Linear and non-linear refractive indices in curved space

Miftachul Hadi¹, Utama Alan Deta² and Andri Sofyan Husein³

¹Physics Research Centre, Indonesian Institute of Sciences (LIPI), Puspiptek, Serpong, Tangerang Selatan 15314, Banten, Indonesia.
²Department of Physics, Universitas Negeri Surabaya, Ketintang, Surabaya, Indonesia.
³Department of Industrial Engineering, University of Muhammadiyah Tangerang, Jalan Perintis Kemerdekaan 1 No.35, Cikokol, Tangerang, Banten 15118, Indonesia.
E-mail: ¹ itpm.id@gmail.com

Abstract. The refractive index and curved space relation is formulated using the Riemann-Christoffel curvature tensor. As a consequence of the fourth rank tensor of the Riemann-Christoffel curvature tensor, we found that the refractive index should be a second rank tensor. The second rank tensor of the refractive index describes a linear optics. It implies naturally that the Riemann-Christoffel curvature tensor is related to the linear optics. In case of a non-linear optics, the refractive index is a sixth rank tensor, if susceptibility is a fourth rank tensor.

1. Introduction

What is really happened if light passes through a medium? This question becomes more interesting nowadays related to conceptual development and technological innovation. One of the very important idea to understand this question is the refractive index, also called index of refraction, i.e. measure of the bending of a ray of light when passing from one medium into another [1]. The refractive index of a medium is an optical parameter, since it exhibits the optical properties of the material [2]. The refractive index is one of the physicochemical properties of optical medium [3]. It is a function that depends on various parameters, including the frequency of the applied electric field [4].

The refractive index, \( n \), is defined as velocity of light of a given wavelength in empty space or vacuum (\( c \)) divided by its velocity in a substance, \( v \), [3]

\[
n = \frac{c}{v}
\]

(1)

It describes how matter affects light propagation, through the electric permittivity, \( \varepsilon \), and the magnetic permeability, \( \mu \) [5]

\[
n = \sqrt{\frac{\varepsilon}{\varepsilon_0} \frac{\mu}{\mu_0}} = \sqrt{\varepsilon_r \mu_r}
\]

(2)

¹ The sign of the refractive index is often taken as positive, but in 1968 Veselago shows that there are substrates with negative permittivity and negative permeability. In these substrates, the refractive index has a negative value [3].
where \( \varepsilon_0 \) and \( \mu_0 \) are the permittivity and the permeability of vacuum respectively, \( \varepsilon_r \) and \( \mu_r \) is relative permittivity and relative permeability of non-vacuum medium respectively which the values are relative i.e. they depend on the characteristics of medium [5, 6].

In the most substrates, the refractive index decreases by increasing temperature [3]. A denser material generally tends to have a larger refraction index [7]. The refractive index in an fibre optic can be changed due to external forces such as the tensile force, the bending force [8].

Mathematically, the refractive index is a zeroth rank tensor (scalar) and it can not be a first rank tensor (vector), but it can be a second rank tensor, a third rank tensor or a higher rank tensor (which is well known as non-linear phenomena of second order, third order, etc) [9]. The refractive index is the zeroth rank tensor, if the medium or material is isotropic. Generally, the refractive index is written as the second rank tensor, \( n_{ij} \), a \( 3 \times 3 \) matrix, if the material is linear. It can be the third rank tensor or the fourth rank tensor if the material is non-linear [11].

The refractive index has a large number of applications. It is mostly applied to identify a particular substance, to confirm its purity or to measure its concentration. It also can be used in determination of drug concentration in pharmaceutical industry, to calculate a focusing power of lenses and a dispersive power of prisms. Also, it can be applied to estimate a thermophysical properties of hydrocarbons and petroleum mixtures [3].

2. The Linear Refractive Index and the Riemann-Christoffel Curvature Tensor

Let \( f(r) \) be defined and differentiable at a point \( (r) \) in a certain region of space (i.e. \( f \) defines a differentiable scalar field). Then the gradient of \( f \) in the spherical coordinate is defined by

\[
\nabla f \equiv \frac{df}{dr} \hat{r}
\]

(3)

In the tensor form [12]

\[
\nabla f = \nabla f = f_j = \frac{\partial f}{\partial x^j}
\]

(4)

where \( f_j \) is the covariant derivative of \( f \) with respect to \( x^j \). Here, \( \nabla f \) defines a vector field i.e. the gradient of a scalar field is a vector field [12, 13].

