Common dynamics of two Pisot substitutions with the same incidence matrix

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Abstract. The matrix of a substitution is not sufficient to completely determine the dynamics associated, even in simplest cases since there are many words with the same abelianization.
In this paper we study the common points of the canonical broken lines associated to two different Pisot irreducible substitutions \( \sigma_1 \) and \( \sigma_2 \) having the same incidence matrix. We prove that if 0 is inner point to the Rauzy fractal associated to \( \sigma_1 \) these common points can be generated with a substitution on an alphabet of so-called "balanced blocks".

1 Introduction

Let \( \sigma_1 \) and \( \sigma_2 \) be two different Pisot substitutions having the same incidence matrix. Although the fixed points of each substitution have the same letter frequencies, they usually show different dynamical and geometrical properties, e.g., their Rauzy fractals have different properties. (The Rauzy fractals can give a geometric model of the dynamical system defined by the substitution, for more detail see section 2).

A classic example is given by the Tribonacci substitution and the flipped Tribonacci substitution, i.e.,
\[ \sigma_1 : \begin{cases} 
  a \rightarrow ab \\
  b \rightarrow ac \\
  c \rightarrow a
\end{cases} \quad \text{and} \quad \sigma_2 : \begin{cases} 
  a \rightarrow ab \\
  b \rightarrow ca \\
  c \rightarrow a
\end{cases} \]

The incidence matrix of \( \sigma_1 \) and \( \sigma_2 \) is \( \begin{pmatrix} 1 & 1 & 1 \\
 1 & 0 & 0 \\
 0 & 1 & 0 \end{pmatrix} \). The dominant eigenvalue satisfies the relation \( X^3 - X^2 - X - 1 = 0 \), hence the name Tribonacci for the substitution.

The Rauzy fractal of the first substitution is a topological disc [1], simply connected, while it is a well known fact that the second fractal is not simply connected, compare Figure[1].

![Figure 1: The Rauzy fractals of \( \sigma_1 \) and \( \sigma_2 \)](image)

We consider another simple example of substitutions \( \tau_1 \) and \( \tau_2 \), i.e.,

\[ \tau_1 : \begin{cases} 
  a \rightarrow aba \\
  b \rightarrow ab
\end{cases} \quad \text{and} \quad \tau_2 : \begin{cases} 
  a \rightarrow aab \\
  b \rightarrow ba
\end{cases} \]

The Rauzy fractal of \( \tau_2 \) is the closure of a countable union of disjoint intervals and the Rauzy fractal of \( \tau_1 \) is an interval, see [1] and Figure[6].

We can deduce from one matrix we can obtain many different substitutions, so many different Rauzy fractals. We are interested to studies commons dynamics of these Rauzy fractals, we are interested to characterize their intersection, for this we need to define a new object. prove that we can consider their intersection as a substitutive set.
**Definition 1.1.** A substitutive set is the closure of the projection of a canonical stepped line associated to a primitive substitution on a contracting space associated to the restriction of a positive integer matrix. For more detail see section 2.

The main result of this paper is the following:

**Theorem 1.1.** Let $\sigma_1$ and $\sigma_2$ be two irreducible unimodular Pisot substitutions with the same incidence matrix. Let $X_{\sigma_1}$ and $X_{\sigma_2}$ the two associated Rauzy fractals; suppose that $0$ is inner point to $X_{\sigma_1}$.
Then the intersection of $X_{\sigma_1}$ and $X_{\sigma_2}$ has non-empty interior, and it is substitutive. There is an algorithm to obtain the substitution for intersection.

**2 Substitutions and Rauzy fractals**

**2.1 General setting**

Let $A := \{a_1, ..., a_d\}$ be a finite set of cardinal $d$ called alphabet. The free monoid $A^*$ on the alphabet $A$ with empty word $\varepsilon$ is defined as the set of finite words on the alphabet $A$, this is $A^* := \bigcup_{k \in \mathbb{N}} A^k$, endowed with the concatenation map. We denote by $A^N$ and $A^Z$ the set of one and two-sided sequences on $A$, respectively. The topology of $A^N$ and $A^Z$ is the product topology of discrete topology on each copy of $A$. Both spaces are metrizable.

