Curved Domain Walls of Five Dimensional Gauged Supergravity

A. H. Chamseddine [1] and W. A. Sabra [1]

Center for Advanced Mathematical Sciences (CAMS) and Physics Department, American University of Beirut, Lebanon.

Abstract

We study curved domain wall solutions for gauged supergravity theories obtained by gauging some of the isometries of the manifold spanned by the scalars of vector and hypermultiplets. We first consider the case obtained by compactifying M-theory on a Calabi-Yau threefold in the presence of G-fluxes. It is found that supersymmetry allows for the construction of domain wall configurations with curved worldvolume and a cosmological constant. However it turns out that the equations of motion, if one insists on the supersymmetric ansatz for the scalars and warp factor, rule out solutions with a cosmological constant and allows only for Ricci-flat worldvolumes. Moreover, in the absence of flux, there are non-supersymmetric solutions with worldvolumes given by Einstein manifolds. Our results are then generalized to all five dimensional gauged supergravity theories with vector and hypermultiplets.

*email: chams@aub.edu.lb
†email: ws00@aub.edu.lb
1 Introduction

Recently domain walls and black holes as solutions of five-dimensional gauged supergravity have been a subject of intensive research. This to a large extent has been motivated by the suggestion of Randall and Sundrum [1] that four dimensional Einstein gravity can be recovered provided we live on a domain wall embedded in anti-de Sitter space. Ultimately the aim is to embed such a model in string or $\mathcal{M}$ theory. The difficulty so far in achieving this is due to the need of a specific model which gives rise to a supersymmetric flow connecting two stable infra-red fixed points with the same cosmological constant. In addition, explicit solutions of ungauged and gauged supergravity theory provide the foundation for a microscopic understanding of black hole physics and also play an important role in the conjectured AdS/CFT correspondence [2]. Here, the anti-de Sitter geometry arises as the vacuum of gauged supergravities in various dimensions. This correspondence may provide the possibility to study the nonperturbative structure of the field theories living on the boundary by means of classical supergravity solutions. Supersymmetric black holes and strings, [3] as well as non-supersymmetric generalizations [4], have been constructed for the $U(1)$ gauged $N = 2$ supergravity [5]. A specific model of these theories, namely the STU model with three vectormultiplets constitutes a consistent truncation of the $N = 8$ theory.

These results motivate a further research in this field, in particular it is of interest to find new solutions to general gauged supergravity theories with non trivial vector and hypermultiplets scalars, such as black holes and domain walls preserving some of the original supersymmetries. Also, one would like to investigate generalizations of the gauged supergravity models in order to find a model which may incorporate Randall-Sundrum scenario in a supersymmetric setting. Domain walls and black hole solutions for gauged theories with gauged isometries of the hypermultiplets have been discussed very recently in [6, 7]. Curved domain wall solutions have also been addressed in [8].

In this paper we are mainly interested in the study of curved domain wall solutions of gauged supergravity with vector and hypermultiplets. We consider the models obtained from the compactification of $\mathcal{M}$ theory on Calabi-Yau threefolds in the presence of background $G$-fluxes. In the absence of fluxes, it is well known that the effective field theory obtained is $N = 2$ five-dimensional supergravity with hyper and vectormultiplets whose number depends on the Betti numbers of the Calabi-Yau threefold. The scalar fields of these multiplets parametrize a manifold $\mathcal{M} = \mathcal{M}_H \otimes \mathcal{M}_V$. The scalars of the hypermultiplets live on a quaternionic manifold $\mathcal{M}_H$, and those of the vectormultiplets live on a very special real manifold $\mathcal{M}_V$. In the presence of a non-trivial $G$-flux, the axion becomes charged in the effective five dimensional theory and one obtains a gauged supergravity model with a scalar potential which depends on the volume of the Calabi-Yau as well as the scalars of the vectormultiplets.

