Density-metric unimodular gravity: vacuum maximal symmetry

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Abstract: We have investigated the vacuum maximally symmetric solutions of recently proposed density-metric unimodular gravity theory, the results are widely different from inflationary scenario. The exponential dependence on time in deSitter space is substituted by a power law. Open space-times with non-zero cosmological constant are excluded.

Keywords: Unimodular gravity, Modified gravity, Empty cosmological models.
1. Introduction

Unimodular gravity is a very modest modification of Einstein general theory of relativity which leads to the same field equations with the cosmological constant emerging as an integration constant. In unimodular theory one apparently reduces the dynamical components of the metric by one. This is accomplished by discharging the determinant of the metric as dynamical variable. Conventionally this demand is granted by imposing \(|g| = 1\) as a constraint[1-9]. However, it was shown that one indeed recovers the same number of dynamical degrees of freedom as in GR via a tertiary constraint[10]. Even though a covariant expression on variation of the metric, \(g_{\mu\nu}\delta g_{\mu\nu} = 0\) emerges from the unitary condition \(|g| = 1\), but the primary relation itself is not established on a covariant form. Determinant of the metric is a scalar density and cannot be kept equal to one in each arbitrary coordinate system. Evidently, this is not compatible with our notion of general covariance. This disadvantage has been removed in density-metric unimodular gravity proposed recently[11]. In metric theories of gravity the gravitational field is represented by the spacetime metric which is a symmetric tensor field of rank \((0,2)\) and signature of \(+2\) defined on a four dimensional manifold. In density-metric theory the gravitational field is represented by a symmetric tensor density field of rank \((0,2)\) and weight \(-\frac{1}{2}\) which its determinant is equal to one by definition. Since the determinant of a tensor density of weight \(-\frac{1}{2}\) in a four dimensional spacetime is a scalar under coordinate transformations, so the unitary condition may be imposed consistently with the principle of general covariance. Using this density metric we may equip the manifold by a specific connection. It turns out
that introducing torsion-free and metric compatibility conditions are not sufficient to fix uniquely all components of the connection. Actually, the components of trace of connection remain undetermined. Hence the suppositions are:

$$g_{\mu \nu} = g_{\nu \mu}$$ Symmetric tensor density of weight $-\frac{1}{2}$ (1.1)
$$|g| = 1$$ Unitary condition (1.2)
$$\Gamma^\rho_{\mu \nu} = \Gamma^\rho_{\nu \mu}$$ Torsion free (1.3)
$$\nabla_{\rho} q_{\mu \nu} = 0$$ Metric compatible (1.4)

By expanding out the equation (1.4) for three different permutations of indices and subtracting the second and third of these form the first we obtain:

$$\Gamma^\lambda_{\mu \nu} = \frac{1}{2} g^{\lambda \rho} (\partial_\mu q_{\nu \rho} + \partial_\nu q_{\mu \rho} - \partial_\rho q_{\mu \nu}) + \frac{1}{4} (\Gamma^\rho_{\rho \mu} \delta^\lambda_{\nu} + \Gamma^\rho_{\rho \nu} \delta^\lambda_{\mu} - \Gamma^\rho_{\rho \kappa} g^{\kappa \lambda} q_{\mu \nu})$$ (1.5)

Contraction of indices $\lambda$ and $\mu$ in (1.5) gives

$$\Gamma^\lambda_{\lambda \nu} = \frac{1}{2} g^{\lambda \rho} \partial_\nu q_{\lambda \rho} + \Gamma^\rho_{\rho \nu}$$ (1.6)

Obviously (1.6) brings us to

$$g^{\lambda \rho} \partial_\nu q_{\lambda \rho} = 0$$ (1.7)

So four relations of (1.7) reduce ten independent components of density-metric to six. Therefore components of connection typically are functions of six independent components of density-metric and four independent components of the trace of connection $\Gamma^\rho_{\rho \mu}$. Using the connection(1.5), we may introduce Riemann tensors and Ricci tensor and scalar respectively by:

$$R^\rho_{\sigma \mu \nu} = \partial_\nu \Gamma^\rho_{\sigma \mu} - \partial_\mu \Gamma^\rho_{\sigma \nu} + \Gamma^\rho_{\rho \lambda} \Gamma^\lambda_{\mu \sigma} - \Gamma^\rho_{\mu \lambda} \Gamma^\lambda_{\nu \sigma}$$ (1.8)
$$R_{\rho \sigma \mu \nu} = q_{\rho \lambda} R^\lambda_{\sigma \mu \nu}$$ (1.9)
$$R_{\sigma \nu} = R^\rho_{\sigma \rho \nu}$$ (1.10)
$$R = q^{\sigma \nu} R_{\sigma \nu}$$ (1.11)

Here (1.8) is a tensor of rank $(1, 3)$, (1.9) is a tensor density of rank $(0, 4)$ and weight $-\frac{1}{2}$, (1.10) is a tensor of rank $(0, 2)$ and (1.11) is a scalar density of weight $+\frac{1}{2}$. Let us notice some algebraic properties of Riemann tensor. It is antisymmetric in its first two indices

$$R_{\rho \sigma \mu \nu} = -R_{\rho \sigma \nu \mu}$$ (1.12)

and it is not invariant under interchange of the first pair of indices with the second:

$$R_{\rho \sigma \mu \nu} \neq R_{\mu \nu \rho \sigma}$$ (1.13)

