Twisted $\Gamma$-Lie algebras and their vertex operator representations

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Abstract. Let $\Gamma$ be a generic subgroup of the multiplicative group $\mathbb{C}^*$ of nonzero complex numbers. We define a class of Lie algebras associated to $\Gamma$, called twisted $\Gamma$-Lie algebras, which is a natural generalization of the twisted affine Lie algebras. Starting from an arbitrary even sublattice $Q$ of $\mathbb{Z}^N$ and an arbitrary finite order isometry of $\mathbb{Z}^N$ preserving $Q$, we construct a family of twisted $\Gamma$-vertex operators acting on generalized Fock spaces which afford irreducible representations for certain twisted $\Gamma$-Lie algebras. As application, this recovers a number of known vertex operator realizations for infinite dimensional Lie algebras, such as twisted affine Lie algebras, extended affine Lie algebras of type $A$, trigonometric Lie algebras of series $A$ and $B$, unitary Lie algebras, and $BC$-graded Lie algebras.

1. Introduction

Affine Kac-Moody Lie algebras, which are a family of infinite-dimensional Lie algebras, have played an important role in mathematics and mathematical physics. One striking discovery in the representation theory of affine Kac-Moody algebras was the vertex operator construction of the basic representations. The first such construction, called principal, was discovered in [LW] and generalized in [KKLW]. Another construction, called homogeneous, was given in [FK] and [S] independently. Besides affine Kac-Moody algebras, there are some other interesting infinite dimensional Lie algebras such as extended affine Lie algebras, trigonometric Lie algebras, unitary Lie algebras and root graded Lie algebras. And the vertex operator representations also play an important role in the study of representation theory of these algebras. To explain our motivation, we first give a brief introduction to these algebras and mention some of their vertex operator constructions which are closely related to our work.

- **Twisted affine Lie algebras** The vertex operator representations for twisted affine Lie algebras were obtained in [L] and [KP] in a very general setting. In [L], the author gave a generalization of all the known vertex...
operator constructions for affine Kac-Moody algebras by introducing the twisted vertex operators associated with an arbitrary isometry of an arbitrary even lattice.

- **Extended affine Lie algebras of type** $A$. Extended affine Lie algebras were first introduced in [H-KT] and systematically studied in [AABGP]. The vertex operator representations for extended affine Lie algebras of type $A$ coordinated by quantum tori have been given in [BS] and [G2] for the principal realizations, and in [G1] in the homogeneous realization. Later in [BGT], a unified treatment was given.

- **Trigonometric Lie algebras.** As a natural generalization of the Sine Lie algebra, four series of $\mathbb{Z}$-graded trigonometric subalgebras $\hat{A}_h - \hat{D}_h$ of $\hat{A}_\infty - \hat{D}_\infty$ were introduced in [G-KL1, G-KL2]. Moreover, the vertex operator constructions for the Lie algebras $\hat{A}_h$ and $\hat{B}_h$ were also presented in [G-KL1, G-KL2].

- **Unitary Lie algebras.** Unitary Lie algebras were first introduced in [AF]. The compact forms of certain intersection matrix algebras developed by Slodowy can be identified with some Steinberg unitary Lie algebras. The vertex operator constructions for a class of unitary Lie algebras coordinated by skew Laurent polynomial rings were presented in [CGJT].

- **BC-graded Lie algebras.** Root graded Lie algebras were first introduced in [BM] and the BC-graded Lie algebras were studied and classified in [ABG]. The homogeneous vertex operator constructions for a class of BC-graded Lie algebras coordinated by skew Laurent polynomial rings were given in [CT].

Though these Lie algebras were introduced for different purposes and were defined by different approaches, we observe that their vertex operator constructions arise from certain known vertex operators for affine Lie algebras of type $A$ or $D$ and depend on certain nonzero complex numbers. It is natural for us to give a general construction so that it contains these vertex operator constructions as special cases. This is the main motivation of our work.

We first give the definition of twisted $\Gamma$-Lie algebras which simultaneously generalize those Lie algebras mentioned above. More precisely, let $\Gamma$ be a subgroup of the multiplicative group $\mathbb{C}^*$ satisfying the following assumption:

(A1). $\Gamma$ is generic, i.e., $\Gamma$ is isomorphic to a free abelian group.

We start from any finite order automorphism of an involutive associative algebra to construct the twisted $\Gamma$-Lie algebra. In particular, we use a lattice to construct an involutive associative algebra, then a twisted $\Gamma$-Lie algebra is defined based on this involutive associative algebra. To explain this process more precisely, we let $N$ be a positive integer, and $P = \mathbb{Z}^N = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_N$ be a lattice of rank $N$ equipped with a $\mathbb{Z}$-bilinear form given by $\langle e_i, e_j \rangle = \delta_{ij}, 1 \leq i, j \leq N$. We take a triple $(Q, \nu, m)$ satisfying the following conditions:
(A2). $Q$ is a sublattice of $P$ such that $\langle \alpha, \alpha \rangle \in 2\mathbb{Z}$ for $\alpha \in Q$.

(A3). $\nu$ is an isometry of $P$ preserving $Q$. That is, $\nu$ is an automorphism of $P$ such that $\langle \nu \alpha, \nu \beta \rangle = \langle \alpha, \beta \rangle$ for $\alpha, \beta \in P$ and $\nu(Q) = Q$.

(A4). $m$ is a positive integer such that $\nu^m = \text{Id}$, the identity map on $P$.

(A5). If $m$ is even, $\langle \nu^{m/2} \alpha, \alpha \rangle \in 2\mathbb{Z}$ for $\alpha \in Q$.

Note that if the triple $(Q, \nu, m)$ satisfies the assumptions (A2)-(A4), then the assumption (A5) can always be arranged by doubling $m$ if necessary. Now we give some examples of the triples $(Q, \nu, m)$ which satisfy assumptions (A2)-(A5). Let $Q = Q(A_{N-1})$, $N \geq 2$ or $Q(D_N)$, $N \geq 5$ be the root lattices of type $A_{N-1}$ and $D_N$ respectively, and let $\nu$ be an isometry of $Q$. It is known that $\nu$ can be lifted to be an isometry of $P$ and has finite order. Thus there exists a positive integer $m$ such that $(Q, \nu, m)$ satisfies assumptions (A2)-(A5).

Starting from a quadruple $(Q, \nu, m, \Gamma)$ which satisfies (A1)-(A5), we construct a twisted $\Gamma$-Lie algebra $\hat{G}(Q, \nu, m, \Gamma)$ and define a family of twisted $\Gamma$-vertex operators acting on a generalized Fock space. By computing the commutator relations of these twisted $\Gamma$-vertex operators, we obtain a class of irreducible representations for the twisted $\Gamma$-Lie algebra $\hat{G}(Q, \nu, m, \Gamma)$. One will see that it is very subtle and technical to determine the commutator relations for the twisted $\Gamma$-vertex operators. To do this, we develop a highly non-trivial generalization of the combinatorial identity presented in Proposition 4.1 of [L]. As applications, we show that with different choices of quadruples $(Q, \nu, m, \Gamma)$, we recover the vertex operator constructions of the twisted affine Lie algebras, extended affine Lie algebras of type $A$ (both homogeneous and principal constructions), trigonometric Lie algebras of series $A$ and $B$, unitary Lie algebras and $BC$-graded Lie algebras given in [L, G-KL1, G-KL2, BS, G1, G2, BGT, CGJT, CT] respectively. Moreover, we also present vertex operator representations for a new twisted $\Gamma$-Lie algebras.

The paper is organized as follows. In Section 2 we give the definition of the twisted $\Gamma$-Lie algebra $\hat{G}(Q, \nu, m, \Gamma)$ for any quadruple $(Q, \nu, m, \Gamma)$ satisfying the assumptions (A1)-(A5). In Section 3 we define the generalized Fock space and give the twisted $\Gamma$-vertex operators. In Section 4 we establish a crucial identity (see (4.10)) by using certain combinatorial identities, which allows us to compute the commutator relations of the twisted $\Gamma$-vertex operators. In Section 5 we prove our main result in Theorem 5.2 which shows that those twisted $\Gamma$-vertex operators acting on generalized Fock spaces give representations for the Lie algebra $\hat{G}(Q, \nu, m, \Gamma)$. Finally, in Section 6 we present the applications of our main result.

Conventions and notations:
1. We denote the sets of integers, positive integers, complex numbers, real numbers and the quotient group $\mathbb{Z}/m\mathbb{Z}$ by $\mathbb{Z}, \mathbb{Z}_+, \mathbb{C}, \mathbb{R}$ and $\mathbb{Z}_m$, respectively.
2. For short, if $\alpha, \beta \in P$, we often write
   \[ \sum_{p \in \mathbb{Z}_m} \nu^p \alpha := \sum_{p \in \mathbb{Z}_m} \nu^p \alpha, \quad \text{and} \quad \sum_{p \in \mathbb{Z}_m} \langle \alpha, \nu^p \beta \rangle := \sum_{p \in \mathbb{Z}_m} \langle \alpha, \nu^p \beta \rangle. \]
3. For \(1 \leq i \neq j \leq N\) and \(c \in \mathbb{C}\), we set \((1 - c)^{\pm \delta_{ij}} = 1\) even if \(c = 1\).

4. Since the fraction powers of nonzero complex numbers will arise in the construction of our vertex operators, we need a convention for this. For an index set \(I\), we fix a set of free generators \(\{q_i, i \in I\}\) of \(\Gamma\), and fix a choice of \(q_i^{1/2}\) for each \(i \in I\). For any \(c = q_{i_1}^{n_1} \cdots q_{i_t}^{n_t} \in \Gamma\), where \(i_1, \cdots, i_t \in I, n_1, \cdots, n_t \in \mathbb{Z}\), we set

\[
c_i := (q_{i_1}^{1/2})^{n_1} \cdots (q_{i_t}^{1/2})^{n_t}.
\]

Then we have \(c_1^2 = (c_1 c_2)^{1/2}\) for all \(c_1, c_2 \in \Gamma\) and \(n \in \mathbb{Z}\).

5. Set \(Q(D_1) = 2\mathbb{Z}e_1\), and for \(N \geq 2\), let

\[
Q(A_{N-1}) = \mathbb{Z}(\epsilon_1 - \epsilon_2) \oplus \cdots \oplus \mathbb{Z}(\epsilon_{N-1} - \epsilon_N),
\]

\[
Q(D_N) = \mathbb{Z}(\epsilon_1 - \epsilon_2) \oplus \cdots \oplus \mathbb{Z}(\epsilon_{N-1} - \epsilon_N) \oplus \mathbb{Z}(\epsilon_{N-1} + \epsilon_N)
\]

be the root lattices of type \(A_{N-1}\) and \(D_N\), respectively.

2. Twisted \(\Gamma\)-Lie algebras

In this section, we give the definition of the twisted \(\Gamma\)-Lie algebra and present some examples. In particular, a twisted \(\Gamma\)-Lie algebra \(\mathcal{G}(Q, \nu, m, \Gamma)\) associated to any quadruple \((Q, \nu, m, \Gamma)\) satisfying (A1)-(A5) is constructed.

2.1. Twisted \(\Gamma\)-Lie algebras. Let \(\mathbb{C}_\Gamma\) be an associative algebra with base elements of the form \(t^n c, n \in \mathbb{Z}, c \in \Gamma\), and multiplication given by

\[
(t^n c_1)(t^n c_2) = c_1^r t^{n+r} c_{1c_2}, \quad n, r \in \mathbb{Z}, c_1, c_2 \in \Gamma.
\]

Let \(\mathcal{A}\) be another associative algebra equipped with an invariant symmetric bilinear form \(\langle \cdot, \cdot \rangle_{\mathcal{A}}\). Suppose that \(\theta\) is a finite order automorphism of \(\mathcal{A}\) and preserves the bilinear form \(\langle \cdot, \cdot \rangle_{\mathcal{A}}\). Let \(m\) be a positive integer such that \(\theta^m = \text{Id}\) and fix a primitive \(m\)-th root of unity \(\omega\). Viewing \(\mathcal{A} \otimes \mathbb{C}_\Gamma\) as a Lie algebra, denoted by \(\mathcal{A}(\Gamma)\), with Lie product \([a \otimes b, a' \otimes b'] = a a' \otimes \omega b b' - a a' \otimes b b'\) for \(a, a' \in \mathcal{A}\) and \(b, b' \in \mathbb{C}_\Gamma\). Extend the automorphism \(\theta\) of \(\mathcal{A}\) to be a Lie automorphism of \(\mathcal{A}(\Gamma)\), still denoted by \(\theta\), by letting

\[
\theta(a \otimes t^n c) := \omega^{-n} \theta(a) \otimes t^n c, \quad a \in \mathcal{A}, n \in \mathbb{Z}, c \in \Gamma.
\]

We denote by \(\mathcal{A}(\theta, m, \Gamma)\) the subalgebra of \(\mathcal{A}(\Gamma)\) fixed by the automorphism \(\theta\). For any \(a \in \mathcal{A}\) and \(n \in \mathbb{Z}\), set

\[
a_{(n)}^\theta = a_{(n)} := m^{-1} \sum_{p \in \mathbb{Z}_m} \omega^{-np} \theta^p(a), \quad \mathcal{A}_{(n)} := \{a_{(n)} | a \in \mathcal{A}\}.
\]

Then we have \(\theta(a_{(n)}) = \omega^n a_{(n)}\) and \(\mathcal{A} = \bigoplus_{p \in \mathbb{Z}_m} \mathcal{A}_{(p)}\). Obviously, the Lie algebra \(\mathcal{A}(\theta, m, \Gamma)\) is spanned by the following elements

\[
a(c, n) := a_{(n)} \otimes t^n c, \quad a \in \mathcal{A}, c \in \Gamma, n \in \mathbb{Z}.
\]
We define a bilinear form $\langle \cdot , \cdot \rangle$ on the Lie algebra $\mathcal{A}(\theta, m, \Gamma)$ as follows

$$\langle a(c_1, n), b(c_2, r) \rangle := \frac{n}{2m} \delta_{n+r,0} \langle a, b \rangle \mathcal{A} \delta_{c_1c_2,1} c_1^r, \quad a, b \in \mathcal{A}, c_1, c_2 \in \Gamma, n, r \in \mathbb{Z}.$$  

It is easy to see that the bilinear form $\langle \cdot , \cdot \rangle$ is a 2-cocycle on the Lie algebra $\mathcal{A}(\theta, m, \Gamma)$. We define a Lie algebra $\hat{\mathcal{A}}(\theta, m, \Gamma) := \mathcal{A}(\theta, m, \Gamma) \oplus \mathbb{C}c$ to be the 1-dimensional central extension of $\mathcal{A}(\theta, m, \Gamma)$ associated to this 2-cocycle.

