Variational-Bayesian Single-Image Devignetting*

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SUMMARY
Vignetting is a common type of image degradation that makes peripheral parts of an image darker than the central part. Single-image devignetting aims to remove undesirable vignetting from an image without resorting to calibration, thereby providing high-quality images required for a wide range of applications. Previous studies into single-image devignetting have focused on the estimation of vignetting functions under the assumption that degradation other than vignetting is negligible. However, noise in real-world observations remains unremoved after inversion of vignetting, and prevents stable estimation of vignetting functions, thereby resulting in low-quality images. In this paper, we introduce a methodology of image restoration based on variational Bayes (VB) to devignetting, aiming at high-quality devignetting in the presence of noise. Through VB inference, we jointly estimate a vignetting function and a latent image free from both vignetting and noise, using a general image prior for noise removal. Compared with state-of-the-art methods, the proposed VB approach to single-image devignetting maintains effectiveness in the presence of noise, as we demonstrate experimentally.

key words: vignetting, devignetting, single-image, variational Bayes

1. Introduction

Vignetting is a type of image degradation where peripheral parts of an image get darker than the central part, as shown in Fig. 1. While vignetting is due to the difference in the amounts of light received by an image sensor, there are several types of vignetting with different causes [1]: natural vignetting, i.e., light falloff with respect to angles of incident light from lenses; pixel vignetting due to variation in angles where light strikes different pixels on digital image sensors; optical vignetting caused by lens components blocking incident light; and mechanical vignetting, which arises from the use of unsupported camera components. Regardless of type, vignetting has a negative impact on image quality, since it impairs the visibility of important image content and alters the distribution of image brightness, thereby causing trouble for detailed image analysis by both humans and computers.

Although vignetting is undesirable for high-quality imaging, it is difficult to eliminate all possible causes of vignetting before capturing an image with a camera. Thus, it is useful to reduce the vignetting effect in a postprocess. While calibration is a basic approach to vignetting correction [1], it requires additional work during image capturing, such as taking reference images [2], and cannot be applied to previously captured images. In this work, we address the problem of single-image devignetting using no other images, thereby affording broader applicability than calibration techniques.

The vignetting effect in an image is commonly modeled with a vignetting function that describes the radial decrease of brightness from the optical center of an imaging system. In devignetting, it is essential to estimate this function accurately. Previous work on single-image devignetting [2]–[6] basically assumed a rather ideal imaging condition, i.e., degradation other than vignetting is absent or negligible. In this case, we can often determine a correct vignetting function from a vignetted image, and then obtain a desired vignetting-free image by inverting the vignetting process of that function. In reality, however, images captured by cameras are often affected by noise, which is inherent in imaging systems due to their optical, electrical, and electronic nature. When noise is dominant in a vignetted image, previous devignetting methods that perform naïve inversion of vignetting can remove only vignetting but not noise, thereby limiting the quality of the restored image. Furthermore, estimation of a vignetting function from a noisy image can become unstable due to noise, which leads to poor devignetting performance.

In this paper, we tackle the problem of single-image devignetting in the presence of noise. To consider noise in devignetting, we make full use of a family of statistical techniques called variational Bayes (VB), which has been successful recently in image restoration [7]–[9]. Specifically, we introduce a general image prior that exploits natural image statistics, whose effectiveness for noise removal has been confirmed in image restoration. Then, in Bayesian inference enabled by VB, we jointly estimate a vignetting function and a latent image without vignetting or noise from an observed image, considering their relationship. Our
prior-driven VB devignetting can achieve high image quality even in the presence of noise because it seeks a latent image free from both vignetting and noise, which further benefits stable estimation of the vignetting function through the joint inference. In experiments, we confirmed the effectiveness of the proposed VB approach to single-image devignetting, especially when noise is strong, comparing the proposed method with state-of-the-art methods both qualitatively and quantitatively.

The remainder of this paper is organized as follows. First, we review previous work related to devignetting in Sect. 2. Next, we describe the construction of a prior-based Bayesian model for devignetting in Sect. 3. In Sect. 4, we derive a VB inference algorithm from this model, and then explain details about our implementation in Sect. 5. In Sect. 6, we present experimental results to examine the effectiveness of the proposed devignetting method. Finally, we conclude this paper with a discussion of future work in Sect. 7.

2. Related Work

While various calibration methods have been proposed in the vignetting literature, such methods require that users follow complicated observation procedures in controlled environments, e.g., taking reference images with known camera configurations to estimate vignetting functions [2], [5]. Goldman [1] showed that vignetting functions can be estimated without special calibration objects or controlled lighting conditions; however, his method still requires multiple images with the same vignetting pattern. In this work, we take a single-image approach to devignetting, aiming at a broader range of applications.

Generally, previous single-image devignetting methods attempt to estimate vignetting functions by minimizing cost functions, employing different criteria to distinguish vignetted and vignetting-free images, and then inverting the vignetting process using estimated functions. For example, Zheng et al. proposed methods to estimate vignetting functions using image segmentation [3], and without segmentation but by exploiting the symmetry of the gradients of vignetted images [4]. Cho et al. [2] proposed a more effective method based on the radial bright channel of an image (i.e., a one-dimensional function composed of maximum values at each radius around an optical center), which is nearly constant for vignetting-free images but decreasing for vignetted images. Meanwhile, Lopez-Fuentes et al. proposed methods that minimize different kinds of image entropies, i.e., log-entropy [5] and negative fuzzy entropy [6] to distinguish vignetted and vignetting-free images. However, these methods have limited ability to improve the quality of noisy images with vignetting because they simply invert vignetting by naive pixelwise division of a vignetted image using an estimated vignetting function [2], [5], which does not remove noise. Furthermore, noise can also make vignetting estimation unstable, thereby leading to poor performance of subsequent vignetting removal, particularly in methods that ignore noise in vignetting estimation [5], [6]. Although some methods incorporate simple noise models in their cost functions for vignetting estimation [2], [4], these methods require additional parameters that depend on noise distributions, whose optimal values can differ between inputs, and their performance in the presence of noise has not been examined experimentally. In this paper, we enable the use of a general image prior for noise removal in devignetting via VB inference, thereby improving the quality of restored images in the presence of noise, as we demonstrate in our experiment. In addition, by establishing a VB devignetting methodology, we can automatically adjust all important image-dependent parameters through joint inference, thereby eliminating the need for parameter tuning and achieving higher applicability.

