Extended Mittag-Leffler function, series and its sum

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Abstract. This paper introduces first order extended Mittag-Leffler function by using generalized polynomial factorials. By applying generalized $\ell$–difference operator, we obtain a formula for the series having extended Mittag-Leffler functions which has several applications in fractional calculus. Suitable examples and numerical verification by MATLAB are inserted to validate our findings.

1. Introduction

Though the discovery of Mittag-Leffler functions have its origin more than 10 decades, for the past two decades the Mittag-Leffler functions has come into limelight. Mittag-Leffler function gained its momentum because of the numerous applications in the fields of physical, biological, engineering, and earth sciences [4]. With usual notations, the Mittag-Leffler function is arrived from a special function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \alpha k)}, \alpha \in \mathbb{C}, \mathbb{R}(\alpha) > 0, z \in \mathbb{C}. \quad (1)$$

and its general form is

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\beta + \alpha k)}, \alpha, \beta \in \mathbb{C}, \mathbb{R}(\alpha) > 0, \mathbb{R}(\beta) > 0, z \in \mathbb{C}. \quad (2)$$

The ordinary and generalized Mittag-Leffler functions alter between the exponential law and power law monitored by ordinary kinetic equations and their fractional counterparts (refer Hilfer [5, 6], Lang [7] and Saxena [9]).

We apply difference operators on fractional order Extended Mittag-Leffler (EML) function, this EML function is an extension of recently established extorial function [1] defined as

$$e_\nu(k_\ell) = 1 + \frac{k_\ell^{(\nu)}}{\Gamma(1\nu + 1)} + \frac{k_\ell^{(2\nu)}}{\Gamma(2\nu + 1)} + \frac{k_\ell^{(3\nu)}}{\Gamma(3\nu + 1)} + \cdots + \infty. \quad (3)$$

The formal definition of Extended Mittag-Leffler (EML) function is given in Section (3). The Extended Mittag-Leffler (EML) function plays a vital role and gives more applications in fractional calculus and fractional order difference equation. The main purpose is to find solutions...
of certain type of fractional difference equation and a summation formula for finding the exact values of certain series of Mittag-Leffler functions. For the first time, with the introduction of EML function, we establish four types of $\ell$-factorials of fractional order in this research using inverse of $\ell$–difference operator.

2. Basic Definitions and Preliminaries

This section includes some important definitions of polynomial factorials, extorial functions and lemmas for the subsequent sections and in the theory of EML functions in fractional calculus.

**Definition 2.1** [8] Let $\ell > 0$ and $\nu \in (−\infty, \infty)$. For $k \in (−\infty, \infty)$, the $\ell$-polynomial factorial is defined by

$$ k_{\ell}^{(\nu)} = \ell^{\nu} \frac{\Gamma(k/\ell + 1)}{\Gamma(k/\ell + 1 - \nu)}, \quad k/\ell + 1 - \nu \notin -N(0), $$

where $-N(0) = \{0, -1, -2, \cdots\}$ and $\Gamma$ is the gamma function.

**Special Cases:**

(i) If $\nu = r \in N(0)$, then (4) can be expressed as

$$ k_{\ell}^{(r)} = \ell^{r} \frac{\Gamma(k/\ell + 1)}{\Gamma(k/\ell + 1 - r)}, \quad k/\ell + 1 - r \notin -N(0), $$

which is same as

$$ k_{\ell}^{(r)} = k(k - \ell)(k - 2\ell) \cdots (k - (r - 1)\ell) = \prod_{j=0}^{r-1} (k - j\ell). \quad (5) $$

For example, $k_{3}^{(5)} = 3^{(5)} \frac{\Gamma(k/3 + 1)}{\Gamma(k/3 + 1 - 5)} = k(k - 3)(k - 6)(k - 9)(k - 12)$. (ii) If $\nu = -r \in N(0)$, i.e., $\nu = -r$, from (4), $k_{\ell}^{(-r)} = \ell^{-r} \frac{\Gamma(k/\ell + 1)}{\Gamma(k/\ell + 1 + r)}$.

