EXTREMAL KÄHLER METRICS ON PROJECTIVE BUNDLES OVER A CURVE

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Abstract. Let $M = P(E)$ be the complex manifold underlying the total space of the projectivization of a holomorphic vector bundle $E \to \Sigma$ over a compact complex curve $\Sigma$ of genus $\geq 2$. Building on ideas of Fujiki [28], we prove that $M$ admits a Kähler metric of constant scalar curvature if and only if $E$ is polystable. We also address the more general existence problem of extremal Kähler metrics on such bundles and prove that the splitting of $E$ as a direct sum of stable subbundles is necessary and sufficient condition for the existence of extremal Kähler metrics in sufficiently small Kähler classes. The methods used to prove the above results apply to a wider class of manifolds, called rigid toric bundles over a semisimple base, which are fibrations associated to a principal torus bundle over a product of constant scalar curvature Kähler manifolds with fibres isomorphic to a given toric Kähler variety. We discuss various ramifications of our approach to this class of manifolds.

1. Introduction

Extremal Kähler metrics were first introduced and studied by E. Calabi in [13, 14]. Let $(M, J)$ denote a connected compact complex manifold of complex dimension $m$. A Kähler metric $g$ on $(M, J)$, with Kähler form $\omega = g(J\cdot, \cdot)$, is extremal if it is a critical point of the functional $g \mapsto \int_M s_g^2 \omega^n / m!$, where $g$ runs over the set of all Kähler metrics on $(M, J)$ within a fixed Kähler class $\Omega = [\omega]$, and $s_g$ denotes the scalar curvature of $g$. As shown in [13], $g$ is extremal if and only if the symplectic gradient $K := \nabla g s_g = J \nabla g s_g$ of $s_g$ is a Killing vector field (i.e. $L_K g = 0$) or, equivalently, a (real) holomorphic vector field (i.e. $L_K J = 0$). Extremal Kähler metrics include Kähler metrics of constant scalar curvature — CSC Kähler metrics for short — in particular Kähler–Einstein metrics. Clearly, if the identity component $\text{Aut}_0(M, J)$ of the automorphism group of $(M, J)$ is reduced to $\{1\}$, i.e. if $(M, J)$ has no non-trivial holomorphic vector fields, any extremal Kähler metric is CSC, whereas a CSC Kähler metric is Kähler–Einstein if and only if $\Omega$ is a multiple of the (real) first Chern class $c_1(M, J)$. In this paper, except for Theorem 1 below, we will be mainly concerned with extremal Kähler metrics of non-constant scalar curvature.

The Lichnerowicz–Matsushima theorem provides an obstruction to the existence of CSC Kähler metrics on $(M, J)$ in terms of the structure of $\text{Aut}_0(M, J)$, which must be reductive whenever $(M, J)$ admits a CSC Kähler metric; in particular, for any CSC Kähler metric $g$, the identity component $\text{Isom}_0(M, g)$ of the group of isometries of $(M, g)$ is a maximal compact subgroup of $(M, J)$ [55, 19]. The latter fact remains true for any extremal Kähler metric (although $\text{Aut}_0(M, J)$ is then no longer reductive in general) and is again an obstruction to the existence of extremal Kähler metrics [14, 48]. Another well-known obstruction to the existence of CSC Kähler metrics within a given class $\Omega$ involves the Futaki character [30, 14], of which a symplectic version, as developed in [47], will be used in this paper (cf. Lemma 2). Furthermore, it is now known that extremal

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Kähler metrics within a fixed Kähler class $\Omega$ are unique up to the action of the reduced automorphism group $\tilde{\text{Aut}}_0(M, J)$.

It was suggested by S.-T. Yau [73] that a complete obstruction to the existence of extremal Kähler metrics in the Kähler class $\Omega = c_1(L)/2\pi$ on a projective manifold $(M, J)$ polarized by an ample holomorphic line bundle $L$ should be expressed in terms of stability of the pair $(M, L)$. The currently accepted notion of stability is the $K$-(poly)stability introduced by G. Tian [68] for Fano manifolds and by S. K. Donaldson [21] for general projective manifolds polarized by $L$. The Yau–Tian–Donaldson conjecture can then be stated as follows. A polarized projective manifold $(M, L)$ admits a CSC Kähler metric if and only if it is $K$-polystable. This conjecture is still open, but the implication ‘CSC $\Rightarrow$ K-polystable’ in the conjecture is now well-established, thanks to work by S. K. Donaldson [20], X. Chen–G. Tian [15], J. Stoppa [64], and T. Mabuchi [53, 54]. In order to account for extremal Kähler metrics of non-constant scalar curvature, G. Székelyhidi introduced [66, 67] the notion of relative $K$-(poly)stability with respect to a maximal torus of the automorphism group of the pair $(M, L)$ — which is the same as the reduced automorphism group $\tilde{\text{Aut}}_0(M, J)$ — and the similar implication ‘extremal $\Rightarrow$ relatively K-polystable’ was recently established by G. Székelyhidi–J. Stoppa [65].

The Yau–Tian–Donaldson conjecture was inspired by and can be regarded as a non-linear counterpart of the well-known equivalence for a holomorphic vector bundle over a compact Kähler manifold $(M, J, \omega)$ to be polystable with respect to $\omega$ on the one hand and to admit a hermitian–Einstein metric [40, 72, 18] on the other. In the case when $(M, J)$ is a Riemann surface this is the celebrated theorem of Narasimhan and Seshadri [56], which, in the geometric formulation given in [9, 17, 28] can be stated as follows. Let $E$ be a holomorphic vector bundle over a compact Riemann surface $\Sigma$. Then, $E$ is polystable if and only if it admits a hermitian metric whose Chern connection is projectively-flat.

This paper is mainly concerned with the existence of extremal Kähler metrics on ruled manifolds, i.e. when $(M, J)$ is the total space of projective fibre bundles $P(E)$ where $E$ is a holomorphic vector bundle over a compact Kähler manifold $(M, J, \omega)$ to be polystable with respect to $\omega$ on the one hand and to admit a hermitian–Einstein metric [40, 72, 18] on the other. In the case when $(M, J)$ is a Riemann surface this is the celebrated theorem of Narasimhan and Seshadri [56], which, in the geometric formulation given in [9, 17, 28] can be stated as follows. Let $E$ be a holomorphic vector bundle over a compact Riemann surface $\Sigma$. Then, $E$ is polystable if and only if it admits a hermitian metric whose Chern connection is projectively-flat.

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While we principally focus on projective bundles, for which sharper results can be obtained (Theorems 1, 2, 4 below), the techniques and a number of results presented in this paper actually address a much wider class of manifolds, called rigid toric bundles.

$\tilde{\text{Aut}}_0(M, J)$ is the unique linear algebraic subgroup of $\text{Aut}_0(M, J)$ such that the quotient $\text{Aut}_0(M, J)/\tilde{\text{Aut}}_0(M, J)$ is a torus, namely the Albanese torus of $(M, J)$ [27]; its Lie algebra is the space of (real) holomorphic vector fields whose zero-set is non-empty [27, 40, 45, 52].
over a semisimple base, which were introduced in our previous paper \cite{4}. Section 3 of this paper is devoted to recalling the main features of this class of manifolds and proving a general existence theorem (Theorem 3).

The simplest situation considered in this paper is the case of a projective bundle over a curve, i.e. a compact Riemann surface. In this case, the existence problem for CSC Kähler metrics can be resolved.

**Theorem 1.** Let \((M, J) = P(E)\) be a holomorphic projective bundle over a compact complex curve of genus \(\geq 2\). Then \((M, J)\) admits a CSC Kähler metric in some (and hence any) Kähler class if and only if the underlying holomorphic vector bundle \(E\) is polystable.

**Remark 1.** The ‘if’ part follows from the theorem of Narasimhan and Seshadri: if \(E\) is a polystable bundle of rank \(m\) over a compact curve (of any genus), then \(E\) admits a hermitian–Einstein metric which in turn defines a flat \(PU(m)\)-structure on \(P(E)\) and, therefore, a family of locally-symmetric CSC Kähler exhausting the Kähler cone of \(P(E)\), see e.g. \cite{40}, \cite{28}. Note also that in the case when \(P(E)\) fibres over \(\mathbb{C}P^1\), \(E\) splits as a direct sum of line bundles, and the conclusion of Theorem 1 still holds by the Lichnerowicz–Matsushima theorem, see e.g. \cite{1} Prop. 3.

**Remark 2.** On all manifolds considered in Theorem 1 rational Kähler classes form a dense subset in the Kähler cone. By LeBrun–Simanca stability theorem \cite[Thm. A]{16} and Lemma 3 below it is then sufficient to consider the existence problem only for an integral Kähler class (or polarization). In this setting, it was shown by Ross–Thomas that any projective bundle \(M = P(E)\) over a compact complex curve of genus \(\geq 1\) is K-polystable (with respect to some polarization) if and only if \(E\) is poly(semi)stable \cite[Thm. 5.13]{61}. In view of this theorem, the “only if” part of Theorem 1 can therefore be alternatively recovered — for any genus \(\geq 1\) — as a consequence of recent papers by T. Mabuchi \cite{53} \cite{54}.

By the de Rham decomposition theorem, an equivalent differential geometric formulation of Theorem 1 is that any CSC Kähler metric on \((M, J) = P(E)\) must be locally symmetric (see \cite[Lemma 8]{28} and \cite{14}). It is in this form that we are going to achieve our proof of Theorem 1 building on the work of A. Fujiki \cite{28}. In fact, \cite{28} already proves Theorem 1 in the case when the underlying bundle \(E\) is simple, modulo the uniqueness of CSC Kähler metrics, which is now known \cite{15} \cite{20} \cite{52}.

In view of this, the main technical difficulty in proving Theorem 1 is related to the existence of automorphisms on \((M, J) = P(E)\) \(\rightarrow \Sigma\). The way we proceed is by fixing a maximal torus \(T\) (of dimension \(\ell\)) in the identity component \(\text{Aut}_0(M, J)\) of the automorphism group, and showing that it induces a decomposition of \(E = \bigoplus_{i=0}^{\ell} E_i\) as a direct sum of \(\ell + 1\) indecomposable subbundles \(E_i\), such that \(T\) acts by scalar multiplication on each \(E_i\) (see Lemma 1 below). By computing the Futaki invariant of the \(S^1\) generators of \(T\), we show that the slopes of \(E_i\) must be all equal, should a CSC Kähler metric exist on \(P(E)\) (see Lemma 3 below)\cite{4}. Then, following the proof of \cite[Theorem 3]{28}, we consider small analytic deformations \(E_t\) of \(E_i = E_i(0)\) with \(E_t\) being stable bundles for \(t \neq 0\). This induces a \(T\)-invariant Kuranishi family \((M, J_t) \cong P(E(t))\), where \(E(t) = \bigoplus_{i=0}^{\ell} E_i(t)\), with \((M, J)\) being the central fibre \((M, J_0)\). We then generalize in Lemma 4 the stability-under-deformations results of \cite{15} \cite{46} \cite{29}, by using the crucial fact that our family is invariant under a fixed maximal torus. This allows us to show that any CSC (or more generally extremal) Kähler metric \(\omega_0\) on \((M, J_0)\) can be included
into a smooth family $\omega_t$ of extremal Kähler metrics on $(M, J_t)$. As $E(t)$ is polystable for $t \neq 0$, the corresponding extremal Kähler metric $\omega_t$ must be locally symmetric, by the uniqueness results \cite{15, 20, 51}. This implies that $\omega_0$ is locally symmetric too, and we conclude as in \cite[Lemma 8]{28}.

We next consider the more general problem of existence of extremal Kähler metrics on the manifold $(M, J) = P(E) \to \Sigma$. Notice that the deformation argument explained above is not specific to the CSC case, but also yields that any extremal Kähler metric $\omega_0$ on $(M, J) = P(E) \to \Sigma$ can be realized as a smooth limit (as $t \to 0$) of extremal Kähler metrics $\omega_t$ on $(M, J_t) = P(E(t))$, where $E(t) = \bigoplus_{i=0}^\ell E_i(t)$ with $E_i(t)$ being stable (and thus projectively-flat and indecomposable) bundles over $\Sigma$ for $t \neq 0$, and where $\ell$ is the dimension of a maximal torus $T$ in the identity component of the group of isometries of $\omega_0$. Unlike the CSC case (where $E_i(t)$ must all have the same slope and therefore $E(t)$ is polystable), the existence problem for extremal Kähler metrics on the manifolds $(M, J_t)$ is not solved in general. The main working conjecture here is that such a metric $\omega_t$ must always be compatible with the bundle structure (in a sense made precise in Sect. 3 below). As we observe in Sect. 5 if this conjecture were true it would imply that the initial bundle $E$ must also split as a direct sum of stable subbundles (and that $\omega_0$ must be compatible too). We are thus led to believe the following general statement would be true.

**Conjecture 1.** A projective bundle $(M, J) = P(E)$ over a compact curve of genus $\geq 2$ admits an extremal Kähler metric in some Kähler class if and only if $E$ decomposes as a direct sum of stable subbundles.

**Remark 3.** This conjecture turns out to be true in the case when $E$ is of rank 2 and $\Sigma$ is a curve of any genus, cf. \cite{7} for an overview.

A partial answer to Conjecture 1 is given by the following result which deals with Kähler classes far enough from the boundary of the Kähler cone.

**Theorem 2.** Let $p : P(E) \to \Sigma$ be a holomorphic projective bundle over a compact complex curve $\Sigma$ of genus $\geq 2$ and $[\omega_S]$ be a primitive Kähler class on $\Sigma$. Then there exists a $k_0 \in \mathbb{R}$ such that for any $k > k_0$ the Kähler class $\Omega_k = 2\pi c_1(O(1)_E) + kp^*[\omega_S]$ on $(M, J) = P(E)$ admits an extremal Kähler metric if and only if $E$ splits as a direct sum of stable subbundles.

In the case when $E$ decomposes as the sum of at most two indecomposable subbundles\footnote{This is equivalent to requiring that the automorphism group of $P(E)$ has a maximal torus of dimension $\leq 1$.} the conclusion holds for any Kähler class on $P(E)$.

The proof of Theorem 2 given in Section 5 will be deduced from a general existence theorem established in the much broader framework of rigid and semisimple toric bundles introduced in \cite{4}, whose main features are recalled in Section 3 below. As explained in Remark 4 this class of manifolds is closely related to the class of multiplicity-free manifolds recently discussed in Donaldson’s paper \cite{25}. Our most general existence result can be stated as follows.

**Theorem 3.** Let $(g, \omega)$ be a compatible Kähler metric on $M$, where $M$ is a rigid semisimple toric bundle over a CSC locally product Kähler manifold $(S, g_S, \omega_S)$ with fibres isomorphic to a toric Kähler manifold $(W, \omega_W, g_W)$, as defined in Sect. 3. Suppose, moreover, that the fibre $W$ admits a compatible extremal Kähler metric. Then, for any $k \gg 0$, the Kähler class $\Omega_k = [\omega] + kp^*[\omega_S]$ admits a compatible extremal Kähler metric.
The terms of this statement, in particular the concept of a compatible metric, are introduced in Section 3. Its proof, also given in Section 3, uses in a crucial way the stability under small perturbations of existence of compatible extremal metric (Proposition 2) which constitutes the delicate technical part of the paper. Another important consequence of Proposition 2 is the general openness theorem given by Corollary 4.

A non-trivial assumption in the hypotheses of Theorem 3 above is the existence of compatible extremal Kähler metric on the (toric) fibre \( W \). This is solved when \( W \cong \mathbb{C}P^r \) and \( M = P(E) \) with \( E \) being holomorphic vector bundle of rank \( r + 1 \), which is the sum of \( \ell + 1 \) projectively-flat hermitian bundles, as a consequence of the fact that the Fubini–Study metric on \( \mathbb{C}P^r \) admits a non-trivial hamiltonian 2-form of order \( \ell \leq r \) (cf. [3]). We thus derive in Sect. 4 the following existence result.

**Theorem 4.** Let \( p : P(E) \to S \) be a holomorphic projective bundle over a compact Kähler manifold \((S, J_S, \omega_S)\). Suppose that \((S, J_S, \omega_S)\) is covered by the product of constant scalar curvature Kähler manifolds \((S_j, \omega_j), \ j = 1, \ldots, N\), and \( E = \bigoplus_{i=0}^r E_i \) is the direct sum of projectively-flat hermitian bundles. Suppose further that for each \( i \) \( c_1(E_i)/\text{rk}(E_i) - c_1(E_0)/\text{rk}(E_0) \) pulls back to \( \sum_{j=1}^N p_{ji}[\omega_j] \) on \( \prod_{j=1}^N S_j \) (for some constants \( p_{ji} \)). Then there exists a \( k_0 \in \mathbb{R} \) such that for any \( k > k_0 \) the Kähler class \( \Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_S] \) admits a compatible extremal Kähler metric.

**Remark 4.** Theorem 1 is closely related to the results of Y.-J. Hong in [36, 37] who, under a technical assumption on the automorphism group of \( S \), that for any hermitian–Einstein (i.e. polystable) bundle \( E \) over a CSC Kähler manifold \( S \), the Kähler class \( \Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_S] \) for \( k \gg 1 \) admits a compatible CSC Kähler metric if and only if the corresponding Futaki invariant \( \mathfrak{F}_{\Omega_k} \) vanishes. However, in the case when \( E \) is not simple (i.e. has automorphisms other than multiples of identity) the condition \( \mathfrak{F}_{\Omega_k} \equiv 0 \) is not in general satisfied for these classes, see [5, Sect. 3.4 & 4.2] for specific examples. Thus, studying the existence of extremal rather than CSC Kähler metrics in \( \Omega_k \) is essential. Another useful remark is that although the hypothesis in Theorem 1 that \( E \) is the sum of projectively-flat hermitian bundles over \( S \) is rather restrictive when \( S \) is not a curve, our result strongly suggests that considering \( E \) to be a direct sum of stable bundles (with not necessarily equal slopes) over a CSC Kähler base \( S \) would be the right general setting for seeking extremal Kähler metrics in \( \Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp^*[\omega_S] \) for \( k \gg 1 \).

In the final Sect. 5, we develop further our approach by extending the leading conjectures [21, 66] about existence of extremal Kähler metrics on toric varieties to the more general context of compatible Kähler metrics that we consider in this paper. Thus motivated, we explore in a greater detail examples when \( M \) is a projective plane bundle over a compact complex curve \( \Sigma \). We show that when the genus of \( \Sigma \) is greater than 1, Kähler classes close to the boundary of the Kähler cone of \( M \) do not admit any extremal Kähler metric. In Appendix A we introduce the notion of a compatible extremal almost Kähler metric (the existence of which is conjecturally equivalent to the existence of a genuine extremal Kähler metric) and show that if the genus of \( \Sigma \) is 0 or 1, then any Kähler class on \( M \) admits an explicit compatible extremal almost Kähler metric.

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2. **Proof of Theorem 1**

As we have already noted in Remark 1, the 'if' part of the theorem is well-known. So we deal with the 'only if' part.
Let \((M,J) = P(E)\), where \(\pi : E \to \Sigma\) is a holomorphic vector bundle of rank \(m\) over a compact curve \(\Sigma\) of genus \(\geq 2\). We want to prove that \(E\) is polystable if \((M,J)\) admits a CSC Kähler metric \(\omega\). Also, by Remark \([1]\), we will be primarily concerned with the case when the connected component of the identity \(\text{Aut}_0(M,J)\) of the automorphisms group of \((M,J)\) is not trivial. Note that, as the normal bundle to the fibres of \(P(E) \to \Sigma\) is trivial and the base is of genus \(\geq 2\), the group \(\text{Aut}_0(M,J)\) reduces to \(H^0(\Sigma,PGL(E))\), the group of fibre-preserving automorphisms of \(E\), with Lie algebra \(\mathfrak{h}(M,J) \cong H^0(\Sigma,\mathfrak{s}(E))\). As any holomorphic vector field in \(\mathfrak{h}(M,J)\) has zeros, the Lichnerowicz–Matsushima theorem \([49, 55]\) implies \(\mathfrak{h}(M,J) = i(M,g) \oplus \mathfrak{ji}(M,g)\), where \(i(M,g)\) is the Lie algebra of Killing vector fields of \((M,J,\omega)\). Thus, \(\text{Aut}_0(M,J) \neq \{\text{Id}\}\) iff \(\mathfrak{h}(M,J) \neq \{0\}\) iff \(i(M,g) \neq \{0\}\). We will fix from now on a maximal torus \(T\) (of dimension \(\ell\)) in the connected component of the group of isometries of \((M,J,\omega)\). Note that \(T\) is a maximal torus in \(\text{Aut}_0(M,J)\) too, by the Lichnerowicz–Matsushima theorem cited above.

We will complete the proof in three steps, using several lemmas.

We start with following elementary but useful observation which allows us to relate a maximal torus \(T \subset \text{Aut}_0(M,J)\) with the structure of \(E\).

**Lemma 1.** Let \((M,J) = P(E) \to S\) be a projective bundle over a compact complex manifold \(S\), and suppose that the group \(H^0(S,PGL(E))\) of fibre-preserving automorphisms of \((M,J)\) contains a circle \(S^1\). Then \(E\) decomposes as a direct sum \(E = \bigoplus_{i=0}^{\ell} E_i\) of subbundles \(E_i\) with \(\ell \geq 1\), \(S^1\) acts on each factor \(E_i\) by a scalar multiplication.

In particular, any maximal torus \(T \subset H^0(S,PGL(E))\) arises from a splitting as above, with \(E_i\) indecomposable and \(\ell = \dim(T)\).