\[\text{For example, the AdS}_5 \times S^5 \text{ compactification of type IIB theory gives } D = 5, N = 8 \text{ gauged supergravity. The isometries of } S^5 \text{ lead to a } SO(6) \text{ gauging, which in turn may be identified with the } SO(6) \text{ } R\text{-symmetry of the } D = 4, N = 4 \text{ super-Yang-Mills theory on the boundary.}\]
The main result of this work is that one may find supersymmetric field configurations which preserve a fraction of supersymmetry but which do not constitute solutions of the equations of motion. This should not come as a surprise since it has already been established that in searching for supersymmetric bosonic vacua, it is not enough to check for the existence of parallel or Killing spinors. For instance, it was demonstrated in [19] that for Lorentzian spaces, Ricci-flatness is not an integrability condition for the existence of parallel spinors. Moreover, from the analysis of the equations of motion, it turns out that a domain wall solution with a flat worldvolume, can be generalized to a solution where the worldvolume is replaced with a Ricci-flat metric for the same values of the scalar fields and warp factors. In the absence of flux, (i.e., ungauged five dimensional supergravity) the equations of motion allow for non-supersymmetric solutions with a positive cosmological constant. These results are then generalized to any gauged supergravity model in five dimensions.

The paper is organized as follows. In the next section we study curved domain wall solutions of M theory on a Calabi-Yau threefold in the presence of background flux. These solutions correspond to M5-branes wrapped over holomorphic curves in Calabi-Yau space. In section three we generalize our results to all gauged supergravity theories with vector and hypermultiplets. Finally we summarize and discuss our results.

2 Curved Domain Walls

In the following, we will study curved domain wall solutions of the low-energy effective theory of M-theory on a Calabi-Yau threefold with background flux. For details of the reduction, the reader is referred to [10, 11]. This theory can also be obtained by gauging both the $U(1)$ subgroup of the $R$–symmetry and the axionic shift present in the universal hypermultiplet [12, 13]. We will consider the case where we only keep the universal hypermultiplet. A hypermultiplet is present in any Calabi-Yau compactification of M-theory and type IIA string theory. For example, compactifying M-theory or type IIA string theory on a rigid Calabi-Yau threefold (i.e., $h_{2,1} = 0$) leads to an $N = 2$ theory with a single hypermultiplet, the so-called universal hypermultiplet [13]. Classically, the scalar fields of the universal hypermultiplet parameterize the group coset manifold $SU(2,1)/U(2)$ [14, 15]. Define the complex coordinates

\[ S = e^{-2\phi} + ia + \chi_1^2 + \chi_2^2, \quad C = \chi_1 + i\chi_2. \] (2.1)

The moduli space is Kähler with Kähler potential

\[ \mathcal{K} = \phi = -\frac{1}{2} \ln \left( \frac{S + \bar{S}}{2} - |C|^2 \right). \] (2.2)

\footnotetext[2]{the term “universal” is slightly misleading for compactifications with $h_{2,1} > 0$, see [8] section 4.4.3.}

\footnotetext[3]{In the above action, $\phi$ is associated with the volume of the Calabi-Yau, $a$ comes from the dual of the four-form of eleven dimensional supergravity, $F = dA_3$, and $C$ corresponds to the expectation values of $A_3$.}
Using the coordinates \( q^X = (S, \bar{S}, C, \bar{C}) \), the metric components are

\[
\begin{align*}
g_{S\bar{S}} &= \frac{1}{4} e^{4\phi}, \\
g_{S\bar{C}} &= -\frac{1}{2} Ce^{4\phi}, \\
g_{S\bar{C}} &= -\frac{1}{2} Ce^{4\phi}, \\
g_{C\bar{C}} &= e^{2\phi} + CC e^{4\phi}.
\end{align*}
\] 

(2.3)

The metric can be written as

\[
ds^2 = u \otimes \bar{u} + v \otimes \bar{v},
\]

(2.4)

where we have introduced the vielbein forms by

\[
\begin{align*}
u &= e^{\phi} dC, \\
v &= e^{2\phi} \left( \frac{dS}{2} - \bar{C} dC \right).
\end{align*}
\]

(2.5)

In addition to the hypermultiplet, the theory also contains the \( N = 2 \) supergravity multiplet and \( n_V \) vectormultiplets. The scalars \( \phi^x, \ x = 1, \ldots, n_V \) of the vectormultiplets parametrize a very special real manifold \( \mathcal{M}_V \) described by an \( n_V \)-dimensional cubic hypersurface

\[
C_{IJK} h^I(\phi^x) h^J(\phi^x) h^K(\phi^x) = 1
\]

(2.6)

of an ambient space parametrized by \( n_V + 1 \) coordinates \( h^I = h^I(\phi^x) \), where \( C_{IJK} \) are the intersection numbers of the Calabi-Yau threefold.