The sum of the cyclic permutations of the last three indices of Riemann tensor vanishes:

$$R_{\rho \sigma \mu \nu} + R_{\rho \mu \nu \sigma} + R_{\rho \nu \sigma \mu} = 0$$ (1.14)
The Bianchi identity holds too. The sum of cyclic permutation of the first three indices of covariant derivatives of the Riemann tensor is zero:

\[ R_{\rho\sigma\nu,\lambda} + R_{\rho\lambda\mu;\nu} + R_{\rho\sigma\nu,\lambda\mu} = 0 \quad (1.15) \]

The Ricci tensor associated with the Christoffel connection is symmetric but in our case this is not generally true. Twice contraction of Bianchi identity (1.15) with \( g^\rho_\mu \) and \( g^\sigma_\nu \) plus doing some calculations in Riemann normal coordinates result in:

\[ \nabla^\sigma \left( \frac{1}{4} R_{\lambda\sigma} + \frac{3}{4} R_{\sigma\lambda} - \frac{1}{2} g_{\sigma\lambda} R \right) = 0 \quad (1.16) \]

In density-metric unimodular gravity figured that \( R^2 \) was the simplest possible choice for a Lagrangian density and proposed

\[ I = \int \kappa R^2 dx^4 \quad (1.17) \]

for the action of gravitational fields in the absence of matter and other kinds of sources. Variation of the action (1.17) with respect to \( \Gamma^\rho_\mu_\lambda \) leads to the field equation

\[ \nabla_\lambda R = 0 \quad (1.18) \]

Also variations of the action (1.17) with respect to \( g^{\mu\nu} \) and applying the unimodular condition

\[ g_{\mu\nu} \delta g^{\mu\nu} = 0 \quad (1.19) \]

by the method of Lagrange undetermined multipliers and inserting field equation (1.18) lead to

\[ R_{[\mu|\nu]} - \frac{1}{4} g_{\mu\nu} R = 0 \quad (1.20) \]

We should mention that density-metric is not eligible to carry out the whole tasks of a common metric. First its operation on two vectors gives a scalar density instead of a scalar. Second the density-metric has six independent components. In accordance with affine theories of gravity we may equip the spacetime with a metric by defining

\[ g_{\mu\nu} = \kappa R_{[\rho\mu\nu]} \quad (1.21) \]

where \( R \) is given by eq.(1.11) and of course should be nonzero. In addition it depends on four independent components of trace of connection,\( \Gamma^\rho_\mu_\lambda \). Thus \( g_{\mu\nu} \) depends on ten independent dynamical variables. Given the equation (1.21) in principle, we can work out density-metric and trace of connection in terms of ten independent components of \( g_{\mu\nu} \) and formally write:

\[ \delta_{\mu\nu} = \kappa_{\mu\nu}(g_{\alpha\beta}) \quad \Gamma^\rho_\mu_\lambda = \Gamma^\rho_\mu_\lambda(g_{\alpha\beta}) \quad (1.22) \]

We have already worked out the solution of this theory for vacuum spherically symmetric space-time:

\[ ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2) \quad (1.23) \]
with
\[ B(r) = (1 + \frac{C'}{r^3})^{-\frac{2}{3}} \left[ 1 - \frac{2GM - \Lambda C'}{r} (1 + \frac{C'}{r^3})^{-\frac{1}{3}} + \Lambda r^2 (1 + \frac{C'}{r^3})^{-\frac{1}{3}} \right] \]  (1.24)
and
\[ A(r) = (1 + \frac{C'}{r^3})^{-2} \left[ 1 - \frac{2GM - \Lambda C'}{r} (1 + \frac{C'}{r^3})^{-1} + \Lambda r^2 (1 + \frac{C'}{r^3})^{-1} \right]^{-1} \]  (1.25)

In these relations \( M \) is the mass of point object which is the source of spherically symmetry and \( \Lambda \) has dimensions \([L^{-2}]\) where may be identified as cosmological constant. \( C' \) is a new constant which appears in this theory with dimensions \([L^{-3}]\).

The constant \( C' \) has a physical interpretation, a convenient way to show its real nature is when we manipulate the post-post-Newtonian formalism; just as the same way that the Schwarzschild radius is fixed by comparison with Newtonian limits. It should be noticed that a spherically symmetric matter distribution has a total mass, but certainly has a vanishing quadrupole moment. Such a matter distribution could have three equal principal moments of inertia \( I \). It seems very probable that \( C' \propto \frac{GI}{c^2} \). In this way \( C' \) is an individual property of the source object. Observational data from double pulsar systems can be used to verify this issue. Here we can offer an upper bound for it.

In the next section we explicitly work out these results and correctly predict what are observed in three classical tests of gravitation.

