Atomic Theory of the Two-fluid Model: Broken Gauge Symmetry in Bose-Einstein condensation

S. J. Han
P.O. Box 4671, Los Alamos, NM 87544-4671

Abstract

We discuss the collective excitations in a spatially inhomogeneous (cylindrically symmetric) Bose-Einstein condensation (BEC) at low temperature ($T \ll T_{\lambda}$). The main result is the dispersion relation for a (first) sound wave that is obtained by describing the perturbation as a Lagrangian coordinate. The dispersion curve is in good agreement with the Bogoliubov phonon spectrum $\omega = ck$, where $k = k_{\theta} = m/r$, the wave number and $c = [4\pi a\rho \hbar^{2}]^{1/2}/M$, the speed of first sound. Based on Bohm’s quantum theory, a spontaneously broken gauge symmetry in a quantum fluid is discussed in terms of the quantum fluctuation-dissipation, from which it is shown that the symmetry breaking takes place at the free surface of BEC in an external field.

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I. INTRODUCTION

The Bose-Einstein condensation (BEC) is a remarkable quantum phenomena occurring in a macroscopic Bose system [1]. London has proposed that the \( \lambda \) transition between He I and He II is a result of the same process which causes the condensation of an ideal Bose gas at low temperatures [2]. In 1941, Landau studied the collective excitation spectrum in He II based on his phenomenological theory of two-fluid model [3], and has shown that the excitation spectrum \( \omega(k) \) of a wave number \( k \) should rise linearly with slope \( c \) [i.e., \( \omega = ck \)] with the speed of (first) sound \( c = 238 \text{m/sec} \) for small \( k \), pass through a maximum, drop to a local minimum at some value \( k_0 \), and then rise again for \( k > k_0 \), which has been confirmed in great detail by Henshaw and Woods [4]. For small \( k \), the excitations are called phonons (quantized sound wave).

In the absence of a satisfactory microscopic theory, there has been considerable development of phenomenological theories following Landau’s two fluid model. Of the various attempts to account for the \( \lambda \) transition, the best known is that of Feynman [5]. Based on the exact quantum partition function as an integral over particle trajectories by his path-integral approach to quantum mechanics, Feynman studied the two-fluid model of He II in a strongly interacting \(^4\text{He} \) gas and pointed out that London’s view on BEC is essentially correct. The point in his argument is that, in a liquid like quantum system, the symmetry of a Bose system plays far more important a role than that of strong inter-atomic forces which do not prevent these particles move freely in the system; yet the pair-interaction which brings about \( \lambda \)-transition also ensures that He II possess the collective excitations (phonons and rotons) [5]. It is also the essential point of Bogoliubov’s theory of superfluidity which has laid the basis for much of our theoretical understanding about superfluidity of He II [6, 7].

Penrose and Onsager [8] further extended the concept of Bose condensation to a strongly interacting Bose gas of He II based on the properties of a ground state wave function derived from the assumption that there is no long-range order interaction. They proposed that one could essentially define the Bose condensed state of a system such as He II as a state in which the reduced density matrix of the system can be factorized in a certain special way, now known as the off-diagonal long range order (ODLRO) after Yang [9].

Although the previous theories [3, 5, 6] have been very successful in explaining the collective phenomena in He II, the dispersion relation for a longitudinal (first) sound wave \( \omega = ck \):
is given only in the case of a spatially homogeneous system for mathematical convenience. Bogoliubov has used an ingenious mathematical method based directly on quantum field theory for his calculation of the excitation spectrum in a weakly interacting Bose system.

The formal extension of the Bogoliubov method to a finite inhomogeneous system is difficult, since one can quantize a scalar field of quasi-particles only in a Hilbert space. Furthermore, it is impossible to make use of the elegant mathematical technique developed in his quantum field theory approach, because the theory of collective excitations in a finite space problem reduces to an initial-boundary value problem for which one must find a correct boundary condition in terms of a dynamical variable. It is, therefore, important to use an appropriate mathematical technique to develop a theory of collective excitations in BEC in a trap.

Recently, however, the study of a new BEC in a trap has helped to revive interest in the study of collective excitations, but it has faced an almost insurmountable challenge to derive a correct dispersion relation for collective excitations in BEC in a trap. The problem with the boundary conditions appearing in the dynamical equations represents the major stumbling block in obtaining a proper dispersion relation for phonon spectrum at low temperature \( T \ll T_\lambda \).

With proper boundary conditions a theory must be capable of computing the excitation spectrum for phonons, and yet explains in a natural way why a symmetry braking takes place and how the sound wave dissipates by the fluctuation-dissipation process at the free surface. These are uniquely new phenomena in a Bose system that is confined by an external potential. Such a theory is the subject of this paper.

In this paper we consider an imperfect Bose gas of \( N \) identical particles in a trap. Between each pair of any particles in the system, there is a hard sphere repulsion of range \( a \) that balances the external potential which determines the size of the condensation in the trap. This pair-interaction is the basic, and almost the only, assumption to describe the weakly interacting Bose particles in a trap. Moreover, this assumption is reasonable to give results which are satisfactory in a dilute Bose gas in trap; the results are also consistent with the theory of superfluidity. The main contributions of the present paper are a semi-classical calculation of the excitation spectrum for phonons and a study of the symmetry breaking along with the fluctuation-dissipation process of sound waves to conserve the energy in an isolated system. The study of collective excitations is essential for a complete understanding
of certain properties of a Bose condensed system, in particular, the longitudinal collective excitations (first sound) at temperature near absolute zero. This provides the course of experiments that confirm BEC in a trap [8].

