Multiple-high-order pole solutions for the NLS equation with quartic terms

Li-Li Wen¹, En-Gui Fan² and Yong Chen¹,³∗

Abstract

The aim of this article is to investigate the multiple-high-order pole solutions to the focusing NLS equation with quartic terms (QNLS) under the non-vanishing boundary conditions (NVBC) via the Riemann-Hilbert (RH) method. The determinant formula of multiple-high-order pole soliton solutions for NVBC is given. Further the double 1nd-order, mixed 2nd- and 1nd-order pole solutions are obtained.

Keywords: the focusing QNLS; NVBC; RH method; multiple-high-order poles soliton solution.

1 Introduction

The NLS equation can well describe the wave evolution of deep water and optical fibers [1, 2]. For some multiplicity of nonlinear phenomena in plasma waves, Bose-Einstein condensates and other physical phenomena, the NLS equation is also a generic model [3, 4]. Nonetheless, several phenomena can not be described by low-order dispersion NLS equation. The NLS equation with high-order dispersion is particularly important for describing the more complex phenomena.

In this article, we investigate the multiple-high-order pole soliton solution of focusing QNLS equation

\[ iq_t + \frac{1}{2} q_{xx} + q^2 q + \gamma (q_{xxxx} + 6|q|^4 q + 4q\bar{q}_x q_x + 2q^2 \bar{q}_x + 6q\bar{q}_x^2 + 8|q|^2 q_{xx}) = 0. \] (1.1)

Soliton solutions as one of the most important consequence of integrable systems, which can be constructed by many different methods. Recently, the Riemann-Hilbert (RH) method as another
form of inverse scattering transform (IST) method is favorite to study the soliton solutions with VBC \[6, 7\], as well as NVBC \[8–14\]. The \(N\) simple poles and one \(N\)-order pole soliton of QNLS equation for VBC \[15\]. In the case of NVBC, the multiple-high-order pole solution not was given just investigated the pure one soliton in \[15\].

The major contributions of this article are to study the QNLS equation with NVBC via RH method. We present the multiple-high-order pole soliton solutions and construct the formula for soliton solution to be a determinant form. For reflection-less and the scatter data \(a(k)\) with multiple-high-order pole, the Talyor series on \(a(k)\) and sectionally meromorphic matrix \(M(x, t, z)\) are fully utilized. The advantage of this method is that the residue conditions of high-order poles are not the necessary conditions.

This work is organized as follows. In Section 2, we investigate IST method of the QNLS equation with NVBC. And we show the analyticities, symmetries and asymptotic of Jost solution and scatter matrix. In section 3, the formulas of single-high-order pole and multiple-high-order pole solutions are shown. We detailed given the double 1\(^{nd}\)-order, mixed 2\(^{nd}\)- and 1\(^{nd}\)-order pole solutions.

2 Inverse scattering transform for NVBC

The QNLS equation admits the Lax pair

\[
\psi_x = \mathbf{h}\psi = (i\lambda\sigma_3 + iQ)\psi, \quad \psi_t = \mathbf{l}\psi = (-8i\gamma\sigma_3\lambda^4 - 8i\gamma Q\lambda^3 + A_2\lambda^2 + A_1\lambda + A_0)\psi, \quad (2.1)
\]

where

\[
A_2 = i(1 + 4\gamma Q^2)\sigma_3 - 4\gamma\sigma_3 Q_x, \quad A_1 = 2\gamma|Q_x, Q|\sigma_3 + i(Q + 4\gamma Q^3 + 2Q_{xx}),
\]

\[
A_0 = -3i\gamma\sigma_3 Q^4 + i\gamma\sigma_3 Q_x^2 - i\gamma\sigma_3(QQ_{xx} + Q_{xx}Q) - \frac{i}{2}\sigma_3 Q^2 + \frac{1}{2}\sigma_3 Q_x + 6\gamma\sigma_3 Q^2 Q_x + \gamma\sigma_3 Q_{xxx},
\]

\[
Q = \begin{pmatrix} 0 & \chi(x, t) \\ \eta(x, t) & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Considering the NVBC \(\lim_{|x|\to\infty} q(x, t) = q_\pm = q_0 e^{i\theta} \) where \(|q_\pm| = q_0\). Naturally, \(\mathbf{h}_\pm = \lim_{x\to\pm\infty} \mathbf{h}, \quad \mathbf{l}_\pm = \lim_{x\to\pm\infty} \mathbf{l}, \) and \(Q_\pm = \lim_{x\to\pm\infty} Q\). The eigenvalues of \(\mathbf{h}_\pm\) and \(\mathbf{l}_\pm\) are \(\pm i\sqrt{\lambda^2 + q_0^2}\) and \(\pm i\sqrt{\lambda^2 + q_0^2}(\lambda - 8\gamma\lambda^3 + 4\gamma\lambda q_0^3)\), respectively. Since the eigenvalues are doubly branched. One can transform the \(\lambda\)-plane to \(z\)-plane, where \(k = \frac{1}{2}(z + \frac{q_0}{z})\) and \(\lambda = \frac{1}{2}(z - \frac{q_0}{z})\).

