Averaged Dynamics Associated with the Lorentz Force Equation

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Abstract

An averaged version of the Lorentz force differential equation is discussed. Once the geometric setting has been introduced, we show that the Lorentz force equation is equivalent to the auto-parallel equation $<L\nabla>\dot{x} = 0$ of a projective linear connection $L\nabla$ on the pullback vector bundle $\pi^*TM$. Using a geometric averaging procedure, we obtain the associated averaged connection $<\tilde{L}\nabla>$ and we consider its auto-parallel equation $<\tilde{L}\nabla>\dot{\tilde{x}} = 0$. We prove that in the ultra-relativistic limit and for narrow one-particle probability distribution functions whose support is invariant under the flow of the Lorentz’s force equation, the auto-parallel curves of the averaged connection $<L\nabla>$ remain close to the auto-parallel curves of $L\nabla$. We discuss some applications of these result. The relation of the present work with other geometric approaches to the classical electrodynamics of point particles is briefly discussed. Finally, in relation with the geometric formulation of the Lorentz force, we introduce the notion of almost-connection, which is a non-trivial generalization of the notion of connection.

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1 Introduction

In this paper several aspects of two different topics are discussed. The first one is the notion of Randers space in the category of metrics with indefinite signature. The second topic is a geometric interpretation of the Lorentz force law and the associated averaged Lorentz dynamics.
Both topics are related. Indeed, one can better discuss the second one after the notion of *semi-Randers space* has been established. This is the scheme that we follow in this work.

The notion of Randers space can be traced back to the original work of G. Randers [1]. He investigated an example of asymmetric space-time structure capable of incorporating the asymmetry of time in a geometric way. One of the consequences was a unifying theory of gravity and electromagnetism of charged point particles. However, the non-degeneracy of the associated *metric tensor* was not discussed in detail. This has been one problem in the mathematical definition of a Randers space, in the case of indefinite signatures. Indeed, while for positive definite Randers metrics there is a satisfactory treatment (for instance [2, chapter 11]) for indefinite signatures, the theory of semi-Randers spaces is not universally accepted and several definitions are currently being used in the literature.

There is the problem of defining a Randers-type structure which is invariant under *gauge transformations*. This is a natural requirement from a physical point of view. However, the combination of this with the non-degeneracy criterion of the fundamental tensor $g$ is non-trivial. Combining both has lead us to define Randers spaces in the context of sheaf and pre-sheaf theory. The reasons to use this general framework are:

1. To use an established mathematical formalism which is very natural to be used in dealing local constructions. This theory has been of fundamental importance in algebraic geometry [41] and in the theory of complex manifolds [42]. Grothendieck formulated a theory of fiber bundles with a sheaf structure [43].

2. One can make contact with other mathematical disciplines. In particular, one expects some connection between Randers-type spaces and some algebraic varieties. This point is work in progress.

There are two general formalisms for indefinite Finsler spaces (that we call semi-Finsler structures), Asanov’s formalism [3] and Beem’s formalism [4, 5]. In this paper, we argue why both frameworks and the corresponding physical interpretations are unsatisfactory, in particular when we try to apply them to Randers-type spaces. The main problem with Asanov’s definition arises when one considers gauge invariance issues in Randers-type metrics; the main problem with Beem’s formalism is that there is not known a natural definition of Randers-type metric in that formalism with a natural physical interpretation.

In section 3 we provide a definition of semi-Randers space which is gauge invariant [6]. It has the advantage that it is taken directly from the Lorentz force equation. However, this definition does not correspond to a Finsler or Lagrange structure. This interpretation is natural in the framework for Randers-type space. All relevant geometric data is extracted from the semi-Riemannian metric $\eta$ and from the Lorentz force equation, which in an arbitrary local coordinate system reads:

$$\frac{d^2 \sigma^i}{d\tau^2} + \eta^{ij} \Gamma_{jk}^i \frac{d\sigma^j}{d\tau} \frac{d\sigma^k}{d\tau} + \eta^{ij} (dA)_{jk} \frac{d\sigma^k}{d\tau} \sqrt{\eta(\frac{d\sigma}{d\tau}, \frac{d\sigma}{d\tau})} = 0, \quad i, j, k = 0, 1, 2, 3,$$

where $\sigma : \mathbb{I} \rightarrow \mathbb{M}$ is a solution curve for $\tau \in \mathbb{I}$, $\eta^{ij} \Gamma_{jk}^i$ are the coefficients of the Levi-Civita connection $\eta \nabla$ of $\eta$ and $dA$ is the exterior derivative of the 1-form $A$.

**Remark.** From the differential equations (1.1) one does not have enough information to extract a connection. Indeed, the natural object which is extracted is what we have called *almost-connection*. Nevertheless this subtlety, we use through almost all the paper the name connection (strictly speaking projective connection), since most of the calculations that we
perform are also suitable for connections (or projective connections). We formalize this notion later in section 11.

In section 4 we introduce the notions of non-Linear connection associated with a second order differential equation.

Section 5 is devoted to introduce a geometric formulation of the Lorentz force equation.

Sections 6 and 7 are devoted to introduce the averaging mechanism. We have follow reference [7, section 4].

In section 8 we apply the averaging procedure to the Lorentz force equation.

In section 9 we compare the difference of the solutions of the original system (1.1) and those of the auto-parallel curves of the averaged connection (starting with the same initial conditions). The result is that for the same initial conditions, in the ultra-relativistic limit and for narrow 1-particle probability distribution functions, the solutions of both differential equations remain near each other, even after a long time evolution. Therefore the original Lorentz dynamics can be approximated by the averaged dynamics.

In section 10 the limits of applicability of the averaged Lorentz model as an approximation of the original Lorentz force equation is discussed. Some applications of this approximation are also briefly mentioned.

1. There are applications of the methods introduced here in fluid modeling of plasmas [9]. These applications can be of interest in accelerator physics [10] and for the description of the dynamics of non-neutral plasmas using fluid models [11].

2. The properties of a bunch of particles in an accelerator machine are measured indirectly. The outcomes of the measurements are associated with an averaged description of the bunch. Therefore, to have a theoretical model for the time evolution of these averaged quantities is relevant, since they are observable quantities. In some circumstances, for the averaged energy and momentum this is achieved by the averaged Lorentz force equation that we present in this work. In particular we also note that although the solutions of the averaged dynamics are not physical (they do not corresponds to trajectories on the unit hyperboloid bundle over the space-time manifold $\mathbf{M}$), they can be used as a definition of the reference trajectory in beam dynamics [10].

Finally, Section 11 is devoted to formalization of the notion of almost-connection. We include this to complete the present work.

Remark. In this work we are considering the problem of the mathematical description of the evolution of charged particles in an external Maxwell field. Therefore we are considering that the Maxwell equations and the Lorentz force equation hold. This differs from other geometric approaches to electromagnetism, where the whole theory electromagnetic theory is reconsidered and reformulated in a geometric way [16]. Our point of view is more conservative. Indeed, we have rewritten the standard theory in such a way that the averaging procedure can be applied. It is a surprise that in some situations, one can substitute the original Lorentz force equation by an averaged version. This has potential interest for several applications in beam dynamics.
2 Criticism of the notion of Semi-Randers Space as Space-Time Structure

2.1 Randers Spaces as Space-Time Structures

Before moving to the more specific problem of defining semi-Randers spaces, let us discuss the notion of semi-Finsler structure. Let $M$ be a $C^\infty$ $n$-dimensional manifold, $TM$ its tangent bundle manifold with $TM \supset N$ with projection $\pi : N \to M$, the restriction to $N$ of the canonical projection $\pi : TM \to M$. Therefore $(N, \pi)$ is sub-bundle of $TM$.

Let us consider the following two standard definitions of semi-Finsler structures currently being used in the literature:

1. Asanov’s definition [3],

**Definition 2.1** A semi-Finsler structure $F$ defined on the $n$-dimensional manifold $M$ is a positive, real function $F : N \to ]0, \infty[$ such that:

(a) It is smooth in $N$,
(b) It is positive homogeneous of degree 1 in $y$,
$c) The vertical Hessian matrix
$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \quad (2.1)$$
is non-degenerate over $N$.

$g_{ij}(x, y)$ is the matrix of the fundamental tensor. The set $N_x$ is called the admissible set of tangent vectors at $x$; the disjoint union $N = \bigsqcup_{x \in M} N$ is the admissible set of vectors over $M$.

In the particular case when the manifold is 4-dimensional and $(g_{ij})$ has signature $(+, -, -, -)$, the pair $(M, F)$ is called Finslerian space-time.

**Remark**. The strong convexity condition is fundamental for comparison results, as well as for the construction of connections [24, 45].

2. Beem’s definition [4,5]

**Definition 2.2** A semi-Finsler structure defined on the $n$-dimensional manifold $M$ is a real function $L : TM \to \mathbb{R}$ such that

(a) It is smooth in the slit tangent bundle $\tilde{N} := TM \setminus \{0\}$
(b) It is positive homogeneous of degree 2 in $y$,
$c) The Hessian matrix
$$g_{ij}(x, y) := \frac{1}{2} \frac{\partial^2 L^2(x, y)}{\partial y^i \partial y^j} \quad (2.2)$$
is non-degenerate on $\tilde{N}$.

In the particular case when the manifold is 4-dimensional and $g_{ij}$ has signature $(+, -, -, -)$, the pair $(M, L)$ is a Finslerian space-time.
2.2 Comparison of Asanov’s and Beems’s Formalism

Some differences between the above definitions are highlighted below:

1. In Beem’s framework there is a geometric definition of light-like vectors and it is possible to derive Finslerian geodesics, including light-like geodesics, from a variational principle [12]. By construction, in Asanov’s formalism it is not possible to do that in an invariant way, because light-like vectors are excluded in the formalism from the beginning, since nothing is said about how to extend the function $F^2$ from $\tilde{N}$ to $TM$.

2. In Asanov’s framework, given a parameterized curve $\sigma : I \to M$, $I \subset \mathbb{R}$ on the semi-Finsler manifold $(M, F)$ such that $\dot{\sigma} \in N$ for all $t \in I$, the length functional acting on $\sigma$ is given by the line integral

$$E_A(\sigma) := \int_{t_{\min}}^{t_{\max}} F(\sigma(t), \dot{\sigma}(t)) dt, \quad I = [t_{\min}, t_{\max}]. \quad (2.3)$$

Due to the homogeneity condition of the Finsler function $F$, $E_A(\sigma)$ is a re-parametrization invariant functional. On the other hand, if we consider Beem’s definition, the energy functional is given by the following expression [12]:

$$E_B(\sigma) := \int_{t_{\min}}^{t_{\max}} L(\sigma(t), \dot{\sigma}(t)) dt, \quad I = [t_{\min}, t_{\max}]. \quad (2.4)$$

Formulated in this way, Beem’s energy functional is not re-parametrization invariant, because the fundamental function $L$ is homogeneous of degree two in $y$.

3. A third difference emerges when we consider the category of Randers-type spaces:

**Definition 2.3 (semi-Randers Space as semi-Finsler Space)**

In Asanov’s framework, a semi-Randers space is characterized by a semi-Finsler function of the form:

$$F(x, y) = \sqrt{\eta(x)_{ij}y^iy^j} + A(x, y), \quad (2.5)$$

where $\eta_{ij}(x)dx^i \otimes dx^j$ is a semi-Riemannian metric defined on $M$ and $A(x, y) := A_i(x)y^i$ is the value of the action of the 1-form $A(x) = A_i dx^i$ acting on $y \in T_xM$.

In the positive definite case and when $\eta$ is a Riemannian metric, the requirement that $(g_{ij})$ is not non-degenerate implies that the 1-form $(A_1, ..., A_n)$ is bounded by $\eta$:

$$A_iA_j\eta^{ij} < 1, \quad \eta^{ik}\eta_{kj} = \delta^i_j.$$

The indefinite case is quite different, since there is not a natural Riemannian metric that induces a norm in the space of homomorphisms. Therefore, the criterion for non-degeneracy becomes unclear.

Secondly, for both the positive and indefinite metric, only the variation of the length functional (2.3) (not directly the integrand itself) is invariant under the gauge transformation $A \to A + d\lambda$: the Finsler function (2.5) is not gauge invariant as well. Even in the case that we could define a metric and norm, transforming the 1-form $A$ by a gauge transformation can change the norm and therefore the hessian $(g_{ij})$ can become degenerated. This fact suggests that while the variation of the length functional has a physical meaning, the fundamental function $F$ does not allow this interpretation.
On the other hand, the notion of Randers space in Beem’s formalism is even more problematic. In this case there is not a formulation of semi-Randers space (because eq. (2.5) is of first order in $y$). This suggests that a proper formulation of the notion of semi-Randers space requires going beyond metric structures.

4. It is interesting to have a definition of semi-Randers structure capable of taking into account light-like trajectories for charged particles. As we have seen, Asanov’s treatment is not capable of considering this problem, since there is not a known Randers-type structure in Beem’s formalism. However, the asymptotic expansion of the ultra-relativistic charged cold fluid model presented in [13] is an example where this light-like trajectories appear naturally. In that model, the leading order contribution to the main velocity field of the charged cold fluid model is a light-like velocity vector field, interacting with the external electromagnetic field; perturbative corrections change the velocity vector field to a time-like vector field.

The above observations make it reasonable to introduce a non-metric interpretation for semi-Randers spaces. The option that we have adopted has been to formalize a geometric structure from the geometric and physical data that we have: the Lorentz force equation and the Lorentzian metric $\eta$.

### 3 Non-Lagrangian Notion of Semi-Randers Space

#### 3.1 Non-Lagrangian Notion of Semi-Randers Space

Given a manifold $M$, let us assume the existence of a smooth semi-Riemannian structure $\eta$ on $M$. This implies that the function

$$\eta : TM \times TM \rightarrow \mathbb{R}$$

$$(X,Y) \mapsto \eta_{ij}(x)X^iY^j, \quad X,Y \in T_xM$$

is smooth on the variables $x, X^i, Y^j$. Since we will use the square root $\sqrt{\eta_{ij}(x)X^iY^j}$, we also require that:

1. $\sqrt{\eta_{ij}(x)X^iY^j}$ is smooth in $\eta N := \bigcup_{x \in M} \{ y \in T_xM, \eta_{ij}(x)X^iY^j > 0 \}$.
2. $\sqrt{-\eta_{ij}(x)X^iY^j}$ is smooth in $\eta \bar{N} := \bigcup_{x \in M} \{ y \in T_xM, \eta_{ij}(x)X^iY^j < 0 \}$.

The null-cone is $\eta NC := \bigcup_{x \in M} \{ y \in T_xM, \eta(y,y) = 0 \}$. Let us assume the existence of a closed 2-form $F$ on $M$. We propose a notion of semi-Randers space based on the following

**Definition 3.1** A semi-Randers space consists of a triplet $(M, \eta, F)$, where $M$ is a space-time manifold, $\eta$ is a semi-Riemannian metric continuous on $M$ and smooth on $TM \setminus \eta NC := \eta N \cup \eta \bar{N}$ and a 2-form $F \in \Lambda^2M$ such that $dF = 0$.

$F$ is on the second de Rham cohomology group $H^2(M)$. Due to Poincaré’s lemma, there is a locally smooth 1-form $A$ such that $dA = F$. Any pair of locally smooth 1-forms $\tilde{A}$ and $A$ such that $\tilde{A} = A + d\lambda$, with $\lambda$ a locally smooth real function defined on the given open neighborhood, are said to be equivalent and produce under exterior derivative the same cohomology class.
A semi-Randers space consists of a triplet $(M, \eta, \mathcal{A})$, where $M$ is a space-time manifold, $\eta$ is a semi-Riemannian metric continuous on $M$ and smooth on $TM \setminus \text{LC}$ and the class of locally smooth 1-forms $A \in [\mathcal{A}]$ is defined such that $dA = F$ for any $A \in \mathcal{F}$.

