Coherence effects in the transition radiation spectrum and practical consequences

Gian Luca Orlandi
Istituto Superiore di Sanità, Physics Laboratory, and Istituto Nazionale di Fisica Nucleare, Gruppo Sanità Roma1
Viale Regina Elena 299, 00161 Roma, Italy

Abstract

In the framework of the pseudophotons method and under the hypothesis of the far field approximation the formal evaluation of the transition radiation spectrum produced by a three-dimensional charged beam interacting with a metallic target is carried out. When spatial incoherent conditions are occurring, the transverse size of a three-dimensional beam affects the spectrum via a suitable dependence on the Fourier transform of the transverse distribution, while the temporal coherent enhance of the spectrum depends only on the longitudinal beam size via the longitudinal form factor in the same manner as in the case of a one-dimensional beam. Spatial incoherent effects, arising at short wave-length in the spectrum, on the one hand allow a better comprehension of some aspects of the transition radiation phenomenology - e.g. an analytical formula for the total radiated energy - and on the other hand offer new perspectives in the practical application of the transition radiation as diagnostics method of the transverse beam size in a particle accelerator.

1Submitted to Nuclear Instruments and Methods in Physics Research Section A
2Tel.: +39-0649903609; Fax: +39-0649387075;
E-mail Gianluca.Orlandi@iss.infn.it
1 Introduction

Transition radiation theory describes those radiative phenomena produced by high energy charged particles crossing the interface between two media with different dielectric properties. Typical transition radiation events are represented by relativistic charged particles crossing in the vacuum a thin metallic layer. As natural extension, such a category of events also includes the diffraction radiation phenomena, i.e., the radiative mechanisms observed when a charged beam passes through a circular or rectangular slit over a metallic screen.

In the past few years the transition radiation theory, based on the single particle formalism describing a point-like charge colliding with an ideal metallic target of infinite size [1, 2, 3, 4], attracted much interest in those aspects concerning the diffractive modifications affecting such formalism as the finite size of the screen is considered [5, 6, 7, 8]. The aim of the present paper is to contribute to an extension of the transition radiation theory by including in the formalism the source size effects arising as the charged beam can be realistically described by a three-dimensional probability distribution.

The pseudophotons method, describing the transition radiation produced by a high energy beam as a radiation field scattered off a metallic screen, is a powerful formalism [1, 9] allowing to calculate the diffractive modifications in the spectrum due to the finite size of the metallic target and/or to the presence of a slit on it. In the framework of such a formalism and supposing - as reasonable in practical situations - the linear size of the detector surface much smaller than the distance between the scattering surface and the detector itself (far field approximation), the complete formal expression of the transition radiation spectrum produced by a three-dimensional electron bunch colliding with a metallic target is obtained in this paper. As different from the classical articles [10, 11, 12], which are dedicated to the collective effects observable at long wave-length in the synchrotron radiation spectrum emitted by one-dimensional electron beam with zero transverse dimension moving on a circular orbit, the present paper treats transition radiative phenomena concerning three-dimensional charged beam and, consequently, de-
rives physical consequences.

2 Transition radiation spectrum produced by a three-dimensional charged beam in the framework of the pseudophotons method

It has been shown \[11, 12\] that the synchrotron radiation spectrum emitted by \(N\) electrons, collected in a longitudinally extended bunch with zero transverse dimension moving on a circular orbit, is described by the following expression

\[
\frac{dI(\omega)}{d\omega} = \frac{dI_e(\omega)}{d\omega} [N + N(N - 1)F(\omega)],
\]

(1)

where \(\frac{dI_e(\omega)}{d\omega}\) - the intensity per frequency unit - is related to the radiative mechanism produced by the single electron in the bunch (single particle spectrum). The longitudinal form factor \(F(\omega)\) can be stated - in the limit of a continuous charge distribution - as the squared module of the Fourier transform of the longitudinal probability distribution function \(\rho_z(z)\):

\[
F(\omega) = \left| \int_{-\infty}^{+\infty} dz \rho_z(z)e^{i(\omega z/c)} \right|^2.
\]

(2)

Due to its relativistic origin, Eq.(1) continues to maintain its validity for one-dimensional beams if, instead of considering the synchrotron radiative mechanism, we consider the transition one. Then, we assume that the \(N\) electrons - collected in a three-dimensional bunch and in linear uniform motion with common velocity \(\vec{w}\) - hit normally a flat metallic screen producing forward and backward transition radiation. Whether the procedure of substituting in (1) the longitudinal form factor (2) with a three-dimensional one

\[
F(\vec{k}) = \left| \int d\vec{r} \rho(\vec{r})e^{i\vec{k}\cdot\vec{r}} \right|^2
\]

