THE DISTRIBUTION OF FUNCTIONS RELATED TO INTEGER OPTIMIZATION

TIMM OERTEL*, JOSEPH PAAT†, AND ROBERT WEISMANTEL †

Abstract. We consider the asymptotic distribution of the IP sparsity function, which measures the minimal support of optimal IP solutions, and the IP to LP distance function, which measures the distance between optimal IP and LP solutions. To this end, we create a framework for studying the asymptotic distribution of general functions related to integer optimization. While there has been a significant amount of research focused around the extreme values that these functions can attain, little is known about their typical values. Each of these functions is defined for a fixed constraint matrix and objective vector while the right hand sides are treated as input. We show that the typical values of these functions are smaller than the known worst case bounds by providing a spectrum of probability-like results that govern their overall asymptotic distributions.

1. Introduction. Let $A \in \mathbb{Z}^{m \times n}$ have rank($A$) = $m$ and $c \in \mathbb{Z}^n$. We view $A$ as a matrix and as a set of column vectors in $\mathbb{Z}^m$, so $B \subseteq A$ means $B$ is a subset of the columns of $A$. For $b \in \mathbb{Z}^m$ define the parametrized integer program

$$\text{IP}(b) := \max\{c^Tz : Az = b \text{ and } z \in \mathbb{Z}^n_{\geq 0}\}.$$ 

We consider $A$ and $c$ to be fixed and assume that

(1.1) there does not exist $x \in \mathbb{R}^n_{\geq 0}$ such that $Ax = 0$ and $c^Tx > 0$.

This implies that if IP($b$) is feasible, then it has an optimal solution. The study of IP($b$) as $b$ varies is referred to as parametric integer programming, and it is important for designing algorithms for solving IP($b$), see, e.g., Papadimitriou [28], Eisenbrand and Shmonin [16], or Eisenbrand and Weismantel [17]. The motivation of this paper is to understand IP($b$) for various $b \in \mathbb{Z}^m$ by studying general functions whose input is IP($b$), or equivalently, whose input is a vector $b$ in $\mathbb{Z}^m$.

Let $f : \mathbb{Z}^m \to \mathbb{R}_{\geq 0} \cup \{\infty\}$. Our goal is to understand the distribution of $f$ under the assumption

(1.2) $f(b) < \infty$ if and only if IP($b$) is feasible.

We assume (1.2) throughout the paper. One property of $f$ that can be quantified is the maximum finite value that $f$ can attain. Other properties are common values of $f$ that are attained, and these are the focus of this paper. One can quantify these common values through the use of asymptotic proportions. For $t \in \mathbb{Z}_{\geq 1}$ and $E \subseteq \mathbb{R}^m$ define

$$\Pr_t(E) := \frac{|\{b \in \mathbb{Z}^m : \|b\|_\infty \leq t \text{ and } f(b) < \infty\} \cap E|}{|\{b \in \mathbb{Z}^m : \|b\|_\infty \leq t \text{ and } f(b) < \infty\}|}.$$ 

The asymptotic proportion of $E$ is defined to be

$$\Pr(E) := \liminf_{t \to \infty} \Pr_t(E).$$

The value $\Pr_t(E)$ is the probability that $E$ occurs in a bounded set. Similarly, $\Pr(E)$ can be viewed as the probability that $E$ occurs anywhere in $\mathbb{Z}^m$. Note that $\Pr(E) \in$...
[0, 1] and $\Pr(E) \geq \Pr(F)$ if $F \subseteq E$. However, $\Pr(\cdot)$ is not necessarily a probability distribution because it only satisfies the weaker condition $\Pr(E \cup F) \geq \Pr(E) + \Pr(F)$ for disjoint $E$ and $F$. The functional $\Pr(\cdot)$ was introduced by Bruns and Gubeladze [9] and later used by Oertel et al. [26] to study sparse solutions of $\text{IP}(b)$.

In order to examine common values of $f$, we are interested in asymptotic proportions of the form

$$\Pr(f \leq M) := \Pr(\{b \in \mathbb{Z}^m : f(b) \leq M\}),$$

for $M \in \mathbb{R}$. It is difficult to compute (or even upper bound) $\Pr(E)$ for general $f$ and $E$. However, there are functions that are important in the context of integer optimization for which we can provide such an analysis. To the best of our knowledge, this refined asymptotic analysis does not appear in the literature in generality.

In this paper we analyze the asymptotic proportions of the sparsity and distance functions, which we will denote by $\sigma$ and $\pi$, respectively (see Sections 1.1 and 1.2 for precise definitions). However, there are other functions of relevance for integer optimization that also fit into our framework. Such functions include the integrality gap function [2, 14] and the optimum value function [18, 33]. Moreover, with slight generalizations to $\mathbb{R}^d$-valued functions $f : \mathbb{Z}^m \rightarrow (\mathbb{R}_{\geq 0} \cup \{\infty\})^d$ one can also consider functions such as the flatness direction [6, VII].

One main contribution of this paper is a set of conditions to bound $\Pr(f \leq M)$ for general functions $f$ and various $M$ (see Theorem 2.4). This result is presented in Section 2 as it requires a brief introduction into groups and lattices. A second contribution of this paper is an application of Theorem 2.4 to bound the values $\Pr(\sigma \leq M)$ and $\Pr(\pi \leq M)$ for various $M \in \mathbb{Z}$. These bounds are in terms of $m$ and the determinants of the submatrices of $A$. We denote the largest absolute value of these determinants and their greatest common divisor, respectively, by

$$\delta := \max \{|\det(B)| : B \subseteq A \text{ is invertible}\}, \text{ and}$$

$$\gamma := \gcd\{|\det(B)| : B \subseteq A \text{ is invertible}\}.$$ 

### 1.1. The sparsity function $\sigma$.

The minimum sparsity of an optimal solution of $\text{IP}(b)$ is

$$\sigma(b) := \min\{|\text{supp}(z)| : z \text{ is an optimal feasible solution of } \text{IP}(b)\},$$

where $\text{supp}(z) := \{i \in \{1, \ldots, n\} : z_i > 0\}$. If $\text{IP}(b)$ is infeasible, then $\sigma(b) := \infty$. The function $\sigma$ is also a measure of the distance between linear codes [5, 32] and has been studied for identifying sparse solutions in combinatorial problems [11, 24].

It was shown in Aliev et al. [4] and Aliev et al. [3] that if $\sigma(b) < \infty$, then

$$\sigma(b) \leq m + \log_2(\gamma^{-1} \cdot \sqrt{\det(A^TA)}) \leq 2m \log_2(2\sqrt{m} \cdot \|A\|_\infty),$$

where $\|A\|_\infty$ denotes the largest absolute entry of $A$. See also Eisenbrand and Shmonin [15] for bounds on the maximum finite value of $\sigma$. There is not much room to improve the upper bound (1.3), in general. In fact, for any $\epsilon > 0$ Aliev et al. [3] provide an example of $A$ and $b$ for which $m + \log_2(\|A\|_\infty)^{1/(1+\epsilon)} \leq \sigma(b)$.

A special case of sparsity is when $c = 0$ in which case $\sigma(b)$ quantifies the sparsest solution among all feasible solutions to $\text{IP}(b)$. For a general matrix $A$, Aliev et al. [4] proved the qualitative result that $\sigma$ is asymptotically periodic, which can be used to show that there exists $M \in \mathbb{R}_{\geq 0}$ such that $\Pr(\sigma \leq M) = 1$. Oertel et al. [26] showed that asymptotic proportions of $\sigma$ can be bounded using the minimum absolute determinant of $A$ or also the ‘number of prime factors’ of the determinants.
If in addition $A$ has the Hilbert basis property and $c = 0$ (i.e., if the columns of $A$ correspond to a Hilbert basis of the cone generated by $A$), then upper and lower bounds on the feasible support value $\sigma(b)$ can be given in terms of only $m$. Cook et al. [12] showed that if $\sigma(b) < \infty$, then $\sigma(b) \leq 2m - 1$, and Sebő improved this to $\sigma(b) \leq 2m - 2$ [30]. As for lower bounds, Bruns et al. [10] gave an example such that $\sigma(b) \geq \frac{2}{3}m$. In this setting asymptotic proportions of the feasible support function $\sigma$ were considered by Bruns and Gubeladze [9], who showed $Pr(\sigma \leq 2m - 3) = 1$.

We show that $\sigma(b)$ is often smaller than the best known universal bound (1.3).

**Theorem 1.1.** If $k \in \{0, \ldots, \lfloor \log_2(\gamma^{-1} \cdot \delta) \rfloor \}$, then

$$Pr(\sigma \leq m + k) \geq \min \left\{ 1, \frac{2^k}{\gamma^{-1} \cdot \delta} \right\}.$$  

In particular, $Pr(\sigma \leq m + \log_2(\gamma^{-1} \cdot \delta)) = 1$.

