QUANTUM MECHANICS OF TIME-DEPENDENT SYSTEMS

CONSTRUCTION OF PURE STATES

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Abstract. For time-dependent systems the wavefunction depends explicitly on time and it is not a pure state of the Hamiltonian. We construct operators for which the above wavefunction is a pure state. The method is based on the introduction of conserved quantities $Q$ and the pure states are defined by $\hat{Q}\psi = q\psi$. The conserved quantities are constructed using parametrised mechanics and the Noether theorem.

1. Introduction

The quantum mechanics of time-dependent systems has been the subject of several investigations and in spite of the many efforts done on the subject it seems that still there is not even agreement on what quantum mechanics of time-dependent systems means.

There are several ways in which one can abord the quantum mechanics of time-dependent systems. For our purposes we will restrict our considerations to systems which can be described by means of a phenomenological effective Lagrangian, $L(q, \dot{q}, t)$, or Hamiltonian, $H(q, p, t)$. In this case one can directly write down a Schrödinger equation

$$\hat{H}\psi = H\left(q, -i\hbar\frac{\partial}{\partial q}, t\right)\psi = i\hbar\frac{\partial\psi}{\partial t}. \quad (1.1)$$

The wave function, $\psi(q, t)$, is explicitly time-dependent and therefore one looses the familiar concept of stationary states. Furthermore, one looses the concept of pure states (those states having a well-defined quantum number with respect to a given observable) and it is

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not clear if there exist other observables with respect to which the wavefunction is a pure state.

This situation can however be corrected if we are able to find conserved quantities $Q$ for the time-dependent system. In this case one can implement the quantum conditions

$$\hat{Q} \psi = q \psi.$$  \hspace{1cm} (1.2)

Therefore, even when the wave function is time-dependent, there would be observables with respect to which the wavefunction behaves as a pure state. The problem is reduced therefore to the construction of conserved quantities for time-dependent systems.

In this work we introduce a novel method for the construction of conserved quantities for time-dependent systems based on parametrised mechanics. In this approach the time $t$ is treated as a configuration variable. The system therefore looks as a time-independent system and conserved quantities can be found by means, in our case for example, of the Noether theorem.

Let us now describe our method more in detail. First of all we will restrict our considerations to systems for which there exist a description in terms of a time-dependent effective Lagrangian, $L(q, \dot{q}, t)$. This restriction is in view of the clear prescription for quantisation which exist in this case.

For time-independent Lagrangians, $L(q, \dot{q})$, the time $t$ can be considered, in some suitable sense, as an ignorable variable. In fact, the Hamiltonian is time-independent, $H(q, p)$, and is furthermore a conserved quantity, $dH/dt = 0$. Furthermore the Schrödinger equation can be solved in terms of a complete set of time-independent wavefunctions $\psi_n(q)$. This leads furthermore to the concept of pure states.

The situation is quite different for time-dependent Lagrangians, $L(q, \dot{q}, t)$. In fact, the Hamiltonian is time-dependent, $H(q, p, t)$, and is no more a conserved quantity, $dH/dt \neq 0$. Furthermore, the solution to the Schrödinger equation is an explicitly time-dependent wavefunction $\psi(q, t)$. Finally, one looses the concept of pure state.

Therefore, for time-dependent Lagrangians the time $t$ can no more be ignored so easily, it is no more an ignorable variable. However, it is neither a variable at the same level than the usual configuration variables $q$. The ideal situation would be to have the time $t$ at the same level than configuration variables $q$, since in that case (only configuration variables and an “ignorable” time) we would be able to construct pure states. Therefore, we must raise the time $t$ to the level of a configuration variable, $t \to q^0$. This is achieved by introducing a new variable $\tau$ playing the role of the old time. In this case the Lagrangian becomes homogeneous at the first-order in the derivatives with respect to the new time variable

$$\bar{L} = L \left( q^i, \dot{q}^i, q^0, \dot{q}^0 \right) \dot{q}^0. \hspace{1cm} (1.3)$$