**Proposition 3.3** Both definitions of semi-Randers space are equivalent.

**Proof.** We showed already one of the directions of the equivalence. To show the other direction, one needs to construct locally 1-forms which produce the required 2-form $F$. This is achieved by the Poincaré lemma in a start shape domain [27, pg 155-156]:

$$A(x) = \left( \int_0^1 t \sum_{k=0}^{n-1} x^k F_{k|}(tx) \, dt \right) dx^j.$$

We adopt definition 3.2, since it has the advantage that it allows us to discuss some local issues related with the inverse variational problem of the Lorentz force equation. Essentially, this is the reason that even if the topology of $M$ is trivial, the 1-forms $A$ are only locally smooth. For two neighborhoods $\mu U$ and $\nu U$, the potentials are $\mu A$ and $\nu A$. Since they are in the same equivalence class $[\mathcal{A}]$ in the intersection $\nu U \cap \mu U \neq \emptyset$ they are related by $\mu A = d(\mu \nu \lambda) + \nu A$, where $\mu \nu \lambda$ is also a locally smooth function.

**Remark.** Note that this construction does not imply any additional restriction on the group $H^1(M)$.

Let us denote the pre-sheaf of locally smooth functions over $M$ by $\bigwedge^p_{\text{loc}} M$. For each of the representatives $A \in [\mathcal{A}] \in \bigwedge^1_{\text{loc}} M$ there is on $M$ a function $F_A$ defined by the following expression:

$$F_A(x, y) = \begin{cases} \sqrt{\eta_{ij}(x)y^iy^j} + A_i(x)y^i & \text{for } \eta_{ij}(x)y^iy^j \geq 0, \\ -\sqrt{-\eta_{ij}(x)y^iy^j} + A_i(x)y^i & \text{for } \eta_{ij}(x)y^iy^j \leq 0. \end{cases} \quad (3.1)$$

The following properties follow easily from definition (3.1) and from the definition $F_A$:

**Proposition 3.4** Let $(M, \eta, [\mathcal{A}])$ be a semi-Randers space with $\eta$ a semi-Riemannian metric, $A \in [\mathcal{A}]$ and $F_A$ given by equation (3.1). Then,

1. On the null cone $\text{NC}_x := \{ y \in T_x M | \eta_{ij}(x)y^iy^j = 0 \}$ $F_A$ is of class $C^0$, for $\eta \in C^0$ and $A \in \bigwedge^1_{\text{loc}} M$. 

[8]
2. The subset where $\eta_x(y, y) \neq 0$ is an open subset of $T_xM$ and $F_A$ is smooth on $T_xM \setminus NC_x$, for $\eta$ smooth and $A \in \bigwedge^1_{\text{loc}} M$.

3. There is no constraint on the non-degeneracy of the fundamental tensor $g_{ij}$. Therefore, it is not required that any representative $A$ of $[A]$ be bounded by 1.

4. The function $F_A$ is positive homogeneous of degree 1 in $y$.

**Remark.** Without the requirement that the fundamental tensor $g_{ij}$ be non-degenerate, it is not possible to define a geodesic equation from a variational principle [14].

### 3.2 Variational Principle Based on semi-Randers Spaces

Let $\hat{\Omega}(M)$ be the set of smooth curves on $M$ with time-like tangent vector field. The functional acting on $\sigma$ is given by the integral:

$$E_{F_A} : \Gamma(M) \to \mathbb{R}$$

$$\sigma \mapsto \int_{\tau} F_A(\sigma(\tau), \dot{\sigma}(\tau)) d\tau$$

(3.2)

with $\tau$ the proper time associated with $\eta$ along the curve $\sigma$. The functional is gauge invariant up to a constant: if we choose another gauge potential $\tilde{A} = A + d\lambda$, then $E_{F_{\tilde{A}}}(\sigma) = E_{F_A}(\sigma) + \text{constant}$, the constant coming from the boundary terms of the integral. Therefore, the variation of the functional is well defined on a given semi-Randers space $(\mathbb{M}, \eta, [A])$. We denote the corresponding functional acting on $\sigma$ by $E_F(\sigma)$.

**Remark.** The constant can be arbitrary large. Also note that the entire variation of the curve of the geodesic $\sigma$ must be inside the domain of a given constant.

The condition that guarantees the construction of the first variation formula and the existence and uniqueness of the corresponding geodesics is that the vertical Hessian $g_{ij}$ must be non-degenerate [14]. However, given a particular potential $A \in [A]$ one cannot guarantee that the Hessian of $F_A$ is non-degenerate. However, due to the possibility of doing gauge transformations in the gauge potential $A(x) \mapsto A(x) + d\lambda(x)$ we have:

**Proposition 3.5** Let $(\mathbb{M}, \eta, [A])$ be a semi-Randers space. Assume that the image of each solution $\sigma$ on the manifold $\mathbb{M}$ is a compact subset. Then

1. There is a representative $\bar{A} \in [A]$ such that the Hessian of the functional $F_{\bar{A}}$ is non-degenerate.

2. The functional (3.2) is well defined on the Randers space $(\mathbb{M}, \eta, [A])$, except for a constant depending on the representative $\bar{A} \in [A]$.

3. If the geodesic curves are parameterized by the proper time associated with the Lorentzian metric $\eta$, the Euler-Lagrange equations of the functional $F_A$ are the Lorentz force equations.

**Proof:**

1. Using the gauge invariance of $E_F(\sigma)$ up to a constant, we can obtain locally an element $\tilde{A} \in [A]$ such that $\tilde{A}_i \tilde{A}_j \eta^{ij} < 1$ in an open neighborhood in the following way. Consider that we start with a 1-form $A$ which is not bounded by 1. The 1-form $\tilde{A}(x) = A(x) + d\lambda(x)$
is also a representative of $[A]$. The requirement that the hessian of $\bar{A}$ is non-degenerate is, using a generalization of the condition [2, pg 289]:

$$0 < \left| 2 + \bar{A}_i(x)y^i + \frac{\sqrt{\eta_{ij}(x)y^jy^j} (\eta^{ij} \bar{A}_i(x)\bar{A}_j(x))}{1 + \bar{A}_i(x)y^i} \right|.$$ 

On the unit tangent hyperboloid $\Sigma_x$ this condition reads:

$$0 < \left| 2 + \frac{\bar{A}_i(x)y^i + \eta^{ij}(x) \bar{A}_i(x)\bar{A}_j(x)}{1 + \bar{A}_i(x)y^i} \right|.$$ 

Let us assume that (if it is negative, the treatment is similar) that

$$\epsilon^2(x, y) := 2 + \frac{\bar{A}_i(x)y^i + \eta^{ij}(x) \bar{A}_i(x)\bar{A}_j(x)}{1 + \bar{A}_i(x)y^i} > 0.$$ 

In order to be able to write down this condition one needs that $1 + \bar{A}_i(x)y^i \neq 0$; the region where this does not hold is the intersection of the hyperplane

$$\mathbf{P}_x := \{ y \in T_x\mathbf{M} \mid 1 + \bar{A}_i(x)y^i = 0 \} \cap \Sigma_x$$

with the unit hyperboloid $\Sigma_x$. The intersection is such that $\mathbf{P}_x \cap \Sigma_x \subset \{ y \in \Sigma_x \mid F(x, y) = 0 \}$.

Let us write in detail the above condition of positiveness ($\bar{A}_i = A_i(x) + \partial_i\lambda(x)$):

$$\epsilon^2(x, y) = 2 + \frac{\left( A_i(x) + \partial_i\lambda(x) \right)y^i + \eta^{ij}(x) \left( A_i(x) + \partial_i\lambda(x) \right) \left( A_j(x) + \partial_j\lambda(x) \right)}{1 + \left( A_i(x) + \partial_i\lambda(x) \right)y^i}.$$ 

If $y \in \Sigma_x$, then for $\beta \neq 1$, $\beta y$ is not on the unit hyperboloid. The situation is different if $y \in N\Sigma_x$. Then, $\beta y \in N\Sigma_x$. Now we make the approximation of $\Sigma_x \rightarrow N\Sigma_x$ in the asymptotic limit $y^0 \rightarrow \infty$. In this approximation, one can perform limits in the expression for $\epsilon^2(x, y)$. In particular, one can consider $\epsilon^2(x, \beta y)$ for large $\beta$:

$$\lim_{\beta \rightarrow \infty} \epsilon^2(x, \beta y) = 2 + \lim_{\beta \rightarrow \infty} \frac{\left( A_i(x) + \partial_i\lambda(x) \right)y^i + \eta^{ij}(x) \left( A_i(x) + \partial_i\lambda(x) \right) \left( A_j(x) + \partial_j\lambda(x) \right)}{1 + \left( A_i(x) + \partial_i\lambda(x) \right)y^i} =$$

$$= 2 + \lim_{\beta \rightarrow \infty} \frac{\left( A_i(x) + \partial_i\lambda(x) \right)y^i}{1 + \left( A_i(x) + \partial_i\lambda(x) \right)y^i} = 3.$$ 

That this limit is well defined for large enough $y^0$ can be seen in the following way. Let us assume that $1 + \left( A_i(x) + \partial_i\lambda(x) \right)y^i = 0$. Then, we consider the expression $1 + \left( A_i(x) + \partial_i\lambda(x) \right)y^i$ for $\beta >> 1$. It is impossible that the second expression is zero except if $\beta = 1$ and $1 + \left( A_i(x) + \partial_i\lambda(x) \right)y^i = 0$.

One should prove that $\epsilon^2(x, y)$ is positive for any $y \in \Sigma_x$. This is achieved because $\epsilon^2(x, y)$ is invariant. Therefore, we can change to a coordinate system where $y^0$ is arbitrary and the value of the bound does not change.

2. When the curve $\sigma$ whose image is defined over several open sets, for instance $^{\mu}\mathbf{U}$ and $^{\nu}\mathbf{U}$, it is evaluated using several local potentials. If they are representatives $^{\mu}A$ $^{\nu}A$ and there is a contribution coming from boundary terms, due to the evaluation of the $d\lambda$ terms on some boundary points in the intersection $^{\mu}\mathbf{U} \cap ^{\nu}\mathbf{U}$. This differences does not contribute to the first variation of the functional. Therefore, the first variation of (3.2) exists and does not depends on the representative. The corresponding extremal curves exist and they are unique. They correspond to the solutions of the Lorentz force equation.
3. If the image is a compact set, the number of sets \( \mu \mathcal{U} \) that we need is finite. Then we obtain a globally defined section of \( \Lambda^1_{\text{loc}} \mathcal{M} \).

4. Once a global locally differentiable 1-form with the required properties is obtained globally over the variation \( \text{Var}(\sigma) \) of \( \sigma : \mathcal{I} \rightarrow \mathcal{M} \), one can follow the standard proof of the deduction of the Lorentz force equation from the variation of a functional [1], [36, pg 47-52]. However, one of the formal differences with standard approaches is that we restrict the variational approach to compact regions.

Remarks

1. It is important to notice that the non-degeneracy of the metric \( g_{ij} \) not necessarily implies that it has the same signature than the semi-Riemannian metric \( \eta \). Further investigations are required to determine the criteria for conservation.

2. If the curve \( \sigma \) is parameterized with respect to a parameter such that the Finslerian arc-length \( F(\sigma, \dot{\sigma}) \) is constant along the geodesic, the geodesic equations have a complicated form (see for instance [2, pg. 296]) and they are not invariant under arbitrary gauge transformations of the 1-form \( A \rightarrow A + d\lambda, \lambda \in \mathcal{F}_{\text{loc}}(\mathcal{M}) \).

3. The fact that we are speaking of local data makes natural to consider notions from pre-sheaf theory [27 chapter 6] as the basic ingredient in the definition of Randers spaces. Sheaf theory is a theory that allows to treat local objects defined on sheafs, for example germs of locally smooth functions or locally smooth differential forms [28, chapter 2]. In this case, the definition of the class \([A]\) is on the second class of the sheaf cohomology on the manifold \( \mathcal{M} \).

With definition (3.2) at hand the problem of how to introduce the gauge symmetry in a Randers geometry is solved. The price to pay is:

1. The class \([A]\) and the Riemannian metric \( \eta \) are unrelated geometric objects. This is in contradiction with the spirit of Randers spaces as a space-time asymmetric structure.

2. One needs to consider locally smooth sections instead of globally defined sections. This happens even if the topology is trivial.

We have seen that given a 1-form \( A \in [A] \) one has to work hard to find another representative \( \bar{A} \in [A] \) such that Douglas’s theorem [14] holds almost

On the other hand, light-like trajectories can not be considered in this formalism for semi-Randers spaces in a natural way. Beem’s formalism allows the treatment of light-like geodesics as extremal curves of an energy functional for some examples of indefinite Finsler space-times. However, it is not known a semi-Randers functional in Beem’s formalism.

4 Non-Linear Connection Associated with a Second Order Differential Equation

Let \( \mathcal{M} \) be an \( n \)-dimensional smooth manifold. A natural coordinate system on the tangent bundle \( \pi : \mathcal{T}\mathcal{M} \rightarrow \mathcal{M} \) is constructed in the following way. Let \((x, U)\) be a local coordinate system on \( \mathcal{M} \), where \( U \subset \mathcal{M} \) is an open sub-set \( \mathcal{M} \) and \( x : U \rightarrow \mathbb{R}^n \) a local coordinate system. An arbitrary tangent vector at the point \( p \in U \) is of the form \( X_p := X^k \frac{\partial}{\partial x^k}|_p \). The
local coordinates associated with the tangent vector $X_p \in T_xM \subset TM$ are $(x^k, y^k)$. From the imbedding $e : N \hookrightarrow TM$ $e(N)$ acquires the induced differential structure from $TM$; $e(N)$ is denote by $N$. Indeed, $N$ is a sub-bundle of the tangent bundle $TM$ of maximal rank.

We recall the following notion of connection [15, pg 314]. Let $\pi : N \rightarrow M$ the differential function $d\pi : TN \rightarrow TM$. Then the vertical bundle is $V = \ker d\pi$.

Definition 4.1 A connection in the sense of Ehresmann is a distribution $\mathcal{H} \subset TN$ such that:

1. There is a decomposition at each point $u \in N$, $T_uN = H_u \oplus \mathcal{V}_u$.

2. The horizontal lift exists for any curve $t \mapsto \sigma(t) \in M$, $t_1 \leq t \leq t_2$ and is defined for each $t \in \mathbb{R}$.

