ARITHMETIC HODGE STRUCTURE AND HIGHER
ABEL-JACOBI MAPS

MASANORI ASAKURA

Abstract. In this paper, we show some applications to algebraic cycles by using higher Abel-Jacobi maps which were defined in [the author: Motives and algebraic de Rham cohomology]. In particular, we prove that the Beilinson conjecture on algebraic cycles over number fields implies the Bloch conjecture on zero-cycles on surfaces. Moreover, we construct a zero-cycle on a product of curves whose Mumford invariant vanishes, but not higher Abel-Jacobi invariant.

1. Introduction

In [A1], we defined a certain Hodge-theoretic structure for an arbitrary variety $X$ over the complex number field $\mathbb{C}$ by using the theory of mixed Hodge module due to Morihiko Saito. We call it an arithmetic Hodge structure of $X$. We defined the higher Abel-Jacobi maps from Bloch’s higher Chow groups $\mathrm{CH}^r(X, m; \mathbb{Q}) = \mathrm{CH}^r(X, m) \otimes \mathbb{Q}$ of $X$ to the extension groups in the category of arithmetic Hodge structures. (In this paper we will simply write $\mathrm{CH}^r(X, m)$ instead of $\mathrm{CH}^r(X, m; \mathbb{Q})$).

The purpose of this paper is to show some applications of our higher Abel-Jacobi maps to algebraic cycles. In particular, we will give the correct proofs of the results announced in [A1].

We are interested in zero-cycles on surfaces, in particular, the kernel $T(X)$ of an Albanese map

$$\mathrm{CH}_0(X)_{\deg=0} \longrightarrow \mathrm{Alb}(X).$$

If the geometric genus $p_g$ is not zero, it was shown by D. Mumford that $T(X)$ is enormous ([M]). Conversely, S. Bloch conjecture:

Conjecture 1.1 ([B1] Lecture 1). Let $X$ be a nonsingular projective surface with $p_g = 0$. Then $T(X) = 0$.

If our higher Abel-Jacobi maps for zero-cycles on surfaces are injective, the Bloch conjecture 1.1 is true. More generally, the injectivity of higher Abel-Jacobi maps (Conjecture 3.5) implies the Bloch-Beilinson conjecture, that is, the existence of motivic filtrations on Chow groups (Proposition 3.6).

Our first main result is to reduce the Bloch conjecture 1.1 to the Beilinson conjecture on the algebraic cycles over number fields:

Theorem 1.2 (Theorem 5.2 cf. [A1] Theorem 4.10). The Beilinson conjecture 5.1 (1), (2) for codimension $r = 2$ implies the Bloch conjecture 1.1.

Another remarkable advantage of our higher Abel-Jacobi maps is that we can do “explicit calculations”. Moreover, it gives a stronger invariant than higher normal functions (cf. [AS]), although our higher Abel-Jacobi maps are defined along the
idea of them. The reason of it is that our arithmetic Hodge structure contains a
data of $\mathbb{Q}$-coefficient perverse sheaf. Let us explain this more precisely. In $\left[A\right]$, we defined the spaces of Mumford invariants $\Xi^{p,q}_{X}(r)$ and $\Lambda^{p,q}_{X}(r)$, and constructed the natural maps $\text{Ext}^{p}_{M(C)}(Q(0),\mathcal{H}^{r}(X)(r)) \to \Xi^{r-p-q+r+p}_{X}(p) \to \Lambda^{r-p-q+r+p}_{X}(p)$. When $X$ is a surface and $p=q=r=2$, it gives the classical Mumford invariant (see $\left[M\right], \left[V\right]$). So far, many results on algebraic cycles of codimension $r \geq 2$ have been obtained only by observing it precisely. There exist, however, algebraic cycles which cannot be captured by only the Mumford invariants. Even for such a cycle, our higher Abel-Jacobi maps work quite effectively. The following is our second main result, which gives one example to the above:

**Theorem 1.3** (Theorem 4.2, cf. $\left[A\right]$ Theorem 4.5). Let $C$ be a projective nonsingular curve over $\overline{\mathbb{Q}}$ with genus $g \geq 2$, such that rankNS($C \times C$) = 3. Let $O \in C(\overline{\mathbb{Q}})$ be a $\overline{\mathbb{Q}}$-valued point such that $K_{C} - (2g - 2) \cdot O$ is not $\mathbb{Q}$-linearly equivalent to 0. Let $P \in C(C) \setminus C(\overline{\mathbb{Q}})$ be any $C$-valued point which is not $\overline{\mathbb{Q}}$-valued one. Put $X := C_{C} \times C_{C}$ and $z := (P, P) - (P, O) - (O, P) + (O, O) \in T(X)$. Then we have $\xi^{2}_{X}(z) = 0$ in $\Xi^{0,2}_{X}(2)$, but $\rho^{2}_{X}(z) \neq 0$ in $\text{Ext}^{2}_{M(C)}(Q(0), \mathcal{H}^{2}(X)(2))$.

For the proof of this theorem we use the modified diagonal cycle and through the proof of it, we see that images of algebraic cycles under our higher Abel-Jacobi map can be calculated explicitly by reducing it to the calculations of the usual Abel-Jacobi map.

Although M. Green and P. Griffiths defined arithmetic Hodge structure independently to the author (the “arithmetic Hodge structure” was named by them which seems quite nice, so the author also uses it in this paper), there are some differences in definition between theirs and ours. The most essential difference is that their definitions do not consider the datum of the $\mathbb{Q}$-structure which is taken in ours. Therefore, the category of their arithmetic Hodge structures does not become an abelian category, but only an exact category. Nevertheless, extension groups can be defined also in their category, and showed that cycle maps have non-trivial images. However, the conjecture 3.5 does not hold for their category perhaps from the lack of considering $\mathbb{Q}$-structure in definition.

We will explain how this paper consists. In §2, we review Carlson’s description about extension groups in the category of graded polarizable mixed Hodge structures, and usual Abel-Jacobi maps. In §3, we review arithmetic Hodge structures, higher Abel-Jacobi maps and Mumford invariants of algebraic cycles, which are constructed in $\left[A\right]$. In §4, we will prove Theorem 1.3. In §5, we will prove Theorem 1.2.

**Acknowledgement**

The author would like to express his sincere gratitude to Professor Shuji Saito for many helpful suggestions and stimulating discussions.

He also thanks to Professor Shin-ichi Mochizuki for teaching him the proof of Lemma 4.1, and to Doctor Ken-ichiro Kimura for teaching him about Griffiths groups. Finally, he would like to thank to Professor Sampei Usui and Professor Takeshi Saito for fruitful discussions and unceasing encouragement.
The author is supported by JSPS Research Fellowships for Young Scientists.

**Notation and Conventions**

1. A **variety** means a quasi-projective algebraic variety over a field. We mainly work with algebraically closed fields of characteristic 0 (e.g. \( \mathbb{C}, \mathbb{Q} \)).
2. For a variety \( X \) over a field \( k \), we denote \( X(S) = \text{Hom}_k(S,X) \) the set of \( S \)-valued points of \( X \).
3. For a variety \( X \) over \( \mathbb{C} \), \( X^{an} \) denotes the associated analytic space: \( X^{an} = X(\mathbb{C}) \).
4. We denote the kernel of the cycle map \( \text{CH}^r(X) \to H^{2r}(X) \) by \( \text{CH}^r(X)_{\text{hom}} \) the subgroup of homologically trivial cycles. Here \( H^*(X) \) is a Weil cohomology (e.g. Betti cohomology, algebraic de Rham cohomology, étale cohomology, and so on). We also write \( \text{CH}_0(X)_{\text{hom}} = \text{CH}_0(X)_{\text{deg}=0} \).
5. In this paper, we fix an embedding \( \bar{\mathbb{Q}} \hookrightarrow \mathbb{C} \).

2. **The Carlson isomorphism on the extensions of mixed Hodge structures**

The category \( \text{MHS} \) of graded polarizable \( \mathbb{Q} \)-mixed Hodge structures is an abelian category, but not, semi-simple. The extension groups in \( \text{MHS} \) have the well-known explicit description due to J. Carlson ([Ca]).

**Theorem 2.1.** Let \( H = (H_Q,W_\bullet,F^\bullet) \) be a graded polarizable \( \mathbb{Q} \)-mixed Hodge structure. We write \( H_C = H_Q \otimes \mathbb{C} \). Then

\[
\begin{align*}
0 &\to W_0H \to \bar{H} \to Q(0) \to 0,
\end{align*}
\]

where \( \bar{H} = (\bar{H}_Q,W_\bullet,F^\bullet) \) is the mixed Hodge structure with \( \bar{H}_Q = W_0H_Q \otimes \mathbb{Q}(0) \), and the weight filtration \( W_\ell \bar{H}_Q = \bar{H}_Q, W_1\bar{H}_Q = W_1H_Q \) (\( \ell \leq -1 \)), and the Hodge filtration \( F^pH_C = F^pW_0H_C \) (\( p \geq 1 \)), \( F^0H_C = F^0W_0H_C + F^0H_Q \) (\( q \leq -1 \)) and

\[
F^0\bar{H}_C = F^0W_0H_C + \mathbb{C} \cdot (\xi,1).
\]

1. \( \text{Ext}^1_{\text{MHS}}(Q(0),H) = \text{Ext}^1_{\text{MHS}}(Q(0),W_0H) \simeq W_{-1}H_C/W_{-1}H_C \cap (F^0W_0H_C + W_0H_Q) \). Here an element \( \xi \in W_{-1}H_C \) corresponds to the following extension of mixed Hodge structures:

\[
0 \to W_0H \to \bar{H} \to Q(0) \to 0,
\]

where \( \bar{H} = (\bar{H}_Q,W_\bullet,F^\bullet) \) is the mixed Hodge structure with \( \bar{H}_Q = W_0H_Q \otimes \mathbb{Q}(0) \), and the weight filtration \( W_\ell \bar{H}_Q = \bar{H}_Q, W_1\bar{H}_Q = W_1H_Q \) (\( \ell \leq -1 \)), and the Hodge filtration \( F^pH_C = F^pW_0H_C \) (\( p \geq 1 \)), \( F^0H_C = F^0W_0H_C + F^0H_Q \) (\( q \leq -1 \)) and

\[
F^0\bar{H}_C = F^0W_0H_C + \mathbb{C} \cdot (\xi,1).
\]

2. \( \text{Ext}^p_{\text{MHS}}(Q(0),H) = 0 \) for \( p \geq 2 \). (This is a formal consequence of the fact that the functor \( \text{Ext}^1_{\text{MHS}}(Q(0),- \) is right exact).

