Generalized scalar field cosmologies: a global dynamical systems formulation

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Abstract
Local and global phase-space descriptions and averaging methods are used to find qualitative features of solutions for the FLRW and the Bianchi I metrics in the context of scalar field cosmologies with arbitrary potentials and arbitrary couplings to matter. The stability of the equilibrium points in a phase-space as well as the dynamics in the regime where the scalar field diverges are studied. Equilibrium points that represent some solutions of cosmological interest such as: several types of scaling solutions, a kinetic dominated solution representing a stiff fluid, a solution dominated by an effective energy density of geometric origin, a quintessence scalar field dominated solution, the vacuum de Sitter solution associated to the minimum of the potential, and a non-interacting matter dominated solution are obtained. All of which reveal a very rich cosmological phenomenology.

Keywords: scalar field cosmology, asymptotic behavior, global phase-space descriptions, averaging methods

(Some figures may appear in colour only in the online journal)

1. Introduction
There are several alternative scalar field theories of gravity of special interest, which include [1, 2], Horndeski theories [4], teleparallel analogue of Horndeski theories [5–7], inflationary models [3], extended quintessence, modified gravity, Horava–Lifschitz and the Galileons, etc, [8–15, 18–69]. There are also several studies in the literature which provide global and/or local dynamical systems analysis for scalar field cosmologies with arbitrary potentials and arbitrary couplings, see [70–93] and discussions therein.

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In [93], the action for a general class of scalar tensor theory of gravity written in the so-called Einstein’s frame was studied. This action is given by [94, 95]:

\[
\mathcal{L} = \int \! d^4x \sqrt{|g|} \left\{ \frac{1}{2} R - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) - \Lambda + \chi(\phi)^{-2} \mathcal{L}(\mu, \nabla \mu, \chi(\phi)^{-1} g_{\alpha\beta}) \right\},
\]

(1)

where a system of units in which $8\pi G = c = \hbar = 1$ is used. In this equation $R$ is the curvature scalar, $\phi$ is the scalar field, $\nabla_\alpha$ is the covariant derivative, $V(\phi)$ is the self-interaction potential, $\Lambda \geq 0$ is the cosmological constant, $\chi(\phi)^{-2}$ is the coupling function. $\mathcal{L}$ is the matter Lagrangian, and $\mu$ is the collective name for the matter degrees of freedom.

For the action (1), the matter energy-momentum tensor is defined by:

\[
T_{\alpha\beta} = -\frac{2}{\sqrt{|g|}} \frac{\delta}{\delta g^{\alpha\beta}} \left\{ \sqrt{|g|} \chi^{-2} \mathcal{L}(\mu, \nabla \mu, \chi^{-1} g_{\alpha\beta}) \right\}.
\]

(2)

The 'energy exchange' vector is defined by:

\[
Q_\beta \equiv \nabla^\alpha T_{\alpha\beta} = -\frac{1}{\chi(\phi)} \frac{d\gamma(\phi)}{d\phi} \nabla_\beta \phi, \quad T = T^\alpha_\alpha,
\]

(3)

where $T$ is the trace of the energy-momentum tensor. The matter is assumed in the form of a perfect fluid with energy density $\rho_m \geq 0$ and pressure $p_m = (\gamma - 1)\rho_m$, $1 \leq \gamma \leq 2$. The expression (3) implies that the scalar field $\phi$ is non-minimally coupled to matter with coupling function $\chi(\phi)$. The geometric properties of the metric are encoded in the function:

\[
G_0(\alpha) = \begin{cases} 
-\frac{3}{\sigma^2} k \alpha \gamma, & \text{spatial curvature of FLRW metrics,} \\
\frac{1}{\sigma^2} \alpha, & \text{anisotropies of Bianchi I metric.}
\end{cases}
\]

(4)

Within the above theory of gravity, the field equations are given by the dynamical system:

\[
\dot{H} = -\frac{1}{2} (\gamma \rho_m + \gamma^2) + \frac{1}{6} a G_0(\alpha),
\]

(5a)

\[
\dot{\rho}_m = -3\gamma H \rho_m - \frac{1}{2} (4 - 3\gamma) \rho_m \frac{d\gamma(\phi)}{d\phi},
\]

(5b)

\[
\dot{a} = a H,
\]

(5c)

\[
\dot{y} = -3 H y - \frac{dV(\phi)}{d\phi} + \frac{1}{2} (4 - 3\gamma) \rho_m \frac{d\gamma(\phi)}{d\phi},
\]

(5d)

\[
\dot{\phi} = y,
\]

(5e)

\[
3H^2 = \rho_m + \frac{1}{2} y^2 + V(\phi) + \Lambda + G_0(\alpha).
\]

(5f)

The phase-space can be defined using equation (5f) as follows:

\[
\left\{ (H, \rho_m, a, y, \phi) \in \mathbb{R}^5 : 3H^2 = \rho_m + \frac{1}{2} y^2 + V(\phi) + \Lambda + G_0(\alpha) \right\}.
\]

(6)

In paper [93] some theorems related to the asymptotic behavior of a very general cosmological model given by system (5) were presented. In particular, the conditions of a
Theorem 2.2.3 in [93]). Mild conditions under the potential for having \( \lim_{t \to \infty} \phi(t) = 0 \) and \( \lim_{t \to +\infty} \phi(t) = +\infty \) (stated as theorem 2.3.3 in [93]) were also considered. It was examined to which extent the hypotheses of the theorems proved in [93] can be relaxed in order to obtain the same conclusions, or to provide a counterexample, by means of generalized harmonic self-interacting potentials: \( V(\phi) = \mu^3 \left[ \frac{\chi_0}{\mu} + b f \cos \left( \frac{\phi}{\sqrt{\mu}} \right) \right] \), \( b \neq 0 \) and \( V_2(\phi) = \mu^3 \left[ b f \left( \cos(\delta) - \cos \left( \frac{\phi}{\sqrt{\mu}} \right) \right) + \phi^3 \right], b \neq 0 \). Harmonic potentials plus cosine corrections were introduced in the context of inflation in loop-quantum cosmology in [96].

Finally, the Hubble-normalized formulation for a scalar field non-minimally coupled to matter, with generalized harmonic potential \( V(\phi) = \frac{\phi^3}{2} + f \left( 1 - \cos \left( \frac{\phi}{\sqrt{\mu}} \right) \right), f > 0 \), and with coupling function \( \chi(\phi) = \chi_0 e^{\frac{\lambda \phi}{\sqrt{\mu}}} \), where \( \lambda \) is a constant and \( 0 \leq \gamma < 2, \ \gamma \neq \frac{3}{2} \), was performed for FLRW metrics and for the Bianchi I metric. Using theorem 2.4 in [93], the late time attractors correspond to the non zero local minimums of the potential, i.e., \( \phi = \phi_* \), satisfying \( \sin(\phi^*/f) + \phi^* = 0, 0 < |\phi^*| \leq 1, \ \cos \left( \frac{\phi^*}{\sqrt{\mu}} \right) + f > 0 \). Whenever \( \phi^* \) is a local non zero minimum of \( V(\phi) \) it follow \( \lim_{t \to \infty} \phi(t) = \phi^* \), \( \lim_{t \to \infty} \dot{\phi}(t) = 0 \), \( \lim_{t \to \infty} H(t) = \frac{V(\phi^*)}{\gamma} \). And \( lim_{t \to \infty} G_0(\alpha) = 0 \). The global minimum of \( V(\phi) \) at \( \phi = 0 \) is unstable to curvature perturbations for \( \gamma > \frac{3}{2} \) in the case of negatively curved FLRW model. This confirms the result in [97] that in a non-degenerate minimum with zero critical value, the curvature will eventually dominate both the perfect fluid and the scalar field densities on the late evolution of the Universe for \( \gamma > 2/3 \). For the Bianchi I model the global minimum \( V(0) = 0 \) is unstable to shear perturbations.

In this companion paper, complementary formulations based on [98] will be presented, including the analysis of the oscillations entering the system via the Klein–Gordon equation [99].

It is useful to define the quantities: \( \lambda = -\frac{V(\phi_0)}{V(\phi_0)}, \ f = \frac{V(\phi_0)}{V(\phi_0)} - \frac{V(\phi_0)}{V(\phi_0)} \) to obtain qualitative information about the past and future asymptotic structure of the equation’s solutions in scalar field theories with potential \( V(\phi) \). This is the basis of the so-called method of \( f \)-divisers. For common scalar field potentials the function \( f(\lambda) \) is found as follows: a monomial potential \( V(\phi) = \frac{1}{\mu}(\mu_\phi)^n, \mu > 0, n = 1, 2, \ldots [100] \) has \( f(\lambda) = -\frac{l}{\mu^n} \). The so-called \( E \)-model [101] with potential \( V(\phi) = V_0(1 - e^{-\sqrt{\mu} \phi})^{2n} \), has \( f(\lambda) = -\frac{l}{\mu^{2n}} \), where \( \mu = \frac{v}{\sqrt{\lambda}} \). An exponential potential \( V(\phi) = V_0 e^{-\lambda \phi} + V_1 \) [102–104] has \( f(\lambda) = -\lambda(\lambda - 1), V_1 \) is the cosmological constant. The hyperbolic potentials: \( V(\phi) = V_0 (\cosh(\xi \phi) - 1) \) [103, 105–114] with \( f(\lambda) = -\frac{1}{2}(\lambda^2 - \xi^2) \) and \( V(\phi) = V_0 \sinh(\gamma \phi) \) [103, 105, 106, 108, 109, 112, 115] with \( f(\lambda) = -\frac{\lambda^2}{\alpha} - \alpha \beta^2 \). The double exponential potential \( V(\phi) = V_0 (e^{\alpha \phi} + e^{\beta \phi}) \) [25, 116, 117] has \( f(\lambda) = -(\lambda + \alpha)(\lambda + \beta) \).

Our procedure for the qualitative analysis of a scalar field non-minimally coupled to matter relies on defining: \( \lambda = -\frac{W(\phi)}{W(\phi)}, \ f = \frac{W(\phi)}{W(\phi)} - \frac{W(\phi)}{W(\phi)} \), and \( g = -\frac{W(\phi)}{W(\phi)} \), where \( W(\phi) = \Lambda + V(\phi), W(\phi) \geq 0 \). It is then assumed that \( f \) and \( g \) can be explicitly written as functions of \( \lambda \) as in the original \( f \)-divisers method. This is a significant advantage due to the
analysis for an arbitrary potential can be performed as a first step, and then just substitute the desired potential’s forms instead of repeating the whole procedure for every distinct potential. Similar approaches were used in [58, 68] for two-field Jordan-Brans-Dicke cosmologies.

This paper is organized as follows. A local dynamical systems analysis using Hubble normalized equations is performed in section 2. Then, a dynamical systems formulation using global dynamical systems variables based on Alho & Uggla’s approach [98] is given in section 3. Section 3.1 is devoted to asymptotic analysis as \( \phi \to \infty \) for arbitrary \( V(\phi) \) and \( \chi(\phi) \). In section 3.2, the physical interpretation of solutions that were found are discussed. Section 4 is devoted to analysis of a scalar field model with potential \( V(\phi) = V_0 e^{-\lambda \phi} \) in a vacuum. In section 5, the so-called \( E \)-model with potential \( V(\phi) = V_0 \left( 1 - e^{-\sqrt{2} \phi} \right)^2 n \) is investigated. This potential was discussed in [101] for a conventional scalar field cosmology and in [59] for the Horava–Lifshitz cosmology. In sections 6 and 7, scalar field cosmologies under the potentials \( V_1(\phi) = \mu^3 \left[ e^{2\phi} + b f \cos \left( \delta + \frac{\phi}{\sqrt{2}} \right) \right] \), \( b \neq 0 \) and \( V_2(\phi) = \mu^3 \left[ b f \left( \cos(\delta) - \cos \left( \delta + \frac{\phi}{\sqrt{2}} \right) \right) + \phi^2 \right] \), \( b \neq 0 \) are respectively studied. Sections 6.1 and 7.1 are devoted to dynamical analysis as \( \phi \to \infty \). In sections 6.2 and 7.2, the study of the oscillatory behavior is presented. In section 8, the main results are summarized. Finally, section 9 is devoted to conclusions.

2. Local dynamical systems analysis for arbitrary \( V(\phi) \) and \( \chi(\phi) \)

For the analysis of the system (5) for an arbitrary \( V(\phi) \) and arbitrary \( \chi(\phi) \) we defined the quantities:

\[
\lambda(\phi) = -\frac{W'(\phi)}{W(\phi)}, \quad f(\phi) = \frac{W''(\phi)}{W(\phi)} - \frac{W'(\phi)^2}{W(\phi)^2}, \quad g(\phi) = -\frac{\chi'(\phi)}{\chi(\phi)}, \quad (7)
\]

where \( W(\phi) = \Lambda + V(\phi), W(\phi) \geq 0 \). The expressions (7) can be alternatively written as:

\[
\frac{d\lambda}{d\phi} = -f(\phi), \quad \frac{dW}{d\phi} = -\lambda(\phi)W(\phi), \quad \frac{d\chi}{d\phi} = -g(\phi)\chi(\phi). \quad (8)
\]

The idea is to assume that \( f \) and \( g \) can be explicitly written as functions of \( \lambda \). Hence, equation (8) are written as:

\[
\frac{d\phi}{d\lambda} = \frac{1}{f(\lambda)}, \quad \frac{dW}{d\lambda} = \frac{\lambda W(\lambda)}{f(\lambda)}, \quad \frac{d\chi}{d\lambda} = \frac{g(\lambda)\chi(\lambda)}{f(\lambda)}. \quad (9)
\]

which can be integrated in quadrature as:

\[
\phi(\lambda) = \phi(1) - \int_1^\lambda \frac{1}{f(\xi)} d\xi, \quad W(\lambda) = W(1)e^{\int_1^\lambda \frac{f(\xi)}{f(\xi)}} d\xi, \quad \chi(\lambda) = \chi(1)e^{\int_1^\lambda \frac{g(\xi)\chi(\xi)}{f(\xi)}} d\xi, \quad (10)
\]

which can be used to generate the potentials and couplings by giving \( f(\lambda) \) and \( g(\lambda) \) as inputs. Defining new variables:

\[
x = \frac{\dot{\phi}}{\sqrt{6H}}, \quad \Omega_m = \frac{\rho_m}{3H^2}, \quad \Omega_0 = \frac{G(\delta)a}{3H^2}, \quad \lambda = -\frac{V'(\phi)}{\Lambda + V(\phi)}, \quad (11)
\]
where $G_0(a) = qa^{-p}$, $p \geq 0$, and a new time variable $\tau = \ln a$, the following dynamical system is obtained:

\[
\frac{dx}{d\tau} = \frac{1}{2} x (3\gamma \Omega_m + p\Omega_0 + 6x^2 - 6) - \sqrt{\frac{3}{2}} \lambda (x^2 + \Omega_0 + \Omega_m - 1)
+ \frac{\sqrt{6}}{4} (3\gamma - 4)\Omega_0 f(\lambda),
\]

\[
\frac{d\Omega_0}{d\tau} = \frac{1}{2} \Omega_m \left( \sqrt{6} (4 - 3\gamma) x g(\lambda) + 2 (3\gamma (\Omega_m - 1) + p\Omega_0 + 6x^2) \right),
\]

\[
\frac{d\Omega_m}{d\tau} = \Omega_0 (3\gamma \Omega_m + p(\Omega_0 - 1) + 6x^2),
\]

\[
\frac{d\lambda}{d\tau} = -\sqrt{6} x f(\lambda),
\]

with the restriction

\[
x^2 + \Omega_m + \Omega_0 = 1 - \frac{W(\phi)}{3H^2} \leq 1.
\]

The equilibrium points of the system (12) are the following:

$A_1(\lambda): (x, \Omega_m, \Omega_0, \lambda) = \left( \frac{4(4 - 3\gamma)\Omega_m}{3(4 - 3\gamma)^2 g(\lambda)^2}, 1 - \frac{4(4 - 3\gamma)^3 g(\lambda)^2}{3(4 - 3\gamma)^2 g(\lambda)^2}, 0, \lambda \right)$, where $\lambda$ denotes any value of $\lambda$ satisfying $f(\lambda) = 0$, represents a matter-kinetic scaling solution. For $1 \leq \gamma < 2$, it follows that $A_1(\lambda)$ is a sink for one of the conditions (a)–(j) in appendix A. If it exists, it will never be a source.

$A_2(\lambda): (x, \Omega_m, \Omega_0, \lambda) = \left( \frac{\sqrt{6} (4 - 3\gamma) x g(\lambda)}{4(4 - 3\gamma)^2 g(\lambda)^2 + 2\lambda}, \frac{2(4 - 3\gamma) x g(\lambda)^2 + 4(\lambda - 3\gamma)}{(4 - 3\gamma) x g(\lambda)^2 + 2\lambda} + 1, \lambda \right)$ represents a matter-scalar field-geometric ‘fluid’ scaling solution. For $1 \leq \gamma < 2$, it follows that $A_2(\lambda)$ is a sink for one of the conditions (a)–(h) in appendix A. It is non-hyperbolic for $p = 6$, and a saddle for $p = 2$. If exists, it will never be a source.

$A_3(\lambda): (x, \Omega_m, \Omega_0, \lambda) = \left( \frac{\sqrt{6} x g(\lambda)^2 + 2\lambda}{4(4 - 3\gamma)^2 g(\lambda)^2 + 2\lambda}, \frac{2(4 - 3\gamma) x g(\lambda)^2 + 4(\lambda - 3\gamma)}{(4 - 3\gamma) x g(\lambda)^2 + 2\lambda} + 1, \lambda \right)$ represents a matter-scalar field scaling solution. In this case, it can proceed semi-analytically. That is, for non-minimal coupling $g \equiv 0$ and assuming $1 \leq \gamma < 2$, $A_3(\lambda)$ is a sink for:

(a) $1 \leq \gamma < 2$, $p > 3\gamma$, $-\frac{4(4 - 3\gamma)}{3(4 - 3\gamma)^2 g(\lambda)^2} \leq \lambda < -\sqrt{3\gamma}$, $f'(\lambda) < 0$, or

(b) $1 \leq \gamma < 2$, $p > 3\gamma$, $\sqrt{3\gamma} \leq \lambda \leq \frac{4(4 - 3\gamma)}{3(4 - 3\gamma)^2 g(\lambda)^2}$, $f'(\lambda) > 0$.

Otherwise, it is a saddle. For non-minimal coupling the analysis has to be done numerically.

$A_4(\lambda): (x, \Omega_m, \Omega_0, \lambda) = \left( -1, 0, 0, \lambda \right)$ is a kinetic dominated solution representing a stiff fluid. $A_4(\lambda)$ is a source for:

(a) $1 \leq \gamma < \frac{4}{3}$, $0 \leq p < 6$, $g(\lambda) < \frac{\sqrt{6} x g(\lambda)^2 + 2\lambda}{3(4 - 3\gamma)^2 g(\lambda)^2}$, $f'(\lambda) > 0$, $\lambda > -\sqrt{6}$, or

(b) $\gamma = \frac{4}{3}$, $0 \leq p < 6$, $f'(\lambda) > 0$, $\lambda > -\sqrt{6}$, or

(c) $\frac{4}{3} < \gamma < 2$, $0 \leq p < 6$, $g(\lambda) > \frac{\sqrt{6} x g(\lambda)^2 + 2\lambda}{3(4 - 3\gamma)^2 g(\lambda)^2}$, $f'(\lambda) > 0$, $\lambda > -\sqrt{6}$.

$A_4(\lambda)$ is a sink for:

(a) $1 \leq \gamma < \frac{4}{3}$, $p > 6$, $g(\lambda) > \frac{\sqrt{6} x g(\lambda)^2 + 2\lambda}{3(4 - 3\gamma)^2 g(\lambda)^2}$, $f'(\lambda) < 0$, $\lambda < -\sqrt{6}$, or
(b) $\frac{4}{3} < \gamma < 2$, $p > 6$, $g(\lambda) < -\frac{\sqrt{6(\gamma-2)}}{3-\gamma}$, $f'(\lambda) < 0$, $\hat{\lambda} < -\sqrt{6}$.

$A_3(\hat{\lambda})$: $(x, \Omega_m, \Omega_0, \lambda) = \left(0, 0, 1, \hat{\lambda}\right)$ is a solution dominated by the effective energy density of $G_0(a)$. It is non-hyperbolic with a 3D unstable manifold for $p > 6$.

$A_4(\hat{\lambda})$: $(x, \Omega_m, \Omega_0, \lambda) = \left(1, 0, 0, \hat{\lambda}\right)$ is a kinetic dominated solution representing a stiff fluid.

$A_5(\hat{\lambda})$ is a source for:
(a) $1 \leq \gamma < \frac{4}{3}$, $p < 6$, $g(\lambda) > -\frac{\sqrt{6(\gamma-2)}}{3-\gamma}$, $f'(\lambda) < 0$, $\hat{\lambda} < \sqrt{6}$, or
(b) $\gamma = \frac{4}{3}$, $p < 6$, $f'(\lambda) < 0$, $\hat{\lambda} < \sqrt{6}$, or
(c) $\frac{4}{3} < \gamma < 2$, $p < 6$, $g(\lambda) < -\frac{\sqrt{6(\gamma-2)}}{3-\gamma}$, $f'(\lambda) < 0$, $\hat{\lambda} < \sqrt{6}$.

