Coherent States of Non-Linear Algebras: Applications to Quantum Optics.

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Abstract

We present a general unified approach for finding the coherent states of polynomially deformed algebras such as the quadratic and Higgs algebras, which are relevant for various multiphoton processes in quantum optics. We give a general procedure to map these deformed algebras to appropriate Lie algebras. This is used, for the non-compact cases, to obtain the annihilation operator eigenstates, by finding the canonical conjugates of these operators. Generalized coherent states, in the Perelomov sense, also follow from this construction. This allows us to explicitly construct coherent states associated with various quantum optical systems.

1 Introduction

Till recently, in quantum optics, only linear Lie algebras have been used to give multiphoton coherent (including squeezed) states [1, 2]. It is well-known that if we have bilinear Hamiltonians for two mode radiation fields characterized by operators $a$, $b$, $a^\dagger$ and $b^\dagger$, then the simplest types of coherent states that can be constructed are the product states $|\alpha > |\beta >$ where $|\alpha >$ and $|\beta >$ are the eigenstates corresponding to $a$ and $b$ respectively. However, if the system has an added symmetry or conservation law, then, a set of coherent states restricted by the extra symmetry can be constructed, by a suitable projection from the ordinary product states. Examples of such coherent states include the coherent states of a radiation field with arbitrary polarization such that $a^\dagger a + b^\dagger b$ is conserved. Here, the symmetry algebra is $SU(2)$, and the corresponding states are the $SU(2)$ coherent states [3]. In the frequency conversion of photons of a given frequency $\Omega$ into two photons of frequencies $\omega_a$ and $\omega_b$, when the two photons are created or destroyed together such that the operator $Q = a^\dagger a - b^\dagger b$ is conserved, the relevant states are the ‘pair coherent states’ or the $SU(1,1)$ ‘Barut-Girardello (BG)’ states [4, 5, 6]. The symmetry algebra in this case is $SU(1, 1)$, defined by $K_+ = a^\dagger b^\dagger$, $K_-$ = ...


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ab, \quad K_0 = \frac{1}{2}(a^\dagger a + b^\dagger b + 1) \quad \text{and the Casimir operator} \quad C = \frac{1}{2}(1 - (a^\dagger a - b^\dagger b)^2) = \frac{1}{2}(1 - Q^2).

The coherent states are the simultaneous eigenstates of \( K_- \) and \( C \). Other coherent states are the SU(1,1) Perelomov states, \( |\lambda> = e^{(\lambda K_+ - \lambda^* K_-)} |0,0> \), which are the ‘Caves-Schumaker’ states that represent two-mode squeezing. A third set of eigenstates \( A |\psi> = (uK_0 + vK_- + cK_+) |\psi> = \tilde{\lambda} |\psi> \) called algebraic coherent states have also been constructed by Agarwal et al., Shanta et al., and others. The BG states can be produced by a dissipative process of the master equation \( \frac{d\rho}{dt} = [\rho, H_{int}] \). The SU(1,1) Perelomov states represent the time evolution of the same Hamiltonian. Thus, the corresponding Lie algebraic structure has proved instrumental in studying the quantum optical properties of two mode radiation fields.

We show, in this paper, that when one has Hamiltonians representing interactions of multimode radiation fields, i.e., three or more modes, then the dynamical symmetry algebra of the Hamiltonian becomes a polynomially deformed algebra. The deformation is quadratic for the three mode case and cubic for the four mode case. The quadratic algebra was discovered by Sklyanin, in the context of statistical physics and field theory. It was shown to be the symmetry algebra of a two-dimensional anisotropic harmonic oscillator and the isotropic harmonic oscillator in curved space. The well-known Higgs algebra, a cubic algebra, occurs in the study of the dynamical symmetries of the Coulomb problem in a space of constant curvature. These algebras have now found a place in quantum optics with the observation that quantum optical Hamiltonians describing multiphoton processes have symmetries which can be described by polynomially deformed SU(1,1) and SU(2) algebras.

A polynomial deformation of a Lie algebra is defined in the following fashion in the Cartan-Weyl basis,

\[
[H, E_\pm] = \pm E_\pm \quad , \quad [E_+, E_-] = f(H) ,
\]

where \( f(H) \) is a polynomial function of \( H \). The corresponding Casimir can be written in the form,

\[
C = E_- E_+ + g(H) ,
\]

\[
= E_+ E_- + g(H - 1) ,
\]

where,

\[
f(H) = g(H) - g(H - 1).
\]

The form of \( g(H) \) can be determined up to the addition of a constant. The eigenstates are characterized by the values of the Casimir operator and the Cartan subalgebra \( H \).