(3)

is the correct one when the charged beam is described by a three-dimensional continuous probability distribution function \(\rho(\vec{r})\), represents the subject of the following part of this paper.
The frequency distribution of the transition radiation intensity can be obtained describing the radiative mechanism in terms of the pseudophotons method \[1, 9\], which allows to directly include in the calculations also screen size effects. According to the Huygens-Fresnel principle, the transition radiation field can be therefore imagined as the diffractive propagation of a pseudophotons field scattered off the metallic screen. In practice the harmonic components of the transition radiation field, meant as scattered components of the transverse electromagnetic (e.m.) field moving with the charge, can be analytically expressed via the integral theorem of Helmholtz and Kirchhoff \[13\]. Such integrals, under the assumption of the far field approximation, can be simply calculated \[1, 14\] by Fourier transforming the pseudophotons field components with respect to the coordinates \((x, y)\) of the metallic target surface \(S\). Therefore, at large distance \(R\) from the target the transition radiation electric field components \((E_{tx}^{tr}, E_{ty}^{tr})\), expressed in terms of the corresponding components \((E_{tx}^{ps}, E_{ty}^{ps})\) of the scattered pseudophotons field, are:

\[
E_{tx,y}^{tr}(k_x, k_y, \omega) = \frac{k}{2\pi R} \times \left( \int_S dxdy E_{tx,y}^{ps}(x, y, \omega) e^{-i(k_x x + k_y y)} \right),
\]

where \(k = \omega/v = \sqrt{k_x^2 + k_y^2 + k_z^2}\) is the module of the transition radiation wave-vector in a non-dispersive medium with phase-velocity \(v = c/\sqrt{\epsilon\mu}\).

### 2.1 Charge electromagnetic field as pseudophotons field

Before expressing the pseudophotons field \((\vec{E}^{ps})\) in terms of the the e.m. field \((\vec{E}^{eh})\) travelling with the electron bunch, it is better to make clear the scenario.

It is assumed that a charged beam, moving along the \(z\) axis with constant velocity \(\vec{w} = (0, 0, w)\), hits a flat surface \(S\), made of an ideal conductor (dielectric constant \(\epsilon_{cond} = \infty\)) and lying on the \((x, y, z = 0)\) plane of the reference frame. A non-dispersive medium with permeability \(\mu = 1\) and characteristic phase velocity \(v = c/\sqrt{\epsilon}\) surrounds the surface. Furthermore it is assumed that \(N\) electrons, forming a three-dimensional bunch at rest with
respect to the center-of-mass frame, are described by a distribution function \( \rho(\vec{r}) \) “frozen” in time:

\[
\rho(\vec{r}, t) = \rho(\vec{r}, 0) = \int \frac{d\vec{k}}{(2\pi)^3} e^{i\vec{k} \cdot \vec{r}} \rho(\vec{k}).
\] (5)

This assumption, in accordance with the observation that the radiative process at a given time only depends on those electrons in the bunch hitting at the same time the metallic screen, corresponds to treat the \( N \) electrons as uncorrelated and described by a probability distribution function and a corresponding Fourier transform expressed as: \( \rho(\vec{r}) = \rho_x(x)\rho_y(y)\rho_z(z) \) and \( \rho(\vec{k}) = \rho_x(k_x)\rho_y(k_y)\rho_z(k_z) \) respectively.

It can be demonstrated that the Fourier transform of the electric field moving with the charge is

\[
\vec{E}_{ch}(\vec{k}, \omega) = (8\pi^2 Ne) \times \frac{\left[-i(\vec{k}/\epsilon) + i(\omega \vec{w}/c^2)\right]}{[k_x^2 + k_y^2 + k_z^2 - (\omega/v)^2]} \rho(\vec{k}) \delta(\omega - \vec{k} \cdot \vec{w}),
\] (6)

The longitudinal field component \( E_z \), proportional to the reciprocal of the square of the relativistic Lorentz factor \( \gamma = E/mc^2 = (1 - (w/v)^2)^{-1/2} \), can be neglected with respect to the electric field components \( E_{x,y} \) transverse to the charge velocity \( \vec{w} \).

