Differential cohomology

Ulrich Bunke

September 27, 2012

Abstract

These course notes first provide an introduction to secondary characteristic classes and differential cohomology. They continue with a presentation of a stable homotopy theoretic approach to the theory of differential extensions of generalized cohomology theories including products and Umkehr maps.

Contents

1 Introduction 2

2 Characteristic forms 3
   2.1 Linear Connections ............................................. 3
   2.2 Invariant connections and quotients .............................. 6
   2.3 Characteristic forms, classes and transgression - theory ........ 10
   2.4 Characteristic forms, classes and transgression - examples ...... 14
   2.5 Metrics and unitarity ........................................... 18
   2.6 Integrality .................................................... 20
   2.7 Integral refinement ............................................ 28

3 Smooth Deligne cohomology 32
   3.1 Recollections on sheaf theory .................................. 32
   3.2 Deligne cohomology .............................................. 35
   3.3 Differential refinements of integral characteristic classes ...... 42
   3.4 Multiplicative structure ......................................... 56
   3.5 Cheeger-Simons differential characters .......................... 66
   3.6 Integration .................................................... 68

∗NWF I - Mathematik, Universität Regensburg, 93040 Regensburg, GERMANY, ulrich.bunke@mathematik.uni-regensburg.de
1 Introduction

This is a script based on my introductory course on differential cohomology taught in spring 2012 at the Regensburg University. The course was designed for students having a profound background in differential geometry, algebraic topology and homotopy theory. The course starts with a detailed introduction to characteristic forms for vector bundles. I explain how one can use locality and integrality of these forms to construct secondary invariants. The main emphasis in this part is put on explicit calculations of examples. I think that this experience is important to understand and work with the fine structure of differential cohomology to be introduced later.

To capture these secondary invariants for complex vector bundles systematically and refine them even more I first introduce differential integral cohomology. More specifically,

| 4 Differential extensions of generalized cohomology theories | 71 |
|-------------------------------------------------------------|----|
| 4.1 The $\infty$-categorical black-box                      | 71 |
| 4.2 Eilenberg-MacLane and de Rham                           | 81 |
| 4.3 Differential function spectra and differential cohomology| 85 |
| 4.4 Differential $K$-theory                                 | 92 |
| 4.5 Differential Bordism theory                             | 99 |
| 4.6 Multiplicative structures                               | 104|
| 4.7 Relative sites                                           | 111|
| 4.8 Thom classes                                            | 120|
| 4.9 Orientation and integration - the topological case      | 132|
| 4.10 Orientation and integration - the differential case     | 137|
| 4.11 Higher $e$-invariants and index theorems               | 143|
| 4.12 Geometrization                                         | 156|

| 5 Exercises                                                 | 164|
|-------------------------------------------------------------|----|
| 5.1 Sheet Nr 1.                                             | 165|
| 5.2 Sheet Nr 2.                                             | 166|
| 5.3 Sheet Nr 3.                                             | 167|
| 5.4 Sheet Nr 4.                                             | 168|
| 5.5 Sheet Nr 5.                                             | 169|
| 5.6 Sheet Nr 6.                                             | 170|
| 5.7 Sheet Nr 7.                                             | 171|
| 5.8 Sheet Nr 8.                                             | 172|
| 5.9 Sheet Nr 9.                                             | 173|
| 5.10 Sheet Nr 10.                                           | 174|
| 5.11 Sheet Nr 11.                                           | 175|
| 5.12 Sheet Nr 12.                                           | 176|
| 5.13 Sheet Nr 13.                                           | 177|
| 5.14 Sheet Nr 14.                                           | 178|
I explain the sheaf theoretic definition of differential cohomology usually called smooth Deligne cohomology. The main theoretical result here is the construction of the differential refinement of the Chern-Weyl homomorphism due to Cheeger-Simons. I give a detailed discussion of various structures like integration and products. Up to this point the course reviews the classical part of the theory. The theory is complemented by a variety of examples. It is again important to do these exercises to get an idea how the theory work, and to learn some calculational tricks.

The course then continues with a presentation of a full-fledged theory of differential extensions of generalized cohomology theories including products and integration. This stable homotopy theoretic approach is new and complements many special constructions for particular cases. The details of the homotopy theoretic approach are developed in Section [4] which can be read independently of the previous sections. I again tried to add, as much as possible, interesting explicit calculations. As examples of differential extensions of generalized cohomology theories I discuss the cases of complex $K$-theory and complex bordism in some detail.

The script contains a variety of exercises ranging from explicit calculations with numerical outputs to verifications of statements made without detailed proofs in the course. While these theoretical pieces should be more or less straight-forward, some of the explicit calculations are more involved and were major pieces of original works or PhD theses. This case will be indicated by giving appropriate references. For many problems I have added a sketch of a proof. The exercise sheets 1-13 were discussed in the weekly exercise classes which complemented the course. The Sheet 14 (cf. [5,14]) collects some problems which were left open during the course. Some of them may be interesting research projects.

Acknowledgements: I am grateful to the participants of the course (M. Ruderer, M. Spitzweck, J. Sprang, A. Straak, G. Tamme, M, Voelkl) for their patience and critical questions which lead to many crucial corrections. Furthermore, I thank Th. Nikolaus and D. Gepner for various helpful discussions, in particular about $\infty$-categorical aspects.

2 Characteristic forms

2.1 Linear Connections

We consider a smooth manifold $M$. By $\Omega(M)$ we denote the de Rham complex of $M$. For its complexification we use the notation $\Omega(M, \mathbb{C})$. Let $E \to M$ be a complex vector bundle. Then we have the $\Omega(M, \mathbb{C})$-module $\Omega(M, E)$ of differential forms with coefficients in $E$. In particular, $\Omega^0(M, E)$ is the space of sections of $E$. Locally, an element of $\Omega(M, E)$ can be written as a finite sum of elements of the form $\omega \otimes \phi$ with $\phi \in \Omega(M, \mathbb{C})$ and $\phi \in \Omega^0(M, E)$. The product of $\alpha \in \Omega(M, \mathbb{C})$ with $\omega \otimes \phi$ is then given by $\alpha \wedge (\omega \otimes \phi) := (\alpha \wedge \omega) \otimes \phi$.

Definition 2.1. A connection on $E$ is a map

$$\nabla : \Omega^0(M, E) \to \Omega^1(M, E)$$
satisfying the Leibnitz rule

\[ \nabla(f \wedge \phi) = df \wedge \phi + f \wedge \nabla \phi , \quad \forall f \in \Omega^0(M, \mathbb{C}) , \forall \phi \in \Omega^0(M, E) . \]

In the following we discuss some constructions with connections and observe that connections always exist.

A trivialization \( \phi \) of \( E \) is a collection of sections \( \phi = (\phi_\alpha)_{\alpha=1}^k \) of \( E \) whose evaluations at each point \( m \in M \) form a basis of the fibre \( E_m \) of \( E \) at \( m \). A trivialization \( \phi \) induces a connection \( \nabla^\phi \) as follows.

**Problem 2.2.** Show that there exists a unique connection \( \nabla^\phi \) on \( E \) characterized by \( \nabla^\phi(\phi_\alpha) = 0 \) for all \( \alpha \in \{1, \ldots, k\} \).

Connections can be glued with the help of a partition of unity. Let \( (\nabla_i)_{i \in I} \) be a collection of connections on \( E \) and \( (\chi_i)_{i \in I} \) be a partition of unity. For a function \( \chi \in \Omega^0(M, \mathbb{C}) \) and a connection \( \nabla \) we write \( \chi \nabla \) for the composition of the operators \( \nabla \) and multiplication by \( \chi \).

**Problem 2.3.** Show that the sum \( \sum_{i \in I} \chi_i \nabla_i \) is a connection on \( E \).

**Lemma 2.4.** Every vector bundle admits a connection

**Proof.** Since a vector bundle is locally trivial, it locally admits connections. We get a global connection by glueing the local ones.

**Problem 2.5.** Work out the details.

We consider the bundle of algebras \( \text{End}(E) \to M \). It induces a graded algebra \( \Omega(M, \text{End}(E)) \) over \( \Omega(M, \mathbb{C}) \) with product given by

\[ (\alpha \otimes \Psi)(\beta \otimes \Phi) := (\alpha \wedge \beta) \otimes \Phi \circ \Psi , \]

This algebra acts on \( \Omega(M, E) \) by

\[ (\alpha \otimes \Psi) \wedge (\omega \otimes \phi) := (\alpha \wedge \omega) \otimes \Psi(\phi) . \]

**Problem 2.6.** Show that the set of connections on \( E \) has the structure of an affine space over \( \Omega^1(M, \text{End}(E)) \) given by

\[ (\nabla + \omega)(\phi) = \nabla \phi + \omega \wedge \phi , \quad \forall \omega \in \Omega^1(M, \text{End}(E)) , \forall \phi \in \Omega^0(M, E) . \]

**Problem 2.7.** Show that a connection on \( E \) uniquely extends to a connection on \( \text{End}(E) \) which is characterized by

\[ (\nabla \omega) \wedge \phi = \nabla(\omega \wedge \phi) - \omega \wedge \nabla \phi \quad (1) \]

for \( \omega \in \Omega^0(M, \text{End}(E)) \), \( \phi \in \Omega^0(M, E) \).
Problem 2.8. Show that a connection on $E$ uniquely extends to a linear map $\nabla : \Omega(M, E) \to \Omega(M, E)$ satisfying the Leibnitz rule

$$\nabla(\omega \wedge \phi) = d\omega \wedge \phi + (-1)^k \omega \wedge \nabla \phi , \quad \forall \omega \in \Omega^k(M, \mathbb{C}) , \forall \phi \in \Omega(M, E) .$$

Using this extension we consider $\nabla^2 := \nabla \circ \nabla : \Omega(M, E) \to \Omega(M, E)$.

Problem 2.9. Show that $\nabla^2$ is given by multiplication by a uniquely determined element $R \nabla \in \Omega^2(M, \text{End}(E))$.

Definition 2.10. The element $R \nabla \in \Omega^2(M, \text{End}(E))$ characterized in Problem 2.9 is called the curvature of $\nabla$. A connection is called flat if its curvature vanishes.

The connection associated to a trivialization of $E$ (Problem 2.2) is flat.

Note that $\nabla$ is an odd operator. Hence we can write $\nabla^2 = \frac{1}{2} [\nabla, [\nabla, \nabla]] = 0$.

Let $f : M' \to M$ be a smooth map and consider the pull-back diagram

$$
\begin{array}{c}
E' \\
\downarrow F \\
M' \\
\downarrow f \\
M
\end{array}
$$

of vector bundles. We have a pull-back operation of sections $(F, f)^* : \Omega(M, E) \to \Omega(M', E')$ such that

$$F(((F, f)^* \omega)(m')) = \omega(m) \circ \Lambda^k df(m)$$

as linear maps $\Lambda^k T_{m'} M' \to E_m$ for $\omega \in \Omega^k(M, E)$, $m' \in M'$ and $m := f(m')$.

We consider a connection $\nabla$ on $E$.

Problem 2.11. Show that there is a unique connection $\nabla' := (F, f)^* \nabla$ on $E'$ which is characterized by

$$\nabla'((F, f)^* \phi) = (F, f)^* (\nabla \phi) , \quad \forall \phi \in \Omega^0(M, E) .$$

If $f = \text{id}$, then we write $F^* := (F, \text{id})^*$. If $E' = f^* E$ and $F$ is the canonical map, then we write $f^* \nabla := (F, f)^* \nabla$.

Problem 2.12. Verify that if $f = \text{id}_M$ and $F \in \Omega^0(M, \text{End}(E))$ is an automorphism of $E$, then we have

$$F^* \nabla = \nabla - F^{-1} \wedge \nabla F .$$

Proof. Note that $F^* \phi = F^{-1} \wedge \phi$. 
We let $\iota_F : f^* \text{End}(E) \iso \text{End}(E')$ be the isomorphism of endomorphism bundles induced by $F$. 

\[ \square \]
Problem 2.13. Show that the curvature satisfies
\[ R(F\ast f)\nabla = \iota_F(F, f)\ast R\nabla. \]
A connection on \( E \) induces a parallel transport \( \|\gamma : E_{\gamma(0)} \to E_{\gamma(1)} \| \) along curves \( \gamma : [0, 1] \to M \). This material is assumed as a prerequisite.
Later we need the following variants.

Example 2.14. If \( E \to M \) is a real vector bundle, then a connection is a map
\[ \nabla : \Omega^0(M, E) \to \Omega^1(M, E) \]
satisfying the Leibnitz rule for real functions \( f \in C^\infty(M) \). Here the real vector space \( \Omega(M, E) = \Gamma(M; \Lambda^*T^*M \otimes E) \) is a \( \Omega(M) \)-module.
Later in this course (e.g. in 3.72) we will assume knowledge of the theory of connections on principal bundles. We refer to [KN96] for details. There is a parallel theory of existence, glueing, and curvature.

2.2 Invariant connections and quotients

Let \( G \) be a Lie group which acts on \( M \). Assume that this action lifts to an action on the bundle \( E \to M \). For every \( g \in G \) we get a pull-back diagram
\[
\begin{array}{ccc}
E & \xrightarrow{\tilde{g}} & E \\
\downarrow & & \downarrow \\
M & \xrightarrow{g} & M
\end{array}
\]
and the map \( g \mapsto \tilde{g} \) satisfies an associativity rule.

Definition 2.15. A connection \( \nabla \) on \( E \) is called \( G \)-invariant if \((\tilde{g}, g)^*\nabla = \nabla\) for all \( g \in G \).

Problem 2.16. Show that if \( G \) acts properly on \( M \), then there exists a \( G \)-invariant connection on \( E \).

Proof. By assumption the map \((\mu, \text{pr}_M) : G \times M \to M \times M\) is proper. There exists a function \( \chi \in C^\infty(M) \) such that \((\mu, \text{pr}_M)^{-1}(\text{supp}(\chi) \times K)\) is compact for all compact \( K \subseteq M \) and \( \int_G g^*\chi \, dg \equiv 1 \). Let \( \tilde{\nabla} \) be any connection on \( E \). Then
\[ \nabla := \int_G (\tilde{g}, g)^*(\chi \tilde{\nabla}) \, dg \]
is an invariant connection. Observe the convergence of this integral.
For the construction of the function \( \chi \) one could proceed as follows. Let \((\phi_i)_{i \in I}\) be a collection of non-negative compactly supported functions whose supports cover \( M \) and set \( \phi_i := \int_G g^*\phi_i \, dg \). The supports of these functions cover \( M \) and by paracompactness we
can choose a subset $J \subseteq I$ such that $(\text{supp}(\phi_i))_{i \in J}$ is a locally finite covering of $M$. We set
\[
\chi := \frac{1}{\sum_{i \in J} \chi_i} \sum_{i \in J} \chi_i .
\]
\[\square\]

Assume that $G$ acts freely and properly on $M$. Then we have a $G$-principal bundle $M \to \bar{M} := G \backslash M$. Furthermore, $G$ acts freely on $E$ such that $\bar{E} := G \backslash E \to \bar{M}$ is a vector bundle which fits into pull-back diagram
\[
\begin{array}{ccc}
E & \xrightarrow{P} & \bar{E} \\
\downarrow & & \downarrow \\
M & \xrightarrow{p} & \bar{M}
\end{array}
\]

**Problem 2.17.** Show that there is a natural isomorphism
\[
(p, P)^* : \Omega^0(\bar{M}, \bar{E}) \cong \Omega^0(M, E)^G .
\]

**Definition 2.18.** We say that an invariant connection $\nabla$ on $E$ descends of there exists a connection $\nabla$ on $\bar{E} \to \bar{M}$ such that $\nabla = (P, p)^* \nabla$.

**Problem 2.19.** If $\nabla$ descends, then $\nabla$ is uniquely determined.

**Problem 2.20.** Assume that $G$ is discrete. Then every invariant connection $\nabla$ on $E$ descends.

If $G$ is not discrete, then there is an obstruction against descending an invariant connection $\nabla$. Let $\mathfrak{g}$ be the Lie algebra of $G$. By $\mathcal{X}(M)$ we denote the set of vector fields on $M$.

**Definition 2.21.** For $A \in \mathfrak{g}$ the fundamental vector field $A^\sharp \in \mathcal{X}(M)$ is defined by
\[
A^\sharp(m) = \frac{d}{dt} \exp(tA)m .
\]

**Definition 2.22.** For $A \in \mathfrak{g}$ the Lie derivative $\mathcal{L}_A : \Omega(M, E) \to \Omega(M, E)$ is defined by
\[
\mathcal{L}_A \phi := \frac{d}{dt} \big|_{t=0} (\exp(tA) \cdot \exp(tA))^* \phi , \phi \in \Omega(M, E) .
\]

Let $\nabla$ be an invariant connection.

**Definition 2.23.** The momentum map $\mu^\nabla \in \text{Hom}(\mathfrak{g}, \Omega^0(M, \text{End}(E))^G$ is defined by
\[
\mu^\nabla(A) \wedge \phi := \nabla_{A^\sharp} \phi - \mathcal{L}_A \phi , \quad \phi \in \Omega^0(M, E) .
\]

**Problem 2.24.** Show that $\mu^\nabla$ is well-defined. Further show that $\nabla$ descends if and only if $\mu^\nabla = 0$. 

7
Example 2.25. Assume that $G$ is a compact Lie group. We consider the action of $G$ on $TG \to G$ by left multiplication. We consider a biinvariant Riemannian metric on $G$ and let $\nabla$ be the Levi-Civita connection on $TG$. Furthermore let $\nabla^l$ be the connection defined by the trivialization of $TG$ by left-invariant vector fields.

Problem 2.26. Calculate the moment maps of $\nabla$ and $\nabla^l$.

We shall see that $\nabla^l$ descends, while $\nabla$ does not.

Example 2.27. Let $\rho : G \to \text{End}(V)$ be a linear representation of $G$ on a finite-dimensional complex vector space $V$ and assume that $G$ acts freely and properly on $M$. Then we get an action of $G$ on the trivial bundle $E := M \times V$ by $g(m, v) := (gm, \rho(g)v)$. The trivial connection on $E \to M$ is $G$-invariant.

Problem 2.28. Calculate the moment map.

We shall see that $\nabla$ descends of and only if $d\rho = 0$.

Example 2.29. In this example we discuss the relation of the moment map in the sense of Definition 2.23 with the moment map in symplectic geometry. Let $(M, \omega)$ be a symplectic manifold such that $\frac{1}{2\pi i} \omega$ is integral (Definition 2.84). For simplicity we assume that $M$ is simply connected. Then there exists a line bundle $L \to M$ with connection $\nabla$ such that $R^\nabla = -\omega$. In fact, $(L, \nabla)$ is uniquely determined up to isomorphism by these conditions.

We now assume that $G$ acts on $M$ in a Hamiltonian fashion with symplectic moment map $\mu \omega \in \text{Hom}(\mathfrak{g}, C^\infty(M))^G$. It is characterized up to an element of $(\mathfrak{g}^*)^G$ by

$$d\mu \omega(A) = i_A \omega.$$

We now observe that $\text{End}(L) = M \times \mathbb{C}$ so that $\mu \nabla \in \text{Hom}(\mathfrak{g}, C^\infty(M, \mathbb{C}))^G$. The action of $G$ on $M$ extends to a action of a $\mathbb{C}^*$-central extension $\tilde{G} \to G$ on $L \to M$. For simplicity we assume that this extension splits. Two splits differ by a character $\chi : G \to \mathbb{C}^*$. Note that the corresponding moment maps differ by $d\chi \in (\mathfrak{g}^* \otimes \mathbb{C})^G$. We calculate for $A \in \mathfrak{g}$ and $X \in \mathfrak{X}(M)$

$$d\mu \nabla(A)(X) = [\nabla_X, \nabla_A - \mathcal{L}_A] = -\omega(X, A^2) = d\mu \omega(A)(X).$$

We conclude that we can adjust the action of $G$ on $L$ such that

$$\mu \nabla = \mu \omega.$$

In the theory of quantization one wants to define a quotient $\tilde{L} \to \tilde{M}$ with connection $\tilde{\nabla}$ whose curvature is symplectic. This is obstructed exactly by the moment map. The way out is to restrict to its zero set $M_0 := (\mu \omega)^{-1}(0)$ and to take $\tilde{L}_0 \to \tilde{M}_0$ with the induced connection (assuming that $M_0$ is smooth and $G$ acts freely on $M_0$). This construction is called symplectic reduction.
Example 2.30. Let $M$ be a compact connected manifold with base point $m \in M$ and $\tilde{M} \to M$ be the associated universal covering. Then $\pi := \pi_1(M, m)$ is finitely generated and $J(M) := \text{Hom}(\pi, U(1))$ is finite-dimensional dimensional abelian Lie group, the Jacobian of $M$. Its connected component of the identity is a torus. We consider the action of $\pi$ on $E := J(M) \times \tilde{M} \times \mathbb{C}$ given by $\gamma(\rho, \tilde{m}, z) := (\rho, \tilde{m}\gamma^{-1}, \rho(\gamma)z)$. This action covers the free and proper action of $\pi$ on $J(M) \times M$ (on the second factor). Hence we have a quotient $P \to J(M) \times M$ which is called the Poincaré bundle.

For $\gamma \in \pi$ we consider the $U(1)$-valued function $\rho \mapsto \rho(\gamma)$ on $J(M)$. We have $\rho(\gamma)^{-1}d\rho(\gamma) \in \Omega^1_{cl}(J(M), \mathbb{C})$. This is actually a homomorphism $\rho^{-1}d\rho \colon \pi \to \Omega^1_{cl}(J(M), \mathbb{C})$, hence a cohomology class $[\theta] \in H^1_{dR}(M; \mathbb{C}) \otimes \Omega^1_{cl}(J(M), \mathbb{C})$ represented by a closed form $\theta \in \Omega^1(M; \mathbb{C}) \otimes \Omega^1_{cl}(J(M), \mathbb{C}) \subseteq \Omega^2_{cl}(J(M) \times M; \mathbb{C})$. Since $H^1(M; \mathbb{C}) = 0$ we can choose a function $\alpha \in \Omega^0(M; \mathbb{C}) \otimes \Omega^1_{cl}(J(M), \mathbb{C}) \subseteq \Omega^0_{cl}(J(M) \times M, \mathbb{C})$ such that $d\alpha$ is $\pi$-invariant and its descent to $M$ equals $\theta$. Then we define the connection $\nabla$ on the trivial bundle $E$ by

$$\nabla = d - \alpha.$$  

**Problem 2.31.** Show that this connection is $\pi$-invariant.

**Proof.** Let $[\gamma] \in \pi$ and $\phi$ be the constant section of $E$ with value 1. Then $([\gamma]^*\nabla)[\gamma]^*(\phi) = [\gamma]^*(\nabla\phi)$. We shall first compute the r.h.s. We have $\nabla\phi = -\alpha$. Furthermore $[\gamma]^*\alpha - \alpha = \int_{[\gamma]} d\alpha = \rho(\gamma)^{-1}d\rho(\gamma)$. Hence $[\gamma]^*(\nabla\phi) = -\rho^{-1}(\gamma)\alpha - \rho(\gamma)^{-2}d\rho(\gamma)$. On the other hand $\nabla([\gamma]^*\phi) = -\rho(\gamma)^{-2}d\rho(\gamma) - \rho(\gamma)^{-1}\alpha$. Hence $\nabla = [\gamma]^*\nabla$. \qed

Hence the choice of $\alpha$ determines a connection $\nabla^{P,\alpha}$ on the Poincaré bundle.

**Problem 2.32.** Calculate the curvature of $\nabla^P$.

**Proof.** We have $R^\nabla = -d\alpha = -\alpha \wedge \theta$. It follows that $R^{\nabla^P} = -\theta \in \Omega^2(J(M) \times M, \mathbb{C})$, where we identify $M \times \mathbb{C} \cong \text{End}(P)$. \qed

**Problem 2.33.** Make this construction explicit in the case where $M$ itself is a torus $T^n := \mathbb{R}^n/\mathbb{Z}^n$. In this case we can take $\alpha$ to be linear.

**Proof.** We identify $\mathbb{R}^n$ with its dual with respect to the standard scalar product. Then $J(T^n) \cong \mathbb{R}^n/\mathbb{Z}^n$ with the identification $[y](u) = \exp(2\pi i \langle y, u \rangle)$ for $u \in \mathbb{Z}^n \cong \pi_1(T^n)$, $y \in \mathbb{R}^n$. We have $\rho^{-1}d\rho(u) = 2\pi i \langle dy, u \rangle$. We can take $\theta := 2\pi i \langle dy \wedge dx \rangle$. Finally, for $\alpha$ we chose $\alpha := 2\pi i \langle dy, x \rangle$. The curvature of $\nabla^P$ is given by $R^{\nabla^P} = -2\pi i \langle dy \wedge dx \rangle$.

**Example 2.34.** Let $M$ be a compact connected manifold with base point $m_0$. We consider the functor $F : \text{MF}^p \to \text{Set}$ which associates to a manifold $T$ the set of isomorphism classes of line bundles $L \to T \times M$ with a flat partial connection $\nabla : \Gamma(T \times M, L) \to \Gamma(T \times M, T^*M \otimes L)$ along the foliation given by the fibres of the projection $T \times M \to T$ and a trivialization of these structures restricted to $T \times \{m_0\}$. For $f : T' \to T$ we let $F(f) : F(T) \to F(T')$ be given by pull-back.
Problem 2.35. Show that this functor is representable. More concretely, consider the Poincaré bundle $P \to J(M) \times M$ with the partial connection $\nabla$ induced by $\nabla^P$. Observe that this determines an object of $F(J(M))$ which is independent of the choices involved in the construction. It induces a natural transformation

$$\text{Hom}_{MF}(\ldots, J(M)) \to F$$

which turns out to be an isomorphism of functors.

Example 2.36. We have a natural central inclusion $U(1) \subset SU(2)$. We have an identification $SU(2)/U(1) \cong \mathbb{CP}^1$ and $SU(2) \times_{U(1), \text{id}} \mathbb{C} \cong L$, where $L \to \mathbb{CP}^1$ is the tautological bundle. Since $U(1)$ is central, the group $SU(2)$ still acts on the quotient $L \to \mathbb{CP}^1$.

Problem 2.37. Show that there is a unique $SU(2)$-invariant connection $\nabla^L$ on $L \to \mathbb{CP}^1$. Calculate its curvature form explicitly.

Proof. Existence is ensured by 2.16. The difference of two invariant connections is an element of $\Omega^1(\mathbb{CP}^1, \mathbb{C})^{SU(2)}$. Show that this space is trivial. The result of the curvature calculation is

$$R^{\nabla^L} = 2\pi i \text{vol}_{\mathbb{CP}^1},$$

where $\text{vol}_{\mathbb{CP}^1}$ is the unique normalized $S^3$-invariant volume form on $\mathbb{CP}^1$.

Problem 2.38. Generalize 2.37 to $\mathbb{CP}^n$ with action of $SU(n+1)$.

2.3 Characteristic forms, classes and transgression - theory

We consider characteristic forms for complex vector bundles.

Definition 2.39. A characteristic form $\omega$ of degree $n$ associates to each connection $\nabla$ on a complex vector bundle bundle $E \to M$ a closed form $\omega(\nabla) \in \Omega^n_d(M, \mathbb{C})$ such hat $\omega((F, f)^*\nabla) = f^*\omega(\nabla)$ for all pull-back diagrams

$$\begin{array}{ccc}
E' & \xrightarrow{F} & E \\
\downarrow & & \downarrow \\
M' & \xrightarrow{f} & M
\end{array}$$

Problem 2.40. Let $\omega$ be a characteristic form of degree $\geq 1$. Show that $\omega(\nabla) = 0$ if $\nabla$ is flat.

Proof. Reduce to a local question. Locally a flat bundle is pulled back from a point. On a point there are no higher-degree forms.

Example 2.41. The form $\dim$ which associates to $(E, \nabla)$ the function $\dim(E) \in \Omega^0_d(M, \mathbb{C})$ is a characteristic form of degree 0.
Our main examples are Chern classes and the components of the Chern character. We start with Chern classes. The determinant is a fibrewise polynomial function \( \det : \text{End}(E) \to M \times \mathbb{C} \) and therefore extends to
\[
\det : \Omega^{ev}(M, \text{End}(E)) \to \Omega^{ev}(M, \mathbb{C}) .
\]
For example,
\[
\det \left( dx \wedge dy \quad dy \wedge dz \quad dz \wedge du \right) = dx \wedge dy \wedge dz \wedge du .
\]
The curvature \( R^\nabla \) (cf. Definition 2.9) of the connection \( \nabla \) is an element
\[
R^\nabla \in \Omega^2(M, \text{End}(E)) \subset \Omega^{ev}(M, \text{End}(E)) .
\]
\textbf{Definition 2.42.} The homogeneous pieces of the total Chern form which is defined by
\[
c(\nabla) := \det(1 - \frac{1}{2\pi i} R^\nabla) = 1 + c_1(\nabla) + \cdots + c_n(\nabla) , \quad c_n(\nabla) \in \Omega^{2n}(M, \mathbb{C})
\]
are called the Chern forms of \( \nabla \).

For example
\[
c_1(\nabla) = -\frac{1}{2\pi i} \text{Tr}(R^\nabla) .
\]
\textbf{Lemma 2.43.} The Chern forms are characteristic forms.

\begin{proof}
Use 2.13 for compatibility with pull-back. In order to see closedness we use that for \( \Phi \in \Omega^{\geq 2}(M, \text{End}(E)) \) we have
\[
d \det(1 + \Phi) = \text{Tr} \left( (1 + \Phi)^{-1}[\nabla, \Phi] \right) .
\]
\end{proof}

\textbf{Problem 2.44.} Prove this formula.

\begin{proof}
Note that an element \( \Phi \in \Omega^{\geq 1}(M, \text{End}(E)) \) is nilpotent. We understand \( \exp(\Phi) \) or \( \log(1 + \Phi) \) in the sense of formal power series. We write
\[
\det(1 + \Phi) = \exp(\text{Tr}(\log(1 + \Phi))) .
\]
Using that \( d \) is a derivation we get
\[
d \det(1 + \Phi) = d \text{Tr}(\log(1 + \Phi)) .
\]
We now use \( d \text{Tr}(\Psi) = \text{Tr}([\nabla, \Psi]) \) which implies \( d \text{Tr}(\Phi^n) = n \text{Tr}(\Phi^{n-1}[\nabla, \Phi]) \), and finally
\[
(1 + \Phi)^{-1} = \sum_{n=0}^{\infty} (-1)^n \Phi^n .
\]
If we insert \( \Phi := -\frac{1}{2\pi i} R^\nabla \) and the Bianchi identity {3} we get \( dc(\nabla) = 0 \).

If \( (E, \nabla^E) \) and \( (F, \nabla^F) \) are two vector bundles with connection on \( M \), then we obtain an induced connection \( \nabla^{E \oplus F} \) on \( E \oplus F \). It is given by
\[
\nabla^{E \oplus F}(\phi \oplus \psi) = \nabla^E \phi \oplus \nabla^F \psi , \quad \phi \in \Omega^0(M, E) , \quad \psi \in \Omega^0(M, F) .
\]
Problem 2.45. Show that \( c(\nabla^{E \oplus F}) = c(\nabla^{E}) \wedge c(\nabla^{F}) \).

Definition 2.46. The Chern character form is given by

\[
\text{ch}(\nabla) := \text{Tr} \exp(-\frac{1}{2\pi i} R^{\nabla}) \in \Omega^{ev}(M)
\]

We write

\[
\text{ch} = \text{ch}_0 \oplus \text{ch}_2 \oplus \ldots, \quad \text{ch}_{2n} \in \Omega^{2n}(M, \mathbb{C})
\]

for its decomposition into homogeneous pieces.

Lemma 2.47. For all \( k \geq 0 \) the homogeneous piece \( \text{ch}_{2k} \) of the Chern character form is a characteristic form.

Proof. We argue as in the proof of Lemma 2.3. For closedness we use again the Bianchi identity and the relation

\[
d\text{Tr} (\exp(\Phi)) = \text{Tr}(\exp(\Phi)[\nabla, \Phi]).
\]

In the following we show that the Chern character forms are compatible with the tensor operations. If \((E, \nabla^{E})\) and \((F, \nabla^{F})\) are two vector bundles with connection on \(M\), then the tensor product \(E \otimes F\) and the bundle \(\text{Hom}(E, F)\) have induced connections \(\nabla^{E \otimes F}\) and \(\nabla^{\text{Hom}(E, F)}\).

Problem 2.48. Show that \(\nabla^{E \otimes F}\) and \(\nabla^{\text{Hom}(E, F)}\) are uniquely determined by the conditions that

\[
\nabla^{E \otimes F} (\phi \otimes \psi) = \nabla^{E} \phi \otimes \psi + \phi \otimes \nabla^{F} \psi, \quad \phi \in \Omega^{0}(M, E), \ \psi \in \Omega^{0}(M, F),
\]

\[
\nabla^{\text{Hom}(E, F)} \Psi = \nabla \circ \Psi - \Psi \circ \nabla, \quad \Psi \in \Omega^{0}(M, \text{Hom}(E, F)).
\]

We equip \(M \times \mathbb{C} \to \mathbb{C}\) with the trivial connection and define \(\nabla^{E^*}\) as the connection induced by \(\nabla^{E}\) on \(E^* := \text{Hom}(E, M \times \mathbb{C})\).

Problem 2.49. Show that \(R^{\nabla^{E \otimes F}} = R^{\nabla^{E}} \otimes \text{id}_{F} + \text{id}_{E} \otimes R^{\nabla^{F}}\) and conclude that

\[
\text{ch}(\nabla^{E \otimes F}) = \text{ch}(\nabla^{E}) + \text{ch}(\nabla^{F}), \quad \text{ch}(\nabla^{E^*}) = \text{ch}(\nabla^{E}) \wedge \text{ch}(\nabla^{F}).
\]

Further show that

\[
\text{ch}_{2i}(\nabla^{E^*}) = (-1)^{i} \text{ch}_{2i}(\nabla^{E}), \quad \text{ch}(\nabla^{\text{Hom}(E, F)}) = \text{ch}(\nabla^{E^*}) \wedge \text{ch}(\nabla^{F}).
\]

The identity

\[
\text{det}(1 + X) = \exp(\text{Tr}(\log(1 + X))) = \exp\left(\sum_{n=1}^{\infty} (-1)^{n-1} \frac{\text{Tr}(X^{n})}{n}\right)
\]

12
implies
\[ c(\nabla) = \exp \left( \sum_{n=1}^{\infty} (-1)^{n-1} (n-1)! \text{ch}_{2n}(\nabla) \right). \]

Therefore \( c_i(\nabla) \) can be expressed as a rational polynomial in the \( \text{ch}_k \) for \( k \leq i \).

**Problem 2.50.** Show that one can reverse this and express \( \text{ch}_{2i}(\nabla) \) as a rational polynomial in the \( c_k(\nabla) \) for \( k \leq i \). For example, in degree 2 we have
\[ c_1(\nabla) = \text{ch}_2(\nabla), \]

Determine these expressions in degree 4 and 6. Give a formula for \( \text{ch}_{2i}(\nabla) \) in terms of the Chern forms of \( \nabla^E \) and \( \nabla^F \).

We now consider the dependence of characteristic forms on the connection. The answer is given in terms of transgression. Consider two connections \( \nabla, \nabla' \) on a bundle \( E \). Then we can make a connection \( \tilde{\nabla} \) on \( \text{pr}^*_M E \to \mathbb{R} \times M \) such that \( \nabla = \tilde{\nabla}_{\{0\} \times M} \) and \( \nabla' = \tilde{\nabla}_{\{1\} \times M} \). For example, take
\[ \tilde{\nabla} = t \text{pr}^*_M \nabla' + (1-t) \text{pr}^*_M \nabla. \]

We say that \( \tilde{\nabla} \) is a path between \( \nabla \) and \( \nabla' \).

Let \( \omega \) be a characteristic form of degree \( n \). Then we define the transgression form
\[ \tilde{\omega}(\nabla', \nabla) := \int_{[0,1] \times M/M} \omega(\tilde{\nabla}) \in \Omega^{n-1}(M; \mathbb{C}). \]

**Lemma 2.51.** The class
\[ [\tilde{\omega}(\nabla', \nabla)] \in \Omega^{n-1}(M; \mathbb{C})/\text{im}(d) \]
is independent of the choice of the path \( \tilde{\nabla} \) between \( \nabla \) and \( \nabla' \). We have
\[ d\tilde{\omega}(\nabla', \nabla) = \omega(\nabla') - \omega(\nabla) \]  

(4)
and
\[ [\tilde{\omega}((\nabla''), \nabla')] + [\tilde{\omega}(\nabla', \nabla)] = [\tilde{\omega}(\nabla''), \nabla)] \]

(5)

**Proof.** The relation (4) follows immediately from Stokes’ theorem. We consider the hyperplane \( A^2 = \{x_0 + x_1 + x_2 = 1\} \subset \mathbb{R}^3 \). Its intersection with the positive quadrant its the standard simplex \( \Delta^2 \). Assume that we are given connections \( \nabla_i \) on \( E \) for \( i = 0, 1, 2 \). Then we consider a connection \( \tilde{\nabla} \) on \( \text{pr}^*_M E \to A^2 \times M \) such that \( \nabla_i = \tilde{\nabla}_{\{x_i=1\} \times M} \).

We can further arrange that the restrictions \( \tilde{\nabla}_{\{x_i=0\} \times M} \) coincide with previously given interpolations. By Stokes’ theorem
\[ \tilde{\omega}(\nabla_1, \nabla_0) + \tilde{\omega}(\nabla_2, \nabla_1) - \tilde{\omega}(\nabla_2, \nabla_0) = d \int_{\Delta^2} \tilde{\omega}(\tilde{\nabla}). \]

(6)

This implies (4) provided we have shown the first assertion. For this we use (6) where we take \( \nabla_1 = \nabla_2 \) and the constant interpolation \( \text{pr}^*_M \nabla \) between them. Note that in this case \( \tilde{\omega}(\nabla_2, \nabla_1) = 0 \).
Definition 2.52. The map 

$$(\nabla', \nabla) \mapsto [\tilde{\omega}(\nabla', \nabla)] \in \Omega^{n-1}(M; \mathbb{C})/\text{im}(d)$$

is called the transgression of $\omega$.

Corollary 2.53. The cohomology class of $\omega(\nabla)$ only depends on the bundle $E$.

Definition 2.54. We define the characteristic class associated to the characteristic form $\omega$ such that it maps the bundle $E \to M$ to the class $\omega(E) := [\omega(\nabla)] \in H^n_{dR}(M; \mathbb{C})$, where $\nabla$ is any choice of connection on $E$.

Problem 2.55. Show that 

$$[\tilde{\text{ch}}(\nabla_1 \otimes \nabla'_1, \nabla_2 \otimes \nabla'_2)] = [\text{ch}(\nabla_1, \nabla_2) \wedge \text{ch}(\nabla'_1)] + [\text{ch}(\nabla_2) \wedge \tilde{\text{ch}}(\nabla'_1, \nabla'_2)] .$$

### 2.4 Characteristic forms, classes and transgression - examples

We start with some examples in which Chern classes are calculated explicitly.

Example 2.56. The following problem is important since it shows that Chern classes are non-trivial. In the axiomatic approach (e.g. in [MS74]) its solution is used to normalize the Chern classes.

Problem 2.57. Show that the class $c_1(L)$ is non-trivial, where $L \to \mathbb{CP}^n$ is the tautological bundle.

**Proof.** Use Example 2.36 and calculate $\langle c_1(L), [\mathbb{CP}^1] \rangle$. □

Consider a complex vector bundle $E \to M$ and let $\mathbb{P}(E) \to M$ be the associated bundle of projective spaces. Recall that a point $p \in \mathbb{P}(E)_m$ is a line in $E_m$. The tautological bundle $L \to \mathbb{P}(E)$ is the subbundle $L \subset p^*E$ given by $L \to \{(H, e) \in p^*(E) \subset \mathbb{P}(E) \times E | e \in H \}$.

Problem 2.58. (Leray-Hirsch theorem) Show that $(c_1(L)^i)_i=0,...,\dim(E)-1$ forms a basis of $H^*(\mathbb{P}(E); \mathbb{C})$ as a $H^*(M; \mathbb{C})$-module.

**Proof.** Use Example 2.56 and the fact that $H^*(\mathbb{CP}^n; \mathbb{C}) \cong \mathbb{C}[c]/(c^{n+1})$ for a generator $c \in H^2(\mathbb{CP}^n; \mathbb{C})$ to show that the classes $(c_1(L)^i)_i=0,...,\dim(E)-1$ restrict to a basis of $H^*(\mathbb{P}(E)_m; \mathbb{C})$ for all $m \in M$. Then argue by induction over the cells of the basis $M$. □

It is known that, as a ring, $H^*(\mathbb{CP}^n; \mathbb{C}) \cong \mathbb{C}[c]/(c^{n+1})$ with $c := -c_1(L)$.

Problem 2.59. Calculate the Chern classes of $T\mathbb{CP}^n$. 

14
Proof. To this end construct a sequence

\[ 0 \to L^* \otimes T^* \mathbb{C}P^n \to \mathbb{C}P^n \times \mathbb{C}^{n+1} \to L^* \to 0 \]

and deduce that

\[ c(T\mathbb{C}P^n) = (1 + c)^{n+1}. \]

\[ \square \]

**Example 2.60.** We reconsider the Poincaré bundle \( P \to J(M) \times M \) (see 2.30). We have a canonical isomorphism \( a : H_1(M; \mathbb{C}) \cong H^1_{dR}(J(M)^0; \mathbb{C}) \). Let \( x \in H_1(M; \mathbb{C}) \) be represented by a homotopy class \( \gamma \in \pi_1(M, m) \). Then \( \frac{1}{2\pi i} \gamma \in \Omega^1_{dR}(J(M)) \) represents \( a(x) \). We consider the map \( a \) as a class \( \tilde{a} \in H^1_{dR}(M; \mathbb{C}) \otimes \mathbb{C} \Omega^1_{dR}(J(M)^0, \mathbb{C}) \subset H^2_{dR}(J(M)^0 \times M; \mathbb{C}) \)

**Lemma 2.61.** We have \( c_1(P) = \tilde{a} \).

**Proof.** We use the notation introduced in 2.30. The curvature of \( \nabla' \) lifts to the \( \pi \)-invariant form \( -d\alpha \). By construction the curvature represents the class \( -\theta \in H^1_{dR}(M; \mathbb{C}) \otimes \mathbb{C} \Omega^1_{dR}(J(M)^0, \mathbb{C}) \). Its total cohomology class is thus \( -2\pi i a \).

We observe that \( c_1(P) \) is non-zero if \( H^1_{dR}(M; \mathbb{C}) \neq 0 \).

We now investigate the transgression of a characteristic form between two flat connections and how it gives rise to a secondary characteristic class. If \( \nabla \) and \( \nabla' \) are flat, then \( d\tilde{\omega}(\nabla', \nabla) = 0 \) by Problem 2.40.

**Definition 2.62.** The secondary characteristic class associated to \( \omega \) maps the pair of flat connections \( \nabla', \nabla \) to the cohomology class

\[ [\tilde{\omega}(\nabla', \nabla)] \in H^{n-1}_{dR}(M; \mathbb{C}). \]

Note that \( \omega(\nabla, \nabla) = 0 \).

**Example 2.63.** We consider the trivial bundle \( S^1 \times \mathbb{C} \) with trivial connection \( d \). For \( \alpha \in \Omega^1(S^1, \mathbb{C}) \) we consider the connection \( d + \alpha \).

**Problem 2.64.** Calculate \( \tilde{c}_1(d + \alpha, d) \).

The result of this calculation is

\[ \tilde{c}_1(d + \alpha, d) = [-\frac{1}{2\pi i} \alpha] \in H^1_{dR}(S^1; \mathbb{C}). \]

This class is non-zero in general. It determines the holonomy of \( d + \alpha \) by

\[ \text{hol}_{d+\nabla}(S^1) = \exp(-\int_{S^1} \alpha) = \exp(-\langle [\alpha], [S^1] \rangle) = \exp(2\pi i \langle \tilde{c}_1(d + \alpha, d), [S^1] \rangle). \]
By naturality, the calculation determines the class $\tilde{c}_1(\nabla', \nabla)$ for any pair of flat connections on a line bundle $L \to M$. We have

$$\exp(2\pi i \langle \tilde{c}_1(\nabla', \nabla), [\gamma] \rangle) = \text{hol}_{\nabla'}(\gamma)\text{hol}_{\nabla}(\gamma)^{-1}. \quad (8)$$

Like the holonomy the class $\tilde{c}_1(\nabla', \nabla)$ can vary continuously with the flat connections in a non-trivial way. On the other hand we have the following.

**Lemma 2.65.** If $\nabla$ and $\nabla'$ belong to the same path component of the space of flat connections, then $\tilde{c}_n(\nabla', \nabla) = 0$ for all $n \geq 2$. A similar statement holds true for all characteristic forms of degree $\geq 3$.

**Proof.** We only consider the case of $c_n$. We consider a connection $\nabla$ on $pr^*_M E \to \mathbb{R} \times M$ such that $\nabla_{[t]}$ is flat for all $t$. Then $R^{\nabla'} = dt \wedge \iota_{\partial_t} R^{\nabla}$. This yields the following formula for the total Chern class $c(\nabla) = 1 - dt \wedge \frac{1}{2\pi i} \text{Tr} R^{\nabla}$. The vanishing of the higher degree components implies the assertion. \qed

The classes $\tilde{c}_n(\nabla', \nabla)$ can thus be used to show that two flat connections belong to different path components. Examples for this will be given later, see Examples 2.83.

**Example 2.66.** We consider a vector bundle $E \to M$. For an automorphism $F \in \text{Aut}(E)$ of $E$ we define the form $\omega(F) := [\tilde{\omega}(F^* \nabla, \nabla)] \in \Omega^{n-1}(M; \mathbb{C})/\text{im}(d)$. We have

$$d\omega(F) = 0$$

and therefore $\omega(F) \in H^{n-1}_{\text{dR}}(M; \mathbb{C})$. The class $\omega(F)$ is independent of the choice of $\nabla$ and only depends on the homotopy class of $F$. It satisfies

$$\omega(F \circ F') = \omega(F) + \omega(F'). \quad (9)$$

**Proof.** The first assertion follows from $\omega(\nabla) = \omega((F, \text{id})^* \nabla)$. The identity (9) follows from (5). A homotopy can be understood as an element $\tilde{F} \in \text{Aut}(pr^*_M E)$, where $pr_M : \mathbb{R} \times M \to M$ denotes the projection. Then $\omega(\tilde{F})$ is closed. Hence its restriction to $\{t\} \times M$ does not depend on $t \in \mathbb{R}$. \qed

For every characteristic form of degree $n$ we have defined a group homomorphism

$$\omega : \pi_0(\text{Aut}(E)) \to H^{n-1}_{\text{dR}}(M; \mathbb{C}).$$

Here is an alternative interpretation of the class $\omega(F)$. We consider the action of $\mathbb{Z}$ on $pr^*_M E \to \mathbb{R} \times M$ given by $n(t, e) = (t+1, F^{-1}(e))$ and let $E(F) \to S^1 \times M$ be the quotient.

**Definition 2.68.** The complex vector bundle $E(F) \to S^1 \times M$ is called the suspension of $E$ with respect to $F$. \newpage
We then have
\[ \omega(F) = \int_{S^1 \times M/M} \omega(E(F)) . \]

**Problem 2.69. Prove this!**

Let \( G \) be a Lie group and \( \rho : G \to GL(n, \mathbb{C}) \) be a representation. Let \( G \times \mathbb{C}^n \to G \) be the \( n \)-dimensional trivial bundle. Then we have a tautological element \( F_\rho \in \text{Aut}(G \times \mathbb{C}^n) \) given by \( F_\rho(g, v) := (g, \rho(g)v) \). We therefore get a class
\[ \omega(F_\rho) \in H^{n-1}_{dR}(G; \mathbb{C}) . \]

Recall that a cohomology class \( x \in H^*_{dR}(G; \mathbb{C}) \) is called primitive, if
\[ \mu^* x = x \cup 1 + 1 \cup x , \]

where \( \mu_G \times G \to G \) is the multiplication map.

**Lemma 2.70.** \( \omega(F_\rho) \) is primitive.

**Proof.** On \( G \times G \times \mathbb{C}^n \to G \times G \) we have the automorphisms \( F_i \) given by \( F_0(h, g, v) := (h, g, \rho(h)v) \), \( F_1(h, g, v) := (h, g, v) = (h, g, \rho(hg)v) \), and \( F_2(h, g, v) := (g, g, \rho(g)v) \). Note that \( F_2 \circ F_0 = F_1 \). We have \( F_0 = \text{pr}_0^* F_\rho \), \( F_1 = \mu^* F_\rho \) and \( F_2 = \text{pr}_1^* F_0 \), where \( \text{pr}_0, \text{pr}_1, \mu : G \times G \to G \) are the projections and multiplication. We thus have \( \mu^* \omega(F_\rho) = \text{pr}_0^* \omega(F_\rho) + \text{pr}_1^* \omega(F_\rho) \) as required. \( \square \)

If we apply this construction to the Chern classes \( c_k \) and the standard representation \( \text{id} \) of the group \( U(n) \) we get primitive classes \( c_{2k-1} = c_k(F_{\text{id}}) \in H^{2k-1}_{dR}(U(n); \mathbb{C}) \). It is known that
\[ H^{2k}_{dR}(U(n); \mathbb{C}) = \Lambda_C(c_1, \ldots, c_{2n-1}) . \] (10)

We can calculate the forms \( \text{ch}_{2n}(F_{\text{id}}) = \hat{\text{ch}}_{2n}((F_{\text{id}}, \text{id})^* \nabla, \nabla) \) explicitly. We consider the Mauer-Cartan form \( g^{-1} dg \in \Omega^1(U(n)) \otimes \text{Mat}(n, \mathbb{C}) \).

**Lemma 2.71.** We have
\[ \text{ch}_{2n}(F_{\text{id}}) = \frac{(-1)^{n-1}(n-1)!}{(2\pi i)^n(2n-1)!} \text{Tr}(g^{-1} dg)^{2n-1} . \]

**Proof.** Explicit calculation following the definitions.

**Problem 2.72.** Do this calculation!

**Problem 2.73.** Calculate \( H^*(U(n); \mathbb{Z}) \) inductively using the Serre spectral sequences associated to the fibrations
\[ U(n-1) \to U(n) \to S^{2n-1} . \]

Deduce (10).
2.5 Metrics and unitarity

We consider a complex vector bundle $E \rightarrow M$ with a hermitean metric $h$. Let $\nabla$ be a connection on $E$.

**Problem 2.74.** Show that there is a unique connection $\nabla^*$ (called the adjoint of $\nabla$ w.r.t $h$), which is characterized by

$$dh(\phi, \psi) = h(\nabla\phi, \psi) + h(\phi, \nabla^*\psi), \quad \forall \phi, \psi \in \Omega^0(M, E).$$

For $\alpha \in \Omega^1(M, \text{End}(E))$ we have

$$(\nabla + \alpha)^* = \nabla^* - \alpha^*,$$

where here * only acts on the endomorphism part.

**Definition 2.75.** A connection on a metrized bundle $(E, h)$ is called unitary if $\nabla^* = \nabla$.

We write $u(E) \subset \text{End}(E)$ for the subbundle of antihermitean endomorphisms. If a connection $\nabla$ is unitary, then so is $\nabla + \alpha$ for all $\alpha \in \Omega^1(M, u(V))$. For any connection $\nabla$ define

$$\omega := \nabla^* - \nabla \in \Omega^1(M, \text{End}(E)).$$

Then

$$\nabla^u := \nabla + \frac{1}{2}\omega$$

is unitary. It is called the symmetrization of $\nabla$.

**Problem 2.76.** Show this result. If $A \in \Omega^0(M, \text{End}(E))$ is invertible and symmetric, then we define $h_A(\phi, \psi) := h(A\phi, \psi)$. Calculate $\nabla^* h - \nabla^* h_A$. Assume that $A$ is scalar and calculate $R^\nabla h - R^\nabla h_A$ in this case.

**Definition 2.77.** A connection is called unitarizable if it is hermitean for some choice of hermitean metric.

There are various obstructions against unitarizability as the following exercise shows.

**Problem 2.78.** We consider the trivial bundle $\mathbb{R}^2 \times \mathbb{C}^2 \rightarrow \mathbb{R}^2$ with connections

$$\nabla := d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dx, \quad \nabla' := d + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} ydx$$

Show that $\nabla$ is unitarizable, while $\nabla'$ is not. The connection $\nabla$ descends to the quotient $\mathbb{R}^2/\mathbb{Z}^2$, but this descent is not unitarizable.

**Problem 2.79.** Show that

$$R^\nabla = -(R^\nabla)^*.$$

Hence, if $\nabla$ is unitary, then $R^\nabla \in \Omega^2(M, u(E))$. Conclude that for unitary connections $\text{ch}(\nabla)$ and $c(\nabla)$ are real forms.
Assume now that $\nabla$ is flat. Then $\nabla^*$ is flat, too. For a characteristic form $\omega$ of degree $n$ we therefore can consider the associated secondary class Def. 2.62
\[ \tilde{\omega}(\nabla^*, \nabla) \in H^{n-1}_{dR}(M; \mathbb{C}) . \]
A priori this is a characteristic class for metrized bundles.

**Lemma 2.80.** The class $\tilde{\omega}(\nabla^*, \nabla) \in H^{n-1}_{dR}(M, \mathbb{C})$ does not depend on the choice of metric.

**Proof.** If $\tilde{h}$ is a metric on the flat bundle $\text{pr}_M^* E \to \mathbb{R} \times M$, then $\tilde{\omega}((\text{pr}_M^* \nabla)^*, \text{pr}_M^* \nabla)$ is closed. Hence its restriction to $\{t\} \times M$ is independent of $t$. This shows the assertion, since every two metrics can be connected. □

**Definition 2.81.** We write $\tilde{\omega}(\nabla) \in H^{n-1}_{dR}(M, \mathbb{C})$ for the characteristic class for flat connections obtained from $\omega$.

If $\nabla$ is unitarizable, then $\tilde{\omega}(\nabla) = 0$. If $\deg(\omega) \geq 3$, then it follows from Lemma 2.65 that $\tilde{\omega}(\nabla)$ only depends on deformation class of the flat connection $\nabla$. This is the rigidity result [BG01].

**Example 2.82.** Let $\lambda \in \mathbb{C} \setminus \{0\}$. We consider the action of $\mathbb{Z}$ on $\mathbb{R} \times \mathbb{C} \to \mathbb{R}$ by $n(t, z) := (t + 1, \lambda^n z)$. It preserves the trivial connection. Hence, by taking the quotient, we get a flat bundle $(E_\lambda, \nabla_\lambda) \to S^1$. We want to calculate $\tilde{c}_1(\nabla_\lambda) \in H^1_{dR}(S^1; \mathbb{C})$. The holonomy of $\nabla_\lambda$ along $S^1$ is multiplication $\lambda^{-1}$. The holonomy of $\nabla_\lambda^*$ is thus $\bar{\lambda}^{-1}$. Note that $E_\lambda$ is trivial. Hence we have
\[ \bar{c}_1(\nabla_\lambda) = \bar{c}_1(\nabla_\lambda^{\text{triv}}) + \bar{c}_1(\nabla^{\text{triv}}, \nabla_\lambda) . \]
We now use (7) and get
\[ \exp(2\pi i \langle \bar{c}_1(\nabla_\lambda), [S^1] \rangle) = |\lambda|^{-2} . \]
It follows that
\[ \langle \bar{c}_1(\nabla_\lambda), [S^1] \rangle = \frac{i}{\pi} \log |\lambda| . \]
This calculation shows that $\bar{c}_1(\nabla)$ varies continuously with the flat connection $\nabla$. In contrast, by Lemma 2.65 the classes $\tilde{c}_n(\nabla)$ for $n \geq 2$ only depend on the path components of $\nabla$.

The following is a very non-trivial example.

**Example 2.83.** We consider a number ring $R$ and an embedding $\sigma : R \to \mathbb{C}$. We fix $k, n \in \mathbb{N}$ and consider a manifold $M$ together with an $n$-equivalence $f : M \to BGL(R, k)$. Since $BGL(R, k)$ is a countable CW-complex its skeleta can be realized by smooth manifolds which gives the existence of $M$. The map $f$ classifies $GL(R, k)$-bundle $\tilde{M} \to M$, and we let $E := \tilde{M} \times_{GL(R, k), \sigma} \mathbb{C}^k \to M$ be the associated bundle with flat connection $\nabla$. The class
\[ \text{ch}_2(\nabla) \in H^{2n-1}(M; \mathbb{C}) \]
is called the Kamber-Tondeur class (compare \[BL95\]). In \[Bor74\] Borel has shown the following: If \(\sigma\) is real and \(j\) is even, then the class \(\overline{\text{ch}}_{2j}(\nabla)\) is non-zero for sufficiently large \(k,n\). If \(\sigma\) is a complex embedding, then given \(j\) the class \(\overline{\text{ch}}_{2j}(\nabla)\) is non-zero for sufficiently large \(k,n\). A proof is beyond the scope of this course.

The non-triviality of \(\overline{\text{ch}}_{6}(\nabla)\) shows that \(\nabla\) can not be connected with any adjoint connection \(\nabla^*\) by a path of flat connections on \(E\).

### 2.6 Integrality

Let

\[
\epsilon_C : H^n(M;\mathbb{Z}) \rightarrow H^n_{dR}(M;\mathbb{C})
\]

be the map induced by the inclusion \(\mathbb{Z} \subset \mathbb{C}\) and the de Rham isomorphism.

**Definition 2.84.** A class \(x \in H^n_{dR}(M;\mathbb{C})\) is called integral if it belongs to the image of \(\epsilon_C\).

Equivalently, a class \(x \in H^n_{dR}(M;\mathbb{C})\) is integral if and only if

\[
\langle x, z \rangle \in \mathbb{Z}
\]

for all smooth cycles \(z \in Z_n(S_\infty(M))\). We shall use the fact that the subset of integral de Rham cohomology classes is closed under pull-back along smooth maps and products. All this follows from the fact that (11) comes from a natural transformation of multiplicative cohomology theories.

**Definition 2.85.** A characteristic form for complex vector bundles \(\omega\) is called integral if \(\omega(E)\) is integral for all complex vector bundles \(E \rightarrow M\).

**Proposition 2.86.** The Chern classes \(c_n\) are integral.

**Proof.** We first check integrality of \(c_1(H)\) for line bundles \(H\). We use Example 2.36 in order to see that \(c_1(L)\) is integral for the tautological bundle \(L \rightarrow \mathbb{C}P^n\). Given a line bundle \(H\) on a manifold \(M\) there exists \(n\) and a map \(f : M \rightarrow \mathbb{C}P^n\) such that \(H \cong f^*L\). Hence \(c_1(H) = f^*c_1(L)\) is integral.

We now discuss \(c_n\). If the bundle \(E\) has a decomposition \(E \cong H_1 \oplus \cdots \oplus H_k\) into line bundles, then

\[
c(E) = \prod_{i=1}^{k} c(H_i) = \prod_{i=1}^{k} (1 + c_1(H_i))
\]

has integral homogeneous components. Let now \(E_0 := E \rightarrow M\) be general of dimension \(r\). In this case we argue by the splitting principle which goes as follows. We can choose a decomposition \(\pi^*E_0 \cong L_1 \oplus E_1\), where \(\pi_1 : \mathbb{P}(E_0) \rightarrow M\) is the projective bundle of \(E\) and \(L_1 \rightarrow \mathbb{P}(E_0)\) is the canonical bundle. We apply the same construction to \(E_1 \rightarrow \mathbb{P}(E_0)\) and then inductively. We obtain a bundle \(q : F(E) \rightarrow M\) and a decomposition \(q^*E \cong H_1 \oplus \cdots \oplus H_r\), where \(q = \text{pr}_1 \circ \text{pr}_2 \circ \cdots \circ \text{pr}_{r-1}\) and e.g. \(H_1 = (\text{pr}_2 \circ \cdots \circ \text{pr}_{r-1})^*L_1\). Hence \(q^*c(E)\) has integral components. It is known that the integral cohomology of \(F(E)\)
is a free module over the integral cohomology ring of $M$. This follows from a Leray-Hirsch argument similar to (2.58). We finally use the following assertion:

**Problem 2.87.** Show that if $x \in H^n_{dR}(M, \mathbb{C})$ and $q^*x$ is integral, then so is $x$. \hfill $\square$

It follows that integral polynomials in the $c_i$ are integral.

**Example 2.88.** The following exercise gives a direct integral interpretation of the highest non-trivial Chern class. Let $E \to M$ be a complex vector bundle of dimension $n$ and $S(E) \to M$ be its sphere bundle. We consider the associated Serre spectral sequence. Its second term has two rows

$$E_2^{*,0} \cong H^*(M; \mathbb{Z}), \quad E_2^{*,2n-1} \cong H^*(M; \mathbb{Z}) \text{or} \mathbb{S}^{2n-1}.$$ 

The only non-trivial differential is $d_{2n-1}$. The corresponding edge sequence is called Gysin sequence. We define the Euler class

$$\chi := d_{2n-1}(\mathbb{S}^{2n-1}) \in H^{2n}(M; \mathbb{Z}).$$

**Problem 2.89.** Show that $c_n(E) = \chi$. 

*Proof.* Show that this formula is compatible with sums and reduce to the one-dimensional case. \hfill $\square$

**Lemma 2.90.** The Newton classes $s_n := n! \text{ch}_n$ are integral.

*Proof.* For line bundles $H$ we have $\text{ch}(H) = \exp(c_1(H))$. Hence $s_n(H)$ is integral by 2.86. If $E$ decomposes into line bundles $E \cong H_1 \oplus \cdots \oplus H_k$, then $s_n(E) = \sum_{i=1}^k s_n(H_i)$ is integral. The general case is reduced to the decomposable case by the splitting principle as in the proof of 2.86. \hfill $\square$

If the characteristic form $\omega$ is integral, then $\omega(F) \in H^{n-1}_{dR}(M)$ (see Example 2.66) is integral for every $F \in \text{Aut}(E)$. This allows to define an absolute invariant of flat connections $\nabla$ on trivializable bundles $E$. Let $\phi, \phi'$ be trivializations. Then we have $F(\phi) = \phi'$ for a suitable automorphism $F$. Hence by (5)

$$[\omega(\nabla, \nabla^\phi)] = [\omega(\nabla, \nabla^\phi')] + \omega(F).$$

It follows that the class

$$[\tilde{\omega}(\nabla, \nabla^\phi)] \in \frac{H^{n-1}_{dR}(M; \mathbb{C})}{\text{im}_C}$$

is independent of the choice of the trivialization $\phi$. 

21
Definition 2.91. We define the Chern-Simons invariant of a flat connection $\nabla$ on a trivializable bundle $E$ associated to the characteristic form $\omega$ as

$$\tilde{\omega}(\nabla) := [\tilde{\omega}(\nabla, \nabla^\phi)] \in \frac{H^{n-1}_{2R}(M; \mathbb{C})}{\text{im} \epsilon},$$

where $\phi$ is some choice of trivialization of $E$. If $M$ is compact oriented of dimension $n - 1$, then we set

$$\text{cs}_\omega(\nabla) := [(\tilde{\omega}(\nabla, \nabla^\phi), [M])] \in \mathbb{C}/\mathbb{Z}.$$

Problem 2.92. Calculate $\tilde{c}_1(\nabla)$.

Proof. We have by (8)

$$\exp(2\pi i \langle \tilde{c}_1(\nabla), [\gamma] \rangle) = \text{hol}_{\nabla}(\gamma).$$

Example 2.93. Let $\omega$ be an integral characteristic form of degree $n$. Let $B$ be a space with a complex vector bundle $V^\delta \to B$ with structure group reduced to $GL(k, \mathbb{C}^\delta)$. Below we refer to $V^\delta$ as a flat bundle. By $V \to B$ we denote the associated bundle with structure group $GL(k, \mathbb{C})$. We assume that $V$ is trivializable.

Let $f : M \to B$ represent an oriented bordism class in $\text{MSO}_{n-1}(B)$. In particular $M$ is closed oriented of dimension $n - 1$. We consider the Chern Simons invariant $\text{cs}_\omega(\nabla) := \text{cs}_\omega(f^*\nabla^{V^\delta}) \in \mathbb{C}/\mathbb{Z}.$

Lemma 2.94. The Chern-Simons invariant $\text{cs}_\omega(f)$ only depends on the bordism class of $f$. It gives a homomorphism $\text{cs}_\omega^{V^\delta} : \text{MSO}_{n-1}(B) \to \mathbb{C}/\mathbb{Z}.$

Proof. Additivity under disjoint union is clear. If $F : Z \to B$ is a zero bordism, then by Stokes' Theorem

$$\langle \tilde{\omega}(f^*\nabla^{V^\delta}, \nabla^{\text{triv}}), [M] \rangle = \int_M \tilde{\omega}(f^*\nabla^{V^\delta}, \nabla^{\text{triv}}) = \int_Z d\tilde{\omega}(F^*\nabla^{V^\delta}, \nabla^{\text{triv}}) = 0.$$ 

This shows bordism invariance.

The orientation $\kappa : \text{MSO} \to H\mathbb{Z}$ induces a natural transformation of homology groups $\kappa : \text{MSO}_{n-1}(B) \to H_{n-1}(B)$. Geometrically, if $f : M \to B$ represents $[f] \in \text{MSO}_{n-1}(B)$, then $\kappa([f]) = f_*[M]$. Later in 3.55 we will observe that $\text{cs}_\omega$ factorizes over this transformation.

The following is a consequence of a generalization of 2.65 from Chern classes to $\omega$.

Problem 2.95. If $\deg(\omega) \geq 3$, then the Chern-Simons invariant $\text{cs}_\omega^{V^\delta}$ only depends on the deformation class of $V^\delta$.

Let $\lambda \in \mathbb{C}$ determine the character $\mathbb{Z} \to \mathbb{C}$ and therefore a flat bundle $V^\delta_\lambda \to S^1$. 

22
Problem 2.96. Calculate the composition

\[ Z \cong \pi_1(S^1) \xrightarrow{\text{can}} \text{MSO}_1(S^1) \xrightarrow{\text{cs}_2} \mathbb{C}/\mathbb{Z}. \]

Use the result to conclude that the homomorphism can is an isomorphism.

