ON ESTIMATES OF DEVIATION OF CONJUGATE FUNCTIONS FROM MATRIX OPERATORS OF THEIR FOURIER SERIES BY SOME EXPRESSIONS WITH R-DIFFERENCES OF THE ENTRIES

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Abstract. We extend the results of the authors from [Abstract and Applied Analysis, Volume 2016, Article ID 9712878] to the case conjugate Fourier series.

1. Introduction

Let \( X = L^p \) or \( X = C \), where \( L^p \) (\( 1 \leq p \leq \infty \)) or \( C \) be the class of all \( 2\pi \)-periodic real-valued functions, integrable in the Lebesgue sense with the \( p \)-th power when \( p \geq 1 \) and essentially bounded when \( p = \infty \) or continuous over \( Q = [-\pi, \pi] \) with the norms

\[
\|f\|_{L^p} := \|f(\bullet)\|_{L^p} = \begin{cases} \left( \int_Q |f(t)|^p dt \right)^{1/p} & \text{when } 1 \leq p < \infty, \\ \text{ess sup}_{t \in Q} |f(t)| & \text{when } p = \infty, \end{cases}
\]

\[
\|f\|_C := \|f(\bullet)\|_C = \sup_{t \in Q} |f(t)|
\]

and consider the trigonometric Fourier series

\[
Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x)
\]

with the partial sums \( S_k f \) and the conjugate one

\[
\tilde{S}f(x) := \sum_{\nu=1}^{\infty} (a_\nu(f) \sin \nu x - b_\nu(f) \cos \nu x)
\]

with the partial sums \( \tilde{S}_k f \). We know that if \( f \in L^1 \), then

\[
\tilde{f}(x) := -\frac{1}{\pi} \int_0^\pi \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt = \lim_{\epsilon \to 0^+} \tilde{f}(x, \epsilon),
\]

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We will also use the modulus of continuity of $f$ defined by

\[
\bar{\omega} := \sup \{ \omega(t) : t \in [0,\pi] \}
\]

with $\psi_x(t) := f(x + t) - f(x - t)$, exists for almost all $x$ \text{[6]} Th.(3.1)IV.

Let $A := (a_{n,k})$ be an infinite matrix of real numbers such that $a_{n,k} \geq 0$ when $k, n = 0, 1, 2, \ldots$, $\lim_{n \to \infty} a_{n,k} = 0$ and $\sum_{k=0}^{\infty} a_{n,k} = 1$. We will use the notation $A_{n,r} = \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}|$, for $r \in \mathbb{N}$.

The $A$-transformation of $S_k f$ and of $\tilde{S}_k f$ be defined, by a matrix convention, as follows

\[
\left( \frac{T_n A f (x)}{T_n A f (x)} \right) := \sum_{k=0}^{\infty} a_{n,k} \left( \frac{S_k f (x)}{S_k f (x)} \right) \quad (n = 0, 1, 2, \ldots)
\]

provided the series are convergent. In this paper, we study the upper bounds of $\|T_n A f - f\|_X$ and $\|T_n A f (\bullet) - f(\bullet, \epsilon)\|_X$ by the modulus of continuity of $f$ in the space $X$ defined by the formula

\[
\tilde{\omega}(f, \delta)_X = \sup_{0 < t \leq \delta} \|\psi_x(t)\|_X.
\]

We will also use the modulus of continuity of $f$ in the space $X$ defined by $\omega(f, \delta)_X := \sup_{0 < t \leq \delta} \|\varphi_x(t)\|_X$, where $\varphi_x(t) := f(x + t) + f(x - t) - 2f(x)$.

We will consider a function $\omega$ of modulus of continuity type on the interval $[0, 2\pi]$, i.e., a nondecreasing continuous function having the properties: $\omega(0) = 0$, $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$ for any $0 \leq \delta_1 \leq \delta_2 \leq \delta_1 + \delta_2 \leq 2\pi$.

The deviation $T_n A f - f$ was estimated in \text{[2]} (see also \text{[1]} Theorems 3.4, p. 290) and \text{[5]} as follows:

**Theorem A.** Let $f \in \{ f \in X : \omega(f, \delta)_X = O(\omega(\delta))$ when $\delta \in [0, 2\pi] \}$ and $r \in \mathbb{N}$. Then

\[
\|T_n A f - f\|_X = O_r \left( H \left( \frac{\pi}{n+1} \right) \left( \frac{\pi}{n+1} + A_{n,r} \right) \right),
\]

where a function of modulus of continuity type $\omega$ satisfies the condition

\[
\int_{0}^{\pi} t^{-2} \omega(t) \, dt = O(H(u)) \quad \text{when } u \in [0, \pi],
\]

with $H(u) \geq 0$, such that

\[
\int_{0}^{u} H(t) \, dt = O(uH(u)) \quad \text{when } u \in [0, \pi].
\]