Let us consider a set of $n = \dim(M)$ second order differential equations. The solutions are parameterized curves on $M$. Assume that the system of differential equations describes the flow of a vector field $G\chi \in \Gamma TN$. In particular, the system of differential equations has the following form:

$$\frac{d^2 x^i}{dt^2} - G^i(x, \frac{dx^j}{dt}) = 0, \quad i, j = 1, \ldots, n, \quad (4.1)$$

for a given parameter $t$. This system of differential equations is equivalent to a first order system of differential equations on $N$,

$$\begin{cases}
\frac{dy^i}{dt} - G^i(x, y) = 0, \\
\frac{dx^j}{dt} = y^j, \quad i, j = 1, \ldots, n.
\end{cases} \quad (4.2)$$

The coefficients $G^i(x, y)$ are called spray coefficients if they are homogeneous functions of degree one on the coordinate $y$; the general case where they are not homogeneous are called semi-spray coefficients. $G^i(x, y)$ are such that they transform under a change of natural local coordinates on $N$, induced from changes of coordinates on $M$, in such a way that the system of differential equations (4.1) remains invariant in form. Explicitly, if the change in the local natural coordinates on the manifold $N$ is of the form:

$$\begin{cases}
\tilde{x}^i = \tilde{x}^i(x), \\
\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j.
\end{cases}$$

The associated co-frame transforms in the following way:

$$\begin{cases}
d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^j} dx^j, \\
d\tilde{y}^i = \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^k} y^k dx^j + \frac{\partial \tilde{x}^i}{\partial y^j} dy^j.
\end{cases}$$

The induced transformation in the associated system of differential equations are

$$\frac{dy^i}{dt} - G^i(x, y) = 0 \Rightarrow \frac{d\tilde{y}^i}{dt} - \tilde{G}^i(\tilde{x}, \tilde{y}) = 0,$$

which implies the transformation of the coefficients $G^i(x, y)$,

$$\tilde{G}^i(\tilde{x}, \tilde{y}) = \sum_{j,k} \left( \frac{\partial \tilde{x}^i}{\partial x^j} \right) y^j \left( \frac{\partial \tilde{x}^s}{\partial x^k} \right) y^k \frac{\partial^2 \tilde{x}^i}{\partial x^j \partial x^s} - \sum_j \left( \frac{\partial \tilde{x}^i}{\partial y^j} \right) G^j(x, y).$$
The vertical distribution $\mathcal{V}$ admits a local holonomic basis given by
\[
\left\{ \frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^n} \right\}, \quad i, j = 1, \ldots, n.
\] (4.3)

Using these spray coefficients it is possible to define a horizontal $n$-dimensional distribution of the fiber bundle $\mathbb{T}N \rightarrow N$,
\[
\left\{ \frac{\delta}{\delta x^1}, \ldots, \frac{\delta}{\delta x^n} \right\}, \quad \frac{\delta}{\delta x^k} := \frac{\partial}{\partial x^k} - \frac{\partial G^i}{\partial y^k} \frac{\partial}{\partial y^i}, \quad i, j = 1, \ldots, n
\] (4.4)
such that it generates a supplementary distribution to the vertical distribution. The non-linear connection coefficients $N^i_j(x, y)$ are defined by the relation
\[
G^i(x, y) := y^k N^i_k(x, y).
\]

For a spray, the connection coefficients of the non-linear connection are:
\[
N^i_k(x, y) = \frac{\partial G^i(x, y)}{\partial y^k}.
\]

Since the spray coefficients $G^i$ transform in a well defined way, the non-linear connection coefficients $N^i_j(x, y)$ also transform in a characteristic form [2, 21]:
\[
\left( \tilde{x}^i = \tilde{x}^i(x), \ y^i = \frac{\partial \tilde{x}^i}{\partial x^j} y^j \right) \Rightarrow \tilde{N}^i_m(x, y) \frac{\partial \tilde{x}^m}{\partial x^j}(x) = N^m_j(x, y) \frac{\partial \tilde{x}^i}{\partial x^j}(x) - \frac{\partial^2 \tilde{x}^i}{\partial x^k \partial x^j}(x) y^j.
\]

Given a spray $G^i(x, y)$, the non-zero coefficients correspond to the covariant derivative of horizontal sections of $\Gamma \mathbb{T}N$ along horizontal sections of $\mathbb{T}N$ and they are given by the Hessian of the spray:
\[
\chi^i_{jk} := \frac{1}{2} \frac{\partial^2 G^i(x, y)}{\partial y^j \partial y^k}.
\]

All the other coefficients are zero. This type of connection resembles the Berwald connection used in Finsler geometry. From the geometry of sprays, it is a natural connection. Note that, while $\chi^i_{jk}$ can be associated with a linear connection in a given vector bundle, the connection coefficients $N^i_j(x, y)$ are not (this is why they are called non-linear connections).

Given the non-linear connection, one can define the horizontal lift of the tangent vectors; the horizontal lift of $X = X^i \partial_i \in \mathbb{T}_x M$ to the space $\mathbb{T}_u \mathbb{N}$ is defined to be $h(X) = X^i \frac{\delta}{\delta x^i}$. This lifting is defined here using local coordinates. However, an intrinsic definition can be found in [40]. From this basic definition, one can define the horizontal lift of vector sections and in general, tensor sections of the corresponding bundles.

The pull-back $\pi^*\mathbb{T}M \rightarrow \mathbb{N}$ of the bundle $\mathbb{T}M$ is the minimal sub-manifold of the cartesian product $\mathbb{N} \times \mathbb{T}M$ such that the following equivalence relation holds: for every $u \in \mathbb{N}$ and $(u, \xi) \in \pi^{-1}_1(u)$, $(u, \xi) \in \pi^*\mathbb{T}M$ iff $\pi \circ \pi_2(u, \xi) = \pi(u)$; the projection on the first and second factors are denoted by
\[
\pi_1 : \pi^*\mathbb{T}M \rightarrow \mathbb{N}, \quad (u, \xi) \rightarrow u,
\]
\[
\pi_2 : \pi^*\mathbb{T}M \rightarrow \mathbb{T}M, \quad (u, \xi) \rightarrow \xi.
\]
The pull-back bundle \( \pi^*\text{T}M \to N \) is such that the following diagram commutes,

\[
\begin{array}{c}
\pi^*\text{T}M \\
\downarrow \pi_1 \\
N \\
\downarrow \pi \\
\text{M.}
\end{array}
\]

\( \pi^*\text{T}M \to N \) is a real vector bundle, with fibers homeomorphic to \( \text{T}_x\text{M} \). Similar pull-back bundles can be constructed from other tensor bundles over \( \text{M} \), for instance \( \pi^*\text{T}^*\text{M} \to N \).

Given the non-linear connection on the bundle \( \text{T}N \to \text{N} \), there are several related linear connections on the pull-back bundle \( \pi^*\text{T}M \to N \). All these connection coefficients are based on the coefficients \( \chi^{\Gamma}_{ijk} \). In particular we can stipulate the following connection on \( \pi^*\text{T}M \):

\[
\nabla^\delta \delta x^j y_i \pi^* \mathbf{Z} := \chi^{\Gamma}_{x, y}^{ijk} y_j \pi^* e_i, \quad \nabla^V y^k := 0, \quad V \in \mathcal{V};
\]

(4.5)

here \( \{\pi^*e_i, i = 0, \ldots, n - 1\} \) is a local frame for sections \( \Gamma(\pi^*\text{T}M) \). The can be generalized this construction to generic tensor bundles over \( \text{M} \).

5 Geometric Formulation of the Lorentz Force Equation

5.1 The non-linear Connection Associated with the Lorentz Force Equation

Let us consider a semi-Randers space \((\text{M}, \eta, [A])\) and the bundle \( \text{N} \to \text{M}, \text{N} := \bigcup_{x \in \text{M}} \{y \in \text{T}_x\text{M}, \eta(y, y) \geq 0\} \subset \text{T}\text{M} \). Then, the following diagram commutes:

\[
\begin{array}{c}
\text{T}\text{M} \\
\downarrow \pi \\
\text{N} \\
\downarrow \pi \\
\text{M.}
\end{array}
\]

where \( \iota \) is the following natural embedding

\[ \iota : \Sigma \to \text{T}\text{M} \]

\((x, y) \mapsto (x, y), x \in \text{M}, y \in \Sigma_x. \]

\( \hat{\pi} \) is the restriction of \( \pi \) to the sub-bundle \( \text{N} \). Let us consider the differential map \( d\hat{\pi} : \text{T}\text{N} \to \text{T}\text{M} \). Recall that the vertical bundle is defined as the kernel \( \mathcal{V} := ker(d\hat{\pi}) \); at each point one has \( ker(d\hat{\pi}|_u) := \mathcal{V}_u, \) with \( u \in \text{N} \).

The system of second order differential equations (1.1) determines a special type of vector field on \( \text{N} \) called spray. It is well known that a spray defines an Ehresmann connection [16]. Let us denote by \( \eta(Z, Y) := \eta_{ij}(x)Z^iY^j. \)

**Definition 5.1** Let \((\text{M}, \eta, [A])\) be a semi-Randers space. For each tangent vector \( y \in \text{T}_x\text{M} \) with \( \eta(y, y) > 0 \), there are defined the following functions:

\[
L^{\Gamma} y^k(x, y) = \eta^{\Gamma} y^k + \frac{1}{2\sqrt{\eta(y, y)}}(\mathbf{F}^i \mathbf{F}^j y^m \eta_{mk} + \mathbf{F}^i \mathbf{F}^j y^m \eta_{mj}) +
\]

\[
+ \mathbf{F}^m \mathbf{F}^i(x) \frac{y^m}{\sqrt{\eta(y, y)}}(\eta_{jk} - \frac{1}{\eta(y, y)} \eta_{js} \eta_{kd} y^d y^j), \quad (5.1)
\]

where \( \eta(y, y) \) is a short way to write \( \eta_{ij}(x)y^i y^j \), \( \eta^{\Gamma} y^k \), \((i, j, k = 0, 1, 2, \ldots, n) \) are the connection coefficients of the Levi-Civita connection \( \nabla \) in a local frame, \( \mathbf{F}^i_j := \partial_i A_j - \partial_j A_i \) and \( \mathbf{F}^i_j = \eta^{kj} \mathbf{F}^k_j, \) for any representative \( A \in [A] \).
The unit hyperboloid sub-bundle \( \Sigma \) acquires an induced connection, which connection coefficients are
\[
\Gamma^i_{jk}(x, y)|_\Sigma = \eta^i_{jk} + \frac{1}{2} (F^i_j(x)y^m_\eta_{mk} + F^i_k(x)y^m_\eta_{mj}) + \\
\frac{1}{2} F^i_m(x) y^m_\eta = \eta^i_{jk} + \frac{1}{2} y^m_\eta_{kl} y^k y^l.
\]

The structure of this function is clear: \( \eta^i_{jk} \) are the connection coefficients of the Lorentzian metric \( \eta \); the other two terms are tensorial. Indeed, one can define the following expressions:
\[
L^i_{jk} = \frac{1}{2 \sqrt{\eta(y, y)}} (F^i_j y^m_\eta_{mk} + F^i_k y^m_\eta_{mj}),
\]
\[
T^i_{jk} = \frac{1}{2 \sqrt{\eta(y, y)}} \left( \eta_{jk} - \frac{1}{\eta(y, y)} \eta_{js} y^s y^l y^l \right).
\]

Therefore,
\[
L^i_{jk} = \eta^i_{jk} + T^i_{jk} + L^i_{jk}.
\]

With the functions \( L^i_{jk} \) and \( T^i_{jk} \), one can construct the following maps:
\[
L_u : T_u N \times T_u N \longrightarrow T_u N \\
(X, Y) \mapsto L^i_{jk}(x, y)X^j Y^k \frac{\delta}{\delta x^i}.
\]

The second operator that we define is:
\[
T_u : T_u N \times T_u N \longrightarrow T_u N \\
(X, Y) \mapsto T^i_{jk}(x, y)X^j Y^k \frac{\delta}{\delta x^i}.
\]

\( u = (x, y) \) and \( X, Y \) are arbitrary tangent vectors \( X, Y \in T_u N \). This can be generalized to homomorphisms acting on vector sections.

Note the following elementary property:
\[
T_u (Y, Y) = 0, \ \forall y \in T_x M, \ Y = y^i \frac{\delta}{\delta x^i}, \ u = (x, y).
\]

However, \( T_u (\cdot, Y) \neq 0 \) and that \( T_u (Z, Z) \neq 0 \) for arbitrary \( Z \).

5.2 The Koszul Connection on \( TN \) Associated with the Lorentz Force Equation

Let \( \{ e_1, ..., e_n \} \) be a local basis for the sections of the frame bundle associated with the tangent bundle \( \Gamma TM \) and let us assume that each \( e_i \) is a time-like tangent vector at \( x \) (therefore, the metric cannot be diagonal in this basis, since \( \eta \) is a semi-Riemannian metric). Then \( \{ \pi^* e_1, ..., \pi^* e_n \} \) is a local frame for the fiber \( \pi^{-1}_u \subset \pi^* TM, \ u \in N; \ \{ h_1, ..., h_n \} \) is the local frame of the horizontal distribution \( H_u \subset T_u N \) obtained by horizontal lift \( h_i = h(e_i) \) and \( \{ v_1, ..., v_n \} \) is a local frame for the vertical distribution \( V_u \subset T_u N \).

Given the set of functions \( \{ L^i_{jk}, i, j, k = 0, ..., n-1 \} \) and the non-linear connection associated with the system of differential equations, there is an associated Koszul connection on \( TN \) [17].

**Proposition 5.2** Let \( (M, \eta, [A]) \) be a semi-Randers space and \( M \) be of dimension \( n \). There is defined a covariant derivative \( L^D \) on \( TN \) determined by the following conditions:
1. For each $X \in \mathcal{H}_u$ and $Z \in \Gamma \mathcal{H}_u$

$$L^D X Z = X^k L^i_{jk}(x,y)Z^j h_i, \quad X = X^i h_i|_u, \quad Z = Z^i h_i|_v,$$

with $\{h_i\}$ a local frame for the horizontal distribution, $u = (x,y)$ and $v$ an arbitrary point of an open set $\mathbf{O} \subset \mathbf{M}$ containing $u$.

2. The covariant derivative of arbitrary sections $Z \in \Gamma \mathbf{TN}$ along vertical direction is zero:

$$L^D V Z = 0, \quad \forall V \in \mathcal{V}, \; Z \in \Gamma \mathbf{TN}.$$

3. The covariant derivative $L^D$ is symmetric, i.e, has zero horizontal torsion:

$$L^D U V - L^D V U - [U, V] = 0, \quad \forall U, V \in \mathcal{H}.$$

4. For all $X \in \mathcal{H}_u$ and $Z \in \Gamma \mathcal{V}_u$, $L^D X Z = 0$.

**Proof:** Let us consider the Finsler geodesic equation associated with the semi-Randers space $F_A = \sqrt{\eta_j(x)y^jy^l + A_i(x)y^i}$ but parameterized using the arc-length of the Lorentzian metric $\eta$:

$$\frac{d^2 x^i}{dt^2} + \eta^i_{jk}(x) \frac{dx^j}{dt} \frac{dx^k}{dt} + \eta^{ij}(dA)_jk(x) \sqrt{\eta(x,y)}y^1 \frac{dx^i}{dt} = 0,$$

where $F = dA$ is the exterior differential of the 1-form $A$ and $\eta(X, Z) = \eta_{ij}(x)X^i Z^j$. From these equations, we can read the value of the semi-spray coefficients [34]:

$$G^i(x,y) = \eta^i_{jk}(x) y^j y^k + \eta^{ij}(dA)_jk(x) \sqrt{\eta(x,y)} y^k.$$

Taking the first and second derivatives with respect to $y$, we obtain:

$$\frac{1}{2} \frac{\partial}{\partial y^j} G^i(x,y) = \eta^i_{kj}(x) y^j + \eta^i_{lj}(x) (dA)_lm(x) \frac{1}{\sqrt{\eta(x,y)}} \eta^s y^s y^m + \eta^i_{lm}(x) \eta^j l_j(x) \sqrt{\eta(x,y)}.$$

$$\frac{1}{2} \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^k} G^i(x,y) = \eta^i_{kj}(x) - \eta^i_{lj}(x) (dA)_lm(x) \frac{1}{2(\sqrt{\eta(x,y)})^3} \eta^s y^s \eta_{lp}(x) y^p +$$

$$+ \eta^i_{lm}(x) \eta^j l_j(x) \frac{1}{2(\sqrt{\eta(x,y)})^2} \eta^s y^s \eta_{ks}(x) y^s +$$

$$+ \eta^i_{lm}(x) (dA)_lj(x) \frac{1}{2(\sqrt{\eta(x,y)})} \eta^s y^s.$$

One can check that $\frac{1}{2} \frac{\partial}{\partial y^j} \frac{\partial}{\partial y^p} G^i(x,y) = L^i_{jk}(x,y)$.

From the definition of these coefficients one can check (following a standard procedure, for instance in [16]) that there is a splitting of the tangent vector spaces $T_u \mathbf{N}$ for each $u \in \mathbf{N}$. In addition, one can check that the connection coefficients are symmetric $L^i_{jk} = L^i_{kj}$. The fact that the covariant derivative along the vertical directions is zero is an additional hypothesis used to make the covariant derivative unique.

**Proposition 5.3** The following properties hold:

1. The Lorentz connection $L^D$ is invariant under gauge transformations $A \rightarrow A + d\lambda$ of the 1-form $A(x) = A_i(x)dx^i$. 

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2. Given a point \( x \in M \), \( L^D \) admits a normal coordinate system centered at \( x \) which coincides with the normal coordinate system associated with \( \eta \nabla \) centered at \( x \) iff \( F(x) = 0 \).

**Proof:**

1. The first property comes from the fact that all the geometric objects appearing in the connection coefficients \( L^i_{jk} \) are gauge invariant.

2. If \( L^D \) is affine, for any point \( x \in M \) there is a coordinate system where the connection coefficients are zero \( L^i_{jk} = 0 \). This implies that \( L^i_{jk}y^jy^k = 0 \) at the point \( x \). Due to the decomposition \( L^i \nabla_{jk} = \eta^i \nabla_{jk} + T^i_{jk} + L^i_{jk} \) this is equivalent to:

\[
0 = (\eta^i_{jk} + T^i_{jk} + L^i_{jk})y^jy^k, \quad \forall y \in \Sigma_x.
\]

Since the transversality condition holds \( (T^i_{jk}y^jy^k = 0) \), one obtains:

\[
0 = (\eta^i_{jk} + L^i_{jk})y^jy^k, \quad \forall y \in \Sigma_x.
\]

Assume that there is a normal coordinate system centered at \( x \) for \( L\nabla \) and that this coordinate system coincides with the normal coordinate system associated with \( \eta\nabla \). Then at the point \( x \), one has the relation

\[
(L^i_{jk}y^jy^k) = F^i_{y^j} = 0, \forall y \in T_xM.
\]

This last condition is strong enough to imply \( F = 0 \).

