Pseudodifferential Operators on Variable Lebesgue Spaces

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To Professor Vladimir Rabinovich on the occasion of his 70th birthday

Abstract. Let $\mathcal{M}(\mathbb{R}^n)$ be the class of bounded away from one and infinity functions $p : \mathbb{R}^n \to [1, \infty]$ such that the Hardy-Littlewood maximal operator is bounded on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$. We show that if $a$ belongs to the Hörmander class $S^\alpha_{\rho,d}$ with $0 < \rho \leq 1$, $0 \leq \delta < 1$, then the pseudodifferential operator $Op(a)$ is bounded on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ provided that $p \in \mathcal{M}(\mathbb{R}^n)$. Let $\mathcal{M}^*(\mathbb{R}^n)$ be the class of variable exponents $p \in \mathcal{M}(\mathbb{R}^n)$ represented as $1/p(x) = \theta/p_0 + (1 - \theta)/p_1(x)$ where $p_0 \in (1, \infty)$, $\theta \in (0, 1)$, and $p_1 \in \mathcal{M}(\mathbb{R}^n)$. We prove that if $a \in S^\alpha_{1,0}$ slowly oscillates at infinity in the first variable, then the condition

$$\lim_{R \to \infty} \inf_{|x| + |\xi| \geq R} |a(x, \xi)| > 0$$

is sufficient for the Fredholmness of $Op(a)$ on $L^{p(\cdot)}(\mathbb{R}^n)$ whenever $p \in \mathcal{M}^*(\mathbb{R}^n)$. Both theorems generalize pioneering results by Rabinovich and Samko [23] obtained for globally log-Hölder continuous exponents $p$, constituting a proper subset of $\mathcal{M}^*(\mathbb{R}^n)$.

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1. Introduction

We denote the usual operators of first order partial differentiation on $\mathbb{R}^n$ by $\partial_{x_j} := \partial/\partial_{x_j}$. For every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with non-negative integers $\alpha_j$, we write $\partial^\alpha := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$. Further, $|\alpha| := \alpha_1 + \cdots + \alpha_n$, and for each vector $\xi = \ldots$
where the symbol ξ for the Euclidean norm of β and called variable Lebesgue spaces was started by Rabinovich and Samko [23, 24].

Let \( C^\infty_0(\mathbb{R}^n) \) denote the set of all infinitely differentiable functions with compact support. Recall that, given \( u \in C^\infty_0(\mathbb{R}^n) \), a pseudodifferential operator \( \text{Op}(a) \) is formally defined by the formula

\[
(\text{Op}(a)u)(x) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} d\xi \int_{\mathbb{R}^n} a(x,\xi)u(y)e^{i(x-y,\xi)}dy,
\]

where the symbol \( a \) is assumed to be smooth in both the spatial variable \( x \) and the frequency variable \( \xi \), and satisfies certain growth conditions (see e.g. [25 Chap. VI]). An example of symbols one might consider is the class \( S_{\rho,\delta}^m \), introduced by Hörmander [12], consisting of \( a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n) \) with

\[
|\partial^\alpha_x \partial^\beta_\xi a(x,\xi)| \leq C_{\alpha,\beta} (\xi)^{m-\rho|\alpha|+\delta|\beta|} \quad (x,\xi \in \mathbb{R}^n),
\]

where \( m \in \mathbb{R} \) and \( 0 \leq \delta, \rho \leq 1 \) and the positive constants \( C_{\alpha,\beta} \) depend only on \( \alpha \) and \( \beta \).

The study of pseudodifferential operators \( \text{Op}(a) \) with symbols in \( S_{\rho,\delta}^0 \) on so-called variable Lebesgue spaces was started by Rabinovich and Samko [23, 24].