Additionally, if

\[
\sum_{l=0}^{n} \sum_{k=0}^{r+l-1} a_{n,k}^{-1} = O_r(1),
\]

then

\[
\|T_n A f - f\|_X = O_r \left( H \left( \frac{\pi}{n+1} \right) A_{n,r} \right).
\]
but if
\begin{equation}
(1.4)
\sum_{k=0}^{\infty} (k + 1) a_{n,k} = O(n + 1),
\end{equation}
then
\[ \|T_{n,A} f - f\|_X = O_r \left( \omega \left( \frac{\pi}{n + 1} \right) + H \left( \frac{\pi}{n + 1} \right) A_{n,r} \right). \]

**Theorem B.** If \( f \in X \) and a matrix \( A \) is such that \( 1.4 \) holds, then for \( r \in \mathbb{N} \)
\[ \| T_{n,A} f - f \|_X = O_r \left( \omega \left( f, \frac{\pi}{n + 1} \right) + \sum_{k=0}^{\infty} \omega \left( f, \frac{\pi}{\mu} \right) X \sum_{k=0}^{\infty} a_{n,k} \right) + \sum_{\mu=1}^{n} \omega \left( f, \frac{\pi}{\mu} \right) X \sum_{k=\mu}^{\infty} |a_{n,k} - a_{n,k + r}|. \]

From our theorems we also derived a corollary for a matrix \( A \) satisfying the condition \( \sum_{k=m}^{\infty} |a_{n,k} - a_{n,k + r}| = O_r(1) \sum_{k=m/c}^{\infty} a_{n,k} + 1 \) with some \( c > 1 \) and \( r \in \mathbb{N} \).

### 2. Statement of the results

Let \( X_\omega = \{ f \in X : \hat{\omega}(f, \delta)_X = O(\omega(\delta)) \} \) when \( \delta \in [0,2\pi] \). We present the estimates of the quantities \( \| \tilde{T}_{n,A} f(\bullet) - \tilde{f}(\bullet) \|_X \) and \( \| \tilde{T}_{n,A} f(\bullet) - \tilde{f}(\bullet, \epsilon) \|_X \) simultaneously. Finally, we give a corollary and a remark.

**Theorem 2.1.** If \( f \in X_\omega \), where \( \omega \) satisfies condition \( 1.1 \) such that \( 1.2 \) holds and \( r \in \mathbb{N} \), then
\[ \| \tilde{T}_{n,A} f(\bullet) - \tilde{f}(\bullet) \|_X = O_r \left( H \left( \frac{\pi}{n + 1} \right) \left( \frac{\pi}{n + 1} + A_{n,r} \right) \right). \]
Additionally, if a matrix \( A \) is such that \( 1.3 \) is true, then
\[ \| \tilde{T}_{n,A} f(\bullet) - \tilde{f}(\bullet) \|_X = O_r \left( H \left( \frac{\pi}{n + 1} \right) A_{n,r} \right). \]

**Theorem 2.2.** If \( f \in X_\omega \), where \( \omega \) satisfies condition \( 1.1 \) such that \( 1.2 \) holds, \( r \in \mathbb{N} \) and a matrix \( A \) is such that \( 1.3 \) is true, then
\[ \| \tilde{T}_{n,A} f(\bullet) - \tilde{f}(\bullet) \|_X = O_r \left( H(A_{n,r}) A_{n,r} \right). \]

**Theorem 2.3.** If \( f \in X_\omega \), where \( \omega \) satisfies condition \( 1.1 \) such that \( 1.2 \) holds and \( r \in \mathbb{N} \), then
\[ \| \tilde{T}_{n,A} f(\bullet) - \tilde{f}(\bullet) \|_X = O_r \left( \omega \left( \frac{\pi}{n + 1} \right) + H \left( \frac{\pi}{n + 1} \right) A_{n,r} \right), \]
where in the case of the first estimate \( \omega \) satisfies the extra condition
\begin{equation}
(2.1)
\int_0^u t^{-1} \omega(t) dt = O(\omega(u)) \text{ when } u \in [0,2\pi],
\end{equation}
but in the case of the second estimate a matrix $A$ is such that (1.4) is true.

**Theorem 2.4.** If $f \in X$ and $r \in \mathbb{N}$, then

$$
\left\| \tilde{T}_{n,A}f(\bullet) - \tilde{f}(\bullet, \frac{\pi}{r(n+1)}) \right\|_X = O_r \left( \tilde{\omega} \left( f, \frac{\pi}{n+1} \right) + \sum_{\mu=1}^{n} \tilde{\omega} \left( f, \frac{\pi}{\mu} \right) \sum_{k=0}^{\mu+1} a_{n,k} \\
+ \sum_{\mu=1}^{n} \tilde{\omega} \left( f, \frac{\pi}{\mu} \right) X \sum_{k=\mu}^{\infty} |a_{n,k} - a_{n,k+r}| \right),
$$

were in the case of the first estimate $\tilde{\omega}$ instead of $\omega$ satisfies the extra condition (2.1), but in the case of the second estimate a matrix $A$ is such that (1.4) is true.

