A non extensive approach to the entropy of symbolic sequences

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Symbolic sequences with long-range correlations are expected to result in a slow regression to a steady state of entropy increase. However, we prove that also in this case a fast transition to a constant rate of entropy increase can be obtained, provided that the extensive entropy of Tsallis with entropic index \( q \) is adopted, thereby resulting in a new form of entropy that we shall refer to as Kolmogorov-Sinai-Tsallis (KST) entropy. We assume that the same symbols, either 1 or \(-1\), are repeated in strings of length \( l \), with the probability distribution \( p(l) \propto \frac{1}{l^\alpha} \). The numerical evaluation of the KST entropy suggests that at the value \( \mu = 2 \) a sort of abrupt transition might occur. For the values of \( \mu \) in the range \( 1 < \mu < 2 \) the entropic index \( q \) is expected to vanish, as a consequence of the fact that in this case the average length \( <l> \) diverges, thereby breaking the balance between determinism and randomness in favor of determinism. In the region \( \mu \geq 2 \) the entropic index \( q \) seems to depend on \( \mu \) through the power law expression \( q = (\mu - 2)^\alpha + 1 \) with \( \alpha \approx 0.13 \) \((q = 1 \text{ with } \mu > 3)\). It is argued that this phase-transition like property signals the onset of the thermodynamical regime at \( \mu = 2 \).

It has been recently pointed out [1] that power law spectra are observed in many disciplines of science ranging from astronomy, geography and physics to electronics, acoustic, linguistic and music. It is also interesting to establish a connection between these observed properties and their algorithmic complexity. This is important not only from a conceptual point of view [2]: It also might result in methods for the detection itself of correlations. In this respect, we want to mention the search for correlations in DNA sequences based on the adoption of entropic indicators [3,4].

It has been remarked [5], however, that something intermediate between periodic and chaotic dynamical behavior exists and that suitable tools to analyze these processes must be built up. These conclusions are widely shared in literature. For instance, also the authors of Refs. [1,4,10] as well as those of Ref. [8], show that the entropy of symbolic sequences in the case of long-range correlations exhibits a regression to the condition of constant Kolmogorov entropy which turns out to be very slow. Analogous results are found in many other papers [11,13] as well as in earlier papers [14].

We shall refer ourselves to the Kolmogorov entropy applied to the symbolic sequences as metric entropy (ME) [15] to keep it distinct from the Kolmogorov-Sinai entropy (KSE) [16,17]. The two entropies are closely related to one another, since both entropies are expressed in terms of the Shannon-Gibbs entropy. However, the latter, the KSE, refers to individual trajectories and, in principle, does not imply any coarse-graining if the assumption is made that cells and time steps of arbitrarily small size can be used. The former applies to symbolic sequences and consequently might be affected by a so large coarse-graining process as to lose a direct connection with the rules, either stochastic or deterministic, from which the sequence is generated. This aspect will be made more transparent by the discussion of the numerical experiment described in this paper.

The main purpose of this paper is that of discussing the consequences of expressing the ME in terms of the Tsallis entropy [18] rather than of the Shannon entropy. This is a form of ME that we shall refer to as Kolmogorov-Sinai-Tsallis (KST) entropy. The Tsallis entropy reads

\[
H_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1}.
\]

Note that this entropy is characterized by the index \( q \) whose departure from the conventional value \( q = 1 \) signals the thermodynamic effects of either long-range correlations in fractal dynamics or the non-local character of quantum mechanics [19]. The increasing interest for Tsallis’ non-extensive entropy is testified by the exponentially growing list of publications on this hot issue [20].

Of remarkable interest for the subject of fractal dynamics is the discovery recently made by Tsallis et al. [21] that the entropic index \( q \) also determines the specific analytical form illustrating the trajectory instability.

