AN EXCURSION APPROACH TO MAXIMA OF THE BROWNIAN BRIDGE

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ABSTRACT. Functionals of Brownian bridge arise as limiting distributions in nonparametric statistics. In this paper we will give a derivation of distributions of extrema of the Brownian bridge based on excursion theory for Brownian motion. Only the Poisson character of the excursion process will be used. Particular cases of calculations include the distributions of the Kolmogorov-Smirnov statistic, the Kuiper statistic, and the ratio of the maximum positive ordinate to the minimum negative ordinate.

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1. Introduction.

Functionals of Brownian bridge arise naturally as asymptotic distributions in nonparametric statistics. For an overview see Shorack and Wellner (1986). The discussion in this paper will be limited to extrema of Brownian bridges even though the techniques are applicable to a wider class of homogeneous functionals. See Carmona et al. (1999), Pitman and Yor (1999), Pitman and Yor (1998) and Perman and Wellner (1996).

Let \((U_t : 0 \leq t \leq 1)\) be the standard Brownian bridge. Define

\[
M^+ = \max_{0 \leq t \leq 1} U_t, \quad M^- = -\min_{0 \leq t \leq 1} U_t
\]

and

\[
m = \min\{M^+, M^-\} \quad \text{and} \quad M = \max\{M^+, M^-\}.
\]

The distribution of \(M^+\) was first computed by Smirnov (1939). For an elementary derivation see Govindarajulu and Klotz (1973). The derivation of the distribution of \(M^-\) was given by Kolmogorov (1933).

Here we will give a derivation based on excursion theory for Brownian motion. It will be shown that the Laplace transform of \(M\) can be derived from that of \(M^+\), and the distribution of \(M^+\) can be derived by an elementary application of the reflection principle for Brownian motion. The Laplace transforms can be inverted explicitly as infinite series. In a similar fashion one can derive the joint distribution of \((M^+, M^-)\).

Let \((B_t : t \geq 0)\) be standard Brownian motion and denote the last exit time from 0 before time \(t\) by \(g_t = \sup\{s \leq t : B_s = 0\}\). Furthermore, define \((L(t) : t \geq 0)\) to be the local time process at level 0 for Brownian motion normalized so that

\[
L(t) \overset{d}{=} \max_{0 \leq s \leq t} B_s.
\]

See Revuz and Yor (1999) for definitions.

The following lemma is well known. See Lévy (1948), Dynkin (1961), Barlow et al. (1989).

**Lemma 1.** The distribution of \(g_1\) is Beta\((1/2, 1/2)\). Given \(g_1\), the process \((B_t : 0 \leq t \leq g_1)\) is a Brownian bridge of length \(g_1\), and the rescaled process \((B(u/g_1) : 0 \leq u \leq 1)\) is a Brownian bridge independent of \(g_1\).

If \(S_\theta\) is an exponential random variable with parameter \(\theta\) independent of \(B\) define \(g_{S_\theta} = \sup\{u \leq S_\theta : B_u = 0\}\). Scaling properties of \(B\)
imply that the rescaled process
\[
\tilde{B} = \left( \frac{B_t S_\theta}{\sqrt{S_\theta}}, \quad 0 \leq t \leq 1 \right)
\]
is Brownian motion on the unit interval independent of \(S_\theta\) and the corresponding last exit time \(\tilde{g}_1 = \sup\{t \leq 1 : \tilde{B}_t = 0\}\) is by Lemma 1 a Beta(1/2, 1/2) random variable independent of \(S_\theta\). It follows that \(g_{S_\theta} = \tilde{g}_1 \cdot S_\theta\) is \(\Gamma(1/2, \theta)\) random variable independent of \(S_\theta\). Furthermore, given \(g_{S_\theta}\), the process \((B_t : 0 \leq t \leq g_{S_\theta})\) is a Brownian bridge of length \(g_{S_\theta}\) and the rescaled process defined by
\[
\tilde{U} = \left( \frac{B(tg_{S_\theta})}{\sqrt{g_{S_\theta}}}, \quad 0 \leq t \leq 1 \right)
\]
is standard Brownian bridge on the unit interval independent of \(g_{S_\theta}\) and \(S_\theta\).

