STABILITY OF SPECTRAL CHARACTERISTICS AND BARI BASIS PROPERTY OF BOUNDARY VALUE PROBLEMS FOR 2 × 2 DIRAC TYPE SYSTEMS

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ABSTRACT. The paper is concerned with the stability property under perturbation \( Q \to \tilde{Q} \) of different spectral characteristics of a boundary value problem associated in \( L^2([0,1]; \mathbb{C}^2) \) with the following \( 2 \times 2 \) Dirac type equation

\[
L_U(Q)y = -iB^{-1}y' + Q(x)y = \lambda y, \quad B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad b_1 < b_2, \quad y = \text{col}(y_1, y_2),
\]

with a potential matrix \( Q \in L^p([0,1]; \mathbb{C}^{2 \times 2}) \) and subject to the regular boundary conditions \( Uy := \{U_1, U_2\}y = 0 \). If \( b_2 = -b_1 = 1 \) this equation is equivalent to one dimensional Dirac equation. Our approach to the spectral stability relies on the existence of the triangular transformation operators for system \((0.1)\) with \( Q \in L^1 \), which was established in our previous works. The starting point of our investigation is the Lipshitz property of the mapping \( Q \to K_Q^\pm \), where \( K_Q^\pm \) are the kernels of transformation operators for system \((0.1)\). Namely, we prove the following uniform estimate:

\[
\|K_Q^\pm - K_{\tilde{Q}}^\pm\|_{X^2_{\infty,1}} + \|K_Q^\pm - K_{\tilde{Q}}^\pm\|_{X^2_{1,1}} \leq C \cdot \|Q - \tilde{Q}\|_p, \quad Q, \tilde{Q} \in U^{2 \times 2}_p, \quad p \in [1, \infty],
\]

on balls \( U^{2 \times 2}_p \) in \( L^p([0,1]; \mathbb{C}^{2 \times 2}) \). It is new even for \( \tilde{Q} = 0 \). Here \( X^2_{\infty,p}, X^2_{1,p} \) are the special Banach spaces naturally arising in such problems. We also obtained similar estimates for Fourier transforms of \( K_Q^\pm \). Both of these estimates are of independent interest and play a crucial role in the proofs of all spectral stability results discussed in the paper. For instance, as an immediate consequence of these estimates we get the Lipshitz property of the mapping \( Q \to \Phi_Q(\cdot, \lambda) \), where \( \Phi_Q(x, \lambda) \) is the fundamental matrix of the system \((0.1)\).

Assuming the spectrum \( \Lambda_Q = \{\lambda_{Q,n}\}_{n \in \mathbb{Z}} \) of \( L_U(Q) \) to be asymptotically simple, denote by \( F_Q = \{f_{Q,n}\}_{|n| > N} \) a sequence of corresponding normalized eigenfunctions, \( L_U(Q)f_{Q,n} = \lambda_{Q,n}f_{Q,n} \). Assuming boundary conditions (BC) to be strictly regular, we show that the mapping \( Q \to \Lambda_Q - \Lambda_0 \), sends \( L^p([0,1]; \mathbb{C}^{2 \times 2}) \) either into the weighted \( l^p \)-space \( P(\{(1 + |n|)^{p-2}\}) \); we also establish its Lipshitz property on compacts in \( L^p([0,1]; \mathbb{C}^{2 \times 2}), \ p \in [1,2] \). The proof of the second estimate involves as an important ingredient inequality that generalizes classical Hardy-Littlewood inequality for Fourier coefficients. It is also shown that the mapping \( Q \to F_Q - F_0 \) sends \( L^p([0,1]; \mathbb{C}^{2 \times 2}) \) into the space \( l^p(\mathbb{Z}; C([0,1]; \mathbb{C}^2)) \) of sequences of continuous vector-functions, and has the Lipshitz property on compacts in \( L^p([0,1]; \mathbb{C}^{2 \times 2}), \ p \in [1,2] \).

Certain modifications of these spectral stability results are also proved for balls \( U^{2 \times 2}_p \) in \( L^p([0,1]; \mathbb{C}^{2 \times 2}), \ p \in [1,2] \).

Note also that the proof of the Lipshitz property of the mapping \( Q \to F_Q - F_0 \) involves the deep Carleson-Hunt theorem for maximal Fourier transform, while the proof of this property for the mapping \( Q \to \Lambda_Q - \Lambda_0 \) relies on the estimates of the classical Fourier transform and is elementary in character.

In the case of \( Q \in L^2 \) we establish a criterion for the system of root vectors of \( L_U(Q) \) to form a Bari basis in \( L^2([0,1]; \mathbb{C}^2) \). Under a simple additional assumption this system forms a Bari basis if and only if BC are self-adjoint.

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1. Introduction

In this paper we continue our investigation of the following first order system of differential equations

\[ \mathcal{L}y = -iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0, 1], \]  

(1.1)

where

\[ B = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}, \quad b_1 < 0 < b_2 \quad \text{and} \quad Q = \begin{pmatrix} 0 & Q_{12} \\ Q_{21} & 0 \end{pmatrix} \in L^1([0, 1]; \mathbb{C}^{2 \times 2}). \]  

(1.2)

If $B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ system (1.1) is equivalent to the Dirac system (see the classical monographs [27, 38]).

Let us associate linearly independent boundary conditions (BC)

\[ U_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + a_{j3}y_1(1) + a_{j4}y_2(1) = 0, \quad j \in \{1, 2\}, \]  

(1.3)

with system (1.1), and denote as $L(Q) := L_U(Q)$ an operator, associated in $L^2([0, 1]; \mathbb{C}^2)$ with the boundary value problem (BVP) (1.1)–(1.3). It is defined by differential expression $\mathcal{L}$ on the domain

\[ \text{dom}(L_U(Q)) = \{ f \in AC([0, 1]; \mathbb{C}^2) : \mathcal{L}f \in L^2([0, 1]; \mathbb{C}^2), \quad U_1(f) = U_2(f) = 0 \}. \]  

(1.4)

The completeness property of the root vectors system of BVP for general $n \times n$ system of the form (1.1) with a nonsingular diagonal $n \times n$ matrix $B$ with complex entries and a potential matrix $Q(\cdot)$ of the form

\[ B = \text{diag}(b_1, b_2, \ldots, b_n) \in \mathbb{C}^{n \times n} \quad \text{and} \quad Q(\cdot) = (q_{jk}(\cdot))_{j,k=1}^n \in L^1([0, 1]; \mathbb{C}^{n \times n}). \]  

(1.5)

was established in [37] for a wide class of BVPs, although for $2 \times 2$ Dirac system with $Q \in C([0, 1]; \mathbb{C}^{2 \times 2})$ it was proved earlier in [38] Chapter 1.3. In [37] the authors also found completeness conditions for non-regular and even degenerate BCs. In these papers it was also established the Riesz basis property of the root vectors systems (with and without parentheses) for different classes of BVPs for $n \times n$ system with arbitrary $B$ (1.1) and $Q \in L^\infty([0, 1]; \mathbb{C}^{n \times n})$. Note also that BVP for $2m \times 2m$ Dirac equation ($B = \text{diag}(-I_m, I_m)$) were investigated in [33] (Bari-Markus property for Dirichlet BVP with $Q \in L^2([0, 1]; \mathbb{C}^{2m \times 2m})$) and in [23] [24] (Bessel and Riesz basis properties on abstract level).

The Riesz basis property in $L^2([0, 1]; \mathbb{C}^2)$ of BVP (1.1)–(1.3), i.e. of the operator $L_U(Q)$, for $2 \times 2$ Dirac system with various assumptions on the potential matrix $Q$ was investigated in numerous papers (see [52] [53] [41] [42] [19] [3] [11] [10] [9] [12] [28] [29] [48] [31] and references therein). At that time the most strong result was obtained by P. Djakov and B. Mityagin [4] and A. Baskakov, A. Derbushev, A. Shcherbakov [3] who proved under the assumption $Q \in L^2([0, 1]; \mathbb{C}^{2 \times 2})$ that the root vectors system of the BVP (1.1)–(1.3) with strictly regular BCs forms a Riesz basis and forms a Riesz basis with parentheses whenever BC are only regular. Note however that the methods of these papers are substantially rely on $L^2$-technique (like Parseval equality, Hilbert-Schmidt operators, etc.) and cannot be applied to $L^1$-potentials.

Later the case $Q \in L^1([0, 1]; \mathbb{C}^{2 \times 2})$ was treated independently and with different methods by the authors [29] [31], on the one hand, and by A.M. Savchuk and A.A. Shkalikov [48], on the other hand. Namely, it was proved that
BVP \(1.1-1.3\) with \(Q \in L^1([0,1]; \mathbb{C}^{2 \times 2})\) and strictly regular boundary conditions has the Riesz basis property while BVP with only regular BC has the property of Riesz basis with parentheses.

Recall in this connection that BC \(1.3\) are regular, if and only if they are equivalent to the following conditions
\[
\hat{U}_1(y) = y_1(0) + by_2(0) + ay_1(1) = 0, \quad \hat{U}_2(y) = dy_2(0) + cy_1(1) + y_2(1) = 0,
\]
with certain \(a, b, c, d \in \mathbb{C}\) satisfying \(ad - bc \neq 0\). Recall also that regular BC \(1.3\) are called strictly regular, if the sequence \(\lambda_0 = \{\lambda_n^0\}_{n \in \mathbb{Z}}\) of the eigenvalues of the unperturbed \((Q = 0)\) BVP \(1.1-1.3\) (of the operator \(L_U(0)\)), is asymptotically separated. In particular, the eigenvalues \(\{\lambda_n^0\}_{|n| > n_0}\) are geometrically and algebraically simple.

It is well known that non-degenerate separated BC are always strictly regular. Moreover, conditions \(1.6\) are strictly regular for Dirac operator if and only if \((a - d)^2 \neq -4bc\). In particular, antiperiodic (periodic) BC are regular but not strictly regular for Dirac system while they may be strictly regular for Dirac type system with certain \(b_1 \neq -b_2\).

To describe our approach to Riesz basis property used in \([29, 31]\) let us denote by \(e_{\pm}(\cdot, \lambda)\) solution to system \((1.1)\) satisfying the initial conditions \(e_{\pm}(0, \lambda) = (\frac{1}{\pm i})\). Our investigation in \([29, 31]\) substantially relies on the following representation of \(e_{\pm}(\cdot, \lambda)\) by means of triangular transformation operators:
\[
e_{\pm}(x, \lambda) = (I + K^\pm_{Q})e^0_{\pm}(x, \lambda) = e^0_{\pm}(x, \lambda) + \int_0^x K^0_{Q}(t, x)e^0_{\pm}(t, \lambda)dt, \quad \text{where} \quad e^0_{\pm}(x, \lambda) = \left(\rho_{\pm}e^{ib_1\lambda x}, b_{\pm}e^{ib_2\lambda x}\right),
\]
and the kernels \(K^\pm = K^\pm_{Q} = (K^\pm_{jk})_{j,k=1}^{2} \in X^0_{1,1}(\Omega; \mathbb{C}^{2 \times 2}) \cap X^0_{\infty,1}(\Omega; \mathbb{C}^{2 \times 2})\) (see \((2.1), (2.2)\) for definitions of these spaces).

In turn, representation \(1.7\) immediately leads to the following key formula for the characteristic determinant \(\Delta(\cdot)\) of the problem \(1.1-1.3\):
\[
\Delta_Q(\lambda) = \Delta_0(\lambda) + \int_0^1 g_{1,Q}(t)e^{ib_1\lambda t} dt + \int_0^1 g_{2,Q}(t)e^{ib_2\lambda t} dt,
\]
Here \(\Delta_0(\cdot)\) is the characteristic determinant of problem \((1.1) - (1.3)\) with \(Q = 0\) and \(g_{l,0}(\cdot) \in L^1[0,1], \ l \in \{1, 2\}\) yields estimate of the difference \(\Delta_Q(\lambda) - \Delta_0(\lambda)\) from above. Combining this estimate with the classical estimate of \(\Delta_0(\cdot)\) from below and applying the Rouché theorem one arrives at the asymptotic formula
\[
\lambda_n = \lambda_n^0 + o(1), \quad \text{as} \quad n \to \infty, \quad n \in \mathbb{Z},
\]
relating the eigenvalues \(\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}\) and \(\Lambda_0 = \{\lambda_n^0\}_{n \in \mathbb{Z}}\) of the operators \(L_U(Q)\) and \(L_U(0)\) (with regular BC), respectively, (see \([29, 31]\) for details and also \([43]\) where formula \((1.8)\) was obtained by using different method). Note also that representation \(1.3\) for the determinant \(\Delta_Q(\cdot)\) was substantially used in a recent papers by A. Makin \([33, 34]\).

In \([29, 31]\) we also applied representation \(1.7\) to obtain asymptotic formulas for solutions to equation \((1.1)\) as well as eigenfunctions of the problem \((1.1) - (1.3)\).

In the present paper we continue our preceding investigation from \([29, 31]\) of BVPs \((1.1)-(1.3)\) and transformation operators for systems \((1.1)\). In the first chapter we prove Lipshitz property of the mappings \(Q \to K^\pm\) on the balls
\[
U^{2 \times 2}_{p, r} := \{F \in L^p([0,1]; \mathbb{C}^{2 \times 2}) : \|F\|_p := \|F\|_{L^p([0,1]; \mathbb{C}^{2 \times 2})} \leq r\}, \quad r > 0,
\]
in \(L^p([0,1]; \mathbb{C}^{2 \times 2})\). Namely, our first main result reads as follows: there is a constant \(C = C(B, p, r)\), not dependent on \(Q, \bar{Q} \in U^{2 \times 2}_{p, r}\), such that for any \(p \in [1, \infty)\) the following uniform estimate holds
\[
\|K^\pm_Q - K^\pm_{\bar{Q}}\|_{X_{\infty,p}(\Omega; \mathbb{C}^{2 \times 2})} + \|K^\pm_Q - K^\pm_{\bar{Q}}\|_{X_{1,p}(\Omega; \mathbb{C}^{2 \times 2})} \leq C \cdot \|Q - \bar{Q}\|_p, \quad Q, \bar{Q} \in U^{2 \times 2}_{p, r}.
\]
Here \(K^\pm_Q\) are the kernels from representation \((1.7)\) for \(\bar{e}\) being a solution of \((1.1)\) with \(\bar{Q}\) in place of \(Q\) and the spaces \(X_{\infty,p}, X_{1,p}\) are introduced in \((2.1), (2.2)\).

Combining uniform estimate \((1.11)\) with representation \((1.8)\) we obtain the following statement meaning the Lipshitz property of the map \(Q \to g_{l,Q}\) on the \(L^p\)-balls and playing a crucial role in our approach to subsequent estimates.
Lemma 1.1. Let \( Q, \overline{Q} \in U^2_{p,r} \) with \( p \in [1, \infty] \). Then the difference of characteristic determinants \( \Delta_Q(\lambda) = \Delta_{Q,U}(\lambda) \) and \( \Delta_{\overline{Q}}(\lambda) = \Delta_{\overline{Q},U}(\lambda) \) of the problem \((1.1) - (1.3)\) admits the following representation

\[
\Delta_Q(\lambda) - \Delta_{\overline{Q}}(\lambda) = \int_0^1 g_1(t)e^{ib_1\lambda t}dt + \int_0^1 g_2(t)e^{ib_2\lambda t}dt.
\]

with \( g_j := g_j,Q - g_j,\overline{Q} \in L^p[0,1], j \in \{1,2\} \). Moreover, there is a constant \( C = C(p,r,B) > 0 \) such that

\[
\|g_1\|_p + \|g_2\|_p = \|g_1,Q - g_1,\overline{Q}\|_p + \|g_2,Q - g_2,\overline{Q}\|_p \leq C \cdot \|Q - \overline{Q}\|_p.
\]

To demonstrate our first applications of estimate \((1.1)\) and Lemma 1.1 let us denote by \( \Lambda_Q := \Lambda_{U,Q} = \{\lambda_{Q,n}\}_{n \in \mathbb{Z}} \) the spectrum of the operator \( L_U(Q) \) assuming it to be asymptotically simple.

As an immediate application of Lemma 1.1 we complete formula \((1.9)\) by establishing \( c_0 \)-Lipschitz property on compacts: for each compact \( K \subset U^2_{1,r} \) and any \( \varepsilon > 0 \) there exists \( N_\varepsilon > 0 \), not dependent on \( Q \in \mathcal{K} \), such that

\[
\sup_{|n| > N_\varepsilon} |\lambda_{Q,n} - \lambda_{n,\overline{Q}}| \leq \varepsilon, \quad Q \in \mathcal{K}.
\]

In the case of Dirac system \((-b_1 = b_2 = 1)\) this result was established earlier in [46, Theorem 3].

Starting with Lemma 1.1 and assuming boundary conditions \((1.3)\) to be strictly regular we establish the Lipschitz property of the mapping \( Q \to \Lambda_Q \) in different norms: there exists an enumeration of the spectra \( \{\lambda_{Q,n}\}_{n \in \mathbb{Z}} \) and \( \{\lambda_{\overline{Q},n}\}_{n \in \mathbb{Z}} \) of the operators \( L_U(Q) \) and \( L_U(\overline{Q}) \), and a set \( \mathcal{I}_{Q,\overline{Q}} \subset \mathbb{Z} \), such that with certain constants \( C = C(B,p,r), N = N(B,p,r) > 0 \), not dependent on \( Q, \overline{Q} \in U^2_{p,r} \) with \( p \in (1,2] \), the following estimates hold:

\[
\text{card} \left( \mathbb{Z} \setminus \mathcal{I}_{Q,\overline{Q}} \right) \leq N,
\]

\[
\sum_{n \in \mathcal{I}_{Q,\overline{Q}}} |\lambda_{Q,n} - \lambda_{\overline{Q},n}|^{p'} \leq C \cdot \|Q - \overline{Q}\|_p^{p'}, \quad 1/p' + 1/p = 1,
\]

\[
\sum_{n \notin \mathcal{I}_{Q,\overline{Q}}} (1 + |n|)^{-2} |\lambda_{Q,n} - \lambda_{\overline{Q},n}|^p \leq C \cdot \|Q - \overline{Q}\|_p^p.
\]

On a compact set \( \mathcal{K} \) in \( L^p([0,1]; \mathbb{C}^{2 \times 2}) \) subsets \( \mathcal{I}_{Q,\overline{Q}} \subset \mathbb{Z} \) can be chosen independent of the pair \( \{Q, \overline{Q}\} \) and in view of \((1.15)\) the summation in \((1.16)-(1.17)\) takes the form: \( \sum_{|n| > N_1} \). Here \( N_1 (\in \mathbb{N}) \) does not depend on \( Q, \overline{Q} \in \mathcal{K} \).

Relation \((1.14)\) is also valid for regular \( BC \) and extends Theorem 3 from [46] to the case of Dirac type system \((-b_1 \neq -b_2)\). When \( \overline{Q} = 0 \) estimates \((1.16)-(1.17)\) give (uniform on balls) \( L^p \)-estimates of the remainder in the asymptotic formula \((1.9)\) for the eigenvalues of the regular problem \((1.3)\) for Dirac type system. For Dirac operator \((-b_1 = b_2 = 1)\) estimate \((1.16)\) with \( \overline{Q} = 0 \) generalizes the corresponding result obtained firstly in [48, Theorems 4.3, 4.5] for compact \( \mathcal{K} = \{Q,0\} \) consisting of two entries. For Dirac operator \((-b_1 = b_2 = 1)\) estimate \((1.16)\) with \( \overline{Q} = 0 \) generalizes the corresponding result obtained firstly in [48, Theorems 4.3, 4.5] with a constant \( C \) that depends on \( Q \) (i.e. for two points compact \( \mathcal{K} = \{Q,0\} \)) and later on in [17] for arbitrary compacts \( \mathcal{K} \) in \( L^1([0,1]; \mathbb{C}^{2 \times 2}) \). Note in this connection, that in the very recent papers, A. Gomilko and L. Rzepnicki [17] and A. Gomilko [44] obtained new, sharp asymptotic formulas for eigenfunctions of Sturm–Liouville operators with singular potentials, and for eigenvalues and eigenfunctions of Dirichlet BVP for Dirac system with \( Q \in L^p([0,1]; \mathbb{C}^{2 \times 2}), 1 \leq p < 2 \), respectively.

Weighted estimate \((1.17)\) is new even for Dirac system with \( Q \in U^2_{p,r} \) and \( \overline{Q} = 0 \) and even for trivial compact \( \mathcal{K} = \{Q,0\} \).

(II) In Sections 6 and 7 we investigate Fourier transform of transformation operators kernels \( K_Q^\pm \) from representations \((1.7)\). Our main result in this direction plays a key role in the sequel and reads as follows. Its proof relies on a new estimation technique for underlying integral equation on the kernels of the transformation operators.
Theorem 1.2. Let $Q, \tilde{Q} \in \mathbb{U}_1^{2 \times 2}$ for some $r > 0$ and let $K^\pm_Q$ and $K^\pm_{\tilde{Q}}$ be the kernels from integral representation (3.2). Then there is a constant $C = C(r, B) > 0$, not dependent on $Q$, $\tilde{Q}$, such that the following uniform estimate holds:

$$\sum_{j,k=1}^2 \int_0^x \left| (K^\pm_Q - K^\pm_{\tilde{Q}})_{jk} (x, t) e^{ib_k \lambda t} \right| dt \leq Ce^{2(b_2 - b_1) |\lambda|} \sum_{j,k \neq k} \left( \sup_{s \in [0, x]} \int_0^s \left| (Q_{jk}(t) - \tilde{Q}_{jk}(t)) e^{i(b_k - b_j) \lambda t} \right| dt \right)^2,$$

$$+ \|Q - \tilde{Q}\|_1 \sup_{s \in [0, x]} \int_0^s \tilde{Q}_{jk}(t) e^{i(b_k - b_j) \lambda t} dt \right), \quad x \in [0, 1], \quad \lambda \in \mathbb{C}. \quad (1.18)$$

First, we apply estimate (1.18) to the fundamental matrix solution $\Phi_Q(x, \lambda)$ of system (1.1) by establishing Lipschitz property of the mapping $Q \to \Phi_Q(x, \lambda)$ on the $L^1$-balls $\mathbb{U}_1^{2 \times 2}$. Namely, we show that there exists a constant $C = C(r, B) > 0$, not dependent on $Q, \tilde{Q} \in \mathbb{U}_1^{2 \times 2}$, such that the following uniform estimate holds

$$\left| \Phi_Q(x, \lambda) - \Phi_{\tilde{Q}}(x, \lambda) \right| \leq Ce^{2(b_2 - b_1) |\lambda|} \sum_{j,k \neq k} \left( \sup_{s \in [0, x]} \int_0^s \left| (Q_{jk}(t) - \tilde{Q}_{jk}(t)) e^{i(b_k - b_j) \lambda t} \right| dt \right)^2,$$

$$+ \|Q - \tilde{Q}\|_1 \sup_{s \in [0, x]} \int_0^s \tilde{Q}_{jk}(t) e^{i(b_k - b_j) \lambda t} dt \right), \quad x \in [0, 1], \quad \lambda \in \mathbb{C}. \quad (1.19)$$

To state the next application of Theorem 1.2 let us assume the spectrum $\Lambda_{U,Q} = \{\lambda_{Q,n}\}_{n \in \mathbb{Z}}$ of $L_U(Q)$ to be asymptotically simple, and introduce a sequence $\{f_{Q,n}\}_{|n| > N}$ of corresponding normalized eigenfunctions, $L_U(Q)f_{Q,n} = \lambda_{Q,n}f_{Q,n}$. Now we are ready to state $L^p$-stability properties of eigenfunctions of the operators $L_U(Q)$.

Theorem 1.3. Let $Q, \tilde{Q} \in \mathbb{U}_{p,r}^{2 \times 2}$, $p \in (1, 2], r > 0$, and let $BC \cup \{U_j\}_{j=1}^2$ of the form (1.3) be strictly regular. Then there exists an enumeration of the spectra $\{\lambda_{Q,n}\}_{n \in \mathbb{Z}}$ and $\{\lambda_{\tilde{Q},n}\}_{n \in \mathbb{Z}}$ of the operators $L_U(Q)$ and $L_U(\tilde{Q})$, and the set $\mathcal{I}_{Q,\tilde{Q}} \subset \mathbb{Z}$, such that with some constants $C, N > 0$, not dependent on matrices $\{Q, \tilde{Q}\}$, the following estimates hold

$$\sum_{n \in \mathcal{I}_{Q,\tilde{Q}}} \| f_{Q,n} - f_{\tilde{Q},n} \|_{C([0,1];\mathbb{C})}^p \leq C \cdot \|Q - \tilde{Q}\|_p^p,$$

$$\sum_{n \in \mathcal{I}_{Q,\tilde{Q}}} (1 + |n|)^{-p-2} \| f_{Q,n} - f_{\tilde{Q},n} \|_{C([0,1];\mathbb{C})}^p \leq C \cdot \|Q - \tilde{Q}\|_p^p. \quad (1.20, 1.21)$$

On compact sets $\mathcal{K}$ in $L^p$ estimates (1.20)–(1.21) are simplified because subsets $\mathcal{I}_{Q,\tilde{Q}} \subset \mathbb{Z}$ can be chosen to be independent of the pair $\{Q, \tilde{Q}\}$, and in view of (1.15) the summation in (1.19)–(1.21) can be replaced with the following: $\sum_{|n| > N}$. Here $N_1 (\in \mathbb{N})$ does not depend on $Q$ and $\tilde{Q}$. Inequality (1.21) generalizes the classical Hardy-Littlewood inequality [54, Theorem XII.3.19] for Fourier coefficients (see Example 7.18).

The proof of estimates (1.20)–(1.21) substantially relies on the uniform estimate (1.19). Moreover, for efficient estimate of the “maximal” Fourier transform (see definition (1.23)) we use classical theorems by Carleson-Hunt ([18, Theorem 6.2.1]), Hausdorff-Young, and Hardy-Littlewood (see [54, Chapter XII]).

Estimates (1.16)–(1.18) with $\tilde{Q} = 0$ give (uniform on balls) $L^p$-estimates of the remainder in the asymptotic formula for the eigenvalues of the problem (1.1) for Dirac type systems. In this case ($\tilde{Q} = 0$) for Dirac system ($b_1 = -b_2$) estimate (1.16) was recently and by another method obtained by A. Savchuk [49].

Note in conclusion that periodic and antiperiodic (necessarily non-strictly regular) BVP for $2 \times 2$ Dirac and Sturm-Liouville equations have also attracted considerable attention during the last decade. For instance, a criterion for the system of root functions of the periodic BVP for $2 \times 2$ Dirac equation to contain a Riesz basis (without parentheses!) was obtained by P. Djakov and B. Mityagin in [9] (see also recent surveys [13] and recent papers [33, 34] by A.S. Makin and the references therein). It is also worth mentioning that F. Gesztesy and V.A. Tkachenko [13, 15] for $q \in \mathbb{L}^2(0, \pi)$ and P. Djakov and B.S. Mityagin [9] for $q \in \mathbb{L}^{-1,2}(0, \pi]$ established by different methods a criterion for the system of root functions to contain a Riesz basis for Sturm-Liouville operator $-\frac{d^2}{dx^2} + q(x)$ on $[0, \pi]$ (see also survey [32]).
Finally, we mention that the Riesz basis property for abstract operators is investigated in numerous papers. Due to the lack of space we only mention [21, 39, 40, 1], the recent survey [50], and the references therein.

The paper is organized as follows.

In Section 2 we introduce the Banach spaces $X_{1,p} := X_{1,p}^0(\Omega)$ and $X_{\infty,p} := X_{\infty,p}^0(\Omega)$ as well as their separable subspaces $X_{\infty,p}^0(\Omega)$ and $X_{0,p}^0(\Omega)$ being the closures of $C(\Omega)$ in respective norms, and investigate Volterra type operators of the form (1.7) with kernels from these spaces. Besides, we show that for each $a \in [0,1]$ the trace mappings $N(x,t) \to N(a,t)$ and $N(x,t) \to N(x,a)$ originally defined on $C(\Omega)$ are extended as continuous mappings $X_{0,p}^0(\Omega) \to L^p[0,a]$ and $X_{0,p}^0(\Omega) \to L^p[a,1]$, respectively. So, the functions $g_t$ in (1.12), being traces of $K_Q^X$, are well defined.

In Section 3 we show that $K_Q^X \in X_{0,p}^0(\Omega) \cap X_{\infty,p}^0(\Omega)$ and prove the main estimate (1.11) for the difference of transformation operators kernels, i.e. establish the Lipshitz property of the mapping $Q \to K_Q^X$ on the $L^p$-balls $\mathbb{U}_{p,r}^{2\times 2}$. In passing we prove several auxiliary statements on kernels $K_Q^\pm$, useful in Subsection 7.1 for proving important estimate (1.18).

In Section 4 we apply the main uniform estimate (1.11) to characteristic determinants $\Delta_Q$. In particular, we prove here Lemma 1.1 as well as clarify asymptotic formula (1.9). We also indicate here (see Remark 4.18) certain classes of strictly regular BC in purely algebraic terms. Emphasize that as distinguished from the case of Dirac operators such a description is non-trivial for Dirac type $(b_1 \neq b_2)$ operators. For instance, if $bc = 0$ and $ad \neq 0$, then BC (1.6) are strictly regular whenever $b_1 \ln |d| + b_2 \ln |a| \neq 0$. Besides, antiperiodic boundary conditions (1.1) $a = d = 1, b = c = 0$ are strictly regular provided that

$$b_1 = -n_1\beta, \quad b_2 = n_2\beta, \quad \text{with} \quad n_1, n_2 \in \mathbb{N}, \quad \beta \in \mathbb{C}, \quad \text{and} \quad n_1 - n_2 = 2p + 1 \in \mathbb{Z}. \quad (1.22)$$

Therefore the previous results imply the following surprising statement.

**Corollary 1.4.** Let $Q, \tilde{Q} \in \mathbb{U}_{p,r}^{2\times 2}$, $p \in (1,2]$. Assume also that $b_1, b_2$ satisfy (1.22). Then antiperiodic BC are strictly regular, hence the operator $L_U(Q)$ has Riesz basis property. Moreover, the corresponding eigenvalues and eigenvectors satisfy uniform Lipshitz type estimates (1.10)–(1.14) and (1.20)–(1.21), respectively.

This result demonstrates substantial difference between Dirac and Dirac type operators.

In Section 5 we investigate Fourier transform and “maximal” Fourier transform

$$F[g](\lambda) := \int_0^1 g(t)e^{i\lambda t} dt \quad \text{and} \quad \mathcal{F}[g](\lambda) := \sup_{x \in [0,1]} \left| \int_0^x g(t)e^{i\lambda t} dt \right|, \quad \lambda \in \mathbb{C}, \quad (1.23)$$

in order to generalize certain classical Hausdorff-Young and Hardy-Littlewood theorems for Fourier coefficients ([54,Chapter XII]). In particular, we prove here that for any incompressible sequence $\Lambda = \{\mu_n\}_{n \in \mathbb{Z}}$ in the strip $\Pi_h$, there is a constant $C = C(p,h,d) > 0$, not dependent on $\Lambda$, such that the following estimate holds uniformly in $g$ and $\Lambda$:

$$\sum_{n \in \mathbb{Z}}(1 + |n|)^{p-2} |F[g]|^p(\mu_n) \leq \sum_{n \in \mathbb{Z}}(1 + |n|)^{p-2} \mathcal{F}[g]^p(\mu_n) \leq C \cdot \|g\|_{p}, \quad g \in L^p[0,1], \quad p \in (1,2]. \quad (1.24)$$

Here and throughout the paper $\Pi_h := \{z \in \mathbb{C} : |\Im z| \leq h\}$ denotes a symmetrical horizontal strip of semi-width $h \geq 0$.

Inequality (1.24) generalizes Hardy-Littlewood theorem ([54,Theorem XII.3.19]) and coincides with it for ordinary Fourier transform and $\mu_n = 2\pi n$. In turn, this inequality is an important ingredient in proving the estimate (1.17).

We also consider “maximal” Fourier transform $\mathcal{F}[K_Q^X]$ given by (1.23) with kernels $K_Q^X(x,t)$ in place of $g$ and applying the main estimate (1.11) establish the following uniform Lipshitz estimate on $L^p$-balls:

$$\|\mathcal{F}[K_Q^X] - \mathcal{F}[K_{\tilde{Q}}^X]\|_{C(\Pi_h)} \leq e^{[b_1h \cdot C(B,p,r) \cdot \|Q - \tilde{Q}\|_{L^p}}, \quad Q, \tilde{Q} \in \mathbb{U}_{p,r}^{2\times 2}, \quad p \in [1,\infty). \quad (1.25)$$

In Section 6 we prove estimates (1.13)–(1.17). Here we substantially use the following important statement that can be extracted from Lemma 1.1 assuming the spectra $\Lambda_{U,Q} = \{\lambda_{Q,n}\}_{n \in \mathbb{Z}}$ and $\Lambda_{U,\tilde{Q}} = \{\tilde{\lambda}_{Q,n}\}_{n \in \mathbb{Z}}$ of operators $L_U(Q)$ and $L_U(\tilde{Q})$, respectively, to be asymptotically simple, we show that for any compact subset $\mathcal{K}$ of $\mathbb{U}_{p,r}^{2\times 2}$ there exist constants $C$ and $N > 0$, not dependent on $\{Q, \tilde{Q}\} \subset \mathcal{K}$, such that the following uniform two-sided estimates hold

$$C^{-1} \cdot |\Delta_{\tilde{Q}}(\lambda_{Q,n})| \leq |\lambda_{Q,n} - \tilde{\lambda}_{Q,n}| \leq C \cdot |\Delta_Q(\lambda_{Q,n})|, \quad |n| > N, \quad Q, \tilde{Q} \in \mathcal{K}. \quad (1.26)$$
Emphasize that in proving (1.16)–(1.17) we use only evaluation of the ordinary Fourier transform and do not use deep Carleson-Hunt result. In particular, the proof of (1.17) relies on the uniform estimate between the first and third terms in [1.24] only with classical \( F \) (not \( \mathcal{F} \)). This fact makes the proof of estimates (1.16)–(1.17) elementary in character.

In Section 7 we prove Theorems 1.2. and 1.3. We also establish uniform estimates similar to (1.20)–(1.21), where the ball \( U_{p,r}^2 \) is replaced with a compact \( K \subset L^p \), and summation is over \(|n| > N\) (see Theorem 7.14). Proposition 7.17 shows that in the case \( Q_{12} = 0 \) summation in (1.20)–(1.21) can be extended to the entire \( \mathbb{Z} \) and the estimate remains uniform for all \( Q_{21} \in L^p[0,1], p \in (1,2) \).

In Section 8 we apply results of Section 7 in the case of \( Q \in L^2 \) to establish a criterion for the system of root vectors of BVP (1.1)–(1.3) to form a Bari basis in \( L^2([0,1];\mathbb{C}^2) \). To the best of our knowledge this problem was not investigated before. Inequality (1.20) reduces the general case to a simplest case of \( Q = 0 \). In Proposition 8.7 we establish a criterion in terms of the eigenvalues of the unperturbed operator \( L_0 \). If either \( b_1 b_2^{-1} \in \mathbb{Q} \) or \( abed = 0 \), then the criterion can be simplified as follows: the system of root vectors of such BVP (1.1), (1.6) forms a Bari basis if and only if BC (1.6) are self-adjoint.

2. The Banach spaces \( X_{1,p} \) and \( X_{\infty,p} \)

Let \( p \in [1,\infty] \). Following [35] denote by \( X_{1,p} := X_{1,p}(\Omega) \) and \( X_{\infty,p} := X_{\infty,p}(\Omega) \) the linear spaces composed of (equivalent classes of) measurable functions defined on \( \Omega := \{(x,t) : 0 \leq t \leq x \leq 1\} \) satisfying

\[
\|f\|_{X_{1,p}}^p := \text{ess sup}_{x \in [0,1]} \int_0^x |f(x,t)|^p \, dx < \infty, \quad p < \infty, \tag{2.1}
\]

\[
\|f\|_{X_{\infty,p}}^p := \text{ess sup}_{x \in [0,1]} \int_0^x |f(x,t)|^p \, dt < \infty, \quad p < \infty, \tag{2.2}
\]

respectively, and \( \|f\|_{X_{1,\infty}} = \|f\|_{X_{\infty,\infty}} = \text{ess sup}_{(x,t) \in \Omega} |f(x,t)| \). It can easily be shown that the spaces \( X_{1,p} \) and \( X_{\infty,p} \) equipped with the norms (2.1) and (2.2) form Banach spaces that are not separable. Denote by \( X_{1,p}^0 := X_{1,p}(\Omega) \) and \( X_{\infty,p}^0 := X_{\infty,p}(\Omega) \) the subspaces of \( X_{1,p}(\Omega) \) and \( X_{\infty,p}(\Omega) \), respectively, obtained by taking the closure of continuous functions \( f \in C(\Omega) \). Clearly, the set \( C'(\Omega) \) of smooth functions is also dense in both spaces \( X_{1,p}^0 \) and \( X_{\infty,p}^0 \). Note also that the following embeddings hold and are continuous

\[
X_{1,p_1} \subset X_{1,p_2} \subset X_{1,1} \quad \text{and} \quad X_{\infty,p_1} \subset X_{\infty,p_2} \subset X_{\infty,1}, \quad p_1 > p_2 \geq 1. \tag{2.3}
\]

The following simple properties of the class \( X_{\infty,1}^0(\Omega) \) will be important in the sequel.

**Lemma 2.1.** Let \( p \geq 1 \). For each \( a \in [0,1] \), the trace mappings

\[
i_{a,\infty} : C(\Omega) \to C[0,a], \quad i_{a,\infty}(N(x,t)) := N(a,t), \quad \text{and} \quad i_{a,1} : C(\Omega) \to C[a,1], \quad i_{a,1}(N(x,t)) := N(x,a) \tag{2.4}
\]

originally defined on \( C(\Omega) \) admit continuous extensions (also denoted by \( i_{a,\infty} \) and \( i_{a,1} \), respectively) as mappings from \( X_{\infty,p}^0(\Omega) \) onto \( L^p[0,a] \) and \( X_{1,p}^0(\Omega) \) onto \( L^p[a,1] \), respectively.

**Proof.** The proof is immediate by combining the corresponding results for \( X_{1,1}^0(\Omega) \) and \( X_{1,1}^0(\Omega) \) (see [31] Lemma 2.2) with the continuity of embeddings (2.3). \( \square \)

Going over to the vector case let us recall some notations. For \( u = \text{col}(u_1,\ldots,u_n) \in \mathbb{C}^n \) we denote

\[
|u|_\alpha := |u_1|^\alpha + \ldots + |u_n|^\alpha, \quad 0 < \alpha < \infty, \quad |u|_\infty = \max\{|u_1|,\ldots,|u_n|\}. \tag{2.5}
\]

When there is no confusion, we will denote \( |u| = |u|_\alpha \). \( \mathbb{C}^n \) equipped with the norm \( \| \cdot \|_\alpha \) will be denoted as \( \mathbb{C}_n^\alpha \). We will also use notation \( u^* := (\frac{1}{u_1} \quad \ldots \quad \frac{1}{u_n}) \). Clearly, the product \( u_1 u_2^* \in \mathbb{C} \).

Further, for \( A = (a_{jk})_{j,k=1}^{n,n} \in \mathbb{C}^{n \times n} \) we denote \( |A|_{\alpha \to \beta} \) a norm of the linear operator from \( \mathbb{C}^n_\alpha \) to \( \mathbb{C}_\beta^\alpha \) with a matrix \( A \), where \( \alpha, \beta \in [1,\infty] \),

\[
|A|_{\alpha \to \beta} = \sup\{|Au|_\beta : u \in \mathbb{C}^n, |u|_\alpha = 1\}. \tag{2.6}
\]
Similarly, when there is no confusion, we will denote $|A| = |A|_{\alpha \to \beta}$. We will use the following well-known inequalities in the sequel:

\[ |u_1 u_2^*| \leq |u_1|_{\alpha} \cdot |u_2^*|_{\alpha'}, \quad 1/\alpha + 1/\alpha' = 1, \quad u_1, u_2 \in \mathbb{C}^n, \]
\[ |Au|_{\beta} \leq |A|_{\alpha \to \beta} |u|_{\alpha}, \quad \alpha, \beta \in [1, \infty], \quad A \in \mathbb{C}^{n \times n}, \quad u \in \mathbb{C}^n, \]
\[ |A_1 A_2|_{\alpha \to \beta} \leq |A_1|_{\alpha \to \beta} \cdot |A_2|_{\alpha \to \beta}, \quad 1 \leq \alpha \leq \beta \leq \infty, \quad A_1, A_2 \in \mathbb{C}^{n \times n}. \]

Now we are ready to introduce the Banach spaces

\[ X_1^p := X_1(p; \Omega; \mathbb{C}^{n \times n}) := X_1(p; \Omega) \otimes \mathbb{C}^{n \times n} \quad \text{and} \quad X_\infty^p := X_\infty(p; \Omega; \mathbb{C}^{n \times n}) := X_\infty(p; \Omega) \otimes \mathbb{C}^{n \times n}, \]

consisting of $n \times n$ matrix-functions $F = (F_{jk})_{j,k=1}^n$ with $X_1^p$ and $X_\infty^p$-entries, respectively, equipped with the norms

\[ \|F\|_{X_1^p} := \sup_{x \in [0,1]} \int_0^1 |F(x,t)|_{1 \to p}^p dx < \infty, \quad p \in [1, \infty), \]
\[ \|F\|_{X_\infty^p} := \sup_{x \in [0,1]} \int_0^1 |F(x,t)|_{\infty \to \infty}^{1/p'} dt < \infty, \quad p \in [1, \infty), \quad 1/p + 1/p' = 1, \]

and $\|F\|_{X_1^\infty} = \|F\|_{X_\infty^\infty} = \sup_{(x,t) \in \Omega} |F(x,t)|_{1 \to \infty}$. Besides, we introduce subspaces

\[ X_0^p := X_0^p(\Omega; \mathbb{C}^{n \times n}) := X_0^p(\Omega) \otimes \mathbb{C}^{n \times n} \quad \text{and} \quad X_\infty^0 := X_\infty^0(\Omega; \mathbb{C}^{n \times n}) := X_\infty^0(\Omega) \otimes \mathbb{C}^{n \times n}, \]

being separable parts of $X_1^p$ and $X_\infty^p$, respectively.

For brevity, throughout the section we denote

\[ L^s := L^s([0,1]; \mathbb{C}^n), \quad C := C([0,1]; \mathbb{C}^n), \quad s \in [1, \infty), \]

Equip the space $L^s$ of vector functions with the following norm

\[ \|f\|_s := \|\text{col}(f_1, \ldots, f_n)\|_s := \|f_1\|_s + \ldots + \|f_n\|_s := \|f_1\|_{L^s([0,1])} + \ldots + \|f_n\|_{L^s([0,1])}, \quad s \in [1, \infty), \]

and $\|f\|_\infty = \max\{\|f_1\|_\infty, \ldots, \|f_n\|_\infty\}$. It is clear, that $\|f\|_s = \int_0^1 |f(x)|_s^s dx, \ s \in [1, \infty)$. With each measurable matrix kernel $N(x,t) = (N_{jk}(x,t))_{j,k=1}^n$ one associates a Volterra type operator

\[ \mathcal{N} : f \to \int_0^x N(x,t) f(t) dt. \]

Denote by $\|N\|_{\alpha \to \beta} := \|N\|_{L^\alpha \to L^\beta}, \ \alpha, \beta \in [1, \infty]$, the norm of the operator $\mathcal{N}$ acting from $L^\alpha$ to $L^\beta$, provided that it is bounded, $\mathcal{N} \in \mathcal{B}(L^\alpha, L^\beta)$. The following result motivates introduction of the spaces $X_1^p$ and $X_\infty^p$. In particular, the second statement sheds light on the interpolation role of these spaces (cf. [35]). This result substantially completes Lemma 2.3 from [33].

Recall that a Volterra operator on a Banach space is a compact operator with zero spectrum.

**Lemma 2.2.** Let $\mathcal{N}$ be a Volterra type operator given by (2.10) for a measurable matrix-function $N(\cdot, \cdot)$.

(i) Let either $q = 1$ or $q = \infty$. Then $\mathcal{N} \in \mathcal{B}(L^q, L^q)$ if and only if $N \in X_{q,1}^n$, in which case

\[ \|\mathcal{N}\|_{q \to q} = \|N\|_{X_{q,1}^n}. \]

If $N \in X_{q,1}^n$, then the operator $\mathcal{N}$ is a Volterra operator in $L^q$, and the inverse $(I + \mathcal{N})^{-1}$ is given by

\[ (I + \mathcal{N})^{-1} = I + S : f \to f + \int_0^x S(x,t) f(t) dt \quad \text{with} \quad S \in X_{q,1}^0. \]

(ii) Let $N \in X_{1,1}^n \cap X_{\infty,1}^n$. Then $\mathcal{N} \in \mathcal{B}(L^s, L^s)$ for each $s \in [1, \infty]$, and

\[ \|\mathcal{N}\|_{s \to s} \leq \|N\|_{X_{1,1}^n}^{1/s} \cdot \|N\|_{X_{\infty,1}^n}^{1 - 1/s}. \]

Moreover, if $N \in X_{1,1}^0 \cap X_{\infty,1}^0$, then $\mathcal{N}$ is a Volterra operator in $L^s$ for each $s \in [1, \infty]$. 
Proof. (i) Relation (2.17) was proved in Lemma 2.3 from [31] (see also [35]). In particular, it implies equivalence of inclusions $\mathcal{N} \subset \mathcal{B}(L^q) := \mathcal{B}(L^q, L^q)$ and $N \in X^0_{q,1}$. Let us prove representation (2.18). Since $N \in X^0_{q,1}$, one finds a sequence $N_k \subset C(\Omega; \mathbb{C}^{n \times n})$ that approaches $N$ in $X^0_{q,1}$ as $k \to \infty$. It is well-known, that the corresponding operators $N_k$ are Volterra operators in $L^q$, such that $(I + N_k)^{-1} = I + S_k$, where $S_k$ is the Volterra operator in $L^q$ of the form (2.16) with $S_k \in C(\Omega; \mathbb{C}^{n \times n})$. Identity (2.17) implies that

$$\lim_{k \to \infty} ||N_k - N||_{q \to q} = \lim_{k \to \infty} ||N_k - N||_{X^0_{q,1}} = 0,$$

(2.20)

which yields that $\mathcal{N}$ is a Volterra operator in $L^q$, and existence of the inverse $(I + \mathcal{N})^{-1}$. Further, continuity of the mapping $T \to T^{-1}$ in the group of invertible operators in the Banach algebra $\mathcal{B}(L^q)$, relation (2.20) and identity (2.17) imply

$$I + S_k = (I + N_k)^{-1} \to (I + \mathcal{N})^{-1} \quad \text{as} \quad k \to \infty \quad \text{in} \quad L^q,$$

(2.21)

$$\|S_k - S_m\|_{X^0_{q,1}} = \|S_k - S_m\|_{q \to q} = \|S_k - S_m\|_{X^0_{q,1}} \to 0 \quad \text{as} \quad k, m \to \infty.$$

(2.22)

Thus, there exists $S \in X^0_{q,1}$ such that $\lim_{k \to \infty} ||S_k - S||_{X^0_{q,1}} = 0$. As it has been already proved for the kernel $N \in X^0_{q,1}$, the kernel $S$ generates Volterra operator $S \in \mathcal{B}(L^q)$ of the form (2.16). Moreover, $\lim_{k \to \infty} ||S_k - S||_{q \to q} = \lim_{k \to \infty} ||S_k - S||_{X^0_{q,1}} = 0$. Combining this with (2.21) implies equality (2.18).

(ii) The proof is immediate from relations (2.17) combined with Riesz-Torin theorem (see [31] and [35]).

The following result demonstrates natural occurrence of the spaces $X^0_{1,p}$ and $X^0_{\infty,p}$ in the study of the integral operators acting from $L^q$ to $L^p$ with the special $\alpha, \beta$.

Proposition 2.3. Let $\mathcal{N}$ be a Volterra type operator given by (2.16) for a measurable matrix-function $N(\cdot, \cdot)$. Let also $p \in [1, \infty]$ and $1/p + 1/p' = 1$. Then:

(i) The inclusion $\mathcal{N} \subset \mathcal{B}(L^1, L^p)$ holds if and only if $N \in X^0_{1,p}$, in which case

$$\|\mathcal{N}\|_{1 \to p} = \|N\|_{X^0_{1,p}}.$$  

(2.23)

Moreover, if $N \in X^0_{0,n}$, then the operator $\mathcal{N}$ is compact from $L^1$ to $L^p$, and the following relation holds:

$$-\mathcal{N} \cdot (I + \mathcal{N})^{-1} = S \in \mathcal{B}(L^1, L^p), \quad \text{where} \quad S : f \to \int_0^x S(x,t)f(t)\,dt \quad \text{with} \quad S \in X^0_{1,p},$$

(2.24)

where operator $(I + \mathcal{N})^{-1}$ is treated as an operator from $\mathcal{B}(L^1, L^1)$, and exists due to Lemma 2.2/i).

(ii) The inclusion $\mathcal{N} \subset \mathcal{B}(L^p, L^\infty)$ holds if and only if $N \in X^0_{\infty,p}$, in which case

$$\|\mathcal{N}\|_{p' \to \infty} = \|N\|_{X^0_{\infty,p}}.$$  

(2.25)

If $N \in X^0_{0,n}$ then the operator $\mathcal{N}$ sends $L^p$ to $C = C([0,1]; \mathbb{C}^n)$ and is compact. Let $\mathcal{N}_C : C \to C$, be a restriction of the operator $\mathcal{N}$. Then the inverse operator $(I + \mathcal{N}_C)^{-1} \in \mathcal{B}(C, C)$, and the following relation holds:

$$-(I + \mathcal{N}_C)^{-1} \cdot \mathcal{N} = S \in \mathcal{B}(L^p, C), \quad \text{where} \quad S : f \to \int_0^x S(x,t)f(t)\,dt \quad \text{with} \quad S \in X^0_{\infty,p}.$$  

(2.26)

(iii) Let $N \in X^0_{1,p} \cap X^0_{\infty,p}$. Then a triangular operator $\mathcal{N}$ is a Volterra operator in $L^s$ for each $s \in [1, \infty]$ and the inverse operator $(I + \mathcal{N})^{-1}$ is given by

$$(I + \mathcal{N})^{-1} = I + S : f \to f + \int_0^x S(x,t)f(t)\,dt, \quad S \in X^0_{1,p} \cap X^0_{\infty,p}.$$  

(2.27)

Proof. (i) Let $f \in L^1$, $g \in L^{p'}$, and $N \in X^0_{1,p}$. Then applying Hölder’s inequality one gets

$$|\langle Nf, g \rangle_{p,p'}| = \int_0^1 \left( \int_0^x N(x,t)f(t)\,dt \right) g^*(x)\,dx \leq \int_0^1 \left( \int_0^x |N(x,t)f(t)|_{p'}\,dt \right) |g^*(x)|_{p'}\,dx$$

$$\leq \int_0^1 |f(t)|_1 \, dt \int_0^1 |N(x,t)|_{1 \to p'} \cdot |g^*(x)|_{p'}\,dx \leq \|f\|_1 \cdot \|N\|_{X^0_{1,p}} \cdot \|g\|_{p'}.$$  

(2.28)
It follows that \( N \in \mathcal{B}(L^1, L^p) \) and \( \|N\|_{1\rightarrow p} \leq \|N\|_{X_{1,p}^n} \). Besides, estimate (2.28) yields \( N^* \in \mathcal{B}(L^{p'}, L^\infty) \), i.e. \( N^*g = \int_1^\infty N(x,t)^*g(x)\,dx \in L^\infty \) for each \( g \in L^{p'} \). From this fact one extracts the opposite inequality \( \|N\|_{1\rightarrow p} \geq \|N\|_{X_{1,p}^n} \), which yields equality (2.24).