Now, we introduce the Jost solutions as the simultaneous solutions of Lax pair such that

\[
\Psi_\pm(x, t, z) \sim \Omega_\pm(z)e^{i\theta(x, t, z)}\sigma_3, \quad \Omega \in \Sigma, \quad x \to \pm\infty, \quad (2.3)
\]
where \( \theta(x, t, z) = kx + k(\lambda - 8\gamma\lambda^3 + 4\gamma\lambda^3)\mu \) and \( \Sigma = \mathbb{R} \cup \mathcal{C}_0 \). \( \mathcal{C}_0 \) is the circle of radius \( q_0 \) in the \( z \)-plane. For factoring the asymptotic exponential oscillations, we define a modified Jost solutions

\[
\mu_{\pm}(x, t, z) = \Psi_{\pm}(x, t, z)e^{-i\theta(x, t, z)\sigma_3}, \quad (2.4)
\]

The modified Jost solutions \( \mu_{\pm}(x, t, z) \) satisfy the asymptotic properties \( \lim_{|x| \to \infty} \mu_{\pm}(x, t, z) = \Omega_{\pm}(z) \). And the modified Jost solution admits the Volterra integral equations

\[
\mu_{\pm}(x, t, z) = \Omega_{\pm}(z) + \int_{\pm\infty}^{x} \Omega_{\pm}(z)e^{ik(x-y)\sigma_3}\Omega_{\pm}^{-1}(y)\Delta Q_{\pm}(y, t, z)dy, \quad (2.5)
\]

where \( \Delta Q_{\pm} = iQ - iQ_{\pm} \). And one can obtain the analyticities of \( \mu_{\pm,j}(x, t, z), (j = 1, 2) \): \( \mu_{+1}(x, t, z) \) and \( \mu_{-2}(x, t, z) \) analytic in \( D_+ = \{ z \in \mathbb{C} \ | \ (|z|^2 - q_0^2)\text{Im}z > 0 \} \), \( \mu_{-1}(x, t, z) \) and \( \mu_{+2}(x, t, z) \) analytic in \( D_- = \{ z \in \mathbb{C} \ | \ (|z|^2 - q_0^2)\text{Im}z < 0 \} \). The subscripts ‘1’ and ‘2’ identify the columns of matrix.

Since \( \Psi_{+}(x, t, z) \) and \( \Psi_{-}(x, t, z) \) are the fundamental solutions of the spectral problem \( (2.1) \), so there exists a scatter matrix \( S(z) = (s_{ij})_{2 \times 2} \) which satisfies the linear relationship

\[
\mu_{+}(x, t, z) = \mu_{-}(x, t, z)e^{i\theta_{\sigma_3}}S(z)e^{-i\theta_{\sigma_3}} , \quad z \in \Sigma \setminus \{ \pm iq_0 \}, \quad (2.6)
\]

Moreover, \( (2.6) \) imply that \( \det S(z) = 1 \). Equation \( (2.6) \) implies that \( s_{11}(z) \) is analytic in \( D_+ \) and \( s_{22}(z) \) is analytic in \( D_- \). However, \( s_{12}(z) \) and \( s_{21}(z) \) are just continuous on \( \Sigma \).

For the QNLS equation with NVBC, there exist two kinds of symmetries for the Jost solution \( \Psi_{\pm}(z) \) and scattering matrix \( S(z) \) in \( z \)-plane:

1. The symmetries of up-half and low-half of \( z \)-plane (\( z \sim \sim \overline{z} \))

\[
\Psi_{\pm}(z) = -\sigma\Psi_{\pm}(\overline{z})\sigma, \quad \Psi_{\pm,j} = (-1)^{j-1}\sigma\Psi_{\pm,(3-j)}(\overline{z}), \quad S(z) = -\sigma\overline{S}(\overline{z})\sigma. \quad (2.7)
\]

2. The symmetries of outside and inside of \( \mathcal{C}_0 \) (\( z \sim \sim -\frac{q_0^2}{z} \))

\[
\Psi_{\pm}(z) = -\frac{1}{z}\Psi_{\pm}(-\frac{q_0^2}{z})\sigma_3 Q_{\pm}, \quad \Psi_{\pm,j}(z) = (-1)^{j-1}\frac{1}{z}q_0^2\Psi_{\pm,(3-j)}(-\frac{q_0^2}{z}), \quad S(z) = Q^{-1}\sigma_3 S(-\frac{q_0^2}{z})\sigma_3 Q_. \quad (2.8)
\]

According to the above symmetries, the scattering matrix \( S(z) \) can be rewritten as a new version

\[
S(z) = \begin{pmatrix} a(z) & -b(z) \\ b(z) & a(z) \end{pmatrix}.
\]

