ONE-PARAMETER HOMOTHETIC MOTION
IN THE HYPERBOLIC PLANE AND
EULER-SAVARY FORMULA

Soley ERSOY, Mahmut AKYIGIT
sersoy@sakarya.edu.tr, makyigit@sakarya.edu.tr
Department of Mathematics, Faculty of Arts and Sciences
Sakarya University, 54187 Sakarya/TURKEY

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Abstract

In [10] one-parameter planar motion was first introduced and the relations between absolute, relative, sliding velocities (and accelerations) in the Euclidean plane \( \mathbb{E}^2 \) were obtained. Moreover, the relations between the Complex velocities one-parameter motion in the Complex plane were provided by [10]. One-parameter planar homothetic motion was defined in the Complex plane, [9]. In this paper, analogous to homothetic motion in the Complex plane given by [9], one-parameter planar homothetic motion is defined in the Hyperbolic plane. Some characteristic properties about the velocity vectors, the acceleration vectors and the pole curves are given. Moreover, in the case of homothetic scale \( h \) identically equal to 1, the results given in [15] are obtained as a special case. In addition, three hyperbolic planes, of which two are moving and the other one is fixed, are taken into consideration and a canonical relative system for one-parameter planar hyperbolic homothetic motion is defined. Euler-Savary formula, which gives the relationship between the curvatures of trajectory curves, is obtained with the help of this relative system.

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1 Preliminaries

Before proceeding any further, we require a definition for the set of hyperbolic number and assume the existence of any number \( j \) which has the property \( j \neq \pm 1 \). In terms of the standard basis \( \{1, j\} \), the hyperbolic number can be written as

\[ z = x + jy \]
where \( j (j^2 = 1) \) is the unipotent (hyperbolic) imaginary unit and the real numbers \( x \) and \( y \) are called the real and unipotent (or hallucinatory) parts of the hyperbolic number \( z \), respectively, [2]-[4],[6]-[8],[11],[12]. The hyperbolic numbers

\[
\mathbb{H} = \mathbb{R} [j] = \{ z = x + jy | x, y \in \mathbb{R}, j^2 = 1 \}
\]

are the real numbers extended to include the unipotent \( j \) in the same manner that \( \mathbb{C} = \mathbb{R} [i] \) are the complex numbers extended to include the imaginary \( i \), \((i^2 = -1)\), [13].

The hyperbolic numbers are also called perplex numbers [6], split-complex numbers [1] or double numbers [1],[7],[11]. The hyperbolic number systems serve as the coordinates in the Lorentzian plane in the same way as the complex numbers serve as coordinates in the Euclidean plane. The role played by the complex numbers in Euclidean space is played by the hyperbolic number systems in the pseudo-Euclidean space, [12].

Addition and multiplication of the hyperbolic numbers are

\[
(x + jy) + (u + jv) = (x + u) + j(y + v),
\]

\[
(x + jy) (u + jv) = (xu + yv) + j(xv + yu).
\]

respectively. This multiplication is commutative, associative and distributes over addition. The hyperbolic conjugate of \( z = x + jy \) is defined by \( \bar{z} = x - jy \).

The hyperbolic inner product is

\[
\langle z, w \rangle = \text{Re}(zw) = \text{Re}(\bar{z}w) = xu - yv
\]

where; \( z = x + jy \) and \( w = u + jv \). Hyperbolic numbers \( z \) and \( w \) are hyperbolic (Lorentzian) orthogonal if \( \langle z, w \rangle = 0 \), [12].

The hyperbolic modulus of \( z = x + jy \) is

\[
\|z\|_h = \sqrt{|\langle z, z \rangle|} = \sqrt{|z\bar{z}|} = \sqrt{|x^2 - y^2|}
\]

and it is the hyperbolic distance of the point \( z \) from the origin. This is the Lorentz invariant of two-dimensional special relativity and their unimodular multiplicative group (the group composed of quadratic matrices determinant of which equals to 1) is the special relativity Lorentz group, [14]. These relations have been used to extend special relativity. Furthermore, by using the functions of the hyperbolic variable, two-dimensional special relativity has been generalized, [3]. These applications make the hyperbolic numbers appropriate for physics and the application of hyperbolic numbers is similar to the application of complex numbers to the Euclidean plane geometry, [14].

