Extremality of Disordered Phase of λ-Model on Cayley Trees

Farrukh Mukhamedov 1,2

1 Department of Mathematical Sciences, College of Science, United Arab Emirates University, Al Ain P.O. Box 15551, United Arab Emirates; far75m@yandex.ru or farrukh.m@uaeu.ac.ae
2 Department of Algebra and Analysis, Institute of Mathematics Named after V.I.Romanovski, 4, University Str., Tashkent 100125, Uzbekistan

Abstract: In this paper, we consider the λ-model for an arbitrary-order Cayley tree that has a disordered phase. Such a phase corresponds to a splitting Gibbs measure with free boundary conditions. In communication theory, such a measure appears naturally, and its extremality is related to the solvability of the non-reconstruction problem. In general, the disordered phase is not extreme; hence, it is natural to find a condition for their extremality. In the present paper, we present certain conditions for the extremality of the disordered phase of the λ-model.

Keywords: extremality; splitting Gibbs measure; Cayley tree

1. Introduction

In communication theory [1], if a process on a tree $T$ is considered, it is crucial to be informed of the information transmitted from the root of the tree (given there is one) to all the nodes of the tree. In this setting, each node inherits information from its parent node with some probability of error. The transmission process is assumed to have an identical distribution on all the edges and different edges of the tree are assumed to act independently [2,3]. Similar kinds of processes have been investigated in statistical physics [4,5]. The main problem that arises with this theory is related to whether the configuration obtained at level $n$ of $T$ typically contains significant information on the root variable. We point out that such a problem arose independently in biology, information theory and statistical physics. For models of statistical physics on trees, the problem is related to the extremality of the disordered Gibbs measure [6,7]. This problem is also related to the solvability of the reconstruction problem of Markov fields on trees (see [8]). The simplest model in statistical physics is the Ising model, for which the extremality of the disordered phase has been investigated [9–11]. It is known that the $q$-state Potts model [12] is a natural generalization of the Ising model to more than two components and encompasses a number of problems in statistical physics [13]. Therefore, in [14,15], the reconstruction problem is investigated for the Potts models over trees. The $q$-state Potts model has a rich structure to illustrate almost every conceivable nuance of the subject. Furthermore, the Potts models became one of the important models in statistical mechanics. It has been known [16] for a long time that, at sufficiently low temperatures, such a $q$-state Potts model on the Cayley tree has at least $q + 1$ translation-invariant Gibbs measures which are also tree-indexed Markov chains. Such translation-invariant tree-indexed measures are equivalently called translation-invariant splitting Gibbs measures (TISGMs). The $q$ measures with homogeneous boundary conditions are always extremal in the set of all Gibbs measures. However, the Gibbs measure associated with the free boundary condition measure is extremal in an intermediate temperature interval below the transition temperature and loses its extremality for even lower temperatures [17] (see [18] for a recent review of the Potts models).

On the other hand, it is natural to consider more general models than the Potts model. In this connection, a so-called λ-model (i.e., a model with nearest neighbors in which the interactions depend on the function $\lambda$) was proposed in [19,20]. This model includes the
Potts model as a special case if the $\lambda$-function is taken as $\lambda(x,y) = J\delta_{xy}$, where $\delta$ is the Kronecker symbol. If one takes $\lambda(x,y) = \cos(2\pi(x - y)/q)$, then we obtain the Clock model. This model, from the physics perspective, is equivalent to the Potts model in the case $q = 3$. Both models undergo a phase transition between a ferromagnetic phase and a paramagnetic phase belonging to the same universality class. In the present paper, we are going to investigate extremity and non-extremity of translation-invariant Gibbs measures of the $\lambda$-model. At this point, we stress that the indicated $\lambda$-model includes various possible interactions that cannot be described by the Potts models, including, in particular, the Widom–Rowlinson model or Potts-SOS model, with the interaction described by the following function [21–23]. Recently, in [24,25], the existence of the phase transition for the $\lambda$-model on the Cayley tree with an order of two has been established. However, the existence of the disordered phase and its extremality were not investigated. In this paper, we are going to fill that gap, i.e., the extremality of such a Gibbs measures will be studied. To prove the main result, the Kesten–Stigum condition [26] is employed for the non-extremality of a Gibbs measure. Furthermore, to establish the extremality, we use so called non-re-constructability in information-theoretic language. Our findings have a significant role in information sciences and percolation theory [8,14] since the model that we considered is much more general than all previously known ones.

