Fixed speed competition on the configuration model with infinite variance degrees

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Complex networks are large data-sets without obvious structure
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- elements are represented by **vertices**
Networks

Complex networks are large data-sets without obvious structure

- elements are represented by *vertices*
- their relationship/interaction are represented by *edges*
Complex networks are large data-sets without obvious structure

- elements are represented by vertices
- their relationship/interaction are represented by edges
- additional information can be added to vertices and edges
Complex networks

IP level internet network, 2003
from the OPTE project, opte.org
Information spread and competition on networks

Some examples

- **Marketing:**
  - companies compete for customers
  - word-of-mouth recommendations on the acquaintance network
  - ‘online’ word-of-mouth: tweets, Facebook posts, etc.
Information spread and competition on networks

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- **Epidemiology:**
  - bacteria and viruses spread among population
  - different strains of a pathogen compete
Information spread and competition on networks

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  - ‘online’ word-of-mouth: tweets, Facebook posts, etc.

- Epidemiology:
  - bacteria and viruses spread among population
  - different strains of a pathogen compete

Coexistence?

Can competitive spreading processes coexist on the network?
Can they both get linear proportion of the vertices?
The underlying graph: the configuration model
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\[ v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \quad v_8 \]
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Degree assumptions

Degrees are i.i.d. copies of $D$, $P(D \geq 2) = 1$ and
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**Power law assumption**

For $\tau \in (2, 3)$,

$$\frac{c_1}{x^{\tau-1}} \leq P(D > x) \leq \frac{C_1}{x^{\tau-1}}$$
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**Power law assumption**

For $\tau \in (2, 3)$,

$$\frac{c_1}{x^{\tau-1}} \leq P(D > x) \leq \frac{C_1}{x^{\tau-1}}$$

This means, $E[D] < \infty$, but $E[D^2] = \infty$!
Competition starts!
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\[ t = 0 \]

\[ v_1, v_2, v_3, v_4, v_5, v_8, v_7, v_6, v_3 \]
Competition starts!

\[ t = 1 \]

\[ v_1 \]
\[ v_2 \]
\[ v_3 \]
\[ v_4 \]
\[ v_5 \]
\[ v_6 \]
\[ v_7 \]
\[ v_8 \]
Competition starts!

$t = 2$

\[ v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_6 \quad v_7 \quad v_8 \]
Competition starts!

\[ t = 3 \]

\[ v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5 \quad v_8 \quad v_7 \quad v_6 \]
Competition starts!

\[ t = 4 - \varepsilon \]

\[ v_1 \quad v_2 \quad v_3 \quad v_7 \quad v_8 \]

\[ v_5 \quad v_6 \quad v_4 \]

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Competition starts!

\( t = 4 \)

\[ v_1 \quad v_2 \quad v_3 \]

\[ v_4 \quad v_5 \quad v_6 \]

\[ v_7 \quad v_8 \]
Competition starts!

\[ t = 4 \]

\[ v_1 \quad v_2 \quad v_3 \quad v_4 \quad v_5\quad v_8 \quad v_7 \quad v_6 \]
From uniformly chosen vertices

- red spreads at speed 1
- blue spreads at speed $1/\lambda$, $\lambda \geq 1$.
- $R_t, B_t$ are the set of vertices colored red/blue at time $t$
- $R_\infty, B_\infty$ are the set of vertices colored red/blue eventually

**Question**

How large is $R_\infty, B_\infty$?
No coexistence when $\lambda > 1$

- Red gets $n - o(n)$ many vertices.
- Blue gets subpolynomially many vertices:

**Heuristic statement**

$$B_\infty \approx \exp \left\{ (\log n) \frac{2^{\lambda+1}}{\lambda+1} \cdot f(n, \lambda; Y_{b}^{(n)}, Y_{r}^{(n)}) \right\}$$ (1)

- $f$ is an oscillating, positive function,
- $Y_{b}^{(n)}$ and $Y_{r}^{(n)}$ are rv-s that tell us how ‘good’ the starting positions are.
No coexistence when $\lambda > 1$

The precise result:

**Theorem (Baroni, Hofstad, K)**

Fix $\lambda > 1$. Then, there exists a bounded and strictly positive random function $C_n(Y_r^{(n)}, Y_b^{(n)})$ such that as $n \to \infty$

\[
\frac{\log(B_\infty)}{(\log n)^{\frac{2}{\lambda+1}} C_n(Y_r^{(n)}, Y_b^{(n)})} \overset{d}{\to} \left(\frac{Y_b^\lambda}{Y_r}\right)^{\frac{1}{\lambda+1}}.
\]

(2)
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\[
\frac{\log(B_\infty)}{(\log n)^{1+\frac{2}{\lambda+1}}} C_n(Y_r^{(n)}, Y_b^{(n)}) \xrightarrow{d} \left(\frac{Y_b^\lambda}{Y_r}\right)^{1 \over \lambda+1}.
\]  

(2)
No coexistence when $\lambda > 1$

The precise result:

**Theorem (Baroni, Hofstad, K)**

Fix $\lambda > 1$. Then, there exists a bounded and strictly positive random function $C_n(Y^{(n)}_r, Y^{(n)}_b)$ such that as $n \to \infty$

$$
\frac{\log(B_\infty)}{(\log n)^{\frac{2}{\lambda+1}} C_n(Y^{(n)}_r, Y^{(n)}_b)} \xrightarrow{d} \left( \frac{Y^\lambda_b}{Y_r} \right)^{\frac{1}{\lambda+1}}.
$$

- $Y^{(n)}_r, Y^{(n)}_b$ are approximating rv-s of the limit rv-s $Y_b, Y_r$
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- $Y_r^{(n)}, Y_b^{(n)}$ are approximating rv-s of the limit rv-s $Y_b, Y_r$
- $Y_b, Y_r$ have exponential tails
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$$

- $Y_r^{(n)}, Y_b^{(n)}$ are approximating rv-s of the limit rv-s $Y_b, Y_r$
- $Y_b, Y_r$ have exponential tails
- $Y_b, Y_r$ depend only on the local neighbourhoods of the starting point of red and blue
- $C_n(Y_r^{(n)}, Y_b^{(n)})$ is *oscillating* and a deterministic function of $n, Y_r^{(n)}, Y_b^{(n)}, \lambda, \tau$:
No coexistence when $\lambda > 1$

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Fix $\lambda > 1$. Then, there exists a bounded and strictly positive random function $C_n(Y_r^{(n)}, Y_b^{(n)})$ such that as $n \to \infty$

$$\frac{\log(B_\infty)}{(\log n)^{\frac{1}{\lambda+1}} C_n(Y_r^{(n)}, Y_b^{(n)})} \xrightarrow{d} \left( \frac{Y_b^\lambda}{Y_r} \right)^{\frac{1}{\lambda+1}}. \quad (2)$$

$$\left( \frac{(\tau - 2)}{\lambda+1} \right)^{\frac{1}{\lambda+1}} \left( \frac{3 - \tau}{(\tau - 1)^2 1 - (\tau - 2)^\lambda} \right)^{\frac{1}{\lambda+1}} < C_n(Y_r^{(n)}, Y_b^{(n)}) < (\tau - 2)^{-2} \cdot \frac{4 - \tau}{1 - (\tau - 2)^\lambda}. $$
Coexistence when $\lambda = 1$?

If one of the neighbourhoods is much better than the other (say red), then red wins, and blue gets $B_\infty \approx n_f(n, Y(n))$.

$f$ is an oscillating, positive function, $Y(n)$ is a rv that tells us how 'good' the starting positions are.

Heuristic for coexistence
If the neighbourhoods are about the same, there is coexistence.
Coexistence when $\lambda = 1$?