Note that the points \( z \neq 0 \) on the lines \( y = x \) are isotropic in the sense that they are nonzero vectors with \( \|z\|_h = 0 \). By this way, the hyperbolic distance creates Lorentzian geometry in \( \mathbb{R}^2 \). This is different from the usual Euclidean geometry of the complex plane, where \( \|z\|_h = 0 \) only if \( z = 0 \) in the complex plane. The set of all points in the hyperbolic plane that satisfy the equation \( \|z\|_h = r > 0 \) is a four-branched hyperbola of hyperbolic radius \( r \), [12].
The hyperbolic number \( z = x + jy \) can be written as follows:

While the hyperbolic number \( z \) is on H-I or H-III plane, then

\[
z = \pm r \left( \cosh \varphi + j \sinh \varphi \right) = \pm re^{j\varphi},
\]

While the hyperbolic number \( z \) is on H-II or H-IV plane, then

\[
z = \pm r \left( \sinh \varphi + j \cosh \varphi \right) = \pm rje^{j\varphi},
\]

[See Figure 1.]

![Hyperbolic Plane](Figure 1)

This formula can be derived by a power series expansion due to the fact that \( \cosh \) has only even powers whereas \( \sinh \) has odd powers. For all real values of the hyperbolic angle \( \varphi \), the hyperbolic number \( e^{j\varphi} \) has norm 1 and lies on the right branch of the unit hyperbola. [12].

A hyperbolic rotation defined by \( e^{j\varphi} \) corresponds to multiplication by the matrix, [12]:

\[
\begin{bmatrix}
\cosh \varphi & \sinh \varphi \\
\sinh \varphi & \cosh \varphi
\end{bmatrix}.
\]

Another property of the hyperbolic inner product is

\[
\langle ze^{j\varphi}, we^{j\varphi} \rangle = \langle z, w \rangle.
\]

In addition, a vector multiplied by \( j \) is a hyperbolic orthogonal vector. This is similar to the role played by the multiplication \( i = e^{i(\pi/2)} \) in the complex plane, [12].
2 One-Parameter Homothetic Motion in the Hyperbolic Plane

In this section, we will define one-parameter homothetic motion in the hyperbolic plane and obtain the relation between the velocities and the accelerations of a point under one-parameter homothetic planar motions.

Homothetic motion of a moving hyperbolic plane $\mathbb{H}$ with respect to a fixed hyperbolic plane $\mathbb{H}'$ will be considered, in that, the orthonormal coordinate systems $\{O; h_1, h_2\}$ and $\{O'; h_1', h_2'\}$ being on the moving and fixed hyperbolic planes $\mathbb{H}$ and $\mathbb{H}'$, respectively, will be analyzed with respect to each other. As the vector $\overrightarrow{OO'}$ represented by the hyperbolic number $u$ determines the distance between the origin point of the moving system and the origin point of the fixed system, the vectorial representation is as follows

$$x' = hx - u$$

(1)

where, $h = h(t) \neq \text{constant}$ is the homothetic scale (Figure 2. and Figure 3.).

Figure 2. $\mathbb{H}/\mathbb{H}'$ hyperbolic homothetic motion that rotates with central angle $\varphi$
Let a fixed point $X$ chosen on the plane $\mathbb{H}$ be represented by the hyperbolic numbers $x = x_1 + jx_2$ and $x' = x_1' + jx_2'$ on the planes $\mathbb{H}$ and $\mathbb{H}'$, respectively. Thus, one-parameter homothetic hyperbolic motion of the moving coordinate system $\{O, h_1, h_2\}$ with respect to the fixed coordinate system $\{O', h_1', h_2'\}$ represented by $\mathbb{H}/\mathbb{H}'$ is defined as the following transformation:

$$x' = u' + hx e^{j\phi}.$$  \hspace{1cm} (2)

Here, $\phi$ is the Lorentzian (either central or hyperbolic) rotation angle of the motion $\mathbb{H}/\mathbb{H}'$, and the hyperbolic number $u'$ represents the origin point of the moving system in the fixed system. The rotation angle $\phi$, the homothetic scale $h$ and $x, x', u$ will be regarded as the differentiable functions of a real parameter.
$t$ from the class $C^{\infty}$. Generally, this parameter $t$ will be used as time and at the moment $t = 0$, the coordinate systems will be accepted as coincident. The hyperbolic number $u = u_1 + ju_2$ represents the origin point $O'$ of the fixed system in the moving system. At this case, if $X' = O'$, then $x' = 0$ and $x = u$. Thus, from the equation (2)

$$u' = -ue^{j\varphi}$$

(3)
is found. Using (2) and (3) the following is obtained.

$$x' = (hx - u)e^{j\varphi}.$$  

(4)

As $\dot{\varphi}(t)$ would give only the translation, we will assume $\frac{d\varphi}{dt} = \dot{\varphi}(t) \neq 0$ during the motion $H / H'$ and call it as the angular velocity of the motion.