**Proof.** Any \(S^1\) in \(H^0(S,PGL(E))\) defines a \(\mathbb{C}^\times\) holomorphic action on \((M,J)\), generated by an element \(\Theta \in \mathfrak{h}(M,J) \cong H^0(S,\mathfrak{s}(E))\). For any \(x \in S\), \(\exp (t\Theta(x))\), \(t \in \mathbb{C}\) generates a \(\mathbb{C}^\times\) subgroup of \(SL(E_x)\) and so \(\Theta(x)\) must be diagonalizable. The coefficients of the characteristic polynomial of \(\Theta(x)\) are holomorphic functions of \(x \in S\), and therefore are constants. It then follows that \(\Theta\) gives rise to a direct sum decomposition \(E = \bigoplus_{i=0}^{\ell} E_i\) where \(E_i\) correspond to the eigenspaces of \(\Theta\) at each fibre.

The second part of the lemma follows easily. \(\square\)

Because of this result and the discussion preceding it, we consider the decomposition \(E = \bigoplus_{i=0}^{\ell} E_i\) as a direct sum of indecomposable subbundles over \(\Sigma\), corresponding to a fixed, maximal \(\ell\)-dimensional torus \(T\) in the connected component of the isometry group of \((g,J,\omega)\). We note that the isometric action of \(T\) is *hamiltonian* as \(T\) has fixed points (on any fibre).

Our second step is to understand the condition that that the Futaki invariant \([30]\), with respect to the Kähler class \(\Omega = [\omega]\) on \((M,J)\), restricted to the generators of \(T\) is zero. Hodge theory implies that any (real) holomorphic vector field with zeros on a compact Kähler \(2m\)-manifold \((M,J,\omega,g)\) can be written as \(X = \text{grad}_g f - J\text{grad}_g h\), where \(f + ih\) is a complex-valued smooth function on \(M\) of zero integral (with respect to the volume form \(\omega^m\)), called the *holomorphy potential* of \(X\), and where \(\text{grad}_g f\) stands for the hamiltonian vector field associated to a smooth function \(f\) via \(\omega\). Then the (real) Futaki invariant associates to \(X\) the real number \(\mathfrak{f}_\omega(X) = \int_M f \text{Scal}_g \omega^m\), where \(\text{Scal}_g\) is the scalar curvature of \(g\). Futaki shows \([30]\) that \(\mathfrak{f}_\omega(X)\) is independent of the choice of \(\omega\) within a fixed Kähler class \(\Omega\), and that (trivially) \(\mathfrak{f}_\omega(X) = 0\) if \(\Omega\) contains a CSC Kähler metric. A related observation will be useful to us: with a fixed symplectic
form \( \omega \), the Futaki invariant is independent of the choice of compatible almost complex structure within a path component.

**Lemma 2.** Let \( J_t \) be a smooth family of integrable almost-complex structures compatible with a fixed symplectic form \( \omega \), which are invariant under a compact group \( G \) of symplectomorphisms acting in a hamiltonian way on the compact symplectic manifold \((M, \omega)\). Denote by \( \mathfrak{g}_\omega \subset C^\infty(M) \) the finite dimensional vector space of smooth functions \( f \) such that \( X = \nabla_\omega f \in \mathfrak{g} \), where \( \mathfrak{g} \) denotes the Lie algebra of \( G \). Then the \( L^2 \)-orthogonal projection of the scalar curvature \( \text{Scal}_{g_t} \) of \((J_t, \omega, g_t)\) to \( \mathfrak{g}_\omega \) is independent of \( t \).

**Proof.** By definition, any \( f \in \mathfrak{g}_\omega \) defines a vector field \( X = \nabla_\omega f \) which is in \( \mathfrak{g} \), and is therefore Killing with respect to any of the Kähler metrics \( g_t = \omega(\cdot, J_t \cdot) \). To prove our claim, we have to show that \( \int_M f \text{Scal}_{g_t} \omega^m \) is independent of \( t \). Using the standard variational formula for scalar curvature (see e.g. [11, Thm. 1.174]), we compute

\[
\frac{d}{dt} \text{Scal}_{g_t} = \Delta(\text{tr}_{g_t} h) + \delta h g_t(r, h) = \delta \delta h,
\]

where \( h \) denotes \( \frac{d}{dt} g_t \), while \( \Delta, \delta \) and \( r \) are the riemannian laplacian, the codifferential and the Ricci tensor of \( g_t \), respectively. Note that to get the last equality, we have used the fact that \( h \) is \( J_t \)-anti-invariant (as all the \( J_t \)'s are compatible with \( \omega \)) while the metric and the Ricci tensor are \( J_t \)-invariant (on any Kähler manifold). Integrating against \( f \), we obtain

\[
\frac{d}{dt} \left( \int_M f \text{Scal}_{g_t} \omega^m \right) = \int_M (\delta \delta h) f \omega^m = \int_M g_t(h, Ddf) \omega^m,
\]

where \( D \) is the Levi–Civita connection of \( g_t \); however, as \( f \) is a Killing potential with respect to the Kähler metric \((g_t, J_t)\), it follows that \( Ddf \) is \( J_t \)-invariant, and therefore \( \int_M f \text{Scal}_{g_t} \omega^m \) is independent of \( t \). \( \square \)

**Remark 5.** One can extend Lemma 2 for any smooth family of (not necessarily integrable) \( G \)-invariant almost complex structures \( J_t \) compatible with \( \omega \). Then, as shown in [47], the \( L^2 \)-projection to \( \mathfrak{g}_\omega \) of the *hermitian scalar curvature* of the almost Kähler metric \((\omega, J_t)\) (see Appendix A for a precise definition) is independent of \( t \). This gives rise to a symplectic Futaki invariant associated to a compact subgroup \( G \) of the group of hamiltonian symplectomorphisms of \((M, \omega)\).

Lemma 2 will be used in conjunction with the Narasimhan–Ramanan approximation theorem (see [57, Prop. 2.6] and [58, Prop. 4.1]), which implies that any holomorphic vector bundle \( E \) over a compact curve \( \Sigma \) of genus \( \geq 2 \) can be included in an analytic family of vector bundles \( E_t, t \in D_\varepsilon \) (where \( D_\varepsilon = \{ t \in \mathbb{C}, \quad |t| < \varepsilon \} \) over \( \Sigma \), such that \( E_0 := E \) and \( E_t \) is stable for \( t \neq 0 \). Such a family will be referred to in the sequel as a small stable deformation of \( E \).

**Lemma 3.** Suppose that the vector bundle \( E = U \oplus V \to \Sigma \) splits as a direct sum of two subbundles, \( U \) and \( V \). Consider the holomorphic \( S^1 \)-action on \((M, J) = P(E)\), induced by fibrewise scalar multiplication by \( \exp(it\Theta) \) on \( V \), and let \( X \in \mathfrak{h}(M, J) \) be the (real) holomorphic vector field generating this action. Then the Futaki invariant of \( X \) with respect to some (and therefore any) Kähler class \( \Omega \) on \((M, J)\) vanishes if and only if \( U \) and \( V \) have the same slope.

**Proof.** We take some Kähler form \( \omega \) on \((M, J) = P(E)\) and, by averaging it over \( S^1 \), we assume that \( \omega \) is \( S^1 \)-invariant. As the \( S^1 \)-action has fixed points, the corresponding real

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4We will tacitly identify throughout the Lie algebra \( \mathfrak{g} \) of a group \( G \) acting effectively on \( M \) with the Lie algebra of vector fields generated by the elements of \( \mathfrak{g} \).
vector field $X$ is $J$-holomorphic and $\omega$-hamiltonian, i.e., $X = \mathbf{grad}_\omega f$ for some smooth function $f$ with $\int_M f \omega^m = 0$.

We now consider small stable deformations $U_t, V_t$, $t \in D_{\varepsilon}$ of $U$ and $V$, and put $E_t = U_t \oplus V_t$. Considering the projective bundle $P(E_t)$, we obtain a non-singular Kuranishi family $(M, J_t)$ with $J_0 = J$. By the Kodaira stability theorem (see e.g. [11]) one can find a smooth family of Kähler metrics $(\omega_t, J_t)$ with $\omega_0 = \omega$. Using the vanishing of the Dolbeault groups $H^{2,0}(M, J_t) = H^{0,2}(M, J_t) = 0$, Hodge theory implies that by decreasing the initial $\varepsilon$ if necessary, we can assume $[\omega_t] = [\omega]$ in $H^2_{\text{DR}}(M)$. Note that any $J_t$ is $S^1$-invariant so, by averaging over $S^1$, we can also assume that $\omega_t$ is $S^1$-invariant.

Applying the equivariant Moser lemma, one can find $S^1$-equivariant diffeomorphisms, $\Phi_t$, such that $\Phi_t^* \omega_t = \omega$. Considering the pullback of $J_t$ by $\Phi_t$, the upshot from this construction is that we have found a smooth family of integrable complex structures $J_t$ such that: (1) each $J_t$ is compatible with the fixed symplectic form $\omega$ and is $S^1$-invariant; (2) $J_0 = J$; (3) for $t \neq 0$, the complex manifold $(M, J_t)$ is equivariantly biholomorphic to $P(U_t \oplus V_t) \rightarrow \Sigma$ with $U_t$ and $V_t$ stable (and therefore projectively-flat) hermitian bundles.

If $U$ and $V$ have equal slopes, then $E_t = U_t \oplus V_t$ becomes polystable for $t \neq 0$, and $(M, J_t)$ has a CSC Kähler metric in each Kähler class. It follows that the Futaki invariant of $X$ on $(M, J_t, \omega)$ is zero for $t \neq 0$.

Conversely, if $U$ and $V$ have different slopes, it is shown in [3, Sect. 3.2] that the Futaki invariant of $X$ is different from zero for any Kähler class on $(M, J_t)$, $t \neq 0$.

We conclude using Lemma 2.

This lemma shows that all the factors in the decomposition $E = \bigoplus_{i=0}^t E_i$ have equal slope, should a CSC Kähler metric exists. As in the proof of Lemma 3 we consider small stable deformations $E_i(t)$ of $E_i$ and our assumption for the slopes insures that $E(t) = \bigoplus_{i=0}^t E_i(t)$ is polystable for $t \neq 0$; furthermore, by acting with $\mathbb{T}$-equivariant diffeomorphisms, we obtain a smooth family of $\mathbb{T}$-invariant complex structures $J_t$ compatible with $\omega$, such that for $t \neq 0$, the complex manifold $(M, J_t)$ has a locally-symmetric CSC Kähler metric in each Kähler class: by the uniqueness of the extremal Kähler metrics modulo automorphisms [13, 52], any extremal Kähler metric on $(M, J_t)$ is locally-symmetric when $t \neq 0$. The third step in the proof of Theorem 1 is then to show that the initial CSC Kähler metric $(J_0, \omega)$ must be locally symmetric too. This follows from the next technical result, generalizing arguments of [10, 29].

**Lemma 4.** Let $J_t$ be a smooth family of integrable almost-complex structures compatible with a symplectic form $\omega$ on a compact manifold $M$, which are invariant under a torus $\mathbb{T}$ of hamiltonian symplectomorphisms of $(M, \omega)$. Suppose, moreover, that $(J_0, \omega)$ define an extremal Kähler metric and that $\mathbb{T}$ is a maximal torus in the reduced automorphism group of $(M, J_0)$. Then there exists a smooth family of extremal Kähler metrics $(J_t, \omega_t, g_t)$, defined for sufficiently small $t$, such that $\omega_0 = \omega$ and $[\omega_t] = [\omega]$ in $H^2_{\text{DR}}(M)$.

**Proof.** Recall that [32, 43] on any compact Kähler manifold $(M, J)$, the reduced automorphism group, $\text{Aut}_0(M, J)$, is the identity component of the kernel of the natural group homomorphism from $\text{Aut}_0(M, J)$ to the Albanese torus of $(M, J)$; it is also the connected closed subgroup of $\text{Aut}_0(M, J)$, whose Lie algebra $\mathfrak{h}_0(M, J) \subset \mathfrak{h}(M, J)$ is the ideal of holomorphic vector fields with zeros.

We denote by $t$ the Lie algebra of $\mathbb{T}$ and by $\mathfrak{h}$ (resp. $\mathfrak{h}_0$) the Lie algebra of the complex automorphism group (resp. reduced automorphism group) of the central fibre $(M, J_0)$. As $\mathbb{T}$ acts in a hamiltonian way, we have $t \subset \mathfrak{h}_0$. By assumption, $t$ is a maximal abelian subalgebra of $\mathfrak{h}_0(M, g_0) = (i(M, g_0) \cap \mathfrak{h}_0$, where $i(M, g_0)$ is the Lie algebra of Killing vector fields of $(M, J_0, \omega, g_0)$. 


As in the Lemma 2 above, we let $t_0 \subset C^\infty(M)$ be the finite dimensional space of smooth functions which are hamiltonians of elements of $t$. As the Kähler metric $(J_0, \omega, g_0)$ is extremal (by assumption), its scalar curvature $\text{Scal}_{g_0}$ is hamiltonian of a Killing vector field $X = \text{grad}_\omega(\text{Scal}_{g_0}) \in \mathfrak{i}_0(M, g_0)$. Clearly, such a vector field is central, so $X \in t$ (by the maximality of $t$) and therefore $\text{Scal}_{g_0} \in t_0$.

For any $\mathbb{T}$-invariant Kähler metric $(\tilde{J}, \tilde{\omega}, \tilde{g})$ on $M$, we denote by $t_0 \subset C^\infty(M)$ the corresponding space of Killing potentials of elements of $t$ (noting that any $X \in t$ has zeros, so that $\mathbb{T}$ belongs to the reduced automorphism group of $(M, \tilde{J})$), and by $\Pi_\mathbb{Z}_t$ the $L^2$-orthogonal projection of smooth function to $t_0$, with respect to the volume form $\tilde{\omega}^m$. Obviously, if the scalar curvature $\text{Scal}_{\tilde{g}}$ of $\tilde{g}$ belongs to $t_0$, then $\tilde{g}$ is extremal.

Following [46], let $C^\infty_1(M)\mathbb{T}$ denote the Fréchet space of $\mathbb{T}$-invariant smooth functions on $M$, which are $L^2$-orthogonal (with respect to the volume form $\omega^m$) to $t_0$, and let $U$ be an open set in $\mathbb{R} \times C^\infty_1(M)\mathbb{T}$ of elements $(t, f)$ such that $\omega + dd^c f$ is Kähler with respect to $J_t$ (here $dd^c f$ denotes the $d^c$-differential corresponding to $J_t$). We then consider the map $\Psi : U \rightarrow \mathbb{R} \times C^\infty_1(M)\mathbb{T}$, defined by

$$\Psi(t, f) = \left(t, (\text{Id} - \Pi_\omega) \circ (\text{Id} - \Pi_{\tilde{\omega}})(\text{Scal}_{\tilde{g}})\right),$$

where $\tilde{\omega} := \omega + dd^c f$ and $\text{Scal}_{\tilde{g}}$ is the scalar curvature of the Kähler metric $\tilde{g}$ defined by $(J_t, \tilde{\omega})$. One can check that this map is $C^1$ and compute (as in [45]), by also using (1) that its differential at $(0, 0) \in U$ is

$$(T_{(0,0)}\Psi)(t, f) = (t, t\delta\delta h - 2\delta\delta(Ddf)^-),$$

where $D$ and $\delta$ are respectively the Levi–Civita connection and the codifferential of $g_0$, $h = \left(\frac{\partial u}{\partial t}\right)_{t=0}$ and $(Ddf)^-$ denotes the $J_0$-anti-invariant part of $Ddf$. Note that $L(f) := \delta\delta((Ddf)^-)$ is a 4-th order (formally) self-adjoint $\mathbb{T}$-invariant elliptic linear operator (known also as the Lichnerowicz operator, see e.g. [32]). When acting on smooth functions, $L$ annihilates $t_0$ (because any Killing potential $f$ satisfies $(Ddf)^- = 0$).

It then follows that $L$ leaves $C^\infty_1(M)\mathbb{T}$ invariant and, by standard elliptic theory, we obtain an $L^2$-orthogonal splitting $C^\infty_1(M)\mathbb{T} = \ker(L) \oplus \text{im}(L)$. However, any smooth $\mathbb{T}$-invariant function $f$ in $\ker(L)$ gives rise to a Killing field $X = \text{grad}_\omega f$ in the centralizer of $t \subset \mathfrak{i}_0(M, g_0)$. As $t$ is a maximal abelian subalgebra of $\mathfrak{i}_0(M, g_0)$ we must have $X \in t$, i.e. $f \in t_0$. It follows that the kernel of $L$ restricted to $C^\infty_1(M)\mathbb{T}$ is trivial, and therefore $L$ is an isomorphism of the Fréchet space $C^\infty_1(M)\mathbb{T}$.

This understood, we are in position to apply standard arguments, using the implicit function theorem for the extension of $\Psi$ to the Sobolev completion $L^2_{1,k}(M)^\mathbb{T}$ (with $k \gg 1$) of $C^\infty_1(M)\mathbb{T}$, together with the regularity result for extremal Kähler metrics, precisely as in [45] 46. 29. We thus obtain a family $(t, \omega_t)$ of smooth, $\mathbb{T}$-invariant extremal Kähler metrics $(J_t, \omega_t)$ (defined for $t$ in a small interval about 0) which converge to the initial extremal Kähler metric $(J_0, \omega)$ (in any Sobolev space $L^2_{2,k}(M)$, $k \gg 1$, and hence, by the Sobolev embedding, in $C^\infty(M)$).

The uniqueness argument thus also applies at $t = 0$, and the initial metric is locally symmetric. We can now conclude the proof of Theorem 1 by a standard argument using the de Rham decomposition theorem (see [28] Lemma 8) and [13]). This realizes the fundamental group of $\Sigma$ as a discrete subgroup group of isometries of the hermitian symmetric space $\mathbb{C}P^{m-1} \times \mathbb{H}$ and thus defines a projectively flat structure on $P(E) \rightarrow \Sigma$. 

□

Theorem 1
3. Rigid toric bundles and the generalized Calabi construction

In this section, we recall the notion of a semisimple and rigid isometric hamiltonian action of a torus on a compact Kähler manifold \((M, g, J, \omega)\) (introduced in [4]), as well as the construction of compatible Kähler metrics (given by the generalized Calabi construction of [4]) on such manifolds. This provides a framework for the search of extremal compatible metrics on rigid toric bundles over a semisimple base, which parallels (and extends) the theory of extremal toric metrics developed in [21, 22, 24]. We then apply the construction of this section to projective bundles of the form \(P(E_0 \oplus \cdots \oplus E_\ell) \to S\), where \(E_i\) is a projectively-flat hermitian bundle over a Kähler manifold \((S, \omega_S)\). In all cases, we prove the existence of compatible extremal Kähler metrics in “small” Kähler classes, cf. Theorems 3 and 4.

3.1. Rigid torus actions. Most of the material in this section is taken from [4, Sect. 2] and we refer the Reader to this article for further details.

Definition 1. Let \((M, g, J, \omega)\) be a connected Kähler \(2m\)-manifold with an effective isometric hamiltonian action of an \(\ell\)-torus \(T\) with momentum map \(z: M \to t^*\). We say the action is rigid if for all \(x \in M\), \(R_x^* g\) depends only on \(z(x)\), where \(R_x: T \to T : x \in M\) is the orbit map.

In other words, the action is rigid if, for any two generators \(X_\xi, X_\eta\) of the action — \(\xi, \eta \in t\) — the smooth function \(g(X_\xi, X_\eta)\) is constant on the levels of the momentum map \(z\).

Henceforth, we suppose that \(M\) is compact.

Obvious and well-known examples of rigid torus actions are provided by toric Kähler manifolds. A key feature of toric Kähler manifolds is actually shared by rigid torus actions, namely the fact that the image of \(M\) by the momentum map is a Delzant polytope \(\Delta \subset t^*\) (see [4, Prop. 4]) and that the regular values of \(z\) are the points in the interior \(\Delta^0\). Thus, to any compact Kähler manifold endowed with a rigid isometric hamiltonian action of an \(\ell\)-torus \(T\), one can associate a smooth compact toric symplectic \(2\ell\)-manifold \((V, \omega_V, T)\), via the Delzant correspondence [16]. Note that the Delzant construction also endows \(V\) with the structure of a complex toric variety \((V, J_V, T^\circ)\).

Another smooth variety is associated to a rigid torus action, namely the complex — or stable — quotient \(\tilde{S}\) of \((M, J)\) by the complexified action of \(T^\circ\). For a general torus action, \(\tilde{S}\) is a \(2(m-\ell)\)-dimensional complex orbifold, but when the torus action is rigid, it is shown in [4, Prop. 5] that \(\tilde{S}\) is smooth, and \(M^0 := z^{-1}(\Delta^0)\) is then a principal \(T^\circ\)-bundle over \(\tilde{S}\). Denote by \(\tilde{M} := M'^0 \times_{T^\circ} V \to \tilde{S}\) the associated fibre bundle in toric manifolds. Then, either \((M, J)\) is \(T^\circ\)-equivariantly biholomorphic to \(\tilde{M}\) or it is obtained by \((T^\circ\)-equivariantly\) blowing down the inverse image in \(M\) of some codimension one faces of \(\Delta\). Thus, \(M\) and \(\tilde{M}\) are (different in general) \(T^\circ\)-equivariant compactifications of the same principal \(T^\circ\)-bundle \(M'^0\) over \(\tilde{S}\).

In either case, by a convenient abuse of notation, we call \(M\) or any complex manifold \(T^\circ\)-equivariantly biholomorphic to \(M\), a rigid toric bundle. In the case when there is no blow-down, then \(M = \tilde{M}\) is a genuine fibre bundle over \(\tilde{S}\) with fibre the toric manifold \(V\), associated to a principal \(T\)-bundle over \(\tilde{S}\), whereas, in the general case, the Kähler metric \(g\) on \(M\) will be described, via its pullback on \(\tilde{M}\), in terms of the toric bundle structure of \(\tilde{M}\), thus allowing to introduce the notion of compatible Kähler metrics on a general rigid toric bundle, cf. [13, 3].