Two trajectories, moving from infinitely close but distinct initial conditions, depart from one another with a law more general than the exponential prescription. The exponential instability is a sort of singularity, namely, a special case of a more general, non-exponential, prescription. This important result is based on the generalization of the KSE [16,17] and consequently of the theorem of Pesin [22]. Palatella and Grigolini [19] have recently corroborated the conclusions of Miller and Sarkar [23] who

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prove that in the quantum case the Von Neumann entropy is linearly proportional to the KSE. Furthermore, the results of these authors have been extended to the case where the quantum expression for the entropy (the von Neumann entropy) is expressed in terms of the Tsallis prescription. The interesting conclusion is that $q < 1$: Palatella and Grigolini argue that this result is a reflection of the occurrence of the Anderson localization.

The present paper is devoted to discussing the convenience of the KST entropy to reveal whether or not a symbolic sequence does have or not a thermodynamical nature. The discussion rests on a key experiment, planned for the specific purpose of establishing correlations in sequences of symbols. The sequence of symbols is established as follows. Two computer generators of random numbers, $x$ and $z$, are used. The former generates random numbers distributed with equal probability in the interval $[0, 1]$ and the latter is the generator of the fluctuations $z = +1$ and $z = -1$, with the same statistical weight. The uncertainty associated with each drawing of the numbers $x$ is

$$h_x = \ln W_x,$$  

where $W_x = 1/\Delta x$ and $\Delta x$ denotes the resolution of the former random generator. The drawing of the numbers $z$ is equivalent to tossing a coin, and consequently is associated with the uncertainty

$$h_z = \ln 2.$$  

Let us imagine now that at regular intervals of time, with the time step $\Delta t = 1$, we draw a number $x$ and a number $z$. The uncertainty $H(N)$ grows as a linear function of the number of drawings $N$,

$$H(N) = N(\ln 2 + \ln W_x).$$  

We introduce a deterministic rule into this totally stochastic picture. This is done by replacing the variable $x$ with the variable $y$ related to $x$ by

$$y = A\left[\frac{1}{(1-x)^{\mu-1}} - 1\right].$$

The probability distribution of the variable $y$ is given by

$$p(y) = (\mu - 1)\frac{A^{\mu-1}}{(A+y)^\mu}.$$  

Note that the first moment of the variable $y$ is given by

$$< y >= \frac{A}{\mu - 2}.$$  

This means that the value $\mu = 2$ is a critical point at which the first moment of the new variable $y$ diverges.

The symbolic sequence is obtained by drawing the number $x$ first. This number determines the number $y$ according to Eq. (1) and fixes the number of sites $N_y = [y] + 1$, with $[y]$ denoting the integer part of $y$, to fill with the same symbol (either 1 or −1). Then we draw the number $z$, and we fill these sites either with +1 or with −1 according to whether we get $z = 1$ or $z = -1$. Note that, according to the treatment of [24], the length of the strings with the same symbols (either +1 or −1) is proportional to the length of the laminar regions generated by the nonlinear maps that are currently used to mimic turbulent phenomena [25]. For this reason we shall refer to them as laminar strings. We can thus provide further support to our conviction that critical properties have to be expected at $\mu = 2$. In fact we notice that the divergence of Eq. (1) at $\mu = 2$ implies that the mean length of the laminar strings is infinite, and that, consequently, once one symbol is known, the chances of guessing correctly a large number of symbols coming afterwards are high. Perhaps, a more proper way of illustrating the region with $1 < \mu \leq 2$, where all the moments of the distribution $p(y)$ of Eq. (3) diverge, is that of referring to it as the region where the balance between randomness and determinism is broken and determinism prevails [26].

In conclusion, we think that the deterministic nature of this region can be properly denoted by the entropic index $q = 0$. On the other hand, we expect that the region $\mu > 3$ is characterized by $q = 1$. This is so because the region $\mu > 3$ implies that the second moment, as well as the first moment of $p(y)$, is finite, thereby ensuring the validity of the central limit theorem, and with it, of ordinary statistical mechanics. This means that $\mu > 3$ is expected to yield $q = 1$.