The length of a word $w \in A^n$ with $n \in \mathbb{N}$ is defined as $|w| = n$. For any letter $a \in A$, we define the number of occurrences of $a$ in $w = w_1w_2 \ldots w_{n-1}w_n$ by $|w|_a = \sharp\{i | w_i = a\}$.

Let $l : A^* \rightarrow \mathbb{Z}^d : w \mapsto (|w|_a)_{a \in A} \in \mathbb{N}^d$ be the natural homomorphism obtained by abelianization of the free monoid, called the abelianization map.

A substitution over the alphabet $A$ is an endomorphism of the free monoid $A^*$ such that the image of each letter of $A$ is a nonempty word.

A substitution $\sigma$ is primitive if there exists an integer $k$ such that, for each pair $(a, b) \in A^2$, $|\sigma^k(a)|_b > 0$. We will always suppose that the substitution is primitive, this implies that for all letter $j \in A$ the length of the successive iterations $\sigma^k(j)$ tends to infinity.

A substitution naturally extends to the set of two sided sequences $A^Z$. We associate to every substitution $\sigma$ its incidence matrix $M$ which is the $n \times n$ matrix obtained by abelianization, i.e. $M_{i,j} = |\sigma(j)|$. It holds that $l(\sigma(w)) = MI(w)$ for all $w \in A^*$.

**Remark.** The incidence matrix of a primitive substitution is a primitive matrix, so with the Perron-Frobenius theorem, it has a simple real positive dominant eigenvalue $\beta$.

**2.2 Rauzy fractals**

**Definition 2.1.** A Pisot number is an algebraic integer $\beta > 1$ such that each Galois conjugate $\beta^{(i)}$ of $\beta$ satisfies $|\beta^{(i)}| < 1$. 

From now, we will suppose that all the substitutions that we consider are irreducible of Pisot type and unimodular. This means that the characteristic polynomial of its incidence matrix is irreducible, its determinant is equal to \( \pm 1 \) and its dominant eigenvalues is a Pisot number. We can prove that any irreducible Pisot substitution is primitive (see [8]).

**Remark.** Note that there exist substitution whose largest eigenvalue is Pisot but whose incidence matrix has eigenvalues that are not conjugate to the dominant eigenvalue. Example is \( 1 \rightarrow 12, 2 \rightarrow 3, 3 \rightarrow 4, 4 \rightarrow 5, 5 \rightarrow 1 \). The characteristic polynomial is reducible. Such substitutions are called Pisot reducible.

**Definition 2.2.** Let \( \sigma \) a substitution and \( u \in A^\mathbb{N} \), \( u \) is a fixed point of \( \sigma \) if \( \sigma(u) = u \). The infinite word \( u \) is a periodic point of \( \sigma \) if there exist \( k \in \mathbb{N} \) such that \( \sigma^k(u) = u \).

Let \( \sigma \) be a primitive substitution, then there exist a finite number of periodic points (see [7]). We associate to the fixed point \( u \) of the substitution a symbolic dynamical system \( (\Omega_u, S) \) where \( S \) is the shift map on \( A^\mathbb{N} \) given by \( S(a_0a_1\ldots) = a_1a_2\ldots \) and \( \Omega_u \) is the closure of \( \{S^m(u) : m \geq 0\} \) in \( A^\mathbb{N} \).

**Remark.** If \( \sigma \) is a primitive substitution then the symbolic dynamical system \( (\Omega_u, S) \) does not depend on \( u \); we denote it by \( (\Omega_\sigma, S) \).

We say that a dynamical system \( (X, f) \) is semiconjugate to another dynamical system \( (Y, g) \) if there exists a continuous surjective map \( \Theta : X \rightarrow Y \) such that \( \Theta \circ f = g \circ \Theta \). An important question is whether and how the symbolic dynamical system \( (\Omega_\sigma, S) \) admit a geometric model. By geometrically realizable we mean there exists a dynamical system \( (X, f) \) defined on a geometrical structure, such that \( (\Omega_\sigma, S) \) is semiconjugate to \( (X, f) \).

In [10], G.Rauzy proves that the dynamical system generated by the substitution \( \sigma(1) = 12, \sigma(2) = 13, \sigma(3) = 1 \), is measure-theoretically conjugate to an exchange of domains in a compact set \( \mathcal{R} \) of the complex plane. This compact subset has a self-similar structure : using method introduced by F.M.Dekking in [6], S.Ito and Pierre Arnoux obtain in [1] an alternative construction of \( \mathcal{R} \) and prove that each of exchanged domains has fractal boundary. We will use the projection method to obtain the Rauzy fractal.