2. Classical Tests

For this purpose we may choose \( \Lambda = 0 \). The geodesic equations imply that the quantity
\[ \epsilon = -g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \]  (2.1)
is constant along the path. For massive particles we typically choose \( \lambda \) so that \( \epsilon = 1 \). While for massless particles we always have \( \epsilon = 0 \). Invariance under time translations leads to conservation of energy while invariance under spatial rotations leads to conservation of the three components of angular momentum. Conservation of direction of angular momentum means that particle moves on a plane. So we may choose \( \theta = \frac{\pi}{2} \). The two remaining Killing vectors correspond to the energy and magnitude of angular momentum. The energy arises from time-like Killing vector \( (\partial_t)^\mu = (1, 0, 0, 0) \) while the Killing vector whose conserved quantity is the magnitude of angular momentum is \( (\partial_\phi)^\mu = (0, 0, 0, 1) \). The two conserved quantities are

\[ E = -(1 + \frac{C'}{r^3})^{-\frac{2}{3}} \left[ 1 - \frac{2GM}{r} (1 + \frac{C'}{r^3})^{-\frac{1}{3}} \right] \frac{dt}{d\lambda} \]  (2.2)

and

\[ L = r^2 \frac{d\phi}{d\lambda} \]  (2.3)

From eq.(2.1)by using eqs.(2.2) and (2.3) we can arrive at the following result:

\[ -E^2 + (1 + \frac{C'}{r^3})^{-\frac{2}{3}} \left( \frac{dr}{d\lambda} \right)^2 + \]
\[ (1 + \frac{C'}{r^3})^{-\frac{2}{3}} \left( 1 - \frac{2GM}{r} (1 + \frac{C'}{r^3})^{-\frac{1}{3}} \right) (\frac{L^2}{r^2} + \epsilon) = 0 \]  (2.4)
Then expanding in powers of \( r \), we shall content ourselves here with only the lowest order of approximation in eq.(2.4). So we have

\[
\left( \frac{dr}{d\lambda} \right)^2 + (1 - \frac{2GM}{r})(L^2 + e) - E^2(1 + \frac{8C'}{3r^3}) + \frac{2EC'}{r^3} = 0 \tag{2.5}
\]

Now we are prepared to discuss the classical tests one by one.

1) **Precession of Perihelia**: With \( \epsilon = 1 \) in eq.(2.5) and multiplying it by \( \left( \frac{d\phi}{d\lambda} \right)^{-2} = \frac{r^4}{L^2} \) and defining \( x = \frac{L^2}{GM} \) we have

\[
\left( \frac{dx}{d\phi} \right)^2 - 2x + x^2 - \frac{2G^2M^2}{L^2}x^3 + \frac{GM}{L^4}(2C' - \frac{8E^2C'}{3})x^3 = \frac{E^2L^2}{G^2M^2} - \frac{L^2}{G^2M^2} \tag{2.6}
\]

Differentiating eq.(2.6) with respect to \( \phi \) gives

\[
\frac{d^2x}{d\phi^2} - 1 + x = \frac{3G^2M}{L^2} - \frac{3GM}{L^4}(C' + \frac{4GME^2C'}{L^4})x^2 \tag{2.7}
\]

After some manipulation [12] we obtain that the perihelion advances by an angle

\[
\Delta \phi = \frac{6\pi G^2M^2}{L^2}(1 - \frac{C'}{GML^2} + \frac{4C'E^2}{3GML^2}) \tag{2.8}
\]

On the other hand we have

\[
E^2 = 1 + \frac{G^2M^2}{L^2}(e^2 - 1) \quad \text{and} \quad L^2 \approx GM(1 - e^2)a \tag{2.9}
\]

where \( e \) is the eccentricity and \( a \) is the semi-major axis of the ellipse. Putting eq.(2.9) in eq.(2.8) yields

\[
\Delta \phi = \frac{6\pi GM}{(1 - e^2)a}(1 + \frac{C'}{3G^2M^2(1 - e^2)a} + \frac{4C'E^2}{3GM(1 - e^2)a^2}) \tag{2.10}
\]

The third term in the parenthesis is smaller than the second term by a \( \frac{GM}{a} \) factor and may be neglected.

\[
\Delta \phi = \frac{6\pi GM}{(1 - e^2)a}(1 + \frac{C'}{3G^2M^2(1 - e^2)a}) \tag{2.11}
\]

Observational data put an upper bound on the \( C' \).

2) **Deflection of Light**: Putting \( \epsilon = 0 \) in eq.(2.5) and multiplying it by \( \left( \frac{d\phi}{d\lambda} \right)^{-2} = \frac{r^4}{L^2} \) and defining \( u = \frac{1}{r} \) we have

\[
\left( \frac{du}{d\phi} \right)^2 + u^2 - 2GMu^3 = \frac{E^2}{L^2}(1 + 8C'u^3) \tag{2.12}
\]

Differentiation of eq.(2.12) with respect to \( \phi \) makes

\[
\frac{d^2u}{d\phi^2} + u = (3GM + \frac{12C'E^2}{L^2})u^2 \tag{2.13}
\]
By some straightforward calculations [13] the deflection angle is

$$\delta = \frac{4}{r_0} [GM + \frac{4C'E^2}{L^2}]$$

(2.14)

where \(r_0\) is the distance of closest approach to the origin. Since \(\frac{E}{L}\) is \(\frac{1}{r_0}\) then eq.(2.14) becomes

$$\delta = \frac{4GM}{r_0} + \frac{16C'}{r_0^3}$$

(2.15)

An upper bound on \(C'\) may be obtained by comparison with observational data.