Hence if \(\gamma \sim \Gamma(1/2, \theta)\) is a random variable independent of the Brownian bridge \(U\) then
\[
(1.4) \quad \sqrt{\gamma}(M^+, M^-) \triangleq \left( \max_{0 \leq t \leq g_{S_\theta}} B_t, - \min_{0 \leq t \leq g_{S_\theta}} B_t \right).
\]

In Section 2 excursion theory will be used to derive the joint distribution of the triple
\[
(1.5) \quad \left( \max_{0 \leq t \leq g_{S_\theta}} B_t, - \min_{0 \leq t \leq g_{S_\theta}} B_t, L(g_{S_\theta}) \right).
\]

Only the fact that certain point processes are Poisson processes will be used rather than the more involved descriptions of the Itô excursion law. This will make it possible to derive distributions related to \(\sqrt{\gamma}(M^+, M^-)\) by identifying this latter pair to the pair on the right in (1.4) and conditioning on \(L(g_{S_\theta})\). In particular it will follow that given \(L(g_{S_\theta})\) the random pair on the right in (1.4) are conditionally independent. Conditional independence will lead to formulae for the distribution of the minimum, the maximum, the sum, the difference and joint distribution of the random pair on the left in (1.4). The distributions related to \((M^+, M^-)\) can be derived from those for \(\sqrt{\gamma}(M^+, M^-)\) by inverting Laplace transforms.

2. Brownian excursions.

Let \(B\) be standard Brownian motion. Let \(U = \{w \in C[0, \infty), w(0) = 0, \exists R > 0, w(t) \neq 0 \text{ iff } t \in (0, R)\} \cup \{\delta\}\) with \(\delta\) being the coffin state be the excursion space equipped with the \(\sigma\)-field generated by all coordinate maps. Let \(\tau_s = \inf\{u : L(u) > s\}\) be the right continuous inverse of the local time process at level 0 of \(B\) and define the point process
Let \( e = ((s, e_s) : 0 \leq s \leq L(S_\theta)) \) of excursions of the randomly rescaled Brownian bridge on the space \((0, \infty) \times U\) by
\[
e_s(u) = 1_{[0 \leq u \leq \tau_s - \tau_{s-}]}(u) \, B_{\tau_s + u}
\]
for \(0 \leq u \leq \tau_s - \tau_{s-}\) when \(\tau_s - \tau_{s-} > 0\) and \(e_s = \delta\) if \(\tau_s - \tau_{s-} = 0\) where \(s \leq L(g S_\theta)\).

For \(w \in U\) and \(w \neq \delta\) define the duration of an excursion as
\[
R(w) = \sup\{u : w(u) \neq 0\}
\]

**Lemma 2.** Let \( e \) be the point process defined in (2.1).

(i) The random variable \(L(S_\theta)\) is exponential with parameter \(\sqrt{2\theta}\).

(ii) Conditionally on \(L(g S_\theta) = t\) the point process \(e\) is a Poisson process in the space \((0, t) \times U\) with mean measure \(\lambda \times n \cdot e^{-\theta R(w)}\) where \(\lambda\) is the Lebesgue measure on \((0, t)\) and \(n\) is Itô’s excursion law for Brownian motion.

(iii) Positive and negative excursions of \((B_t : 0 \leq t \leq g S_\theta)\) are independent Poisson processes conditionally on \(L(g S_\theta) = t\).

**Proof:** Let \(\tilde{e}\) be the excusion process of Brownian motion \(B\). The excursion of Brownian motion straddling \(S_\theta\) can be interpreted as the first one in the process of marked Brownian excursions where an excursion \(w\) is marked with probability \(1 - e^{-\theta R(w)}\) and marks are given independently. The marked and unmarked excursions are independent Poisson processes on \((0, \infty) \times U\) with mean measures
\[
\lambda \times n \cdot (1 - e^{-\theta R(w)}) \quad \text{and} \quad \lambda \times n \cdot e^{-\theta R(w)}
\]
respectively where \(\lambda\) is the Lebesgue measure and \(n\) is Itô’s excursion law on \(U\). See Rogers and Williams (1987), p. 418 for details. The point process \(e\) is the initial portion of \(\tilde{e}\) up to but not including the first marked excursion.

