On M-injective and M-projective Modules
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Abstract
A left $R$-module $M$ is called max-injective (or m-injective for short) if for any maximal left ideal $I$, any homomorphism $f : I \to M$ can be extended to $g : R \to M$, if and only if $\text{Ext}^1_R(R/I,M) = 0$ for any maximal left ideal $I$. A left $R$-module $M$ is called max-projective (or m-projective for short) if $\text{Ext}^1_R(M,N) = 0$ for any max-injective left $R$-module $N$. We prove that every left $R$-module has a special m-projective precover and a special m-injective preenvelope. We characterize $C$-rings, SF rings and max-hereditary rings using m-projective and m-injective modules.

Keywords: M-injective modules; m-projective modules; max-hereditary rings.

AMS Subject Classification (2020): Primary: 16D40 ; Secondary: 18G25.

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Throughout, $R$ will denote an associative ring with identity, and modules will be unital $R$-modules, unless otherwise stated. As usual, we denote by $\text{Mod}-R$ the category of right $R$-modules. For a module $M$, $E(M)$, $M^+$ denote the the injective hull and the character module $\text{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$ of $M$, respectively.

Given a class $\mathcal{C}$ of right $R$-modules, we will denote by $\mathcal{C}^\perp = \{X : \text{Ext}^1_R(C,X) = 0 \text{ for all } C \in \mathcal{C}\}$ the right orthogonal class of $\mathcal{C}$, and by $\perp\mathcal{C} = \{X : \text{Ext}^1_R(X,C) = 0 \text{ for all } C \in \mathcal{C}\}$ the left orthogonal class of $\mathcal{C}$. Let $M$ be a right $R$-module. A homomorphism $\phi : M \to F$ with $F \in \mathcal{C}$ is called a $\mathcal{C}$-preenvelope of $M$ ([2]) if for any homomorphism $f : M \to G$ with $G \in \mathcal{C}$, there is a homomorphism $g : F \to G$ such that $g\phi = f$. Moreover, if the only such $g$ are automorphisms of $F$ when $F = G$ and $f = \phi$, the $\mathcal{C}$-preenvelope is called a $\mathcal{C}$-envelope of $M$. Following [2], a monomorphism $\alpha : M \to C$ with $C \in \mathcal{C}$ is said to be a special $\mathcal{C}$-preenvelope of $M$ if $\text{coker}(\alpha) \in \perp\mathcal{C}$. Dually, we have the definitions of a (special) $\mathcal{C}$-precover and a $\mathcal{C}$-cover. $\mathcal{C}$-envelopes ($\mathcal{C}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism. A pair $(\mathfrak{g}, \mathcal{C})$ of classes of left $R$-modules is called a cotorsion theory ([2]) if $\mathfrak{g}^\perp = \mathcal{C}$ and $\perp\mathcal{C} = \mathfrak{g}$. A cotorsion theory $(\mathfrak{g}, \mathcal{C})$ is called perfect (complete) if every left $R$-module has a $\mathfrak{g}$-envelope and an $\mathfrak{g}$-cover (a special $\mathcal{C}$-preenvelope and a special $\mathfrak{g}$-precover). A cotorsion theory $(\mathfrak{g}, \mathcal{C})$ is said to be hereditary ([3]) if whenever $0 \to L' \to L \to L'' \to 0$ is exact with $L, L'' \in \mathfrak{g}$, then $L'$ is also in $\mathfrak{g}$. By [3], $(\mathfrak{g}, \mathcal{C})$ is hereditary if and only if $0 \to C' \to C \to C'' \to 0$ is exact with $C, C' \in \mathcal{C}$, then $C''$ is also in $\mathcal{C}$.

A left $R$-module $M$ is called max-injective (or m-injective for short) if for any maximal left ideal $I$, any homomorphism $f : I \to M$ can be extended to $g : R \to M$, if and only if $\text{Ext}^1_R(R/I,M) = 0$ for any maximal left ideal $I$. A ring $R$ is said to be left m-injective if $R$ is m-injective as a left $R$-module [10]. m-injective modules have been studied further in [11]. In this paper, the concept of max-projective (or m-projective for short) modules is introduced. A left $R$-module $M$ is said to be m-projective if $\text{Ext}^1_R(M,N) = 0$ for any m-injective left $R$-module $N$. In what follows, $m - pr, m - in$ stands for the class of all m-projective (resp. all m-injective) left $R$-modules. We prove that $(m - pr, m - in)$ is a complete cotorsion theory. We prove that $R$ is a left C-ring (i.e. for every essential left ideal $I$ of $R$, $R/I$ has a simple submodule) if and only if every cyclic left $R$-module is m-projective if and only if every m-injective left $R$-module is injective if and only if $(m - pr, m - in)$ is hereditary, and every m-injective left $R$-module is m-projective. We also prove that $R$ is left max-hereditary (i.e. if every maximal left ideal is projective) if and only if every quotient of an m-injective left $R$-module is m-injective if and only if every m-projective left $R$-module has projective dimension at most 1. It is also shown that, $R$ is a left SF-ring if and only if every left $R$-module is m-injective if and only if every cotorsion left $R$-module is m-injective if and only if $(m - pr, m - in)$ is hereditary and

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Received : 07–10–2019, Accepted : 14–02–2020
every m-projective left R-module is m-injective.

