HOMOLOGY OF LINEAR GROUPS VIA CYCLES IN
\(BG \times X\)

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1. INTRODUCTION

Homology theories for algebraic varieties are often constructed using simplicial sets of algebraic cycles. For example, Bloch’s higher Chow groups and motivic cohomology, as defined by Suslin and Voevodsky [17], are given in this fashion. In this paper, we construct homology groups \(H_i(X, G)\), where \(G\) is an algebraic group and \(X\) is a variety, by considering cycles on the simplicial scheme \(BG \times X\), an idea first suggested by Andrei Suslin. If \(X = \text{Spec}(R)\) is an affine scheme, then there is a natural map

\[ H_i(G(R), A) \to H_i(X, G; A) \]

whose source is the usual group homology of the discrete group \(G(R)\) of \(R\)-points of the algebraic group \(G\). Moreover, this map is an isomorphism if \(R = k\) and \(k\) is algebraically closed, so that these groups capture the homology of the discrete group \(G(k)\). The functors \(H_i(\cdot, G; A)\) are naturally equipped with transfer maps in the sense of Suslin–Voevodsky [16]. We also have cohomological versions \(H^i(X, G; A)\) and by applying these to the standard cosimplicial variety \(\Delta^\bullet_k\) we deduce a spectral sequence

\[ E_1^{s,t} = H^s(\Delta^t, G; \mathbb{Z}/n) \Rightarrow H_{et}^{s+t}(BG_k, \mathbb{Z}/n) \]

for \(k\) algebraically closed with \(\text{char}(k)\) not dividing \(n\). Since \(H^i(\Delta^0, G; A) \cong H^i(G(k), A)\), the bottom row of the spectral sequence is just \(H^\bullet(G(k), \mathbb{Z}/n)\).

Thus, this spectral sequence provides a mechanism for comparing the group cohomology of the discrete group \(G(k)\) with the étale cohomology of the simplicial scheme \(BG_k\).

We recall the following conjecture of E. Friedlander [4].

**Conjecture 1.1 (Friedlander’s Generalized Isomorphism Conjecture).** If \(G\) is an algebraic group over an algebraically closed field \(k\), then the natural map of simplicial schemes \(BG(k) \to BG_k\) (with the first one being “discrete”) induces an isomorphism

\[ H_{et}^\bullet(BG_k, \mathbb{Z}/n) \to H^\bullet(BG(k), \mathbb{Z}/n) = H^\bullet(G(k), \mathbb{Z}/n) \]

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provided \( \text{char}(k) \) does not divide \( n \).

If \( k = \mathbb{C} \), then the Isomorphism Conjecture reduces to the claim that the identity map \( G^\delta \rightarrow G^{\text{top}} \), where \( G^\delta \) is the Lie group \( G^{\text{top}} \) viewed as a discrete group, induces an isomorphism in cohomology with finite coefficients:

\[
H^\bullet(BG^{\text{top}}, \mathbb{Z}/n) \rightarrow H^\bullet(BG^\delta, \mathbb{Z}/n) = H^\bullet(G^\delta, \mathbb{Z}/n).
\]

In this context, it also makes sense to consider the field \( k = \mathbb{R} \) and assert this isomorphism for real Lie groups.

The Isomorphism Conjecture is known to hold in the following cases:

1. \( k = \mathbb{F}_p \), \( G \) arbitrary [4];
2. \( G \) solvable [4], [7], [11];
3. \( G \) is one of the stable groups \( GL, SL, Sp, O \) [7], [8], [14];
4. \( H^2(G, \mathbb{Z}/n) \) for \( G \) a real Chevalley group [13];
5. \( H^3(GL_m(k), \mathbb{Z}/n) \) (\( k \) arbitrary), \( H^3(SL_2(\mathbb{C}), \mathbb{Z}/n) \), [9], [12], [15].

Using the spectral sequence described above, we are able to deduce all the known cases of the Isomorphism Conjecture immediately (except for the results about real Lie groups). Moreover, we can explicitly identify the group \( H^4\text{\text{\acute{e}t}}(BG_m, \mathbb{Z}/n) \).

**Corollary 3.5** There is an exact sequence

\[
0 \rightarrow H^4\text{\text{\acute{e}t}}(BGL_m, \mathbb{Z}/n) \rightarrow H^4(BGL_m(k), \mathbb{Z}/n) \rightarrow \mathcal{H}^4(\Delta^1, GL_m; \mathbb{Z}/n)
\]

where \( \mathcal{H} \) is the difference of the maps induced by the two face maps \( \Delta^0 \rightarrow \Delta^1 \).

Conjecturally, the map \( \mathcal{H} \) is the zero map, of course.

This paper is organized as follows. In Section 2 we construct the functors \( \mathcal{H}_i(X, G; A) \) and discuss their basic properties. Section 2.3 contains some relatively easy examples. The spectral sequence is constructed in Section 3. There is a relationship between the \( \mathcal{H}_i(\text{Spec}(k), G; A) \) and the homology groups \( H_i(BG(\bar{k}); \Gamma; A) \), where \( \Gamma = \text{Gal}(\bar{k}/k) \). We discuss this in Section 4 and prove the following results.

**Theorem 4.1** Let \( G \) be an algebraic group over \( \mathbb{R} \) and let \( A = \mathbb{Q} \) or \( \mathbb{Z}/n \) where \( n \) is odd. Then there are canonical isomorphisms

\[
\mathcal{H}_i(\text{Spec}(\mathbb{R}), G; A) \cong H_i(G(\mathbb{C}), A)^{\mathbb{Z}/2}
\]

for all \( i \geq 0 \).

If \( A = \mathbb{Z}/2 \), then the structure of \( \mathcal{H}_i(\text{Spec}(\mathbb{R}), G; A) \) is much more complicated. However, we can deal with the case \( G = G_m \).

**Theorem 4.8** For all \( k \geq 1 \),

\[
\mathcal{H}_{2k}(\text{Spec}(\mathbb{R}), G_m; \mathbb{Z}/2) \cong \mathcal{H}_{2k+1}(\text{Spec}(\mathbb{R}), G_m; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{k-1}.
\]

This is proved by noting that an equivariant version of Friedlander’s conjecture holds for \( G_m \), and then using the calculation of the Bredon cohomology of \( BG_m(\mathbb{C})^{\text{top}} \cong \mathbb{CP}^\infty \) given in [3].
We are also able to handle unipotent groups over $\mathbb{R}$. We discuss this at the end of Section 4.