2. Preliminaries

Let $\Gamma^k_+ = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$, with the root $x^0$ (with each vertex having exactly $k + 1$ edges, except for the root $x^0$, which has $k$ edges). Here, $V$ is the set of vertices and $L$ is the set of edges. The vertices $x$ and $y$ are called nearest neighbors and they are denoted by $l = \langle x, y \rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1 \rangle, \ldots, \langle x_{d-1}, y \rangle$ is called a path from point $x$ to point $y$. The distance $d(x,y), x, y \in V$, on the Cayley tree, is the length of the shortest path from $x$ to $y$.

$$W_n = \{ x \in V \mid d(x,x^0) = n \}, \quad V_n = \bigcup_{m=1}^{n} W_m, \quad L_n = \{ l = \langle x, y \rangle \in L \mid x, y \in V_n \}.$$  

The set of direct successors of $x$ is defined by

$$S(x) = \{ y \in W_{n+1} : d(x, y) = 1 \}, x \in W_n.$$  

Observe that any vertex $x \neq x^0$ has $k$ direct successors and $x^0$ has $k + 1$.

In this paper, we consider the models where the spin takes values in the set $\Phi = \{ 1, 2, \ldots, q \}$ and is assigned to the vertices of the tree. A configuration $\sigma$ on $V$ is then defined as a function $\sigma : V \rightarrow \sigma(x) \in \Phi$; the set of all configurations coincides with $\Omega = \Phi^V$. The Hamiltonian of the $\lambda$-model has the following form

$$H(\sigma) = \sum_{\langle x, y \rangle \in L} \lambda(\sigma(x),\sigma(y))$$  

where the sum is taken over all pairs of nearest-neighbor vertices $\langle x, y \rangle, \sigma \in \Omega$. From a physical point of view, the interactions between particles do not depend on their locations; therefore, from now on we will assume that $l$ is a symmetric function, i.e., $\lambda(u,v) = \lambda(v,u)$ for all $u, v \in \mathbb{R}$.

We note that the $\lambda$-model of this type can be considered as a generalization of the Potts model. The Potts model corresponds to the choice $\lambda(x,y) = -J\delta_{xy}$, where $x, y, J \in \mathbb{R}$.

3. Splitting Gibbs Measures

In this section, we recall the definition of splitting Gibbs measures.
Now, let us define a finite-dimensional distribution of probability measure \( \mu^{(n)} \) on \( \Omega_n := \Phi V_n \) as follows

\[
\mu^{(n)}(\sigma_n) = Z_n^{-1} \exp \{-\beta H_n(\sigma_n) + \sum_{\omega \in W_n} h(x(x), x)\}, \quad \sigma_n \in \Omega_n
\]  

(2)

where \( \beta = 1/T, T > 0 \) is the temperature, and

\[
Z_n^{-1} = \sum_{\sigma \in \Phi V_n} \exp \{-\beta H_n(\sigma_n) + \sum_{x \in W_n} h(x(x), x)\}
\]

is the normalizing factor. Moreover, in (2), \( \{h_x = (h_1, x, \ldots, h_q, x) \in \mathbb{R}^q, x \in V\} \) is the set of vectors and

\[
H_n(\sigma_n) = \sum_{(x, y) \in E_n} \lambda(\sigma(x), \sigma(y)).
\]

We say that a sequence of a probability distribution \( \{\mu^{(n)}\} \) is consistent if, for all \( n \geq 1 \) and \( \sigma_{n-1} \in \Omega_{n-1} \), one has

\[
\sum_{\omega_n \in \Phi V_n} \mu^{(n)}(\sigma_{n-1} \lor \omega_n) = \mu^{(n-1)}(\sigma_{n-1}).
\]  

(3)

Here, \( \sigma_{n-1} \lor \omega_n \) is the concatenation of the configurations. In this setting, there is a unique measure \( \mu \) on \( \Omega \), such that for all \( n \) and \( \sigma_n \in \Omega_n \),

\[
\mu(\{\sigma| V_n = \sigma_n\}) = \mu^{(n)}(\sigma_n).
\]

Such a measure is called a splitting Gibbs measure (SGM), corresponding to the Hamiltonian (1) and a vector-valued function \( h_x, x \in V \) (for more information, see [27]).

