1}) \left( \sup_{0 \leq t \leq T} \| \hat{v} \|_{s} + \sup_{0 \leq t \leq T} \| \hat{v} \|_{s} \right)
\leq B T^{\frac{1}{2}} (b^{p_1 - 1} + b^{p_2 - 1}) \left( \sup_{0 \leq t \leq T} \| \hat{v} \|_{s} + \sup_{0 \leq t \leq T} \| \hat{v} \|_{s} \right) \leq \frac{1}{2} \| \hat{v} \|_{Y_T},
\]
where \( B = 8 c' \), if \( B T^{\frac{1}{2}} (b^{p_1 - 1} + b^{p_2 - 1}) \leq \frac{1}{2} \). Thus, \( F \) is a contraction on \( S_{T,b} \), which implies that the map \( F \) has a fixed point \( u \in Y_T \) and (5.1) has a unique solution. Since \( u_t \in C(0,T; H^s_{\alpha, \beta}) \) and \( u = iu_{xx} + i f(u) \), we have \( u \in C(0,T; H^{s+2}) \). In addition, \( u \) satisfies the boundary conditions, which implies that \( u \in C(0,T; H^2_{\alpha, \beta}) \).

Therefore, \( u \in C(0,T; H^{s+2} \cap H^2_{\alpha, \beta}) \).

To prove \( K \) is Lipschitz continuous, let \( u_1 = Kw_1, u_2 = Kw_2 \) and \( u = u_1 - u_2 \). Then,
\[
\| u \|_{Y_T} = \| S(t)(w_1 - w_2) + F u_1 - F u_2 \|_{Y_T}
\leq \| S(t)(w_1 - w_2) \|_{Y_T} + \| F u_1 - F u_2 \|_{Y_T}
= \sup_{0 \leq t \leq T} \| S(t)(w_1 - w_2) \|_{s} + \sup_{0 \leq t \leq T} \left\| \frac{\partial}{\partial t} S(t)(w_1 - w_2) \right\|_{s} + \| F u_1 - F u_2 \|_{Y_T},
\]

well-posedness, and give the behavior of the solution as \( t \to \infty \) with \( b > H_0 \). In this section, we prove that for small initial data, Theorem 6.2. There exist positive numbers \( \delta, b \) depending upon the size of initial value \( w_0 \). Consider the map \( F \) defined on \( S_{b, \infty} \) as follows:

\[
Fv = S(t)w_0 + \int_0^t S(t - \tau)(if(v))(x, \tau)d\tau.
\]

Then, Propositions 4.1 and 4.3 imply that

\[
\sup_{0 \leq t < \infty} \|Fv\|_s = \sup_{0 \leq t < \infty} \left\| S(t)w_0 + \int_0^t S(t - \tau)(if(v))(x, \tau)d\tau \right\|_s
\]

\[
\leq \sup_{0 \leq t < \infty} \|S(t)w_0\|_s + \sup_{0 \leq t < \infty} \left\| \int_0^t S(t - \tau)(if(v))(x, \tau)d\tau \right\|_s
\]

\[
\leq c \|w_0\|_s + c \sup_{0 \leq t < \infty} \|if(v)\|_s
\]

\[
\leq c \|w_0\|_s + c \left( \sup_{0 \leq t < \infty} \|v\|_s^{p_1} + \sup_{0 \leq t < \infty} \|v\|_s^{p_2} \right).
\]

6. Global existence and exponential decay of small amplitude solutions.

Section 5 provides the proof of existence and uniqueness of the solution of (5.1) in a finite time interval \([0, T]\) with \( T \) depending upon the size of initial value \( w_0 \). In this section, we prove that for small initial data, \( T \) can be infinite, called global well-posedness, and give the behavior of the solution as \( t \to \infty \).

**Theorem 6.1.** Let \( w_0 \in H^s_{\alpha, \beta} \) with \( \frac{1}{2} < s \leq 1 \) be given. If the solution does not exist for all \( t > 0 \), then there is a finite \( T^* > 0 \) such that (5.1) has a unique solution \( w \in C \left( [0, T^*) ; H^s_{\alpha, \beta} \right) \) with \( \lim_{t \to T^*} \|w(x, t)\|_s = \infty \). Here, \( T^* \) is called the lifespan of the solution.

**Proof.** For any given \( w_0 \in H^s_{\alpha, \beta} \), Theorem 5.2 implies that there exists a \( T = T(\|w_0\|_s) > 0 \) such that (5.1) has a unique solution \( u \) on \((0, T)\). For bounded \( \|u\|_s \), a standard extension argument can be used to extend the time interval of existence of the solution. Thus, if the solution does not exist for all time \( t > 0 \) with \( \|u\|_s \) finite at each \( t \), then \( \|u\|_s \) must blow up at some \( T^* > 0 \).