By applying the property $\Gamma(y + 1) = y \Gamma(y)$ for $r$ times to the denominator of the above expression,

$$ k_{\ell}^{(-r)} = \prod_{j=1}^{r} (k + j\ell)^{-1}, (k + j\ell) \neq 0. \quad (6) $$

For example, $k_{3}^{(-5)} = 3^{(-5)} \frac{\Gamma(k/3 + 1)}{\Gamma(k/3 + 1 - 5)} = \prod_{j=1}^{5} (k + 3j)^{-1}$.

(iii) Now, replacing the $\ell$ by $-\ell$ in the equation (6), we get

$$ k_{-\ell}^{(r)} = k(k + \ell)(k + 2\ell) \cdots (k + (r - 1)\ell), $$

which motivates to define

$$ k_{(-\ell)}^{(\nu)} = \ell^{\nu} \frac{\Gamma(k/(-\ell) + \nu)}{\Gamma(k/(-\ell))}, \quad k/(-\ell) \notin -N(0), \nu \in (−\infty, \infty). \quad (7) $$

For example, $k_{3}^{(4)} = (4) \frac{\Gamma(k/3 + 4)}{\Gamma(k/3)} = k(k + \ell)(k + 2\ell)(k + 3\ell)$.

(iv) By replacing $\nu$ by $-\nu$ in (7), we get $k_{(-\ell)}^{(-r)} = \ell^{r} \Gamma(k/(-\ell) - \nu)/\Gamma(k/(-\ell))$ for $\nu, \ell > 0$.

For $\nu = r \in N(1)$, $k_{(-\ell)}^{(-r)} = \prod_{j=1}^{r} (k - j\ell)^{-1}, (k - j\ell) \neq 0$.

**Lemma 2.2** If $r \in N(0), \ell > 0$, then $(k - \ell)^{(r)} k_{(-\ell)}^{(-r)} = 1$.

**Proof:** The proof follows by replacing $k$ by $k - \ell$ in (5) and $k_{(-\ell)}^{(-r)} = \prod_{j=1}^{r} (k - j\ell)^{-1}$. 

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**Definition 2.3** [1] For $-1 < \ell < 1$ and $k, \nu \geq 0$, the $\ell$-extorial function denoted as $e_\nu(k_\ell)$ is defined as

$$e_\nu(k_\ell) = 1 + \frac{k^{(\nu)}}{\Gamma(\nu + 1)} + \frac{k^{(2\nu)}}{\Gamma(2\nu + 1)} + \frac{k^{(3\nu)}}{\Gamma(3\nu + 1)} + \cdots + \infty. \quad (8)$$

**Definition 2.4** [1] If $k^{(r\nu)} \neq 0$, for $r = 1, 2, \ldots$, and $\nu > 0$, $|\ell| \geq 1$, then the extorial function for negative index is defined as

$$e_{-\nu}(k_\ell) = 1 + \frac{1}{\Gamma(\nu + 1)} \frac{1}{(k + \ell)^{\nu}} + \frac{1}{\Gamma(2\nu + 1)} \frac{1}{(k + \ell)^{(2\nu)}} + \cdots + \infty. \quad (9)$$

### 3. Extended Mittag-Leffler Function

In this section, we define the Extended Mittag-Leffler (EML) function by introducing $c$ and $\ell$ in the existing Mittag-Leffler function and it is defined for the negative index with some special cases. This new function satisfies many difference equations, which will yield several applications.

**Definition 3.1** For $c \in [0, 1]$, $|\lambda| < 1$ and $|\ell| < 1$, the Extended Mittag-Leffler (EML) function for $k \in (-\infty, \infty)$ and $j\nu + 1 \notin N(0)$, is defined as

$$e_\nu(\lambda, k_\ell, c) = \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j\nu + 1)} (k - (a + \ell) + cj\lambda\nu)^{(j\nu)}_{\ell}. \quad (10)$$

**Special Cases:**

(i) When $\ell = 0$, $a = -1$, $\lambda = \nu = 1$, $e_1(1, k_0, c) = e^k$.

(ii) $e_\nu(\lambda, k_\ell, 0)$ is Extorial Function.