We now specialize the above construction, in particular the blow-down procedure, in the case when the (rigid) torus action is, in addition, semisimple, according to the following general definition.
Definition 2. An isometric hamiltonian torus action on a Kähler manifold \((M, g, J, \omega)\) is **semisimple** if for any regular value \(z_0\) of the momentum map, the derivative with respect to \(z\) of the family \(\omega_S(z)\) of Kähler forms on the complex (stable) quotient \(S\) of \((M, J)\) (induced by the symplectic quotient construction at \(z\)) is parallel and diagonalizable with respect to \(\omega_S(z_0)\).\(^3\)

For a **semisimple and rigid** isometric hamiltonian torus action the Kähler metrics \(\omega_S(z)\), parametrized by \(z\) in \(\Delta^0\), on the stable quotient \(S\) are simultaneously diagonal and have the same Levi–Civita connection. There then exists a Kähler metric \((g_S, \omega_S)\) on \(S\), such that the Kähler forms \(\omega_S(z)\) are simultaneously diagonalizable with respect to \(g_S\) and parallel with respect to the Levi–Civita connection of \(g_S\), so that the universal cover of \((\hat{S}, \omega_S)\) is a product \(\prod_{j=1}^{N}(S_j, \omega_j)\) of Kähler manifolds \((S_j, \omega_j)\) of dimensions \(2d_j, j = 1, \ldots, N\), in such a way that the restriction to \(S_j\) of the pullback of \(\omega_S(z)\) is a multiple of \(\omega_j\) by an affine function of \(z\). Moreover, to any face of codimension one, \(F_b\), of \(\Delta\) involved in the blow-down process corresponds a factor \((S_b, \omega_b)\) in the product \(\prod_{j=1}^{N}(S_j, \omega_j)\), which is isomorphic to the standard complex projective space \((\mathbb{C} \mathbb{P}^{d_b}, \omega_b)\) of (positive) complex dimension \(d_b\) equipped with a Fubini-Study metric of holomorphic sectional curvature equal to 2 — equivalently of scalar curvature equal to 2\(^{6}\) — so that \(\omega_b = 2\pi c_1(\Omega_{\mathbb{C} \mathbb{P}^{d_b}}(1))\).

Conversely, let \(\hat{S}\) be a compact Kähler manifold, whose universal cover is a Kähler product of the form \(\prod_{j=1}^{N}(S_j, \omega_j) = \prod_{a \in A}(S_a, \omega_a) \times \prod_{b \in B}(\mathbb{C} \mathbb{P}^{d_b}, \omega_b)\), where each \(\omega_b\) is the Kähler form of a Fubini-Study metric of holomorphic sectional curvature equal to 2 \((A \text{ or } B \text{ may possibly be empty})\). We moreover assume that \(\pi_1(\hat{S})\) acts diagonally by Kähler isometries on the universal cover, so that \(\hat{S}\) has the structure of a fibre product of flat unitary \(\mathbb{C} \mathbb{P}^{d_b}\)-bundles, \(b \in B\), over a compact Kähler manifold \(S\), covered by the product \(\prod_{a \in A}(S_a, \omega_a)\). Let \(T\) a real (compact) torus of dimension \(\ell\), of Lie algebra \(\mathfrak{t}\), \(\Delta\) be a Delzant polytope in the dual space \(\mathfrak{t}^*\), and \((V, J_V, \omega_V, T)\) a \(T\)-toric Kähler \(2\ell\)-manifold, with momentum polytope \(\Delta\). Among the \(n\) codimension one faces \(F_i\), \(i = 1, \ldots, n\), of \(\Delta\), with inward normals \(u_i\) in \(\mathfrak{t}\), we distinguish a subset \(\{F_b : b \in B\}\) (possibly empty) with inward normals \(u_b\). Let \(P\) be a principal \(T\)-bundle over \(\hat{S}\), such that \(-2\pi c_1(\tilde{P})\), as a \(t\)-valued 2-form, is **diagonalizable** with respect to the local product structure of \(\hat{S}\), i.e. is of the form \(\sum_{j=1}^{N}[\omega_j] \otimes p_j = \sum_{a \in A}[\omega_a] \otimes p_a + \sum_{b \in B}[\omega_b] \otimes u_b\), where all \(p_j\) are (constant) elements of \(\mathfrak{t}\) and, we recall, \(u_b\) denotes the inward normal of the distinguished codimension one face of \(\Delta\) associated to the factor \((\mathbb{C} \mathbb{P}^{d_b}, \omega_b)\) in the universal cover of \(\hat{S}\). We denote by \(\hat{M} = \hat{P} \times_{\mathfrak{t}} V\) the associated toric bundle over \(\hat{S}\).

With these data in hand, the blow-down process relies on the general **restricted toric quotient construction**, introduced in our previous work \(^4\), which, in the current situation, goes as follows.

Consider the product manifold \(S_0 = \prod_{b \in B} \mathbb{C} \mathbb{P}^{d_b}\) equipped with a principal \(T\)-bundle \(P_0\) with \(c_1(P_0) = \sum_{b \in B} c_1(\Omega_{\mathbb{C} \mathbb{P}^{d_b}}(-1)) \otimes u_b\), and the corresponding bundle of toric Kähler manifolds \(W = P_0 \times_{\mathfrak{t}} V\) over \(S_0\), with momentum map \(z : W \rightarrow \Delta \subset \mathfrak{t}^*\). Then the restricted toric quotient construction associates to \(W\) a toric manifold \((W, J_W, T)\), of the same dimension \(2(\ell + \sum_{b \in B} d_b)\) as \(W\), obtained from \(W\) by collapsing \(z^{-1}(F_b)\), \(b \in B\). Recall that, whereas \(V\) is obtained, via the Delzant construction, as a symplectic reduction of \(\mathbb{C}^n\) by the \((n - \ell)\)-dimensional torus \(G\), kernel of the map \((a_1, \ldots, a_n) \mod \mathbb{Z}^n \mapsto \sum_{i=1}^{n} a_i u_i \mod \Lambda\) from \(\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n\) onto \(T\), \(W\) is similarly obtained as a symplectic reduction of \(\oplus_{b \in B} \mathbb{C}^{d_b+1} \oplus \mathbb{C}^{n-|B|}\) by \(G \subset \mathbb{T}^n\), via the natural diagonal action.

\(^3\)In general, \(\hat{S}\) is well-defined as a complex orbifold for \(z \in \text{the connected component } U_{a_0} \text{ of } z_0 \text{ in the regular values.}
of $\mathbb{T}^n$ on $\bigoplus_{b \in \mathcal{B}} \mathbb{C}^{d_b+1} \oplus \mathbb{C}^{n-|\mathcal{B}|}$ (where $|\mathcal{B}|$ is the cardinality of $\mathcal{B}$); in this picture, the $(\ell + \sum_{b \in \mathcal{B}} d_b)$-dimensional torus $T$ acting on $W$ is identified with the quotient $\mathbb{T}^n / \sum_{b \in \mathcal{B}} d_b / G$, whereas the restricted subtorus $T$ is identified with the subtorus $\mathbb{T}^n / G$ of $T$, cf. [4, Sect. 1.6] for details, in particular for the identification of $W$ with a blow-down of $W^\mathcal{B}$.

We denote by $b : W \to W$ the $(T, T)$-equivariant blow-down map of $\hat{W}$ onto $W$. Along the inclusion $T \subset T$. We then have the following definition.

**Definition 3.** A blow-down of $\hat{M} = \hat{P} \times_T V \to \hat{S} \to S$ is a (locally trivial) fibre bundle $(M, J)$ over $S$, with fibre the Kähler manifold $(W, J_W)$, endowed with a global fibrewise (restricted) holomorphic action of the $\ell$-dimensional torus $T$, and a $T$-equivariant holomorphic map $\hat{M} \to M$, equal to the blow-down map $b$ on the corresponding Kähler fibres over $S$. We summarize this definition in a $T$-equivariant commutative diagram

$$
\begin{array}{ccc}
\hat{M} = \hat{P} \times_T V & \longrightarrow & \hat{S} \\
\downarrow & & \downarrow \\
M & \longrightarrow & S,
\end{array}
$$

and refer to the manifold $(M, J, T)$ as a rigid toric bundle over a semisimple base.

Using this construction, the blow-down was introduced in [4] under the simplifying assumption that the local product structure of $\hat{S}$ consists of global factors for $b \in \mathcal{B}$ (i.e. $\hat{S} \to S$ is a trivial fibre bundle). In particular, the blow-down was expressed in [4, Sect. 2.5] in terms of the universal covers of $M$ and $\hat{M}$. In fact, in this case there exists a diagonalizable principal $T$-bundle $P$ over $S$ with first Chern class $2\pi c_1(P) = \sum_{a \in A} [\omega_a] \otimes p_a$ and we can identify $\hat{M} = \hat{P} \times_T V \cong P \times_T W$. Then, $M := P \times_T W$ clearly satisfies the definition [3] above.

We now illustrate the blow-down construction in the case of projective bundles.

**3.2. Projective bundles as rigid toric bundles.** In this paragraph, we specialize the previous discussion to the case when the Delzant polytope $\Delta$ is a simplex in $t' \cong \mathbb{R}^\ell$, with codimension one faces $F_0, \ldots, F_j$; the associated complex toric variety $V$ is then the complex projective space $V \cong (\mathbb{CP}^j, T^c)$ and $\hat{M}$ is then $T^c$-equivariantly biholomorphic to a $\mathbb{CP}^j$-bundle over a Kähler manifold $\hat{S}$ of the type discussed in [3.1] since $\hat{M}$ comes from a principal $T^c$-bundle, $M$ is actually of the form $P(\mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_j) \to \hat{S}$, where $\mathcal{L}_i$ are hermitian holomorphic line bundles (the $T^c$ action is then induced by scalar multiplication on $\mathcal{L}_i$).

According to the discussion in [3.1], a blow-down process on $\hat{M}$ is encoded by the realization of $\hat{S}$ as a fibre product of flat projective unitary $\mathbb{CP}^{d_{j}}$-bundles over a Kähler manifold $S$. We here only consider flat projective bundles of the form $P(E)$, where $E$ is a rank $r + 1$ projectively-flat hermitian vector bundle over $S$ (in general the obstruction to the existence of $E$ is given by a torsion element of $H^2(S, \mathcal{O}^*)$; in particular, such an $E$ always exists if $S = \Sigma$ is a Riemann surface). We then have $\hat{S} = P(E_0) \times_S \cdots \times_S P(E_\ell) \to S$, where each $E_i \to S$ is a projectively-flat hermitian bundle of rank $d_i + 1$, and we assume that $c_1(E_i)/(d_i + 1) - c_1(E_0)/(d_0 + 1)$ pulls back to $\sum_{a \in A} p_{ia} [\omega_a]$ on the covering space $\prod_{a \in A} (S_a, \omega_a)$.

In this case, we have that $\hat{M} = P(\mathcal{O}(-1)_{E_0} \oplus \cdots \oplus \mathcal{O}(-1)_{E_\ell}) \to \hat{S}$, where $\mathcal{O}(-1)_{E_i}$ is the (fibrewise) tautological line bundle over $P(E_i) \to S$ — trivial over the other factors of $\hat{S}$ — whereas $M = P(E_0 \oplus \cdots \oplus E_\ell) \to S$, the blow-down process being, over each point of $S$, the standard blow-down process from $P(\bigoplus_{j=0}^\ell \mathcal{O}(-1)_{V_j}) \to \prod_{j=0}^\ell (V_j)$.

A simple illustration of this construction is $W = \mathbb{CP}^2$ seen as a fibrewise $S^1$-equivariant blow-down of the first Hirzebruch surface $\hat{W} = P(\mathcal{O} \oplus \mathcal{O}(-1)) \to \mathbb{CP}^1$. 
to $P(V)$, for any splitting $V = \bigoplus_{j=0}^{\ell} V_j$ of a complex vector space $V$ into a direct sum of $\ell + 1$ $(d_j + 1)$-dimensional vector subspaces, $d_j > 0$, $\ell > 0$, cf. \[1\].

To go further into the geometry of the situation, we next fix a hermitian metric on $E_i$ whose Chern connection has curvature $\Omega_i \otimes \operatorname{Id}_{E_i}$ with

$$\Omega_i - \Omega_0 = \sum_{a \in A} p_{ia} \omega_a, \quad i \geq 1,$$

where $p_a = (p_{1a}, \ldots, p_{\ell a}) \in \mathbb{R}^\ell \cong t$ will be the constants of our construction. Let $\tilde{\theta}_i$ be a connection 1-form for the principal $U(1)$-bundle over $\tilde{S}$, associated to the line bundle $\mathcal{O}(-1)_{E_i}$, with curvature $d\tilde{\theta}_i = -\omega_i + \Omega_i$, where $\omega_i$ pulls back to the Fubini–Study metric of scalar curvature $2d_i(d_i + 1)$ on the universal cover of $P(E_i)$ when $d_i \geq 1$, and is zero when $d_i = 0$. We then put $\tilde{\theta}_j = \tilde{\theta}_j - \theta_0$ to define a principal $T$-connection $\theta = (\tilde{\theta}_1, \ldots, \tilde{\theta}_\ell)$ associated with the principal $T^\ell$-bundle $M^0$ over $\tilde{S}$.

3.3. The generalized Calabi construction on rigid toric bundles over a semi-simple base. As recalled in \[3.1\] any compact Kähler manifold $(M, J, \omega, g)$ endowed with a rigid and semisimple isometric hamiltonian action of an $\ell$-torus $T$, is equivariantly biholomorphic to a rigid toric bundle over a semisimple base, obtained by a blow-down process from an associated bundle in $T$-toric manifolds $\hat{M}$. It still remains to describe Kähler structure $(g, \omega)$ on $M$: according to \[1\] Thm. 2], this is done by using the generalized Calabi construction which we now recall, following \[4\], with slightly different notation. We freely use the notation of \[3.1\].

The generalized Calabi construction is made of three main building blocks — only two if there is no blow-down — and produces a family of (smooth) singular Kähler structures on $\hat{M}$, which descend to genuine Kähler metrics on $M$, called compatible: for any Kähler manifold $(M, J, \omega, g)$ endowed with a rigid and semisimple isometric hamiltonian action of an $\ell$-torus $T$, the Kähler structures $(g, \omega)$ is compatible.

The first building block of the construction is the choice of a compatible $T$-invariant Kähler metric $g_V$ on the symplectic toric manifold $(V, \omega_V, T)$. This part is well-known (see e.g. \[1\] [2] [22] [34]): let $z \in C^\infty(V, t^\epsilon)$ be the momentum map of the $T$ action with image $\Delta$ and let $V^0 = z^{-1}(\Delta^0)$ be the union of the generic $T$ orbits. On $V^0$, orthogonal to the $T$ orbits is a rank $\ell$ distribution spanned by commuting holomorphic vector fields $JX_\xi$ for $\xi \in t$. Hence there is a function $t: V^0 \to t/2\pi \Lambda$, defined up to an additive constant, such that $dt(JX_\xi) = 0$ and $dt(X_\xi) = \xi$ for $\xi \in t$. The components of $t$ are ‘angular variables’, complementary to the components of the momentum map $z: V^0 \to t^\ast$, and the symplectic form in these coordinates is simply

$$\omega_V = \langle dz \wedge dt \rangle,$$

where the angle brackets denote contraction of $t$ and $t^\ast$. These coordinates identify each tangent space with $t \oplus t^\ast$, so any $T$-invariant $\omega_V$-compatible Kähler metric must be of the form

$$g_V = \langle dz, G, dz \rangle + \langle dt, H, dt \rangle,$$

where $G$ is a positive definite $S^2t$-valued function on $\Delta^0$, $H$ is its inverse in $S^2t^\ast$—observe that $G$ and $H$ define mutually inverse linear maps $t^\ast \to t$ and $t \to t^\ast$ at each point—and $\langle \cdot, \cdot, \cdot \rangle$ denotes the pointwise contraction $t^\ast \times S^2t \times t^\ast \to \mathbb{R}$ or the dual contraction. The corresponding almost complex structure is defined by

$$Jdt = -\langle G, dz \rangle$$

from which it follows that $J$ is integrable if and only if $G$ is the hessian of a function $U$ (called symplectic potential) on $\Delta^0$ \[34\].
Necessary and sufficient conditions for $U$ to come from a globally defined $\mathbb{T}$-invariant $\omega_V$-compatible Kähler metric on $V$ were obtained in [2, 3, 22]. We state here the first-order boundary conditions obtained in [1] Prop. 1: for any face $F \subset \Delta$, denote by $t_F \subset t$ the vector subspace spanned by the inward normals $u_i \in t$ to all codimension one faces of $\Delta$, containing $F$; as $\Delta$ is Delzant, the codimension of $t_F$ equals the dimension of $F$. Furthermore, the annihilator $t_F^0$ of $t_F$ in $t^*$ is naturally identified with $(t/t_F)^*$. Then a smooth strictly convex function $U$ on $\Delta^0$ corresponds to a $\mathbb{T}$-invariant, $\omega_V$-compatible Kähler metric $g_V$ via [2] if and only if the $S^2t^*$-valued function $H = \text{Hess}(U)^{-1}$ on $\Delta^0$ verifies the following boundary conditions:

- [smoothness] $H$ is the restriction to $\Delta^0$ of a smooth $S^2t^*$-valued function on $\Delta$;
- [boundary values] for any point $z$ on the codimension one face $F_i \subset \Delta$ with inward normal $u_i$, we have

$$H_z(u_i, \cdot) = 0 \quad \text{and} \quad (dH)_z(u_i, u_i) = 2u_i,$$

where the differential $dH$ is viewed as a smooth $S^2t^* \otimes t$-valued function on $\Delta$;

- [positivity] for any point $z$ in the interior of a face $F \subseteq \Delta$, $H_z(\cdot, \cdot)$ is positive definite when viewed as a smooth function with values in $S^2(t/t_F)^*$.

These conditions can be formulated in the following alternative way, cf. [2, 22]: (i) $U$ is smooth and strictly convex on the interior, $F^0$, of each face $F$ of $\Delta$; (ii) if $F = \bigcap_i F_i$, where $F_i$ is a codimension one face of $\Delta$ on which $\langle u_i, z \rangle + c_i = 0$, then, in some neighbourhood of $F^0$ in $\Delta$, $U$ is equal to $\frac{1}{2}\sum_i (\langle u_i, z \rangle + c_i) \log(\langle u_i, z \rangle + c_i)$ up to a smooth function.

We denote by $\mathcal{S}(\Delta)$ the space of all symplectic potentials on $\Delta$ defined either way. For any $U \in \mathcal{S}(\Delta)$, we thus get a $\mathbb{T}$-invariant, $\omega_V$ compatible, Kähler metric $g_V$ on $V$.

The second building block of the generalized Calabi construction consists in using $g_V$ to construct a Kähler metric $g_W$ on the variety $W$, with respect to which the restricted $\mathbb{T}$-action is rigid and semisimple. This part of the construction only appears in the situation “with blow-down” and relies in a crucial way on [1] Prop. 2. Recall that $W$ was obtained by a restricted symplectic quotient process, which ultimately amounts to a blow-down of $\hat{W} = P_0 \times_\mathbb{T} V$, where $P_0$ is a $\mathbb{T}$-principal bundle over $\prod_{b \in B} \mathbb{C}P^{d_b}$, cf. [3.1]. The construction of $g_W$ then requires the choice of a connection 1-form $\theta_0$ on $P_0$, with curvature $d\theta_0 = \sum_{b \in B} \omega_b \otimes u_b$ where, we recall, $\omega_b$ is the (normalized) Fubini–Study metric on $\mathbb{C}P^{d_b}$ of scalar curvature $2d_b(d_b + 1)$, and $u_b \in t$ is the inward normal to the codimension one face $F_b \subset \Delta$ (satisfying $\langle u_b, z \rangle + c_b = 0$). We still denote by $\theta_0 \in \Omega^1(W^0, t)$ the induced 1-form on the open dense subset $W^0 := P_0 \times_\mathbb{T} V^0$ of $\hat{W}$ and we consider the Kähler structure on $W^0$ defined by:

$$g_W = \sum_{b \in B} (\langle u_b, z \rangle + c_b) g_b + \langle dz, G, dz \rangle + \langle \theta_0, H, \theta_0 \rangle,$$

$$\omega_W = \sum_{b \in B} (\langle u_b, z \rangle + c_b) \omega_b + \langle dz \wedge \theta_0 \rangle,$$

$$d\theta_0 = \sum_{b \in B} \omega_b \otimes u_b,$$

with $G = \text{Hess}(U) = H^{-1}$. Clearly, the Kähler structure $(g_W, \omega_W)$ is well-defined on $W^0 = P_0 \times_\mathbb{T} V^0$. As shown in [4], the pair $(g_W, \omega_W)$ smoothly extends to $\hat{W}$ — not as a Kähler structure however — and descends to a smooth, $\mathbb{T}$-invariant, Kähler structure on $W$.