Let us analyse the equation below [14, 15, 19]

\[
\frac{1}{R} = \hat{N} \cdot \nabla \ln n(r)
\]

(5)

where \( R \) is a radius of curvature, \( \hat{N} \) is an unit vector along the principal normal or has the same direction with \( \nabla \ln n(r) \) and \( n(r) \) is the space dependent refractive index. Eq.(5) tells us that the rays are therefore bent in the direction of increasing refractive index [14].

Let us define that

\[
\hat{N} \equiv \frac{\nabla n(r)}{|\nabla n(r)|}
\]

(6)

2 Isotropy comes from the Greek words isos (equal) and tropos (way): uniform in all directions [10]. An isotropic material is a material that has the same optical properties, regardless of the direction in which light propagates through the material [4, 10].

3 Linear material is a material that when exposed to light at a certain frequency will generate light with the same frequency [6].
Using notation of gradient operator in (3) and substituting (6) into (5), we obtain

\[
\frac{1}{R} = \frac{\overset{\wedge}{\nabla} n(r)}{|\overset{\wedge}{\nabla} n(r)|} \cdot \overset{\wedge}{\nabla} \ln n(r) = \frac{\frac{dn(r)}{dr}}{\left| \frac{dn(r)}{dr} \right|} \cdot \frac{d}{dr} \overset{\hat{r}}{\int} \frac{1}{n(r)} dn(r) \tag{7}
\]

Because \( \overset{\hat{r}}{\hat{r}} \cdot \overset{\hat{r}}{\hat{r}} = |\overset{\hat{r}}{\hat{r}}||\overset{\hat{r}}{\hat{r}}| \cos 0 = 1 \), \( \overset{\hat{r}}{\hat{r}} \) is a unit vector which its magnitude i.e. \( |\overset{\hat{r}}{\hat{r}}| = 1 \), then eq.(7) becomes

\[
\frac{1}{R} = \frac{1}{n(r)} \left( \frac{dn(r)}{dr} \right)^2 \left( \left| \frac{dn(r)}{dr} \right| \right)^{-1} \tag{8}
\]

If we assume that the derivative of a function \( n(r) \) always takes a positive value then

\[
\left| \frac{dn(r)}{dr} \right| = \frac{dn(r)}{dr} \tag{9}
\]

So, eq.(8) becomes

\[
\frac{1}{R} = \frac{1}{n(r)} \frac{dn(r)}{dr} \tag{10}
\]

where \( n(r) \) can be e.g. exponential, logarithmic, quadratic, linear functions.

What is precisely the form of function \( n(r) \)? We assume that the curvature of space is produced by a spherically symmetric body with mass at rest. So, the mass of this body produces the static spherically symmetric gravitational field. The static condition means that with a static coordinate system, the metric tensors, \( g_{\mu\nu} \), are independent of time, \( x^0 \) or \( t \), and also \( g_{tm} = 0, \text{ for } m = r, \theta, \phi \) \[16\]. Let us choose an isotropic and static spherically symmetric metric in two-dimensional space-time in the form as below \[15, 16\]

\[
ds^2 = e^{2\nu(r)} c^2 dt^2 - e^{2\lambda(r)} dr^2 \tag{11}
\]

where \( r \) is the spatial (radial) coordinate, measured as the circumference divided by \( 2\pi \), of a sphere centered around the massive body. The isotropic space is shown by the invariance of the spherically symmetric metric under space rotation. Eq.(11) shows that the metric tensors are

\[
g_{tt} = e^{2\nu(r)} c^2, \quad g_{rr} = -e^{2\lambda(r)}, \quad g_{\mu\nu} = 0, \quad \mu \neq \nu \tag{12}
\]

The choice of the exponential functions for the metric tensors in eqs.(11), (12), will makes the solution of Einstein field equation simpler \[18\].

