BRAIDS, METALLIC RATIOS AND PERIODIC SOLUTIONS OF THE $2n$-BODY PROBLEM

YUIKA KAJIHARA, EIKO KIN, AND MITSURU SHIBAYAMA

Abstract. Periodic solutions of the planar $N$-body problem determine braids through the trajectory of $N$ bodies. Braid types can be used to classify periodic solutions. According to the Nielsen-Thurston classification of surface automorphisms, braids fall into three types: periodic, reducible and pseudo-Anosov. To a braid of pseudo-Anosov type, there is an associated stretch factor greater than 1, and this is a conjugacy invariant of braids. In 2006, the third author discovered a family of multiple choreographic solutions of the planar $2n$-body problem. We prove that braids obtained from the solutions in the family are of pseudo-Anosov type, and their stretch factors are expressed in metallic ratios. New numerical periodic solutions of the planar $2n$-body problem are also provided.

1. Introduction

Consider the motion of $m$ points in the plane $\mathbb{R}^2$

$$\mathbf{x}(t) = (x_1(t), \ldots, x_m(t)),$$

where $x_i(t) \in \mathbb{R}^2$ is the position of the $i$th point at $t \in \mathbb{R}$. Let $Q_m(t) = \{x_1(t), \ldots, x_m(t)\}$. We assume the following.

- $\mathbf{x}(t)$ is collision-free, i.e., for any $t \in \mathbb{R}$, $x_i(t) \neq x_j(t)$ if $i \neq j$.
- There exists $t_0 > 0$ such that

$$Q_m(t + t_0) = Q_m(t).$$

Then we have a (geometric) braid

$$b(\mathbf{x}(t), [0, t_0]) = \bigcup_{t \in [0, t_0]} \{(x_1(t), t), \ldots, (x_m(t), t)\} \subset \mathbb{R}^2 \times [0, t_0]$$

with base points $Q_m(0) (= Q_m(t_0))$. The actual location of base points is irrelevant for the study of braids. To remove the data of the location, we

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consider its braid type \( \langle b(x(t), [0, t_0]) \rangle \) instead of the braid (See Section 3.1 for the definition of braid types).

We investigate periodic solutions of the planar \( N \)-body problem given by the following ODEs.

\[
(1.1) \quad m_i \ddot{x}_i = - \sum_{j \neq i} m_i m_j \frac{x_i - x_j}{|x_i - x_j|^3}, \quad x_i \in \mathbb{R}^2, \quad m_i > 0 \ (i = 1, \ldots, N).
\]

Suppose that \( x(t) = (x_1(t), \ldots, x_N(t)) \) is a periodic solution with period \( T \) of (1.1). The solution \( x(t) \) determines a (pure) braid \( b(x(t), [0, T]) \) and its braid type \( \langle b(x(t), [0, T]) \rangle \). Braid types can be used to classify periodic solutions of the planar \( N \)-body problem.

**Question 1.1** (Montgomery [Mon], (cf. Moore [Moo93])). For any pure braid \( b \) with \( N \) strands, is there a periodic solution of the planar \( N \)-body problem whose braid type is equal to \( \langle b \rangle \)?

Question 1.1 is wide open for every \( N > 3 \). In the case of \( N = 3 \) Question 1.1 is true by work of Moeckel-Montgomery [MM15]. For other studies on braids obtained from periodic solutions, see a pioneer work by Moore [Moo93]. See also [FGA21, MS13, Mon98].

**Remark 1.2.** We consider the following Newton equations

\[
(1.2) \quad m_i \ddot{x}_i = - \sum_{j \neq i} m_i m_j \frac{x_i - x_j}{|x_i - x_j|^\alpha}, \quad x_i \in \mathbb{R}^2, \quad m_i > 0 \ (i = 1, \ldots, N),
\]

where \( \alpha \geq 1 \). The case \( \alpha = 2 \) corresponds to (1.1) describing the motion of \( n \) bodies under the influence of the gravitation. One can ask the same question as Question 1.1 for the planar \( N \)-body problem given by (1.2). It is known by Montgomery [Mon98] that Question 1.1 is true for any “tied” braid type when \( \alpha \geq 3 \) (i.e., under the assumption that the force is strong).

According to the Nielsen-Thurston classification of surface automorphisms [FLP79], braids fall into three types: periodic, reducible and pseudo-Anosov. (See Section 3.3.) To a braid \( b \) of pseudo-Anosov type, there is an associated stretch factor \( \lambda(b) > 1 \), and this is a conjugacy invariant of pseudo-Anosov braids. Since the Nielsen-Thurston type is also a conjugacy invariant, one can define the stretch factor \( \lambda(\langle b \rangle) := \lambda(b) \) for the pseudo-Anosov braid type \( \langle b \rangle \) of \( b \). See (3.1) in Section 3.3.

The stretch factor tells us a dynamical complexity of pseudo-Anosov braids. In this paper we ask the following question related to Question 1.1.

**Question 1.3.** Let \( b \) be a pure braid with \( N \) strands. Suppose that \( b \) is of pseudo-Anosov type. Is there a periodic solution of the planar \( N \)-body problem whose braid type is equal to \( \langle b \rangle \)?
The stretch factor of each pseudo-Anosov braid with 3 strands is a quadratic irrational (Section 3.4). This is not necessarily true for pseudo-Anosov braids with more than 3 strands. Moore [Moo93] and Chenciner-Montgomery [CM00] found a simple choreographic solution to the 3-body problem such that the three bodies chase one another along a figure-8 curve. The braid type of the solution is pseudo-Anosov and its stretch factor is the 6th power of the 1st metallic ratio \( s_1 \) (Example 3.6), i.e., golden ratio, where the \( k \)th metallic ratio \( s_k \) \((k \in \mathbb{N})\) is given by
\[
s_k = \frac{1}{2}(k + \sqrt{k^2 + 4}) = k + \frac{1}{k + \frac{1}{k + \frac{1}{k + \cdots}}}.
\]

The study of braid types of the periodic solutions has been relatively less investigated. We hope that the following result sheds some light on Question 1.3. Let \( \lfloor \cdot \rfloor \) be the floor function.

**Theorem 1.4.** For \( n \geq 2 \) and \( p \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), there exists a periodic solution \( x_{n,p}(t) \) of the planar 2n-body problem with equal masses whose braid type \( \tilde{X}_{n,p} \) is pseudo-Anosov with the stretch factor \((s_{2p})^{2n/d}\), where \( d = \gcd(n, p) \).

A representative of the braid type \( \tilde{X}_{n,p} \) in Theorem 1.4 is the \((\frac{n}{d})\)th power \((\beta_{n,p})^{\frac{n}{d}}\) of the 2n-braid \( \beta_{n,p} \) introduced in Section 4. In 2006, the third author proved the existence of a family of multiple choreographic solutions \( x_{n,p}(t) \) of the planar 2n-body problem with equal masses [Shi06]. Some of the solutions in the family had already found by Chen [Che01] and Ferraio-Terracini [FT04]. The orbit of the periodic solution \( x_{n,p}(t) \) consists of \( 2d \) closed curves, each of which is the trajectory of \( \frac{n}{d} \) bodies. The braid types \( X_{n,p}(t) \) in Theorem 1.4 are realized by \( x_{n,p}(t) \) given in [Shi06]. More precisely, for \( n \geq 2 \) and \( p \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), there exists a periodic solution
\[
x_{n,p}(t) = (x_1(t), \ldots, x_{2n}(t))
\]
with period \( T > 0 \) of the planar 2n-body problem such that
\[
x_i(t + \frac{d}{n}T) = x_{\sigma_{n,p}(i)}(t) \quad \text{for} \quad i = 1, \ldots, 2n,
\]
where \( \sigma_{n,p} := (1, 3, \ldots, 2n - 1)^p(2, 4, \ldots, 2n)^{-p} \in \mathfrak{S}_{2n} \) is a permutation of \( 2n \) elements. Thus, the braid \( y_{n,p} := b(x_{n,p}(t), [0, \frac{d}{n}T]) \) and the braid type \( Y_{n,p} := \langle y_{n,p} \rangle \) are obtained from the solution \( x_{n,p}(t) \), and the \((\frac{n}{d})\)th power \((y_{n,p})^{\frac{n}{d}}\) represents the braid type \( X_{n,p} \). See Figure 11 for periodic solutions \( x_{n,p}(t) \) for \( 0 \leq t \leq \frac{d}{n}T \). Theorem 1.4 follows from the following (see Remark 3.4).

**Theorem 1.5.** For \( n \geq 2 \) and \( p \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), the braid type \( Y_{n,p} \) is pseudo-Anosov with the stretch factor \((s_{2p})^{2n/d}\). In particular, the braid type \( X_{n,p} \) of the solution \( x_{n,p}(t) \) is pseudo-Anosov with the stretch factor \((s_{2p})^{2n/d}\).
Corollary 1.6. Let $X_{n,p}$ be the braid type as in Theorem 1.5. For $n \geq 2$ and $p, p' \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \}$ with $p < p'$, we have the following.