Now let \( N \in X_{1,p}^n \). Due to the inclusion (2.3) one has \( N \in X_{1,1}^{0,n} \). By Lemma 2.2(ii), \( N \) is a Volterra operator in \( L^1 \) and the inverse \( (I + N)^{-1} \in \mathcal{B}(L^1, L^1) \) is given by (2.18), with \( S \in X_{1,1}^{0,n} \). Hence \( S = -(I + N)^{-1} \in \mathcal{B}(L^1, L^p) \) because \( N \in \mathcal{B}(L^1, L^p) \). This yields the desired relation (2.24), but only with \( S \in X_{1,1}^{0,n} \cap X_{1,p}^n \).

Let us show that \( S \in X_{1,p}^n \). Since \( N \in X_{1,1}^{0,n} \), one finds a sequence \( N_k \in C(\Omega; C^{\alpha,\infty}) \) that approaches \( N \) in \( X_{1,p}^n \) as \( k \to \infty \). Clearly, it also approaches \( N \) in \( X_{1,1}^{0,n} \). With account of (2.22) and identity (2.24) this implies,

\[
\|N_k - N_m\|_{X_{1,p}^n} = \|N_k - N_m\|_{1\rightarrow p} \to 0 \quad \text{and} \quad \|(I + N_k)^{-1} - (I + N_m)^{-1}\|_{1\rightarrow 1} \to 0 \quad \text{as} \quad k, m \to \infty. \tag{2.29}
\]

Clearly, \( N_k \) is a sequence of compact operators from \( L^1 \) to \( L^p \) approaching \( N \) in \( \| \cdot \|_{1\rightarrow p} \)-norm. Hence \( N \) is a compact operator from \( L^1 \) to \( L^p \). It is clear that \( S_k = -N_k(I + N_k)^{-1} \), \( k \in \mathbb{Z} \). Hence, identity (2.22) and relation (2.24) imply

\[
\|S_k - S_m\|_{X_{1,p}^n} = \|S_k - S_m\|_{1\rightarrow p} = \|N_k(I + N_k)^{-1} - N_m(I + N_m)^{-1}\|_{1\rightarrow p}
\leq \|N_k - N_m\|_{1\rightarrow p} \|(I + N_k)^{-1} - (I + N_m)^{-1}\|_{1\rightarrow 1} \to 0 \quad \text{as} \quad k, m \to \infty. \tag{2.30}
\]

Thus, there exists \( S \in X_{1,p}^n \) such that \( \lim_{k \to \infty} \|S_k - S\|_{X_{1,p}^n} = 0 \). As it has been already proved for the kernel \( N \in X_{1,1}^{0,n} \), the kernel \( S \) generates a compact operator \( S \in \mathcal{B}(L^1, L^p) \) of the form (2.16). Moreover, \( \lim_{k \to \infty} \|S_k - S\|_{1\rightarrow p} = \lim_{k \to \infty} \|S_k - S\|_{X_{1,p}^n} = 0 \) and relation (2.19) also implies that \( S_k = -N_k(I + N_k)^{-1} \to -N(I + N)^{-1} = S \) as \( k \to \infty \) in the norm \( \| \cdot \|_{1\rightarrow 1} \), which implies that \( S = S \) and \( S \in X_{1,p}^n \).

(i) Inequality \( \|N\|_{p'\rightarrow \infty} \leq \|N\|_{X_{1,p}^n} \) is immediate from Hölder’s inequality. The opposite one is proved as in the previous step. The inclusion \( \text{ran}(N) \subset C(\Omega; C^{\alpha,\infty}) \) is obvious for \( N \in C(\Omega; C^{\alpha,\infty}) \). Given an arbitrary \( N \in X_{1,1}^{0,n} \), take a sequence of continuous functions \( N_k \in C(\Omega; C^{\alpha,\infty}) \) approaching \( N \) in \( X_{1,1}^{0,n} \) as \( k \to \infty \). Then for any \( f \in L^{p'} \) the sequence \( N_k f \in C \) approaches \( N f \) uniformly, hence \( N f \in C \). It follows from the equality \( \lim_{k \to \infty} \|N - N_k\|_{p'\rightarrow \infty} = \lim_{k \to \infty} \|N - N_k\|_{X_{1,p}^n} = 0 \) that \( N \) is compact from \( L^p \) to \( C \).

Now let \( N \in X_{1,1}^{0,n} \subset X_{1,1}^{0,n} \). Lemma 2.2(i) implies that \( N \) is a Volterra operator in \( L^\infty \) and, thus, \( N \) is a Volterra operator in \( C \) with \( (I + N_C)^{-1} \in \mathcal{B}(C, C) \). It follows that \( S := -(I + N_C)^{-1}N \in \mathcal{B}(L^p, C) \). The proof is finished in the same way as in part (i), where all the norms \( \| \cdot \|_{X_{1,p}^n}, \| \cdot \|_{1\rightarrow p} \) and \( \| \cdot \|_{1\rightarrow 1} \) are replaced with \( \| \cdot \|_{X_{1,p}^n}, \| \cdot \|_{p'\rightarrow \infty} \) and \( \| \cdot \|_{\infty} \), respectively.

(ii) The statement is immediate by combining parts (i) and (ii) with Lemma 2.2(ii).

\[ \square \]

**Remark 2.4.** In connection with Proposition 2.28 let us recall Theorems XI.1.5 and XI.1.6 from [20], concerning integral representations of bounded linear operators. Namely, let \( p \in (1, \infty), p' \in [1, \infty) \) and \( 1/p + 1/p' = 1 \), and let \( R \) and \( S \) be bounded linear operators from \( L^1[0,1] \) to \( L^p[0,1] \) and \( L^{p'}[0,1] \) to \( C[0,1] \), respectively. Then they admit the following integral representations:

\[
(Rf)(x) = \int_0^1 R(x,t)f(t)\,dt, \quad \text{and for } p < \infty \text{ its norm is } \|R\|_{1\rightarrow p} = \text{ess sup}_{t \in (0,1]} \int_0^1 |R(x,t)|^p\,dx < \infty, \tag{2.31}
\]

\[
(Sf)(x) = \int_0^1 S(x,t)f(t)\,dt, \quad \text{and for } p < \infty \text{ its norm is } \|S\|_{p'\rightarrow \infty} = \text{ess sup}_{x \in (0,1]} \int_0^1 |S(x,t)|^p\,dt < \infty. \tag{2.32}
\]

Denote by \( B_{1,p}(r) \) (\( B_{1,p}(r) \)) the ball centered at zero of radius \( r \) in \( X_{1,p}^n \) (\( X_{1,1}^{0,n} \)).

**Lemma 2.5.** Let either \( q = 1 \) or \( q = \infty \) and \( p \in [2, \infty) \). Let \( N \in X_{q,p}^n \) and let \( S \) be the kernel from (2.24) or (2.26) if \( q = 1 \) or \( q = \infty \), respectively, i.e. the kernel of “inverse” operator \( S = -(I + N)^{-1} \) or \( S = -(I + N_C)^{-1}N \).

(i) The kernel \( S \) satisfies the following inequality

\[
\|S\|_{X_{1,p}^n} \leq 2^{1-1/p} \cdot \|N\|_{X_{1,p}^n} \cdot \exp \left( p^{-1}2^{-p-1}\|N\|_{X_{1,p}^n} \right). \tag{2.33}
\]
(ii) Moreover, if \( N, \tilde{N} \in B_{q,p}(r) \subset X_{r,p}^{0,n} \), and \( \tilde{S} \) is the kernel defined for \( \tilde{N} \) similarly as above from (2.24) or (2.26), then the following uniform estimate holds on the ball \( B_{q,p}(r) \):

\[
\|S - \tilde{S}\|_{X_{r,p}^{0,n}} \leq 3^{1-1/p} \exp(p^{-1}3^{p/p}) \cdot \|N - \tilde{N}\|_{X_{r,p}^{0,n}}.
\] (2.34)

Proof. (i) Let first \( q = 1 \), i.e. \( N \in X_{1,p}^{0,n} \). By Proposition 2.3(i), \( S \in X_{1,p}^{0,n} \), and corresponding Volterra type operators are related by \( S = -(1 + N_N)^{-1} \), where \( N, S \in B(L^1, L^p) \) and inverse operator \( (I + N_N)^{-1} \) is treated as an operator in \( L^1 \). Since \( N \) acts from \( L^1 \) to \( L^p \), we also have \( S = -(I + N_p)^{-1}N \), where \( N_p \) is a restriction of \( N \) acting in \( L^p \). Hence \( N + S + N_pS = 0 \) on \( L^1 \). This implies that the kernels \( N \) and \( S \) are related by the equation

\[
N(x,t) + S(x,t) + \int_t^\infty N(x,\xi)S(\xi,t)\,d\xi = 0, \quad \text{for almost all } (x,t) \in \Omega.
\] (2.35)

Since \( p \geq 2 \), combining Hölder’s inequality with power-mean inequality gives for almost all \((x,t) \in \Omega\),

\[
\left| \int_t^\infty N(x,\xi)S(\xi,t)\,d\xi \right|^p \leq \left( \int_t^\infty |S(x,\xi)|^p\,d\xi \right)^{p/p'} \int_t^\infty |N(\xi,t)|^p\,d\xi \leq \|N\|_{X_{r,p}^p}^p \cdot \int_t^\infty |S(x,\xi)|^p\,d\xi.
\] (2.36)

Here and throughout the rest of the proof \(|A| = |A|_{1,p} \) for \( A \in \mathbb{C}^{n \times n} \). Since \( N, S \in X_{1,p}^{0,n} \), then, according to Lemma 2.1 for any \( t \in [0,1] \) the traces \( N(\cdot,t) \) and \( S(\cdot,t) \) are well defined and belong to \( L^p[0,1] \). Therefore, fixing \( t \in [0,1] \), integrating (2.35) over \( x \in [t,1] \), applying inequality \( 2^{1-p}|a + b|^p \leq |a|^p + |b|^p \) and estimate (2.30) yields

\[
2^{1-p}S_p(t) := 2^{1-p} \int_t^1 |S(x,t)|^p\,dx \leq 2^{1-p} \int_t^1 |N(x,t)|^p\,dx + \|N\|_{X_{r,p}^p}^p \int_t^1 \left( \int_t^\infty |S(x,\xi)|^p\,d\xi \right)\,dx
\]

\[
\leq \|N\|_{X_{r,p}^p}^p + \|N\|_{X_{r,p}^p}^p \int_t^1 \left( \int_t^\infty |S(x,\xi)|^p\,d\xi \right)\,dx = \|N\|_{X_{r,p}^p}^p \left( 1 + \int_t^1 S_p(\xi)\,d\xi \right), \quad t \in [0,1].
\] (2.37)

Applying Grönwall’s inequality here implies

\[
S_p(t) \leq 2^{p-1} \cdot \|N\|_{X_{r,p}^p}^p \cdot \exp \left( (1-t)2^{p-1}\|N\|_{X_{r,p}^p}^p \right), \quad t \in [0,1].
\] (2.38)

Noting that \( \|S\|_{X_{r,p}^p}^p = \sup_{t \in [0,1]} S_p(t) \), we arrive at (2.33) with \( q = 1 \).

The case \( q = \infty \) is treated similarly but using identity \( N + S + S N = 0 \) instead of (2.35).

(ii) Let again \( q = 1 \). Setting \( \tilde{S}(x,t) := S(x,t) - \tilde{S}(x,t) \) and \( \tilde{N}(x,t) := N(x,t) - \tilde{N}(x,t) \) we easily derive from equation (2.35) and similar equation related \( N(x,t) \) and \( \tilde{S}(x,t) \), that

\[
|\tilde{S}(x,t)| \leq |\tilde{N}(x,t)| + \int_t^\infty |\tilde{N}(x,\xi)| \cdot |S(\xi,t)|\,d\xi + \int_t^\infty |\tilde{N}(x,\xi)| \cdot |\tilde{S}(\xi,t)|\,d\xi.
\] (2.39)

Using inequality (2.30) and performing the same transformations as in (2.37) we arrive at

\[
3^{1-p} \tilde{S}_p(t) := 3^{1-p} \int_t^1 |\tilde{S}(x,t)|^p\,dx \leq \|\tilde{N}\|_{X_{r,p}^p}^p + \|\tilde{N}\|_{X_{r,p}^p}^p \int_t^1 S_p(\xi)\,d\xi + \|\tilde{N}\|_{X_{r,p}^p}^p \int_t^1 \tilde{S}_p(\xi)\,d\xi, \quad t \in [0,1].
\] (2.40)

Note that \( \|N\|_{X_{r,p}^p} \leq r \), hence, in accordance with estimate (2.35),

\[
1 + \int_t^1 S_p(\xi)\,d\xi \leq 1 + \int_t^1 \mu e^{(1-\xi)\mu} \,d\xi = e^{(1-t)\mu} = e^\mu = \exp(2^{p-1}r^p), \quad \mu := 2^{p-1}r^p.
\] (2.41)

With account of (2.41) and inequality \( \|\tilde{N}\|_{X_{r,p}^p} \leq r \), estimate (2.40) turns into

\[
\tilde{S}_p(t) \leq 3^{p-1} \exp(2^{p-1}r^p) \cdot \|\tilde{N}\|_{X_{r,p}^p}^p + 3^{p-1}r^p \int_t^1 \tilde{S}_p(\xi)\,d\xi, \quad t \in [0,1].
\] (2.42)

Now Grönwall’s inequality applies and gives

\[
\tilde{S}_p(t) \leq 3^{p-1} \exp(2^{p-1}r^p) \cdot \|N - \tilde{N}\|_{X_{r,p}^p}^p \cdot \exp \left( (1-t)3^{p-1}r^p \right) \leq 3^{p-1} \exp(3^p r^p) \cdot \|N - \tilde{N}\|_{X_{r,p}^p}^p, \quad t \in [0,1].
\] (2.43)

In turn, this inequality yields (2.34). \( \square \)
Lemma 2.6. For each $p \in [1, \infty]$ the spaces $X_{1,p}^n$ and $X_{\infty,p}^n$ form Banach algebras with respect to the product (“composition of kernels”)

$$N(x,t) = (N_1 * N_2)(x,t) := \int_{t}^{x} N_1(x,\xi)N_2(\xi,t) \, d\xi.$$  \hfill (2.44)

Proof. (i) Let $N_1, N_2 \in X_{1,p}^n$. Assuming that $p \in (1, \infty)$ and applying Hölder’s inequality yields

$$\int_{t}^{1} |N(x,t)|^p \, dx \leq \int_{t}^{1} \left( (x-t)^{p/p'} \int_{t}^{x} |N_1(x,\xi)|^p |N_2(\xi,t)|^p \, d\xi \right) \, dx \leq \int_{t}^{1} |N_2(\xi,t)|^p \left( \int_{\xi}^{1} |N_1(x,\xi)|^p \, dx \right) \, d\xi \leq \|N_1\|_{X_{1,p}^n} \cdot \|N_2\|_{X_{1,p}^n}. \hfill (2.45)$$

It follows that $N_1 * N_2 \in X_{1,p}^n$ and the inequality $\|N\|_{X_{1,p}^p} \leq \|N_1\|_{X_{1,p}^n} \cdot \|N_2\|_{X_{1,p}^n}$ holds. This proves the statement for $p > 1$. The cases $p = 1$ and $p = \infty$ are treated simpler without using Hölder’s inequality.

(ii) The case of the space $X_{\infty,p}^n$ is treated similarly. \hfill $\square$

Lemma 2.7. Let $N \in X_{\infty,p}^n$, $p \geq 1$, let $N_p(x) := \sup_{\xi \in [0,\omega]} \int_{0}^{\xi} |N(\xi,t)|^p \, dt$, and let $S$ be given by (2.24). Then the following implication holds

$$N_p(x) < (2x)^{1-p} \implies S_p(x) \leq \frac{2^{p-1}N_p(x)}{1 - (2x)^{p-1}N_p(x)}.$$  \hfill (2.46)

In particular, if $\|N\|_{X_{\infty,p}^n} \leq r := 2^{1/p-1}R$ where $R < 1$, then $\|S\|_{X_{\infty,p}^n} \leq R \cdot (1 - R^p)^{-1/p}$.

Proof. Applying Hölder’s inequality and changing order of integration, yields

$$\int_{0}^{x} \left| \int_{t}^{x} S(x,\xi) N(\xi,t) \, d\xi \right|^p \, dt \leq x^{p-1} \int_{0}^{x} \int_{0}^{1} \left| S(x,\xi) N(\xi,t) \right|^p \, dt \, d\xi \hfill (2.47)$$

It easily follows from relation (2.36) with account of (2.47) that

$$2^{1-p} \int_{0}^{x} |S(x,t)|^p \, dt \leq \int_{0}^{x} |N(x,t)|^p \, dt + x^{p-1} \int_{0}^{x} |S(x,\xi)|^p \, d\xi \int_{0}^{\xi} |N(\xi,t)|^p \, dt \leq N_p(x) \left( 1 + x^{p-1} \int_{0}^{x} |S(x,t)|^p \, dt \right). \hfill (2.48)$$

This estimate implies (2.46). \hfill $\square$

3. Triangular transformation operators

Consider $2 \times 2$ Dirac type system (3.1)

$$L(Q)y := -iB^{-1}y' + Q(x) y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0,1],$$  \hfill (3.1)

with the diagonal matrix $B$ and potential matrix $Q(\cdot)$ given by (1.2).

The existence of a triangular transformation operator for system (3.1) with summable potential matrix $Q \in L^1([0,1]; \mathbb{C}^{2\times 2})$ was established in our previous paper [31] (the case $Q \in L^\infty([0,1]; \mathbb{C}^{2\times 2})$ was treated earlier in [36]). The purpose of this section is to prove Lipshitz dependence (in respective norms) of the kernels of transformation operators on $Q \in L^p([0,1]; \mathbb{C}^{2\times 2})$. We start with the following result from [31].

Theorem 3.1. [31] Theorem 2.5 Let $Q = \text{codag}(Q_{12}, Q_{21}) \in L^1([0,1]; \mathbb{C}^{2\times 2})$. Assume that $e_{\pm}(\cdot, \lambda)$ are the solutions to the system (3.1) satisfying the initial conditions $e_{\pm}(0, \lambda) = \left( \frac{1}{2} \right)$. Then $e_{\pm}(\cdot, \lambda)$ admits the following representation by means of the triangular transformation operator

$$e_{\pm}(x, \lambda) = (I + K^\pm) e_{\pm}^0(x, \lambda) = e_{\pm}^0(x, \lambda) + \int_{0}^{x} K^\pm(x,t)e_{\pm}^0(t,\lambda) \, dt,$$  \hfill (3.2)

where

$$e_{\pm}^0(x, \lambda) = \begin{pmatrix} e^{ib_1x}\lambda & e^{ib_2x}\lambda \\ -e^{ib_2x}\lambda & e^{ib_1x}\lambda \end{pmatrix}, \quad \text{and} \quad K^\pm = \begin{pmatrix} k^\pm_{jk} \end{pmatrix}_{j,k=1} \in X_{1,1}^0(\Omega; \mathbb{C}^{2\times 2}) \cap X_{\infty,1}^0(\Omega; \mathbb{C}^{2\times 2}).$$  \hfill (3.3)
Moreover, it is shown in [36] that if \( Q = \text{codiag}(Q_{12}, Q_{21}) \in C^1([0,1]; \mathbb{C}^{2 \times 2}) \), then the matrix kernel in triangular representation (3.2) is smooth, \( K^\pm = (K^\pm_{jk})_{j,k=1}^2 \in C^1(\Omega, \mathbb{C}^{2 \times 2}) \), and it is the unique solution of the following boundary value problem

\[
B^{-1} D_x K^\pm(x,t) + D_t K^\pm(x,t) B^{-1} + iQ(x) K^\pm(x,t) = 0, \quad x \in [0,1],
\]

\[
K^\pm(x,0)B^{-1}\begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} = 0, \quad x \in [0,1].
\]

The proof of this result in [36] is divided in two steps. At first it is proved that there exists the smooth unique solution \( R(x,t) = (R_{jk}(x,t))^2_{j,k=1} \in C^1(\Omega, \mathbb{C}^{2 \times 2}) \) of the problem (3.1)-(3.5) satisfying instead of (3.6) the following conditions:

\[
R_{11}(x,0) = R_{22}(x,0) = 0, \quad x \in [0,1].
\]

Using this result it is proved in [31] the following result being a starting point of our investigation here.

**Proposition 3.2.** [31] Let \( Q \in L^1([0,1]; \mathbb{C}^{2 \times 2}) \) and let \( K^\pm \) be the kernels of the corresponding transformation operators from representation (3.2). Then there exist

\[
R = (R_{jk})^2_{j,k=1} \in X^0_{1,1}(\Omega; \mathbb{C}^{2 \times 2}) \cap X^0_{\infty,1}(\Omega; \mathbb{C}^{2 \times 2}) \quad \text{and} \quad P^\pm = \text{diag}(P^\pm_1, P^\pm_2) \in L^1([0,1]; \mathbb{C}^{2 \times 2}),
\]

such that

\[
K^\pm(x,t) = R(x,t) + P^\pm(x-t) + \int_t^x R(x,s)P^\pm(s-t)ds, \quad 0 \leq t \leq x \leq 1.
\]

Moreover, \( R(\cdot, \cdot) \) is a unique solution of the following system of integral equations for \( 0 \leq t \leq x \leq 1, \)

\[
R_{kk}(x,t) = -\frac{i}{a_k} \int_{x-t}^x Q_{kj}(\xi) R_{jk}(\xi, \xi - x + t) d\xi, \quad j = 2/k, \quad k \in \{1,2\},
\]

\[
R_{jk}(x,t) = \frac{i}{a_k - a_j} Q_{jk}(\alpha_k x + \alpha_j t) - \frac{i}{a_j} \int_{\alpha_k x + \alpha_j t}^x Q_{jk}(\xi) R_{kk}(\xi, \gamma_k (\xi - x) + t) d\xi, \quad j = 2/k, \quad k \in \{1,2\}.
\]

Here and in what follows the following notations are used:

\[
a_k := b_k^{-1}, \quad \gamma_k := \frac{a_k}{a_j} = \frac{b_j}{b_k}, \quad \alpha_k := \frac{a_k}{a_k - a_j} = \frac{b_j}{b_j - b_k}, \quad j = 2/k, \quad k \in \{1,2\}.
\]

Note that \( \alpha_1 + \alpha_2 = 1 \). Since \( b_1 < 0 < b_2 \), it is clear that \( \alpha_k \in (0,1) \) and \( \gamma_k < 0, k \in \{1,2\} \).

Note also that for smooth \( Q(\in C^1([0,1]; \mathbb{C}^{2 \times 2})) \) system (3.10)-(3.11) is equivalent to the system (3.4)-(3.5) and denote by \( \tilde{e}_\pm(\cdot, \lambda) \) the solutions of the system (3.11) satisfying the initial conditions \( \tilde{e}_\pm(0,0) = (\frac{1}{\pm 1}) \). In accordance with Theorem 3.1 it admits the representation

\[
\tilde{e}_\pm(x,\lambda) = (I + \tilde{K}^\pm) e_0^\pm(x,\lambda) = e_0^\pm(x,\lambda) + \int_0^x \tilde{K}^\pm(x,t) e_0^\pm(t,\lambda) dt.
\]

The main result of this section shows the Lipshitz of the mapping \( Q \to K^\pm := K^\pm_Q \) on the balls in \( L^p([0,1]; \mathbb{C}^{2 \times 2}) \) and reads as follows.

**Theorem 3.3.** Let \( Q = \text{codiag}(Q_{12}, Q_{21}) \), \( \tilde{Q} = \text{codiag}(\tilde{Q}_{12}, \tilde{Q}_{21}) \in L^{2p,r}_p \) for some \( p \in [1, \infty) \) and \( r > 0 \). Let also \( K^\pm_Q \) and \( K^\pm_{\tilde{Q}} \) be the kernels of the corresponding transformation operators from representations (3.2) and (3.13). Then

\[
K^\pm := K^\pm_Q, \quad \tilde{K}^\pm := K^\pm_{\tilde{Q}} \in X^0_{1,p}(\Omega; \mathbb{C}^{2 \times 2}) \cap X^0_{\infty,p}(\Omega; \mathbb{C}^{2 \times 2}),
\]

and there exists a constant \( C = C(B,p,r) \) that does not depend on \( Q \) and \( \tilde{Q} \) such that the following estimate holds

\[
\|K^\pm - \tilde{K}^\pm\|_{X^0_{\infty,p}(\Omega; \mathbb{C}^{2 \times 2})} + \|K^\pm - \tilde{K}^\pm\|_{X^0_{1,p}(\Omega; \mathbb{C}^{2 \times 2})} \leq C \cdot \|Q - \tilde{Q}\|_p.
\]
Let $\tilde{R}(\cdot, \cdot)$ be the kernel satisfying the system of integral equations (3.10)–(3.11) with $\tilde{Q}(\cdot)$ in place of $Q(\cdot)$. Alongside definition (3.9) the kernels $\tilde{R}^\pm(\cdot, \cdot)$ and $\tilde{R}(\cdot, \cdot)$ are related by the similar equation
\[ \tilde{R}^\pm(x, t) = \tilde{R}(x, t) + \tilde{P}^\pm(x - t) + \int_t^x \tilde{R}(x, s)\tilde{P}^\pm(s - t) ds \]
with a certain diagonal matrix function $\tilde{P}^\pm = \text{diag}(\tilde{P}_1^\pm, \tilde{P}_2^\pm) \in L^1([0, 1]; C^{2\times2})$. For brevity we will systematically use the following notations below,
\[ \tilde{Q} := Q - \tilde{Q}, \quad \tilde{R}^\pm := K^\pm - \tilde{K}^\pm, \quad \tilde{R} := R - \tilde{R}, \quad \tilde{P}^\pm := P^\pm - \tilde{P}^\pm. \]

In the following two auxiliary results we show that the (non-linear) mappings
\[ \tilde{R}_{Q, \infty} : L^p([0, 1]; C^{2\times2}) \to X_{\infty,p}(\Omega; C^{2\times2}), \quad \tilde{R}_{Q, 1} : L^p([0, 1]; C^{2\times2}) \to X_{1,p}(\Omega; C^{2\times2}), \quad \tilde{R}_{Q, 1} : L^p([0, 1]; C^{2\times2}) \to R = R_Q, \]
are uniformly Lipshitz on each ball of the radius $r$ in $L^p([0, 1]; C^{2\times2})$.

**Lemma 3.4.** Let $Q, \tilde{Q} \in U_{p,r}^{2\times2}$ for some $p \in [1, \infty)$ and $r > 0$. Let also $R = R_Q$ be the solution of problem (3.10)–(3.11) and let $\tilde{R} = \tilde{R}_Q$ be the solution of problem (3.10)–(3.11) with $\tilde{Q}$ in place of $Q$. Then, $R, \tilde{R} \in X_{\infty,p}(\Omega; C^{2\times2})$ and there exists a constant $C(\infty) = C(\infty)(B, p, r)$ not dependent on $Q$, $\tilde{Q}$, and such that the following uniform estimate holds
\[ \|R - \tilde{R}\|_{X_{\infty,p}(\Omega; C^{2\times2})} = \|R_Q - \tilde{R}_Q\|_{X_{\infty,p}(\Omega; C^{2\times2})} \leq C(\infty)\|Q - \tilde{Q}\|_p. \]

In particular, kernels $\tilde{R}_Q$ are uniformly bounded on each $L^p$-ball, i.e.
\[ \|R_Q\|_{X_{\infty,p}(\Omega; C^{2\times2})} \leq C(\infty)\|Q\|_p \quad \text{for} \quad Q \in U_{p,r}^{2\times2}. \]

**Proof.** We put
\[ \tilde{R}_{jk} = R_{jk} - \tilde{R}_{jk}, \quad \tilde{Q}_{jk} = Q_{jk} - \tilde{Q}_{jk}, \quad j, k \in \{1, 2\}. \]
and
\[ \tilde{\mathcal{J}}_{jk}(p, x) := \int_0^x |\tilde{R}_{jk}(x, t)|^p dt, \quad \tilde{\mathcal{J}}_{jk}(p, x) := \int_0^x |R_{jk}(x, t)|^p dt, \quad j, k \in \{1, 2\}. \]
It is easily seen that for any $f \in L^p[0, 1] \subset L^1[0, 1]$ and $0 \leq t_1 \leq t_2 \leq 1$, one has due to Hölder’s inequality
\[ \left| \int_{t_1}^{t_2} f(\xi) d\xi \right|^p \leq (t_2 - t_1)^{p/p'} \int_{t_1}^{t_2} |f(\xi)|^p d\xi \leq \int_{t_1}^{t_2} |f(\xi)|^p d\xi, \quad 1/p' + 1/p = 1. \]

Writing down the difference $R_{11} - \tilde{R}_{11}$ as a sum of two terms containing $\tilde{Q}_{12}$ and $\tilde{R}_{21}$ as the factors, then evaluating each summand with the help of inequality (3.23), and making use of the change of variables $\xi = u$, $\xi - x + t = v$, we obtain
\[ \tilde{\mathcal{J}}_{11}(p, x) := \int_0^x |\tilde{R}_{11}(x, t)|^p dt \leq 2^{p-1}|b_1|^p \left( \int_0^x dt \int_{x-t}^x |\tilde{Q}_{12}(\xi)| \tilde{R}_{21}(\xi, \xi - x + t)|^p d\xi + \int_0^x dt \int_{x-t}^x |\tilde{Q}_{12}(\xi)| \tilde{R}_{21}(\xi, \xi - x + t)|^p d\xi \right) \leq 2^{p-1}|b_1|^p \left( \int_0^x |\tilde{Q}_{12}(u)|^p du \int_0^u |R_{21}(u, v)|^p dv + \int_0^x |\tilde{Q}_{12}(u)|^p du \int_0^u |\tilde{R}_{21}(u, v)|^p dv \right) \leq 2^{p-1}|b_1|^p \int_0^x |\tilde{Q}_{12}(u)|^p \tilde{J}_{21}(p, u) du + 2^{p-1}|b_1|^p \int_0^x |\tilde{Q}_{12}(u)|^p \tilde{J}_{12}(p, u) du. \]
Similarly, it follows from (3.11) and (3.22)\n\n$$\mathcal{J}_{21}(p, x) = \int_0^x |\tilde{R}_{21}(x, t)|^p dt \leq \frac{3^{p-1}}{|a_1 - a_2|^p} \int_0^x |\tilde{Q}_{21}(\alpha_1 x + \alpha_2 t)|^p dt + \frac{3^{p-1}}{|a_2|^p} \int_0^x dt \int_{\alpha_1 x + \alpha_2 t}^x |\tilde{Q}_{21}(\xi)R_{11}(\xi, \gamma_1(\xi - x) + t)|^p d\xi + \frac{3^{p-1}}{|a_2|^p} \int_0^x dt \int_{\alpha_1 x + \alpha_2 t}^x |\tilde{Q}_{21}(\xi)\tilde{R}_{11}(\xi, \gamma_1(\xi - x) + t)|^p d\xi.$$ \n(3.25)\n
Making use the change of variables $\xi = u, \frac{a_1}{a_2}(\xi - x) + t = v$, in the two last integrals, noting that $\frac{a_1}{a_2} > 0$ and $\frac{a_1}{a_2}(u - x) > 0$, and setting\n\n$$C_1(B, p) := \max \left\{ \frac{3^{p-1}}{|a_1 - a_2|^{p-1}}, \frac{3^{p-1}}{|a_2|^p} \right\}$$ \n(3.26)\n
we obtain\n\n$$\mathcal{J}_{21}(p, x) \leq \frac{3^{p-1}}{|a_2|} \cdot |a_1 - a_2|^{p-1} \int_0^x |\tilde{Q}_{21}(u)|^p du + \frac{3^{p-1}}{|a_2|^p} \int_0^x |\tilde{Q}_{21}(u)|^p du \int_{\frac{a_1}{a_2} u - x}^u |R_{11}(u, v)|^p dv + C_1(B, p) \int_0^x |\tilde{Q}_{21}(u)|^p du \int_{\frac{a_1}{a_2} u - x}^u |\tilde{R}_{11}(u, v)|^p dv \leq C_1(B, p) \int_0^x |\tilde{Q}_{21}(u)|^p du,$$ \n(3.27)\n
Now we are ready to evaluate the functions $R_{21}(\cdot, \cdot)$ and $R_{21}(\cdot, \cdot)$ in $X_{\infty, p}(\Omega)$-norm. We show that these norms depend only on radius $r$ of the ball in $L^p([0, 1]; \mathbb{C}^{2 \times 2})$ contained $Q$ and do not depend on $Q$ itself. This fact plays a crucial role in what follows.

If $Q = 0$ the estimates (3.27) and (3.24) are simplified to\n\n$$\mathcal{J}_{21}(p, x) = \int_0^x |R_{21}(x, t)|^p dt \leq C_1(B, p) \left\{ \int_0^x |Q_{21}(t)|^p dt + \int_0^x \mathcal{J}_{11}(p, u) Q_{21}(u)|^p du \right\}$$ \n(3.28)\n
and\n\n$$\mathcal{J}_{11}(p, x) = \int_0^x |R_{11}(x, t)|^p dt \leq |b_1|^p \int_0^x |Q_{12}(u)|^p \mathcal{J}_{21}(p, u) du,$$ \n(3.29)\n
respectively. Inserting the second estimate into (3.28) one gets\n\n$$\mathcal{J}_{21}(p, x) \leq C_1(B, p) \left\{ \|Q_{21}\|_{L^p[0,1]} + |b_1|^p \int_0^x |Q_{21}(u)|^p du \int_0^u |Q_{12}(s)|^p \mathcal{J}_{21}(p, s) ds \right\} \leq C_1(B, p) \left\{ \|Q_{21}\|_{L^p[0,1]} + |b_1|^p \int_0^1 |Q_{21}(u)|^p du \int_0^x |Q_{12}(s)|^p \mathcal{J}_{21}(p, s) ds \right\}$$ \n(3.30)\n
Now the Grönwall’s inequality applies and leads to the first desired estimate\n\n$$\mathcal{J}_{21}(p, x) \leq C_1(B, p) \|Q_{21}\|_{L^p[0,1]} \cdot \exp \left\{ |b_1|^p \|Q_{21}\|_{L^p[0,1]} \int_0^x |Q_{12}(s)|^p ds \right\}$$ \n(3.31)\n
In turn, inserting this estimate in (3.24) yields the second desired estimate\n\n$$\mathcal{J}_{11}(p, x) \leq |b_1|^p C_1(B, p) \|Q_{21}\|_{L^p[0,1]} \cdot \exp \left\{ |b_1|^p \|Q_{21}\|_{L^p[0,1]} \int_0^x |Q_{12}(s)|^p ds \right\} du \leq C_1(B, p) \exp \left\{ |b_1|^p \|Q_{21}\|_{L^p[0,1]} \int_0^x |Q_{12}(s)|^p ds \right\} du - 1$$ \n(3.32)
For potential matrices $Q$ belonging the ball of $L^p([0, 1]; C^2)$ of the radius $r$ these estimates imply
\[
\|R_{11}(\cdot, \cdot)\|_{X_{p, p}^2(\Omega)}^p \leq C_1(B, p) \left( \exp \left( |b_1|^p \cdot r^{2p} \right) - 1 \right) \leq |b_1|^p r^{2p} C_1(B, p) \exp \left( |b_1|^p \cdot r^{2p} \right),
\]
\[
\|R_{21}(\cdot, \cdot)\|_{X_{p, p}^2(\Omega)}^p \leq C_1(B, p)r^p \exp \left( |b_1|^p \cdot r^{2p} \right).
\]
We set
\[
C_1(B, p, r) := C_1(B, p)r^p \exp \left( |b_1|^p \cdot r^{2p} \right) \cdot \max \{1, |b_1|^p r^p\}.
\]
Next using estimates (3.33) we return to evaluation of the functions $\tilde{\mathcal{H}}_{11}(p, \cdot)$ and $\tilde{\mathcal{H}}_{21}(p, \cdot)$. Combining estimate (3.24) with estimate (3.33) (for $\tilde{\mathcal{H}}_{21}(p, \cdot)$) and using definition (3.34) yields
\[
\tilde{\mathcal{H}}_{11}(p, x) \leq 2^{p-1}|b_1|^p C_1(p, r)\|\tilde{Q}_{12}\|_{L^p}^p + 2^{p-1}|b_1|^p \int_0^x \tilde{\mathcal{H}}_{21}(p, t)|\tilde{Q}_{12}(t)|^p dt.
\]
Inserting this inequality into (3.27) and taking into account estimate (3.33) (for $\tilde{\mathcal{H}}_{11}(p, \cdot)$), using definition (3.34), and noting that $\|\tilde{Q}_{12}\|_{L^p} \leq r$ we derive
\[
\tilde{\mathcal{H}}_{21}(p, x) \leq C_1(B, p) (1 + C_1(p, r)) \int_0^x |\tilde{Q}_{21}(u)|^p du + C_1(B, p) \int_0^x \tilde{\mathcal{H}}_{11}(p, u)|\tilde{Q}_{21}(u)|^p du
\]
\[
\leq C_1(B, p) (1 + C_1(p, r)) \|\tilde{Q}_{21}\|_{L^p}^p + 2^{p-1}|b_1|^p C_1(p, r)\|\tilde{Q}_{12}\|_{L^p}^p \cdot \|\tilde{Q}_{21}\|_{L^p}^p
\]
\[
+ 2^{p-1}|b_1|^p C_1(B, p) \int_0^x |\tilde{Q}_{21}(u)|^p du \int_0^u \tilde{\mathcal{H}}_{21}(p, t)|\tilde{Q}_{12}(t)|^p dt
\]
\[
\leq C_2(B, p, r) \left( \|\tilde{Q}_{21}\|_{L^p} + \|\tilde{Q}_{12}\|_{L^p}^p \right) + 2^{p-1}|b_1|^p C_1(B, p) \int_0^x \tilde{\mathcal{H}}_{21}(p, t)|\tilde{Q}_{12}(t)|^p dt \int_0^x |\tilde{Q}_{21}(u)|^p du
\]
\[
\leq C_2(B, p, r) \left( \|\tilde{Q}_{21}\|_{L^p} + \|\tilde{Q}_{12}\|_{L^p}^p \right) + 2^{p-1}|b_1|^p C_1(B, p) \cdot \|\tilde{Q}_{21}\|_{L^p} \int_0^x \tilde{\mathcal{H}}_{21}(p, t)|\tilde{Q}_{12}(t)|^p dt,
\]
where
\[
C_2(B, p, r) = \max \left\{ C_1(B, p)(1 + C_1(p, r)), 2^{p-1}|b_1|^p C_1(p, r)r \right\}.
\]
Applying Grönwall’s inequality to (3.36) and noting that $\|\tilde{Q}_{21}\|_{L^p}, \|\tilde{Q}_{12}\|_{L^p} \leq r$, one easily deduces
\[
\tilde{\mathcal{H}}_{21}(p, x) \leq C_2(B, p, r) \left( \|\tilde{Q}_{12}\|_{L^p} + \|\tilde{Q}_{21}\|_{L^p} \right) \exp \left( 2^{p-1}|b_1|^p C_1(B, p) \cdot \|\tilde{Q}_{21}\|_{L^p} \int_0^x |\tilde{Q}_{12}(t)|^p dt \right)
\]
\[
\leq C_2(B, p, r) \left( \|\tilde{Q}_{12}\|_{L^p} + \|\tilde{Q}_{21}\|_{L^p} \right) \exp \left( 2^{p-1}|b_1|^p C_1(B, p) \cdot r^{2p} \right).
\]
Implementing this inequality into (3.35) we arrive at the estimate
\[
\tilde{\mathcal{H}}_{11}(p, x) \leq \left( \|\tilde{Q}_{12}\|_{L^p} + \|\tilde{Q}_{21}\|_{L^p} \right) \left( C_1(p, r) + C_2(B, p, r) \|\tilde{Q}_{21}\|_{L^p} \exp \left( 2^{p-1}|b_1|^p C_1(B, p) \cdot r^{2p} \right) \right)
\]
\[
\leq 2^{p-1}|b_1|^p \|\tilde{Q}\|_{L^p([0, 1]; C^2 \times 2)} \left( C_1(p, r) + C_2(B, p, r)r^p \cdot \exp \left( 2^{p-1}|b_1|^p C_1(B, p) \cdot r^{2p} \right) \right).
\]
Similar reasoning leads to similar estimates for $\tilde{\mathcal{H}}_{12}$ and $\tilde{\mathcal{H}}_{22}$. Combining these estimates with (3.37) and (3.38) we arrive at (3.24). □

**Lemma 3.5.** Let $Q, \tilde{Q} \in U_{p, r}^{2 \times 2}$ for some $p \in [1, \infty)$ and $r > 0$. Assume also that $R = R_Q$ and $\tilde{R} = R_{\tilde{Q}}$ are the (unique) solutions of the problem (3.10) – (3.11) with $Q$ and $\tilde{Q}$ in place of $Q$, respectively. Then $R, \tilde{R} \in X_{p, r}^1(\Omega; C^2 \times 2)$ and there exists a constant $C(1) = C(1)(B, p, r)$ not dependent on $Q$ and $\tilde{Q}$ and such that the following uniform estimate holds
\[
\|R - \tilde{R}\|_{X_{p, r}^1(\Omega; C^2 \times 2)} = \|R_Q - R_{\tilde{Q}}\|_{X_{p, r}^1(\Omega; C^2 \times 2)} \leq C(1)\|Q - \tilde{Q}\|_{p}.
\]
In particular, one has the uniform estimate: $\|R_Q\|_{X_{p, r}^1(\Omega; C^2 \times 2)} \leq C(1)\|Q\|_p$ for $Q \in U_{p, r}^{2 \times 2}$. □
Proof. (i) As in the proof of Lemma 5.4 we put (see (5.21)) \( \tilde{R}_{jk} = R_{jk} - \tilde{R}_{jk} \) and \( \tilde{Q}_{jk} = Q_{jk} - \tilde{Q}_{jk}, j, k \in \{1, 2\} \). First we prove estimate (5.39) for the case \( j \neq k \). Since \( \alpha_1, \alpha_2 \in (0, 1) \) and \( \alpha_1 + \alpha_2 = 1 \), it follows that

\[
0 \leq t \leq \alpha_k + \alpha_j t \leq 1. \tag{3.40}
\]

Making a change of variable \( u = \alpha_k x + \alpha_j t \) and taking into account (3.40) and (3.29), one has

\[
\frac{1}{\alpha_k - \alpha_j} \int_t^1 \tilde{Q}_{jk}(\alpha_k x + \alpha_j t) \, dx = \left[ \frac{1}{\alpha_k} \int_t^1 \tilde{Q}_{jk}(u) \, du \right]^{\alpha_k + \alpha_j t} \tilde{Q}_{jk}(u) \, du \leq C_b \int_t^1 \left( \tilde{Q}_{jk}(u) \right)^{\alpha_k + \alpha_j t} \tilde{Q}_{jk}(u) \, du, \tag{3.41}
\]

where \( C_b := \max\{|b_1|, |b_2|\} = 1/\min\{|a_1|, |a_2|\} \). Further, note that alongside \( R_{jk} \) the kernel \( \tilde{R}_{jk} \) satisfies equation (3.11) with \( \tilde{Q}_{jk} \) in place of \( Q_{jk} \), \( j, k \in \{1, 2\} \). Taking difference of these equations, then integrating the result with respect to \( x \in [t, 1] \), making use of boundedness of \( u = \xi, v = (\xi - x) \gamma_{ik} + t = (\xi - x) a_{ij} + t \), and using inequalities (3.41), (3.23) we obtain

\[
3^{1-p}C_b^{-p} \int_t^1 |\tilde{R}_{jk}(x, t)|^p \, dx = 3^{1-p}C_b^{-p} \int_t^1 |R_{jk}(x, t) - \tilde{R}_{jk}(x, t)|^p \, dx
\]

\[
\leq \int_t^1 |\tilde{Q}_{jk}(u)|^p \, du + \int_t^1 \left( \int_{\alpha_k x + \alpha_j t}^x |\tilde{Q}_{jk}(\xi)| \tilde{R}_{kk}(\xi, \gamma_k(\xi - x) + t) \right)^p \, d\xi
\]

\[
+ \int_t^1 \left( \int_{\alpha_k x + \alpha_j t}^x |\tilde{Q}_{jk}(\xi)| \tilde{R}_{kk}(\xi, \gamma_k(\xi - x) + t) \right)^p \, d\xi
\]

\[
= \int_t^1 |\tilde{Q}_{jk}(u)|^p \, du + \int_t^1 \left( \int_{\alpha_k x + \alpha_j t}^x |\tilde{Q}_{jk}(u)| \tilde{R}_{kk}(u, v) \right)^p \, dv
\]

\[
+ \int_t^1 \left( \int_{\alpha_k x + \alpha_j t}^x |\tilde{Q}_{jk}(u)| \tilde{R}_{kk}(u, v) \right)^p \, dv
\]

\[
\leq \|\tilde{Q}_{jk}\|_{p,1}^p + \|\tilde{Q}_{jk}\|_{p,1}^p \cdot \|\tilde{R}_{kk}\|_{p,1} \cdot \|\tilde{R}_{kk}\|_{p,1} \cdot \|\tilde{Q}_{jk}\|_{p,1}^p + \|\tilde{Q}_{jk}\|_{p,1}^p \cdot \|\tilde{R}_{kk}\|_{p,1}^p. \tag{3.42}
\]

Here we use simple inequalities \( \alpha_k + \alpha_j t \leq 1 \) and \( (v - t)^{\alpha_k} + 1 \leq 1 \). The latter holds because \( t \leq v \) and \( \alpha_j \alpha_k < 0 \). It follows from (3.42) with account of definitions (2.11) and (2.2) that

\[
\|\tilde{R}_{jk}\|_{X_{1,p}} \leq 3C_b \left( \|\tilde{Q}_{jk}\|_p + \|\tilde{Q}_{jk}\|_p \cdot \|\tilde{R}_{kk}\|_{X_{\infty,p}(\Omega)} + \|\tilde{Q}_{jk}\|_p \cdot \|\tilde{R}_{kk}\|_{X_{\infty,p}(\Omega)} \right), \quad j \neq k. \tag{3.43}
\]

Further, in accordance with Lemma 5.4 the following estimates hold

\[
\|\tilde{R}_{kk}\|_{X_{\infty,p}(\Omega)} \leq C(\infty) \|\tilde{Q}\|_p \quad \text{and} \quad \|\tilde{R}_{kk}\|_{X_{\infty,p}(\Omega)} \leq C(\infty) \|Q\|_p \leq C(\infty) r,
\]

with the constant \( C(\infty) = C(\infty)(r, B, p) \) not dependent on \( Q \) and \( \tilde{Q} \) running through the ball \( U_{p,2}^{2 \times 2} \). Inserting these estimates into (3.43) yields

\[
\|\tilde{R}_{jk}\|_{X_{1,p}} \leq 3C_b \left( \|\tilde{Q}_{jk}\|_p (1 + \|\tilde{R}_{kk}\|_{X_{\infty,p}}) + r \|\tilde{R}_{kk}\|_{X_{\infty,p}} \right) \leq 3C_b \left( 1 + 2rC(\infty) \right) \|\tilde{Q}\|_p, \quad j \neq k. \tag{3.44}
\]
(ii) Going over to the case \( j = k \) we start with equation (3.10) and similar equation for \( \tilde{R}_{kk} \) which holds with \( \tilde{Q}_{jk} \) in place of \( Q_{jk} \), \( j, k \in \{1, 2\} \). Taking difference of (3.10) and this equation, then integrating the difference with respect to \( x \in [t, 1] \), and then making use the change of variables \( \xi = u, \xi - x + t = v \), applying inequality (3.23) one obtains

\[
2^{1-p}|aj|^p \int_t^1 |\tilde{R}_{jj}(x, t)|^p dx = 2^{1-p}|aj|^p \int_t^1 |R_{jj}(x, t) - \tilde{R}_{jj}(x, t)|^p dx
\]

\[
\leq \int_t^1 dx \int_{x-t}^x |\tilde{Q}_{jk}(\xi, \xi - x + t)|^p d\xi + \int_t^1 dx \int_{x-t}^x |\tilde{Q}_{jk}(\xi)R_{kk}(\xi, \xi - x + t)|^p d\xi
\]

\[
= \int_0^t dv \int_0^{v-t+1} |\tilde{Q}_{jk}(u)\tilde{R}_{kk}(u, v)|^p du + \int_0^t dv \int_0^{v-t+1} |\tilde{Q}_{jk}(u)R_{kk}(u, v)|^p du
\]

\[
\leq \int_0^1 dv \int_0^1 |\tilde{Q}_{jk}(u)\tilde{R}_{kk}(u, v)|^p du + \int_0^1 dv \int_0^1 |\tilde{Q}_{jk}(u)R_{kk}(u, v)|^p du
\]

\[
= \int_0^1 |\tilde{Q}_{jk}(u)|^p du \int_0^1 v |\tilde{R}_{kk}(u)\tilde{R}_{kk}(u, v)|^p dv + \int_0^1 |\tilde{Q}_{jk}(u)|^p du \int_0^1 v |\tilde{R}_{kk}(u)R_{kk}(u, v)|^p dv.
\]  \quad (3.45)

It follows with account of definitions (2.1) and (2.2) that

\[
\|R_{jj} - \tilde{R}_{jj}\|_{X_{1, p}(\Omega)} \leq 2|b_j| \left( \|\tilde{Q}_{jk}\|_p \cdot \|\tilde{R}_{kk}\|_{X_{\infty, p}(\Omega)} + \|\tilde{Q}_{jk}\|_p \cdot \|R_{kk}\|_{X_{\infty, p}(\Omega)} \right), \quad j \in \{1, 2\}.
\]  \quad (3.46)

Further, in accordance with Lemma 3.4 the following estimates hold

\[
\|\tilde{R}_{kk}\|_{X_{\infty, p}(\Omega)} \leq C(\infty)\|\tilde{Q}\|_p \quad \text{and} \quad \|R_{kk}\|_{X_{\infty, p}} \leq C(\infty)\|Q\|_p \leq C(\infty)r,
\]

with the constant \( C(\infty) = C(\infty)(r, b_1, b_2, p) \) not dependent on \( Q \) and \( \tilde{Q} \) from the ball in \( L^p \) of the radius \( r \). Inserting these estimates into (3.45) yields

\[
\|\tilde{R}_{jj}\|_{X_{1, p}(\Omega)} \leq 2|b_j| \left( r\|\tilde{R}_{kk}\|_{X_{\infty, p}} + \|\tilde{Q}\|_p \cdot \|R_{kk}\|_{X_{\infty, p}} \right) \leq 4C_nrC(\infty) \cdot \|\tilde{Q}\|_p, \quad j \in \{1, 2\}.
\]  \quad (3.47)

Combining (3.47) with (3.47) one arrives at the desired estimate (3.9).

Combining Lemma 3.4 with Lemma 3.5 we arrive at a part of the following result.

**Proposition 3.6.** Let \( Q = \text{codig}(Q_{12}, Q_{21}) \), \( \tilde{Q} = \text{codig}(\tilde{Q}_{12}, \tilde{Q}_{21}) \in \mathbb{T}^{2 \times 2} \) for some \( p \geq 1 \) and \( r > 0 \). Then:

(i) The system of integral equations (3.10) - (3.11) has the unique solution \( R = R_Q = (R_{jk})^2_{j,k=1} \) belonging to \( X^1_{1, p}(\Omega; \mathbb{C}^{2 \times 2}) \cap X^0_{\infty, p}(\Omega; \mathbb{C}^{2 \times 2}) \). Moreover, this solution is unique in the space \( X^0_{\infty, p}(\Omega; \mathbb{C}^{2 \times 2}) \).