Because of a finite, inhomogeneous density of BEC, the collective excitation spectrum depends on the spatial density profile and the boundary conditions. In this paper we shall study the existence and excitation energy spectrum of a cylindrical (first) sound wave in a cylindrically symmetric inhomogeneous Bose condensate (CSIBC) which is an extension of the previous study [18]. The present analysis, in principle, can be extended to an ellipsoidal Bose condensate in a trap. In practice, however, the mathematical analysis of the collective excitations in an ellipsoidal Bose-Einstein condensation in prolate spheroidal coordinates is mathematically more challenging due to the complications of the ellipsoidal wave propagations, and the well-known special cases may also play an important role to check the general case [19, 20, 21].

To present this work as a self-contained paper, we first review the concept of ODLRO in which Penrose and Onsager suggested that the reduced density matrix of a Bose condensed state can be factorized,

$$\rho(r, r') = \psi^\dagger(r)\psi(r) + \gamma(|r - r'|).$$  \hspace{1cm} (1)

where $\gamma \to 0$ as $|r - r'| \to \infty$. The single particle wave function $\psi(r)$ represents the condensed state in ODLRO and is viewed as a function of macroscopic dynamical variables. The single particle state function $\psi(r)$ is also interpreted as the mean value of a quantum particle field [10].

With the hard sphere approximation, one can show that the mean field satisfies the nonlinear Schrödinger equation (Gross-Pitaevskii),

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2M} \nabla^2 \psi + [V(x)_{\text{ext}} + g_1|\psi|^2] \psi,$$  \hspace{1cm} (2)

where $V_{\text{self}} = g_1|\psi|^2 = 4\pi \hbar^2 a/M|\psi|^2$ and $a$ is the s-wave scattering length. It is a self-consistent Hartree equation for the Bose condensed wave function [11]. In our model of an imperfect Bose gas, the wave function describes a near perfect gas modified as little as possible by the presence of the pair-interactions. Basic to the theory of superfluidity is the idea of Bogoliubov [6] that this pair-interaction which brings about the Bose condensed state also ensures that the system possess the longitudinal collective excitations (first sound) in BEC. We must therefore use the effective potential that includes the self-interaction term in
Schrödinger equation, which might be thought of as a generalization of Bohm’s theory to the theory of superfluidity, since the nonlinear term \( g_1 |\psi|^2 \) is invariant under a \( U(1) \) group transformation.

II. BASIC EQUATIONS

In order to derive the density profile for CSIBC, we write the equations for the ensemble average energy in the usual quantum theory:

\[
\mathcal{H} = \int \psi^\dagger \left( -\frac{\hbar^2}{2M} \nabla^2 + V(x)_{\text{ext}} + \frac{g_1}{2} |\psi|^2 \right) \psi \, dx,
\]

\[
\mathcal{E}_{\text{ave}} = \int \left( \frac{\hbar^2}{2M} |\nabla \psi|^2 + V(x)_{\text{ext}} |\psi|^2 + \frac{g_1}{2} |\psi|^4 \right) \, dx, \quad \int \psi^\dagger \psi \, dx = N.
\]

For a BEC in a trap, the ground state density can be obtained in terms of an external potential and the chemical potential by minimizing the energy functional Eq. (4) for a fixed number of particles in the system with the condition \( p = 0 \) (Penrose-Onsager criterion for BEC):

\[
\rho(x) = |\psi(x)|^2 = \frac{M}{4\pi\hbar^2a} [\mu - V(x)_{\text{ext}}],
\]

where \( \mu \) is a Lagrangian multiplier and is the chemical potential. It should be noted that the ground-state wave function Eq. (5) has a nodal surface at which the density becomes zero and is the boundary of CSIBC.

As we have emphasized above, the most serious question in regard to a possible application of Bogoliubov’s quantum field theory technique (or an extended version, the Bogoliubov-de Gennes equation) to a finite Bose system is that the theory remains valid only in an infinite uniform system or a system in a box with periodic boundary conditions but not a finite inhomogeneous system of BEC in a trap.

Both for this reason, and because it is essential to show that a symmetry breaking takes place at the nodal surface of CSIBC, we employ Bohm’s quantum theory with emphasis on the ensemble of particle trajectories. Briefly, Bohm writes \( \psi(r) \) in the form

\[
\psi(x, t) = f(x, t) \exp\left[ \frac{i}{\hbar} S(x, t) \right] = \rho^{1/2}(x, t) \exp\left[ \frac{i}{\hbar} S(x, t) \right],
\]

where \( S(x, t) \) is a phase (or an action). We then rewrite Eq. (2) to obtain,

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \frac{\nabla S}{M}) = 0
\]
\[
\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2M} + V(x) - \frac{\hbar^2}{4M} \left[ \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right] = 0,
\] (6b)

where \( V(x) = V(x)_{ext} + gp \). Here the potential \( V(x) \) includes \( V_{self} = gp \). The last term \( U_{eqmp} \) of Eq. (6b) is the effective quantum-mechanical potential (EQMP) which plays a crucial role in the discussion of a symmetry breaking in a finite spatially inhomogeneous Bose system.