1. $\lambda(X_{n,p}) < \lambda(X_{n,p'})$ if $\gcd(n, p) = \gcd(n, p')$. In particular $X_{n,p} \neq X_{n,p'}$.
2. $\lambda(X_{n,p}) < \lambda(X_{n,p'})$ if $n$ is prime. In particular $X_{n,p} \neq X_{n,p'}$.

The $2n$ bodies for the solution $x_{n,p}(t)$ form a regular $2n$-gon at the initial time $t = 0$, and the next first time is $t = \left(\frac{d}{2n}\right)T$ when the $2n$ bodies form a regular $2n$-gon again. See Figure 1 for $x_{n,p}(t)$ at $t = \left(\frac{d}{2n}\right)T$. From the viewpoint of the configuration of the “next” regular $2n$-gon, it is proved in [Shi06] that $x_{n,p}(t)$ and $x_{n,p'}(t)$ are distinct solutions for distinct $p, p' \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \}$ (Remark 2.1). On the other hand, from the viewpoint of braid types, Corollary 1.6 tells us that $X_{n,p}$ is different from $X_{n,p'}$ if $\gcd(n, p) = \gcd(n, p')$.

Table 1 shows the stretch factor $\lambda_{n,p} = \lambda(X_{n,p})$ and the entropy $\log(\lambda_{n,p})$ for several pairs $(n, p)$. One can see from this table that $\lambda(X_{n,p}) \neq \lambda(X_{n,p'})$ if $p \neq p'$ up to $n = 11$. Therefore the braid types $X_{n,p}$ and $X_{n,p'}$ of the solutions for $p \neq p' \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \}$ are distinct up to $n = 11$. 
Because of an intriguing formula of metallic ratios, \( s_k^3 = s_{k^3+3k} \) for example, stretch factors \( \lambda(X_{n,p}) \) happen to coincide with the ones for different pairs \((n,p')\) occasionally (see Example 4.1). Nevertheless, we conjecture that \( X_{n,p} \neq X_{n,p'} \) for all \( p, p' \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \) with \( p \neq p' \).

The organization of the paper is as follows. In Section 2 we introduce a family of periodic solutions \( x_{n,p}(t) \) in [Shi06] of the planar 2n-body problem. In Section 3 we briefly review the necessarily background on braid groups. We prove Theorem 1.5 in Section 4. In Section 5 we give new numerical periodic solutions \( x_{n,p}(t) \) of the planar 2n-body problem when \( p > \lfloor \frac{n}{2} \rfloor \).

2. Periodic solutions of the planar 2n-body problem

This section is devoted to explain the periodic solutions \( x_{n,p}(t) \). The existence was proven with the variational method. They have high symmetries because they can be represented as elements of a functional space limited by several group actions. The minimizers of the action functional under the symmetry correspond to those solutions. They are also regarded as orbits on the shape sphere. They are constructed through minimizing methods, and
Table 1. Some examples obtained from Theorem 1.4

| $(n, p)$ | $d$ | $X_{n, p} = ((\beta_{n, p})^{\pi})$ | $\lambda_{n, p} = (\bar{s}_{2p})^{\bar{\pi}}$ | $\log(\lambda_{n, p}) = (\frac{2n}{d}) \log(\bar{s}_{2p})$ |
|---------|-----|------------------------------------|--------------------------------------|------------------------------------------------|
| (2, 1)  | 1   | $(\beta_{2, 1})^2$                 | $(\bar{s}_{2})^{4}$                  | 3.525494348078172                                 |
| (3, 1)  | 1   | $(\beta_{3, 1})^4$                 | $(\bar{s}_{2})^{6}$                  | 5.288241522117257                                  |
| (4, 1)  | 1   | $(\beta_{4, 1})^{4}$               | $(\bar{s}_{2})^{8}$                  | 7.05098696156343                                   |
| (4, 2)  | 2   | $(\beta_{4, 2})^{2}$               | $(\bar{s}_{4})^{4}$                  | 5.774541900715241                                  |
| (5, 1)  | 1   | $(\beta_{5, 1})^{10}$              | $(\bar{s}_{2})^{10}$                 | 8.813735870195430                                  |
| (5, 2)  | 1   | $(\beta_{5, 2})^{10}$              | $(\bar{s}_{4})^{10}$                 | 14.436354757188103                                 |
| (6, 1)  | 1   | $(\beta_{6, 1})^{16}$              | $(\bar{s}_{2})^{16}$                 | 10.576483044234514                                 |
| (6, 2)  | 2   | $(\beta_{6, 2})^{4}$               | $(\bar{s}_{4})^{8}$                  | 8.661812851072861                                  |
| (6, 3)  | 3   | $(\beta_{6, 3})^{2}$               | $(\bar{s}_{6})^{4}$                  | 7.27378536928267                                   |
| (7, 1)  | 1   | $(\beta_{7, 1})^{14}$              | $(\bar{s}_{2})^{14}$                 | 12.339230218273601                                 |
| (7, 2)  | 1   | $(\beta_{7, 2})^{14}$              | $(\bar{s}_{4})^{14}$                 | 20.210896652503344                                 |
| (7, 3)  | 1   | $(\beta_{7, 3})^{14}$              | $(\bar{s}_{6})^{14}$                 | 25.458250429248935                                 |
| (8, 1)  | 1   | $(\beta_{8, 1})^{8}$               | $(\bar{s}_{2})^{16}$                 | 14.101977392312687                                 |
| (8, 2)  | 2   | $(\beta_{8, 2})^{4}$               | $(\bar{s}_{4})^{8}$                  | 11.549083801430482                                 |
| (8, 3)  | 1   | $(\beta_{8, 3})^{8}$               | $(\bar{s}_{6})^{16}$                 | 29.095143347713069                                 |
| (8, 4)  | 4   | $(\beta_{8, 4})^{2}$               | $(\bar{s}_{4})^{4}$                  | 8.378850189044405                                  |
| (9, 1)  | 1   | $(\beta_{9, 1})^{9}$               | $(\bar{s}_{2})^{18}$                 | 15.864724566351773                                 |
| (9, 2)  | 1   | $(\beta_{9, 2})^{9}$               | $(\bar{s}_{4})^{18}$                 | 25.985438553218586                                 |
| (9, 3)  | 3   | $(\beta_{9, 3})^{9}$               | $(\bar{s}_{6})^{9}$                  | 10.91678755392400                                 |
| (9, 4)  | 1   | $(\beta_{9, 4})^{9}$               | $(\bar{s}_{8})^{18}$                 | 37.704825850699920                                 |
| (10, 1) | 1    | $(\beta_{10, 1})^{16}$            | $(\bar{s}_{2})^{20}$                 | 17.627471740390860                                 |
| (10, 2) | 2    | $(\beta_{10, 2})^{2}$             | $(\bar{s}_{4})^{10}$                 | 14.436354751788103                                 |
| (10, 3) | 1    | $(\beta_{10, 3})^{16}$            | $(\bar{s}_{6})^{120}$                | 36.368929184641338                                 |
| (10, 4) | 2    | $(\beta_{10, 4})^{4}$             | $(\bar{s}_{8})^{10}$                 | 20.947125472611013                                 |
| (10, 5) | 5    | $(\beta_{10, 5})^{2}$             | $(\bar{s}_{10})^{4}$                 | 9.249753365091010                                  |
| (11, 1) | 1    | $(\beta_{11, 1})^{11}$            | $(\bar{s}_{2})^{22}$                 | 19.390218914299944                                 |
| (11, 2) | 1    | $(\beta_{11, 2})^{11}$            | $(\bar{s}_{4})^{22}$                 | 31.7598845393328                                   |
| (11, 3) | 1    | $(\beta_{11, 3})^{11}$            | $(\bar{s}_{6})^{22}$                 | 40.005822103105473                                 |
| (11, 4) | 1    | $(\beta_{11, 4})^{11}$            | $(\bar{s}_{8})^{22}$                 | 46.083676039744226                                 |
| (11, 5) | 1    | $(\beta_{11, 5})^{11}$            | $(\bar{s}_{10})^{22}$                | 50.873643508000555                                 |

we omit analytic techniques for the proof and describe geometric properties of $x_{n, p}(t)$ including the group actions and shape sphere.

