A Non-Degenerate Neutrino Mass Signature in the Galaxy Bispectrum

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In the Standard Model, neutrinos are massless, yet oscillation experiments show in fact they do have a small mass. Currently only the differences of the masses’ squares are known, and an upper bound on the sum. However, upcoming surveys of the Universe’s large-scale structure (LSS) can probe the neutrino mass by exposing how neutrinos modulate galaxy clustering. But these measurements are challenging: in looking at the clustering of galaxy pairs, the effect of neutrinos is degenerate with galaxy formation, the details of which are unknown. Marginalizing over them degrades the constraints. Here we show that using correlations of galaxy triplets—the 3-Point Correlation Function or its Fourier-space analog the bispectrum—can break the degeneracy between galaxy formation physics (known as biasing) and the neutrino mass. Specifically, we find a signature of neutrinos in the bispectrum’s dipole moment (with respect to triangle opening angle) that is roughly orthogonal to the contribution of galaxy biases. This signature was missed in previous works by failing to account for how neutrinos alter mode-coupling between perturbations on different scales. Our proposed signature will contribute to upcoming LSS surveys’ such as DESI making a robust detection of the neutrino mass. We estimate that it can offer several-σ evidence for non-zero \( m_\nu \) with DESI from the bispectrum alone, and that this is independent from information in the galaxy power spectrum.

I. INTRODUCTION

With the recent discovery of the Higgs boson, the last missing piece of the Standard Model (SM)—or the first hint of beyond-SM physics—is the neutrino mass scale and hierarchy. Terrestrial experiments such as \( \beta \)-decay \[1\] struggle to reveal either: cosmology offers a nearly unique avenue forward. Our current paradigm predicts a relic neutrino background with number density of \( 112 \text{ cm}^{-3} \) per neutrino flavor. According to neutrino oscillation experiments \[2, 3, 4\] neutrinos are massive with a lower bound on the mass sum of \( m_\nu > 50 \text{ meV} \) and an upper bound around \( m_\nu < 150 \text{ meV} \) \[5\]. Consequently, the relic neutrino background must become non-relativistic at some point in the Universe’s history, as its kinetic energy is diluted by cosmic expansion and eventually becomes subdominant to the rest-mass energy. The time of this transition is set by the mass and in turn imprints a scale on the clustering of matter, potentially enabling large-scale structure (LSS) surveys to detect \( m_\nu \) \[6, 7, 8\].

Given the tremendous volume of data to become available in the next decade via efforts such as Dark Energy Spectroscopic Instrument (DESI: 2019-2024), Euclid (2022-2028), and Roman Observatory (WFIRST: 2025-2030), a robust detection of \( m_\nu \) should be within the reach of near-term cosmology. However, extracting it with certainty from galaxy surveys is not trivial. The typical approach is to measure a decrement in the galaxy power spectrum (Fourier Transform (FT) of the pair correlation function) on scales within the horizon neutrinos reach while they are still relativistic. The neutrinos are homogeneous below this scale and so suppress density fluctuations. However, the decrement must be measured relative to the large-scale amplitude of the power spectrum. This latter cannot be directly obtained from LSS surveys, but rather is taken from the Cosmic Microwave Background (CMB). The large-scale power spectrum amplitude turns out to be degenerate with suppression of CMB fluctuations by free electrons after the Universe is re-ionized. This degeneracy limits the precision available via the power spectrum. Furthermore, the fact that galaxies do not perfectly trace the matter additionally reduces the power spectrum precision on \( m_\nu \). We do not have a full predictive model of how galaxies form. Instead we take an approximate model where the galaxy fluctuations trace different nonlinear functions of the matter density with unknown amplitudes (biases) to be fit for from the data. These biases can mimic the neutrinos’ effect on the power spectrum, so marginalizing over them degrades the \( m_\nu \) constraints.

In this paper we show how we can break the degeneracy of neutrino mass with galaxy biasing by using a unique neutrino signature imprinted on the redshift-space galaxy bispectrum. The bispectrum is the FT of the 3-Point Correlation Function (3PCF), which quantifies excess clustering of galaxy triplets over and above that of a spatially random distribution. We motivate our argument by calculating the neutrino corrections to the
real-space bispectrum. We then proceed to show how the degeneracy between \( m_\nu \) and bias is broken in redshift-space. Redshift space is the actual observable for LSS galaxy surveys. It stems from the fact that while we may observe galaxies two angular coordinates, their line-of-sight position is inferred from a redshift, and this redshift also includes peculiar velocities.