Let \( p : \mathbb{R}^n \to [1, \infty] \) be a measurable a.e. finite function. By \( L^{p(\cdot)}(\mathbb{R}^n) \) we denote the set of all complex-valued functions \( f \) on \( \mathbb{R}^n \) such that

\[
I_{p(\cdot)}(f/\lambda) := \int_{\mathbb{R}^n} |f(x)/\lambda|^{p(x)}dx < \infty
\]

for some \( \lambda > 0 \). This set becomes a Banach space when equipped with the norm

\[
||f||_{p(\cdot)} := \inf \left\{ \lambda > 0 : I_{p(\cdot)}(f/\lambda) \leq 1 \right\}.
\]

It is easy to see that if \( p \) is constant, then \( L^{p(\cdot)}(\mathbb{R}^n) \) is nothing but the standard Lebesgue space \( L^p(\mathbb{R}^n) \). The space \( L^{p(\cdot)}(\mathbb{R}^n) \) is referred to as a variable Lebesgue space.

**Lemma 1.1.** (see e.g. [14 Theorem 2.11] or [9 Theorem 3.4.12]) If \( p : \mathbb{R}^n \to [1, \infty] \) is an essentially bounded measurable function, then \( C^\infty_0(\mathbb{R}^n) \) is dense in \( L^{p(\cdot)}(\mathbb{R}^n) \).

We will always suppose that

\[
1 < p_- := \inf_{x \in \mathbb{R}^n} p(x), \quad \text{ess sup}_{x \in \mathbb{R}^n} p(x) := p_+ < \infty.
\]

Under these conditions, the space \( L^{p(\cdot)}(\mathbb{R}^n) \) is separable and reflexive, and its dual space is isomorphic to \( L^{p'(\cdot)}(\mathbb{R}^n) \), where

\[
1/p(x) + 1/p'(x) = 1 \quad (x \in \mathbb{R}^n)
\]

(see e.g. [14 or 9 Chap. 3]).

Given \( f \in L^1_{loc}(\mathbb{R}^n) \), the Hardy-Littlewood maximal operator is defined by

\[
Mf(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy
\]
where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing $x$ (here, and throughout, cubes will be assumed to have their sides parallel to the coordinate axes). By $\mathcal{M}(\mathbb{R}^n)$ denote the set of all measurable functions $p : \mathbb{R}^n \to [1, \infty]$ such that (1.1) holds and the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$.

Assume that (1.1) is fulfilled. Diening [7] proved that if $p$ satisfies
\begin{equation}
|p(x) - p(y)| \leq \frac{c}{\log(e + 1/|x - y|)} \quad (x, y \in \mathbb{R}^n)
\end{equation}
and $p$ is constant outside some ball, then $p \in \mathcal{M}(\mathbb{R}^n)$. Further, the behavior of $p$ at infinity was relaxed by Cruz-Uribe, Fiorenza, and Neugebauer [5, 6], where it was shown that if $p$ satisfies (1.2) and there exists a $p_\infty > 1$ such that
\begin{equation}
|p(x) - p_\infty| \leq \frac{c}{\log(e + |x|)} \quad (x \in \mathbb{R}^n),
\end{equation}
then $p \in \mathcal{M}(\mathbb{R}^n)$. Following [9, Section 4.1], we will say that if conditions (1.2)–(1.3) are fulfilled, then $p$ is globally log-Hölder continuous.

Conditions (1.2) and (1.3) are optimal for the boundedness of $M$ in the pointwise sense; the corresponding examples are contained in [20] and [5]. However, neither (1.2) nor (1.3) is necessary for $p \in \mathcal{M}(\mathbb{R}^n)$. Nekvinda [18] proved that if $p$ satisfies (1.1)–(1.2) and
\begin{equation}
\int_{\mathbb{R}^n} |p(x) - p_\infty|^{1/p(x) - p_\infty} \, dx < \infty
\end{equation}
for some $p_\infty > 1$ and $c > 0$, then $p \in \mathcal{M}(\mathbb{R}^n)$. One can show that (1.3) implies (1.4), but the converse, in general, is not true. The corresponding example is constructed in [2]. Nekvinda further relaxed condition (1.3) in [19]. Lerner [15] (see also [9, Example 5.1.8]) showed that there exist discontinuous at zero or/and at infinity exponents, which nevertheless belong to $\mathcal{M}(\mathbb{R}^n)$. We refer to the recent monograph [9] for further discussions concerning the class $\mathcal{M}(\mathbb{R}^n)$.

Our first main result is the following theorem on the boundedness of pseudo-differential operators on variable Lebesgue spaces.

