Branching annihilating random walk with long-range repulsion: logarithmic scaling, reentrant phase transitions, and crossover behaviors

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Abstract
We study absorbing phase transitions in the one-dimensional branching annihilating random walk with long-range repulsion. The repulsion is implemented as hopping bias in such a way that a particle is more likely to hop away from its closest particle. The bias strength due to long-range interaction has the form $\epsilon x^{-\sigma}$, where $x$ is the distance from a particle to its closest particle, $0 \leq \sigma \leq 1$, and the sign of $\epsilon$ determines whether the interaction is repulsive (positive $\epsilon$) or attractive (negative $\epsilon$). A state without particles is the absorbing state. We find a threshold $\epsilon_s$, such that the absorbing state is dynamically stable for small branching rate $q$ if $\epsilon < \epsilon_s$. The threshold differs significantly, depending on parity of the number $\ell$ of offspring. When $\epsilon > \epsilon_s$, the system with odd $\ell$ can exhibit reentrant phase transitions from the active phase with nonzero steady-state density to the absorbing phase, and back to the active phase. On the other hand, the system with even $\ell$ is in the active phase for nonzero $q$ if $\epsilon > \epsilon_s$. Still, there are reentrant phase transitions for $\ell = 2$. Unlike the case of odd $\ell$, however, the reentrant phase transitions can occur only for $\sigma = 1$ and $0 < \epsilon < \epsilon_s$. We also study the crossover behavior for $\ell = 2$ when the interaction is attractive (negative $\epsilon$), to find the crossover exponent $\phi = 1.123(13)$ for $\sigma = 0$.

Keywords Branching annihilating random walk · Long-range repulsion · Reentrant phase transition · Crossover

1 Introduction

The branching annihilating random walk (BAW) [1, 2] is a nonequilibrium reaction–diffusion system with pair annihilation, $A + A \rightarrow \emptyset$, and branching, $A \rightarrow (\ell + 1)A$. Throughout this paper, we reserve $\ell$ to denote the number of offspring. In a typical setting, particles diffuse symmetrically on a $d$-dimensional lattice and pair annihilation occurs when two particles happen to occupy the same site. The vacuum state without particles is an absorbing state in that probability current from other states to the vacuum state is nonzero, while transition from the vacuum state to any other state is prohibited.

If branching rarely occurs, then the steady-state particle density (of an infinite system) is zero, which is a key characteristic of an absorbing phase. For sufficiently large branching rate, the system can be in an active phase with nonzero steady-state particle density. The absorbing phase transition from the absorbing phase to the active phase occurs at a certain critical branching rate. In this paper, we are interested in critical behaviors of one-dimensional systems, and in the following, the dimension is always assumed 1.

The universality class to which the BAW belongs differs according to parity of $\ell$ [3–6]. The BAW with odd $\ell$ belongs to the directed percolation (DP) universality class, as is consistent with the DP conjecture [7, 8]. The BAW with even $\ell$ can be mapped to a kinetic Ising model with two symmetric absorbing states [9] (see Sect. 5.1 for a mapping) and belongs to the directed Ising (DI) universality class [10]. For a review of these two universality classes, see, e.g., Refs. [11–13]. In the literature, the DI class is also called the parity-conserving universality class [14–16], DP2 class [11], or the generalized voter class [17], depending on which property of this universality class is emphasized.

Recently, a modified version of the BAW was suggested by introducing hopping bias in such a way that a particle is attracted by its closest particle [18, 19]. The strength
of the bias depends on the distance $x$ between a particle and its closest particle in a power-law fashion $x^{-\sigma}$. For odd $\ell$, this bias does not alter the critical behavior [18]. On the other hand, for even $\ell$, the modified model in one dimension does not belong to the DI class if $\sigma < 1$. Rather, the critical exponents vary with $\sigma$ [19]. In Ref. [20], the different critical behavior for $\sigma < 1$ from the DI class is attributed to the long-range nature of the bias, by studying the crossover that occurs for large but finite range of attraction. In this paper, we further generalize the BAW by allowing the long-range interaction to be repulsive and study its critical phenomena.

The structure of this paper is as follows. The definition of the generalized model will be provided in Sect. 2. If particles repel each other, then pair annihilation becomes inefficient to remove particles [21]. Accordingly, one may anticipate that the model with repulsion is always in the active phase for nonzero branching rate. This anticipation turns out to be right if $\ell$ is even. However, a careful analysis to be presented in Sect. 3 shows that the critical branching rate can be nonzero for odd $\ell$. Rather unexpectedly, the model exhibits intriguing reentrant phase transitions, which will be analyzed in Sect. 4 for $\ell = 1$ and in Sect. 5 for $\ell = 2$. The results are summarized in Sect. 6.

## 2 Model

This section defines the model to be studied and summarizes established results that are relevant to later discussion. We consider a one-dimensional lattice of size $L$ with periodic boundary conditions. Each site is either occupied by at most one particle or vacant. A particle branches with rate $q$ or hops to one of its nearest neighbors with rate $1-q$ ($0 \leq q \leq 1$). When a particle at site $i$ branches, its $\ell$ offspring are placed at consecutive sites $i + \xi, i + 2\xi, \ldots, i + \ell \xi$, where $\xi$ is a random variable that takes 1 or $\ell$ with equal probability. When a particle at site $i$ hops, it moves to site $i \pm 1$ with probability $(1 + B_i)/2$, where

$$B_i = \epsilon \text{sgn} (R_i - L_i) \min \{R_i, L_i\} + \mu)^{-\sigma}.$$  

(1)

In the above equation, $\text{sgn} (x)$ is the sign of $x$ with $\text{sgn} (0) = 0$, $\sigma$ is nonnegative, and $\epsilon, \mu$ are constants with restriction $|\epsilon| < (1 + \mu)^\sigma$ to ensure $-1 < B_i < 1$. If two particles happen to occupy a same site by any transition event, then pair annihilation $(A + A \rightarrow \emptyset)$ of the two particles occurs immediately.

