Integral inequalities for some convex functions via generalized fractional integrals

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Abstract
In this paper, we obtain the Hermite–Hadamard type inequalities for \( s \)-convex functions and \( m \)-convex functions via a generalized fractional integral, known as Katugampola fractional integral, which is the generalization of Riemann–Liouville fractional integral and Hadamard fractional integral. We show that through the Katugampola fractional integral we can find a Hermite–Hadamard inequality via the Riemann–Liouville fractional integral.

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1 Introduction
A function \( f : I \rightarrow \mathbb{R} \), where \( I \) is an interval of real numbers, is called convex if the following inequality holds:

\[
f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b)
\]

for all \( a, b \in I \) and \( t \in [0,1] \). Function \( f \) is called concave if \(-f\) is convex.

The Hermite–Hadamard inequality [4] for convex functions \( f : I \rightarrow \mathbb{R} \) on an interval of real line is defined as

\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{b - a} \int_{a}^{b} f(x) \, dx \leq \frac{f(a) + f(b)}{2},
\]

where \( a, b \in I \) with \( a < b \).

Since the Hermite–Hadamard inequality has many applications, many authors generalized this inequality. The Hermite–Hadamard inequality is also established for several kinds of convex functions. For more results and generalizations, see [2, 6, 10–14]. The Hermite–Hadamard inequality (2) is not only established for the classical integral but also for fractional integrals (e.g., see [1, 7, 18, 22]), for conformable fractional integrals (e.g., see [19, 21]), and recently for generalized fractional integrals (e.g., see [8, 9]).
**Definition 1.1** ([5]) Let $s \in (0, 1]$. A function $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$, where $\mathbb{R} = [0, \infty)$, is called $s$-convex function in the second sense if

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b)$$

for all $a, b \in I$ and $t \in [0, 1]$.

**Definition 1.2** ([3, 23]) A function $f : [0, b] \rightarrow \mathbb{R}$, with $b > 0$, is said to be $m$-convex if the following inequality holds:

$$f(ta + m(1-t)c) \leq tf(a) + m(1-t)f(c)$$

for all $a, c \in [0, b]$ and $t \in [0, 1]$ and for all $m \in [0, 1]$. $f$ is $m$-concave if $-f$ is $m$-convex.

**Definition 1.3** ([15]) Let $\alpha > 0$ with $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, and $1 < x < b$. The left- and right-hand side Riemann–Liouville fractional integrals of order $\alpha$ of function $f$ are given by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) \, dt,$$

and

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) \, dt,$$

respectively, where $\Gamma(\alpha)$ is the gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} \, dt$.

**Definition 1.4** ([16]) Let $\alpha > 0$ with $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, and $1 < x < b$. The left- and right-hand side Hadamard fractional integrals of order $\alpha$ of function $f$ are given by

$$H_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left( \ln \frac{x}{t} \right)^{\alpha-1} \frac{f(t)}{t} \, dt,$$

and

$$H_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \left( \ln \frac{t}{x} \right)^{\alpha-1} \frac{f(t)}{t} \, dt.$$

**Definition 1.5** ([9]) Let $[a, b] \subset \mathbb{R}$ be a finite interval. Then the left- and right-hand side Katugampola fractional integrals of order $\alpha(>0)$ of $f \in \mathcal{C}_{\rho}^\alpha (a, b)$ are defined by

$$\rho J_{a+}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^\rho - t^\rho)^{\alpha-1} t^{\rho-1} f(t) \, dt,$$

and

$$\rho J_{b-}^\alpha f(x) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b (t^\rho - x^\rho)^{\alpha-1} t^{\rho-1} f(t) \, dt,$$
with \( a < x < b \) and \( \rho > 0 \), where \( X^p_c(a, b) \) \((c \in \mathbb{R}, 1 \leq p \leq \infty)\) is the space of those complex-valued Lebesgue measurable functions \( f \) on \([a, b]\) for which \( \|f\|_{X^p_c} < \infty \), where the norm is defined by
\[
\|f\|_{X^p_c} = \left( \int_a^b \left| \frac{t^{\rho} f(t)}{t} \right|^p \frac{dt}{t} \right)^{1/p} < \infty
\]
for \( 1 \leq p < \infty \), \( c \in \mathbb{R} \) and for the case \( p = \infty \),
\[
\|f\|_{X^\infty_c} = \text{ess sup}_{a \leq t \leq b} \left| t^{\rho} f(t) \right|
\]
where ess sup stands for essential supremum.

**Theorem 1.6** ([9]) Let \( \alpha > 0 \) and \( \rho > 0 \). Then, for \( x > a \),
1. \( \lim_{\rho \to 1} I_\alpha^\rho a + f(x) = I_\alpha a + f(x) \)
2. \( \lim_{\rho \to 0^+} I_\alpha^\rho a + f(x) = H_\alpha a + f(x) \).

**Lemma 1.7** ([20]) For \( 0 < \alpha \leq 1 \) and \( 0 \leq a < b \), we have
\[
|a^\alpha - b^\alpha| \leq (b - a)a.
\]

We recall the classical beta functions:
\[
\beta(a, b) = \int_0^1 x^{a-1}(1-x)^{b-1} \, dx.
\]

We introduce the following generalization of beta function:
\[
\rho \gamma(a, b) = \int_0^1 (x^\rho)^{a-1}(1-x^\rho)^{b-1} x^{a-1} \, dx.
\]

Note that as \( \rho \to 1 \) then \( \rho \gamma(a, b) \to \beta(a, b) \).