2.1. Symmetry. Let $G$ be a finite group. We consider a 2-dimensional orthogonal representation $\rho : G \to O(2)$, a homomorphism $\sigma : G \to S_{2n}$, to the symmetric group on $2n$ elements, and another 2-dimensional orthogonal representation $\tau : G \to O(2)$. We will denote by $\Lambda$, the set of $T$-periodic
orbits. The action of $G$ to $\Lambda$ is defined by
\[
g \cdot ((x_1, \ldots, x_{2n})(t)) = (\rho(g)x_{\sigma(g^{-1})}(1), \ldots, \rho(g)x_{\sigma(g^{-1})}(2n))(\tau(g^{-1})(t))
\]
for $g \in G$ and $x(t) = (x_1, \ldots, x_{2n})(t) \in \Lambda$, where the above $\rho, \sigma, \tau$ represent respectively actions of $G$ on $\mathbb{R}^2$ by orthogonal transformations, on indices \( \{1, 2, \ldots, 2n\} \) by permutations, and on the circle $\mathbb{R}/T\mathbb{Z}$. Specifically, we take $G$ as the group $G_{n,p} := \langle g_n, h_{n,p} \rangle$ generated by the two elements $g_n$ and $h_{n,p}$, where
\[
\rho(g_n) = \begin{pmatrix} \cos(\frac{\pi}{n}) & -\sin(\frac{\pi}{n}) \\ \sin(\frac{\pi}{n}) & \cos(\frac{\pi}{n}) \end{pmatrix},
\sigma(g_n) = (1, 2, \ldots, 2n),
\tau(g_n) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
and
\[
\rho(h_{n,p}) = 1,
\sigma(h_{n,p}) = (1, 3, \ldots, 2n - 1)^{-p}(2, 4, \ldots, 2n)^p,
\tau(h_{n,p}) = \begin{pmatrix} \cos(\frac{2\pi d}{n}) & -\sin(\frac{2\pi d}{n}) \\ \sin(\frac{2\pi d}{n}) & \cos(\frac{2\pi d}{n}) \end{pmatrix}
\]
\(d := \text{gcd}(n, p)\).

Let us denote by $\Lambda^{G}_{n,p}$, the invariant set under the action of $G_{n,p}$ in $\Lambda$, i.e.,
\[
\Lambda^{G}_{n,p} = \{ x(t) \in \Lambda \mid x_i(t) = \rho(g)x_{\sigma(g^{-1})}(i)(\tau(g^{-1})(t)) \}
\]
\(i = 1, 2, \ldots, 2n, g \in G_{n,p}, t \in \mathbb{R}\} \).

We now check the properties of $\Lambda^{G}_{n,p}$. First, from the invariance under $g_n$, we have
\[
g_n \cdot ((x_1, x_2, \ldots, x_{2n})(t)) = (e^{\frac{2\pi}{n}}x_2, e^{\frac{2\pi}{n}}x_1, \ldots, e^{\frac{2\pi}{n}}x_{2n-1})(-t).
\]
Here we identify $\mathbb{R}^2$ with $\mathbb{C}$. In particular, $x_1(t), x_2(-t), \ldots, x_{2n-1}(t), x_{2n}(-t)$ forms a regular $2n$-gon, and $n$ bodies with odd indices and $n$ bodies with even indices rotate in mutually opposite directions.

Second, since
\[
\rho(g_n^2) = \begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix},
\sigma(g_n^2) = (1, 3, \ldots, 2n - 1)(2, 4, \ldots, 2n)
\]
\(\text{and}
\[
\tau(g_n^2) = 1,
\]
the configuration always consists of two regular $n$-gons, which are formed by $n$ bodies $x_1(t), x_3(t), \ldots, x_{2n-1}(t)$ of odd indices and $n$ bodies $x_2(t), x_4(t), \ldots, x_{2n}(t)$ of even indices. Thus, to determine the positions of $2n$ bodies $x_1, \ldots, x_{2n}$, it is sufficient to know the positions of two bodies $x_1$ and $x_2$. In fact for each $k \in \{1, \ldots, n\}$ and $t \in \mathbb{R}$,
\[
x_{2k-1}(t) = \omega^{(k-1)}x_1(t), \ x_{2k}(t) = \omega^{(k-1)}x_2(t),
\]
where $\omega = e^{2\pi i/n}$. This enables us to use the shape sphere (introduced in Section 2.2) which represents configurations of $2n$ bodies in the periodic solutions.

Lastly, the invariance under $h_{n,p}$ tells us that

$$h_{n,p} \cdot ((x_1, x_2, \ldots, x_{2n})(t)) = (x_{\sigma(h_{n,p}^{-1})(1)}, x_{\sigma(h_{n,p}^{-1})(2)}, \ldots, x_{\sigma(h_{n,p}^{-1})(2n)})(t - \frac{dT}{n}),$$

and hence

$$x_i \left( t + \frac{dT}{n} \right) = x_{\sigma(h_{n,p}^{-1})(i)}(t) \quad (i = 1, 2, \ldots, 2n),$$

where

$$(2.1) \quad \sigma(h_{n,p}^{-1}) = (1, 3, \ldots, 2n - 1)^p(2, 4, \ldots, 2n)^{-p} \in \mathcal{S}_{2n}.$$ 

This implies that $x_{n,p}(t)$ consists of $2d$ closed curves and $\frac{2}{d}$ bodies chase one another along each closed curve. See Figure 1.

2.2. The shape sphere. We consider the group action on the circle $S^1$ to $\mathbb{C}^2$ by

$$z \cdot (x_1, x_2) = (zx_1, zx_2), \quad z \in S^1, (x_1, x_2) \in \mathbb{C}^2.$$ 

The quotient space $(\mathbb{C}^2 - \{0\})/S^1$ under the above action is realized by the following projection:

$$\pi: \mathbb{C}^2 - \{0\} \longrightarrow \mathbb{R}^3 - \{0\} \cong (\mathbb{C}^2 - \{0\})/S^1$$

$$(x_1, x_2) \longmapsto u(t) = (u_1, u_2, u_3)$$

where

$$(u_1, u_2, u_3) = (|x_1|^2 - |x_2|^2, 2\text{Re}(x_1\bar{x}_2), 2\text{Im}(x_1\bar{x}_2))$$

$$= (r_1^2 - r_2^2, 2r_1r_2\cos(\theta_1 - \theta_2), 2r_1r_2\sin(\theta_1 - \theta_2)).$$

Here $x_1 = r_1e^{i\theta_1}$ and $x_2 = r_2e^{i\theta_2}$. Set the rays

$$A_\pm = \{(\pm s, 0, 0) \mid s \in \mathbb{R}_{>0}\},$$

$$B_{2k} = \left\{ \left(0, s\cos\left(\frac{2\pi k}{n}\right), s\sin\left(\frac{2\pi k}{n}\right) \mid s \in \mathbb{R}_{>0}\right) \right\} \quad (k \in \mathbb{Z}) \text{ and}$$

$$B_{2k-1} = \left\{ \left(0, s\cos\left(\frac{2k-1}{n}\pi\right), s\sin\left(\frac{2k-1}{n}\pi\right) \mid s \in \mathbb{R}_{>0}\right) \right\} \quad (k \in \mathbb{Z}).$$

In the quotient space $(\mathbb{C}^2 - \{0\})/S^1$, the sets $A_\pm$ and $B_{2k}$ correspond to collisions of the original $2n$ bodies. If $u(t) \in A_+$ (resp. $A_-$), then all bodies with odd (resp. even) indeces collide at $t \in \mathbb{R}$ and if $u(t) \in B_{2k}$, then two regular $n$-gons fit. See Figure 1 for the configurations of 8 bodies corresponding to $B_0, B_2, B_4$ and $B_6$.

Let $u(t) = u_{n,p}(t)$ be a curve corresponding to the solution $x_{n,p}(t)$. As a result, $u(t)$ passes through neither $A_\pm$ nor $B_{2k}$, because $x_{n,p}(t)$ has no collision (Shi06, Proposition 3). On the other hand, each $B_{2k-1}$ represents a configuration where $2n$ bodies form a regular $2n$-gon. See Figure 1 for the configurations of 8 bodies corresponding to $B_{-1}, B_1, B_3$ and $B_5$. 
Set

\[
M(k) = \begin{pmatrix}
-1 & 0 & 0 \\
0 & \cos(\frac{2\pi k}{n}) & \sin(\frac{2\pi k}{n}) \\
0 & \sin(\frac{2\pi k}{n}) & -\cos(\frac{2\pi k}{n})
\end{pmatrix}.
\]

It is easy to see that \( M(k) \) is an orthogonal matrix with eigenvalues \( \lambda = 1, -1 \). The eigenvector for \( \lambda = 1 \) is \( B_k \), and hence \( M(k) \) represents \( \pi \)-rotation with respect to \( B_k \). The invariance under \( g_n \) is associated with

\[
\begin{pmatrix}
u_1(-t) \\
u_2(-t) \\
u_3(-t)
\end{pmatrix} = M(-1) \begin{pmatrix}
u_1(t) \\
u_2(t) \\
u_3(t)
\end{pmatrix}
\]

and it implies that \( u(t) \) and \( u(-t) \) are symmetric with respect to \( B_{-1} \). In other words, rotating this curve \( \pi \) with respect to \( B_{-1} \), \( u(t) \) coincides with \( u(-t) \). In particular \( u(0) \in B_{-1} \). Similarly, the invariance under \( h_{n,p} \) is associated with

\[
\begin{pmatrix}
u_1(t + 2\bar{T}) \\
u_2(t + 2\bar{T}) \\
u_3(t + 2\bar{T})
\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 \\
0 & \cos(\frac{4\pi p}{n}) & -\sin(\frac{4\pi p}{n}) \\
0 & \sin(\frac{4\pi p}{n}) & \cos(\frac{4\pi p}{n})
\end{pmatrix} \begin{pmatrix}
u_1(t) \\
u_2(t) \\
u_3(t)
\end{pmatrix},
\]

where \( \bar{T} = \frac{d\bar{T}}{2n} \). Substituting \(- (t + \bar{T}) \) into \( t \) and applying the invariance under \( g_n \), we obtain

\[
\begin{pmatrix}
u_1(-t + \bar{T}) \\
u_2(-t + \bar{T}) \\
u_3(-t + \bar{T})
\end{pmatrix} = M(2p - 1) \begin{pmatrix}
u_1(t + \bar{T}) \\
u_2(t + \bar{T}) \\
u_3(t + \bar{T})
\end{pmatrix}.
\]

Taking \( t = 0 \) gives

\[
\begin{pmatrix}
u_1(\bar{T}) \\
u_2(\bar{T}) \\
u_3(\bar{T})
\end{pmatrix} = M(2p - 1) \begin{pmatrix}
u_1(\bar{T}) \\
u_2(\bar{T}) \\
u_3(\bar{T})
\end{pmatrix},
\]

and hence \( u(t + \bar{T}) \) and \( u(-t + \bar{T}) \) are symmetric with respect to \( B_{2p-1} \) and \( u(\bar{T}) \in B_{2p-1} \). It means that the original configuration of \( 2n \) bodies forms a regular \( 2n \)-gon again at \( t = \bar{T} \). Other symmetries with respect to \( B_{2jp-1} \) for \( j = 2, 3, \ldots \) can be seen in the same manner.

