MODULES WITH FINITE COUSIN COHOMOLOGIES HAVE
UNIFORM LOCAL COHOMOLOGICAL ANNIHILATORS

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Abstract. Let $A$ be a Noetherian ring. It is shown that any finite $A$–module $M$
of finite Krull dimension with finite Cousin complex cohomologies has a uniformlocal cohomological annihilator. The converse is also true for a finite module $M$satisfying $(S_2)$ which is over a local ring with Cohen–Macaulay formal fibres.

1. Introduction

Throughout let $A$ denote a commutative Noetherian ring and $M$ a finite (i.e.
finitely generated) $A$-module. Recall that an $A$-module $M$ is called equidimensional(or unmixed) if $\text{Min}_A(M) = \text{Assh}_A(M)$ (i.e. for each minimal prime $p$ of $\text{Supp}_A(M)$,$\dim_A(M) = \dim(A/p)$). For an ideal $a$ of $A$, write $H^i_a(M)$ for the $i$th local coho-mology module of $M$ with support in $V(a) = \{ p \in \spec(A) : p \supseteq a \}$. An element$x \in A$ is called a uniform local cohomological annihilator of $M$ if $x \in A \backslash \bigcup_{p \in \text{Min}_A(M)} p$ and for each maximal ideal $m$ of $A$, $xH^i_m(M) = 0$ for all $i < \dim_A(M_m)$. The exis-tence of a local cohomological annihilator is studied by Hochster and Huneke [6]and proved its importance for the existence of big Cohen–Macaulay algebras and auniform Artin–Rees theorem [7].

In [12], Zhou studied rings with a uniform local cohomological annihilator.
Hochster and Huneke, in [5], proved that if $A$ is locally equidimensional (i.e. $A_m$ is-equidimensional for every maximal ideal $m$ of $A$) and is a homomorphic image of aGorenstein ring of finite dimension, then $A$ has a strong uniform local cohomological annihilator (i.e. $A$ has an element which is a uniform local cohomological annihilator of $A_p$ for each $p \in \spec(A)$). In [12], Zhou showed that if a locally equidimensional ring $A$ of positive dimension is a homomorphic image of a Cohen–Macaulay ring offinite dimension (or an excellent local ring), then $A$ has a uniform local cohomological annihilator.

Cousin complexes were introduced by Hartshorne in [4] and have a commutativealgebra analogue given by Sharp in [10]. Recently, Cousin complexes have been stud-ied by several authors. In [2], [3], and [8], Dibaei, Tousi, and Kawasaki studied finiteCousin complexes (i.e. the Cousin complexes with finitely generated cohomologies).In [9], Proposition 9.3.5], Lipman, Nayak, and Sastry generalized these results to complexes on formal schemes.

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In section 2, it is proved that any finite $A$–module of finite Krull dimension with finite Cousin complex cohomologies has a uniform local cohomological annihilator (Theorem 2.7). As a result it follows that if $(A, \mathfrak{m})$ is local, satisfies Serre’s condition $(S_2)$, and such that all of its fibres of $A \to \hat{A}$ are Cohen–Macaulay, then $A$ has a uniform local cohomological annihilator (Corollary 2.10). For a finite module $M$ over a local ring $(A, \mathfrak{m})$ satisfying $(S_2)$ and with Cohen Macaulay formal fibres, it is proved that the following conditions are equivalent: (i) $\hat{M}$, the completion of $M$ with respect to $\mathfrak{m}$–adic topology, is equidimensional; (ii) $\mathcal{C}_A(M)$, the Cousin complex of $M$ is finite; (iii) $M$ has a uniform local cohomological annihilator (Theorem 2.13).

In section 3, for certain modules $M$, the relationship between the cohomology modules of the Cousin complex of $M$ and the local cohomology modules of $M$ with respect to an arbitrary ideal of $A$ is studied. It is shown that the $M$–height of $a$ is equal to the infimum of numbers $r$ for which $0 : A \to H^r_a(M)$ does not contain the product of all the annihilators of the Cousin cohomologies of $M$ (Theorem 3.2).