In this paper, we give the Hermite–Hadamard type inequalities for \( s \)-convex functions and for \( m \)-convex functions via generalized fractional integral. Throughout the paper, \( X^p_c(a, b) \) \((c \in \mathbb{R}, 1 \leq p \leq \infty)\) is the space as defined in Definition 1.5 and \( L_1[a, b] \) stands for the space of Lebesgue integrable over the closed interval \([a, b]\) where \( a, b \) are some real numbers with \( a < b \).

## 2 Hermite–Hadamard type inequalities for \( s \)-convex function

In this section we give Hermite–Hadamard type inequalities for \( s \)-convex function.

**Theorem 2.1** Let \( \alpha > 0 \) and \( \rho > 0 \). Let \( f : [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in X^p_c(a^\rho, b^\rho) \). If \( f \) is also an \( s \)-convex function on \([a^\rho, b^\rho]\), then the following inequalities hold:
\[
2^{s-1} f \left( \frac{a^\rho + b^\rho}{2} \right) \leq \frac{\rho^s \Gamma(\alpha + 1)}{2(b^\rho + a^\rho)^s} \left[ \rho \left( \frac{a^\rho f(a^\rho) + \rho I^\rho_a f(a^\rho)}{2} \right) \right]
\]
\[
\leq \left[ \frac{\alpha}{\alpha + s} + \alpha \beta(\alpha, s + 1) \right] \frac{f(a^\rho) + f(b^\rho)}{2}, \tag{5}
\]

\( \leq \left[ \frac{\alpha}{\alpha + s} + \alpha \beta(\alpha, s + 1) \right] \frac{f(a^\rho) + f(b^\rho)}{2}, \tag{5}
\]
where the fractional integrals are considered for the function \( f(x^*) \) and evaluated at \( a \) and \( b \), respectively.

**Proof** Let \( t \in [0,1] \). Consider \( x, y \in [a, b], a \geq 0 \), defined by \( x^* = t^a a^o + (1 - t^o) b^o, \ y^* = t^b b^o + (1 - t^o) a^o \). Since \( f \) is an \( s \)-convex function on \([a^o, b^o]\), we have

\[
f\left(\frac{x^* + y^*}{2}\right) \leq \frac{f(x^*) + f(y^*)}{2^s}.
\]

Then we have

\[
2f\left(\frac{a^o + b^o}{2}\right) \leq f\left(t^a a^o + (1 - t^o) b^o\right) + f\left(t^b b^o + (1 - t^o) a^o\right).
\]

Multiplying both sides of (6) by \( t^{a^o - 1} \), \( a > 0 \) and then integrating the resulting inequality with respect to \( t \) over \([0,1]\), we obtain

\[
\frac{2s}{a^o} f\left(\frac{a^o + b^o}{2}\right) \leq \int_0^1 t^{a^o - 1} f\left(t^a a^o + (1 - t^o) b^o\right) dt + \int_0^1 t^{a^o - 1} f\left(t^b b^o + (1 - t^o) a^o\right) dt
\]

\[
= \int_a^b \frac{b^o - a^o}{b^o - a^o} f(x^*) \frac{x^o - a^o}{a^o - b^o} dx
\]

\[
+ \int_a^b \frac{b^o - a^o}{b^o - a^o} f(y^*) \frac{y^o - a^o}{b^o - a^o} dy
\]

\[
= \frac{\rho^{a^o - 1} \Gamma(a)}{(b^o + a^o)^a} \left[ \rho_{a^o} f(b^o) + \rho^{a^o} f(a^o) \right].
\]

This establishes the first inequality. For the proof of the second inequality in (5), we first observe that for an \( s \)-convex function \( f \), we have

\[
f\left(t^a a^o + (1 - t^o) b^o\right) \leq (t^o)^s f\left(a^o\right) + (1 - t^o)^s f\left(b^o\right)
\]

and

\[
f\left(t^b b^o + (1 - t^o) a^o\right) \leq (t^o)^s f\left(b^o\right) + (1 - t^o)^s f\left(a^o\right).
\]

By adding these inequalities, we get

\[
f\left(t^a a^o + (1 - t^o) b^o\right) + f\left(t^b b^o + (1 - t^o) a^o\right) \leq (t^o)^s + (1 - t^o)^s [f\left(a^o\right) + f\left(b^o\right)]
\]

Multiplying both sides of (8) by \( t^{a^o - 1} \), \( a > 0 \) and then integrating the resulting inequality with respect to \( t \) over \([0,1]\), we obtain

\[
\frac{\rho^{a^o - 1} \Gamma(a)}{(b^o + a^o)^a} \left[ \rho_{a^o} f(b^o) + \rho^{a^o} f(a^o) \right]
\]

\[
\leq \int_0^1 t^{a^o - 1} \left[(t^o)^s + (1 - t^o)^s \right][f\left(a^o\right) + f\left(b^o\right)] dt.
\]
Since
\[ \int_0^1 t^{\alpha p + s p - 1} \, dt = \frac{1}{\rho (\alpha + s)}, \]
and by choosing the change of variable \( t^\rho = z \), we have
\[ \int_0^1 t^{\alpha p - 1} (1 - t^\rho)^s \, dt = \frac{\beta(\alpha, s + 1)}{\rho}. \]
Thus (9) becomes
\[ \frac{\rho^{\alpha - 1} \Gamma(\alpha)}{(b^\rho + a^\rho)^\rho} \left[ r_{a^\rho}^\rho f'(b^\rho) + \rho r_{b^\rho}^\rho f'(a^\rho) \right] \leq \frac{1}{\rho} \left[ \frac{1}{\alpha + s} + \beta(\alpha, s + 1) \right] \left( f(a^\rho) + f(b^\rho) \right). \]
Thus (7) and (10) give (5). \( \Box \)