This corresponds to a particular kind of constrained systems. One can check that the Hamiltonian is identically zero, $\bar{H} \equiv 0$. Furthermore there exists a first-class constraint

$$\phi = \bar{p}_0 + H(q^i, q^0, \bar{p}_i) \approx 0. \hspace{1cm} (1.4)$$
For constrained systems quantum mechanics is done in terms of first-class constraints, \( \dot{\phi} \psi = 0 \), which in our case reduces to
\[
-i \hbar \frac{\partial \psi}{\partial t} + H \left( q^i, t, -i \hbar \frac{\partial}{\partial q^i} \right) \psi = 0 ,
\]
which we recognise as the original Schrödinger equation. Therefore the Lagrangian \( \bar{L} \) is dynamically equivalent to the original \( L \).

The conserved quantities associated to \( \bar{L} \) are constructed with the Noether theorem. We introduce the Noether variation of the Lagrangian
\[
\xi(\bar{L}) = \xi^0 \frac{\partial \bar{L}}{\partial q^0} + \xi^i \frac{\partial \bar{L}}{\partial q^i} + \dot{\xi}^0 \frac{\partial \bar{L}}{\partial \dot{q}^0} + \dot{\xi}^i \frac{\partial \bar{L}}{\partial \dot{q}^i}.
\]
(1.6)

If \( \xi(\bar{L}) = 0 \) then
\[
Q(q^i, q^0, \bar{p}_i) = -\xi^0(q^i, q^0) H(q^i, q^0, \bar{p}_i) + \xi^i(q^i, q^0) \bar{p}_i .
\]
(1.7)
is a conserved quantity. Now we can use the conserved quantity above to obtain pure states. The corresponding equation is
\[
\hat{Q} \psi = \left[ -\xi^0(q^i, q^0) H(q^i, q^0, \bar{p}_i) + \xi^i(q^i, q^0) \bar{p}_i \right] \psi = q \psi .
\]
(1.8)

As an example we consider the damped harmonic oscillator described by the equation
\[
\ddot{x} + 2 \Gamma \dot{x} + \omega_0^2 x = 0 ,
\]
(1.9)
The corresponding effective Lagrangian is
\[
L = \frac{m}{2} (\dot{x}^2 - \omega_0^2 x^2) e^{2 \Gamma t} .
\]
(1.10)
The corresponding wave functions are
\[
\psi_n(x, t) = N_n H_n \left( \left( \frac{m \omega}{\hbar} \right)^{1/2} e^{\Gamma t} x \right) \times \exp \left( -\frac{m}{2 \hbar} (\omega + i \Gamma) e^{2 \Gamma t} x^2 \right) \exp \left( -i \left( n + \frac{1}{2} \right) \omega t + \frac{\Gamma}{2} t \right) .
\]
(1.11)
where \( N_n = (m \omega/\pi \hbar)^{1/4}/(2^n n!)^{1/2} \). From the parametrised Lagrangian we find the conserved quantity
\[
Q = \frac{m}{2} \left[ \dot{x}^2 + 2 \Gamma \dot{x} x + \omega_0^2 x^2 \right] e^{2 \Gamma t} = \frac{m}{2} \left[ (\dot{x} + \Gamma x)^2 + \omega^2 x^2 \right] e^{2 \Gamma t} .
\]
(1.12)
It is easy to check that the wavefunctions (1.11) are eigenfunctions of \( \hat{Q} \)
\[
\hat{Q} \psi = q \psi ,
\]
(1.13)
where the eigenvalue $q$ is given by

$$q = \left( n + \frac{1}{2} \right) \hbar \omega. \quad (1.14)$$

Therefore, the procedure of parametrisation and use of the Noether theorem allows the construction of conserved quantities which can be used for the construction of pure states.