In particular, a polynomial deformation of \( SL(2, R) \) is of the form \( N_0 = J_0, N_+ = F(J_0, J) J_+, N_- = F(J_0, J) J_- \), where the \( J_i \)'s are the ordinary \( SL(2, R) \) generators. The commutation relations are \([N_0, N_\pm] = \pm N_\pm \) and \([N_+, N_-] = F(N_0) \). When \( F(N_0) \) is quadratic in \( N_0 \) the algebra is called a quadratic algebra and if it is cubic in \( N_0 \) the ”Higgs” algebra results.

As an example of occurrence of non-linear algebras in quantum optics, consider the triboson Hamiltonian

\[
H = \omega_a a^\dagger a + \omega_b b^\dagger b + \omega c^\dagger c + \kappa ab^\dagger c + \kappa^* a^\dagger bc.
\]
Physically, for this Hamiltonian, $a$, $b$ and $c$ represent the pump, signal and idler modes for parametric amplification and the idler, pump and signal modes in frequency conversion. Raman and Brillouin scattering can be described by $H$, if $a$, $b$ and $c$ represent input, vibrational and Stokes modes for a Stokes process and anti-Stokes, input and vibrational modes for an anti-Stokes process. It also represents the interaction of $N$ identical two-level atoms with a single mode radiation field. This has been considered by many authors [24, 25, 26, 27] by ordinary linear Lie algebraic methods leading to approximate results for specific cases. Infinite dimensional Lie algebraic techniques have also been attempted and the physics has been extracted by a truncation of these algebras, hence the results obtained have again been approximate, with a number of assumptions [27].

In this paper, we show that this Hamiltonian and its generalizations have a non-linear algebra as its dynamical symmetry algebra and the construction of the coherent states is straightforward using the representation theory of these algebras. Furthermore, all three types of coherent states can be constructed on the basis of our method. We shall show that this Hamiltonian is formed by operators which obey a finite quadratic polynomial deformation of $SL(2, R)$ and the construction of CS for this Hamiltonian is a fairly straightforward process based on the undeformed algebra.

For the Hamiltonian given in equation (4), let $a$ represent a pump system and $b$ and $c$ represent the signal and idler variables. The interaction Hamiltonian between the pump and signal-idler subsystem is given by

$$H_{int} = \kappa a^\dagger b c^\dagger + \kappa^* a^\dagger bc. \quad (5)$$

Energy conservation requires that $\omega_a = \omega_b + \omega_c$. If the signal and idler frequencies are equal then $\omega_b = \frac{\omega_a}{2}$ and $\omega_c = \frac{\omega_a}{2}$ with, $H_{0}^{free} = \omega_a (a^\dagger a + \frac{b^\dagger b + c^\dagger c}{2})$.

The generators of the polynomial quadratic algebra are defined by the operators

$$J_0 = \frac{1}{2}(a^\dagger a - K_0) \quad (6)$$

$$J_- = a b^\dagger c^\dagger = a K_+ \quad (7)$$

$$J_+ = a^\dagger bc = a^\dagger K_- \quad (8)$$

where $K_0$, $K_-$ and $K_+$ form $SU(1,1)$ generators. The algebra closes only if we define an additional conserved quantity $H_0$ given by :

$$H_0 = \frac{a^\dagger a + K_0}{2}. \quad (9)$$

Since $H_0$ is related to $H_{0}^{free}$ through $H_{0}^{free} = 2H_0 - \frac{1}{2}$, we see that physically this condition is satisfied. In fact, $H_0$ can also be related to the Manley-Rowe invariants of the system. The algebra is given by:

$$[J_+, J_-] = -3J_0^2 + (2H_0 - 1)J_0 - C_{bc}(K_0) + H_0(H_0 + 1) \quad (10)$$

Where $C_{bc} = \frac{1}{4} - \frac{(b^\dagger b - c^\dagger c)^2}{4} = \frac{1-Q^2}{4}$ is the Casimir operator for the idler-signal system , for which $Q$ is a conserved quantity. For the case special $b^\dagger b - c^\dagger c = 0$, $C_{bc} = \frac{1}{4}$.

The two commuting generators are then $H_0$ and $J_0$ and a general eigenstate of the system is labelled by the quantum numbers corresponding to their eigenvalues and is given by $|h_0, j_0 >$. Similarly, the symmetry algebra for four photon processes is a Higgs algebra.
For general multiphoton Hamiltonians:

\[ H = H_0 + \kappa (a_0)^m (a_1^n)^n + c.c \]  

we can define \( N_0, N_-, N_+ \) in an analogous way

\[
N_+ = a_0^m (a_1^n)^n \\
N_- = a_1^n (a_1^n)^m \\
N_0 = \frac{1}{m+n} (a_1^1 a_1 - a_0^1 a_0)
\]

and show that we get n-dimensional polynomial algebras as the symmetry algebra if \( H_0 = \frac{1}{m+n} (a_0^1 a_0 + a_1^1 a_1) \) is conserved.