**Remark 2.2** By letting \( \rho \to 1 \) in (5) of Theorem 2.1, we get Theorem 3 of [22].

**Theorem 2.3** Let \( \alpha > 0 \) and \( \rho > 0 \). Let \( f : [a^\rho, b^\rho] \subset \mathbb{R} \to \mathbb{R} \) be a differentiable mapping on \([a^\rho, b^\rho]\) with \( 0 \leq a < b \). If \(|f'|\) is \( s \)-convex on \([a^\rho, b^\rho]\), then the following inequality holds:
\[ \left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^{\alpha - 1} \Gamma(\alpha)}{(b^\rho + a^\rho)^\rho} \left[ r_{a^\rho}^\rho f'(b^\rho) + \rho r_{b^\rho}^\rho f'(a^\rho) \right] \right| \leq \frac{b^\rho - a^\rho}{2} \left[ \frac{1}{\alpha + s + 1} + \beta(\alpha + 1, s + 1) \right] \left( |f'(a^\rho)| + |f'(b^\rho)| \right). \]

**Proof** From (7) one can have
\[ \frac{\rho^{\alpha - 1} \Gamma(\alpha)}{(b^\rho + a^\rho)^\rho} \left[ r_{a^\rho}^\rho f'(b^\rho) + \rho r_{b^\rho}^\rho f'(a^\rho) \right] = \int_0^1 t^{\alpha p - 1} f(t^\rho a^\rho + (1 - t^\rho)b^\rho) \, dt + \int_0^1 t^{\alpha p - 1} f(t^\rho b^\rho + (1 - t^\rho)a^\rho) \, dt. \]
Integrating by parts, we get
\[ \frac{f(a^\rho) + f(b^\rho)}{2} = \frac{\rho^{\alpha - 1} \Gamma(\alpha)}{(b^\rho + a^\rho)^\rho} \left[ r_{a^\rho}^\rho f'(b^\rho) + \rho r_{b^\rho}^\rho f'(a^\rho) \right] = \frac{b^\rho - a^\rho}{\alpha} \int_0^1 t^{\alpha(s+1)-1} \left[ f(t^\rho b^\rho + (1 - t^\rho)a^\rho) - f(t^\rho a^\rho + (1 - t^\rho)b^\rho) \right] \, dt. \]
By using the triangle inequality and \( s \)-convexity of \(|f'|\) and the change of variable \( t^\rho = z \), we obtain
\[ \left| \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^{\alpha - 1} \Gamma(\alpha)}{(b^\rho + a^\rho)^\rho} \left[ r_{a^\rho}^\rho f'(b^\rho) + \rho r_{b^\rho}^\rho f'(a^\rho) \right] \right| \leq \frac{b^\rho - a^\rho}{\alpha} \int_0^1 t^{\alpha(s+1)-1} \left| f(t^\rho b^\rho + (1 - t^\rho)a^\rho) - f(t^\rho a^\rho + (1 - t^\rho)b^\rho) \right| \, dt \]
\[ \leq \frac{b^\rho - a^\rho}{\alpha} \int_0^1 t^{\alpha(s+1)-1} \left( |f(t^\rho b^\rho + (1 - t^\rho)a^\rho)| + |f(t^\rho a^\rho + (1 - t^\rho)b^\rho)| \right) \, dt \]
\[
\frac{b^\rho - a^\rho}{\alpha} \int_0^1 t^\rho(\alpha + 1) - 1 \left[ t^\rho \left( |f'(b^\rho)| + (1 - t^\rho)^\rho |f'(a^\rho)| \right) + (t^\rho)^\rho |f'(a^\rho)| + (1 - t^\rho)^\rho |f'(b^\rho)| \right] dt \\
= \frac{b^\rho - a^\rho}{\alpha} \int_0^1 t^\rho(\alpha + 1) - 1 \left[ (t^\rho)^\rho + (1 - t^\rho)^\rho \right] [f'(a^\rho)] + [f'(b^\rho)]] dt \\
= \frac{b^\rho - a^\rho}{\alpha \rho} \left[ \frac{1}{\alpha + s + 1} + \beta(\alpha + 1, s + 1) \right] [f'(a^\rho)] + [f'(b^\rho)]].
\]

(14)

Corollary 2.4 Under the same assumptions of Theorem 2.3.

1. If \( \rho = 1 \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\rho + 1)^\alpha} \left[ I^\rho_{1, a} f(b) + I^\rho_{1, b} f(a) \right] \right| \\
\leq \frac{b - a}{2} \left[ \frac{1}{\alpha + s + 1} + \beta(\alpha + 1, s + 1) \right] \left( |f'(a)| + |f'(b)| \right).
\]

(15)

2. If \( \rho = s = 1 \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\rho + 1)^\alpha} \left[ I^\rho_{1, a} f(b) + I^\rho_{1, b} f(a) \right] \right| \\
\leq \frac{b - a}{2} \left[ \frac{1}{\alpha + 2} + \beta(\alpha + 1, 2) \right] \left( |f'(a)| + |f'(b)| \right).
\]

(16)

3. If \( \rho = s = \alpha = 1 \), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b + a} \int_a^b f(x) \, dx \right| \leq \frac{b - a}{4} \left( |f'(a)| + |f'(b)| \right).
\]

(17)

In order to prove our further results, we need the following lemma.