**Corollary 2.1.** If $f \in X_\omega$, where $\omega$ satisfies the condition (1.1) such that (1.2) is true and

$$
\sum_{k=m}^{\infty} |a_{n,k} - a_{n,k+r}| = O_r(1) \sum_{k=m/c}^{\infty} a_{n,k} \frac{k}{k+1},
$$

with some $c > 1$ and $r \in \mathbb{N}$ holds, then

$$
\left\| \tilde{T}_{n,A}f(\bullet) - \tilde{f}(\bullet, \frac{\pi}{r(n+1)}) \right\|_X = O_r \left( \frac{H(\frac{\pi}{n+1})}{n+1} + \sum_{k=0}^{n} a_{n,k} \frac{H(\frac{\pi}{k+1})}{k+1} \right),
$$

were in the case of the first estimate $\tilde{\omega}$ instead of $\omega$ satisfies extra condition (2.4), but in the case of the second estimate a matrix $A$ is such that (1.4) is true.

**Remark 2.1.** We note that our extra conditions (1.3) and (1.4) for a lower triangular infinite matrix $A$ always hold.

### 3. Auxiliary results

We begin this section by some notations from [4] and [6] Section 5 of Chapter II. Let for $r = \pm 1, \pm 2, \ldots$

$$
D_{\sigma,r}(t) = \frac{\sin(2k+r)t}{2 \sin \frac{\pi}{2}}, \quad \tilde{D}_{r,k}(t) = \frac{\cos(2k+r)t}{2 \sin \frac{\pi}{2}}, \quad \tilde{D}_{k,r}(t) = \frac{\cos \frac{\pi}{2} - \cos (2k+r)t}{2 \sin \frac{\pi}{2}}.
$$

It is clear by [6] that $\tilde{S}_k f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \tilde{D}_{k,1}(t) dt$, whence

$$
\tilde{T}_{n,A}f(x) - \tilde{f}(x) = \frac{1}{\pi} \int_{0}^{\pi} \tilde{\psi}_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}(t) dt
$$

and

$$
\tilde{T}_{n,A}f(x) - \tilde{f}(x, \frac{\pi}{r(n+1)}) = \frac{1}{\pi} \int_{0}^{\pi} \tilde{\psi}_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}_{k,1}(t) dt
$$

Next, we present the known estimates and relations.
Lemma 3.1. [6] If $0 < |t| \leq \pi$, then $|\tilde{D}^\delta_{k,1}(t)| \leq \frac{\pi}{2|t|}$, $|\tilde{D}_{k,1}(t)| \leq \frac{\pi}{|t|}$ and, for any real $t$, we have $|\tilde{D}_{k,1}(t)| \leq \frac{1}{2}k(k + 1)|t|$ and $|\tilde{D}^\delta_{k,1}(t)| \leq k < k + 1$.

Lemma 3.2. [4] Let $r \in N$, $l \in Z$ and $a := (a_n) \subset C$. If $x \neq \frac{2l\pi}{r}$, then for every $m \geq n$

$$\sum_{k=n}^{m} a_k \sin kx = - \sum_{k=n}^{m} (a_k - a_{k+r}) \tilde{D}^\delta_{k,1}(t) + \sum_{k=m+1}^{m+r} a_k \tilde{D}^\delta_{k,1}(t) - \sum_{k=n}^{n+r-1} a_k \tilde{D}^\delta_{k,1}(t),$$

and

$$\sum_{k=n}^{m} a_k \cos kx = \sum_{k=n}^{m} (a_k - a_{k+r}) D^\delta_{k,1}(t) - \sum_{k=m+1}^{m+r} a_k D^\delta_{k,1}(t) + \sum_{k=n}^{n+r-1} a_k D^\delta_{k,1}(t).$$

We additionally need two estimates with a function of modulus of continuity type $\omega$.

Lemma 3.3. [2] If (1.1) and (1.2) hold, then for $c \geq 1$ and $\beta > \alpha > 0$

$$\int_{\alpha}^{\beta} t^{-\gamma} \omega(t)dt = O((\beta - \alpha)H(c(\beta - \alpha)))$$

when $(\beta - \alpha) \in [0, 2\pi]$.

Lemma 3.4. [2] If (1.1) and (1.2) hold, then for $b \geq 1$,

$$\int_{u}^{\pi} t^{-\gamma} \omega(t)dt = O(H(b\pi))$$

when $u \in [0, \pi]$.

Finally, we present a very useful trivial property of a function of modulus of continuity type $\omega$.

Lemma 3.5. A function $\omega$ of modulus of continuity type on the interval $[0, 2\pi]$ satisfies the following conditions $\delta_0^{-1} \omega(\delta_2) \leq 2\delta_1^{-1} \omega(\delta_1)$ for $\delta_2 \geq \delta_1 > 0$ and $\omega(\pi) \leq \pi (\pi)$ for $\delta > 0, n \in N$.