### A. Perturbation Theory Bispectrum

Cosmological perturbation theory (PT) expresses the late-time matter density field as, essentially, a Taylor-expansion around the initial conditions. Each term corresponds to a higher power of the linear density fluctuation \( \delta_1 \equiv \rho_{\text{lin}} / \rho_{\text{lin}} - 1 \), with \( \rho_{\text{lin}} \) the density at the time we initialize with linear theory, \( \rho_{\text{lin}} \) the average density at that time, and \( \delta_1 \ll 1 \) for PT to be valid. Typically we work in Fourier space such that any density (linear or nonlinear) is expanded as:

\[
\delta_\ell(\vec{k}, \tau) = \int \frac{d^3 \vec{k}}{(2\pi)^3} e^{-i\vec{k} \cdot \vec{x}} \delta(\vec{x}, \tau)
\]

In a matter-dominated (Einstein-de Sitter) universe, we may expand the matter density fluctuation \( \delta_\ell \) in a series ordered by powers of the scale factor \( a(\tau) \), with \( \tau \) conformal time. This is known as Eulerian Standard Perturbation Theory (SPT), and we have (e.g. [4]):

\[
\delta_m(\vec{k}, \tau) = \sum_{n=1}^\infty a^n(\tau) \delta_n(\vec{k});
\]

the FT of the matter velocity divergence, \( \tilde{\theta}_m(\vec{k}, \tau) \), can be expanded similarly. The matter is treated as a fluid (though see [10]), and solving the perturbed fluid equations (continuity, Euler, and Poisson) using the ansatz Eq. (2) results in higher-order density and velocity divergence terms as:

\[
\delta_n(\vec{k}) = \int d^3 \vec{q}_1 \cdots d^3 \vec{q}_n \, \delta_D^{[3]} \left( \vec{k} - \sum_{i=1}^n \vec{q}_i \right) \, F_n(\vec{q}_1, \ldots, \vec{q}_n),
\]

\[
\times \, \delta_1(\vec{q}_1) \cdots \delta_1(\vec{q}_n)
\]

\[
\tilde{\theta}_n(\vec{k}) = \int d^3 \vec{q}_1 \cdots d^3 \vec{q}_n \, \delta_D^{[3]} \left( \vec{k} - \sum_{i=1}^n \vec{q}_i \right) \, G_n(\vec{q}_1, \ldots, \vec{q}_n),
\]

\[
\times \, \delta_1(\vec{q}_1) \cdots \delta_1(\vec{q}_n);
\]

where \( F_n \) and \( G_n \) are respectively the SPT density and velocity kernels and \( \delta_D^{[3]} \) is a 3D Dirac delta distribution enforcing momentum conservation. \( F_1 = G_1 = 1 \) and at \( n = 2 \) we have

\[
F_2(\vec{q}_1, \vec{q}_2) = \frac{17}{21} L_0(\vec{q}_1 \cdot \vec{q}_2) + \frac{1}{2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) L_1(\vec{q}_1 \cdot \vec{q}_2)
\]

\[
+ \frac{4}{21} L_2(\vec{q}_1 \cdot \vec{q}_2),
\]

\[
G_2(\vec{q}_1, \vec{q}_2) = \frac{13}{21} L_0(\vec{q}_1 \cdot \vec{q}_2) + \frac{1}{2} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) L_1(\vec{q}_1 \cdot \vec{q}_2)
\]

\[
+ \frac{8}{21} L_2(\vec{q}_1 \cdot \vec{q}_2),
\]

where \( L_\ell \) is a Legendre polynomial of order \( \ell \). The leading-order (tree-level) matter bispectrum represents correlations of matter density fluctuations at three different wave-vectors:

\[
B_m(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \langle \delta_m(\vec{k}_1) \delta_m(\vec{k}_2) \delta_m(\vec{k}_3) \rangle
\]

\[
\times (2\pi)^3 \delta_D^{[3]} \left( \sum_{i=1}^3 \vec{k}_i \right)
\]

where \( \langle \rangle \) means taking an expectation value over many possible draws from the distribution of the initial density fluctuations, which according to the ergodic hypothesis we take as equivalent to averaging over many different spatial regions of the Universe.

We note that PT does not predict the location of each galaxy, but rather the statistical properties of the density field. Therefore, all the information is encoded in the N-Point Correlation Functions (NPCFs) of the density field (though on very small scales see [11]). At linear order, since the density field is believed to be a Gaussian field, all the odd-point correlation functions vanish and even-point functions can be expressed in terms of the linear power spectrum \( P(k) \):

\[
P(k) = \langle \delta_1(\vec{k}_1) \delta_1(\vec{k}_2) \rangle / (2\pi)^3 \delta_D^{[3]}(\vec{k}_1 + \vec{k}_2).
\]