**Remark.** In the positive case, it is well known that the requirement that the Chern connection of a Randers space lives on the manifold \( M \) is that the 1-form \( A \) must be parallel \( \eta\nabla A = 0 \). This is a stronger condition than \( dA = 0 \). The parallel condition indicates that the space is Berwald; the closeness condition indicates that the space is Douglas [2, pg 304]; Douglas’s spaces are such that they have the same geodesics than the underlying Riemannian metric \( \eta \). One can easily move the proofs to the Lorentzian and indefinite setting, if one adopt Asanov’s framework.

**Corollary 5.4** Let \((M, \eta, [A])\) be a semi-Randers space. Then the Lorentz force equation can be written as

\[
L^D \dot{x} = 0,
\]

where \( x : I \to M \) is a time-like curve parameterized with respect to the proper time of the Lorentzian metric \( \eta \), \( x \) is the horizontal lift on \( N \) and \( L^\nabla \) is the non-linear connection determined by the system of differential equations (1.1).

**Proof:** A solution of the auto-parallel condition of the Lorentz connection defines a curve on \( N \) given by \((x, y)(t) = (x(\tau), \dot{x}(\tau))\). Projecting this curve into \( M \) using \( \pi \), one obtains a curve \( x(\tau) \) which is a solution of the Lorentz law.

5.3 Lorentz Connection on the Pull-back bundle \( \pi^*TM \)

We introduce the third framework, which will be directly used later to define the averaged connection. Given the non-linear connection \( L^D \) on \( TN \to N \), there is a natural linear connection on the pull-back bundle \( \pi^*TM \to N \) that we denote by \( L^\nabla \) characterized by the following:
Proposition 5.5 The linear connection $\nabla^L$ on the pull-back bundle $\pi^*TM \to N$ is such that:

1. $\nabla^L$ on $\pi^*TM \to N$ is a symmetric connection:

$$L^\nabla X^*Y - L^\nabla Y^*X - X^*[X,Y] = 0,$$

where $X,Y \in \Gamma TM$, $\tilde{X},\tilde{Y} \in \Gamma TN$ are horizontal lifts of $X,Y \in \Gamma TM$ to $\Gamma TN$, with $\eta(X,X) > 0$ and $\eta(Y,Y) > 0$.

2. The covariant derivative along vertical directions of sections of $\pi^*TM$ are zero:

$$L^\nabla v^j_{\pi^*e_k} = 0, \quad (j,k = 1,\ldots,n).$$

3. The covariant derivative along horizontal directions is given by the formula:

$$L^\nabla h^j_{\pi^*e_k} = L^\Gamma_{i j k}^i (x,y) \pi^*e_i, \quad (i,j,k = 1,\ldots,n).$$

4. By definition the covariant derivative of a function $f \in \mathcal{F}(N)$ is given by

$$L^\nabla_X f := \dot{X}(f), \quad \forall \dot{X} \in T_uN.$$

Proof: One checks by direct computation that these relations define a covariant derivative on $\pi^*TM$ and that they are self-consistent. A general covariant derivative can be expressed in terms of the connection 1-forms:

$$\omega_{ij} = L^\Gamma_{jik}^i dx^k + L^\Upsilon_{ij k}^i \delta y^k,$$

where we have used a local frame of 1-forms \{dx^1,\ldots,dx^n, \delta y^1,\ldots,\delta y^n\}. Since the covariant derivative of sections on $\pi^*TM$ along vertical directions is zero, one obtains

$$L^\Upsilon_{ijk}^i \delta y^k = 0 \Rightarrow L^\Gamma_{ijk}^i = 0$$

at each point $(x,y) \in N$. Since the torsion tensor is zero:

$$L^\Gamma_{ijk}^i = L^\Gamma_{jik}^i.$$

Therefore, we only have to provide the rule of how to derive sections along horizontal directions. Since the coefficients given by formula (5.1) are symmetric, this rule is consistent with the torsion-free condition. Finally, we want that the covariant derivative to be a local operator. This is satisfied by (5.5), which guarantees that $\nabla^L$ satisfies the Leibnitz rule.

Corollary 5.6 Let $M, N, \pi^*TM$ and $\nabla^L$ be as before. Then the auto-parallel curves of the linear Lorentz connection $\nabla^L$ are in one to one correspondence with the solutions of the Lorentz force equation:

$$L^\nabla \dot{x}^* = 0 \Leftrightarrow L D_{\dot{x}} = 0, \quad \dot{x} = \frac{d\sigma}{d\tau}.$$

Proof: If in some coordinate system the Lorentz connection $\nabla^L$ has the connection coefficients $L^\Gamma_{ijk}^i$:

$$\left(\pi^* \left( L^\nabla_{dx(\tau)} \right) \frac{dx(\tau)}{d\tau} \right)^i = \left( \frac{d^2 x^1(\tau)}{d\tau^2} + L^\Gamma_{ijk}^i (x, \frac{dx(\tau)}{d\tau}, \frac{dx^j(\tau)}{d\tau}, \frac{dx^k(\tau)}{d\tau}) \right) =$$

$$= \left( \frac{d^2 x^1(\tau)}{d\tau^2} + \frac{\eta^i_{kj} - \eta^i l (dA)_{lm} \frac{1}{2(\eta(dx(\tau),dx(\tau)))^{3/2}} \eta^j_{lm} dx^m(\tau) dx^p(\tau)}{\eta_{kp} dx^i(\tau)} \right)$$
\[
\frac{1}{2} \left( \frac{dx(\tau)}{d\tau} \right)^2 + \left( \eta \Gamma^i_{jk} + \eta^i (dA)_{lj} \frac{dx^j(\tau)}{d\tau} + \eta^i (dA)_{ls} \frac{dx^s(\tau)}{d\tau} \right) \frac{dx^j(\tau)}{d\tau} \frac{dx^k(\tau)}{d\tau}.
\]

6 The Average Operator Associated with a Family of Automorphisms

6.1 Average of a Family of Automorphisms

The averaged connection was introduced in the context of positive definite Finsler geometry in [7]. However, in this work we need to formulate the theory for arbitrary linear connections on the bundle \( \pi^* TM \to N \), where \( N \to M \) is a sub-bundle of the tangent bundle \( TM \to M \).

Let \( \pi^*, \pi_1, \pi_2 \) be the canonical projections of the pull-back bundle \( \pi^* T^{(p,q)} M \to N \), \( T^{(p,q)} M \) being the tensor bundle of type \((p,q)\) over \( M \); \( \pi^*_u T^{(p,q)} M \) the fiber over \( u \in N \), \( T^{(p,q)} M \) the tensor space over \( x \in M \) and \( S_x \) a generic element of \( T^{(p,q)} M \); \( S_u \) is the evaluation of the section \( S \in \Gamma(\pi^* T^{(p,q)} M) \) at the point \( u \in N \).

The averaging operation requires the following structure:

1. A family of non-intersecting, oriented sub-manifolds

\[ \Sigma^{(p,q)} := \{ \Sigma^{(p,q)}_x \subset T_x^{(p,q)} M, \ x \in U \subset M \} \subset T^{(p,q)} M; \]

the particular case \( \Sigma = \Sigma^{(1,0)}, \Sigma_x = \Sigma^{(1,0)}_x \) is relevant case for us.

2. A measure at each point \( x \), which is an element \( f(x, y) \omega_x(y) \in \wedge^m \Sigma^{(p,q)}_x \), where \( m \) is the dimension of \( \Sigma_x \), \( f_x := f(x, y) \) has compact support; \( \text{supp}(f_x) \subset \Sigma_x \); \( f_x : \Sigma_x \to [0, \infty] \) is required to have compact support.

\( \Sigma^{(p,q)} \to M \) is a bundle over \( M \); the fiber \( \Sigma_x \) can be non-compact. However, using local bump functions one can reduce the domain of \( f \) to be compact.

We need to introduce some additional notation. For each tensor \( S_x \in T^{(p,q)} M \) and \( v \in \pi^{-1}(z) \), \( z \in U \subset M \) the following isomorphisms are defined:

\[ \pi^2_{|v} : \pi^*_v T^{(p,q)} M \to T^{(p,q)} M, \quad S_v \to S_z \]

\[ \pi^*_v : T^{(p,q)} M \to \pi^*_v T^{(p,q)} M, \quad S \to \pi^*_v S_z. \]
Definition 6.1 Consider the family of automorphisms,
\[ \{ A_w : \pi^*_w \mathcal{T}M \to \pi^*_w \mathcal{T}M, \ w \in \pi^{-1}(x) \} \].

The averaged operator of this family is the automorphism
\[ < A >_x : \mathcal{T}_x M \to \mathcal{T}_x M \]
\[ S_x \mapsto \frac{1}{\text{vol}(\Sigma_x)} \left( \int_{\Sigma_x} \pi^*_2 |u A_u \pi^*_u \right) \cdot S_x, \]
\[ u \in \pi^{-1}(x), \ S_x \in \Gamma_x M. \]

The integral operation means the following:
\[ \left( \int_{\Sigma_x} \pi^*_2 |u A_u \pi^*_u \right) \cdot S_x := \int_{\Sigma_x} \left( \pi^*_2 |u A_u \pi^*_u S_x \right) f(x, u) d\text{vol}(x, u), \]
where \( d\text{vol}(x, u) \) is the measure on the fiber \( \pi^{-1}(x) \subset N \) and must have compact support. We will denote the averaged automorphisms by symbols between brackets.

6.2 Examples of Geometric Structures which Provide an Averaging Procedure

1. Lorentz Structure [34]. The geometric data is a Lorentzian metric \( \eta \) on \( M \). The disjoint union of the family of sub-manifolds \( \Sigma_x \subset \mathcal{T}_x M \) defines the fibre bundle \( \pi : \Sigma \to M \), which we call the unit hyperboloid bundle over \( x \in M \):
\[ \Sigma := \bigsqcup_{x \in M} \{ \Sigma_x \subset \mathcal{T}_x M \}, \quad \Sigma_x := \{ y \in \mathcal{T}_x M | \eta(y, y) = 1 \} \]

The manifold \( \Sigma_x \) is non-compact and oriented. The measure on \( \Sigma_x \) is given by the following \((n - 1)\)-form
\[ \omega_x(y) := f(x, y) \sqrt{\eta} \frac{1}{y^0} dy^1 \wedge \cdots \wedge dy^{n-1}, \quad y^0 = y^0(x^0, x^1, \ldots, x^{n-1}, y^1, \ldots, y^{n-1}), \]
where \( f(x, y) = f(x, y) \) is positive and with compact support on \( \Sigma_x \); the function \( y^0 \) defines the parameterized hypersurface \( \Sigma_x \subset \mathcal{T}_x M \).

Note that the Lorentzian metric \( \eta \) does not determine the manifolds \( \{ \Sigma_x, x \in M \} \). For instance, one can consider \( \Sigma \) to be the collection of null cones over \( M \):
\[ \mathcal{N} := \bigsqcup_{x \in M} \{ \mathcal{N}_x \subset \mathcal{T}_x M \}, \quad \mathcal{N}_x := \{ y \in \mathcal{T}_x M | \eta(y, y) = 0 \} \]

\( \pi : \mathcal{N} \to M \) is the light-cone bundle over \( M \) and \( \mathcal{N}_x \) is the light-cone over \( x \); on the other hand, \( e : \mathcal{N} \to \mathcal{T}M \) is a sub-bundle of \( \mathcal{T}M \to M \).

2. Finsler Structure [2, 32]. In this case, the Finsler function \( F(x, y) \) defines a fundamental tensor \( g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2(x, y)}{\partial y^i \partial y^j} \) which is positive definite, homogeneous of degree zero on \( y \), smooth and living on the sub-bundle \( N := \mathcal{T}M \setminus \{0\} \). In this context, \( \Sigma \) is defined as the disjoint union,
\[ \Sigma := \bigsqcup_{x \in M} \{ I_x \subset \mathcal{T}_x M \}, \quad I_x := \{ y \in \mathcal{T}_x M | F(y, y) = 1 \} \]
Σ is the indicatrix bundle over M; the manifold Ix is compact and strictly convex for each x ∈ M and is indicatrix at x. The volume form is
\[ d\text{vol}(x, y) := f(x, y) \sqrt{g} \frac{1}{y^0} dy^1 \wedge \cdots \wedge dy^{n-1}, \quad y^0 = y^0(y^1, \ldots, y^{n-1}). \]

3. Symplectic Structures [25]. In this case, there is defined on T*M a non-degenerate, closed 2-form ω. Due to Darboux’s theorem [25, pg 246], there is a canonical local coordinate system of T*M such that the symplectic form ω can be written as
\[ \omega = \sum_{i=1}^{n} dp^i \wedge dq^i. \]
Associated with ω there is defined on the dual tangent bundle T*M a volume 2n-form:
\[ S = \omega \wedge \cdots \wedge \omega. \]
In the canonical coordinates (q, p), the 2n-differential form is
\[ S(p, q) = dp^1 \wedge \cdots dp^n \wedge dq^1 \wedge \cdots dq^n. \]
Therefore, let us explore the following construction. Let us assume the existence of a non-where zero vector field V (therefore the Euler characteristic of TM must be different than zero). Then we can construct the following form (2n − 1)-form,
\[ \omega_q(p) = \iota_V S, V \in \Gamma(T(T^*M)) \]
which is a non-degenerate (2n − 1)-form, whose value on V is zero. Let us choose a distribution of commuting vector fields, \( \{X_i, [X_i, X_j] = 0, \quad i, j = 1, \ldots, 2n - 1\} \) locally complementary to V and such that \( \{X_1, \ldots, X_{2n-1}, V\} \) is a local frame of TM. The distribution \( \{X_1, \ldots, X_{2n-1}\} \) is integrable and dvol(x, y) is a volume form on the integral manifold \( S^\perp \). On the other hand, it is a fibered manifold:
\[ \pi : S^\perp := \bigsqcup_{x \in M} S^\perp_x \longrightarrow M. \]
The interesting thing about this example is that we can only construct local averaging procedures; the overlapping of the averaging operation in different open sets is non-trivial and in general one needs more structure to define consistently the averaging procedure globally.

4. Hermitian Vector Bundles [35]. The construction is similar to the one in the Finslerian case. The sub-manifolds \( \Sigma_x \) are defined as:
\[ \Sigma_x = \{y \in T_x M \mid H(y, y) = 1\}, \]
where H is the hermitian structure on M. Therefore
\[ \Sigma := \bigsqcup_{x \in M} \Sigma_x. \]
Let us assume that the hermitian structure is of the form \( H = \eta + \omega \), where \( \eta \) is a Riemannian structure and \( \omega \) is a complex structure. To define the measure and the volume form we can use either the complex structure \( \omega \) (which defines a measure on T*M) or the Riemannian structure \( \eta \) (which defines a volume form on the tangent bundle TM).
6.3 Average Operator Acting on Sections

The averaging operation can be extended to a family of operators acting on sections of tensor bundles. This is especially important for the next section. In particular, let $\pi^*, \pi_1, \pi_2, \pi^*T^{(p,q)}M$ and $T^{(p,q)}M$ as before. Let us consider the sections $S \in \Gamma(T^{(p,q)}M)$ and $\pi^*S \in \Gamma(\pi^*T^{(p,q)}M)$ and the following isomorphisms:

\[
\pi_2|: \Gamma(\pi^*T^{(p,q)}M) \rightarrow \Gamma(T^{(p,q)}M), \quad S_v \mapsto S_z,
\]

\[
\pi^*: \Gamma(T^{(p,q)}M) \rightarrow \Gamma(\pi^*T^{(p,q)}M), \quad S_z \mapsto \pi^*S_z.
\]

Both isomorphisms are defined pointwise

**Definition 6.2** Consider the family of automorphims

\[
\{A(W) : \Gamma(\pi^*TM) \rightarrow \Gamma(\pi^*TM) , \ W \in \pi^{-1}(U), U \in M\}.
\]

The averaged operator of this family is the automorphism

\[
<A> : \Gamma(TU) \rightarrow \Gamma(TU)
\]

such that at each point $x \in U$ it is given by:

\[
(A \cdot S)(x) := \frac{1}{\text{vol}(\Sigma_x)} \left( \int_{\Sigma_x} \pi_2|_u (A\pi^* \cdot S)(u) \right),
\]

\[
u \in \pi^{-1}(x), S \in \Gamma TM,
\]

where $(A\pi^* \cdot S)(u)$ is the evaluation of the section $(A\pi^* \cdot S)(u)$ at $u$.