Example 2.97. Trivializability of the bundle \( V \to B \) underlying \( V^\delta \) is an annoying condition. One can extend the range of the definition of the Chern-Simons invariant \( \text{cs}_{2n-1} \) as follows. We are going to define an invariant of flat bundles on \( M \) which extend as bundles to some zero bordism: Assume that \( \nabla \) is a flat connection on \( E \to M \), and that \( Z \) is a zero-bordism with an extension \( F \to Z \) of \( E \). We choose some extension \( \tilde{\nabla} \) of \( \nabla \) to \( Z \), not necessarily flat. Then we define

\[ \text{cs}_\omega(\nabla) := \left[ \int_Z \omega(\tilde{\nabla}) \right] \in \mathbb{C}/\mathbb{Z}. \]  

(12)

Problem 2.98. Show that this is a well-defined invariant of the flat connection \( \nabla \).

The common domain of both definitions of \( \text{cs}_\omega \) are flat connections on bundles \( E \to M \) where \( E \) is trivializable and \( M \) is zero-bordant.

Problem 2.99. Show that both constructions coincide on this common domain.

Let \((L, \nabla^L)\) be the tautological line bundle of \( \mathbb{C}P^n \). For \( k \in \mathbb{N} \) we consider the power \( H := L^\otimes k \). Let \( \pi : M \to \mathbb{C}P^n \) be the unit sphere bundle of \( H \). The bundle \( \pi^*H \to M \) is canonically trivialized. Let \( \nabla^{H,\text{triv}} \) be the associated trivial connection. The bundle \( \pi^*L \) acquires a canonical flat connection \( \tilde{\nabla} \) characterized as follows. For a local section \( \psi \) of \( \pi^*L \) we have \( \nabla \psi = 0 \) if and only if \( \nabla^{H,\text{triv}} \psi \otimes k = 0 \).

Problem 2.100. Calculate \( \text{cs}_{c_{n+1}}(\nabla) \in \mathbb{C}/\mathbb{Z} \).

Proof. Let \( q : Z \to \mathbb{C}P^n \) be the disc bundle of \( H \). Then \( q^*L \) is an extension of \( \pi^*L \) across \( Z \). We let \( \tilde{\nabla} \) be any extension of \( \nabla \) over \( Z \). Then we have

\[ \int_Z c_1^{n+1}(\tilde{\nabla}) = \int_Z c_1^{n+1}(q^*\nabla^L) + \int_Z dc_1^{n+1}(\tilde{\nabla}, q^*\nabla^L) = \int_M c_1^{n+1}(\nabla, \pi^*\nabla^L). \]

Here we use Stocke’s theorem and that the integral over \( Z/\mathbb{C}P^n \) of a form pulled back from \( \mathbb{C}P^n \) vanishes. We now must calculate the transgression explicitly. Let \( \alpha \in \Omega^1(M; \mathbb{C}) \) be defined by

\[ \nabla + \alpha = q^*\nabla^L. \]

Then we form the connection

\[ \tilde{\nabla} := \text{pr}_M^*\nabla + t\alpha \]

on \( \text{pr}_M^*\pi^*L \to \mathbb{R} \times M \). Its curvature is given by

\[ R^{\tilde{\nabla}} = dt \wedge \alpha + t\text{pr}_{\mathbb{C}P^n}^*R^L, \]
where we use that \( \pi^* R^\nabla = [\nabla, \alpha] \). We have

\[
\tilde{c}_1^{n+1}(\nabla, \pi^* \nabla^L) = \frac{(-1)^{n+1}}{(2\pi i)^{n+1}(n+1)} \int_0^1 t^n dt \wedge \alpha \wedge \pi^* (R^\nabla)^n = \frac{-1}{2\pi i} \alpha \wedge \pi^* c_1(\nabla^L)^n.
\]

We now calculate \( \int_{M/\mathbb{CP}^n} \alpha \). We fix a base point and identify the fibre of \( H \) with \( \mathbb{C} \). The fibre of \( M \) is then identified with \( U(1) \), and a typical parallel local section of \( \pi^* L \) is given by a branch of \( \psi(u) := u^k \). We get

\[
d \frac{du}{ku} \psi(u) = (d \psi)(u) = (\pi^* \nabla^L \psi)(u) = ((\nabla + \alpha) \psi)(u) = \alpha \psi(u)
\]

and conclude that \( \alpha = \frac{du}{ku} \). It follows that

\[
\int_{M/\mathbb{CP}^n} \alpha = \frac{2\pi i}{k}.
\]

We conclude that

\[
\text{cs}_{c_1^{n+1}}(\nabla) = \left[ \int_M \tilde{c}_1^{n+1}(\nabla, \pi^* \nabla^L) \right] = \left[ -\frac{1}{k} \int_{\mathbb{CP}^n} c_1(\nabla^L)^n \right] = \left[ -\frac{1}{k} \right].
\]

\[ \square \]

**Problem 2.101.** Show that \( \text{cs}_{c_1^{n+1}}(\nabla^{\otimes r}) = [-\frac{r}{k}] \).

We can now modify the construction of the bordism invariant \( \text{2.94} \) as follows. Let \( V^\delta \to B \) be flat and its underlying continuous bundle \( V \) be classified by \( v : B \to BGL(k, \mathbb{C}) \). Previously \( \text{cs}_V^{\delta} : \text{MSO}_{n-1}(B) \to \mathbb{C}/\mathbb{Z} \) was defined under the condition that \( v \) is homotopic to a constant map. Using the construction (12) we can now extend the definition to get a homomorphism

\[
\text{cs}_V^{\delta} : \ker (v_* : \text{MSO}_{n-1}(B) \to \text{MSO}_{n-1}(BGL(k, \mathbb{C})))
\]

by setting

\[
\text{cs}_V^{\delta}(f) := \text{cs}_{\omega}(f^* \nabla).
\]

In \( 3.53 \) we will further extend this homomorphism to all of \( \text{MSO}_{n-1}(B) \) using differential cohomology.

Consider the map \( v : BSL(k, \mathbb{C}^\delta) \to BGL(k, \mathbb{C}) \) and observe that we then have \( 0 = v_* : \text{MSO}_3(\text{BSL}(k, \mathbb{C}^\delta)) \to \text{MSO}_3(\text{BGL}(k, \mathbb{C})) \). Let \( V^\delta \to BSL(k, \mathbb{C}^\delta) \) be associated to the standard representation of \( SL(k, \mathbb{C}^\delta) \).

**Problem 2.102.** Show that \( \text{cs}_{c_1^{n+1}}(\nabla^{\otimes r}) \to \mathbb{C}/\mathbb{Z} \) vanishes. Furthermore show that \( \text{cs}_{c_2}^{\delta} : \text{MSO}_3(\text{BSL}(k, \mathbb{C}^\delta)) \to \mathbb{C}/\mathbb{Z} \) is non-trivial by calculating examples.
Proof. Calculate $\text{cs}_\omega(f^*\nabla) - \text{cs}_\omega((f^*\nabla)^*)$ for suitable $f : M \to BSL(k, \mathbb{C})$ for some metric on $f^*V$, compare this with the Kamber-Tondeur class and use Borel’s result that the latter generates $H^3(BSL(k, \mathbb{C}^d); \mathbb{R})$. \hfill \Box

The Chern-Simons invariants for Seifert manifolds with maps to $BSp(1)^d$ have been determined in D. Auckly [Auc94b]. More calculations can be found in [Auc94a], [KK90].

Here is a special case. Let $M \to S^1$ be a 2-torus bundle. Then we have a sequence

$$0 \to \pi_1(T^2) \to \pi_1(M) \to \pi_1(S^1) \to 0 . \quad (13)$$

We consider generators $A, B \in \pi_1(T^2)$ and an element $T \in \pi_1(M)$ which maps to a generator of $\pi_1(S^1)$. We consider a representation $\rho : \pi_1(M) \to Sp(1)$ with $\rho(T) = J$ (quaternionic notation) and $\rho(A) = \exp(2\pi i \phi)$, $\rho(B) = \exp(2\pi i \psi)$ for some $\phi, \psi \in \mathbb{R}$. Let $(V \to M, \nabla)$ be the associated two-dimensional flat bundle.

**Problem 2.103.** Calculate $\tilde{c}_2(\nabla)$ in terms of the data $\psi, \phi$ and the extension $(13)$. Discuss, under which conditions $\rho$ exists.

**Example 2.104.** Let $\omega$ be an integral characteristic class of degree $n$. Then we can define an invariant of connections on $n - 1$-dimensional manifolds. Assume that $\nabla$ is a connection on a bundle $E \to M$ over an oriented closed $n - 1$-dimensional manifold $M$ which extends to a bundle $F \to Z$ on an oriented zero bordism $Z$ of $M$. Then we can choose an extension $\tilde{\nabla}$ of $\nabla$ to $F$ with product structure near to the boundary and define

$$\text{cs}_\omega(\nabla) := \int_z \omega(\tilde{\nabla}) \in \mathbb{C}/\mathbb{Z} .$$

**Problem 2.105.** Show that $\text{cs}_\omega(\nabla)$ is independent of the choice of the zero bordism and extension.

**Proof.** Two choices can be glued along $M$. The difference of the corresponding two integrals is integral by the integrality of $\omega$. See also the proof of [2.111]. \hfill \Box

The assumption of a product structure simplifies the argument for [2.105] but is not necessary.

**Problem 2.106.** Verify this assertion.

**Proof.** See [3.42] for a more general case. \hfill \Box

Let $\omega$ be an integral characteristic form of degree $n$. We consider a hermitean bundle $(E, h)$ on a manifold $M$ which extends to a zero bordism of $M$. Let $\mathcal{A}(E)^{u,0}$ denote the space of unitary connections on $E \to M$ with trivial determinant (see [2.123]). This is an affine space over $\Omega^1(M, su(E))$, (compare [2.6]), where $su(E)$ denotes the anti-hermitean trace-free bundle endomorphisms. Observe that if $M$ is closed oriented of dimension $n - 1$, then we get a function $\text{cs}_\omega : \mathcal{A}(E)^{u,0} \to \mathbb{R}/\mathbb{Z}$, the Chern-Simons functional.

**Problem 2.107.** Calculate $d\text{cs}_\omega$ in general.
Assume that $n = 3$ and characterize the critical points of the Chern-Simons functional $\text{cs}_{c_2}$.

Proof. Notice that $d\text{cs}_\omega(\nabla)$ is a linear map $\Omega^1(M, \text{su}(E)) \to \mathbb{R}$. Let $P \to B$ be a $U(1)$-principal bundle, $E \to B$ the associated line bundle, and $\nabla$ be a connection on $E$. For $\kappa \in \mathbb{Z}$ we let $p : M \to B$ be the unit sphere bundle of

$$E^k := E \otimes \cdots \otimes E \quad \text{k times}.$$

The bundle $p^*E^k \to M$ is canonically trivialized. Let $q : Z \to B$ be the unit disc bundle and $\tilde{\nabla}$ be any connection on $q^*E^k$ which extends the trivial connection on $M$.

**Problem 2.108.** Show that for all $n \geq 1$

$$\left[ \int_{Z/B} c_1(\tilde{\nabla})^n \right] = -\left[ \kappa^n c_{1,n-1}(\nabla) \right] \in \Omega^{2n-2}(B; \mathbb{C}) / \text{im}(d).$$

If $V \to M$ is a real vector bundle with connection $\nabla$, then we can form the complex vector bundle $V \otimes \mathbb{R} \mathbb{C}$ with induced connection $\nabla \otimes \mathbb{C}$.

**Definition 2.109.** If $V \to M$ is a real vector bundle, then we define its Pontrjagin classes by

$$p_k(V) := (-1)^k c_{2k}(V \otimes \mathbb{C}).$$

If $\nabla$ is a connection, then we define the Pontrjagin form $p_k(\nabla) := (-1)^k c_{2k}(\nabla \otimes \mathbb{C})$.

The Pontrjagin forms are integral characteristic forms for real vector bundles.

It is known that $\pi_3(\text{MSO}) = 0$. Given a compact oriented Riemannian three manifold $M$ we thus can find a zero bordism $Z$. We choose a Riemannian metric on $Z$ extending the metric on $M$ with a product structure near $\partial Z = M$ and let $\nabla$ be the Levi-Civita connection on $TZ$.

**Definition 2.110.** We define the Chern-Simons invariant of $(M, g)$ by

$$\text{CS}(M, g) := [\int_Z p_1(\nabla)] \in \mathbb{C}/\mathbb{Z}.$$ 

**Lemma 2.111.** $\text{CS}(M, g)$ is well-defined independently of the choice of the Riemannian metric and the zero bordism.

Proof. Let $Z'$ be another zero bordism. We equip $Z'$ with a metric extending $g$. Then we can obtain a compact Riemannian manifold $-Z \cup_M Z'$. We have

$$\int_{Z'} p_1(\nabla) - \int_Z p_1(\nabla) = \int_{-Z \cup_M Z'} p_1(\nabla) \in \mathbb{Z}$$

by the integrality of $c_2$. Hence $\text{CS}(M, g)$ only depends on the Riemannian manifold $M$. \qed
Note that one can improve the Chern-Simons invariant using the following additional integrality of $p_1$ of tangent bundles given by the signature theorem (see [BGV92]).

$$\langle p_1(TN), [N] \rangle = 3 \text{sign}(N) \in 3\mathbb{Z}$$

for every oriented closed four manifold $N$. Hence we can define, using the notation as above,

$$\text{CS}_{\text{refined}}(M, g) := \left[ \frac{1}{3} \int_Z p_1(\nabla) \right].$$

(14)

**Problem 2.112.** Show that the assumption of a product structure is important here.

**Proof.** The product structures makes sure that the restriction of the Levi-Civita connection to the boundary induces the Levi-Civita connection of the boundary. Without a product structure the corresponding difference is measured by the second fundamental form. \qed

**Problem 2.113.** We consider the lens space $L^3_k$. Calculate $\text{CS}(L^3_k, g)$ and $\text{CS}_{\text{refined}}(L^3_k, g)$, where $g$ is the metric induced from the round metric on $S^3$.

The result is

$$\text{CS}(L^3_k, g) = \left[ \frac{1}{k} \right].$$

Here is a trick for the calculation. The $U(1)$-principal bundle structure on $p : L^3_k \to \mathbb{CP}^1$ gives a decomposition $TL^3_k \cong p^*T\mathbb{CP}^1 \oplus H$, where $H$ is trivialized by the $U(1)$-action. We let $\nabla' := p^*\nabla^{T\mathbb{CP}^1} \oplus \nabla^H$ an be adapted connection. We extend $\nabla'$ to the disc bundle $q : D \to \mathbb{CP}^1$ with boundary $L^3_k$ in the form $q^*\nabla^{T\mathbb{CP}^1} \oplus \tilde{\nabla}$, where $\tilde{\nabla}$ extends $\nabla^H$. Then we have

$$\text{CS}(L^3_k, g) = \left[ \int_D p_1(\nabla') \right] + \left[ \int_{L^3_k} \tilde{p}_1(\nabla^g, \nabla') \right].$$

(15)

Note that $\tilde{\nabla}$ is not trivial since the trivialization of $H$ does not extend to the disc bundle. We rather have $\int_D c_1^2(\tilde{\nabla}) = -k^2 \in \mathbb{Z}$ independently of the extension, see Problem 2.108. Together with the addivity of the Pontrjagin form

$$p_1(q^*\nabla^{T\mathbb{CP}^1} \oplus \tilde{\nabla}) = q^*c_1^2(\nabla^{T\mathbb{CP}^1}) + c_1^2(\tilde{\nabla}^H)$$

we get $\left[ \int_D p_1(\nabla') \right] = [-k^2] = 0$. In order to calculate the second integral in (15) we go to the $k$-fold covering $S^3 \to L^3_k$. The family of connections interpolating between $\nabla^g$ and $\nabla'$ pulls back to a family interpolating between the Levi-Civita connection on $TS^3$ associated to the round metric and the lift of $\nabla'$. We can extend the round metric from $S^3$ (considered as equator of $S^4$) as round metric to the upper half of $S^4$. Its Pontrjagin form vanishes. On the other hand we can extend the lift of $\nabla'$ similarly as above preserving the decomposition. We get

$$0 = \langle p_1(TS^4), [S^4] \rangle = k \int_{L^3_k} \tilde{p}_1(\nabla^g, \nabla') - 1,$$
hence
\[ \int_{L^3_k} \tilde{p}_1(\nabla^g, \nabla') = \frac{1}{k}. \]

For the refined version we get
\[ \text{CS}_{\text{refined}}(L^3_k, g) = \left[ \frac{1 - k^3}{3k} \right]. \]

\[ \square \]

**Example 2.114.** For Riemannian manifold \((M, g)\) with Levi-Civita connection \(\nabla\) the Pontrjagin form \(p_i(g) := p_i(\nabla)\) only depends on the Weyl tensor (see [Ave70] for an argument). In particular, we have \(p_i(g) = p_i(g')\) if \(g\) and \(g'\) are conformally equivalent.

At the moment we can define, generalizing 2.110 a Chern-Simon invariant \(\text{CS}(M, g)\) for \(4n - 1\)-dimensional manifolds for which there exists a zero bordism \((Z, h)\) by
\[ \text{CS}(M, g) = \left[ \int_Z p_n(h) \right] \in \mathbb{C}/\mathbb{Z}. \]

\(\text{CS}(M, g)\) is a conformal invariant of \(M\). It is, for example, an obstruction against finding a locally conformally flat zero bordism.

**Corollary 2.115.** \((L^3_k, g)\) does not bound totally umbilically a locally conformally flat manifold.

Below, in 3.60 we will drop the assumption that \(M\) is zero bordant.

### 2.7 Integral refinement

Let \(\omega\) be an integral characteristic form of degree \(n\) and \(E \to M\) be a vector bundle. Then we have
\[ \omega(E) \in \text{im}(\epsilon_\mathbb{C} : H^n(M; \mathbb{Z}) \to H^n_{\text{dR}}(M; \mathbb{C})). \]

This suggests the following definition.

**Definition 2.116.** An integral refinement \(\omega^Z\) of \(\omega\) associates to each bundle \(E \to M\) a class \(\omega^Z(E) \in H^n(M; \mathbb{Z})\) with \(\epsilon_\mathbb{C}(\omega^Z(E)) = \omega(E)\) such that for every pull-back diagram
\[ E' \xrightarrow{F} E \]
\[ M' \xrightarrow{f} M \]

we have
\[ f^*\omega^Z(E) = \omega^Z(E'). \]

**Theorem 2.117.** An integral characteristic class for complex vector bundles has a unique integral refinement.
Proof. The functor which associates to each manifold $M$ the set of isomorphism classes of complex vector bundles on $M$ is represented by the space

$$BU := \bigsqcup_{n \geq 0} BU(n).$$

The calculation of the integral cohomology of this space is a basic result in algebraic topology, see (18). It could be used to give a short conceptual proof of the theorem. For pedagogical reasons in order to present a basic technique in the field of differential cohomology we give another argument which only invests the following consequence:

**Corollary 2.118.** The integral cohomology of $BU$ is concentrated in even degrees and torsion-free.

We have a Bockstein sequence

$$\cdots \to H^k(BU; \mathbb{C}) \to H^k(BU; \mathbb{C}/\mathbb{Z}) \to H^{k+1}(BU; \mathbb{Z}) \to H^{k+1}(BU; \mathbb{C}) \to \cdots.$$ 

We see that $H^{\text{odd}}(BU; \mathbb{C}/\mathbb{Z}) \cong 0$, $H^{\text{odd}}(BU; \mathbb{C}) \cong 0$ and $H^{\text{ev}}(BU; \mathbb{Z}) \to H^{\text{ev}}(BU; \mathbb{C})$ is injective.

Recall that a map of spaces $f : X \to Y$ is called $n + 1$-connected if it induces a bijection on $\pi_0$ and the homotopy fibre of $f$ at each point of $Y$ has trivial homotopy groups up to degree $n$. Note that this implies (use Serre’s spectral sequence) that the pull back $f^* : H^k(Y; A) \xrightarrow{\sim} H^k(X, A)$ is an isomorphism for all $k \leq n$ and every abelian group $A$. We assume that $\omega$ is of degree $n$ and $M$ is $m$-dimensional. We define $r := \max(n, m) + 1$.

If $E \to M$ is a vector bundle, then there exists a manifold $N$ with a vector bundle bundle $F \to N$ classified by an $r$-connected map $u : N \to BU$ and a map $h : M \to N$ such that $E \cong h^*F$. For $N$ we can take, for example, an approximation of an $r$-skeleton of $BU$. The map $u$ then induces an isomorphism in complex and integral cohomology in degree $\leq r$. If $n$ is odd, then $\omega(F) = 0$ and hence $\omega(E) = h^*\omega(F) = 0$. If $n$ is odd, then we can take $\omega^Z = 0$.

From now on assume that $n$ is even. Then there exists a unique class $w \in H^n(N; \mathbb{Z})$ such that $\epsilon_C(w) = \omega(F)$. We define $\omega^Z(E) := h^*w \in H^n(M; \mathbb{Z})$. We must show that $\omega^Z(E)$ is well-defined independently of the choices of $f$ and $F \to N$.

Assume that $F' \to N'$, $h' : M \to N'$ constitutes another choice. Then we produce a diagram

\[
\begin{array}{ccc}
N & \xrightarrow{u} & N'' \\
\downarrow h & & \downarrow u'' \\
M & & BU \\
\downarrow h' & & \downarrow u'
\end{array}
\]

which commutes up to homotopy, where $u''$ is an $r$-connected map, as well. We conclude that $g^*w'' = w$ and $g^*w'' = w'$ and therefore $h^*w = h'^*w'$.

This finishes the verification that $\omega^Z$ is well-defined. Finally we show that $\omega^Z$ is natural.
Given a diagram (16) we consider the extension

\[
\begin{array}{ccc}
E' & \xrightarrow{F} & E \xrightarrow{h} F \\
\downarrow & & \downarrow \\
M' & \xrightarrow{f} & M \xrightarrow{h} N
\end{array}
\]

(17)

We can assume that \( N \to BU \) is a \( r' \)-connected map. Then by construction \( \omega^Z(E') = f^* \omega^Z(E) \).

If \( u : N \to BU \) is \( n + 1 \)-connected, then we have an isomorphism \( u^* : H^n(BU; \mathbb{Z}) \to H^n(N; \mathbb{Z}) \). Given an integral characteristic form \( \omega \) of degree \( n \) we thus get a class \( \omega^Z \in H^n(BU; \mathbb{Z}) \) such that \( u^* \omega^Z = \omega^Z(F) \). The argument above shows that \( \omega^Z \) is well-defined. The class \( \omega^Z \) is called the universal characteristic class associated to the integral characteristic form \( \omega \).

We apply this to the Chern forms \( c_n \) and get integral classes \( c_n^Z \in H^{2n}(BU; \mathbb{Z}) \). We have a decomposition

\[
H^*(BU; \mathbb{Z}) = \prod_{n \geq 0} H^*(BU(n); \mathbb{Z}) .
\]

We let

\[
c_n = (c_{n,1}, c_{n,2}, \ldots , )
\]

be the corresponding decomposition of the Chern classes. Then it is known that the integral cohomology of the classifying spaces are given as polynomial rings by

\[
H^*(BU(n); \mathbb{Z}) \cong \mathbb{Z}[c_{1,n}, \ldots , c_{n,n}] .
\]

Problem 2.119. Use the Gysin sequence of the sphere bundles \( S^{2n-1} \to BU(n-1) \to BU(n) \) in order to show (18) by induction w.r.t \( n \).

In general the integral refinement of an integral characteristic form contains more information than the associated complex valued cohomology class.

Example 2.120. We consider an \( U(1) \) principal bundle \( \pi : M \to \mathbb{C}P^n \) with first Chern class \( pc \in H^2(\mathbb{C}P^n; \mathbb{Z}) \) for a prime \( p \in \mathbb{N} \)

Problem 2.121. Show that \( \pi_1(M) \cong \mathbb{Z}/p\mathbb{Z} \).

We have a non-canonical isomorphism \( J(M) \cong \mathbb{Z}/p\mathbb{Z} \). Consider the Poincaré bundle \( P \to J(M) \times M \) (see Example 2.30). For \( \rho \in J(M) \) we consider the restriction \( P_\rho := P|_{\{\rho\} \times M} \to M \).

Problem 2.122. Show that \( \rho \mapsto c_1^Z(P_\rho) \) provides an isomorphism of abelian groups

\[
\mathbb{Z}/p\mathbb{Z} \cong J(M) \xrightarrow{\sim} H^2(M; \mathbb{Z}) .
\]
Proof. We identify $M$ with the unit sphere bundle of $L^p$. Then we observe that $\pi^*L^p$ is canonically trivialized. Hence we can define a unique flat connection $\nabla^{\pi^*L}$ which induces the trivial connection on $\pi^*L^p$. This connection is characterized by $\nabla^{\pi^*L} = \nabla^{\triv} = 0$ for local sections $\phi$ such that $\nabla^{\triv} \phi = 0$. We further observe that $\nabla^{\pi^*L}$ generates $H^2(M;\mathbb{Z})$. We get the isomorphism $J(M) \ni x = \rho_0^k \mapsto c_1^Z(\pi^*L^k) = c_1^Z(L_{\rho_0^k}) \in H^2(M;\mathbb{Z})$.

Note that $\epsilon_C(c_1^Z(P_{\rho})) = 0$.

Let $E \to M$ be a vector bundle with connection. Then we can form its determinant $\det(E) \to M$ (a line bundle) which has an induced connection $\nabla^{\det(E)}$.

**Problem 2.123.** Show that $c_1(\nabla) = c_1(\nabla^{\det(E)})$ and $c_1^Z(E) = c_1^Z(\det(E))$.

The integral total Chern class is defined by

$$c^Z(E) = 1 + c_1^Z(E) + c_2^Z(E) + \ldots.$$ 

Recall the Newton classes from Lemma 2.90

**Problem 2.124.** Show that

$$s_4^Z(E) = c_1^Z(E)^2 - 2c_2^Z(E).$$

Show that for two complex vector bundles $E, F$ we have the relation

$$c^Z(E \oplus F) = c^Z(E) \cup c^Z(F)$$

of total Chern classes.

**Example 2.125.** For a real vector bundle $V \to M$ we define the integral Pontrjagin classes by

$$p_i^Z(V) := (-1)^i c_{2i}^Z(V \otimes \mathbb{C}) \in H^{4i}(M;\mathbb{Z}).$$

The integral total Pontrjagin class is defined by

$$p^Z(V) := 1 + p_1^Z(V) + p_2^Z(V) + \ldots.$$ 

**Problem 2.126.** Show for a real vector bundle $V \to M$ we have $2c_1^Z(V \otimes \mathbb{C}) = 0$ for all odd $i \geq 1$. Show further that for two real bundles $V, W$ we have

$$p^Z(V \oplus W) = p^Z(V) \cup p^Z(W) + 2\text{-torsion}.$$ 

See [MS74, Ch. 15] for more information.
Problem 2.127. If $E_R$ is the underlying real vector bundle of a complex vector bundle, then $c_j^Z (E_R \otimes \mathbb{C}) = 0$ for all odd $j \geq 1$. Conclude that
\[ p^Z (E_R \oplus F_R) = p^Z (E_R) \cup p^Z (F_R) \]
and calculate $p(T \mathbb{PC}_R^n)$ explicitly.

Problem 2.128. Calculate $c_1^Z (L \otimes \mathbb{C})$, where $L \to \mathbb{RP}^\infty$ is the universal real line bundle.

Problem 2.129. We let $M_r \to S^4$ be the $SU(2)$-principal bundle classified by $c_2^Z (M_r) = r \sigma_{S^4} \in H^4 (S^4; \mathbb{Z})$, $r \in \mathbb{N}$. Calculate $H^4 (M_r; \mathbb{Z})$ and $p_1^Z (TM_r)$.

3 Smooth Deligne cohomology

3.1 Recollections on sheaf theory

We assume basic knowledge of sheaf theory. We consider the category of smooth manifolds $\text{Mf}$ as a site with the topology given by open coverings. It comes with a structure sheaf $\mathcal{C}^\infty$ which associates to every manifold $M$ its algebra of complex-valued smooth functions. An abelian Lie group $A$ represents a sheaf of abelian groups
\[ A \in \text{Sh}_{\text{Ab}} (\text{Mf}) . \]

If $A$ is not discrete, then $A^\delta$ denotes $A$ with the discrete topology. We have a natural map of sheaves $A^\delta \to A$ which is far from being an isomorphism.

Problem 3.1. Show that an exact sequence
\[ 0 \to A \to B \to C \to 0 \]
of abelian Lie groups induces an exact sequence of sheaves
\[ 0 \to A \to B \to C \to 0 . \quad (19) \]

Proof. The main point it to show that $B \to C$ is surjective. Use that $B \to C$ has local sections. \hfill \Box

Note that it is important to consider the sequence (19) in sheaves. As a sequence of presheaves it is not exact in general.

Problem 3.2. Discuss this assertion in the example of the exponential sequence
\[ 0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R} / \mathbb{Z} \to 0 \]
The difference of the notions of exactness in the categories of sheaves and presheaves is the source of sheaf cohomology. In particular we let $H^* (M; A)$ denote the cohomology of $M$ with coefficients in $A$, where $H^* (M; \ldots)$ is the higher derived image of the functor
\[ \Gamma (M, \ldots) : \text{Sh}_{\text{Ab}} (\text{Mf}) \to \text{Ab} \]
of evaluation at $M$. These cohomology groups can be calculated using $\Gamma(M,\ldots)$-acyclic resolutions.

If $A$ is an abelian group, then we can define the cohomology of $M$ with coefficients in $A$ using simplicial or homotopy theoretic means, e.g. by

$$H^n(M; A) := [M, K(A, n)]$$

or

$$H^n(M; A) := H^p(\text{Hom}(C_*(M), A)),$$

where $K(A, n)$ is the $n$th Eilenberg-MacLane space of $A$ and $C_*(M)$ is the simplicial chain complex of $M$. We shall use the fact that there are canonical isomorphisms between these definitions and

$$H^n(M; A) := H^n(M; A^\delta).$$

**Problem 3.3.** Show that the three versions of the cohomology of $M$ with coefficients in a discrete abelian group $A$ mentioned above are canonically isomorphic.

**Problem 3.4.** Let $F$ be a sheaf on $Mf$ and consider the presheaf $\mathcal{H}^k(F)$ which associates to $M \in Mf$ the cohomology group $H^k(M, F)$. Let $\mathcal{H}^k(F)$ be its sheafification and show that

$$\mathcal{H}^k(F) \cong \begin{cases} F & k = 0 \\ 0 & k \geq 0 \end{cases}.$$

For applications we must know examples of $\Gamma(M,\ldots)$-acyclic sheaves.

**Lemma 3.5.** A sheaf $F$ of $C^\infty$-modules is $\Gamma(M,\ldots)$-acyclic.

**Proof.** The existence of smooth partitions of unity shows that $F$ is fine. A fine sheaf if $\Gamma(M,\ldots)$-acyclic. In the following we give the details of the argument.

To an open covering $U$ of $M$ we associate the Čech complex $\check{C}(U; F)$. We let $\check{C}(M; F)$ be the colimit of Čech complexes over a cofinal system of open coverings. For every $n \geq 0$ the functor $\text{Sh}_{\text{Ab}} \to \text{Ch}, F \to \check{C}^n(M; F)$, is exact and $H^0(\check{C}^n(M; F)) = \Gamma(M, F)$.

We thus get a $\delta$-functor of Čech-cohomology $(H^*(\check{C}^n(M; \ldots)), \delta)$ extending $\Gamma(M,\ldots)$. Uniqueness of such $\delta$-functor extensions implies that $H^*(\check{C}^n(M; F)) \cong H^*(M; F)$. Hence we can calculate $H^*(M; F)$ as Čech cohomology.

For the Lemma it thus suffices to see that the Čech complex of a sheaf $F$ of $C^\infty$-modules is exact. We use the $C^\infty$-module structure through the following. Let $V \subseteq U$ be an inclusion of open subsets, $x \in F(V)$, and $\chi \in C_c^\infty(V)$. The sheaf axioms imply that there exists an extension by zero $(\chi x)_0 \in F(U)$ of $\chi x$.

**Problem 3.6.** Prove this!

Let $U = (U_i)_{i \in I}$ be a covering and $(\chi_i)_{i \in I}$ be a partition of unity. Then we define a homotopy $h : \check{C}^*(U; F) \to \check{C}^{*-1}(U; F)$ by

$$h(x)_{j_0,\ldots,j_{n-1}} = (-1)^{n-1} \sum_{i \in I} (\chi_i x_{j_0,\ldots,j_{n-1},i})_0.$$
Problem 3.7. Check that $\partial h + h \partial = \text{id}$.
It follows that $H^{\geq 1}(\check{\mathcal{C}}(\mathcal{U}; \mathcal{F})) = 0$ for every covering $\mathcal{U}$, and thus $H^{\geq 1}(\check{\mathcal{C}}(M; \mathcal{F})) = 0$ by the exactness of the colimit. \qed

Example 3.8. The complex de Rham complex can be interpreted as a sheaf of complexes $\Omega_{\mathcal{C}} \in \text{Ch}(\text{Sh}_{\text{Ab}}(\text{Mf}))$. The sheaves $\Omega^n_{\mathcal{C}}$ are sheaves of $C^\infty$-modules and therefore $\Gamma(M; \ldots)$-acyclic by Lemma 3.5. By the Poincaré Lemma $\Omega_{\mathcal{C}}$ resolves the sheaf $\mathbb{C}^\delta$. Hence we have a canonical isomorphism

$$H^d_{\text{dR}}(M; \mathbb{C}) \cong H^*(M; \mathbb{C}^\delta).$$

Problem 3.9. Show that for all $n > 0$ we have $H^n(M; \mathbb{R}) = 0$.

Problem 3.10. Calculate and compare $H^*(T^n; \mathcal{U}(1))$ and $H^*(T^n; \mathcal{U}(1)^\delta)$.

Problem 3.11. Let $\Omega^k_{\text{cl},\mathcal{C}}$ be the sheaf of closed complex $k$-forms. Calculate $H^*(M; \Omega^k_{\text{cl},\mathcal{C}})$.

Up to this point we have discussed the cohomology of sheaves. We now introduce the notion of cohomology for complexes of sheaves called hyper cohomology.

Definition 3.12. If $\mathcal{F} \in \text{Ch}(\text{Sh}_{\text{Ab}}(\text{Mf}))$ is a lower bounded complex of sheaves of abelian groups, then by $H^*(M; \mathcal{F})$ we mean the hypercohomology of $\mathcal{F}$. Let $\mathcal{F} \to \mathcal{C}$ be a quasi-isomorphism of lower bounded complexes of sheaves such that $\mathcal{C}$ consists of injective sheaves, then by definition

$$H^*(M; \mathcal{F}) := H^*(\mathcal{C}(M)) .$$

Problem 3.13. Show that the hyper cohomology of $\mathcal{F}$ is well-defined (independent of the resolution), and that it suffices to resolve by complexes of $\Gamma(M; \ldots)$-acyclic sheaves.

Problem 3.14. Show that the notion of hypercohomology extends the definition of cohomology for sheaves if we consider a sheaf as a complex of sheaves concentrated in degree 0.

Problem 3.15. Calculate the hyper cohomology groups of the complexes

$$\mathbb{Z} \to \mathbb{C}^\delta , \mathbb{Z} \to \mathbb{C}$$

in terms of homotopy theoretic data.

From now on we will omit the prefix “hyper”.

Problem 3.16. For $p \geq 1$ consider the complex of sheaves on $\text{Mf}$

$$\mathcal{K}(p) : 0 \to \mathbb{C}^\delta \overset{\text{dlog}}{\to} \Omega^1_{\mathcal{C}} \to \ldots \to \Omega^p_{\mathcal{C}} \to 0 .$$

Calculate $H^*(M; \mathcal{K}(p))$ in terms of homotopy theoretic and differential geometric data of $M$. 

34
Let
\[ C : C^0 \to C^1 \to C^2 \to \ldots \]
be a complex of sheaves (or more general a complex in some abelian category). Then we define an increasing filtration by subcomplexes
\[ F^0C : Z^0, \quad F^1C : C^0 \to Z^1, \quad \ldots, \quad F^kC : C^0 \to \cdots \to C^{k-1} \to Z^k \]
We have quasi-isomorphisms
\[ F^iC/F^{i-1}C \cong H^i(\mathcal{C})[-i]. \]

**Definition 3.17.** The spectral sequence associated to this filtration
\[ (E_r,d_r) \Rightarrow H^*(M,\mathcal{C}) \]
with
\[ E_2^{p,q} = H^p(M,H^q(\mathcal{C})) \]
is called the hypercohomology spectral sequence.

**Problem 3.18.** Discuss the hypercohomology spectral sequence for the complexes \( K(p) \) introduced in 3.16.

### 3.2 Deligne cohomology

A map of cochain complexes \( f : C \to D \) of abelian groups (or more general, of objects in an abelian category like sheaves) can be extended to an exact triangle
\[ \ldots \text{Cone}(f)[-1] \to C \to D \to \text{Cone}(f) \to \ldots. \]
Here
\[ \text{Cone}(f)^i = C^{i+1} \oplus D^i, \quad \partial(x \oplus y) = (-\partial x \oplus (\partial y - f(x)) . \]
The triangle induces a long exact sequence in cohomology
\[ H^{n-1}(\text{Cone}(f)) \to H^n(\mathcal{C}) \to H^n(D) \to H^n(\text{Cone}(f)) \to \ldots. \]

We let \( \sigma^{\geq n} : \text{Ch} \to \text{Ch} \) be the stupid truncation functor given by
\[ (\sigma^{\geq n}C)^k := \begin{cases} C^k & k \geq n \\ 0 & k < n \end{cases} . \]
Note that
\[ H^k(\sigma^{\geq n}C) = \begin{cases} 0 & k < n \\ Z^n(\mathcal{C}) & k = n \\ H^k(\mathcal{C}) & k > n \end{cases} . \]
In a similar manner we define
\[
(\sigma <^n C)^k := \left\{ \begin{array}{ll}
C^k & k < n \\
0 & k \geq n \end{array} \right\}.
\]

Note that
\[
H^k(\sigma <^n C) = \left\{ \begin{array}{ll}
0 & k \geq n \\
C^{n-1}/\text{im}(d) & k = n - 1 \\
H^k(C) & k < n - 1 \end{array} \right\}.
\]

All this applies to complexes of sheaves.

We now come to Deligne cohomology. We consider the natural map
\[
\mathbb{Z} \oplus \sigma \geq n \Omega_C \to \Omega_C, \quad (z, x) \mapsto z - x
\]
using the not written inclusions \( \mathbb{Z} \to \mathbb{C} \) and \( \sigma \geq n \Omega_C \to \Omega_C \).

**Definition 3.19.** We define the \( n \)th Deligne complex \( \mathcal{D}(n) \in \text{Ch}(\text{Sh}_{\text{Ab}}(\text{Mf})) \) by
\[
\mathcal{D}(n) := \text{Cone}(\mathbb{Z} \oplus \sigma \geq n \Omega_C \to \Omega_C)[-1].
\]

We define the \( n \)th Deligne cohomology by
\[
\hat{H}^n_{\text{Del}}(M; \mathbb{Z}) := H^n(M; \mathcal{D}(n)).
\]

Note that we take different complexes \( \mathcal{D}(n) \) for different \( n \). The Deligne complex fits into an exact triangle
\[
\cdots \to \mathcal{D}(n) \to \mathbb{Z} \oplus \sigma \geq n \Omega_C \to \Omega_C \to \mathcal{D}(n)[1] \to \cdots.
\]

The interesting piece of the associated long exact sequence is
\[
\to H^{n-1}(M; \mathbb{Z}) \to H^{n-1}_{dR}(M; \mathbb{C}) \to \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \overset{(1,R)}{\to} H^n(M, \mathbb{Z}) \oplus \Omega^n_{cl}(M; \mathbb{C}) \to H^n_{dR}(M; \mathbb{C}) \to \cdots.
\]

**Problem 3.20.** Show that \( \hat{H}^n(M; \mathbb{Z}) \) naturally fits into an exact sequence
\[
H^{n-1}(M; \mathbb{Z}) \to H^{n-1}_{dR}(M; \mathbb{C}) \underset{a}{\to} \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \overset{(1,R)}{\to} H^n(M; \mathbb{Z}) \times_{H^n(M; \mathbb{C})} \Omega^n_{cl}(M; \mathbb{C}) \to 0.
\]

Calculate \( H^i(M; \mathcal{D}(n)) \) for all \( i \neq n \). The result is
\[
\hat{H}^i_{\text{Del}}(M; \mathbb{Z}) \cong \left\{ \begin{array}{ll}
H^{i-1}(M; \mathbb{C}/\mathbb{Z}) & i < n \\
H^i(M; \mathbb{Z}) & i > n \end{array} \right\}.
\]

Show that
\[
H^{n-1}(M; \mathbb{C}/\mathbb{Z}) \cong \text{ker}(R: \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \to \Omega^n_{cl}(M)),
\]
and that the composition
\[
H^{n-1}(M; \mathbb{C}/\mathbb{Z}) \to \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \overset{1}{\to} H^n(M, \mathbb{Z})
\]
is the negative of the Bockstein operator associated to the sequence of coefficients
\[
0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}/\mathbb{Z} \to 0.
\]
Proof. We have the quasi-isomorphisms

$$\text{Cone}(\mathbb{Z} \to \Omega_C)[-1] \cong \text{Cone}(\mathbb{Z} \to \mathbb{C}[\delta])[-1] \cong \mathbb{C}/\mathbb{Z}[\delta][-1].$$

We consider the short exact sequence of complexes of sheaves

$$0 \to \text{Cone}(\mathbb{Z} \to \Omega_C)[-1] \to \mathcal{D}(n) \to \text{Cone}(\sigma^{\geq n}\Omega_C \to 0)[-1] \to 0 \quad (21)$$

which is induced by the natural inclusion and projection. Since

$$\text{Cone}(\sigma^{\geq n}\Omega_C \to 0)[-1] \cong \sigma^{\geq n}\Omega_C$$

and

$$H^i(M; \sigma^{\geq n}\Omega_C) = 0 \ \forall i < n, \quad H^n(M; \sigma^{\geq n}\Omega_C) = \Omega^n\cl(M; \mathcal{C})$$

we get from the long exact cohomology sequence associated to (21) that

$$H^{i-1}(M; \mathbb{C}/\mathbb{Z}) \cong H^i(M; \mathcal{D}(n))$$

for all $i < n$ and

$$0 \to H^{n-1}(M; \mathbb{C}/\mathbb{Z}) \to \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \overset{R}{\to} \Omega^n\cl(M; \mathbb{C}) \ . \quad (22)$$

For $i > n$ we obtain from the long exact sequence for the cone $\mathcal{D}(n)$ that

$$I : \hat{H}^i_{\text{Del}}(M; \mathbb{Z}) \to H^i(M; \mathbb{Z})$$

is an isomorphism. Finally, the assertion for $i = n$ follows from exact sequence of the cone $\mathcal{D}(n)$, too.

The long exact sequence of the cone $\text{Cone}(\mathbb{Z} \to \Omega_C)[-1]$ is the Bockstein sequence down shifted by $-1$. The first map in (21) induces a map from the Bockstein sequence to the long exact sequence of the cone $\mathcal{D}(n)$. In particular we get

$$H^{n-1}(M; \mathbb{Z}) \longrightarrow H^{n-1}(M; \mathbb{C}) \longrightarrow H^{n-1}(M; \mathbb{C}/\mathbb{Z}) \overset{-\beta}{\longrightarrow} H^n(M; \mathbb{Z}) \longrightarrow H^n(M; \mathbb{C}) \ ,$$

$$H^{n-1}(M; \mathbb{Z}) \longrightarrow H^{n-1}(M; \mathbb{C}) \longrightarrow \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \overset{I \otimes R}{\longrightarrow} H^n(M; \mathbb{Z}) \oplus \Omega^n\cl(M) \longrightarrow H^n(M; \mathbb{C})$$

where $\beta$ is the Bockstein operator. The minus sign comes from the shift. \qed

It turns out to be useful to have different representations of Deligne cohomology. We define

$$\mathcal{E}(n) := \text{Cone}(\mathbb{Z} \to \sigma^{< n}\Omega_C)[-1], \quad (23)$$

i.e.

$$\mathcal{E}(n) \ : \ 0 \to \mathbb{Z} \to \Omega_C^0 \overset{-d}{\longrightarrow} \cdots \overset{-d}{\longrightarrow} \Omega_C^{n-1} \to 0$$
where \( \mathbb{Z} \) sits in degree 0 and hence \( \Omega_i^C \) in degree \( i+1 \). This complex sits in the short exact sequence of complexes of sheaves

\[
0 \to \text{Cone}(\sigma^{>n} \Omega_C \Rightarrow \sigma^{>n} \Omega_C)[-1] \to \mathcal{D}(n) \to \mathcal{E}(n) \to 0.
\]

The left map is the natural inclusion and the right map the projection. Since the cone of an isomorphism is acyclic the projection map is a quasi-isomorphism. We conclude that

\[
\hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \cong H^n(M; \mathcal{E}(n)). \tag{24}
\]

We now consider the exact sequence

\[
0 \to \sigma^{<n} \Omega_C[-1] \to \mathcal{E}(n) \to \mathbb{Z} \to 0.
\]

It induces a long exact sequence in cohomology. Its most interesting piece is

\[
H^{n-1}(M; \mathbb{Z}) \to \Omega^{n-1}(M; \mathbb{C})/\text{im}(d) \xrightarrow{\alpha} H^n(M, \mathcal{E}(n)) \xrightarrow{I} H^n(M; \mathbb{Z}) \to 0.
\]

The notation \( I \) is used since this map corresponds to \( I \) under the isomorphism \( \text{(24)} \).

**Definition 3.21.** We define

\[
a : \Omega^{n-1}(M; \mathbb{C})/\text{im}(d) \to \hat{H}^n_{\text{Del}}(M; \mathbb{Z})
\]

to be the map given by

\[
\Omega^{n-1}(M; \mathbb{C})/\text{im}(d) \xrightarrow{\alpha} H^n(M, \mathcal{E}(n)) \xrightarrow{[24]} \hat{H}^n_{\text{Del}}(M; \mathbb{Z})
\]

**Corollary 3.22.** We have an exact sequence

\[
H^{n-1}(M; \mathbb{Z}) \to \Omega^{n-1}(M; \mathbb{C})/\text{im}(d) \xrightarrow{\alpha} \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \xrightarrow{I} H^n(M; \mathbb{Z}) \to 0. \tag{25}
\]

**Problem 3.23.** Show that \( a : \Omega^{n-1}(M; \mathbb{C})/\text{im}(d) \to \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \) extends the map \( a : H^{n-1}(M; \mathbb{C}) \to \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \) in \( \text{(20)} \) and that \( R \circ a = d \).

**Proof.** The first assertion follows from the commutativity

\[
\begin{array}{ccc}
\Omega_C[-1] & \xrightarrow{\text{D}(n)} & \mathcal{D}(n) \\
\downarrow & & \downarrow \\
\sigma^{<n} \Omega_C[-1] & \xrightarrow{\text{E}(n)} & \mathcal{E}(n)
\end{array}
\]

The upper horizontal map induces the map \( a : H^{n-1}(M; \mathbb{C}) \to \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \), while the lower horizontal map induces \( a : \Omega^{n-1}(M; \mathbb{C})/\text{im}(d) \to \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \).
For the second assertion we consider the following web of exact sequences

\[
\begin{array}{ccccccccc}
0 & \to & \text{Cone}(0 \to \Omega^n_{\mathbb{C}})[-1] & \to & \text{Cone}(\mathbb{Z} \to \Omega^n_{\mathbb{C}})[-1] & \to & \text{Cone}(\mathbb{Z} \to \Omega^{<n}_{\mathbb{C}})[-1] & \to & 0 \\
0 & \to & \text{Cone}(\sigma^{\geq n}_{\Omega_{\mathbb{C}}} \to \sigma^{\geq n}_{\Omega_{\mathbb{C}}})[-1] & \to & \mathcal{D}(n) & \to & \mathcal{E}(n) & \to & 0 \\
0 & \to & \text{Cone}(\sigma^{\geq n}_{\Omega_{\mathbb{C}}} \to 0)[-1] & \to & \text{Cone}(\sigma^{\geq n}_{\Omega_{\mathbb{C}}} \to \Omega^{<n}_{\mathbb{C}})[-1] & \to & 0 & \to & 0 \\
0 & \to & 0 & \to & 0 & \to & 0 & \to & 0 \\
\end{array}
\]

Let \( \delta_v \) and \( \delta_h \) denote the boundary maps associated to the right vertical and the upper horizontal exact sequence. Then we know from homological algebra (see Exercise 3.25) that

\[
\delta_v \circ R = -\delta_h \circ \pi.
\]

Let now

\[
\omega \in \Omega^{n-1}(M, \mathbb{C})/\text{im}(d) \cong H^n(M, \sigma^{<n}_{\Omega_{\mathbb{C}}}[−1]).
\]

Then in \( H^{n+1}(M, \text{Cone}(0 \to \sigma^{\geq n}_{\Omega_{\mathbb{C}}})[-1]) \) we have

\[
[0 \oplus R(\omega)] = \delta_v(R(a(\pi^{-1}(\omega))) = -\delta_h(0 \ominus \omega) = [0 \oplus d\omega].
\]

We read off that \( R(a(\omega)) = d\omega. \)

Let us collect all this information in the following commuting diagram, called the differential cohomology diagram.

**Proposition 3.24.** The Deligne cohomology fits into the differential cohomology diagram

\[
\begin{array}{cccccc}
\Omega^{n-1}(M; \mathbb{C})/\text{im}(d) & \xrightarrow{d} & \Omega^n(M; \mathbb{C}) \\
H^n_{\text{dR}}(M; \mathbb{C}) & \xrightarrow{a} & H^n_{\text{Del}}(M; \mathbb{Z}) & \xleftarrow{R} & H^n_{\text{dR}}(M; \mathbb{C}) \\
H^n(M; \mathbb{C}) & \xleftarrow{\delta} & H^n(M; \mathbb{Z}) & \xrightarrow{\delta} & H^n(M; \mathbb{Z}) \\
\end{array}
\]

where the diagonal compositions are exact and the part marked in red is a segment of the long exact Bockstein sequence.
**Example 3.25.** Here is an exercise in homological algebra. We consider a web of short exact complexes

\[
\begin{array}{c}
0 \\
\downarrow \\
0 \rightarrow A \rightarrow D \rightarrow G \rightarrow 0 \\
\downarrow \\
0 \rightarrow B \rightarrow E \rightarrow H \rightarrow 0 \\
\downarrow \\
0 \rightarrow C \rightarrow F \rightarrow I \rightarrow 0 \\
\downarrow \\
0 \\
0 0 0
\end{array}
\]

of cochain complexes. Let \( e \in H^n(E) \) be such that its image in \( H^n(I) \) vanishes. Then we can lift the image of \( e \) in \( H^n(H) \) to a class of \( H^n(G) \) and apply the boundary operator \( \delta_{uh} \) of the upper horizontal sequence to get a well-defined class

\[
U(e) \in \frac{H^{n+1}(A)}{\delta_{uh}\delta_{rv}H^{n-1}(I)}.
\]

Similarly we can lift the image of \( e \) in \( H^n(F) \) to \( H^n(C) \) and apply the boundary operator \( \delta_{lv} \) of the left vertical sequence to get a class

\[
V(e) \in \frac{H^{n+1}(A)}{\delta_{lv}\delta_{lh}H^{n-1}(I)}.
\]

**Problem 3.26.** Show that

\[
[U(e)] = [-V(e)] \in \frac{H^{n+1}(A)}{\delta_{uh}\delta_{rv}H^{n-1}(I) + \delta_{lv}\delta_{lh}H^{n-1}(I)}.
\]

**Problem 3.27.** Relate the structure of \( \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \) with the hypercohomology spectral sequence of the complex of sheaves \( \mathcal{E}(n) \).

Deligne cohomology \( \hat{H}^n_{\text{Del}}(\ldots; \mathbb{Z}) \) is a contravariant functor from \( \text{Mf} \) to \( \text{Ab} \). It is not homotopy invariant, but its deviation from homotopy invariance is measured by the homotopy formula. Let \( i_t : M \rightarrow [0,1] \times M \) be the inclusion determined by \( t \in [0,1] \).

**Proposition 3.28.** If \( \hat{x} \in \hat{H}^n_{\text{Del}}([0,1] \times M; \mathbb{Z}) \), then

\[
i_t^* \hat{x} - i_0^* \hat{x} = a(\int_{[0,1] \times M/M} R(\hat{x}))
\]

**Proof.** By homotopy invariance of integral cohomology we know that there exists \( y \in H^n_{dR}(M; \mathbb{Z}) \) such that \( I(\hat{x}) = \text{pr}_M^*y \). We can choose a lift \( \hat{y} \in \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \) such that \( I(\hat{y}) = y \). Then \( I(\text{pr}_M^*\hat{y} - \hat{x}) = 0 \) and therefore \( \hat{x} = \text{pr}_M^*\hat{y} + a(\omega) \) for some \( \omega \in \Omega^{n-1}([0,1] \times M; \mathbb{C}) \).
Note that $R(\tilde{x}) = \text{pr}_M^*y + d\omega$. Using $\text{pr}_M \circ i_0 = \text{pr}_M \circ i_1$, Stokes’ theorem, and that 
\[
\int_{[0,1] \times M/M} \circ \text{pr}_M^* = 0
\]
we get
\[
i_1^*\dot{x} - i_0^*\dot{x} = a(i_1^*\omega - i_0^*\omega) = a\left(\int_{[0,1] \times M/M} d\omega\right) = a\left(\int_{[0,1] \times M/M} R(\tilde{x})\right).
\]

Let $M = U \cup V$ be a decomposition into open submanifolds.

**Problem 3.29.** Show that there exists a Mayer-Vietoris sequence of the form
\[
\cdots \to H^{n-2}(U \cap V; \mathbb{C}/\mathbb{Z}) \to \hat{H}^n_{\text{Del}}(M; \mathbb{Z}) \to \hat{H}^n_{\text{Del}}(U; \mathbb{Z}) \oplus \hat{H}^n_{\text{Del}}(V; \mathbb{Z}) \to \hat{H}_{\text{Del}}(U \cap V; \mathbb{Z}) \to H^{n+1}(M; \mathbb{Z}) \to \cdots
\]
which extends to the left and right by the Mayer-Vietoris sequences of $H^*(\ldots; \mathbb{C}/\mathbb{Z})$ and $H^*(\ldots; \mathbb{Z})$.

**Proof.** It is clear that sheaf cohomology $H^*(M; \mathcal{D}(n))$ has a Mayer-Vietoris sequence. The rest follows from the calculations in Proposition 3.20. □

**Example 3.30.** We have canonical isomorphisms $\hat{H}^0_{\text{Del}}(pt; \mathbb{Z}) \cong \mathbb{Z}$ and $\text{ev} : \hat{H}^1_{\text{Del}}(pt; \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}$. Let $M$ be any smooth manifold. We define a map
\[
\phi : \hat{H}^1_{\text{Del}}(M; \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}^M
\]
by $\phi(x)(m) = \text{ev}(m^*x)$, where we consider $m$ as a map $m : pt \to M$.

**Problem 3.31.** Show that $\phi(x) \in C^\infty(M; \mathbb{C}/\mathbb{Z})$ and $d\phi(x) = R(x)$. Show that $\phi : \hat{H}^1_{\text{Del}}(M; \mathbb{Z}) \to C^\infty(M, \mathbb{C}/\mathbb{Z})$ is an isomorphism of groups.

**Proof.** We use the Five Lemma and that $C^\infty(M, \mathbb{C}/\mathbb{Z})$ fits into an exact sequence
\[
H^0(M; \mathbb{Z}) \to \Omega^0(M; \mathbb{C}) \to C^\infty(M, \mathbb{C}/\mathbb{Z}) \to H^1(M; \mathbb{Z}) \to 0
\]
which is compatible with (25) by $\phi$. □

**Problem 3.32.** Calculate $\hat{H}^1_{\text{Del}}(S^1; \mathbb{Z})$. Show that there is a unique class $\tilde{e} \in \hat{H}^1_{\text{Del}}(S^1; \mathbb{Z})$ with $R(\tilde{e}) = dt$ (here $t$ is a coordinate given by $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$) which vanishes after restriction to the base point $\{t = 0\} \in S^1$. Calculate the restriction $\tilde{e}|_{\{t\}}$ for every point $\{t\} \in S^1$. Show that the class $\tilde{e}$ is primitive with respect to the group structure of $S^1$.

**Problem 3.33.** Use 3.32 in order to define a map $\text{ev} : \hat{H}^2_{\text{Del}}(M; \mathbb{Z}) \to C^\infty(M^{S^1}; \mathbb{C}/\mathbb{Z})$ for every manifold $M$. Here we understand smooth functions from the free loop space $M^{S^1}$ in the diffeological sense.

Let $G$ be a simple simply-connected compact Lie group.
**Problem 3.34.** Show that there is a unique (up to sign) biinvariant class \(x \in \hat{H}^3_{\text{Del}}(G; \mathbb{Z})\) such that \(I(x) \in H^3(G; \mathbb{Z})\) is a generator. Show that this class is primitive. Discuss the non-simply connected case.

**Example 3.35.** If \(M\) is a \(n-1\)-dimensional connected closed oriented manifold, then we have a canonical identification 

\[ \text{ev} : \hat{H}^n\text{Del}(M; \mathbb{Z}) \cong H^{n-1}_d(M; \mathbb{C})/\text{im}(\epsilon_{\mathbb{C}}) \cong \mathbb{C}/\mathbb{Z}. \]

This follows from the exact sequence (20). Explicitly, the identification is given by \(\hat{x} \mapsto [\int_M \omega]\), where \(\omega \in \Omega^{n-1}(M; \mathbb{C})\) is such that \(a(\omega) = \hat{x}\). Indeed, such form exists by (25) since \(I(\hat{x}) = 0\) for dimensional reasons, and \(\omega\) is well-defined up to integral forms. Therefore the class in \(\mathbb{C}/\mathbb{Z}\) of the integral does not depend on the choice.

Recall the complexes introduced in 3.16. We consider the map

\[ E(p) \to K(p-1)[-1] \]

given by

\begin{align*}
0 & \text{ degree zero} \\
2\pi i \text{id}_{\Omega^{q}} & q \geq 1 \\
\Omega^{q}_{C}(M) \in f & \mapsto \exp(2\pi if) \in \mathbb{C}^{*}(M) \text{ degree one}
\end{align*}

**Problem 3.36.** Show that (26) is a quasi-isomorphism so that 

\[ \hat{H}^p_{\text{Del}}(M; \mathbb{Z}) \cong H^{p-1}(M; K(p-1)). \]

### 3.3 Differential refinements of integral characteristic classes

Let \(\omega\) be an integral characteristic form for complex vector bundles of degree \(n\). It has a unique integral refinement \(\omega^Z\) by Theorem 2.117.

**Definition 3.37.** A differential refinement of \(\omega\) associates to every vector bundle with connection \((E, \nabla)\) on \(M\) a class \(\hat{\omega}(\nabla) \in \hat{H}_{\text{Del}}(M; \mathbb{Z})\) such that

\[ R(\hat{\omega}(\nabla)) = \omega(\nabla), \quad I(\omega(\nabla)) = \omega^Z(E) \]

and for every map \(f : M' \to M\) we have \(f^*\hat{\omega}(\nabla) = \hat{\omega}(f^*\nabla)\).

A longer, but in some cases clearer notation for evaluation of the differential refinement on \((E, \nabla)\) would be \(\hat{\omega}(E, \nabla)\). Note that

\[ \ker(R) \cap \ker(I) = H^{n-1}_{dR}(M; \mathbb{C})/\text{im}(\epsilon_{\mathbb{C}}) \]

is non-trivial. Therefore the differential refinement \(\hat{\omega}(\nabla)\) can potentially contain finer information than the pair of the form \(\omega(\nabla)\) and \(\omega^Z(E)\).

The homotopy formula Proposition 3.28 determines how \(\hat{\omega}(\nabla)\) depends on the connection \(\nabla\).
Lemma 3.38. If $\nabla$ and $\nabla'$ are two connections on the same bundle, then we have

$$\hat{\omega}(\nabla') - \hat{\omega}(\nabla) = a(\hat{\omega}(\nabla', \nabla)) .$$

Problem 3.39. Prove this Lemma.

Theorem 3.40. An integral characteristic form of degree $n$ admits a unique differential refinement.

Proof. We can assume that $n$ is even. Let $\omega$ be an integral characteristic form and $\omega^Z$ be its unique integral refinement by Theorem 2.117. Let $E \rightarrow M$ be a vector bundle with connection $\nabla$. Assume that $n = \dim(M)$. Then we choose an $n + 1$-connected map $u : N \rightarrow BU$ classifying $F \rightarrow M$ and a connection $\nabla^F$, a map $f : M \rightarrow N$ and an isomorphism $E \cong f^*F$. Since $H^{n-1}_{{\text{dR}}} (N; \mathbb{C}) = 0$ we are forced to define $\hat{\omega}(\nabla^F) \in \hat{H}^n_{{\text{Del}}}(M; \mathbb{Z})$ uniquely such that $R(\hat{\omega}(\nabla^F)) = \omega(\nabla^F)$ and $I(\hat{\omega}(\nabla^F)) = \omega^Z(F)$. By naturality we are forced to define $\hat{\omega}(f^*\nabla^F) = f^*\hat{\omega}(\nabla^F)$. By Lemma 3.38 we are forced to define

$$\hat{\omega}(\nabla) = f^*\hat{\omega}(\nabla^F) + a(\hat{\omega}(\nabla, f^*\nabla^F)) .$$

This already shows the uniqueness clause. It remains to show that $\hat{\omega}$ is well-defined and natural. We argue as in the proof of Theorem 2.117 using the same notation. We must show that

$$f^*\hat{\omega}(\nabla^F) + a(\hat{\omega}(\nabla, f^*\nabla^F)) - f^*\hat{\omega}(\nabla^F) - a(\hat{\omega}(\nabla, f^*\nabla^F)) = 0 .$$

We have

$$\hat{\omega}(\nabla^F) = g^*\hat{\omega}(\nabla'^F) + a(\hat{\omega}(\nabla^F, g^*\nabla'^F))$$

and

$$\hat{\omega}(\nabla'^F) = g'^*\hat{\omega}(\nabla'^F) + a(\hat{\omega}(\nabla'^F, g'^*\nabla'^F)) .$$

Hence we must see that

$$f^*g^*\hat{\omega}(\nabla'^F) + f^*a(\hat{\omega}(\nabla^F, g^*\nabla'^F)) - f^*g'^*\hat{\omega}(\nabla'^F) - f^*a(\hat{\omega}(\nabla^F, g'^*\nabla'^F)) + a(\hat{\omega}(\nabla, f^*\nabla^F)) - a(\hat{\omega}(\nabla, f^*\nabla^F)) = 0 .$$

This reduces to

$$f^*g^*\hat{\omega}(\nabla'^F) - f^*g'^*\hat{\omega}(\nabla'^F) - \hat{\omega}(f^*g^*\nabla'^F, f^*g'^*\nabla'^F) = 0$$

which holds true by the homotopy formula, Proposition 3.28 and the additivity 5 of the transgression.

Naturality of $\hat{\omega}$ is now easy and left to the reader. 

Example 3.41. In these examples we discuss some general properties of differential refinements of integral characteristic forms.

We consider the differential refinement $\hat{\omega}$ of an integral characteristic form $\omega$ of degree $n$. Let $Z$ be a compact oriented $n$-manifold with boundary $M$ and $E \rightarrow Z$ be a complex vector bundle with connection $\nabla$. 

43
Problem 3.42. Show that
\[ \text{ev}(\hat{\omega}(\nabla_M)) = \int_Z \omega(\nabla) \in \mathbb{C}/\mathbb{Z}. \]

Proof. Since \( H^n(Z; \mathbb{Z}) = 0 \) we have \( \hat{\omega}(\nabla) = a(\alpha) \) for some form \( \alpha \in \Omega^{n-1}(Z, \mathbb{C}) \). Note that \( d\alpha = \omega(\nabla) \). It follows by Stoke’s theorem that
\[ \text{ev}(\hat{\omega}(\nabla_M)) = \left[ \int_M \alpha \right] = \left[ \int_Z d\alpha \right] = \left[ \int_Z \omega(\nabla) \right]. \]

Problem 3.43. If \( \omega \) has degree \( n \geq 1 \), \( E \) is trivialized, and \( \nabla \) is the trivial connection, then \( \hat{\omega}(\nabla) = 0 \).

Proof. Indeed, we can assume that \( n \geq 2 \). A trivial bundle with trivial connection can be obtained as pull-back from a point. We now use that \( \hat{H}^2_{Del}(pt; \mathbb{Z}) = 0 \).

Problem 3.44. Let \( E \to M \) be a \( k \)-dimensional vector bundle with connection \( \nabla \). Show \( \hat{c}_n(\nabla) = 0 \) for all \( n > k \).

Proof. One can show by adapting the proof of Theorem 3.40 that an integral characteristic form for \( k \)-dimensional vector bundles has a unique differential refinement. Since for \( n > k \) the restriction of \( c_n \) to \( k \)-dimensional bundles vanishes, so does its differential refinement. \( \square \)

Let \( E \to M \) be a vector bundle, \( \nabla \) a connection on \( E \) and \( h \) be a hermitean metric.

Problem 3.45. Show that \( \hat{c}_n(\nabla) = \hat{c}_n(\nabla^*) \).

Proof. Verify this one the level of curvatures and then use uniqueness of differential extensions. \( \square \)

Let \( E \to M \) be a vector bundle with connection \( \nabla^E \) and \( (\det(E), \nabla^{\det(E)}) \in \text{Line}_\nabla(M) \) be its determinant.

Problem 3.46. Show that \( \hat{c}_1(\nabla^E) = \hat{c}_1(\nabla^{\det(E)}) \).

Proof. Use the uniqueness of differential refinements. \( \square \)

Example 3.47. In the following examples we consider \( \hat{c}_1 \) and the classification of line bundles with connection using \( \hat{H}^2_{Del}(\ldots, \mathbb{Z}) \).

We have (see Exercise 3.32) an isomorphisms
\[ \hat{H}^2_{Del}(S^1; \mathbb{Z}) \cong H^1(S^1; \mathbb{C}/\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}. \]
We consider the connection $\nabla = d + \alpha$ on the trivial one-dimensional bundle on $S^1$ where $\alpha \in \Omega^1(S^1; \mathbb{C})$. Then we have

$$\hat{c}_1(\nabla) = \left[-\frac{1}{2\pi i} \int_{S^1} \alpha \right] \in \mathbb{C}/\mathbb{Z}.$$  

We have by 3.43 and 3.38

$$c_1(\nabla) = a(\hat{c}_1(\nabla, d)).$$  

We now use 2.64.