3) Gravitational Redshift: Let us consider an observer with four-velocity \(U^\mu\) who is stationary \((U^i = 0)\), we have:

$$U^0 = (B(r))^{-\frac{1}{2}}$$

(2.16)

Any such observer measures the frequency of a photon following along a null geodesic \(\chi^\mu(\lambda)\) to be

$$\omega = -g_{\mu\nu} \frac{dx^\nu}{d\lambda} U^\mu = B(r)^{\frac{1}{2}} \frac{dt}{d\lambda}$$

(2.17)

By considering eq.(2.2) and (2.2) we conclude that for a photon emitted at \(r_1\) and observed at \(r_2\) the observed and emitted frequencies are related by

$$\frac{\omega_2}{\omega_1} = \left(\frac{B(r_1)}{B(r_2)}\right)^{\frac{1}{2}}$$

(2.18)

In the interval \(\Lambda^{-\frac{1}{2}} \gg r \gg 2GM\) and \(C'^{-\frac{1}{2}}\) we get

$$\frac{\omega_2}{\omega_1} = 1 - GM\left(\frac{1}{r_1} - \frac{1}{r_2}\right) - \frac{C'}{3} \left(\frac{1}{r_1^3} - \frac{1}{r_2^3}\right) + \frac{\Lambda}{2} (r_1^2 - r_2^2)$$

(2.19)

This is also consistent with observational constraints.

Now we want to work out the solutions of these field equations for the vacuum maximal symmetry space-times. In other words our goal is to illustrate the analogue of Minkowski, deSitter and anti-deSitter space-times in the density metric unimodular theory of gravity. The deSitter-Schwarzschild analogue in static form may be obtained by putting \(M = C' = 0\) in eqs.(1.24) and (1.25).

3. Maximal Symmetry

A rigorous mathematical analysis may be applied by taking \(N\) dimensional manifold equipped with a density metric and a trace of connection. Assuming the density metric of rank \((0, 2)\) and weight \(-\frac{1}{2}\) being form invariant under a given coordinate transformation generated by a Killing vector \(\xi^\mu\) leads to:

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu - \frac{1}{2} g_{\mu\nu} \nabla_\lambda \xi^\lambda = 0$$

(3.1)
This means that $\nabla_\mu \xi^\nu - \frac{1}{4} g_{\mu\nu} \nabla \xi^\lambda$ is an anti-symmetric tensor. Since the trace of connection is taken as dynamical variable the condition to be form invariant provide us:

$$\nabla_\nu \nabla_\lambda \xi^\lambda = \xi^\lambda \left( \frac{\partial}{\partial x^\nu} \Gamma^\rho_{\mu\lambda} - \frac{\partial}{\partial x^\lambda} \Gamma^\rho_{\mu\nu} \right) \equiv \hat{R}^\rho_{\nu\lambda} \xi^\lambda$$ \hspace{1cm} (3.2)

It can be easily shown that

$$\nabla_\sigma \nabla_\rho \xi_\mu = -R^\lambda_{\rho\sigma\mu} \xi_\lambda - \frac{1}{4} g_{\mu\rho} \hat{R}^\lambda_{\nu\sigma\lambda} + \frac{1}{4} g_{\sigma\mu} \hat{R}^\lambda_{\rho\nu\lambda} + \frac{1}{4} g_{\nu\rho} \hat{R}^\lambda_{\sigma\lambda\mu}$$ \hspace{1cm} (3.3)

Following the same argument as given by Weinberg[14], we may conclude that the manifold should have $\frac{N(N+1)}{2}$ Killing vectors to be maximally symmetric. We start from general formula for commutator of covariant derivatives of tensors

$$\nabla_\sigma \nabla_\nu (\nabla_\mu \xi_\rho - \frac{1}{4} g_{\mu\rho} \nabla_\eta \xi^\eta) - \nabla_\nu \nabla_\sigma (\nabla_\mu \xi_\rho - \frac{1}{4} g_{\mu\rho} \nabla_\eta \xi^\eta) =$$

$$-R^\lambda_{\rho\sigma\mu} (\nabla_\mu \xi_\lambda - \frac{1}{4} g_{\mu\lambda} \nabla_\eta \xi^\eta) - R^\lambda_{\mu\sigma\nu} (\nabla_\lambda \xi_\rho - \frac{1}{4} g_{\lambda\rho} \nabla_\eta \xi^\eta)$$ \hspace{1cm} (3.4)

Then apply eqs.(3.1),(3.2) and (3.3) into eq.(3.4). It can be shown that for a maximally symmetric manifold we should have:

$$\hat{R}_{\mu\nu} = 0$$ \hspace{1cm} (3.5)

This is an important result, since it causes the Riemann curvature tensor to resume the same algebraic property just as Riemann geometry, i.e. if Christoffel connections were used. Following Weinberg’s procedure we obtain

$$(N - 1)R_{\lambda\rho\sigma\nu} = R_{\nu\rho} g_{\lambda\sigma} - R_{\sigma\rho} g_{\lambda\nu}$$ \hspace{1cm} (3.6)

and

$$R_{\sigma\rho} = \frac{1}{N} g_{\sigma\rho} R$$ \hspace{1cm} (3.7)

Putting (3.7) into (1.16) gives

$$\nabla_\lambda R = 0$$ \hspace{1cm} (3.8)

Of course this does not mean $R$ is constant since it is a density scalar.