The Cauchy-Binet formula (see, e.g., [22]) implies that $\delta \leq \sqrt{\det(AA^\top)}$ with strict inequality if $A$ has at least two invertible submatrices. Hence, Theorem 1.1 gives tighter bounds on the asymptotic proportions of $\sigma$ than (1.3) does, and the differences are strict if $A$ has at least two invertible submatrices.

**1.2. The distance function $\pi$.** Another important function related to integer optimization measures distance between optimal solutions of $\text{IP}(b)$ and optimal solutions of its linear relaxation

$$\text{LP}(b) := \max \{ c^\top x : Ax = b \text{ and } x \in \mathbb{R}_{\geq 0}^n \}.$$  

For ease of presentation we will assume that $c$ has the property that the optimum solution $\text{LP}(b)$ is unique for all feasible $b$. Note that this can always be achieved by perturbing $c$. See Remark 5.1 in Section 5 for further discussion on this assumption and its implications. Let $x^*(b)$ denote the optimal solution of $\text{LP}(b)$.

Define the distance function to be

$$\pi(b) := \min \{ \| x^*(b) - z^* \|_1 : z^* \text{ is an optimal solution of } \text{IP}(b) \}.$$  

If $\text{IP}(b)$ is infeasible, then $\pi(b) := \infty$.

The distance between $\text{IP}(b)$ and $\text{LP}(b)$ solutions is a classic question in IP theory. It has been used to measure the sensitivity of optimal IP solutions [7, 8, 13] and to create efficient dynamic programs for solving integer programs [17, 23]. Eisenbrand and Weismantel [17] showed that if $\pi(b) \leq \infty$, then $\pi(b) \leq m(2m \| A \|_\infty + 1)^m$. By modifying their proof\(^1\) it can be shown that if $\pi(b) \leq \infty$, then

$$\pi(b) \leq m(2m + 2)^m \delta.$$  

See also Cook et al. [13], Aliev et al. [2], and Paat et al. [27] for bounds on the maximum finite value of the distance function. It is not known if the bound in (1.4) is tight. In the case $m = 1$, Aliev et al. [2] provide a tight upper bound on the related distance function $\pi^\infty$ (see below).

Gomory [18] used the group structure of $A$ to study the value function of $\text{IP}(b)$, and he proved that this function is asymptotically periodic (see also Wolsey [33]). Using his results along with Theorem 2.4, one can show that $Pr(\pi \leq (m+1) \gamma^{-1} \cdot \delta) = 1$.

\(^1\)The proof of (1.4) is exactly the same as [17, Theorem 3.1] except the $\| \cdot \|_\infty$-norm is replaced by the norm $\| x \|_* := \|(B^{-1}x)\|_\infty$, where $B \subseteq A$ satisfies $|\det(B)| = \delta$. 

However, this can be extended to a more refined distribution, which we present in Theorem 1.2 (a). Theorem 1.2 (b) modifies the proof of Theorem 1.2 (a) to consider IP to LP distance in the \( \ell^\infty \)-norm. Distance in the \( \ell^\infty \)-norm was studied in [7, 8, 13, 27]. Theorem 1.2 (b) partially resolves [27, Conjecture 1], which states that the IP to LP distance in terms of the \( \ell^\infty \)-norm can be bounded by a function of \( \delta \). Define \( \pi^\infty \) analogously to \( \pi \) using the \( \ell^\infty \)-norm rather than the \( \ell^1 \)-norm.

**Theorem 1.2.** If \( k \in \{0, \ldots, \gamma^{-1} \cdot \delta - 1\} \), then

(a) \( \Pr(\pi \leq m \gamma^{-1} \cdot \delta + k) \geq \frac{k + 1}{\gamma^{-1} \cdot \delta} \), and

(b) \( \Pr(\pi^\infty \leq \gamma^{-1} \cdot \delta + k) \geq \frac{k + 1}{\gamma^{-1} \cdot \delta} \).

In particular, \( \Pr(\pi \leq (m + 1) \gamma^{-1} \cdot \delta - 1) = 1 \) and \( \Pr(\pi^\infty \leq 2 \gamma^{-1} \cdot \delta - 1) = 1 \).

A corollary of Theorem 1.2 is a bound on the typical distance between IP(\( b \)) and LP(\( b \)) solutions in terms of \( \|A\|_{\infty} \) rather than \( \delta \). The result [17, Theorem 3.1] implies \( \Pr(\pi \leq m(2m \|A\|_{\infty} + 1)^m) = 1 \). Theorem 1.2 and Hadamard’s inequality (see, e.g., [22]) give the following improvement.

**Corollary 1.3.** The function \( \pi \) satisfies

\[ \Pr(\pi \leq (m + 1)(\sqrt{m} \|A\|_{\infty})^m - 1) = 1. \]

**1.3. Outline of the paper and notation.** Section 2 provides a general framework for studying the values \( \Pr(f \leq M) \) and proves the fundamental Theorem 2.4. Preliminaries about optimal LP(\( b \)) solutions are in Section 3. We use these in Sections 4 and 5 to prove results about \( \sigma \) and \( \pi \), respectively.

For \( K \subseteq \mathbb{R}^m \) and \( d \in \mathbb{R}^m \) define \( K + d := \{b + d : b \in K\} \). The \( k \)-dimensional vector of all zeros is \( \mathbf{0}^k \) and the vector of all ones is \( \mathbf{1}^k \). When multiplying a matrix \( B \subseteq \mathbb{Z}^m \) and a vector \( y \in \mathbb{R}^B \) as \( By \), we use \( y_b \) to denote the component of \( y \) corresponding to \( b \in B \). For a set \( B \subseteq \mathbb{Z}^m \), we use \( \text{cone}(B) \) to denote the convex cone generated by the elements in \( B \). The interior of a convex set \( P \subseteq \mathbb{R}^m \) is denoted by \( \text{int}(P) \). A set \( \Lambda \subseteq \mathbb{Z}^m \) is a **lattice** if \( 0^m \in \Lambda \), \( b + d \in \Lambda \) for \( b, d \in \Lambda \), and if \( b \in \Lambda \) then \( -b \in \Lambda \). An affine lattice is a set \( \overline{\Lambda} \subseteq \mathbb{R}^m \) of the form \( \overline{\Lambda} = b + \Lambda \), where \( b \in \mathbb{Z}^m \) and \( \Lambda \subseteq \mathbb{Z}^m \) is a lattice. The dimension of an affine lattice is largest number of linearly independent vectors in \( \Lambda \). For more background on these concepts, we refer to [29] and [6, Chapter VII].

**2. Asymptotic proportions for general functions.** We first present a few auxiliary lemmata and notations before proving the main theorem of this section. The first lemma follows from standard results on triangulations and subdivisions. Thus, we omit the proof. For more on subdivisions see [6, Page 332] or [34, Chapter 9].

**Lemma 2.1.** Let \( A^1, \ldots, A^s \in \mathbb{Z}^{m \times m} \) be square matrices of rank \( m \). Then there exist finite sets \( B^1, \ldots, B^s \subseteq \mathbb{Z}^m \) such that

(a) \( \bigcup_{i=1}^s \text{cone}(A^i) = \bigcup_{j=1}^f \text{cone}(B^j) \),

(b) \( \text{int}(\text{cone}(B^i)) \cap \text{int}(\text{cone}(B^j)) = \emptyset \) for distinct \( i, j \in \{1, \ldots, \ell\} \), and

(c) for each tuple \( (i, j) \in \{1, \ldots, s\} \times \{1, \ldots, \ell\} \), either \( \text{cone}(B^j) \subseteq \text{cone}(A^i) \) or \( \text{int}(\text{cone}(B^j)) \cap \text{cone}(A^i) = \emptyset \).
Lemma 2.2. Let $C \subseteq \mathbb{R}^m$ be an $m$-dimensional cone and $b \in \mathbb{R}^m$. There exists $d \in \mathbb{R}^m$ such that

$$(d + C) \subseteq C \cap (b + C).$$

Proof. The recession cone of $C \cap (b + C)$ is $C$. Hence, for every $d \in C \cap (b + C)$ the set $d + C$ is contained in $C \cap (b + C)$. Thus, it is enough to show $C \cap (b + C) \neq \emptyset$. Assume to the contrary that $C \cap (b + C) = \emptyset$. By the separating hyperplane theorem (see, e.g., [19]), there exists an $\alpha \geq 0$ and a hyperplane $\{d \in \mathbb{R}^m : \mathbf{h}^\top d = \alpha\}$ such that $\mathbf{h}^\top d > \alpha \geq 0$ for all $d \in \text{int}(C)$ and $\alpha > \mathbf{h}^\top \mathbf{d}$ for all $\mathbf{d} \in \text{int}(b + C)$. Let $\mathbf{d} \in \text{int}(C)$ and $\mathbf{d} \in \text{int}(b + C)$ for each $t \in \mathbb{Z}_{\geq 0}$ and

$$\mathbf{h}^\top (td + \mathbf{d}) = t \mathbf{h}^\top \mathbf{d} + \mathbf{h}^\top \mathbf{d} > t\alpha + \mathbf{h}^\top \mathbf{d}.$$ 