**Remark 2.1.** It is proved in [Shi06, Proposition 5] that \( u(t) \notin B_{2k-1} \) for all \( t \in (0, \bar{T}) \) and \( k \in \mathbb{Z} \). It implies that \( x_{n,p}(t) \) and \( x_{n,p'}(t) \) are distinct smooth solution for \( p \neq p' \) in the sense that \( u(\bar{T}) \) belongs to \( B_{2p-1} \), that is in the sense that the configuration of the first regular \( 2n \)-gon lives in the distinct \( B_{2p-1} \) for each \( p \).

Consider the projection from \( \mathbb{R}^3 - \{0\} \) to the 2-sphere \( \mathbb{S}^2 \). The projective space is called the shape sphere. The image of \( u(t) \in \mathbb{R}^3 - \{0\} \) under the projection is also denoted by the same notation \( u(t) \), and we call a family \( \{u(t)\}_t \) (on the shape sphere) the shape curve. Determining the shape curve \( u(t) \) for \( t \in (0, \bar{T}) \), we obtain the shape curve \( u(t) \) for all \( t \in \mathbb{R} \) from the above symmetries. For example, we show the shape curves \( u(t) \) for \( t \in \mathbb{R} \).
when \((n, p) = (3, 1)\) and \((n, p) = (4, 2)\) in Figure 2. Each point \(B_i\) in Figure 2 indicates the projection of the ray \(B_i\) onto the shape sphere. The solid arrows (resp. the dotted arrows) illustrate the shape curve \(u(t)\) in the front side on the shape sphere, i.e., \(u_1(t) = |x_1(t)|^2 - |x_2(t)|^2 > 0\), (resp. the back side, i.e., \(u_1(t) < 0\)). The dotted arrow of label 2 follows from symmetry of the solid arrow of label 1 with respect to \(B_1\). The remaining cases are treated in the same fashion.

\[
\begin{align*}
(1) \quad (n, p) &= (3, 1) \\
(2) \quad (n, p) &= (4, 2)
\end{align*}
\]

**Figure 2.** The shape curve \(u(t)\) for \(t \in \mathbb{R}\) when (1) \((n, p) = (3, 1)\) and (2) \((n, p) = (4, 2)\).

The point \(B_i\) in the figure is the projection of the ray \(B_i\) onto the shape sphere.

**Remark 2.2.** Though the set \(\mathcal{L}_{n,p}^C\) does not determine how the shape curve \(u(t)\) moves on the shape sphere for \(t \in (0, \bar{T})\), we can prove through variational arguments that it does not happen like Figure 3(1) or (2). See [Shi06, Propositions 5 and 6] for the proof.

Figure 3 illustrates the projection of the shape curve \(u(t)\) onto the \(u_2u_3\)-plane together with the configuration of 8 bodies corresponding to each \(B_i\) when \((n, p) = (4, 1)\) and \((4, 2)\).

3. **Braid groups and mapping class groups**

3.1. **Geometric braids.**

In this section, we recall definitions of (geometric) braids and the braid types. For the basics on braid groups, see Birman [Bir74]. Let \(D\) be a closed disk in the plane \(\mathbb{R}^2\) and \(Q_m = \{q_1, \ldots, q_m\}\) be a set of \(m\) points in the interior of \(D\). Let \(\gamma_1, \ldots, \gamma_m\) be mutually disjoint \(m\) arcs in \(D \times [0, 1]\) with the following properties.

- \(\partial(\gamma_1 \cup \cdots \cup \gamma_m) = \{(q_1, t), \ldots, (q_m, t) \mid t \in \{0, 1\}\} \subset D \times \{0, 1\}^^m\),
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Error type 1: Across the $u_2u_3$-plane

Error type 2: Non-monotonicity

Figure 3. Error types of the shape curve $u(t)$ ($0 \leq t \leq \bar{T}$) when $(n, p) = (4, 2)$. (1) Error type 1: $u(t)$ ($0 \leq t \leq \bar{T}$) is across the $u_2u_3$-plane. (2) Error type 2: $u(t)$ is not monotone. For the shape curve $u(t)$ ($0 \leq t \leq \bar{T}$), see the solid arrow with label 1 in Figure 2.

- $\gamma_i$ ($i = 1, \ldots, m$) starts at $(q_i, 0) = \gamma_i \cap (D \times \{0\})$ and it goes up monotonically with respect to the $[0, 1]$-factor. In particular $\gamma_i \cap (D \times \{t\})$ consists of a single point for $0 \leq t \leq 1$.

We call $b = \gamma_1 \cup \cdots \cup \gamma_m \subset D \times [0, 1]$ a (geometric) braid with base points $Q_m$ and call each $\gamma_i$ a strand of the braid $b$. We say that braids $b$ and $b'$ with base points $Q_m$ are equivalent if there is a 1-parameter family of braids with base points $Q_m$ deforming $b$ to $b'$. By abuse of notations, the equivalence class $[b]$ is also denote by $b$.

For braids $b$ and $b'$ with base points $Q_m$, the product $bb'$ is defined as follows. We first stuck $b$ on $b'$ and concatenate them to get disjoint arcs properly embedded in $D \times [0, 2]$. By normalizing its height, we obtain a braid (in $D \times [0, 1]$) with the same base points $Q_m$ and this is the braid $bb'$. The set of all braids with base points $Q_m$ with this product gives a group structure. The group is called the (geometric) braid group with base points $Q_m$ and it is denoted by $B(Q_m)$. Note that the identity element $1_{Q_m} \in B(Q_m)$ is given by a braid consisting of straight arcs.

Let $A_m = \{a_1, \ldots, a_m\}$ be a set of $m$ points in the interior of $D$ such that $a_1, \ldots, a_m$ lie on a segment in this order. We write $B_m = B(A_m)$ and call $B_m$ the $m$-braid group. The isomorphism class of the above braid group $B(Q_m)$ with base points $Q_m$ does not depend on the location of base points, and $B(Q_m)$ is isomorphic to $B_m$. To define braid types of geometric braids with arbitrary base points $Q_m$, we now take an isomorphism between $B(Q_m)$ and $B_m$. We first choose an orientation preserving homeomorphism $f : D \to D$ such that $f(A_m) = Q_m$. Then take an isotopy $\{f_t\}_{0 \leq t \leq 1}$ on $D$ between the identity map $id_D$ and $f$, i.e., $f_0 = id_D$ and $f_1 = f$. We consider two kinds of mutually disjoint $m$ arcs $\gamma^+$ and $\gamma^-$ properly embedded in $D \times [0, 1]$ as
Figure 4. The projection of the shape curve $u(t)$ for $t \in \mathbb{R}$ onto the $u_2u_3$-plane when (1) $(n, p) = (4, 1)$ and (2) $(n, p) = (4, 2)$. The configuration of 8 bodies corresponding to $B_i$ is illustrated.
follows.

\[ \gamma^+ = \bigcup_{t \in [0,1]} \{(f_t(a_1), t), \ldots, (f_t(a_m), t)\}, \]

\[ \gamma^- = \bigcup_{t \in [0,1]} \{(f_{1-t}(a_1), t), \ldots, (f_{1-t}(a_m), t)\}. \]

Note that \( Q_m = f_1(A_m) = \{f(a_1), \ldots, f(a_m)\} \) and \( A_m = f_0(A_m) = \{a_1, \ldots, a_m\} \).

Because of this, it makes sense to stack a braid \( b \in B(Q_m) \) on \( \gamma^+ \), and we obtain the resulting disjoint \( m \) arcs \( b \cdot \gamma^+ \subset D \times [0,2] \). Then we stack \( \gamma^- \)
on \( b \cdot \gamma^+ \). As a result we have disjoint \( m \) arcs

\[ \gamma^- \cdot b \cdot \gamma^+ \subset D \times [0,3]. \]

By normalizing the height of the arcs, we obtain a braid (in \( D \times [0,1] \)) with base points \( A_m \), and we still denote it by the same notation \( \gamma^- \cdot b \cdot \gamma^+ \).