**Heuristic for winning**

If one of the neighbourhoods is much better than the other (say red), then red wins, and blue gets

$$B_\infty \approx n^f(n, Y_r^{(n)}, Y_b^{(n)})$$  \hspace{1cm} (3)

- $f$ is an oscillating, positive function,
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If the neighbourhoods are about the same, there is coexistence.
The precise result:
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**Theorem (Hofstad, K)**

Fix $\lambda = 1$. Then, if $Y_r^{(n)}/Y_b^{(n)} \notin [\tau - 2, 1/(\tau - 2)]$, then there exists a bounded and strictly positive random function $C_n(Y_r^{(n)}, Y_b^{(n)}) < 1$ such that as $n \to \infty$

$$\frac{\log(B_{\infty})}{\log n \cdot C_n(Y_r^{(n)}, Y_b^{(n)})} \xrightarrow{d} \sqrt{\frac{Y_b}{Y_r}}. \quad (4)$$
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$$\frac{\log(\mathcal{B}_\infty)}{\log n \cdot C_n(Y_r^{(n)}, Y_b^{(n)})} \xrightarrow{d} \sqrt{\frac{Y_b}{Y_r}}.$$  \hspace{0.5cm} (4)

**Work in progress (Hofstad, K)**

Fix $\lambda = 1$. Then, if $Y_r^{(n)}/Y_b^{(n)} \in [\tau - 2, 1/(\tau - 2)]$, then there is coexistence.
Related work on competition on graphs

- (Häggström and Pemantle, ‘98) $\mathbb{Z}^d$, $\text{Exp}(1)$ passage times: coexistence for $\lambda = 1$,
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$\lambda > 1$: slower color occupies $n^{\beta(\lambda)}$ many vertices, $\beta(\lambda) < 1$ is a deterministic function of $\lambda$ (Deijfen, van der Hofstad, ‘13)

$\text{CM}_n(D)$, $D$ power law $\tau \in (2,3)$, $\text{Exp}(1)$ passage times, $\forall \lambda$: NO coexistence, winner is random winning probability (explicitly computable) loser color paints $O(1)$ many vertices
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- (Antunovic, Dekel, Mossel and Peres, ‘11) Random regular graphs, $\text{Exp}(1)$ passage times:
  - $\lambda = 1$: coexistence
  - $\lambda > 1$: slower color occupies $n^{\beta(\lambda)}$ many vertices, $\beta(\lambda) < 1$ is a deterministic function of $\lambda$
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- (Deijfen, van der Hofstad, ‘13) $\text{CM}_n(D)$, $D$ power law $\tau \in (2, 3)$, $\text{Exp}(1)$ passage times,
  - $\forall \lambda$: NO coexistence, winner is random
  - winning probability (explicitly computable)
  - loser color paints $O(1)$ many vertices
Proof
Key observation: ∃ joint construction
Key observation: \( \exists \) joint construction

\[
t = 0
\]

\[
\begin{align*}
v_1 & \quad v_2 & \quad v_3 & \quad v_4 & \quad v_5 & \quad v_6 & \quad v_7 & \quad v_8 \\
& & & & & & &
\end{align*}
\]
Key observation: ∃ joint construction

\[ t = 1 - \varepsilon \]
Key observation: ∃ joint construction
Key observation: $\exists$ joint construction

$t = 2 - \varepsilon$
Key observation: ∃ joint construction
Key observation: $\exists$ joint construction

$t = 3 - \varepsilon$
Key observation: \exists \text{ joint construction}

\begin{align*}
t &= 3 \\
v_1 &

\begin{array}{c}
\text{\textcolor{red}{v}_8} \\
\text{\textcolor{red}{v}_2} \\
\text{\textcolor{red}{v}_1} \\
\text{\textcolor{blue}{v}_6} \\
\text{\textcolor{black}{v}_4} \\
\text{\textcolor{black}{v}_3} \\
\text{\textcolor{red}{v}_7} \\
\end{array}
\end{align*}
Key observation: \( \exists \) joint construction
Key observation: $\exists$ joint construction

$t = 4 - \varepsilon/2$
Key observation: \( \exists \) joint construction
Branching process approximation

**Key observation**
The local neighbourhood of the red and blue sources look like branching processes.

- Let $t(n^\rho)$ be the time to reach cluster size $n^\rho$ for red.
- Let $D^*$ be the size biased version of $D$ (note that $\mathbb{E}[D^*] = \infty$ because $\tau \in (2, 3)$).
Branching process approximation

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Lemma (Coupling to BP)

There exists $\rho > 0$ s.t. $\mathcal{R}_{t(n^\rho)} \cap \mathcal{B}_{t(n^\rho)} = \emptyset$ whp, and the forward degrees in both the red and the blue process can be coupled to i.i.d. degrees $\sim D^* - 1$. 
Branching process approximation

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There exists $\rho > 0$ s.t. $R_{t(n^\rho)} \cap B_{t(n^\rho)} = \emptyset$ whp, and the forward degrees in both the red and the blue process can be coupled to i.i.d. degrees $\sim D^* - 1$.

