Infinite-dimensional 3-algebra and integrable system

Min-Ru Chen\textsuperscript{a,b,1}, Shi-Kun Wang\textsuperscript{c,2}, Ke Wu\textsuperscript{a,3} and Wei-Zhong Zhao\textsuperscript{a,d, 4}

\textsuperscript{a}School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
\textsuperscript{b}College of Mathematics and Information Sciences, Henan University, Kaifeng 475001, China
\textsuperscript{c}Institute of Applied Mathematics, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China
\textsuperscript{d}Institute of Mathematics and Interdisciplinary Science, Capital Normal University, Beijing 100048, China

Abstract

The relation between the infinite-dimensional 3-algebras and the dispersionless KdV hierarchy is investigated. Based on the infinite-dimensional 3-algebras, we derive two compatible Nambu Hamiltonian structures. Then the dispersionless KdV hierarchy follows from the Nambu-Poisson evolution equation given the suitable Hamiltonians. We find that the dispersionless KdV system is not only a bi-Hamiltonian system, but also a bi-Nambu-Hamiltonian system. Due to the Nambu-Poisson evolution equation involving two Hamiltonians, more intriguing relationships between these Hamiltonians are revealed. As an application, we investigate the system of polytropic gas equations and derive an integrable gas dynamics system in the framework of Nambu mechanics.

KEYWORDS: 3-algebra, Integrable system, Nambu mechanics.

PACS numbers: 02.30.Ik, 02.20.Sv, 11.25.Hf

\textsuperscript{1}cmr@henu.edu.cn
\textsuperscript{2}wsk@amss.ac.cn
\textsuperscript{3}wuke@mail.cnu.edu.cn
\textsuperscript{4}Corresponding author: zhaowz100@163.com
1 Introduction

Infinite-dimensional algebras have attracted a lot of interest from physical and mathematical points of view. They have been extensively studied in the past several decades. In the context of integrable systems, the infinite-dimensional algebras play a very important role. The Virasoro algebra is an important infinite-dimensional algebra. The intriguing relation between the Virasoro algebra and the Korteweg-de Vries (KdV) equation was found by Gervais and Neveu [1, 2]. They considered the equivalent Poisson brackets of the Fourier transform field (FTF) corresponding to the classical Virasoro algebra and derived the KdV equation with respect to the FTF from the Poisson evolution equation given a suitable Hamiltonian. The similar construction for the super Virasoro algebra and super KdV equation has also been exploited [3]-[8]. As a higher-spin extension of the Virasoro algebra, the $W_\infty$ algebra has been found to be associated to the generalized KdV hierarchies [9, 10].

To construct the generalized Hamiltonian dynamics, Nambu [11] first proposed the notion of 3-bracket. Thus Nambu dynamics is described by the phase flow given by Nambu-Hamilton equations of motion which involves two Hamiltonians. Bayen and Flato [12] pointed out that in the classical case Nambu mechanics is equivalent to a singular Hamiltonian mechanics. The connection between Nambu mechanics and conventional Hamiltonian ideas has also been explored by Mukunda and Sudarshan [13]. A notion of a Nambu 3-algebra was introduced in [14] as a natural generalization of a Lie algebra for higher-order algebraic operations. Recently a world-volume description of multiple M2-branes using the 3-algebra was proposed independently by Bagger and Lambert [15], and Gustavsson [16] (BLG). Since then, great attention has been paid to the BLG theory and 3-algebras [17]-[24]. The study of the infinite-dimensional 3-algebras has made big progress in recent years. Curtright et al. [25] constructed a 3-bracket variant of the Virasoro-Witt (V-W) algebra through the use of $su(1,1)$ enveloping algebra techniques. The Kac-Moody 3-algebra was investigated by Lin [26]. More recently, the $w_\infty$ 3-algebra which satisfies the fundamental identity (FI) condition of 3-algebra was derived from two different ways. Chakrabortty et al. [27] gave a $w_\infty$ 3-algebra by applying a double scaling limits on the generators of the $W_\infty$ algebra. Chen et al. [28] started from the investigation of the high-order V-W 3-algebra where the parameter $\hbar$ is introduced in the generators. By requiring the classical limit $\{ , , \} = \lim_{\hbar \to 0} \frac{1}{\hbar}[ , , ]$ to hold, the $w_\infty$ 3-algebra follows from the high-order V-W 3-algebra.

We have mentioned above the role of infinite-dimensional algebras in the integrable systems. An interesting open question is whether there is a relation between the infinite-dimensional 3-algebras and the integrable equations. The goal of this paper is to establish this kind of relation and apply it to the physics system. We shall present two new infinite-dimensional 3-algebras which are the $SDiff(T^2)$ and classical Heisenberg 3-algebras and show how these two 3-algebras and already known $w_\infty$ 3-algebra are related to the dispersionless KdV hierarchy. More intriguing properties related to the dispersionless KdV hierarchy will be revealed.

