Maxwell equations in homogeneous spaces with solvable groups of motions.

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Keywords: Maxwell equations, Klein-Gordon-Fock equation, algebra of symmetry operators, theory of symmetry, linear partial differential equations.

1 Introduction

A special place in mathematical physics is occupied by the problem of the exact integration of the field equations for electromagnetic and gravitational fields. The problem can be successful solved if the space and the electromagnetic fields possess some symmetry. Homogeneous spaces are one of the important examples of the space manifolds with symmetry. Stackel spaces are another example of such spaces. Both of these sets of spaces are applied in the theory of electromagnetism and gravitation due to the fact that, in these spaces, methods of commutative and noncommutative integration of equations of motion of single test particles can be applied.

The methods of commutative integration is based on the use of a commutative algebra of symmetry operators (integrals of motion) that form a complete set. The complete set includes first- and second-degree linear operators in momentum formed from complete sets of geometric objects consisting of vector and tensor Killing fields. The method is known as the method of the complete separation of variables. The theory of the complete separation of variables was mainly constructed in the works [1, 2, 3, 4, 5, 6, 7]. A description of the theory and detailed bibliography can be found in [8, 9, 10]. Examples of applications of the theory of complete separation of variables in the theory of gravitation can be found in the works [11, 12, 13, 14, 15, 16]. The methods of non-commutative integration is based on the use of the algebra of symmetry operators, which are linear in momenta and constructed using noncommutative Killing vector fields forming noncommutative groups of motion of spacetime $G_3$. Among these spacetime manifolds, the homogeneous spaces are of greatest interest for the theory of gravity (see, for example, [17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27]). The theory of the noncommutative integration method and development of the theory can be found in the works [29, 30, 31, 32, 33, 34].
Thus, these two methods are essentially complementary and have similar classification problems (by solving a classification problem, we mean enumerating all metrics of the corresponding spaces that are not equivalent in terms of admissible transformations of privileged coordinate systems; likewise, all electromagnetic potentials of admissible electromagnetic fields that are not equivalent in terms of admissible gradient transformations). Among these classification problems, the most important are the following.

The classification of all metrics of the Stackel and homogeneous spaces in privileged coordinate systems. For Stackel spaces, this problem was solved in the papers cited above. For homogeneous spaces, this problem was solved in the work of Petrov (see [28]).

The classification of all (admissible) electromagnetic fields to which these methods can be applied. For the Hamilton–Jacobi and Klein–Gordon–Fock equations, this problem is completely solved in homogeneous spaces (see [30, 31, 32, 33]). In Stackel spaces, it is completely solved for the Hamilton–Jacobi equation (see [8, 9, 10]) and partially solved for the Klein–Gordon–Fock equation.

The classification of all vacuum and electrovacuum solutions of the Einstein equations with metrics of Stackel and homogeneous spaces in admissible electromagnetic fields. This problem is completely solved for the Stackel metric (see, for example, [5, 12, 13] and bibliography in [8, 9, 10]). For homogeneous spaces, this classification problem has not yet been studied.

Thus, for the complete solution of the problem of uniform classification, it remains to integrate the Einstein–Maxwell vacuum equations using the previously found potentials of admissible electromagnetic fields and the known metrics of homogeneous spaces in privileged (canonical) coordinate systems. This problem can also be divided into two stages. In the first stage, all solutions of Maxwell vacuum equations for the potentials of admissible electromagnetic fields should be found.

In the paper [35], the first problem was decided for the case where there exist groups $G_3(II–VI)$ in the homogeneous spaces. The present work is devoted to the homogeneous spaces with groups of motion $G_3(VII)$. Thus, the classification problem for solvable groups of motions will be solved.