The next statement describes the condition on \( h_x \) ensuring that the sequence \( \{\mu^{(n)}\} \) is consistent.

**Theorem 1** ([19,24]). The measures \( \mu^{(n)}, n = 1, 2, \ldots \) satisfy the consistency condition if and only if for any \( x \in V \) the following equation holds:

\[
u_{k,x} = \prod_{y \in S(x)} \frac{\sum_{j=1}^{q-1} \exp \{\beta \lambda(j, x)\} u_{j,y} + \exp \{\beta \lambda(q, x)\}}{\sum_{j=1}^{q-1} \exp \{\beta \lambda(j, x)\} u_{j,y} + \exp \{\beta \lambda(q, x)\}}
\]  

(4)

where \( u_{k,x} = \exp \{h_{k,x} - h_{q,x}\}, k = 1, q - 1 \).

In what follows, we restrict ourself to the case \( \Phi = \{1, 2, 3\} \) (i.e., \( q = 3 \)), and we consider the following function:

\[
\lambda(i, j) = \begin{cases} \pi, & \text{if } |i - j| = 2, \\ \bar{\pi}, & \text{if } |i - j| = 1, \\ \bar{\tau}_1, & \text{if } i = j = 1, \\ \bar{\tau}_2, & \text{if } i = j = 2, \\ \bar{\tau}_3, & \text{if } i = j = 3, \\ \end{cases}
\]  

(5)

where \( \pi, \bar{\pi}, \bar{\tau}_1, \bar{\tau}_2, \bar{\tau}_3 \in \mathbb{R} \) for some given numbers.

**Remark 1.** We point out that the considered model is more general than the well known Potts model [12], since if \( \bar{\pi} = \bar{\pi} \) and \( \bar{\pi} \neq \bar{\tau}_1 = \bar{\tau}_2 = \bar{\tau}_3 \), then this model reduces to the mentioned one. Namely, in this setting, the function \( \lambda \) is defined by \( \lambda(i, j) = \bar{\pi} + (\bar{\tau}_1 - \bar{\pi})\delta_{ij} \). The case of \( \bar{\tau}_1 = \bar{\tau}_2 = \bar{\tau}_3 \) has been investigated in [24,25]. If \( \lambda(i, j) = |i - j| \), then the model reduces to the Solid-on-SOlid (SOS) model [28]. If \( \lambda(i, j) = |i - j| + \delta_{ij} \), then the model becomes the Potts-SOS model [23].
Now, we are going to find the disordered phase of the model which is related to the translation-invariant splitting Gibbs measure (for short, SGM). Here, through the use of the translation-invariant SGM, it is an SGM that corresponds to a solution \( u_x \) of (4), which satisfies \( u_x = u_y \) for all \( x, y \in V \). This means that \( u_x = u \), where \( u = (u_1, u_2) \), \( u_1, u_2 > 0 \). Due to Theorem 1, \( u_1 \) and \( u_2 \) must satisfy the following system equations:

\[
\begin{align*}
  u_1 &= \left( C_1 u_1 + B u_2 + A \right) \frac{k}{A u_1 + B u_2 + C_3}, \\
  u_2 &= \left( B u_1 + C_2 u_2 + B \right) \frac{k}{A u_1 + B u_2 + C_3},
\end{align*}
\]

where \( A = \exp{\{\beta a\}} \), \( B = \exp{\{\beta b\}} \), \( C_1 = \exp{\{\beta c_1\}} \), \( k = 1, 2, 3 \).

We recall that an SGM is called the disordered phase of the \( \lambda \)-model if \( u_1 = 1, u_2 = 1 \) is a solution of (6). This means that the corresponding SGM does not have external boundary conditions.