### 2.1 Velocities and the Composition of Velocities

Let the point $X$ on the moving plane $H$ change its location depending on a parameter $t$ while undergoing one-parameter homothetic motion of plane $H$ with regards to the plane $H'$. At this case, two motions belonging to the point $X$ occur. Let’s explore what kind of relation exists between these homothetic motions and their velocities.

The velocity vector of the point $X$ with respect to the plane $H$, that is, the vectorial velocity which the point has while drawing the trajectory curve on is called the relative velocity of the point and is represented as $V_r$. The relative velocity $V_r$ of $X$ is

$$V_r = h\dot{x}e^{j\varphi}.$$  

(5)

The velocity of the point $X$ with respect to the fixed plane $H'$ is called the absolute velocity of the point $X$ and is represented by $V_a$. The absolute velocity of $X$ is

$$V_a = \left(h + jh\dot{\varphi}\right)x e^{j\varphi} - (\dot{u} + ju\dot{\varphi}) e^{j\varphi} + h\dot{x}e^{j\varphi}.$$  

(6)

If the differential of equation (5) is substituted in the last equation

$$V_a = \dot{u}' + \left(h + jh\dot{\varphi}\right)x e^{j\varphi} + h\dot{x}e^{j\varphi}$$

(7)
can be written. If

$$V_f = \left(h + jh\dot{\varphi}\right)x e^{j\varphi} - (\dot{u} + ju\dot{\varphi}) e^{j\varphi}$$

(8)

the following equation is obtained

$$V_a = V_f + V_r.$$  

(9)

Here, $V_f$ is the sliding velocity of the one-parameter planar homothetic motion $H / H'$. If the point $X$ is a fixed point on the moving plane $H$, then $V_f = 0$. So it is easily seen that

$$V_a = V_r.$$  

(10)
2.2 The Rotation Pole and the Pole Trajectories

Studying the points of the one-parameter homothetic hyperbolic motion $H/H'$ where the $V_f$ sliding velocity equals to zero in every moment $t$ will reveal the rotation pole term. Thus, by taking $V_f = 0$ in the equation \(\text{(8)}\), the pole point $P = (p_1, p_2) \in \mathbb{H}$ is the hyperbolic number as

$$p = \frac{\dot{u} + j\dot{\varphi}u}{\dot{h} + jh\dot{\varphi}} \quad \text{(11)}$$

or

$$p = p_1 + jp_2 = \frac{\dot{h}u - h\dot{\varphi}^2u}{h^2 - h^2\dot{\varphi}^2} + \frac{j\dot{h}\dot{\varphi}u - h\dot{\varphi}\dot{u}}{h^2 - h^2\dot{\varphi}^2}.$$  

In the special case of $h(t) = 1$, the following equation exists:

$$p = p_1 + jp_2 = u + j\dot{\varphi}$$

which was given in \[15\].

Let the rotation pole of the homothetic hyperbolic motion $H/H'$ be $P$ and a moving point on $H'$ be $X$. Given this condition, the pole ray $\overrightarrow{PX}$ from the pole $P$ to the point $X$ is expressed by the following equation

$$\overrightarrow{PX} = (hx - p)e^{j\varphi}.$$  

(12)

In addition, if the equations (8) and (11) are considered together

$$V_f = \left(\dot{h} + jh\dot{\varphi}\right)(x - p)e^{j\varphi}$$

(13)

is found. As in \[15\], $V_f$ and $\overrightarrow{PX}$ are seen to be orthogonal to each other given condition of $h(t) = 1$.

The length of the vector obtained from equation (13) is

$$\|V_f\|_h = \sqrt{\left(\dot{h}^2 - h^2\dot{\varphi}^2\right)\left[(x_1 - p_1)^2 + (x_1 - p_1)^2\right]}.$$  

In special case of $h(t) = 1$ as given in \[15\],

$$\|V_f\|_h = |\dot{\varphi}|\left\|\overrightarrow{PX}\right\|_h$$

is found.

