A variant of Schur’s product theorem 
and its applications

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Abstract

We show the following version of the Schur’s product theorem. If 
\[ M = (M_{j,k})_{j,k=1}^n \in \mathbb{R}^{n \times n} \] is a positive semidefinite matrix with all entries 
on the diagonal equal to one, then the matrix 
\[ N = (N_{j,k})_{j,k=1}^n \] with the 
entries 
\[ N_{j,k} = M_{j,k}^2 - \frac{1}{n} \] is positive semidefinite. As a corollary of this 
result, we prove the conjecture of E. Novak on intractability of numerical 
integration on a space of trigonometric polynomials of degree at most one 
in each variable. Finally, we discuss also some consequences for Bochner’s 
theorem, covariance matrices of \( \chi^2 \)-variables, and mean absolute values of 
trigonometric polynomials.

Keywords: Schur’s theorem, positive definite matrices, Bochner’s theorem, 
numerical integration, tractability

1 Introduction

Over twenty years ago, motivated by tractability studies of numerical inte-
gration, E. Novak made the following conjecture:

Conjecture 1 (E. Novak). The matrix

\[ \left\{ \prod_{i=1}^d \left( \frac{1 + \cos(x_{j,i} - x_{k,i})}{2} - \frac{1}{n} \right) \right\}_{j,k=1}^n \]

is positive semidefinite for all \( n, d \geq 2 \) and all choices of \( x_1, \ldots, x_n \in \mathbb{R}^d \).

Erich Novak published this conjecture also in NA Digest in November 1997 and 
tested it numerically. It also appeared as Open Problem 3 in [5]. Never-
theless, it seems that up to now the problem remained unsolved.

Further extensive numerical tests were provided by A. Hinrichs and the 
author in [3], all supporting the belief that Conjecture 1 is true. The main

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difficulty in actually proving this conjecture turned out to be to identify the important properties of the function \( f(t) = (1 + \cos t)/2 \), which play a role in this question. Led by other numerical tests, \( \textit{3} \) conjectures that the same property is true for all positive positive-definite functions with the value at zero equal to one. Unfortunately, even in this form, the conjecture remained unsolved.

Our proof of this conjecture, which we present in this note, is based on a certain simple but rather unexpected and apparently unknown property of the Hadamard product. To state it we need few simple notations. If \( M \in \mathbb{R}^{n \times n} \) is symmetric, we say that it is positive semidefinite, if \( c^T M c \geq 0 \) for all \( c \in \mathbb{R}^n \). Similarly, \( M \in \mathbb{C}^n \) is called positive semidefinite, if it is selfadjoint and \( c^* M c \geq 0 \) for all \( c \in \mathbb{C}^n \). If \( M, N \in \mathbb{C}^{n \times n} \) are two matrices, we denote by \( M \circ N \) their Hadamard product \( \textit{4} \), i.e. a matrix with entries \((M \circ N)_{j,k} = M_{j,k} \cdot N_{j,k}\) for all \( j, k = 1, \ldots, n \). Furthermore, the partial ordering \( M \succeq N \) means that \( M - N \) is positive semidefinite. Finally, \( E_n \in \mathbb{R}^{n \times n} \) is a matrix with all entries equal to one.

Using this notation, the main result of this paper then reads as follows.

**Theorem 1.** Let \( M \in \mathbb{R}^{n \times n} \) be a positive semidefinite matrix with \( M_{j,j} = 1 \) for all \( j = 1, \ldots, n \). Then

\[
M \circ M \succeq \frac{1}{n} \cdot E_n.
\]

We give the proof of Theorem \( \textit{1} \) in Section \( \textit{2} \). Let us note that it actually resembles the original work of Schur \( \textit{9} \). This section also includes a complex version of Theorem \( \textit{1} \) as well as a variant for matrices with general diagonal. Sections \( \textit{3} \) and \( \textit{4} \) discuss the connections with Bochner’s theorem and with covariance matrices of multivariate \( \chi^2 \) random variables. Finally, Section \( \textit{5} \) gives an account on the numerical integration, which was the original motivation of E. Novak, and shows, how Theorem \( \textit{1} \) implies Conjecture \( \textit{4} \).