Let $\tau$ be an anti-involution of $\mathcal{A}$ which preserves $\langle \cdot , \cdot \rangle$. We simply call the pair $(\mathcal{A}, \tau)$ an involutive associative algebra. Suppose that $\theta$ is also an automorphism of the involutive associative algebra $(\mathcal{A}, \tau)$, i.e., it commutes with $\tau$. Define an anti-involution $\hat{\tau}$ on $\mathbb{C}_\Gamma$ by

$$\hat{\tau}(c) := c^{-n} t^n T_c^{-1}, \quad n \in \mathbb{Z}, c \in \Gamma.$$  

Consider the linear map $\hat{\tau}$ on the Lie algebra $\hat{\mathcal{A}}(\theta, m, \Gamma)$ given by

$$\hat{\tau}(a(n) \otimes t^n T_c) := -\tau(a(n)) \otimes t^n T_c, \quad \hat{\tau}(c) = c, \quad a \in \mathcal{A}, n \in \mathbb{Z}, c \in \Gamma,$$

which is a Lie involution of $\hat{\mathcal{A}}(\theta, m, \Gamma)$. Now we define the twisted $\Gamma$-Lie algebra $\hat{\mathcal{A}}_\tau(\theta, m, \Gamma)$ to be the set of fixed-points of $\hat{\mathcal{A}}(\theta, m, \Gamma)$ under the involution $\hat{\tau}$. Then one can see that the following elements

$$\hat{a}(c, n) := a(c, n) + \hat{\tau}(a(c, n)), \quad a \in \mathcal{A}, c \in \Gamma, n \in \mathbb{Z},$$

together with the central element $c$, span the algebra $\hat{\mathcal{A}}_\tau(\theta, m, \Gamma)$.

**Remark 2.1.** In the Lie algebra $\hat{\mathcal{A}}_\tau(\text{Id}, 1, \Gamma)$, we define a formal power series

$$\hat{a}(c, z) = \sum_{n \in \mathbb{Z}} \hat{a}(c, n) z^n$$

for $a \in \mathcal{A}$ and $c \in \Gamma$. Note that the formal power series satisfying the following so-called “$\Gamma$-locality” ($[G-KK], [Li]$)

$$(z_1 - z_2)(c_1z_1 - z_2)(z_1 - c_2z_2)(c_1z_1 - c_2z_2)\hat{a}_1(c_1, z_1), \hat{a}_2(c_2, z_2) = 0,$$

where $a_1, a_2 \in \mathcal{A}$ and $c_1, c_2 \in \Gamma$. Motivated by the notion of $\Gamma$-conformal algebra defined in $[G-KK]$ and $\Gamma$-vertex algebra defined in $[Li]$, we call the Lie algebra $\hat{\mathcal{A}}_\tau(\theta, m, \Gamma)$ the twisted $\Gamma$-Lie algebra.

**Remark 2.2.** Let $\mathcal{A}^{\text{op}}$ be the opposite algebra of $\mathcal{A}$ and $\text{ex}$ be the exchange involution of $\mathcal{A} \oplus \mathcal{A}^{\text{op}}$, i.e., $\text{ex}(a, a') = (a', a)$. Extend the automorphism $\theta$ and the bilinear form $\langle \cdot , \cdot \rangle$ of $\mathcal{A}$ to be an automorphism and a bilinear form of $(\mathcal{A} \oplus \mathcal{A}^{\text{op}}, \text{ex})$ by $\theta(a, a') := (\theta(a), \theta(a'))$ and $\langle (a, a'), (b, b') \rangle := \langle a, b \rangle_\mathcal{A} + \langle a', b' \rangle_\mathcal{A}$, respectively. Then we have $\hat{\mathcal{A}}(\theta, m, \Gamma) \cong \hat{\mathcal{A}} \oplus \mathcal{A}^{\text{op}}_{\text{ex}}(\theta, m, \Gamma)$.

**2.2. Examples.** We denote by $\mathcal{M}_N$ the $N \times N$-matrix algebra over $\mathbb{C}$, and denote by $E_{i,j}, 1 \leq i, j \leq N$ the unit matrices in $\mathcal{M}_N$. In what follows, if $\mathcal{A} = \mathcal{M}_N$, then the invariant bilinear form $\langle \cdot , \cdot \rangle_\mathcal{A}$ is always taken to be the trace form.

**1. Twisted affine Lie algebras.** Let $(\mathcal{A}, \tau)$ be a finite dimensional simple involutive associative algebra and $\Gamma = \{1\}$, then the Lie algebra $\hat{\mathcal{A}}_\tau(\theta, m, \Gamma)$ is a twisted affine Lie algebra. Moreover, all the classical affine Lie algebras, that is,
the affine Lie algebras of type $X_l^{(r)}$, $r = 1, 2$ and $X = A, B, C, D$, can be realized by this way.

2. **Extended affine Lie algebras** $\hat{\mathfrak{g}}_N(C_q)$. Recall the Lie algebra $\hat{\mathfrak{g}}_N(C_q)$ defined in [BGT], which is a 1-dimensional central extension of the matrix algebra over the quantum torus $C_q$ associated to $q = (q_1, \ldots, q_l) \in (C^*)^l$. Explicitly, it has a basis $E_{i,j}t_0^{a}t^n$ and $c$, for $1 \leq i, j \leq N$, $n \in \mathbb{Z}^l$, $n_0 \in \mathbb{Z}$, subject to the Lie relation

$$[E_{i,j}t_0^{a}t^n, E_{k,p}t_0^{b}t^r] = q^{\delta_{ij}}\delta_{jk}E_{i,k}t_0^{a+b}t^{n+r} - q^{\delta_{ip}}\delta_{jp}E_{i,p}t_0^{a+r}t^{n+b} + n_0q^{\delta_{ip}}\delta_{jk}\delta_{n_0+r,0}\delta_{c_0+r,0}c,$$

where $t^n = t_1^{a_1} \cdots t_l^{a_l}$ and $q^n = q_1^{a_1} \cdots q_l^{a_l}$, and $c$ is central. Assume that the subgroup $\Gamma_q$ of $\mathbb{C}$ generated by $q_1, \ldots, q_l$ is a free abelian group of rank $k$. Then the Lie algebra $\tilde{\mathcal{M}}_N \oplus \tilde{\mathcal{M}}_N^\text{op} (Id, 1, \Gamma_q)$ is isomorphic to $\hat{\mathfrak{g}}_N(C_q)$ via the isomorphism $\tilde{E}_{i,j}(q^n, n_0) \mapsto E_{i,j}t_0^{a}t^n$, $c \mapsto c$, $1 \leq i, j \leq N$, $(n_0, n) \in \mathbb{Z}^{|l+1|}$.

3. Trigonometric Lie algebras (cf. [G-KL1, G-KL2]). Given a $l$-tuple $h = (h_1, \ldots, h_l) \in \mathbb{R}^l$, we set $\Gamma_h = \{e^{2\sqrt{-1}t(h,n)}| n = (n_1, \ldots, n_l) \in \mathbb{Z}^l\}$, where $(h, n) = h_1n_1 + \cdots + h_ln_l$. Assume that $h_1, \ldots, h_l$ are $\mathbb{Z}$-linearly independent so that $\Gamma_h$ is generic. Set $\mathcal{A} = \mathbb{C} \oplus \mathbb{C}$, $\tau = \text{ex}$ and $\Gamma = \Gamma_h$. Then in $\tilde{\mathcal{A}}(1, 1, \Gamma_h)$, one has

$$[A_{n,n_0}, A_{r,r_0}] = 2\sqrt{-1}\sin(n_0(h, r) - r_0(h, n))A_{n+r,n_0+r_0} + n_0\delta_{n_0+r_0,0}\delta_{c_0+r_0,0}c,$$

where $A_{n,n_0} := e^{-n_0\sqrt{-1}(\tau(h,n))}(1,0)(e^{-\sqrt{-1}(\tau(h,n))}, n_0)$. These are the commutator relations of the trigonometric Lie algebra of series $A_h$ defined in [G-KL1]. Furthermore, in $\tilde{\mathcal{A}}(2, 2, \Gamma_h)$, one has

$$[B_{n,n_0}, B_{r,r_0}] = 2\sqrt{-1}\sin(n_0(h, r) - r_0(h, n))B_{n+r,n_0+r_0} + (-1)^{n_0}2\sqrt{-1}\sin(n_0(h, r) - r_0(h, n))B_{n+r,n_0+r_0} + n_0\delta_{n_0+r_0,0}\delta_{c_0+r_0,0}c,$$

where $B_{n,n_0} := 2e^{-n_0\sqrt{-1}(\tau(h,n))}(1,0)(e^{-\sqrt{-1}(\tau(h,n))}, n_0)$. Then the Lie algebra $\tilde{\mathcal{A}}(2, 2, \Gamma_h)$ is isomorphic to the trigonometric Lie algebra of series $B_h$ (cf. [G-KL2]).

4. **Unitary Lie algebras** $\hat{\mathfrak{u}}_N(C_T)$. The Lie algebra $\hat{\mathfrak{u}}_N(C_T)$ defined in [CGJT] is is spanned by the elements $u_{i,j}(c, n), c, 1 \leq i, j \leq N, c \in \Gamma, n \in \mathbb{Z}$ with the relation $u_{i,j}(c, n) = -(c)^{-n}u_{j,i}(c^{-1}, 1)$, and subject to

$$[u_{i,j}(c_1, n), u_{k,l}(c, n + r)] = \delta_{jk}c^r_{i,k}u_{i,j}(c_1c_2, n + r) + \delta_{jk}(-c_1)^{n-r}c_{i,j}(c_1^{-1}c_2^{-1}, n + r) - \delta_{ik}(-1)^{n_0}c_1^{-n-r}u_{i,j}(c_1c_2^{-1}, n + r)\delta_{n_0+n,0}c - \delta_{ik}(-c_1^{-1}c_2^{-1}, n + r) + n\delta_{n_0+n,0}\delta_{c_0,0}c,$$

where $c$ is a central element. Choose $\mathcal{A} = \mathcal{M}_N \oplus \mathcal{M}_N^\text{op}$, $\tau = \text{ex}$, $\theta : (A, B) \mapsto (B^t, A^t)$ and $m = 2$, where $A, B \in \mathcal{M}_N$, $A^t$ is the transpose of $A$. It is easy to check that the following linear map gives an isomorphism from the unitary Lie algebra $\hat{\mathfrak{u}}_N(C_T)$
to the twisted $\Gamma$-Lie algebra $\widehat{A}_r(\theta, 2, \Gamma)$:

$$u_{i,j}(c, n) \mapsto 2(E_{i,j}, 0)(c, n), \; c \mapsto c, \; 1 \leq i, j \leq N, c \in \Gamma, n \in \mathbb{Z}.$$

5. **BCN-graded Lie algebras** $\widehat{\mathfrak{g}}_{2N}(\mathbb{C}_\Gamma)$. Following [CT], the Lie algebra $\widehat{\mathfrak{g}}_{2N}(\mathbb{C}_\Gamma)$ is spanned by the elements $f_{\rho_i, \rho_j}(c, n), 1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1, c \in \Gamma, n \in \mathbb{Z}$, with relation $f_{\rho_i, \rho_j}(c, n) = -(c)^{-n}f_{-\rho_j, -\rho_i}(c^{-1}, n)$. The Lie bracket in $\widehat{\mathfrak{g}}_{2N}(\mathbb{C}_\Gamma)$ is given by

$$[f_{\rho_i, \rho_j}(c_1, n), f_{\rho_k, \rho_l}(c_2, r)] = \delta_{\rho_j, \rho_k}c_1^1f_{\rho_i, \rho_l}(c_1c_2, n + r) + \delta_{\rho_i, \rho_l}c_1^{-n-r}c_2^r$$

$$f_{-\rho_j, -\rho_i}(c_1^{-1}, n + r) - \delta_{\rho_i, -\rho_k}c_1^{-n-r}f_{-\rho_j, -\rho_l}(c_1c_2, n + r) - \delta_{-\rho_j, \rho_i}(c_1/c_2)^r$$

$$f_{\rho_i, -\rho_k}(c_1c_2^{-1}, n + r) + \delta_{\rho_j, \rho_k}\delta_{\rho_i, \rho_l}(c_1^{-1}c_2)c_1^{-1}c_2c_2^{-1}n\mathbf{c} - \delta_{\rho_i, -\rho_k}\delta_{\rho_j, -\rho_l}c_1^{-1}c_2^{-1}c_2^{-1}n\mathbf{c}.$$

Choose $\mathcal{A} = \mathcal{M}_{2N}, \tau : E_{\rho_i, \rho_j} \mapsto E_{-\rho_j, -\rho_i}, \theta = \text{Id}$ and $m = 1$, where $1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1$. Then the following linear map gives an isomorphism from the BCN-graded Lie algebra $\widehat{\mathfrak{g}}_{2N}(\mathbb{C}_\Gamma)$ to the twisted $\Gamma$-Lie algebra $\widehat{A}_r(\text{Id}, 1, \Gamma)$:

$$f_{\rho_i, \rho_j}(c, n) \mapsto E_{\rho_i, \rho_j}(c, n), \; c \mapsto c, \; 1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1, c \in \Gamma, n \in \mathbb{Z}.$$

2.3. **Lattice construction of involutive associative algebras.** One of the most important steps in the vertex operator representation theory of affine Lie algebras is the lattice construction for semi-simple Lie algebras. In this section, we will use the lattice construction for involutive associative algebras, which will be used to present vertex operator representations for the twisted $\Gamma$-Lie algebras.

Given a triple $(Q, \nu, m)$ which satisfies assumptions (A2)-(A5). We define a set

$$\mathcal{J} = \{(\rho_i, \rho_j)|\rho_i\epsilon_i - \rho_j\epsilon_j \in Q, 1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1\}.$$  

Let $\mathcal{G}(Q) = \bigoplus_{(\rho_i, \rho_j) \in \mathcal{J}} \mathbb{C} e_{\rho_i, \rho_j}$ be a vector space over $\mathbb{C}$, where $\{e_{\rho_i, \rho_j}\}_{(\rho_i, \rho_j) \in \mathcal{J}}$ is a set of symbols. We define multiplication on $\mathcal{G}(Q)$ by

$$e_{\rho_i, \rho_j}e_{\rho_k, \rho_l} = \delta_{\rho_j, \rho_k}\epsilon(\rho_i\epsilon_i, \rho_j\epsilon_j, \rho_k\epsilon_k, \rho_l\epsilon_l),$$

where $(\rho_i, \rho_j), (\rho_k, \rho_l) \in \mathcal{J}$ and $\epsilon : Q \times Q \rightarrow \mathbb{C}$ is a normalized 2-cocycle on $Q$ associated with the function $(-1)^{\alpha\beta}$, that is, it satisfies the conditions

$$\epsilon(\alpha, \beta)\epsilon(\alpha + \beta, \gamma) = \epsilon(\beta, \gamma)\epsilon(\alpha, \beta + \gamma),$$

$$\epsilon(0, 0) = 1, \; \epsilon(\alpha, \beta)/\epsilon(\beta, \alpha) = (-1)^{\alpha\beta}.$$

Define a linear map $\tau$ and a bilinear form $\langle , \rangle_{\mathcal{G}}$ on $\mathcal{G}(Q)$ as follows

$$\tau e_{\rho_i, \rho_j} = (-1)^{\delta_{\rho_i, \rho_j}(-\rho_i)}$$

$$\langle e_{\rho_i, \rho_j}, e_{\rho_k, \rho_l}\rangle_{\mathcal{G}} = \delta_{\rho_j, \rho_k}\delta_{\rho_i, \rho_l}\epsilon(\rho_i\epsilon_i, \rho_j\epsilon_j, \rho_k\epsilon_k, \rho_l\epsilon_l)$$

for $(\rho_i, \rho_j), (\rho_k, \rho_l) \in \mathcal{J}$. It is easy to see that $(\mathcal{G}(Q), \tau)$ is an involutive associative algebra and $\langle , \rangle_{\mathcal{G}}$ is an invariant symmetric bilinear form on it.
Recall that $\nu$ is an isometry of $P$. For any $1 \leq i \leq N$, since $\langle \nu(e_i), \nu(e_i) \rangle = 1$, there exist $i_i = \pm 1$ and a permutation $\sigma \in S_N$ such that $\nu(e_i) = i_i e_{\sigma(i)}$. We introduce the following notation for later used

$$i_0 := i \quad \text{and} \quad i_r := \left( \prod_{p=0}^{r-1} i_{\sigma(p)} \right) \sigma(r) \quad 1 \leq i \leq N, \quad 1 \leq r \leq m - 1.$$  

We say an automorphism $\theta$ of $(G(Q), \tau)$ is compatible with the isometry $\nu$ if there exists a function $\eta: \mathbb{Z}_m \times Q \to \mathbb{C}$ such that

$$\theta^r(e_{\rho_i,\rho_j}) = \eta(r, \rho_i \epsilon_i - \rho_j \epsilon_j) e_{\rho_i,\rho_j}, \quad r \in \mathbb{Z}_m, \quad (\rho_i, \rho_j) \in J.$$  

Given an automorphism $\theta$ of $(G(Q), \tau)$ which satisfies $\theta^m = \text{Id}$, preserves the bilinear form $\langle , \rangle_G$, and is compatible with the isometry $\nu$. Then we have a twisted $\Gamma$-Lie algebra $G(Q)_{\theta}(\theta, m, \Gamma)$, which is also denoted by $\hat{G}(Q, \nu, m, \Gamma)$.