However, since gravitational structure formation induces non-linearity the matter bispectrum offers additional information. Yet galaxy redshift surveys observe not the matter bispectrum but the galaxy bispectrum. Galaxies are biased tracers of the matter, where the bias coefficients encode unknown details of galaxy formation physics that we marginalize over when fitting models. We have (e.g. [12])

\[
\delta_g = b_1 \delta_m + b_2 \left( \delta_m^2 - \langle \delta_m^2 \rangle \right) + b_3 \vec{S}_m.
\]

\( \delta_g \) is the Fourier-space galaxy density fluctuation, \( b_1 \) the linear bias, \( b_2 \) the nonlinear bias, \( b_3 \) the tidal tensor bias, and \( \vec{S}_m \) the matter’s tidal tensor. We adopt this bias model here because the three biases above are the only ones for which there is robust observational evidence in bispectrum or 3PCF measurements ([13, 14]), and we note that we neglect stochastic contributions [12].

Using Eq. (7) we may compute the galaxy bispectrum
as
\[ B_g(\vec{k}_1, \vec{k}_2, \vec{k}_3) = 2b_1^2 P(k_1) P(k_2) \]
\[ \times \left[ f_2(\vec{k}_1, \vec{k}_2) + \gamma \mathcal{L}_0(\vec{k}_1 \cdot \vec{k}_2) + \frac{2\gamma'}{3} \mathcal{L}_2(\vec{k}_1 \cdot \vec{k}_2) \right] + \text{cyc.} \tag{8} \]
where \( \gamma \equiv 2b_2/b_1 \) and \( \gamma' \equiv 2b_3/b_1 \); “cyc.” means to add the two additional permutations corresponding to the first term evaluated at \((\vec{k}_2, \vec{k}_3)\) and \((\vec{k}_3, \vec{k}_1)\).

**B. Massive Neutrinos: Linear Theory**

Neutrinos of mass \( m_\nu \) became non-relativistic at redshift \( 1 + z_\nu \approx 1894 \) (\( m_\nu / \text{eV} \)) which means they contribute to the matter energy density \[ \Omega_\nu \text{.} \] They have energy density \( \Omega_\nu \) and mass fraction \( f_\nu \) \[ \Omega_\nu = \frac{m_\nu}{93.14 \, h^2 \, \text{eV}}, \quad f_\nu = \frac{\Omega_\nu}{\Omega_m} \tag{9} \]
where \( \Omega_\nu \) and \( \Omega_m \) are the neutrino and matter energy densities in units of the critical density.

In what follows we adopt the treatment of [17]. The fluid equation for the velocity divergence in the presence of massive neutrinos can be written as:
\[ \left[ \frac{\partial}{\partial \tau} + \mathcal{H}(\tau) \right] \theta = - \frac{3}{2} \mathcal{H}^2(\tau) - k^2 c_s^2(\tau) \delta \tag{10} \]
where \( \mathcal{H} \equiv d \ln a / d \tau \) is the conformal expansion rate and \( c_s(\tau) \) is the sound speed of massive neutrinos, which can be approximated by their root-mean square thermal velocity \( \sigma_\nu \) (e.g. [15] §9). Similarly to the Jeans length, there is now a new time-dependent scale, the free-streaming scale \[ k_{FS}(\tau) = \sqrt{\frac{3}{2} \mathcal{H}(\tau)} \approx 3 \sqrt{a(\tau)} \Omega_m / 2 \frac{m_\nu}{\text{eV}} } \approx \frac{h}{\text{Mpc}}. \tag{11} \]

On scales larger than \( k_{FS} \) (i.e. \( k < k_{FS} \)), the neutrinos essentially behave like dark matter, contributing a source term to the growth of matter perturbations. In contrast, on smaller scales (i.e. \( k > k_{FS} \)), the neutrinos tend to wash out the matter density perturbations on smaller scales because of neutrinos’ thermal velocities [17].

**II. REAL-SPACE MOTIVATION**

Since the neutrinos change the fluid equations, we expect that they also alter both the linear power spectrum and the PT kernels. Hence they will affect the bispectrum. Now, on scales below the free-streaming scale \( (k > k_{FS}) \), the neutrinos suppress the power spectrum so we have the well-known result \( P \rightarrow P(1 - 8f_\nu) \), and we get an overall suppression of the bispectrum as \( \sim (1 - 16f_\nu) \), in rough agreement with the more careful calculation of [20] for the equilateral limit of the bispectrum. This suppression acts equally on all of the multi-scale moments of the pre-cyclic bispectrum as can be easily seen from Eq. (8), and is perfectly degenerate with the linear bias value. Furthermore, the suppression in the power spectrum will already be used to search for the neutrino mass in the 2-point statistics; consequently remeasuring its effect using the bispectrum is unlikely to add any independent information when combined with 2-point statistics. In this work, we therefore do not explicitly show any of the effects from the power spectrum rescaling. Anywhere \( P(k) \) appears in what follows, it can be taken to include this effect implicitly if one wishes.