Lemma 2.5 Let \( \alpha > 0 \) and \( \rho > 0 \). Let \( f : [a^\rho, b^\rho] \subset \mathbb{R}_+ \rightarrow \mathbb{R} \) be a differentiable mapping on \( (a^\rho, b^\rho) \) with \( 0 \leq a < b \). Then the following equality holds if the fractional integrals exist:

\[
f(a^\rho) + f(b^\rho) = \frac{\rho^\rho \Gamma(\alpha + 1)}{2(\rho + 1)^\alpha} \left[ I^\rho_{1, a} f(b^\rho) + I^\rho_{1, b} f(a^\rho) \right] \\
= \frac{\rho(b^\rho - a^\rho)}{2} \int_0^1 \left[ (1 - t^\rho)^\alpha - (t^\rho)^\alpha \right] t^{\rho - 1} f(t^\rho a^\rho + (1 - t^\rho) b^\rho) \, dt.
\]

(18)

Proof By using the similar arguments as in the proof of Lemma 2 in [18]. First consider

\[
\int_0^1 (1 - t^\rho)^\alpha t^{\rho - 1} f(t^\rho a^\rho + (1 - t^\rho) b^\rho) \, dt \\
= \frac{(1 - t^\rho)^\alpha f(t^\rho a^\rho + (1 - t^\rho) b^\rho)}{\rho(a^\rho - b^\rho)} \bigg|_0^1 \\
+ \frac{\alpha}{a^\rho - b^\rho} \int_0^1 (1 - t^\rho)^{\alpha - 1} t^{\rho - 1} f(t^\rho a^\rho + (1 - t^\rho) b^\rho) \, dt
\]
Let \( \rho(\rho - a^\rho) \geq 1 \), throughout all other results we denote

\[
I_f(\alpha, \rho, a, b) = \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^{\alpha-1}\Gamma(\alpha + 1)}{(b^\rho - a^\rho)^{\alpha+1}} \cdot \rho I_{x}^{\alpha} f(x^\rho) \bigg|_{x=a}.
\]

Similarly, we can show that

\[
\int_{0}^{1} t^{\rho\alpha} \cdot t^{\rho-1} f'(t^\rho a^\rho + (1-t^\rho)b^\rho) \, dt
\]

\[
= -\frac{f(a^\rho)}{\rho(\rho - a^\rho)} + \frac{\rho^{\alpha-1}\Gamma(\alpha + 1)}{(b^\rho - a^\rho)^{\alpha+1}} \cdot \rho I_{x}^{\alpha} f(x^\rho) \bigg|_{x=b}.
\]

Thus from (19) and (20) we get (18).

\[\square\]

Remark 2.6 By taking \( \rho = 1 \) in (18) of Lemma 2.5, we get Lemma 2 in [17].

Throughout all other results we denote

\[
I_f(\alpha, \rho, a, b) = \frac{f(a^\rho) + f(b^\rho)}{2} - \frac{\rho^{\alpha-1}\Gamma(\alpha + 1)}{(b^\rho - a^\rho)^{\alpha+1}} \cdot \rho I_{x}^{\alpha} f(x^\rho) \bigg|_{x=a}.
\]

Theorem 2.7 Let \( \alpha > 0 \) and \( \rho > 0 \). Let \( f : [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R} \) be a differentiable mapping on \( [a^\rho, b^\rho] \) such that \( f' \in L_1[a, b] \) with \( 0 \leq a < b \). If \( |f'|^q \) is \( s \)-convex on \( [a^\rho, b^\rho] \) for some fixed \( q \geq 1 \), then the following inequality holds:

\[
|I_f(\alpha, \rho, a, b)| \leq \frac{\rho(b^\rho - a^\rho)}{2} \left( \frac{1}{\rho(\alpha + 1)} \right)^{1-1/q} \times \left( \frac{\rho \gamma(s + 1, \alpha + 1) + \frac{1}{\rho(\alpha + s + 1)}}{s} \right)^{|q|} \left( \frac{\rho \gamma(1, \alpha + s + 1)}{b^\rho \gamma(\alpha + 1, s + 1)} \right)^{1/q}.
\]

Proof Using Lemma 2.5 and the power mean inequality and \( s \)-convexity of \( |f'|^q \), we obtain

\[
|I_f(\alpha, \rho, a, b)| = \frac{\rho(b^\rho - a^\rho)}{2} \left( \frac{1}{\rho(\alpha + 1)} \right)^{1-1/q} \times \left( \frac{\rho \gamma(s + 1, \alpha + 1) + \frac{1}{\rho(\alpha + s + 1)}}{s} \right)^{|q|} \left( \frac{\rho \gamma(1, \alpha + s + 1)}{b^\rho \gamma(\alpha + 1, s + 1)} \right)^{1/q}.
\]
\[
\times \left( \left( \beta (s+1, \alpha +1) + \frac{1}{\alpha + s + 1} \right) \right) \frac{f''(a)}{q} \\
+ \left( \beta (1, \alpha + s + 1) + \beta (\alpha + 1, s + 1) \right) \frac{f''(b)}{q} \right)^{1/q}.
\]

(22)

Hence the proof is completed. \(\Box\)

**Corollary 2.8** Under the similar conditions of Theorem 2.7.

1. If \(\rho = 1\), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma'(a+1)}{2(b+a)^\alpha} \left[ f'_\alpha(b) + f'_\beta(a) \right] \right|
\leq \frac{(b-a)}{2} \left( \frac{1}{\alpha + 1} \right) \times \left( \left( \beta (s+1, \alpha +1) + \frac{1}{\alpha + s + 1} \right) \frac{f''(a)}{q} \\
+ \left( \beta (1, \alpha + s + 1) + \beta (\alpha + 1, s + 1) \right) \frac{f''(b)}{q} \right)^{1/q}.
\]

2. If \(\rho = s = 1\), then

\[
\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma'(a+1)}{2(b+a)^\alpha} \left[ f'_\alpha(b) + f'_\beta(a) \right] \right|
\leq \frac{(b-a)}{2} \left( \frac{1}{\alpha + 1} \right) \times \left( \left( \beta (2, \alpha +1) + \frac{1}{\alpha + 2} \right) \frac{f''(a)}{q} \\
+ \left( \beta (1, \alpha + 2) + \beta (\alpha + 1, 2) \right) \frac{f''(b)}{q} \right)^{1/q}.
\]