Recall the elements $e_{\rho_i,\rho_j}(c, n), (\rho_i, \rho_j) \in J, n \in \mathbb{Z}, c \in \Gamma$ defined in (2.1), which together with $c$ span the Lie algebra $\hat{G}(Q, \nu, m, \Gamma)$. For any $(\rho_i, \rho_j) \in J, c \in \Gamma$ and $n \in \mathbb{Z}$, we set $G_{\rho_i,\rho_j}(c, z) = \sum_{n \in \mathbb{Z}} e_{\rho_i,\rho_j}(c, n) z^{-n}$. Then we have the following proposition whose verification is straightforward.

**Proposition 2.3.** Let $(\rho_i, \rho_j), (\rho_k, \rho_l) \in J, r \in \mathbb{Z}_m$ and $c_1, c_2 \in \Gamma$, then

$$G_{\rho_i,\rho_j}(c_1, z) = (-1)^{\delta_{ij}} G_{-\rho_j,-\rho_i}(c_1^{-1}, c_1 z),$$

$$G_{\rho_i,\rho_j}(c_1, \omega^{-r} z) = \eta(r, \rho_i \epsilon_i - \rho_j \epsilon_j) G_{\rho_i,\rho_j}(c_1, z).$$

and

$$[G_{\rho_k,\rho_l}(c_1, z_1), G_{\rho_k,\rho_l}(c_2, z_2)] = m^{-1} \sum_{r \in \mathbb{Z}_m} \delta_{\rho_k,-\rho_k} \xi_r(\alpha, \beta) G_{-\rho_j,-\rho_i}(c_1^{-1}, c_1 z_1) \delta(\omega^r z_2 / z_1)$$

$$+ m^{-1} \sum_{r \in \mathbb{Z}_m} \delta_{\rho_j,\rho_k} \xi_r(\alpha, \beta) G_{\rho_i,\rho_l}(c_1 c_2, z_1) \delta(\omega^r z_2 / c_1 z_1)$$

$$+ m^{-1} \sum_{r \in \mathbb{Z}_m} \delta_{\rho_i,\rho_l} \xi_r(\alpha, \beta) G_{-\rho_j,-\rho_k}(c_1^{-1} c_2^{-1}, c_1 z_1) \delta(\omega^r c_2 z_2 / z_1)$$

$$+ m^{-1} \sum_{r \in \mathbb{Z}_m} \delta_{\rho_j,-\rho_k} \xi_r(\alpha, \beta) G_{\rho_i,-\rho_k}(c_1 c_2^{-1}, z_1) \delta(\omega^r c_2 z_2 / c_1 z_1)$$

$$+ m^{-2} \sum_{r \in \mathbb{Z}_m} \delta_{c_1 c_2} \delta_{\rho_j,\rho_k} \delta_{\rho_i,\rho_l} \xi_r(\alpha, \beta) (D \delta) (\omega^r z_2 / c_1 z_1)$$

$$+ m^{-2} \sum_{r \in \mathbb{Z}_m} \delta_{c_1 c_2} (\alpha, \beta) \xi_r(\alpha, \beta) (D \delta) (\omega^r z_2 / z_1),$$

where $\alpha := \rho_i \epsilon_i - \rho_j \epsilon_j$, $\beta := \rho_k \epsilon_k - \rho_l \epsilon_l$, $\xi_r(\alpha, \beta) = \epsilon(\alpha, \nu^r \beta) \eta(r, \beta)$, and $\delta(z) = \sum_{n \in \mathbb{Z}} z^n$, $(D \delta)(z) = \sum_{n \in \mathbb{Z}} n z^n$. \hfill $\square$
3. Generalized Fock space and twisted \( \Gamma \)-vertex operators

In this section we define the generalized Fock space and then construct a family of twisted \( \Gamma \)-vertex operators. We also present some properties of these operators.

Recall that \( P = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_N \) is a lattice of rank \( N \), and \((Q, \nu, m, \Gamma)\) is a quadruple satisfying the assumptions (A1)-(A5).

First we introduce a Heisenberg algebra \( \mathcal{H}(\nu) \) associated to the pair \((P, \nu)\). Let \( \mathcal{H} = P \otimes_{\mathbb{Z}} \mathbb{C} \). We extend the isometry \( \nu \) of \( P \) to a linear automorphism of \( \mathcal{H} \), and extend the bilinear form \( \langle \cdot, \cdot \rangle \) on \( P \) to a \( \mathbb{C} \)-bilinear non-degenerate symmetric form on \( \mathcal{H} \). For any \( n \in \mathbb{Z} \) and \( h \in \mathcal{H} \), we define \( h_{(n)} = \sum_{p \in \mathbb{Z}_m} \omega^{-np} \nu^p(h) \) and \( \mathcal{H}_{(n)} = \{ h_{(n)} | h \in \mathcal{H} \} \). Define a Heisenberg algebra
\[
\mathcal{H}(\nu) = \text{span}_{\mathbb{C}} \{ x(n), 1 | n \in \mathbb{Z} \setminus \{0\}, x \in \mathcal{H}_{(n)} \}
\]
with Lie bracket
\[
[x(n), y(r)] = m^{-1} \langle x, y \rangle n \delta_{n+r,0}, \ n, r \in \mathbb{Z} \setminus \{0\}, x \in \mathcal{H}_{(n)}, y \in \mathcal{H}_{(r)}.
\]
Let \( S := S(\mathcal{H}(\nu)^{-}) \) be the symmetric algebra over the commutative subalgebra \( \mathcal{H}(\nu)^{-} \) of \( \mathcal{H}(\nu) \) spanned by the elements \( x(n) \) for \( x \in \mathcal{H}_{(n)} \) and \( n < 0 \). It is well-known that the Heisenberg algebra \( \mathcal{H}(\nu) \) has a canonical irreducible representation on the symmetric algebra \( S \). Let \( z, z_1, z_2 \) be formal variables and \( \alpha \in \mathcal{H} \). We set
\[
E^{\pm}(\alpha, z) = \exp(-\sum_{\pm n \in \mathbb{Z}^+} m^\alpha(n) z^{-n}) \in \text{End}(S)[[z^{\pm 1}]].
\]
 Recall that \( \omega \) is a primitive \( m \)-th root of unity. Let \( \omega_0 \) be a primitive \( m_0 \)-th root of unity such that \( \omega_0^{m_0} = \omega \), where \( m_0 = m \) if \( m \) is even, and \( m_0 = 2m \) if \( m \) is odd. Let \( \langle \omega_0 \rangle \) be the cyclic subgroup of \( \mathbb{C}^* \) generated by \( \omega_0 \). Then we have 
\( -1, \omega \in \langle \omega_0 \rangle \). Following [L], we define a function \( C : Q \times Q \rightarrow \langle \omega_0 \rangle \) by
\[
C(\alpha, \beta) = \prod_{p \in \mathbb{Z}_m} (-\omega^{-p})^{\langle \alpha, \nu^p \beta \rangle}, \ \alpha, \beta \in Q.
\]
Let \( \varepsilon_C : Q \times Q \rightarrow \langle \omega_0 \rangle \) be a normalized 2-cocycle associated with the function \( C \). That is, \( \varepsilon_C \) satisfies the following conditions
\[
\varepsilon_C(\alpha, \beta) \varepsilon_C(\alpha + \beta, \gamma) = \varepsilon_C(\beta, \gamma) \varepsilon_C(\alpha, \beta + \gamma),
\]
\[
\varepsilon_C(0, 0) = 1, \ \varepsilon_C(\alpha, \beta)/\varepsilon_C(\beta, \alpha) = C(\alpha, \beta),
\]
for \( \alpha, \beta, \gamma \in Q \). Now we define a twisted group algebra \( \mathbb{C}[Q, \varepsilon_C] = \oplus_{\alpha \in Q} \mathbb{C}e_\alpha \) with multiplication \( e_\alpha e_\beta = \varepsilon_C(\alpha, \beta) e_{\alpha + \beta}, \ \alpha, \beta \in Q \). Such a twisted group algebra can also be obtained in the following way. Let \( \hat{Q} \) be the unique (up to equivalence) central extension of \( Q \) by the cyclic group \( \langle \kappa_0 \rangle \) of order \( m_0 \) such that
\[
aba^{-1}b^{-1} = \prod_{p \in \mathbb{Z}_m} (\kappa_0^{m_0p} - m_0^{\alpha p} \kappa_0^{b p})^{\langle \alpha, \nu^p \beta \rangle}, \ a, b \in \hat{Q},
\]
where $\tilde{\rho} : \hat{\mathbb{Q}} \to \mathbb{Q}$ is the natural homomorphism. It is known that, by choosing a section of $\hat{\mathbb{Q}}$, the algebra $\mathbb{C}[\mathbb{Q}, \varepsilon_C]$ is isomorphic to the quotient algebra $\mathbb{C}\{\mathbb{Q} \} := \mathbb{C}[\hat{\mathbb{Q}}]/(\kappa_0 - \omega_0)\mathbb{C}[\hat{\mathbb{Q}}]$ ( $\mathbb{C}[\cdot]$ denotes the group algebra).

Now we are in a position to give the definition of the generalized Fock space. Let $\hat{\nu}$ be an automorphism of $\mathbb{C}[\mathbb{Q}, \varepsilon_C]$ such that
\begin{equation}
(3.2) \quad \hat{\nu}(e_{\alpha}) \in \mathbb{C}e_{\nu(\alpha)} \text{ and } \hat{\nu}^m = \text{Id}.
\end{equation}

Let $T$ be a $\mathbb{C}[\mathbb{Q}, \varepsilon_C]$-module. Following [DL], we assume that $\mathcal{H}(0)$ acts on $T$ in such a way that
\[ T = \bigoplus_{\alpha \in \mathbb{Q}} T_{\alpha(0)}, \quad \text{where} \quad T_{\alpha(0)} = \{ t \in T | h.t = \langle h, \alpha(0) \rangle, \ h \in \mathcal{H}(0) \}, \]
and that the actions of $\mathbb{C}[\mathbb{Q}, \varepsilon_C]$ and $\mathcal{H}(0)$ on $T$ are compatible in the sense that
\begin{equation}
(3.3) \quad e_{\alpha}.b \in T_{(\alpha+\beta)(0)}, \quad e_{\alpha}^{-1}\hat{\nu}(e_{\alpha}).b = \omega^{-\langle \sum \nu^\alpha \beta, \hat{\alpha} \rangle}b,
\end{equation}
for $\alpha, \beta \in \mathbb{Q}$ and $b \in T_{\beta(0)}$. The following remark shows the existence of the automorphism $\hat{\nu}$ and module $T$.

**Remark 3.1.** It was proved in Section 5 of [L] that $\nu$ can be lifted to be an automorphism $\hat{\nu}$ of $\hat{\mathbb{Q}}$ such that
\[ \hat{\nu}(\kappa_0) = \kappa_0, \quad \hat{\nu}^m = 1 \text{ and } (\hat{\nu}(a))^{-1} = \nu(\bar{a}), \ a \in \hat{\mathbb{Q}}, \]
which implies that $\hat{\nu}$ induces an automorphism, still called $\hat{\nu}$, of the quotient algebra $\mathbb{C}\{\mathbb{Q} \}$. Note that $\mathbb{C}\{\mathbb{Q} \} \cong \mathbb{C}[\mathbb{Q}, \varepsilon_C]$, thus $\hat{\nu}$ is an automorphism of $\mathbb{C}[\mathbb{Q}, \varepsilon_C]$ satisfying (3.2). Next we show the existence of the module $T$. Set $\mathcal{N} = \{ \alpha \in \mathbb{Q}|\langle \alpha, \beta(0) \rangle = 0, \forall \beta \in \mathbb{Q} \}$. Let $\hat{\mathcal{N}} \subset \hat{\mathbb{Q}}$ be the pull back of $\mathcal{N}$ in $\mathbb{Q}$. In [L], a certain class of induced $\hat{\mathbb{Q}}$-module $T = \mathbb{C}[\hat{\mathbb{Q}}] \otimes \mathbb{C}[\hat{\mathcal{N}}]$ ( $T'$ is an $\hat{\mathcal{N}}$ module), on which $\kappa_0$ acts by $\omega_0$ and $a^{-1}\hat{\nu}(a)$ acts by $\omega^{-\langle \sum \nu^\alpha a, \hat{\alpha} \rangle}/2$ for $a \in \hat{\mathbb{Q}}$, was constructed. We define a $\mathcal{H}(0)$-action on $T$ by $h.(b \otimes t) = \langle h, \bar{b} \rangle b \otimes t$ for $h \in \mathcal{H}(0), b \in \hat{\mathbb{Q}}$ and $t \in T'$. One easily checks $\langle \mathcal{H}(0), \alpha \rangle = 0, \alpha \in \mathcal{N}$, which implies the action is well-defined. For $\alpha \in \mathbb{Q}$, set $T_{\alpha(0)} = \text{span}_\mathbb{C}\{ \alpha(0) \otimes t' | t' \in T' \}$. Then we have $h.t = \langle h, \alpha(0) \rangle t$ for $h \in \mathcal{H}(0), t \in T_{\alpha(0)}$. Finally, it is easy to check that the action of $\mathcal{H}(0)$ is compatible with the natural action of $\mathbb{C}[\mathbb{Q}, \varepsilon_C]$ on $T$. \qed