**Remark 2.2.**

1. It is easy to see that \( \text{Ext}^1_{\text{MHS}}(H_1,H_2) \simeq \text{Ext}^1_{\text{MHS}}(Q(0),H_1 \otimes H_2) \).
2. The above description (=Theorem 2.1 (1), which I was learned from Morihiko Saito) is different from the one of the extension groups in the category \( \text{MHS} \) of mixed Hodge structures which are not necessarily graded polarizable.

Originally, J. Carlson calculated the extensions in \( \text{MHS} \) ([Ca]):

\[
\text{Ext}^1_{\text{MHS}}(Q(0),H) \simeq W_0H_C/F^0W_0H_C + W_0H_Q.
\]

Let \( X \) be a projective nonsingular variety \( X \) over \( \mathbb{C} \). By Theorem 2.1 (1), we have

\[
\text{Ext}^1_{\text{MHS}}(Q(0),H^{2r-1}(X,Q(r))) \simeq J^r(X).
\]
Here $J^r(X) = H^{2r-1}(X^{an}, \mathbb{C})/F^r + H^{2r-1}(X^{an}, \mathbb{Q}(r))$ is the $r$-th intermediate Jacobian of $X$ (modulo torsion). It is isomorphic to the Picard variety $\text{Pic}^0(X)(\mathbb{C})$ if $r = 1$, and the Albanese variety $\text{Alb}(X)(\mathbb{C})$ if $r = \dim X$.

Let us recall the Abel-Jacobi map

$$\rho : \text{CH}^r(X)_{\text{hom}} \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2r-1}(X, \mathbb{Q}(r))).$$

Due to the formalism of mixed Hodge modules ([SaM1], [SaM2]), there is the cycle map

$$\text{CH}^r(X) \to \text{Ext}^{2r}_{\text{MHM}(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(r)).$$

It induces the following commutative diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
\text{CH}^r(X)_{\text{hom}} & \xrightarrow{\rho} & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2r-1}(X, \mathbb{Q}(r))) \\
\downarrow & & \downarrow \\
\text{CH}^r(X) & \to & \text{Ext}^{2r}_{\text{MHM}(X)}(\mathbb{Q}_X(0), \mathbb{Q}_X(r)) \\
\downarrow & & \downarrow \\
\text{CH}^r(X)/\text{CH}^r(X)_{\text{hom}} & \to & H^{2r}(X, \mathbb{Q}) \cap H^{r,r} \\
\downarrow & & \downarrow \\
0 & & 0.
\end{array}
$$

The top horizontal arrow $\rho$ is the Abel-Jacobi map.

On the other hand, there is the classical definition of Abel-Jacobi maps due to Griffiths and Weil. Let $Z \in \text{CH}^r(X)_{\text{hom}}$ be an algebraic cycle. There is a topological $(2\dim X - 2r + 1)$-cycle $\Gamma$ whose boundary is $Z$: $\partial \Gamma = Z$. Then the classical Abel-Jacobi class is defined as follows:

$$\rho(Z) = \sum_{j=1}^g \left( \int_{\Gamma} \omega_j \right) \omega_j^* \in J^r(X),$$

where $\omega_1, \ldots, \omega_g \in F^{n-r+1}H^{2n-2r+1}(X, \mathbb{C})$ is a basis, and $\omega_1^*, \ldots, \omega_g^* \in H^{2r-1}(X, \mathbb{C})/F^r$ denotes the Serre dual class of those: $\langle \omega_i, \omega_j^* \rangle = \delta_{ij}$.

The following is well-known (cf. [EZ]):

**Proposition 2.3.** The classical Abel-Jacobi map (2.4) coincides with the previous one (2.2) under the Carlson isomorphism (2.1). □

Let $k$ be an algebraically closed subfield of $\mathbb{C}$. Put $\mathcal{M}_k = \text{MHM}(\text{Spec} k)$ which consists of objects $H = (H_Q, H_k, F^*, W_Q, W_k, i)$ where

- $H_Q$ is a finite $\mathbb{Q}$-vector space,
- $H_k$ is a finite $k$-vector space,
- $F^*$ is a finite decreasing filtration on $H_k$ (called the Hodge filtration),
- $W_Q$ is a finite increasing filtration on $H_Q$ (called the weight filtration),
- $W_k$ is a finite increasing filtration on $H_k$ (called the weight filtration),
- $i : H_k \to H_C := H_Q \otimes \mathbb{C}$ is a $k$-linear map (called the comparison map) such that $i_C : H_k \otimes_k \mathbb{C} \to H_C$ is bijective,

which satisfy

1. $W_Q$ and $W_k$ is compatible under the comparison map $i$,
2. $(H_Q, W_Q, i(F^*))$ is a mixed Hodge structure.
(3) There is a polarization form on each graded component $\text{Gr}_W H = (\text{Gr}_W^W H, \text{Gr}_W^W H, F^\bullet, i)$ defined over $k$.

We have an analogous description of the extension groups in $\mathbb{M}_k$ (the proof is similar to the one of Theorem 2.1).

**Proposition 2.4.** (1) $\text{Ext}^1_{\mathbb{M}_k} (\mathbb{Q}(0), H) = \text{Ext}^1_{\mathbb{M}_k} (\mathbb{Q}(0), W_0 H)$

$$= W_{-1} H_C / W_{-1} H_C \cap (i(F^0 W_0 H_k) + W_0 H_Q).$$

(2) $\text{Ext}_p^1 (\mathbb{Q}(0), H) = 0$ if $p \geq 2$.

(3) $\text{Ext}^p_{\mathbb{M}_k} (H_1, H_2) = \text{Ext}^p_{\mathbb{M}_k} (\mathbb{Q}(0), H_1^* \otimes H_2)$.

3. Arithmetic Hodge structure: review of [A1]

We review the notion of arithmetic Hodge structures, higher Abel-Jacobi maps and the Mumford invariants which are introduced in [A1].

3.1. Arithmetic Hodge structure. Let $X$ be a quasi-projective nonsingular variety over $\mathbb{C}$. Then $X$ is defined by finitely many equations which possess finitely many coefficients. By considering the coefficients as parameters of a space $S$, we can obtain a model $f: X \rightarrow S$ and the Cartesian diagram:

$$\begin{array}{ccc}
X_S & \xrightarrow{J} & X \\
\downarrow & & \downarrow \\
S & \xrightarrow{a} & \text{Spec} \mathbb{C},
\end{array}$$

where $S$ is a nonsingular variety over $\mathbb{Q}$, and the map $a$ factors through the generic point $\text{Spec} \mathbb{Q}(S) \rightarrow S$. We define the abelian category

$$\mathbb{M}(X) = \lim_{\xrightarrow{\leftarrow} X_S} \text{MHM}(X_S)$$

Here $\text{MHM}(X_S)$ is the category of mixed Hodge modules on the variety $X_S$ over $\mathbb{Q}$. In the above limit, $X_S$ runs over all models (3.1), and for a morphism $j: X_{S'} \rightarrow X_S$ of the models, we take the pull-back $j^*: \text{MHM}(X_{S'}) \rightarrow \text{MHM}(X_{S'})$. We call it the category of *arithmetic Hodge modules*. In particular, we call $\mathbb{M}(\mathbb{C}) := \mathbb{M}($Spec$\mathbb{C})$ the category of *arithmetic Hodge structures*.

The *realization functor* $r_X: \mathbb{M}(X) \rightarrow \text{Perv}(X)$ are constructed as follows. In the diagram (3.1), the morphism $a: \text{Spec} \mathbb{C} \rightarrow S$ induces the $\mathbb{C}$-morphism $\mathcal{O}_S \otimes \mathbb{C} \rightarrow \mathbb{C}$, which defines a closed point $s \in S^{an} = S \otimes_{\mathbb{Q}} \mathbb{C}$. Let $i_s: \text{Spec} \mathbb{C} \hookrightarrow S^{an}$ be the corresponding inclusion. Then a factors as Spec $\mathbb{C} \xrightarrow{i} S^{an} \rightarrow S$, and similarly $X \xrightarrow{i_X} X_{S}^{an} \rightarrow X_S$. We define $r_{X_S}: \text{MHM}(X_S) \rightarrow \text{Perv}(X)$ the functor which maps the perverse sheaf $K^*_S$ of a mixed Hodge module to $^pH^{H^*_i} i_X(K^*_S)$. These functors are compatible for any transition $X_{S'} \rightarrow X_S$. Therefore, passing to the limit, we get the realization functor $r_X = \lim_{\xrightarrow{\leftarrow} X_S} r_{X_S}$.

Although the model (3.1) is not uniquely determined, any two models $X_{S_1}$ and $X_{S_2}$ can be imbedded into the following diagram:
Therefore the Tate Hodge module $Q_{X_S}(r)$ defines a well-defined object $Q_X(r)$ in $\mathcal{M}(X)$, which we call the arithmetic Tate Hodge module.