For our domain wall solutions, the gauge fields are irrelevant and therefore our Lagrangian is given by\footnote{In this paper, the indices \( A, B \) represent five-dimensional flat indices, \( A = (a, 5) \). Curved indices are represented by \( M = (\mu, z) \).}

\[
\mathcal{L} = E \left( \frac{1}{2} R - \partial_M \phi \partial^M \phi - \frac{3}{4} G_{IJ} \partial_M h^I h^J - \mathcal{V}(\phi, q) \right)
\]

(2.7)

where

\[
\mathcal{V} = \frac{g^2}{8} e^{4\phi} G^{IJ} \alpha_I \alpha_J
\]

(2.8)

is the scalar potential of the theory, \( E = \sqrt{-\det g_{MN}} \) and \( \alpha_I \) are the flux vectors. We also have the following relations coming from the underlying very special geometry\footnote{In this paper, the indices \( A, B \) represent five-dimensional flat indices, \( A = (a, 5) \). Curved indices are represented by \( M = (\mu, z) \).}

\[
\begin{align*}
h^I h^I &= 0, \\
h^I h^J G_{IJ} &= g_{xy}, \\
G_{IJ} &= h_I h_J + h^I h^J g_{xy}, \\
\partial_x h_I &= \sqrt{\frac{2}{3}} h_{Ix}, \\
\partial_x h^I &= -\sqrt{\frac{2}{3}} h^I.
\end{align*}
\]

(2.9)
Note that only the field $\phi$ of the hypermultiplet is kept as a dynamical variable. The supersymmetry transformations of the fermionic fields (gravitini, gaugini and hyperini) in this model allow for the splitting of the spinors components \[7\] and for simplicity we will concentrate on the equations for the spinor $\epsilon^1$ and drop the spinor indices. Therefore the supersymmetry transformations we wish to study in a bosonic background (and the absence of gauge fields) are

\[
\delta \psi_M = \left( \partial_M + \frac{1}{4} \Omega_M^{AB} \Gamma_{AB} + \frac{1}{4 \sqrt{6}} g e^{2\phi} \Gamma_M W \right) \epsilon, \quad (2.10)
\]

\[
\delta \zeta = -i \frac{e^{2\phi}}{4} \left( \Gamma^M \partial_M e^{-2\phi} + \frac{\sqrt{6}}{2} g W \right) \epsilon, \quad (2.11)
\]

\[
\delta \lambda_x = i \frac{3}{4} \left( 3 \partial_M h_I \Gamma^M + \sqrt{\frac{3}{2}} g e^{2\phi} \alpha_I \right) \partial_x h^I \epsilon. \quad (2.12)
\]

Here $W = \alpha_I h^I$ and $\Omega_M^{AB}$ are the spin connections. Our purpose is to find supersymmetric domain wall solutions with curved worldvolumes, this is achieved by solving the equations obtained by setting the supersymmetry transformations of the fermi fields to zero. Solutions with flat worldvolumes have been considered first in \[10\]. The metric of our curved brane can be put in the form

\[
ds^2 = e^{2U(z)} g_{\mu\nu}(x) dx^\mu dx^\nu + dz^2 \quad (2.13)
\]

and we assume that all the dynamical scalar fields of the theory depend only on the fifth coordinate $z$. The non vanishing spin connections for our metric are given by

\[
\Omega_{\mu a b}(x, z) = \omega_{\mu a b}(x),

\Omega_{\mu a 5}(x, z) = U' e^U e_{a \mu}(x). \quad (2.14)
\]