Consequence: initial phases are BP trees. How fast do they grow?
Growth rate of BP

$Z_k$: size of the $k$th generation in the BP

**Heuristic meaning**

$Z_k$ grows double exponentially, i.e., $\exists$ limiting rv $Y$ s.t.,

$$Z_k \approx e^{Y\left(\frac{1}{\tau-2}\right)^k}$$
Growth rate of BP

\[ Z_k: \text{size of the } k\text{th generation in the BP} \]

**Heuristic meaning**

\[ Z_k \text{ grows double exponentially, i.e., } \exists \text{ limiting rv } Y \text{ s.t.,} \]

\[ Z_k \approx e^{Y\left(\frac{1}{\tau-2}\right)^k} \]

**Theorem (Growth rate of BP, Grey '73)**

\[ (\tau - 2)^k \log(Z_k) \xrightarrow{a.s.} Y \text{ and } Y \text{ has an exponential tail.} \]
Choose a $\rho \ll 1$. Then $k^*$, the last generation with $Z_k > n^\rho$, is

$$k^* = \left\lceil \log \log n + \log \left( \frac{\rho}{Y_r} \right) \right\rceil \left\lceil \log (\tau - 2) \right\rceil$$
Choose a $\rho \ll 1$. Then $k^*$, the last generation with $Z_k > n^\rho$, is

$$k^* = \left\lceil \frac{\log \log n + \log (\varrho/Y_r)}{|\log(\tau - 2)|} \right\rceil$$

Therefore,

$$Z_{k^*} \approx \exp\{Y(\tau - 2)^{-k^*}\} = n^{\rho(\tau - 2)^{a_n - 1}}$$

with $a_n := \left\{ \frac{\log \log n + \log (\varrho/Y_r)}{|\log(\tau - 2)|} \right\}$. 
Choose a $\rho \ll 1$. Then $k^*$, the last generation with $Z_k > n^\rho$, is

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Therefore,

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with $a_n := \left\{ \frac{\log \log n + \log(\varrho/Y_r)}{|\log(\tau - 2)|} \right\}$. 

**Question**

What is the maximal degree in the last generation?
Maximal degree in last generation

Lemma

Let $X_i, \ i = 1, \ldots, m$ be i.i.d. random variables with power-law tail exponent $\alpha$. Then for $K > 0$

$$
P\left( \max_{i=1,\ldots,m} X_i < \left( \frac{m}{K \log n} \right)^{1/\alpha} \right) \leq \frac{1}{n^{c_1 K}},
$$

(5)

Recall:

- from GBA’s lecture: $\max_{i=1,\ldots,m} X_i/m^{1/\alpha} \xrightarrow{d} \xi_\alpha$,
- $Z_{k^*} \approx n^{\rho'/(\tau-2)^a n^{-1}}$

Corollary (The maximal degree in generation $k^*$)

$$
u_0 := \left( \frac{n^{\rho'/(\tau-2)^a n^{-1}}}{C \log n} \right)^{1/(\tau-2)} \text{ whp}
$$
After the coupling fails

Key idea

Structure the high-degree part of the graph in layers of roughly equal degree (on a log log scale).
2nd step: mountain climbing

Define recursively,

- $u_0 \approx n^{\rho(\tau-2)a n^{-2}}$,
- $u_{i+1} := \left(\frac{u_i}{C \log n}\right)^{1/(\tau-2)}$
- and the nested layers: $\Gamma_i := \{v \in [n] : D_v \geq u_i\}$.
- Note that $u_i$ grows double-exponentially: $u_i \approx \left(n^{\rho(\tau-2)a n^{-2}}\right)^{1/(\tau-2)^i}$
2nd step: degree-mountain

\[ N(S) := \text{neighbors of } S \]