4. Vacuum Cosmic Dynamics

The homogeneity and isotropy of a space-time imply that it is maximally symmetric. In co-moving coordinates the metrics are given by Robertson-Walker line element,

$$ds^2 = dt^2 - a^2(t) \left\{ \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right\}$$ \hspace{1cm} (4.1)

To prepare the appropriate density metric consider the following scenario: Multiply the components of Robertson-Walker metric by inverse of $\sqrt{|g|}$, where $|g|$ is determinant of the
metric. This results in a density tensor of rank \((0,2)\) with the desired weight \(-\frac{1}{2}\). The nonzero components of the new density metric are,

\[
g_{tt} = \frac{(1 - kr^2)^{\frac{1}{4}}}{a^2 r \sin^{\frac{1}{2}} \theta}, \quad g_{rr} = \frac{a^4}{r \sin^{\frac{1}{2}} \theta (1 - kr^2)^{\frac{1}{4}}},
\]

\[
g_{\theta\theta} = \frac{a^2 r (1 - kr^2)^{\frac{1}{4}}}{\sin^{\frac{1}{2}} \theta}, \quad g_{\phi\phi} = a^4 r \sin^{\frac{1}{2}} \theta (1 - kr^2)^{\frac{1}{4}} \tag{4.2}
\]

and the nonzero components of its inverse are,

\[
g^{tt} = -\frac{a^3 r \sin^{\frac{1}{2}} \theta}{(1 - kr^2)^{\frac{1}{4}}}, \quad g^{rr} = \frac{r \sin^{\frac{1}{2}} \theta (1 - kr^2)^{\frac{3}{4}}}{a^2},
\]

\[
g^{\theta\theta} = -\frac{\sin^{\frac{1}{2}} \theta}{a^4 r (1 - kr^2)^{\frac{1}{4}}}, \quad g^{\phi\phi} = \frac{1}{a^4 r \sin^{\frac{3}{2}} \theta (1 - kr^2)^{\frac{1}{4}}} \tag{4.3}
\]

In terms of the unknown components of trace of connection, the nonzero components of the connection are determined by using eqs.(4.2),(4.3) and (1.5). The results are presented in A.1. The off diagonal components of the field equation (1.10) simply are:

\[
R_{\mu\nu} = 0 \quad \text{for} \quad \mu \neq \nu \tag{4.4}
\]

Inserting the connection from A.1 into eqs.(1.8), and (1.10), then \(R_{rt} = 0\) gives

\[
(\Gamma^\rho_{\rho t} + \frac{\dot{a}}{a})(\Gamma^\rho_{\rho r} - \frac{2}{r} - \frac{kr}{2(1 - kr^2)}) = 0 \tag{4.5}
\]

Equation (4.5) derives a manifestly unique expression for the radial component of trace of connections.

\[
\Gamma^\rho_{\rho r} = \frac{2}{r} + \frac{kr}{1 - kr^2} \tag{4.6}
\]

Similarly the polar component of trace of connection will be obtained by considering the equation \(R_{t\theta} = 0\). It gives:

\[
(\Gamma^\rho_{\rho t} + \frac{\dot{a}}{a})(\Gamma^\rho_{\rho \theta} - \cot \theta) - 2 \frac{\partial}{\partial t} \Gamma^\rho_{\rho \theta} = 0 \tag{4.7}
\]

Eq. (4.7) imposes a unique expression for polar component of connection trace

\[
\Gamma^\rho_{\rho \theta} = \cot \theta \tag{4.8}
\]

The azimuth component of trace of connection will emerge immediately by considering \(R_{t\phi} = 0\). This simply reads

\[
\Gamma^\rho_{\rho \phi}(\Gamma^\rho_{\rho t} + \frac{\dot{a}}{a}) = 0 \tag{4.9}
\]

Therefore eq.(4.9) at last determines azimuth component of connection trace

\[
\Gamma^\rho_{\rho \phi} = 0 \tag{4.10}
\]
The other off-diagonal components of field equations, $R_{r\theta} = R_{r\phi} = R_{\theta\phi} = 0$, are satisfied automatically by the given equations (4.6), (4.8) and (4.10). Plugging equations (4.6), (4.8) and (4.10) into the given components of the connection in A.1 reduce the non-vanishing components of the connection. They are given in A.2. These results may be used to compute the diagonal components of Ricci tensor:

$$R_{tt} = \frac{3}{4} \left( \frac{\ddot{a}}{a} + \frac{\dot{a}}{a} \Gamma^\rho_{\rho t} + \ddot{\Gamma}^\rho_{\rho t} \right)$$

$$R_{rr} = -\frac{2k}{1 - kr^2} - \frac{a^2}{4(1 - kr^2)} \left( \frac{\ddot{a}}{a} + \frac{1}{2} \left( \frac{\dot{a}}{a} \right)^2 + 2 \frac{\dot{a}}{a} \Gamma^\rho_{\rho t} + \ddot{\Gamma}^\rho_{\rho t} + \frac{1}{2} \Gamma^\rho_{\rho t} \right)^2$$

$$R_{\theta\theta} = -\frac{2kr^2}{8a^2} - \frac{\dot{a}}{a}^2 [2 \frac{\ddot{a}}{a} + \frac{4}{a} \frac{\dot{a}}{a} \Gamma^\rho_{\rho t} + \Gamma^\rho_{\rho t}]$$

$$R_{\phi\phi} = R_{\theta\theta} \sin^2 \theta$$

Let us write the diagonal components of the field equation (1.20):

$$\frac{3}{2} g^{tt} R_{tt} - \frac{1}{2} g^{rr} R_{rr} - g^{\theta\theta} R_{\theta\theta} = 0$$

$$\frac{3}{2} g^{rr} R_{rr} - \frac{1}{2} g^{tt} R_{tt} - g^{\theta\theta} R_{\theta\theta} = 0$$

$$g^{\theta\theta} R_{\theta\theta} - \frac{1}{2} g^{tt} R_{tt} - \frac{1}{2} g^{rr} R_{rr} = 0$$