This paper is organized as follows. In section 2, we investigate the infinite-dimensional 3-algebras. In section 3, we discuss the relation between the infinite-dimensional algebras and the dispersionless KdV hierarchy in the framework of Hamiltonian mechanics. Nambu mechanics is a generalization of classical Hamiltonian mechanics. In section 4, we establish the relation between the infinite-dimensional 3-algebras and the dispersionless KdV system in the framework of Nambu mechanics. We point out that the dispersionless KdV system is not only a bi-Hamiltonian system, but also a bi-Nambu-Hamiltonian system. An application in the gas dynamics system is given in section 5. We end this paper with the concluding remarks in section 6.
The Nambu mechanics [11] is a generalization of classical Hamiltonian mechanics, where the Nambu 3-bracket is defined for a triple of classical observables on the three-dimensional phase space $\mathbb{R}^3$ with coordinates $\dot{x}, \dot{y}, \dot{z}$ by the following formula:

$$\{f, g, h\} = \frac{\partial (f, g, h)}{\partial (\dot{x}, \dot{y}, \dot{z})} = \frac{\partial f}{\partial \dot{x}} \left( \frac{\partial g}{\partial \dot{y}} \frac{\partial h}{\partial \dot{z}} - \frac{\partial h}{\partial \dot{y}} \frac{\partial g}{\partial \dot{z}} \right)$$

$$+ \frac{\partial g}{\partial \dot{x}} \left( \frac{\partial h}{\partial \dot{y}} \frac{\partial f}{\partial \dot{z}} - \frac{\partial f}{\partial \dot{y}} \frac{\partial h}{\partial \dot{z}} \right) + \frac{\partial h}{\partial \dot{x}} \left( \frac{\partial f}{\partial \dot{y}} \frac{\partial g}{\partial \dot{z}} - \frac{\partial g}{\partial \dot{y}} \frac{\partial f}{\partial \dot{z}} \right),$$

which satisfies the properties of skewsymmetry, the Leibniz rule and the following FI condition:

$$\{\{f, g, h, i\} + \{f, g, i, h\} + \{g, h, f, i\} + \{h, i, f, g\} + \{i, f, g, h\}, \{f, g, h\}\} = 0.$$  \(\text{(5)}\)

The generators of $w_\infty$ 3-algebra are given by

$$L^r_m = \sqrt{z} \exp[(r - \frac{1}{2})\dot{x} - 2m\dot{y}],$$

where $m, r \in \mathbb{Z}$. The generators (3) can be identified with the modes of the deformations of 2-torus [27]. Substituting the generators (3) into (1), we obtain the $w_\infty$ 3-algebra

$$\{L^r_m, L^s_n, L^h_k\} = (h(n - m) + s(m - k) + r(k - n))L^{r+s+h-1}_{m+n+k}.$$  \(\text{(4)}\)

It is well-known that the Nambu bracket structure of order $n$ on phase space induces infinite family of subordinated Nambu structures of orders $n - 1$ and lower, including the family of Poisson structures [14]. Let us define the Poisson bracket parametrized by $L^0_0$ as $\{L^r_m, L^s_n\} = \{L^r_m, L^s_n, L^0_0\}$. This parametrized Poisson bracket satisfies the Jacobi identity, it can be regarded as the generalized $w_\infty$ algebra

$$\{L^r_m, L^s_n\} = (ms - nr)L^{r+s-1}_{m+n},$$  \(\text{(5)}\)

where $m, n, r, s \in \mathbb{Z}$. When $r = s = 1$, (5) leads to the well-known V-W algebra

$$\{L_m, L_n\} = (m - n)L_{m+n}.$$  \(\text{(6)}\)

It should be pointed out that the $w_\infty$ algebra [29] is equivalent to the algebra of smooth area-preserving diffeomorphisms of the cylinder $S^1 \times R^1$. Its Poisson bracket algebra is also given by (5), but the superscripts $r, s$ in generators are the conformal spin with $r, s \geq 1$.

Let us take

$$T^r_m = \sqrt{z} \exp[\sqrt{2}m\dot{x} + \sqrt{2}r\dot{y}],$$  \(\text{(7)}\)

where $m, r \in \mathbb{Z}$. Substituting the generators (7) into (1), we obtain the following infinite-dimensional 3-algebra:

$$\{T^r_m, T^s_n, T^h_k\} = (h(n - m) + s(m - k) + r(k - n))T^{r+s+h}_{m+n+k}.$$  \(\text{(8)}\)

From (8), we have

$$\{T^r_m, T^s_n\}T^0_0 = \{T^r_m, T^s_n, T^0_0\} = (ms - nr)T^{r+s}_{m+n}.$$  \(\text{(9)}\)

Note that (9) is nothing but a subalgebra of the centerless algebra of diffeomorphisms of the torus [30, 31]. Thus we call (8) the $SDiff(T^2)$ 3-algebra.
3 Dispersionless KdV system

The dispersionless KdV system is an important integrable model which has many applications in physics, such as unstable fingering patterns of two-dimensional flows of viscous fluids [32], dynamics of interacting cold atomic gases [33] and polytropic gas dynamics [34]. The infinite number of conserved charges $\hat{H}_n$ of the dispersionless KdV system [35, 36] are given by

$$\hat{H}_n = \frac{(2n - 3)!!}{(2n - 2)!!n} \int_0^{2\pi} u^n(x) dx,$$  \hspace{1cm} (10)

where $n \in \mathbb{Z}_+$ and $(-1)!! = 0!! = 1$.