The original uncertainty of Eq. (1) is deformed by the nonlinear transformation of Eq. (3). However, as we have seen, this can force $q$ to depart from the usual statistical value $q = 1$ only in the region $\mu < 3$. The region $\mu < 2$ is expected to yield $q = 0$. We are thus only left with the problem of establishing the dependence of $q$ on $\mu$ in the region $2 < \mu \leq 3$. This can be done evaluating numerically the KST entropy as follows. After defining a given sequence, we fix a window of length $N$. Note that this length $N$ from now on will be referred to as time. Then we move this window along the sequence generated according to the rules earlier illustrated. For any position of this window we find a given configuration $A_1 \bullet A_2 \bullet \ldots \bullet A_N$, where the $A_i$’s have either the value +1 or the value −1. We count the number of configurations of the same kind obtained moving the window along the chain, then we divide the number of these configurations by the total number of possible configurations $W(N)$, thereby determining the probabilities $p_i$. Finally, we use Eq. (1) to evaluate the entropy corresponding to this window of length $N$. It is convenient to study all this in the specific case where the symbolic sequence is generated with no correlation among the distinct sites. In this specific case
different values of the entropic index \( q \) denotes the behavior corresponding to \( q \). The middle line denotes the behavior corresponding to \( q = 1 \). Let us identify, therefore, \( q = 1 \) with \( q_{\text{true}} \), namely, the entropic index properly reflecting a given statistical condition, the total absence of correlations, in this case. Then we see that for \( q < q_{\text{true}} \), the time derivative of entropy tends to increase upon increase of the time \( N \). We see also that if the probing index \( q \) is larger than the correct entropic index, namely, \( q > q_{\text{true}} \), the time evolution of \( H_q(N) \) is characterized by a rate of increase smaller than the increase linear in time. It is plausible that the same qualitative behavior is present even if \( q_{\text{true}} \neq 1 \). In fact, if we assume this behavior to be valid in general, namely, even in the case where \( q_{\text{true}} < 1 \), we predict that the entropy growth for \( q = 1 \) is slower than that of a linear function of time, thus fitting the observation made by several authors (see, for instance, the work of Ref. [24]). This is an important remark since our main purpose here is to apply our statistical analysis to the case where the symbolic sequence is characterized by extended correlations, and consequently the entropic index is expected to depart from the normal value \( q_{\text{true}} = 1 \).

However, before addressing this challenging problem, it is convenient to recall some important properties. First of all, it is worth noticing that the rules earlier adopted are equivalent to those used in recent papers [27] to build up sequences that turn out to be statistically equivalent to the real DNA sequences. The distributions of +1, corresponding to purines, and of −1, corresponding to pyrimidines, was actually established adopting a nonlinear map [31]. The nonlinear map adopted, in turn, was the same as that widely used in the recent few years to generate anomalous diffusion. In a more recent paper [24], it has been shown that these nonlinear maps produce effects statistically equivalent to a stochastic generator which is, in fact, the same as that earlier illustrated as a generator of long-range correlations in the symbolic sequences under study in this paper.

Let us focus now our attention on the fact that any finite string \( A_1 \cdots A_N \) can be associated to an erratic trajectory moving from the “time” \( i = 1 \) to the “time” \( i = N \) on an one-dimensional lattice. The correspondence is established using the following prescription. At the time \( i \) the random walker makes a jump of unit length to the right or to the left according to whether \( A_i = 1 \) or \( A_i = -1 \). It is shown [24] that in the case of a random walker with correlations infinitely extended in time there is a significant probability that the random walker might make \( N \) steps in the same direction. Thus, in a process of diffusion, with all the walkers initially concentrated in the same site, the distribution will split into two ballistic peaks moving in opposite directions. With the increase of \( N \) an increasing number of walkers belonging to a peak moving in a given direction will make jumps in the opposite direction. Thus the intensity of the side peaks of the distribution is a decreasing function of time, known [24] to be proportional to the correlation function \( \Phi(k) \equiv \langle A_i A_{i+k} \rangle / \langle A_i \rangle \). It is evident that the strings \( A_1 \cdots A_2 \cdots \cdots \cdots \cdots \cdots \cdots A_N \) with all the \( A_i \)s equal to either +1 or −1, have the same intensity as these side peaks, to which these strings are equivalent. For this reason we shall refer ourselves to these side strings, with the same length as that of the exploring window of size \( N \) and with all the symbols \( A_i \) corresponding to the same letter, as border strings.