$A_6(\hat{\lambda})$ is a sink for:
(a) $1 \leq \gamma < \frac{4}{3}$, $p > 6$, $g(\lambda) > -\frac{\sqrt{6(\gamma-2)}}{3-\gamma}$, $f'(\lambda) > 0$, $\hat{\lambda} > \sqrt{6}$, or
(b) $\gamma > \frac{4}{3}$, $p > 6$, $g(\lambda) > -\frac{\sqrt{6(\gamma-2)}}{3-\gamma}$, $f'(\lambda) > 0$, $\hat{\lambda} > \sqrt{6}$.

$A_7(\hat{\lambda})$: $(x, \Omega_m, \Omega_0, \lambda) = \left(\frac{4}{3}, 0, 1 - \frac{2}{3\gamma}, \hat{\lambda}\right)$ is a scaling solution where the energy density of the scalar field and the effective energy density from the $G_0(a)$ scale with the same order of magnitude. For $1 \leq \gamma \leq 2$, $A_7(\hat{\lambda})$, is a sink for one of the conditions (a)–(ar) in appendix A. It will never be a source.

$A_8(\hat{\lambda})$: $(x, \Omega_m, \Omega_0, \lambda) = \left(\frac{4}{3}, 0, 0, \hat{\lambda}\right)$ represents the typical quintessence scalar field dominated solution. Assuming $1 \leq \gamma \leq 2$, $A_8(\hat{\lambda})$ is a sink for:
(a) $1 \leq \gamma < \frac{4}{3}$, $p > \hat{\lambda}^2$, $f'(\hat{\lambda}) > 0$, $0 < \hat{\lambda} < \sqrt{6}$, $g(\hat{\lambda}) < \frac{6\gamma-2\hat{\lambda}^2}{4\hat{\lambda}-3\gamma\hat{\lambda}}$, or
(b) $1 \leq \gamma < \frac{4}{3}$, $p > \hat{\lambda}^2$, $g(\hat{\lambda}) > \frac{6\gamma-2\hat{\lambda}^2}{4\hat{\lambda}-3\gamma\hat{\lambda}}$, $-\sqrt{6} < \hat{\lambda} < 0$, $f'(\hat{\lambda}) < 0$, or
(c) $\gamma = \frac{4}{3}$, $p > \hat{\lambda}^2$, $-2 < \hat{\lambda} < 0$, $f'(\hat{\lambda}) < 0$, or
(d) $\gamma = \frac{4}{3}$, $p > \hat{\lambda}^2$, $f'(\hat{\lambda}) > 0$, $0 < \hat{\lambda} < 2$, or
(e) $\frac{4}{3} < \gamma < 2$, $p > \hat{\lambda}^2$, $g(\hat{\lambda}) > \frac{6\gamma-2\hat{\lambda}^2}{4\hat{\lambda}-3\gamma\hat{\lambda}}$, $f'(\hat{\lambda}) > 0$, $0 < \hat{\lambda} < \sqrt{6}$, or
(f) $\frac{4}{3} < \gamma < 2$, $p > \hat{\lambda}^2$, $-\sqrt{6} < \hat{\lambda} < 0$, $g(\hat{\lambda}) < \frac{6\gamma-2\hat{\lambda}^2}{4\hat{\lambda}-3\gamma\hat{\lambda}}$, $f'(\hat{\lambda}) < 0$.

It will never be a source.

$A_9$: $(x, \Omega_m, \Omega_0, \lambda) = (0, 0, 0, 0)$ represents the vacuum de Sitter solution associated to the minimum of the potential. It is a sink for $p > 0$, $\gamma > 0$, $f(0) > 0$. Otherwise, it is a saddle.

$A_{10}(\hat{\lambda})$: $(x, \Omega_m, \Omega_0, \lambda) = \left(0, 1, 0, \hat{\lambda}\right)$, where we denote by $\hat{\lambda}$, the values of $\lambda$ for which $g(\lambda) = 0$, represents a non-interacting matter dominated solution. It is a saddle.

3. Global dynamical analysis for arbitrary $V(\phi)$ and $\chi(\phi)$

In section 2, the stability of the equilibrium points using Hubble-normalized equations, which is essentially based on the Copeland, Liddle & Wands’s approach [69] was investigated. It is well-known that this procedure is well-suited to investigate local stability features of the equilibrium points. However, it does not provides a global description of the phase space when generically $\phi$ diverges or when $H \rightarrow 0$. In these cases the method fails. For this reason, a new global systems analysis for arbitrary $V(\phi)$ and $\chi(\phi)$ as $|\phi| \rightarrow \infty$ is presented based on Alho and Uggla’s approach [98]. Doing so, we set $L = 0$ for simplicity.
Now we assume that the following limits exist:

\[ N = \lim_{\phi \to +\infty} \frac{V'(\phi)}{V(\phi)}, \quad M = \lim_{\phi \to +\infty} \frac{\chi'(\phi)}{\chi(\phi)} \]  

(14)

This permits the definition of the new functions \( W_V(\phi) \), \( W_\chi(\phi) \):

\[ W_V(\phi) = \frac{\chi'(\phi)}{V(\phi)} - N, \quad W_\chi(\phi) = \frac{\chi'(\phi)}{\chi(\phi)} - M, \]  

(15)

where the constants \( N \) and \( M \) are such that:

\[ \lim_{\phi \to +\infty} W_V(\phi) = 0, \quad \lim_{\phi \to +\infty} W_\chi(\phi) = 0. \]  

(16)

Afterwards, we define the following variables:

\[
T = \frac{m}{m + H}, \quad \theta = \tan^{-1}\left(\frac{\dot{\phi}}{\sqrt{2V(\phi) + 2\rho_m + 2G_0(a)}}\right), \\
\Omega_m = \frac{\rho_m}{3H^2}, \quad \Omega_0 = \frac{G_0(a)}{3H^2}, \quad m > 0,
\]

where \( G_0(a) = qa^{-p}, p \geq 0 \), such that:

\[ V(\phi) = \frac{3m^2(1 - T)^2(\cos(2\theta) - 2\Omega_0 - 2\Omega_m + 1)}{2T^2}, \]

\[ H = m\left(\frac{1 - T}{T}\right), \quad \dot{\phi} = \frac{\sqrt{6m(1 - T)\sin(\theta)}}{T}, \quad \rho_m = \frac{3m^2(1 - T)^2\Omega_m}{T^2}. \]  

(18)

Finally, the time variable \( \tau = \ln a \) is defined resulting in the following unconstrained dynamical system:

\[
\frac{dT}{d\tau} = \frac{1}{2}(1 - T)T \left(3\gamma \Omega_m + 6 \sin^2(\theta) + p\Omega_0\right), \\
\frac{d\theta}{d\tau} = -\sec(\theta) \left\{3\sqrt{6} \sin(3\theta) + 6(\cos(2\theta) - 2\Omega_0 + 1)(N + W_V(\phi)) + \Omega_m \left(-6\sqrt{\gamma} \sin(\theta) - 12(2M + N + W_V(\phi) + 2W_\chi(\phi)) + 18\gamma(M + W_\chi(\phi))\right) + \sqrt{6} \sin(\theta)(3 - 2p\Omega_0)\right\}, \\
\frac{d\Omega_m}{d\tau} = -\frac{1}{2}\Omega_m \left\{6 \cos(2\theta) - \sqrt{6}(3\gamma - 4) \sin(\theta)(M + W_\chi(\phi)) - 2(3\gamma(\Omega_m - 1) + p\Omega_0 + 3)\right\}, \\
\frac{d\Omega_0}{d\tau} = -\Omega_0(3\gamma\Omega_m + \Omega_0(3\gamma - 4) \cos(2\theta) - p\Omega_0 + p - 3), \\
\frac{d\phi}{d\tau} = \sqrt{6} \sin(\theta). \]  

(19a-d)

Rather than discussing all the equilibrium points of (19) (which is a complementary formulation of the system (12)), the focus will be to study what happens in the region of \( \phi \to +\infty \).
Due to the symmetries of the model, it is not necessary to study the limit $\phi \to -\infty$, which can be accomplished by taking the reversal $\phi \to -\phi$.

3.1. Asymptotic analysis as $\phi \to \infty$

In this section the stability of the equilibrium points of (19) as $\phi \to \infty$ (for functions $V$ and $\chi$ which are well-behaved at infinity of exponential orders of $N$ and $M$, respectively) is discussed. This analysis at infinity complements the analysis of section 2.

**Definition 1. (Definition 1 [70])** Let $V: \mathbb{R} \to \mathbb{R}$ be a $C^2(\mathbb{R})$ nonnegative function. Let there exist some $\phi_0 > 0$ for which $V(\phi) > 0$ for all $\phi > \phi_0$ and some number $N < \infty$, such that the function:

$$W_V: [\phi_0, \infty) \to \mathbb{R}, \quad \phi \to \frac{V'(\phi)}{V(\phi)} - N$$

is well-defined and satisfies:

$$\lim_{\phi \to \infty} W_V(\phi) = 0.$$  \hspace{1cm} (21)

Then $V$ is said to be well-behaved at infinity (WBI) of exponential order $N$.

**Definition 2. (Definition 2 [70])** A $C^k(\mathbb{R})$ function $V(\phi)$ is a class $k$ WBI function, if it is WBI of exponential order $N$, and there are $\phi_0 > 0$, and a coordinate transformation $\varphi = h(\phi)$ which maps the interval $[\phi_0, \infty)$ onto $(0, \epsilon]$, where $\epsilon = h(\phi_0)$, satisfying $\lim_{\phi \to +\infty} h(\phi) = 0$, and has the following additional properties:

(a) $h$ is $C^{k+1}$ and strictly decreasing.

(b) The functions:

$$W_V(\varphi) = \begin{cases} V'(h^{-1}(\varphi)) \frac{V'(h^{-1}(\varphi))}{V(h^{-1}(\varphi))} - N, & \varphi > 0, \\ 0, & \varphi = 0 \end{cases}$$ \hspace{1cm} (22)

and

$$\tilde{h}'(\varphi) = \begin{cases} h'(h^{-1}(\varphi)), & \varphi > 0, \\ \lim_{\varphi \to \infty} h'(\phi), & \varphi = 0 \end{cases}$$ \hspace{1cm} (23)

are $C^k$ on the closed interval $[0, \epsilon]$ and

(c)

$$\frac{dW_V}{d\varphi}(0) = \frac{d\tilde{h}'}{d\varphi}(0) = 0.$$ \hspace{1cm} (24)

The last hypotheses are equivalent to:

$$\lim_{\varphi \to 0} \frac{W_V'(h^{-1}(\varphi))}{\tilde{h}'(h^{-1}(\varphi))} \quad \text{and} \quad \lim_{\varphi \to 0} \frac{W_V''(h^{-1}(\varphi))}{\tilde{h}''(h^{-1}(\varphi))} = 0.$$  

Assuming the $C^2$ functions $V(\phi)$ and $\chi(\phi)$ are class 2 WBI, the unconstrained dynamical system is obtained:

$$\frac{dT}{dt} = \frac{1}{2}(1 - T)T \left(3\gamma \Omega_m + 6 \sin^2(\theta) + p\Omega_0 \right),$$ \hspace{1cm} (25a)
\[
\frac{d\theta}{d\tau} = -\frac{\sec(\theta)}{4\sqrt{6}} \left( 3\sqrt{6} \sin(3\theta) + 6(\cos(2\theta) - 2\Omega_0 + 1)(N + W_\chi(\varphi)) ight) \\
+ \Omega_m \left( -6\sqrt{6} \sin(\theta) - 12(2M + N + W_\chi(\varphi) + 2W_\chi(\varphi)) ight) \\
+ 18\gamma(M + W_\chi(\varphi)) + \sqrt{6} \sin(\theta)(3 - 2p\Omega_0) \right),
\]
(25b)
\[
\frac{d\Omega_m}{d\tau} = -\frac{1}{2} \Omega_m \left\{ 6 \cos(2\theta) - \sqrt{6}(3\gamma - 4) \sin(\theta)(M + W_\chi(\varphi)) \\
- 2(3\gamma(\Omega_m - 1) + p\Omega_0 + 3) \right\},
\]
(25c)
\[
\frac{d\Omega_0}{d\tau} = -\Omega_0(-3\gamma\Omega_m + 3 \cos(2\theta) - p\Omega_0 + p - 3),
\]
(25d)
\[
\frac{d\varphi}{d\tau} = \sqrt{6}h'(\varphi) \sin(\theta),
\]
(25e)

defined on the phase space:
\[\{(T, \theta, \Omega_m, \Omega_0, \varphi) \in \mathbb{R}^5 : 0 \leq T \leq 1, \quad \frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \varphi \leq h(\phi_0), \quad 2T^2 V(h^{-1}(\varphi)) = 3m^2(1 - T)^2(\cos(2\theta) - 2\Omega_0 - 2\Omega_m + 1)\}. \]
(26)

\(\theta\) is unique modulo 2\(\pi\). These have been chosen such that \(\cos \theta \geq 0\). In the following list \(\tan^{-1}[x, y]\) gives the arc tangent of \(y/x\), taking into account on which quadrant the point \((x, y)\) is in. When \(x^2 + y^2 = 1\), \(\tan^{-1}[x, y]\) gives the number \(\theta\) such that \(x = \cos \theta\) and \(y = \sin \theta\).

The equilibrium points of system (25) with \(\varphi = 0\) (i.e., corresponding to \(\phi \to \infty\)) are the following:

\(B_1: (0, \tan^{-1}\left(\sqrt{1 - \frac{N^2}{M^2}}, -\frac{N}{\sqrt{6}M}\right) + 2\pi c_1, 0, 0, 0)\), \(c_1 \in \mathbb{Z}\), represents a scalar field dominated solution satisfying \(H \to \infty\). It is always a non-hyperbolic saddle.

\(B_2: (1, \tan^{-1}\left(\sqrt{1 - \frac{N^2}{M^2}}, -\frac{N}{\sqrt{6}M}\right) + 2\pi c_1, 0, 0, 0)\), \(c_1 \in \mathbb{Z}\), represents a scalar field dominated solution, satisfying \(H \to 0\). The case of physical interest is when \(B_2\) is non-hyperbolic with a 4D stable manifold for one of the conditions (a)–(j) of appendix B.

\(B_3: (0, 2\pi c_1, 0, 1, 0)\), \(c_1 \in \mathbb{Z}\), with eigenvalues \(\{0, \frac{\pi}{2}, \frac{\pi}{2}, p, p - 3\gamma\}\). It represents a geometric ‘fluid’ dominated solution with \(H \to \infty\). The physical interesting situation occurs when \(B_3\) is non-hyperbolic with a 4D unstable manifold for \(p > 6\), \(1 \leq \gamma \leq 2\). It is a non-hyperbolic saddle otherwise.

\(B_4: (1, 2\pi c_1, 0, 1, 0)\), \(c_1 \in \mathbb{Z}\), represents a geometric ‘fluid’ dominated solution with \(H \to 0\). It is a non-hyperbolic saddle.

\(B_5: (0, \tan^{-1}\left(\sqrt{1 - \frac{p^2}{6\gamma}}, -\frac{p}{\sqrt{6}\gamma}\right) + 2\pi c_1, 0, 1 - \frac{p}{\sqrt{6}\gamma}, 0)\), \(c_1 \in \mathbb{Z}\), represents a scaling solution where neither the energy density of the geometric ‘fluid’, nor the energy density of the scalar field completely dominates, which satisfies \(H \to \infty\). It is a non-hyperbolic saddle.

\(B_6: (1, \tan^{-1}\left(\sqrt{1 - \frac{p^2}{6\gamma}}, -\frac{p}{\sqrt{6}\gamma}\right) + 2\pi c_1, 0, 1 - \frac{p}{\sqrt{6}\gamma}, 0)\), \(c_1 \in \mathbb{Z}\), represents a scaling solution where neither the energy density of geometric ‘fluid’ nor the energy density of the scalar field completely dominates, which satisfies \(H \to 0\). The situation of physical interest is
when $B_6$ is non-hyperbolic with a 4D stable manifold for one of the conditions (a)–(an) in appendix B.

$B_7$: \( 0, \tan^{-1}\left( \frac{\sqrt{6(2-\gamma^2)-4(4-\gamma)\gamma M^2}}{\sqrt{6(2-\gamma)}} \right) \), \( c_1 \in \mathbb{Z} \),

represents a matter-scalar field—geometric ‘fluid’ scaling solution with $H \to \infty$. It is a non-hyperbolic saddle.

$B_8$: \( (1, \tan^{-1}\left( \frac{\sqrt{6(2-\gamma^2)-4(4-\gamma)\gamma M^2}}{\sqrt{6(2-\gamma)}} \right) \), \( c_1 \in \mathbb{Z} \),

represents a matter-scalar field—geometric ‘fluid’ scaling solution with $H \to 0$. The situation of physical interest is when $B_8$ is non-hyperbolic with a 4D stable manifold for one of the conditions (a)–(k) of appendix B.

$B_9$: \( 0, \tan^{-1}\left( \frac{\sqrt{6(2-\gamma^2)-4(4-\gamma)\gamma M^2}}{\sqrt{6(2-\gamma)}} \right) \), \( c_1 \in \mathbb{Z} \),

represents a matter-scalar field scaling solution with $H \to \infty$. It is a non-hyperbolic saddle.

$B_{10}$: \( 1, \tan^{-1}\left( \frac{\sqrt{6(2-\gamma^2)-4(4-\gamma)\gamma M^2}}{\sqrt{6(2-\gamma)}} \right) \), \( c_1 \in \mathbb{Z} \),

represents a matter-scalar field scaling solution with $H \to 0$. The situation of physical interest is when $B_{10}$ is non-hyperbolic with a 4D stable manifold for:

(a) \( \frac{\pi}{4} < \gamma < \frac{\pi}{2} \), \( N < -\sqrt{6} \), \( \frac{N}{3\gamma} < M \frac{6\gamma^2-12c+6N^2}{(3\gamma-4)^2} < M < \frac{\sqrt{6(2-\gamma^2)}-2\sqrt{6}}{3\gamma-4} \), \( p > 0 \),

(b) \( \frac{\pi}{4} < \gamma < 2 \), \( N < -\sqrt{6} \), \( \frac{2\sqrt{6}-\sqrt{6(2-\gamma^2)}}{3\gamma-4} \) \( \frac{N}{3\gamma} < M \frac{6\gamma^2-12c+6N^2}{(3\gamma-4)^2} < M < \frac{\sqrt{6(2-\gamma^2)}-2\sqrt{6}}{3\gamma-4} \), \( p > 0 \),

(c) \( \frac{\pi}{4} < \gamma < \frac{\pi}{2} \), \( N > \sqrt{6} \), \( \frac{N}{3\gamma} < M \frac{6\gamma^2-12c+6N^2}{(3\gamma-4)^2} < M < \frac{2\sqrt{6}+\sqrt{6(2-\gamma^2)}}{3\gamma-4} \), \( p > 0 \),

(d) \( \frac{\pi}{4} < \gamma < 2 \), \( N > \sqrt{6} \), \( \frac{N}{3\gamma} \), \( \frac{N}{3\gamma} < M \frac{6\gamma^2-12c+6N^2}{(3\gamma-4)^2} < M < \frac{2\sqrt{6}+\sqrt{6(2-\gamma^2)}}{3\gamma-4} \), \( p > 0 \).

$B_{11}$: \( 0, \tan^{-1}\left( \frac{\sqrt{6\gamma^2}}{2N+M(4-3\gamma)^2} \right) \), \( c_1 \in \mathbb{Z} \),

represents a matter-scalar field scaling solution with $H \to \infty$. It is a hyperbolic saddle.

$B_{12}$: \( 1, \tan^{-1}\left( \frac{\sqrt{6\gamma^2}}{2N+M(4-3\gamma)^2} \right) \), \( c_1 \in \mathbb{Z} \),

represents a matter-scalar field scaling solution with $H \to 0$. The situation of physical interest occurs when $B_{12}$ is non-hyperbolic with a 4D stable manifold for one of the conditions (a)–(az) in appendix B.

3.2. Discussion

Firstly, a dynamical system analysis of the system (5) for arbitrary $V(\phi)$ and arbitrary $\chi(\dot{\phi})$ formulated as system (12) was provided. Secondly, to complement the analysis of system (12), the dynamics of the system (5) at the limit $\phi \to +\infty$ formulated as system (25) was analyzed. In both cases, the equilibrium points in the finite region of the phase space (local analysis) as well as in the limit $\phi \to +\infty$ (global analysis at infinity) were exhaustively examined. Equilibrium points which represent some solutions of cosmological interest, such as several types of scaling solutions, were obtained. In particular, such solutions include: a kinetic dominated solution representing a stiff fluid, a solution dominated by an effective energy density of geometric origin, a quintessence scalar field dominated solution, the vacuum de Sitter solution associated to the minimum of the potential, and a non-interacting matter dominated solution.
4. Example: the scalar field model with potential $V(\phi) = V_0 e^{-\lambda \phi}$ in vacuum

In this section, a scalar field model with potential $V(\phi) = V_0 e^{-\lambda \phi}$ in a vacuum for the flat FLRW metric is considered. That is, $N = -\lambda, M = 0, W_\gamma \equiv 0, W_\chi \equiv 0, \Omega_m \equiv 0, \Omega_0 \equiv 0, H_0 \equiv 0, G_0 \equiv 0$, and $\chi \equiv 1$.

The traditional formulation of the stability analysis of the equilibrium points in a cosmological setup uses Hubble-normalized equations, which is essentially based on the Copeland, Liddle and Wands’s approach [69]. It is well-known that this procedure is well-suited to investigate local stability features of the equilibrium points.