The following is straightforward due to the formalism of mixed Hodge modules.

**Proposition 3.1.** (1) There are the standard operations on the derived category of bounded complex of arithmetic Hodge modules:

$$f_*, f^!, f^*, f^!; \mathbb{D}, \otimes, \text{Hom}.$$ 

Those functors satisfy the adjointness, projection formulas, and are compatible with the ones on perverse sheaves under the realization functor $r_X: \mathcal{M}(X) \to \text{Perv}(X)$. 

(2) Let $f: X \to Y$ be a proper morphism of nonsingular varieties, and $M$ a pure object of $\mathcal{M}(X)$. Then the decomposition theorem of type Beilinson-Bernstein-Deligne-Gabber

$$f_* M \simeq \oplus H^k f_* M[-k]$$ holds.

We denote the arithmetic Hodge structure $H^k(X, Q(r)) = H^k(X)(r) := H^k f_* Q_X(r)$ for the structure morphism $f: X \to \text{Spec} \mathbb{C}$. As an immediate corollary of Proposition 3.1, we have the Leray spectral sequence

$$E_2^{pq} = \text{Ext}^p_{\mathcal{M}(Y)}(Q_Y(0), H^q f_* Q_X(r)) \implies \text{Ext}^{p+q}_{\mathcal{M}(X)}(Q_X(0), Q_X(r)),$$

for a morphism $f: X \to Y$ of nonsingular varieties. If $f$ is proper, it degenerates at $E_2$-terms.

### 3.2. The space of Mumford invariants.

Recall the spaces of Mumford invariants $\Xi_{X}^{p,q}(r)$ and $\Lambda_{X}^{p,q}(r)$ for a projective nonsingular variety $X$ over $\mathbb{C}$ (see for details, [1] §3).

Let $H^q_{\text{dR}}(X/C) := H^q(X, \Omega^\bullet_{X/C})$ be the algebraic de Rham cohomology of $X$, and $F^r H^q_{\text{dR}}(X/C) := H^q(X, \Omega^{r+1}_{X/C})$ the Hodge filtration. There is the Gauss-Manin connection

$$\nabla: H^q_{\text{dR}}(X/C) \to H^q_{\text{dR}}(X/C) \otimes_C \Omega^1_C/\mathbb{Q}.$$

It satisfies the Griffiths transversality $(\nabla F^r \subset F^{r-1} \otimes \Omega^1_C/\mathbb{Q})$. Then we define the space of the Mumford invariants $\Xi^{p,q}_X(r)$ (resp. $\Lambda^{p,q}_X(r)$) as the cohomology at the middle term of the following complex:

$$F^{p+1} H^q_{\text{dR}}(X/C) \otimes \Omega^{r-1}_{C/\mathbb{Q}} \nabla H^q_{\text{dR}}(X/C) \otimes \Omega^r_{C/\mathbb{Q}} \nabla H^{p+1-q}_{\text{dR}}(X/C) \otimes \Omega^{r+1}_{C/\mathbb{Q}},$$

(resp. $H^{q-1}(\Omega^p_{X/C}) \otimes \Omega^{r-1}_{C/\mathbb{Q}} \nabla H^{q}(\Omega^p_{X/C}) \otimes \Omega^r_{C/\mathbb{Q}} \nabla H^{q+1}(\Omega^p_{X/C}) \otimes \Omega^{r+1}_{C/\mathbb{Q}}$.}
Proposition 3.2 ([A1] Proposition 3.4, 3.6). There are the following natural maps:
\[
\text{Ext}^p_{\text{MHM}(S)}(Q_S(0), H^r(X)(r)) \to \Xi^{-p,q-r+p}_X(p) \to \Lambda^{-p,q-r+p}_X(p).
\]

Proof. (sketch) Note that \(\text{Ext}^p_{\text{MHM}(S)}(Q_S(0), H^r(X)(r)) = \lim_{\to} \text{Ext}^p_{\text{MHM}(S)}(Q_S(0), H^q f_* Q_{X_S}(r)).\)

There is the forgetful functor \(\text{MHM}(S) \to \text{MF}_{rh}(S),\) which is exact and faithful (see [A1] §2). Therefore we have the well-defined map between the Yoneda extension groups
\[
\text{Ext}^p_{\text{MHM}(S)}(Q_S(0), H^q f_* Q_{X_S}(r)) \to \text{Ext}^p_{\text{MF}_{rh}(S)}(O_S(0), R^q f_* \Omega^*_{X_S/S}).
\]

Using the Koszul resolution of \(O_S,\) we can show that the right hand side is isomorphic to the cohomology at the middle term of the following complex (cf. [A1], Lemma 3.3):
\[
F^{r-p+1} H_S^r \otimes \Omega^{p-1}_{S/Q} \to F^{r-p} H_S^r \otimes \Omega^{p}_{S/Q} \to F^{r-p-1} H_S^r \otimes \Omega^{p+1}_{S/Q},
\]
where we put \(F^k H_S^r := R^q f_* \Omega^{k}_{X_S/S}.\) Thus we have the natural map
\[
\text{Ext}^p_{\text{MHM}(S)}(Q_S(0), H^r(X)(r)) \to \lim_{\to} \text{Ext}^p_{\text{MF}_{rh}(S)}(O_S(0), R^q f_* \Omega^*_{X_S/S}) \simeq \Xi^{-p,q-r+p}_X(p).
\]

The other natural map to \(\Lambda^{-p,q-r+p}_X(p)\) can be constructed similarly. \(\square\)

3.3. Filtrations on Chow groups and higher Abel-Jacobi maps. Let \(X\) be a nonsingular projective variety over \(\mathbb{C}.\) We can construct the cycle map from Bloch’s higher Chow groups ([B2]) to the extension groups in the category of arithmetic Hodge modules ([A1] §4):
\[
\lim_{\to} \text{CH}^r(X_S, m) \to \lim_{\to} \text{Ext}^{2r-m}_{\text{MHM}(X_S)}(Q_{X_S}(0), Q_{X_S}(r)) = \text{Ext}^{2r-m}_{\text{MHM}(X)}(Q_X(0), Q_X(r)). \tag{3.3}
\]

We denote the above map ([B3]) by \(c_X.\)

By the Leray spectral sequence ([B2]), there is the filtration \(F^\bullet\) on the right hand side in (3.3). We define the filtration on Chow groups as follows:
\[
F^r \text{CH}^r(X, m) := c^{-1}_X(F^r \text{Ext}^{2r-m}_{\text{MHM}(X)}(Q_X(0), Q_X(r))).
\]

Since the Leray spectral sequence ([B2]) degenerates at \(E_2\)-terms, the cycle map \(c_X\) induces the following map.
\[
\rho_X^r : \text{Gr}^p_F \text{CH}^r(X, m) \to \text{Ext}^p_{\text{MHM}(S)}(Q_S(0), H^{2r-m-r}(X)(r)). \tag{3.4}
\]

We call the above the \(\nu\)-th higher Abel-Jacobi map. These are injective by definition of \(F^\bullet \text{CH}^r(X, m).\)

Proposition 3.3. (1) \(F^1 \text{CH}^r(X) = \text{CH}^r(X)_{\text{hom}}.\) \(F^1 \text{CH}^r(X, m) = \text{CH}^r(X, m)\) for \(m \geq 1.\)

(2) \(F^2 \text{CH}^r(X, m)\) is contained in the kernel of the cycle map to the Deligne cohomology group. In particular, we have \(F^2 \text{CH}^1(X) = 0.\)

(3) \(F^2 \text{CH}_0(X) = T(X).\)

(4) \(F^{r+1} \text{CH}^r(X, m) = F^{r+2} \text{CH}^r(X, m) = \cdots,\) for all \(r.\)

(5) \(F^r \text{CH}^r(X, m) \cdot F^n \text{CH}^r(X, n) \subset F^{r+n} \text{CH}^{r+s}(X, m+n).\)
(6) \( F^* \) is respected by any algebraic correspondence. Each correspondence \( \Gamma_* \) on \( Gr^p CH^r(X, m) \) depends only on the Künneth \((2 \dim X - 2r + m + \nu, \tau)\)-component of the Betti cohomology class \([\Gamma] \in H^*(X \times Y)\).

Proof. (1). The former follows from that \( r_C : M(C) \rightarrow \{ \mathbb{Q} \text{-vector space} \} \) is faithful. The latter follows from the fact that the cycle class map from \( CH^r(X, m) \) to ordinary Betti cohomology is zero by a standard weight argument.

(2). It follows from that the realization functor \( r_C \) factors through the category of mixed Hodge structures.

(3). Let \( \pi = \pi_{2n-1} \in CH^n(X \times X) \) be the algebraic cycle in [Mur2] 4.1. Theorem 2, which has the following properties: (i) \( \pi_2^* = \pi_* \), (ii) the cycle class of \( \pi \) is the Künneth \((1, 2n-1)\) component of the diagonal cycle \( \Delta_X \), (iii) \( \ker(\pi_* : CH_0(X)_{\deg=0} \rightarrow CH_0(X)) = T(X) \). Consider the commutative diagram

\[
\begin{array}{cccc}
F^1CH_0(X) & \xrightarrow{c'} & \text{Ext}^1_{M(C)}(\mathbb{Q}(0), H^{2n-1}(X)(n)) \\
\pi_* & & \downarrow \pi_* \\
F^1CH_0(X) & \xrightarrow{} & \text{Ext}^1_{M(C)}(\mathbb{Q}(0), H^{2n-1}(X)(n)).
\end{array}
\]

Since the property (ii), the right vertical arrow is bijective. Hence \( F^2CH_0(X) = \ker c' \) contains the kernel of the left \( \pi_* \), that is, \( T(X) \) by the property (iii). On the other hand, the Hodge realization functor induces the natural map \( \text{Ext}^1_{M(C)}(\mathbb{Q}(0), H^{2n-1}(X)(n)) \rightarrow \text{Ext}^1_{MHS}(\mathbb{Q}(0), H^{2n-1}(X)(n)) \cong \text{Alb}(X)(C) \) which induces the usual Albanese map, so \( F^2CH_0(X) \subset T(X) \). Therefore \( F^2CH_0(X) = T(X) \).