2 Maxwell Equations in the Homogeneous Spaces

Homogeneous Spaces

By definition, a space–time manifold $V_4$ is a homogeneous space if a three-parameter group of motions acts on it whose transitivity hypersurface $V_3$ is endowed with the Euclidean space signature. A semi-geodesic coordinate system $[u^i]$ is used. The metric $V_4$ has the form:

$$ds^2 = g_{ij}du^idu^j = -du^0^2 + g_{\alpha\beta}du^\alpha du^\beta, \quad \text{det}|g_{\alpha\beta}| > 0.$$ (1)

Coordinate indices of the variables of the semi-geodesic coordinate system are denoted by lower-case Latin letters: $i, j, \ldots = 0, 1 \ldots 3$. The coordinate indices of the variables of the local coordinate system on the hypersurface $V_3$ are denoted by lower-case Greek letters: $\alpha, \beta, \gamma, \ldots = 1, \ldots 3$. The temporal variable is indexed by 0. Group indices and indices of a non-holonomic frame are denoted by $a, d, c \ldots = 1, \ldots 3$. The letters p, q denote the indices varying from 2 to 3. Summation is performed over repeated upper and lower indices within the index range.
We consider the Maxwell equations with zero sources for electromagnetic potential (4):

\[ g_{\alpha \beta} = e_{\alpha}^{a} e_{\beta}^{b} \eta_{ab}(u^{0}), \quad e_{\alpha,0} = 0, \quad \eta_{ab} = \eta_{ab}(u^{0}). \] (2)

while the form

\[ \omega^{a} = e_{a}^{\alpha} du^{\alpha} \]

is invariant with respect to transformations of the group \( G_{3} \). The vectors of the frame \( e_{\alpha}^{a} \) define a non-holonomic coordinate system in \( V_{3} \). The dual triplet of vectors \( e_{\alpha}^{a} \) (\( e_{\alpha}^{a} e_{\alpha}^{b} = \delta_{b}^{a}, \ e_{\alpha}^{a} e_{\beta}^{a} = \delta_{\beta}^{a} \)) constructs the operators of the \( G_{3} \) algebra group:

\[ \hat{Y}_{a} = e_{a}^{\alpha} \partial_{\alpha}, \quad [\hat{Y}_{a}, \hat{Y}_{b}] = C_{ab}^{c} \hat{Y}_{c}. \] (3)

In the following, this definition of homogeneous spaces is used. The electromagnetic field is invariant with respect to transformations of the group acting in the space. It has the form:

\[ A_{i} = l_{i}^{a} \alpha_{a}, \ \alpha_{a} = \alpha_{a}(u^{0}). \] (4)

### 3 Maxwell Equations

We consider the Maxwell equations with zero sources for electromagnetic potential (4):

\[ \frac{1}{\sqrt{-g}} (g_{ij} F^{ij})_{,j} = 0. \] (5)

\[ \frac{1}{\sqrt{-g}} (g g^{\alpha \beta} F_{0 \beta})_{,\alpha} = \frac{1}{l}(l_{a}^{\alpha} \eta_{ab} \dot{\alpha}_{b})_{,\alpha} = \frac{l_{a,\alpha}}{l} \eta^{ab} \beta^{a} \beta^{b} = \eta_{ab} \eta_{b} \] (6)

Notation used:

\[ f_{a} = l_{a}^{\alpha} f_{\alpha}, \quad g = - \det |g_{\alpha \beta}| = - (\eta l)^{2}, \quad (\eta_{a}^{2} = \det |\eta_{\alpha \beta}|), \quad l = \det |l_{a}^{\alpha}|. \]

The dots denote the time derivatives. Then, we have the first equation in the form:

\[ (l_{a,\alpha}^{\alpha} + l_{[a]}^{\alpha}) \beta^{a} = 0. \] (7)

If \( i = \alpha \), from Equation (5), it follows that:

\[ \frac{1}{\eta} (\eta g^{\alpha \beta} F_{0 \beta})_{,\beta} = \frac{l}{l} (l g^{\alpha \beta} g^{\gamma \delta} F_{\beta \gamma})_{,\nu} \Rightarrow \frac{1}{\eta} (\eta g^{\alpha \beta} l_{a}^{\alpha} \dot{\alpha}_{b})_{,\beta} = \frac{l}{l} (l g^{\alpha \beta} l_{a}^{\alpha} l_{a}^{\gamma} \delta_{b}^{\beta} F_{\beta \gamma})_{,\nu} \Rightarrow \] (8)

\[ \frac{l_{a}^{\alpha}}{l_{a}^{\alpha}} \beta^{a} = \frac{l}{l} (l g^{\alpha \beta} l_{a}^{\alpha} l_{b}^{\gamma} F_{\beta \gamma})_{,\alpha} \eta_{\alpha b} \eta_{\beta b}. \] (9)