**Notation.** Throughout, $k$ denotes a field and $p$ denotes the exponential characteristic of $k$; that is, $p = 1$ if $\text{char}(k) = 0$ and $p = \text{char}(k)$ if $\text{char}(k) > 0$.

**Acknowledgments.** The construction of the $\mathcal{H}_i(X, G; A)$ was suggested to us by Andrei Suslin. We thank him for putting us on this path.

**Conventions:** Throughout this paper, given a field $k$, a $k$-scheme is a separated scheme of finite type over $k$.

## 2. Construction of the functors $\mathcal{H}_i(X, G; A)$

Let $G$ be a linear algebraic group over $k$. Consider the simplicial classifying scheme $BG$:

$$
\star \xrightarrow{} G \xrightarrow{} G^2 \xrightarrow{} G^3 \xrightarrow{} \cdots \xrightarrow{} G^n \xrightarrow{} \cdots.
$$

For any $k$-scheme $X$, we may form the product $BG \times X$:

$$
X \xrightarrow{} G \times X \xrightarrow{} G^2 \times X \xrightarrow{} G^3 \times X \xrightarrow{} \cdots \xrightarrow{} G^n \times X \xrightarrow{} \cdots.
$$

If $U$ and $V$ are smooth $k$-schemes, define $C_0(U \times V/V)$ as

$$
C_0(U \times V/V) = \mathbb{Z} \left\{ \text{closed integral subschemes } Z \subset U \times V \text{ such that } Z \to V \text{ is finite and surjective over some connected component of } V \right\}.
$$

This construction was first studied by Suslin-Voevodsky in [16].

The functor $C_0(U \times V/V)$ is covariant in $U$ via pushforward of cycles. In more detail, if $p : U \to U'$ is a morphism of $k$-schemes and if $Z \subset U \times V$ is a closed integral subscheme that is finite and surjective over some connected component of $V$, then $Z' = (p \times \text{id}_V)(Z)$ is finite and surjective over some component of $V$. The map $p_*$ is defined by sending $[Z]$ (the cycle associated to $Z$) to $d[Z']$, where $d$ is the degree of the finite, dominant map $Z \to Z'$, and then extending $p_*$ to all of $C_0(U \times V/V)$ by linearity.

The functor $C_0(U \times V/V)$ is contravariant in $V$ via pullback of cycles. Given a morphism $f : V' \to V$ of smooth $k$-schemes, for a closed, integral subscheme $Z$ of $U \times V$ that is finite and surjective over a component of $V$, the pullback $f^{-1}(Z) = V' \times_f Z$ is a closed subscheme of $U \times V'$ each integral component of which is finite and surjective over a component of $V'$. One defines $f^*(\{Z\})$ to be the cycle in $U \times V'$ associated to $f^{-1}(Z)$ by taking multiplicities of its integral components in the usual fashion. The definition of $f^*$ is then extended to all of $C_0(U \times V/V)$ by linearity.

In particular, we apply the covariant functor $C_0(\cdot \times X/X)$ degreewise to the simplicial scheme $BG$ to obtain a simplicial abelian group $C_0(BG \times \cdots)$.
of sheaves in the $qfh$ topology. Recall [16] that a morphism $\phi : X \to Y$ is a topological epimorphism if the underlying Zariski topological space of $Y$ is a quotient space of the underlying Zariski topological space of $X$. The map $q$ is a universal topological epimorphism if for any $Z \to Y$ the morphism $q_Z : X \times_Y Z \to Z$ is a topological epimorphism. An $h$-covering of a scheme $X$ is a finite family of morphism of finite type $\{p_i : X_i \to X\}$ such that $\coprod p_i : \coprod X_i \to X$ is a universal topological epimorphism. A $qfh$-covering of $X$ is an $h$-covering $\{p_i\}$ such that all the morphisms $p_i$ are quasi-finite.

If $X$ is a smooth separated scheme of finite type over $k$, let $\mathbb{Z}[1/p]_{qfh}(X)$ denote the sheaf in the $qfh$ topology associated to the presheaf

$$T \mapsto \mathbb{Z}[1/p]\text{Hom}_{\text{Sch}/k}(T, X),$$

2.1. **Variance.** The notation $\mathcal{H}_i(X, G; A)$ was chosen to suggest the same variance in $X$ and $G$ as the bi-functor $\text{Hom}(-, -)$. Namely, if $f : G \to H$ is a morphism of linear algebraic groups over $k$, then there is an induced map $f_* : \mathcal{H}_i(X, G; A) \to \mathcal{H}_i(X, H; A)$ for any scheme $X$. Indeed, the push-forward maps $f_*^{X^n} : C_0(G^n \times X/X) \to C_0(H^n \times X/X)$, $n \geq 0$, are compatible with the simplicial structures so that we obtain a map of simplicial abelian groups $f_* : C_0(BG \times X/X) \to C_0(BH \times X/X)$ and hence a map on homotopy groups as desired. Likewise, if $\varphi : X \to Y$ is a morphism of smooth $k$-schemes, then pull-back of cycles determines maps $\varphi^* : C_0(G^n \times X/X) \to C_0(G^n \times Y/Y)$, $n \geq 0$, that are again compatible with simplicial structures so that we have an induced map $\varphi^* : \mathcal{H}_i(Y, G; A) \to \mathcal{H}_i(X, G; A)$ on homotopy groups.

Of course, we have the opposite variance for the functors $\mathcal{H}^i(X, G; A)$ — these are covariant in $G$ and contravariant in $X$.

2.2. **Sheaf-theoretic interpretation of $C_0(BG \times X/X)$**. We now present an equivalent definition, due to Suslin-Voevodsky. [16] of $\mathcal{H}_i(X, G)$ in terms of sheaves in the $qfh$ topology. Recall [16] that a morphism $q : X \to Y$ is a topological epimorphism if the underlying Zariski topological space of $Y$ is a quotient space of the underlying Zariski topological space of $X$. The map $q$ is a universal topological epimorphism if for any $Z \to Y$ the morphism $q_Z : X \times_Y Z \to Z$ is a topological epimorphism. An $h$-covering of a scheme $X$ is a finite family of morphism of finite type $\{p_i : X_i \to X\}$ such that $\coprod p_i : \coprod X_i \to X$ is a universal topological epimorphism. A $qfh$-covering of $X$ is an $h$-covering $\{p_i\}$ such that all the morphisms $p_i$ are quasi-finite.