### 2 Proof of Theorem \( \textit{1} \) and its variants

**Proof of Theorem \( \textit{1} \).** Using the singular value decomposition of \( M \), we can write \( M = AA^T \), where \( A \in \mathbb{R}^{n \times n} \). We denote the rows of \( A \) by \( A_j, j = 1, \ldots, n \). Then \( M_{j,k} = \langle A_j, A_k \rangle \) and \( \|A_j\|_2^2 = \langle A_j, A_j \rangle = M_{j,j} = 1 \) for every \( j = 1, \ldots, n \).

We need to show that

\[
\sum_{j,k=1}^n c_j c_k M_{j,k}^2 \geq \frac{1}{n} \left( \sum_{j=1}^n c_j \right)^2 \tag{1}
\]
for every $c \in \mathbb{R}^n$. We start with a reformulation of the left-hand side of (1)

$$
\sum_{j,k=1}^{n} c_j c_k M_{j,k}^2 = \sum_{j,k=1}^{n} c_j c_k (AA^T)_{j,k}^2 = \sum_{j,k=1}^{n} c_j c_k (A_j, A_k)^2
$$

$$
= \sum_{j,k=1}^{n} c_j c_k \left( \sum_{u=1}^{n} A_{j,u} A_{k,u} \right)^2 = \sum_{j,k=1}^{n} c_j c_k \sum_{u,v=1}^{n} A_{j,u} A_{k,u} A_{j,v} A_{k,v}
$$

$$
= \sum_{u,v=1}^{n} \left( \sum_{j=1}^{n} c_j A_{j,u} A_{j,v} \right) \cdot \left( \sum_{k=1}^{n} c_k A_{k,u} A_{k,v} \right)
$$

(2)

$$
= \sum_{u,v=1}^{n} \left( \sum_{j=1}^{n} c_j A_{j,u} A_{j,v} \right)^2.
$$

We leave out the terms with $u \neq v$ and apply a variant of the inequality between arithmetic and quadratic mean

$$
\sum_{u=1}^{n} \xi_u^2 \geq \frac{1}{n} \left( \sum_{u=1}^{n} \xi_u \right)^2
$$

for

$$
\xi_u = \sum_{j=1}^{n} c_j A_{j,u}^2.
$$

In this way, we extend (2) and finish the proof of (1) by using $\|A_j\|_2 = 1$

$$
\sum_{j,k=1}^{n} c_j c_k M_{j,k}^2 \geq \sum_{u=1}^{n} \left( \sum_{j=1}^{n} c_j A_{j,u}^2 \right)^2 \geq \frac{1}{n} \left( \sum_{u=1}^{n} \sum_{j=1}^{n} c_j A_{j,u}^2 \right)^2 = \frac{1}{n} \left( \sum_{j=1}^{n} c_j \right)^2 = \frac{1}{n} \langle c, \text{diag} M \rangle^2.
$$

(3)

Theorem 1 can be extended to the setting, where the entries on the diagonal are not identically equal to one. For this sake, we denote by $\text{diag} M = (M_{1,1}, \ldots, M_{n,n})^T$ the diagonal entries of $M$ whenever $M \in \mathbb{R}^{n \times n}$.