However, $t\alpha + \mathbf{h}^\top \mathbf{d} > \alpha$ for large enough $t$. This contradicts $td + \mathbf{d} \in \text{int}(b + C)$. \[\square\]

For two functions $g, h : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ we write $g \sim h$ if $\lim_{t \rightarrow \infty} g/h = 1$ and $g \prec h$ if $\lim_{t \rightarrow \infty} g/h \leq 1$. For a $k$-dimensional set $P \subseteq \mathbb{R}^m$, we denote the $k$-dimensional Lebesgue measure by $\text{vol}_k(P)$. If $\Lambda \subseteq \mathbb{Z}^m$ is an $m$-dimensional lattice, then the determinant of $\Lambda$ is $\det(\Lambda) := |\det(B)|$, where $B \in \mathbb{Z}^{m \times m}$ is any matrix such that $\Lambda = B \cdot \mathbb{Z}^m$. The following result is a variation of classic known results in Ehrhart theory, see for instance [25, Theorem 7] and [21, Theorem 1.2]

Lemma 2.3. Let $P \subseteq \mathbb{R}^m$ be a $k$-dimensional rational polytope and $\Lambda \subseteq \mathbb{Z}^m$ be an $m$-dimensional affine lattice. There exists a constant $c_{P, \Lambda} > 0$ such that

$$(2.1) \quad |tP \cap \Lambda| \sim c_{P, \Lambda} t^k.$$ 

If $k = m$, then $c_{P, \Lambda} = \text{vol}_m(P)/\det(\Lambda)$ and

$$(2.2) \quad |tP \cap \Lambda| \sim c_{P, \Lambda} t^m.$$ 

Let $A \in \mathbb{Z}^{m \times n}$ have rank($A$) = $m$ and let $f : \mathbb{Z}^m \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ as in (1.2). By assumption, the function value $f(b)$ is finite only if IP($b$) is feasible, i.e., if there is a vector $z \in \mathbb{Z}^m_{\geq 0}$ such that $Az = b$. In particular, the choices of $b$ that we need to consider to bound $\Pr(f \leq M)$ are those $b$ in cone($A$) that also lie on the lattice generated by $A$, which we denote by

$$(2.3) \quad \Lambda := A \cdot \mathbb{Z}^m.$$ 

The lattice $\Lambda$ induces an equivalence relationship $\equiv_A$ on $\mathbb{Z}^m$, where $b \equiv_A d$ if and only if $b - d \in \Lambda$. There are $\gamma$ many equivalence classes generated by $\equiv_A$, see, e.g., [20, page 22].

In Theorem 2.4 we bound $\Pr(f \leq M)$ by dividing cone($A$) into subcones and analyzing the corresponding equivalence classes generated by each subcone. Let

$$(2.4) \quad A^1, \ldots, A^s \subseteq A \quad \text{be invertible matrices such that} \quad \text{cone}(A) = \bigcup_{i=1}^s \text{cone}(A^i).$$

These matrices exist by Carathéodory’s theorem. Define the lattices

$$(2.5) \quad \Gamma^i := A^i \cdot \mathbb{Z}^m \quad \forall \quad i \in \{1, \ldots, s\}.$$
For each $i \in \{1, \ldots, s\}$ the lattice $\Gamma^i$ induces an equivalence relationship $\equiv_{\Gamma^i}$ on $\Lambda$: for every $b, d \in \Lambda$ we say $b \equiv_{\Gamma^i} d$ if $b - d \in \Gamma^i$. Moreover, $\Gamma^i$ is a sublattice of $\Lambda$ and therefore the relation $\equiv_{\Gamma^i}$ induces a quotient group $\Lambda/\Gamma^i$ with cardinality

$$|\Lambda/\Gamma^i| = \det(\Gamma^i)/\det(\Lambda) = \gamma^{-1} \cdot |\det(A^i)|,$$

that is, $\equiv_{\Gamma^i}$ partitions $\Lambda$ into $\gamma^{-1} \cdot |\det(A^i)|$ many different equivalence classes.

If $b \in \Lambda \cap \operatorname{cone}(A^i)$ and $\operatorname{IP}(b)$ is feasible, then $b \equiv_{\Gamma^i} g$ for some $g$ in $\Lambda/\Gamma^i$. The converse does not hold: if $b \in \Lambda \cap \operatorname{cone}(A^i)$ and $b \equiv_{\Gamma^i} g$ for some $g \in \Lambda/\Gamma^i$, then $\operatorname{IP}(b)$ is not necessarily feasible. This is due to the fact that the set $\{b \in \mathbb{Z}^m : \operatorname{IP}(b) \text{ is feasible}\}$ only becomes structured for large values of $b$, which is a phenomenon related to the Frobenius number, see, e.g., [1, 31]. In order to reach a partial converse, we consider not only the points $b \in \Lambda \cap \operatorname{cone}(A^i)$ for which $b \equiv_{\Gamma^i} g$ for some $g \in \Lambda/\Gamma^i$, but also those $b$ that are ‘far enough’ in $\operatorname{cone}(A^i)$. To formalize far enough, we choose a suitable vector

$$d^i \in A^i \cdot \mathbb{Z}_{\geq 0}^m \text{ for each } i \in \{1, \ldots, s\}$$

and consider the vectors $b \in \Lambda \cap [\operatorname{cone}(A^i) + d^i]$.

After structuring the points in $\Lambda \cap \operatorname{cone}(A)$, we assume that the values of $f(b)$ can be bounded over the points in $b \in \Lambda \cap [\operatorname{cone}(A^i) + d^i]$. If enough equivalence classes in $\Lambda/\Gamma^i$ can be uniformly bounded in this set, then this is enough to bound $\Pr(f \leq M)$. We denote the set of equivalence classes that are uniformly bounded by $M$ using the following notation:

$$X^i(f, M) := \left\{ g \in \Lambda/\Gamma^i : \max\{f(b) : b \equiv_{\Gamma^i} g \text{ and } b \in \mathbb{Z}^m \cap \operatorname{cone}(A^i) + d^i\} \leq M \right\}.$$

**Theorem 2.4.** Let $k \in \{0, 1, \ldots, \gamma^{-1} \cdot \delta\}$ and $|X^i(f, M)|$ be defined as in (2.8). If $|X^i(f, M)| \geq \min\{\gamma^{-1} \cdot |\det A^i|, k\}$ for all $i \in \{1, \ldots, s\}$, then

$$\Pr(f \leq M) \geq \frac{k}{\gamma^{-1} \cdot \delta}.$$

**Proof.** Recall $\Lambda$ defined in (2.3). If $b \in \mathbb{Z}^m$ and $f(b) < \infty$, then $b \in \Lambda \cap \operatorname{cone}(A)$. Therefore,

$$\Pr_t(f \leq M) = \frac{|\{b \in \Lambda \cap \operatorname{cone}(A) : \|b\|_\infty \leq t \text{ and } f(b) \leq M\}|}{|\{b \in \Lambda \cap \operatorname{cone}(A) : \|b\|_\infty \leq t \text{ and } f(b) < \infty\}|}$$

for all $t \in \mathbb{Z}_{\geq 0}$. By Lemma 2.1 we can partition $\operatorname{cone}(A)$ into rational cones $C^j := \operatorname{cone}(B^j)$, where $j \in \{1, \ldots, \ell\}$, such that $\operatorname{int}(C^i) \cap \operatorname{int}(C^j) = \emptyset$ for all distinct $i, j \in \{1, \ldots, \ell\}$. Also, either $C^j \subseteq \operatorname{cone}(A^i)$ or $\operatorname{int}(C^j) \cap \operatorname{cone}(A^i) = \emptyset$ for all $i \in \{1, \ldots, s\}$ and $j \in \{1, \ldots, \ell\}$. Define the set

$$F := \{b \in \operatorname{cone}(A) : b \in C^i \cap C^j \text{ for some pair of distinct } i, j \in \{1, \ldots, \ell\}\},$$

and observe that

$$\Lambda \cap \operatorname{cone}(A) = [\Lambda \cap F] \cup \bigcup_{j=1}^\ell [\Lambda \cap \operatorname{int}(C^j)].$$
The sets in the latter union are pairwise disjoint. Therefore, by (2.9) it follows that
\[
\Pr_t(f \leq M) = \frac{|\{b \in A \cap F : \|b\|_{\infty} \leq t \text{ and } f(b) \leq M\}|}{|\{b \in A \cap \text{cone}(A) : \|b\|_{\infty} \leq t \text{ and } f(b) < \infty\}|}
\]
\[
+ \sum_{j=1}^{\ell} \frac{|\{b \in A \cap \text{int}(C^j) : \|b\|_{\infty} \leq t \text{ and } f(b) \leq M\}|}{|\{b \in A \cap \text{cone}(A) : \|b\|_{\infty} \leq t \text{ and } f(b) < \infty\}|}.
\]  
(2.10)