**Lemma**

\[ \Gamma_i \subset N(\Gamma_{i+1}) \text{ whp} \]
2nd step: degree-mountain

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$N(S)$: = neighbors of $S$

**Lemma**

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2nd step: degree-mountain

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**Lemma**

\( \Gamma_i \subset N(\Gamma_{i+1}) \quad \text{whp} \)
2nd step: red climber
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\[ t = k^* \]

\[ n^{1/(\tau - 1)} \]
2nd step: red climber

\[ t = k^* + 1 \]
2nd step: red climber

\[ t = k^* + 2 \]

\[ n^{1/(\tau - 1)} \]
2nd step: red climber

\[ t = k^* + 3 \]
2nd step: red climber

\[ t = k^* + 4 \]

\[ n^{1/(\tau-1)} \]
2nd step: red climber

\[ t = k^* + 5 \]

\[ n^{1/(\tau - 1)} \]
Time it takes to reach the top

- Maximal degree in the graph: \( M = \max_{i \in [n]} D_i \approx n^{1/(\tau-1)} \)
Time it takes to reach the top

- Maximal degree in the graph: $M = \max_{i \in [n]} D_i \approx n^{1/(\tau - 1)}$
- recall $u_i \approx \left(n^{\rho'(\tau - 2)a^{n-2}}\right)^{1/(\tau - 2)^i}$
Time it takes to reach the top

- Maximal degree in the graph: $M = \max_{i \in [n]} D_i \approx n^{1/(\tau - 1)}$
- Recall $u_i \approx \left(n^{\rho' (\tau - 2) a_n - 2}\right)^{1/(\tau - 2)^i}$
- $i^*$ is the number of layers needed (so $k^* + i^*$ is the total time needed).
Time it takes to reach the top

- Maximal degree in the graph: $M = \max_{i \in [n]} D_i \approx n^{1/(\tau - 1)}$
- Recall $u_i \approx \left(n^{\rho'/(\tau - 2)}a_{n-2}\right)^{1/(\tau - 2)}$
- $i^*$ is the number of layers needed (so $k^* + i^*$ is the total time needed).
- $i^*$ defined by $M^{\tau - 2} < u_{i^*} \leq M$

$$i^* = \left\lfloor -1 - \frac{\log((\tau - 1)\rho(\tau - 2)a_{n - 1})}{|\log(\tau - 2)|} \right\rfloor$$

where $b_n = \left\{\frac{-\log((\tau - 1)(\tau - 2)a_{n - 1})}{|\log(\tau - 2)|}\right\}$
Time to reach the top

- Number of layers needed is

\[ i^* = -1 + \frac{-\log((\tau - 1)\varrho' (\tau - 2)^{a_n - 1})}{|\log(\tau - 2)|} - b_n \]
Time to reach the top

- Number of layers needed is

\[ i^* = -1 + \frac{-\log((\tau - 1)\rho'(\tau - 2)^{a_n-1})}{|\log(\tau - 2)|} - b_n \]

- Time until the coupling fails is

\[ k^* = \left\lceil \frac{\log \log n + \log(\rho'/Y_r^{(n)})}{|\log(\tau - 2)|} \right\rceil = \frac{\log \log n + \log(\rho'/Y_r^{(n)})}{|\log(\tau - 2)|} + (1 - a_n) \]
Time to reach the top

- Number of layers needed is
  \[ i^* = -1 + \frac{-\log((\tau - 1) \varrho' (\tau - 2)^{a_n-1})}{|\log(\tau - 2)|} - b_n \]

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  \[ k^* = \left\lceil \frac{\log \log n + \log(\varrho'/Y_r^{(n)})}{|\log(\tau - 2)|} \right\rceil = \frac{\log \log n + \log(\varrho'/Y_r^{(n)})}{|\log(\tau - 2)|} + (1 - a_n) \]

- Add them together: the time to reach just below the top is
  \[ T_r := k^* + i^* = \frac{\log \log n - \log((\tau - 1) Y_r^{(n)})}{|\log(\tau - 2)|} - 1 - b_n, \]
Time to reach the top

- Number of layers needed is
  \[ i^* = -1 + \frac{-\log((\tau - 1)\varrho'\tau - 2)^{a_n-1})}{|\log(\tau - 2)|} - b_n \]