During one-parameter homothetic hyperbolic motion $H/H'$ the geometric locus of the pole points $P$ in each $t$ moment is the moving pole curve ($P'$) on the plane $H$ and the fixed pole curve ($P'$) on the plane $H'$, respectively. Due to equation (11), the following is written:

$$p' = (h\dot{p} - u)e^{j\dot{\varphi}}.$$
and from the differentiation of this last equation with respect to $t$

$$\dot{\mathbf{p}}' = \left[ \left( \hat{h} + jh\dot{\varphi} \right) \mathbf{p} - (\dot{\mathbf{u}} + j\mathbf{u}\dot{\varphi}) + h\dot{\mathbf{p}} \right] e^{j\varphi}$$

is obtained. Here, if the equation of the pole point given by equation (11) is substituted in the last equation,

$$\dot{\mathbf{p}}' = h\dot{\mathbf{p}} e^{j\varphi}$$

(14)

is found.

Thus, the tangent vectors at the contact points of the pole curves coincide with each other after the Lorentzian rotation $\varphi$ and the translation $h$.

Let the arc elements of the moving and the fixed pole curves be $ds$ and $ds'$, respectively. In this case

$$ds = \|\dot{\mathbf{p}}\|_h \, dt \quad \text{and} \quad ds' = \|\dot{\mathbf{p}}'\|_h \, dt$$

can be written. With the help of this last equation and equation (14), we get

$$ds' = |h| \, ds.$$

Thus, the following theorem can be given.

**Theorem 1** In one-parameter planar homothetic hyperbolic motion $\mathbb{H}/\mathbb{H}'$, the moving pole curve $(P)$ in the plane $\mathbb{H}$ rolls by sliding on the fixed pole curve $(P')$ on the plane $\mathbb{H}'$. The coefficient of this sliding, rolling motion is the homothetic scale $h$.

**Special Case** In the special case of $h = 1$, we get $ds' = ds$, that is, the pole curves roll on each other without sliding, which is given in [15].

### 2.3 Accelerations and the Composition of Accelerations

Let $X$ be a moving point on the moving hyperbolic plane $\mathbb{H}$. In this case, the acceleration of the point $X$ with respect to $\mathbb{H}$ is called relative acceleration and is defined by $\frac{d^2 \mathbf{x}}{dt^2} = \ddot{\mathbf{x}}$. In addition, the relative acceleration $\mathbf{b}_r$ with respect to fixed hyperbolic plane $\mathbb{H}'$ can be written as

$$\mathbf{b}_r = h\ddot{\mathbf{x}} e^{j\varphi}.$$  

(15)

The acceleration of $X'$ with respect to $\mathbb{H}'$ is called the absolute acceleration $\mathbf{b}_a$, and with the help of the differentiation of $\mathbf{V}_a$ with respect to $t$,

$$\mathbf{b}_a = \frac{d\mathbf{V}_a}{dt} = \dot{\mathbf{V}}_a = (\mathbf{x} - \mathbf{p}) \left[ \hat{h} + h\dot{\varphi}^2 + j \left( 2h\dot{\varphi} + h\ddot{\varphi} \right) \right] e^{j\varphi} - \dot{\mathbf{p}} \left( \hat{h} + jh\dot{\varphi} \right) e^{j\varphi} + 2\ddot{\mathbf{x}} \left( \hat{h} + jh\dot{\varphi} \right) e^{j\varphi} + h\ddot{\mathbf{x}} e^{j\varphi}$$

(16)

is obtained and in this last equation, the expression

$$\mathbf{b}_r = (\mathbf{x} - \mathbf{p}) \left[ \hat{h} + h\dot{\varphi}^2 + j \left( 2h\dot{\varphi} + h\ddot{\varphi} \right) \right] e^{j\varphi} - \dot{\mathbf{p}} \left( \hat{h} + jh\dot{\varphi} \right) e^{j\varphi}$$

(17)
is called the sliding acceleration and

\[ b_c = 2\dot{x} \left( \dot{h} + jh \dot{\phi} \right) e^{j\phi} \]  

(18)

is called the Coriolis acceleration.

Thus, the following theorem can be given.

**Theorem 2** In \( \mathbb{H}/\mathbb{H}' \) one-parameter planar homothetic hyperbolic motion, there is the following relation between the accelerations

\[ b_a = b_f + b_c + b_r. \]

The acceleration pole is known by the vanishing of the sliding acceleration under one-parameter planar motion. Thus, the following theorem is obtained, given the condition of \( b_f = 0 \).