The stochastic dynamics of the model can be represented as

$$1_i \prod_{k=1}^{\ell} c_{i+k} \rightarrow 1_i \prod_{k=1}^{\ell} \bar{c}_{i+k} \text{ with rate } \frac{q}{2},$$

$$1_i \prod_{k=1}^{\ell} c_{i-k} \rightarrow 1_i \prod_{k=1}^{\ell} \bar{c}_{i-k} \text{ with rate } \frac{q}{2},$$

(2)

$$1_i c_{i+1} \rightarrow 0_i \bar{c}_{i+1} \text{ with rate } (1 - q) \frac{1 + B_i}{2},$$

$$1_i c_{i-1} \rightarrow 0_i \bar{c}_{i-1} \text{ with rate } (1 - q) \frac{1 - B_i}{2},$$

where $c_j$ stands for the occupation number at site $j$ with $\bar{c}_j \equiv 1 - c_j$ and $1_i(0_i)$ means that site $i$ is occupied (vacant). Monte Carlo simulations for the rule (2) were performed in the following way. Assume that there are $M$ particles at time $t$. Choose one among $M$ particles at random with equal probability. Assume that a particle at site $i$ is chosen. After the choice, we generate a random number $\theta$ that is uniformly distributed in $0 < \theta < 1$. If $\theta < q/2$, then $\ell$ offspring are placed on the right-hand side of $i$. Else if $\theta < q$, then $\ell$ offspring are placed on the left-hand side of $i$. Else if $\theta < q + (1 - q)(1 + B_i)/2$, then the chosen particle hops to the right. Otherwise, the chosen particle hops to the left. Pair annihilation occurs if applicable. After the change of a configuration, time increases by $1/M$.

The model will be called the BAW with long-range interaction (BAWL). For positive (negative) $\epsilon$, a particle is in a sense repulsed (attracted) by its closest particle located at $\min\{R_i, L_i\}$. Thus, we will refer to the model with positive (negative) $\epsilon$ as the BAW with long-range repulsion (attraction) to be abbreviated as BAWLR (BAWLA).

We define the particle density $\rho(t)$ as

$$\rho(t) = \lim_{L \rightarrow \infty} \frac{1}{L} \sum_{i=1}^{L} \langle c_i \rangle,$$

(3)

where $c_i$ is the occupation number at site $i$ at time $t$ and $\langle \cdots \rangle$ stands for average over ensemble. In actual simulations, the system size $L$ is so large that a finite-size effect is negligible up to the observation time and the system evolves from the fully-occupied initial condition with $\rho(0) = 1$.

The BAWL with $q = 0$ is the annihilating random walk with long-range interaction (AWL), extensively studied in Ref. [21] (see also Refs. [19, 22] for the AWL with attractive interaction). We summarize some results of Ref. [21] that are relevant to later discussion. For the two-particle initial condition in which there are only two particles in a row at $t = 0$, the probability $P_s$ that the two particles are never annihilated is

$$P_s^{-1} = 1 + \sum_{i=1}^{\infty} \prod_{k=1}^{i} \frac{1 - \epsilon(k + \mu)^{-\sigma}}{1 + \epsilon(k + \mu)^{-\sigma}}.$$  

(4)

For $\sigma = 0$ and $\sigma = 1$, $P_s$ becomes
where \( \Theta(\cdot) \) is the Heaviside step function. For \( 0 < \sigma < 1 \), \( P_s \) is nonzero for \( \varepsilon > 0 \) and 0 for \( \varepsilon \leq 0 \). When necessary, one can numerically calculate \( P_s \) for \( 0 < \sigma < 1 \) using Eq. (4).

When there are only three particles at sites \( i - r, i, i + r \) and particles at \( i \pm r \) are assumed immobile, the time \( T_0(\varepsilon) \) is defined as the rate of the density decrease per particle at time \( t \) as

\[
T_0 \sim \begin{cases} \frac{r^\sigma \exp (C_s r^{1-\sigma})}{(1+\varepsilon)^2}, & \sigma < 1, \\ \frac{r^2 \ln r}{(1+\varepsilon)^2}, & \sigma = 1 \text{ and } 2\varepsilon > 1, \\ \frac{r^2}{(1+\varepsilon)^2}, & \sigma = 1 \text{ and } 2\varepsilon = 1, \\ \frac{r^2}{(1+\varepsilon)^2}, & \text{otherwise.} \end{cases}
\]  

(7)

for positive \( \varepsilon \) and

\[
T_0 \sim \begin{cases} \frac{r^\sigma}{(1+\varepsilon)^2}, & \sigma \leq 1, \\ \frac{r^2}{(1+\varepsilon)^2}, & \sigma > 1, \end{cases}
\]  

(9)

for negative \( \varepsilon \).