In section 2 we present standard results for systems described by time-independent Lagrangians. In section 3 we show how all the above nice prescriptions fail for a time-dependent system. Section 4 illustrates how parametrised mechanics works. Section 5 is devoted to the example. The conclusions and comments on future developments are contained in section 6.

2. Time-Independent Systems

Here we present the general results for mechanics, conservation laws and quantum mechanics for systems described by a time-independent Lagrangian.

2.1. Classical Mechanics

We will restrict our considerations to mechanical systems described by a Lagrangian

$$L = L(q, \dot{q}). \quad (2.1)$$

The equations of motion are given by

$$\frac{\delta L}{\delta q^i} = \frac{\partial L}{\partial q^i} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) = 0. \quad (2.2)$$

Contraction of the equations of motion with $\dot{q}^i$ gives

$$\frac{dE}{dt} = \dot{q}^i \frac{\delta L}{\delta \dot{q}^i} = 0, \quad (2.3)$$

where $E$ is the energy of the system

$$E = \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L. \quad (2.4)$$

The canonical momenta are defined by

$$p_i = \frac{\partial L}{\partial \dot{q}^i}, \quad (2.5)$$

and the canonical Hamiltonian is given by
\[ H = q^i p_i - L. \quad (2.6) \]

The Hamiltonian is a function of \( q \)'s and \( p \)'s only, \( H = H(q, p) \), and furthermore is a conserved quantity. The equations of motion (2.2) are replaced by the Hamilton equations

\[
\begin{align*}
\dot{q}^i &= \frac{\partial H}{\partial p_i}, \\
\dot{p}_i &= -\frac{\partial H}{\partial q^i}. 
\end{align*} \quad (2.7)
\]

The time derivative of a phase space function \( F = F(q, p) \) is given by

\[
\dot{F} = \{F, H\}, \quad (2.8)
\]

where \( \{,\} \) is the Poisson bracket defined by

\[
\{F, G\} = \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q^i} \frac{\partial F}{\partial p_i}. \quad (2.9)
\]

### 2.2. The Noether theorem

The most systematic way for finding conserved quantities is through the Noether theorem. Let us define the Noether variation of the Lagrangian

\[
\xi(L) = \xi^i \frac{\partial L}{\partial q^i} + \dot{\xi}^i \frac{\partial L}{\partial \dot{q}^i}. \quad (2.10)
\]

We furthermore define the charge

\[
Q(\xi) = \xi^i \frac{\partial L}{\partial \dot{q}^i} = \xi^i p_i. \quad (2.11)
\]

The time derivative of \( Q(\xi) \) is given by

\[
\dot{Q}(\xi) = \xi(L) - \xi^i \frac{\delta L}{\delta q^i}. \quad (2.12)
\]

Therefore, if the equations of motion (2.2) hold and if

\[
\xi(L) = 0, \quad (2.13)
\]

then \( Q(\xi) \) is a conserved quantity.

### 2.3. Quantum Mechanics
For our purposes we can consider quantum mechanics just as an operator realisation of the Poisson bracket algebra. In the coordinate representation

\[ q^i \rightarrow q^i, \]
\[ p_i \rightarrow -i\hbar \frac{\partial}{\partial q^i}. \]  

(2.14)

Furthermore, all classical functions are transformed in differential operators,

\[ F \rightarrow \hat{F} = F(\hat{q}, \hat{p}), \]  

(2.15)

with the additional restriction

\[ [\hat{F}, \hat{G}] = i\hbar \{\hat{F}, \hat{G}\}. \]  

(2.16)

There is furthermore a wave function \( \psi = \psi(q, t) \) satisfying the Schrödinger equation

\[ \hat{H} \psi = H \left( q, -i\hbar \frac{\partial}{\partial q} \right) \psi = i\hbar \frac{\partial \psi}{\partial t}. \]  

(2.17)

Since the Hamiltonian is time independent we can write

\[ \psi(q, t) = \varphi(q) e^{-i\omega t}, \]  

(2.18)

and the Schrödinger equation reduces to

\[ \hat{H} \varphi = H \left( q, -i\hbar \frac{\partial}{\partial q} \right) \varphi = \hbar \omega \varphi = E \varphi. \]  

(2.19)

The above is nothing more than the quantum mechanical manifestation of the classical conservation law for the energy, (2.4). The solutions to eq. (2.19) define pure states, i.e. states with a well defined quantum number with respect to a given observable, the Hamiltonian in this case.

