Trading-off Bias and Variance in Stratified Experiments and in Matching Studies, Under a Boundedness Condition on the Magnitude of the Treatment Effect

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Abstract

I consider estimation of the average treatment effect (ATE), in a population composed of \( S \) groups or units, when one has unbiased estimators of each group’s conditional average treatment effect (CATE). These conditions are met in stratified experiments and in matching studies. I assume that each CATE is bounded in absolute value by \( B \) standard deviations of the outcome, for some known \( B \). This restriction may be appealing: outcomes are often standardized in applied work, so researchers can use available literature to determine a plausible value for \( B \). I derive, across all linear combinations of the CATEs’ estimators, the minimax estimator of the ATE. In two stratified experiments, my estimator has twice lower worst-case mean-squared-error than the commonly-used strata-fixed effects estimator. In a matching study with limited overlap, my estimator achieves 56% of the precision gains of a commonly-used trimming estimator, and has an 11 times smaller worst-case mean-squared-error.

Keywords: stratified randomized controlled trial, matching estimator, limited overlap, trimming, bias-variance trade-off, average treatment effect, minimax-linear estimator, bounded normal mean model, shrinkage.

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1 Introduction

In stratified randomized controlled trials (SRCTs), strata of similar units are formed, and a randomization takes place within each stratum. Stratifying always weakly reduces the variance, which may explain why SRCTs are popular. From 2015 to 2018, the *American Economic Journal: Applied Economics* published 50 articles presenting RCT results. 21 were based on non-paired SRCTs. Researchers analyzing SRCTs commonly use two estimators. The first is the treatment coefficient in a regression of the outcome on the treatment and strata fixed effects, hereafter referred to as the strata-fixed-effects estimator (SFE). The second, hereafter referred to as the unbiased ATE estimator (UATE), is a weighted average, across strata, of the difference between the mean outcomes of treated and control units within each stratum, where each stratum-specific conditional average treatment effect (CATE) estimator is weighted by the proportion the stratum accounts for in the RCT population. As is well-known (see Angrist [1998]), when the treatment probability does not vary across strata, the two estimators are numerically equivalent. But when the treatment probability does vary, the two estimators are not numerically equivalent. In practice, the SFE estimator is much more popular among applied researchers. Of the 21 aforementioned SRCTs, 17 use the SFE, two use the UATE, and two use a third estimator (a regression of the outcome on the treatment alone).

Assuming homoscedasticity and constant effects across strata, the SFE is efficient. However, constant effects across strata is a strong assumption in many SRCTs. Strata often correspond to geographic locations, and treatment effects often vary across space. For instance, in an SRCT stratified at the Head-Start-center level, Walters (2015) finds substantial variation of Head Start across centers. SRCTs are also often stratified by gender, and treatment effects may vary along that dimension as well (see e.g. Anderson [2008]). With heterogeneous effects across strata, the SFE may be biased for the ATE, unlike the UATE, but it remains more efficient if the outcome is homoscedastic (see Theorem 5.4 of Crump et al. [2006]). This leads to a bias-variance trade-off between the two estimators.

In this paper, I investigate this bias-variance trade-off. Specifically, I consider the class of linear combinations of the CATEs’ estimators, a class both the UATE and SFE belong to, and I derive the minimax-linear estimator within that class, assuming homoscedasticity. To allow for non-trivial bias-variance trade-offs, I assume that each CATE is bounded in absolute value by $B$ standard deviations of the outcome $\sigma$, for some known constant $B$. One

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1 I exclude paired RCTs, because they are such that by construction, the treatment probability does not vary across strata, so the bias-variance trade-off I introduce below is absent.
could of course consider other restrictions on the parameter space. For instance, one could bound the expected outcome without treatment. However, the restriction I propose may be appealing to practitioners. Outcomes are often normalized by their standard deviations in applied work. Thus, researchers can readily use the available literature studying related treatments to determine a plausible value for $B$. They may find it more difficult to form a prior on, say, a plausible range of values for the expected outcome without treatment, simply because that quantity is less likely to be consistently reported across studies. Under the assumption that each CATE is bounded in absolute value by $B\sigma$, I show that the minimax-linear estimator is a weighted sum of the CATEs estimators, with positive weights that sum to less than 1. The most precisely estimated CATEs receive a weight equal to the share of the population their stratum accounts for, like in the UATE. The least precisely estimated CATEs receive a weight proportional to one over the estimator’s variance, like in the SFE, and shrunk towards zero. Thus, the minimax-linear estimator shares features with both estimators. Yet, in the two SRCTs I revisit, the minimax-linear estimator is much closer to the UATE than to the SFE. Finally, in an SRCT, the minimax-linear estimator is feasible given $B$: it only depends on known quantities.

