Non trivial critical exponents for finite temperature chiral transitions at fixed total fermion number

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Abstract

We analyze the finite temperature chiral restoration transition of the \((D = d + 1)\)-dimensional Gross-Neveu model for the case of a large number of flavors and fixed total fermion number. This leads to the study of the model with a nonzero imaginary chemical potential. In this formulation of the theory, we have obtained that, in the transition region, the model is described by a chiral conformal field theory where the concepts of dimensional reduction and universality do apply due to a transmutation of statistics which makes fermions act as if they were bosons, having zero energy. This result should be generic for theories with dynamical symmetry breaking, such as Quantum Chromodynamics.

Key-words: Dynamical symmetry breaking; Finite temperature QFT; Dimensional reduction; Phase transitions; Chemical potential.

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1 Introduction

Recently, Kocić and Kogut [1] have shown that, in the limit of a large number of flavors ($N$), the universality class of the finite temperature chiral symmetry restoration transition in the $3D$ Gross-Neveu model is mean field theory, in contrast to the "standard" sigma model scenario which predicts the $2D$ Ising model universality class. The responsible for the breakdown of the "standard" scenario, dimensional reduction plus universality [2], would be the absence of canonical scalar fields in the model, since the $\sigma$ auxiliary field has a composite nature. Indeed, as explained in [1], both the density and the size of the $\sigma$ meson increase with the temperature in such a way that close to the restoration temperature the system is densely populated with overlapping composites.

In opposition to what happens in the dynamics of BCS superconductors, here the constituent fermions are essential degrees of freedom, even in the scaling region. Thus, we conclude that, for this model, the effect of compactifying the Euclidean time direction is simply to regulate the infrared behavior and suppress fluctuations. This fermionic model was first analyzed in [3] and it was further argued that the results would not be an artifact of the large $N$ limit [4]. (In addition, it was shown how the Ising point is recovered in 4D Yukawa models beyond the leading order in the $1/N$ expansion and why this should not happen in Gross-Neveu models.) Lattice simulations of this 3D model have verified the predictions of the large $N$ expansion at zero temperature, at nonzero temperature, and at nonzero chemical potential separately [5]. The results for the critical indices have been checked and improved by larger scale simulations enhanced by histogram methods [6]. The conclusion was that the data are in excellent agreement with mean field theory and rule out
the Ising model values. However, the question concerning the nature of the transition for large but finite $N$ in Gross-Neveu models remained open until the recent work of Reisz [7] where he suggests that the mean field exponents are indeed an artifact of the large $N$ limit. By a combination of different techniques, like $1/N$ expansion, dimensional reduction and high-temperature series, it is shown how the 2D Ising model exponents are recovered beyond the leading order in the $1/N$ expansion in such a way that the ”standard” scenario is still valid, if $N$ is not large enough.

In this letter we shall study further the large $N$ limit of the Gross-Neveu model. We will show that, if we fix the particle number, it is possible to get non-trivial exponents even if $N$ is infinitely large. This third set of exponents is consistent with the ”standard” scenario with the difference that, as we will see, the reduced theory is not the linear sigma model but, instead, again a four fermi theory for massless fermions. According to the ”standard” linear sigma model scenario, if dimensional reduction plus universality arguments hold, the finite temperature transition of the $D$-dimensional Gross-Neveu model would lie in the same universality class of the $(d = D - 1)$-dimensional linear sigma model. This is not what is found and the reason why is easy to understand in the finite temperature Matsubara formalism: fermions do not have a zero frequency Matsubara mode. (The introduction of a real chemical potential would not change this results since it would simply cause a shift of the frequencies in the imaginary axis.) On the other hand, we will show that the imposition of a constraint which fixes the total fermion number gives rise to a fermionic zero frequency Matsubara mode. In this case, the transition has neither mean field nor scalar critical indices but rather chiral indices, a feature of theories with long range interactions. The
The basic idea is to introduce a pure imaginary chemical potential in the partition function of the model. The imaginary chemical potential arises when we impose a constraint which fixes the particle number \( \mathbb{N} \). This is generally done by inserting in the functional integral a functional delta function which enforces exactly the constraint. As a consequence, an auxiliary field is introduced in the problem. Usually, saddle-point solutions for this auxiliary field are imaginary implying in this way a real chemical potential. (This happens for example in the non-linear sigma model.)