For the sake of some preliminary remarks we make the simplifying assumption that all the strings but the border strings have the same probability \( p(N) \). The dependence of \( p(N) \) on \( N \) is established by setting the normalization condition which yields

\[
p(N) = \frac{1 - 2\Pi(N)}{2N - 2}.
\]

As earlier remarked, according to Ref. [24] \( \Pi(N) = \frac{\Phi(N)}{2} \).

Thus, under the assumption of equal probability for all the strings but the border strings, we can write

\[
H_q(N) = \frac{2\Pi(N)^q + (2^N - 2)^{1-q}(1 - 2\Pi(N))^q - 1}{1 - q},
\]

with \( \Pi(N) \) given by Eq. (10). We note that, in principle, the rules adopted to establish long-range correlations in the sequences under study in this paper, make the correlation function \( \Phi(N) \) read

\[
\Phi(N) = \frac{A^\beta}{(A + N)^\beta},
\]

where \( \beta = \mu - 2 \). Consequently, the decay of these border strings is extremely slow and it dominates the entropy time evolution for a long time. On the other hand, for times so long as to make the contribution of the central part more important than that resulting from the border strings, the correct entropic index is given by \( q = 1 \), in accordance to the fact that in such a condition the statistical properties of the sequences become indistinguishable from that of totally uncorrelated sequences [21]. This is in line with the fact that the diffusion process [24] resulting from these rules is characterized by two distinct rescaling properties, the ballistic rescaling of the peaks and the Lévy rescaling of the central part of the diffusion. This means that the interesting statistical properties are
blurred by the presence of the border strings. For this reason, we decided to disregard the border strings and to set the normalization condition only on the other strings.

In principle, if no length limitation were set on the analysis of data, it would be possible to derive the correct statistical properties by examining suitably large windows. However, for the sake of computational simplicity we set the maximum length of the window to be \( N_{\text{max}} = 10 \). On the other hand, as we shall see, the approach based on disregarding the border strings makes it possible to reveal the effect of correlations on the entropic index with sequences of relatively small length.

It has to be stressed that the detection of the proper entropic index becomes more and more difficult as the power index \( \mu \) comes closer and closer to the critical value \( \mu = 2 \). In fact, the probabilities of given strings of length \( N \) are closely related to the correlation functions. The correlation function \( \Phi(N) \), for instance, on the basis of the Shannon-McMillan-Breiman theorem \([32, 33]\), is the probability of a string of length equal to 2. This makes it possible to explain why the finite length of the sequence yields an error on the numerical evaluation of the probability of a given sequence and suggests how to correct this error. In fact, the finite length of the sequence causes the truncation of the longer strings and, consequently, the decay of correlation function \( \Phi(N) \) becomes faster than theoretically expected on the basis of the prescription of Eq. \( (3) \). Thus, rather than expressing the entropic index \( q \) in terms of the parameter \( \mu \) corresponding to the prescription of Eq. \( (3) \), we relate \( q \) to an effective \( \tilde{\mu} \), obtained from the numerical evaluation of the correlation function \( \Phi(N) \). More precisely, we determine numerically the parameter \( \beta \) and from it \( \tilde{\mu} = 2 + \beta \). The numerical results show that for values of \( \tilde{\mu} \approx 2.3 \) or larger, the effective power index coincides with the value that theoretically should correspond to Eq. \( (8) \).

These numerical expedients make it possible for us to bring the determination of \( q \) as a function of \( \mu \), much closer to the critical region \( \mu = 2 \). The method adopted is illustrated by Fig. 3. As expected, on the basis of the results illustrated by Fig. 2 and concerning the theoretical prescription of Eq. \( (5) \), there exists a crucial value of \( q \), which results in a linear dependence of \( H_q(N) \) on \( N \). In Fig. 3, for instance, we see that at \( \tilde{\mu} \approx \mu = 2.5 \) the solid line, corresponding to \( q \approx 0.89 \) fits very well a straight line.