I make an homoscedasticity assumption to ensure that the SFE is more efficient than the UATE, thus leading to a bias-variance trade-off. I still show that without that assumption, the minimax-linear estimator has a lower worst-case MSE than the UATE whenever the treated outcome’s variance is larger than that of the untreated outcome. When the opposite holds, the minimax-linear estimator still has a lower worst-case MSE, provided the ratio of the treated and untreated outcomes variances is not below a computable bound.

I use my results to revisitBehaghel et al. (2017) and Blattman & Annan (2016), two SRCTs where the treatment probability varies substantially across strata. This variation leads to non-trivial differences between the UATE and SFE, as large as 15% in relative terms for 5 of the 11 outcomes I study, and larger than 25% for two outcomes. For those two outcomes, the two estimators are significantly different at the 10% level. The minimax-linear estimator is at most 11% away from the UATE. Across the 11 outcomes I study, the standard error of the minimax-linear estimator is on average 3% smaller than that of the UATE, and 2% larger than that of the SFE. Its worst-case mean-squared error is 4% smaller than that of the UATE, and 60% smaller than that of the SFE.

Overall, in SRCTs where the treatment probability varies substantially across strata, I recommend that researchers either use the minimax-linear estimator or the UATE. My results show that with the ATE as the target parameter, a bias-variance trade-off cannot rationalize the use of the SFE. Rationalizing the use of this estimator requires changing
the target parameter, or assuming that the CATEs are constant across strata.

The minimax-linear estimator I propose can also be used in matching studies with limited overlap, meaning that units’ propensity scores are not bounded away from zero and one. Limited overlap can lead to poor finite sample performance of matching estimators (see Busso et al., 2014 and Rothe, 2017) and may lead those estimators to converge at a slower rate than the standard parametric rate (see Khan & Tamer, 2010). To address this issue, Crump et al. (2009) assume homoscedasticity and redefine the target parameter as the ATE in the subpopulation whose ATE can be estimated most precisely. In practice, this often leads to dropping observations with an estimated propensity score outside of the [0.1, 0.9] interval, thus motivating the well-known rule of thumb proposed the authors. This approach minimizes variance, but unlike my minimax-linear estimator it does not control bias with respect to the original target (the ATE in the full population). I revisit Connors et al. (1996), a famous example of a matching study with limited overlap, that was also revisited by Crump et al. (2009). The estimator of Crump et al. (2009) trims 18% of the sample, and its standard error is 17% lower than that of the estimator without trimming. My minimax-linear estimator downweights 3% of the sample, and its standard error is 10% lower than that of the estimator without trimming. Thus, the minimax-linear estimator achieves 56% of the precision gains achieved by the estimator of Crump et al. (2009), without changing the target parameter, but at the expense of assuming a bound on the CATEs. The worst-case MSE of the estimator of Crump et al. (2009) is eleven times larger than that of my estimator.

Related Literature

On top of the aforementioned literature on matching estimators with limited overlap, this paper is related to a vast literature in statistics, that has studied linear- and affine-minimax estimators. The setting I consider can be cast as a bounded normal mean model, where realizations of normally distributed random variables are used to estimate a linear combination of their means, which are assumed to be bounded. Donoho (1994), who considers a more general setup than the bounded normal mean model, characterizes the risk of the affine-minimax estimator, and shows that it cannot be more than 25% larger than that of the minimax estimator. Armstrong & Kolesár (2018) consider a similar set-up as Donoho (1994), and show how to construct optimal confidence intervals. The closed-

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2I do not assume that CATEs estimators are normally distributed, but as noted by Armstrong & Kolesár (2021b), this distributional assumption is not of essence to derive the minimax-linear estimator.
form expression of the minimax-linear estimator I derive is, to my knowledge, new. In their Section 4.2., Ibragimov & Khas’minskii (1985) derive a closed-form expression of the minimax-linear estimator in a setting that can accommodate an RCT with one strata, or a matching estimator where covariates only take one value. My result generalizes theirs to the case with two or more strata, or two or more covariates’ values. The bias-variance trade-off this paper is concerned with only arises with two or more strata or covariates’ values, so this extension is of essence to the problem at hand.

My paper is also related to a growing econometrics literature that has applied the set-up in Donoho (1994) to other estimation problems. Within that literature, my paper is most closely related to Armstrong & Kolesár (2021a), who study ATE estimation under unconfoundedness when the mean outcome conditional on the covariates is a Lipschitz function with a bounded Lipschitz constant. Bounding the CATEs by $B$ standard deviations of the outcome is appealing given that outcomes are often normalized by their standard deviation in applied research, so researchers can use prior literature to chose a reasonable value for $B$. Finding reasonable values for the bound on the Lipschitz constant might be more difficult for practitioners. Moreover, in SCRTs the minimax-linear estimator I propose is feasible given $B$. Computing it does not require estimating the outcome’s variance in a first step, unlike the estimator proposed in Armstrong & Kolesár (2021a).

**Organization of the paper.** Section 2 introduces the setup. Section 3 presents the paper’s main result. Section 4 presents an extension. Section 5 presents three empirical applications. All proofs are relegated to the appendix.