In Eq. (6), Bohm suggested that one may regard Eq. (6a) as the conservation of current if one identifies \( \rho \) as the probability density and \( \mathbf{v} = \nabla S(x)/M \). \( S(x,t) \) the solution of Eq. (6b) is the action. However, in the limit \( \hbar \to 0 \), \( S(x,t) \) is a solution of the Hamilton-Jacobi equation and becomes a phase of the wave function \( \psi(x,t) \) [14, 15]. In general, the solution of the quantum Hamilton-Jacobi equation Eq. (6b) defines an ensemble of possible particle trajectories, which can be obtained in principle from the Hamilton-Jacobi function, \( S(x) \), by integrating the velocity, \( \mathbf{v}(x) = \nabla S(x)/M \). The equation for \( S \) implies, however, that the particle is acted on, not only by a potential \( V(x) \) but also by the effective quantum mechanical potential \( U_{eqmp} \). Moreover Eq. (6b) suggests us how we choose stable particle trajectories about which we may linearize the equations of motion. For if, as was done in the previous study of collective excitations in He II [5, 6], a system is described as an infinite uniform medium, \( U_{eqmp} \) becomes zero. On the other hand, in a finite system the effective quantum mechanical potential diverges at the nodal surface,

\[
U_{eqmp} = -\frac{\hbar^2}{4M} \left[ \frac{\nabla^2 \rho}{\rho} - \frac{1}{2} \frac{(\nabla \rho)^2}{\rho^2} \right] = -\frac{\hbar^2}{M} \frac{\nabla^2 f}{f},
\] (7)

where \( \rho(r) \) is zero at the nodal surface.

As emphasized by Bohm [14], a particle experiences the force from \( U_{eqmp} \) and fluctuates with its momentum \( \mathbf{p} = \nabla S \) and energy near the surface with the degree of divergence \( U_{eqmp} = M\omega_0^2\mu_0/D^2 \) as \( D = [\mu_0 - (1/2)M\omega_0^2 r^2] \to 0 \) near the surface. Hence \( U_{eqmp} \) breaks up the phase coherence of a sound wave in the surface layer, the thickness of which is in the order of mean free path.

**III. FIRST-ORDER EQUATIONS**

Now we come to the question of why we must find an alternate approach such as Bohm’s quantum theory to the quantum field theory method by which Bogoliubov has already ob-
tained a correct energy spectrum of collective excitations $\omega = ck$ with $c = [4\pi a\rho \hbar^2]^{1/2}/M$ in the long wavelength limit. The reason is obvious: one can quantize the scalar field of quasi-particles only in a Hilbert space, but not in a finite inhomogeneous system such as CSIBC. By the same reason, an extended version of the Bogoliubov theory with the mean field, the Bogoliubov-de Gennes equation, is not also applicable to a finite inhomogeneous system and it would lead to erroneous results. This observation provides a deeper appreciation of Feynman’s path-integral approach in his atomic theory of the two-fluid model although his analysis was confined to a uniform system. The basic approach in both Feynman’s two-fluid model and our present approach is that a Bose system with self-consistent interactions can be studied by the ensemble of particle trajectories to which Bohm’s quantum theory is more useful for describing a system with a well-defined boundary. More importantly, by Bohm’s interpretation of quantum mechanics, the solution of the Hamilton-Jacobi equation can be separated from the solution of the quantum Hamilton-Jacobi equation in the classical limit ($i.e.$, $\hbar \to 0$), and the action of the quantum Hamilton-Jacobi equation is used only for the measurement process to reconcile with the uncertainty principle by statistical ensemble of particle trajectories. This unique feature allows one to apply perturbations to the solution of the Hamilton-Jacobi equation, and at the same time the effective quantum mechanical potential is useful to describing the fluctuation-dissipation process to explain the spontaneously broken local gauge symmetry in the system.

As in the previous work, the perturbation to a particle trajectory is treated as a Lagrangian coordinate to the semi-classical solution. This leads a set of the linearized dynamical equations from which we derive a second-order partial differential equation in terms of a displacement vector. We solve this differential equation for the solution of a problem in CSIBC with correct initial-boundary conditions.

Since the present calculation of collective excitation spectrum in CSIBC is new, and is of some interest in itself, we present it here in full detail. We now introduce a symmetry breaking perturbation to the particle trajectories in CSIBC in a Lagrangian coordinate; it is defined as $x = x_0 + \xi(x_0, t)$, where $\xi(x_0, t)$ is a function of the unperturbed position of a particle in the condensate, and remains attached to the particle as it moves.

In addition to the particle displacements, we introduce phase coherence that relates the Bose condensed wave function (mean field) in ODLRO to a many-body ground-state wave function, $\xi_i \cdot \nabla S(x_0, t) = \sum_i \xi_i \cdot \nabla_i S_{0,i}(x_{0,i}, t)$, where $S_{0,i}(x_{0,i}, t)$ is the phase of a single Bose
particle in the system. This is a necessary and sufficient condition for the phase coherence. It also gives a simple interpretation of the mean field and quasi-particles (coherent excited states) in BEC. It is therefore obvious that a single particle excitation does not occur in our study as in Feynman’s work \[5\]. Moreover, the collective excitations we study in this paper are phase coherent sound waves.