3. If \(\rho = s = \alpha = 1\), then

\[
\left| \frac{f(a) + f(b)}{2} - \int_a^b f(x) \, dx \right| \leq \frac{(b-a)}{2} \left( \frac{1}{\alpha + 1} \right) \times \left( \frac{f''(a)}{q} + \frac{f''(b)}{q} \right)^{1/q}.
\]

**Theorem 2.9** Let \(\alpha > 0\) and \(\rho > 0\). Let \(f : [a^\rho, b^\rho] \subset \mathbb{R}_+ \rightarrow \mathbb{R}\) be a differentiable mapping on \([a^\rho, b^\rho]\) such that \(f^\prime \in L^1[a, b]\) with \(0 \leq a < b\). If \(|f'|^q\) is \(s\)-convex on \([a^\rho, b^\rho]\) for some fixed \(q \geq 1\), then the following inequality holds:

\[
\left\vert I_f(\alpha, \rho, a, b) \right\vert \leq \frac{\rho^{\frac{1}{2}} (b^\rho - a^\rho)}{2} \left[ \left( \beta (s+1, \alpha +1) + \frac{1}{\alpha + s + 1} \right) \left( |f''(a)|^q + |f''(b)|^q \right) \right]^{1/q}.
\]

(23)

**Proof** Using Lemma 2.5, the property of modulus, the power mean inequality, and the fact that \(|f'|^q\) is an \(s\)-convex function, we have

\[
\left\vert I_f(\alpha, \rho, a, b) \right\vert \leq \frac{\rho (b^\rho - a^\rho)}{2} \left( \int_0^1 \left( (1-t^\rho)^\alpha - (t^\rho)^\alpha \right) t^{\rho-1} |f''(t^\rho a^\rho + (1-t^\rho)b^\rho)| \, dt \right)
\leq \frac{\rho (b^\rho - a^\rho)}{2} \left( \int_0^1 t^{\rho-1} \, dt \right)^{1/q}.
\]
\[ \frac{1}{2} \rho (b^\rho - a^\rho) \left( \int_0^1 \left( (1 - t^\rho)^a + (t^\rho)^a \right) ((1 - t^\rho)^{\alpha+s+1} + (t^\rho)^{\alpha+s+1}) \, dt \right)^{1/q} \]

By using the change of variable \( t^\rho = z \), we get

\[ A = \int_0^1 \left( (1 - t^\rho)^a (t^\rho)^s (t^\rho)^s \right) \, dt = \beta(s + 1, \alpha + 1) + \frac{1}{\alpha + s + 1} \]

and

\[ B = \int_0^1 \left( (1 - t^\rho)^a (1 - t^\rho)^s (t^\rho)^s \right) \, dt = \beta(\alpha + 1, s + 1) + \frac{1}{\alpha + s + 1}. \]

Thus substituting the values of \( A \) and \( B \) in (24) and applying the fact that \( \beta(a, b) = \beta(b, a) \), we get the desired result.

**Corollary 2.10** Under the similar conditions of Theorem 2.7.

1. If \( \rho = 1 \), then

\[ \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b + a)^{\alpha}} \left[ f'_a(b) + f'_b(a) \right] \right| \leq \frac{(b - a)}{2} \left( \beta(s + 1, \alpha + 1) + \frac{1}{\alpha + s + 1} \right) \left[ |f'(a)|^q + |f'(b)|^q \right]^{1/q}. \]

2. If \( \rho = s = 1 \), then

\[ \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b + a)^{\alpha}} \left[ f'_a(b) + f'_b(a) \right] \right| \leq \frac{(b - a)}{2} \left( \beta(2, \alpha + 1) + \frac{1}{\alpha + 2} \right) \left[ |f'(a)|^q + |f'(b)|^q \right]^{1/q}. \]

3. If \( \rho = s = \alpha = 1 \), then

\[ \left| \frac{f(a) + f(b)}{2} - \frac{1}{b + a} \int_a^b f(x) \, dx \right| \leq \frac{(b - a)}{2} \left( \frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{1/q}. \]
3 Hermite–Hadamard type inequalities for m-convex function

In this section we give Hermite–Hadamard type inequalities for m-convex function.

Theorem 3.1 Let \( \alpha > 0 \) and \( \rho > 0 \). Let \( f : [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in X_{\rho}(a^\rho, b^\rho) \). If \( f \) is also an m-convex function on \([a^\rho, b^\rho]\), then the following inequalities hold:

\[
\begin{align*}
 f \left( \frac{m^\rho(a^\rho + b^\rho)}{2} \right) & \leq \frac{\rho^\rho \Gamma(\alpha + 1)}{2((mb)^\rho - (ma)^\rho)a}\rho^{\alpha} \int_{ma}^{mb} f((mb)^\rho) + \frac{m^\rho \rho^\rho \Gamma(\alpha + 1)}{2(b^\rho - a^\rho)a}\rho^{\alpha} \int_{ma}^{mb} f(a^\rho) \\
 & \leq \frac{m^\rho}{2} (f(a^\rho) + f(b^\rho)).
\end{align*}
\]

(25)

Proof Since \( f \) is m-convex, we have

\[
\begin{align*}
 f \left( \frac{x^\rho + m^\rho y^\rho}{2} \right) & \leq \frac{f(x^\rho) + m^\rho f(y^\rho)}{2}.
\end{align*}
\]