(ii) Let also \( \tilde{R} = R_{\tilde{Q}} \) be the (unique) solution of the problem (3.10) - (3.11) with \( \tilde{Q} \) in place of \( Q \). Then \( \tilde{R} \in X^1_{1, p}(\Omega; \mathbb{C}^{2 \times 2}) \cap X^0_{\infty, p}(\Omega; \mathbb{C}^{2 \times 2}) \) and the following uniform estimate holds

\[
\|R - \tilde{R}\|_{X_{1, p}(\Omega; \mathbb{C}^{2 \times 2})} + \|R - \tilde{R}\|_{X_{\infty, p}(\Omega; \mathbb{C}^{2 \times 2})} \leq C(1, \infty)\|Q - \tilde{Q}\|_p,
\]  \quad (3.48)

where the constant \( C(1, \infty) := C(1)(B, p, r) + C(\infty)(B, p, r) \) does not depend on \( Q, \tilde{Q} \in \mathbb{T}^{2 \times 2} \).

(iii) The following uniform estimate holds (with \( b := \max\{\|b_1\|, \|b_2\|\} \))

\[
\|R_{Q}(\cdot, \cdot)\|_{X_{1, p}(\Omega; \mathbb{C}^{2 \times 2})} + \|R_{\tilde{Q}}(\cdot, \cdot)\|_{X_{1, p}(\Omega; \mathbb{C}^{2 \times 2})} \leq C_1(B, p, r) r^p \cdot \exp \left( \|b\|_p \cdot r^{2p} \right) \cdot \max\{1, \|b\|_r^p\}.
\]

(iv) The operator

\[
\mathcal{R}_Q : \left( \frac{f_1}{f_2} \right) = \int_0^x R_Q(x, t)\left( \frac{f_1(t)}{f_2(t)} \right) dt = \int_0^x R_{11}(x, t) R_{12}(x, t) R_{21}(x, t) R_{22}(x, t) \left( \frac{f_1(t)}{f_2(t)} \right) dt
\]  \quad (3.49)

is a Volterra operator in the scale \( L^1((0, 1], \mathbb{C}^2) \), \( s \in [1, \infty] \). Moreover, there exists a constant \( C_3 = C_3(B, p, r) > 0 \) not dependent on \( s \in [1, \infty] \) and \( Q, \tilde{Q} \in \mathbb{T}^{2 \times 2} \), and such that the following uniform estimate holds

\[
\|(I + \mathcal{R}_Q)^{-1} - (I + \mathcal{R}_{\tilde{Q}})^{-1}\|_{s \to s} \leq C_3\|\tilde{Q} - Q\|_p.
\]  \quad (3.50)
Proof. (i), (ii). These statements are immediate by combining Lemma 3.4 with Lemma 3.5.

(iii) This statement is immediate from estimates (3.35) and definition (3.34).

(iv) In accordance with (i) the kernel $R = (R_{jk})_{j,k=1}^{p} \in X_{C}^{0}(\Omega; C^{2x2}) \cap X_{p}^{1}(\Omega; C^{2x2})$. Therefore, Lemma 2.2(ii) ensures that $\mathcal{R} := \mathcal{R}_{Q}$ is Volterra operator in each $L^{p}(\Omega; C^{2})$ and $(I + \mathcal{R})^{-1} = I + S$, where $(Sf)(x) = \int_{0}^{\infty} S(x,t)f(t)\,dt$, $f \in L^{1}(\Omega; C^{2})$, and $S$ is also a Volterra operator, where $S \in X_{\infty}^{0}(\Omega; C^{2x2}) \cap X_{1}^{1}(\Omega; C^{2x2})$.

Let us first assure that $Q \in C^{1}(\Omega; C^{2x2})$. Now the kernel $R_{Q} = (R_{jk})_{j,k=1}^{p} \in C^{1}(\Omega; C^{2x2})$, hence so is the kernel $S = S_{Q}$, i.e. $(S_{jk})_{j,k=1}^{p} \in C^{1}(\Omega; C^{2x2})$. Moreover, according to 3.9 the operator $I + \mathcal{R}$ intertwines the operators $L_{0}(Q)$ and $L_{0}(0)$ generated by equation (3.11) with potential matrices $Q$ and $0$, respectively, subject to the Cauchy boundary condition $y(0) = 0$, i.e. $L_{0}(Q)(I + \mathcal{R}) = (I + \mathcal{R})L_{0}(0)$. It follows that $(I + S)L_{0}(Q) = L_{0}(0)(I + S)$. Starting with this equation and repeating the proof of 3.9, one rewrites it as an equation on the matrix kernel $S = (S_{jk})_{j,k=1}^{p}$ of the operator $S$, similar to (3.4)–(3.8), which in turn leads to the system of integral equations similar to (3.10)–(3.14).

Setting $J_{jk}(t) := \int_{0}^{t} |S_{jk}(x,t)|^{p}dx$, $p \in [1, \infty)$, and following the reasoning of Lemma 3.4 with $X_{1,p}$-norm instead of $X_{\infty,p}$-norm one derivates that for $Q \in U_{2x2}^{p} \cap C^{1}(\Omega; C^{2x2})$ the following uniform estimate holds

$$\|S\|_{X_{1,1}(\Omega; C^{2x2})} \leq \|S\|_{X_{1,p}(\Omega; C^{2x2})} \leq C_{1}\|Q\|_{p}. \quad (3.51)$$

Performing similar computatons as in Lemma 3.5 and using estimate (3.51) we arrive at the similar estimate

$$\|S\|_{X_{\infty,1}(\Omega; C^{2x2})} \leq \|S\|_{X_{\infty,p}(\Omega; C^{2x2})} \leq C_{\infty}\|Q\|_{p}. \quad (3.52)$$

As usual, both constants $C_{1} = C_{1}(B,p,r)$ and $C_{\infty} = C_{\infty}(B,p,r)$ do not depend on $Q$.

Going over to the case $Q = \text{codiag}(Q_{12}, Q_{21}) \in L^{p}(\Omega; C^{2x2})$, choose a sequence $Q_{n} = \text{codiag}(Q_{12,n}, Q_{21,n}) \in C^{1}(\Omega; C^{2x2})$ approaching $Q$ in $L^{p}(\Omega; C^{2x2})$-norm. So, we have estimates (3.51)–(3.52) with $Q_{n}$ instead of $Q$. Passing in these inequalities as the limit that $n \to \infty$ we arrive at the required estimates (3.51)–(3.52) with $Q \in L^{p}(\Omega; C^{2x2})$. Using relations (2.17) one rewrites these estimates in the operator form

$$\|I + \mathcal{R}\|_{1 \to 1} = \|I + S\|_{1 \to 1} \leq 1 + C_{1}\|Q\|_{p}, \quad (3.53)$$

$$\|I + \mathcal{R}\|_{\infty \to \infty} = \|I + S\|_{\infty \to \infty} \leq 1 + C_{\infty}\|Q\|_{p}. \quad (3.54)$$

Combining (3.53)-(3.54) with estimate (2.19) from Lemma 2.2 one has

$$\|I + \mathcal{R}\|_{s \to s} = \|I + S\|_{s \to s} \leq 1 + C_{1}^{1/s}C_{1}^{-1/s}\|Q\|_{p} \leq C_{2}, \quad s \in [1, \infty), \quad (3.55)$$

$$\|\mathcal{R}_{Q} - \mathcal{R}_{Q}\|_{s \to s} \leq C_{1}^{s}C_{1}^{-s/2}\|Q - \tilde{Q}\|_{p}, \quad s \in [1, \infty), \quad (3.56)$$

where $C_{2} := 1 + r \max\{C_{1}, C_{\infty}\}$. Combining (3.55)–(3.56) we arrive at

$$\|I + \mathcal{R}_{Q}^{-1} - (I + \mathcal{R}_{Q})^{-1}\|_{s \to s} = \|\mathcal{R}_{Q}^{-1}(\mathcal{R}_{Q} - \mathcal{R}_{Q})(I + \mathcal{R}_{Q})^{-1}\|_{s \to s} \leq \|I + \mathcal{R}_{Q}^{-1}\|_{s \to s}\|\mathcal{R}_{Q} - \mathcal{R}_{Q}\|_{s \to s} \leq C_{2}^{2}C_{1}^{1/s}C_{1}^{-s/2}\|Q - \tilde{Q}\|_{p}. \quad (3.57)$$

Setting $C_{3} := C_{2}^{2}\max\{C_{1}, C_{\infty}\}$ we arrive at (3.50).

Proof of Theorem 3.3. Let $Q = \text{codiag}(Q_{12}, Q_{21}), \tilde{Q} = \text{codiag}(\tilde{Q}_{12}, \tilde{Q}_{21})$ and let $Q, \tilde{Q} \in U_{p,2}^{2x2}$ for some $p \in [1, \infty)$ and $r > 0$. In accordance with Theorem 3.1 for any $Q = \text{codiag}(Q_{12}, Q_{21}) \in L^{1}(\Omega; C^{2x2})$ representation (3.2)–(3.3) holds with the matrix kernel $K_{\pm} \in X_{\infty}^{0}(\Omega; C^{2x2}) \cap X_{1}^{1}(\Omega; C^{2x2})$. To evaluate the difference $K_{\pm} - \tilde{K}_{\pm}$ in respective norms we start with representations (3.9) and (3.10) for $K_{\pm}$ and $\tilde{K}_{\pm}$, respectively.

First, one should find $P_{\pm} = \text{diag}(P_{1}^{\pm}, P_{2}^{\pm}) \in L^{p}(\Omega; C^{2x2})$ so that $K_{\pm}$ satisfies condition (3.6), i.e.

$$a_{1}K_{j_{1}}^{\pm}(x,0) \pm a_{2}K_{j_{2}}^{\pm}(x,0) = 0, \quad j \in \{1,2\}. \quad (3.58)$$

Inserting representation (3.9) for $K_{\pm}$ in these relations leads to the following system of Volterra type integral equations

$$\begin{cases}
    a_{1}P_{1}^{\pm}(x) + \int_{0}^{\infty} [a_{1}R_{11}(x,t)P_{1}^{\pm}(t) \pm a_{2}R_{12}(x,t)P_{2}^{\pm}(t)]dt = \mp a_{2}R_{12}(x,0) =: g_{1}^{\pm}(x), \\
    \pm a_{2}P_{2}^{\pm}(x) + \int_{0}^{\infty} [a_{1}R_{12}(x,t)P_{1}^{\pm}(t) \pm a_{2}R_{22}(x,t)P_{2}^{\pm}(t)]dt = -a_{1}R_{21}(x,0) =: g_{2}^{\pm}(x),
\end{cases} \quad (3.59)$$

where $g_{1}^{\pm}(x), g_{2}^{\pm}(x)$ are as defined in (3.9).
By Lemma 3.5, \( R = R_Q \in X^0_{1,p}(\Omega; \mathbb{C}^{2 \times 2}) \). Therefore, Lemma 2.1 applies and ensures that the trace map \( i_{0,1} : X^0_{1,p}(\Omega) \to L^p[0,1], \) \( i_{0,1}(N(x,t)) := N(x,0) \), is a contraction, hence the functions \( R_{jk}(x,0), j, k \in \{1, 2\} \), are well defined and \( g^+_k(\cdot) \in L^p[0,1], j \in \{1, 2\} \).

On the other hand, by Proposition 3.6(iv), the operator \( R \) of the form \( (3.39) \) is a Volterra operator in \( L^p([0,1]; \mathbb{C}^{2 \times 2}) \). Therefore, system \( (3.59) \), being a second kind system of Volterra equations in \( L^p([0,1]; \mathbb{C}^2) \) with respect to \( \text{col}\{a_1 P_1^k(\cdot), \pm a_2 P_2^k(\cdot)\} \), has the unique solution in \( L^p([0,1]; \mathbb{C}^2) \).

Similar conclusion with respect to \( \bar{R} = R_Q \) and \( \bar{g}^\pm(x) := \text{col}(\bar{g}^+_1, \bar{g}^+_2) := \text{col}(\pm a_2 \bar{R}_{12}(x,0), -a_1 \bar{R}_{21}(x,0)) \).

Taking the difference of \( (3.9) \) and \( (3.10) \), then taking \( p \)-th power with account of estimate \( (3.23) \), and integrating the obtained inequality with respect to \( t \in [0,x] \) one gets

\[
4^{-p+1} \int_0^x |K^\pm(x,t) - \bar{K}^\pm(x,t)|^p dt \leq \int_0^x |R(x,t) - \bar{R}(x,t)|^p dt + \int_0^x |P^\pm(t) - \bar{P}^\pm(t)|^p dt
\]

\[
+ \int_0^x |R(x,s) - \bar{R}(x,s)|^p ds \int_0^s |\bar{P}^\pm(t)|^p dt + \int_0^x |R(x,s)|^p ds \int_0^s |P^\pm(s-t) - \bar{P}^\pm(s-t)|^p dt
\]

\[
\leq \|R - \bar{R}\|_{L^p([0,x]; \mathbb{C}^{2 \times 2})} \left( 1 + \|\bar{P}^\pm\|_{L^p}\right) + \|P^\pm - \bar{P}^\pm\|_{L^p} \left( 1 + \|R\|_{L^p([0,x]; \mathbb{C}^2)}\right).
\]  

(3.60)

Combining Lemma 3.5 (see 3.39) with Lemma 2.1 yields the following estimate

\[
\|g^\pm - \bar{g}^\pm\|_{L^p([0,x]; \mathbb{C}^2)} \leq \|g^+_1 - \bar{g}^+_1\|_{L^p[0,1]} + \|g^+_2 - \bar{g}^+_2\|_{L^p[0,1]} = |a_2| \cdot \|R_{12}(x,0) - \bar{R}_{12}(x,0)\|_{L^p[0,1]} + |a_1| \cdot \|R_{21}(x,0) - \bar{R}_{21}(x,0)\|_{L^p[0,1]}
\]

\[
\leq |a_2| \cdot \|R_{12} - \bar{R}_{12}\|_{L^p[0,1]} + |a_1| \cdot \|R_{21} - \bar{R}_{21}\|_{L^p[0,1]} \leq C_1 \|Q - \bar{Q}\|_p.
\]  

(3.61)

where \( C_1 \) does not depend on \( Q, \bar{Q} \in \mathbb{C}^{2 \times 2} \).

Further, it follows from the system \( (3.39) \) and similar system for a vector \( \bar{P} \) that

\[
\left( a_1 P_1^\pm(\cdot) \pm a_2 P_2^\pm(\cdot) \right) = (I + \bar{R})^{-1} \left( g^+_1 \pm a_2 \bar{g}^+_2(\cdot) \right).
\]  

(3.62)

It follows from \( (3.61) \) with \( \bar{Q} = 0 \) that

\[
\|g^\pm\|_{L^p([0,x]; \mathbb{C}^2)} \leq C_1 \|Q\|_p \leq r C_1, \quad Q \in \mathbb{C}^{2 \times 2}.
\]  

(3.63)

Combining (3.62) with estimates (3.61), (3.63) and estimates (3.54), (3.55) from Proposition 3.6(iv), implies

\[
|a_k| \cdot \left\| P^\pm_k - \bar{P}^\pm_k \right\|_p \leq \left\| (I + \bar{R})^{-1} g^\pm - (I + \bar{R})^{-1} \bar{g}^\pm \right\|_p \leq \left\| (I + \bar{R})^{-1} g^\pm + (I + \bar{R})^{-1} (g^\pm - \bar{g}^\pm) \right\|_p
\]

\[
\leq \left\| (I + \bar{R})^{-1} - (I + \bar{R})^{-1} \right\|_{p \to p} \left\| g^\pm + \left\| (I + \bar{R})^{-1} \right\|_{p \to p} \left\| g^\pm - \bar{g}^\pm \right\|_p \leq C_4 \|Q - \bar{Q}\|_p, \quad k \in \{1, 2\},
\]  

(3.64)

where \( C_4 = C_1 \cdot (C_2 + r C_3) \) and does not depend on \( Q, \bar{Q} \in \mathbb{C}^{2 \times 2} \). Given that \( b_k = a_k^{-1} \), this in turn yields estimates

\[
\left\| P^\pm - \bar{P}^\pm \right\|_p \leq C_5 \|Q - \bar{Q}\|_p, \quad \left\| P^\pm \right\|_{L^p} \leq C_5 r, \quad \left\| Q, \bar{Q} \right\|_{L^p[0,1]} \in \mathbb{C}^{2 \times 2},
\]  

(3.65)

where \( C_5 = (|b_1|^p + |b_2|^p)^{1/p} C_4 \). Inserting estimates \( (3.65) \) and \( (3.48) \) into \( (3.00) \), we arrive at the estimate

\[
\left\| K^\pm - \bar{K}^\pm \right\|_{X_\infty, p(\Omega; C^2)} \leq C \cdot \left\| Q - \bar{Q}\right\|_p
\]  

(3.66)

being a part of the required estimate \( (3.15) \). The second inequality in \( (3.15) \) is proved similarly. \( \square \)

**Remark 3.7.** (i) For Dirac \( 2 \times 2 \) system \( (B = \text{diag}(-1, 1)) \) with continuous \( Q \) the triangular transformation operators have been constructed in [27] Ch.10.3 and [38] Ch.1.2. For \( Q \in (L^1[0,1]; \mathbb{C}^{2 \times 2}) \) it is proved in [2] by an appropriate generalization of Marchenko’s method.

(ii) Let \( J : f \to \int_0^x f(t) dt \) denote the Volterra integration operator on \( L^p[0,1] \). Note that the similarity of integral Volterra operators given by (2.16) to the simplest Volterra operator of the form \( B \otimes J \) acting in the spaces \( L^p([0,1]; \mathbb{C}^2) \)
has been investigated in [36, 45]. The technique of investigation of integral equations for the kernels of transformation operators in the spaces $X_{\infty,1}(\Omega)$ and $X_{2,1}(\Omega)$ goes back to the paper [35].

(iii) The proof of Proposition 3.6(ii) can be significantly simplified in multiple cases without the use of the intricate intertwining property. Namely, if $p \geq 2$, then inequalities (3.51)–(3.52) are immediate by combining Lemma 2.6(i) with Proposition 3.6(ii). Similarly, if radius $r$ is sufficiently small, then Lemma 2.7 implies these estimates.

Finally, let us consider a compact set $\mathcal{K}$ instead of the ball $U_{p,r}^2$. Now, estimate (3.48) and Lemma 2.2(ii) ensure the compactness of the set of operators $\{I + R_{Q}\}_{Q \in \mathcal{K}}$ in each $\mathcal{B}(L^n) := \mathcal{B}(L^n, L^n)$. It remains to note that the set $\{(I + R_{Q})^{-1}\}_{Q \in \mathcal{K}}$ is compact in $\mathcal{B}(L^n)$, as an image of a compact set under the continuous mapping $T \to T^{-1}$, defined on the open set of invertible elements of the Banach algebra $\mathcal{B}(L^n)$. The proof is finished in the same way using (3.57).

4. General properties of a $2 \times 2$ Dirac-type BVP

Here we consider $2 \times 2$ Dirac-type equation (1.1)  
\[- iB^{-1}y' + Q(x)y = \lambda y, \quad y = \text{col}(y_1, y_2), \quad x \in [0, 1], \]  
subject to the following general boundary conditions  
\[U_j(y) := a_{j1}y_1(0) + a_{j2}y_2(0) + a_{j3}y_1(1) + a_{j4}y_2(1) = 0, \quad j \in \{1, 2\}. \]  
Let us also set  
\[A := \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}, \quad A_{jk} := \begin{pmatrix} a_{1j} & a_{1k} \\ a_{2j} & a_{2k} \end{pmatrix}, \quad J_{jk} := \det(A_{jk}), \quad j, k \in \{1, \ldots, 4\}. \]  
With the system (1.1), one associates, in a natural way, the maximal operator $L_{\max} = L_{\max}(Q)$ defined in $L^2([0, 1]; \mathbb{C}^n)$ by the differential expression (1.1) on the domain  
\[\text{dom}(L_{\max}) = \{y \in W^1_1([0, 1]; \mathbb{C}^n) : L_{\max}y \in L^2([0, 1]; \mathbb{C}^n)\}. \]  
Next we denote by $L := L(Q, U_1, U_2)$ the operator associated in $L^2([0, 1]; \mathbb{C}^2)$ with the BVP (1.1)–(1.2). It is defined as the restriction of the maximal operator $L_{\max} = L_{\max}(Q)$ (4.4) to the domain  
\[\text{dom}(L) = \text{dom}(L(Q, U_1, U_2)) = \{y \in \text{dom}(L_{\max}) : U_1(y) = U_2(y) = 0\}. \]  
Let  
\[\Phi(\cdot, \lambda) = \begin{pmatrix} \varphi_{11}(\cdot, \lambda) & \varphi_{12}(\cdot, \lambda) \\ \varphi_{21}(\cdot, \lambda) & \varphi_{22}(\cdot, \lambda) \end{pmatrix} := \begin{pmatrix} \Phi_1(\cdot, \lambda) & \Phi_2(\cdot, \lambda) \end{pmatrix}, \quad \Phi(0, \lambda) = I_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]  
be a fundamental matrix solution of the system (1.1). Here $\Phi_k(\cdot, \lambda)$ is the $k$th column of $\Phi(\cdot, \lambda)$.

The eigenvalues of the problem (4.1)–(4.2) counting multiplicity are the zeros (counting multiplicity) of the characteristic determinant  
\[\Delta(Q)(\lambda) := \det \begin{pmatrix} U_1(\Phi_1(\cdot, \lambda)) & U_1(\Phi_2(\cdot, \lambda)) \\ U_2(\Phi_1(\cdot, \lambda)) & U_2(\Phi_2(\cdot, \lambda)) \end{pmatrix}. \]  
Inserting (4.6) and (4.7) into (4.9), setting $\varphi_{jk}(\lambda) := \varphi_{jk}(1, \lambda)$, and taking notations (4.3) into account we arrive at the following expression for the characteristic determinant  
\[\Delta(Q)(\lambda) = J_{12} + J_{34}e^{it_1+b_2}J_{12} + J_{13}\varphi_{11}(\lambda) + J_{14}\varphi_{12}(\lambda) + J_{12}\varphi_{12}(\lambda) + J_{14}\varphi_{22}(\lambda). \]  
Alongside the problem (4.4)–(4.2) we consider the same problem with $Q$ in place of $Q$. Denote the corresponding fundamental solution matrix, its entries, and the corresponding characteristic determinant as $\tilde{\Phi}(\cdot, \lambda)$, $\tilde{\varphi}_{jk}(\cdot, \lambda)$, $j, k \in \{1, 2\}$, and $\tilde{\Delta}(\lambda)$, respectively. If $Q = 0$ we denote a fundamental matrix solution as $\Phi^0(\cdot, \lambda)$. Clearly  
\[\Phi^0(x, \lambda) = \begin{pmatrix} e^{ib_1x} & 0 \\ 0 & e^{ib_2x} \end{pmatrix} := \begin{pmatrix} \varphi^0_{11}(x, \lambda) & \varphi^0_{12}(x, \lambda) \\ \varphi^0_{21}(x, \lambda) & \varphi^0_{22}(x, \lambda) \end{pmatrix} := \begin{pmatrix} \Phi^0_1(x, \lambda) & \Phi^0_2(x, \lambda) \end{pmatrix}, \quad x \in [0, 1], \quad \lambda \in \mathbb{C}. \]  
Here $\Phi^0_k(\cdot, \lambda)$ is the $k$th column of $\Phi^0(\cdot, \lambda)$. In particular, the characteristic determinant $\Delta_0(\cdot)$ becomes  
\[\Delta_0(\lambda) = J_{12} + J_{34}e^{i(b_1+b_2)} + J_{32}e^{ib_1} + J_{14}e^{ib_2}. \]
In the case of Dirac system \((B = \text{diag}(-1, 1))\) this formula is simplified to
\[
\Delta_0(\lambda) = J_{12} + J_{34} + J_{32}e^{-i\lambda} + J_{14}e^{i\lambda}.
\]

4.1. **Representation of the characteristic determinant.** Our investigation of the perturbation determinant relies on the following result clarifying our Proposition 3.1 from \([31]\) and coinciding with it for \(Q \in L^1([0, 1]; \mathbb{C}^{2 \times 2})\).

**Lemma 4.1.** Let \(Q \in L^p([0, 1]; \mathbb{C}^{2 \times 2})\) for some \(p \in [1, \infty)\). Then the functions \(\varphi_{jk}(\cdot, \lambda), j, k \in \{1, 2\}\), admit the following representations
\[
\varphi_{jk}(x, \lambda) = \delta_{jk}e^{ib_k\lambda x} + \int_0^x K_{j, k}(x, t)e^{ib_k\lambda t}dt + \int_0^x K_{j, 2k}(x, t)e^{ib_2\lambda t}dt, \quad x \in [0, 1], \lambda \in \mathbb{C},
\]
where
\[
K_{jl,k} := 2^{-1} \left( K_{jl,k}^+ + (-1)^{j+k} K_{jl,k}^- \right) \in X_{1,p}^0(\Omega) \cap X_{2,p}^0(\Omega), \quad j, k, l \in \{1, 2\}.
\]

**Proof.** Comparing initial conditions and applying the Cauchy uniqueness theorem one easily gets \(2 \Phi_1(\cdot, \lambda) = 2(\varphi_{11}(\cdot, \lambda)) = e_+ (\cdot, \lambda) + e_- (\cdot, \lambda)\). Inserting in place of \(e_+ (\cdot, \lambda)\) and \(e_- (\cdot, \lambda)\) their expressions from (4.2) one arrives at (4.12)−(4.13) for \(k = 1\). Case \(k = 2\) is treated similarly.

**Remark 4.2.** Let \(Q_{12} = 0\) and \(Q_{21} \in L^1([0, 1])\). Straightforward calculations show that in this case
\[
\Phi(x, \lambda) = \begin{pmatrix} e^{ib_1\lambda x} & 0 \\ -ib_2e^{ib_2\lambda x} \int_0^x Q_{21}(t)e^{ib_2\lambda t}dt & e^{ib_2\lambda x} \end{pmatrix}.
\]

Let us demonstrate formulas (4.12) in this model example. Set for brevity \(q(t) := -ib_2Q_{21}(t)\) and recall that \(\alpha_1 = \frac{b_2}{2b_2-b_1}\) and \(\alpha_2 = \frac{b_1}{2b_2-b_1}\). It is easy to verify using (3.10), (3.11), (3.59) and (3.9) that
\[
R(x, t) = \begin{pmatrix} 0 & 0 \\ \alpha_2q(\alpha_1x + \alpha_2t) & 0 \end{pmatrix}, \quad P^\pm(x) = \begin{pmatrix} 0 & \pm\alpha_1q(\alpha_1x) \\ 0 & 0 \end{pmatrix},
\]
\[
K^\pm(x, t) = \begin{pmatrix} 0 & \pm\alpha_1q(\alpha_1x - \alpha_1t) \\ \alpha_2q(\alpha_1x + \alpha_2t) & 0 \end{pmatrix}.
\]

Relation (4.13) now implies that
\[
K_{21,1}(x, t) = \alpha_2q(\alpha_1x + \alpha_2t), \quad K_{22,1}(x, t) = \alpha_1q(\alpha_1x - \alpha_1t),
\]
while \(K_{jl,k} = 0\) for other triples \((j, k, l) \in \{1, 2\}^3\). Hence (4.12) implies that \(\varphi_{jk}(x, \lambda) = \delta_{jk}e^{ib_k\lambda x}\) for all pairs \((j, k) \in \{1, 2\}^2\) but (2, 1) and that
\[
\varphi_{21}(x, \lambda) = \int_0^x \alpha_2q(\alpha_1x + \alpha_2t)e^{ib_2\lambda t}dt + \int_0^x \alpha_1q(\alpha_1x - \alpha_1t)e^{ib_2\lambda t}dt, \quad x \in [0, 1], \lambda \in \mathbb{C}.
\]

Making changes of variables \(u = \alpha_1x + \alpha_2t, \nu = \alpha_1x - \alpha_1t\) in two integrals in (4.18) we arrive at (4.13). Namely, first integral turns into integral \(e^{ib_2\lambda x} \int_{\alpha_1x}^x q(t)e^{ib_2\lambda t}dt\), while the second one turns into similar integral with integration over interval \([0, \alpha_1x]\).

**Lemma 4.3.** Let \(Q, \tilde{Q} \in \mathbb{U}^2_{p, r, \times 2}\) for some \(p \in [1, \infty)\) and \(r > 0\). Then the following representation takes place
\[
\varphi_{jk}(x, \lambda) = \int_0^x \tilde{K}_{j, k}(x, t)e^{ib_k\lambda t}dt + \int_0^x \tilde{K}_{j, 2k}(x, t)e^{ib_2\lambda t}dt, \quad x \in [0, 1], \lambda \in \mathbb{C},
\]
where
\[
\tilde{K}_{jl,k} := K_{jl,k} - \tilde{K}_{jl,k} \in X_{1,p}(\Omega) \cap X_{2,p}(\Omega), \quad j, k, l \in \{1, 2\},
\]
and for some \(C = C(p, r, B)\) the following uniform estimate holds
\[
\|\tilde{K}_{jl,k}\|_{X_{1,p}(\Omega)} + \|\tilde{K}_{jl,k}\|_{X_{2,p}(\Omega)} \leq C \cdot \|Q - \tilde{Q}\|_p, \quad j, k, l \in \{1, 2\}.
\]

**Proof.** Subtracting formula (4.12) for \(\tilde{Q}\) from the same formula for \(Q\) we arrive at (4.19). Taking into account formulas (4.13) and applying Theorem 3.3 (estimate (3.15)) we arrive at uniform estimate (4.21).
Lemma 4.4. Let $Q \in L^p(\{0,1\}; \mathbb{C}^{2 \times 2})$ for some $p \in [1, \infty)$. Then the characteristic determinant $\Delta_Q(\cdot)$ of the problem (1.11-1.12) is an entire function of exponential type and admits the following representation:

$$
\Delta_Q(\lambda) = \Delta_0(\lambda) + \int_0^1 g_{1,Q}(t)e^{ib_1\lambda t}dt + \int_0^1 g_{2,Q}(t)e^{ib_2\lambda t}dt,
$$

(4.22) 

$$
g_{i,Q}(\cdot) = J_{32}K_{11,1}(1,\cdot) + J_{42}K_{21,1}(1,\cdot) + J_{13}K_{11,2}(1,\cdot) + J_{14}K_{21,2}(1,\cdot) \in L^p[0,1], \quad l \in \{1,2\}.
$$

(4.23) 

Proof. Consider representations (4.12) for $\varphi_{jk}(\cdot, \cdot)$, $j,k \in \{1,2\}$. By Lemma 2.1, $K_{jl,k}(\cdot,\cdot) \in X^0_{1,p}(\Omega) \cap X^0_{\infty,p}(\Omega)$, $j,k,l \in \{1,2\}$. Therefore by Lemma 2.1 the trace functions $K_{jl,k}(1,\cdot)$ are well defined and $K_{jl,k}(1,\cdot) \in L^p[0,1]$, $j,k,l \in \{1,2\}$. Therefore inserting $x = 1$ in (4.12) we obtain special representations for $\varphi_{jk}(\cdot)$, $j,k \in \{1,2\}$,

$$
\varphi_{jk}(1,\lambda) = \delta_{jk}e^{ib_k\lambda} + \int_0^1 K_{11,k}(1,t)e^{ib_1\lambda t}dt + \int_0^1 K_{21,k}(1,t)e^{ib_2\lambda t}dt.
$$

(4.24) 

Inserting these expressions in (4.3) and taking formula (4.10) for $\Delta_0(\cdot)$ into account we arrive at (4.22) and (4.23). □

Lemma 4.5. Let $Q, \bar{Q} \in U^{2\times 2}_{p,r}$ for some $p \in [1, \infty)$ and $r > 0$. Then the following representation takes place

$$
\Delta_Q(\lambda) - \Delta_{\bar{Q}}(\lambda) = \int_0^1 \hat{g}_1(t)e^{ib_1\lambda t}dt + \int_0^1 \hat{g}_2(t)e^{ib_2\lambda t}dt,
$$

(4.25) 

where $\hat{g}_l = g_{Q,l} - g_{\bar{Q},l} \in L^p[0,1]$, $l \in \{1,2\}$, and for some $\hat{C} = \hat{C}(p, r, B, A)$, the following uniform estimate holds

$$
\|\hat{g}_l\|_p + \|\hat{g}_2\|_p = \|g_{Q,1} - g_{\bar{Q},1}\|_p + \|g_{Q,2} - g_{\bar{Q},2}\|_p \leq \hat{C} \cdot \|Q - \bar{Q}\|_p, \quad Q, \bar{Q} \in U^{2\times 2}_{p,r}.
$$

(4.26) 

Proof. Subtracting formula (4.22) for $\bar{Q}$ from the same formula for $Q$ we arrive at (4.25) with

$$
\hat{g}_l(\cdot) = g_{l}(\cdot) - g_{\bar{l}}(\cdot) = J_{32}\hat{K}_{11,1}(1,\cdot) + J_{42}\hat{K}_{21,1}(1,\cdot) + J_{13}\hat{K}_{11,2}(1,\cdot) + J_{14}\hat{K}_{21,2}(1,\cdot), \quad l \in \{1,2\}.
$$

(4.27) 

Combining formulas (4.26) with estimate (4.11), and Lemma 2.1 implies the following uniform estimate

$$
\|\hat{K}_{jl,k}(1,\cdot)\|_p \leq \|K_{jl,k}\|_{X_{\infty,r}(\Omega)} \leq C(p, r, B) \cdot \|Q - \bar{Q}\|_p, \quad j,k,l \in \{1,2\}, \quad Q, \bar{Q} \in U^{2\times 2}_{p,r}.
$$

(4.28) 

In turn, combining formulas (4.28) and (4.27) we arrive at uniform estimate (4.26) with

$$
\hat{C} = \hat{C}(p, r, B, A) = 2 \cdot (|J_{32}| + |J_{42}| + |J_{13}| + |J_{14}|) \cdot C(p, r, B).
$$

(4.29) 

This completes the proof. □

Lemma 4.6. Let $Q \in U^{2\times 2}_{p,r}$ for some $p \in [1, \infty)$ and $r > 0$. Then there exists a constant $C = C(r, B, A) > 0$ not dependent on $Q$ and $p$ and such that the following uniform estimate holds

$$
|\Delta_Q(\lambda)| \leq C \cdot (e^{-b_1\text{Im }\lambda} + e^{-b_2\text{Im }\lambda}), \quad \lambda \in \mathbb{C}.
$$

(4.30) 

Proof. Since $b_1 < 0 < b_2$ then

$$
e^{-b_1\text{Im }\lambda} + e^{-b_2\text{Im }\lambda} > 1, \quad e^{-b_1\text{Im }\lambda} + e^{-b_2\text{Im }\lambda} > \left|e^{i(b_1 + b_2)\lambda}\right|, \quad \lambda \in \mathbb{C}.
$$

(4.31) 

Inserting these inequalities into (4.10) implies estimate (4.30) for $\Delta_0$ with $C_0 = |J_{12}| + |J_{34}| + |J_{32}| + |J_{14}|$. Since $U^{2\times 2}_{p,r} \subset U^{2\times 2}_{1,r}$, it follows from (4.22), (4.26), and estimate (4.30) for $\Delta_0$ that

$$
|\Delta_Q(\lambda)| \leq |\Delta_0(\lambda)| + \|g_1\|_1 \cdot \max\{e^{-b_1\text{Im }\lambda}, 1\} + \|g_2\|_1 \cdot \max\{e^{-b_2\text{Im }\lambda}, 1\} \leq (C_0 + \hat{C}\|Q\|_1)(e^{-b_1\text{Im }\lambda} + e^{-b_2\text{Im }\lambda}), \quad \lambda \in \mathbb{C}.
$$

(4.32) 

Since $\|Q\|_1 \leq \|Q\|_p \leq r$, this implies (4.30) with $C = C_0 + \hat{C}r$. □
4.2. Regular and strictly regular boundary conditions. Let us recall the definition of regular boundary conditions.

**Definition 4.7.** Boundary conditions \([4.2]\) are called regular if
\[
J_{14}J_{32} \neq 0.
\]

Let us recall one more definition (cf. \([22]\)).

**Definition 4.8.** Let \(\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}}\) be a sequence of complex numbers. It is called incompressible if for some \(d \in \mathbb{N}\) every rectangle \([t-1, t+1] \times \mathbb{R} \subset \mathbb{C}\) contains at most \(d\) entries of the sequence, i.e.
\[
\text{card}\{n \in \mathbb{Z} : |\text{Re}\lambda_n - t| \leq 1\} \leq d, \quad t \in \mathbb{R}.
\]

To emphasize parameter \(d\) we will sometimes call \(\Lambda\) an incompressible sequence of density \(d\).

Recall that \(D_r(z) \subset \mathbb{C}\) denotes the disc of radius \(r\) with a center \(z\).

Let us recall the following simple property of incompressible sequences that improves Lemma 4.3 from \([31]\).

**Lemma 4.9.** Let \(\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}\) be an incompressible sequence of density \(d\). Then for any \(\varepsilon \leq (2d)^{-1}\) every connected component of the union of discs \(\cup_{n \in \mathbb{Z}} D_{\varepsilon}(\lambda_n)\) has at most \(d\) discs \(D_{\varepsilon}(\lambda_n)\).

**Proof.** Let \(2d \varepsilon \leq 1\) and assume that some connected component \(C\) of \(\cup_{n \in \mathbb{Z}} D_{\varepsilon}(\lambda_n)\) has more than \(d\) discs \(D_{\varepsilon}(\lambda_n)\). Let \(D_0 = D_{\varepsilon}(\lambda_{n_0}), n_0 \in \mathbb{Z}\), be one of the discs in \(C\).

Consider the sequence \(C\) of all discs \(D_{\varepsilon}(\lambda_n)\) in \(C\) that have graph distance at most \(d\) from \(D_0\). Let us show that \(C\) has more than \(d\) discs. Indeed, if no disc in \(C\) has graph distance more than \(d\) from \(D_0\), then \(C\) contains all discs from \(C\) which has more than \(d\) discs. Otherwise, \(C\) has some disc \(D\) with distance \(d\) from \(D_0\). All discs on the path from \(D_0\) to \(D\) belong to \(C\) and hence \(C\) has at least \(d+1\) discs.

Let \(D = D_{\varepsilon}(\lambda_0)\) be a disc in \(C\). Since graph distance from \(D\) to \(D_0\) is at most \(d\) and disc radii are \(\varepsilon\) we have \(|\lambda_n - \lambda_{n_0}| < 2d \varepsilon \leq 1\). Thus centers of all discs in \(C\) lie in the rectangle \([t_0 - 1, t_0 + 1] \times \mathbb{R}\), where \(t_0 = \text{Re}\lambda_{n_0}\). Since there are more than \(d\) discs in \(C\) it contradicts the fact that \(\Lambda\) is an incompressible sequence of density \(d\). \(\Box\)

In Section 6 we will also need the following additional property of incompressible sequences.

**Lemma 4.10.** Let \(\Lambda_0 = \{\lambda_n^0\}_{n \in \mathbb{Z}}\) be an incompressible sequence of density \(d_0\). Let \(\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}\) be a sequence of complex numbers such that \(|\lambda_n - \lambda_n^0| < M, n \in \mathbb{Z}\), for some \(M > 0\). Then \(\Lambda\) is an incompressible sequence of density \(d = d_0 [M + 1]\). Here \([x]\) denotes the smallest integer \(k\) such that \(x \leq k\).

**Proof.** Let \(t \in \mathbb{R}\) and let
\[
\mathcal{N}_t := \{n \in \mathbb{Z} : \lambda_n \in [t-1, t+1] \times \mathbb{R}\}.
\]

We need to show that \(\text{card}(\mathcal{N}_t) \leq d = d_0 [M + 1]\). Since \(|\lambda_n - \lambda_n^0| < M, n \in \mathbb{Z}\), it follows that
\[
\lambda_n^0 \in R_{t, M} := [t-1 - M, t+1 + M] \times \mathbb{R}, \quad n \in \mathcal{N}_t,
\]
which in turn yields
\[
\text{card}(\mathcal{N}_t) \leq \text{card}\{n \in \mathbb{Z} : \lambda_n^0 \in R_{t, M}\}.
\]

Let \(t_k := t - M + 2k, k \in \mathbb{N}\). It is clear from the definition of \([x]\) that
\[
R_{t, M} \subset \bigcup_{k=0}^{[M]} \left( [t_k - 1, t_k + 1] \times \mathbb{R}\right).
\]
Combining \((4.34)\) with \((4.35)\) implies that \(\text{card}\{n \in \mathbb{Z} : \lambda_n^0 \in R_{t, M}\} \leq d_0 [M + 1]\). Relation \((4.35)\) now yields desired inequality \(\text{card}(\mathcal{N}_t) \leq d\). \(\Box\)

Let us recall certain important properties from \([31]\) of the characteristic determinant \(\Delta(\cdot)\) in the case of regular boundary conditions.
Proposition 4.11. [31] Proposition 4.6] Let the boundary conditions (4.2) be regular and let \( \Delta_Q(\cdot) \) be the characteristic determinant of the problem (4.11) (4.12) given by (4.18). Then the following statements hold:

(i) The function \( \Delta_Q(\cdot) \) has infinitely many zeros \( \Lambda := \{ \lambda_n \}_{n \in \mathbb{Z}} \) counting multiplicities and \( \Lambda \subset \Pi_h \) for some \( h \geq 0 \).

(ii) The sequence \( \Lambda \) is incompressible.

(iii) For any \( \varepsilon > 0 \) there exists \( C_{\varepsilon} > 0 \) such that the determinant \( \Delta_Q(\cdot) \) admits the following estimate from below

\[
|\Delta_Q(\lambda)| \geq C_{\varepsilon}(e^{-b_1 \text{Im} \lambda} + e^{-b_2 \text{Im} \lambda}), \quad \lambda \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} D_{\varepsilon}(\lambda_n).
\]  

(4.39)

Clearly, the conclusions of Proposition 4.11 are valid for the characteristic determinant \( \Delta_0(\cdot) \) given by (4.10).

Let \( \Lambda_0 = \{ \lambda^0_n \}_{n \in \mathbb{Z}} \) be the sequence of its zeros counting multiplicity. Let us order the sequence \( \Lambda_0 \) in a (possibly non-unique) way such that \( \text{Re} \lambda^0_n \leq \text{Re} \lambda^0_{n+1}, \ n \in \mathbb{Z} \).

Let us recall an important result from [29, 31] and [18] concerning asymptotic behavior of eigenvalues.

Proposition 4.12. [31] Proposition 4.7] Let \( Q \in L^1([0,1]; \mathbb{C}^2 \times \mathbb{C}) \) and let boundary conditions (4.2) be regular. Then the sequence \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) of zeros of \( \Delta_Q(\cdot) \) can be ordered in such a way that the following asymptotic formula holds

\[
\lambda_n = \lambda^0_n + o(1), \quad \text{as} \quad |n| \to \infty, \quad n \in \mathbb{Z}.
\]  

(4.40)

Let us refine this ordering to have some additional important properties.

Proposition 4.13. Let \( Q \in L^1([0,1]; \mathbb{C}^2 \times \mathbb{C}) \) and let boundary conditions (4.2) be regular. Then:

(i) For any \( \varepsilon > 0 \) there exist constants \( M_{\varepsilon} = M_{\varepsilon}(Q,B,A) > 0 \) and \( C_{\varepsilon} = C_{\varepsilon}(B,A) > 0 \) such that

\[
|\Delta_Q(\lambda) - \Delta_0(\lambda)| < |\Delta_0(\lambda)|, \quad \lambda \notin \tilde{\Omega}_{\varepsilon},
\]  

(4.41)

\[
|\Delta_Q(\lambda)| > C_{\varepsilon}(e^{-b_1 \text{Im} \lambda} + e^{-b_2 \text{Im} \lambda}), \quad \lambda \notin \tilde{\Omega}_{\varepsilon}.
\]  

(4.42)

where

\[
\tilde{\Omega}_{\varepsilon} := \mathbb{D}_{\varepsilon}(0) \cup \Omega^0_{\varepsilon}, \quad \Omega^0_{\varepsilon} := \bigcup_{n \in \mathbb{Z}} \mathbb{D}_{\varepsilon}(\lambda^0_n).
\]  

(4.43)

(ii) The sequence \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) can be ordered in such a way that for any \( \varepsilon > 0 \) and \( n \in \mathbb{Z} \) numbers \( \lambda_n \) and \( \lambda_n^0 \) belong to the same connected component of \( \tilde{\Omega}_{\varepsilon} \). In addition, relation (4.39) also holds for this ordering.

Proof. (i) By Proposition 4.11(ii), the sequence \( \Lambda_0 = \{ \lambda^0_n \}_{n \in \mathbb{Z}} \) of zeros of \( \Delta_0 \) is incompressible, is of density \( d_0 \) and lies in the strip \( \Pi_h \) for some \( d_0 = d_0(B,A) \in \mathbb{N} \) and \( h_0 = h_0(B,A) > 0 \). Set \( \varepsilon_0 := (2d_0)^{-1} = \varepsilon_0(B,A) \) and let \( 0 < \varepsilon \leq \varepsilon_0 \).

By Proposition 4.11(iii), there exists \( C_{\varepsilon} = C_{\varepsilon}(B,A) > 0 \) such that the estimate (4.39) for \( \Delta_0 \) holds,

\[
|\Delta_0(\lambda)| > C_{\varepsilon}(e^{-b_1 \text{Im} \lambda} + e^{-b_2 \text{Im} \lambda}), \quad \lambda \notin \Omega^0_{\varepsilon}.
\]  

(4.44)

Since functions \( g_1 \) and \( g_2 \) from the representation (4.22) are summable, it follows from [31] Lemma 3.5 with \( \delta = C_{\varepsilon}^{-1}/6 = \delta(B,A) \), Lemma 4.4 and estimate (4.31) that

\[
|\Delta_0(\lambda) - \Delta_Q(\lambda)| \leq \int_0^1 g_1(t) e^{ib_1 \lambda t} dt + \int_0^1 g_2(t) e^{ib_2 \lambda t} dt \\
\leq 6^{-1} C_{\varepsilon}^{-1}(e^{-b_1 \text{Im} \lambda} + 1 + e^{-b_2 \text{Im} \lambda} + 1) \\
< 2^{-1} C_{\varepsilon}^{-1}(e^{-b_1 \text{Im} \lambda} + e^{-b_2 \text{Im} \lambda}), \quad |\lambda| > M_{\varepsilon}.
\]  

(4.45)

with certain \( M_{\varepsilon} = M_{\varepsilon}(Q,B,A) > 0 \). Combining estimate (4.44) with (4.45) yields (4.41) and (4.42) with \( C_{\varepsilon} = C_{\varepsilon}/2 \).

Estimate (4.42) yields that all zeros of \( \Delta(\cdot) \) belong to \( \Omega_{\varepsilon} \), \( 0 < \varepsilon \leq \varepsilon_0 \).

(ii) Let \( \Omega^0_{\varepsilon,k}, k \in \mathbb{Z} \), be the sequence of all connected components of \( \Omega^0_{\varepsilon} \). Since \( \Lambda_0 \) is an incompressible sequence of density \( d_0 \) and \( \varepsilon \leq (2d_0)^{-1} \), Lemma 4.3 implies that each connected component of \( \Omega^0_{\varepsilon} \) contains at most \( d_0 \) discs \( \mathbb{D}_{\varepsilon}(\lambda^0_n) \). Hence

\[
diam(\Omega^0_{\varepsilon,k}) \leq 2\varepsilon d_0 \leq 1, \quad k \in \mathbb{Z}.
\]  

(4.46)

Let \( \mathcal{C}_{\varepsilon} \) be a connected component of \( \tilde{\Omega}_{\varepsilon} \) that contains \( \mathbb{D}_{M_{\varepsilon}}(0) \). Clearly, \( \mathcal{C}_{\varepsilon} \) is the union of the disc \( \mathbb{D}_{M_{\varepsilon}}(0) \) and all connected components \( \Omega^0_{\varepsilon,k} \) that intersect with this disc. Hence inequality (4.40) implies that

\[
\mathcal{C}_{\varepsilon} \subset \mathbb{D}_{M_{\varepsilon}+1}(0).
\]  

(4.47)
We can cover $\mathbb{D}_{M_{\varepsilon}+1}(0)$ with $\lceil M_{\varepsilon}+1 \rceil$ rectangles $[t-1,t+1] \times \mathbb{R}$. Hence, inequality (4.34) yields that $\mathbf{c}_\varepsilon$ contains at most $d_0 \cdot \lceil M_{\varepsilon}+1 \rceil$ entries of $\Lambda_0$. Due to estimate (4.31) the Rouche theorem applies and ensures that in every (necessary bounded) connected component of $\Omega_\varepsilon$, the functions $\Delta_0$ and $\Delta_Q = \Delta_0 + (\Delta_Q - \Delta_0)$ have the same number of zeros counting multiplicity. Hence, $\Delta_Q(\cdot)$ has at most $d_0 \cdot \lceil M_{\varepsilon}+1 \rceil$ zeros in $\mathbf{c}_\varepsilon$ counting multiplicity and at most $d_0$ zeros in any other connected component of $\Omega_\varepsilon$.

Due to this for each $\varepsilon$ we can choose ordering of the sequence $\{\lambda_n\}_{n \in \mathbb{Z}}$, in such a way that for each $n \in \mathbb{Z}$, numbers $\lambda_n$ and $\lambda_n^0$ belong to the same connected component of $\Omega_\varepsilon$. Note that this ordering is different for different $\varepsilon$. Let’s analyze how this ordering changes when $\varepsilon$ tends to zero. Without loss of generality we can assume that $M_{\varepsilon}$ is a non-decreasing function of $\varepsilon$. We have $\varepsilon = \varepsilon_0$ as our initial state of the ordering. Each connected component has finite number of $\lambda_n^0$ and ordering is fixed up to a permutation within the component. With $\varepsilon$ tending to zero each connected component of $\Omega_0^0$ which is not yet part of $\mathbf{c}_\varepsilon$ will have a finite number of events happening. One type of event is a split. This is when some of the discs $\mathbb{D}_\varepsilon(\lambda_n^0)$ and $\mathbb{D}_\varepsilon(\lambda_m^0)$ no longer intersect each other. For each split we will refine the ordering after the split based on the newly formed components and will track the “lifetime” of each new component independently. Note that equal eigenvalues $\lambda_n^0$ and $\lambda_m^0$ will always stay in one component together for all $\varepsilon > 0$. The second type of event is when the disc $\mathbb{D}_\varepsilon(M_{\varepsilon})$ “reaches” one of the discs $\mathbb{D}_\varepsilon(\lambda_n^0)$ from the component we track. In this case this component is consumed by $\mathbf{c}_\varepsilon$ and we no longer need to change ordering. It is clear that after consumption ordering will satisfy the property of $\lambda_n$ and $\lambda_n^0$ being in $\mathbf{c}_\varepsilon$ simultaneously. Note, that for each $n \in \mathbb{Z}$ this process will end in a finite number of steps: either component will split into individual discs $\mathbb{D}_\varepsilon(\lambda_n^0)$ that no longer will split and ordering will be fixed for all multiple eigenvalues equal to $\lambda_n^0$, or component $\mathbf{c}_\varepsilon$ will consume the component of $\mathbb{D}_\varepsilon(\lambda_n^0)$ and so ordering will not change after that. Hence this process defines a single ordering of the sequence $\Lambda$ for which $\lambda_n$ and $\lambda_n^0$ are in the same connected component of $\Omega_\varepsilon$ for all $\varepsilon > 0$ and $n \in \mathbb{Z}$. Combining this with the fact that for connected components $\mathbf{c}_{\varepsilon,k}$ we have $\text{diam}(\mathbf{c}_{\varepsilon,k}) \to 0$ as $\varepsilon \to 0$ uniformly at $k$, yields relation (4.40).