(iii) $e_\nu(\lambda, k_\ell, 1)$ is the existing Mittag-Leffler function.

**Example 3.2** (i) If $(k - (a + \ell))$ is the multiple of $\ell$ and $\nu$ is a positive integer, then we can take any value for $\lambda$.

(ii) If $c = -1, \nu = 1, a = -\ell$

$$e_\nu(\lambda, k_{-\ell}, 1) = 1 + \frac{\lambda^1}{\Gamma(1)} (k - \ell)^{(1)}_{-\ell} + \frac{\lambda^2}{\Gamma(2)} (k - 2\ell)^{(2)}_{-\ell} + \cdots = \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j)} (k - j\ell)^{(j)}_{-\ell}.$$

(iii) $e_1(\lambda, k_\ell, c) = \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j)} (k + j\lambda\nu)^{(j)}_{\ell}$, $|\lambda| < 1, |\ell| < 1$.

**Example 3.3** $e_1(\lambda, 10_{0.5}, 2) = \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j)} (10 + 2(0.5))_{0.5} = \sum_{j=0}^{\infty} \frac{\lambda^j}{\Gamma(j)} (10 + j)^{(j)}_{0.5}$

$$= 1 + \frac{\lambda^1}{\Gamma(1)} (11)_{0.5} + \frac{\lambda^2}{\Gamma(2)} (12)_{0.5} + \frac{\lambda^3}{\Gamma(3)} (13)_{0.5} + \cdots = 1 + \frac{\lambda^1}{\Gamma(1)} (11.5)_{0.5} + \frac{\lambda^2}{\Gamma(2)} (12.5)_{0.5} + \cdots$$

**Lemma 3.4** (Extended Mittag-Leffler Product Formula) For $k_1, k_2 \in (-\infty, \infty)$, if $|\ell| < 1$, $|\lambda| < 1$, $a = -\ell, c = 1$ and $\nu = 1$, then

$$e_1(\lambda, (k_1)_\ell, 1) e_1(\lambda, (k_2)_\ell, 1) = e_1(\lambda, (k_1 + k_2)_\ell, 1). \quad (11)$$

**Proof:** From the definition of EML function,

$$e_1(\lambda, (k_1)_\ell, 1) e_1(\lambda, (k_2)_\ell, 1) = \{1 + \frac{\lambda^1}{\Gamma(1)} (1 + k_1)^{2}_{\ell} + \frac{\lambda^2}{\Gamma(2)} (1 + k_2)^{3}_{\ell} + \cdots\}$$
Proof: (i) Since 

\[ u = \frac{\lambda(k+\ell)}{\Gamma(k+\ell)} \]

which is the same as

\[ \text{R.H.S} \]

In general, this theorem can be proved for \( \nu \in N(1) = \{1, 2, 3, \cdots\} \).

Example 3.5 Take \( k_1 = 2, k_2 = 1, \lambda = 0.1, \ell = 0.5 \) in the equation (11), we get

\[
L.H.S = 1 + (0.1)^{(3)}\frac{1}{2^1} + (0.01)^{(3)}(2)\frac{1}{2^1} + (0.001)\frac{1}{3^1} + (0.0001)\frac{1}{3^1} + (0.00001)\frac{1}{3^1} + (0.000001)\frac{1}{3^1} = 1.34095641.
\]

\[
R.H.S = \left[ 1 + (0.1)^{(2)}\frac{1}{2^1} + (0.01)^{(2)}(2)\frac{1}{2^1} + (0.001)^{(2)}(1)\frac{1}{3^1} + (0.0001)^{(2)}(1)\frac{1}{3^1} + (0.00001)\frac{1}{3^1} \right] \times \left[ 1 + (0.1)^{(1)}\frac{1}{2^1} + (0.01)^{(1)}(0)\frac{1}{3^1} \right] = (1.21550625)(1.1025) = 1.34095641.
\]