On the local time scale \(L(g S_\theta)\) is the time of the occurrence of a point in a linear Poisson process hence exponential. By the reflection principle the density of \(\max_{0 \leq s \leq t} B_s\) is given by
\[
f(x) = \sqrt{\frac{2}{\pi t}} e^{-x^2/2t}.
\]
Hence the density of \(\max_{0 \leq s \leq S_\theta} B_s\) is given by
\[
\frac{\sqrt{2\theta}}{\sqrt{\pi}} \int_0^\infty e^{-x^2/2t} t^{-1/2} e^{-\theta t} \, dt = \sqrt{2\theta} e^{-\sqrt{2\theta} x}.
\]
For a computation of this Laplace transform see Oberhettinger and Badii (1973), p. 41, formula 5.28. By (1.3) this is also the density of \(L(S_\theta)\).
The unmarked excursions up to \( L(g_{S\theta}) \) are the excursions of the rescaled Brownian bridge \( (B_t : 0 \leq t \leq g_{S\theta}) \). It follows that given \( L(S\theta) = t \) the process \( e \) is a Poisson process on \((0, t) \times U\) with mean measure \( \lambda \times n \cdot e^{-\theta R(w)} \) where \( \lambda \) is the Lebesgue measure on \((0, t)\) and \( n \) is Itô’s excursion law for standard Brownian motion. This proves (ii).

Positive and negative excursions in the point process \( e \) are conditionally independent given \( L(S\theta) = t \) by independence properties of Poisson processes for disjoint sets which proves (iii).

Let \( M^+, M^- \) and \( M \) be as defined in Section 1. Let \( U \) be Brownian bridge and \( \gamma \) an independent \( \Gamma(1/2, \theta) \) random variable. By the definition of the Brownian bridge as Brownian motion conditioned to be 0 at time \( t = 1 \) and an application of the reflection principle for Brownian motion one derives that for \( x > 0 \)

\[
P(M^+ \geq x) = \lim_{\epsilon \to 0} P\left( \max_{0 \leq t \leq 1} B_t \geq x \mid |B_1| \leq \epsilon \right)
\]

\[
= \lim_{\epsilon \to 0} P\left( \max_{0 \leq t \leq 1} B_t \geq x \mid |B_1| \leq \epsilon \right) / P(|B_1| \leq \epsilon)
\]

\[
= \lim_{\epsilon \to 0} P(2x - \epsilon \leq B_1 \leq 2x + \epsilon) / P(|B_1| \leq \epsilon)
\]

\[
(2.2)
\]

It follows from the distribution of \( M^+ \) given by (2.2) that

\[
P(\sqrt{\gamma}M^+ \geq x) = \sqrt{\theta/\pi} \int_0^{\infty} \exp(-2x^2/s) s^{-1/2} e^{-\theta s} ds
\]

\[
= \exp(-2x\sqrt{2\theta}).
\]

(2.4)

For a computation of this Laplace transform see Oberhettinger and Badii (1973), p. 41, formula 5.28.

Turning to excursions recall that \( R(w) \) stands for the length of the excursion and denote \( w^+ = \max_u w(u) \). Define

\[
m(x) = \int_{\{w^+ \geq x\}} e^{-\theta R(w)} n(dw).
\]

The point process of the maxima \( (e^+_s : 0 \leq s \leq L(S\theta)) \) is a measurable map of the process \( e \) and hence conditionally on \( L(g_{S\theta}) = t \) a Poisson process on \((0, \infty)\); see Kingman (1993). The conditional mean measure
of the set \([x, \infty)\) is \(t \cdot m(x)\). This implies that

\[
P\left(\max_{0 \leq t \leq g_{S \theta}} B_t \geq x \mid L(g_{S \theta}) = t\right) = 1 - e^{-t \cdot m(x)}
\]

This in connection with Lemma 2 (i) and (2.3) gives that

\[
P\left( \max_{0 \leq t \leq g_{S \theta}} B_t \geq x \mid L(g_{S \theta}) = t \right) = 1 - e^{-t \cdot m(x)}
\]

Comparing (2.3) and (2.6) we obtain that

\[
m(x) = \sqrt{2 \theta} e^{-2x \sqrt{2 \theta}} / \left(1 - e^{-2x \sqrt{2 \theta}}\right).
\]

Furthermore, given \(L(g_{S \theta}) = t\), by Lemma 2, (iii), the positive and negative excursions of the point process \(e\) are conditionally independent Poisson processes. This implies for \(x, y > 0\)

\[
P\left( \max_{0 \leq t \leq g_{S \theta}} B_t \leq x, - \min_{0 \leq t \leq g_{S \theta}} B_t \leq y \mid L(g_{S \theta}) = t \right) = e^{-t \cdot m(x)} e^{-t \cdot m(y)}.
\]

This also gives the joint distribution of the triple (1.5). In particular (2.8) implies that two random variables

\[
\max_{0 \leq t \leq g_{S \theta}} B_t \quad \text{and} \quad - \min_{0 \leq t \leq g_{S \theta}} B_t
\]

are conditionally independent given \(L(S_{\theta}) = t\).