1. M-projective Modules

In this section several properties and characterization of m-projective modules are given. First we recall the definition.

**Definition 1.1.** A left R-module M is said to be m-projective if \( \text{Ext}_R^1(M, N) = 0 \) for any m-injective left R-module N. The right version can be defined similarly.

Recall that a ring \( R \) is called a left coherent ring if every finitely generated left ideal of \( R \) is finitely presented. Following [11], a ring \( R \) is said to be left max-coherent if every maximal left ideal is finitely presented. Obviously, Noetherian rings are max-coherent. But, left coherent rings need not be max-coherent.

**Example 1.1.** Obviously, any projective module is m-projective. By the definition, any simple left \( R \)-module is m-projective. Since over a left max-coherent ring, FP-injective modules are m-injective, m-projective modules are FP-projective. (a left \( R \)-module \( M \) is called FP-injective (resp. FP-projective) provided that \( \text{Ext}_R^1(F, M) = 0 \) (resp. \( \text{Ext}_R^1(F, M) = 0 \)) for any finitely presented (resp. FP-injective) left \( R \)-module \( F \).

**Proposition 1.1.** The following are equivalent for a left m-injective left max-coherent ring \( R \) and a left \( R \)-module \( M \).

1. \( M \) is m-projective.
2. \( M \) is projective with respect to every exact sequence \( 0 \to K \to T \to L \to 0 \) with \( K \) is m-injective.
3. For every exact sequence \( 0 \to A \to E \to M \to 0 \), where \( E \) is m-projective, \( A \to E \) is an m-injective preenvelope of \( A \).
4. \( M \) is a cokernel of an m-injective preenvelope \( A \to E \) with \( E \) projective.

**Proof.** (1) \( \Rightarrow \) (2) and (1) \( \Rightarrow \) (3) are trivial.

(2) \( \Rightarrow \) (1) Let \( N \) be an m-injective left \( R \)-module. There exists an exact sequence \( 0 \to N \to E \to L \to 0 \) with \( E \) injective. This induces an exact sequence \( \text{Hom}(M, E) \to \text{Hom}(M, L) \to \text{Ext}_R^1(M, N) \to 0 \). Since \( \text{Hom}(M, E) \to \text{Hom}(M, L) \) is exact by (2), \( \text{Ext}_R^1(M, N) \). So \( M \) is m-projective.

(3) \( \Rightarrow \) (4) Let \( 0 \to A \to E \to M \to 0 \) be an exact sequence with \( E \) projective. Since \( R \) is left max-coherent left m-injective ring, \( E \) is m-injective by [11, Proposition 2.4(2)]. Thus, \( A \to E \) is an m-injective preenvelope.

(4) \( \Rightarrow \) (1) Let \( M \) be a cokernel of an m-injective preenvelope \( A \to E \) with \( E \) projective. Then, there is an exact \( 0 \to A \to E \to M \to 0 \). For each m-injective left \( R \)-module \( N \), \( \text{Hom}(E, N) \to \text{Hom}(A, N) \to \text{Ext}_R^1(M, N) \to 0 \). Note that \( \text{Hom}(E, N) \to \text{Hom}(A, N) \) is epic by (4). Thus \( \text{Ext}_R^1(M, N) \), and so \( M \) is m-projective.

Now we have the following Lemma.

**Lemma 1.1.** Let \( R \) be a ring. \( (m - \text{pr}, m - \text{in}) \) is a complete cotorsion theory.

**Proof.** (3) Let \( \mathcal{C} \) be the set of representatives of simple left \( R \)-modules. Thus \( m - \text{in} = \mathcal{C}^\perp \). Since \( m - \text{pr} = \perp(\mathcal{C}^\perp) \), the result follows from [2, Definition 7.1.5] and [4, Theorem 10].

A ring \( R \) is said to be a left \( C \)-ring if for every essential left ideal \( I \) of \( R \), \( R/I \) has a simple submodule. Right perfect rings, left semiartinian rings are well known examples of left \( C \)-rings ([1, 10.10]).

**Corollary 1.1.** Let \( R \) be a ring. Then the following are equivalent.

1. \( R \) is a left \( C \)-ring.
2. Every left \( R \)-module is m-projective.
3. Every cyclic left \( R \)-module is m-projective.
4. Every m-injective left \( R \)-module is injective.
5. \( (m - \text{pr}, m - \text{in}) \) is hereditary, and every m-injective left \( R \)-module is m-projective.