To obtain valid vignetting functions and reduce computational cost, previous devignetting methods employed parameterization of one-dimensional vignetting functions. The Kang-Weiss model [10], which is based on the physical properties of vignetting, is a standard parameterization scheme used in various methods [2], [4]. Another common scheme is the polynomial model [1], whose parameters can be adjusted via naive hill-climbing optimization [5], [6]. In contrast, we do not employ any explicit parameterization. Instead, we estimate a one-dimensional vignetting function directly under certain constraints, thereby enabling more flexible vignetting estimation than the previous parameterization approach, while utilizing a state-of-the-art parametric method [2] to obtain rough estimates for initialization.

3. Model

In single-image devignetting, we aim to estimate a latent image with no vignetting from an observed image affected by vignetting. Each of these digital images consists of pixels with light intensity values arranged as a two-dimensional array. We can flatten this array into a vector by stacking its pixel values in the lexicographical order. Here, let \( x, y \in \mathbb{R}^n \) be vectors of the latent and observed images, respectively, where \( n \) denotes the total number of pixels per image. The vignetting effect is commonly described using a one-dimensional vignetting function [2], [4], [5], since natural vignetting is mostly radial, i.e., it is constant at each radius around an optical center, which we assume to be known as in the majority of previous studies [1]–[3]. After discretization, we can represent this function as another vector \( w \in \mathbb{R}^m \), where \( m \) is the number of discrete radii and the first element at the radial origin corresponds to the optical center.

In the following, we construct a Bayesian model to describe the relationship between parameters, i.e., the observed and latent images, the vignetting function, and other weight parameters, regarding them as variables and defining the probability distribution of each random variable. An overview of this model is shown in Fig. 2.

(1) Observed Image

Unlike previous vignetting studies, we assume that each
pixel of the observed image is affected by additive zero-mean Gaussian noise. Note that this is a standard assumption in image restoration [8], [11]. By denoting the vector of pixelwise noise intensity values by $n \in \mathbb{R}^n$, we obtain the following relationship between parameters:

$$y = Pw \circ x + n,$$

where $P \in \mathbb{R}^{n \times m}$ is a matrix that prolongs the one-dimensional vignetting function into a two-dimensional function, as shown in Fig. 3, and $\circ$ denotes elementwise multiplication. This degradation model is shown in Fig. 4.

Using the degradation model in Eq. (1), we define the distribution of the observed image $y$ as the following independent Gaussian distributions:

$$p(y|x, w) \propto \prod_{i=1}^{n} \sqrt{b} \exp \left( - \frac{b}{2} (y - Pw \circ x)^2 \right),$$

where index $i$ in a subscript denotes the $i$th element, and $b \in \mathbb{R}$ is the inverted noise variance, which we refer to as a fidelity parameter because it measures the reliability of the noisy observation.

(2) Latent Image

As seen from the degradation model in Eq. (1), we cannot recover the latent image $x$ by simply applying the inverse of the vignetting $Pw$ to the observed image $y$, since noise $n$ is present (unlike in previous devignetting methods that ignore noise). This implies that complete information on $x$ is already lost in $y$ due to noise. Hence, we need to provide additional information on $x$ to remove noise. To this end, we exploit the statistical regularity of natural images, i.e., an image in the real world is generally smooth except for edges [12]. This means that local variations in the image, i.e., the magnitudes of image gradients, should be small at most pixels, but can be large at a few discontinuities.

To minimize local image variations, we define the following independent zero-mean Gaussian distributions on the gradient magnitudes of the latent image $x$:

$$p(x) \propto \prod_{i=1}^{n} \sqrt{a_i} \exp \left( - \frac{a_i}{2} \left( (D_h x)^2 + (D_v x)^2 \right) \right),$$

where $D_h$ and $D_v \in \mathbb{R}^{n \times n}$ are matrices of horizontal and vertical differentiation for $x$, respectively, and $a \in \mathbb{R}^n$ is a local smoothness parameter, where each element $a_i$ measures the smoothness of $x$ at the $i$th pixel. Thus, we can achieve edge-preserving smoothing by adapting $a$ to the structure of $x$ through inference. This type of prior has already been employed in VB image restoration [13] because its Gaussian form is convenient for VB inference. In addition, the strong sparsity-promoting power of this prior is effective for tasks where sharp images are desired, e.g., blur removal [14]. In devignetting, this property helps distinguish the smooth variation of the vignetting function from the variation of the latent image, which can be more discontinuous due to edges.

(3) Vignetting Function

To regularize the estimation of the vignetting function without losing flexibility, we perform smoothing along radii by imposing a Gaussian distribution on the magnitudes of the radial gradients of $w$ as follows:

$$p(w) \propto \prod_{i=1}^{m} \sqrt{z} \exp \left( - \frac{z}{2} (D_i w)^2 \right),$$

where $D_i \in \mathbb{R}^{n \times n}$ is a matrix of radial differentiation for $w$, and $z \in \mathbb{R}$ is a global smoothness parameter for $w$. Note that this prior is not sparsity-promoting, unlike the edge-preserving image prior in Eq. (3), since the smoothness parameter $z$ is globally unique and not adaptive to individual elements of $D_i w$.