Let us begin with the gravitino supersymmetry transformation \[2.10\]. Using our ansatz, this gives for the spatial components

\[
\delta \psi_\mu = \left( D_\mu + \frac{1}{2} U' e^U \gamma_\mu \gamma_5 + \frac{1}{4 \sqrt{6}} g e^{2\phi} e^U \gamma_\mu W \right) \epsilon, \quad (2.15)
\]

where we have used

\[
\Gamma_\mu = e^U \gamma_\mu, \quad \Gamma_5 = \gamma_5, \quad D_\mu = \partial_\mu + \frac{1}{4} \omega_\mu^{ab} \gamma_{ab}. \quad (2.16)
\]

Here the prime denotes differentiation with respect to $z$. From the fifth component, we obtain
\[
\delta \psi_z = \left( \partial_z + \frac{1}{4\sqrt{6}}g e^{2\phi} e^U \gamma_5 W \right) \varepsilon. 
\]

(2.17)

For the hyperino (2.11), we obtain

\[
\delta \zeta = -\frac{i}{4} e^{2\phi} \left( \gamma_5 \partial_z e^{-2\phi} + \frac{\sqrt{6}}{2} g W \right) \varepsilon.
\]

(2.18)

Assuming the following projection condition

\[
\gamma_5 \varepsilon = -\varepsilon,
\]

(2.19)

then the vanishing of the hyperino supersymmetry transformation implies the following constraint

\[
g W e^{2\phi} = -\frac{4}{\sqrt{6}} \phi',
\]

(2.20)

which in turn implies for the vanishing of the gravitino transformations

\[
\left( D_\mu - \frac{1}{6} e^U (\phi' + 3U') \gamma_\mu \right) \varepsilon = 0,
\]

\[
\left( \partial_z + \frac{1}{6} \phi' \right) \varepsilon = 0.
\]

(2.21)

This implies that \( \varepsilon = e^{-\frac{1}{6} \phi} \varepsilon(x) \varepsilon_0 \), where \( \varepsilon_0 \) is a constant spinor satisfying (2.19) and \( \varepsilon(x) \) depends only on the worldvolume coordinates. Integrability of the above equations implies that

\[
\frac{1}{6} e^U (\phi' + 3U') = c,
\]

(2.22)

where \( c \) is a constant, we thus obtain

\[
D_\mu \varepsilon(x) = \left( \partial_\mu + \frac{1}{4} \omega_\mu^{ab}(x) \gamma_{ab} - c \gamma_\mu \right) \varepsilon(x) = 0.
\]

(2.23)

This is the Killing spinor equation for a purely gravitational background in four dimensions. Therefore to obtain supersymmetric domain walls in five dimensions, the four dimensional worldvolume must be a Lorentzian manifold admitting Killing spinors. We shall come to this point later.

The vanishing of the gaugino supersymmetry variation yields

\[
\left( \partial_z h_I - \frac{g}{\sqrt{6}} e^{2\phi} \alpha_I \right) \partial_z h^I = 0.
\]

(2.24)
This implies that the quantity in the bracket must be proportional to \( h^I \) due to very special geometry. This will then imply that

\[
\left( \partial_z h_I - \frac{g}{\sqrt{6}} e^{2\phi} \alpha_I \right) = -\frac{g}{\sqrt{6}} e^{2\phi} W h_I. \tag{2.25}
\]

This gives using (2.20), a simple differential equation

\[
\partial_z (e^{-\frac{2}{3} \phi} h_I) = \sqrt{\frac{1}{6}} g e^{\frac{4}{3} \phi} \alpha_I, \tag{2.26}
\]

which can be easily integrated if we perform the following change of variable

\[
\frac{dw}{dz} = e^{\frac{4}{3} \phi}
\]

and the solution is given by

\[
e^{-\frac{2}{3} \phi} h_I = \frac{w}{\sqrt{6}} g \alpha_I + q_I \tag{2.27}
\]

where \( q_I \) are constants. The above equation is normally referred to as the attractor equation. Therefore the scalars of the theory can now be determined in terms of algebraic equations. To fix the solution completely (at least in terms of algebraic equations), we need to solve the differential equation (2.22),

\[
(\phi' + 3U') = 6ce^{-U}.
\]

This can be easily integrated using the solutions of the scalar fields to give

\[
e^{U(w)} = e^{-\frac{2}{3} \phi} \left( 2c \int e^{-\phi(w')} dw' + \text{const} \right).
\]