In this case, if \( R \) is left max-coherent, \( R \) is left Noetherian.
we characterize left max-hereditary rings over a left max-coherent ring. Hence there is

\begin{equation}
\text{Theorem 1.1.}
\end{equation}

Thus M is injective, as desired.

(5) \Rightarrow (2) By Lemma 1.1, for any left R-module M, there is a short exact sequence
0 \rightarrow M \rightarrow F \rightarrow L \rightarrow 0, where F is m-injective and L is m-projective. So (2) follows from (5).

In this case, if R is a left max-coherent ring, every FP-injective left R-module is m-injective, and so injective. This means R is left Noetherian by [6, Theorem 3].

\begin{corollary}
Let R be a left max-coherent ring. Then the following are equivalent.

1. Every m-injective left R-module is FP-injective.
2. Every finitely presented left R-module is m-projective.

In this case, R is a left coherent ring.
\end{corollary}

Proof. (1) \Leftrightarrow (2) follow from Lemma 1.1, since every module has a special m − pr-precover and a special m − in-preenvelope.

To prove the last statement, let M be an FP-injective left R-module with N a pure submodule, then M/N is m-injective by [11, Proposition 2.6] since R is left max-coherent. Therefore M/N is FP-injective by (1), and hence R is a left coherent ring by [7, Theorem 3.7].

A ring R will be called left max-hereditary if every maximal left ideal is projective. Recall that a ring R is said to be left PP if every principal left ideal of R is projective. Then any left PP-ring with every maximal left ideal principal is left max-hereditary. Now we have the following characterizations of left max-hereditary rings.

\begin{proposition}
Let R be a ring. The following are equivalent.

1. R is left max-hereditary.
2. Every quotient of an m-injective left R-module is m-injective.
3. Every m-projective left R-module has projective dimension at most 1.

Proof. (1) \Rightarrow (2) Let M be an m-injective left R-module and N a submodule of M. We shall show that M/N is m-injective. To this end, let I be a maximal left ideal of R and i : I \rightarrow R the inclusion and \pi : M \rightarrow M/N the canonical map. For any f : I \rightarrow M/N, then there exists g : I \rightarrow M such that \pi g = f since I is projective by (1). Hence there is h : R \rightarrow M such that hi = g since M is m-injective. It follows that (\pi h)i = f, and so M/N is m-injective.

(2) \Rightarrow (3) Let M be an m-projective left R-module and N a left R-module, then there is a short exact sequence
0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0 with E injective. Note that L is m-injective by (2), and so we have the exact sequence
0 = Ext^1_R(M, L) \rightarrow Ext^2_R(M, N) \rightarrow Ext^2_R(M, E) = 0. Thus Ext^2_R(M, N) = 0 and hence M has projective dimension at most 1.

(3) \Rightarrow (1) holds since every simple left R-module is m-projective.

A left R-module M is said to be MI-injective [11] if Ext^1_R(N, M) = 0 for any m-injective left R-module N. Next we characterize left max-hereditary rings over a left max-coherent ring.

\begin{theorem}
Let R be a left max-coherent ring. The following are equivalent.

1. R is left max-hereditary.
2. Every MI-injective left R-module is injective.
3. (m − pr, m − in) is hereditary, and every m-projective left R-module has a monic m-injective cover.

Proof. (1) \Rightarrow (2) is clear by [11, Proposition 3.4].

(2) \Rightarrow (1) Let N be a quotient of an m-injective left R-module M. Suppose f : F \rightarrow N is a m-injective cover of N by [11, Remark 2.10(1)]. Then there exists a homomorphism h : M \rightarrow F such that fh = \pi, where \pi : M \rightarrow N. Hence f is an epimorphism. By [11, Remark 3.2(1)], ker(f) is MI-injective, and so it is injective by (2). So, N is m-injective.
(1) \Rightarrow (3) holds by [5, Proposition 4] since the class of m-injective left \( R \)-modules is closed under direct sums over a left max-coherent ring by [11, Proposition 2.4(2)].