3. Examples of calculations

3.1 Distributions of \(M\) and \(m\).

Let \(m\) and \(M\) be defined as in (1.2). From (2.6) we have

\[
P(\sqrt{\gamma} M \leq x) = \frac{\sqrt{2 \theta}}{\sqrt{2 \theta} + 2m(x)} = \tanh(x \sqrt{2 \theta}).
\]

Let \(F_M\) be the cumulative distribution function of \(M\). Writing out (3.1), taking into account the independence of \(M\) and \(\gamma\) and dividing both sides by \(\sqrt{\theta}\) we get

\[
\frac{1}{\sqrt{\pi}} \int_0^{\infty} F_M(x/\sqrt{s}) s^{-1/2} e^{-\theta s} ds = \frac{\tanh(x \sqrt{2 \theta})}{\sqrt{\theta}}.
\]
Oberhettinger and Badii (1973), p. 294, formula 8.51, give the inverse of the Laplace transform on the right of (3.2) as

\[
\frac{1}{\sqrt{\pi}} F_M(x/\sqrt{s}) s^{-1/2} = \frac{1}{\sqrt{2x}} \theta_2(0|s/(2x^2)) = \frac{1}{\sqrt{2x}} \frac{\sqrt{2x}}{\sqrt{\pi s}} \sum_{n=-\infty}^{\infty} (-1)^n \exp(-2n^2x^2/s).
\]

This yields

\[
F_M(z) = \sum_{n=-\infty}^{\infty} (-1)^n \exp(-2n^2z^2),
\]
or

\[
1 - F_M(z) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \exp(-2n^2z^2),
\]

which is the formula for the distribution of the Kolmogorov–Smirnov test statistic.

Turning to \(m = \min\{M^+, M^-\}\) observe that by (2.8)

\[(3.3)\quad P(\sqrt{\gamma} m > x) = \sqrt{2\theta} \int_{0}^{\infty} e^{-\sqrt{2\theta} t} (1 - e^{-tm(x)})^2 \, dt.\]

Integration yields

\[(3.4)\quad P(\sqrt{\gamma} m > x) = 1 - \frac{2\sqrt{2\theta}}{\sqrt{2\theta} + m(x)} + \frac{\sqrt{2\theta}}{\sqrt{2\theta} + 2m(x)}.\]

Denote by \(F_m\) the distribution function of \(m\). From (3.4), independence of \(m\) and \(\gamma\) and dividing both sides by \(\sqrt{\theta}\) we get

\[
\frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \left(1 - F_m(x/\sqrt{s})\right) s^{-1/2} e^{-\theta s} \, ds = \frac{\tanh(x\sqrt{2\theta})}{\sqrt{\theta}} + \frac{2e^{-2x\sqrt{2\theta}}}{\sqrt{\theta}} - \frac{1}{\sqrt{\theta}}.
\]

The first term has been inverted above, the second is given by Oberhettinger and Badii (1973), p. 258, formula 5.87, and the third is elementary. Substituting \(z\) for \(x/\sqrt{s}\) one gets

\[
1 - F_m(z) = \sum_{n=-\infty}^{\infty} (-1)^n \exp(-2n^2z^2) + 2e^{-2z^2} - 1
\]

\[(3.5)\quad = 2 \sum_{n=2}^{\infty} (-1)^n \exp(-2n^2z^2).\]
3.2 Joint distributions, sums, differences, quotients.

Integrating out the conditioning variable in (2.8) we get

\( P(\sqrt{\gamma M^+} \leq x, \sqrt{\gamma M^-} \leq y) = \)

\[
\frac{\sqrt{2\theta}}{127}
\]

\[
\frac{1}{2}\theta + m(x) + m(y)
\]

\[
= \frac{(1 - e^{-2x\sqrt{2\theta}})(1 - e^{-2y\sqrt{2\theta}})}{1 - e^{-2x\sqrt{2\theta}}e^{-2y\sqrt{2\theta}}}
\]

\[
= \frac{(e^{x\sqrt{2\theta}} - e^{-x\sqrt{2\theta}})(e^{y\sqrt{2\theta}} - e^{-y\sqrt{2\theta}})}{e^{(x+y)\sqrt{2\theta}} - e^{-(x+y)\sqrt{2\theta}}}
\]

\[
= 2\frac{\sinh(x\sqrt{2\theta})\sinh(y\sqrt{2\theta})}{\sinh((x+y)\sqrt{2\theta})}
\]

\[
= \cos((x+y)\sqrt{2\theta}) - \cosh((x-y)\sqrt{2\theta})\cosh((x+y)\sqrt{2\theta})
\]

\[
= \coth((x+y)\sqrt{2\theta}) - \cosh((x-y)\sqrt{2\theta})\csch((x+y)\sqrt{2\theta})
\]

Let \( F(z, w) \) be the joint distribution function of the pair \((M^+, M^-)\).