It is worthwhile to hope that the effect of the neutrinos on the PT kernels will be a more distinctive signature that is not degenerate with galaxy biasing. So motivated, we now outline how the neutrinos modify the PT kernels to leading order in \( f_\nu \). We denote the modified kernels \( \mathcal{F} \) and \( \mathcal{G} \). We follow [17] but our ultimate goal here is to obtain simple corrections to the bispectrum valid at order \( f_\nu \). We have
\[ \mathcal{F}_2(\vec{q}_1, \vec{q}_2) = \frac{15A_1 + 2A_4 \mathcal{L}_0(\vec{q}_1 \cdot \vec{q}_2)}{21} \tag{12} \]
\[ + \frac{1}{2} \left( A_2 q_1 q_2 + A_3 q_2 q_1 \right) \mathcal{L}_1(\vec{q}_1 \cdot \vec{q}_2) + \frac{A_1}{21} \mathcal{L}_2(\vec{q}_1 \cdot \vec{q}_2), \]
\[ \mathcal{G}_2(\vec{q}_1, \vec{q}_2) = \frac{12C_1 + 4C_4 \mathcal{L}_0(\vec{q}_1 \cdot \vec{q}_2)}{21} \tag{13} \]
\[ + \frac{1}{2} \left( C_2 q_1 q_2 + C_3 q_2 q_1 \right) \mathcal{L}_1(\vec{q}_1 \cdot \vec{q}_2) + \frac{8C_4}{21} \mathcal{L}_2(\vec{q}_1 \cdot \vec{q}_2). \]

The \( A \) and \( C \) coefficients are:
\[ A_1 = \frac{7}{10} \sigma^{(2)}_{11}(q_1, q_2) [f(q_1) + f(q_2)], \]
\[ A_2 = f(q_2) [\sigma^{(2)}_{11}(q_1, q_2) + \sigma^{(2)}_{12}(q_1, q_2) f(q_2)], \]
\[ A_3 = f(q_1) [\sigma^{(2)}_{11}(q_1, q_2) + \sigma^{(2)}_{12}(q_1, q_2) f(q_1)], \]
\[ A_4 = \frac{7}{2} \sigma^{(2)}_{12}(q_1, q_2) f(q_1) f(q_2), \]
\[ C_1 = \frac{7}{6} \sigma^{(2)}_{21}(q_1, q_2) [f(q_1) + f(q_2)], \]
\[ C_2 = f(q_2) [\sigma^{(2)}_{21}(q_1, q_2) + \sigma^{(2)}_{12}(q_1, q_2) f(q_2)], \]
\[ C_3 = f(q_1) [\sigma^{(2)}_{21}(q_1, q_2) + \sigma^{(2)}_{12}(q_1, q_2) f(q_1)], \]
\[ C_4 = \frac{7}{4} \sigma^{(2)}_{22}(q_1, q_2) f(q_1) f(q_2), \]
where \( f(q) = \partial \ln D(q, \tau) / \partial \tau \) is the logarithmic derivative of the linear growth rate \( D(q, \tau) \) and we have suppressed its dependence on \( \tau \). \( \sigma^{(2)} \) is a \( 2 \times 2 \) matrix:
\[ \sigma^{(2)}_{11}(q_1, q_2) = \frac{1}{N^{(2)}} \left[ 2\omega^{(2)} + 1 - 2 \right] \tag{13} \]
\[ \omega^{(2)}(q_1, q_2) = f(q_1) + f(q_2), \]
\[ N^{(2)} = (2\omega^{(2)} + 3)(\omega^{(2)} - 1) + 3f_\nu. \]

Given the growth rate \( D(q, \tau) \), we can obtain \( f \), thence \( N^{(2)} \) and \( \omega^{(2)} \), and finally the matrix \( \sigma \). This in turn
allows computation of the $A_{i}$ and $C_{i}$ and thence of $F_2$ and $G_2$.

Since the neutrino fraction $f_{\nu} \ll 1$, we expand $D(q, \tau)$ to $O(f_{\nu})$ as \cite{21}:

$$D(k, \tau) = \left(1 - \frac{3}{5} f_{\nu}\right) D(\tau)$$

where $D(\tau)$ is the linear growth rate in an EdS cosmology in the absence of neutrinos. The modified kernels are then:

$$F_2(q_1, q_2) = F_2(q_1, q_2) + \frac{4}{245} \left[ L_0(q_1 \cdot \hat{q}_1) - L_2(q_1 \cdot \hat{q}_1) \right] f_{\nu},$$

$$G_2(q_1, q_2) = G_2(q_1, q_2) - \frac{83}{245} \left[ L_0(q_1 \cdot \hat{q}_1) \right] f_{\nu}$$

$$+ \frac{3}{10} \left( \frac{q_1}{q_2} + \frac{q_2}{q_1} \right) L_1(q_1 \cdot \hat{q}_1) + \frac{64}{245} L_2(q_1 \cdot \hat{q}_1) \right] f_{\nu}. \tag{15}$$

We notice that the $F_2$ kernel receives no correction at the dipole ($\ell = 1$) whereas the $G_2$ kernel does.