If $X$ is a smooth separated scheme of finite type over $k$, let $\mathbb{Z}[1/p]_{qfh}(X)$ denote the sheaf in the $qfh$ topology associated to the presheaf

$$T \mapsto \mathbb{Z}[1/p]\text{Hom}_{\text{Sch}/k}(T, X),$$

By Definition 2.1. Let $A$ be an abelian group. The group $\mathcal{H}_i(X, G; A)$ is defined to be the $i$-th homotopy group of the simplicial abelian group $C_0(BG \times X/X) \otimes A$; that is,

$$\mathcal{H}_i(X, G; A) = h_i(C_0(BG \times X/X) \otimes A, d),$$

where $d = \sum (-1)^kp_k$ and $p_k : G^i \to G^{i-1}$ is the map

$$p_k(g_0, g_1, \ldots, g_{i-1}) = \begin{cases} (g_1, \ldots, g_{i-1}) & k = 0 \\ (g_0, \ldots, g_{k-1}g_k, \ldots, g_{i-1}) & 0 < k < i - 1 \\ (g_0, \ldots, g_{i-2}) & k = i - 1. \end{cases}$$

We also define groups $\mathcal{H}^i(X, G; A)$ by

$$\mathcal{H}^i(X, G; A) = h^i(\text{Hom}(C_0(BG \times X/X), A)).$$


where \( \mathbb{Z}[1/p]S \) denotes the free \( \mathbb{Z}[1/p] \)-module on the set \( S \) and \( p \) is the exponential characteristic of \( k \).

**Theorem 2.2** ([16], 6.7). If \( Y \) is a separated scheme, then there is an isomorphism

\[
C_0(Y \times X/X) \otimes \mathbb{Z}[1/p] \cong \Gamma(X, \mathbb{Z}[1/p]_{qfh}(Y))
\]

that is natural in both \( X \) and \( Y \).

Note that this theorem implicitly asserts that \( C_0(Y \times -/-)[1/p] \) is a sheaf in the qfh-topology. As shown in [16, §5], any sheaf \( \mathcal{F} \) in the qfh topology admits transfer maps; that is, for any finite surjective map \( f : X \to S \), where \( X \) is reduced and irreducible and \( S \) is irreducible and regular, there is a transfer homomorphism

\[
\text{Tr}_{X/S} : \mathcal{F}(X) \longrightarrow \mathcal{F}(S)
\]
satisfying certain expected properties (cf. [16, 4.1]). In the case \( \mathcal{F} = C_0(Y \times -/-) \), the transfer homomorphism is defined by pushforward of cycles in the evident manner.

By Theorem 2.2, if \( p \) is invertible in \( A \), the complex \((C_0(BG \times -/-) \otimes A, d)\) is a complex of qfh sheaves and so is equipped with degreewise transfer maps. These commute with \( d \) and hence for any finite surjective map \( X \to S \), we have transfer maps

\[
\text{Tr}_{X/S} : \mathcal{H}_i(X, G; A) \longrightarrow \mathcal{H}_i(S, G; A).
\]

Thus we have proved the following proposition.

**Proposition 2.3.** Suppose that \( p \) is invertible in the abelian group \( A \). Then the functor \( \mathcal{H}_i(-, G; A) \) is a presheaf with transfers. \( \square \)

### 2.3. First examples.

**Proposition 2.4.** Let \( G \) be a finite group. For all \( i \geq 0 \), \( \mathcal{H}_i(X, G; \mathbb{Z}) = H_i(G, \mathbb{Z}) \) for any integral scheme \( X \).

**Proof.** An irreducible subscheme \( Z \subset G^i \times X \) that is finite and surjective over \( X \) must be isomorphic to \( X \) since the scheme \( G^i \times X \) is simply the disjoint union of copies of \( X \). It follows that the group \( C_0(G^i \times X/X) \) is the free abelian group on the closed points of \( G^i \). Thus, \( \mathcal{H}_i(X, G; A) \) is the \( i \)-th homology of the standard complex for computing \( H_\bullet(G, \mathbb{Z}) \). \( \square \)

In this case the pullback and transfer maps are easy to describe. For any morphism \( f : X \to S \), \( f^* : \mathcal{H}_i(S, G; \mathbb{Z}) \to \mathcal{H}_i(X, G; \mathbb{Z}) \) is induced by pullback of cycles, which in this case is the map sending \( \{g\} \times S \) to \( \{g\} \times X \). Thus \( f^* \) is simply the identity map (after making the identification of Proposition 2.4). Suppose \( f : X \to S \) is a finite surjective morphism between integral schemes, and set

\[
d = \deg(f) = [k(X) : k(S)].
\]
Then $f_*$ is induced by pushforward of cycles, which in this case is the map sending $\{g\} \times X$ to $d\{g\} \times S$. Thus $f_* : \mathcal{H}_i(X, G; \mathbb{Z}) \to \mathcal{H}_i(S, G; \mathbb{Z})$ is simply multiplication by $d$.

**Proposition 2.5.** Let $k$ be an algebraically closed field and let $A$ be an abelian group. Then we have natural (in $G$ and $A$) isomorphisms

$$\mathcal{H}_i(\text{Spec}(k), G; A) \cong H_i(G(k), A)$$

$$\mathcal{H}_i(\text{Spec}(k), G; A) \cong H^i(G(k), A),$$

for all $i \geq 0$, where $G(k)$ is the discrete group of $k$-rational points of $G$.

**Proof.** This follows from the observation that $C_0(G/\text{Spec}(k))$ is the free abelian group on $k$-points of $G$. □

If $k$ is not algebraically closed, then the situation is more complicated.

**Proposition 2.6.** Let $k$ be a field and $\overline{k}$ be an algebraic closure of $k$. Let $\Gamma$ denote the absolute Galois group $\text{Gal}(\overline{k}/k)$. Then for all $i \geq 0$,

$$\mathcal{H}_i(\text{Spec}(k), G; \mathbb{Z}) \cong H_i(BG(\overline{k})/\Gamma).$$

**Proof.** Observe that $C_0(G/\text{Spec}(k))$ is the free abelian group on the closed points of $G$, which are in one-to-one correspondence with the orbits of the $\Gamma$ action on $G(\overline{k})$. We thus have

$$C_0(G/\text{Spec}(k)) \cong \mathbb{Z}\{G(\overline{k})/\Gamma\}. \quad \square$$

This point of view allows us to make some calculations in Section 4.