4. Proofs of the results

Proof of Theorem 2.1. It is clear that for an odd $r$

$$\left( \tilde{T}_{n,A}(x) - \tilde{f}(x) \right)$$

$$\left( \tilde{T}_{n,A}(x) - \tilde{f}(x, \frac{\pi}{2}) \right)$$

$$= \left( + \right) \frac{1}{\pi} \int_{0}^{\pi} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \left( \tilde{D}^\delta_{k,1}(t) \right) dt + \frac{1}{\pi} \int_{\pi}^{2\pi} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}^\delta_{k,1}(t) dt$$

$$+ \frac{1}{\pi} \sum_{m=1}^{\lfloor r/2 \rfloor} \int_{\frac{2m\pi}{r}}^{\frac{2m\pi}{r} + \frac{\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}^\delta_{k,1}(t) dt + \frac{1}{\pi} \sum_{m=0}^{\lfloor r/2 - 1 \rfloor} \int_{\frac{2m\pi}{r} + \frac{\pi}{r}}^{\frac{2m\pi}{r} + \frac{\pi}{r} + \frac{\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}^\delta_{k,1}(t) dt$$

$$= \left( J_1(x) \right) + J_2(x) + J_3(x) + J_4(x)$$

and for an even $r$
\[
\begin{align*}
&\left( \tilde{T}_nAf(x) - \tilde{f}(x) \right)
- \left( \tilde{T}_nAf(x) - \tilde{f}(x, \pi n) \right)
= \left( + \frac{1}{\pi} \int_0^{\pi n} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \left( \tilde{D}^{k,1}_{k,1}(t) \right) dt + \frac{1}{\pi} \int_{1/\pi}^{\pi n} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}^{k,1}_{k,1}(t) dt \\
&\quad + \frac{1}{\pi} \sum_{m=1}^{[r/2]-1} \int_{2m\pi}^{2m\pi + \frac{\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}^{k,1}_{k,1}(t) dt + \frac{1}{\pi} \sum_{m=0}^{2(m+1)\pi} \int_{2m\pi}^{2m\pi + \frac{\pi}{r}} \psi_x(t) \sum_{k=0}^{\infty} a_{n,k} \tilde{D}^{k,1}_{k,1}(t) dt
\right)
\end{align*}
\]

Then
\[
\left( \frac{\|\tilde{T}_nAf(\bullet) - \tilde{f}(\bullet, \pi n)\|_X}{\|\tilde{T}_nAf(\bullet) - \tilde{f}(\bullet, \pi n)\|_X} \right)
\leqslant \left( \|J_1 + J_2 + I_1''\|_X + \|J_1 + J_2 + I_1''\|_X \right) + \|\tilde{f}\|_X.
\]

By Lemma 3.1
\[
\|J_1''\|_X \leqslant \frac{1}{\pi} \int_0^{\pi n} \|\psi_x(t)\|_X \sum_{k=0}^{\infty} a_{n,k} \tilde{D}^{k,1}_{k,1}(t) dt
\leqslant \frac{1}{2\pi} \int_0^{\pi n} \|\psi_x(t)\|_X \sum_{k=0}^{\infty} a_{n,k} \frac{\pi}{t} dt \leqslant \frac{1}{2} \int_0^{\pi n} O(\omega(t)) \frac{dt}{t}.
\]

Since, by Lemma 3.2,
\[
\sum_{k=0}^{\infty} a_{n,k} \tilde{D}^{k,1}_{k,1}(t) = \sum_{k=0}^{\infty} a_{n,k} \cos \frac{(2k+1)t}{2} \sin \frac{t}{2}
\]
\[
= \frac{1}{2} \sin \frac{t}{2} \left( \sum_{k=0}^{\infty} a_{n,k} \cos kt \cos \frac{t}{2} - \sum_{k=0}^{\infty} a_{n,k} \sin kt \sin \frac{t}{2} \right)
\]
\[
= \frac{\cos \frac{t}{2}}{2} \sin \frac{t}{2} \left( \sum_{k=0}^{\infty} \left( a_{n,k} - a_{n,k+r} \right) D_{k,r}^{k,1}(t) + \sum_{k=0}^{r-1} a_{n,k} D_{k,r}^{k,1}(t) \right)
\]
\[
- \frac{1}{2} \left( - \sum_{k=0}^{\infty} \left( a_{n,k} - a_{n,k+r} \right) \tilde{D}^{k,1}_{k,1}(t) - \sum_{k=0}^{r-1} a_{n,k} \tilde{D}^{k,1}_{k,1}(t) \right),
\]
whence
\[
\left| \sum_{k=0}^{\infty} a_{n,k} \tilde{D}^{k,1}_{k,1}(t) \right| \leqslant \frac{1}{2} \sin \frac{t}{2} \sum_{k=0}^{r-1} a_{n,k} \leqslant \frac{1}{\sin \frac{t}{2} \sin \frac{\pi}{2}} A_{n,r}.
\]

Hence and by Lemma 3.1,
\[
\left( \frac{\|J_1 + J_2 + I_1''\|_X + \|J_1 + J_2 + I_1''\|_X}{\|J_2 + I_1''\|_X + \|J_2 + I_1''\|_X} \right)
\]
and therefore

\[ \| J_1 | x + \| J_1 + J_2 + I''_r | x \bigg\| \leq \sum_{m=0}^{[r/2]-\kappa} \int \frac{2 \pi}{\sin \left( \frac{\pi}{2} - \frac{\pi}{4 (m+1)} \right)} O(\omega(t)) \cdot \frac{\pi}{t} + \frac{[r/2]-\kappa}{\pi} \int \frac{2 \pi}{\sin \left( \frac{\pi}{2} - \frac{\pi}{4 (m+1)} \right)} O(\omega(t)) \cdot \frac{\pi}{t} dt \]