By (3.6)

(3.7)

\[
P(\sqrt{\gamma M^+} \leq x, \sqrt{\gamma M^-} \leq y) = \sqrt{\theta/\pi} \int_{0}^{\infty} F(x/\sqrt{s}, y/\sqrt{s})s^{-1/2}e^{-\theta s}ds.
\]

The right side is given in (3.6). Oberhettinger and Badii (1973), p. 294, formula 8.52, give the inverse of the first term of the transform as

\[
\frac{1}{\sqrt{2(x+y)}} \theta_3 \left( 0 \Big| \frac{s}{2(x+y)^2} \right) = \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^{\infty} e^{-2k^2(x+y)^2/s}
\]

The inverse of the second term of the transform can be obtained from Oberhettinger and Badii (1973), formula 8.8.60: we find that the inverse is

\[
\frac{1}{\sqrt{2(x+y)}} \theta_4 \left( \frac{(x-y)/2}{x+y} \Big| \frac{s}{2(x+y)^2} \right) =
\]

\[
= \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^{\infty} \exp \left( -2(x+y)^2 \left[ \frac{(x-y)/2}{x+y} + k + 1 \right]^2 / s \right)
\]

\[
= \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^{\infty} \exp(-2[k(x+y) + x]^2 / s).
\]

Combining these yields
\[ P(M^+ \leq z, M^- \leq w) = \sum_{k=-\infty}^{\infty} \exp(-2k^2(z + w)^2) - \sum_{k=-\infty}^{\infty} \exp(-2[k(z + w) + z]^2), \]
in agreement with Shorack and Wellner (1986), formula (2.2.22), page 39.

Another statistic whose distribution can be obtained from formula (2.8) is the Kuiper statistic defined as \( K = M^+ + M^- \). It seems cumbersome to proceed from the joint distribution of \( M^+ \) and \( M^- \) so we use directly the formula (2.8). Denote \( U = \sqrt{\gamma}M^+ \) and \( V = \sqrt{\gamma}M^- \). The joint cumulative distribution function of \( U \) and \( V \) has been shown to be
\[ G(u, v) = \frac{2 \sinh(u \sqrt{2\theta}) \sinh(v \sqrt{2\theta})}{\sinh((u + v)\sqrt{2\theta})} \]
A simple calculation shows that the cumulative distribution function of \( U + V \) is given by
\[ P(U + V \leq z) = \int_{0}^{z} G_u(u, z - u) \, du \]
where \( G_u \) is the partial derivative of \( G \) with respect to \( u \). A straightforward calculation yields
\[ G_u(u, z - u) = \frac{2\sqrt{2\theta} \sinh^2((z - u)\sqrt{2\theta})}{\sinh^2(z\sqrt{2\theta})} \]
and integration gives
\[ P(U + V \leq z) = \coth(z\sqrt{2\theta}) - \frac{z\sqrt{2\theta}}{\sinh^2(z\sqrt{2\theta})}. \]
Using the fact that \( \gamma \) and \( M^+ + M^- \) are independent we obtain the Laplace transform of the cumulative distribution function \( F_K \) of the Kuiper statistic as
\[ \frac{\sqrt{\theta}}{\sqrt{\pi}} \int_{0}^{\infty} F_K(z/\sqrt{s}) s^{-1/2} e^{-\theta s} \, ds = \coth(z\sqrt{2\theta}) - \frac{z\sqrt{2\theta}}{\sinh^2(z\sqrt{2\theta})}. \]
After dividing by \( \sqrt{\theta} \) it remains to invert the two terms on the right and substitute for \( z/\sqrt{s} \). The first term has been inverted above when deriving the joint distribution of \( M^+ \) and \( M^- \). We get
\[
\frac{1}{\sqrt{2\pi s}} \theta_3 \left( 0 \left| \frac{s}{2z^2} \right. \right) = \frac{1}{\sqrt{2\pi s}} \sum_{k=-\infty}^{\infty} e^{-2k^2z^2/s}.
\]