2. Cousin complexes

Let $M$ be an $A$–module and let $\mathcal{H} = \{H_i : i \geq 0\}$ be the family of subsets of $\text{Supp}_A(M)$ with $H_i = \{p \in \text{Supp}_A(M) : \dim_A(M_p) \geq i\}$. The family $\mathcal{H}$ is called the $M$–height filtration of $\text{Supp}_A(M)$. Define the Cousin complex of $M$ as the complex

\[(*) \quad C_A(M) : 0 \overset{d_{-2}}{\to} M^{-1} \overset{d_{-1}}{\to} M^0 \overset{d_0}{\to} M^1 \overset{d_1}{\to} \cdots \overset{d_{i-1}}{\to} M^i \overset{d_i}{\to} M^{i+1} \to \cdots ,\]

where $M^{-1} = M$, $M^i = \bigoplus_{p \in H_i \setminus H_{i+1}} (\text{Coker} d^{i-2})_p$ for $i > -1$. The homomorphism $d^{i} : M^{i} \to M^{i+1}$ has the following property: for $m \in M^i$ and $p \in H_i \setminus H_{i+1}$, the component of $d^{i}(m)$ in $(\text{Coker} d^{i-2})_p$ is $\overline{m}/1$, where $\overline{m} : M^i \to \text{Coker} d^{i-1}$ is the natural map (see [10] for details).

Throughout, for the Cousin complex $(*)$, we use the following notations:

\[K^i := \text{Ker} d^i, D^i := \text{Im} d^{i-1}, H^i := K^i/D^i, i = -1, 0, \ldots .\]

We call the Cousin complex $C_A(M)$ finite if, for each $i$, the cohomology module $H^i$ is finite. Recall that for an ideal $\mathfrak{a}$ of $A$ and an $A$–module $M$, the $M$–height of $\mathfrak{a}$ is defined by $\text{ht}_M(\mathfrak{a}) := \inf\{\dim M_p : p \in \text{Supp}_A(M) \cap V(\mathfrak{a})\}$. Note that $\text{ht}_M(\mathfrak{a}) \geq 0$ whenever $M \neq aM$. If $M$ is finitely generated then $\text{ht}_M(\mathfrak{a}) = \text{ht}(\overline{\mathfrak{a}M})$, where $I = \text{Ann}_A(M)$.

We begin the following lemma which for the first part we adopt the argument in [11], Theorem).

Lemma 2.1. Let $M$ be an $A$-module. For any integer $k$ with $0 \leq k < \text{ht}_M(\mathfrak{a})$, the following statements are true.

(a) $H^s_a(M^k) = 0$ for all integers $s \geq 0$.
(b) $\text{Ext}^s_A(A/\mathfrak{a}, M^k) = 0$ for all integers $s \geq 0$.

Proof. (a). Set $C_{k-1} := \text{Coker} d^{k-2} = M^{k-1}/D^{k-1}$ so that $M^k = \bigoplus_{p \in \text{Supp}_A(M)} (C_{k-1})_p$. For each $k < \text{ht}_M(\mathfrak{a})$ and each $p \in \text{Supp}_A(M)$ with $\text{ht}_M(p) = k$, there exists an element $x \in a \setminus p$. Thus the multiplication map $(C_{k-1})_p \overset{x}{\to} (C_{k-1})_p$ is an automorphism and so the multiplication map $H^s_a((C_{k-1})_p) \overset{x}{\to} H^s_a((C_{k-1})_p)$ is also an
automorphism for all integers $s$. One may then conclude that $H^s_n((C_{k-1})_p) = 0$. Now, from additivity of local cohomology functors, it follows that $H^s_n(M^k) = 0$.