**Definition 2.3.** A stepped line \( L = (x_n) \) in \( \mathbb{R}^d \) is a sequence (it could be finite or infinite) of points in \( \mathbb{R}^d \) such that \( x_{n+1} - x_n \) belong to a finite set. A canonical stepped line is a stepped line such that \( x_0 = 0 \) and for all \( n \geq 0 \), \( x_{n+1} - x_n \) belong to the canonical basis of \( \mathbb{R}^d \).

Using the abelianization map, to any finite or infinite word \( W \), we can associate a canonical stepped line in \( \mathbb{R}^d \) as a sequence \( (l(P_n)) \), where \( P_n \) are the prefix of length \( n \) of \( W \).

An interesting property of the canonical stepped line associated to a fixed point of primitive Pisot substitution is that it remains within bounded distance from the expanding direction (given by the right eigenvector of Perron-Frobenius
We denote by $E_s$ the stable space (or contracting space) and $E_u$ the unstable space (or expanding direction). We denote by $\pi_s$ the linear projection in the contracting plane, parallel to the expanding direction and $\pi_u$ the projection in the expanding direction parallel to the contracting plane. We will project the stepped line on the contracting space in the direction of the right Perron-Frobenius eigenvector. We obtain a bounded set in $(d-1)$-dimensional vector space.

**Definition 2.4.** Let $\sigma$ an irreducible Pisot substitution. The Rauzy fractal associated to $\sigma$ is the closure of the projection of the canonical stepped line associated to any fixed point of $\sigma$ in the contracting plane parallel to the expanding direction.

We note the projection $\pi$ of the orbit of the fixed point associated to a Pisot irreducible substitution $\sigma$ on a contracting space associated to its incidence matrix.

**Proposition 2.1.** The projection $\pi$ of the symbolic dynamical system $\Omega_\sigma$ associated to a Pisot irreducible substitution $\sigma$ to the Rauzy fractal is a continuous map.

Proof. The proof is given in [7].

We denote by $X_\sigma$ the Rauzy fractal (Central tile) associated to $\sigma$:

$$X_\sigma := \{\pi_s(l(u_0\ldots u_{k-1}), k \in \mathbb{N})\}.$$  

with $u_0\ldots u_{k-1}$ is a prefix of the fixed point of length $k$. Subtiles of the central tile $X_\sigma$ are naturally defined, depending on the letter associated to the vertex of the stepped line that is projected. On this sets for $i \in A$:

$$X_\sigma(i) := \{\pi_s(l(u_0\ldots u_{k-1}), k \in \mathbb{N}, u_k = i)\}.$$

**Proposition 2.2.** Let $\sigma$ a Pisot substitution and $X_\sigma$ its associated Rauzy fractal. The boundary of $X_\sigma$ has zero measure.

Proof. See [4] and [16].

2.3 Central tiles viewed as a graph directed iterated function

The tiles $X_\sigma(i)$ can be written as a so-called graph iterated function system (GIFS).

**Definition 2.5.** (GIFS)

Let $G$ be a finite directed graph with set of vertices $\{1, \ldots, q\}$ and set of edges $E$. Denote the set of edges leading from $i$ to $j$ by $E_{ij}$. To each $e \in E$ associated a contractive mapping $\tau_e : \mathbb{R}^n \to \mathbb{R}^n$. If for each $i$ there is some outgoing edge we call $(G, \{\tau_e\})$ a GIFS.

**Definition 2.6.** (Prefix-suffix automaton)

Let $\sigma$ be a substitution over the alphabet $A$ and let $P$ be the finite set
The prefix-suffix automaton of \( \sigma \) has \( A \) as a set of vertices and \( P \) as a set of label edges: there is an edge labeled by \((p, a, s)\) from \( a \) to \( b \) if and only if \( \text{pas} = \sigma(b) \).