The world line corresponding to the propagation of light is called null geodesic which is defined as

\[
ds^2 \equiv 0 \tag{13}
\]

Substituting (13) into (11), we obtain

\[
0 = e^{2\nu(r)} c^2 dt^2 - e^{2\lambda(r)} dr^2 \tag{14}
\]

If we define the speed of light in the present of a static spherically symmetric gravitational field, \( v \), as below \[15\]

\[
v \equiv \frac{dr}{dt} \tag{15}
\]
where this speed of light is therefore depends on the spatial coordinate, \( v(r) \), through the space dependent metric elements, then from eqs.(14), (15), we obtain

\[
v(r) = \frac{e^{\nu(r)}}{e^{\lambda(r)}} c
\]  

(16)

So, the form of function \( n(r) \) is

\[
n(r) = \frac{c}{v(r)} = \frac{e^{\lambda(r)}}{e^{\nu(r)}}
\]

(17)

Eq.(17) implies that, geometrically, the vacuum space (where the value of refractive index is equal to 1) is a flat space.

The metric tensors in eq.(11) can be used to solve the Einstein field equation in empty space [16]. Simply, the Einstein field equation can be defined as [17]

\[
G_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu} = -\kappa T_{\mu\nu}
\]

(18)

where \( G_{\mu\nu} \) is the Einstein tensor, \( T_{\mu\nu} \) is the energy-momentum tensor and \( \kappa \) is a constant. Here, the Einstein tensor is defined as

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R
\]

(19)

where \( R_{\mu\nu} \) is Ricci tensor and \( R \) is Ricci scalar, \( R = g^{\mu\nu} R_{\mu\nu} = g_{\mu\nu} R^{\mu\nu} \).

The Einstein field equation in vacuum (empty) space means that the energy-momentum tensor, \( T_{\mu\nu} \), is equal to zero [18]. So, the consequence is \( R_{\mu\nu} = 0 \). The complete solution of the Einstein field equation for vacuum space in (1+1) dimensions is

\[
ds^2 = \left( 1 - \frac{2GM}{c^2 r} \right) c^2 dt^2 - \left( 1 - \frac{2GM}{c^2 r} \right)^{-1} dr^2
\]

(20)

where \( M \) is the mass of the central body which exerts the gravitational field. Eq.(20) is nothing but the Schwarzschild solution of the Einstein field equation for vacuum space.

The solutions, i.e. eqs.(11), (20), mean that [16, 18]

\[
e^{2\nu(r)} = 1 - \frac{2GM}{c^2 r}
\]

(21)

and

\[
e^{2\lambda(r)} = \left( 1 - \frac{2GM}{c^2 r} \right)^{-1}
\]

(22)

Substituting (21), (22) into (17), we obtain

\[
n(r) = \left( 1 - \frac{2GM}{c^2 r} \right)^{-1}
\]

(23)

Eq.(23) shows us that the refractive index, \( n(r) \), increases with the increase in the magnitude of gravitational field strength, \( GMr^{-2} \) [15].

Using eq.(23), let us calculate the values of the function \( n(r) \) for some values of \( A(r) = 2GMc^{-2}r^{-1} \) as below. We see from Table 1, the slope of refractive index function, \( n(r) \), always
Table 1. The graph of refractive index function, \( n(r) \), versus \( A(r) \) some values of \( A(r) \).

| \( A(r) \) | \( n(r) \) |
|---|---|
| 0  | 1   |
| 0.25 | 1.33 |
| 0.5  | 2   |
| 0.75 | 4   |
| 1    | \(+\infty\) |

The table shows the behavior of the refractive index function, \( n(r) \), with respect to different values of \( A(r) \).

The refractive index tends to rise (positive). So, the refractive index function, \( n(r) \), satisfies the assumption given in eq.(9).