The third and last building block of the generalized Calabi construction similarly consists in constructing a suitable Kähler structure on $M^0 = \hat{P} \times_\mathbb{T} V^0$, via the choice of

\[\text{In [2] the strict convexity condition on the interior of the proper faces is realized equivalently as a condition on the determinant of Hess } U.\]
a connection 1-form $\hat{\theta}$ on $\hat{P}$, with curvature (covered by) $\sum_{a \in A} \omega_a \otimes p_a + \sum_{b \in B} \omega_b \otimes u_b$. Then the restriction of $(\hat{P}, \hat{\theta})$ to each fibre of $\hat{S} \to S$ is isomorphic to $(P, \theta_0)$ over $\prod_{b \in B} \mathbb{C}P^d_b$. Still denoting by $\hat{\theta} \in \Omega^1(M^0)$ the induced 1-form on $M^0 = \hat{P} \times_{\mathbb{T}} V^0$, we consider the Kähler structure $(g, \omega)$ on $M^0$ defined by:

$$g = \sum_{j=1}^N \langle (p_j, z) + c_j \rangle g_j + \langle dz, G, dz \rangle + \langle \hat{\theta}, H, \hat{\theta} \rangle,$$

$$\omega = \sum_{j=1}^N \langle (p_j, z) + c_j \rangle \omega_j + \langle dz \wedge \hat{\theta} \rangle,$$

$$d\hat{\theta} = \sum_{j=1}^N \omega_j \otimes p_j,$$

where:

- $G = \text{Hess}(U) = H^{-1}$, where $U$ is the symplectic potential of the chosen toric Kähler structure $g_V$ on $V$;
- for each $b \in B$, $p_b = u_b$ and the real number $c_b$ is such that $\langle p_b, z \rangle + c_b = 0$ on the codimension one face $F_b$;
- for each $a \in A$, $\langle p_a, z \rangle + c_a$ is positive on $\Delta$.

Clearly, (6) defines a smooth tensor on $\hat{M}$ and it is shown in [4] Thm. 2] that it is the pullback of a smooth metric on the blow-down $\hat{M}$. Indeed, this is obvious in the case when the fibre bundle $\hat{S} \to S$ is trivial (for example taking $\hat{M}$ be simply connected, as in [4]). Then, there exists a principal $\mathbb{T}$-bundle over $S$ with connection form $\theta$ and curvature $d\theta = \sum_{a \in A} \omega_a \otimes p_a$ and the restriction of $\theta$ to $S_0 = \prod_{b \in B} \mathbb{C}P^d_b$ gives rise to a principal $\mathbb{T}$-bundle $P_0$ over $S^0$ with connection 1-form $\theta_0$ and curvature $d\theta_0 = \sum_{b \in B} \omega_b \otimes u_b$.

Thus, $M^0 \cong P \times_{\mathbb{T}} W$, $M^0 \cong P \times_{\mathbb{T}} W^0$ where $W^0 = P_0 \times_{\mathbb{T}} V^0$. It follows that the metric (6) restricts on each $W^0$ fibre to the metric $(g_W, \omega_W)$ defined by (5); as $(g_W, \omega_W)$ compactifies smoothly on $W$, and $\langle p_a, z \rangle + c_a$ are strictly positive on $M$, (6) defines a Kähler structure on $M$. To handle the general case, one can consider the universal covers of $\hat{M}$ and $M$ and use the previous argument, noting that the smooth extension of the metric is a local property; a direct argument in the case of the projective bundles described in Sect. 3.2 can be given along the lines of [5] § 1.3. This completes the generalized Calabi construction according to [4].

Assuming that the metrics $(g_j, \omega_j)$, the connection 1-form $\hat{\theta}$, the polytope $\Delta$ and the constants $(p_j, c_j)$ are all fixed, (6) defines a family of Kähler metrics parametrized by symplectic potentials $U \in S(\Delta)$ (or, equivalently, by toric Kähler metrics on $(V, \omega_V, \mathbb{T})$).

We note that for this family, the symplectic 2-form $\omega$ remains unchanged, so we obtain a family of $\mathbb{T}$-invariant $\omega$-compatible Kähler metrics corresponding to different complex structures. However, any two such complex structures are biholomorphic, under a $\mathbb{T}$-equivariant diffeomorphism in the identity component: this is well-known in the case of a symplectic toric manifolds (i.e., on $(V, \omega_V, \mathbb{T})$) see [2, 21], and the same argument holds (fibrewise) on $W$ and $M$, see [5] § 1.4. The pullbacks of the symplectic form $\omega$ under such diffeomorphisms introduce a Kähler class $\Omega$ on a fixed complex manifold $(M, J)$ (we can take $J$ to be the complex structure on $M$ introduced in Definition 4) it corresponds to the standard symplectic potential $U_0$, see [2, 34].

**Definition 4.** Kähler structures $(g, \omega)$ arising from the generalized Calabi construction on a rigid toric bundle $M$, depending on the choice of a symplectic potential $U$ on the corresponding Delzant polytope $S(\Delta)$, whose explicit expression is given by (6) on $M^0$, are called compatible. The corresponding Kähler classes are accordingly called compatible Kähler classes.
We shall further assume that the metrics \((g_j, \omega_j)\) are fixed and have constant scalar curvature \(\text{Scal}_j\) (with \(\text{Scal}_b = 2d_b(d_b + 1)\) for \(b \in \mathcal{B}\)) and that \(\Delta\) and \(p_j\) are fixed. Recall that for \(b \in \mathcal{B}\), the constants \(c_b\) are also fixed by requiring \(\langle u_b, z \rangle + c_b = 0\) on the codimension one face \(F_b \subset \Delta\). The real constants \(c_a, a \in \mathcal{A}\) can vary (on a given manifold \((M, J)\)) and they parametrize the compatible Kähler classes.

### 3.4. The isometry Lie algebra

For a compact Kähler manifold \((M, g)\), we denote by \(\mathfrak{i}_0(M, g)\) the Lie algebra of all Killing vector fields with zeros; this is equivalently the Lie algebra of all hamiltonian Killing vector fields.

The following result has been established in the case \(\ell = 1\) in [5, Prop. 3] and its proof generalizes to the general case. For the convenience of the Reader, we reproduce the argument from [5].

**Lemma 5.** Let \((g, \omega)\) be a compatible Kähler metric on \(M\), where the stable complex quotient \(\hat{S}\) is equipped with the local product Kähler metric \((g_{\hat{S}}, \omega_{\hat{S}})\) covered by \(\prod_{j=1}^{N}(S_j, \omega_j)\).

Denote by \(\hat{p}: M^0 \rightarrow \hat{S}\) the principal \(\mathbb{T}^c\)-fibre structure of the regular part \(M^0\) of \(\mathbb{T}\) action on \(M\). Let \(\mathfrak{g}(\mathbb{T}, g)\) be the centralizer in \(\mathfrak{i}_0(M, g)\) of the \(\ell\)-torus \(\mathbb{T}\).

Then the vector space \(\mathfrak{g}(\mathbb{T}, g)\) is the direct sum of a lift of \(\mathfrak{i}_0(\hat{S}, g_{\hat{S}})\) and the Lie algebra \(\mathfrak{t} \subset \mathfrak{i}_0(M, g)\) of \(\mathbb{T}\) in such a way that the natural homomorphism \(\hat{p}_*: \mathfrak{g}(\mathbb{T}, g) \rightarrow \mathfrak{i}_0(\hat{S}, g_{\hat{S}})\) is a surjection.

**Proof.** Denote by \(K = \text{grad}_{\omega}z \in C^\infty(M, TM) \otimes \mathfrak{t}^*\) the family of hamiltonian Killing vector fields generated by \(\mathbb{T}\): thus, the span of \(K\) realizes the Lie algebra \(\mathfrak{t}\) of \(\mathbb{T}\) as a subalgebra of \(\mathfrak{i}_0(M, g)\).

Let \(X\) be a holomorphic vector field on \(\hat{S}\) which is hamiltonian with respect to \(\omega_{\hat{S}}\); then the projection \(X_j\) of \(X\) onto the distribution \(\mathcal{H}_j\) (induced by \(TS_j\) on the universal cover \(\prod_{j=1}^{N}S_j\) of \(S\)) is a Killing vector field with zeros, so \(\iota_{X_j}\omega_{\hat{S}} = -df_j\) for some function \(f_j\) (with integral zero). Thus \(\sum_{j=1}^{N}f_jp_j\) is a family of hamiltonians for \(X\) with respect to the family of symplectic forms covered by \(\sum_{j=1}^{N}\omega_j \otimes p_j\): since this is the curvature \(d\tilde{\theta}\) of the connection on \(M^0\), \(X\) lifts to a holomorphic vector field \(\tilde{X} = X_H + \sum_{j=1}^{N}f_j(p_j, K)\) on \(M^0\), which is hamiltonian with potential \(\sum_{j=1}^{N}(\langle p_j, z \rangle + c_j)f_j\) and commutes with the components of \(K\). (Here \(X_H\) is the horizontal lift to \(M^0\) with respect to \(\tilde{\theta}\).) As the metric \(g\) extends to \(M\) and \(\tilde{X}\) is Killing with respect to \(g\), it extends to \(M\) too (note that \(M \setminus M^0\) has codimension \(\geq 2\)). It is not difficult to see that \(\tilde{X}\) has zeros on \(M\) (in fact, if \(s_0 \in \hat{S}\) is a zero of \(X\) then \(\tilde{X} - \sum_{j=1}^{N}f_j(s_0, p_j, K)\) vanishes on \(M^0\)) so that \(\tilde{X}\) is an element of \(\mathfrak{i}_0(M, g)\). Of course, this shows that the Killing potential \(\sum_{j=1}^{N}(\langle p_j, z \rangle + c_j)f_j\) extends as a smooth function on \(M\).

Conversely, any \(\tilde{X} \in \mathfrak{g}(\mathbb{T}, g)\) is a \(\mathbb{T}^c\)-invariant holomorphic vector field, so its restriction to \(M^0\) is projectable to a holomorphic vector field \(X \in \mathfrak{i}_0(\hat{S})\). This allows to reverse the above arguments: for \(\tilde{X} = X_H + f(p, K) + hJ(q, K)\) (where \(p, q \in \mathfrak{t}\) and \(f, g \in C^\infty(\hat{S})\)) be Killing with respect to the metric \(\mathfrak{g}(\mathbb{T}, g)\), we must have \(q = 0\) and \(X\) be Killing with respect to \(g_{\hat{S}}\). Such a vector field maps to zero iff it comes from a constant multiple of \(K\). This gives a projection to \(\mathfrak{i}_0(\hat{S}, g_{\hat{S}})\) splitting the inclusion just defined.

This is the main ingredient in the proof of the following result.

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8 Presumably, the Kähler metrics \((g_j, \omega_j)\) must be CSC in order to obtain an extremal Kähler metric \((g, \omega)\) as above. We do not prove this here, but this fact has been established for \(\ell = 1\) in [3, Prop. 14].
Proposition 1. Let \((J, g, \omega)\) be a compatible Kähler metric on \(M\) where the stable quotient \(\hat{S}\) is endowed with a local product Kähler structure \((g_{\hat{S}}, \omega_{\hat{S}})\), covered by \(\prod_{j=1}^{N} (S_j, \omega_j)\) with \((S_j, \omega_j)\) having constant scalar curvature.

Then \(g\) is invariant under a maximal torus \(G\) of the reduced automorphism group \(\text{Aut}_0(M, J)\).

Proof. Let \(G\) be a maximal torus in the group of hamiltonian isometries \(\text{Isom}_0(M, g)\), containing the \(\ell\)-torus \(T\). By Lemma 5 \(G\) is the product of a maximal torus in the group of hamiltonian isometries \(\text{Isom}_0(S, g_{\hat{S}})\) and the \(\ell\)-torus. Denote by \(\mathfrak{g} \subset i_0(M, g)\) the corresponding Lie algebra. We are going to show that \(\mathfrak{g}^C = \mathfrak{g} + J\mathfrak{g}\) is a maximal abelian subalgebra of \(\mathfrak{h}_0(M, J)\).

As in the proof of Lemma 5 we consider natural homomorphism \(\hat{p}_* : \hat{z}(T, J) \to \mathfrak{h}_0(\hat{S})\) from the centralizer \(\hat{z}(T, J)\) of \(T\) in \(\mathfrak{h}_0(M, J)\) to \(\mathfrak{h}_0(\hat{S})\). The proof of Lemma 5 shows that the restriction of \(\hat{p}_*\) to \(\hat{z}(T, g)\) is surjective onto \(i_0(\hat{S}, g_{\hat{S}})\).

By assumption, the induced Kähler metric \((g_{\hat{S}}, \omega_{\hat{S}})\) on \(\hat{S}\) is of constant scalar curvature, so by the Lichnerowicz–Matsushima theorem [49, 55], \(\mathfrak{h}_0(\hat{S})\) is the complexification of \(i_0(\hat{S}, g_{\hat{S}})\). It follows that \(\hat{p}_* : \hat{z}(T, J) \to \mathfrak{h}_0(S)\) is also surjective. As \(\mathfrak{g} \subset \hat{z}(T, g)\) is a maximal abelian subalgebra, its projection to \(i_0(S, g_{\hat{S}})\) must also be a maximal abelian subalgebra, so is then the image \(\hat{p}_*(\mathfrak{g}^C) \subset \mathfrak{h}_0(\hat{S})\) (by using the Lichnerowicz–Matsushima theorem again). It follows that \(\mathfrak{g}^C \subset \mathfrak{h}_0(M, J)\) is maximal abelian iff \(\mathfrak{g}^C \cap \mathfrak{h}_0(S) = 0\) is a maximal abelian subalgebra of the complex algebra of fibre-preserving holomorphic vector fields \(\mathfrak{h}_0(\hat{S})\). But the fibre \(V\) is a toric variety under \(T\), so \(\mathfrak{g}^C \cap \mathfrak{h}_0(\hat{M}) = \mathfrak{t}^C = t + Jt\), which is clearly a maximal abelian subalgebra of \(\mathfrak{h}(V, J\mathfrak{V})\) and hence also of \(\mathfrak{h}_0(\hat{M})\). □

3.5. The extremal vector field. For convenience, we will introduce at places a basis of \(t\) (resp. of \(t^*\)), for example by taking \(\ell\) generators of the lattice \(\Lambda\) (where \(T = t/2\pi \Lambda\)). This identifies the vector space \(t\) with \(\mathbb{R}^\ell\) (and \(t^*\) with \((\mathbb{R}^\ell)^*\)), and fixes a basis of Poisson commuting hamiltonian Killing fields \(K_1, \ldots, K_\ell\) in \(K\). Thus, a \(S^2t\)-valued function \(H\) on \(\Delta\) can be seen as an \(\ell \times \ell\)-matrix of functions \((H_{rs}) = H\) on \(\Delta\). Similarly, we write \(z = (z_1, \ldots, z_\ell)\) for the momentum coordinates with respect to \(K_1, \ldots, K_\ell\).

An important technical feature of the Kähler metrics given by the generalized Calabi construction (20) is the simple expression of their scalar curvature in terms of the geometry of \((V, g_V)\) and \((\hat{S}, g_{\hat{S}})\) (see e.g. [3] p. 380):

\[
(7) \quad \text{Scal}_g = \sum_{j=1}^{N} \frac{\text{Scal}_j}{\langle p_j, z \rangle} + c_j + \frac{1}{p(z)} \sum_{r,s=1}^{\ell} \frac{\partial^2}{\partial z_r \partial z_s} (p(z) H_{rs}),
\]

where \(p(z) = \prod_{j=1}^{N} (p_j, z) + c_j d_j\). This formula generalizes the expression obtained by Abreu [1] in the toric case (when \(\hat{S}\) is a point).

Another immediate observation is that the volume form \(\text{Vol}_\omega = \omega^m\) is given by

\[
(8) \quad \omega^m = p(z) \left( \omega_{\hat{S}}^d \wedge \langle dz \wedge \hat{\theta} \rangle^\wedge \ell \right) = p(z) \left( \bigwedge_j \omega_j^{\wedge d_j} \right) \wedge \langle dz \wedge \hat{\theta} \rangle^\wedge \ell,
\]

where \(\sum_{j=1}^{N} d_j = d = m - \ell\). It follows that integrals over \(M\) of functions of \(z\) (pullbacks from \(\Delta\)) are given by integrals on \(\Delta\) with respect to the volume form \(p(z) \, dv\), where \(dv\) is the (constant) euclidean volume form on \(t\), obtained by wedging any generators of the lattice \(\Lambda\).

We now recall the definition in [31] of the extremal vector field of a compact Kähler manifold \((M, J, g, \omega)\). Let \(G\) be a maximal connected compact subgroup of the reduced
group of automorphisms $\text{Aut}_0(M, J)$\footnote{By a well-known result of Calabi \cite{13}, any extremal Kähler metric must be invariant under such a $G$.}. Following \cite{31}, the \textit{extremal vector field} of a $G$-invariant Kähler metric $(g, J, \omega)$ on $M$ is the Killing vector field whose Killing potential is the $L^2$-projection of the scalar curvature $\text{Scal}_g$ of $g$ to the space $\mathfrak{g}_\omega$ of all Killing potentials (with respect to $g$) of elements of the Lie algebra $\mathfrak{g}$. Futaki and Mabuchi \cite{31} showed that this definition is independent of the choice of a $G$-invariant Kähler metric within the given Kähler class $\Omega = [\omega]$ on $(M, J)$. Since the extremal vector field is necessarily in the centre of $\mathfrak{g}$, it can be equally defined if we take $G$ to be only a maximal torus in $\text{Aut}_0(M, J)$. This remark is relevant to the Kähler metrics \cite{8} as we have already shown in Proposition \ref{prop} that they are automatically invariant under such a torus $G$. In this case, by Lemma \ref{lem} $\mathfrak{g}_\omega$ is the direct sum of $\mathfrak{t}_\omega$ (which in turn is identified to the space of affine functions of $z$) and a subspace of Killing potentials of zero integral of lifts of Killing vector fields on $(\hat{S}, g_{\hat{S}})$. We have shown in the proof of Lemma \ref{lem} that the later potentials are all of the form $\sum_j (\langle p_j, z \rangle + c_j) f_j$ where $f_j$ is a function on $\hat{S}$ of zero integral with respect to $\omega_{\hat{S}}^d$. As the scalar curvature of a compatible metric is a function of $z$ only (see \cite{17}, we assume $\text{Scal}_j$ are constant) it follows from \cite{18} that the $L^2$-projection of $\text{Scal}_g$ to $\mathfrak{g}_\omega$ lies in $\mathfrak{t}_\omega$. This shows that the extremal vector field lies in $\mathfrak{t}$ and that the projection of $\text{Scal}_g$ orthogonal to the Killing potentials of $g$ takes the form:

$$\text{Scal}^\perp_g = \langle A, z \rangle + B + \text{Scal}_g,$$

where

$$\begin{cases}
\sum_s \alpha_s A_s + \alpha B + 2\beta = 0, \\
\sum_s \alpha_{rs} A_s + \alpha_r B + 2\beta_r = 0,
\end{cases}
\tag{9}$$

with

$$\alpha = \int_\Delta p(z) dv, \quad \alpha_r = \int_\Delta z_r p(z) dv, \quad \alpha_{rs} = \int_\Delta z_r z_s p(z) dv,$$

$$\beta = \frac{1}{2} \int_\Delta \text{Scal}_g p(z) dv = \int_{\partial \Delta} p(z) d\sigma + \frac{1}{2} \int_\Delta \left(\sum_j \frac{\text{Scal}_j}{\langle p_j, z \rangle + c_j}\right) p(z) dv,$$

$$\beta_r = \frac{1}{2} \int_\Delta \text{Scal}_g z_r p(z) dv = \int_{\partial \Delta} z_r p(z) d\sigma + \frac{1}{2} \int_\Delta \left(\sum_j \frac{\text{Scal}_j}{\langle p_j, z \rangle + c_j}\right) z_r p(z) dv.$$

Here $d\sigma$ is the $(\ell - 1)$-form on $\partial \Delta$ with $u_i \wedge d\sigma = -dv$ on the face $F_i$ with normal $u_i$. These formulæ are immediate once one applies the divergence theorem and the boundary conditions \cite{4} for $H$, noting that the normals are inward normals, which introduces a sign compared to the usual formulation of the divergence theorem.

The extremal vector field of $(M, g, J, \omega)$ is $-(A, K)$, where $K \in C^\infty(M, TM) \otimes \mathfrak{t}^*$ is the generator of the $\mathbb{T}$ action.

### 3.6. The extremal equation and stability of its solutions under small perturbation.

It follows from the considerations in Sect. 3.3 that on a given manifold $M$ of the type we consider, finding a \textit{compatible} extremal Kähler metric $(g, \omega)$ of the form \cite{8} reduces to solving the equation (for a unknown symplectic potential $U \in S(\Delta)$)

$$\langle A, z \rangle + B + \sum_{j=1}^N \frac{\text{Scal}_j}{c_j + \langle p_j, z \rangle} - \frac{1}{p(z)} \sum_{r,s} \frac{\partial^2}{\partial z_r \partial z_s} (p(z) H_{rs}) = 0,$$

where

- $(H_{rs}) = H = (\text{Hess}(U))^{-1}$;
• $(c_j, p_j, \text{Scal}_j)$ are fixed constants;
• $p(z) = \prod_{j=1}^N (c_j + \langle p_j, z \rangle)^{d_j}$ is strictly positive on $\Delta^0$ but vanishes on the blow-down faces $F_b$, $b \in B$;
• $A$ and $B$ are expressed in terms of $(c_j, p_j, \text{Scal}_j)$ by (9).

Recall from Sect. 3.3 that the real constants $c_a$, $a \in A$ parametrize compatible Kähler classes on a given manifold $M$. A general result of LeBrun–Simanca [46] affirms that Kähler classes admitting extremal Kähler metric form an open subset of the Kähler cone. We want to obtain a relative version of this result, by showing that compatible Kähler classes which admit a compatible extremal Kähler metric is an open condition on the parameters $c_a$.