In particular if \( b = 1_{Q_m} \in B(Q_m) \), then \( \gamma^- 1_{Q_m} \gamma^+ = 1_{A_m} \in B_m \). The correspondence \( b \mapsto \gamma^- \cdot b \cdot \gamma^+ \) gives us an isomorphism between \( B(Q_m) \) and \( B_m \).

For an element \( b \in B_m \), we put indices \( 1, \ldots, m \) at the bottoms of strands so that the index \( i \) indicates \((a_i, 0) \in D \times \{0\}\). Let \( \sigma_i \) be an element of \( B_m \) as in Figure 5(1). The braid group \( B_m \) is generated by \( \sigma_1, \sigma_2, \ldots, \sigma_{m-1} \), and it has the following braid relations.

(B1) \( \sigma_i \sigma_j = \sigma_j \sigma_i \ (|i - j| \geq 2) \).

(B2) \( \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \ (1 \leq i \leq m - 2) \).

See Figure 5(2)–(6) for some braids. There is a surjective homomorphism

\[ \hat{\sigma} : B_m \rightarrow \mathfrak{S}_m \]

from \( B_m \) to the symmetry group \( \mathfrak{S}_m \) of \( m \) elements sending each \( \sigma_j \) to the transposition \((j, j + 1)\). The kernel of \( \hat{\sigma} \) is called the pure braid group (or colored braid group) \( P_m < B_m \). An element of \( P_m \) is called a pure braid. See Figure 5(3)(4)(5) for some pure braids.
Let $Z(B_m)$ be the center of $B_m$ which is an infinite cyclic group generated by a full twist $\Delta^2$, where a half twist $\Delta = \Delta_m \in B_m$ is given by

$$\Delta_m = (\sigma_1 \sigma_2 \ldots \sigma_{m-1})(\sigma_1 \sigma_2 \ldots \sigma_{m-2}) \cdots (\sigma_1 \sigma_2) \sigma_1.$$ 

See Figure 5(6) for a half twist $\Delta \in B_4$.

Given a braid $b \in B_m$, consider the projection $\overline{b}$ in the quotient group

$$B_{2n} = B_{2n}/Z(B_{2n}).$$

The braid type $\langle b \rangle$ of $b$ is a conjugacy class of $\overline{b}$ in $B_{2n}$.

In the case of the braid group $B(Q_m)$ with base points $Q_m$, the braid type $\langle b \rangle$ of $b \in B(Q_m)$ is defined by the braid type $\langle \gamma^- \cdot b \cdot \gamma^+ \rangle$ of the braid $\gamma^- \cdot b \cdot \gamma^+ \in B_m$ (with base points $A_m$), where $\gamma^+$ and $\gamma^-$ are arcs as above. The braid type $\langle b \rangle$ is well-defined, i.e., it does not depend on the above orientation preserving homeomorphism $f : D \to D$ and the isotopy $\{f_t\}_{0 \leq t \leq 1}$.

**Example 3.1.**

1. For the 3-braid $\sigma_1 \sigma_2^{-1}$, it follows that

$$\hat{\sigma}(\sigma_1 \sigma_2^{-1}) = \hat{\sigma}(\sigma_1) \hat{\sigma}(\sigma_2^{-1}) = (12)(23) = (123) \in \mathfrak{S}_3,$$

see Figure 4(2). Hence $\hat{\sigma}((\sigma_1 \sigma_2^{-1})^3) = 1 \in \mathfrak{S}_3$ which means that $(\sigma_1 \sigma_2^{-1})^3 \in P_3$.

2. For the 3-braid $\sigma_1^2 \sigma_2^{-2}$, it follows that

$$\hat{\sigma}(\sigma_1^2 \sigma_2^{-2}) = \hat{\sigma}(\sigma_1^2) \hat{\sigma}(\sigma_2^{-2}) = 1 \cdot 1 = 1 \in \mathfrak{S}_3,$$

see Figure 3(4). Hence $\sigma_1^2 \sigma_2^{-2} \in P_3$.

**Example 3.2.** For the Euler’s periodic solution of the planar 3-body problem, three bodies are collinear at every instant. A full twist $\Delta^2 = (\sigma_1 \sigma_2 \sigma_1)^2 = (\sigma_1 \sigma_2)^3 \in B_3$ (Figure 3(5)) represents the braid type of the solution. Since $Z(B_3)$ is generated by $\Delta^2$, the braid type of the Euler’s periodic solution is trivial. Similarly, it is the trivial braid type for the Lagrange’s periodic solution of the planar 3-body problem, since the triangle formed by the three bodies is equilateral for all time.

### 3.2. Mapping class groups.

Let $X_1, \ldots, X_n$ be possibly empty subspaces of an orientable manifold $M$. For instance $M$ is a connected orientable surface $\Sigma_{g,m}$ of genus $g \geq 0$ with $m$ punctures (possibly $m = 0$) and $X_i$ ($i = 1, \ldots, n$) is a finite set in $\Sigma_{g,m}$. Let $\text{Homeo}_+(M, X_1, \ldots, X_n)$ be the group of orientation-preserving self-homeomorphisms of $M$ that map $X_i$ onto $X_i$ for each $i = 1, \ldots, n$. We do not require that homeomorphisms fix the boundary $\partial M$ pointwise. The mapping class group $\text{MCG}(M, X_1, \ldots, X_n)$ is defined by

$$\text{MCG}(M, X_1, \ldots, X_n) = \pi_0(\text{Homeo}_+(M, X_1, \ldots, X_n)),$$

that is the group of isotopy classes of elements of $\text{Homeo}_+(M, X_1, \ldots, X_n)$. When $X$ is an empty subspace of $M$, then we write $\text{MCG}(M) = \text{MCG}(M, X)$. 
We apply elements of mapping class groups from right to left, i.e., we apply \( g \) first for the product \( fg \).

Let \( D_m = D \setminus A_m \) be an \( m \)-punctured disk, where \( A_m = \{a_1, \ldots, a_m\} \) is the set of \( m \) points in the interior of \( D \) as in Section 3.1. By definition, \( \text{MCG}(D_m) \) is the group of isotopy classes of elements of \( \text{Homeo}_+(D_m) \) which fix \( \partial D \) setwise. In this paper, we mainly consider an \( m \)-punctured disk \( D_m \) or an \( m \)-punctured sphere \( \Sigma_{0,m} \) as an orientable manifold \( M \) for the mapping class groups. We take a point in \( \Sigma_{0,m} \) and call it \( \infty \). An element \( f \in \text{Homeo}_+(\Sigma_{0,m}, \{\infty\}) \) means that \( f \) fixes the point \( \infty \). Puncturing the point \( \infty \), we think of \( \text{MCG}(\Sigma_{0,m}, \{\infty\}) \) as a subgroup of \( \text{MCG}(\Sigma_{0,m+1}) \). Also we may regard \( \text{MCG}(\Sigma_{0,m}, \{\infty\}) \) as the mapping class group of an \( m \)-punctured plane. See Figure 6(1)(2).

The mapping class group \( \text{MCG}(D_m) \) is generated by \( h_1, \ldots, h_{m-1} \), where \( h_i \) is the right-handed half twist about a segment \( s_i \) connecting the \( i \)th and \((i + 1)\)th punctures, see Figure 6(3). More precisely, let \( D_i \subset \text{int}(D) \) be a closed disk such that \( D_i \) contains the two points \( a_i \) and \( a_{i+1} \) together with a segment \( s_i \) between the punctures \( a_i \) and \( a_{i+1} \). Moreover \( D_i \) contains no other points of \( A_m \). Then the right-handed half-twist \( h_i \in \text{MCG}(D_m) \) is a mapping class that fixes the exterior of \( D_i \) and rotates \( s_i \) in \( D_i \) by \( \pi \) in the counter-clockwise direction. Hence \( h_i \) interchanges the \( i \)th puncture with the \((i + 1)\)th puncture.