Now we define the generalized Fock space $V_T = T \otimes S$ to be the tensor product of the $\mathbb{C}[\mathbb{Q}, \varepsilon_C]$-module $T$ and the $\mathcal{H}(\nu)$-module $S$. For $\alpha \in \mathcal{H}(0), \beta \in \mathbb{Q}$ and $h \in \mathcal{H}(\nu)$, we also define certain operators acting on $V_T$ as follows
\[ \alpha(0).(t \otimes s) = \langle \alpha, \gamma \rangle t \otimes s, \quad e_\beta.(t \otimes s) = (e_\beta.t) \otimes s, \quad h.(t \otimes s) = t \otimes (h.s), \]
where $t \in T_{\alpha(0)}, \gamma \in \mathbb{Q}$ and $s \in S$. In case $\alpha \in \mathcal{H}(0)$ such that $\langle \alpha, \mathbb{Q} \rangle \in \mathbb{Z}$, for a formal variable $z$ and a nonzero complex number $c$, we define operators
\[ z^\alpha.(t \otimes s) = z^{(\alpha, \gamma)} t \otimes s, \quad e^\alpha.(t \otimes s) = e^{(\alpha, \gamma)} t \otimes s. \]
Lemma 3.2. For \( \alpha, \beta \in Q, \gamma, \gamma_1, \gamma_2 \in \mathcal{H}, c \in \Gamma \) and \( r \in \mathbb{Z}_m \), we have
\[
\hat{v}_\alpha = e_\alpha \omega^{-\sum \nu^\alpha \sigma - (\alpha, \sum \nu^\alpha \sigma)/2}, \quad \hat{v}^r \gamma(z) = \gamma(\omega^{-r}z), \quad E^\pm(\nu^r \gamma, z) = E^\pm(\gamma, \omega^{-r}z),
\]
\[
\sum \nu^\sigma e_\beta = e_\beta \sum \nu^\sigma + \sum (\nu^\sigma, \beta), \quad e_\beta e_\gamma = e_\beta \sum \nu^\sigma + \sum (\nu^\sigma, \beta),
\]
\[
e_\alpha e_\beta = C(\alpha, \beta) e_\beta e_\alpha, \quad [\gamma(0)(0), e_\beta] = m^{-1} \sum (\nu^\sigma \gamma, \beta) e_\beta,
\]
\[
[\gamma_1(z_1), E^\pm(\gamma_2, z_2)] = \sum_{p \in \mathbb{Z}_m} \frac{\langle \gamma_1, \nu^p \gamma_2 \rangle}{m} \left( \sum_{n > 0} (\omega^p z_2 / z_1)^n E^\pm(\gamma_2, z_2) \right),
\]
\[
E^+(\gamma_1, z_1) E^-(\gamma_2, z_2) = E^-(\gamma_2, z_2) E^+(\gamma_1, z_1) \prod_{p \in \mathbb{Z}_m} (1 - \omega^p z_2 / z_1)^{\langle \gamma_1, \nu^p \gamma_2 \rangle}. \quad \square
\]

Before giving the twisted \( \Gamma \)-vertex operators, we define two constants:
\[
\zeta(\alpha) = \begin{cases} \zeta'(\alpha)2^{(\alpha, \nu^m/2\alpha)/2}, & \text{if } m \in 2\mathbb{Z}, \\ \zeta'(\alpha), & \text{if } m \in 2\mathbb{Z} + 1, \end{cases}
\]
\[
\kappa(p_i, p_j, c) = \prod_{0 \leq p < m} (1 - c \omega^p)^{\langle \rho e_i, p_i \sigma \rho e_j \rangle} \prod_{0 < p < m} (1 - \omega^p)^{\langle \rho e_i, p_i \sigma \rho e_j \rangle},
\]
where \( \alpha \in Q, \zeta'(\alpha) = \prod_{0 \leq p < m/2} (1 - \omega^p)^{\langle \alpha, \nu^\alpha \rangle}, \) and \( (p_i, p_j) \in \mathcal{J} \) (see (2.2)), \( c \in \Gamma \) such that either \( p_i \neq p_j \) or \( p_i = p_j, c \neq 1. \)

Now we define the twisted \( \Gamma \)-vertex operators \( Y_{p_i, p_j}(c, z) \) on \( V_T \) by
\[
Y_{p_i, p_j}(c, z) = \begin{cases} p_i e_i(z), & \text{if } p_i = p_j, c = 1, \\ m^{-1} \zeta(p_i e_i - p_j e_j) \kappa(p_i, p_j, c) X_{p_i, p_j}(c, z), & \text{otherwise}, \end{cases}
\]
where \( (p_i, p_j) \in \mathcal{J}, c \in \Gamma, \) and \( X_{p_i, p_j}(c, z) \) is defined as follows
\[
X_{p_i, p_j}(c, z) = e_{p_i e_i - p_j e_j} E^-(p_i e_i, z) E^-(p_j e_j, cz) E^+(p_i e_i, z) E^+(p_j e_j, cz) z^2 \sum \nu^p (p_i e_i - p_j e_j)^2 / 2 - \sum \nu^p (p_i e_i - p_j e_j)^2 / 2.
\]

In what follows we describe the relations among the twisted \( \Gamma \)-vertex operators \( Y_{p_i, p_j}(c, z) \) defined above. Let \( \eta(r, \alpha) \) be the complex numbers determined by
\[
\hat{v}^r(e_\alpha) = \eta(r, \alpha) e_{\nu^r(\alpha)}, \quad r \in \mathbb{Z}_m, \alpha \in Q.
\]
Recall the notation \( i_r \) introduced in (2.4). Then by definition we have \( \zeta(\nu^r(\alpha)) = \zeta(\alpha) \) and \( \kappa(p_i r, p_j r, c) = \kappa(p_i, p_j, c). \) It follows from this and the first three identities in Lemma 3.2 that

**Proposition 3.3.** For \( (p_i, p_j) \in \mathcal{J}, c \in \Gamma \) and \( r \in \mathbb{Z}_m, \) we have
\[
Y_{p_i, p_j}(c, z) = (-1)^{\delta_{p_i} p_j} Y_{-p_j, -p_i}(c^{-1}, cz),
\]
\[
Y_{p_i, p_j}(c, \omega^{-r}z) = \eta(r, p_i e_i - p_j e_j) Y_{p_i, p_j}(c, z). \quad \square
\]
4. An identity

In this section we prove an identity (see (4.10)) which will be used in the next section to calculate the commutator relations for the twisted Γ-vertex operators. As we have mentioned that this identity is a nontrivial generalization of the combinatorial identity given by Lepowsky in [L]. Throughout this section, the notations $s$ and $t_i, i = 1, 2, \cdots$ denote some distinct nonzero complex numbers.

First we recall some well-known identities:

$$
(4.1) \quad \sum_{i=1}^{n} \left( \prod_{j \neq i}^{n} \frac{t_i}{t_i - t_j} \right) \frac{s}{s - t_i} = \prod_{i=1}^{n} \frac{s}{s - t_i}, \quad \sum_{i=1}^{n} \left( \prod_{j \neq i}^{n} \frac{t_j}{t_i - t_j} \right) \frac{t_i}{s - t_i} = \prod_{i=1}^{n} \frac{t_i}{s - t_i},
$$

$$
(4.2) \quad \sum_{i=1}^{n} \left( \prod_{j \neq i}^{n} \frac{t_i}{t_i - t_j} \right) \left( \frac{s}{s - t_i} \right)^2 = \left( 1 + \sum_{j=1}^{n} \frac{t_j}{s - t_j} \right) \prod_{i=1}^{n} \frac{s}{s - t_i}.
$$

$$
(4.3) \quad \sum_{i=1}^{n} \left( \prod_{j \neq i}^{n} \frac{t_j}{t_i - t_j} \right) \left( \frac{t_i}{s - t_i} \right)^2 = \left( \sum_{j=1}^{n} \frac{s}{s - t_j} - 1 \right) \prod_{i=1}^{n} \frac{t_i}{s - t_i}.
$$

Now we use these identities to prove the following result:

**Lemma 4.1.** Let $a_i = 1$ or 2 for $i \in I := \{1, \cdots, n\}$. Set $I(t) = \{i \in I | a_i = t\}$ for $t = 1$ or 2. Then we have

$$
(4.4) \quad \prod_{i=1}^{n} \left( \frac{s}{s - t_i} \right)^{a_i} = \sum_{i \in I(1)} \left( \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i - t_j} \right)^{a_j} \right) \frac{s}{s - t_i}
$$

$$
+ \sum_{i \in I(2)} \left( \prod_{j \in I, j \neq i} \left( \frac{t_j}{t_i - t_j} \right)^{a_j} \right) \left[ \frac{st_i}{(s - t_i)^2} + \left( 1 + \sum_{j \in I, j \neq i} \frac{a_j t_j}{t_j - t_i} \right) \frac{s}{s - t_i} \right],
$$

$$
(4.5) \quad \prod_{i=1}^{n} \left( \frac{t_i}{s - t_i} \right)^{a_i} = \sum_{i \in I(1)} \left( \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i - t_j} \right)^{a_j} \right) \frac{t_i}{s - t_i}
$$

$$
+ \sum_{i \in I(2)} \left( \prod_{j \in I, j \neq i} \left( \frac{t_j}{t_i - t_j} \right)^{a_j} \right) \left[ \frac{st_i}{(s - t_i)^2} + \left( \sum_{j \in I, j \neq i} \frac{a_j t_i}{t_j - t_i} - 1 \right) \frac{t_i}{s - t_i} \right].
$$

In particular, let $z$ be a formal variable, one has

$$
(4.6) \quad \prod_{i=1}^{n} \left( \frac{1}{1 - t_i z} \right)^{a_i} = \sum_{i \in I(1)} \left( \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i - t_j} \right)^{a_j} \right) \frac{1}{1 - t_i z}
$$

$$
+ \sum_{i \in I(2)} \left( \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i - t_j} \right)^{a_j} \right) \left[ \frac{zt_i}{(1 - t_i z)^2} + \left( 1 + \sum_{j \in I, j \neq i} \frac{a_j t_j}{t_j - t_i} \right) \frac{1}{1 - t_i z} \right],
$$
\[
\prod_{i=1}^{n} \left( \frac{t_i z}{1 - t_i z} \right)^a_i = \sum_{i \in I(1)} \left( \prod_{j \in I, j \neq i} \left( \frac{t_j}{t_i - t_j} \right)^a_j \right) \frac{t_i z}{1 - t_i z} \\
+ \sum_{i \in I(2)} \left( \prod_{j \in I, j \neq i} \left( \frac{t_j}{t_i - t_j} \right)^a_j \right) \left[ \frac{t_i z}{(1 - t_i z)^2} + \left( \sum_{j \in I, j \neq i} \frac{a_j t_i}{t_j - t_i - 1} \right) \frac{t_i z}{1 - t_i z} \right].
\]

**Proof.** For the identity (4.1), by applying the first identity in (4.1), we have

\[
\prod_{i=1}^{n} \left( \frac{s}{s - t_i} \right)^a_i = \left( \sum_{i \in I} D_i \frac{s}{s - t_i} \right) \left( \sum_{k \in I(2)} E_k \frac{s}{s - t_k} \right) = G_1 + G_2 + G_3 + G_4,
\]

where \(D_i = \prod_{j \in I, j \neq i} \frac{t_i}{t_i - t_j}, \ E_k = \prod_{j \in I(2), j \neq i} \frac{t_k}{t_k - t_j}, \) and

\[
G_1 = \sum_{i \in I(1), k \in I(2)} D_i E_k \frac{t_i}{t_i - t_k} \frac{s}{s - t_i}, \quad G_2 = \sum_{i, k \in I(2), i \neq k} D_i E_k \frac{t_i}{t_i - t_k} \frac{s}{s - t_i},
\]

\[
G_3 = \sum_{i \in I(1), k \in I(2), i \neq k} D_i E_k \frac{t_k}{t_k - t_i} \frac{s}{s - t_k}, \quad G_4 = \sum_{i \in I(1)} D_i E_i \left( \frac{s}{s - t_i} \right)^2.
\]

By using the first identity in (4.1) and identity (4.2), we get

\[
G_1 = \sum_{i \in I(1)} D_i \left( \frac{\prod_{k \in I(2), k \neq i} \frac{t_k}{t_k - t_i}}{\frac{t_i}{t_i - t_k} - \frac{s}{s - t_i}} \right) \frac{s}{s - t_i} = \sum_{i \in I(1)} D_i E_i \frac{s}{s - t_i},
\]

\[
G_2 = \sum_{i \in I(2)} D_i \left( \frac{\sum_{k \in I(2), k \neq i} \prod_{j \in I(2), j \neq i, k} \frac{t_k}{t_k - t_j}}{\frac{t_i}{t_i - t_k} \frac{t_k}{t_k - t_i} - \frac{s}{s - t_i}} \right) \frac{s}{s - t_i}
\]

\[
= \sum_{i \in I(2)} D_i \left[ \sum_{k \in I(2), k \neq i} \prod_{j \in I(2), j \neq i, k} \frac{t_k}{t_k - t_j} \frac{t_i}{t_i - t_k} \frac{s}{s - t_i} \right] \frac{s}{s - t_i}
\]

\[
G_3 = \sum_{i \in I(2)} E_i \left( \frac{\sum_{k \in I, k \neq i} \prod_{j \in I, j \neq k} \frac{t_k}{t_k - t_j}}{\frac{A_k}{t_k - t_i} - \frac{t_i}{t_i - t_k}} \right) \frac{s}{s - t_i}
\]

\[
= \sum_{i \in I(2)} E_i \left[ \sum_{k \in I, k \neq i} \frac{t_k}{t_k - t_j} \frac{t_i}{t_i - t_k} \frac{s}{s - t_i} \right] \frac{s}{s - t_i}
\]

\[
= \sum_{i \in I(2)} D_i E_i \left( \frac{\sum_{j \in I, j \neq i} \frac{t_j}{t_j - t_i}}{s - t_i} \right) \frac{s}{s - t_i}.
\]
It is clear that the expression of $G_1 + G_2 + G_3 + G_4$ coincides with the right hand-side of (4.4), as desired. The identity (4.5) can be proved similarly by using the second identity in (4.1) and identity (4.3), which is omitted. □

According to the formal power identities (4.6) and (4.7), we have

**Lemma 4.2.** Let $I, a_i, i \in I, I(t)$ be the same as in Lemma 4.1. Then

$$
\prod_{i=1}^{n} \left( \frac{1}{1-t_i z} \right)^{a_i} - \prod_{i=1}^{n} \left( \frac{-t_i^{-1} z^{-1}}{1-t_i^{-1} z^{-1}} \right)^{a_i} = \sum_{i \in I(1)} \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i-t_j} \right)^{a_j} \delta(t_i z) 
$$

\begin{align*}
+ \sum_{i \in I(2)} \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i-t_j} \right)^{a_j} \left[ (D\delta)(t_i z) + \left( \sum_{j \in I \neq i} \frac{a_j t_j}{t_j-t_i} + 1 \right) \delta(t_i z) \right].
\end{align*}

**Proof.** From the formal power identities (4.6) and (4.7), we know

$$
\prod_{i=1}^{n} \left( \frac{1}{1-t_i z} \right)^{a_i} - \prod_{i=1}^{n} \left( \frac{-t_i^{-1} z^{-1}}{1-t_i^{-1} z^{-1}} \right)^{a_i} 
$$

\begin{align*}
= \sum_{i \in I(1)} \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i-t_j} \right)^{a_j} \left( \frac{1}{1-t_i z} + \frac{t_i^{-1} z^{-1}}{1-t_i^{-1} z^{-1}} \right) 
\end{align*}

\begin{align*}
+ \sum_{i \in I(2)} \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i-t_j} \right)^{a_j} \left( \frac{t_i z}{(1-t_i z)^2} - \frac{t_i^{-1} z^{-1}}{(1-t_i^{-1} z^{-1})^2} \right) 
\end{align*}

\begin{align*}
+ \sum_{i \in I(2)} \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i-t_j} \right)^{a_j} \left( \sum_{j \in I \neq i} \frac{a_j t_j}{t_j-t_i} + 1 \right) \left( \frac{1}{1-t_i z} + \frac{t_i^{-1} z^{-1}}{1-t_i^{-1} z^{-1}} \right)
\end{align*}

\begin{align*}
= \sum_{i \in I(1)} \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i-t_j} \right)^{a_j} \delta(t_i z) 
\end{align*}

\begin{align*}
+ \sum_{i \in I(2)} \prod_{j \in I, j \neq i} \left( \frac{t_i}{t_i-t_j} \right)^{a_j} \left[ (D\delta)(t_i z) + \left( \sum_{j \in I \neq i} \frac{a_j t_j}{t_j-t_i} + 1 \right) \delta(t_i z) \right]
\end{align*}

as desired. □

Let $f(z) \in \mathbb{C}[z, z^{-1}]$ and $a \in \mathbb{C}^*$. The following formal power series identities (cf. [FLM]) are well-known.