Defining the variables:

$$x = \frac{\dot{\phi}}{\sqrt{6} H}, \quad y = \frac{\sqrt{V(\phi)}}{\sqrt{3} H}, \quad (27)$$

and $\tau \equiv \ln a$, through $d\tau = H dt$, $H > 0$, the following dynamical system is obtained:

$$\frac{dx}{d\tau} = 3x^3 - 3x + \sqrt{\frac{3}{2}} \lambda y^2, \quad \frac{dy}{d\tau} = 3xy \left( x - \frac{\sqrt{6}}{6} \right). \quad (28)$$

This system can be reduced in one dimension using the relation $x^2 + y^2 = 1$:

$$\frac{dx}{d\tau} = f(x) := -3 \left(1 - x^2\right) \left( x - \frac{\sqrt{6}}{6} \right). \quad (29)$$

Equation (29) is integrable and leads to

$$\tau := \ln a = c_1 + \ln \left[ (1 - x)\frac{1}{\sqrt{6}} (x + 1)\frac{1}{\sqrt{6}} \right] 6x - \sqrt{6} \left[ \frac{\sqrt{6}}{6} \right]. \quad (30)$$

That is, the scaling factor $a$ can be expressed as a function of $x$ by

$$a(x) = e^{c_1} (1 - x)^{\frac{1}{12}} (x + 1)^{\frac{1}{12}} \left[ 6x - \sqrt{6} \left( \frac{\sqrt{6}}{6} \right) \right], \quad x \in [-1, 1]. \quad (31)$$

Additionally,

$$\frac{d\phi}{dx} = \frac{d\phi}{d\tau} = -\frac{2\sqrt{6}x}{(1 - x^2) \left( 6x - \sqrt{6} \right)}. \quad (32)$$

from which it follows by integration that:

$$\phi(x) = c_2 + \ln \left[ (1 - x)^{\frac{1}{\sqrt{6} - 3}} (x + 1)^{\frac{1}{\sqrt{6} + 3}} \right] 6x - \sqrt{6} \left[ \frac{\sqrt{6}}{6} \right]. \quad (33)$$

On the other hand, from (27) with $V(\phi) = V_0 e^{-\lambda \phi}$ holds:

$$H(x) = e^{\frac{c_2}{2}} \sqrt{\frac{V_0}{3 - 3x^2}} (1 - x)^{\frac{1}{\sqrt{6} - 3}} (x + 1)^{\frac{1}{\sqrt{6} + 3}} \left[ 6x - \sqrt{6} \left( \frac{\sqrt{6}}{6} \right) \right]. \quad (34)$$

Finally, it follows that:

$$\frac{dt}{dx} = \frac{2\sqrt{3} e^{c_2} (1 - x)^{\frac{1}{\sqrt{6} - 3}} (x + 1)^{\frac{1}{\sqrt{6} + 3}} 6x - \sqrt{6} \left( \frac{\sqrt{6}}{6} \right)}{\sqrt{V_0(1 - x^2)} \left( 6x - \sqrt{6} \right)}. \quad (35)$$
Figure 1. $a(x)$ given by (31) with $c_1 = 0$, $\phi(x)$ given by (33) with $c_2 = 0$ and $H(x)$ given by (34) with $c_2 = 0$, $V_0 = 1$, for several choices of $\lambda$.

$$t = 2\sqrt{3} e^{\frac{c_1}{\lambda}} \int \frac{(1 - x)^{c_1}}{(x + 1)^{c_1}} \left[ 6x - \sqrt{6} \lambda \right] \frac{-\frac{1}{\lambda^2} \sqrt{6}}{\sqrt{V_0(1 - x^2)(6x - \sqrt{6}\lambda)}} \, dx.$$ \hspace{1cm} (36)

Figure 1 show $a(x)$, given by (31) for $c_1 = 0$, represented by the solid thick red line; $\phi(x)$ given by (33) for $c_2 = 0$, represented by solid thinner blue line; and $H(x)$, given by (34) for $c_2 = 0$, $V_0 = 1$, represented by the dashed black line for several choices of $\lambda$. The cosmological time $t$ is related with $x$ by the quadrature (36).

Figure 1(a) shows:

$$\lim_{x \to -1} (a(x), \phi(x), H(x)) = (0, +\infty, +\infty),$$ \hspace{1cm} (37)

$$\lim_{x \to 1} (a(x), \phi(x), H(x)) = (0, -\infty, +\infty),$$ \hspace{1cm} (38)

$$\lim_{x \to \sqrt{\frac{\lambda}{6}}} (a(x), \phi(x), H(x)) = (+\infty, -\infty, 0).$$ \hspace{1cm} (39)

Figure 1(b) shows:

$$\lim_{x \to -1} (a(x), \phi(x), H(x)) = (0, +\infty, +\infty),$$ \hspace{1cm} (40)

$$\lim_{x \to 1} (a(x), \phi(x), H(x)) = (0, -\infty, +\infty),$$ \hspace{1cm} (41)
whenthescalarfield is divergent or the Hubble parameter can be either zero, infinity or it tends to a finite non-zero number. But, the above limits show that at the equilibrium points the scalar field eventually diverges and although this procedure is well-suited to investigate local stability features of the equilibrium points, it does not provide a global description of the phase space when generically \( \phi \) diverges or when \( H \to 0 \). In such case the method fails. For this reason, a new global systems analysis motivated by Alho & Uggla’s approach [98] is presented.

Using

\[
T = \frac{m}{m + H}, \quad \theta = \tan^{-1} \left( \frac{\dot{\phi}}{\sqrt{2V(\phi)}} \right),
\]

the following unconstrained dynamical system is deduced:

\[
\frac{dT}{d\tau} = 3(1 - T)T \sin^2(\theta), \quad \frac{d\theta}{d\tau} = \frac{1}{2} \cos(\theta) \left( \sqrt{6\lambda} - 6 \sin(\theta) \right),
\]

defined in the finite cylinder \( S \) with boundaries \( T = 0 \) and \( T = 1 \).

The variable \( T \) is suitable for global analysis [98], due to

\[
\frac{dT}{d\tau} \bigg|_{\sin \theta = 0} = 0, \quad \frac{d^2T}{d\tau^2} \bigg|_{\sin \theta = 0} = 0, \quad \frac{d^3T}{d\tau^3} \bigg|_{\sin \theta = 0} = 9\lambda^2(1 - T)T.
\]

From the first equation of (50) and equation (51), \( T \) is a monotonically increasing function on \( S \). As a consequence, all orbits originate from the invariant subset \( T = 0 \) (which contains the \( \alpha \)-limit), which is classically related to the initial singularity with \( H \to \infty \), and ends on the invariant boundary subset \( T = 1 \), which corresponds asymptotically to \( H = 0 \).

The equilibrium points of equation (50) are the following:

(a) \( P_1 : (T, \theta) = (0, -\frac{\pi}{3} + 2c_1\pi), \quad c_1 \in \mathbb{Z} \), with eigenvalues \( \left\{ 3, \sqrt{\frac{4}{3}\lambda} + 3 \right\} \). It represents a kinetic dominated solution with \( H \to \infty \). The stability conditions of \( P_1 \) are the
following:

1. Saddle for $\lambda < -\sqrt{6}$.
2. Non-hyperbolic for $\lambda = -\sqrt{6}$.
3. Source for $\lambda > -\sqrt{6}$.

(b) $P_2 : (T, \theta) = \left(0, \frac{\pi}{2} + 2c_1\pi\right)$, $c_1 \in \mathbb{Z}$ with eigenvalues $\left\{3, 3 - \frac{2}{3}\lambda\right\}$. It represents a kinetic dominated solution with $H \to \infty$. The stability conditions of $P_2$ are the following:

1. Source for $\lambda < \sqrt{6}$.
2. Non-hyperbolic for $\lambda = \sqrt{6}$.
3. Saddle for $\lambda > \sqrt{6}$.

(c) $P_3 : (T, \theta) = \left(0, \arcsin\left(\frac{\lambda}{\sqrt{6}}\right)\right)$ with eigenvalues $\left\{\frac{\lambda^2}{2}, \frac{1}{2} (\lambda^2 - 6)\right\}$. It represents a scalar field dominated solution with $H \to \infty$. The stability conditions of $P_3$ are the following:

1. Non-hyperbolic for $\lambda \in \{-\sqrt{6}, 0, \sqrt{6}\}$.
2. Saddle for $-\sqrt{6} < \lambda < 0$ or $0 < \lambda < \sqrt{6}$.

(d) $P_4 : (T, \theta) = \left(1, -\frac{\pi}{2} + 2c_1\pi\right)$, $c_1 \in \mathbb{Z}$ with eigenvalues $\left\{-3, \sqrt{\frac{1}{3} \lambda + 3}\right\}$. It represents a kinetic dominated solution with $H \to 0$. The stability conditions of $P_4$ are the following:

1. Sink for $\lambda < -\sqrt{6}$.
2. Non-hyperbolic for $\lambda = \sqrt{6}$.
3. Saddle for $\lambda > -\sqrt{6}$.

(e) $P_5 : (T, \theta) = \left(1, \pi + 2c_1\pi\right)$, $c_1 \in \mathbb{Z}$ with eigenvalues $\left\{-3, 3 - \sqrt{\frac{1}{3} \lambda}\right\}$. It represents a kinetic dominated solution with $H \to 0$. The stability conditions of $P_5$ are the following:

1. It is a sink for $\lambda > \sqrt{6}$.
2. Non-hyperbolic for $\lambda = \sqrt{6}$.
3. Saddle for $\lambda < \sqrt{6}$.

(f) $P_6 : (T, \theta) = \left(1, \arcsin\left(\frac{\lambda}{\sqrt{6}}\right)\right)$ with eigenvalues $\left\{-\frac{\lambda^2}{2}, \frac{1}{2} (\lambda^2 - 6)\right\}$. It represents a scalar field dominated solution with $H \to 0$. The stability conditions of $P_6$ are the following:

1. Non-hyperbolic for $\lambda \in \{-\sqrt{6}, 0, \sqrt{6}\}$.
2. Sink for $-\sqrt{6} < \lambda < 0$ or $0 < \lambda < \sqrt{6}$.

In table 1, the existence conditions and stability conditions of the equilibrium points of equation (50) are summarized.

In the figure 2, some orbits of the flow of equation (50) (left panel) and a projection over the cylinder $S$ (right panel) for different values of $\lambda$ are presented.
Table 1. Existence conditions and stability conditions of the equilibrium points of equation (50), \( c_1 \in \mathbb{Z} \).

| Label | \((T, \theta)\) | Existence | Stability |
|-------|-----------------|-----------|-----------|
| \(P_1\) | \((0, -\varphi + 2c_1\pi)\) | \(\forall \lambda\) | Saddle for \(\lambda < -\sqrt{6}\). Non-hyperbolic for \(\lambda = -\sqrt{6}\). Source for \(\lambda > -\sqrt{6}\). |
| \(P_2\) | \((0, \varphi + 2c_1\pi)\) | \(\forall \lambda\) | Saddle for \(\lambda > \sqrt{6}\). Non-hyperbolic for \(\lambda = \sqrt{6}\). Source for \(\lambda < \sqrt{6}\). |
| \(P_3\) | \(\left(0, \arcsin \left(\frac{\lambda}{\sqrt{6}}\right)\right)\) | \(-\sqrt{6} \leq \lambda \leq \sqrt{6}\) | Non-hyperbolic for \(\lambda \in \{-\sqrt{6}, 0, \sqrt{6}\}\). Saddle for \(-\sqrt{6} < \lambda < 0\) or \(0 < \lambda < \sqrt{6}\). |
| \(P_4\) | \((1, -\varphi + 2c_1\pi)\) | \(\forall \lambda\) | Sink for \(\lambda < -\sqrt{6}\). Non-hyperbolic for \(\lambda = -\sqrt{6}\). Saddle for \(\lambda > -\sqrt{6}\). |
| \(P_5\) | \((1, \varphi + 2c_1\pi)\) | \(\forall \lambda\) | Sink for \(\lambda > \sqrt{6}\). Non-hyperbolic for \(\lambda = \sqrt{6}\). Saddle for \(\lambda < \sqrt{6}\). |
| \(P_6\) | \(\left(1, \arcsin \left(\frac{\lambda}{\sqrt{6}}\right)\right)\) | \(-\sqrt{6} \leq \lambda \leq \sqrt{6}\) | Non-hyperbolic for \(\lambda \in \{-\sqrt{6}, 0, \sqrt{6}\}\). Sink for \(-\sqrt{6} < \lambda < 0\) or \(0 < \lambda < \sqrt{6}\). |

Figure 2. Unwrapped solution space (left panel)—projection over the cylinder \(S\) (right panel) solution space of system (50) for different values of \(\lambda\).

5. Example: the scalar field model with \(E\)-potential \(V(\phi) = V_0 \left(1 - e^{-\sqrt{\frac{1}{3}} \phi}\right)^{2n}\) in vacuum

Considering the \(E\)-model with potential \(V(\phi) = V_0 \left(1 - e^{-\sqrt{\frac{1}{3}} \phi}\right)^{2n}\). This is a nonnegative potential with a single minimum located at \((\phi, V(\phi)) = (0, 0)\). Therefore, the model admits
the Minkowski solution represented by the equilibrium point \((H, \dot{\phi}, \phi) = (0, 0, 0)\). The potential has a plateau \(V = V_0\), when \(\phi \rightarrow +\infty\), while \(V \sim V_0 \ e^{-2n \sqrt{\phi}}\) as \(\phi \rightarrow -\infty\) [101]. At small \(\phi\) the \(E\)-potential behaves as \(\phi^{2n}\).

Defining the Hubble normalized variables:

\[
x = \frac{\dot{\phi}}{\sqrt{6H}}, \quad Y = \left(\frac{V(\phi)}{3H^2}\right)^{\frac{1}{3}} = \tilde{T} \left(1 - e^{-\sqrt{6\phi}}\right), \quad \tilde{T} = \left[\frac{V_0}{3H^2}\right]^{\frac{1}{3}}, \tag{52}
\]

the dynamical system:

\[
\frac{dx}{dN} = 6 \mu \dot{Y}^{2n-1}(Y - \tilde{T}) + (q - 2)x, \quad \tag{53a}
\]

\[
\frac{dY}{dN} = \frac{Y(-6 \mu x + q + 1) + 6 \mu x \tilde{T}}{n}, \quad \tag{53b}
\]

\[
\frac{d\tilde{T}}{dN} = \frac{(q + 1)\tilde{T}}{n}, \quad \tag{53c}
\]

is studied, where \(\mu = \frac{n}{\sqrt{6n}}\) and \(q = 2 - 3Y^{2n}\). This system has been extensively studied in [101] in the context of a canonical scalar field cosmology, and it was extended in [59] to Horava–Lifshitz cosmology. Now, the most relevant features of the solution space according to [59] will be discussed. It can be easily proven that \(\tilde{T}\) is monotonically increasing towards the future and decreasing towards the past. The phase space is limited to the past by the invariant subset \(\tilde{T} = 0\) for \(Y \leq 0\), and by \(\tilde{T} - Y = 0\) for \(Y \geq 0\). The state space is bounded when \(\tilde{T} > 0\), \(\tilde{T} - Y > 0\). The two past boundaries are intersected at the two massless scalar field points \(M_\pm = (\Sigma, Y) = (\pm 1, 0)\). The subset \(\tilde{T} - Y = 0\), on the other hand, is divided in two disconnected regions separated by the de Sitter equilibrium point \(dS^2 = (T, Y) = (1, 1)\).

Introducing the complementary global transformation (which for \(n = 1\) is exactly the definition of \(\theta\) in section 4):

\[
x = F(\theta) \sin(\theta), \quad Y = \cos(\theta), F(\theta) = \sqrt{\frac{1 - \cos^{2n}(\theta)}{1 - \sin^{2n}(\theta)}} = \sqrt{\sum_{k=0}^{n-1} \cos^{2k}(\theta)}, \tag{54}
\]

the following regular unconstrained 2D dynamical system:

\[
\frac{d\theta}{dN} = -6 \mu F(\theta)(\tilde{T} - \cos(\theta)) - \frac{3F(\theta)^2 \sin(2\theta)}{2n}, \tag{55a}
\]

\[
\frac{d\tilde{T}}{dN} = \frac{3\tilde{T}(1 - \cos^{2n}(\theta))}{n}, \tag{55b}
\]

is obtained, and the deceleration parameter becomes

\[
q = 2 - 3 \cos^{2n}(\theta). \tag{56}
\]

In figure 3, the unwrapped solution space (left panel) and the projection over the cylinder \(S\) (right panel) of the system (57a) and (57b) for some values of \(n, \mu, \alpha\) are represented. This plot clearly shows that the future boundary is \(T = 1\) which corresponds to \(H = 0\), and the final state is the Minkowski point given by the limit cycle. Introducing new compact variable \(T = \frac{T}{1 + \tilde{T}}\)
and a new time derivative \( \frac{d\tau}{d\ln n} = 1 + \tilde{T} = (1 - T)^{-1} \); the regular system:

\[
\frac{d\theta}{d\tau} = \frac{3(T - 1)F(\theta)^2 \sin(\theta)\cos(\theta)}{n} - \frac{6\mu F(\theta)((T - 1)\cos(\theta) + T)}{n}, \tag{57a}
\]

\[
\frac{dT}{d\tau} = \frac{3(T - 1)^2 T (\cos^{2n}(\theta) - 1)}{n}, \tag{57b}
\]

is obtained.

The past boundary is attached to the phase-space, and in the new variables \((\theta, T)\) is defined by \( \{ T = 0, \cos(\theta) \leq 0 \} \cup \{ T - (1 - T) \cos \theta = 0, \cos(\theta) > 0 \} \). It is also included the future boundary \( T = 1 \) which corresponds to \( H = 0 \), and the final state is the Minkowski point. The region \( \{ T - (1 - T) \cos \theta < 0, \cos(\theta) > 0 \} \) is forbidden.

The equilibrium points of (57a) and (57b) are given by:

**M±:** \( \tilde{T} = T = 0; \quad x = \pm 1; \quad Y = 0; \quad \theta = \pm \frac{\pi}{2} + 2k\pi, \quad k = 0, 1, 2, \ldots \) They are massless scalar field solutions. They are saddle and source, respectively, as it is confirmed in figure 3.

**dS:** \( \tilde{T} = 1; \quad T = \frac{1}{2}; \quad x = 0; \quad Y = 1; \quad \theta = 2k\pi, \quad k = 0, 1, 2, \ldots \) They are de Sitter solutions.

**PL:** \( \tilde{T} = T = 0; \quad x = 2 \mu; \quad Y = -\frac{(1 - 4 \mu^2)}{2}; \quad \theta = \pm \arccos Y. \) It exists for \( \mu < 1/2 \), and corresponds to a powerlaw selfsimilar solution for the exponential potential.

### 6. Example: a scalar-field cosmology with generalized harmonic potential

\( V(\phi) = \mu^3 \left[ \frac{\phi^2}{\mu^2} + b f(\delta + \frac{\phi}{f}) \right], \quad b \neq 0 \) in vacuum

In this section the asymptotic analysis as \( \phi \to \infty \) of a scalar-field cosmology with generalized harmonic potential

\[
V(\phi) = \mu^3 \left[ \frac{\phi^2}{\mu} + b f(\delta + \frac{\phi}{f}) \right], \quad b \neq 0, \tag{58}
\]
in a vacuum is presented for a flat FLRW model, where we let $N = 0, M = 0, W = 0$, and $\chi \equiv 1$.

Furthermore,

$$W_\nu(\phi) = \frac{2\phi - b\mu \sin \left( \delta + \frac{\phi}{2} \right)}{b\mu \cos \left( \delta + \frac{\phi}{2} \right) + \phi^2}.$$  \hfill (59)

### 6.1. Analysis as $\phi \to \infty$

In this section the dynamics as $\phi \to \infty$ of a scalar-field cosmology with generalized harmonic potential (58) in a vacuum is analyzed.

Defining the transformation:

$$\varphi = h(\phi) = \varphi = \left( \delta + \frac{\phi}{2} \right)^{-\frac{1}{2}},$$

it follows that $V(\phi)$ is 2 WBI with exponential order $N = 0$.

$$W_V(\varphi) = \begin{cases} 
- b\mu \varphi^8 \sin \left( \frac{\varphi}{\varphi^*} \right) - 2f\varphi^8 + 2f \varphi^4 
& \text{if } \varphi > 0, \\
0, 
& \text{if } \varphi = 0
\end{cases}$$

$$\bar{h}'(\varphi) = \begin{cases} 
- \varphi^5 
& \text{if } \varphi > 0, \\
0, 
& \text{if } \varphi = 0
\end{cases}$$

which satisfies conditions (b) and (c) of definition 2. Hence, the dynamical system:

$$\frac{dT}{d\tau} = 3(1 - T)T \sin^2(\theta),$$

$$\frac{d\theta}{d\tau} = - \frac{1}{2} \cos(\theta) \left( 6 \sin(\theta) + \sqrt{6}W_V(\varphi) \right),$$

$$\frac{d\varphi}{d\tau} = \sqrt{6}\bar{h}'(\varphi) \sin(\theta),$$

is obtained, where we have used the new time variable $\tau = \ln a$, defined on a phase space which consists of the vector product $S \times J$ of the finite cylinder $S$ with boundaries $T = 0$ and $T = 1$ with the interval $J = \left[ 0, \left( \delta + \frac{\phi}{2} \right)^{-\frac{1}{2}} \right]$. The variable $T$ is suitable for global analysis [98], due to:

$$\left. \frac{dT}{d\tau} \right|_{\sin \theta = 0} = 0, \quad \left. \frac{d^2 T}{d\tau^2} \right|_{\sin \theta = 0} = 0,$$

$$\left. \frac{d^3 T}{d\tau^3} \right|_{\sin \theta = 0} = \frac{9(1 - T)T\varphi^8 \sin \left( \frac{\varphi}{\varphi^*} \right) + 2f \left( \delta \varphi^4 - 1 \right)}{f^2 \left( b\mu \varphi^8 \cos \left( \frac{\varphi}{\varphi^*} \right) + f \left( \delta \varphi^4 - 1 \right) \right)^2}.$$  \hfill (64)

From equations (63a) and (64), $T$ is a monotonically increasing function on $S \times J$. As a consequence, all orbits originate from the invariant subset $T = 0$ (which contains the $\alpha$-limit), which
Figure 4. Evaluation of $(\cos \theta, \sin \theta, T)$ at some orbits of the system (65) for $(b, f, \delta, \mu) = (0.1, 0.33, 0, 1)$. In the plot, the points (iii) and (iv) are sources; the points (v) and (vi) are saddles. The vertical plane represents the invariant set \{$(T, \theta, \varphi)$: $\varphi = 0$, $\sin(\theta) = 0$\}. The vertical sides (i) and (ii) of this rectangle are the local sinks.

is classically related to the initial singularity with $H \to \infty$ and ends at the invariant boundary subset $T = 1$, which corresponds to the asymptotically $H = 0$.

