Realization of the minimal extended seesaw mechanism and $TM_2$ type neutrino mixing in the light of $A_4 \times C_4$ symmetry group

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Abstract

We study the phenomenology of an $A_4 \times C_4$ based neutrino mass model accommodating a light sterile neutrino in the minimal extended seesaw scheme. Mass terms consisting of the Standard Model neutrinos, the right-handed heavy neutrinos and the sterile neutrinos are obtained in terms of the vacuum alignments of a set of flavons transforming under $A_4 \times C_4$. The corresponding mass matrices when incorporated into the MES formula, give rise to the TM$_2$ mixing pattern having non-zero reactor angle. We express all the active and the sterile oscillation observables in terms of only four real model parameters. Using this highly constrained scenario we predict $\sin^2 \theta_{23} = 0.545^{+0.003}_{-0.004}$, $\sin \delta = -0.911^{+0.006}_{-0.005}$, $|U_{e4}|^2 = 0.029^{+0.009}_{-0.008}$, $|U_{\mu 4}|^2 = 0.010^{+0.003}_{-0.003}$ and $|U_{\tau 4}|^2 = 0.006^{+0.002}_{-0.002}$ which are consistent with the current data.

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Observations made in the neutrino oscillation experiments have confirmed that neutrinos have mass, albeit tiny. The Standard Model (SM) of particle physics cannot accommodate the neutrino mass due to the absence of right-handed neutrinos, unlike the case for the charged leptons and the quarks. The inclusion of additional right-handed neutrino fields along with the seesaw mechanism \([1–4]\) plays a vital role in modelling properties of massive neutrinos. The well known PMNS matrix encodes the mixing between the neutrino flavour eigenstates and their mass eigenstates. This matrix is parametrised in terms of three mixing angles and three CP phases (in a three flavoured paradigm),

\[
U_{\text{PMNS}} = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\
    -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13}
\end{pmatrix} \cdot U_{\text{Maj}},
\]

where \(c_{ij} = \cos \theta_{ij}, \ s_{ij} = \sin \theta_{ij}\). The diagonal matrix, \(U_{\text{Maj}} = \text{diag}(1, e^{i\alpha}, e^{i(\beta + \delta)})\), contains the Majorana CP phases \(\alpha, \beta\) which become observable if the neutrinos behave as Majorana particles.

Although the last two decades of neutrino oscillation experiments made tremendous progress in determining the three flavour mixing angles, efforts are underway to measure these parameters more precisely. We do not yet know whether the atmospheric mixing is maximal or not. If it is not, the octant of the atmospheric mixing angle, \(\theta_{23}\), is to be determined. Measurement of the Dirac CP phase, \(\delta\), will confirm CP violation in the leptonic sector and may explain the observed baryon asymmetry via leptogenesis. The nature of the neutrinos, i.e. whether they are Dirac or Majorana, is still an open question which can not be settled with the help of the oscillation experiments. On the other hand, the observation of neutrino-less double-beta decays (0\(\nu\beta\beta\)) will establish the Majorana nature. Such decays are yet to be observed. The oscillation experiments have determined the mass-squared differences (solar: \(\Delta m_{21}^2\) and atmospheric: \(\Delta m_{31}^2\)), but they are not sensitive to the absolute neutrino mass scale. Data from the Planck satellite provides an upper bound on the sum of neutrino masses, \(\sum_{i} m_i \leq 0.16 \text{ eV} \ [5]\). Experimental searches are also being made to directly measure the electron neutrino mass using the kinematics of beta decays. Recently, the KATRIN collaboration has announced its first result on the effective electron antineutrino mass using the tritium beta decay, \(^3\text{H} \rightarrow ^3\text{He} + e^- + \bar{\nu}_e\), and reported the upper bound for
the effective antineutrino mass [6, 7], \( m_{\bar{\nu}_e} < 1.1 \) eV at the 90\% confidence level (CL).

Although the three flavour paradigm of neutrino oscillation is well established, there are some experimental results that motivate us to go beyond this and postulate the existence of one or more sterile neutrinos. This possibility has gained considerable attention in recent years. In principle, the presence of a fourth neutrino can impressively explain several sets of experimental anomalies. The first indication came from the LSND experiment which showed evidence of oscillation with mass scale \( \sim eV^2 \) [8–10] in \( \bar{\nu}_\mu \leftrightarrow \bar{\nu}_e \) channel. Later MiniBooNE experiment also confirmed it [11]. The Reactor Anomaly involves a deficit of reactor antineutrinos detected in short-baseline (<500 m) experiments with recalculated neutrino fluxes [12]. The short-baseline neutrino oscillations can also explain the so-called Gallium Anomaly observed during the calibrations runs of the radiochemical experiments, GALLEX and SAGE. The ratio of the experimental flux to the theoretical estimate was found to be 0.86 ± 0.05. The resolution of both the Reactor and the Gallium anomalies with the help of the active-sterile oscillations point towards a common region of the parameter space with the sterile neutrino having mass in the \( \sim eV \) scale [13, 14].

The proposed sterile neutrino is an SM singlet which does not participate in the weak interactions, but they can mix with the active neutrinos enabling them to be probed in the oscillation experiments. The addition of a single sterile neutrino field leads to an oscillation parameter space consisting of a 4 \times 4 unitary mixing matrix along with three independent mass-squared-differences. Among them, the preferred scenario, often called the 3+1 scheme [15–18], has three active neutrinos and one sterile neutrino in the sub-eV and eV scale respectively. The 2+2 scheme, in which two pairs of neutrino mass states differ by \( O(eV) \), is not consistent with the solar and the atmospheric data [19]. The 1+3 scheme in which the three active neutrinos are in eV scale and the sterile neutrino is lighter than the active neutrinos is disfavoured by cosmology. Therefore, in this paper, we assume the 3+1 scenario. The recently proposed Minimal Extended Seesaw (MES) [20, 21] has many appealing features. The active-sterile mixing obtained in MES is suppressed by the ratio of masses of the active and sterile sectors. With the active neutrino mass of the order of \( \sim 0.01 \) eV and the sterile neutrino mass of the order of eV, this suppression is consistent with the active-sterile mixing as observed in LSND and MiniBooNe.

A large number of neutrino mass models based on discrete flavour symmetry groups have been proposed [22–25] in the last decade. These models generate various mixing patterns
such as the well known tribimaximal mixing (TBM) [26–32]. Since the non-zero value of the reactor mixing angle [33–37] has ruled out TBM, one of the popular ways to achieve realistic mixings is through either its modifications or extensions [38–43]. Unlike the active-only mixing scenarios, realising the minimal extended type-I seesaw with the help of discrete groups is somewhat recent and limited [44–46]. It is in this context that we propose a model to implement the MES and obtain oscillation observables consistent with the latest experiments. Our model produces an extension of the TBM called the TM$_2$ [47–54] in which the second column of the TBM is preserved. We use $A_4 \times C_4$ as the flavour group for our model. We propose several scalar fields, often called the flavons, which couple with the charged-lepton fields as well as the various neutrino fields. The inherent properties of $A_4$ and $C_4$ as well as the residual symmetries of the vacuum alignments of the flavons, determine the structure and the symmetries of the mass matrices.

The content of this paper is organised as follows. The features of the MES scheme are outlined in Section II. In Section III, we briefly explain the representation theory of the flavour group and move on to construct the Yukawa Lagrangian based on the proposed flavon content of the model. We also provide the Vacuum Expectation Values (VEVs) of these flavons. The flavon potentials which lead to these VEVs are constructed in the Appendix A. In Section IV, the mass matrices are constructed in terms of the VEVs. We provide the formulae for various experimental observables as functions of the model parameters. In Section V, we compare these formulae with the experimental results and make predictions. Section VI is kept for drawing the conclusion of the work.