Next, we consider the asymptotic of Jost solution and scattering matrix. For convenient, we introduce the following notations \( A_d = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \) and \( A_o = \begin{pmatrix} 0 & a_{12} \\ a_{21} & 0 \end{pmatrix} \). Consider the
asymptotic expansion of \( \mu_{\pm}(x, t, z) \)

\[
\mu_{\pm}(x, t, z) = \sum_{n=0}^{\infty} \mu_{\pm}^{(n)}(x, t, z),
\]

where

\[
\mu_{\pm}^{(0)}(x, t, z) = \Omega_{\pm}(z) = I - \frac{1}{z} \sigma_3 Q_{\pm}, \quad \mu_{\pm}^{(n+1)}(x, t, z) = \int_{-\infty}^{x} \Omega_{\pm}(z) e^{ik(z-y)\sigma_3} \Omega_{\pm}^{-1}(z) \Delta Q_{\pm}(y, t) \mu_{\pm}^{(n)}(y, t, z) dy.
\]

So

\[
\mu_{\pm,d}^{(n+1)} = \int_{-\infty}^{x} \left( \Omega_{\pm,0}^{-1} \Delta Q_{\pm} \mu_{\pm,d}^{(n)} + \Omega_{\pm,0}^{-1} \Delta Q_{\pm} \mu_{\pm,o}^{(n)} \right) dy
+ \int_{-\infty}^{x} \Omega_{\pm,0} e^{ik(z-y)\sigma_3} \left( \Omega_{\pm,0}^{-1} \Delta Q_{\pm} \mu_{\pm,d}^{(n)} + \Omega_{\pm,0}^{-1} \Delta Q_{\pm} \mu_{\pm,o}^{(n)} \right) e^{-ik(z-y)\sigma_3} dy,
\]

\[
\mu_{\pm,o}^{(n+1)} = \int_{-\infty}^{x} \left( \Omega_{\pm,0}^{-1} \Delta Q_{\pm} \mu_{\pm,d}^{(n)} + \Omega_{\pm,0}^{-1} \Delta Q_{\pm} \mu_{\pm,o}^{(n)} \right) dy
+ \int_{-\infty}^{x} e^{ik(z-y)\sigma_3} \left( \Omega_{\pm,0}^{-1} \Delta Q_{\pm} \mu_{\pm,d}^{(n)} + \Omega_{\pm,0}^{-1} \Delta Q_{\pm} \mu_{\pm,o}^{(n)} \right) e^{-ik(z-y)\sigma_3} dy.
\]

If \( z \to \pm \infty \), we note the fact that \( k = \frac{1}{2}(z + \frac{n^2}{z}) \) and \( \frac{n^2}{z} = 1 + \frac{n^2}{z^2} + \cdots \). And taking the integration by part for the last two terms of \( \mu_{\pm,d}^{(1)} \) and \( \mu_{\pm,o}^{(1)} \), one can derive the following results

\[
\mu_{\pm,d}^{(1)} = O\left(\frac{\mu_{\pm,d}^{(0)}}{z}\right) + O\left(\frac{\mu_{\pm,o}^{(0)}}{z^2}\right) + O\left(\frac{\mu_{\pm,o}^{(0)}}{z^3}\right) = O\left(\frac{1}{z}\right), \quad \mu_{\pm,o}^{(1)} = O\left(\frac{\mu_{\pm,d}^{(0)}}{z^2}\right) + O\left(\frac{\mu_{\pm,o}^{(0)}}{z^3}\right) + O\left(\frac{\mu_{\pm,o}^{(0)}}{z^4}\right) = O\left(\frac{1}{z}\right).
\]

Iterate in the same method, one can obtain the asymptotic

\[
\mu_{\pm,d}^{(2n)} = O\left(\frac{1}{z^{2n}}\right), \quad \mu_{\pm,o}^{(2n)} = O\left(\frac{1}{z^{2n+1}}\right), \quad \mu_{\pm,d}^{(2n+1)} = O\left(\frac{1}{z^{2n+1}}\right), \quad \mu_{\pm,o}^{(2n+1)} = O\left(\frac{1}{z^{2n+1}}\right).
\]

For \( z \to 0 \), one can obtain the asymptotic in the similar way

\[
\mu_{\pm,d}^{(2n)} = O(\frac{1}{z^n}), \quad \mu_{\pm,o}^{(2n)} = O(\frac{1}{z^{n+1}}), \quad \mu_{\pm,d}^{(2n+1)} = O(\frac{1}{z^{n+1}}), \quad \mu_{\pm,o}^{(2n+1)} = O(\frac{1}{z^{n+1}}).
\]

So we obtain the following asymptotic by some calculations for the equation (2.9)

\[
\mu_{\pm}(x, t, z) = I + O\left(\frac{1}{z}\right), \quad z \to \pm \infty, \quad \mu_{\pm}(x, t, z) = -\frac{1}{2} \sigma_3 Q_{\pm} + O(1), \quad z \to 0. \tag{2.10}
\]

And one can derive the solution of QNLS equation \( q(x,t) = \lim_{z \to \infty} z(\mu_{\pm})_{21} \). By substituting the asymptotic of \( \mu_{\pm} \) into (2.6), one can obtain the asymptotic behaviors of the scattering matrix \( S(z) \)

\[
S(z) = I + O\left(\frac{1}{z}\right), \quad z \to \pm \infty, \quad S(z) = \frac{q_+}{q_-} I + O(z), \quad z \to 0. \tag{2.11}
\]
3 The Riemann-Hilbert problem