**Remark 2.** We notice that, for symmetric models (interactions), the disordered Gibbs measure corresponds to the zero solution of the recurrent equations (Equation (4)). However, if the model is not symmetric, then the corresponding recurrent equation has no zero solution. In this case, the disordered measure is defined as a translation-invariant Gibbs measure that corresponds to the free boundary condition. Therefore, in some studies (see, for example, [29]), the disordered measure is called a free measure.

Since \( u_1 = 1, u_2 = 1 \) is a solution of (6), which yields

\[
B + C_2 = A + C_3, \quad C_1 = C_3.
\]

**Remark 3.** It is worth noting that (7) is satisfied for the Potts model. However, if one considers the SOS model (see Remark 1) the condition (7) is not satisfied, i.e., the model does not possess the disordered phase. Now, let us provide a model that is different from the Potts ones and for which the disordered phase exists.

**Example 1.** Let \( \lambda(i, j) = |i - j| + J_1(\delta_{2j} + \delta_{2j}) \), where \( i, j \in \{1, 2, 3\} \) and \( J > 0, J_1 > 0 \). Since \( \lambda(1, 1) = \lambda(3, 3) \), then the first condition of (7) is reduced to

\[
e^{2J} + 1 = e^{(l+1)l} + e^{2l+1}.
\]

From this, we easily obtain

\[
e^{l+1} = \frac{1}{2} \left(e^{l} + \sqrt{5e^{2l} - 4}\right).
\]

Hence, if (8) is satisfied, then the considered model possesses the disordered phase. Clearly, the model is totally different from the Potts and SOS models.

Hence, in the sequel, we assume that (7) is satisfied. Therefore, one infers that the line \( u_1 = 1 \) in (6) is invariant, so the equation over this invariant line reduces to

\[
u_2 = \left( \frac{C_2 u_2 + 2B}{B u_2 + B + C_2} \right)^k.
\]

By denoting

\[
X = \frac{C_2 u_2}{2B} , \quad a = \frac{1}{8B^3} , \quad b = \frac{C_2 (B + C_2)}{2B^2},
\]

where \( A = \exp{\{\beta a\}} \), \( B = \exp{\{\beta b\}} \).
we rewrite (9) as follows:

$$aX = \left( \frac{1}{b} + X \right)^2. \tag{11}$$

One can see that $X > 0$, $k \geq 1$, $a > 0$, and $b > 0$. Then, through the use of [4] (Proposition 10.7), one finds the following:

**Lemma 1 ([24]).**

(i) If $b \leq \left(\frac{k+1}{k-1}\right)^2$, then a solution to (11) is unique.

(ii) If $b > \left(\frac{k+1}{k-1}\right)^2$, then there are $\zeta_1(b)$ and $\zeta_2(b)$, such that $0 < \zeta_2 < \zeta_1$, and if $\zeta_1 < a < \zeta_2$, then (11) has three solutions.

(iii) If $a = \zeta_1$ and $a = \zeta_2$, then (11) has two solutions. The quantity $\zeta_1$ and $\zeta_2$ are determined from the formula

$$\zeta_i(b) = \frac{1}{x_i} \left(\frac{1}{b} + \frac{1}{x_i}\right)^k, \quad i = 1, 2, \tag{12}$$

where $x_1$ and $x_2$ are solutions to the equation $z^2 + (3-b)z + b = 0$.

The condition (i) of Lemma 1 reduces to

$$\frac{C_2(B + C_2)}{2B^2} < \left(\frac{k+1}{k-1}\right)^2$$

which is equivalent to

$$\left(\frac{C_2}{B}\right)^2 + \frac{C_2}{B} - 2\left(\frac{k+1}{k-1}\right)^2 < 0 \tag{13}$$

by solving the last one, we obtain the following fact.

**Theorem 2.** Assume that, for the $\lambda$-model given by (5), the condition (7) is satisfied. Then, the following statements hold:

(i) If

$$\frac{C_2}{B} < \frac{1}{2} \left(\sqrt{1 + 8\left(\frac{k+1}{k-1}\right)^2} - 1\right), \tag{14}$$

then the disordered phase is unique;

(ii) If

$$\frac{C_2}{B} > \frac{1}{2} \left(\sqrt{1 + 8\left(\frac{k+1}{k-1}\right)^2} - 1\right), \tag{15}$$

then the phase transition occurs.