Moreover, to obtain a valid vignetting function, we constrain $w$ such that it takes values between zero and one, decreases with respect to radii, and is one at the origin corresponding to the optical center:
\[0 \leq \mathbf{w} \leq 1,\]  
\[D_i \mathbf{w} < 0,\]  
\[w_1 = 1,\]  
(5)  
(6)  
(7)

where vector inequalities are elementwise; in Eq. (7), the index 1 in the subscript denotes the first element of \( \mathbf{w} \). These constraints reflect the natural characteristics of vignetting, which reduces image brightness gradually in radial directions from the optical center. Here, the constraint in Eq. (7) is necessary to determine the value scale of the latent image uniquely, since Eq. (1) holds for any scalar \( c \) and vectors \( \mathbf{x}', \mathbf{w}' \) such that \( \mathbf{x} = c \mathbf{x}' \) and \( \mathbf{w} = \frac{1}{c} \mathbf{w}' \). Note that the smoothing prior in Eq. (4) does not alleviate this issue, since it only constrains the relative differences between the values of \( \mathbf{w} \), but not their absolute magnitudes. Without fixing the central value with Eq. (7), \( \mathbf{w} \) can become arbitrarily small, which in turn makes \( \mathbf{x} \) brighter than expected. While these constraints were implied in previous devignetting studies via stricter parameterization of vignetting functions [2], [4], [5], we do not impose any further restriction, thereby achieving greater flexibility.

4. Algorithm

Under the model defined in Sect. 3, the devignetting problem is reduced to the estimation of the most probable latent image \( \mathbf{x} \) given the observed image \( \mathbf{y} \). This can be achieved by maximizing the marginal posterior probability of \( \mathbf{x} \) given \( \mathbf{y} \) while marginalizing out \( \mathbf{w} \) as follows:

\[
\hat{\mathbf{x}} = \arg \max_{\mathbf{x}} p(\mathbf{x} | \mathbf{y}) = \arg \max_{\mathbf{x}} \int p(\mathbf{x}, \mathbf{w} | \mathbf{y}) d\mathbf{w}. \tag{8}
\]

Here, the joint posterior distribution of the latent variables can be obtained as the product of the distributions in our model:

\[
p(\mathbf{x}, \mathbf{w} | \mathbf{y}) \propto p(\mathbf{y} | \mathbf{x}, \mathbf{w})p(\mathbf{x})p(\mathbf{w}), \tag{9}
\]

where we have used Bayes’ theorem [15].

Although \( \mathbf{w} \) follows a simple Gaussian prior in Eq. (4), its domain is also constrained by Eqs. (5) to (7), and thus we cannot perform exact marginalization in Eq. (8) easily (e.g., by integration using Gaussian properties). In other words, it is difficult to impose these constraints during analytical marginalization. Hence, we employ a VB technique called mean-field approximation [16] to perform approximate marginalization, so that we can impose the constraints during optimization. Specifically, we factorize the exact joint posterior distribution into approximate posterior distributions of the individual variables as follows:

\[
p(\mathbf{x}, \mathbf{w} | \mathbf{y}) \approx q(\mathbf{x}) q(\mathbf{w}). \tag{10}
\]

To obtain a tractable distribution, we also restrict each approximate posterior as independent Gaussian distributions, whose covariance matrix is diagonal, as in previous studies into VB image restoration [8], [9]. We denote the mean and elementwise variance of \( q(\mathbf{x}) \) by vectors \( \mu_\mathbf{x} \) and \( \sigma_\mathbf{x}^2 \), respectively (i.e., the covariance is \( \text{diag}(\sigma_\mathbf{x}^2) \)), and those of \( q(\mathbf{w}) \) by \( \mu_\mathbf{w} \) and \( \sigma_\mathbf{w}^2 \). After this VB approximation, estimates of \( \mathbf{w} \) are explicitly available as the posterior mean parameter \( \mu_\mathbf{w} \), on which we can easily impose the constraints. Another merit of this approximation is that we can bypass the need to deal with the complicated exact posterior distribution of \( \mathbf{x} \), which would arise when \( \mathbf{w} \) is analytically marginalized; instead, we have simple Gaussian posteriors of \( \mathbf{x} \) and \( \mathbf{w} \), which can be separately optimized with respect to their parameters. Once we obtain the approximate posterior \( q(\mathbf{x}) \), we can take its mode as the maximum-posterior (MAP) solution \( \hat{\mathbf{x}} \), which coincides with the mean \( \mu_\mathbf{x} \) in the Gaussian case.

To make approximation as accurate as possible, we consider the Kullback-Leibler divergence from the exact to the approximate joint posterior, i.e., from the left-hand to the right-hand sides of Eq. (10), defined as follows:

\[
D_{\text{KL}}(q || p) = \int q(\mathbf{x}) q(\mathbf{w}) \ln \frac{q(\mathbf{x}) q(\mathbf{w})}{p(\mathbf{x}, \mathbf{w})} d\mathbf{x} d\mathbf{w}. \tag{11}
\]

Negating Eq. (11) and substituting Eq. (9) along with Eqs. (2) to (4) as detailed in Appendix A, we obtain the following objective to be maximized, which we refer to as the evidence lower bound (ELBO):

\[
\text{ELBO} = \frac{1}{2} \ln b - \frac{1}{2} \mathbf{b}^T \left\{ \frac{(\mathbf{y} - \mathbf{P} \mu_\mathbf{w} \circ \mu_\mathbf{x})^2}{\mathbf{P}^2 \sigma_\mathbf{w}^2 \circ \sigma_\mathbf{x}^2} + (\mathbf{P} \mu_\mathbf{w})^2 \circ \sigma_\mathbf{x}^2 + \mathbf{P}^2 \sigma_\mathbf{w}^2 \circ \mu_\mathbf{x}^2 \right\}
+ \frac{1}{2} \mathbf{j}^T \ln \mathbf{a} - \frac{1}{2} \mathbf{a}^T \left\{ \frac{D_h \mu_\mathbf{x}}{\sigma_\mathbf{x}^2} - D_y \mathbf{D}_h \sigma_\mathbf{x}^2 + D_y \sigma_\mathbf{x}^2 \right\}
+ \frac{m}{2} \ln z - \frac{1}{2} \mathbf{z}^T \left\{ \frac{(\mathbf{D}_h \mu_\mathbf{w})^2}{\sigma_\mathbf{w}^2} + (\mathbf{D}_h \sigma_\mathbf{w}^2 + \mathbf{D}_y \sigma_\mathbf{w}^2) \right\}
+ \frac{1}{2} \mathbf{I}^T \ln \sigma_\mathbf{x}^2 + \frac{1}{2} \mathbf{I}^T \ln \sigma_\mathbf{w}^2 + \text{const.}, \tag{12}
\]

where \( \circ \) in a superscript denotes elementwise power, vector logarithms are elementwise, and \( \mathbf{j} \in \mathbb{R}^n \) and \( \mathbf{i} \in \mathbb{R}^m \) are vectors of ones used to express summation in terms of matrix multiplication.