Let us introduce the FTF as follows:

$$u(x) = -i \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} L_m e^{-imx},$$  \hspace{1cm} (11)

where $i = \sqrt{-1}$, the field $u(x)$ is periodic in $x$ with period $2\pi$ and has continuous $x$ derivative. Thus the Poisson bracket realization of the V-W algebra (6) can be expressed as

$$\{u(x), u(y)\}_2 = u_x \delta(x - y) + 2u \delta_x(x - y),$$  \hspace{1cm} (12)

where the subscript $x$ denotes the derivative with respect to the variable $x$. By means of (12), it can be shown that the Hamiltonians (10) are in involution, i.e., $\{\hat{H}_n, \hat{H}_m\}_2 = 0$. Substituting the Hamiltonians (10) into the Poisson evolution equation

$$\frac{\partial u(t, x)}{\partial t} = \{u, \hat{H}_n\}$$  \hspace{1cm} (13)

and using the Poisson bracket (12), we obtain the dispersionless KdV hierarchy

$$\frac{\partial u(t, x)}{\partial t} = A_{n-1} u^{n-1} u_x$$

where $A_n$ is given by

$$A_n = \frac{(2n + 1)!!}{(2n)!!}. \hspace{1cm} (15)$$

It has been well known that integrability of many systems is closely related to their bi-Hamiltonian property [37]. The dispersionless KdV system is a bi-Hamiltonian system. One
can associate it to the two kinds of Poisson brackets called the first and second Hamiltonian structures. We note that the second Hamiltonian structure (12) of the dispersionless KdV system can be derived from the classical V-W algebra. To give its first Hamiltonian structure, let us turn to another infinite-dimensional algebra, i.e., Heisenberg algebra. Its Poisson bracket realization [38] is given by

$$\{a_m, a_n\} = m\delta_{m+n,0}. \quad (16)$$

Taking $u(x) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{+\infty} a_m e^{-imx}$, then we can rewrite (16) as

$$\{u(x), u(y)\}_1 = 2\delta_x (x - y). \quad (17)$$

It is worth noting that (17) is the first Hamiltonian structure of the dispersionless KdV system [35]. Based on (17), it is not difficult to derive the dispersionless KdV hierarchy (14) from the Poisson evolution equation (13). By means of (12) and (17), we obtain the ratio between $\{u, \hat{H}_m\}_2$ and $\{u, \hat{H}_m\}_1$ as follows:

$$\frac{\{u, \hat{H}_m\}_2}{\{u, \hat{H}_m\}_1} = \frac{2m - 1}{2m - 2} e^{-iry}. \quad (18)$$

4 Infinite-dimensional 3-algebras and Dispersionless KdV system

In the previous section, we have investigated the relation between the infinite-dimensional algebras and the dispersionless KdV system. We turn our attention back to the case of the infinite-dimensional 3-algebras. Let us introduce the FTF $u(x, y)$ as follows:

$$u(x, y) = \frac{1}{4\pi^2} \sum_{m,r} L_{m,r} e^{-imx - iry}. \quad (19)$$

It is assumed that $u$ has continuous $x$ and $y$ derivatives and satisfies the periodic boundary condition $u(t, x + 2k\pi, y + 2l\pi) = u(t, x, y), (k, l \in \mathbb{Z})$. Using (19), we can rewrite $w_\infty$ 3-algebra (4) as the following Nambu 3-bracket relation of the FTF:

$$\{u(x_1, y_1), u(x_2, y_2), u(x_3, y_3)\}_2 = [u_{x_1}(x_1, y_1)\delta(x_1 - x_2)\delta(x_1 - x_3)\delta(y_1 - y_2)\delta(y_1 - y_3) - u_{x_1}(x_1, y_1)\delta(x_1 - x_2)\delta(y_1 - y_2)\delta(y_1 - y_3)]e^{-iry}.$$  

where the subscripts $x_i$ and $y_i$ throughout this paper denote the partial derivatives with respect to the variables $x_i$ and $y_i$, respectively.

Let us consider the following extended Hamiltonians of (10) with respect to the variable $y$:

$$H_n = \frac{(2n - 3)!!}{(2n - 2)!!} \int_0^{2\pi} \int_0^{2\pi} u^n(x, y) dx dy, \quad (21)$$
where \( u(x, y) = \dot{u}(x)v(y) \).