To summarize, our domain wall configurations are given by

\[
\begin{align*}
 ds^2 &= e^{2U(w)} g_{\mu\nu}(x) dx^\mu dx^\nu + e^{-\frac{2}{3} \phi} dw^2, \\
 Y_I &= h_I e^{-\frac{2}{3} \phi} = H_I = \frac{w}{\sqrt{6}} g \alpha_I + q_I, \\
 e^{-\phi} &= C_{1JK} Y^I Y^J Y^K, \quad Y^I = e^{-\frac{2}{3} \phi} h^I. \tag{2.28}
\end{align*}
\]

This configuration is BPS if the four dimensional metric \( g_{\mu\nu}(x) \) admits Killing spinors.

We now turn to the study of the equations of motion for our model and see whether the above supersymmetric configuration is indeed a solution of the theory. The Einstein equations obtained from the action (2.7) for our ansatz read

\[
\begin{align*}
 R^{(5)}_{\mu\nu} &= R^{(4)}_{\mu\nu} - g_{\mu\nu} e^{2U} (U'' + 4U'^2) = \frac{2}{3} g_{\mu\nu} \mathcal{V} e^{2U}, \\
 R_{zz} &= 2\phi'^2 + \frac{3}{2} G_{IJ} \partial_z h^I \partial_z h^J + \frac{2}{3} \mathcal{V} = -4(U'' + U'^2). \tag{2.29}
\end{align*}
\]
where $(R_{\mu\nu}^{(5)}, R_{zz})$ and $R_{\mu\nu}^{(4)}$ are the Ricci-tensors of the domain wall and its four dimensional worldvolume respectively. For the scalar field $\phi$, related to the Calabi-Yau volume, one gets

$$\frac{1}{E} \partial_M (E g^{MN} \partial_N \phi) = \frac{1}{2} \frac{\partial \mathcal{V}}{\partial \phi}.$$  \hfill (2.30)

To analyze Einstein equations of motion, we plug in our ansatz for the scalar fields, this gives

$$R_{\mu\nu}^{(4)} = 3 g_{\mu\nu} e^{2U} (U'^2 - \frac{1}{9} \phi'^2).$$ \hfill (2.31)

If we turn to the $\phi$ equation of motion, we obtain from (2.30)

$$\phi'' + 4\phi'U' = 2\mathcal{V},$$ \hfill (2.32)

where we have used $E = \sqrt{-\det g_{MN}} = e^{4U} \sqrt{-\det(g_{\mu\nu})}$. Using (2.20), (2.26) and (2.22) we conclude that $c = 0$, this yields

$$\phi' + 3U' = 0,$$
$$R_{\mu\nu}^{(4)} = 0.$$ \hfill (2.33)

and hence the worldvolume metric of the domain wall solutions is Ricci-flat. We conclude that supersymmetry allows for the possibility of supersymmetric four dimensional worldvolume configurations with a negative cosmological constant. However, these configurations satisfy the equations of motion only if the cosmological constant is zero and the worldvolume is Ricci-flat.

A close inspection of the equations of motion reveals that if we consider the supergravity cases where the potential $\mathcal{V}$ is zero, such as ungauged supergravity models, then in the study of supersymmetric domain walls, the scalar fields of the theory decouple and Einstein equations of motion give,

$$R_{\mu\nu}^{(5)} = R_{\mu\nu}^{(4)} = g_{\mu\nu} e^{2U} (U'' + 4U'^2) - 4(U'' + U'^2) = 0,$$

In this case, the vanishing of the gravitino supersymmetry variation gives

$$\delta \psi_\mu = \left( \partial_\mu + \frac{1}{4} \omega_{\mu}^{ab} \gamma_{ab} - \frac{1}{2} U' e^U \gamma_\mu \right) \epsilon = 0,$$
$$\delta \psi_z = \partial_z \epsilon = 0$$ \hfill (2.35)

and integrability implies

$$U' e^U = 2c,$$
which when plugged into the equations of motion gives

\[ R_{\mu\nu}^{(4)} = 12c^2 g_{\mu\nu}. \]

Therefore the equations of motion allow for Einstein spaces with positive cosmological constant, clearly these solutions are not supersymmetric. Curved supersymmetric solutions can be obtained in these cases if the cosmological constant is zero.