(b). Assume in general that $N$ is an $A$-module such that $H^s_n(N) = 0$ for all $s \geq 0$. We show, by induction on $i, i \geq 0$, that $\text{Ext}_A^i(A/a, N) = 0$. For $i = 0$, one has $\text{Hom}_A(A/a, N) = H^0_n(A/a, H^0_n(N))$ which is zero. Assume that $i > 0$ and the claim is true for any such module $N$ and all $j \leq i - 1$. Choose $E$ to be an injective hull of $N$ and consider the exact sequence $0 \rightarrow N \rightarrow E \rightarrow N' \rightarrow 0$, where $N' = E/N$. As $H^0_n(E) = 0$, it follows that $H^s_n(N') = 0$ for all $s \geq 0$. Thus $\text{Ext}^i_A(A/a, N') = 0$, by our induction hypothesis. As, by the above exact sequence $\text{Ext}^i_A(A/a, N') \cong \text{Ext}^i_A(A/a, N)$, the result follows. \hfill \Box

The following technical result is important for the rest of the paper.

**Proposition 2.2.** Let $M$ be an $A$-module and let $a$ be an ideal of $A$ such that $aM \neq M$. Then, for each non-negative integer $r$ with $r < \text{ht}_M(a)$,

$$\prod_{i=0}^r (0 : A \text{Ext}_A^{r-i}(A/a, H^{i-1})) \subseteq 0 : A \text{Ext}_A^r(A/a, M).$$

Here $\prod$ is used for product of ideals.

**Proof.** For each $j \geq -1$, there are the natural exact sequences

(1) $$0 \rightarrow M^{j-1}/K^{j-1} \rightarrow M^j \rightarrow M^j/D^j \rightarrow 0,$$

(2) $$0 \rightarrow H^{j-1} \rightarrow M^{j-1}/D^{j-1} \rightarrow M^{j-1}/K^{j-1} \rightarrow 0.$$

Let $0 \leq r < \text{ht}_M(a)$.

We prove by induction on $j$, $0 \leq j \leq r$, that

(3) $$\prod_{i=0}^j (0 : A \text{Ext}_A^{r-i}(A/a, H^{i-1})) \cdot (0 : A \text{Ext}_A^{r-i}(A/a, M^{j-1}/K^{j-1})) \subseteq 0 : A \text{Ext}_A^r(A/a, M).$$

In case $j = 0$, the exact sequence (2) implies the exact sequence

$$\text{Ext}_A^r(A/a, H^{-1}) \rightarrow \text{Ext}_A^r(A/a, M) \rightarrow \text{Ext}_A^r(A/a, M^{-1}/K^{-1})$$

so that

$$(0 : A \text{Ext}_A^r(A/a, H^{-1})) \cdot (0 : A \text{Ext}_A^r(A/a, M^{-1}/K^{-1})) \subseteq 0 : A \text{Ext}_A^r(A/a, M)$$

and thus the case $j = 0$ is justified.

Assume that $0 \leq j < r$ and formula (3) is settled for $j$. Therefore, by Lemma 2.1 (b), formula (1) implies that

(4) $$\text{Ext}_A^{r-j}(A/a, M^{j-1}/K^{j-1}) \cong \text{Ext}_A^{r-j-1}(A/a, M^j/D^j).$$

On the other hand the exact sequence (2) implies the exact sequence

$$\text{Ext}_A^{r-j-1}(A/a, H^j) \rightarrow \text{Ext}_A^{r-j-1}(A/a, M^j/D^j) \rightarrow \text{Ext}_A^{r-j-1}(A/a, M^j/K^j),$$

from which it follows that

(5) $$(0 : A \text{Ext}_A^{r-j-1}(A/a, H^j)) \cdot (0 : A \text{Ext}_A^{r-j-1}(A/a, M^j/K^j)) \subseteq 0 : A \text{Ext}_A^{r-j-1}(A/a, M^j/D^j).$$
Now (4) and (5) imply that
(6) \[ (0 : A \operatorname{Ext}^{r-j-1}_A(A/a, H^j)) \cdot (0 : A \operatorname{Ext}^{r-j-1}_A(A/a, M^j_{K^j})) \subseteq 0 : A \operatorname{Ext}^{r-j}_A(A/a, M^{j-1}_{K^{j-1}}). \]