The set $F$ is a finite union of sets each of which has a dimension strictly less than $m$. Therefore, the set $\{b \in F : \|b\|_{\infty} \leq t\}$ is a finite union of polytopes of dimension strictly less than $m$. It follows from Lemma 2.3 that there exists a constant $c_{F,A}$ such that $\{b \in A \cap F : \|b\|_{\infty} \leq t\} \sim c_{F,A} t^k$. Similarly, $\{b \in \text{cone}(A) : \|b\|_{\infty} \leq t\}$ is an $m$-dimensional polytope, and there exists a constant $c_{A,A}$ such that $\{b \in A \cap \text{cone}(A) : \|b\|_{\infty} \leq t\} \sim c_{A,A} t^m$. Therefore,
\[
\lim_{t \to \infty} \frac{|\{b \in A \cap F : \|b\|_{\infty} \leq t \text{ and } f(b) \leq M\}|}{|\{b \in A \cap \text{cone}(A) : \|b\|_{\infty} \leq t \text{ and } f(b) < \infty\}|} = \lim_{t \to \infty} \frac{c_{F,A} t^k}{c_{A,A} t^m} = 0.
\]
(2.11)

For each $j \in \{1, \ldots, \ell\}$ define the truncated cone $P^j := C^j \cap [-1,1]^m$. To finish the proof of the theorem it is enough to show
\[
\lim_{t \to \infty} \frac{|\{b \in A \cap tP^j : f(b) \leq M\}|}{|A \cap tP^j|} \geq \frac{k}{\gamma^{-1} \delta}.
\]  
(2.12)

Indeed, if (2.12) is true, then we obtain
\[
\Pr(f \leq M) = \lim_{t \to \infty} \sum_{j=1}^{\ell} \frac{|\{b \in A \cap tP^j : f(b) \leq M\}|}{|\{b \in A \cap \text{cone}(A) : \|b\|_{\infty} \leq t, f(b) < \infty\}|}
\]
\[
\geq \lim_{t \to \infty} \min_{j=1, \ldots, \ell} \frac{|\{b \in A \cap tP^j : f(b) \leq M\}|}{|A \cap tP^j|}
\]
\[
\geq \frac{k}{\gamma^{-1} \delta},
\]
where the first equation comes from (2.10) and (2.11), the first inequality follows because $C^1, \ldots, C^\ell$ partition $\text{cone}(A)$ and $\{b \in A \cap tP^j : f(b) < \infty\} \subseteq A \cap tP^j$, and the final inequality comes from (2.12).

We prove (2.12). Without loss of generality $C^j \subseteq \text{cone}(A^j)$. By Lemma 2.3 $|tP^j \cap A| \sim t^m \text{vol}(P^j)/\det(A)$. Let $g^1, \ldots, g^{\gamma^{-1} \det(P^j)}$ denote the distinct elements in $A/G^j$. Without loss of generality assume that $g^1, \ldots, g^k \in X^j(f,M)$. Using the fact that every $b \in tP^j$ belongs to exactly one equivalence class with the relation $\equiv_{G^j}$, it follows that
\[
|\{b \in tP^j \cap A : f(b) \leq M\}| = \sum_{i=1}^{\gamma^{-1} \det(P^j)} |\{b \in tP^j \cap (g^i + G^j) : f(b) \leq M\}|.
\]

Lemma 2.2 and $C^j \subseteq \text{cone}(A^j)$ imply that there exists $h \in \mathbb{Z}^m$ with $h + C^j \subseteq d^j + \text{cone}(A^j)$. Also, there exists a finite number of hyperplanes $H^1, \ldots, H^r \subseteq \mathbb{R}^m$ parallel to the facets of $C^j$ such that
\[
(C^j \setminus [h + C^j]) \cap G^j \subseteq \bigcup_{i=1}^{r} [H^i \cap G^j].
\]
By Lemma 2.3 \( |tP^j \cap [h + tP^j] \cap [g^i + \Gamma^j]| \sim t^m \text{vol}_m(P^j)/\text{det}(\Gamma^j) \) for each \( i \in \{1, \ldots, \gamma^{-1} \cdot \text{det}(\Gamma^j)\} \). In particular, it follows from the main assumption in the theorem that \( |\{b \in tP^j \cap (g^i + \Gamma^j) : f(b) \leq M\}| \sim t^m \text{vol}_m(P^j)/\text{det}(\Gamma^j) \). Hence,

\[
\lim_{t \to \infty} \frac{|\{b \in tP^j \cap A : f(b) \leq M\}|}{|tP^j \cap A|} \geq \lim_{t \to \infty} \sum_{i=1}^{k} \frac{|\{b \in tP^j \cap (g^i + \Gamma^j) : f(b) \leq M\}|}{|tP^j \cap A|} = \frac{k}{\gamma^{-1} \cdot \text{det}(\Gamma^j)}.
\]

This completes the proof as \( \text{det}(\Gamma^j) = |\text{det}(A^j)| \leq \delta \).

3. Preliminaries for results on optimal IP solutions. An invertible matrix \( B \subseteq A \) is an optimal LP basis matrix if there exists \( b \in \mathbb{Z}^m \) such that LP\((b)\) has an optimal solution \( x^* \in \mathbb{R}_{\geq 0}^n \) with \( \{a \in A : x^*_a > 0\} = B \). It is well known that for every feasible \( b \in \mathbb{Z}^m \) there exists an optimal LP basis matrix \( B \) such that LP\((b)\) has an optimal solution \( x^* \in \mathbb{R}_{\geq 0}^n \) with \( \{a \in A : x^*_a > 0\} \subseteq B \); we refer to such a \( B \) as an optimal LP\((b)\) basis matrix. In order to apply Theorem 2.4 to \( \sigma \) and \( \pi \), we choose \( A^1, \ldots, A^s \) in (2.4) to be optimal LP basis matrices. This section collects properties of these matrices. Lemma 3.1 is a folklore result.

**Lemma 3.1.** If \( B \subseteq A \) is an optimal LP\((b)\) basis matrix for \( b \in \mathbb{Z}^m \), then \( B \) is an optimal LP\((b + Bu)\) basis matrix for any \( u \in \mathbb{R}^m \) satisfying \( B^{-1}b + u \geq 0^m \).

Let \( B \subseteq A \) be an optimal LP\((b)\) basis matrix. Let \( \equiv_B \) denote the equivalence relation \( b \equiv_B d \) if and only if \( b - d \in \Gamma^B \), where \( \Gamma^B := B \cdot \mathbb{Z}^m \). Gomory showed in [18, Theorem 2] that if \( b \) is ‘far away’ from the boundary of \( \text{cone}(B) \), that is if \( b \) is in the set\(^2\)

\[
F = F(B) := \{b \in \mathbb{Z}^m : B^{-1}b \geq 3\delta \cdot 1^m \text{ and IP}(b) \text{ is feasible}\},
\]

then there exists an optimal solution \( z^* \) to IP\((b)\) whose support is contained in \( B \) and at most \( |\text{det}(B)| \) many additional non-basic columns. More precisely, \( z^* = z^B + z^N \), where \( \{a \in A : z^B_a > 0\} \subseteq B \), \( \{a \in A : z^N_a > 0\} \cap B = \emptyset \) and \( |\{a \in A : z^N_a > 0\}| \leq |\text{det}(B)| \).

Observe that the equation \( Az^* = Az^B + Az^N \) and the inclusion \( \{a \in A : z^B_a > 0\} \subseteq B \) implies that the relations \( Az^B \equiv_B 0^m \) and \( Az^N \equiv_B b \) hold. Hence, \( z^N \) is the subvector of \( z^* \) that ensures \( Az^* \) is in the same equivalence class as \( b \) modulo \( \equiv_B \). Gomory also argued that \( z^N \) can be chosen to be a minimal subvector that ensures \( Az^* \) is in the same equivalence class as \( b \). By minimal, we mean that there does not exist a vector \( z^N \nleq \tilde{z}^N \) such that \( Az^N \equiv_B b \). We denote the set of these minimal vectors \( z^N \) by

\[
N = N(B) := \left\{ \begin{array}{l}
\exists b \in F \text{ and } z^B \in \mathbb{Z}_{\geq 0}^n \text{ such that } \\
(i) \{a \in A : z^B_a > 0\} \subseteq B, \\
(ii) z^B + z \text{ is an optimal solution of IP}(b), \\
(iii) Az \equiv_B b, \text{ and } \\
(iv) \text{ if } 0^n \leq \bar{z} \leq z \text{ and } Az \equiv_B 0^m, \text{ then } \bar{z} = 0^n
\end{array} \right\}.\]

\(^2\)In fact, the set of far away points defined by Gomory is in terms of the distance from \( b \) to the boundary of \( \text{cone}(B) \), and this set contains \( F \). The choice of \( 3\delta \) is chosen to simplify the proofs in this section.
Our next lemma shows that vectors in $N$ are not too large, i.e., an optimal solution of $\text{IP}(b)$ only uses a few non-basic columns. Lemma 3.2 can be proved using techniques in [18], but we provide a proof for completeness. We use the following observation:

\begin{equation}
\text{(3.3)} \quad \text{If } B \subseteq A \text{ is an optimal LP basis matrix, then } \|B^{-1}A\|_\infty \leq \frac{\delta}{|\det(B)|}.
\end{equation}

In order to show (3.3), assume to the contrary that there exists $a \in A$, $d \in B$, and $y \in \mathbb{R}^n$ such that $a = By$ and $y_d > \delta/|\det(B)|$. It follows that $|\det(B \cup \{d\} \setminus \{a\})| = |y_d| \cdot |\det(B)| > \delta$, which contradicts the definition of $\delta$.