Therefore, the harmonic components of the transverse electric field traveling with the charge follows from Eq.(6):

\[
E_{x,y}^{ch}(\vec{r}, \omega) = -\frac{ie^{i(\omega z/w)} Ne\rho_z(\omega/w)}{\pi w} \times \\
\times \int dk_x dk_y \frac{k_x\rho_x(k_x)\rho_y(k_y)e^{i(k_x x + k_y y)}}{\epsilon[k_x^2 + k_y^2 + \alpha^2]},
\] (7)

where \( \alpha = \omega/(w\gamma) \). The pseudophoton field can be finally derived from the constraint of the boundary conditions over the metallic surface imposed on the transverse component of the total field:

\[
E_{x,y}^{ps}(x, y, z = 0) + E_{x,y}^{ch}(x, y, z = 0) = 0.
\] (8)
Then from Eq.(8) the explicit expression of the harmonic component $s$ of the pseudophoton field is:

$$E_{ps,x,y}(\vec{r}, \omega) = \frac{ie^{-i(\omega z/c)}N \rho_z(\omega/w)}{\pi w} \times \int dk_x dk_y \frac{k_{x,y} \rho_x(k_x) \rho_y(k_y) e^{i(k_x x + k_y y)}}{\epsilon[k_x^2 + k_y^2 + \alpha^2]}.$$  

(9)

### 2.2 Transition radiation field as scattered pseudophotons field

With respect to a spherical $(r, \theta, \phi)$ coordinates reference and assuming to observe the radiation far enough from the metallic screen along the polar $z$ axis ($z = R \cos \theta \approx R$), the harmonic components of the transition field Eq.(4) can be explicitly expressed via Eq.(9) as

$$E_{tr,x,y}(\vec{\tau}, \omega) = \frac{ikN \epsilon \rho_z(\omega/w)e^{-i(\omega R/c)}}{2\pi^2 R w} \times \int \int d\kappa_x d\kappa_y \frac{k_{x,y} \rho_x(\kappa_x) \rho_y(\kappa_y) e^{-i(\vec{\kappa} \cdot \vec{r})}}{[\kappa_x^2 + \kappa_y^2 + \alpha^2]}.$$  

(10)

where the vector $\vec{\xi} = (x, y)$ indicates the coordinates over the target surface $S$, while $\vec{\kappa} = (\kappa_x, \kappa_y)$ and $\vec{\tau} = (k_x, k_y)$ indicate the transverse components of the wave-vectors corresponding to the pseudophotons and the transition radiation fields respectively.

As evident from the different role played in (10) by the Fourier transform of the transverse distribution functions $\rho_u(k_u)$ ($u = x, y$) with respect to the longitudinal one $\rho_z(k_z)$, the three-dimensional generalization of Eq.(4) by substituting the longitudinal form factor with a three-dimensional one such as in Eq.(3) appears to be incorrect. It is also obvious to notice that Eq.(10) reduces to the well known single particle transition radiation field [1] when $N = 1$ and $\rho_u(k_u) = 1$ ($u = x, y, z$).
2.3 Transition radiation spectrum: from the discrete three-dimensional distribution to the continuous limit

In practical situations, as those occurring in an accelerator machine, the number of the electrons composing a bunch is so large that it can be reasonably described by means of a continuous probability distribution function (continuous density limit). Therefore, in order to derive the expression of the actually detected transition radiation spectrum, an average procedure has to be carried out:

\[
\frac{d^2 I}{d\Omega d\omega} = \left\langle \frac{d^2 I_{\text{dis}}}{d\Omega d\omega} \right\rangle_{\text{ens}} \tag{11}
\]

where in the second term of the previous statement the energy radiated - per solid angle and frequency units - by a particular discrete charge distribution is averaged over the ensemble of the infinite series of the equivalent discrete particle configurations [12].