### 2.1 Supersymmetric curved solutions

In this section, we return to the Killing spinor equation\(^5\) that must be satisfied in the four dimensional world

\[ D_{\mu} \varepsilon(x) = \left( \partial_{\mu} + \frac{1}{4} \omega^{ab}_{\mu}(x) \gamma_{ab} \right) \varepsilon(x) = 0. \] (2.36)

If one assumes static worldvolume, i.e., metrics admitting covariantly constant time-like vector,

\[ g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + ds_3^2, \] (2.37)

then the only possibility for a supersymmetric solution is flat space \([17]\). On the other hand, considering metrics with a covariantly constant light-like vector (parallel null-vector), then the most general \(d\)-dimensional Lorentzian metric in this case is known and was given many years ago by Brinkmann \([18]\). This is referred to as the Brinkmann metric or the so-called \(pp\)-wave. In four dimensions, the general form of this metric can be written in the following form

\[ ds^2 = 2 dx^+ dx^- + a(dx^+)^2 + b_i dx^i dx^+ + g_{ij} dx^i dx^j. \] (2.38)

Here \(i = 1, \ldots, d-2\). The null vector is \(\partial_-\), \((\partial_- a = \partial_- b_i = \partial_- g_{ij} = 0)\). A subclass of these solutions with covariantly constant spinors has been given in \([19]\). The metric of this subclass takes the form

\[ ds^2 = 2 dx^+ dx^- + a(dx^+)^2 + b \epsilon_{ij} x^i dx^j dx^+ + dx^i dx^i \] (2.39)

where \(a = a(x^+, x^i)\) and \(b = b(x^+)\) are any two functions. An mentioned in the introduction, it is important to note that the existence of parallel spinors does not imply Ricci-flatness \([19]\). In fact for the above metric one finds that

\[ R_{++} = -\frac{1}{2} \Delta a + 2b^2. \] (2.40)

\(^5\)In the Mathematics literature, a Killing spinor equation refers to the case with non-zero cosmological constant. In the absence of a cosmological constant, our Killing spinors are known as parallel spinors.
where $\Delta = \partial_i \partial_i$. In summary, the subclass (2.39) of Brinkmann spaces have covariantly constant spinors without having to satisfy Einstein equations of motion. Therefore it must be stressed that in order to fix the supersymmetric bosonic solutions completely, one has to solve the equations of motion and thus impose extra conditions. In this case one must set $\Delta a = 4b^2$.

3 General case

Gauged supergravity models are obtained when some global isometries of the ungauged theory including $R$-symmetry are made local. Minimal $N = 2$ supergravity theories, i.e., those with eight real supercharges, contain an arbitrary number $n_V$ of vectormultiplets and $n_H$ hypermultiplets (in this work tensor multiplets are ignored). The fermionic fields of the theory are the two gravitini $\psi^i_M$ which are symplectic Majorana spinors ($i = 1, 2$ are $SU_R(2)$ indices), the gaugini $\lambda^i \hat{a}$ and the hyperini $\zeta^\alpha$ ($\alpha = 1, \ldots, 2n_H$). The bosonic fields consist of the graviton, vector bosons $A^\mu_I$ ($I = 0, 1, \ldots, n_V$), the real scalar fields $\phi^x$ of the vectormultiplets and the scalars $q^X$ ($X = 1, \ldots, 4n_H$) of the hypermultiplet matter fields. The scalars of the theory parametrize a manifold $M$ which is the direct product of a very special and a quaternionic manifold

$$
M = M_V \otimes M_H.
$$

The scalars $\phi^x$, $x = 1, \ldots, n_V$, parametrize the target space $M_V$. Note that for the quaternionic manifold, there are two types of indices $\alpha$ and $i$, corresponding to fundamental representations of $USp(2n_H)$ and $SU(2)$. The target manifold $M_V$ of the scalar fields of the vectormultiplets is a very special manifold described by an $n_V$–dimensional cubic hypersurface

$$
C_{IJK} h^I(\phi^x) h^J(\phi^x) h^K(\phi^x) = 1
$$

of an ambient space parametrized by $n_V + 1$ coordinates $h^I = h^I(\phi^x)$, where $C_{IJK}$ is a completely symmetric constant tensor defining the Chern–Simons couplings of the vector fields. For more details concerning the classification of the allowed homogeneous manifolds we refer the reader to [16], [20].