Remark 3.6 Let \( \nu \) be a non-negative integer and \( \lambda, k_1, k_2, \ell \) be positive real numbers. Then for \( k \) in \( N(1) \), we have

\[ \Delta_{\ell}^{-1} e_{\nu}(\lambda, k_1, k_2, \ell) = \sum_{r=1}^{[\frac{k}{\ell}]} e_{\nu}(\lambda, k_1, k_2, \ell - r\ell). \]

Theorem 3.7 (2) If \( u(k) \) is a real valued function defined in \( \mathbb{R} \), then

\[ \Delta_{\ell}^{-1} u(\ell) \bigg|_{\ell = j} = \sum_{r=1}^{[\frac{j}{\ell}]} u(k - r\ell), \quad \text{where } j = k - \lfloor \frac{k}{\ell} \rfloor \ell. \]

Theorem 3.8 Let \( e_{\nu}(\lambda, k_1, k_2, \ell), k \in [0, \infty) \) be a real valued function. Then for \( k \in \ell, \infty \), and \( j = k - \lfloor \frac{k}{\ell} \rfloor \ell \), we have

\[
\Delta_{\ell}^{-1} e_{\nu}(\lambda, k_1, k_2, \ell) \bigg|_{\ell = j} = \sum_{r=1}^{[\frac{j}{\ell}]} e_{\nu}(\lambda, k_1, k_2, \ell - r\ell). \tag{12}
\]

Proof: Taking \( u(k) = e_{\nu}(\lambda, k_1, k_2, \ell) \) in Theorem (3.7), we get the proof.

Lemma 3.10 For \( r \in N(1) \) and positive \( \ell \), we have the following identities.

(i) \( \Delta_{\ell} k_{\ell}^{(r)} = r(k+\ell)^{(r-1)} \), (ii) \( \Delta_{\ell} k_{\ell}^{(r)} = r\ell(k+\ell)^{(r-1)} \),

(iii) \( \Delta_{\ell} k_{\ell}^{(-r)} = -r\ell(k+\ell)^{\ell-(r+1)} \) and (iv) \( \Delta_{\ell} k_{\ell}^{(-r)} = -r\ell(k+\ell)^{\ell-(r+1)}. \)

Proof: (i) Since \( r \in N(1) \), from the Definition (2.1),

\[
\Delta_{\ell} k_{\ell}^{(r)} = \Delta_{\ell} \left[ \ell^{r\ell} \Gamma(k+\ell) \Gamma(k+\ell-1) \right] = \Delta_{\ell} \left[ \ell^{r\ell} \Gamma(k+\ell-(r+1)) \Gamma(k+\ell-(r+1)) \right] \]

which is same of RHS of (i).

The proofs of (ii), (iii), (iv) are simple and similar as the proof of (i).
Theorem 3.11 If \( m \in N(1), \ell \in (0, \infty), k \in [0, \infty) \) and \( c \in -N(1) \), then

\[
\left. \Delta^{-m}_\ell e_\nu(\lambda, k_\ell, c) \right|_{(m-1)\ell+j}^k = \sum_{r=m}^{\left\lfloor \frac{k}{\ell} \right\rfloor} \frac{(r-1)(m-1)}{(m-1)!} e_\nu(\lambda, (k-r\ell)_\ell, c).
\]

(13)

Proof: Taking \( \Delta^{-1}_\ell \) on (12) and applying (12) for \( \Delta^{-1}_\ell e_\nu(\lambda, (k-r\ell)_\ell, c) \), we get

\[
\Delta^{-2}_\ell e_\nu(\lambda, k_\ell, c) \left|_{\ell+j}^{k} \right. = \sum_{r=2}^{\left\lfloor \frac{k}{\ell} \right\rfloor} \frac{(r-1)(1)}{(1)!} e_\nu(\lambda, (k-r\ell)_\ell, c).
\]

(14)

Again taking \( \Delta^{-1}_\ell \) on (14), by (12) for \( \Delta^{-1}_\ell e_\nu(\lambda, (k-r\ell)_\ell, c) \), we get

\[
\Delta^{-3}_\ell e_\nu(\lambda, k_\ell, c) \left|_{2\ell+j}^{k} \right. = \sum_{r=3}^{\left\lfloor \frac{k}{\ell} \right\rfloor} \frac{(r-1)(2)}{(2)!} e_\nu(\lambda, (k-r\ell)_\ell, c).
\]

(15)

Now (13) will be obtained by continuing this process for \( m \) times.