Let \( x^\rho = m^\rho t^\alpha a^\rho + m^\rho (1 - t^\alpha) b^\rho, \ y^\rho = t^\alpha b^\rho + (1 - t^\alpha)a^\rho \) with \( t \in [0, 1] \). Then we obtain

\[
\begin{align*}
 f \left( \frac{m^\rho(a^\rho + b^\rho)}{2} \right) & \leq \frac{f(m^\rho t^\alpha a^\rho + m^\rho (1 - t^\alpha) b^\rho) + m^\rho f(t^\alpha b^\rho + (1 - t^\alpha)a^\rho)}{2}.
\end{align*}
\]

(26)

Multiplying both sides of (26) by \( t^{\rho - 1}, \alpha > 0 \) and then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
\begin{align*}
 & \frac{2}{\rho \alpha} \int_{0}^{1} t^{\rho - 1} f \left( \frac{m^\rho(a^\rho + b^\rho)}{2} \right) dt \\
& \leq \int_{0}^{1} t^{\rho - 1} f \left( m^\rho t^\alpha a^\rho + m^\rho (1 - t^\alpha) b^\rho \right) dt + m^\rho \int_{0}^{1} t^{\rho - 1} f \left( t^\alpha b^\rho + (1 - t^\alpha)a^\rho \right) dt \\
& = \int_{mb}^{ma} \left( x^\rho - (mb)^\rho \right)^{\alpha - 1} \frac{x^{\rho - 1} dx}{(ma)^\rho - (mb)^\rho} \\
& + m^\rho \int_{x}^{b} \left( y^\rho - a^\rho \right)^{\alpha - 1} \frac{y^{\rho - 1} dy}{b^\rho - a^\rho} \\
& = \frac{\rho^{\rho - 1} \Gamma(\alpha)}{(mb)^\rho - (ma)^\rho a}\rho^{\alpha} \int_{ma}^{mb} f((mb)^\rho) + \frac{m^\rho \rho^{\rho - 1} \Gamma(\alpha)}{(b^\rho - a^\rho)a}\rho^{\alpha} \int_{ma}^{mb} f(a^\rho).
\end{align*}
\]

(27)

Now by multiplying both sides of (27) by \( \frac{\alpha^\rho}{2} \), we get the first inequality of (25). For the second inequality, using m-convexity of \( f \), we have

\[
\begin{align*}
 f \left( m^\rho t^\alpha a^\rho + m^\rho (1 - t^\alpha) b^\rho \right) + m^\rho f \left( (1 - t^\alpha)a^\rho + t^\alpha b^\rho \right) \leq m^\rho f(a^\rho) + f(b^\rho).
\end{align*}
\]

(28)

Multiplying both sides of (28) by \( t^{\rho - 1}, \alpha > 0 \) and then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain

\[
\begin{align*}
 & \frac{\rho^{\rho - 1} \Gamma(\alpha)}{(mb)^\rho - (ma)^\rho a}\rho^{\alpha} \int_{ma}^{mb} f((mb)^\rho) + \frac{m^\rho \rho^{\rho - 1} \Gamma(\alpha)}{(b^\rho - a^\rho)a}\rho^{\alpha} \int_{ma}^{mb} f(a^\rho) \\
& \leq \frac{m^\rho}{\rho \alpha} (f(a^\rho) + f(b^\rho)).
\end{align*}
\]

(29)

Now, by multiplying both sides of (29) by \( \frac{\alpha^\rho}{2} \), we get the second inequality of (25). \( \square \)
Corollary 3.2 Under the assumptions of Theorem 3.1, we have
1. For \( \rho = 1 \), then
   \[
   f \left( \frac{m(a + b)}{2} \right) 
   \leq \frac{\Gamma(\alpha + 1)}{2(mb - ma)^\alpha} \int_{ma}^{mb} f(mb) + \frac{m\Gamma(\alpha + 1)}{2(b - a)^\alpha} \int_{a}^{b} f(a) 
   \leq \frac{m}{2} (f(a) + f(b)).
   \]

2. For \( \rho = \alpha = 1 \), then
   \[
   f \left( \frac{m(a + b)}{2} \right) \leq \frac{1}{2(mb - ma)} \int_{ma}^{mb} f(x) \, dx + \frac{m}{2(b - a)} \int_{a}^{b} f(x) \, dx 
   \leq \frac{m}{2} (f(a) + f(b)).
   \]

Remark 3.3 If we take \( m = 1 \) in (31) of Corollary (3.2)(2), then we get (2).

Theorem 3.4 Let \( \alpha > 0 \) and \( \rho > 0 \). Let \( f : [a^\rho, b^\rho] \subset \mathbb{R}_+ \to \mathbb{R} \) be a positive function with \( 0 \leq a < b \) and \( f \in X^\rho_1(a^\rho, b^\rho) \). If \( f \) is also an \( m \)-convex function on \( [a^\rho, b^\rho] \). Let \( F(x^\rho, y^\rho)_\nu : [0, 1] \to \mathbb{R} \) be defined as
   \[
   F(x^\rho, y^\rho)_\nu = \frac{1}{2} \left[ f(t^\nu x^\rho + m^\rho (1 - t^\nu) y^\rho) + f((1 - t^\nu)x^\rho + m^\rho t^\nu y^\rho) \right].
   \]

Then we have
   \[
   \frac{1}{(b^\rho - a^\rho)\alpha} \int_{a}^{b} (b^\rho - u^\rho)^{\alpha-1} u^\rho - 1 f(u^\rho, \frac{a^\rho + b^\rho}{2}, (b^\rho - a^\rho)^{\rho\alpha} f(b^\rho) + \frac{m}{2\alpha^\rho} f(\frac{a^\rho + b^\rho}{2}).
   \]