A similar definition application if the operators act on cartesian products of $\Gamma(\pi^*T^{(p,q)}M)$.

7 Averaged Connection of a Linear Connection on $\pi^*TM$

We adopt a differential volume form $\omega(x,y)$ such that $(d\omega(x,y))|_{\Sigma_x} = 0$. Therefore we denote $(\omega(x,y))|_{\Sigma_x} = d\text{vol}(x,y)$.

**Definition 7.1** Let $M$ be a $n$-dimensional smooth manifold, $\pi(u) = x$ and consider a differentiable real function $f \in FM$. Then $\pi^*f \in \mathcal{F}(N)$ is defined by the condition

\[
\pi^*f = f(x).
\]

The horizontal lift using a non-linear connection defined on the bundle $TN \rightarrow N$ is

\[
\iota : \Gamma TM \rightarrow \Gamma TN
\]

\[
X^i \frac{\partial}{\partial x^i}|_x \mapsto X^i \frac{\delta}{\delta x^i}|_u, \quad u \in \pi^{-1}(x).
\]

**Proposition 7.2** Let $M$ be a $n$-dimensional manifold and assume that $N$ is endowed with a non-linear connection, $u \in \pi^{-1}(x) \subset N$, with $x \in M$ and let us consider a linear connection $\nabla$ defined on the vector bundle $\pi^*TM \rightarrow N$. Then there is defined on $M$ a linear covariant derivative along $X$, $<\nabla>_X$ determined by the following conditions:
1. \( \forall X \in T_xM \) and \( Y \in TM \), the covariant derivative of \( Y \) in the direction \( X \), is given by the following averaging operations:

\[
< \nabla > X Y := < \pi_2|_u \nabla_{\iota_u(X)} \pi_v^* Y >_u, \forall v \in U_u, \quad (7.3)
\]

where \( U_u \) is a open neighborhood of \( u \).

2. For every smooth function \( f \in F_M \) the covariant derivative is given by the following average:

\[
< \nabla > X f := < \pi_2|_u \nabla_{\iota_u(X)} \pi_v^* f >_u, \forall v \in U_u. \quad (7.4)
\]

3. The distribution function is such that \( < f > < \infty \).

**Proof**: it is shown in reference [7, section 4].

For the physical example that we are interested the averaging operation and the notion of volume that we are using is obtained by isometric embedding of the ambient Lorentzian structure \( \eta \) on the unit hyperboloid times the positive weight function \( f \) (which will be the one-particle distribution function). Since in particular the function \( f_x \) will be required later to be \( L^1(\Sigma_x) \) and with compact support,

\[
vol(\Sigma_x) := \int_{\Sigma_x} f(x, y) \, dvol(x, y) < \infty.
\]

The manifold \( \Sigma_x \) is oriented. In particular, the integration is performed in the unit tangent hyperboloid,

\[
\Sigma_x := \{ y \in T_xM, \mid \eta(y, y) = 1, y^0 > 0 \}.
\]

The measure that we will use is a solution of the Vlasov equation, which is obtained for a particular Liouville vector field.

**Definition 7.3 (Generalized Torsion)** Let \( \nabla \) be a linear connection on \( \pi^*TM \rightarrow N \), then the generalized torsion tensor acting on the vector fields \( X, Y \in TM \) is defined as

\[
\text{Tor} : \Gamma \pi^*TM \times \Gamma \pi^*TM \rightarrow \Gamma \pi^*TM
\]

\[
(\pi^*X, \pi^*Y) \rightarrow \text{Tor}(\nabla)(\pi^*X, \pi^*Y) = \nabla_{\iota(Y)\pi^*Y} \pi^*X - \nabla_{\iota(X)\pi^*Y} \pi^*X - \pi^*[X, Y]. \quad (7.5)
\]

This tensor mimics the usual torsion tensor \( \text{Tor} \),

\[
\text{Tor} : \Gamma TM \times \Gamma TM \rightarrow \Gamma TM
\]

\[
(X, Y) \rightarrow \text{Tor}(\nabla)(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]. \quad (7.6)
\]

**Proposition 7.4** The averaged connection \( < \nabla > \) has a torsion \( \text{Tor}(< \nabla >) \) such that

\[
\text{Tor}(< \nabla >) = < \text{Tor}(\nabla) >. \quad (7.7)
\]

**Proof**: It is shown in reference [7].

**Corollary 7.5** Let \( M \) be an \( n \)-dimensional manifold and \( \nabla \) a linear connection on the bundle \( \pi^*TM \rightarrow M \) with \( \text{Tor}(\nabla) = 0 \). Then \( \text{Tor}(< \nabla >) = 0 \).
Proof: It is direct from the proof of proposition (3.3.4). \(\square\)

If \(\text{Tor}(<\nabla>) = 0\) we say that the connection \(<\nabla>\) is torsion free.

Corollary 7.6 Let \(M\) be an \(n\)-dimensional manifold. If the connection \(\nabla\) in \(\pi^*TM\) has the connection coefficients \(\Gamma^i_{jk}\). Then the average connection \(<\nabla>\) has the coefficients

\[
<\Gamma^i_{jk}(x)> = \frac{1}{\text{vol}(\Sigma_x)} \int_{\Sigma_x} \Gamma^i_{jk}(x, y) \, d\text{vol}(x, y).
\]

(7.8)

Proof: Let \(\{e_i\}, \{\pi^*e_i\}, \{\iota(e_i)\}\) be local frames for the sections of the vector bundles \(TM, \pi^*TM\) and the horizontal bundle \(H\) such that the covariant derivative is defined through the relations

\[
\nabla_h(e_j) = \pi^*e_i = \Gamma^i_{jk}\pi^*e_i, \quad i, j, k = 1, 2, ..., n.
\]

Then, let us take the covariant derivative

\[
<\nabla>e_j e_k = \frac{1}{\text{vol}(\Sigma_x)} \int_{\Sigma_x} \pi^2(\nabla_{\iota(e_j)}\pi^*e_k) = \frac{1}{\text{vol}(\Sigma_x)} \int_{\Sigma_x} \pi^2\Gamma^i_{jk}\pi^*e_i = \frac{1}{\text{vol}(\Sigma_x)} \int_{\Sigma_x} \Gamma^i_{jk}e_i.
\]

The relation (7.8) follows from the definition of the connection coefficients of the averaged connection,

\[
<\nabla>e_j e_k = <\Gamma>^i_{jk}e_i(x).
\]

\(\square\)

8 The Averaged Lorentz Connection

Given the linear connection \(L\nabla\) on the bundle \(\pi^*TM \to \Sigma\) we can obtain an associated averaged connection.

Usually the measure is given as \(f(x, y) \, d\text{vol}(x, y)\), where \(f(x, y)\) is the one-particle probability distribution function of a kinetic model. It must be gauge invariant and such that the low order momentum moments must be finite. The volume form is

\[
d\text{vol}(x, y) = \sqrt{-\det \eta} \, \frac{1}{y^0} dy^1 \wedge \cdots \wedge dy^{n-1}, \quad y^0 = y^0(x^0, ..., x^{n-1}, y^1, ..., y^n).
\]

Then one can prove the following:

Proposition 8.1 The averaged connection of the Lorentz connection \(L\nabla\) on the pull-back bundle \(\pi^*TM \to \Sigma\) is an affine, symmetric connection on \(M\). The connection coefficients are given by the formula:

\[
< L\Gamma^i_{jk} > := \eta \Gamma^i_{jk} + (F^i_j < y^m > \eta_{mk} + F^i_k < y^m > \eta_{mj}) + F^i_m \left( < y^m >^{3/2} < \eta_{jk} - \eta_{jm} > < y^m y^s y^l > \right).
\]

(8.1)

Each of the integrations is equal to the \(y\)-integration along the fiber:

\[
\text{vol}(\Sigma_x) = \int_{\Sigma_x} f(x, y) \, d\text{vol}(x, y), \quad < y^i > := \frac{1}{\text{vol}(\Sigma_x)} \int_{\Sigma_x} y^i f(x, y) \, d\text{vol}(x, y),
\]

\[
< y^m y^s y^l > := \frac{1}{\text{vol}(\Sigma_x)} \int_{\Sigma_x} y^m y^s y^l f(x, y) \, d\text{vol}(x, y).
\]
Proof: Equation (8.1) follows easily from the definition of the averaged connection by linearity. We only need to prove that \( < y^i > \) and the other moments are given by the corresponding integrals and the identity operator \( I_d : \pi^*TM \rightarrow \pi^*TM, (x, y) \mapsto (x, y) \):

\[
<y^i> = \frac{1}{\text{vol}(\Sigma_x)} \int_{\Sigma_x} \pi_2 Id y^i \pi^*(y^i)f(x, y) \text{dvol}(x, y) = \frac{1}{\text{vol}(\Sigma_x)} \int_{\Sigma_x} y^i f(x, y) \text{dvol}(x, y)
\]

and similarly for other moments. Note that since \( y \in \Sigma_x \), \( \eta(y, y) = 1 \) and it does not appear in the moments. \( \square \)

Remarks.

1. If we consider the bundle \( \pi^*TM \rightarrow N \), the coefficients of the averaged Lorentz connection are

\[
< L \nabla > := \eta \Gamma^i_{kj}(x) - \eta^i(x)(dA)_{lm}(x) < 1 \frac{1}{2(\eta(y, y))} \eta_{js}(x) y^s y^m \eta_{kp}(x) y^p > + \eta^i(x)(dA)_{lm}(x) < 1 \frac{1}{2(\eta(y, y))} \eta_{jk}(x) y^m > + \eta^i(x)(dA)_{lj}(x) < 1 \frac{1}{2(\eta(y, y))} \eta_{ks}(x) y^s >.
\]

2. The definition given by formula (8.1) does not depend on the natural coordinates.

The following proposition enumerates basic properties of the averaged connection. The proof is straightforward; one needs to check the coefficients given by the formula (8.1):

**Proposition 8.2** Let \((M, \eta, [A])\) be a semi-Randers space, \( f : \Sigma \rightarrow \mathbb{R} \) which is non-negative with compact support and \( < L \nabla > \) be the averaged Lorentz connection. Then

1. \( < L \nabla > \) is an affine, symmetric connection on \( M \). Therefore, for any point \( x \in M \), there is a normal coordinate system such that the averaged coefficients are zero.
2. \( < L \nabla > \) is determined by the first, second and third moments of the distribution function \( f(x, y) \).

**Remark** While the first property is a general property of the averaged connection, the second one is a specific property of the averaged Lorentz connection and that we are considering trajectories whose velocity fields are in the unit hyperboloid \( \Sigma \).

9 Comparison Between the Geodesics of \( L \nabla \) and \( < L \nabla > \)

9.1 Basic Geometry in the Space of Connections

Let us consider the Lorentzian manifold \((M, \eta)\) with signature \((+, -, ..., -)\) and \( M \) \( n \)-dimensional. If there is defined on \( M \) a time-like vector field \( U \) normalized such that \( \eta(U, U) = 1 \), one can define the Riemannian metric \( \bar{\eta} \) [19]:

\[
\bar{\eta}(X, Y) := -\eta(X, Y) + 2\eta(X, U)\eta(Y, U).
\]
\( \eta \) determines a Riemannian metric on the vector space \( T_x M \) \( \tilde{\eta}_x = \tilde{\eta}_i(x) dy^i \otimes dy^j \). Therefore the pair \((T_x M, \tilde{\eta}_x)\) is a Riemannian manifold. It induces a distance function \( d_\eta \) on the manifold \( T_x M, d_\eta : T_x M \times T_x M \rightarrow \mathbb{R}; \ (y, z) \mapsto inf \{ \int_0^{1} \tilde{\eta}_x(\dot{\sigma}, \dot{\sigma}) d\tau, \ \dot{\sigma}(0) = y, \dot{\sigma}(1) = z \} \).

We assume that \( f(x, y) \) has compact support on the unit hyperboloid bundle \( \Sigma \). Then the diameter of the distribution \( f_x := f(x, \cdot) : \Sigma_x \rightarrow \mathbb{R} \); \ (x, y) \mapsto f(x, y) \) is \( \alpha_x := sup \{ d_\eta(y_1, y_2) \mid y_1, y_2 \in supp(f_x) \} \). We define the parameter \( \alpha := sup \{ \alpha_x, x \in M \} \).

We choose as vector field \( U(x) \) in the definition of the Riemannian metric (9.1) the following:

\[
U(x) = \begin{cases} 
\frac{\langle \hat{y}(x) \rangle}{\sqrt{\eta_{ij}(x) \langle \hat{y}'(x) \rangle \langle \hat{y}'(x) \rangle}}, & \eta(\langle y >, \langle y >) (x) > 0 \\
0, & \text{otherwise.}
\end{cases}
\]

\( U(x) \) is not continuous on the boundary \( \partial(\pi(supp(f))) \), since in the interior of \( \pi(supp(f)) \) is time-like and outside the closure of \( \pi(supp(f))) \) is zero. In particular one has,

\[
\| \langle y > (x) \|_\eta = \begin{cases} 
\eta(\langle y >, \langle y >) (x) > 1, & x \in int(supp(\pi(f(x, y)))) \\
0, & x \in M \setminus int(supp(\pi(f(x, y))))
\end{cases}
\]

One can approximate the main velocity field \( \langle y > \) by another vector field \( \langle y > (x) \) which is smooth in the whole space-time \( M \) and time-like in an open sub-set of the interior of \( \pi(supp(f)) \) (basically in \( \pi(supp(f)) \setminus \partial(\pi(supp(f))) \)). In this way, one can extend the definition of the vector field \( U(x) \) to the whole \( supp(f) \) in a smooth way.

**Proposition 9.1** Let the distribution function \( f(x, y) \) be of class \( C^1 \) and \( \eta \) a \( C^\infty \) Lorentzian metric on \( M \). Then there is a \( C^1 \) vector field \( \langle y > \) which coincides with the time-like vector field \( \langle y > \), and such the following relation holds:

\[
\| \langle y > - \langle \hat{y} > \|_\eta = O(\alpha^3),
\]

with \( \alpha \) the diameter of the support of the distribution \( f_x \) and with \( \langle \hat{y} > \) the associated Riemannian metric to the vector field \( \langle y > \).

**Proof:** Let us consider the \( C^\infty \) function

\[
b(t) = \begin{cases} 
e^{-t}, & t > 0 \\
0, & t \leq 0
\end{cases}
\]

\( k \) is a positive parameter. Now consider an arbitrary open subset \( O^c \subset int(supp(\pi(f(x, y)))) \) and the restriction of the vector field \( \langle y > \) to \( O^c \) denoted by \( \langle y > \mid O^c \). Then, we consider the distance function between each point \( z \in \partial(supp(\pi(f))) \) to the open set \( O \). This distance function, which we denote by \( t(z) \) is realized by a point \( w \in O \). We can consider the geodesic segment between \( z \) and \( w \), for a similar enough \( O \) to \( int(supp(\pi(f(x, y)))) \) in such a way that these segments exists. After this, we can define the following interpolating vector field:

\[
\langle v(x) = \begin{cases} 
0, & x \in M \setminus int(supp(\pi(f(x, y)))) \\
\frac{1}{t(z)} t(z) v(w(z)), & x = \hat{z} \in int(supp(\pi(f(x, y)))) \setminus O^c \)
\]

This vector field is still not smooth but it is continuous. One can smooth it using bump functions based on the function \( b(t) \) in such a way that the resulting smooth field is zero in \( M \setminus int(supp(\pi(f))) \) and where the parameter \( t \) depends on the point \( z \in \partial(supp(f)) \)
and corresponds to the geodesic distance (using $\bar{\eta}$, which we now it is smooth on $\text{int}(\text{supp}(\pi(f(x,y))))$.