**Theorem 2.** Let $M \in \mathbb{R}^{n \times n}$ be a positive semidefinite matrix. Then

$$
M \circ M \succeq \frac{1}{n} (\text{diag} M)(\text{diag} M)^T.
$$

**Proof.** The proof follows in the same manner as the proof of Theorem 1. Indeed, writing again $M = AA^T$, (2) and (3) becomes

$$
\sum_{j,k=1}^{n} c_j c_k M_{j,k}^2 \geq \frac{1}{n} \left( \sum_{u=1}^{n} \sum_{j=1}^{n} c_j A_{j,u}^2 \right)^2 = \frac{1}{n} \left( \sum_{j=1}^{n} c_j \cdot \|A_j\|_2^2 \right)^2
$$

$$
= \frac{1}{n} \left( \sum_{j=1}^{n} c_j M_{j,j} \right)^2 = \frac{1}{n} \langle c, \text{diag} M \rangle^2.
$$

\[\square\]
Without much additional effort, Theorem 1 allows also a complex version.

**Theorem 3.** Let \( M \in \mathbb{C}^{n \times n} \) be a positive semidefinite Hermitian matrix with \( M_{j,j} = 1 \) for all \( j = 1, \ldots, n \). Let \( N \in \mathbb{R}^{n \times n} \) be a matrix with entries \( N_{j,k} = |M_{j,k}|^2 \). Then

\[
N \succeq \frac{1}{n} \cdot E_n.
\]

**Proof.** Let \( M = AA^* \) with the rows of \( A \) again denoted by \( A_j \in \mathbb{C}^n \), \( j = 1, \ldots, n \).

Then \( M_{j,k} = \langle A_j, A_k \rangle = \sum_{u=1}^n A_{j,u} \overline{A_{k,u}} \). Furthermore, let \( c \in \mathbb{C}^n \) be arbitrary. Then the analogue of (2) becomes

\[
\sum_{j,k=1}^n c_j c_k |M_{j,k}|^2 = \sum_{j,k=1}^n c_j c_k \sum_{u=1}^n A_{j,u} \overline{A_{k,u}}^2 = \sum_{j,k=1}^n c_j c_k \sum_{u,v=1}^n A_{j,u} \overline{A_{j,v}} A_{k,u} \overline{A_{k,v}} \\
\geq \frac{1}{n} \sum_{u,v=1}^n c_j |A_{j,u}|^2 \geq \frac{1}{n} \sum_{j=1}^n c_j = \frac{c^* E_n c}{n}.
\]

\( \square \)

**Remark 1.** Let us point out that Theorem 3 fails if one replaces the condition \( N_{j,k} = |M_{j,k}|^2 \) by \( N_{j,k} = M_{j,k}^2 \) (which we checked by numerical simulations).

### 3 Bochner’s theorem

If \( \mu \) is a finite Borel measure on \( \mathbb{R}^d \), then its Fourier transform is given by

\[
g(\xi) = \hat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x \rangle} \, d\mu(x), \quad \xi \in \mathbb{R}^d,
\]

where \( \xi \cdot x = \langle \xi, x \rangle \) is the inner product of \( x \in \mathbb{R}^d \) and \( \xi \in \mathbb{R}^d \). Classical Bochner’s theorem (actually its easy part), cf. [1, 2], states that the matrix

\[
(\hat{\mu}(\xi_j - \xi_k))_{j,k=1}^n
\]

is positive semidefinite for every choice of \( \xi_1, \ldots, \xi_n \in \mathbb{R}^d \).

The proof follows by a simple calculation, as we have for every \( c \in \mathbb{C}^d \)

\[
\sum_{j,k=1}^n c_j c_k g(\xi_j - \xi_k) = \sum_{j,k=1}^n c_j c_k \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi_j - \xi_k, x \rangle} \, d\mu(x) \\
= \int_{\mathbb{R}^d} \sum_{j,k=1}^n c_j c_k e^{-2\pi i \xi_j \cdot x} e^{2\pi i \xi_k \cdot x} \, d\mu(x) \\
= \int_{\mathbb{R}^d} \left| \sum_{j=1}^n c_j e^{-2\pi i \xi_j \cdot x} \right|^2 \, d\mu(x) \geq 0.
\]
We refer to [10] for a classical overview of positive definite functions. Theorem 1 leads to the following modification of one part of Bochner’s Theorem.