**Example.** For the Fibonacci substitution \( 1 \mapsto 2 \) and \( 2 \mapsto 1 \), one gets:

\[
P = \{(e, 1, 2), (1, 2, e), (e, 1, e)\}.
\]

The prefix-suffix automaton of the Fibonacci substitution is:

![Fibonacci substitution automaton](image)

**Theorem 2.1.** Let \( \sigma \) be a primitive unit Pisot substitution over the alphabet \( A \). The central tile \( X_\sigma \) is a compact subset with nonempty interior. Each subtile is the closure of its interior. The subtiles of \( X_\sigma \) are solution of the GIFS

\[
\forall i \in A, X_\sigma(i) = \bigcup_{j \in A, (p, i, s) \rightarrow j} MX_\sigma(j) + \pi_s l(p).
\]

**Proof.** The proof is given in [13].

\[ \square \]
2.4 Disjointness of the subtiles of the central tile

To ensure that the subtiles are disjoint, we introduce the following combinatorial condition in substitutions.

Definition 2.7. (Strong coincidence condition).

A substitution $\sigma$ over the alphabet $\mathcal{A}$ satisfies the strong coincidence condition if for every pair $(b_1, b_2) \in \mathcal{A}^2$, there exist $k \in \mathbb{N}$ and $a \in \mathcal{A}$ such that $\sigma^k(b_1) = p_1a_{s_1}$ and $\sigma^k(b_2) = p_2a_{s_2}$ with $l(p_1) = l(p_2)$ or $l(s_1) = l(s_2)$.

Remark. The strong coincidence condition is satisfied by every unit Pisot substitution over two letter alphabet [3]. It is conjectured that every substitution of Pisot type satisfies the strong coincidence condition.

Theorem 2.2. Let $\sigma$ be a primitive unit Pisot substitution. If $\sigma$ satisfies the strong coincidence condition, then the subtiles of the central tiles have disjoint interiors.

Proof. The proof for the disjointness is given in [1].

Remark. If 0 is inner point to the Rauzy fractal associated to a Pisot substitution then the subtiles of the central tiles have disjoint interiors (see [13]).

2.5 Substitutive sets

A substitutive set is the closure of the projection of a canonical stepped line associated to substitution on a contracting space of a restriction of a positive integer matrix. In particular a Rauzy fractal is a substitutive set since it is the projection of canonical stepped line associated to a fixed point on the contracting space associated to the matrix of substitution. So we can expand the definition of Rauzy fractal to substitutive set. In particular a substitutive set can be expressed as the attractor of some graph directed iterated function system (IFS). See [2]

3 Intersection of Rauzy fractals

Let $\sigma_1$ and $\sigma_2$ two Pisot irreducible substitutions with the same incidence matrix, we consider $X_{\sigma_1}$ and $X_{\sigma_2}$ their associated Rauzy fractals respectively. The intersection of $X_{\sigma_1}$ and $X_{\sigma_2}$ is non-empty since it contains 0, and it is a compact set (intersection of two compacts).

Proposition 3.1. Let $\sigma_1$ and $\sigma_2$ be two Pisot irreducible substitutions with the same incidence matrix. We consider $L_1$ and $L_2$ the canonical broken lines associated to a fixed point of $\sigma_1$ and $\sigma_2$ respectively, let $P_1$ and $P_2$ two points from $L_1$ and $L_2$ respectively. Then $\pi_\sigma(P_1) = \pi_\sigma(P_2)$ implies $P_1 = P_2$. 

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Proof. The Perron Frobenius eigenvalues is irrational in the irreducible case. We project $P_1$ and $P_2$ in the contracting space parallel to the expanding space (the direction of the Perron Frobenius eigenvectors). If $\pi_s(P_1) = \pi_s(P_2)$ then $(P_1 P_2)$ is parallel to the expanding direction. This implies that expanding direction is rational.

Proposition 3.2. Let $\sigma_1$ and $\sigma_2$ be two substitutions with the same incidence matrix, we consider $X_{\sigma_1}$ (resp. $X_{\sigma_2}$) the Rauzy fractal associated to $\sigma_1$ (resp. $\sigma_2$) and $X_{\sigma}$ the common point of $X_{\sigma_1}$ and $X_{\sigma_2}$. Then the boundary of $X_{\sigma}$ is included in the union boundary of $X_{\sigma_1}$ and $X_{\sigma_2}$ and has zero measure.