(4). Let \( \nu \geq r + 1 \) and \( n = \dim X \). Then there is the following commutative diagram:

\[
\begin{array}{cccc}
Gr^p CH^r(X, m) & \xrightarrow{L} & \text{Ext}^r_{M(C)}(\mathbb{Q}(0), H^{2r-m-\nu}(X)(r)) \\
\downarrow & & \downarrow \rho^r
\end{array}
\]

where \( L \) is the Lefschetz operator, that is, \( x \mapsto x.H \) for a hyperplane section \( H \subset X \). By the hard Lefschetz theorem, the right vertical arrow is bijective. On the other hand, the left vertical arrow is 0 because \( n - r + m + \nu \geq n + m + 1 \). Therefore \( Gr^p CH^r(X) = 0 \).

(5). Clear by definition of \( F^r CH^r(X, m) \).

(6). It follows from the injectivity of the higher Abel-Jacobi maps \([3, 4]\).

Example 3.4. Let \( r = \dim X = n, \nu = 2 \) and \( m = 0 \). Then the higher Abel-Jacobi map gives the following map:

\[
\rho^\nu_X : F^2CH_0(X) = T(X) \rightarrow \text{Ext}^2_{M(C)}(\mathbb{Q}(0), H^{2n-2}(X)(n)).
\]

We call the above \([3, 3]\) the second Albanese map.

We conjecture that the filtration \( F^* CH^r(X, m) \) terminates:

Conjecture 3.5. \( F^N CH^r(X, m) = 0 \) for some \( N \gg 0 \). By Proposition \([3, 3]\), it is equivalent to \( F^{r+1} CH^r(X, m) = 0 \). In other words, the higher Abel-Jacobi map

\[
\rho^r_X : F^r CH^r(X, m) \rightarrow \text{Ext}^r_{M(C)}(\mathbb{Q}(0), H^{r^c+m}(X)(r))
\]

is injective for each \( X, r \) and \( m \).
The above conjecture is true for $r = 1$. However, when $r \geq 2$, it is a very difficult problem.

**Proposition 3.6.** If the conjecture 3.5 is true, $F^rCH^r(X, m)$ gives the conjectural filtration on Chow groups (cf. [11], [Mur1], [SaS]). In particular, it implies the Bloch conjecture 1.1.

We remark that in order to the Bloch conjecture 1.1, we need only the injectivity of the second Albanese maps (3.5) for any surfaces. We will show that Conjecture 3.5 for $r = 2, m = 0$ follows from the Beilinson conjecture 5.1 for $r = 2$ in §5.

**Definition 3.7.** Composing the higher Abel-Jacobi map (3.4) and the maps in Proposition 3.2, we have the following maps:

$$
\xi_X^r : Gr^rCH^r(X, m) \rightarrow \Xi^r_{X}(\nu, r, -m),
$$

$$
\delta_X^r : Gr^rCH^r(X, m) \rightarrow \Lambda^r_{X}(\nu, r, -m).
$$

We call $\xi_X^r(z)$ (resp. $\delta_X^r(z)$) for a cycle $z \in Gr^rCH^r(X, m)$, the Mumford $\xi$-invariant (resp. $\delta$-invariant) of $z$.

For more about the Mumford invariants and its applications, we are preparing another paper [A2].

**4. A calculation of a higher Abel-Jacobi invariant**

The Mumford invariant is not enough to capture all algebraic cycles. We hope that the invariant obtained from our higher Abel-Jacobi maps will be stronger and capture all cycles.

In this section, we give an example of a 0-cycle on a surface whose image by the second Albanese map does not vanish, but so does the Mumford $\xi$-invariant of it.

**4.1.** We use the following curve constructed by Shin-ichi Mochizuki:

**Lemma 4.1** (S.Mochizuki). There is a projective nonsingular curve $C$ over $\bar{\mathbb{Q}}$ of genus $g \geq 2$, such that the rank of Neron Severi group of $C \times C$ is 3, or equivalently $\text{End}(J(C)) \otimes \mathbb{Q} = \mathbb{Q}$.

We will give the proof of above lemma at the end of this section.

**Theorem 4.2.** Let $C$ be a curve as in Lemma 4.1. Let $O \in C(\bar{\mathbb{Q}})$ be a $\bar{\mathbb{Q}}$-point of $C$ such that the divisor $K_C - (2g - 2) \cdot O$ is not $\mathbb{Q}$-linearly equivalent to 0, and $P \in C(\mathbb{C}) \setminus C(\mathbb{Q})$ be any $\mathbb{C}$-valued point which is not contained in the set of $\mathbb{Q}$-points. Put $C_{\mathbb{C}} := C \otimes \mathbb{C}$, $X := C_{\mathbb{C}} \times C_{\mathbb{C}}$ and $z := (P, P) - (O, O) + (O, O) \in T(X)$. Then we have

$$
\xi_X^2(z) = 0 \quad \text{in} \quad \Xi^0_{X}(2),
$$

but

$$
\rho_X^2(z) \neq 0 \quad \text{in} \quad \text{Ext}^2(\mathbb{C}(0), H^2(X)(2)).
$$

**Remark 4.3.** For any curve $C$ over $\bar{\mathbb{Q}}$ of genus $\geq 2$, there are always $\mathbb{Q}$-valued points $P$ and $O$ of $C$ such that the divisor $P - O$ is not $\mathbb{Q}$-linearly equivalent to 0. In fact, by the Mumford-Manin conjecture (=Raynaud’s theorem), the set $C(\mathbb{Q}) \cap J(C)(\mathbb{Q})_{\text{tor}}$ is finite, which means that the divisor $P - O$ is not torsion except for finitely many points $P$. 
4.2. Proof of Theorem 4.2. First we prove the vanishing $\xi^2_S(z) = 0$. Let $z_S \in \text{CH}^2(S \times C \times C)$ be any model of $z$, that is, there is a morphism $r : \text{Spec} C \to S$ such that $(r \times 1 \times 1)^*(z_S) = z \in \text{CH}^2(C_C \times C_C)$. By shrinking $S$, we may assume that $z_S|_{(t) \times C \times C}$ is contained in the kernel of the Albanese map for each $t \in S_C$. Then, by the following commutative diagram, $\xi^2_S(z)$ is contained in the image of the map $a_S$.

\[
\begin{array}{ccc}
T(X) & \xrightarrow{\xi^2_S} & H^2_{\text{dR}}(X/C) \otimes \Omega^2_{C/Q}/\nabla(F^1H^2_{\text{dR}}(X/C) \otimes \Omega^1_{C/Q}) \\
\uparrow & & \uparrow a_S \\
F^2_S \text{CH}(S \times C \times C) & \longrightarrow & R^2p_*\Omega^*_{S \times C \times C/S} \otimes \Omega^2_{S/Q}/\nabla(F^1R^2p_*\Omega^*_{S \times C \times C/S} \otimes \Omega^1_{S/Q}),
\end{array}
\]

where $p_S : S \times C \times C \to C \times C$ denotes the projection. We can choose the variety $S$ to be 1-dimensional (see the below). Therefore the image of $a_S$ is zero. Thus we have $\xi^2_S(z) = 0$.

Next we show the non-vanishing of the second Albanese class $\rho^2_S(z)$. By the definition of $\mathcal{M}(C)$, the extension group can be written as an inductive limit of the one in $\text{MHM}(S)$:

\[
\text{Ext}^2_{\mathcal{M}(C)}(Q(0), H^2(X)(2)) = \lim_{\rightarrow} \text{Ext}^2_{\text{MHM}(S)}(Q_S(0), R^2p_*Q_{S \times C \times C}(2)).
\]

Therefore $\rho^2_S(z)$ does not vanish if and only if so does not in $\text{Ext}^2_{\text{MHM}(S)}(Q_S(0), R^2p_*Q(2))$ for any sufficiently dominant $S \to C$. The latter condition is equivalent to the non-vanishing in $\text{Ext}^4_{\text{MHM}(S \times C \times C)}(Q(0), Q(2))$ by the decomposition theorem of mixed Hodge modules.

Let $f_P : \text{Spec} C \to C$ be the associated morphism of the point $P$. Note that $f_P$ factors through a generic point of $C$, because $P$ is not a $Q$-valued point. We will take a good lifting $z_0 \in \text{CH}^2(C_C \times C_C)$ such that $(f_P \times 1 \times 1)^*(z_0) = z$, and prove that for any dominant $j_S : S \to C$, the cycle class of $z_S = (j_S \times 1 \times 1)^*(z_0)$ does not vanish in $\text{Ext}^4_{\text{MHM}(S \times C \times C)}(Q(0), Q(2))$.