However, as we will see, real solutions are also allowed in the Gross-Neveu model with constrained particle number. Such a solution can be thought as an imaginary chemical potential. In this new "physical" system, we do have a zero frequency Matsubara mode associated to fermions affecting the infrared sensitiveness of thermodynamic quantities and, consequently, changing its critical indices. We observed that the new critical exponents are exactly the same as those corresponding to a zero temperature dimensionally reduced effective theory. In other words, after the decoupling of nonzero frequency Matsubara modes (dimensional reduction) the effective theory for the light degrees of freedom has a set of critical indices identical to that of the original zero temperature one (universality), where one simply has to replace \( D = d + 1 \) by \( d \).
2 The Gross-Neveu model at fixed total fermion number

We start by considering a \((D = d + 1)\)-dimensional fermionic model with fixed total fermion number \(B\) described by the canonical partition function

\[
Z = \text{Tr} \left\{ e^{-\beta H} \delta(\hat{N} - B) \right\},
\]

where \(\beta\) is the inverse of the temperature, \(H\) is some Hermitian Hamiltonian and

\[
\hat{N} = \int d^d x (\psi^\dagger \psi)
\]

is the fermion number operator. We can rewrite the partition function as a functional integral over Grassmann fields

\[
Z = \int_{B.C.} D\psi \delta \left( \int d^d x (\psi^\dagger \psi) - B \right) e^{-A(\psi^\dagger,\psi)}
\]

\[
= \int_{B.C.} D\psi \delta D\theta e^{-\tilde{A}(\psi^\dagger,\psi,\theta)}
\]

where we have used the integral representation of the functional delta function. In the above equation \(A\) is the action of the unconstrained model while \(\tilde{A}\) is given by

\[
\tilde{A} = A + \int d^d x \int_0^\beta d\tau \frac{i}{\beta} \gamma_0 \psi + \frac{iB}{\beta} \int_0^\beta d\tau \theta.
\]

where the Euclidean gamma matrices are antihermitian and satisfy \(\{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu}\). As usual, the Grassmann fields are integrated with antiperiodic boundary conditions (B.C.) with respect to the Matsubara time. Note that the Bose field \(\theta\) is a function of \(\tau\) but not of \(x\). This is expected
since \(\theta\) is an auxiliary field related to a global constraint with respect to space rather than a local one.

We will use as fermionic action, the action for the massive Gross-Neveu model \([9]\) with \(N\) components. By rewriting it in its standard auxiliary field version we get the following Euclidean action for the fixed fermion number model

\[
S_E(\psi, \bar{\psi}, \sigma, \theta) = \int_0^\beta d\tau \int d^d x \left[ \bar{\psi} \left( -\gamma^0 \frac{\partial}{\partial \tau} + i\vec{\gamma} \cdot \vec{\nabla} + m + g\sigma + i\frac{\theta}{\beta} \gamma^0 \right) \psi + \frac{1}{2} \sigma^2 + \delta^d(\mathbf{x}) \frac{iB^\beta}{\beta} \theta \right].
\]  

(5)

For bare fermion mass \(m = 0\), there is a discrete chiral symmetry \(\psi \rightarrow \gamma_s \psi, \bar{\psi} \rightarrow -\bar{\psi} \gamma_s\) which is spontaneously broken whenever a non-vanishing condensate \(\langle \bar{\psi} \psi \rangle\) is generated. The condensate

\[
\langle \bar{\psi}(x)\psi(x) \rangle \propto \int \mathcal{D}\theta \mathcal{D}\sigma \mathcal{D}\bar{\psi} \mathcal{D}\psi \left[ \bar{\psi}(x)\psi(x) \right] e^{-S_E(\psi, \bar{\psi}, \sigma, \theta)},
\]  