The construction of the metric $\imath\bar{\eta}$ is based in the $k$-parameter family of bump functions based on the function $b(t)$. The idea is to fit the parameter $k$ such that in the smoothing process using bump function $[27]$, we obtain $\|\imath\hat{y} > - \imath\bar{\eta} > \| = O(\alpha^3)$. The construction is similar to Steenrod’s construction $[39, \text{section 6.7}]$.

Given a continuous operator $A_x : T_xM \rightarrow T_xM$, its operator norm is defined by

$$\|A\|_{\eta}(x) := \sup \left\{ \frac{\|A(y)\|_{\eta}(x)}{\|y\|_{\eta}(x)}, y \in T_xM \setminus \{0\} \right\}.$$  

Let us denote the space of linear connections on $T\Sigma$ by $\nabla_{\Sigma}$. This space is a finite dimensional manifold which points are coordinated by the set of functions $\{\Gamma^i_{jk}(x,y), \ i,j,k = 0, 1, ..., n-1\}$.

**Proposition 9.2** On the space $\nabla_{\Sigma}$, there is a distance function. The distance between two points $1\nabla, 2\nabla \in \nabla_{\Sigma}$ is given by:

$$d_{\eta}(1\nabla, 2\nabla)(x) := \sup \left\{ \frac{\sqrt{\eta(x)(1\nabla X - 2\nabla X, 1\nabla X - 2\nabla X)}}{\eta(X,X)} \right\}, \ X \in \Gamma T\Sigma,$$

$$1\nabla, 2\nabla \in \nabla_{\Sigma} \right\}. \quad (9.2)$$

**Proof:** The function $(9.2)$ is symmetric and non-negative. The distance between two arbitrary connections is zero iff

$$\sqrt{\eta(1\nabla X - 2\nabla X, 1\nabla X - 2\nabla X)} = 0$$

for all $X \in \Sigma_x$. This happens iff $1\nabla X = 2\nabla X$ for any $X \in \Sigma_x$. The triangle inequality also holds, using the triangle inequality for $\bar{\eta}$:

$$d_{\eta}(\nabla_1, \nabla_3) = \sup \left\{ \frac{\sqrt{\eta(1\nabla X - 3\nabla X, 1\nabla X - 3\nabla X)}}{\eta(X,X)} \right\} \leq$$

$$\leq \sup \left\{ \frac{\sqrt{\eta(1\nabla X - 2\nabla X, 1\nabla X - 2\nabla X)}}{\eta(X,X)} \right\} +$$

$$+ \sup \left\{ \frac{\sqrt{\eta(2\nabla X - 3\nabla X, 2\nabla X - 3\nabla X)}}{\eta(X,X)} \right\} \leq$$

$$\leq d_{\eta}(1\nabla, 2\nabla) + d_{\eta}(2\nabla, 3\nabla).$$

For an arbitrary 1-form $\omega$ we denote by $\omega^\sharp := \eta^{-1}(\omega, \cdot)$ the vector obtained by duality, using the Lorentzian metric $\eta$. Similarly, given a vector field $X$ over $M$, one can define the dual one form $X^\sharp := \eta(X, \cdot)$; $\iota_X \omega$ is the inner product of the vector $X$ with the form $\omega$.

**Proposition 9.3** Let $f(x,y)$ be the one-particle probability distribution function such that each function $f_x$ has compact and connected support $\text{supp}(f_x) \subset \Sigma_x$. Then

$$(L\nabla y - \langle L\nabla, y \rangle)(x) = -(\imath_3 F)^\sharp(x) \cdot (\iota_Y(\delta(y)(x,y)) + O_2(\delta^2(y))(x,y) + O_3(\delta^3(y))(x,y), \quad (9.3)$$

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where \( \delta^m(x, y) = \langle \hat{y}^m > (x) - y^m \) does not depend on \( y \in \text{supp}(f_x) \) or the 2-form \( F \). The tensors \( \mathcal{O}_i(x, y) \) are given by the following expressions:

\[
\mathcal{O}_2^m(\delta^j(y))(x, y) = \frac{1}{2} \left( \langle \hat{y}^m > (x) \delta^s(x, y) \delta^d(x, y) > + \langle \dot{y}^m > \langle \hat{\dot{y}}^m > \delta^s(x, \dot{y}) \delta^d(x, \dot{y}) > \right) \eta_{sj} \eta_{ik} y^i y^k,
\]

\[
\mathcal{O}_3^m(\delta^j(y))(x, y) = \frac{1}{2} \langle \delta^m(x, \dot{y}) \delta^s(x, \dot{y}) \delta^d(x, \dot{y}) > \eta_{sj} \eta_{ik} y^i y^k.
\]

**Proof**: From the expressions for the connection coefficients,

\[
< L \nabla > y - L \nabla_y y = \frac{1}{2} \left( \langle F^j_i(x) < \hat{y}^m > (x) - y^m \rangle \eta_{mk} + \langle \dot{F}^i_i(k)(x) < \hat{y}^m > (x) - y^m \rangle \eta_{mj} \right) +
\]

\[
+ \langle F^i_m(x) ((< \hat{y}^m > (x) - y^m) \eta_{jk} - \eta_{js} \eta_{kl} < \hat{y}^m \hat{y}^s \hat{y}^d > (x) - y^m y^s y^d) \rangle y^i y^j \frac{\partial}{\partial x^l},
\]

since \( \eta_{ij} y^i y^j = 1 \). The difference between the two connections can be expressed in terms of the following tensors:

\[
\delta^m(x, y) = \langle \hat{y}^m > (x) - y^m, \quad \delta^{mkl}(x, y) y_s y_l = \langle \hat{y}^m \hat{y}^s \hat{y}^d > (x) \eta_{sj} \eta_{ik} y^i y^k - y^m
\]

and is given by the following expression:

\[
\left( < L \nabla > y - L \nabla_y y \right)(x) = \frac{1}{2} \left( \langle F^j_i(x) (\delta^m(x, y)) \eta_{mk} + \langle \dot{F}^i_i(k)(x) (\delta^m(x, y)) \eta_{mj} \right) +
\]

\[
+ \langle F^i_m(x) \left( \delta^m(x, y) \eta_{jk} - \eta_{js} \eta_{kl} \delta^{mkl}(x, y) \right) \rangle y^i y^j \frac{\partial}{\partial x^l} =
\]

\[
= \left( \langle F^j_i y^j \delta^k(x, y) y_k + \langle F^i_m(x) (\delta^m(x, y) - y_j y_k \delta^{mkl}(x, y)) \rangle \right) \frac{\partial}{\partial x^l},
\]

where \( y_j = \eta_{jk} y^k \).

In the above subtractions the second contribution is of the same order in \( \delta(y) \) as the first one. To show this, recall from the definitions:

\[
\delta^{mkl}(x, y) y_s y_l = \langle \hat{y}^m(x) \hat{y}^s(x) \hat{y}^d(x) > \eta_{sj} \eta_{ik} y^i y^k - y^m
\]

where the hat-notation is used to distinguish integrated variables from the coordinates \( y^i \). Then we can use the following relations:

\[
\dot{y}^s = \langle \hat{\dot{y}}^s > (x) - \delta(x, \dot{y}^s), \quad \ddot{y}^s = \langle \hat{\dot{y}}^s > (x) - \delta^s(x, \dot{y}), \quad \ddot{y}^m = \langle \hat{\dot{y}}^m > (x) - \delta^m(x, \dot{y}).
\]

One substitutes this relation and, taking into account that \( \langle \delta(x, \dot{y}) >= 0 \), one gets

\[
\hat{\delta}^{mkl}(x, y) y_s y_l = \left( \langle \hat{y}^m > (x) < \hat{y}^s > (x) < \hat{y}^d > (x) + < \hat{\dot{y}}^m > (x) < \hat{\dot{y}}^s > (x) < \hat{\dot{y}}^d > (x, \dot{y}) +
\]

\[
+ < \dot{y}^l > (x) < \delta^l(x, \dot{y}) \delta^m(x, \dot{y}) > + < \dot{y}^s > (x) < \delta^m(x, \dot{y}) \delta^d(x, \dot{y}) > -
\]

\[
- < \delta^m(x, \dot{y}) \delta^s(x, \dot{y}) \delta^d(x, \dot{y}) > \right) \eta_{sj} \eta_{ik} y^i y^k - y^m.
\]

Now we use a similar relation to go further:

\[
y^s = \langle \hat{y}^s > (x) - \delta(x, y^s), \quad y^l = \langle \hat{y}^l > (x) - \delta^l(x, y).
\]
We introduce these expressions in the calculation of $\delta^{mst}$:

$$\delta^{mst}(x, y)y^3 y^k = \left( \langle \hat{y}^m > (x) (y^4 + \delta^4(x, y)) (y^4 + \delta^4(x, y)) \right) + \langle \hat{y}^m > (x) < \delta^4(x, y) \delta^4(x, y) + (y^4 + \delta^4(x, y)) < \delta^4(x, y) \delta^4(x, y) > -$$

$$- \langle y^4 + \delta^4(x, y) \delta^4(x, y) > - (y^4 + \delta^4(x, y)) \delta^4(x, y) + (y^4 + \delta^4(x, y)) \delta^4(x, y) > -$$

Using again the fact that $y^4_{\eta_3 y^k} = 1$ we get (again using that $\langle \hat{y}^m > = y^m + \delta(x, y)$ to recombine the first and last term):

$$\delta^{mst}(x, y)y^3 y^k = \delta^m(x, y) + z \langle y^m > (x) \delta^m(x, y) + 2 \langle y^m > (x) \delta^m(x, y) \eta_{xy} y^j - 2 \langle y^m > (x) \delta^m(x, y) \eta_{xy} y^j + 2 \Omega^m_2 (\delta^2) + 2 \Omega^m_3 (\delta^3).$$

The tensors $O_i$ are given by the formulas (9.5) and (9.6).

The transversal contribution to the difference between the connections is given by:

$$\frac{1}{2} \nabla^4_{m(x)}(\delta^m(x, y) \eta_{jk} - \eta_{i} \eta_{j} \delta^{mst}(x, y) y^j y^k) = \frac{1}{2} \nabla^4_{m(x)}(\delta^m(x, y) - \delta^m(x, y)$$

$$- 2 \langle \hat{y}^m > (x) \delta^m(x, y) \eta_{xy} y^j - \Omega^m_2 (\delta^2) - \Omega^m_3 (\delta^3)$$

$$= - \nabla^4_{m(x)}(\langle y^m > (x) \delta^m(x, y) \eta_{xy} y^j) - \Omega^m_2 (\delta^2) - \Omega^m_3 (\delta^3).$$

The longitudinal contribution is

$$\frac{1}{2}(\nabla^4_{j(x)}(\langle \hat{y}^m > (x) - y^m) \eta_{mk} + \nabla^4_{k(x)}(\langle \hat{y}^m > (x) - y^m) \eta_{mj}) y^j y^k = \nabla^4_{j(x)} y^j \delta^k(x, y) y^k.$$

Adding together the longitudinal and transversal contributions and taking into account the formula $\delta^m(x, y) = \langle \hat{y}^m > (x) - y^m$ we get the following expression:

$$(\nabla^4_{j(x)}(\delta^m(x, y) y^j) - \nabla^4_{m(x)}(\langle \hat{y}^m > (x) \delta^m(x, y) \eta_{xy} y^j) + \Omega^m_2 (\delta^2) + \Omega^m_3 (\delta^3))$$

$$= - \left( \nabla^4_{j(x)} (\delta^k(x, y) y^j) - \Omega^m_2 (\delta^2) - \Omega^m_3 (\delta^3).$$

Let us consider a frame $\{e_i, i = 0, ..., n - 1\}$ such that $e_i a$ is diagonal at the point $x \in M$. After calculating the distance of the connections using the formula (4.6.3), the leading term in $\delta$ is quadratic.

**Proposition 9.4** Let $(M, \eta, [A])$ and $L\nabla$ be as before and assume that $f_x$ has compact and connected support for each fixed $x \in M$ and that $\alpha := \sup \{\alpha_x, x \in M\} << 1$. Then the following holds:

$$d_{\eta}(L\nabla, L\nabla) > (x) \leq \|F\|_{\eta(x)} C(x) \alpha^2 + 2 \Omega^2_2 (x) \alpha^2 (1 + \alpha) + \Omega^3_2 (x) \alpha^3 (1 + \alpha), \quad (9.6)$$

with $C(x), C_2(x), C_3(x)$ being functions depending only on $x$ with value of the order of unity.

**Proof:** From equation (4.6.3) one obtains:

$$\|L\nabla y - L\nabla > y \|_{\eta} = \|F^i j(x) \delta^i (x, y) \delta^k (x, y) y^k e_i \|_{\eta} + \Omega^m_2 (\delta^2) (x, y) + \Omega^m_3 (\delta^3) (x, y) \|_{\eta} \leq$$

$$\leq \|F^i j(x, y) \delta^i (x, y) \delta^k (x, y) y^k e_i \|_{\eta} + \| \Omega^m_2 (\delta^2) (x, y) \|_{\eta} + \| \Omega^m_3 (\delta^3) (x, y) \|_{\eta}.$$

Each of these terms can be bounded.

The bound of the first term comes as follow, since $\|e_i\| = 1$:

$$\|F^i j(x, y) \delta^i (x, y) \delta^k (x, y) y^k e_i \|_{\eta} \leq \|F\|_{\eta} \cdot \| \delta^k (x, y) y^k \|_{\eta} \leq \|F\|_{\eta} \cdot \| \delta (x, y) \|_{\eta} \cdot \| \delta^k (x, y) y^k \|_{\eta}.$$
Since the support of the distribution function $f(x,y)$ is compact and connected, one can write the decomposition $\langle \hat{\gamma} \rangle (x) = \epsilon (x) + z(x)$ with the property that $z(x) \in \text{supp}(f_x)$. In the case of the Minkowski metric one can check by geometric inspection that $\Vert \epsilon(x) \Vert_\eta \leq \alpha$.

This bound of $\epsilon(x)$ follows from the shape of the unit hyperboloid in the following way. First, note that the domain $\Sigma_x := \{ y \in T_xM \mid \eta(y,y) \geq 1 \}$ is a convex set. Indeed, we note that $\partial(\Sigma_x) = \Sigma_x$ and that $\langle \hat{\gamma} \rangle (x) \in \Sigma_x$. Secondly, each $(\Sigma_x, \tilde{\eta}_x)$ is a Riemannian structure with zero curvature tensor. Therefore it is isometric to a piece of the Euclidean space. Therefore, we can use the standard definition of center of mass, in this case with an invariant measure given by $f(x,y,\delta) \text{vol}(x,y)$. The function $f(x,y,\delta)$ is such that

$$\int_0^1 ds f(x,y,\delta) = f(x,y),$$

where $s$ is the parameter of the convex line connecting $\delta \in \text{supp}(f_x)$ with $\langle \hat{\gamma} \rangle$. Let us denote by $\text{supp}(f_x)$ the convex envelope of $\text{supp}(f_x)$ By construction $\langle \hat{\gamma} \rangle \in \text{supp}(f_x)$. Indeed one can check that $\langle \hat{\gamma} \rangle$ is the center of mass of the convex set $\text{supp}(f_x)$ [36].