- Time until the coupling fails is
  \[ k^* = \left[ \log \frac{\log n + \log(\varrho'/Y_r^{(n)})}{|\log(\tau - 2)|} \right] = \log \frac{\log n + \log(\varrho'/Y_r^{(n)})}{|\log(\tau - 2)|} + (1 - a_n) \]

- Add them together: the time to reach just below the top is
  \[ T_r := k^* + i^* = \frac{\log \log n - \log((\tau - 1)Y_r^{(n)})}{|\log(\tau - 2)|} - 1 - b_n, \]

Key observation

\( T_r \) does not depend on \( \rho \)!
4th step: red avalanche

\[ t = k^* + 5 \]
4th step: red avalanche

\[ t = T_r + 1 \]

\[ n^{1/(\tau-1)} \]
4th step: red avalanche

\[ t = T_r + 2 \]

\[ n^{1/(\tau - 1)} \]

\[ n^{\frac{\tau - 2}{\tau - 1}} \]
4th step: red avalanche

\[ t = T_r + 3 \]
4th step: red avalanche

\[ t = T_r + 4 \]

\[ n^{1/(\tau-1)} \]

\[ u_0, u_1, u_2, u_3, u_4, u_5 \]

\[ n^{\frac{\tau-2}{\tau-1}} \]

\[ n^{\frac{\tau-2}{\tau-1}} \]

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4th step: red avalanche

\[ t = T_r + 5 \]
4th step: red avalanche

\[ t = T_r + 6 \]

\[ n^{1/(\tau-1)} \]

\[ u_0 \]
\[ u_1 \]
\[ u_2 \]
\[ u_3 \]
\[ u_4 \]
\[ u_5 \]

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Crossing the peak

Key heuristic

The highest degree vertices induce a complete graph whp, so red crosses the peak in a single step.
Crossing the peak

Key heuristic
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Lemma

For two sets of vertices \( A, B \subset [n] \), if the total degree of \( A \),
\[ S_A := \sum_{v \in A} \deg(v) \], is \( o(n) \) and if \( S_AS_B > n \cdot h(n) \), then

\[
P_n(A \text{ and } B \text{ do not share an edge}) < e^{-\frac{h(n)}{4E[D]}}.
\]
Lemma

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- $v^*$ is a red vertex with degree $D_{v^*} \geq u_{i^*} \approx n^{(\tau-2)b_n}/(\tau-1)$,
Lemma

For two sets of vertices $A, B \subset [n]$, if the total degree of $A$, $S_A := \sum_{v \in A} \deg(v)$, is $o(n)$ and if $S_A S_B > n \cdot h(n)$, then

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- $v^\star$ is a red vertex with degree $D_{v^\star} \geq u^\star \approx n^{(\tau-2)bn / (\tau-1)}$,
- define $\tilde{u}_1 := n \cdot \log n / u_{i^\star} \approx n^{1-(\tau-2)bn / (\tau-1)}$
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- \( v^* \) is a red vertex with degree \( D_{v^*} \geq u_{i^*} \approx n^{(\tau-2)b_n}/(\tau-1) \),
- define \( \tilde{u}_1 := n \cdot \log n / u_{i^*} \approx n^{1-(\tau-2)b_n}/(\tau-1) \)
- and the new layer \( \tilde{\Gamma}_1 := \{ v \in [n], D_v > \tilde{u}_1 \} \).
Lemma

For two sets of vertices $A, B \subset [n]$, if the total degree of $A$, $S_A := \sum_{v \in A} \deg(v)$, is $o(n)$ and if $S_A S_B > n \cdot h(n)$, then

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- $v^*$ is a red vertex with degree $D_{v^*} \geq u_{i^*} \approx n^{(\tau-2)bn/\tau}$,
- define $\tilde{u}_1 := n \cdot \log n / u_{i^*} \approx n^{1-(\tau-2)bn/\tau}$
- and the new layer $\tilde{\Gamma}_1 := \{v \in [n], D_v > \tilde{u}_1\}$.