Proof. Let $x$ a point from the boundary of $X_{\sigma}$. We suppose that $x$ is not on the boundary of $X_{\sigma_1}$. Then there exist $r_1 > 0$ such that $B(x, r_1) \subset X_{\sigma_1}$. If $x$ is not in the boundary of $X_{\sigma_2}$ then there exist $r_2 > 0$ such that $B(x, r_2) \subset X_{\sigma_2}$. Then there exist $r = \min(r_1, r_2)$ such that $B(x, r) \subset X_{\sigma_1} \cap X_{\sigma_2}$, $x$ is in the boundary of $X_{\sigma}$ then $x$ is in the boundary of $X_{\sigma_2}$. Then $\partial X_{\sigma} \subset \partial X_{\sigma_1} \cup \partial X_{\sigma_2}$. Since $\partial X_{\sigma_1}$ and $\partial X_{\sigma_2}$ have zero measure then $\partial X_{\sigma}$ has zero measure.

Figure 4: Sets of common points of the fractals of Tribonacci and the flipped Tribonacci.

3.1 The main result: Morphism generating the common points of two Pisot substitutions with the same incidence matrix

In this section we consider $\sigma_1$ and $\sigma_2$ be two unimodular irreducible Pisot substitutions with the same incidence matrix. We denote $X_{\sigma_1}$ and $X_{\sigma_2}$ their associated
Rauzy fractals respectively. We suppose that 0 is an inner point to \(X_\sigma_1\). We note \(X_\sigma\) the closure of the intersection of the interior of \(X_\sigma_1\) and the interior of \(X_\sigma_2\). Let \((\Omega_\sigma_1, S)\) and \((\Omega_\sigma_2, S)\) the symbolic dynamical systems associated to \(\sigma_1\) and \(\sigma_2\) respectively. We consider \(\pi_1\) (resp. \(\pi_2\)) the projection map from the symbolic dynamical system \((X_\sigma_1, S)\) into the Rauzy fractal (rep. \(\pi_2\)).

We will prove that \(X_\sigma\) is a substitutive set, and it can be generated by a substitution obtained with algorithm generating the common point of the interior of \(X_\sigma_1\) and \(X_\sigma_2\).

**Definition 3.1.** For a dynamical system \((X, T)\) if \(A\) is a subset of \(X\), and \(x \in A\), we define the first return time of \(x\) as \(n_x = \inf\{n \in \mathbb{N}^*|T^n(x) \in A\}\) (it is infinite if the orbit of \(x\) does not come back to \(A\)). If the first return time is finite for all \(x \in A\), we define the induced map of \(T\) on \(A\) (or first return map) as the map \(x \mapsto T^{n_x}(x)\), and we denote this map by \(T_A\).

**Definition 3.2.** A sequence \(u = (u_n)\) is minimal (or uniformly recurrent) if every word occurring in \(u\) occurs in an infinite number of positions with bounded gaps, that is, if for every factor \(W\), there exist \(s\) such that for every \(n\), \(W\) is a factor of \(u_n \ldots u_{n+s-1}\).

**Lemma 3.1.** The closure of the intersection \(X_\sigma\) has non empty interior and non-zero Lebesgue measure.

**Proof.** We suppose that 0 is an inner point to \(X_\sigma_1\). Then there exist an open set \(U\) such that \(0 \in U \subset X_\sigma_1\). The Rauzy fractal is the closure of its interior and 0 is a point of \(X_\sigma_2\), hence there exist a sequence of points \((x_n)_{n \in \mathbb{N}}\) from the interior of \(X_\sigma_2\) which converges to 0. Then there exist open sets \(V_n\) such that \(x_n \in V_n \subset X_\sigma_2\). Since \((x_n)\) converge to 0, there exists \(N \in \mathbb{N}\) such that \(x_N \in U\). We denote by \(W\) the open set \(W = U \cap V_N\), \(W\) is non-empty and \(W \subset X_\sigma_1 \cap X_\sigma_2\). The intersection of \(X_\sigma_1\) and \(X_\sigma_2\) contains a non empty open set, hence it has non-zero Lebesgue measure. \(\square\)

We define the subgroup \(\Gamma\) of \(\mathbb{Z}^d\) as :

\[
\Gamma = \left\{ \sum_{i=1}^{d} n_i e_i / \sum_{i=1}^{d} n_i = 0, n_i \in \mathbb{Z} \right\}
\]

with \(e_i\) is the canonical bases of \(\mathbb{R}^d\).