The first step in developing this alternate approach is to associate with a Bose particle having precise stable particle trajectories which are a function of position and momentum. In this connection, it is worthwhile to note that the use of the Hamilton-Jacobi equation in solving for the motion of a particle is only matter of convenience and that, in principle, we can always solve it directly by the conventional dynamical equations with the correct initial-boundary conditions for a finite space problem. In order to study the modified excitations in a trap, we find it convenient to write down the basic dynamical equations.

This is most simply done by writing \( \mathbf{v}(x_0, t) = \nabla S(x_0, t)/M \), then Eqs. (5) and (6) yield the following dynamical equations:

the equation of motion for a single particle

\[
M \left( \frac{\partial}{\partial t} \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla \mu, \tag{8}
\]

the equation of continuity,

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{v}) = 0, \tag{9}
\]

where \( \rho \) is henceforth interpreted as the number density,

the equation of state,

\[
\mu(x, t) = \mu_{\text{loc}}[\rho(x, t)] + V_{\text{ext}}, \tag{10}
\]

where \( \mu_{\text{loc}}[\rho(x, t)] \) is the chemical potential in the local density approximation \[26\]. These dynamical equations allow us to show that the symmetry breaking takes place at the nodal surface and yield a correct modified phonon spectrum in CSIBC.

A little algebra with Eqs. (8), (9) and (10) gives the following linearized equations:

\[
S(x, t) = S(x_0, t) + \mathbf{\xi} \cdot \nabla_0 S(x_0, t) \tag{11a}
\]

\[
\mathbf{v}(x, t) = \frac{\partial}{\partial t} \mathbf{\xi} \tag{11b}
\]

\[
\rho(x, t) = \rho(x_0) - \nabla_0 \cdot [\rho(x_0) \mathbf{\xi}] = \rho(x_0) + \delta \rho \tag{11c}
\]

\[
\mu(x, t) = \mu_{\text{loc}}(x_0) + V_{\text{ext}} - \frac{\partial}{\partial \rho}(\mu_{\text{loc}})[\nabla_0 \cdot (\rho_0 \mathbf{\xi})], \tag{11d}
\]
where $p(x_0) = 0$ in BEC and $\nabla_0$ denotes the partial derivative with respect to $x_0$ with $\nabla \to \nabla_0 - \nabla_0 \xi \cdot \nabla_0$. Eq. (11c) was derived by substituting Eq. (11b) to Eq. (9), then integrating over time.

These full equations of motion describe the ensemble of stable particle trajectories from which we may the study of the perturbed particle trajectories as we have done previously for a spherically symmetric condensate [18]. The perturbed particle orbits are obtained by the same method as Weinberg’s analysis of electron orbits [25].

It may also be worth of noticing, though less obviously, that Feynman’s atomic displacements with the back-flow in his intuitive description of phonons in $^4$He are mathematically equivalent to our approach with the Lagrangian displacements and the phase coherence in ODLRO; we arrive at the almost same dispersion relation for a sound (longitudinal) wave. Hence our analysis of collective excitations in BEC is essentially equivalent to that of atomic theory of the two-fluid model of Feynman since both approaches study the ensemble of particle trajectories.

In order to make a contact with a possible future experiment, we make the following assumptions: the condensate is a long and thin cylinder, i.e., $L \gg b$, where $L$ is the length of the condensate, $b$ the radius. We therefore assume $kb \ll 1$, where $k$ is the wave number along the z-axis. We may relax the last assumption, but with considerable amount of additional algebra. We further assume, for simplicity, a cylindrically symmetric condensate and $\lambda = \omega_0^2/\omega_\perp^2 \ll 1$ which is consistent with $L \gg b$, and also $kb \ll 1$. These assumptions reduce the problem effectively to a two-dimensional problem in the cylindrical polar coordinates.

If we take the linear momentum $p_0 = 0$ in BEC, the equation of motion gives

$$\mu_{local}(\rho_0(x_0), t) + V_{ext} = \text{constant} = \mu_{local}[\rho_0(0)] = \mu_0, \quad (12)$$

which shows that the peak density at the center of a trap plays a pivotal role - an essential point, since the speed of (first) sound $c = [4\pi a\rho(0)\hbar^2/\hbar^2/2/M$ must be finite at the center and dissipates to zero giving rise to a surface energy at the nodal surface as we shall see.

Now we are ready to derive the first-order equation of motion in terms of $\xi$, which describes the collective excitations with initial-boundary conditions. To show this we linearize Eq. (8) using Eq. (10), Eqs. (11) and Eq. (12) and the result is

$$\frac{\partial^2}{\partial t^2} \xi = \frac{\mu_0}{M} \nabla \sigma - \omega_0^2 [\xi \cdot \nabla]r + (r \cdot \nabla \xi) - \omega_0^2 [\sigma r + \frac{1}{2}r^2 \nabla \sigma]. \quad (13)$$
Here we have taken \( V_{\text{ext}}(r) = M \omega_0^2 r^2 / 2 \), \( \partial \mu_{\text{loc}} / \partial \rho = 4 \pi \hbar^2 a / M \) \([12]\), \( \sigma = \nabla \cdot \xi \), and \( \omega_0 \) is the radial trap frequency, and have also dropped the subscript in \( \rho_0 \). It is also understood henceforth that \( r = r_\perp \), \( \nabla = \nabla_\perp \), and \( \xi = (\xi_r, \xi_\theta) e^{i m \theta} \).