Using this numerical method to determine \( q \) we find the interesting results illustrated in Fig. 3. On the basis of the earlier remarks making plausible that \( q = 0 \) at \( \mu = 2 \), we have been led to fit the numerical data with

\[
q = (\mu - 2)^\alpha, \quad (13)
\]

for \( \mu \geq 2 \) and

\[
q = 1 \quad (14)
\]

for \( \mu \geq 3 \).

We see from Fig. 3 that the fitting function of Eq. \( (13) \) results in a satisfactory agreement with the numerical result if we set \( \alpha \approx 0.13 \). This means that the critical value \( q = 0 \) is reached with an infinite derivative, reinforcing our conviction that \( \mu = 2 \) is a critical point of transition to thermodynamics. The disorder in the region \( 2 < \mu < 3 \) is partial, and localized to the transition from one laminar string to another. However, this is enough to generate a thermodynamic behavior. The traditional wisdom would confine thermodynamics to the region \( \mu > 3 \), which is where the conventional central limit theorem applies. In a sense this analysis shows that thermodynamics is possible also in the region where the central limit theorem holds in the generalized form established by Lévy \([35, 36]\). It is interesting to remark that earlier research work \([37, 38]\) has established that the dynamical approach to diffusion, based on the stationary assumption on the fluctuations responsible for diffusion, is incompatible with the condition \( \mu < 2 \). In this region a diffusion process must rest on a continuous-time random walk method implying the breakdown of the stationary assumption \([37]\). The region \( 2 \leq \mu < 3 \) is compatible with stationary diffusion even if the diffusion process departs from ordinary Brownian diffusion and takes the shape of a Lévy process \([24]\). Therefore we conclude that the stationary diffusion processes have the same regime of validity as the non-extensive thermodynamics of Tsallis. In fact, this paper shows that the new perspective of Tsallis extends the regime of validity of thermodynamics to regions earlier imagined as being non thermodynamic, in this case, to \( \mu < 3 \). However, thermodynamics, even within this new perspective, cannot overcome the border \( \mu = 2 \). In other words, it seems that the Tsallis thermodynamics has the same regime of validity as the dynamic approach to diffusion, which is based on the assumption that fluctuations are characterized by a stationary correlation function \([39]\).

We have seen that the numerical calculations rests on both the expedient of adopting \( \tilde{\mu} \) rather than \( \mu \) and that of neglecting the border strings. The latter method is not only an expedient to extend the regime of validity of our numerical calculations. It reflects a property that probably deserves further studies. In fact the border strings correspond to the peaks that appear in the dynamical approach to the Lévy diffusion. As discussed in Ref. \([24]\), these peaks are a consequence of the dynamic approach and the Lévy statistics are recovered only in the time asymptotic limit. On the other hand, as pointed out by the authors of \([11]\), a satisfactory agreement between the entropic properties of trajectories and the general probabilistic arguments of \([11, 13]\) is obtained in the long-time regime, where the peak intensity tends to vanish. This means, in other words, that the peaks seem to be dynamic properties incompatible with the thermodynamic treatment, in accordance with the observation made in...
this paper that the emergence of a $H_q(N)$ linearly dependent on $N$ would be blurred by the presence of the two border strings.

We would be tempted to stress that the detection of $q < 1$ is expected on the basis of the theoretical analysis made by Lyra and Tsallis on the dynamics of logistic map. These authors show indeed that the generalization of the Pesin theorem to the case of the logistic map implies $q < 1$ at the chaos threshold. However, by the same token we should conclude that these results disagree with those of Ref. [30], which rests, on the contrary, on a condition physically much closer to that discussed in the present paper. Actually, some caution must be exerted in establishing a straight connection between the results of this paper and the research work of [40,44], for the reasons pointed out earlier in this paper. Here we are dealing with the ME that might imply a so strong coarse graining as to lose a close relation with the KST theorem [21] and consequently with the generalization of the important theorem of Pesin [22]. This is made evident also by the fact that the correlated sequences are here generated by a stochastic approach, even if this turns out to be equivalent to the adoption of a nonlinear map [31].