In the following, we derive an update formula for each of the parameters to be estimated, e.g., the parameters of the approximate posteriors, and the weight parameters in our model. Details about this derivation are given in Appendix A.

1. Latent Image

Setting the derivative of Eq. (12) with respect to \( \mu_\mathbf{x} \) and \( \sigma_\mathbf{x}^2 \) to zero, we obtain the parameters of \( q(\mathbf{x}) \) as follows:

\[
\mu_\mathbf{x} = \left\{ \begin{array}{c}
\text{diag} \left( (\mathbf{P} \mu_\mathbf{w})^2 + \mathbf{P}^2 \sigma_\mathbf{w}^2 \right) \\
+ \frac{1}{\mathbf{b}} \mathbf{b}^T \left( \mathbf{D}_h \text{diag}(\mathbf{a}) \mathbf{D}_h + \mathbf{I}^T \text{diag}(\mathbf{a}) \mathbf{D}_y \right) \\
\end{array} \right\}^{-1} \tag{13}
\]

\[
\sigma_\mathbf{x}^2 = \left( b (\mathbf{P} \mu_\mathbf{w})^2 + \mathbf{P}^2 \sigma_\mathbf{w}^2 \right) + \left( \mathbf{D}_h^2 + \mathbf{D}_y^2 \right) \mathbf{a}^{-1}. \tag{14}
\]
(2) Vignetting Function

Setting the derivative of Eq. (12) with respect to \( \mu_w \) and \( \sigma_w^2 \) to zero, we obtain the parameters of \( q(\mathbf{w}) \) as follows:

\[
\mu_w = \left( P^T \text{diag} \left( \mu_x^2 + \sigma_x^2 \right) P + \frac{z}{b} D_f^T D_f \right)^{-1} P^T (\mu_x \circ y), \tag{15} \]

\[
\sigma_w^2 = \left( b P^T (\mu_x^2 + \sigma_x^2) + D_f^T D_f \right)^{-1}. \tag{16} \]

Since it is difficult to analytically impose the constraints in Eqs. (5) to (7) on \( q(\mathbf{w}) \), we numerically impose them on the mean parameter \( \mu_w \) during its update. This is done by solving Eq. (15) via constrained optimization, which we detail in Sect. 5.

(3) Weights

We can also obtain the weight parameters in our model, i.e., the fidelity \( b \), the local smoothness of the latent image \( a_i \), and the global smoothness of the vignetting function \( \zeta \). Setting the derivative of Eq. (12) with respect to \( b, a_i \), and \( \zeta \) to zero, we obtain these parameters as follows:

\[
b = \frac{n}{\sum_{i=1}^{n} \left( (y - P \mu_w \circ \mu_x^2 + P^2 \sigma_w^2 \circ \sigma_x^2) + (P \mu_w)^2 \circ \sigma_x^2 + P^2 \sigma_w^2 \circ \mu_x^2 \right)}, \tag{17} \]

\[
a_i = \frac{1}{\sum_{i=1}^{m} \left( (D_i \mu_x)^2 + (D_i \mu_w)^2 + D_i^2 \sigma_x^2 + D_f^2 \sigma_w^2 \right)}, \tag{18} \]

\[
\zeta = \frac{m}{\sum_{i=1}^{m} \left( (D_i \mu_w)^2 + D_f^2 \sigma_w^2 \right)}. \tag{19} \]

Here, \( a_i \) depends on the inverse of the local image variation at the \( i \)th pixel; thus, it becomes small around an edge, i.e., it has large variation and weakens smoothing, thereby preserving the edge.

Using the update formulas obtained above, we analyze the properties of the derived VB inference to discuss the advantages of the proposed VB devignetting approach. First, we can naturally consider the relationship of the parameters in VB inference, since they are mutually dependent in the update formulas. In particular, the mean of the vignetting function \( \mu_w \) depends on the latent image estimate \( \mu_x \) in Eq. (3). Thus, the vignetting function is determined by considering the noise-free version of the observed image \( y \), unlike previous methods that depend only on the observed noisy image [2], [4], [5]. This consideration leads to more stable vignetting estimation because noise can alter the brightness distribution of the vignette and thus confuse the estimation of the vignetting function if ignored. Second, since we obtain full posterior estimates rather than point estimates, we can access not only the means but also the variances of the parameters treated as Gaussian random variables. Note that these variances represent uncertainty in parameter estimation [11], which is unavailable to traditional techniques such as MAP but beneficial to VB [17].

For example, terms dependent on the variance of the latent image \( \sigma_x^2 \) appear in the denominator of Eq. (18), i.e., the update of the local smoothness parameter. These positive values effectively work as additional regularizers [18] that prevent the weight from reaching infinity, thereby contributing to stability. Finally, we can adjust the weight parameters automatically through VB inference, i.e., using the update formulas in Eqs. (17) to (19), whereas traditional image restoration methods often require manual tuning of such parameters [11]. Since the optimal values of such parameters vary greatly among input images, this property grants wider applicability to the proposed VB method than existing non-VB methods.

Since the parameters of the approximate posteriors and weight parameters depend on each other, we cannot obtain them simultaneously. Thus, we update them iteratively, i.e., we vary one while fixing the others, starting with reasonable initial estimates. Note that such a VB inference algorithm is guaranteed to converge [17]. This algorithm is summarized in Algorithm 1.