Observe that a map $X \to G^i$ naturally defines an element of $C_0(G^i \times X/X)$, and hence there is a map of chain complexes

$$\mathbb{Z}\text{Hom}(X, BG) \to C_0(BG \times X/X).$$

For example, if $X = \text{Spec}(R)$ is affine, then a morphism $X \to G^i$ is simply an $R$-point of $G^i$ and hence we have the map of chain complexes

$$C_\bullet(G(R)) \to C_0(BG \times X/X).$$

**Proposition 2.7.** Let $X = \text{Spec}(R)$ be an affine scheme. Then there is a natural map

$$H_i(G(R), \mathbb{Z}) \to \mathcal{H}_i(X, G; \mathbb{Z})$$

for each $i \geq 0$.

□

These maps are isomorphisms in the contexts of Propositions 2.4 and 2.5, but in general they need not be either injective or surjective. In fact, we have the following result.

**Proposition 2.8.** There is an isomorphism

$$\mathcal{H}_1(\text{Spec}(R), G_m; \mathbb{Z}) \cong \mathbb{R}^\times$$

and the map $H_1(G_m(R); \mathbb{Z}) \to \mathcal{H}_1(\text{Spec}(R), G_m; \mathbb{Z})$ is surjective with kernel $\{\pm 1\}$. 
Proof. The abelian group \( H_1(\text{Spec}(R), \mathbb{G}_m; \mathbb{Z}) \) is generated by classes of non-zero complex numbers, \([z]\) for \( z \in \mathbb{C}^\times\), modulo the relations \([z] = [\overline{z}]\) and \([zw] = [z] + [w]\), for all \( z, w \in \mathbb{C}^\times\). The homomorphism \( \mathbb{Z}(\mathbb{C}^\times) \to \mathbb{R}_{>0}^\times \) induced by sending \( z \in \mathbb{C}^\times\) to \(|z|\) annihilates these relations and hence induces a map

\[
H_1(\text{Spec}(R), \mathbb{G}_m; \mathbb{Z}) \to \mathbb{R}_{>0}^\times.
\]

Likewise, these relations show that the function \( \mathbb{R}_{>0} \to H_1(\text{Spec}(R), \mathbb{G}_m; \mathbb{Z}) \) given by \( r \mapsto [r] \) is a homomorphism. The composition \( \mathbb{R}_{>0} \to H_1(\text{Spec}(R), \mathbb{G}_m; \mathbb{Z}) \to \mathbb{R}_{>0}^\times \) is clearly the identity. For \( w \in \mathbb{C}^\times\), we have \( w = z^2\), for some \( z\), and hence \([w] = [z^2] = 2[z] = [z] + [\overline{z}] = [|w|]\). This shows that the composition

\[
H_1(\text{Spec}(R), \mathbb{G}_m; \mathbb{Z}) \to \mathbb{R}_{>0} \to H_1(\text{Spec}(R), \mathbb{G}_m; \mathbb{Z})
\]

is also the identity.

Finally, the composition \( \mathbb{R}^\times \cong H_1(\mathbb{G}_m(R); \mathbb{Z}) \to H_1(\text{Spec}(R), \mathbb{G}_m; \mathbb{Z}) \to \mathbb{R}_{>0}^\times \) is given by \( r \mapsto |r|\). □

3. The Spectral Sequence and Friedlander’s Isomorphism Conjecture

Let \( \Delta^q \) denote the linear hypersurface in \( \mathbb{A}^{q+1} \) defined by the equation \( t_0 + t_1 + \cdots + t_q = 1 \). There are obvious coface and codegeneracy maps

\[
\delta^i : \Delta^{q-1} \to \Delta^q \quad \sigma^i : \Delta^q \to \Delta^{q-1}
\]

making \( \Delta^\bullet \) a cosimplicial scheme.

Fix \( s \geq 0 \). Applying the functor \( C_0(G^s \times -/-) \) to \( \Delta^\bullet \) yields a simplicial abelian group \( C_0(G^s \times \Delta^\bullet/\Delta^\bullet) \). We therefore have a double complex \( A_{\bullet\bullet} \) with

\[
A_{s,t} = C_0(G^s \times \Delta^t/\Delta^t).
\]

Applying the functor \( \text{Hom}(-, \mathbb{Z}/n) \) to \( A_{\bullet\bullet} \) yields a double cochain complex \( E_{\bullet\bullet}^s \) with

\[
E_{s,t}^s = \text{Hom}(C_0(G^s \times \Delta^t/\Delta^t), \mathbb{Z}/n).
\]

As usual, we have two first quadrant spectral sequences converging to the cohomology of the total complex. Taking horizontal cohomology first, we obtain

\[
E_1^{s,t} = \mathcal{H}^s(\Delta^t, G; \mathbb{Z}/n).
\]

Using the work of Suslin-Voevodsky and the second spectral sequence, we may identify the abutment. If \( \mathcal{F} \) is a presheaf with transfers, define a presheaf \( \mathcal{F}_t \) by \( \mathcal{F}_t(X) = \mathcal{F}(X \times \Delta^t) \). Then \( \mathcal{F}_\bullet \) is a simplicial presheaf of abelian groups. Also, we let \( C_\bullet(\mathcal{F}) \) denote the simplicial abelian group \( \mathcal{F}(\Delta^\bullet) \). There are obvious maps of simplicial presheaves \( C_\bullet(\mathcal{F}) \to \mathcal{F}_\bullet \) and \( \mathcal{F} \to \mathcal{F}_\bullet \) (where \( C_\bullet(\mathcal{F}) \) is regarded as a degree-wise constant presheaf).
Corollary 7.7 of [16] asserts that both of these maps induce isomorphisms upon applying the functor $\text{Ext}^\bullet_{\text{qfh}}(-, \mathbb{Z}/n)$, provided $k$ is algebraically closed and $n$ is relatively prime to the exponential characteristic of $k$. Moreover, these qfh Ext groups coincide with the étale Ext groups $\text{Ext}^\bullet_{\text{ét}}(-, \mathbb{Z}/n)$ by [16, 10.10].