Corollary 3.12 Let \( 0 < \ell < 1, |\lambda| < 1 \). An exact solution of the second linear \( \ell \)-difference equation \( \Delta^{-2}_\ell u(k) = e_1(\lambda, k_\ell, c) \) is given by

\[
\Delta^{-2}_\ell e_1(\lambda, k_\ell, c) \left|_{\ell+j}^{k} \right. = \left[ \frac{1}{(\lambda\ell)^2} e_1(\lambda, (k-2c)_\ell, c) - \frac{1}{\lambda\ell} e_1(\lambda, (j-c)_\ell, c) \right] \left( \frac{1}{1,\ell} \right)^k \left|_{\ell+j}^{k} \right.
\]

(16)

Proof: By applying the operator \( \Delta_\ell \) on the EML function, \( \Delta e_1(\lambda, k_\ell, c) = \Delta \left\{ 1 + \frac{\lambda}{\ell}(k-(a+\ell) + c\ell)_\ell^{(1)} + \frac{\lambda^2}{2!}(k-(a+\ell) + 2c\ell)_\ell^{(2)} + \frac{\lambda^3}{3!}(k-(a+\ell) + 3c\ell)_\ell^{(3)} + \cdots \right\} \).

Since \( \Delta_\ell \) is a linear operator, we have

\[
\Delta e_1(\lambda, k_\ell, c) = \left\{ 1 + \frac{\lambda}{\ell} \Delta_\ell(k-(a+\ell) + c\ell)_\ell^{(1)} + \frac{\lambda^2}{2!} \Delta_\ell(k-(a+\ell) + 2c\ell)_\ell^{(2)} + \frac{\lambda^3}{3!} \Delta_\ell(k-(a+\ell) + 3c\ell)_\ell^{(3)} + \cdots \right\}.
\]

(17)

Now consider,

\[
\Delta_\ell(k-(a+\ell) + c\ell)^{(1)}_\ell = (k-\ell-(a+\ell) + c\ell)^{(1)}_\ell - (k-(a+\ell) + c\ell)^{(1)}_\ell = k + \ell - (a+\ell) + c\ell - k + (a+\ell) - c\ell = \ell(1)
\]

Similarly,

\[
\Delta_\ell(k-(a+\ell) + 2c\ell)^{(2)}_\ell = (k+\ell-(a+\ell) + 2c\ell)^{(2)}_\ell - (k-(a+\ell) + 2c\ell)^{(2)}_\ell = 2\ell(k-(a+\ell) + 2c\ell)_\ell^{(1)}.
\]

In general, we find \( \Delta_\ell(k-(a+\ell) + r\ell c)^{(r)}_\ell = r\ell(k-(a+\ell) + r\ell c)_\ell^{(r-1)} \).

Substituting all the values in (17), we get

\[
\Delta e_1(\lambda, k_\ell, c) = \left\{ 0 + \frac{\lambda}{\ell}\ell(k-(a+\ell) + c\ell)^{(0)}_\ell + \frac{\lambda^2}{2!} 2\ell(k-(a+\ell) + 2c\ell)_\ell^{(1)} + \frac{\lambda^3}{3!} 3\ell(k-(a+\ell) + 3c\ell)_\ell^{(2)} + \cdots \right\}
\]

\[
\Delta^{-1}_\ell e_1(\lambda, k_\ell, c) = \lambda e_1(\lambda, (k+c)_\ell, c) \Rightarrow \Delta^{-1}_\ell e_1(\lambda, k_\ell, c) = \frac{1}{\lambda\ell} e_1(\lambda, (k-c)_\ell, c)
\]

\[
\Delta^{-1}_\ell e_1(\lambda, k_\ell, c) \left|_{\ell+j}^{k} \right. = \frac{1}{\lambda\ell} \left[ e_1(\lambda, (k-c)_\ell, c) - e_1(\lambda, (j-c)_\ell, c) \right]
\]
\[
\Delta_{\ell}^{-1}(e_1(\lambda,k_{\ell},c)^{k})_{\ell+j} = \frac{1}{\lambda^k}\left[\Delta_{\ell}^{-1}e_1(\lambda,(k-c)_{\ell},c) - e_1(\lambda,(j-c)_{\ell},c)\Delta_{\ell}^{-1}(1)_{\ell+j}\right]
\]
which gives (16).