**Theorem 3** Let the pole point be \( P \) of the one-parameter homothetic hyperbolic motion \( \mathbb{H}/\mathbb{H}' \). During this motion the acceleration pole point \( Q = (q_1, q_2) \in \mathbb{H} \) is the hyperbolic number as

\[ q = p + \frac{\dot{p} \left( \dot{h} + jh \dot{\phi} \right)}{\dot{h} + h \dot{\phi}^2 + j \left( 2h \dot{\phi} + h \dot{\phi} \right)} \]  

(19)

where \( \dot{h} + h \dot{\phi}^2 \neq \mp \left( 2h \dot{\phi} + h \dot{\phi} \right) \).

**Special Case** In the special case of \( h = 1 \), given the condition that \( \dot{\phi}^2 - \phi^4 \neq 0 \), the acceleration pole point of one-parameter hyperbolic motion is

\[ q = p + \frac{\dot{p} \left( \dot{\phi} - j\phi^3 \right)}{\dot{\phi}^2 - \phi^4} \]

which was given in \([15]\).

### 3 Canonical Relative System for Homothetic Motion in the Hyperbolic Plane

Let’s consider an \( A \) plane which moves with regard to \( \mathbb{H} \) and \( \mathbb{H}' \) hyperbolic planes, first one moving and the second one fixed. Let’s examine the motion of the coordinate system \( \{B; a_1, a_2\} \) which defines hyperbolic plane \( A \), and hyperbolic planes \( \mathbb{H} \) and \( \mathbb{H}' \) with regard to the coordinate systems \( \{O; h_1, h_2\} \) and \( \{O'; h'_1, h'_2\} \). [See Figure 4. and 5.] If the vector \( \overrightarrow{OB} \) is defined by the hyperbolic number \( b = b_1 + jb_2 \), by applying the hyperbolic inner product, \( b_1^2 - b_2^2 > 0 \) or \( b_1^2 - b_2^2 < 0 \) can be obtained. As seen in Figure 3.1. and Figure 3.2.
respectively, the vector $\overrightarrow{OB}$ can be on the plane H-I or H-II in hyperbolic motion.

Figure 4. $\overrightarrow{OB}$ vector is on H-I plane

Figure 5. $\overrightarrow{OB}$ vector is on H-II plane
The rotation angles of the one-parameter planar hyperbolic motion $\mathbb{A}/\mathbb{H}$ and $\mathbb{A}/\mathbb{H}'$ are $\varphi$ and $\psi$, respectively. If the origin points of $O$, $B$ and $O'$, $B$ are coincident, then there exists following relations:

\[
\begin{align*}
    a_1 &= \cosh \varphi h_1 + \sinh \varphi h_2 \\
    a_2 &= \sinh \varphi h_1 + \cosh \varphi h_2
\end{align*}
\]

and

\[
\begin{align*}
    a_1' &= \cosh \psi h'_1 + \sinh \psi h'_2 \\
    a_2' &= \sinh \psi h'_1 + \cosh \psi h'_2
\end{align*}
\]

respectively [See Figure 6.].

Let $X$ be a point with coordinates $x_1$, $x_2$ on the moving plane $\mathbb{A}$. If we denote the vectors $\overrightarrow{BX}$, $\overrightarrow{OB}$ and $\overrightarrow{OB'}$ with the hyperbolic numbers $\tilde{X} = x_1 + jx_2$, $b = b_1 + jb_2$ and $b' = b'_1 + jb'_2$, respectively; then we can write

\[
    x = (b + h\tilde{x}) e^{j\varphi}
\]

and

\[
    x' = (b' + h\tilde{x}') e^{j\varphi}
\]

where, $h = h(t) \neq$ constant is the homothetic scale of the motion and the hyperbolic numbers $x$ and $x'$ denote the point $X$ with respect to the coordinate systems of $\mathbb{H}$ and $\mathbb{H}'$, respectively.

The velocities of the motion with the help of the differentiation of the equations (20) and (21) can be found. By differentiating the equation (20)

\[
    d\mathbf{x} = (\sigma + (dh + j\tau) \tilde{x} + hd\tilde{x}) e^{j\varphi}
\]

is obtained, in which

\[
    \sigma = \sigma_1 + j\sigma_2 = db + jbd\varphi, \quad \tau = d\varphi
\]

and the relative velocity vector of $X$ (with respect to $\mathbb{H}$) is $\mathbf{V}_r = \frac{dx}{dt}$.