Now, let us take derivation of eq.(23), we obtain that

\[
\frac{dn(r)}{dr} = -\left(1 - \frac{2GM}{c^2 r}\right)^{-2} \left(\frac{2GM}{c^2 r^2}\right)
\]

(24)

Substituting (23), (24), into (10), we obtain

\[
\frac{1}{R} = -\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \left(\frac{2GM}{c^2 r^2}\right)
\]

(25)

or

\[
\frac{1}{R} = -\frac{A(r)}{1 - A(r)} r^{-1}
\]

(26)

The values of the curvature, \( 1/R \), eqs.(25), (26), for some values of \( A(r) \) are listed in Table 2 below. In Table 2, if we take \( 2GMc^{-2}r^{-1} = 1 \) or \( 2GMc^{-2} = r \) then the curvature is (minus)

| \( A(r) \) | \( 1/R \) |
|---|---|
| 0  | 0   |
| 0.25 | \(-\frac{1}{2}r^{-1}\) |
| 0.5  | \(-r^{-1}\) |
| 0.75 | \(-3r^{-1}\) |
| 1    | \(-\infty\) |

Table 2. The graph of the curvature, \( 1/R \), versus \( A(r) \) for some values of \( A(r) \).

In tensor form, by using relation in eq.(4) and tensor notation \( N_k \), i.e. the first rank tensor for \( \hat{N} \), the right hand side of eq.(5) can be written as

\[ N_k \frac{\partial}{\partial x^j}[\ln n(r)] \]

(27)

Because of the Riemann-Christoffel curvature tensor is the fourth rank tensor, so the refractive index in eq.(27) should be written as the second rank tensor. We obtain the relation between the curvature tensor and the refractive index tensor as below [20]

\[ \frac{R_{mijk}}{g} = N_k \frac{\partial \ln n_{mi}}{\partial x^j} \]

(28)

Eq.(28) implies that the curved space which is described by the Riemann-Christoffel curvature tensor related naturally to linear optics.
3. The Non-Linear Refractive Index and the Riemann-Christoffel Curvature Tensor

How about the form of the non-linear refractive index i.e. the refractive index related to the non-linear optics? In optics, non-linear properties of materials are usually described by non-linear susceptibilities [21]. Mathematically, the optical non-linear response can be expressed as a relationship between the polarization density, $\vec{P}$, and the electric field, $\vec{E}$.

In the linear case, a relation between the polarization density and the electric field is simply expressed [23, 24]

$$\vec{P} = \varepsilon_0 \chi^{(1)} \vec{E}$$

(29)

where $\varepsilon_0$ is the permittivity of vacuum space, $\chi^{(1)}$ is the first order susceptibility or linear susceptibility and it is a scalar, whereas the polarization and the electric field are vectors.

In the non-linear case, the polarization density can be modelled as a power series of the electric field, $\vec{E}$, as below [23, 24, 27]

$$\vec{P} = \varepsilon_0 \left[ \chi^{(1)} \vec{E} + \chi^{(2)} \vec{E}^2 + \chi^{(3)} \vec{E}^3 + \ldots \right]$$

(30)

where $\vec{E}^2 = \vec{E} \cdot \vec{E}$, $\vec{E}^3 = \vec{E} \cdot \vec{E} \cdot \vec{E}$ etc. $\vec{P}^1$ is called the linear polarization while $\vec{P}^2$, $\vec{P}^3$, are called the second and third non-linear polarizations, respectively. Thus, the polarization is composed of linear and non-linear components [24]. The first susceptibility term, $\chi^{(1)}$, corresponds to the linear susceptibility (dimensionless) and the subsequent non-linear susceptibilities, $\chi^{(a)}$, have units of (meter/volt)$^{a-1}$ [28, 29]. The quantities $\chi^{(2)}$ and $\chi^{(3)}$ are known as the second order and third order susceptibilities, respectively. These electric susceptibilities, $\chi^{(1)}$, $\chi^{(2)}$, $\chi^{(3)}$, are the second, third and fourth rank tensors, respectively [23]. In general, the non-linear susceptibilities depend on the frequencies of the applied fields [30]. In optical Kerr effect, the third order susceptibility, $\chi^{(3)}$, related to the non-linear refractive index [4].