The curves of the equilibrium points of (65) are the following:

(a) $(T, \theta, \varphi) = (T_c, 2n\pi, 0)$ with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.
(b) $(T, \theta, \varphi) = (T_c, \pi + 2n\pi, 0)$ with eigenvalues $\{-3, 0, 0\}$. It is non-hyperbolic.
(c) $(T, \theta, \varphi) = (0, -\frac{\pi}{2} + 2n\pi, 0)$ with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.
(d) $(T, \theta, \varphi) = (0, \frac{\pi}{2} + 2n\pi, 0)$ with eigenvalues $\{3, 3, 0\}$. It is non-hyperbolic.
(e) $(T, \theta, \varphi) = (1, -\frac{\pi}{2} + 2n\pi, 0)$ with eigenvalues $\{-3, 3, 0\}$. It behaves as saddle.
(f) $(T, \theta, \varphi) = (1, \frac{\pi}{2} + 2n\pi, 0)$ with eigenvalues $\{-3, 3, 0\}$. It behaves as saddle.

In figure 4, functions $(\cos \theta, \sin \theta, T)$ are evaluated at some orbits of the system (65) for $(b, f, \delta, \mu) = (0.1, 0.33, 0, 1)$. In the plot points (iii) and (iv), which are sources, and points (e) and (f), which are saddles, are represented. A vertical plane represents the invariant set \{$(T, \theta, \varphi)$: $\varphi = 0$, $\sin(\theta) = 0$\}. The vertical sides (a) and (b) of this rectangle are the local sinks.

6.2. Oscillating regime.

In the reference [118], oscillating scalar field models with potential $\frac{1}{4} \phi^2$ and potentials $\frac{1}{4} \phi^2 + W(\phi)$ with $W$ smooth and $W(\phi) = o(\phi^3)$ were studied. Improved asymptotic expansions for the solution in homogeneous and isotropic spaces were derived. Various generalizations for non-linear massive scalar fields, $k$-essence models and $f(R)$-gravity were obtained. In this section the potential $V(\phi) = \mu^3 \left[ \frac{b}{n} + bf \cos \left( \delta + \frac{\varphi}{n} \right) \right]$, $b \neq 0$ we look for oscillatory behavior, as expected from numerical investigations. Asymptotic expansions are derived as well. Noticing that cosine corrections are $O(bf\mu^3)$, they therefore do not fall in the potential class studied by [118].
The pair
\[
\left( \frac{\sqrt{2 \mu \phi}}{\sqrt{\phi^2 + 2 \mu \phi^2}}, \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + 2 \mu \dot{\phi}^2}} \right)
\]
defines a function that depends on the time variable \( t \), with values in the unit circle. Hence, an angular function \( \vartheta(t) \), that is unique under identification module \( 2\pi \), can be defined by
\[
\vartheta = \tan^{-1} \left( \frac{\dot{\phi}}{\sqrt{2 \mu \phi}} \right),
\]
together with the radial variable
\[
r = \sqrt{\dot{\phi}^2 + 2 \mu \dot{\phi}^2},
\]
related by:
\[
-b f \mu^3 \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2 f \mu}} \right) + 3H^2 - \frac{r^2}{2} = 0.
\]
The inverse transformation \((r, \vartheta) \mapsto (\phi, \dot{\phi})\) is
\[
\phi = \frac{r \cos(\vartheta)}{\sqrt{2 \mu}}, \quad \dot{\phi} = r \sin(\vartheta).
\]
For expanding universes \((H > 0)\) the following hold:
\[
\dot{r} = -\sqrt{\frac{3}{2} r} \sin^2(\vartheta) \sqrt{2 b f \mu^3 \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2 f \mu}} \right) + r^2 + b \mu^3 \sin(\vartheta)} \times \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2 f \mu}} \right),
\]
\[
\dot{\vartheta} + \sqrt{2 \mu} = \frac{b \mu^3 \sin(\delta) \cos(\vartheta)}{r} + \left( \frac{b \mu^2 \cos(\delta) \cos^2(\vartheta)}{\sqrt{2 f}} - \sqrt{3} \sin(\vartheta) \cos(\vartheta) \sqrt{b f \mu^3 \cos(\delta)} \right) + r \left( \sqrt{6 f} \tan(\delta) \sin(\vartheta) \cos^2(\vartheta) \sqrt{b f \mu^3 \cos(\delta)} - b \mu^2 \sin(\delta) \cos^3(\vartheta) \right) \frac{1}{4 f^2 \mu} + O(r^2).
\]
As \( b \to 0 \) the last equations reduce to:
\[
\dot{r} = -\sqrt{\frac{3}{2} r^3} \sin^2(\vartheta), \quad \dot{\vartheta} = -\sqrt{2 \mu} - \sqrt{\frac{3}{2} r} \sin(\vartheta) \cos(\vartheta).
\]
The solutions of the limiting equations admit the asymptotic expansions [118]
\[
r(t) = \frac{4}{\sqrt{6 t}} + O(t^{-2} \ln t), \quad \vartheta(t) = -\sqrt{2 \mu} + O(\ln t).
\]
Hence,
\[
\phi(t) = \frac{4 \cos t}{\sqrt{6 t}} + O(t^{-2} \ln t), \quad \dot{\phi}(t) = \frac{4 \sin t}{\sqrt{6 t}} + O(t^{-2} \ln t).
\]
These expansions can be improved to the order $O(t^{-3} \ln t)$ as in [118]. Here, asymptotic expansions of the full problem ($b \neq 0$) are derived using averaging techniques.

Note that:

\[
\dot{r} = b \mu^3 \sin(\delta) \sin(\vartheta) + \left( \frac{b \mu^2 \cos(\delta) \cos(\vartheta) \sin(\vartheta)}{\sqrt{2f}} - \sqrt{3} \sqrt{bf \mu^3 \cos(\delta) \sin^2(\vartheta)} \right) r + O \left( r^2 \right),
\]

\[
\dot{\vartheta} + \sqrt{2} \mu = \frac{b \mu^3 \sin(\delta) \cos(\vartheta)}{r} + \left( \frac{b \mu^2 \cos(\delta) \cos^2(\vartheta)}{\sqrt{2f}} - \sqrt{3} \sin(\vartheta) \cos(\vartheta) \sqrt{bf \mu^3 \cos(\delta)} \right)
\]

\[
\times \left( \sqrt{6f} \tan(\delta) \sin(\vartheta) \cos^2(\vartheta) \sqrt{bf \mu^3 \cos(\delta)} - b \mu^2 \sin(\delta) \cos^3(\vartheta) \right)
\]

\[
\times \frac{r + O \left( r^2 \right)}{4f^2 \mu}.
\]

In order to obtain an approximated solution near the oscillatory regime, the average with respect to $\vartheta$ over any orbit of period $2\pi$ given by

\[
\langle f \rangle = \frac{1}{2\pi} \int_{c-2\pi}^{c+2\pi} f(\vartheta) d\vartheta, \quad \cos(c) \geq 0,
\]

is applied to $f = (\dot{r}, \dot{\vartheta})$, leading to

\[
\dot{r} = -\frac{1}{2} \sqrt{3}r \sqrt{bf \mu^3 \cos(\delta)}, \quad \dot{\vartheta} = \frac{b \mu^2 \cos(\delta)}{2\sqrt{2f}} - \sqrt{2} \mu.
\]

The averaged equations have solutions:

\[
\begin{align*}
    r(t) &= r_0 e^{-\frac{1}{2} \sqrt{3}r_0 \sqrt{bf \mu^3 \cos(\delta)}}, \\
    \vartheta(t) &= \left( \frac{b \mu^2 \cos(\delta)}{2\sqrt{2f}} - \sqrt{2} \mu \right) t + \vartheta_0.
\end{align*}
\]

Introducing along with $\vartheta$ and $r$, the new variable

\[
\varepsilon = \frac{H}{\mu + H},
\]

with inverse

\[
H = \frac{\mu \varepsilon}{1 - \varepsilon},
\]

satisfying:

\[
-bf \mu^3 (\varepsilon - 1)^2 \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2f} \mu} \right) - \frac{1}{2} r^2 (\varepsilon - 1)^2 + 3 \mu^2 \varepsilon^2 = 0,
\]

along with the time derivative $\dot{\varepsilon}$ given by

\[
\frac{d\varepsilon}{dt} = \mu + H,
\]

the following equations hold:

\[
\varepsilon' = -\frac{r^2 (1 - \varepsilon)^3 \sin^2(\vartheta)}{2 \mu^2}.
\]
\[ r' = -3r \varepsilon \sin^2(\vartheta) + b\mu^2(1 - \varepsilon) \sin(\vartheta) \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2}/f} \right) \]  
(82b)

\[ \vartheta' = -\sqrt{2}(1 - \varepsilon) - 3\varepsilon \sin(\vartheta) \cos(\vartheta) + \frac{b\mu^2(1 - \varepsilon) \cos(\vartheta)}{r} \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2}/f} \right). \]  
(82c)

To obtain an approximated solution near the oscillatory regime the average with respect to \( \vartheta \) over any orbit of period \( 2\pi \), \( \langle (\varepsilon', r', \vartheta') \rangle \) is taken, leading to:

\[ \varepsilon' = \frac{r^2(\varepsilon - 1)^3}{4 \mu^2}, \]  
(83a)

\[ r' = -\frac{3r \varepsilon}{2}, \]  
(83b)

\[ \vartheta' = \sqrt{2}(\varepsilon - 1) - \left( \frac{b\mu^2(\varepsilon - 1) \cos(\vartheta)}{r} \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2}/f} \right) \right). \]  
(83c)

However, as \( r \to 0 \),

\[
\frac{1}{2\pi} \int_c^{c+2\pi} \frac{b\mu^2(\varepsilon - 1) \cos(\vartheta) \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2}/f} \right)}{r} \cos(\vartheta) \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2}/f} \right) \vartheta \, d\vartheta \\
\sim \frac{b\mu(\varepsilon - 1) \cos(\delta)}{2\sqrt{2} f} - \frac{r^2(b(\varepsilon - 1) \cos(\delta))}{32 \left( \sqrt{2}/f^3 \mu \right)} + O \left( r^3 \right). \]  
(84)

Finally, it follows that:

\[ \varepsilon' = -\frac{r^2(1 - \varepsilon)^3}{4 \mu^2}, \]  
(85a)

\[ r' = -\frac{3r \varepsilon}{2}, \]  
(85b)

\[ \vartheta' = \left[ -\sqrt{2} + \frac{b\mu \cos(\delta)}{2\sqrt{2} f} - \frac{r^2(b \cos(\delta))}{32 \left( \sqrt{2}/f^3 \mu \right)} \right] (1 - \varepsilon), \]  
(85c)

with the averaged constraint:

\[ \mu^2 \left( 3\varepsilon^2 - b f \mu(1 - \varepsilon)^2 \cos(\delta) \right) + \frac{r^2(1 - \varepsilon)^2(b \mu \cos(\delta) - 4f)}{8f} = 0, \]  
(86)

as \( r \to 0 \).

The above system is integrable yielding:

\[ r(\varepsilon) = \frac{\sqrt{2} \sqrt{c_1(\varepsilon - 1)^2 + \mu^2(6\varepsilon - 3)}}{1 - \varepsilon}, \]  
(87a)

\[ \vartheta(\varepsilon) = \mu \text{ tanh}^{-1} \left( \frac{\sqrt{c_1(\varepsilon - 1)^2 + \mu^2}}{\sqrt{9 \mu^2 - 3c_1}} \right) \frac{(b\mu \cos(\delta) - 4f)}{f \sqrt{18 \mu^2 - 6c_1}} + \frac{b\mu \cos(\delta)}{8\sqrt{2}/f^3(1 - \varepsilon)} + c_2, \]  
(87b)
Figure 5. Phase portrait of equation (76) (left panel). Projection over the cylinder $S$ (right panel) for $(b, f, \delta) = (0.1, 0.33, 0)$ and different values of $\mu$.

and

$$3(t - t_0) = \ln \left( \frac{(1 - \varepsilon)^2 \left( \frac{\mu \sqrt{\sqrt{g} \rho^2 - 3 \rho_1} + \sqrt{g} \rho_1 \rho^2} {\rho^2 - 3} \right) \sqrt{\rho^2 - 4}} {c_1 (c-1)^2 + \rho^2 (8c - 3)} \right)$$

$$\sim \ln \left( \frac{2 \rho \left( \sqrt{g} \rho^2 - 3 \rho_1 + 1 \right) - c_1} {c_1} \sqrt{\rho^2 - 4} \right) - \frac{6 \rho^2 \varepsilon^2} {c_1 - 3 \rho^2} + O \left( \varepsilon^3 \right),$$

as $\varepsilon \to 0$. 23
7. Example: a scalar-field cosmology with generalized harmonic potential

\[ V(\phi) = \mu^3 \left[ b f \left( \cos(\delta) - \cos \left( \delta + \frac{\phi}{f} \right) \right) + \frac{\phi^2}{\mu^2} \right], \quad b \neq 0, \text{ in vacuum} \]

In this section an asymptotic analysis as \( \phi \to \infty \) of a scalar-field cosmology with generalized harmonic potential:

\[ V(\phi) = \mu^3 \left[ b f \left( \cos(\delta) - \cos \left( \delta + \frac{\phi}{f} \right) \right) + \frac{\phi^2}{\mu^2} \right], \quad b \neq 0, \quad (88) \]

A phase portrait of equation (76) (left panel) and a projection over the cylinder \( S \) (right panel) for \((b, f, \delta) = (0.99, 0.09, 0)\) and different values of \( \mu \). The plots show the oscillatory behavior of solutions.
in a vacuum for a flat FLRW model is performed. Let be $N = 0, M = 0, W = 0, \Omega_m = 0, \Omega_0 = 0, \rho_m = 0, G_0 = 0$ and $\chi \equiv 1$.

Furthermore,

$$W_V(\phi) = \frac{b \mu \sin \left( \delta + \frac{\phi}{T} \right) + 2\phi}{b \mu f \left( \cos(\delta) - \cos \left( \delta + \frac{\phi}{T} \right) \right)} + \phi^2.$$  \hspace{1cm} (89)

7.1. Analysis as $\phi \to \infty$

In this section, the dynamics as $\phi \to \infty$ of a scalar-field cosmology with generalized harmonic potential (88) in a vacuum is analyzed.

Using again transformation (60) and a new time variable $\tau = \ln a$, a system of the form (65) holds with definitions:

$$\bar{\bar{W}}_V(\varphi) = \begin{cases} 
\frac{b \mu \varphi^8 \sin \left( \varphi^2 \right) - 2\delta f \varphi^8 + 2f \varphi^4}{f \left( b \mu \varphi^8 \left( \cos(\varphi^2) - \cos \left( \frac{1}{\varphi^2} \right) \right) + f \left( \delta \varphi^4 - 1 \right)^2 \right)}, & \varphi > 0, \\
0, & \varphi = 0
\end{cases} \hspace{1cm} (90)$$

$$\bar{\bar{h}}(\varphi) = \begin{cases} 
\frac{-\varphi^5}{4f}, & \varphi > 0, \\
0, & \varphi = 0
\end{cases} \hspace{1cm} (91)$$

which satisfy conditions (b) and (c) of definition 2. The phase space is a vector product $S \times H$ of the finite cylinder $S$ with boundaries $T = 0$ and $T = 1$ with the interval $J = \left[ 0, \left( \delta + \frac{2}{T} \right) \right]$. The variable $T$ is suitable for global analysis [98], due to

$$\frac{dT}{d\tau} \bigg|_{\theta = 0} = 0, \quad \frac{d^2T}{d\tau^2} \bigg|_{\theta = 0} = 0, \quad \frac{d^3T}{d\tau^3} \bigg|_{\theta = 0} = \frac{9(1 - T)T \varphi^8 \sin \left( \frac{1}{\varphi^2} \right) + 2f \left( \delta \varphi^4 - 1 \right)^2}{f^2 \left( b \mu \varphi^8 \cos \left( \frac{1}{\varphi^2} \right) + f \left( \delta \varphi^4 - 1 \right)^2 \right)^2}. \hspace{1cm} (92)$$

As in section 6.1, $T$ is a monotonically increasing function on $S \times J$. As a consequence, all orbits are originated from the invariant subset $T = 0$ (which contains the $\alpha$-limit), which is classically related to an initial singularity with $H \to \infty$, and ends on the invariant boundary subset $T = 1$, which corresponds asymptotically to $H = 0$.

The curves of the equilibrium points are the same as in (65) with essentially the same dynamics as in figure 4. That is, for $(b, f, \delta, \mu) = (0.1, 0.33, 0, 1)$ points (iii) and (iv) are sources, and points (v) and (vi) are saddles. A vertical plane represents the invariant set $\{(T, \theta, \varphi): \varphi = 0, \sin(\theta) = 0\}$. The vertical sides (i) and (ii) of this rectangle are the local sinks.