**Remark 3.13** Corollary (3.12) is used to obtain formula for finding the value of finite EML series.

**Example 3.14** Consider the summation formula (16) for finding the value of finite series. By taking \(m = 2, \nu = \ell = \lambda = 1, a = -4, c = -1, k = 4, j = 0, k = 4\) (multiple of \(\ell\)), (14) becomes,

\[
\Delta_{\ell}^{-2}e_1(\lambda,k_{\ell},c)^{k}_{\ell+j} = \sum_{r=2}^{[\frac{r}{2}]} \frac{(r-1)_{(1)}}{(1)!}e_1(\lambda,(r-\ell)_{\ell},c)
\]

and the value of \(\Delta_{\ell}^{-2}e(\lambda,k_{\ell},c)^{k}_{\ell+j}\) is given in (16).

Substituting the values in the L.H.S of the summation formula (14), we get\[
\frac{1}{(1)^2}[e(1,6_{(1)},-1)] - 1[e(1,1_{(1)},-1)] = \frac{1}{4^{(1)}} \frac{4^{(1)}}{1^{(1)}} = 55
\]

Now consider,

\[
e(1,6_{(1)},-1) = 1 + \frac{8^{(1)}}{1^{(1)}} + \frac{7^{(2)}}{2^{(2)}} + \frac{6^{(3)}}{3^{(3)}} + \frac{5^{(4)}}{4^{(4)}} = 55, e(1,1_{(1)},-1) = 5, \frac{4^{(1)}}{1^{(1)}} = 4.
\]

\[
e(1,3_{(1)},-1) = 13, \frac{1^{(1)}}{1^{(1)}} = 1 and L.H.S of (19) = 55 - 20 - 13 + 5 = 27.
\]

Similarly by substituting the values, R.H.S of (19) = 8 + 10 + 9 = 27.

Note that LHS of (18) is an exact solution of the difference equation \(\Delta_{\ell}^2u(k) = e_\nu(\lambda,k_{\ell},c)\) and RHS is a numerical solution of the above difference equation.

**Theorem 3.15** Let the EML function \(e_\nu(\lambda,k_{\ell},c)\) and the polynomial factorial function \(k_{\ell}^{(n)}\) be two real valued functions. Then we have

\[
\Delta_{\ell}^{-1}[k_{\ell}^{(n)}e_\nu(\lambda,k_{\ell},c)] = k_{\ell}^{(n)}\Delta_{\ell}^{-1}e_\nu(\lambda,k_{\ell},c) - \Delta_{\ell}^{-1}[\Delta_{\ell}^{-1}e_\nu(\lambda,(k+\ell)_{\ell},c)\Delta_{\ell}k_{\ell}^{(n)}]
\]

**Proof:** From \(\Delta_{\ell}u(k) = u(k+\ell) - u(k)\), we find that

\[
\Delta_{\ell}[k_{\ell}^{(n)}\frac{1}{\Delta_{\ell}}e_\nu(\lambda,(k-c)_{\ell},c)] = \frac{1}{\Delta_{\ell}}e_\nu(\lambda,(k+\ell-c)_{\ell},c)\Delta_{\ell}k_{\ell}^{(n)} + k_{\ell}^{(n)}\frac{1}{\Delta_{\ell}}\Delta_{\ell}e_\nu(\lambda,(k-c)_{\ell},c).
\]