4. Three-Indexed Markov Chains of the Disordered Phase

A tree-indexed Markov chain is defined as follows. Suppose we are given a tree with vertices set $V$, and a probability measure $\nu$ and a transition matrix $P = (P_{ij})_{i,j \in \Phi}$ on the single-site space, which is the finite set $\Phi = \{1, 2, \ldots, q\}$ here. We can obtain a tree-indexed Markov chain $X : V \to \Phi$ by choosing $X(x^0)$ according to $\nu$ and choosing $Xv$, for each vertex $v \neq x^0$, using the transition probabilities given the value of its parent, independently of everything else (see Definition 12.2 in [6] for a detailed definition).
Now, let us consider the \( \lambda \)-model given by (5) with condition (7). Then, the model has the disordered phase, which corresponds to a solution \((1, 1)\) of (4). Thus, the transition stochastic matrix is given by

\[
P = \frac{1}{2B + C_2} \begin{pmatrix} C_1 & B & A \\ B & C_2 & B \\ A & B & C_1 \end{pmatrix}. \tag{16}
\]

It is easy to calculate that the matrix \( P \) has three eigenvalues: 1 and

\[
\alpha_1 = \frac{C_1 - A}{2B + C_2}, \quad \alpha_2 = \frac{C_2 - B}{2B + C_2}.
\]

Using (7), one can see that

\[
\alpha_1^2 - \alpha_2^2 = \frac{1}{(2B + C_2)^2} \left( (C_1 - A)^2 - (C_2 - B)^2 \right)
= \frac{1}{(2B + C_2)^2} (C_1 - A - C_2 + B)(C_1 - A + C_2 - B)
= \frac{4(B - A)(C_1 - B)}{(2B + C_2)^2}. \tag{17}
\]

4.1. Conditions of Non-Extremality

In this subsection, we are going to find the regions of the parameters, the disordered phase is not extreme in the set of all Gibbs measures (including the non-translation invariant ones).

It is known that a sufficient condition (Kesten–Stigum condition \([26]\)) for the non-extremality of a Gibbs measure \( \mu \) corresponding to the matrix \( P \) on the Cayley tree of order \( k \geq 1 \) is that \( k|\alpha_{\text{max}}|^2 > 1 \), where \( |\alpha_{\text{max}}| \) is the second largest (in absolute value) eigenvalue of \( P \) \([22,26]\). Furthermore, we are going to employ this condition for the non-extremity of the disordered phase.

Let us consider

\[
|\alpha_{\text{max}}| = \max\{|\alpha_1|, |\alpha_2|\}.
\]

Through the use of (17), we have

\[
|\alpha_{\text{max}}| = \begin{cases} 
|\alpha_1|, & \text{if } (B - A)(C_1 - B) \geq 0 \\
|\alpha_2|, & \text{if } (B - A)(C_1 - B) < 0,
\end{cases}
\]

Hence, by checking the non-extremality condition, we arrive at the following result.

**Theorem 3.** Assume that, for the \( \lambda \)-model given by (5), the condition (7) is satisfied. Suppose that one of the following conditions is held:

(i) if \((B - A)(C_1 - B) \geq 0\) and \( \frac{|C_1 - A|}{2B + C_2} > \frac{1}{\sqrt{k}} \)

(ii) if \((B - A)(C_1 - B) < 0\) and \( \frac{|C_2 - B|}{2B + C_2} > \frac{1}{\sqrt{k}} \),

then the disordered phase is not extreme.
4.2. Conditions for Extremality

From the previous subsection, we infer the non-extremality conditions for the disordered phase. However, in the present subsection, we are going to find sufficient conditions for its extremality (or non-reconstructability in information-theoretic language [8,14,15,30]).

If, from the tree $\Gamma^k_x$, one removes an arbitrary edge $(x^l, x^l) = l \in L$, then it is divided into two components $\Gamma^k_{x^l}$ and $\Gamma^k_{x^l}$, each of which is called a *semi-infinite Cayley tree* or *Cayley subtree*.