5. Implementation

To initialize Algorithm 1, we utilize the parametric method proposed by Cho et al. [2]. This method is relatively robust against noise and also efficient, because it reduces an input image to a single-dimensional feature and estimates a small number of vignetting parameters; thus, it is a suitable method to obtain rough initial estimates. Note that, however, this method itself cannot remove noise, since it simply divides the input image by the estimated vignetting function. We initialize \( \mu_x \) and \( \mu_w \) with the estimates of the vignetting-free image and vignetting function obtained by Cho et al.’s method, respectively, while setting \( \sigma_x^2 \) and \( \sigma_w^2 \) to zero. Note that, while we may initialize the variance parameters with larger values, such a setting may produce large positive terms (relatively to other terms dependent on mean parameters) in the denominators of Eqs. (17) to (19). Then, the resulting weight parameters can affect other parameters too quickly in a few iterations after initialization, leading to numerical instability; thus, we just initialize these variances with zero, and avoid potential zero division numerically by value clamping as described below. Since we have no prior information about the weight parameters, we initially set \( b, a_i, \) and \( \zeta \) to one.

Although a VB inference algorithm is theoretically

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**Algorithm 1 Algorithm of VB inference for devignetting.**

**Input:** \( y \)

**Output:** \( \hat{x} \)

1: Initialize \( \mu_x, \sigma_x^2, \mu_w, \sigma_w^2, b, a, \zeta \).
2: repeat
3:   Update \( \mu_x \) and \( \sigma_x^2 \) with Eqs. (13) and (14).
4:   Update \( \mu_w \) and \( \sigma_w^2 \) with Eqs. (15) and (16).
5:   Update \( b, a, \) and \( \zeta \) with Eqs. (19) to (17).
6: until convergence.
7: return \( \hat{x} = \mu_x \).
guaranteed to converge, it does not necessarily converge numerically, mainly due to accumulation of errors in floating-point operations; even if the algorithm approaches to convergence, it sometimes takes a prohibitively long time in practice. Thus, we need stopping criteria to terminate the algorithm at some iteration. Specifically, we check the variation of $\mu_k$ between iterations in terms of mean squared error (MSE), and terminate the algorithm when the error is less than 0.1, or after 16 iterations, which we empirically found enough to obtain reasonable restoration results.

The algorithm presented in Sect. 4 can be implemented approximately in terms of operations on dense vectors and sparse matrices. We employ forward finite difference to construct the differentiation matrices $D_h$, $D_c$, and $D_r$, and use linear interpolation to construct the prolongation operator $P$. For the number of discrete radii $m$, i.e., the number of elements of the one-dimensional vignetting function $w$, we employ the rounded maximum radius in the discretized image domain, which is calculated by converting the Cartesian coordinates of pixels into polar coordinates. To solve the large-sparse linear system in Eqs. (13) and (15) under the constraints in Eqs. (5) to (7), we employed the primal-dual interior-point method [19]. In elementwise inversion in Eqs. (14) and (16) to (19), we perform value clamping of denominators (minimum $10^{-4}$) to avoid potential numerical instability due to zero division.

6. Experiments

To evaluate the effectiveness of the proposed devignetting method experimentally, we used synthetic images to assess image quality quantitatively with respect to ground truth, and also real images to examine real-world performance qualitatively.

We compared the proposed method with the state-of-the-art method proposed by Cho et al. [2], which outperforms more classical methods, such as the method proposed by Zheng et al. [4]. In addition, we examined the recent method proposed by Lopez-Fuentes et al. [5], which is based on minimization of a certain type of image entropy. While Lopez-Fuentes et al. proposed several entropies and found that the best-performing entropy depends on image sizes [6], we used the basic version [5] because they did not detail the implementation of alternative entropies. For the parameters of these previous methods, we used the default values suggested in their respective papers [2], [5].

6.1 Synthetic Images

We prepared observed images by synthetically degrading three standard test images as ground-truth latent images, which are shown in Fig. 5 and referred to as pattern, pepper, and Lena. Here, the size of each image was $n = 512 \times 512$ pixels. First, following previous devignetting studies [2], [4], [5], we applied vignetting based on the off-axis illumination component of the Kang-Weiss model [10] defined as follows:

$$A(r) = \frac{1}{(1 + (\frac{r}{h})^2)^2},$$

where $r$ is a radius from an optical center, and $f$ is a focal length parameter; here, we assumed the optical center of the vignetting function to be the center of each image. Then, as in typical restoration studies [8], [11], [13], we added zero-mean Gaussian noise, whose level was measured via the following signal-to-noise ratio (SNR) [20]:

$$SNR = 10 \log_{10} \sum_{i=1}^{n} \frac{(Pw \circ x)_i^2}{\sigma^2} \quad [\text{dB}],$$

where $Pw \circ x$ is the vignetted image with $n$ pixels, and $\sigma = \frac{1}{\sqrt{m}}$ is the standard deviation of the noise. In this experiment, we set $f$ to 250 (the heaviest vignetting setting in the previous devignetting study [2]) to obtain visually noticeable amount of vignetting, for which we can easily evaluate devignetting performance, and chose $\sigma$ to obtain SNR values of 40 and 20, which correspond to weak and strong noise, respectively. Treating each of these degraded images as an observed image, we restored it with each devignetting method to obtain an estimate of the original latent image.