(6)

serves as an order parameter of the transition. It is well known that this is the role played by the auxiliary field \(\sigma\) in eq. (5). Thus, we will study the phase structure of the chiral symmetry restoration transition with the aid of the order parameter \(\sigma\). Note that eq. (5) is invariant by a partially local gauge transformation. By this we mean a gauge transformation which is local in the Matsubara time and global in the space. In fact, the transformation \(\psi \rightarrow e^{i\phi} \psi, \bar{\psi} \rightarrow e^{-i\phi} \bar{\psi}\), leaves the action invariant provided that \(\theta \rightarrow \theta + \beta \partial \phi / \partial \tau\). Thus, \(\theta\) plays the role of a ”gauge field”. It should be observed that, due to the periodicity of the \(\theta\) with respect to the Matsubara time, the last term in eq. (5) is also gauge invariant: \(\int_0^\beta d\tau \partial \phi / \partial \tau = \phi(\beta) - \phi(0) = 0\). Later we shall discuss the role of this gauge symmetry with respect to the zero mode of the fermion field.
3 Saddle-point evaluation of the functional integral

Integrating out exactly the fermions in eq. (3) we obtain the effective action

\[
S_{\text{eff}} = -N \tr \log \left( -\gamma^0 \frac{\partial}{\partial \tau} + i \vec{\gamma} \cdot \vec{\nabla} + m + g \sigma + i \frac{\theta}{\beta} \gamma^0 \right)
+ \int_0^{\beta} d\tau \left( \int d^d x \frac{1}{2} \sigma^2 + i B \theta \right).
\]  

(7)

Now we perform a saddle-point evaluation of the functional integral. From the structure of \( S_{\text{eff}} \) we know that the saddle-point solution is exact at large \( N \). If \( N \) is large enough, the fluctuations of the \( \sigma \) and \( \theta \) auxiliary fields average out and for this reason we look for uniform saddle-point solutions. By imposing stationarity with respect to \( \sigma \) we obtain the gap equation

\[
\Sigma = m + \frac{4g^2}{\beta} \sum_{n=-\infty}^{\infty} \int^\Lambda d^d q \frac{\Sigma}{(2\pi)^d q^2 + (\rho - \bar{\omega}_n)^2 + \Sigma^2},
\]

(8)

where \( \bar{\omega}_n = \frac{2\pi}{\beta} (n + \frac{1}{2}) \), and we have used the fact that, to the leading order, the fermion self-energy \( \Sigma \) comes from the \( \sigma \) auxiliary field tadpole: \( \Sigma = m - g^2 < \bar{\psi}\psi > \). Here \( \Lambda \) is an ultraviolet cutoff and we have defined \( \rho = \theta / \beta \). To find the temperature dependence of the order parameter \( \sigma \) one has to solve the gap equation near criticality. The critical temperature is determined by

\[
1 = \frac{4g^2}{\beta_c} \sum_{n=-\infty}^{\infty} \int^\Lambda d^d q \frac{1}{(2\pi)^d q^2 + (\rho_c - \bar{\omega}_{nc})^2},
\]

(9)

with \( \bar{\omega}_{nc} = \frac{2\pi}{\rho_c} (n + \frac{1}{2}) \). On the other hand, stationarity with respect to \( \theta \) gives us

\[
\frac{N}{\beta} \tr [\gamma^0 \cdot S_F(\tau - \tau'; x - x')] = B,
\]

(10)
where $S_F$ denotes the fermion propagator given by the inverse of the argument of the log in Eq.(7).