(4.8) $f(z)\delta(az) = f(a^{-1})\delta(az)$,

(4.9) $f(z)(D\delta)(az) = f(a^{-1})(D\delta)(az) - (D_z f)(a^{-1})\delta(az)$,

where $D_z = \frac{d}{dz}$. Now we state and prove our main result in this section.
Proposition 4.3. Let $t_i, i \in I$ be distinct nonzero complex numbers and $I = \{1, \cdots, n\}$. Let $a_i \in \mathbb{Z}$ and $a_i \geq -2, i \in I$. For $t \in \mathbb{Z}$ and $t \geq -2$, set $I(t) = \{i \in I| a_i = t\}$. Then we have

\begin{equation}
(4.10) \prod_{i \in I}(1 - t_i z)^{a_i} - \prod_{i \in I}(1 - t_i^{-1} z^{-1})^{a_i}(-t_i z)^{a_i} = \sum_{i \in I (-1)} \prod_{I, j \neq i} (1 - t_j t_i^{-1})^{a_j} \delta(t_i z) + \sum_{i \in I (-2)} \prod_{I, j \neq i} (1 - t_j t_i^{-1})^{a_j}[(D\delta)(t_i z) + (1 + \sum_{j \in I, j \neq i} a_j(t_j t_i^{-1} - 1)^{-1}) \delta(t_i z)].
\end{equation}

Proof. Note that $(1 - t_i z)^{a_i} = (1 - t_i^{-1} z^{-1})^{a_i}(-t_i z)^{a_i}$ if $a_i \geq 0$. Set $I_+ = \{i \in I| a_i \geq 0\}$, and $I_- = I (-1) \cup I (-2)$. From Lemma 4.2 and (4.8), (4.9), one has that

\begin{equation}
\prod_{i \in I}(1 - t_i z)^{a_i} - \prod_{i \in I}(1 - t_i^{-1} z^{-1})^{a_i}(-t_i z)^{a_i} = (\prod_{k \in I_+}(1 - t_k z)^{a_k}) \left\{ \sum_{i \in I (-1)} \prod_{I, j \neq i} (1 - t_j t_i^{-1})^{a_j} \delta(t_i z) + \sum_{i \in I (-2)} \prod_{I, j \neq i} (1 - t_j t_i^{-1})^{a_j}[(D\delta)(t_i z) + (1 + \sum_{j \in I, j \neq i} a_j(t_j t_i^{-1} - 1)^{-1}) \delta(t_i z)] \right\}
\end{equation}

\begin{align}
&= \sum_{i \in I (-1)} \prod_{I, j \neq i} (1 - t_j t_i^{-1})^{a_j} \delta(t_i z) + \sum_{i \in I (-2)} \prod_{I, j \neq i} (1 - t_j t_i^{-1})^{a_j}[(D\delta)(t_i z) + (1 + \sum_{j \in I, j \neq i} a_j(t_j t_i^{-1} - 1)^{-1}) \delta(t_i z)]
&\quad + \sum_{j \in I_+} (1 - t_j z)^{-1}(-t_j z)^{a_k} \left[ \sum_{i \in I (-2)} \prod_{I, j \neq i} (1 - t_j t_i^{-1})^{a_j} \delta(t_i z) \right]
&\quad + \sum_{i \in I (-1)} \prod_{I, j \neq i} (1 - t_j t_i^{-1})^{a_j}[(D\delta)(t_i z) + (1 + \sum_{j \in I, j \neq i} a_j(t_j t_i^{-1} - 1)^{-1}) \delta(t_i z)]
&\quad + \sum_{i \in I (-2)} \prod_{I, j \neq i} (1 - t_j t_i^{-1})^{a_j}[(D\delta)(t_i z) + (1 + \sum_{j \in I, j \neq i} a_j(t_j t_i^{-1} - 1)^{-1}) \delta(t_i z)]
\end{align}

as desired.

\[\Box\]

Remark 4.4. If one takes $t_i = \omega^{-i}, a_i = \langle \nu^i \alpha, \beta \rangle$ for $1 \leq i \leq m, \alpha, \beta \in Q$ in (4.10), and notes that

\[\sum_{p \in \mathbb{Z}_m, p \neq r} \langle \nu^p \alpha, \beta \rangle = \sum_{p \in \mathbb{Z}_m, p \neq 0} \frac{\langle (\nu^p + \nu^{-p}) \nu^r \alpha, \beta \rangle}{1 - \omega^p},\]

where $r \in \mathbb{Z}_m$ satisfies $\langle \nu^r \alpha, \beta \rangle = -2$, then the identity (4.10) becomes the identity given in Proposition 4.1 of [L].
5. Vertex operator realizations of twisted $\Gamma$-Lie algebras

In this section we first state the main result of this paper, and then prove it by calculating the commutator relations for the twisted $\Gamma$-vertex operators $Y_{\rho_i,\rho_j}(c_1, z_1)$ and $Y_{\rho_k,\rho_l}(c_2, z_2)$, where $(\rho_i, \rho_j), (\rho_k, \rho_l) \in J$ and $c_1, c_2 \in \Gamma$. For the sake of convenience, we write $\alpha := \rho_i \epsilon_i - \rho_j \epsilon_j, \ \beta := \rho_k \epsilon_k - \rho_l \epsilon_l$.

5.1. The main theorem. For $\alpha, \beta \in Q$, set
\[
\varepsilon'(\alpha, \beta) = \prod_{-m/2 < p < 0} (-\omega^p)^{[\alpha, \mu^p \beta]}, \ \varepsilon(\alpha, \beta) = \varepsilon'(\alpha, \beta)\varepsilon_C(\alpha, \beta).
\]
It follows from (3.1) that $\varepsilon$ is a normalized 2-cocycle on $Q$ associated with the function $(-1)^{[\alpha, \beta]}$ (cf. (2.3)). Then by the lattice construction given in Section 2.4, we have an involutive associative algebra $(G(Q), \tau)$ and a bilinear form $\langle \cdot, \cdot \rangle_G$.

Recall the constants $\eta(r, \alpha), r \in \mathbb{Z}_m, \alpha \in Q$ defined in (3.4). We define a linear map $\tilde{\nu}$ on $G(Q)$ as follows
\[
\tilde{\nu}(e_{\rho_i,\rho_j}) = \eta(1, \rho_i \epsilon_i - \rho_j \epsilon_j) e_{\rho_i,\rho_j}, \ (\rho_i, \rho_j) \in J.
\]

Lemma 5.1. The linear map $\tilde{\nu}$ is an automorphism of the involutive associative algebra $(G(Q), \tau)$, and $\tilde{\nu}^m = \text{Id}$. Moreover, $\tilde{\nu}$ preserves the form $\langle \cdot, \cdot \rangle_G$.

Proof. Recall that $\tilde{\nu}$ is an automorphism of $\mathbb{C}[Q, \varepsilon_C]$ (see (3.2)). Then we have
\[
\varepsilon_C(\alpha, \beta)\eta(r, \alpha + \beta)e_{\nu^r(\alpha + \beta)} = \eta(r, \alpha)\eta(r, \beta)\varepsilon_C(\nu^r(\alpha), \nu^r(\beta))\varepsilon_{\nu^r(\alpha + \beta)}.
\]
This, together with the fact $\varepsilon'(\alpha, \beta) = \varepsilon'(\nu^r \alpha, \nu^r \beta)$, gives
\[
\varepsilon(\alpha, \beta)\eta(r, \alpha + \beta) = \varepsilon(\nu^r(\alpha), \nu^r(\beta))\eta(r, \alpha)\eta(r, \beta), \ \forall\alpha, \beta \in Q, r \in \mathbb{Z}_m.
\]
One can conclude from this fact that $\tilde{\nu}$ is an automorphism of $(G(Q), \tau)$ and preserves the form $\langle \cdot, \cdot \rangle_G$. \qed

By definition and the previous lemma, we see that the automorphism $\tilde{\nu}$ of $(G(Q), \tau)$ is compatible with the isometry $\nu$ (cf. (2.5)). Therefore, we have a twisted $\Gamma$-Lie algebra $\tilde{G}(Q, \nu, m, \Gamma)$ with Lie bracket given in Proposition 2.3.

Now we state our main theorem of this paper. For $(\rho_i, \rho_j) \in J$ and $c \in \Gamma$, let $y_{\rho_i,\rho_j}(c, n)$ be the $-n$-component of $Y_{\rho_i,\rho_j}(c, z)$. Set $Q' = \{\alpha \in Q : \langle \alpha, \alpha \rangle = 2\}$, and $Q'' = \{\rho_i \epsilon_i - \rho_j \epsilon_j : 1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1\} \cap Q$. Then we have

Theorem 5.2. The generalized Fock space $V_T$ affords a representation of the twisted $\Gamma$-Lie algebra $\tilde{G}(Q, \nu, m, \Gamma)$ with the actions given by
\[
\varepsilon_{\rho_i,\rho_j}(c, n) \mapsto y_{\rho_i,\rho_j}(c, n), \ c \mapsto 1,
\]
where $(\rho_i, \rho_j) \in J, c \in \Gamma$ and $n \in \mathbb{Z}$. Moreover, if $\Gamma \neq \{1\}$ and span$_zQ'' = Q$, or if $\Gamma = \{1\}$ and span$_zQ' = Q$. Then the $\tilde{G}(Q, \nu, m, \Gamma)$-module $V_T$ is irreducible if and only if the $\mathbb{C}[Q, \varepsilon_C]$-module $T$ is irreducible.
5.2. Some lemmas. To prove Theorem 5.2, we need the following four lemmas. The first lemma is well-known (cf. [FLM]).

Lemma 5.3. Let $F(z_1, z_2)$ be a formal power series in $z_1, z_2$ with coefficients in a vector space such that $\lim_{z_2 \to c z_1} F(z_1, z_2)$ exists for some $c \in \mathbb{C}^*$. Then

\begin{align}
(5.1) & \quad F(z_1, z_2)\delta(z_2/cz_1) = F(z_1, cz_1)\delta(z_2/cz_1), \\
(5.2) & \quad F(z_1, z_2)(D\delta)(z_2/cz_1) = F(z_1, cz_1)(D\delta)(z_2/cz_1) + (D_{z_2}F)(z_1, z_2)\delta(z_2/cz_1). 
\end{align}

The following two lemmas can be shown by applying Lemma 3.2 and a straightforward computation.

Lemma 5.4.

\[ [X_{\rho_i, \rho_j}(c_1, z_1), X_{\rho_k, \rho_l}(c_2, z_2)] = B(z_1, z_2) \cdot C(z_1, z_2), \]

where

\[
B(z_1, z_2) = E^-(\rho_i, c_1 z_1) E^-(-\rho_j, c_1 z_1) E^-(-\rho_k, c_2 z_2) E^-(-\rho_l, c_2 z_2) E^+(\rho_i, z_1) E^+(\rho_k, z_2)
\]

\[
\sum_{\nu} \frac{\hat{c}_1^{\nu} \hat{c}_2^{\nu}}{\nu} \left( \frac{\hat{c}_1^{\nu} \hat{c}_2^{\nu}}{\nu} \right) \frac{1}{2} \sum_{\nu} \frac{\hat{c}_1^{\nu} \hat{c}_2^{\nu}}{\nu} \left( \frac{\hat{c}_1^{\nu} \hat{c}_2^{\nu}}{\nu} \right)
\]

and

\[
C(z_1, z_2) = \prod_{p \in \mathbb{Z}_m} \left( 1 - \omega^p z_2 / z_1 \right)^{\rho_i \rho_k (\nu_i, \nu_k)} \left( 1 - \omega^p z_2 / c_1 z_1 \right)^{-\rho_j \rho_k (\nu_j, \nu_k)} \\
\times \left( 1 - \omega^p c_2 z_2 / z_1 \right)^{-\rho_i \rho_k (\nu_i, \nu_k)} \left( 1 - \omega^p c_2 z_2 / c_1 z_1 \right)^{\rho_j \rho_k (\nu_j, \nu_k)}
\]

Lemma 5.5. Let $r \in \mathbb{Z}_m$, if $\langle \rho_i, \rho_k \nu^r \epsilon_k \rangle = -1$. Then

\begin{align}
(5.3) & \quad B(z_1, z_2) = c_1^{(\rho_i, \rho_j, \sum \nu^r \epsilon_j)} \varepsilon_C(\alpha, \nu^r \beta) \eta(r, \beta) X_{-\rho_j, \rho_l} (c_1^{-1} c_2, c_1 z_1);

(5.4) & \quad B(z_1, z_2) = \varepsilon_C(\alpha, \nu^r \beta) \eta(r, \beta) X_{\rho_i, \rho_l} (c_1 c_2, z_1);

(5.5) & \quad B(z_1, z_2) = c_1^{(\rho_i, \rho_j, \sum \nu^r \epsilon_j)} c_2^{(\rho_k, \rho_l, \sum \nu^r \epsilon_l)} \varepsilon_C(\alpha, \nu^r \beta) \eta(r, \beta) X_{-\rho_j, \rho_l} (c_1^{-1} c_2, c_1 z_1);
\end{align}
If \( \langle \rho_i \epsilon_j, \rho \nu^r \epsilon_l \rangle = -1 \), then
\[
B(z_1, z_2)|_{z_2 = -r_{c_1 c_2}^{-1} z_1} = c_2^{\langle \rho_i, \nu^r \epsilon_l \rangle} \varepsilon_c(\alpha, \nu^r \beta) \eta(r, \beta) X_{\rho_i, -p_k l}(c_1 c_2^{-1}, z_1).
\]

Finally, by definition, one immediately has

**Lemma 5.6.** Let \( \alpha, \beta \in Q \) and \( r \in \mathbb{Z}_m \), then
\[
(\zeta(\alpha) \zeta(\beta)) = (\alpha, \nu^r \beta)^{-1} \zeta(\alpha + \nu^r \beta)^{-1} = \prod_{0 < p < m} (1 - \omega^p)^{-\langle \alpha, \nu^p (\nu^r \beta) \rangle}.
\]

### 5.3. Proof of Theorem 5.2

To prove the first part of the theorem, we only need to prove that the commutator relation for the twisted \( \Gamma \)-vertex operators \( Y_{\rho_i, \rho_j}(c_1, z_1) \) with \( Y_{\rho_k l, \rho_l l}(c_2, z_2) \) is the same as \( 2.6 \) under the correspondence \( Y_{\rho_i, \rho_j}(c, z) \rightarrow G_{\rho_i, \rho_j}(c, z) \), for \( (\rho_i, \rho_j), (\rho_k l, \rho_l l) \in J \) and \( c_1, c_2, c \in \Gamma \). From the definition of \( Y_{\rho_i, \rho_j}(c, z) \), \( X_{\rho_i, \rho_j}(c, z) \), and note that the identity given in Lemma 5.4, we need to work on the product \( B(z_1, z_2) \cdot C(z_1, z_2) \). For \( C(z_1, z_2) \), we apply Proposition 4.3 to rewrite it into a summation of \( \delta \) and \( D(\delta) \) functions. For this purpose we divide the argument into the following seven cases.