Let $z_0$ be the modified diagonal cycle $\Delta_O \in \text{CH}^2(C \times C \times C)$. Recall the definition of it ([GS]). Put

- $\Delta_{\text{xxx}} = \{(x, x, x) \in C \times C \times C \mid x \in C\}$,
- $\Delta_{O\text{xx}} = \{(O, x, x) \in C \times C \times C \mid x \in C\}$,
- $\Delta_{xOx} = \{(x, O, x) \in C \times C \times C \mid x \in C\}$,

and the subvarieties of $C \times C \times C$ of codimension 2. Then the modified diagonal cycle $\Delta_O$ is defined as:

\[
\Delta_{\text{xxx}} - \Delta_{O\text{xx}} - \Delta_{xOx} - \Delta_{\text{xxO}} + \Delta_{O\text{xx}} + \Delta_{OxO} + \Delta_{xOO}.
\]

It is easy to see that $z_0(= \Delta_O)$ is homologically trivial, and therefore so is $z_S$. Therefore it defines an Abel-Jacobi class $\rho(z_S) \in \text{Ext}^1_{\text{MHS}}(Q(0), H^1(S_C) \otimes H^1(C_C)^{\otimes 2}(2))$. In order to prove the non-vanishing of the class of $z_S$ in $\text{Ext}^4_{\text{MHM}(S \times C \times C)}(Q(0), Q(2))$, it suffices to show the non-vanishing of the class $\rho(z_S)$. To do this, we may assume $\dim S = 1$ by taking suitable hyperplane sections. Thus we have reduced the proof of Theorem 4.2 to the following:
Lemma 4.4. Let $C_C$, $O$ and $z_0$ as above. Then for any nonsingular curve $S$ over $C$ with a dominant morphism $j_S : S \to C_C$, the image of the Abel-Jacobi class $\rho(z_0)$ under the following map

$$(j_S \times 1 \times 1)^* : \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C_C)^{\otimes 3}(2)) \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(S) \otimes H^1(C_C)^{\otimes 2}(2))$$

does not vanish.

**Proof.** We write $C_C$ by $C$ simply. We prove the assertion in the following step:

(4.4.1) The Abel-Jacobi class $\rho(z_0) \in \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C)^{\otimes 3}(2))$ is not zero.

(4.4.2) The image of the Abel-Jacobi class $\rho(z_0)$ under the following map

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C)^{\otimes 3}(2)) \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C) \otimes \text{Sym}^2(H^1(C))(2))$$

does not vanish. Here $\text{Sym}^2(H) = H^{\otimes 2}/\{a \otimes b - b \otimes a\}$ denotes the symmetric product.

(4.4.3) The natural map

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C) \otimes \text{Sym}^2(H^1(C))(2)) \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(S) \otimes \text{Sym}^2(H^1(C))(2))$$

is injective.

**Proof of (4.4.4).** Let $p_i : C \times C \times C \to C \times C$ be the $i$-th projection, and $p_{ij} : C \times C \times C \to C \times C$ the projection into $(i,j)$-th component. We note that $\Delta_{xxx} = p_{12}^* \Delta p_{23}^* \Delta, \Delta_{Oxx} = p_{23}^* \Delta p_1^* O, \cdots$, where $\Delta \subset C \times C$ the diagonal cycle.

Let $i : C \times C \hookrightarrow C \times C \times C$ be an inclusion such as: $(x,y) \mapsto (x,y,y)$. We can easily show that the pull-back of the modified diagonal cycle $i^* z_0$ is equal to $\Delta^2 + O \times K_C$ in $\text{CH}_0(C \times C)$. We want to show the non-vanishing of $i^*(\rho(z_0))$. It is the image of $\rho(z_0)$ under the map

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C)^{\otimes 3}(2)) \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C) \otimes H^2(C)(2))$$

induced from the cup-product $H^1(C)^{\otimes 2} \to H^2(C)$. Under the isomorphism $H^1(C) \otimes H^2(C)(2) \cong H^1(C)(1), i^*(\rho(z_0))$ coincides with the Abel-Jacobi class of $q_1^* i^* z_0 = -K_C + (2g-2) \cdot O \in \text{CH}_0(C)$ where $q_1 : C \times C \to C$ denotes the 1st projection. Since $K_C - (2g-2) \cdot O$ is not $\mathbb{Q}$-linearly equivalent to 0, it does not vanish.

**Proof of (4.4.2).** More generally, we show the image of $\rho(z_0)$ under the map

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C)^{\otimes 3}(2)) \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), \text{Sym}^3(H^1(C))(2))$$

do not vanish. Put $\rho(z_0)$ be the image. Let $\alpha : \text{Sym}^3(H^1(C)) \to H^1(C)^{\otimes 3}$ be a morphism of Hodge structures defined as: $v_1 \cdot v_2 \cdot v_3 \mapsto \sum_{\sigma \in \mathfrak{S}_3} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes v_{\sigma(3)}$. Then we want to show $\alpha(\rho(z_0)) \neq 0$. Since the modified diagonal cycle is invariant under the action of the symmetric group $\mathfrak{S}_3$, we have $\alpha(\rho(z_0)) = \sum_{\sigma \in \mathfrak{S}_3} \rho(z_0^\sigma) = 6 \rho(z_0)$. Therefore the assertion follows from (4.4.1).

**Proof of (4.4.3).** Let $\overline{S}$ be a smooth completion of $S$, and $\overline{j}_S : \overline{S} \to C$ be the extension of the morphism $j_S$.

Since the category of polarizable pure Hodge structures is semi-simple, the map

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C) \otimes \text{Sym}^2(H^1(C))(2)) \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(\overline{S}) \otimes \text{Sym}^2(H^1(C))(2))$$

is injective. Therefore it suffices to show that

$$\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(\overline{S}) \otimes \text{Sym}^2(H^1(C))(2)) \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(S) \otimes \text{Sym}^2(H^1(C))(2))$$
is injective. There is the exact sequence of mixed Hodge structures:

\[ 0 \rightarrow H^1(\mathcal{F}, \mathbb{Q}) \rightarrow H^1(S, \mathbb{Q}) \rightarrow \oplus \mathbb{Q}(-1) \rightarrow 0. \]

Therefore, applying \( \mathbb{R} \text{Hom}_{\text{MHS}}(\mathbb{Q}(0), (-) \otimes \text{Sym}^2(H^1(C)))(2) \) to this short exact sequence, our assertion follows from that \( \text{Hom}_{\text{HS}}(\mathbb{Q}(0), \text{Sym}^2(H^1(C))(1)) = 0 \). Consider the surjection:

\[ \text{Hom}_{\text{HS}}(\mathbb{Q}(0), (H^1(C))^\otimes 2(1)) \rightarrow \text{Hom}_{\text{HS}}(\mathbb{Q}(0), \text{Sym}^2(H^1(C))(1)). \]

The left hand side is isomorphic to \( \text{End}(J(C)) \otimes \mathbb{Q} \), which is a 1-dimensional vector space generated by the diagonal cycle class \([\Delta]\) because of \( \text{rankNS}(C \times C) = 3 \). The class \([\Delta]\) vanishes into the right hand side, which means \( \text{Hom}_{\text{HS}}(\mathbb{Q}(0), \text{Sym}^2(H^1(C))(1)) = 0 \).

Thus we have proved all steps. \( \square \)

Remark 4.5 (A “heuristic proof” of Theorem 4.2). The most technical point of the above proof is to choose the modified diagonal cycle as a lifting of the 0-cycle \( z \). The reader may have a question why we chose it. We give a heuristic answer to this question.

Let \( z_0 \in \text{CH}^2(C \times C \times C) \) be a lifting of \( z \) which is homologically trivial. Then we have the Abel-Jacobi class of \( \rho(z_0) \in \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C))^\otimes 3(2) \). The essential part of the above proof is Lemma 4.4 that is, to show that the image of \( \rho(z_0) \) under the natural map

\[ \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C))^\otimes 3(2) \rightarrow \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(S) \otimes H^1(C)^\otimes 2(2)) \]

does not vanish for all dominant maps \( S \rightarrow C \) from nonsingular curves \( S \).

Consider the following commutative diagram:

\[
\begin{array}{ccc}
\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C)^\otimes 3(2)) & \longrightarrow & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(S) \otimes H^1(C)^\otimes 2(2)) \\
\downarrow^\alpha & & \downarrow \\
\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(C)^\otimes 3/N^1(2)) & \longrightarrow & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^1(\mathcal{F}) \otimes H^1(C)^\otimes 2/N^1(2)),
\end{array}
\]

where \( N^\cdot \) denotes the coniveau filtration (cf. [11] p.162). The map \( \beta \) is injective. Therefore we only have to choose a lifting \( z_0 \) satisfying \( \alpha(\rho(z_0)) \neq 0 \). This is a well-known problem concerning with the Griffiths groups. The Griffiths group of a projective nonsingular variety \( X \) (denoted by \( \text{Griff}^r(X) \)) is defined to be the group of homologically trivial cycle modulo algebraic equivalence: \( \text{Griff}^r(X) := \text{CH}^r(X)_{\text{hom}}/\text{CH}^r(X)_{\text{alg}} \). In the conjectural theory of mixed motives, there is the following isomorphism:

\[ \text{Griff}^r(X) \simeq \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2r-1}(X)/N^{r-1}(2)). \]

The right hand side is conjectured to be injected into the extension group of mixed Hodge structures \( \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^{2r-1}(X)/N^{r-1}(2)) \). Therefore to find a cycle \( z_0 \) such that \( \alpha(\rho(z_0)) \neq 0 \) is conjecturally equivalent to find a non-trivial element in the Griffiths group. I was taught from K.Kimura that the modified diagonal cycle for certain curves gives the non-trivial element of the Griffiths groups (cf. [K]). But we do not necessarily need to prove it for Lemma 4.4. I don’t know whether the modified diagonal cycle gives a non-trivial element of the Griffiths group for Mochizuki’s curve (Lemma 4.4).
Remark 4.6. When $C$ is an elliptic curve, the 0-cycle $z = (P, P) - (P, O) - (O, P) + (O, O)$ on $C \times C$ is rationally equivalent to 0. (Easy exercise. Hint: consider the embedding $C \hookrightarrow C \times C$, $x \mapsto (x, -x + P)$.)

4.3. **Proof of Lemma 4.1.** We can easily find the desired curve at least over $\mathbb{C}$:

**Lemma 4.7.** There is a nonsingular projective curve $X$ of genus $g \geq 2$ defined over $\mathbb{C}$ such that $\text{End}(J(X)) = \mathbb{Z}$.