**Theorem 4.** Let $\mu$ be a finite Borel measure on $\mathbb{R}^d$ and let $g$ be its Fourier transform. Then

$$
\left| g(\xi_j - \xi_k) \right|^2 \geq \frac{g^2(0)}{n} \cdot E_n
$$

for every choice of $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$.

**Proof.** By the easy part of Bochner’s theorem, cf. (4), we know that the matrix $M = (M_{j,k})_{j,k=1}^n$ with $M_{j,k} = g(\xi_j - \xi_k)$ is positive semidefinite. Furthermore, $M_{j,j} = g(0) = \int_{\mathbb{R}^d} 1 \, d\mu(x)$ is real for every $j = 1, \ldots, n$. The result then follows by invoking Theorem 3. \qed

Theorem 4 can be generalized to other locally compact abelian groups by just applying the corresponding version of Bochner’s theorem. In this way, one can prove for example the following version for torus $\mathbb{T}$, cf. Conjecture 3 in [3].

**Theorem 5.** Let $\alpha = (\alpha_j)_{j \in \mathbb{Z}}$ be a non-negative summable sequence, i.e. $\alpha_j \geq 0$ for every $j \in \mathbb{Z}$ and $\alpha \in \ell^1(\mathbb{Z})$. Let $g(x) = \sum_{j \in \mathbb{Z}} \alpha_j e^{ijx}$ for every $x \in \mathbb{T}$. Then

$$
\left| g(\xi_j - \xi_k) \right|^2 \geq \frac{g^2(0)}{n} \cdot E_n
$$

for every choice of $\xi_1, \ldots, \xi_n \in \mathbb{T}$.

Theorem 4 allows an interesting reformulation in a language of independent random variables.

**Theorem 6.** Let $\mu$ be a Borel probability measure on $\mathbb{R}^d$. Let $\omega_1$ and $\omega_2$ be two independent random vectors, both distributed with respect to $\mu$. Then

$$
\mathbb{E} \left\| \sum_{j=1}^n c_j e^{-2\pi i x_j (\omega_1 - \omega_2)} \right\|^2 \geq \frac{1}{n} \sum_{j=1}^n |c_j|^2
$$

(5)

for every choice of $c \in \mathbb{C}^n$ and $x_1, \ldots, x_n \in \mathbb{R}^d$.

**Proof.** We denote again by $g$ the Fourier transform of $\mu$. The proof follows from Theorem 4 and the following direct calculation

$$
\mathbb{E} \left\| \sum_{j=1}^n c_j e^{-2\pi i x_j (\omega_1 - \omega_2)} \right\|^2 = \mathbb{E} \sum_{j,k=1}^n c_j \overline{c_k} e^{-2\pi i (x_j - x_k) \cdot (\omega_1 - \omega_2)}
$$

$$
= \sum_{j,k=1}^n c_j \overline{c_k} \mathbb{E} e^{-2\pi i (x_j - x_k) \cdot \omega_1} \cdot \mathbb{E} e^{-2\pi i (x_j - x_k) \cdot \omega_2}
$$

$$
= \sum_{j,k=1}^n c_j \overline{c_k} |g(x_j - x_k)|^2.
$$

\qed
Remark 2. If \( d = 1 \) and \( \omega_1, \omega_2 \) are i.i.d. standard normal variables, then \( \omega_1 - \omega_2 \) is again a normal variable. After rescaling, (5) gives for every \( y_1, \ldots, y_n \in \mathbb{R} \) and every \( c \in \mathbb{C}^n \)

\[
\mathbb{E}\left| \sum_{j=1}^{n} c_j e^{i y_j \omega} \right|^2 \geq \frac{1}{n} \left| \sum_{j=1}^{n} c_j \right|^2.
\] (6)