**Theorem 1.2.** Let $0 < \rho \leq 1$, $0 \leq \delta < 1$, and $a \in S^{n(\rho - 1)}_{\rho, \delta}$. If $p \in \mathcal{M}(\mathbb{R}^n)$, then $\text{Op}(a)$ extends to a bounded operator on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

The respective result for $a \in S^0_{0, 0}$ and $p$ satisfying (1.1)–(1.3) was proved by Rabinovich and Samko [23, Theorem 5.1].

Following [23, Definition 4.5], a symbol $a \in S^m_{1, 0}$ is said to be slowly oscillating at infinity in the first variable if
\begin{equation}
|\partial_\xi^\alpha \partial_x^\beta a(x, \xi)| \leq C_{\alpha, \beta}(x) |\xi|^{m - |\alpha|},
\end{equation}
where
\begin{equation}
\lim_{x \to \infty} C_{\alpha, \beta}(x) = 0
\end{equation}
for all multi-indices $\alpha$ and $\beta \neq 0$. We denote by $SO^m$ the class of all symbols slowly oscillating at infinity. Finally, we denote by $SO^m_{0}$ the set of all symbols
a ∈ SO$_m$, for which (1.5) holds for all multi-indices $\alpha$ and $\beta$. The classes SO$_m$ and SO$_0^m$ were introduced by Grushin [11].

We denote by $\mathcal{M}^*(\mathbb{R}^n)$ the set of all variable exponents $p \in \mathcal{M}(\mathbb{R}^n)$ for which there exist constants $p_0 \in (1, \infty)$, $\theta \in (0, 1)$, and a variable exponent $p_1 \in \mathcal{M}(\mathbb{R}^n)$ such that

$$\frac{1}{p(x)} = \frac{\theta}{p_0} + \frac{1 - \theta}{p_1(x)}$$

for almost all $x \in \mathbb{R}^n$. Rabinovich and Samko observed in the proof of [23, Theorem 6.1] that if $p$ satisfies (1.1)–(1.3), then $p \in \mathcal{M}^*(\mathbb{R}^n)$. It turns out that the class $\mathcal{M}^*(\mathbb{R}^n)$ contains many interesting exponents which are not globally log-Hölder continuous (see [13]). In particular, there exists $\varepsilon > 0$ such that for every $\alpha, \beta$ satisfying $0 < \beta < \alpha \leq \varepsilon$ the function

$$p(x) = 2 + \alpha + \beta \sin \left( \log(\log |x|) \chi_{\{x \in \mathbb{R}^n : |x| > e\}}(x) \right) \quad (x \in \mathbb{R}^n)$$

belongs to $\mathcal{M}^*(\mathbb{R}^n)$.

As usual, we denote by $I$ the identity operator on a Banach space. Recall that a bounded linear operator $A$ on a Banach space is said to be Fredholm if there is an (also bounded linear) operator $B$ such that the operators $AB - I$ and $BA - I$ are compact. In that case the operator $B$ is called a regularizer for the operator $A$.

Our second main result is the following sufficient condition for the Fredholmness of pseudodifferential operators on variable Lebesgue spaces.

**Theorem 1.3.** Suppose $p \in \mathcal{M}^*(\mathbb{R}^n)$ and $a \in SO^0$. If

$$\lim_{R \to \infty} \inf_{|x| + |\xi| \geq R} |a(x, \xi)| > 0,$$

then the operator $\text{Op}(a)$ is Fredholm on the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$.

As it was the case with Theorem 1.2, for $p$ satisfying (1.1)–(1.3) this result was established by Rabinovich and Samko [23, Theorem 6.1]. Notice that for such $p$ condition (1.6) is also necessary for the Fredholmness (see [23, Theorems 6.2 and 6.5]). Whether or not the necessity holds in the setting of Theorem 1.3 remains an open question.