Proof Since \( f \) is an \( m \)-convex function, we have
   \[
   F(x^\rho, y^\rho)_\nu \leq \frac{1}{2} \left[ t^\nu f(x^\rho) + m^\rho (1 - t^\nu)f(y^\rho) + (1 - t^\nu)f(x^\rho) + m^\rho t^\nu f(y^\rho) \right] 
   = \frac{1}{2} \left[ f(x^\rho) + m^\rho f(y^\rho) \right],
   \]

and also
   \[
   F \left( x^\rho, \frac{a^\rho + b^\rho}{2} \right)_\nu \leq \frac{1}{2} \left[ f(x^\rho) + m^\rho f \left( \frac{a^\rho + b^\rho}{2} \right) \right].
   \]

Take \( x^\rho = t^\nu a^\rho + (1 - t^\nu)b^\rho \), we have
   \[
   F \left( t^\nu a^\rho + (1 - t^\nu)b^\rho, \frac{a^\rho + b^\rho}{2} \right)_\nu \leq \frac{1}{2} \left[ f \left( t^\nu a^\rho + (1 - t^\nu)b^\rho \right) + m^\rho f \left( \frac{a^\rho + b^\rho}{2} \right) \right].
   \]
Multiplying both sides of (33) by \( t^{\rho-1} \), \( \alpha > 0 \) and then integrating the resulting inequality with respect to \( t \) over \([0, 1]\), we obtain
\[
\int_0^1 t^{\rho-1} F\left(t^{\rho} a^{\rho} + (1 - t^{\rho}) b^{\rho}, \frac{a^{\rho} + b^{\rho}}{2}\right) dt \\
\leq \frac{1}{2} \int_0^1 t^{\rho-1} \left[f(t^{\rho} a^{\rho} + (1 - t^{\rho}) b^{\rho}) + \frac{m^\rho f\left(\frac{a^{\rho} + b^{\rho}}{2}\right)}{2}\right] dt. \tag{34}
\]
Then, by the change of variable \( u^{\rho} = t^\rho a^\rho + (1 - t^\rho) b^\rho \), we get the desired inequality (32). □

Remark 3.5 By taking \( \rho = 1 \) in (32) of Theorem 3.4, we get Theorem 6 in [22].

4 Applications to special means

In this section, we consider some applications to our results. Here we consider the following means:

(1) The arithmetic mean:
\[
A(a, b) = \frac{a + b}{2}; \quad a, b \in \mathbb{R}.
\]

(2) The logarithmic mean:
\[
L(a, b) = \frac{\ln |b| - \ln |a|}{b - a}; \quad a, b \in \mathbb{R}, |a| \neq |b|, a, b \neq 0.
\]

(3) The generalized log mean:
\[
L_n(a, b) = \left[\frac{b^{n+1} - a^{n+1}}{(n+1)(b - a)}\right]^{1/n}; \quad a, b \in \mathbb{R}, n \in \mathbb{Z} \setminus \{1, 0\}, a, b \neq 0.
\]

Proposition 4.1 Let \( a, b \in \mathbb{R}, a < b, 0 \notin [a, b], \) and \( n \in \mathbb{Z}, |n| \geq 2 \), then
\[
\left|A(a^n, b^n) - \frac{b - a}{b + a} L_n(a, b)\right| \leq \frac{|n|(b - a)}{2} A\left(|a|^{n-1}, |b|^{n-1}\right). \tag{35}
\]

Proof By taking \( f(x) = x^n \) in Corollary 2.4(3), we get the required result. □

Proposition 4.2 Let \( a, b \in \mathbb{R}, a < b, 0 \notin [a, b], \) and \( n \in \mathbb{Z}, |n| \geq 2 \). Then, for \( q \geq 1 \), we have
\[
\left|A(a^n, b^n) - \frac{b - a}{b + a} L_n(a, b)\right| \leq \frac{|n|(b - a)}{2^{2-1/q}} A^{1/q}\left(|a|^{q(n-1)}, |b|^{q(n-1)}\right). \tag{36}
\]

Proof By taking \( f(x) = x^n \) in Corollary 2.8(3), we get the required result. □

Proposition 4.3 Let \( a, b \in \mathbb{R}, a < b, 0 \notin [a, b], \) and \( n \in \mathbb{Z}, |n| \geq 2 \). Then, for \( q \geq 1 \), we have
\[
\left|A(a^n, b^n) - \frac{b - a}{b + a} L_n(a, b)\right| \leq \frac{|n|(b - a)}{2} A^{1/q}\left(|a|^{q(n-1)}, |b|^{q(n-1)}\right). \tag{37}
\]

Proof By taking \( f(x) = x^n \) in Corollary 2.10(3), we get the required result. □
Proposition 4.4 Let \( a, b \in \mathbb{R}, a < b, 0 \notin [a, b], \) and \( n \in \mathbb{Z}, m \in [0, 1], \) then we have
\[
f(mA(a, b)) \leq \frac{1}{2} L^m_n(a, mb) + \frac{m}{2} L^n_m(a, b) \leq mA(a^n, b^n). \tag{38}
\]

Proof By taking \( f(x) = x^n \) in Corollary 3.2(2), we get the required result. \( \square \)

Proposition 4.5 Let \( a, b \in \mathbb{R}, a < b, 0 \notin [a, b], \) then
\[
A(a^{-1}, b^{-1}) - \frac{b-a}{b+a} L(a, b) \leq \frac{b-a}{2} A(|a|^{-2}, |b|^{-2}). \tag{39}
\]

Proof By taking \( f(x) = \frac{1}{x} \) in Corollary 2.4(3), we get the required result. \( \square \)

Proposition 4.6 Let \( a, b \in \mathbb{R}, a < b, 0 \notin [a, b] \) and \( m \in [0, 1], \) then we have
\[
A(a^{-1}, b^{-1}) - \frac{b-a}{b+a} L(a, b) \leq \frac{b-a}{2^2-1} A^1(1, |a|^{-2}, |b|^{-2}). \tag{40}
\]