Interpreting \( 2/T_0 \) with \( r = 1/\rho(t) \) as the rate of the density decrease per particle at time \( t \), it was found [21] that

\[
\rho(t) \sim \begin{cases} (\ln t)^{-1/(1-\sigma)}, & \sigma < 1 \text{ and } \varepsilon > 0, \\ r^{-1/(1+2\sigma)}, & \sigma = 1 \text{ and } 2\varepsilon > 1, \\ r^{-1/2}, & \text{otherwise,} \end{cases}
\]  

(10)

for \( \varepsilon > 0 \) and

\[
\rho(t) \sim \begin{cases} r^{-1/(1+\sigma)}, & \sigma \leq 1, \\ r^{-1/2}, & \sigma \geq 1, \end{cases}
\]  

(11)

for \( \varepsilon < 0 \).

We would like to comment on a recent numerical work [23] of the AWL for \( \sigma = 0 \) and positive \( \varepsilon \), which claims that \( \rho(t) \) behaves as \( (\ln t)^{-b} \) with \( b \) to vary continuously, depending on \( \varepsilon \). We think, however, the neglect of \( \ln \ln r \) correction [21] erroneously gives this conclusion.

### 3 Stability of the absorbing state

Since \( P_s \) for \( \sigma < 1 \) and \( \varepsilon > 0 \) is positive as seen in Sect. 2, an offspring has a chance to avoid being annihilated with its parent, which might lead us to claim that the critical point of the BAWLR with \( \sigma < 1 \) is zero and the system is in the active phase as long as \( q \) is nonzero. To check the validity of this claim, we investigate the stability of the absorbing state for small \( q \).

To this end, we consider an initial condition that each site is occupied with probability \( \rho_0 \). Assuming both \( \rho_0 \) and \( q \) are small \((0 < \rho_0 \ll 1, q \ll 1)\), we can treat any particle-number changing event (branching or pair annihilation) as independent, because a typical distance between sites at which two consecutive such events in time domain occur should be large in space. We first consider the case with \( \varepsilon = 1 \) and general cases will be discussed later.

The rate of density decrease due to pair annihilation is roughly \( 2/T_0(\varepsilon) \) with \( r = 1/\rho(t) \) [21]. To estimate the rate of density increase by branching, we assume that a particle at site \( i \), to be called the parent, begets its offspring at \( t = 0 \). With probability \( P_s \), the offspring will escape from the parent and the number of particles increases by 1, while with probability \( 1 - P_s \), the parent and the offspring undergo pair annihilation, which makes the number of particles decrease by 1. Hence, the rate of density increase by branching is \( q[P_s - (1 - P_s)] = q(2P_s - 1) \). The independence argument finally gives

\[
\frac{d \ln \rho}{dr} \approx q(2P_s - 1) - \frac{2}{T_0(1/\rho)}.
\]  

(12)

If \( P_s < 1/2 \), then the branching triggers even faster (exponential) density decay, which makes the absorbing state stable. On the other hand, if \( P_s > 1/2 \), then the absorbing state is unstable, because \( 1/T_0 \to 0 \) as \( \rho_0 \to 0 \); see Eq. (7). Hence, the condition for the absorbing state to be stable is \( P_s < 1/2 \). As we will see soon, the condition will differ significantly for \( \varepsilon = 2 \).

Unlike the claim at the beginning of this section, there should be a positive threshold \( \varepsilon_s \), determined by \( P_s(\varepsilon_s) = \frac{1}{2} \), such that for \( 0 < \varepsilon < \varepsilon_s \), the steady-state density for small \( q \) is zero and, in turn, the critical point \( q_c \) should be nonzero. From Eq. (4), \( \varepsilon_s \) is found as

\[
\varepsilon_s = \begin{cases} 1/3, & \sigma = 0, \\ 0.438213752, & \sigma = 0.5, \mu = 1, \\ (\mu + 2)/3, & \sigma = 1, \end{cases}
\]  

(13)

where the numerical value of \( \varepsilon_s \) for \( \sigma = 0.5 \) is presented for later purpose.

When \( P_s < 1/2 \), Eq. (12) suggests that the density decreases exponentially for small but nonzero \( q_c \), which is a typical feature of the absorbing phase of systems belonging to the DP class. This is usually interpreted as the generation of spontaneous annihilation by the chain of reaction \( A \to 2A \to 0 \). Hence, we anticipate that the BAWL with \( \varepsilon = 1 \) belongs to the DP class if \( P_s < 1/2 \).
To confirm this scenario, we performed simulations for \( \sigma = 0.8, \epsilon = 0.5, \) and \( \mu = 0 \) with \( L = 2^2 \). \( P_s \) for this case is about 0.445. If this case does indeed belong to the DP class, then the density at the critical point should decay as

\[
\rho(t) \sim t^{-\delta_{DP}}, \tag{14}
\]

with the critical decay exponent \( \delta_{DP} = 0.15946 \) [24] of the DP class. In Fig. 1, we depict \( \rho(t) \) versus \( t \) on a semilogarithmic scale for \( q = 0.7025, 0.7026, \) and 0.7027. To reduce statistical error, we averaged over 40 independent runs for each parameter set. The curve for \( q = 0.7025 \) (0.7027) eventually veers down (up), while the curve for \( q = 0.7026 \) remains flat for long time. Therefore, we conclude that the critical point is \( q_c = 0.7026(1) \), where the number in parentheses indicates the uncertainty of the last digit, and this case belongs to the DP class as anticipated.