Lemma

All the vertices in $\tilde{\Gamma}_1$ are occupied by red at time $T_r + 1$, i.e.,

$$\tilde{\Gamma}_1 \subset R_{T_r+1} \quad \text{whp.} \quad (6)$$
4th step: red avalanche

\[ t = k^* + 5 \]
4th step: red avalanche

\[ t = T_r + 1 \]

\[ n^{1/(\tau-1)} \]

\[ u_0, u_1, u_2, u_3, u_4, u_5 \]

\[ n^{\frac{\tau-2}{\tau-1}} \]

\[ \tau - 2 \]

\[ \tau - 1 \]
4th step: red avalanche

\[ t = T_r + 2 \]
4th step: red avalanche

t = T_r + 3

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4th step: red avalanche

\[ t = T_r + 4 \]
4th step: red avalanche

\[ t = T_r + 5 \]
4th step: red avalanche

\[ t = T_r + 6 \]

\[ n^{1/(\tau - 1)} \]
4th step: red avalanche

Key idea

Same analysis as for the climbing phase, but now moving down on the mountain. Since $k^*$ is already high up in the mountain, the avalanche goes much further down than the climbing phase went up.
4th step: red avalanche

Key idea

Same analysis as for the climbing phase, but now moving down on the mountain. Since $k^*$ is already high up in the mountain, the avalanche goes much further down than the climbing phase went up.

- Define $\tilde{u}_{\ell+1} = C \log n \cdot (\tilde{u}_\ell)^{\tau-2}$,
4th step: red avalanche

**Key idea**

Same analysis as for the climbing phase, but now moving down on the mountain. Since $k^*$ is already high up in the mountain, the avalanche goes much further down than the climbing phase went up.

- Define $\tilde{u}_{\ell+1} = C \log n \cdot (\tilde{u}_\ell)^{\tau-2}$,
- the layers ‘on the way down’

$$\tilde{\Gamma}_\ell := \{v : D_v > \tilde{u}_\ell\},$$

(note that these are shifted wrt the layers on the way up)
4th step: red avalanche

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\[
\tilde{\Gamma}_\ell := \{ v : D_v > \tilde{u}_\ell \},
\]

(note that these are shifted wrt the layers on the way up)

Corollary (red avalanche)

for all \( \ell < (1 - \varepsilon) \log \log n / |\log(\tau - 2)| \)

\[
\tilde{\Gamma}_{\ell+1} \subset N(\tilde{\Gamma}_\ell) \quad \text{and} \quad \tilde{\Gamma}_\ell \subset R_{Tr+\ell} \quad \text{whp.} \quad (7)
\]
5th step: Blue climber

\[ t = k^* + 5 \]

\[ u_0 \]
\[ u_1 \]
\[ u_2 \]
\[ u_3 \]
\[ u_4 \]
\[ u_5 \]
\[ n^{1/(\tau - 1)} \]
\[ n^{\tau - 2} \]
\[ n^{\tau - 1} \]
5th step: Blue climber

\[ t = T_r + 1 \]

\[ n^{1/(\tau-1)} \]

\[ u_0 \]

\[ u_1 \]

\[ u_2 \]

\[ u_3 \]

\[ u_4 \]

\[ u_5 \]
5th step: Blue climber

\[ t = T_r + 2 \]
5th step: Blue climber

\[ t = T_r + 3 \]

\[ n^{\frac{1}{\tau - 1}} \]

\[ u_0, u_1, u_2, u_3, u_4, u_5 \]

\[ n^{\frac{\tau - 2}{\tau - 1}} \]
5th step: Blue climber

\[ t = T_r + 4 \]

\[ n^{1/(\tau - 1)} \]
5th step: Blue climber

\[ t = T_r + 5 \]

\[ n^{1/(\tau-1)} \]
5th step: Blue climber

\[ t = T_r + 5 \]

\[ n^{1/(\tau-1)} \]

\[ u_0 \]
\[ u_1 \]
\[ u_2 \]
\[ u_3 \]
\[ u_4 \]
\[ u_5 \]
Proposition

Let $D_{\max}^{(b,n)}(t)$ denote the degree of the maximal degree blue vertex at time $t$.