II. MINIMAL EXTENDED SEESAW

In the Standard Model, the left-handed charged-lepton fields, $l_L = (e_L, \mu_L, \tau_L)^T$, and the neutrino fields, $\nu_L = (\nu_e, \nu_\mu, \nu_\tau)^T$, transform as the $SU(2)$ doublet, $L = (\nu_L, l_L)^T$. They couple with the right-handed charged-lepton fields, $l_R = (e_R, \mu_R, \tau_R)^T$, to form the charged-lepton mass term,

$$\bar{L}y_l l_R H,$$

(2)

where $y_l$ are the Yukawa couplings. In general, $y_l$ is a $3 \times 3$ complex matrix. The electroweak symmetry is spontaneously broken when the Higgs acquires the VEV,

$$\langle H \rangle = (0, v)^T.$$

(3)
Subsequently, the mass term, Eq. (2), becomes
\[ \bar{l}_LM_lR, \] (4)
where \( M_l = vy_l \) is the charged-lepton mass matrix.

In the type-I seesaw framework, we add extra right-handed neutrino fields, \( \nu_R \), to the SM. We may assume that three families of such fields exist, i.e. \( \nu_R = (\nu_{R1}, \nu_{R2}, \nu_{R3})^T \). They couple with the left-handed fields, \( L \), forming the Dirac neutrino mass term,
\[ \bar{L}y_L \nu R \tilde{H}, \] (5)
where \( \tilde{H} = i\sigma_2 H \). As a result of the Spontaneous Symmetry Breaking (SSB), this term becomes
\[ \bar{\nu}_L M_D \nu_R, \] (6)
where \( M_D = vy_\nu \) is the Dirac neutrino mass matrix. The right-handed neutrino fields can couple with themselves resulting in the Majorana mass term,
\[ \frac{1}{2} \bar{\nu}_R M_R \nu_R, \] (7)
where \( M_R \) is the \( 3 \times 3 \) Majorana neutrino mass matrix which is assumed to have a value of the order of \( 10^{15} \) GeV. The canonical type-I seesaw can be extended to accommodate an eV-scale sterile neutrino at the cost of no fine-tuning of the Yukawa coupling. To implement this MES scheme we need to include an extra gauge singlet field, \( \nu_s \), which couples with the heavy neutrino fields, \( \nu_R \), leading to
\[ \bar{\nu}_s M_s \nu_R, \] (8)
where \( M_s \) is a \( 1 \times 3 \) mass matrix. We assume that the coupling of the sterile field \( (\nu_s) \) with itself as well as with the left-handed fields \( (L) \) is forbidden.

Combining Eqs. (6, 7, 8), we obtain the Lagrangian containing the neutrino mass matrices relevant to the MES:
\[ \mathcal{L}_\nu = \bar{\nu}_L M_D \nu_R + \frac{1}{2} \bar{\nu}_R M_R \nu_R + \bar{\nu}_s M_s \nu_R + h.c. \] (9)

The Lagrangian, Eq. (9), leads to the following \( 7 \times 7 \) neutrino mass matrix in the \( (\nu_L, \nu_R^c, \nu_s^c) \) basis:
\[ M_\nu^{7 \times 7} = \begin{pmatrix} 0 & M_D & 0 \\ M_D^T & M_R & M_s^T \\ 0 & M_s & 0 \end{pmatrix}. \] (10)
Being analogous to the canonical type I seesaw, the MES scheme allows us to have the hierarchical mass spectrum assuming $M_R >> M_s > M_D$. The right-handed neutrinos are much heavier compared to the electroweak scale enabling them to be decoupled at the low scale. As a result, Eq.(10) can be block diagonalized to obtain the effective neutrino mass matrix in the $(\nu_L, \nu_c)$ basis,

$$M^4_{\nu} = \begin{pmatrix} -M_D M^{-1}_R M^T_D & M_D M^{-1}_R M^T_s \\ M_s (M^{-1}_R)^T M^T_D & M_s M^{-1}_R M^T_s \end{pmatrix}.$$  

(11)

This particular type of model is a minimal extension of the type I seesaw in the sense that only an extra sterile field is added whose mass is also suppressed along with that of the three active neutrinos. Since $M^4_{\nu}$ has rank three, one of the active neutrinos becomes massless\(^1\).

Assuming $M_s > M_D$, we may apply a further seesaw approximation on Eq.(11) to get the active neutrino mass matrix,

$$M^3_{\nu} \simeq M_D M^{-1}_R M^T_s (M_s M^{-1}_R M^T_s)^{-1} M_s M^{-1}_R M^T_D - M_D M^{-1}_R M^T_D.$$  

(12)

It is worth mentioning that the RHS of Eq. (12) remains non-vanishing since $M_s$ is a row vector $1 \times 3$ rather than a square matrix. Under the approximation $M_s > M_D$, we also obtain the mass of the 4th mass eigenstate\(^2\),

$$m_4 \simeq M_s M^{-1}_R M^T_s.$$  

(13)

The charged-lepton mass matrix, $M_l$, Eq. (4), is a $3 \times 3$ complex matrix in general. Its diagonalisation leads to the charged-lepton masses,

$$U_L M_l U_R^\dagger = \text{diag}(m_e, m_\mu, m_\tau),$$  

(14)

where $U_L$ and $U_R$ are unitary matrices. The low energy effective $3 \times 3$ neutrino mass matrix, $M^3_{\nu}$, Eq. (12), is complex symmetric. Its diagonalisation is given by

$$U^\dagger_{\nu} M^3_{\nu} U^*_\nu = \text{diag}(m_1, m_2, m_3),$$  

(15)

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\(^1\) If we want to accommodate more than one sterile neutrino at the eV scale, we need to increase the number of heavy neutrinos as well. Otherwise, more than one active neutrino becomes massless which is ruled out experimentally.

\(^2\) Since the active-sterile mixing is small, the 4th mass eigenstate ($\nu_4$) more or less corresponds to the sterile state ($\nu_s$).
where $U_\nu$ is a unitary matrix and $m_1$, $m_2$ and $m_3$ are the light neutrino masses$^3$. Using $U_L$ and $U_\nu$, we obtain the $4 \times 4$ light neutrino mixing matrix,

$$U \simeq \begin{pmatrix} U_L(1 - \frac{1}{2}RR^\dagger)U_\nu & U_LR \\ -R^\dagger U_\nu & 1 - \frac{1}{2}R^\dagger R \end{pmatrix},$$

(16)

where the three-component column vector $R$ is given by

$$R = M_D M_R^{-1} M_s^T (M_s M_R^{-1} M_s^T)^{-1}.$$

(17)

$U$, Eq. (16), relates the neutrino flavour eigenstates with the neutrino mass eigenstates,

$$U(\nu_e, \nu_\mu, \nu_\tau, \nu_s)^T = (\nu_1, \nu_2, \nu_3, \nu_4)^T,$$

(18)

in the basis where the charged-lepton mass matrix is diagonal. From Eq. (16), it is evident that the strength of the active-sterile mixing is governed by

$$U_L R = (U_{e4}, U_{\mu4}, U_{\tau4})^T.$$

(19)

Note that $R$ is suppressed by the ratio $O(M_D)/O(M_s)$. The $3 \times 3$ mixing matrix involving the three active neutrinos, $(\nu_e, \nu_\mu, \nu_\tau)$, and the three lightest mass eigenstates, $(\nu_1, \nu_2, \nu_3)$, can be approximately given by

$$U_{PMNS} \simeq U_L U_\nu.$$

(20)

III. FLAVOUR STRUCTURE OF THE MODEL

A. The flavour group: $A_4 \times C_4$

We construct the model in the framework of the discrete group $A_4 \times C_4$. $A_4$, which is the smallest group with a triplet irreducible representation, has been studied extensively in the literature [29, 55–61]. Here we briefly mention the essential features of this group in the context of model building. $A_4$ is the rotational symmetry group of the regular tetrahedron. It has the group presentation,

$$\langle S, T \mid S^2 = T^3 = (ST)^3 = I \rangle.$$

(21)

$^3$ In the MES framework we have $m_1 = 0.$
|   | (1) | (12)(34) | (123) | (132) |
|---|-----|----------|-------|-------|
| 1 | 1   | 1        | 1     | 1     |
| ω | 1   | 1        | ω     | ω̄    |
| ω̄| 1   | 1        | ω̄    | ω     |
| 3  | 3   | −1       | 0     | 0     |

TABLE I: The character table of the $A_4$ group. $ω$ and $ω̄$ are the complex cube roots of unity $e^{i\frac{2\pi}{3}}$ and $e^{-i\frac{2\pi}{3}}$ respectively.