The statistical analysis in terms of the ME is insensitive to the adoption of a nonlinear map [31] or a stochastic approach, even if this turns out to be equivalent to the adoption of a nonlinear map [31]. However, the peaks are dynamic properties that in all cases seem to be incompatible with the adoption of a merely entropic approach. This reinforces the need for carrying out the ME analysis by disregarding the border strings, as we propose in this paper.

In summary, this paper sheds light into the breakdown of extensivity caused by time correlations. There are at the least two important sources of non extensivity: nonlocality in space [15] and nonlocality in time. For the spatial case, and up to now, there is a no clearcut connection in the literature between $q$ and the critical index characterizing spatial correlations (although there is a variety of strong indications). For the temporal case this manuscript establishes, for the first time, the analogous connection between $q$ and $\mu$ (see Eq. [3] and Eq. [14]). This is an interesting result and some efforts should be made to establish theoretically the critical exponent $\alpha$.

[1] V.S. Anishchenko, W. Ebeling, A.B. Neiman, Chaos, Soliton & Fractals 4 (1994) 69.
[2] W. Ebeling, Physica A 194 (1993) 563.
[3] W. Ebeling, R. Feistel, H. Herzl, Physica Scripta 35 (1987) 761.
[4] W. Li, K. Kaneko, Europhys. Lett. 17 (1992) 655.
[5] H. Herzl, W. Ebeling, A.O. Schmitt, Phys. Rev. E 50 (1994) 5061.
[6] A.O. Schmitt, H. Herzl, J. Theor. Biol. 188 (1997) 369.
[7] P. Liò, A. Politi, M. Buiatti, S. Ruffo, J. Theor. Biol. 180 (1996) 151.
[8] P. Gaspard, X.-J. Wang, Proc. Natl. Acad. Sci. USA 85 (1988) 4591.
[9] W. Ebeling, G. Nicolis, Chaos, Solitons & Fractals 2 (1992) 635.
[10] W. Ebeling, G. Nicolis, Europhys. Lett. 14 (1991) 191.
[11] X.-J. Wang, Phys. Rev. A 45 (1992) 8407.
[12] J. Freund, Phys. Rev. E 53 (1996) 5793.
[13] J. Freund, W. Ebeling, K. Rateitschack, Phys. Rev. E 54 (1996) 5561.
[14] P. Sze´ papalusy, G. Györgyi, Phys. Rev. A 33 (1986) R2852.
[15] C. Beck, F. Schlögl, Thermodynamics of chaotic systems, Cambridge University Press, Cambridge (1993).
[16] A. N. Kolmogorov, Dok. Acad. Nauk. SSSR 119 (1958) 861.
[17] Ya.G. Sinai, Dok. Acad. Nauk. SSSR 124 (1959) 768.
[18] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[19] L. Palatella, P. Grigolini, [quant-ph/9810083]
[20] http://tsallis.cat.cbpf.br/biblio.htm
[21] C. Tsallis, A.R. Plastino, W.-M. Zheng, Chaos, Soliton & Fractals 8 (1997) 885.
[22] Ya.B. Pesin, Russian Mathematical Surveys (4)32 (1977) 55.
[23] P.A. Miller, S. Sarkar, [chaos-dyn/9812003]
[24] P. Allegrini, P. Grigolini, B.J. West, Phys. Rev. E 54 (1996) 4769.
[25] J. Klafter, G. Zumofen, M.F. Shlesinger, Lecture Notes in Physics (Springer) 457 (1995) 183.