Now, we have a question: if the non-linear refractive index related to the third order susceptibility and the third order susceptibility is the fourth rank tensor [23] then how to define the non-linear refractive index related to the fourth rank tensor of the third order susceptibility? Usually, the optical frequencies are too large as compared to the vibrational frequencies of a material, so $\Re \chi^{(3)} >> \Im \chi^{(3)}$. For linearly polarized monochromatic light in an isotropic medium or a cubic crystal, the non-linear refractive index, $n_2$, can be expressed by [27]

$$n_2 = \frac{12\pi}{n_0} \Re \chi^{(3)}$$

(31)

or

$$n_0 = \frac{12\pi}{n_2} \Re \chi^{(3)} = 12\pi (n_2)^{-1} \Re \chi^{(3)}$$

(32)

4 Light is an electromagnetic wave, and the electric field of this wave oscillates perpendicularly to the direction of light propagation. If the direction of the electric field of light is well defined, it is polarized light. The most common source of polarized light is a laser [22].
5 A non-linear system is a system in which the change of the output is not proportional to the change of the input [25]. In optics, the non-linearity is typically observed only at very high intensities (field strength) of light such as those provided by lasers [26].
6 In general, susceptibility is a complex quantity. The real part is related to the refraction, while the imaginary part is related to the absorption.
where \( n_0 \) is linear refractive index and \( \text{Re} \chi^{(3)} \) is a real part of the third order non-linear susceptibility. We see from eqs. (31), (32), the non-linear refractive index is a function of the linear refractive index.

Values of the linear refractive index, \( n_0 \), and non-linear refractive index, \( n_2 \), of some oxides are shown in Table 3. Also, the relation between the refractive index, \( n_0 \), and the non-linear refractive index, \( n_2 \), can be listed in Table 3 and Figure 1 below [27].

| Oxide      | \( n_0 \) | \( n_2 \times 10^{14} \) esu |
|------------|-----------|-------------------------------|
| Li\(_2\)O  | 1.644     | 30.6                          |
| CuO        | 2.63      | ...                           |
| MgO        | 1.736     | 29.8                          |
| CaO        | 1.838     | 32.4                          |
| SrO        | 1.810     | 35.1                          |
| BaO        | 1.980     | 39.0                          |
| ZnO        | 2.008     | 42.8                          |
| CdO        | 2.49      | 95.2                          |
| B\(_2\)O\(_3\) | 1.64 | 4.8                        |
| Al\(_2\)O\(_3\) | 1.760 | 16.9                       |
| Ga\(_2\)O\(_3\) | 1.952 | 40.9                       |
| In\(_2\)O\(_3\) | ... | 43.3                      |
| Sc\(_2\)O\(_3\) | ... | ...                      |
| Y\(_2\)O\(_3\) | 1.92 | 27.1                       |
| SiO\(_2\)(q) | 1.544 | 9.8                       |
| GeO\(_2\) | 1.65      | 45.0                          |
| SnO\(_2\) | 1.997     | ...                          |
| PbO        | 2.51      | 185.0                         |
| TiO\(_2\)(a) | 2.554 | 132.0                       |
| TiO\(_2\)(r) | 2.616 | 49.3                       |
| ZrO\(_2\) | 2.12      | 93.7                          |
| CeO\(_2\) | ...       | ...                          |
| Sb\(_2\)O\(_3\) | 2.087 | 134.4                      |
| Bi\(_2\)O\(_3\) | ... | 54.5                      |
| V\(_2\)O\(_5\) | ... | ...                      |
| Nb\(_2\)O\(_5\) | ... | 114.4                     |
| Ta\(_2\)O\(_5\) | ... | 127.8                     |
| TeO\(_2\) | 2.27      | ...                          |
| MoO\(_3\) | 2.21      | ...                          |
| WO\(_3\)  | 2.50      | ...                          |
| MnO       | 2.16      | ...                          |
| Fe\(_2\)O\(_3\) | 3.01 | ...                      |
| CoO       | ...       | ...                          |
| NiO       | 2.1818    | ...                          |

Table 3. The linear refractive index, \( n_0 \), and the non-linear refractive index, \( n_2 \), of some oxide [27].
Fig. 1. The graph of the linear refractive index, $n_0$, and the non-linear refractive index, $n_2$, relation of some oxides [27].