We have the following bound

$$\Vert \delta(x,y) \Vert_\bar{\eta} \leq \Vert \langle \hat{\gamma} \rangle (x) - y \Vert_\bar{\eta} \leq \Vert \epsilon + z(x) - y \Vert_\bar{\eta} \leq \Vert \epsilon \Vert_\bar{\eta} + \Vert z(x) - y \Vert_\bar{\eta} \leq \alpha + \alpha = 2\alpha.$$

For the third factor, one has the following bound $(\delta(x,y) = \langle y \rangle - y)$:

$$|\delta^k(x,y) y_k| = |\langle \hat{\gamma} \rangle^k (x) y_k - 1| = |\langle \hat{\gamma} \rangle^k (x) (y_k - \langle \hat{\gamma} \rangle^k (x) + \langle \hat{\gamma} \rangle^k (x)) - 1| \leq |\langle \hat{\gamma} \rangle^k (x) (y_k - \langle \hat{\gamma} \rangle^k (x))| + |\langle \hat{\gamma} \rangle^k (x) - \langle \hat{\gamma} \rangle^k (x) - 1|.$$

Using Cauchy-Schwarz inequality for $\bar{\eta}$ we obtain:

$$|\delta^k(x,y) y_k| \leq \Vert \langle \hat{\gamma} \rangle^k (x) \Vert_\bar{\eta} \Vert (y_k - \langle \hat{\gamma} \rangle^k (x)) \Vert_\bar{\eta} + |\langle \hat{\gamma} \rangle^k (x) - \langle \hat{\gamma} \rangle^k (x) - 1| \leq \sqrt{1 + \Vert \epsilon \Vert_\bar{\eta} \alpha} + \sqrt{1 + \Vert \epsilon \Vert_\bar{\eta} - 1} \leq \sqrt{1 + \alpha} \alpha + (1 + \alpha - 1) \leq (1 + \alpha) \alpha + (1 + \alpha - 1) = 2\alpha + \alpha^2.$$

Therefore, one obtains:

$$\Vert (F_i^j \delta^j(x,y)) (\delta^k(x,y) y_k) \Vert_\bar{\eta} \leq \Vert F \Vert_\bar{\eta} (x) C(x) \alpha^2.$$

The function $C(x)$ in equation (4.6.6) is bounded by the constant 2.

Using homogeneity properties, one can see that the following relations hold:

$$\Vert \mathcal{O}_2(\delta^2) \Vert_\bar{\eta} \leq C_2^2(x) \alpha^2 (1 + B_2(x) \alpha), \quad \text{(9.7)}$$

and

$$\Vert \mathcal{O}_3(\delta^3) \Vert_\bar{\eta} \leq C_3^3(x) \alpha^3 (1 + B_3(x) \alpha). \quad \text{(9.8)}$$

The functions $C_i(x)$ depend on the particular shape of the support of the distribution function $f$ and on the curvature of the metric $\eta$. Using geometric arguments (and in particular compactness and connectedness of the $\text{supp}(f_x)$), one can bound the function $|\delta^0(x,y)|$ in terms of $\alpha$ in a similar way as we did for $C(x)$. The reason why the constants will be of order 1 is that this was the case for $C(x)$ and there is not new divergences in the bound of the functions $B_i(x)$ and $C_i(x)$. \hfill \Box
Remark 1. For arbitrary Lorentzian manifolds the same ideas can also be applied to bound the functions \(C_i(x)\), as we have directly seen for \(C(x)\). A homogeneity argument implies that these functions are also bounded by constants of order 1.

Remark 2. In this proof it is not essential that \(\alpha \ll 1\). However it simplifies the calculation.

Corollary 9.5 Let \((M, \eta, [A])\) be a Randers space with \(\bigcup_{x \in M} \text{supp}(f_x)\) compact, with \(L^\nabla\) and \(< L^\nabla >\) as before. Then there is a global bound:

\[
d_\eta(L^\nabla, < L^\nabla >)(x) \leq C_\|F\|_\|\alpha^2 + 2C_2^2\alpha^2(1 + B_2\alpha) + C_3^3\alpha^3(1 + B_3\alpha), \quad \forall x \in M.
\]

where the constants \(C, C_2, C_3, B_2, B_3\) are of order 1.

Proof: It follows from proposition (9.4) and compactness of \(\bigcup_{x \in M} \text{supp}(f_x)\).

9.2 Comparison Between the Geodesics of \(L^\nabla\) and \(< L^\nabla >\)

There are several ways of defining the energy of a bunch of particles. We have choose one which will be useful to get the desired comparison. We define the energy function \(E\) of a distribution \(f\) to be the real function:

\[
E : M \rightarrow \mathbb{R}
\]

\[
x \mapsto E(x) := \inf\{y_0, y \in \text{supp}(f_x)\},
\]

where \(y_0\) is the 0-component of a tangent vector of a possible trajectory of a point particle, measured in the laboratory coordinate frame.

Let us restrict ourselves to the case that the Lorentzian metric is the Minkowski metric in dimension \(n\). Let us denote by \(\bar{\gamma}(t)\) the gamma factor of the Lorentz transformation from the local frame, defined by the vector field \(U\) to the laboratory frame, at the instant \(t\) defined by the local time, the coordinate time defined by the laboratory frame. Let us define \(\theta^2(t) = \bar{y}^2(t) - < \bar{y}^2 > (t)\) and \(\bar{\theta}^2(t) = < \bar{y}^2 > (t) - \bar{y}^2(t)\). Here \(\bar{y}(t)\) is the velocity tangent vector field along a solution of the Lorentz force equation and \(\bar{y}(t)\) is the spatial component of the tangent vector field along a solution of the averaged equation, with both solutions having the same initial conditions. The maximal values of these quantities on the compact space-time manifold are denoted by \(\theta^2\) and \(\bar{\theta}^2\).

Theorem 9.6 Let \((M, \eta, [A])\) be a semi-Randers space and \(\eta\) the Minkowski metric. Let us assume that:

1. The auto-parallel curves of unit velocity of the connections \(L^\nabla\) and \(< L^\nabla >\) are defined for time \(t\), the time coordinate measured in the laboratory frame.

2. The ultra-relativistic limit holds: \(E(x) \gg 1\) for all \(x \in M\).

3. The distribution function is narrow in the sense that \(\alpha \ll 1\) for all \(x \in M\).

4. The following inequality holds:

\[
|\theta^2 - \bar{\theta}^2| \ll 1,
\]

5. The support of the distribution function \(f\) is invariant under the flow of the Lorentz force equation.

6. The change in the energy function is adiabatic: \(\frac{d}{dt} \log E \ll 1\).
Then for the same arbitrary initial condition \((x(0), \dot{x}(0))\), the solutions of the equations
\[
L \nabla x \dot{x} = 0, \quad < L \nabla > \dot{x} = 0
\]
are such that:
\[
\| \ddot{x}(t) - x(t) \| \leq 2(C(x)\|F\|(x) + C_2^2(x)(1 + B_2(x)\alpha))\alpha^2 E^{-2}(x) t^2,
\]
where the functions \(C(x), C_1(x)\) and \(B_2(x)\) are bounded by constants of order 1 and the norm \(\| \cdot \|\) is the spatial norm in the lab. frame.

**Proof:** At the instant \(t\) we calculate the distance measured in the laboratory frame between \(x(t)\) and \(\ddot{x}(t)\), solutions of the geodesic equations of the corresponding connections \(L \nabla\) and \(< L \nabla >\), when both geodesics have the same initial conditions \((x(0), \dot{x}(0))\). For this, we use the general formula for the solution:
\[
x^i(t) = x^i(0) + \int_0^t ds \dot{x}(0) + \int_0^t ds \ddot{x}(l).
\]
Since the initial conditions for both connections are the same, the equivalent relation for the geodesics of the averaged connection is
\[
\ddot{x}^i(t) = x^i(0) + \int_0^t ds \dot{x}(0) + \int_0^t ds \ddot{x}(l).
\]
We have to estimate the distance between both solutions. Since we know the distance between both connections, it is possible to give a natural bound for the distance between solutions. The main property is the smoothness theorem on the dependence of solutions of differential equations on the external parameters (for instance [20, Appendix 1] or [30, chapter 1]).

Let us consider the family of connections depending on the distance between the two connections
\[
\xi_{\text{max}} = d_{\eta}(L \nabla, < L \nabla >)
\]
given by the convex sum:
\[
\xi \nabla := \frac{1}{\xi_{\text{max}}} (\xi_{\text{max}} - \xi) L \nabla + \frac{1}{\xi_{\text{max}}} \xi < L \nabla >, \quad \xi \in [0, \xi_{\text{max}}].
\]
For \(\xi = 0\) one has \(\xi \nabla = L \nabla\), while for \(\xi = \xi_{\text{max}}\) one has the averaged connection. Using the result of the smoothness of the solutions of the differential equations, one can expand the solution \(\xi \dot{x}^i\) of the geodesic equation for the connection with parameter \(\xi\) in the following way.

In particular, the second derivative with respect to the coordinate time \(t\) reads:
\[
\xi \ddot{x}^i = \dot{0} \dot{x}^i + (\partial_\xi \xi \ddot{x}^i)_{\xi=0} \cdot \xi + \mathcal{O}(\xi^2).
\]
We need to evaluate the derivative \((\partial_\xi \xi \ddot{x}^i)_{\xi=0}\). From the formula (4.11.14), one obtains:
\[
(\partial_\xi \xi \ddot{x}^i)_{\xi=0} = \frac{1}{\xi_{\text{max}}} \cdot (L \nabla_x \dot{x} - < L \nabla_x \dot{x} >),
\]
where \(\dot{x}(t)\) is the solution of \(L \nabla_x \dot{x} = 0\) with the given initial conditions. This is because the support of the distribution function \(f\) is invariant under the flow of the Lorentz force equation. Therefore, since we are dividing by the distance \(\xi_{\text{max}}\), the derivative is such that its norm is 1:
\[
\|(\partial_\xi \xi \ddot{x})\|_{\xi=0} = \frac{1}{\xi_{\text{max}}} \cdot \|(L \nabla_x \dot{x} - < L \nabla_x \dot{x} >)\|_{\eta} = 1,
\]
because of the definition of $\xi_{mas}$ and the formula (4.6.14). On the other hand the relations between proper times and coordinate time in the laboratory frame are

$$d\tau = \gamma^{-1} dt, \quad d\tilde{\tau} = \tilde{\gamma}^{-1} dt.$$  

This implies the following relation between derivatives:

$$\frac{d}{dt} = \frac{\gamma^{-1}}{d\tau} \frac{d}{d\tau}, \quad \frac{d}{dt} = \tilde{\gamma}^{-1} \frac{d}{d\tilde{\tau}}.$$  

Using the hypotheses $|\theta^2(t) - \bar{\theta}^2(t)| << 1$ and $\frac{d}{dt}\log E << 1$, one obtains the following relation,

$$\|\tilde{x}(t) - x(t)\|_{\bar{\eta}} \leq 2t \int_0^t dt E^{-2} \| \frac{d^2 \tilde{x}(l)}{dl^2} - \frac{d^2 x(l)}{dl^2}\|_{\bar{\eta}}.$$  

The factor 2 comes from the bound of the term where appears the derivative of the energy. We have use the adibatic hypothesis.

Therefore, in the ultra-relativistic regime,

$$\|\tilde{x}(t) - x(t)\|_{\bar{\eta}} \leq 2(\alpha \| F \|_{\bar{\eta}}(x) + C_2^2(x) + B_2(x) \alpha) \alpha^2 \tilde{\gamma}^{-1}(x) E^{-2}(x) t^2.$$  

The norm of the electromagnetic field also changes when it transforms from one coordinate system to another coordinate system. The way of changing is given by a Lorentz transformation, which gamma factor is $\tilde{\gamma}$. Therefore:

$$\| F \| = \tilde{\gamma} \| F \|_{\bar{\eta}}.$$  

Finally we get,

$$\|\tilde{x}(t) - x(t)\| \leq 2(C \| F \|_{\bar{\eta}}(x) + C_2^2(x)(1 + B_2(x) \alpha)) \alpha^2 E^{-2}(x) t^2.$$  

**Remark.** It is important to notice that the above bound is Lorentz covariant. Although we have performed the calculation in a particular local coordinate frame, both the left and right sides of (9.10) obey the same transformation law under Lorentz transformations. For instance, if we express this bound in a coordinate system in the co-moving frame of the bunch, one has that:

$$\|\tilde{x}(t) - x(t)\|_{\bar{\eta}} \leq 2(C \| F \|_{\bar{\eta}}(x) + C_2^2(x)(1 + B_2(x) \alpha)) \alpha^2 m^{-2} \tilde{t}^2,$$

for a mass $m = 1$ in the units we use. One can check that formula (9.11) is Lorentz invariant by direct inspection of the transformation rule of each of the factors that appear there:

1. $\alpha$ is Lorentz invariant.
2. The product $E^{-2}\tilde{t}^2$ is Lorentz invariant.
3. The functions \( C(x), C_1(x) \) and \( B_1(x) \) are Lorentz invariant.

4. The function \( \|F\|_{\bar{\eta}} \) transforms in the same way as \( \|\bar{x}(t) - x(t)\|_{\bar{\eta}}^{-1} \) under Lorentz transformations.

5. The condition \( \frac{d\log E}{dt} \ll 1 \) does not appear explicitly in expression (9.10). Therefore we do not need to check it explicitly.

**Corollary 9.7** Let \((M, \bar{\eta})\) be as before and such that there is a global bound for \( \|F\|_{\bar{\eta}}(x) \leq \|F\|_{\bar{\eta}} < \infty \), the energy function is bounded from below by a constant \( E \) and the curves \( x(t) \) and \( \bar{x}(t) \) are compact. Then there are some constants \( C_i \) such that \( C_i(x) < C_i \) and the following relation holds
\[
\|\bar{x}(t) - x(t)\| \leq 2(C(x)\|F\|(x) + C^2(1 + \alpha)) \alpha^2 E^{-2} t^2. \tag{9.15}
\]

**Remarks**

1. In the above result the 2-form \( F \) is physically interpreted as the Faraday form.

2. These bounds happen at least in two physical scenarios:
   
   (a) That the manifold \( M \) is compact. In this case, one has to consider spatial boundaries, since it is well-known that if the space-time is compact, they exists closed time-like curve ([34, pg 58]). This can broke up causality.

   (b) The trajectories that we consider are compact. In this case, one can define an effective compact space-time manifold with a spurious boundary and apply the first case, although excluding closed time-like curves, which do not happen in physical situations.

3. The external field \( F \) does not depend on the energy of the beam of particles. However, this is not necessarily the case in some situations. For instance, there are effects which could be reduce the beam size (adiabatic damping and Landau Damping [10]). If this happens, there is a strong reduction of the size in energy and momenta of the beam. This implies that one can describe this as an effective exponent \( E^{-2+\beta} \), with \( \beta \) negative.

4. That the curves \( x(t) \) and \( \bar{x}(t) \) have compact image has a physical interpretation: all the trajectories start in a source region and finish in a target region.

**Theorem 9.8** Under the same hypothesis as in theorem 4.6.6, the difference between the tangent vectors is given by
\[
\|\dot{x}(t) - \dot{\bar{x}}(t)\| \leq \left( K(x)\|F\|(x) + K^2(1 + D_2(x)\alpha)\alpha^2 \right) E^{-1} t, \tag{9.16}
\]

with \( K_i \) and \( D_2(x) \) functions bounded by constants of order 1.

**Proof:** The proof of this theorem is similar to the proof of Theorem (9.6), although based on the following formula for the tangent velocity field along a curve:
\[
\dot{x}(t) = \dot{x}(0) + \int_0^t \dot{x}(l)dl. \tag{9.17}
\]

By the same argument the relation (4.6.16) is Lorentz invariant.

**Corollary 9.9** Under the same hypothesis than in Corollary 9.7, there are some constants of order 1, \( K, K_i \) and \( D_i \), such that \( K(x) \leq K, K_i(x) \leq K_i, D_i(x) \leq D_i \) and the following relation holds:
\[
\|\dot{x}(t) - \dot{\bar{x}}(t)\| \leq \left( K\|F\|(x) + K^2(1 + D_2(x)\alpha)\alpha^2 \right) E^{-1} t, \tag{9.18}
\]

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10 Applications of the Averaged Lorentz Dynamics and Discussion

*Theorem 9.6* and *Theorem 9.8* are proved under some hypotheses which have to be motivated from a mathematical or physical point of view.

1. The first hypothesis asserts the existence of solutions of some geodesic equations at least in an interval of time $[0, T] \ni T$. If the space-time manifold $\mathbf{M}$ is compact and with boundary, this hypothesis is strong, since there is a time $T$ where any geodesics finish in a boundary. If we are in a situation such that the solutions of the auto-parallel equations are compact sub-sets on $\mathbf{M}$, we must guarantees the existence of the solution up to time $t \leq T$.

2. Ultra-relativistic dynamics implies (as an assumption) that the minimal value of $y^0$ on the support of the distribution is much larger as 1. The definiton of ultra-relativistic limit is a gauge invariant notion and it is also independent of the frame of reference.

3. The condition that states that $|\theta^2(t) - \bar{\theta}^2(t)| \ll 1$. If the motion is ultra-relativistic this condition implies $\|\dot{x}\|_{\eta}^2 = 1 + \epsilon$, with $\epsilon \ll 1$. Therefore the above hypothesis is the statement that the integral curves corresponding to the averaged connection are in the ultra-relativistic regime too.