Similarly n-photon Dicke Models with Hamiltonians of the form:

\[ H = H_0 + \kappa \sum_i \sigma_-(i) (a_1^n)^n + \kappa^* \sum_i \sigma_+(i) (a_1^n)^n \]

with

\[
N_0 = \sum_i \sigma_0(i) - a_1^1 a_1 \\
N_- = \sum_i \sigma_-(i) (a_1^n)^n \\
N_+ = \sum_i \sigma_+(i) (a_1^n)^n
\]

satisfying a polynomial Lie Algebra of order n if \( H_0 = \epsilon \sum_i \sigma_0(i) + \omega_1 a_1^1 a_1 \) is conserved.

We present a unified approach for finding the coherent states (CS) of these algebras. Apart from its application to quantum optics, the method of construction presented here is quite general and will greatly facilitate the physical applications of these algebras to many quantum mechanical problems. This method is a generalisation to non-linear algebras of the procedure for constructing multiphoton states outlined in reference [28]. For ordinary Lie algebras, the construction of the CS for the non-compact cases, was shown to be a two step procedure. First, the canonical conjugate of the lowering operator were found and the CS of these algebras were obtained by the action of the exponential of the respective conjugate operators on the vacuum [28, 29, 30]. This method was in complete parallel to the one used for constructing the coherent states for harmonic oscillator algebras. Another CS, dual to the first one, naturally follows from the above construction. Here, we generalise the above construction to non-linear algebras and provide a mapping between the deformed algebras and their undeformed counterparts. This mapping is utilized to find the CS in the Perelomov sense [2]. Apart from obtaining the known CS of the \( SU(1, 1) \) algebra, we construct the CS for the quadratic and cubic polynomial algebras. Other coherent states in the literature
which are essentially special cases of this construction are the ‘f-oscillator states’ \[32\] and the non-linear states \(f(N_0)a|\lambda \rangle = \lambda |\lambda \rangle,\)\[31\] which have been shown to be useful for the trapped ion problem. While these are non-linear harmonic oscillator coherent states, the CS that we construct may be called non-linear SU(1,1) (or SU(2)) coherent states. These states would give a multi-mode generalization of the type \(f(n_a,n_b)a^n b^m|\lambda \rangle = \lambda |\lambda \rangle,\) as one of the possible coherent states. Thus our construction encompasses existing non-linear states and allows for the construction of new physical states. One such state, for example, is the case \(n=1\) and \(m=1,\) which is a two-mode realization of the non-linear coherent states. Our method is quite general and encompasses q-deformations \[33\] of linear Lie algebras. In this work, we concentrate on finite, polynomial SU(2) and SU(1,1) algebras, in view of their importance in quantum optics.

2 Construction of Coherent States of Non-Linear Algebras: Formalism.

Having seen that polynomially deformed algebras occur in a large class of systems, we now give the formalism for the construction of coherent states of these algebras. For the purpose of clarity, we start with Lie Algebras and then extend the method to the deformed algebras in a straightforward way. In the next section, we shall show how this formalism can be used to explicitly construct the coherent states for application to multiphoton processes.

We introduce our method by first considering SU(1,1) for which the generators satisfy the commutation relations

\[
[K_+, K_-] = -2K_0, \quad [K_0, K_\pm] = \pm K_\pm.
\] (15)

Thus for this case one finds, \(f(K_0) = -2K_0\) and \(g(K_0) = -K_0(K_0 + 1).\) The quadratic Casimir operator is given by \(C = K_-K_+ + g(K_0) = K_-K_+ - K_0(K_0 + 1).\) \(\tilde{K}_+\), the canonical conjugate of \(K_-\), satisfying

\[
[K_-, \tilde{K}_+] = 1,
\] (16)

can be written in the form,

\[
\tilde{K}_+ = K_+ F(C, K_0).
\] (17)

Eq.(17) then yields,

\[
F(C, K_0)K_-K_+ - F(C, K_0 - 1)K_+K_- = 1;
\] (18)

making use of the Casimir operator relation given earlier, one can solve for \(F(C, K_0)\) in the form,

\[
F(C, K_0) = \frac{K_0 + \alpha}{C + K_0(K_0 + 1)}.
\] (19)

The constant, arbitrary, parameter \(\alpha\) in \(F\) can be determined by demanding that Eq.(17) be valid in the entire Hilbert space.