The self-interacting scalars of the hypermultiplets in an $N = 2$, $D = 5$ theory live on a quaternionic Kähler manifold [21], with a quaternionic metric tensor which we denote by $g_{XY}(q)$.

The bosonic Lagrangian of the gauged theory for vanishing gauge fields is given by

$$
\mathcal{L} = E \left( \frac{1}{2} R - \frac{1}{2} g_{XY} \partial_M q^X \partial^M q^Y - \frac{1}{2} g_{xy} \partial_M \phi^x \partial^M \phi^y - V(\phi, q) \right),
$$

where the scalar potential is given by

---

[6]the index $\hat{a}$ is the flat index of the tangent space group $SO(n_V)$ of the scalar manifold $M_V$
\[ V = -g^2 \left[ 2P_{ij}P^{ij} - P_{ij}^\alpha P^{\alpha ij} \right] + 2g^2 \mathcal{N}_{i\alpha} \mathcal{N}^{i\alpha}. \]  

(3.4)

Here

\[ P_{ij} \equiv h^\mathcal{I} P_{\mathcal{I}ij}, \]

\[ P_{ij}^\alpha \equiv h^\mathcal{I} P_{\mathcal{I}ij}, \]

\[ \mathcal{N}^{i\alpha} \equiv \frac{\sqrt{6}}{4} h^\mathcal{I} K^X_i f^{\mathcal{I}i}. \]  

(3.5)

Here \( K^X_i, P_i \) are the Killing vectors and prepotentials respectively. The vielbeins \( f_i^X \) obey the following relation \( g_{XY} f_i^X f_{j\beta}^X = \epsilon_{ij} C_{\alpha\beta} \), where \( \epsilon_{ij} \) and \( C_{\alpha\beta} \) are the \( SU(2) \) and \( USp(2n_H) \) invariant tensors respectively. For details of the gauging and notations we refer the reader to [12].

The bosonic part of the supersymmetry transformations of the fermi fields in the gauged theory, after dropping the gauge fields contribution, are given by

\[ \delta \varepsilon \psi_{M_i} = D_M \varepsilon_i + \frac{i}{\sqrt{6}} g \Gamma_M \varepsilon^j P_{ij}, \]

\[ \delta \varepsilon \lambda^a_i = -i f_i^a \Gamma^M \varepsilon_i \partial_M \phi^x + g \varepsilon^j P_{ij}^a, \]

\[ \delta \varepsilon \zeta^\alpha = -i f_i^\alpha \Gamma^M \varepsilon_i \partial_M q^X + g \varepsilon^i \mathcal{N}^\alpha. \]  

(3.6)

A general form for scalar potentials which guarantees stability [22] is given by

\[ V = g^2 \left[ -6W^2 + \frac{9}{2} g^{\Lambda \Sigma} \partial_\Lambda W \partial_\Sigma W \right] \]  

(3.7)

where \( \Lambda, \Sigma \) run over all the scalars of the theory. The transition from (3.4) to (3.7) can be achieved by writing

\[ h^\mathcal{I} P_{\mathcal{I}r} = \sqrt{\frac{3}{2}} W Q^r, \quad Q^r Q^r = 1 \]  

(3.8)

and imposing the condition \( \partial_x Q^r = 0 \). This condition is in general satisfied only on a submanifold of the total scalar manifold and is also required for the existence of flat BPS domain wall solutions of the theory [3, 23].