Figure 1 shows "the exponential relation" between the linear refractive index, $n_0$, and the non-linear refractive index, $n_2$. It means that the non-linear refractive index increases with increasing the linear refractive index.

We see from eq.(28), the linear refractive index is the second rank tensor and if the third order susceptibility is the fourth rank tensor [23], then we can write eq.(32) as

$$n_{mi} = 12\pi n^{pqrs}_{mi} \chi^{(3)}_{pqrs} \tag{33}$$

where $\chi^{(3)}_{pqrs}$ is the fourth rank tensor of the real part of third order susceptibility.

As we see from eqs.(32), (33)

$$(n_2)^{-1} = n^{pqrs}_{mi} \rightarrow n_2 = n^{pqrs}_{mi} \tag{34}$$

It means that the non-linear refractive index, $n_2$, should be the sixth rank tensor (a mixed tensor of second rank contravariant and fourth rank covariant), $n^{pqrs}_{mi}$.

Substituting (33) into (28), we obtain

$$R_{mijk} = g \left\{ N_k \frac{\partial}{\partial x^j} \left[ \ln \left( 12\pi n^{pqrs}_{mi} \chi^{(3)}_{pqrs} \right) \right] \right\} \tag{35}$$

Eq.(35) shows that the non-linear optics relates the sixth rank tensor of the refractive index with the curvature tensor.

4. The Refractive Index and a Mass in Curved Space

How to formulate the space dependent linear (the second rank tensor) and non-linear (the sixth rank tensor) refractive indices related to the mass of an object as expressed in eq.(23)? In order to answer this question, we need to understand the quantities $G$, $c$ in eq.(23). The simple
understanding of the quantities of $G$, $c$ is coming from the Einstein field equation, eq.(18). We are informed from eq.(18) that the gravitational constant, $G$, is a scalar (because the speed of light in vacuum space, $c$, is a scalar).

As previously stated, in case of the linear optics, we take the space dependent refractive index as the second rank tensor. Because of gravitational constant, $G$, the speed of light, $c$, the spatial coordinate (distance), $r$, are scalars, then eq.(23) can be written as

\[ n_{mi} = \left(1 - \frac{2G}{c^2 r} M^{mi}\right)^{-1} \]  

(36)

where $M^{mi}$ is the second rank tensor of mass [33, 34].

In case of the non-linear optics, we substitute eq.(33) into (36), then we obtain

\[ n_{pqrs}^{mi} = \frac{1}{12\pi} (\chi_{pqrs}^{(3)} - 1 - \frac{2G}{c^2 r} M^{mi})^{-1} \]  

(37)

or, in analogy with (36)

\[ n_{pqrs}^{mi} = \left(1 - \frac{2G}{c^2 r} M^{mi}_{pqrs}\right)^{-1} \]  

(38)

We obtain from (37), (38), that

\[ M^{mi}_{pqrs} = \frac{c^2 r}{2G} \left\{ 1 - \frac{12\pi}{1} \chi_{pqrs}^{(3)} \left(1 - \frac{2G}{c^2 r} M^{mi}\right) \right\} \]  

(39)

What is the value of $M^{mi}_{pqrs}$ compared to $M^{mi}$? Substituting the various values of the linear and non-linear refractive indices for some oxides in Table 3 into eq.(31) or (32), we obtain the third order susceptibility for various oxides. Using eq.(36), by substituting the values of the gravitational constant, the speed of light in the vacuum space and taking certain value of $r$, we obtain "linear mass". Substituting the linear mass and the third order susceptibility into eq.(39), we obtain "non-linear mass".

If we substitute eq.(23) into (5), we obtain "the mass-curvature equation" for one spatial dimension as below

\[ \frac{1}{R} = \hat{N} . \nabla \ln \left(1 - \frac{2G}{c^2 r} M\right)^{-1} \]  

(40)

How about "the generalized mass-curvature equation" for the second rank tensor of the linear refractive index and the sixth rank tensor of the non-linear refractive index?

If we substitute eq.(36) into (28), we obtain

\[ R_{mijk} = g \left\{ N_k \frac{\partial}{\partial x^j} \left[ \ln \left(1 - \frac{2G}{c^2 r} M^{mi}\right)^{-1} \right] \right\} \]  

(41)

We call this equation (41) as "the generalized linear mass-curvature equation".