Now consider the case $\mathcal{F} = C^0(G^s \times -/ -)$. Taking vertical homology in the double complex yields $s$-th column

$$\text{Ext}^\bullet_{\text{qfh}}(C^0(G^s \times -/ -), \mathbb{Z}/n) = \text{Ext}^\bullet_{\text{ét}}(C^0(G^s \times -/ -), \mathbb{Z}/n).$$

In turn, the latter is $\text{Ext}^\bullet_{\text{ét}}(\mathbb{Z}\text{Hom}(-, G^s), \mathbb{Z}/n)$ since $C^0(G^s \times -/ -)[1/p]$ is the qfh sheafification of $\mathbb{Z}[1/p]\text{Hom}(-, G^s)$. But these groups are precisely the étale cohomology groups $H^s_{\text{ét}}(G^s, \mathbb{Z}/n)$ (Corollary 7.8 of [16]). What we conclude, then, is that the second spectral sequence is simply the usual spectral sequence for computing the étale cohomology of the simplicial scheme $BG_k$:

$$E^{s,t}_1 = H^t_{\text{ét}}(G^s, \mathbb{Z}/n) \Rightarrow H^{s+t}(BG_k, \mathbb{Z}/n)$$

(see [5], p. 16). We have thus proved the following result.

**Theorem 3.1.** Let $k$ be an algebraically closed field and $n$ a positive integer relatively prime to the exponential characteristic of $k$. Then there is a first quadrant spectral sequence

$$E^{s,t}_1 = H^s(\Delta^t, G; \mathbb{Z}/n) \Rightarrow H^{s+t}(BG_k, \mathbb{Z}/n).$$

□

**Remark 3.2.** The results of [16] are stated only for fields of characteristic zero. Work of de Jong [2] allows us to replace this assumption with the assumption that $n$ is relatively prime to the exponential characteristic of $k$.

Observe that we have isomorphisms

$$E^{s,0}_1 = H^s(\text{Spec}(k), G; \mathbb{Z}/n) \cong H^s(G(k), \mathbb{Z}/n) = H^s(BG(k), \mathbb{Z}/n)$$

by Proposition 2.5, and under this isomorphism, the edge homomorphism

$$H^s_{\text{ét}}(BG_k, \mathbb{Z}/n) \to E^{s,0}_1 \cong H^s(BG(k), \mathbb{Z}/n)$$

is the map conjectured to be an isomorphism by Friedlander. Consequently, we have:

**Corollary 3.3.** Friedlander’s Isomorphism Conjecture for an algebraically closed field $k$, an algebraic group $G$ defined over $k$, and a positive integer $n$ relatively prime to the exponential characteristic of $k$ is equivalent to the assertion that the map of complexes

$$\text{Tot}(\text{Hom}(C^0(BG \times \Delta^\bullet/\Delta^\bullet), \mathbb{Z}/n)) \to \text{Hom}(C^0(BG/\text{Spec}(k)), \mathbb{Z}/n)$$

is a quasi-isomorphism.
For example, one might optimistically conjecture that for each fixed \( t \geq 0 \) the map

\[
\mathcal{H}^i(\Delta^t, G; \mathbb{Z}/n) \rightarrow \mathcal{H}^i(\text{Spec}(k), G; \mathbb{Z}/n)
\]

is an isomorphism, a result which would imply that \( \mathcal{K} \) is a quasi-isomorphism and hence the Isomorphism Conjecture. Though we have no counter-example to this stronger assertion, it seems unlikely to hold.

The evident generalization to arbitrary coefficients of the above assertion — i.e., the assertion that the map of complexes

\[
C_0(BG/\text{Spec}(k)) \rightarrow \text{Tot}(C_0(BG \times \Delta^s/\Delta^s))
\]

is a quasi-isomorphism — turns out to be false: Take \( G = \mathbb{G}_m \) and for each fixed \( t \) let \( C_0(\mathbb{G}_m^\times \times \Delta^t/\Delta^t) \) denote the normalized chain complex obtained from \( C_0(\text{BG}_m \times \Delta^t/\Delta^t) \) by modding out degeneracies. Then the bicomplexes \( C_0(BG_m^\times \times \Delta^s/\Delta^s) \) and \( C_0(BG_m \times \Delta^s/\Delta^s) \) are quasi-isomorphic, and by \( 17 \), we have that for each fixed \( s \), the complex \( C_0(\mathbb{G}_m^\wedge s \times \Delta^s/\Delta^s) \) computes the weight \( s \) motivic cohomology of the field \( k \):

\[
h_i(C_0(\mathbb{G}_m^\wedge s \times \Delta^s/\Delta^s)) \cong H_{\mathcal{M}}^{s-i}(k, \mathbb{Z}(s)).
\]

It follows that \( h_2 \) of the total complex associated to \( C_0(\mathbb{G}_m^\times \times \Delta^s/\Delta^s) \) is the second \( K \)-group of the field \( k \):

\[
h_2(C_0(\mathbb{G}_m^\times \times \Delta^s/\Delta^s)) \cong K_2(k) = \bigwedge^2(k^\times)/\langle a \wedge (1-a) | a, 1-a \in k^\times \rangle.
\]

By contrast, we have \( h_2(C_0(\text{BG}_m/\text{Spec}(k))) \cong \bigwedge^2(k^\times) \), so that \( \mathcal{K} \) has a kernel at \( h_2 \) (given by the Steinberg relations).

The map \( \mathcal{K} \) is a quasi-isomorphism for \( G = \mathbb{G}_m \) (with \( \mathbb{Z}/n \) coefficients), however, since the Isomorphism Conjecture holds for this algebraic group (see Corollary \( 13.5 \) below). This is connected to the fact that the motivic cohomology of \( k \) with \( \mathbb{Z}/n \) coefficients coincides with the singular cohomology of a point:

\[
h_i(C_0(\mathbb{G}_m^\wedge s \times \Delta^s/\Delta^s)) \cong H_{\mathcal{M}}^{s-i}(k, \mathbb{Z}/n(s)) \cong \begin{cases} \mathbb{Z}/n, & \text{if } s = i, \\ 0, & \text{otherwise}. \end{cases}
\]

The spectral sequence also allows us to deduce the following criterion for verifying Friedlander’s Isomorphism Conjecture:

**Theorem 3.4.** Let \( G \) be an algebraic group over an algebraically closed field \( k \) and let \( n \) be a positive integer relatively prime to the exponential characteristic of \( K \). Assume that for all smooth \( k \)-varieties \( X \) and closed points \( x \) of \( X \), the map

\[
H_i(G(k), \mathbb{Z}/n) \rightarrow H_i(G(\overline{R}), \mathbb{Z}/n)
\]

is an isomorphism for all \( i \leq r \), where \( \overline{R} \) is the absolute integral closure of \( R = \mathcal{O}_{X,x}^{\text{hens}}, \) the henselization of the local ring of \( X \) at \( x \). Then the map

\[
H_i(BG_k, \mathbb{Z}/n) \rightarrow H_i(BG(k), \mathbb{Z}/n)
\]
is an isomorphism for all \( i \leq r \) and, moreover, there is an exact sequence

\[
0 \rightarrow H^{r+1}_{\text{ét}}(BG_k, \mathbb{Z}/n) \rightarrow H^{r+1}(BG(k), \mathbb{Z}/n) \xrightarrow{d} H^{r+1}(\Delta^1, G; \mathbb{Z}/n),
\]

where \( d \) is the difference of the maps induced by the two face maps \( \Delta^0 \rightarrow \Delta^1 \).

**Proof.** The first implication is well-known under the stronger hypothesis that the map \( \mathcal{F} \) is an isomorphism with \( \mathcal{R} \) replaced by \( R \) (see e.g. [7]). Our machinery allows us to use this weaker hypothesis. Consider the spectral sequence

\[
E_1^{s,t} = \mathcal{H}^s(\Delta^t, G; \mathbb{Z}/n) \Rightarrow H^{s+t}_{\text{ét}}(BG_k, \mathbb{Z}/n)
\]

of Theorem 3.1. Since \( \text{Spec}(\mathcal{R}) \) is a limit of qfh neighborhoods of \( x \in X \), our hypothesis implies that the functors \( \mathcal{H}_i(-, G; \mathbb{Z}/n) \) are locally constant for the qfh topology. If \( \mathcal{F} \) is a presheaf on the category of schemes over \( k \), let \( \mathcal{F}_{\text{qfh}} \) denote the sheafification of \( \mathcal{F} \) in the qfh topology. We then have the following chain of natural isomorphisms (where \( \text{Ab} \) denotes the category of abelian groups):

\[
\text{Ext}^i_{\text{Ab}}(\mathcal{H}_i(\text{Spec}(k), G; \mathbb{Z}/n), \mathbb{Z}/n) \cong \text{Ext}^i_{\text{qfh}}(\mathcal{H}_i(-, G; \mathbb{Z}/n)_{\text{qfh}}, \mathbb{Z}/n)
\]

\[
\cong \text{Ext}^i_{\text{qfh}}(\mathcal{H}_i(- \times \Delta^i, G; \mathbb{Z}/n)_{\text{qfh}}, \mathbb{Z}/n)
\]

\[
\cong \text{Ext}^i_{\text{qfh}}(\mathcal{H}_i(\Delta^i, G; \mathbb{Z}/n), \mathbb{Z}/n)
\]

\[
\cong \text{Ext}^i_{\text{Ab}}(\mathcal{H}_i(\Delta^i, G; \mathbb{Z}/n), \mathbb{Z}/n)
\]

where the first holds because the functor \( \mathcal{H}_i(-, G; \mathbb{Z}/n) \) is locally constant in the qfh topology, the middle two follow from Theorem 7.6 of [10], and the last holds by definition. Since the functors \( \mathcal{H}_i(-, G; \mathbb{Z}/n) \) are \( n \)-torsion, the map \( \mathcal{H}_i(\Delta^i, G; \mathbb{Z}/n) \rightarrow \mathcal{H}_i(\text{Spec}(k), G; \mathbb{Z}/n) \) must be a weak equivalence for \( i \leq r \) and hence the same is true of the map

\[
\mathcal{H}_i(\text{Spec}(k), G; \mathbb{Z}/n) \rightarrow \mathcal{H}_i(\Delta^i, G; \mathbb{Z}/n)
\]

for \( i \leq r \). Thus, for all \( s \leq r \),

\[
E_2^{s,t} = \begin{cases} 
H^s(BG(k), \mathbb{Z}/n) & t = 0 \\
0 & t > 0.
\end{cases}
\]

So we see that the map \( H^1_{\text{ét}}(BG_k, \mathbb{Z}/n) \rightarrow H^i(BG(k), \mathbb{Z}/n) \) is an isomorphism for \( i \leq r \). Moreover, we have \( E_r^{r+1,0} = E_2^{r+1,0} \) and therefore we obtain the exact sequence

\[
0 \rightarrow H^{r+1}_{\text{ét}}(BG_k, \mathbb{Z}/n) \rightarrow H^{r+1}(BG(k), \mathbb{Z}/n) \xrightarrow{d} H^{r+1}(\Delta^1, G; \mathbb{Z}/n).
\]

\[\square\]

**Corollary 3.5.** The natural map

\[
H^i_{\text{ét}}(BG_k, \mathbb{Z}/n) \rightarrow H^i(BG(k), \mathbb{Z}/n)
\]

is an isomorphism in the following cases:

1. \( G \) finite, solvable, or the normalizer of a maximal torus in a reductive group;
(2) \( G = GL_m \) in cohomological degrees \( i \leq 3 \).

Moreover, there is an exact sequence
\[
0 \rightarrow H^4_{\text{ét}}(BGL_m, \mathbb{Z}/n) \rightarrow H^4(BGL_m(k), \mathbb{Z}/n) \rightarrow H^4(\Delta^1, GL_m; \mathbb{Z}/n).
\]

Proof. Let \( X \) be a smooth \( k \)-variety, \( x \in X \) a closed point, and \( \overline{R} \) the absolute integral closure of \( O_{X,x}^{\text{ben}} \).