**Proof.** Let $S_0$ be a nonsingular projective surface over $\overline{\mathbb{Q}}$ with irregularity $q = 0$ (e.g. $S_0 = \mathbb{P}^2_{\overline{\mathbb{Q}}}$), and $L$ be a very ample line bundle on $S_0$. We assume that general smooth member of the linear system $|L|$ is not hyper-elliptic. We consider a Lefschetz pencil $f : S \to \mathbb{P}^1_{\overline{\mathbb{Q}}}$ obtained from $L$. We will show that the geometric generic fiber $C = f^{-1}(\overline{\mathbb{Q}})$ satisfies $\text{End}(J(C_{\overline{\mathbb{Q}}})) = \mathbb{Z}$.

Note that $\text{End}(J(C_{\overline{\mathbb{Q}}})) \subset \text{End}(H^1_{\acute{e}t}(C_{\overline{\mathbb{Q}}}, \mathbb{Q}_\ell))$, and $\gamma \in \text{End}(J(C_{\overline{\mathbb{Q}}}))$ satisfies $\gamma \cdot T_j^{n_j} = T_j^{n_j} \gamma$ for some $n_j > 0$, where $T_j \in \pi_1(\overline{\eta}, \overline{\eta})$ is the local monodromy generator. Let $\delta_j$ be the corresponding vanishing cycle. Then by the Picard-Lefschetz formula, we have

$$T_j(x) = x - (x, \delta_j)\delta_j \quad \text{for } x \in H^1_{\acute{e}t}(\mathbb{C})_\overline{\mathbb{Q}}.$$ 

Therefore we have $(x, \delta_j)\gamma(\delta_j) = (\gamma(x), \delta_j)\delta_j$ for all $x \in H^1_{\acute{e}t}(\mathbb{C})_\overline{\mathbb{Q}}$. We put

$$\gamma(\delta_j) = \varepsilon_j\delta_j \quad (4.1)$$

for some $\varepsilon_j \in \mathbb{Q}_\ell$. Moreover, applying $T_i^{n_i}$ to $(4.1)$, we have $(\delta_i, \delta_j)(\varepsilon_i - \varepsilon_j) = 0$ for all $i, j$. Since every vanishing cycles are conjugate under the action of $\pi_1(\overline{\eta}, \overline{\eta})$, (that is, for any $\delta_i$ and $\delta_j$, there is a $\sigma \in \pi_1(\overline{\eta}, \overline{\eta})$ such that $\sigma(\delta_i) = a \cdot \delta_j$ for some $a \in \mathbb{Q}_\ell$), we can see that for any $\delta_i$ and $\delta_j$, there are $\delta_{k_1}, \ldots, \delta_{k_l}$ such that $(\delta_i, \delta_{k_1}) \neq 0, (\delta_{k_1}, \delta_{k_2}) \neq 0, \ldots, (\delta_{k_l}, \delta_j) \neq 0$. Therefore all the $\varepsilon_i$ coincide.

The vanishing cycles generate $H^1_{\acute{e}t}(\mathbb{C})_\overline{\mathbb{Q}}$ because of $q = 0$. Therefore we have $\gamma = n \cdot \text{id}$, which implies the assertion. □

**Lemma 4.8.** There is an absolutely unramified $p$-adic number field $K$, and a proper curve

$$\pi : X \longrightarrow \text{Spec} \mathcal{O}_K \quad (4.2)$$

over the integer ring $\mathcal{O}_K$ such that,

(1) $\pi$ is smooth over $K$, and $\text{End}(J(X_{\bar{K}})) = \mathbb{Z}$,

(2) The special fiber $X_0$ is a geometrically connected stable curve $E_1 \cup \cdots \cup E_g$ over $\mathcal{F}_q$ (the residue field, $q = p^n$ ) such that

(a) each $E_i$ is an ordinary elliptic curve, that is, $\text{End}(E_i) \otimes \mathbb{Q}$ is a quadratic number field, and the rational prime $p$ is complete split : $p = p_1p_2$, (Then $\text{End}(E_i) = \text{End}(E_i, \mathcal{F}_q)$ )

(b) $E_i$ and $E_{i+1}$ intersect at one point for $i = 1, \ldots, g - 1$, and $E_i$ and $E_j$ do not intersect if $|j - i| \geq 2$,

(c) $\text{Hom}(E_i, E_j) = 0$ for $i \neq j$.

**Proof.** Let $E_{1,F}, \ldots, E_{g,F}$ be geometrically connected CM-elliptic curves over a number field $F$ such that

(i) $\text{Hom}(E_{i,F}, E_{j,F}) = 0$ for $i \neq j$;

(ii) $\text{End}(E_{i,F}) = \text{End}(E_{i,\overline{\mathbb{Q}}})$,

(iii) there is the smooth model $E_{i,\mathcal{O}_F} \to \text{Spec} \mathcal{O}_F$ over the integer ring $\mathcal{O}_F$ of $F$. 
Let $Y_{O_F} = E_1^{O_F} \cup \cdots \cup E_g^{O_F}$ be a chain of the above elliptic curves such as in the condition \([\ref{def:E_1}].\)

Let $\mathcal{M}_g$ be the moduli scheme of nonsingular proper curves of genus $g \geq 2$ over $O_F$, and $\overline{\mathcal{M}}_g$ the compactification of it in the sense of \([\ref{DM}].\) Note that $\overline{\mathcal{M}}_g$ is not a scheme but a stack. There is an integer $N \neq 0$, an étale open $U \to \overline{\mathcal{M}}_g$ where $U$ is a regular scheme, and a point $y : \text{Spec} O := \text{Spec} O_F[1/N] \to U$ which associates with $Y_O := Y_{O_F} \otimes_{O_F} O_F[1/N]$. We assume that $\text{Spec} O_F[1/N] \to \text{Spec} \mathbb{Z}$ is étale by replacing $N$ by a suitable one.

We denote $F_p$ the completion of $F$ by a prime $p$, and $O_{F_p}$ its integer ring.

**Claim 4.9.** Let $p \in \text{Spec} O$ be any prime, and $O_y(p) = O_{U,y(p)}$ the stalk of $O_U$ at the point $y(p)$. Then, there is an embedding

$$\alpha : O_y(p) \hookrightarrow O_{F_p}$$

of $O_F$-local rings which induces the isomorphism of the residue fields.

**Proof.** Let $\hat{O}_{y(p)}$ be the completion of $O_y(p)$ by the maximal ideal $m$. Since $O_y(p)$ is a regular local ring, there is an isomorphism $\hat{O}_{y(p)} \simeq O_{F_p}[t_1, \ldots, t_{3g-3}]$. We may assume that each $t_j$ is contained in $O_y(p)$. In fact, we can replace $t_j$ by $t_j'$ such that $t_j - t_j' \in m^2$.

Since the transcendental degree of $F_p$ over $F$ is infinite, there are $(3g-3)$-elements $q_1, \ldots, q_{3g-3} \in pO_{F_p}$ which are algebraically independent. Then the $O_{F_p}$-algebra homomorphism $\hat{O}_{y(p)} = O_{F_p}[t_1, \ldots, t_{3g-3}] \to O_{F_p}$ defined by $t_j \mapsto q_j$ induces the $O_F$-algebra homomorphism $\alpha : O_y(p) \to O_{F_p}$.

The remaining part of the proof is to show that $\alpha$ is injective. Let $I$ be the kernel of $\alpha$. Since $O_y(p) \otimes_{O_F} F \to O_{F_p} \otimes_{O_F} F$ is a homomorphism of fields (and hence injective), we have $I \otimes_{O_F} F = 0$. On the other hand, $I \subset O_y(p) \subset O_{F_p}[t_1, \ldots, t_{3g-3}]$ is torsion free over $O_F$. Therefore we have $I = 0$.

Let

$$X_{O_{F_p}} \to \text{Spec} O_{F_p} \quad (p \in \text{Spec} O)$$

be the proper curve over $O_{F_p}$ which is the pull-back of the family $X_U \to U$ by $\alpha$. By Lemma \([\ref{lem:alpha}].\), the generic fiber satisfies the condition \([\ref{def:genericfiber}].\). The special fiber satisfies \([\ref{def:specialfiber}].\) and \([\ref{def:specialfiber2}].\). Moreover, there are infinite many prime $p$ such that each irreducible component of the special fiber is ordinary. In fact, the density of such primes is larger than or equal to $1/2^g$. So the condition \([\ref{def:generalcondition}].\) is also satisfied.

This completes the proof. \(\square\)

**Lemma 4.10.** Let $A \to \text{Spec} R$ be a proper smooth abelian scheme over a discrete valuation ring $R$. Put $A_n := A \otimes_R R/m^{n+1}$ where $m$ is the maximal ideal. Then the natural maps $\text{End}(A) \to \text{End}(A_0)$ and $\text{End}(A_n) \to \text{End}(A_0)$ are injective.

**Proof.** Due to

$$\text{End}(A) \subset \text{End}(A \otimes_R \hat{R}) = \varprojlim_n \text{End}(A_n) \quad (\hat{R} := \varprojlim_n R/m^{n+1}),$$

it suffices to show the latter assertion.

Let $\ell$ be a rational prime which is invertible in $R/m$. Then the group scheme $\ell^* (A_n)$ of the $\ell^*$-torsion points is finite étale over $\text{Spec} R/m^{n+1}$, and therefore so is
the endomorphism scheme $\text{End}(\ell^e(A_n))$. By the formal étaleness (EGA IV §17), we have

$$\text{End}(\ell^e(A_n)) = \text{Hom}(\text{Spec} R/\mathfrak{m}^{n+1}, \text{End}(\ell^e(A_n)))$$

$$\simeq \text{Hom}(\text{Spec} R/\mathfrak{m}, \text{End}(\ell^e(A_n))) = \text{End}(\ell^e(A_0)).$$

Therefore it suffices to show that the natural map $\text{End}(A_n) \to \prod_i \text{End}(\ell^e(A_n))$ is injective. It follows from the fact that the subgroup of the $\ell$-primary torsion points is schematically dense.