Since the absorbing state is unstable if \( P_s > 1/2 \), the steady state should have nonzero density for small but nonzero \( q \). Setting the time derivative of \( \rho \) in Eq. (12) to be 0, the steady-state density \( \rho_s \) can be estimated by

\[
T_0(1/P_s) \sim 1/q, \tag{15}
\]

which, along with Eq. (7), gives

\[
\rho_s \sim \begin{cases} 
(-\ln q)^{-1/(1-\sigma)}, & \sigma < 1, \\
q^{1/(1+2\epsilon)}, & \sigma = 1,
\end{cases} \tag{16}
\]

for small \( q \) and \( \epsilon > \epsilon_c \). Hence, there is a continuous transition with \( q_c = 0 \).

Now, we make an ansatz for a scaling function that describes the absorbing phase transition with \( q_c = 0 \). From the above discussion, \( 1/q \) should be a characteristic time scale of the system. For \( \sigma = 1 \), the density would be described by a scaling function of the form

\[
\rho(t;q) = q^{1/(1+2\epsilon)}G_1(qt), \tag{17}
\]

To ensure that \( \rho(t;q) \) has the desired behavior as \( t \to \infty \), the scaling function \( G_1 \) should have the following asymptotic behavior: \( G_1(x) \sim x^{-1/(1-\sigma)} \) for small \( x \) and \( G_1(x) \sim x^{-1/(1+2\epsilon)} \) as \( x \to 0 \) (actually, the limit \( x \to 0 \) should be understood as a joint limit \( t \to \infty \) and \( qt \to 0 \)).

For \( \sigma < 1 \), the logarithmic behavior should be incorporated properly into the scaling function. Since the characteristic time scale is \( 1/q \) and there is a logarithmic density decay at \( q = 0 \), a proper scaling parameter should be \(-\ln t/\ln q \). Thus, we write

\[
\rho(t;q) = (-\ln q)^{-1/(1-\sigma)}G_\sigma(-\ln t/\ln q). \tag{18}
\]

for \( \sigma < 1 \). The desired asymptotic behaviors for large \( t \) are reproduced if \( G_\sigma(x) \sim x^{-1/(1-\sigma)} \) for small \( x \) and \( G_\sigma(x) \sim 1 \) for large \( x \). Notice that if we choose \( qt \) as a scaling parameter as in Eq. (17), we cannot reproduce the desired asymptotic behavior in the small \( qt \) limit.

Before confirming the validity of the scaling ansatz by numerical simulations, we will discuss the stability of the absorbing state for \( \ell > 1 \).

Right after a particle branches, it and its offspring form a cluster of size \( \ell + 1 \). Particles in the middle of the cluster are likely to be annihilated in short time. If \( \ell \) is odd, only two particles in the cluster are likely to remain in short time and the fate of the cluster is almost identical to the case of \( \ell = 1 \). Hence, the scenario for \( \ell = 1 \) is applicable to the BAWL with odd \( \ell \), though the value of \( \epsilon_c \) can differ depending on \( \ell \).

If \( \ell \) is even, then the size of the cluster shrinks to 3 in short time. By \( \bar{P}_s \), we denote the probability that no pair annihilation happens in the AWL if the system evolves from a configuration only with three particles in a row. That is, with probability \( \bar{P}_s \), all the three particles survive, and with probability \( 1 - \bar{P}_s \), only one particle survives. Since at least one particle always survives, the rate of the density increase by branching is \( 2\bar{P}_s \), which gives

\[
\frac{d\ln \rho}{dt} \approx 2\bar{P}_s - \frac{2}{T_0}. \tag{19}
\]

Now, we discuss the relation between \( P_s \) and \( \bar{P}_s \). Since the particle in the middle is more likely to be annihilated than particles in the two-particle initial condition, \( \bar{P}_s \) cannot be larger than \( P_s \). In the mean time, if \( P_s > 0 \), the three particles should survive with nonzero probability once they happen to be separated by a large distance. Since

![Fig. 1](image-url) Semilogarithmic plots of \( \rho(t) \) vs. \( t \) for \( \sigma = 0.8, \epsilon = 0.5, \mu = 0, \) and \( \ell = 1 \) with \( q = 0.7025, 0.7026, \) and 0.7027 (bottom to top), where \( \delta_{DP} \approx 0.15946 \) is the critical decay exponent of the DP class. The curve for \( q = 0.7026 \) is flat for more than 2 logarithmic decades, while the other curves eventually veer up (0.7027) or down (0.7025), indicating that the transition point is \( q_c = 0.7026(1) \) and this case belongs to the DP class.
probability of being separated by a large distance cannot be zero as far as the distance is finite, \( P_s > 0 \) should entail \( \tilde{P}_s > 0 \). Therefore, \( \tilde{P}_s \) is positive if and only if \( P_s \) is positive. Accordingly, if \( P_s \) is positive and \( \epsilon \) is even, then the absorbing state is unstable and the scaling functions (17) and (18) are expected to describe the critical phenomena. If \( P_s = \tilde{P}_s = 0 \) and \( q \) is small, then Eq. (19) shows that only pair annihilation governs the long-time behavior of the particle density.

To confirm the scaling ansatz as well as the scenario for even \( \epsilon \), we performed Monte Carlo simulations for two cases, \( \epsilon = 1 \) and \( \epsilon = 2 \). First, we present simulations results for \( \sigma = 1 \) and \( \mu = 0 \). Figure 2 depicts data-collapse plots using Eq. (17) for \( \epsilon = 0.6 \) (\( \epsilon = 2 \)) and \( \epsilon = 0.8 \) (\( \epsilon = 1 \) and \( \epsilon = 2 \)). Notice that \( \epsilon \), for \( \epsilon = 1 \) and \( \mu = 0 \) is \( \frac{2}{3} \approx 0.67 \). The system size in the simulations is \( L = 2^{20} \) and number of independent runs ranges from 8 to 40. The collapse is almost perfect, which confirms the scaling ansatz (17).