**Definition 4.14.** Let $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ be the sequence of zeros of the characteristic determinant $\Delta_Q(\cdot)$ of the Dirac-type operator $L(Q)$ with summable potential and regular boundary conditions. Let $\Omega_\varepsilon$ be defined in (4.43). The ordering of $\Lambda$ for which $\lambda_n$ and $\lambda_n^0$ belong to the same connected component of $\Omega_\varepsilon$ for all $\varepsilon > 0$ and $n \in \mathbb{Z}$, will be called a **canonical ordering**. Proposition 4.13 implies its existence.

In the sequel we need the following definitions.

**Definition 4.15.** (i) **A sequence** $\Lambda := \{\lambda_n\}_{n \in \mathbb{Z}}$ of complex numbers is said to be **separated** if for some positive $\tau > 0$, 
\[ |\lambda_j - \lambda_k| > 2\tau \quad \text{whenever} \quad j \neq k. \]  
(4.48)

In particular, all entries of a separated sequence are distinct.

(ii) The sequence $\Lambda$ is said to be **asymptotically separated** if for some $N \in \mathbb{N}$ the subsequence $\{\lambda_n\}_{|n| > N}$ is separated.

Let us recall a notion of strictly regular boundary conditions.

**Definition 4.16.** **Boundary conditions** (4.42) are called **strictly regular**, if they are regular, i.e. $J_{14}J_{32} \neq 0$, and the sequence of zeros $\lambda_0 = \{\lambda_n^0\}_{n \in \mathbb{Z}}$ of the characteristic determinant $\Delta_Q(\cdot)$ is asymptotically separated. In particular, there exists $n_0$ such that zeros $\{\lambda_n^0\}_{|n| > n_0}$ are geometrically and algebraically simple.

It follows from Proposition 4.12 that the sequence $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ of zeros of $\Delta_Q(\cdot)$ is asymptotically separated if the boundary conditions are strictly regular.

Assuming boundary conditions (4.42) to be regular, let us rewrite them in a more convenient form. Since $J_{14} \neq 0$, the inverse matrix $A_{14}^{-1}$ exists. Therefore writing down boundary conditions (4.42) as the vector equation $(U_1(y) \quad U_2(y)) = 0$ and multiplying it by the matrix $A_{14}^{-1}$ we transform these conditions as follows

\[ \begin{cases} \hat{U}_1(y) = y_1(0) + by_2(0) + ay_1(1) = 0, \\ \hat{U}_2(y) = dy_2(0) + cy_1(1) + y_2(1) = 0, \end{cases} \]  
(4.49)
with some \(a, b, c, d \in \mathbb{C}\). Now \(J_{14} = 1\) and the boundary conditions \((4.49)\) are regular if and only if \(J_{32} = ad - bc \neq 0\).

Thus, the characteristic determinants \(\Delta_0(\cdot)\) and \(\Delta(\cdot)\) take the form

\[
\Delta_0(\lambda) = d + ae^{ib_1 + ib_2}\lambda + (ad - bc)e^{ib_1}\lambda + e^{ib_2}\lambda, \quad (4.50)
\]

\[
\Delta(\lambda) = d + ae^{ib_1 + ib_2}\lambda + (ad - bc)\varphi_{11}(\lambda) + \varphi_{22}(\lambda) + c\varphi_{12}(\lambda) + b\varphi_{21}(\lambda). \quad (4.51)
\]

Recall that \(a_n \approx b_n\), \(n \in \mathbb{Z}\), means that there exists \(C_2 > C_1 > 0\) such that \(C_1|b_n| \leq |a_n| \leq C_2|b_n|, n \in \mathbb{Z}\). We will need the following simple property of zeros of \(\Delta_0(\cdot)\).

**Lemma 4.17.** Let boundary conditions \((4.49)\) be strictly regular and \(\Delta_0 = \{\lambda_n^0\}_{n \in \mathbb{Z}}\) be the sequence of zeros of \(\Delta_0(\cdot)\).

Then the following asymptotic relation holds

\[
|1 + de^{-ib_2}\lambda_n^0|^2 + |1 + ae^{ib_1}\lambda_n^0|^2 \approx 1, \quad n \in \mathbb{Z}. \quad (4.52)
\]

Moreover, if either \(b_1/b_2 \in \mathbb{Q}\) or \(bc = 0\), then the sequence \(\Delta_0\) is separated. I.e. for some \(\tau > 0\) we have

\[
|\lambda_n^0 - \lambda_m^0| > 2\tau, \quad n \neq m, \quad n, m \in \mathbb{Z}. \quad (4.53)
\]

**Proof.** Note that

\[
\Delta_0(\lambda) = (d + e^{ib_2}\lambda) (1 + ae^{ib_1}\lambda) - bc \cdot e^{ib_1}\lambda = (1 + e^{-ib_2}\lambda) (1 + ae^{ib_1}\lambda) - bc \cdot e^{ib_1}\lambda, \quad \lambda \in \mathbb{C}. \quad (4.54)
\]

Hence

\[
(1 + de^{-ib_2}\lambda_n^0) (1 + ae^{ib_1}\lambda_n^0) = bc \cdot e^{i(b_1 - b_2)} \lambda_n^0, \quad n \in \mathbb{Z}. \quad (4.55)
\]

According to Proposition \((4.40)\), there exists \(h \geq 0\) such that \(\{\lambda_n^0\}_{n \in \mathbb{Z}} \subset \Pi_h\). Hence

\[
e^{ib_j}\lambda_n^0 \approx 1, \quad n \in \mathbb{Z}, \quad j \in \{1, 2\}. \quad (4.56)
\]

First assuming that \(bc \neq 0\) and combining \((4.55)\) with \((4.56)\) yields the following estimate with some \(C_2 > C_1 > 0\)

\[
C_2 > |1 + de^{-ib_2}\lambda_n^0|^2 + |1 + ae^{ib_1}\lambda_n^0|^2 \geq 2\left|\left(1 + de^{-ib_2}\lambda_n^0\right)\left(1 + ae^{ib_1}\lambda_n^0\right)\right| = 2|bc| \cdot |e^{i(b_1 - b_2)}\lambda_n^0| > C_1, \quad |n| \in \mathbb{Z}. \quad (4.57)
\]

which proves \((4.52)\) in this case.

Now let \(bc = 0\). In this case \(ad \neq 0\) and \(\Delta_0(\lambda) = e^{ib_2}\lambda (1 + de^{-ib_2}\lambda) (1 + ae^{ib_1}\lambda)\). Let \(\Lambda^1 = \{\lambda_{m,n}^1\}_{n \in \mathbb{Z}}\) and \(\Lambda^2 = \{\lambda_{m,n}^2\}_{n \in \mathbb{Z}}\) be the sequences of zeros of the first and second factor, respectively. Clearly, these sequences constitute arithmetic progressions lying on the lines, parallel to the real axis. More precisely,

\[
\lambda_{1,n}^0 = \frac{\arg(-d) + 2\pi n}{b_2} - i\frac{\ln|d|}{b_2}, \quad \lambda_{2,n}^0 = \frac{\arg(-a^{-1}) + 2\pi n}{b_1} + i\frac{\ln|a|}{b_1}, \quad n \in \mathbb{Z}. \quad (4.58)
\]

Since boundary conditions \((4.49)\) are strictly regular, the arithmetic progressions \(\Lambda^1\) and \(\Lambda^2\) are asymptotically separated. Let us prove relation \((4.59)\). If \(b_1/b_2 \in \mathbb{Q}\), then the entire sequence \(\{\lambda_{m,n}^0\}_{n \in \mathbb{Z}}\) is periodic, and hence it will be separated not just asymptotically (this is valid only if \(bc = 0\), see Remark \((4.18)\) case 3, below). If \(b_1/b_2 \notin \mathbb{Q}\), then the set \(\{2\pi n/b_2 - 2\pi m/b_1\}_{m,n \in \mathbb{Z}}\) is everywhere dense on \(\mathbb{R}\). Hence the arithmetic progressions \(\Lambda^1\) and \(\Lambda^2\) should lie on different lines, parallel to the real axis, and will be separated as well. Relation \((4.53)\) implies the following asymptotic relations

\[
1 + de^{-ib_2}\lambda_{m,n}^2 \approx 1, \quad 1 + ae^{ib_1}\lambda_{m,n}^1 \approx 1, \quad n \in \mathbb{Z}. \quad (4.59)
\]

Relations \((4.59)\) trivially imply \((4.52)\). \(\square\)

**Remark 4.18.** Let us list some types of strictly regular boundary conditions \((4.49)\). In all of these cases except \(4b\) the set of zeros of \(\Delta_0\) is a union of finite number of arithmetic progressions.

1. **Regular BC** \((4.49)\) for Dirac operator \((-b_1 = b_2 = 1)\) are strictly regular if and only if \((a - d)^2 \neq -4bc\).
2. **Separated BC** \((a = d = 0, bc \neq 0)\) are always strictly regular.
3. **Let** \(b_1/b_2 \in \mathbb{Q}\), i.e. \(b_1 = -n_1\beta, b_2 = n_2\beta, n_1, n_2 \in \mathbb{N}, \beta > 0\) and \(\gcd(n_1, n_2) = 1\). Since \(ad \neq bc\), \(\Delta_0(\cdot)\) is a polynomial at \(e^{i\beta}\lambda\) of degree \(n_1 + n_2\). Hence, BC \((4.49)\) are strictly regular if and only if this polynomial does not have multiple roots. Let us list some cases with explicit conditions.
(a) \[31\] Lemma 5.3 \ Let \(ad \neq 0\) and \(bc = 0\). Then \(BC (4.49)\) are strictly regular if and only if 
\[b_1 \ln |d| + b_2 \ln |a| \neq 0 \quad \text{or} \quad n_1 \arg(-d) - n_2 \arg(-a) \notin 2\pi \mathbb{Z}. \] (4.60)

(b) In particular, antiperiodic \(BC (a = d = 1, b = c = 0)\) are strictly regular if and only if \(n_1 - n_2\) is odd. Note that these \(BC\) are not strictly regular in the case of a Dirac system.

(c) \[31\] Proposition 5.6 \ Let \(a = 0, bc \neq 0\). Then \(BC (4.49)\) are strictly regular if and only if 
\[n_1^{n_1} n_2^{n_2} (-d)^{n_1 + n_2} \neq (n_1 + n_2)^{n_1 + n_2} (-bc)^{n_2}. \] (4.61)

(4) Let \(\alpha := -b_1/b_2 \notin \mathbb{Q}\). Then the problem of strict regularity of \(BC\) is generally much more complicated. Let us list some known cases:

(a) \[31\] Lemma 5.3 \ Let \(ad \neq 0\) and \(bc = 0\). Then \(BC (4.49)\) are strictly regular if and only if 
\[b_1 \ln |d| + b_2 \ln |a| \neq 0. \] (4.62)

(b) \[31\] Proposition 5.6 \ Let \(a = 0, bc, d \in \mathbb{R} \setminus \{0\}\). Then \(BC (4.49)\) are strictly regular if and only if 
\[d \neq -(\alpha + 1) \left(\frac{|bc|\alpha^{-\alpha}}{\alpha + 1}\right). \] (4.63)

5. Fourier transform estimates

5.1. Generalization of Hausdorff-Young and Hardy-Littlewood theorems. To evaluate deviations of eigenvalues of operators \(L(Q)\) and \(L(\bar{Q})\) we extend here classical Hausdorff-Young and Hardy-Littlewood interpolation theorems for Fourier coefficients (see \[51\] Theorem XII.2.3 and \[51\] Theorem XII.3.19, respectively) to the case of arbitrary incompressible sequence \(\Lambda = \{\mu_n\}_{n \in \mathbb{Z}}\) instead of \(\Lambda = \{2\pi n\}_{n \in \mathbb{Z}}\).

For efficient estimate of eigenvectors deviations in Section 7 we will use the following (sublinear) Carleson transform (the maximal version of the classical Fourier transform)
\[E[f](\lambda) := \sup_{N>0} \left| \int_{-N}^{N} F[f](t)e^{-i\lambda t} dt \right|, \quad \lambda \in \mathbb{R}. \] (5.1)

where \(F[f]\) denotes the classical Fourier transform, \(F[f](\lambda) = \lim_{N \to \infty} \int_{-N}^{N} f(t)e^{i\lambda t} dt\). The most important property of such transforms is the following Carleson-Hunt theorem (see \[18\] Theorems 6.2.1, 6.3.3)).

Theorem 5.1. For any \(p \in (1, \infty)\) the Carleson operator \(E\) is a bounded operator from \(L^p\) to itself, i.e. there exists a constant \(C_p > 0\) such that
\[\|E[f]\|_{L^p} \leq C_p \|f\|_{L^p}, \quad f \in L^p(\mathbb{R}).\]

For our considerations it is more convenient to consider the following version (with \(g\) in place of \(F[g]\)) of \(E\):
\[\mathcal{F}[g](\lambda) := \sup_{x \in [0,1]} \left| \int_{0}^{x} g(t)e^{i\lambda t} dt \right|, \quad g \in L^p[0,1], \quad \lambda \in \mathbb{C}. \] (5.2)

We also put for brevity \(\mathcal{F}[g]^\theta(\lambda) := (\mathcal{F}[g](\lambda))^\theta\).

Combining Carleson-Hunt Theorem \[5.1\] with Hausdorff-Young theorem leads to the following result (see e.g. \[49\] for details).

Proposition 5.2. For any \(p \in (1, 2]\) the maximal Fourier transform \(\mathcal{F}\) maps \(L^p[0,1]\) into \(L^{p'}[0,1]\) boundedly, i.e. the following estimate holds
\[\int_{-\infty}^{\infty} \mathcal{F}[g]^p(x) dx \leq \gamma_p \cdot \|g\|^p_p, \quad g \in L^p[0,1], \quad 1/p + 1/p' = 1, \] (5.3)

where \(\gamma_p > 0\) does not depend on \(g \in L^p[0,1]\).

In the sequel we will need the following lemma which proof substantially relies on estimate \(5.3\).
Lemma 5.3. Let \( g \in L^p[0, 1] \) for some \( p \in (1, 2] \) and \( h \geq 0 \). Let us set
\[
g_n := \sup \{ \mathcal{F}[g](\lambda) : \lambda \in \Pi_{h,n} \}, \quad \Pi_{h,n} := [n, n + 1] \times [-h, h] \subset \mathbb{C}. \quad n \in \mathbb{Z}. \tag{5.4}
\]
Then the following inequality holds
\[
\sum_{n \in \mathbb{Z}} g_n^p \leq C_{p,h} \cdot \|g\|_p^{p'}, \quad C_{p,h} := \gamma_p \cdot e^{\gamma_p(h+1)}, \quad 1/p + 1/p' = 1. \tag{5.5}
\]
Proof. By definition \( \mathcal{F}[g](\lambda) = \sup_{x \in [0, 1]} |G_x(\lambda)| \), where \( G_x(\lambda) := \int_0^x g(t)e^{\lambda t}dt, \ x \in [0, 1] \), is an entire function. Hence the function \( |G_x(\cdot)|^{p'} \) is subharmonic, \( x \in [0, 1] \).

Let \( n \in \mathbb{Z} \) be fixed. It is clear that \( D_{1}(\lambda) \subset [n - 1, n + 2] \times [-h - 1, h + 1] \) for \( \lambda \in \Pi_{h,n} \). Combining subharmonic property with definition \( \Pi_{h,n} \) and this inclusion yields
\[
|G_x(\lambda)|^{p'} \leq \frac{1}{\pi} \int_{|t+iy-\lambda| \leq h} |G_x(t+iy)|^{p'} dt dy \leq \frac{1}{\pi} \int_{-h}^{h} \int_{-h}^{h} \mathcal{F}[g](\lambda)^{p'} dt dy, \quad x \in [0, 1], \ \lambda \in \Pi_{h,n}. \tag{5.6}
\]
Taking supremum over \( x \) and \( \lambda \) in \( \Pi_{h,n} \) with account of \( \Pi_{h,n} \), and then summing up resulting inequalities, and taking into account that the union of intervals \( [n - 1, n + 2] \), \( n \in \mathbb{Z} \), covers every point of \( \mathbb{R} \) exactly 3 times, we get
\[
\sum_{n \in \mathbb{Z}} g_n^p = \sum_{n \in \mathbb{Z}} \sup_{x \in [0, 1]} |G_x(\lambda)|^p \leq \frac{1}{\pi} \sum_{n \in \mathbb{Z}} \int_{-h}^{h} \int_{-h}^{h} \mathcal{F}[g](\lambda)^{p'} dt dy = \frac{3}{\pi} \int_{-h}^{h} \int_{-h}^{h} \mathcal{F}[g](\lambda)^{p'} dt dy. \tag{5.7}
\]
It is clear from the definition \( \Pi_{h,n} \) that
\[
\mathcal{F}[g](t+iy) = \mathcal{F}[\tilde{g}_y](t), \quad \tilde{g}_y(x) := g(x)e^{yx}, \quad t, y \in \mathbb{R}, \ x \in [0, 1]. \tag{5.8}
\]
Hence inequality \( \Pi_{h,n} \) implies
\[
\int_{-\infty}^{\infty} \mathcal{F}[g](\lambda)^{p'} dt = \int_{-\infty}^{\infty} \mathcal{F}[\tilde{g}_y](t) dt \leq \gamma_p \cdot \|\tilde{g}_y\|_p^{p'} \leq \gamma_p \cdot e^{\gamma_p|y|} \cdot \|g\|_p^{p'}, \quad y \in \mathbb{R}. \tag{5.9}
\]
Inserting \( \Pi_{h,n} \) into \( \Pi_{h,n} \) we arrive at the inequality
\[
\sum_{n \in \mathbb{Z}} g_n^p \leq \frac{3\gamma_p}{\pi} \cdot \|g\|_p^{p'} \int_{-h}^{h} e^{\gamma_p|y|} dy = C \cdot \|g\|_p^{p'}, \quad C := \frac{6\gamma_p}{\pi p'} \left( e^{\gamma_p(h+1)} - 1 \right) < \gamma_p e^{\gamma_p(h+1)} = C_{p,h}. \tag{5.10}
\]
To estimate \( C \) we used inequalities \( \pi > 3 \) and \( p' > 2 \). This finishes the proof of the desired estimate \( \Pi_{h,n} \). \( \square \)

Now we are ready to state the main result of this section being a generalization of the Hausdorff-Young and Hardy-Littlewood theorems to the case of non-harmonic series with exponents forming an incompressible sequence \( \Lambda = \{\mu_n\}_{n \in \mathbb{Z}} \) of density \( d(\mu) \) instead of \( \Lambda = \{2\pi n\}_{n \in \mathbb{Z}} \).

Theorem 5.4. Let \( p \in (1, 2] \) and let \( \Lambda = \{\mu_n\}_{n \in \mathbb{Z}} \) be an incompressible sequence of density \( d \in \mathbb{N} \) lying in the strip \( \Pi_{h} \). Then there exists \( C = C(p, h, d) > 0 \) that does not depend on \( \Lambda \) and such that the following estimates hold uniformly with respect to \( g \) and \( \Lambda \)
\[
\sum_{n \in \mathbb{Z}} \mathcal{F}[g](\mu_n) \leq C \cdot \|g\|_p^{p'}, \quad g \in L^p[0, 1], \quad 1/p' + 1/p = 1. \tag{5.11}
\]
\[
\sum_{n \in \mathbb{Z}} (1 + |n|)^{p-2} \mathcal{F}[g](\mu_n) \leq C \cdot \|g\|_p^p, \quad g \in L^p[0, 1]. \tag{5.12}
\]
Proof. For brevity let us set
\[
\mathcal{F}_{f} := \mathcal{F}[f], \quad \mathcal{F}_{\lambda}(\cdot) := \left( \mathcal{F}_{f}(\lambda \cdot) \right)^{\alpha}, \quad f \in L^1[0, 1], \ \lambda \in \mathbb{C}, \ \alpha > 0. \tag{5.13}
\]
(1) Set \( m_n := \lfloor \text{Re} \mu_n \rfloor, \ n \in \mathbb{Z}, \) where \( \lfloor x \rfloor \) denotes the integer part of \( x \). Since \( \{\mu_n\}_{n \in \mathbb{Z}} \) is an incompressible sequence of density \( d \), it follows that for every \( k \in \mathbb{Z} \) there are at most \( d \) indexes \( n \in \mathbb{Z} \) with \( m_n = k \). It is clear from the definition \( \Pi_{h,n} \) of \( g_n \) that \( \mathcal{F}_{g}(\mu_n) \leq g_{m_n}, \ n \in \mathbb{Z} \). Summing up the \( p' \)th powers of these inequalities, setting
\[ C := d \cdot C_{p,h}, \text{ then applying Lemma } 5.3 \text{ (inequality } 5.5 \text{) and taking observation on multiplicities of } m_n \text{ into account, we arrive at} \]
\[ \sum_{n \in \mathbb{Z}} \mathcal{F}_g^p(\mu_n) \leq \sum_{n \in \mathbb{Z}} g_{m_n}^p \leq d \sum_{k \in \mathbb{Z}} g_{m_k}^p \leq C \cdot \|g\|_p^p. \] (5.14)

(ii) Now let us prove (5.12). Consider a mapping
\[ T : f \rightarrow \{n \mathcal{F}_f(\mu_n)\}_{n \in \mathbb{Z}} \] (5.15)
defined on \( L^1[0,1] \). Equip the set of integers \( \mathbb{Z} \) with a (finite) discrete measure \( \nu \) by setting \( \nu(n) = (1 + |n|)^{-2} \), \( n \in \mathbb{Z} \). Inequality (5.11) applied with \( g = f \) and \( p = 2 \) implies that
\[ \|Tf\|_{l^2(\mathbb{Z};\nu)}^2 = \sum_{n \in \mathbb{Z}} |(Tf)(n)|^2 = \sum_{n \in \mathbb{Z}} \frac{n^2}{(1 + |n|)^2} \mathcal{F}_{\mathcal{F}}^2(\mu_n) \leq \sum_{n \in \mathbb{Z}} \mathcal{F}_{\mathcal{F}}^2(\mu_n) \leq N_2^2 \|f\|_2^2, \] (5.16)
where \( N_2 = C(2,h,d) \), i.e. the mapping \( T \) boundedly maps \( L^2[0,1] \) into the weighted space \( l^2(\mathbb{Z};\nu) \) and the norm of this mapping is at most \( N_2 \).

Next denote by \( l^1_w(\mathbb{Z};\nu) \) a weak \( l^1(\mathbb{Z};\nu) \) space. We show that the mapping
\[ T : L^1[0,1] \rightarrow l^1_w(\mathbb{Z};\nu) \] (5.17)
is well defined. To this end we set \( E_t := \{n \in \mathbb{Z} : |n| \mathcal{F}_f(\mu_n) > t\} \). Since \( |\mathcal{F}_f(\mu_n)| \leq h, n \in \mathbb{Z} \), it follows from (5.14) that \( \mathcal{F}_f(\mu_n) \leq e^h \cdot \|f\|_1, \) \( n \in \mathbb{Z} \). Therefore one gets
\[ \nu(E_t) = \sum_{|n| \mathcal{F}_f(\mu_n) > t} (1 + |n|)^{-2} \leq \sum_{|n| > t/(e^h \cdot \|f\|_1)} (1 + |n|)^{-2} \leq \frac{2 e^h \|f\|_1}{t}. \] (5.18)

This estimate means that \( Tf \in l^1_w(\mathbb{Z};\nu) \) for each \( f \in L^1[0,1], \) i.e. the mapping \( T \) has a weak type \((1,1)\) with a norm not exceeding \( N_1 := 2 e^h \). Since for each \( \lambda \in \mathbb{C} \), the functional \( \mathcal{F}_f(\lambda) \) is sublinear,
\[ \mathcal{F}_{f_1+f_2}(\lambda) \leq \mathcal{F}_{f_1}(\lambda) + \mathcal{F}_{f_2}(\lambda), \quad f_1,f_2 \in L^1[0,1], \] (5.19)
it follows that \( T \) is a quasilinear operator with parameter \( \kappa \leq 2 \) in both cases we considered. Combining estimates (5.16) and (5.18) and applying the Marcinkiewicz theorem ([4, Theorem 1.3.1], [54, Theorem XII.4.6]) we conclude that the mapping \( T \) is of type \((p,p)\) for each \( p \in (1,2] \) with a norm not exceeding \( c_{\kappa,p}N_1^{2p-1}N_2^{-2/p} \) for some constant \( c_{\kappa,p} > 0 \), which proves (5.12).

Corollary 5.5. Let \( \Lambda = \{\mu_n\}_{n \in \mathbb{Z}} \) be sequence of zeros of a sine-type function \( \Phi(\cdot) \) with the width of indicator diagram 1. Then for any \( p \in (1,2] \) estimates (5.11) and (5.12) hold uniformly in \( g \in L^p[0,1] \) and \( \Lambda \).

Proof. The proof is immediate from Theorem 5.4 if one notes that the null set of sine-type function \( \Phi(\cdot) \) is always incompressible (see [23], [22], and Proposition 4.11(ii)).

Next we present a version of Bessel type inequalities, where the maximal version \( \mathcal{F} \) of Fourier transform is replaced by the classical one. It is an immediate consequence of Theorem 5.4. However, we present a direct proof which is elementary in character because it does not involve Carleson-Hunt Theorem 5.1.

Proposition 5.6. Let \( p \in (1,2], \) \( 1/p' + 1/p = 1, \) let \( \Lambda = \{\mu_n\}_{n \in \mathbb{Z}} \) be an incompressible sequence of density \( d \in \mathbb{N} \) lying in the strip \( \Pi_h, \) and \( G(\lambda) = G_0(\lambda) := \int_0^1 g(t)e^{ib\lambda t} \, dt. \) Then there exists \( C = C(b,p,h,d) > 0 \) that does not depend on \( \Lambda \) and such that the following estimates hold uniformly with respect to \( g \in L^p[0,1] \) and \( \Lambda \)
\[ \sum_{n \in \mathbb{Z}} |G(\mu_n)|^p = \sum_{n \in \mathbb{Z}} \left| \int_0^1 g(t)e^{ib\mu_nt} \, dt \right|^p \leq C \cdot \|g\|_p^p, \quad g \in L^p[0,1]. \] (5.20)
\[ \sum_{n \in \mathbb{Z}} (1 + |n|)^{p-2} |G(\mu_n)|^p \leq C \cdot \|g\|_p^p, \quad g \in L^p[0,1]. \] (5.21)
Proof. (i) It follows from the definition of $G(\cdot)$ that it is an entire function of exponential type not exceeding $|b|$. Moreover, since $p \in (1, 2]$, the Hausdorff-Young inequality for the Fourier transform (see e.g. [4 Theorem 1.2.1], [51 Chapter 5.1]) ensures that $G(\cdot) \in L^p(\mathbb{R})$, and

$$\|G\|_{L^p} := \left( \int_{-\infty}^{+\infty} |G(\lambda)|^p \, d\lambda \right)^{1/p} \leq C_1(b, p) \cdot \|g\|_p, \quad p \in (1, 2],$$

where $C_1(b) > 0$ does not depend on $p$ and $g$. Therefore $G \in L^p_\sigma$, with $\sigma := |b|$.

To evaluate the left-hand side of (5.20) we repeat the reasoning of [22, Lemma 2](see also [26, Section 20.1]) paying attention to the dependence of the constant $C$ on parameters involved. Since $\Lambda$ is an incompressible sequence of the density $d$ lying in the strip $\Pi_\delta$, then every point of $\mathbb{C}$ is covered by at most $d$ closed discs $D_1(\mu_n), n \in \mathbb{Z}$. Combining this fact with the subharmonicity of $|G(z)|^{p'}$ implies

$$\sum_{n \in \mathbb{Z}} |G(\mu_n)|^{p'} \leq \frac{1}{\pi} \sum_{n \in \mathbb{Z} | \lambda_n | \leq 1} \iint |G(x+iy)|^{p'} \, dx \, dy \leq \frac{d}{\pi} \iint |G(x+iy)|^{p'} \, dx \, dy \leq \frac{2d}{\pi} \int_0^{h+1} e^{t|b|y} \, dy \cdot \|G\|_{L^p}^{p'}.$$  

Combining estimate (5.22) with (5.23) and setting

$$C_2(b, p, h, d) := \frac{2d}{\pi} \int_0^{h+1} e^{t|b|y} \, dy = \frac{2d(e^{p'(b(h+1))} - 1)}{p'(|b|)} \quad \text{and} \quad C(b, p, h, d) = C_2(b, p, h, d) \cdot C_1(b, p)^{p'},$$

we arrive at (5.20).

(ii) Now we define the operator $T$ by formula (5.15) but with $F$ instead of $\mathcal{F}$, i.e. we put

$$T : f \to \{nG_f(\mu_n)\}_{n \in \mathbb{Z}}.$$  

The rest of reasoning is just a repetition of that in the proof of Theorem 5.4 while instead of (5.11) we apply estimate (5.20) with $p = 2$.

Next we also extend inverse statements of the Hausdorff-Young and Hardy-Littlewood theorems to the case of non-harmonic exponentials series with exponents $\Lambda = \{\mu_n\}_{n \in \mathbb{Z}}$ forming the null set of a sine type entire function instead of $\Lambda = \{2\pi n\}_{n \in \mathbb{Z}}$.

**Proposition 5.7.** Let $p \in (1, 2]$, $p' = p/(p-1)$. Let $F(\cdot)$ be a sine-type function with the width of indicator diagram $1$. Assume also that a sequence of its zeros $\Lambda = \{\mu_n\}_{n \in \mathbb{Z}}$ is separated. Then there exists $C = C(p, \Lambda) > 0$ such that the following statements hold:

(i) For any sequence $\{a_n\}_{n \in \mathbb{Z}} \in l^p(\mathbb{Z})$ the series

$$\sum_{n \in \mathbb{Z}} a_ne^{i\mu_n x} (= : f(x))$$

converges in $L^p[0, 1]$ to a certain function $f \in L^p[0, 1]$ and the following estimate holds

$$\|f\|_{L^p[0, 1]} \leq C \cdot \|a\|_{l^p(\mathbb{Z})}.$$  

(ii) For any sequence $\{a_n\}_{n \in \mathbb{Z}} \in l^p'(\mathbb{Z}; (1 + |n|)^{p'-2})$ the series (5.22) converges in $L^{p'}[0, 1]$ to a certain function $f \in L^{p'}[0, 1]$ and the following estimate holds

$$\|f\|_{L^{p'}[0, 1]} \leq C \cdot \sum_{n \in \mathbb{Z}} (1 + |n|)^{p'-2} |a_n|^{p'}.$$  

**Proof.** (i) Consider a mapping

$$a = \{a_n\}_{n \in \mathbb{Z}} \to f(x) = \sum_{n \in \mathbb{Z}} a_ne^{i\mu_n x}.$$  


Since \( \Lambda \) is a separated set of zeros of a sine-type function, Levin’s theorem (see [26, Theorem 23.2]) ensures that the sequence of exponentials \( \{ e^{i\mu_n t} \}_{n \in \mathbb{Z}} \) forms a Riesz basis in \( L^2[0, 1] \). It implies, in particular, that for each sequence \( \{ a_n \}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z}) \) the series (5.26) converges in \( L^2[0, 1] \) and the following estimate holds

\[
\| f \|_{L^2[0, 1]} \leq C(2, \Lambda) \left( \sum_{n \in \mathbb{Z}} |a_n|^2 \right)^{1/2} = C(2, \Lambda) \| a \|_{l^2(\mathbb{Z})}.
\]  

(5.30)

Further, let \( p = 1 \) and \( \{ a_n \} \in l^1(\mathbb{Z}) \). Since \( F(\cdot) \) is a sine-type function, the sequence \( \Lambda \) of its zeros lie in a horizontal strip \( \Pi_h \) for some \( h \geq 0 \). Therefore one gets that the series (5.26) converges absolutely (he uniformly) and determines a bounded (in fact, continuous) function satisfying

\[
|f(x)| \leq \exp(hx) \cdot \| a \|_{l^1(\mathbb{Z})} \leq \exp(h) \cdot \| a \|_{l^1(\mathbb{Z})}, \quad x \in [0, 1].
\]  

(5.31)

Thus estimate (5.27) holds with \( p = 1 \) and \( p = 2 \). It remains to apply Riesz-Torin theorem (see [53, Theorem XII.1.11], [4, Theorem 1.1.1], [51, Theorem V.1.3])

(ii) By Levin-Golovin theorem (26, Theorem 23.2), the system of exponentials \( \{ e^{i\mu_n x} \}_{n \in \mathbb{Z}} \) forms a Riesz basis in \( L^2[0, 1] \). Noting that \( \{ a_n \}_{n \in \mathbb{Z}} \in l^p(\mathbb{Z}; (1 + |n|)^{p-2}) \subset l^2(\mathbb{Z}) \) one gets that the series \( \sum_{n \in \mathbb{Z}} a_n e^{i\mu_n x} \) converges in \( L^2[0, 1] \). Setting \( s_m = \sum_{|n| \leq m} a_n e_n \) we show that, in fact, this series converges in \( L^q[0, 1] \) for \( q = p' \in [2, \infty) \).

Let also \( \{ \chi_j \}_{j \in \mathbb{Z}} \) be biorthogonal in \( L^2(0,1) \) to the sequence \( \{ e^{i\mu_n x} \}_{n \in \mathbb{Z}} \), i.e.

\[
(e_n, \chi_j) = \int_0^1 e^{i\mu_n x} \chi_j(t) dt = F_j(\mu_n) = \delta_{n,j}, \quad F_j(z) := \frac{F(z)}{F'(\mu_n)(z - \mu_n)}.
\]

This sequence forms a Riesz basis in \( L^2[0,1] \) alongside with \( \{ e_n \} \). Therefore, any \( f \in L^2[0,1] \) admits a decomposition \( g = \sum_{n \in \mathbb{Z}} b_n \chi_n \) with \( \{ b_n \}_{n \in \mathbb{Z}} \in l^2(\mathbb{Z}) \), and applying Hölder’s inequality and Theorem 5.4 (ii) one gets

\[
\left| \int_0^1 s_m(t)g(t) dt \right| = \left| \left( \sum_{|n| \leq m} a_n e_n, \sum_{j \in \mathbb{Z}} b_j \chi_j \right) \right| = \left| \sum_{|n| \leq m} a_n b_n \right| \leq \left( \sum_{|n| \leq m} |a_n|^q (1 + |n|)^{q-2} \right)^{1/q} \cdot \left( \sum_{|n| \leq m} |b_n|^p (1 + |n|)^{p-2} \right)^{1/p} \leq C_p \| g \|_p \left( \sum_{|n| \leq m} |a_n|^q (1 + |n|)^{q-2} \right)^{1/q}.
\]

(5.32)

Since \( L^2[0,1] \subset L^p[0,1] \) for \( p \in (1,2] \), this inequality is extended to \( g \in L^p[0,1] \). Taking then supremum over \( g \) running through the unit ball in \( L^p[0,1] \) one derives \( \| s_m \|_q^q \leq C_p \sum_{|n| \leq m} |a_n|^q (1 + |n|)^{q-2} \). It follows with account of the condition \( \{ a_n \}_{n \in \mathbb{Z}} \in l^q(\mathbb{Z}; (1 + |n|)^{q-2}) \) that

\[
\| s_k - s_m \|_q^q \leq C_p \sum_{k \leq |n| \leq m + 1} |a_n|^q (1 + |n|)^{q-2} \to 0 \quad \text{as} \quad k, m \to \infty,
\]  

(5.33)

i.e. the sequence \( \{ s_m \}_{m \in \mathbb{Z}} \) is a Cauchy sequence. Thus, there exists \( f \in L^q[0,1] \) such that \( \lim_{m \to \infty} \| s_m - f \|_q = 0 \). Passing to the limit in (5.32) as \( m \to \infty \) we arrive at (5.27). \( \square \)

**Remark 5.8.** The proof of Lemma 5.3 extends the classical reasoning on estimates of Hardy space functions and \( L^p_\sigma \)-classes of entire functions (see [26, Lectures 20-21, 22, Lemma 2]) to the case of maximal Fourier transform.

In the proof of Theorem 5.4(ii) (inequality (5.12)) and Proposition 5.7(ii) we generalize the reasoning of the proof of [54, Theorem 12.3.19]. When the paper was almost ready we found out that inequality (5.11) was proved in the recent paper [49]. We have kept the proof for reader’s convenience.
5.2. Uniform versions of Riemann-Lebesgue Lemma. The following result will be needed in the sequel and easily follows by combining Lemma 5.3 with Chebyshev’s inequality. It can be treated as a uniform (with respect to \( g \in U_{p,r} \)) version of the classical Riemann-Lebesgue Lemma.

**Lemma 5.9.** Let \( g \in U_{p,r} \) for some \( p \in (1,2) \) and \( r > 0 \). Let also \( b \in \mathbb{R} \setminus \{0\} \) and \( h \geq 0 \). Then for any \( \delta > 0 \) there exists a set \( I_{g,\delta} \subset \mathbb{Z} \) such that the following inequalities hold uniformly with respect to \( g \in U_{p,r} \),

\[
\text{card}(\mathbb{Z} \setminus I_{g,\delta}) \leq N_{\delta} := C \cdot (r/\delta)^{p'}, \quad 1/p' + 1/p = 1,
\]

\[
\mathcal{F}[g](b\lambda) = \sup_{x \in [0,1]} \left| \int_0^x g(t)e^{ib\lambda t} dt \right| < \delta, \quad \lambda = \bigcup_{n \in I_{g,\delta}} [n,n+1) \times [-h,h].
\]

Here \( C = C(p,b,h) > 0 \) does not depend on \( g, r \) and \( \delta \).

**Proof.** Note that the proof of Lemma 5.3 remains the same if we replace \( \mathcal{F}[g](x+iy) \) with \( \mathcal{F}[g](b(x+iy)) \) in the definition (5.3) of \( g_n \). The only change is that the constant \( C_{p,h} \) depends also on \( b \) now. Below we assume this adjusted definition of \( g_n \).

Let \( \delta > 0 \). Let us prove that \( I_{g,\delta} := \{ n \in \mathbb{Z} : g_n < \delta \} \) satisfies (5.34)–(5.35). Inequality (5.35) immediately follows from the definition of \( g_n \), \( \mathcal{F}[g] \), and \( I_{g,\delta} \). Applying inequality (5.5) to estimate the cardinality of \( I_{g,\delta} \) we derive

\[
Cp' \geq ||g||_p^{p'} \geq \sum_{n \in \mathbb{Z}} g_n^{p'} \geq \sum_{n \notin I_{g,\delta}} g_n^{p'} \geq \sum_{n \notin I_{g,\delta}} \delta^{p'} = \text{card}(\mathbb{Z} \setminus I_{g,\delta}) \cdot \delta^{p'}.
\]

In turn, this yields (5.34). \( \square \)

Let us emphasize that the term “uniformity” in Lemma 5.3 does not relate to the set \( I_{g,\delta} \) that depends on \( g \), but only to the “size” of its complement (see (5.34)). Note also that Lemma 5.9 for regular Fourier transforms can be proved easier without the use of the deep Carleson-Hunt theorem.

**Lemma 5.10.** Let \( g \in U_{p,r} \) for some \( p \in (1,2) \) and \( r > 0 \). Let also \( b \in \mathbb{R} \setminus \{0\} \) and \( h \geq 0 \). Then for any \( \delta > 0 \) there exists a set \( I_{g,\delta} \subset \mathbb{Z} \) such that the following inequalities hold uniformly with respect to \( g \in U_{p,r} \),

\[
\text{card}(\mathbb{Z} \setminus I_{g,\delta}) \leq N_{\delta} := C \cdot (r/\delta)^{p'}, \quad 1/p' + 1/p = 1,
\]

\[
\mathcal{F}[g](b\lambda) := \left| \int_0^1 g(t)e^{ib\lambda t} dt \right| < \delta, \quad \lambda = \bigcup_{n \in I_{g,\delta}} [n,n+1) \times [-h,h].
\]

Here \( C = C(p,b,h) > 0 \) does not depend on \( g, r \) and \( \delta \).

**Proof.** First we can prove the version of Lemma 5.3 with \( F[g](t+iy) \) in place of \( \mathcal{F}[g](t+iy) \) and use classical Hausdorff-Young theorem [51, Chapter 5.1] for Fourier transforms instead of the deep inequality (5.3) in transition (5.3). After that the proof follows the proof of Lemma 5.9. \( \square \)

Next we investigate the maximal version of Fourier transform defined on the space \( X_{\infty,1}(\Omega) \) by:

\[
\mathcal{F}[G](\lambda) := \sup_{x \in [0,1]} \left| \int_0^x G(x,t)e^{ib\lambda t} dt \right|, \quad \lambda \in \mathbb{C}.
\]

First we present the following “uniform” version of the Riemann-Lebesgue lemma for the space \( X_{\infty,1}^0(\Omega) \). To this end for any \( h \geq 0 \) we let

\[
C_0(\Pi_h) := \{ \varphi \in C(\Pi_h) : \lim_{t \to \pm \infty} \varphi(t \pm iy) = 0 \text{ uniformly in } y \in [-h,h] \}.
\]

**Proposition 5.11.** Let \( h \geq 0 \) and let \( F \) be given by (5.39). Then:

(i) The nonlinear mapping \( F : X_{\infty,1}^0(\Omega) \to C(\Pi_h) \) is well defined and it is Lipschitz, i.e.

\[
\|F[G] - F[\tilde{G}]\|_{C(\Pi_h)} \leq e^{\|b\|h} \cdot \|G - \tilde{G}\|_{X_{\infty,1}^0(\Omega)}, \quad G, \tilde{G} \in X_{\infty,1}^0(\Omega).
\]

(ii) For any \( h \geq 0 \) the mapping \( F \) continuously maps \( X_{\infty,1}^0(\Omega) \) into \( C_0(\Pi_h) \).

(iii) For any compact set $X$ in $X_{0,1}^0(\Omega)$ the following relation holds

$$\lim_{\lambda \to \infty} \mathcal{F}[G](\lambda) = 0 \quad \text{uniformly in} \quad G \in X \quad \text{and} \quad \lambda \in \Pi_h.$$  

(5.41)

**Proof.** (i) Let $G \in X_{0,1}^0(\Omega)$. By Proposition 2.3(ii) clearly valid in the scalar case too, the operator $f \to \int_0^x G(x,t)f(t) \, dt$ maps $L^\infty[0,1]$ into $C[0,1]$. To prove the continuity of $\mathcal{F}[G](\cdot)$, given an $\varepsilon > 0$ one finds $\delta > 0$, such that $\max_{x \in [0,1]} |1 - \exp(ib\mu t)| < \varepsilon$ whenever $|\mu| < \delta$. Clearly,

$$\max_{x \in [0,1]} \left| \int_0^x G(x,t)e^{ib\lambda x} \, dt \right| \leq \max_{x \in [0,1]} \left| \int_0^x G(x,t)e^{ib\lambda x} \, dt \right| + \max_{x \in [0,1]} \left| \int_0^x G(x,t)(e^{ib\lambda x} - e^{ib\lambda x}) \, dt \right|. \quad \text{(5.42)}$$

Interchanging $\lambda_1$ and $\lambda_2$, combining both inequalities, and setting $C(\lambda_1, \lambda_2) := \min\{e^{-b\lambda_1}, e^{-b\lambda_2}\}$, one arrives at

$$|\mathcal{F}[G](\lambda_2) - \mathcal{F}[G](\lambda_1)| \leq \max_{x \in [0,1]} \left| \int_0^x G(x,t)(e^{ib\lambda_2 x} - e^{ib\lambda_1 x}) \, dt \right| \leq \varepsilon C(\lambda_1, \lambda_2) \cdot \|G\|_{X_{0,1}^0(\Omega)}, \quad \text{(5.43)}$$

whenever $|\lambda_2 - \lambda_1| < \delta$. Thus, the function $\mathcal{F}[G](\cdot)$ belongs to $C(\Pi_h)$ and is uniformly continuous.

Next, let us establish the Lipschitz property of $\mathcal{F} : X_{0,1}^0(\Omega) \to C(\Pi_h)$. Given $G, \tilde{G} \in X_{0,1}^0(\Omega)$, one gets

$$|\mathcal{F}[G](\lambda) - \mathcal{F}[	ilde{G}](\lambda)| \leq \max_{x \in [0,1]} \left| \int_0^x (G(x,t) - \tilde{G}(x,t)) e^{ib\lambda x} \, dt \right| \leq \|G - \tilde{G}\|_{X_{0,1}^0(\Omega)}. \quad \text{(5.44)}$$

(ii) Recall, that in accordance with the definition of $X_{0,1}^0(\Omega)$, the set $C(\Omega)$, hence the space $C^1(\Omega)$, is dense in $X_{0,1}^0(\Omega)$. Fix $G \in C^1(\Omega)$ and integrating by parts one gets

$$\left| \int_0^x G_1(x,t)e^{ib\lambda t} \, dt \right| \leq \frac{1}{|b|} \left( |G_1(x,x)| \cdot |e^{ib\lambda x}| + |G_1(x,0)| + \int_0^x |D_t G_1(x,t)e^{ib\lambda t}| \, dt \right) \leq \frac{1}{|b|} \left( |G_1(x,x)| + |G_1(x,0)| + \int_0^x |D_t G_1(x,t)| \, dt \right) \cdot (e^{-b\lambda} + 1) \leq \frac{3\|G_1\|_{C^1(\Omega)}}{|b|} \cdot \frac{(e^{-b\lambda} + 1)}{|\Re \lambda|}, \quad x \in [0,1]. \quad \text{(5.45)}$$

Thus, $\lim_{\lambda \to \infty} \mathcal{F}[G](t + iy) = 0$ uniformly in $y \in [-h, h]$, and $\mathcal{F}[G_1](\cdot) \in C_0(\Pi_h)$. Combining this relation with estimate (5.44) yields similar relation for $\mathcal{F}[G]$ with any $G \in X_{0,1}^0(\Omega)$, i.e. $\mathcal{F}[G](\cdot) \in C_0(\Pi_h)$ for any $G \in X_{0,1}^0(\Omega)$.

(iii) By (ii), the mapping $\mathcal{F}$ continuously maps $X_{0,1}^0(\Omega)$ into $C_0(\Pi_h)$. Therefore the image $\mathcal{F}(X)$ of a compact set $X$ is also compact in $C_0(\Pi_h)$. To derive uniform relation (5.41) it remains to apply the necessary condition of compactness in $C_0(\Pi_h)$ (uniform smallness of “tails”).

Proposition 5.11(iii) contains as a special case the following “uniform” version of the classical Riemann-Lebesgue Lemma. Namely, for any compact $K$ in $L^1[0,1]$ one has:

$$\sup_{g \in K} \left| \int_0^1 g(t)e^{i\lambda t} \, dt \right| = o(1) \quad \text{as} \quad \lambda \to \infty \quad \text{uniformly in} \quad g \in K \quad \text{and} \quad \lambda \in \Pi_h. \quad \text{(5.46)}$$

Next we complete Proposition 5.11 by evaluating the “maximal” Fourier transform $\mathcal{F}[G](\cdot)$ in the plane instead of a strip. This statement will be useful in Section 6 when applying the Roué theorem.

**Lemma 5.12.** Let $X$ be a compact set in $X_{0,1}^0(\Omega), \ b \in \mathbb{R} \setminus \{0\}$ and $\delta > 0$. Then there exists a constant $C = C(X, b, \delta) > 0$ such that the following uniform in $G \in X$ estimate takes place

$$\mathcal{F}[G](\lambda) \leq \delta (e^{-b\lambda} + 1), \quad |\lambda| > C, \quad G \in X. \quad \text{(5.47)}$$

**Proof.** Since $C^1(\Omega)$ is dense in $X_{0,1}^0(\Omega)$ and $X \subset X_{0,1}^0(\Omega)$ is compact, there exists a finite $\delta/2$-net $\{G_1, \ldots, G_n\}$ for $X$, such that $G_j \in C^1(\Omega)$. Let $G \in X$, then for some $j \in \{1, \ldots, n\}$ we have $\|G - G_j\|_{X_{0,1}^0(\Omega)} < \delta/2$. It is clear that

$$\left| \int_0^x (G(x,t) - G_j(x,t)) e^{ib\lambda t} \, dt \right| \leq \|G - G_j\|_{X_{0,1}^0(\Omega)} \max_{t \in [0,1]} |e^{ib\lambda t}| < \frac{\delta}{2} (e^{-b\lambda} + 1), \quad x \in [0,1], \quad \lambda \in \mathbb{C}. \quad \text{(5.48)}$$
Repeating the deduction of estimate (5.45) one derives
\[
\left| \int_0^x G_j(x,t)e^{ib\lambda t} \, dt \right| \leq \frac{3\|G_j\|_{C^1(T)}}{|b\lambda|} (e^{-b\lambda\epsilon_0} + 1) < \frac{\delta}{2} (e^{-b\lambda\epsilon_0} + 1), \quad x \in [0, 1], \quad |\lambda| > C,
\]
with \( C = \frac{6}{\epsilon_0} \max \{ \|G_k\|_{C^1(T)} : k \in \{1, \ldots, n\} \} \). Clearly, \( C \) only depends on \( \lambda, b, \) and \( \delta \). Combining (5.48) with (5.49) we arrive at (5.47).

Finally we apply Proposition 5.11 and Lemma 5.12 to transformation operators.

**Corollary 5.13.** Let \( K_Q^\pm \) be the kernel of transformation operator from representation (3.2). Then the composition \( Q \to K_Q^\pm \to \mathcal{F}[K_Q^\pm] \) continuously maps \( L^p([0, 1]; \mathbb{C}^{2 \times 2}) \) into \( C_0(\Pi_h; \mathbb{C}^{2 \times 2}) \), \( h \geq 0 \), and is a Lipschitz mapping on balls in \( L^p([0, 1]; \mathbb{C}^{2 \times 2}) \), \( p \in [1, \infty) \), i.e.
\[
\|\mathcal{F}[K_Q^\pm] - \mathcal{F}[K_Q^\pm]\|_{C(\Pi_h)} \leq e^{\|h\| \cdot C(p, r) \cdot \|Q - \bar{Q}\|_{L^p}}, \quad Q, \bar{Q} \in \mathbb{U}_{p,r}^{2 \times 2}.
\]

**Proof.** The proof is immediate by combining Proposition 5.11(iii) with Theorem 5.3.

**Lemma 5.14.** Let \( K \) be a compact set in \( L^1([0, 1]; \mathbb{C}^{2 \times 2}) \) and \( Q \in K \). Let also \( K^\pm = K_Q^\pm \) be the kernel of the transformation operator from representation (3.2). Then for any \( \delta > 0 \) there exists a constant \( M = M(K, B, \delta) > 0 \) such that the following estimate takes place uniformly in \( Q \in K \)
\[
\mathcal{F}[K^\pm](\lambda) = \sup_{x \in [0, 1]} \left| \int_0^x K^\pm_{jk}(x,t)e^{ib\lambda t} \, dt \right| \leq \delta (e^{-b\lambda\epsilon_0} + 1), \quad |\lambda| > M, \quad j, k \in \{1, 2\}.
\]

In particular, for any \( h \geq 0 \) one has: \( \sup_{Q \in K} \mathcal{F}[K^\pm](\lambda) \to 0 \) as \( |\lambda| \to \infty \) and \( \lambda \in \Pi_h \).

**Proof.** By Theorem 5.3 the mapping \( T^\pm : Q \to K_Q^\pm(\cdot, \cdot) \) continuously maps \( L^1([0, 1]; \mathbb{C}^{2 \times 2}) \) into \( X^0_{\infty,1}(\Omega; \mathbb{C}^{2 \times 2}) \). Hence the image \( \{K^\pm(\cdot, \cdot) : Q \in K\} \) is compact in \( X^0_{\infty,1}(\Omega; \mathbb{C}^{2 \times 2}) \). Lemma 5.12 completes the proof.