Eq.(10) is equivalent to the statement $<\psi^\dagger \psi> = B$ which fixes the mean number of fermions. Note that some care is needed to evaluate the trace in (10) since $\gamma^0 \cdot S_F$ is divergent for $\tau = \tau'$ (for $x = x'$ there is no problem due to the presence of the ultraviolet cutoff $\Lambda$). The same kind of difficulty is encountered in the theory of fermion systems, for instance Fermi liquid theory, when defining expectation values of particle number operators using the one-particle propagator $\Sigma$. In order to remedy this problem it is sufficient to evaluate $\gamma^0 \cdot S_F(\tau - \tau'; 0)$ with a temporal point-splitting $\tau - \tau' = \eta$ where $\eta$ is a small positive number. After we let $\eta \to 0^+$ getting a finite result. This means that if we write the Fourier representation of $\gamma^0 \cdot S_F(\tau - \tau'; 0)$, the limit $\tau - \tau' \to 0$ does not commute with the Matsubara sum. This is the standard procedure used in many-particle systems to evaluate expectation values of one-body operators $\Sigma$. Evaluating explicitly the trace in (11), we obtain

$$\frac{1}{\beta} \sum_{n=-\infty}^{\infty} \int_{-\Lambda}^{\Lambda} \frac{d^d q}{(2\pi)^d} e^{i\bar{\omega}_n}(\rho - \bar{\omega}_n) = \frac{-ib}{4},$$

where $\eta \to 0^+$ and we have introduced the barion density $b = B/(N\Omega)$, $\Omega$ being the volume. Note that the sum is divergent without the convergence factor $e^{i\bar{\omega}_n}$ even if $\rho = 0$. It is divergent simply because it is not defined (like the series $1 - 1 + 1 - 1 + ...$).

The Matsubara sums in eqs.(8) and (11) are done by contour integration and we obtain

$$\Sigma = m + 2g^2 \Sigma \int_{-\Lambda}^{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{\omega} \left( \frac{1}{e^{\beta\omega} + 1} - \frac{1}{e^{\beta\omega} - 1} \right),$$

where $\eta \to 0^+$ and we have introduced the barion density $b = B/(N\Omega)$, $\Omega$ being the volume. Note that the sum is divergent without the convergence factor $e^{i\bar{\omega}_n}$ even if $\rho = 0$. It is divergent simply because it is not defined (like the series $1 - 1 + 1 - 1 + ...$).
for eq. (8) and

\[ 2 \int_{\Lambda} \frac{d^d q}{(2\pi)^d} \left( \frac{1}{e^{i\theta} e^{\beta\omega} + 1} + \frac{1}{e^{i\theta} e^{-\beta\omega} + 1} \right) = b, \]  

(13)

for eq. (11) and we have defined \( \omega = \sqrt{q^2 + \Sigma^2} \). One saddle-point solution for \( \theta \) is \( \theta = i\beta\mu \), \( \mu \) playing the role of an ordinary chemical potential. This solution is non-singular in the infrared since it does not generate any zero mode. In this situation the exponents are mean field. However, the solution \( \theta = (2m + 1)\pi \), where \( m \) is an integer, is also possible if \( B \) is even. (We need \( B \) even in order to ensure that the free energy is a real quantity.) Note that every solution of this form corresponds to the same free energy and therefore we can choose any of them, say \( \theta = \pi \). This particular solution is related to any other solution \( \theta = (2m + 1)\pi \) by a gauge transformation. The existence of such solutions is a consequence of the partially local gauge symmetry we have discussed in the beginning of this paper.