Case 1: \( c_1 \neq 1, c_2 \neq 1, c_1 c_2 \neq 1 \) and \( c_1 \neq c_2 \);
Case 2: \( c_1 = 1 \) and \( c_2 \neq 1 \);
Case 3: \( c_2 = 1 \) and \( c_1 \neq 1 \);
Case 4: \( c_1 = c_2 = 1 \);
Case 5: \( c_1 \neq 1, c_2 \neq 1, c_1 c_2 = 1 \) and \( c_1 \neq c_2 \);
Case 6: \( c_1 \neq 1, c_2 \neq 1, c_1 = c_2 \) and \( c_1 c_2 \neq 1 \);
Case 7: \( c_1 \neq 1, c_2 \neq 1, c_1 c_2 = 1 \) and \( c_1 = c_2 \).

For Case 1. Recall that \( \Gamma \) is generic, which implies that the numbers \( \omega^p, \omega^p c_1^{-1}, \omega^p c_2, \omega^p c_2 c_1^{-1}, p \in \mathbb{Z}_m \) are distinct. Thus, by applying Proposition 4.3 we obtain
\[
C(z_1, z_2) = \sum_{r \in \mathbb{Z}_m} \left[ \delta_{\rho_i, -p_k l} H_{1, r} \delta(\omega^r z_2 / c_1 z_1) + \delta_{\rho_j, p_k l} H_{2, r} \delta(\omega^r z_2 / c_1 z_1) \right.
\]
\[
+ \delta_{\rho_i, p_k l} H_{3, r} \delta(\omega^r c_2 z_2 / c_1 z_1) + \delta_{\rho_j, -p_k l} H_{4, r} \delta(\omega^r c_2 z_2 / c_1 z_1) \right],
\]

where
\[
H_{1, r} = \prod_{0 < p < m} (1 - \omega^p)^{\rho_i p_k \epsilon \epsilon_l} \prod_{p \in \mathbb{Z}_m} (1 - \omega^p)^{\rho_j p_k \epsilon \epsilon_l} \cdot (1 - \omega^p \omega^p c_2) \cdot (1 - \omega^p \omega^p c_2 c_1),
\]
\[
H_{2, r} = \prod_{0 < p < m} (1 - \omega^p)^{-\rho_i p_k \epsilon \epsilon_l} \prod_{p \in \mathbb{Z}_m} (1 - \omega^p)^{-\rho_j p_k \epsilon \epsilon_l} \cdot (1 - \omega^p \omega^p c_2) \cdot (1 - \omega^p \omega^p c_2 c_1),
\]
\[
H_{3, r} = \prod_{0 < p < m} (1 - \omega^p)^{-\rho_i p_k \epsilon \epsilon_l} \prod_{p \in \mathbb{Z}_m} (1 - \omega^p)^{\rho_j p_k \epsilon \epsilon_l} \cdot (1 - \omega^p \omega^p c_2) \cdot (1 - \omega^p \omega^p c_2 c_1),
\]
\[
H_{4, r} = \prod_{0 < p < m} (1 - \omega^p)^{\rho_i p_k \epsilon \epsilon_l} \prod_{p \in \mathbb{Z}_m} (1 - \omega^p)^{-\rho_j p_k \epsilon \epsilon_l} \cdot (1 - \omega^p \omega^p c_2) \cdot (1 - \omega^p \omega^p c_2 c_1).
\]
Therefore, by using identities (5.1) and (5.3.5.7), we have

\[ [Y_{\rho_1,\rho_1}(c_1, z_1), Y_{\rho_2,\rho_2}(c_2, z_2)] \]

\[ = m^{-1} \sum_{r \in \mathbb{Z}_m} \zeta(\alpha) \zeta(\beta) \kappa(\rho_1, \rho_2) \kappa(\rho_1, \rho_2)[\delta_{\rho_1, -\rho_2} H_{1,r} B(z_1, \omega^{-r} z_1) \]

\[ \cdot (\omega^{r} z_2/11) + \delta_{\rho_1, -\rho_2} H_{2,r} B(z_1, \omega^{-r} z_1) \delta_{\rho_2, z_2/11} + \delta_{\rho_1, -\rho_2} H_{3,r} \]

\[ \cdot B(z_1, \omega^{-r} z_2/11) + \delta_{\rho_2, -\rho_1} H_{4,r} B(z_1, \omega^{-r} z_2/11) \delta_{\rho_2, z_2/11}] \]

\[ = m^{-1} \sum_{r \in \mathbb{Z}_m} \kappa(\rho_1, \rho_2) \kappa(\rho_1, \rho_2,c_2) \left( \prod_{0 < p < m} \left( 1 - \omega^{p} \right)^{-\alpha \theta(p \theta)} \right) \]

\[ \cdot \delta_{\rho_1, -\rho_2} H_{1,r} c_1^{(\rho \epsilon_1, \rho \sum \theta^2)} \kappa(-\rho_1, \rho_2, c_2 \omega^{-r} c_1^{-1} \xi_{r}(\alpha, \beta) Y_{\rho_1, -\rho_2} c_2^{-1} c_1 z_1) \]

\[ \cdot B(z_1, \omega^{-r} z_2/11) + \delta_{\rho_1, -\rho_2} H_{2,r} c_2^{(\rho \epsilon_2, \rho \sum \theta^2)} \kappa(-\rho_2, -\rho_1, c_2 \omega^{-r} c_1^{-1} 1) \delta_{\rho_2, z_2/11} \]

\[ \cdot \kappa(\rho_1, \rho_2, c_2) \delta_{\rho_2, -\rho_1} H_{3,r} c_1^{(\rho \epsilon_1, \rho \sum \theta^2)} \kappa(-\rho_1, -\rho_2, c_2 \omega^{-r} c_1^{-1} 1) \delta_{\rho_2, z_2/11} \] .

Comparing the above commutator relation with (2.6), we see that the result for case 1 follows from the the following identities, which can be checked directly.

\[ \kappa(\rho_1, \rho_2, c_2) \prod_{0 < p < m} \left( 1 - \omega^{p} \right)^{-\alpha \theta(p \theta)} \]

\[ \begin{cases} \left( -1 \right)^{\delta_{ij}} c_1^{(\rho \epsilon_1, \rho \sum \theta^2)} \kappa(-\rho_1, -\rho_2, c_2 \omega^{-r} c_1^{-1} 1) H_{1,r}^{-1}, & \text{if } \rho_1 + \rho_2 = 0, \\ \kappa(\rho_1, \rho_2, c_2) H_{2,r}^{-1}, & \text{if } \rho_1 = \rho_2 k_r, \\ \left( -1 \right)^{\delta_{ij}} c_1^{(\rho \epsilon_1, \rho \sum \theta^2)} c_2^{(\rho \epsilon_2, \rho \sum \theta^2)} \kappa(-\rho_1, -\rho_2, c_2 \omega^{-r} c_1^{-1} 1) H_{3,r}^{-1}, & \text{if } \rho_1 = \rho_2 k_r, \\ \left( -1 \right)^{\delta_{ij}} c_1^{(\rho \epsilon_1, \rho \sum \theta^2)} \kappa(\rho_1, \rho_2, c_2) H_{4,r}^{-1}, & \text{if } \rho_1 = \rho_2 k_r. \end{cases} \]

For Case 2. We divide the proof of this case into two subcases. First we consider the subcase for \( \rho_1 = \rho_2 \). Since \( c_1 = 1 \), we have \( Y_{\rho_1, -\rho_2}(1, z_1) = \rho_1 \epsilon_1(z_1) \) and \( \alpha = 0 \). The proof of this subcase is straightforward, and is omitted for shortness.

Next, we consider the other subcase for \( \rho_1 \neq \rho_2 \). If moreover \( \rho_1 + \rho_2 = 0 \), by using (3.5), we see that the result follows from the fact \( Y_{\rho_1, \rho_2}(1, z_1) = 0 \). On
the other hand, if \( \rho_i + \rho_j \neq 0 \), then \( i \neq j \), and hence \( |\langle \gamma, \rho_k \sigma \rangle| \leq 1 \) and \( |\langle \gamma, \rho \sigma \rangle| \leq 1 \). Together with (4.10) this gives 

\[
C(z_1, z_2) = \sum_{r \in \mathbb{Z}_m} (\delta_{\rho_1, -\rho_k} + \delta_{\rho_j, \rho_k}) L_{1,r} \delta(\omega^r z_2/z_1) + (\delta_{\rho_1, -\rho_k} + \delta_{-\rho_j, \rho_k}) L_{2,r} \delta(\omega^r c_2 z_2/z_1),
\]

where 

\[
L_{1,r} = \prod_{0 < p < m} (1 - \omega^p)^{\langle \rho, \rho_k \sigma \rangle_{\epsilon_k}} \prod_{p \in \mathbb{Z}_m} (1 - \omega^p c_2)^{-\langle \rho, \rho \sigma \rangle_{\epsilon_l}},
\]

\[
L_{2,r} = \prod_{0 < p < m} (1 - \omega^p)^{-\langle \rho, \rho \sigma \rangle_{\epsilon_l}} \prod_{p \in \mathbb{Z}_m} (1 - \omega^p c_2^{-1})^{\langle \rho, \rho \sigma \rangle_{\epsilon_k}}.
\]

Similar to the proof in Case 1, by applying Lemmas 5.3, 5.6, and (5.1), we get 

\[
[Y_{\rho_i, \rho_j} (1, z_1), Y_{\rho_k, \rho_l} (c_2, z_2)] = m^{-1} \sum_{r \in \mathbb{Z}_m} \kappa(\rho_k, \rho_l, c_2) \prod_{0 < p < m} (1 - \omega^p)^{-\langle \rho, \rho \sigma \rangle_{\epsilon_l}} \{ L_{1,r} \xi_r(\alpha, \beta) \delta(\omega^r z_2/z_1) \\
\cdot \left[ \delta_{\rho_1, -\rho_k} \kappa(-\rho_j, \rho_l, c_2)^{-1} Y_{-\rho_j, \rho_l} (c_2, z_1) + \delta_{\rho_1, -\rho_k} \kappa(\rho_i, \rho_l, c_2)^{-1} Y_{\rho_i, \rho_l} (c_2, z_1) \right] \\
+ L_{2,r} c_2 \xi_r(\rho_1, \rho_l, c_2^{-1}) \delta(\omega^r c_2 z_2/z_1) \left[ \delta_{\rho_1, -\rho_k} \kappa(-\rho_j, -\rho_k, c_2^{-1})^{-1} Y_{-\rho_j, -\rho_k} (c_2^{-1}, z_1) \right] \}.
\]

A direct computation shows the following identity: 

\[
\prod_{0 < p < m} (1 - \omega^p)^{-\langle \rho, \rho \sigma \rangle_{\epsilon_l}} \kappa(\rho_k, \rho_l, c_2)
\]

\[
\left\{ \begin{array}{ll}
\kappa(-\rho_j, \rho_l, c_2) L_{1,r}^{-1}, & \text{if } \rho_i + \rho_k = 0, \\
\kappa(\rho_i, \rho_l, c_2) L_{1,r}^{-1}, & \text{if } \rho_j = \rho_k, \\
(-1)^{\delta_k} c_2^{-\langle \rho, \rho \sigma \rangle_{\epsilon_l}} \kappa(-\rho_j, -\rho_k, c_2^{-1}) L_{2,r}^{-1}, & \text{if } \rho_i = \rho_l, \\
(-1)^{\delta_k} c_2^{-\langle \rho, \rho \sigma \rangle_{\epsilon_l}} \kappa(\rho_i, -\rho_k, c_2^{-1}) L_{2,r}^{-1}, & \text{if } \rho_j = \rho_l = 0.
\end{array} \right.
\]

The result for Case 2 then follows from the above two identities. It is clear that the result for Case 3 follows from Case 2 and Proposition 3.3.

For Case 4, we divide the argument into two subcases. The first one is for \( \rho_i = \rho_j \) or \( \rho_k = \rho_l \), the other one is for \( \rho_i \neq \rho_j \) and \( \rho_k \neq \rho_l \). In fact, if \( \rho_i = \rho_j \) or \( \rho_k = \rho_l \), then the proof is the same as that for Case 2. Next we suppose \( \rho_i \neq \rho_j \) and \( \rho_k \neq \rho_l \). If moreover \( \rho_i + \rho_j = 0 \) or \( \rho_k + \rho_l = 0 \), then by (3.5), we see that \( Y_{\rho_i, \rho_j} (1, z_1) = 0 \) or \( Y_{\rho_k, \rho_l} (1, z_2) = 0 \). Otherwise, \( i \neq j \) and \( k \neq l \), then our vertex operator \( Y_{\rho_i, \rho_j} (1, z) \) coincides with the one defined in \([L]\), and thus Theorem 5.2 follows from Theorem 8.2 in \([L]\).
For Case 5. In this case, the numbers $\omega^p, \omega^{p_1}c_1^{-1}, \omega^{p_2}c_2^{-1}c_2, p \in \mathbb{Z}_m$ are distinct. For $t = 0, \pm 1, \pm 2$, we set $M(t) = \{ r \in \mathbb{Z}_m | -\langle \rho_j \epsilon_j, \rho_k \nu^p \epsilon_k \rangle - \langle \rho_i \epsilon_i, \rho_l \nu^p \epsilon_l \rangle = t \}$. Then by (4.10), we get

$$C(z_1, z_2) = \sum_{r \in \mathbb{Z}_m} \sum_{r \in M(-1)} N_{1,r} \delta(\omega^r z_2/c_1 z_1) + \sum_{r \in M(-2)} N_{2,r} \delta(\omega^r z_2/c_1 z_1),$$

where

$$N_{1,r} = \prod_{0<p<m} (1 - \omega^p)^{-\langle \rho_j \epsilon_j, \rho_k \nu^p \epsilon_k \rangle - \langle \rho_i \epsilon_i, \rho_l \nu^p \epsilon_l \rangle} \cdot \prod_{p \in \mathbb{Z}_m} (1 - \omega^p c_1)^{\langle \rho_i \epsilon_i, \rho_k \nu^p \epsilon_k \rangle} (1 - \omega^p c_1)^{\langle \rho_j \epsilon_j, \rho_k \nu^p \epsilon_l \rangle},$$

$$N_{2,r} = 1 - \sum_{p \in \mathbb{Z}_m} \left( \frac{\langle \rho_j \epsilon_j, \rho_k \nu^p \epsilon_k \rangle}{1 - \omega^p c_1} + \frac{\langle \rho_j \epsilon_j, \rho_l \nu^p \epsilon_l \rangle}{1 - \omega^p c_1} \right) - \sum_{p \neq r \in \mathbb{Z}_m} \frac{-\langle \rho_j \epsilon_j, \rho_k \nu^p \epsilon_k \rangle - \langle \rho_i \epsilon_i, \rho_l \nu^p \epsilon_l \rangle}{1 - \omega^p}$$

and $H_{i,r}$, for $i = 1, 4$ are defined in the proof for Case 1. This implies

$$[Y_{\rho_i, \rho_j}(c_1, z_1), Y_{\rho_k, \rho_l}(c_2, z_2)] = O_1 + O_2 + O_3 + O_4,$$

where

$$O_1 = D(z_1, z_2) \sum_{r \in \mathbb{Z}_m} \delta_{\rho_i, -\rho_k r} H_{1,r} \delta(\omega^r z_2/c_1),$$

$$O_2 = D(z_1, z_2) \sum_{r \in \mathbb{Z}_m} \delta_{\rho_j, -\rho_l r} H_{4,r} \delta(\omega^r c_2 z_2/c_1 z_1),$$

$$O_3 = D(z_1, z_2) \sum_{r \in M(-1)} N_{1,r} \delta(\omega^r z_2/c_1 z_1),$$

$$O_4 = D(z_1, z_2) \sum_{r \in M(-2)} N_{2,r} \delta(\omega^r z_2/c_1 z_1),$$

and $D(z_1, z_2) = m^{-2} \zeta(\alpha) \zeta(\beta) \kappa(\rho_i, \rho_i, \rho_j, \rho_k, \rho_l, c_2) B(z_1, z_2).$

Similar to the proof given in Case 1, we have

$$O_1 = m^{-1} \sum_{r \in \mathbb{Z}_m} \delta_{\rho_i, -\rho_k r} (-1)^{\delta_{\rho_i, -\rho_k r}} Y_{\rho_i, \rho_l, c_1^{-2}}(c_1 z_1) \delta(\omega^r z_2/c_1)$$

for $t = 0, \pm 1, \pm 2$.
For the term $O_3$, note that $r \in M(-1)$, we know that either $\rho_i \epsilon_i = \rho_l \nu_l$ or $\rho_j \epsilon_j = \rho_k \epsilon_k$. If $\rho_i \epsilon_i = \rho_l \nu_l$, then $N_{1,r} = H_{3,r}$ as $1 - \omega^{p}(\rho_j \epsilon_j \rho_k \epsilon_k) = 1$. Similarly, if $\rho_j \epsilon_j = \rho_k \epsilon_k$, then $N_{1,r} = H_{2,r}$. Thus, by the same proof as that of Case 1, we have

$$O_3 = m^{-1} \sum_{r \in M(-1)} \delta_{\rho_j \epsilon_j \rho_k} \xi_{r}(\alpha, \beta) Y_{p_i \epsilon_i \rho_l}(1, z_1) \delta(\omega^r z_2 / c_1 z_1)$$

(5.10) + $m^{-1} \sum_{r \in M(-1)} \delta_{\rho_i \epsilon_i \rho_l}(1) \sum_{\alpha \beta} \xi_{r}(\alpha, \beta) Y_{-\rho_j \epsilon_j \rho_k}(1, c_1 z_1) \delta(\omega^r z_2 / c_1 z_1).