**Problem 3.48.** Conclude that for a line bundle with connection $(L, \nabla^L)$ on a manifold $M$ we have

$$\text{ev}(\hat{c}_1(\nabla))(\gamma) = \text{hol}_\nabla(\gamma), \quad \gamma \in M^{S^1},$$

where $\text{ev}$ is the evaluation map found in 3.33.

Let $\text{Line}_\nabla(M)$ be the group of isomorphism classes of line bundles with connection under the tensor product operation.

**Problem 3.49.** Verify the existence of inverses in $\text{Line}_\nabla$. Show that

$$\hat{c}_1 : \text{Line}_\nabla(M) \to \hat{H}^2_{\text{Del}}(M; \mathbb{Z})$$

is a natural isomorphism.

**Proof.** Note that $BU(1)$ classifies line bundles. We know that $BU(1) \cong K(\mathbb{Z}, 2)$ and $H^*(BU(1); \mathbb{Z}) \cong \mathbb{Z}[c^2_1]$. Here the universal first Chern class $c^2_1$ is such that if $L \to M$ is classified by $l : M \to BU(1)$, then $c^2_1(L) = l^*c^2_1$. Every class $x \in H^2(M; \mathbb{Z})$ can be written as $x = l^*c^2_1$ for some map $l : M \to BU(1)$ and therefore is the first Chern class of a line bundle. Furthermore, the tensor product of line bundles induces the $h$-space structure on $BU(1)$ and $c^2_1$ is primitive. This has the effect that $c^2_1(L \otimes L') = c^2_1(L) + c^2_1(L')$.

Given $(L, \nabla^L) \in \text{Line}_\nabla(M)$ we choose a bundle $H$ such that $L \otimes H^{-1}$ is trivializable, e.g. such that $c^2_1(H) = -c^2_1(L)$.

Let $\nabla^H$ be any connection on $H$. Then we can find an $\alpha \in \Omega^1(M; \mathbb{C})$ such that $c_1(\nabla^L) + c_1(\nabla^H) = d\alpha$. Then $(H, \nabla^H + 2\pi i\alpha)$ is the inverse of $(L, \nabla^L)$.

Note that $\hat{c}_1(L, \nabla^L) \in \hat{H}^2_{\text{Del}}(M; \mathbb{Z})$ is characterized completely by its holonomy function $\text{ev}(\hat{c}_1(L, \nabla^L)) \in C^\infty(M^{S^1}; \mathbb{C}/\mathbb{Z})$ (see Exercise 3.33). Since the holonomy of $(L, \nabla^L) \otimes (H, \nabla^H)$ is the product of holonomies of the factors we see that $\hat{c}_1$ is additive.

Assume that $\hat{c}_1(L, \nabla^L) = 0$. Then $(L, \nabla)$ has trivial holonomy along every path and thus can be trivialized (including the connection).

On the other hand, let $x \in \hat{H}^2_{\text{Del}}(M; \mathbb{Z})$ be given. Then we choose a line bundle $(L, \nabla^L)$ such that $\hat{c}_1(L) = I(x)$. We can further adjust $\nabla$ such that $R^\nabla = -2\pi i R(x)$. Then $x - \hat{c}_1(L, \nabla) = a(\alpha)$ for some $\alpha \in \Omega^1(M; \mathbb{C})$. We consider the connection $\nabla' := d - 2\pi i \alpha$ on the trivial bundle $L^{\text{triv}} \to M$. Then $\hat{c}_1((L, \nabla) \otimes (L^{\text{triv}}, \nabla')) = x$. \hfill $\square$

Let $\mathcal{U}$ be a covering of $M$. Recall the definition 3.16 of the complex of sheaves $\mathcal{K}(1)$.
Problem 3.50. Show that a Čech cocycle \( c \in C^1(U, K(1)) \) can naturally be identified with the glueing data for a line bundle with connection. Use this to construct the isomorphism

\[
\hat{c}_1 : \text{Line}_\nabla(M) \cong H^1(M; K(1))
\]

explicitly on the level of Čech cohomology. Verify compatibility with the isomorphism 3.36.

Proof. Note that

\[
K(1) : \mathbb{C}^* \xrightarrow{d \log} \Omega^1 \to 0.
\]

Let \( \mathcal{U} := (U_\alpha) \) be a covering. Then a one-cocycle is given by \( x = (\omega_\alpha, g_{\alpha,\beta}) \), where \( \omega_\alpha \in \Omega^1(U_\alpha, \mathbb{C}) \) and \( g_{\alpha,\beta} \in C^\infty(U_\alpha \cap U_\beta; \mathbb{C}^*) \). The relation \( \delta x = 0 \) is equivalent to

\[
\omega_\beta - \omega_\alpha = d \log g_{\alpha,\beta} \text{ on } U_\alpha \cap U_\beta \text{ and } g_{\alpha,\beta} g_{\beta,\gamma} = g_{\alpha,\gamma} \text{ on the triple intersections } U_\alpha \cap U_\beta \cap U_\gamma.
\]

This is exactly the cocycle condition for a line bundle locally trivialized by sections \( s_\alpha \) on \( U_\alpha \) such that \( s_\alpha = g_{\alpha,\beta} s_\beta \) on \( U_\alpha \cap U_\beta \) and a connection such that \( \nabla \log s_\alpha = \omega_\alpha \).

This has a higher-degree analog. Čech cocycles for classes in \( H^2(M; K(2)) \cong \hat{H}^3_{\text{Del}}(M; \mathbb{Z}) \) correspond to Hitchin’s descent data for geometric gerbes with band \( \mathbb{C}^* \). The group \( \hat{H}^3_{\text{Del}}(M; \mathbb{Z}) \) classifies isomorphisms classes of geometric gerbes. See [Hit01] and [Bry08] for more information.

Example 3.51. In the following examples we generalize the Chern-Simons invariants from relative to absolute invariants.

Let \( \nabla \) be a flat connection on a trivializable bundle \( E \to M \). Then

\[
\hat{\omega}(\nabla) \in \frac{H^{n-1}(M; \mathbb{C})}{\text{im}(\epsilon_C)} = \ker(R) \cap \ker(I).
\]

With this identification \( \hat{\omega}(\nabla) \) coincides with the Chern-Simons invariant of the flat connection \( \nabla \) introduced in Definition 2.91. Indeed, if \( \nabla^{\text{triv}} \) is a trivial connection, then \( \hat{\omega}(\nabla) = a(\hat{\omega}(\nabla, \nabla^{\text{triv}})) \) and \( \hat{\omega}(\nabla) = \hat{\omega}(\nabla, \nabla^{\text{triv}}) \). More generally, if \( \nabla \) is just flat, then we get

\[
\hat{\omega}(\nabla) \in H^{n-1}(M; \mathbb{C}/\mathbb{Z})
\]

This is the generalization of the Chern-Simons invariant dropping the trivializability condition.

Let \( \nabla \) on \( E \to M \) have finite holonomy.

Problem 3.52. Show that \( \hat{\omega}(\nabla) \in H^{n-1}(M; \mathbb{C}/\mathbb{Z}) \) is a torsion class (determine and fill in the missing details of the argument below).

Proof. There exists a reduction of structure groups of \( E \) to a finite group \( H \). Let \( f : M \to BH \) a classifying map so that \( E \cong f^*F \) for some bundle \( F \to BH \). Since higher-degree rational cohomology of \( BH \) is trivial we have an isomorphism

\[
B : H^{n-1}(BH; \mathbb{C}/\mathbb{Z}) \cong H^n(BH; \mathbb{Z})
\]

46
from the Bockstein sequence. We now observe that
\[ \hat{\omega}(\nabla) = f^* B^{-1}(\omega(F)) . \]

\[ \square \]

**Example 3.53.** We generalize the theory developed in 2.93. Let \( \omega \) be an integral characteristic form of degree \( n \). We consider a space \( B \) with a flat bundle \( V \to B \) (a complex vector bundle with structure group \( GL(n, \mathbb{C}) \)). If \( f : M \to B \) is a map, then \( f^* V \to M \) becomes a complex vector bundle with a flat connection \( \nabla \). If \( M \) is closed, oriented, \( n - 1 \)-dimensional, then we define

\[ cs^V_\omega(f) := \hat{\omega}(\nabla) \in \hat{H}^n_{Del}(M; \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z} . \]

**Problem 3.54.** Show that \( cs^V_\omega(f) \) only depends on the oriented bordism class of \( f \) and that we get a homomorphism

\[ cs^V_\omega : MSO_{n-1}(B) \to \mathbb{C}/\mathbb{Z} . \]

**Proof.** Use 3.42 \( \square \)

**Lemma 3.55.** There exists a class \( u \in H^{n-1}(B; \mathbb{C}/\mathbb{Z}) \cong \text{Hom}(H_{n-1}(B; \mathbb{Z}); \mathbb{C}/\mathbb{Z}) \) such that

\[ cs^V_\omega([f : M \to B]) = u(f_*[M]) . \]

In particular, \( cs^V_\omega \) factorizes over the natural transformation \( \kappa : MSO_{n-1}(B) \to H_{n-1}(B; \mathbb{Z}) \).

**Proof.** Let \( g : \tilde{B} \to B \) be an \( n + 1 \)-connected approximation of \( B \) and \( \tilde{V} := g^*V \). Then we have isomorphisms \( g_* : MSO_{n-1}(\tilde{B}) \to MSO_{n-1}(B) \) and \( g^* : H^{n-1}(B; \mathbb{C}/\mathbb{Z}) \to H^{n-1}(B; \mathbb{C}/\mathbb{Z}) \), and

\[ cs^V_\omega \circ g_* = cs^\tilde{V}_\omega . \]

From now on we can assume that \( B \) is a smooth manifold and \( V \) has a flat connection \( \nabla^V \). We have \( \hat{\omega}(\nabla^V) \in H^{n-1}(B; \mathbb{C}/\mathbb{Z}) \subset \hat{H}^n_{Del}(M; \mathbb{Z}) \) and

\[ cs^V_\omega(f) = \text{ev}(f^* \hat{\omega}(\nabla^V)) = \hat{\omega}(\nabla^V)(f_*[M]) . \]

\[ \square \]

On \( B\mathbb{Z}/k\mathbb{Z} \) we consider the flat line bundle \( V \) given by the character \( \mathbb{Z}/k\mathbb{Z} \to U(1), [1] \mapsto \exp(2\pi ik^{-1}) \). Let \( L^{2n-1} := S^{2n-1}/\mu_k \), where the group of \( k \)th roots of unity \( \mu_k \) acts by multiplication on \( S^{2n-1} \subset \mathbb{C}^{2n} \). We consider the canonical class \([f : L^{2n-1} \to B\mathbb{Z}/k\mathbb{Z}] \in MSO_{2n-1}(B\mathbb{Z}/k\mathbb{Z}) \)

**Problem 3.56.** Calculate \( cs^{V_1}_{C_1}(f) \)!
Example 3.57. Let \((M, g)\) be a closed oriented connected Riemannian three-manifold. Then we can form the class \(-\hat{c}_2(\nabla) \in \hat{H}_{Del}^4(M; \mathbb{Z})\), where \(\nabla\) is the Levi-Civita connection on \(TM \otimes \mathbb{C}\). It corresponds to the Chern-Simons invariant \(\text{CS}(M, g)\) (Definition 2.110) under the identification \(\hat{H}_{Del}^4(M; \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}\) (Example 3.35).

Problem 3.58. Show this assertion.

Proof. Use 3.42

We consider the Hopf fibration \(S^7 \to S^4\). We realize \(S^7\) as the unit sphere in \(H^2 \cong \mathbb{R}^8\). The group of unit quaternions \(Sp(1)\) acts on \(S^7\) by right multiplication. This turns the Hopf fibration into a \(Sp(1)\)-principal bundle. We define a connection such that the horizontal distribution is the orthogonal complement of the vertical bundle in the round geometry of \(S^7\). Let \(E \to S^4\) be the complex vector bundle with induced connection \(\nabla\) associated to the representation of \(Sp(1)\) on \(\mathbb{C}^2 \cong H\) by left multiplication. We consider the class

\[
\hat{c}_2(\nabla) \in \hat{H}_{Del}^4(S^4; \mathbb{Z}) .
\]

For \(r > 0\) let \(S(r) \subset S^4\) be the distance sphere centered at the north pole.

Problem 3.59. Calculate

\[
\text{ev}(\hat{c}_2(\nabla)|_{S(r)}) \in \mathbb{C}/\mathbb{Z} .
\]

Proof. We first show that \(c_2(\nabla) = \text{vol}_{S^4}\) (the normalized volume form). Then by 3.42

\[
\text{ev}(\hat{c}_2(\nabla)|_{S(r)}) = [\text{vol}(B(r))] \in \mathbb{C}/\mathbb{Z} .
\]

Example 3.60.

Definition 3.61. If \(\nabla\) is a connection on a real vector bundle, then we define the differential lift of the Pontrjagin form by

\[
\hat{p}_i(\nabla) := (-1)^k \hat{c}_2(\nabla \otimes \mathbb{C}) \in \hat{H}_{Del}^{4n}(M; \mathbb{Z}) .
\]

If \((M, g)\) is a Riemannian manifold and \(\nabla\) is the Levi-Civita connection on \(TM\), then we set \(\hat{p}_k(g) := \hat{p}_k(\nabla)\). We can now extend the Definition of the Chern-Simons invariant to the non-bounding case.

Definition 3.62. If \((M, g)\) is a closed oriented connected \(4n-1\)-dimensional Riemannian manifold, then we define

\[
\text{CS}(M, g) := \hat{p}_n(g) \in \hat{H}_{Del}^{4n}(M; \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z} .
\]

Problem 3.63. Verify, that this extends 2.114.

Lemma 3.64. \(\text{CS}(M, g)\) is a conformal invariant. It vanishes e.g. if \(M\) bounds (with product structure) a locally conformally flat manifold.
Proof. We consider a metric on \( \mathbb{R} \times M \) of the form \( \tilde{g} = f(dt^2 + g) \), where \( f \in C^\infty(\mathbb{R} \times M) \). Then we have \( p_n(\tilde{g}) = 0 \). It follows from the homotopy formula that \( \hat{p}_n(\tilde{g})_{|\{t\} \times M} \) is independent of \( t \).

**Problem 3.65.** Let \( \mathbb{Z}/p\mathbb{Z} \) act on \( S^{4n-1} \subset \mathbb{C}^n \) diagonally by \( [1] \mapsto (\zeta^{q_1}, \ldots, \zeta^{q_{2n}}) \), where \( \zeta = \exp(2\pi ip^{-1}) \) is a primitive root of unity and the numbers \( g.g.T(q_1, \ldots, q_{2n}, p) = 1 \) and set \( L_{p}^{4n-1}(q_1, \ldots, q_{2n}) := S^{4n-1}/(\mathbb{Z}/p\mathbb{Z}) \) with the induced round metric \( g \). Calculate \( \text{CS}(L_{p}^{4n-1}(q_1, \ldots, q_{2n}), g) \in \mathbb{C}/\mathbb{Z} \).

**Example 3.66.** If \( V \to M \) is a real vector bundle with connection \( \nabla \), then \( c_{2n+1}(\nabla \otimes V) = 0 \) for all \( n \geq 0 \). It follows that \( \hat{c}_{2n+1}(\nabla \otimes \mathbb{C}) \in H^{4n+1}(M; \mathbb{C}/\mathbb{Z}) \subseteq \hat{H}_{D}^{4n+2}(M; \mathbb{Z}) \).

In particular, this class is independent of the connection and only depends on the real bundle \( V \).

**Problem 3.67.** Calculate \( \beta(\hat{c}_1(\nabla \otimes C)) \in H^2(\mathbb{R}P^n; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \) for the canonical bundle \( (V, \nabla) \) of \( \mathbb{R}P^n \), where \( \beta : H^1(\mathbb{R}P^n; \mathbb{Z}/2\mathbb{Z}) \to H^2(\mathbb{R}P^n; \mathbb{Z}) \) is the Bockstein.

**Proof.** \( \beta(\hat{c}_1(\nabla \otimes C)) \) is the generator. Indeed, we have an embedding \( \mathbb{R}P^n \to \mathbb{C}P^n \) such that \( V \otimes \mathbb{C} \) is the restriction of the canonical bundle \( L \to \mathbb{C}P^n \). We have \( \beta(\hat{c}_1(\nabla \otimes C)) = \hat{c}_1(L)|_{\mathbb{R}P^n} \). The right-hand side is known to restrict to a generator. \( \square \)

We have a natural map \( i : H^*(M; \mathbb{Z}/2\mathbb{Z}) \to H^*(M; \mathbb{C}/\mathbb{Z}) \to H_D^*(M; \mathbb{Z}) \) induced by \( \mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{C}/\mathbb{Z} \), \( [1] \to [\frac{1}{2}] \). With this notation \( \hat{c}_1(\nabla \otimes \mathbb{C}) = i(w_1(V)) \).

**Example 3.68.** We consider the bottom of the Whitehead tower of \( BO \): 

\[
\begin{array}{c}
BFivebrane = BO(9) \\
\downarrow \\
BString = BO(8) \xrightarrow{\mathbb{Z}/6} K(\mathbb{Z}, 8) \\
\downarrow \pi_8 \\
BSpin = BO(4) \xrightarrow{\mathbb{Z}/4} K(\mathbb{Z}, 4) \\
\downarrow \pi_4 \\
BSO = BO(2) \xrightarrow{w_2} K(\mathbb{Z}/2\mathbb{Z}, 2) \\
\downarrow w_1 \\
BO \xrightarrow{w_1} K(\mathbb{Z}/2\mathbb{Z}, 1)
\end{array}
\]

In order to understand this discuss the following topological exercise.
Problem 3.69. We have the Pontrjagin classes \( p_i \in H^4(BO; \mathbb{Z}) \). Let \( \pi_k : BO(k) \to BO \) be the projection. Observe that there are unique classes \( \frac{p_1}{2} \in H^4(BSpin; \mathbb{Z}) \) and \( \frac{p_2}{6} \in H^8(BString; \mathbb{Z}) \) (these symbols are names!) such that \( \pi_k^*p_1 = 2\frac{p_1}{2} \) and \( \pi_k^*p_2 = 6\frac{p_2}{6} \).

**Definition 3.70.** A (stable) \( O(k-1) \)-structure on a real vector bundle \( V \to M \) is a lift of the classifying map \( v : M \to BO \) of the stabilization of \( V \) to \( \tilde{v} : M \to BO(k) \).

We use special names (oriented, spin string and fivebrane) in the first few cases.

Problem 3.71. 1. Let \((V, \nabla)\) be a real vector bundle with spin structure. Show that \( \hat{c}_1(\nabla \otimes \mathbb{C}) = 0 \).

2. Show that the characteristic form \( \frac{1}{2}p_1 \) for real spin vector bundles with connection is integral and has unique differential refinement \( \hat{p}_2 \). Show that \( \hat{p}_2 \) is additive under direct sum.

3. Show that the characteristic form \( \frac{1}{6}p_2 \) for real string vector bundles with connection is integral and has unique differential refinement \( \hat{p}_6 \). Show that \( \hat{p}_6 \) is additive under direct sum.

**Example 3.72.** We refer to \([KN96]\) for more details on the following material. Let \( G \) be a Lie group with Lie algebra denoted by \( g \) and \( \pi : P \to M \) be a \( G \)-principal bundle. By definition this means that \( P \) admits a free right \( G \)-action such that \( M \cong P/G \) and \( \pi \) has local sections. A connection on \( P \) is a \( G \)-invariant decomposition \( TP \cong T^v\pi \oplus T^h\pi \), where \( T^v\pi := \ker(d\pi) \) is the vertical bundle generated by the fundamental vector fields \( A^\sharp \in \mathcal{X}(P), X \in g \), of the \( G \)-action and \( T^h\pi \) is a complementary bundle. Equivalently, a connection determines and is determined through

\[
T^h\pi = \ker(\omega)
\]

by a form \( \omega \in \Omega^1(P) \otimes g \) which satisfies

1. \( R^g_\omega = \text{Ad}(g)^{-1}\omega \)
2. \( \omega(A^\sharp(p)) = A \) for all \( p \in P \) and \( A \in g \).

The curvature of the connection \( \omega \) is defined by

\[
R^\omega := d\omega + \frac{1}{2}[\omega, \omega]
\]

We define the adjoint bundle \( \text{Ad}(P) := P \times_{G, \text{Ad}} g \). Then we can consider the curvature as a horizontal invariant two-forms

\[
R^\omega \in (\Omega^2(P) \otimes g)^{\text{hor}, G} \cong \Omega^2(M, \text{Ad}(P))
\]

It satisfies the Bianchi identity

\[
\nabla^\omega R^\omega := d\Omega + [\omega, \Omega] = 0
\]
Let us now assume that the Lie group $G$ has finitely many components. By $I^*(G) \subseteq S^*(g^*_C)$ we denote the algebra of $G$-invariant complex-valued symmetric polynomials on $g$. An element $\phi \in I^k(G)$ induces a characteristic form

$$cw(\phi) : (P \to M, \omega) \mapsto cw(\phi)(\omega) \in \Omega^k(M; \mathbb{C})$$

for $G$-principal bundles with connection by

$$cw(\phi)(\omega) = \phi\left(\frac{R^\omega}{2\pi i}\right),$$

where we apply $\phi$ fibrewise.

**Problem 3.73.** Show the Bianchi-identity $\nabla^\omega R^\omega = 0$ and use it in order to verify that $cw(\phi)(\omega)$ is closed.

We get a characteristic class for $G$-principal bundles

$$cw(\phi)(P) := [cw(\phi)(\omega)] \in H^{2k}(M; \mathbb{C})$$

(for some choice of connection $\omega$ on $P$) and therefore a universal class

$$cw(\phi) \in H^{2k}(BG; \mathbb{C}).$$

**Problem 3.74.** Use transgression in order to show that $cw(\phi)$ is well-defined.

**Definition 3.75.** The homomorphism of graded rings

$$cw : I^*(G) \to H^{2*}(BG; \mathbb{C})$$

constructed above is called the Weyl homomorphism.

In fact, for compact $G$ the map $cw$ is an isomorphism by a theorem of H. Cartan.

**Problem 3.76.** Prove the theorem of Cartan that for compact Lie groups the Weyl homomorphism is an isomorphism.

*Proof.* Here are the main steps. Compare e.g. [Dup78] for a complete argument.

Let $T \subseteq G$ be a maximal torus and $W := W(G, T)$ be the Weyl group.

Show that the restriction induces an isomorphism

$$I^*(G) \simto S^*(t_C^*)^W.$$  

The main point here is that for every $A \in g$ there exists $g \in G$ such that $\text{Ad}(g)(A) \in t$. Further we identify

$$S^*(t_C^*)^W \simto H^{2*}(BT; \mathbb{C})^W$$

by an explicit calculation.
Then we have the diagram

\[
\begin{array}{c}
I(G) \xrightarrow{\text{cw}} H(BG; \mathbb{Z}) \\
\cong \downarrow \cong \\
S^*(t^*_C)W \xrightarrow{\text{cw}} H(BT; \mathbb{Z})
\end{array}
\]

It suffices to show that \(i^* : H(BG; \mathbb{Z}) \to H(BT; \mathbb{Z})\) is injective. We have a fibration

\[B/T \to BT \to BG.\]

It suffices to find an element \(\alpha \in H^{\dim(G/T)}(BT; \mathbb{C})\) such that \(f!(\alpha) = 1\). Indeed, then we have

\[\beta = f!(\alpha \cup i^*\beta), \quad \forall \beta \in H^*(BG; \mathbb{C}).\]

Let \(\Delta^+(g, t) \subset \text{Hom}(T, \mathbb{C}^*)\) be set of positive roots. For every \(\alpha \in \Delta^+(g, t)\) let \(L_\alpha \to BT\) be the associated Line bundle. One checks that

\[\alpha := c \prod_{\alpha \in \Delta^+(g, t)} c_1(L_\alpha)\]

doess the job for an appropriate choice of the normalization \(c\).

As before we say that \(\phi \in I^k(G)\) is integral if there exists a class \(z \in H^{2k}(BG; \mathbb{Z})\) such that \(\epsilon_C(z) = \text{cw}(\phi)\). Unlike the case \(G = U(n)\) we can not expect that \(z\) is uniquely determined by \(\phi\). We define the graded ring \(\tilde{I}^*(G)\) as the pull-back

\[
\begin{array}{c}
\tilde{I}^*(G) \xrightarrow{\text{cw}} I^*(G) \\
\downarrow \downarrow \\
H^{2*}(BG; \mathbb{Z}) \xrightarrow{\epsilon_C} H^{2*}(BG; \mathbb{C})
\end{array}
\]

For \(\tilde{\phi} \in \tilde{I}^*(G)\) we let \(\phi \in I^*(G)\) and \(\phi_Z \in H^{2k}(BG; \mathbb{Z})\) denote the underlying invariant polynomial and integral cohomology class. The following theorem is due to Cheeger-Simons [CS85, Thm. 2.2].

**Theorem 3.77.** Assume that \(G\) is a Lie group with finitely many components. For every \(\tilde{\phi} \in \tilde{I}^k(G)\) there exists a unique \(\tilde{H}^k_{\text{Del}}(\ldots; \mathbb{Z})\)-valued characteristic class for \(G\)-principal bundles with connection

\[\tilde{c}_W(\tilde{\phi} : (P \to M, \omega) \mapsto \tilde{c}_W(\tilde{\phi})(\omega) \in \tilde{H}^k_{\text{Del}}(M; \mathbb{Z})\]

such that

\[R(\tilde{c}_W(\tilde{\phi})(\omega)) = \text{cw}(\phi)(\omega), \quad I(\tilde{c}_W(\tilde{\phi})(\omega)) = \phi_Z(P).\]
Proof. Observe that $H^{\text{odd}}(BG; \mathbb{C}) = 0$. We can now proceed as in the proof of Theorem 3.40. Let $\tilde{\phi} \in \tilde{I}^k(G)$. If $N \to BG$ is a $2k + 1$-connected approximation, then $H^{2k-1}(N; \mathbb{C}) = 0$. If $(P \to N, \omega)$ is a $G$-principal bundle with connection, then the class
\[
\hat{c}_w(\tilde{\phi})(\omega) \in \hat{H}^{2k}_{\text{Del}}(M; \mathbb{Z})
\]
is uniquely determined by
\[
R(\hat{c}_w(\tilde{\phi})(\omega)) = c_w(\phi)(\omega), \quad I(\hat{c}_w(\tilde{\phi})(\omega)) = \tilde{\phi}_Z(P).
\]

\[ \square \]

**Problem 3.78.** Find out where we have used the assumption that $G$ is finitely many connected components.

**Proof.** We use that $H^{\text{odd}}(BG; \mathbb{C}) = 0$. This is not true without this assumption. For example, we have
\[
H^1(BGL(n; \mathbb{Q}^\delta); \mathbb{C}) \cong \text{Hom}(\mathbb{Q}^*, \mathbb{C}) \cong \prod_{p \in \mathbb{N}, \text{prime}} \mathbb{C}.
\]
Assume now that $G$ is a Lie group with finitely many components. Since we consider the cohomology with coefficients in a rational vector space we can reduce to the case that $G$ is connected. Now one has to go through the structure theory of connected Lie groups which says that it can be reduced to product of simple groups by iteratively factoring out normal abelian subgroups. A simple group can further be reduced to its maximal compact Lie group. For the compact group we apply the surjectivity part of Cartan’s theorem, or rather its substep, the injectivity of $H^*(BG; \mathbb{C}) \to H^*(BT; \mathbb{C})$ for the restriction to a maximal torus. Vanishing of the odd cohomology is then preserved if one argues backwards through the previous reduction steps. \[ \square \]

**Problem 3.79.** View the assertions in 3.71 as special cases of Theorem 3.77.

**Example 3.80.** Let $\tilde{\phi} = (\phi, \phi_Z) \in \tilde{I}^*(G)$ and assume that $\omega$ has finite holonomy $H \subseteq G$. The bundle $P \to M$ has an $H$-reduction classified by a map $h : M \to BH$. Consider further the map
\[
B : H^*(BG; \mathbb{Z}) \to H^*(BH; \mathbb{Z}) \cong H^{*-1}(BH; \mathbb{C}/\mathbb{Z}).
\]
The following is [CS85, Prop. 2.10].

**Problem 3.81.** Show that $\hat{c}_w(\tilde{\phi})(\omega) = h^*B(\phi_Z)$.

We consider Lie group $G$ with finitely many component and its discrete version $G^\delta$.

**Problem 3.82.** Show that for every $\tilde{\phi} \in \tilde{I}^k(G)$ there exists an element $\phi \in H^{2k-1}(BG^\delta; \mathbb{C}/\mathbb{Z})$ which represents the universal class $\hat{c}_w(\tilde{\phi})$ for $G$-bundles with flat connection.
Example 3.83. In this example we study the Euler class. We identify
\[ \text{so}(2n)_C \cong \Lambda^2 \mathbb{C}, \quad X \cong \frac{1}{2} \sum_{i,j=1}^{2n} \langle e_i, X e_j \rangle e^i \wedge e^j \]
and define the Pfaffian \( \text{Pf} \in S^n(\text{so}(2n)_C^*) \) by the identity
\[ \text{Pf}(X) \text{vol}_{\mathbb{R}^n} = X \wedge \cdots \wedge X \quad \forall X \in \text{so}(2n)_C. \]
We further introduce the normalized version \( \bar{\text{Pf}}(A) := \text{Pf}(-iA) \). (27)

Let \( S(\xi_m) \to BSO(m) \) be the unit sphere bundle of the universal bundle \( \xi_m \to BSO(m) \).
We consider the associated Leray-Serre spectral sequence. The differential \( d_2 : E_2^{0,m-1} \to E_2^{m,0} \) is a map
\[ d : H^{m-1}(S^{m-1}; \mathbb{Z}) \to H^m(BSO(m); \mathbb{Z}), \]
and we define the universal Euler class \( \chi \in H^m(BSO(m); \mathbb{Z}) \) by \( \chi := d_2(\text{or}_{S^{m-1}}) \).

Problem 3.84. 1. Show that \( \bar{\text{Pf}} \in I^n(SO(2n)) \).

2. Show that \( 2\chi = 0 \) if \( m \) is odd.

3. Show that \( \text{cw}(\bar{\text{Pf}}) = \epsilon_C(\chi) \) if \( m \) is even.

Proof. For 2. use the fibrewise antipodal map.
For 3. we argue as follows. The map \( U(n) \to SO(2n) \) induces an isomorphism of maximal tori and therefore an injection \( i^* : H^*(BSO(2n); \mathbb{C}) \to H^*(BU(n); \mathbb{C}) \). It therefore suffices to show the identity as an identity of characteristic classes for \( n \)-dimensional complex vector bundles. This has been done in Example 2.88.

We define
\[ \bar{\text{Pf}} := \left\{ \begin{array}{ll} (\chi, \text{Pf}) & m \text{ even} \\ (\chi, 0) & m \text{ odd} \end{array} \right\} \in \tilde{I}^n(SO(m)) \]
and
\[ \hat{\chi} := \text{cw}(\bar{\text{Pf}}). \]
If \( V \to M \) is a real oriented \( m \)-dimensional vector bundle with connection \( \nabla \), then we write
\[ \hat{\chi}(\nabla) := \hat{\chi}(\omega), \]
where \( \omega \) is the corresponding connection on the oriented frame bundle \( SO(V) \to M \).

Definition 3.85. The characteristic class \( \hat{\chi} \) is the differentially refined version of the Euler class.
Let $V \to M$ be a real oriented euclidean vector bundle of dimension $m$ with a flat euclidean connection $\nabla^V$. Let $\beta$ be the fibrewise volume $m - 1$-form (induced by the metric) on the sphere bundle $\pi : S(V) \to M$ normalized to have integral one. Using the connection we extend this to a form $\beta \in \Omega^{m-1}(S(V); \mathbb{C})$. Flatness of the connection implies $d\beta = 0$. The segment of the homological Gysin sequence

$$\ldots H_0(M; \mathbb{Z}) \to H_m(S(V); \mathbb{Z}) \xrightarrow{\pi^*} H_m(M; \mathbb{Z}) \to 0$$

shows:

1. For every $x \in H_m(M, \mathbb{Z})$ we can find a preimage $y \in H_m(S(V); \mathbb{Z})$ such that $\pi^*(y) = x$.

2. The class of the evaluation $f(\nabla)(x) := \langle [\beta], y \rangle \in \mathbb{C}/\mathbb{Z}$ does not depend on the choice of the preimage.

We therefore get a class

$$f(\nabla) \in \text{Hom}(H_{m-1}(M); \mathbb{C}/\mathbb{Z}) \cong H^{m-1}(M; \mathbb{C}/\mathbb{Z}).$$

**Problem 3.86.** Show that

$$a(f) = \hat{\chi}(\nabla).$$

Assume in addition that $M$ is closed, oriented and $m - 1$-dimensional. Choose a triangulation $(\sigma_i : \Delta^{m-1} \to M)_{i \in I}$ of $M$. Let $(m_j)_{j \in J}$ be the set of vertices. For every $j \in J$ choose a unit vector $v_j \in V_{m_j}$. Let $b_i$ denote the baricenter of the simplex $\sigma_i$. Parallel transport along straight lines in $\sigma_i$ of the vectors associated to the vertices of $\sigma_i$ gives collections of unit vectors $u_i := (w_i(0), \ldots, w_i(m - 1)) \in V_{b_i}$. We assume that these collections $u_i$ are bases for all $i \in I$. The basis $u_i$ spans a geodesic simplex in the unit sphere $S(V_{b_i})$ of oriented volume $\text{vol}(u_i)$. The following is due to Cheeger-Simons [CS85, Thm. 8. 14].

**Problem 3.87.** Show that

$$\text{ev}(\hat{\chi}(V)) = \sum_{i \in I} \text{vol}(u_i)] \in \mathbb{C}/\mathbb{Z}.$$

If $(E, \nabla)$ is a complex vector bundle with connection of dimension $n$, then we let $(E|_{\mathbb{R}}, \nabla|_{\mathbb{R}})$ denote the underlying oriented real vector bundle with connection.

**Problem 3.88.** Show that for a complex vector bundle $E \to M$ with connection $\nabla$ we have $\hat{c}_n(\nabla) = \hat{\chi}(\nabla|_{\mathbb{R}})$.

**Proof.** Since we consider characteristic classes of complex vector bundles it suffices to show that the underlying curvatures coincide. We check this first in the case $n = 1$. At a fixed point $x \in M$ we can identify $E_x \cong \mathbb{C}$ and assume that $R^\nabla = i\alpha$ for some $\alpha \in \Lambda^2 T^*_x M$. Then we have

$$R^\nabla z = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}$$
with respect to the standard basis \( \{ e_1, e_2 \} \) of \( \mathbb{C} \cong \mathbb{R}^2 \). This implies at this point

\[
cw(\tilde{\Phi})\left( R^\nabla_{|R} \right) = Pf(-i \frac{R^\nabla_{|R}}{2\pi i}) = -\frac{\alpha}{2\pi} .
\]

On the other hand,

\[
c_1(\nabla) = -\frac{R^\nabla}{2\pi i} = -\frac{\alpha}{2\pi} .
\]

We extend this to the general case using diagonalization. \( \Box \)

**Example 3.89.** Let \( P \to M \) be a \( G \)-principal bundle with connection \( \omega \). Let \( \tilde{\phi} = (\phi_Z, \phi) \in \tilde{I}^n(G) \).

**Problem 3.90.** Show that there exists a natural (under pull-back of principal bundles with connection) form \( \theta(\omega) \in \Omega^{2n-1}(P; \mathbb{C}) \) such that \( d\theta(\omega) = \phi(\nabla) \). Show that

\[
\pi^* \tilde{c}_w(\tilde{\phi})(\omega) = a(\theta(\omega)) .
\]

**Proof.** Note that \( \pi^*P \to P \) is canonically trivialized and has the trivial connection \( \omega_0 \). We define \( \theta := \tilde{c}_w(\phi)(\omega, \omega_0) \) using transgression along the linear path from \( \omega_0 \) to \( \pi^*\omega \). The second assertion is a consequence of the homotopy formula 3.28. \( \Box \)

We consider \( G := SU(2) \) and \( \tilde{\phi} \in I(SU(2)) \) such that \( \phi_Z = c_2 \).

**Problem 3.91.** Give an explicit formula for \( \theta(\omega) \) in 3.90.

### 3.4 Multiplicative structure

Let \( R \in \text{Sh}_{\text{Rings}}(\text{MF}) \) be a sheaf of commutative rings. Then the sheaf cohomology \( H^*(M; R) \) becomes a graded commutative ring. Here are the basic steps to see this. The notion of sheaf of rings can be formalized using the symmetric monoidal structure on \( \text{Sh}_{\text{Ab}}(\text{MF}) \). The evaluation \( \Gamma(M; \ldots) \) is lax symmetric monoidal. Using the fact that the category of sheaves contains sufficiently many flat sheaves we see that these properties descend to the derived category \( D^{\geq 0}(\text{Sh}_{\text{Ab}}(M)) \) so that the derived functor of evaluation preserves rings, too. Differential graded algebras (abbreviated by dga’s) are ring objects in \( \text{Ch}(\text{Ab}) \) and in particular present rings in the derived category \( D^{\geq 0}(\text{Sh}_{\text{Ab}}(M)) \).

The sheaf of de Rham complexes \( \Omega_C \) is a sheaf of dga’s which resolves the sheaf of rings \( \mathbb{C}^6 \). The wedge product of forms induces a product on \( H^*_{\text{dR}}(M; \mathbb{C}) \). The de Rham isomorphism \( \text{Rham} : H^*(M; \mathbb{C}^6) \sim H^*_{\text{dR}}(M; \mathbb{C}) \) is induced by the quasi isomorphism of sheaves of dga’s \( \mathbb{C}^6 \to \Omega_C \) and is hence multiplicative. Moreover, the map \( \epsilon_C : H^*(M; \mathbb{Z}) \to H^*_{\text{dR}}(M; \mathbb{C}) \) is multiplicative since it is induced by the composition \( \mathbb{Z} \to \mathbb{C}^6 \to \Omega_C \) of maps of sheaves of dga’s. This leaves open the question why the multiplicative structures on integral cohomology defined using sheaf theory or simplicial cohomology coincide. An argument will be given in Lemma 4.111.
Definition 3.92. A product on Deligne cohomology is the datum of a graded commutative ring structure (denoted by $\cup$) on $\hat{H}^*_{\text{Del}}(M; \mathbb{Z})$ for every manifold $M$ such that

1. $f^*: \hat{H}^*_{\text{Del}}(M; \mathbb{Z}) \to \hat{H}^*_{\text{Del}}(M'; \mathbb{Z})$ is a homomorphism of rings for every smooth map $f: M' \to M$,
2. $R: \hat{H}^*_{\text{Del}}(M; \mathbb{Z}) \to \Omega^*_{\text{cl}}(M; \mathbb{C})$ is multiplicative for all $M$,
3. $I: \hat{H}^*_{\text{Del}}(M; \mathbb{Z}) \to H^*(M; \mathbb{Z})$ is multiplicative for all $M$,
4. and $a(\alpha) \cup x = a(\alpha \cup R(x))$ for all $\alpha \in \Omega^*(M; \mathbb{C})/\text{im}(d)$ and $x \in \hat{H}^*_{\text{Del}}(M; \mathbb{Z})$.

Proposition 3.93. There exists a unique product on Deligne cohomology.

Proof. We first show existence. We shall use the complexes $\mathcal{E}(p)$ defined in (23) in order to represent Deligne cohomology, see (24). We construct products $\cup: \mathcal{E}(p) \otimes \mathcal{E}(q) \to \mathcal{E}(p + q)$ (28)

by

$$x \cup y := \begin{cases} xy & \text{deg}(x) = 0 \text{ or } \text{deg}(y) = 0 \\ x \wedge (-dy) & \text{deg}(x) > 0 \text{ and } \text{deg}(y) = q > 0 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3.94. Show that this is a morphism of complexes. Furthermore verify associativity.

Proof. We let $\partial$ denote the differential of the complexes, while $d$ is the de Rham differential. deg$(x)$ denotes the degree of $x$ as an element in $\mathcal{E}(p)$, not the degree as a form. We show that the $\cup$-product is a morphism of complexes. We consider the case that $p, q > 0$. There are many cases:

1. If deg$(x) = 0$ and deg$(y) = 0$, then we have in $\Omega^0(M; \mathbb{C}) = \mathcal{E}(p + q)^1$ that $\partial(x \cup y) = xy$ and $\partial x \cup y + x \cup \partial y = xy$.
2. If $0 < \text{deg}(y) < q$, then in $\Omega^{\text{deg}(y)}(M, \mathbb{C})$ we have $\partial(x \cup y) = -xdy$ and $\partial x \cup y + x \cup \partial y = -xdy$.
3. If deg$(y) = q > 0$, then in $\Omega^q(M, \mathbb{C})$ we have $\partial(x \cup y) = -xdy$ and $\partial x \cup y + x \cup \partial y = -xdy$.
4. If deg$(x) > 0$ and $0 < \text{deg}(y) < q$, then we have $\partial(x \cup y) = 0$ and $\partial x \cup y + (-1)^{\text{deg}(x)} x \cup \partial y = 0$.
5. If deg$(x) > 0$ and deg$(y) = q > 0$, then $\partial(x \cup y) = -d(x \wedge -dy)) = dx \wedge dy$ and $\partial x \cup y + (-1)^{\text{deg}(x)} x \wedge dy = dx \wedge dy$.

Let us check associativity in some cases. We consider the product $\mathcal{E}(p) \otimes \mathcal{E}(q) \otimes \mathcal{E}(r) \to \mathcal{E}(p + q + r)$. 

57
1. If $\deg(x) > 0$ and $0 < \deg(y) < q$, and $\deg(z) = r$. Then we have $(x \cup y) \cup z = 0$ and $x \cup (y \cup z) = 0$.

2. If $\deg(x) > 0$ and $\deg(y) = q$, and $\deg(z) = r$, then $(x \cup y) \cup z = xdy \wedge dz$ and $x \cup (y \cup z) = x \wedge dy \wedge dz$. If $\deg(x) = \deg(y) = 0$, then $(x \cup y) \cup z = xyz$ and $x \cup (y \cup z) = xyz$. The remaining cases are similar.

We define $H : \mathcal{E}(p) \otimes \mathcal{E}(q) \rightarrow \mathcal{E}(p + q)[-1]$ by

$$H(x \otimes y) = \begin{cases} 0 & \deg(x) = 0 \text{ or } \deg(y) = 0 \\ (-1)^{\deg(x)} x \wedge y & \text{otherwise} \end{cases}$$

Let $s : \mathcal{E}(p) \otimes \mathcal{E}(q) \rightarrow \mathcal{E}(q) \otimes \mathcal{E}(p)$ be the symmetry.

**Problem 3.95.** Show that $H$ is a homotopy between $\cup$ and $\cup \circ s$.

**Proof.** Again we consider several cases.

1. Assume that $0 < \deg(x) < p$ and $0 < \deg(y) < q$. Then we have

$$\partial H(x \otimes y) + H(\partial(x \otimes y)) = -d(-1)^{\deg(x)}(x \wedge y) + H(-dx \otimes y + (-1)^{\deg(x)}x \otimes (-dy))$$

$$= -(-1)^{\deg(x)}dx \wedge y + x \wedge dy$$

$$+ (-1)^{\deg(x)}dx \wedge y - x \wedge dy$$

$$= 0$$

$(-1)^{\deg(x)} \deg(y) y \cup x - x \cup y = 0$

2. If $0 < \deg(x) < p$ and $\deg(y) = q > 0$, then

$$\partial H(x \otimes y) + H(\partial(x \otimes y)) = -d(-1)^{\deg(x)}(x \wedge y) + H(-dx \otimes y)$$

$$= -(-1)^{\deg(x)}dx \wedge y + x \wedge dy$$

$$+ (-1)^{\deg(x)}dx \wedge y$$

$$= x \wedge dy$$

$(-1)^{\deg(x)} \deg(y) y \cup x - x \cup y = x \wedge dy$

3. For $\deg(x) = 0 = \deg(y)$ we have

$$\partial H(x \otimes y) + H(\partial(x \otimes y)) = 0$$

$$(-1)^{\deg(x)} \deg(y) y \cup x - x \cup y = xy - xy$$

$$= 0$$
If deg($x$) = $p$ and 0 < deg($y$) < $q$ then

$$
\partial H(x \otimes y) + H(\partial(x \otimes y)) = -d(-1)^{\text{deg}(x)}(x \wedge y) = -(-1)^{\text{deg}(x)}dx \wedge y + x \wedge dy
$$

$$
= -x \wedge dy
$$

$$
= -(-1)^{\text{deg}(x)}dx \wedge y
$$

$$
(-1)^{\text{deg}(x) \text{deg}(y)}y \cup x - x \cup y = -(-1)^{\text{deg}(x) \text{deg}(y)}y \wedge dx
$$

$$
= -(-1)^{\text{deg}(x)}dx \wedge y
$$

We conclude that (28) induces a product

$$
\cup : \hat{H}^p_{\text{Del}}(M; \mathbb{Z}) \otimes \hat{H}^q_{\text{Del}}(M; \mathbb{Z}) \to \hat{H}^{p+q}_{\text{Del}}(M; \mathbb{Z})
$$

which is natural, associative and graded commutative.

We now show that this product is compatible with the structure maps. First of all we observe that $R$ is induced by the map

$$
\tilde{R} : \mathcal{E}(p) \to \Omega^p_{\mathbb{C}} ,
$$

$$
y \mapsto -dy , \text{deg}(y) = p , \quad y \mapsto 0 \text{ if deg}(y) < p .
$$

Indeed, if deg($y$) = $p$, then $((0 \oplus -dy) \oplus y) \in \mathcal{D}(p)$ is a lift, and $R$ is induced by the projection onto the $-dy$-component. We see that

$$
\tilde{R}(x \cup y) = R(x) \wedge R(y).
$$

Similarly, $I$ is induced by the projection

$$
\tilde{I} : \mathcal{E}(p) \to \mathbb{Z},
$$

$$
y \mapsto y , \text{deg}(y) = 0 , \quad y \mapsto 0 \text{ if deg}(y) > 0 .
$$

Indeed, if deg($y$) = 0, then $((y \oplus 0) \oplus 0) \in \mathcal{D}(p)$ is a lift and $I$ is induced by the projection onto the first component.

**Problem 3.96.** Show the compatibility with $a$.

We now show uniqueness. Assume that $\cup'$ is a second product. Then we consider

$$
B := \cup' - \cup : \hat{H}^p_{\text{Del}}(\ldots; \mathbb{Z}) \otimes \hat{H}^q_{\text{Del}}(\ldots; \mathbb{Z}) = \hat{H}^{p+q}(\ldots; \mathbb{Z}) .
$$

This is a bilinear natural transformation.

**Problem 3.97.** Show that $B$ factorizes over a bilinear transformation

$$
H^p(\ldots; \mathbb{Z}) \otimes H^q(\ldots; \mathbb{Z}) \to H^{n-1}(\ldots; \mathbb{C}/\mathbb{Z}) .
$$
We now argue that such a transformation is necessarily zero. A natural transformation of functors
\[ H^p(\ldots; \mathbb{Z}) \times H^q(\ldots; \mathbb{Z}) \to H^{n-1}(\ldots; \mathbb{C}/\mathbb{Z}) \]
is represented by a map of Eilenberg-Mac Lane spaces
\[ K(\mathbb{Z}, p) \times K(\mathbb{Z}, q) \to K(\mathbb{C}/\mathbb{Z}, p + q - 1) \] .
If the transformation is bilinear, then this map factorizes over
\[ K(\mathbb{Z}, p) \wedge K(\mathbb{Z}, q) \to K(\mathbb{C}/\mathbb{Z}, p + q - 1) \] .
Since this smash product \( p + q - 1 \)-connected this latter map is homotopic to a constant map. We conclude that \( B = 0 \). It follows \( \cup = \cup' \).

**Example 3.98.** We identify \( J(S^1) \cong S^1 \). Consider the Poincaré bundle \( P \to S^1 \times S^1 \) with its canonical connection \( \nabla_P \) described in 2.33.

**Problem 3.99.** Show that in \( \hat{\mathbb{H}}^2_{\text{Del}}(S^1 \times S^1; \mathbb{Z}) \)
\[ \hat{c}_1(\nabla^P) = \text{pr}_1^* \hat{e} \cup \text{pr}_2^* \hat{e} \] .

**Proof.** Define
\[ \delta := \hat{c}_1(\nabla^P) - \text{pr}_1^* \hat{e} \cup \text{pr}_2^* \hat{e} \in \hat{\mathbb{H}}^2_{\text{Del}}(S^1 \times S^1; \mathbb{Z}) \] .
We first see that \( R(\delta) = 0 \). Hence
\[ \delta \in H^1(S^1 \times S^1; \mathbb{C}/\mathbb{Z}) \cong \text{Hom}(H_1(S^1 \times S^1; \mathbb{Z}), \mathbb{C}/\mathbb{Z}) \] .
Therefore we must calculate the holonomy of \( \delta \) along the two basis cycles \( S^1 \times \{1\} \) and \( \{1\} \times S^1 \). The holonomies of both terms vanish separately. \( \square \)

**Example 3.100.** We consider two classes \( x, y \in \hat{\mathbb{H}}^1_{\text{Del}}(S^1; \mathbb{Z}) \) and want to calculate \( x \cup y \in \hat{\mathbb{H}}^2_{\text{Del}}(S^1; \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z} \). Note that we can write \( x = \hat{n} \hat{e} + a(f) \) and \( y = \hat{m} \hat{e} + a(g) \) for some \( f, g \in C^\infty(S^1; \mathbb{C}) \). These functions are determined up to integral constants by the differential equations
\[ R(x) - ndt = df, \quad [f(0)] = [x_{\mid (0)}], \quad R(y) - mdt = dg, \quad [g(0)] = [y_{\mid (0)}] . \]
Then
\[ x \cup y = n \hat{m} \hat{e} \cup \hat{e} + a((mf - ng)dt + f \wedge dg) . \]
This gives
\[ \text{ev}(x \cup y) = n \hat{m} \text{ev}(\hat{e} \cup \hat{e}) + \int_{S^1} (fm - ng)dt + f \wedge dg . \]
We must calculate \( \text{ev}(\hat{e} \cup \hat{e}) \). Let \( \text{diag} : S^1 \to S^1 \times S^1 \) be the diagonal. By Exercise 3.98 we have
\[ \text{ev}(\hat{e} \cup \hat{e}) = \text{ev}(\text{diag}^* \hat{c}_1(\nabla^P)) = \text{hol}_\gamma(\text{diag}) . \]
Since \( \text{diag} \) is bordant to the union of the cycles \( S^1 \times \{1\} \) and \( \{1\} \times S^1 \) on which \( \nabla^P \) is trivial, we can calculate the holonomy along \( \text{diag} \) by a curvature integral over the bordism, a triangle covering half of the torus. We get

\[
\text{ev}(\hat{e} \cup \hat{e}) = \left[ \frac{1}{2} \right].
\]

The final formula is

\[
\text{ev}(x \cup y) = \left[ \frac{nm}{2} + \int_{S^1} (fm - ng)dt + f \wedge dg \right].
\]

**Example 3.101.** Note that \( H^*(M; \mathbb{C}/\mathbb{Z}) \) is an \( H^*(M; \mathbb{Z}) \)-module in the natural way. Let \( i : H^*(M; \mathbb{C}/\mathbb{Z}) \to \hat{H}^{n+1}_{\text{Del}}(M; \mathbb{Z}) \) be the inclusion. The following identity extends 3.92, 4.

**Problem 3.102.** Show that \( i(x) \cup y = i(x \cup I(y)) \).

**Proof.** First observe that the formula holds true if \( x = a(\alpha) \). Furthermore observe that \( i(x) \cup y \) only depends on \( I(y) \). Both assertions follow from Definition 3.92, 4. From this conclude that if \( \deg(x) = n \) and \( H^n(M; \mathbb{Z}) \) is torsion free, then the formula holds true. For the general case observe that there exists a smooth map \( f : M \to \tilde{M} \) and classes \( \tilde{x} \in H^n(M; \mathbb{C}/\mathbb{Z}) \) and \( \tilde{y} \in H^*(\tilde{M}; \mathbb{Z}) \) such that \( H^n(\tilde{M}; \mathbb{Z}) \) is torsion-free and \( f^*\tilde{x} = x \) and \( f^*\tilde{y} = I(y) \). Since the identity in question holds true for \( \tilde{x} \) and any differential lift of \( \tilde{y} \), it also holds true for \( x \) by naturality of the product. \( \Box \)

**Example 3.103.** Let \( H^*(M; \mathbb{Z}) \ni z \mapsto \overline{z} \in H^*(M; \mathbb{Z}/2\mathbb{Z}) \) denote the mod-2 reduction. We have a Steenrood square

\[
\text{Sq}^{2n} : H^{2n+1}(M; \mathbb{Z}/2\mathbb{Z}) \to H^{4n+1}(M; \mathbb{Z}/2\mathbb{Z}).
\]

Finally we have a natural map

\[
i : H^{4n+1}(M; \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\mathbb{Z}/2\mathbb{Z} \to \mathbb{C}/\mathbb{Z}} H^{4n+1}(M; \mathbb{C}/\mathbb{Z}).
\]

The following has first been observed in [Gom08]. Our argument below differs from Gomi’s and seems to be much simpler.

**Problem 3.104.** Show that for \( \hat{x} \in \hat{H}^{2n+1}_{\text{Del}}(M; \mathbb{Z}) \) we have \( \hat{x} \cup \hat{x} = i(\text{Sq}^{2n}(I(\overline{x}))) \).

**Proof.** We first discuss the case that \( n \geq 1 \). We can assume that the classifying map \( x \) of \( I(\hat{x}) \) has a factorization

\[
\begin{array}{ccc}
N & \xrightarrow{f} & X \\
\downarrow y & & \downarrow \tilde{x} \\
M & \xrightarrow{\hat{x}} & K(\mathbb{Z}, 2n + 1)
\end{array}
\]

where \( y \) is at least \( \max(4n+2, \dim(M)) \)-connected. We can assume that \( f \) is the inclusion of a submanifold and that there exists a class \( \tilde{y} \in \hat{H}^{2n+1}_{\text{Del}}(N; \mathbb{Z}) \) such that \( f^*\tilde{y} = \hat{x} \). To see this proceed as follows:
1. replace $N$ by $N \times \mathbb{R}^k$, $y$ by $y \circ \text{pr}_N$, and $f$ by $(f, j)$, where $j : M \to \mathbb{R}^k$ is an embedding.

2. choose some class $\hat{y}_0 \in \check{H}^{2n+1}_{\text{Del}}(N; \mathbb{Z})$ with $I(\hat{y}_0)$ classified by $y$ and a form $\alpha \in \Omega^{2n}(M; \mathbb{C})$ such that $f^*\hat{y}_0 = \hat{x} + a(\alpha)$.

3. Since $f$ is an embedding of a submanifold we can choose a form $\beta \in \Omega^{2n}(N; \mathbb{C})$ such that $f^*\beta = \alpha$.

4. We set $\hat{y} := \hat{y}_0 + a(\beta)$.

If $n \geq 1$, then the Bockstein operator induces an isomorphism

$$B : H^{4n+1}(K(\mathbb{Z}, 2n + 1); \mathbb{C}/\mathbb{Z}) \xrightarrow{\sim} H^{4n+2}(K(\mathbb{Z}, 2n + 1); \mathbb{Z})$$

and if $u \in H^{2n+1}(K(\mathbb{Z}, 2n + 1); \mathbb{Z})$ is the universal element, then we have

$$i(\text{Sq}^n(\bar{u})) = B^{-1}(u \cup u).$$

The same holds true on $N$. Since $R(\hat{y} \cup \hat{y}) = R(\hat{y}) \cup R(\hat{y}) = 0$ we have $\hat{y} \cup \hat{y} \in H^{4n+1}(N; \mathbb{C}/\mathbb{Z})$ so that

$$B(\hat{y} \cup \hat{y}) = I(\hat{y} \cup \hat{y}) = B(i(\text{Sq}^n(\overline{\hat{y}}))).$$

Since $B$ is injective we get the result in the case $n \geq 1$.

If $n = 1$, then the universal calculation is $\hat{e} \cup \hat{e} = a(\frac{dt}{2})$. The assertion now follows from $\text{Sq}^0 = \text{id}$ and $i(\text{Sq}^1) = a(\frac{dt}{2})$. \hfill \Box

**Example 3.105.** The following examples concern the calculation of products of degree-one classes.

We consider the maps $\text{pr}_1, \text{pr}_2, \mu : T^2 \to S^1$.

**Problem 3.106.** Show that

$$\text{pr}_1^*\hat{e} \cup \text{pr}_2^*\hat{e} \cup \mu^*\hat{e} = 0.$$  

**Proof.** Use that $\hat{e}$ is primitive and Gomi’s result 3.104. \hfill \Box

Let $f, g \in C^\infty(M; \mathbb{C}/\mathbb{Z})$ and $x, y \in H^1_{\text{Del}}(M; \mathbb{Z})$ be the corresponding classes (see 3.31).

We are interested in the product $x \cup y$.

**Problem 3.107.** For every map $\gamma : S^1 \to M$ calculate $\text{ev}(\gamma^*(x \cup y)) \in \mathbb{C}/\mathbb{Z}$.

**Proof.** If $f = \exp(2\pi i h)$ for some function $h \in C^\infty(M; \mathbb{C})$ (i.e. $I(x) = 0$), then

$$\text{ev}(x \cup y) = \left[ \int_{S^1} \gamma^*hg^{-1}dg \right].$$

62
For the general case we can proceed as follows. We consider \((f, g) : M \to T^2\) and observe that \(x \cup y = (f, g)^*(pr_1^* \hat{e} \cup pr_2^* \hat{e})\). By exercise 3.98 we must calculate \(\text{hol}_{(f, g)^* \nabla^g} (\gamma)\). Or we write \(\text{ev}(\gamma^*(x \cup y)) = \gamma^* x \cup \gamma^* y\) and use Problem 3.100.

We consider the manifold \(\mathbb{C}^* \setminus \{1\}\) and the functions

\[
\left[\frac{1}{2\pi i} \ln z, \frac{1}{2\pi i} \ln (1 - z)\right] \in C^\infty(\mathbb{C}^* \setminus \{1\}, \mathbb{C}/\mathbb{Z})
\]

as elements \(f, g \in \hat{H}^1_{\text{del}}(\mathbb{C}^* \setminus \{1\}; \mathbb{Z})\).

**Problem 3.108.** Calculate \(f \cup g\).

**Proof.** The result is

\[
f \cup g = 0.
\]

First observe by calculation that \(R(f \cup g) = 0\). Furthermore, we have \(I(f \cup g) = 0\). It follows that \(f \cup g = a([\alpha])\) for some \([\alpha] \in H^1(\mathbb{C}^* \setminus \{1\}; \mathbb{C})/H^1(\mathbb{C}^* \setminus \{1\}; \mathbb{Z})\). We choose a basis \(u, v \in H_1(\mathbb{C}^* \setminus \{1\}; \mathbb{Z})\) given by small counterclock wise circles around 0 and 1. This basis induces an identification \(H^1(\mathbb{C}^* \setminus \{1\}; \mathbb{C})/H^1(\mathbb{C}^* \setminus \{1\}; \mathbb{Z}) \cong (\mathbb{C}/\mathbb{Z})^2\). We thus have to calculate \(\int_u \alpha\) and \(\int_v \alpha\). We now use the result of Example 3.100.

**Example 3.109.** In the next examples we calculate products involving higher-dimensional classes. The calculation is usually easy if one of the classes is topologically trivial. The general case is usually difficult and requires some tricks.

Let \(M\) be a closed oriented surface, \((L, \nabla) \in \text{Line}_\nabla(M)\) be a line bundle with connection on \(M\) and \(u \in C^\infty(M; \mathbb{C}/\mathbb{Z})\). Then we have classes \(x := \hat{c}_1(L, \nabla) \in H^2_{\text{del}}(M; \mathbb{Z})\) and \(y := [u] \in \hat{H}^1_{\text{del}}(M; \mathbb{Z})\) (see 3.31).

**Problem 3.110.** Calculate \(\text{ev}(x \cup y) \in \mathbb{C}/\mathbb{Z}\).

**Proof.** Assume that \(L\) is trivial \(\nabla^L = d + 2\pi i \alpha\). Then \(\hat{c}_1(L) = a(-\alpha)\) and

\[
\text{ev}(x \cup y) = [-\int_M \alpha \wedge du].
\]

The general case is complicated.

**Problem 3.111.** On the two-torus \(T^2\) with affine coordinates \(s, t\) calculate the product \(\hat{c}_1(L, \nabla) \cup [u]\) for \((L, \nabla) \in \text{Line}_\nabla(T^2)\) and \(u \in C^\infty(T^2, \mathbb{C}/\mathbb{Z})\).

**Proof.** We first let \((L, \nabla)\) be such that \(\hat{c}_1(L, \nabla) = kpr_1^* \hat{e} \cup pr_2^* \hat{e}\). We further assume that \(u : T^2 \to \mathbb{C}/\mathbb{Z}\) is a homomorphism so that \([u] = mpr_1^* \hat{e} + npr_2^* \hat{e}\). Then

\[
\hat{c}_1(L, \nabla) \cup [u] = -mkpr_1^*(\hat{e} \cup \hat{e}) \cup pr_2^* \hat{e} + nkpr_1^* \hat{e} \cup pr_2^*(\hat{e} \cup \hat{e}) = a([k(n + m) ds \wedge dt]/2) .
\]
The general case now follows by pertubation with forms.

Let $M$ be a an oriented connected closed surface and $y \in H^1(M;\mathbb{Z})$. Then there exists a map $f : M \to T^2$ of degree one and a class $u \in H^1(T^2;\mathbb{Z})$ such that $f^*u = y$. Without loss of generality we can assume that $y$ is primitive. Then there exists a dual element $x \in H^1(M;\mathbb{Z})$ such that $\langle x \cup y, [M] \rangle = 1$ and $(x, y)$ span a hyperbolic summand of the first cohomology. The classes $x, y$ can be represented by closed integral forms $\alpha, \beta$. The map $f : M \to T^2$ can then be obtained as the period map associated to $x, y$.

We can now first calculate $\text{ev}(\hat{c}_1(H, \nabla^H) \cup \hat{u})$ for some function $u : T^2 \to \mathbb{C}/\mathbb{Z}$ representing $\hat{u}$. The pairing $\hat{c}_1(L, \nabla^L) \cup \hat{y}$ is then obtained from $\text{ev}(\hat{c}_1(H, \nabla^H) \cup \hat{u})$ and a perturbation by forms.

Example 3.112. In the following examples we consider the duality pairing in Deligne cohomology induced by the product and evaluation. We consider a closed oriented $n-1$-dimensional manifold. We define a pairing $\langle \ldots, \ldots \rangle : \hat{H}^p_{\text{Del}}(M;\mathbb{Z}) \otimes \hat{H}^{n-p}_{\text{Del}}(M;\mathbb{Z}) \to \mathbb{C}/\mathbb{Z}$ by

$$\langle x, y \rangle := \text{ev}(x \cup y) .$$

We have the following differential version of Poincaré duality.

**Problem 3.113.** Show that this pairing is non-degenerated in the sense that $\langle x, y \rangle = 0$ for all $y \in \hat{H}^{n-p}_{\text{Del}}(M;\mathbb{Z})$ implies $x = 0$.

**Proof.** See e.g. [FMS07] for details.

**Problem 3.114.** Let $M$ be a two-dimensional and orientable. Show that

$$\cup : \hat{H}^1_{\text{Del}}(M;\mathbb{Z}) \otimes \hat{H}^1_{\text{Del}}(M;\mathbb{Z}) \to \hat{H}^2_{\text{Del}}(M;\mathbb{Z})$$

is surjective.

**Example 3.115.** In the following we discuss the compatibility of differential refinements of integral characteristic forms with products. Let $\omega$ and $\omega'$ be integral characteristic forms for complex vector bundles of degree $n$ and $m$. Then $\omega \wedge \omega'$ is a characteristic form of degree $n + m$.

**Problem 3.116.** Show that $\overline{\omega \wedge \omega'} = \overline{\omega} \cup \overline{\omega'}$.

Conclude that $\hat{s}_4 = \hat{c}_1 - 2\hat{c}_2$.
Proof. We first consider the difference \((\omega \wedge \omega')^Z - \omega^Z \cup \omega'^Z\). This would be an integral refinement of zero and hence vanishes by the uniqueness of integral refinements. We now consider the difference \(\omega \wedge \omega' - \hat{\omega} \cup \hat{\omega}'\). This would be a differential refinement of zero and again vanishes, this time by the uniqueness of differential refinements. \(\square\)

**Problem 3.117.** Show that \(c\hat{\omega}\) (see Theorem 3.77) is additive and multiplicative. More precisely, for a \(G\)-principal bundle with connection \((P \to M, \omega)\) and \(\hat{\phi}, \hat{\psi} \in \tilde{I}^*(G)\) we have e.g.\[
c\hat{\omega}(\hat{\phi})(\omega) \cup c\hat{\omega}(\hat{\psi})(\omega) = c\hat{\omega}(\hat{\phi} \hat{\psi})(\omega).
\]
Let \((E, \nabla) \cong (E_1, \nabla_1) \oplus (E_2, \nabla_2)\) be a decomposition of a complex vector bundle with connection.

**Problem 3.118.** Show that \(\hat{c}(\nabla) = \hat{c}(\nabla_1) \cup \hat{c}(\nabla_2)\).

**Problem 3.119.** Conclude that for a real vector bundle \(V \to M\) with connection \(\nabla\) we have for all \(n \geq 0\) that \(2\hat{c}_{2n+1}(\nabla \otimes \mathbb{C}) = 0\).

Show by example that for every \(n\) there exists a real bundle \((V, \nabla)\) such that \(\hat{c}_{2n+1}(\nabla \otimes \mathbb{C}) \neq 0\).

**Proof.** Use the splitting principle, 3.67 and 3.102. \(\square\)

We conclude that for a decomposition of real vector bundles \((V, \nabla) \cong (V_1, \nabla_1) \oplus (V_2, \nabla_2)\) we have \(\hat{p}(\nabla) - \hat{p}(\nabla_1) \cup \hat{p}(\nabla_2) = 2\)-torsion and that the r.h.s is non-trivial in general.

**Problem 3.120.** Let \(n_i = \text{dim}(V_i)\) be even for \(i = 1, 2\). Show that \(\hat{p}_{n_1}(\nabla_1) \cup \hat{p}_{n_2}(\nabla_2) = \hat{p}_{n_1+n_2}(\nabla)\).