Let us first give some necessary definitions from [30]. We consider finite complete subtrees $T$ that are initial points of the Cayley tree $\Gamma^k_{x^l}$. The boundary $\partial T$ of a subtree $T$ consists of the neighbors which are on $\Gamma^k_{x^l} \setminus T$. We identify subgraphs of $T$ with their vertex sets and write $E$ for the edges within a subset $A$ and $\partial A$.

Following [30], there are two key ingredients which are the quantities $\kappa$ and $\gamma$. Both of them are properties of the collection of Gibbs measures $\{\mu^T_{\gamma}\}$, where the boundary condition $\gamma$ is fixed and $T$ ranges over all initial finite complete subtrees of $\Gamma^k_{x^l}$. For a given subtree $T$ of $\Gamma^k_{x^l}$ and a vertex $x \in T$, we write $\gamma_x$ for the (maximal) subtree of $T$ rooted at $x$. When $x$ is not the root of $T$, let $\mu^T_{\gamma_x}$ denote the (finite-volume) Gibbs measure in which the parent of $x$ has its spin fixed to $s$ and the configuration on the bottom boundary of $\gamma_x$ (i.e., on $\partial \gamma_x \setminus \{\text{parent of } x\}$) is specified by $\gamma$.

For two measures $\mu_1$ and $\mu_2$ on $\Omega$, $\|\mu_1 - \mu_2\|_x$ denotes the variation distance between the projections of $\mu_1$ and $\mu_2$ onto the spin at $x$, i.e.,

$$\|\mu_1 - \mu_2\|_x = \frac{1}{2} \sum_{i \in \{1, 2, 3\}} | \mu_1(\sigma(x) = i) - \mu_2(\sigma(x) = i) | .$$

Let $\eta^{x,a}$ be the configuration $\eta$ with the spin at $x$ set to $s$.

Following [30], the quantities $\kappa(\mu)$ and $\gamma(\mu)$ are defined by

$$\kappa(\mu) = \sup_{x \in \Gamma^k_x} \max_{s,s'} \| \mu^T_{\gamma_x} - \mu^T_{\gamma_x} \|_x,$$

$$\gamma(\mu) = \sup_{A \subset \Gamma^k_x} \max \| \mu_A^{\eta,s} - \mu_A^{\eta,s'} \|_x,$$

where the maximum is taken over all boundary conditions $\eta$, all sites $y \in \partial A$, all neighbors $x \in A$ of $y$ and all spins $s, s' \in \{1, 2, 3\}$.

It is known that a sufficient condition for the extremality of the translation-invariant Gibbs measure is $\kappa(\mu) \gamma(\mu) < 1$ [30] (Theorem 9.3).

In what follows, the measure $\mu$ is taken as the disordered phase; therefore, $\kappa(\mu)$ and $\gamma(\mu)$ are denoted by $\kappa$ and $\gamma$.

Note that $\kappa$ has the particularly simple form

$$\kappa(\mu) = \frac{1}{2} \max_{u \in \{1, 2, 3\}} \sum_{i \in \Omega} | P_{\gamma u} - P_{\gamma u} | .$$

Hence, from (16), one has

$$\kappa = \frac{1}{2(2B + C_2)} \max \left\{ |C_1 - B| + |C_2 - B| + |A - B|, 2 |C_1 - A| \right\}$$ (18)

We shall calculate $\gamma$.

$$\gamma = \max \left\{ \| \mu_A^{\eta,1} - \mu_A^{\eta,2} \|_x, \| \mu_A^{\eta,2} - \mu_A^{\eta,3} \|_x, \| \mu_A^{\eta,3} - \mu_A^{\eta,1} \|_x \right\} ,$$
where
\[
\| \mu_A^{\eta \epsilon_{ij}} - \mu_A^{\eta \epsilon_{ij}} \|_x = \frac{1}{2} \sum_{s \in \{1,2,3\}} |\mu_A^{\eta \epsilon_{i1}}(\sigma(x) = s) - \mu_A^{\eta \epsilon_{i2}}(\sigma(x) = s)|
\]
\[
= \frac{1}{2} (|P_{i1} - P_{i2}| + |P_{i2} - P_{i3}| + |P_{i3} - P_{i3}|)
\]
\[
= \frac{1}{2(2B + C_2)} (|C_1 - B| + |C_2 - B| + |A - B|)
\]

Hence, \( \gamma = \kappa \). Hence, the extremality condition reduces to \( k\kappa^2 < 1 \). By (18), we have the following result.