\[ \| J'_1 | x + \| J_1 + J_2 + I''_r | x \bigg\| \leq O(1)(r/2) + \frac{\pi}{n+1} H \left( \frac{\pi}{n+1} \right) + 2 A_{n,r} \sum_{m=0}^{[r/2]-\kappa} \int \frac{2 \pi}{\sin \left( \frac{\pi}{2} - \frac{\pi}{4 (m+1)} \right)} O(\omega(t)) \cdot \frac{\pi}{t} dt \]

\[ \| J'_1 | x + \| J_1 + J_2 + I''_r | x \bigg\| \leq O(1) \left[ \frac{\pi}{n+1} H \left( \frac{\pi}{n+1} \right) + \| J_1 + J_2 + I''_r | x \bigg\| \right] \]

Similarly, by Lemma 3.1, Lemmas 3.3, 3.4, with \( c = b = r \) and the estimates \( | \sin \frac{t}{2} | \geq \frac{| t |}{\pi} \), \( | \sin \frac{rt}{2} | \geq 2(m+1) - \frac{rt}{\pi} \) for \( t \in \left[ \frac{2(m+1)\pi}{r}, \frac{2(m+1)\pi}{r} - \frac{\pi}{r(n+1)} \right] \subset [0, \pi] \), where \( m \in \{0, \ldots, [r/2] - 1\} \), we get

\[ \| I_2 \bigg\| \leq \frac{[r/2]-1}{\pi} \sum_{m=0}^{[r/2]-1} \int \frac{2 \pi}{\sin \left( \frac{\pi}{2} - \frac{\pi}{4 (m+1)} \right)} \| \psi(t) \bigg\| \cdot \frac{\pi}{t} + \frac{[r/2]-1}{\pi} \sum_{m=0}^{[r/2]-1} \int \frac{2 \pi}{\sin \left( \frac{\pi}{2} - \frac{\pi}{4 (m+1)} \right)} \| \psi(t) \bigg\| \cdot \frac{\pi}{t} dt \]
Consider an odd and the second result also follows.

Analogously, as in the proof of Theorem 2.1, we prove Theorem 2.2.

Applying condition (1.3) we have

\[ T_{r/n,A} \sum_{m=0}^{[r/2]-1} \int_{r/(n+1)}^{r/(n+1)} O(\omega(t)) t dt + \int_{r/(n+1)}^{r/(n+1)} O(\omega(t)) t dt \]

\[ = \int_{r/(n+1)}^{r/(n+1)} O(\omega(t)) t dt + A_{n,r} \sum_{m=0}^{[r/2]-1} \int_{r/(n+1)}^{r/(n+1)} O(\omega(t)) t dt \]

Thus

\[ \|I_2\|_X = O(1) \left[ \frac{\pi}{n+1} H \left( \frac{\pi}{n+1} \right) + A_{n,r} H \left( \frac{\pi}{n+1} \right) \right]. \]

Collecting these estimates we obtain the first result.

Applying condition (1.3) we have

\[ \left( n + 1 \right) \sum_{k=0}^{\infty} \left\| a_{n,k} - a_{n,k+r} \right\|^{-1} = \left[ \sum_{l=0}^{n} \sum_{k=0}^{\infty} \left\| a_{n,k} - a_{n,k+r} \right\| \right]^{-1} \]

and the second result also follows. \( \square \)

Proof of Theorem 2.2. Analogously, as in the proof of Theorem 2.1, we consider an odd \( r \) and an even \( r \). Then,

\[ \left( \hat{T}_{r,n,A} f(x) - \hat{f}(x) \right) \left( \hat{T}_{r,n,A} f(x) - \hat{f}(x) \right) \]

\[ = \left( \hat{A}_{n,r} \right) \int_{0}^{\infty} \psi_2(t) \sum_{k=0}^{\infty} a_{n,k} \left( \hat{D}_{k,1}(t) \right) dt + \int_{0}^{\infty} \psi_2(t) \sum_{k=0}^{\infty} a_{n,k} \hat{D}_{k,1}(t) dt \]
\[ + \frac{1}{\pi} \sum_{m=1}^{[r/2]} \frac{2m+1}{2m} \int_0^{\pi} \psi_x(t) \sum_{k=0}^{\infty} a_n,k \tilde{D}^2_{k,1}(t) dt + \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \frac{2m+1}{2m} \int_0^{\pi} \psi_x(t) \sum_{k=0}^{\infty} a_n,k \tilde{D}^2_{k,1}(t) dt \]

or

\[ \left( \frac{\tilde{T}_{n,A} f(x) - \tilde{f}(x)}{\tilde{T}_{n,A} f(x) - \tilde{f}(x, \frac{1}{2} A_{n,r})} \right) \]

\[ = \left( \frac{1}{\pi} \sum_{m=1}^{[r/2]} \frac{2m+1}{2m} \int_0^{\pi} \psi_x(t) \sum_{k=0}^{\infty} a_n,k D^2_{k,1}(t) dt + \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \frac{2m+1}{2m} \int_0^{\pi} \psi_x(t) \sum_{k=0}^{\infty} a_n,k \tilde{D}^2_{k,1}(t) dt \]

respectively. Since \( A_{n,r} \leq 2 \), we can estimate our terms analogously as in the proof of Theorem 2.1 with \( A_{n,r} \) instead of \( \frac{2}{\pi+1} \) and thus we obtain the desired estimate.