**Proof.** Use 3.44 in order to exclude disturbing terms. \(\square\)

**Problem 3.121.** If \((V, \nabla^V)\) and \((W, \nabla^W)\) are real oriented vector bundles with connection, then we have \(\hat{\chi}(\nabla^V) \cup \hat{\chi}(\nabla^W) = \hat{\chi}(\nabla^V \oplus \nabla^W)\).

**Proof.** Show that \(\chi(V) \cup \chi(W) = \chi(V \oplus W)\). Then extend this to the differential refinements using unicity considerations as in similar cases.
Problem 3.122. Show the identity $\hat{\chi}^2 = \hat{p}_n$ of differential characteristic classes for real oriented $2n$-dimensional vector bundles with connection.

Proof. We must show that the corresponding elements in $\tilde{I}(SO(2n))$ coincide. The main point is the equality $\chi_Z = p^n$. One can use the splitting principle and 3.120 in order to reduce to the case $n = 1$. □

3.5 Cheeger-Simons differential characters

The goal of this section is to relate the Deligne cohomology $\hat{H}^*_{Del}(M; \mathbb{Z})$ with the predating classical definition of the group of Cheeger-Simons differential characters [CS85]. We let $\text{sing}_\infty(M)$ be the simplicial complex of smooth simplices of $M$ and $C_*(\text{sing}_\infty(M))$ be the associated chain complex. Integration over simplices induces the de Rham map

$$\text{Rham} : \Omega^*_c(M) \to \text{Hom}(C_*(\text{sing}_\infty(M)); \mathbb{C}), \quad \omega \mapsto \{z \mapsto \int_z \omega\}.$$ 

A version of the de Rham Lemma says that this map induces an isomorphism

$$\text{Rham} : H^*_{dR}(M; \mathbb{C}) \sim H^*(M; \mathbb{C})$$

between de Rham cohomology and the smooth singular cohomology.

If $c = \sum_{\sigma \in C_n(\text{sing}_\infty(M))} n_\sigma \sigma$ is a smooth chain, then we define its support by

$$|c| := \bigcup_{\{\sigma \in C_n(\text{sing}_\infty(M))| n_\sigma \neq 0\}} \sigma(\Delta^n) \subseteq M.$$ 

It is the compact subset of $M$ which looks like something at most $n$-dimensional. More precisely, we have:

Lemma 3.123. There exists an open neighbourhood $U \subseteq M$ of $|c|$ such that $H^k(U; \mathbb{Z}) = 0$ for all $k > n$.

Proof. I would like to see an argument.

We let $Z^n_{n-1}(M) := Z_{n-1}(C_*(\text{sing}_\infty(M)))$ be the group of smooth $n-1$-cycles on $M$. For $z \in Z^n_{n-1}(M)$ we choose an open neighbourhood $U$ of $|z|$ such that $H^n_{dR}(M; \mathbb{C}) = 0$. For $\hat{x} \in \hat{H}^n_{Del}(M; \mathbb{Z})$ we can find a form $\omega \in \Omega^{n-1}(U; \mathbb{C})$ such that $a(\omega) = \hat{x}|_U$.

Definition 3.124. We define $\text{ev}_{\hat{x}} \in \text{Hom}(Z^n_{n-1}(M), \mathbb{C}/\mathbb{Z})$ by

$$\text{ev}_{\hat{x}}(z) := \int_z \omega.$$ 

Lemma 3.125. This evaluation is well-defined. It induces a homomorphism

$$\text{ev} : \hat{H}^n_{Del}(M; \mathbb{Z}) \to \text{Hom}(Z^n_{n-1}(M), \mathbb{C}/\mathbb{Z}).$$ 

If $z = \delta c$ for some $c \in C_n(\text{sing}_\infty(M))$, then $\text{ev}_{\hat{x}}(z) = [\int_c R(\hat{x})]$. 

66
Proof. In order to prove well-definedness the main observation is that if \( \omega' \) is another choice, then the difference \( \omega' - \omega \) is integral. This implies \( \int_z (\omega - \omega') = 0 \). It is easy to see that we get a homomorphism. For the last assertion note that \( R(\hat{x})|_{U} = d\omega \). By Stoke’s theorem \( \int_z \omega = \int_c R(\hat{x}) \).

**Definition 3.126.** A Cheeger-Simons differential character of degree \( n > 0 \) is a homomorphism

\[
\phi : Z_{n-1}^\infty(M) \to \mathbb{C}/\mathbb{Z}
\]

such that there exists a form \( R(\phi) \in \Omega^n_{cl}(M, \mathbb{C}) \) with

\[
\phi(\delta c) = \left[ \int_c R(\phi) \right].
\]

We let \( \hat{H}^n_{CS}(M; \mathbb{Z}) \) denote the group of Cheeger-Simons differential characters of degree \( n > 0 \). We define \( \hat{H}^0_{CS}(M; \mathbb{Z}) := \mathbb{Z}(M) \).

In Lemma 3.125 we have constructed a natural map

\[
\text{ev} : \hat{H}^n_{Del}(M; \mathbb{Z}) \to \hat{H}^n_{CS}(M; \mathbb{Z}).
\]

**Lemma 3.127.** The group of Cheeger-Simons characters fits into the differential cohomology diagram

\[
\begin{array}{cccc}
\Omega^{n-1}(M; \mathbb{C})/\text{im}(d) & \xrightarrow{d} & \Omega^n_{cl}(M; \mathbb{C}) \\
\text{} & \xrightarrow{R} & \text{ } \\
H^{n-1}_{dR}(M; \mathbb{C}) & \xrightarrow{a} & \hat{H}^n_{CS}(M; \mathbb{Z}) & \xrightarrow{I} & H^n(M; \mathbb{Z}) \\
\text{ } & \xrightarrow{\text{-Bockstein}} & \text{ } \\
H^{n-1}(M; \mathbb{C}/\mathbb{Z}) & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ }
\end{array}
\]

Proof. We first construct the structure maps. If \( \phi \in \hat{H}^n_{CS}(M; \mathbb{Z}) \), then we first observe that the form \( R(\phi) \) is uniquely determined. To this end evaluate \( \phi \) on the cycles given by the boundaries of small \( n \)-simplices. Using that \( Z_{n-1}^\infty(M) \) is a free \( \mathbb{Z} \)-module we choose a lift \( \hat{\phi} : Z_{n-1}^\infty(M) \to \mathbb{C} \). Then the cocycle \( d\hat{\phi} - \text{Rham}(R(\phi)) \in Z^n(C^n(M; \mathbb{Z})) \) represents \( I(\phi) \in H^n(M; \mathbb{Z}) \). Finally, for \( \alpha \in \Omega^{n-1}(M; \mathbb{C}) \) we define \( a(\alpha)(z) := \text{Rham}(\alpha)(z) \).

**Problem 3.128.** Verify the uniqueness of \( R(\phi) \) in detail (see [CS85] for an argument). Show that \( I \) is well-defined. Further verify the properties of a differential cohomology diagram. Show that \( \text{ev} \) is compatible with the maps \( R, a, I \).

We have an exact sequence

\[
H^{n-1}(M; \mathbb{Z}) \to \Omega^{n-1}(M, \mathbb{C})/\text{im}(d) \xrightarrow{a} \hat{H}^n_{CS}(M; \mathbb{Z}) \xrightarrow{(I, R)} H^n(M; \mathbb{Z}) \times_{H^n_{dR}(M; \mathbb{C})} \Omega^n_{cl}(M; \mathbb{C}) \to 0
\]

which is compatible via \( \text{ev} \) with (20). We can thus apply the Five Lemma and see
Corollary 3.129.

\[ \text{ev} : \hat{H}^n_{Del}(M; \mathbb{Z}) \to \hat{H}^n_{CS}(M; \mathbb{Z}) \]

is an isomorphism.

Example 3.130. Let \( M \) be closed, connected, oriented and \( n - 1 \)-dimensional. The identification \( \hat{H}^n_{Del}(M; \mathbb{Z}) \cong \mathbb{C}/\mathbb{Z} \) \((3.35)\) is given by \( \hat{x} \mapsto \text{ev}_x(\{M\}) \), where \( \{M\} \in Z_{n-1}(M) \) is some representative of the fundamental cycle \( [M] \) of \( M \).

Note that in [CS85] also a product for differential characters was defined. By the uniqueness of products \(3.93\) the evaluation \( \text{ev} : \hat{H}^n_{Del}(M; \mathbb{Z}) \to \hat{H}^n_{CS}(M; \mathbb{Z}) \) is automatically multiplicative.

### 3.6 Integration

We now consider a proper submersion \( \pi : W \to M \) of dimension \( n \).

**Problem 3.131.** Observe that a proper submersion is the same as a locally trivial fibre bundle with closed fibres.

An orientation of \( \pi \) is an orientation of the vertical bundle \( T^v\pi := \ker(d\pi) \). If \( \pi \) is oriented, then we have an integration of forms

\[ \int_{W/M} : \Omega^*(W) \to \Omega^{*-n}(M) \]

such that \( d\int_{W/M} \omega = \int_{W/M} d\omega \). It induces an integration

\[ \pi^!_{\text{Rham}} : H^*_{dR}(W; \mathbb{C}) \to H^*_{dR}(M; \mathbb{C}) \; , \quad \pi^!_{\text{Rham}}[\omega] := \left[ \int_{W/M} \omega \right] \]

in de Rham cohomology. Moreover, we have an integration

\[ \pi^!: H^*(W; \mathbb{Z}) \to H^{*-n}(M; \mathbb{Z}) \]

in integral cohomology which is compatible with \( \pi^!_{\text{Rham}} \) via the map \( \epsilon_\mathbb{C} \). All these integrations are natural for pull-back diagrams

\[
\begin{array}{ccc}
W' & \xrightarrow{F} & W \\
\downarrow{\pi'} & & \downarrow{\pi} \\
M' & \xrightarrow{f} & M
\end{array}
\]

and functorial for compositions

\[ W \xrightarrow{\pi} M \xrightarrow{\kappa} N \]

of oriented proper submersions.
Example 3.132. We assume that the fibres $F$ of $\pi$ are connected and consider the Serre spectral sequence associated to the bundle $\pi$ for integral or complex cohomology. Every cohomology class $x \in H^*(W; \mathbb{Z})$ has a symbol $\sigma(x) \in E^*_{\infty} \subseteq E^*_{2n} \cong H^{* - n}(M; \mathbb{Z})$, where the last identification uses the orientation of the fibres. The integration is given by $\pi_! = \sigma$.

Problem 3.133. Use this description of the integration in order to prove that $\pi_!$ is compatible with $\mathbf{R}$am and $\epsilon_{\mathbb{C}}$, and verify the remaining assertions about pull-back and composition made above.

We now define the notion of an integration structure for the differential cohomology $\hat{H}^{*}_{\text{Del}}(\ldots, \mathbb{Z})$.

Definition 3.134. An integration for $\hat{H}^{*}_{\text{Del}}(\ldots, \mathbb{Z})$ is given by maps $\hat{\pi}_! : \hat{H}^{*}_{\text{Del}}(W; \mathbb{Z}) \to H^{* - n}_{\text{Del}}(M; \mathbb{Z})$ for all proper oriented submersions $\pi : W \to M$ such that

\[
\begin{align*}
\Omega^{*-1}(W; \mathbb{C})/\text{im}(d) &\xrightarrow{a} \hat{H}^{*}_{\text{Del}}(W; \mathbb{Z}) \xrightarrow{\hat{\pi}_!} H^{*}(W; \mathbb{Z}) \xrightarrow{I} \Omega^{*}_{\text{cl}}(W; \mathbb{C}) \\
\Omega^{* - n - 1}(M; \mathbb{C})/\text{im}(d) &\xrightarrow{a} \hat{H}^{* - n}_{\text{Del}}(M; \mathbb{Z}) \xrightarrow{\pi_!} H^{* - n}(M; \mathbb{Z}) \xrightarrow{I} \Omega^{*}_{\text{cl}}(M; \mathbb{C})
\end{align*}
\]

commutes, where $n = \dim(W) - \dim(M)$. Furthermore, for every diagram (29) we have

$\hat{\pi}_! \circ F^* = f^* \circ \hat{\pi}_!$.

Theorem 3.135. There exists a unique integration structure for $\hat{H}^{*}_{\text{Del}}(\ldots, \mathbb{Z})$. It is functorial for compositions.

Proof. We construct the integration structure in the picture of Cheeger-Simons characters. Let $\phi \in \check{H}^{k}_{CS}(W; \mathbb{Z})$. Then we define $\hat{\pi}_!(\phi) \in \check{H}^{k-n}_{CS}(W; \mathbb{Z})$ as follows. Let $z \in Z^{k-n}_{k-n-1}(M)$. We choose an open neighbourhood of $U$ of $|z|$ which is homotopy equivalent to a $k - n - 1$-dimensional complex. Then $\pi^{-1}(U)$ is equivalent to a $k - 1$-dimensional complex. In particular there exists a form $\alpha \in \Omega^{k-1}(\pi^{-1}(U); \mathbb{C})$ such that $a(\alpha) = \phi|_{\pi^{-1}(U)}$. By naturality of the integration and its compatibility with $a$ we are forced to define

$\hat{\pi}_!(\phi)(z) := \left[ \int_{z} \int_{\pi^{-1}(U)/U} alpha \right]$.

Problem 3.136. Show that this construction is compatible with $a$, $R$, pull-backs, and that it is functorial. Further show that the integration is compatible with Mayer-Vietoris sequences induced by decompositions of the base.
In order to show compatibility with $I$ we argue as follows. The integration $\hat{\pi}$ induces an integration $\pi'$ in integral cohomology which is compatible with pull-back along diagrams (29) and complexification $\epsilon_C$. It coincides with $\hat{\pi}$ if $H^*(M; \mathbb{Z})$ is torsion-free, in particular if $M$ is a point.

**Problem 3.137.** Show that there is at most one functorial integration for integral cohomology with is compatible with pull-back along diagrams (29), Mayer-Vietoris sequences associated to decompositions of the base, and which coincides with $\pi$ for $M = pt$.

By this exercise we have $\pi' = \pi$.

Alternatively, the existence of an integration for $\hat{H}_{Del}^*(\ldots; \mathbb{Z})$ follows from 4.228.

**Problem 3.138.** Show the projection formula

$$\hat{\pi}(\pi^* x \cup y) = x \cup \hat{\pi}(y) , \quad x \in \hat{H}_{Del}^*(M; \mathbb{Z}) , \quad y \in \hat{H}_{Del}^*(W; \mathbb{Z}) .$$

**Proof.** This is not easy. See [DL05] or [BKS10] for arguments based on different models. Alternatively, this follows from 4.230.

**Example 3.139.** Let $(P, \nabla^P)$ be the Poincaré bundle over $J(S^1) \times S^1$.

**Problem 3.140.** Calculate the integrals of $\hat{c}_1(\nabla^P)$ over the two projections.

**Proof.** Note that these integrals are classes in $\hat{H}_{Del}^1(\ldots; \mathbb{Z})$ and thus $\mathbb{C}/\mathbb{Z}$-valued functions.

Consider now a surface $M$ of genus $g$ and again the Poincaré bundle $(P, \nabla^P)$ over $J(M) \times M$.

**Problem 3.141.** Calculate the integral of $\hat{c}_1(\nabla^P)^{g+1}$ along the projection $J(M) \times M \to M$.

**Proof.** Note that the result can be interpreted as an element of Line$_\nabla(M)$. The question is to characterize this isomorphism class of line bundles.

**Example 3.142.** Let $p : W \to B$ be an oriented bundle of compact manifolds with boundary $\partial W \to B$. Let $q : \partial W \to B$ denote the restriction of $p$ to the boundary. We consider $x \in \hat{H}^n_{Del}(W; \mathbb{Z})$.

**Problem 3.143.** Show that

$$q_!(x) = a(\int_{W/B} R(x)) .$$
4 Differential extensions of generalized cohomology theories

4.1 The ∞-categorical black-box

In the following we will use some ∞-categorical language as a black box. General references are [Lur09] and [Lur]. Our philosophy is to work, as much as possible, in a model independent way. We try to highlight the places where models are required for calculations. The ∞-categorical black box provides the set-up and the space where our objects live. It ensures correctness of the constructions.

The input and output of calculations belong to the classical world, and the dependence of the output from the input does not depend on the details of interior machine. In particular, as we try to emphasize, this relation can be revealed in praxis just using the formal properties of the ∞-categorical machine.

First of all, we use the word ∞-category as an abbreviation for (∞, 1)-category. If \( C \) is a category, then its nerve \( N(C) \) is an ∞-category. If \( C \) is an ∞-category and \( W \) is a collection of morphisms, then we can form its localization \( \iota: C \to C[W^{-1}] \). It is characterized by an obvious universal property that

\[
\text{Fun}_{W^{-1}}(C, D) \cong \text{Fun}(C[W^{-1}], D),
\]

where \( \text{Fun}_{W^{-1}}(C, D) \) denotes the full subcategory of functors which map morphisms from \( W \) to equivalences. If \( X \) is an object of \( C \), then usually we use the same symbol in order to denote the corresponding object in \( C[W^{-1}] \). If it is confusing not to distinguish, then we use the notation \( \iota X \) or \( \iota(X) \) for the latter.

Let \( sSet, \ Top, \ Sp \) and \( Ch \) denote the categories of simplicial sets, topological spaces, spectra or chain complexes. Then \( N(sSet)[W^{-1}], N(Top)[W^{-1}], N(Sp)[W^{-1}] \) or \( N(Ch)[W^{-1}] \) denote the localizations of the associated ∞-categories at weak equivalences, stable equivalences or quasi-isomorphisms, respectively. All these ∞-categories have symmetric monoidal versions, the first with respect to the cartesian product, the second with respect to \( \wedge \), and the last with respect to \( \otimes \).

For two objects \( X, Y \in C \) we let \( \text{map}(X, Y) \in N(sSet)[W^{-1}] \) be the mapping space. The categories \( N(Sp)[W^{-1}] \) or \( N(Ch)[W^{-1}] \) are stable and therefore have mapping spectra which will be denoted by \( \text{Map}(X, Y) \). Note that \( \text{map}(X, Y) \cong \Omega^\infty \text{Map}(X, Y) \).

These ∞-categories are complete and cocomplete. They are hence tensored and cotensored over \( N(sSet)[W^{-1}] \), or the equivalent ∞-category \( N(Top)[W^{-1}] \). We will write the tensor of an object \( C \) with a space \( X \) by \( C \otimes X \), and the cotensor will be denoted by \( C^X \).

The stable ∞-categories \( N(Ch)[W^{-1}] \) and \( N(Sp)[W^{-1}] \) are tensored and cotensored over spectra \( N(Sp)[W^{-1}] \), and we use a similar notation for these structures with a subscript \( s \) added. Note that for a space \( X \) we have \( C \otimes X \cong C \otimes_s \Sigma^\infty X \) and \( C^X \cong C^s \Sigma^\infty X \).

We will often use the following argument. Let \( \Phi \) and \( \Psi \) be colimit preserving functors \( \text{Fun}(N(Top)[W^{-1}], C) \).

71
Lemma 4.1. An transformation (equivalence) $\Phi(\text{pt}) \to \Psi(\text{pt})$ extends naturally to a transformation (equivalence) $\Phi \to \Psi$. There are corresponding versions for contravariant functors which map colimits to limits.

Proof. Indeed, for every space $X \in N(\text{Top})[W^{-1}]$ we can form its category of simplices $\text{Simp}(X)$ and get a natural equivalence

$$\text{colim}_n(\text{Simp}(X)) \text{pt} \sim X.$$ 

We obtain

$$\Phi(X) \cong \text{colim}_n(\text{Simp}(X)) \Phi(\text{pt}) \to \text{colim}_n(\text{Simp}(X)) \Psi(\text{pt}) \cong \Psi(X).$$

Example 4.2.

Problem 4.3. Show that the tensor and cotensor for spectra can be written in terms of the smash product and internal mapping object $\text{Map}$ as follows:

$$E \otimes X \cong E \wedge \Sigma^\infty_+ X, \quad E^X \cong \text{Map}(\Sigma^\infty_+ X, E), \quad X \in N(\text{sSet})[W^{-1}], E \in N(\text{Sp})[W^{-1}].$$

Proof. We discuss the cotensor. Observe that both sides map colimits in the $X$-variable to limits and are canonically equivalent for $X = \text{pt}$. Lemma 4.1 gives the assertion.

Problem 4.4. Show that the tensor and cotensor structure with $N(\text{sSet})[W^{-1}]$ on $N(\text{Ch})[W^{-1}]$ is given in terms of the chain complex $C_*(X)$ of $X \in N(\text{sSet})[W^{-1}]$ by

$$A \otimes X = A \otimes C_*(X), \quad A^X \cong \text{Map}(C_*(X), A), \quad A \in N(\text{Ch})[W^{-1}],$$

where $\text{Map}$ denotes the mapping chain complex.

Proof. We discuss the tensor. Both sides coincide for $X = \ast$ and preserve colimits.

For a chain complex $A \in N(\text{Ch})[W^{-1}]$ let $H_k(A)$ denote the homology group in degree $n$.

Problem 4.5. Show that for a space $X \in N(\text{sSet})[W^{-1}]$, and integer $k \geq 0$ and a chain complex $A \in N(\text{Ch})[W^{-1}]$ we have

$$H_0(A) \cong \pi_0(\text{map}(\mathbb{Z}, A))$$

and more generally

$$H_k(A^X) \cong \pi_k(\text{map}(C_*(X), A)).$$
Proof. We assume the first equivalence (30) and deduce the second. We get

\[ H_k(A^X) \cong \pi_0(\text{map}(\mathbb{Z}[k], A^X)) \]
\[ \cong \pi_0(\text{map}(\mathbb{Z}[k] \otimes X, A)) \]
\[ \cong \pi_0(\text{map}(C_\ast(X)[k], A)) \]
\[ \cong \pi_k(\text{map}(C_\ast(X), A)) \].

In order to show (30) it seems to be necessary to use a model for the \( \infty \)-category \( \mathbb{N}(\text{Ch})[W^{-1}] \).

Let \( C \) be an \( \infty \)-category.

**Definition 4.6.** We call \( \text{Sm}(C) := \text{Fun}(\mathbb{N}(\text{Mf}^{\text{op}}), C) \) the \( \infty \)-category of smooth objects in \( C \).

Recall that \( \text{Mf} \) has a topology given by open coverings.

**Definition 4.7.** We say that a smooth object \( X \in \text{Sm}(C) \) satisfies descent, if for every open covering \( U \) of \( M \) the augmentation map \( X(M) \to \lim_{\text{N}(\Delta)} X(U^\cdot) \) is an equivalence in \( C \), where \( U^\cdot \in \text{Mf}^{\Delta^{\text{op}}} \) is the nerve of \( U \).

For the following we make the technical assumption that \( C \) is presentable. This holds true for all examples considered here. We let \( \text{Sm}^{\text{desc}}(C) \subseteq \text{Sm}(C) \) denote the full subcategory of objects satisfying descent. Then by [Lur09, 6.2.2.7] we have an adjunction

\[ L : \text{Sm}(C) \rightleftarrows \text{Sm}^{\text{desc}}(C) \].

The functor \( L \) is called sheafification. In general it seems very difficult to evaluate sheafifications explicitly. See Example 4.8 for a method which can be applied in some interesting cases.

**Example 4.8.** Let \( f : X \to Y \) be a morphism in \( \text{Sm}(C) \). Sometimes we need a criterion ensuring that \( L(f) : L(X) \to L(Y) \) in \( \text{Sm}^{\text{desc}}(C) \) is an equivalence (see Lemma 4.11). We define the functor \( L : \text{Sm}(C) \to \text{Sm}(C) \) which on objects acts as

\[ L(X)(M) := \text{colim}_{\text{N}(\Delta)} X(U^\cdot) , \]

where the first colimit is over the filtered system of open coverings of \( M \).

**Problem 4.9.** Describe this precisely as a functor between \( \infty \)-categories.
Proof. We consider the functor $\text{Mf}^{\text{op}} \to \text{Filt}$ which associates to every manifold $M$ the filtered partially ordered set of open coverings of $M$ indexed by the points of $M$. Then we let $\tilde{\text{Mf}}$ be the transport category of pairs $(M, U)$ of manifolds and coverings whose morphisms $(M, U) \to (M', U')$ are smooth maps $f : M \to M'$ such that $f^{-1}(U'_{f(m)}) \subseteq U_m$ for all $m$. We have the Čech nerve functor $\tilde{\text{Mf}} \to \text{Mf}^{\Delta^{\text{op}}}$, $(M, U) \mapsto \mathcal{U}$. Precomposition with this functor gives the functor $\tilde{\mathcal{L}} : \text{Sm}(\mathcal{C}) \to \text{Fun}(\mathcal{N}(\tilde{\text{Mf}})^{\text{op}}, \text{Fun}(\mathcal{N}(\Delta), \mathcal{C}))$. Finally we compose with $\lim_{\Delta}$ and get

$$\text{Sm}(\mathcal{C}) \to \text{Fun}(\mathcal{N}(\tilde{\text{Mf}})^{\text{op}}, \mathcal{C}).$$

We have an adjunction

$$\Phi^* : \text{Fun}(\mathcal{N}(\tilde{\text{Mf}})^{\text{op}}, \mathcal{C}) \rightleftarrows \text{Fun}(\mathcal{N}(\text{Mf})^{\text{op}}, \mathcal{C}) : \Phi^*,$$

where $\Phi : \tilde{\text{Mf}} \to \text{Mf}$ forgets the coverings. We define $\mathcal{L} := \Phi^* \circ \tilde{\mathcal{L}}$. Note that on objects

$$\mathcal{L}(X)(M) = \text{colim}_U \text{lim}_{\mathcal{N}(\Delta)} X(\mathcal{U}^\bullet).$$

There is a natural morphism $\text{id} \to \mathcal{L}$. We define

$$\mathcal{L}^\infty := \text{colim}(\text{id} \to \mathcal{L} \to \mathcal{L}^2 \to \mathcal{L}^3 \to \ldots) : \text{Sm}(\mathcal{C}) \to \text{Sm}(\mathcal{C}).$$

By construction the natural morphism $\mathcal{L}^\infty \to (\mathcal{L}^\infty)^2$ is an equivalence. We let

$$\text{Sm}^{\mathcal{L}^\infty}(\mathcal{C}) \subseteq \text{Sm}(\mathcal{C})$$

be the essential image of $\mathcal{L}^\infty$. By the recognition principle [Lur09, Prop. 5.2.7.4], since condition 3. is satisfied, this is a localization. If $F$ satisfies descent, then we observe that $F \to \mathcal{L}^\infty F$ is an equivalence. Hence we have a sequence of localizations

$$\text{Sm}^{\text{desc}}(\mathcal{C}) \subseteq \text{Sm}^{\mathcal{L}^\infty}(\mathcal{C}) \subseteq \text{Sm}(\mathcal{C}).$$

In particular, the natural transformation

$$L \to L \circ \mathcal{L}^\infty$$

is an equivalence.

**Problem 4.10.** Do we have $\text{Sm}^{\text{desc}}(\mathcal{C}) \cong \text{Sm}^{\mathcal{L}^\infty}(\mathcal{C})$?

**Lemma 4.11.** If $f : X \to Y$ is a morphism in $\text{Sm}(\mathcal{C})$ such that $\mathcal{L}(f) : \mathcal{L}(X) \to \mathcal{L}(Y)$ is an equivalence, then $L(f) : L(X) \to L(Y)$ is an equivalence.
Proof. The condition implies that $L_\infty(f)$ is an equivalence. Now by [31] we have an equivalence $L(L_\infty(f)) \cong L(f)$ so that $L(f)$ is an equivalence, too. \hfill \Box

Here is an application of Lemma 4.11. Let $C(\Omega^n) \in Sm(N(Ch)[W^{-1}])$ be the functor which associates to every manifold $M$ the complex which is concentrated in degree zero and given there by $\Omega^n(M)$. Since $(\sigma^{\geq n}\Omega)[n]$ satisfies descent the natural inclusion map $C(\Omega^n) \to (\sigma^{\geq n}\Omega)[n]$ extends to $i : L(C(\Omega^n)) \to (\sigma^{\geq n}\Omega)[n]$.

**Problem 4.12.** Show that $i$ is an equivalence.

**Proof.** We check that $L(C(\Omega^n)) \to L((\sigma^{\geq n}\Omega)[n]) \cong (\sigma^{\geq n}\Omega)[n]$ is an equivalence and apply Lemma 4.11. Indeed, if $U$ is a good covering of $M$, then

$$\tilde{C}(U, C(\Omega^n)) \to \tilde{C}(U, (\sigma^{\geq n}\Omega)[n])$$

is an equivalence. Since good coverings are cofinal in all coverings we get the desired result. \hfill \Box

We have a functor

$$t : N(Mf) \to N(Top)[W^{-1}]$$

which associates to smooth manifold its underlying topological space. It satisfies codescent in the sense that for every open covering $U$ of a manifold $M$ we have an equivalence

$$\text{colim}_{N(\Delta)^{op}} t(U^\bullet) \to t(M).$$

We now assume that the $\infty$-category $C$ is cotensored over $N(Top)[W^{-1}]$.

**Definition 4.13.** We define the smooth function object functor

$$Sm : C \to Sm^{desc}(C)$$

by

$$Sm(X)(M) := X^{t(M)}.$$

More formally we should write the functor as the adjoint of the composition

$$C \times N(Mf)^{op} \overset{1d \times t}{\to} C \times N(Top)[W^{-1}]^{op} \to C.$$

Since $t$ satisfies codescent we see that $Sm(X)$ indeed satisfies descent.

**Example 4.14.** We consider a spectrum $E \in N(Sp)[W^{-1}]$.

**Lemma 4.15.** We have

$$\pi_k(Sm(E)(M)) = \mathbb{E}^{-k}(M).$$
Proof. We have

\[
\pi_k(\text{Sm}(E)(M)) \overset{def}{=} \pi_k(E^{t(M)}) \\
\overset{\text{?}}{=} \pi_k(\text{Map}(\Sigma_+^\infty t(M), E)) \\
\cong \pi_0(\text{Map}(\Sigma_+^{\infty+k} t(M), E)) \\
\overset{def}{=} E^{-k}(M).
\]

\[\square\]

Example 4.16. We consider a complex of sheaves \(A \in \text{Ch}(\text{Sh}_{\text{Ab}}(\text{Mf}))\) as an object of \(\text{Sm}(\mathbb{N}(\text{Ch}))\).

Problem 4.17. Show that \(A \in \text{Sm}(\mathbb{N}(\text{Ch}))\) satisfies descent. Furthermore, show by example that its image \(\infty A \in \text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}])\) may not satisfy descent. Finally show that if \(A\) is a complex of fine sheaves, then \(\infty A\) satisfies descent.

Proof. Descent in \(\text{Sm}(\mathbb{N}(\text{Ch}))\) is equivalent to the degree-wise sheaf condition. In the case of \(\text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}])\), for an open covering \(U\) of \(M\), the desired limit is represented by the Čech complex (see Problem 4.23 for an argument)

\[\hat{C}(U, A) \cong \lim_{\Delta \to} \text{lim}_{\mathbb{N}(\Delta)} A(U^*)\,.
\]

Let \(A = \mathbb{Z}\). Then \(\infty A\) does not satisfy descent. To this end consider a good (i.e. all multiple intersections are empty or contractible) covering of a manifold with higher-degree integral cohomology which is calculated by \(\hat{C}(U, \mathbb{Z})\). Of course, \(\infty \mathbb{Z}(M)\) lives in degree 0. Since the Čech complex of a fine sheaf is acyclic (Lemma 3.5) we get descent in this case. \[\square\]

Example 4.18. Let * denote the final category. It has one object and one morphism. There is a natural functor \(p : \text{Mf} \to *\). We have an equivalence \(\text{Fun}(N(*), C) \cong C\).

Problem 4.19. Let \(X \in C\) and \(p^* X \in \text{Sm}(C)\) be the constant smooth object. Show that \(L(p^* X)(*) \cong X\).

Proof. We have a natural morphism

\[c : p^* X \to \text{Sm}(X)\,.
\]

We use homotopy invariance of \(p^* X\) and \(\text{Sm}(X)\) to show that the induced map \(L(p^* X) \to L(\text{Sm}(X))\) is an equivalence. This implies by [4.11] that \(L(p^* X) \to L(\text{Sm}(X))\) is an equivalence. Since \(\text{Sm}(X)\) satisfies descent we have \(L(\text{Sm}(X))(*) \cong \text{Sm}(X)(*) \cong X\). \[\square\]

Definition 4.20. A smooth object \(X \in \text{Sm}^{\text{desc}}(C)\) is called constant if there exists an object \(X_* \in C\) such that \(L(p^* X_*) \cong X\).
It is clear by 4.19 that \( \ast \) must be the evaluation of \( \ast \) on the manifold \( \ast \). The following reproduces a result of Dugger, [Dug01].

**Problem 4.21.** Show that a smooth space \( X \in \text{Sm}^{\text{desc}}(N(\text{Top})[W^{-1}]) \) is constant if and only if \( X = \text{Sm}(X_\ast) \) for some space \( X_\ast \in N(\text{Top})[W^{-1}] \).

**Proof.** There is a canonical map \( L(p^\ast X(\ast)) \to \text{Sm}(X(\ast)) \). Show that it is an equivalence. Show that both \( L(p^\ast X(\ast)) \) and \( \text{Sm}(X(\ast)) \), are homotopy invariant (see Problem 4.31 for the first). Then use coverings and descent in order to reduce to the problem to show that \( L(p^\ast X(\ast))(U) \to \text{Sm}(X(\ast))(U) \) is an equivalence for all contractible \( U \). Finally, this holds true since by homotopy invariance both evaluations are equivalent to \( X(\ast) \).

**Example 4.22.** The following exercise provides a tool for the explicit computation of some limits in \( N(\text{Ch})[W^{-1}] \). We consider a cosimplicial chain complex \( A \in \text{Fun}(\Delta, \text{Ch}) \) and its version \( \infty A \in \text{Fun}(N(\Delta), N(\text{Ch})[W^{-1}]) \). We let \( \text{tot}(A) \in \text{Ch}[W^{-1}] \) be the chain complex obtained as total complex of the double complex associated to \( A \) by normalizing the simplicial direction.

**Problem 4.23.** Show that there is an equivalence

\[
\lim_{N(\Delta)\infty} A \cong \text{tot}(A) .
\]

**Proof.** We first show that \( \text{tot}(A) \cong R\lim_\Delta(A) \). We use the projective model category on \( \text{Ch} \) in order to present \( N(\text{Ch})[W^{-1}] \). In order to present \( \text{Fun}(N(\Delta), N(\text{Ch})[W^{-1}]) \) we then use the injective model category structure on \( \text{Ch}^{\Delta} \). We consider the object \( C_\ast(\Delta^\bullet) \in \text{Ch}^{\Delta} \) and form for \( A \in \text{Ch}^{\Delta} \) the internal \( \text{Hom}(C_\ast(\Delta^\bullet), A) \in \text{Ch}^{\Delta} \). We write \( \text{hom}(A, B) := \lim_\Delta \text{Hom}(A, B) \). By an explicit calculation we observe that

\[
\text{tot}(A) = \text{hom}(C_\ast(\Delta^\bullet), A) .
\]

With the chosen model category structures the functor

\[
\text{hom} : (\text{Ch}^{\Delta})^{\text{op}} \times \text{Ch}^{\Delta} \to \text{Ch}
\]

is bi-Quillen. The projection \( \Delta^\bullet \to \ast \) induces the map \( C_\ast(\Delta^\bullet) \to \mathbb{Z} \), and dually

\[
\lim_\Delta A \cong \text{hom}(\mathbb{Z}, A) \to \text{hom}(C_\ast(\Delta^\bullet), A) \cong \text{tot}(A) .
\]

(32)

A level-free lower bounded chain complex is projectively cofibrant in \( \text{Ch} \). Hence \( \mathbb{Z} \) and \( C_\ast(\Delta) \) are injectively cofibrant in \( \text{Ch}^{\Delta} \). It follows that (32) is an equivalence if \( A \) is fibrant. Moreover, \( \text{tot} \) preserves equivalences. Let \( A \to R(A) \) be a fibrant replacement. We get a chain of equivalences

\[
R\lim_\Delta(A) \overset{\text{def}}{=} \lim_\Delta R(A) \overset{32}{=} \text{tot}(R(A)) \cong \text{tot}(A) .
\]

77
The final step is to observe that the natural map

$$\text{Fun}(N(\Delta), N(\text{Ch}))[W^{-1}] \to \text{Fun}(N(\Delta), N(\text{Ch})[W^{-1}])$$

induces an equivalence which maps $A$ to $\infty A$ such that

$$\lim_{\Delta} A \cong R\lim_{\Delta}(A).$$

This rigidification result uses the fact that the class of quasi isomorphisms $W$ is the class of weak equivalences of a combinatorial model category structure on $\text{Ch}$ (see [Lur, Prop. 1.3.3.12]).

**Example 4.24.** The following is similar to a theory developed in a different context in [MV99]. Let $C$ be a presentable $\infty$-category and $I := [0, 1] \in \text{Mf}$ be the unit interval.

**Definition 4.25.** We say that $X \in \text{Sm}(C)$ is homotopy invariant, if $\text{pr}^* : X(M) \to X(I \times M)$ is an equivalence for all manifolds $X$.

We let $\text{Sm}^I(C) \subseteq \text{Sm}(C)$ denote the full subcategory of homotopy invariant objects. We have an adjunction

$$\mathcal{I} : \text{Sm}(C) \leftrightarrows \text{Sm}^I(C).$$ (33)

Let $i_0, i_1 : * \to I$ be the inclusions of the endpoints of the interval.

**Definition 4.26.** We say that $X \in \text{Sm}(C)$ is elementary homotopy invariant, if for every manifold $M$ the morphisms $i_0^*, i_1^* : X(I \times M) \to X(M)$ are equivalent.

**Problem 4.27.** Show that elementary homotopy invariance is equivalent to homotopy invariance.

**Proof.** Assume that $X$ is homotopy invariant. Then the compositions

$$X(M) \xrightarrow{\text{pr}^*} X(I \times M) \xrightarrow{i_j^*} X(M), \quad j = 0, 1$$

are both equivalent to $\text{id}_{X(M)}$. Since the first map is an equivalence we conclude that $i_0^*$ and $i_1^*$ are equivalent.

Now assume that $X$ is elementary homotopy invariant. The composition

$$X(M) \xrightarrow{\text{pr}^*} X(I \times M) \xrightarrow{i_0^*} X(M)$$

is equivalent to the identity. We must show that

$$X(I \times M) \xrightarrow{i_j^*} X(M) \xrightarrow{\text{pr}^*} X(I \times M)$$
is also equivalent to the identity. To this end we consider the map $\mu : I \times I \to I$, $\mu(s,t) := st$, and the diagram

$$
\begin{array}{c}
I & \xrightarrow{\ast} & I \\
\downarrow_{i_0 \times 1_I} & & \downarrow_{i_0} \\
I \times I & \xrightarrow{\mu} & I \\
\downarrow_{i_1 \times 1_I} & & \downarrow_{1_I} \\
I & \xrightarrow{id_I} & I \\
\end{array}
$$

We insert this diagram into $X(\cdots \times M)$ in order to obtain the desired result. \qed

We define a functor

$$
\tilde{s} : \text{Sm}(C) \to \text{Sm}(C)
$$

as the composition of

$$
s : \text{Sm}(C) \to \text{Sm}(\text{Fun}(\mathbb{N}(\Delta^{op}), C))
$$

and

$$
colim_{\mathbb{N}(\Delta^{op})} : \text{Sm}(\text{Fun}(\mathbb{N}(\Delta^{op}), C)) \to \text{Sm}(C),
$$

where $s$ is precomposition by

$$
Mf \to \text{Fun}(\Delta, Mf), \quad M \mapsto \Delta^* \times M.
$$

We have a natural map of cosimplicial manifolds $\text{pr} : \Delta^* \times M \to M$ (the target is the constant cosimplicial manifold) which induces a morphism $id \to \tilde{s}$.

**Problem 4.28.** If $X$ is homotopy invariant, then $X \to \tilde{s}(X)$ is an equivalence.

**Proof.** We show that the map of simplicial objects $X(M) \to X(\Delta^* \times M)$ induced by the projection is an equivalence. Hence we must show that $\text{pr}^* : X(M) \to X(\Delta^n \times M)$ is an equivalence for every $M$ and $n \geq 0$.

Let $i_{(0,\ldots,0)} : * \to \Delta^n$ be the inclusion of the zero corner. It suffices to show that $(i_{(0,\ldots,0)} \times id_M)^*$ is an inverse of $\text{pr}^*$. The non-trivial part is the verification that $\text{pr}^* \circ i_{(0,\ldots,0)}^* \cong id_{X(\Delta^n \times M)}$.

We let $\mu : I \times \Delta^n \to \Delta^n$ be the map

$$(s, (0 \leq t_0 \leq \cdots \leq t_n \leq 1)) \mapsto (0 \leq st_0 \leq \cdots \leq st_n \leq 1).$$

Then we have a commutative diagram

$$
\begin{array}{c}
\Delta^n & \xrightarrow{\ast} & \Delta^n \\
\downarrow_{i_0 \times 1_{\Delta^n}} & & \downarrow_{i_{(0,\ldots,0)}} \\
I \times \Delta^n & \xrightarrow{\mu} & \Delta^n \\
\downarrow_{i_1 \times 1_{\Delta^n}} & & \downarrow{id_{\Delta^n}} \\
\Delta^n & \xrightarrow{id_{\Delta^n}} & \Delta^n \\
\end{array}
$$
If we insert this into $X(\cdots \times M)$ and use that $X$ is elementary homotopy invariant, we get the desired equivalence.

**Problem 4.29.** If a morphism $f : X \to Y$ in $\text{Sm}(C)$ induces an equivalence $\bar{s}(f) : \bar{s}(X) \to \bar{s}(Y)$, then $I(f) : I(X) \to I(Y)$ is an equivalence.

*Proof.* This is similar to 4.11. We define

$$\bar{s}^\infty := \text{colim}(\text{id} \to \bar{s} \to \bar{s}^2 \to \cdots).$$

Then $\bar{s}^\infty \to (\bar{s}^\infty)^2$ is an equivalence. Use [Lur09, Prop. 5.2.7.4] (Condition 3.) in order to see that the essential image $\text{Sm}^{\bar{s}^\infty}(C)$ of $\bar{s}^\infty$ is a localization

$$\bar{s}^\infty : \text{Sm}^{\bar{s}^\infty}(C) \subseteq \text{Sm}(C).$$

Since $\bar{s}^\infty$ preserves homotopy invariant objects we have a chain of localizations

$$\text{Sm}^I(C) \subseteq \text{Sm}^{\bar{s}^\infty}(C) \subseteq \text{Sm}(C)$$

and therefore $I \circ \bar{s}^\infty \cong I$. If $\bar{s}(f)$ is an equivalence, then so is $\bar{s}^\infty(f)$ and hence $I \circ \bar{s}^\infty(f) \cong I(f)$. \hfill \Box

**Problem 4.30.** Is $\text{Sm}^I(C) \cong \text{Sm}^{\bar{s}^\infty}(C)$?

**Problem 4.31.** Show that for $X \in C$ the object $L(p^*C)$ is homotopy invariant.

*Proof.* We have a diagram

$$\begin{array}{ccc}
\text{Sm}^{I,\text{desc}}(C) & \longrightarrow & \text{Sm}^{\text{desc}}(C) \\
\downarrow & & \downarrow \\
\text{Sm}^I(C) & \longrightarrow & \text{Sm}(C)
\end{array}$$

of inclusions of full subcategories. Taking left adjoints we get

$$\begin{array}{ccc}
\text{Sm}^{I,\text{desc}}(C) & \leftarrow & \text{Sm}^{\text{desc}}(C) \\
L|_{\text{Sm}^I(C)} & & L
\end{array}$$

Note that for $X \in C$ we have obviously $p^*X \in \text{Sm}^I(C)$ so that

$$L(p^*X) = L(I(p^*X)) = I(L(p^*X)),$$

in other words, $L(p^*X)$ is homotopy invariant. \hfill \Box
4.2 Eilenberg-MacLane and de Rham

Let $A \in \text{Ch}$ be a chain complex of complex vector spaces. It gives rise to a constant sheaf of chain complexes $A \in \text{Ch}(\mathbf{Sh}_{\text{Ab}}(\text{Mf}))$. Recall that $\Omega \in \text{Ch}(\mathbf{Sh}_{\text{Ab}}(\text{Mf}))$ denotes the sheaf of complex de Rham complexes.

**Definition 4.32.** The sheaf of differential forms with coefficients in $A$ is defined by

$$\Omega A := \Omega \otimes_{\mathbb{C}} A \in \text{Ch}(\mathbf{Sh}_{\text{Ab}}(\text{Mf})).$$

**Problem 4.33.** If $M$ is compact, then we have

$$\Omega A(M) = \Omega \otimes_{\mathbb{C}} A.$$

Show by example that the inclusion

$$\Omega \otimes_{\mathbb{C}} A \hookrightarrow \Omega A(M)$$

may be strict without the compactness assumption.

Let $H^*(A)$ be the cohomology of the chain complex $A$. We consider $H^*(A)$ as a chain complex with trivial differential.

**Problem 4.34.** Show that for every manifold $M$ there exists an isomorphism

$$H^*(\Omega H^*(A)(M)) \cong H^*(\Omega A(M)).$$

**Proof.** We can choose a quasi-isomorphism $H^*(A) \to A$. Hence we get a quasi-isomorphism of complexes of sheaves $\Omega H^*(A) \to \Omega A$ which induces the asserted isomorphism on the level of cohomology. □

Recall that $\mathbb{N}(\text{Ch})[W^{-1}]$ is a stable $\infty$-category and hence has mapping spectra. We consider the group of integers $\mathbb{Z}$ as an object of $\mathbb{N}(\text{Ch})[W^{-1}]$ by viewing it as a chain complex concentrated in degree zero. The Eilenberg-MacLane spectrum is defined by

$$H \mathbb{Z} := \text{Map}(\mathbb{Z}, \mathbb{Z}).$$

It can be considered as a commutative algebra in $\mathbb{N}(\text{Sp})[W^{-1}]$ so that we can form its module category $\text{Mod}(H \mathbb{Z})$. The homotopy groups of $H \mathbb{Z}$ are given by

$$\pi_*(H \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \ast = 0 \\ 0 & \ast \neq 0 \end{cases}.$$

This looks innocent at a first glance but seems difficult to show without going to a concrete realization of the $\infty$-category $\mathbb{N}(\text{Ch})[W^{-1}]$.

**Fact 4.35.** There exists a unique symmetric monoidal equivalence

$$H : \mathbb{N}(\text{Ch})[W^{-1}] \simeq \text{Mod}(H \mathbb{Z})$$

such that $H(\mathbb{Z}) = H \mathbb{Z}$. We call $H$ the Eilenberg-MacLane spectrum functor.
Proof. The domain and target are stable symmetric monoidal ∞-categories. The domain is generated as a stable ∞-category under colimits by the chain complex \( Z \) in degree zero, and \( \text{Mod}(HZ) \) is generated in a similar way by \( HZ \). We obtain \( H \) as the unique colimit preserving functor which maps \( Z \) to \( HZ \). See [Lur, Prop. 7.1.2.7] for a general argument of this type. \qed

**Lemma 4.36.** We have a canonical isomorphism

\[
H_\ast(A) \cong \pi_\ast(H(A))
\]

Proof. We can write

\[
H(A) \cong \text{Map}(Z, A)
\]

Indeed, the functor \( \text{Map}(Z, \ldots) \) preserves colimits, commutes with shifts, and produces \( HZ \) for \( A = Z \) by the definition of \( HZ \). Using the resenation of \( H(A) \) we can calculate the homotopy groups as follows:

\[
\pi_k(H(A)) \cong \pi_k(\text{Map}(Z, A))
\]
\[
\cong \pi_0(\text{Map}(Z[k], A))
\]
\[
\cong \pi_0(\text{map}(Z[k], A))
\]
\[
\cong H_k(A)
\]

For a \( HZ \)-module spectrum \( E \in \text{Mod}(HZ) \) we form the smooth object

\[
\text{Sm}(E) \in \text{Sm}(\text{Mod}(HZ))
\]

in \( \text{Mod}(HZ) \). We now consider a chain complex \( A \in \text{Ch} \) of complex vector spaces and the object \( \Omega A \in \text{Sm}(\text{N}(\text{Ch})) \) defined in [4.32]. It induces an object \( \Omega A \in \text{Sm}(\text{N}(\text{Ch})[W^{-1}]) \) and therefore a smooth spectrum

\[
H(\Omega A) \in \text{Sm}(\text{Mod}(HZ))
\]

On the other hand we can form the smooth spectrum

\[
\text{Sm}(H(A)) \in \text{Sm}(\text{Mod}(HZ))
\]

Assume for example that \( A \) is concentrated in degree 0. Then we have

\[
\pi_k(\text{Sm}(H(A))(M)) \cong H(A)^{-k}(M) \cong H^{-k}(M; A) \cong H^{-k}(\Omega A(M))
\]

This isomorphism extends to general complexes. More precisely, we have the following version of the de Rham isomorphism.
Proposition 4.37. There exists an equivalence of smooth spectra
\[ \text{Rham} : H(\Omega A) \to \text{Sm}(H(A)) . \]

Proof. In the following we give an argument which is modeled on the classical proof using integration over simplices. An independent, more functorial proof will be given in [4.111].

Let \( C_*(M) \) be the smooth singular complex of \( M \). Integration over simplices gives a quasi-isomorphism
\[ \int : \Omega A(M) \xrightarrow{\sim} \text{Map}(C_*(M), A) , \]
where \( \text{Map} \) denotes the internal mapping object of \( \text{Ch} \). We therefore get an equivalence of smooth spectra
\[ H(\int) : H(\Omega A(\ldots)) \xrightarrow{\sim} H(\text{Map}(C_*(\ldots), A)) . \]

We now use \[ H(\text{Map}(C_*(\ldots), A)) \cong \text{Map}_{\text{Mod}(HZ)}(H(C_*(\ldots)), H(A)) \]
\[ \cong \text{Map}_{\text{Mod}(HZ)}(\Sigma_+^\infty(\ldots) \wedge HZ, H(A)) \]
\[ \cong \text{Map}_{\text{Mod}(HZ)}(\Sigma_*^\infty(\ldots), H(A)) \]
\[ \cong \text{Sm}(H(A))(\ldots) , \]
where the marked equivalence is given in the following Lemma.

Lemma 4.38. There exists an equivalence in \( \text{Fun}(\mathbb{N}(\text{Mf}), \text{Mod}(HZ)) \)
\[ H(C_*(\ldots)) \xrightarrow{\sim} HZ \wedge \Sigma_+^\infty(\ldots) . \]

Proof. We use that \( H \) preserves the tensor structure and get
\[ H(C_*(\ldots)) \cong H(Z \otimes \ldots) \cong H(Z) \otimes \cdots \cong HZ \wedge \Sigma_+^\infty(\ldots) . \]

As mentioned above an alternative proof of the de Rham equivalence will be given in [4.111].

Note that \( \Omega A \) is a complex of fine sheaves, i.e. it admits an action of partitions of unity. By Problem 4.17 we gave
\[ \Omega A \in \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch})[W^{-1}]) . \]

Since \( H : \mathbb{N}(\text{Ch})[W^{-1}] \to \mathbb{N}(\text{Sp})[W^{-1}] \) is the composition of an equivalence and a right-adjoint, it preserves the descent property. In particular we have
\[ H(\Omega A) \in \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch})[W^{-1}]) . \]

(34)
Example 4.39. We consider the sheaf $\Omega^n_{cl} \in \text{Sh}_{\text{Ab}}(M_{f})$ of closed $n$-forms.

**Problem 4.40.** Calculate the homotopy and cohomology groups of the evaluations on $M$ of the following presheaves and sheaves derived from $\Omega^n_{cl}$ in terms of the de Rham cohomology on $M$.

1. We let $C(\Omega^n_{cl}) \in \text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}])$ be the presheaf of chain complexes given by $\Omega^n_{cl}$ located in degree 0. Calculate $H^*(C(\Omega^n_{cl}))(M))$.

2. Calculate $H^*(L(C(\Omega^n_{cl}))(M))$, where $L : \text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}]) \to \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch})[W^{-1}])$ is the sheafification.

3. Similarly let $C^{\leq 0}(\Omega^n_{cl}) \in \text{Sm}(\text{Ch}^{\leq 0}[W^{-1}])$ be the negatively graded chain complex represented by $\Omega^n_{cl}$ and $L : \text{Sm}(\mathbb{N}(\text{Ch}^{\leq 0})[W^{-1}]) \to \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch}^{\leq 0})[W^{-1}])$. Calculate $H^*(L(C^{\leq 0}(\Omega^n_{cl}))(M))$.

4. Calculate the groups $H^*(\overline{s}C(\Omega^n_{cl}))(M))$, $H^*(\overline{s}C^{\leq 0}(\Omega^n_{cl}))(M))$, $H^*(L(\overline{s}C(\Omega^n_{cl}))(M))$ and $H^*(L(\overline{s}C^{\leq 0}(\Omega^n_{cl}))(M))$, where $s$ is as in 4.24.

5. We consider $\Omega^n_{cl}$ as a smooth constant simplicial abelian group $S(\Omega^n_{cl}) \in \text{Sm}(\mathbb{N}(s\text{Ab})[W^{-1}])$. Calculate $\pi_* (S(\Omega^n_{cl}))(M))$.

6. Calculate $\pi_* (\overline{s}S(\Omega^n_{cl}))(M))$.

**Proof.** These calculations can be found, with different notation, in [HS05]. In order to understand sheafifications use 4.12.

1. 
\[
H^*(C(\Omega^n_{cl}))(M)) = \begin{cases} 
\Omega^n_{cl}(M) & * = 0 \\
0 & * \neq 0 
\end{cases}
\]

2. 
\[
H^*(L(C(\Omega^n_{cl}))(M)) = \begin{cases} 
\Omega^n_{cl}(M) & * = 0 \\
H^{n+*}_{dR}(M) & * > 0 \\
0 & * < 0 
\end{cases}
\]

3. 
\[
H^*(L(C^{\leq 0}(\Omega^n_{cl}))(M)) = \begin{cases} 
0 & * \geq 0 \\
H^{n+*}_{dR}(M) & * < 0 
\end{cases}
\]

4. 
\[
H^*(\overline{s}C(\Omega^n_{cl}))(M)) = H^*(\overline{s}C^{\leq 0}(\Omega^n_{cl}))(M)) = \begin{cases} 
H^{n+*}_{dR}(M) & * \leq 0 \\
0 & * > 0 
\end{cases}
\]
\[
H^*(L(\overline{s}C(\Omega^n_{cl}))(M)) = H^*_{dR}(M)
\]
\[
H^*(L(\overline{s}C^{\leq 0}(\Omega^n_{cl}))(M)) = \begin{cases} 
H^{n+*}_{dR}(M) & * < 0 \\
0 & * \geq 0 
\end{cases}
\]
5. \[ \pi_*(S(\Omega^n_{cl})(M)) = \begin{cases} \Omega^n_{cl}(M) & * = 0 \\ 0 & * \neq 0 \end{cases} \]

6. \[ \pi_*(\overline{S}\Omega^n_{cl})(M)) = \begin{cases} H^{n-*}_{dR}(M) & * \geq 0 \\ 0 & * < 0 \end{cases} \]

4.3 Differential function spectra and differential cohomology

A homotopy theoretic construction of a differential extension of a generalized cohomology has first been given in the foundational paper by Hopkins and Singer \cite{HS05}. In that paper the differential cohomology was defined in terms of the homotopy groups of differential function spaces. The approach in the present paper uses the differential function spectrum. Such objects have already been built in \cite{HS05} from differential function spaces. In our present course we give a direct construction of differential function spectra. It emphasizes the stable aspect of differential cohomology. This will be important for the construction of operations like integration or transfer.

Before we start with the construction we define the notion of a differential extension of a generalized cohomology theory. This is important since there are many examples where a differential extension has been constructed by other methods, e.g. of geometric or analytic nature. The axioms list the properties which one expects to hold in every model of a differential refinement of a generalized cohomology theory.

We start with a description of the data which goes into the definition. For an abelian group \( A \) we can form the Moore spectrum \( MA \). It is a connective spectrum characterized by \( HZ \wedge MA \cong H(A) \).

For a spectrum \( E \) we write \( EA := E \wedge MA \). Note that \( MZ \cong S \) and that we have a canonical map \( MZ \to MA \) inducing \( E \to EA \).

**Problem 4.41.** Show that for every \( i \in \mathbb{Z} \) there is an exact sequence

\[ 0 \to \pi_i(E) \otimes A \to \pi_i(EA) \to \text{Tor}(\pi_{i-1}(E) \otimes A) \to 0 . \]

**Proof.** Start with a presentation

\[ 0 \to \bigoplus_{\alpha} \mathbb{Z} \to \bigoplus_{\beta} \mathbb{Z} \to A \to 0 \]

of the group \( A \). Model this by a map between wedges of sphere spectra and obtain \( MA \) as cofibre

\[ \bigvee_{\alpha} S \to \bigvee_{\beta} S \to MA . \]

Smash with \( E \) and discuss the associated long exact sequence in homotopy. \( \square \)
Problem 4.42. Show that $HC \cong MC$. Furthermore, if $F$ is a $HC$-module spectrum, then every map $c : E \to F$ of spectra uniquely extends to a map $c_C : EC \to F$ of $HC$-module spectra.

The construction of the differential extension of $E^*$ will depend on the choice of differential data.

Definition 4.43. A differential data is a triple $(E, A, c)$ consisting of

1. a spectrum $E \in N(Sp)[W^{-1}]$,
2. a chain complex $A \in Ch$ of complex vector spaces, and
3. a map $c : E \to H(A)$ in $N(Sp)[W^{-1}]$.

We say that $(E, A, c)$ is strict if $c$ induces an equivalence $c_C : EC \to H(A)$. A morphism of differential data $(E, A, c) \to (E', A', c')$ is a commutative diagram in $N(Sp)[W^{-1}]$

$$
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow{c} & & \downarrow{c'} \\
H(A) & \xrightarrow{H(\phi)} & H(A')
\end{array}
$$

for some morphisms $f : E \to E'$ and $\phi : A \to A'$.

Given the spectrum $E$ we can take $A := \pi_* E \otimes \mathbb{C}$ and let $c : E \to H(A)$ be a map uniquely determined up to homotopy such that that it induces $\pi_*(E) \to \pi_*(A) \cong \pi_*(E) \otimes \mathbb{C}, \ x \mapsto x \otimes 1$. This datum is strict.

Definition 4.44. We call a differential data of this form the canonical differential data.

Problem 4.45. Analyse the problem whether one can define a functor from spectra to differential data which maps a spectrum to its canonical differential data.

Proof. The answer is "no". The problem is that the map $c$ is unique up to homotopy, but the homotopy is not unique. $\square$

Definition 4.46. A differential extension of the cohomology theory $E^*$ associated to a differential data $(E, A, c)$ is a tuple $(\hat{E}^*, R, I, a)$ consisting of

1. a functor $\hat{E}^* : Mf^{op} \to \{\mathbb{Z}\text{-graded abelian groups}\}$,
2. a transformation $R : \hat{E}^* \to \Omega A^*_c$,
3. a transformation $I : \hat{E}^* \to E^*$, and
4. a transformation $a : \Omega A^{*-1}/\text{im}(d) \to \hat{E}^*$,

such that
1. \( \text{Rham} \circ R = c \circ I \),
2. \( R \circ a = d \),
3. 
   \[
   E^{*-1} \xrightarrow{\varepsilon} \Omega A^{*-1}/\text{im}(d) \xrightarrow{a} \hat{E}^* \xrightarrow{I} E^* \rightarrow 0
   \]  
   \hspace{1cm} (35)

is exact.

There is an obvious notion of a morphism between differential extensions associated to the same differential data. In view of (35) a morphism is automatically an isomorphism by the Five Lemma.

**Example 4.47.** Observe that \( (\hat{H}^*_\text{Del}, R, I, a) \) is a differential extension of \( HZ^* \) associated to the canonical data.

Let \( (\hat{E}^*, R, I, a) \) be a differential extension. The homotopy formula measures the deviation of the functor \( \hat{E} \) from homotopy invariance.

**Lemma 4.48 (Homotopy formula).** For \( x \in \hat{E}^*(\mathbb{R})([0,1] \times M) \) we have the equality

\[
i_1^*x - i_0^*x = a(\int_{[0,1] \times M/M} R(x))
\]

in \( \hat{E}^*(M) \).

**Proof.** The same argument as in Proposition 3.28. \(\square\)

In the following we give the general construction of differential extensions using differential function spectra. Recall that the smooth objects \( \text{Sm}(E), H(\sigma^\geq n \Omega A) \) and \( H(\Omega A) \) satisfy descent. Modelled on the construction of Deligne cohomology we make the following definition.

**Definition 4.49.** We define the \( n \)'th differential function spectrum

\[
\text{Diff}^n(E, A, c) \in \text{Sm}^{\text{desc}}(\mathcal{M}(\text{Sp})[W^{-1}])
\]

by

\[
\text{Diff}^n(E, A, c) := \text{Cone} \left( \text{Sm}(E) \vee H(\sigma^\geq n \Omega A) \xrightarrow{c - \text{Rham}} \text{Sm}(H(A)) \right) [-1].
\]

Recall the definition of the Deligne complex 3.19. The following problem continues Example 4.47

**Problem 4.50.** Let \( E = H\mathbb{Z} \) and \( A = \mathbb{C} \) and \( c : H\mathbb{Z} \rightarrow H(\mathbb{C}) \) be the canonical map. Show that there is a natural equivalence

\[
H(D(n)) \cong \text{Diff}^n(H\mathbb{Z}, \mathbb{C}, c).
\]

We now come back to the general case and define the differential cohomology groups in terms of the homotopy groups of the differential function spectrum.
**Definition 4.51.** The differential cohomology functor associated to \((E, A, c)\) is defined by

\[
\hat{E}^n := \pi_{-n}(\text{Diff}^n(E, A, c)).
\]

Note that in degree \(n\) we use the differential function spectrum indexed by \(n\). This situation is similar as in the construction of Deligne cohomology, Definition 3.19.

We now discuss the dependence of the differential function spectrum on the data. One can describe the \(\infty\)-category of strict differential data as a pull-back in \(\infty\)-categories \(\infty\text{Cat}\)

\[
\begin{array}{ccc}
\text{Data}^{str} & \overset{}{\longrightarrow} & \mathbb{N}(\text{Ch}) \\
\downarrow & & \downarrow H \\
\mathbb{N}(\text{Sp})[W^{-1}] & \overset{}{\longrightarrow} & \text{Mod}(H \mathbb{C})
\end{array}
\]

**Problem 4.52.** Show that there exists a natural map of monoids in \(\mathbb{N}(\text{sSet})[W^{-1}]\)

\[
\Omega \text{map}(E, H(A)) \to \text{end}(E, A, c).
\]

**Proof.** Let \(x\) be an object of an \(\infty\)-category \(C\). Then we can identify the group of automorphisms of \(x\) as a pull-back

\[
\begin{array}{ccc}
\text{aut}(x) & \longrightarrow & \text{Fun}(BZ, C) \\
\downarrow & & \downarrow x \\
\bullet & \longrightarrow & C
\end{array}
\]

where the right vertical map is induced by the base point \(\bullet \to BZ\), and the group structure comes from the cogroup structure of \(BZ\). Therefore

\[
\begin{array}{ccc}
\text{aut}(E, A, c) & \longrightarrow & \text{Fun}(BZ, \text{Data}^{str}) \\
\downarrow & & \downarrow \text{Data}^{str} \\
\bullet & \longrightarrow & \text{Data}^{str}
\end{array}
\]

We now insert the definition of \(\text{Data}^{str}\) as a pull-back, move the pull-back outside the functor \(\text{Fun}(BZ, \ldots)\), and restrict to \(\text{id}_E\) and \(\text{id}_A\) in the lower left and upper right corners. This gives a diagram with \(\text{Fun}(BZ, \text{map}(EC, H(A)))\) in the upper left corner and therefore a map

\[
\Omega \text{map}(EC, H(A)) \cong \text{Fun}(BZ, \text{map}(EC, H(A))) \to \text{aut}(E, A, c).
\]

Finally we precompose use the canonical map \(\text{map}(E, H(A)) \to \text{map}(EC, H(A))\).

We now show how one can interpret the construction of the differential function spectrum as a functor from the \(\infty\)-category of data to spectra. Let \(A^2_2\) be the category of the shape

\[
\bullet \to \bullet \leftarrow \bullet.
\]
Then we get a natural functor

\[ \tilde{P}^n : \text{Data}^{\text{str}} \to \text{Fun}(N(\Lambda^2_2), \text{Sm}^{\text{desc}}(N(\text{Sp})[W^{-1}])) , (E, A, c) \mapsto (\text{Sm}(E) \to \text{Sm}(EC) \leftarrow H(\sigma^\geq_n(\Omega A))). \]

We have

\[ \text{Diff}^n(E, A, c) \cong \lim_{n(\Lambda^2_2)} \tilde{P}^n(E, A, c). \]

**Problem 4.53.** Give a precise \( \infty \)-categorical description of the functor \( P^n : \text{Data}^{\text{str}} \to \text{Sm}^{\text{desc}}(N(\text{Sp})[W^{-1}]) \) explained above.

**Proof.** We have a natural transformation in \( \text{Fun}(\Lambda^2_2, \infty \text{Cat}) \)

\[
\begin{array}{ccc}
N(\text{Sp})[W^{-1}] & \xrightarrow{\text{Ch}} & N(\text{Sp})[W^{-1}] \\
\downarrow \text{Sm} & & \downarrow \text{Sm} \\
\text{Sm}^{\text{desc}}(N(\text{Sp})[W^{-1}]) & \xrightarrow{\text{Id}} & \text{Sm}^{\text{desc}}(N(\text{Sp})[W^{-1}]) \\
\end{array}
\]

Taking the limit over \( N(\Lambda^2_2) \) we get the functor \( P^n : \text{Data}^{\text{str}} \to \text{Sm}(N(\text{Sp})[W^{-1}]). \)

For completeness let us mention that in order to capture non-strict data we can form the limit of

\[
\begin{array}{ccc}
\text{Data} & \xrightarrow{\text{H}} & N(\text{Ch}) \\
\downarrow & & \downarrow \text{H} \\
N(\text{Sp})[W^{-1}] & \xrightarrow{\text{Id}} & N(\text{Sp})[W^{-1}]^N[1] \\
\end{array}
\]

We now calculate the homotopy groups of the differential function spectrum.