The azimuth component of eq. (1.20) reproduces the same equation as eq. (4.17). To achieve the final result it is sufficient to substitute eqs. (4.11)-(4.13) into eqs. (4.15)-(4.17) which all merely get to the same equation as,

$$\frac{\ddot{a}}{a} - \frac{1}{4} \left( \frac{\dot{a}}{a} \right)^2 - \frac{4k}{a^2} = 0$$

The field equation (1.18) does not express a new equation and indeed relation (4.18) already satisfies equation (1.18). In the case of Minkowski space-time, $a$ is constant and $k$ is equal to zero. These are in agreement with relation (4.18) if we distinguish,

$$\Gamma^\rho_{\rho t} = 0$$

According to arguments expounded in [11], concerning gauge fixing, we regard that generally relation (4.19) rigorously holds. This is consistent with our result for vacuum spherical symmetry and $M = 0$ presented in the mentioned reference. Finally substituting relation (4.19) into eq. (4.18) yields

$$\frac{\ddot{a}}{a} - \frac{1}{4} \left( \frac{\dot{a}}{a} \right)^2 - \frac{4k}{a^2} = 0$$

It is relevant to compare the obtained equation (4.20) with its counterpart in general theory of relativity which is Friedmann equation. We have [15],

$$\dot{a}^2 = \frac{\Lambda}{3} a^2 - k$$

$$\frac{2\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} - \Lambda = 0$$
Eliminating $\Lambda$ between equations (4.21) and (4.22) we get,

$$\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2 - \frac{k}{a^2} = 0$$  \hspace{1cm} (4.23)

It is interesting to notice that in eqs. (4.20) and (4.23) identical terms appear with merely different numerical coefficients and both equations are invariant under time reversal transformation. So it is plausible while they have probably the same roots we predict to obtain results with different behavior than the existing models. We are going to solve the eq.(4.20). To do this task, we assume in accordance with eq.(4.21) it has a solution of general form

$$\dot{a}^2 = f(a)$$  \hspace{1cm} (4.24)

Differentiating eq.(4.24) with respect to time and taking $\dot{a} \neq 0$ yields

$$\ddot{a} = \frac{1}{2} \frac{df(a)}{da}$$  \hspace{1cm} (4.25)

Plugging eqs.(4.24) and (4.25) into eq.(4.20) gives

$$\frac{a}{2} \frac{df(a)}{da} - \frac{1}{4} f(a) - 4k = 0$$  \hspace{1cm} (4.26)

Eq.(4.24) has the general solution,

$$\dot{a}^2 \equiv f(a) = \Lambda \sqrt{a} - 16k$$  \hspace{1cm} (4.27)

where $\Lambda$ is constant of integration and we have intended to call it so to show its resemblance with cosmological constant in Friedmann model. Eq.(4.27) may be integrated and has the general solution

$$\pm t = \frac{4(\Lambda \sqrt{a} + 32k)}{3\Lambda^2} \sqrt{\Lambda \sqrt{a} - 16k + C}$$  \hspace{1cm} (4.28)

where $C$ is the second constant of integration and may be put equal to zero without loose of generality. To find scale factor as a function of time from eq.(4.28) we will face with a cubic equation. In the next section we will discuss some features of the solutions of this equation in different situations.

5. Empty Cosmological Models

Reality condition imposed on (4.27) and (4.28) imply the following limits on the domains of $a$ and $t$ without solving the equations.

| Case | $\Lambda > 0$ | $k$ | $a \geq \left(\frac{16}{\Lambda}\right)^2$ | $t \in R$ | deSitter | (5.1) |
|------|--------------|-----|--------------------------------|-----------|-----------|------|
| I    | $k = 1$      |     |                                  |           | deSitter  | (5.2) |
| II   | $k = 0$      |     |                                  |           | deSitter  | (5.3) |
| III  | $k = -1$     |     |                                  |           | no solution | (5.4) |
| IV   | $k = 1$      |     |                                  |           | Minkowski | (5.5) |
| V    | $k = 0$      |     |                                  |           | Minkowski | (5.5) |
\[ VI \quad \Lambda = 0 \quad k = -1 \quad a \propto t \quad t \in R \quad \text{Milne (5.6)} \]

\[ VII \quad \Lambda < 0 \quad k = +1 \quad \text{no solution (5.7)} \]

\[ VIII \quad \Lambda < 0 \quad k = 0 \quad \text{no solution (5.8)} \]

\[ IX \quad \Lambda < 0 \quad k = -1 \quad 0 \leq a \leq \left(\frac{16}{\Lambda}\right)^2 \quad |t| \leq \frac{2}{3}\left(\frac{16}{\Lambda}\right)^2 \quad \text{anti-deSitter (5.9)} \]

Now we express some points for each case in more detail.

I. \( \Lambda > 0, k = 1 \)

All closed models with positive lambda have a lower bound on \( a \) equal to \( a_{\min} = \frac{256}{\Lambda^2} \).