By means of (20), we find that the Hamiltonians (21) are in involution with the Nambu 3-bracket structure, i.e., \( \{ H_k, H_m, H_n \}_2 = 0 \). In the framework of Nambu mechanics, the Nambu-Hamilton equations of motion involve two Hamiltonians. Let us consider the following Nambu-Poisson (N-P) evolution equation:

\[
\frac{\partial u(t, x, y)}{\partial t} = \{ u, H_m, H_n \},
\]  

(22)

where \( m, n \in \mathbb{Z}_+ \) and \( m \neq n \). Substituting the Hamiltonians (21) into (22) and using the Nambu 3-bracket relation (20), we obtain

\[
\frac{\partial u(t, x, y)}{\partial t} = \{ u, H_m, H_n \}_2 = S_{mn}A_{m+n-2}\hat{u}^{m+n-2}u_x,
\]

(23)

where \( S_{mn} = \frac{(m-n)(2m-3)(2m-4)n}{(2m-2)(2m-2)!!}ie^{-iy} \). Let us take \( \hat{t}_{mn} = S_{mn}v^{m+n-2}(y)t \) and require \( \frac{\partial \hat{v}(y)}{\partial t} = 0 \), we obtain the dispersionless KdV hierarchy

\[
\frac{\partial \hat{u}(t, x)}{\partial \hat{t}_{mn}} = A_{m+n-2}\hat{u}^{m+n-2}u_x
\]

(24)

from (23), where \( m + n \geq 3 \). Note that (24) matches with the dispersionless KdV hierarchy (14) except for the first equation in (14).

Let us consider the FTF (19) with substitution of the generators \( T^r_m \) of \( SDiff(T^2) \) 3-algebra for \( L^r_m \) of \( w_\infty \) 3-algebra, it is easy to rewrite the \( SDiff(T^2) \) 3-algebra as the Nambu 3-bracket structure of the FTF \( u \). Then based on this equivalent Nambu 3-bracket relation, after a straightforward calculation, we find \( \frac{\partial u(t, x, y)}{\partial t} = \{ u, H_m, H_n \} = 0 \).

The central extensions of the infinite-dimensional algebras have been well investigated. The non-trivial central extensions of various infinite-dimensional algebras have been determined, such as \( w_\infty \) and \( W_\infty \) algebras [29, 30] and \( SDiff(T^2) \) algebra [30, 31]. An open question is whether there exist the non-trivial central extensions for the infinite-dimensional 3-algebras. Even though we do not know the non-trivial central extension of \( SDiff(T^2) \) 3-algebra (8), nevertheless it still admits the trivial central extensions.

Let us consider the \( SDiff(T^2) \) 3-algebra (8) with the following trivial central extension:

\[
\{ T^r_m, T^s, T^h_k \} = [h(n - m) + s(m - k) + r(k - n)](T^r_{m+n+k} + 4\pi^2\delta_{m+n+k,0}\delta_{r+s+h,1})
\]

\[
= [h(n - m) + s(m - k) + r(k - n)]T^r_{m+n+k}
\]

\[
+ 4\pi^2[3(ms - nr) + n - m]\delta_{m+n+k,0}\delta_{r+s+h,1}.
\]

(25)

It is easy to verify that (25) satisfies the FI condition (2) and skewsymmetry. By means of the FTF \( u(x, y) = \frac{1}{4\pi^2}\sum_{m,r} T^r_m e^{-imx-iry} \), we may rewrite (25) as

\[
\{ u(x_1, y_1), u(x_2, y_2), u(x_3, y_3) \}_1
\]

\[
= [u_{x_1}(x_1, y_1)\delta(x_1 - x_2)\delta(y_1 - y_2)\delta(y_1 - y_3) - u_{x_2}(x_1, y_1)\delta(x_1 - x_2)]
\]

\[
\delta(x_1 - x_3)\delta(y_1 - y_3) + u_{y_1}(x_1, y_1)\delta(x_1 - x_2)\delta(x_1 - x_3)\delta(y_1 - y_2)
\]

\[
\delta(y_1 - y_3) - u_{y_2}(x_1, y_1)\delta(x_1 - x_2)\delta(x_1 - x_3)\delta(y_1 - y_2)\delta(y_1 - y_3) + 3u(x_1, y_1)
\]

\[
\delta(x_1 - x_2)\delta(x_1 - x_3)\delta(y_1 - y_2)\delta(y_1 - y_3) - 3u(x_1, y_1)\delta(x_1 - x_2)\delta(x_1 - x_3)
\]

\[
\delta(y_1 - y_2)\delta(y_1 - y_3) + A,
\]