Now applying the inverse of generalized difference operator, we obtain

\[
\Delta_{\ell}^{-1}[k_{\ell}^{(n)}\frac{1}{\Delta_{\ell}}\Delta_{\ell}e_\nu(\lambda,(k-c)_{\ell},c)] = k_{\ell}^{(n)}\frac{1}{\Delta_{\ell}}\Delta_{\ell}e_\nu(\lambda,(k-c)_{\ell},c)
\]

\[
-\Delta_{\ell}^{-1}[\frac{1}{\lambda_{\ell}}e_\nu(\lambda,(k+\ell-c)_{\ell},c)\Delta_{\ell}k_{\ell}^{(n)}]
\]

The proof follows by taking \(e_\nu(\lambda,k_{\ell},c) = \frac{1}{\Delta_{\ell}}\Delta_{\ell}e_\nu(\lambda,(k-c)_{\ell},c)\) and \(\Delta_{\ell}^{-1}e_\nu(\lambda,k_{\ell},c) = \frac{1}{\Delta_{\ell}}e_\nu(\lambda,(k-c)_{\ell},c)\) in equation (21).

The extended Mittag-Leffler function is validated through the summation formula numerically by the above example. Here we have the graph of EML functions through MATLAB by adjusting the values of parameters in the Extended Mittag-Leffler function defined in (2.3). The First figure gives the graph of Extended Mittag-Leffler function for the values \(a = -\ell, c = \nu = \ell = 1, \lambda = 2, j\) varies from 0 to 10, \(k\) varies from 0 to 50. Similarly for the second figure the value of
$k$ varies from $-500$ to $500$. Here we assumed $k$ is a multiple of $\ell$.

4. Applications

Consider a very long rod with $v(k_1, k_2)$ as temperature at position $k_1$ and at time $k_2$. Even though we have assumed that the flow of heat is instant, but in reality the flow will take time from one point $k_1$ to its neighbouring points $k_1 - \ell_1$ and $k_1 + \ell_1$ in one dimensional flow. In that case one can consider partial delay difference equation [3]. By Newton’s law of cooling, the Partial difference equation of the heat flow of rod is

$$u(k_1, k_2 + \ell_2) = au(k_1 - \ell_1, k_2) + bu(k_1, k_2) + cu(k_1 + \ell_1, k_2).$$

(22)

We shall show that $u(k_1, k_2) = e_{\nu}(\lambda, k_1, 1), e_{\nu}(\lambda, k_2, 1)$ is a solution of the equation (22) if

$$e_{\nu}(\lambda, \ell_2) = ae_{\nu}(\lambda, -\ell_1) + b + ce_{\nu}(\lambda, \ell_1), \nu \in \mathbb{N}(1).$$

(23)
Substituting \( u(k_1, k_2) = e_\nu(\lambda, k_1, 1).e_\nu(\lambda, k_2, 1) \) in (22), we get
\[
e_\nu(\lambda, k_1, 1).e_\nu(\lambda, k_2 + \ell_1, 1) = a e_\nu(\lambda, k_1 - \ell_1, 1).e_\nu(\lambda, k_2, 1) + b e_\nu(\lambda, k_1, 1).e_\nu(\lambda, k_2, 1)
+ c e_\nu(\lambda, k_1 + \ell_1, 1).e_\nu(\lambda, k_2, 1).
\]
(24)

By the Lemma (3.4) and canceling \( e_\nu(\lambda, k_1, 1).e_\nu(\lambda, k_2, 1) \) both sides of the (24), we get the condition (23). Thus heat flow is controlled by the condition (23) which is independent of \( k_1 \) and \( k_2 \). Thus we conclude that heat flow can be controlled by proper selection of \( \lambda \), satisfying the control equation(23).

5. Conclusions
This paper presents definition of extension of Mittag-Leffler function. By this we are able to obtain a formula for the series of the Extended Mittag-Leffler functions using inverse of generalized difference operator. Here, the summation formula for the extended Mittag-Leffler function, arrived by first order difference equations, can be extended to the \( n^{th} \) order difference equations in future. We derived the summation formula for the finite series by equating the summation form solution and exact solution of certain difference equation with the extended Mittag-Leffler functions. This Extended ML function obeys as solution of several difference equations like heat equations. Obtaining EML function as solution of partial difference equation is the significance of this research work.

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