3. Time-Dependent Systems

We now present the classical mechanics, conservation laws and quantum mechanics for systems described by time-dependent Lagrangians.

3.1. Classical Mechanics

Let us now consider a mechanical systems described by a time-dependent Lagrangian

\[ L = L(q, \dot{q}, t). \]  

(3.1)
The equations of motion are given, as before, by (2.2). The energy $E$ is defined as in (2.4). Now, however, the time derivative of the energy is

$$\frac{dE}{dt} = \dot{q}^i \frac{\delta L}{\delta q^i} - \frac{\partial L}{\partial t} \neq 0. \quad (3.2)$$

The Hamiltonian formalism remains almost unchanged. The momenta are defined as in (2.5) and the Hamiltonian as in (2.6), however, now the Hamiltonian is a function also of $t$, $H = H(q, p, t)$. The Hamilton equations (2.7) are unchanged. This time, however, the time derivative of a phase space function $F = F(q, p, t)$ is given by

$$\dot{F} = \frac{\partial F}{\partial t} + \{F, H\}, \quad (3.3)$$

where $\{, \}$ is the Poisson bracket defined in (2.9).

The results of Section 2.2, concerning the Noether theorem, remain unchanged for time-dependent systems. However, the Noether theorem does not provide useful conserved quantities to be used for quantisation.

### 3.2. Quantum Mechanics

Once again we can consider quantum mechanics just as an operator realisation of the Poisson bracket. In the coordinate representation eqs. (2.10) are still valid, and all classical functions are transformed in differential operators, as in (2.11) with the additional restriction (2.12).

There is furthermore a wave function $\psi = \psi(q, t)$ satisfying the Schrödinger equation

$$\hat{H} \psi = H \left( q, -i \hbar \frac{\partial}{\partial q}, t \right) \psi = i \hbar \frac{\partial \psi}{\partial t}. \quad (3.4)$$

Now, however, the simple prescription (2.18) does not work anymore. The solutions will be, therefore, unavoidably of the form $\psi = \psi(q, t)$. In this case it is no more possible to talk of stationary states. Furthermore the wavefunction is not a pure state with respect to the Hamiltonian. Finally, it is not clear if there exist other observables with respect to which the wavefunction is a pure state.

The problem is therefore reduced to the construction of conserved quantities, if any, with respect to which the wave function $\psi(q, t)$ is a pure state. The solution to this problem is provided by parametrised mechanics.

### 4. Parametrised Mechanics

Here we present parametrised classical mechanics, conservation laws, and quantum mechanics.

#### 4.1. Parametrising unparametrised mechanics
In parametrised mechanics the time \( t \) is put at the same level than the configuration variables \( q, t \rightarrow q^0 \). In this case the role of the time is played by a new parameter \( \tau \). The new Lagrangian is obtained by requiring invariance of the action

\[
S = \int L(q, \bar{q}, t) \, dt = \int L(q, \theta, q^0) \, \dot{q}^0 \, d\tau ,
\]

(4.1)

where now derivatives with respect to the old time are denoted by a bar, and dots denote derivatives with respect to \( \tau \); furthermore

\[
\theta^i = \frac{\dot{q}^i}{q^0}.
\]

(4.2)

Therefore, the new Lagrangian is given by

\[
\bar{L} = L(q, \theta, q^0) \dot{q}^0.
\]

(4.3)