7.2. Oscillating regime.

In this section, the potential $V(\phi) = \mu^3 \left[ b f \left( \cos(\delta) - \cos \left( \delta + \frac{\phi}{T} \right) \right) + \frac{\phi^2}{T^2} \right], b \neq 0$ is analyzed; looking for oscillatory behavior as expected from numerical investigations. Asymptotic
expansions are derived as well. As before, the pair
\[
\left( \frac{\sqrt{2 \mu \phi}}{\sqrt{\dot{\phi}^2 + 2 \mu \phi^2}}, \frac{\dot{\phi}}{\sqrt{\dot{\phi}^2 + 2 \mu \phi^2}} \right),
\]
(93)
defines a function of \( t \) with values in the unit circle. Therefore, the angular function \( \vartheta(t) \), which is unique under identification modulo \( 2\pi \), can be defined by
\[
\vartheta = \tan^{-1} \left( \frac{\dot{\phi}}{\sqrt{2 \mu \phi}} \right),
\]
(94)
together with the radial function
\[
r = \sqrt{\dot{\phi}^2 + 2 \mu^2 \phi^2},
\]
(95)
related by:
\[
b f \mu^3 \left[ \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2} f \mu} \right) - \cos(\delta) \right] + 3H^2 - \frac{r^2}{2} = 0.
\]
(96)
The inverse transformation \((r, \vartheta) \mapsto (\phi, \dot{\phi})\) is
\[
\phi = \frac{r \cos(\vartheta)}{\sqrt{2} \mu}, \quad \dot{\phi} = r \sin(\vartheta).
\]
(97)
For expanding universes \((H > 0)\) the following equations hold:
\[
\dot{r} = -\sqrt{\frac{3}{2}} r \sin^2(\vartheta) \sqrt{2bf \mu^3 \left[ \cos(\delta) - \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2} f \mu} \right) \right]} + \dot{r}^2
\]
(98a)
\[
- b \mu^3 \sin(\vartheta) \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2} f \mu} \right),
\]
\[
\dot{\vartheta} = -\sqrt{\frac{3}{2}} \mu \sin(\vartheta) \cos(\vartheta) \sqrt{2bf \mu^3 \left[ \cos(\delta) - \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2} f \mu} \right) \right]} + \dot{\vartheta}^2
\]
(98b)
\[
- \frac{b \mu^3 \cos(\vartheta) \sin \left( \delta + \frac{r \sin(\vartheta)}{\sqrt{2} f \mu} \right)}{r}.
\]
In the limit \( b \to 0 \) the solutions of the limiting equation admit the asymptotic expansions [118]:
\[
\dot{\vartheta}(t) = -\sqrt{2} \mu t + O(\ln t), \quad r(t) = \frac{4}{\sqrt{6t}} + O(t^{-2 \ln t}).
\]
(99)
Hence, when \( b = 0 \):
\[
\phi(t) = \frac{4 \cos t}{\sqrt{6t}} + O(t^{-2 \ln t}), \quad \dot{\phi}(t) = \frac{4 \sin t}{\sqrt{6t}} + O(t^{-2 \ln t}).
\]
(100)
The asymptotic expansions of the full problem ($b \neq 0$) are derived as follows. Note that:

\[
\begin{align*}
\dot{r} &= -b\mu^3 \sin(\delta) \sin(\theta) - \frac{b\mu^2 \cos(\delta) \cos(\theta) \sin(\theta) r}{\sqrt{2}f} \\
&\quad - \frac{\sqrt{3}\sqrt{b\mu^2} \cos(\theta) \sin(\theta) \sin^2(\theta) r^{3/2}}{\sqrt{2}} \\
&\quad + \frac{b\mu \cos^2(\theta) \sin(\delta) \sin(\theta) r^2}{4f^2} + O \left(r^{5/2}\right),
\end{align*}
\]
(101a)

\[
\begin{align*}
\dot{\theta} + \sqrt{2} \mu &= -\frac{\sqrt{3}\cos(\theta) \sqrt{b\mu^2} \cos(\theta) \sin(\theta) \sin(\theta) \sqrt{r}}{\sqrt{2}} + \frac{b\mu \cos^3(\theta) \sin(\delta) r^{3/2}}{4f^2} \\
&\quad - \frac{(\sqrt{3}\cos(\theta) (b\mu \cos(\delta) \cos^2(\theta) + 2f) \sin(\theta)) r^{3/2}}{4 \left(2^{3/4} f \sqrt{b\mu^2} \cos(\theta) \sin(\delta)\right)} \\
&\quad + \frac{b \cos(\delta) \cos^4(\theta) r^2}{12\sqrt{2}f^3} + O \left(r^{5/2}\right).
\end{align*}
\]
(101b)

To obtain an approximated solution near the oscillatory regime, the average with respect to $\theta$ over any orbit of period $2\pi$, $\langle (\dot{r}, \dot{\theta}) \rangle$ is taken, leading to

\[
\begin{align*}
\dot{r} &= kr^{3/2}, \quad k = \frac{2^{3/4} \sqrt{3} \sqrt{b\mu} (E \left(\frac{\pi}{2}\right) - E \left(\frac{\pi}{2} + \pi\right)) \sqrt{\sin(\delta)}}{5\pi},
\end{align*}
\]
(102a)

\[
\begin{align*}
\dot{\theta} + \sqrt{2} \mu &= \frac{b \cos(\delta) (r - 4f\mu)(r + 4f\mu)}{32\sqrt{2}f^3}.
\end{align*}
\]
(102b)

where $E \left(\phi| m\right)$ gives an elliptic integral of the second kind:

\[
E \left(\phi| m\right) = \int_0^\phi \left(1 - m \sin^2(\theta)\right) \frac{d\theta}{\sin(\theta)}, \quad -\frac{\pi}{2} < \phi < \frac{\pi}{2}
\]
(103)

The averaged equations have solutions

\[
\begin{align*}
r(t) &= \frac{4}{(c_1 + kt)^2},
\end{align*}
\]
(104)

\[
\begin{align*}
\dot{\theta}(t) &= -\frac{b \cos(\delta) \left(3f^2\mu^2 (c_1 + kt) + \frac{1}{(c_1 + kt)^2}\right) + 12f^3 \mu (c_1 + kt)}{6\sqrt{2}f^3 k} + \phi_0.
\end{align*}
\]
(105)

Introducing along with $\theta$ and $r$, the new variable

\[
\varepsilon = \frac{H}{\mu + H},
\]
(106)

with inverse

\[
H = \frac{\mu \varepsilon}{1 - \varepsilon},
\]
(107)
related by:

$$bf\mu^3(\varepsilon - 1)^2 \left( \cos \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2}f_{\mu}} \right) - \cos(\delta) \right) - \frac{1}{2}r^2(\varepsilon - 1)^2 + 3 \mu^2 \varepsilon^2 = 0, \quad (108)$$

along with the time derivative \( \dot{\varphi} \) given by

$$\frac{d\dot{\varphi}}{dt} = \mu + H, \quad (109)$$

the following equations are obtained:

$$\varepsilon' = \frac{r^2(\varepsilon - 1)^3 \sin^2(\vartheta)}{2 \mu^2}, \quad (110a)$$

$$r' = -3\varepsilon r \sin^2(\vartheta) + b \mu^2(\varepsilon - 1) \sin(\vartheta) \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2}f_{\mu}} \right), \quad (110b)$$

$$\vartheta' = -\sqrt{2}(1 - \varepsilon) - 3\varepsilon \sin(\vartheta) \cos(\vartheta) + \frac{b \mu^2(\varepsilon - 1) \cos(\vartheta)}{r} \sin \left( \delta + \frac{r \cos(\vartheta)}{\sqrt{2}f_{\mu}} \right). \quad (110c)$$

In order to obtain an approximated solution near the oscillatory regime, the average with respect to \( \vartheta \) over any orbit of period \( 2\pi \), \( (\varepsilon', r', \vartheta') \), is taken, leading to:

$$\varepsilon' = \frac{(\varepsilon - 1)^3 r^2}{4 \mu^2}, \quad (111a)$$

$$r' = -\frac{3\varepsilon r}{2}, \quad (111b)$$

$$\vartheta' = \frac{(\varepsilon - 1)(4f + b \mu \cos(\delta))}{2\sqrt{2}f} \frac{b(\varepsilon - 1) \cos(\delta) r^2}{32 \left( \sqrt{2}f^3 \mu \right)}, \quad (111c)$$

with the averaged constraint:

$$3\mu^2 \varepsilon^2 = \frac{r^2((\varepsilon - 1)^2(b \mu \cos(\delta) + 4f))}{8f} = 0, \quad (112)$$

as \( r \to 0 \).

The above system is integrable yielding:

$$r(\varepsilon) = \sqrt{\frac{2}{2} \sqrt{c_1(\varepsilon - 1)^2 + \mu^2(6 \varepsilon - 3)}}, \quad (113a)$$

$$\mu = \frac{b \cos(\delta)}{\varepsilon - 1} - \frac{8f^2 \tanh^{-1} \left( \frac{c_1(\varepsilon - 1)^2 + \mu^2}{\mu^2(b \mu \cos(\delta) + 4f)} \right)}{2\sqrt{b} \mu^3 - b c_1} \left( \frac{b \mu \cos(\delta) + 4f}{\sqrt{b} \mu^2 - b c_1} \right) + c_2, \quad (113b)$$

28
Figure 7. Phase portrait of equation (100) (left panel). Projection over the cylinder $S$ (right panel) for $(b, f, \delta) = (0.1, 0.33, 0)$ and different values of $\mu$.

and

\[
3(t - t_0) = \ln \left( \frac{(1-\varepsilon)^2 \left( \frac{2 \mu}{c_1^{3/2}} - \frac{\sqrt{9 \mu^2 - 3 c_1 + 3 \varepsilon}}{c_1} \right) \sqrt{\mu^2 - \varepsilon}}{c_1^{3/2} \varepsilon \mu} \right) \sim \ln \left( \frac{2 \mu}{c_1^{3/2} \mu^2} \sqrt{\mu^2 - \varepsilon} \right) = \frac{6 \mu^2 \varepsilon}{c_1^{3/2} \mu^2} + O(\varepsilon^2), \]

as $\varepsilon \to 0$. 

Figure 8. Phase portrait of equation (100) (left panel). Projection over the cylinder $S$ (right panel) for $(b, f, \delta) = (0.99, 0.09, 0)$ and different values of $\mu$.

In figure 7, the phase portrait of equation (100) (left panel) and the projection over the cylinder $S$ (right panel) for $(b, f, \delta) = (0.1, 0.33, 0)$ and different values of $\mu$ are presented. In figure 8, the phase portrait of equation (100) (left panel) and the projection over the cylinder $S$ (right panel) for $(b, f, \delta) = (0.99, 0.09, 0)$ with different values of $\mu$ are shown. The plots show the periodic nature of solutions.

8. Discussion

In this research, a local dynamical systems analysis for arbitrary $V(\phi)$ and $\chi(\phi)$ using Hubble normalized equations was provided. The analysis relies on two arbitrary functions $f(\lambda)$ and $g(\lambda)$ which encode a potential and a coupling function through the quadrature

$$\phi(\lambda) = \phi(1) - \int_{1}^{\lambda} \frac{1}{f(s)} \, ds, \quad V(\lambda) + \Lambda = W(1)e^{\int_{1}^{\lambda} \frac{1}{f(s)} \, ds}, \quad \chi(\lambda) = \chi(1)e^{\int_{1}^{\lambda} \frac{a(s)}{s^2} \, ds}.$$
Afterwards, in section 3 a global dynamical systems formulation using the Alho & Uggla’s approach [98] was implemented. The equilibrium points that represent some solutions of cosmological interest were obtained. In particular, a matter-kinetic scaling solution, a matter-scalar field scaling solution, a kinetic dominated solution representing a stiff fluid, a solution dominated by the effective energy density of the geometric term \( G_0(a) \), a scaling solution where the kinetic term and the effective energy density from \( G_0(a) \) scales with the same order of magnitude, a quintessence scalar field dominated solution, the vacuum de Sitter solution associated to the minimum of the potential and a non-interacting matter dominated solution. All of which reveal very rich cosmological behavior.

For the exponential potential \( V(\phi) = V_0 e^{-\lambda \phi} \) in a vacuum analyzed in section 4, the asymptotic states are the same using either the Hubble-normalized equations or the Alho & Uggla’s approach [98], but in some examples the asymptotic behavior can be better explained using Alho & Uggla’s approach [98]. Examples are the E-model with \( V(\phi) = V_0 \left( 1 - e^{-\sqrt{\frac{\lambda}{s}}} \right)^{2n} \) [59, 101] and the (generalized) harmonic potential.

When Hubble-normalized quantities are used, typically the evolution equation for \( H \), which is given by the Raychaudhuri equation, decouples. In particular, this is always the case for a scalar field with exponential potential. This is due to the fact the exponential potential has symmetry such that its derivative is also an exponential function. The asymptotic of the remaining reduced system is then typically given by the equilibrium points and often it can be determined by a dynamical system analysis [15–17]. For other potentials that do not satisfy the above symmetry, like the harmonic potential \( V(\phi) = \mu^2 \phi^2 \), the Raychaudhuri equation fails to decouple [98]. Hubble-normalized equations are very often difficult to be analyze using the standard dynamical systems approach due to oscillations entering the system via the Klein–Gordon equation [99]. In reference [99], oscillations and future asymptotics of locally rotationally symmetric Bianchi type III cosmologies with a massive scalar field with potential \( V(\phi) = \frac{1}{2} \phi^2 \) were studied. According to previous discussions, the Alho & Uggla’s approach [98] is used to study scalar-field cosmologies with generalized harmonic potentials of the type \( V(\phi) = \mu^2 \phi^2 + \text{cosine corrections} \).

In section 5, the potential of the so-called E-model: \( V(\phi) = V_0 \left( 1 - e^{-\sqrt{\frac{\lambda}{s}}} \right)^{2n} \) with 
\[ f(s) = \frac{e^{\left( s - \sqrt{\frac{5}{2} n} \right)}}{2n}, \mu = \frac{2n}{\sqrt{5}} \]
discussed in [101] for a conventional scalar field cosmology and for the Horava–Lifshitz cosmology in [59], was studied. The dynamics is equivalent to that of the exponential potential plus a cosmological constant, \( V = V_0 e^{-\sqrt{\frac{5}{2} n} \phi} + \Lambda \) having the f-deviser: \( f(s) = -s \left( s - \sqrt{\frac{5}{2} n} \right) \), up to a rescaling in the independent variable.

In sections 6 and 7, scalar field cosmologies under harmonic potentials \( V_1(\phi) = \mu^3 \left( \frac{\phi^2}{2} + b f \cos \left( \delta + \frac{\phi}{2} \right) \right) \), \( b \neq 0 \) and \( V_2(\phi) = \mu^3 \left( b f \left( \cos(\delta) - \cos \left( \delta + \frac{\phi}{2} \right) \right) \right) + \frac{\phi^2}{2}, \ b \neq 0 \) were respectively studied. The Alho and Uggla’s approach [98] was used later in sections 6.1 and 7.1 to find qualitative features and also to present asymptotic analysis as \( \phi \to \infty \) for the harmonic potentials \( V_1(\phi) \) and \( V_2(\phi) \) in a vacuum, respectively. In section 6.2, the oscillatory regime for a scalar field under the potential \( V_1(\phi) \) was investigated. Meanwhile, in section 7.2, oscillations of a scalar field under potential \( V_2(\phi) \) were studied.

9. Conclusions

In this paper, both local and global phase-space descriptions and averaging methods were used to find qualitative features of solutions for the FLRW and the Bianchi I metrics in the context of scalar field cosmologies with arbitrary potentials and arbitrary couplings to matter. The stability of the equilibrium points in a phase-space, as well as the regime where the scalar field
diverges, were studied. Equilibrium points which represent some solutions of cosmological interest such as: several types of scaling solutions, a kinetic dominated solution representing a stiff fluid, a solution dominated by an effective energy density of geometric origin, a quintessence scalar field dominated solution, the vacuum de Sitter solution associated to the minimum of the potential and a non-interacting matter dominated solution were also found.

All these reveal a very rich cosmological phenomenology.

The preliminary analysis of oscillations in scalar-field cosmologies with generalized harmonic potentials of type \( V(\phi) = \mu^2 \phi^2 + \cos(\phi) \) (implemented in sections 6.2 and 7.2) will be improved in a forthcoming paper using averaging techniques similar to those used in [99] for a family of generalized harmonic potentials when \( H \) monotonically tends to zero. In this approach, the Hubble scalar plays a role of a time dependent perturbation parameter which controls the magnitude of the error between full-system and time-averaged solutions. These oscillations can be viewed as perturbations that can be smoothed out, with the benefit that the averaged Raychaudhuri equation decouples in the averaged system. At the end, the analysis of the system is reduced to the study of corresponding averaged equations.

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Appendix A. Existence and stability conditions of the equilibrium points of the system (12) as \( \phi \to \infty \)

The equilibrium points of system (12) are the following:

\[ A_1(\hat{\lambda}) = (\chi, \Omega_m, \Omega_0, \lambda) = \left( \frac{(4 - 3\gamma)\mu^2(\hat{\lambda})^2}{4(\gamma - 2)}, 1 - \frac{(4 - 3\gamma)\mu^2(\hat{\lambda})^2}{6(\gamma - 2)^2}, 0, \hat{\lambda} \right), \]

where \( \hat{\lambda} \) denotes the values of \( \lambda \) for which \( f(\lambda) = 0 \). It exists for \( 1 - \frac{(4 - 3\gamma)^2\mu^2((\hat{\lambda})^2)}{6(\gamma - 2)^2} \geq 0 \). The eigenvalues are

\[ \left\{ \begin{array}{c}
\frac{3(\gamma - 2)}{2} - \frac{(4 - 3\gamma)^2\mu^2(\hat{\lambda})^2}{4(\gamma - 2)}, \quad 3\gamma - \frac{(4 - 3\gamma)^2\mu^2(\hat{\lambda})^2}{2(\gamma - 2)}, \quad \frac{6(\gamma - 2)\gamma + (3\gamma - 4)\mu^2((\hat{\lambda})^2) + 2\lambda}{2(\gamma - 2)}, \quad \frac{3\gamma - 4\mu^2((\hat{\lambda})^2)}{\gamma - 2} \end{array} \right. \]

For \( 1 \leq \gamma < 2 \), it follows that \( A_1(\hat{\lambda}) \) is a sink for:

(a) \( 3 < p \leq 4, \; 1 \leq \gamma < \frac{4}{3}, \; -\sqrt{2} \sqrt{\frac{(1 - 2\gamma)(\gamma - 1)}{(\gamma - 4)(\gamma - 2)}} < g(\hat{\lambda}) < 0, \; f'(\hat{\lambda}) > 0, \; \hat{\lambda} > \left( \frac{3\gamma - 2}{\gamma - 2} \right) g(\hat{\lambda}) - \frac{3(\gamma - 2)\gamma}{(\gamma - 4)(\gamma - 2)}, \) or

(b) \( 3 < p \leq 4, \; 1 \leq \gamma < \frac{4}{3}, \; 0 < g(\hat{\lambda}) < \sqrt{2} \sqrt{\frac{(1 - 2\gamma)(\gamma - 1)}{(\gamma - 4)(\gamma - 2)}}, \; f'(\hat{\lambda}) < 0, \; \hat{\lambda} < \left( \frac{3\gamma - 2}{\gamma - 2} \right) g(\hat{\lambda}) - \frac{3(\gamma - 2)\gamma}{(\gamma - 4)(\gamma - 2)}, \) or

(c) \( 4 < p \leq 6, \; 1 \leq \gamma < \frac{4}{3}, \; -\sqrt{2} \sqrt{\frac{(1 - 2\gamma)(\gamma - 1)}{(\gamma - 4)(\gamma - 2)}} < g(\hat{\lambda}) < 0, \; f'(\hat{\lambda}) > 0, \; \hat{\lambda} > \left( \frac{3\gamma - 2}{\gamma - 2} \right) g(\hat{\lambda}) - \frac{3(\gamma - 2)\gamma}{(\gamma - 4)(\gamma - 2)}, \) or

(d) \( 4 < p \leq 6, \; 1 \leq \gamma < \frac{4}{3}, \; 0 < g(\hat{\lambda}) < \sqrt{2} \sqrt{\frac{(1 - 2\gamma)(\gamma - 1)}{(\gamma - 4)(\gamma - 2)}}, \; f'(\hat{\lambda}) < 0, \; \hat{\lambda} < \left( \frac{3\gamma - 2}{\gamma - 2} \right) g(\hat{\lambda}) - \frac{3(\gamma - 2)\gamma}{(\gamma - 4)(\gamma - 2)}, \) or

(e) \( 4 < p \leq 6, \; \frac{4}{3} < \gamma < \frac{5}{3}, \; -\sqrt{2} \sqrt{\frac{(1 - 2\gamma)(\gamma - 1)}{(\gamma - 4)(\gamma - 2)}} < g(\hat{\lambda}) < 0, \; f'(\hat{\lambda}) < 0, \; \hat{\lambda} < \left( \frac{3\gamma - 2}{\gamma - 2} \right) g(\hat{\lambda}) - \frac{3(\gamma - 2)\gamma}{(\gamma - 4)(\gamma - 2)}, \) or
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(a) $1 \leq \gamma \leq 2$, $p > 0$, $\hat{\lambda} \leq \frac{1}{3} \left(-4g(\hat{\lambda}) - \sqrt{48\gamma + 9\gamma^2}g(\hat{\lambda})^2 - 24\gamma g(\hat{\lambda})^2 + 16g(\hat{\lambda})^2 + 3\gamma g(\hat{\lambda})\right)$.

(b) $1 \leq \gamma \leq 2$, $p > 0$, $\hat{\lambda} \geq \frac{1}{3} \left(-4g(\hat{\lambda}) + \sqrt{48\gamma + 9\gamma^2}g(\hat{\lambda})^2 - 24\gamma g(\hat{\lambda})^2 + 16g(\hat{\lambda})^2 + 3\gamma g(\hat{\lambda})\right)$.

The eigenvalues are
\[
\left\{ \frac{p(4 - 3\gamma)g(\hat{\lambda}) + 2(p - 3\gamma)\hat{\lambda}}{(-4 + 3\gamma)g(\hat{\lambda}) - 2\hat{\lambda}}, \frac{1}{(-8 + 6\gamma)g(\hat{\lambda}) - 4\hat{\lambda}} \left(12 - 9\gamma\right)g(\hat{\lambda}) - 3(-2 + \gamma)\hat{\lambda} - \sqrt{3}\sqrt{\left(2(-4 + 3\gamma)^3g(\hat{\lambda})\hat{\lambda} + (4 - 3\gamma)^2g(\hat{\lambda})^2 \left(3 + 12\gamma - 8\lambda^2\right) - 2(-4 + 3\gamma)g(\hat{\lambda})\hat{\lambda} (6 - 3\gamma)\right)}\right\\ + 6\hat{\gamma}^2 - 4\hat{\lambda}^2 - 3(-2 + \gamma) \left(24\gamma^2 + (2 - 9\gamma)\hat{\lambda}^2\right) - 2(-4 + 3\gamma)g(\hat{\lambda})\hat{\lambda} \left(6 - 3\gamma + 6\hat{\gamma}^2 - 4\hat{\lambda}^2\right) - 3(-2 + \gamma) \left(24\gamma^2 + (2 - 9\gamma)\hat{\lambda}^2\right)\right\}.
\]

In this case using a semi-analytically procedure for non-minimal coupling $g \equiv 0$ the eigenvalues are reduced to
\[
\left\{ \frac{-3(2 - 3\gamma)\gamma(24\gamma^2 + (2 - 9\gamma)\hat{\lambda}^2) - 3(\gamma - 2)\hat{\lambda}}{4\hat{\lambda}}, \frac{-3(2 - 3\gamma)\gamma(24\gamma^2 + (2 - 9\gamma)\hat{\lambda}^2) - 3(\gamma - 2)\hat{\lambda}}{4\hat{\lambda}} \right\}.
\]

Hence, assuming that $1 \leq \gamma < 2$, $A_3(\hat{\lambda})$ is a sink for:

(a) $1 \leq \gamma < 2$, $p > 3\gamma - \frac{2\sqrt{5}}{\sqrt{3} - 2} \leq \hat{\lambda} < -\sqrt{3}\sqrt{\gamma} f'(\hat{\lambda}) < 0$, or

(b) $1 \leq \gamma < 2$, $p > 3\gamma \sqrt{3}\sqrt{\gamma} < \hat{\lambda} < \frac{2\sqrt{5}}{\sqrt{3} - 2} f'(\hat{\lambda}) > 0$.