Next, we present simulation results for \( \sigma = 0 \) (\( \epsilon = 1 \)) and \( \sigma = 0.5 \) (\( \epsilon = 2 \)). For both cases, we set \( \mu = 0 \) and \( \epsilon = 0.5 \). The values of \( q \) are \( 10^{-8}, 10^{-7} \), and \( 10^{-6} \). The system sizes in the simulations are \( 2^{14} \) and \( 2^{15} \) for \( \sigma = 0 \) and \( \sigma = 0.5 \), respectively. The number of independent runs is 8 for all cases. Figure 3 depicts data-collapse plots according to the scaling ansatz (18), which shows a nice agreement.

Now, we discuss absorbing phase transitions of the BAWLR with \( \epsilon = 2 \) for \( \sigma = 1 \) and \( 0 < \epsilon < 0.5 \). In this regime, \( P_s = 0 \) and the density decays as \( t^{-1/2} \) for small \( q \); see Eq. (19) along with Eq. (10). Hence, it can be anticipated that there should be a phase transition at nonzero \( q_c \), and this case should belong to the DI class.

![Fig. 2 Data collapse plots of \( q^{-1/(1+2\epsilon)} \rho \) vs. \( qt \) for \( \epsilon = 0.6 \) top, \( \epsilon = 0.8 \) middle, and \( \epsilon = 0.8 \) bottom](image)

![Fig. 4 Semilogarithmic plots of \( \rho^{\text{BAW}} \) vs. \( t \) for \( q = 0.165 \), 0.167, and 0.169](image)
that the critical point is \( q_c = 0.167(2) \) and, as expected, this case belongs to the DI class.

4 Single offspring

In this section, we continue studying absorbing phase transitions of the BAWLR with \( \epsilon = 1 \). When it comes to the universal behavior, nothing new beyond the result in Sect. 3 is expected. Rather, we are interested in the behavior of the phase boundary as \( \epsilon \) approaches \( \epsilon_c \).

To this end, we performed Monte Carlo simulations for \( \sigma \leq 1 \). Differently from the numerical studies in Sect. 3, \( q \) is now fixed and \( \epsilon \) is a tuning parameter. The critical point will be denoted by \( \epsilon_c \). Since the model belongs to the DP class, the critical point was found using the method as in Fig. 1 (details not shown here).

We simulated the BAWLR for three sets of parameters, \( \{ \sigma = 0, \mu = 0 \} \), \( \{ \sigma = 0.5, \mu = 1 \} \), and \( \{ \sigma = 1, \mu = 1 \} \). The threshold values for these parameter sets are given in Eq. (13). The critical points are tabulated in Table 1. It is intriguing that \( \epsilon_c \) approaches \( \epsilon_s \) from above. Since \( \epsilon_c \) becomes smaller than \( \epsilon_s \), the variability of the last digits of the BAWLR with phase boundaries.

To illustrate the reentrant phase transitions, we depict phase boundaries (\( \epsilon_c \) versus \( q \)) in Fig. 5a. There are two salient features of the phase boundaries. First, the reentrant phase transitions indeed occur for \( \epsilon_s < \epsilon < \epsilon_b \). We roughly estimate \( \epsilon_b \) as 0.521, 0.605, and 1.26, for \( \sigma = 0, 0.5 \), and 1, respectively, from Table 1. Second, the behavior of the phase boundary for small \( q \) varies with \( \sigma \). To be concrete, we analyzed how \( \epsilon_c \) approaches \( \epsilon_s \) as \( q \to 0 \). In Fig. 5b, \( \epsilon_c - \epsilon_s \) is drawn against \( q \) on a double logarithmic scale, along with power-law fitting results for small \( q \).

\[
\epsilon_c - \epsilon_s = c q^\phi
\]

with \( \phi = 1.15, 0.66, \) and 0.18 for \( \sigma = 0, 0.5 \), and 1, respectively. This power-law behavior is reminiscent of a crossover behavior.

Now, we provide a hand-waving argument as to why there are reentrant phase transitions. Let us revisit the initial condition with low density as in Sect. 3. Assume that a particle at site \( i \) branches its offspring at site \( i + 1 \) at time \( t = 0 \). If these two particles survive, then next branching occurs approximately at time \( 1/q \). We assume that the parent is at site \( j \) and it branches a second offspring at site \( j + 1 \). Since the mean distance of the surviving samples with the two-particle initial condition increases as \( t^{1/(1+\sigma)} \) [21], the average distance between the parent and its first offspring at the time of the second birth is \( d \approx q^{-1/(1+\sigma)} \).

Now, there are three particles at site \( j \), \( j + 1 \), and \( j + d \).