To find \( a \) as a function of time from eq.(4.28) we face with a cubic equation which its discriminant is positive for this model. This means it merely has one real root. It is:

\[
a = \left\{ \frac{16}{\Lambda} \sqrt[3]{1 + 9\left(\frac{\Lambda}{16}\right)^4 t^2} + \sqrt{-1 + (1 + 9\left(\frac{\Lambda}{16}\right)^4 t^2)^2} \right. \\
\left. + \sqrt[3]{1 + 9\left(\frac{\Lambda}{16}\right)^4 t^2} - \sqrt{-1 + (1 + 9\left(\frac{\Lambda}{16}\right)^4 t^2)^2 - 1} \right\}^2
\]

(5.10)

Models of class I possess event horizon which can be find by considering

\[
\int_0^{r_H} \frac{dr}{\sqrt{1 - r^2}} = \int_{t_0}^{+\infty} \frac{dt}{a(t)}
\]

(5.11)

We infer from eq.(5.10) that \( a \to +\infty \) as \( t \to +\infty \). Taking \( a(t_0) = a_0 \), eq.(5.11) gives

\[
\sin^{-1}(r_H) = \int_{t_0}^{+\infty} \frac{da}{a\dot{a}}
\]

(5.12)

Calculating the integral in eq.(5.12) by inserting eq.(4.27), we will have

\[
r_H = \cos(tan^{-1} \sqrt{\frac{\Lambda\sqrt{a_0}}{16} - 1})
\]

(5.13)

II. \( \Lambda > 0, k = 0 \)

In flat models with positive lambda eq.(4.27) reduces to

\[
\frac{\dot{a}}{\sqrt{a}} = \pm \sqrt{\Lambda}
\]

(5.14)

and integration of eq.(5.14) with respect to time gives

\[
a = \left(\frac{3}{4} \sqrt{\Lambda |t|}\right)^{\frac{4}{3}}
\]

(5.15)

The integration constant has been chosen so that we have \( a = 0 \) at \( t = 0 \). While Hubble and deceleration parameters in Friedmann empty flat model are constant, eq.(5.15) implies

\[
H \equiv \frac{\dot{a}}{a} = \frac{4}{3t}
\]

(5.16)
and
\[ q \equiv -\frac{\ddot{a}}{aH^2} = -\frac{1}{4t^2} \]  
(5.17)

Deceleration parameter eq.(5.17) stands for an accelerating Universe which its acceleration rate decreases by passing of time.

The exponential growth of deSitter space in models I and II is replaced by a power law which practically is more plausible to match the physical reality. This behavior is in no way inflationary. Model II also possesses an event horizon

\[ r_H = \int_{t_0}^{+\infty} \frac{dt}{a(t)} = \int_{t_0}^{+\infty} \frac{1}{a_0} \left( \frac{t_0}{t} \right)^{\frac{4}{3}} dt = \frac{1}{3a_0} \]  
(5.18)

where \( a_0 = a(t_0) \). It should be noted this model is singular at \( t = 0 \).

III. \( \Lambda > 0, k = -1 \)

Eq.(4.28) for an open model with positive lambda has a real root

\[
a = \left\{ \frac{16}{\Lambda} \left[ \sqrt[3]{-1 + 9 \left( \frac{\Lambda}{16} \right)^4t^2} + \sqrt{-1 + (-1 + 9 \left( \frac{\Lambda}{16} \right)^4t^2)^2} \right] + \sqrt[3]{-1 + 9 \left( \frac{\Lambda}{16} \right)^4t^2} - \sqrt{-1 + (-1 + 9 \left( \frac{\Lambda}{16} \right)^4t^2)^2 - 1} \right\}^2 \]  
(5.19)

which is defined merely for \(|t| > \frac{2}{3}(\frac{16}{\Lambda})^2\). While it is singular at \( t = \frac{2}{3}(\frac{16}{\Lambda})^2 \) is not defined for long intervals of time. For this reason this model can not present a physically acceptable space-time.

IV. \( \Lambda = 0, k = 1 \)

There is no solution for a closed model with zero lambda.

V. \( \Lambda = 0, k = 0 \)

The flat model with zero lambda presents the well-known Minkowski space.

VI. \( \Lambda = 0, k = -1 \)

The open model with zero lambda presents Milne space.

VII. \( \Lambda < 0, k = +1 \)

There is no solution for closed models with negative lambda.

VIII. \( \Lambda < 0, k = 0 \)

There is no solution for flat models with negative lambda.

IX. \( \Lambda < 0, k = -1 \)

In this case eq.(4.28) has a negative lambda has a negative discriminant which means there are three real roots. Since this does not give a unique solution and the definition of function is not fulfilled, so should be rejected.
6. Concluding Remarks

1- There exists no solution with oscillating nature.

2- Models II and III are singular and the nature of their singularities could not be explored by constructing a scalar of Riemann tensor. It is illustrative to compute the density scalar $R$ by applying eqs.(4.25) and (4.26), we get

$$R = -\frac{3}{4} \frac{r \sin^{\frac{1}{2}} \theta}{(1 - k r^2)^{\frac{3}{2}}} \Lambda$$  \hspace{1cm} (6.1)

The action may be rewritten by inserting eq.(6.1) into eq.(1.17)

$$I = \int \kappa \frac{r^2 \sin \theta}{\sqrt{1 - k r^2}} \left[ -\frac{9}{16} \Lambda^2 \right] dtdrd\theta d\phi$$  \hspace{1cm} (6.2)

This does reminisce a non-flat geometry with constant curvature. So we may hope with some certain that singularity is non-intrinsic.