The paper is organized as follows. In Section 2.2, the Diening-Ružička generalization (see [10]) of the Fefferman-Stein sharp maximal theorem to the variable exponent setting is stated. Further, Diening’s results [8] on the duality and left-openness of the class $\mathcal{M}(\mathbb{R}^n)$ are formulated. In Section 2.4 we discuss a pointwise estimate relating the Fefferman-Stein sharp maximal operator of $\text{Op}(a)u$ and $M_q u := M(|u|^q)^{1/q}$ for $q \in (1, \infty)$ and $u \in C_0^\infty(\mathbb{R}^n)$. Such an estimate for the range of parameters $\rho, \delta$, and $m = n(\rho - 1)$ as in Theorem 1.2 was recently obtained by Michalowski, Rule, and Staubach [16]. Combining this key pointwise estimate with the sharp maximal theorem and taking into account that $M_q$ is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ for some $q \in (1, \infty)$ whenever $p \in \mathcal{M}(\mathbb{R}^n)$, we give the proof of Theorem 1.2 in Section 2.5.
Section 3 is devoted to the proof of the sufficient condition for the Fredholm-ness of a pseudodifferential operator with slowly oscillating symbol. In Section 3.1, we state analogues of the Riesz-Thorin and Krasnoselskii interpolation theorems for variable Lebesgue spaces. Section 3.2 contains the composition formula for pseudodifferential operators with slowly oscillating symbols and the compactness result for pseudodifferential operators with symbols in $SO_0^{-1}$. Both results are essentially due to Grushin [11]. Section 3.3 contains the proof of Theorem 1.3. Its outline is as follows. From (1.6) it follows that there exist symbols $b_R \in SO_0$ and $\phi_R + c \in SO_0^{-1}$ such that $I - \text{Op}(a) \text{Op}(b_R) = \text{Op}(\phi_R + c)$. Since $\phi_R + c \in SO_0^{-1}$, the operator $\text{Op}(\phi_R + c)$ is compact on all standard Lebesgue spaces. Its compactness on the variable Lebesgue space $L^p(\cdot)(\mathbb{R}^n)$ is proved by interpolation, since it is bounded on the variable Lebesgue space $L^{p_1}(\cdot)(\mathbb{R}^n)$, where $p_1$ is the variable exponent from the definition of the class $M^*(\mathbb{R}^n)$. Actually, the class $M^*(\mathbb{R}^n)$ is introduced exactly for the purpose to perform this step. Therefore $\text{Op}(b_R)$ is a right regularizer for $\text{Op}(a)$ on $L^p(\cdot)(\mathbb{R}^n)$. In the same fashion it can be shown that $\text{Op}(b_R)$ is a left regularizer for $\text{Op}(a)$. Thus $\text{Op}(a)$ is Fredholm.

2. Boundedness of the operator $\text{Op}(a)$

2.1. Lattice property of variable Lebesgue spaces

We start with the following simple but important property of variable Lebesgue spaces. Usually it is called the lattice property or the ideal property.

Lemma 2.1. (see e.g. [9, Theorem 2.3.17]) Let $p : \mathbb{R}^n \to [1, \infty]$ be a measurable a.e. finite function. If $g \in L^p(\cdot)(\mathbb{R}^n)$, $f$ is a measurable function, and $|f(x)| \leq |g(x)|$ for a.e. $x \in \mathbb{R}^n$, then $f \in L^p(\cdot)(\mathbb{R}^n)$ and $\|f\|_{p(\cdot)} \leq \|g\|_{p(\cdot)}$.

2.2. The Fefferman-Stein sharp maximal function

Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$. For a cube $Q \subset \mathbb{R}^n$, put

$$f_Q := \frac{1}{|Q|} \int_Q f(x)dx.$$

The Fefferman-Stein sharp maximal function is defined by

$$M^# f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(x) - f_Q|dx,$$

where the supremum is taken over all cubes $Q$ containing $x$.

It is obvious that $M^# f$ is pointwise dominated by $Mf$. Hence, by Lemma 2.1

$$\|M^# f\|_{p(\cdot)} \leq \text{const} \|f\|_{p(\cdot)}$$

whenever $p \in M(\mathbb{R}^n)$. The converse is also true. For constant $p$ this fact goes back to Fefferman and Stein (see e.g. [25, Chap. IV, Section 2.2]). The variable exponent analogue of the Fefferman-Stein theorem was proved by Diening and Růžička [10].
Theorem 2.2. (see [10, Theorem 3.6] or [9, Theorem 6.2.5]) If \( p, p' \in \mathcal{M}(\mathbb{R}^n) \), then there exists a constant \( C_\#(p) > 0 \) such that for all \( f \in \mathcal{L}^{p(\cdot)}(\mathbb{R}^n) \),
\[
\| f \|_{p(\cdot)} \leq C_\#(p) \| M_\# f \|_{p(\cdot)}.
\]