Supposing \( N \) electrons moving with a common uniform velocity \( \vec{w} \), the following expression

\[
\rho(\vec{k}) = \frac{1}{N} \sum_{j=1}^{N} e^{-i\vec{k} \cdot \vec{r}_j} \tag{12}
\]

is the Fourier transform of the spatial discrete distribution

\[
\rho(\vec{r}, t) = \frac{1}{N} \sum_{j=1}^{N} \delta(\vec{r} - \vec{r}_j). \tag{13}
\]

A useful notation, distinguishing in the Fourier transform of the charge distribution the contribution \( \tilde{\rho}_j(k_x, k_y) = e^{-i(k_x x_j + k_y y_j)} \) referred to electrons belonging to the same slice (common \( z \) coordinate)

\[
\rho(\vec{k}) = \frac{1}{N} \sum_{j=1}^{N} e^{-ik_z z_j} \tilde{\rho}_j(k_x, k_y), \tag{14}
\]

allows also to outline that spatial coherent effects are expected to be observed in the spectrum, when the beam can be considered - with respect to the observed wave-length - as a discrete sequence of transverse continuous charge slices. In addition temporal coherent effects are expected to arise in the spectrum when the wave-length is so long that the slices sequence has to
be considered as a continuous charge distribution also in the longitudinal direction.

As more extensively described in the Appendix, the complete formal expression of the transition radiation spectrum - produced by a three-dimensional electron beam colliding normally with a metallic screen being at rest in the vacuum - can be expressed as

\[
\frac{d^2 I(\vec{\tau}, \omega)}{d\Omega d\omega} = \frac{d^2 I^s(\vec{\tau}, \omega)}{d\Omega d\omega} [N + N(N - 1)F(\omega)],
\]

(15)

where

\[
\frac{d^2 I^s(\vec{\tau}, \omega)}{d\Omega d\omega} = \frac{c(ke)^2}{(2\pi)^4(\pi w)^2} \times
\]

\[
\sum_{u=x,y} \left| \int_S d\vec{\xi} \int d\vec{\kappa} \frac{\kappa_u \rho_x(\kappa_x) \rho_y(\kappa_y) e^{i(\vec{\tau} - \vec{\kappa}) \cdot \vec{\xi}}}{\kappa_x^2 + \kappa_y^2 + \alpha^2} \right|^2.
\]

(16)

under the hypothesis of the far field approximation and \(F(\omega)\) is the longitudinal form factor [Eq.(3)].

Some comments about the last statements. (1) When temporal coherent conditions are occurring, the transition radiation spectrum in the three-dimensional case shows the same dependence on the longitudinal form factor [Eq.(3)] as in the one-dimensional case [Eq.(1)]. (2) The results stated by Eqs.(15, 16) are consistent with the ones valid for one-dimensional charged beam when \(\rho_x(\kappa_x) = \rho_y(\kappa_y) = 1\) or when the observed wave-length is longer than the transverse size of the beam. (3) Conversely, when the observed wave-length is shorter than the transverse size of the beam, source size effects appear evident in the spectrum via a dependence on the Fourier transforms of the transverse distribution of the beam \((\rho_u(\kappa_u), u = x, y)\). Therefore, both the measurements of the energy and of the transverse size of the beam are not independent each other when the spectrum shows spatial incoherent characteristics.
3 Gaussian beam colliding with an infinitely extended metallic screen

In the ideal and handy case of a three-dimensional gaussian beam normally hitting a metallic screen $S$ of infinite size, the analytical expression for the transition radiation spectrum can be derived from Eqs. (16):

$$
\frac{d^2 I_s(k_x, k_y, \omega)}{d\Omega d\omega} = \frac{c(ke)^2 (k_x^2 + k_y^2) |\rho_x(k_x)|^2 |\rho_y(k_y)|^2}{(\pi w)^2 [k_x^2 + k_y^2 + \alpha^2]^2}.
$$

(17)

It is easy to check that for a point-like charge ($\rho_x(k_x) = \rho_y(k_y) = 1$) Eq. (17) reduces to the well known frequency independent single particle spectrum [1, 2, 3]:

$$
\frac{d^2 I_e}{d\theta d\omega} = \frac{2(e\beta)^2}{c\pi} \frac{\sin^3 \theta}{(1 - \beta^2 \cos^2 \theta)^2},
$$

(18)

where $k_x^2 + k_y^2 = (k \sin \theta)^2$ and the polar angle $\theta$ is referred to the $z$ axis picked to be normal to the target surface.