For the purpose of illustration, we demonstrate our method, by using the one oscillator realization of the \(SU(1,1)\) generators \(K_- = a^2, K_+ = a^{\dagger 2}, K_0 = \frac{1}{2}(aa^{\dagger} + a^{\dagger}a).\) Since the coherent states of this realization have been studied extensively in the literature \[10, 13, 14\], this provides a good testing
ground for our method. The ground states defined by $K_- |0> = \frac{1}{2}a^2 |0> = 0$, are, $|0>$ and $|1> = a^\dagger |0>$, in terms of the oscillator Fock space.

$$K_0 |0> = \frac{1}{4}(2a^\dagger a + 1) |0> = \frac{1}{4} |0> ,$$

and

$$C |0> = \frac{3}{16} |0> .$$

Thus $[K_-, \tilde{K}_+] |0> = K_- \tilde{K}_+ |0> \text{ yields } \alpha = \frac{3}{4}$.

Similarly, for the other ground state $|1>$,

$$[K_-, \tilde{K}_+] |1> = |1> ,$$

leads to $\alpha = \frac{1}{4}$.

Hence, there are two disjoint sectors characterized by the $\alpha$ values $\frac{3}{4}$ and $\frac{1}{4}$, respectively. These results match identically with the earlier known ones $[28]$, once we rewrite $F$ as,

$$F(C, K_0) = \frac{K_0 + \alpha}{C + K_0(K_0 + 1)} ,$$

$$= \frac{K_0 + \alpha}{K_-K_+} .$$

The unnormalized coherent state $|\beta>\text{, which is the annihilation operator eigenstate, i.e, } K_- |\beta> = \beta |\beta>$, is given in the vacuum sector $|0>$ by

$$|\beta> = e^{\beta K_+} |0> .$$

For the vacuum sector $|1>$, where $\alpha = \frac{1}{4}$, a similar construction holds. These states, which provide a realization of the Cat states $[35]$, play a prominent role in quantum measurement theory. The canonical conjugate $\tilde{K}_+$ such that:

$$[\tilde{K}_+, K_+] = 1 .$$

can be constructed, as in ref. $[28]$ from this, one can find the eigenstate of $\tilde{K}_+$ operator, in the form,

$$|\gamma> = e^{\gamma \tilde{K}_+} |0> .$$

This CS, after proper normalization, is the well-known Yuen (squeezed) state: $e^{\mu a^\dagger - \mu^* a^2}$, with $\gamma = \frac{\mu}{|\mu|} tanh(|\mu|) [2, 36]$. Our construction can be easily generalized to various other realizations of the $SU(1, 1)$ algebra, such as the two mode realization, where the corresponding states are the Pair coherent and Perelomov(Cave-Schumaker) states.
We now extend the above procedure to the quadratic algebra, which is the relevant algebra in considering the coherent states of trilinear boson Hamiltonians\cite{33}. The algebra is given by:

\[ [J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = +(2H_0 - 1)J_0 - 3J_0^2 - \frac{1 - q^2}{4} + H_0(H_0 + 1) \quad . \tag{28} \]

where the positive or negative sign of \( (2H_0 - 1) \) determines whether the algebra is a quadratic deformation of SU(2) or SU(1, 1) respectively.

In this case,

\[
f_1(J_0) = (2H_0 - 1)J_0 - 3J_0^2 - \frac{1 - Q^2}{4} + H_0(H_0 + 1)
= g_1(J_0) - g_1(J_0 - 1), \tag{29} \]

where

\[
g_1(J_0) = J_0[H_0(H_0 + 1) - \frac{1 - Q^2}{4} + (H_0 - \frac{1}{2})(J_0 + 1)] - J_0(J_0 + 1)(J_0 + \frac{1}{2}) \quad . \tag{30} \]

In this case we have three different vacua, \(| h_0, h_0 + \frac{q}{2} >, | h_0, h_0 - \frac{q}{2} > \) and \(| h_0, -h_0 > \) where \( h_0 \) is the eigenvalue of the operator \( H_0 \) and \( q \) is the eigenvalue of \( Q = b^\dagger b - c^\dagger c \).

This is a special case of the general quadratic algebra:

\[
[N_0, N_\pm] = \pm N_\pm, \quad [N_+, N_-] = \pm 2bN_0 + aN_0^2 + c \quad . \tag{31} \]

In this case, \( f_1(N_0) = \pm 2bN_0 + aN_0^2 + c = g_1(N_0) - g_1(N_0 - 1) \). with \( g_1(N_0) = \frac{a}{2}N_0(N_0 + 1)(N_0 + \frac{1}{2}) + N_0(c \pm b(N_0 + 1)) \).