(26)
where $A$ is the contribution of the trivial central extension term and given by

$$A = e^{-i\gamma} \left[ 3\delta(x_1 - x_2)\delta_x(x_1 - x_3)\delta_{y_1}(y_1 - y_2)\delta(y_1 - y_3) - 3\delta_{x_1}(x_1 - x_2)
\delta(x_1 - x_3)\delta(y_1 - y_2)\delta_{y_1}(y_1 - y_3) + i\delta_{x_1}(x_1 - x_2)\delta_x(x_1 - x_3)\delta(y_1 - y_2)\right.$$  
$$\delta(y_1 - y_3) - i\delta(x_1 - x_2)\delta_{x_1}(x_1 - x_3)\delta(y_1 - y_2)\delta(y_1 - y_3)].$$  

(27)

Due to the contribution of this trivial central extension term, the N-P evolution equation of the FTF $u$ becomes

$$\frac{\partial u(t, x, y)}{\partial t} = \{u, H_m, H_n\}_1 = \tilde{S}_{mn}A_{m+n-3}u^{m+n-3}ux,$$

(28)

where $\tilde{S}_{mn} = (m-n)(2m-3)!!(2n-3)!!(2m+2n-6)!!/(2m-2)!!(2n-2)!!(2m+2n-5)!!$, $\text{ie}^{-iy}$. By taking $\dot{i}_{mn} = \tilde{S}_{mn}v^{m+n-3}(y)t$ and $\frac{\partial v(y)}{\partial t} = 0$, (28) leads to the dispersionless KdV hierarchy (24). We also observe that the Hamiltonians (21) satisfy $\{H_k, H_m, H_n\}_1 = 0$.

After understanding the crucial importance of this trivial central extension, we may only consider its contribution. Thus we have

$$\{u(x_1, y_1), u(x_2, y_2), u(x_3, y_3)\}_1 = e^{-i\gamma} \left[ 3\delta(x_1 - x_2)\delta_x(x_1 - x_3)\delta_{y_1}(y_1 - y_2)\delta(y_1 - y_3) - 3\delta_{x_1}(x_1 - x_2)
\delta(x_1 - x_3)\delta(y_1 - y_2)\delta_{y_1}(y_1 - y_3) + i\delta_{x_1}(x_1 - x_2)\delta_x(x_1 - x_3)\delta(y_1 - y_2)\right.$$  
$$\delta(y_1 - y_3) - i\delta(x_1 - x_2)\delta_{x_1}(x_1 - x_3)\delta(y_1 - y_2)\delta(y_1 - y_3)].$$  

(29)

Based on (29), we may directly derive (28). Comparing (23) with (28), we have $\{u, H_m, H_n\}_1 = u$. The form of this ratio is the same as (18) except for the coefficient. Introducing $u(x, y) = \frac{1}{\sqrt{4\pi t}} \sum_{m,r} a^r_m e^{-imx-iry}$, we obtain the following 3-algebra from (29):

$$\{a^r_m, a^s_n, a^h_k\} = \frac{1}{4} \left[ 3(ms - nr) + (n - m) \right] \delta_{m+n+k,0} \delta_{r+s+h,1}.$$

(30)

When $r = s = 1$, $k = 0$ and $h = -1$, (30) reduces to the classical Heisenberg algebra (16). For this reason we call (30) the classical Heisenberg 3-algebra.

We have associated the dispersionless KdV system to the two kinds of Nambu 3-bracket which are (29) and (20) called the first and second Nambu Hamiltonian structures, respectively. These Nambu Hamiltonian structures can be derived from the classical Heisenberg and $w_\infty$ 3-algebras, respectively. It can be very easily observed that their sum is again a Nambu Hamiltonian structure. It implies that these two Nambu Hamiltonian structures are compatible. Thus the dispersionless KdV system is not only a bi-Hamiltonian system, but also a bi-Nambu-Hamiltonian system.

For the Hamiltonians (21), we note that they are in involution with the first and second Nambu Hamiltonian structures, respectively. To achieve a better understanding of the relationship between these Hamiltonians, let us consider the Hamiltonian pairs $(H_n, H_m)$ and $(H_k, H_l)$ with $n + m = k + l$. We find that the ratios between $\{u(x, y), H_m, H_n\}_i$ and $\{u(x, y), H_k, H_l\}_i$, $i=1,2$, are

$$\frac{\{u(x, y), H_m, H_n\}_1}{\{u(x, y), H_k, H_l\}_1} = \phi \frac{2m - k - l}{k - l},$$

(31)

where $\phi$ is given by the hypergeometric function

$$\phi = 5F_4 \left[ \begin{array}{c} \frac{1}{2}m + \frac{1}{2}, \ m - 1, \ m - k, \ m - l, \ \frac{1}{2} \\
\frac{1}{2}m - \frac{1}{2}, \ k, \ l, \ m - \frac{1}{2} \end{array} \right].$$

(32)
Although the ratios (31) are not equal to one, we note that the corresponding N-P evolution equation may lead to the same dispersionless KdV equation with the appropriate rescaling. It implies that there is an intrinsic equivalent relation between these Hamiltonian pairs. When $s = m + n$ is even and odd for the Hamiltonian pairs $(H_m, H_n)$, for a given Nambu Hamiltonian structure, there are $\frac{s-2}{2}$ and $\frac{s-1}{2}$ different Hamiltonian pairs which may derive to the same dispersionless KdV equation from the N-P evolution equation, respectively. This can be regarded as the generic property of the dispersionless KdV hierarchy from the viewpoint of Nambu mechanics.