Together with (5.11) this implies

(5.11) $N_{2,r} = \frac{1}{2} \langle \beta, \sum_{l} \nu^p \beta \rangle$, if $r \in M(-2)$.

By using (5.2) and (5.4), we have

$$B(z_1, z_2)(D\delta)(\omega^r z_2 / c_1 z_1)$$

$$= B(z_1, \omega^{-r} c_1 z_1)(D\delta)(\omega^r z_2 / c_1 z_1) - m(\rho_k \epsilon_k (z_2) - \rho_l \nu_l (z_2)$$

$$+ (\beta, \sum_{\beta} \nu^p \beta / 2) B(z_1, \omega^{-r} c_1 z_1) \delta(\omega^r z_2 / c_1 z_1).$$

Together with (5.11) this gives

(5.12) $B(z_1, z_2) [ (D\delta)(\omega^r z_2 / c_1 z_1) + N_{2,r} \delta(\omega^r z_2 / c_1 z_1) ]$

$$= \varepsilon_C(\alpha, \nu \beta) [ (D\delta)(\omega^r z_2 / c_1 z_1) + (\rho_i \epsilon_i (z_1) - \rho_l \nu_l (c_1 z_1)) \delta(\omega^r z_2 / c_1 z_1) ].$$

It follows from (5.7) that

$$\zeta(\alpha) \zeta(\beta) \kappa(\rho_i, i, \rho_j, j, c_1) \kappa(\rho_k, k, c_1^{-1}) N_{1,r} = \varepsilon'(\alpha, \nu \beta)$$

if $r \in M(-2)$. Together with (5.12) this implies

(5.13) $O_4 = m^{-1} \sum_{r \in M(-2)} \xi_{r}(\alpha, \beta) (\rho_i \epsilon_i (z_1) - \rho_j \epsilon_j (c_1 z_1)) \delta(\omega^r z_2 / c_1 z_1)$

$$+ m^{-2} \sum_{r \in M(-2)} \xi_{r}(\alpha, \beta)(D\delta)(\omega^r z_2 / c_1 z_1).$$

Combining (5.8), (5.9), (5.10) and (5.13), and comparing the expression of $O_1 + O_2 + O_3 + O_4$ with (2.6), one sees that the result for the first part of the theorem under the condition of Case 5 follows from the following identity.

$$O_3 + m^{-1} \sum_{r \in M(-2)} \xi_{r}(\alpha, \beta) (\rho_i \epsilon_i (z_1) - \rho_j \epsilon_j (c_1 z_1)) \delta(\omega^r z_2 / c_1 z_1)$$

$$= m^{-1} \sum_{r \in \mathbb{Z}_m} \delta_{\rho_j \epsilon_j \rho_k} \xi_{r}(\alpha, \beta) Y_{p_i \epsilon_i \rho_l}(1, z_1) \delta(\omega^r z_2 / c_1 z_1)$$

$$+ m^{-1} \sum_{r \in \mathbb{Z}_m} \delta_{\rho_i \epsilon_i \rho_l} (1) \sum_{\alpha \beta} \xi_{r}(\alpha, \beta) Y_{-\rho_j \epsilon_j \rho_k}(1, c_1 z_1) \delta(\omega^r z_2 / c_1 z_1),$$
It is also clear that the result for Case 6 follows from Case 5 and Proposition 3.3. Finally, for Case 7, we note that in this case one has $c_1 = c_2 = -1$, but this is impossible as the group $\Gamma$ is generic. Therefore, we have finished the proof of the first part of the theorem. The proof of the second part of the theorem is standard, which is omitted.

\[ Q, \nu, m, \]

\[ \Box \]

which is omitted.

\[ \Gamma \)

We also provide a vertex operator representation for the $BC_{N-1}$-graded Lie algebra $\hat{\mathfrak{g}}_{2N}^{(2)}(\mathbb{C}_\Gamma)$.

6. Applications

This section is devoted to the application of Theorem 5.2. By choosing some special quadruples $(Q, \nu, m, \Gamma)$, we recover vertex operator representations presented in [L, G1, G2, BS, BGT, G-KL1, G-KL2, CGJT, CT].

6.1. Realization of twisted affine Lie algebras. Following [L], we define a Lie algebra $\mathfrak{g} = \mathcal{H} \oplus \sum_{\alpha \in Q'} \mathbb{C}x_\alpha$, where $\mathcal{H} = P \otimes \mathbb{C}$ and $Q' = \{\alpha \in Q | \langle\alpha, \alpha\rangle = 2\}$, with Lie bracket

\[ [\mathcal{H}, \mathcal{H}] = 0, \quad [h, x_\alpha] = \langle h, \alpha \rangle x_\alpha = -[x_\alpha, h], \]

\[ [x_\alpha, x_\beta] = \begin{cases} \varepsilon(\alpha, -\alpha)\alpha, & \text{if } \alpha + \beta = 0, \\ \varepsilon(\alpha, \beta)x_{\alpha+\beta}, & \text{if } \langle\alpha, \beta\rangle = -1, \\ 0, & \text{if } \langle\alpha, \beta\rangle \geq 0, \end{cases} \]

for $h \in \mathcal{H}, \alpha, \beta \in Q'$. Extend the bilinear form $\langle, \rangle$ of Cartan subalgebra $\mathcal{H}$ of $\mathfrak{g}$ to be an invariant form on $\mathfrak{g}$ by

\[ \langle h, x_\alpha \rangle = 0, \quad \langle x_\alpha, x_\beta \rangle = \delta_{\alpha+\beta,0}\varepsilon(\alpha, -\alpha), \quad h \in \mathcal{H}, \alpha, \beta \in Q'. \]

And extend the linear automorphism $\nu$ of $\mathcal{H}$ to be a Lie automorphism of $\mathfrak{g}$ by

\[ \nu(x_\alpha) = \eta(1, \alpha)x_{\nu(\alpha)}, \quad \alpha \in Q'. \]

For $n \in \mathbb{Z}$ and $x \in \mathfrak{g}$, we set $x_{(n)} = n^{-1}\sum_{p \in \mathbb{Z}} \omega^{-np}\nu^p(x)$ and $\mathfrak{g}_{(n)} = \{x_{(n)} | x \in \mathfrak{g}\}$. Consider the twisted affine Lie algebra $\hat{\mathfrak{g}}(\nu) = \sum_{n \in \mathbb{Z}} \mathfrak{g}_{(n)} \otimes t^n \oplus \mathbb{C}c$, with Lie bracket

\[ [x \otimes t^n, y \otimes t^r] = [x, y] \otimes t^{n+r} + n^{-1}n(x, y)\delta_{n+r,0}c \]

for $x \in \mathfrak{g}_{(n)}, y \in \mathfrak{g}_{(r)}, n, r \in \mathbb{Z}$, and $c$ is central.

Take $\Gamma = \{1\}$. One can check that the Lie algebra $\hat{\mathcal{G}}(Q, \nu, m, \{1\})$ is isomorphic to $\hat{\mathfrak{g}}(\nu)$ via the isomorphism

\[ \bar{e}_{i, \rho, j}^{(-1)}(1, n) \mapsto (x_{i, -\rho, j})_{(n)} \otimes t^n, \quad \bar{e}_{k, k}^{(-1)}(1, n) \mapsto (\epsilon_k)_{(n)} \otimes t^n, \quad c \mapsto c, \]

for $(i, \rho, j) \in J, i \neq j, 1 \leq k \leq N$ and $n \in \mathbb{Z}$. Comparing this with Theorem 5.2 we obtain the following result which was given in [L].

Corollary 6.1. The generalized Fock space $V_T$ affords a representation of the $\nu$-twisted affine Lie algebra $\hat{\mathfrak{g}}(\nu)$ with action given by

\[ (x_{i, -\rho, j})_{(n)} \otimes t^n \mapsto y_{i, \rho, j}(1, n), \quad (\epsilon_k)_{(n)} \otimes t^n \mapsto y_{k, k}(1, n), \quad c \mapsto 1, \]
for $(i, \rho, j) \in \mathcal{J}, i \neq j, 1 \leq k \leq N$ and $n \in \mathbb{Z}$. In particular, if $\text{span}_\mathbb{C} Q_{(2)} = Q$, then $V_T$ is irreducible if and only if the $\mathbb{C}[Q, \varepsilon_C]$-module $T$ is irreducible.

6.2. Realization of extended affine Lie algebras of type $A_{N-1}$. In this section, we present the homogenous and principal vertex operator representations of the Lie algebra $\hat{\mathfrak{g}}_N(\mathbb{C}_q)$. Let $Q = Q(A_{N-1}), N \geq 2, \nu = \text{Id}, m = 1$ and $\Gamma = \Gamma_q$ (cf. Section 2.2). Note that in this case we have $C(\alpha, \beta) = (-1)^{[\alpha, \beta]}$, thus there is a 2-cocycle $\varepsilon^* : P \times P \to \{\pm 1\}$ associated with $C$ determined by

$$\varepsilon^*(\sum m_i \varepsilon_i, \sum n_j \varepsilon_j) = \prod_{i,j} (\varepsilon^*(\varepsilon_i, \varepsilon_j))^{m_in_j},$$

where $\varepsilon^*(\varepsilon_i, \varepsilon_j) = 1$ if $i \leq j$ and $= -1$ if $i > j$.

Define a $\mathbb{C}[Q, \varepsilon^*]$-module structure and $H(0)$-action on the group algebra $\mathbb{C}[Q] = \bigoplus_{\alpha \in Q} \mathbb{C} \varepsilon^\alpha$ as follows

$$(6.1) \quad e_\alpha e_\beta = \varepsilon^*(\alpha, \beta) e_{\alpha+\beta}, \quad h e_\alpha = \langle h, \alpha \rangle e_\alpha, \quad h \in H(0) = H(\alpha, \beta) \in Q,$$

which is obviously compatible in the sense of [5.3]. Thus we may take $\varepsilon_C = \varepsilon^*$ and let $V_T = \mathbb{C}[Q] \otimes S$. One can check that the Lie algebra $\hat{\mathfrak{g}}(Q(A_{N-1}), \text{Id}, 1, \Gamma_q)$ is isomorphic to $\mathfrak{g}_N(\mathbb{C}_q)$ via the isomorphism

$$\varepsilon^*(\varepsilon_i, \varepsilon_j) e_i^{n_j}(q^j, n_0, n) \mapsto E_{i,j} t_0^{n_0} t^n, \quad c \mapsto c, \quad 1 \leq i, j \leq N, (n_0, n) \in \mathbb{Z}^{l+1}.$$

Comparing this with Theorem 5.2 we obtain the following result, which was given in [G1] [BGT].

**Corollary 6.2.** There is an irreducible representation of the Lie algebra $\hat{\mathfrak{g}}_N(\mathbb{C}_q)$ on $V_T = \mathbb{C}[Q(A_{N-1})] \otimes S$ in the homogeneous picture. The representation is given by the mapping

$$E_{i,j} t_0^{n_0} t^n \mapsto \varepsilon^*(\varepsilon_i, \varepsilon_j) g_i,j(q^n, n_0), \quad c \mapsto c, \quad 1 \leq i, j \leq N, n \in \mathbb{Z}, n \in \mathbb{Z}^l. \quad \square$$

In what follows we present the principal realization of $\hat{\mathfrak{g}}_N(\mathbb{C}_q^n)$, where $q^N = (q_1^N, \cdots, q^N)$. Set $E = E_1, \cdots, E_N, E_{N+1}, F = \sum_{i=1}^N \omega E_i$, where $\omega$ is a primitive $N$-th root of unity. It was proved in [G2] that the subalgebra of $\mathfrak{g}_N(\mathbb{C}_q)$ spanned by the elements $F^i E^{n_0} t_0^{n_0} t^n$, $c, \quad i, n_0 \in \mathbb{Z}, n \in \mathbb{Z}^l$, is isomorphic to the Lie algebra $\hat{\mathfrak{g}}_N(\mathbb{C}_q^n)$. For $i, j, n_0, r_0 \in \mathbb{Z}, n, r \in \mathbb{Z}^l$, one has that

$$(6.2) \quad [F^i E^{n_0} t_0^{n_0} t^n, F^j E^{r_0} t_0^{n_0} t^r] = \omega^{jn_0} q^{-n} F^{i+j} E^{n_0+r_0} t_0^{n_0+r_0} t^{n+r} + n_0 q^{-r_0} \omega^{jn_0} F^{i+j}E^{n_0+r_0} t_0^{n_0+r_0} t^{n+r} + n_0 q^{-r_0} \omega^{jn_0} F^{i+j}E^{n_0+r_0} t_0^{n_0+r_0} t^{n+r},$$

where $\bar{i}$ is the unique integer in $\{1, \cdots, N\}$ such that $\bar{i} \equiv i \pmod{N}$.

Choose $Q = Q(A_{N-1}), N \geq 2, \nu = \nu_C, m = N$ and $\Gamma = \Gamma_q$, where $\nu_C$ is the isometry of $P$ defined by $\nu_C(\varepsilon_i) = \varepsilon_{\sigma(i)}$, $1 \leq i \leq N, \sigma = (12 \cdots N)$. Thus $\nu_C$ is the
Coxeter isometry of $Q(A_{N-1})$ and satisfies the following conditions
\[
\sum_{p \in \mathbb{Z}_m} \nu_p^2 \alpha = 0, \forall \alpha \in Q, \sum_{p \in \mathbb{Z}_m} \langle \nu_p^2 \alpha, \beta \rangle = m \mathbb{Z}, \forall \alpha, \beta \in Q.
\]
This implies $C(\alpha, \beta) = 1$ for all $\alpha, \beta \in Q$. Thus we may take $\varepsilon_C = 1$ and $\eta(p, \alpha) = 1$ for all $p \in \mathbb{Z}_n, \alpha \in Q$. Then the trivial $\mathbb{C}[Q, \varepsilon_C]$-module $\mathbb{C}$ satisfies the condition (3.3) and hence we may take the generalized Fock space to be $S$.