If we assume the differentiation of the equation (21),

\[
    d'\mathbf{x} = (\sigma' + (dh + j\tau') \tilde{x} + d\tilde{x}) e^{j\psi}
\]
can be obtained along with the equation
\[ \sigma' = \sigma_1' + j\sigma_2' = d'b + jb'd\psi, \quad \tau' = d\psi. \] (25)

Also, the absolute velocity vector, that is, the velocity vector of \(X\) with respect to \(H'\), is \(V_a = \frac{d\tilde{x}}{dt}\).

Here, \(\sigma_i\), \(\sigma'_i\), \((i = 1, 2)\), \(\tau\), \(\tau'\) are linear differential forms of \(t\) and are called Lorentzian Pfaffian forms of one-parameter homothetic hyperbolic motion. The real parameter \(t\) represents time.

If \(V_f = 0\) and \(V_a = 0\), the point \(X\) is fixed on the hyperbolic planes \(H\) and \(H'\), respectively. Thus, the conditions of \(X\) being fixed on the \(H\) and \(H'\) planes are

\[ d\tilde{x} = \frac{1}{h}(\sigma + (dh + jh\tau)\tilde{x}) \] (26)

and

\[ d\tilde{x} = \frac{1}{h}(\sigma' + (dh + jh\tau')\tilde{x}) \] (27)

respectively. If the equation (26) is substituted into equation (24),

\[ d\tilde{x} = [(\sigma' - \sigma) + jh(\tau' - \tau)\tilde{x}] e^{j\psi} \] (28)

can be obtained, where the sliding velocity vector of the point \(X\) is \(V_f = \frac{dx}{dt}\).

Thus, following can be easily obtained:

\[ d'x = d_f x + dx \] (29)

This satisfies the relation between velocities which is given in equation (29).

Just to avoid translation, it is assumed that \(\dot{\varphi} \neq 0\) and \(\dot{\psi} \neq 0\). The rotation pole of the motion \(H/\mathbb{H}'\) is characterized by the sliding velocity \(P\) being zero.

For that reason, if \(d_f x = 0\), from the equation (28), the pole point \(P\) of the one-parameter planar hyperbolic homothetic motion is obtained as

\[ p = j\frac{\sigma' - \sigma}{\tau' - \tau} \] (30)

and if Lorentzian coordinates are preferred on the condition that \(\overrightarrow{BP} = p = p_1 + jp_2\), it can be written

\[ p_1 = \frac{\sigma_2' - \sigma_2}{\tau - \tau'}, \quad p_2 = \frac{\sigma_1' - \sigma_1}{\tau - \tau'} \] (31)

which is given in [5].

In the \(\mathbb{H}/\mathbb{H}'\) one-parameter planar hyperbolic homothetic motion, moving and fixed pole curves determine the geometric locus of the point \(P\) in \(\mathbb{H}\) and \(\mathbb{H}'\) planes, respectively. In other words; \((P)\) and \((P')\) are the representation of the moving and fixed pole curves, respectively. Also, the pole tangents can be either on the plane H-I or H-II [See Figure 7].
Let’s first choose the pole tangents of the pole curves \((P)\) and \((P')\) on the plane \(H-II\) because the same results would be obtained by following similar operations on the plane \(H-I\).

### 3.1 The Euler-Savary Formula for One-Parameter Planar Hyperbolic Homothetic Motion

Let’s choose the moving plane \(\mathcal{A}\), represented by the coordinate system \(\{B; a_1, a_2\}\), in such way to meet the following conditions:

i) The origin of the system \(B\) coincides with the instantaneous rotation pole \(P\)

ii) The axis \(\{B; a_2\}\) is the pole tangent, that is, it coincides with the common tangent of the pole curves \((P)\) and \((P')\) (on the plane \(H-II\)) [See Figure 8.].

Figure 8.