$A_4$ has 12 elements which fall under four conjugacy classes. Its conjugacy classes and irreducible representations are listed in Table I.

For the triplet representation, $3$, we choose the following basis,

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (22)$$

The representations $ω$ and $ω̄$ transform as $ω$ and $ω̄$ respectively under the generator $T$ and trivially under the generator $S$. The tensor product of two triplets, $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$, leads to

$$1 \equiv x_1 y_1 + x_2 y_2 + x_3 y_3, \quad (23)$$
$$ω \equiv x_1 y_1 + ω x_2 y_2 + ω x_3 y_3, \quad (24)$$
$$ω̄ \equiv x_1 y_1 + ω̄ x_2 y_2 + ω x_3 y_3, \quad (25)$$
$$3 \equiv (x_2 y_3 + x_3 y_2, x_3 y_1 + x_1 y_3, x_1 y_2 + x_2 y_1)^T, \quad (26)$$
$$3 \equiv (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)^T. \quad (27)$$

Along with $A_4$, we also utilise the group $C_4$ in our model. $C_4$ is the cyclic group with four elements. As an irreducible singlet representation, these elements can be obtained as \{i, −1, −i, 1\} with $i$ acting as the generator. The elements \{−1, 1\} form a $C_2$ subgroup of $C_4$. $C_4$ plays a crucial role in determining the vacuum alignments of the flavons proposed in our model.
B. Field assignments and the Lagrangian

We extend the SM particle sector by the inclusion of three right-handed neutrinos, \( \nu_R = (\nu_{R1}, \nu_{R2}, \nu_{R3})^T \), and an eV scale sterile neutrino \( \nu_s \) in the fermion sector and four flavon multiplets, \( \phi_l, \eta, \phi, \phi_s \), in the scalar sector. The field content of the model, along with the irreducible representations of \( A_4 \times C_4 \) which they belong to, are given in Table II.

Under \( A_4 \), the three flavours of the lepton doublets, \( L = (L_e, L_\mu, L_\tau)^T \), as well as the three right-handed neutrinos \( \nu_R = (\nu_{R1}, \nu_{R2}, \nu_{R3})^T \), transform as triplets \( (3) \) whereas the three right-handed charged leptons, \( e_R, \mu_R \) and \( \tau_R \), transform as singlets, \( (1), (\bar{\omega}), (\bar{\bar{\omega}}) \) respectively. The sterile neutrino is an invariant singlet. The flavons \( \phi_l, \phi, \phi_s \) transform as triplets and the flavon \( \eta \) is an invariant singlet. These fields are also assigned various charges under the \( C_4 \) group which facilitates the required couplings among them.

Given the above-mentioned field assignments, we obtain the following Yukawa Lagrangian:

\[
\mathcal{L}_Y = Y_e \left( \bar{L}_e \frac{\phi_l}{\Lambda} \right)_1 e_R H + Y_\mu \left( \bar{L}_\mu \frac{\phi_l}{\Lambda} \right) \mu_R H + Y_\tau \left( \bar{L}_\tau \frac{\phi_l}{\Lambda} \right) \tau_R H \\
+ Y_\eta \left( \bar{L}_\nu_R \right)_1 \frac{\eta}{\Lambda} \tilde{H} + Y_{\bar{\nu}} \left( \bar{L}_\nu_R \right)_{3s} \frac{\phi}{\Lambda} \tilde{H} + Y_{\phi_a} \left( \bar{L}_\nu_R \right)_{3a} \frac{\phi}{\Lambda} \tilde{H} \\
+ Y_s \bar{\nu}_s^c ( \nu_R \phi_s )_1 + M ( \bar{\nu}_R^c \nu_R )_1
\]

where \( Y_e, Y_\mu, Y_\tau, Y_{\eta}, Y_s, Y_{\phi}, Y_{\phi_s} \) are the Yukawa-like dimensionless coupling constants, \( M \) is a constant of mass dimension one and is of the order of \( 10^{15} \) GeV comparable to the GUT scale, \( \Lambda \) is the cut off scale of the theory. \( (1), (\bar{\omega}), (\bar{\bar{\omega}}) \) are the tensor products given in Eqs. (23-27) respectively. We assign an additional \( C_4 \) charge to \( \nu_s \) i.e. \( \nu_s \rightarrow i\nu_s \), so that the terms \( (\bar{\nu}_s^c \nu_s)_1 \) and \( \bar{L}_\nu_R \frac{\phi}{\Lambda} \tilde{H} \) are forbidden. Under this \( C_4 \), \( \phi_s \) is assigned to transform as \( \phi_s \rightarrow -i\phi_s \), so that the term \( \bar{\nu}_s^c ( \nu_R \phi_s )_1 \) is allowed.

In the above Lagrangian, the first three terms are responsible for the charged-lepton mass generation. \( \bar{L}_e_R, \bar{L}_\mu_R \) and \( \bar{L}_\tau_R \) are invariants under \( C_4 \). Therefore the flavon \( \phi_l \) which is
also an invariant under $C_4$ couples with these terms. The rest of the terms involve the right-handed neutrino triplet $\nu_R$ and they contribute to the neutrino mass generation. The terms in the second line of Eq. (28) contain $\bar{L}\nu_R$ which transforms as $-i$ under $C_4$. Therefore the flavons $\phi$ and $\eta$ which transform as $i$ couple with $\bar{L}\nu_R$. These terms results in the Dirac mass matrix for the neutrinos. The sterile neutrino mass term consists of $\bar{\nu}_c\nu_R$ and the flavon $\phi_s$ which transform as $-1$ under $C_4$. Finally, we have the Majorana mass term with the large $M$ generating the tiny neutrino masses through the type-1 seesaw mechanism.

Like the Higgs field, the flavon fields also acquire VEVs through SSB. The Higgs VEV breaks the gauge symmetry while the flavon VEVs break the discrete flavour symmetry, $A_4 \times C_4$. In the Appendix, we construct the flavon potentials invariant under the flavour group. The flavon potentials have discrete sets of minima and through SSB one of these sets becomes the VEVs for our model. These VEVs are given below:

$$\langle \phi_l \rangle = v_l(1,1,1)^T, \quad (29)$$
$$\langle \eta \rangle = v_\eta, \quad (30)$$
$$\langle \phi \rangle = v_\phi (0,-i,0)^T, \quad (31)$$
$$\langle \phi_s \rangle = v_s (1,0,1)^T. \quad (32)$$

IV. MASS MATRICES AND OBSERVABLES

Substituting the Higgs VEV, Eq. (3), and the flavon VEV, Eqs. (29), in the Lagrangian for the charged-lepton sector (first line of Eq. (28)), we obtain the charged-lepton mass matrix, Eq. (4),

$$M_l = v^2 \frac{v_l}{A} \begin{pmatrix}
Y_e & Y_\mu & Y_\tau \\
Y_e & \omega Y_\mu & \bar{\omega} Y_\tau \\
Y_e & \bar{\omega} Y_\mu & \omega Y_\tau
\end{pmatrix}. \quad (33)$$

This mass matrix is diagonalised using the transformation,

$$U_L M_l = \text{diag}(m_e, m_\mu, m_\tau), \quad (34)$$

where

$$U_L = \frac{1}{\sqrt{3}} \begin{pmatrix}
1 & 1 & 1 \\
1 & \bar{\omega} & \omega \\
1 & \omega & \bar{\omega}
\end{pmatrix}. \quad (35)$$
is the $3 \times 3$ trimaximal magic matrix and

$$m_e = Y_e v \frac{v_l}{\Lambda}, \quad m_\mu = Y_\mu v \frac{v_l}{\Lambda}, \quad m_\tau = Y_\tau v \frac{v_l}{\Lambda} \quad (36)$$

are the masses of the charged leptons.