The entire discussion of collective excitations is based on the first-order equation Eq. (13) which must be solved by initial-boundary conditions \([18]\). We proceed further in two stages. First, we discuss the surface waves on the free surface of CSIBC in a trap and show that the free surface of the BEC in equilibrium behaves like a classical fluid. Secondly, we solve the same equation, but remove the condition of incompressibility to derive the dispersion relation for a compressional wave, an ordinary sound wave (phonons) which shows that the presence of collective excitations with excitation energies that are linear \( \omega = c k \) with respect to both the speed of first sound \( c = (4 \pi a \rho \hbar^2)^{1/2} / M \), and the phonon wave number \( k = k_\theta = m / r \), consistent with the Bogoliubov theory \([6]\).

**IV. DISPERSION RELATION FOR SURFACE WAVES IN CSIBC**

In this section we carry out the first part of the analysis for the collective excitations in CSIBC. The excitation of surface waves on a free surface can be initiated by perturbations of the external trapping potential just like gravity waves on the surface of a fluid in a gravitational field.

For the surface waves, we may impose the following two boundary conditions on \( \xi \) at the free surface of the condensate: one of which is incompressibility of a fluid at the free surface and the other of which is the condition of irrotational motion at the free surface \([3, 20]\). Since we intend to relax the first condition later in the discussion of a compressional wave which is our main objective in this paper, no extensive discussion is necessary. In any case, they are mathematically \( \sigma \equiv \nabla \cdot \xi = 0 \) and \( \omega \equiv \nabla \times \xi = 0 \). These two conditions allow one to rewrite the first-order equation, Eq. (13) as

\[
\frac{\partial^2}{\partial t^2} \xi = -\omega_0^2 \xi - \omega_0^2 (r \cdot \nabla) \xi. \tag{14}
\]

Also, the above two conditions imply that there exists a scalar function \( \chi \) that satisfies the Laplace equation, \( \nabla^2 \chi = 0 \) and \( \xi_s = -\nabla \chi \). Here the subscript \( s \) stands for the surface
waves. The general solution for $\chi$ is given in the cylindrical coordinates by

$$\chi(r, t) = [Q_+^m(t)r^m + Q_-^m(t)r^{-(m)}] e^{im\theta} \tag{15}$$

where we may set $Q_-^m(t) = 0$ for a solid cylindrical condensate.

It is a simple algebra, taking the gradient on Eq. (15) and substituting it into Eq. (14), to obtain the time-dependent equation for $Q_+^m$,

$$\frac{d^2}{dt^2} Q_+^m + m\omega_0^2 Q_+^m = 0, \tag{16}$$

from which one can write down the dispersion relation at once

$$\omega_{\text{surf}}^2 = m\omega_0^2. \tag{17}$$

The dispersion relation for the surface waves is a function of the radial trapping frequency (external force) $\omega_0$ and the mode number $m$. There are two aspects to the dispersion relation. First, the dispersion relation is independent of the the pair-interaction potential (i.e., internal dynamics) of a trapped Bose gas and of the radius of the condensate. A priori, the condition of irrotational flow is assumed to derive the dispersion relation. Since this condition is valid for a surface wave both in a classical fluid and in a superfluid [20, 22], the dispersion relation Eq. (17) is a manifestation of broken symmetry in a Bose system. Moreover, it is independent of $\hbar$ in spite of the quantum ground state density given by Eq. (5) which is a function of $\hbar$. Second, the surface waves are driven solely by the perturbations of an external trapping force on the surface of BEC just like a gravity wave under the action of the force of gravity in a gravitational field [22].

Unlike the spherical wave [18], the dispersion relation Eq. (17) has an interesting geometrical interpretation: for $m = 0$, it corresponds the sausage mode for which the fluid column makes a radial oscillation, but this mode is absent in CSIBC on account of $\omega = 0$; for $m = 1$, the hose mode, the column makes a garden hose like motion without changing the shape of the cross-section. For higher mode $m \gg 1$, $k_\theta = m/b$ corresponds a short waves along the circumference of the column with the radius of $b$. Moreover, it is fairly simple a task to observe the surface waves by applying the perturbation along the waist line in a long ellipsoidal condensate. The experimental data [27] agree well with Eq. (17).