\( F_{\alpha \beta} \) can be found using the relations (2)–(4):

\[ F_{\alpha \beta} = (l_{a}^{\alpha}, l_{a}^{\alpha} - l_{a}^{\alpha}) \alpha_{a} = l_{a}^{\alpha} l_{a}^{\gamma} l_{b}^{\gamma} (l_{a}^{\alpha} l_{b}^{\gamma} - l_{b}^{\alpha} l_{a}^{\gamma}) \alpha_{a} = l_{a}^{\alpha} l_{a}^{\gamma} C_{ba}^{c} \alpha_{c} \Rightarrow \] (10)
Structural constants of a group $G$ can be represented in the form:

$$C^c_{ab} = C^c_{12} \varepsilon_{ab} + C^c_{p} \varepsilon_{ab}^p,$$

where

$$\varepsilon_{ab}^A = \delta^A_a \delta^B_b - \delta^A_b \delta^B_a.$$

From the relations:

$$(\varepsilon_{ab}^A \eta^a \eta^b) = (\eta^a \eta^b) - (\eta^a \eta^b),$$

it follows that:

$$\eta^2 \varepsilon_{cd}^A \eta^a \eta^b = (\eta_{h3} \varepsilon_{12}^A + \eta_{23} \varepsilon_{31}^A + \eta_{13} \varepsilon_{23}^A),$$

Equations (11) take the form:

$$\eta \dot{\beta}^a = \delta^a_1 (\gamma_1 (C^1_{32} - \gamma_2 (C^1_{23} + \omega_3) + \gamma_3 (C^1_{21} + \omega_2)) + \delta^a_2 (\gamma_1 (C^2_{32} + \omega_3) +$$

$$\gamma_2 C^2_{13} - \gamma_3 (C^2_{12} + \omega_1)) + \delta^a_3 (-\gamma_1 (C^3_{23} + \omega_2) + \gamma_2 (C^3_{13} + \omega_1) + \gamma_3 C^3_{21}),$$

$$\omega_{ab} \beta^b = \eta \dot{\alpha}^b_a,$$

$$\omega_{a} \beta^a = 0, \quad \omega_{a} = t^a_{a/a} + l_{a/l},$$

where

$$\gamma_1 = \sigma_1 \eta_{11} + \sigma_2 \eta_{12} + \sigma_3 \eta_{13}, \quad \gamma_2 = \sigma_1 \eta_{12} + \sigma_2 \eta_{22} + \sigma_3 \eta_{23},$$

$$\gamma_3 = \sigma_1 \eta_{13} + \sigma_2 \eta_{23} + \sigma_3 \eta_{33}, \quad \sigma_1 = C^a_{11} \alpha_a, \quad \sigma_2 = C^a_{21} \alpha_a, \quad \sigma_3 = C^a_{12} \alpha_a.$$

Let us find sets of the Maxwell Equations (11)–(16) for all solvable groups.

**Groups $G_3(I–VII)$**

The components of the metric tensor and structural constants $C^c_{ab}$ were found by Petrov (see [28]). The components of the vector $t^a_{l/a}$ were found in our work [33]:

$$e^a_{a} = \delta^a_1 \delta^a_1 \exp(-k u^3) + \delta^a_2 (-\delta^a_1 \varepsilon_3 \exp(-k u^3) + \delta^a_2 \exp(-nu^3)) + \delta^a_3 \varepsilon_3^3,$$

$$e^a_{a} = \delta^a_1 \delta^a_1 \exp(k u^3) + \delta^a_2 (\delta^a_1 \varepsilon_3 \exp nu^3 + \delta^a_2 \exp nu^3)) + \delta^a_3 \varepsilon_3^3,$$

$$C^c_{ab} = k \delta^c_1 \varepsilon_{ab}^1 + (\varepsilon^c_{ab} + n \delta^c_2) \varepsilon_{ab}^2.$$
Solution of the Maxwell Equations (14)–(16) has the form:

\[ \beta^a = \text{const}, \quad \alpha_a = \beta^b \int_0^1 \eta_{ab} du^0; \]  

(19)