4 Scaling properties at fixed fermion number

Let us discuss the implications of the solution \( \theta = \pi \) to the critical behavior of the model. First, by using eq. (9) together with eq. (8) we can rewrite the gap equation for the order parameter \( \sigma \) as

\[
\sum \frac{m}{\Sigma} + \frac{4g^2}{\beta_c} \sum_{n=-\infty}^{\infty} \int_{\Lambda} \frac{d^d q}{(2\pi)^d} \frac{(q^2 + (\rho_c - \bar{\omega}_n)^2) \left( \frac{\beta_c}{\beta} \right) - (q^2 + (\rho - \bar{\omega}_n)^2 + \Sigma^2) (q^2 + (\rho - \bar{\omega}_n)^2 + \Sigma^2)}{(q^2 + (\rho_c - \bar{\omega}_n)^2) (q^2 + (\rho - \bar{\omega}_n)^2 + \Sigma^2)} = 0. \]  

(14)

This form of the gap equation is particularly well suited for extracting critical indices since the problem reduces to the power counting of the infrared divergences in the integral over \( q \). For
\[ \theta = \pi \] we see that the above integral is infrared divergent as \( \Sigma \to 0 \). The critical indices are defined by \( \langle \bar{\psi} \psi \rangle \big|_{m \to 0} \sim t^\beta \), \( \langle \bar{\psi} \psi \rangle \big|_{t \to 0} \sim m^{1/\delta} \), \( \langle \Sigma \rangle \big|_{m \to 0} \sim t^\nu \) etc and, since \( \Sigma \sim \langle \bar{\psi} \psi \rangle \), \( \beta = \nu \) to the leading order. Above four dimensions the integral is infrared finite and the scaling is mean field. Below four dimensions, however, the \( \Sigma \to 0 \) limit is singular and the integral scales as \( (\bar{\omega}_n^2 t^2 + \Sigma^2)^{\frac{4-D}{2}} \). At a critical point \( t = 0 \) away from the chiral limit we have \( m \sim \Sigma^{d-1} \). Thus, we can easily see that, below four dimensions the \( \beta \) and \( \delta \) exponents are

\[ \beta = \frac{1}{d-2}, \quad \delta = d - 1. \]  

(15)

The remaining exponents are easily obtained as well, \( \eta = 4 - d \) and \( \gamma = 1 \), and one can check that they obey hyperscaling. If we compare these indices with those obtained from the zero temperature case: \( \beta = \frac{1}{D-2}, \delta = D - 1, \eta = 4 - D, \gamma = 1 \) (for a recent review see [13]), we conclude that both sets define a chiral conformal field theory in \( d \) and \( (D = d + 1) \) dimensions respectively. We should also note that the exponents (15) are the same as for a zero temperature dimensionally reduced Gross-Neveu model in which the chiral symmetry restoration transition will occur as we approach a thermally renormalized coupling constant obtained during the reduction procedure.

It should be noted that the real solution \( \theta = \pi \) makes the model to behave as a bosonic model. Thus, this solution transmute the statistics of the fermions into bosons. Unlike to statistical transmutation in Chern-Simons models, the present situation is not confined to \( d = 3 \).
5 Conclusions

We have shown that, even in the absence of canonical scalar fields, we do obtain non mean field critical exponents for the finite temperature chiral restoration transition in a four-fermion theory provided we introduce a global constraint fixing the particle number. At the saddle-point level, it amounts to introducing an imaginary chemical potential which plays the role of a "gauge" field. This procedure gives rise to a zero frequency Matsubara mode which is interpreted, after the limit of high temperatures, as the only relevant degree of freedom of the reduced theory. The reduced theory is still a four-fermi theory at zero temperature and in a lower dimension whose parameters carry a temperature dependence resulting from the reduction procedure [14]. It also exhibits dynamical breaking of discrete chiral symmetry which is restored near the critical thermally renormalized coupling constant, in such a way that the universality class is the same as the original finite temperature model.

This behavior should also be expected in any model with dynamical symmetry breaking, such as QCD, and at finite density, since, on general grounds, the (bulk) behavior of the system at real (fixed) chemical potential $\mu$ is identical to that at a fixed fermion number $B$, provided $B$ is the mean fermion number for the system at chemical potential $\mu$. 
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