Based on the analyticity and asymptotic of eigenfunctions $\mu_{\pm}$ and $S(z)$, the solutions can be derived by RH problem. Firstly, we define a sectionally meromorphic matrix $M(x, t, z)$:

\[
M^+(x, t, z)|_{z \in D_+} = \left( \frac{\mu_{+, 1}(x, t, z)}{a(z)} \mu_{-, 2}(x, t, z) \right), \quad M^-(x, t, z)|_{z \in D_-} = \left( \mu_{-, 1}(x, t, z) \frac{\mu_{+, 2}(x, t, z)}{a(z)} \right).
\]

**Proposition 3.1.** The sectionally meromorphic matrix $M(x, t, z)$ satisfying the following RH problem:

- **Analyticity:** $M(x, t, z)$ is meromorphic in $D_+ \cup D_-$.

- **Jump condition:** $M^+(x, t, z) = M^-(x, t, z) \left( \frac{1 - \rho \tilde{\rho}}{e^{2i\theta} \rho} - \frac{e^{2i\theta} \tilde{\rho}}{1} \right)$, $z \in \Sigma$.

- **Asymptotic behavior:** $M(x, t, z) = I + O(\frac{1}{z})$, $z \to \infty$, $M(x, t, z) = -\frac{1}{2}\sigma_3 Q_- + O(1)$, $z \to 0$.

According to the asymptotic of $\mu_{\pm}$ and $S(z)$, the solution of QNLS can be rewritten as $q(x, t) = \lim_{z \to \infty} z(M(x, t, z))_{21}$. For $a(z)$ with $N$ high order zeros, not only the residue conditions are useful but the coefficients of negative power should be considered. However, these coefficients are not straightforward derived. Moreover, if $z_n$ is the $N$-order zero point of $a(z)$, so is $-\frac{Q_n^2}{z_n}$. This situation can be equivalent to the zero points of $a(z)$ always paired in $D_+$.

We firstly consider single-high-order pole solutions. We assume $a(z)$ with one $N$-order zero $z_0 \in \{D_+ \cap \text{Im}z_0 > 0\}$. According to the symmetry properties of $S(z)$, one can derive $-\frac{Q^2}{z_0}$ also is the zero point of $a(z)$. For convenient, we make the following notations $\{\nu_1 \equiv z_0, \nu_2 \equiv -\frac{Q^2}{z_0}, \nu_1 \equiv \bar{z}_0, \nu_2 \equiv -\frac{Q^2}{z_0}\}$, where $\nu_1, \nu_2 \in D_+$, and $\bar{\nu}_1, \bar{\nu}_2 \in D_-$. So $a(z)$ can be expanded as the Taylor series

\[
a(z) = a_0(z) \prod_{j=1}^{2} (z - \nu_j)^N, \quad j = 1, 2,
\]

and $a_0(z) \neq 0$ for all $z \in D_+$. Reflection coefficient $\rho(z)$ and $\rho(\bar{z})$ with the Laurent expansion

\[
\rho(z) = \rho_{j, 0}(z) + \sum_{m_j=1}^{N} \frac{\rho_{j, m_j}}{(z - \nu_j)^{m_j}}, \quad \rho(\bar{z}) = \rho_{j, 0}(\bar{z}) + \sum_{m_j=1}^{N} \frac{\bar{\rho}_{j, m_j}}{(z - \bar{\nu}_j)^{m_j}}, \quad j = 1, 2,
\]

where $\rho_{j, m_j} = \lim_{z \to \nu_j} \frac{1}{(z - \nu_j)} \frac{d^{N-m_j}}{dz^{N-m_j}} [(z - \nu_j)^N \rho(z)]$, $(m_j = 1, 2, \cdots, N)$, $\rho_{j, 0}(z)$ is analytic for all $z \in D_+$. According to the definition of $M(x, t, z)$, one can obtain that $z = \nu_j$ are the $N$-order poles.
of $M_1(x,t,z)$ and $z = \nu_j$ are the $N$-order poles of $M_2(x,t,z)$. $M_2(x,t,z)$ analytic as $z = \nu_j$ and $M_1(x,t,z)$ analytic as $z = \nu_j$. So we have the following expand

$$M_{21}(z) = \frac{q}{x} + \frac{2}{j=1} \sum_{p=1}^{N} G_{j,p}(x,t) (z-\nu_j)^p,$$

$$M_{22}(z) = 1 + \frac{2}{j=1} \sum_{p=1}^{N} F_{j,p}(x,t) (z-\nu_j)^p. \quad (3.1)$$

$G_{j,p}(x,t)$ and $F_{j,p}(x,t)$ are undetermined. According to the analyticity one can get the Taylor expansion

$$e^{-2i\theta(z)} = \sum_{s_j=0}^{+\infty} f_{j,s_j}(x,t)(z-\nu_j)^{s_j}, \quad M_{21}(z) = \sum_{s_j=0}^{+\infty} \zeta_{j,s_j}(x,t)(z-\nu_j)^{s_j}, \quad M_{22}(z) = \sum_{s_j=0}^{+\infty} \mu_{j,s_j}(x,t)(z-\nu_j)^{s_j},$$

where

$$f_{j,s_j}(x,t) = \lim_{z \to \nu_j} \frac{1}{s_j!} \frac{\partial^{s_j}}{\partial z^{s_j}} e^{-2i\theta(z)}, \quad \mu_{j,s_j}(x,t) = \lim_{z \to \nu_j} \frac{1}{s_j!} \frac{\partial^{s_j}}{\partial z^{s_j}} M_{21}(z).$$