We now recall a relation between \( B_m \) and \( \text{MCG}(D_m) \). There is a surjective homomorphism

\[
\Gamma : B_m \rightarrow \text{MCG}(D_m)
\]

which sends \( \sigma_i \) to \( h_i \) for \( i = 1, \ldots, m-1 \). The kernel of \( \Gamma \) is the center \( Z(B_m) \), and hence \( B_m = B_m/Z(B_m) \) is isomorphic to \( \text{MCG}(D_m) \). Collapsing \( \partial D \) to the point \( \infty \) in the sphere, we have a homomorphism

\[
c : \text{MCG}(D_n) \rightarrow \text{MCG}(\Sigma_{0,m}, \{\infty\}).
\]

By abuse of notations, we simply denote by \( b \), the mapping class \( c(\Gamma(b)) \in \text{MCG}(\Sigma_{0,m}, \{\infty\}) \). Also we denote by \( \langle b \rangle \), the conjugacy class \( \langle c(\Gamma(b)) \rangle \) of \( c(\Gamma(b)) \in \text{MCG}(\Sigma_{0,m}, \{\infty\}) \). Note that this notation \( \langle b \rangle \) is the same as the braid type of \( b \in B_m \).
3.3. Nielsen-Thurston classification. According to the Nielsen-Thurston classification [Thu88], mapping classes fall into three types: periodic, reducible and pseudo-Anosov. Assume that $3g - 3 + m \geq 1$. A mapping class $\phi \in \text{MCG}(\Sigma_{g,m})$ is periodic if $\phi$ is of finite order. A mapping class $\phi \in \text{MCG}(\Sigma_{g,m})$ is reducible if there is a collection of mutually disjoint and non-homotopic essential simple closed curves $C_1, \ldots, C_j$ in $\Sigma_{g,m}$ for $j \geq 1$ such that $C_1 \cup \cdots \cup C_j$ is preserved by $\phi$. Here a simple closed curve $C$ in $\Sigma_{g,m}$ is essential if each component of $\Sigma_{g,m} \setminus C$ has negative Euler characteristic. (There is a mapping class that is periodic and reducible.) A mapping class $\phi \in \text{MCG}(\Sigma_{g,m})$ is pseudo-Anosov if $\phi$ is neither periodic nor reducible. Note that the Nielsen-Thurston type is a conjugacy invariant, i.e., two mapping classes are conjugate to each other in $\text{MCG}(\Sigma_{g,m})$, then their Nielsen-Thurston types are the same.

Pseudo-Anosov mapping classes have many important properties for the study of dynamical systems. For more details which we describe below, see [FLP79, FM12]. A homeomorphism $\Phi : \Sigma_{g,m} \to \Sigma_{g,m}$ is pseudo-Anosov if there exist a constant $\lambda = \lambda(\Phi) > 1$ and a pair of transverse measured foliations $(F^+, \mu^+)$ and $(F^-, \mu^-)$ such that

$$\Phi((F^+, \mu^+)) = (F^+, \lambda \mu^+) \quad \text{and} \quad \Phi((F^-, \mu^-)) = (F^-, \frac{1}{\lambda} \mu^-).$$

This means that $\Phi$ preserves both foliations $F^+$ and $F^-$, and it contracts the leaves of $F^-$ by $\frac{1}{\lambda}$ and it expands the leaves of $F^+$ by $\lambda$. The invariant foliations $F^+$ and $F^-$ are called the unstable and stable foliations for $\Phi$, and $\lambda > 1$ is called the stretch factor for $\Phi$.

Remark 3.3. The invariant foliations $F^+$ and $F^-$ for the pseudo-Anosov homeomorphism $\Phi$ are singular foliations which mean that they have common singularities in the interior of $\Sigma_{g,m}$ or at punctures of $\Sigma_{g,m}$. The number of singularities is finite. A 1-pronged singularity may occur at a puncture of $\Sigma_{g,m}$, yet there are no 1-pronged singularities in the interior of $\Sigma_{g,m}$.

Each pseudo-Anosov mapping class $\phi \in \text{MCG}(\Sigma_{g,m})$ contains a pseudo-Anosov homeomorphism $\Phi$ as a representative of $\phi$. We set $\lambda(\phi) = \lambda(\Phi)$ and call it the stretch factor of the mapping class $\phi = [\Phi]$. The stretch factor $\lambda(\phi)$ is a conjugacy invariant of pseudo-Anosov mapping classes. Moreover $\lambda(\phi)$ is the largest eigenvalue of a Perron-Frobenius integral matrix. Thus $\lambda(\phi)$ is an algebraic integer which is a real number grater than 1 and $|\lambda'| < \lambda(\phi)$ holds for each conjugate element $\lambda' \neq \lambda(\phi)$. The logarithm $\log(\lambda(\phi))$ of the stretch factor $\lambda(\phi)$ is called the entropy of $\phi$.

Remark 3.4. If $\phi \in \text{MCG}(\Sigma_{g,m})$ is pseudo-Anosov, then $\phi^k$ is pseudo-Anosov for all $k \geq 1$ and the equality $\lambda(\phi^k) = (\lambda(\phi))^k$ holds.

Recall the two homomorphisms $\Gamma : B_m \to \text{MCG}(D_m)$ and $c : \text{MCG}(D_m) \to \text{MCG}(\Sigma_{0,m}, \{\infty\}) < \text{MCG}(\Sigma_{0,m+1})$. We say that a braid $b \in B_m$ is periodic
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Figure 7. (1) The figure-8 solution $x(t)$ with period $T$. (2) A representative braid $\sigma_1^{-1}\sigma_2 \in \langle b(x(t), [0, \frac{T}{3}]) \rangle$.

(resp. reducible, pseudo-Anosov) if the mapping class $c(\Gamma(b))$ is of the corresponding type. When $b$ is a pseudo-Anosov braid, the stretch factor $\lambda(b)$ of $b$ is defined by the stretch factor $\lambda(c(\Gamma(b)))$ of the mapping class $c(\Gamma(b))$. In this case, it makes sense to say that the braid type $\langle b \rangle$ is pseudo-Anosov, and we can define the stretch factor $\lambda(\langle b \rangle)$ of the braid type $\langle b \rangle$ by

$$\lambda(\langle b \rangle) = \lambda(b) = \lambda(c(\Gamma(b))),$$

since both Nielsen-Thurston type and the stretch factor are conjugacy invariants.

3.4. Pseudo-Anosov $3$-braids. It is well-known that for positive integers $k_j$'s, $\ell_j$'s and $r$, the $3$-braid

$$\sigma_1^{k_1}\sigma_2^{-\ell_1} \cdots \sigma_1^{k_r}\sigma_2^{-\ell_r}$$

is pseudo-Anosov. Moreover any pseudo-Anosov $3$-braid $\alpha$ is conjugate to a braid $\sigma_1^{k_1}\sigma_2^{-\ell_1} \cdots \sigma_1^{k_r}\sigma_2^{-\ell_r}$ in $B_3$ which is unique up to a cyclic permutation. See Murasugi [Mur74] for example. Then the stretch factor $\lambda(\alpha)$ is the eigenvalue greater than 1 of

$$M_{(k_1, \ell_1, \ldots, k_r, \ell_r)} = \left(\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right)^{k_1} \left(\begin{array}{ccc} 1 & 0 & \ell_1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right)^{k_r}.$$  

See Handel [Han97] for example.

Example 3.5 (Metallic $3$-braids (Appendix A in [FT11])). For $p \geq 1$, the $3$-braid $\sigma_1^{2p}\sigma_2^{-2p}$ is pseudo-Anosov, and the stretch factor $\lambda(\sigma_1^{2p}\sigma_2^{-2p})$ is the eigenvalue greater than 1 of $M_{(2p, 2p)} = \left(\begin{array}{ccc} 1 + 4p^2 & 2p \\ 2p & 1 \end{array}\right)$. Thus

$$\lambda(\sigma_1^{2p}\sigma_2^{-2p}) = (p + \sqrt{p^2 + 1})^2 = \left(\frac{1}{2}(2p + \sqrt{4p^2 + 4})\right)^2 = (s_{2p})^2.$$
Example 3.6. Let us consider the figure-8 solution \( x(t) = (x_1(t), x_2(t), x_3(t)) \) by Moore \[\text{Moo93}\] and Chenciner-Montgomery \[\text{CM00}\], see Figure 7. The periodic solution \( x(t) \) has a property such that
\[
\begin{align*}
x_1(t + \frac{T}{3}) &= x_2(t), \\
x_2(t + \frac{T}{3}) &= x_3(t), \\
x_3(t + \frac{T}{3}) &= x_1(t),
\end{align*}
\]
where \( T > 0 \) is the period of \( x(t) \). This property tells us that \( x(t) \) determines a braid \( b(x(t), [0, \frac{T}{3}]) \). One sees that \( \sigma_1^{-1}\sigma_2 \in B_3 \) is a representative of \( \langle b(x(t), [0, \frac{T}{3}]) \rangle \) and \( (\sigma_1^{-1}\sigma_2)^3 \) represents the braid type \( \langle b(x(t), [0, T]) \rangle \) of the solution \( x(t) \). It is easy to see that \( \sigma_1^{-1}\sigma_2 \) is conjugate with \( \sigma_1\sigma_2^{-1} \) in \( B_3 \). By \[\text{[3.2]}\], \( \sigma_1\sigma_2^{-1} \) is a pseudo-Anosov braid with the stretch factor \( (s_1)^2 \). Thus the braid type of the figure-8 solution is pseudo-Anosov with the stretch factor \( (s_1)^6 \) \[Remark 3.4\], and hence it is a non-trivial braid type in contrast with the Euler’s solution and Lagrange’s solution \( \text{Example 3.2} \).