We will state and prove our stability result in a slightly more general setting, by considering $(10)$ as a family of differential operators on $S(\Delta)$, parametrized by $\lambda \in \{ (c_a, p_a, \text{Scal}_a), a \in A \}$ (thus $\lambda$ takes values in a $(2 + \ell)|A|$-dimensional euclidean vector space). For any $\lambda$ such that $\langle p_a, z \rangle + c_a > 0$ on $\Delta$, we consider

$$(11) \quad P_\lambda(U) = \langle A_\lambda, z \rangle + B_\lambda + \sum_{j=1}^N \frac{\text{Scal}_j}{c_j + \langle p_j, z \rangle} - \frac{1}{p_\lambda p_0} \sum_{r,s=1}^{\ell} \frac{\partial^2}{\partial z_r \partial z_s} \left( p_\lambda p_0 H_{rs} \right),$$

where $(H_{rs}) = \text{Hess}(U)^{-1}$, $p_\lambda(z) = \prod_{a \in A} (\langle p_a, z \rangle + c_a)^{d_a}$, $p_0(z) = \prod_{b \in B} (\langle u_b, z \rangle + c_b)^{d_b}$, and $A_\lambda, B_\lambda$ are introduced by (9). The central result of this section is the following one.

**Proposition 2.** Let $(g_0, \omega_0)$ be a compatible extremal Kähler on $M$, with symplectic potential $U_0$ and parameters $\lambda_0 = (c_a^0, p_a^0, \text{Scal}_a^0)$, $a \in A$. Then there exists $\varepsilon > 0$ such that for any $\lambda$ with $|\lambda - \lambda_0| < \varepsilon$ there exists a symplectic potential $U_\lambda \in S(\Delta)$ such that $P_\lambda(U_\lambda) = 0$ on $\Delta^0$.

The proof of this proposition has several steps and will occupy the rest of this section.

It is not immediately clear from (11) that $P_\lambda$ is a well-defined differential operator; in the presence of blow-downs, the terms $\frac{\text{Scal}_j}{c_j + \langle p_a, z \rangle}$ and $\frac{1}{p_\lambda p_0}$ become degenerate on the boundary of $\Delta$. Of course, for $\lambda = \lambda_0$ we know from (10) that $P_\lambda(U) = \text{Scal}_j^+$ where $g$ is the compatible metric on $M$ corresponding to $U$, and $\text{Scal}_j^+$ is the $L^2$-projection of the scalar curvature to the space of functions orthogonal to the Killing potentials of $g$. However, for generic values of $\lambda$ the data $(c_a, p_a, \text{Scal}_a)$ are not longer associated with a compatible Kähler class on a smooth manifold: for this to be true $p_a$ and $\text{Scal}_a$ must satisfy integrality conditions. To overcome this technical difficulty, we are going to rewrite our equation on the smooth compact manifold $W$. (Note that for $b \in B$, $p_b = u_b, c_b, \text{Scal}_b = 2d_b(d_b + 1)$ are fixed in our construction.)

Recall from Sect. 3.3 that any symplectic potential $U \in S(\Delta)$ introduces a compatible Kähler metric $(g_W, \omega_W)$ on the manifold $W$ obtained by blowing down $\hat{W} = P_0 \times_T V$. Thus, $(W, g_W, \omega_W)$ itself is obtained by the generalized Calabi construction with $S$ being a point.

By a well-known result of G. W. Schwarz [63], the space $C^\infty(V)^T$ of $T$-invariant smooth functions on the toric symplectic manifold $(V, \omega_V, T)$ is identified with the space of pullbacks (via the momentum map $z$) of smooth functions $C^\infty(\Delta)$ on $\Delta$; similarly, the space of smooth $T$-invariant functions on $W$ (resp. on $M$) which are constant on the inverse images of the momentum map $z$ is identified with the space $C^\infty(\Delta)$. We will use implicitly these identification throughout. Occasionally, when we want to emphasize the
dependence of this identification on \( z \), we will denote these isomorphisms by \( S_z \). With this convention, we have

**Lemma 6.** Let \( U \in S(\Delta) \) be a symplectic potential of a compatible Kähler metric \( g_W \) on \((V,\omega_V,\nabla)\) and \((g_W,\omega_W)\) be the corresponding compatible Kähler metric on \( W \). Then, for any \( \lambda \) such that \( \langle p_\lambda, z \rangle + c_\lambda > 0 \) on \( \Delta \),

\[
\begin{aligned}
P_\lambda(U) &= \langle A_\lambda, z \rangle + B_\lambda + \sum_{a \in A} c_a \frac{\text{Scal}_a}{\langle p_\lambda, z \rangle} + \text{Scal}_W \\
&\quad - \frac{1}{p_\lambda(z)} \sum_{r,s=1}^\ell \left( \frac{\partial^2 p_\lambda}{\partial z_r \partial z_s} (z) g_W(K_r,K_s) \right) \\
&\quad + \frac{2}{p_\lambda(z)} \sum_{r=1}^\ell \left( \frac{\partial p_\lambda}{\partial z_r} (z) \Delta_W z_r \right),
\end{aligned}
\]

where \( \text{Scal}_W \) and \( \Delta_W \) respectively denote the scalar curvature and the Riemannian Laplacian of \( g_W \), and \( dz_r = -\omega_W(K_r, \cdot) \).

**Proof.** We work on the open dense subset \( W^0 = P_0 \times V^0 \) where the compatible metric \((g_W,\omega_W)\) takes the explicit form \( \Theta \). The formula \( (7) \) for the scalar curvature of the compatible metric \( g_W \) then specifies to

\[
\text{Scal}_W = \sum_{b \in B} \frac{\text{Scal}_b}{\langle p_\lambda, z \rangle + c_b} - \frac{1}{p_\lambda(z)} \sum_{r,s=1}^\ell \frac{\partial^2}{\partial z_r \partial z_s} \left( p_0(z) H_{rs} \right).
\]

Still using the explicit form \( (5) \) of the Kähler structure, we calculate that for the pullback to \( W \) of a smooth function \( f(z) \) on \( \Delta \)

\[
\begin{aligned}
d\delta_W^* f &= d \left( \sum_{r,s=1}^\ell \frac{\partial f}{\partial z_s} H_{rs}(\theta_0)_r \right) \\
&= \sum_{k,r,s=1}^\ell \frac{\partial}{\partial z_k} \left( \frac{\partial f}{\partial z_s} H_{rs} \right) dz_k \wedge (\theta_0)_r + \sum_{b \in B} \left( \sum_{r,s=1}^\ell \frac{\partial f}{\partial z_s} H_{rs} p_{br} \right) \omega_b,
\end{aligned}
\]

where the decompositions \( \theta_0 = (\theta_0)_1, \ldots, (\theta_0)_\ell \) and \( p_b = (p_{b1}, \ldots, p_{b\ell}) \) are with respect to the chosen basis of \( t \) and \( t^* \). Wedging with \( \omega_W \), we obtain the following expression for the Laplacian

\[
\Delta_W f = -\frac{1}{p_0(z)} \sum_{r,s=1}^\ell \frac{\partial}{\partial z_r} \left( p_0(z) \frac{\partial f}{\partial z_s} H_{rs} \right).
\]

Specifying \( (13) \) to \( f = z_r \) and putting the above formulae back in \( (11) \) implies the lemma. \( \square \)

Note that \( \frac{1}{p_\lambda(z)} \) and \( \frac{\text{Scal}_a}{\langle p_\lambda, z \rangle} \) pull back to smooth functions on \( W \) for \( \lambda \) such that \( c_a + \langle p_\lambda, z \rangle > 0 \) on \( \Delta \), and \( A_\lambda \) and \( B_\lambda \) are well-defined and depend smoothly on \( \lambda \) (at least for \( \lambda \) close to \( \lambda_0 \)). Thus, Lemma \( (8) \) implies that \( P_\lambda \) is a fully non-linear 4-th order differential operator which depends smoothly on \( \lambda \) (for \( \lambda \) sufficiently close to \( \lambda_0 \)). It follows that \( P_\lambda(U) \in C^\infty(\Delta) \) for any \( U \in S(\Delta) \).

Our problem is formulated in terms of compatible Kähler metrics on \( V \) (or, equivalently, on \( W \) and \( M \)) with respect to a fixed symplectic form \( \omega_V \) (resp. \( \omega_W \) and \( \omega \)). This introduces the space of symplectic potentials \( S(\Delta) \) where we have to work with smooth
functions on $\Delta^0$ which have a prescribed boundary behaviour on $\partial\Delta$. Our lack of understanding of the convergence in this space (with respect to suitable Sobolev norms) leads us to make an additional technical step and reformulate our initial problem as an existence result on a suitable subspace of the space $\mathcal{M}_\Omega(M)^G \cong \{f \in C_0^\infty(M)^G : \omega_0 + dd^c f > 0\}$ of $G$-invariant Kähler metrics in the Kähler class of $(g_0, J_0, \omega_0)$, where $C_0^\infty(M)^G$ denotes the space of $G$-invariant smooth functions on $M$ of zero integral with respect to $\omega_0^m$ (thus $\mathcal{M}_\Omega(M)^G$ is viewed as an open set in $C_0^\infty(M)^G$ with respect to $\| \cdot \|_{C^2}$). Once this interpretation is achieved, we will apply the implicit function theorem along the lines of the proof of Lemma 4.

First of all, note that the Frechét space $C^\infty(\Delta)$ pulls back via $z$ to a closed subspace in $C^\infty(\overline{V})^T$, $C^\infty(\overline{W})^T$ and $C^\infty(M)^G$, where $T$ (resp. $G$) is a maximal torus in $\text{Aut}_0(\overline{W})$ (resp. $\text{Aut}_0(M)$) containing $\mathbb{T}$, as in Proposition 11 this follows easily from the description of the Lie algebras of $T$ and $G$ given in Lemma 5. Furthermore, by (3), the corresponding normalized subspaces of functions with zero integral for the measures $p_\lambda(z)p_0(z)\text{Vol}_{\omega_0}$, $p_\lambda(z)\text{Vol}_{\omega_0}$, respectively, are identified with the space $C_0^\infty(\Delta)$ of smooth functions of zero integral with respect to the volume form $d\mu_0 = p_\lambda(z)p_0(z)dv$ on $\Delta^0$: this normalization will be used throughout.

Secondly, to adopt the classical point of view of Kähler metrics on a fixed complex manifold, we consider the Fréchet space $\mathcal{M}_\Omega(V)^T \cong \{f \in C_0^\infty(\Delta) : \omega_0^V + dd^c_V f > 0\}$ of $\mathbb{T}$-invariant Kähler metrics in the Kähler class $\Omega = [\omega_0^V]$, where the complex structure on $V$ (resp. on $W$ and $M$) is determined (and will be fixed throughout) by the initial compatible metric $(g_0, \omega_0)$; similarly, we introduce the spaces $\mathcal{M}_\Omega(W)^T$ and $\mathcal{M}_\Omega(M)^G$ of Kähler metrics in the given Kähler class which are invariant under a maximal torus (see Proposition 11). These three spaces are interrelated by the generalized Calabi construction as follows.

**Lemma 7.** Let $\tilde{\omega}_V = \omega_0^V + dd^c_V f$ be a Kähler metric in $\mathcal{M}_\Omega(V)^T$. Then $\tilde{\omega}_W = \omega_0^W + dd^c_W f$ and $\tilde{\omega} = \omega_0 + dd^c_M f$ define Kähler metrics in $\mathcal{M}_\Omega(W)^T$ and $\mathcal{M}_\Omega(M)^G$, respectively, such that $\tilde{\omega}_V, \tilde{\omega}_W$ and $\tilde{\omega}$ are linked by the generalized Calabi construction on $M$, with respect to the data $(\Delta, \hat{S}, \hat{\theta}, \omega_j)$ of the initial metric $\omega_0$, but with momentum co-ordinate $\hat{\xi} = z + d\tilde{\nu}(f(K))$.

**Proof.** A direct calculation based on the expressions of $dd^c_V f, dd^c_W f$ and $dd^c_M f$, see (12); we leave the details to the reader. $\square$

Lemma 7 allows us to introduce subspace of compatible Kähler metrics $\mathcal{M}_\Omega^{\text{comp}}(W) = \mathcal{M}_\Omega(W)^T \cap C_0^\infty(\Delta)$ and $\mathcal{M}_\Omega^{\text{comp}}(M) = \mathcal{M}_\Omega(M)^G \cap C_0^\infty(\Delta)$ (within a fixed Kähler class $\Omega$) and identify each of them with the space $\mathcal{M}_\Omega(V)^T$. The correspondence which associates to any $\tilde{\omega}_W \in \mathcal{M}_\Omega^{\text{comp}}(W)$ (resp. $\tilde{\omega} \in \mathcal{M}_\Omega^{\text{comp}}(M)$) the corresponding symplectic potential $\tilde{U} \in \mathcal{S}(\Delta)^{\Gamma}$ allows us to reformulate our existence problem on the space $\mathcal{M}_\Omega^{\text{comp}}(W)$ as follows: for any $\lambda$ sufficiently close to $\lambda_0$ (so that $A_\lambda, B_\lambda$ are well-defined

\footnote{For a metric $\tilde{\omega}_V = \omega_0^V + dd^c_V f \in \mathcal{M}_\Omega(V)^T$ the corresponding symplectic potential $\tilde{U}$ is linked to $f$ by a Legendre transform [2 34]; this is true fibrewise for metrics in $\mathcal{M}_\Omega^{\text{comp}}(W)$ and $\mathcal{M}_\Omega^{\text{comp}}(M)$.}
and \( \langle p_a, \bar{z} \rangle + c_a > 0 \) on \( \Delta \), we consider the family of differential operators on \( \mathcal{M}_\Omega(W)^T \)

\[
Q_\lambda(\tilde{\omega}_W) = \frac{\lambda}{p_\lambda(\bar{z})} \langle A_\lambda, \bar{z} \rangle + B_\lambda + \sum_{j=1}^N \frac{\text{Scal}_j}{c_j + \langle p_j, \bar{z} \rangle}
\]

(14)

\[
+ \tilde{\text{Scal}}_W - \frac{1}{p_\lambda(\bar{z})} \sum_{r,s} \left( \left( \frac{\partial^2 p_\lambda}{\partial z_r \partial z_s} \right)(\bar{z}) \tilde{g}_W(K_r, K_s) \right)
\]

\[
+ \frac{2}{p_\lambda(\bar{z})} \sum_r \left( \left( \frac{\partial p_\lambda}{\partial z_r} \right)(\bar{z}) \tilde{\Delta}_W(\bar{z}_r) \right),
\]

where \( \bar{z} = z + d\bar{w}.f(K) \) is the momentum map of \( T \) with respect to the Kähler form \( \tilde{\omega}_W = \omega_0^W + d\bar{w}.f \) of the Kähler metric \( \tilde{g}_W \), and \( \tilde{\text{Scal}}_W \) (resp. \( \tilde{\Delta}_W \)) denote the scalar curvature (resp. laplacian) of \( \tilde{g}_W \). Thus, by Lemmas 6 and 7, any Kähler metric \( \tilde{\omega}_W \in \mathcal{M}_\Omega^{\text{comp}}(W) \) for which \( Q_\lambda(\tilde{\omega}_W) = 0 \) gives rise to a symplectic potential \( U \in S(\Delta) \) solving \( P_\lambda(U) = 0 \).

The positive factor \( \frac{\lambda}{p_\lambda(\bar{z})} \) in front of \( Q_\lambda \) is introduced so that for any compatible metric \( \tilde{\omega}_W \in \mathcal{M}_\Omega^{\text{comp}}(W) \), the function \( S_\lambda(Q_\lambda(\tilde{\omega}_W)) \) is \( L^2 \)-orthogonal with respect to the measure \( d\mu_0 = p_\lambda p_0 dv \) on \( \Delta \) to the space of affine functions on \( \lambda^* \), where, we recall, \( S_\lambda \) denotes the identification of \( T \)-invariant smooth functions on \( W \) which are constant on the inverse images of \( \bar{z} \) (equivalently of \( z \)) with pullbacks via \( \bar{z} \) of smooth functions on \( \Delta \). Indeed, by Lemma 6 \( p_\lambda(\bar{z})p_0(\bar{z})Q_\lambda(\tilde{\omega}_W) = P_\lambda(U)p_\lambda(\bar{z})p_0(\bar{z}) \), so integrating by parts the r.h.s. of (14) and using (1) we get

\[
\int_\Delta P_\lambda(U)f(z)p_\lambda(z)p_0(z)dv = -\int_\Delta \langle H, \text{Hess}(f) \rangle p_\lambda(z)p_0(z)dv
\]

\[
+ \int_\Delta \left( \langle A, z \rangle + B + \sum_{j=1}^N \frac{\text{Scal}_j}{c_j + \langle p_j, z \rangle} \right)f(z)p_\lambda(z)p_0(z)dv
\]

\[
+ 2\int_{\partial\Delta} f(z)p_\lambda(z)p_0(z)d\sigma,
\]

which holds for any smooth function \( f(z) \). When \( f \) is affine, the first term in the r.h.s is clearly zero, while by the definition \( A_\lambda \) of \( A \) and \( B \) the sum of the two other terms is zero too; our claim then follows by Lemma 7 and the expression \( \text{Vol} \) for the volume form of the compatible metric \( \tilde{\omega}_W \).

Let \( \Pi_0 \) denote the orthogonal \( L^2 \)-projection of \( C^\infty(\Delta) \) to the finite dimensional subspace of affine functions on \( \lambda^* \) with respect to the measure \( d\mu_0 = p_\lambda p_0 dv \) on \( \Delta \), and \( C^\infty_\lambda(\Delta) \) be the kernel of \( \Pi_0 \). We then consider the map \( \Psi : \mathcal{U} \rightarrow \mathbb{R}^{(2+|\lambda|)} \times C^\infty_\lambda(\Delta) \), defined in a small neighbourhood \( \mathcal{U} \) of \( (\alpha_0, 0) \in \mathbb{R}^{(2+|\lambda|)} \times C^\infty_\lambda(\Delta) \) by

\[
\Psi(\lambda, f) = \left( \lambda, (\Pi_0(S_\lambda(Q_\lambda(\tilde{\omega}_W)))) \right),
\]

where \( \tilde{\omega}_W = \omega_0^W + d\bar{w}.f \) is a compatible metric on \( \mathcal{M}_\Omega^{\text{comp}}(W) \). Note that if \( f \) has sufficiently small \( C^1 \)-norm, the equation \( (\Pi_0) \circ (S_\lambda(Q_\lambda(\tilde{\omega}_W))) = 0 \) is satisfied if and only if \( Q_\lambda(\tilde{\omega}_W) = 0 \): this follows from the fact that \( \Pi_0 \circ S_\lambda \circ S_\lambda^{-1} \) defines a continuous family of linear endomorphisms of the finite dimensional space of affine functions on \( \lambda^* \), with the identity corresponding to \( \tilde{\omega}_W = \omega_0^W \); thus \( \Pi_0 \circ S_\lambda \circ S_\lambda^{-1} \circ \Pi_0 \) is invertible for \( \tilde{\omega}_W \) close to \( \omega_0^W \) and, hence (by using that \( \Pi_0(S_\lambda(Q_\lambda(\tilde{\omega}_W))) = 0 \)) we get

\[
\Pi_0 \circ S_\lambda \circ S_\lambda^{-1} \circ (\Pi_0) \left( S_\lambda(Q_\lambda(\tilde{\omega}_W)) \right) = -\Pi_0 \circ S_\lambda \circ S_\lambda^{-1} \circ \Pi_0 \left( S_\lambda(Q_\lambda(\tilde{\omega}_W)) \right)
\]

which is zero if \( S_\lambda(Q_\lambda(\tilde{\omega}_W)) = 0 \) i.e. \( Q_\lambda(\tilde{\omega}_W) = 0 \).
By the discussion above, we are in position to complete the proof of Proposition \[2\] by applying the inverse function theorem to the extension of \( \Psi \) to suitable Sobolev spaces, together with elliptic regularity (as in \[40\], see also the proof of Lemma \[1\]) in order to find a family \( \tilde{\omega}_0^\lambda = \omega_0^\lambda + d\bar{d}_W f_\lambda \) of smooth compatible metrics satisfying \( \Psi(\lambda, f_\lambda) = (\lambda, 0) \) for \( |\lambda - \lambda_0| < \varepsilon \).

Let us first introduce the functional spaces we will work on. Recall that \( C_+^\infty(\Delta) \) is seen as a (closed) Fréchet subspace of the space of \( T \)-invariant smooth functions on \( W \) (resp. \( G \)-invariant smooth functions on \( M \)) which are constant on the inverse images of the momentum map \( z \) for the sub-torus \( T \). It follows from the description of the Lie algebra of \( T \) (resp. \( G \)) given in Lemma \[5\] that \( C_+^\infty(\Delta) \) is precisely the intersection of \( C^\infty(\Delta) \) with the space \( C_+^\infty(W) \) of \( T \)-invariant smooth functions on \( W \) which are \( L^2 \)-orthogonal with respect to \( p_{\lambda_0} \text{Vol}_{\omega_0^0} \) to Killing potentials of \( g_0^0 \) (resp. the space \( C_+^\infty(M) \) of \( G \)-invariant smooth functions on \( M \) which are \( L^2 \)-orthogonal with respect to \( \text{Vol}_{\omega_0} \) to Killing potentials of \( g_0 \)).

We let \( L^{2k}(W, \Delta) \) (resp. \( L^{2k}(M, \Delta) \)) be the closure of \( C_+^\infty(\Delta) \) with respect to the Sobolev norm \( \| \cdot \|_{2, \Delta} \) on \( W \) for the measure \( p_{\lambda_0}(z) \text{Vol}_{\omega_0^0} \) and riemannian metric \( g_0^0 \) (resp. the Sobolev norm \( \| \cdot \|_{2, \Delta} \) on \( M \) with respect to \( \text{Vol}_{\omega_0} \) and \( g_0 \)). For \( k \gg 1 \), the Sobolev embedding \( L^{2k+4}(W, \Delta) \subset C^3_\Delta(\Delta) \) allows us to extend the differential operator \( \Psi \) to a \( C^1 \)-map from a neighbourhood of \( (\lambda_0, 0) \in \mathbb{R}^{(2+\theta)|A|} \times L^{2k+4}_{\psi}(W, \Delta) \) into \( L^{2k}(W, \Delta) \), such that \( \Psi(\lambda_0, 0) = 0 \); furthermore, as the principal part of \( Q_\lambda \) is concentrated in the term \( \text{Scal}_W \), one can see that \( \Psi \) is a fourth-order quasi-elliptic operator \[40\].