(3) \Rightarrow (1) Let \( M \) be any m-injective left \( R \)-module and \( N \) any submodule of \( M \). We have to prove that \( M/N \) is m-injective. In fact, there exists an exact sequence \( 0 \rightarrow N \rightarrow E \rightarrow L \rightarrow 0 \) with \( E \) is m-injective and \( L \) is \( R \)-module by Lemma 1.1. Since \( L \) has a m-projective \( R \)-module \( \alpha : E \rightarrow F \) such that \( \pi = \phi \alpha \). Thus \( \phi \) is epic, and hence it is an isomorphism. So \( L \) is m-injective. For any simple left \( R \)-module \( S \), we have the exact sequence \( 0 = \Ext^1_R(S, L) \rightarrow \Ext^1_R(S, N) \rightarrow \Ext^1_R(S, E) \). Note that \( \Ext^1_R(S, E) = 0 \) by [3, Proposition 1.2] since \((m - \pr, m - \in)\) is hereditary, and hence \( \Ext^1_R(S, N) = 0 \). On the other hand, the short exact sequence \( 0 \rightarrow N \rightarrow M \rightarrow M/N \rightarrow 0 \) induces the exactness of the sequence \( 0 = \Ext^1_R(S, M) \rightarrow \Ext^1_R(S, M/N) \rightarrow \Ext^1_R(S, N) = 0 \). Therefore \( \Ext^1_R(S, M/N) = 0 \), as desired.

Finally, we give some new characterizations of left SF-rings. Recall that a ring \( R \) is called a left SF-ring if each simple left \( R \)-module is flat.

**Theorem 1.2.** Let \( R \) be a ring. The following are equivalent.

1. \( R \) is a left SF-ring.
2. Every left \( R \)-module is m-injective.
3. Every m-projective left \( R \)-module is projective.
4. Every cotorsion left \( R \)-module is m-injective.
5. \((m - \pr, m - \in)\) is hereditary and every m-projective left \( R \)-module is m-injective.

**Proof.** (2) \Rightarrow (4) and (2) \Rightarrow (5) are trivial.

(4) \Rightarrow (1) Let \( S \) be any simple left \( R \)-module and \( M \) be a right \( R \)-module. Since \( M^+ \) is pure-injective and hence cotorsion, \( M^+ \) is m-injective by (4). So \( 0 = \Ext^1_R(S, M^+) \cong \Tor^R_1(M, S)^+ \). Thus, \( \Tor^R_1(M, S) = 0 \) for any right \( R \)-module \( M \). Hence \( S \) is flat, and so \( R \) is a left SF-ring.

(1) \Rightarrow (2) Let \( M \) be a left \( R \)-module. Then for any simple left \( R \)-module \( S \), \( 0 = \Tor^R_1(M^+, S) \cong \Ext^1_R(S, M)^+ \). So \( \Ext^1_R(S, M) = 0 \). Hence \( M \) is m-injective.

(2) \Rightarrow (3) Let \( M \) be any m-projective left \( R \)-module. Since every left \( R \)-module is m-injective by (2), \( \Ext^1_R(M, N) = 0 \) for any left \( R \)-module \( N \). Hence \( M \) is projective.

(3) \Rightarrow (2) Let \( M \) be a left \( R \)-module. There exists an exact sequence \( 0 \rightarrow M \rightarrow E \rightarrow P \rightarrow 0 \) with \( E \) m-injective and \( P \) m-projective by Lemma 1.1. By (3), \( P \) is projective, and so \( M \) is m-injective.

(5) \Rightarrow (2) Let \( M \) be any left \( R \)-module. By Lemma 1.1, there is a short exact sequence \( 0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0 \) with \( F \) m-projective and \( K \) m-injective. Then \( F \) is m-injective and hence \( M \) is m-injective by (5).

**References**

[1] Clark, J., Lomp, C., Vanaja, N., Wisbauer, R.: Lifting modules. Frontiers in Mathematics, Birkhaauser Verlag, Basel (2006).

[2] Enochs, E.E., Jenda, O.M.G.: Relative homological algebra. Berlin: Walter de Gruyter (2000).

[3] Enochs, E.E., Jenda, O.M.G., Lopez-Ramos, J.A.: The existence of Gorenstein flat covers. Math. Scand. 94(1), 46-62 (2004).

[4] Eklof, P.C., Trlifaj, J.: How to make Ext vanish, Bull. London Math. Soc. 33(12), 41-51 (2001).

[5] Garcia Rozas, J.R., Torrecillas, B.: Relative injective covers, Comm. Algebra, 22(8), 2925-2940 (1994).

[6] Megibben, C.: Absolutely pure modules. Proc. Amer. Math. Soc. 18, 155-158 (1967).

[7] Moradzadeh-Dehkordi, A., Shojae, S.H.: Rings in which every ideal is pure-projective or FP-projective. J. Algebra, 478, 419-436 (2017).

[8] Ramamurthi, V.S.: On the injectivity and flatness of certain cyclic modules. Proc. Amer. Math. Soc. 48, 21-25 (1975).
[9] Smith, P.F.: *Injective modules and prime ideals*. Comm. Algebra, 9(9), 989-999 (1981).

[10] Wang, M.Y., Zhao, G.: *On maximal injectivity*. Acta Math. Sin. 21(6), 1451-1458 (2005).

[11] Xiang, Y.: *Max-injective, max-flat modules and max-coherent rings*. Bull. Korean Math. Soc. 47(3), 611-622 (2010).

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