The equations of motion are given by

\[
\delta \bar{L} \delta q^i = \frac{\partial \bar{L}}{\partial q^i} - \frac{d}{d\tau} \left( \frac{\partial \bar{L}}{\partial \dot{q}^i} \right) = \frac{\partial L}{\partial q^i} \dot{q}^0 - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{\theta}^i} \right) = 0 ,
\]

\[
\frac{\delta \bar{L}}{\delta q^0} = \frac{\partial \bar{L}}{\partial q^0} - \frac{d}{d\tau} \left( \frac{\partial \bar{L}}{\partial \dot{q}^0} \right) = \frac{\partial L}{\partial q^0} \dot{q}^0 - \frac{d}{d\tau} \left( -\theta^i \frac{\partial L}{\partial \theta^i} + L \right) = 0 .
\]

(4.4)

The momenta are given by

\[
\bar{p}_0 = \frac{\partial \bar{L}}{\partial \dot{q}^0} = -\frac{\partial L}{\partial \theta^i} \theta^i + L = -H(q^i, q^0, \theta^i) ,
\]

(4.5a)

\[
\bar{p}_i = \frac{\partial \bar{L}}{\partial \dot{q}^i} = \frac{\partial L}{\partial \theta^i} ,
\]

(4.5b)

The Hamiltonian is given by

\[
\bar{H} = q^0 \bar{p}_0 + q^i \bar{p}_i - \bar{L} \equiv 0 .
\]

(4.6)

If the original Lagrangian was regular then, such as \( \dot{q}^i \)'s are solved in terms of \( p^i \)'s, eq. (4.5b) can be solved for \( \theta^i \)'s in terms of \( \bar{p}^i \)'s. Therefore (4.5a) can be rewritten as

\[
\bar{p}_0 = -H(q^i, q^0, \bar{p}_i) .
\]

(4.7)

The above relation is the primary first-class constraint associated to the parametrisation invariance of the Lagrangian [1]

\[
\phi = \bar{p}_0 + H(q^i, q^0, \bar{p}_i) \approx 0 .
\]

(4.8)

As is well known, for constrained systems quantum mechanics is implemented in terms of first-class constraints. In the configuration representation
\[ q^i \rightarrow q^i, \]
\[ q^0 \rightarrow t, \]
\[ \bar{p}_i \rightarrow -\imath \hbar \frac{\partial}{\partial q^i}, \]
\[ \bar{p}_0 \rightarrow -\imath \hbar \frac{\partial}{\partial t}. \]  

(4.9)

For the first-class constraint we obtain
\[ \hat{\phi} = -\imath \hbar \frac{\partial}{\partial t} + H \left( q^i, t, -\imath \hbar \frac{\partial}{\partial q^i} \right) \approx 0. \]  

(4.10)

The physical states \( \psi \) are those annihilated by (4.10)
\[ \hat{\phi} \psi = -\imath \hbar \frac{\partial \psi}{\partial t} + H \left( q^i, t, -\imath \hbar \frac{\partial}{\partial q^i} \right) \psi = 0, \]  

which we recognise as the original Schrödinger equation.

Therefore, the parametrised theory contains the standard results. However, this does not provide a solution to our problem: the search for conserved quantities.

### 4.2. The Noether theorem for parametrised mechanics

The Noether theorem applies unchanged to parametrised mechanics. In this case however, the configuration space is extended by the addition of the old time variable as a configuration variable. Therefore, it is convenient to write down explicitly all the corresponding formulae for this case.

In this case the Noether variation of the Lagrangian is given by
\[
\xi(\bar{L}) = \xi^0 \frac{\partial \bar{L}}{\partial q^0} + \xi^i \frac{\partial \bar{L}}{\partial q^i} + \dot{\xi}^0 \frac{\partial \bar{L}}{\partial q^0} + \dot{\xi}^i \frac{\partial \bar{L}}{\partial q^i} \\
= \xi^0 \frac{\partial L}{\partial q^0} q^0 + \xi^i \frac{\partial L}{\partial q^i} \dot{q}^0 + \xi^0 \left( -\dot{\theta}^i \frac{\partial L}{\partial \theta^i} + L \right) + \dot{\xi}^i \frac{\partial L}{\partial \theta^i} .
\]  