For quantitative evaluation, we assessed the quality of images in terms of two metrics: peak signal-to-noise ratio (PSNR), which is a standard metric that summarizes pixelwise fidelity of images; and structural similarity (SSIM) [21], which reflects human visual perception. For each observed and restored image, both metrics were calculated with reference to the corresponding latent image as the ground truth. In Tables 1 to 3, we show the PSNR and SSIM values of the observed and restored images for pattern, pep-
From these results, we can see that all methods successfully improved the quality of each observed image in terms of PSNR. In terms of SSIM, however, the two previous methods degraded the quality of some of the observed images, e.g., pattern with strong noise (Table 1). In contrast, the proposed method always improved the quality of the observed images in terms of both PSNR and SSIM. Moreover, it consistently achieved higher image quality than the previous methods when noise was strong. Looking at the restored images in Figs. 6 (d), 7 (d) and 8 (d), we can verify that the proposed method can remove not only vignetting but also noise. These results demonstrate the stability of the proposed VB approach, which exhibits good restoration performance regardless of variable factors, such as image content and noise. Although Cho et al.’s method achieved a slightly higher PSNR value than that of the proposed method for Lena with weak noise (Table 3), this can be attributed to its naïve devignetting scheme, i.e., pixelwise division with an estimated vignetting function without using any image priors, which does not remove and even preserves noise in ground-truth images. Confirming this, the result for pattern, which originally contains no noise, showed an opposite trend, i.e., the proposed method achieved higher PSNR and SSIM values than Cho et al.’s method even when noise was weak (Table 1). When noise was increased, Cho et al.’s
method overestimated pixel values, as observed in the corners of the pattern and Lena images shown in Figs. 9 (a) and 10 (a), respectively. Note that, while the whitened-out pixels in these regions could result from underestimated values of vignetting functions, they were successfully corrected by the proposed method after refining these estimates through joint inference, as shown in Fig. 9 (b) and Fig. 10 (b). Although Lopez-Fuentes et al.’s method achieved very high PSNR values for pepper and Lena with weak noise (Tables 2 and 3), it performed poorly for pattern (Table 1) compared with the other methods. These results suggest that its entropy-based vignetting estimation is not robust, i.e., its performance is easily affected by image content and noise. Overall, the performance of the proposed method was stable and comparable with the previous methods for weak noise, and better than the previous methods for strong noise, as we intended in designing the prior-based VB method.

To further analyze the performance of the proposed method, the accuracy of the estimated vignetting functions (in terms of MSE between the prolonged two-dimensional versions of ground-truth and estimated vignetting functions) are shown in Tables 4 to 6. We can see that the proposed method could estimate vignetting functions with smaller errors than Cho et al.’s previous method (i.e., the initial estimates of the proposed method), except for the Lena images. Note that our goal is to improve the quality of restored images rather than the accuracy of estimated vignetting functions; indeed, these MSE values do not reflect the incorrect devignetting by the previous method and the correction by the proposed method (observed through whitelout in Fig. 10). This suggests that the MSE accuracy of vignetting estimation does not necessarily measure its contribution to restoration. Despite globally higher errors, the proposed method, owing to its joint consideration of vignetting and noise, could still improve vignetting estimates locally (e.g., around the whitelout), thereby achieving higher image quality.

While our aim in this work is not efficiency (computation speed) but effectiveness (image quality), to facilitate a fair comparison between the different methods and discuss the additional cost of the VB approach, we also present the computational time taken for devignetting, as in previous work [2]. In Table 7, the average computational time for all input images is summarized for our implementation of each method running on an Intel Xeon E5-2660 CPU; here, we excluded one result (pepper with strong noise) of the proposed method from averaging, for which the maximum number of iterations was reached before the image variation fell below the threshold (described in Sect. 5). Lopez-Fuentes et al.’s method took significantly longer than the other two methods, due to its entropy computation, which must be performed for all candidates of vignetting parameters in the hill-climbing optimization at each iteration. While the proposed method obviously required more time than the efficient method by Cho et al. used for initialization, this can be considered as the price of the higher image quality realized by the proposed method, which estimates a vignetting function and a latent image iteratively.

In addition, we varied the focal length parameter \( f \) of vignetting for Lena to examine the stability of the proposed method for different amounts of vignetting. The results are summarized in Table 8; here, \( f = 100 \) and 500 correspond to stronger and weaker vignetting, respectively, than the case

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**Fig. 9** Close-ups of the bottom-right corners of restored pattern images with strong noise (Fig. 6): (a) restored by the method of Cho et al. [2]; and (b) restored by the proposed method.

**Fig. 10** Close-ups of the top-left corners of restored Lena images with strong noise (Fig. 8): (a) restored by the method of Cho et al. [2]; and (b) restored by the proposed method.

**Table 4** Accuracy of vignetting estimation for pattern (MSE \( \times 10^6 \)).

| Noise | Cho [2] | Proposed |
|-------|---------|-----------|
| Weak  | 707     | 689       |
| Strong| 3717    | 3716      |

**Table 5** Accuracy of vignetting estimation for pepper (MSE \( \times 10^6 \)).

| Noise | Cho [2] | Proposed |
|-------|---------|-----------|
| Weak  | 8644    | 7589      |
| Strong| 6706    | 5244      |

**Table 6** Accuracy of vignetting estimation for Lena (MSE \( \times 10^6 \)).

| Noise | Cho [2] | Proposed |
|-------|---------|-----------|
| Weak  | 1020    | 1341      |
| Strong| 716     | 1479      |

**Table 7** Average computational time [seconds].

| Method         | Lopez-Fuentes [5] | Cho [2] | Proposed |
|----------------|-------------------|---------|-----------|
|                | 428.1             | 1.4     | 21.2      |

---
of $f = 250$ in Table 3. The trend of the results was basically the same as before: while the performance of the proposed method was comparable with Cho et al.’s method for weak noise (especially in terms of SSIM), it was consistently better for strong noise, confirming the stability in the presence of severe degradation.

To demonstrate the advantage of the proposed joint removal of noise and vignetting, we also performed devignetting after naïve denoising. Specifically, we applied bilateral filtering [22] (with width parameter 9 and standard deviation parameter 8) to the Lena images with vignetting focal length $f = 250$, and then applied Cho et al.’s and the proposed methods to the denoised images. The results are summarized in Table 9. Comparing this with Table 3, we can see that both methods scored higher PSNR values when used after denoising; thus, explicit denoising before devignetting can be effective, at least in terms of PSNR. In terms of SSIM, however, the previous method resulted in lower quality when noise was weak, possibly due to oversmoothing by denoising, while the SSIM performance of the proposed method was comparable to the original case without explicit denoising. The proposed method also consistently achieved higher PSNR and SSIM values than the previous method, indicating that its effectiveness was not compromised by denoising. Note that, in practice, such a two-step approach (i.e., devignetting after denoising) requires additional parameter tuning for each image, in order to successfully remove noise without oversmoothing. By contrast, direct application of the proposed method can bypass this difficulty (while achieving equivalent SSIM quality), which is a distinct advantage of our VB approach.