From (6), it follows that
\[
\prod_{i=0}^{j+1} (0 : A \operatorname{Ext}^{r-i}_A(A/a, H^{i-1})) \cdot (0 : A \operatorname{Ext}^{r-j-1}_A(A/a, M^j_{K^j})) = 0
\]
\[
\prod_{i=0}^{j} (0 : A \operatorname{Ext}^{r-i}_A(A/a, H^{i-1})) \cdot (0 : A \operatorname{Ext}^{r-j-1}_A(A/a, M^j_{K^j})) \subseteq 0 : A \operatorname{Ext}^r_A(A/a, M).
\]

and, by the induction hypothesis (3), it follows that
\[
\prod_{i=0}^{j+1} (0 : A \operatorname{Ext}^{r-i}_A(A/a, H^{i-1})) \cdot (0 : A \operatorname{Ext}^{r-j-1}_A(A/a, M^j_{K^j})) \subseteq 0 : A \operatorname{Ext}^r_A(A/a, M).
\]

This is the end of the induction argument. Putting \( j = r \) in (3) gives the result, because \( \operatorname{Ext}^0_A(A/a, M^r) = 0 \) by Lemma 2.1 (b) and, as by (1) for \( j = r \) there is an embedding \( \operatorname{Ext}^0_A(A/a, M^r_{K^r}) \hookrightarrow \operatorname{Ext}^0_A(A/a, M^r) \), it follows that \( \operatorname{Ext}^0_A(A/a, M^r_{K^r}) = 0 \).

An immediate corollary to the above result is the following.

**Corollary 2.3.** Assume that \( M \) is a finite \( A \)-module and that \( a \) is an ideal of \( A \) such that \( aM \neq M \). Then, for each integer \( r \) with \( 0 \leq r < \text{ht}(a) \),
\[
\prod_{i=0}^{r-1} (0 : A H^i) \subseteq \cap_{i=0}^{r} (0 : A \operatorname{Ext}^i_A(A/a, M)).
\]

**Proof.** It follows by Proposition 2.2 and the fact that the extension functors are linear. \( \square \)

**Corollary 2.4.** Let \( M \) be a finite \( A \)-module of dimension \( n \) and let \( a \) be an ideal of \( A \) such that \( aM \neq M \). Assume that \( x \) is an element of \( A \) such that \( xH^i = 0 \) for all \( i \). Then \( x^n \) annihilates all the modules \( \operatorname{Ext}^i_A(A/a, M) \), \( r = 0, 1, \ldots, \text{ht}(a) - 1 \) for all ideals \( a \) of \( A \).

**Proof.** It follows clearly from Corollary 2.3. \( \square \)

The following lemma states an easy but essential property of annihilators of Cousin cohomologies.

**Lemma 2.5.** Assume that \( M \) is a finite \( A \)-module of finite dimension \( \operatorname{dim}_A(M) = n \) and that \( \mathcal{C}_A(M) \) is finite, then \( \cap_{i \geq -1} (0 : A H^i) \not\subseteq \cup_{p \in \operatorname{Min}_A(M)} p \).

**Proof.** By (2.7), vii), \( V(0 : A H^i) = \operatorname{Supp}_A(H^i) \subseteq \{ p \in \operatorname{Supp}_A(M) : \operatorname{dim}_A(M_p) \geq i + 2 \} \) for all \( i \geq -1 \). Hence \( (0 : A H^i) \not\subseteq \cup_{p \in \operatorname{Min}_A(M)} p \). Now Prime Avoidance Theorem implies that \( \cap_{i \geq -1} (0 : A H^i) \not\subseteq \cup_{p \in \operatorname{Min}_A(M)} p \). \( \square \)
We are now in a position to prove that the modules with finite Cousin complexes have uniform local cohomological annihilators. But one can state more.