**Corollary 5.15.** Let \( K \) be a ball either in the Sobolev spaces \( W^1_2([0, 1]) \) with \( s \in \mathbb{R}_+ \) or in the Lipschitz space \( L^1_\alpha([0, 1]) \), with \( \alpha \in (0, 1] \), or in the space \( V^1_0([0, 1]) \) of functions of bounded variation. Then relations (5.41) and (5.51) hold true uniformly in \( Q \in K \).

**Proof.** It well known that the balls in \( W^1_2([0, 1]) \) and \( L^1_\alpha([0, 1]) \) are relatively compact in \( L^p([0, 1]) \) because of compact embedding \( W^1_2([0, 1]) \to L^p([0, 1]) \) and \( L^1_\alpha([0, 1]) \to L^p([0, 1]) \). Besides, balls in \( V^1_0([0, 1]) \) are relatively compact in \( L^p([0, 1]) \) due to the second Helly’s theorem and Lebesgue’s dominated convergence theorem. It remains to apply Proposition 5.11 and Lemma 5.12.

**Remark 5.16.** (i) Let us present a simple example of the non-compact set in \( L^p([0, 1]) \) for which uniform relation (5.41) is violated. Consider the following set of functions
\[
\mathcal{G} := \{ g_\mu(x) := g_0(x)e^{-i\mu x} : \mu \in \mathbb{R} \}, \quad g_0 \in L^p([0, 1]), \quad c_0 := \int_0^1 g_0(t) \, dt > 0.
\]

It is clear that
\[
\mathcal{F}[g_\mu](\mu) \geq \int_0^1 g_0(t)e^{-i\mu t} \, dt = c_0 \neq 0 \quad \text{and} \quad \lim_{|\lambda| \to \infty} \mathcal{F}[g_\mu](\lambda) = 0.
\]

The last relation is satisfied not uniformly on \( \mathcal{G} \). Moreover, inequality (5.35) holds on sets \( \mathcal{I}_{\mu,\delta} = \mathbb{Z} \setminus (\mu - N_\delta, \mu + N_\delta) \) that depends on \( \mu \), and their complements “tend to infinity” when \( \mu \to \infty \), but have uniformly bounded “sizes”, \( \text{card}(\mathbb{Z} \setminus \mathcal{I}_{\mu,\delta}) \leq 2N_\delta \). We are indebted to V.P. Zastavnyi who has informed us about this example.

(ii) One can complete the scalar version of Proposition 2.3(ii) by proving that for any \( g(\cdot, \cdot) \in X_{\infty,1}(\Omega) \) the function \( \int_0^x G(x,t)e^{ib\lambda t} \, dt \) is continuous in two variables \( (x, \lambda) \), in particular, \( \mathcal{F}[G](\cdot) \in C(\Pi_h) \) for any \( h > 0 \).
6. Stability property of eigenvalues

6.1. Uniform localization of spectrum. In this subsection we will obtain uniform with respect to \( Q \in \mathcal{K} \) version of the asymptotic formula \((4.44)\), where \( \mathcal{K} \) is either a compact in \( L^1([0,1];\mathbb{C}^{2\times 2}) \) or \( \mathcal{K} = \bigcup_{p_r}^{2\times 2}, p \in (1,2) \).

First, we enhance Proposition \( 4.12 \) to get uniform estimates for \( Q \in \mathcal{K} \), where \( \mathcal{K} \) is compact in \( L^1 \). The following result generalizes \( [10, \text{Theorem 3}] \) to the case of Dirac-type system and regular boundary conditions.

**Proposition 6.1.** Let \( \mathcal{K} \) be compact in \( L^1([0,1];\mathbb{C}^{2\times 2}) \) and \( Q \in \mathcal{K} \). Let boundary conditions \((4.42)\) be regular, let \( \Delta(\cdot) := \Delta_Q(\cdot) \) be the corresponding characteristic determinant, and let \( \Lambda := \Lambda_Q := \{ \lambda_n \}_{n \in \mathbb{Z}} \) be canonically ordered sequence of its zeros. Let also \( \Lambda_0 = \{ \lambda^0_n \}_{n \in \mathbb{Z}} \) be the sequence of zeros \( \Delta_0 \). Then the following estimates hold:

(i) There exists a constant \( M = M(\mathcal{K}, B, A) > 0 \) that does not depend on \( Q \) and such that
\[
\sup_{n \in \mathbb{Z}} |\lambda_{Q,n} - \lambda^0_n| \leq M, \quad Q \in \mathcal{K}.
\]

(ii) For any \( \varepsilon > 0 \) there exists a constant \( N_\varepsilon = N_\varepsilon(\mathcal{K}, B, A) \in \mathbb{N} \) such that
\[
\sup_{|n| > N_\varepsilon} |\lambda_{Q,n} - \lambda^0_n| \leq \varepsilon, \quad Q \in \mathcal{K}.
\]

Proof. Recall, that the sequence \( \Lambda_0 = \{ \lambda^0_n \}_{n \in \mathbb{Z}} \) of zeros of \( \Delta_0 \) is incompressible, is of density \( d_0 \) and lies in the strip \( \Pi_{h_0} \) for some \( d_0 = d_0(B,A) \in \mathbb{N} \) and \( h_0 = h_0(B,A) > 0 \). Let \( 0 < \varepsilon \leq \varepsilon_0 = (2d_0)^{-1} = \varepsilon_0(B,A) \). Note that functions \( g_1 \) and \( g_2 \) from representation \((4.22)\) are linear combinations of kernel traces \( K_j^{\pm}(1, j), k \in \{1,2\} \). Repeating the proof of estimate \((4.45)\) but using Lemma \(6.14\) with \( x = 1 \) instead of \([31, \text{Lemma 3.5}]\), we see that this estimate is valid with a constant \( M_\varepsilon = M_\varepsilon(\mathcal{K}, B, A) > 0 \) that does not depend on \( Q \).

(i) For brevity we set \( \lambda_n := \lambda_{Q,n}, \in \mathbb{Z} \). Recall that for each \( n \in \mathbb{Z} \) and \( \varepsilon > 0 \) numbers \( \lambda_n \) and \( \lambda^0_n \) belong to the same connected component of \( \overline{\Omega} \), defined in \((4.43)\). Also recall that connected components of \((4.43)\) satisfy properties \((4.46)-(4.47)\). Hence for \( \lambda_n, \lambda^0_n \in \mathcal{C}_\varepsilon \) we have \( |\lambda_n - \lambda^0_n| < 2(M_\varepsilon + 1) \). All other components have diameter at most \( 1 \) and hence \( |\lambda_n - \lambda^0_n| < 1 \) for such \( n \). By fixing \( \varepsilon = \varepsilon_0 \) this yields \((6.1)\) with \( M = 2(M_\varepsilon + 1) \) that does not depend on \( Q \). In turn, Lemma \(6.10\) implies that \( \Delta_Q \) is an incompressible sequence of density \( d = d_0 \) \( [M_\varepsilon + 1] \). Since \( \Lambda_0 \subset \Pi_{h_0} \), inequality \((6.1)\) yields that \( \Delta_Q \subset \Pi_h \) for \( h = h_0 + M \).

(ii) Set \( \varepsilon := \min\{\varepsilon/(2d_0), \varepsilon_0\} \) and \( N_\varepsilon := \sup\{|n| : |\lambda^0_n| < M_\varepsilon + 1\} \). Since \( M_\varepsilon \) does not depend on \( Q \), then \( N_\varepsilon \) also does not depend on \( Q \). Let \( |n| > N_\varepsilon \). In this case, due to the choice of \( N_\varepsilon \) and inclusion \((4.47)\), numbers \( \lambda_n \) and \( \lambda^0_n \) do not belong to \( \overline{\mathcal{C}}_\varepsilon \). Recall that the sequence \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{N}} \) is canonically ordered. It means that \( \lambda_n, \lambda^0_n \in \mathcal{C}_{\varepsilon,k}^0 \) for some \( k \in \mathbb{Z} \). Inequality \((4.46)\) now yields that \( |\lambda_n - \lambda^0_n| < 2\varepsilon d_0 \leq \varepsilon, |n| > N_\varepsilon \), and finishes the proof of this part.

Finally, let us prove estimate \((6.3)\). Since boundary conditions \((4.22)\) are strictly regular, the sequence \( \Lambda_0 \) is asymptotically separated. Namely, for some \( \tau_0 = \tau_0(B,A) > 0 \) and \( N_0 = N_0(B,A) \in \mathbb{N} \) we have
\[
|\lambda^0_n - \lambda^0_m| > 2\tau_0, \quad n \neq m, \quad |n|, |m| > N_0.
\]

Let \( 0 < \varepsilon \leq \varepsilon_0 := \tau_0/3 \) and \( N_\varepsilon := \max\{N_0, \sup\{|n| : |\lambda^0_n| < M_\varepsilon + 3\varepsilon\}\} \). Here we redefined values of \( \varepsilon_0 \) and \( N_\varepsilon \) that we set above. From definition \((4.43)\) of \( \Omega_\varepsilon \), inequality \((6.3)\) and the choice of \( N_\varepsilon \) it follows that the discs \( \mathbb{D}_{\varepsilon,\lambda^0_n}(\lambda^0_n), |n| > N_\varepsilon \), are disjoint and do not intersect with \( \mathbb{D}_{\varepsilon,\lambda_n}(0) \). Hence rings \( \mathbb{D}_{\varepsilon,\lambda^0_n} - \mathbb{D}_{\varepsilon,\lambda^0_n} \subset \mathbb{C} \setminus \overline{\Omega}_\varepsilon, \quad |n| > N_\varepsilon \).

(6.5)

In particular, \( \mathcal{T}_\varepsilon(\lambda^0_n) \subset \mathbb{C} \setminus \overline{\Omega}_\varepsilon, \quad |n| > N_\varepsilon \). Combining \((4.44)\) and the Rouche theorem, implies that each disc \( \mathbb{D}_{\varepsilon,\lambda^0_n}, \quad |n| > N_\varepsilon \), contains exactly one (simple) zero of \( \Delta_Q(\cdot) \), i.e.,
\[
|\lambda_n - \lambda^0_n| < \varepsilon \leq \tau_0/3, \quad |n| > N_\varepsilon.
\]

(6.6)
It follows from (6.6) that
\[ \mathbb{D}_2(\lambda_0^n) \subset \mathbb{D}_2(\lambda_n) \subset \mathbb{D}_2(\lambda_0^n), \quad |n| > N_{\varepsilon}. \] (6.7)
Since the discs \( \mathbb{D}_2(\lambda_0^n), |n| > N_{\varepsilon}, \) are disjoint, inclusion (6.7) implies that the discs \( \mathbb{D}_2(\lambda Q, n), |n| > N_{\varepsilon}, \) are also disjoint. Inclusions (6.6) and (6.7) imply that \( \mathbb{T}_2(\lambda_n) \subset \mathbb{C} \setminus \Omega_{\varepsilon}. \) Hence combining (4.12) with (4.31) yields (6.3). \( \Box \)

Next we extend Proposition 6.1 to the case \( K = \mathbb{U}^{2\times 2}_{p,r}, \ p \in (1, 2). \) Part (i) remains valid but we substantially rely on the fact that \( p > 1. \) Part (ii) only remains valid if we relax inequality \( |n| > N_{\varepsilon} \) to an inclusion \( n \in I_{Q,\varepsilon} \), where complements of the sets \( I_{Q,\varepsilon} \) have uniformly bounded cardinalities over \( Q \in \mathbb{U}^{2\times 2}_{p,r}. \)

**Proposition 6.2.** Let \( Q \in \mathbb{U}^{2\times 2}_{p,r} \) for some \( p \in (1, 2) \) and \( r > 0. \) Let boundary conditions (4.22) be regular, let \( \Delta() := \Delta_Q() \) be the corresponding characteristic determinant, and let \( \Lambda := \Lambda_Q = \{\lambda_Q n\}_{n \in \mathbb{Z}} \) be a canonically ordered sequence of its zeros. Then the following statements hold true:

(i) There exists a constant \( M = M(p,r,B,A) > 0, \) not dependent on \( Q, \) such that
\[ \sup_{n \in \mathbb{Z}} |\lambda_Q n - \lambda^n_0| \leq M, \quad Q \in \mathbb{U}^{2\times 2}_{p,r}. \] (6.8)

In particular, there exist constants \( h = h(p,r,B,A) \geq 0 \) and \( d = d(p,r,B,A) > 0, \) not dependent on \( Q, \) such that \( \Lambda_Q \) is an incompressible sequence of density \( d \) and lying in the strip \( \Pi_k. \)

(ii) For any \( \varepsilon > 0 \) there exists \( N_{\varepsilon} = N_{\varepsilon}(p,r,B,A) \in \mathbb{N} \) that do not depend on \( Q, \) and a set \( I_{Q,\varepsilon} \subset \mathbb{Z} \) such that,
\[ \text{card}(\mathbb{Z} \setminus I_{Q,\varepsilon}) \leq N_{\varepsilon}, \quad |\lambda_n - \lambda^n_0| < \varepsilon, \quad n \in I_{Q,\varepsilon}. \] (6.9)

(iii) If boundary conditions (4.22) are strictly regular then there exists \( \varepsilon_0 = \varepsilon_0(B,A) \) such that for any \( \varepsilon \in (0, \varepsilon_0) \) the discs \( \mathbb{D}_2(\lambda_n), n \in I_{Q,\varepsilon}, \) are disjoint, and there exists \( \tilde{C}_\varepsilon = \tilde{C}_\varepsilon(B,A) > 0, \) not dependent on \( Q, p \) and \( r, \) such that
\[ \min_{|\lambda - \lambda_0^n| = 2\varepsilon} |\Delta(\lambda)| \geq \tilde{C}_\varepsilon, \quad n \in I_{Q,\varepsilon}. \] (6.10)

where \( I_{Q,\varepsilon} \subset \mathbb{Z} \) satisfies (6.9) - (6.10).

**Proof.** (i) By Lemma 4.4, the difference \( \Delta_Q(\lambda) - \Delta_0(\lambda) \) admits representation (4.22) and hence
\[ |\Delta_Q(\lambda) - \Delta_0(\lambda)| \leq \left| \int_0^1 g_1(t)e^{ib_1\lambda t} dt \right| + \left| \int_0^1 g_2(t)e^{ib_2\lambda t} dt \right|, \quad \lambda \in \mathbb{C}. \] (6.12)

In accordance with (4.22) the uniform estimate \( \|g_1\| |p| + \|g_2\| |p| \leq \tilde{C}_{\varepsilon} \|Q\| |p| \leq \tilde{C}_\varepsilon |p| =: R \) holds, where \( \tilde{C} = \tilde{C}(p, r, B, A) > 0. \) Since \( p > 1, \) it follows from (6.12) and Hölder’s inequality that
\[ |\Delta_Q(\lambda) - \Delta_0(\lambda)| \leq \sum_{j=1}^2 \|g_j\|_p \|e^{b_j p'1/\lambda} - 1\|_{1/p'} \leq R \cdot \frac{e^{-b_1 \lambda} + e^{-b_2 \lambda}}{|b_0 p' \lambda|^{1/p'}}, \quad \text{Re} \lambda \neq 0, \quad 1/p + 1/p' = 1, \] (6.13)
where \( b_0 := \min\{|b_1|, |b_2|\}. \) Recall, that the sequence \( \Lambda_0 = \{\lambda_0^n\}_{n \in \mathbb{Z}} \) of zeros of \( \Delta_0 \) is incompressible, has density \( d_0 \) and lies in the strip \( \Pi_0, \) for some \( d_0 = d_0(B,A) \in \mathbb{N} \) and \( h_0 = h_0(B,A) > 0. \)

Let \( 0 < \varepsilon \leq \varepsilon_0 := (2d_0)^{-1} = \varepsilon_0(B,A). \) By Proposition 4.11(iii), there exists \( C_\varepsilon^0 = C_\varepsilon^0(B,A) > 0 \) such that the estimate (4.14) for \( \Delta_Q(\cdot) \) holds:
\[ |\Delta_0(\lambda)| > C_\varepsilon^0 \left( e^{-b_1 \lambda} + e^{-b_2 \lambda} \right) > C_\varepsilon^0, \quad \lambda \notin \Omega_\varepsilon^0 = \bigcup_{n \in \mathbb{Z}} \mathbb{D}_\varepsilon(\lambda_0^n). \] (6.14)

We can also assume that \( C_\varepsilon^0 \) is a non-decreasing function of \( \varepsilon. \) Combining (6.13) with estimate (6.14) for \( \varepsilon = \varepsilon_0 \) and setting
\[ h = h(p,r,B,A) := \max \left\{ h_0 + \varepsilon_0, \left( 2R/C_\varepsilon^0 \right)^{p'} (b_0 p')^{-1} \right\}, \] (6.15)
one has
\[ |\Delta_Q(\lambda)| \geq |\Delta_0(\lambda)| - |\Delta_Q(\lambda) - \Delta_0(\lambda)| > 2^{-1} C_\varepsilon^0 \left( e^{-b_1 \lambda} + e^{-b_2 \lambda} \right) > 0, \quad |\text{Im} \lambda| \geq h. \] (6.16)
This implies the inclusion \( \Lambda_Q \subset \Pi_k \) for all \( Q \in \mathbb{U}^{2\times 2}_{p,r}. \)
Set $\delta = C_0^0/4 (< C_{\varepsilon_0}/2)$. As established above, $g_j \in U_{p/R}^{j \times 2}$, $j \in \{1, 2\}$, $R = r \cdot \tilde{C}(p, r, B, A)$. Hence by Lemma 5.10 there exists $C_j = C(p, h, b_j) > 0$ such that uniform inequalities (5.37)–(5.38) hold for $g := g_j$, $j \in \{1, 2\}$, i.e.

$$
\text{card}(Z \setminus \mathcal{T}_{g_j, \delta}) \leq N_{j, \delta} := C_j \cdot (R/\delta)^d, \quad \lambda \in \bigcup_{n \in \mathbb{Z}_j, \varepsilon} [n, n + 1] \times [-h, h].
$$

(6.17)

Set $\tilde{\mathcal{T}}_{Q, \varepsilon} := \mathcal{T}_{g_1, \delta} \cap \mathcal{T}_{g_2, \delta}$. Combining (6.12), (6.14), (6.16) and (6.18) we arrive at

$$
|\Delta_Q(\lambda)| \geq |\Delta_0(\lambda)| - |\Delta_Q(\lambda) - \Delta_0(\lambda)| > C_0^0/4, \quad \lambda \notin \Omega_{h, Q, \varepsilon} := \Omega^0 e \cup \Pi_{h, Q, \varepsilon}, \quad \Pi_{h, Q, \varepsilon} := \bigcup_{n \notin \mathcal{I}_{Q, \varepsilon}} (n, n + 1) \times (-h, h).
$$

(6.19)

It is clear from (6.17) that $\text{card}(Z \setminus \tilde{\mathcal{T}}_{Q, \varepsilon}) \leq \tilde{N}_\varepsilon := N_{g_1, \delta} + N_{g_2, \delta}$, which implies that

$$
\Pi_{h, Q, \varepsilon} := \bigcup_{k=1}^{m_x} \left((\alpha_k, \beta_k) \times (-h, h)\right) \subseteq \Pi_h, \quad m := m_x \leq \tilde{N}_\varepsilon,
$$

(6.20)

$$
\alpha_k, \beta_k \in \mathbb{Z}, \quad k \in \{1, \ldots, m\}, \quad \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_m < \beta_m, \quad \sum_{k=0}^{m}(\beta_k - \alpha_k) \leq \tilde{N}_\varepsilon, \quad \lambda \in \mathbb{Z}_{\varepsilon, k}^{\varepsilon}
$$

(6.21)

i.e. segments $[\alpha_k, \beta_k], k \in \{1, \ldots, m\}$, with integer ends are disjoint with a total length not exceeding $\tilde{N}_\varepsilon$, that does not depend on $Q$.

As in the proof of Proposition 4.13 let $\mathcal{C}^0_{\varepsilon, j}, j \in \mathbb{Z}$, be the sequence of all connected components of $\Omega^0_{\varepsilon}$. Recall, that since $\varepsilon \leq (2d_0)^{-1}$, then each such connected component contains at most $d_0$ discs $D(\lambda_0^0)$, and according to (4.46) has diameter at most one, diam$(\mathcal{C}^0_{\varepsilon, j}) \leq 2d_0 \varepsilon \leq 1$, $j \in \mathbb{Z}$.

Let $k \in \{1, \ldots, m\}$. Consider a connected component $\mathcal{C}_k$ of $\Omega_{h, Q, \varepsilon}$ that contains the rectangle $(a_k, \beta_k) \times (-h, h)$. Clearly, $\mathcal{C}_k$ is the union of this rectangle and a finite set of connected components $\mathcal{C}^0_{\varepsilon, j}$. Since diam$(\mathcal{C}^0_{\varepsilon, j}) \leq 1, j \in \mathbb{Z}$, it follows that $\mathcal{C}_k \subset (\alpha_k - 1, \beta_k + 1) \times (-h, h)$. And, thus,

$$
\text{diam}(\mathcal{C}_k) \leq \beta_k - \alpha_k + 2 + 2h \leq \tilde{N}_\varepsilon + 2 + 2h.
$$

(6.22)

This implies that all connected components of $\Omega_{h, Q, \varepsilon}$ has uniformly bounded diameters for all $Q \in U_{p/R}^{j \times 2}$.

Due to (6.19) the Rouché theorem applies and ensures that in every connected component of $\Omega_{h, Q, \varepsilon}$ the functions $\Delta_0$ and $\Delta_Q = \Delta_0 + \Delta_Q - \Delta_0$ have the same number of zeros counting multiplicity. At this point, we need to enhance the canonical ordering to satisfy the following property: for any $\varepsilon > 0$ and $n \in \mathbb{Z}$ numbers $\lambda_n$ and $\lambda_n^0$ belong to the same connected component of $\Omega_{h, Q, \varepsilon}$. It is clear from (6.18) and from the fact that $C_0^0$ is non-decreasing function of $\varepsilon$, that the sets $\mathcal{I}_{g_j, C_0^0/4}$ monotonically increase as $\varepsilon$ tends to 0, i.e., $\mathcal{I}_{g_j, C_0^0/4} \subset \mathcal{I}_{g_j, C_0^0/4}$ if $\varepsilon_1 > \varepsilon_2$. Hence, the same is true for $\tilde{\mathcal{T}}_{Q, \varepsilon}$. Now the proof finished in the same way as in Proposition 4.13 (ii) by tracking the “lifetime” of connected components of $\Omega_{h, Q, \varepsilon}$. This property, in particular, implies

$$
|\lambda_n - \lambda_n^0| \leq \text{diam}(\mathcal{C}_{\varepsilon, n}), \quad n \in \mathbb{Z}, \quad \varepsilon > 0,
$$

(6.23)

where $\mathcal{C}_{\varepsilon, n}$ is a connected component of $\Omega_{h, Q, \varepsilon}$ that contains $\lambda_n^0$ (and, thus, also contains $\lambda_n$). Now set $\varepsilon = \varepsilon_0$. Since connected components have uniformly bounded diameters, inequality (6.23) implies inequality (6.8). In turn, Lemma 4.10 yields that $\Delta_Q$ is an impressible sequence of density $d$, that does not depend on $Q$.

(ii) Set $\tilde{\varepsilon} := \min\{\varepsilon/(2d_0), \varepsilon_0\}$. Let $\mathcal{I}_{Q, \varepsilon}$ be the set of integers $n$, for which connected component $\mathcal{C}^0_{\varepsilon, j_n}$ of $\Omega_{\varepsilon}$ that contains $D(\lambda^0_n)$ does not intersect with $\Pi_{h, Q, \varepsilon}$. Let $n \in \mathcal{I}_{Q, \varepsilon}$. In this case, $\mathcal{C}^0_{\varepsilon, j_n}$ is also a connected component of $\Omega_{h, Q, \varepsilon}$. Hence inequality (6.23) yields that

$$
|\lambda_n - \lambda_n^0| < \text{diam}(\mathcal{C}^0_{\varepsilon, j_n}) \leq 2 \cdot \tilde{\varepsilon} \cdot d_0 \leq \varepsilon, \quad n \in \mathcal{I}_{Q, \varepsilon},
$$

(6.24)

and proves inequality (6.10).
Let us estimate \( \text{card}(\mathbb{Z} \setminus I_{Q,\varepsilon}) \). Let \( n \notin I_{Q,\varepsilon} \). Then the connected component \( \mathfrak{c}_{Q,\varepsilon, n}^0 \) intersects with \( \Pi_{h, Q, \varepsilon} \). Since \( \text{diam}(\mathfrak{c}_{Q,\varepsilon, n}^0) \leq 1 \), it follows from (6.20) that \( \mathfrak{c}_{Q,\varepsilon, n}^0 \subset (\alpha_k - 1, \beta_k + 1) \times (-h, h) \) for some \( k \in \{1, \ldots, m_2\} \). Thus,

\[
\lambda_n^0 \in \tilde{\Pi}_{h, Q, \varepsilon} := \bigcup_{k=1}^{m_2} \left( (\alpha_k - 1, \beta_k + 1) \times (-h, h) \right), \quad n \in \mathbb{Z} \setminus I_{Q,\varepsilon}.
\] (6.25)

Since \( \Lambda_0 = \{\lambda_n^0\}_{n \in \mathbb{Z}} \) is an incompressible sequence of density \( d_0 \), then for each \( k \in \{1, \ldots, m_2\} \) the rectangle \((\alpha_k - 1, \beta_k + 1) \times (-h, h)\) contains at most \( d_0 \left( (\beta_k - \alpha_k + 2)/2 \right) \) entries of \( \Lambda_0 \). Inclusion (6.25) and inequality (6.21) now implies

\[
\text{card}(\mathbb{Z} \setminus I_{Q,\varepsilon}) \leq \sum_{k=1}^{m_2} \frac{d_0}{2} \sum_{k=1}^{m_2} (\beta_k - \alpha_k) + \frac{3d_0 m_2^2}{2} \leq 2d_0 \tilde{N}_\varepsilon =: N_\varepsilon,
\] (6.26)

which finishes the proof since \( \tilde{N}_\varepsilon \) does not depend on \( Q \).

(iii) Since boundary conditions (4.2) are strictly regular, then relation (6.17) holds. Let us redefine \( \varepsilon_0 \) defined above as \( \varepsilon_0 := \tau_0/3 \). Let \( 0 < \varepsilon \leq \varepsilon_0 \). Let us redefine the set \( I_{Q,\varepsilon} \) defined in the proof of part (ii) above:

\[
I_{Q,\varepsilon} := \left\{ n \in \mathbb{Z} : |n| > N_0, \quad D_{3\varepsilon}(\lambda_n^0) \cap \Pi_{h, Q, \varepsilon} = \emptyset \right\},
\] (6.27)

where \( \Pi_{h, Q, \varepsilon} \) is defined in (6.20). Using the same reasoning as in the proof of part (ii) we can prove relation (6.29).

Let \( n \in I_{Q,\varepsilon} \). Since \( |n| > N_0 \), then discs \( D_{\varepsilon}(\lambda_n^0) \) are disjoint due to (6.4). Hence each such a disc is a standalone connected component of \( \Omega_0 \) that does not intersect with \( \Pi_{h, Q, \varepsilon} \). Hence the new set \( I_{Q,\varepsilon} \) is a subset of previously defined set \( I_{Q,\varepsilon} \), which implies (6.11) due to the proof of part (ii). Due to the construction (6.24) of \( I_{Q,\varepsilon} \), it is clear that the inequality (6.13) holds for \( \lambda \in D_{3\varepsilon}(\lambda_n^0) \) \( \cap \mathcal{D}_\varepsilon(\lambda_n^0) \). The proof of the estimate (6.11) is now finished in the same way as in the proof of Proposition 6.11(ii). \( \square \)

**Proposition 6.3.** Let \( \mathcal{K} \) be compact in \( L^1([0, 1]; \mathbb{C}^{2 \times 2}) \) and \( Q, \tilde{Q} \in \mathcal{K} \). Let boundary conditions (4.2) be strictly regular, and let \( \Lambda_Q = \{\lambda_{Q, n}\}_{n \in \mathbb{Z}} \) and \( \Lambda_{\tilde{Q}} = \{\lambda_{\tilde{Q}, n}\}_{n \in \mathbb{Z}} \) be canonically ordered sequences of zeros of characteristic determinants \( \Delta := \Delta_Q \) and \( \tilde{\Delta} := \Delta_{\tilde{Q}} \) respectively. Then there exists constants \( N \in \mathbb{N}, C \geq 1 \) that do not depend on \( Q \) and \( \tilde{Q} \) and depend on \( \mathcal{K}, A, B \) only and such that the following uniform estimate holds

\[
C^{-1} \cdot |\Delta_{\tilde{Q}}(\lambda_{Q, n})| \leq |\lambda_{Q, n} - \lambda_{\tilde{Q}, n}| \leq C \cdot |\Delta_{\tilde{Q}}(\lambda_{Q, n})|, \quad |n| > N, \quad Q, \tilde{Q} \in \mathcal{K}.
\] (6.28)

**Proof.** For brevity set \( \lambda_n := \lambda_{Q, n}, \tilde{\lambda}_n := \lambda_{\tilde{Q}, n} \), and \( \Delta := \Delta_Q, \tilde{\Delta} := \Delta_{\tilde{Q}} \). Let \( \Lambda_0 = \{\lambda_n^0\}_{n \in \mathbb{Z}} \) be the sequence of zeros of the characteristic determinant \( \Delta_0 \). Since boundary conditions (4.2) are strictly regular then the sequence \( \Lambda_0 \) is asymptotically separated, i.e. the estimate (6.1) holds for some \( \tau_0 > 0 \) and \( N_0 \in \mathbb{N} \). Applying Proposition 6.1(ii) with \( \varepsilon = \min\{\varepsilon_0, \tau_0/3\} = \varepsilon(\mathcal{K}, B, A) \) we see that the discs \( D_{2\varepsilon}(\tilde{\lambda}_n) \), \( |n| > N_\varepsilon \), are disjoint, and

\[
|\lambda_n - \lambda_n^0| < \varepsilon, \quad |\tilde{\lambda}_n - \lambda_n^0| < \varepsilon, \quad |n| > N_\varepsilon.
\] (6.29)

In particular, \( \lambda_n \in D_{2\varepsilon}(\tilde{\lambda}_n) \), \( |n| > N_\varepsilon \). Hence for each \( |n| > N_\varepsilon \) the function \( f(z) := \tilde{\Delta}(z)/(z - \lambda_n) \) is non-zero holomorphic in \( D_{2\varepsilon}(\tilde{\lambda}_n) \) with \( f(\lambda_n) := \tilde{\Delta}^*(\lambda_n) \neq 0 \). If \( \lambda_n = \tilde{\lambda}_n \) then relation (6.28) is trivial as all parts are equal to zero. If \( \lambda_n \neq \tilde{\lambda}_n \) then combining the Minimum Principle for analytic functions with (6.3) we have

\[
\frac{|\tilde{\Delta}(\lambda_n)|}{|\lambda_n - \lambda_n^0|} \geq \min_{|z - \lambda_n^0| = 2\varepsilon} \frac{|\tilde{\Delta}(z)|}{|z - \lambda_n^0|} \geq \frac{C_{2\varepsilon}}{2\varepsilon}, \quad |n| > N_\varepsilon, \quad Q, \tilde{Q} \in \mathcal{K}.
\] (6.30)

Relation (6.30) now yields the second inequality in (6.28) with \( C = \frac{C_{2\varepsilon}}{2\varepsilon} \) and \( N = N_\varepsilon \).

On the other hand, by Proposition 6.1(i), \( \Lambda_Q \subset \Pi_A \) with \( h := h(\mathcal{K}, \tilde{\mathcal{B}}, A) \), not dependent on \( Q \in \mathcal{K}, \) i.e. \( |\text{Im} \lambda_n| \leq h \) for \( n \in \mathbb{Z} \) and \( Q \in \mathcal{K} \). Moreover, Lemma 4.10 (see estimate (4.30)), ensures the uniform estimate \( |\tilde{\Delta}(\lambda)| = |\Delta_{\tilde{Q}}(\lambda)| \leq C_h, \lambda \in \Pi_A \), for any \( \tilde{Q} \in \mathcal{U}_{1/2} \). Applying the Maximum Principle similarly to (6.30) yields the first uniform inequality in (6.28). \( \square \)

Recall that notation \( x_n \asymp y_n \) as \( |n| \to \infty \), means that there exists \( N \in \mathbb{N} \) and \( C_2 > C_1 > 0 \) such that two-sided estimate \( C_1 |y_n| \leq |x_n| \leq C_2 |y_n| \), \( |n| > N \), holds.
Corollary 6.4. Let \( Q, \tilde{Q} \in L^1([0,1]; \mathbb{C}^{2\times2}) \) and let boundary conditions (4.2) be strictly regular. Let \( \{\lambda_n\}_{n \in \mathbb{Z}} \) and \( \{	ilde{\lambda}_n\}_{n \in \mathbb{Z}} \) be canonically ordered sequences of zeros of characteristic determinants \( \Delta := \Delta_Q \) and \( \tilde{\Delta} := \Delta_{\tilde{Q}} \) respectively. Then the following estimates hold
\[
|\lambda_n - \tilde{\lambda}_n| \asymp |\Delta(\lambda_n)|, \quad |\lambda_n - \lambda_0^n| \asymp |\Delta_0(\lambda_n)| \quad \text{as} \quad |n| \to \infty.
\]

6.2. Stability property of eigenvalues for \( Q \in L^p \). Here we apply abstract results from Section 5 to establish stability of the mapping \( Q \to \Lambda_Q := \{\lambda_{Q,n}\}_{n \in \mathbb{Z}} \) in different norms. Proposition 6.3 shows that to this end it suffices to evaluate the sequences \( \{\Delta(\lambda_n)\}_{n \in \mathbb{Z}} = \{\Delta_{\tilde{Q}}(\lambda_{Q,n})\}_{n \in \mathbb{Z}} \) when \( Q \) runs through either the ball \( \mathbb{U}^{2\times2}_{p,r} \) or a compact \( \mathcal{K} \subset L^1 \). Our main result in this direction reads as follows.

Lemma 6.5. Let \( Q, \tilde{Q} \in L^1([0,1]; \mathbb{C}^{2\times2}) \), let boundary conditions (4.2) be regular, and let \( \Lambda_Q = \{\lambda_{Q,n}\}_{n \in \mathbb{Z}} \) be a sequence of zeros of characteristic determinant \( \Delta_Q \).

(i) Let \( p \in (1,2] \) and \( r > 0 \). Then there exists \( C = C(p,r,B) > 0 \) such that the following inequalities hold:
\[
\sum_{n \in \mathbb{Z}} \left| \Delta_{\tilde{Q}}(\lambda_{Q,n}) \right|^p \leq C \cdot \|Q - \tilde{Q}\|_p^p, \quad Q, \tilde{Q} \in \mathbb{U}^{2\times2}_{p,r},
\]
\[
\sum_{n \in \mathbb{Z}} (1 + |n|)^{p-2} \left| \Delta_{\tilde{Q}}(\lambda_{Q,n}) \right|^p \leq C \cdot \|Q - \tilde{Q}\|_p^p, \quad Q, \tilde{Q} \in \mathbb{U}^{2\times2}_{p,r}.
\]

(ii) Let \( \mathcal{K} \subset L^1([0,1]; \mathbb{C}^{2\times2}) \) be a compact. Then the following holds:
\[
\sup_{n \in \mathbb{Z}} \left| \Delta_{\tilde{Q}}(\lambda_{Q,n}) \right| \leq C \cdot \|Q - \tilde{Q}\|_1, \quad Q, \tilde{Q} \in \mathcal{K},
\]
\[
\sup_{Q,\tilde{Q} \in \mathcal{K}} \left| \Delta_{\tilde{Q}}(\lambda_{Q,n}) \right| \to 0 \quad \text{as} \quad n \to \infty.
\]

In other words, the set of sequences \( \{\{\Delta_{\tilde{Q}}(\lambda_{Q,n})\}_{n \in \mathbb{Z}}\}_{Q,\tilde{Q} \in \mathcal{K}} \) forms a compact set in \( c_0(\mathbb{Z}) \).

Proof. In accordance with Lemma 6.5 the difference of determinants admits representation (4.25), i.e.
\[
\Delta_{\tilde{Q}}(\lambda) - \tilde{\Delta}_{\tilde{Q}}(\lambda) = \int_0^1 \tilde{g}_1(t)e^{ib_1\lambda t} dt + \int_0^1 \tilde{g}_2(t)e^{ib_2\lambda t} dt,
\]
where \( \tilde{g}_j = g_{Q,j} - g_{\tilde{Q},j} \in L^p[0,1], j \in \{1,2\}, \) and satisfies uniform estimate (4.20).

Since \( \Delta_{\tilde{Q}}(\cdot) \) is an entire sine-type function, its zero set \( \Lambda_Q = \{\lambda_{Q,n}\}_{n \in \mathbb{Z}} \) is incompressible (see Proposition 4.11(ii)). Moreover, with Proposition 6.1, all null sequences \( \{\lambda_{Q,n}\}_{n \in \mathbb{Z}}, Q \in \mathcal{K}, \) are incompressible of density \( d = d(K,B,A) \in \mathbb{N} \) and lie in the strip \( \Pi_h \) with \( h = h(K,B,A) > 0 \), where neither \( d \) nor \( h \) depends on \( Q \in \mathcal{K} \).

Let us prove estimate (6.32). Applying Bessel type inequality (5.24) with \( g = \tilde{g}_j, j \in \{1,2\}, \) and \( \mu_n = \lambda_n = \lambda_{Q,n}, n \in \mathbb{Z}, \) and taking uniform (with respect to \( Q, \tilde{Q} \in \mathbb{U}^{2\times2}_{p,r} \)) estimate (4.26) into account, yields
\[
\sum_{n \in \mathbb{Z}} \left| \Delta_{\tilde{Q}}(\lambda_n) \right|^{p'} = \sum_{n \in \mathbb{Z}} \left| \Delta_Q(\lambda_n) - \tilde{\Delta}_Q(\lambda_n) \right|^{p'} \leq 2^{p'-1} \sum_{n \in \mathbb{Z}} \left( \int_0^1 \tilde{g}_1(t)e^{ib_1\lambda_n t} dt \right)^{p'} + \left( \int_0^1 \tilde{g}_2(t)e^{ib_2\lambda_n t} dt \right)^{p'} \leq 2^{p'-1} C_1 \parallel \tilde{g}_1 \parallel_p^{p'} \|Q - \tilde{Q}\|_p^{p'}, \quad Q, \tilde{Q} \in \mathbb{U}^{2\times2}_{p,r}.
\]

Here the constants \( C_1 = C_1(b,p,h,d) > 0 \) and \( \tilde{C} = \tilde{C}(b,p,h,d) > 0 \) are taken from the uniform estimates (5.20) and (4.26), respectively. Setting \( C = 2^{p'-1} C_1 \tilde{C} \) we arrive at (6.32).

Weighted estimate (6.33) is proved similarly but using inequality (5.21). Estimate (6.34) follows from (6.36) and the estimate
\[
\int_0^1 \tilde{g}_j(t)e^{ib_1\lambda_n t} dt \leq e^{b_1|\text{Im} \lambda_n|} \cdot \|\tilde{g}_j\|_1 \leq \tilde{C} e^{b_1|\text{Im} \lambda_n|} \cdot \|Q - \tilde{Q}\|_1, \quad j \in \{1,2\}, \quad \lambda \in \Pi_h, \quad Q, \tilde{Q} \in \mathbb{U}^{2\times2}_{1,r}.
\]
Further, according to (2.23) and (1.13) functions \( \hat{g}_1(t) \) and \( \hat{g}_2(t) \) are linear combinations of sixteen well-defined summable trace functions \( K_{\delta,j,k}(1, t) \) and \( K_{\hat{\delta},j,k}(1, t), j, k \in \{1, 2\} \). Lemma 5.14 now implies relation (6.35) due to inclusion \( \lambda_n \in \Pi_h \). The last statement of the lemma follows from the well-known criteria of compactness in \( c_0(\mathbb{Z}) \).

Next we enhance and complete Proposition 6.12 in the case of \( Q \in L^p([0, 1]; C^{2 \times 2}) \) with \( p \in [1, 2] \). Our first result restricts the set \( K \) of potentials matrices to be a compact.

**Theorem 6.6.** Let \( K \) be compact in \( L^p([0, 1]; C^{2 \times 2}) \) for some \( p \in [1, 2] \), and \( Q, \widetilde{Q} \in K \). Let boundary conditions (1.2) be strictly regular, and let \( \Lambda_Q := \{ \lambda_{Q,n} \}_{n \in \mathbb{Z}} \) and \( \Lambda_{\widetilde{Q}} := \{ \lambda_{\widetilde{Q},n} \}_{n \in \mathbb{Z}} \) be canonically ordered sequences of zeros of characteristic determinants \( \Delta(\cdot) := \Delta_Q(\cdot) \) and \( \widetilde{\Delta}(\cdot) := \Delta_{\widetilde{Q}}(\cdot) \), respectively. Then there exist constants \( N = N(K, A, B) \in \mathbb{N} \) and \( C = C(p, K, A, B) > 0 \) that do not depend on \( Q \) and \( \widetilde{Q} \) and such that the following estimates hold:

\[
\sum_{|n| > N} \left| \lambda_{Q,n} - \lambda_{\widetilde{Q},n} \right|^{p'} \leq C \cdot \| Q - \widetilde{Q} \|_{p'}, \quad Q, \widetilde{Q} \in K, \quad p \in (1, 2], \quad 1/p' + 1/p = 1, \tag{6.39}
\]

\[
\sum_{|n| > N} (1 + |n|)^{p-2} \left| \lambda_{Q,n} - \lambda_{\widetilde{Q},n} \right|^p \leq C \cdot \| Q - \widetilde{Q} \|_{p}, \quad Q, \widetilde{Q} \in K, \quad p \in (1, 2]. \tag{6.40}
\]

If \( p = 1 \) then the following holds:

\[
\sup_{|n| > N} \left| \lambda_{Q,n} - \lambda_{\widetilde{Q},n} \right| \leq C \cdot \| Q - \widetilde{Q} \|_1, \quad Q, \widetilde{Q} \in K, \tag{6.41}
\]

\[
\sup_{Q, \widetilde{Q} \in K} \left| \lambda_{Q,n} - \lambda_{\widetilde{Q},n} \right| \to 0 \quad \text{as} \quad n \to \infty. \tag{6.42}
\]

In other words, the set of sequences \( \left\{ \left\{ \left| \lambda_{Q,n} - \lambda_{\widetilde{Q},n} \right| \right\}_{n \in \mathbb{Z}} \right\}_{Q, \widetilde{Q} \in K} \) forms a compact set in \( c_0(\mathbb{Z}) \).

**Proof.** Clearly, \( K \) is compact in \( L^1([0, 1]; C^{2 \times 2}) \) as well, and \( K \subset U_{p,r}^{2 \times 2} \) for some \( r = r(p, K) > 0 \). First, note that relation (6.42) was proved in Proposition 6.1(ii). Further, in accordance with Proposition 6.3 there exist constants \( C, N > 0 \), not dependent on \( Q, \widetilde{Q} \), such that the uniform estimate (6.28) holds, i.e.

\[
|\lambda_{Q,n} - \lambda_{\widetilde{Q},n}| \leq C \cdot |\Delta_Q(\lambda_{Q,n})|, \quad |n| > N, \quad Q, \widetilde{Q} \in K. \tag{6.43}
\]

Combining this estimate with all the statements of Lemma 6.5 finishes the proof. \( \square \)

Applying Theorem 6.6 with a two-point compact \( K = \{ Q, 0 \} \) we can complete Proposition 6.12 as follows.

**Corollary 6.7.** Let \( Q \in L^p([0, 1]; C^{2 \times 2}) \) for some \( p \in (1, 2] \). Let boundary conditions (1.2) be strictly regular, and let \( \Delta(\cdot) \) be the corresponding characteristic determinant. Then the sequence \( \Lambda = \{ \lambda_n \}_{n \in \mathbb{Z}} \) of its zeros can be ordered in such a way that the following inequalities take place

\[
\sum_{n \in \mathbb{Z}} |\lambda_n - \lambda_0|^p < \infty, \quad 1/p' + 1/p = 1, \tag{6.44}
\]

\[
\sum_{n \in \mathbb{Z}} (1 + |n|)^{p-2} |\lambda_n - \lambda_0|^p < \infty. \tag{6.45}
\]

Note also that relation (6.44) in the case of \( 2 \times 2 \) Dirac system was obtained firstly in [18, Th. 4.3, 4.5].

**Remark 6.8.** Here we will show that inequalities (6.39) and (6.40) generally can not be derived from each other. Let \( p = 3/2 \) and, thus, \( p' = 3 \). First we assume that

\[
\alpha_n := \lambda_n - \bar{\lambda}_n = \left( (1 + |n|) \ln^2 (1 + |n|) \right)^{-1/3}.
\]

It is clear that

\[
\{ \alpha_n \}_{n \in \mathbb{Z}} \in l^3(\mathbb{Z}) \text{ while } \left\{ (1 + |n|)^{-1/3} \alpha_n \right\}_{n \in \mathbb{Z}} \notin l^{3/2}(\mathbb{Z}),
\]
which shows that the inequality (6.39) holds while (6.40) is not true. Now let
\[ \alpha_n = \begin{cases} k^{-1/3}, & n = k^2 \text{ for some } k \in \mathbb{N}, \\ 0, & n \neq k^2. \end{cases} \]

In this case it is clear that the opposite relations hold,
\[ \{\alpha_n\}_{n \in \mathbb{Z}} \notin l^3(\mathbb{Z}) \text{ while } \left\{ (1 + |n|)^{-1/3} \alpha_n \right\}_{n \in \mathbb{Z}} \in l^{3/2}(\mathbb{Z}). \]

Note also that under rather general condition \( \alpha_n = o(n^{-1/p'}) \) as \( n \to \infty \), inequality (6.39) does imply (6.40). Indeed, in this case \( |\alpha_n|^{p'} = \frac{\beta}{1 + |n|} \) and \( \beta_n < 1 \) for \( |n| > N \). Then since \( p' = p/(p-1) > p \),
\[
(1 + |n|)^{p-2} |\alpha_n|^p = \frac{\beta_n^{p/p'}}{1 + |n|} \leq \beta_n = |\alpha_n|^{p'}, \quad |n| > N. \tag{6.46}
\]

Hence, if \( \{\alpha_n\}_{n \in \mathbb{Z}} \in l^{p'}(\mathbb{Z}) \) then \( \left\{ (1 + |n|)^{|1-2/p} \alpha_n \right\}_{n \in \mathbb{Z}} \in l^p(\mathbb{Z}). \)

Next we extend Theorem 6.6 to the case \( K = \mathbb{U}^{2 \times 2}_{p,r} \). Similarly to Proposition 6.2 we cannot select a universal constant \( N \) serving all potentials. Instead, we need to sum over the sets of integers, the complements of which have uniformly bounded cardinality.

**Theorem 6.9.** Let \( \tilde{Q}, \tilde{Q} \subseteq \mathbb{U}^{2 \times 2}_{p,r} \) for some \( p \in (1,2] \) and \( r > 0 \). Let boundary conditions (4.2) be strictly regular, and let \( \Lambda_Q = \{\lambda_Q,n\}_{n \in \mathbb{Z}} \) and \( \Lambda_{\tilde{Q}} = \{\lambda_{\tilde{Q}},n\}_{n \in \mathbb{Z}} \) be canonically ordered sequences of zeros of characteristic determinants \( \Delta := \Delta_Q \) and \( \tilde{\Delta} := \Delta_{\tilde{Q}} \), respectively. Then the following holds:

(i) There exists constants \( N \in \mathbb{N}, C_1, C_2, C > 0, \) not dependent on \( Q, \tilde{Q} \), and a set \( \mathcal{I} := \mathcal{I}_{Q,\tilde{Q}} \subseteq \mathbb{Z} \), such that the following estimates hold
\[
\text{card} \left( \mathbb{Z} \setminus \mathcal{I}_{Q,\tilde{Q}} \right) \leq N, \quad Q, \tilde{Q} \subseteq \mathbb{U}^{2 \times 2}_{p,r}, \tag{6.47}
\]
\[
C_1 \cdot \left| \Delta_{\tilde{Q}}(\lambda_{Q,n}) \right| \leq |\lambda_{Q,n} - \lambda_{\tilde{Q},n}| \leq C_2 \cdot \left| \Delta_Q(\lambda_Q,n) \right|, \quad n \in \mathcal{I}_{Q,\tilde{Q}}, \quad Q, \tilde{Q} \subseteq \mathbb{U}^{2 \times 2}_{p,r}, \tag{6.48}
\]
\[
\sum_{n \in \mathcal{I}_{Q,\tilde{Q}}} \left| \lambda_{Q,n} - \lambda_{\tilde{Q},n} \right|^{p'} \leq C \cdot \|Q - \tilde{Q}\|^{p'}, \quad Q, \tilde{Q} \subseteq \mathbb{U}^{2 \times 2}_{p,r}, \quad 1/p + 1/p' = 1, \tag{6.49}
\]
\[
\sum_{n \in \mathcal{I}_{Q,\tilde{Q}}} (1 + |n|)^{p-2} \left| \lambda_{Q,n} - \lambda_{\tilde{Q},n} \right|^p \leq C \cdot \|Q - \tilde{Q}\|^p, \quad Q, \tilde{Q} \subseteq \mathbb{U}^{2 \times 2}_{p,r}. \tag{6.50}
\]

(ii) For any \( \varepsilon > 0 \) there exists a set \( \mathcal{I}_\varepsilon := \mathcal{I}_{Q,\tilde{Q},\varepsilon} \subseteq \mathbb{Z} \) and a constant \( N_\varepsilon = N_\varepsilon(p,r,A,B) \in \mathbb{N} \) that does not depend on \( Q \) and \( \tilde{Q} \), such that the following uniform estimates hold
\[
\text{card} \left( \mathbb{Z} \setminus \mathcal{I}_{Q,\tilde{Q},\varepsilon} \right) \leq N, \quad Q, \tilde{Q} \subseteq \mathbb{U}^{2 \times 2}_{p,r}, \tag{6.51}
\]
\[
\sup_{n \in \mathcal{I}_{Q,\tilde{Q},\varepsilon}} |\lambda_{Q,n} - \lambda_{\tilde{Q},n}| \leq \varepsilon \|Q - \tilde{Q}\|, \quad Q, \tilde{Q} \subseteq \mathbb{U}^{2 \times 2}_{p,r}. \tag{6.52}
\]

**Proof.** As in the proof of Proposition 6.3 we set for brevity \( \lambda_n := \lambda_{Q,n}, \tilde{\lambda}_n := \lambda_{\tilde{Q},n}, \) and \( \tilde{\Delta} := \Delta_{\tilde{Q}}, \Delta := \Delta_Q \). Recall, that \( \Lambda_0 = \{\lambda_0^n\}_{n \in \mathbb{Z}} \) is a sequence of zeros of the characteristic determinant \( \Delta_0 \).

(i) Applying Proposition 6.2(iii) with \( \varepsilon = \varepsilon_0 = \varepsilon(K,B,A) \) we see that the discs \( \mathbb{D}_{2\varepsilon}(\tilde{\lambda}_n), n \in \mathcal{I}_{Q,\tilde{Q},\varepsilon} \), are disjoint, and
\[
|\lambda_n - \lambda_0^n| < \varepsilon, \quad |\tilde{\lambda}_n - \lambda_0^n| < \varepsilon, \quad n \in \mathcal{I}_{Q,\tilde{Q}} := \mathcal{I}_{Q,\varepsilon} \cap \mathcal{I}_{\tilde{Q},\varepsilon}. \tag{6.53}
\]
In particular, \( \lambda_n \in \mathbb{D}_{2\varepsilon}(\tilde{\lambda}_n), n \in \mathcal{I}_{Q,\tilde{Q}} \). The proof of inequality (6.38) is finished in the same way as in Proposition 6.3 Further, note that
\[
\text{card}(\mathbb{Z} \setminus \mathcal{I}_{Q,\tilde{Q}}) \leq \text{card}(\mathbb{Z} \setminus \mathcal{I}_{Q,\varepsilon}) + \text{card}(\mathbb{Z} \setminus \mathcal{I}_{\tilde{Q},\varepsilon}) \leq 2N_\varepsilon := N(p,r,B,A) := N, \tag{6.54}
\]
which proves (6.47). Combining inequality (6.48) with Lemma 6.5(i) implies inequalities (6.49) – (6.50).