Now, in order to apply the inverse function theorem, it is enough to establish the following

**Lemma 8.** Let \( T_0 : C_+^\infty(\Delta) \rightarrow C_+^\infty(\Delta) \) be the linearization at \( \omega_0^0 \in \mathcal{M}_\Omega^\text{comp}(W) \) of \( Q_{\lambda_0} \). Then \( T_0 \) is an isomorphism of Fréchet spaces.

**Proof.** Let \((g_0, J_0, \omega_0)\) be the compatible extremal Kähler metric on \( M \) corresponding to the initial value \( \lambda = \lambda_0 \). For any function \( f \in C_+^\infty(\Delta) \) we consider the compatible Kähler metric \( \tilde{g} \) on \( M \), with Kähler form \( \tilde{\omega} = \omega_0 + dd^c_M f \) and the compatible Kähler metric \( \tilde{g}_W \) on \( W \) with Kähler form \( \tilde{\omega}_W = \omega_0^0 + dd^c_W f \). We saw already in Sect. \[35\] that for \( \lambda = 0 \), \( Q_{\lambda_0} (\tilde{\omega}_W) = P_{\lambda_0}(U) = \text{Scal}_g^\perp U \), where \( U \) and \( \text{Scal}_g^\perp \) are the symplectic potential and normalized scalar curvature of \( \tilde{g} \). It then follows from \[32\] [45] that the linearization \( T_0 \) of \( Q_{\lambda_0} \) (at \( \omega_0^0 \)) is equal to \(-2\) times the Lichnerowicz operator \( L \) of \((g_0, \omega_0)\) acting on the space of pullbacks (via \( z \)) of functions in \( C_+^\infty(\Delta) \). We have already observed in the proof of Lemma \[3\] that \( L \) is an isomorphism when restricted to the space \( C_+^\infty(M) \) of \( G \)-invariant smooth functions \( L^2 \)-orthogonal to Killing potentials of \( g_0 \). The main point here is to refine this by showing that \( L \) is an isomorphism when restricted to subspace \( C_+^\infty(\Delta) \), the only missing piece being the surjectivity.

Suppose for a contradiction that \( L : C_+^\infty(\Delta) \rightarrow C_+^\infty(\Delta) \) is not surjective. Considering the extension of \( L \) to an operator between the Sobolev spaces \( L^{2k}(M, \Delta) \rightarrow L^{2k}(M, \Delta) \) (by elliptic theory \( L \) is a closed operator), our assumption is then equivalent to the existence of a non-zero function \( u \in L^2_0(M, \Delta) \) such that, for any \( \phi \in C_+^\infty(\Delta) \), \( L(\phi) \) is \( L^2 \) orthogonal to \( u \). As any sequence of functions converging in \( L^2(M) \) has a point-wise converging subsequence, \( u = u(z) \) is (the pullback to \( M \) of) a \( L^2 \)-function on \( \Delta \), and using \[45\] we have

\[
\int_{\Delta} L(\phi) u_0^m = \int_{\Delta} L(\phi) u(z)p(z)dv = 0.
\]
We claim that (15) implies

\[ \int_M L(f)u_0^\alpha = 0 \]

for any \( f \in C_\infty^0(M)^G \). This would be a contradiction because \( L \) extends to an isomorphism between the closures \( L^2_\perp(M)^G \) and \( L^2_\perp(M)^G \) of \( C_\infty^0(M)^G \) in the corresponding Sobolev spaces on \( M \).

It is enough to establish (16) by integrating on \( M^0 = z^{-1}(\Delta^0) \) (which is the complement of the union of submanifolds of real codimension at least 2).

The Lichnerowicz operator \( L \) has the following general equivalent expression \[32, 45\]

\[ L(f) = \frac{1}{2} \Delta g_0 f + g_0(dd^c f, \rho_{g_0}) + \frac{1}{2} g_0(df, d\text{Scal}_{g_0}), \]

where \( \rho_{g_0} \) is the Ricci form of \((g_0, J_0)\) and \( \Delta_{g_0} \) is its laplacian. We will use the specific form (6) of \( g_0 \) to express the r.h.s of the above equality in terms of the geometry of \((V, g_{N_0})\) and \((\hat{S}, g_{\hat{S}})\).

Let \( f \) be any \( G \)-invariant (and hence \( T \)-invariant) smooth function on \( M \). It can be written on \( M^0 \) as a smooth function depending on \( z \) and \( \hat{S} \) and, for any \( s \in \hat{S} \), we will denote by \( f_s(z) = f(z, s) \) the corresponding smooth function of \( z \) (Note that, as the pullback of \( f \) to \( \hat{M} \) is smooth, \( f_s(z) \) is a smooth function on \( \Delta \), not only on \( \Delta^0 \).) Similarly, for any \( z \in \Delta^0 \), \( f_s(z) = f(z, s) \) stands for the corresponding smooth function on \( \hat{S} \).

Using \[4\] Prop. 7] and the specific form (6) of \( g_0 \), it is straightforward to check that on \( M^0 \) we have

\[
\begin{align*}
dd^c f &= \sum_{k, r, t = 1}^\ell \frac{\partial}{\partial z_k} (\frac{\partial f}{\partial z_t} H_{rt}) d z_k \wedge \hat{\theta}_r + \sum_{j = 1}^N \left( \sum_{r, t = 1}^\ell \frac{\partial f}{\partial z_t} H_{rt} p_{jr} \right) \omega_j \\
&\quad + \sum_{r = 1}^\ell \left( d^c_{\hat{S}} \left( \sum_{s = 1}^\ell \frac{\partial f}{\partial z_t} H_{rt} \right) \wedge \hat{\theta}_r + d^c_{\hat{S}} \left( \sum_{t = 1}^\ell \frac{\partial f}{\partial z_t} H_{rt} \right) \wedge J \hat{\theta}_r \right) \\
&\quad + d^c f \wedge \Delta_{g_0} f = \Delta_{S, z} f + \Delta_{g_0} f, \\
\rho_{g_0} &= \sum_{j = 1}^N \rho_j - \sum_{k, r, t = 1}^\ell \frac{\partial}{\partial z_k} \left( \frac{1}{2p(z)} \frac{\partial (p(z) H_{rt})}{\partial z_t} \right) d z_k \wedge \hat{\theta}_r \\
&\quad - \frac{1}{2p(z)} \sum_{j = 1}^N \left( \sum_{r, t = 1}^\ell \frac{\partial (p(z) H_{rt})}{\partial z_t} p_{jr} \right) \omega_j, \\
\text{Scal}_{g_0} &= \text{Scal}_{\hat{S}, z} - \frac{1}{p(z)} \sum_{r, t = 1}^\ell \frac{\partial^2}{\partial z_r \partial z_t} (p(z) H_{rt}),
\end{align*}
\]

where

- \( p(z) = \prod_{j = 1}^N ((p_j, z) + c_j)^{d_j} \);
- \( \hat{\theta} = (\hat{\theta}_1, \ldots, \hat{\theta}_\ell) \) and \( p_j = (p_{j1}, \ldots, p_{j\ell}) \) with respect to the chosen basis of \( t \);
- \( d_{\hat{S}} \) and \( d_{\hat{S}}^c \) are the differential and the \( d^c \)-operator acting on functions and forms on \( \hat{S} \);
- \((g_j, \omega_j)\) are the product CSC Kähler factors of the Kähler metric \((g_{\hat{S}}, \omega_{\hat{S}})\), with respective Ricci forms \( \rho_j \) and laplacians \( \Delta_{g_j} \).
\[ g_{\tilde{S},z} = \sum_{j=1}^{N} (g_{j}, z) + c_j g_j \]
is the quotient Kähler metric on \( \tilde{S} \) at \( z \), and \( \omega_{\tilde{S},z}, \text{Scal}_{\tilde{S},z} \) and \( \Delta_{\tilde{S},z} \) denote its Kähler form, scalar curvature and laplacian, respectively.

Substituting back in (17), we obtain
\[
L(f) = L(f_0) + \Delta_{\tilde{S},z}((\Delta_{g_0} f_0) z) + \Delta_{g_0}((\Delta_{\tilde{S},z} f_0) z)
+ \sum_{j=1}^{N} R_j(z) \Delta_{g_j}(f_0),
\]
where \( L_{\tilde{S},z} \) is the Lichnerowicz operator of \( g_{\tilde{S},z} \), and \( R_j(z) \) are coefficients (that can be found explicitly from the above formulae) depending only on \( z \), and such that \( p(z) R_j(z) \) are smooth on \( \Delta \).

If we integrate the above expression for \( L(f) \) against \( u(z) \) (by using (8)) we get that \( \int_M L(f) u \omega_0^n \) is a non-zero constant multiple of
\[
\int_{\tilde{S}} \left( \int_{\Delta^0} (\int_{\Delta^0} L(f_0) u(z) p(z) dv) \omega_0^d + \int_{\tilde{S}} \left( \int_{\Delta^0} \Delta_{g_0}((\Delta_{\tilde{S},z} f_0) z) u(z) p(z) dv \right) \omega_0^d + \int_{\tilde{S}} \left( \int_{\Delta^0} \Delta_{g_0}((\Delta_{\tilde{S},z} f_0) z) \omega_0^d \right) u(z) dv
+ \sum_{j=1}^{N} \int_{\Delta^0} \left( \int_{\Delta^0} \Delta_{g_j}(f_0) \omega_0^d \right) p(z) R_j(z) u(z) dv.\]

To see that all the terms vanish, note that the first term is zero by (15); the third and fourth terms are zero because \( L_{\tilde{S},z} \) and \( \Delta_{\tilde{S},z} \) are self-adjoint (with respect to \( \omega_{\tilde{S},z} \)) and therefore their images are \( L^2 \)-orthogonal to constants on \( \tilde{S} \). The fifth term is also zero because \( \Delta_{g_j}(f) \) is \( L^2 \)-orthogonal to constants on \( \tilde{S} \) with respect to \( \omega_{\tilde{S}} \): this follows easily from the local product structure of \( g_{\tilde{S}} \). For the second term one uses that \( \Delta_{g_0} \) defines a self-adjoint operator on \( C^\infty(\Delta) \) with respect to the measure \( p(z) dv \): thus, for any smooth function \( \phi(z) \) on \( \Delta \),
\[
\int_{\tilde{S}} \left( \int_{\Delta^0} \Delta_{g_0}(((\Delta_{\tilde{S},z} f_0) z) \phi(z) p(z) dv \right) \omega_0^d = \int_{\Delta^0} \left( \int_{\tilde{S}} \Delta_{g_0}(f_0) \omega_0^d \right) (\Delta_{g_0} \phi) dv = 0
\]
because \( \Delta_{\tilde{S},z} f_0 \) is \( L^2 \)-orthogonal to constants on \( \tilde{S} \); as \( u \) is in the closure in \( L^2 \) of pullbacks of smooth functions on \( \Delta \), the second term vanishes too.

This concludes the proof of the lemma. \( \square \)

An immediate consequence of Proposition 2 is the following

**Corollary 1.** The existence of a compatible extremal Kähler metric is an open condition on the set of admissible Kähler classes on \( M \).

**Proof.** As we have already observed, the admissible Kähler classes are parametrized by the real constants \( c_a \) for \( a \in A \). We thus apply Proposition 2 by taking \( \lambda = (c_a, p_0, \text{Scal}_0) \). \( \square \)

### 3.7. Proof of Theorem 3

To deduce Theorem 3 from Proposition 2 we observe that the differential operators (11) satisfy \( P_\lambda = P_\lambda t \) for any real number \( t \neq 0 \).

On any Kähler manifold \((M, g, \omega)\) obtained by the generalized Calabi construction with data \( \lambda = (c_a, p_a, \text{Scal}_a) \), we can consider the sequence of differential operators \( P_\lambda k \) where \( \lambda_k = (c_a + k, p_a, \text{Scal}_a) \). The differential operator \( P_\lambda k \) is the same as \( P_\lambda \) and \( \frac{\lambda_k}{k} \) converges when \( k \to \infty \) to the data corresponding to the extremal Kähler metric equation for a compatible Kähler metrics on \( W \). We then readily infer Theorem 3 from Proposition 2.
Remark 6. As any invariant Kähler metric on a toric manifold is compatible, Theorem 3 implies the existence of (compatible) extremal metrics on a rigid semisimple toric bundles $M$ over a CSC locally product Kähler manifold, in the case when there are no blow-downs and $W = V$ is a toric extremal Kähler manifold.

Remark 7. An interesting class of rigid toric bundles comes from the theory of multiplicity-free manifolds recently discussed in [25]. A typical example is obtained by taking a compact connected semisimple Lee group $G$ and a maximal torus $T \subset G$ with Lie algebra $t$; if we pick a positive Weyl chamber $t_+ \subset t$ (and identify $t$ with its dual space $t^*$ via the Killing form), for any Delzant polytope $\Delta$ contained in the interior of $t_+$, one can consider the manifold $M = p : G \times T V \rightarrow S = G/T$, where $V$ is the toric manifold with Delzant polytope $\Delta$. Note that $G$ has a structure of principal $T$-bundle over the flag manifold $S = G/T$ with a connection 1-form $\theta \in \Omega^1(G, t)$ whose curvature $\omega(z) = \langle d\theta, z \rangle$ defines a family of symplectic forms on $S$ (the Kirillov–Kostant–Souriau forms); identifying $S \cong G^\circ/B$, where $B$ is a Borel subgroup of the complexification $G^\circ$ of $G$, each $\omega(z)$ defines a homogeneous Kähler metric $g(z)$ on the complex manifold $S$ (which is therefore of constant scalar curvature); the Ricci form $\omega_S$ of $\omega(z)$ is independent of $z$, giving rise to the normal (Kähler–Einstein) metric $g_S$ on $S$. Now, for any toric Kähler metric on $V$, corresponding to a symplectic potential $U \in S(\Delta)$, one considers the Kähler metric on $M$

$$g = p^*(g(z) + kgs) + \langle dz, G, dz \rangle + \langle \theta, H, \theta \rangle, \quad \omega = p^*(\omega(z) + k\omega_S) + (dz \wedge d\theta),$$

where $G = \text{Hess}(U)$, $H = G^{-1}$, $z \in \Delta$ and $k > 0$. In this case, $G \rightarrow S = G/T$ is not necessarily a diagonalizable principal $T$-bundle over $S = G/T$ (in other words, $M = G \times T V \rightarrow S = G/T$ is a rigid but not in general semisimple toric bundle). However, most parts of the discussion in Sect. 3 do extend to this case too (see also [3]), with some obvious modifications. The key points are that (a) the volume form of $g(z) + kgs$ is a multiple $p(z)$ (depending only on $z$) of $\text{Vol}_{g_S}$: this allows to extend the curvature computations (see [3, Prop. 7]) and formula (3) to this case, (b) for any $z \in \Delta$, $g(z) + kgs$ is a CSC Kähler metric on $S$: this allows to extend the results in Sects. 3.4 and c) there is a similar formula to (7) for the scalar curvature of $g$, found by Raza [52], which allows to reduce the extremal equation for the Kähler metrics in the above form to (10) with $p_a$ being essentially the positive roots of $G$, $c_a = k$ and $\text{Scal}_a$ positive constants. Proposition 2 and its corollaries (Corollary 1 and Theorem 3) extend to this setting too. We thus get both openness and existence of extremal Kähler metrics of the above form when $V$ is an extremal toric Kähler variety and $k \gg 0$.

4. PROOF OF THEOREM 4

As another application of Theorem 3, we derive Theorem 4 from the introduction. This is the case when $V = \mathbb{C}P^\ell$ and $W = \mathbb{C}P^r$, $r \geq \ell \geq 1$ and $M = P(E_0 \oplus \cdots \oplus E_\ell) \rightarrow S$ (see Sect. 3.2). It follows from the general theory of hamiltonian 2-forms [3, 4] that any Fubini–Study metric on $\mathbb{C}P^r$ admits a rigid semisimple isometric action of an $\ell$-dimensional torus $T$, for any $1 \leq \ell \leq r$ (see in particular [3, Prop. 17] and [4, Thm. 5]): thus, $W = \mathbb{C}P^r$ admits a compatible extremal Kähler metric.

Let $\omega$ be a compatible Kähler on $M$: as the fibre is $\mathbb{C}P^r$, by re-scaling, we can assume without loss that $[\omega] = 2\pi c_1(O(1)_E) + p^*\alpha$, where $\alpha$ is a cohomology class on $S$. The form (3) of $\omega$ and the assumption on the first Chern classes $c_1(E_i)$ imply that $\alpha$ is diagonal with respect to the product structure of $S$, in the sense that it pulls back to the covering product space as $\alpha = \sum_{a \in A} q_a [\omega_a]$ for some real constants $q_a$. Therefore, $\Omega_a = 2\pi c_1(O(1)_E) + kps[\omega_S] = [\omega] + \sum_{a \in A} (k - q_a)p^* [\omega_a]$. If we choose $q$ with $q > q_a$, then $\tilde{\omega} = \omega + \sum_{a \in A} (q - q_a)p^* [\omega_a]$ is clearly a compatible Kähler metric too.
Thus, \( \Omega_k = [\tilde{\omega}] + (k - q)p^*\omega_S \) with \([\tilde{\omega}]\) compatible, and we derive Theorem 4 from the introduction as a particular case of Theorem 3.

5. Proof of Theorem 2

Suppose that \((g, \omega)\) is an extremal Kähler metric in \( \Omega_k = 2\pi c_1(O(1)_E) + kp^*\omega_S \) on \((M, J) = P(E_0 \oplus \ldots \oplus E_t) \to \Sigma\), where \(E_i\) are indecomposable holomorphic vector bundles over a compact curve \(\Sigma\) of genus \(g \geq 2\). We can assume without loss that \(\omega_S\) is the Kähler form of a constant curvature metric on \(\Sigma\) and, by virtue of Theorem 1 that the scalar curvature of \(g\) is not constant. In particular, \(\ell \geq 1\).

We have seen in Lemma 4 that the \(\ell\)-dimensional torus \(T\) acting by scalar multiplication on each \(E_i\) is maximal in the reduced automorphism group \(\widetilde{\text{Aut}}_0(M, J) \cong H^0(\Sigma, PGL(E))\). By a well-known result of Calabi [14], the identity component of any \(\Sigma\) of Kähler isometries of an extremal Kähler metric is a maximal compact subgroup of \(\text{Aut}_0(M, J)\), so we can assume without loss that \((g, \omega)\) is \(T\)-invariant.

By considering small stable deformations \(E_i(t)\) and applying Lemma 4, we can find a smooth family of extremal \(T\)-invariant Kähler metrics \((J_t, g_t, \omega_t)\), converging to \((J, \omega)\) in any \(C^k(M)\), such that \((M, J_t) \cong P(\bigoplus_{i=1}^\ell E_i(t))\), and \([\omega_t] = [\omega]\) in \(H^2_{dr}(M)\). By the equivariant Moser lemma, we can assume without loss that \(\omega_t = \omega_t\).

It is not difficult to see that for any Kähler class on \((M, J_t)\) (for \(t \neq 0\)) is compatible; this follows from the fact that the cohomology \(H^2(M) \cong H^{1,1}(M, J)\) is generated by any compatible Kähler class on \((M, J_t)\) and the pullback \(p^*\omega_S\). By Theorem 3 and the uniqueness of the extremal Kähler metrics up to automorphisms [15], for any \(t \neq 0\) we can take \(k > 0\) such that the extremal Kähler metric \((g_t, \omega)\) on \((M, J_t)\) is compatible with respect to the rigid semisimple action of the maximal torus \(T\). Strictly speaking, Theorem 4 produces a lower bound \(k_0\) for such \(k\), depending on \(J_t\). However, in our case \(|\mathcal{A}| = 1\), the simplex \(\Delta\), the moment map \(z\) and the metric on \(\Sigma\) are fixed, and the parameter \(\lambda = (c, p, \text{Scalar})\) defining the corresponding extremal equation (10) for a compatible metric on \((M, J_t, [\omega])\) is independent of \(t\); indeed, the constants \(p \in \mathbb{T}\) and \(c \in \mathbb{R}\) are determined by the first Chern classes \(c_1(E_i)\) and the cohomology class \(\Omega_k = [\omega] \in H^2_{dr}(M)\). Thus, the deformation argument used in Sect. 5.7 produces a lower bound \(k_0\) independent of \(t\), such that for any \(k > k_0\) and \(t \neq 0\), \((g_t, \omega)\) is an extremal Kähler metric in \(\Omega_k\) with respect to which the maximal torus \(T\) acts in a rigid and semisimple way.