The terms in the second line of the Lagrangian, Eq. (28), generate the Dirac mass matrix for the neutrinos. The VEV of the conjugate Higgs in these terms picks out the left-handed neutrinos from the $SU(2)$ lepton doublets. Substituting Higgs VEV and the VEVs of the flavons $\eta$ and $\phi$, Eqs. (30, 31), in these terms, we obtain the Dirac neutrino mass matrix, Eq. (6),

$$M_D = v \frac{1}{\Lambda} \begin{pmatrix} v_\eta Y_\eta & 0 & -i v_\phi (Y_\phi - Y_{\phi a}) \\ 0 & v_\eta Y_\eta & 0 \\ -i v_\phi (Y_\phi + Y_{\phi a}) & 0 & v_\eta Y_\eta \end{pmatrix}. \quad (37)$$

Substituting the VEV of $\phi_s$, Eq. (32), in the mass term for the sterile neutrino, $Y_s \bar{\nu}_s (\nu_R \phi_s)_1$, we obtain the mass matrix representing the couplings between $\nu_s$ and $\nu_R$, Eq. (8),

$$M_s = v_s Y_s \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}. \quad (38)$$

Finally, from the term $M (\bar{\nu}_R \nu_R)_1$, we obtain the mass matrix for the heavy right-handed neutrinos, Eq. (7),

$$M_R = MI, \quad (39)$$

where $I$ is the $3 \times 3$ identity matrix.

We implement the MES scheme, Eq. (11), using the neutrino mass matrices, $M_D$, $M_s$, $M_R$, Eqs. (37, 38, 39), and obtain the following effective neutrino mass matrix:

$$M^4_{\nu} = \begin{pmatrix} \frac{1}{m} \begin{pmatrix} 1 - (\kappa_s - \kappa_a)^2 & 0 & -2i \kappa_s \\ 0 & 1 & 0 \\ -2i \kappa_s & 0 & 1 - (\kappa_s + \kappa_a)^2 \end{pmatrix} \frac{\sqrt{m m_s}}{\sqrt{2}} \begin{pmatrix} 1 - i(\kappa_s - \kappa_a) \\ 0 \\ 1 - i(\kappa_s + \kappa_a) \end{pmatrix} \\ \frac{\sqrt{m m_s}}{\sqrt{2}} \begin{pmatrix} 1 - i(\kappa_s - \kappa_a) \\ 0 \\ 1 - i(\kappa_s + \kappa_a) \end{pmatrix} \end{pmatrix}, \quad (40)$$

where

$$m = \frac{v^2 v_\eta Y_\eta^2}{M \Lambda^2}, \quad m_s = \frac{2v_s^2 Y_s^2}{M}, \quad \kappa_s = \frac{v_\phi Y_\phi}{v_\eta Y_\eta}, \quad \kappa_a = \frac{v_\phi Y_{\phi a}}{v_\eta Y_\eta}. \quad (41)$$
Here, the masses $m$ and $m_s$ are suppressed by the very high value of $M \simeq 10^{15}$ GeV as expected in the seesaw mechanism. We assume that the ratio of $m$ to $m_s$, $m/m_s \sim \frac{v^2 \nu^2}{2 \eta^2 Y^2}$, is small. Under this assumption, we use Eq. 12 to obtain the $3 \times 3$ neutrino mass matrix,

$$M_\nu^{3\times3} = \frac{m}{2} \begin{pmatrix} \left(\kappa_s - \kappa_a - i\right)^2 & 0 & \kappa_a^2 - \left(\kappa_s - i\right)^2 \\ 0 & -2 & 0 \\ \kappa_a^2 - \left(\kappa_s - i\right)^2 & 0 & \left(\kappa_s + \kappa_a - i\right)^2 \end{pmatrix}.$$  \tag{42}

Using the unitary matrix,

$$U_\nu = \frac{1}{\sqrt{2\kappa}} \begin{pmatrix} i + \kappa_s + \kappa_a & 0 & -i + \kappa_s - \kappa_a \\ 0 & i\sqrt{2\kappa} & 0 \\ i + \kappa_s - \kappa_a & 0 & i - \kappa_s - \kappa_a \end{pmatrix} \text{ with } \kappa = \sqrt{1 + \kappa_s^2 + \kappa_a^2},  \tag{43}$$

we diagonalise $M_\nu^{3\times3}$, Eq. (42),

$$U_\nu^\dagger M_\nu^{3\times3} U_\nu^* = m \begin{pmatrix} 0, 1, 1 + \kappa_s^2 + \kappa_a^2 \end{pmatrix},  \tag{44}$$

to obtain the light neutrino masses,

$$m_1 = 0, \quad m_2 = m, \quad m_3 = m(1 + \kappa_s^2 + \kappa_a^2).  \tag{45}$$

Substituting the expressions of $U_L$ and $U_\nu$, Eqs. (35, 43), in Eq. (20), we obtain the PMNS mixing matrix in terms of the parameters $\kappa_s$ and $\kappa_a$,

$$U_{\text{PMNS}} = \frac{1}{\sqrt{6\kappa}} \begin{pmatrix} 2(i + \kappa_s) & i\sqrt{2\kappa} & -2\kappa_a \\ (i + \kappa_s)(1 + \omega) + \kappa_a(1 - \omega) & i\sqrt{2\kappa}\bar{\omega} & (-i + \kappa_s)(1 - \omega) - \kappa_a(1 + \omega) \\ (i + \kappa_s)(1 + \bar{\omega}) + \kappa_a(1 - \bar{\omega}) & i\sqrt{2\kappa}\omega & (-i + \kappa_s)(1 + \bar{\omega}) - \kappa_a(1 + \bar{\omega}) \end{pmatrix}.  \tag{46}$$

The absolute values of the elements of the middle column of this mixing matrix are equal to $\frac{1}{\sqrt{3}}$, i.e. the mixing has the TM$_2$ form. Since $\kappa_s$ and $\kappa_a$ are real parameters, we have $m < m(1 + \kappa_s^2 + \kappa_a^2)$ in Eq. (45). Therefore, we obtain $m_2 < m_3$ which is consistent with the normal hierarchy of the neutrino masses. This also implies that the model forbids inverted hierarchy under the condition that the mixing is TM$_2$.

From Eq. (46), we extract the three mixing angles in the active sector,

$$\sin^2 \theta_{13} = \frac{2\kappa_a^2}{3(1 + \kappa_s^2 + \kappa_a^2)},  \tag{47}$$
$$\sin^2 \theta_{12} = \frac{1 + \kappa_s^2 + \kappa_a^2}{3 + 3\kappa_s^2 + \kappa_a^2},$$  
(48)

$$\sin^2 \theta_{23} = \frac{3 + 3\kappa_s^2 + 2\sqrt{3}\kappa_a + \kappa_a^2}{2(3 + 3\kappa_s^2 + \kappa_a^2)}.$$  
(49)

We also calculate the Jalskog’s CP-violation parameter [62] in the active sector,

$$J = \text{Im}(U_{e2}U_{\mu3}U_{e3}^*U_{\mu2}^*) = -\frac{\kappa_s\kappa_a}{3\sqrt{3}(1 + \kappa_s^2 + \kappa_a^2)}.$$  
(50)

Given the three mixing angles and J in terms of the model parameters, we can obtain $\sin \delta$ using the following expression:

$$\sin \delta = J / (\sin \theta_{13} \sin \theta_{12} \sin \theta_{23} \cos^2 \theta_{13} \theta_{12} \theta_{23}).$$  
(51)

Note that the $4 \times 4$ mixing matrix is parametrised using six mixing angles ($\theta_{13}$, $\theta_{12}$, $\theta_{23}$, $\theta_{14}$, $\theta_{24}$, $\theta_{34}$) and three Dirac CP phases ($\delta_{13}$, $\delta_{14}$, $\delta_{24}$) with the help of the parametrization mentioned in [20]. However, we used the parametrisation for the $3 \times 3$ mixing matrix, Eq. (1), to extract the mixing angles ($\theta_{13}$, $\theta_{12}$, $\theta_{23}$) and the CP phase ($\delta = \delta_{13}$) given in Eqs. (47, 48, 49, 51). This approximation is valid since the active-sterile mixing is quite small.