(ii) If \( Q = \tilde{Q} \) then inequality (6.52) is trivial. Assume that \( Q \neq \tilde{Q} \). We will apply Chebyshev’s inequality as it was done in Lemma 5.9. Let \( \varepsilon > 0 \) and let us set
\[
\mathcal{I}_{Q,\tilde{Q},\varepsilon} := \{ n \in \mathbb{Z} : |\lambda_{Q,n} - \lambda_{\tilde{Q},n}| \leq \varepsilon \|Q - \tilde{Q}\|_p \cap \mathcal{I}_{Q,\tilde{Q}}. \tag{6.55}
\]
It is clear that (6.52) follows from (6.55). Further, inequality (6.49) implies
\[
C \cdot \|Q - \tilde{Q}\|_p' \geq \sum_{n \in I_{Q,\tilde{Q}}} |\lambda_{Q,n} - \lambda_{\tilde{Q},n}|^p' \geq \sum_{n \in I_{Q,\tilde{Q}} \setminus I_{Q,\tilde{Q},\varepsilon}} |\lambda_{Q,n} - \lambda_{\tilde{Q},n}|^p' \geq \sum_{n \in I_{Q,\tilde{Q}} \setminus I_{Q,\tilde{Q},\varepsilon}} \varepsilon^p Q - \tilde{Q} ||p' = \varepsilon^p \|Q - \tilde{Q}\|_p' \text{ card } (I_{Q,\tilde{Q}} \setminus I_{Q,\tilde{Q},\varepsilon}). \tag{6.56}
\]
Since \( Q \neq \tilde{Q} \) and \( I_{Q,\tilde{Q},\varepsilon} \subset I_{Q,\tilde{Q}} \), inequalities (6.56) and (6.47) imply
\[
\text{card } (\mathbb{Z} \setminus I_{Q,\tilde{Q},\varepsilon}) = \text{card } (I_{Q,\tilde{Q}} \setminus I_{Q,\tilde{Q},\varepsilon}) + \text{card } (\mathbb{Z} \setminus I_{Q,\tilde{Q}}) \leq C \varepsilon^{-p'} + N =: N_\varepsilon, \tag{6.57}
\]
which proves (6.51). \[\Box\]

Building up on the example in Remark 6.10(i) we will show that Proposition 6.1(ii) is not valid for balls in \( L^p \) and demonstrate significance of introducing subsets \( I_{Q,\tilde{Q}} \) in Proposition 6.2(ii). We also show that the constant \( C \) in (6.41) can not be arbitrary small in the case of compacts in \( L^1 \).

**Proposition 6.10.** Let \( Q_{12} = 0 \). Let boundary conditions (6.49) be regular (and thus \( \{\lambda^0_n\}_{n \in \mathbb{Z}} \subset \Pi_\mu \) for some \( h \geq 0 \)) and let \( \Lambda_Q = \{\lambda_{Q,n}\}_{n \in \mathbb{Z}} \) be canonically ordered sequence of zeros of the characteristic determinant \( \Delta := \Delta_Q \). Let also
\[
\mathcal{K} = \left\{ G_\mu := \begin{pmatrix} 0 & 0 \\ \mu & 0 \end{pmatrix} : \mu \in \Pi_\mu \right\}, \quad g_\mu(x) = g_0(x)e^{-ix(b_1 - b_2)x}, \quad x \in [0, 1], \quad \mu \in \Pi_\mu, \tag{6.58}
\]
where \( g_0 \in \mathbb{U}_{p,r} \), \( c_0 := \int_0^1 g_0(t)dt \neq 0 \), for some \( p \geq 1 \) and \( r > 0 \).

(i) Let \( b \neq 0 \). Then relation (6.2) is not valid for \( \mathcal{K} \). More precisely, there exists \( \varepsilon_0 > 0 \) such that
\[
|\lambda_{Q,n,m} - \lambda^0_n| \geq \varepsilon_0, \quad m, n, \in \mathbb{Z}, \quad \text{where} \quad Q_n = G_{\lambda^0_n}. \tag{6.59}
\]

(ii) Let again \( b \neq 0 \). Then for \( Q = G_\mu \in \mathcal{K} \) relations (6.9) – (6.10) are valid with \( I_{Q,\varepsilon} = \{n_\mu + 1, \ldots, n_\mu + N_\varepsilon\} \), where \( n_\mu \in \mathbb{Z} \) and \( N_\varepsilon = N_\varepsilon(90, \mathbf{B}, A) \in \mathbb{N} \) does not depend on \( \mu \).

(iii) Let \( b = 0 \). Then \( \lambda_{Q,n} = \lambda^0_n \) for all \( n \in \mathbb{Z} \) and \( Q \in L^1([0, 1]; C^2 \times \mathbb{C}) \) with \( Q_{12} = 0 \).

(iv) Let \( X = \{G_\mu/\mu : \mu \in \Pi_\mu, |\mu| > 1\} \cup \{0\} \). Then \( X \) is compact in \( L^p([0, 1]; C^2 \times \mathbb{C}) \). Let \( b \neq 0 \). Then there exists \( N_0, \varepsilon_0 > 0 \) such that
\[
|\lambda_{Q,n,m} - \lambda^0_n| \geq \varepsilon_0 \|Q_n\|_p, \quad |n| > N_0, \quad \text{where} \quad Q_n = G_{\lambda^0_n}/\lambda^0_n \in X. \tag{6.60}
\]

**Proof.** Since \( Q_{12} = 0 \), then explicit formula (4.14) holds. Inserting it into (4.51) we arrive at
\[
\Delta_Q(\lambda) = \Delta_0(\lambda) - ib_2e^{ib_2\lambda} \int_0^1 Q_{21}(t)e^{i(b_1 - b_2)\lambda t}dt. \tag{6.61}
\]
Hence \( \Delta_Q(\lambda) = 0 \) is equivalent to
\[
F_0(\lambda) = J(\lambda, Q), \quad \text{where} \quad F_0(\lambda) := e^{-ib_2\lambda}\Delta_0(\lambda) \quad \text{and} \quad J(\lambda, Q) := ib_2 \int_0^1 Q_{21}(t)e^{i(b_1 - b_2)\lambda t}dt. \tag{6.62}
\]

(i) Since boundary conditions (6.49) are regular, Proposition 6.11 implies that \( \{\lambda^0_n\}_{n \in \mathbb{Z}} \) is an incompressible sequence lying in the strip \( \Pi_\mu \). For \( n \in \mathbb{Z} \) set \( Q_n := G_{\lambda^0_n} \), i.e. \( Q_{21} = Q_{n,21} = g_{\lambda^0_n} \). Since \( bco \neq 0 \), definition (6.58) of \( g_\mu \) yields
\[
J(\lambda, Q_n) = ib_2 \int_0^1 g_0(t)e^{i(b_1 - b_2)(\lambda - \lambda^0_n)t}dt, \quad J(\lambda^0_n, Q_n) = ib_2bc_0 =: \alpha \neq 0, \quad n \in \mathbb{Z}. \tag{6.63}
\]
Set $\delta := |\alpha|/2 > 0$. Since $F_0^\alpha(\lambda)$ is uniformly bounded in the strip $\Pi_{h+1}$, i.e. $|F_0^\alpha(\lambda)| \leq M_0$, $\lambda \in \Pi_{h+1}$, then

$$|F_0^\alpha(\lambda)| \leq |F_0^\alpha(\lambda_0^0)| + M_0|\lambda - \lambda_0^0| = M_0|\lambda - \lambda_0^0| < \delta, \quad |\lambda - \lambda_0^0| < \varepsilon_1, \quad n \in \mathbb{Z}. \quad (6.64)$$

with some $\varepsilon_1 \in (0, 1)$. At the same time since $I(\mu) := \int_0^1 g_0(t)e^{i(b_1 - b_2)\mu t}dt$ is continuous at $\mu$, it follows from (6.63) that for some $\varepsilon_2 > 0$ we have

$$|J(\lambda, Q_n) - \alpha| < \delta, \quad |\lambda - \lambda_0^0| < \varepsilon_2, \quad n \in \mathbb{Z}. \quad (6.65)$$

Setting $\varepsilon_0 := \min(\varepsilon_1, \varepsilon_2)$, taking into account that $2\delta = |\alpha|$ and combining (6.64), (6.65) we arrive at

$$|F_0(\lambda)| < |J(\lambda, Q_n)|, \quad |\lambda - \lambda_0^0| < \varepsilon_0, \quad n \in \mathbb{Z}. \quad (6.66)$$

Remarks (6.31), (6.32) and (6.60) now imply that for $n \in \mathbb{Z}$ determinant $\Delta_{Q_n}(\lambda)$ has no zeros in $D_{\varepsilon_0}(\lambda_0^0)$ which implies desired inequality (6.59).

(ii) Let $\varepsilon > 0$, $\mu \in \mathbb{R}$ and $Q = G_{\mu}$. For $M > 0$ we set $T_{\mu,M} := \{n \in \mathbb{Z} : |\text{Re} \lambda_0^0 - \mu| \leq M\}$. Similarly to the proof of Proposition [1,13] we can apply Riemann-Lebesgue lemma and Rouche theorem to the relation (6.62) with $Q_{21} = g_{\mu}$. This yields relations (6.9)–(6.10) with $T_{Q,\varepsilon} = T_{\mu, M}$, for some $M_\varepsilon = M_\varepsilon(g_{\mu}, B, A) > 0$ that does not depend on $\mu$. Since $\{\text{Re} \lambda_0^0\}_{n \in \mathbb{Z}}$ is an incompressible non-decreasing sequence, it is clear that $T_{\mu, M} \subset \{n_\mu + 1, \ldots, n_\mu + N_\varepsilon\}$, for some $n_\mu \in \mathbb{Z}$ and $N_\varepsilon = N_\varepsilon(g_{\mu}, B, A) \in \mathbb{N}$ that does not depend on $\mu$, which finishes the proof.

(iii) Since $b = 0$, it follows from (6.61) that $Q(\lambda) = \Delta_0(\lambda)$, $\lambda \in \mathbb{C}$, whenever $Q_{12} = 0$. This implies that spectra of the operators $L(Q)$ and $L(0)$ coincide and finishes the proof.

(iv) First let us show that $X$ is compact. Let $\varepsilon > 0$ and let us build a finite $\varepsilon$-net for $X$. Set $R = \|g_0\|e^{(b_2 - b_1)h}/\varepsilon$. It is clear that $X_R = \{G_{\mu}/\mu : \mu \in \Pi_B, 1 \leq |\mu| \leq R\}$ is a compact in $L^p([0, 1]; \mathbb{C}^{2 \times 2})$ since $G_{\mu}/\mu$ is a continuous function of $\mu$ to $L^p$ and the set of $\mu$ in the definition of $X_R$ is a compact in $\mathbb{C}$. On the other hand definition (6.58) of $G_{\mu}$ implies that $\|G_{\mu}/\mu\| \leq \|g_0\|e^{(b_2 - b_1)h}/|\mu| < \varepsilon$, for $|\mu| > R$. Hence adding zero to the $\varepsilon$-net of $X_R$ gives the $\varepsilon$-net for $X$.

Proposition (6.3) implies that $|\lambda_\nu - \lambda_0^0| \geq C^{-1}|\Delta_0(\lambda_0^0)|$, $|n| > N$, for some $C, N > 0$, not dependent on $Q$. Let $|n| > N$ and $Q := Q_n = G_{\lambda_0^0}/\lambda_0^0$. Definition (6.58) of $G_{\mu}$ and $c_0$, and relation (6.61) imply

$$|\lambda_n - \lambda_0^0| \geq C^{-1}|\Delta_{Q_n}(\lambda_0^0) - \Delta_0(\lambda_0^0)| = C^{-1}|b_2b_2e^{i(b_1 - b_2)}\lambda_n^0| \left|\int_0^1 g_0(t)e^{-i(b_1 - b_2)}\lambda_n^0 + e^{i(b_1 - b_2)}\lambda_n^0 dt\right| \leq C^{-1}|b_2b_2e^{i(b_1 - b_2)}\lambda_0^0|.$$  

(6.67)

On the other hand $\|Q_n\| \leq \|g_0\|e^{(b_2 - b_1)h}/|\lambda_0^0|$, and $e^{i(b_1 - b_2)}\lambda_0^0| \geq e^{-b_2h}$. Hence setting

$$\varepsilon_0 = C^{-1}|b_2b_0|\|g_0\|^{-1}e^{(b_1 - b_2)h} > 0$$

and combining (6.67) with the estimate on $\|Q_n\|_p$, we arrive at the desired estimate (6.60). \qed

Remark 6.11. (i) Emphasize that the proofs of all results in this Section including the proofs of Theorems 6.6 and 6.9 rely on Bessel type inequalities (5.20), (5.21) (see Proposition 5.6) for ordinary Fourier transform, not for its maximal version described in Theorem 5.4 which proof relies on the deep Carleson-Hunt result 5.1.

(ii) Theorem 6.3 remains valid if $K$ is compact in $L^1([0, 1]; \mathbb{C}^{2 \times 2})$ and bounded in $L^p([0, 1]; \mathbb{C}^{2 \times 2})$ which is slightly wider class of sets than compacts in $L^p([0, 1]; \mathbb{C}^{2 \times 2})$.

(iii) The case of Dirac system $(b_1 = -b_2 = 1)$ and $Q = 0$ has been extensively studied in multiple recent papers by Sadovnichaya, Savchuk and Shkalikov. In particular, estimate (6.39) was established earlier in [18] Theorems 4.3, 4.5 with the constant $C$ that depends on $Q$, while estimate (6.49) of Theorem 6.6 with $Q = 0$ was established in [17].

Weighted estimates (6.40) and (6.50), as well as estimate (6.39), which establish stability property of the spectrum under perturbation $Q \to Q$, are new even for Dirac system.

(iv) Emphasize that the uniform estimates (6.1)–(6.2) in Proposition 6.1 are valid in the case of regular BC, which generalizes Theorem 3 in [46] even in the case of Dirac system $(b_1 = -b_2 = 1)$.

(v) In a very recent preprint [41] L. Rzepnicki obtained sharp asymptotic formulas for deviations $\lambda_n - \lambda_0^0 = \delta_n + \rho_n$ in the case of Dirichlet BVP for Dirac system with $Q \in L^p([0, 1]; \mathbb{C}^{2 \times 2})$, $1 \leq p < 2$. Namely, $\delta_n$ is explicitly expressed via Fourier coefficients and Fourier transforms of $Q_{12}$ and $Q_{21}$, while $\{\rho_n\}_{n \in \mathbb{Z}} \in L^{p/2}(\mathbb{Z})$, i.e. has “twice” better convergence to zero than what formula (6.39) guarantees for $\lambda_n - \lambda_0^0$. Similar result was obtained for eigenfunctions.
(vi) We mention also the papers [6], [7], and [5] where different spectral properties of \( j \)-selfadjoint Dirac operators were investigated.

7. Stability property of eigenfunctions

Throughout the section we will use the following notation for the “maximal” Fourier transform of the potential matrix \( Q \). Namely, let us set for \( x \in [0,1] \), \( \lambda \in \mathbb{C} \) and \( k \in \{1,2\} \),

\[
\mathcal{F}_k(x,\lambda) := \mathcal{F}_k(Q)(x,\lambda) := \sup_{s \in [0,x]} \left| \int_0^s Q_{jk}(t)e^{i(b_k-b_j)\lambda t} \, dt \right|, \quad j = 2/k. \tag{7.1}
\]

Note that this notation is generally valid for any matrix-function \( W \in L^1([0,1];\mathbb{C}^{2 \times 2}) \).

7.1. Estimates of Fourier transforms of transformation operators. In this subsection we study “Fourier” transforms of the kernels of the corresponding transformation operators from representation (3.2) of the form \( \int_0^x K_{jk}^+(x,t)e^{i\lambda b_k b_j t} \, dt \). The motivation comes from the formula (4.12) for the entries of the fundamental matrix solution of the system (4.11) where these integrals appear. We will estimate these integrals with the “maximal” Fourier transforms \( \mathcal{F}_1(x,\lambda) \), \( \mathcal{F}_2(x,\lambda) \) of the potential matrix.

In what follows we heavily use notation (4.12), in particular, \( a_k = b_k^{-1} \), \( k \in \{1,2\} \). Note also that

\[
\alpha_j = a_j/(a_j - a_k), \quad b_k\alpha_j^{-1} = b_k - b_j, \quad b_k\alpha_j^{-1}\alpha_k = -b_j, \quad k \in \{1,2\}, \quad j = 2/k. \tag{7.2}
\]

As a first step we study “Fourier” transforms of the auxiliary kernels \( R \) from the representation (3.9) for the kernels of the transformation operators \( K^\pm \). The first auxiliary result estimates generalized “Fourier” transforms with an arbitrary bounded function \( f \) instead of the exponential function in the integral.

**Lemma 7.1.** Let \( Q \in L^1([0,1];\mathbb{C}^{2 \times 2}) \), and let \( R = (R_{jk})_{j,k=1}^2 \in \left( X^0_{1,1}(\Omega) \cap X^0_{\infty,1}(\Omega) \right) \otimes \mathbb{C}^{2 \times 2} \) be a (unique) solution of the system of integral equations (3.10)–(3.11). Let \( x \in [0,1] \) be fixed and let \( f \in L^\infty(\mathbb{R}) \) be such that \( f(t) = 0 \) for \( t \notin [0,x] \). Let us set

\[
F_{jk}(s; f) := \sup_{u \in [0,s]} \left| \int_0^u R_{jk}(s,t)f(t+v) \, dt \right|, \quad s \in [0,x], \quad j,k \in \{1,2\}. \tag{7.3}
\]

Then the following estimates hold for \( s \in [0,x], \ k \in \{1,2\}, \ j = 2/k \):

\[
F_{kk}(s; f) \leq |b_k| \int_0^s \left| Q_{kj}(t) \right| \cdot F_{jk}(t; f) \, dt, \tag{7.4}
\]

\[
F_{jk}(s; f) \leq |b_j| \sup_{u \in [0,s]} \left( \alpha_j \int_0^u Q_{jk}(\alpha_k s + \alpha_j t) \cdot f(t+v) \, dt \right) + 2|b_j b_k| \cdot \|Q_{kj}\|_{L^1([0,s])} \cdot \int_0^s |Q_{kj}(t)| \cdot F_{jk}(t; f) \, dt. \tag{7.5}
\]

**Proof.** Let us set

\[
J_{jk}(s,u,v) := \int_0^u R_{jk}(s,t)f(t+v) \, dt, \quad 0 \leq u \leq s \leq x, \quad v \in \mathbb{R}, \quad j,k \in \{1,2\}. \tag{7.6}
\]

Note that since \( f(t) = 0 \), \( t \notin [0,x] \), then \( J_{jk}(s,u,v) = 0 \) for \( 0 \leq u \leq s \leq x \) and \( v \notin [-u,x] \). Hence definition (7.3) of \( F_{jk}(s; f) \) implies that

\[
|J_{jk}(s,u,v)| \leq F_{jk}(s; f), \quad 0 \leq u \leq s \leq x, \quad v \in \mathbb{R}, \quad j,k \in \{1,2\}. \tag{7.7}
\]
Let $k \in \{1, 2\}$ and $j = 2/k$. It follows from (3.10) that

$$J_{kk}(s, u, v) = \int_0^u R_{kk}(s, t) f(t + v) dt$$

$$= -\frac{i}{a_k} \int_0^u f(t + v) dt \int_{s + t}^s Q_{kj}(\xi) R_{jk}(\xi, \xi + t) d\xi$$

$$= -ib_k \int_{s - u}^s Q_{kj}(\xi) d\xi \int_{s - \xi}^u R_{jk}(\xi, \xi + s) f(t + v) dt$$

$$= -ib_k \int_{s - u}^s Q_{kj}(\xi) d\xi \int_{0}^{u + \xi - s} R_{jk}(\xi, \eta) f(\eta + s - \xi + v) d\eta$$

$$= -ib_k \int_{s - u}^s Q_{kj}(\xi) J_{jk}(\xi, u + \xi - s, v + s - \xi) d\xi, \quad 0 \leq u \leq s \leq x, \quad v \in \mathbb{R}. \quad (7.8)$$

Changing notation and taking into account (7.7), relation (3.11) yields that

$$|J_{kk}(\xi, u, v)| \leq |h_k| \int_0^\xi |Q_{kj}(t)| \cdot F_{jk}(t; f) dt, \quad 0 \leq u \leq \xi \leq x, \quad v \in \mathbb{R}, \quad (7.9)$$

Taking supremum over $v \in [-u, x]$ and $u \in [0, \xi]$ in (7.9) now yields (7.4).

Taking into account notation (3.12), relation (3.11) yields for $0 \leq u \leq s \leq x$ and $v \in \mathbb{R}$ that

$$J_{jk}(s, u, v) = \int_0^u R_{jk}(s, t) f(t + v) dt = -\frac{i}{a_j} \cdot (J_{1,jk}(s, u, v) + J_{2,jk}(s, u, v)), \quad (7.10)$$

$$J_{1,jk}(s, u, v) := -\frac{a_j}{a_k - a_j} \int_0^u Q_{jk} \left( \frac{a_k s - a_j t}{a_k - a_j} \right) f(t + v) dt = a_j \int_0^u Q_{jk}(a_k s + a_j t) \cdot f(t + v) dt, \quad (7.11)$$

$$J_{2,jk}(s, u, v) := \int_0^u f(t + v) dt \int_{s + a_j t}^s Q_{jk}(\xi) R_{kk}(\xi, v_x + t) d\xi, \quad v_x := \gamma_k(\xi - s). \quad (7.12)$$

Changing order of integration in (7.12) and then making the change of variable $\eta = t + v_x$ we arrive at

$$J_{2,jk}(s, u, v) = \int_{a_j s}^s Q_{jk}(\xi) d\xi \int_0^{\min\{u, \xi - v_x\}} R_{kk}(\xi, v_x + t) \cdot f(t + v) dt$$

$$= \int_{a_j s}^s Q_{jk}(\xi) d\xi \int_{v_x}^{\min\{u + v_x, \xi\}} R_{kk}(\xi, \eta) \cdot f(\eta - v_x + v) d\eta. \quad (7.13)$$

Combining (7.10), (7.13), (7.15) and (7.9) we get:

$$|J_{2,jk}(s, u, v)| \leq \int_{a_j s}^s |Q_{jk}(\xi)| \left( |J_{kk}(\xi, v_x, v - v_x)| + |J_{kk}(\xi, \min\{u + v_x, \xi\}, v - v_x)| \right) d\xi$$

$$\leq 2|b_k| \int_0^s |Q_{jk}(\xi)| d\xi \int_0^\xi |Q_{kj}(t)| \cdot F_{jk}(t; f) dt$$

$$= 2|b_k| \int_0^s |Q_{kj}(t)| \cdot F_{jk}(t; f) dt \int_0^{\min\{u, \xi - v_x\}} |Q_{jk}(\xi)| d\xi$$

$$\leq 2|b_k| \cdot \|Q_{jk}\|_{L^1[0,u]} \int_0^s |Q_{kj}(t)| \cdot F_{jk}(t; f) dt, \quad 0 \leq u \leq s \leq x, \quad v \in \mathbb{R}. \quad (7.14)$$

Inserting (7.11) and (7.14) into (7.10) we arrive at the following inequality for $0 \leq u \leq s \leq x, \quad v \in \mathbb{R}$

$$|J_{jk}(s, u, v)| \leq b_j a_j \int_0^u Q_{jk}(a_k s + a_j t) \cdot f(t + v) dt + 2|b_k b_k| \cdot \|Q_{jk}\|_{L^1[0,u]} \int_0^s |Q_{kj}(t)| \cdot F_{jk}(t; f) dt. \quad (7.15)$$

Taking supremum over $v \in [-u, x]$ and $u \in [0, s]$ in (7.15) yields (7.5).
Corollary 7.2. Let $Q \in \mathbb{U}^{2 \times 2}$ for some $r > 0$, and let $R = (R_{jk})_{j,k=1}^{2} \in (X_{1,1}^{0}(\Omega) \cap X_{k,1}^{0}(\Omega)) \otimes \mathbb{C}^{2 \times 2}$ be a (unique) solution of the system of integral equations (3.10) – (3.11). Then the following uniform estimate holds for $k,l \in \{1,2\}$.

$$
\sup_{s \in [0,t]} \left| \int_{0}^{t} R_{jk}(x,t) e^{i\lambda b_{k}t} dt \right| \leq C e^{(b_{2} - b_{1})|\text{Im}\lambda|x} \cdot F_{k}(x,\lambda), \quad x \in [0,1], \quad \lambda \in \mathbb{C},
$$

(7.16)

where $C = C(B,r) > 0$ does not depend on $Q$, $x$ and $\lambda$, and $F_{k}(x,\lambda)$ is defined in (7.1).

Proof. Let $\lambda \in \mathbb{C}$, $x \in [0,1]$, $k \in \{1,2\}$ and $j = 2/k$ be fixed for the rest of the proof. We will apply Lemma 7.1 with $f(t) = e^{ib_{k}t}$, $t \in [0,1]$, and $f(t) = 0$, $t \notin [0,1]$. Let us estimate the first summand in the r.h.s. of (7.14) for such $f$. To this end, for $0 \leq u \leq s \leq x$ and $v \in [-u,x]$ we have

$$
\left| \alpha_{j} \int_{0}^{u} Q_{jk}(\alpha_{k}s + \alpha_{j}t) \cdot f(t + v) dt \right| = \alpha_{j} \int_{\min(\{0,-v\})}^{\max(\{0,-v\})} Q_{jk}(\alpha_{j}t + \alpha_{k}s) e^{ib_{k}\lambda(t+v)} dt
$$

$$
\leq 2 \left| \exp (\lambda b_{k}t) \right| \sup_{t \in [0,s]} \left| \int_{0}^{t} Q_{jk}(\xi) \exp (i(b_{k} - b_{j})\lambda\xi) d\xi \right|
$$

$$
\leq 2 e^{(b_{k} - b_{j})|\text{Im}\lambda|x} \cdot F_{k}(x,\lambda).
$$

(7.17)

Here we used the change of variable $\xi = \alpha_{j}t + \alpha_{k}s$, identity $b_{k}\alpha_{j}^{-1} = (b_{k} - b_{j})\xi$ and $-b_{k}\alpha_{j}^{-1}\alpha_{k}s = b_{j}s$, and definition (7.1) of $F_{k}(x,\lambda)$. Inserting (7.17) into (7.15) now yields

$$
F_{jk}(s;f) \leq |b_{j}| \cdot F_{k}(x,\lambda) + 2|b_{j}b_{k}| \int_{0}^{s} |Q_{kj}(t)| \cdot F_{jk}(t;f) dt, \quad s \in [0,x].
$$

(7.18)

Applying Grönwall’s inequality to (7.18) and taking into account that $\int_{0}^{s} |Q_{kj}(u)| du \leq \|Q_{kj}\|_{L^{1}[0,1]} \leq r$, implies that

$$
\left| \int_{0}^{s} R_{jk}(s,t) e^{i\lambda b_{k}t} dt \right| = |J_{jk}(s,u,0)| \leq F_{jk}(s;f) \leq |b_{j}| \cdot F_{k}(x,\lambda) \cdot \exp \left( 2|b_{j}b_{k}| \int_{0}^{s} |Q_{kj}(u)| du \right),
$$

$$
\leq C_{j} \cdot F_{k}(x,\lambda), \quad 0 \leq u \leq s \leq x,
$$

(7.19)

where $C_{j} := |b_{j}| \exp (2|b_{2}b_{2}| r^{2})$. Setting $s = x$ and taking supremum over $u \in [0,x]$ in (7.19) implies (7.16) for $l \neq k$. Inserting the estimate $F_{jk}(t;f) \leq C_{j} \cdot F_{k}(x,\lambda)$, $t \in [0,x]$, from (7.19) into (7.14) yields that

$$
\left| \int_{0}^{u} R_{kk}(s,t) e^{i\lambda b_{k}t} dt \right| = |J_{kk}(s,u,0)| \leq F_{kk}(s;f) \leq |b_{k}| \cdot C_{j} \int_{0}^{u} |Q_{kj}(t)| \cdot F_{k}(x,\lambda) dt
$$

$$
= |b_{k}| \cdot C_{j} \cdot \|Q_{kj}\|_{L^{1}[0,s]} \cdot F_{k}(x,\lambda) \leq C \cdot F_{k}(x,\lambda), \quad 0 \leq u \leq s \leq x,
$$

(7.20)

where $C = |b_{1}b_{2}| r \cdot \exp (2|b_{2}b_{2}| r^{2})$. Setting $s = x$ and taking supremum over $u \in [0,x]$ in (7.20) implies (7.16) for $l = k$ and completes the proof. \(\square\)

We also need to estimate Fourier transforms of the auxiliary functions $P_{k}^{\pm}$ from the representation (3.8) – (3.9).

Lemma 7.3. Let $Q \in \mathbb{U}^{2 \times 2}$ for some $r > 0$ and let $K^{\pm}$ be the kernels from the integral representation (3.2). Let matrix-function $P^{\pm}$ be given by (3.8) – (3.9) for $K^{\pm}$. Then the following uniform estimate holds for $x \in [0,1]$ and $\lambda \in \mathbb{C}$

$$
\left| \int_{0}^{x} P_{k}^{\pm}(t) e^{ib_{k}\lambda(x-t)} dt \right| \leq C \cdot e^{(b_{2} - b_{1})|\text{Im}\lambda|x} \cdot (\mathcal{F}_{1}(x,\lambda) + \mathcal{F}_{2}(x,\lambda)), \quad k \in \{1,2\},
$$

(7.21)

where $C = C(B,r) > 0$ does not depend on $Q$, $x$ and $\lambda$, and $\mathcal{F}_{1}(x,\lambda)$, $\mathcal{F}_{2}(x,\lambda)$ are defined in (7.1).
Proof. Recall that \( \gamma_1 = a_1 a_2^{-1} \), \( \gamma_2 = a_2 a_1^{-1} \). It follows from (3.59) that

\[
P_{1\pm}^r (x) = \mp \gamma_2 R_{12} (x, 0) - \int_0^x \left( R_{11} (x, t) P_{1\pm}^r (t) \pm \gamma_2 R_{12} (x, t) P_{2\pm}^r (t) \right) dt, \tag{7.22}
\]

\[
P_{2\pm}^r (x) = \mp \gamma_1 R_{21} (x, 0) - \int_0^x \left( \mp \gamma_2 R_{21} (x, t) P_{1\pm}^r (t) + R_{22} (x, t) P_{2\pm}^r (t) \right) dt. \tag{7.23}
\]

Let \( k \in \{1, 2\} \) and \( j = 2/k \) be fixed for the rest of the proof. Let us set \( \gamma := \max \{|\gamma_1|, |\gamma_2|\} \). It follows from (7.22)–(7.23) that for \( x \in [0, 1] \) and \( \lambda \in \mathbb{C} \) we have

\[
\left| \int_0^x P_{k\pm}^r (t) e^{ib_k \lambda (x-t)} dt \right| \leq \gamma \left| \int_0^x R_{kj} (t, 0) e^{ib_k \lambda (x-t)} dt \right| + \gamma \sum_{l=1}^2 \left| \int_0^x P_{l\pm}^r (s) \int_s^x R_{kl} (t, s) e^{ib_k \lambda (x-t)} dt \right| \bigg| \left| \int_0^x P_{k\pm}^r (t) e^{ib_k \lambda (x-t)} dt \right| \\
= \gamma |Z_{kj} (x, \lambda)| + \gamma \sum_{l=1}^2 \left| \int_0^x P_{l\pm}^r (s) Z_{kl} (x, s, \lambda) ds \right|, \quad Z_{kl} (x, s, \lambda) := \int_s^x R_{kl} (t, s) e^{ib_k \lambda (x-t)} dt. \tag{7.24}
\]

It follows from (3.10) and Corollary 7.2 that for \( 0 \leq s \leq x \leq 1 \) and \( \lambda \in \mathbb{C} \) we have

\[
|Z_{kk} (x, s, \lambda)| = \left| b_k \int_s^x e^{ib_k \lambda (x-t)} dt \int_{t-s}^t Q_{kj} (\xi) R_{jk} (\xi, \xi - t + s) d\xi \right| = \left| b_k \int_0^x Q_{kj} (\xi) d\xi \int_{\min \{x, s+\xi\}}^{\max \{s, \xi\}} R_{jk} (\xi, \xi - t + s) e^{ib_k \lambda (x-t)} dt \right| = \left| b_k \int_0^x Q_{kj} (\xi) d\xi \int_{\min \{s, \xi\}}^{\max \{x+s-x, 0\}} R_{jk} (\xi, \eta) e^{ib_k \lambda (x+\xi-\xi-x-s)} \eta \right| \leq 2 |b_k| \int_0^x \left| Q_{kj} (\xi) e^{ib_k \lambda (x-\xi-x)} \right| \sup_{u \in [0, \xi]} \left| \int_0^u R_{jk} (\xi, \eta) e^{ib_k \lambda \eta} d\eta \right| d\xi \leq 2 |b_k| \cdot C \int_0^x \left| Q_{kj} (\xi) \cdot e^{ib_k \lambda (x-x)} \cdot e^{(b_j-b_k) \Im \lambda} \cdot \mathcal{F}_k (\xi, \lambda) \right| d\xi \leq 2 |b_k| \cdot C \cdot \|Q_{kj}\|_1 \cdot e^{2(b_2-b_1) \Im \lambda} \cdot \mathcal{F}_k (x, \lambda). \tag{7.25}
\]

Relation (3.11) implies for \( 0 \leq s \leq x \leq 1 \) and \( \lambda \in \mathbb{C} \)

\[
Z_{kj} (x, s, \lambda) = \int_s^x R_{kj} (t, s) e^{ib_k \lambda (x-t)} dt = \frac{i}{a_j} \left( Z_{kj,1} (x, s, \lambda) - Z_{kj,2} (x, s, \lambda) \right), \tag{7.26}
\]

\[
Z_{kj,1} (x, s, \lambda) := \frac{\alpha_j}{a_j - a_k} \int_s^x Q_{kj} (\alpha_j t + \alpha_k s) e^{ib_k \lambda (x-t)} dt, \tag{7.27}
\]

\[
Z_{kj,2} (x, s, \lambda) := \frac{\alpha_j}{a_k} \int_s^x e^{ib_k \lambda (x-t)} dt \int_{\alpha_j + \alpha_k s}^t Q_{kj} (\xi) R_{jj} (\xi, \gamma_j (\xi - t) + s) d\xi. \tag{7.28}
\]

Making a change of variable \( \xi = \alpha_j t + \alpha_k s \) in (7.27) combined with identities \( \alpha_j/a_j = b_j/a_j - (b_k - b_j) \xi \) and \( b_k \alpha_j^{-1} \alpha_k s = -b_j s \), and definition (7.31) of \( \mathcal{F}_j (x, \lambda) \) we get for \( 0 \leq s \leq x \leq 1 \) and \( \lambda \in \mathbb{C} \)

\[
\left| Z_{kj,1} (x, s, \lambda) \right| = \left| \int_s^{\alpha_j x + \alpha_k s} Q_{kj} (\xi) \exp \left( ib_k \lambda (x + \alpha_j^{-1} (\xi + \alpha_k s)) \right) d\xi \right| = \left| e^{ib_k (b_j-b_k) \lambda s} \int_s^{\alpha_j x + \alpha_k s} Q_{kj} (\xi) e^{ib_j \lambda s} d\xi \right| \leq 2 e^{ib_2(b_2-b_1) \Im \lambda} \mathcal{F}_j (x, \lambda). \tag{7.29}
\]
Changing order of integration in (7.28), then making a change of variable \( \eta = \gamma_j (\xi - t) + s \) combined with the identities \( t = \xi - \gamma_k \eta + \gamma_k s, \gamma_j (\xi - \alpha_j^{-1} \xi - \gamma_k s) + s = \xi \) and \( b_k \gamma_k = b_j \), and applying Corollary 7.2 yields

\[
|Z_{kj,2}(x, s, \lambda)| = \left| \gamma_j \int_{\xi}^{x} Q_{kj}(\xi) d\xi \int_{\xi}^{\min\{x, \alpha_j^{-1}(\xi + \gamma_k s)\}} R_{jj}(\xi, \gamma_j (\xi - t) + s) e^{ib_k \lambda(x-t)} dt \right|
\]

\[
= \left| \int_{\xi}^{x} Q_{kj}(\xi) d\xi \int_{\xi}^{\min\{\gamma_j(\xi-x)+s, \xi\}} R_{jj}(\xi, \eta) \exp(\{ib_k \lambda(x-\xi + \gamma_k \eta - \gamma_k s)\}) d\eta \right|
\]

\[
\leq 2 \int_{\xi}^{x} \left| Q_{kj}(\xi) \right| e^{\left( (\lambda(b_k(x-\xi)-b_k s)) \right)} \sup_{w \in [0, \xi]} \left| \int_{0}^{w} R_{jj}(\xi, \eta) e^{ib_{j} \lambda \eta} d\eta \right| d\xi
\]

\[
\leq 2C \int_{\xi}^{x} \left| Q_{kj}(\xi) \right| e^{\left( (\lambda(b_k(x-\xi)-b_k s)) \right)} \cdot F_{j}(x, \lambda) \int_{\xi}^{x} \left| Q_{kj}(\xi) \right| d\xi
\]

\[
\leq 2C \cdot e^{(2(b_k-b_s)) \lambda x} \cdot F_{j}(x, \lambda) \int_{\xi}^{x} \left| Q_{kj}(\xi) \right| d\xi
\]

Inserting (7.29) and (7.30) into (7.20), we arrive at

\[
|Z_{kj}(x, s, \lambda)| \leq C_{j} \cdot e^{(2(b_k-b_s)) \lambda x} \cdot F_{j}(x, \lambda), \quad 0 \leq s \leq x \leq 1, \quad \lambda \in \mathbb{C},
\]

where \( C_{j} = 2|b_j| (1 + Cr) \). Finally, inserting (7.25) and (7.31) into (7.24), using estimate (3.65) from the proof of Theorem 3.3 for \( p = 1 \), and inequalities \( |Q_{kj}| \leq 1 \), \( |Q_{1}| \leq r \), yields

\[
\left| \int_{0}^{x} P_{k}^{\pm}(t) e^{ib_{k} \lambda(x-t)} dt \right| \leq \gamma \cdot \| P_{k}^{\pm} \|_{L[0,x]} \sup_{s \in [0,x]} |Z_{kk}(x, s, \lambda)| + \gamma \cdot (1 + \| P_{j}^{\pm} \|_{L[0,x]}) \sup_{s \in [0,x]} |Z_{kj}(x, s, \lambda)|
\]

\[
\leq \gamma \cdot C(1) (B, r) \left( \| Q_{1} \|_{L[0,x]} \sup_{s \in [0,x]} |Z_{kk}(x, s, \lambda)| + (1 + \| Q_{1} \|_{L[0,x]}) \sup_{s \in [0,x]} |Z_{kj}(x, s, \lambda)| \right)
\]

\[
\leq C_{3} \cdot e^{2(b_k-b_s) \lambda x} \cdot (F_{k}(x, \lambda) + F_{j}(x, \lambda)),
\]

where \( C_{3} = C_{3}(B, r) > 0 \) does not depend on \( Q, x \) or \( \lambda \), which finishes the proof.

Now we are ready to prove the first main result of the section about kernels \( K_{\pm} \) of the transformation operators.

**Theorem 7.4.** Let \( Q \in U_{1,r}^{2x2} \) for some \( r > 0 \), and let \( K_{\pm} \) be the kernels of the corresponding transformation operators from representation (3.2). Then the following uniform estimate holds

\[
\left| \int_{0}^{x} K_{jk}^{\pm}(x, t) e^{ib_{k} \lambda (x-t)} dt \right| \leq C \cdot e^{2(b_k-b_s) \lambda x} (F_{1}(x, \lambda) + F_{2}(x, \lambda)), \quad x \in [0, 1], \quad \lambda \in \mathbb{C}, \quad j, k \in \{1, 2\},
\]

where \( C = C(B, r) > 0 \) does not depend on \( Q, x, \lambda \), and \( F_{1}(x, \lambda) \) is defined in (7.11). In other words,

\[
\left| \int_{0}^{x} K_{jk}^{\pm}(x, t) e^{ib_{k} \lambda (x-t)} dt \right| \leq C \cdot e^{2(b_k-b_s) \lambda x} \sum_{l \neq m} \sup_{h \in [0,x]} \int_{0}^{h} Q_{lm}(t) e^{i(b_m-b_l) \lambda t} dt, \quad x \in [0, 1], \quad \lambda \in \mathbb{C}, \quad j, k \in \{1, 2\},
\]

**Proof.** It follows from (3.9) that

\[
K_{jk}^{\pm}(x, t) = R_{jk}(x, t) + \delta_{jk} P_{k}^{\pm}(x-t) + \int_{t}^{x} R_{jk}(x, s) P_{k}^{\pm}(s-t) ds, \quad 0 \leq t \leq x \leq 1, \quad j, k \in \{1, 2\},
\]

where \( P_{k}^{\pm} \in L[0, 1], k \in \{1, 2\} \). This in turn yields for \( x \in [0, 1], \lambda \in \mathbb{C} \) and \( j, k \in \{1, 2\} \),

\[
\left| \int_{0}^{x} K_{jk}^{\pm}(x, t) e^{ib_{k} \lambda (x-t)} dt \right| \leq \left| \int_{0}^{x} R_{jk}(x, t) e^{ib_{k} \lambda (x-t)} dt \right| + \left| \int_{0}^{x} P_{k}^{\pm}(t) e^{i\lambda (x-t)} dt \right| + \left| \int_{0}^{x} R_{jk}(x, t) \int_{0}^{t} P_{k}^{\pm}(s) e^{ib_{k} \lambda (t-s)} ds dt \right|.
\]
Proof. Then the following estimates hold for equation (3.1) for $Q$ and $X$

\[ \int_0^t R_{jk}(x,t) \int_0^t P_k^\pm(s) e^{ib_k \lambda(t-s)} ds \, dt \leq \sup_{t \in [0,x]} \int_0^t P_k^\pm(s) e^{ib_k \lambda(t-s)} ds \leq C \sup_{t \in [0,x]} \left( e^{2(b_2-b_1)\lambda t} \right) \| R_{jk} \|_{L^\infty(\Omega)} \]

\[ \leq C_1 e^{2(b_2-b_1)\lambda x} \left( \mathcal{F}_1(x,\lambda) + \mathcal{F}_2(x,\lambda) \right), \quad (7.37) \]

where $C_1 = C_1(B,r)$ does not depend on $Q$, $x$ and $\lambda$. Putting (7.10), (7.21) and (7.37) in (7.36) we arrive at (7.33). $\square$

7.2. Stability of Fourier transforms of transformation operators. Alongside equation (4.1) we consider similar Dirac type equation with the same matrix $B$ but with a different potential matrix $Q \in L^1([0,1];\mathbb{C}^{2\times 2})$. In this subsection we apply results of the previous subsection to study stability of “Fourier” transforms of the kernels of the corresponding transformation operators from representation (3.2). Namely, we establish analogue of Lipshitz property for the deviation

\[ \int_0^x (K_Q^\pm - K_Q^\mp)_{jk}(x,t) e^{ib_k \lambda t} dt. \quad (7.38) \]

Theorem 7.8 will play crucial role in the study of deviations of the root functions of operators $L(Q)$ and $L(\tilde{Q})$.

Below we systematically use notation (3.17). Let us recall it:

\[ \tilde{Q} := Q - \tilde{Q}, \quad K^\pm := K_Q^\pm, \quad \tilde{K}^\pm := \tilde{K}_Q^\pm, \quad R := R_Q, \quad \tilde{R} := \tilde{R}_Q, \quad P^\pm := P_Q^\pm, \quad \tilde{P}^\pm := \tilde{P}_Q^\pm. \quad (7.39) \]

To estimate the deviation (7.38) we first need to extend auxiliary results of the previous subsection about $R$ and $P^\pm$ to $\tilde{R}$ and $\tilde{P}^\pm$.

Lemma 7.5. Let $Q, \tilde{Q} \in L^1([0,1];\mathbb{C}^{2\times 2})$ and let $R, \tilde{R} \in (X_1^0(\Omega) \cap X_\infty^0(\Omega)) \otimes \mathbb{C}^{2\times 2}$ be (unique) solutions of the system of integral equations (3.10) - (3.11) for $Q$ and $\tilde{Q}$ respectively. Let $x \in [0,1]$ be fixed and let $f \in L^\infty(\mathbb{R})$ be such that $f(t) = 0$ for $t \notin [0,x]$. Let us set

\[ \tilde{F}_{jk}(s;f) := \sup_{s \in [0,x]} \left| \int_0^u (R - \tilde{R})_{jk}(s,t) f(t+v) dt \right|, \quad s \in [0,x], \quad j,k \in \{1,2\}. \quad (7.41) \]

Then the following estimates hold for $s \in [0,x], \ j,k \in \{1,2\}, \ j = 2/k$:

\[ \tilde{F}_{kk}(s;f) \leq |b_k| \int_0^s \left( |Q_{kj}(t)| \cdot \tilde{F}_{jk}(t;f) + |\tilde{Q}_{kj}(t)| \cdot \tilde{F}_{jk}(t;f) \right) dt, \quad (7.42) \]

\[ \tilde{F}_{jk}(s;f) \leq \Theta_{jk}(s;f) + 2|b_j b_k| \cdot \|Q_{kj}\|_{L^1[0,s]} \int_0^s |Q_{kj}(t)| \cdot \tilde{F}_{jk}(t;f) dt, \quad (7.43) \]

\[ \Theta_{jk}(s;f) := |b_j| \sup_{s \in [0,x]} \left| \alpha_j \int_0^u \tilde{Q}_{jk}(\alpha_k s + \alpha_j t) \cdot f(t+v) dt \right| 
\]

\[ + 2|b_j| \int_0^s \left| \tilde{Q}_{kj}(t) \cdot \tilde{F}_{kk}(t;f) \right| dt + 2|b_j b_k| \cdot \|Q_{kj}\|_{L^1[0,s]} \int_0^s |\tilde{Q}_{kj}(t)| \cdot \tilde{F}_{jk}(t;f) dt. \quad (7.44) \]

Here $\tilde{Q} = Q - \tilde{Q}$ and $\tilde{F}_{jk}(s;f), \ j,k \in \{1,2\}$, is defined in (7.3) with $\tilde{R}_{jk}$ in place of $R_{jk}$.

Proof. We will follow the schema of the proof of Lemma 7.1 step by step and will use the following identity

\[ X \cdot Y - \tilde{X} \cdot \tilde{Y} = X \cdot (Y - \tilde{Y}) + (X - \tilde{X}) \cdot \tilde{Y} = X \cdot \tilde{Y} + \tilde{X} \cdot \tilde{Y}, \quad (7.45) \]

every time we encounter a product difference of the form $X \cdot Y - \tilde{X} \cdot \tilde{Y}$, where $X, Y$ are some objects associated with equation (3.1) for $Q$ and $\tilde{X}, \tilde{Y}$ are the same objects associated with equation (3.1) for $\tilde{Q}$. 
Recall that $\tilde{R} = R - \bar{R}$. Let us set
\[
\tilde{J}_{jk}(s, u, v) := \int_0^u \hat{R}_{jk}(s, t)f(t + v) \, dt, \quad 0 \leq u \leq s \leq x, \quad v \in \mathbb{R}, \quad j, k \in \{1, 2\}. \tag{7.46}
\]
Let $k \in \{1, 2\}$ and $j = 2/k$. Similarly to (7.13) with account of (7.35) we have for $0 \leq u \leq s \leq x$ and $v \in \mathbb{R}$,
\[
\tilde{J}_{kk}(s, u, v) = \int_0^u \hat{R}_{kk}(s, t)f(t + v) \, dt \\
= -ib_k \int_0^u f(t + v) \, dt \int_{s-t}^s \left( Q_{kj}(\xi) \hat{R}_{kj}(\xi, \xi - s + t) - \hat{Q}_{kj}(\xi) \tilde{R}_{kj}(\xi, \xi - s + t) \right) \, d\xi \\
= -ib_k \int_0^u f(t + v) \, dt \int_{s-t}^s \left( Q_{kj}(\xi) \hat{R}_{kj}(\xi, \xi - s + t) + \hat{Q}_{kj}(\xi) \tilde{R}_{kj}(\xi, \xi - s + t) \right) \, d\xi \\
= -ib_k \int_0^s \left( Q_{kj}(\xi) \tilde{J}_{jk}(\xi, u, \xi - s, v + s - \xi) + \hat{Q}_{kj}(\xi) \tilde{J}_{jk}(\xi, u, \xi - s, v + s - \xi) \right) \, d\xi. \tag{7.47}
\]
Similarly to (7.35) we get
\[
|\tilde{J}_{kk}(\xi, \eta, v)| \leq |b_k| \int_0^\xi \left( |Q_{kj}(t)| \cdot \hat{F}_{jk}(t; f) + |\hat{Q}_{kj}(t)| \cdot \tilde{F}_{jk}(t; f) \right) \, dt, \quad 0 \leq \eta \leq \xi \leq x, \quad v \in \mathbb{R}, \tag{7.48}
\]
which yields (7.42).