4 Covariance matrices

Theorem 1 has an interesting consequence for covariance matrices of multivariate \( \chi^2 \)-distributions (with one degree of freedom). Let \( M \in \mathbb{R}^{n \times n} \) be a positive semidefinite matrix with ones on the diagonal and let \( X_1, \ldots, X_n \) be Gaussian random variables with zero mean, unit variance and \( \text{Cov}(X_j, X_k) = M_{j,k} \) for all \( j, k = 1, \ldots, n \). Using Wick’s theorem, cf. [5, 6, 11], we observe that

\[
\text{Cov}(X_j^2, X_k^2) = \mathbb{E}[X_j^2 X_k^2] - \mathbb{E}[X_j^2] \cdot \mathbb{E}[X_k^2] = 2(\mathbb{E}[X_j X_k])^2 = 2M_{j,k}^2.
\]

Applying Theorem 1, we obtain

\[
(\text{Cov}(X_j^2, X_k^2))_{j,k=1}^{n} = (2M_{j,k}^2)_{j,k=1}^{n} \geq \frac{2}{n} \cdot E_n.
\]

Thus we have proven

Theorem 7. Let \( X = (X_1, \ldots, X_n) \) be a vector of standard normal variables. Then

\[
(\text{Cov}(X_j^2, X_k^2))_{j,k=1}^{n} \geq \frac{2}{n} \cdot E_n.
\]

5 Numerical integration

In this section, we summarize the approach of [7], where E. Novak studied, how well the quadrature formulas

\[
Q_n(f) = \sum_{i=1}^{n} c_i f(x_i), \quad c_i \in \mathbb{R}, \quad x_i \in [0, 1]^d
\] (7)

approximate the integral \( \text{INT}_d(f) = \int_{[0,1]^d} f(x) dx \). Here \( f \) belongs to a unit ball of a Hilbert space \( F_d \), which is defined as a \( d \)-fold tensor product of a space \( F_1 \), which in turn is a three dimensional Hilbert space with an orthonormal basis given by the functions

\[
e_1(x) = 1, \quad e_2(x) = \cos(2\pi x), \quad e_3(x) = \sin(2\pi x), \quad x \in [0, 1].
\]

Hence \( F_d \) is a \( 3^d \)-dimensional Hilbert space. The point evaluation \( \delta_x : f \to f(x) \) may be written in the form

\[
f(x) = \langle f, \delta_x \rangle_{F_d} \quad \text{with} \quad \delta_x(z) = \prod_{j=1}^{d} \left[ 1 + \cos(2\pi(x_j - z_j)) \right].
\]
In this way, $F_d$ becomes a reproducing kernel Hilbert space with the kernel

$$K_d(x, y) = \langle \delta_x, \delta_y \rangle_{F_d} = \prod_{j=1}^{d} [1 + \cos(2\pi(x_j - y_j))], \quad x, y \in [0, 1]^d.$$ 

This allows to compute the worst-case error of $Q_n$ given by (7) as

$$e_{\text{wor}}(Q_n)^2 = \sup_{\|f\|_{F_d} \leq 1} |\text{INT}_d f - Q_n(f)|^2 = \left\| 1 - \sum_{j=1}^{n} c_j \delta_{x_j} \right\|^2_{F_d} = 1 - 2 \sum_{j=1}^{n} c_j + \sum_{j,k=1}^{n} c_j c_k K_d(x_j, x_k).$$

If all $c_j$’s are positive, we can use the positivity of $K_d$ and obtain

$$e_{\text{wor}}(Q_n)^2 \geq 1 - 2 \sum_{j=1}^{n} c_j + \sum_{j=1}^{n} c_j^2 2^d.$$ 

For the optimal choice $c_j = 2^{-d}$ this becomes

$$e_{\text{wor}}(Q_n)^2 \geq \max(1 - n 2^{-d}, 0). \quad (8)$$

This estimate shows the intractability of numerical integration on $F_d$ with quadrature formulas with positive weights since for a fixed error the number $n$ of sample points needs to grow exponentially with the dimension $d$.