If we substitute eq.(38) as "the generalized linear mass-curvature equation".

If we substitute eq.(38) into (35), we obtain

\[ R_{mijk} = g \left( N_k \frac{\partial}{\partial x^j} \left[ \ln \left[12\pi \left(1 - \frac{2G}{c^2 r} M^{mi}_{pqrs}\right)^{-1} \chi_{pqrs}^{(3)} \right] \right] \right) \]  

(42)

This equation (42) is called "the generalized non-linear mass-curvature equation".

What is the curvature (42) compared to (41)? In linear optics, the generalized curvature of space can be obtained by substituting the linear mass into eq.(41). In non-linear optics, the generalized curvature of space can be obtained by substituting the non-linear mass and third order susceptibility into eq.(42).
5. Discussion and Conclusion
In one and two dimensional spaces, the Gaussian curvatures are $1/R$ (a circle) and $1/R^2$ (a sphere), respectively. Because of the homogeneous and isotropic spaces can be spherical [35] then the Gaussian curvature can be related with the homogeneous and isotropic spaces. This homogeneous and isotropic spaces have a constant curvature [36, 37]. It means that the Gaussian curvature has a constant curvature. A sphere is an example of a surface of constant (positive) curvature. Georg Friedrich Bernhard Riemann, a student of Johann Carl Friedrich Gauss, generalize the Gauss curvature of space for more than two dimensions. The result is the Riemann-Christoffel curvature tensor where the Christoffel symbol is used in the formulation of the generalized curvature.

Related with the refractive index, because of the refractive index is related to the curvature of space for one and two dimensions, and this curvature of space can be generalized to more than two dimensions, then the refractive index should be able to be formulated in more than two dimensional curved space. It gives the second rank tensor of the refractive index as the consequence of the fourth rank tensor of the Riemann-Christoffel curvature tensor. The second rank tensor of the refractive index describes the linear optics. It implies that the Riemann-Christoffel curvature tensor is related naturally to the linear optics.

Because of the non-linear refractive index can be expressed as a function of the linear refractive index and the third order of the susceptibility, where the linear refractive index is the second rank tensor and the third order of susceptibility is the fourth rank tensor then the non-linear refractive index should be the sixth rank tensor. It means that the Riemann-Christoffel curvature tensor can be related to the non-linear optics.

If the linear and non-linear refractive indices can be formulated using "linear and non-linear masses", then the Riemann-Christoffel curvature tensor can be formulated using linear and non-linear masses. The interesting question is how if we replace mass in the curvature-mass relation with mass (or energy) of a topological object, e.g. kink, a one dimensional topological soliton? How about the formulation of the linear and non-linear refractive indices related with mass (energy) of the kink?