The Nambu 3-bracket can also be generalized to the bracket of $n$ functions $f_i = f_i(\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)$ defined by [14]

\[ \{f_1, f_2, \cdots, f_n\} = \frac{\partial (f_1, f_2, \cdots, f_n)}{\partial (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n)}. \tag{33} \]

Substituting the generators (3) and (7) into (33), we obtain the $w_\infty$ and $SDiff(T^2)$ n-algebras as follows:

\[ \{L_{\tau_1}^{m_1}, L_{\tau_2}^{m_2}, \cdots, L_{\tau_n}^{m_n}\} = \{T_{\tau_1}^{m_1}, T_{\tau_2}^{m_2}, \cdots, T_{\tau_n}^{m_n}\} = 0, \tag{34} \]

where $n \geq 4$. Due to these two null n-algebras, it is not difficult to verify that the Hamiltonians (21) satisfy

\[ \{H_{i_1}, H_{i_2}, \cdots, H_{i_n}\} = 0, \tag{35} \]

and the generalized N-P evolution equation is

\[ \frac{\partial u(t, x, y)}{\partial t} = \{u, H_{i_1}, H_{i_2}, \cdots, H_{i_{n-1}}\} = 0. \tag{36} \]

5 Application

As an application, let us consider the 2+1 dimension isentropic polytropic gas dynamics equation [39]

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + (\rho u)_x + (\rho v)_y &= 0, \\
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + k\gamma \rho \gamma^2 \rho_x &= F_1, \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + k\gamma \rho \gamma^2 \rho_y &= F_2,
\end{align*}
\tag{37}
\]

where $\rho(t, x, y)$ is the gas density, $(u, v)$ is the velocity vector of gas flow, $(F_1, F_2)$ is the body force vector per unit mass, $\gamma$ and $k$ are the constants, $\gamma$ denotes the ratio of the specific heats. The system of polytropic gas equations has attracted a great deal of attention due to its wide applications in physics. Let us focus on a special gas motion with $\gamma = 3$, $v = 0$, $F_2 = 3k\rho\rho_y$ and $F_1 = 0$, thus (37) becomes

\[
\begin{align*}
\frac{\partial \rho(t, x, y)}{\partial t} &= -(\rho u)_x, \\
\frac{\partial u(t, x, y)}{\partial t} &= -\frac{1}{2}(u^2 + 3k\rho^2)_x.
\end{align*}
\tag{38}
\]
Under the dimensional reduction of $y = 0$, (38) reduces to the well-investigated gas dynamics equation in $1 + 1$ dimensions, which admits two infinite sets of conserved charges, i.e., Eulerian and Lagrangian conserved charges, respectively [40, 41].

Let us take

$$u = -\frac{1}{4}ie^{-iy}(A_+ + A_-),$$
$$\rho = -\frac{1}{4\sqrt{3k}}ie^{-iy}(A_+ - A_-).$$

(39)

Thus (38) can be rewritten as

$$\frac{\partial A_\pm(t, x, y)}{\partial t} = i^{2}e^{-iy}A_\pm A_{\pm x}. \quad (40)$$

We immediately recognize that (40) is nothing but the first equation of the hierarchy (23). It implies that we may derive (38) in the framework of Nambu mechanics. To achieve this result, we take the Hamiltonians

$$H_{\pm n} = \frac{(2n - 3)!!}{(2n - 2)!!n} \int \int A^n_{\pm}(x, y)dx\,dy$$
$$= \frac{(2n - 3)!!}{(2n - 2)!!n} \int \int (2i)^n e^{iy(2\pm 1)\sqrt{3k\rho})^n dx\,dy. \quad (41)$$

When $y = 0$, the Hamiltonians (41) are the combination of the Eulerian and Lagrangian conserved charges of $1 + 1$ dimensional gas dynamics equation.