6.2 Real Images

Following previous studies [2], [4], we used three images with real vignetting effects from the Berkeley Segmentation Dataset [23], which we refer to as bird, diver, and waterfall. The optical center of each image was estimated by a previously proposed technique [5] and fed to each method as a known parameter. Since we did not have degradation-free images as ground truth, only qualitative evaluation by visual comparison was possible for these real images.

The results for the bird, diver, and waterfall images are shown in Figs. 11 to 13, respectively. For bird, which has a simple image structure suitable for vignetting estimation and contains little noise, all methods successfully removed vignetting in the peripheral parts of the observed image, and their results were comparable. For diver and waterfall, Lopez-Fuentes et al.’s method failed to remove vignetting, and produced nearly the same images as the observed images, possibly due to their complex structures. In contrast, Cho et al.’s method and the proposed method yielded images whose peripheral parts were brighter than the corresponding observed images, which indicates successful devignetting.
Looking closer at diver as in Fig. 14, we can see the proposed method fixed whiteout artifacts produced by Cho et al.’s method, as already observed for synthetic images in Sect. 6.1. In addition, as shown in the close-ups of the waterfall images in Fig. 15, we can see that the proposed method successfully corrected whiteout, and also removed noise in the background sky and compression artifacts around trees, thereby demonstrating the effectiveness of the proposed VB approach for real-world applications.

7. Conclusion

In this paper, we have proposed a method of single-image devignetting that can effectively deal with noise. Through prior-based VB inference, the proposed method jointly estimates both a vignetting function and a latent image without vignetting or noise. Thus, it maintains good performance even when noise is not negligible, as we confirmed experimentally.

One possible future extension of this work would be to integrate the estimation of unknown optical centers of vignetting functions [4], [5] into the proposed VB framework, which can estimate multiple parameters in consideration of their relationship. Supporting non-radial vignetting functions [24] will also extend the capability of the proposed VB vignetting estimation. In addition, applying the proposed method to various techniques, such as panorama image stitching [1], where effective devignetting is required for high-quality image generation, is also of practical interest.

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Appendix A: Derivation of the VB Inference Algorithm

In this section, we detail the derivation of the algorithm discussed in Sect. 4.

First, we derive the ELBO in Eq. (12). Substituting Eq. (9) into the negative of the right-hand of Eq. (11) and separating terms, we obtain the following ELBO:

\[ \text{ELBO} = \mathbb{E} \left[ \ln p(y|x, w) \right] + \mathbb{E} \left[ \ln p(x) \right] + \mathbb{E} \left[ \ln p(w) \right] + H[x] + H[w], \] (A.1)

where \( \mathbb{E} \) and \( H \) denote expectation and entropy with respect to the approximate posteriors, respectively. We can evaluate each term as follows. Taking the logarithmic expectations of Eqs. (2) to (4), we obtain

\[ \mathbb{E} \left[ \ln p(y|x, w) \right] = \frac{n}{2} \ln b - \frac{b}{2} \mathbb{E} \left[ (y - Pw \circ x)^2 \right] + \text{const.}, \] (A.2)

\[ \mathbb{E} \left[ \ln p(x) \right] = \frac{1}{2} a^T \mathbb{E} \left[ (D_n x)^2 \right] + \mathbb{E} \left[ (D_n x)^2 \right] + \text{const.}, \] (A.3)

\[ \mathbb{E} \left[ \ln p(w) \right] = \frac{m}{2} \ln \zeta - \frac{z}{2} a^T \mathbb{E} \left[ (D_w w)^2 \right] + \text{const.}, \] (A.4)

where we have used the linearity of expectation [15]. Furthermore, we decompose the expectations in Eq. (A.2) as follows:

\[ \mathbb{E} \left[ (y - Pw \circ x)^2 \right] = \mathbb{E} \left[ (y - Pw \circ x)^2 \right] - 2 \mathbb{E} \left[ (Pw \circ x) \right] \mathbb{E} \left[ x \right] + \mathbb{E} \left[ (Pw \circ x)^2 \right] \mathbb{E} \left[ x^2 \right], \] (A.5)

where we have used the independence of \( x \) and \( w \) due to the mean-field approximation in Eq. (10). We evaluate the expectations in this equation as follows:

\[ \mathbb{E} \left[ x \right] = \mu_x, \] (A.6)

\[ \mathbb{E} \left[ x^2 \right] = \mu_x^2 + \sigma_x^2, \] (A.7)

\[ \mathbb{E} \left[ Pw \right] = P \mu_w, \] (A.8)

\[ \mathbb{E} \left[ (Pw)^2 \right] = (P \mu_w)^2 + P^2 \sigma_w^2, \] (A.9)

where we have used the linearity and quadratic identity of expectation [15], and our definition of the Gaussian parameters, i.e., the mean and variance of \( x \) are \( \mu_x \) and \( \sigma_x^2 \), and those of \( w \) are \( \mu_w \) and \( \sigma_w^2 \). Substituting Eqs. (A.6) to (A.9) into Eq. (A.5) and completing the square, we obtain

\[ \mathbb{E} \left[ (y - Pw \circ x)^2 \right] = (y - P \mu_w \circ \mu_x)^2 + P^2 \sigma_w^2 \circ \sigma_x^2 + \left( P \mu_w \right)^2 \circ \sigma_w^2 + \mu_x^2. \] (A.10)
Similarly, we evaluate the expectations in Eqs. (A.3) and (A.4) as follows:

\[
E[(D_x x)^2] = (D_x \mu_x)^2 + D_h^2 \sigma_h^2, \quad (A.11)
\]

\[
E[(D_x x)^2] = (D_x \mu_x)^2 + D_v^2 \sigma_v^2, \quad (A.12)
\]

\[
E[(D_w w)^2] = (D_w \mu_w)^2 + D_i^2 \sigma_i^2. \quad (A.13)
\]

Moreover, since \( q(x) \) and \( q(w) \) are independent Gaussian distributions with variances \( \sigma_h^2 \) and \( \sigma_w^2 \), we obtain the entropies of \( x \) and \( w \) using the Gaussian entropy [15] as follows:

\[
H[x] = \frac{1}{2} J^T \ln \sigma_h^2 + \text{const.}, \quad (A.14)
\]

\[
H[w] = \frac{1}{2} I^T \ln \sigma_w^2 + \text{const.} \quad (A.15)
\]

Substituting Eqs. (A.2) to (A.4) with the evaluated expectations in Eqs. (A.10) to (A.13) and the entropies in Eqs. (A.14) and (A.15) into Eq. (A.1), we obtain the ELBO as in Eq. (12).