Let the notations as in Lemma 4.8. The Jacobian variety $J(X_0)$ of $X_0$ is isomorphic to $E_1 \times \cdots \times E_g$, and $\text{End}(J(X_0))$ is isomorphic to $\text{End}(E_1) \times \cdots \times \text{End}(E_g)$.

**Lemma 4.11.** Consider an arbitrary lifting $X' \to \text{Spec} \mathcal{O}_K$ of $X_0$. Let $\varphi$ be an endomorphism of $J(X_0)$. If there is a positive integer $m > 0$ such that $m \cdot \varphi$ can be lifted on an endomorphism of $J(X'_R)$, then so does $\varphi$:

$$\text{End}(J(X'_R)) = \text{End}(J(X'_R) \otimes \mathbb{Q}) \cap \text{End}(J(X_0)).$$

**Proof.** Let $\psi := \tilde{\varphi} \in \text{End}(J(X'_R))$ be the lifting of $m \cdot \varphi$. We want to show that

$$\psi^*(H^1_{\text{cris}}(J(X'_R), \mathbb{Z}_\ell)) \subset m \cdot H^1_{\text{cris}}(J(X'_R), \mathbb{Z}_\ell)$$

for all rational prime $\ell$. If $\ell \neq p$, it follows from the isomorphism $H^1_{\text{dR}}(J(X'_R), \mathbb{Z}_\ell) \simeq H^1_{\text{cris}}(J(X'_R), \mathbb{Z}_\ell)$.

Firstly, we note that $\psi$ is defined over $K$. In fact, assume that $\psi$ is defined over a finite extension $K'$ over $K$. We can assume that $K'/K$ is a Galois extension. Then, for any $\sigma \in \text{Gal}(K'/K)$, $\psi^\sigma$ is also a lifting of $m \cdot \varphi$, which coincides with $\psi$ by Lemma 4.10.

Since the crystalline cohomology depends only on the special fiber, we have

$$\psi^*(H^1_{\text{cris}}(J(X_0)/\mathcal{O}_K)) \subset m \cdot H^1_{\text{cris}}(J(X_0)/\mathcal{O}_K).$$

Moreover, by the isomorphism $H^1_{\text{cris}}(J(X_0)/\mathcal{O}_K) \simeq H^1_{\text{dR}}(J(X'/\mathcal{O}_K)$ (because $K$ is absolutely unramified), we have

$$\psi^*(H^1_{\text{dR}}(J(X'/\mathcal{O}_K)) \subset m \cdot H^1_{\text{dR}}(J(X'/\mathcal{O}_K).$$

Therefore the assertion (4.3) follows by applying the Dieudonné functor ([Fal], FL 9.11).}

**Lemma 4.12.** There are finitely many subalgebras $A_0, \cdots, A_m$ of $\text{End}(J(X_0))$ such that, for any lifting $X' \to \text{Spec} \mathcal{O}_K$ of $X_0$, the image of $\text{End}(J(X'_R))$ is $A_i$ for some $i$.

**Proof.** By Lemma 4.11, we have

$$\text{End}(J(X'_R)) = (\text{End}(J(X'_R) \otimes \mathbb{Q}) \cap \text{End}(J(X_0)),$$

where $\text{End}(J(X'_R) \otimes \mathbb{Q})$ is a $\mathbb{Q}$-subalgebra of $\text{End}(J(X_0) \otimes \mathbb{Q}) = \text{End}(J(X_0) \otimes \mathbb{Q})$.

Since $\text{End}(J(X_0) \otimes \mathbb{Q})$ is a reduced and finite-dimensional $\mathbb{Q}$-algebra, there are at most finitely many $\mathbb{Q}$-subalgebras of it. Thus we have the assertion.

Let us prove Lemma 4.1.

Consider a deformation $X' \to \text{Spec} \mathcal{O}_K$ for each integer $m \geq 0$ such that

• $X' \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^{m+1} = X \otimes_{\mathcal{O}_K} \mathcal{O}_K/p^{m+1}$,
the generic fiber is defined over a number field.

By Lemma 4.12, \( \text{End}(J_{X_K}^i) = A_i \) for some \( i \geq 0 \). We may assume that \( A_0 = \mathbb{Z} \). We want to show that \( \text{End}(J_{X_K}^i) = A_0 = \mathbb{Z} \) if \( m \) is sufficiently large.

There is a finite extension \( K'/K \) such that \( \text{End}(J_{X_{K'}}^i) = \text{End}(J_{X_K}^i) \). We put \( X_n := X \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}/p^{n+1} \) and \( X'_n := X' \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}/p^{n+1} \). Choose an endomorphism \( \varphi_i \in A_i \setminus \mathbb{Z} \) for each \( i \geq 1 \). Since \( \text{End}(J_{X_{K'}}^i) = \cap (\text{End}(J_{X_n}^i)) \), there is an integer \( N > 0 \) such that \( \varphi_i \notin \text{End}(J_{X_n}^i) \) for all \( i \geq 1 \). Let \( m \geq N \). Since \( \text{End}(J_{X_{K'}}^i) \subseteq \text{End}(J_{X_n}^i) \) for all \( i \geq 1 \), we have \( \varphi_i \notin \text{End}(J_{X_{K'}}^i) \), which means \( \text{End}(J_{X_{K'}}^i) = A_0 = \mathbb{Z} \).

This completes the proof of Lemma 4.1.

5. The Beilinson conjecture

5.1. Recall the Beilinson conjecture ([1], 11.4, c)):

Conjecture 5.1 (Beilinson). Let \( r \geq 2 \) be an integer.

1. The Abel-Jacobi map
   \[
   \rho : \text{CH}^r(X)_{\text{hom}} \to J^r(X) \tag{5.1}
   \]
   is injective for any nonsingular projective variety \( X \) over \( \overline{\mathbb{Q}} \).

2. Let \( X \) be a nonsingular projective variety over \( \overline{\mathbb{Q}} \), and \( z \in \text{CH}^r(X)_{\text{hom}} \) be any algebraic cycle on \( X = X \otimes \mathbb{Q} \mathbb{C} \). Then, for each \( \sigma \in \text{Aut}(\mathbb{C}/\mathbb{Q}) \), \( \rho(z) = 0 \) if and only if \( \rho(z^\sigma) = 0 \).

For a projective nonsingular variety \( X \) over \( \mathbb{C} \), it is conjectured that the motivic filtration \( F_M^* \text{CH}^r(X) \) coincides with the kernel of the Abel-Jacobi map \( \rho : \text{CH}^r(X)_{\text{hom}} \to J^r(X) \). The Beilinson conjecture 5.1 (1) asserts that \( F_M^2 \text{CH}^r(X) = 0 \) if \( X \) is defined over a number field. The Beilinson conjecture 5.1 (2) asserts that the kernel of Abel-Jacobi maps should be “algebraic”. It holds, at least, when \( r = 1 \) or \( \dim X \). (I do not know whether it holds in case \( r \neq 1, \dim X \).)

Theorem 5.2. The Beilinson conjecture 5.1 (1), (2) for \( r = 2 \) implies the Bloch conjecture 5.1.

Proof. By Proposition 5.6 and the remark after it, we show Conjecture 5.3 for \( r = 2, m = 0 \) under the Beilinson conjecture 5.1 (1), (2) for \( r = 2 \). By the definition of the cycle map 5.3, it suffices to show that the following map
   \[
   \text{CH}^2(X_S) \to \text{Ext}^4_{\text{MHS}(X_S)}(\mathbb{Q}_{X_S}(0), \mathbb{Q}_{X_S}(2))
   \]
   is injective for any model \( X_S \). If \( X_S \) is projective, it follows directly from the Beilinson conjecture 5.1 (1). For an open case, we will use the Beilinson conjecture 5.1 (2).

Let \( U \) be a nonsingular quasi-projective (not necessarily complete) variety over \( \overline{\mathbb{Q}} \), and \( Y \) be a smooth completion of \( U \) such that \( D = \cup D_i = Y - U \) is a simple normal crossing divisor. We want to show that the map
   \[
   \text{CH}^2(U) \to \text{Ext}^4_{\text{MHS}(U_{\mathbb{C}})}(\mathbb{Q}(0), \mathbb{Q}(2))
   \]
   is injective. Since any extension groups of degree \( \geq 2 \) in MHS vanishes, the right hand side is an extension of \( W_2H^4(U_{\mathbb{C}}, \mathbb{Q}) \cap F^2 \) by \( \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^3(U_{\mathbb{C}}, \mathbb{Q}(2))) \).

Put \( \text{CH}^2(U)_{\text{hom}} := \ker(\text{CH}^2(U) \to H^4(U_{\mathbb{C}}, \mathbb{Q})) \). We will show that
   \[
   \text{CH}^2(U)_{\text{hom}} \to \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^3(U_{\mathbb{C}}, \mathbb{Q}(2)) \tag{5.2}
   \]
is injective.