The Taylor expand of $e^{2i\theta(z)}$ also can be obtained. By considering the equation (2.6) and the definition of $M(x,t,z)$, comparing the corresponding coefficients of $(z-\nu_j)^p$ and $(z-\nu_j)^{-p}$. One can derived $G_{j,p}(x,t)$ and $F_{j,p}(x,t)$ as the form

$$F_{j,p}(x,t) = - \sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \eta_{j,m_j} \eta_{j,m_j-p-s_j}(x,t) \zeta_{j,s_j}(x,t), \quad G_{j,p}(x,t) = \sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \rho_{j,m_j} \eta_{j,m_j-p-s_j}(x,t) \mu_{j,s_j}(x,t),$$

where $p = 1, 2, \cdots, N$. For $N = 1$, $F_{j,p}(x,t)$ and $G_{j,p}(x,t)$ degenerate into the residue conditions. In addition, $\mu_{j,s_j}(x,t)$ and $\zeta_{j,s_j}(x,t)$ can be expressed as $F_{j,p}(x,t)$ and $G_{j,p}(x,t)$ via direct calculation

$$\zeta_{j,s_j}(x,t) = \left( \frac{(-1)^s q}{\nu_j^{s+1}} + \sum_{p=1}^{N} \left( p + s_j - 1 \right) \left( \frac{-1)^s G_{j,p}(x,t)}{(\nu_j-\nu)^{s+1}} \right) \right), \quad s_j = 0, 1, 2, \cdots$$

$$\mu_{j,s_j}(x,t) = \left\{ \begin{array}{ll}
1 + \sum_{p=1}^{N} \frac{F_{j,p}(x,t)}{\nu_j (\nu_j-\nu_j)^{p}}, & s_j = 0, \\
\sum_{p=1}^{N} \left( p + s_j - 1 \right) \left( \frac{-1)^s F_{j,p}(x,t)}{(\nu_j-\nu_j)^{s+1}} \right), & s_j = 1, 2, \cdots.
\end{array} \right.$$
Then one can obtain the following system

\[
F_{j,p}(x,t) = -\sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \frac{(\nu_j x)_{s_j} \overline{F}_{j,m_j} \overline{F}_{j,m_j-p-s_j}}{\nu_j^{s_j+1}} (x,t) q_-
- \sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} N \left( q + s_j - 1 \right) \frac{(\nu_j x)_{s_j} \overline{F}_{j,m_j} \overline{F}_{j,m_j-p-s_j}}{(\nu_j - \nu_j)^{s_j+q}} (x,t) G_{j,q}(x,t),
\]

\[
G_{j,p}(x,t) = \sum_{m_j=p}^{N} \rho_j m_j f_{j,m_j-p}(x,t)
+ \sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} N \left( q + s_j - 1 \right) \frac{(\nu_j x)_{s_j} \overline{F}_{j,m_j} \overline{F}_{j,m_j-p-s_j}}{(\nu_j - \nu_j)^{s_j+q}} (x,t) F_{j,q}(x,t).
\]

For convenient, we introduce the following symbols \((s,p = 1,2,\cdots,N)\):

\[
|F\rangle = (F_1 F_2 \cdots F_N)_{T}, \quad |G\rangle = (G_1 G_2 \cdots G_N)_{T}, \quad |\beta\rangle = (\beta_1 \beta_2)_{T},
\]

\[
\beta_j = (\beta_{j,1} \beta_{j,2} \cdots \beta_{j,N}), \quad |\eta\rangle = (\eta_1 \eta_2)_{T}, \quad \eta_j = (\eta_{j,1} \eta_{j,2} \cdots \eta_{j,N}),
\]

\[
\beta_{j,l} = -\sum_{m_j=l}^{N} \sum_{s_j=0}^{m_j-l} \frac{(-1)^{s_j} \overline{F}_{j,m_j} \overline{F}_{j,m_j-l-s_j}}{\nu_j^{s_j+1}} (x,t) q_-
\eta_{j,l}(x,t) = \sum_{m_j=l}^{N} \rho_j m_j f_{j,m_j-l}(x,t),
\]

\[
[\mathcal{M}_{j,l}] = ([\mathcal{M}_{j,l}]_{p,q})_{N \times N} = \left( -\sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \left( q + s_j - 1 \right) \frac{(\nu_j x)_{s_j} \overline{F}_{j,m_j} \overline{F}_{j,m_j-p-s_j}}{(\nu_j - \nu_j)^{s_j+q}} (x,t) \right)_{N \times N}.
\]

and \(\chi = \begin{bmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{bmatrix}\). So the equations (3.3) can be rewritten as

\[
|F\rangle = |\beta\rangle + \chi|G\rangle, \quad |G\rangle = |\eta\rangle - \chi|F\rangle.
\]