Proof By taking \( f(x) = \frac{1}{x} \) in Corollary 2.8(3), we get the required result. \( \square \)

Proposition 4.7 Let \( a, b \in \mathbb{R}, a < b, 0 \notin [a, b] \) and \( m \in [0, 1], \) then we have
\[
A(a^{-1}, b^{-1}) - \frac{b-a}{b+a} L(a, b) \leq \frac{b-a}{2} A^1(|a|^{-2q}, |b|^{-2q}). \tag{41}
\]

Proof By taking \( f(x) = \frac{1}{x} \) in Corollary 2.10(3), we get the required result. \( \square \)

Proposition 4.8 Let \( a, b \in \mathbb{R}, a < b, 0 \notin [a, b], \) and \( m \in [0, 1], \) then we have
\[
f(mA(a^{-1}, b^{-1})) \leq \frac{1}{2} L(ma, mb) + \frac{m}{2} L(a, b) \leq mA(a^{-1}, b^{-1}). \tag{42}
\]

Proof By taking \( f(x) = \frac{1}{x} \) in Corollary 3.2(2), we get the required result. \( \square \)

5 Conclusion
In Sect. 2, some Hermite–Hadamard type inequalities for \( s \)-convex functions in a generalized fractional form were obtained. In Corollaries 2.4, 2.8, and 2.10, we obtained some new results related to \( s \)-convex functions, convex functions via Riemann–Liouville fractional integrals and via classical integrals. In Sect. 3, we established a Hermite–Hadamard type inequality for \( m \)-convex functions in generalized fractional integrals. In Corollary 3.2, a new Hermite–Hadamard type inequality for \( m \)-convex functions via Riemann–Liouville fractional integrals and via classical integrals was proved.

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References
1. Chen, F.: Extension of the Hermite–Hadamard inequality for harmonically convex functions via fractional integrals. Appl. Math. Comput. 268, 121–128 (2015)
2. Dragomir, S.S.: Hermite–Hadamard’s type inequality for operator convex functions. Appl. Math. Comput. 218(3), 766–772 (2011)
3. Dragomir, S.S., Toader, G.H.: Some inequalities for m-convex functions. Stud. Univ. Babeş–Bolyai, Math. 38, 21–28 (1993)
4. Hadamard, J.: Etude sur les propriétés des fonctions entières et en particulier d’une fonction considérée par Riemann. J. Math. Pures Appl. 58, 171–215 (1893)
5. Hudzik, H., Maligranda, L.: Some remarks on s-convex functions. Aequ. Math. 48(1), 100–111 (1994)
6. Ilican, I.: Hermite–Hadamard type inequalities for harmonically convex functions. Hacet. J. Math. Stat. 43(6), 935–942 (2014)
7. Ilican, I., Wu, S.: Hermite–Hadamard type inequalities for harmonically convex functions via fractional integrals. Appl. Math. Comput. 237, 237–244 (2014)
8. Jleli, M., O’Regan, D., Samet, B.: On Hermite–Hadamard type inequalities via generalized fractional integrals. Turk. J. Math. 40, 1221–1230 (2016)
9. Katugampola, U.N.: New approach to generalized fractional derivatives. Bull. Math. Anal. Appl. 6(4), 1–15 (2014)
10. Latif, M.A., Shoaib, M.: Hermite–Hadamard type integral inequalities for differentiable m-preinvex and α, m-preinvex functions. J. Egypt. Math. Soc. 23(2), 236–241 (2015)
11. Niculescu, C.P.: The Hermite–Hadamard inequality for log-preinvex functions. Nonlinear Anal. 75(2), 662–669 (2012)
12. Noor, M.A.: On Hadamard integral inequalities involving two log-convex functions. J. Inequal. Pure Appl. Math. 8(3), 1–14 (2007)
13. Noor, M.A.: On Hadamard integral inequalities for product of two preinvex functions. Nonlinear Anal. Forum 14, 167–173 (2009)
14. Ozdemir, M.E., Avci, M., Kavurmaci, H.: Hermite–Hadamard-type inequalities via α, m-convexity. Comput. Math. Appl. 61(9), 2614–2620 (2011)
15. Podlubny, I.: Fractional Differential Equations: Mathematics in Science and Engineering. Academic Press, San Diego (1999)
16. Samko, S.G., Kilbas, A.A., Marichev, O.I.: Fractional Integrals and Derivatives, Theory and Applications. Gordon & Breach, Amsterdam (1993)
17. Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, H.: Hermite–Hadamard inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. 57, 2403–2407 (2013)
18. Sarikaya, M.Z., Set, E., Yaldiz, H., Basak, N.: Hermite–Hadamard’s inequalities for fractional integrals and related fractional inequalities. Math. Comput. Model. 57(9), 2403–2407 (2013)
19. Set, E., Akindemir, A.O., Mumcu, I.: The Hermite–Hadamard’s inequality and its extensions for conformable fractional integrals of any order α > 0 (2016). Preprint
20. Set, E., Ozdemir, M.E., Dragomir, S.S.: On Hadamard-type inequalities involving several kinds of convexity. J. Inequal. Appl. 2010, 12 (2010)
21. Set, E., Sarikaya, M.Z., Gozpinar, A.: Some Hermite–Hadamard type inequalities for convex functions via conformable fractional integrals and related inequalities (2016). Preprint
22. Set, E., Sarikaya, M.Z., Ozdemir, M.E., Yaldirim, H.: The Hermite–Hadamard’s inequality for some convex functions via fractional integrals and related results. J. Appl. Math. Stat. Inform. 10(2), 69–83 (2014)
23. Toader, G.H.: Some generalisations of the convexity. In: Proc. Colloq. Approx. Optim, pp. 329–338. Cluj-Napoca, Romania (1984)