(4.12)

The Noether charge is defined as
\[ Q(\xi) = \xi^0 \frac{\partial \bar{L}}{\partial q^0} + \xi^i \frac{\partial \bar{L}}{\partial q^i} = \xi^0 \bar{p}_0 + \xi^i \bar{p}_i . \]  

(4.13)

Once again, if the equations of motion hold, and if
\[ \xi(\bar{L}) = 0, \]  

(4.14)
then the Noether charge $Q(\xi)$ is a conserved quantity

$$\frac{dQ}{d\tau} = 0.$$  \hfill (4.15)

It is also a conserved quantity with respect to the original time

$$\frac{dQ}{dt} = \frac{d\tau}{dt} \frac{dQ}{d\tau} = 0.$$  \hfill (4.16)

By using the constraint (4.8) we observe that $Q$ can be written as

$$Q(q^i, q^0, \bar{p}_i) = -\xi^0(q^i, q^0) H(q^i, q^0, \bar{p}_i) + \xi^i(q^i, q^0) \bar{p}_i.$$  \hfill (4.17)

Therefore it is something more than the Hamiltonian alone. The corrective term $\xi^i \bar{p}_i$ is able to kill the effect of the time dependence and give a conserved quantity.

### 4.3. Parametrised Quantum Mechanics

Quantum mechanics is performed in terms of the quantum version of $Q$, namely

$$\hat{Q} = Q(q^i, q^0, \hat{p}_i) = -\xi^0(q^i, q^0) H(q^i, q^0, \hat{p}_i) + \xi^i(q^i, q^0) \hat{p}_i.$$  \hfill (4.18)

Of course, in (4.18) may appear operator ordering problems. The corresponding “Schrödinger” equation is

$$\hat{Q} \psi = \left[ -\xi^0(q^i, q^0) H(q^i, q^0, \hat{p}_i) + \xi^i(q^i, q^0) \hat{p}_i \right] \psi = q \psi.$$  \hfill (4.19)

Therefore, the Schrödinger equation is modified to an equation for pure states.

### 5. The Damped Harmonic Oscillator

The damped harmonic oscillator is the simplest system described by a time-dependent Lagrangian. Some of the results presented here are standard. We refer the reader to [2] for further details.

#### 5.1. Classical mechanics of the damped harmonic oscillator

The damped harmonic oscillator is described by the equation

$$\ddot{x} + 2\Gamma \dot{x} + \omega_0^2 x = 0,$$  \hfill (5.1)

where $\Gamma > 0$. The usual Lagrangian used to describe the damped harmonic oscillator is the Bateman Lagrangian [3]

$$L = \frac{m}{2} (\dot{x}^2 - \omega_0^2 x^2) e^{2\Gamma t}.$$  \hfill (5.2)
The energy would be
\[ E = \frac{m}{2} (\dot{x}^2 + \omega_0^2 x^2) e^{2\Gamma t}. \] (5.3)

The momentum is given by
\[ p = m \dot{x} e^{2\Gamma t}. \] (5.4)

The Hamiltonian is then
\[ H = \frac{p^2}{2m} e^{-2\Gamma t} + \frac{m}{2} \omega_0^2 x^2 e^{2\Gamma t}. \] (5.5)

5.2. Quantum mechanics of the damped harmonic oscillator

The Schrödinger equation associated to the Hamiltonian (5.5) is
\[ -\hbar^2 \frac{\partial^2}{\partial x^2} + m\omega_0^2 x^2 \] \[ \psi = i \hbar \frac{\partial \psi}{\partial t}. \] (5.6)