**Theorem 4.54.** For \( i > n \) we have

\[ \pi_{-i}(\text{Diff}^n(E, A, c)) \cong E^i(M). \]

For \( i < n \) we have

\[ \pi_{-i}(\text{Diff}^n(E, A, c)) \cong \text{Cone}(c)^{i-1}(M), \]

and if \( (E, A, c) \) is strict, then

\[ \pi_{-i}(\text{Diff}^n(E, A, c)) \cong EC/Z^{i-1}(M). \]

Finally, for \( i = n \) we have an exact sequence

\[ E^{n-1}(M) \to H(A)^{n-1}(M) \to E^n(M) \oplus \Omega A^n_3(M) \to H(A)^n(M) \to 0. \quad (36) \]

**Proof.** We have a long exact sequence of homotopy groups associated to the cone which for \( i > n \) reads as

\[
\begin{array}{c}
E^{i-1}(M) \oplus H(A)^{i-1}(M) \to H(A)^{i-1}(M) \leftarrow \pi_{-i}(\text{Diff}^n(E, A, c)) \\
\end{array}
\]

\[
\begin{array}{c}
\to E^i(M) \oplus H(A)^i(M) \to H(A)^i(M) \\
\end{array}
\]
This gives the assertions for $i > n$. A similar reasoning using $\pi_{-n}(H(\sigma^{\geq n}\Omega A) = \Omega A^{n}_{cl}$ gives the case $i = n$.

We have a triangle

$$
\rightarrow \text{Cone}\left(\text{Sm}(E) \xrightarrow{\sigma} \text{Sm}(H(A))\right) \left[-1\right] \rightarrow \text{Diff}(E, A, c) \rightarrow H(\sigma^{\geq n}\Omega A) \rightarrow \Omega \sigma_{\geq n}\Omega A_n_{cl}
$$

If $(E, A, c)$ is strict, then we have an equivalence

$$
\text{Cone}\left(\text{Sm}(E) \xrightarrow{\sigma} \text{Sm}(H(A))\right) \left[-1\right] \cong \text{Sm}(E\mathbb{C}/\mathbb{Z})[-1].
$$

Since for $i < n$ we have $\pi_{-i}H(\sigma^{\geq n}\Omega A) = 0$ we conclude the assertion for $i < n$. \qed

We now consider the fibre sequence defining the spectrum $E(n)$

$$
\text{Cone}(H(\sigma^{\geq n}\Omega A) \xrightarrow{\sigma} H(\sigma^{\geq n}\Omega A))[-1] \rightarrow \text{Diff}^{n}(E) \xrightarrow{\pi} \mathcal{E}(n),
$$

where the first map is the composition of the de Rham equivalence

$$
\text{Cone}(H(\sigma^{\geq n}\Omega A) \rightarrow H(\Omega A))[-1] \xrightarrow{\sim} \text{Cone}(H(\sigma^{\geq n}\Omega A) \rightarrow \text{Sm}(H(A)))
$$

with the two obvious inclusions. The map $\pi$ induces an equivalence. We can identify

$$
\mathcal{E}(n) \cong \text{Cone}(\text{Sm}(E) \rightarrow H(\sigma^{< n}\Omega A))[-1].
$$

The map $H(\sigma^{< n}\Omega A)) \rightarrow \mathcal{E}(n)$ induces a map

$$
a : \Omega A^{n-1}(M)/\text{im}(d) \rightarrow \hat{E}^{n}(M) .
$$

The same arguments as for Deligne cohomology show:

**Proposition 4.55.** The tuple $(\hat{E}^{*}, R, I, a)$ is a differential extension of $E^{*}$. In detail, the map $a : \Omega A^{n-1}(M)/\text{im}(d) \rightarrow \hat{E}^{n}(M)$ extends the map $H(A)^{n-1}(M) \rightarrow \hat{E}^{n}(M)$. We have

$$
R \circ a = d .
$$

The sequence

$$
E^{n-1}(M) \rightarrow \Omega A^{n-1}(M)/\text{im}(d) \rightarrow \hat{E}^{n}(M) \rightarrow E^{n}(M) \rightarrow 0 \quad (37)
$$

is exact. If $(E, A, c)$ is strict, then we have the differential cohomology diagram

$$
\begin{align*}
\Omega A^{n-1}(M)/\text{im}(d) & \xrightarrow{d} \Omega A^{n}_{cl}(M) \\
EC^{n-1}(M) & \xrightarrow{a} \hat{E}^{n}(M) \\
EC/\mathbb{Z}^{n-1}(M) & \xrightarrow{-\text{Bockstein}} E^{n}(M)
\end{align*}
$$

90
Example 4.56. We consider the differential data $(HZ, C, HZ \rightarrow HC)$ in order to define the differential extension $(\hat{HZ}, R, I, a)$.

Problem 4.57. Show that there is a unique isomorphism of differential extensions between $(\hat{HZ}, R, I, a)$ and $(\hat{H}_{\text{Del}}, R, I, a)$.

Proof. See [BS10] or [SS08a].

Example 4.58. Recall the action 4.52 of $\Omega \text{map}(E, H(A))$ on the datum $(E, A, c)$.

Problem 4.59. Describe the action of $\pi_0(\Omega \text{map}(E, H(A))) \cong H(A)^{-1}(E)$ on $\hat{E}^n$.

Proof. Let $\phi \in \Omega \text{map}(E, H(A))$ and $\hat{x} \in \hat{E}^n(M)$. We write $\nu_\phi(\hat{x})$ for the action of $\phi$ on $\hat{x}$. Note that $\phi(I(\hat{x})) \in H(A)^{-1}(M)$. We get

$$\nu_\phi(\hat{x}) = a(\phi(I(\hat{x}))).$$

\[\square\]

Unlike generalized cohomology theories, differential cohomology is not homotopy invariant. The deviation from homotopy invariance is measured by the homotopy formula 4.48. But it still has an interesting Mayer-Vietoris sequence as a consequence of the descent property of the differential function spectrum.

Problem 4.60. Construct the Mayer-Vietoris sequence in differential cohomology. Assume that the data is strict.

Proof. If $M = U \cup V$ is a decomposition into two open submanifolds, then by the descent property of the differential function spectrum we get a pull-back

$$\begin{array}{ccc}
\text{Diff}^n(E, A, c)(M) & \longrightarrow & \text{Diff}^n(E, A, c)(U) \\
\downarrow & & \downarrow \\
\text{Diff}^n(E, A, c)(V) & \longrightarrow & \text{Diff}^n(E, A, c)(U \cap V)
\end{array}$$

in $\mathbb{N}(\text{Sp})[W^{-1}]$. The interesting segment of the long exact sequence is

$$\cdots \rightarrow E\mathbb{C}/\mathbb{Z}^{n-2}(U \cap V) \rightarrow \hat{E}^n(M) \rightarrow \hat{E}^n(U) \oplus \hat{E}^n(V) \rightarrow \hat{E}^n(U \cap V) \rightarrow E^{n+1}(M) \rightarrow \cdots.$$

It extends by the Mayer-Vietoris sequence of $E\mathbb{C}/\mathbb{Z}^*$ and $E^*$ to the left- and right-hand sides. \[\square\]

Example 4.61. Let $V$ be some abelian group and $G$ be a monoid acting on $V$.

Problem 4.62. Show that there is a unique extension differential extension of the cohomology theory $H^*(\ldots; V)$, and that the action of $G$ extends to an action on this differential extension.
Example 4.63. A differential cohomology class \( x \in \hat{E}^n(M) \) is essentially the datum of an underlying class \( I(x) \in E^n(M) \) together with form data \( R(x) \in \Omega A_d^n(M) \) combined in an appropriate homotopical manner. This leads to the following two very suggestive alternative descriptions of differential cohomology suggested to me by Bruce Williams. These models make the nature of the structure maps \( R, I, a \) completely clear.

Problem 4.64. Show that there are natural equivalences between \( \text{Diff}^n(E, A, c) \) and the pull-backs in \( \text{Sm} (\mathbb{N}(\text{Sp})[W^{-1}]) \)

\[
\begin{align*}
\text{Diff}^n(E, A, c)' & \longrightarrow H(L(\Omega A_d^n)) , & \text{Diff}^n(E, A, c)'' & \longrightarrow H(L(\Omega A_d^{n-1}/\text{im}(d))) . \\
\text{Sm}(E) & \longrightarrow \text{Sm}(H(A)) & \text{Sm}(E) & \longrightarrow \text{Sm}(H(A)) \\
\end{align*}
\]

Proof. Use 4.12. \( \square \)

Let \( \iota : \text{Mod}(HZ) \rightarrow \mathbb{N} (\text{Sp})[W^{-1}] \) be the forgetful functor.

Problem 4.65. Do we have an equivalence \( L \circ \iota \cong \iota \circ L \) of transformations \( \text{Sm}(\text{Mod}(HZ)) \rightarrow \text{Sm}^{\text{desc}} (\mathbb{N}(\text{Sp})[W^{-1}]) \).

A positive answer would allow to interchange \( H \) and \( L \) in right-upper corners of the pull-back diagrams in 4.64.

4.4 Differential \( K \)-theory

One of the most important examples of differential cohomology theories is the differential extension of complex \( K \)-theory \( \text{KU}^* \). It was first used in string theory and \( M \)-theory in order to model the global topology of fields with differential form field strength, see e.g. [Fre00], [FH00]. It is also closely related to the local index theory of Dirac operators and can be used to formulate refined index theorems capturing secondary information, see 4.247, [FL10]. To some extent, the consideration of differential complex \( K \)-theory motivated the development of differential cohomology theory as a field of mathematical research. Predating the constructions of full differential extension of complex \( K \)-theory there were geometric constructions of the functor \( \text{KU}R/Z \), starting with Karoubi [Kar90], [Kar94], [Kar87] and Lott [Lot91]. There are various topological, geometric and analytic models of differential complex \( K \)-theory, see e.g. [HS05], [BS09], [BSSW09], [SS08b], [Ort09], [BM06]. This variety of models called for the investigation of uniqueness in [BS10].

Here we present differential complex \( K \)-theory as an example of our general construction. The main tool to relate it with geometric problems is the cycle map defined in 4.70. We consider a ring spectrum \( \text{KU} \) representing the generalized cohomology theory \( \text{KU}^* \), called complex \( K \) theory. In order to be able to connect with geometry we fix an identification

\[
\Omega^\infty \text{KU} \cong \mathbb{Z} \times \text{BU}
\] (38)
as $h$-spaces. If $KU$ is defined using the classical approach via vector bundles, then this identification is built in into the construction. On the other hand there are more abstract constructions of $KU$ e.g. from the multiplicative formal group law using the Landweber exact functor theorem. In this case the relation with vector bundles is not obvious.

We fix the Bott element $b \in \pi_2(KU)$ represented by the map

$$S^2 \cong \mathbb{C}P^1 \xrightarrow{L^*} BU(1) \hookrightarrow BU \rightarrow \{0\} \times BU \hookrightarrow \mathbb{Z} \times BU \cong \Omega^\infty KU,$$

where $L^* \rightarrow \mathbb{C}P^1$ is the dual of the tautological bundle (see [2.36]). Note that the cohomological degree of the Bott element $b$ is $-2$. The choice of $b \in \pi_2(KU)$ determines an isomorphism of rings $\pi_*(KU) \cong \mathbb{Z}[b, b^{-1}]$. We define the chain complex $A := \mathbb{C}[b, b^{-1}] \in \text{Ch}$ with trivial differential. There is a map

$$c : KU \rightarrow H(A)$$

uniquely determined up to a unique homotopy class of homotopies which induces the inclusion

$$\pi_* (KU) \cong \mathbb{Z}[b, b^{-1}] \hookrightarrow \mathbb{C}[b, b^{-1}] \cong \pi_* (H(A))$$

on the level of homotopy groups.

**Problem 4.66.** Verify this assertion.

We let $(KU, A, c)$ be our strict differential data for the differential extension $\widehat{KU}^*$.  

**Problem 4.67.** Calculate as abelian groups $\widehat{KU}^*(\ast)$ and $\widehat{KU}^*(S^1)$.

A complex vector bundle $V \rightarrow M$ represents a class $KU^0(M)$. In order to make this precise we have to use the identification (38). Note that $\bigsqcup_{n \geq 0} BU(n)$ is the classifying space for complex vector bundles. Let

$$v : M \rightarrow \bigsqcup_{n \geq 0} BU(n)$$

be the classifying map of $V$.

**Definition 4.68.** We define the topological cycle $\text{cycl}(V) \in KU^0(M)$ of the complex vector bundle $V \rightarrow M$ as the homotopy class of the composition

$$M \xrightarrow{v} \bigsqcup_{n \geq 0} BU(n) \rightarrow \mathbb{Z} \times BU \cong \Omega^\infty KU.$$

Let $\text{Vect} : Mf^{op} \rightarrow \text{CommMon} (\text{Set})$ be functor which associates to a manifold $M$ the commutative monoid of isomorphism classes of vector bundles with respect to the direct sum. Then we can interpret the topological cycle map

$$\text{cycl} : \text{Vect} \rightarrow KU^0$$
as a map of smooth monoids $\text{Sm}(\text{CommMon}(\text{Set}))$. This uses the additional precision that \((38)\) is a map of $h$-spaces.

Let $\nabla$ be a connection on the complex vector bundle $V \to M$. Then we can define a Chern character form

$$\text{ch}(\nabla) := \text{Tr} \exp(-\frac{bR^\nabla}{2\pi i}) \in \Omega A^0_{\text{cl}}(M).$$

This is essentially (2.46). We add the factor $b$ of cohomological degree $-2$ in order to get an element of degree zero.

**Problem 4.69.** Show that $\text{Rham}([\text{ch}(\nabla)]) = c(\text{cycl}(V))$ in $H(A)^0(M)$.

**Proof.** The natural transformation $\text{Vect}(M) \ni V \mapsto \text{Rham}([\text{ch}(\nabla)]) \in H(A)^0(M)$ is additive and hence factorizes over a transformation $\text{KU}^0(M) \to H(A)^0(M)$. The restriction of this transformation to compact manifolds is thus represented by a map of $h$-spaces $\Omega^\infty \text{KU} \to \Omega^\infty H(A)$, i.e. a primitive cohomology class in $H(A)^0(\Omega^\infty \text{KU})$. We know that $H^\text{ev}(\Omega^\infty \text{KU}; \mathbb{C})^{\text{prim}}$ is spanned by the classes $\text{ch}_{2n}$ for $n \geq 0$. It follows that $H(A)^0(\Omega^\infty \text{KU})^{\text{prim}}$ is spanned by the classes $b^n \text{ch}_{2n}$ for $n \geq 0$. We must only check that $\text{Rham}(\text{ch}(\nabla)) = b^n \text{or}_{S^{2n}} \in H(A)^0(S^{2n})$ when the (virtual) bundle $V \to S^{2n}$ represents the element in $\widetilde{\text{KU}}^0(S^{2n})$ which corresponds to $b^n \in \text{KU}^{2n}(\ast)$. Using multiplicativity of the Chern character it suffices to check this for $n = 1$. The Bott generator is $\text{cycl}(L^\ast) - \text{cycl}(1)$. In this case, by 2.36

$$\text{Rham}(\text{ch}((\text{cycl}(L^\ast)))) - \text{Rham}(\text{ch}(\text{cycl}(1))) = \text{bor}_{S^2}$$

as required. \qed

We let

$$\text{Vect}^\text{geom} : \text{Mf}^{\text{op}} \to \text{CommMon}$$

be the functor which associates to a manifold $M$ the commutative monoid (under direct sum) of isomorphism classes of vector bundles with connection.

**Definition 4.70.** A differential refinement of the topological cycle map is a transformation

$$\text{cycl} : \text{Vect}^\text{geom} \to \text{KU}^0$$

of semigroup valued functors satisfying

1. $R(\text{cycl}(V, \nabla)) = \text{ch}(\nabla) \in \Omega A^0_{\text{cl}}$

2. $I(\text{cycl}(V, \nabla)) = \text{cycl}(V) \in \text{KU}^0$.

**Theorem 4.71.** There exists a unique differential refinement of the topological cycle map.
Proof. Let \((V, \nabla^V)\) be a \(k\)-dimensional vector bundle with connection on a manifold \(M\) of dimension \(n\). Then we can find a \(n + 1\)-connected approximation \(N \to BU(k)\) with a bundle with connection \((W, \nabla^W)\) and a smooth map \(h : M \to N\) such that \(h^*(W, \nabla^W) \cong (V, \nabla^V)\). Observe that \(0 = h^*: H(A)^{-1}(N) \to H(A)^{-1}(M)\). We choose some element \(w \in \widehat{KU}^0(N)\) with \(R(w) = \text{ch}(\nabla^W)\) and \(I(w) = \text{cycl}(W)\). Such an element exists by (36) and is uniquely determined up to elements in \(a(H(A)^{-1}(N))\). By naturality we are forced to define
\[
\widehat{\text{cycl}}(V, \nabla^V) := h^*(w).
\]
If we can show that this class is well-defined then the remaining properties (naturality, additivity) follow easily.

Given a second choice we argue as in the proof of Theorem 2.117. Using the notation introduced there we can assume that \((W, \nabla^W) \cong g^*(W'', \nabla^{W''})\) and \((W', \nabla^{W'}) \cong g'^*(W'', \nabla^{W''})\).

**Problem 4.72.** Give more details.

We must show that \(h^* g^* w'' = h'^* g'^* w''\). Let \(H : I \times M \to N''\) be the homotopy from \(g \circ h\) to \(g' \circ h'\). The bundle \(H^*(W'', \nabla^{W''})\) can be glued to a bundle \((\bar{W}, \nabla^{\bar{W}})\) on \(S^1 \times M\). By the homotopy formula
\[
h^* g^* w'' = h'^* g'^* w'' = a(R(H^*(w''))) = a(\int_{S^1 \times M} \text{ch}(\nabla^{\bar{W}})) = 0
\]
since
\[
\int_{S^1 \times M} \text{ch}(\nabla^{\bar{W}}) \in \text{im}(\text{ch} : KU^{-1}(M) \to H(A)^{-1}(M)).
\]

Let us add a remark on the role of cycle maps. First of all it is obvious that the cycle map provides a means to write down elements in differential complex \(K\)-theory. Reflecting on what it means to calculate an element in differential \(K\)-theory one can get several possibilities. One can map the element to some known group. For example, in the odd case, one can restrict to a point and study the resulting element in \(\mathbb{C}/\mathbb{Z}\). But often a very satisfactory way to calculate an element in differential \(K\)-theory is to exhibit it as a cycle of an explicit vector bundle with connection. This makes even more sense in view of the following result.

We let \(\text{Vect}_{\text{geom virtual}}\) denote the smooth set of virtual (i.e. formal differences of) vector bundles.

**Example 4.73.** Let \(M\) be a compact manifold.

**Problem 4.74.** Show that the cycle map \(\widehat{\text{cycl}} : \text{Vect}_{\text{geom virtual}}(M) \to \widehat{\text{KU}}^0(M)\) is surjective.

**Proof.** See [SS08b].

\[\square\]
Example 4.75. For $n \geq 1$ the group $\text{Spin}(n + 1)$ acts on $S^n$. Note that by suspension $\text{K}U^n(S^n) \cong \mathbb{Z}g$ for some generator $g$.

Problem 4.76. Show that for $n \geq 2$ there exists a unique $x \in \text{K}U^n(S^n)$ with $I(x) = g$ which is $\text{Spin}(n + 1)$-invariant and evaluates trivially on points. What goes wrong in the case $n = 1$?

See [BS09, Ch. 5.7] for an argument.

Problem 4.77. Note that $\text{K}U^0(S^{2n}) \cong b^n \text{K}U^{2n}(S^{2n}) \cong b^n \mathbb{Z}$. Calculate the class $b^n$. More precisely find a virtual bundle with connection $S^n$. Note that Problem 4.77.

Example 4.78. To a map $f : M \rightarrow U(k)$ we can associate the suspension bundle $T(f) \rightarrow S^1 \times M$. It is a $k$-dimensional complex vector bundle which represents a class $\text{cyc1}(T(f)) \in \text{K}U^0(S^1 \times M)$. This class only depends on the homotopy class $[f] \in [M, U(k)]$ of $f$. The set of homotopy classes $[M, U(k)]$ has a natural group structure induced by the group structure on $U(k)$ and one can check that the map $[f] \mapsto \text{cyc1}(T(f))$ is a homomorphism. The projection $\text{pr} : S^1 \times M \rightarrow M$ is $\text{K}U$-oriented and we have an Umkehr map $\text{pr}_! : \text{K}U^0(S^1 \times M) \rightarrow \text{K}U^{-1}(M)$. We define an odd version of a topological cycle map

$$\text{cyc1}^{-1} : [M, U(k)] \rightarrow \text{K}U^{-1}(M)$$

which maps the homotopy class $[f]$ of the map $f : M \rightarrow U(k)$ to the class

$$\text{cyc1}^{-1}([f]) := \text{pr}_!(\text{cyc1}(T(f))) \in \text{K}U^{-1}(M).$$

This map preserves the group structures. We now ask whether one can refine this construction to differential $K$-theory.

Problem 4.79. Is there a cycle map

$$\widehat{\text{cyc1}}^{-1} : C^\infty(\ldots, U(k)) \rightarrow \widehat{\text{K}U}^{-1}(\ldots)$$

which preserves the group structure.

$$I(\widehat{\text{cyc1}}^{-1}(f)) = \text{cyc1}^{-1}(f).$$

Proof. The answer is yes for $k = 1$ and no for $k \geq 2$.

Every cycle map $\widehat{\text{cyc1}}^{-1}$ is induced by a universal class $\hat{u}_k \in \widehat{\text{K}U}^{-1}(U(k))$. It preserves the group structure if and only if $\hat{u}_k$ is primitive, i.e. satisfies $\mu^*\hat{u}_k = \text{pr}_!^1\hat{u}_k + \text{pr}_!^2\hat{u}_k$ on $U(k) \times U(k)$. We first consider the case $k = 1$. One can show that there exists a unique primitive class $\hat{u}_1 \in \widehat{\text{K}U}^{-1}(U(1))$ whose curvature $R(\hat{u}_1)$ is the normalized invariant volume form. For $k \geq 2$ the universal class can not be primitive. Otherwise, the higher-degree components of its curvature would be a primitive forms of degree $\geq 2$, but such forms do not exist. \qed

96
Example 4.80. The $n$th Chern class can be considered as a transformation of smooth sets
\[ c_n : \text{Vect} \to H\mathbb{Z}^{2n}. \]

Problem 4.81. Observe that $c_n$ extends to a transformation
\[ c_n : \text{KU}^0 \to H\mathbb{Z}^{2n}. \]
Furthermore show, that there exists a unique extension
\[ \hat{c}_n : \text{KU}^0 \to \hat{H}_D(M; \mathbb{Z}) \]
such that
\[ \hat{c}_n(\text{cycl}(V, \nabla)) = \hat{c}_n(\nabla) \]
(where $\hat{c}_n$ on the r.h.s. is the differential refinement of $c_n$ according to Theorem 3.40)

Proof. If $M$ is compact, then we can identify the set $\text{KU}^0(M)$ with the set of stable equivalence classes of vector bundles on $M$. Since the Chern class $c_n$ is well-defined on stable equivalence classes we get the factorization.
In order to construct the differential refinement we again use manifold approximations of $BU$. See [Bun10b] for details.

Example 4.82. Consider a multiplicatively closed subset $S \subset \mathbb{N}$ and the spectrum $\text{KU}[S^{-1}]$. We have a natural extension
\[ \text{KU} \xrightarrow{c} H(\mathbb{C}[b, b^{-1}]) \]
\[ \text{KU}[S^{-1}] \]
We define $\text{KU}[S^{-1}]$ using the differential data $(\text{KU}[S^{-1}], A, c)$.

Problem 4.83. Show that there exists a natural transformation $\text{KU} \to \text{KU}[S^{-1}]$ which induces an isomorphism $\text{KU}^*(M) \otimes \mathbb{Z}[S^{-1}] \to \text{KU}[S^{-1}]^*(M)$ for compact manifolds $M$.

Why do we assume that $M$ is compact?

For every $k \in S$ we have the Adams operations $\Psi^k$ on $\text{KU}[S^{-1}]$ and $\Psi^k_H : \mathbb{C}[b, b^{-1}] \to \mathbb{C}[b, b^{-1}], b \mapsto k^{-1}b$ such that the following diagram commutes canonically.
\[ \text{KU}[S^{-1}] \xrightarrow{c} H(\mathbb{C}[b, b^{-1}]) \]
\[ \text{KU}[S^{-1}] \]
\[ \text{KU}[S^{-1}] \xrightarrow{\Psi^k_H} H(\mathbb{C}[b, b^{-1}]) \]

Hence for every $k \in S$ we have a map of data
\[ \Psi^k : (\text{KU}[S^{-1}], A, c) \to (\text{KU}[S^{-1}], A, c) \]
which induces differential Adams operation $\hat{\Psi}^k$ on $\text{KU}[S^{-1}]$. 

Problem 4.84. Verify the relation $\hat{\Psi}^k \hat{\Psi}^l = \hat{\Psi}^{kl}$.

More one Adams operations in differential $K$-theory can be found in [Bun10a]. We consider the torus $T^3$ with its action on itself.

Problem 4.85. Calculate $\hat{KU}^*(\{5\}^{-1}(T^3))$ and determine the action of $\hat{\Psi}^5$.

Let $(L, \nabla)$ be a complex line bundle over a manifold $M$.

Problem 4.86. Show that
\[ \hat{\text{cycl}}((L, \nabla)^{\otimes k}) = \hat{\Psi}^k(\hat{\text{cycl}}(L, \nabla)) \]
in $\hat{KU}[(k)^{-1}](M)$.

Example 4.87. Recall the notion of a geometric family $\mathcal{E}$ over $M$ [BS09, Def. 2.2]. It consists of
1. a proper submersion $f : W \to M$,
2. a Riemannian structure $(T^h f, g^{T^v f})$ consisting of a horizontal subbundle and a vertical Riemannian metric,
3. a family of Dirac bundles $V$,
4. an orientation of $T^v \pi$.

We say that a geometric family is of degree 0 or $-1$ if the fibre dimension is even or odd. Isomorphism classes of geometric families on $M$ form a graded semi-group $\text{GeomFam}^*(M)$ under fibrewise disjoint sum. In [BS09, Def. 2.2] we have constructed a model $\hat{KU}^*_{\text{GeomFam}}$ of a differential extension $\hat{KU}^*$ as a group of equivalence classes of geometric families. In particular we have a tautological cycle map
\[ \hat{\text{cycl}}_{\text{GeomFam}} : \text{GeomFam}^*(M) \to \hat{KU}^*_{\text{GeomFam}}(M) , \quad * \in \{0, -1\} \]
which maps a geometric family to its equivalence class. A vector with $V \to M$ with hermitean metric and metric connection $\nabla$ can naturally be considered as geometric family $\mathcal{E}$ over $M$, where $f = \text{id} : M \to M$.

The following exercise shows that the differential cycle map can be extended to geometric families.

Problem 4.88. Show that there is a unique isomorphism of group-valued functors
\[ \iota : \hat{KU}^0_{\text{GeomFam}} \cong \hat{KU}^0 \]
which is compatible with the structures $a, R, I$. Show further, that for a hermitean vector bundle $V$ with connection $\nabla$ we have
\[ \iota(\hat{\text{cycl}}_{\text{GeomFam}}(V, \nabla)) = \hat{\text{cycl}}(V, \nabla) . \]

Proof. For the first statement we refer to [BS10]. The second follows from the unicity of the cycle map.
4.5 Differential Bordism theory

The complex bordism theory $\text{MU}^*$ plays a fundamental role in stable homotopy theory. Its coefficient ring carries the universal formal group law. The relation between formal group laws and complex oriented cohomology theories governs the structure of the stable homotopy category. More specifically, via the Landweber exact functor theorem, complex bordism gives rise to a variety of complex oriented generalized cohomology theories with Landweber exact formal group laws. This correspondence generalizes to the differential case. In [BSSW09] (cf. 4.135) we observed that a differential extension of $\hat{\text{MU}}$ gives rise to differential extensions of complex oriented generalized cohomology theories with Landweber exact formal group laws. In this sense the differential extension of $\text{MU}$ plays a fundamental role in the theory.

It turned out that for bordism theories in general there are geometric constructions of differential extensions in which additional structures like multiplications or integration are easy to built in. This has been used e.g. in [BKST10] to deliver a bordism model for the differential extension of ordinary integral cohomology in which one has integration and products and a simple verification the projection formula.

The discussion of the cycle map for complex bordism is in many aspects parallel to the cycle map in the holomorphic and algebraic cases where one associates to Arakelov cycles classes in absolute Hodge cohomology or Arakelov Chow groups. We will not try to review the vast literature in this direction.

In the present course we discuss the differential extension of $\text{MU}$ because it is fundamental and simple. The constructions for other bordism theories are simple modifications of the case of $\text{MU}$.

Let $\text{MU}$ be a spectrum representing the complex bordism theory. We let $A := \text{MU} \otimes \mathbb{C}$ and fix an equivalence $c : \text{MU} \mathbb{C} \to H(A)$ so that

$$\text{MU} \to \text{MU} \mathbb{C} \xrightarrow{c} H(A)$$

induces the complexification map in homotopy. We obtain a strict differential data $(\text{MU}, A, c)$ and therefore a differential extension $(\hat{\text{MU}}, R, I, a)$.

In the following we want to connect $\text{MU}$ with the geometric picture. To this end we realize $\text{MU}$ as the Thom spectrum associated to the map $BU \to BO$.

Cycles for complex cobordism classes are maps between manifolds with complex stable normal bundle. In the following we give some details which are needed to generalize to the geometric situation. Let $f : W \to M$ be a smooth map of manifolds.

**Definition 4.89.** A representative $N$ of the stable normal bundle of $f$ is an exact sequence

$$0 \to TW \xrightarrow{df \oplus \alpha} f^*TM \oplus \mathbb{R}^k \to N \to 0.$$  

We call $N \to W$ the underlying bundle of $N$. 
The sequence $\mathcal{N}$ is just the infinitesimal model of an embedding $\iota$ over $M$

\[ W \xrightarrow{\iota} M \times \mathbb{R}^k \xrightarrow{f} M \]  

Given $\iota$ we let $\alpha$ be the second component of $d\iota$.

**Definition 4.90.** A cycle for a class in $\text{MU}(M)$ of degree $n$ is a triple $(f, N, J)$ of a proper smooth map $f : W \to M$ such that $\dim(M) - \dim(W) = n$, a representative of a stable normal bundle $N$, and a complex structure $J$ on its the underlying bundle.

There is a natural notion of an isomorphism between cycles. It is very useful to add homotopies of representatives of the stable normal bundle with fixed underlying bundle to the isomorphism relation. This point of view is also adopted in [BSSW09].

**Definition 4.91.** We let $\text{Cycle}^*_\text{MU}(M)$ denote the graded semigroup of isomorphism classes of cycles with respect to disjoint sum.

If we realize $\text{MU}$ as the Thom spectrum of the map $\text{BU} \to \text{BO}$, then the Thom-Pontrjagin construction gives a map of semigroups $\text{cycl} : \text{Cycle}^*_\text{MU}(M) \to \text{MU}^*(M)$.

**Problem 4.92.** Understand the details. One can actually put an equivalence relation on $\text{Cycle}^*_\text{MU}$ involving stabilization of the representative of the stable normal bundle and bordism such that $\text{cycl}$ induces an isomorphism on quotients.

**Proof.** See [BSSW09] for details. \qed

A complex vector bundle has an oriented underlying real vector bundle and therefore a Thom class for $HZ$. Putting the $HZ$-Thom isomorphisms of the universal bundles $\xi_n \to BU(n)$ for all $n \geq 0$ together we get a Thom isomorphism $\Phi : M^*(BU) \xrightarrow{\cong} M^*(MU)$ for every $HZ$-module spectrum $M$. Note that on the right-hand side we consider the spectrum cohomology of $\text{MU}$ with coefficients in $M$. The shifts involved in the construction of $\text{MU}$ ensure that there is no degree-shift in the Thom isomorphism. In particular, we have the Thom isomorphism $\Phi : H(A)^0(BU) \xrightarrow{\cong} H(A)^0(MU)$.

Under this isomorphism the spectrum cohomology class $c : \text{MU} \xrightarrow{\cong} H(A)$ corresponds to a cohomology class $u := \Phi^{-1}(c) \in H(A)^0(BU)$. 

100
The space $BU$ classifies stable complex vector bundles. For a stable complex vector bundle $N \to W$ classified by a map $\rho : W \to BU$ we write

$$u(N) := \rho^* u \in H(A)^0(W).$$

(40)

If $(f, N, J)$ is a cycle, then $f$ is proper and oriented for ordinary cohomology. Hence we have an Umkehr map

$$f_! : H(A)^*(W) \to H(A)^{*+n}(M),$$

(see 4.9 for more details on Umkehr maps which are denotes there by $I(\iota, \nu)_!$).

**Problem 4.93.** Show that $c(\text{cycl}(f, M, J)) = f_!(u(N)) \in H(A)^n(M)$.

We know that

$$HZ^*(BU) \cong \mathbb{Z}[c_1, c_2, \ldots]$$

is a polynomial ring in the universal Chern classes. Hence we can interpret

$$u \in H(A)^0(BU) \cong A[[c_1, c_2, \ldots]]^0$$

as a formal power series in the universal Chern classes with coefficients in $A$. Given a connection $\nabla$ on the complex vector bundle $N$ we can define the form

$$u(\nabla) = u(c_1(\nabla), \ldots) \in \Omega A_d^0(W)$$

(41)

by replacing the universal Chern classes by the corresponding Chern forms. Observe that

$$\text{Rham}(u(\nabla)) = u(N).$$

**Example 4.94.** The following exercise illustrates the nature of $u$. Note that

$$A = \pi_* (\text{MUC}) \cong \mathbb{C}[[\mathbb{CP}^1], [\mathbb{CP}^2], \ldots]$$

is a polynomial ring in the complex bordism classes $[\mathbb{CP}^n] \in \text{MU}^*_{2n}$ of the complex projective spaces.

**Problem 4.95.** Find an explicit formula for $u$ in terms of the generators $[\mathbb{CP}^n]$ and $c_k$.

**Proof.** Suitable tools for this calculation can be found in [HBJ92].

Let $W$ be a $k$-dimensional manifold. Then we define the complex of distributional forms on $W$ as the topological dual

$$\Omega^*_{\infty}(W) = \Omega^*_{\infty}(W, \text{or}_W)' ,$$

where $\Omega^*_{\infty}(W, \text{or}_W)$ denotes the complex of Fréchet spaces of compactly supported smooth forms with coefficients in the orientation bundle of $W$.

We fix a manifold $W$ and consider the sheaf $\Omega_{\infty}$ of distributional forms on $W$. Then we define the sheaf

$$\Omega A_{\infty} := \Omega_{\infty} \otimes A$$

and let $\Omega A_{\infty}(W)$ denote its complex of global sections.
Problem 4.96. Understand the difference between $\Omega_{-\infty}(W) \otimes \mathbb{C} A$ and $\Omega A_{-\infty}(W)$.

If $f : W \to M$ is an oriented proper smooth map and $n := \dim(M) - \dim(W)$, then we can define a push-forward

$$f_! : \Omega^*_c(W) \to \Omega^{*+n}_{-\infty}(M)$$

as the adjoint of the pull-back

$$f^* : \Omega_c(M, \text{or}_M) \to \Omega_c(W, f^* \text{or}_M) \xrightarrow{\sim} \Omega_c(W, \text{or}_W),$$

where the identification $f^* \text{or}_M \cong \text{or}_W$ is fixed by the orientation of $f$. We extend the push-forward to the tensor product with $A$

$$f_! : \Omega^*_A(W) \to \Omega^{*+n}_{-\infty}(M).$$

Problem 4.97. Fill in some details.

Let $(f, \mathcal{N}, J)$ be a $\text{MU}$-cycle and $\nabla$ be a connection on the underlying complex bundle of $\mathcal{N}$. Then we get a form $u(\nabla) \in \Omega A^0_{\text{cl}}(W) \subseteq \Omega A^0_{-\infty, \text{cl}}(W)$.

We know that the inclusion

$$\Omega A(M) \to \Omega A_{-\infty}(M)$$

is a quasi-isomorphism. We use this to extend the de Rham isomorphism to distributional forms

$$\text{Rham} : H^*(\Omega A_{-\infty}(M)) \xrightarrow{\sim} H(A)^*(M)$$

(see [dR84] for details). The following proposition is the bordism analog of 4.69.

Problem 4.98. Show that

$$\text{Rham}(f_! u(\nabla)) = c(\text{cycl}(f, \mathcal{N}, J)).$$

Definition 4.99. A geometric $\text{MU}$-cycle of degree $n$ is a tuple $(f, \mathcal{N}, J, \nabla, \eta)$, where $(f, \mathcal{N}, J)$ is a $\text{MU}$-cycle of degree $n$, $\nabla$ is a connection on the underlying complex bundle of $\mathcal{N}$, and $\eta \in \Omega A^{n-1}_{-\infty}(M)/\text{im}(d)$ is such that $f_!(u(\nabla)) - d\eta \in \Omega A^n_{-\infty}(M)$. We let $\text{Cycle}^{\text{geom, }*}_{\text{MU}}(W)$ denote the graded semigroup of isomorphism classes of geometric $\text{MU}$-cycles.

The sum operation is given by disjoint union of the geometric pieces and the sum of the forms.

If $h : W' \to W$, then we can define $h^*(f, \mathcal{N}, J, \nabla, \eta)$ if $f$ is transverse to $h$.

Problem 4.100. Fill in the details. Note that the pull-back of $\eta$ needs an argument.

Proof. See [BSSW09] for details.
Problem 4.101. Show that there exists unique additive differential refinement
\[ \widehat{\text{cycl}} : \text{Cycle}^{\text{geom}}_{\text{MU}} \to \text{MU}^*(M) \]
such that
\[ I \circ \widehat{\text{cycl}}(f, N, J, \nabla, \eta) = \text{cycl}(f, N, J) \quad R(\widehat{\text{cycl}}(f, N, J, \nabla, \eta)) = f_i(u(\nabla)) - d\eta \]
which is compatible with the partially defined pull-back.

Example 4.102. Let \((M, g)\) be a closed compact Riemannian manifold. Let \((f, N, J)\) be an \text{MU}-cycle of degree \(n\). Let us choose a connection \(\nabla\) on \(N\). Then there exists a unique \(\eta \in \Omega^{A^n-1}_{\infty}(M)\) which is orthogonal to \(\ker(d)\) and such that \(f_i(u(\nabla)) - d\eta\) is harmonic. We form
\[ \widehat{\text{cycl}}(f, N, J, \nabla, \eta) \in \text{MU}^n(M) . \]

Problem 4.103. Show that \(\text{cycl}^*_{\text{MU}}(M) \to \text{MU}^*(M)\) is additive.

We consider the manifold \((S^2, g)\) with the standard metric. Let \(\gamma : S^1 \to S^2\) be a smooth embedding. The normal bundle of \(\gamma\) is trivialized. We get a representative
\[ N : 0 \to TS^1 \to f^*TS^2 \oplus \mathbb{R} \to S^1 \to S^1 \times \mathbb{C} \to 0 \]
of the stable normal bundle with complex structure \(J := i\) on the quotient. Then
\[ \text{harm}(\gamma, N, J) \in \text{MU}^1(S^2) . \]
For \(x \in S^2\) we get
\[ \text{harm}(\gamma, N, J)|_x \in \text{MU}^1(*) \cong \mathbb{C}/\mathbb{Z} . \]

Problem 4.106. Calculate the element \(\text{harm}(\gamma, N, J)|_x \in \mathbb{C}/\mathbb{Z}\) for every \(x \in S^2\).

We consider the circle \((S^1, g)\). A point \(x \in S^1\) gives naturally rise to topological cycle \(c(x) \in \text{Cycle}^1_{\text{MU}}(S^1)\).

Problem 4.107. Describe the dependence of \(\widehat{\text{cycl}}(c(x))\) on \(x\).
4.6 Multiplicative structures

In this section we give a general construction of a multiplicative differential extension of a multiplicative generalized cohomology theory. It was a common belief that this is possible, but the details were an open problem for a while. Multiplicative extensions have been known in special cases like ordinary cohomology, $KU$, $MU$ and other bordism theories, or Landweber exact theories. See [BSSW09] and [Upm11]. In the present section we restrict attention to the commutative case which captures most examples. There should be an analogous theory for the solely associative case.

The input for the construction is a multiplicative refinement of differential data.

**Definition 4.108.** A multiplicative differential data is a differential data

$$(E, A, c)$$

where

1. $E \in \text{CommMon}(\mathbb{N}(\text{Sp})[W^{-1}]^\wedge)$ is a commutative ring spectrum,
2. $A \in \text{CommMon}(\text{Ch}^\otimes)$ is a commutative DGA over $\mathbb{C}$, and
3. $c : EC \to H(A)$ is a morphism of commutative ring spectra.

A morphism of differential data $(E, A, c) \to (E', A', c')$ is a commutative diagram in $\text{CommMon}(\mathbb{N}(\text{Sp})[W^{-1}]^\wedge)$

$$
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow{c} & & \downarrow{c'} \\
H(A) & \xrightarrow{H(\phi)} & H(A')
\end{array}
$$

for some morphisms $\phi : E \to E'$ in $\text{CommMon}(\mathbb{N}(\text{Sp})[W^{-1}]^\wedge)$ and $f : A \to A'$ of DGA’s.

As before one can describe the $\infty$-category of strict multiplicative differential data as a pull-back in $\infty$-categories $\infty\text{Cat}$

$$
\text{Data}^{str,mult} \xrightarrow{\text{N}} \text{N}(\text{CommMon}(\text{Ch}^\otimes)) \xrightarrow{H} \text{CommMon}(\mathbb{N}(\text{Sp})[W^{-1}]^\wedge \wedge M\mathcal{C}^\otimes) \xrightarrow{\text{Mod}(H\mathbb{C})}
$$

Again, before we turn to the construction of a multiplicative differential cohomology theory in terms of a multiplicative differential function spectrum we first define what we mean by a multiplicative differential extension of a multiplicative cohomology theory. In this way we capture examples constructed by other means.

**Definition 4.109.** A multiplicative differential extension of the multiplicative cohomology theory $E^*$ associated to a multiplicative data $(E, A, c)$ is a tuple $(\hat{E}^*, R, I, a)$ which is a differential extension of $E$ such that $\hat{E}^*$ has values in graded commutative rings, $R$ and $I$ are transformations of ring-valued functors, and for $x \in \Omega A(M)/\text{im}(d)$ and $y \in \hat{E}(M)$ we have

$$a(x) \cup y = a(x \wedge R(y))$$

104
Note that the notion of a multiplicative differential extension only depends on a homotopy theoretic part of the data \((E,A,c)\), namely the multiplicative cohomology theory \(E^*\), the differential graded algebra \(A\), and the transformation of cohomology theories \(E^* \to H(A)^*\) induced by \(c\). Therefore we can talk about multiplicative extensions having only fixed this coarser part datum. This is the way how previous definitions are related to the one given here.

Let \(E \in \text{CommMon}(\mathbb{N}([\text{Sp}[\mathcal{W}^{-1}]]))\) be a commutative ring spectrum. It is a natural question when the canonical strict data (Definition 4.44) \((E,A,c)\) with \(A := \pi_*(E) \otimes \mathbb{C}\) can be refined to a multiplicative data.

**Definition 4.110.** We say that \(E\) is formal (over \(\mathbb{C}\)), if there exists an morphism of ring spectra \(c : E \to H(A)\) refining the canonical morphism.

Most of our examples are formal: \(HZ, \text{MU}\) (see 4.130), \(\text{KU}\) (see 4.121). One can also show that the algebraic \(K\)-theory spectrum of a number ring is formal, see [BT12].

Observe that \((HZ, R, HZ \to H\mathbb{C})\) is a multiplicative data and \((\hat{H}^*_{\text{det}}, R, I, a)\) is an associated multiplicative extension.

In the following we give a general construction of multiplicative extensions. If \(A\) is a commutative DGA, then \(\Omega A\) becomes a sheaf of commutative DGA’s.

**Lemma 4.111.** The de Rham equivalence refines to an equivalence

\[
\text{Rham} : H(\Omega A) \to \text{Sm}(H(A))
\]

in \(\text{Sm}(\text{CommMon}(\mathbb{N}([\text{Sp}[\mathcal{W}^{-1}]])))\).

**Proof.** In principle we can repeat the proof of Proposition 4.37. The main point is to observe that the integration map

\[
\int : \Omega A(M) \to \text{Map}(C_*(M), A)
\]

refines to an equivalence in \(\text{CommMon}([\text{Ch}[\mathcal{W}^{-1}]]))\). This seems to be true (compare [AS10]) but I do not know a good reference for this precise statement.

Therefore we give an alternative argument which does not involve integration of forms. We know that

\[
H(\Omega A) \in \text{Sm}^{\text{desc}}(\text{CommMon}(\mathbb{N}([\text{Sp}[\mathcal{W}^{-1}]])))
\]

is constant. We have a natural morphism

\[
p^*H(\Omega A(*)) \to H(\Omega A)
\]

in \(\text{Sm}(\text{CommMon}(\mathbb{N}([\text{Sp}[\mathcal{W}^{-1}]])))\), where \(p : \text{Mf} \to *\) is as in 4.18. Hence the natural morphism

\[
L(p^*(H(A))) = L(p^*(H(\Omega A(*)))) \to H(\Omega A)
\]

is an equivalence in \(\text{Sm}^{\text{desc}}(\text{CommMon}(\mathbb{N}([\text{Sp}[\mathcal{W}^{-1}]])))\). On the other hand, in this category we have the equivalence (see Problem 4.21)

\[
L(p^*(H(A))) \cong \text{Sm}(H(A)) .
\]
We therefore get an equivalence

\[ R_{\text{Rham}} : H(\Omega A) \xrightarrow{\sim} \text{Sm}(H(A)). \]

\[ \square \]

**Problem 4.112.** Verify that the equivalences obtained in 4.37 and 4.111 coincide.

Note that the cone in Definition 4.49 can be written as a pull-back. The presentation of the differential function spectrum can be refined to the multiplicative case by interpreting that pull-back in smooth commutative ring spectra. We will actually refine

\[ \text{Diff}^\bullet(E, A, c) := \prod_{n \in \mathbb{Z}} \text{Diff}^n(E, A, c) \]

to a commutative ring spectrum.

For a commutative ring spectrum \( R \) we define the commutative ring spectrum \( R[z, z^{-1}] := \prod_{n \in \mathbb{Z}} R \). We further let \( \sigma^\geq \Omega A := \prod_{n \in \mathbb{Z}} \sigma^\geq n \Omega A \). If \( A \) is a commutative DGA, then \( \sigma^\geq \Omega A \) is a commutative DGA as well.

**Definition 4.113.** We define \( \text{Diff}^\bullet(E, A, c) \in \text{Sm}(\text{CommMon}(\mathbb{N}(\text{Sp})[W^{-1}]^\wedge)) \) as the pull-back

\[ \begin{array}{ccc}
\text{Diff}^\bullet(E, A, c) & \xrightarrow{c} & H(\sigma^\geq \Omega A) \\
\downarrow & & \downarrow R_{\text{Rham}} \\
\text{Sm}(E[z, z^{-1}]) & \xrightarrow{c} & H(\Omega A)[z, z^{-1}]
\end{array} \]

Note that

\[ \pi_n(\text{Diff}^\bullet(E, A, c)) = \prod_{k \in \mathbb{Z}} \pi_n(\text{Diff}^k(E, A, c)) \]

forms a bigraded ring. In particular, by taking the diagonal, we get the graded ring-valued functor

\[ \hat{E}^* := \prod_{k \geq 0} \pi_k(\text{Diff}^k(E, A, c)). \]

By construction, the transformations

\[ R : \hat{E}^* \to \Omega A^*_{\text{cl}}, \quad I : \hat{E}^* \to E^* \]

become transformations between graded commutative ring valued functors.

**Problem 4.114.** Show that for \( x \in \Omega A^{n-1}(M)/\text{im}(d) \) and \( y \in \hat{E}^m(M) \)

\[ a(x) \cup y = a(x \cup R(y)) \]

**Example 4.115.** If we apply the above construction to the data \( (H\mathbb{Z}, \mathbb{R}, c) \), where \( c : H\mathbb{Z} \to H\mathbb{C} \) is the canonical map, then we get the multiplicative structure on \( \hat{H}\mathbb{Z}^* \).
Problem 4.116. Show that the canonical isomorphism $\hat{H}^*_\text{Del} \cong \hat{H}\mathbb{Z}$ from Problem 4.57 is multiplicative.

Example 4.117. The sphere spectrum $S$ is a commutative ring spectrum which comes with a canonical map of ring spectra $c : S \to H\mathbb{C}$. We get the differential function ring spectrum $\text{Diff}^*(S, \mathbb{C}, c)$. Note that every spectrum is an $S$-module.

Problem 4.118. Let $(E, A, c)$ be a multiplicative data. Work out the definition of a module data $(F, B, d)$. Construct $\text{Diff}(F, B, d)$ as a $\text{Diff}(E, A, c)$-module.

Problem 4.119. Show that for every differential data $(E, A, c)$ we get a $\text{Diff}^*(S, \mathbb{C}, c)$-module structure on $\text{Diff}^*(E, A, c)$.

Example 4.120. We can fix an isomorphism of commutative ring spectra $c : KU \cong H(\mathbb{C}[b, b^{-1}])$.

Problem 4.121. Show this!

Proof. We first consider connective $K$-theory $\text{ku}$. Similarly as in Problem 4.130 we produce an equivalence of ring spectra

$$\text{ku} \wedge H(\mathbb{C}) \cong H(\mathbb{C}[b]) .$$

It is useful to write the localization $\iota : \mathbb{N}(\text{Ch}) \to \mathbb{N}(\text{Ch})[W^{-1}]$ explicitly. So more precisely we should write $\text{ku} \wedge H(\mathbb{C}) \cong H(\iota(\mathbb{C}[b]))$. Then we invert the multiplication by $b$. We get

$$\text{KU} \cong \text{ku}[b^{-1}] \cong H(\iota(\mathbb{C}[b])[b^{-1}] \cong H(\iota(\mathbb{C}[b])[b^{-1}]) .$$

Here $\iota(\mathbb{C}[b])[b^{-1}]$ is the inversion of $b$ in the $\infty$-category algebras in over $\iota(\mathbb{C}[b])$. As a final step one has to check that $\iota(\mathbb{C}[b])[b^{-1}] \cong \iota(\mathbb{C}[b, b^{-1}])$.

Alternatively one can use the existence of a strictly multiplicative Chern form for vector bundles with connection. Details can be found in [BT, Sec. 3.6]. The datum $(\text{KU}, \mathbb{C}[b, b^{-1}], c)$ is a strict multiplicative differential datum. We therefore get a multiplicative differential extension $(\text{KU}, R, I, a)$. This is not the extension is not unique if one does not require multiplicativity.

Problem 4.122. Show that there exists a unique multiplicative differential extension associated to the datum described above. Observe that the extension is not unique if one does not require multiplicativity.

Proof. Compare with [BS10].

Recall that $\text{Vect}^\text{geom}$ is a semiring valued functor with product induced by the tensor product.

Problem 4.123. Show that the geometric cycle map constructed in Theorem 4.71

$$\text{cycl} : \text{Vect}^\text{geom} \to \text{KU}^0$$

is a natural transformation of semiring valued functors.
Proof. Observe that the deviation from multiplicativity gives rise to a natural transfor-
mation
\[ \text{Vect} \times \text{Vect} \to KU/C/Z^{-1}. \]
Now use that \( KU/C/Z^{-1}(BU(n) \times BU(m)) = 0 \) for all \( n, m \geq 0 \). \( \square \)

Example 4.124. The following exercise concerns aspects of the \( \hat{S}^* \)-module structure
found in \( \ref{1.19} \). Let \( \hat{S}^* \to \hat{KU}^* \) be the unit. We consider the tautological bundle \( L \to \C P^1 \cong S^2 \) with its invariant connection \( \nabla \). Then we get a class \( \hat{cyc}(L, \nabla) \in \hat{KU}^0(S^2) \). Note that \( \pi_1(S) \cong \Z/(2\Z) \cong S^{-1}(*) \). The generator of this group is the class \( \eta \) of the
framed manifold \( S^4 \) with the non-bounding framing coming from the group structure. It
has a canonical lift \( \hat{\eta} \in \hat{S}^{-1}(*) \). Let \( p_{S^k} : S^k \to * \) be the projection.

Problem 4.125. Calculate \( \hat{c}(\hat{\eta}) \in \hat{KU}^{-1} \cong \C/Z \) and \( p_{S^2} \hat{\eta} \cup \hat{cyc}(L, \nabla_L) \in \hat{KU}^{-1}(S^2) \).

Proof. Similarly, we consider the tautological \( \H \)-bundle \( E \to \H P^1 \cong S^4 \) with its invariant
connection \( \nabla^E \). We get the element
\[ \hat{cyc}(L, \nabla) \in \hat{KU}^0(S^4). \]
The group \( SU(2) \) with its right-invariant framing represents a generator
\[ \sigma \in \pi_3 = S^{-3}(*) \cong \Z/(24\Z). \]
It again lifts canonically to \( \hat{\sigma} \in \hat{S}^{-3}(*) \).

Problem 4.126. Calculate \( \hat{c}(\hat{\sigma}) \in \hat{KU}^{-3} \cong \C/Z \) and \( p_{S^4} \hat{\sigma} \cup \hat{cyc}(E, \nabla^E) \in \hat{KU}^{-3}(S^4) \).

Proof. In the following we show how to reduce the calculation of \( \hat{c} : \hat{S}^*(*) \to \hat{KU}^*(*) \)
to a topological problem. For \( k \geq 1 \) we have an isomorphism
\[ \text{SC}/\Z^{-k-1}(*) \cong \hat{S}_{\text{flat}}^{-k}(*) \to S^{-k}(*) \]
Let \( u \in S^{-k}(*) \), \( \hat{u} \in \hat{S}^{-k}(*) \), and \( u_{\C/Z} \in \text{SC}/\Z^{-k-1}(*) \) be its preimages under these
isomorphisms. Then \( \hat{c}(\hat{u}) = \epsilon_{\C/Z}(u_{\C/Z}) \in KU/C/Z^{-k-1}(*) \subseteq \hat{KU}^k(*) \). The map
\[ e : \pi_k(S) \cong S^{-k}(*) \to KU/C/Z^{-k-1}(*) \]
is the complex variant of Adam’s \( e \)-invariant (cf \( \ref{1.11} \) and \( \ref{1.23} \) for more details). It
is known that \( e(\eta) \) has order 2 and \( e(\sigma) \) has order 12. We conclude that \( \hat{c}(\hat{\eta}) \) has order 2
and \( \hat{c}(\hat{\sigma}) \) has order 12. We refer to \( \ref{1.24} \) for the calculations.

Example 4.127. Let \( S \subset \N \) be multiplicatively closed. Recall the construction of the
Adams operations from Example 4.82.
Problem 4.128. Show that there exists a unique multiplicative differential extension of $\hat{KU}[S^{-1}]$ and that the Adams operations $\Psi^k$, $k \in S$, extend to multiplicative operations $\hat{\Psi}^k$ on $\hat{KU}[S^{-1}]$.

Example 4.129. The data $(MU, A, MU \cong H(A))$ is a multiplicative.

Problem 4.130. Show this!

Proof. The point is to show that the map $c : MU \to H(A)$ refines to a morphism of commutative ring spectra. Note that $A \cong \pi_*(MU \wedge H(\mathbb{C})) \cong \pi_0(\text{map}_{\text{Mod}(H(\mathbb{C}))}(H(\mathbb{C}), MU \wedge H(\mathbb{C})))$
is a free commutative $\mathbb{C}$-algebra in generators $(x_i)_{i \in \mathbb{N}}$, cf. 4.94. We use the last incarnation of the generators to construct a map of $H\mathbb{C}$-modules

$$M := H(\mathbb{C})\langle x_1, \ldots \rangle \cong \bigoplus_{i \in \mathbb{N}} H(\mathbb{C})[- \deg(x_i)] \to MU \wedge H\mathbb{C}.$$ 

It induces an equivalence of commutative algebras in $\text{Mod}(H(\mathbb{C}))$

$$\text{Free}_{\text{Mod}(H(\mathbb{C}))}(M) \simto MU \wedge H\mathbb{C}.$$ 

Since $\mathbb{C}$ is a $\mathbb{Q}$-algebra the classical free commutative $\mathbb{C}$-algebra over the graded vector space $\mathbb{C}\langle x_1, \ldots \rangle$ (which is $A$) coincides with the free commutative $\mathbb{C}$-algebra taken in the $\infty$-category $\text{Mod}(\mathbb{C})$. Hence

$$\text{Free}_{\text{Mod}(H(\mathbb{C}))}(M) \cong H(\text{Free}_{\text{Mod}(\mathbb{C})}\mathbb{C}\langle x_1, \ldots \rangle) \cong H(A).$$

\qed

Problem 4.131. Use the arguments given in the proofs of 4.130 and 4.121 to show that a commutative ring spectrum $E$ is formal over $\mathbb{C}$ provided one of the following conditions holds true:

1. $\pi_*(E) \otimes \mathbb{C}$ is a free commutative algebra.
2. $E$ is periodic and $\pi_*(E[0, \ldots, \infty]) \otimes \mathbb{C}$ is a free commutative algebra, where $E[0, \ldots, \infty]$ denotes the connective cover of $E$.

The multiplicative data $(MU, A, c)$ gives rise to a multiplicative extension $(\hat{MU}^*, R, I, c)$. The graded semigroup $\text{Cycle}_{MU}^{\text{geom}}(W)$ has a partially defined product. More precisely, $(f, N, J, \nabla, \eta) \cup (f', N', J', \nabla', \eta')$ can be defined if $f$ and $f'$ are transverse. Details can be found in [BSSW09].

Problem 4.132. Fill in the details. Show that the differential cycle map $\hat{\text{cycl}}$ obtained in 4.101 is multiplicative.
Proof. Consider the deviation from multiplicativity. Use the compatibility of the cycle map with the curvature to show that it induces a transformation $\text{MU}^l \otimes \text{MU}^k \to \text{MUC}/\mathbb{Z}^{k+l-1}$. Then argue that such a transformation is zero since $\text{MU}$ is an even spectrum.

**Problem 4.133.** Assume that $(M,g)$ is a compact Kähler manifold or a compact symmetric space. Show that the harmonic cycle map

$$\text{harm} : \text{Cycle}_{\text{MU}}^*(M) \to \hat{\text{MU}}^*(M)$$

is multiplicative.

**Proof.** Use formality, or more precisely, that the product of harmonic forms is harmonic. $\square$

Let $\gamma_0, \gamma_1 : S^1 \to L^3_{\mathbb{Z}/p\mathbb{Z}}$ define a link. Observe that they have canonical refinements to topological cycles $c(\gamma_i) \in \text{Cycle}_{\text{MU}}^2(S^3)$. Let $\hat{c}(\gamma_i) \in \text{Cycle}_{\text{MU}}^{\text{geom},2}$ be geometric refinements.

**Problem 4.134.** Calculate

$$\text{harm}(\hat{c}(\gamma_0)) \cup \text{harm}(\hat{c}(\gamma_1)) \in \text{MU}^4(L^3_{\mathbb{Z}/p\mathbb{Z}}) \cong \mathbb{C}/\mathbb{Z}.$$

**Example 4.135.** In this example we use the notation $A_E := \pi_*(E) \otimes \mathbb{C}$. We have a morphism of commutative ring spectra $\alpha : \text{MU} \to \text{KU}$. It induces a morphism of rings $A_{\text{MU}} \to A_{\text{KU}}$ and a morphism of multiplicative data

$$\hat{\alpha} : (\text{MU}, A_{\text{MU}}, c_{\text{MU}}) \to (\text{KU}, A_{\text{KU}}, c_{\text{KU}}).$$

Note that the formal group law over $\pi_*(\text{KU})$ associated to $\alpha$ is Landweber exact [Lan76]. As a consequence we have the Conner-Floyd theorem [CF66],[HH92] stating that for a compact manifold $M$ the map $\alpha$ induces an isomorphism

$$(\text{MU}(M)^* \otimes_{\text{MU}} \text{KU})^* \cong \text{KU}^*(M).$$

This extends to differential cohomology. Note that $\hat{\text{MU}}(M)^*$ is a $\hat{\text{MU}}(0) \cong \text{MU}$-module.

**Problem 4.136.** Show that if $M$ is compact, then $\hat{\alpha}$ induces an isomorphism

$$(\hat{\text{MU}}(M)^* \otimes_{\text{MU}} \text{KU})^* \cong \hat{\text{KU}}^*(M).$$

**Proof.** See [BSSW09] for an argument. $\square$

Let $* \to S^2$ be the inclusion of a point. It extends canonically to a cycle $* \in \text{Cycle}_{\text{MU}}^2(S^2)$.

**Problem 4.137.** Calculate $\hat{\alpha}(\text{harm}(*)) \in \hat{\text{KU}}^2(S^2)$. More precisely, find a vector bundle $(V, \nabla)$ with connection such that $b \text{cycl}(V, \nabla) = \hat{\alpha}(\text{harm}(*))$. 

110
4.7 Relative sites

In this subsection we introduce some language which is mainly used to capture, in the \( \infty \)-categorical context, the functoriality of constructions like evaluations with proper support and integration maps. If one is solely interested in the latter constructions for a specific map of manifolds, the one should skip most of this subsection and just look how these constructs look like when evaluated on a specific map.

The main application of this theory is the definition of the notion of differential cohomology \( \hat{E}_K^\ast(W) \) with support in a closed subset \( K \subset W \), or with proper support \( \hat{E}_{prop/M}^\ast(W) \) over a map \( f : W \rightarrow M \). Relative versions of \( \hat{H}_Z \) in the Cheeger-Simons and Hopkins-Singer pictures have been considered and compared in [BT06]. See also [Zuc03].

For a manifold \( B \) we let \( Mf/B \) be the site of manifolds over \( B \) equipped with the topology induced from \( Mf \). We write objects in the form \((M \rightarrow B)\). A morphism \((M \rightarrow B) \rightarrow (M' \rightarrow B)\) is a smooth map \( f : M \rightarrow M' \) which preserves the structure maps to \( B \). We let

\[
\text{Sm}_B(C) := \text{Fun}(\mathbb{N}(Mf/B)^{op}, C)
\]

denote the \( \infty \)-category of smooth objects in \( C \) over \( B \).

**Example 4.138.** Here are typical examples which one should to have in mind. Let \( V \rightarrow B \) be a complex vector bundle. Then we obtain a smooth abelian group \( \Gamma(V) \in \text{Sm}^{\text{desc}}_B(\mathbb{N}(\text{Ab})) \) which associates to \( f : M \rightarrow B \) the space of sections of the pull-back \( f^\ast V \rightarrow M \). Assume further that \( V \) has a flat connection \( \nabla \). Then we could form the sheaf of chain complexes

\[
\Omega(V, \nabla) \in \text{Sm}^{\text{desc}}_B(\mathbb{N}(\text{Ch})[W^{-1}])
\]

which associates to \( f : M \rightarrow B \) the twisted de Rham complex \( \Omega(M, f^\ast V) \) with differential \( d \otimes 1 + 1 \otimes f^\ast \nabla \).

For a smooth map \( f : W \rightarrow B \) we have a functor \( f^\sharp : Mf/W \rightarrow Mf/B \) given by

\[
f^\sharp(M \rightarrow W) := (M \rightarrow W \overset{f}{\rightarrow} B).
\]

It induces an adjunction

\[
f^\ast := (f^\sharp)^* : \text{Sm}_B(C) \rightleftarrows \text{Sm}_W(C) : f_*. \]

Note that \( f^\ast X(M \rightarrow W) \cong X(M \rightarrow W \rightarrow B) \). In order to describe the right-adjoint \( f_* \) we consider the category \( Mf/(W \rightarrow B) \) of diagrams of the form

\[
D := \begin{array}{ccc}
V & \longrightarrow & W \\
\downarrow & & \downarrow \\
M & \longrightarrow & B
\end{array}
\]

(44)
A morphism $D \to D'$ is a pair of smooth maps $V \to V'$ and $M \to M'$ which preserve the structure maps. We have forgetful functors

$$p : \text{Mf}/(W \to B) \to \text{Mf}/W, \quad p(D) := (V \to W)$$

and

$$q : \text{Mf}/(W \to B) \to \text{Mf}/B, \quad q(D) := (M \to B).$$

Both, $p$ and $q$ induce adjunctions

$$p^* : \text{Fun}(\text{Mf}^/(W\to B)^\text{op}, C) \leftrightarrows \text{Fun}(\text{Mf}/W^/(M\to B), C) : p_*,$$

and

$$q^* : \text{Fun}(\text{Mf}/B^\text{op}, C) \leftrightarrows \text{Fun}(\text{Mf}/(W \to B)^\text{op}, C) : q_*,$$

and we have

$$f_* = q_* p^*.$$  

It is now easy to see that

$$(f_* X)(M \to B) = \lim_{\text{Mf}/(W \to B)//(M\to B)} X(V \to W).$$

(45)

**Problem 4.139.** Show that $f^*$ and $f_*$ preserve the subcategories of sheaves.

**Proof.** The assertion is clear for $f^*$. For $f_*$ we can use formula (45).  \qed

**Example 4.140.** Let $p : B \to *$ be the canonical map. It gives rise to a functor $p^* : \text{Sm}(C) \to \text{Sm}_B(C)$. For example, for $\Omega_C \in \text{Sm}(\text{N}(\text{Ch}[W^{-1}])$ we get $p^* \Omega(M \to B) = \Omega(B)$. Continuing example 4.138, we have natural isomorphisms

$$f^* \Gamma(V) \cong \Gamma(f^* V), \quad f^* \Omega(V, \nabla) \cong \Omega(f^* V, f^* \nabla).$$

If $X \in \text{Sm}(C)$, the we often use the notation $X_B := p^* X \in \text{Sm}_B(C)$.

If $f : W \to B$ is a submersion, then we have a functor $f^\sharp : \text{Mf}/B \to \text{Mf}/W$ given by $f^\sharp(M \to B) := M \times_B W \to W$.