(3.5)

\(|F\rangle\) and \(|G\rangle\) are solved as

\[
|F\rangle = \chi(I + \chi\chi)^{-1}|\eta\rangle + (I - \chi(I + \chi\chi)^{-1}\chi)|\beta\rangle, \quad |G\rangle = (I + \chi\chi)^{-1}(|\eta\rangle - \chi|\beta\rangle).
\]

(3.6)

So the expansions of \(M_{21}(x,t,\chi)\) can be given as

\[
M_{21}(x,t,\chi) = \frac{q_+}{z} + \sum_{j=1}^{2} \sum_{s=1}^{N} \frac{G_{j,s}(x,t)}{(z - \nu_j)^s} = \frac{q_+}{z} + \frac{\det[I + \chi\chi + (|\eta\rangle - \chi|\beta\rangle)\langle Y(\chi)\rangle]}{\det[I + \chi\chi]} - 1.
\]

(3.7)

where

\[
\langle Y(z)\rangle = \begin{bmatrix} \frac{1}{z - \nu_1} & \frac{1}{z - \nu_1} & \cdots & \frac{1}{z - \nu_1} \\ \frac{1}{z - \nu_2} & \frac{1}{z - \nu_2} & \cdots & \frac{1}{z - \nu_2} \\ \frac{1}{z - \nu_3} & \frac{1}{z - \nu_3} & \cdots & \frac{1}{z - \nu_3} \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}.
\]

One can derive the solutions of equation QNLS with one \(N\)-order pole

\[
q(x,t) = q_+ + \left( \frac{\det[I + \chi\chi + (|\eta\rangle - \chi|\beta\rangle)\langle Y(\chi)\rangle]}{\det[I + \chi\chi]} - 1 \right),
\]

(3.8)
where

\[ |\eta| = (\eta_1 \quad \eta_2 \cdots \eta_N)^T, \quad \beta = (\beta_1 \quad \beta_2 \cdots \beta_N)^T, \quad (Y_0) = (Y_1^0 \quad Y_2^0 \cdots Y_N^0), \quad \eta_j = (\eta_{j1} \quad \eta_{j2} \cdots \eta_{jN}), \]

\[ \eta_j(t) = \sum_{m_j=1}^{N_j} \rho_j \nu_j f_j(x,t), \quad \beta_j = (\beta_{j1} \quad \beta_{j2} \cdots \beta_{jN}), \quad Y_j^0 = (1 \quad 0 \cdots 0)_{1 \times N_j}, \]

\[ \beta_{j,l} = -\sum_{m_j=1}^{N_j} \sum_{s_j=0}^{m_j-1} \frac{(-1)^{s_j} \bar{\rho}_j \nu_{j,l} \bar{f}_j \nu_{j,l}^{-t-s_j}(x,t)q_{j}}{\nu_{j,l}^{s_j+1}}, \quad \chi = \begin{pmatrix} \bar{w}_{11} & \bar{w}_{12} \\ \bar{w}_{21} & \bar{w}_{22} \end{pmatrix}, \]

and \([\bar{w}_{j,l}](j, l = 1, 2)\) are \(N \times N\) matrix

\[ [\bar{w}_{j,l}] = ([\bar{w}_{j,l}]_{p,q})_{N \times N} = \left(-\sum_{m_j=p}^{N} \sum_{s_j=0}^{m_j-p} \left(\frac{q + s_j - 1}{s_j}\right) (-1)^{s_j} \bar{\rho}_j \nu_{j,l} \bar{f}_j \nu_{j,l}^{-t-s_j}(x,t)\right)_{N \times N}. \]

Next we consider the multiple-high-order pole solutions, \(a(z)\) with \(N\) high-order zeros \(z_1, z_2, \cdots, z_N\), and the powers are \(n_1, n_2, \cdots, n_N\), respectively. Then \(-\frac{\partial^2}{\partial z^2}, -\frac{\partial^2}{\partial z} \cdots, -\frac{\partial^2}{\partial z}\) also are the zeros of \(a(z)\).

And the powers are \(n_1, n_2, \cdots, n_N\), too. We also make the following notations \(|\nu_j = z_i, \nu_j|_{j=N+i} \approx -\frac{\partial^2}{\partial z^2}, \nu_j|_{j=N+i} \approx -\frac{\partial^2}{\partial z}\}_{i=1}^{N}.\) So \(a(z)\) can be expanded as

\[ a(z) = a_0(z)(z-\nu_1)^{n_1}(z-\nu_2)^{n_2}(z-\nu_{N+1})^{n_N}(z-\nu_{N+1})^{n_1}(z-\nu_{N+2})^{n_N}, \]

Similar to the one high-order pole, \(\rho(z)\) can be expanded as the Laurent series

\[ \rho(z) = \rho_{00}(z) + \sum_{m_j=1}^{n_j} \frac{\rho_{j,m_j}}{(z-\nu_j)^{m_j}}, \quad \rho(z) = \rho_{00}(z) + \sum_{m_j=1}^{n_j} \frac{\bar{\rho}_j \nu_{j,m_j}}{(z-\bar{\nu}_j)^{m_j}}, \]

where \(\rho_{j,m_j} = \lim_{z \rightarrow \nu_j} \frac{1}{(z-\nu_j)^{m_j}} \bigg[ (z-\nu_j)^{m_j} \rho(z) \bigg], \) and \(\rho_{00}(z)\) is analytic for all \(z \in D_+\) and \(j = 1, \cdots, 2N\). So, in the similar method, one can derive the soliton solution formula with multiple-high-order poles:

**Theorem 3.1.** For the NVBC, if \(a(z)\) with multiple-high-order zeros, then the soliton solutions of equation (3.10) is

\[ q(x,t) = q_0 - \frac{\det (I + \chi \eta + (|\beta| - \chi |\beta|) (Y_0))}{\det (I + \chi \eta)} - 1, \]

where

\[ |\eta| = (\eta_1 \quad \eta_2 \cdots \eta_{2N})^T, \quad |\beta| = (\beta_1 \quad \beta_2 \cdots \beta_{2N})^T, \quad (Y_0) = (Y_1^0 \quad Y_2^0 \cdots Y_N^0), \]

\[ Y_j^0 = (1 \quad 0 \cdots 0)_{1 \times n_j}, \eta_j = (\eta_{j1} \quad \eta_{j2} \cdots \eta_{jN}), \beta_j = (\beta_{j1} \quad \beta_{j2} \cdots \beta_{jN}), \]
\[ \eta_{j,l} = \sum_{m_j=l}^{n_j} \rho_{j,m_j} f_{j,m_j-l}(x,t), \quad \beta_{j,l} = \sum_{m_j=l}^{n_j} \sum_{s_j=0}^{m_j-l} (-1)^s \frac{\overline{\rho}_{j,m_j} \overline{f}_{j,m_j-l-s_j}(x,t) q_s}{q_j^{s+1}}. \]

\[
\chi = \begin{pmatrix}
\begin{bmatrix} \varpi_{11} & \varpi_{12} & \cdots & \varpi_{1(2N)} \\
\varpi_{21} & \varpi_{22} & \cdots & \varpi_{2(2N)} \\
\vdots & \vdots & \ddots & \vdots \\
\varpi_{(2N)1} & \varpi_{(2N)2} & \cdots & \varpi_{(2N)(2N)}
\end{bmatrix}
\end{pmatrix},
\]

and \([\varpi_{j,l}]_{(j,l) = 1, 2, \cdots, 2N}\) are \(n_j \times n_l\) matrix

\[
[\varpi_{j,l}] = ([\varpi_{j,l}]_{p,q})_{n_j \times n_l} = \left( - \sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \left( q + s_j - 1 \right) \frac{(-1)^s \overline{\rho}_{j,m_j} \overline{f}_{j,m_j-p-s_j}(x,t)}{(q_j - q)_{s_j+q}} \right)_{n_j \times n_l}.
\]

**Double 1\(^{\text{nd}}\) – order pole solution**

Let \(\nu_{1}\) and \(\nu_{2}\) are the 1-order zeros point of \(a(z)\), so is \(\nu_{3}\) and \(\nu_{4}\). The reflection coefficient \(\rho(z)\) can be expand as Laurent series \(\rho(z) = \rho_{j,0}(z) + \frac{\rho_{j,1}}{z-\nu_{j}}\). Now \(\chi\) in (3.10) is defined as

\[
\chi = \begin{pmatrix}
\begin{bmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} & \varpi_{14} \\
\varpi_{21} & \varpi_{22} & \varpi_{23} & \varpi_{24} \\
\varpi_{31} & \varpi_{32} & \varpi_{33} & \varpi_{34} \\
\varpi_{41} & \varpi_{42} & \varpi_{43} & \varpi_{44}
\end{bmatrix}
\end{pmatrix},
\]

where \(n_1 = n_2 = n_2 = n_4 = 1, \quad j, l = 1, 2, 3, 4\)

\[
|\eta\rangle = (\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4)^T, \quad |\beta\rangle = (\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4)^T, \quad |Y_0\rangle = (1 \quad 1 \quad 1 \quad 1),
\]

\[
\beta_{j,l} = - \sum_{m_j=l}^{n_j} \sum_{s_j=0}^{m_j-l} (-1)^s \frac{\overline{\rho}_{j,m_j} \overline{f}_{j,m_j-l-s_j}(x,t) q_s}{q_j^{s+1}}, \quad \eta_{j,l} = \sum_{m_j=l}^{n_j} \rho_{j,m_j} f_{j,m_j-l}(x,t).
\]

\[
[\varpi_{j,l}] = [\varpi_{j,l}]_{p,q} = - \sum_{m_j=p}^{n_j} \sum_{s_j=0}^{m_j-p} \left( q + s_j - 1 \right) \frac{(-1)^s \overline{\rho}_{j,m_j} \overline{f}_{j,m_j-p-s_j}(x,t)}{(q_j - q)_{s_j+q}},
\]