4. A similar interpretation for the narrowness condition is clear: the diameter of the distribution is much smaller than the energy of the system. The condition of narrowness is gauge and Lorentz invariant. We can compare our definition of narrowness (which is that $E(x) >> \alpha$) with the invariant warm fluid condition, which is that $<\hat{y}_k - \bar{\hat{y}}_k, \hat{y}_j - \bar{\hat{y}}_j> < \eta_{jk} >> 1$.

5. Our condition $|\theta^2(t) - \bar{\theta}^2(t)| \ll 1$ is equivalent to the invariant warm fluid condition, because $<\hat{y}_k - \bar{\hat{y}}_k, \hat{y}_j - \bar{\hat{y}}_j> = |\theta^2(t) - \bar{\theta}^2(t)|$ when everything is applied on the unit hyperboloid.

6. The hypothesis on the invariance of $\text{supp}(f)$ by the flow of the Lorentz force equation guarantees that the distribution functions that we consider are related with the Lorentz force equation and therefore that they are physically significant. One possible relation is through the Vlasov equation, which is an equation describing a distribution function in kinetic theory [18]. In Vlasov’s model, the one-particle probability distribution function $f$ is constant along the flow of the Lorentz force. From this perspective, this is the most simple case, since it could be that $\text{supp}(f)$ is invariant by the flow but not constant. Therefore, our theorems (9.6) and (9.8) apply to more general kinetic models.

There are two situations where the results of section 9 are applicable:

1. The ultra-relativistic regime, $E \to \infty$. In this case, $\|x(t) - \bar{x}(t)\| \to 0$ for distributions with finite $\alpha$ and for finite times $t$. The requirement of $|\theta^2 - \bar{\theta}^2| \ll 1$ is naturally fulfilled for common distributions and in particular for smoothly truncated Gaussian distributions.
2. For the Dirac delta distribution:

\[ f(x, y) = \delta(y - V(x))\Psi(x), \quad x \in M. \] (10.1)

This delta distribution function corresponds to the charged cold fluid model solution of the Vlasov equation.

Since the width in the space of tangent velocities of the distribution is zero, \( \alpha = 0 \). The trajectories, however, are constrained to follow the Lorentz force equation, since by hypothesis the support of the distribution function is invariant under the flow of the Lorentz force equation. Note that in this example, the distribution is not smooth. However, one can check that the Lorentz and averaged Lorentz connections coincide. This fact suggests that we can extend our results to a bigger category of functional spaces for the distribution function \( f \).

Let us discuss the limits of applicability of the averaged dynamics and in particular of the formula (9.10). We make the assumption that during the time of evolution the \( \gamma \) factors are constant. This is not realistic in general (since it can happen acceleration of the beam of particles) but it is useful in order to provide an estimate of the range of validity of the time \( t \) of the approximating the Lorentz dynamics by the averaged dynamics. Therefore, equation (9.11) takes the form

\[ \|\tilde{x}(t) - x(t)\| \leq C \alpha^2 E^{-2}(t_0)\|F\| t^2. \]

Let us assume a natural maximal distance \( L_0 \) between two particles in a bunch. For instance, in an accelerator machine \( L_0 \) could be the diameter of the pipe. The averaged model is applicable until the difference \( \|\tilde{x}(t) - x(t)\| \) is of order \( L_0 \). This can start to happen after a time evolution such that the difference \( \|\tilde{x}(t) - x(t)\| = L_0 \). The characteristic time where the averaged model loses validity is:

\[ t_{max} \sim E \left( \frac{1}{\alpha} \frac{L_0}{C\|F\|} \right)^{\frac{1}{2}}. \] (10.2)

**Corollary 10.1** In the limit \( E \to \infty, \frac{d}{dt}\log E << 1 \) and (or) the limit \( \alpha \to 0 \) if all the other parameters remain finite, \( t_{max} \to \infty \).

**Applications of the Averaged Lorentz Dynamics**

There are some applications of the averaged Lorentz force equation:

1. The solutions of the averaged dynamics can be used as a definition of the reference trajectory in beam dynamics. There are two reasons for adopting this perspective:

   (a) As Theorem 9.6 states, for the same initial conditions, in the ultra-relativistic limit and for narrow probability distributions, the difference between the original trajectory and the averaged trajectory is small.

   (b) For narrow distributions in the ultra-relativistic limit, the integral curves of the averaged velocity field coincide at least up to order \( \alpha^2 \) with auto-parallel vector field of the averaged connection. This is proved in the companion paper [9].

The reference trajectory is a fundamental notion of beam dynamics [10]. It is usually understood that the reference trajectory does not necessarily coincide with the trajectory of a real particle. Our criticism of this idea comes from the point that in this case, the reference trajectory is not observable. Therefore, referring all the optic design and other
parameters of an accelerator machine to this trajectory can be a dangerous strategy in extreme high density and energy conditions, since one is assuming an ideal behavior which in principle can or cannot be the right description of the dynamics in such situations.

Based on this, we propose that the reference trajectory is directly linked with a solution of the averaged Lorentz dynamics. This method has the advantage that because the solutions of the averaged dynamics are observable, its solutions can be directly tested and experimentally controlled.

2. The averaged dynamics is a fundamental tool in a new justification of the use of fluid models as an approximation of a kinetic model. There are two reasons for this:

   (a) It is a general belief in beam dynamics and plasma physics that fluid models appear from kinetic theory as an averaged description (for instance a standard reference is [11], [29] and [31]). However, derivations are based on assumptions on the higher order moments of the distribution function. These are hypotheses which can or cannot hold at very extreme conditions. We have approach that problem without making hypothesis on the higher moments.

   (b) Since the averaged Lorentz connection is affine and symmetric, one can make use of normal coordinates to simplify some calculations. This technical point is fundamental in our approach to the problem. Indeed, this is a consequence of the fact that an averaged dynamics is technically simple compared with the original dynamics.

3. In order to write down the geodesic equations of the averaged dynamics, one only needs the first, second and third order moments. This makes the averaged dynamics insensitive to higher order moments of the original distribution $f$. We think this is an advance which respect to standard treatments [11], [29], [31]. The approximation of the Lorentz dynamics by the averaged dynamics requires narrowness of the distribution function and that the process happens in the ultra-relativistic regime. Since the averaged model is insensitive to higher moments, it is not harmful to assume that moments of higher order as three are zero, as soon as one proves this is consistent with the premises of the approximation. We think that this is a general principle by which to understand the validity of fluids models as an approximate description of kinetic models.

4. The averaged dynamics defined by the averaged connection $< L \nabla >$ can be used in numerical simulations of the bunch dynamics in accelerator physics. Since the averaged equation is simpler, it can be used to improve the calculational speed in simulations of the evolution of bunches composed by a large number of charged particles. This is an idea, however, that still needs to be tested.

Relation of this work to previous work.

The averaging procedure used in this work was first introduced in reference [7] by the author, although in the context of Finsler geometry and formulated using a geometric formalism. In this paper we have adopted a local description using frames and local coordinates.

At first glance, there is a resemblance between the averaged procedure presented here and the procedure discovered by Zalaletdinov and co-workers [21]. However we think that these procedures are intrinsically different, although unfortunately the use of a common notation and terminology makes it easy to confuse them. In fact, the averaging considered by those authors is very different from the one here and also the one appearing in [7]; their averaging procedure is taken on open sets of the space-time itself, while our averaging procedure is an integration along the fiber over a pull-back bundle over the unit hyperboloid. In local coordinates, it corresponds
to integration on the space of allowed velocities, at fixed point of the space-time. Indeed, our
procedure has some resemblance to the averaging along the fiber in classical mechanics [22] or
the averaging along the fiber that appears in algebraic topology, through a generalization of
Poincaré’s lemma [23]. These two examples have been formalized in a general framework [8].

In this work we are considering a geometrization of the dynamics of charged point particles
with external fields and not a geometrization of the whole electromagnetic theory. Indeed, one
of the main points discussed in this work is that the natural interpretation of the classical
electrodynamics of point charged point particles in an external electromagnetic field does not
 correspond to a space-time metric structure. There are theories which are a geometrization of
Classical Electrodynamics and where Finsler and Lagrangian structures appear in a natural
way. For such theories, one can see, for example, the approach due to R. Miron and co-workers
[16], [24]. However, the aim of the present work is very different and must not be confuse with
the kind of theories developed by these authors.

There exists a previous geometrization of Lorentz force equation discovered by Miron and
co-workers [37]. The connection that they define coincides with our ∇ connection. We have
seen that both are affine equivalent connections (ones the parameter has been fixed to be the
proper-time parameter associated with ¯η). However, we have not yet clarified what is the relation
between the two approaches.

11 Notion of Almost-Connection

During our analysis of the Lorentz force equation, we have considered the associated non-linear
connection on T N and the associated linear connections on π∗TM → Σ. However, strictly
speaking the system of differential equations (1.1) does not define a non-linear connection on
T Σ. The reason is the following. Let us fix a point u ∈ Σ. There exists a natural embedding
e : Σ ֒→ TM. Then one can consider the sub-bundle e(Σ) and the  covariant derivative LD acting on elements of Γ T e(Σ). This covariant derivative can be extended to derive sections of the extended bundle ∪ u∈Σ T uN, with is a sub-bundle of TM:

\[ L\hat{D} : \Gamma(\bigcup_{u\in \Sigma} T_uN) \times \Gamma(\bigcup_{u\in \Sigma} T_uN) \rightarrow \Gamma(\bigcup_{u\in \Sigma} T_uN), \]

\[ L\hat{D}\hat{X}\hat{Y} = \hat{X}^i \frac{\partial\hat{Y}^j}{\partial x^i} \frac{\partial}{\partial x^j} + L\hat{\Gamma}^j_{ik} \hat{X}^k \frac{\partial}{\partial x^i}, \quad \hat{X}, \hat{Y} \in \Gamma(\bigcup_{u\in \mathcal{N}} T_uTM). \]

From the definition of LD, this operator cannot be extended in a smooth and natural way to be a covariant derivative on TM. This is because of the appearance of factors η(y, y) in the
connection coefficients: the function \( \sqrt{\eta(y, y)} \) is not defined in an open neighborhood. The
non-extendibility of this function is the problem to extend the covariant derivative along those
directions.

The above fact suggests the existence of a mathematical object which is a generalization of the
ordinary notion of covariant derivative in the sense that allows covariant derivatives along outer
directions to the manifold, but that are not obtained as a restriction of an ambient covariant
derivative.

Example. The Lorentz connection was obtained from the Lorentz force equation assuming
that the torsion is zero. We obtained the following connection coefficients:

\[ L\hat{\Gamma}^i_{jk} = \eta\hat{\Gamma}^i_{jk} + T^i_{jk} + L^i_{jk}, \]
\[ L^i_{jk} = \frac{1}{2\eta(y,y)}(F^i_{jk}y^m\eta_{mk} + F^i_{jkm}\eta_{mj}), \]
\[ T^i_{jk} = F^i_{m} \frac{y^m}{\sqrt{\eta(y,y)}}(\eta_{jk} - \frac{1}{\eta(y,y)}\eta_{js}\eta_{kl}y^s y^l). \]

This defines a rule to derive sections of \( T\Sigma \) along directions of \( T\Sigma \). The same coefficients provide a rule to derive sections of \( \bigcup_{u \in \Sigma} T_u TM \) along directions of \( TM \), but we can extend the definition of the covariant derivative to the whole manifold \( TM \).

A related issue is the following. On the unit hyperboloid recall that \( T^i_{jk}y^j y^k = 0 \) (this is why it was called *transverse component*). Therefore, if one considers instead an alternative connection \( L\tilde{\nabla} \), determined by the connection coefficients

\[ L\tilde{\Gamma}^i_{jk} = \eta\Gamma^i_{jk} + L^i_{jk}, \]

the corresponding geodesic equation (parameterized by the proper time of the Lorentzian metric \( \eta \)) is again the Lorentz force equation [37]. Therefore we see that both \( L\nabla \) and \( \tilde{\nabla} \) reproduce the Lorentz force and both are torsion-free (the connection coefficients are symmetric in the lower indices). This is in contradiction with the fact that a connection of Berwald type (linear or non-linear) is determined by the set of all geodesics as parameterized curves on the base manifold \( M \) and the torsion tensor (for linear connections the procedure can be seen in [17]; for non-linear connections, a procedure to define the connection is described for example in [16].

A solution to this dilemma comes from the fact that the Lorentz force equation applies only to time-like trajectories for which the tangent velocity vector fields live on the unit hyperboloid \( \Sigma \); that is, we cannot extend the operator

\[ L\tilde{D} : \Gamma(\bigcup_{u \in \Sigma} T_u TM) \times \Gamma(\bigcup_{u \in \Sigma} T_u TM) \rightarrow \Gamma(\bigcup_{u \in \Sigma} T_u TM) \]

to a genuine operator of the form

\[ L\tilde{D} : \Gamma(TM) \times \Gamma(TM) \rightarrow \Gamma(TM). \]

In other words, we cannot extract enough information from the Lorentz force equation to determine a non-linear *projective connection* ([38]), because on the base manifold, we do not have information about all the possible geodesics. This is why we can not extend the connection.

One can be tempted to extend the Lorentz force equation to another equation valid for any kind of trajectory. This is partially accomplished by the averaged connection. However, these kind of extensions could be non-natural or unique and indeed hide a natural object such as the *almost-connection*.

**Preliminary definition of Almost-Connection**

Let \( M \) be a manifold of dimension \( n \). An almost-connection is a second order vector field \( \chi \) defined on a sub-bundle \( D \hookrightarrow TTM \), with \( D \) of arbitrary rank. Associated with \( D \) is the corresponding almost projective covariant derivative on \( \pi^*TM \).

**Examples.**

1. A projective connection (in the sense of Cartan [38]) is an almost connection such that \( D = TM \).

2. The Lorentz connection \( L\nabla \) provides an example where \( \Sigma = D \neq TM \). One can define a *Koszul connection* acting on \( \Gamma(\bigcup_{u \in \Sigma} T_u TM) \).
3. The averaged covariant derivative $<L\nabla>$ is an affine connection and therefore an almost connection in the above sense.

One can consider the corresponding linear connections $L\nabla$ and $\tilde{L}\nabla$ on $\pi^*TM$. Generally, their averaged connections $<L\nabla>$ and $<\tilde{L}\nabla>$ are not the same. However, we would like to have an averaged operation which is well defined for the objects in a given category. By definition this will be the category of almost-connections and the corresponding morphisms. We require that the result of the averaging operation be the same for each representative belonging to the same almost connection.

**Definition 11.1** Let $M$ be a manifold of dimension $n$. An almost non-linear connection is the maximal set of semi-sprays $\chi$ defined on a sub-bundle $D \subset TTM$ such that they have the same averaged linear covariant derivative $<\nabla>$ and the same torsion tensor:

$$\hat{T}(\hat{X}, \hat{Y}) := L:\hat{D}_X\hat{Y} - L:\hat{D}_Y\hat{X} - [\hat{X}, \hat{Y}], \quad \hat{X}, \hat{Y} \in \Gamma D. \quad (11.1)$$

Associated with $\chi$ over $D$ is the corresponding almost-covariant derivative on the bundle $\pi^*TM \to D$. Any of the connections in the same almost-connection has the same averaged connection. The distance function (9.3) can also be defined for almost-connections.

The general properties of almost connections are being explored in another separated work. Some of these properties are based on straightforward generalizations of the quantities associated with Koszul connections. For instance, the generalization of the curvature tensor is:

$$\hat{R}(\hat{X}, \hat{Y}, \hat{Z}) := L:\hat{D}_X L:\hat{D}_Y \hat{Z} - L:\hat{D}_Y L:\hat{D}_X \hat{Z} - L:\hat{D}_{[\hat{X}, \hat{Y}]} \hat{Z}, \quad \hat{X}, \hat{Y}, \hat{Z} \in \Gamma D. \quad (11.2)$$

The notion of auto-parallel curve of an almost-connection is not so easy to handle for almost-connections, since in general there will be initial conditions such that the auto-parallel curve goes out from the sub-bundle $D$. Indeed, for auto-parallel curves whose initial velocity vector is not on $TD$, the trajectory goes out from $D$ after any finite time.

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