Otherwise, it is a saddle. For non-minimal coupling the analysis is more complicated, so a numerical study is more reliable.

$A_4(\hat{\lambda})$: $(x, \Omega_m, \Omega_0, \lambda) = \left(-1, 0, 0, \hat{\lambda}\right)$. The eigenvalues are
\[
\left\{ 6 - p, -3\gamma + \sqrt{\frac{4}{3}(3\gamma - 4)g(\hat{\lambda}) + 6, \sqrt{6}\hat{\lambda} + 6, \sqrt{6}f'(\hat{\lambda})} \right\}.
\]

$A_4(\hat{\lambda})$ is a source for:

(a) $1 \leq \gamma < \frac{4}{3} \leq \hat{\lambda} < 0 < 6 g(\hat{\lambda}) < \frac{\sqrt{2}(-2)}{\sqrt{3} - 2}, f'(\hat{\lambda}) > 0 > -\sqrt{6}$.

(b) $1 \leq \gamma < \frac{4}{3} \leq \hat{\lambda} > 0 \leq 6 f'(\hat{\lambda}) > 0 > -\sqrt{6}$, or

(c) $\frac{4}{3} < \gamma < 2 \leq \hat{\lambda} < 0 < 6 g(\hat{\lambda}) < \frac{\sqrt{2}(-2)}{\sqrt{3} - 2}, f'(\hat{\lambda}) > 0 > -\sqrt{6}$.

$A_4(\hat{\lambda})$ is a sink for:

(a) $1 \leq \gamma < \frac{4}{3} \leq \hat{\lambda} > 0 \leq 6 g(\hat{\lambda}) < \frac{\sqrt{2}(-2)}{\sqrt{3} - 2}, f'(\hat{\lambda}) < 0 = -\sqrt{6}$.

(b) $\frac{4}{3} < \gamma < 2 \leq \hat{\lambda} < 0 \leq 6 f'(\hat{\lambda}) < 0 = -\sqrt{6}$.

$A_5(\hat{\lambda})$: $(x, \Omega_m, \Omega_0, \lambda) = \left(0, 0, 1, \hat{\lambda}\right)$. The eigenvalues are $\left\{ 0, \frac{p}{2}, p, p - 3\gamma \right\}$. It has a 3D unstable manifold for $p > 6$.

$A_6(\hat{\lambda})$: $(x, \Omega_m, \Omega_0, \lambda) = \left(1, 0, 0, \hat{\lambda}\right)$. The eigenvalues are
\[
\left\{ 6 - p, -3\gamma + \sqrt{\frac{4}{3}(4 - 3\gamma)g(\hat{\lambda}) + 6, \frac{\sqrt{6}\lambda}{\sqrt{6}f'(\hat{\lambda})}} \right\}.
\]

$A_6(\hat{\lambda})$ is a source for: 34
(a) $1 \leq \gamma < \frac{4}{9}$, $p < 6$, $g(\lambda) = -\frac{\sqrt{6(\gamma^2 - 2)}}{3\sqrt{\gamma}} + f'(\lambda) < 0$, $\lambda < \sqrt{6}$, or
(b) $\gamma = \frac{4}{9}$, $p < 6$, $f'(\lambda) < 0$, $\lambda < \sqrt{6}$, or
(c) $\frac{4}{9} < \gamma < 2$, $p < 6$, $g(\lambda) = -\frac{\sqrt{6(\gamma^2 - 2)}}{3\sqrt{\gamma}} + f'(\lambda) < 0$ $\lambda < \sqrt{6}$.

$A_0(\lambda)$ is a sink for:

(a) $1 \leq \gamma < \frac{4}{9}$, $p > 6$ $g(\lambda) = -\frac{\sqrt{6(\gamma^2 - 2)}}{3\sqrt{\gamma}} + f'(\lambda) > 0$, $\lambda > \sqrt{6}$, or
(b) $\gamma > \frac{4}{9}$, $p > 6$ $g(\lambda) = -\frac{\sqrt{6(\gamma^2 - 2)}}{3\sqrt{\gamma}} + f'(\lambda) > 0$, $\lambda > \sqrt{6}$.

$A_\gamma(\lambda)$: $(x, \Omega, 0, \lambda) = \left( \frac{1}{\sqrt{6\lambda}}, 0, 1 - \frac{\sqrt{6}}{\lambda^2}, \hat{\lambda} \right)$. It exists for $\lambda < 0$, $0 \leq p \leq 6$, or $\hat{\lambda} > 0$, $0 \leq p \leq 6$. The eigenvalues are

\[-\frac{1}{4} \left( 6 - p - \sqrt{\frac{(p-6)(\frac{1}{27}\gamma p - 8\lambda^2)}{\lambda}} \right), -3\gamma + \frac{4\sqrt{(p-6)(\frac{1}{27}\gamma p - 8\lambda^2)}}{\lambda} + p - \frac{v'(\lambda)}{\lambda} \].

For $1 \leq \gamma \leq 2$, $A_\gamma(\lambda)$ is a sink for:

(a) $1 \leq \gamma < \frac{4}{9}$, $\lambda > -\sqrt{6}$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $g(\lambda) > \frac{2\lambda(p-3\lambda)}{(3-4p)}$, $f'(\lambda) < 0$, or
(b) $1 \leq \gamma < \frac{4}{9}$, $\lambda \geq \sqrt{6}$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $g(\lambda) < \frac{2\lambda(p-3\lambda)}{(3-4p)}$, $f'(\lambda) > 0$, or
(c) $1 \leq \gamma < \frac{4}{9}$, $\lambda > \frac{8}{3\sqrt{3}}$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $g(\lambda) < \frac{2\lambda(p-3\lambda)}{(3-4p)}$, $f'(\lambda) > 0$, or
(d) $1 \leq \gamma < \frac{4}{9}$, $\sqrt{6} \leq \lambda < -\frac{8}{3\sqrt{3}}$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $g(\lambda) > \frac{2\lambda(p-3\lambda)}{(3-4p)}$, $f'(\lambda) < 0$, or
(e) $1 \leq \gamma < \frac{4}{9}$, $-\frac{8}{3\sqrt{3}} \leq \lambda < 0$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $g(\lambda) > \frac{2\lambda(p-3\lambda)}{(3-4p)}$, $f'(\lambda) < 0$, or
(f) $1 \leq \gamma < \frac{4}{9}$, $-\sqrt{6} \leq \lambda < -\frac{8}{3\sqrt{3}}$, $\frac{1}{16} \left( 9\lambda^2 + \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right) \leq p \leq \lambda^2$, $g(\lambda) > \frac{2\lambda(p-3\lambda)}{(3-4p)}$, $f'(\lambda) < 0$, or
(g) $1 \leq \gamma < \frac{4}{9}$, $0 \leq \lambda < \frac{8}{3\sqrt{3}}$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 + \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $g(\lambda) < \frac{2\lambda(p-3\lambda)}{(3-4p)}$, $f'(\lambda) > 0$, or
(h) $1 \leq \gamma < \frac{4}{9}$, $\frac{8}{3\sqrt{3}} \leq \lambda \leq \sqrt{6}$, $\frac{1}{16} \left( 9\lambda^2 + \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right) \leq p < \lambda^2$, $g(\lambda) < \frac{2\lambda(p-3\lambda)}{(3-4p)}$, $f'(\lambda) > 0$, or

(i) $\gamma = \frac{4}{9}$, $\lambda \geq \frac{1}{\sqrt[3]{15}}$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $f'(\lambda) > 0$, or
(j) $\gamma = \frac{4}{9}$, $\lambda > 2$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $f'(\lambda) > 0$, or
(k) $\gamma = \frac{4}{9}$, $\lambda \leq -\frac{8}{3\sqrt{3}}$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $f'(\lambda) < 0$, or
(l) $\gamma = \frac{4}{9}$, $-\frac{8}{3\sqrt{3}} \leq \lambda < 2$, $0 \leq p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\lambda^2 \left( 27\lambda^2 - 64 \right) \right)$, $f'(\lambda) > 0$, or
\( \gamma = \frac{1}{4}, \quad 0 < \lambda \leq \frac{8}{\sqrt{3}}, \quad 0 < p < \lambda^2, \quad f'(\lambda) > 0, \) or

\( \gamma = \frac{1}{4}, \quad 2 < \lambda < \frac{8}{\sqrt{15}}, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right) \leq p < 4, \quad f'(\lambda) > 0, \) or

\( \gamma = \frac{1}{4}, \quad \frac{8}{\sqrt{15}} < \lambda \leq 2, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right) \leq p < \lambda^2, \quad f'(\lambda) > 0, \) or

\( \gamma = \frac{1}{4}, \quad -\frac{8}{\sqrt{15}} < \lambda < -2, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right), \quad f'(\lambda) < 0, \) or

\( \gamma = \frac{1}{4}, \quad -\frac{8}{\sqrt{3}} < \lambda < 0, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right), \quad f'(\lambda) < 0, \) or

\( \gamma = \frac{1}{2}, \quad -\frac{8}{\sqrt{3}} < \lambda < -2, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right), \quad f'(\lambda) < 0, \) or

\( \gamma = \frac{1}{2}, \quad -\frac{8}{\sqrt{15}} < \lambda < 0, \quad 0 < p < \lambda^2, \quad f'(\lambda) < 0, \) or

\( \gamma = \frac{1}{2}, \quad \frac{8}{\sqrt{15}} < \lambda < 2, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right) \leq p \leq \lambda^2, \quad f'(\lambda) > 0, \) or

\( \gamma = \frac{1}{2}, \quad -\frac{8}{\sqrt{3}} < \lambda < \frac{8}{\sqrt{3}}, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right), \quad f'(\lambda) > 0, \) or

\( \gamma = \frac{1}{2}, \quad -\frac{8}{\sqrt{3}} < \lambda < 0, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right), \quad f'(\lambda) > 0, \) or

\( \gamma = \frac{1}{2}, \quad \frac{8}{\sqrt{3}} < \lambda < 2, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right) \leq p < \lambda^2, \quad g(\lambda) < \frac{2\lambda(p - \lambda)}{(3\gamma - 4p)}, \quad f'(\lambda) > 0, \) or

\( \gamma = \frac{1}{2}, \quad -\frac{8}{\sqrt{3}} < \lambda < -2, \quad 0 < p < \frac{1}{16} \left( 9\lambda^2 - \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right), \quad g(\lambda) > \frac{2\lambda(p - \lambda)}{(3\gamma - 4p)}, \quad f'(\lambda) < 0, \) or

\( \gamma = \frac{1}{2}, \quad -\frac{8}{\sqrt{27}} < \lambda < 0, \quad \lambda^2, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right), \quad f'(\lambda) < 0, \) or

\( \gamma = \frac{1}{2}, \quad \frac{8}{\sqrt{27}} < \lambda < 2, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right) \leq p < \lambda^2, \quad g(\lambda) > \frac{2\lambda(p - \lambda)}{(3\gamma - 4p)}, \quad f'(\lambda) > 0, \) or

\( \gamma = \frac{1}{2}, \quad -\frac{8}{\sqrt{3}} < \lambda < \frac{8}{\sqrt{3}}, \quad 0 < p < \lambda^2, \quad g(\lambda) > \frac{2\lambda(p - \lambda)}{(3\gamma - 4p)}, \quad f'(\lambda) > 0, \) or

\( \gamma = \frac{1}{2}, \quad -\frac{8}{\sqrt{3}} < \lambda < 0, \quad \lambda^2, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right), \quad f'(\lambda) < 0, \) or

\( \gamma = \frac{1}{2}, \quad \frac{8}{\sqrt{3}} < \lambda < 2, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3} \sqrt{\lambda^2 \left( 27\lambda^2 - 64 \right)} \right) \leq p < \lambda^2, \quad g(\lambda) > \frac{2\lambda(p - \lambda)}{(3\gamma - 4p)}, \quad f'(\lambda) > 0, \) or
\[
\begin{align*}
\text{(af)} & \quad \frac{4}{3} < \gamma < 2, \quad -\sqrt{6} < \hat{\lambda} < -2\sqrt{6}\sqrt{\frac{\sqrt{\lambda}}{\hat{\lambda}^2}}, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right), \quad g(\hat{\lambda}) < \frac{2(\mu - \lambda)}{\lambda - 4\lambda} f'(\lambda) < 0, \text{ or} \\
\text{(ag)} & \quad \frac{4}{3} < \gamma < 2, \quad -\frac{8}{\sqrt{3}\lambda} < \hat{\lambda} < -\frac{8}{\sqrt{3}\lambda}, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right), \quad g(\hat{\lambda}) < \frac{2(\mu - \lambda)}{\lambda - 4\lambda} f'(\lambda) < 0, \text{ or} \\
\text{(ah)} & \quad \frac{4}{3} < \gamma < 2, \quad -\frac{8}{\sqrt{3}\lambda} \leq \hat{\lambda} < 0, \quad 0 < p < \lambda^2, \quad g(\hat{\lambda}) < \frac{2(\mu - \lambda)}{\lambda - 4\lambda}, \quad f'(\lambda) < 0, \text{ or} \\
\text{(ai)} & \quad \frac{4}{3} < \gamma < 2, \quad -\sqrt{6} < \hat{\lambda} < -2\sqrt{6}\sqrt{\frac{\sqrt{\lambda}}{\hat{\lambda}^2}}, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right) \leq p < \\
& \quad \hat{\lambda}^2, \quad g(\hat{\lambda}) < \frac{2(\mu - \lambda)}{\lambda - 4\lambda}, \quad f'(\lambda) < 0, \text{ or} \\
\text{(aj)} & \quad \frac{4}{3} < \gamma < 2, \quad -2\sqrt{6}\sqrt{\frac{\sqrt{\lambda}}{\hat{\lambda}^2}} < \hat{\lambda} < -\frac{8}{\sqrt{3}\lambda}, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right) \leq p < \\
& \quad \hat{\lambda}^2, \quad g(\hat{\lambda}) < \frac{2(\mu - \lambda)}{\lambda - 4\lambda}, \quad f'(\lambda) < 0, \text{ or} \\
\text{(ak)} & \quad \gamma = 2, \quad \hat{\lambda} \geq \sqrt{6}, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right), \quad g(\hat{\lambda}) > \frac{2(\mu - \lambda)}{\lambda - 4\lambda}, \quad f'(\lambda) > 0, \text{ or} \\
\text{(al)} & \quad \gamma = 2, \quad \hat{\lambda} \geq \sqrt{6}, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right), \quad g(\hat{\lambda}) > \frac{2(\mu - \lambda)}{\lambda - 4\lambda}, \quad f'(\lambda) > 0, \text{ or} \\
\text{(am)} & \quad \gamma = 2, \quad \lambda \leq -\sqrt{6}, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right), \quad g(\hat{\lambda}) < \\
& \quad \frac{2(\mu - \lambda)}{\lambda - 4\lambda} f'(\lambda) < 0, \text{ or} \\
\text{(an)} & \quad \gamma = 2, \quad 0 < \lambda \leq \frac{8}{\sqrt{3}\lambda}, \quad 0 < p < \lambda^2, \quad g(\hat{\lambda}) > \frac{2(\mu - \lambda)}{\lambda - 4\lambda}, \quad f'(\lambda) > 0, \text{ or} \\
\text{(ao)} & \quad \gamma = 2, \quad \frac{8}{\sqrt{3}\lambda} < \lambda < \sqrt{6}, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right) \leq p < \lambda^2, \quad g(\hat{\lambda}) > \\
& \quad \frac{2(\mu - \lambda)}{\lambda - 4\lambda}, \quad f'(\lambda) > 0, \text{ or} \\
\text{(ap)} & \quad \gamma = 2, \quad -\sqrt{6} < \lambda < -\frac{8}{\sqrt{3}\lambda}, \quad 0 < p \leq \frac{1}{16} \left( 9\lambda^2 - \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right), \quad g(\hat{\lambda}) < \\
& \quad \frac{2(\mu - \lambda)}{\lambda - 4\lambda}, \quad f'(\lambda) < 0, \text{ or} \\
\text{(aq)} & \quad \gamma = 2, \quad -\frac{8}{\sqrt{3}\lambda} \leq \lambda < 0, \quad 0 < p < \lambda^2, \quad g(\hat{\lambda}) < \frac{2(\mu - \lambda)}{\lambda - 4\lambda}, \quad f'(\lambda) < 0, \text{ or} \\
\text{(ar)} & \quad \gamma = 2, \quad -\sqrt{6} < \lambda < -\frac{8}{\sqrt{3}\lambda}, \quad \frac{1}{16} \left( 9\lambda^2 + \sqrt{3}\sqrt{\frac{\lambda^2}{\hat{\lambda}^2}} \right) \leq p < \lambda^2, \quad g(\hat{\lambda}) < \\
& \quad \frac{2(\mu - \lambda)}{\lambda - 4\lambda} f'(\lambda) < 0.
\end{align*}
\]
\[ \frac{4}{3} < \gamma < 2, \quad p > \hat{\lambda}^2, \quad g(\hat{\lambda}) > \frac{6\hat{\lambda} - 2\hat{\lambda}^2}{4\hat{\lambda} - 3\hat{\lambda}}, \quad f'(\hat{\lambda}) > 0, \quad 0 < \hat{\lambda} < \sqrt{6}, \text{ or} \]

\[ \frac{4}{3} < \gamma < 2, \quad p > \hat{\lambda}^2, \quad -\sqrt{\gamma} < \hat{\lambda} < 0, \quad g(\hat{\lambda}) < \frac{6\hat{\lambda} - 2\hat{\lambda}^2}{4\hat{\lambda} - 3\hat{\lambda}}, \quad f'(\hat{\lambda}) < 0. \]

It will never be a source.

\[ A_9: (x, \Omega_m, \Omega_i, \lambda) = (0, 0, 0, 0). \] The eigenvalues are

\[ \left\{ -p, -3\gamma, -\frac{1}{4} \left( 3 + \sqrt{9 - 12f(0)} \right), -\frac{1}{4} \left( 3 - \sqrt{9 - 12f(0)} \right) \right\}. \]

\[ A_9 \text{ is a sink for } p > 0, \gamma > 0, f(0) > 0. \] Otherwise, it is a saddle.

\[ A_{10}(\hat{\lambda}): (x, \Omega_m, \Omega_i, \lambda) = (0, 0, 0, \hat{\lambda}), \] where the values of \( \lambda \) for which \( g(\lambda) = 0 \) are denoted by \( \tilde{\lambda} \). The eigenvalues are

\[ \frac{1}{4} \left( 3\gamma - 6 - \sqrt{24(4 - 3\gamma)f(\hat{\lambda})g'(\hat{\lambda}) + 9(\gamma - 2)^2} \right), \]

\[ \frac{1}{4} \left( 3\gamma - 6 + \sqrt{24(4 - 3\gamma)f(\hat{\lambda})g'(\hat{\lambda}) + 9(\gamma - 2)^2} \right), 3\gamma, 3\gamma - p \]. \( A_{10}(\hat{\lambda}) \) is a saddle.