Take a regular value \(z_0\) of the momentum map \(z\) associated to the hamiltonian action of \(T\) on \((M, \omega)\) and consider the family of Kähler quotient metrics \((\tilde{g}_t, \tilde{J}_t)\) on the symplectic quotient \(\tilde{\mathcal{S}}\). By identifying the symplectic quotient with the stable quotient, we see that \((\tilde{\mathcal{S}}, \tilde{J}_t) \cong P(E_0(t)) \times \Sigma \cdots \times \Sigma P(E_t(t)) \to \Sigma\) (see Sect. 3.2). As for \(t \neq 0\) the action of \(T\) is rigid and semisimple, \(g_t\) is compatible, and the quotient Kähler metric \((\tilde{g}_t, \tilde{J}_t)\) must be locally a product of CSC Kähler metrics. By the de Rham decomposition theorem \(\tilde{g}_t\) must be a locally-symmetric metric modelled on the hermitian-symmetric space \(\mathbb{C}P^{d_1} \times \cdots \times \mathbb{C}P^{d_t} \times \mathbb{H}\), where \(d_i + 1 = \text{rk}(E_i)\) (so that \(\mathbb{C}P^{d_i}\) is a point if \(d_i = 1\)) and \(\mathbb{H}\) is the hyperbolic plane. By continuity, \((\tilde{g}_0, \tilde{J}_0)\) is a locally-symmetric Kähler metric on \(\tilde{\mathcal{S}}\) of the same type. By the de Rham decomposition theorem and considering the form of the covering transformations we obtain representations \(\rho_i: \pi_1(\Sigma) \to PU(d_i + 1)\), and therefore \(E_i\) must be stable by the standard theory [56].

In the case when \(\ell = 1\), we can assume without loss by Theorem 1 that \(E\) is not polystable, and we can then use instead of Theorem 4 the stronger results [5] Thm. 1 & 6 which affirm that any extremal Kähler metric on \((M, J_t)\) (for \(t \neq 0\)) must be compatible with respect to the natural \(S^1\)-action.
6. Further observations

6.1. Relative K-energy and the main conjecture. Leaving aside the specific motivation of this paper to study projective bundles over a curve, the theory of rigid semisimple toric bundles which we reviewed in Sect. 3 extends the theory of extremal Kähler metrics on toric manifolds \([21, 22, 24, 66, 74, 75]\) to this more general context.

To recast the leading conjectures \([21, 66]\) in the toric case to this setting, recall from \([21]\) that if we parametrize compatible Kähler metrics on toric manifolds \([21, 22, 24, 66, 74, 75]\) to this more general context.

Further observations from computations in \([21, 66, 74]\) lead us to introduce the linear functional

\[
\mathcal{F}^\Omega \left( f \right) := \int_{\Delta} f(z) p(z) d\sigma + \frac{1}{2} \int_{\Delta} \left( \langle A, z \rangle + B + \sum_{j=1}^{N} \frac{\text{Scal}_j}{(p_j, z) + c_j} \right) f(z) p(z) dv.
\]

The above calculation of \(d\mathcal{E}_{g}^\Omega\) shows that \(\mathcal{F}^\Omega(f) = 0\) if \(f\) is an affine function of \(z\).

Furthermore, using the fact that the derivative of \(\log \det H\) is \(\text{tr} H^{-1} dH\), we obtain the following generalization of Donaldson’s formula for \(\mathcal{E}^\Omega\):

\[
\mathcal{E}^\Omega(U) = 2 \mathcal{F}^\Omega(U) - \int_{\Delta} \left( \log \det \text{Hess} U(z) \right) p(z) dv.
\]

(In case of doubt about the convergence of the integrals, one can introduce a reference potential \(U_c\) and a relative version \(\mathcal{E}_{g}^\Omega\) of \(\mathcal{E}^\Omega\), but in fact, as Donaldson shows, the convexity of \(U\) ensures that the positive part of \(\log \det \text{Hess} U(z)\) is integrable, hence \(- \log \det \text{Hess} U(z)\) has a well defined integral in \((-\infty, \infty)\).

According to \([21, 66]\), the existence of a solution \(U \in \mathcal{S}(\Delta)\) to \(10\) should be entirely governed by properties of the linear functional \(18\):

**Conjecture 2.** Let \(\Omega\) be a compatible class on \(M\). Then the following conditions should be equivalent:

1. \(\Omega\) admits an extremal Kähler metric.
2. \(\Omega\) admits a compatible extremal Kähler metric (i.e. \(10\) has a solution in \(\mathcal{S}(\Delta)\)).
3. \(\mathcal{F}^\Omega(f) \geq 0\) for any piecewise linear convex function \(f\) on \(\Delta\), and is equal to zero if and only if \(f\) is affine\(^\text{12}\).
Of course, by the proof of Theorem 2, Conjecture 2 would imply Conjecture 1. Our formula (19) can be used to show as in [21, Prop. 7.1.3] that $F^\Omega(f) \geq 0$ if the relative K-energy is bounded from below. However, according to Chen–Tian [15], the boundedness from below of $E^\Omega$ is a necessary condition for the existence of an extremal Kähler metric.

If $\Omega$ admits a compatible extremal Kähler metric with symplectic potential $U$ and inverse hessian $H$, one can use (10) and integration by parts (taking into account (4)) in order to re-write (18) as

$$(20) F^\Omega(f) = \int_\Delta \langle H, \text{Hess} f \rangle p(z) dv.$$  

This formula makes sense for smooth functions $f(z)$, but can also be used to calculate $F^\Omega(f)$ in distributional sense for any piecewise linear convex function as in [75], showing that the second statement of Conjecture 2 implies the third.

We thus have the following partial result.

**Proposition 3.** If $\Omega$ admits an extremal Kähler metric then $F^\Omega(f) \geq 0$ for any convex piecewise linear function. If $\Omega$ admits a compatible extremal metric then, furthermore, $F^\Omega(f) = 0$ if and only if $f$ is an affine function on $\Delta$.

Of course, the most difficult part of Conjecture 2 is to prove $(3) \Rightarrow (2)$. So far the Conjecture 2 has been fully established in the cases when $\ell = 1$ [5] and when $M$ is a toric surface (i.e. $\ell = 2$ and $S$ is a point) with vanishing extremal vector field [24].

6.2. Computing $F^\Omega$. It is natural to consider (following Donaldson [21]) the space of $S^{2\mathbb{R}}^*$-valued functions $H$ on $\Delta$ satisfying just the boundary conditions (4). If such a function satisfies the (underdetermined, linear) equation (10), then formula (20) holds, and it can be used to compute the action of $F^\Omega$ (in distributional sense) on piecewise linear functions.

Note that if a solution to (10) exists, then so do many because the double divergence is underdetermined.

If a solution $H$ of (10) happens to be positive definite on each face of $\Delta$, i.e. if it verifies the positivity condition in Sect. 3.3, then formulae (6) introduce an almost Kähler metric on $M$ (see e.g. [3]) and one can show that (7) computes its hermitian scalar curvature (see Appendix A). Thus, positive definite solutions of (10) correspond to compatible extremal almost Kähler metrics. If such extremal almost Kähler metrics exist, it then follows from (20) (see [75] and Proposition 3 above) that the condition (3) of Conjecture 2 is verified. Thus, the existence of a positive definite solution $H$ of (10) (and verifying the boundary conditions (4)) is conjecturally equivalent to the existence of a compatible extremal Kähler metric (corresponding to another positive definite function $H^\Omega$ with inverse equal to the hessian of a function $U_\Omega$). In fact, following [21], as log det is strictly convex on positive definite matrices, the functional $\int_\Delta (\log \det H)p(z)dv$ is strictly convex on the space of positive definite solutions of (10), and therefore has at most one minimum $H^\Omega$. Such a minimum would automatically have its inverse equal to the hessian of a function $U_\Omega$ (see [24]). Thus, $H^\Omega$ would then give the extremal Kähler metric in the compatible Kähler class $\Omega$.

Thus motivated, it is natural to wonder if on the manifolds we consider in this paper a (not necessarily positive definite) solution $H$ of (10) exists, thus generalizing the extremal polynomial introduced in [5] on $M = P(E_0 \oplus E_1) \to S$ (in fact $P(z) = p(z)H(z)$ would be the precise generalization).
6.3. Example: projective plane bundles over a curve. We illustrate the above discussion by explicit calculations on the manifold \( M = P^3(C \oplus L_1 \oplus L_2) \to \Sigma \), where \( L_1 \) and \( L_2 \) are holomorphic line bundles over a compact complex curve \( \Sigma \) of genus \( g \).

We put \( p_1 = \deg(L_1) \) and assume without loss that \( p_2 \geq p_1 \geq 0 \). Note that in the case \( p_1 = p_2 = 0 \), the vector bundle \( E = C \oplus L_1 \oplus L_2 \) is polystable, and therefore the existence of extremal Kähler metrics is given by Theorem 11. The cases \( p_1 = p_2 > 0 \) and \( p_2 > p_1 = 0 \), on the other hand, are solved in [3]. We thus assume furthermore that \( p_2 > p_1 > 0 \).

To recast our example in the set up of Sect. 3, we take a riemannian metric \( g_\Sigma \) of constant scalar curvature \( 4(1-g) \) on \( \Sigma \). To ease the notation, we put \( C = 4(1-g) \). Let \( z_i \) be the momentum map of the natural \( S^1 \)-action by multiplication on \( L_i \). Thus, without loss, for a compatible Kähler metric on \( M \), the momentum coordinate \( z = (z_1, z_2) \) takes values in the simplex \( \Delta = \{(z_1, z_2) \in \mathbb{R}^2 | z_1 \geq 0, z_2 \geq 0, 1 - z_1 - z_2 \geq 0 \} \) (which is the Delzant polytope of the fibre \( CP^2 \) viewed as a toric variety).

It is shown in [5] App. A2 that in this case there are no extremal compatible Kähler metrics with a hamiltonian 2-form of order 2 while Theorem 2 does imply existence of compatible extremal Kähler metrics in small Kähler classes. Therefore, we do not have an explicit construction of these extremal Kähler metrics. Instead, we will now attempt to find explicit extremal almost Kähler metrics (see the preceding section and the Appendix below). We thus want to find a smooth matrix function \( H(z) = (H_{rs}(z)) \) satisfying the boundary conditions (14) and which solves the linear equation (10). Motivated by the explicit form of such a matrix in the case when a hamiltonian 2-form does exist [3], we look for solutions of a ‘polynomial’ form \( H_{rs} = \frac{P_{rs}}{(c+p_1 z_1 + p_2 z_2)} \), where \( P_{rs}(z) \) are fourth degree polynomials in \( z_1 \) and \( z_2 \), and the constant \( c \) is such that \( c + p_1 z_1 + p_2 z_2 > 0 \) on \( \Delta \) (recall that \( c \) parametrizes compatible Kähler classes on \( M \)). The boundary conditions are then solved by

\[
P_{11} = 2(c + p_1 z_1 + p_2 z_2) z_1 (1 - z_1) + z_1^2 (x_0 z_2^2 + x_2 (1 - z_1 - z_2)^2 + 2 y_1 z_2 (1 - z_1 - z_2)),
\]

\[
P_{12} = -2(c + p_1 z_1 + p_2 z_2) z_1 z_2 + z_1 z_2 (y_0 (1 - z_1 - z_2)^2 - x_0 z_1 z_2 - (1 - z_1 - z_2) (y_1 z_1 + y_2 z_2)),
\]

\[
P_{22} = 2(c + p_1 z_1 + p_2 z_2) z_2 (1 - z_2) + z_2^2 (x_0 z_2^2 + x_1 (1 - z_1 - z_2)^2 + 2 y_2 z_1 (1 - z_1 - z_2)),
\]

where \( x_0, x_1, x_2, y_0, y_1, y_2 \) are free parameters. The extremal condition (10) corresponds to the linear equations

\[
y_0 = -x_0 - x_2 + v_0,
\]

\[
y_1 = -x_1 - x_2 + v_1,
\]

\[
y_2 = -x_0 - x_1 + v_2,
\]

with

\[
v_0 = (-12c + C + 4p_1 + 4p_2) (5c p_2^2 + p_1^2 + 5c p_2 p_1 + 5p_2^2 p_1 + 5c p_1^2 + 5p_1 p_2^2 + 5p_1 p_2 + 5p_1) \over (2(5c^2 + 50c^2 p_1 + 13c p_1^2 + p_1^2 + 50c p_2^2 + 37c p_1 p_2 + 5p_2^2 + 13c p_2^2 + 5p_1 p_2 + 5p_2 + 5p_1))
\]

\[
v_1 = (-12c + C + 4p_1 + 4p_2) (15c p_1^2 + 3p_2^2 + 15c p_1^2 + 3p_2^2 + 5c p_2^2 + 5p_1 p_2^2 + 5p_1 p_2 + 5p_2 + 5p_1) \over (2(5c^2 + 50c^2 p_1 + 13c p_1^2 + p_1^2 + 50c p_2^2 + 37c p_1 p_2 + 5p_2^2 + 13c p_2^2 + 5p_1 p_2 + 5p_2 + 5p_1))
\]

\[
v_2 = (-12c + C + 4p_1 + 4p_2) (5c p_1^2 + p_1^2 + 50c p_1 p_2 + 5c p_1^2 + 5p_1 p_2^2 + 5p_1 p_2 + 5p_2 + 5p_1) \over (2(5c^2 + 50c^2 p_1 + 13c p_1^2 + p_1^2 + 50c p_2^2 + 37c p_1 p_2 + 5p_2^2 + 13c p_2^2 + 5p_1 p_2 + 5p_2 + 5p_1)).
\]

Thus, given a compatible Kähler class on \( M \), we have a 3-parameter family of smooth ‘polynomial’ solutions \( H(z) \) to (10), which verify the boundary conditions (11) on \( \Delta \).
Now, investigating the integrability condition (that $H^{-1}$ be a hessian of a smooth function on $\Delta^0$, see the previous section), we find out that it is equivalent to the following five algebraic equations on the parameters $(x_0, x_1, x_2, y_0, y_1, y_2)$:

\begin{align}
 x_0 &= y_1 + y_2, \\
 x_1 &= y_2 + y_0, \\
 x_2 &= y_0 + y_1,
\end{align}

(22)

\begin{align}
 2(p_2 - p_1)y_0 + 2p_2y_1 - y_0y_1 &= 0, \\
 2(p_1 - p_2)y_0 + 2p_1y_2 - y_0y_2 &= 0.
\end{align}

(23)

The problem is over-determined, but there is a unique solution $H_0^\Omega$ satisfying the linear system \(22\) (additionally to \(21\)): we compute that this solution is given by

\begin{align*}
 x_0 &= \frac{1}{10}(-2v_0 + 3v_1 + 3v_2), \\
 x_1 &= \frac{1}{10}(3v_0 - 2v_1 + 3v_2), \\
 x_2 &= \frac{1}{10}(3v_0 + 3v_1 - 2v_2), \\
 y_0 &= \frac{1}{10}(4v_0 - v_1 - v_2), \\
 y_1 &= \frac{1}{10}(-v_0 + 4v_1 - v_2), \\
 y_2 &= \frac{1}{10}(-v_0 - v_1 + 4v_2).
\end{align*}

(24)

Substituting back in \(23\), one sees that the full integrability conditions can be solved if \(12c + C + 4p_1 + 4p_2 = 0\) (a constraint that is never satisfied for \(p_2 > p_1 \geq 1\), \(C = 4(g - 1)\), and \(c > 0\)); this observation is consistent with the non-existence result in [5, App. A2].

We now investigate the positivity condition for our distinguished solution $H_0^\Omega$ of \(4\) and \(10\). First of all, when $c \to \infty$, the $v_i$’s tend to 0, so $H_0^\Omega$ tends to the matrix associated to a Fubini–Study metric on $\mathbb{CP}^2$. It follows that $H_0^\Omega$ becomes positive-definite on each face for sufficiently small Kähler classes, and therefore $H_0^\Omega$ defines an explicit extremal (non-Kähler) almost Kähler metric in $\Omega$ (see Appendix A below). This is of course consistent (via Conjecture 2) with the existence of a (non-explicit) extremal Kähler metric in $\Omega$, given by Theorem 2. Furthermore, if $g = 0, 1$ (i.e. $C < 0$), a computer assisted verification shows that, in fact, $H_0^\Omega$ is positive definite on each face of $\Delta$ for all Kähler classes. We thus obtain the following result.

**Proposition 4.** Let $M = P(E) - p \to \Sigma$ with $E = \mathcal{O} \oplus \mathcal{L}_1 \oplus \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are holomorphic line bundles of degrees $1 \leq p_1 < p_2$ over a compact complex curve $\Sigma$ of genus $g$.

If $g = 0, 1$, then $M$ admits a compatible extremal almost Kähler metric for the Kähler form of any compatible Kähler metric on $M$. In particular, for every Kähler class on $M$ the condition (3) of Conjecture 2 is verified.

If $g \geq 2$, then the same conclusion holds for the compatible Kähler forms in sufficiently small Kähler classes $\Omega_k = 2\pi c_1(\mathcal{O}(1)_E) + kp[\omega_\Sigma]$, $k \gg 0$.

As speculated in the previous section, the explicit solution $H_0^\Omega$ of \(4\) and \(10\) can be used to compute the action of the functional $\mathcal{F}^\Omega$ on piecewise linear convex functions (by extending formula \(20\) in a distributional sense, after integrating by parts and using \(4\)). As a simple illustration of this, let us take a simple crease function $f_a$ with crease along the segment $S_a = \{(t, a - t), 0 < t < a\}$ for some $a \in (0, 1)$ (thus as $a \to 0$, the crease moves to the lower left corner of the simplex $\Delta$). A normal of the crease is $u = (1, 1)$ and one easily finds that

\[
 \mathcal{F}^\Omega(f_a) = \int_{S_a} H_0^\Omega(u, u) d\sigma,
\]

(24)

\[
 = \int_0^a ((H_{11} + 2H_{12} + H_{22})(t, a - t))(c + pt + p_2(a - t)) dt,
\]

where $H_{ij}$ are the components of $H_0^\Omega$.

where \( d\sigma \) is the contraction of the euclidian volume \( dv \) on \( \mathbb{R}^2 \) by \( u \). Note that the integrand (being a rational function of \( c \) with a non-vanishing denominator at \( c = 0 \)), and hence the integral, is continuous near \( c = 0 \); for \( c = 0 \) the integral equals

\[
\frac{1}{6}(1 - a)a^3(-C + 2(p_1 + p_2) + a(C + 4(p_1 + p_2))),
\]

which is clearly negative for \( a \in (0, 1) \) sufficiently small as long as \( C = 4(g - 1) > 2(p_1 + p_2) \). If we take \( g > 2 \), such \( p_1 \) and \( p_2 \) do exist. By Proposition 3 this implies a non-existence result of extremal Kähler metrics when \( p_1 \) and \( p_2 \) satisfy the above inequality and \( c \) is small enough. (As a special case, for \( p_1 = p_2 \) we have recast the non-existence part of \([5, \text{Thm. 6}]\).)

**Proposition 5.** Let \( M \) be as in Proposition 4 with \( g > 2 \) and \( p_1, p_2 \) satisfying \( 2(g - 1) > p_1 + p_2 \). Then all sufficiently ‘big’ Kähler classes do not admit any extremal Kähler metric.

**Appendix A. Compatible extremal almost Kähler metrics**

In this appendix, we calculate the hermitian scalar curvature of a compatible almost Kähler metric and extend the notion of extremal Kähler metrics to the more general almost Kähler case.

Recall that on a general almost Kähler manifold \((M^{2m}, g, J, \omega)\), the *canonical hermitian connection* \( \nabla \) is defined by

\[
\nabla_X Y = D_X Y - \frac{1}{2} J(D_X J)(Y),
\]

where \( D \) is the Levi-Civita connection of \( g \). Note that

\[
g((D_X J)Y, Z) = \frac{1}{2} g(N(X, Y), JZ)
\]

where \( N(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y] \) is the Nijenhuis tensor of \( J \). The *Ricci form*, \( \rho^\nabla \), of \( \nabla \) represents \( 2\pi c_1(M, J) \) and its trace \( s^\nabla \) (given by \( 2m\rho^\nabla \wedge \omega^{m-1} = s^\nabla \omega^m \)) is called *hermitian scalar curvature* of \((g, J, \omega)\).

The hermitian scalar curvature plays an important role in a setting described by Donaldson [10] (see also [32]), in which \( s^\nabla \) is identified with the momentum map of the action of the group \(
\text{Ham}(M, \omega)\) of hamiltonian symplectomorphisms of a compact symplectic manifold \((M, \omega)\) on the (formal) Kähler Fréchet space of \( \omega \)-compatible almost Kähler metrics \( \mathcal{AK}_{\omega} \). It immediately follows from this formal picture [26] [47] that the critical points of the functional on \( \mathcal{AK}_{\omega} \)

\[
g \mapsto \int_M (s^\nabla)^2 \omega^m
\]

are precisely the \( \omega \)-compatible almost Kähler metrics for which \( \text{grad}_{\omega}s^\nabla \) is a Killing vector field. This provides a natural extension of the notion of an extremal Kähler metric to the more general almost Kähler context.

**Definition 5.** An almost Kähler metric \((g, \omega)\) for which \( \text{grad}_{\omega}s^\nabla \) is a Killing vector field is called *extremal*.

Now let \( M \) be a manifold obtained by the generalized Calabi construction of Sect. 3.3. In the notation of this section, for any \( S^1 \)-valued function \( H \) on \( \Delta \), satisfying the boundary and positivity conditions, formulae [3] introduce a pair \((g, \omega)\) of a smooth metric \( g \) and a symplectic form \( \omega \) on \( M \), such that the field of endomorphisms \( J \) defined by \( \omega(\cdot, \cdot) = g(J\cdot, \cdot) \) is an almost complex structure, i.e., \((g, \omega)\) is an almost Kähler
structure on $M$. We shall refer to such pairs $(g, \omega)$ as compatible almost Kähler metrics on $M$.