The representation theory of the quadratic algebra has been studied in the literature\cite{22, 42}. It shows a rich structure depending on the values of ‘\( a \)’. In the non-compact case, i.e, for polynomial deformations of SU(1, 1), the unitary irreducible representations (UIREP) are either bounded below or above, we can construct the canonical conjugate \( \tilde{N}_+ \) of \( N_- \) such that \( [N_-, N_+] = 1 \). It is given by \( \tilde{N}_+ = N_+F_1(C, N_0) \), with

\[
F_1(C, N_0) = \frac{N_0 + \delta}{C(N_0) - \frac{a}{2}N_0(N_0 - 1)(N_0 + \frac{1}{2}) - N_0(c \pm b(N_0 + 1))} \quad . \tag{32} \]

As can be seen easily, in the case of the finite dimensional UIREP, \( \tilde{N}_+ \) is not well defined since \( F_1(C, N_0) \) diverges on the highest state. The values of \( \delta \) can be fixed by demanding that the relation, \( [N_-, \tilde{N}_+] = 1 \), holds in the vacuum sector \(| v_i >, \) where \(| v_i >, \) are annihilated by \( N_- \). This gives \( N_-\tilde{N}_+ | v_i > = | v_i >, \) which leads to \( (N_0 + \delta) | v_i > = | v_i >. \) The value of the Casimir operator, \( C = N_-N_+ + g_1(N_0), \) can then be calculated. Hence, the unnormalized coherent state \(| \alpha >, \) such that \( N_- | \alpha > = \alpha | \alpha > \) is given by \( e^{\alpha\tilde{N}_+} | v_i >. \) We can define the canonical conjugate of \( N_+ \) by \( [\tilde{N}_+, N_+] = 1 \). The other coherent state is \(| \gamma > = e^{\gamma\tilde{N}_+} | \tilde{v}_i >, \) where \( \tilde{N}_+ | \tilde{v}_i > = 0 \). Depending on whether the UIREP is infinite or finite dimensional, this quadratic algebra can also be mapped onto the SU(1, 1) and SU(2) algebras, respectively. Leaving aside the commutators not affected by this mapping, one gets,

\[
[N_+, \tilde{N}_+] = -2bN_0 \quad . \tag{33} \]
where $b = 1$ corresponds to the $SU(1, 1)$ and $b = -1$ gives the $SU(2)$ algebra. Explicitly,

$$\bar{N}_- = N_-G_1(C, N_0) , \quad (34)$$

and

$$G_1(C, N_0) = \frac{(N_0^2 - N_0)b + \epsilon}{C - g_1(N_0 - 1)} , \quad (35)$$

$\epsilon$ being an arbitrary constant. One can immediately construct CS in the Perelomov sense (see page 73-74 in ref[2]) as $|\xi > = U|v_i >$, where $U = e^{\eta N_+ - \eta^* N_-}$, with $\xi = \frac{2\eta}{\eta} \tanh(|\eta|)$. For the compact case, the CS are analogous to the spin and atomic coherent states[3, 37].

The cubic algebra, which is also popularly known as the Higgs algebra in the literature, appears in the study of the Coulomb problem in a curved space[18] and in quantum optics for quadrilinear boson Hamiltonians. The generators satisfy,

$$[M_0, M_\pm] = \pm M_\pm , \quad [M_+, M_-] = 2cM_0 + 4hM_0^3 , \quad (36)$$

where, $f_2(M_0) = 2cM_0 + 4hM_0^3 = g_2(M_0) - g_2(M_0 - 1)$, and

$$g_2(M_0) = cM_0(M_0 + 1) + hM_0^3(M_0 + 1)^2 . \quad (37)$$

Analysis of its representation theory yields a variety of UIREP’s, both finite and infinite dimensional, depending on the values of the parameters $c$ and $h$[39]. In the non-compact case the canonical conjugate is given by,

$$\bar{M}_+ = M_+ F_2(C, M_0) , \quad (38)$$

where,

$$F_2(C, M_0) = \frac{M_0 + \zeta}{C - cM_0(M_0 + 1) - hM_0^3(M_0 + 1)^2} . \quad (39)$$

As before, the annihilation operator eigenstate is given by

$$|\rho > = e^{\rho \bar{M}_+} |p_\rho > , \quad (40)$$

where, $|p_\rho >$ are the states annihilated by $M_-$. Like the previous cases, the dual algebra yields another coherent state. This algebra can also be mapped in to $SU(1, 1)$ and $SU(2)$ algebras, as has been done for the quadratic case:

$$[M_+, \bar{M}_-] = -2dM_0 , \quad (41)$$

where, $d = 1$ and $d = -1$ correspond to the $SU(1, 1)$ and $SU(2)$ algebras respectively. Here,

$$\bar{M}_- = M_- G_2(C, M_0) , \quad (42)$$

where,

$$G_2(C, M_0) = \frac{(M_0^2 - M_0)d + \sigma}{C - g_2(M_0 - 1)} , \quad (43)$$

$\sigma$ being a constant. The coherent state in the Perelomov sense is then $|\zeta > = U|p_\zeta >$, where, $U = e^{\zeta M_+ - \zeta^* M_-}$. In earlier works on non-linear algebras, the generators of the deformed algebra have been written in terms of the undeformed ones[22, 23]. However, in our approach the undeformed $SU(1, 1)$ and $SU(2)$ generators are constructed from the deformed generators.
3 Explicit Construction of the Coherent States for Physical Application.