(2) For the group \( G_1(II) \) \((k = n = 0, \quad \varepsilon = 1)\):

\[ \dot{\beta}^a = -\delta_1^a \alpha_1 \eta_{11}, \quad \dot{\alpha}_a = \frac{1}{\eta} \eta_{ab} \beta^b; \]  

(20)

(3) For the group \( G_1(III) \) \((k = 1, \quad n = \varepsilon = 0)\):

\[ \dot{\beta}^a = -\delta_2^a \alpha_2 \eta_{22}, \quad \beta^3 = 0, \quad \dot{\alpha}_a = \frac{1}{\eta} \eta_{ab} \beta^b; \]  

(21)

(4) For the group \( G_1(IV) \) \((k = n = \varepsilon = 1)\):

\[ \dot{\beta}^a = -\delta_1^a ((\alpha_1 + \alpha_2) \eta_{11} + \alpha_2 \alpha_1 \eta_{12} - \alpha_1 \eta_{22}) + \delta_2^a ((\alpha_1 + \alpha_2) \eta_{11} - \alpha_1 \eta_{12}); \]  

\[ \beta^3 = 0, \quad \dot{\alpha}_a = \frac{1}{\eta} \eta_{ab} \beta^b; \]  

(22)

(5) For the group \( G_1(V) \) \((k = n = 1, \quad \varepsilon = 0)\):

\[ \dot{\beta}^a = \delta_1^a (-\alpha_2 \eta_{12} + \alpha_1 \eta_{22}) + \delta_2^a (\alpha_1 \eta_{12} - \alpha_2 \eta_{11}), \quad \beta^3 = 0, \quad \dot{\alpha}_a = \frac{1}{\eta} \eta_{ab} \beta^b; \]  

(23)

(6) For the group \( G_1(VI) \) \((k = 1, \quad n = 2, \quad \varepsilon = 0)\):

\[ \dot{\beta}^a = -\delta_1^a (2\alpha_2 \eta_{12} - \alpha_1 \eta_{22}) + \delta_2^a (2\alpha_2 \eta_{11} - \alpha_1 \eta_{12}), \quad \beta^3 = 0, \quad \dot{\alpha}_a = \frac{1}{\eta} \eta_{ab} \beta^b. \]  

(24)

Equations (20), (24) were integrated into our work [35]. In the present paper, the solutions for the group \( G(VII) \) were found.

II. Group \( G(VII) \).

When obtaining the Maxwell equations for the groups \( G_3(I–VI) \), the components of vector fields \( l^a_\alpha \) could be constructed directly from the components of the metric tensor (see [35]). For the group \( G(VII) \), this cannot be performed. Therefore, the vectors \( l^a_\alpha \) must be found directly from the conditions (2). Consider these conditions for the structural constants of the group \( G_3(VII) \):

\[ C^a_{23} = -\delta_1^a + 2\delta_2^a \cos \alpha, \quad C^2_{13} = 1, \quad \alpha = \text{const}. \]

By coordinate transformation of the form \( \tilde{u}^a = \tilde{u}^a(u^3) \) the vector field \( l^a_3 \) can be diagonalized:

\[ l^a_3 = \delta^a_3. \]

From the commutation relations, it follows that:

\[ X_{1,3} = -X_2; \quad X_{2,3} = X_1 - 2X_2 \cos \alpha \Rightarrow l^a_2 = -l^a_{1,3}; \quad l^a_{2,3} = l^a_{2,33} + 2l^a_{1,3} \cos \alpha + l^a_1 = 0. \]  

(25)

Solution of the Equation (25) has the form:

\[ l^a_1 = \exp (-q_3)(a^a_1(u^p) \sin p_3 + b^a_1(u^p) \cos p_3), \]
\[ l^2_p = - \exp (-q_3)(a^3_2(u^p) \sin (p_3 - \alpha) + b^2_2(u^p) \cos (p_3 - \alpha)), \]

where \( p, q = 1, 2, q_3 = u^3 \cos \alpha, p_3 = u^3 \sin \alpha \). Since the operators \( X_p \) commute, the vectors \( a^p_q, a^p_q \) can be simultaneously diagonalized by coordinate transformations of the form \( \tilde{u}^p = \tilde{u}^p(u^q) \):

\[ a^p_q = \delta^p_q, \quad b^p_q = \delta^p_q, \]

From the commutation relations it follows that: \( a^3_3 = 0, b^3_3 = 0 \).