A few final remarks will close this section. First, it is important to note that the dispersion relation for the surface waves in BEC [Eq. (17)] also implies that Osborne’s experimental
observation on the contour of a rotating He II \(^{28}\) remains indeed correct, a point which appears to be quite contrary to what one expects from Landau’s two-fluid model \(^{3}\) and which has been extensively studied \(^{29, 30}\) since Osborne’s first observation on the surface contour of rotating \(^4\)He \(^{28}\) in 1950. In short, the free surface of a superfluid behaves like a classical fluid under the external force; this is a peculiarly universal behavior of a superfluid. It is also evident that Feynman’s vortex line model \(^{30}\) cannot explain the Osborne’s observation as emphasized by Meservey \(^{31}\). Second, Rayfield and Reif \(^{32}\) observed that, apart from the quantization of circulation, a vortex moves like a classical fluid with an ion probe at core, which implies a break-down of superfluidity at a vortex core \(^{34}\). This peculiar universal classical fluid-like behavior of the free surface of CSIBC is essentially in agreement with the breakdown of superfluidity at a nodal surface of a vortex core. This observation also resolves the recent controversy over the Magnus force \(^{35}\). The classical Magnus force employed by Vinen \(^{36}\) in his analysis of vortex quantization is indeed the correct one which gave the value \(\kappa = \oint \mathbf{v}_s \cdot d\mathbf{l}\), where \(\kappa = \hbar / M = 0.997 \times 10^{-3} \text{cm}^2 / \text{sec}\) \(^{32, 33}\).

V. DISPERSION RELATION FOR (FIRST) SOUND WAVES IN CSIBC

One problem that requires the introduction of the collective excitations for its solution is that of the superfluidity of CSIBC. Our aim here is to understand the (first) sound wave propagation and its modified excitation spectrum in CSIBC. This calculation gives the excitation spectrum almost identical to that of Bogoliubov’s phonon spectrum, \(\omega = ck\) with \(c = [4\pi a \rho \hbar^2]^{1/2}/M\) and \(k = m/r\).

It may be useful to recall Feynman’s theory of phonons at this point. In a series of papers \(^{5}\), Feynman has laid out an elegantly simple theory of collective excitations (phonons and rotons) in \(^4\)He based on his path-integral approach to quantum mechanics based on the exact quantum partition function. Feynman has shown that, at temperature sufficiently below the \(\lambda\) point, the ground state wave function has a positive amplitude for any configuration, since the ground state wave function is symmetric and thus has no nodal surface. A density fluctuation involving a large number of particles (or long wavelengths) creates a back-flow by the conservation of particle density; the compressed density in one part of the system is then left with the rarefaction adjacently. Feynman goes on to argue that, with the back-flow
and the conservation of the momentum, the atomic displacements do not lead to a single particle excitation, but to extreme low-energy excitations of compressional waves with no nodes (phonons), since a nodal surface breaks the symmetry property of a Bose system as we have shown in Sec II of this paper. In spite of a simple picture of sound waves in Feynman’s analysis just like Landau’s two-fluid model [3], the speed of (first) sound is not given as a function of the pair-interaction potential which he presented at the outset, but is shown to be identical with the velocity of sound defined by the usual macroscopic considerations [3].

In contrast, the above description of our method implies that, without any reference to a classical fluid, one should be able to explicitly calculate the modified excitation spectrum as a function of the speed of sound and the wave number with the condition of irrotational motion of a superfluid. We now define the compressibility of the fluid as $\sigma = \nabla \cdot \xi$ in Eq. (11c). It is a mathematical description of small-amplitude density waves (phonons) through the equation of continuity and the phase-coherence, and is essentially equivalent to Feynman’s picture of phonons in superfluid He II. The energy spectrum of collective excitations in the phonon regime can be obtained from Eq. (13) by taking the divergence on both sides. With the vector identity $\nabla \cdot (r \cdot \nabla)\xi = r \cdot \nabla \sigma + \sigma$, where $\nabla = \nabla_\perp$ and $r = r_\perp$, and the condition of irrotational motion $\nabla \times \xi = 0$ [3], it is a straightforward algebra to obtain the first-order equation,

$$\frac{\partial^2}{\partial t^2} \sigma(r, t) = \frac{1}{2} \omega_0^2 (\alpha^2 - r^2) \nabla^2 \sigma - 3 \omega_0^2 r \frac{\partial}{\partial r} \sigma - 4 \omega_0^2 \sigma,$$

where $\alpha^2 = \frac{8\pi a n_0(0) h^2}{(M \omega_0)^2}$ and $\nabla^2 = \nabla_\perp^2$.

Writing $\sigma(r, t) = S(t) W(r) e^{im\theta}$, we obtain the variable separated equations:

$$\frac{d^2}{dt^2} S(t) + \lambda_n S(t) = 0.$$  

$$\left(\alpha^2 - r^2\right) \left[ \frac{1}{r} \frac{d}{dr} \left( r \frac{d}{dr} W_n(r) \right) - \frac{m^2}{r^2} W_n(r) \right] \\
- 6r \frac{d}{dr} W_n(r) \\
+ 2\left( \lambda_n/\omega_0^2 - 4 \right) W_n(r) = 0,$$

where $\lambda_n$ is a constant of separation.