Comparing Eqs. (11, 13) with Eq. (40), it is clear that the model parameter $m_s$ corresponds to the mass of the 4th mass eigenstate,

$$m_4 = m_s.$$  
(52)

Using Eq. (17), we obtain the three-component column vector $R$,

$$R = \sqrt{\frac{m}{2m_s}} \begin{pmatrix} 1 - i(\kappa_s - \kappa_a) \\ 0 \\ 1 - i(\kappa_s + \kappa_a) \end{pmatrix}.$$  
(53)

Substituting Eqs. (35, 53) in Eq. (19) we obtain

$$U_{e4} = \frac{\sqrt{2m}}{\sqrt{3m_s}}(1 - i\kappa_s), \quad |U_{e4}|^2 = \frac{2}{3} \frac{m}{m_s} (1 + \kappa_s^2),$$  
(54)

$$U_{\mu4} = -\frac{\bar{\omega}}{\sqrt{6m_s}}(1 - i\kappa_s + \sqrt{3}\kappa_a), \quad |U_{\mu4}|^2 = \frac{1}{6} \frac{m}{m_s} (1 + \kappa_s^2 + 2\sqrt{3}\kappa_a + 3\kappa_a^2),$$  
(55)

$$U_{\tau4} = -\frac{\omega}{\sqrt{6m_s}}(1 - i\kappa_s - \sqrt{3}\kappa_a), \quad |U_{\tau4}|^2 = \frac{1}{6} \frac{m}{m_s} (1 + \kappa_s^2 - 2\sqrt{3}\kappa_a + 3\kappa_a^2).$$  
(56)
Using Eqs. (54, 55, 56), we can write the three active-sterile mixing angles in terms of the model parameters,

$$\sin^2\theta_{14} = \frac{2}{3} \frac{m}{m_s} (1 + \kappa_s^2),$$

(57)

$$\sin^2\theta_{24} = \frac{1}{6} \frac{m}{m_s} \frac{1 - \frac{2}{3} \frac{m}{m_s} (1 + \kappa_s^2)}{(1 + \kappa_s^2 + 2\sqrt{3}\kappa_a + 3\kappa_a^2)},$$

(58)

$$\sin^2\theta_{34} = \frac{1}{6} \frac{m}{m_s} \frac{(1 + \kappa_s^2 - 2\sqrt{3}\kappa_a + 3\kappa_a^2)}{1 - \frac{2}{3} \frac{m}{m_s} (1 + \kappa_s^2) - \frac{1}{6} \frac{m}{m_s} (1 + \kappa_s^2 + 2\sqrt{3}\kappa_a + 3\kappa_a^2)}.$$  

(59)

For the extraction of $\delta_{14}$ and $\delta_{24}$, we need to calculate the Jarlskog-like rephasing invariants from the active-sterile sector. In this context, we refer the readers to a recent work [63], in which nine independent rephasing invariants in terms of the six mixing angles and the three Dirac phases have been evaluated in the context of the $4 \times 4$ mixing matrix. With the help of these invariants, we may extract $\delta_{14}$ and $\delta_{24}$.

The effective neutrino mass applicable to the neutrinoless double-beta decay [64, 65] is given by

$$m_{\beta\beta} = |m_1 U^2_{e1} + m_2 U^2_{e2} + m_3 U^2_{e3} + m_s U^2_{e4}|.$$  

(60)

Substituting the values of the neutrinos masses, Eq. (45), the elements of the first row of the mixing matrix, Eq. (46), and the expression for $U_{e4}$, Eq. (54), in the above equation, we obtain

$$m_{\beta\beta} = \frac{m}{3} (1 - 2\kappa_a^2 + 2\kappa_a^2 - 4i\kappa_s) = \frac{m}{3} \sqrt{(1 - 2\kappa_a^2 + 2\kappa_a^2)^2 + 16\kappa_s^2}.$$  

(61)

V. PHENOMENOLOGY AND PREDICTIONS

| $3\sigma$ range |
|--------------------|
| $\sin^2\theta_{13}$ | 0.02044 → 0.02437 |
| $\Delta m^2_{21}$ | $6.79 \times 10^{-5}$ eV$^2$ → $8.01 \times 10^{-5}$ eV$^2$ |
| $\Delta m^2_{31}$ | $2.431 \times 10^{-3}$ eV$^2$ → $2.622 \times 10^{-3}$ eV$^2$ |
| $\Delta m^2_{41}$ | 0.87 eV$^2$ → 2.04 eV$^2$ |

TABLE III: The mixing observables which are used to evaluate the model parameters $\kappa_s$, $\kappa_a$, $m$ and $m_s$. The $3\sigma$ ranges of $\sin^2\theta_{13}$, $\Delta m^2_{21}$ and $\Delta m^2_{31}$ are taken from Ref. [66] and that of $\Delta m^2_{41}$ is taken from Ref. [67–69].
Our model allows only four degrees of freedom in the neutrino Yukawa sector, denoted by the free parameters $\kappa_s$, $\kappa_a$, $m$, $m_s$. There are eleven independent experimentally measured quantities, $\sin^2 \theta_{12}$, $\sin^2 \theta_{23}$, $\sin^2 \theta_{13}$, $\sin \delta$, $\Delta m^2_{21}$, $\Delta m^2_{31}$, $m_{\beta\beta}$, $|U_{e4}|^2$, $|U_{\mu 4}|^2$, $|U_{\tau 4}|^2$ all of which can be expressed in terms of the above mentioned four model parameters. So, it is clear that the model is extremely constrained. In this section, we calculate the model parameters using the experimental data and also make predictions. To calculate the allowed ranges of the model parameters, $\kappa_s$, $\kappa_a$, $m$ and $m_s$, we utilise the observables $\sin^2 \theta_{13}$, $\Delta m^2_{21}$, $\Delta m^2_{31}$ and $\Delta m^2_{41}$ whose experimental values are given in Table III. These values are obtained from the global fit data published by the nufit group [66] and the active-sterile mixing data from Ref. [67–69].

The expression for the reactor mixing angle is given by Eq. (47). From Table III, we have $0.02044 \leq \sin^2 \theta_{13} \leq 0.02437$ at the $3\sigma$ level. Therefore, using Eq. (47), we obtain

$$0.02044 \leq \frac{2\kappa_a^2}{3(1 + \kappa_s^2 + \kappa_a^2)} \leq 0.02437.$$ (62)

Using Eq. (45), we calculate the ratio of the mass-squared differences of the active neutrinos,

$$\frac{\Delta m^2_{31}}{\Delta m^2_{21}} = (1 + \kappa_s^2 + \kappa_a^2)^2.$$ (63)

The mass-squared differences have experimental $3\sigma$ ranges (Table III),

$$6.79 \times 10^{-5} \text{ eV}^2 \leq \Delta m^2_{21} \leq 2.431 \times 10^{-3} \text{ eV}^2, \quad 6.79 \times 10^{-5} \text{ eV}^2 \leq \Delta m^2_{31} \leq 2.622 \times 10^{-3} \text{ eV}^2.$$ (64)

Substituting these ranges in the ratio, Eq. (63), we obtain

$$30.3 \leq (1 + \kappa_s^2 + \kappa_a^2)^2 \leq 38.6.$$ (65)

We use Eqs. (62, 65) to constrain the parameters $\kappa_s$ and $\kappa_a$. In this analysis, we chose $\sin^2 \theta_{13}$ and $\frac{\Delta m^2_{31}}{\Delta m^2_{21}}$ because these are the most precisely measured quantities that can be used for constraining $\kappa_s$ and $\kappa_a$. The results are shown in Figure 1. The lower and the upper blue curves correspond to the relationships $\frac{2\kappa_s^2}{3(1 + \kappa_s^2 + \kappa_a^2)} = 0.02044$ and $\frac{2\kappa_a^2}{3(1 + \kappa_s^2 + \kappa_a^2)} = 0.02437$ respectively. The inner and the outer red circular arcs correspond to the relationships $(1 + \kappa_s^2 + \kappa_a^2)^2 = 30.3$ and $(1 + \kappa_s^2 + \kappa_a^2)^2 = 38.6$ respectively. The allowed range of $\kappa_s$ and $\kappa_a$ is given by the intersection of the red and the blue regions.
FIG. 1: The parameters $\kappa_s$ and $\kappa_a$ constrained using the reactor mixing angle and the ratio of the mass-squared differences of the active neutrinos.