While for a three-dimensional bunch, having a gaussian transverse distribution ($\rho_u(k_u) = e^{-\frac{(k_u \sigma_u)^2}{2}}, u = x, y, z$), the Eq. (17) can be written as

$$
\frac{d^2 I_s}{d\Omega d\omega} = \frac{(e\beta)^2 \sin^2(\theta)e^{-(k \sin \theta)^2(\sigma_x^2 \cos^2 \phi + \sigma_y^2 \sin^2 \phi)}}{c\pi^2 (1 - \beta^2 \cos^2 \theta)^2},
$$

(19)

which, integrated over $(0, 2\pi)$ with respect to the azimuthal angle $\phi$, reduces to

$$
\frac{d^2 I_s}{d\theta d\omega} = \frac{d^2 I_e}{d\theta d\omega} e^{-(k \sin \theta)^2(\sigma_x^2 + \sigma_y^2)} I_0[(k \sin \theta)^2(\sigma_x^2 - \sigma_y^2)],
$$

(20)

where $I_\nu(z) = e^{-\frac{i\pi\nu}{2}} J_\nu(e^{\frac{i\pi}{2}}z)$ ($-\pi < argz \leq \pi/2$) and $J_\nu(z)$ is the Bessel function of the first kind [15]. In fig.1 the angular distribution [Eq. (20)] corresponding to a cylindrical symmetric gaussian beam ($\sigma = \sigma_x = \sigma_y$) is compared with the single particle spectrum [Eq. (18)] for different values of the beam energy ($E = 500MeV, 5GeV, 50GeV$) and of the $\chi = \sigma/\lambda$ parameter, where $\lambda$ is the observed wave-length.
The shift of the polar angle distribution of the transition radiation intensity - shown in fig.1 - is also meaningful and instructive of the circumstance that beam energy measurements, based on the analysis of the angular distribution of the spectrum, can be - depending on the observed wave-length - strongly affected by the beam transverse size at the low energy regime. In fact, according to the single particle theory, the angular position $\theta_{max}$ of the intensity peak is typically considered as a rough estimate of the beam energy ($\theta_{max} \approx mc^2/E$).

Finally - with regard to Eq.(20) - the complete expression for the transition radiation spectrum corresponding to a three dimensional gaussian beam passing through an infinite metallic screen is

$$\frac{d^2I}{d\theta d\omega} = [N + N(N - 1)F(\omega)] \frac{d^2I_s}{d\theta d\omega}$$

(21)

with $F(\omega) = e^{-(\frac{\omega_0}{\omega})^2}$.

4 Source size dependence of the spectrum and main experimental consequences

From Eqs.(17, 19) it can be observed that the spectral angular distribution carries information about the transverse beam size. Then, as far as the arguments treated in this paper can be considered valid, the analysis of the angular distribution of the transition radiation intensity offers information not only about the beam energy, but also about the beam size. From such a circumstance several consequences concerning beam diagnostics applications of the transition radiation and deeper understanding of its phenomenology can be drawn.

First of all the beam energy measurement based on the usual single particle analysis of the angular distribution of the transition radiation in the optical region can be affected by a systematic error as larger as longer is the transverse beam size with respect to the observed wave-length. Moreover, such a dependence of the spectrum on the Fourier transform of the transverse charge distribution offers interesting and suggestive opportunities in the field of the beam diagnostic applications of the transition radiation. In
fact, although the technique to measure the transverse size of the beam by analyzing the angular distribution of the intensity offers a poor appealing as beam diagnostic tool in the optical region (OTR, [14, 17, 18]), nevertheless it appears to be promising in the infrared-micrometer analysis [19, 20, 21] of those transition radiation spectra produced by intense and high energy charged beams passing through a hole in a metallic screen, which are properly referred as diffraction radiation phenomena [1].

Related to the understanding of the phenomenology, it is to be noticed that, according to Eq.(17), the spectral angular distribution decreases to zero when the frequency tends to infinity, and that such a behaviour is in agreement with the high frequency features of the experimentally observed transition radiation spectra [4, 9]. Moreover, it is to be stressed that such agreement - in this context necessarily qualitative - with the experimental results does not follow from any assumption on the frequency dependence of the dielectric constant of the metallic screen. Conversely it is well known that, in order to explain such a feature in the measured transition radiation spectra, it is necessary to dismiss in the high frequency regime the hypothesis of considering ideal conductor properties for the metallic screen (dielectric constant $\epsilon_{\text{cond}} = \infty$) and to take into account the real dielectric properties ($\epsilon_{\text{cond}} \approx 1$) when the frequency is higher than the plasma frequency $\omega_p$ [9], according to:

$$
\epsilon_{\text{cond}}(\omega) = 1 - \left(\frac{\omega_p}{\omega}\right)^2.
$$