**Proposition 2.6.** Assume that $M$ is a finite $A$–module of finite $\dim_A(M) = n$ and that $\mathcal{C}_A(M)$ is finite. Then there exists an element $x \in A \setminus \bigcup_{p \in \text{Min}_A(M)} p$ such that $x\text{Ext}_A^j(A/m^j, M) = 0$ for all $i < \text{ht}_M(m)$, $j \geq 0$ and all maximal ideals $m$ in $\text{Supp}_A(M)$.

*Proof.* It follows by Lemma 2.5 and Corollary 2.4.

**Theorem 2.7.** Assume that $M$ is a finite $A$–module of finite $\dim_A(M) = n$ and that $\mathcal{C}_A(M)$ is finite, then $M$ has a uniform local cohomological annihilator.

*Proof.* By Proposition 2.6, there is an element $x \in A \setminus \bigcup_{p \in \text{Min}_A(M)} p$ such that $x\text{Ext}_A^j(A/m^j, M) = 0$ for all $i < \text{ht}_M(m)$, $j \geq 0$ and all maximal ideals $m$ in $\text{Supp}_A(M)$. Choose a maximal ideal $m$ in $\text{Supp}_A(M)$ and $i < \text{ht}_M(m)$. As $x \in \text{Ann}_A(\text{Ext}_A^j(A/m^j, M))$ for all $j$, we have $x \in \text{Ann}_A(\lim_{j \to \infty} (\text{Ext}_A^j(A/m^j, M)))$, i.e. $xH_m^i(M) = 0$ for all $i < \text{ht}_M(m)$. □

**Corollary 2.8.** Assume that $A$ has finite dimension and that $\mathcal{C}_A(A)$ is finite. Then $A$ has a uniform local cohomological annihilator, and so $A$ is locally equidimensional and is universally catenary.

*Proof.* It is clear from Theorem 2.6 and [12, Theorem 2.1]. □

In [12, Corollary 3.3], Zhou proved that any locally equidimensional Noetherian ring has a uniform local cohomological annihilator provided it is a homomorphic image of a Cohen–Macaulay ring of finite dimension. Here we have the following result:

**Corollary 2.9.** Assume that $(A, m)$ is local with Cohen–Macaulay formal fibres. Let $M$ be a finite $A$–module such that it satisfies $(S_2)$ and that $\text{Min}_A(M) = \text{Assh}_A(M)$. Then $M$ has a uniform local cohomological annihilator.

*Proof.* By [2, Theorem 2.1], $\mathcal{C}_A(M)$ is finite. Now Theorem 2.7 implies the result. □

**Corollary 2.10.** (Compare with [12, Corollary 3.3 (i)]). Assume that $(A, m)$ is local and that it satisfies $(S_2)$ and all of its formal fibres are Cohen–Macaulay. Then $A$ has a uniform local cohomological annihilator.

*Proof.* See [2, Corollary 2.2]. □

**Proposition 2.11.** Let $M$ be a finite $A$–module such that it has a uniform local cohomological annihilator. Then $M$ is locally equidimensional.

*Proof.* Let $m \in \text{Max} \text{Supp}_A(M)$. We will show that $\dim_{A_m}(M_m) = \dim A_m/pA_m$ for all $p \in \text{Spec} A$ with $p \in \text{Min}_A(M)$ and $p \subseteq m$. By assumption, there exists an element $x \in A \setminus \bigcup_{p \in \text{Min}_A(M)} p$ such that $xH^i_m(M) = 0$ for all $i < \dim_{A_m}(M_m)$. As $x \in A_m \setminus \bigcup_{pA_m \in \text{Min}_{A_m}(M_m)} pA_m$, and $H^i_m(M) \cong H^i_{mA_m}(M_m)$ by using the definition of local cohomology, we may assume that $(A, m)$ is local with the maximal ideal $m$ and write $d := \dim_A(M)$.