2.3. Duality and left-openness of the class \( \mathcal{M}(\mathbb{R}^n) \)

Let \( 1 \leq q < \infty \). Given \( f \in \mathcal{L}^{q\text{-loc}}(\mathbb{R}^n) \), the \( q \)-th maximal operator is defined by
\[
M_q f(x) := \sup_{Q \ni x} \left( \frac{1}{|Q|} \int_Q |f(y)|^q \, dy \right)^{1/q},
\]
where the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \) containing \( x \). For \( q = 1 \) this is the usual Hardy-Littlewood maximal operator. Diening \[8\] established the following deep duality and left-openness result for the class \( \mathcal{M}(\mathbb{R}^n) \).

Theorem 2.3. (see [8, Theorem 8.1] or [9, Theorem 5.7.2]) Let \( p : \mathbb{R}^n \to [1, \infty] \) be a measurable function satisfying (1.1). The following statements are equivalent:

(a) \( M \) is bounded on \( \mathcal{L}^{p(\cdot)}(\mathbb{R}^n) \);
(b) \( M \) is bounded on \( \mathcal{L}^{p'(\cdot)}(\mathbb{R}^n) \);
(c) there exists an \( s \in (1/p - 1, 1) \) such that \( M \) is bounded on \( \mathcal{L}^{sp(\cdot)}(\mathbb{R}^n) \);
(d) there exists a \( q \in (1, \infty) \) such that \( M_q \) is bounded on \( \mathcal{L}^{p(\cdot)}(\mathbb{R}^n) \).

2.4. The crucial pointwise estimate

One of the main steps in the proof of Theorem 1.2 is the following pointwise estimate.

Theorem 2.4. (see [10, Theorem 3.3]) Let \( 1 < q < \infty \) and \( a \in S^m_{\rho, \delta} \) with \( 0 < \rho \leq 1, 0 \leq \delta < 1 \), and \( m = n(\rho - 1) \). For every \( u \in C^\infty_0(\mathbb{R}^n) \),
\[
M_\#(\text{Op}(a)u)(x) \leq C(q, a)M_q u(x) \quad (x \in \mathbb{R}^n),
\]
where \( C(q, a) \) is a positive constant depending only on \( q \) and the symbol \( a \).

This theorem generalizes the pointwise estimate by Miller \[17\,\text{Theorem 2.8}\] for \( a \in S^1_{0,0} \) and by Álvarez and Hounie \[1\,\text{Theorem 4.1}\] for \( a \in S^m_{\rho, \delta} \) with the parameters satisfying \( 0 < \delta \leq \rho \leq 1/2 \) and \( m \leq n(\rho - 1) \).

Let \( 0 < s < 1 \). One of the main steps in the Rabinovich and Samko’s proof \[23\,\text{Corollary 3.4}\] of the boundedness on \( \mathcal{L}^{p(\cdot)}(\mathbb{R}^n) \) of the operator \( \text{Op}(a) \) with \( a \in S^1_{1,0} \) is another pointwise estimate
\[
M_\#(|\text{Op}(a)u|^s)(x) \leq C[Mu(x)]^s \quad (x \in \mathbb{R}^n)
\]
for all \( u \in C^\infty_0(\mathbb{R}^n) \), where \( C \) is a positive constant independent of \( u \). It was proved in \[23\,\text{Corollary 3.4}\] following the ideas of Álvarez and Pérez \[2\], where the same estimate is obtained for the Calderón-Zygmund singular integral operator in place of the pseudodifferential operator \( \text{Op}(a) \).
2.5. Proof of Theorem 1.2
Suppose \( p \in \mathcal{M}(\mathbb{R}^n) \). Then, by Theorem 2.4, \( p' \in \mathcal{M}(\mathbb{R}^n) \) and there exists a number \( q \in (1, \infty) \) such that \( M_q \) is bounded on \( L^{p'}(\mathbb{R}^n) \). In other words, there exists a positive constant \( \tilde{C}(p, q) \) depending only on \( p \) and \( q \) such that for all \( u \in L^{p'}(\mathbb{R}^n) \),

\[
\|M_qu\|_{p(\cdot)} \leq \tilde{C}(p, q)\|u\|_{p(\cdot)}. \tag{2.1}
\]