Let \( P_m(d) \) be the probability that three particles survive forever if the system (with \( q = 0 \)) evolves from the initial condition with only three particles at sites \( j \), \( j + 1 \), and \( j + d \). Obviously, \( P_m(d) \) should be an increasing function of \( d \) with \( P_m(d) \to P_c \) as \( d \to \infty \). Accordingly, it is

| \( q \) | \( \epsilon_c \) | \( \sigma = 0 \) | \( \sigma = 0.5 \) | \( \sigma = 1 \) |
|------|-----------------|-----------------|-----------------|-----------------|
| 0.0001 | 0.3335(1) | 0.4447(1) | 1.0845(5) |
| 0.0003 | 0.3339(1) | 0.4528(5) | 1.0135(2) |
| 0.001 | 0.3356(1) | 0.4697(2) | 1.1285(1) |
| 0.003 | 0.3419(1) | 0.4933(1) | 1.1553(1) |
| 0.01 | 0.3634(2) | 0.5262(3) | 1.1883(1) |
| 0.03 | 0.4030(3) | 0.5592(2) | 1.2199(1) |
| 0.1 | 0.4639(1) | 0.5923(1) | 1.2498(1) |
| 0.2 | 0.5000(5) | 0.6044(1) | 1.2559(1) |
| 0.3 | 0.5163(5) | 0.6044(3) | 1.2465(5) |
| 0.4 | 0.5207(1) | 0.5960(5) | 1.2228(1) |
| 0.5 | 0.5136(5) | 0.5780(1) | 1.1814(1) |
| 0.6 | 0.4911(1) | 0.5450(1) | 1.1109(1) |
| 0.7 | 0.4410(1) | 0.4838(1) | 0.9838(1) |
| 0.8 | 0.3251(1) | 0.3531(1) | 0.7165(5) |
| 0.9 | 0.0496(1) | 0.0533(1) | 0.1081(1) |

Numbers in parentheses indicate uncertainty of the last digits.
not impossible for \( P_m(d) \) to be smaller than \( \frac{1}{2} \) even if \( P_s \) is larger than \( \frac{1}{2} \).

The rate \( w \) of particle-number change due to the two branching events is

\[
w = -q(1 - P_s) + qP_m\left[1 - q(1 - P_m) + qP_m\right]
= q(2P_s - 1) - q^2P_s(1 - 2P_m),
\]

which modifies Eq. (12) as

\[
\frac{d\ln \rho}{dt} \approx q(2P_s - 1) - q^2P_s(1 - 2P_m) - \frac{2}{F_0}.
\]

If \( P_s > 1/2 \) and \( q \) is extremely small but nonzero, the absorbing state is unstable as we have seen. Since \( P_m \) is a decreasing function of \( q \) (recall that \( d \) is a decreasing function of \( q \)), \( P_m \) may be smaller than \( \frac{1}{2} \) from certain \( q \) and the stability of the absorbing state can change at, say, \( q_1 \), which can be found approximately as a solution of

\[
q(1 - 2P_m) = 2P_s - 1,
\]

where \( P_m \) should be regarded as a function of \( q \). Since the system should be in the active phase if \( q \) is close to 1, the stability of the absorbing state changes again at, say, \( q_2 \) \((q_2 > q_1)\), which explains the existence of reentrant phase transitions.

In mathematics literature, a model is called attractive if, roughly speaking, more particles induce longer life time (and/or larger chance of survival) of the system. A typical example of an attractive model is the contact process [26, 27]. The BAW is not attractive [1] and neither is the BAWL. Thus, larger branching rate does not necessarily imply more chance of survival. In the BAWL, we have shown that larger branching rate sometimes makes the system fall into an absorbing state. To our knowledge, the BAWL is the first model that exhibits a reentrant phase transition due to the lack of attractiveness. It is worth commenting that the reentrant phase transition observed in Ref. [28] should be attributed to the absence of the active phase in the branching annihilating process [2], because the reentrant phase transition disappears when dynamic branching is employed [28].

5 Two offspring

In this section, \( c \) is always set to be 2 and we do not explicitly mention the value of \( c \) when we refer to the model. If \( \sigma < 1 \) and \( \epsilon > 0 \) or if \( \sigma = 1 \) and \( 2\epsilon > 1 \), that is, if \( P_s > 0 \), then the absorbing state is dynamically unstable and the critical phenomena are completely described by the scaling functions (17) and (18). Hence, this section is interested in phase transitions in the cases with \( P_s = 0 \).

For \( \sigma < 1 \), the condition \( P_s = 0 \) is applicable only to the BAWLA. Recall that the BAWLA with nonzero \( \epsilon \) does not belong to the DI class [18, 19]. Thus, there should be a crossover behavior for small \(|\epsilon|\), which is the topic of Sect. 5.1.

For \( \sigma = 1 \), \( P_s \) can be zero even if \( \epsilon > 0 \) and this case, as shown in Sect. 3, belongs to the DI class. Quite unexpectedly, another reentrant phase transitions occur in a certain region of \( \epsilon \). These reentrant phase transitions will be studied in Sect. 5.2.

5.1 Crossover in the BAWLA with \( \sigma = 0 \)

The crossover behavior is described by a scaling function

\[
\rho(t; q, \epsilon) = t^{-\delta_{DI}}R\left[t|\epsilon|^{\nu_1}/\phi, (q - q_0)t^{1/\nu_1}\right],
\]

where \( q_0 \) is the critical point for \( \epsilon = 0 \); \( \delta_{DI} \) and \( \nu_1 \) are the critical exponent of the DI class; and \( \phi \) is the crossover exponent. In a recent study [29], \( \nu_1 \) is found to be 3.55 and we will use this value for a data-collapse plot.

To find the crossover exponent, we analyze how the phase boundary behaves when \(|\epsilon|\) is small. If we denote the critical point for negative \( \epsilon \) by \( q_\epsilon \), Eq. (24) gives

\[
|q_\epsilon - q_0| \sim |\epsilon|^{1/\phi},
\]

which will be used to find the crossover exponent \( \phi \).