Then, at time $T_r + t$,

$$D_{\max}^{(b,n)}(T_r + t) = \exp\left(\frac{Y_b}{\tau - \frac{2}{2}}\right)^{\left\lfloor \frac{T_r + t}{\lambda} \right\rfloor + 1} \left(1 + o_p(1))\right),$$

as long as $t$ is s.t. the rhs is less than the location of the red avalanche at time $t$. 

(8)
5th step: the maximal degree of blue

Key idea

The intersection time $t_c$ (defined as time when the maximum degree of blue is at the same location as the red avalanche) satisfies

$$D_{\max}^{(b,n)}(T_r + t_c) = \tilde{u}_{t_c}$$  \hspace{1cm} (9)

Solve for $t_c$ and plug it back to find $D_{\max}^{(b,n)}(T_r + t_c)$.

Solve

$$\exp \left( Y_b(\tau - 2) - \frac{T_r + t_c}{\lambda} - 1 \right) = n^{(\tau - 2)t_c^{-1}}.$$  \hspace{1cm} (10)
5th step: the maximal degree of blue

Lemma

The maximum degree of blue before hitting red is

\[
\log \left( D_{\text{max}}^{(b,n)}(T_r + t_c) \right) = \left( \log n \right) \frac{2}{\lambda + 1} \left( \frac{Y_b^\lambda}{Y_r} \frac{\alpha}{\tau - 1} \right) \frac{1}{\lambda + 1} \left( \tau - 2 \right) \frac{1 + b_n^{(r)}}{\lambda + 1} - 1 + \left\{ \frac{T_r + t_c}{\lambda} \right\},
\]

This is just before the intersection time, i.e., if blue jumps first after \( T_r + t_c \), he might increase its exponent ‘by a bit’. (rhs changes by a constant factor).
6th step: Competing with the avalanche

Key idea

Whichever uncolored vertex is ‘close’ to a high degree blue vertex, has a chance to become blue. All other vertices become red.

Method used:
6th step: Competing with the avalanche

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Whichever uncolored vertex is ‘close’ to a high degree blue vertex, has a chance to become blue. All other vertices become red.

Method used:

- count restricted paths, do first and second moment method on their number,
6th step: Competing with the avalanche

Key idea

Whichever uncolored vertex is ‘close’ to a high degree blue vertex, has a chance to become blue. All other vertices become red.

Method used:

- count restricted paths, do first and second moment method on their number,
- for the lower bound, show that red cannot bite out too much from these close enough vertices.
Open problems

- $\tau \in (2, 3)$:
  - If the BPs $(D^*, T_r)$ and $(D^*, T_b)$ are both explosive, NO coexistence for all $\lambda$, and both colors can win.
  - If one color is explosive, the other one is conservative, then the explosive one wins for all $\lambda$.
  - If both underlying branching processes are non-explosive, and further assume $T_r \xrightarrow{d} \lambda T_b$, then there is no coexistence if $\lambda \neq 1$ (the fastest color wins).
  - previous setting, $\lambda = 1$: the outcome might sensitively depend on the transmission time distribution.

- $\tau > 3$:
  - If $T_r, T_b$ both have continuous distribution, and the branching process approximations of them have different Malthusian parameters, then there is no coexistence, and the number of vertices painted by the slower color is $n^{\beta(\lambda)}$ for some deterministic $\beta(\lambda) \in (0, 1)$.
  - If the Malthusian parameters agree, we suspect that there is asymptotic co-existence.
Thank you for the attention!
Thank you for the attention!
Thank you for the attention!

\[ t = k^* + 1 \]

\[ n^{1/(\tau-1)} \]
Thank you for the attention!

\[ t = k^* + 2 \]

\[ n^{1/(\tau-1)} \]
Thank you for the attention!

\[ t = k^* + 3 \]

\[ n^{1/(\tau-1)} \]

\[ u_0 \quad u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \]

\[ n^{\tau-2} \quad n^{\tau-1} \]

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Thank you for the attention!
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\[ t = k^* + 5 \]
Thank you for the attention!

\[ t = T_r + 1 \]

\[ n^{1/(\tau-1)} \]
Thank you for the attention!
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t = Tr + 3

$u_0 \to u_1 \to u_2 \to u_3 \to u_4 \to u_5$

$n^{1/(\tau-1)}$
Thank you for the attention!
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