Furthermore, other and more qualified confirmations to the present approach to the transition radiation theory for real electron beams follow from a comparison with some other well-known features of the experimental spectra. In fact, it is possible to derive the analytical expression for the total energy emitted during the radiative mechanism by integrating Eq.(20) with respect to the polar angle $\theta$ and the wave-number $k$. For such a purpose the transverse size of the beam is assumed so large that the ideal conductor hypothesis ($\epsilon_{\text{cond}} = \infty$) can be considered valid in that part of the frequency band effectively of interest in the integration. Moreover, assuming for simplicity a gaussian cylindrical symmetry ($\sigma_x = \sigma_y = \sigma$) for an electron beam with energy $E$, the total - “forward” and “backward” - energy emitted during the
collision can be expressed as:

\[ W^{tr} = \frac{\sqrt{\pi}(e\beta)^2}{2\sigma} \frac{E}{mc^2}. \]  

(23)

In the range of validity of the just referred hypothesis and without introducing any frequency cut-off, the analytical formula expressed by Eq.(23) allows a plain explanation of the experimental main features observed in the emitted transition radiation energy in both the low and the high beam energy limits. The previous statement is a particular case of the most general expression valid for an asymmetric gaussian distribution \((\sigma_x \neq \sigma_y)\):

\[ W^{tr} = \frac{\sqrt{\pi}(e\beta)^2}{2\sqrt{\sigma_x\sigma_y}} \frac{E}{mc^2} F\left(\frac{1}{4}, \frac{1}{4}, 1; -\frac{(\sigma_x^2 - \sigma_y^2)^2}{4\sigma_x^2\sigma_y^2}\right) \]  

(24)

where \(F(\alpha, \beta; \gamma; z)\) is the Hypergeometric function \([15]\).

As comment to the last result, it should be noticed that, approaching the beam axis over the boundary surface, the pseudophotons field [Eq.(9)] diverges to infinity in the case of a single particle, while it maintains a finite value in the case of a three-dimensional charge distribution, as expected for the electric field inside the core of a spatially extended charge distribution. In fact, using spherical coordinates and referring the polar angle \(\theta\) to the z axis normal to the target surface lying on the \((x, y)\) plane, it can be demonstrated that, for a gaussian beam with a cylindrical symmetry, the pseudophotons electric field [Eq.(9)] over the boundary surface is

\[ E^{ps}(\rho) \propto \left| \int d\kappa \frac{\kappa^2 e^{-\frac{i\kappa\sigma^2}{2}} J_1(\kappa\rho)}{\kappa^2 + \alpha^2} \right| \]  

(25)

where \(\rho = \sqrt{x^2 + y^2}\) and \(\kappa = k\sin\theta\). It can be easily verified that for a point-like charge \((\sigma = 0)\) the field Eq.(25) diverges to infinity \((\sim 1/\rho)\) as \(\rho \to 0\), while in the case of a transverse gaussian beam [Eq.(23)] continues to maintain a finite and bounded value \((\sim 1/\sigma)\).

5 Conclusions

In the framework of the pseudophotons method and under the hypothesis of the far field approximation it has been demonstrated that, in the case
of a three-dimensional charged beam colliding with a metallic target, the
temporal coherent part of the spectrum depends on the longitudinal form
factor in the same manner as in the one-dimensional case. It has been also
demonstrated that the transverse beam size affects the spectrum via the
Fourier transform of the transverse charge distribution and that the spectrum
becomes insensitive to it - reducing to the well known single particle spectrum
- when the observed wave-length is larger than the transverse beam size. As
reasonable to expect, the single particle spectrum can be interpreted as the
spatial coherent limit of the transition radiation spectrum corresponding to
a three-dimensional beam. The demonstrated evidence in the spectrum of
the spatial incoherent effects due to the transverse size of the beam - on the
one hand - allows a better comprehension of some aspects of the transition
radiation phenomenology - on the other hand - offers new perspectives in the
use of the transition radiation as beam diagnostic tool.

In fact, as the transverse size of the beam is increasing, the high frequency
damping feature observed in the experimental transition radiation spectra
should depend more on the source size effects than on the frequency depen-
dence of the dielectric constant of the metallic screen. Moreover, supposing
as dominant such a mechanism, it is also possible to derive an analytical
formula for the total radiated energy showing an unexpected dependence on
the reciprocal of the transverse beam size besides the expected one on both
the square of the charge velocity and energy.