Next, we derive the update formula for each parameter.

1) Latent Image

First, we collect terms with respect to \( \mu_x \) in Eq. (12):

**ELBO**

\[
\begin{align*}
&= -\frac{b}{2} J^T \left( (y - P \mu_w \circ \mu_x)^2 + P^2 \sigma_w^2 \circ \mu_x \right) \\
&- \frac{1}{2} a^T (D_h \mu_x)^2 + \text{const.} \\
&= -\frac{1}{2} J^T \begin{bmatrix} b \text{diag} \left( (P \mu_x)^2 + P^2 \sigma_w^2 \right) \\
+ D_h \text{diag}(a) D_h \\
+ D_i \text{diag}(a) D_i \\
+ b(P \mu_w \circ y)^T \mu_x + \text{const.}
\end{bmatrix} \mu_x \\
&= b(P \mu_w \circ y). \quad (A.16)
\end{align*}
\]

Then, we take the derivative of the right-hand side with respect to \( \mu_x \) and set it to zero:

\[
\begin{align*}
&= b \left( (P \mu_w)^2 + P^2 \sigma_w^2 \right) \\
&+ D_h^T \text{diag}(a) D_h + D_i^T \text{diag}(a) D_i \\
&= b(P \mu_w \circ y). \quad (A.17)
\end{align*}
\]

Solving this with respect to \( \mu_x \), we obtain Eq. (13).

Next, we collect terms with respect to \( \sigma_h^2 \) in Eq. (12):

**ELBO**

\[
\begin{align*}
&= -\frac{b}{2} J^T \left( (y - P \mu_w \circ \mu_x)^2 + P^2 \sigma_w^2 \circ \sigma_h \right) \\
&- \frac{1}{2} a^T (D_h^2 \sigma_h^2 + D_v^2 \sigma_v) + \frac{1}{2} J^T \ln \sigma_h^2 + \text{const.} \\
&= -\frac{1}{2} \begin{bmatrix} b \left( (P \mu_w)^2 + P^2 \sigma_w^2 \right)^T \\
+ a^T (D_h^2 + D_v^2) \sigma_h \\
+ \frac{1}{2} J^T \ln \sigma_h^2 + \text{const.}
\end{bmatrix} \sigma_h \\
&= \frac{1}{2} (\sigma_h^{-1} - 1). \quad (A.18)
\end{align*}
\]

Solving this with respect to \( \sigma_h^2 \) and set it to zero:

\[
\frac{1}{2} \sigma_h^{-1} = \frac{1}{2} \left( b(P \mu_w \circ y)^2 + P^2 \sigma_w^2 \circ \sigma_h \right) \\
+ D_h^2 + D_i^2. \quad (A.19)
\]

(3) Weights

First, we collect terms with respect to \( b \) in Eq. (12):

**ELBO**

\[
\begin{align*}
&= \frac{n}{2} \ln b \\
&- \frac{b}{2} J^T \left( (y - P \mu_w \circ \mu_x)^2 \\
&+ P^2 \sigma_w^2 \circ \sigma_h \\
&+ (P \mu_w)^2 \circ \sigma_h \right) + \text{const.} \quad (A.20)
\end{align*}
\]
Then, we take the derivative of the right-hand side with respect to \( b \) and set it to zero:

\[
\frac{n}{2} b^{-1} = \frac{1}{2} \mathbf{J}^T \left( (\mathbf{y} - \mathbf{P} \mu_w \circ \mu_x)^2 + \mathbf{P}^2 \sigma_w^2 \circ \sigma_x^2 + (\mathbf{P} \mu_w)^2 \circ \sigma_x^2 + \mathbf{P}^2 \sigma_w^2 \circ \mu_x^2 \right). \tag{A·25}
\]

Solving this with respect to \( b \), we obtain Eq. (17).

Second, we collect terms with respect to \( a_i \) for each \( i \) in Eq. (12):

\[
\text{ELBO} = \frac{1}{2} \ln a_i - \frac{a_i}{2} \left( (\mathbf{D}_h \mu_x)^2 + (\mathbf{D}_v \mu_x)^2 \right) + \text{const}. \tag{A·26}
\]

Then, we take the derivative of the right-hand side with respect to \( a_i \) and set it to zero:

\[
\frac{1}{2} a_i^{-1} = \frac{1}{2} \left( (\mathbf{D}_h \mu_x)^2 + (\mathbf{D}_v \mu_x)^2 \right). \tag{A·27}
\]

Solving this with respect to \( a_i \), we obtain Eq. (18).

Third, we collect terms with respect to \( z \) in Eq. (12):

\[
\text{ELBO} = \frac{1}{2} m \ln z - \frac{z}{2} \mathbf{i}^T \left( (\mathbf{D}_z \mu_w)^2 + \mathbf{D}_z \sigma_w^2 \right) + \text{const}. \tag{A·28}
\]

Then, we take the derivative of the right-hand side with respect to \( z \) and set it to zero:

\[
\frac{m}{2} z^{-1} = \frac{1}{2} \mathbf{i}^T \left( (\mathbf{D}_z \mu_w)^2 + \mathbf{D}_z \sigma_w^2 \right). \tag{A·29}
\]

Solving this with respect to \( z \), we obtain Eq. (19).