### Appendix B. Existence and stability conditions of the equilibrium points of system (25) as \( \phi \rightarrow \infty \)

The equilibrium points of system (25) with \( \varphi = 0 \) (i.e., corresponding to \( \phi \rightarrow \infty \)) are the following:

\[ B_1: \left( 0, \tan^{-1} \left[ \sqrt{1 - \frac{N^2}{6}}, -\frac{N}{\sqrt{6}} \right] + 2\pi c_1, 0, 0, 0 \right), \quad c_1 \in \mathbb{Z} \text{ with eigenvalues} \]

\[ \left\{ 0, \frac{\gamma}{2}, \frac{1}{2} (N^2 - 6), N^2 - p, N(2M + N) - \frac{3}{4} \gamma (MN + 2) \right\}. \] It exists for \( N^2 \leq 6 \). It is always a non-hyperbolic saddle for \( N^2 \leq 6 \).

\[ B_2: \left( 1, \tan^{-1} \left[ \sqrt{1 - \frac{N^2}{6}}, -\frac{N}{\sqrt{6}} \right] + 2\pi c_1, 0, 0, 0 \right), \quad c_1 \in \mathbb{Z} \text{ with eigenvalues} \]

\[ \left\{ 0, -\frac{\gamma}{2}, \frac{1}{2} (N^2 - 6), N^2 - p, N(2M + N) - \frac{3}{4} \gamma (MN + 2) \right\}. \] It exists for \( N^2 \leq 6 \). The case of physical interest is when it is non-hyperbolic with a 4D stable manifold for:

(a) \( M \in \mathbb{R}, \ -2 < N < 0, \ p > N^2, \ \gamma = \frac{1}{4}, \) or

(b) \( M \in \mathbb{R}, \ 0 < N < 2, \ p > N^2, \ \gamma = \frac{1}{4}, \) or

(c) \( -\sqrt{6} < N \leq -2, \ p > N^2, \ 1 < \gamma < \frac{1}{4}, \ M > \frac{2(N^2 - 3\gamma)}{N(5\gamma - 4)}, \) or

(d) \( -2 < N < 0, \ p > N^2, \ 1 < \gamma < \frac{1}{4}, \ M > \frac{2(N^2 - 3\gamma)}{N(5\gamma - 4)}, \) or

(e) \( 0 < N < 2, \ p > N^2, \ \frac{1}{4} < \gamma < 2, \ \lambda > \frac{2(N^2 - 3\gamma)}{N(5\gamma - 4)}, \) or

(f) \( 2 < N < \sqrt{6}, \ p > N^2, \ \frac{1}{4} < \gamma < 2, \ M > \frac{2(N^2 - 3\gamma)}{N(5\gamma - 4)}, \) or

(g) \( -\sqrt{6} < N \leq -2, \ p > N^2, \ \frac{1}{4} < \gamma < 2, \ M > \frac{2(N^2 - 3\gamma)}{N(5\gamma - 4)}, \) or

(h) \( 0 < N < 2, \ p > N^2, \ 1 < \gamma < \frac{1}{4}, \ M > \frac{2(N^2 - 3\gamma)}{N(5\gamma - 4)}, \) or

(i) \( 2 < N < \sqrt{6}, \ p > N^2, \ 1 < \gamma < \frac{1}{4}, \ M > \frac{2(N^2 - 3\gamma)}{N(5\gamma - 4)}, \) or

(j) \( -2 < N < 0, \ p > N^2, \ \frac{1}{4} < \gamma < 2, \ M > \frac{2(N^2 - 3\gamma)}{N(5\gamma - 4)}. \)

\[ B_3: \left( 0, 2\pi c_1, 0, 1, 0 \right), \quad c_1 \in \mathbb{Z} \text{ with eigenvalues} \left\{ 0, \frac{1}{2}, \frac{1}{2}, p, p - 3\gamma \right\}. \] The physical interesting situation is when it is non-hyperbolic with a 4D unstable manifold for \( p > 6, 1 \leq \gamma \leq 2 \). It is a non-hyperbolic saddle otherwise.

\[ B_4: \left( 1, 2\pi c_1, 0, 1, 0 \right), \quad c_1 \in \mathbb{Z} \text{ with eigenvalues} \left\{ 0, -\frac{1}{2}, -\frac{1}{2}, p, p - 3\gamma \right\}. \] It is a non-hyperbolic saddle.

\[ B_5: \left( 0, \tan^{-1} \left[ \sqrt{1 - \frac{p^2}{6}}, -\frac{p}{\sqrt{6p}} \right] + 2\pi c_1, 0, 1 - \frac{1}{\sqrt{p}}, 0 \right), \quad c_1 \in \mathbb{Z} \text{ with eigenvalues} \]
\[
\left\{ 0, \frac{\pi(4-3\gamma)M+2N}{2N}, -3\gamma, \frac{1}{4} \left( p - 6 + \frac{\sqrt{p^2 - 6(N^2(9p - 6) - 8p^2)}}{N} \right) \right\}
\]

It exists for \( p > 0 \), \( N^2 > \frac{p^2}{6} \). Furthermore, it satisfies \( \Omega_0 \geq 0 \) for \( 0 < p < 6, N^2 \geq p \), or \( p \geq 6, N^2 > \frac{p^2}{6} \). It is a non-hyperbolic saddle.

\[ B_6: \left( 1, \tan^{-1} \left[ \sqrt{1 - \frac{2}{\sqrt{N}}, -\frac{2}{\sqrt{N}} \sqrt{N}} \right] + 2\pi c_1, 0, 1 - \frac{p}{N} \right), \quad c_1 \in \mathbb{Z} \text{ with eigenvalues} \]

\[
\left\{ 0, -4 \frac{\pi(4-3\gamma)M+2N}{2N}, -3\gamma, \frac{1}{4} \left( p - 6 + \frac{\sqrt{p^2 - 6(N^2(9p - 6) - 8p^2)}}{N} \right) \right\}
\]

It exists for \( p > 0 \), \( N^2 > \frac{p^2}{6} \). Furthermore, it satisfies \( \Omega_0 \geq 0 \) for \( 0 < p < 6, N^2 \geq p \), or \( p \geq 6, N^2 > \frac{p^2}{6} \). The situation of physical interest is when it is non-hyperbolic with a 4D stable manifold for:

- (a) \( N > \frac{1}{\sqrt{15}} \), \( 0 < p \leq \frac{1}{16} \left( 9N^2 - 3\sqrt{N^2 (27N^2 - 64)} \right) \), \( \gamma = \frac{3}{4} \), or
- (b) \( N < -\frac{8}{3\sqrt{15}} \), \( 0 < p \leq \frac{1}{16} \left( 9N^2 - 3\sqrt{N^2 (27N^2 - 64)} \right) \), \( \gamma = \frac{3}{4} \), or
- (c) \( N = \frac{8}{\sqrt{15}} \), \( 0 < p \leq \frac{1}{4} \), \( \gamma = \frac{4}{7} \), or
- (d) \( 0 < N \leq \frac{8}{3\sqrt{15}} \), \( 0 < p < N^2 \), \( \gamma = \frac{4}{7} \), or
- (e) \( -\frac{8}{3\sqrt{15}} < N < 0 \), \( 0 < p < N^2 \), \( \gamma = \frac{1}{3} \), or
- (f) \( \frac{8}{3\sqrt{15}} < N < \frac{8}{3\sqrt{15}} \), \( 0 < p \leq \frac{1}{16} \left( 9N^2 - 3\sqrt{N^2 (27N^2 - 64)} \right) \), \( \gamma = \frac{4}{7} \), or
- (g) \( \frac{8}{3\sqrt{15}} < N \leq 2, \frac{1}{16} \left( 9N^2 + \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \leq p < N^2 \), \( \gamma = \frac{4}{7} \), or
- (h) \( -2 < N < -\frac{8}{3\sqrt{15}} \), \( \frac{1}{16} \left( 9N^2 + \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \leq p < N^2 \), \( \gamma = \frac{4}{7} \), or
- (i) \( 2 < N < \frac{8}{\sqrt{15}} \), \( \frac{1}{16} \left( 9N^2 + \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \leq p < 4 \), \( \gamma = \frac{4}{7} \), or
- (j) \( \frac{8}{\sqrt{15}} < N \leq -2, \frac{1}{16} \left( 9N^2 + \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \leq p < 4 \), \( \gamma = \frac{4}{7} \), or
- (k) \( \frac{8}{3\sqrt{15}} < N < \frac{8}{\sqrt{15}} \), \( 0 < p \leq \frac{1}{16} \left( 9N^2 - \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \), \( 1 \leq \gamma < \frac{4}{7} \), \( M < \frac{2}{(3p - 4p)} \), or
- (l) \( \frac{8}{3\sqrt{15}} < N < \frac{8}{\sqrt{15}} \), \( 0 < p \leq \frac{1}{16} \left( 9N^2 - \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \), \( \frac{1}{4} < \gamma \leq 2 \), \( M > \frac{2}{(3p - 4p)} \), or
- (m) \( N < -\frac{8}{3\sqrt{15}} \), \( 0 < p \leq \frac{1}{16} \left( 9N^2 - \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \), \( 1 \leq \gamma < \frac{3}{4} \), \( M > \frac{2}{(3p - 4p)} \), or
- (n) \( N < -\frac{8}{3\sqrt{15}} \), \( 0 < p \leq \frac{1}{16} \left( 9N^2 - \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \), \( \frac{3}{4} < \gamma \leq 2 \), \( M < \frac{2}{(3p - 4p)} \), or
- (o) \( N > \frac{8}{\sqrt{15}} \), \( 0 < p \leq \frac{1}{16} \left( 9N^2 - \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \), \( 1 \leq \gamma < \frac{3}{4} \), \( M < \frac{2}{(3p - 4p)} \), or
- (p) \( N > \frac{8}{\sqrt{15}} \), \( 0 < p \leq \frac{1}{16} \left( 9N^2 - \sqrt{3} \sqrt{N^2 (27N^2 - 64)} \right) \), \( \frac{3}{4} < \gamma \leq 2 \), \( M > \frac{2}{(3p - 4p)} \), or
- (q) \( N = \frac{8}{\sqrt{15}} \), \( 0 < p \leq \frac{4}{7} \), \( \frac{1}{4} \leq \gamma < \frac{3}{4} \), \( M > \frac{16}{(3p - 4p)} \), or
- (r) \( N = \frac{8}{\sqrt{15}} \), \( 0 < p \leq \frac{4}{7} \), \( \frac{3}{4} < \gamma \leq 2 \), \( M > \frac{16}{(3p - 4p)} \), or
- (s) \( N = \frac{8}{\sqrt{15}} \), \( 4 \leq p \leq \frac{64}{15} \), \( \frac{1}{4} \leq \gamma < \frac{3}{4} \), \( M > \frac{16}{(3p - 4p)} \), or
- (t) \( N = \frac{8}{\sqrt{15}} \), \( 4 \leq p \leq \frac{64}{15} \), \( \frac{3}{4} < \gamma \leq 2 \), \( M > \frac{16}{(3p - 4p)} \), or
(u) \(-2 < N < -\frac{8}{\sqrt{3}}, \quad \frac{1}{16} \left(9 N^2 + 27 \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < N^2, \quad 1 \leq \gamma < \frac{4}{3}, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(v) \(-\sqrt{6} < N < -\frac{8}{\sqrt{27}}, \quad \frac{1}{16} \left(9 N^2 + 3 \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < N^2, \quad 1 \leq \gamma < \frac{4}{3}, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(w) \(\frac{8}{\sqrt{3}} < N < 2, \quad \frac{1}{16} \left(9 N^2 + 3 \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < N^2, \quad \frac{4}{3} < \gamma < 2, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(x) \(\frac{8}{\sqrt{6}} < N < \sqrt{6}, \quad \frac{1}{16} \left(9 N^2 + \sqrt{3} \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < N^2, \quad \frac{4}{3} < \gamma < 2, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(y) \(\frac{8}{\sqrt{3}} < N < 2, \quad \frac{1}{16} \left(9 N^2 + 3 \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < N^2, \quad 1 \leq \gamma < \frac{4}{3}, \quad M < \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(z) \(\frac{8}{\sqrt{6}} < N < \sqrt{6}, \quad \frac{1}{16} \left(9 N^2 + \sqrt{3} \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < N^2, \quad 1 \leq \gamma < \frac{4}{3}, \quad M < \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(aa) \(-2 < N < -\frac{8}{\sqrt{3}}, \quad \frac{1}{16} \left(9 N^2 + 3 \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < N^2, \quad \frac{4}{3} < \gamma < 2, \quad M < \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(ab) \(-\sqrt{6} < N < -\frac{8}{\sqrt{27}}, \quad \frac{1}{16} \left(9 N^2 + \sqrt{3} \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < N^2, \quad \frac{4}{3} < \gamma < 2, \quad M < \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(ac) \(2 < N < \frac{1}{\sqrt{3}}, \quad \frac{1}{16} \left(9 N^2 + \sqrt{3} \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < 4, \quad \frac{4}{3} < \gamma < 2, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(ad) \(-\frac{8}{\sqrt{15}} < N < -2, \quad \frac{1}{16} \left(9 N^2 + \sqrt{3} \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < 4, \quad 1 \leq \gamma < \frac{4}{3}, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(ac) \(2 < N < \frac{1}{\sqrt{3}}, \quad \frac{1}{16} \left(9 N^2 + \sqrt{3} \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < 4, \quad 1 \leq \gamma < \frac{4}{3}, \quad M < \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(af) \(-\frac{8}{\sqrt{15}} < N < -2, \quad \frac{1}{16} \left(9 N^2 + \sqrt{3} \sqrt{N^2 (27 N^2 - 64)} \right) \leq p < 4, \quad \frac{4}{3} < \gamma < 2, \quad M < \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(af) \(-\frac{8}{\sqrt{3}} \leq N < 0, \quad 0 < p < N^2, \quad 1 \leq \gamma < \frac{4}{3}, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(ah) \(0 < N < \frac{8}{\sqrt{3}}, \quad 0 < p < N^2, \quad \frac{4}{3} < \gamma < 2, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(aii) \(2 < N < \frac{1}{\sqrt{15}}, \quad 4 \leq p < N^2, \quad \frac{4}{3} < \gamma < 2, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(aij) \(-\frac{8}{\sqrt{15}} < N < -2, \quad 4 \leq p < N^2, \quad 1 \leq \gamma < \frac{4}{3}, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(ak) \(-\frac{8}{\sqrt{15}} \leq N < 0, \quad 0 < p < N^2, \quad \frac{4}{3} < \gamma < 2, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(al) \(0 < N < \frac{8}{\sqrt{3}}, \quad 0 < p < N^2, \quad 1 \leq \gamma < \frac{4}{3}, \quad M > \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(am) \(2 < N < \frac{8}{\sqrt{15}}, \quad 4 \leq p < N^2, \quad 1 \leq \gamma < \frac{4}{3}, \quad M < \frac{2N(p-3\gamma)}{(3-4p)}, \) or

(an) \(-\frac{8}{\sqrt{15}} < N < -2, \quad 4 \leq p < N^2, \quad \frac{4}{3} < \gamma < 2, \quad M < \frac{2N(p-3\gamma)}{(3-4p)}.

B_7: \(\{0, \tan^{-1} \left[ \frac{\sqrt{1 - \frac{2(p-3\gamma)^2}{3M(4-3\gamma)^2}}}{\sqrt{6M(4-3\gamma)}} \right]+2\pi c_1, \frac{26 - 8(p-3\gamma)^2}{3(4-3\gamma)^2 M^2}, \frac{(4-3\gamma)^2 M^2 + 2(2(p-3\gamma)^2)}{(4-3\gamma)^2 M^2} \}, \)

\(c_1 \in \mathbb{Z}, \) with eigenvalues

\(\left\{ 0, \frac{1}{4} \left( \frac{-3M(p-6\gamma+2M^2)}{M(4-3\gamma)} \right), \frac{1}{4} \left( \frac{-2M^2 - 6(p-3\gamma)^2}{M(4-3\gamma)} \right) \right\}. \) It exists for:

(a) \(1 \leq \gamma < \frac{4}{3}, \quad p = 3\gamma, \quad M > 0, \) or
(b) $\frac{4}{3} < \gamma \leq 2$, $p = 3\gamma$, $M > 0$, or
(c) $1 \leq \gamma < \frac{4}{3}$, $p = 3\gamma$, $M > 0$, or
(d) $\frac{4}{3} < \gamma \leq 2$, $p = 3\gamma$, $M > 0$, or
(e) $1 \leq \gamma < \frac{4}{3}$, $p = 3\gamma$, $M < 0$, or
(f) $\frac{4}{3} < \gamma \leq 2$, $p = 3\gamma$, $M < 0$, or
(g) $1 \leq \gamma < \frac{4}{3}$, $p = 3\gamma$, $M < 0$, or
(h) $\frac{4}{3} < \gamma \leq 2$, $p = 3\gamma$, $M < 0$, or

(i) $1 \leq \gamma < \frac{4}{3}$, $3\gamma < p \leq 6$, $M \geq \sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(j) $\frac{4}{3} < \gamma < 2$, $3\gamma < p \leq 6$, $M \geq -\sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(k) $1 \leq \gamma < \frac{4}{3}$, $3\gamma < p \leq 6$, $M \geq \sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(l) $\frac{4}{3} < \gamma < 2$, $3\gamma < p \leq 6$, $M \geq -\sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(m) $1 \leq \gamma < \frac{4}{3}$, $3\gamma < p \leq 6$, $M \leq -\sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(n) $\frac{4}{3} < \gamma < 2$, $3\gamma < p \leq 6$, $M \leq -\sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(o) $\frac{4}{3} < \gamma < 2$, $3\gamma < p \leq 6$, $M \leq -\sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or

It is a non-hyperbolic saddle.

With eigenvalues

$$\begin{align*}
0, \quad \frac{\sqrt{\frac{3}{4} M p + 4 M (p - 3 \gamma) - 2 (p - 3 \gamma)}}{M (4 - 3 \gamma)} + \frac{4 c_1}{3 (4 - 3 \gamma)^2 M^2}, \quad (4 - 3 \gamma)^2 M^2, \quad (4 - 3 \gamma)^2 M^2, \quad (4 - 3 \gamma)^2 M^2
\end{align*}$$

with

$$c_1 \in \mathbb{Z}.$$

(a) $1 \leq \gamma < \frac{4}{3}$, $p = 3\gamma$, $M > 0$, or
(b) $\frac{4}{3} < \gamma < 2$, $p = 3\gamma$, $M > 0$, or
(c) $1 \leq \gamma < \frac{4}{3}$, $p = 3\gamma$, $M > 0$, or
(d) $\frac{4}{3} < \gamma < 2$, $p = 3\gamma$, $M > 0$, or
(e) $1 \leq \gamma < \frac{4}{3}$, $p = 3\gamma$, $M < 0$, or
(f) $\frac{4}{3} < \gamma < 2$, $p = 3\gamma$, $M < 0$, or
(g) $1 \leq \gamma < \frac{4}{3}$, $p = 3\gamma$, $M < 0$, or
(h) $\frac{4}{3} < \gamma < 2$, $p = 3\gamma$, $M < 0$, or

(i) $1 \leq \gamma < \frac{4}{3}$, $3\gamma < p \leq 6$, $M \geq \sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(j) $\frac{4}{3} < \gamma < 2$, $3\gamma < p \leq 6$, $M \geq \sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(k) $1 \leq \gamma < \frac{4}{3}$, $3\gamma < p \leq 6$, $M \geq \sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(l) $\frac{4}{3} < \gamma < 2$, $3\gamma < p \leq 6$, $M \geq \sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(m) $1 \leq \gamma < \frac{4}{3}$, $3\gamma < p \leq 6$, $M \leq -\sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(n) $\frac{4}{3} < \gamma < 2$, $3\gamma < p \leq 6$, $M \leq -\sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(o) $1 \leq \gamma < \frac{4}{3}$, $3\gamma < p \leq 6$, $M \leq -\sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
(p) $\frac{4}{3} < \gamma < 2$, $3\gamma < p \leq 6$, $M \leq -\sqrt{\frac{4(3\gamma - p)}{3\gamma - 4}}$, or
The situation of physical interest is when it is non-hyperbolic with a 4D stable manifold for:

(a) \( N < -3 \sqrt{6} \), \( p = \frac{1}{16} \left( 27N^2 - \sqrt{3}N^2 (243N^2 - 1216) \right) \), \( \frac{4}{7} < \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} < M < -\sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} \), or

(b) \( N < -3 \sqrt{6} \), \( N^2 - \sqrt{3}N^2 (N^2 - 6) < p \leq \frac{1}{7} \left( 9N^2 - \sqrt{3}N^2 (27N^2 - 160) \right) \), \( \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} < M < -\sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} \), or

(c) \( N < -3 \sqrt{6} \), \( \frac{1}{7} \left( 3N^2 - \sqrt{3}N^2 (3N^2 - 16) \right) < p < \frac{1}{16} \left( 27N^2 - \sqrt{3}N^2 (243N^2 - 1216) \right) \), \( \frac{4}{7} < \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} < M < -\sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} \), or

(d) \( N < -3 \sqrt{6} \), \( \frac{1}{7} \left( 9N^2 - \sqrt{3}N^2 (27N^2 - 160) \right) < p < \frac{1}{16} \left( 27N^2 - \sqrt{3}N^2 (243N^2 - 1216) \right) \), \( \frac{4}{7} < \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} < M < -\sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} \), or

(e) \( -3 \sqrt{6} \leq N < -\sqrt{6} \), \( p = \frac{1}{16} \left( 27N^2 - \sqrt{3}N^2 (243N^2 - 1216) \right) \), \( \frac{4}{7} < \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} < M < -\sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} \), or

(f) \( -3 \sqrt{6} \leq N < -\sqrt{6} \), \( N^2 - \sqrt{3}N^2 (N^2 - 6) < p \leq \frac{1}{7} \left( 9N^2 - \sqrt{3}N^2 (27N^2 - 160) \right) \), \( \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} < M < -\sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} \), or

(g) \( -3 \sqrt{6} \leq N < -\sqrt{6} \), \( \frac{1}{7} \left( 3N^2 - \sqrt{3}N^2 (3N^2 - 16) \right) < p < \frac{1}{16} \left( 27N^2 - \sqrt{3}N^2 (243N^2 - 1216) \right) \), \( \frac{4}{7} < \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} < M < -\sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} \), or

(h) \( -3 \sqrt{6} \leq N < -\sqrt{6} \), \( \frac{1}{7} \left( 9N^2 - \sqrt{3}N^2 (27N^2 - 160) \right) < p < \frac{1}{16} \left( 27N^2 - \sqrt{3}N^2 (243N^2 - 1216) \right) \), \( \frac{4}{7} < \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} < M < -\sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} \), or

(i) \( N > \sqrt{6} \), \( p = \frac{1}{16} \left( 27N^2 - \sqrt{3}N^2 (243N^2 - 1216) \right) \), \( \frac{4}{7} < \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} < M < \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} \), or

(j) \( N > \sqrt{6} \), \( N^2 - \sqrt{3}N^2 (N^2 - 6) < p \leq \frac{1}{7} \left( 9N^2 - \sqrt{3}N^2 (27N^2 - 160) \right) \), \( \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} < M < \sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} \), or

N > \sqrt{6}, \( \frac{1}{7} \left( 3N^2 - \sqrt{3}N^2 (3N^2 - 16) \right) < p < \frac{1}{16} \left( 27N^2 - \sqrt{3}N^2 (243N^2 - 1216) \right) \), \( \frac{4}{7} < \gamma < \frac{2p(N^2 - p)}{6N^2 - p^2} \), \( \sqrt{2} \sqrt{\frac{(\gamma - 2)(3\gamma - p)}{(3\gamma - 4p)}} < M < \frac{2N(p-3\gamma)}{\gamma(3\gamma-4p)} \).
with eigenvalues
\[
\left\{ 0, \frac{\sqrt{6(2-\gamma)^2(4-3\gamma)^2M^2}}{\sqrt{(4-3\gamma)^2(4\gamma^2-4)}} \right\}.
\]
It exists for:
(a) \( \gamma = \frac{2}{3}, p > 0, \) or
(b) \( \frac{1}{3} < \gamma < \frac{2}{3}, \frac{8\sqrt{3}-6\gamma}{\gamma^2-4} < \frac{M}{\sqrt{6(2-\gamma)^2}}, p > 0, \) or
(c) \( \frac{2}{3} < \gamma < 2, \frac{8\sqrt{3}-6\gamma}{\gamma^2-4} < \frac{M}{\sqrt{6(2-\gamma)^2}}, p > 0. \)