Assume, to the contrary, that there exists $p \in \text{Min}_A(M)$ with $e := \dim A/p < d$. Set $S = \{q \in \text{Min}_A(M) : \dim A/q \leq e\}$ and $T = \text{Ass}_A(M) \setminus S$. There exists
a submodule \( N \) of \( M \) such that \( \text{Ass}_A(N) = T \) and \( \text{Ass}_A(M/N) = S \). Note that \( \dim_A(M/N) = c \) and that \( \dim_A(N) = d \). As \( \sqrt{0 :_A N} = \cap_{q \in T} q \), it follows that there exists an element \( y \in 0 :_A N \setminus \cup_{q \in S} q \). Thus, trivially, \( yH^i_m(N) = 0 \) for all \( i \geq 0 \). The exact sequence \( 0 \to N \to M \to M/N \to 0 \) implies the exact sequence \( H^i_m(M) \to H^i_m(M/N) \to H^{i+1}_m(N) \). As \( xH^i_m(M) = 0 \) for all \( i < d \), it follows that \( xyH^i_m(M/N) = 0 \) for all \( i < d \). In particular, \( xyH^c_m(M/N) = 0 \). Thus \( xy \in \bigcap_{q \in \text{Assh}_A(M/N)} q \) (c.f. [1, Proposition 7.2.11 and Theorem 7.3.2]). Therefore \( xy \in \mathfrak{p} \) by the choice of \( \mathfrak{p} \). As \( \mathfrak{p} \in S \cap \text{Min}_A(M) \), this is a contradiction.

Now we can state the following result which partially extends Corollary 2.8.

**Corollary 2.12.** Let \( M \) be a finite \( A \)-module such that its Cousin complex \( C_A(M) \) is finite. Then \( M \) is locally equidimensional.

**Proof.** The proof is clear from Theorem 2.7 and Proposition 2.11.

Now it is easy to provide an example of a module whose Cousin complex has at least one non–finite cohomology.

**Example.** Consider a Noetherian local ring \( A \) of dimension \( d > 2 \). Choose any pair of prime ideals \( \mathfrak{p} \) and \( \mathfrak{q} \) of \( A \) with conditions \( \dim A/\mathfrak{p} = 2 \), \( \dim A/\mathfrak{q} = 1 \), and \( \mathfrak{p} \nsupseteq \mathfrak{q} \). Then \( \text{Min}_A(A/\mathfrak{p}\mathfrak{q}) = \{ \mathfrak{p}, \mathfrak{q} \} \) and so \( A/\mathfrak{p}\mathfrak{q} \) is not an equidimensional \( A \)-module and thus its Cousin complex is not finite.

We are now ready to present the following result which, for a finite module \( M \), shows connections of finiteness of its Cousin complex, existence of a uniform local cohomological annihilator for \( M \), and equidimensionality of \( \widehat{M} \).

**Theorem 2.13.** Let \( A \) be a local ring with Cohen–Macaulay formal fibres. Assume that \( M \) is a finite \( A \)-module which satisfies the condition \((S_2)\) of Serre. Then the following statements are equivalent.

(i) \( \text{Min}_{\widehat{A}}(\widehat{M}) = \text{Assh}_{\widehat{A}}(\widehat{M}) \).

(ii) The Cousin complex of \( M \) is finite.

(iii) \( M \) has a uniform local cohomological annihilator.

**Proof.** (i) \( \Rightarrow \) (ii) by [2, Theorem 2.1].

(ii) \( \Rightarrow \) (iii). This is Theorem 2.7.