We first find \( q_0 \) for \( \epsilon = 0 \). In the literature, the value of \( q_0 \) is available (for example, see Refs. [28, 30], but we will provide more accurate value in this paper. To compare our estimate to the literature, one should notice that \( p \) is usually used to denote the hopping probability, which is 1 - \( q \) in this paper.

To find \( q_0 \), we performed simulations with the two-particle initial condition. At the critical point, the number \( N(t) \) of particles averaged over all ensemble behaves for large \( t \) as

\[
N(t) \sim t^{\eta}.
\]

Since \( \eta \approx 0 \) for the DI class [25], \( N(t) \) drawn against \( t \) should veer up (down) if the system is in the active (absorbing) phase. Thus, we will find \( q_0 \) by plotting \( N \) as a function of \( t \) on a semilogarithmic scale around the critical point.

In Fig. 6a, we present the simulation results. The number of independent runs for each parameter is \( 5 \times 10^4 \). Since the graph for \( q = 0.489 \) 532 (0.489 53) veers up (down) in the long-time limit, we conclude that \( q_0 = 0.489 \) 531(1).

Note that this estimate is more accurate than previous estimates in the literature (for example, in Ref. [30], the critical point was estimated as \( 1 - p_c = 0.489 \) 65(5), but this estimate was based on somewhat wrong value of the critical decay exponent).
When $\varepsilon < 0$ and $\sigma = 0$, the BAWLA does not belong to the DI class \[18, 19\]. For example, the critical decay exponent for this case is $\delta_{\text{DI}} = 0.2872$. Let us first check whether the BAWLA with $\sigma = 0$ indeed forms a universality class. In Fig. 6b, we present simulation results for $\varepsilon = -0.16$ at $q = 0.61225, 0.61227, 0.61229$. The system size in the simulation is $L = 2^{21}$ and the numbers of independent runs are 160, 320, and 160 for $q = 0.61225, 0.61227, 0.61229$, respectively. We depict $\rho t^\nu$ as a function of $t$. At $q = 0.61227$, the curve is almost flat for more than two logarithmic decades, which gives $q_c = 0.61227(2)$ and confirms that the critical behavior of the BAWLA with $\sigma = 0$ is indeed universal.

We found critical points by extensive Monte Carlo simulations for various $\varepsilon$'s, which are tabulated in Table 2. In Fig. 7a, we plot $q_c - q_0$ vs. $|\varepsilon|$ for the BAWLA with $\sigma = 0$ and $\varepsilon = 2$ on a double logarithmic scale. The sizes of error bars are smaller than the symbol size. The straight line with slope 0.89 shows the result of power-law fitting. b Scaling-collapse plot of $\rho t^\nu$ vs. $t|\varepsilon|^{\nu/\phi}$ at $q = 0.612531$ for $\varepsilon = -5 \times 10^{-4}, -10^{-3}, -2 \times 10^{-3},$ and $-4 \times 10^{-3}$ on a double logarithmic scale. All data collapse nicely onto a single curve for large $t$.

Table 2 Critical points $q_c$ of the BAWLA with $\varepsilon = 2$ for $\sigma = 0$

| $\varepsilon$  | $q_c$         |
|---------------|--------------|
| 0             | 0.489531(1)  |
| $-0.0025$     | 0.4936(2)    |
| $-0.005$      | 0.49715(15)  |
| $-0.01$       | 0.5036(1)    |
| $-0.02$       | 0.5151(1)    |
| $-0.04$       | 0.5346(1)    |
| $-0.08$       | 0.5658(1)    |
| $-0.16$       | 0.61227(2)   |

Numbers in parentheses indicate uncertainty of the last digits.
Table 3  A mapping of the BAW ($\epsilon = 2$) to a nonequilibrium kinetic Ising model

| BAW       | NEKIM   | Transition rate |
|-----------|---------|----------------|
| $A_+ A_+ \Rightarrow \emptyset A_+$ | ↑↓↑⇒↑↑↓ | $D_1$          |
| $\emptyset A_+ \Rightarrow A_+ \emptyset$ | ↑↑↑⇒↑↑↑ | $D_1$          |
| $A_- A_0 \Rightarrow \emptyset A_-$ | ↑↑↑⇒↑↑↑ | $D_2$          |
| $A_+ A_0 \Rightarrow \emptyset A_-$ | ↑↑↑⇒↑↑↑ | $D_2$          |
| $A_+ A_- \Rightarrow \emptyset \emptyset$ | ↑↑↑⇒↑↑↑ | $2D_1$         |
| $A_- A_+ \Rightarrow \emptyset \emptyset$ | ↑↑↑⇒↑↑↑ | $2D_2$         |
| $A_\pm A_\mp \Rightarrow A_\pm A_\mp$ | ↑↓↑⇒↑↓↓ | $\lambda$      |
| $A_\pm A_\pm \Rightarrow A_\pm A_\pm$ | ↑↑↑⇒↑↑↑ | $\lambda$      |
| $A_+ A_\pm \Rightarrow \emptyset A_\pm$ | ↑↑↑⇒↑↑↑ | $\lambda$      |
| $\emptyset A_\pm \Rightarrow A_\pm A_\pm$ | ↑↑↑⇒↑↑↑ | $\lambda$      |
| $A_\mp A_\pm \Rightarrow A_\pm A_\mp$ | ↑↑↑⇒↑↑↑ | $\lambda$      |
| $\emptyset A_\mp \Rightarrow A_\pm A_\pm$ | ↑↑↑⇒↑↑↑ | $\lambda$      |
| $A_\mp A_\mp \Rightarrow A_\mp A_\mp$ | ↑↑↑⇒↑↑↑ | $\lambda$      |

$\dagger$ and $\ddagger$ stand for Ising spins with spin 1 and −1, respectively. $A_+$ and $A_-$ are interpreted as domain walls $\dagger\ddagger$ and $\ddagger\dagger$, respectively. In branching events, a parent is indicated by a bold face symbol. If $D_1 = D_2$, $A_+$ and $A_-$ are dynamically indistinguishable.

preserved by dynamics. Note that the BAW in this paper corresponds to the rate $D_1 = D_2 = p/2$ and $\lambda = (1 - p)/2$.