Supersymmetric domain wall solutions with flat worldvolume of the form (2.13) have been recently discussed in [3]. The Killing spinors of these solutions satisfy the projection condition

\[ \gamma_5 \varepsilon_i = \sigma^r_i Q^r \varepsilon^j. \]  

(3.9)

\[ ^7 \text{Here } W \text{ is the ‘superpotential’ and } Q^r \text{ are } SU(2) \text{ phases} \]
It was found that for the flat BPS domain wall, the warp factor and the scalar fields of the theory satisfy

$$
\phi^\Lambda = -3gg^\Lambda \Sigma \partial_\Sigma W, \quad \phi^A = (\phi^x, q^X), \quad (3.10)
$$

$$
U' = gW, \quad (3.11)
$$

We now generalize the results of the previous sections to all gauged supergravity models with vector and hypermultiplets. It can be easily seen that supersymmetry transformations allow for a negative cosmological constant if one modifies (3.11) to take the form

$$
U' = 2ce^{-U} + gW, \quad (3.12)
$$

The Einstein equations of motion give for the worldvolume Ricci-tensor the following equation

$$
R^{(4)}_{\mu\nu} = g_{\mu\nu} e^{2U} (4U'^2 + U'' + \frac{2}{3} V), \quad (3.13)
$$

One also obtains for the Ricci-tensor zz-component,

$$
R_{zz} = g_{\Lambda\Sigma} \partial_z \phi^\Lambda \partial_z \phi^\Sigma + \frac{2}{3} V = -4(U'' + U'^2) \quad (3.14)
$$

Using (3.10) and (3.7) we then obtain

$$
(U'' + U'^2) = -3g^2 g^\Lambda \Sigma \partial_\Lambda W \partial_\Sigma W + g^2 W^2 \quad (3.15)
$$

The relations (3.15) and (3.7), when substituted in (3.13) give

$$
R^{(4)}_{\mu\nu} = 3g_{\mu\nu} e^{2U} (U'^2 - g^2 W^2). \quad (3.16)
$$

Using (3.12), we obtain that

$$
U'' + U'^2 = -3g^2 g^\Lambda \Sigma \partial_\Lambda W \partial_\Sigma W + g^2 W^2 + 2cgW \quad (3.17)
$$

Comparing with (3.13), we deduce for non-vanishing $W$ that $c = 0$ and therefore

$$
R^{(4)}_{\mu\nu} = 0. \quad (3.18)
$$

One can easily check that the conditions (3.10) and (3.11) solve the scalar fields equations of motion.

4 Discussion

We found general domain wall solutions of five dimensional gauged supergravity theories coupled to vector and hypermultiplets. The examples obtained from the compactification
of M-theory on a Calabi-Yau threefold were first examined and it was found that Killing spinor equations are satisfied for configurations with anti-de Sitter worldvolumes. However, it turns out that the equations of motion imply Ricci-flatness for the worldvolume metric and thus exclude the case of non-zero cosmological constant. Therefore, contrary to the common belief, supersymmetric solutions are not necessarily solutions of the theory. Similar observation were previously made in the literature. For example in [19], it was shown that purely gravitational backgrounds can have parallel spinors without having to satisfy Einstein equations of motion, i.e., not Ricci-flat. In spaces with Lorentzian signature, having parallel spinors does not imply Ricci flatness. Therefore extra conditions have to be imposed in order for the space to be also a solution of Einstein gravity. Moreover, it is known in the study of black holes (see [24]) that solving the Killing spinors equations does not fix the bosonic supersymmetric solution completely. There, one has to fix the solution in terms of harmonic functions by solving for the gauge field equations of motion. Clearly, harmonic functions can not be obtained from the first order differential equations imposed by supersymmetry.

BPS curved domain wall solutions are obtained by searching for worldvolume metrics with parallel spinors. The classification of space-times admitting parallel spinors is a holonomy problem. In our case, i.e., for four dimensional worldvolumes, one looks for subgroups of $\text{Spin}(3, 1) = SL(2, \mathbb{C})$ which fix a spinor and then identify the Lorentzian manifolds which admit these subgroups as their holonomy groups. This implies that the holonomy group of the supersymmetric four-dimensional worldvolume is $\mathbb{R}^2$. Metrics with such a holonomy group are known and are given by the supersymmetric $pp$-wave [19]. Moreover, one can construct non-supersymmetric curved domain wall solutions by having a worldvolume metric which does not admit parallel spinors such as Schwarzschild black hole. Non-supersymmetric solutions with Einstein worldvolumes with a positive cosmological constant are possible in the cases of the ungauged five dimensional supergravity models. It will be interesting to investigate domain wall solutions with worldvolumes given by solutions with non-trivial gauge fields [25].

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