Not as the Poisson evolution equation, the N-P evolution equation involves two Hamiltonians. Let us focus on the Hamiltonian pairs $(H_{\pm 1}, H_{\pm 2})$ and $(H_{\pm 1}, H_{\pm 3})$. Substituting these two Hamiltonian pairs into the N-P evolution equation and using the Nambu Hamiltonian structures (20) and (29) with substitution of $A_{\pm}(x, y)$ for $u(x, y)$, respectively, we may derive (40). From applying (20) and (29), we see that the classical Heisenberg and $w_\infty$ 3-algebras play an crucial role in the derivation of (40). Moreover the deep insight into the relationship between $H_{\pm i}, i = 1, 2, 3$, has been achieved. Since the isentropic gas dynamics equation (38) can be expressed as the two non-interacting dispersionless KdV equations (40), its solution can be obtained through the implicit solution of (40) $A_{\pm}(t, x, y) = \phi_{\pm}(\frac{1}{2}ie^{-iy}A_{\pm t+x})\psi_{\pm}(y)$, where $\phi_{\pm}$ and $\psi_{\pm}$ are the arbitrary differentiable functions. Due to no dispersive term in (40), it can be observed a transition of wave shape from conservative to dissipative behaviour. A remarkable feature of the system (38) is that the body force $F_2$ should depend on the gas density such that the velocity along the $y$ direction is zero.

6 Concluding Remarks
We have investigated the relation between the infinite-dimensional 3-algebras and the dispersionless KdV hierarchy. By introducing the FTFs, we rewrote the classical Heisenberg and $w_\infty$ 3-algebras as the first and second Nambu 3-bracket structures of the FTFs, respectively. By choosing the suitable extended Hamiltonians of the dispersionless KdV system, we found that these Hamiltonians are in involution for these two Nambu 3-bracket structures. The dispersionless KdV hierarchy follows from the N-P evolution equation with these Hamiltonians. Note that
these two Nambu Hamiltonian structures are compatible. Thus the dispersionless KdV system is not only a bi-Hamiltonian system, but also a bi-Nambu-Hamiltonian system. Due to the N-P evolution equation involving two Hamiltonians, more intriguing relationships between these Hamiltonians have been revealed. As an application of the infinite-dimensional 3-algebras, we derived an integrable generalized gas dynamics system in the framework of Nambu mechanics. It should be pointed out that we just deal with a simple nonlinear evolution equation. For the future investigations, it is of interest to derive other integrable nonlinear evolution equations related with the infinite-dimensional 3-algebras. Thus more applications of the infinite-dimensional 3-algebras in physics will be possible. Furthermore the connection between Nambu mechanics and Hamiltonian mechanics [12, 13] deserves further study with the help of the infinite-dimensional 3-algebras. We believe that the infinite-dimensional 3-algebras may lead to a better understanding of the integrable systems. This may shed new light on the integrable systems and Nambu mechanics.

Acknowledgements

The authors are indebted to Prof. Y.K. Lau for valuable discussion. We would like to thank the referees for their helpful comments. This work is partially supported by NSF projects (10975102 and 11031005), KZ201210028032 and PHR201007107.

References

[1] J.L. Gervais and A. Neveu, Dual string spectrum in Polyakov’s quantization (II). Mode separation, Nucl. Phys. B 209 (1982) 125.

[2] J.L. Gervais, Infinite family of polynomial functions of the Virasoro generators with vanishing Poisson brackets, Phys. Lett. B 160 (1985) 277.

[3] B. A. Kupershmidt, A super Korteweg-de Vries equation: An integrable system, Phys. Lett. A 102 (1984) 213.

[4] M. Chaichian and P. Kulish, Superconformal algebras and their relation to integrable nonlinear systems, Phys. Lett. B 183 (1987) 169.

[5] A. Bilal and J.L. Gervais, Superconformal algebra and super-kdv equation: two infinite families of polynomial functions with vanishing poisson brackets, Phys. Lett. B 211 (1988) 95.

[6] Q. Ho-Kim, (Super) Korteveg de Vries equation as a (super) conformal field theory, Phys. Rev. D 36 (1987) 3829.

[7] P. Mathieu, Superconformal algebra and supersymmetric Korteweg-de Vries equation, Phys. Lett. B 203 (1988) 287.

[8] K. Tanaka, Solvable two-dimensional supersymmetric models and the supersymmetric Virasoro algebra, Phys. Rev. D 42 (1990) 2745.

[9] P. Mathieu, Extended classical conformal algebras and the second hamiltonian structure of Lax equations, Phys. Lett. B 208 (1988) 101.
[10] K. Yamagishi, The KP hierarchy and extended Virasoro algebras, Phys. Lett. B 205 (1988) 466.

[11] Y. Nambu, Generalized Hamiltonian dynamics, Phys. Rev. D 7 (1973) 2405.

[12] F. Bayen and M. Flato, Remarks concerning Nambu’s generalized mechanics, Phys. Rev. D 11 (1975) 3049.

[13] N. Mukunda and E.C.G. Sudarshan, Relation between Nambu and Hamiltonian mechanics, Phys. Rev. D 13 (1976) 2846.

[14] L. Takhtajan, On foundation of the generalized Nambu mechanics, Commun. Math. Phys. 160 (1994) 295 [hep-th/9301111].