Comparing with Eq. \cite{8} we see that if $\gamma = -4 f_{\nu} / 245$ and $\gamma' = 6 f_{\nu} / 245$, the nonlinear and tidal tensor biases could perfectly mask the effects of the neutrinos. Put another way, at the level of the real-space bispectrum, nonlinear and tidal tensor biasing are perfectly degenerate with the impact of massive neutrinos.

Importantly, we observe that the nonlinear and tidal tensor biases, at least at the pre-cyclic level, do not enter the $\ell = 1$ (dipole) moment of the bispectrum. Hence, if the neutrinos modified this moment, it would be a non-degenerate signature of their mass. While cyclic summing complicates this picture, the mixing of pre-cyclic multipoles into post-cyclic multipoles is dominantly such that a pre-cyclic multipole contributes most to post-cyclic multipoles at that $\ell$ and higher (as shown in \cite{22} Figure 6 modeling the 3PCF). The angular mixing structure of the 3PCF and of the bispectrum is the same so the conclusion holds for our case as well \cite{24}.

Thus we can expect that the $\ell = 2$ tidal tensor term, which tends to be significant, will minimally mix into the $\ell = 1$ post-cyclic term (see \cite{22} Figure 6, (2, 1) panel). While the $\ell = 0$ contribution of nonlinear bias could mix into post-cyclic $\ell = 1$, it turns out to be a relatively broadband contribution there with rather modest, smooth scale-dependence (see \cite{22} Figure 6, (0, 1) panel).

So motivated, we now investigate whether the redshift-space bispectrum contains a neutrino correction to its dipole. We also check whether the nonlinear and tidal tensor biases enter the dipole once we go to redshift space.

III. REDSHIFT-SPACE BISPECTRUM AND NEUTRINO SIGNATURE

Following \cite{22}, the redshift-space bispectrum may be written as a function of the “internal” triangle parameters $k_1$, $k_2$, and $k_1 \cdot k_2$ and then the “external” parameters $\mu \equiv \hat{k}_1 \cdot \hat{n}$, with $\hat{n}$ the line of sight, and $\omega$, the azimuthal angle of $\vec{k}_2$ about $\vec{k}_1$. After averaging over $\omega$, we can write

$$B_{\omega}(\vec{k}_1, \vec{k}_2, \vec{k}_3) = \sum_{\ell=0}^{\infty} B_{\omega}^{(\ell)}(k_1, k_2, \hat{k}_1 \cdot \hat{k}_2) L_\ell(\mu), \tag{16}$$

with

$$B_{\omega}^{(\ell)}(k_1, k_2, \hat{k}_1 \cdot \hat{k}_2) = \left[ F_2(\vec{k}_1, \vec{k}_2) D_{SQ1}^{\ell} + G_2(\vec{k}_1, \vec{k}_2) D_{SQ2}^{\ell} + D_{NLB}^{\ell} + D_{FOG}^{\ell} \right] b_1^2 P(k_1) P(k_2) + \text{cyc.} \tag{17}$$

$D_{SQ1}$ and $D_{SQ2}$ are respectively the linear and second-order squashing terms, $D_{NLB}$ is the contribution from non-linear biasing, and $D_{FOG}$ is what \cite{26} terms “finger of God” but in fact is a deterministic term due to the Kaiser formula and does not include smearing due to thermal velocities.

In this work, we focus on the isotropic bispectrum, so we average over all possible triangle orientations (i.e. we integrate over $\omega$, which picks out the external $L = 0$ (monopole) moment of the bispectrum by orthogonality).