(iii) \( \Rightarrow \) (i). There exists an element \( x \in A \setminus \bigcup_{p \in \text{Min}_A(M)} \mathfrak{p} \) such that \( xH^i_m(M) = 0 \) for all \( i < \dim_A(M) \), and, by artinian–ness of local cohomology modules, \( xH^i_m(\widehat{M}) = 0 \) for all \( i < \dim_{\widehat{A}}(\widehat{M}) \). Assume that \( Q \) is an element of \( \text{Min}_{\widehat{A}}(\widehat{M}) \). Note that \( 0 :_A M \subseteq Q \cap A \) and, by Going Down Theorem, \( Q \cap A \in \text{Min}_A(M) \). Hence \( x \notin Q \). Therefore \( \widehat{M} \) has a uniform local cohomological annihilator. Now, Proposition 2.11 implies that \( \text{Min}_{\widehat{A}}(\widehat{M}) = \text{Assh}_{\widehat{A}}(\widehat{M}) \).

We end this section by showing that any finite \( A \)-module \( M \) which has a uniform local cohomological annihilator is universally catenary, that is the ring \( A/(0 :_A M) \) is universally catenary.
Theorem 2.14. Let $M$ be a finite $A$–module that has a uniform local cohomological annihilator. Then $A/(0 :_A M)$ has a uniform local cohomological annihilator and so $A/(0 :_A M)$ is universally catenary.

Proof. By Proposition 2.11, $A/(0 :_A M)$ is locally equidimensional. By [12, Theorem 3.2], it is enough to show that $A / _p\bigcap \frac{A}{p} \cong A/p$ has a uniform local cohomological annihilator for each minimal prime ideal $p$ of $M$. We prove it by using the ideas given in the proof of [12, Theorem 3.2].

Assume that $p \in \text{Min}_A(M)$ and that $m$ is a maximal ideal containing $p$. As $M_p$ is an $A_p$–module of finite length we set $t := l_{A_p}(M_p)$. Then there exists a chain of submodules $0 \subset N_1 \subset N_2 \subset \cdots \subset N_t \subset M$ such that

$$0 \longrightarrow A/p \longrightarrow M \longrightarrow M/N_0 \longrightarrow 0,$$

$$0 \longrightarrow A/p \longrightarrow M/N_0 \longrightarrow M/N_1 \longrightarrow 0,$$

$$\vdots$$

$$0 \longrightarrow A/p \longrightarrow M/N_{t-2} \longrightarrow M/N_{t-1} \longrightarrow 0,$$

$$0 \longrightarrow A/p \longrightarrow M/N_{t-1} \longrightarrow M/N_t \longrightarrow 0.$$

Since $M_m$ is equidimensional, $\text{ht}_M(m/p) = \text{ht}_M(m)$. As, by definition of $t$, $p \not\in \text{Ass}_A(M/N_i)$, it follows that $0 :_A (M/N_i) \not\subseteq p$. Localizing the above exact sequences at $m$ implies the following exact sequences.

$$0 \longrightarrow (A/p)_m \longrightarrow M_m \longrightarrow (M/N_0)_m \longrightarrow 0,$$

$$0 \longrightarrow (A/p)_m \longrightarrow (M/N_0)_m \longrightarrow (M/N_1)_m \longrightarrow 0,$$

$$\vdots$$

$$0 \longrightarrow (A/p)_m \longrightarrow (M/N_{t-2})_m \longrightarrow (M/N_{t-1})_m \longrightarrow 0,$$

$$0 \longrightarrow (A/p)_m \longrightarrow (M/N_{t-1})_m \longrightarrow 0.$$

Choose an element $y \in 0 :_A (M/N_i) \setminus p$. By assumption, there is an element $x \in A \setminus \bigcup_{q \in \text{Min}_A(M)} q$ such that $xH^i_{m_{A_n}}(M_m) = 0$ for all $i < \text{ht}_M(m)$. Now, with a similar technique as in the proof of [12, Lemma 3.1 (i)] one can deduce that $(xy)^jH^i_{m}(A/p)_m = 0$ for all $i < \text{ht}_M(m)$ and for some integer $l > 0$. \[\Box\]

Corollary 2.15. Let $M$ be a finite $A$–module of finite dimension such its Cousin complex $C_A(M)$ is finite. Then the ring $A/0 :_A M$ is universally catenary.