**Theorem 6.2.** There exist positive numbers \( \delta \) and \( b \) such that for any \( w_0 \in H^s_{\alpha, \beta} \) with \( \|w_0\|_s \leq \delta \), (5.1) has a unique solution \( u \in C(R^+ ; H^s_{\alpha, \beta}) \) satisfying

\[
\sup_{0 \leq t < \infty} \|u(x, t)\|_s \leq b.
\]

**Proof.** Define

\[
S_{b, \infty} = \left\{ v \in C(R^+ ; H^s_{\alpha, \beta}) : \sup_{0 \leq t < \infty} \|v(x, t)\|_s \leq b \right\}
\]

with \( b > 0 \) to be determined. Consider the map \( F \) defined on \( S_{b, \infty} \) as follows:

\[
Fv = S(t)w_0 + \int_0^t S(t - \tau)(if(v))(x, \tau)d\tau.
\]

Then, Propositions 4.1 and 4.3 imply that

\[
\sup_{0 \leq t < \infty} \|Fv\|_s = \sup_{0 \leq t < \infty} \left\| S(t)w_0 + \int_0^t S(t - \tau)(if(v))(x, \tau)d\tau \right\|_s
\]

\[
\leq \sup_{0 \leq t < \infty} \|S(t)w_0\|_s + \sup_{0 \leq t < \infty} \left\| \int_0^t S(t - \tau)(if(v))(x, \tau)d\tau \right\|_s
\]

\[
\leq c \|w_0\|_s + c \sup_{0 \leq t < \infty} \|if(v)\|_s
\]

\[
\leq c \|w_0\|_s + c \left( \sup_{0 \leq t < \infty} \|v\|_s^{p_1} + \sup_{0 \leq t < \infty} \|v\|_s^{p_2} \right).
\]
Choosing $b > 0$ and $\delta > 0$ such that $c(b^{p_1} + b^{p_2}) \leq \frac{1}{2} b$ and $c\delta \leq \frac{1}{2} b$, we obtain that 
$$\sup_{0 \leq t < \infty} \| Fv \|_s \leq b \text{ if } \| w_0 \|_s \leq \delta.$$ 
Thus, $F$ is a mapping defined on $S_{b, \infty}$.

Let $v = v_1 - v_2$. Proposition 4.3 yields

$$\sup_{0 \leq t < \infty} \| Fv_1 - Fv_2 \|_s = \sup_{0 \leq t < \infty} \left\| \int_0^t S(t - \tau)i(f(v_1) - f(v_2))(x, \tau)d\tau \right\|_s \leq c \sup_{0 \leq t < \infty} \left\| f(v_1) - f(v_2) \right\|_s \leq c \left( \sup_{0 \leq t < \infty} \| v_1 \|_{s}^{p_1 - 1} + \sup_{0 \leq t < \infty} \| v_1 \|_{s}^{p_2 - 1} \right) \sup_{0 \leq t < \infty} \| v \|_s \leq c(b^{p_1 - 1} + b^{p_2 - 1}) \sup_{0 \leq t < \infty} \| v \|_s \leq \frac{1}{2} \sup_{0 \leq t < \infty} \| v_1 - v_2 \|_s,$$

if $c(b^{p_1 - 1} + b^{p_2 - 1}) \leq \frac{1}{2}$. Thus, the contraction property of the map $F$ is obtained, which gives a desired solution. The proof is completed.

**Theorem 6.3.** If $s \in \left(\frac{1}{2}, 1\right]$, then there is a $\eta > 0$ such that for any given $w_0 \in H^s_{\alpha, \beta}$ with $\| w_0 \|_s < \eta$, the unique solution of (5.1) satisfies

$$\| u(x, t) \|_{L^2} \leq ce^{-\rho t} \| w_0 \|_{L^2}, \quad t \geq 0 \quad (6.1)$$

where $c > 0$ and $\rho > 0$ are constants and independent of $w_0$.

**Proof.** First, we define an operator $Y : L^2 \to L^2$ by

$$Y = \sum_k Y_k, \quad Y_k = \psi_k \psi_k^*,$$

where the series is strongly convergent and $\psi_k$ is the normalized eigenvector of $A^*$ given in Section 3. Let $w \in L^2$ and $w = \sum_j c_j \phi_j$. Then,

$$w^*Yw = \left( \sum_j c_j \phi_j \right) \left( \sum_k \psi_k \psi_k^* \right) \left( \sum_j c_j \phi_j \right) = \sum_j c_j^* \phi_j^* \psi_j \psi_j^* c_j \phi_j$$

with $\phi_k^* \psi_j = \delta_{kj}$ implies

$$w^*Yw = \sum_j |c_j|^2 \geq D_0^2 \left\| \sum_j c_j \phi_j \right\|_{L^2}^2 = D_0^2 \| w \|_{L^2}^2,$$