**Lemma 9.** The hermitian scalar curvature $s^\nabla$ of a compatible almost Kähler metric corresponding to $H = (H_{rs})$ is given by

$$s^\nabla = \sum_{j=1}^{N} \frac{\text{Scal}_j}{c_j} + \langle p_j, z \rangle - \frac{1}{p(z)} \sum_{r,s=1}^{\ell} \frac{\partial^2}{\partial z_r \partial z_s} (p(z)H_{rs}).$$

**Proof.** The result is local and we work on the open dense subset $M^0$ where the $\ell$-torus $T$ acts freely. Recall that $M^0$ is a principal $T^c$ bundle over $\hat{S}$. Let $V$ be the foliation defined by the $T^c$ fibres and $K_r = J\text{grad}_g z_r$ be the Killing vector fields generating $T$; then $TV$ is spanned by $K_r, JK_r$ at each point of $M^0$ and, by construction,

$$(27) \quad \mathcal{L}_{K_r} J = 0, \quad K^0_r = \sum_s H_{rs} \hat{\theta}_s, \quad JK^0_r = -dz_r.$$ 

In order to compute the hermitian Ricci tensor, we take a local non-vanishing holomorphic section $\Phi_S$ of the anti-canonical bundle $K^{-1}_S = \wedge^{d,0}(\hat{S})$ of $\hat{S}$ (which pulls back to a $(d,0)$-form on $M$) and wedge it with the $(\ell,0)$-form $\Phi_V = (K^0_1 - \sqrt{-1}JK^0_1) \wedge \cdots \wedge (K^0_\ell - \sqrt{-1}JK^0_\ell)$. Thus, $\Phi = \Phi_S \wedge \Phi_V$ is a non-vanishing section of $K^{-1}_{M^0}$ and the hermitian Ricci form $\rho^\nabla$ is then given by

$$\rho^\nabla = -d\Im(\alpha),$$

where $\nabla \Phi = \alpha \otimes \Phi$.

Denote by $T\mathcal{H}$ the $g$-orthogonal complement of $TV$; the spaces $T\mathcal{H}$ and $TV$ then define the decomposition of $TM^0$ as the sum of horizontal and vertical spaces and, therefore

$$(28) \quad \nabla_X \Phi = (\nabla^H_X \Phi_S) \wedge \Phi_V + \Phi_S \wedge \nabla^V_X \Phi_V,$$

where $\nabla_X Y = \nabla^H_X Y + \nabla^V_X Y$ denotes the decomposition into horizontal and vertical parts.

Our first observation is that \textsuperscript{[3]} Prop. 8] generalizes in the non-integrable case in the following sense: The foliation $\mathcal{V}$ is totally-geodesic with respect to both the Levi–Civita and hermitian connections. Indeed, with respect to the Levi–Civita connection $D$ we have $\langle D_{JK_r}K_s, X \rangle = \langle D_{JK_s}K_r, X \rangle = 0$ for any $X \in T\mathcal{H}$; using $[K_r, JK_s] = 0$, our claim reduces to check that $\langle D_{JK_r}JK_s, X \rangle = 0$. We take $X$ be the horizontal lift of a basic vector field and use the Koszul formula

$$2\langle D_{JK_r}JK_s, X \rangle = \mathcal{L}_{JK_r} \langle JK_s, X \rangle + \mathcal{L}_{JK_s} \langle JK_r, X \rangle - \mathcal{L}_X \langle JK_r, JK_s \rangle$$

$$+ \langle [JK_r, JK_s], X \rangle + \langle XJK_r, JK_s \rangle + \langle XJK_s, JK_r \rangle$$

$$= \langle L_X JK_r, JK_s \rangle + \langle L_X JK_s, JK_r \rangle$$

$$= \langle L_X g \rangle (JK_r, JK_s) = \langle L_X g(z) \rangle (JK_r, JK_s) = 0,$$

where $(\hat{g} = \hat{g}(z), \hat{\omega} = \hat{\omega}(z))$ denote the Kähler quotient structure on $\hat{S}$ (also identified with the horizontal part of $(g, \omega)$). Considering the hermitian connection $\nabla$, by (29) and (26), our claim reduces to showing that $N(K_r, X)$ is horizontal for any $X \in T\mathcal{H}$; using (26) and the fact that $\mathcal{V}$ is totally-geodesic with respect to $D$, we get $\langle N(K_r, X), U \rangle = 2\langle (DU)J(K_r), X \rangle = 0$, for any $U \in TV$.

\footnote{It is easily seen as in \textsuperscript{[1]} that $J$ is integrable, i.e. $(g, \omega)$ defines a Kähler metric, if and only if $H^{-1}$ is the hessian of a smooth function on $\Delta^0$.}
Using that $\nabla$ we can write
\[ V_b(31) \]
so that
\[ r \]
\[ \phi \]
\[ \det \]
\[ X \]
any horizontal lift
\[ (29) \]
\[ (42)–(46) \]
in $\{3, \text{Prop. 9}\}$ hold true in the non-integrable case too, i.e. we have
\[
D_X Y = D_X^R Y - C(X, Y) \\
D_X U = \langle C(X, \cdot), U \rangle + [X, U]^V \\
D_U X = [U, X]^H + \langle C(X, \cdot), U \rangle \\
D_U V = D_U^V V,
\]
where $X, Y \in TH$, $U, V \in TV$ and $C(\cdot, \cdot)$ is the O’Neill tensor given by
\[
2C(X, Y) = \sum_{r=1}^\ell \left( \Omega_c(X, Y) K_r + \Omega_r(JX, Y) J K_r \right)
\]
with $\Omega_r = d\hat{\theta}_r = \sum_{j=1}^N p_{jr} \otimes \omega_j$. Using (20), it follows that the horizontal lift of $\nabla H$ coincides with the projections of both $D$ and $\nabla$ to horizontal vectors. In particular, for any horizontal lift $X$, $\nabla_X^H \Phi_S = \frac{1}{2} \left( (d_S - \sqrt{-1}d_S^\ast) \log ||\Phi_S||_g^2 \right) (X)\Phi_S$. On the other hand, as $K_r$ are Killing and $V$ is totally geodesic, $\nabla_X^V \Phi_V = 0$, so that we get from (28)
\[
\alpha(X) = \frac{1}{2} \left( (d_S - \sqrt{-1}d_S^\ast) \log ||\Phi_S||_g^2 \right) (X), \quad \forall X \in TH.
\]
To compute $\alpha(U)$ for $U \in TF$, consider first $\nabla_U \Phi_V$. As $V$ is totally geodesic, we can write $\nabla_U \Phi_V = (a(U) - \sqrt{-1}b(U))\Phi_V$. It follows from the very definition of $\Phi_V$ (and the fact that $\text{span}(K_1, \ldots, K_\ell)$ is $\omega$-Lagrangian) that $\Phi_V(K_1, K_2, \ldots, K_\ell) = \det g(K_r, K_s) = \det H$, and therefore
\[
(\nabla_U \Phi_V)(K_1, K_2, \ldots, K_\ell) = \left( a(U) - \sqrt{-1}b(U) \right) \det H.
\]
Using the definition of $\Phi_V$ again, we obtain
\[
b(U) = \text{trace}(H^{-1} \circ A_U), \quad (A_U)_{rs} = -\langle \nabla_U K_r, J K_s \rangle.
\]
Using that $K_r$ is Killing, (26) and (27) we further calculate
\[
(A_U)_{rs} = -\langle D_U K_r, J K_s \rangle + \frac{1}{2} \langle (D_U J)(K_r), K_s \rangle
\]
\[
= \frac{1}{2} \left( dK_r^2(J K_s, U) - \frac{1}{2} \omega(N(K_r, K_s), U) \right)
\]
\[
= \frac{1}{2} \left( \sum_{p,k} H_{rk,p} d\omega_p (JK_s) \hat{\theta}_k(U) - \frac{1}{2} \sum_k d\omega([JK_r, J K_s]) \hat{\theta}_k(U) \right)
\]
\[
= \frac{1}{2} \left( -\sum_{k,p} H_{rk,p} \hat{H}_{ps} \hat{\theta}_k(U) - \frac{1}{2} \sum_k d\omega([JK_r, J K_s]) \hat{\theta}_k(U) \right)
\]
\[
= -\frac{1}{4} \sum_{k,p} (H_{rk,p} \hat{H}_{ps} + H_{sk,p} \hat{H}_{ps}) \hat{\theta}_k(U),
\]
so that
\[
b(U) = \sum_{r,s} H^{rs} (A_U)_{rs} = -\frac{1}{2} \sum_{r,k} H_{rk,r} \hat{\theta}_k(U).
\]
Finally, in order to compute $\nabla_U \Phi_S$, note that $\mathcal{L}_U \Phi_S = 0$, and therefore

$$(\nabla_U \Phi_S)(X_1, \cdots, X_d) = \sum_{k=1}^{d} \left( \frac{1}{2} \Phi_S(X_1, \cdots, X_{k-1}, J(D_U J)(X_k), X_{k+1}, \cdots, X_d) - \Phi_S(X_1, \cdots, X_{k-1}, (D^H_{X_k} U), X_{k+1}, \cdots, X_d) \right),$$

where $X_k \in T\mathcal{H}$. Now, using (29), we further specify

$$(D^H_{X_k} U) = \frac{1}{2} \sum_{r=1}^{\ell} \sum_{j=1}^{N} \left( K^r_j(U)p_{jr}(J_j X^r_k) - JK^r_j(U)p_{jr} X^r_k \right),$$

where $X^r_k$ (resp. $J_j$) denote the $g_S$-orthogonal projection (resp. restriction) of $X_k$ (resp. $J$) to the subspace $TS_j \subset T\hat{S}$ (recall that the universal cover of $(\hat{S}, g_{\hat{S}})$ is the Kähler product of $(S_j, g_j, \omega_j)$, so that the projection of $TS_j$ to $T\hat{S}$ is a well-defined $D$-parallel subbundle of $T\hat{S}$). Using (27), and the expressions (30) and (31), we eventually find that

$$\text{Im}(\alpha) = -\frac{1}{2} d^f_S \log \|\Phi_S\|^2_g + \frac{1}{2} d^f \log p(z) + \frac{1}{2} \sum_{k,r} H_{kr,k} \theta_r$$

$$= -\frac{1}{2} d^f_S \log \|\Phi_S\|^2_g + \frac{1}{2} \sum_{k,r} \left( \frac{\partial p}{\partial z_k} H_{kr} + p(z) \frac{\partial H_{kr}}{\partial z_k} \right) \theta_r$$

$$\rho^\nabla = \sum_{j=1}^{N} \rho_j - \sum_{i, r, k} \frac{\partial}{\partial z_k} \left( \frac{1}{2p(z)} \frac{\partial (p(z) H_{ir})}{\partial z_i} \right) dz_k \wedge \theta_r$$

$$- \frac{1}{2p(z)} \sum_{i, r} \left( \frac{\partial p(z) H_{ir}}{\partial z_i} \frac{\partial \hat{\omega}}{\partial z_r} + \frac{\partial p(z) H_{ir}}{\partial z_r} \frac{\partial \hat{\omega}}{\partial z_i} \right),$$

where, we recall, $\rho_j$ is the Ricci form of $(S_j, g_j, \omega_j)$, $\hat{\omega}(z) = \sum_{j=1}^{N} \left( \sum_{r=1}^{\ell} (p_{jr} \varpi_r + c_j) \omega_j \right)$, and $p(z) = \prod_{j=1}^{N} \left( \sum_{r=1}^{\ell} (p_{jr} \varpi_r + c_j) \right)$. The formula for $s^\nabla$ follows easily. \hfill \Box

References

[1] M. Abreu, Kähler geometry of toric varieties and extremal metrics, Internat. J. Math 9 (1998), 641–651.
[2] M. Abreu, Kähler metrics on toric orbifolds, J. Differential Geom. 58 (2001), 151–187.
[3] V. Apostolov, D. M. J. Calderbank and P. Gauduchon, Hamiltonian 2-forms in Kähler Geometry I General Theory, J. Differential Geom. 73 (2006), 359–412.
[4] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. Tønnesen-Friedman, Hamiltonian 2-forms in Kähler Geometry II: Global Classification, J. Differential Geom. 68 (2004), 277–345.
[5] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. Tønnesen-Friedman, Hamiltonian 2-forms in Kähler geometry, III. Extremal metrics and stability, Invent. Math. 173 (2008), 547–601.
[6] V. Apostolov, D. M. J. Calderbank, P. Gauduchon and C. Tønnesen-Friedman, Extended Kähler metrics on ruled manifolds and stability, in: Géométrie différentielle, physique mathématique, mathématiques et société (II)– Volume en l’honneur de Jean-Pierre Bourguignon, (ed. O. Hijazi) Astérisque 322 (2008), 93–150.
[7] V. Apostolov and C. Tønnesen-Friedman, A remark on Kähler metrics of constant scalar curvature on ruled complex surfaces, Bull. London Math. Soc. 38 (2006), 494–500.
[8] M. F. Atiyah, Convexity and commuting hamiltonians, Bull. London Math. Soc. 4 (1982), 1–15.
[9] M. F. Atiyah and R. Bott, The Yang–Mills equations over Riemann surfaces, Philos. Trans. Roy. Soc. London Ser. A, 308 (1982), 524–615.
[10] S. Bando and T. Mabuchi, *Uniqueness of Einstein Kähler metrics modulo connected group actions*, Algebraic geometry, Sendai, 1985, Adv. Stud. Pure Math., 10, North-Holland, Amsterdam (1987), 11–40.
[11] A. Besse, *Einstein Manifolds*, Springer-Verlag, 1987.
[12] D. Burns and P. de Bartolomeis, *Stability of vector bundles and extremal metrics*, Invent. Math. **92** (1988), 403–407.
[13] E. Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, Princeton Univ. Press (1982), 259–290.
[14] E. Calabi, *Extremal Kähler metrics, II*, in: Differential Geometry and Complex Analysis (eds. I. Chavel and H.M. Farkas), Springer-Verlag, 1985.
[15] X. X. Chen and G. Tian, *Geometry of Kähler metrics and holomorphic foliation by discs*, Publ. Math. Inst. Hautes Étud. Sci. **107** (2008), 1–107.
[16] T. Delzant, *Hamiltoniens périodiques et image convexe de l’application moment*, Bull. Soc. Math. France **116** (1988), 315–339.
[17] S. K. Donaldson, *A new proof of a theorem of Narasimhan and Seshadri*, J. Differential Geom. **18** (1983), 269–277.
[18] S. K. Donaldson, *Infinite determinants, stable bundles and curvature*, Duke Math. J. **54** (1987), 231–247.
[19] S. K. Donaldson, *Remarks on gauge theory, complex geometry and 4-dimensional topology*, Fields Medalists’ Lectures, 384–403, World Sci. Ser. 20th Century Math. 5, World. Sci. Publishing, River Edge, NJ, 1997.
[20] S. K. Donaldson, *Scalar curvature and projective embeddings, I*, J. Differential Geom. **59** (2001), 479–522.
[21] S. K. Donaldson, *Scalar curvature and stability of toric varieties*, J. Differential Geom. **62** (2002), 289–349.
[22] S. K. Donaldson, *Interior estimates for solutions of Abreu’s equation*, Collect. Math. **56** (2005) 103–142.
[23] S. K. Donaldson, *Lower bounds on the Calabi functional*, J. Differential Geom. **70** (2005), 453–472.
[24] S. K. Donaldson, *Constant scalar curvature metrics on toric surfaces*, Geom. Funct. Anal. **19**, 83–136.
[25] S. K. Donaldson, *Kähler geometry on toric manifolds, and some other manifolds with large symmetry*, in ‘Handbook of geometric analysis’, No. 1, 29–75, Adv. Lect. Math. (ALM), 7, Int. Press, Somerville, MA, 2008.
[26] T. Drăghici, private communication to the first author, 1999.
[27] A. Fujiki, *On Automorphism Groups of Compact Kähler Manifolds*, Inventiones math. **44** (1978), 225–258.
[28] A. Fujiki, *Remarks on extremal Kähler metrics on ruled manifolds*, Nagoya Math. J. **126** (1992), 89–101.
[29] A. Fujiki and G. Schumacher, *The moduli space of extremal compact Kähler manifolds and generalized Weil-Petersson metrics*, Publ. Res. Inst. Math. Sci. **26** (1990), 101–183.
[30] A. Futaki, *Kähler–Einstein metrics and Integral Invariants*, Springer-LNM **1314**, Springer-Verlag (1988).
[31] A. Futaki and T. Mabuchi, *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*, Math. Ann. **301** (1995), 199–210.
[32] P. Gauduchon, Calabi’s extremal metrics: An elementary introduction, in preparation.
[33] D. Guan, *Existence of extremal metrics on compact almost homogeneous Kähler manifolds with two ends*, Trans. Amer. Math. Soc. **347** (1995), 2255–2262.
[34] V. Guillemin, *Kähler structures on toric varieties*, J. Differential Geom. **40** (1994), 285–309.
[35] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.
[36] Y.-J. Hong, *Constant Hermitian scalar curvature equations on ruled manifolds*, J. Differential Geom. **53** (1999), 465–516.
[37] Y.-J. Hong, *Gauge-fixing constant scalar curvature equations on ruled manifolds and the Futaki invariants*, J. Differential Geometry **60** (2002), 389–453.
[38] A. D. Hwang, *On existence of Kähler metrics with constant scalar curvature*, Osaka J. Math. **31** (1994), 561–595.
[39] A. D. Hwang and M. A. Singer, *A momentum construction for circle-invariant Kähler metrics*, Trans. Amer. Math. Soc. **354** (2002), 2285–2325.
[40] S. Kobayashi, Differential geometry of complex vector bundles. Publications of the Mathematical Society of Japan, 15. Memorial Lectures, 5. Princeton University Press, Princeton, NJ; Iwanami Shoten, Tokyo, 1987.

[41] K. Kodaira and J. Morrow, Complex Manifolds, Holt, Rinehart and Winston, New York, 1971.

[42] N. Koiso and Y. Sakane, Non-homogeneous Kähler–Einstein metrics on compact complex manifolds, in Curvature and Topology of Riemannian Manifolds (Katakai, 1985), Lecture Notes in Math. 1201, Springer-Verlag, Berlin, 1986, 165–179.

[43] N. Koiso and Y. Sakane, Non-homogeneous Kähler–Einstein metrics on compact complex manifolds II, Osaka J. Math. 25 (1988), 933–959.

[44] C. R. LeBrun, Polarized 4-manifolds, extremal Kähler metrics, and Seiberg-Witten theory, Math. Res. Lett. 2 (1995), 653–662.

[45] C. R. LeBrun and S. Simanca, Extremal Kähler metrics and complex deformation theory, Geom. Funct. Anal. 4 (1994), 298–336.

[46] C. R. LeBrun and S. Simanca, On the Kähler classes of extremal metrics, in ‘Geometry and global analysis’ (Sendai, 1993), 255–271, Tohoku Univ., Sendai, 1993.

[47] M. Lejmi, Extremal almost-Kähler metrics, to appear in Internat. J. Math, available at arXiv:0908.0859.

[48] M. Levine, A remark on extremal Kähler metrics, J. Differential Geom. 21 (1985), 73–77.

[49] A. Lichnerowicz, Sur les transformations analytiques des variétés kählériennes, C.R. Acad. Sci. Paris 244 (1957), 3011–3014.

[50] M. Lübke and A. Teleman, The Kobayashi–Hitchin Correspondence, World Scientific, 1995.

[51] T. Mabuchi, An energy-theoretic approach to the Hitchin-Kobayashi correspondence for manifolds, II, to appear.

[52] T. Mabuchi, Uniqueness of extremal Kähler metrics for an integral Kähler class, Internat. J. Math. 15 (2004), 531–546.

[53] T. Mabuchi, K-stability of constant scalar curvature polarization, preprint 2008, [arXiv:0812.4993](http://arxiv.org/abs/0812.4993).

[54] Y. Matsushima, Sur la structure du groupe d’homéomorphismes analytiques d’une certaine variété kählérienne, Nagoya Math. J. 11 (1957), 145–150.

[55] M. Narasimhan and S. Ramanan, Deformations and moduli space of vector bundles over an algebraic curve, Ann. Math. 85 (1965), 540–567.

[56] A. Ramanathan, Stable principal bundles on a compact Riemann surface, Math. Ann. 101 (1975), 391–417.

[57] G. Schwarz, Smooth functions invariant under the action of a compact Lie group, Topology 14 (1975), 63–68.

[58] J. Stoppa, K-stability of constant scalar curvature polarization, preprint 2008, [arXiv:0803.4095](http://arxiv.org/abs/0803.4095).

[59] G. Székelyhidi, Extremal metrics and K-stability, Ph.D. thesis, arXiv:math/0611002.

[60] G. Tian, Kähler–Einstein metrics with positive scalar curvature, Invent. Math. 130 (1997), 1–37.

[61] C. Tønnesen-Friedman, Extremal Kähler metrics on minimal ruled surfaces, J. reine angew. Math. 502 (1998), 175–197.

[62] C. Tønnesen-Friedman, Extremal Kähler metrics and hamiltonian functions II, Glasgow Math. J. 44 (2002), 241–253.

[63] C. Tønnesen-Friedman, Extremal Kähler metrics and hamiltonian functions, Math. Scand. 90 (2002), 180–190.

[64] K. Uhlenbeck and S.-T. Yau, On the existence of Hermitian-Yang-Mills connections in stable vector bundles. Frontiers of the mathematical sciences: 1985 (New York, 1985). Comm. Pure Appl. Math. 39 (1986), no. S, suppl., S257–S293.
[73] S.-Y. Yau, *Open problems in Geometry*, in ‘Differential geometry: partial differential equations on manifolds’ (Los Angeles, CA, 1990), 1–28, Proc. Sympos. Pure Math. 54, AMS.
[74] B. Zhou and X. Zhu, *Relative K-stability and modified K-energy on toric manifolds*, Adv. Math. 219 (2008), 1327–1362.
[75] B. Zhou and X. Zhu, *K-stability on toric manifolds*, Proc. Amer. Math. Soc. 136 (2008), 3301–3307.

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