We now give an outline of the method of explicit construction of coherent states for general multiphoton processes for which the generators satisfy the algebra \([11]\): \([N_0, N_\pm] = \pm N_\pm\) and \([N_+ N_-] = g(N_0) - g(N_0 - 1)\).

The action on eigenstates of \(N_0\) is given by

\[
N_0 | j, m > = (j + m) | j, m >
\]

\[\text{(44)}\]

\[
N_+ | j, m > = \sqrt{C(j) - g(j + m)} | j, m + 1 >
\]

\[\text{(45)}\]

\[
N_- | j, m > = \sqrt{C(j) - g(j + m - 1)} | j, m - 1 >
\]

\[\text{(46)}\]

where \(C(j) = g(j - 1)\).

Depending on the order of the polynomial algebra \(n\), there will be \(n+1\) degenerate states annihilated by \(N_-\). We denote these as \(|j, 0 >_i\). For each, the value of \(\delta = \delta_i\) is appropriately chosen as shown earlier.

The coherent state is given by

\[
| \alpha > = A e^{\alpha N_+} | j, 0 >_i
\]

\[\text{(47)}\]

\[
= A \sum_n \frac{\alpha^n}{n!} (N_+)^n \frac{N_0 + \delta_i}{g(j - 1) - g(N_0)} \ldots \frac{N_0 + n - 1 + \delta_i}{g(j - 1) - g(N_0 + n - 1)} | j, 0 >
\]

\[
= A \sum_n \frac{\alpha^n}{\sqrt{(g(j - 1) - g(j)) \ldots (g(j - 1) - g(j + n - 1))}} | j, n >
\]

\[\text{(47)}\]

‘A’ being the normalization constant.

A discussion of coherent states is incomplete without showing that these states do give a resolution of the identity and that they are overcomplete. From the resolution of the identity we have:

\[
\int d\sigma(\alpha^*, \alpha) |\alpha\rangle\langle\alpha| = 1
\]

\[\text{(48)}\]

Within the polar decomposition ansatz

\[
d\sigma(\alpha^*, \alpha) = \sigma(r) d\theta dr
d
\]

\[\text{(49)}\]

with \(r = |\alpha|\) and an yet unknown positive density \(\sigma\) which provides the measure. For the general case we have:

\[
2\pi \int_0^\infty dr \sigma(r) r^{2n+1} = A(g(j - 1) - g(j)) \ldots (g(j - 1) - g(j + n - 1))
\]

\[\text{(50)}\]

For the various cases the substitution of the explicit value of \(g(j)\) then reduces the expression on the R.H.S to a rational function of Gamma Functions and the measure \(\sigma\) can be found by an inverse
Mellin transform. For the general case the measure is a Meijer’s G-function. The fact that these states are overcomplete (i.e.; \( \beta |\alpha| \neq 0 \)) can be shown for explicit examples. This we shall later show in the case of a quadratic algebra.

We now construct the state explicitly for purposes of application. First we show that, this method indeed, gives us the well known SU(1,1) Barut-Girardello (pair coherent) states for SU(1,1) in the familiar form [3]. The action on Hilbert Space of the generators is given in the original BG representation by:

\[
K_0 | \phi \rangle = (-\phi + m) | \phi, m \rangle, \\
K_+ | \phi, m \rangle = \frac{1}{\sqrt{2}} \sqrt{(m+1)(-2\phi + m)} | \phi, m + 1 \rangle, \\
K_- | \phi, m \rangle = \frac{1}{\sqrt{2}} \sqrt{m(-2\phi + m - 1)} | \phi, m - 1 \rangle.
\]

There are two vacuua annihilated by \( K_- \), they correspond to \( | \phi, 0 \rangle \) and \( | \phi, 2\phi + 1 \rangle \). The coherent state \( | \alpha \rangle \) constructed on the vacuum \( | \phi, 0 \rangle \) gives us, \( \delta = \phi + 1 \), so that \( (N_0 + \delta) | \phi, 0 \rangle = | \phi, 0 \rangle \) and the resulting coherent state is:

\[
| \alpha \rangle = e^{\alpha K_+} | \phi, 0 \rangle
\]

where, \([K_-, K_+] = 1\) and \( K_+ = K_+ F(C, K_0) \) with

\[
F(C, K_0) = \frac{K_0 + \delta}{C - g(K_0)} = \frac{K_0 + \delta}{C + \frac{1}{2} K_0 (K_0 + 1)}.
\]