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References
[1] Encyclopedia Britannica, Refractive index, https://www.britannica.com/science/refractive-index.
[2] Shyam Singh, Refractive Index Measurement and Its Applications, Physica Scripta, Vol. 65, 167-180, 2002.
[3] Fardad Koohyar, Refractive Index and Its Applications, J Thermodyn Catal 4:e117, 2013.
[4] Karsten Rottwitt, Peter Tidemand-Lichtenberg, Nonlinear Optics: Principles and Applications, CRC Press, 2015.
[5] The interaction of light with matter http://www1.udel.edu/chem/sneal/shn_tchng/CHEM620.bak/Resources/Chi/4LightDielectrics.pdf
[6] Andri Sofyan Husein, Analisis Teoritis Pemantulan dan Pembiasan Gelombang Elektromagnetik pada Medium Linier, Anisotrop dan Tak Homogen, B.Sc. Thesis, University of Lampung, 2009.
[7] Yangang Liu, Peter H. Daum, Relationship of refractive index to mass density and self-consistency of mixing rules for multicomponent mixtures like ambient aerosols, Journal of Aerosol Science 39, 974-986, 2008.
[8] Kazuo Nagano, Shojiro Kawalami, and Shigeo Nishida, Change of the refractive index in an optical fiber due to external forces, Applied Optics Vol. 17, No. 13, 1978.
[9] Roniyus Marjunus, private communications.
[10] Isotropy: Definition & Materials
https://study.com/academy/lesson/isotropy-definition-materials.html
[11] Doddy Hardhienata, private communications.
[12] Murray R. Spiegel, Vector Analysis, McGraw-Hill, 1959.
[13] Kelly, Mechanics Lecture Notes Part III: Foundations of Continuum Mechanics
http://homepages.engineering.auckland.ac.nz
[14] L.D. Landau, E.M. Lifshitz, Electrodynamics of Continuous Media, Butterworth-Heinemann, Oxford, 1975.
[15] Soma Mitra, Sonameth Chakrabarty, Fermat’s Principle in Curved Space-Time, No Emission from Schwarzschild Black Hols as Total Internal Reflection and Black Hole Unruh Effect, arXiv:1512.03885v1 [gr-qc]. 2015.
[16] P.A.M. Dirac, General Theory of Relativity, John Wiley & Sons, 1975.
[17] Miftachul Hadi, Solution of the Gravitational Field Equation of a Twisted Baby Skrymion String, PhD Thesis, Universiti Brunei Darussalam, 2019.
[18] Taufiq Hidayat, Teori Relativitas Einstein: Sebuah Pengantar, Penerbit ITB, 2010.
[19] Miftachul Hadi, Andri Husein, Utama Alan Deta, A refractive index in bent fibre optics and curved space, IOP Conf. Series: Journal of Physics: Conf. Series 1171 (2019) 012016.
[20] Miftachul Hadi, A Refractive Index of a Kink, and all references therein, https://osf.io/preprints/inarxiv/x2qtw, 28 Nov 2019.
[21] Takahiro Morimoto, Naoto Nagaosa, Topological nature of nonlinear optical effects in solids, Sci. Adv. 2016; 2 : e1501524 20 May 2016.
[22] Introduction to Polarization
https://www.edmundoptics.com/resources/application-notes/optics/introduction-to-polarization/
[23] E.S. Jatirian-Foltides, J.J. Escobedo-Alatorre, P.A. Marquez-Aguilar, H. Hardhienata, K. Hingerl, A. Alejo-Molina, About the calculation of the second-order susceptibility $\chi^{(2)}$ tensorial elements for crystals using group theory, Rev. mex. Fís. E vol.62 no.1 Mexico ene./Jun. 2016.
[24] S. Choudhary, R.W. Boyd, Tutorial on nonlinear optics, Proceedings of the International School of Physics "Enrico Fermi" Course 190 "Frontiers in Modern Optics" edited by D. Faccio, J. Dudley and M. Clerici (IOS Amsterdam; SIF, Bologna) 2016.
[25] Wikipedia, Nonlinear system.
[26] Wikipedia, Nonlinear optics.
[27] Vesselin Dimitrov, Sumio Sakka, Linear and nonlinear optical properties of simple oxides, J. Appl. Phys. 79 (3), 1996.
[28] Wikipedia, Electric susceptibility.
[29] Francois Cardarelli, Materials Handbook: A Concise Desktop Reference, Springer, 2017.
[30] Robert W. Boyd, Nonlinear Optics, Third Edition, Academic Press, Inc. 2008.
[31] Frank Morgan, Riemannian Geometry: A Beginner’s Guide, Jones and Bartlett Publishers, 1993.
[32] S. Batz, U. Peschel, Linear and nonlinear optics in curved space, Physical Review A 78, 043821, 2008.
[33] Edward B. Rockower, A relativistic mass tensor with geometric interpretation, Am. J. Phys. 55 (1), January 1987.
[34] Fikhtengol’ts, Properties of the Mass Tensor, Soviet Physics JETP, Volume 34, Number 2, February 1972.
[35] Etchihiro Komatsu, Curvature
http://www.as.utexas.edu/astronomy/education/spring05/komatsu/lecture15.pdf.
[36] Sascha Batz, Ulf Peschel, Solitons in curved space of constant curvature, Physical Review A 81, 053806, 2010.
[37] L.D. Landau, E.M. Lifshitz, The Classical Theory of Fields, Pergamon, 1994.