**Problem 4.141.** Show that for a submersion $f$ we have $f_*(X) := (f^\sharp)^*$. In particular, $f_*$ is exact.

**Proof.** In this case $(\text{Mf}/(W \to B))/(M \to B)$ has a final object

$$\begin{array}{ccc}
M \times_B W & \longrightarrow & W \\
\downarrow & & \downarrow \\
M & \longrightarrow & B
\end{array}$$

\qed

112
We now assume that $C$ is pointed by the zero object $\ast$, i.e. an object which is both, final and initial. Our examples are stable $\infty$-categories, but also $\mathbb{N}(\text{Ch})$. We have a functor

$$\text{kernel} : \text{Fun}(\mathbb{N}([1]), C) \rightarrow C$$

which takes the fibre $\text{kernel}(f)$ of a map $f : C \rightarrow D$ at $\ast \rightarrow D$.

**Example 4.142.** Let $\iota : \mathbb{N}((\text{Ch}) \rightarrow \mathbb{N}((\text{Ch})[W^{-1}]$ be the localization and $A \rightarrow B$ an object of $\text{Fun}(\mathbb{N}([1]), \mathbb{N}((\text{Ch})))$.

**Problem 4.143.** Show that there is a natural morphism

$$\phi : \iota(\text{kernel}(A \rightarrow B)) \rightarrow \text{kernel}(\iota(A) \rightarrow \iota(B))$$

Analyse when it is an equivalence.

**Proof.** Since $\iota$ preserves the zero object we get the natural map $\phi$ by the universal property of the pull-back defining the kernel. It is an equivalence for example, if $A \rightarrow B$ is level wise surjective.

Note $\text{kernel}(A \rightarrow B)$ is given by the complex of level wise kernels of the map $A \rightarrow B$. On the other hand, $\text{kernel}(\iota(A) \rightarrow \iota(B))$ is represented by the cone $\text{Cone}(A \rightarrow B)[-1]$. If $A \rightarrow B$ is level wise surjective, then we get a map of long exact sequences which shows that the map $\phi$ is a quasi-isomorphism. \hfill \square

Using these constructs we can define relative versions of smooth objects. Let $K \subset M$ be closed and $X \in \text{Sm}(C)$.

**Definition 4.144.** We define the evaluation of $X$ with support in $K$ by

$$X_K(M) = X(M, M \setminus K) := \text{kernel}(X(M) \rightarrow X(M \setminus K))$$

**Example 4.145.** We get

$$(\Omega^n_C)_K(M) \cong \{ \omega \in \Omega^n(M, \mathbb{C}) | \omega|_{M \setminus K} = 0 \} = \{ \omega \in \Omega^n(M, \mathbb{C}) | \text{supp}(\omega) \subseteq K \}$$

If we consider $\Omega_C \in \text{Sm}(\mathbb{N}(\text{Ch})[W^{-1}])$, then $(\Omega_C)_K$ is not so simple since we must take the kernel in the $\infty$-categorical sense. We can not simply take all forms supported in $K$. See \ref{4.158} for similar discussion.

Definition \ref{4.144} gives the evaluation of a functor $X_K$ on objects $X_K(M)$. In order to fully construct this functor we proceed as follows. Consider the category $\text{Mf}_{\text{rel}}$ of pairs $(M, U)$ of manifolds and open subsets $U \subseteq M$. A morphism $(M, U) \rightarrow (M', U')$ is a smooth map $f : M \rightarrow M'$ such that $f(U) \subseteq U'$.

**Problem 4.146.** Define the extension

$$\mathcal{E} : \text{Sm}(C) \rightarrow \text{Fun}(\mathbb{N}(\text{Mf}_{\text{rel}})^{op}, C)$$

which on objects acts as $\mathcal{E}(X)(M, U) = X(M, U)$. 113
Proof. We have a forgetful functor $r : \text{Mf}_{rel} \to \text{Mf}$, $(M,U) \mapsto M$. Furthermore, we have a functor
\[
d : \text{Fun}(\mathbb{N}(\text{Mf}_{rel})^{op}, \mathcal{C}) \to \text{Fun}(\mathbb{N}(\text{Mf}_{rel})^{op}, \text{Fun}(\mathbb{N}([1]), \mathcal{C}))
\]
which maps $X$ to the diagram $X(M,\emptyset) \to X(U,\emptyset)$. Then we can write
\[
E := \ker \circ d \circ r^* : \text{Sm}(\mathcal{C}) \to \text{Fun}(\mathbb{N}(\text{Mf}_{rel})^{op}, \mathcal{C})
\]
\[\square\]

Problem 4.147. Assume that $X$ satisfies descent, $K \subseteq M$ is closed, $V \subseteq M$ is open and contains $K$. Then we have excision: The natural map $X_K(M) \to X_K(V)$ is an equivalence.

Definition 4.148. We define the compactly supported evaluation of $X$ by
\[
X_c(M) := \colim_K X_K(M),
\]
where the colimit is taken over all compact subsets $K \subseteq M$.

Problem 4.149. Define a compactly supported extension functor
\[
\mathcal{E}_c : \text{Sm}(\mathcal{C}) \to \text{Sm}(\mathcal{C})
\]
such that
\[
\mathcal{E}_c(X)(M) \cong X_c(M).
\]

Example 4.150. We get
\[
\mathcal{E}_c(\Omega^n_c)(M) = \{\omega \in \Omega^n(M,\mathcal{C}) \mid \text{supp}(\omega) \text{ is compact}\}.
\]

Example 4.151. We write $\pi_*(\text{Sm}(E)^c) =: E^{-*}_c$ for the compactly supported $E$-cohomology functor. This notation coincides with the common usage, but conflicts the use of the symbol $c$ otherwise in the present paper. The group $E^{-*}_c(M)$ is in general not the group of cohomology classes in $E^{-*}(M)$ which vanish outside of some compact subset.

Problem 4.152. Show that for every $k \in \mathbb{Z}$ there exists a natural isomorphism
\[
E^k_c(\mathbb{R}^n) \cong E^{k-n}(\ast).
\]
(46)

We now define the push-forward with proper support along a smooth map $W \to B$. We let $\text{Mf}/(W \to B)$ be the category of objects $(D,K)$ with $D$ a diagram as in (44) and $K \subseteq V$ closed such that the induced map $K \to M$ is proper. Morphisms $f : D \to D'$ are structure preserving maps such that $f(M \setminus K) \subseteq M' \setminus K'$. We define
\[
\tilde{p}^* : \text{Fun}(\mathbb{N}(\text{Mf}/W)^{op}, \mathcal{C}) \to \text{Fun}(\mathbb{N}(\text{Mf}/(W \to B))^{op}, \mathcal{C})
\]
in a natural way so that on objects
\[
\tilde{p}^*(X)(D,K) := X_K(V \to W).
\]
Problem 4.153. Make this precise.

Proof. Similar to 4.146.

Let \( r : \tilde{\text{Mf}}/(W \to B) \to \text{Mf}/(W \to B) \) be the functor which maps \((D, K)\) to \(D\). Then we have an adjunction

\[
r_{\flat} : \text{Fun}(\text{N}(\tilde{\text{Mf}}/(W \to B))^{\text{op}}, \text{C}) \rightleftharpoons \text{Fun}(\text{N}(\text{Mf}/(W \to B))^{\text{op}}, \text{C}) : r^{*}.
\]

Explicitly,

\[
(r_{\flat} X)(D) = \text{colim}_K X(D, K).
\]

Definition 4.154. We define the push-forward with proper support by

\[
 f!: = q_{\ast} \circ r_{\flat} \circ \tilde{p}^{\ast}.
\]

For \( X \in \text{Sm}_B(\text{C}) \) and \((p : M \to B) \in \text{Mf}/B\) we write

\[
 X_{\text{prop}/p}(M \to B) = X_{\text{prop}/B}(M \to B) := ((\text{id}_B) ; \text{id}^{\ast}_B X)(M \to B).
\]

Example 4.155. Assume that \( f : W \to B \) is an oriented submersion and \( n = \text{dim}(B) - \text{dim}(W) \). Consider \( \Omega_C \in \text{Sm}_B(\text{N}(\text{Ch})) \). Then

\[
 (f_{\ast} f^{\ast}(\Omega_C)_{B})(M \to B) = \Omega_{C, \text{prop}/M}(M \times_B W)
\]

is the space of forms on \( M \times_B W \) with proper support over \( M \). Note that \( M \times_B W \to M \) is naturally an oriented submersion.

Problem 4.156. Show that integration over the fibre defines a transformation

\[
 \int_{W/B} : f_{\ast} f^{\ast}(\Omega_C)_{B} \to (\Omega_C[n])_{B}.
\]

Example 4.157. Let \( \iota : \text{N}(\text{Ch}) \to \text{N}(\text{Ch})[W^{-1}] \) be the localization. Let \( \Omega \in \text{Sm}_B^{\text{desc}}(\text{N}(\text{Ch})) \) be a complex of soft sheaves. We have a natural map (see 4.143)

\[
 \iota(f_{\ast} f^{\ast} \Omega) \to f_{\ast} f^{\ast} \iota(\Omega).
\]

Problem 4.158. Show that the natural map

\[
 \iota(f_{\ast} f^{\ast} \Omega) \to f_{\ast} f^{\ast} \iota(\Omega)
\]

is an equivalence.

Proof. We fix \( M \to B \). Then we must show that

\[
 \iota(\Omega_{\text{prop}/M}(M \times_B W \to M)) \to \iota(\Omega)_{\text{prop}/M}(M \times_B W \to M)
\]
is an equivalence. We have

\[ \Omega_{\text{prop}/M}(M \times_B W \to M) = \text{colim}_K \Omega_K(M \times_B W \to M) \]

where \( K \subseteq M \times_B W \) is proper over \( M \). On the other hand

\[ \iota(\Omega)_{\text{prop}/M}(M \times_B W \to M) = \text{colim}_K \iota(\Omega)_K(M \times_B W \to M) . \]

The filtered colimit commutes with the limit defining the support. We therefore get and

\[ \Omega_{\text{prop}/M}(M \times_B W \to M) = \text{kernel}(\Omega(M \times_B W \to M) \to \text{colim}_K \Omega((M \times_B W \setminus K) \to M)) \]

and

\[ \iota(\Omega)_{\text{prop}/M}(M \times_B W \to M) = \text{kernel}(\iota(\Omega(M \times_B W \to M)) \to \text{colim}_K \iota(\Omega(M \times_B W \setminus K) \to M)). \]

Since a filtered colimit commutes with cohomology we see that

\[ \text{colim}_K \iota(\Omega((M \times_B W \setminus K) \to M)) \cong \iota(\text{colim}_K \Omega((M \times_B W \setminus K) \to M)). \]

Since \( \Omega \) is soft the map \( \Omega(M \times_B W) \to \text{colim}_K \Omega(M \times_B W \setminus K) \) is surjective. This implies by 4.143 that its kernel also represents the kernel taken after localization. \( \square \)

**Example 4.159.** We consider a differential data \((E, A, c)\). Define \( \text{Diff}(E, A, c)_K^n(W) \) as in 4.144.

**Problem 4.160.** Assume that the data \((E, A, c)\) is strict. Make the long exact sequence of the pair \((W, W \setminus K)\) in differential E-cohomology explicit.

**Proof.** For every \( n \) we have sequence

\[ \cdots \to E^C/\mathbb{Z}^{n-2}(W) \to E^C/\mathbb{Z}^{n-2}(W \setminus K) \to \hat{E}^n_R(W) \to \hat{E}^n(W) \to \hat{E}^n(W \setminus K) \to E^{n+1}(K) \to \cdots \]

which continues with the long exact pair sequences of \( E^C/\mathbb{Z}^* \) and \( E^* \) to the left and the right. \( \square \)

**Problem 4.161.** Derive the compactly supported version of the exact sequence (37).

**Proof.** We have

\[ \to E^{k-1}_c(M) \to \Omega A^{k-1}_{c^{-1}}/\text{im}(d_{|\Omega A^{k-2}_c}) \to \hat{E}^k_c(M) \to E^k_c \to 0 . \]

To see this consider the filtered colimit over the compact subsets of \( M \) of the pull-back diagrams defining \( \text{Diff}^k(E, W, c)_K \). Commute the colimit inside and use 4.143 to commute the colimit with the localization on the de Rham side. \( \square \)
**Problem 4.162.** Assume that the data \((E, A, c)\) is multiplicative. Let \(K, K' \subseteq W\) be closed. Show that \(\hat{E}_K^* (M)\) is an \(\hat{E}^*(W)\)-module. Refine this structure to a product
\[
\hat{E}_K^* (W) \otimes \hat{E}_{K'}^* (W) \to \hat{E}_{K \cap K'}^* (W).
\]

**Example 4.163.** We consider the compact subset \(\{0\} \subset \mathbb{R}\) and calculate \(\hat{H}_Z^* (R)\) using the exact sequences. We write out the interesting pieces:
\[
\begin{align*}
\mathbb{C}/\mathbb{Z} &\to \mathbb{C}/\mathbb{Z} \oplus \mathbb{C}/\mathbb{Z} \to \hat{H}_Z^2 (\{0\}; \mathbb{R}) \to 0 \\
0 &\to \hat{H}_Z^1 (\{0\}; \mathbb{R}) \to \Omega^0 (\mathbb{R}, \mathbb{C})/\mathbb{Z} \to \Omega^0 (\mathbb{R} \setminus \{0\}, \mathbb{C})/\mathbb{Z} \oplus \mathbb{Z} \\
0 &\to \hat{H}_Z^0 (\{0\}; \mathbb{R}) \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}
\end{align*}
\]
We conclude that
\[
\hat{H}_Z^k (\{0\}; \mathbb{R}) \cong \begin{cases} 
\mathbb{C}/\mathbb{Z} & k = 2 \\
0 & k \neq 2 
\end{cases}.
\]

**Problem 4.164.** Calculate \(\hat{H}_Z^k ([0, 1]; \mathbb{R})\).

*Proof.* Use again the exact sequences. The result is
\[
\hat{H}_Z^k ([0, 1]; \mathbb{R}) \cong \begin{cases} 
\mathbb{C}/\mathbb{Z} & k = 2 \\
0 & k \neq 2 
\end{cases}.
\]
Here \(\Omega^0 ((0,1], \mathbb{R}) \subset \Omega^0 ((0,1], \mathbb{C})\) are those functions on the interval which extend smoothly by zero to \(\mathbb{R}\). \(\square\)

**Problem 4.165.** Calculate \(\hat{H}_Z^* (S^1)\).

*Proof.* We write out the relevant sequences.
\[
\begin{align*}
\mathbb{C}/\mathbb{Z} &\to \mathbb{C}/\mathbb{Z} \to \hat{H}_Z^2 (S^1) \to \mathbb{C}/\mathbb{Z} \to 0 \\
0 &\to \hat{H}_Z^1 (S^1) \to \hat{H}_Z^1 (S^1) \to \Omega^0 (S^1 \setminus \{1\}, \mathbb{C})/\mathbb{Z} \to 0 \\
0 &\to \hat{H}_Z^0 (S^1) \to \mathbb{Z} \to \mathbb{Z}
\end{align*}
\]
We conclude that
\[
\hat{H}_Z^k (S^1) \cong \begin{cases} 
0 & k \neq 2 \\
\mathbb{C}/\mathbb{Z} & k = 2 
\end{cases}.
\]

The coincidence \(\hat{H}_Z^* (\mathbb{R}) \cong \hat{H}_Z^* (S^1)\) is of course expected by excision.
Example 4.166. We calculate $\hat{H}Z_c^k(\mathbb{R})$ using 4.161. We again write out the relevant sequences
\[
0 \to \mathbb{C}/\mathbb{Z} \to \hat{H}Z_c^2(\mathbb{R}) \to 0 \\
0 \to \Omega_c^0(\mathbb{R}) \to \hat{H}Z_c^1(\mathbb{R}) \to \mathbb{Z} \to 0 \tag{47} \\
0 \to \hat{H}Z_c^0(\mathbb{R}) \to 0
\]
We conclude that
\[
\hat{H}Z_c^k(\mathbb{R}^1) \cong \begin{cases} 
\mathbb{C}/\mathbb{Z} & k = 2 \\
\mathbb{C}/\mathbb{Z} & k = 1 \\
0 & k \notin 1, 2
\end{cases}.
\]
Alternatively we could use 4.164

Example 4.167. Let $W$ be a smooth oriented $n$-manifold with boundary $\partial W$.

Problem 4.168. Calculate $\hat{H}Z_{\partial W}^n(W)$, $\hat{H}Z_{\partial W}^{n+1}(W)$ and $\hat{H}Z_c^{n+1}(W \setminus \partial W)$.

Proof. For the calculation we use appropriate exact sequences, e.g. the pair sequence obtained in 4.160. We get $\hat{H}Z_{\partial W}^n(W) \cong \mathbb{C}/\mathbb{Z}$ and $\hat{H}Z_{\partial W}^{n+1}(W) \cong 0$ and $\hat{H}Z_c^{n+1}(W \setminus \partial W) \cong \mathbb{C}/\mathbb{Z}$.

Problem 4.169. If $N \subseteq M$ is a closed embedded submanifold of non-zero codimension, then we have for every $k \in \mathbb{Z}$ a natural isomorphism $\hat{E}_N^k(M) \cong E/\mathbb{Z}^{k-1}(M, M \setminus N)$.

Proof. Use 4.160

Problem 4.170. If $N \subseteq M$ is an embedded submanifold of codimension zero, then we have for every $k \in \mathbb{Z}$ a natural injection $\hat{E}_N^k(N \setminus \partial N) \subseteq \hat{E}_N^k(M)$.

Proof. Note that we can not expect an isomorphism since the elements of $\hat{E}_N^k(N \setminus \partial N)$ have curvature compactly supported in $N \setminus \partial N$ while the curvatures of the elements in $\hat{E}_N^k(M)$ can be smoothly extended by zero to $M$, but need not be supported properly in the interior of $N$. We have seen this effect already in the calculation 4.164.

Example 4.171. We have the de Rham equivalence in $\text{Sm}^{\text{desc}}(\mathbb{N}(\mathbb{Sp})[W^{-1}])$
\[
\text{Rham} : H(\Omega A) \to \text{Sm}(H(A))
\]
which as an equivalence of sheaves of ring spectra if $A$ is a commutative dga.

Problem 4.172. Show that it naturally induces an equivalence
\[
\text{Rham} : H(\Omega A_{\text{prop}/M}) \to \text{Sm}(H(A))_{\text{prop}/M}
\]
in $\text{Sm}(\text{Mod}(HZ))$ (or $\text{Sm}(\text{Mod}(H(\Omega A)))$ if $A$ is a commutative dga).
Proof. Use the naturality of $Rham$. □

If the differential data $(E, A, c)$ is multiplicative we can interpret the construction of $Diff^*(E, A, c)_{prop/M}$ in $p^*Diff^*(E, A, c)$-module spectra. In particular we get a $p^*\hat{E}^*$-module structure on $\hat{E}^*_{prop/M}$, where we define

$$\hat{E}^*_{prop/M} := \pi_*(Diff^*(E, A, c)_{prop/M}).$$

Again this notation is sloppy in the sense note earlier in 4.151.

Problem 4.173. Analyse the difference between $\hat{E}^*_{prop/M} = \pi_*(Diff^*(E, A, c)_{prop/M})$ and the group $\pi_*(Diff^*(E, A, c))_{prop/M}$ of differential cohomology classes having proper support over $M$.

Proof. $\hat{E}^*_{prop/M}$ is a subgroup of $E^*_M$ consisting of those elements which vanish after restriction to the complement of a sufficiently large subset which is proper over $M$. $\hat{E}^*_{prop/M}$ maps to $\pi_*(Diff^*(E, A, c))_{prop/M}$, but not injectively in general. □

Example 4.174. Let $f : W \to M$ be a smooth map. In order to explicitly construct differential cohomology classes which are properly supported over $M$ we need versions of the cycle maps which respect supports. We first consider the case of bordism.

Definition 4.175. A geometric $\mathbf{MU}$-cycle (cf. Def. 4.99) $(g, N, J, \nabla, \eta)$ on $W$ is properly supported over $M$ if the composition $f \circ g$ is proper and $\eta$ has proper support over $M$. We let $\text{Cycle}_{\text{geom},*,\mathbf{MU},prop/M}(W)$ denote the graded semigroup of isomorphism classes of geometric $\mathbf{MU}$-cycles.

Problem 4.176. Show that there exists natural refinement of the cycle map $\widehat{\text{cycl}}$ introduced in 4.101 to a cycle map

$$\widehat{\text{cycl}} : \text{Cycle}_{\text{geom},*,\mathbf{MU},prop/M}(W) \to \widehat{\mathbf{MU}}^*_{prop/M}(W).$$

Proof. Let $c := (g : E \to W, N, J, \nabla, \eta)$ be a geometric $\mathbf{MU}$-cycle which is properly supported over $M$. Then there exists a closed subset $K \subseteq W$ which is proper over $M$ such that $g(E)$ and $\text{supp}(\eta)$ are contained in the interior of $K$. The restriction of the cycle $c$ to $W \setminus K$ is the zero cycle. We can consider $(g, N, J)$ as a relative cycle for the pair $(W, W \setminus K)$ and therefore get a class

$$y := \text{cycl}(g, N, J) \in \mathbf{MU}^*(W, W \setminus K) \cong \mathbf{MU}^*_K(W).$$

We have an exact sequence (cf. 4.160)

$$\cdots \mathbf{MU}C/Z^* \to \widehat{\mathbf{MU}}^*_K(W) \to \widehat{\mathbf{MU}}^*(W) \to \widehat{\mathbf{MU}}^*(W \setminus K).$$

Since $\widehat{\text{cycl}}(x)|_{W \setminus K} = 0$ we conclude that there exists a class $\hat{y} \in \widehat{\mathbf{MU}}_K(W)$ which maps to $\widehat{\text{cycl}}(c) \in \widehat{\mathbf{MU}}^*(W)$. The class $\hat{y} \in \widehat{\mathbf{MU}}_K(W)$ is unique up to elements which come
from $\text{MU}C/\mathbb{Z}^{* - 2}(W \setminus K)$. The image of $\text{MU}C/\mathbb{Z}^{* - 2}(W \setminus K) \to \widehat{\text{MU}}_K^*(W)$ is detected by the composition with $I$. Hence we can choose $\hat{y}$ uniquely such that $I(\hat{y}) = y$. We define
$$\widehat{\text{cycl}}(c) := \hat{y}.$$ 

A cycle for a differential $K$-theory class on $W$ which is properly supported over $M$ is given by the following data $(V_0, V_1, \nabla_0, \nabla_1, \phi, K)$, where $(V_i, \nabla_i)$, $i = 0, 1$ are complex vector bundles with connection on $W$, $K \subseteq W$ is closed and proper over $M$, and $\phi : (V_0|_{W\setminus K} \to (V_1|_W\setminus K$ is a connection-preserving vector bundle isomorphism.

**Problem 4.177.** Show that there exists a unique natural and additive way to associate to the data $(V_0, V_1, \nabla_0, \nabla_1, \phi, K)$ a class $\widehat{\text{cycl}}(V_0, V_1, \nabla_0, \nabla_1, \phi, K) \in \widehat{\text{KU}}_K(W)$.

### 4.8 Thom classes

Let $E \in \text{CommMon}(\text{Sp}[W^{-1}])$ be a commutative ring spectrum which represents the multiplicative cohomology theory $E^*$. Assume that $f : W \to M$ is a real vector bundle of dimension $n$. A Thom class of $W$ is a class $\nu \in E^n_{\text{prop}/M}(W)$ whose restriction to each fibre $W_m$, $m \in M$ maps to $\pm 1$ under the isomorphism
$$E^n_{\text{prop}/m}(m) \cong E^n_*(\mathbb{R}^n) \xrightarrow{\pi_*} E^n_0(*) \quad (48).$$

The first isomorphism depends on an identification of the fibre $W_m$ with $\mathbb{R}^n$. If the characteristic of the ring $\pi_*(E)$ is not 2, and if we fix a Thom class on $W \to M$, then the bundle $W$ acquires an induced ordinary orientation. An oriented frame of the fibre $W_m$, i.e. an orientation-preserving isomorphism $W_m \cong \mathbb{R}^n$, is characterized by the requirement that $\nu$ goes to 1 under (48).

**Definition 4.178.** A differential Thom class on a real $n$-dimensional vector bundle $W \to M$ is a class $\hat{\nu} \in \hat{E}^n_{\text{prop}/M}(W)$ such that $I(\hat{\nu}) := \nu \in E^n_{\text{prop}/M}(W)$ is a Thom class. We say that $\hat{\nu}$ refines $\nu$. We define
$$\text{Td}(\hat{\nu}) := \int_{W/M} R(\hat{\nu}) \in \Omega A^0_{\text{cl}}(W), \quad \text{Td}(\nu) := \text{Rham}(\text{Td}(\hat{\nu})) \in H(A)^0(W)^\times.$$

**Problem 4.179.** Show that every Thom class has a differential refinement.

**Problem 4.180.** If $\hat{\nu}$ is a differential Thom class on $W \to M$, then show that
$$\text{Td}(\hat{\nu}) = 1 + \sum_j e^j \omega_j$$

with homogeneous elements $e^j \in E^{<0}$ and $\omega_j \in \Omega_{\text{cl}}^{-\deg(e_j)}(M, \mathbb{C})$. Conclude that $\text{Td}(\hat{\nu})$ is a unit.
Proof. Note that $\sum_j e^j \omega_j$ is nilpotent. \hfill \Box

The topological Thom class gives a Thom isomorphism
\[ \Phi_\nu := \nu \cup f^*(\ldots) : E^*(M) \to E^{*+n}_{prop/M}(W) \]
of $E^*(M)$-modules.

**Problem 4.181.** Assume that $(E, A, c)$ is strict. Show that the differential version
\[ \hat{\nu} \cup f^*(\ldots) : \hat{E}^*(M) \to \hat{E}^{*+n}_{prop/M}(W) \]
is an injective morphism of $\hat{E}^*(M)$ modules.

**Proof.** Let $x \in \hat{E}^k(M)$ and assume that $\hat{\nu} \cup f^*(x) = 0$. Then $R(\hat{\nu} \cup f^*(x)) = R(\hat{\nu}) \wedge f^*R(x) = 0$. We conclude that $\int_{W/M} R(\hat{\nu}) \wedge R(x) = 0$. Since $\int_{W/M} R(\hat{\nu})$ is a unit by 4.180, we see that $R(x) = 0$. Hence $x$ is flat. For a flat class $x$ the class $\hat{\nu} \cup f^*(x)$ is the image of $x$ under the Thom isomorphism $E^*\mathbb{C}/\mathbb{Z}^{k-1}(M) \to E^*\mathbb{C}/\mathbb{Z}^{k-1+n}_{prop/M}(W)$. Therefore $x = 0$. \hfill \Box

**Definition 4.182.** We say that two differential Thom classes $\hat{\nu}', \hat{\nu}$ on a bundle $N \to W$ are homotopic if there exists a differential Thom class $\tilde{\nu}$ on the pull-back $\tilde{N} := \text{pr}^*N \to I \times W$ which connects $\hat{\nu}'$ and $\hat{\nu}$ and satisfies $\text{Td}(\tilde{\nu}) = \text{pr}^*\text{Td}(\hat{\nu})$.

Note that because of the condition on the $\text{Td}$-forms homotopy in the sense of differential Thom classes is a stronger condition than just homotopy as differential cohomology classes. In particular, if $\hat{\nu}$ and $\hat{\nu}'$ are homotopic, then we have the equality $\text{Td}(\hat{\nu}) = \text{Td}(\hat{\nu}')$.

**Problem 4.183.** We assume that $(E, A, c)$ is strict. Show that homotopy classes of differential Thom classes refining a given underlying topological Thom class $\nu$ and with fixed form $\text{Td}(\hat{\nu})$ are classified by the group
\[ H(A)^{-1}(W) / \text{Td}(\nu) \cup c(E^{-1}(W)) \]

**Proof.** Let $\hat{\nu}$ and $\hat{\nu}'$ be two differential Thom classes on a bundle $\pi : N \to W$ refining the same underlying topological Thom class $\nu$ and such that we have an equality of integrals
\[ \text{Td}(\hat{\nu}) := \int_{N/W} R(\hat{\nu}') = \int_{N/W} R(\hat{\nu}) . \]

Then there exists a form $\eta \in \Omega A_{prop/W}^{n-1}(N)/\text{im}(d)$ uniquely determined up to $c(E_{prop/W}^{n-1}(N))$ such that $\hat{\nu}' - \hat{\nu} = a(\eta)$. We calculate that $d \int_{N/W} \eta = 0$. By the Thom isomorphism in cohomology we have
\[ \int_{N/W} c(E_{prop/W}^{n-1}(N)) = \text{Td}(\nu) \cup c(E^{-1}(W)) . \]

121
We therefore define the difference class
\[
\delta(\hat{\nu}', \hat{\nu}) := \text{Rham}(\int_{N/W} \eta) \in \frac{H(A)^{-1}(W)}{\text{Td}(\nu) \cup c(E^{-1}(W))}.
\] (49)

We claim that \(\delta(\hat{\nu}', \hat{\nu})\) exactly detects whether the two Thom classes \(\hat{\nu}\) and \(\hat{\nu}'\) are homotopic.

First we show that if \(\delta(\hat{\nu}', \hat{\nu}) = 0\), then \(\hat{\nu}'\) and \(\hat{\nu}\) are homotopic. Assume that \(\delta(\hat{\nu}', \hat{\nu}) = 0\).

Then we can find an element \(y \in E^{-1}(W)\) such that
\[
\text{Rham}(\int_{N/W} \eta) = \text{Td}(\nu) \cup c(y).
\]

We can find a closed form \(\beta \in \Omega^{n-1}_{\text{prop/W}, \text{cl}}(N)/\text{im}(d)\) such that
\[
\text{Rham}(\beta) = c(z) = \nu \cup \pi^*y \text{ and } \int \beta = \int \eta.
\]
Indeed, if \(\beta_0\) is some representative of the class \(c(z)\), then we have
\[
\int_{N/W} \eta - \int_{N/W} \beta_0 = d\alpha.
\]
We then set
\[
\beta := \beta_0 + d(R(\hat{\nu}) \wedge \pi^*(\alpha \cup \text{Td}(\hat{\nu})^{-1})).
\]

We now have \(\hat{\nu}' - \hat{\nu} = a(\eta) = a(\eta - \beta)\). We consider the homotopy
\[
\tilde{\nu} := \hat{\nu} + t(\eta - \beta).
\]

We must check the curvature condition. We have
\[
R(\tilde{\nu}) = \text{pr}^* R(\hat{\nu}) + dt \wedge (\eta - \beta) + t \wedge d\eta.
\]
Since \(\int_{N/W} d\eta = 0\) and \(\int_{N/W} (\eta - \beta) = 0\) we conclude that
\[
\text{Td}(\tilde{\nu}) = \text{pr}^* \text{Td}(\hat{\nu}).
\]

We assume now that \(\hat{\nu}'\) and \(\hat{\nu}\) are connected by a homotopy \(\tilde{\nu}\). By the homotopy formula
\[
\hat{\nu}' - \hat{\nu} = a(\int_{I \times N/N} R(\tilde{\nu})).
\]

Hence we get
\[
\delta(\hat{\nu}', \hat{\nu}) = \int_{I \times N/W} R(\tilde{\nu}) = \int_{I \times W/W} \text{pr}^* R(\hat{\nu}) = 0.
\]

Finally, let \(y \in H(A)^{-1}(W)\) and \(\hat{\nu}\) be given. Then we define
\[
\hat{\nu}' := \hat{\nu} + a(R(\hat{\nu}) \wedge \pi^*(\text{Td}(\hat{\nu})^{-1} \cup y))
\]

With this choice
\[
\delta(\hat{\nu}', \hat{\nu}) = [y] \in \frac{H(A)^{-1}(W)}{\text{Td}(\nu) \cup c(E^{-1}(W))}.
\]
The solution of Problem 4.183 allows to state the following two-out-of-three principle. Assume that \((E, A, c)\) is strict. Let \(N = N_0 \oplus N_1\) be a decomposition of the bundle \(N \to W\) and \(p_i : N \to N_i\) be the projections. Then we consider the relation

\[
\hat{\nu} = p_0^* \hat{\nu}_0 \cup p_1^* \hat{\nu}_1
\] (50)

between differential Thom classes on \(N\) and the summands \(N_i\).

**Problem 4.184.** Show that two of these classes determine the third uniquely up to homotopy such that (50) holds true.

**Proof.** Assume that \(\hat{\nu}\) and \(\hat{\nu}_1\) are given. The equality

\[
\text{Td}(\hat{\nu}_1) = \frac{\text{Td}(\hat{\nu})}{\text{Td}(\hat{\nu}_0)}
\]

fixes the \(\text{Td}\)-form of \(\hat{\nu}_1\). If we choose any differential Thom class \(\hat{\nu}_1'\) with this \(\text{Td}\)-form, then we have a class

\[
\delta := \delta(\hat{\nu}, p_0^* \hat{\nu}_0 \cup p_1^* \hat{\nu}_1') \in \frac{H(A)^{-1}(W)}{\text{Td}(\nu) \cup c(E^{-1}(W))}.
\]

We are forced to define

\[
\hat{\nu}_1 := \hat{\nu}_1' + a(R(\hat{\nu}_1') \wedge \text{pr}^*(\text{Td}(\hat{\nu}_1)^{-1} \cup \delta))
\] .

\[
\square
\]

We now discuss the special case of the Thom isomorphism for trivial bundles. In particular we relate it with the suspension isomorphism on the one hand, and with the integration of differential forms on the other. Let \(\text{pr} : \mathbb{R}^n \to *\) be the projection. Then we can construct a commutative diagram in \(\mathbb{N}(Sp)[W^{-1}]\). It turns out to be useful not to hide the localization \(i : \mathbb{N}(Ch) \to \mathbb{N}(Ch)[W^{-1}]\).

\[
\begin{array}{cccccc}
\text{pr}_1 \text{pr}_* \text{Sm}(E) & \overset{\text{c}}{\longrightarrow} & \text{pr}_1 \text{pr}_* \text{Sm}(H(A)) & \overset{\text{Rham}}{\longrightarrow} & \text{pr}_1 \text{pr}_* H(\Omega A) & \leftrightarrow & \text{pr}_1 \text{pr}_* H(\iota(\sigma^{\geq k} \Omega A)) \\
& & & & & (i) & (ii) \\
\text{Sm}(E[-n]) & \overset{\text{c}}{\longrightarrow} & \text{Sm}(H(A)[-n]) & \overset{\text{Rham}}{\longrightarrow} & H(\iota(\Omega A[-n])) & \leftrightarrow & H(\iota(\sigma^{\geq k-n} \Omega A[-n])) \\
\end{array}
\] (51)

The two vertical maps in (51) marked by (i) are induced by the integration of forms \(\int_{\mathbb{R}^n}\) (see 4.155). Note that we can not define the integration on \(\text{pr}_1 \text{pr}_* \iota(\Omega A)\) directly since the elements of its evaluation are not really properly supported forms.
The arrows marked by (\textit{ii}) are equivalences, similar to 4.158. The lower right square and the upper right square commute. We explain the construction of the left vertical maps in the evaluation at $M$. For $r > 0$ let $B^c(r) \subset \mathbb{R}^n$ be the complement of the closed $r$-ball centered at zero. Then we have a canonical identification
\[
\text{Sm}_{\text{prop}/M}(E)(M \times \mathbb{R}^n) \cong \text{colim}_r \text{Map}(\Sigma^\infty(M_+ \wedge (\mathbb{R}^n / B^c(r))), E)
\]
If we identify $S^n$ with the one-point compactification of $\mathbb{R}^n$, then we get a compatible system of natural maps $S^n \to \mathbb{R}^n / B^c(r)$ for all $r > 0$ and therefore a map
\[
\text{colim}_r \text{Map}(\Sigma^\infty(M_+ \wedge (\mathbb{R}^n / B^c(r))), E) \to \text{Map}(\Sigma^\infty M_+ \wedge \Sigma^\infty S^n, E) \cong \text{Map}(\Sigma^\infty M_+, E[-n])
\]
were
\[
E[-n] := \text{Map}(\Sigma^\infty S^n, E).
\]
The composition of these maps gives the arrow
\[
\text{Sm}_{\text{prop}/M}(E)(M \times \mathbb{R}^n) \to \text{Sm}(E[-n])(M)
\]
and, similarly,
\[
\text{Sm}_{\text{prop}/M}(H(A))(M \times \mathbb{R}^n) \to \text{Sm}(H(A)[-n])(M).
\]
The left square in (51) commutes.

\textbf{Problem 4.185.} The middle square in (51) commutes. If $A$ is a commutative dga, then we can interpret this square in $H(\Omega A(M))$-modules.

\textit{Proof.} Note that $A$ is a $\mathbb{C}$-module. Moreover, the vertical arrow and the composition $i \circ (\text{ii})^{-1}$ are isomorphisms whose inverse is given by the multiplication with a representative of the Thom class $H\Omega^n_{c,cl}(\mathbb{R}^n, \mathbb{C})$ and its image under $\text{Rham}$. The square with the vertical arrows inverted thus commutes since $\text{Rham}$ is multiplicative and preserves the Thom classes. \hfill \square

If one takes the product of these diagrams (51) over all $k$ and uses the fact that the diagram of ring spectra $\text{Diff}^\bullet(E, A, c)$ acts naturally on all entries one can refine this product to a diagram in $\text{Mod}(\text{Diff}^\bullet(E, A, c))$.

\textbf{Problem 4.186.} Show that the diagram (51) induces a desuspension map
\[
\text{desusp} : \text{pr}_! \text{pr}^* \text{Diff}^\bullet(E, A, c) \to \text{Diff}^\bullet_{-n}(E, A, c)
\]
(of $\text{Diff}^\bullet(E, A, c)$-modules).

\textit{Proof.} We must interchange the order of $\text{pr}_! \text{pr}^*$ and the finite limit defining $\text{Diff}^\bullet(E, A, c)$. This works since $\text{pr}_!$ involves filtered colimits, only. \hfill \square
Corollary 4.187. We have a desuspension map

\[ \text{desusp} : E^*_\text{prop}/M(M \times \mathbb{R}^n) \to E^{*-n}(M) \]

of \( E^*(M) \)-modules.

In order to verify functorial properties of differential integration maps we need the following statement about desuspension in stages.

Problem 4.188. We have a refinement of the desuspension map

\[ \text{desusp} : \hat{E}^*_\text{prop}/M(M \times \mathbb{R}^l \times \mathbb{R}^n) \to \hat{E}^{*-n}_\text{prop}/M(M \times \mathbb{R}^l) , \]

and the diagram

\[ \begin{array}{ccc} E^*_\text{prop}/M(M \times \mathbb{R}^l \times \mathbb{R}^n) & \xrightarrow{\text{desusp}} & \hat{E}^{*-n}_\text{prop}/M(M \times \mathbb{R}^l) \\ \downarrow & & \downarrow \text{desusp} \\ \hat{E}^*_\text{prop}/M(M \times \mathbb{R}^l) & \xrightarrow{\text{desusp}} & \hat{E}^{*-n}_\text{prop}/M(M \times \mathbb{R}^l) \end{array} \]

commutes.

Proof. Write out the diagram (51) as a composition of two similar, appropriately adapted diagrams. \( \square \)

Example 4.189. Let \( \hat{\nu} \in \hat{S}^n_c(\mathbb{R}^n) \) be a differential Thom class in differential stable cohomotopy refining the canonical class in \( S^n_c(\mathbb{R}^n) \). Exterior multiplication by \( \hat{\nu} \) (see 4.119) induces a map

\[ \hat{\nu} \times \ldots : \hat{E}^*(M) \to \hat{E}^*_\text{prop}/M(M \times \mathbb{R}^n) . \]

Problem 4.190. Show that on \( \hat{E}^*(M) \) we have the identity

\[ \text{desusp} \circ (\hat{\nu} \times \ldots) = \text{id} . \]

Example 4.191. In this example we discuss the geometric construction of differential \( \text{MU} \)-Thom classes for complex vector bundles with connection. Let \( p : V \to M \) be a real vector bundle of dimension \( 2n \) with a complex structure \( I \). We consider the zero section \( 0_V : M \to V \). We have a canonical inclusion \( i : p^*V \to TV \) by the linear structure and an isomorphism \( TM \oplus V \overset{d0_V + 0_V}\to \overset{0_V}\to TV \). In particular we get an exact sequence

\[ \mathcal{N}_{0_V} : 0 \to TM \overset{d0_V}\to 0_V TV \to V \to 0 \]

which turns \( V \) into the normal bundle of \( 0_V \). We therefore get an \( \text{MU} \)-cycle \((0_V, \mathcal{N}_{0_V}, I)\) of degree \( 2n \) (cf. Definition 4.90) and a class \( \nu_{\text{MU}} := \text{cycl}(0_V, \mathcal{N}_{0_V}, I) \in \text{MU}^{2n}_{\text{prop}/M}(V) \) (see the proof of 4.176 for the support condition).

Problem 4.192. 1. Show that \( \nu_{\text{MU}} \) is a \( \text{MU} \)-Thom class of \( V \).
2. Show that \( \nu_{MU} \) can be refined to a class in \( MU_{0\nu}^{2n}(V) \).

Let now \( \nabla \) be a connection on \( V \) preserving the complex structure \( I \). Then we can define the form \( u(\nabla) \in \Omega A^0(M) \) as in \( [11] \).

**Problem 4.193.** Show that there exists a differential Thom class \( \hat{\nu} \in \widehat{MU}_{prop/M}^{2n}(V) \) such that

\[
\text{Td}(\hat{\nu}) = u(\nabla).
\]

**Proof.** Let \( \eta \in \Omega A^{2n-1}_\infty(V)_{prop/M} \) be some distributional form such that \( 0_{V^!}(u(\nabla)) - d\eta = 0 \). Then we have by \( [4.176] \) a differential Thom class

\[
\hat{\nu}_0 := \text{cycl}(0_V, N_{0\nu}, I, \nabla, \eta) \in \widehat{MU}_{prop/M}^{2n}(V).
\]

Even without having a solution of Problem \( [4.176] \) it is clear that there exists a differential Thom class \( \hat{\nu}_0 \in \widehat{MU}_{prop/M}^{2n}(V) \) with \( \text{Td}(\hat{\nu}_0) = u(\nabla) - d\int_{V/M} \eta \) since this form represents the correct cohomology class. Now \( d\int_{V/M} \eta \) is smooth and exact. Hence there exists a smooth form \( \mu \in \Omega A^{-1}(M) \) such that \( d\mu = d\int_{V/M} \eta \). We can further choose a form \( \tilde{\mu} \in \Omega A^{2n-1}_{prop/M}(V) \) such that \( \int_{V/M} \tilde{\mu} = \mu \). We define a corrected Thom form \( \hat{\nu} := \hat{\nu}_0 + a(\tilde{\mu}) \) which satisfies

\[
\text{Td}(\hat{\nu}) = u(\nabla).
\]

Note that the homotopy class of the differential Thom class \( \hat{\nu} \in \widehat{MU}_{prop/M}^{2n}(V) \) is not uniquely determined by the condition \( \text{Td}(\hat{\nu}) = u(\nabla) \). In fact, by the solution of Problem \( [4.183] \) the set of such Thom classes forms a torsor over

\[
\frac{H(A)^{-1}(M)}{u(V) \cup c(MU^{-1}(M))},
\]

see \( [40] \) for the characteristic class \( u(V) \). One can fix this ambiguity by requiring naturality.

**Problem 4.194.** Show that there is a unique way to associate a homotopy class of differential Thom classes \( \hat{\nu}(\nabla^V) \in \widehat{MU}_{prop/M}^{2n}(V) \) to a \( n \)-dimensional complex vector bundle with connection \( (V, \nabla^V) \) on a manifold \( M \) which is natural under pull-back and such that \( \text{Td}(\hat{\nu}(\nabla^V)) = u(\nabla^V) \).

**Proof.** We use the same technique which was already successful in the proofs of \( [3.40] \), \( [3.77] \) and \( [4.71] \). For the notation see also \( [2.117] \). The classifying space for \( n \)-dimensional complex vector bundles is \( BU(n) \). Let \( m := \dim(M) \). We can find a factorization

\[
\begin{align*}
V & \xrightarrow{H} W \xrightarrow{\xi_n} \xi \\
M & \xrightarrow{h} N \xrightarrow{u} BU(n)
\end{align*}
\]
of the classifying map of the bundle $V \to M$, where $u : N \to BU(n)$ is $m + 1$-connected and $\xi_n \to BU(n)$ is the universal $n$-dimensional complex vector bundle. We can assume that $W$ has a connection $\nabla^W$ which pulls back to $\nabla^V$.

The odd degree cohomology of $N$ is concentrated in degrees which exceed the dimension of $M$ so that the pull-back

$$h^* : \frac{H(A)^{-1}(N)}{u(W) \cup c(MU^{-1}(N))} \to \frac{H(A)^{-1}(M)}{u(V) \cup c(MU^{-1}(M))}$$

vanishes. If we choose $\nu(W)$ such that $Td(\nu(W)) = u(\nabla^W)$, then the pull-back $H^*\nu(W)$ satisfies $Td(\nu(\nabla^V)) = u(\nabla^V)$ as required and is independent of the choice of $\nu(W)$. We are forced to define $\nu(\nabla^V) := H^*\nu(W)$.

It remains to show that $\nu(\nabla^V)$ is well-defined independently of the choices, and to verify that our construction of the differential Thom class is natural. For well-definedness we argue as before. Two choices can be related with a third by a diagram

$$\begin{array}{ccc}
N & \xrightarrow{h} & M \\
| & \searrow & \nearrow \\
| & \uparrow g & \downarrow u \\
N' & \rightarrow & BU(n) \\
| & \searrow & \nearrow \\
| & \uparrow g' & \downarrow u' \\
N'' & \rightarrow & M \\
| & \searrow & \nearrow \\
| & \uparrow g' & \downarrow u' \\
N'' & \rightarrow & BU(n)
\end{array}$$

where $g \circ h$ and $g' \circ h'$ are homotopic. Let $H : I \times M \to N''$ be such a homotopy. We can lift this homotopy to an identification of vector bundles $\tilde{H} : pr_M^* V \to H^*W''$. We can further assume that under this identification $\tilde{H}^*\nabla^{W''} = pr_M^*\nabla^V$. To this end we just have to make sure that $H$ is an embedding. We can reach this situation by modifying $N''$. The connection $\nabla^{W''}$ is the defined by an extension of $pr_M^*\nabla^V$ on the image of $H$.

With these choices

$$\int_{I \times M/M} u(H^*\nabla^{W''}) = 0.$$ 

The homotopy formula gives

$$H'^*\nu(W'') - H^*\nu(W'') = a(\int_{I \times V/V} R(\tilde{H}^*\nu(W''))) .$$

The difference class (49) between the two Thom classes is now given by

$$\delta(H'^*\nu(W''), H^*\nu(W'')) = \left[ \int_{V/M} \int_{I \times V/V} R(\tilde{H}^*\nu(W'')) \right] \in \frac{H(A)^{-1}(M)}{u(V) \cup c(MU^{-1}(M))} .$$

But by Fubini

$$\left[ \int_{V/M} \int_{I \times V/V} R(\tilde{H}^*\nu(W'')) \right] = \left[ \int_{I \times M/M} u(H^*\nabla^{W''}) \right] = 0$$

127
Naturality is now easy to check. \hfill \square

Note that the argument for Problem 4.194 only depends on the fact that the rational cohomology of the classifying space $BU(n)$ is concentrated in even degrees. So a similar argument shows e.g.:

**Corollary 4.195.** There is a unique way to associate a homotopy class of differential Thom classes $\hat{\nu}(\nabla^V) \in \text{KU}_{prop/M}^n(V)$ to a $n$-dimensional Spin$^c$ vector bundle with Spin$^c$-connection $(V, \nabla^V)$ on a manifold $M$ which is natural under pull-back and such that $\text{Td}(\hat{\nu}(\nabla^V)) = \text{Td}(\tilde{\nabla}^V)^{-1}$ (see 4.199 for notation).

**Example 4.196.** In the following example we consider cannibalistic classes. They arise from the non-compatibility of Thom isomorphisms with cohomology operations. The exercises should clarify how the differential refinement of the theory of cannibalistic classes works. We consider the examples of the Adams operations on complex $K$-theory and the Chern character between complex $K$-theory and ordinary cohomology.

We consider a real vector bundle $V \to M$ of dimension $n$. We assume that $V$ has a Spin$^c$-structure, i.e. we have a Spin$^c(n)$-principal bundle $Q \to M$ and an identification $Q \times_{\text{Spin}^c(n)} \mathbb{R}^n \cong V$. By definition, a Spin$^c$-connection on $V$ is a connection $\tilde{\nabla}$ on $Q$. The Spin$^c$-structure provides a Thom class $\nu \in \text{KU}^n(MV)$ and a Thom isomorphism $\Phi_\nu : \text{KU}^*(M) \to \text{KU}^{*+n}(MV)$.

This follows from the existence of the Atiyah-Bott-Shapiro orientation \text{ABS}:

$$\text{ABS} : \text{MSpin}^c \to \text{KU}.$$ 

This orientation is in fact multiplicative (see \text{Joa04} for an $E_\infty$-version) and compatible with the complex orientation of $c : \text{MU} \to \text{KU}$ given by the multiplicative formal group law $x + \text{KU} y = x + y + bxy$, i.e. the following diagram commutes

$$\begin{array}{ccc}
\text{MU} & \longrightarrow & \text{MSpin}^c \\
\downarrow^c & & \downarrow_{\text{ABS}} \\
\text{KU} & \rightarrow & \\
\end{array}$$

where the upper map is induced by the maps $\beta : U(n) \to \text{Spin}^c(2n)$, see e.g. (52). This observation will help in calculations.

We fix $k \in \mathbb{N}$. We refer to Example 4.82 for the notation related to Adams operations.

**Definition 4.197.** We define the cannibalistic class

$$\rho^k(V) := \Phi_\nu^{-1}(\Psi^k(\Phi_\nu(1))) \in \text{KU}[[k]^{-1}]^0(M).$$

Let $\text{Vect}_{\text{Spin}^c}^R \in \text{Sm(CommMon(Set))}$ be the smooth monoid of real vector bundles with Spin$^c$-structure with respect to the sum and $\text{KU}[[k]^{-1}]^0(M)^\times$ denote the units in the ring $\text{KU}[[k]^{-1}]^0(M)$. 

128
Problem 4.198. 1. Show that \( V \mapsto \rho^k(V) \) is an exponential (sums go to products) natural transformation
\[
\text{Vect}_{\mathbb{R}}^{\text{Spin}^c} \to (KU[[k]]^{-1})^\times
\]
in \( \text{Sm}((\text{CommMon}(\text{Set})).) \).

2. One can twist a \( \text{Spin}^c \)-structure by a complex line bundle \( V \). Let \( V \in \text{Vect}_{\mathbb{R}}^{\text{Spin}^c}(M) \) denote the twist of \( V \) by \( L \). Calculate \( \rho^k(V^L) \cup \rho^k(V)^{-1} \).

3. A complex line bundle \( E \to M \) has a canonical \( \text{Spin}^c \)-structure. Calculate \( \rho^k(E) \).

4. Let \( V \to B \) be a two-dimensional \( \text{Spin}^c \)-bundle. Since the underlying real vector bundle is associated to \( SO(2) \cong U(1) \) it has a complex structure. Calculate \( \rho^k(V) \).

Proof. For 1. one uses the multiplicativity of the ABS-orientation which relates the Thom classes for the sum of \( \text{Spin}^c \)-bundles with the Thom classes of the product.

For 2. one uses that \( \nu_{V_L} := \nu_V \cup \text{cyl}(L) \). We get
\[
\rho^k(V^L) = \Phi_{\nu_{V_L}}^{-1}(\Psi^k(\nu_{V_L})) = \Phi_{\nu_{V_L}}^{-1}(\Psi^k(\nu_V)\text{cyl}(L^k)) = \rho^k(V)\text{cyl}(L^{k-1})
\]
It follows that \( \rho^k(V^L) \cup \rho^k(V)^{-1} = \text{cyl}(L^{k-1}) \).

For 3. we first calculate in the universal example \( E \to BU(1) \). Since \( KU^*(BU(1)) \) is torsion-free the Chern character is injective. We therefore first calculate \( \text{ch}(\rho^k(E)) \). We know that \( b_z := 0_E^r \nu_E \in KU^2(BU(1)) \) is a coordinate for the multiplicative formal group law. The first Chern class \( x \in H^2(BU(1); \mathbb{C}) \) of \( E \) is the coordinate of the additive formal group law. It follows that \( \exp(bx) - 1 = \text{ch}(b_z) \). We apply \( \text{ch} \circ 0_E^r \) to
\[
\Psi^k(\nu_E) = \rho^k(E) \cup \nu_E
\]
and get
\[
\Psi^k_H(\text{ch}(z)) = \text{ch}(\rho^k(E)) \cup \text{ch}(z).
\]
We multiply with \( b \) and rearrange the terms:
\[
k^{-1}\Psi^k_H(\text{ch}(b_z)) = \text{ch}(\rho^k(E)) \cup \text{ch}(bz).
\]
We now substitute \( z \) by the \( x \)-variable and carry out the cohomological Adams operation which multiplies \( b \) by \( k \) in order to obtain
\[
\text{ch}(\rho^k(E)) = \frac{1}{k} \frac{\exp(kbx) - 1}{\exp(bx) - 1} = \frac{1 + e^{bx} + \cdots + e^{(k-1)bx}}{k}.
\]
It follows that
\[
\rho^k(E) = \frac{\text{cyl}(E^0) + \text{cyl}(E^1) + \cdots + \text{cyl}(E^{k-1})}{k}.
\]

1Let \( P \to M \) be the \( U(1) \)-principal bundle associated to \( L \). Then the twisted \( \text{Spin}^c \)-structure is given by the \( \text{Spin}^c \)-principal bundle \( Q^L := Q \times_{U(1)} P \). The group \( \text{Spin}^c(n) \) acts on the first factor (note that \( U(1) \) is central), and the isomorphism \( Q^L \times_{\text{Spin}^c(n)} \mathbb{R}^n \cong V \) is induced from \( Q \times_{\text{Spin}^c(n)} \mathbb{R}^n \cong V \) in the obvious way.
We now discuss 4. Note that the $\text{Spin}^c$-structure associated to the complex bundle $V$ differs from the original one. It can be written in the form $V^L$ for a certain line bundle $L \to M$. Using the formulas above we have

$$\rho^k(V) = 1 + \text{cycl}(V) + \cdots + \text{cycl}(V^{k-1}) \frac{\text{cycl}(L^{k-1})}{k}.$$ 

In order to understand $L$ we discuss the structure of $\text{Spin}^c(2)$ in detail. The identification $\text{SO}(2) \cong \text{U}(1)$ together with the map $\beta$ in (52) define a split $s : \text{SO}(2) \to \text{Spin}^c(2)$ of the exact sequence

$$0 \to \text{U}(1) \to \text{Spin}^c(2) \xrightarrow{\pi} \text{SO}(2) \to 0.$$ 

Let us make this explicit. We identify $\text{Spin}(2) \cong \tilde{\text{U}}(1)$. We can write $\text{Spin}^c(2) = \text{Spin}(2) \times_{\mathbb{Z}/2\mathbb{Z}} \tilde{\text{U}}(1)$. Let $t \in \text{SO}(2)$. Then take a preimage $\tilde{t} \in \text{Spin}(2) \cong \tilde{\text{U}}(1)$ and define the element $s(t) := [\tilde{t}, \tilde{t}] \in \text{Spin}^c(2)$. The split induces a character $\chi : \text{Spin}^c(2) \to \tilde{\text{U}}(1)$, $[\tilde{a}, \tilde{b}] \mapsto \tilde{b}^{-1}\tilde{a}$. If $V \to M$ is a 2-dimensional $\text{Spin}^c$-bundle, then using $\chi$ we can associate the complex bundle $V(\chi) \to M$. The homomorphism $c : \text{Spin}^c(2) \to \text{U}(1)$, $[\tilde{a}, \tilde{b}] \mapsto \tilde{b}^2 = \tilde{a}^2(\tilde{b}\tilde{a}^{-1})^2$ can be written as $c = \pi\chi^2$. We have $c \circ \beta = \text{id}$. Hence we must take $L = V(\chi)^{-1}$. This is Problem 4.199.

The map $\text{MSpin}^c \to \text{MSO} \to \mathbb{H}\mathbb{Z}$ provides an ordinary orientation for every $\text{Spin}^c$-bundle $V \to M$ and hence a Thom class $\nu_\mathbb{C} \in H\mathbb{C}^n(M^V)$. We get a characteristic class

$$\text{Td}(V) := [\Phi_\nu^{-1}(\text{ch}(\nu))]^{-1} = [\Phi_\nu^{-1}(\text{ch}(\Phi_\nu(1)))]^{-1} \in HA^0(M).$$

Do not confuse this characteristic class of $\text{Spin}^c$-vector bundles with the class $\text{Td}(\nu)$ (cf. 4.178) associated to the $K$-theory Thom class $\nu$. They are inverse to each other.

**Problem 4.199.** 1. Show that $V \to \text{Td}(V)$ is an exponential characteristic class

$$\text{Td} : \text{Vect}^{\text{Spin}^c}_\mathbb{R} \to HA^0.$$ 

2. By Chern-Weyl theory there exists a natural characteristic form

$$\text{Td}(\nabla) \in \Omega A^0_d(M).$$

**Calculate the corresponding invariant polynomial in $I(\text{Spin}^c(n))$.**

3. Give a formula for $\text{Td}(\nabla)$ in terms of the curvature $R\nabla$. 

130
Proof. We discuss briefly. We consider the diagram of Lie groups connecting $Spin^c(2n)$ with other classical groups.

\[
\begin{array}{ccc}
U(1) & \xrightarrow{\det} & U(n) \\
& \beta \downarrow & \downarrow \alpha \\
Spin^c(2n) & \xrightarrow{\pi} & SO(2n)
\end{array}
\]  \hspace{1cm} (52)

Let us explain the map $\beta$. We start with the presentation of $Spin(2n) \to SO(2n)$ as the two-fold connected covering (universal if $n \geq 2$). Then we have an isomorphism

$$Spin^c(2n) = Spin(2n) \times_{\mathbb{Z}/2\mathbb{Z}} U(1)$$

(can be taken as a definition here). We have a natural map $U(n) \to SO(2n)$ which induces a surjective map $\pi_1(U(n)) \to \pi_1(SO(2n))$. We form the right pull-back-square in the diagram

\[
\begin{array}{ccc}
U(1) & \xrightarrow{\sqrt{\det}} & \tilde{U}(n) \\
\downarrow \alpha & & \downarrow p \\
U(1) & \xleftarrow{\det} & U(n)
\end{array}
\]

where the vertical maps are two-fold coverings. As indicated, on $\tilde{U}(n)$ we can form a square-root of the determinant.

We now define the map $\beta$ such that it maps $u \in U(n)$ to the class $[a(\tilde{u}), \sqrt{\det(\tilde{u})}] \in Spin^c(2n)$, where $\tilde{u} \in \tilde{U}(n)$ is a lift of $u$ under $p$. Note that this class is independent of the choice of the lift $\tilde{u}$.

If we identify $I(U(n)) \cong \mathbb{C}[[bx_1, \ldots, bx_n]]^\Sigma_n$ and $I(U(1)) \cong \mathbb{C}[[bt]]$, then the image under $\alpha^*$ of $I(SO(2n))$ in $I(U(n))$ is given by $\mathbb{C}[[((bx_1)^2, \ldots, (bx_n)^2)]^\Sigma_n$. Moreover $\det^*(bt) = \sum_{i=1}^n bx_i$. In particular, the restriction

$$\beta^* : I(Spin^c(2n)) \to I(U(n))$$

is injective.

We consider the universal line bundle $L \to \mathbb{C}P^\infty$ with its Thom classes $\nu_C$ and $\nu$ for $H^C$ and $K^U$. Let $0_L : \mathbb{C}P^\infty \to L$ be the zero section. Then $x_C := 0_C^L \nu_C$ generates the additive formal group law, while $x := 0_L^C \nu$ generates the multiplicative formal group law. Hence we have the relation $\text{ch}(bx) = \exp(bx_C) - 1$. Let $Q \in HA^0(\mathbb{C}P^\infty)$ be the class such that $\text{ch}(bv) = Qb \nu_C$, i.e. $Q = Td(L)^{-1}$. Then $\exp(bx_C) - 1 = \text{ch}(bx) = Qbx_C$. We see that

$$Q = \frac{\exp(bx_C) - 1}{bx_C}.$$
Using
\[
e^{x} - 1 = \frac{\sinh(x/2)}{x/2} e^{x/2}
\]
we conclude that
\[
\beta^* Td = e^{-\frac{1}{2} \sum_{i=1}^{n} bx_i} \prod_{i=1}^{n} \frac{bx_i/2}{\sinh(bx_i/2)}.
\]

If we define \( \hat{A} \in I(BSO(2n)) \) such that
\[
\alpha^* \hat{A} = \prod_{i=1}^{n} \frac{bx_i/2}{\sinh(bx_i/2)},
\]
then we see that
\[
Td = \pi^* \hat{A} \exp \left( -\frac{c_{\text{et}}}{2} \right).
\]

We let \( \text{Vect}_{\mathbb{R}}^{\text{Spin}^c, \text{geom}} \) be the smooth monoid of \( \text{Spin}^c \)-bundles with \( \text{Spin}^c \)-connection with respect to the direct sum.

**Problem 4.200.** 1. Show that there exists a unique differential refinement
\[
\hat{\rho}^k : \text{Vect}_{\mathbb{R}}^{\text{Spin}^c, \text{geom}} \to (\mathbf{KU}[[\{k\}^{-1}]]^0 \times
\]

such that
\[
I(\hat{\rho}^k(V, \tilde{V})) = \rho^k(V), \quad R(\hat{\rho}^k(V, \tilde{V})) = \Psi_H^k(Td(\tilde{V}))^{-1} \wedge Td(\tilde{V}).
\]

2. Show that \( \hat{\rho}^k \) is exponential.

3. Calculate \( \hat{\rho}^k((V, \tilde{V})^L, \nabla^L) \cup \hat{\rho}^k(V, \tilde{V})^{-1} \).

For some of the solutions see [Bun10a].

4.9 Orientation and integration - the topological case

We first recall the homotopy theoretic construction of the Umkehr map. For a space \( X \in \text{Top} \) and \( k \in \mathbb{Z} \) we abbreviate by \( X^k_+ := \Sigma^{\infty+k} X \in \mathbb{N}(Sp)[W^{-1}] \) the suspension spectrum of \( X \) shifted by \( k \). For a real vector bundle \( N \to X \) we let \( X^N \in \mathbb{N}(Sp)[W^{-1}] \) denote the Thom spectrum of \( N \). Note that \( X^{X \times \mathbb{R}^k} \cong X_k^k \).

Let \( f : W \to M \) be a proper smooth map. For simplicity we assume that \( M \) is connected and all components of \( W \) have the same dimension. We set \( n := \dim(M) - \dim(W) \). For sufficiently large \( k \) we can choose a fibrewise embedding \( \iota : W \hookrightarrow M \times \mathbb{R}^l \). Its differential provides a representative of the stable normal bundle (see Definition 4.89)
\[
\mathcal{N} : 0 \to TW \xrightarrow{d} f^* TM \oplus \mathbb{R}^l \to N \to 0.
\]
We can extend $\iota$ to an open embedding also denoted by $\iota : N \to M \times \mathbb{R}^l$. If $f : W \to M$ is a submersion, then we can require that this extension is a map over $M$. This condition will play an important role when we consider the pull-back of representatives of orientations. In any case, it gives rise to a map of Thom spectra

$$c_1(\iota) : M_+^l \to W^N$$

(53)
called the clutching map whose description we recall in the following. We let $\tilde{N} = N \cup \partial N$ be the fibrewise one-point compactification of $N$. Then we define a map $(M \times \mathbb{R}^l)_+ \to \tilde{N}/\partial N$ which is the inverse $\iota^{-1}$ on $\iota(N)$ and sends the complement of $\iota(N)$ and the basepoint to the basepoint represented by the contracted boundary $\partial N$. Stabilization gives the map of spectra $c_1(\iota) : M_+^l \to W^N$.

Let $E$ be a commutative ring spectrum. For any real vector bundle $N \to W$ over a compact base we have an identification $E^\ast_{prop/W}(N) \cong E^\ast(W^N)$.

Problem 4.201. Give an argument.

Hence we can talk about a representative $\nu : W^N \to E[l + n]$ of a Thom class.

Definition 4.202. A representative $(\iota, \nu)$ of an $E$-orientation on the map $W \to M$ is given by the data of an embedding $\iota : N \to M \times \mathbb{R}^l$ and a representative of a Thom class $W^N \to E[l + n]$.

If $N \to W$ is a real vector bundle, then we have the Thom diagonal $\text{diag} : W^N \to W^N \wedge W_+$. The representative of an $E$-orientation gives rise to a map in $\text{Mod}(E)$

$$I(\iota, \nu) : E \wedge M_+^l \xrightarrow{c_1(\iota)} E \wedge W^N \xrightarrow{\text{diag}} E \wedge W^N \wedge W_+ \xrightarrow{\nu} E \wedge E[l + n] \wedge W_+ \xrightarrow{\mu} E[l + n] \wedge W_+.$$

Definition 4.203. For every $E$-module spectrum $F \in \text{Mod}(E)$ the map

$$I(\iota, \nu) ! : \text{Map}_{\text{Mod}(E)}(I(\iota, \nu), F) : \text{Sm}_M(F)(W) \to \text{Sm}_M(F)(M)[n]$$

and the induced map on the level of homotopy groups

$$\bar{I}(\iota, \nu) ! : F^\ast(W) \to F^{\ast+n}(M)$$

are called the Umkehr or integration maps.

The usual notation for the Umkehr map is $f ! := \bar{I}(\iota, \nu) !$. But in the present section this conflicts with the notation for the proper push-forward of a smooth object so it will be avoided.