**Lemma 3.2.** Let $B \subseteq A$ be an optimal LP basis. If $z \in N(B)$, then

$$
\|B^{-1}Az\|_\infty \leq \|B^{-1}A\|_\infty \cdot \|z\|_1 \leq \frac{\delta}{|\det(B)|} \cdot \|z\|_1 \leq \gamma^{-1} \cdot \delta.
$$

**Proof.** Write $z$ as $z = \sum_{i=1}^{t} x^i$, where $x^1, \ldots, x^t \in \mathbb{Z}_{\geq 0}$ are standard unit vectors and satisfy $\|x^i\|_1 = \ldots = \|x^t\|_1 = 1$. There do not exist nested sets $I \subsetneq J \subseteq \{1, \ldots, t\}$ with $\sum_{i \in J} A x^i \equiv_{B} \sum_{i \in J} A x^i$ because $z \in N$. Hence, the equivalence class of $\sum_{i \not\in J} A x^i$ modulo $\equiv_B$ is distinct for nested subsets of $\{1, \ldots, t\}$. For each $J \subseteq \{1, \ldots, t\}$, the problem $\text{IP}(\sum_{i \not\in J} A x^i)$ is feasible. By (2.6) there are $\gamma^{-1} \cdot |\det(B)|$ many equivalence classes in $\Lambda/\Gamma^B$ that can be attained by the sums $\sum_{i \not\in J} A x^i$, where $\Lambda = A \cdot \mathbb{Z}^n$ is the lattice defined in (2.3). Hence,

$$
\|z\|_1 = t \leq \gamma^{-1} \cdot |\det(B)|.
$$

Using the latter inequality along with (3.3), it follows that

$$
\|B^{-1}Az\|_\infty \leq \|B^{-1}A\|_\infty \cdot \|z\|_1 \leq \frac{\delta}{|\det(B)|} \cdot \|z\|_1 \leq \gamma^{-1} \cdot \delta.
$$

Our next result states that if $b \in F$, then we can construct an optimal solution of $\text{IP}(b)$ using the matrix $B$ and the set $N$. Lemma 3.3 can also be proved using the techniques in [18], but we provide a proof for completeness.

**Lemma 3.3.** Let $B \subseteq A$ be an optimal LP basis matrix. If $b \in F(B)$, then there exists an optimal solution of $\text{IP}(b)$ of the form $z^B + z$, where $z \in N(B)$, $z^B \in \mathbb{Z}_{\geq 0}$, and $\{a \in A : z^B_a > 0\} \subseteq B$.

**Proof.** Let $z^*$ be optimal for $\text{IP}(b)$. Let $z \leq z^*$ satisfy $Az \equiv_B b$ and minimize $\|z\|_1$. Note that $z \in N$ and by Lemma 3.2 we have $\|B^{-1}Az\|_\infty \leq \gamma^{-1} \cdot \delta$. Using this and the fact that $b \in F$, we have

\begin{equation}
\text{(3.4)} \quad B^{-1} (b - Az) \geq 3\delta \cdot 1^m - \gamma^{-1} \cdot \delta \cdot 1^m = (3 - \gamma^{-1}) \delta \cdot 1^m \geq 0^m.
\end{equation}

We define $z^B \in \mathbb{Z}_{\geq 0}$ component-wise using the columns of $A$. Define $z^B_a := [B^{-1}(b - Az)]_a$ for $a \in B$ and $z^B_a := 0$ for $a \in A \setminus B$. Note that $\{a \in A : z^B_a > 0\} \subseteq B$ and $z^B \in \mathbb{Z}_{\geq 0}$ because $Az \equiv_B b$. By Lemma 3.1 and (3.4), $B$ is an optimal LP basis matrix. Hence, $z^B$ is optimal for $\text{LP}(b - Az)$. Also, $z^* - z$ is feasible for $\text{IP}(b - Az)$, so $c^T(z^* - z) \leq c^T z^B$. Thus, $z^B + z$ is optimal for $\text{IP}(b)$.

Lemma 3.3 states that the set $N$ and the basis $B$ can be used as building blocks to construct optimal solutions for $\text{IP}(b)$ when $b \in F$. It turns out that these building
blocks satisfy additional structural properties that we will use to prove Theorems 1.1 and 1.2. In particular, the converse of Lemma 3.3 holds: if \( z \in N \) and \( b \in F \) such that \( Az \equiv_B b \), then there exists an optimal solution to IP(b) of the form \( z^B + z \), see Lemma 3.4 (a). Also, \( N \) is down-closed, that is if \( z \in N \) and \( z \) satisfies \( \zeta \leq z \), then \( \zeta \in N \), see Lemma 3.4 (b). Finally, we can construct vectors in \( N \) from other vectors in \( N \) by swapping subvectors that are equivalent modulo \( \equiv_B \), see Lemma 3.4 (c).

**Lemma 3.4.** Let \( B \) be an optimal LP basis matrix and \( z \in N(B) \).

(a) If \( b \in F(B) \) satisfies \( Az \equiv_B b \), then there exists \( z^B \in \mathbb{Z}_{\geq 0}^n \) such that \( \{a \in A : z_a^B > 0\} \subseteq B \) and \( z + z^B \) is an optimal solution of IP(b).

(b) Each vector \( z \in \mathbb{Z}^n \) with \( 0^n \leq z \leq z \) is in \( N(B) \).

(c) For every \( \zeta, \eta \in N(B) \) with \( \zeta \leq z \) and \( A\eta \equiv_B A\zeta \) there exists \( y \in N(B) \) with \( y \leq z + (\eta - \zeta) \) and \( Ay \equiv_B A\zeta \).

**Proof.** The definition of \( N \) implies that there exists a right hand side \( b \in F \) and an optimal solution to IP(b) of the form \( z^B + z \), where \( \{a \in A : z_a^B > 0\} \subseteq B \). For the proofs of (a), (b), (c), consider any \( \zeta \in \mathbb{Z}^n \) and \( b \in F \) with \( 0^n \leq \zeta \leq z \) and \( A\zeta \equiv_B b \). By Lemma 3.3 there is an optimal solution to IP(b) of the form \( \eta^B + \eta \), where \( \eta \in N \) and \( \eta^B \) satisfies \( \{a \in A : \eta_a^B > 0\} \subseteq B \). Note that \( A\zeta \equiv_B A\eta \).

The relation \( A\zeta \equiv_B A\eta \) implies that there exists \( u \in \mathbb{Z}^n \) such that

\[
\{a \in A : u_a \neq 0\} \subseteq B \quad \text{and} \quad A(\zeta - \eta + u) = 0^n.
\]

We have

\[
\|u\|_\infty = \|B^{-1}A(\zeta - \eta)\|_\infty \leq \|B^{-1}A\zeta\|_\infty + \|B^{-1}A\eta\|_\infty \\
\leq \|B^{-1}A\|_\infty \|\zeta\|_1 + \|B^{-1}A\eta\|_\infty \\
\leq \|B^{-1}A\|_\infty \|\zeta\|_1 + \|B^{-1}A\eta\|_\infty \\
\leq 2\gamma^{-1} \cdot \delta,
\]

where the last inequality follows from (3.3) and Lemma 3.2. The fact that \( b, b \in F \) implies that \( |B^{-1}(b - Az)|_a \geq (3 - \gamma^{-1})\delta \) and \( |B^{-1}(B - A\eta)|_a \geq (3 - \gamma^{-1})\delta \) for every \( a \in B \). Hence, for every \( a \in B \) it follows that

\[
z_a^B - u_a = |B^{-1}(b - Az)|_a - u_a \geq (3 - \gamma^{-1})\delta - 2\gamma^{-1} \cdot \delta \geq 0 \quad \text{and}
\]

\[
\eta_a^B + u_a = |B^{-1}(B - A\eta)|_a + u_a \geq (3 - \gamma^{-1})\delta - 2\gamma^{-1} \cdot \delta \geq 0.
\]

Equations (3.5) and (3.6) imply that \( (z + z^B) - (\zeta - \eta + u) = (z^B - u) + (z + (\eta - \zeta)) \) is feasible for IP(b) and \( (\eta^B + \eta) + (\zeta - \eta + u) = (\eta^B + u) + z \) is feasible for IP(b). If \( c^T(\zeta - \eta + u) < 0 \), then \( z + z^B \) is not optimal for IP(b), which is a contradiction. If \( c^T(\zeta - \eta + u) > 0 \), then \( \eta^B + \eta \) is not optimal for IP(b), which is also a contradiction. Hence, \( c^T(\zeta - \eta + u) = 0 \). It follows that

\[
(\eta^B + u) + (z - \eta + u) = (\eta^B + u) + z \quad \text{is optimal for IP(b)}
\]

and

\[
(z + z^B) - (\zeta - \eta + u) = (z^B - u) + (z + (\eta - \zeta)) \quad \text{is optimal for IP(b)}.
\]