In the NEKIM, there are two absorbing states with all spins either up or down. If $D_1 > D_2$, then hopping of $A_+$ ($A_-$) particle is more likely to hop to the right (left), which effectively enlarges a domain of upspins in the NEKIM. This bias can be interpreted as an effect of external magnetic field coupled to Ising spins, which breaks the symmetry between the two absorbing states. In the presence of the symmetry breaking field, the model now belongs to the DP class and its crossover exponent is $1/\phi \approx 0.47$ or $\phi \approx 2.1$ [30–33].

Since the symmetry between two absorbing states (when interpreted in terms of a equivalent NEKIM) is not broken by long-range attraction, the bias due to a symmetry breaking field should be regarded as a different perturbation from that due to long-range interaction. This can be clearly explained by the different values of the corresponding crossover exponents, $1/\phi = 0.89$ for the long-range attraction and $1/\phi = 0.47$ for the symmetry breaking field. Besides, changing the direction of the external magnetic field ($h \rightarrow -h$) does not induce new phenomena, because the dynamics are invariant under the simultaneous transformations $h \rightarrow -h$ and $A_+ \rightarrow A_-$. This should be contrasted with sign change of $\epsilon$ in the BAWL, which has a drastic effect as we observed.

5.2  Reentrant phase transition for $\sigma = 1$

Now, we move on to the BAWLR with $\sigma = 1$ and $\epsilon < 0.5$. As shown in Fig. 4, this case belongs to the DI class. An intriguing phenomenon was observed as $q$ varies for $\epsilon = 0.35$ and $\mu = 0.2$. In Fig. 8, we depict the behavior of the density for $q = 10^{-4}$, $10^{-2}$, $0.1$, and $0.2$. Since we have found $q_c \approx 0.165$ in Fig. 4, it is natural to expect that the density decays as $t^{-0.5}$ for $q = 0.1$ and saturates for $q = 0.2$, which is indeed the case. A rather surprising result is the behavior for $q = 10^{-2}$, which has nonzero steady-state density. Since the absorbing state is stable for very small $q$ as the curve for $q = 10^{-4}$ in Fig. 8 shows, there should be different transitions points in three regions, $0 < q < 0.01$, $0.01 < q < 0.1$, and $0.1 < q < 0.2$. We did not dare to find the phase boundary with accuracy. Our preliminary simulations showed that the reentrant phase transitions occur for some narrow region of $\epsilon$, which is roughly $0.34 < \epsilon < 0.36$.

We think the reentrant phase transitions should be accounted for by the nonattractiveness of the BAWLR as in Sect. 4. For extremely small $q$, the discussion in Sect. 3 shows that the absorbing state is stable. As $q$ increases, the steady-state density starts to be nonzero from a certain $q_1$. And then competition between the nonattractiveness and the branching results in a reentrant phase transitions in the region $q > q_1$. However, we do not have a good quantitative argument about the region of $\epsilon$, where the reentrant phase transitions occur. We would like to leave this intriguing question to later research.

6 Summary

In this paper, we have studied absorbing phase transitions in the branching annihilating random walk with long-range interaction with main focus on the effect of repulsion. The results in Ref. [21], especially $P_\sigma$, are importantly relied on to understand the behavior of the BAWL.
When the number $\ell$ of offspring per branching is 1, the condition that the absorbing state is stable against small perturbation of branching is found to be $P_s \leq \frac{1}{2}$. Even though $P_s > \frac{1}{2}$, there can be reentrant phase transitions from the active phase, to the absorbing phase, and back to the active phase as the branching rate $q$ increases from zero. This reentrant phase transitions are explained, based on the nonattractiveness [1] of the BAWLR. The critical phenomena at nontrivial transition point are described by the DP scaling. We also argued that a similar conclusion should be arrived at for any odd number of $\ell$.

When $\ell = 2$, unlike the case with $\ell = 1$, there is only the active phase for $q > 0$ if $P_s > 0$. Only when $P_s = 0$, the transition point $q_c$ can be nonzero. For the case with $P_s = 0$, we studied two cases; $\sigma = 0$ with $\epsilon < 0$ and $\sigma = 1$ with $\epsilon < \frac{1}{2}$. For the first case, we studied the crossover from the DI class to the class with $\sigma = 0$. We found the crossover exponent to be $\phi = 1.23(13)$. For the second case, we have observed reentrant phase transitions that occur in the order; absorbing, active, absorbing, and then active phases. All the three transitions share critical behavior with the DI class.

When $q_c = 0$, we suggested the scaling functions around $q = 0$. The scaling function for $\sigma < 1$ is different from that for $\sigma = 1$ in that a logarithmic scaling appears for $\sigma < 1$; see Eqs. (18) and (17) for $\sigma < 1$ and $\sigma = 1$, respectively. These scaling functions are confirmed by numerical simulations.

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