**Theorem 4.** Assume that, for the \( \lambda \)-model given by (5), the condition (7) is satisfied. Then, the following statements hold:

(i) If \( |C_1 - B| + |C_2 - B| + |A - B| > 2|C_1 - A| \) and
\[
\frac{|C_1 - B| + |C_2 - B| + |A - B|}{2B + C_2} < \frac{2}{\sqrt{k}'}
\]
then the disordered phase is extreme.

(ii) If \( |C_1 - B| + |C_2 - B| + |A - B| \leq 2|C_1 - A| \) and
\[
\frac{|C_1 - A|}{2B + C_2} < \frac{1}{\sqrt{k}'}
\]
then the disordered phase is extreme.

**Remark 4.** We point out that if \( A = B \), then by (7) we infer that \( C_1 = C_2 \), which by Remark 1 the model reduces to the Potts model. The conditions of the extremality of the disordered phase of this model has been investigated in [8,14,29,31]. Such a condition from Theorem 4 is given as follows
\[
\frac{|C_1 - A|}{2A + C_1} < \frac{1}{\sqrt{k}}.
\]

Due to Theorem 2, if
\[
\frac{C_2}{B} < \frac{1}{2} \left( \sqrt{1 + 8 \left( \frac{k + 1}{k - 1} \right)^2} - 1 \right)
\]
then the disordered phase is unique, and so it is extreme. Hence, we are interested in its extremality in the case of
\[
\frac{C_2}{B} > \frac{1}{2} \left( \sqrt{1 + 8 \left( \frac{k + 1}{k - 1} \right)^2} - 1 \right),
\]
where the phase transition exists. The last condition results in \( C_2 > B \) (if \( k \geq 2 \)).

Now, let us consider several other possibilities for the values of \( A, B, C_1, C_2 \).

**Case I.** Assume that \( B > A \); then, due to (7), one has \( C_1 - C_2 = B - A \), which implies \( C_1 > C_2 > B > A \). Then, \((B - A)(C_1 - B) > 0\), which, from Theorem 3, means that if
\[
\frac{C_1 - A}{2B + C_2} > \frac{1}{\sqrt{k}'}
\]
the disordered phase is not extreme.
On the other hand, we have
\[
|C_1 - B| + |C_2 - B| + |A - B| = C_1 - B + C_2 - B + B - A
\]
\[
= B - A + C_2 + C_2 + B - A
\]
\[
= 2(C_2 - A)
\]

which, together with
\[
C_2 - A < C_1 - A
\]
and (ii) Theorem 4, implies that the disordered phase is extreme if
\[
\frac{C_1 - A}{2B + C_2} < \frac{1}{\sqrt{k}}
\]

The last condition can be rewritten as follows by using (7)
\[
\frac{C_1 - A}{2B + C_2} = \frac{B - 2A + C_2}{2B + C_2}
\]
\[
= \frac{C_2/B - 2\tau + 1}{2 + C_2/B}
\]
\[
< \frac{1}{\sqrt{k}}
\]

where \( \tau = \frac{4}{T} \). Hence, we obtain
\[
\frac{C_2}{B} < \frac{(2\tau - 1)\sqrt{k} + 2}{\sqrt{k} - 1}
\]

Therefore, one has the following result.

**Theorem 5.** Assume that for the \( \lambda \)-model given by (5), the condition (7) is satisfied and \( C_2 > B > A \). Then, the disordered phase is extreme if and only if
\[
\frac{C_2}{B} < \frac{(2\tau - 1)\sqrt{k} + 2}{\sqrt{k} - 1},
\]
where \( \tau = \frac{4}{T} \).

**Case II.** Assume that \( A > B \); then, we have \( C_1 < C_2 \). However, this case has five subcases. We are going to consider all of them one by one.