Recall that \( z \) was any vector satisfying \( \zeta \leq z \). In particular, if \( \zeta = z \), then we have \( b = b \). By (3.7) \( (\eta^B + u) + z \) is optimal for IP(b). This proves (a). We can
also use the fact that \((\mathbf{y}^B + \mathbf{u}) + \mathbf{z}\) is optimal for \(\text{IP}(\mathbf{b})\) to prove (b). Indeed, the relationship \(\mathbf{z} \leq \mathbf{z}\) and the vectors \(\mathbf{b}\) and \((\mathbf{y}^B + \mathbf{u}) + \mathbf{z}\) certify that \(\mathbf{z} \in N\). Hence, (b) holds. Finally, (3.8) shows that \((\mathbf{z}^B - \mathbf{u}) + (\mathbf{y} + (\mathbf{y} - \mathbf{z}))\) is optimal for \(\text{IP}(\mathbf{b})\). Take any vector \(\mathbf{y} \in \mathbb{Z}_{\geq 0}^n\) that minimizes \(\|\mathbf{y}\|\) while satisfying \(\mathbf{y} \leq \mathbf{z} + (\mathbf{y} - \mathbf{z})\) and \(A\mathbf{y} \equiv_B \mathbf{b}\). The vector \(\mathbf{y}\) is in \(N\), which proves (c).

4. Results about the sparsity function \(\sigma\).

Proof of Theorem 1.1. Let \(\Lambda = A \cdot \mathbb{Z}^m\) as in (2.3). Let \(k \in \{0, \ldots, \lfloor \log_2(\gamma^{-1} \cdot \delta) \rfloor \}\). We find \(A^1, \ldots, A^s\) satisfying (2.4) and \(d_1, \ldots, d_s\) satisfying (2.7), such that

\[(4.1) \quad |X^i(\sigma, m + k)| \geq \min\{\gamma^{-1} \cdot |\det(A^i)|, 2^k\} \quad \forall \, i \in \{1, \ldots, s\},\]

where \(X^i(\sigma, m + k)\) is defined in (2.8). The result then follows from Theorem 2.4.

Assumption (1.1) implies that \(\text{LP}(\mathbf{b})\) has an optimal vertex solution for every \(\mathbf{b} \in \mathbb{Z}^m \cap \text{cone}(A)\), and the supports of these vertex solutions are contained in optimal LP bases. Hence,

\[\text{cone}(A) = \bigcup_{B \text{ an optimal LP basis matrix}} \text{cone}(B).\]

Let \(A^1, \ldots, A^s \subseteq A\) be any set of optimal LP basis matrices that satisfy (2.4). For each \(i \in \{1, \ldots, s\}\), let \(F^i = F(A^i)\) and \(N^i = N(A^i)\) be defined in (3.1) and (3.2).

Let \(i \in \{1, \ldots, s\}\). Recall the lattice \(\Gamma^i = A^i \cdot \mathbb{Z}^m\) defined in (2.5) and the corresponding equivalence relation \(\equiv_{\Gamma^i}\). For each \(g \in \Lambda / \Gamma^i\) apply Lemma 2.2 with \(C = \text{cone}(A^i)\) to obtain \(b^{i,g} \in \mathbb{Z}^m\) such that

\[(4.2) \quad b^{i,g} \equiv_{\Gamma^i} g \quad \text{and} \quad b^{i,g} \in \text{cone}(A^i) \cap \bigcap_{z \in N^i} \left[ \text{cone}(A^i) + [A^i + A^i(4\delta \cdot 1^m)] \right].\]

Note that \(0^m \in N^i\). Therefore, \(b^{i,g} \in \text{cone}(A^i) + A^i(4\delta \cdot 1^m) \subseteq F^i\). This implies \(\text{cone}(A^i) + b^{i,g} \subseteq F^i\). Let \(d^i \in \mathbb{Z}^m\) be any vector satisfying

\[d^i \equiv_{\Gamma^i} 0^m \quad \text{and} \quad d^i \in \bigcap_{g \in \Lambda / \Gamma^i} \left[ \text{cone}(A^i) + b^{i,g} \right].\]

The vectors \(d^1, \ldots, d^s\) satisfy (2.7).

We complete the proof of (4.1) in two cases.

Case 1. For every \(g \in \Lambda / \Gamma^i\) assume that there exists \(z^g \in N^i\) such that \(A(z^g) \equiv_{\Gamma^i} g\) and \(|\text{supp}(z^g)| < k\). We will show that each \(g\) is in \(X^i(\sigma, m + k)\).

Let \(g \in \Lambda / \Gamma^i\) and \(b \in \mathbb{Z}^m \cap \text{cone}(A^i) + d^i\) such that \(b \equiv_{\Gamma^i} g\). By Lemma 3.3 with \(B = A^i\), there exists an optimal solution of \(\text{IP}(\mathbf{b})\) of the form \(z^{A^i} + z^g\), where \(\{a \in A : z_{a}^{A^i} > 0\} \subseteq A^i\). Thus,

\[\sigma(b) \leq |\text{supp}(z^{A^i})| + |\text{supp}(z^g)| < m + k.\]

By (2.6) we have \(|\Lambda / \Gamma^i| = \gamma^{-1} \cdot |\det(A^i)|\). Thus, \(g \in X^i(\sigma, m + k)\). As \(g\) was arbitrarily chosen in \(\Lambda / \Gamma^i\), we have

\[|X^i(\sigma, m + k)| = |\Lambda / \Gamma^i| = \gamma^{-1} \cdot |\det(A^i)| \geq \min\{\gamma^{-1} \cdot |\det(A^i)|, 2^k\}.\]

Case 2. Assume that there exists \(g \in \Lambda / \Gamma^i\) such that \(|\text{supp}(z^g)| \geq k\) for every vector \(z^g \in N^i\) satisfying \(A(z^g) \equiv_{\Gamma^i} g\). Choose \(z^g \in N^i\) that minimizes \(|\text{supp}(z^g)|\) over all vectors in \(N^i\) satisfying \(A(z^g) \equiv_{\Gamma^i} g\). We claim that

\[(4.4) \quad \text{if } \mathbf{y}, \mathbf{z} \in \mathbb{Z}_{\geq 0}^n \text{ are distinct vectors satisfying } \mathbf{y}_n \leq \mathbf{z}, \text{ then } A\mathbf{y} \not\equiv_{\Gamma^i} A\mathbf{z}.\]
Assume to the contrary that there are distinct vectors \( \mathbf{y}, \mathbf{z} \leq \mathbf{z}^* \) such that \( A\mathbf{y} \equiv A\mathbf{z} \). Without loss of generality \( 0^n \leq \mathbf{z} \). We also assume that \( \text{supp}(\mathbf{y}) \cap \text{supp}(\mathbf{z}) = \emptyset \) (this can be enforced by replacing \( \mathbf{y} \) by the vector \( \tilde{\mathbf{y}} \) defined by \( \tilde{\mathbf{y}}_a = \mathbf{y}_a = 0 \) if \( \mathbf{z}_a > 0 \)). Lemma 3.4 (b) states that \( \mathbf{y}, \mathbf{z} \in N^i \). Thus, we can apply Lemma 3.4 (c) with \( \mathbf{z} = \mathbf{z}^* \) to conclude that there exists \( \mathbf{y}^1 \in N^i \) such that \( \mathbf{y}^1 \leq \mathbf{z}^* + (\mathbf{y} - \mathbf{z}) \) and \( A\mathbf{y}^1 \equiv A\mathbf{z} \). Note that \( \text{supp}(\mathbf{y}^1) \subseteq \text{supp}(\mathbf{z}^*) \) because \( 0 \leq \mathbf{z} \) and \( \text{supp}(\mathbf{y}) \cap \text{supp}(\mathbf{z}) = \emptyset \), there exists some index \( j \in \text{supp}(\mathbf{z}) \) such that \( y^1_j \leq |\mathbf{z}^* + (\mathbf{y} - \mathbf{z})|_j = |\mathbf{z}^* - \mathbf{z}|_j < \mathbf{z}^*_j \). The previous argument can be repeated with \( \mathbf{y}^1 \) in place of \( \mathbf{z}^* \) to obtain a vector \( \mathbf{y}^2 \in N^i \) such that \( \text{supp}(\mathbf{y}^2) \subseteq \text{supp}(\mathbf{y}^1) \), \( \mathbf{y}^2 \leq \mathbf{y}^1 + (\mathbf{y} - \mathbf{z}), A\mathbf{y}^2 \equiv A\mathbf{y}^1 \), and \( \mathbf{y}^2 < \mathbf{y}^1 \). Repeating this argument \( t = k \) many times yields a vector \( \mathbf{y}^i \in N^i \) with \( A\mathbf{y}^i \equiv A\mathbf{z} \), \( \text{supp}(\mathbf{y}^i) \subseteq \text{supp}(\mathbf{z}^*) \), and \( \mathbf{y}^i = 0 \). However, this contradicts the minimal support property of \( \mathbf{z}^* \). This proves (4.4).