Substituting this range of values in the expression of the solar mixing angle, Eq. (48), we predict

$$0.340 \leq \sin^2 \theta_{12} \leq 0.342.$$

(66)

TM$_2$ mixing fixes $|U_{e2}|^2$ to be $\frac{1}{3}$. We also have $\sin^2 \theta_{12} \cos^2 \theta_{13} = |U_{e2}|^2$. Therefore, TM$_2$ scheme strongly constrains $\theta_{12}$ given the precise experimental determination of $\theta_{13}$. This constraint, Eq. (66), is consistent with the 3$\sigma$ experimental range $0.275 \leq \sin^2 \theta_{12} \leq 0.350$, Table IV.

Substituting the allowed range of $\kappa_s$ and $\kappa_a$ in the expressions of the atmospheric mixing angle, the Jarlskog invariant and the Dirac CP phase, Eqs. (49-51), we predict

$$0.541 \leq \sin^2 \theta_{23} \leq 0.548,$$

(67)

$$-0.916 \leq \sin \delta \leq -0.905 \quad \text{with} \quad -0.0329 \leq J \leq -0.299.$$

(68)
TABLE IV: The values of the observables predicted by the model in comparison to their experimental ranges [66–74]. $m_{\beta\beta} < 0.05$ eV is the most stringent bound from the KamLAND-Zen experiment [70].

| Observable | Prediction | Experimental range |
|------------|------------|--------------------|
| $\sin^2 \theta_{12}$ | $0.340 \rightarrow 0.342$ | $0.275 \rightarrow 0.350$ |
| $\sin^2 \theta_{23}$ | $0.541 \rightarrow 0.548$ | $0.428 \rightarrow 0.624$ |
| \(\sin \delta\) | $-0.916 \rightarrow -0.905$ | $-1 \rightarrow 0.707$ |
| $|U_{e4}|^2$ | $0.021 \rightarrow 0.038$ | $0.012 \rightarrow 0.047$ |
| $|U_{\mu4}|^2$ | $0.007 \rightarrow 0.013$ | $0.005 \rightarrow 0.03$ |
| $|U_{\tau4}|^2$ | $0.004 \rightarrow 0.008$ | $< 0.16$ |
| $m_{\beta\beta}$ | $0.0302$ eV $\rightarrow$ $0.0371$ eV | $< 0.05$ eV |

These predictions are also consistent with the experimental ranges, Table IV. Note that the determination of the octant of $\theta_{23}$ is still an open problem experimentally. If the $\mu$-$\tau$ reflection symmetry [75–80] is broken, we have $\theta_{23}$ either in the first or the second octant. The model predicts it to be in the second octant. The global fit [66] of oscillation data gives hints for CP violation. Even though the measurement is not precise, $135^\circ \leq \delta \leq 366^\circ$, it favours a relatively large negative value for $\sin \delta$. Our prediction, Eq. (68), supports this scenario. In Figure 2, we have shown the predictions for $\sin^2 \theta_{23}$ and $\sin \delta$.

FIG. 2: The predicted ranges of $\sin^2 \theta_{23}$ and $\sin \delta$ as constrained by the parameters $\kappa_s$ and $\kappa_a$. 

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Under the MES scheme, the mass of the lightest neutrino, $m_1$, vanishes. As a result, the experimental mass-squared differences directly lead to the masses of the other neutrino states,

$$m_2 = \sqrt{\Delta m_{21}^2}, \quad m_3 = \sqrt{\Delta m_{31}^2},$$  \hspace{1cm} \text{(69)}

Given the experimental ranges, Eqs. (64), MES models predict

$$8.24 \times 10^{-3} \text{ eV} \leq m_2 \leq 8.95 \times 10^{-3} \text{ eV}, \quad 4.93 \times 10^{-2} \text{ eV} \leq m_3 \leq 5.12 \times 10^{-2} \text{ eV}. \hspace{1cm} \text{(70)}$$

The model parameter $m$ corresponds to the neutrino mass $m_2$, so its allowed range is the same as that of $m_2$ given above,

$$8.24 \times 10^{-3} \text{ eV} \leq m \leq 8.95 \times 10^{-3} \text{ eV}. \hspace{1cm} \text{(71)}$$

The mass-squared difference, $\Delta m_{41}^2 = m_4^2 - m_1^2$, has the experimental range (Table III),

$$0.87 \text{ eV}^2 \leq \Delta m_{41}^2 \leq 2.04 \text{ eV}^2. \hspace{1cm} \text{(72)}$$

Since $m_1$ vanishes in MES models, we have

$$m_4 = \sqrt{\Delta m_{41}^2}. \hspace{1cm} \text{(73)}$$

Using Eqs. (52, 72, 73), we obtain the allowed range of the model parameter $m_s$,

$$0.93 \text{ eV} \leq m_s \leq 1.42 \text{ eV}. \hspace{1cm} \text{(74)}$$

The active-sterile mixing observables, Eqs. 54-56, depend on all the four model parameters, $\kappa_s$, $\kappa_a$, $m$, and $m_s$. By varying these parameters within their respective ranges we predict the values of these observables,

$$0.021 \leq |U_{e4}|^2 \leq 0.038, \hspace{1cm} \text{(75)}$$

$$0.007 \leq |U_{\mu 4}|^2 \leq 0.013, \hspace{1cm} \text{(76)}$$

$$0.004 \leq |U_{\tau 4}|^2 \leq 0.008. \hspace{1cm} \text{(77)}$$

In Figure 3, we have plotted them against the parameter, $m_s$. These predictions are well within their corresponding experimental ranges, Table IV.

Substituting the allowed ranges of $\kappa_s$, $\kappa_a$, $m$, and $m_s$ in Eq. (61), we predict the value of the neutrinoless double-beta decay mass,

$$0.03020 \text{ eV} \leq m_{\beta\beta} \leq 0.03712 \text{ eV}, \hspace{1cm} \text{(78)}$$
which is also shown in Fig. 4. This range is quite narrow because the model strongly constrains the first row of the mixing matrix through the model parameters $\kappa_s$ and $\kappa_a$, which effectively constrains the Majorana phases as well.

![Graphs showing $(a) |U_{e4}|^2$ vs $m_s$, $(b) |U_{\mu4}|^2$ vs $m_s$, and $(c) |U_{\tau4}|^2$ vs $m_s$.](image)

FIG. 3: The active-sterile mixing observables predicted by the model plotted against $m_s$.

![Graph showing the prediction of the effective neutrino mass, $m_{\beta\beta}$, in relation to the active-sterile mixing strength.](image)

FIG. 4: The prediction of the effective neutrino mass, $m_{\beta\beta}$, in relation to the active-sterile mixing strength.