From the point of view of the beam diagnostic implications, the depen-
dence on transverse form factor appears to be an useful tool to derive in-
formation about the transverse beam size from the analysis of the spectral
angular distribution, even observing the radiation in conditions of tempo-
ral incoherence and in non-optical wave-length regions, when the radiative
mechanism is produced in a non-destructive way by a charged beam pass-
ing through a slit in a metallic target (diffraction radiation). In fact from
Eqs.(20), supposing as known the beam energy, it appears in principle pos-
sible to measure the beam size by observing the radiation in conditions of
spatial incoherence, i.e., by selecting - by means of a monochromator - from
the spectrum a wave-length much shorter than the transverse size itself and
by analyzing the corresponding angular distribution.

**Acknowledgement**

This work was partially developed after the accomplishment of the Ph.D. in Physics at the "Tor Vergata" University of Rome during a period of hospitality at Laboratori Nazionali di Frascati of INFN, where I took advantage of useful discussions with Dr. M. Castellano, Prof. S. Tazzari, Dr. F. Tazzioli and Dr. V. Verzilov. For encouragement, precious comments and helpful discussions I would like to express my acknowledgments to Dr. A. Aiello, Prof. C. Bernardini, Dr. C. Curceanu (Petrascu), Dr. G. Dattoli, Dr. S. Frullani, Prof. L. Paoluzi, Dr. L. Picardi and Prof. S. Segre.

**A Appendix**

The formal expression of the transition radiation spectrum produced by a three-dimensional charged beam [Eqs.(13, 14)] is obtained supposing as infinite the number of the electrons composing the bunch. According to the continuous density limit [12] one can assume: (1) The ensemble average of Eq.(11) does not depend on $N$. (2) All the sum operations over $N$ can be extended to infinity and, thus, substituted with integral operations, while the charge discrete spatial distribution [Eq.(13)] can be identified with the corresponding mean continuous probability distribution function [Eq.(14)].

Disregarding the common phase factor and indicating the integrand in Eq.(14) as

$$H_{x,y}(\tau, \kappa, \xi) = \frac{kNe}{2\pi^2Rw} \times \kappa_{x,y}e^{i[(k_x-\kappa_x)x+(k_y-\kappa_y)y]} \times \frac{\epsilon[k_x^2 + k_y^2 + \alpha^2]}{[\kappa_x^2 + \kappa_y^2 + \alpha^2]}.$$

and according to Eq.(14), the energy radiated - per solid angle and frequency units - by a particular discrete charge distribution can be expressed as:

$$\frac{d^2I_{dis}(\tau)}{d\Omega d\omega} = \frac{cR^2}{4\pi^2N^2} \times$$
\[ \sum_{u=x,y} [N |S_u(\vec{\tau})|^2 + N(N - 1) |T_u(\vec{\tau})|^2], \] 

(27)

where

\[ |S_u(\vec{\tau})|^2 = \int d\xi d\kappa d\xi' d\kappa' H_u(\vec{\tau}, \vec{\kappa}, \vec{\xi}) H_u^*(\vec{\tau}, \vec{\kappa}', \vec{\xi}') \times \] 

\[ \sum_{j=1}^{N} \frac{\tilde{\rho}_j(\vec{\kappa}) \tilde{\rho}_j^*(\vec{\kappa}')}{N} \] 

(28)

and

\[ |T_u(\vec{\tau})|^2 = \int d\xi d\kappa d\xi' d\kappa' H_u(\vec{\tau}, \vec{\kappa}, \vec{\xi}) H_u^*(\vec{\tau}, \vec{\kappa}', \vec{\xi}') \times \] 

\[ \sum_{j,m(j\neq m)=1}^{N} \frac{e^{i(\omega/w)(z_m-z_j)}}{N(N-1)} \tilde{\rho}_j(\vec{\kappa}) \tilde{\rho}_m^*(\vec{\kappa}') \] 

(29)

The average over the ensemble of the discrete configurations of the first term in Eq.(27) is then proportional to

\[ \langle |S_u(\vec{\tau})|^2 \rangle_{\text{ens}} = \left| \int d\xi d\kappa H_u(\vec{\tau}, \vec{\kappa}, \vec{\xi}) \tilde{\rho}(\vec{\kappa}) \right|^2 \] 

(30)

where \( \tilde{\rho}(\vec{\kappa}) = \rho_\xi(\kappa_x) \rho_\eta(\kappa_y) \) is the Fourier transform of the continuous probability distribution \( \rho_\xi(x) \rho_\eta(y) \) in the transverse plane. While the temporal coherent contribution to the transition radiation spectrum is

\[ \langle |T_u(\vec{\tau})|^2 \rangle_{\text{ens}} = F(\omega) \left| \int d\xi d\kappa H_u(\vec{\tau}, \vec{\kappa}, \vec{\xi}) \tilde{\rho}(\vec{\kappa}) \right|^2 \] 

(31)

where \( F(\omega) \) is the longitudinal form factor [Eq.(2)] under the hypothesis of an ultra-relativistic electron beam traveling in the vacuum (\( \epsilon = 1, w \approx c \)).