To invert the second term rewrite it as

\[
\frac{\sqrt{2z}}{\sinh^2(z\sqrt{2\theta})} = \frac{4\sqrt{2z}e^{-2z\sqrt{2\theta}}}{(1 - e^{-2z\sqrt{2\theta}})^2} = 4\sqrt{2z} \sum_{k=1}^{\infty} ke^{-2kz\sqrt{2\theta}}
\]

for \( z > 0 \) and \( \theta > 0 \). The inverse Laplace transforms of the terms in the sum are known. One can either recall that they are Laplace transforms of the hitting time of level \( 2kz \) by standard BM, or use formula 5.85 on p. 258 in Oberhettinger and Badii (1973) to show that for \( a > 0 \)

\[
\int_0^{\infty} \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s} e^{-\theta s} \, ds = e^{-a\sqrt{2\theta}}.
\]

Since all the terms are non-negative functions the order of summation and integration can be changed. Hence for \( z > 0 \) the inverse Laplace transform of the second term is

\[
4\sqrt{2z} \sum_{k=1}^{\infty} \frac{2k^2z}{\sqrt{2\pi s^3}} \exp(-2k^2z^2/s) = \frac{1}{\sqrt{\pi s}} \sum_{k=-\infty}^{\infty} \frac{4k^2z^2}{s} \exp(-2k^2z^2/s).
\]

Substitute \( x = z/\sqrt{s} \) to get

\[
(3.9) \quad F_K(x) = \sum_{k=-\infty}^{\infty} e^{-2k^2x^2} - \sum_{k=-\infty}^{\infty} 4k^2x^2 e^{-2k^2x^2} = \sum_{k=-\infty}^{\infty} (1 - 4k^2x^2)e^{-2k^2x^2}.
\]

**Remark:** Vervaat (1979) gives a construction of standard Brownian excursion from a Brownian bridge. Let \( U \) be a Brownian bridge \([0,1]\) and let \( \sigma \) be the time when \( U \) attains its minimum on \([0,1]\) (\( \sigma \) is a.s. unique). Then the process \( (e(t) : 0 \leq t \leq 1) \) defined by

\[
e(t) = U_{\sigma + t \mod 1} - U_\sigma(t)
\]

is a standard Brownian excursion. It is a simple consequence of this transformation that the Kuiper statistic has the distribution of the maximum of the standard Brownian excursion and (3.9) is another derivation of the distribution of this maximum.

For the difference \( U - V \) a straightforward computation yields for \( z > 0 \)
(3.10) \[ P(U - V \geq z) = \int_{z}^{\infty} G_u(u, u - z) \, du \]

provided \( P(U > 0) = 1 \) and \( P(V > 0) = 1 \) which is the case for the variables in question. Using the joint cumulative distribution function yields

\[ G_u(u, u - z) = 2\sqrt{2\theta} \frac{\sinh^2(\sqrt{2\theta}(u - z))}{\sinh^2(2\sqrt{2\theta}u)} . \]

From (3.10) it follows

\[ P(U - V \geq z) = 2\sqrt{2\theta} \int_{z}^{\infty} \frac{\sinh^2(\sqrt{2\theta}(u - z))}{\sinh^2(2\sqrt{2\theta}u)} \, du \]

\[ = 2\sqrt{2\theta} \int_{0}^{\infty} \frac{\sinh^2(\sqrt{2\theta}u)}{\sinh^2(2\sqrt{2\theta}(u + z))} \, du . \]

Since \( \gamma \) and \((M^+, M^-)\) are independent the above identity, after cancelling \( \sqrt{\theta} \) can be rewritten as

\[ \frac{1}{\sqrt{\pi}} \int_{0}^{\infty} \frac{1}{\sqrt{s}} P(M^+ - M^- \geq \frac{z}{\sqrt{s}} e^{-\theta s} \, ds = 2 \int_{0}^{\infty} \frac{\sinh^2(\sqrt{2\theta}u)}{\sinh^2(\sqrt{2\theta}(2u + z))} \, du . \]

This gives the Laplace transform of \( P(M^+ - M^- \geq \frac{z}{\sqrt{s}}) / \sqrt{s} \) as a function of \( s \) for fixed \( z \).