When the condition (i) is considered: by using the equation (31),

\[
\sigma_1 = \sigma_1', \quad \sigma_2 = \sigma_2
\]

(32)
are obtained. From the equations (23) and (25),
\[
\begin{align*}
d\mathbf{b} &= (d\mathbf{b} + j\mathbf{b}d\varphi)e^{j\varphi} = \sigma e^{j\varphi} \\
d'\mathbf{b} &= (d'b' + j\mathbf{b}'d\varphi)e^{j\psi} = \sigma'e^{j\psi}
\end{align*}
\] (33)
are found. If the equation (22) and the last equation are took into consideration:
\[
d\mathbf{p} = d'\mathbf{p} = db = d'b
\] (34)
is found. Thus, the moving pole curve \((P)\), the pole tangent of which is given, and the fixed pole curves \((P')\) are rolling on each other without sliding.
The second condition, that is, the condition that the pole tangent coincides with \(a_2\), requires the coefficient of \(a_1\) to be zero. Here, \(\sigma_1 = \sigma'_1 = 0\) and \(\sigma = j\sigma_2 = j\sigma'_2\) can be written. Consequently, the derivative equations of the canonical relative system \(\{P; a_1, a_2\}\) are
\[
\begin{align*}
da_1 &= \tau a_2 = j\tau e^{j\varphi}, \quad da_2 = \tau a_1 = -\tau e^{j\varphi}, \quad dp = j\sigma_2 a_1 = \sigma e^{j\varphi}
\end{align*}
\] (35)
and
\[
\begin{align*}
da'_1 &= \tau' a_2 = j\tau'e^{j\psi}, \quad da'_2 = \tau' a_1 = j\tau'e^{j\psi}, \quad dp' = j\sigma'_2 a_1 = \sigma e^{j\psi}.
\end{align*}
\] (36)
Here \(\sigma = ds\) is the scalar arc element of the pole curves \((P)\) and \((P')\). \(\tau\) is the hyperbolic cotangent angle, that is, two neighboring tangent angles of \((P)\). Thus, the curvature of \((P)\) on the point \(P\) is represented by \(\frac{\tau}{\sigma} = \frac{d\varphi}{ds}\).
Similarly, the curvature of the fixed pole curve \((P')\) on the point \(P\) is \(\frac{\tau'}{\sigma} = \frac{d\psi}{ds}\) where \(\tau'\) is the hyperbolic cotangent angle.
The inverse values of these ratios
\[
r = \frac{\sigma}{\tau}
\] (37)
and
\[
r' = \frac{\sigma}{\tau'}
\] (38)
give the curvature radius of the pole curves \((P)\) and \((P')\), respectively.
When \(d\nu = \tau' - \tau\) is the infinitesimal small hyperbolic instantaneous rotation angle, the moving hyperbolic plane \(\mathbb{H}\), with respect to the fixed plane \(\mathbb{H}'\), rotates around the rotation pole \(P\) as much as this hyperbolic angle in the \(dt\) time scale. Thus, the hyperbolic angular velocity of the rotational motion of \(\mathbb{H}\) with respect to \(\mathbb{H}'\) is
\[
\frac{\tau' - \tau}{dt} = \frac{d\nu}{dt} = \nu
\] (39)
From the equations (37), (38), and the last equation, the following can be written:
\[
\frac{\tau' - \tau}{dt} = \frac{d\nu}{dt} = \frac{1}{r'} - \frac{1}{r}
\] (40)
Let the direction of the unit tangent vector $\mathbf{a}_2$ be in the direction determined by time-based pole curves ($P$) and ($P'$). Let’s choose the vector $\mathbf{a}_2$ in such way to ensure that $\frac{d\theta}{dt} > 0$. In this case, $r > 0$ as the curvature center of the moving pole ($P$) curve is at the right side of the directed pole tangent \{\(P; \mathbf{a}_2\)\}. Similarly, $r' > 0$.

According to the canonical relative system, the differentiation $\mathbf{x}$- the coordinates of which are $x_1$, $x_2$- with respect to the planes $\mathbb{H}$ and $\mathbb{H}'$ are

\[
d\mathbf{x} = (\sigma + (dh + jh\tau) \mathbf{x} + hd\mathbf{x})e^{j\phi}\tag{41}
\]

and

\[
d'\mathbf{x} = (\sigma' + (dh + jh\tau') \mathbf{x} + hd\mathbf{x})e^{j\phi'}\tag{42}
\]

respectively. If

\[
hd\mathbf{x} = -\sigma - (dh + jh\tau) \mathbf{x} \tag{43}
\]

then the point $X$ is fixed on the hyperbolic plane $\mathbb{H}$. Similarly, if

\[
hd\mathbf{x} = -\sigma - (dh + jh\tau') \mathbf{x} \tag{44}
\]

then the point $X$ is fixed on the hyperbolic plane $\mathbb{H}'$. Also, the sliding velocity $\mathbf{V}_f$ of the movement $\mathbb{H}/\mathbb{H}'$ corresponds to the differentiation

\[
df \mathbf{x} = jh(\tau' - \tau) \mathbf{x}e^{j\phi}. \tag{45}
\]