Define the set

\[
H := \{ \mathbf{h} \in \Lambda / \Gamma^i : \mathbf{h} \equiv A\mathbf{z} \text{ for some } \mathbf{z} \in \mathbb{N}^n \text{ with } 0^n \leq \mathbf{z} \leq \mathbf{z}^* \text{ and } |\text{supp}(\mathbf{z})| \leq k \}.
\]

By (4.4), there are at least \( 2^k \) elements in \( H \). Let \( \mathbf{b} \in \mathbb{N}^n \cap \{ \text{cone}(A^i) + \mathbf{d}^i \} \) such that \( \mathbf{b} \equiv A\mathbf{h} \) for some \( \mathbf{h} \in H \). There exists \( \mathbf{z} \in \mathbb{N}^n \) such that \( 0^n \leq \mathbf{z} \leq \mathbf{z}^* \) and \( A\mathbf{z} \equiv \mathbf{b} \). The definition of \( N^i \) and Lemma 3.4 (a) with \( B = A^i \) imply that there exists an optimal solution to \( \text{IP}(\mathbf{b}) \) of the form \( \mathbf{z} + \mathbf{z}^* \), where \( \{ \mathbf{a} \in A : \mathbf{z}^*_a > 0 \} \subseteq A^i \). Hence,

\[
\sigma(\mathbf{b}) \leq |\text{supp}(\mathbf{z} + A^i)| \leq |\text{supp}(\mathbf{z}^*)| + |\text{supp}(\mathbf{z})| \leq m + k.
\]

This implies that \( H \subseteq X^i(\sigma, m + k) \) and

\[
|X^i(\sigma, m + k)| \geq |H| \geq 2^k \geq \min\{\gamma^{-1} \cdot |\det(A^i)|, 2^k\}.
\]

This completes the proof of (4.1).

\[\square\]

5. Results about the proximity function \( \pi \).

Proof of Theorem 1.2. Let \( \Lambda = A \cdot \mathbb{N}^m \) as in (2.3). Let \( k \in \{0, \ldots, \gamma^{-1} \cdot \delta - 1\} \).

We find \( A^1, \ldots, A^s \) satisfying (2.4) and \( \mathbf{d}^1, \ldots, \mathbf{d}^s \) satisfying (2.7), such that

\[
|X^i(\pi, \gamma^{-1} \cdot \delta + k)| \geq \min\{\gamma^{-1} \cdot |\det(A^i)|, k + 1\} \quad \forall \ i \in \{1, \ldots, s\}.
\]

where \( X^i(\pi, m + k) \) is defined in (2.8). The result then follows from Theorem 2.4. Similarly, in order to prove (b) it is enough to show

\[
|X^i(\pi^*, \gamma^{-1} \cdot \delta + k)| \geq \min\{\gamma^{-1} \cdot |\det(A^i)|, k + 1\} \quad \forall \ i \in \{1, \ldots, s\}.
\]

We choose \( A^1, \ldots, A^s \) to be the optimal LP basis matrices of \( A \). Set \( \Gamma^i = A^i \cdot \mathbb{N}^m \) as in (2.5). For each \( i \in \{1, \ldots, s\} \), let \( F^i = F(A^i) \) and \( N^i = N(A^i) \) be as in (3.1) and (3.2). Let \( \{ \mathbf{b}^i : \mathbf{g} \in \Lambda / \Gamma^i \} \) be the vectors defined in (4.2) and \( \mathbf{d}^i \) the vector defined in (4.3). It was shown in the proof of Theorem 1.1 that \( A^1, \ldots, A^s \) satisfy (2.4) and \( \mathbf{d}^1, \ldots, \mathbf{d}^s \) satisfy (2.7). It is left to show (5.1) and (5.2).

Let \( i \in \{1, \ldots, s\} \). Let \( \mathbf{g} \in \Lambda / \Gamma^i, \mathbf{z}^* \in N^i \) satisfy \( A\mathbf{z}^* \equiv A\mathbf{g} \), and \( \mathbf{b} \in \text{cone}(A^i) + \mathbf{d}^i \) satisfy \( \mathbf{b} \equiv A\mathbf{g} \). The definition of \( N^i \) and Lemma 3.4 (a) with \( B = A^i \) imply that there exists an optimal solution to \( \text{IP}(\mathbf{b}) \) of the form \( \mathbf{z}^* + \mathbf{z}^* \), where \( \{ \mathbf{a} \in A : \mathbf{z}^*_a > 0 \} \subseteq A^i \). Let \( \mathbf{x}^* \) be the optimal vertex solution of the linear program \( \text{LP}(\mathbf{b}) \) with
\{a \in A : x^*_a > 0\} \subseteq A^i. Observe that
\[
\pi(b) \leq \|x^* - (z^{A^i} + z^g)\|_1 \leq \|x^* - z^{A^i}\|_1 + \|z^g\|_1 \\
= \|(A^i)^{-1}[A(x^* - z^{A^i})]\|_1 + \|z^g\|_1 \\
= \|(A^i)^{-1}Az^g\|_1 + \|z^g\|_1 \\
\leq m \cdot \|(A^i)^{-1}Az^g\|_\infty + \|z^g\|_1 \\
\leq m \cdot \frac{\delta}{|\det(A^i)|} + \|z^g\|_1 \\
\leq m \cdot \gamma^{-1} \cdot \delta + \|z^g\|_1,
\]
(5.3)
where the first equation follows from the inclusions \(\{a \in A : x^*_a > 0\} \subseteq A^i\), and the last two inequalities follow from Lemma 3.2 with \(B = A^i\). If we replace the \(\ell^i\)-norm in (5.3) with the \(\ell^\infty\)-norm, then we obtain the similar inequalities
\[
\pi^\infty(b) \leq \|(A^i)^{-1}Az^g\|_\infty + \|z^g\|_1 \leq \|z^g\|_1 \cdot \frac{\delta}{|\det(A^i)|} + \|z^g\|_1 \leq \gamma^{-1} \cdot \delta + \|z^g\|_1.
\]
(5.4)
We prove (5.1) and complete the proof of Theorem 1.2 (a) with two cases. We omit the details of the proof of (5.2) as they are the same with (5.3) replaced with (5.4).

**Case 1.** Assume that for each \(g \in \Lambda/T^i\) there exists \(z^g \in N^i\) such that \(Az^g \equiv_{T^i} g\) and \(\|z^g\|_1 \leq k\). We claim that \(\Lambda/T^i = X^i(\pi, m\gamma^{-1} \cdot \delta + k)\). Let \(g \in \Lambda/T^i\) and \(b \in \mathbb{Z}^m \cap [\text{cone}(A^i) + d^i]\) such that \(b \equiv_{T^i} g\). Equation (5.3) implies
\[
\pi(b) \leq m\gamma^{-1} \cdot \delta + k.
\]
Hence, \(g \in X^i(\pi, m\gamma^{-1} \cdot \delta + k)\). By (2.6) we have
\[
|X^i(\pi, m\gamma^{-1} \cdot \delta + k)| = |\Lambda/T^i| = \gamma^{-1} \cdot |\det(A^i)| \geq \min\{\gamma^{-1} \cdot |\det(A^i)|, k + 1\}.
\]
This proves (5.1).

**Case 2.** Assume that there exists \(g^* \in \Lambda/T^i\) such that \(\|z^{g^*}\|_1 > k\) for all \(z^g \in N^i\) with \(Az^g \equiv_{T^i} g^*\). Fix \(z \in \mathbb{Z}^n\) to be any vector satisfying \(0^n \leq z \leq z^{g^*}\) and \(\|z\|_1 = k\). Consider the set
\[
H := \{g \in \Lambda/T^i : h \equiv_{T^i} A\mathbf{z} \text{ for } \mathbf{z} \in \mathbb{Z}^n \text{ with } 0^n \leq \mathbf{z} \leq z\}.
\]
By Lemma 3.4 (b) and the fact that \(z \leq z^{g^*}\), if \(\mathbf{z} \in \mathbb{Z}^n\) and \(0^n \leq \mathbf{z} \leq z\), then \(\mathbf{z} \in N^i\). By the definition of \(N^i\), if distinct \(\mathbf{z}, \tilde{\mathbf{z}} \in \mathbb{Z}^n\) satisfy \(0^n \leq \mathbf{z} \leq \tilde{\mathbf{z}} \leq z\), then \(A\mathbf{z} \equiv_{T^i} A\tilde{\mathbf{z}}\). Hence, \(H\) has at least \(|z|_1 + 1\) many elements.