**Lemma 3.2.** Let \(\sigma\) be an irreducible Pisot substitution, and \(X_\sigma\) its associated Rauzy fractal. If 0 is inner point to \(X_\sigma\) then \(X_\sigma\) is a fundamental domain of \(E_s\) for the projection of \(\Gamma\) on the stable space.

**Proof.** The proof is given in [13]. \(\square\)

**Lemma 3.3.** Let \(W\) be a non-empty open set in \(X_\sigma\). Let \(V_1 = \pi_1^{-1}(W)\) and \(V_2 = \pi_2^{-1}(W)\) from \(\Omega_\sigma_1\) and \(\Omega_\sigma_2\) respectively. If \(n\) is a first return time in \(V_2\) then \(n\) is a return time in \(V_1\).
such that second block obtain this morphism (or substitution). Since defined on the set of the minimal balanced blocks. There exist an algorithm to

\[ \sigma \]

time in
two minimal initial word

Proof. We have

\[ \sigma \]

X

Theorem 3.1. Let

Definition 3.3. Let \( U \) and \( V \) two finite words, we say that \( \begin{pmatrix} U \\ V \end{pmatrix} \) is balanced block if \( l(U) = l(V) \), where \( l \) is the abelianization map from \( A^* \) in \( \mathbb{Z}^d \).

Definition 3.4. A minimal balanced block is a balanced block, such for every strict prefix \( U_k \), \( V_k \) of \( U \) and \( V \) respectively of length \( k \), \( l(U_k) \neq l(V_k) \).

Lemma 3.4. Let \( u \) and \( v \) be tow fixed points of \( \sigma_1 \) and \( \sigma_2 \) respectively, then we can decompose \( u \) and \( v \) on a finite minimal balanced blocks.

Proof. Let \( u \) and \( v \) be tow fixed points of \( \sigma_1 \) and \( \sigma_2 \) respectively. We have \( 0 \in X_\sigma \) then there exist \( v_1 \) and \( v_2 \) two prefix of \( u \) and \( v \) respectively such that \( x = \pi(v_1) = \pi(v_2) \) and \( l(v_1) = l(v_2) \). We obtain a balanced block: \( \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \), we can decompose it with minimal balanced blocks and we consider the image of each new minimal balanced block with \( \sigma_1 \) and \( \sigma_2 \). Then there exist new minimal balanced blocks which appear, we consider the image of each new blocks by \( \sigma_1 \) and \( \sigma_2 \). Since every word appears with a bounded distance, all the minimal balanced blocks will appear after a finite time. Then we can obtain a decomposition of \( u \) and \( v \) with a finite number of minimal balanced blocks. A simple case appears when \( u \) and \( v \) begin with the same letter \( i \), then the first minimal balanced block is \( \begin{pmatrix} i \\ i \end{pmatrix} \).

Theorem 3.1. \( X_\sigma \) is a substitutive set.

Proof. We have \( X_\sigma \) is the closure of the projection of points associated to balanced blocks, from the two stepped lines associated to the fixed points of \( \sigma_1 \) and \( \sigma_2 \). These common points can be obtained as a fixed point of a new substitution defined on the set of the minimal balanced blocks. There exist an algorithm to obtain this morphism (or substitution). Since \( 0 \) is a point from \( X_\sigma \) there exist two minimal initial word \( v_1 \) and \( v_2 \) from the language of \( \sigma_1 \) and \( \sigma_2 \) respectively such that \( l(v_1) = l(v_2) \)

We denote the block \( \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \) and we consider \( \sigma_1(v_1) \) and \( \sigma_2(v_2) \) we obtain a second block \( \begin{pmatrix} \sigma_1(v_1) \\ \sigma_2(v_2) \end{pmatrix} \) with the property \( l(\sigma_1(v_1)) = l(\sigma_2(v_2)) \) because \( \sigma_1 \) and \( \sigma_2 \) have the same matrix. These blocks have a finite length, because the return time in \( X_\sigma \) is bounded. We consider the decomposition of this balanced block
\[
\begin{pmatrix}
\sigma_1(v_1) \\
\sigma_2(v_2)
\end{pmatrix}
\text{ with minimal balanced blocks.}
\]
This mean we can write \[
\begin{pmatrix}
\sigma_1(v_1) \\
\sigma_2(v_2)
\end{pmatrix} = \begin{pmatrix} u_1 \ldots u_k \\
 w_1 \ldots w_k
\end{pmatrix}
\text{ with the property } l(u_1) = l(w_1), \ldots, l(u_n) = l(v_n).
\]