Now, let’s examine the curvature centers of the trajectory curves drawn on their fixed plane by the points of moving planes in the motion of $\mathbb{H}/\mathbb{H}'$. In the canonical relative system, the points $X$, $X'$ having the coordinates $x_1$, $x_2$ and $x'_1$, $x'_2$, respectively, are situated, together with the instantaneous rotation pole $P$ in every $t$ moment on the instantaneous trajectory normal, which belongs to $X$. Moreover, this curvature center can be considered as the limit of the meeting point of the normals of the two neighboring points on the curve. Thus,

\[
\overrightarrow{PX} = x_1 + jx_2 = \mathbf{x},
\]

\[
\overrightarrow{PX'} = x'_1 + jx'_2 = \mathbf{x}' \tag{46}
\]

vectors have the same direction which passes through $P$. Then, for the points $X$ and $X'$, the equation is

\[
\frac{\mathbf{x}}{\mathbf{x}'} = \frac{x_1 + jx_2}{x'_1 + jx'_2} = \lambda \in \mathbb{R}. \tag{47}
\]

If the differential of this last equation is taken, then we get

\[
d\mathbf{x}\mathbf{x}' - \mathbf{x}d\mathbf{x}' = 0. \tag{48}
\]

If the conditions that the point $X$ be fixed on the plane $\mathbb{H}$ and the point $X'$ be fixed on the plane $\mathbb{H}'$ are provided, then

\[
\sigma [\mathbf{x} - \mathbf{x}'] + jh\mathbf{x}\mathbf{x}'(\tau' - \tau) = 0 \tag{49}
\]
can be obtained. As the vectors $\overrightarrow{PX}$, $\overrightarrow{PX}'$ are on the plane H-II, 

$$x = aje^{j\alpha}$$  \hspace{1cm} (50) 

and

$$x' = a'je^{j\alpha}.$$  \hspace{1cm} (51) 

That is, $a$ and $a'$, respectively, represent the distance of the points $X$ and $X'$ on the plane H-II from the rotation pole $P$. Also, the angle $\alpha$ is bounded by the pole curves $\overrightarrow{PX} = \overrightarrow{PX}'$, [See Figure 9.]

If the equations (50) and (51) are substituted into equation (49), then

$$j\sigma(a - a') + jhaa'e^{j\alpha}(\tau' - \tau) = 0$$  \hspace{1cm} (52) 

can be obtained, and if the equation (40) is considered together with this last equation,

$$\left(\frac{1}{a} - \frac{1}{a'}\right) e^{-j\alpha} = h \left(\frac{1}{r'} - \frac{1}{r}\right) = h \frac{d\nu}{ds}. $$  \hspace{1cm} (53) 

is found. Here, $r$ and $r'$ are the radii of curvature of the pole curves $P$ and $P'$, respectively. $ds$ represents the scalar arc element and $d\nu$ represents the infinitesimal hyperbolic angle of the motion of the pole curves.

The equation (53) is called the Euler-Savary formula for one-parameter plane hyperbolic homothetic motion. Consequently, the following theorem can be given.

Figure 9.
Theorem 4 Let $\mathbb{H}$ and $\mathbb{H}'$ be the moving and fixed hyperbolic planes, respectively. A point $X$, assumed on $\mathbb{H}$, draws a trajectory whose instantaneous center of curvature is $X'$ on the plane $\mathbb{H}'$. In the inverse homothetic motion of $\mathbb{H}/\mathbb{H}'$, a point $X'$ assumed on $\mathbb{H}'$ draws a trajectory whose center of curvature is $X$ on the plane $\mathbb{H}$. The relation between the points $X$ and $X'$ is given by the Euler-Savary formula given in the equation (53).

Remark Let's choose the moving plane $A$ represented by the coordinate system $\{B; a_1, a_2\}$ in such way to meet following conditions:

i) The origin of the system $B$ and the instantaneous rotation pole $P$ coincide with each other, i.e. $B = P$. [See Figure 10.]

ii) The axis $\{B; a_1\}$ is the pole tangent, that is, it coincides with the common tangent of the pole curves $(P)$ and $(P')$ (on the plane H-I)

Thus, if the operations in section 3.1. are performed considering the conditions i) and ii), the Euler-Savary formula for one-parameter planar hyperbolic homothetic motion remains unchanged, that is, it is the same as in the equation (53) [See Figure 11.]
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