**Proof of Theorem 2.3.** Similarly, as in the proof of Theorem 2.1

\[ \left( \left\| \tilde{T}_{n,A} f(\bullet) - \tilde{f}(\bullet) \right\|_X \right) \leq \left( \left\| J_1 \right\|_X \right) + \left\| J_2 + J_1'' \right\|_X + \left\| J_2 + J_1'' \right\|_X + \left\| J_2 \right\|_X. \]

By Lemma 3.1 and \eqref{eq:2.12}

\[ \left\| J_1'' \right\|_X \leq \frac{1}{\pi} \int_0^{\pi/n+1} \max\left\{ \frac{2m+1}{2m} \right\} \left( \sum_{k=0}^{\infty} a_n,k D^2_{k,1}(t) \right) dt \leq \frac{1}{\pi} \sum_{k=0}^{\infty} (k+1) a_n,k \int_0^{\pi/n+1} \omega(t) dt \]

\[ = O(n+1) \int_0^{\pi/n+1} \omega(t) dt = O(1) \omega\left( \frac{\pi}{n+1} \right) = O\left( \frac{\pi}{n+1} \right) \]

and by Lemma 3.1 and \eqref{eq:2.11}

\[ \left\| J_1 \right\|_X \leq \frac{1}{\pi} \int_0^{\pi/n+1} \max\left\{ \frac{2m+1}{2m} \right\} \left( \sum_{k=0}^{\infty} a_n,k D^2_{k,1}(t) \right) dt \leq \frac{1}{2\pi} \int_0^{\pi/n+1} \max\left\{ \frac{2m+1}{2m} \right\} \left( \sum_{k=0}^{\infty} a_n,k \frac{\pi}{t} \right) dt \]

\[ \leq \frac{1}{2} \int_0^{\pi/n+1} \frac{\omega(t)}{t} dt = O\left( \omega\left( \frac{\pi}{n+1} \right) \right) = O\left( \omega\left( \frac{\pi}{n+1} \right) \right). \]

Further, by the same lemmas and conditions as in the above proofs and Lemma 3.5, we obtain with \( \kappa = 1 \) when \( r \) is even, and \( \kappa = 0 \) when \( r \) is odd, that

\[ \left\| J_2 + J_1'' \right\|_X + \left\| J_2 + J_1'' \right\|_X. \]
\[
\begin{align*}
\|a\|_{2} & \leq \frac{2}{\pi} \left( \sum_{m=1}^{[r/2]-\kappa} \left( \int_{0}^{2m\pi} + \frac{\pi}{2} \right) \|\psi_{*}(t)\| \sum_{k=0}^{\infty} \left| a_{n,k} \overline{D}_{c,k,1}^{-1} \right| dt \right) \\
\|a\|_{2} & \leq \frac{2}{\pi} \left( \sum_{m=1}^{[r/2]-\kappa} \left( \int_{0}^{2m\pi} + \frac{\pi}{2} \right) \|\psi_{*}(t)\| \sum_{k=0}^{\infty} \left| a_{n,k} \overline{D}_{c,k,1}^{-1} \right| dt \right) \\
\|a\|_{2} & \leq \frac{2}{\pi} \sum_{m=1}^{[r/2]-\kappa} \left( \int_{0}^{2m\pi} \frac{O(\omega(t))}{t} dt + \frac{2}{\pi} \sum_{m=0}^{\infty} \left( \int_{0}^{2m\pi} + \frac{\pi}{2} \right) \frac{O(\omega(t))}{\sin \frac{t}{2} \sin \frac{\pi}{2t}} \right) A_{n,r} dt \\
\|a\|_{2} & \leq \frac{2}{\pi} \sum_{m=1}^{[r/2]-\kappa} \left( \int_{0}^{2m\pi} \frac{O(\omega(t))}{t} dt + 2A_{n,r} \sum_{m=0}^{\infty} \left( \int_{0}^{2m\pi} + \frac{\pi}{2} \right) \frac{O(\omega(t))}{t \left( \frac{t^{2}}{\pi} - 2m \right)} \right) dt \\
\|a\|_{2} & \leq 4 \sum_{m=1}^{[r/2]-\kappa} \frac{O \left( \left( \frac{2m\pi}{\pi} \right) \right)}{r(n+1)} \frac{\pi}{\left( [r/2] + 1 \right) A_{n,r}} \int_{\frac{t^{2}}{\pi}}^{\frac{\pi}{t}} \left( \frac{\pi}{n+1} \right) dt \\
\|a\|_{2} & \leq 4 \sum_{m=1}^{[r/2]-\kappa} \frac{O \left( \left( \frac{2m\pi}{\pi} \right) \right)}{r(n+1)} \frac{\pi}{\left( [r/2] + 1 \right) A_{n,r}} + O(1) A_{n,r} H \left( \frac{\pi}{n+1} \right) \\
\|a\|_{2} & \leq 4 \sum_{m=1}^{[r/2]-\kappa} \frac{O \left( \left( \frac{2m\pi}{\pi} \right) \right)}{r(n+1)} \frac{\pi}{\left( [r/2] + 1 \right) A_{n,r}} + O(1) A_{n,r} H \left( \frac{\pi}{n+1} \right) \\
\|a\|_{2} & = O(1) \left( \frac{\pi}{n+1} + A_{n,r} H \left( \frac{\pi}{n+1} \right) \right)
\end{align*}
\]