4. Proof of Theorem 1.5

For \( n \geq 2 \) and \( p \geq 1 \), we define braids \( u_n, v_n, \beta_{n,p} \in B_{2n} \) as follows.
\[
\begin{align*}
u_n &= (\sigma_1\sigma_2 \cdots \sigma_{2n-1})(\sigma_1\sigma_3 \cdots \sigma_{2n-1})^{-1}, \\
v_n &= (\sigma_1\sigma_2 \cdots \sigma_{2n-1})^{-1}(\sigma_1\sigma_3 \cdots \sigma_{2n-1}) \text{ and} \\
\beta_{n,p} &= u_n^pv_n^p.
\end{align*}
\]
See also Figure 8 together with the braid relation \( (B1) \) in Section 3.1. It is easy to check that \( \tilde{\sigma}(u_n) = (1, 3, \ldots, 2n - 1) \) and \( \tilde{\sigma}(v_n) = (2, 4, \ldots, 2n)^{-1} \). Hence by \[\text{[3.1]}\], we have
\[
(4.1) \quad \tilde{\sigma}(\beta_{n,p}) = \tilde{\sigma}(u_n^pv_n^p) = (1, 3, \ldots, 2n - 1)^p(2, 4, \ldots, 2n)^{-p} = \sigma(h_{n,p}^{-1}).
\]
Figure 9. Case \( n = 4 \). (1) An \( n \)-fold branched cover \( p : \Sigma_{0,2n} \to \Sigma_{0,2} \) with branched points 0 and \( \infty \), where punctures lie on the equators. (2) The upper hemisphere for \( \Sigma_{0,2n} \). (3) The upper hemisphere for \( \Sigma_{0,2} \).

Proof of Theorem 1.5. The proof consists of the following two steps. In Step 1, we prove that for \( n \geq 2 \) and any \( p \geq 1 \), the braid \( \beta_{n,p} \) is pseudo-Anosov with \( \lambda(\beta_{n,p}) = (s_{2p})^2 \). (We have no restriction on \( p \) in Step 1.) In Step 2, we prove that for any \( n \geq 2 \) and \( p \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \), the braid types of \( \beta_{n,p} \) and \( y_{n,p} = b(x_{n,p}(t), [0, \frac{d}{n}]) \) are the same. In other words, \( \beta_{n,p} \in \langle y_{n,p} \rangle \). Since \( X_{n,p} = \langle (y_{n,p})^{\frac{d}{n}} \rangle \), it follows that \( X_{n,p} = \langle (\beta_{n,p})^{\frac{d}{n}} \rangle \). Hence by Step 1 together with Remark 3.4, \( X_{n,p} \) is a pseudo-Anosov braid type with the stretch factor

\[
\lambda(X_{n,p}) = \lambda((\beta_{n,p})^{\frac{d}{n}}) = (\lambda(\beta_{n,p}))^{\frac{d}{n}} = (s_{2p})^{\frac{d}{n}}.
\]

Step 1. For \( n \geq 2 \) and \( p \geq 1 \), the braid \( \beta_{n,p} \) is pseudo-Anosov with \( \lambda(\beta_{n,p}) = (s_{2p})^2 \). In particular \( \lambda(\beta_{n,p}) < \lambda(\beta_{n,p'}) \) if \( p < p' \).

Proof of Step 1. We consider a 2-punctured sphere \( \Sigma_{0,2} \) and denote the two punctures of \( \Sigma_{0,2} \) by \( \odot \) and \( \infty \). We pick two points in \( \Sigma_{0,2} \) and call them 0 (the north pole) and \( \infty \) (the south pole). Given \( n \geq 2 \), we take an \( n \)-fold branched cover

\[ p : \Sigma_{0,2n} \to \Sigma_{0,2} \]

with branched points 0 and \( \infty \). (We cut a longitude of \( \Sigma_{0,2} \) between 0 and \( \infty \), take \( n \) copies of the resulting surface, and past them to make an \( 2n \)-punctured sphere.) We denote lifts of 0, \( \infty \) in \( \Sigma_{0,2} \) by 0, \( \infty \) in \( \Sigma_{0,2n} \) respectively. Let \( x_1, x_2, \ldots, x_{2n} \) be punctures of \( \Sigma_{0,2n} \) such that \( p \) sends \( x_{2k} \)}
Figure 10. Case $n = 4$. (1) A lift $\tilde{a} = \tilde{a}_n \in \text{MCG}(\Sigma_{0,2n}, \{0\}, \{\infty\})$ of (2) $a = \sigma_1^2 \in \text{MCG}(\Sigma_{0,2}, \{0\}, \{\infty\})$. (3) A lift $\tilde{b} = \tilde{b}_n \in \text{MCG}(\Sigma_{0,2n}, \{0\}, \{\infty\})$ of (4) $b = \sigma_2^{-2} \in \text{MCG}(\Sigma_{0,2}, \{0\}, \{\infty\})$. For (2) and (4), $a$ and $b$ (as elements of $B_3$) have base points $x_1, 0$, and $x_2$.

Figure 11. Case $n = 4$. (1) An arc in the equator. (2) A segment in the plane. (3) The braid $u_n$ corresponding to $\tilde{a}^* \in \text{MCG}(\Sigma_{0,2n}, \{\infty\})$. (The arc in (1) is identified with the segment in (2).) For (1) and (2), arrows indicate the image of the punctures under $\tilde{a}$.

(1) $x_{2k-1}$ to $x_2$ (resp. $x_1$). In the view from $0 \in \Sigma_{0,2n}$ in the upper hemisphere, we may assume that $x_1, \ldots, x_{2n}$ lie on the equator counterclockwise and these $2n$ punctures form the regular $2n$-gon. See Figure 9.

Let $a = \sigma_1^2, b = \sigma_2^{-2} \in B_3$. Since $a$ and $b$ are pure 3-braids, we can regard $a$ and $b$ as elements of $\text{MCG}(\Sigma_{0,2}, \{0\}, \{\infty\})$, see Figure (2)(4). We lift $a$
and \( b \) to \( \Sigma_{0,2n} \), and call them
\[
\tilde{a}, \tilde{b} \in \text{MCG}(\Sigma_{0,2n}, \{0\}, \{\infty\}) < \text{MCG}(\Sigma_{0,2n+1}, \{\infty\}).
\]
(Clearly both \( \tilde{a} \) and \( \tilde{b} \) fix the two points 0 and \( \infty \).) We have
\[
\tilde{a}(x_{2k-1}) = x_{2k+1} \text{ and } \tilde{a}(x_{2k}) = x_{2k} \text{ for } k = 1, \ldots, n,
\]
\[
\tilde{b}(x_{2k-1}) = x_{2k-1} \text{ and } \tilde{b}(x_{2k}) = x_{2k-2} \text{ for } k = 1, \ldots, n,
\]
where we interpret indices modulo 2n. Notice that \( \tilde{a} \) rotates the regular \( n \)-gon \( x_1 x_3 \ldots x_{2n-1} \) by \( \frac{i}{n} \) counterclockwise about the north pole 0; \( \tilde{b} \) rotates the regular \( n \)-gon \( x_2 x_4 \ldots x_{2n} \) by \( \frac{i}{n} \) clockwise about the same point 0, see Figure 10(1)(3). In other words, under the action of \( \tilde{a} \), each puncture \( x_{2i-1} \) (\( i = 1, \ldots, 2n \)) with odd index is passing through in front of the puncture \( x_{2i} \) with even index from the view of the north pole 0 \( \in \Sigma_{0,2n} \). Similarly, under the action of \( \tilde{b} \), each puncture \( x_{2i} \) (\( i = 1, \ldots, 2n \)) with even index is passing through in front of the puncture \( x_{2i-1} \) with odd index.

Forgetting the point 0 \( \in \Sigma_{0,2n} \), we think of \( \tilde{a} \) and \( \tilde{b} \) as elements, say \( \tilde{a}^* \) and \( \tilde{b}^* \) of \( \text{MCG}(\Sigma_{0,2n}, \{\infty\}) \) respectively. To find the planar 2n-braids for \( \tilde{a}^* \) and \( \tilde{b}^* \), we cut the equator of \( \Sigma_{0,2n} \) at a point between the consecutive punctures \( x_{2n} \) and \( x_1 \) (in the cyclic order) into an arc, and we regard the arc as a segment in the plane containing the punctures \( x_1, \ldots, x_{2n} \) in this order, see Figure 11(1)(2). Then from the actions of \( \tilde{a}^* \) and \( \tilde{b}^* \) on 2n punctures in the plane, one sees that 2n-braids corresponding to \( \tilde{a}^*, \tilde{b}^* \) are given by \( u_n, v_n \in B_{2n} \) respectively. See Figure 11(3). (Although we do not need braid representatives corresponding to \( \tilde{a} \) and \( \tilde{b} \) in the proof of Step 1, Figure 10(1) and (3) illustrate these representatives for \( \tilde{a} \) and \( \tilde{b} \) respectively in case \( n = 4 \).)

We define
\[
\phi_p = (\tilde{a})^p (\tilde{b})^p \in \text{MCG}(\Sigma_{0,2n}, \{0\}, \{\infty\}) < \text{MCG}(\Sigma_{0,2n+1}, \{\infty\}).
\]
It follows that \( \phi_p \) is a lift of \( a^p b^p = \sigma_1^{2p} \sigma_2^{-2p} \in \text{MCG}(\Sigma_{0,2}, \{0\}, \{\infty\}) \). Recall that \( a^p b^p \) is a pseudo-Anosov mapping class with the stretch factor \( (s_p)^2 \), see Example 3.5. Since \( \phi_p \) is a lift of \( a^p b^p \), \( \phi_p \) is also pseudo-Anosov with the same stretch factor as \( a^p b^p \). Hence \( \lambda(\phi_p) = (s_p)^2 \).