Hence

\[
| \alpha \rangle = \sum_n \frac{\alpha^n}{n!} (K_+ F(C, K_0))^n | \phi, 0 \rangle = \sum_n \frac{\alpha^n}{n!} (K_+)^n F(C, K_0) \ldots F(C, K_0 + n - 1) | \phi, 0 \rangle
\]

substituting the values of \( F \) we get:

\[
| \alpha \rangle = A \sum_n \frac{\alpha^n}{n!} (K_0 + \delta)(K_0 + \delta + 1)\ldots(K_0 + \delta + n - 1) (-\phi + \frac{1}{2})(-\phi + \frac{3}{2})\ldots(-\phi + \frac{n-1}{2}) (K_+)^n | \phi, 0 \rangle
\]

\[
= A \sum_n (2\alpha)^n \frac{\Gamma(-2\phi)}{\Gamma(n+1) \Gamma(-2\phi + n)} \sqrt{n!(-2\phi + n - 1)!} (\sqrt{2})^n \sqrt{\Gamma(-2\phi)} | \phi, n \rangle
\]

\[
= A \sqrt{\Gamma(-2\phi)} \sum_n (\sqrt{2\alpha})^n \frac{1}{(\Gamma(n+1) \Gamma(-2\phi + n))^{\frac{1}{2}}} | \phi, n \rangle,
\]

which is precisely the well-known state of Barut and Girardello upto the normalization coefficient \( A \). For example for the SU(1,1), \( g(j) = -\frac{1}{2} j(j+1) \) then the right hand side becomes in the BG representation

\[
2\pi \int_0^\infty dr \sigma(r) r^{2n+1} = A \frac{\Gamma(n+1) \Gamma(-2\phi + n)}{\Gamma(-2\phi)}
\]
where $A$ is a numerical constant and from the inverse Mellin transform, we get $\sigma(r) = A r^{-2\phi + 1} K_{\frac{1}{2} + \phi}(2r)$.

The second state annihilated by $K_-$ is the state $|\phi, 2\phi + 1>$ and this corresponds to $\delta = -\phi$. The coherent state is:

$$|\alpha> = A' \sqrt{\Gamma(2\phi)} \frac{1}{\sqrt{\Gamma(n+1)\Gamma(2\phi+n)}} \left| \phi, 2\phi + 1 + n > \right. (59)$$

The third state given by Eq. (29) is:

$$|\gamma> = B(\gamma) \frac{\Gamma(n-2\phi)}{\sqrt{\Gamma(n+1)\Gamma(-2\phi)}} |\phi, n> . (60)$$

This is the state constructed by Perelomov [2], up to a normalisation constant $B(\gamma)$.

For the quadratic case, we take an illustrative algebra relevant to the trilinear boson cases described in the introduction. For convenience we rewrite the three boson algebra as: $[N_0, N_{\pm}] = N_{\pm}$ and $[N_+ N_-] = -3N_0^2 + 4\epsilon N_0 - \epsilon^2$ with $\epsilon = 2H_0 - 1$. We define $n = (h_0 + j_0)$, where $h_0$ and $j_0$ are the quantum numbers associated with $H_0$ and $N_0$ respectively. The state $|n>$ corresponds to the state $|\epsilon, n >$ and the three states annihilated by $N_-$ are given by $|\epsilon, 0 >, |\epsilon, \epsilon - \frac{1}{2} >, |\epsilon, \epsilon + \frac{1}{2} >$.

The action of the operators on eigenfunctions of $N_0$ is given by:

$$N_0 |\epsilon, n> = (n) |\epsilon, n>, (61)$$

$$N_+ |\epsilon, n> = \sqrt{(n + \frac{3}{2} - \epsilon)(n+1)(n + \frac{1}{2} - \epsilon)} |\epsilon, n+1>, (62)$$

$$N_- |\epsilon, n> = \sqrt{(n - \frac{1}{2} - \epsilon)n(n + \frac{1}{2} - \epsilon)} |\epsilon, n-1>. (63)$$

We give the explicit construction of the coherent state for the case $|v_i> = |\epsilon, 0>$, for which $\delta = 1$. Suitable choices of $\delta$ will give the other two coherent states. Here $g(N_0) = -(N_0 + \frac{3}{2} - \epsilon)(N_0 + \frac{1}{2} - \epsilon)$ and $g(N_0) - g(N_0 - 1) = 3N_0^2 - 4\epsilon N_0 + \epsilon^2$.