and

\[
\begin{align*}
\|b\|_{2} & \leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \left( \int_{0}^{2m\pi} + \frac{\pi}{2} \right) \left( \int_{0}^{2m\pi} + \frac{\pi}{2} \right) \|\psi_{*}(t)\| \sum_{k=0}^{\infty} \left| a_{n,k} \overline{D}_{c,k,1}^{-1} \right| dt \\
\|b\|_{2} & \leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \left( \int_{0}^{2m\pi} + \frac{\pi}{2} \right) \left( \int_{0}^{2m\pi} + \frac{\pi}{2} \right) \frac{O(\omega(t))}{\sin \frac{t}{2} \sin \frac{\pi}{2t}} A_{n,r} dt + \frac{1}{2} \sum_{m=0}^{[r/2]-1} \left( \int_{0}^{2m\pi} + \frac{\pi}{2} \right) \frac{O(\omega(t))}{\sin \frac{t}{2} \sin \frac{\pi}{2t}} \right) dt
\end{align*}
\]
Thus our proof is complete.

Proof of Theorem 2.4. Let us above

\[ \left\| \mathcal{T}_{n,a}f(\bullet) - \mathcal{F}(\bullet) \right\|_X \leq \left( \|J_1\|_X + \|J_2 + I''_2\|_X + \|J_2 + I''_1\|_X \right), \]

\[ \|J_1\|_X \leq \frac{1}{\pi} \int_0^{\pi/2} \|\psi(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} \hat{D}_k(t) \right| dt \leq \frac{1}{2} \sum_{k=0}^{\infty} a_{n,k} \int_0^{\pi/2} \frac{\bar{\omega}(f,t) x}{t} dt \]

\[ = O(1) \bar{\omega}\left(f, \frac{r}{n+1}\right) x = O\left(\bar{\omega}\left(f, \frac{r}{n+1}\right) x\right), \] by (2.4),

and

\[ \|J'_1\|_X \leq \frac{1}{\pi} \int_0^{\pi/2} \|\psi(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} \hat{D}_k(t) \right| dt \leq \frac{1}{2} \sum_{k=0}^{\infty} a_{n,k} (k+1) dt \]

\[ \leq O(n+1) \int_0^{\pi/2} \bar{\omega}(f,t) x dt = O\left(\bar{\omega}\left(f, \frac{r}{n+1}\right) x\right), \] by (2.1).

Further, taking \( t_m = \left\lfloor \frac{\pi}{r-2\pi n^2} \right\rfloor \) and \( t = \left\lfloor \frac{\pi}{r} \right\rfloor \), using Lemma 3.5, and with \( r = 1 \) when \( r \) is even, and \( r = 0 \) when \( r \) is odd, we obtain

\[ \|J_2 + I''_2\|_X + \|J_2 + I''_1\|_X \]

\[ \leq \frac{2}{\pi} \left( \sum_{m=1}^{\mid r/2 \mid} \int_0^{\pi/2} \left| \frac{a_{n,m}}{2m-\pi/2} \right| + \int_{\pi/2}^{\pi/2} \left| \frac{a_{n,m}}{2m-\pi/2} \right| \|\psi(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} \hat{D}_k(t) \right| dt \right) \]

\[ = \frac{2}{\pi} \left( \sum_{m=1}^{\mid r/2 \mid} \int_0^{\pi/2} + \sum_{m=1}^{\mid r/2 \mid} \int_{\pi/2}^{\pi/2} \left| \frac{a_{n,m}}{2m-\pi/2} \right| \|\psi(t)\|_X \left| \sum_{k=0}^{\infty} a_{n,k} \hat{D}_k(t) \right| dt \right) \]
Next, taking \( \tau_m = \left[ \frac{\pi}{\pi + 2(r/2) + 1} \right] \), we obtain

\[
\|I_2\| \leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi + \pi}^{(2m+2)\pi} \left\| \psi_t(t) \right\| X \sum_{k=0}^{\infty} \left| a_n, k \right| \left| D^{\circ}_{k, 1}(t) \right| \, dt
\]

\[
\leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \left( \int_{2m\pi + \pi}^{(2m+2)\pi} + \int_{(2m+2)\pi}^{(2m+4)\pi} - \int_{2m\pi + \pi}^{(2m+4)\pi} \right) \left\| \psi_t(t) \right\| X \sum_{k=0}^{\infty} \left| a_n, k \right| \left| D^{\circ}_{k, 1}(t) \right| \, dt
\]