The explicit expression of transition radiation spectrum stated by Eqs.(15, 16) follows from Eqs.(30, 31) according to Eq.(26).

References

[1] M. L. Ter-Mikaelian, *High-energy electromagnetic processes in condensed media* (John Wiley and Sons, New York, 1972).
[2] G. M. Garibian, Sov. Phys. JETP 6 (33), 6 (1958) 1079.

[3] F. G. Bass and V. M. Yakovenko, Sov. Phys. Usp. 8, 3 (1965) 420.

[4] V. L. Ginzburg and V. N. Tsyтович, Transition radiation and transition scattering (Adam Hilger, Bristol, 1990).

[5] W. Barry, Proceedings of the 7th Beam Instrumentation Workshop, AIP Conf. Proc. 390 (Argonne, 1996) 173.

[6] N. F. Shul’ga, S. N. Dobrovol’sky and V. G. Syshchenko, Nucl. Instr. and Methods in Phys. Res. B 145 (1998) 180-184.

[7] A. P. Potylitsyn, Nucl. Instr. and Methods in Phys. Res. B 145 (1998) 169-179.

[8] M. Castellano, A. Cianchi, G. Orlandi and V. A. Verzilov, Nucl. Instr. and Methods in Phys. Res. A 435 (1999) 297-307.

[9] J. D. Jackson, Classical Electrodynamics (John Wiley and Sons, New York, 1975).

[10] L. I. Schiff, Rev. Sci. Instrum. 17, 1 (1946) 6.

[11] J. S. Nodvick and D. S. Saxon, Phys. Rev. 96, 1 (1954).

[12] C. J. Hirschmugl, M. Sagurton and G. P. Williams, Phys. Rev. A 44, 2 (1991).

[13] M. Born, E. Wolf, Principles of optics (Pergamon Press, Oxford, 1965).

[14] M. Castellano, V. A. Verzilov, Phys. Rev. ST-AB 1, 062801, (1998).

[15] I. S. Gradshteyn, I. M. Ryzhik Table of Integrals, Series and Products (Academic Press, Orlando, 1980)

[16] L. Wartski, S. Roland, J. Lasalle, M. Bolore and G. Filippi, J. Appl. Phys. 46, 8 (1975).
[17] D. W. Rule, Nucl. Instr. and Methods in Phys. Res. B 24/25, 901-904 (1987).

[18] D. W. Rule and R. B. Fiorito, AIP Conference Proceedings 229 (1991).

[19] Y. Shibata, S. Hasebe, K. Ishi, T. Takahashi, T.Ohsaka, M. Ikezawa, T. Nakazato, M. Oyamada, S. Urasawa, T.Yamakawa and Y. Kondo, Phys. Rev. E 52, 6787 (1995).

[20] M. Castellano, L. Catani, A. Cianchi, M. Geitz, G. Orlandi and V. A. Verzilov, Phys. Rev. E 63, 056501 (2001).

[21] R.B. Fiorito, D.W. Rule, Nucl. Instr. and Methods in Phys. Res. B 173 (2001) 67-82.
Figure Captions

FIGURE 1. Intensity distribution (A.U.) vs polar angle θ (rad) for several values of the beam energy (E=0.5, 5 and 50 GeV). The single particle distribution (a) is compared with the ones corresponding to a cylindrical symmetrical gaussian beam (σₓ = σᵧ = σ) for different values of the χ = σ/λ parameter: χ = 2 · 10² (b), χ = 10³ (c), χ = 2 · 10³ (d). The value χ = 2 · 10³ corresponds to a transverse size σ = 1 mm and to an observed wave-length λ = 0.5 μm.
\[ \frac{dI}{d\omega d\theta} \]

- **E = 500 MeV**
- **E = 5 GeV**
- **E = 50 GeV**

\[ \theta \]