To estimate $e_{\text{wor}}(Q_n)^2$ from below for quadrature rules with general weights $c$, we use the fact that the projection of any $y \in F_d$ onto the ray generated by $x \in F_d$ is given by $\frac{\langle y, x \rangle_{F_d} \delta_x}{\|x\|_{F_d}}$ and its norm is equal to $\frac{\|y\|_{F_d}}{\|x\|_{F_d}}$. In this way we obtain

$$\inf_{c_j, x_j} \left\| 1 - \sum_{j=1}^{n} c_j \delta_{x_j} \right\|^2_{F_d} = \inf_{c_j, x_j} \inf_{\alpha \in \mathbb{R}} \left\| 1 - \alpha \sum_{j=1}^{n} c_j \delta_{x_j} \right\|^2_{F_d} = \inf_{c_j, x_j} \left\{ \left\| 1 \right\|^2_{F_d} - \frac{\left\| 1, \sum_{j=1}^{n} c_j \delta_{x_j} \right\|^2_{F_d}}{\left\| \sum_{j=1}^{n} c_j \delta_{x_j} \right\|^2_{F_d}} \right\} = 1 - \sup_{c_j, x_j} \sum_{j,k=1}^{n} c_j c_k K_d(x_j, x_k) \right\}. \quad (9)$$

Erich Novak conjectured, that the estimate (8) applies also for quadrature formulas (7) with general weights, which is by (9) equivalent to Conjecture 1.
Finally, let us show how Theorem 1 implies the positive answer to Conjecture 1. We define matrices $M^1, \ldots, M^d$ by

$$M^i_{j,k} = \cos\left(\frac{x_{j,i} - x_{k,i}}{2}\right), \quad i = 1, \ldots, d, \quad j, k = 1, \ldots, n.$$ 

By Bochner’s theorem, the matrices $M^i$ are all positive semidefinite and so is their Hadamard product $M = M^1 \circ \cdots \circ M^d$. Obviously, $M$ has all its diagonal elements equal to one. Finally, Theorem 1 shows that the matrix $M \circ M - \frac{1}{n} E_n$ with entries

$$M^2_{j,k} - \frac{1}{n} = \prod_{i=1}^d \cos^2\left(\frac{x_{j,i} - x_{k,i}}{2}\right) - \frac{1}{n} = \prod_{i=1}^d \frac{1 + \cos(x_{j,i} - x_{k,i})}{2} - \frac{1}{n}$$

is also positive semidefinite.

Hence, the integration problem on $F_d$ is intractable even when we allow negative weights $c_j$’s in the quadrature formula (7).

Remark 3. Theorem 1 opens an interesting and (most likely) largely unexplored area of research of the partial matrix ordering given by $\succeq$. For example, the classical calculation (4) can be reformulated as the statement that the zero matrix is a lower bound of the set

$$\{ (\hat{\mu}(\xi_j - \xi_k))^n_{j,k=1} : \mu \text{ is a finite Borel measure} \},$$

where Theorem 4 states that the matrix $E_n/n$ is a lower bound of the set

$$\{ (|\hat{\mu}(\xi_j - \xi_k)|^2)^n_{j,k=1} : \mu \text{ is a probability Borel measure} \}.$$

In general, one may try to prove other non-trivial bounds for the infimum of other classes of matrices.

Other interesting questions in connection with Theorem 1 include

- Is there a version of Theorem 1 which would deal with the Hadamard product of two different matrices, i.e. with $M_1 \circ M_2$ instead of $M \circ M$?
- Is there a variant of Theorem 1 for higher Hadamard powers of $M$?
- Does Theorem 1 hold also when $|g|^2$ gets replaced by a general positive definite function?
- And more generally, is there some converse of Theorem 1?

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