It follows from (3.7) and Proposition 2.3 that, for $i, j \in \mathbb{Z}$ and $n, r \in \mathbb{Z}_l$,
\[
\begin{align*}
[G^i(n, z_1), G^j(r, z_2)] &= G^{i+j}(n + r, \omega^{-j} z_1) \delta(\omega^j z_2/Q^n z_1) \\
- G^{i+j}(n + r, \omega^{-j} z_2) \delta(\omega^j z_1/Q^n z_2) + \delta_{i+j,N}(D\delta)(\omega^j z_2/Q^n z_1)
\end{align*}
\]
where $G^i(n, z) := N \zeta(\epsilon_{N-1+i} - \epsilon_1)^{-1}(1 - \omega^{-i})^{q_{N-1,N}^{-1}} G^{1,j,1}(q^n, z)$. Let $G^i(n, z) = \sum_{n_0 \in \mathbb{Z}_l} g^i(n, n_0) z^{-n_0}$, then by comparing the identity (6.3) with (6.2), we obtain the following isomorphism between the Lie algebra $\hat{\mathfrak{g}}_{N}^\ast(\mathbb{C}_q)$ and $\hat{\mathfrak{g}}(Q(A_{N-1}), \nu_c, N, \Gamma_q)$

\[
F^i \stackrel{E_{n_0}^j}{\mapsto} t^{n_0} \mapsto g^i(n, n_0), \ c \mapsto c, i, n_0 \in \mathbb{Z}, n \in \mathbb{Z}_l.
\]

Set $y^i(n, n_0) := N \zeta(\epsilon_{N-1+i} - \epsilon_1)^{-1}(1 - \omega^{-i})^{q_{N-1,N}^{-1}} B^{1,j,1}(q^n, n_0)$ for $i, n_0 \in \mathbb{Z}$ and $n \in \mathbb{Z}_l$. Thus from Theorem 5.2, we obtain the following result which was given in [BS] for the case $N = 2$ and in [G2, BGT] for all $N \geq 2$.

**Corollary 6.3.** There is an irreducible representation of the Lie algebra $\hat{\mathfrak{g}}_{N}^\ast(\mathbb{C}_q)$ on $V_T = S$ in the principal picture. The representation is given by the mapping

\[
F^i \stackrel{E_{n_0}^j}{\mapsto} t^{n_0} \mapsto g^i(n, n_0), \ c \mapsto c, i, n_0 \in \mathbb{Z}, n \in \mathbb{Z}_l. \quad \square
\]

### 6.3. Realization of trigonometric Lie algebras.

Let $N = 1, Q = \{0\}, \nu = \text{Id}$ or $-\text{Id}, m = 1$ or 2 and $\Gamma = \Gamma_\text{h}$ (cf. Section 2.2). Note that in this case the generalized Fock space is $V_T = S(\mathcal{H}(\nu)^\sim)$. Set $g(n, n_0) = e^{-\pi \sqrt{-1}}(q^n, \eta_{1,1}(e^{-2\sqrt{-1}}(q^n, n_0))$ for $(n_0, n) \in \mathbb{Z}_l^+$. From the third example in Section 2.2 and Proposition 2.3, one can check that the trigonometric Lie algebra of series $\hat{A}_h$ (resp. $\hat{B}_h$) is isomorphic to $\hat{\mathfrak{g}}(\{0\}, \text{Id}, 1, \Gamma_\text{h})$ (resp. $\hat{\mathfrak{g}}(\{0\}, -\text{Id}, 2, \Gamma_\text{h})$) via the isomorphism

\[
A_{n, n_0} (\text{resp. } 2B_{n, n_0}) \mapsto g(n, n_0), \ c \mapsto c, (n_0, n) \in \mathbb{Z}_l^+.
\]

Set $y(n, n_0) = e^{-\pi \sqrt{-1}}y_{1,1}(e^{-2\sqrt{-1}}(q^n, n_0))$ for $(n_0, n) \in \mathbb{Z}_l^+$. From the isomorphisms given above and Theorem 5.2, we have the following result, which was given in [G-KL1, G-KL2].

**Corollary 6.4.** The generalized Fock spaces $S(\mathcal{H}(\text{Id})^\sim)$ and $S(\mathcal{H}(\text{Id})^\sim)$ afford irreducible representations for the trigonometric Lie algebras of series $\hat{A}_h$ and $\hat{B}_h$ respectively. The representations are respectively given by the mapping

\[
A_{n, n_0} \mapsto y(n, n_0), \ c \mapsto 1, \quad \text{and} \quad B_{n, n_0} \mapsto 2y(n, n_0), \ c \mapsto 1, (n_0, n) \in \mathbb{Z}_l^+. \quad \square
\]
6.4. Realization of unitary Lie algebras. Choose \( Q = Q(A_{N-1}) \), \( \nu = -\text{Id} \), \( m = 2 \). Then \( C(\alpha, \beta) = (-1)^{(\alpha, \beta)} \) for \( \alpha, \beta \in Q \), and we may take \( \varepsilon_C = \varepsilon^* \), and \( \eta(p, \alpha) = 1 \) for all \( \alpha \in Q, p \in \mathbb{Z}_m \) as \( \nu \) preserves \( \varepsilon^* \). Note that \( \mathcal{H}_{(0)} = 0 \) in this case and hence the condition (3.3) is equivalent to that of \( e_{2\alpha} \) acting as the identity operator on \( T \) for any \( \alpha \in Q \). Let \( \mathbb{C}[P/2P] = \bigoplus_{\alpha \in P} \mathbb{C}e_\alpha, \alpha = \alpha + 2P \) be the group algebra over the quotient group \( P/2P \). Define a \( \mathbb{C}[Q, \varepsilon^*] \)-module structure on \( \mathbb{C}[P/2P] \) by

\[
e_\alpha e_\beta = \varepsilon^*(\alpha, \beta)e_{\alpha + \beta}, \quad \alpha \in Q, \beta \in P.
\]

Obviously, \( e_{2\alpha} \) acts as the identity operator on \( \mathbb{C}[P/2P] \). Therefore, we take \( V_T = \mathbb{C}[P/2P] \otimes S \). One can check that the Lie algebra \( \hat{\mathcal{G}}(Q(A_{N-1}), -\text{Id}, 2, \Gamma) \) is isomorphic to the unitary Lie algebra \( \hat{\mathfrak{n}}_N(\mathbb{C}_\Gamma) \) via the isomorphism

\[
2\varepsilon^*(\epsilon_i, \epsilon_j)e_{-ij}(c, n) \mapsto \epsilon_i \epsilon_j f_{\rho_i, \rho_j}(c, n), \quad c \mapsto c, \quad 1 \leq i, j \leq N, c \in \Gamma, n \in \mathbb{Z}.
\]

Comparing this with Theorem 5.2, we have the following result obtained in [CGJT].

**Corollary 6.5.** The generalized Fock space \( V_T = \mathbb{C}[P/2P] \otimes S \) affords a representation for the unitary Lie algebra \( \hat{\mathfrak{n}}_N(\mathbb{C}_\Gamma) \) with the actions given by

\[
u_{ij}(c, n) \mapsto 2\varepsilon^*(\epsilon_i, \epsilon_j)y_{ij}(c, n), \quad c \mapsto c, \quad 1 \leq i, j \leq N, n \in \mathbb{Z}, c \in \Gamma. \quad \Box
\]

6.5. Realization of the \( BC_N \)-graded Lie algebra \( \mathfrak{d}_{2N}(\mathbb{C}_\Gamma) \). Choose \( Q = Q(D_N) \), \( \nu = \text{Id} \) and \( m = 1 \). By a similar argument as we did in Section 6.2, we may take \( \varepsilon_C = \varepsilon^* \) and the \( \mathbb{C}[Q, \varepsilon^*] \)-module \( T \) to be \( \mathbb{C}[Q(D_N)] \) with action defined in (6.1). Note that the Lie algebra \( \hat{\mathcal{G}}(Q(D_N), \text{Id}, 1, \Gamma) \) is isomorphic to the Lie algebra \( \hat{\mathfrak{d}}_{2N}(\mathbb{C}_\Gamma) \) with the mapping given by

\[
\varepsilon^*(\epsilon_i, \epsilon_j)e_{\rho_i, \rho_j}(c, n) \mapsto f_{\rho_i, \rho_j}(c, n), \quad c \mapsto c, \quad 1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1, n \in \mathbb{Z}, c \in \Gamma.
\]

Thus, by Theorem 5.2 we have the following result, which was given in [CT].

**Corollary 6.6.** There is an irreducible \( \hat{\mathfrak{d}}_{2N}(\mathbb{C}_\Gamma) \)-module structure on the generalized Fock space \( V_T = \mathbb{C}[Q(D_N)] \otimes S \) with action given by

\[
f_{\rho_i, \rho_j}(c, n) \mapsto \varepsilon^*(\epsilon_i, \epsilon_j)y_{\rho_i, \rho_j}(c, n), \quad c \mapsto c, \quad 1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1, n \in \mathbb{Z} \text{ and } c \in \Gamma. \quad \Box
\]

6.6. Realization of the \( BC_{N-1} \)-graded Lie algebra \( \hat{\mathfrak{d}}_{2N}(2)(\mathbb{C}_\Gamma) \). In this section we give a homogeneous vertex operator construction for the \( BC_{N-1} \)-graded Lie algebras \( \hat{\mathfrak{d}}_{2N}(2)(\mathbb{C}_\Gamma) \) with grading subalgebra of type \( B_{N-1} \) defined in [ABC]. In what follows we take \( Q = Q(D_N) \), \( \nu = \nu_d \) and \( m = 2 \), where \( \nu_d \) is the diagram automorphism of \( Q(D_N) \). Recall that \( \nu_d(\epsilon_i) = \epsilon_i \) for \( i = 1, \cdots, N-1 \) and \( \nu_d(\epsilon_N) = -\epsilon_N \). Then we have \( i_1 = i \) for \( i = 1, \cdots, N-1 \) and \( N_1 = -N \).

Define an involution \( * \) of the Lie algebra \( \hat{\mathfrak{d}}_{2N}(\mathbb{C}_\Gamma) \) (cf. Section 2.2) as follows

\[
f_{\rho_i, \rho_j}(c, n)^* = (-1)^n f_{\rho_i, \rho_j}(c, n), \quad c \mapsto c,
\]
where \(1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1, c \in \Gamma\) and \(n \in \mathbb{Z}\). We denote by
\[\hat{\mathfrak{o}}_{2N}^{(2)}(\mathbb{C}_{\Gamma})\]
the subalgebra consisting of fixed-points of \(\hat{\mathfrak{o}}_{2N}(\mathbb{C}_{\Gamma})\) under the involution \(\hat{\cdot}\). One can easily check that the Lie algebra \(\hat{\mathfrak{o}}_{2N}^{(2)}(\mathbb{C}_{\Gamma})\) is a \(BC_{N-1}\)-graded Lie algebra with grading subalgebra of type \(B_{N-1}\) in the sense of \(\text{ABC}\). We remark that if \(\Gamma = \{1\}\), then this fixed-point subalgebra is nothing but the affine Lie algebra \(\hat{\mathfrak{o}}_{2N}^{(2)}(\mathbb{C})\).

Note that in the case \(\nu = \nu_d\) and \(m = 2\), we have \(C(\alpha, \beta) = (-1)^{(\alpha, \beta)}\) for \(\alpha, \beta \in Q(D_N)\). Thus we may choose \(\varepsilon_C = \varepsilon^*\) and \(\eta(p, \alpha) = 1\) for \(p \in \mathbb{Z}_m\) and \(\alpha \in Q(D_N)\). Let \(\mathbb{C}[P/2\mathbb{Z}_\varepsilon_{N}] = \oplus_{\alpha \in P} \mathbb{C}_{\hat{\alpha}}\), \(\hat{\alpha} = \alpha + 2\mathbb{Z}_\varepsilon_N\) be the group algebra over the quotient group \(P/2\mathbb{Z}_\varepsilon_N\). Define a \(\mathbb{C}[Q(D_N), \varepsilon^*]\)-module structure and an \(\mathcal{H}_{(0)}\)-action on \(\mathbb{C}[P/2\mathbb{Z}_\varepsilon_{N}]\) by
\[e_{\alpha}, e_{\beta} = \varepsilon^*(\alpha, \beta)e_{\alpha + \beta}; \quad h.e_{\beta} = \langle h, \beta \rangle e_{\beta}, \quad \alpha \in Q(D_N), \beta \in P, \ h \in \mathcal{H}_{(0)},\]
which is compatible in the sense of \((5.3)\). Therefore, we may take the generalized Fock space to be \(\mathbb{C}[P/2\mathbb{Z}_\varepsilon_{N}] \otimes S\). For \(j = 0, 1\), set
\[\mathbb{C}[P/2\mathbb{Z}_\varepsilon_{N}]^j = \{e_\alpha | \alpha = \sum_{i=1}^{N} a_i \varepsilon_i, a_1, \cdots, a_{N-1} \in \mathbb{Z}, a_N = 0, 1, \sum_{i=1}^{N} a_i \in 2\mathbb{Z} + j\},\]
which are irreducible \(\mathbb{C}[Q(D_N), \varepsilon^*]\)-submodules of \(\mathbb{C}[P/2\mathbb{Z}_\varepsilon_{N}]\).

One can check that the Lie algebra \(\hat{\mathfrak{o}}_{2N}^{(2)}(\mathbb{C}_{\Gamma})\) is isomorphic to \(\hat{\mathfrak{g}}(Q(D_N), \nu_d, 2, \Gamma)\) with the isomorphism given by
\[g_{\rho_i, \rho_j}(c, n) \mapsto \varepsilon^*(\varepsilon_i, \varepsilon_j) e_{\rho_i, \rho_j}(c, n), \quad 2c \mapsto c, \ 1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1, c \in \Gamma, n \in \mathbb{Z}.\]
This together with Theorem 5.2 gives us the following result.

**Theorem 6.7.** There is an \(\hat{\mathfrak{o}}_{2N}^{(2)}(\mathbb{C}_{\Gamma})\)-module structure on \(\mathbb{C}[P/2\mathbb{Z}_\varepsilon_{N}] \otimes S\) by the mapping
\[g_{\rho_i, \rho_j}(c, n) \mapsto \varepsilon^*(\varepsilon_i, \varepsilon_j) y_{\rho_i, \rho_j}(c, n), \quad 2c \mapsto 1,\]
for \(1 \leq i, j \leq N, \rho_i, \rho_j = \pm 1, n \in \mathbb{Z}\) and \(c \in \Gamma\). Moreover, the \(\hat{\mathfrak{o}}_{2N}^{(2)}(\mathbb{C}_{\Gamma})\)-module \(\mathbb{C}[P/2\mathbb{Z}_\varepsilon_{N}]^0 \otimes S\) is completely reducible and the irreducible components are \(\mathbb{C}[P/2\mathbb{Z}_\varepsilon_{N}]^1 \otimes S\) and \(\mathbb{C}[P/2\mathbb{Z}_\varepsilon_{N}]^\dagger \otimes S\). \(\square\)

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