Cosmological observations set upper bounds to the sum of the neutrino masses for three generations of neutrinos, $\Sigma m_i = m_1 + m_2 + m_3$. However, in the presence of the sterile neutrino, the bound gets affected. At the same time, some recent cosmological models offer an explanation in favour of the existence of the sterile neutrino via its self-interaction. For more detail in this regard please refer to the references [81–83].
VI. DISCUSSION AND CONCLUSION

In this paper, we construct the leptonic mass matrices in terms of the VEVs of a set of flavon fields transforming under $A_4 \times C_4$. In the charged-lepton sector, we obtain a non-diagonal mass matrix. In the neutrino sector, we use the MES formula, Eq. (12), to construct the effective $3 \times 3$ seesaw mass matrix. The unitary matrices $U_L$ and $U_\nu$ diagonalise the charged-lepton and the neutrino mass matrices respectively. Their product determines the mixing in the active sector, i.e $U_{PMNS} = U_L U_\nu$, Eq. (20). In our model, the unitary contribution from the charged-lepton sector ($U_L$) has a $3 \times 3$ trimaximal form, Eq. (35). On the other hand, the contribution from the active neutrino sector ($U_\nu$) has the form which corresponds to the second flavour eigenstate being equal to the second mass eigenstate as evident from the off-diagonal zeros in $U_\nu$, Eq. (43). Consequently, the second column of $U_L$ is preserved in the product $U_L U_\nu$ and as a result, we obtain the TM$_2$ mixing.

The symmetries of $U_L$ and $U_\nu$ can be traced back to the symmetries of the flavon VEVs. The VEV of the flavon in the charged-lepton sector, $\langle \phi_l \rangle$, Eq. (29), has the residual $C_3$ symmetry generated by $T$ of $A_4$, Eq. (22),

$$T\langle \phi_l \rangle = \langle \phi_l \rangle. \quad (79)$$

The trimaximal symmetry of $U_L$ is nothing but the manifestation of the above mentioned $C_3$ symmetry. The VEV of the flavon in the active neutrino sector, $\langle \phi \rangle$, Eq. (31), has the $C_2$ residual symmetry generated by $T^2ST$,

$$T^2ST\langle \phi \rangle = \langle \phi \rangle. \quad (80)$$

The off-diagonal zeros in the neutrino mass matrix, $M_D$, Eq. (37), emerges because of the above mentioned $C_2$ residual symmetry. This structure is maintained in the seesaw mass matrix, Eq. (42), as well and hence the diagonalising matrix ($U_\nu$) attains the form with off-diagonal zeros, Eq. (43).

$U_\nu$ obtained in the model contains two parameters $\kappa_s$ and $\kappa_a$. These parameters correspond to the symmetric and the antisymmetric parts of the neutrino mass matrix, $M_D$, which in turn originate from the symmetric and the antisymmetric parts of the tensor product of
triplets of $A_4$. If $\kappa_a$ vanishes, $U_\nu$ becomes bimaximal, i.e.

$$\kappa_a \rightarrow 0 \quad \Rightarrow \quad U_\nu \rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

which will lead to tribimaximal (TBM) mixing. The observation of non-zero reactor angle has ruled out TBM. Hence, the parameter $\kappa_a$ plays the vital role of the generation of the non-zero reactor angle in the model. The role of the antisymmetric part of the $A_4$ product rule in generating a non-zero value of the reactor angle has also been emphasized in [84].

We obtain CP violation even though all the free parameters in the model are real. The charged-lepton mass matrix, $M_l$, Eq. (33), turns out to be complex because of the presence of complex Clebsch-Gordon coefficients, i.e. the CP violation originating from the charged-lepton sector is geometric in nature [85–87]. The neutrino mass matrix, $M_D$, Eq. (37), is complex on account of the VEV, $\langle \phi \rangle$, being complex, i.e. in the neutrino sector, CP is spontaneously broken. This is achieved with the help of the $C_4$ group (Table II) as explained in Appendix A. Since $M_l$ and $M_D$ are complex, the corresponding diagonalising matrices $U_L$ and $U_\nu$ also become complex and they generate the complex mixing matrix, $U_{PMNS} = U_L U_\nu$. It can be shown that if $U_\nu$ were real, the resulting mixing matrix $U_L U_\nu$ would be $\mu$-$\tau$ symmetric implying $\theta_{23} = \frac{\pi}{4}$. In such a scenario, despite $U_\nu$ being real, CP would be maximally broken ($\delta = \pm \frac{\pi}{2}$) because of the complex contribution from the charged-lepton sector ($U_L$) alone. Our model, with $U_\nu$ also being complex, breaks $\mu$-$\tau$ symmetry and we obtain $\theta_{23} \neq \frac{\pi}{4}$. The complex $U_\nu$ also shifts $\delta$ away from its maximal value, i.e. $\delta \neq \pm \frac{\pi}{2}$. Therefore, the origin of the non-maximal values of the atmospheric mixing as well as the CP phase is the complex VEV, $\langle \phi \rangle$.

LSND and MiniBooNE observations suggest the existence of sterile neutrinos. The observed active-sterile mixing (\(|U_{e4}|^2, |U_{\mu 4}|^2\)) is found to be of the order of $\frac{\sqrt{\Delta m_{21}^2}}{\sqrt{\Delta m_{31}^2}}$. The Minimal Extended Seesaw provides a natural framework to achieve this relationship. It is in this context that we built the model to explain both the active and the sterile mixing observables. In the model, these observables are given in terms of four parameters, $\kappa_s, \kappa_a, m$ and $m_s$. We use the experimental ranges of the reactor mixing angle, $\sin^2 \theta_{13}$, and the mass-squared differences, $\Delta m_{21}^2$ and $\Delta m_{31}^2$, to extract the allowed values of $\kappa_s, \kappa_a$ and $m$, as we obtain $m_2 < m_3$ corresponding to normal hierarchy. The extracted values of $\kappa_s$ and $\kappa_a$ are used
to predict $\theta_{23}$ and $\delta$. These predictions can be tested when these observables are measured more precisely in future oscillation experiments. The model parameter, $m_s$, corresponds to the sterile neutrino mass and is determined by the active-sterile mass-squared difference, $\Delta m^2_{41}$. The three model parameters, $\kappa_s$, $\kappa_a$ and $m$ (constrained using $\sin^2 \theta_{13}$, $\Delta m^2_{21}$ and $\Delta m^2_{31}$), as well as the fourth parameter, $m_s$ (constrained using $\Delta m^2_{41}$), are used to evaluate the active-sterile mixing. We find that these values are consistent with the experimental results. We also obtain strong constraints on the range of the effective mass governing the neutrinoless double-beta decay.

**Appendix A: Flavon potentials and vacuum alignments**

First, we consider the flavon $\phi_l = (\phi_{l1}, \phi_{l2}, \phi_{l3})$. Using Eq. (26), we construct a triplet at the quadratic order,

$$ (\phi_l \phi_l)_3 = (\phi_{l2} \phi_{l3}, \phi_{l3} \phi_{l1}, \phi_{l1} \phi_{l2})^T. \quad (A1) $$

Using the triplets $\phi_l$ and $(\phi_l \phi_l)_3$, we construct the invariant,

$$ |v_l \phi_l - (\phi_l \phi_l)_3|^2 = (v_l \phi_l - (\phi_l \phi_l)_3)^T (v_l \phi_l - (\phi_l \phi_l)_3), \quad (A2) $$

where $v_l$ is a positive constant with mass dimension one. With this invariant, we construct the potential,

$$ V(\phi_l) = k_l |v_l \phi_l - (\phi_l \phi_l)_3|^2, \quad (A3) $$

where $k_l$ is a dimensionless positive constant. The potential is positive semidefinite, i.e. $V(\phi_l) \geq 0$. It vanishes when $v_l \phi_l = (\phi_l \phi_l)_3$ which corresponds to four discrete points, $\phi_l = v_l(1,1,1)^T, v_l(1,-1,-1)^T, v_l(-1,1,-1)^T, v_l(-1,-1,1)^T$. They form the vertices of a tetrahedron as evident from the $A_4$ symmetry of the potential. Through the mechanism of Spontaneous Symmetry Breaking (SSB), the flavon acquires one of these minima as its Vacuum Expectation Value (VEV),

$$ \langle \phi_l \rangle = v_l(1,1,1)^T. \quad (A4) $$

Consider the flavon $\eta$. This flavon is invariant under $A_4$ and it transforms as $i$ under $C_4$. We express $\eta$ in terms of its real and imaginary parts, $\eta = \eta_r + i\eta_i$, and construct the quadratic invariant,

$$ |\eta|^2 = \eta^\ast \eta = \eta_r^2 + \eta_i^2. \quad (A5) $$
Under $C_4$, we have

$$\eta \rightarrow i\eta \quad \Rightarrow \quad \eta_r \rightarrow -\eta_i, \eta_i \rightarrow \eta_r, \quad (A6)$$

$$\eta \rightarrow -\eta \quad \Rightarrow \quad \eta_r \rightarrow -\eta_r, \eta_i \rightarrow -\eta_i, \quad (A7)$$

$$\eta \rightarrow -i\eta \quad \Rightarrow \quad \eta_r \rightarrow \eta_i, \eta_i \rightarrow -\eta_r. \quad (A8)$$