To invert this Laplace transform note that for fixed \( u > 0 \) and \( z > 0 \) the integrand on the right as a function of \( \theta \) is a Laplace transform. The inversion proceeds by first inverting the integrand for fixed \( u \) and then integrating with respect to \( u \). For \( \theta > 0 \) and \( u > 0 \) write

\[ \frac{\sinh^2(\sqrt{2\theta}u)}{\sinh^2(\sqrt{2\theta}(2u + z))} = \frac{e^{2\sqrt{2\theta}u} - 2 + e^{-2\sqrt{2\theta}u}}{e^{2\sqrt{2\theta}(2u + z)} - 2 + e^{-2\sqrt{2\theta}(2u + z)}} \]

\[ = \frac{(e^{2\sqrt{2\theta}u} - 2 + e^{-2\sqrt{2\theta}u}) \cdot e^{-2\sqrt{2\theta}(2u + z)}}{1 - 2e^{\sqrt{2\theta}(2u + z)} + e^{-4\sqrt{2\theta}(2u + z)}} \]

\[ = (e^{2\sqrt{2\theta}u} - 2 + e^{-2\sqrt{2\theta}u}) \sum_{k=1}^{\infty} k e^{-2k\sqrt{2\theta}(2u + z)} . \]

The series expansion follows from noting that for \(|x| < 1\)

\[ \frac{x}{(1 - x)^2} = \sum_{k=1}^{\infty} kx^k \]
and setting \( x = e^{-2\sqrt{2}\theta(2u+z)} \). Multiplying out one obtains

\[
(3.11) \quad \sum_{k=1}^{\infty} ke^{-(2kz+(4k-2)u)\sqrt{2\theta}} - 2 \sum_{k=1}^{\infty} ke^{-(2k(2u+z))\sqrt{2\theta}} + \sum_{k=1}^{\infty} ke^{-(2kz+(4k+2)u)\sqrt{2\theta}}.
\]

All the terms in the sums can be inverted using (3.8). From left to right in (3.11) the inverses of the sums are

\[
(3.12) \quad \frac{1}{\sqrt{2\pi}s^3} \sum_{k=1}^{\infty} k(2kz + (4u - 2)) \cdot e^{-(2kz+(4k-2)u)^2/2s},
\]

\[
(3.13) \quad -2 \frac{1}{\sqrt{2\pi}s^3} \sum_{k=1}^{\infty} 2k^2(2u + z) \cdot e^{-(2k(2u+z))^2/2s},
\]

and

\[
(3.14) \quad \frac{1}{\sqrt{2\pi}s^3} \sum_{k=1}^{\infty} k(2kz + (4u + 2)) \cdot e^{-(2kz+(4k+2)u)^2/2s}.
\]

The sums are absolutely integrable so the order of summation and inversion can be interchanged. To derive the distribution of \( M^+ - M^- \) multiply (3.12), (3.13) and (3.14) by \( 2\sqrt{2\pi} \) and integrate with respect to \( u \). Again the change in order of summation and integration is justified by absolute integrability and Fubini’s theorem. Summing up yields

\[
(3.15) \quad \frac{1}{\sqrt{s}} \left( \sum_{k=1}^{\infty} \frac{2k}{4k - 2} e^{-2k^2z^2/s} - \sum_{k=1}^{\infty} e^{-2k^2z^2/s} + \sum_{k=1}^{\infty} \frac{2k}{4k + 2} e^{-2k^2z^2/s} \right).
\]

Finally note that (3.15) gives

\[
(3.16) \quad \frac{1}{\sqrt{s}} P(M^+ - M^- \geq \frac{z}{\sqrt{s}}) = \frac{1}{\sqrt{s}} \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \cdot e^{-2k^2z^2/s}
\]
or

\[
P(M^+ - M^- \geq z) = \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \cdot e^{-2k^2z^2}.
\]

For a different approach see Kosorok and Lin (1999). Turning to quotients define \( Q = M^+/M^- \). We can multiply the numerator and denominator by \( \sqrt{\gamma} \) and choose \( \theta = 1/2 \). Conditionally on \( L(S_0) = t \) an elementary calculation gives the conditional distribution function of \( Q \) as

\[
F_{Q|L(S_0)=t}(z) = -\int_0^\infty tm'(x)e^{-tm(x)}e^{-tm(xz)} \, dx.
\]
Unconditioning and changing the order of integration gives

\[(3.17) \quad F_Q(z) = - \int_0^\infty \frac{m'(x) \, dx}{(1 + m(x) + m(xz))^2}.\]