\[
\leq \frac{1}{\pi} \sum_{m=0}^{[r/2]-1} \int_{2m\pi + \pi}^{(2m+2)\pi} \left( \frac{\tilde{\omega}(f, t) X}{2\sin \frac{\pi}{2}} \sum_{k=0}^{\tau_m^2} + \frac{\tilde{\omega}(f, t) X}{\sin \frac{\pi}{2} \sin \frac{\omega}{2}} \sum_{k=\tau_m^{r_2}}^{\infty} \left| a_n, k - a_n, k+r \right| \right) \, dt
\]
\[
\sum_{k=0}^{\infty} a_{n,k} dt
\]
\[
\sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| dt
\]
\[
\sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| dt
\]
\[
O_r(1) \sum_{\mu=1}^{n} \sum_{k=0}^{\infty} a_{n,k} + O_r(1) \sum_{\mu=1}^{n} \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}|
\]
\[
O_r(1) \omega \left( f, \frac{\pi}{n+1} \right)_X.
\]
Thus the result follows. \(\square\)

**Proof of Corollary 2.1.** Theorem 2.3 implies that
\[
\|\tilde{T}_{n,A}f(x) - \tilde{f}(x)\|_X = O_r \left( \omega \left( f, \frac{\pi}{n+1} \right)_X + \sum_{\mu=1}^{n} \sum_{k=0}^{\infty} a_{n,k} \right)
\]
\[
+ O_r(1) \omega \left( f, \frac{\pi}{n+1} \right)_X \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}|.
\]
Since (2.2)
\[
\sum_{\mu=1}^{n} \omega \left( f, \frac{\pi}{\mu} \right)_X \sum_{k=0}^{\infty} |a_{n,k} - a_{n,k+r}| = O_r(1) \sum_{\mu=1}^{n} \omega \left( f, \frac{\pi}{\mu} \right)_X \left( \sum_{k=0}^{\mu} + \sum_{k=\mu}^{\infty} a_{n,k} \right)
\]
If (1.1) and (1.2) hold, then one has

\[
\begin{align*}
\lesssim \tilde{O}_r(1) \left( \sum_{\mu=1}^{n} \tilde{\omega} \left( f, \frac{\pi}{\mu} \right) X \left( \sum_{k=\mu}^{\infty} \frac{a_{n,k}}{k+1} \right) + \tilde{O}_r(1) \left( \sum_{\mu=1}^{n} \tilde{\omega} \left( f, \frac{\pi}{\mu} \right) X \left( \sum_{k=\mu+1}^{\infty} \frac{a_{n,k}}{k+1} \right) \right) \right) \\
\lesssim \tilde{O}_r(1) \left( \sum_{\mu=1}^{n} \tilde{\omega} \left( f, \frac{\pi}{\mu} \right) X \sum_{k=0}^{\mu} a_{n,k} + \tilde{O}_r(1) \left( \sum_{\mu=1}^{n} \tilde{\omega} \left( f, \frac{\pi}{\mu} \right) X \left( \sum_{k=\mu}^{\infty} \frac{a_{n,k}}{k+1} \right) \right) \right)
\end{align*}
\]

one has

\[
\left\| \tilde{T}_{n,A}f(\bullet) - \tilde{f}(\bullet) \right\|_X \leqslant \tilde{O}_r(1) \left( \sum_{\mu=1}^{n} \tilde{\omega} \left( f, \frac{\pi}{\mu} \right) X \sum_{k=\mu}^{\infty} \frac{a_{n,k}}{k+1} \right) + \tilde{O}_r(1) \left( \sum_{\mu=1}^{n} \tilde{\omega} \left( f, \frac{\pi}{\mu} \right) X \left( \sum_{k=\mu}^{\infty} \frac{a_{n,k}}{k+1} \right) \right)
\]

If (1.11) and (1.12) hold, then

\[
\tilde{\omega} \left( f, \frac{\pi}{n+1} \right) X \leqslant \frac{1}{n+1} \sum_{\mu=0}^{n} \tilde{\omega} \left( f, \frac{\pi}{\mu+1} \right) X = O(1) \frac{H(\frac{\pi}{n+1})}{n+1}.
\]
\[ \sum_{\mu=k}^{n} \frac{\hat{\omega}(f, \frac{\pi}{n+1})}{\mu + 1} X = \frac{1}{\pi^2} \int_{\pi}^{\pi+1} \frac{\hat{\omega}(f, t) X}{t} dt = O(1) \]

and therefore
\[ \left\| \tilde{T}_{n,A} f(\bullet) - \tilde{f}(\bullet) \right\|_{X} = O_{r} \left( \frac{H\left( \frac{\pi}{n+1} \right)}{n+1} \right) + O_{r} \left( \sum_{k=0}^{n} a_{n,k} \frac{H\left( \frac{k+1}{n+1} \right)}{k+1} \right) + O_{r} \left( \frac{\pi}{n+1} \sum_{k=n+1}^{\infty} a_{n,k} \right). \]

Since
\[ \sum_{k=n+1}^{\infty} \frac{a_{n,k}}{k+1} \leq \frac{1}{n+1} \sum_{k=n+1}^{\infty} a_{n,k} \leq \frac{1}{n+1} \]
the result follows. □

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