Thus, the vectors of the frame of the homogeneous space of type \( VII \) according to Bianchi can be represented in the form:

\[ l^3_p = \exp (-q_3)(\delta^3_1 \sin p_3 + \delta^3_2 \cos p_3), \]

\[ l^2_p = \exp (-q_3)(\delta^2_1 \sin (p_3 - \alpha) + \delta^2_2 \cos (p_3 - \alpha)), \quad l^3_p = \delta^3_3. \]

The Maxwell Equations will take the form:

\[ \eta \hat{\beta}_a = \delta^1_1(\gamma_1 - 2\gamma_2 \cos \alpha) + \delta^3_3 \gamma_2, \quad \Rightarrow \quad \gamma_2 = \eta \hat{\beta}_2, \gamma_1 = \eta(\hat{\beta}_1 + 2\hat{\beta}_2 \cos \alpha). \]

The system of Maxwell’s equations can be represented in the form:

\[ \begin{align*}
\sigma \eta_{11} - \alpha_2 \eta_{12} &= \gamma_1, \\
\sigma \eta_{12} - \alpha_2 \eta_{22} &= \gamma_2 (\sigma = 2\alpha_2 \cos \alpha - \alpha_1); \\
\beta_1 \eta_{11} + \beta_2 \eta_{12} &= \eta \hat{\alpha}_1, \\
\beta_1 \eta_{12} + \beta_2 \eta_{22} &= \eta \hat{\alpha}_2, \beta_3 = 0; \\
\eta \hat{\alpha}_3 &= \beta_1 \eta_{13} + \beta_2 \eta_{23} \quad \Rightarrow \quad \alpha_3 = \frac{\int (\beta_1 \eta_{13} + \beta_2 \eta_{23}) du_0}{\eta}. 
\end{align*} \]

From Equations (28) and (29), it follows that:

\[ \begin{align*}
\eta_{11}(\alpha_2 \hat{\alpha}_2 - \sigma \hat{\alpha}_1)(\alpha_2 \hat{\beta}_1 + \sigma \hat{\beta}_2) &= \gamma_1 \beta_2(\alpha_2 \hat{\alpha}_2 - \sigma \hat{\alpha}_1) - \alpha_2 \hat{\alpha}_2(\beta_1 \gamma_1 + \beta_2 \gamma_2), \\
\alpha_1 \hat{\alpha}_2(\eta(\alpha_2 \hat{\alpha}_2 - \sigma \hat{\alpha}_1) + \beta_1 \gamma_1 + \beta_2 \gamma_2) &= 0. 
\end{align*} \]

When solving the system of equations (31), (32), the variants that need to be considered are:

**A.** \( \alpha_2 \neq 0 \). From the system of Equation (29), it follows:

\[ \eta_{11}(\alpha_2 \beta_1 + \sigma \beta_2) = \eta(\alpha_1 \alpha_2 + \beta_1 \beta_2), \quad \eta_{12} = \frac{1}{\alpha_2}(\sigma_1 \eta_{11} - \eta \tilde{\beta}_1), \quad \eta_{22} = \frac{1}{\alpha_2}(\sigma_1^2 \eta_{11} - \eta(\sigma_1 \tilde{\beta}_1 + \alpha_2 \beta_2)). \]

When solving the set of Equations (31) and (33), the following variants must be consider:

1. \( (\alpha_2 \hat{\alpha}_2 - \sigma \hat{\alpha}_1) \neq 0 \) \( \Rightarrow \) \( \eta_{11} = \eta(\alpha_2 \hat{\alpha}_2 + \beta_1 \beta_2) \). We consider Equation (32): Let us use the following notations:

\[ \alpha_1 = \sqrt{\rho} \sin (\omega/2), \quad \alpha_2 = \sqrt{\rho} \cos (\omega/2), \quad \Omega = (2\beta_2 \hat{\beta}_1 \cos \alpha + \beta_1 \hat{\beta}_1 + \beta_2 \hat{\beta}_2), \quad \omega = \omega(u^0), \]