From Eqs. (A6-A8) it is clear that $\eta_r^2 \eta_i^2$ is an invariant at the quartic order. Using the invariants, $\eta_r^2 + \eta_i^2$ and $\eta_r^2 \eta_i^2$, we construct the potential for $\eta$,

$$\mathcal{V}(\eta) = k_{\eta_1} (|\eta|^2 - v_\eta^2)^2 + k_{\eta_2} \eta_r^2 \eta_i^2, \quad (A9)$$

where $k_{\eta_1}, k_{\eta_2}$ are dimensionless positive constants and $v_\eta$ has mass dimension one. This potential is also positive semidefinite and it has four minima where it vanishes. They correspond to $\eta = v_\eta, iv_\eta, -v_\eta, -iv_\eta$ as evident from the $C_4$ symmetry of the potential. Through SSB, we obtain the VEV,

$$\langle \eta \rangle = v_\eta. \quad (A10)$$

The flavon $\phi$ transforms as a triplet under $A_4$ and $i$ under $C_4$. Unlike $\phi_i$, $\phi$ is a complex triplet. We express $\phi$ in terms of its components, $\phi = (\phi_1, \phi_2, \phi_3)^T = (\phi_{1r} + i\phi_{1i}, \phi_{2r} + i\phi_{2i}, \phi_{3r} + i\phi_{3i})^T$, and construct the quadratic invariant,

$$|\phi|^2 = \phi^\dagger \phi = \phi_{1r}^2 + \phi_{1i}^2 + \phi_{2r}^2 + \phi_{2i}^2 + \phi_{3r}^2 + \phi_{3i}^2. \quad (A11)$$

Using $\phi$ and $\phi^*$, we also construct the triplets

$$(\phi^* \phi)_3s = (\phi_{2r}, \phi_{3r} + \phi_{2i}, \phi_{3i}, \phi_{1r} + \phi_{3i}, \phi_{1i}, \phi_{1r} + \phi_{2i}, \phi_{2i} + \phi_{1i}, \phi_{1r} + \phi_{2i}, \phi_{2i} + \phi_{1i}, \phi_{1r} + \phi_{2i})^T, \quad (A12)$$

$$(\phi^* \phi)_3a = (\phi_{2r}, \phi_{3i} - \phi_{2i}, \phi_{3r}, \phi_{1r} - \phi_{1i}, \phi_{1i} - \phi_{1r}, \phi_{1r} - \phi_{2i}, \phi_{2i} - \phi_{1i}, \phi_{1i} - \phi_{1r}, \phi_{1r} - \phi_{2i})^T, \quad (A13)$$

where $(\phi^* \phi)_{3s}$ and $(\phi^* \phi)_{3a}$ are the symmetric and the antisymmetric products as given in Eq. (26) and Eq. (27) respectively. These triplets are invariant under $C_4$. Using them, we obtain the quartic invariants,

$$| (\phi^* \phi)_{3s} |^2 = (\phi^* \phi)_{3s}^T (\phi^* \phi)_{3s}, \quad (A14)$$

$$| (\phi^* \phi)_{3a} |^2 = (\phi^* \phi)_{3a}^T (\phi^* \phi)_{3a}. \quad (A15)$$

The real part and the imaginary part of $\phi$, i.e. $\phi_r = (\phi_{1r}, \phi_{2r}, \phi_{3r})^T$ and $\phi_i = (\phi_{1i}, \phi_{2i}, \phi_{3i})^T$ respectively, transform individually as triplets under $A_4$. With them, we construct the $A_4$ invariant,

$$\phi_{1r}^T \phi_{1i} + \phi_{2r} \phi_{2i} + \phi_{3r} \phi_{3i}. \quad (A16)$$
Under $C_4$, $\phi^T_i \phi_i$ transforms as $-1$. Therefore, we square it to obtain the quartic invariant $(\phi^T_i \phi_i)^2$.

Using the invariants discussed above, we construct the potential for the flavon $\phi$,

$$
\mathcal{V}(\phi) = k_{\phi_1} (|\phi|^2 - v^2_\phi)^2 + k_{\phi_2} |(\phi^* \phi)_3\phi|^2 + k_{\phi_3} |(\phi^* \phi)_3\phi|^2 + k_{\phi_4} (\phi^T_i \phi_i)^2,
$$

(A17)

where $k_{\phi_1}, k_{\phi_2}, k_{\phi_3}, k_{\phi_4}$ are dimensionless positive constants and $v_\phi$ has mass dimension one. This potential is also positive semidefinite and it has twelve minima where it vanishes. They are $\phi = v_\phi(\pm 1, 0, 0)^T$, $v_\phi(0, \pm 1, 0)^T$, $v_\phi(0, 0, \pm 1)^T$, $v_\phi(\pm i, 0, 0)^T$, $v_\phi(0, \pm i, 0)^T$, $v_\phi(0, 0, \pm i)^T$.

The six real minima correspond to the six edge-centres of the tetrahedron as evident from the $A_4$ symmetry of the potential. Their six complex counterparts (containing $i$) result from the additional $C_4$ symmetry. Through SSB, we obtain one among these minima as the VEV,

$$
\langle \phi \rangle = v_\phi(0, -i, 0)^T.
$$

(A18)

Finally, we consider the triplet flavon, $\phi_s = (\phi_{s1}, \phi_{s2}, \phi_{s3})^T$, which transforms as $-1$ under $C_4$. We construct the quadratic invariant,

$$
|\phi_s|^2 = \phi^*_{s1} \phi_{s1} + \phi^*_{s2} \phi_{s2} + \phi^*_{s3} \phi_{s3}.
$$

(A19)

We couple $\phi_s$ with $\phi$ to obtain

$$
\phi^T_s \phi = \phi_{s1} \phi_1 + \phi_{s2} \phi_2 + \phi_{s3} \phi_3.
$$

(A20)

The term $\phi^T_s \phi$ is invariant under $A_4$, but transforms as $-i$ under $C_4$. By multiplying it with its conjugate, we obtain the quartic invariant,

$$
| (\phi^T_s \phi) |^2 = (\phi^T_s \phi)^* (\phi^T_s \phi).
$$

(A21)

Using Eqs. (A19, A21), we construct the following potential involving $\phi_s$ and $\phi$:

$$
\mathcal{V}(\phi_s, \phi) = k_s (|\phi_s|^2 - 2v^2_s)^2 + k_1 | (\phi^T_s \phi) |^2,
$$

(A22)

where $k_s, k_1$ are dimensionless positive constants. This potential attains its minimum value when both of its constituent terms vanish. We have already obtained the VEV for $\phi$, Eq. (A18). Given this VEV, the potential, Eq. (A22), vanishes when $\phi_s$ is aligned along $v_s(1, 0, 1)^T$ which becomes its VEV,

$$
\langle \phi_s \rangle = v_s(1, 0, 1)^T.
$$

(A23)
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