From Theorem 2.5 it follows that there exists a constant \( C_\#(p) \) such that for all \( u \in C_0^\infty(\mathbb{R}^n) \),

\[
\|\text{Op}(a)u\|_{p(\cdot)} \leq C_\#(p)\|M^\#(\text{Op}(a)u)\|_{p(\cdot)}. \tag{2.2}
\]

On the other hand, from Theorem 2.6 and Lemma 2.7 we obtain that there exists a positive constant \( C(q, a) \), depending only on \( q \) and \( a \), such that

\[
\|M^\#(\text{Op}(a)u)\|_{p(\cdot)} \leq C(q, a)\|M_qu\|_{p(\cdot)}. \tag{2.3}
\]

Combining (2.1), (2.2), and (2.3), we arrive at

\[
\|\text{Op}(a)u\|_{p(\cdot)} \leq C_\#(p)C(q, a)\tilde{C}(p, q)\|u\|_{p(\cdot)}
\]

for all \( u \in C_0^\infty(\mathbb{R}^n) \). It remains to recall that \( C_0^\infty(\mathbb{R}^n) \) is dense in \( L^{p'}(\mathbb{R}^n) \) (see Lemma 1.1). \( \square \)

3. Fredhollness of the operator \( \text{Op}(a) \)

3.1. Interpolation theorem

For a Banach space \( X \), let \( B(X) \) and \( K(X) \) denote the Banach algebra of all bounded linear operators and its ideal of all compact operators on \( X \), respectively.

**Theorem 3.1.** Let \( p_j : \mathbb{R}^n \to [1, \infty], \ j = 0, 1, \) be a.e. finite measurable functions, and let \( p_\theta : \mathbb{R}^n \to [1, \infty] \) be defined for \( \theta \in [0, 1] \) by

\[
\frac{1}{p_\theta(x)} = \frac{\theta}{p_0(x)} + \frac{1-\theta}{p_1(x)} \quad (x \in \mathbb{R}^n).
\]

Suppose \( A \) is a linear operator defined on \( L^{p_0}(\mathbb{R}^n) \cup L^{p_1}(\mathbb{R}^n) \).

(a) If \( A \in B(L^{p_j}(\mathbb{R}^n)) \) for \( j = 0, 1 \), then \( A \in B(L^{p_\theta}(\mathbb{R}^n)) \) for all \( \theta \in [0, 1] \) and

\[
\|A\|_{B(L^{p_\theta}(\mathbb{R}^n))} \leq 4\|A\|_{B(L^{p_j}(\mathbb{R}^n))}\|A\|_{B(L^{p_\theta}(\mathbb{R}^n))}^{1-\theta}. \tag{2.4}
\]

(b) If \( A \in K(L^{p_0}(\mathbb{R}^n)) \) and \( A \in B(L^{p_1}(\mathbb{R}^n)) \), then \( A \in K(L^{p_\theta}(\mathbb{R}^n)) \) for all \( \theta \in (0, 1) \).

Part (a) is proved in [11 Corollary 7.1.4] under the more general assumption that \( p_j \) may take infinite values on sets of positive measure (and in the setting of arbitrary measure spaces). Part (b) was proved in [23 Proposition 2.2] under the additional assumptions that \( p_j \) satisfy [1.1]–[1.3]. It follows without these assumptions from a general interpolation theorem by Cobos, Kühn, and Schonbeck [11 Theorem 3.2] for the complex interpolation method for Banach lattices satisfying the Fatou property. Indeed, the complex interpolation space
Let $m \in \mathbb{Z}$ and $OPSO^m$ be the class of all pseudodifferential operators $Op(a)$ with $a \in SO^m$. By analogy with [11, Section 2] one can get the following composition formula (see also [21, Theorem 6.2.1] and [22, Chap. 4]).