[15] J. Bagger and N. Lambert, Modeling multiple M2’s, Phys. Rev. D 75 (2007) 045020 [hep-th/0611108]; Gauge symmetry and supersymmetry of multiple M2-branes, Phys. Rev. D 77 (2008) 065008 [arXiv:0711.0955]; Comments on multiple M2-branes, JHEP 02 (2008) 105 [arXiv:0712.3738].

[16] A. Gustavsson, Algebraic structures on parallel M2-branes, Nucl. Phys. B 811 (2009) 66 [arXiv:0709.1260].

[17] S.A. Cherkis and C. Sämann, Multiple M2-branes and Generalized 3-Lie algebras, Phys. Rev. D 78 (2008) 066019 [arXiv:0807.0808].

[18] G. Papadopoulos, M2-branes, 3-Lie algebras and Plücker relations, JHEP 05 (2008) 054 [arXiv:0804.2662].

[19] S. Mukhi and C. Papageorgakis, M2 to D2, JHEP 05 (2008) 085 [arXiv:0803.3218].

[20] M. Van Raamsdonk, Comments on the Bagger-Lambert theory and multiple M2-branes, JHEP 05 (2008) 105 [arXiv:0803.3803].

[21] J. Gomis, G. Milanesi and J.G. Russo, Bagger-Lambert theory for general Lie algebras, JHEP 06 (2008) 075 [arXiv:0805.1012].

[22] P.M. Ho and Y. Matsuo, M5 from M2, JHEP 06 (2008) 105 [arXiv:0804.3629].

[23] P.M. Ho, Y. Imamura, Y. Matsuo and S. Shiba, M5-brane in three-form flux and multiple M2-branes, JHEP 08 (2008) 014 [arXiv:0805.2898].

[24] C.H. Chen, K. Furuuchi, P.M. Ho and T. Takimi, More on the Nambu-Poisson M5-brane theory: scaling limit, background independence and an all order solution to the Seiberg-Witten map, JHEP 10 (2010) 100 [arXiv:1006.5291].

[25] T.L. Curtright, D.B. Fairlie and C.K. Zachos, Ternary Virasoro-Witt algebra, Phys. Lett. B 666 (2008) 386 [arXiv:0806.3515].

[26] H. Lin, Kac-Moody extensions of 3-algebras and M2-branes, JHEP 07 (2008) 136 [arXiv:0805.4003].

[27] S. Chakraborrty, A. Kumar and S. Jain, $w_\infty$ 3-algebra, JHEP 09 (2008) 091 [arXiv:0807.0284].
[28] M.R. Chen, K. Wu and W.Z. Zhao, Super $w_\infty$ 3-algebra, JHEP 09 (2011) 090 [arXiv:1107.3295].

[29] C.N. Pope, L.J. Romans and X. Shen, The complete structure of $W_\infty$, Phys. Lett. B 236 (1990) 173.

[30] E.G. Floratos and J. Iliopoulos, A note on the classical symmetries of the closed bosonic membranes, Phys. Lett. B 201 (1988) 237.

[31] I. Antoniadis, P. Ditsas, E. Floratos and J. Iliopoulos, New realizations of the Virasoro algebra as membrane symmetries, Nucl. Phys. B 300 (1988) 549.

[32] R. Teodorescu, P. Wiegmann and A. Zabrodin, Unstable Fingering Patterns of Hele-Shaw Flows as a Dispersionless Limit of the Kortweg-de Vries Hierarchy, Phys. Rev. Lett. 95 (2005) 044502 [cond-mat/0502179].

[33] M. Kulkarni and A.G. Abanov, Hydrodynamics of cold atomic gases in the limit of weak nonlinearity, dispersion, and dissipation, Phys. Rev. A 86 (2012) 033614 [arXiv:1205.5917].

[34] A. Das and Z. Popowicz, Supersymmetric polytropic gas dynamics, Phys. Lett. A 296 (2002) 15 [hep-th/0109223].

[35] A. Choudhuri, B. Talukdar and U. Das, Lagrangian Approach to Dispersionless KdV Hierarchy, SIGMA 3 (2007) 096 [arXiv:0706.0314].

[36] J.C. Brunelli, Hamiltonian Structures for the Generalized Dispersionless KdV Hierarchy, Rev. Math. Phys. 8 (1996) 1041 [solv-int/9601001].

[37] F. Magri, A simple model of the integrable Hamiltonian equation, J. Math. Phys. 19 (1978) 1156.

[38] H. Awata, H. Kubo, S. Odake and J. Shiraishi, Virasoro-type Symmetries in Solvable Models, hep-th/9612233.

[39] G.B. Whitham, Linear and Nonlinear Waves, Wiley, New York (1974).

[40] H. Gümral and Y. Nutku, MultiHamiltonian structure of equations of hydrodynamic type, J. Math. Phys. 31 (1990) 2606.

[41] P.J. Olver and Y. Nutku, Hamiltonian structures for systems of hyperbolic conservation laws, J. Math. Phys. 29 (1988) 1610.