3- The Riemann tensor made of the metric (4.2) while satisfying (4.27) is consistent with eq.(3.6). So we may say with confidence that the obtained space-times are maximally symmetric.

4- Open models admit merely $\Lambda = 0$.

5- The inflationary behavior of deSitter models are replaced by a power law character which reserve to be physically interpreted.

6- The event horizon $r_H$ given by eq.(5.18) goes to infinity if $t_0$ approaches to zero. We may hope while we have improved the inflationary scenario also have resolved the existence of event horizon.

A. 1

Non-zero components of the connection are:

$$\Gamma^t_{tt} = \frac{1}{4} (\Gamma^t_{\rho t} - \frac{3}{a} \dot{\rho}) , \quad \Gamma^t_{tr} = \frac{1}{4} (\Gamma^t_{\rho r} - \frac{2}{r} - \frac{kr}{1 - k r^2}) , \quad \Gamma^t_{t\theta} = \frac{1}{4} (\Gamma^t_{\rho \theta} - \cot \theta) , \quad \Gamma^t_{t\phi} = \frac{1}{4} \Gamma^t_{t\phi}$$

$$\Gamma^r_{rr} = \frac{1}{4(1 - k r^2)} (\Gamma^r_{\rho r} + \frac{\dot{a}}{a}) , \quad \Gamma^r_{r\theta} = \frac{a^2 r^2}{4} (\Gamma^r_{\rho \theta} + \frac{\dot{\rho}}{\rho}) , \quad \Gamma^r_{r\phi} = \frac{a^2 r^2 \sin^2 \theta}{4} (\Gamma^r_{\rho \phi} + \frac{\dot{\phi}}{\phi})$$

$$\Gamma^\theta_{rt} = \frac{1}{4} (\Gamma^\theta_{r \theta} - \cot \theta) , \quad \Gamma^\theta_{\theta t} = \frac{1}{4} (\Gamma^\theta_{\rho t} + \frac{\dot{\rho}}{\rho}) , \quad \Gamma^\theta_{\theta r} = \frac{1}{4} (\Gamma^\theta_{\rho r} - \frac{2}{r} + \frac{3kr}{1 - k r^2})$$

$$\Gamma^\phi_{rt} = \frac{1}{4} (\Gamma^\phi_{r \phi} - \cot \theta) , \quad \Gamma^\phi_{\phi t} = \frac{1}{4} (\Gamma^\phi_{\rho t} + \frac{\dot{\rho}}{\rho}) , \quad \Gamma^\phi_{\phi r} = \frac{1}{4} (\Gamma^\phi_{\rho r} - \frac{2}{r} + \frac{3kr}{1 - k r^2})$$

$$\Gamma^\theta_{rr} = \frac{1}{4 a^2 r^2} (\Gamma^\theta_{\rho r} - \frac{2}{r} - \frac{kr}{1 - k r^2}) , \quad \Gamma^\theta_{r \phi} = \frac{1}{4} (\Gamma^\theta_{\rho \phi} - \frac{3 \cot \theta}{r})$$
\[ \Gamma^\phi_{\theta\theta} = -\frac{\Gamma^\rho_{\theta\rho}}{4\sin^2\theta}, \quad \Gamma^\phi_{\phi\theta} = \frac{1}{4}(\Gamma^\rho_{\rho\theta} + 3\cot\theta), \quad \Gamma^\phi_{\phi\phi} = \frac{1}{4}\Gamma^\rho_{\rho\phi}. \]

(-) represents differentiation with respect to time.

A. 2

The non-zero components of the connection after inserting eqs.(4.6),(4.8) and (4.10) in the given components of connection in A.1 are:

\[ \Gamma^r_{tt} = \frac{1}{4}(\Gamma^\rho_{\rho t} - 3\frac{\dot{a}}{a}), \quad \Gamma^r_{rr} = \frac{a^2}{4(1 - kr^2)}(\Gamma^\rho_{\rho t} + \frac{\dot{a}}{a}), \quad \Gamma^r_{\theta\theta} = \frac{a^2 r^2}{4(1 - kr^2)}(\Gamma^\rho_{\rho t} + \frac{\dot{a}}{a}), \quad \Gamma^r_{\phi\phi} = \Gamma^r_{\theta\theta}\sin^2\theta \]

\[ \Gamma^\theta_{rt} = \frac{1}{4}(\Gamma^\rho_{\rho t} + \frac{\dot{a}}{a}) , \quad \Gamma^\theta_{rr} = \frac{kr}{1 - kr^2}, \quad \Gamma^\theta_{\theta\theta} = -r(1 - kr^2), \quad \Gamma^\theta_{\phi\phi} = \Gamma^r_{\theta\theta}\sin^2\theta \]

\[ \Gamma^\phi_{\theta t} = \frac{1}{4}(\Gamma^\rho_{\rho t} + \frac{\dot{a}}{a}), \quad \Gamma^\phi_{\theta r} = \frac{1}{r}, \quad \Gamma^\phi_{\phi\theta} = -\sin\theta\cos\theta \]

\[ \Gamma^\phi_{\phi r} = \frac{1}{r}, \quad \Gamma^\phi_{\phi\phi} = \cot\theta. \]

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