Due to the explicit appearance of time in the Hamiltonian, this equation does not admit stationary solutions. The solutions to eq. (5.6) is given by
\[ \psi_n(x, t) = N_n H_n \left( \left( \frac{m\omega}{\hbar} \right)^{1/4} e^{\Gamma t} x \right) \] \[ \times \exp\left( -\frac{m}{2\hbar} (\omega + i\Gamma) e^{2\Gamma t} x^2 \right) \exp\left( -i \left( n + \frac{1}{2} \right) \omega t + \frac{\Gamma}{2} t \right). \] (5.7)

where \( N_n \) is a normalisation constant given by
\[ N_n = \frac{1}{(2n!)^{1/2}} \left( \frac{m\omega}{\pi \hbar} \right)^{1/4}, \] (5.8)

and
\[ \omega^2 = \omega_0^2 - \Gamma^2 > 0, \] (5.9)

The wavefunction depends explicitly on time, therefore it is not stationary. Furthermore it is not a pure state.

5.3. The parametrised damped harmonic oscillator

The parametrised damped harmonic oscillator is described by the Lagrangian
\[ L = \frac{m}{2} \left( \frac{\dot{x}^2}{\tau} - \omega_0^2 x^2 \right) e^{2 \Gamma t}. \] 

(5.10)

Using the Noether theorem as applied to parametrised mechanics (Sec. 4.2) we find the following conserved quantity

\[ Q = \frac{m}{2} \left[ \dot{x}^2 + 2 \Gamma \dot{x} + \omega_0^2 x^2 \right] e^{2 \Gamma t} = \frac{m}{2} \left[ (\dot{x} + \Gamma x)^2 + \omega^2 x^2 \right] e^{2 \Gamma t}. \] 

(5.11)

In fact, the time derivative of this quantity is

\[ \dot{Q} = m (\dot{x} + \Gamma x) (\ddot{x} + 2 \Gamma \ddot{x} + \omega_0^2 x) e^{2 \Gamma t} = 0, \] 

(5.12)

which is zero in virtue of the equations of motion (5.1). Let us finally observe that, for \( \Gamma = 0 \), \( Q \) reduces to the usual energy.

Up to our knowledge the above quantity has not been reported as a conserved quantity associated to the damped harmonic oscillator.

### 5.4. Quantum mechanics of the parametrised damped harmonic oscillator

Now we use \( Q \) for doing quantum mechanics. The corresponding operator is given by

\[ \hat{Q} = -\frac{\hbar^2}{2m} e^{-2 \Gamma t} \frac{\partial^2}{\partial x^2} - i \hbar \Gamma \left( x \frac{\partial}{\partial x} + \frac{1}{2} \right) + \frac{m}{2} \omega_0^2 x^2 e^{2 \Gamma t}. \] 

(5.13)

We can next construct the Schrödinger-like equation

\[ \hat{Q} \psi = \left[ -\frac{\hbar^2}{2m} e^{-2 \Gamma t} \frac{\partial^2}{\partial x^2} - i \hbar \Gamma \left( x \frac{\partial}{\partial x} + \frac{1}{2} \right) + \frac{m}{2} \omega_0^2 x^2 e^{2 \Gamma t} \right] \psi = q \psi. \] 

(5.14)

The factor \( \frac{1}{2} \) in the middle term appears after operator ordering, cf. \[4\]. We can verify that the solution is given exactly by (5.7). The eigenvalue \( q \) is given by

\[ q = \left( n + \frac{1}{2} \right) \hbar \omega. \] 

(5.15)

In conclusion, \( Q \) is the observable with respect to which the wave function \( \psi \), solution of the time dependent Schrödinger equation, behaves as a pure state.

### 6. Conclusions

We have introduced a method which allows to interpret the time-dependent wavefunctions as pure states with respect to another observables. The method is based on the construction of conserved quantities associated to time-dependent systems. The method
itself deserves more consideration since it provides new conserved quantities for time-dependent systems. In fact, (5.11) is a conserved quantity for the damped harmonic oscillator. However, it cannot be found by direct application of the Noether theorem to the unparametrised Lagrangian (5.2). This is an example of a non-Noetherian conserved quantity.

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