Taking into account (3.12) and (7.45), relation (3.11) yields for $0 \leq u \leq s \leq x$ and $v \in \mathbb{R}$,
\[
\tilde{J}_{jk}(s, u, v) = \int_0^u \hat{R}_{jk}(s, t)f(t + v) \, dt = -\frac{i}{a_j} \cdot \left( \tilde{J}_{1,jk}(s, u, v) + \tilde{J}_{2,jk}(s, u, v) \right), \tag{7.49}
\]
\[
\tilde{J}_{1,jk}(s, u, v) := a_j \int_0^u \hat{Q}_{jk}(\alpha_k s + \alpha_j t) \cdot f(t + v) \, dt, \tag{7.50}
\]
\[
\tilde{J}_{2,jk}(s, u, v) := \int_0^u f(t + v) \, dt \int_{s-t}^s \left( Q_{kj}(\xi) \hat{R}_{kk}(\xi, \gamma_k(\xi - s) + t) - \hat{Q}_{kj}(\xi) \tilde{R}_{kk}(\xi, \gamma_k(\xi - s) + t) \right) \, d\xi \\
= \int_0^u f(t + v) \, dt \int_{a_j s + \alpha_j t}^s \left( Q_{kj}(\xi) \hat{R}_{kk}(\xi, \gamma_k(\xi - s) + t) + \hat{Q}_{kj}(\xi) \tilde{R}_{kk}(\xi, \gamma_k(\xi - s) + t) \right) \, d\xi. \tag{7.51}
\]
For brevity we denote,
\[
\xi_1 := \gamma_k(\xi - s), \quad \xi_2 := \min\{u + \gamma_k(\xi - s), \xi\}, \quad v_\xi = v - \gamma_k(\xi - s). \tag{7.52}
\]

Similarly to (7.46) and (7.44), it follows from (7.35), (7.48) and (7.47), that for $0 \leq u \leq s \leq x$ and $v \in \mathbb{R}$,
\[
|\tilde{J}_{2,jk}(s, u, v)| \leq \int_{a_k s + \alpha_j t}^s \left( |Q_{kj}(\xi)| \cdot |\tilde{J}_{kk}(\xi, \xi_1, v_\xi)| \right) \, d\xi \\
\leq 2|b_k| \int_0^s |Q_{kj}(\xi)| \, d\xi \int_0^\xi \left( |Q_{kj}(t)| \cdot |\hat{F}_{jk}(t; f)| + |\hat{Q}_{kj}(t)| \cdot |\tilde{F}_{jk}(t; f)| \right) \, dt + 2 \int_0^s |Q_{kj}(\xi)| \cdot |\hat{F}_{kk}(\xi; f)| \, d\xi \\
\leq 2|b_k| \cdot |Q_{kj}|_{L^1[0, s]} \int_0^s \left( |Q_{kj}(t)| \cdot |\hat{F}_{jk}(t; f)| + |\hat{Q}_{kj}(t)| \cdot |\tilde{F}_{jk}(t; f)| \right) \, dt + 2 \int_0^s |Q_{kj}(t)| \cdot |\tilde{F}_{kk}(t; f)| \, dt. \tag{7.53}
\]

Inserting (7.50) and (7.52) into (7.49) we arrive at (7.43) - (7.44). \hfill \Box

**Corollary 7.6.** Let $Q, \tilde{Q} \in \mathbb{U}^{2 \times 2}_{1, r}$ for some $r > 0$, and let $R, \tilde{R} \in (X^1_{1, 1}(\Omega) \cap X^0_{\infty, 1}(\Omega)) \otimes C^{2 \times 2}$ be (unique) solutions of the system of integral equations (3.10) - (3.11) for $Q$ and $\tilde{Q}$ respectively. Then the following uniform estimate holds for $x \in [0, 1], \lambda \in \mathbb{C}$ and $j, k \in \{1, 2\}$,
\[
\sup_{s \in [0, x]} \left| \int_0^s (R - \tilde{R})_{jk}(x, t) e^{i\lambda t} \, dt \right| \leq C \cdot e^{(b_2 - b_1)|1 + \lambda|x} \left( \mathcal{F}_k|Q - \tilde{Q}|(x, \lambda) + \|Q - \tilde{Q}\|_1 \mathcal{F}_k|\tilde{Q}|(x, \lambda) \right), \tag{7.54}
\]
where \( C = C(B, r) > 0 \) does not depend on \( Q, \bar{Q}, x \) and \( \lambda \), and \( \mathcal{F}_k[W](x, \lambda) \) is defined in (7.1) for \( W \in L^1([0, 1]; C^{2 \times 2}) \).

**Proof.** The proof will follow the proof of Corollary 7.2. Let \( x \in [0, 1], k \in \{1, 2\} \) and \( j = 2/k \) be fixed for the rest of the proof. We will apply Lemma 7.3 with \( f(t) = e^{ib_k \lambda t}, t \in [0, x], \) and \( f(t) = 0, t \notin [0, x] \). Let us estimate \( \Theta_k(s; f) \) in (7.43)–(7.44) for such \( f \). Similarly to (7.17) we have \( 0 \leq u \leq s \leq x, v \in [-u, x] \) and \( \lambda \in \mathbb{C} \)

\[
\left| \alpha_j \int_0^u \hat{Q}_{jk}(a_k s + a_j t) \cdot f(t + v) \, dt \right| \leq 2e^{(b_2 - b_1)|\Im \lambda|x} \cdot \mathcal{F}_k[\hat{Q}](x, \lambda) =: 2\mathcal{A}_k(x, \lambda). \tag{7.55}
\]

Further, estimates (7.19)–(7.20) on \( \mathcal{F}_k(t; f) \) in place of \( \mathcal{F}_k(t; f) \) yield for \( l \in \{1, 2\} \) and \( m = 2/l \)

\[
\int_0^s \hat{Q}_{ml}(t) \cdot \mathcal{F}_k(t; f) \, dt \leq C_2 \cdot \left| \mathcal{Q}_j \right|_1 \cdot \mathcal{A}_k(x, \lambda), \quad \mathcal{A}_k(x, \lambda) := e^{(b_2 - b_1)|\Im \lambda|x} \cdot \mathcal{F}_k[\hat{Q}](x, \lambda). \tag{7.56}
\]

Here and below constants \( C_1, C_2, C_3, \) only depend on \( B, r \) and do not depend on \( Q, \bar{Q}, x \) and \( \lambda \). Taking into account that \( \max \{ \left| Q_{jk} \right|_{L^1([0, 1])}, \left| Q_{kj} \right|_{L^1([0, 1])} \} \leq r \), inserting (7.55)–(7.56) in (7.42)–(7.44) implies

\[
\mathcal{F}_{kk}(s; f) \leq C_2 \left( \left| \mathcal{Q}_j \right|_1 \cdot \mathcal{A}_k(x, \lambda) + \int_0^s \left| Q_{kj}(t) \right| \cdot \mathcal{F}_k(t; f) \, dt \right), \quad s \in [0, x], \quad \lambda \in \mathbb{C}, \tag{7.57}
\]

\[
\mathcal{F}_{jk}(s; f) \leq C_3 \left( \mathcal{A}_k(x, \lambda) + \left| \mathcal{Q}_j \right|_1 \cdot \mathcal{A}_k(x, \lambda) + \int_0^s \left| Q_{kj}(t) \right| \cdot \mathcal{F}_k(t; f) \, dt \right), \quad s \in [0, x], \quad \lambda \in \mathbb{C}. \tag{7.58}
\]

The proof is finished in the same way as in Corollary 7.2 using Grönwall’s inequality. \( \square \)

Following the proof of Lemma 7.3 and using Corollary 7.6 and identity (7.35) in the same way it was done in the proof Lemma 7.5 we can obtain the following result on the auxiliary matrix-function \( \tilde{P}^\pm = P^\pm - \tilde{P}^\pm \).

**Lemma 7.7.** Let \( Q, \bar{Q} \in U^{2 \times 2}_r \) for some \( r > 0 \), and let \( K^\pm, \tilde{K}^\pm \) be the kernels of the corresponding transformation operators from representation (5.32) for \( Q \) and \( \bar{Q} \) respectively. Let matrix-functions \( P^\pm, \tilde{P}^\pm, k \in \{1, 2\} \), be given by (3.38)–(5.39) for \( K^\pm \) and \( \tilde{K}^\pm \) respectively. Then the following uniform estimate holds for \( x \in [0, 1] \) and \( \lambda \in \mathbb{C} \)

\[
\int_0^x \left( P_k^\pm(t) - \tilde{P}_k^\pm(t) \right)e^{ib_k \lambda(x-t)} \, dt \leq C \cdot e^{2(b_2 - b_1)|\Im \lambda|x} \sum_{j=1}^2 \left( \mathcal{F}_j[Q - \tilde{Q}](x, \lambda) + \left| Q - \bar{Q} \right|_1 \mathcal{F}_j[\bar{Q}](x, \lambda) \right), \quad k \in \{1, 2\}, \tag{7.59}
\]

where \( C = C(B, r) > 0 \) does not depend on \( Q, \bar{Q}, x \) and \( \lambda \), and \( \mathcal{F}_j[W](x, \lambda) \) is defined in (7.1) for \( W \in L^1([0, 1]; C^{2 \times 2}). \)

Now we are ready to prove Theorem 1.2 our main result on stability of Fourier transforms of kernels \( K^\pm \) of the transformation operators. We will formulate it again using compact notation (7.1).

**Theorem 7.8.** Let \( Q, \bar{Q} \in U^{2 \times 2}_r \) for some \( r > 0 \), and let \( K^\pm := K^\pm_Q, \tilde{K}^\pm := K^\pm_{\bar{Q}} \) be the kernels of the corresponding transformation operators from representation (5.32) for \( Q \) and \( \bar{Q} \) respectively. Then the following uniform estimate holds for \( x \in [0, 1] \) and \( \lambda \in \mathbb{C} \)

\[
\sum_{j,k=1}^2 \int_{0}^{x} \left( K^\pm_{Q} - K^\pm_{\bar{Q}} \right) \mathcal{F}_j(x, \lambda) e^{ib_k \lambda t} \, dt \leq C \cdot e^{2(b_2 - b_1)|\Im \lambda|x} \sum_{j=1}^2 \left( \mathcal{F}_j[Q - \tilde{Q}](x, \lambda) + \left| Q - \bar{Q} \right|_1 \mathcal{F}_j[\bar{Q}](x, \lambda) \right), \quad (7.60)
\]

where \( C = C(B, r) > 0 \) does not depend on \( Q, \bar{Q}, x \) and \( \lambda \), and \( \mathcal{F}_j[W](x, \lambda) \) is defined in (7.1) for \( W \in L^1([0, 1]; C^{2 \times 2}). \)

**Proof.** The proof is similar to the proof of Theorem 7.4. First we subtract formula (7.35) for \( \bar{Q} \) from the same formula for \( Q \) and apply identity (7.45) to the product of differences \( R_{jk}(x, s)P_k^\pm(s - t) - R_{jk}(x, s)\tilde{P}_k^\pm(s - t) \) in the integral. This yields the following inequality with account of notations (7.39)–(7.40),

\[
\left| \int_0^x \tilde{R}_{jk}(x, t)e^{ib_k \lambda t} \, dt \right| \leq \left| \int_0^x \bar{R}_{jk}(x, t)e^{ib_k \lambda t} \, dt \right| + \left| \int_0^x \tilde{P}_k^\pm(t)e^{ib_k \lambda(x-t)} \, dt \right|
\]

\[
+ \left| \int_0^x R_{jk}(x, t) \int_{0}^{t} \bar{P}_k^\pm(s)e^{ib_k \lambda(t-s)} \, ds \, dt \right| + \left| \int_0^x \bar{R}_{jk}(x, t) \int_{0}^{t} \tilde{P}_k^\pm(s)e^{ib_k \lambda(t-s)} \, ds \, dt \right|. \tag{7.61}
\]
Theorem 7.9. The following uniform estimate of the deviation of fundamental matrix will play important role in studying deviations
More specifically, for the third summand we apply Lemma 7.7 again to estimate “maximal” Fourier transform of $\tilde{P}^\pm_k$ and Lemma 3.4 to estimate $\|\tilde{R}_{jk}\|_{L^\infty(\Omega)}$, while for the fourth summand we apply Lemma 7.3 to estimate “maximal” Fourier transform of $\tilde{P}^\pm_k$ and again Lemma 3.4 to estimate $\|\tilde{R}_{jk}\|_{L^\infty(\Omega)}$.

7.3. Stability property of the fundamental matrix. Throughout the section $\Phi_{Q}(x, \lambda)$ and $\Phi_{\tilde{Q}}(x, \lambda)$ will denote the fundamental matrix solutions of the system (4.1) for $Q$ and $\tilde{Q}$ respectively and notations (4.6), (4.9) will be used. For reader’s convenience we recall them here,

$$
\Phi_{Q}(\cdot, \lambda) = \left( \varphi_{11}(\cdot, \lambda) \quad \varphi_{12}(\cdot, \lambda) \right) = (\Phi_1(\cdot, \lambda) \quad \Phi_2(\cdot, \lambda)), \quad \lambda \in \mathbb{C}, \quad (7.62)
$$

$$
\Phi_{\tilde{Q}}(\cdot, \lambda) = \left( \tilde{\varphi}_{11}(\cdot, \lambda) \quad \tilde{\varphi}_{12}(\cdot, \lambda) \right) = (\tilde{\Phi}_1(\cdot, \lambda) \quad \tilde{\Phi}_2(\cdot, \lambda)), \quad \lambda \in \mathbb{C}, \quad (7.63)
$$

$$
\Phi_{0}(x, \lambda) = \left( e^{ib_2 x} 0 \quad 0 e^{ib_2 x} \right) =: \left( \varphi_{110}(x, \lambda) \quad \varphi_{120}(x, \lambda) \quad \varphi_{210}(x, \lambda) \quad \varphi_{220}(x, \lambda) \right) =: (\Phi_{0}(x, \lambda) \quad \Phi_{0}(x, \lambda)), \quad x \in [0, 1], \quad \lambda \in \mathbb{C}. \quad (7.64)
$$

The following uniform estimate of the deviation of fundamental matrix will play important role in studying deviations of root vectors.

**Theorem 7.9.** Let $Q, \tilde{Q} \in U_{1,r}^{2 \times 2}$ for some $r > 0$. Then the following uniform estimate holds

$$
|\Phi_{Q}(x, \lambda) - \Phi_{0}(x, \lambda)| \leq C \cdot e^{2(b_2-b_1)\|\lambda\|_{\mathbb{C}} x} \sum_{k=1}^{2} \left( \mathcal{F}_k[Q - \tilde{Q}](x, \lambda) + \|Q - \tilde{Q}\|_{L^\infty(K)} \right) \quad (7.65)
$$

where $C = C(B, r) > 0$ does not depend on $Q, \tilde{Q}, x$ and $\lambda$. In particular,

$$
|\Phi_{Q}(x, \lambda) - \Phi_{0}(x, \lambda)| \leq C \cdot e^{2(b_2-b_1)\|\lambda\|_{\mathbb{C}} x} \sum_{k \neq j}^{2} \left( \sup_{s \in [0, x]} \left| \int_{0}^{s} (Q_{jk}(t) - \tilde{Q}_{jk}(t)) e^{i(b_k-b_j)\lambda t} dt \right| \right) + \|Q - \tilde{Q}\|_{L^\infty(K)} \quad (7.66)
$$

**Proof.** It follows from representations (4.19)–(4.20) in Lemma 4.3 with account of formula (4.13), that each deviation $\varphi_{jk}(x, \lambda) - \tilde{\varphi}_{jk}(x, \lambda)$, $j, k \in \{1, 2\}$, is a linear combinations of all eight deviations (7.63). Applying Theorem 4.8 with account of notation (7.1) completes the proof. □

Recall that for $f \in L^s([0, 1]; \mathbb{C}^n)$, $s \in (0, \infty)$, $n \in \mathbb{N}$, we denote $\|f\|_{s} := \|f\|_{L^s([0, 1]; \mathbb{C}^n)}$.

**Proposition 7.10.** Let $K$ be compact in $L^1([0, 1]; \mathbb{C}^{2 \times 2})$ and $h \geq 0$. Then the following uniform estimates hold:

(i) For any $\delta > 0$ there exists $M_5 = M_5(\delta, K, B, h) > 0$ such that

$$
\|\varphi_{jk}(\cdot, \lambda)\|_{\delta} + \|\varphi_{kk}(\cdot, \lambda) - \varphi_{0k}(\cdot, \lambda)\|_{\delta} < \delta, \quad s \in (0, \infty), \quad |\lambda| > M_5, \quad \|\lambda\|_{\mathbb{C}} \leq h, \quad k \in \{1, 2\}, \quad Q \in K. \quad (7.67)
$$

(ii) There exists $\delta_0 = \delta_0(B, h) > 0$ such that for $M_{\delta_0}$ defined in part (i) we have

$$
\|\varphi_{kk}(\cdot, \lambda)\|_{\delta_0} \geq \delta_0, \quad s \in (0, \infty), \quad |\lambda| > M_\delta, \quad \|\lambda\|_{\mathbb{C}} \leq h, \quad k \in \{1, 2\}, \quad Q \in K. \quad (7.68)
$$

(iii) Let $r > 0$ then there exists $C_2 = C_2(r, B, h) > 0$ such that

$$
|\varphi_{jk}(x, \lambda)| + \left| \frac{d}{d\lambda} \varphi_{jk}(x, \lambda) \right| \leq C_2, \quad x \in [0, 1], \quad |\lambda| \leq h, \quad j, k \in \{1, 2\}, \quad Q \in U_{1,r}^{2 \times 2}. \quad (7.69)
$$
Proof. (i) The proof is immediate from representations (4.12 - 4.13), Lemma 5.14 and inequality \( \|f\|_s \leqslant \|f\|_\infty \) valid for \( f \in L^\infty[0,1] \) and \( s \in (0, \infty] \).

(ii) It is clear that

\[
|\varphi_{kk}^0(x, \lambda)| = |e^{ib_k x \lambda}| \geq C, \quad x \in [0,1], \quad |\text{Im}\, \lambda| \leq h, \quad k \in \{1,2\},
\]

with \( C := \exp (-\max\{|b_1|, |b_2|\} \cdot h) \). Inequality (7.70) implies that

\[
\|\varphi_{kk}^0(\cdot, \lambda)\|_s \geq C, \quad s \in (0, \infty], \quad |\text{Im}\, \lambda| \leq h, \quad k \in \{1,2\}.
\]

Setting \( \delta_0 = C/2 \) and combining inequalities (7.71) and (7.67) we arrive at (7.68), since \( C - \delta_0 = 0 \).

(iii) The proof is immediate from representations (4.12 - 4.13) in Lemma 5.14 and Theorem 5.3. Here is how it looks like for the derivative \( \partial_x \varphi_{jk}(x, \lambda) \). We have for \( x \in [0,1], \quad |\text{Im}\, \lambda| \leq h, j, k \in \{1,2\} \) and \( Q \in U_{1,r}^2 \):

\[
\left| \frac{d}{d\lambda} \varphi_{jk}(x, \lambda) \right| \leq \left| b_k e^{ib_k h \lambda} \right| + \sum_{l=1}^2 \int_0^x |b_l t K_{j,l,k}(x, t) e^{ib_l h \lambda}| \, dt
\]

\[
\leq |b_k e^{ib_k h}| + 2 \sum_{l=1}^2 |b_l e^{ib_l h}| \|K_{j,l,k}\|_{X_{\infty,1}(U;C^{2\times 2})} \leq b_0 e^{b_0 h}(1 + 2C\|Q\|_1) \leq b_0 e^{b_0 h}(1 + 2Cr),
\]

where \( b_0 := \max\{|b_1|, |b_2|\} \). \qed

We need similar result for balls \( U_{p,r}^{2\times 2} \), \( p \in (1,2] \).

**Proposition 7.11.** Let \( Q \in U_{p,r}^{2\times 2} \) for some \( p \in (1,2], \quad r > 0 \), and \( h \geq 0 \). Then the following uniform estimates hold:

(i) For any \( \delta > 0 \) there exists a set \( J_{Q,\delta} \subset \mathbb{Z} \) such that for \( s \in (0, \infty], k \in \{1,2\} \) and \( j = 2/k \) we have

\[
\text{card}(\mathbb{Z} \setminus J_{Q,\delta}) \leq N_\delta := C_0 (\delta/r')^{p}, \quad 1/p' + 1/p = 1,
\]

\[
\|\varphi_{j,k}(\cdot, \lambda)\|_s + \|\varphi_{kk}(\cdot, \lambda) - \varphi_{kk}^0(\cdot, \lambda)\|_s < \delta, \quad \lambda \in \Pi_{h,Q,\delta}, \quad \Pi_{h,Q,\delta} := \bigcup_{n \in J_{Q,\delta}} [n, n+1] \times [-h, h].
\]

(ii) There exists a constant \( \delta_0 = \delta_0(B, h) > 0 \) such that

\[
\|\varphi_{kk}(\cdot, \lambda)\|_s \geq \delta_0, \quad \lambda \in \Pi_{h,Q,\delta}, \quad \delta \in (0, \delta_0), \quad s \in (0, \infty], \quad k \in \{1,2\},
\]

where the set \( J_{Q,\delta} \) also satisfies inequalities (7.73) - (7.74).

**Proof.** (i) The proof for \( s = \infty \) is immediate from the estimate (7.66) of Theorem 7.9 and Lemma 5.14 applied with \( g = Q_{jk}, \quad b = b_k - b_k \) and \( \delta' = 6C^{-1}e^{2(b_1 - b_2)h/2} \), where \( C \) is taken from the inequality (7.66). Namely, we need to set \( J_{Q,\delta} = I_{Q_{12},\delta'} \cap I_{Q_{11},\delta'} \). Inequality \( \|f\|_s \leq \|f\|_\infty \) valid for \( f \in L^\infty[0,1] \) and \( s \in (0, \infty] \), finishes the proof.

(ii) The proof is immediate by combining part (i) and inequality (7.67) as it was done in the proof of Proposition 7.11 (ii) with the same \( \delta_0 = \exp(-\max\{|b_1|, |b_2|\} \cdot h) \cdot 2 \).

Recall that for \( F \in L^s([0,1];C^{2\times 2}) \), \( s \in (0, \infty] \), we denote \( \|F\|_s := \|F\|_{L^s([0,1];C^{2\times 2})} \). Combining Theorems 7.9 and 7.14 we obtain an important stability property of the fundamental matrix.

**Proposition 7.12.** Let \( Q, \tilde{Q} \in U_{p,r}^{2\times 2} \) for some \( p \in (1,2] \) and \( r > 0 \). Let \( \Lambda = \{\mu_n\}_{n \in \mathbb{Z}} \) be an incompressible sequence of density \( d \) lying in the strip \( \Pi_\Lambda \). Then for some \( C = C(p, r, B, h, d) > 0 \) that does not depend on \( Q, \tilde{Q} \) and \( \Lambda \) the following uniform estimates hold

\[
\sum_{n \in \mathbb{Z}} \|\Phi_Q(\cdot, \mu_n) - \Phi_{\tilde{Q}}(\cdot, \mu_n)\|_{\infty}^{p'} \leq C \cdot \|Q - \tilde{Q}\|_{p'} \quad Q, \tilde{Q} \in U_{p,r}^{2\times 2}, \quad 1/p + 1/p' = 1,
\]

\[
\sum_{n \in \mathbb{Z}} (1 + |n|)^{p-2} \|\Phi_Q(\cdot, \mu_n) - \Phi_{\tilde{Q}}(\cdot, \mu_n)\|_\infty \leq C \cdot \|Q - \tilde{Q}\|_p, \quad Q, \tilde{Q} \in U_{p,r}^{2\times 2}.
\]
Lemma 7.13. Let $Q, Q_0 \in U_1^{2\times 2}$ for some $r > 0$. Let also

$$F(x, \lambda) := \sum_{j=1}^{2} \left( \alpha_j + \sum_{k,l=1}^{2} \beta_{jkl} \varphi_{kl}(1,\lambda) \right) \Phi_j(x, \lambda), \quad \text{and} \quad \tilde{F}(x, \lambda) := \sum_{j=1}^{2} \left( \alpha_j + \sum_{k,l=1}^{2} \beta_{jkl} \tilde{\varphi}_{kl}(1,\lambda) \right) \tilde{\Phi}_j(x, \lambda), \quad (7.82)$$

where $\alpha := \text{col}(\alpha_1, \alpha_2) \in \mathbb{C}^2$ and $\beta := (\beta_{jkl})_{j,k,l=1}^{2} \in \mathbb{C}^{2\times 2\times 2}$. Let $h \geq 0$. Then there exists $C = C(r, B, \alpha, \beta, h)$ that does not depend on $Q$ and $Q_0$ such that

$$\left\| F(\cdot, \lambda) - \tilde{F}(\cdot, \tilde{\lambda}) \right\|_{\infty} \leq C \cdot \left( \left\| \lambda - \tilde{\lambda} \right\| + \left\| \Phi_Q(\cdot, \lambda) - \Phi_{Q_0}(\cdot, \lambda) \right\|_{\infty} \right), \quad \lambda, \tilde{\lambda} \in \Pi_h. \quad (7.83)$$

Proof. We will split the desired difference into two parts:

$$\left\| F(\cdot, \lambda) - \tilde{F}(\cdot, \tilde{\lambda}) \right\|_{\infty} \leq \left\| F(\cdot, \lambda) - \tilde{F}(\cdot, \tilde{\lambda}) \right\|_{\infty} + \left\| \tilde{F}(\cdot, \lambda) - \tilde{F}(\cdot, \tilde{\lambda}) \right\|_{\infty}, \quad \lambda, \tilde{\lambda} \in \Pi_h. \quad (7.84)$$

The second summand is trivially estimated using Proposition 7.10(iii). Indeed, estimate (7.69) is valid for $\tilde{F}(x, \lambda)$ but with $C_2$ that also depends on matrices $\alpha$ and $\beta$. Hence

$$\left| \tilde{F}(x, \lambda) - \tilde{F}(x, \tilde{\lambda}) \right| = \left| \int_{\lambda}^{\tilde{\lambda}} \frac{d}{dz} \tilde{F}(x, z) \, dz \right| \leq \max_{z \in [\lambda, \tilde{\lambda}]} \left| \frac{d}{dz} \tilde{F}(x, z) \right| \cdot |\lambda - \tilde{\lambda}| \leq C_2 |\lambda - \tilde{\lambda}|, \quad x \in [0, 1], \quad \lambda, \tilde{\lambda} \in \Pi_h. \quad (7.85)$$
Recall that \( \|f\|_{c^2} = \max\{|f_1|, |f_2|\} \) for \( f = \text{col}(f_1, f_2) \in \mathbb{C}^2 \). For the first summand the part of the estimate (7.69) regarding \( |\varphi_{jk}(x, \lambda)| \) implies

\[
|F(x, \lambda) - \overline{F(x, \lambda)}| \leq \sum_{j=1}^{2} \left( |\alpha_j| \cdot |\Phi_j(x, \lambda) - \overline{\Phi_j(x, \lambda)}| + \sum_{k,l=1}^{2} |\beta_{jkl}| \cdot \left| \varphi_{kl}(1, \lambda) \Phi_j(x, \lambda) - \overline{\varphi_{kl}(1, \lambda) \overline{\Phi_j(x, \lambda)}} \right| \right) \\
\leq \sum_{j=1}^{2} \left( |\alpha_j| \cdot |\Phi_j(x, \lambda) - \overline{\Phi_j(x, \lambda)}| \\
+ \sum_{k,l=1}^{2} |\beta_{jkl}| \left( |\varphi_{kl}(1, \lambda) - \overline{\varphi_{kl}(1, \lambda)}| \cdot |\Phi_j(x, \lambda)| + |\overline{\varphi_{kl}(1, \lambda)}| \cdot |\Phi_j(x, \lambda) - \overline{\Phi_j(x, \lambda)}| \right) \right) \\
\leq C_1 \sum_{j=1}^{2} \left( |\Phi_j(x, \lambda) - \overline{\Phi_j(x, \lambda)}| + |\Phi_j(1, \lambda) - \overline{\Phi_j(1, \lambda)}| \right) \\
\leq 4C_1 \|\Phi_Q(\cdot, \lambda) - \Phi_{\overline{Q}}(\cdot, \lambda)\|_{\infty}, \quad x \in [0, 1], \quad \lambda, \overline{\lambda} \in \Pi_k, \tag{7.86}
\]

where \( C_1 > 0 \) only depends on \( \alpha, \beta \) and \( C_2 \) from (7.69). Inserting (7.85) and (7.86) into (7.84) we arrive at (7.83) with \( C = \max\{4C_1, C_2\} \), which finishes the proof. \( \square \)

7.4. Stability property of the eigenfunctions. Now we are ready to formulate and prove main results of this section. The following result in the case of \( p = 1 \) generalizes [40] Theorem 4 for the case of Dirac-type system and extends it with the new results for \( p \in (1, 2] \).

**Theorem 7.14.** Let \( K \) be compact in \( L^p([0, 1]; \mathbb{C}^{2 \times 2}) \) for some \( p \in [1, 2] \) and \( Q, \overline{Q} \in K \). Let boundary conditions (4.2) be strictly regular. Let also \( s \in (0, \infty) \). Then there exist systems \( \{f_{Q,n}\}_{n \in \mathbb{Z}} \) and \( \{f_{\overline{Q},n}\}_{n \in \mathbb{Z}} \) of root vectors of the operators \( L(Q) \) and \( L(\overline{Q}) \) such that the following uniform relations hold

\[
\|f_{Q,n}\|_s = \|f_{\overline{Q},n}\|_s = 1, \quad |n| > N, \quad Q, \overline{Q} \in K, \tag{7.87}
\]

\[
\sup_{|n| > N} \left\| f_{Q,n} - f_{\overline{Q},n} \right\|_{\infty} \leq C \cdot \|Q - \overline{Q}\|_1, \quad Q, \overline{Q} \in K, \quad p = 1, \tag{7.88}
\]

\[
\sum_{|n| > N} \left\| f_{Q,n} - f_{\overline{Q},n} \right\|_{p'}^{p'} \leq C \cdot \|Q - \overline{Q}\|_p^{p'}, \quad Q, \overline{Q} \in K, \quad p \in (1, 2], \quad 1/p' + 1/p = 1, \tag{7.89}
\]

\[
\sum_{|n| > N} (1 + |n|)^{p-2} \left\| f_{Q,n} - f_{\overline{Q},n} \right\|_{p}^{p} \leq C \cdot \|Q - \overline{Q}\|_p^{p'}, \quad Q, \overline{Q} \in K, \quad p \in (1, 2]. \tag{7.90}
\]

Here constants \( N \in \mathbb{N} \) and \( C > 0 \) do not depend on \( Q, \overline{Q} \) and \( s \). If \( p = 1 \) then also

\[
\sup_{Q, \overline{Q} \in K} \left\| f_{Q,n} - f_{\overline{Q},n} \right\|_{\infty} \rightarrow 0 \quad \text{as} \quad |n| \rightarrow \infty. \tag{7.91}
\]

**Proof.** Let \( \Lambda_0 = \{\lambda_0^n\}_{n \in \mathbb{Z}} \) be the sequence of zeros of the characteristic determinant \( \Delta_0 \). Let \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) and \( \overline{\Lambda} = \{\overline{\lambda}_n\}_{n \in \mathbb{Z}} \) be canonically ordered sequences of zeros of characteristic determinants \( \Delta := \Delta_Q \) and \( \overline{\Delta} := \Delta_{\overline{Q}} \) (i.e. eigenvalues of the operators \( L(Q) \) and \( L(\overline{Q}) \)) respectively (see definition 4.14 that involves sequences \( \Lambda \) and \( \Lambda_0 \)). Proposition 6.1(i) ensures that there exist constants \( h = h(K, B, A) > 0 \) and \( d = d(K, B, A) > 0 \), not dependent on \( Q \) and \( \overline{Q} \), such that \( \Lambda_Q \) and \( \overline{\Lambda}_{\overline{Q}} \) an incompressible sequences of density \( d \) lying in the strip \( \Pi_h \). Proposition 6.3 implies
that for some constants $N_0 \in \mathbb{N}$ and $C_0 > 0$, not dependent on $Q$ and $\tilde{Q}$, uniform inequality (6.28) holds,

$$|\lambda_n - \tilde{\lambda}_n| \leq C_0|\Delta(\lambda_n)| \leq C_0|\Delta(\lambda_n) - \tilde{\Delta}(\lambda_n)| \leq C_0 \cdot J \cdot \max_{j,k \in \{1,2\}} |\varphi_{jk}(1, \lambda_n) - \tilde{\varphi}_{jk}(1, \lambda_n)| \leq C_1 \left\| \Phi_Q(\cdot, \lambda_n) - \Phi_{\tilde{Q}}(\cdot, \lambda_n) \right\|_\infty, \quad |n| > N_0. \quad (7.92)$$

Here we applied formula (4.8) for $\Delta$ and $\tilde{\Delta}$, and set $J := |J_{32}| + |J_{13}| + |J_{42}| + |J_{14}|$ and $C_1 = C_0 \cdot J > 0$.

Let $\varepsilon > 0$ be fixed (we will choose it later). By Proposition 6.3 we can choose $N_\varepsilon \geq N_0$ that only depends on $K$, $A$ and $B$ that guarantees uniform inequality (6.22), i.e.

$$|\lambda_n - \lambda^n_0| < \varepsilon, \quad |\tilde{\lambda}_n - \lambda^n_0| < \varepsilon, \quad |n| > N_\varepsilon. \quad (7.93)$$

Let $\delta \in (0, \delta_0]$ be fixed (we will choose it later) and $M_\delta$ be chosen to satisfy uniform inequality (7.67) of Proposition 7.10 on $\varphi_{jk}(\cdot, \lambda)$. Since $\delta \leq \delta_0$ we can assume that $M_\delta > M_{\delta_0}$. Hence inequality (7.65) is also satisfied. Since $\lambda^n_0 \to \infty$ as $n \to \infty$ we can choose $N_{\delta,\varepsilon} \geq N_\varepsilon$ such that $|\lambda^n_0| > M_\delta - \varepsilon$ for $|n| > N_{\delta,\varepsilon}$. Combining this with inequality (7.93) ensures that $|\lambda_n| > M_\delta$ and $|\tilde{\lambda}_n| > M_\delta$ for $|n| > N_{\delta,\varepsilon}$. Due to the choice of $M_\delta$ and inequalities $|\Im \lambda_n| \leq h$, $|\Im \lambda_n| \leq h$, Proposition 7.10 implies that for $s \in (0, \infty)$, $k \in \{1,2\}$, $j = 2/k$, we have with account of notations (7.62)-(7.61),

$$\left\| \varphi_{kk}(\cdot), \lambda_n \right\|_s < \delta, \quad \left\| \varphi_{jk}(\cdot), \lambda_n \right\|_s < \delta, \quad C_2 \leq \left\| \varphi_{kk}(\cdot), \lambda_n \right\|_s \leq C_3, \quad |n| > N_{\delta,\varepsilon}, \quad (7.94)$$

$$\left\| \tilde{\varphi}_{kk}(\cdot), \lambda_n \right\|_s < \delta, \quad \left\| \tilde{\varphi}_{jk}(\cdot), \lambda_n \right\|_s < \delta, \quad C_2 \leq \left\| \tilde{\varphi}_{kk}(\cdot), \lambda_n \right\|_s \leq C_3, \quad |n| > N_{\delta,\varepsilon}, \quad (7.95)$$

where $C_2 = \delta_0, C_3 > 0$ do not depend on $Q, \tilde{Q}$ and $n$.

Since boundary conditions are regular, one can transform them to the form (4.49) with coefficients $a, b, c, d$ satisfying relation $ad \neq bc$.

(i) In this step we assume that $|b| + |c| \neq 0$. Without loss of generality it suffices to consider the case $b \neq 0$. It is easy to verify that the vector-functions

$$f_n(\cdot) := F(\cdot), \lambda_n), \quad F(x, \lambda) := (b + a \varphi_{12}(\lambda)) \Phi_1(x, \lambda) - (1 + a \varphi_{11}(\lambda)) \Phi_2(x, \lambda), \quad (7.96)$$

$$\tilde{f}_n(\cdot) := \tilde{F}(\cdot), \lambda_n), \quad \tilde{F}(x, \lambda) := (b + a \tilde{\varphi}_{12}(\lambda)) \Phi_1(x, \lambda) - (1 + a \tilde{\varphi}_{11}(\lambda)) \tilde{\Phi}_2(x, \lambda), \quad (7.97)$$

are (possibly zero) eigenfunctions of the operators $L(Q)$ and $L(\tilde{Q})$ respectively, corresponding to the eigenvalues $\lambda_n$ and $\tilde{\lambda}_n$ (see also the proof of Theorem 1.1 in [31]). Let us show that with appropriate choice of $\delta > 0$ the following inequalities hold:

$$C_4 \leq \|f_n\|_s \leq C_5, \quad C_4 \leq \|\tilde{f}_n\|_s \leq C_5, \quad |n| > N_{\delta,\varepsilon}, \quad s \in (0, \infty), \quad (7.98)$$

where $C_4, C_5 > 0$ do not depend on $Q$ and $\tilde{Q}$. The estimate from above trivially follows from Proposition 7.10(iii), and is valid uniformly for all $n \in \mathbb{Z}$, $Q, \tilde{Q} \in K$ and $s \in (0, \infty)$. Let $F(x, \lambda) = \text{col}(F_1(x, \lambda), F_2(x, \lambda))$. Since $b \neq 0$, inequalities (7.94) imply that

$$\|f_n\|_s = \|F(\cdot), \lambda_n)\|_s \geq \|F_1(\cdot), \lambda_n)\|_s \geq \|b + a \varphi_{12}(\lambda)\|_s \cdot \|\varphi_{11}(\cdot), \lambda_n)\|_s - |1 + a \varphi_{11}(\lambda)\|_s \cdot \|\varphi_{12}(\cdot), \lambda_n)\|_s$$

$$\geq (|b| - |a|\delta)C_2 - (1 + |a|\delta)\delta = |b|C_2 - \delta(1 + |a|(C_2 + \delta)) \geq |b|C_2/2 =: C_4, \quad |n| > N_{\delta,\varepsilon}, \quad (7.99)$$

with appropriate $\delta = \delta(C_2(K, B), a, b) = \delta(K, B, A) > 0$ that does not depend on $Q$ and $\tilde{Q}$. Inequality (7.98) for $\|\tilde{f}_n\|_s$ is proved in the same way.

Now inequalities (7.98) allow us to set $f_{Q,n} := f_n/\|f_n\|_s$ and $f_{\tilde{Q},n} := \tilde{f}_n/\|\tilde{f}_n\|_s$ for $|n| > N_{\delta,\varepsilon}$. Vectors $f_{Q,n}$ and $f_{\tilde{Q},n}$ are proper, normalized eigenfunctions of the operators $L(Q)$ and $L(\tilde{Q})$ respectively, corresponding to simple eigenvalues $\lambda_n$ and $\tilde{\lambda}_n$, and satisfy (7.87) with $N = N_{\delta,\varepsilon}$. To estimate $\left\| f_{Q,n} - f_{\tilde{Q},n} \right\|_\infty$ we first observe that for any $u, v \in L^\infty([0,1]; C^2)$ and $s \in (0, \infty)$ we have,

$$\left\| \frac{u}{\|u\|_s} - \frac{v}{\|v\|_s} \right\|_\infty \leq \left\| \frac{u}{\|u\|_s} - \frac{v}{\|v\|_s} \right\|_\infty + \left\| \frac{u}{\|u\|_s} - \frac{u}{\|v\|_s} \right\|_\infty + \left\| \frac{u}{\|v\|_s} - \frac{v}{\|v\|_s} \right\|_\infty \leq 2\|v\|_\infty \cdot \left\| \frac{u - v}{\|v\|_s} \right\|_\infty,$$
Here in the last step we applied inequalities \(\|v\|_s \leq \|v\|_\infty\) and \(\|u\|_s - \|v\|_s \leq \|u - v\|_\infty\). Setting \(u = f_n\) and \(v = \bar{f}_n\) in (7.100), taking into account inequalities (7.98) and then applying Lemma 7.13 to functions \(F(\cdot, \lambda)\) and \(\bar{F}(\cdot, \lambda)\) from (7.96)–(7.97) with account of (7.92) we arrive at

\[
\left\| f_{Q,n} - \bar{f}_{Q,n} \right\|_\infty = \left\| \frac{f_n}{\| f_n \|_s} - \frac{\bar{f}_n}{\| \bar{f}_n \|_s} \right\|_\infty \leq 2C_5C_4^{-2} \left\| f_n - \bar{f}_n \right\|_\infty = C_6 \left\| F(\cdot, \lambda_n) - \bar{F}(\cdot, \lambda_n) \right\|_\infty.
\]

\[
\leq C_7|\lambda_n - \bar{\lambda}_n| + C_7 \left\| \Phi_Q(\cdot, \lambda_n) - \Phi_Q(\cdot, \lambda_n) \right\|_\infty \leq C_8 \left\| \Phi_Q(\cdot, \lambda_n) - \Phi_Q(\cdot, \lambda_n) \right\|_\infty, \quad |n| > N_{\delta, \varepsilon},
\]

where \(C_6 = 2C_5C_4^{-2} > 0\), \(C_7 = C_6C(r, B, \alpha, \beta, h) = C_6C(K, B, A) > 0\) and \(C_8 = C_7(C_1 + 1) > 0\). Here \(C(r, B, \alpha, \beta, h)\) is from Lemma 7.13 where \(r = \sup_{Q \in K} \|Q\|_1 < \infty\) and \(\alpha \in \mathbb{C}^2\) and \(\beta \in \mathbb{C}^{2 \times 2}\) are derived from formula (7.98) and only depend on \(a\) and \(b\).

Now inequality (7.101) and Proposition 7.12 imply estimates (7.98)–(7.99) since \(\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}\) is an incompressible sequence of density \(d\) lying in the strip \(\Pi_\varepsilon\), while Proposition 7.10(i) implies relation (7.91) since \(\|f_{Q,n} - \bar{f}_{Q,n}\|_\infty \leq \|f_{Q,n} - f_{Q,0}\|_\infty + \|f_{Q,n} - f_{Q,0}\|_\infty\). Finally, estimate (7.88) is immediate from inequality (7.101) combined with Lemma 7.11 and Theorem 3.3 which finishes the proof in the case \(|b| + |c| > 0\).

(ii) Now assume that \(b = c = 0\). In this case \(ad \neq 0\) and \(\Delta_0(\lambda) = (d + e^{ib_1\lambda})(1 + ae^{ib_1\lambda})\). Let \(\Lambda_0 = \{\lambda_0^n\}_{n \in \mathbb{Z}_2}\) and \(\Lambda_0^2 = \{\lambda_0^n\}_{n \in \mathbb{Z}_2}\) be the sequences of zeros of the first and second factor, respectively, where \(\mathbb{Z}_1 \uplus \mathbb{Z}_2 = \mathbb{Z}\), i.e., \(\lambda_0 = \Lambda_0 \cup \Lambda_0^2\). Clearly, these sequences constitute arithmetic progressions lying on the lines parallel to the real axis. Lemma 4.17 implies that the arithmetic progressions \(\Lambda_0\) and \(\Lambda_0^2\) are separated, i.e., \(|\lambda_0^n - \lambda_0^m| > 2\kappa, n \in \mathbb{Z}_1, m \in \mathbb{Z}_2\) for some \(\kappa > 0\). This implies the following estimates,

\[
|1 + ae^{ib_1\lambda_0^n}| \geq \tau, \quad n \in \mathbb{Z}_1, \quad |d + e^{ib_2\lambda_0^n}| \geq \tau, \quad m \in \mathbb{Z}_2.
\]

(7.102)

for some \(\tau = \tau(A, B) > 0\). It follows from inequalities (7.102), (7.91) and (7.93) that are all valid for \(n \in \mathbb{Z}_1\) and \(|n| > N_{\delta, \varepsilon}\), that

\[
|1 + a\varphi_{11}(\lambda_n)| \geq |1 + ae^{ib_1\lambda_0^n}| - |a| \cdot |\varphi_{11}(\lambda_n)| \geq |1 + a\varphi_{22}(\lambda_n)| - |a| \cdot |\varphi_{22}(\lambda_n)| - |a| \cdot |d + e^{ib_2\lambda_0^n}| \geq \tau - |a| |d + e^{ib_2\lambda_0^n}| - |a| \cdot |\varphi_{22}(\lambda_n)| - |a| \cdot |\varphi_{22}(\lambda_n)| - |a| \cdot |e^{ib_1\lambda_n} - \varphi_{11}(\lambda_n)| - |a| \cdot |d + e^{ib_2\lambda_0^n}| - |a| \cdot |\varphi_{22}(\lambda_n)| - |a| \cdot |\varphi_{22}(\lambda_n)| - |a| \cdot |e^{ib_1\lambda_n} - \varphi_{22}(\lambda_n)| - |a| \cdot |d + e^{ib_2\lambda_0^n}| - |a| \cdot |\varphi_{22}(\lambda_n)| - |a| \cdot |\varphi_{22}(\lambda_n)| - |a| \cdot |e^{ib_1\lambda_n} - \varphi_{11}(\lambda_n)| \geq \tau - |a| |e^{ib_2\lambda_0^n} - \varphi_{22}(\lambda_n)| \geq \tau - |a| (e^{ib_1\lambda_n} - \varphi_{11}(\lambda_n) - |a| (e^{ib_1\lambda_n} - \varphi_{11}(\lambda_n) - \varepsilon) = \tau/2, \quad n \in \mathbb{Z}_1, \quad |n| > N_{\delta, \varepsilon},
\]

if we set \(\delta = \frac{\tau}{4|a|} > 0\) and \(\varepsilon = \log(1 + |a||e^{ib_1\lambda}|)/|b_1| > 0\). Making \(\delta\) and \(\varepsilon\) smaller if needed, we can similarly guarantee the inequality

\[
|d + \varphi_{22}(\lambda_n)| \geq \tau/2, \quad n \in \mathbb{Z}_2, \quad |n| > N_{\delta, \varepsilon}.
\]

(7.104)

As per the step (i) of the proof the vector-functions

\[
f_n(\cdot) := F(\cdot, \lambda_n), \quad \bar{f}_n(\cdot) := \bar{F}(\cdot, \lambda_n), \quad n \in \mathbb{Z}_1,
\]

(7.105)

are (possibly zero) eigenfunctions of the operators \(L(Q)\) and \(L(\tilde{Q})\) respectively, corresponding to the eigenvalues \(\lambda_n\) and \(\bar{\lambda}_n\), \(n \in \mathbb{Z}_1\). Here \(F(\cdot, \lambda)\) and \(\bar{F}(\cdot, \lambda)\) are defined in (7.96)–(7.97). Similarly to inequality (7.98) in the step (i) let us show that with appropriate choice of \(\delta > 0\) the following inequalities hold:

\[
C_9 \leq \|f_n\|_s \leq C_{10}, \quad C_9 \leq \|\bar{f}_n\|_s \leq C_{10}, \quad n \in \mathbb{Z}_1, \quad |n| > N_{\delta, \varepsilon}, \quad s \in (0, \infty),
\]

(7.106)

where \(C_9, C_{10} > 0\) do not depend on \(Q, \tilde{Q}\) and \(s\). As in the step (i) we only need to focus on the estimate from below. To this end inequalities (7.103) and (7.94) imply

\[
\|f_n\|_s = \|F(\cdot, \lambda_n)\|_s \geq \|F_2(\cdot, \lambda_n)\|_s \geq |1 + a\varphi_{11}(\lambda_n)| \cdot \|\varphi_{22}(\cdot, \lambda_n)\|_s - |b + a\varphi_{12}(\lambda_n)| \cdot \|\varphi_{21}(\cdot, \lambda_n)\|_s \geq \tau C_2/2 - (|b| + |a|\delta) \tau C_2/4 = C_9, \quad n \in \mathbb{Z}_1, \quad |n| > N_{\delta, \varepsilon},
\]

(7.107)
with appropriate adjustment to \( \delta \) if needed. Inequality (7.106) on \( \| \tilde{f}_n \|_s \) is established similarly. As in the step (i) we set \( f_{Q,n} := f_n / \| f_n \|_s \), \( \tilde{f}_{Q,n} := \tilde{f}_n / \| \tilde{f}_n \|_s \) for \( n \in I_1 \) and \( |n| > N_{\delta,e} \). Inequalities (7.106) imply inequality (7.101) for \( n \in I_1 \) and \( |n| > N_{\delta,e} \).

Going over to the second branches \( \{ \lambda_n \}_{n \in \mathbb{Z}_2} \) and \( \{ \tilde{\lambda}_n \}_{n \in \mathbb{Z}_2} \) of eigenvalues, we note that the vector-functions

\[
\begin{align*}
f_n(\cdot) &:= G(\cdot, \lambda_n), & G(x, \lambda) := (d + \varphi_{22}(\lambda))\Phi_1(x, \lambda) - \varphi_{21}(\lambda)\Phi_2(x, \lambda), & n \in I_2, \\
\tilde{f}_n(\cdot) &:= \tilde{G}(\cdot, \tilde{\lambda}_n), & \tilde{G}(x, \lambda) := (d + \tilde{\varphi}_{22}(\lambda))\tilde{\Phi}_1(x, \lambda) - \tilde{\varphi}_{21}(\lambda)\tilde{\Phi}_2(x, \lambda), & n \in I_2,
\end{align*}
\]

are (possibly zero) eigenfunctions of the operators \( L(Q) \) and \( L(0) \) respectively, corresponding to the eigenvalues \( \lambda_n \) and \( \tilde{\lambda}_n, \) \( n \in I_2 \). Using inequality (7.104) instead of (7.103) and adjusting \( \delta \) and \( \varepsilon \) if needed, we can obtain inequalities (7.106) but for \( n \in I_2 \) instead of \( n \in I_1 \). Applying Lemma 7.13 to functions \( G(\cdot, \lambda) \) and \( \tilde{G}(\cdot, \lambda) \) we similarly arrive at (7.101) for \( n \in I_2 \) and \( |n| > N_{\delta,e} \).

Now with inequality (7.101) being valid for all \( |n| > N_{\delta,e} \), the proof is finished in the same way as in the step (i). \( \square \)

Next we extend Theorem 7.14 to the case \( \mathcal{K} = \mathbb{U}_{p,r}^{2 \times 2} \). Similarly to Theorem 6.9 we cannot select a universal constant \( N \) serving all potentials. Instead, we need to sum over the sets of integers, the complements of which have uniformly bounded cardinality.

**Theorem 7.15.** Let \( Q, \tilde{Q} \in \mathbb{U}_{p,r}^{2 \times 2} \) for some \( p \in (1, 2] \) and \( r > 0 \). Let boundary conditions (1.2) be strictly regular. Let also \( s \in (0, \infty] \). Then there exist systems \( \{ f_{Q,n} \}_{n \in \mathbb{Z}} \) and \( \{ f_{\tilde{Q},n} \}_{n \in \mathbb{Z}} \) of root vectors of the operators \( L(Q) \) and \( L(\tilde{Q}) \) and the set \( \mathcal{I}_{Q, \tilde{Q}} \subset \mathbb{Z} \) such that the following uniform relations hold

\[
\begin{align*}
\text{card} \left( \mathbb{Z} \setminus \mathcal{I}_{Q, \tilde{Q}} \right) &\leq N, & Q, \tilde{Q} &\in \mathbb{U}_{p,r}^{2 \times 2}, \\
\| f_{Q,n} \|_s = \| f_{\tilde{Q},n} \|_s &= 1, & n \in \mathcal{I}_{Q, \tilde{Q}}, & Q, \tilde{Q} &\in \mathbb{U}_{p,r}^{2 \times 2}, \\
\sum_{n \in \mathcal{I}_{Q, \tilde{Q}}} \left\| f_{Q,n} - f_{\tilde{Q},n} \right\|_{p'} &\leq C \cdot \| Q - \tilde{Q} \|_{p'}, & Q, \tilde{Q} &\in \mathbb{U}_{p,r}^{2 \times 2}, & 1/p' + 1/p &= 1, \\
\sum_{n \in \mathcal{I}_{Q, \tilde{Q}}} (1 + |n|)^{p-2} \left\| f_{Q,n} - f_{\tilde{Q},n} \right\|_\infty &\leq C \cdot \| Q - \tilde{Q} \|_p, & Q, \tilde{Q} &\in \mathbb{U}_{p,r}^{2 \times 2}.
\end{align*}
\]

Here constants \( N \in \mathbb{N} \) and \( C > 0 \) do not depend on \( Q, \tilde{Q} \) and \( s \).