The dispersion relation is determined by the eigenvalues of the radial equation, Eq. (20). In the following we show the eigenvalues are a function of the speed of (first) sound and the
radial trap frequency by transforming Eq. (20) to the Sturm-Liouville problem and thereby obtaining the eigenvalues in terms of the complete set of ortho-normal functions,

\[ \dot{\lambda}_n = \int_0^b r(\alpha^2 - r^2)^3 \left[(\frac{d}{dr}W_n)^2 + \frac{m^2}{r^2} W_n^2(r)\right]dr \]  

(21)

and with

\[ \int_0^b r(\alpha^2 - r^2)^2 W_m(r)W_n(r)dr = \delta_{m,n}. \]  

(22)

Since \( \dot{\lambda}_n = 2(\lambda_n/\omega_0^2 - 4) \) and \( b = [8\pi a\rho_0(0)h^2]^{1/2}/(\omega_0 M) \), Eq. (21) shows that the eigenvalues are indeed a function of the speed of (first) sound at the center of the condensate \( i.e., b = [\sqrt{2}/\omega_0]c_{ctr} \) and \( c_{ctr} = [4\pi a\rho(0)h^2]^{1/2}/M \) for all values of the mode number \( m \). To derive Eq. (21), we have tacitly assumed a point source at the center of CSIBC and thus \( \sigma(r) = 0 \) at both \( r = 0 \) and the free surface \( r = b \). Eqs. (21) and (22) show that a sound wave travels from the center, where it has the peak velocity, to the free surface since the speed of the (first) sound wave varies as the density of CSIBC is not uniform. The eigenvalues are also a function of the mode number. The reason for this is of course that the mode number \( m \) is a constant of the motion.

In principle the eigenvalues can be obtained from Eq. (21), but the integral representation is hopelessly complicated to evaluate the eigenvalues. Nevertheless, it shows that the eigenvalues are positive definite and they are indeed a function of the speed of (first) sound \( c_{ctr} \) for all values of the mode number.

It is helpful to look at Eq. (20) in complex plane and to consider Eq. (20) as a linear eigenvalue equation. This can be done easily by mathematical transformation; it also serves a very useful and important purpose to see how the sound wave travels in CSIBC. If we define the complex variable \( W_n(r) = r^{\pm(m)}Z_n(r) \) and with \( x = r^2/\alpha^2 \), then Eq. (20) transforms to a well-known differential equation,

\[ x(1 - x) \frac{d^2}{dx^2} Z_n + [c - (a + b + 1)x] \frac{d}{dx} Z_n - ab Z_n = 0. \]

(23)

This is the Gauss’s hypergeometric equation [37] with \( c = \pm m + 1, a + b = \pm m + 3, \) and \( ab = (1/4) [\dot{\lambda}_n \mp 6m] \).

The advantage of this transformation is to calculate the eigenvalues by means of a simple numerical method. Moreover, there are none of the end-point singularity problems associated
with $W_m(r)$ that one encounters in the integral representation Eq. (21). And the various numerical methods [38] are available to compute the eigenvalues.

One further point must be made with regard to Eq. (23). The hypergeometric equation Eq. (23) has regular singular points at $x = 0$, $x = 1$ and $x = \infty$. Its solution is the hypergeometric function which is analytic in the complex plane with a cut from 1 to $\infty$ along the real axis. The branch point $x = 1$ corresponds to the location of the free surface of the fluid column of CSIBC. Parenthetically, we also note that the analyticity of the complex variable reflects the fact that a sound wave can travel only inside of the free surface as it must.

Since the present work critically depends on the numerical accuracy, we may pause to examine to what extent one can take the numerical solutions to the eigenvalue problem. One question that arises is whether or not it is possible to see the physical significance of the cut in the complex plane. To better understand the analyticity, it should be emphasized that $\sigma = \nabla \cdot \xi$ has been interpreted as an amplitude of a compressional wave (i.e., collective mode) in CSIBC in which a cylindrically symmetric sound wave travels from the center toward the free surface. The presence of the cut implies that there cannot be a sound wave beyond the branch point.

The collective solution exists in a domain in which $\sigma \neq 0$. Such a solution must be analytic in the complex plane. To check this we must be sure a solution to Eq. (23) is compatible with Eq. (21). It is physically obvious that a sound wave can travel in the domain $[0, 1]$ in which the solution to Eq. (23) is analytic. The branch point reflects the nodal surface at which the sound wave must dissipate to conserve the total energy of an isolated system giving rise a surface energy. Thus our numerical solutions in the domain $[0, 1]$ will always satisfy this physical condition; it is a unique feature of finite systems.

Here we employ a simple numerical method [39] to evaluate the smallest eigenvalues corresponding to the low-energy phonon excitations in the domain $[0, 1]$. This simple numerical method should, however, be quite satisfactory provided the solutions are numerically stable near the branch point [38].

Returning to Eq. (19) and taking $S(t) = e^{i\omega t}$, we obtain the dispersion relation as

$$\omega_{ph} = \pm (\lambda_n)^{1/2} = \pm [2\hat{\lambda}_s + 4]^{1/2} \omega_0. \quad (24)$$

where $\hat{\lambda}_s$ is the smallest eigenvalues from Eq. (23). The ratio $\omega/\omega_0$ with respect to the mode
number $m$ is plotted in Fig.1.