Using Eqs. (24), (26), and (30) of \cite{25} and Taylor-expanding all terms in our Eq. \cite{17} to $O(f_{\nu})$, including the alteration in $f$, which enters via $\beta = f / b_1$ in the equations of \cite{25}, we find that the nonlinear bias and the tidal tensor bias do not contribute to the redshift-space bispectrum “internal” dipole. We then find the neutrino contribution to the dipole as

$$B_{\omega}^{(\ell=0)}(k_1, k_2) \approx - \frac{b_1^2}{5} \left( \frac{4 + b_1}{f_{\nu}} \right) f_{\nu} \left( \frac{k_2}{k_1} + \frac{k_1}{k_2} \right) f^{\nu=0} P(k_1) P(k_2) + \text{cyc.} \tag{18}$$

Eq. \cite{18} is the main result of this paper. $f^{\nu=0}$ is the logarithmic derivative of the linear growth rate in an EdS cosmology in the absence of neutrinos (see Eq. \cite{14}). We note that we here present our result only at $O(\beta)$; since $\beta \approx 1/3$ for e.g. the main cosmology sample (CMASS; Constant stellar Mass Sample) of the Sloan Digital Sky Survey Baryon Oscillation Spectroscopic Survey (SDSS BOSS), the main precursor to DESI, $O(\beta^2)$ and higher terms are suppressed. We did compute them but in the interest of brevity defer presenting them to work in prep. that will more fully explore neutrino effects in the bispectrum and 3PCF. They do not introduce any dependence on nonlinear or tidal-tensor biasing into the bispectrum dipole, and so do not alter the qualitative findings of this work.

Now, at the pre-cyclic level, as we have already noted, the bispectrum dipole has no contribution from either nonlinear or tidal tensor biasing. After cyclic summing and re-projection onto Legendre polynomials, the tidal tensor bias does enter very, very modestly into the $\ell = 1$ term, and the nonlinear bias term will enter with a slightly larger amplitude (see \cite{22} Figure 6). However,
neither of these terms have the unique \(k_1/k_2+k_2/k_1\) scale dependence of the original, pre-cyclic \(\ell = 1\) term, which will also contribute to its post-cyclic analog. Hence, the scale dependence can be further used to break any remaining (likely minimal) degeneracy between the biases and the neutrino mass.

We also note that baryon-dark matter relative velocities \([27]\) can, in principle, couple to galaxy formation \([28]\) and add an additional bias term to the bispectrum \([29]\) or 3PCF \([21]\) that enters the dipole. However, observationally, at least in the 777, 202 Luminous Red Galaxies (LRGs) of the BOSS CMASS sample, this bias has been constrained to \(< 1\%\) using both 3PCF \([31]\) and power spectrum \([32]\). Furthermore, as shown in \([30]\) it has a very unique scale dependence that can be used to disentangle it from the standard 3PCF (or bispectrum) dipole term: thus we expect it would not be a significant degeneracy with neutrino mass measurement. Nonetheless, the fact that it enters the same term argues that careful treatment of the relative velocity in future survey analyses is warranted if robust use of the bispectrum or 3PCF for neutrino mass is desired.

IV. OPTIMAL LINEAR BIAS

Observing that the linear bias enters our neutrino signature Eq. \((\ref{nu})\) in a rather unique way, we can ask what value of \(b_1\) is optimal for detecting this signature. Naively, one might seek to maximize this term’s ratio to the other contributions at the dipole, which scale as \(b_1^2\). However, if we assume that \(b_1\) can be very well-determined both from 2-point statistics or from the \(\ell = 2\) term of the bispectrum, which has a large amplitude (see \([22]\) Figure 7), then we do not need to optimize this ratio, as the non-\(f_p\) piece of the bispectrum dipole can simply be subtracted off as “known” in an analysis.

Rather, we should seek to maximize the signal-to-noise on our signature Eq. \((\ref{nu})\). For simplicity let us assume that we work at some fiducial wave number \(k\), such that \(k_1 \sim k_2 \sim k_3 \approx k\). We focus on the leading-order covariance, which comes from the Gaussian Random Field contribution \(6P_{\text{obs}}P_{\text{obs}}P_{\text{obs}}\). \(P_{\text{obs}}\) is the observed galaxy power spectrum, approximated here as \(b_1^2P + 1/n\) where \(1/n\) is the “shot noise” (or Poisson noise) caused by the discrete nature of galaxies, and \(n\) is the number density of the survey. We then have the noise as the square-root of the covariance, i.e. \(N \sim 2.4\left[b_1^2P(k_*) + 1/n\right]^{3/2}\). Using Eq. \((\ref{nu})\) as our signal, we may optimize the \(S/N\) with respect to \(b_1\) by setting \(\partial(S/N)/\partial b_1 = 0\) and solving for \(b_1\) \([33]\). We find

\[
 b_{1,\text{best}} = \frac{16}{3 + \sqrt{9 + 128nP(k_*)}}. \quad (\ref{nu})
\]

Typically galaxy surveys (such as BOSS and DESI) are designed to have \(nP \sim 1\) at roughly the Baryon Acoustic Oscillation (BAO) scale, \(k_{\text{BAO}} \sim 0.01\ h/\text{Mpc}\). This scale is well within the regime where PT is valid and so we would expect our computed neutrino signature to be a reasonable model on such scales. Taking \(k_* \sim k_{\text{BAO}}\) and hence \(nP \sim 1\) in Eq. \((\ref{nu})\), we obtain \(b_{1,\text{best}} \sim 1\).