**Proposition 3.2.** If $Op(a_1) \in OPSO^{m_1}$ and $Op(a_2) \in OPSO^{m_2}$, then their product $Op(a_1)Op(a_2) = Op(\sigma)$ belongs to $OPSO^{m_1+m_2}$ and its symbol $\sigma$ is given by

$$\sigma(x, \xi) = a_1(x, \xi)a_2(x, \xi) + c(x, \xi), \quad x, \xi \in \mathbb{R}^n,$$

where $c \in SO_0^{m_1+m_2-1}$.

**Proposition 3.3.** Let $1 < q < \infty$. If $c \in SO_0^{-1}$, then $Op(c) \in K(L^q(\mathbb{R}^n))$.

**Proof.** From Theorem [1.2] it follows that $Op(c) \in B(L^q(\mathbb{R}^n))$ for all constant exponents $q \in (1, \infty)$. By [11, Theorem 3.2], $Op(c) \in K(L^2(\mathbb{R}^n))$. Hence, by the Krasnol'skii interpolation theorem (Theorem 5.11(b) for constant $p_j$ with $j = 0, 1$), $Op(c) \in K(L^q(\mathbb{R}^n))$ for all $q \in (1, \infty)$.

### 3.3. Proof of Theorem 1.3

The idea of the proof is borrowed from [11, Theorem 3.4] and [23, Theorem 6.1]. Let $\varphi \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ be such that $\varphi(x, \xi) = 1$ if $|x| + |\xi| \leq 1$ and $\varphi(x, \xi) = 0$ if $|x| + |\xi| \geq 2$. For $R > 0$, put

$$\varphi_R(x, \xi) = \varphi(x/R, \xi/R), \quad x, \xi \in \mathbb{R}^n.$$  

From (1.6) it follows that there exists an $R > 0$ such that

$$\inf_{|x| + |\xi| \geq R} |a(x, \xi)| > 0.$$  

Then it is not difficult to check that

$$b_R(x, \xi) := \begin{cases} 
1 - \varphi_R(x, \xi) & \text{if } |x| + |\xi| \geq R, \\
0 & \text{if } |x| + |\xi| < R,
\end{cases}$$

belongs to $SO^0$. It is also clear that $\varphi_R \in SO^0$.

From Proposition 3.3 it follows that there exists a function $c \in SO_0^{-1}$ such that

$$Op(ab_R) - Op(a)Op(b_R) = Op(c). \quad (3.1)$$

On the other hand, since

$$a(x, \xi)b_R(x, \xi) = 1 - \varphi_R(x, \xi), \quad x, \xi \in \mathbb{R}^n,$$

we have

$$Op(ab_R) = Op(1 - \varphi_R) = I - Op(\varphi_R). \quad (3.2)$$

Combining (3.1) and (3.2), we get

$$I - Op(a)Op(b_R) = Op(\varphi_R) + Op(c) = Op(\varphi_R + c). \quad (3.3)$$
Since \( p \in \mathcal{M}^*(\mathbb{R}^n) \), there exist \( p_0 \in (1, \infty) \), \( \theta \in (0, 1) \), and \( p_1 \in \mathcal{M}(\mathbb{R}^n) \) such that

\[
\frac{1}{p(x)} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1(x)} \quad (x \in \mathbb{R}^n).
\]

From Theorem 1.2 we conclude that all pseudodifferential operators considered above are bounded on \( L^{p_0}(\mathbb{R}^n) \), \( L^{p(\cdot)}(\mathbb{R}^n) \), and \( L^{p_1(\cdot)}(\mathbb{R}^n) \). Since \( \varphi_R + c \in SO_0^{-1} \), from Proposition 3.3 it follows that \( \text{Op}(\varphi_R + c) \in \mathcal{K}(L^{p_0}(\mathbb{R}^n)) \). Then, by Theorem 3.1(b), \( \text{Op}(\varphi_R + c) \in \mathcal{K}(L^{p(\cdot)}(\mathbb{R}^n)) \). Therefore, from (3.3) it follows that \( \text{Op}(b_R) \) is a right regularizer for \( \text{Op}(a) \). Analogously it can be shown that \( \text{Op}(b_R) \) is also a left regularizer for \( \text{Op}(a) \). Thus \( \text{Op}(a) \) is Fredholm on \( L^{p(\cdot)}(\mathbb{R}^n) \). \( \square \)

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