The situation of physical interest is when it is non-hyperbolic with a 4D stable manifold for:
(a) \( 1 \leq \gamma < \frac{4}{3}, N < \sqrt{6}, \frac{N}{\gamma^2-4} < M < \frac{2\sqrt{6(2-\gamma)^2}}{\gamma^2-4}, p > \)
(b) \( \frac{4}{3} < \gamma < 2, N < \sqrt{6}, \frac{2\sqrt{6(2-\gamma)^2}}{\gamma^2-4} < M < \frac{5\gamma^2-12N^2}{(\gamma^2-4)^{3/2}}, p > \)
(c) \( \frac{2}{3} < \gamma < 2, N > \sqrt{6}, \frac{N}{\gamma^2-4} < \frac{5\gamma^2-12N^2}{(\gamma^2-4)^{3/2}}, p > \)
(d) \( \frac{4}{3} < \gamma < 2, N > \sqrt{6}, \frac{N}{\gamma^2-4} < \frac{5\gamma^2-12N^2}{(\gamma^2-4)^{3/2}}, p > \)

The situation of physical interest is when it is non-hyperbolic with a 4D stable manifold for:
(a) \( M \in \mathbb{R}, \gamma = \frac{2}{3}, N < -2, \) or
(b) \( M \in \mathbb{R}, \gamma = \frac{2}{3}, N > 2, \) or
(c) \( \frac{2}{3} < \gamma < 2, N = -\sqrt{6}, M > \frac{2(\gamma^2-4)}{\gamma^2-4}, \) or
(d) \( \frac{2}{3} < \gamma < 2, 0 < N < \frac{2(\gamma^2-4)}{\gamma^2-4}, M > \frac{2(\gamma^2-4)}{\gamma^2-4}, \) or
(e) \( \frac{2}{3} < \gamma < 2, N < -\sqrt{6}, M > \frac{2(\gamma^2-4)}{\gamma^2-4}, \) or

\[ B_{11}: \ 0, \tan^{-1} \left[ \frac{\sqrt{6(2-\gamma)^2(4-3\gamma)^2M^2}}{\sqrt{(4-3\gamma)^2(4\gamma^2-4)}} \right] + 2\pi c_1, \frac{4M^2(2N^2-M^2)(4(3N^2-12\gamma^2)-3(2\gamma^2-3\gamma^2)M^2)}{\sqrt{(2(2N+M(4-3\gamma^2)))}} \right\}, c_1 \in \mathbb{Z},
\]

with eigenvalues
\[
\left\{ 0, \frac{3(2N+M(4-3\gamma^2))}{\sqrt{(2(2N+M(4-3\gamma^2)))}} \right\}.
\]
It exists for:
(a) \( M \in \mathbb{R}, \gamma = \frac{2}{3}, N < 0, \) or
(b) \( M \in \mathbb{R}, \gamma = \frac{2}{3}, N > 0, \) or
(c) \( \frac{2}{3} < \gamma < 2, N = -\sqrt{6}, M > \frac{2(\gamma^2-4)}{\gamma^2-4}, \) or
(d) \( \frac{2}{3} < \gamma < 2, 0 < N < \sqrt{6}, M < \frac{2(\gamma^2-4)}{\gamma^2-4}, \) or
(e) \( \frac{2}{3} < \gamma < 2, N < -\sqrt{6}, M > \frac{2(\gamma^2-4)}{\gamma^2-4}, \) or
(f) $\gamma \leq 2$, $-\sqrt{6} < N < 0$, $M > \frac{2(\gamma^2 - 3\gamma)}{(\gamma - 4)N}$, or

(g) $\frac{4}{3} < \gamma < 2$, $N > \sqrt{6}$, $M \leq \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(h) $1 \leq \gamma < \frac{4}{3}$, $N = -\sqrt{6}$, $M < \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(i) $1 \leq \gamma < \frac{4}{3}$, $N > \sqrt{6}$, $M \geq \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(j) $1 \leq \gamma < \frac{4}{3}$, $N < -\sqrt{6}$, $M \leq \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(k) $1 \leq \gamma < \frac{4}{3}$, $0 < N \leq \sqrt{6}$, $M > \frac{2(N^2 - 3\gamma)}{(\gamma - 4)N}$, or

(l) $1 \leq \gamma < \frac{4}{3}$, $-\sqrt{6} < N < 0$, $M < \frac{2(N^2 - 3\gamma)}{(\gamma - 4)N}$

It is an hyperbolic saddle.

$$B12: \left(1, \tan^{-1} \left[\sqrt{1 - \frac{4\gamma^2}{4\gamma^2 + M(4 - 3\gamma)^2}} - \frac{\sqrt{6}}{2N + M(4 - 3\gamma)}\right] + 2\pi c_1, \frac{4N(3\gamma^2 + 4(M(4 - 3\gamma)^2))}{128 + M(4 - 3\gamma)} \right), c_1 \in \mathbb{Z},$$

with eigenvalues $$\left\{0, \frac{4M + 2N - 3\gamma}{2N + M(4 - 3\gamma)}, \frac{M(4 - 3\gamma)^2 + 2N(3\gamma^2) - 4\gamma + 3\gamma}{2N + M(4 - 3\gamma)}, \frac{4N + M(2\gamma^2 - 3\gamma)}{128 + M(4 - 3\gamma)} \right\}.$$ (3N(-2 + \gamma) + 3M(-4 + 3\gamma) + \sqrt{(-3M^2 (-3 + 8\gamma^2 - 12\gamma) (4 - 3\gamma)^2 + 6M3 N(-4 + 3\gamma)^3 + 6MN(-4 + 3\gamma)}

\((-6 + 4N^2 + 3\gamma - 6\gamma^2) - 9(-2 + \gamma) + (\gamma^2 (2 - 9\gamma) + 24\gamma^2))\}. \ \text{It exists for:}

(a) $M \in \mathbb{R}$, $\gamma = \frac{4}{3}$, $N < 2$, or

(b) $M \in \mathbb{R}$, $\gamma = \frac{4}{3}$, $N > 2$, or

(c) $\frac{4}{3} \leq \gamma < 2$, $N = -\sqrt{6}$, $M > \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(d) $\frac{4}{3} \leq \gamma < 2$, $0 < N \leq \sqrt{6}$, $M < \frac{2(N^2 - 3\gamma)}{(\gamma - 4)N}$, or

(e) $\frac{4}{3} \leq \gamma < 2$, $N < -\sqrt{6}$, $M \geq \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(f) $\frac{4}{3} \leq \gamma < 2$, $-\sqrt{6} < N < 0$, $M > \frac{2(N^2 - 3\gamma)}{(\gamma - 4)N}$, or

(g) $\frac{4}{3} \leq \gamma < 2$, $N > \sqrt{6}$, $M \leq \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(h) $1 \leq \gamma < \frac{4}{3}$, $N = -\sqrt{6}$, $M < \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(i) $1 \leq \gamma < \frac{4}{3}$, $N > \sqrt{6}$, $M \geq \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(j) $1 \leq \gamma < \frac{4}{3}$, $N < -\sqrt{6}$, $M \leq \frac{-\sqrt{6} - 2\gamma}{\gamma - 4}$, or

(k) $1 \leq \gamma < \frac{4}{3}$, $0 < N \leq \sqrt{6}$, $M > \frac{2(N^2 - 3\gamma)}{(\gamma - 4)N}$, or

(l) $1 \leq \gamma < \frac{4}{3}$, $-\sqrt{6} < N < 0$, $M > \frac{2(N^2 - 3\gamma)}{(\gamma - 4)N}$

The situation of physical interest is when it is non-hyperbolic with a 4D stable manifold for:

(a) $\gamma = \frac{4}{3}$, $2 < N \leq 2.06559$, $p > 4$, or

(b) $\gamma = \frac{4}{3}$, $-2.06559 \leq N < -2$, $p > 4$, or

(c) $\gamma = 1$, $N = -2.24061$, $M = M_{1.1}$, $p > \frac{6N}{M + 2N}$, or

(d) $\gamma = 1$, $N = 2.24061$, $M = -2.13014$, $p > \frac{6N}{M + 2N}$, or

(e) $\gamma = 1$, $N = 2.44949$, $\frac{6N}{M + 2N} < N \leq M_{1.3}$, $p > \frac{6N}{M + 2N}$, or

(f) $\gamma = 1$, $N \leq -2.44949$, $M_{1.1} \leq M < -\sqrt{N^2 - 6 - N}$, $p > \frac{6N}{M + 2N}$, or

(g) $\gamma = 1$, $N > 2.44949$, $\sqrt{N^2 - 6 - N} \leq M \leq M_{1.1}$, $p > \frac{6N}{M + 2N}$, or

(h) $\gamma = 1$, $2.24061 < N < 2.44949$, $M_{1.2} \leq M \leq M_{1.3}$, $p > \frac{6N}{M + 2N}$, or

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(i) $\gamma = 1, \quad -2.44949 < N < -2.42061, \quad M_{1,1} \leq M \leq M_{1,2}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(j) $\gamma = 1, \quad -2.42061 < N < 0, \quad M_{1,1} \leq M < \frac{6 - 6N}{N}, \quad p > \frac{6N}{M_{2}^{2} + 2N}$, or

(k) $\gamma = 1, \quad -2.44949 < N < -2.42061, \quad M_{1,3} \leq M < \frac{6 - 6N}{N}, \quad p > \frac{6N}{M_{2}^{2} + 2N}$, or

(l) $\gamma = 1, \quad 0 < N < 2.44949, \quad -\frac{2}{N^{2}} < M < M_{1,1}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(m) $\gamma = 1.62562, \quad N > 2.44949, \quad M_{2,1} \leq M < \frac{N}{\sqrt{N} - \gamma}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(n) $\gamma = 1.62562, \quad N > N_{2,2}, \quad M = M_{2,1}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(o) $1 < \gamma < \frac{9}{8}, \quad N = N_{2,2}, \quad M = M_{2,2}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(p) $\gamma = 1.62562, \quad N < N_{2,1}, \quad \sqrt{\frac{6N - 12 + N^2}{(3y - 4)^2}} + \frac{N}{\sqrt{N} - \gamma}, \quad M \leq M_{2,1}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(q) $\gamma = 1.62562, -2 < N < 0, \quad \frac{2N^2 - 4N}{N - 4N}, \quad M \leq M_{2,1}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(r) $\gamma = 1.62562, \quad N_{3,1} \leq N < -2.44949, \quad \sqrt{\frac{6N - 12 + N^2}{(3y - 4)^2}} + \frac{N}{\sqrt{N} - \gamma}, \quad M \leq M_{2,3}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(s) $\gamma = 1.62562, \quad 0 < N < 2.44949, \quad M_{2,1} \leq M < \frac{2N^2 - 6N}{N(3y - 4)}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(t) $1 < \gamma < \frac{9}{8}, \quad N = N_{2,4}, \quad M_{2,1} \leq M < \frac{2N^2 - 6N}{N(3y - 4)}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(u) $\frac{2}{3} < \gamma < 1.62562, \quad N > 2.44949, \quad M_{2,1} \leq M < \frac{N}{\sqrt{N} - \gamma}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(v) $1.62562 < \gamma < 2, \quad N > 2.44949, \quad M_{2,1} \leq M < \frac{N}{\sqrt{N} - \gamma}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(w) $1 < \gamma < \frac{9}{8}, \quad N < -2.44949, \quad M_{2,1} \leq M < \frac{N}{\sqrt{N} - \gamma}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(x) $\frac{2}{3} < \gamma < \frac{9}{8}, \quad N < -2.44949, \quad M_{2,1} \leq M < \frac{N}{\sqrt{N} - \gamma}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(y) $1 < \gamma < \frac{9}{8}, \quad N = N_{3,2}, \quad M_{2,3} \leq M < \frac{2N^2 - 6N}{N(3y - 4)}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(z) $\gamma = 1.62562, \quad -2.44949 < N \leq -2, \quad \frac{2}{3} < \gamma < \frac{9}{8}, \quad M_{2,3} \leq M < \frac{2N^2 - 6N}{N(3y - 4)}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(aa) $1 < \gamma < \frac{9}{8}, \quad N > N_{3,4}, \quad \sqrt{\frac{6N - 12 + N^2}{(3y - 4)^2}} + \frac{N}{\sqrt{N} - \gamma}, \quad M \leq M_{2,1}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(ab) $1 < \gamma < \frac{9}{8}, \quad N = N_{3,3}, \quad \frac{2N^2 - 6N}{N(3y - 4)}, \quad M \leq M_{2,1}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(ac) $1 < \gamma < \frac{9}{8}, \quad N = 2.44949, \quad \frac{2N^2 - 6N}{N(3y - 4)}, \quad M \leq M_{2,3}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(ad) $\frac{9}{8} < \gamma < \frac{4}{3}, \quad N > N_{3,4}, \quad \sqrt{\frac{6N - 12 + N^2}{(3y - 4)^2}} + \frac{N}{\sqrt{N} - \gamma}, \quad M \leq M_{2,3}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(ae) $\frac{4}{3} < \gamma < 1.62562, \quad N < N_{3,1}, \quad \frac{6N - 12 + N^2}{(3y - 4)^2} + \frac{N}{\sqrt{N} - \gamma}, \quad M \leq M_{2,1}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(af) $1.62562 < \gamma < 2, \quad N < N_{3,1}, \quad \frac{6N - 12 + N^2}{(3y - 4)^2} + \frac{N}{\sqrt{N} - \gamma}, \quad M \leq M_{2,1}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(ag) $\frac{4}{3} < \gamma < 1.62562, \quad N_{3,1} \leq N < -2.44949, \quad \frac{6N - 12 + N^2}{(3y - 4)^2} + \frac{N}{\sqrt{N} - \gamma}, \quad M \leq M_{2,3}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(ah) $1.62562 < \gamma < 2, \quad N_{3,1} \leq N < -2.44949, \quad \frac{6N - 12 + N^2}{(3y - 4)^2} + \frac{N}{\sqrt{N} - \gamma}, \quad M \leq M_{2,3}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(ai) $\frac{4}{3} < \gamma < 1.62562, \quad 0 < N < 2.44949, \quad M_{2,1} \leq M < \frac{2N^2 - 6N}{N(3y - 4)}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(aj) $1.62562 < \gamma < 2, \quad 0 < N < 2.44949, \quad M_{2,1} \leq M < \frac{2N^2 - 6N}{N(3y - 4)}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(ak) $\frac{9}{8} < \gamma < \frac{4}{3}, \quad -2.44949 < N < 0, \quad M_{2,1} \leq M < \frac{2N^2 - 6N}{N(3y - 4)}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or

(ald) $1 < \gamma < \frac{9}{8}, \quad N_{3,2} \leq N < 0, \quad M_{2,1} \leq M < \frac{2N^2 - 6N}{N(3y - 4)}, \quad p > \frac{6N}{M_{4}(3-3y)}$, or
(am) \(1 < \gamma < \frac{9}{8}, \) \(-2.44949 < N < N_{3,2}, \) \(M_{2,1} \leq M \leq M_{2,2}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(an) \(1 < \gamma < \frac{9}{8}, \) \(0 < N < N_{3,3}, \) \(\frac{2N^2-6N}{N(3^3-4^4)} < M \leq M_{2,1}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(ao) \(\frac{9}{8} < \gamma < \frac{17}{8}, \) \(0 < N < N_{3,3}, \) \(\frac{2N^2-6N}{N(3^3-4^4)} < M \leq M_{2,1}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(ap) \(1 < \gamma < \frac{9}{8}, \) \(-2.44949 < N < N_{3,2}, \) \(M_{2,3} \leq M < \frac{2N^2-6N}{N(3^3-4^4)}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(aq) \(1 < \gamma < \frac{9}{8}, \) \(2.44949 < N \leq N_{3,4}, \) \(\sqrt{\frac{6N-N^2-12N^2}{(3^3-4^4)} + \frac{N}{3^7-4^7}} < M \leq M_{2,3}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(ar) \(\frac{9}{8} < \gamma < \frac{17}{8}, \) \(2.44949 < N \leq N_{3,4}, \) \(\sqrt{\frac{6N-N^2-12N^2}{(3^3-4^4)} + \frac{N}{3^7-4^7}} < M \leq M_{2,3}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(as) \(\frac{3}{8} < \gamma < 1.62562, -2.44949 < N \leq N_{3,2}, \) \(\frac{2N^2-6N}{N(3^3-4^4)} < M \leq M_{2,2}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(at) \(1.62562 < \gamma < 2 - 2.44949 < N \leq N_{3,2}, \) \(\frac{2N^2-6N}{N(3^3-4^4)} < M \leq M_{2,2}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(au) \(1 < \gamma < \frac{9}{8}, \) \(N_{3,2} < N < 2.44949, \) \(M_{2,2} \leq M \leq M_{2,3}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(av) \(1 < \gamma < \frac{9}{8}, \) \(N_{3,3} < N < 2.44949, \) \(\frac{2N^2-6N}{N(3^3-4^4)} < M \leq M_{2,1}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(aw) \(\frac{9}{8} < \gamma < \frac{17}{8}, \) \(N_{3,2} < N < 0, \) \(\frac{2N^2-6N}{N(3^3-4^4)} < M \leq M_{2,1}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(ax) \(1.62562 < \gamma < 2 - N_{3,2} < N < 0, \) \(\frac{2N^2-6N}{N(3^3-4^4)} < M \leq M_{2,1}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(ay) \(\frac{9}{8} < \gamma < \frac{17}{8}, \) \(N_{3,3} \leq N < 2.44949, \) \(\frac{2N^2-6N}{N(3^3-4^4)} < M \leq M_{2,3}, \) \(p > \frac{6N}{M(4^{3/2}+2N)}, \) or

(az) \(\gamma = \frac{9}{8}, \) \(N \leq 2.61227, \) \(\frac{1}{3} \sqrt{64N^2 - 378 - \frac{2}{3} N} < M \leq M_{4,1}, \) \(p > \frac{54N}{5M+16N}, \) or

(ba) \(\gamma = \frac{9}{8}, \) \(N \leq -2.44949, \) \(M_{4,1} \leq M \leq -\frac{1}{3} \sqrt{64N^2 - 378 - \frac{2}{3} N}, \) \(p > \frac{54N}{5M+16N}, \) or

(bb) \(\gamma = \frac{9}{8}, \) \(2.44949 < N \leq 2.61272, \) \(\frac{1}{3} \sqrt{64N^2 - 378 - \frac{2}{3} N} < M \leq M_{4,1}, \) \(p > \frac{54N}{5M+16N}, \) or

(bc) \(\gamma = \frac{9}{8}, -2.44949 < N < 0, \) \(M_{4,1} \leq M \leq \frac{(54-16N^2)}{5N}, \) \(p > \frac{54N}{5M+16N}, \) or

(bd) \(\gamma = \frac{9}{8}, 0 < N < 2.44949, \) \(\frac{(54-16N^2)}{5N} < M \leq M_{4,1}, \) \(p > \frac{54N}{5M+16N}, \) or

(be) \(\gamma = 2, \) \(0 < N < 2.44949, \) \(\frac{(8N^2-27)N}{8N} < M \leq \frac{N^2-6}{N}, \) \(p > \frac{6N}{N-M}, \) or

(bf) \(\gamma = 2, -2.44949 < N < 0, \) \(\frac{N^2-6}{N} < M \leq \frac{1}{8} \sqrt{729-48N^2} \frac{N}{N^2} + \frac{(8N^2-27)N}{8N}, \) \(p > \frac{6N}{N-M}, \) or

Where \(M_{1,1}, M_{1,2}, M_{1,3}\) are the first, second and the third root of the polynomial \(P(M) = 2M^3N + M^2(8N^2 - 15) + M(8N^3 - 18N) + 21N^2 - 72,\) respectively.

\(M_{2,1}, M_{2,2}\) and \(M_{2,3}\) are the first, second and the third root of the polynomial \(P_2(M) = -72\gamma^3 + 144\gamma^2 + M^3 (54\gamma^4 - 216\gamma^3 N + 288\gamma^2 N^2 - 128N^3) + M^2 (108\gamma^3 - 261\gamma^2 + 120\gamma - 72\gamma N^2 + 192\gamma N^3 - 128N^2 + 48) + M (24\gamma^3 - 32N^2 - 36\gamma N^2 + 6\gamma^2 N + 24\gamma N + 48N) + 27\gamma^2 N^2 - 60\gamma N^2 + 12N^2,\) respectively. \(N_{3,1}, N_{3,2}, N_{3,3}\) and \(N_{3,4}\) are the first, second, the third and the fourth root of the polynomial: \(P(N) = -1152\gamma^4 + 1440\gamma^3 + 1512\gamma^2 + 414\gamma + 64N^2 + (32\gamma^2 - 1176\gamma + 96)N^2 + 36,\) respectively. Finally, \(M_{4,1}\) is first root and \(M_{4,3}\) is the third root of the polynomial: \(P_4(M) = 125M^3N + M^2 (800N^2 - 1650) + M (1280N^3 - 3270N) + 5460N^2 - 204.12.\)

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