Let $\text{CH}^1_Y(D)_{\text{hom}}$ be the subgroup of $\oplus_i \text{CH}^1(D_i)$ generated by cycles which are homologically equivalent to 0 in $Y$, that is, $\text{CH}^1_Y(D)_{\text{hom}} = \ker(\oplus_i \text{CH}^1(D_i) \rightarrow \text{CH}^2(Y)/\text{CH}^2(Y)_{\text{hom}})$. Consider the following commutative diagram:

$$
\begin{array}{ccc}
0 & \rightarrow & \text{CH}^1_Y(D)_{\text{hom}} \\
& & \downarrow \\
& & J(Y_C, D_C) \\
& & \downarrow \rho_D \\
\text{CH}^2(Y)_{\text{hom}} & \rightarrow & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^3(Y_C, \mathbb{Q}(2))) \\
\downarrow & & \downarrow b \\
\text{CH}^2(U)_{\text{hom}} & \rightarrow & \text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H^3(U, \mathbb{Q}(2))) \\
& & \downarrow \\
& & 0
\end{array}
$$

(5.3)

Here we put $J(Y_C, D_C) = \ker b$. Note that $\rho_Y$ is injective by the Beilinson conjecture. We show that if an algebraic cycle $z \in \text{CH}^2(Y)_{\text{hom}}$ satisfies $b(\rho_Y(z)) = 0$, then there is a cycle $w \in \text{CH}^1_Y(D)_{\text{hom}}$ such that $\rho_Y(z) = a(\rho_D(w)) = \rho_Y(i(w))$.

**Lemma 5.3.** $J(Y_C, D_C) = \rho_{D_C}(\text{CH}^1_{Y_C}(D_C)_{\text{hom}})$, where $\text{CH}^1_{Y_C}(D_C)_{\text{hom}}$ denotes the subgroup of $\oplus_i \text{CH}^1(D_i, C)$ generated by cycles which are homologically equivalent to 0 in $Y_C$.

**Proof.** For a $\mathbb{Q}$-mixed Hodge structure $H$, we write $\text{Ext}^1_{\text{MHS}}(\mathbb{Q}(0), H)$ by $J(H)$ simply.

There are the following exact sequences:

$$
\text{Gr}^W_3 H^3_{D_C}(Y_C) \rightarrow H^3(Y_C) \rightarrow W_3H^3(U_C) \rightarrow 0, \quad (5.4)
$$

$$
0 \rightarrow W_3H^3(U_C) \rightarrow W_4H^3(U_C) \rightarrow \text{Gr}^W_3 H^3(U_C) \rightarrow 0. \quad (5.5)
$$

(Here $H^*(X_C)(r)$ denotes the Betti cohomology $H^*(X_C^{an}, \mathbb{Q}(r))$.) By an easy computation, we can see that there is a natural isomorphism $\text{Gr}^W_3 H^3_{D_C}(Y_C) \simeq \oplus_i \text{H}^3(D_{i,C})(-1)$ of Hodge structure, and a surjection $\text{ker}(\oplus_i \text{H}^2(D_{i,C})(-1) \rightarrow H^4(Y_C)) \rightarrow \text{Gr}^W_3 H^3(U_C)$. We put $V = \ker(\oplus_i \text{H}^2(D_{i,C})(-1) \rightarrow H^4(Y_C))$. We have the exact sequence

$$
\oplus_i J(H^1(D_{i,C})(1)) \rightarrow J(H^3(Y_C)(2)) \rightarrow J(W_3H^3(U_C)(2)) \rightarrow 0
$$

(5.6)

from (5.3), and

$$
\text{Hom}_{\text{MHS}}(\mathbb{Q}(0), V(2)) \rightarrow J(W_3H^3(U_C)(2)) \rightarrow J(W_4H^3(U_C)(2)) = J(H^3(U_C)(2))
$$

(5.7)
Finally we remark:  

**Theorem 5.4** \(\text{(A1) Theorem 4.9)}\). The conjecture \(3.3\) implies the Beilinson conjecture \(5.1\) (1). In particular, it implies the finiteness of the rank of \(\text{CH}_0(X)\) for any projective nonsingular variety \(X\) over a number field.
Proof. Put $\mathbb{M}_\bar{Q} = \text{MHM}(\text{Spec} \bar{Q})$. Then the higher Abel-Jacobi map $\rho_X^{\nu}$ factors as follows:

$$\begin{array}{ccc}
\text{Gr}_p^k \text{CH}^r(X) & \xrightarrow{\rho_X^{\nu}} & \text{Ext}_{\mathbb{M}_\bar{Q}}^k(Q(0), H^{2r-\nu}(X \otimes \bar{Q})(r)) \\
\downarrow & & \downarrow \\
\text{Ext}_{\mathbb{M}_\bar{Q}}^k(Q(0), H^{2r-\nu}(X \otimes \bar{Q})(r)).
\end{array}$$

By Proposition 2.4 (2), the extension groups in $\mathbb{M}_\bar{Q}$ of degree $\geq 2$ vanishes. Therefore we have $F^2CH^r(X) = 0$ by Conjecture 3.5.

The latter assertion follows from the Mordell-Weil theorem because of the isomorphism $\text{CH}_0(X)_{\text{deg}=0} \sim \text{Alb}(X)(k)$.

REFERENCES

[A1] M.Asakura: Motives and algebraic de Rham cohomology. (to appear).
[A2] _______: (in preparation).
[AS] M.Asakura and S.Saito: Filtration on Chow groups and higher Abel-Jacobi maps. (preprint).
[Be] A.Beilinson: On the derived category of perverse sheaves. Lecture note in Math. 1289, 27-41, Springer, 1987.
[B1] S.Bloch: Lectures on algebraic cycles. Duke Univ. Math. Ser. Vol IV, Duke Univ. Durham, NC, 1980.
[B2] _______: Algebraic cycles and higher $K$-theory, Adv. in Math. 61 (1986), 267-304.
[BKL] Bloch, Kas and Lieberman: Zero cycles on surfaces with $p_g = 0$. Compositio. Math. 33 (1976), 135-145.
[BBD] A.Beilinson, J.Bernstein and P.Deligne: Faisceaux Pervers. Astérisque 100, (1983).
[Ca] J.Carlson: Extensions of mixed Hodge structures. Journées de géométrie algébrique d’Angers 1979, Sijthoff and Noordhoff, pp.107-127.
[D1] P.Deligne: Théorie de Hodge. II. Publ. Math. IHES, 40 (1971), 5-58.
[D2] _______: Théorie de Hodge. III. Publ. Math. IHES, 44 (1975), 5-77.
[DM] P.Deligne and D.Mumford: The irreducibility of the space of curves of given genus. Publ. Math. IHES, 36 (1969), 75-110.
[DMOS] P.Deligne et al: Hodge cycles, Motives and Shimura Varieties. Lecture note in Math. 900, Springer, 1982.
[EZ] F.El Zein and S.Zucker: Extendability of normal functions associated to algebraic cycles. in Topics in transcendental algebraic geometry, edited by P.Griffiths, Princeton, 1984 pp.269-288.
[Fal] G.Faltings: Crystalline cohomology and $p$-adic Galois representation.
[FL] Fontaine and Lafaille: Construction de representation $p$-adic, Ecor. (1982).
[G1] M.Green: Griffiths infinitesimal invariant and the Abel-Jacobi map. J. Diff. Geom. 29 (1989), 545-555.
[G2] _______: What comes after Abel-Jacobi maps. (preprint)
[G3] M.Green et al: Algebraic cycles and Hodge theory. Lecture note in Math. 1594, Springer, 1994.
[GS] B.Gross and C.Schoen: The modified diagonal cycle on the triple product of a pointed curve. Ann. Inst. Fourier, Grenoble 45, 3 (1995), 649-679.
[EGA IV] A.Grothendieck and J.Dieudonné: Étude locale des schémas et des morphismes de schémas. Publ.Math.IHES. 20(1964); 24(1965), 28(1966), 32(1967).
[H-dR] R.Hartshorne: On the de Rham cohomology of algebraic varieties. Publ. Math. IHES, 45 (1976), 5-99.
[J1] U.Jannsen: Mixed Motives and Algebraic $K$-theory. Lecture note in Math. 1400, Springer, 1990.
Motivic sheaves and filtration on Chow groups. In Motives I, Proc. Sympos. Pure Math. vol.55, AMS, 1994, pp.245-302.

M.Kashiwara: A study of variation of mixed Hodge structure. Publ. RIMS. Kyoto Univ. 22 (1986), 991-1024.

K.Kimura: On modified diagonal cycles in the triple products of Fermat Quotients. (preprint).

S.L.Kleiman: Algebraic cycles and the Weil conjectures. Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp.359-386.

D.Mumford: Rational equivalence of 0-cycles on surfaces. J. Math. Kyoto Univ. 9 (1969), 195-204.

J.P.Murre: On a conjectural filtration on Chow groups of an algebraic variety. Indag. Math, New Series, 4 (1993), 177-201.

J.P.Murre: On the motives of an algebraic surface. J. Reine. Angew. Math. 409 (1990), 190-204.

M.Saito: Modules de Hodge polarisables. Publ. RIMS. Kyoto Univ. 24 (1988), 849-995.

M.Saito: Mixed Hodge modules. Publ. RIMS. Kyoto Univ. 26 (1990), 221-333.

H.Saito: A generalization of Abel’s theorem and some finiteness property of zero-cycles on surfaces. Composit. Math. 84 (1992), 289-332.

S.Saito: Motives and filtration on Chow groups. Invent. Math. 125 (1996), 149-196.

J.Steenbrink and S.Zucker: Variation of mixed Hodge structure. Invent. math. 80 (1985) no.3, 489-542.

C.Voisin: Variations de structure de Hodge et zéro-cycles sur les surfaces généralis. Math. Ann. 299 (1994) 77-103.

C.Voisin: Some results on Green’s higher Abel-Jacobi maps. Ann. Math 149 (1999).

Research Institute for Mathematics Sciences, Kyoto University, Oiwakecho, Sakyo-ku, Kyoto, 606-8502, JAPAN

E-mail address: asakura@kurims.kyoto-u.ac.jp