If \( G \) is finite, then clearly
\[
H_i(G(k), \mathbb{Z}/n) \cong H_i(G(\overline{R}), \mathbb{Z}/n)
\]
for all \( i \geq 0 \). If \( G \) is solvable, then \( G \) has a descending central series whose graded quotients are either \( G_m \) or \( G_a \). Clearly,
\[
H_i(G_a(k), \mathbb{Z}/n) \rightarrow H_i(G_a(\overline{R}), \mathbb{Z}/n)
\]
is an isomorphism for \( i \geq 0 \). An easy application of Hensel's lemma shows that the same is true for \( G_m \). By iterated use of the Hochschild–Serre spectral sequence we see that the map
\[
H_i(G(k), \mathbb{Z}/n) \rightarrow H_i(G(\overline{R}), \mathbb{Z}/n)
\]
is an isomorphism for all \( i \geq 0 \). If \( T \) is a maximal torus in a reductive group \( S \), then there is a short exact sequence
\[
1 \rightarrow T \rightarrow N_S(T) \rightarrow W \rightarrow 1
\]
where \( W \) is finite. Again, the Hochschild–Serre spectral sequence shows that the map
\[
H_i(N_S(T)(k), \mathbb{Z}/n) \rightarrow H_i(N_S(T)(\overline{R}), \mathbb{Z}/n)
\]
is an isomorphism for \( i \geq 0 \). (All the preceding facts may be found in [7].)

Finally, if \( G = GL_m \), then for \( n \) prime to the characteristic of \( k \) the map
\[
H_i(G(k), \mathbb{Z}/n) \rightarrow H_i(G(\overline{R}), \mathbb{Z}/n)
\]
is an isomorphism for \( i \leq 3 \) ([9], p. 146). Therefore, the isomorphism conjecture holds in this case as well and we obtain the above mentioned exact sequence. \( \Box \)

4. Calculations

For an arbitrary field \( k \), Proposition 2.6 relates the groups \( H_\bullet(\text{Spec}(k), G; \mathbb{Z}) \) with a construction using a quotient of the action of the absolute Galois group \( \Gamma = \text{Gal}(\overline{k}/k) \):
\[
H_i(\text{Spec}(k), G; \mathbb{Z}) \cong H_i(BG(\overline{k})/\Gamma).
\]

Now consider the field \( k = \mathbb{R} \). We have \( \overline{\mathbb{R}} = \mathbb{C} \) and \( \text{Gal}(\overline{k}/k) = \mathbb{Z}/2 \). If \( G \) is an algebraic group over \( \mathbb{R} \), then we have an isomorphism
\[
H_\bullet(\text{Spec}(\mathbb{R}), G; A) = H_\bullet(BG(\mathbb{C})/\Gamma; A).
\]

We therefore have the following result.
Theorem 4.1. Let $G$ be an algebraic group over $\mathbb{R}$ and let $A = \mathbb{Q}$ or $\mathbb{Z}/n$, where $n$ is odd. Then there is a canonical isomorphism

$$H_i(\text{Spec}(\mathbb{R}), G; A) \xrightarrow{\cong} H_i(G(\mathbb{C}), A)^{\mathbb{Z}/2}$$

for each $i \geq 0$.

Proof. This is a standard fact in the homology of quotient spaces, proved using the transfer map. See [1], p. 38. □

If $A = \mathbb{Z}/2$, the calculation is more difficult and we are only able to handle two cases. First, consider the case $G = \mathbb{G}_m$. We are interested in calculating $H_\bullet(B\mathbb{G}_m(\mathbb{C})/\Gamma; \mathbb{Z}/2)$. Since homology and cohomology are dual with field coefficients, it suffices to compute $H^\bullet(B\mathbb{G}_m(\mathbb{C})/\Gamma; \mathbb{Z}/2)$. While it is possible to do this directly using various standard techniques, we proceed as follows.

Suppose $Q$ is a finite group acting on a CW-complex $X$ in such a way that if an element of $Q$ fixes a cell of $X$, then it fixes it pointwise. There is associated to this a cohomology theory, called (ordinary) Bredon cohomology, $H^\bullet_Q(X; M)$, where $M$ is a Mackey functor, with the property that

$$H^\bullet_Q(X; \mathbb{Z}) \cong H^\bullet(X/Q; \mathbb{Z}).$$

Here, $A$ is the constant Mackey functor associated to $A$. For a thorough discussion of Bredon cohomology, we refer the reader to [10]. What is relevant for us is the following result.

Proposition 4.2. Suppose $f : X \rightarrow Y$ is a map of $Q$-CW-complexes such that for each subgroup $H$ of $Q$ the induced map $f^H : X^H \rightarrow Y^H$ of fixed point spaces induces an isomorphism

$$H^\bullet(X^H; \mathbb{Z}/p) \rightarrow H^\bullet(Y^H; \mathbb{Z}/p).$$

Then the induced map

$$f^* : H^\bullet_Q(Y; \mathbb{Z}/p) \rightarrow H^\bullet_Q(X; \mathbb{Z}/p)$$

is an isomorphism.

Proof. See [10], p. 26. □

This suggests the following.

Equivariant Isomorphism Conjecture 4.3. Let $G$ be an algebraic group over $\mathbb{R}$. Let $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$, and let $p$ be a prime number. Then the identity map $G(\mathbb{C}) \rightarrow G(\mathbb{C})^{\text{top}}$ induces an isomorphism

$$H^\bullet_\Gamma(BG(\mathbb{C})^{\text{top}}; \mathbb{Z}/p) \rightarrow H^\bullet_\Gamma(BG(\mathbb{C}; \mathbb{Z}/p)).$$

Proposition 4.4. Friedlander’s isomorphism conjecture for $G(\mathbb{C})$ and $G(\mathbb{R})$ implies the equivariant isomorphism conjecture for $G(\mathbb{C})$. 
Proof. Note that for $\Gamma = \mathbb{Z}/2$, the only subgroups are $\Gamma$ and the trivial subgroup $\{1\}$, and the corresponding fixed point spaces are $BG(\mathbb{R})$ and $BG(\mathbb{C})$, respectively. Thus, if the maps
\[
H_\bullet(BG(\mathbb{R}); \mathbb{Z}/p) \to H_\bullet(BG(\mathbb{R})^{\text{top}}; \mathbb{Z}/p)
\]
and
\[
H_\bullet(BG(\mathbb{C}); \mathbb{Z}/p) \to H_\bullet(BG(\mathbb{C})^{\text{top}}; \mathbb{Z}/p)
\]
are both isomorphisms, then Proposition 4.2 implies that the map
\[
H_\bullet(\Gamma; BG(\mathbb{C})^{\text{top}}; \mathbb{Z}/p) \to H_\bullet(\Gamma; BG(\mathbb{C}); \mathbb{Z}/p)
\]
is an isomorphism. □

Corollary 4.5. Let $G$ be a solvable Lie group. Then the equivariant isomorphism conjecture holds for $G$.