Problem 4.204. In the case that $f : W \to M$ is a proper submersion give an interpretation of the Umkehr map as a transformation

$$I(\iota, \nu) ! : f_! f^\ast \text{Sm}_M(F) \to \text{Sm}_M(F)[n]$$
Proof. Since \( f \) is a submersion we can and will choose the embedding \( \iota : N \to M \times \mathbb{R}^l \) as a map over \( M \). If \( h : M' \to M \) is a manifold over \( M \), then there is a functorial way to construct a pull-back \( h^*(\iota, \nu) \) of representatives of \( E \)-orientations. We have a pull-back diagram

\[
\begin{array}{ccc}
N' & \xrightarrow{\lambda} & N \\
\downarrow \pi' & & \downarrow \\
W' & \xrightarrow{H} & W \\
\downarrow f' & & \downarrow f \\
M' & \xrightarrow{h} & M
\end{array}
\]

with \( W' := M' \times_M W \) and \( N' := H^*N \). We can identify the vector bundle \( N' \to W' \) with the normal bundle of an induced embedding \( \iota' \) as follows. Let \( \iota_2 : N \to \mathbb{R}^l \) be the second component of \( \iota \). We consider the embedding

\[
\iota' : N'(f' \circ \pi') \times N \xrightarrow{\lambda} M' \times N \xrightarrow{\lambda \times \iota_2} M' \times \mathbb{R}^l .
\]

The corresponding representative of the stable normal bundle is

\[
N' : 0 \to TW'/df' \oplus H^*f'*TM' \oplus \mathbb{R}^l \to N' \to 0 .
\]

We define the representative of the Thom class

\[
\nu' : W'^{N'} \xrightarrow{\lambda} W^{N} \xrightarrow{\nu} E[l + n]
\]

and let

\[
h^*(\iota, \nu) := (\iota', \nu') .
\]

If

\[
\begin{array}{ccc}
A & \xrightarrow{g} & B \\
\downarrow p & & \downarrow q \\
M & &
\end{array}
\]

is a smooth map over \( M \), then there is a functorially induced diagram in \( \text{Mod}(E) \)

\[
\begin{array}{ccc}
E \wedge A^l_+ & \longrightarrow & E \wedge B^l_+ \\
\downarrow \text{proj}(\iota, \nu) & & \downarrow \text{proj}(\iota, \nu) \\
E[l + n] \wedge (A \times_M W)_+ & \longrightarrow & E[l + n] \wedge (B \times_M W)_+
\end{array}
\]

We consider this construction as a functor

\[
\mathcal{N}(\text{Mf}/M) \to \text{Fun}(\mathcal{N}([1])^{\text{op}}, \text{Mod}(E)) .
\]

We take the adjoint and compose with \( \text{Map}_{\text{Mod}(E)}(\ldots, F) \). This gives a functor \( \mathcal{N}(\text{Mf}/M) \times \mathcal{N}([1]) \to \text{Mod}(E) \). Its evaluation at \( 0 \in [1] \) can be identified with \( f_* f^* \text{Sm}_M(F) \), while the evaluation at \( 1 \in [1] \) is \( \text{Sm}_M(F)[n] \). Finally, since \( f \) is proper we have \( f_! = f_* . \) \qed

134
Example 4.205. In the definition of a homotopy between two representatives of topological $E$-orientations we need the following extension of the construction of the pull-back. Assume that $M$ decomposes as a product $M = M_1 \times M_2$ and that $h_1 : M'_1 \to M_1$ is a smooth map. For a homotopy, $M_1 = [0,1]$ and $M'_1 = \{0,1\}$. We further assume that there exists an open subset $U \subseteq M_1$ containing the image of $h_1$ such that the projection $pr_1 \circ f : f^{-1}(U \times M_2) \to U$ is a submersion. In this case we can choose $\iota$ such that its restriction $\iota_{|\iota^{-1}(f^{-1}(U \times M_2))}$ is a map over $U$. Under these assumptions we still can define the pull-back $h^*(\iota, \nu)$ by the construction above, where $h = h_1 \times id_{M_2}$.

Definition 4.206. We say that two representatives $(\iota_i, \nu_i)$, $i = 0, 1$, of $E$-orientations are homotopic if there exists a representative of an $E$-orientation $(\iota, \nu)$ on $id \times f : I \times W \to I \times M$ with the property that $\iota : N \to I \times M \times \mathbb{R}^l$ preserves fibres over $I$ in neighbourhoods of the endpoints of the interval, and which pulls back to the representatives of $E$-orientations $(\iota_i, \nu_i)$ at the end-points of the interval $I = [0,1]$.

There is a natural operation of stabilization of the embedding $\iota : N \to M \times \mathbb{R}^l$ to an embedding $\iota^* : N \oplus \mathbb{R}^r \to M \times \mathbb{R}^{l+r}$. Note that $W^{N \oplus \mathbb{R}^r} \cong \Sigma^r W^N$. Let $\nu^r := \Sigma^r \nu$ and set $\Sigma^r I(\iota, \nu) := (\nu^r, \Sigma^r \nu)$. Then we have

$I(\iota, \nu)_1 = I(\Sigma^r(\iota, \nu))_1 : Sm(F)(W) \to Sm(F)(M)[n]$. 

Definition 4.207. An $E$-orientation of $f$ is an equivalence class $[\iota, \nu]$ of representatives of $E$-orientations under the equivalence relation generated by homotopy and stabilization.

It is clear that the Umkehr map in $F$-cohomology

$I(\iota, \nu)_1 : F^*(W) \to F^{*+n}(M)$

as a function of the $E$-orientation is well-defined.

Example 4.208. In this example we discuss a prototypical case of an index theorem. An index theorem is a statement which relates, via cycle maps, a geometric integration with the topological integration map. The prototypical example, which gave the name, is the Atiyah-Singer index theorem which relates the analytic push-forward of vector bundles in terms of twisted Dirac operators with the topological integration in $KU$-theory. Let $f : W \to M$ be a proper submersion with representative

$N_f : 0 \to TW \xrightarrow{df \oplus \beta} f^*TM \oplus \mathbb{R}^l \to N_f \to 0$

of the stable normal bundle coming from an embedding $\iota$, and a complex structure $I_f$ on $N_f$. The complex structure gives rise to a $MU$-Thom class $\nu^{MU} \in MU^{dim(N)}(N_f)_{prop/W}$ as in [4.19]. We let $I(\iota, \nu)_1 : MU^*(W) \to MU^{*+n}(M)$ be the integration, where $n := dim(M) - dim(W)$. We now define a geometric integration

$I : Cycle^{*}_MU(W) \to Cycle^{*+n}_MU(M)
as follows. Let \((g : A \to W, \mathcal{N}_g, I_g)\) be a \(\text{MU}\)-cycle on \(W\). Then we define
\[
\mathcal{N}^f_{\text{fg}} : 0 \to TA \xrightarrow{\partial \otimes \alpha} g^*TW \oplus \mathbb{R}^k \xrightarrow{g^*df \oplus g^*T \oplus \text{id}_{\mathbb{R}^k}} g^*f^*TM \oplus \mathbb{R}^l \oplus \mathbb{R}^k \to \mathcal{N}^f_{\text{fg}} \to 0
\]
(contract the first to injective maps to one). The bundle \(\mathcal{N}^f_{\text{fg}}\) has a natural filtration \(g^*\mathcal{N}_f \subseteq \mathcal{N}^f_{\text{fg}}\) with quotient \(\mathcal{N}_g\). We can therefore obtain a well-defined (up to homotopy) identification \(\mathcal{N}^f_{\text{fg}} \approx g^*\mathcal{N}^f \oplus \mathcal{N}_g\) by chosing a split. Note that in view of the definition \[4.91\] of \(\text{Cycle}_{\text{MU}}\) it is only the homotopy class of the representative of the stable normal which matters. We define the geometric integral of the cycle \((g : A \to W, \mathcal{N}_g, I_g)\) by
\[
I(g : A \to W, \mathcal{N}_g, I_g) := (f \circ g : A \to M, \mathcal{N}^f_{\text{fg}}, g^*I_f \oplus I_g).
\]

**Problem 4.209.** Show the index theorem asserting that the following diagram commutes:
\[
\begin{array}{ccc}
\text{Cycle}^*_\text{MU}(W) & \xrightarrow{\text{cycl}} & \text{MU}^*(W) \\
| & | & | \\
\text{Cycle}^{*+n}_{\text{MU}}(M) & \xrightarrow{\text{cycl}} & \text{MU}^{*+n}(M)
\end{array}
\]

**Proof.** This should follow by a careful analysis of the Thom-Pontrjagin construction. 

**Example 4.210.** In the following we contrast index theorems with Riemann-Roch theorems. A Riemann-Roch theorem is a statement about the compatibility of a natural transformation between cohomology theories and integration maps. In the example below we compare integration in \(\text{E}\)-theory with ordinary cohomology. We assume that the characteristic of the ring \(\pi_*(E)\) is not 2. Then an \(\text{E}\)-Thom class on a vector \(N \to W\) determines an ordinary orientation, hence a Thom class \(\nu_C \in H\mathbb{C}^{\dim(N)}(W^N)\).

**Definition 4.211.** We define the class \(\text{Td}(\nu) \in H(A)^0(W)^\times\) uniquely such that
\[
\text{Td}(\nu) \cup \nu_C \approx c(\nu).
\]

The following proposition is an immediate consequence of the definitions.

**Proposition 4.212.** We have the Riemann-Roch theorem
\[
\begin{array}{ccc}
E^k(W) & \xrightarrow{c} & H(A)^k(W) \\
\downarrow I(\iota, [\nu]) & & \downarrow I(\iota, [c(\nu)]) \\
E^{k+n}(M) & \xrightarrow{c} & H(A)^{k+n}(M)
\end{array}
\]

**Problem 4.213.** Assume that \(W \to M\) be a proper holomorphic map between complex manifolds. Make the Riemann-Roch theorem for complex bordism explicit. In particular consider the cases bundles of curves and surfaces.

**Proof.** The exercise consists in the calculation of \(\text{Td}(\nu)\).
4.10 Orientation and integration - the differential case

Let \((E,A,c)\) be a multiplicative datum. We assume that \(f : W \to M\) is a proper smooth map and set \(n := \dim(M) - \dim(W)\). As in the topological case we choose an embedding \(\iota : W \to M \times \mathbb{R}^l\) over \(M\) and an extension to an open embedding, also denoted by \(\iota : N \to M \times \mathbb{R}^l\), of the normal bundle. We further choose a Thom class \(\nu \in E^k_{prop/W}(N)\), where \(k = l + n\) is the dimension of the normal bundle \(\pi : N \to W\). These structures determine the topological integration map (Definition 4.203)

\[ \bar{I}(\iota,\nu) : F^*(W) \to F^{*+n}(M) \]

for every \(E\)-module spectrum \(F\). It is given by the composition

\[ \bar{I}(\iota,\nu)_! : F^*(W) \xrightarrow{\pi^*} F^*(N) \xrightarrow{\iota^*} F_{prop/W}^{*+k}(W) \xrightarrow{\text{excision}} F_{prop/M}^{*+k}(M \times \mathbb{R}^l) \xrightarrow{\text{desusp}} F^{*+n}(M). \]

In order to generalize this to the differential case we need the notion of a differential \(E\)-orientation.

**Definition 4.214.** A representative of a differential \(E\)-orientation refining \((\iota,\nu)\) is a pair \((\iota,\hat{\nu})\) where \(\hat{\nu}\) is a differential Thom class refining \(\nu\) (see 4.178).

For simplicity we only consider the integration for the differential extension of \(E\) itself.

The generalization to modules is straightforward, see 4.233.

**Definition 4.215.** We define the integration in differential \(E\)-theory

\[ \hat{I}(\iota,\hat{\nu}) : \hat{E}^*(W) \to \hat{E}^{*+n}(M) \]

associated to the representative of the differential \(E\)-orientation \((\iota,\hat{\nu})\) by the following composition

\[ \hat{I}(\iota,\hat{\nu})_! : \hat{E}^*(W) \xrightarrow{\pi^*} \hat{E}^*(N) \xrightarrow{\iota^*} \hat{E}_{prop/W}^{*+k}(N) \xrightarrow{\text{excision}} \hat{E}_{prop/M}^{*+k}(M \times \mathbb{R}^l) \xrightarrow{\text{desusp}} \hat{E}^{*+n}(M). \] (54)

Note that we use 4.162 in the second map.

If \(f\) is a proper submersion, then we can choose the embedding \(\iota : N \to M \times \mathbb{R}^l\) such that it is a map over \(M\), i.e. that the diagram

\[
\begin{array}{ccc}
N & \xrightarrow{\pi} & W \\
\downarrow{\iota} & & \downarrow{f} \\
& M \times \mathbb{R}^l & \xrightarrow{pr_M} M
\end{array}
\]

commutes.

**Problem 4.216.** Check that if \(\iota : N \to M \times \mathbb{R}^l\) is a map over \(M\), then the differential integration \(\bar{I}(\iota,\nu)_!\) is a morphism of \(\hat{E}^*(M)\)-modules.
We define the map of complexes
\[ R(\iota, \hat{\nu}) : \Omega^* A(W) \to \Omega^* A(M)[n] \]
by
\[ R(\iota, \hat{\nu})(\alpha) := \int_{M \times \mathbb{R}^l/M} (\iota^{-1})^* [R(\hat{\nu}) \wedge \pi^* \alpha] , \quad \alpha \in \Omega A(W) . \]
The form \((\iota^{-1})^* [R(\hat{\nu}) \wedge \pi^* \alpha]\) is first defined on \(\iota(N)\). After extension by zero it can be considered as a smooth form on \(M \times \mathbb{R}^l\) which is properly supported over \(M\). So the integral is well-defined. It follows from the closedness of \(R(\hat{\nu})\) and Stokes’ theorem that \(R(\iota, \hat{\nu})\) commutes with the differential. We call \(R(\iota, \hat{\nu})\) the curvature map of the integration associated to \((\iota, \hat{\nu})\). This is justified by the following exercise.

**Problem 4.217.** Check that for \(x \in \hat{E}^*(W)\) and \(\alpha \in \Omega A(W)\) we have
\[ R(\hat{I}(\iota, \nu); (x)) = R(\iota, \hat{\nu})(R(x)) , \quad \hat{I}(\iota, \nu)_! (a(\alpha)) = a(R(\iota, \hat{\nu})(\alpha)) . \]
Sometimes we write the curvature map \(R(\hat{I})\) as a function of the integration \(\hat{I} := \hat{I}(\iota, \hat{\nu})\). Summing up, the differential integration fits into the following commuting diagram:

\[
\begin{array}{ccc}
\Omega A^{* - 1}(W)/\text{im}(d) & \xrightarrow{R} & \hat{E}^*(W) \\
\downarrow{R(\iota, \hat{\nu})} & & \downarrow{\hat{I}(\iota, \hat{\nu})} \\
\Omega A^{* + n - 1}(M)/\text{im}(d) & \xrightarrow{R} & \hat{E}^{* + n}(M) \\
\end{array}
\]

Similar constructions of the integration in differential cohomology have been considered in [HS05] and [Fer12].

**Problem 4.218.** If the embedding \(\iota : N \to M \times \mathbb{R}^l\) is a map over \(M\), then the curvature map of the integration associated to \((\iota, \hat{\nu})\) is given by
\[ R(\iota, \hat{\nu})(\alpha) = \int_{W/M} \text{Td}(\hat{\nu}) \wedge \alpha , \]
where \(\text{Td}(\hat{\nu})\) is defined in 4.178.

We now consider the compatibility of the push-forward with pull-back. We assume that \(f : W \to M\) is a proper submersion and that \(\iota : N \to M \times \mathbb{R}^l\) is a map over \(M\) in order to ensure existence of all pull-backs which are necessary in the construction. Let \(h : M' \to M\) be a smooth map and consider the bundle \(f' : M' \times_M W \to M'\). We use the notation introduced in the proof of 4.204 and define \(\hat{\nu}' := \lambda^* \hat{\nu}\). Then \(h^*(\iota, \hat{\nu}) := (\iota', \hat{\nu}')\) is the pulled-back representative of a differential \(E\)-orientation of \(f'\).
Problem 4.219. Show that a representative of a differential $E$-orientation $(\iota, \hat{\nu})$ on $W \to M$ gives rise to a transformation

$$\hat{I}(\iota, \hat{\nu}) : f \cdot f^* \hat{E}^*_M \to \hat{E}^*_M.$$ 

Proof. The arguments are similar to 4.204. 

This assertion encodes the compatibility of the integration with pull-back diagrams. Again we can extend the definition of the pull-back as in 4.205.

Definition 4.220. We say that two representatives $((\iota_i, \hat{\nu}_i), i=0,1)$ of differential $E$-orientations are homotopic if there exists a representative of a differential $E$-orientation $(\tilde{\iota}, \tilde{\hat{\nu}})$ on $id \times f : I \times W \to I \times M$ with the property that $\iota : \tilde{N} \to I \times M \times \mathbb{R}$ preserves fibres over $I$ in neighbourhoods of the endpoints of the interval, which restricts to $(\iota_i, \hat{\nu}_i)$, $i=0,1$, and whose curvature map satisfies

$$R(\tilde{\iota}, \tilde{\hat{\nu}}) \circ pr_W^* = pr_M^* \circ R(\iota_0, \hat{\nu}_0),$$ 

where $pr_M : I \times M \to M$ and $pr_W : I \times W \to W$ are the projections.

Problem 4.221. Assume that $\iota : N \to M \times \mathbb{R}$ is a map over $M$ and that two differential $E$-Thom classes $\hat{\nu}_i, i=0,1$, are homotopic in the sense of 4.182. Check that the representatives of differential $E$-orientations $(\iota, \hat{\nu}_i)$ are homotopic.

Proof. Use 4.218. 

Problem 4.222. Show that two homotopic representatives of a differential $E$-orientations induce the same integration map $\hat{E}^*(W) \to \hat{E}^{*+n}(M)$.

Proof. Use the homotopy formula 4.48. 

Next we discuss the stabilization of differential $E$-orientations. Recall, that we can define a stabilization $(\iota^r, \Sigma^r \nu)$ of a representative of a topological $E$-orientation such that the topological integration map does not change on the spectrum level, i.e. we have $I(\iota, \nu) = I(\iota^r, \Sigma^r \nu)$. In particular, stabilization does not effect the integration map in cohomology groups.

In the differential case we again stabilize the embedding as before. In addition we must define a stabilization of differential $E$-Thom classes. This definition involves the choice of differential Thom classes for trivial bundles. Unlike the topological case there is no natural choice, but if we assume that our data $(E, A, c)$ is strict, then there is a natural choice up to homotopy of Thom classes. It suffices to choose a differential Thom class $\hat{\nu}_1 \in \hat{E}_c^1(\mathbb{R})$. We fix this choice by the condition that

$$\text{desusp}(\hat{\nu}_1) = 1,$$ 

where $\text{desusp}$ is the desuspension map defined in 4.186. Indeed, if $y \in H(A)^{-1}(*), then

$$\text{desusp}(\hat{\nu}_1^* + a(R(\hat{\nu}_1^*) \wedge y)) = \text{desusp}(\hat{\nu}_1^*) + y.$$
We start with any differential lift $\hat{\nu}_1'$ of the canonical topological Thom class and set $z := \text{desusp}(\hat{\nu}_1')$. Then

$$\hat{\nu}_1 := \hat{\nu}_1' - a(R(\hat{\nu}_1') \wedge z)$$

satisfies (55). Furthermore, by 4.182 it is uniquely determined by this condition up to homotopy of differential Thom classes. We now define

$$\hat{\nu}_l := \text{pr}_1^*\hat{\nu}_1 \cup \cdots \cup \text{pr}_l^*\hat{\nu}_1 \in E_1^l(\mathbb{R}^n).$$

Here we use 4.162 to get compact support of the product. By pull-back we then get Thom classes, also denoted by $\hat{\nu}_l$, on trivialized bundles $M \times \mathbb{R}^l$.

We now define the stabilization of the differential Thom class $\hat{\nu}$ by

$$\Sigma^l \hat{\nu} := p^*\hat{\nu} \cup q^*\hat{\nu}_l,$$

where $p : N \oplus (W \times \mathbb{R}^l) \to N$ and $q : N \oplus (W \times \mathbb{R}^l) \to (W \times \mathbb{R}^l)$ are the projections. This determines a stabilized representative $(\iota, \Sigma^l \hat{\nu})$ of a differential $E$-orientation which is natural up to homotopy of Thom classes.

**Problem 4.223.** Show the equality of differential integration maps $\hat{I}(\iota, \Sigma^l \hat{\nu})_! = \hat{I}(\iota, \hat{\nu})_!$.

**Proof.** This uses 4.187 and desuspension in stages 4.188. \qed

We assume that the datum $(E, A, c)$ is strict in order to have a well-defined notion of stabilization on the level of homotopy classes of representatives of differential $E$-orientations.

**Definition 4.224.** On the set of representatives of differential $E$-orientations we consider the equivalence relation generated by stabilization and homotopy. A differential $E$-orientation is an equivalence class of representatives of differential $E$-orientations.

**Corollary 4.225.** It follows from 4.222 and 4.223 that the integration

$$\hat{I}(\iota, \hat{\nu})_! : \hat{E}^*(W) \to \hat{E}^{*+n}(M)$$

only depends on the equivalence class $[\iota, \hat{\nu}]$ of differential $E$-orientations.

**Problem 4.226.** Assume that $f$ is a proper submersion, $(\iota_0, \hat{\nu}_0)$ is a differential $E$-orientation of $f$ where $\iota_0 : N \to M \times \mathbb{R}^l$ is a map over $M$. Classify differential $E$-orientations refining the topological $E$-orientation $(\iota_0, \nu_0)$ with curvature map $R(\iota_0, \hat{\nu}_0)$.

**Proof.** We can work with embeddings $\iota : N \to M \times \mathbb{R}^l$ which are maps over $M$. Any two become homotopic in this class after suitable stabilization. We now observe using 4.218 that a homotopy of differential orientations with fixed $\iota$ exactly corresponds to a homotopy of differential Thom classes. It suffices therefore to classify stable homotopy classes of differential Thom classes. Here we use 4.183. We finally conclude that the set of differential $E$-orientations $[\iota, \hat{\nu}]$ is a torsor over the group

$$\frac{H(A)^{-1}(W)}{c(E^{-1}(W))).}$$

\qed
Problem 4.227. Drop the assumption that \( f \) is a submersion. Fix a curvature map \( R(t_0, \hat{v}_0) \) and classify differential \( E \)-orientations refining the topological \( E \)-orientation \((\iota, \nu)\) with curvature map \( R(\iota, \hat{\nu}) \).

Proof. I do not know the answer. \qed

In [BS09, Def. 3.5, Cor. 3.6] we introduced a notion of a differential \( \text{KU} \)-orientation and discussed the classification of differential \( \text{KU} \)-orientations refining a given topological one. The definitions and results of [BS09] differ from the theory presented here. See 4.246 for more details. A similar remark applies to the comparison with the definitions and the results of [BSSW09] for the complex bordism \( \text{MU} \).

Example 4.228. Consider the differential cohomology theories \( \hat{\text{H}}_{\mathbb{Z}}^* \) or \( \hat{\text{S}}^* \). In this case \( \text{Td}(\hat{\nu}) = 1 \) for every differential Thom class. Since also \( H(A)^{-1}(W) = 0 \) it follows that there is a unique up to homotopy differential Thom class \( \hat{\nu} \) which refines \( \nu \).

If \( f : W \rightarrow M \) is a proper submersion with a topological orientation \((\iota, \nu)\), then there exists a preferred differential orientation \((\iota, \hat{\nu})\) refining this topological orientation such that \( \iota : N \rightarrow M \times \mathbb{R}^l \) is a map over \( M \). In particular there exists a preferred differential integration maps \( \hat{I}_{\hat{E}} \) and \( \hat{I}_{\hat{S}} \).

Note that \( S^1 \) has a trivialized normal bundle and is therefore \( S \)-oriented. Hence the projection \( \text{pr}_M : S^1 \times M \rightarrow M \) is canonically \( S \)-oriented. We therefore have an integration

\[
\hat{I}_F^* : \hat{F}^*(S^1 \times M) \rightarrow F^{*-1}(M).
\]

The integration over the \( S^1 \)-factor played an important role in the uniqueness considerations in [BS10]. In this reference we required the following additional properties:

Problem 4.229. Show that

1. If \( t : S^1 \rightarrow S^1 \) is inversion, then \( \hat{I}_F^*(t^*x) = -\hat{I}_F^*(x) \).

2. \( \hat{I}_F^*(\text{pr}_M^*x) = 0 \).

Problem 4.230. Verify the following assertions:

1. If \( g : M \rightarrow U \) is a second proper submersion, \( m = \text{dim}(U) - \text{dim}(M) \) and \((\iota_f, \hat{\nu}_f)\), \((\iota_g, \hat{\nu}_g)\) are differential \( E \)-orientation for \( f \) and \( g \), then there is a natural construction of a differential \( E \)-orientation \((\iota_{gof}, \hat{\nu}_{gof})\) such that

\[
\hat{I}(\iota_{gof}, \hat{\nu}_{gof})! = \hat{I}(\iota_g, \hat{\nu}_g)! \circ \hat{I}(\iota_f, \hat{\nu}_f)! : \hat{E}^k(W) \rightarrow \hat{E}^{k+n+m}(U).
\]

2. Composition of differential orientations commutes with pull-back.

3. We consider a morphism of multiplicative differential data \( \phi : (E, A, c) \rightarrow (E', A', c') \).

A differential \( E \)-orientation \((\iota, \hat{\nu})\) naturally induces a differential \( E' \)-orientation \((\iota, \phi_*(\hat{\nu}))\), and we have the equality

\[
\phi_* \circ \hat{I}(\iota, \hat{\nu}) = \hat{I}(\iota, \phi_*(\hat{\nu}))(\circ \phi_* : \hat{E}^k(W) \rightarrow \hat{E}^{k+n}(M).
\]
Proof. For 1. one just writes out the composition of the sequences for $f$ and $g$. Then one adds commuting cells which connect this composition with corresponding sequence for the composition $g \circ f$. Note that the embedding $\iota_{g \circ f} : N_{g \circ f} := N_g \times_M N_f \to U \times \mathbb{R}^{l+k}$ is canonically induced by the embeddings $\iota_g$ and $\iota_f$, and $\hat{\nu}_{g \circ f} = \text{pr}^*_{N_f} \hat{\nu}_f \cup \text{pr}^*_{N_g} \hat{\nu}_g$. In the argument one should further use 4.187 and 4.188. For assertion 2. one should write out an even bigger diagram. The argument for assertion 3. is clear.

Example 4.231. We now introduce the notion of a differentially $E$-oriented zero bordism of a smooth proper map $f : \mathcal{W} \to M$ with a differential $E$-orientation $(\iota, \hat{\nu})$. It is given by a proper smooth map $F : \mathcal{Z} \to I \times M$ with a differential $E$-orientation $(\tilde{\iota}, \tilde{\hat{\nu}})$ such that

1. the composition $\text{pr}_I \circ F : \mathcal{Z} \to M \to I$ is a submersion and the embedding $\tilde{\iota} : \tilde{\mathcal{N}} \to I \times M \times \mathbb{R}^l$ is a map over $I$ near the end-points of the interval,
2. the restriction of the differentially $E$-oriented map $F$ to $\{1\} \times M$ is isomorphic to the differentially $E$-oriented map $f$, and
3. $\{0\} \times M$ does not intersect the image of $F$.

Observe the similarities with the notion of a homotopy 4.220.

Problem 4.232 (Bordism formula). Show that for $x \in \hat{E}^*(\mathcal{Z})$ we have the bordism formula

$$\hat{I}(\iota, \hat{\nu})_!(x|_W) = a \left( \int_{I \times M/M} R(\tilde{i}, \tilde{\hat{\nu}})(R(x)) \right).$$

Proof. This is a consequence of the homotopy formula Prop. 3.28 applied to $\hat{I}(\tilde{i}, \tilde{\hat{\nu}})_!(x)$. Note that $\hat{I}(\tilde{i}, \tilde{\hat{\nu}})_!(x|_{\{0\}} \times M) = 0$ and $\hat{I}(\tilde{i}, \tilde{\hat{\nu}})_!(x|_{\{1\}} \times M) = \hat{I}(\iota, \hat{\nu})_!(x|_W)$.

Example 4.233. Let $E \in \text{CommMon}((\mathbb{N}\text{Sp})[W^{-1}])$ be a commutative ring spectrum. Then we can form the $\infty$-category $\text{Mod}(E)$ of $E$-module spectra. We consider a multiplicative differential data $(E, A, c)$. If $F$ is an $E$-module, then we can define a notion of a differential $(E, A, c)$-module data $(F, B, d)$ in a natural way so that $\text{Diff}^*(F, B, d)$ becomes a $\text{Diff}^*(E, A, c)$-module (cf. 4.118).

Let now $f : \mathcal{W} \to M$ be a proper smooth map. If $(\iota, \hat{\nu})$ is a differential $E$-orientation of $f$, then we obtain a differential $F$-integration $\hat{I}^F(\iota, \hat{\nu})_!$ in a natural way.

Problem 4.234. Work out the details.

If the differential $E$-orientation is clear from the context we often use the shorter notation $\hat{I}^F := \hat{I}^F(\iota, \hat{\nu})_!$.

Every spectrum $F$ is an $S$-module. Assume that $f : \mathcal{W} \to M$ is a proper submersion which is topologically oriented for $S$. Then by 4.228 there exists a preferred differential
orientation \((\iota, \hat{\nu})\) where \(\iota : N \to M \times \mathbb{R}^d\) is a map over \(M\). Hence we get an a preferred differential \(F\)-integration \(\hat{I}_F^F\). The associated curvature map is

\[
R(\hat{I}_F^F)(\alpha) = \int_{W/M} \alpha .
\]

### 4.11 Higher \(\epsilon\)-invariants and index theorems

**Example 4.235** (The higher complex \(\epsilon\)-invariant). In this very long example we provide an explicit calculation of integrals in differential complex \(K\)-theory. From the point of view of topology, what we are going to calculate, is a higher version of Adam’s \(\epsilon\)-invariant. But already the restriction of this higher \(\epsilon\)-invariant to a point is interesting. In this case, the calculation made here provides the solution to previous exercises 4.124.

A closed \(n\)-dimensional manifold \(M\) with a framing of a representative of the stable normal bundle is oriented for \(S\). We use the procedure explained in 4.233 in order to construct an induced differential integration \(\hat{I}_{\text{KU}}^\text{KU} : \text{KU}^0(M) \to \text{KU}^0(*)\). The closed framed \(n\)-manifold \(M\) represents (via Thom-Pontrjagin) a stable homotopy class \([M] \in S_n(*) \cong \pi_n^*\). Assume that \(n\) is odd. Let

\[
e_C : \pi_n^* \to \mathbb{C}/\mathbb{Z}
\]

be the complex version of the \(\epsilon\)-invariant of Adams defined using the unit \(\epsilon\) by

\[
e_C : \pi_n^* = \pi_n(S) \xrightarrow{\sim} \pi_{n+1}(S\mathbb{C}/\mathbb{Z}) \xrightarrow{\epsilon} \pi_{n+1}(\text{KU}\mathbb{C}/\mathbb{Z}) \cong \mathbb{C}/\mathbb{Z}
\]

(cf. 4.126).

**Problem 4.236.** Show that in \(\mathbb{C}/\mathbb{Z}\) we have \(e_C([M]) = \hat{I}_{\text{KU}}^\text{KU}(1)\).

*Proof.* See [BS09]. \(\square\)

We consider the groups \(SO(3)\) and \(SU(2)\) as framed manifolds with respect to their left and right-invariant framings and write \([SO(3)]_l\), \([SO(3)]_r\), \([SU(2)]_l\) and \([SU(2)]_r\) for the corresponding stable homotopy classes.

**Problem 4.237.** Calculate the \(e_C\)-invariant of these stable homotopy classes using Exercise 4.236.

More generally, let \(f : W \to M\) be a proper submersion with a framing \(fr\) of the vertical bundle \(T^v f\). This framing determines an \(S\)-orientation of \(f\). Note that \(T^v f\) is a stable complement of the stable normal bundle, and stable homotopy classes of framings of \(T^v f\) correspond bijectively to stable homotopy classes of framings of representatives of the stable normal bundle. We consider the class

\[
g(f, fr) := I_{\text{S}}^\text{S}(1) \in S^{-n}(M),
\]

where \(n := \dim(M) - \dim(W)\). This stable cohomotopy class is of course difficult to understand, in general.
The S-orientation again can be lifted to a differential S-orientation and the associated integration \( \hat{I}^S \) does not depend on the choices. It again induces a differential KU-integration and therefore a map \( \hat{I}^{KU}_1 : \text{KU}^*(W) \to \text{KU}^{*+n}(M) \).

**Problem 4.238.** Show that \( \hat{I}^{KU}_1(1) \) is flat.

**Definition 4.239.** We define the higher e\(_C\)-invariant of \((f,fr)\) by

\[
e_{\mathbb{C}}(f,fr) := \hat{I}^{KU}_1(1) \in \text{KU}/\mathbb{Z}^{-n-1}(M).
\]

Below we will often simply write \( e_{\mathbb{C}}(f) \) if the framing is clear from the context.

**Problem 4.240.** Show that \( e_{\mathbb{C}}(f) \) only depends on the class \( q(f,fr) \).

As we shall see the higher e\(_C\)-invariant of \( f \) is much easier to calculate than \( q(f,fr) \) and can be used to give some information on the latter.

We consider the special case where \( f : W \to M \) is a \( G \)-principal bundle. The vertical bundle \( T^v f \) has a canonical framing given by the fundamental vector fields of the \( G \)-action.

**Problem 4.241.** Calculate the class \( \hat{I}^{KU}_1(1) \in \text{KU}/\mathbb{Z}^{-n-1}(M) \) in terms of characteristic classes of the principal bundle.

**Proof.** Let \( T \) be the maximal torus of \( G \). We fix an embedding \( \kappa : S^1 \to T \). Let \( D^2 \subset \mathbb{C} \) be the unit disc with the standard action of \( S^1 \). Then \( q : Z := W \times_{S^1} D^2 \to M \) is a fibre-wise zero-bordism. It fits into a diagram

\[
\begin{array}{ccc}
W & \xrightarrow{f} & Z \\
\downarrow \quad & \quad & \downarrow \\
W/S^1 & \quad & M \\
\downarrow \quad & \quad & \quad \\
M
\end{array}
\]

The vertical tangent bundle of \( q \) can be written in the form \( T^v q = s^* T^v r \oplus s^* V_1 \), where \( V_1 := W \times_{S^1} \mathbb{C} \). Let \( t \) denote the Lie algebra of \( T \) and \( \Delta^+ \subset t^*_\mathbb{C} \) be a system of positive roots of the pair \((\mathfrak{g},t)\). Then we have a \( T \)-equivariant decomposition

\[
\mathfrak{g} \cong t \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}_\alpha,
\]

where \( \mathfrak{g}_\alpha \) is a complex one-dimensional representation of \( T \) with weight \( \alpha \). Let \( \bar{t} := t / \kappa_*(\mathbb{R}) \). For a weight \( \beta \) of \( S^1 \) we consider the complex line bundle \( V_\beta := W \times_{S^1,\beta} \mathbb{C} \to
For a weight $\alpha \in t^*_c$ of $T$ we get a weight $\kappa^* \alpha$ of $S^1$. Then $T^r \cong (W/S^1 \times \bar{t}) \oplus \bigoplus_{\alpha \in \Delta^+} V_{\kappa^* \alpha}$. We thus have

$$T^q = (Z \times \bar{t}) \oplus s^*(V_1 \oplus \bigoplus_{\alpha \in \Delta^+} V_{\kappa^* \alpha}) .$$  \hspace{1cm} (56)

The first summand is framed, and the second summand has a natural complex structure. Therefore, $Tq$ has a natural stable complex structure and hence a $Spin^c$-structure.

Let $\nabla_{T^q}$ be a stable complex connection on $T^q$ which respects the sum-decomposition above and which restricts to the trivial connection $\nabla_{triv}$ on $W = \partial Z$ determined by the framing.

The bordism formula \[4.232\] predicts that

$$e_C(f) = a(\int_{Z/M} \text{Td}(\hat{I}(\hat{\nu})) .$$

Here $\hat{\nu}$ is a differential $\text{KU}$-Thom class of the normal bundle of $\nu$ which extends the differential $\text{S}$-Thom class on the boundary. We now use the following general fact. Let $\nabla$ be a complex connection on a complex representative of the stable normal bundle $\tilde{N}$ of $\nu$ which is the trivial connection associated to the framing over $W$. Then by \[4.193\] we can find a Thom class $\hat{\nu}_{\text{MU}} \in \hat{\text{MU}}^{\dim(\tilde{N})}$ with $\text{Td}(\hat{\nu}_{\text{MU}}) = u(\nabla)$. We let $\hat{\nu}$ be the image of $\hat{\nu}_{\text{MU}}$ under the orientation $\text{MU} \to \text{KU}$. Note that with this choice $\text{Td}(\hat{\nu}) = \text{Td}(\nabla)^{-1} = \text{Td}(\hat{I}(\hat{\nu}))$, where $\text{Td}(\nabla)$ is the form representing the characteristic class of $Spin^c$-vector bundles considered in \[4.199\]. We have fixed an isomorphism of complex vector bundles $T^q \oplus \tilde{N} \cong Z \times \mathbb{C}^l$ which extends the isomorphism given by the framings over $W$. We therefore have a transgression formula

$$\text{Td}(\nabla^{T^q}) \wedge \text{Td}(\nabla) = 1 + d\text{Td}(\nabla^{T^q} \oplus \nabla, \nabla_{triv}) ,$$

where $\text{Td}(\nabla^{T^q} \oplus \nabla, \nabla_{triv})$ vanishes on $W$. This gives

$$\text{Td}(\nabla)^{-1} = \frac{\text{Td}(\nabla^{T^q})}{1 + d\text{Td}(\nabla^{T^q} \oplus \nabla, \nabla_{triv})} .$$

It follows that

$$\text{Rham}(\text{Td}(\nabla)^{-1}) = \text{Rham}(\text{Td}(\nabla^{T^q})) \in HA^0(Z, W) .$$

In particular we get

$$\text{Rham}(\int_{Z/M} \text{Td}(\hat{I}(\hat{\nu}))) = \text{Rham}(\int_{Z/M} \text{Td}(\nabla^{T^q})) .$$

Therefore

$$e_C(f) = a(\int_{Z/M} \text{Td}(\nabla^{T^q})) .$$

Let $x := bc_1$, where $c_1 \in H^2(W/S^1; \mathbb{C})$ be the first real Chern class of the $S^1$-bundle $W \to W/S^1$.  

145
Lemma 4.242. We have

\[
Rham(\int_{Z/(W/S^1)} \text{Td}(\nabla^{T^q}) = \frac{1}{x} \left( 1 - \frac{x}{e^x - 1} \prod_{\alpha \in \Delta^+, \kappa^* \alpha \neq 0} \frac{\kappa^* \alpha \ x}{e^{\kappa^* \alpha} \ x - 1} \right) .
\] (57)

Proof. Let \( T^q \) denote some stabilization of \( T^q \). According to the decomposition (56) we choose a complex connection \( \tilde{\nabla}^{T^q} \) as follows. First we let \( T^q := (Z \times (t \oplus \bar{t})) \oplus s^*(V \oplus \bigoplus_{\alpha \in \Delta^+} V_{\kappa^* \alpha}) \), where the first summand has the complex structure \( i(t, t') = (-t', t) \). We fix a connection \( \nabla \) on \( V_1 \). It induces connections on \( V_\beta \) for all weights \( \beta \) of \( S_1 \). We choose a connection \( \tilde{\nabla}_1 \) on \( q^*V_1 \) which is trivialized near \( W \) and coincides with \( q^*\nabla \) near the zero section. This fixes a choice of \( \tilde{\nabla}_\beta \) for every weight \( \beta \) since \( q^*V_\beta = q^*V_1 \otimes n_\beta \). In particular, with this choice we have \( c_1(\tilde{\nabla}) = \beta c_1(\nabla) \). We let \( \nabla^{T^q} \) be the sum of the connections \( \nabla_{\kappa^* \alpha} \) and the trivial connection on \( Z \times (t \oplus \bar{t}) \). Then we have

\[
\text{Td}(\nabla^{T^q}) = \text{Td}(\tilde{\nabla}_1) \prod_{\alpha \in \Delta^+, \kappa^* \alpha \neq 0} \text{Td}(\tilde{\nabla}_{\kappa^* \alpha}) .
\]

Using the generating power series

\[
\text{Td}(x) = \frac{x}{e^x - 1}
\]

(this has been calculated in Problem 4.199) we get

\[
\text{Td}(\nabla^{T^q}) = \frac{bc_1(\tilde{\nabla}_1)}{ebc_1(\tilde{\nabla}_1) - 1} \prod_{\alpha \in \Delta^+, \kappa^* \alpha \neq 0} \frac{\kappa^* \alpha \ bc_1(\tilde{\nabla}_1)}{e^{\kappa^* \alpha} \ bc_1(\tilde{\nabla}_1) - 1}
\]

We get, using the calculation 2.100

\[
\int_{Z/(W/S^1)} \text{Td}(\nabla^{T^q}) \equiv - \frac{1}{bc_1(\nabla)} \left( \frac{bc_1(\nabla)}{ebc_1(\nabla) - 1} \prod_{\alpha \in \Delta^+, \kappa^* \alpha \neq 0} \frac{\kappa^* \alpha \ bc_1(\nabla)}{e^{-\kappa^* \alpha} \ bc_1(\nabla) - 1} - 1 \right)
\]

modulo exact forms. The Lemma follows from the fact that \( x \) is represented by the closed form \( c_1(\nabla) \). \( \square \)

The higher rank case

Lemma 4.243. If \( \text{rk}(G) \geq 2 \), then \( e_C(f) = 0 \).
Proof. It suffices to calculate in the universal example where $f : W \to M$ is the universal bundle $\pi : EG \to BG$. We first observe that $EG/S^1 \cong BS^1$ and therefore $H^*(EG/S^1; \mathbb{C}) \cong \mathbb{C}[x]$ with $x \in H^2(EG/S^1; \mathbb{C})$. If $\text{rk}(G) \geq 2$, then we can choose a decomposition $T \cong \kappa(S^1) \times T'$, where $T'$ is a $\text{rk}(G) - 2$-dimensional torus and $T_1 \cong S^1$. We consider the iterated bundle

$$EG/S^1 \to EG/(S^1 \times T_1) \to BG.$$ 

The right-hand side of (57) is a certain formal power series $P(x) \in \mathbb{C}[[x]]$. We have

$$e_C(\pi) = \int_{(EG/S^1)/BG} P(x) = \int_{(EG/(S^1 \times T_1))/BG} \int_{(EG/S^1)/(EG/(S^1 \times T_1))} P(x).$$

Note that $EG/(S^1 \times T_1) \cong B(S^1 \times T_1)$ so that the cohomology of $EG/(S^1 \times T_1)$ is concentrated in even degrees. But $P(x)$ is of even degree and the fibre of $EG/S^1 \to EG/(S^1 \times T_1)$ is one-dimensional so that $\int_{(EG/S^1)/(EG/(S^1 \times T_1))} P(x)$ is a class of odd-degree and hence trivial.

We now assume that $\text{rk}(G) = 1$. The non-trivial examples are

$$G \cong S^1, \quad G \cong SU(2), \quad G \cong SO(3).$$

The case $G = S^1$

We consider the universal case $f : ES^1 \to BS^1$. In this case $EG/S^1 \cong BS^1$ so that Lemma 57 immediately gives

$$\text{Rham}(\int_{Z/BS^1} Td(\nabla^{T^n} \eta^s)) = \frac{1}{x} \left(1 - \frac{x}{e^x - 1}\right).$$

Therefore, with $x = bc_1$,

$$e_C(f) = b \left[\frac{1}{x} \left(1 - \frac{x}{e^x - 1}\right)\right] \in H^{-2}(BS^1; \mathbb{C}[b,b^{-1}]/\text{im}(\text{ch})).$$

Here are the first terms of this power series

$$\frac{1}{2} - \frac{1}{12} x + \frac{1}{720} x^3 - \frac{1}{30240} x^5 + \frac{1}{1209600} x^7 - \frac{1}{47900160} x^9 + \frac{691}{130767436800} x^{11} + \ldots.$$ 

Already the first two terms are interesting. The constant term $\frac{1}{2}$ is the $e_C$-invariant of $S^1$ as a framed manifold (a bundle over a point). Note that $\pi_1^S = \mathbb{Z}/2\mathbb{Z}$, and the generator $\eta$ is represented by the framed manifold $S^1$ via the Pontrjagin Thom construction. This gives $\text{ord}(e_C(\eta)) = 2$.

We now consider the linear term. To this end we discuss the pull-back of $e_C(f)$ along the canonical generator $S^2 \to BS^1$ of $\pi_2(BS^1) \cong \mathbb{Z}$. The restriction of the universal bundle along the generator $S^2 \to BS^1$ of $\pi_2(BS^1) \cong \mathbb{Z}$ gives the Hopf bundle $\pi : S^3 \to S^2$. 

147
Note that on $S^2$ we know that $\text{im}(\text{ch})$ is the image of the integral cohomology in complex cohomology. Therefore we have

$$KU_C/\mathbb{Z}^{-2}(S^2) \cong H^{-2}(S^2, \mathbb{C}[b, b^{-1}])/\text{im}(\text{ch}) \cong (\mathbb{C}[x]/(x^2))/(\mathbb{Z}^2[x]/(x^2)).$$

The higher $e_C$-invariant of the Hopf bundle is

$$e_C(\pi) = \left[\frac{1}{2} - \frac{x}{12}\right] \in (\mathbb{C}[x]/(x^2))/(\mathbb{Z}^2[x]/(x^2)).$$

This class has order 12. On the other hand, the Hopf map $\pi : S^3 \to S^2$ represents the stable cohomotopy class

$$\kappa \in S^{-1}(S^2) \cong [S^2, S^{-1}] \cong \pi_3 \cong \mathbb{Z}/24\mathbb{Z}.$$

It follows that $e_C(\kappa)$ has order 12.

It remains to interpret the higher terms. See the corresponding discussion in the case of $G = SU(2)$.

**The case $G = SU(2)$**

In this case we have one root $\alpha$ which is twice the fundamental weight of the maximal torus $S^1 \subset SU(2)$. The reason is that the adjoint representation factors over the two-fold covering $SU(2) \to SO(3)$, and the adjoint representation of $SO(3)$ is the standard representation on $\mathbb{R}^3$ on which the maximal torus of $SO(3)$ acts by the fundamental representation. Therefore we have

$$\text{Rham}\left(\int_{Z/(ESU(2)/S^1)} \text{Td}(\nabla^T \varphi)\right) = \frac{1}{x} \left(1 - \frac{x}{e^x - 1} \frac{2x}{e^{2x} - 1}\right).$$

We must integrate this further over the bundle $ESU(2)/S^1 \to BSU(2)$ with fibre $SU(2)/S^1 \cong S^2$. In general, of $G$ is a compact Lie group with maximal torus $T \subset G$, then the pull-back along $BT \to BG$ identifies $H^*(bG; \mathbb{C})$ with the subring $H^*(BT; \mathbb{C}) W \subset H^*(BT; \mathbb{C})$, where $W$ is the Weyl group of the pair $(G, T)$. In the present case the Weyl group is $\mathbb{Z}/2\mathbb{Z}$ which acts by $x \to -x$ on $H^*(BT; \mathbb{C}) \cong \mathbb{C}[[x]]$. The Gysin sequence of the $S^2$-bundle $r : ESU(2)/S^1 \to BSU(2)$ in integral cohomology using the absence of odd cohomology gives

$$0 \to \mathbb{Z}[[x^2]] \xrightarrow{r^*} \mathbb{Z}[[x]] \xrightarrow{f} \mathbb{Z}[[x^2]] \to 0,$$

where

$$\mathbb{Z}[[x]] \cong H^*(BS^1; \mathbb{Z}), \quad \mathbb{Z}[[x^2]] \cong H^*(BSU(2); \mathbb{Z}).$$

It follows that $\int_r x^{2n+1} = x^{2n}$ (immediate at least up to sign). The same formula now holds true in complex cohomology. We conclude that

$$e_C(\pi) = b^2 \left[\frac{1}{x^2} \left(1 - \frac{x}{e^x - 1} \frac{2x}{e^{2x} - 1}\right)\right]_{\text{even}} \in H^{-4}(BSU(2); \mathbb{C}[b, b^{-1}])/\text{im}(\text{ch}).$$
The first few terms of this series are
\[-\frac{11}{12} - \frac{1}{240} x^2 + \frac{1}{6048} x^4 - \frac{1}{172800} x^6 + \frac{1}{5322240} x^8 - \frac{691}{11887948800} x^{10} \ldots .\]

The constant term $-\frac{11}{12}$ is the $e_C$-invariant of the class in $\pi_3^s \cong \mathbb{Z}/24\mathbb{Z}$ represented by the framed manifold $SU(2)$ under the Pontrjagin Thom construction.

In order to discuss the quadratic term we consider the restriction of $e_C(f)$ along the canonical generator $S^4 \to BSU(2)$ of $\pi_4(BSU(2)) \cong \mathbb{Z}$. Note that $\pi_4(BSU(2)) \cong \mathbb{Z}$.

The generator classifies a $SU(2)$-principal bundle $E \to S^4$ whose second Chern class $c_2(E) \in H^4(S^4; \mathbb{Z})$ is a generator. The image of the Chern character in $H^{-4}(S^4; \mathbb{C}[b, b^{-1}])$ is the subgroup $H^{-4}(S^4; \mathbb{Z}[b, b^{-1}])$. Therefore the class $e_C(q) = \left[-\frac{11}{12} - \frac{1}{240} \text{or}_{S^4}\right]$ has order 240. We can conclude that $q$ represents a generator of $S^{-3}(S^4) \cong \pi_4^s \cong \mathbb{Z}/240\mathbb{Z}$.

In order to interpret the higher terms we must understand the image of $\text{ch}$ in greater detail. It is clear that
\[
\text{KU}C/\mathbb{Z}^{-4}(BSU(2)) \cong \prod_{n \geq 0} \mathbb{C}/\mathbb{Z}
\]
but it seems to be not so easy to make this isomorphism explicit. One checks that the Chern character of the standard representation of dimension $n + 1$ is given by
\[
\text{ch}(V_n) = \frac{\sinh((n + \frac{1}{2})x)}{\sinh(\frac{1}{2}x)}
\]
for $n = 0, 1, \ldots$ where $b^2c_2 = -x^2$. It remains to use this calculation to produce the isomorphism (58) explicitly and to derive the corresponding basis decomposition of $e_C(f)$.

**The case $SO(3)$**

In this case we again have one root $\alpha$ which is equal to the fundamental weight of the maximal torus $S^1 \subset SO(3)$ since the adjoint representation of $SO(3)$ is the standard representation on $\mathbb{R}^3$ on which the maximal torus of $SO(3)$ acts by the fundamental representation. Therefore we have
\[
\text{Rham} \left( \int_{Z/(ESO(3)/S^1)} \text{Td}(\nabla^{T^*q'}) \right) = \frac{1}{x} \left( 1 - \left( \frac{x}{e^x - 1} \right)^2 \right).
\]

We must integrate this further over the bundle $ESO(3)/S^1 \to BSO(3)$ with fibre $SO(3)/S^1 \cong S^2$. As in the case of $SU(2)$ we conclude that
\[
e_C(f) = b^2 \left[ \frac{1}{x^2} \left( 1 - \left( \frac{x}{e^x - 1} \right)^2 \right) \right] \in H^{-4}(BSO(3); \mathbb{C}[b, b^{-1}]/\text{im}(\text{ch})).
\]

The first few terms in this expansion are
\[-\frac{5}{12} - \frac{1}{240} x^2 + \frac{1}{6048} x^4 - \frac{1}{172800} x^6 + \frac{1}{5322240} x^8 - \frac{691}{11887948800} x^{10} \ldots .\]
The constant $-\frac{5}{12}$ is again the $e_C$-invariant of the element in $\pi_3^c$ represented by the framed manifold $SO(3)$ under the Thom Pontrjagin construction. The higher terms coincide with those of $SU(2)$, a fact which can also be seen directly.

\[ \square \]

**Example 4.244.** We consider a proper submersion $f : W \to M$ and set $n := \dim(M) - \dim(W)$. The choice of an embedding $\iota$ determines a representative $N_f$ of the stable normal bundle for $f$. Let us fix a complex structure $I$ and a connection $\nabla_f$ on its underlying bundle $N_f$.

The zero section $0_{N_f} : W \to N_f$ has a canonical representative of the normal bundle $N_{0_{N_f}}$ with underlying bundle $N_f$. The datum $(0_{N_f}, N_{0_{N_f}}, I_f, \nabla_f)$ is a topological $\text{MU}$-cycle representing a $\text{MU}$-Thom class $\nu$ on $N_f$ (cf. Example 4.191). By the solution of Problem 4.194 there is a natural choice $\hat{\nu} := \hat{\nu}(\nabla_f)$ of a differential Thom class on $N_f$ refining $\nu$ with $\text{Td}(\hat{\nu}) = u(\nabla_f)$. We can write $\hat{\nu} := \widehat{\text{cycl}}(0_{N_f}, I_{0_{N_f}}, I_f, \nabla_f, \eta_f)$ for a suitable choice of $\eta_f \in \Omega^{-1}A_{\text{prop}/W,-\infty}(N_f)$. The pair $(\iota, \hat{\nu})$ represents a differential $\text{MU}$-orientation for $f$ and hence yields an integration $\hat{I}(\iota, \hat{\nu})$.

We now define a geometric integration

$\hat{I}_{\text{geom}} : \text{Cycle}_{\text{MU},*}(W) \to \text{Cycle}_{\text{MU},*+n}(M)$

as follows. Let

$g := (g : B \to W, N_g, I_N, \nabla_N, \eta_g) \in \text{Cycle}_{\text{MU}}(W)$.

Then we get a canonical representative of the normal bundle $N_{fog}$ whose underlying bundle admits an isomorphism

$N_{fog} \cong N_g \oplus g^*N_f$ (59)

which is unique up to homotopy. Indeed, the underlying bundle $N_{fog}$ admits a filtration $N_f \subset N_{fog}$ with quotient $N_g$, and the isomorphism is fixed by the choice of a split. We choose such an isomorphism and define $I_{fog}$ and $\nabla_{fog}$ using this direct sum decomposition and set

$\eta_{fog} := \int_{W/M} u(\nabla_f) \wedge \eta_g + \int_{W/M} (\int_{N_f/W} \eta_f) \wedge R(\widehat{\text{cycl}}(g))$.

Then we define

$\hat{I}_{\text{geom}}(g) = (f \circ g, N_{fog}, I^g \oplus g^*I_f, \nabla^g \oplus g^*\nabla_f, \eta_{fog})$.

Observe that $\widehat{\text{cycl}}(\hat{I}_{\text{geom}}(g))$ does not depend the choice of the isomorphism (59). The following is the prototype of a differential index theorem. It compares the integration with respect to a differential orientation with a geometric push-forward construction.
Problem 4.245. Show the index theorem:

\[
\begin{array}{ccc}
\text{Cycle}_{\text{MU}, \text{geom}}^*(W) & \xrightarrow{\text{cycl}} & \text{MU}^*(W) \\
\downarrow I_{\text{geom}} & & \downarrow I_{(\iota, \nu)} \\
\text{Cycle}_{\text{MU}, \text{geom, } n}^*(M) & \xrightarrow{\text{cycl}} & \text{MU}^{n}(M)
\end{array}
\]

Proof. I am interested to see the details. \qed

Example 4.246. We consider a proper submersion \( f : W \to M \) with a fiberwise Riemannian metric and a horizontal distribution. We get an induced connection \( \nabla^{f} \) on the vertical bundle \( T^{v}f := \ker(df) \) by \cite[Ch. 9]{BGV92}. We assume that \( T^{v}f \) has a \( Spin^{c} \)-structure and that we have chosen a \( Spin^{c} \)-extension \( \tilde{\nabla} \) of \( \nabla^{f} \). Let \( S(T^{v}f) \) be the fibrewise spinor bundle. It acquires the structure of a family of Dirac bundles. In this way we get a geometric family \( \mathcal{W} \) over \( M \). Let \( V \to W \) be a hermitean vector bundle with metric connection \( \nabla^{V} \). The geometric family determined by the twisted spinor bundle \( S(T^{v}f) \otimes V \) will be denoted by \( \mathcal{W} \otimes V \). Its associated family of Dirac operators will be denoted by \( D(\mathcal{W} \otimes V) \). We refer to \cite{Bun09} for details on geometric families.

The \( Spin^{c} \)-structure on \( T^{v}f \) determines an equivalence class of a \( KU \)-orientation \( [\iota, \nu] \) of \( f \). For a complex vector bundle with connection \( (V, \nabla^{V}) \) on \( W \) we define the topological index by

\[
\text{index}^{\text{top}}(V, \nabla^{V}) := I_{[\iota, \nu]}(\text{cycl}(V)) \in KU^{n}(M),
\]

where \( n = \dim(M) - \dim(W) \). The analytical index is the index of the family of twisted Dirac operators

\[
\text{index}^{\text{an}}(V, \nabla^{V}) := \text{index}(D(\mathcal{W} \otimes V)).
\]

The Atiyah-Singer index theorem for families states that

\[
\text{index}^{\text{top}}(V, \nabla^{V}) = \text{index}^{\text{an}}(V, \nabla^{V}).
\]

Note that both indices do not depend on the connections. The goal of the following is to state a differential refinement of this index theorem which takes the geometric structures into account.

The \( Spin^{c} \)-connection \( \tilde{\nabla} \) determines a form

\[
\text{Td}(\tilde{\nabla}) \in \Omega C[0, b^{-1}]_{\iota, \nu}(W)^{*}.
\]

By \ref{4.195} we can find a natural Thom class \( \hat{\nu}(T^{v}f) := \hat{\nu}(\tilde{\nabla}) \) on \( T^{v}f \) which is unique up to homotopy and satisfies \( \text{Td}(\hat{\nu}(T^{v}f)) = \text{Td}(\nabla)^{-1} \). We now choose an embedding \( \iota \) and get an associated representative of the stable normal bundle \( N \) with underlying bundle \( N \). In particular we have a trivialization of \( T^{v}f \oplus N \cong W \times \mathbb{R}^{l} \).

Let \( k := l + n \) be the dimension of \( N \). Using the two-out-of-three principle \ref{4.184} and the canonical Thom class \( \hat{\iota} \) on the trivial bundle we get a well-defined homotopy class
of differential Thom classes \( \nu \in \tilde{\mathbf{K}}U^k_{prop/W}(N) \) such that \( \nu_{\hat{t}} = \text{pr}_{T^v f}^{*}\nu(T^v f) \cup \text{pr}_N^{*}\nu \). In particular we have \( \text{Td}(\nu) = \text{Td}(\nabla) \). The pair \( (\iota, \nu) \) induces a well-defined integration

\[
\hat{I}_t := \hat{I}(\iota, \nu) : \tilde{\mathbf{K}}U^*(W) \to \tilde{\mathbf{K}}U^{*+n}(M)
\]

with curvature map

\[
R(\iota, \nu)(\alpha) = \int_{W/M} \text{Td}(\nabla) \wedge \alpha.
\]

This differential integration only depends on the \( \text{Spin}^c \)-structure on \( T^v f \) and the choice of the \( \text{Spin}^c \)-connection \( \nabla \). We define the differentially refined topological index of a vector bundle with connection \((V, \nabla V)\) on \( W \) by

\[
\text{index}^{\text{top}}(V, \nabla V) := \hat{I}(\text{cycl}(V, \nabla V)) \in \tilde{\mathbf{K}}U^n(M).
\]  

(60)

We assume that \( n \) is even and that the family of twisted Dirac operators \( D(W \otimes V) \) has a kernel bundle \( U \). This bundle acquires an induced metric and a metric connection \( \nabla^U \) (see [BGV92] Ch. 10] for details). We consider

\[
\text{index}^{\text{an}}(V, \nabla V) := b^{-\frac{n}{2}} \text{cycl}(U, \nabla^U) \in \tilde{\mathbf{K}}U^n(M)
\]

as the differentially refined analytical index. Using the methods of [BS09] one can drop the assumptions that \( n \) is even and that there is a kernel bundle so that the analytic index is defined in general.

The comparison between the differentially refined topological and the analytical indices is the contents of the differential version of the Atiyah-Singer index theorem. The local index [BGV92] theorem asserts that

\[
\text{ch}(\nabla^U) = \int_{W/M} \text{Td}(\nabla) \wedge \text{ch}(\nabla^V) + d\eta(W \otimes V),
\]

where \( \eta(W \otimes V) \in \Omega\mathbb{C}[b, b^{-1}]^{n-1}(M) \) is the \( \eta \)-form of [BC89].

**Problem 4.247.** Show the differential index theorem: In \( \tilde{\mathbf{K}}U^n(M) \) we have

\[
\text{index}^{\text{top}}(V, \nabla V) = \text{index}^{\text{an}}(V, \nabla V) + a(\eta(W \otimes V)).
\]  

(61)

**Proof.** The differential index theorem follows from the following proposition.

**Proposition 4.248.** Let \( \hat{I}_1, \hat{I}_1' \) be two integrations in differential \( K \)-theory defined for differentially \( K \)-oriented (in the sense of [BS09]) proper submersions which satisfy:

1. \( \hat{I}_1 \) and \( \hat{I}_1' \) are compatible with curvature.
2. They are compatible with cartesian diagrams.
3. They are compatible with composition.
4. They satisfy the bordism formula.

5. They coincide with the topological integration $I$ on flat classes.

Then $\hat{I}_t = \hat{I}_t'$.

Indeed, the integration defined in [BS09] coincides with the right-hand side of (61) and satisfies the assumptions of Proposition 4.248. The integration in differential $K$-theory used in the definition of the topological index (60) also satisfies these assumptions by 4.230 and 4.232. □

Proof. (Prop. 4.248) We consider the difference $\delta := \hat{I}_t - \hat{I}_t'$. We must show that $\delta = 0$. The first lemma reduces the proof of Proposition 4.248 to the case where the base of the bundle is a point.

Lemma 4.249. If $\delta = 0$ in the case where $M$ is a point, then $\delta = 0$ in general.

Proof. Let $f : W \to M$ be a proper submersion with a differential $K$-orientation. We let $\hat{I}_{f,!}$ and $\hat{I}_{f,'!}$ denote the associated integrations and write $\delta_f := \hat{I}_{f,!} - \hat{I}_{f,'!}$. We consider a class $x \in \mathring{\hat{K}}U^*(W)$. The assertion that the integrations are compatible with the curvature in detail means that

$$R(\hat{I}_{f,!}(x)) = \int_{W/M} Td(\nabla) \wedge R(x) = R(\hat{I}_{f,'!}(x)) .$$

This implies that $R(\delta_f(x)) = 0$ so that $\delta_f(x) \in \mathring{\hat{K}}U_{flat}^{*+n}(M) \cong \mathring{K}UC/\mathbb{Z}^{*+n-1}(M)$. In particular, $\delta_f(x)$ is homotopy invariant and therefore only depends on the underlying topological class $I(x)$ of $x$. As a consequence of the Anderson selfduality of $\mathring{K}U$ one can detect classes in $\mathring{\hat{K}}U^{*+n}(M)$ by evaluating them against $\mathring{K}U_*(M)$ classes. So if $\langle \delta_f(x), u \rangle = 0$ for all $\mathring{K}U$-homology classes $u \in \mathring{\hat{K}}U_{*+n-1}(M)$, then $\delta_f(x) = 0$. By [BD82] or [Jak98] we can realize $u$ geometrically in the form $u = g_* (v \cap [B])$, where $g : B \to M$ is a smooth map from a closed manifold $B$ of dimension $* + n - 1$ with a $Spin^c$-structure and $v \in \mathring{K}U^0(B)$. The projection $p : B \to *$ is then $\mathring{K}U$-oriented and we have

$$\langle \delta_f(x), u \rangle = I_{p,!} (g^* \delta_f(x) \cup v) \in \mathring{K}UC/\mathbb{Z}^0(*) \cong \mathbb{C}/\mathbb{Z} .$$

We choose a differential refinement of the $\mathring{K}U$-orientation of $p$ and a lift $\tilde{v} \in \mathring{\hat{K}}U^0(B)$ of $v$. We let $\tilde{f}$ and $\tilde{g}$ be defined by the cartesian diagram

$$\begin{array}{ccc}
\tilde{W} & \xrightarrow{\tilde{g}} & W \\
\downarrow \tilde{f} & & \downarrow f \\
B & \xrightarrow{g} & M
\end{array}$$
where $\tilde{f}$ has its induced differential $\textbf{KU}$-orientation. Then we have the sequence of equalities

$$\langle \delta_f(x), u \rangle = I_{p,1}(g^*\delta_f(x) \cup v)$$

$$\overset{!!}{=} \hat{I}_{p,1}(g^*\delta_f(x) \cup \hat{v})$$

$$\overset{!}{=} \hat{I}_{p,1}(g^*\hat{I}_f(x) \cup \hat{v}) - \hat{I}'_{p,1}(g^*\hat{I}'_f(x) \cup \hat{v})$$

$$\overset{!!!}{=} \hat{I}_{p_0\tilde{f},1}(\tilde{g}^*x \cup \tilde{f}^*\hat{v}) - \hat{I}'_{p_0\tilde{f},1}(\tilde{g}^*x \cup \tilde{f}^*\hat{v})$$

$$\overset{!}{=} 0.$$  

At the equalities marked by $!$ we use the assumption that $\delta = 0$ for maps to a point. The equality marked by $!!$ uses the assumption $4.248$, 5. At the equality marked by $!!!$ we use $4.248$, 2. und 3.

In order to finish the proof of Proposition 4.248 we consider the special case where $M = \ast$. Since $\textbf{KUC}/\mathbb{Z}^{\text{odd}}(\ast) = 0$ and by periodicity of $\textbf{KU}$ it remains to consider the case where $\dim(W) = -n$ and $\deg(x) = 1 - n$. We now reduce to the case of odd $n$.

**Lemma 4.250.** If we assume that $\delta = 0$ in the case that $M$ is a point and $n$ is odd, then $\delta = 0$ in general.

**Proof.** Assume that $n$ is even. We consider the map $p : S^1 \to \ast$ with some differential $K$-orientation and a class $e \in \hat{\textbf{KU}}^1(S^1)$ such that $\hat{I}_{p,1}(e) = 1$. Since $\dim(S^1)$ is odd we have by assumption that $\hat{I}'_{p,1}(e) = 1$. We now consider the diagram

$$\begin{array}{ccc}
W \times S^1 & \xrightarrow{\tilde{f} \times \text{id}_{S^1}} & S^1 \\
\downarrow \text{id}_W \times p & & \downarrow p \\
W & \xrightarrow{f} & \ast \\
\downarrow f & & \downarrow \text{id}_W \\
\ast & & \ast
\end{array}$$

By a simple calculation again using the compatibility of the integrations with pull-back, composition and the projection formula we get

$$\delta_f(x) = \delta_{f \circ (\text{id}_W \times p)}(x \cup (f \times \text{id}_{S^1})^*e).$$

The right-hand side vanishes since $\dim(W \times S^1)$ is odd.  