Moreover, for any \( \varepsilon > 0 \) there exist a set \( \mathcal{I}_\varepsilon := \mathcal{I}_{Q, \tilde{Q}, \varepsilon} \subset \mathbb{Z} \) and a constant \( N_\varepsilon = N_{\varepsilon}(p, r, A, B) \in \mathbb{N} \) that does not depend on \( Q \) and \( \tilde{Q} \), such that the following uniform estimates hold

\[
\begin{align*}
\text{card} \left( \mathbb{Z} \setminus \mathcal{I}_{Q, \tilde{Q}, \varepsilon} \right) &\leq N_\varepsilon, & Q, \tilde{Q} &\in \mathbb{U}_{p,r}^{2 \times 2}, \\
\sup_{n \in \mathcal{I}_{Q, \tilde{Q}, \varepsilon}} \left\| f_{Q,n} - f_{\tilde{Q},n} \right\|_\infty &\leq \varepsilon \| Q - \tilde{Q} \|_p, & Q, \tilde{Q} &\in \mathbb{U}_{p,r}^{2 \times 2}.
\end{align*}
\]

**Proof.** Let \( \Lambda_0 = \{ \lambda_n \}_{n \in \mathbb{Z}}, \Lambda = \Lambda_Q = \{ \lambda_n \}_{n \in \mathbb{Z}} \) and \( \tilde{\Lambda} = \Lambda_{\tilde{Q}} = \{ \tilde{\lambda}_n \}_{n \in \mathbb{Z}} \) be the same as in the proof of Theorem 7.14. Proposition 6.2(i) ensures that there exist constants \( h = h(p, r, B, A) > 0 \) and \( d = d(p, r, B, A) > 0 \), not dependent on \( Q \) and \( \tilde{Q} \), such that \( \Delta_Q \) and \( \Delta_{\tilde{Q}} \) an incompressible sequences of density \( d \) lying in the strip \( \Pi_d \). Theorem 6.9(ii) implies that for some constants \( N_0 \in \mathbb{N} \) and \( C_0 > 0 \), not dependent on \( Q \) and \( \tilde{Q} \), uniform inequalities (6.47)–(6.48) hold,

\[
\begin{align*}
\text{card} \left( \mathbb{Z} \setminus \mathcal{I}_{Q, \tilde{Q}} \right) &\leq N_0, \\
|\lambda_n - \tilde{\lambda}_n| &\leq C_0 |\Delta(\lambda_n)| \leq C_1 \left\| \Phi_Q(\cdot, \lambda_n) - \Phi_{\tilde{Q}}(\cdot, \lambda_n) \right\|_\infty, & n \in \mathcal{I}_{Q, \tilde{Q}}.
\end{align*}
\]

Here we applied trivial uniform estimate on \( |\Delta(\lambda_n)| \) from (7.92), which is valid for \( Q, \tilde{Q} \in \mathbb{U}_{p,r}^{2 \times 2} \) and \( n \in \mathbb{Z} \).
Let $\varepsilon > 0$ be fixed (we will choose it later). By Proposition 6.2(ii) there exists $N_\varepsilon = N_\varepsilon (p, r, B, A)$ that do not depend on $Q$ and $\tilde{Q}$, and the sets $I_{Q, \varepsilon}, I_{\tilde{Q}, \varepsilon} \subset \mathbb{Z}$, such that
\begin{equation}
\text{card} (\mathbb{Z} \setminus I_{Q, \varepsilon}) + \text{card} (\mathbb{Z} \setminus I_{\tilde{Q}, \varepsilon}) \leq N_\varepsilon ,
\end{equation}
\begin{equation}
|\lambda_n - \lambda_0^0| < \varepsilon, \quad |\tilde{\lambda}_n - \lambda_0^0| < \varepsilon, \quad n \in I_{Q, \varepsilon} \cap I_{\tilde{Q}, \varepsilon}.
\end{equation}

Let $\delta \in (0, \delta_0]$ be fixed (we will choose it later) and $J_{Q, \delta}$ be chosen to satisfy uniform inequalities (7.73)--(7.75) of Proposition 7.11 on $\varphi_{jk}(\cdot, \lambda)$. Now set,
\begin{equation}
I_{Q, \tilde{Q}, \delta, \varepsilon} := \{ n \in I_{Q, \delta} \cap I_{\tilde{Q}, \delta} \cap I_{Q, \varepsilon} : \lambda_n \in \Pi_{h, Q, \delta}, \tilde{\lambda}_n \in \Pi_{h, \tilde{Q}, \delta} \}.
\end{equation}
It follows from (7.116), (7.118), (7.73) and the fact that $\Lambda_Q$ and $\Lambda_{\tilde{Q}}$ are incomparable sequences of density $d$, not dependent on $Q$ and $\tilde{Q}$, that card $(\mathbb{Z} \setminus I_{Q, \tilde{Q}, \delta, \varepsilon}) \leq N_{\delta, \varepsilon}$, with some $N_{\delta, \varepsilon}$ that does not depend on $Q$ and $\tilde{Q}$.

Combining inequalities (7.73)--(7.75) with Proposition 7.10(iii), implies that for $s \in (0, \infty)$, $k \in \{1, 2\}$, $j = 2/k$, we have with account of notations (7.62)--(7.64),
\begin{align*}
\|\varphi_{kk}(\cdot, \lambda_n) - \varphi_{kk}^0(\cdot, \lambda_n)\|_s < \delta, & \quad \|\tilde{\varphi}_{jk}(\cdot, \lambda_n)\|_s < \delta, \quad 0 \leq \|\varphi_{kk}(\cdot, \lambda_n)\|_s \leq C_2, \quad n \in I_{Q, \tilde{Q}, \delta, \varepsilon}, \\
\|\tilde{\varphi}_{kk}(\cdot, \lambda_n) - \varphi_{kk}^0(\cdot, \lambda_n)\|_s < \delta, & \quad \|\tilde{\varphi}_{jk}(\cdot, \lambda_n)\|_s < \delta, \quad 0 \leq \|\tilde{\varphi}_{kk}(\cdot, \lambda_n)\|_s \leq C_2, \quad n \in I_{Q, \tilde{Q}, \delta, \varepsilon},
\end{align*}
where $C_2 > 0$ does not depend on $Q, \tilde{Q}$ and $n$. Clearly, inequalities (7.117), (7.119) are valid for $n \in I_{Q, \tilde{Q}, \delta, \varepsilon}$ since $I_{Q, \tilde{Q}, \delta, \varepsilon}$ is a subset of each of the three sets $I_{Q, \delta}, I_{\tilde{Q}, \delta}, I_{Q, \varepsilon}$.

From this point, the proof of relations (7.110)--(7.113) is carried out in the same way as in Theorem 7.14 with all inequalities $|n| > N_{\delta, \varepsilon}$ replaced by inclusion $n \in I_{Q, \tilde{Q}, \delta, \varepsilon}$.

Inequality (7.115) easily follows from (7.112) by applying Chebyshev’s inequality technique used in the proof of Theorem 6.3(ii).

In the sequel we need the following definition.

**Definition 7.16.** Let $\mathfrak{F} = \{f_n\}_{n \in \mathbb{Z}}$ be a sequence of elements of a Banach space $X$.

(i) The sequence $\mathfrak{F}$ is called **almost normalized in $X$** if $\|f_n\|_X \approx 1$, $n \in \mathbb{Z}$.

(ii) The sequence $\mathfrak{F}$ is called **asymptotically normalized in $X$** if for some $N \in \mathbb{N}$ we have $\|f_n\|_X = 1$, $|n| > N$.

The following example shows that in some cases we can relax compactness condition and even boundedness of $K$ and sum over all $n \in \mathbb{Z}$ in (7.89)--(7.90). Though due to relation (7.100) to relax boundedness of $K$ we can only normalize eigenfunctions in $L^\infty([0, 1]; \mathbb{C}^2)$. Normalizing in $L^r([0, 1]; \mathbb{C}^2)$ for any $s \in (0, \infty)$ would need, e.g., inclusion $K \subset \mathcal{U}_{1,r}^2$ for some $r > 0$.

**Proposition 7.17.** Let $Q_{12} = \tilde{Q}_{12} = 0$ and $Q_{21}, \tilde{Q}_{21} \in L^p([0, 1])$ for some $p \in (1, 2]$. Let boundary conditions (4.49) be strictly regular with the parameter $b$ in them being zero. Then eigenvalues of operators $L(Q)$ and $L(\tilde{Q})$ are simple and separated and for some systems $\{f_n\}_{n \in \mathbb{Z}}$ and $\{\tilde{f}_n\}_{n \in \mathbb{Z}}$ of their eigenfunctions the following uniform relations hold
\begin{equation}
\|f_n\|_\infty = \|\tilde{f}_n\|_\infty = 1, \quad n \in \mathbb{Z},
\end{equation}
\begin{equation}
\sum_{n \in \mathbb{Z}} \|f_n - \tilde{f}_n\|_\infty^{p'} \leq C \cdot \|Q - \tilde{Q}\|_p^{p'}, \quad 1/p' + 1/p = 1,
\end{equation}
\begin{equation}
\sum_{n \in \mathbb{Z}} (1 + |n|)^{p-2} \|f_n - \tilde{f}_n\|_\infty^{p} \leq C \cdot \|Q - \tilde{Q}\|_p^{p'},
\end{equation}
where $C = C(p, B, A) > 0$ does not depend on $Q$ and $\tilde{Q}$.

**Proof.** Since $Q_{12} = 0$ then explicit formula (4.14) holds. In particular
\begin{equation}
\varphi_{21}(x, \lambda) = -ib_2 e^{ib_2 \lambda x} \int_0^x Q_{21}(t) e^{i(b_1 - b_2) \lambda t} dt, \quad x \in [0, 1], \quad \lambda \in \mathbb{C}.
\end{equation}
Since \( b = 0 \), relation (4.14) implies that vector function \( \alpha_1 \Phi_1(\cdot, \lambda) + \alpha_2 \Phi_2(\cdot, \lambda) \) satisfy boundary conditions (4.49) if and only if
\[
\left( \begin{array}{cc} 1 + ae^{ib_1\lambda} & 0 \\
 ce^{ib_1\lambda} + \varphi_{21}(\lambda) & d + e^{ib_2\lambda} \end{array} \right) \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = 0.
\] (7.127)
Since boundary conditions (4.49) are strictly regular then \( ad \neq 0 \) and (7.127) implies that eigenvalues of \( L(Q) \) are simple and separated, do not depend on \( Q \) and form a union of two arithmetic progressions lying on two lines parallel to the real axis,
\[
\lambda_{1,n} = \tilde{\lambda}_{1,n} = \frac{\arg(-a^{-1}) + 2\pi n}{b_1} + i \frac{\ln|a|}{b_1}, \quad \lambda_{2,n} = \tilde{\lambda}_{2,n} = \frac{\arg(-d) + 2\pi n}{b_2} - i \frac{\ln|d|}{b_2}, \quad n \in \mathbb{Z}.
\] (7.128)
For eigenvalues \( \lambda_{2,n} \) we have \( d + e^{ib_2\lambda_{2,n}} = 0 \) and hence it follows from (7.124) that vector functions
\[
g_{2,n}(x) := \Phi_2(x, \lambda_{2,n}) = \Phi_2^0(x, \lambda_{2,n}) = \text{col}(0, e^{ib_2\lambda_{2,n}})
\] (7.129)
are the eigenfunctions of the operator \( L(Q) \). Hence
\[
\sum_{n \in \mathbb{Z}} \|g_{2,n} - \tilde{g}_{2,n}\|_\infty^p = 0 \quad \text{and} \quad \|g_{2,n}\|_\infty = \|\tilde{g}_{2,n}\|_\infty = \sup_{x \in [0,1]} |e^{ib_2\lambda_{2,n}}| = \max\{1, |d|\} =: C_2 > 0, \quad n \in \mathbb{Z} \quad (7.130)
\]
For eigenvalues \( \lambda_{1,n} \) we have \( 1 + ae^{ib_1\lambda_{1,n}} = 0 \). Since boundary conditions are strictly regular, then for \( d_n := d + e^{ib_1\lambda_{1,n}} \) we have \( D^{-1} \leq |d_n| \leq D, \quad n \in \mathbb{Z} \), for some \( D > 1 \) that only depends on \( a, d, b_1, b_2 \). Hence (7.127) implies that vector functions
\[
g_{1,n}(x) := d_n \Phi_1(x, \lambda_{1,n}) - (ce^{ib_1\lambda_{1,n}} + \varphi_{21}(\lambda_{1,n})) \Phi_2(x, \lambda_{1,n})
\]
(7.131)
are non-zero eigenfunctions of the operator \( L(Q) \) corresponding to the eigenvalues \( \lambda_{1,n} \). It follows from (4.14) that
\[
\|g_{1,n}\|_\infty \geq |d_n| \sup_{x \in [0,1]} |e^{ib_1\lambda_{1,n}}| \geq D^{-1} \max\{1, |a|^{-1}\} =: C_1 > 0, \quad n \in \mathbb{Z} \quad (7.132)
\]
It is also clear that \( \|g_{1,n}\|_\infty \leq C(A, B)(|Q_{21}|^1 + 1) \). Therefore, relations (7.130) and (7.132) imply that
\[
\{g_{2,n}\}_{n \in \mathbb{Z}} := \{g_{1,n}\}_{n \in \mathbb{Z}} \cup \{g_{2,n}\}_{n \in \mathbb{Z}} \quad (7.133)
\]
is almost normalized sequence of eigenfunctions of the operator \( L(Q) \).
Since \( \lambda_{1,n} = \tilde{\lambda}_{1,n} \) it follows from (7.131) and (4.14) that
\[
g_{1,n}(x) - \tilde{g}_{1,n}(x) := d_n(\varphi_{21}(x, \lambda_{1,n}) - \tilde{\varphi}_{21}(x, \lambda_{1,n})) - e^{ib_2\lambda_{1,n}}(\varphi_{21}(1, \lambda_{1,n}) - \tilde{\varphi}_{21}(1, \lambda_{1,n}))
\] (7.134)
Hence (7.126), definition 5.2 of \( \mathcal{F}[f] \) and (7.128) imply for \( n \in \mathbb{Z} \),
\[
\|g_{1,n} - \tilde{g}_{1,n}\|_\infty \leq |b_2| \cdot \sup_{x \in [0,1]} |e^{ib_2\lambda_{1,n}}| \cdot (|d_n| + |e^{ib_2\lambda_{1,n}}|) \cdot \mathcal{F}[Q_{21} - \tilde{Q}_{21}](|b_1 - b_2| \lambda_{1,n})
\] (7.135)
\leq |b_2| \cdot \max\{1, |a|^{-b_2/b_1}\} \cdot (D + |a|^{-b_2/b_1}) \cdot \mathcal{F}[Q_{21} - \tilde{Q}_{21}](|b_1 - b_2| \lambda_{1,n})
\] (7.136)
Combining Theorem 5.4 and (7.135) we arrive at
\[
\sum_{n \in \mathbb{Z}} \|g_{1,n} - \tilde{g}_{1,n}\|_\infty^p \leq C(p, b_1, b_2, a, d) \cdot \|Q_{21} - \tilde{Q}_{21}\|_\infty^p
\] (7.137)
Now set
\[
f_{j,n} := \frac{g_{j,n}}{\|g_{j,n}\|_\infty}, \quad \tilde{f}_{j,n} := \frac{\tilde{g}_{j,n}}{\|g_{j,n}\|_\infty}, \quad n \in \mathbb{Z}, \quad j \in \{1, 2\}.
\] (7.138)
Clearly \( f_{j,n} \) and \( \tilde{f}_{j,n} \) are eigenfunctions of the operators \( L(Q) \) and \( L(\tilde{Q}) \) that satisfy (7.128). Further, similarly to (7.100) for any non-zero elements \( u, v \) of some Banach space we have
\[
\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leq \frac{2}{\|u\|} \|u - v\|.
\] (7.139)
Combining (7.139), (7.140) and (7.141) we arrive at
\begin{equation}
\| f_{j,n} - f_{j,n}^\infty \| \leq 2C_{\lambda}^{-1}\| g_{j,n} - \tilde{g}_{j,n} \|, \quad n \in \mathbb{Z}, \quad j \in \{1, 2\}.
\end{equation}
Inequality (7.123) now immediately follows from (7.139), (7.140) and (7.137). Inequality (7.123) is derived similarly. □

The following example shows that in a special case \( Q_{12} = 0, b = 0, a = 1 \), stability property (7.123) of the eigenfunctions of the operator \( L(Q) \) is equivalent to the abstract inequality (5.11) from Theorem 5.4 with a sequence \( \{\mu_n\}_{n \in \mathbb{Z}} \) being an arithmetic progression.

**Example 7.18.** Let \( p > 0, Q_{12} = 0, Q_{21} \in L^1[0, 1] \), and let boundary conditions (4.19) be strictly regular with \( b = 0 \) and \( a = 1 \). Set \( \mu_n = 2(1 - b_2/b_1)\pi n, n \in \mathbb{Z} \). Assume that
\begin{equation}
\sum_{n \in \mathbb{Z}} \int_0^1 Q_{21}(t)e^{i\mu_n t}dt < \infty,
\end{equation}
which holds whenever \( Q_{21} \in L^p[0, 1], p \in (1, 2] \), due to the classical Hausdorff-Young theorem for Fourier coefficients (see [54, Theorem XII.2.3]). Formula (7.130) for \( \varphi_{21}(x, \lambda) \) easily implies equivalences
\begin{equation}
\sum_{n \in \mathbb{Z}} \sup_{x \in [0, 1]} \left| \int_0^x Q_{21}(t)e^{i\mu_n t}dt \right|^p < \infty \iff \sum_{n \in \mathbb{Z}} |\varphi_{21}(x, \mu_n)|^p < \infty,
\end{equation}
\begin{equation}
\sum_{n \in \mathbb{Z}} \left| \int_0^1 Q_{21}(t)e^{i\mu_n t}dt \right|^p < \infty \iff \sum_{n \in \mathbb{Z}} |\varphi_{21}(1, \mu_n)|^p < \infty.
\end{equation}
Recall that the sequence of eigenvalues of the operator \( L(Q) \) (of BVP (4.1), (4.3)) in this special case is the union of arithmetic progressions (7.128). In particular, they are simple and separated. Furthermore, the sequence \( \{g_{Q,n}\}_{n \in \mathbb{Z}} \) defined in (7.129), (7.131), (7.133) is almost normalized sequence of eigenfunctions of the operator \( L(Q) \). Combining relation (7.134) with the pre-condition (7.144) now implies the following equivalence
\begin{equation}
\sum_{n \in \mathbb{Z}} \sup_{x \in [0, 1]} \left| \int_0^x Q_{21}(t)e^{i\mu_n t}dt \right|^p < \infty \iff \sum_{n \in \mathbb{Z}} \| g_{Q,n} - g_{0,n} \|_\infty < \infty.
\end{equation}
The first condition in (7.144) is equivalent to the abstract inequality (5.11) from Theorem 5.4 for the function \( g = Q_{21} \) and the sequence \( \mu_n = 2(1 - b_2/b_1)\pi n \).

**Corollary 7.19.** Let \( Q \in L^p([0, 1]; \mathbb{C}^{2 \times 2}), p \in (1, 2], p' = p/(p - 1) \), and let boundary conditions (4.2) be strictly regular. Then the systems \( \{f_n\}_{n \in \mathbb{Z}} \) and \( \{f_{n,0}\}_{n \in \mathbb{Z}} \) of root vectors of the operators \( L(Q) \) and \( L(0) \) can be chosen asymptotically normalized in \( L^{p'}([0, 1]; \mathbb{C}^2) \) and satisfying the following uniform estimates
\begin{equation}
\sum_{n \in \mathbb{Z}} \| f_n - f_{n,0} \|_{p'}^p < \infty;
\end{equation}
\begin{equation}
\sum_{n \in \mathbb{Z}} (1 + |n|)^{p' - 2} \| f_n - f_{n,0} \|_p^p < \infty.
\end{equation}

**Definition 7.20.** Two sequences \( \{f_n\}_{n \in \mathbb{Z}} \) and \( \{g_n\}_{n \in \mathbb{Z}} \) in a Banach space \( X \) are called \( \theta \)-close in \( X \) if \( \{\|f_n - g_n\|_X\}_{n \in \mathbb{Z}} \in l^p(\mathbb{Z}) \), i.e.
\begin{equation}
\sum_{n \in \mathbb{Z}} \|f_n - g_n\|_X < \infty.
\end{equation}

If \( X \) is a Hilbert space and \( \theta = 2 \) it is a classic definition of quadratically close sequences in a Hilbert space (see [16, Subsection IV.2.4])

**Corollary 7.21.** Assume conditions of Theorem 7.19. Then systems \( \{f_n\}_{n \in \mathbb{Z}} \) and \( \{f_{n,0}\}_{n \in \mathbb{Z}} \) of root vectors of the operators \( L(Q) \) and \( L(0) \) can be chosen to be asymptotically normalized and \( p' \)-close in \( L^{p'}([0, 1]; \mathbb{C}^2) \).

**Proof.** Since \( \|f\|_{p'} \leq \|f\|_\infty \) for \( f \in L^\infty([0, 1]; \mathbb{C}^2) \), inequality (7.145) yields estimate (7.147) with \( \theta = p' \) and \( X = L^{p'}([0, 1]; \mathbb{C}^2) \). □
8. Criterion for Bari basis property

The Bari-Markus property of quadratic closeness of systems of root vectors of the operators $L(Q)$ and $L(0)$ when $Q \in L^2([0,1];\mathbb{C}^{2\times 2})$ was studied in numerous papers (Mityagin, Baskakov references). But the question whether system of root vectors of the operator $L(Q)$ forms a proper Bari basis was never investigated to the best of our knowledge.

**Definition 8.1.** A sequence of vectors in a Hilbert space $\mathcal{H}$ forms a Bari basis if it is quadratically close to an orthonormal complete sequence of vectors.

Our considerations are largely based on the following abstract criterion for Bari basis property.

**Proposition 8.2.** [10] Theorem VI.3.2] A complete system $\mathfrak{F} = \{f_n\}_{n \in \mathbb{Z}}$ of unit vectors in a Hilbert space $\mathfrak{H}$ forms a Bari basis if and only if there exists a sequence $\{g_n\}_{n \in \mathbb{Z}}$ biorthogonal to $\mathfrak{F}$ that is quadratically close to $\mathfrak{F}$.

In the sequel we will need the following slightly more practical version of this criterion. Throughout the section for any vector $f \neq 0$ in a Hilbert space we will denote as $\hat{f}$ the normalization of $f$, $\hat{f} := f/\|f\|$.\[\]

**Lemma 8.3.** Let $\mathfrak{F} = \{f_n\}_{n \in \mathbb{Z}}$ be a complete system of vectors in a Hilbert space $\mathfrak{H}$. Let also $\{g_n\}_{n \in \mathbb{Z}}$ be “almost biorthogonal” to $\mathfrak{F}$. Namely, $(f_n,g_m) = 0$, $n \neq m$, $(f_n,g_n) \neq 0$, $n, m \in \mathbb{Z}$. Then normalization of $\mathfrak{F}$, $\mathfrak{F} := \{f_n\}_{n \in \mathbb{Z}}$, forms a Bari basis in $\mathfrak{H}$ if and only if
\[\sum_{n \in \mathbb{Z}} \left( \frac{\|f_n\|^2 \|g_n\|^2}{\|f_n \|^2 \|g_n\|^2} - 1 \right) < \infty. \tag{8.1}\]

**Proof.** Put $f'_n = \hat{f}_n = f_n/\|f_n\|$ and $g'_n = \frac{\|f_n\|}{(f_n,g_n)} \cdot g_n$. It is clear that $\mathfrak{F}' := \{f'_n\}_{n \in \mathbb{Z}}$ is complete system of unit vectors in $\mathfrak{H}$ and $\mathfrak{F}' := \{g'_n\}_{n \in \mathbb{Z}}$ is biorthogonal to $\mathfrak{F}'$.\[\]

**Lemma 8.4.** Let $Q \in L^2([0,1];\mathbb{C}^{2\times 2})$ and let boundary conditions $\phi, \psi$ be strictly regular. Then normalized systems of root vectors of the operators $L(Q)$ and $L(0)$ form Bari basis in $L^2([0,1];\mathbb{C}^2)$ only simultaneously.

**Proof.** In accordance with Corollary 7.24 we can choose almost normalized systems of root vectors $\{f_n\}_{n \in \mathbb{Z}}$ and $\{f'_n\}_{n \in \mathbb{Z}}$ of the operators $L(Q)$ and $L(0)$, respectively, that are quadratically close, i.e.
\[\left\{ \|f_n - f'_n\| \right\}_{n \in \mathbb{Z}} \in L^2(\mathbb{Z}). \tag{8.3}\]

Since vector systems $\{f_n\}_{n \in \mathbb{Z}}$ and $\{f'_n\}_{n \in \mathbb{Z}}$ are almost normalized, it follows from inclusion (8.3) and inequality (1.39) that $\left\{ \|f_n - f'_n\| \right\}_{n \in \mathbb{Z}} \in L^2(\mathbb{Z})$. This inclusion in turn implies that normalized root vector systems $\{\hat{f}_n\}_{n \in \mathbb{Z}}$ and $\{\hat{f}'_n\}_{n \in \mathbb{Z}}$ form Bari basis in $L^2([0,1];\mathbb{C}^2)$ only simultaneously.\[\]

**Lemma 8.5.** Let $h \geq 0$ and $\lambda \in \Pi_h$. Denote
\[e_j(x) := e_{j,\lambda}(x) := e^{ib_jx}, \quad e_j := e_{j,\lambda} := e^{ib_j}, \quad E^\pm_{j,\lambda} := \int_0^1 |e_j(x)|^{\pm 2} dx = \int_0^1 e^{\mp 2b_j \text{Im} \lambda x} dx. \tag{8.4}\]
Then the following estimates hold:
\[
E_j^+ E_j^- \geq 1 + \frac{(b_2 \text{Im } \lambda)^2}{3}, \quad \text{in particular } \ E_j^+ E_j^- > 1, \quad \text{if } \text{Im } \lambda \neq 0,
\] (8.5)

\[
\frac{E_1^+ E_2^-}{E_1^- E_2^+} = |e_{21} e_1| + O(|\text{Im } \lambda|^2) = 1 + (b_2 - b_1) \cdot \text{Im } \lambda + O(|\text{Im } \lambda|^2).
\] (8.6)

**Proof.** It is clear that
\[
E_j^\pm = f(\mp 2b_j \text{Im } \lambda), \quad \text{where } f(x) := \frac{e^x - 1}{x} = 1 + \frac{x}{2} + O(x^2), \quad |x| < h.
\] (8.7)

It follows from Taylor expansion of \(e^x\) that
\[
f(x) f(-x) = \frac{e^x - 1}{x} \cdot \frac{e^{-x} - 1}{-x} = \frac{e^x + e^{-x} - 2}{x^2} = 2 \sum_{k=1}^{\infty} \frac{x^{2k-2}}{(2k)!} \geq 1 + \frac{x^2}{12}, \quad x \in \mathbb{R}.
\] (8.8)

Estimate (8.5) now immediately follows from (8.7) and (8.8).

Further, it follows from (8.7) that \(E_j^\pm = 1 \mp b_j y + O(y^2)\), where we set for brevity \(y := \text{Im } \lambda\). Hence
\[
\sqrt{E_1^+ E_2^-} = \sqrt{(1 \mp b_1 y + O(y^2))(1 \pm b_2 y + O(y^2))} = \sqrt{1 \pm (b_2 - b_1) y + O(y^2)} = 1 \mp \frac{b_2 - b_1}{2} y + O(y^2).
\] (8.9)

Further note that
\[
|e_{21} e_1| = |e^{(b_1 - b_2) \text{Im } \lambda}| = e^{(b_2 - b_1) \text{Im } \lambda} = 1 + (b_2 - b_1) \cdot \text{Im } \lambda + O(|\text{Im } \lambda|^2).
\] (8.10)

Relation (8.6) now immediately follows from (8.10) and (8.9). \(\square\)

Notation (8.4) will be used throughout this section.

**Proposition 8.6.** Let \(Q \in L^2([0,1]; \mathbb{C}^{2 \times 2})\) and let boundary conditions (8.11) be of the form
\[
y_1(0) + ay_1(1) = dy_2(0) + y_2(1) = 0, \quad a \neq 0,
\] (8.11)

and are strictly regular. Then the normalized system of root vectors of the operator \(L(Q)\) forms a Bari basis in \(L^2([0,1]; \mathbb{C}^2)\) if and only if \(|a| = |d| = 1\).

**Proof.** Due to Lemma 8.4 it is sufficient to consider the case \(Q = 0\). If \(|a| = |d| = 1\) then it’s easy to verify that the operator \(L(0)\) with boundary conditions (8.11) is self-adjoint. Hence its normalized system of root vectors forms an orthonormal basis in \(L^2([0,1]; \mathbb{C}^2)\) and Bari basis in particular.

Now assume that the normalized system of root vectors of the operator \(L(0)\) forms a Bari basis in \(L^2([0,1]; \mathbb{C}^2)\). According to the proof of Lemma 8.4 the eigenvalues of the operator \(L(0)\) are simple and split into two separated arithmetic progressions \(\Lambda_0^1\) and \(\Lambda_0^2\), where
\[
e^{ib_2 \lambda} = -d, \quad \lambda \in \Lambda_0^1 = \{\lambda_{1,n}^0\}_{n \in \mathbb{Z}}, \quad \text{and} \quad e^{ib_1 \lambda} = -a^{-1}, \quad \lambda \in \Lambda_0^2 = \{\lambda_{2,n}^0\}_{n \in \mathbb{Z}}.
\] (8.12)

It is easy to verify that vectors
\[
f_{1,n}^0(x) = \text{col} \left(0, e^{ib_2 \lambda_{1,n}^0 x} \right), \quad g_{1,n}^0(x) = \text{col} \left(0, e^{-ib_2 \lambda_{1,n}^0 x} \right), \quad n \in \mathbb{Z},
\] (8.13)

are the eigenvectors of the operators \(L(0)\) and \(L^*(0)\) respectively corresponding to the eigenvalues \(\lambda_{1,n}^0\) and \(\lambda_{1,n}^0\), and the vectors
\[
f_{2,n}^0(x) = \text{col} \left(e^{ib_1 \lambda_{2,n}^0 x}, 0 \right), \quad g_{2,n}^0(x) = \text{col} \left(e^{-ib_1 \lambda_{2,n}^0 x}, 0 \right), \quad n \in \mathbb{Z},
\] (8.14)

are the eigenvectors of the operators \(L(0)\) and \(L^*(0)\) respectively corresponding to the eigenvalues \(\lambda_{2,n}^0\) and \(\lambda_{2,n}^0\). It is clear that
\[
(f_{j,n}^0, g_{k,m}^0) = \delta_{j,m}, \quad j, k \in \{1, 2\}, \quad n, m \in \mathbb{Z}.
\] (8.15)

Thus the union system \(\mathcal{F} := \{f_{j,n}^0\}_{n \in \mathbb{Z}} \cup \{g_{j,n}^0\}_{n \in \mathbb{Z}}\) is the system of root vectors of the operator \(L(0)\) and \(\mathcal{G} := \{g_{j,n}^0\}_{n \in \mathbb{Z}} \cup \{g_{j,n}^0\}_{n \in \mathbb{Z}}\) is biorthogonal to it.
Since normalization of the system \( \mathfrak{F} \) forms a Bari basis in \( L^2([0,1]; \mathbb{C}^2) \) then according to Lemma [S.3] we have

\[
\sum_{j=1,2} \sum_{n \in \mathbb{Z}} \alpha_{j,n} < \infty, \quad \alpha_{j,n} := \frac{\|f_{j,n}^0\|^2 - \|g_{j,n}^0\|^2}{(f_{j,n}^0, g_{j,n}^0)} - 1, \quad j \in \{1, 2\}, \; n \in \mathbb{Z}.
\]  

Let \( j = 1, n \in \mathbb{Z} \) be fixed and \( \lambda = \lambda_{1,n}^0 \). Then taking into account Lemma [S.5] and formula [8.16] we have

\[
\alpha_{1,n} = \|f_{1,n}^0\|^2 \cdot \|g_{1,n}^0\|^2 - 1 = E_2^- - 1 \geq \frac{(b_2 \text{Im} \lambda_{1,n}^0)^2}{3}. \tag{8.17}
\]

It follows from (4.58) that \( b_2 \text{Im} \lambda_{1,n}^0 = -\ln |d| \). Since the series in (8.18) converges formula (8.17) implies that \( \ln |d| = 0 \), which means that \( |d| = 1 \).

Similarly considering the case \( j = 2 \) we conclude that \( |d| = 1 \), which finishes the proof.

The following intermediate result plays the crucial role in proving Theorem [8.9]

**Proposition 8.7.** Let boundary conditions (4.2) be of the form (4.49) and be strictly regular. Let \( \{\lambda_n^0\}_{n \in \mathbb{Z}} \) be the sequence of the eigenvalues of the operator \( L(0) \). Then the normalized system of root vectors of the operator \( L(0) \) forms a Bari basis in \( L^2([0,1]; \mathbb{C}^2) \) if and only if the following conditions hold

\[
b_1 |c| + b_2 |b| = 0, \quad \sum_{n \in \mathbb{Z}} |\text{Im} \lambda_n^0|^2 < \infty, \quad \sum_{n \in \mathbb{Z}} (|z_n| - \text{Re} z_n) < \infty, \quad z_n := \frac{1 + de^{-ib_2 \lambda_n^0}}{1 + ae^{ib_2 \lambda_n^0}}. \tag{8.18}
\]

**Proof.** (i) First let \( b = c = 0 \). Then it follows from (4.55) that \( z_n = 0 \). It also follows from (4.58) that \( \text{Im} \lambda_n^0 \to 0 \) if and only if \( |a| = |d| = 1 \). Hence condition (8.18) is equivalent to \( |a| = |d| = 1 \), i.e. that the operator \( L(0) \) is selfadjoint. This in turn is equivalent to Bari basis property of the system of root vectors of the operator \( L(0) \) due to Proposition [S.6]

(ii) Now let \( |b| + |c| > 0 \). Without loss of generality we can assume that \( b \neq 0 \). By definition of strictly regular boundary conditions there exists \( n_0 \in \mathbb{N} \) such that eigenvalues \( \lambda_n^0 \) of \( L(0) \) for \( |n| > n_0 \) are geometrically and algebraically simple and separated from each other. Let \( \mathfrak{F} := \{f_n\}_{n \in \mathbb{Z}} \) be a system of root vectors of the operator \( L(0) \) and \( \mathfrak{F} := \{g_n\}_{n \in \mathbb{Z}} \) be the corresponding system for the adjoint operator \( L^*(0) \) (zero superscript is omitted for convenience). Clearly, \( \mathfrak{F} \) is almost biorthogonal to \( \mathfrak{F} \). Hence Lemma [S.3] implies that normalization of \( \mathfrak{F} \) forms a Bari basis in \( L^2([0,1]; \mathbb{C}^2) \) if and only if condition (8.1) holds.

According to the proof of Theorem 1.1 in [31] vector-functions \( f_n(\cdot) \) and \( g_n(\cdot) \) for \( |n| > n_0 \) are of the following form,

\[
f_n(x) := \text{col} \left( be^{ib_2 \lambda_n^0 x} - (1 + ae^{ib_2 \lambda_n^0})e^{ib_2 \lambda_n^0 x} \right), \tag{8.19}
\]

\[
g_n(x) := \text{col} \left( (1 + de^{-ib_2 \lambda_n^0})e^{-ib_2 \lambda_n^0 x} - kbe^{-ib_2 \lambda_n^0 x} \right), \quad k := -b_2b_1^{-1} > 0. \tag{8.20}
\]

Let \( |n| > n_0 \) be fixed and set \( \lambda = \lambda_n^0 \). Taking into account notation (8.21) and performing straightforward calculations we see that

\[
\|f_n\|^2 = |b|^2 E_1^+ + |1 + ac_1|^2 E_2^+, \tag{8.21}
\]

\[
\|g_n\|^2 = |1 + de_2^{-1}|^2 E_1^- + k^2 |b|^2 E_2^-, \tag{8.22}
\]

\[
(f_n, g_n) = b \left( (1 + de_2^{-1}) + k(1 + ac_1) \right). \tag{8.23}
\]

Since boundary conditions (4.49) are strictly regular, it follows from the proof of Theorem 1.1 in [31] that the following estimate holds

\[
(f_n, g_n) \propto \Delta'(\lambda_n^0) \propto 1, \quad |n| > n_0. \tag{8.24}
\]

Here for \( a_n, b_n \in \mathbb{C}, \; n \in S \subset \mathbb{Z}, \) notation \( a_n \approx b_n, \; n \in S, \) means that \( C_1 |b_n| \leq |a_n| \leq C_2 |b_n|, \; n \in S, \) for some \( C_2 > C_1 > 0 \). Hence condition (8.21) is equivalent to

\[
\sum_{|n| > n_0} \left( \|f_n\|^2 \cdot \|g_n\|^2 - |(f_n, g_n)|^2 \right) < \infty. \tag{8.25}
\]
With account of (8.21)–(8.23) we get

$$\|f_n\|^2 \cdot \|g_n\|^2 - |(f_n, g_n)|^2 = \left( |b|^2 \cdot E_1^+ + |1 + ae_1|^2 : E_2^+ \right) \cdot \left( |1 + de_2^{-1}|^2 E_1^- + k^2|b|^2 E_2^+ \right) - |b|^2 \left( |1 + de_2^{-1}| + k(1 + ae_1) \right) = \tau_{1,n} + \tau_{2,n} + \tau_{3,n}, \quad |n| > n_0, \tag{8.26}$$

where

$$\tau_{1,n} := |b|^2 \cdot |1 + de_2^{-1}|^2 \cdot (E_1^+ E_1^- - 1), \quad \tau_{2,n} := k^2|b|^2 \cdot |1 + ae_1|^2 \cdot (E_2^+ E_2^- - 1), \tag{8.27}$$

$$\tau_{3,n} := k^2|b|^4 E_1^+ E_2^- + |1 + de_2^{-1}|^2 \cdot |1 + ae_1|^2 \cdot E_2^+ E_1^- - 2k|b|^2 \cdot \Re z_n, \tag{8.28}$$

where $z_n$ is defined in (8.18).

According to Proposition 4.11(i), $\lambda_n^0 \in \Pi_h, n \in \mathbb{Z}$, for some $h \geq 0$. Hence it follows from (8.5) that

$$0 \leq E_2^+ E_2^- - 1 \asymp |\Im \lambda_n^0|^2, \quad n \in \mathbb{Z}. \tag{8.29}$$

Since $b \neq 0$ and $k > 0$ relations (8.27) and (8.29) now imply that

$$0 \leq \tau_{1,n} \asymp |1 + de_2^{-1}|^2 \cdot |\Im \lambda_n^0|^2, \quad 0 \leq \tau_{2,n} \asymp |1 + ae_1|^2 \cdot |\Im \lambda_n^0|^2, \quad |n| > n_0. \tag{8.30}$$

Combining (8.30) with Lemma 4.17 we get

$$0 \leq \tau_{1,n} + \tau_{2,n} \asymp |\Im \lambda_n^0|^2, \quad |n| > n_0. \tag{8.31}$$

With account of (8.29) we get

$$k^2|b|^4 E_1^+ E_2^- + |1 + de_2^{-1}|^2 \cdot |1 + ae_1|^2 \cdot E_2^+ E_1^- \geq 2k|b|^2 \cdot \left( |1 + de_2^{-1}|^2 \cdot |1 + ae_1|^2 \right) \sqrt{E_1^+ E_1^- \cdot E_2^+ E_2^-} \geq 2k|b|^2 \cdot \Re z_n, \quad n \in \mathbb{Z}. \tag{8.32}$$

Hence $\tau_{3,n} \geq 0, n \in \mathbb{Z}$. Combining this with (8.31) and (8.26) we see that condition (8.25) holds if and only if

$$\sum_{|n| > n_0} |\Im \lambda_n^0|^2 < \infty, \quad \sum_{|n| > n_0} \tau_{3,n} < \infty. \tag{8.33}$$

Similar to calculations done in (8.32) one can verify that

$$\tau_{3,n} = |r_{4,n}|^2 + \tau_{5,n}, \quad \tau_{4,n} := \sqrt{E_1^+ E_2^- \cdot k|b|^2 - E_1^+ E_1^- \cdot z_n}, \tag{8.34}$$

$$\tau_{5,n} := 2k|b|^2 \cdot \left( \sqrt{E_1^+ E_1^- \cdot E_2^+ E_2^-} - 1 \right) \cdot \Re z_n. \tag{8.35}$$

It follows from (8.29) that for some $C > 0$,

$$0 \leq \sqrt{E_1^+ E_1^- \cdot E_2^+ E_2^-} - 1 \leq C \cdot |\Im \lambda_n^0|^2, \quad n \in \mathbb{Z}. \tag{8.36}$$

Combining the last relation with (8.30) we see that for some $\tilde{C} > 0$,

$$|\tau_{5,n}| \leq \tilde{C} \cdot |\Im \lambda_n^0|^2, \quad |n| > n_0. \tag{8.37}$$

Hence if the first series in (8.33) converges then so is the series $\sum_{|n| > n_0} |\tau_{5,n}|$. Hence in view of (8.34) condition (8.33) is equivalent to

$$\sum_{|n| > n_0} |\Im \lambda_n^0|^2 < \infty, \quad \sum_{|n| > n_0} |\tau_{4,n}|^2 < \infty. \tag{8.38}$$

With account of (8.6) we get from (8.34) that

$$\tau_{4,n} = \sqrt{E_2^+ E_1^- \left( |e_2^{-1} e_1| + O(|\Im \lambda_n^0|^2) \right) \cdot k|b|^2 - z_n}, \quad |n| > n_0. \tag{8.39}$$
Since $\sqrt{E_{1}^{+} E_{1}^{-}} \simeq 1$, $n \in \mathbb{Z}$, it is clear now that condition (8.35) is equivalent to
\[
\sum_{|n| > n_0} |\text{Im} \lambda_n^0|^2 < \infty, \quad \sum_{|n| > n_0} |\tau_{j,n}|^2 < \infty, \quad \tau_{j,n} := k|b|^2|e^{-1}_2 e_1| - z_n.
\] (8.40)

It follows from (4.55) that
\[
|\tau_{j,n}|^2 = k^2|b|^4|e^{-1}_2 e_1|^2 + |z_n|^2 - 2k|b||e^{-1}_2 e_1| \cdot \text{Re } z_n
= k^2|b|^4|e^{-1}_2 e_1|^2 + |b c|^2|e^{-1}_2 e_1|^2 - 2k|b||e^{-1}_2 e_1| \cdot \text{Re } z_n
= |b|^2|e^{-1}_2 e_1| \cdot \left( |e^{-1}_2 e_1|^2 + (k^2|b|^2 + |c|^2) - 2k\text{Re } z_n \right)
= |b|^2|e^{-1}_2 e_1| \cdot \left( |e^{-1}_2 e_1| \cdot |b| - |c|^2 \right) + 2k (|z_n| - \text{Re } z_n).
\] (8.41)

Since $b \neq 0$ and $|e^{-1}_2 e_1| \simeq 1$ it follows from (8.41) that (8.40) is equivalent to (8.13) which finishes the proof. \(\square\)

**Corollary 8.8.** Assume conditions of Proposition 8.7. Let $|n| > n_0$ be fixed and let $f_n$, $g_n$ be eigenvectors defined via (8.19) - (8.20). Then
\[
||f_n||^2 \cdot ||g_n||^2 = |(f_n, g_n)|^2,
\] (8.42)

if and only if
\[
\text{Im } \lambda_n^0 = 0 \quad \text{and} \quad k|b|^2 = \left( 1 + de^{-ib_2 a_n^0} \right) \left( 1 + ae^{ib_1 \lambda_n^0} \right).
\] (8.43)

**Proof.** (i) First let condition (8.43) hold. Since $\text{Im } \lambda_n^0 = 0$ then $E_{1}^\pm = E_{2}^\pm = 1$. Hence it follows from (8.27), (8.34) and (8.35) that $\tau_{1,n} = \tau_{2,n} = \tau_{5,n} = 0$, while
\[
\tau_{4,n} := k|b|^2 - (1 + de^{-1}) (1 + ae^1).
\] (8.44)

Second condition in (8.43) now implies that $\tau_{4,n} = 0$. Hence $\tau_{3,n} = 0$ and formula (8.26) implies (8.42).

(ii) Now let condition (8.42) hold. Since $\tau_{j,n} \geq 0$, $j \in \{1, 2, 3\}$, formula (8.25) and condition (8.42) implies that $\tau_{j,n} = 0$, $j \in \{1, 2, 3\}$. If $\text{Im } \lambda_n^0 \neq 0$ then by Lemma 8.3 $E_{j}^\pm E_{j}^{-1} - 1 > 0$, $j \in \{1, 2\}$. Since $\tau_{1,n} = \tau_{2,n} = 0$, $b \neq 0$, $k > 0$ it follows from (8.27) that $1 + de^{-1} = 1 + ae^1 = 0$. This contradicts Lemma 4.17 since boundary conditions (4.49) are strictly regular. Hence $\text{Im } \lambda_n^0 = 0$, which again implies that $E_{1}^\pm = E_{2}^\pm = 1$. This in turn implies that $\tau_{5,n} = 0$ and formula (8.44) for $\tau_{4,n}$. Since $\tau_{3,n} = \tau_{5,n} = 0$, then $\tau_{4,n} = 0$. Formula (8.44) now implies second condition in (8.43) which finishes the proof. \(\square\)

Now we are ready to formulate the main result of this section.

**Theorem 8.9.** Let $Q \in L^2((0, 1]; \mathbb{C}^{2 \times 2})$ and let boundary conditions (4.49) be strictly regular. Let either $b_1/b_2 \in \mathbb{Q}$ or $abcd = 0$. Then the normalized system of root vectors of the operators $L(Q)$ forms a Bari basis in $L^2((0, 1]; \mathbb{C}^2)$ if and only if the operator $L(0)$ is self-adjoint. The latter holds if and only if the matrix $\left( \begin{array}{cc} a & \mu b \\ \mu^{-1} c & d \end{array} \right)$ with $\mu = \sqrt{-b_2/b_1}$ is unitary.

**Proof.** Due to Lemma 8.4 it is sufficient to consider the case $Q = 0$. Since boundary conditions (4.2) are regular we can transform them to the form (4.49).

(i) If the operator $L(0)$ with boundary conditions (4.49) is self-adjoint then its normalized system of root vectors forms an orthonormal basis in $L^2((0, 1]; \mathbb{C}^2)$ and Bari basis in particular. Further, note that boundary conditions (4.49) are self-adjoint if and only if $A_{12} B A_{12}^\star = A_{34} B A_{34}^\star$, where $A_{12} = \left( \begin{array}{cc} 1 & 0 \\ 0 & a \end{array} \right)$ and $A_{34} = \left( \begin{array}{cc} c & 1 \\ a & 0 \end{array} \right)$. Straightforward calculations show that it is equivalent to unitarity property of the matrix $\left( \begin{array}{cc} a & \mu b \\ \mu^{-1} c & d \end{array} \right)$

(ii) Now assume that the normalized system of root vectors of the operator $L(0)$ forms a Bari basis in $L^2((0, 1]; \mathbb{C}^2)$. If $b = c = 0$ then Proposition 8.6 yields that $|a| = |d| = 1$, in which case operator $L(0)$ is self-adjoint. This finishes the proof in this case.

Now let $|b| + |c| \neq 0$. Proposition 8.7 implies that relations (8.18) take place. Since $|b| + |c| \neq 0$, first condition in (8.18) implies that $b \neq 0$ and $c \neq 0$. 

First, let $b_1/b_2 \in \mathbb{Q}$. In this case $b_1 = -m_1\beta$, $b_2 = m_2\beta$, where $\beta > 0$, $m_1, m_2 \in \mathbb{N}$. Set $m = m_1 + m_2$. Since $ad \neq bc$, $\Delta_0(\cdot)$ is a polynomial at $e^{i\beta x}$ of degree $m$ with non-zero roots $e^{i\mu_k}$, $\mu_k \in \mathbb{C}$, $k \in \{1, \ldots, m\}$. Hence, zeros \(\{\lambda_n^0\}_{n \in \mathbb{Z}}\) of $\Delta_0(\cdot)$ form a union of arithmetic progressions \(\left\{\frac{\mu_k + 2\pi n}{\beta}\right\}_{n \in \mathbb{Z}}\), $k \in \{1, \ldots, m\}$. If $\Im \mu_k \neq 0$, for some $k \in \{1, \ldots, m\}$, then

$$\sum_{n \in \mathbb{Z}} \left| \frac{\Im \mu_k + 2\pi n}{\beta} \right|^2 = \sum_{n \in \mathbb{Z}} \left| \frac{\Im \mu_k}{\beta} \right|^2 = \infty,$$  

which contradicts the first relation in (8.18). Hence $\lambda_n^0 = 0$, $n \in \mathbb{Z}$. This implies that $E_\pm^2 = \int_0^1 |e^{ib_2\lambda_n^0 x}|^2 dx = 1$ and

$$\tau_{4,n} = k|b|^2 = z_n = k|b|^2 - \left(1 + de^{-ib_2\lambda_n^0}\right)\left(1 + ae^{ib_1\lambda_n^0}\right).$$  

According to the proof of Proposition 8.5 condition (8.18) is equivalent to (8.33), i.e. $\sum_{n \in \mathbb{Z}} |\tau_{4,n}|^2 < \infty$. It is clear that $e^{-ib_2\lambda_n^0} = (e^{-i\beta \lambda_n})^m$ for some $k = k_n$. Hence $e^{-ib_2\lambda_n^0}$ attains a finite set of values when $n \in \mathbb{Z}$. Similarly $e^{ib_1\lambda_n^0}$ attains a finite set of values when $n \in \mathbb{Z}$. Hence, $\tau_{4,n}$ attains a finite set of values when $n \in \mathbb{Z}$ and each value is attained infinite times. Now condition (8.33) implies that $\tau_{4,n} = 0$, $n \in \mathbb{Z}$. Corollary 8.8 yields condition (8.42) for eigenvectors $f_n$ and $g_n$ introduced in the proof of Proposition 8.5. Taking into account formula (8.49) we see that normalized eigenvectors $f_n^*$ and $g_n^*$ of the operators $L(0)$ and $L^*(0)$ corresponding to the common eigenvalue $\lambda_n^0 = \lambda_n^0$ are equal for all $n \in \mathbb{Z}$. It follows easily from this that $L(0) = L^*(0)$.

Next, let’s assume that $a = d = 0$ (without extra condition on $b_1/b_2$). In this case $\Delta_0(\lambda) = e^{ib_2\lambda} - bce^{ib_1\lambda}$. Hence $\lambda_n^0 = \frac{\mu_k + 2\pi n}{b_2 - b_1}$, $n \in \mathbb{Z}$, where $bc = e^{i\mu_k}$. Relation $\sum_{n \in \mathbb{Z}} |\lambda_n^0|^2 < \infty$ implies that $\lambda_n^0 = 0$, $n \in \mathbb{Z}$. Further, since $a = d = 0$ then $\tau_{4,n} = k|b|^2 - 1 = |c|$ equals a constant. Since $\sum_{n \in \mathbb{Z}} |\tau_{4,n}|^2 < \infty$ then $\tau_{4,n} = 0$, $n \in \mathbb{Z}$. Now the same reasoning as above implies that the operator $L(0)$ is self-adjoint. Along the way, we see that this is the case if and only if $|b|^2 = k^{-1}$ and $|c| = |b|^{-1}$.

Finally, let $b_1/b_2 \notin \mathbb{Q}$, $a = 0$, $bcd \neq 0$ (the case $d = 0$, $abc \neq 0$ can be treated similarly). Clearly, \(\{z_n\}_{n \in \mathbb{Z}}\) is a bounded sequence, hence \(\sum_{n \in \mathbb{Z}} \left| z_n \right|^2 < \infty\) implies that $\Im z_n \to 0$ as $n \to \infty$. Since $a = 0$ then $\Delta_0(\lambda) = 1 + de^{-ib_2\lambda} - bce^{ib_1\lambda}$. Hence, $z_n = 1 + de^{-ib_2\lambda_n^0} - bce^{ib_1\lambda_n^0}$. Let $\lambda_n^0 = \alpha_n + ib_2\beta_n$, $d = r_2 e^{i\psi_2}$, $bc = r_1 e^{i\psi_1}$, where $\alpha_n, \beta_n, r_1, r_2 > 0$ and $\psi_1, \psi_2 \in [-\pi, \pi]$. Then

$$\Im z_n = r_2 e^{ib_2\beta_n} \sin(\psi_2 - b_2\alpha_n) = r_1 e^{-b_1\beta_n} \sin(\psi_1 + b_1\alpha_n) \to 0 \quad \text{as} \quad n \to \infty.$$  

This implies that $\sin(\psi_2 - b_2\alpha_n) \to 0$ and $\sin(\psi_1 + b_1\alpha_n) \to 0$ as $n \to \infty$. Since $b_1/b_2 \notin \mathbb{Q}$ and $\alpha_n = \frac{2\pi n}{b_2 - b_1}(1 + o(1))$, this contradicts Weyl’s equidistribution theorem and implies that in this case condition (8.18) never holds.

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