Interestingly, this is lower than the bias either of LRGs \((b_1 \approx 2\) for BOSS) or of the Emission Line Galaxies (ELGs) that will be used in addition to LRGs for DESI. However, if voids, which have negative bias, could be combined with galaxies with an appropriate weighting, one could form a sample with the desired linear bias of order unity. \([34]\) proposes that having zero-bias tracers is desirable for constraining Primordial Non-Gaussianity (PNG), and suggests that such a sample can be constructed by using environmental information; such an approach might be another route to constructing the ideal sample for our application. Furthermore, we note that the bias of neutral hydrogen (HI) clouds is thought to be order unity at \(z < 1\) \([35]\): this argues that either direct study of their bispectrum, or indirect study of it via the absorption pattern on quasars (the Lyman-\(\alpha\) forest), may be of significant benefit for constraining the neutrino mass.

V. DETECTABILITY ESTIMATE

We now estimate the detectability of the neutrino signature here identified. Importantly, we can essentially ignore the need to either constrain or marginalize over nonlinear and tidal tensor bias: they are not degenerate with our neutrino signature. To estimate the precision on the \(\ell = 1\) term, we note that this term is also the most significant source of the BAO information in the 3PCF and bispectrum \([22]\) Figure 7). The BAO are very roughly, an order 5\% feature in the bispectrum \([36]\) Figure 3, see also \([37]\) Figure 3). Hence, in the reasonable approximation that all the BAO information is in the dipole, the 5\% BAO detection with BOSS of \([14]\) and \([38]\) implies the 1\% error on the total dipole is 1\%. Our signature’s ratio to the total dipole is roughly \((4 + b_1)\beta f_\nu/\left[5(1 + (4/3)\beta)\right] \approx 0.3\%\), using \(b_1 = 2\) and \(\beta = 0.37\) for BOSS CMASS and \(f_\nu \approx 1\%\). This ratio implies the signature should be detectable at 0.3\% in BOSS. DESI will have roughly \(30\times\) the BOSS volume, and so the detection significance should be roughly \(\sqrt{30}\) larger, to give 2\%. We note that this estimate should not be taken too literally, as it depends on a number of assumptions. More detailed forecasting will appear in future work.

The most appropriate interpretation of our estimate above is simply that it shows that the bispectrum dipole signature is within reach of future surveys. DESI forecasts indicate a 3\% detection of \(m_\nu\) from the power spectrum alone even if it is near its minimum \([35]\); for our fiducial value of \(f_\nu = 1\%\) above this corresponds to 6\%, somewhat better than the bispectrum dipole. However, the bispectrum dipole can be added in independently because we have not incorporated the power spectrum sup-
pression in our signature. Critically, though, the power spectrum forecasts do not include any galaxy biases other than linear bias. Hence realistically we expect the bispectrum dipole to offer a larger fractional improvement on power-spectrum alone than the forecast above suggests.

As noted earlier, to use the dipole to constrain \( m_\nu \), everything else entering it must be perfectly known. In detail, this requires \( f^\nu=0 \approx \Omega_m^{0.55} \) and also \( \sigma_8 \), the rms amplitude of the matter clustering on 8 Mpc/h spheres. Folding in the errorsbars on these quantities from Planck as \( \sigma(\sigma_8) = 0.8\% \) and \( \sigma(\Omega_m) = 2\% \), and noting that \( \sigma_8^x \) enters each power spectrum in Eq. (18), we find \( \sigma(f^\nu) \) would be negligibly degraded. Adding in a 1% error on linear bias (though DESI will do much better) also negligibly inflates the error budget. Finally, we note that if one considers non-General Relativity (GR) models of gravity, then \( f^\nu=0 \approx \Omega_m^\nu \), with \( \gamma \) now to be constrained from the data as well \( \sigma(\gamma) \). This point illustrates that constraining non-GR theories is important for robustly measuring \( m_\nu \) with LSS.