Proof. Friedlander’s isomorphism conjecture holds for $G(\mathbb{C})$ and $G(\mathbb{R})$ [11]. □

In the case $G = G_m$, we see that there is an isomorphism
\[
H_\bullet(\mathbb{C}P^\infty; \mathbb{Z}/2) \to H_\bullet(BG_m(\mathbb{C})^{\text{top}}; \mathbb{Z}/2).
\]
We are trying to calculate the latter; these are the groups $H^\bullet(\text{Spec}(\mathbb{R}), G_m; \mathbb{Z}/2)$. Note, however, that $BG_m(\mathbb{C})^{\text{top}}$ is (equivariantly) homotopy equivalent to $\mathbb{C}P^\infty$. So we must compute the groups
\[
H_\bullet(\mathbb{C}P^\infty; \mathbb{Z}/2).
\]
Associated to a $\mathbb{Z}/2$-CW-complex $X$ is its $\mathbb{Z}/2$-equivariant cohomology, which forms a bigraded ring
\[
H^{\bullet, \bullet}(X; \mathbb{A}),
\]
and extends the Bredon cohomology ring in the sense that we have
\[
H^{k, 0}(X; \mathbb{Z}) \cong H^{k}_{}(X; \mathbb{Z}).
\]
We shall not need the detailed definition of this theory, but the interested reader may consult [10]. What is important for us is the following result.

Proposition 4.6. The cohomology of $\mathbb{C}P^\infty$ is given by
\[
H^{\bullet, \bullet}(\mathbb{C}P^\infty; \mathbb{Z}/2) \cong H^{\bullet, \bullet}(pt; \mathbb{Z}/2)[c],
\]
where $\deg c = (2, 1)$.

Proof. See [3], 5.4, p. 18. □

The $\mathbb{Z}/2$-equivariant cohomology of a point with $\mathbb{Z}$ coefficients has been calculated (see [3], Appendix B), and from it one deduces
\[
H^{p, q}(pt; \mathbb{Z}/2) \cong \begin{cases} 
\mathbb{Z}/2 & q \geq p \geq 0 \text{ or } q + 2 \leq p \leq 0 \\
0 & \text{otherwise.}
\end{cases}
\]
Moreover, there is a commutative product on \( H^{\bullet,\bullet}(pt) \) with the property that the product of any element in degree \((p, q)\), \(q \geq p \geq 0\) with an element of degree \((i, j)\), \(j + 2 \leq i \leq 0\) is zero.

**Proposition 4.7.** The cohomology groups \( H^{s,0}(\mathbb{C}P^\infty; \mathbb{Z}/2) \) satisfy \( H^{1,0} = 0 \) and for \( k \geq 1 \), \( H^{2k,0} \cong H^{2k+1,0} \cong (\mathbb{Z}/2)^{k-1} \).

**Proof.** Observe that elements of \( H^{s,0}(\mathbb{C}P^\infty) \) arise only as products of powers of the generator \( c \) (of degree \((2, 1)\)) with elements of \( H^{\bullet,\bullet}(pt) \) in degrees \((p, q)\) with \(q + 2 \leq p \leq 0\). As there are no elements in degrees \((-1, -1), (0, -1)\), or \((-1, -2)\), we see that \( H^{s,0} \) vanishes for \( s = 1, 2, 3\). Denote the generator of \( H^{s,t}(pt) \) by \( x_{(s,t)} \). Then one sees easily that for \( k \geq 2 \), we have

| group          | generators                                         |
|----------------|----------------------------------------------------|
| \( H^{2k,0} \) | \( x_{(0,-k)} c^k, x_{(-2,-(k+1))} c^{k+1}, \ldots, x_{(4-2k,2-2k)} c^{2k-2} \) |
| \( H^{2k+1,0} \)| \( x_{(-1,-(k+1))} c^{k+1}, x_{(-3,-(k+2))} c^{k+2}, \ldots, x_{(3-2k,1-2k)} c^{2k-1} \) |

This completes the proof. \( \square \)

Recall that we have isomorphisms

\[ H^*_!(\mathbb{C}P^\infty; \mathbb{Z}/2) \cong H^*(BG_m(\mathbb{C}); \mathbb{Z}/2) \cong H^*(\text{Spec}(\mathbb{R}), G_m; \mathbb{Z}/2) . \]

Proposition 4.7 therefore gives us the following result.

**Theorem 4.8.** For all \( k \geq 1 \),

\[ H^k(\text{Spec}(\mathbb{R}), G_m; \mathbb{Z}/2) \cong H^{2k+1}(\text{Spec}(\mathbb{R}), G_m; \mathbb{Z}/2) \cong (\mathbb{Z}/2)^{k-1} . \]

The same is therefore true for \( H_{2k} \) and \( H_{2k+1} \). \( \square \)

Now suppose \( G \) is a unipotent group over \( \mathbb{R} \). Let \( X = BG(\mathbb{C}) \) and \( Y = BG(\mathbb{R}) = X^G \). According to [10], p. 35, there is a long exact sequence (with \( \mathbb{Z}/2 \) coefficients)

\[ \tilde{H}^n((X/Y)/\Gamma) \to H^n(X) \to \tilde{H}^n((X/Y)/\Gamma) \oplus H^n(Y) \to \tilde{H}^{n+1}((X/Y)/\Gamma). \]

Note, however, that

\[ (X/Y)/\Gamma = (X/\Gamma)/Y \]

and

\[ \tilde{H}^*((X/\Gamma)/Y) \cong H^*(X/\Gamma, Y). \]

Thus, this sequence becomes

\[ H^n(X/\Gamma, Y) \to H^n(X) \to H^n(X/\Gamma, Y) \oplus H^n(Y) \to H^{n+1}(X/\Gamma, Y). \]

Since \( \tilde{H}^n(X) = \tilde{H}^n(Y) = 0 \) in this case, we see that \( H^n(X/\Gamma, Y) = 0 \) for all \( n \geq 0 \). The long exact sequence of the pair \((X/\Gamma, Y)\) then shows that

\[ \tilde{H}^n(X/\Gamma) = 0 \]

for all \( n \geq 0 \). We therefore have the following result.
Theorem 4.9. Let $G$ be a unipotent group over $\mathbb{R}$. Then for all $i > 0$, $\mathcal{H}^i(\text{Spec}(\mathbb{R}), G; \mathbb{Z}/2) = 0$. The same is therefore true for $\mathcal{H}_i$. □

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