Forgetting the point 0 \( \in \Sigma_{0,2n} \), we obtain \( \phi_p^* \in \text{MCG}(\Sigma_{0,2n}, \{\infty\}) \) from \( \phi_p = (\tilde{a})^p (\tilde{b})^p \). Note that \( \phi_p^* = (\tilde{a}^*)^p (\tilde{b}^*)^p \) is a mapping class corresponding to the braid \( \beta_{n,p} = u_n^p v_n^p \).

Claim. The stable/unstable foliation \( \mathcal{F}^+/− \) of \( \phi_p \) is not 1-pronged at 0 \( \in \Sigma_{0,2n} \).

For the proof of Step 1, it is enough to prove Claim. The reason is that if \( \mathcal{F}^+/− \) is not 1-pronged at the point 0 \( \in \Sigma_{0,2n} \), then the same singular foliations \( \mathcal{F}^+ \) and \( \mathcal{F}− \) are still invariant foliations for \( \phi_p^* \), see Remark 3.3.
This implies that $\phi^*_p$ (and hence the braid $\beta_{n,p}$) is pseudo-Anosov with the same stretch factor $(s_p)^2$ as $\phi_p$, i.e.,

$$\lambda(\beta_{n,p}) = \lambda(\phi^*_p) = \lambda(\phi_p) = (s_{2p})^2.$$ 

**Proof of Claim.** Let us consider the stable/unstable foliation $E^+$ and $E^-$ for the pseudo-Anosov element $a^p b^p$. Then $E^{+/^-}$ has 1-pronged singularities at each of the two punctures of $\Sigma_{0,2}$ and at each of the two points $0$ and $\infty$. Let $F^+$ and $F^-$ denote lifts of $E^+$ and $E^-$ respectively. It follows that $F^{+/^-}$ is the stable/unstable foliation for $\phi_p$, and $F^{+/^-}$ has a 1-pronged singularity at each of the $2n$ punctures and $F^{+/^-}$ has $n$-pronged singularities ($n \geq 2$) at the points $0$ and $\infty$ in $\Sigma_{0,2n}$. In particular $F^{+/^-}$ is not 1-pronged at $0 \in \Sigma_{0,2n}$. This completes the proof of Claim.

By Claim, we finished the proof of Step 1.

Recall that $y_{n,p} = b(x_{n,p}(t), [0, 2T])$ and $\check{T} = \frac{dT}{2n}$.

**Step 2.** $\beta_{n,p} \in \langle y_{n,p} \rangle$. In particular $(\beta_{n,p})^{\frac{\check{T}}{T}} \in X_{n,p} = \langle (y_{n,p})^{\frac{\check{T}}{T}} \rangle$.

Proof of Step 2. Let us consider the shape curve $u(t) \ (t \in [0, 2T])$ for the solution $x_{n,p}(t)$. By the arguments in Section 2.2, the shape curve $u(t)$ $(t \in [0, 2T])$ satisfies the following properties.

- (s1) $u(0) \in B_{-1,1}$, $u(T) \in B_{2p-1}$ and $u(2T) \in B_{4p-1}$.
- (s2) $u_1(t) > 0$ for $0 < t < T$.
- (s3) $u_1(t) < 0$ for $T < t < 2T$.
- (s4) $x_i(2T) = x_{\sigma(h_{n,p})(i)}(0)$ for $i = 1, \ldots, 2n$.

Recall that $n$ bodies with odd indices and $n$ bodies with even indices rotate in mutually opposite directions. The above (s1) $(u(0) \in B_{-1,1}$, $u(T) \in B_{2p-1}$) and (s2) tell us that each of bodies $x_{2i}(t)$’s $(i = 1, \ldots, n)$ with even indices is passing through in front of bodies with odd indices (in the time interval $(0, T]$ from the view of the origin $0 \in \mathbb{R}^2$. Similarly (s1) $(u(T) \in B_{2p-1}$, $u(2T) \in B_{4p-1}$) and (s3) imply that each of bodies $x_{2i-1}(t)$’s $(i = 1, \ldots, n)$ with odd indices is passing through in front of the bodies with even indices (in the time interval $(T, 2T]$). These properties connect up $(b)^p \in \text{MCG}(\Sigma_{0,2n}, \{0\}, \{\infty\})$ with $u(t)$ for $t \in [0, T]$ (resp. $(a)^p \in \text{MCG}(\Sigma_{0,2n}, \{0\}, \{\infty\})$ with $u(t)$ for $t \in [T, 2T]$), see Figure 10(3)(4). Recalling that the permutation $\sigma(\beta_{n,p})$ of the braid $\beta_{n,p} = u_{n}^{p} v_{n}^{p}$ coincides with $\sigma(h_{n,p}^{-1})$, see Figure 11. Putting these properties together with (s4), we conclude that the $2n$-braid $u_{n}^{p} v_{n}^{p}$ ($= \beta_{n,p}$) is a representative of the braid type $\langle y_{n,p} \rangle$ of $y_{n,p} = b(x_{n,p}(t), [0, 2T])$. This completes the proof of Step 2.

By Steps 1 and 2, we have finished the proof of Theorem 1.5. □

We end this section with an example.

**Example 4.1.** Corollary 1.6 and Table 4 may suggest that $\lambda(X_{n,p})$ does not coincide with $\lambda(X_{n,p'})$ for different pairs $(n, p) \neq (n, p')$. However, the stretch factors happen to coincide for different pairs occasionally: The kth
metallic ratio $s_k$ has a formula $(s_k)^3 = s_{k^3 + 3k}$ for each $k \in \mathbb{N}$. In particular $(s_6)^3 = s_{234}$ when $k = 6$. We now claim that $\lambda(X_{n,3}) = \lambda(X_{n,117})$ for all $n = 3^{2^k}$ with $k \geq 5$. Then $117 \leq \lfloor \frac{n}{2} \rfloor$. By Theorem 1.4, we have $\lambda(X_{n,3}) = (s_6)^{\frac{2n}{3}} = (s_6)^{3 \cdot 2^{k+1}}$ and $\lambda(X_{n,117}) = (s_{234})^{\frac{2n}{9}} = (s_{234})^{2^{k+1}}$. By the equality $(s_6)^3 = s_{234}$, we have $\lambda(X_{n,3}) = ((s_6)^3)^{2^{k+1}} = (s_{234})^{2^{k+1}} = \lambda(X_{n,117})$.

5. NEW NUMERICAL PERIODIC SOLUTIONS OF THE 2n-BODY PROBLEM

We numerically found the periodic solutions $x_{n,p}(t)$ for $p = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ in Figure 1. In order to obtain those, we consider the Fourier series of the solutions and compute the Fourier coefficient by using the steepest descent method. Though the existence of the periodic orbits theoretically guarantees for $p = 1, \ldots, \lfloor \frac{n}{2} \rfloor$, new numerical solutions are obtained for several pairs with $(n, p)$ with $\lfloor \frac{n}{2} \rfloor < p < n$. See Figure 12. Then it is natural to ask the following question.

**Question 5.1.** For $n \geq 2$ and $\lfloor \frac{n}{2} \rfloor < p < n$, does there exist a periodic solution $x_{n,p}(t)$ of the planar 2n-body problem whose braid type $X_{n,p}$ is given by the braid $(\beta_{n,p})^d$ with $d = \gcd(n, p)$?

If the answer of Question 5.1 is positive, then Theorem 1.4 is extended to some pairs $(n, p)$ with $\lfloor \frac{n}{2} \rfloor < p < n$, i.e. if the answer of Question 5.1 is positive, then the braid type $X_{n,p}$ of the periodic solution $x_{n,p}(t)$ of the planar 2n-body problem is pseudo-Anosov with the stretch factor $(s_{2p})^\frac{2n}{d}$.

See also Step 1 of the proof of Theorem 1.5.

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\begin{align*}
t &= \frac{dT}{n} \\
t &= \frac{3dT}{4n} \\
t &= \frac{dT}{2n} \\
t &= \frac{dT}{4n} \\
(1) & \quad x_{3,2}(t) \\
(2) & \quad x_{4,3}(t) \\
(3) & \quad x_{5,3}(t) \\
(4) & \quad x_{5,4}(t) \\
t &= 0
\end{align*}

Figure 12. (1) $x_{3,2}(t)$. (2) $x_{4,3}(t)$. (3) $x_{5,3}(t)$. (4) $x_{5,4}(t)$.

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Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan

Email address: kajihara.yuika.75e@st.kyoto-u.ac.jp

Center for Education in Liberal Arts and Sciences, Osaka University, Toyonaka, Osaka 560-0043, Japan

Email address: kin.eiko.celas@osaka-u.ac.jp

Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan

Email address: shibayama@amp.i.kyoto-u.ac.jp