Substituting (2.7) and observing that

\[m'(x) = -\frac{1}{2} \sinh^2 x\]

gives

\[F_Q(z) = \frac{1}{2} \int_0^\infty \frac{(1 - e^{-2x})^2(1 - e^{-2xz})^2 \, dx}{\sinh^2 x (1 - e^{-2x(z+1)})^2} \]

\[= 2 \int_0^\infty \frac{\sinh^2 zx \, dx}{\sinh^2 ((z + 1)x)} \]

\[= \frac{1}{z + 1} \left(1 - \frac{\pi \text{ctg} \left( \frac{\pi z}{z + 1} \right)}{z + 1} \right)\]

where the last integral is given in Gradshteyn and Ryzhik (1994), formula 3.511.9.

3.3 Covariance and Correlation.

Yet another functional of interest is the correlation between \(M^+\) and \(M^-\). By independence of \(\gamma\) and \((M^+, M^-)\) it will suffice to compute the covariance between \(U = \sqrt{\gamma} M^+\) and \(V = \sqrt{\gamma} M^-\). Starting from

\[(3.18) \quad \text{cov}(U, V) = E(\text{cov}(U, V|L(g_{S_0}))) + \text{cov}(E(U|L(g_{S_0})), E(V|L(g_{S_0}))).\]

we first notice that by (1.4), (1.5), and (2.8) the variables \(U\) and \(V\) can be taken to be conditionally independent, given \(L(g_{S_0})\). The conditional covariance in (3.18) is 0. By independence and symmetry \(E(U|L(g_{S_0})) = E(V|L(g_{S_0}))\). One gets

\[\text{cov}(U, V) = \text{var}(E(U|L(g_{S_0}))).\]

The conditional expectation is computed from (2.8) as

\[(3.19) \quad E(U|L(g_{S_0}) = t) = \int_0^\infty (1 - e^{-tm(x)}) \, dx\]
Denote $\psi(u) = E(U|L(g_{S\theta}) = u)$. Note that for $2\theta = 1$ the local time $L(g_{S\theta})$ is exponential with parameter 1 and compute

$$E(\psi^2(L(g_{S\theta}))) =$$

$$= \int_0^\infty e^{-t} dt \int_0^\infty (1 - e^{-tm(x)}) dx \int_0^\infty (1 - e^{-tm(y)}) dy$$

$$= \int_0^\infty dx \int_0^\infty dy \left[ 1 - \frac{1}{1 + m(x)} - \frac{1}{1 + m(y)} + \frac{1}{1 + m(x) + m(y)} \right]$$

$$= \int_0^\infty dx \int_0^\infty dy \frac{e^{-2(x+y)}(2 - e^{-2x} - e^{-2y})}{1 - e^{-2(x+y)}}$$

$$= \int_0^\infty dt \frac{e^{-2t}}{1 - e^{-2t}} \int_0^t (2 - e^{-2(t-v)} - e^{-2v}) dv$$

$$= \int_0^\infty \frac{e^{-2t}}{1 - e^{-2t}} (2t - (1 - e^{-2t})) dt$$

$$= 2 \int_0^\infty \sum_{k=1}^\infty t e^{-2kt} dt - \frac{1}{2}$$

$$= 2 \sum_{k=1}^\infty \frac{1}{4k^2} - \frac{1}{2}$$

$$= \frac{\pi^2}{12} - \frac{1}{2}$$

The second line follows from Fubini’s theorem and the third by changing variables to $x + y = t$, $y = v$. From (2.8) we have $E(U) = E(\psi(L(g_{S\theta}))) = 1/2$ and hence

$$\text{cov}(U, V) = \text{var}(\psi(L(g_{S\theta})))$$

$$= \frac{\pi^2}{12} - \frac{3}{4}$$

From (2.2) one derives $E(M^+) = E(M^-) = \sqrt{2\pi}/4$ and $\text{var}(M^+) = 1/2 - \pi/8$ so from

$$\text{cov}(U, V) = E(\gamma) E(M^+M^-) - E^2(\sqrt{\gamma}) E(M^+)E(M^-)$$

we have

$$E(M^+M^-) = \frac{\pi^2}{12} - \frac{1}{2}.$$ 

and

$$\text{cov}(M^+, M^-) = \frac{\pi^2}{12} - \frac{1}{2} - \frac{\pi}{8}.$$
yielding

\[\text{corr}(M^+, M^-) = \frac{\pi^2}{12} - \frac{1}{2} - \frac{\pi}{8} = -0.654534.\]
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