From our construction the CS is:

$$|\alpha> = e^{\epsilon N_+} |\epsilon, 0> = \sum_n \frac{\alpha^n}{n!} (N_+)^n |\epsilon, 0> (64)$$

Thus:

$$|\alpha> = A \sum_n \frac{\alpha^n}{n!} (N_+ F(N_0, C)) |0> (65)$$

Constructing the $F's$ from $g(N_0)$ we get:

$$|\alpha> = A \sum_n \frac{\alpha^n}{n!} (N_+)^n F(N_0) F(N_0 + 1) \ldots F(N_0 + n - 1) |0>$$



11
A \sum_n \frac{\alpha^n}{n!} (N_+)^n \frac{N_0 + \delta}{(N_0 - \epsilon)(N_0)(N_0 + 1 - \epsilon)} \cdots \frac{N_0 + n - 1 + \delta}{(N_0 + n - \epsilon)(N_0 + n)(N_0 + n - \epsilon)} | \epsilon, 0 > \\
= A \sum_n \frac{\alpha^n}{(n - \frac{1}{2} - \epsilon)!n!(n + \frac{1}{2} - \epsilon)!} (N_+)^n | \epsilon, 0 > \\
= A \frac{\Gamma(\frac{1}{2} - \epsilon)\Gamma(\frac{3}{2} - \epsilon)}{\Gamma(n + \frac{1}{2} - \epsilon)\Gamma(n + 1)\Gamma(n + \frac{3}{2} - \epsilon)} \sum_n \frac{\alpha^n}{\sqrt{\Gamma(n + \frac{1}{2} - \epsilon)\Gamma(n + 1)\Gamma(n + \frac{3}{2} - \epsilon)}} | \epsilon, n > , (66)

A is the normalization coefficient, which can be easily determined to be

$$ (0_F^2(\frac{1}{2} - \epsilon, \frac{3}{2} - \epsilon | \alpha |^2))^{\frac{1}{2}} $$

These set of states can be shown to be overcomplete:

$$ | < \beta | \alpha > |^2 = \frac{0_F^2(\frac{1}{2} - \epsilon, \frac{3}{2} - \epsilon, \alpha \beta^*)}{(0_F^2(\frac{1}{2} - \epsilon, \frac{3}{2} - \epsilon, | \alpha |^2))^{\frac{1}{2}}(0_F^2(\frac{1}{2} - \epsilon, \frac{3}{2} - \epsilon, | \beta |^2))^{\frac{1}{2}}}. (67) $$

The completeness relation is given by

$$ \int_0^\infty dr \sigma(r) r^{2n+1} = \Gamma(n + 1) \frac{\Gamma(\frac{1}{2} - \epsilon + n)\Gamma(\frac{3}{2} - \epsilon + n)}{\Gamma(\frac{1}{2} - \epsilon)\Gamma(\frac{3}{2} - \epsilon)} (68) $$

and $\sigma(r)$ can be determined to be a confluent hypergeometric function from the inverse Mellin transformation formula. The resolution of the identity can thus be obtained.

$$ \sigma(r) = \frac{1}{\Gamma(\frac{1}{2} - \epsilon)(\Gamma(\frac{3}{2} - \epsilon))} G_{03}^{30}(r | 0^0_{0, -\frac{1}{2} - \epsilon, \frac{3}{2} - \epsilon}). (69) $$

Where $G_{03}^{30}(x)$ is a Meijer’s G-function.

The other two coherent states based on the two vacua, $| \epsilon, \epsilon + \frac{1}{2} >$ and $| \epsilon, \epsilon - \frac{1}{2} >$ can similarly be constructed by chosing $\delta = \frac{3}{2} - \epsilon$ and $\delta = \frac{1}{2} - \epsilon$.

The state corresponding to the Perelomov state is :

$$ | \gamma > = B'(\gamma) \sum_n (\gamma)^n \frac{\Gamma(n + \frac{3}{2} - \epsilon)\Gamma(n + \frac{1}{2} - \epsilon)}{\Gamma(n + 1)\Gamma(\frac{1}{2} - \epsilon)\Gamma(\frac{3}{2} - \epsilon)} | \phi, n > . (70) $$

The normalisation constant can be calculated easily and using a method similar to the one used for Eq.(68), the overcompleteness of these states and the resolution of the identity can also be easily obtained.

### 4 Conclusion

To conclude, we have found a general method for constructing the coherent states for various polynomially deformed algebras for quantum optical systems, whose dynamics are governed by multilinear boson Hamiltonians. Since our method is algebraic and relies on the group structure
of well-known algebras, the precise nature of the non-classical behaviour of these CS can be easily inferred from our construction. It will be of particular interest to see the time development of the system and the role of the deformation parameters in the physical system described in the text. For a system initially in a coherent state, it is fairly straightforward to calculate the time evolution of the system exactly using the methods of reference [13]. Since many of these algebras are related to quantum mechanical problems with non-quadratic, non-linear Hamiltonians, a detailed study of the properties of the CS associated with non-linear and deformed algebras is of physical relevance [12, 13]. This is the subject of our current and future work [33].

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