VI. DISCUSSION & CONCLUSIONS

There are few previous works on the effect of neutrinos on the bispectrum. \[20\] computed the leading-order real-space bispectrum effect in PT. Rather than modifying the kernels themselves, this work expands in terms of CDM and neutrino density fluctuations respectively \( \delta_c \) and \( \delta_\nu \), and treats the latter as a rescaling of the former (their equations 3.43 and 3.44). Hence the neutrino contribution in their work, which looks like \( \delta_c \delta_c \delta_\nu \), is proportional to \( \delta_c \delta_\nu \). This means that the neutrino contribution as they compute it will simply be an overall rescaling of the bispectrum. \[41\] pursues neutrino effects in the real-space bispectrum up to one-loop level, but again do not modify the PT kernels and rather treat neutrino contributions by replacing the power spectrum with a cross power spectrum between CDM and neutrinos (their equations 2.11 and 2.12). This will result in an overall rescaling of the bispectrum much as we have discussed incorporating the power spectrum in our model would do. \[41\] also compared their work with simulations, finding reasonable agreement. \[41\] developed the Effective Field Theory (EFT) of LSS including neutrinos in the power spectrum, and more recently, \[12\] extended this formalism to the bispectrum; they find the dominant neutrino effect (90%) is in the change to the CDM growth rate. However, they do not account for alterations to the PT kernels (see their equation 3.11), so it is not clear the calculation indeed captures the full neutrino effect.

\[43\] is the most recent study of neutrino mass in the redshift-space bispectrum. They used a large suite of N-body simulations to forecast constraining power, meaning that they should have been sensitive to both suppression of the power spectrum and alterations to the mode-coupling (the effect we study). Their focus was breaking the degeneracy between \( \sigma_8 \) and \( m_\nu \). They did not consider nonlinear or tidal tensor bias. Working to \( k_{\text{max}} = 0.5 \text{h/Mpc} \), they found \( \sigma(m_\nu) = 57 \text{meV} \) for a 1 [Gpc/h]^3 survey. Scaling to BOSS using the effective volume of 2.43 [Gpc/h]^3 of \[14\], this implies \( \sigma(m_\nu) = 37 \text{meV} \). This is rather tighter than our expected BOSS constraint. Consequently we conjecture that \[43\] is exploiting information beyond the bispectrum dipole signature. However, how much of that information remains accessible after marginalizing over nonlinear and tidal tensor bias remains for further investigation.

Our projected constraint stems only from the bispectrum dipole, which is protected from these biases. Furthermore, the BAO detection \[14\] we exploited to forecast our expected detection of \( m_\nu \) did not exploit anywhere near the small scales implied by \[43\]'s choice of \( k_{\text{max}} \), but goes very roughly to only about \( k_{\text{max}} \approx 0.3 \text{h/Mpc} \). Realistically, pushing down to the scales harnessed in \[43\] requires detailed modeling of nonlinearity and baryonic effects can also become relevant \[45\]. CMB lensing is another promising technique for studying \( m_\nu \), especially with the advent of next-generation efforts such as CMB-S4 \[46\]. While not degenerate with galaxy biasing, part of the constraint does come from small scales and thus again baryonic effects can be problematic unless carefully modeled \[47\]. Thus we argue it is preferable to adopt a method that only requires relatively large scales that remain under perturbative control.

What distinguishes our work from \[20\] and \[40\] is our analysis of how \( m_\nu \) alters the coupling of different Fourier modes as structure forms, expressed by the perturbation theory kernels. We thus show how, at the pre-cyclic level, \( m_\nu \) produces not just a rescaling but a geometric effect that alters the predicted clustering pattern as a function of the full triangle shape, including the open angle (the power spectra depend only on side lengths). It is the difference in shape dependence of the biasing terms (they depend only on even-parity functions of opening angle, \( \ell = 0 \) and \( \ell = 2 \)) and the neutrino effect (at \( \ell = 1 \)) that enables breaking the degeneracy.

Finally, we should note that the treatment of \[17\] has had modifications suggested to it. First, approximating the neutrinos by their linear contribution violates momentum conservation on small scales \[16\]. However, our approach does not require highly non-linear scales. Since \( k_{\text{FS}} \ll k_{\text{NL}} \) there is a large range of perturbative \( k \) where our approach is valid. We also note that \[48\] identifies new bias terms that enter the redshift-space bispectrum if there are selection effects, e.g. from the way a spectroscopic survey is targeted from an imaging survey. Their Table 1 shows three terms that enter the dipole. Though no such terms have yet been observationally detected, how to protect the dipole against these biases is worth investigating in future work.

Overall, we have shown that the redshift-space bispectrum’s dipole is a promising tool that should significantly strengthen the power of upcoming surveys such as DESI to constrain the neutrino mass. The fact that our signature appears only in redshift space also argues further...
work should be done on anisotropic bispectrum \[49, 50\] and 3PCF \[51\], along the lines of \[52, 53, 54\], and \[55\] but including neutrino mass. Future work will be to perform more detailed calculations of the effect we have found and to convert our bispectrum model to one for the 3PCF, as this latter has advantages regarding application to survey data (e.g. \[56, 57, 14\]), especially in mitigating observational systematics such as fiber assignment (e.g. \[58\]), which will be a particular challenge in DESI.

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using the plane wave expansion into spherical harmonics and spherical Bessel functions, the spherical harmonic addition theorem, and orthogonality.

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