With this method we obtain a finite numbers of blocks with the same abelianization. We consider this set of blocks and we consider the image of each block with the two substitutions \(\sigma_1\) and \(\sigma_2\) and we obtain a morphism witch generate all the common points of the stepped lines.

\[\square\]

4 Examples

4.1 Algorithm to obtain the morphism of the common points of two Rauzy fractals

4.1.1 Example 1

I will take the example of \(\tau_1\) and \(\tau_2\) to show how the algorithm is working. In this example the first minimal balanced block that we consider is the beginning of the two fixed points associated to \(\tau_1\) and \(\tau_2\) it will be \(\begin{pmatrix} a \\ a \end{pmatrix}\).

And we consider the image of the first element of this block by \(\tau_1\) and the second one by \(\tau_2\) so we obtain :
\[
\begin{pmatrix} a \\ a \end{pmatrix} \xrightarrow{\tau_1, \tau_2} \begin{pmatrix} a & b & a \\ a & a & b \end{pmatrix}.
\]

We denote by \(A\) the minimal balanced block \(\begin{pmatrix} a \\ a \end{pmatrix}\) and by \(B\) the minimal balanced block \(\begin{pmatrix} b & a \\ a & b \end{pmatrix}\).

So we obtain \(A \rightarrow AB\).

The second step is to consider the same thing with the new block \(\begin{pmatrix} b & a \\ a & b \end{pmatrix}\).

We consider the image of this block with the two substitution \(\tau_1\) and \(\tau_2\), and we obtain :
\[
\begin{pmatrix} b & a \\ a & b \end{pmatrix} \xrightarrow{\tau_1, \tau_2} \begin{pmatrix} a & b & a \\ a & b & a \\ b & a & b \end{pmatrix}.
\]

We obtain an other block \(\begin{pmatrix} b \\ b \end{pmatrix}\) and we denote by \(C\) the projection over this new block and we obtain the image of \(B\) is \(ABCA\). We continuous with this algorithm and we obtain the image of the block \(\begin{pmatrix} b \\ b \end{pmatrix}\) is the new block \(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\). So we obtain the image of the letter \(C\) is a new letter \(D\). Finally the image of the letter \(D\) is \(DAAC\). So, we obtain an alphabet \(B\) in 4 letters and we can define the morphism \(\phi\) as :

\[
\begin{pmatrix} b \\ b \end{pmatrix} \xrightarrow{\tau_1, \tau_2} \begin{pmatrix} ababa \\ aabba \\ ababa \\ aabba \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \\ ababa \\ aabba \
\end{pmatrix}
\]
\[
\phi : \begin{cases}
  A \rightarrow AB \\
  B \rightarrow ABCA \\
  C \rightarrow D \\
  D \rightarrow DAAC
\end{cases}
\]

And we consider the projection \( \pi \) of the letters \( A, B, C, D \) in the sets of blocks 
\[
\begin{pmatrix} a \\ a \end{pmatrix}, \begin{pmatrix} ba \\ ab \end{pmatrix}, \begin{pmatrix} b \\ b \end{pmatrix} \text{ et } \begin{pmatrix} ab \\ ba \end{pmatrix}
\]

Then we have: 
\[
(\tau^n_1(a) \cap \tau^n_2(a)) = \pi(\phi^n(A)).
\]

The morphism \( \phi \) generate all the common points of the two Rauzy fractals associated to \( \tau_1 \) and \( \tau_2 \).