We have not yet used that both integrations satisfy the bordism formula. Let $Z$ with $W \cong \partial Z$ be a $\text{Spin}^c$-zero-bordim of $W$. We extend the differential $\textbf{KU}$-orientation of $W$ across $Z$. Note that this is possible since we use the notion of a differential orientation.
Using this Lemma we can calculate \( \text{Spin}^c \). Furthermore we assume that \( x = y_W \) for some class \( y \in \text{KU}^{1-n}(Z) \). The assumption \[4.248\] in detail means that

\[
\hat{I}_{f,!}(x) = a(\int_Z Td(\tilde{V}) \wedge R(y)) = \hat{I}_{f,!}'(x) .
\]

It follows that \( \delta_f(x) = 0 \) if \( W \) is zero-bordant and the class \( x \) extends over the zero bordism.

By the Pontrjagin-Thom construction the group of bordism classes of \( \text{Spin}^c \)-manifolds of dimension \(-n\) equipped with \( \text{KU}^{1-n} \)-classes is isomorphic to \( \text{MSpin}^c_{-n}(\Omega^{\infty+n-1}\text{KU}) \). The difference \( \delta \) therefore induces a homomorphism \( \tilde{\delta} : \text{MSpin}^c_{-n}(\Omega^{\infty+n-1}\text{KU}) \to \mathbb{C}/\mathbb{Z} \). In order to finish the proof of Proposition \[4.248\] it suffices to show that \( \tilde{\delta} = 0 \) in the case of odd \( n \).

The Atiyah-Bott-Shapiro orientation \( \text{MSpin}^c \to \text{KU} \) induces a homomorphism

\[
\text{ABS} : \text{MSpin}^c_{-n}(\Omega^{\infty+n-1}\text{KU}) \to \text{KU}_{-n}(\Omega^{\infty+n-1}\text{KU}) .
\]

If \( \tilde{\delta} \) factors over \( \text{ABS} \), then it is induced by a map \( \tilde{\delta} : \text{KU}_{-n}(\Omega^{\infty+n-1}\text{KU}) \to \mathbb{C}/\mathbb{Z} \), i.e. a class \( \tilde{\delta} \in \text{KU}^{1-n}(\Omega^{\infty+n-1}\text{KU}) \). If \( n \) is odd, then \( \text{KU}^{1-n}(\Omega^{\infty+n-1}\text{KU}) \cong \text{KU}^{1}(\mathbb{Z} \times BU) \cong 0 \). Therefore the following Lemma completes the proof of Proposition \[4.248\].

**Lemma 4.251.** The homomorphism \( \tilde{\delta} \) factors over \( \text{ABS} \).

**Proof.** We let \([W, x] \in \text{MSpin}^c_{-n}(\Omega^{\infty+n-1}\text{KU})\) denote the class represented by the closed \(-n\)-dimensional \( \text{Spin}^c \)-manifold \( W \) and the class \( x \in \text{KU}^{1-n+1}(W) \). We have already seen that \( \tilde{\delta}([W, x]) \) only depends on the bordism class of \((W, x)\). Since \( \delta \) is linear we have \( \tilde{\delta}([W, x + x']) = \delta([W, x]) + \delta([W, x']) \). Following \[BD82\], in order to show that \( \tilde{\delta} \) factors over \( \text{ABS} \), it remains to show that \( \tilde{\delta} \) preserves vector bundle modifications.

Let \( g : S \to W \) be the sphere bundle of a real \( \text{Spin}^c \)-vector bundle of dimension \( 2k + 1 \). The \( \text{Spin}^c \)-structure on \( W \) induces a \( \text{Spin}^c \)-structure on \( S \). The map \( \tilde{\delta} \) preserves vector bundle modifications if in this case we have the equality \( \tilde{\delta}([S, b^{-k}g^*x]) = \delta([W, x]) \), where \( b^{-k} \in \text{KU}^{2k} \) is a power of the Bott element. We choose differential orientations of the maps and a differential lift \( \hat{x} \) of \( x \). We assume:

**Lemma 4.252.** We have \( \delta_g(1) = 0 \) in the case that \( g : S \to W \) is a sphere bundle of a \( \text{Spin}^c \)-vector bundle.

Using this Lemma we can calculate

\[
\tilde{\delta}([S, b^{-k}g^*x]) = \hat{I}_{f,g!*}(b^{-k}g^*\hat{x}) - \hat{I}_{f,g!*}'(b^{-k}g^*\hat{x}) = (\hat{I}_{f,!} \circ \hat{I}_{g,!})(b^{-k}g^*\hat{x}) - (\hat{I}_{f,!}' \circ \hat{I}_{g,!}')(b^{-k}g^*\hat{x}) = \hat{I}_{f,!}((\hat{x} \cup b^{-k}\hat{I}_{g,!})(1)) - \hat{I}_{f,!}'((\hat{x} \cup b^{-k}\hat{I}_{g,!}')(1)) = \delta_f(x \cup b^{-k}\hat{I}_{g,!}(1)) - \hat{I}_{f,!}'(x \cup b^{-k}\delta_f(1)) = \delta_f(x) .
\]

\footnote{This part of the argument was suggested by Stephan Stolz.}
The second summand of the last line vanishes by Lemma [4.252]. In the first summand we use the identity of topological $KU$-classes $b^{-k}g_i(1) = 1$ and that $\delta_f$ factors over the underlying topological $KU$-class.

It remains to show Lemma [4.252]. Since $\delta$ is compatible with pull-back the class $\delta_g(1)$ is pulled by from a universal class in $KU\mathbb{C}/\mathbb{Z}^{-2k-1}(BSpin^c(2k+1))$. Since $BSpin^c(2k+1)$ is the classifying space of a compact Lie group its $KU$ and $KU\mathbb{C}/\mathbb{Z}$-theories are concentrated in even degrees by [AH61].

A differentially refined Atiyah-Singer index theorem was first shown by Freed and Lott in [FL10]. The analytic indices of Freed-Lott and in [4.247] essentially coincide. But the construction of the topological indices is different. So [4.247] is not completely equivalent to [FL10]. Differential $KU$-index theorems were also studied in [Klo08] and [Ort09] (even equivariantly).

If one applies the differential lift of the Chern character $\hat{ch} : \hat{KU}^* \to H\mathbb{Q}[b,b^{-1}]$ to both sides of the index formula (61), then one obtains the equality

$$ \hat{ch}(\hat{\text{index}}^\text{top}(V,\nabla^V)) = \hat{ch}(\hat{\text{index}}^\text{an}(V,\nabla^V)) + a(\eta(W \otimes V)) $$

in $H\mathbb{Q}[b,b^{-1}]^n(M)$. The validity of this equality is a mixture if a Riemann-Roch and an index theorem which has been verified in [Bis05], [BS09].

### 4.12 Geometrization

Let $G$ be a Lie group and $R(G)^+ \subset R(G)$ be the semiring of isomorphism classes of finite-dimensional representations under $\oplus$ and $\otimes$. Let $P \to M$ be a $G$-principal bundle and $\omega$ be a connection on $P$. If $(\rho, H_\rho)$ is a finite-dimensional representation of $G$, then we can form the associated vector bundle $P \times_{G,\rho} H_\rho := V_\rho$. It acquires a connection $\nabla_{V_\rho}$ from $\omega$. Below we write $V_\rho := B_\rho(\rho, H_\rho)$ and $(V_\rho, \nabla_{V_\rho}) := B_{P,\omega}(\rho, H_\rho)$. The associated vector bundle construction gives a diagram of morphisms of semirings

![Diagram](image)

The construction $B_P$ can be applied to the universal $G$-bundle $EG \to BG$. Since
$KU^0(BG)$ is a group we get an extension

\[
\begin{array}{ccc}
R(G) & \xrightarrow{\text{BG}} & \text{Vect}(BG) \\
\downarrow & & \downarrow \text{cyl} \\
R(G) & \xrightarrow{\text{BG}} & KU^0(BG)
\end{array}
\]

of $B_{EG}$ to a map $\hat{B}_{EG}$.

Let $E^*$ be a generalized cohomology theory. If $X$ is a space, then $E^*(X)$ has a natural topology such that the open neighbourhoods of 0 are the kernels of restrictions $E^*(X) \to E^*(Y)$ for all maps $Y \to X$ from finite CW-complexes $Y$. We apply this to $KU^0(BG)$.

Let $I := \ker(\dim R(G) \to \mathbb{Z})$ be the augmentation ideal and $R(G)^I$ be the $I$-adic completion of the ring $R(G)$. For the following see e.g. [AS69].

**Theorem 4.253** (Atiyah-Hirzebruch). Assume that $G$ is compact. Then the map $\hat{B}_{EG}$ induces a topological isomorphism

$$\hat{B}_{BG} : R(G)^I \xrightarrow{\sim} KU^0(BG).$$

**Example 4.254.** Let $G = U(1)$. We have $BU(1) \cong \mathbb{C}P^\infty$ and $KU^*(\mathbb{C}P^\infty) \cong \mathbb{Z}[x][b, b^{-1}]$, where $x \in KU^0(\mathbb{C}P^\infty)$ is the class $x = \text{cycl}(L) - 1$ with $L \to \mathbb{C}P^\infty$ being the tautological bundle. We have an identification

$$R(U(1)) \cong \mathbb{Z}[u, u^{-1}],$$

where $u$ is the defining representation $U(1) \xrightarrow{\text{id}} U(1)$. With this identification we have $\text{cycl}(L) = B_{EU(1)}(u)$.

**Problem 4.255.** Verify explicitly that the map $B_{EU(1)} : u \mapsto \text{cycl}(L)$ extends to a topological isomorphism

$$R(U(1))^I \xrightarrow{\sim} KU^0(\mathbb{C}P^\infty).$$

**Proof.** It is natural to consider the restrictions along the subspaces $\mathbb{C}P^n \subset \mathbb{C}P^\infty$. Note that on $\mathbb{C}P^n$ we have $(\text{cycl}(L) - 1)^{n+1} = 0$. \qed

Consider now the group $G = \mathbb{Z}/2\mathbb{Z}$. We have $B\mathbb{Z}/2\mathbb{Z} \cong \mathbb{R}P^\infty$.

**Problem 4.256.** Verify the Atiyah-Hirzebruch theorem for the group $\mathbb{Z}/2\mathbb{Z}$ by explicit calculation.

**Proof.** Show by calculation that $K(\mathbb{R}P^\infty) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ and $R(S^1)^I \cong \mathbb{Z} \oplus \mathbb{Z}_2$. Show then that $B_{EZ/\mathbb{Z}}$ is a topological isomorphism. \qed
We now consider the diagram

\[
\begin{array}{c}
R(G)^+ \\
\downarrow \tilde{B}_{P,\omega} \\
R(G) \\
\end{array}
\begin{array}{c}
\xrightarrow{BP_{\omega}} \\
\xrightarrow{\text{geom}} \\
\xrightarrow{\text{cycl}} \\
\end{array}
\begin{array}{c}
\text{Vect} \\
\downarrow \text{cyc}_1 \\
\text{KU}^0(M) \\
\end{array}
\]

where the factorization \( \tilde{B}_{P,\omega} \) of \( B_{P,\omega} \) exists by the universal property of the ring completion \( R(G) \) of \( R(G)^+ \).

**Problem 4.257.** Assume that \( M \) is compact. Show that \( \tilde{B}_{P,\omega} \) is continuous with respect to the I-adic topology on \( R(G) \) and the discrete topology on \( \text{KU}^0(M) \). We let

\[
\tilde{B}_{P,\omega} : R(G)_I \to \text{KU}^0(M)
\]

be the extension by \( \tilde{B}_{P,\omega} \) by continuity.

**Proof.** It suffices to show that \( \tilde{B}_{P,\omega} \) annihilates \( I^{2n+2} \), where \( n := \dim(M) \). First of all, \( R \circ \tilde{B}_{P,\omega} \) annihilates \( I^{n+1} \). It follows that we have a factorization

\[
\tilde{B}_{P,\omega} : I^{n+1} \to K\mathbb{C}/\mathbb{Z}^{-1}(M) \to \text{KU}^0(M) .
\]

The first map is a morphism of \( R(G) \)-modules. Hence the restriction of \( \tilde{B}_{P,\omega} \) to \( I^{2n+2} \) factors over

\[
I^{n+1} \otimes I^{n+1} \to K\mathbb{C}/\mathbb{Z}^{-1}(M) \otimes \text{KU}^0(M) \to K\mathbb{C}/\mathbb{Z}^{-1}(M) \to \text{KU}^0(M)
\]

which vanishes since the first map vanishes. \( \square \)

We interpret the principal bundle \( P \) as a map (determined up to homotopy) \( P : M \to BG \).

A connection on \( P \) gives rise to a lift

\[
\begin{array}{c}
\text{KU}^0(M) \\
\downarrow I \\
\text{K}^0(BG) \\
\end{array}
\begin{array}{c}
\xrightarrow{\nabla} \\
\xrightarrow{P^*} \\
\xrightarrow{P^*} \\
\end{array}
\begin{array}{c}
\text{KU}^0(M) \\
\end{array}
\]

We complete this diagram as follows:

\[
\begin{array}{c}
\Omega A^0_{cl}(M) \\
\downarrow R \\
H(A)^0(BG) \\
\end{array}
\begin{array}{c}
\xrightarrow{c} \\
\xrightarrow{\nabla} \\
\xrightarrow{c} \\
\end{array}
\begin{array}{c}
\text{KU}^0(M) \\
\downarrow I \\
\text{K}^0(BG) \\
\end{array}
\begin{array}{c}
\xrightarrow{P^*} \\
\xrightarrow{P^*} \\
\xrightarrow{P^*} \\
\end{array}
\begin{array}{c}
\text{KU}^0(M) \\
\end{array}
\]

158
The dashed arrow exists since $c$ is an equivalence after complexification and completion, and $\Omega A_{cl}^0(M)$ is a complex vector space. Recall that $A = \mathbb{C}[b, b^{-1}]$ and note that $H(A)^0(M)$ is bigraded by cohomological and internal degree where $\deg(b) = -2$. In the discussion above we used the total degree. The domain and target of $C_\nabla$ are still graded by the internal degree.

**Problem 4.258.** Show that $C_\nabla$ preserves the internal degree.

*Proof.* This is a consequence of 4.69. 

The condition that $C_\nabla$ preserves the internal degree is equivalent to the condition that $\nabla$ rationally preserves all Adams operations.

We now turn to the general case. Let $B$ be some space not necessarily of the form $BG$ and consider a map $P : M \to B$ from a compact manifold $M$. We equip $\hat{K}U^0(M)$ and $\Omega A_{cl}^0(M)$ with the discrete topology. Let $\nabla$ be a continuous lift

$$
\begin{array}{c}
\Omega A_{cl}^0(M) \\
\downarrow c \quad \quad \quad \quad \quad \quad \downarrow \nabla
\end{array}
\quad
\begin{array}{c}
H(A)^0(B) \\
\downarrow \quad \quad \quad \quad \quad \downarrow
\end{array}
\quad
\begin{array}{c}
\hat{K}U^0(M) \\
\downarrow I
\end{array}
\quad
\begin{array}{c}
K^0(B) \\
\downarrow P^*
\end{array}
\quad
\begin{array}{c}
\hat{K}U^0(M)
\end{array}
\quad
\begin{array}{c}
P^*
\end{array}
$$

The map $C_\nabla$ exists again since $c$ becomes an isomorphism after complexification of the domain and completion.

**Definition 4.259.** We call $C_\nabla$ the cohomological character of $\nabla$.

This leads to the following generalization of the notion of a connection on a $G$-principal bundle.

**Definition 4.260.** Let $B$ be some space. A geometrization of a map $P : M \to B$ from a compact manifold $M$ is a continuous lift $\nabla$ in the diagram of topological groups (62) such that the cohomological character $C_\nabla$ preserves the internal degree.

**Example 4.261.** Let us consider geometrizations of the identity maps $S^{2n} \to S^{2n}$, $n \geq 0$.

1. We first consider the case that $B = \ast$. Then $K^0(\ast) \cong \mathbb{Z}$. We get a geometrization $\nabla$ be setting $\nabla(1) := 1 \in \hat{K}U^0(M)$.

2. The next case is $n = 1$. We have $\hat{K}U^0(S^2) \cong \mathbb{Z} \oplus \mathbb{Z}x$, where $x = \text{cycl}(L) - 1$ and $L \to S^2 \cong \mathbb{C}P^1$ is the tautological bundle. We define a geometrization by

$$
\nabla(1) := 1 , \quad \nabla(x) := \text{cycl}(L, \nabla^L) - 1 .
$$

Here we can take any connection.
3. We now consider the geometrization of the identity map $S^4 \to S^4$. We have $KU^0(S^4) \cong \mathbb{Z}_1 \oplus \mathbb{Z}_x$ with $x := \text{cyc}1(V) - 2$, where $V \to S^4$ is associated to the Hopf $SU(2)$-bundle $S^7 \to S^4$ and the defining 2-dimensional representation of $SU(2)$. Let us choose some connection $\nabla^V$. We define

$$\nabla(1) := 1, \quad \nabla(x) := \widetilde{\text{cyc}}1(V, \nabla^V) - 2.$$  

We now calculate the cohomological character. Note that $\text{ch}(1) = 1$ and $\text{ch}(x) = b^2 \text{or}_{S^4}$. This gives

$$C_{\nabla}(1) = 1, \quad C_{\nabla}(b^2 \text{or}_{S^4}) = \text{ch}(\nabla^V) - 2 = b\text{ch}_2(\nabla^V) + b\text{ch}_4(\nabla^V).$$

This cohomological character preserves the internal degree if any only if $\text{ch}_2(\nabla^V) = 0$. We can achieve this by choosing a $SU(2)$-connection $\nabla^V$.

4. Let us now consider the identity map $S^6 \to S^6$. We have $KU^0(S^6) \cong \mathbb{Z}_1 \oplus \mathbb{Z}_x$, with $x = \text{cyc}1(V) - d$, where $V \to S^6$ is an appropriate bundle. We again choose a connection $\nabla^V$ and set $\nabla_0(1) := 1$ and $\nabla_0(x) := \widetilde{\text{cyc}}1(V, \nabla^V) - d$. We have $\text{ch}(1) = 1$ and $\text{ch}([V] - d) = b^3 \text{or}_{S^6}$. The cohomological character is now given by

$$C_{\nabla_0}(1) = 1, \quad C_{\nabla_0}(b^3 \text{or}_{S^6}) = \text{ch}(\nabla^V) - d = b\text{ch}_2(\nabla^V) + b^2\text{ch}_4(\nabla^V) + b^3\text{ch}_6(\nabla^V).$$

It is not clear whether we can find a connection with $\text{ch}_2(\nabla^V) = 0$ and $\text{ch}_4(\nabla^V) = 0$. But we can choose forms $\alpha_1 \in \Omega^1(S^6, \mathbb{C})$ and $\alpha_3 \in \Omega^3(S^6, \mathbb{C})$ with $d\alpha_1 = \text{ch}_2(\nabla^V)$ and $d\alpha_3 = \text{ch}_4(\nabla^V)$. Then we define

$$\nabla(1) := 1, \quad \nabla(x) = \nabla_0(x) - a(b\alpha_1 + b^2\alpha_3).$$

Then

$$C_{\nabla}(1) = 1, \quad C_{\nabla}(b^3 \text{or}_{S^6}) = b^3\text{ch}_6(\nabla^V)$$

preserves degree.

**Example 4.262.** We now consider the geometrization of the inclusion $f : \mathbb{CP}^1 \to \mathbb{CP}^\infty$. We have $KU^0(\mathbb{CP}^\infty) \cong \mathbb{Z}[[x]]$, where $x = \text{cyc}1(L) - 1$ (see 4.254). We choose any connection $\nabla^{f^*L}$ on $f^*L$ and define a geometrization by $\nabla(x^n) := (\text{cyc}(f^*L, \nabla^{f^*L}) - 1)^n$.

**Problem 4.263.** Show that this prescription defines a geometrization.

**Proof.** We have $R(\nabla(x^n)) = 0$ for $n \geq 2$. It follows that $\nabla(x^n) = 0$ for $n \geq 4$. This implies that $\nabla$ extends by continuity to a map

$$\nabla : KU^0(\mathbb{CP}^\infty) \to \widetilde{KU}^0(\mathbb{CP}^1).$$

We have $\text{ch}(x) = c_1b$ and hence $\text{ch}(x^n) = c_1^n b^n$. It follows that

$$C_{\nabla}(x^n) = b^n c_1(\nabla^{f^*L})^n.$$

Hence the cohomological character preserves degree.  

\[\Box\]
**Problem 4.264.** Construct a geometrization of the inclusion $f : \mathbb{R}P^3 \to \mathbb{R}P^\infty$.

*Proof.* We have (see 4.256)

$$KU^0(\mathbb{R}P^\infty) \cong \mathbb{Z}1 \oplus \mathbb{Z}x.$$  

A topological generator of the $\mathbb{Z}_2$-summand is the class $x := \text{cycl}(L) - 1$, where $L$ is the complexification of the tautological real bundle. Its restriction to $\mathbb{R}P^3$ has a flat connection $\nabla^{f^*L}$.

$$\nabla(1) := 1, \quad \nabla(x) := \text{cycl}(f^*L, \nabla^{f^*L})$$

does the job. Verify! \hfill \square

**Example 4.265.** Let us fix a number $p \in \mathbb{N}$ and consider the space $X$ defined as the homotopy fibre

$$X \to BSU(2) \xrightarrow{pc_2} K(\mathbb{Z}, 4).$$

Let $f : M \to X$ be some map from a compact manifold.

**Problem 4.266.** Construct a geometrization of $f$.

*Proof.* Since $c_2p$ is a rational isomorphism the space $X$ is rationally acyclic. Note that $X$ has finite skeleta. We can choose a smooth compact $\dim(M) + 1$-connected approximation $M' \to X$ and a factorization

$$f : M \xrightarrow{g} M' \xrightarrow{f'} X.$$ Note that

$$0 = g^* : H(A)^{-1}(M') \to H(A)^{-1}(M).$$

We can obtain $X$ as an iterated pull-back

$$\begin{array}{c}
X \xrightarrow{h} K(\mathbb{Z}/p\mathbb{Z}, 3) \xrightarrow{*} \\
BSU(2) \xrightarrow{c_2} K(\mathbb{Z}, 4) \xrightarrow{p} K(\mathbb{Z}, 4) \\
\end{array}$$

This makes clear that the fibre of $h$ is $K(\mathbb{Z}, 3)$. By [AH68] the group $KU^0(K(\mathbb{Z}, 3))$ is divisible. It follows, from the Serre spectral sequence that

$$h^* : R(SU(2))_I \cong KU^0(BSU(2)) \to KU^0(X)$$

is injective and has dense range (see [Bun11] Prop. 5.14] for similar arguments).

We know that $R(SU(2)) = \mathbb{Z}[x]$ and $I = (x)$, where $x + 2$ is the defining representation $SU(2) \hookrightarrow U(2)$. Hence $R(SU(2))_I \cong \mathbb{Z}[x]$.

We choose a bundle $V := f'^*h^*B_{SU(2)}(x)$ with connection $\nabla^V$. Note that $\text{ch}(h^*) = 2$. Hence we can find a form $\alpha \in \Omega A^{-1}(M')$ such that $d\alpha = \text{ch}(\nabla^V) - 2$. Then we define

$$\nabla'(x) := \text{cycl}(V, \nabla^V) - 2 - a(\alpha) \in \widetilde{KU}^0(M')$$

161
and extend this to a ring map
\[ \nabla' : R(SU(2)) \cong \mathbb{Z}[x] \to \hat{KU}^0(M') . \]
Finally we define
\[ \nabla(y) = g^* \nabla'(y) , \quad y \in R(SU(2)) . \]
We check continuity. Let \((y_i)\) be a sequence in \(R(SU(2))\) such that \(h^* y_i \to 0\). We must show that \(\nabla(y_i) \to 0\). We observe that \(R(\nabla'(x)) = 0\) and \(I(h^* y_i) = 0\) for \(i \gg 0\). Hence \(\nabla'(y_i) \in H(A)^{-1}(M')/\text{im}(\text{ch})\) for \(i \gg 0\). This implies \(\nabla(y_i) = 0\) for \(i \gg 0\).
Since \(R(\nabla(x)) = 0\) the cohomological character is determined by \(C_{\nabla}(1) = 1\). It preserves the internal degree. \(\square\)

**Problem 4.267.** If \(B\) is a space of the homotopy type of a \(CW\)-complex with finite skeleta and \(f : M \to B\) is a map from a compact manifold, then \(f\) admits a geometrization. Given \(k \geq \dim(M) + 1\) we can in addition require that the geometrization is pulled-back from a \(k\)-connected approximation \(M' \to B\). Such a geometrization will be called good.

**Proof.** One proceeds as in Example 4.265. We use the same notation. First decompose \(\hat{KU}^0(M')\) into torsion and free part. Define \(\nabla'\) on the torsion part using flat classes. Then deal with the free part by defining \(\nabla'\) on generators.
Similar arguments can be found in [Bun11, Prop. 4.4]. \(\square\)

**Problem 4.268.** Let \(f : M \to B\) as in Prop. 4.267 and \(\nabla_f\) be a good geometrization of \(f\). Let further \(F : Z \to B\) by an extension of \(f\) to a compact zero bordism \(Z\) of \(M\). Show that \(\nabla_f\) extends to a geometrization \(\nabla_F\) of \(F\).

**Problem 4.269.** Show by example that in 4.268 we can not drop the assumption that \(\nabla_f\) is good.

**Example 4.270.** We continue the example 4.265. Let \(M\) be a closed \(2n-1\)-dimensional \(Spin^c\)-manifold and \(f : M \to X\). We choose a good geometrization \(\nabla\) of \(f\). Let \(k \geq 1\). We choose any \(\hat{KU}\)-orientation refining the \(KU\)-orientation of \(M \to \ast\) given by the \(Spin^c\)-structure and define
\[ e_k(M) := \hat{I}_! \nabla(x^k) \in \hat{KU}^{-2n+1}(\ast) \cong \mathbb{C}/\mathbb{Z} \]
using the associated integration map.

**Problem 4.271.** Show that \(e_k(M)\) only depends on \(Spin^c\)-bordism class of \(M \to X\).

**Proof.** Let \(F : Z \to X\) be a \(Spin^c\)-zero bordism. Then we can extend the differential \(KU\)-orientation and \(\nabla\). We have by the bordism formula 4.232
\[ \hat{I}_! \nabla(x^k) = a(\int_Z \text{Td}(\tilde{\nu}) \wedge R(\nabla(x^k))) = 0 , \]
since $R(\nabla(x^k)) = 0$. \hfill \square

We therefore have defined a family of homomorphisms

$$e_k : MSpin^{c}_{2n-1}(X) \to \mathbb{C}/\mathbb{Z}.$$ 

Let $\sigma : S^7 \to S^4$ be the Hopf bundle and $g : S^4 \to BSU(2)$ represent the generator of $\pi_4(BSU(2)) \cong \mathbb{Z}$. The composition $pc_2 \circ g \circ \sigma$ is homotopically constant. In fact, we get a unique lift

$$\begin{array}{ccc}
S^7 & \xrightarrow{g \circ \sigma} & BSU(2) \\
\downarrow f & & \downarrow h \\
S^7 & \xrightarrow{g \circ \sigma} & BSU(2)
\end{array}$$

Note that $S^7$ has a canonical $Spin^c$-structure induced by the stable framing. We get a class $[f : S^7 \to X] \in MSpin^c_7(X)$.

**Problem 4.272.** Calculate

$$e_k([f : S^7 \to X]) \in \mathbb{C}/\mathbb{Z}.$$
5  Exercises
5.1 Sheet Nr 1.

1. Show in detail, that every vector bundle admits a connection.

2. Show, that a connection on a bundle $E \to M$ uniquely extends to a linear map $\Omega(M,E) \to \Omega(M,E)$ which satisfies the Leibniz rule.

3. Develop the details of the construction of the pull-back of a connection and prove the formula for the pull-back of the curvature.

4. Show that an invariant connection descends if and only if its momentum map vanishes. Show that in this case the descent is uniquely determined.

5. Calculate the moment map of the Levi-Civita connection on a compact Lie-group with left action and biinvariant metric.

6. Determine explicitly the connection on the Poincaré bundle on the torus $T^n$ whose curvature is an invariant two-form on the $2n$-dimensional torus $J(T^n) \times T^n$. 

5.2 Sheet Nr 2.

1. Calculate the curvature of the $SU(2)$-invariant connection on the tautological bundle $L \rightarrow \mathbb{C}P^1$.

2. Verify the identities

\[ \text{ch}(\nabla^{E \oplus F}) = \text{ch}(\nabla^E) + \text{ch}(\nabla^F), \quad \text{ch}(\nabla^{E \otimes F}) = \text{ch}(\nabla^E) \wedge \text{ch}(\nabla^F), \]

\[ \text{ch}(\nabla^{\text{Hom}(E, F)}) = \text{ch}(\nabla^F) \wedge \text{ch}(\nabla^{E^*}) \]

and

\[ \text{ch}_{2i}(\nabla^{E^*}) = (-1)^i \text{ch}_{2i}(\nabla^E). \]

3. Show that one can express $\text{ch}_{2k}(\nabla)$ as a rational polynomial in the forms $c_i(\nabla)$ for $i \leq k$. Determines these polynomials explicitly for $k = 4$ and 6.

4. For $F \in \text{Aut}(E)$ show

\[ \tilde{\omega}(F) = \int_{S^1 \times M/M} \omega(E(F)) \]

(see \ref{2.66} for notation)

5. Show that $R^{E^*} = -(R^E)^*$ and conclude that $c_i(\nabla)$ and $\text{ch}_{2i}(\nabla)$ are real forms if $\nabla$ is unitary.

6. Show

\[ \text{ch}_{2n}(F_{id}) = \frac{(-1)^{n-1}(n-1)!}{(2\pi i)^n(2n-1)!} \text{Tr}(g^{-1}dg)^{2n-1}. \]

(see Lemma \ref{2.71} for notation)
5.3 Sheet Nr 3.

1. Calculate $cs_{c_2}(\nabla)$ for all flat connections on line-bundles on the lense space $L_{p}^{2n-1} := S^{2n-1}/\mu_p$, $p \in \mathbb{N}$ prime.

2. Consider the map $v : BSL(k, \mathbb{C}^{\delta}) \to BGL(k, \mathbb{C})$ and observe that we then have $0 = v : MSO_3(BSL(k, \mathbb{C}^{\delta})) \to MSO_3(BGL(k, \mathbb{C}))$. Let $V^{\delta} \to BSL(k, \mathbb{C}^{\delta})$ be associated to the standard representation of $SL(k, \mathbb{C})$.

Show that $cs_{c_2}(V^{\delta}) : MSO_3(BSL(k, \mathbb{C}^{\delta})) \to \mathbb{C}/\mathbb{Z}$ vanishes. Furthermore show that $\text{Im}(cs_{c_2}(V^{\delta})) : MSO_3(BSL(k, \mathbb{C}^{\delta})) \to \mathbb{R}$ is non-trivial. Hint: Calculate $\tilde{c}_2(f^*\nabla) - \tilde{c}_2((f^*\nabla)^*)$ for suitable $f : M \to BSL(k, \mathbb{C})$ for some metric on $f^*V$, compare this with the Kamber-Tondeur class and use Borel’s result that the latter generates $H^3(BSL(k, \mathbb{C}); \mathbb{R})$.

3. Let $M \to S^1$ be a 2-torus bundle. Then we have a sequence

$$0 \to \pi_1(T^2) \to \pi_1(M) \to \pi_1(S^1) \to 0. \tag{63}$$

We consider generators $A, B \in \pi_1(T^2)$ and an element $T \in \pi_1(M)$ which maps to a generator of $\pi_1(S^1)$. We consider a representation $\rho : \pi_1(M) \to Sp(1)$ with $\rho(T) = J$ (quaternionic notation) and $\rho(A) = \exp(2\pi i \phi)$, $\rho(B) = \exp(2\pi i \psi)$ for some $\phi, \psi \in \mathbb{R}$. Let $(V \to M, \nabla)$ be the associated two-dimensional flat bundle. Calculate $\tilde{c}_2(\nabla)$ in terms of the data $\psi, \phi$ and the extension \[13\]. Discuss, under which conditions $\rho$ exists. Conclude that $\text{Re}(cs_{c_2}(V^{\delta}))$ from exercise 2. is non-trivial.

4. Calculate the Chern-Simons invariant of the lense space $CS(S^3/\mu_k, g)$, where $g$ is a Riemannian metric of constant positive curvature.

5. Determine the sets

$$\{CS(S^3, g) \mid g \text{ a Riemannian metric} \} \subset \mathbb{C}/\mathbb{Z}$$

and

$$\{CS_{\text{refined}}(S^3, g) \mid g \text{ a Riemannian metric} \} \subset \mathbb{C}/\mathbb{Z}$$

(see \[14\]).

6. Determine

$$CS(M, g) - \left[ \int p_{1}(g^{TZ}) \right]$$

in the case that $g^{TZ}$ extends $g$, but does not necessarily have a product structure.
5.4 Sheet Nr 4.

1. Let $X$ be a countable $n$-dimensional $CW$-complex. Show that there exists a smooth manifold $M$ and a homotopy equivalence $M \to X$ (see [BS10, Sec. 2] for an argument).

2. Let $f : X \to Y$ be an $n + 1$-connected map and $M$ be an $n$-dimensional manifold. Show that $f_* : [M, X] \to [M, Y]$ is a bijection.

3. Let $M$ be a connected manifold with fundamental group $\pi_1(M) = \mathbb{Z}/p\mathbb{Z}$. For $\rho \in J(M)$ let $P_\rho := P_{\{\rho\} \times M}$ be the restriction of the Poincaré bundle $P \to J(M) \times M$ (see Ex. 2.30). Show that

$$J(M) \ni \rho \mapsto c_1^\mathbb{Z}(P_\rho) \in H^2(M; \mathbb{Z})$$

induces an isomorphism of groups between $J(M)$ and the torsion subgroup of $H^2(M; \mathbb{Z})$.

4. For $r \in \mathbb{N}$ we let $M_r \to S^4$ be the $SU(2)$-principal bundle classified by $c_2^\mathbb{Z}(M_r) = r[S^4] \in H^4(S^4; \mathbb{Z})$. Calculate $H^4(M_r; \mathbb{Z})$ and its element $p_1^\mathbb{Z}(TM_r)$ explicitly.

5. Show that $0 \neq c_1(L \otimes \mathbb{C})$ in $H^2(\mathbb{RP}^\infty; \mathbb{Z})$, where $L \to \mathbb{RP}^\infty$ is the tautological bundle.

6. Find an explicit formula for the numbers

$$\langle p_1(T\mathbb{C}P^n)^2 \cup p_3(T\mathbb{C}P^n), [\mathbb{C}P^{10}] \rangle \in \mathbb{Z} , \quad n \geq 10 .$$

7. For a discrete abelian group $A$ we have the following constructions of the cohomology of a manifold $M$ with coefficients in $A$:

$$H^n(M; A) = \begin{cases} 
H^n(M, A) & \text{sheaf cohomology} \\
[M, K(A, n)] & \text{homotopy theoretic definition} \\
H^n(\text{Hom}(C_*(M), A)) & \text{singular cohomology}
\end{cases}$$

Find equivalences between these definitions which are natural in $M$ and $A$. 

168
5.5 **Sheet Nr 5.**

1. For a lower bounded complex of sheaves $\mathcal{F}$ construct the hypercohomology spectral sequence with

$$H^p(M, H^q(\mathcal{F})) = E_2^{p,q}, \quad (E_r, d_r) \Rightarrow H^{p+q}(M, \mathcal{F}) .$$

2. We consider the Hopf fibration $\pi : S^3 \to \mathbb{C}P^1$ and the complex of sheaves

$$\mathbb{C}P^1 \supset U \mapsto \Omega_C(\pi^{-1}(U)) \in \text{Ch}^+ .$$

Make the hypercohomology spectral sequence explicite.

3. Make the hypercohomology spectral sequence for

$$\mathcal{E}(n) := \text{Cone}(\mathbb{Z} \to \sigma^n \Omega_C)$$

explicite and relate it with the structure of Deligne cohomology.

4. Show that

$$x \mapsto \{ M \ni m \mapsto \text{ev}(m^*x) \in \mathbb{C}/\mathbb{Z} \}$$

defines a natural isomorphism

$$f : \hat{H}^1_{\text{Del}}(M; \mathbb{Z}) \simeq C^\infty(M; \mathbb{C}/\mathbb{Z}) .$$

Show further that $f^{-1}(x)df(x) = R(x)$.

5. Show that

$$x \mapsto \{ L(M) \ni \gamma \mapsto \text{ev}(\gamma^*x) \in \mathbb{C}/\mathbb{Z} \}$$

defines a natural injective homomorphism

$$f : \hat{H}^2_{\text{Del}}(M; \mathbb{Z}) \simeq C^\infty(L(M); \mathbb{C}/\mathbb{Z}) .$$

Show that it is not surjective, in general, and calculate $df$. Here $L(M) := C^\infty(S^1, M)$ is the free loop space and smooth maps out of $L(M)$ are understood in the diffeological sense.

6. Show that the canonical generator $e \in \hat{H}^1_{\text{Del}}(S^1; \mathbb{Z})$ is primitive. Let further $G$ be a compact connected and simply connected simple Lie group. Show that there is a unique (up to sign) bi-invariant class $e \in \hat{H}^3_{\text{Del}}(G; \mathbb{Z})$ such that $I(e) \in H^3(G; \mathbb{Z})$ is a generator. Show that this class is primitive.
5.6 Sheet Nr 6.

1. Consider the complex tangent bundle $T\mathbb{H}\mathbb{C}^n$ of the complex hyperbolic space $\mathbb{H}\mathbb{C}^n$ with the Levi-Civita connection $\nabla$ and a point $x \in \mathbb{H}\mathbb{C}^n$. For $r > 0$ let $S_r \subset \mathbb{H}\mathbb{C}^n$ be the distance sphere at $x$. Calculate $\text{ev}_{S_r}(\hat{c}_n(\nabla)) \in \mathbb{C}/\mathbb{Z}$ as a function of $r$.

2. Let $E \to M$ be a vector bundle, $\nabla$ a connection on $E$ and $h$ be a hermitean metric. Show that $\hat{c}_n(\nabla) = \hat{c}_n(\nabla)$.

3. Let $(E, \nabla)$ be a flat complex vector bundle. Then $\text{Im} \hat{s}_{2n}(\nabla) \in H^{2n-1}(M; \mathbb{R})$. Find the relation between $\text{Im} \hat{s}_n(\nabla)$ and the Kamber-Tondeur class $\text{ch}_{2n}(\nabla)$.

4. Show that a Čech 1-cocycle for the complex $\mathcal{K}(1) : \mathbb{C}^* \xrightarrow{d \log} \Omega^1_{\mathbb{C}}$

can naturally be identified with the glueing data for a line bundle with connection. Use this to construct the isomorphism

$$\hat{c}_1 : \text{Line}_V(M) \xrightarrow{\sim} H^1(M; \mathcal{K}(1))$$

explicitly on the level of Čech cohomology. Verify that this is compatible with the previous construction of $\hat{c}_1$ and the identification $H^1(M; \mathcal{K}(1)) \cong \hat{H}^2_{De}(M; \mathbb{Z})$ described in the course.

5. Let $V \to B$ a vector bundle with structure group $GL(k, \mathbb{C})$ over some space $B$ and $\omega$ be an integral characteristic form. For a map $f : M \to B$ from a closed oriented $n-1$-manifold define

$$\text{cs}_\omega^V(f : M \to B) := \text{ev}(\hat{\omega}(\nabla^f V)) \in \mathbb{C}/\mathbb{Z}.$$ 

Show that this defines a homomorphism $\text{cs}_\omega^V : \text{MSO}_{n-1}(B) \to \mathbb{C}/\mathbb{Z}$ which extends the previous definition made under the assumption that $V$ was trivializable as a bundle with structure group $GL(k, \mathbb{C})$. Show further, that $\text{cs}_\omega^V$ factorizes over the orientation transformation $\text{MSO}_{n-1}(B) \to H_{n-1}(M; \mathbb{Z}), [f : M \to B] \mapsto f_*[M]$.

6. Fix a $p$th root of unity $\xi$ and consider the flat bundle $V \to B\mathbb{Z}/p\mathbb{Z}$ with holonomy $[k] \mapsto \xi^k$. Let $L_p^{2n-1} := S^{2n-1}/(\mathbb{Z}/p\mathbb{Z})$ and $f : L_p^{2n-1} \to B\mathbb{Z}/p\mathbb{Z}$ be the map which induces the canonical identification of fundamental groups. Calculate $\text{cs}_{\xi}^V(f) \in \mathbb{C}/\mathbb{Z}$.

170
5.7 Sheet Nr 7.

1. For a closed oriented Riemannian manifold \((M,g)\) of dimension \(n - 1\) define, using the complexification of the Levi-Cevita connection,

\[
\text{CS}(g) := \text{ev}((-1)^n \hat{c}_{2n}(\nabla^{TM \otimes \mathbb{C}})) \in \mathbb{C}/\mathbb{Z}.
\]

Show that this extends the previous definition of the Chern-Simons invariant dropping the assumption that \(M\) is zero-bordant.

2. Show that \(\text{CS}(g)\) is an invariant of the conformal class of \(g\) which vanishes if \(M\) totally umbilically bounds a locally conformally flat manifold.

3. Let \(\mathbb{Z}/p\mathbb{Z}\) act on \(S^{4n-1} \subset \mathbb{C}^{2n}\) diagonally by \([1] \mapsto (\zeta^{q_1}, \ldots, \zeta^{q_{2n}})\), where \(\zeta = \exp(2\pi ip^{-1})\) is a primitive root of unity and the numbers \(g.g.T(q_1, \ldots, q_{2n}, p) = 1\) and set \(L_p^{4n-1}(q_1, \ldots, q_{2n}) := S^{4n-1}/(\mathbb{Z}/p\mathbb{Z})\) with the induced round metric \(g\). Calculate

\[
\text{CS}(L_p^{4n-1}(q_1, \ldots, q_{2n}), g) \in \mathbb{C}/\mathbb{Z}.
\]

4. Calculate \(\hat{c}_1(\nabla \otimes \mathbb{C}) \in \bar{H}^2_{Del}(\mathbb{RP}^n; \mathbb{Z})\) for the canonical bundle \((V, \nabla)\) of \(\mathbb{RP}^n\) in algebro-topological terms.

5. Show that an oriented real vector bundle \(V\) admits a spin structure if (and only if ?) \(\hat{c}_1(\nabla \otimes \mathbb{C}) = 0\) for some and hence every connection.

6. Show that the integral characteristic form \(\frac{p_1}{2}\) for real spin bundles has a unique differential refinement. Show that this refinement is additive.
5.8 Sheet Nr 8.

1. Let $G$ be a Lie group with finitely many connected components and $P \to M$ be a $G$ principal bundle with connection $\omega$. Let $\tilde{\phi} \in \tilde{I}(G)$. Show that there exists a natural (under pull-back of principal bundles with connection) form $\theta(\omega) \in \Omega^{2n-1}(P; \mathbb{C})$ such that $d\theta(\omega) = \phi(R^\omega)$. Show that

$$\pi^* \hat{\omega}(\tilde{\phi})(\omega) = a(\theta(\omega)).$$

2. We consider $G := SU(2)$ and $\tilde{\phi} \in I(SU(2))$ such that $\phi_Z = c_2$. Give an explicit formula for $\theta(\omega)$ in Exercise 1.

3. Let $(V, h, \nabla)$ be a flat euclidean vector bundle of dimension $2n$ and $\pi : S(V) \to M$ be its sphere bundle. Let $\beta$ be the normalized fibrewise volume form on $S(V)$, extended to $S(V)$ using the connection. We define $f(\nabla) \in H^{2n-1}(M; \mathbb{C}/\mathbb{Z})$ by the condition

$$f(\nabla)(x) = \left[\int_{\tilde{x}} \beta\right] \in \mathbb{C}/\mathbb{Z}$$

for $\tilde{x} \in H_{2n-1}(S(V); \mathbb{Z})$ such that $\pi_*(\tilde{x}) = x \in H_{2n-1}(M; \mathbb{Z})$. Show that $f(\nabla)$ is well-defined and that $f(\nabla) = \hat{\chi}(\nabla)$.

4. Complete the arguments in the construction of the $\cup$-product.

5. On $S^1 \times S^1$ show that $\text{pr}_1^* \hat{\epsilon} \cup \text{pr}_2^* \hat{\epsilon} \cup \mu^* \hat{\epsilon} = 0$, where $\mu$ is the group structure.

6. Consider the smooth $\mathbb{C}/\mathbb{Z}$-valued functions on $\mathbb{C}^* \setminus \{1\}$ given by $f := \left[\frac{1}{2\pi i} \ln(z)\right]$ and $g := \left[\frac{1}{2\pi i} \ln(1 - z)\right]$ as elements in $\tilde{H}_{Del}(\mathbb{C}^* \setminus \{1\}; \mathbb{Z})$. Show that $f \cup g = 0$. 
5.9 Sheet Nr 9.

1. We consider a closed oriented \( n - 1 \)-dimensional manifold \( M \) and define the
\[
\langle \ldots, \ldots \rangle : \check{H}^p_{\text{Del}}(M; \mathbb{Z}) \otimes \check{H}^{n-p}_{\text{Del}}(M; \mathbb{Z}) \to \mathbb{C}/\mathbb{Z}
\]
by \( x \otimes y \mapsto \text{ev}(x \cup y) \). Show that this pairing is non-degenerate.

2. Complete the details of the construction of the differential cohomology diagram for \( \hat{H}_{\text{CS}}^*(M; \mathbb{Z}) \).

3. Show that there is a unique natural integration for \( H(\ldots; \mathbb{Z}) \) along proper oriented submersions which is compatible with Mayer-Vietoris sequences associated to decompositions of the base.

4. Let \( (P, \nabla^P) \) be the Poincaré bundle over \( J(S^1) \times S^1 \). Calculate the integrals of \( \check{c}_1(\nabla^P) \) over the two projections.

5. Let \( p : W \to B \) be an oriented bundle of compact manifolds with boundary \( \partial W \to B \). Let \( q : \partial W \to B \) denote the restriction of \( p \) to the boundary. We consider \( x \in \check{H}^n_{\text{Del}}(W; \mathbb{Z}) \). Show that
\[
q_!(x) = a(\int_{W/B} R(x)) .
\]

6. Let \( Z \in \text{Sm}(\mathcal{N}(\text{Ch})) \) and consider its interpretation \( Z_\infty \in \text{Sm}(\mathcal{N}(\text{Ch})[W^{-1}]) \). Show that \( Z \) satisfies descent, but \( Z_\infty \) does not.
5.10 Sheet Nr 10.

1. Show that the tensor structure with \( \mathbb{N}([\text{sSet}]/[W^{-1}]) \) on \( \mathbb{N}(\text{Ch})/[W^{-1}] \) is given by 
   \[ A \otimes X \cong A \otimes C_\ast(X) \], where \( C_\ast(X) \) is the chain complex associated to \( C \).

2. We define a functor \( \bar{s} : \text{Sm}(C) \to \text{Sm}(C) \) as the composition of 
   \( s : \text{Sm}(C) \to \text{Sm}((\text{Fun}(\mathbb{N}(\Delta^{op}), C)) \) and \( \colim_{\mathbb{N}(\Delta^{op})} : \text{Sm}(\text{Fun}(\mathbb{N}(\Delta^{op}), C)) \to \text{Sm}(C) \), where \( s \) is 
   precomposition by \( \text{Mf} \to \text{Fun}(\Delta, \text{Mf}) \), \( M \mapsto M \times \Delta^\ast \).

   We consider the sheaf \( \Omega^n \in \text{Sh}_{\text{Ab}}(\text{Mf}) \) of closed \( n \)-forms. Calculate the homotopy 
   and cohomology groups of the evaluations on \( M \) of the following presheaves 
   and sheaves derived from \( \Omega^n \) in terms of the de Rham cohomology on \( M \).

   (a) We let \( C(\Omega^n) \in \text{Sm}(\mathbb{N}(\text{Ch})/[W^{-1}]) \) be the presheaf of chain complexes given 
   by \( \Omega^n \) located in degree 0. Calculate \( H^*(C(\Omega^n))(M) \).

   (b) Calculate \( H^*(L(C(\Omega^n))(M)) \), where \( L : \text{Sm}(\mathbb{N}(\text{Ch})/[W^{-1}]) \to \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch})/[W^{-1}]) \) is 
   the sheafification.

   (c) Similarly let \( C^{\leq 0}(\Omega^n) \in \text{Sm}(\mathbb{N}(\text{Ch})^{\leq 0}/[W^{-1}]) \) be the negatively graded chain complex 
   represented by \( \Omega^n \) and \( L : \text{Sm}(\mathbb{N}(\text{Ch})^{\leq 0}/[W^{-1}]) \to \text{Sm}^{\text{desc}}(\mathbb{N}(\text{Ch})^{\leq 0}/[W^{-1}]) \). 
   Calculate \( H^*(L(C^{\leq 0}(\Omega^n))(M)) \)

   (d) Calculate the groups \( H^*(\bar{s}C(\Omega^n))(M)) \), \( H^*(\bar{s}C^{\leq 0}(\Omega^n))(M)) \) and \( H^*(L(\bar{s}C(\Omega^n))(M)) \) 
   and \( H^*(L(\bar{s}C^{\leq 0}(\Omega^n))(M)) \), where \( s \) is as in [4.24]

   (e) We consider \( \Omega^n \) as a smooth constant simplicial abelian group \( S(\Omega^n) \in \text{Sm}(\mathbb{N}(\text{Ab})/[W^{-1}]) \). 
   Calculate \( \pi_*(S(\Omega^n))(M)) \).

   (f) Calculate \( \pi_*(\bar{s}S(\Omega^n))(M)) \).

3. Let \( (HZ, C, c) \) be the canonical datum of \( HZ \) and \( D(n) \) the \( n \)th Deligne complex. 
   Show that \( \text{Diff}^n(HZ, C, c) \cong H(D(n)) \)

4. Let \( (E, A, c) \) be a datum. Provide a natural morphism of monoids 
   \[ \Omega^\text{map}(E, H(A)) \to \text{end}(E, A, c) \]
   and calculate the action of \( \pi_0(\Omega^\text{map}(E, H(A))) \cong H(A)^{-1}(E) \) on \( \hat{E}^\ast \).

5. Let \( (E, A, c) \) be a datum. Show that there are pull-back diagrams 

\[
\begin{array}{ccc}
\text{Diff}^n(E, A, c) & \longrightarrow & L(H(\Omega A^n)) \\
\downarrow & & \downarrow \\
\text{Sm}(E) & \longrightarrow & \text{Sm}(H(A))
\end{array}
\quad \quad \quad 
\begin{array}{ccc}
\text{Diff}^n(E, A, c) & \longrightarrow & L(H(\Omega A^{n-1}/\text{im}(d))) \\
\downarrow & & \downarrow \\
\text{Sm}(E) & \longrightarrow & \text{Sm}(H(A))
\end{array}
\]
5.11 Sheet Nr 11.

1. We consider a complex vector bundle $V \to M$. Show, in the framework of $KU$-theory, that $Rham(ch(\nabla V)) = c(cycl(V))$.

2. Show in detail the existence and uniqueness of the lifts $\hat{c}_n : \hat{KU}^0 \to \hat{HZ}^{2n}$ of the Chern classes.

3. Show that there is a unique $SU(2)$-invariant class $u \in \hat{KU}^1(SU(2))$ which refines a generator of $\hat{KU}^1(SU(2)) \cong \mathbb{Z}$ and restricts to zero at the identity element.

4. We consider the canonical datum $(MU, A, c)$. Let $u \in H(A)^0(BU)$ correspond to $c \in H(A)^0(MU)$ under the Thom isomorphism. Show that for a $MU$-cycle $z := (f : W \to M, N, I)$ of degree $n$ we have $f_iu(N) = c(cycl(z)) \in H(A)^n(M)$.

5. Write (see previous exercise) $u = \sum_{n=0}^{\infty} [CP^n] \Phi_n$ with $\Phi_n \in \mathbb{C}[c_1, \ldots, c_n]^{2n}$. Calculate $\Phi_n$ for small $n$ (e.g. $n \in \{0, 1, 2, 3\}$) explicitly.

6. We consider $S^2$ with the standard metric. A map $f : S^1 \to S^2$ gives rise to a $MU$-cycle $Z(f)$ of degree 1 in a canonical way. Let $x \in S^2$ be some point. Calculate $harm(Z(f))|_x \in \hat{MU}^1(*) \cong \mathbb{C}/\mathbb{Z}$.
5.12 Sheet Nr 12.

1. Show that the cycle map $\text{cyc} : \text{iVect}_{\text{geom}} \to \widehat{\text{KU}}$ is multiplicative.

2. Show that $\text{MU}$ is formal over $\mathbb{C}$.

3. Show that $\text{Diff}(S, C, c)$ fits into a pull-back

$$
\begin{array}{ccc}
\text{Diff}(S, C, c) & \longrightarrow & \text{Diff}(HZ, C, c) \\
\downarrow & & \downarrow \\
\text{Sm}(S) & \longrightarrow & \text{Sm}(HZ)
\end{array}
$$

4. Let $(E, A, c)$ be a multiplicative data. Work out the definition of a module data $(F, B, d)$. Construct $\text{Diff}(F, B, d)$ as a $\text{Diff}(E, A, c)$-module.

5. Let $\eta \in \pi_1(S) \cong \mathbb{Z}/2\mathbb{Z}$ and $\sigma \in \pi_3(S) \cong \mathbb{Z}/24\mathbb{Z}$ be the generators given by the Hopf maps $S^3 \to S^2$ and $S^7 \to S^4$. The have canonical lifts $\hat{\eta} \in \hat{S}^{-1}(\ast)$ and $\hat{\sigma} \in \hat{S}^{-3}(\ast)$. Let $\hat{e} : \hat{S}^* \to \widehat{\text{KU}}^*$ be induced by the unit. Calculate $\hat{e}(\hat{\eta}) \in \widehat{\text{KU}}^{-1} \cong \mathbb{C}/\mathbb{Z}$ and $\hat{e}(\hat{\sigma}) \in \widehat{\text{KU}}^{-3} \cong \mathbb{C}/\mathbb{Z}$.

6. Show that $\text{KO}$ is formal over $\mathbb{C}$. 
5.13 Sheet Nr 13.

1. Let $W$ be a compact oriented $n$-manifold with boundary $\partial W$. Calculate $\hat{H}^n_{\partial W}(W)$, $\hat{H}^{n+1}_{\partial W}(W)$ and $\hat{H}^{n+1}_c(W \setminus \partial W)$.

2. Let $\text{MSpin}^c \to \text{KU}$ be the ABS-orientation. The universal bundle $V \to B\text{Spin}^c(n)$ is thus $\text{KU}$-oriented with Thom class $\nu$. Let $0_V : B\text{Spin}^c(n) \to V$ be the zero section. Describe $\text{H}_C[b, b^{-1}]^*(B\text{Spin}^c(n))$ and calculate $\text{ch}(0_V^*\nu)$.

3. Calculate $\text{ch}(\rho^k(V))$, where $\rho^k(V)$ is the cannibalistic class for the Adams operation $\Psi^k$ and $V$ is as in 1.

4. Show that there exists a unique differential refinement $\hat{\rho}^k : \text{Vect}_{\text{Spin}^c, \text{geom}}^k \to \hat{\text{KU}}[\frac{1}{k}]^0$.

5. Calculate $\hat{\rho}^k(L, \nabla)$, where $(L, \nabla)$ is the tautological bundle on $\mathbb{CP}^n$ with its invariant connection.

6. Let $\hat{\nu} \in \hat{S}_c^n(\mathbb{R}^n)$ be a differential Thom class. Let $x \in \hat{E}^*(M)$. Show that

$$\text{desusp}(\text{pr}_{\mathbb{R}^n}^*\hat{\nu} \cup \text{pr}_M^*x) = x.$$
5.14 Sheet Nr. 14.

These problems are probably more than just exercises. I am interested to see solutions.

1. Show Lemma 3.123
2. Descide 4.10, 4.65 and 4.30
3. Descide 4.79
4. Prove the bordism formula 4.232
5. Solve 4.227
6. Prove the differential MU-index theorem 4.245
References

[ABS64] M. F. Atiyah, R. Bott, and A. Shapiro. Clifford modules. *Topology*, 3(suppl. 1):3–38, 1964.

[AH61] M. F. Atiyah and F. Hirzebruch. Vector bundles and homogeneous spaces. In *Proc. Sympos. Pure Math., Vol. III*, pages 7–38. American Mathematical Society, Providence, R.I., 1961.

[AH68] D. W. Anderson and Luke Hodgkin. The $K$-theory of Eilenberg-MacLane complexes. *Topology*, 7:317–329, 1968.

[AS69] M. F. Atiyah and G. B. Segal. Equivariant $K$-theory and completion. *J. Differential Geometry*, 3:1–18, 1969.

[AS10] C. Arias Abad and F. Schaetz. The $A_{\infty}$ de Rham theorem and integration of representations up to homotopy. *ArXiv e-prints*, November 2010.

[Auc94a] David R. Auckly. Chern-Simons invariants of 3-manifolds which fiber over $S^1$. *Internat. J. Math.*, 5(2):179–188, 1994.

[Auc94b] David R. Auckly. Topological methods to compute Chern-Simons invariants. *Math. Proc. Cambridge Philos. Soc.*, 115(2):229–251, 1994.

[Ave70] André Avez. Characteristic classes and Weyl tensor: Applications to general relativity. *Proc. Nat. Acad. Sci. U.S.A.*, 66:265–268, 1970.

[BC89] Jean-Michel Bismut and Jeff Cheeger. $\eta$-invariants and their adiabatic limits. *J. Amer. Math. Soc.*, 2(1):33–70, 1989.

[BD82] Paul Baum and Ronald G. Douglas. $K$ homology and index theory. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 117–173. Amer. Math. Soc., Providence, R.I., 1982.

[BG01] Jean-Michel Bismut and Sebastian Goette. Families torsion and Morse functions. *Astérisque*, (275):x+293, 2001.

[BGV92] Nicole Berline, Ezra Getzler, and Michèle Vergne. *Heat kernels and Dirac operators*, volume 298 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.

[Bis05] J. M. Bismut. Eta invariants, differential characters and flat vector bundles. *Chinese Ann. Math. Ser. B*, 26(1):15–44, 2005. With an appendix by K. Corlette and H. Esnault.

[BKS10] Ulrich Bunke, Matthias Kreck, and Thomas Schick. A geometric description of differential cohomology. *Ann. Math. Blaise Pascal*, 17(1):1–16, 2010.
[BL95] Jean-Michel Bismut and John Lott. Flat vector bundles, direct images and higher real analytic torsion. *J. Amer. Math. Soc.*, 8(2):291–363, 1995.

[BM06] Moulay-Tahar Benameur and Mohamed Maghfoul. Differential characters in $K$-theory. *Differential Geom. Appl.*, 24(4):417–432, 2006.

[Bor74] Armand Borel. Stable real cohomology of arithmetic groups. *Ann. Sci. École Norm. Sup. (4)*, 7:235–272 (1975), 1974.

[Bry08] Jean-Luc Brylinski. *Loop spaces, characteristic classes and geometric quantization*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2008. Reprint of the 1993 edition.

[BS09] Ulrich Bunke and Thomas Schick. Smooth $K$-theory. *Astérisque*, (328):45–135 (2010), 2009.

[BS10] Ulrich Bunke and Thomas Schick. Uniqueness of smooth extensions of generalized cohomology theories. *J. Topol.*, 3(1):110–156, 2010.

[BSSW09] Ulrich Bunke, Thomas Schick, Ingo Schröder, and Moritz Wiethaup. Landweber exact formal group laws and smooth cohomology theories. *Algebr. Geom. Topol.*, 9(3):1751–1790, 2009.

[BT] U. Bunke and G. Tamme. Regulators and cycle maps in higher-dimensional differential algebraic $K$-theory. in preparation.

[BT06] Mark Brightwell and Paul Turner. Relative differential characters. *Comm. Anal. Geom.*, 14(2):269–282, 2006.

[BT12] Ulrich Bunke and Georg Tamme. Multiplicative regulators and cycle maps. in preparation, 2012.

[Bun09] Ulrich Bunke. Index theory, eta forms, and Deligne cohomology. *Mem. Amer. Math. Soc.*, 198(928):vi+120, 2009.

[Bun10a] Ulrich Bunke. Adams operations in smooth $K$-theory. *Geom. Topol.*, 14(4):2349–2381, 2010.

[Bun10b] Ulrich Bunke. Chern classes on differential $K$-theory. *Pacific J. Math.*, 247(2):313–322, 2010.

[Bun11] U. Bunke. On the topological contents of eta invariants. *eprint arXiv:1103.4217*, March 2011.

[CF66] P. E. Conner and E. E. Floyd. *The relation of cobordism to $K$-theories*. Lecture Notes in Mathematics, No. 28. Springer-Verlag, Berlin, 1966.
[CS85] Jeff Cheeger and James Simons. Differential characters and geometric invariants. In Geometry and topology (College Park, Md., 1983/84), volume 1167 of Lecture Notes in Math., pages 50–80. Springer, Berlin, 1985.

[DL05] Johan L. Dupont and Rune Ljungmann. Integration of simplicial forms and Deligne cohomology. Math. Scand., 97(1):11–39, 2005.

[dR84] Georges de Rham. Differentiable manifolds, volume 266 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1984. Forms, currents, harmonic forms, Translated from the French by F. R. Smith, With an introduction by S. S. Chern.

[Dug01] Daniel Dugger. Universal homotopy theories. Adv. Math., 164(1):144–176, 2001.

[Dup78] Johan L. Dupont. Curvature and characteristic classes. Lecture Notes in Mathematics, Vol. 640. Springer-Verlag, Berlin, 1978.

[Fer12] F. Ferrari Ruffino. Flat pairing and differential homology. ArXiv e-prints, August 2012.

[FH00] Daniel S. Freed and Michael Hopkins. On Ramond-Ramond fields and K-theory. J. High Energy Phys., (5):Paper 44, 14, 2000.

[FL10] Daniel S. Freed and John Lott. An index theorem in differential K-theory. Geom. Topol., 14(2):903–966, 2010.

[FMS07] Daniel S. Freed, Gregory W. Moore, and Graeme Segal. The uncertainty of fluxes. Comm. Math. Phys., 271(1):247–274, 2007.

[Fre00] Daniel S. Freed. Dirac charge quantization and generalized differential cohomology. In Surveys in differential geometry, Surv. Differ. Geom., VII, pages 129–194. Int. Press, Somerville, MA, 2000.

[GM09] Richard Green and Varghese Mathai. Harmonic Cheeger-Simons characters with applications. J. Geom. Phys., 59(5):663–672, 2009.

[Gom08] Kiyonori Gomi. Differential characters and the Steenrod squares. In Groups of diffeomorphisms, volume 52 of Adv. Stud. Pure Math., pages 297–308. Math. Soc. Japan, Tokyo, 2008.

[HBJ92] Friedrich Hirzebruch, Thomas Berger, and Rainer Jung. Manifolds and modular forms. Aspects of Mathematics, E20. Friedr. Vieweg & Sohn, Braunschweig, 1992. With appendices by Nils-Peter Skoruppa and by Paul Baum.

[HH92] Michael J. Hopkins and Mark A. Hovey. Spin cobordism determines real K-theory. Math. Z., 210(2):181–196, 1992.
[Hit01] Nigel Hitchin. Lectures on special Lagrangian submanifolds. In Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), volume 23 of AMS/IP Stud. Adv. Math., pages 151–182. Amer. Math. Soc., Providence, RI, 2001.

[HS05] M. J. Hopkins and I. M. Singer. Quadratic functions in geometry, topology, and M-theory. J. Differential Geom., 70(3):329–452, 2005.

[Jak98] Martin Jakob. A bordism-type description of homology. Manuscripta Math., 96(1):67–80, 1998.

[Joa04] Michael Joachim. Higher coherences for equivariant $K$-theory. In Structured ring spectra, volume 315 of London Math. Soc. Lecture Note Ser., pages 87–114. Cambridge Univ. Press, Cambridge, 2004.

[Kar87] Max Karoubi. Homologie cyclique et $K$-théorie. Astérisque, (149):147, 1987.

[Kar90] Max Karoubi. Théorie générale des classes caractéristiques secondaires. $K$-Theory, 4(1):55–87, 1990.

[Kar94] Max Karoubi. Classes caractéristiques de fibrés feuilletés, holomorphes ou algébriques. In Proceedings of Conference on Algebraic Geometry and Ring Theory in honor of Michael Artin, Part II (Antwerp, 1992), volume 8, pages 153–211, 1994.

[KK90] Paul A. Kirk and Eric P. Klassen. Chern-Simons invariants of 3-manifolds and representation spaces of knot groups. Math. Ann., 287(2):343–367, 1990.

[Klo08] Kevin Robert Klonoff. An index theorem in differential $K$-theory. ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–The University of Texas at Austin.

[KN96] Shoshichi Kobayashi and Katsumi Nomizu. Foundations of differential geometry. Vol. I. Wiley Classics Library. John Wiley & Sons Inc., New York, 1996. Reprint of the 1963 original, A Wiley-Interscience Publication.

[Lan76] Peter S. Landweber. Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$. Amer. J. Math., 98(3):591–610, 1976.

[Lot94] John Lott. $\mathbf{R}/\mathbf{Z}$ index theory. Comm. Anal. Geom., 2(2):279–311, 1994.

[Lur] J. Lurie. Higher algebra. Bookproject, Version May18, 2011.

[Lur09] Jacob Lurie. Higher topos theory, volume 170 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009.

[MS74] John W. Milnor and James D. Stasheff. Characteristic classes. Princeton University Press, Princeton, N. J., 1974. Annals of Mathematics Studies, No. 76.
[MV99] Fabien Morel and Vladimir Voevodsky. $\mathbb{A}^1$-homotopy theory of schemes. *Inst. Hautes Études Sci. Publ. Math.*, (90):45–143 (2001), 1999.

[Ort09] M. L. Ortiz. *Differential equivariant K-theory*. PhD thesis, The University of Texas at Austin, 2009.

[SS08a] James Simons and Dennis Sullivan. Axiomatic characterization of ordinary differential cohomology. *J. Topol.*, 1(1):45–56, 2008.

[SS08b] James Simons and Dennis Sullivan. Structured bundles define differential $K$-theory. *Astérisque*, (321):1–3, 2008. Géométrie différentielle, physique mathématique, mathématiques et société. I.

[Upm11] M. Upmeier. Products in Generalized Differential Cohomology. *ArXiv e-prints*, December 2011.

[Zuc03] Roberto Zucchini. Relative topological integrals and relative Cheeger-Simons differential characters. *J. Geom. Phys.*, 46(3-4):355–393, 2003.