**Subcase (a).** Let us suppose that \( C_1 < B < C_2 < A \). Then, \((B - A)(C_1 - B) > 0\), which from Theorem 3 means that if
\[
\frac{A - C_1}{2B + C_2} > \frac{1}{\sqrt{k}}
\]

the disordered phase is not extreme.

On the other hand, we have
\[
|C_1 - B| + |C_2 - B| + |A - B| = B - C_1 + C_2 - B + A - B
\]
\[
= C_2 - C_1 + A - B
\]
\[
= 2(A - B).
\]
Due to our assumption, we have $-B < -C_1$, which yields $A - B < A - C_1$; therefore, from (ii) Theorem 4, we infer that the disordered phase is extreme if
\[
\frac{A - C_1}{2B + C_2} < \frac{1}{\sqrt{k}}.
\]

Consequently, through (19), we obtain the following result:

**Theorem 6.** Assume that, for the $\lambda$-model given by (5), the condition (7) is satisfied and $C_1 < B < C_2 < A$. Then, the disordered phase is extreme if and only if
\[
\frac{A - C_1}{2B + C_2} < \frac{1}{\sqrt{k}}.
\]

**Subcase (b).** Let us suppose that $C_1 < B < A < C_2$. Then, using the same argument, we obtain the same result as Theorem 6.

**Subcase (c).** Let us suppose that $B < C_1 < C_2 < A$. Then, $(B - A)(C_1 - B) < 0$, which from Theorem 3 means that if
\[
\frac{C_2 - B}{2B + C_2} > \frac{1}{\sqrt{k}},
\]
then the disordered phase is not extreme, which implies
\[
\frac{C_2}{B} > \frac{\sqrt{k} + 2}{\sqrt{k} - 1}.
\]

It is easy to find that
\[
|C_1 - B| + |C_2 - B| + |A - B| = 2(C_2 - B).
\]

Then, by using (7), one has
\[
2(C_2 - B) - 2|C_1 - A| = 2(C_2 - B - A + C_1) = 2(C_2 - B - A + B - A + C_2) = 4(C_2 - A) < 0.
\]

This means that $|C_2 - B| < |C_1 - A|$ which, due to (ii) Theorem 4, implies that the disordered phase is extreme if
\[
\frac{A - C_1}{2B + C_2} < \frac{1}{\sqrt{k}}.
\]

This yields
\[
\frac{C_2}{B} > \frac{(2\tau - 1)\sqrt{k} - 2}{\sqrt{k} + 1},
\]
where $\tau = \frac{A}{B}$.

Now, comparing (21) and the last one, we infer the following result:

**Theorem 7.** Assume that for the $\lambda$-model given by (5), the condition (7) is satisfied and $B < C_1 < C_2 < A$. If
\[
\frac{(2\tau - 1)\sqrt{k} - 2}{\sqrt{k} + 1} < \frac{C_2}{B} < \frac{\sqrt{k} + 2}{\sqrt{k} - 1},
\]
the disordered phase is extreme. Here, as before, $\tau = \frac{A}{B}$.
**Subcase (d).** Let us suppose that \( B < C_1 < A < C_2 \). Then, \((B - A)(C_1 - B) < 0\), which from Theorem 3 means that if
\[
\frac{C_2 - B}{2B + C_2} > \frac{1}{\sqrt{k}}
\]
the disordered phase is not extreme (21).

We have
\[
|C_1 - B| + |C_2 - B| + |A - B| = 2(C_2 - B).
\]

Then, by means of (7), one finds
\[
|C_2 - B| > |C_1 - A|
\]
which, due to (i) Theorem 4, results in the disordered phase being extreme if
\[
\frac{C_2 - B}{2B + C_2} < \frac{1}{\sqrt{k}}.
\]

Hence, by (21), we obtain the following fact.

**Theorem 8.** Assume that for the \( \lambda \)-model given by (5), the condition (7) is satisfied and \( B < C_1 < A < C_2 \). Then, the disordered phase is extreme if and only if
\[
\frac{C_2 - B}{2B + C_2} < \frac{1}{\sqrt{k}}. \tag{23}
\]

**Subcase (e).** Let us suppose that \( B < A < C_1 < C_2 \). Then, using the same argument as subcase (d), we obtain the same result as Theorem 8.

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