![Figure 5: The Rauzy fractals of \( \tau_1 \) and \( \tau_2 \).](image)

![Figure 6: Common points of \( \tau_1 \) and \( \tau_2 \) with distinction of blocs defined with \( \phi \)](image)

### 4.1.2 Example 2

For the two substitutions of Tribonacci and the flipped Tribonacci it is more complicated see Figure[7], we can define the morphism \( \phi \) which generate all the common points as follows:

\[
\phi : \begin{cases}
  A \rightarrow AB \\
  B \rightarrow C \\
  C \rightarrow AD \\
  D \rightarrow AE \\
  E \rightarrow F \\
  F \rightarrow ADDGA \\
  G \rightarrow AH \\
  H \rightarrow ID \\
  I \rightarrow ADJ \\
  J \rightarrow AHK \\
  K \rightarrow IDGA
\end{cases}
\]

and the projection map \( \pi \) :
Corollaire 4.1. Let $\sigma_1$ and $\sigma_2$ be the two substitution Tribonacci and the flipped Tribonacci defined as follows:
We consider $U$ and $V$ their two fixed points, then the letter $c$ doesn’t occur in the same position in $U$ and $V$.

**Proof.** Minimal balanced blocks represents a decomposition of the two fixed points $U$ and $V$. We remark that in these finite minimal blocks there is no $c$ which appears in the same position. One can then deduce that the letter $c$ does not appear in the same position in two fixed points $U$ and $V$. \hfill \square

### 4.1.3 Example 3

Now we will consider more general example defined as follows:

$$
\delta_1^i : \begin{cases}
  a \rightarrow a^ib \\
  b \rightarrow a^{i-1}c \\
  c \rightarrow a
\end{cases} \quad \text{and} \quad \delta_2^i : \begin{cases}
  a \rightarrow aba^{i-1} \\
  b \rightarrow aca^{i-2} \\
  c \rightarrow a
\end{cases}
$$

$\delta_1^i$ and $\delta_2^i$ have the same incidence matrix. We can define the morphism of their common points for all $i \geq 3$ as:

$$
\phi_i : \begin{cases}
  A \rightarrow AB \\
  B \rightarrow AC \\
  C \rightarrow (AAD)^{i-1}[AAE(AAD)^{i-2}AAE(AAD)]^{i-1}A \\
  D \rightarrow AF \\
  E \rightarrow (AAD)^{i-3}A \\
  F \rightarrow (AAD)^{i-1}[AAE(AAD)^{i-3}AAE(AAD)]^{i-1}A.
\end{cases}
$$

Figure 8: The Rauzy fractals of $\delta_1^3$ and $\delta_3^2$. 
Remark. The property 0 is inner point is sufficient, and we have this example of substitutions with the same incidence matrix but the intersection is reduced to the origin.

We can give an example where the intersection is empty. We consider the two substitutions \( \chi_1 \) and \( \chi_2 \) defined as follows:

\[
\chi_1 : \begin{cases} 
    a \rightarrow aab \\
    b \rightarrow ab 
\end{cases} \quad \text{and} \quad \chi_2 : \begin{cases} 
    a \rightarrow baa \\
    b \rightarrow ba 
\end{cases}
\]

Proof. We consider \( u_1 \) and \( u_2 \) the two fixed points associated to \( \chi_1 \) and \( \chi_2 \) respectively. If \( a.x \) is a prefix of \( u_1 \) then \( b.x \) is a prefix of \( u_2 \).

We will reason by induction: for \( x = a \) it is so verified for \( n = 1 \). We suppose now that \( a.x \) is prefix of \( u_1 \) and \( b.x \) is a prefix of \( u_2 \) with \( |x| = n \). \( \chi_1(a.x) = aab\chi_1(x) \) is prefix of \( u_1 \), \( \chi_2(b.x)b = ba\chi_2(x)b \) is a prefix of \( u_2 \) if and
only if $\chi_1(x) = \chi_2(x)b$.

We have for the two letter $a$ and $b$:

- $x = a : b.\chi_1(a) = baab = \chi_2(a)b$.
- $x = b : b.\chi_1(b) = bab = \chi_2(b)b$.

We consider now $x = x_1x_2\ldots x_n$ with $x_i \in \{a, b\}$

$b.\chi_1(x) = b.\chi_1(x_1x_2\ldots x_n) = b.\chi_1(x_1)\ldots \chi_1(x_n)$

$\vdots$

$= \chi_2(x_1)\chi_2(x_2)\ldots \chi_2(x_n)b$

So we prove that there exist an infinite word $u$ such that $u_1 = a.u$ and $u_2 = b.u$. □

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