where Proposition 3.3 has been used. Hence, $Y$ is bounded and positive on $L^2$. Define $X : L^2 \to L^2$ by

$$X = \sum_k \zeta_k Y_k, \quad \zeta_k = \frac{1}{2 \text{Re } \lambda_k} \geq \frac{1}{2\gamma} > 0$$

where $-\gamma \geq \text{Re } \lambda_k \geq -c_0 \gamma$ with $c_0, \gamma > 0$. Thus, by $\zeta_k = -(2 \text{Re } \lambda_k)^{-1}$ and the definition of $A$ in (2.1) and (2.2), it is obtained that

$$A^*X + XA + Y = \sum_k (A^* \zeta_k Y_k + \zeta_k Y_k A + Y_k)$$

$$= \sum_k \left( \zeta_k A^* \psi_k \psi_k^* + \zeta_k \psi_k \psi_k^* A + \psi_k \psi_k^* \right)$$
Let Proposition 3.3, which gives

$$
\frac{d}{dt}(u^* X u) = \left( \frac{d}{dt} u \right)^* X u + u^* X \left( \frac{d}{dt} u \right)
$$

$$
= (Au + if(u))^* X u + u^* X (Au + if(u))
$$

$$
= u^* A^* X u + u^* X A u + (if(u))^* X u + iu^* X f(u)
$$

$$
= u^* (A^* X + X A) u + (if(u))^* X u + iu^* X f(u)
$$

$$
= -u^* Yu + (if(u))^* X u + iu^* X f(u). 
$$

(6.2)

Let $w = if(u)$. For any constant $d > 0$, we have

$$
(w^* X u + u^* X w - d^2 u^* X u - \frac{1}{d^2} w^* X w = -\left( du^* - \frac{1}{d} w^* \right) X \left( du - \frac{1}{d} w \right) \leq 0,
$$

and

$$
(if(u))^* X u + u^* X (if(u)) \leq d^2 u^* X u + \frac{1}{d^2} (f(u))^* X (f(u)).
$$

(6.3)

Choose $d > 0$ so that in the sense of quadratic forms on $L^2$, $d^2 X \leq \frac{1}{2} Y$. By Proposition 3.3,

$$
\frac{1}{d^2} (f(u))^* X (f(u)) = \frac{1}{d^2} (f(u))^* \sum_k \zeta_k \psi_k \psi_k^* (f(u)) = \frac{1}{d^2} \sum_k \zeta_k (f(u))^* \psi_k \psi_k^* (f(u))
$$

$$
\leq \frac{1}{2c_0 d^2} \sum_k \left| \int_0^L f(u) \psi_k dx \right|^2 \leq \frac{1}{d^2} d \left. \| f(u) \|_{L^2}^2
$$

$$
\leq \frac{cD}{2c_0 d^2} \left( \sup_{0 \leq t \leq T} ||u||_{p_1}^{p_1 - 1} + \sup_{0 \leq t \leq T} ||u||_{p_2}^{p_2 - 1} \right)^2 ||u||_{L^2}^2
$$

$$
\leq \frac{c (b^{p_1 - 1} + b^{p_2 - 1})^2 D}{2c_0 d^2} ||u||_{L^2}^2 \leq \frac{1}{4} u^* Yu
$$

where $b$ is chosen in Theorem 6.2 and further satisfies

$$
c (b^{p_1 - 1} + b^{p_2 - 1})^2 D (2c_0 d^2)^{-1} \leq \frac{1}{4}.
$$

From (6.2) we have

$$
\frac{d}{dt} (u^* X u) = -u^* Yu + (if(u))^* X u + u^* X (if(u))
$$

$$
\leq -u^* Yu + \frac{3}{4} u^* Yu = -\frac{1}{4} u^* Yu.
$$
Moreover, (1.5) implies that $\frac{d}{dt}(u^*Iu) \leq 0$ where $I$ is the identity map on $L^2$. Since $Y \geq 2d^2X$ and $Y \geq c_1I$, it is obtained that there is a $c_2 > 0$ such that $u^*Yu \geq c_2u^*(I + X)u$. Hence,

$$\frac{d}{dt}[u^*(I + X)u] \leq \frac{d}{dt}(u^*Xu) \leq -\frac{1}{4}u^*Yu \leq -\frac{c_2}{4}u^*(I + X)u,$$

which yields that

$$u^*(I + X)u \leq w_0^*(I + X)w_0e^{-c_2t/4}$$

or

$$\|u(\cdot, t)\|_{L^2}^2 \leq C\|w_0\|_{L^2}^2e^{-c_2t/4},$$

where $C$ is a positive constant. The proof is completed.

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