(1) Let \( \alpha_1 \neq 0 \). Then the equation (34) can be reduced to the form:

\[ 2\alpha_2 \hat{\alpha}_1 \cos \alpha - \alpha_1 \hat{\alpha}_1 - \alpha_2 \hat{\alpha}_2 = 2\beta_2 \hat{\beta}_1 \cos \alpha + \beta_1 \hat{\beta}_1 + \beta_2 \hat{\beta}_2. \]
(a) $\omega \neq 0$. In this case, we take the function $\omega$ as a new time variable and denote by the point the derivative on this variable. The functions $\beta_p, \rho$ depend on $\omega$. Then the equation (34) can be reduced to the form:

$$\dot{\rho}(\cos \alpha \sin \omega - 1) + \cos \alpha(1 + \cos \omega)\rho = 2\Omega.$$  \hfill (35)

The function $\rho$ can be represented in the form: $\rho = R(\omega)\tau(\omega)$, where

$$R(\omega) = \int \frac{\cos \alpha(1 + \cos \omega)}{1 - \cos \alpha \sin \omega} d\omega,$$

and the function $\tau$ has the form:

$$\tau = (c + 2\int \frac{\Omega}{R(1 - \cos \alpha \sin \omega)} d\omega),$$

(b) $\omega = a = \text{const} \Rightarrow \rho = (c - 2\int \frac{\Omega}{(1 - \cos \alpha \sin \omega)} du^0)$.

2. $\alpha_1 = 0, \quad \eta_{11} = \eta_{22}^2\beta_\alpha, \quad \eta_{12} = -\eta_{22}\beta_\alpha, \quad \eta_{22} = -\eta_{22}^2 + 2\cos \alpha\beta_\alpha \beta_\alpha, \quad \beta = \ln(\beta_1 + 2\cos \alpha\beta_2)$

The final solutions are represented in Solutions.

3. $\alpha_2\beta_1 + \sigma\beta_2 = 0, \eta_{11}$ is an arbitrary function of $u^0$. In this case, there are two variants to consider:

(a) $\alpha_1 = 0 \Rightarrow$ function $\eta_{pq}$ can be found from (33).

(b) $\alpha_1 \neq 0 \Rightarrow \alpha_1\alpha_2 + \beta\beta_2 = \alpha_2\alpha_2 + \beta_2\beta_2 = 0 \Rightarrow \alpha_2\alpha_2 + \beta_2\beta_2 = 0(\beta = 2\beta_2 \cos \alpha + \beta_1)$.

From the last equation it follows that:

$$\alpha_2 = c \sin \omega \beta_2 = c \cos \omega.$$

B. $\alpha_2 = 0$. In this case, from the set of Equations (28) and (29), it follows that:

$$\alpha_1\dot{\alpha}_1 + \beta_1\dot{\beta}_1 + 2\cos \alpha\beta_1\dot{\beta}_2 = 0, \quad \alpha_1 = \sqrt{c - (\beta_1)^2 - 4\cos \alpha \int \beta_1\beta_2 du^0}.$$

The functions $\eta_{ab}$ are determined from Equations (28) and (29). The results are given in the Solutions.

4 Solutions

In this section, all solutions of Maxwell’s vacuum equations for homogeneous Bianchi type VII spaces and electromagnetic fields invariant with respect to the groups of motions $G_3(VII)$ are given. For all solutions, the functions $\alpha_3$ and $\eta_{33}$ have the form:

$$\alpha_3 = \int (\eta_{13}^2\beta_1 + \eta_{23}^2\beta_2) du^0, \quad \eta_{33} = \frac{\eta^2 - 2\eta_{12}\eta_{13}\eta_{23} + \eta_{11}\eta_{23}^2 + \eta_{22}\eta_{13}^2}{\eta_{11}\eta_{22} - \eta_{13}^2}.$$

Other functions that specify solutions are shown below.

4.1 $\alpha_2 \neq 0$. 