**Mixed 2\(^{\text{nd}}\) – and 1\(^{\text{nd}}\) – order pole solution**

Let \(\nu_{1}\) and \(\nu_{3}\) are 2-order zero point of \(a(z)\), \(\nu_{2}\) and \(\nu_{4}\) are 1-order zeros of \(a(z)\). The reflection coefficient \(\rho(z)\) can be expand as Laurent series \(\rho(z) = \rho_{j,0}(z) + \sum_{m_j=1}^{n_j} \frac{\rho_{j,m_j}}{(z-\nu_{j})_{m_j}}\). Now \(\chi\) in (3.10) is defined as

\[
\chi = \begin{pmatrix}
\begin{bmatrix} \varpi_{11} & \varpi_{12} & \varpi_{13} & \varpi_{14} \\
\varpi_{21} & \varpi_{22} & \varpi_{23} & \varpi_{24} \\
\varpi_{31} & \varpi_{32} & \varpi_{33} & \varpi_{34} \\
\varpi_{41} & \varpi_{42} & \varpi_{43} & \varpi_{44}
\end{bmatrix}
\end{pmatrix},
\]

where \(n_1 = n_3 = 2, \quad n_2 = n_4 = 1, \quad j, l = 1, 2, 3, 4\)

\[
|\eta\rangle = (\eta_1 \quad \eta_2 \quad \eta_3 \quad \eta_4)^T, \quad |\beta\rangle = (\beta_1 \quad \beta_2 \quad \beta_3 \quad \beta_4)^T, \quad |Y_0\rangle = (1 \quad 0 \quad 1 \quad 0 \quad 1),
\]

9
\[ \beta_{j,l} = -\sum_{m_j=l}^{n_j} \sum_{s_j=0}^{m_j-l} \frac{(-1)^s_j \rho_{j,m_j} f_{j,m_j-l-s_j}(x,t) q_j}{\nu_j^{s_j+1}}, \quad \eta_{j,l} = \sum_{m_j=l}^{n_j} \rho_{j,m_j} f_{j,m_j-l}(x,t), \]

\[ [\varpi_{j,l}]_{p,q} = -\sum_{m_j=p}^{m_j} \sum_{s_j=0}^{m_j-p} \left( q + s_j - 1 \right) \frac{(-1)^s_j \rho_{j,m_j} f_{j,m_j-p-s_j}(x,t)}{(\nu_j - \nu_l)^{s_j+q}}. \]

**Acknowledgments**

This work is supported by the National Natural Science Foundation of China (No.12175069) and Science and Technology Commission of Shanghai Municipality (No.21JC1402500 and No.18dz2271000).

**References**

[1] A. Osborne, Nonlinear ocean waves and the inverse scattering transform, Elsevier New York, (2010).
[2] G. Agrawal, Nonlinear fiber optics, 4th ed. Academic New York Chap. 4(2006).
[3] H. Bailung, S.K. Sharma, Y. Nakamura, Observation of peregrine solitons in a multicomponent plasma with negative ions, Phys. Rev. Lett. 107 (2011) 255005.
[4] E.P. Gross, Structure of a quantized vortex in Boson systems, Il Nuovo Cimento Series 10. 20 (1961) 454-477.
[5] A. Chowdury, D.J. Kedziora, A. Ankiewicz, N. Akhmediev, Soliton solutions of an integrable nonlinear Schrödinger equation with quintic terms, Phys. Rev. E. 90 (2014) 032922.
[6] J.K. Yang, Nonlinear waves in intergrable and nonintergrable systems, Soc. Industrial Appl. Math. Philadelphia. (2010).
[7] L.L. Wen, N. Zhang, E.G. Fan, N-Soliton solution of the Kundu-type equation via Riemann-Hilbert approach, Acta Math. Sci. 40 (2020) 113-126.
[8] G. Biondini, G. Kovářič, Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions, J. Math. Phys. 55 (2014) 031506.
[9] B. Primari, M.J. Ablowitz, G. Biondini, Inverse scattering transform for the vector nonlinear Schrödinger equation with nonvanishing boundary conditions, J. Math. Phys. 47 (2006) 063508.
[10] L.L. Wen, E.G. Fan, Inverse scattering transformation for the Sasa-Satsuma equation with nonvanishing boundary conditions, arXiv:1911.11944.
[11] Y.S. Zhang, X.X. Tao, S.W. Xu, The bound-state soliton solutions of the complex modified KdV equation, Inverse Probl. 36 (2020) 065003.
[12] Y.S. Zhang, J.G. Rao, Y. Cheng, J.S. He, Riemann-Hilbert method for the Wadati-Konno-Ichikawa equation: N simple poles and one higher-order pole, Physica. D. 399 (2019) 173-185.
[13] Y.S. Zhang, X.X. Tao, T.T. Yao, J.S. He, The regularity of the multiple higher-order poles solitons of the NLS equation, Stud. Appl. Math. 145 (2020) 812-827.
[14] M. Pichler M, G. Biondini, On the focusing non-linear Schrödinger equation with nonzero boundary conditions and double poles, IMA J. Appl. Math. 82 (2017) 131-151.
[15] N. Liu, B.L. Guo, Solitons and rogue waves of the quartic nonlinear Schrödinger equation by Riemann-Hilbert approach, Nonlinear Dyn. 100 (2020) 629-646.