The dispersion relation Eq. (21) (Fig. 1.) shows that for large $m$, $\omega/\omega_0$ is linear with respect to the mode number $m$ (or wave number $k_{\theta} = m/r$). It is almost exactly the same form of the usual dispersion relation $\omega = ck$ of a (first) sound wave with $k_{\theta} = m/r$. This has the following simple interpretation: for large values of the mode number $m$, a wave, whose wavelength is shorter than the size of CSIBC but longer than atomic dimensions, sees the medium as if it is an infinite medium. Thus we recover in an essential manner the Bogoliubov dispersion relation $\omega = ck$ at long wavelengths. This assures us that in fact the present analysis describes the collective excitations correctly at temperature near absolutely zero in CSIBC. For small values of the mode number, the dispersion curve is slightly parabolic with respect to the mode number due to geometrical effects-finite size effects. For $m = 0$ mode, $\omega/\omega_0 = 2.8153$ is a unique value in a finite space problem. It corresponds to a uniform perturbation in polar angle in CSIBC.

As in the case of a spherically symmetric nonuniform condensate, the whole problem of collective excitations in CSIBC is reduced to the question of how one measures the speed of first sound $c_{ctr} = [4\pi an(0)h^2]^{1/2}/M$ (or the peak density) for an outgoing cylindrical sound wave at various radial points in CSIBC. This requires a precise measurement of the peak density, a prerequisite for an experimental confirmation of BEC in a trap.

There is one more point to be made about observing the phonon spectrum in a spatially inhomogeneous condensate. To understand the difficulty, let us take the simplest case $m = 0$ mode. Unlike the surface waves, it is important to perturb the medium at the center to initiate a (first) sound wave uniformly in polar angle. How, then, if we were to confirm BEC in CSIBC by studying the radial propagation of the sound wave, might we try to initiate short waves at the center and to measure the speed of sound wave at $r < b$. As far as we can see, this is a difficult task; it is, nonetheless, essential to see if we can initiate the sound waves at the center and place a probe at $r < b$, inside the nodal surface, since the the sound waves dissipate near the nodal surface as discussed above. The question of whether a systematic perturbation consistent with the present theory is experimentally feasible remains to be seen. Nevertheless, it would be necessary to have a precise dispersion curve on the collective excitations in CSIBC similar to that of Henshaw and Woods in He II. And it is still the only way to quantitatively establish the realization of BEC in a trap.

There is one last point to be made about the break-down of superfluidity in BEC at
the free surface. To understand this, let us consider what has happened to the sound wave propagating toward the free surface. As it approaches the free surface, the particle trajectories rapidly fluctuate due to the effective quantum mechanical potential $U_{eqmp}$ which diverges as $M\omega_0^2\mu_0/D^2$ as $D = [\mu_0 - (1/2)M\omega_0^2r^2] \to 0$ near the free surface \[14\]. This fluctuation breaks up the phase coherence of the sound wave which dissipates at the free surface giving rise a surface energy.

The important mechanism for the symmetry breaking is that the sound wave dissipates by the interaction with the particles in the surface layer which is ensured by the fact that the initial state and the final state of CSIBC always have the same energy. Therefore, the phenomenon of the fluctuation of particles due to $U_{eqmp}$ and the dissipation of sound waves at the surface layer can be understood in terms of Kubo’s fluctuation-dissipation theory \[40\]. One of our main objectives of this paper is to understand how the symmetry of the ground state wave function breaks down at the free surface, i.e., a nodal surface. It is natural to identify this underlying basic mechanism for the symmetry breaking as a spontaneously broken symmetry at the free surface which accompanies phonons as Nambu-Goldstone bosons \[16, 17\].

VI. CONCLUSION AND DISCUSSION

I should like to close with one final remark on the symmetry breaking in an isolated BEC in trap. By describing the perturbation as a Lagrangian coordinate, our analysis on the dynamical equations is a accurate quantum mechanical treatment for a many-body problem just as that of the quantum field theory method by Bogoliubov for a uniform system \[6\], provided a correct quantum ground state function Eq. (5) is given. Not only does the analysis correctly predicts the low-energy excitation spectrum for phonons, consistent with the Bogoliubov spectrum but also shows the correct low-energy spectrum for the surface waves in a finite inhomogeneous system confined by the external potential.

The basis of our reasoning for the symmetry breaking at the nodal surface follows from the observation that, in spite of the quantum ground state given by Eq. (5), the dispersion relation for the surface wave Eq. (17) is independent of $\hbar$ but not for the phonon spectrum of a superfluid Eq. (24). This is a definite proof that the gauge symmetry of a Bose system is broken at the free surface. In the phonon regime the Bogoliubov dispersion relation $\omega = ck$
holds in CSIBC with the small geometrical corrections. One also notices from Fig. 1 that there is a striking similarity between the dispersion curves of the spherical and the cylindrical condensates, although the angular momentum $\ell$ and the mode number $m$ have an entirely different meaning [25]. The salient point in our results is that the modified energy spectrum for the phonons is a function of the speed of first sound, $c_{\text{ctr}} = [4\pi a n(0)\hbar^2]^{1/2}/M$ (or the s-wave scattering length). As Bogoliubov emphasized in his work on a uniform Bose gas at zero temperature [6], the weakly repulsive pair-interaction potential indeed plays a crucial role for the structure of the ground state and the superfluidity [6].

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FIG. 1: The ratio $\omega/\omega_0 = \lambda^{1/2} = [2\lambda + 4]^{1/2}$ is plotted as a function of the mode number $m$. It shows how the energy spectrum of phonons varies with the mode number.