[26] From the work of [11] we derive reliable prescriptions to evaluate $h \equiv \lim_{N \to \infty} (H_1(N)/N)$. Adapting these prescriptions to our notations, we see that this limit is finite if $\mu > 2$ and it vanishes for $\mu \leq 2$. The non vanishing limit of $\mu$ is reached with a very slow regression process, with an inverse power law behavior. We are thus tempted to interpret the region $2 < \mu < 3$ as one where the coarse-grain induced randomness is not totally overcome by the long-range correlations. We cannot rule out the possibility that the entropic index $q$ undergoes a slow transition from $q < 1$ to $q = 1$. Rather, the conclusion of Wang [11] can be made compatible with the result of the numerical investigation of this paper by making the plausible conjecture that the entropic index $q < 1$ is not permanent and undergoes a slow regression to $q = 1$. The theoretical predictions of [11] suggests that the time evolution, in the “time” $N$, of $q(N)$ towards the asymptotic value $q(\infty) = 1$ might take place with an inverse power law behavior.
[27] P. Allegrini, M. Barbi, P. Grigolini, B.J. West, Phys. Rev. E 52 (1995) 5281.
[28] P. Allegrini, P. Grigolini, B.J. West, Phys. Lett. A 211 (1996) 217.
[29] P. Allegrini, M. Buiatti, P. Grigolini, B.J. West, Phys. Rev. E 57 (1998) 4558.
[30] P. Allegrini, M. Buiatti, P. Grigolini, B.J. West, Phys. Rev. E 58 (1998) 3640.
[31] T. Geisel, J. Nierwetberg, A. Zacherl, Phys. Rev. Lett. 54 (1985) 616.
[32] V.I. Arnold, A. Avez, Ergodic Problems of Classical Mechanics (Benjamin, New York, 1968).
[33] P. Billingsley, Ergodic Theory and Information (Wiley,
New York, 1965)
[34] I.P. Cornfeld, S.V. Fomin, Ya.G. Sinai, *Ergodic Theory* (Springer, Berlin, 1982).
[35] P. Lévy, *Théorie de l’addition des variables aléatoires* (Gauthier-Villars, Paris, 1937).
[36] E.W. Montroll, M.F. Shlesinger, in *Nonequilibrium Phenomena II: From Stochastic to Hydrodynamics*, Studies in Statistical Mechanics, Volume XI, edited by J.L. Lebowitz and E.W. Montroll, North-Holland, Amsterdam (1984).
[37] R. Bettin, R. Mannella, B.J. West, P. Grigolini, Phys. Rev. E 51 (1995) 212.
[38] G. Trefán, E. Floriani, B. J. West, P. Grigolini, Phys. Rev. E 50 (1994) 2564.
[39] R. Mannella, P. Grigolini, B.J. West, Fractals 2 (1994) 81.
[40] M. Buiatti, P. Grigolini, A. Montagnini, cond-mat/9809108.
[41] P.A. Alemany, D.H. Zanette, Phys. Rev. E 49 (1994) R956.
[42] D.H. Zanette, P.A. Alemany, Phys. Rev. Lett. 75 (1995) 366.
[43] C. Tsallis, S.V.F. Levy, A.M.C. Souza, R. Maynard, Phys. Rev. Lett. 75 (1995) 3859; Phys. Rev. Lett. 77 (1996) 5442 (Erratum).
[44] M.L. Lyra, C. Tsallis, Phys. Rev. Lett. 80 (1998) 53.
[45] C. Anteneodo, C. Tsallis, Phys. Rev. Lett. 80 (1998) 5313.

FIG. 1. The KST entropy as a function of $N$ in the completely uncorrelated case (corresponding to $\mu = \infty$). In this case the KST entropy is expressed by Eq. (8). For this reason the three curves have been derived from Eq. (8). The upper, middle and bottom lines refer to $q = 0.9$, $q = 1$ and $q = 1.1$, respectively.

FIG. 2. The KST entropy as a function of $N$ with $\tilde{\mu} \geq \mu = 2.5$. The three curves have been obtained using the numerical treatment described in the text. The upper (squares), middle (circles) and bottom (triangles) plots refer to $q = 0.82$, $q = 0.89$ and $q = 0.98$, respectively.

FIG. 3. The entropic index $q$ versus $\tilde{\mu}$. See the text for the definition of $\tilde{\mu}$. The points with error bars are the result of the numerical treatment described in the text, and the line denotes the function $q = (\tilde{\mu} - 2)^{\alpha}$ with $\alpha \approx 0.13$. 