We claim \(H \subseteq X^i(\pi, m\gamma^{-1} \cdot \delta + k)\), which proves
\[
|X^i(\pi, m\gamma^{-1} \cdot \delta + k)| \geq |H| \geq |z|_1 + 1 = k + 1 \geq \min\{\gamma^{-1} \cdot |\det(A^i)|, k + 1\}
\]
and completes the proof of (5.1). Let \(g \in H\) and \(\mathbf{z} \in \mathbb{Z}^n\) satisfy \(0^n \leq \mathbf{z} \leq z\) and \(g \equiv_{T^i} A\mathbf{z}\). Let \(b \in \mathbb{Z}^m \cap [\text{cone}(A^i) + d^i]\) such that \(b \equiv_{T^i} Ag\). By (5.3) with \(z^g = \mathbf{z}\) it follows that
\[
\pi(b) \leq m\gamma^{-1} \cdot \delta + |\mathbf{z}|_1 \leq m\gamma^{-1} \cdot \delta + |z^g|_1 \leq m\gamma^{-1} \cdot \delta + k.
\]
Hence, \(g \in X^i(\pi, m\gamma^{-1} \cdot \delta + k)\).
Remark 5.1. In the beginning of Section 1.2 we made the assumption that the optimal solution LP(b) is unique for all feasible b. If this assumption is dropped, then the definition of proximity would need to be refined into two cases: the minimum distance between an optimal LP vertex solution and an optimal IP solution

\[ \pi_{\text{min}}(b) := \min_{x^*} \min_{z^*} \left\{ \|x^* - z^*\|_1 : x^* \text{ is an optimal vertex solution of LP(b)} \right\}, \]

and the maximum of the minimum distance between LP optimal vertices and IP optimal solutions

\[ \pi_{\text{max}}(b) := \max_{x^*} \min_{z^*} \left\{ \|x^* - z^*\|_1 : x^* \text{ is an optimal vertex solution of LP(b)} \right\}. \]

If IP(b) is infeasible, then \( \pi_{\text{min}}(b) = \pi_{\text{max}}(b) = \infty \). Observe that \( \pi_{\text{min}}(b) \leq \pi_{\text{max}}(b) \) for all feasible b. The value \( \pi_{\text{min}}(b) \) can be bounded by considering only one solution of LP(b) while \( \pi_{\text{max}}(b) \) needs to consider every optimal vertex of LP(b). It follows immediately from Theorem 1.2 that

\[ \Pr(\pi_{\text{min}} \leq m\gamma^{-1} \cdot \delta + k) \geq \frac{k + 1}{\gamma^{-1} \cdot \delta}, \text{ for all } k \in \{0, \ldots, \gamma^{-1} \cdot \delta - 1\}. \]

It is not clear if \( \pi_{\text{max}}(b) \) can be bounded in the same way. However, for the extreme case \( k = \gamma^{-1} \cdot \delta - 1 \) it can be shown that

\[ \Pr(\pi_{\text{max}} \leq (m + 1)\gamma^{-1} \cdot \delta - 1) = 1. \]

The proof is similar to that of Theorem 1.2 and is omitted here.

Acknowledgments. The authors would like to thank Laurence Wolsey for providing references on the asymptotic behavior of the IP value function.

REFERENCES

[1] D. Adjiashvili, T. Oertel, and R. Weismantel, A polyhedral Frobenius theorem with applications to integer optimization, SIAM Journal on Discrete Mathematics, 29 (2015), pp. 1287–1302.

[2] I. Aliev, M. Henk, and T. Oertel, Distances to lattice points in knapsack polyhedra, Mathematical Programming, (2019), pp. 808–816.

[3] I. Aliev, J. De Loera, F. Eisenbrand, T. Oertel, and R. Weismantel, The support of integer optimal solutions, SIAM Journal on Optimization, 28 (2018), pp. 2152–2157.

[4] I. Aliev, J. De Loera, T. Oertel, and C. O’Neil, Sparse solutions of linear diophantine equations, SIAM Journal on Applied Algebra and Geometry, 1 (2017), pp. 239–253.

[5] N. Alon, R. Panigrahy, and S. Yekhanin, Deterministic approximation algorithms for the nearest codeword problem, in Dinur, Jansen K., Naor J., Rolim J. (eds) Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX 2009, RANDOM 2009, vol. 5687, 2009.

[6] A. Barvinok, A Course in Convexity, vol. 54, Graduate Studies in Mathematics, American Mathematical Society, Providence, Rhode Island, 2002.

[7] C. Blair and R. Jeroslow, The value function of a mixed integer program: I, Discrete Mathematics, 19 (1977), pp. 121–138.

[8] C. Blair and R. Jeroslow, The value function of a mixed integer program: II, Discrete Mathematics, 25 (1979), pp. 7–19.

[9] W. Bruns and J. Gubeladze, Normality and covering properties of affine semigroups, Journal für die reine und angewandte Mathematik, 510 (2004), pp. 151 – 178.

[10] W. Bruns, J. Gubeladze, M. Henk, A. Martin, and R. Weismantel, A counterexample to an integer analogue of Carathéodory’s theorem, Journal für die reine und angewandte Mathematik, 510 (1999), pp. 179–185.
[11] J. Cho, Y. Chen, and Y. Ding, *On the (co)girth of a connected matroid*, Discrete Applied Mathematics, 155 (2007), pp. 2456–2470.
[12] W. Cook, J. Fonlupt, and A. Schrijver, *An integer analogue of Carathéodory’s theorem*, Journal of Combinatorial Theory, Series B, 40 (1986), pp. 63–70.
[13] W. Cook, A. Gerards, A. Schrijver, and E. Tardos, *Sensitivity theorems in integer linear programming*, Mathematical Programming, 34 (1986), pp. 251–264.
[14] M. Dyer and A. Frieze, *Probabilistic analysis of the multidimensional knapsack problem*, Mathematics of Operations Research, (1989), pp. 564–568.
[15] F. Eisenbrand and G. Shmonin, *Carathéodory bounds for integer cones*, Operations Research Letters, 34 (2006), pp. 564–568.
[16] F. Eisenbrand and G. Shmonin, *Parametric integer programming in fixed dimension*, Mathematics of Operations Research, 33 (2008).
[17] F. Eisenbrand and R. Weismantel, *Proximity results and faster algorithms for integer programming using the Steinitz lemma*, in Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms, 2018, pp. 808–816.
[18] R. Gomory, *On the relation between integer and noninteger solutions to linear programs*, Proceedings of the National Academy of Sciences, 53 (1965), pp. 260–265.
[19] P. M. Gruber, *Convex and discrete geometry*, vol. 336 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Berlin, 2007.
[20] P. M. Gruber and C. G. Lekkerkerker, *Geometry of numbers*, vol. 37 of North-Holland Mathematical Library, North-Holland Publishing Co., Amsterdam, second ed., 1987.
[21] M. Henk and E. Linke, *Note on the coefficients of rational Ehrhart quasi-polynomials of Minkowski-sums*, Online J. Anal. Comb., (2015), p. 12.
[22] R. A. Horn and C. R. Johnson, *Matrix Analysis 2nd Edition*, Cambridge University Press New York, NY, USA, 2012.
[23] K. Jansen and L. Rohwedder, *On integer programming and convolution*, in 10th Innovations in Theoretical Computer Science (ITCS 2019), 2018, pp. 43:1–43:7.
[24] N. Karmarkar and R. Karp, *An efficient approximation scheme for the one-dimensional bin-packing problem*, in 23rd annual symposium on foundations of computer science (Chicago, Ill. 1982), 1982, pp. 312 – 320.
[25] P. McMullen, *Lattice invariant valuations on rational polytopes*, Arch. Math. (Basel), 31 (1978/79), pp. 509–516, https://doi.org/10.1007/BF01226481.
[26] T. Oertel, J. Paat, and R. Weismantel, *Sparsity of integer solutions in the average case*, in Proceedings of the 20th Integer Programming and Combinatorial Optimization Conference, 2019.
[27] J. Paat, R. Weismantel, and S. Weltge, *Distances between optimal solutions of mixed-integer programs*, Mathematical Programming, https://doi.org/10.1007/s10107-018-1323-z (2018).
[28] C. H. Papadimitriou, *On the complexity of integer programming*, J. Assoc. Comput. Mach., 28 (1981), pp. 765–768.
[29] A. Schrijver, *Theory of linear and integer programming*, John Wiley & Sons, Inc. New York, NY, 1986.
[30] A. Schrijver, *Hilbert bases, Carathéodory’s theorem and combinatorial optimization*, in Proceedings of the 1st Integer Programming and Combinatorial Optimization Conference, 1990, pp. 431–455.
[31] A. Strömbergsson, *On the limit distribution of Frobenius numbers*, Acta Arithmetica, (2012), pp. 81–107.
[32] A. Vardy, *The intractability of computing the minimum distance of a code*, IEEE Transactions on Information Theory, (1997), pp. 1757 – 1766.
[33] L. Wolsey, *The b-hull of an integer program*, Discrete Applied Mathematics, 3 (1981), pp. 193–201.
[34] G. M. Ziegler, *Lectures on Polytopes*, vol. 152 of Graduate Texts in Mathematics, Springer-Verlag New York, 1995.