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The functions $\eta_{12}, \eta_{22}$ have the form:

$$\eta_{12} = \frac{1}{\alpha_2} (\sigma_1 \eta_{11} - \eta \beta), \quad \eta_{22} = \frac{1}{\alpha_2} (\sigma_2^2 \eta_{11} - \eta (\sigma_1 \beta + \alpha_2 \beta)), \quad \sigma_1 = 2 \alpha_2 \cos \alpha - \alpha_1, \quad \beta = 2 \beta^2 \cos \alpha + \beta^1.$$

1. $\beta^1 \alpha_2 + \beta_2 \sigma_1 \neq 0$, $\eta_{11} = \eta \frac{\alpha_1 \eta_{11} + \beta^2 \beta}{\alpha_2 + \beta_2 \sigma_1}$, $\Omega = (\beta^1 \beta + \beta^2 \beta) + 2 \beta^2 \beta^1 \cos \alpha$.

(a) $\alpha_1 = \sqrt{\rho} \sin c$, $\alpha_1 = \sqrt{\rho} \cos c$, $\rho = \int \frac{2 \Omega \eta_{11}}{\cos \alpha \sin c - 1}$.

(b) $\alpha_1 = \sqrt{\rho} \sin \frac{\sigma}{2}$, $\alpha_2 = \sqrt{\rho} \cos \frac{\sigma}{2}$, $\omega = \omega(u^0)$, $\beta^p = \beta^p(\omega)$, $\dot{\beta}^p = \partial \beta^p/\partial \omega$,

$$\rho = \frac{\mathcal{R}}{1 - \cos \alpha \sin \omega} (c - 2 \int \Omega_1 \cos \alpha \sin \omega d\omega), \mathcal{R} = \exp \int \frac{\cos \alpha d\omega}{1 - \cos \alpha \sin \omega}.$$

2. $\eta_{11}$ is an arbitrary function of $u^0$.

(a) $\alpha_1 = 0$, $\eta_{11} = \eta \frac{\beta^2 \beta}{\sigma_2}$, $\eta_{12} = -\eta \frac{\beta^2 + 2 \cos \alpha \beta^2 \beta}{\sigma_2}$, $\eta = \ln(\beta^1 + 2 \cos \alpha \beta^2)$.

(b) $\alpha_1 = ac \sin \omega$, $\alpha_2 = c \sin \omega$, $\beta_2 = c \cos \omega$, $\beta_1 = c(a - 2 \cos \alpha) \cos \omega$. $c, a = \text{const}$

$$\eta_{12} = (2 \cos \alpha - a) \eta_{11} + a \eta, \quad \eta_{22} = (2 \cos \alpha - a)^2 \eta_{11} + \eta(a(2 \cos \alpha - a) + 1).$$

4.2 $\alpha_2 = 0$.

1. $\alpha_1 = \sqrt{c - (\beta^1)^2 - 4 \cos \alpha \int \beta^1 \beta^2 d\omega}$, $\eta_{11} = -\eta \frac{2 \cos \alpha \beta^2 \beta}{\alpha_2}$, $\eta_{22} = -\eta \frac{\beta}{\alpha_1}$.

2. $\beta_2 = 0, \alpha_1 = c \sin \omega, \beta_1 = c \cos \omega$. $\eta_{22}, \omega$ are arbitrary functions of $u^0$.

$$\eta_{12} = 0, \quad \eta_{11} = -\eta \omega, \quad \eta_{13} = \frac{\dot{\alpha}_3}{\beta_1}.$$

All functions included in these expressions that are not additionally described (for example, $\eta, \eta \beta$, and so on) are arbitrary functions of $u^0$.

5 Conclusions

In the paper, the classification of solutions of vacuum Maxwell equations for the case where the electromagnetic fields and the metrics of homogeneous spaces are invariant with respect to solvable groups of motions was completed (for the groups $G_3(I–VI)$, classification was carried out in the paper [35]). Since this classification was carried out in the canonical frame (2), it allows one to proceed with the classification of exact solutions of the vacuum Einstein–Maxwell equations for the found fields. This will be of interest for the study of the early stages of the evolution of the Universe.

FUNDING: The work is supported by Russian Science Foundation, project number N 23-21-00275.

INSTITUTIONAL REVIEW BOARD STATEMENT: Not applicable.

INFORMED CONSENT STATEMENT: Not applicable.

DATA AVAILABILITY STATEMENT: The data that support the findings of this study are available within the article.

ACKNOWLEDGMENTS: The work has no financial support Conflicts of Interest: The author declares no conflict of interest.
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