Proof. By Theorem 2.7, $M$ has a uniform local cohomological annihilator. Now, the result follows by Theorem 2.14. \[\Box\]

### 3. Height of an ideal

As mentioned in Corollary 2.3 and in the proof of Theorem 2.7, we may write the following corollary.

Corollary 3.1. For any finite $A$–module $M$ and any ideal $a$ of $A$ with $aM \neq M$,

$$\prod_{-1 \leq i} (0 :_A H^i) \subseteq 0 :_A H^i_{htM(a)-1}(M).$$
We now raise the question that whether it is possible to improve the upper bound restriction.

**Question.** Does the inequality
\[ \prod_{-1 \leq i} (0 :_A H^i) \subseteq 0 :_A H^0_{ht_M(a)}(M) \]
hold?

It will be proved that the answer is negative for the class of finite \(A\)-modules \(M\) with finite Cousin cohomologies. More precisely,

**Theorem 3.2.** Assume that \(M\) is a finite \(A\)-module of finite dimension and that its Cousin complex \(C_A(M)\) is finite. Then
\[ ht_M(a) = \inf \{ r : \prod_{-1 \leq i} (0 :_A H^i) \nsubseteq 0 :_A H^r_a(M) \}, \]
for all ideals \(a\) with \(aM \neq M\).

**Proof.** By Corollary 2.3, \[ \prod_{i \geq -1} (0 :_A H^i) \subseteq 0 :_A \text{Ext}^r(\mathcal{A}/a^n, M) \] for all \(r, 0 \leq r < ht_M(a)\) and all \(n \geq 0\). Passing to the direct limit, as in the proof of Theorem 2.7, one has \[ \prod_{i \geq -1} (0 :_A H^i) \subseteq 0 :_A H^r_a(M) \] for all \(r < ht_M(a)\). Hence we have
\[ ht_M(a) \leq \inf \{ r : \prod_{-1 \leq i} (0 :_A H^i) \nsubseteq 0 :_A H^r_a(M) \}. \]

Thus it is sufficient to show that \[ \prod_{-1 \leq i} (0 :_A H^i) \nsubseteq 0 :_A H^0_{ht_M(a)}(M). \] By Independence Theorem of local cohomology (c.f. [1, Theorem 4.2.1]), \(H^0_{ht_M(a)}(M) = H^0_{ht_M(b)}(M)\) as \(\overline{A} = A/(0 :_A M)\)-module, where \(b = a + 0 :_A M/0 :_A M\). Note that \(ht_M(a) = ht_M(b)\) and that \(C_A(M) \cong C_{\overline{A}}(M)\) (see [2, Lemma 1.2]).

Hence we may assume that \(0 :_A M = 0\). Set \(h := ht_M(a)\). Let \(x \in 0 :_A H^0_a(M)\). As \(aM \neq M\), there exists a minimal prime \(q\) over \(a\) in \(\text{Supp}_A(M)\) such that \(dim \mathcal{A}_q = ht_M(a)\). Hence \(x/1 \in 0 :_A H^0_a(M_q)\). Thus, by any choice of \(pA_q \in \text{Assh}_A(M_q)\) we have \(x/1 \in pA_q\) (see [1, Proposition 7.2.11(ii) and Theorem 7.3.2]) and so \(x \in p\).

Therefore, one has \(0 :_A H^0_a(M) \subseteq \bigcup_{p \in \text{Min}_A(M)} p\). On the other hand, by Lemma 2.5,
\[ \prod_{i \geq -1} (0 :_A H^i) \nsubseteq \bigcup_{p \in \text{Min}_A(M)} p, \]
from which it follows that
\[ \prod_{i \geq -1} (0 :_A H^i) \nsubseteq 0 :_A H^0_a(M). \]

\[ \square \]

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