## 2 Setup and examples

**Definition 2.1 (ATE and CATEs)** One is interested in estimating an average treatment effect $\tau$, equal to a weighted average of conditional average treatment effects $(\tau_s)_{1 \leq s \leq S}$, with weights $(p_s)_{1 \leq s \leq S}$ that are known, positive, and sum to 1:

$$\tau = \sum_{s=1}^{S} p_s \tau_s.$$  \hspace{1cm} (2.1)

Other papers in this literature include: Armstrong & Kolesár (2021b), who study sensitivity analysis in locally misspecified GMM models; Rambachan & Roth (2019), who study difference-in-differences estimation with bounded departures from parallel trends; Armstrong & Kolesár (2018), Imbens & Wager (2019), and Noack & Rothe (2019), who study estimation in regression discontinuity designs with bounded second derivatives of the mean of the potential outcomes conditional on the running variable.
**Assumption 1** One observes random variables $(\hat{\tau}_s)_{1 \leq s \leq S}$ such that:

1. $E(\hat{\tau}_s) = \tau_s$ for all $s$.
2. $cov(\hat{\tau}_s, \hat{\tau}_{s'}) = 0$ for all $s' \neq s$.
3. There is a strictly positive unknown real number $\sigma$ and positive real numbers $(v_s)_{1 \leq s \leq S}$ such that $V(\hat{\tau}_s) = \sigma^2 v_s$ for all $s$.
4. There is a strictly positive known real number $B$ such that $|\tau_s| \leq B\sigma$ for all $s$.

**Example 1: SRCTs.** Under some assumptions, Definition 2.1 and Assumption 1 are applicable to SRCTs. Let $s$ index strata, and let $S$ be the number of strata. Let $\tau_s$ be the CATE in stratum $s$, $p_s$ be the share of the population stratum $s$ accounts for, and $\tau$ be the ATE. Let $\hat{\tau}_s$ be the difference between the average outcome of treated and control units in stratum $s$. Let $n_s$ be the number of units in stratum $s$, let $n_{1s} \in \{1, ..., n_s - 1\}$ be the number of treated units, let $n_{0s} = n_s - n_{1s}$ be the number of control units, and let $D_s$ be a vector stacking the treatment indicators of all units in stratum $s$. Assume that:

1. for all $s$, for every $(d_1, ..., d_{n_s}) \in \{0, 1\}^{n_s}$ such that $d_1 + ... + d_{n_s} = n_{1s}$,
   
   \[ P(D_s = (d_1, ..., d_{n_s})) = \frac{1}{\binom{n_s}{n_{1s}}} \]

2. The random vectors $(D_s)$ are mutually independent across $s$.

Points 1 and 2 above are standard conditions that hold by design in SRCTs. They imply that Points 1 and 2 of Assumption 1 hold. If one further assumes that the outcome is homoscedastic and that the units in stratum $s$ are randomly drawn from a superpopulation, $V(\hat{\tau}_s) = \sigma^2 \left( \frac{1}{n_{0s}} + \frac{1}{n_{1s}} \right)$. Therefore, Point 3 of Assumption 1 holds, with $v_s = \frac{1}{n_{0s}} + \frac{1}{n_{1s}}$. If the units in stratum $s$ are a convenience rather than a random sample, $\sigma^2 \left( \frac{1}{n_{0s}} + \frac{1}{n_{1s}} \right)$ is a sharp upper bound of $V(\hat{\tau}_s)$, and the analysis below still applies. Finally, Point 4 of Assumption 1 requires that in each stratum, the CATE be bounded in absolute value by $B\sigma$, for a known $B$.

**Example 2: Augmented Inverse Propensity Weighting (AIPW) estimators.** Under some assumptions, the building blocks of the AIPW estimator of Robins et al. (1994) satisfy Assumption 1, conditional on the covariates as in Armstrong & Kolesár.
Let $s$ index independent and identically distributed (iid) units drawn from a superpopulation. Let $Y_s(0)$ and $Y_s(1)$ denote the untreated and treated outcomes of $s$, and let $X_s$ denote $s$’s covariates. Assume that
\[ (Y_s(0), Y_s(1)) \perp \perp D_s | X_s : \tag{2.2} \]
treatment is ignorable conditional on the covariates. Let \( \tau_s = E(Y_s(1) - Y_s(0) | X_s) \), and let \( p_s = 1/S \). Then, \( \tau = 1/S \sum_{s=1}^S E(Y_s(1) - Y_s(0) | X_s) \) is the sample average treatment effect, a parameter also considered by Crump et al. (2009). Letting \( Y_s \) and \( D_s \) denote the outcome and treatment of \( s \), we let \( e(X_s) = P(D_s = 1 | X_s) \) denote their propensity-score, and for all \( d \in \{0, 1\} \), we let \( \mu_d(X_s) = E(Y_s | D_s = d, X_s) \). Then, let
\[ \hat{\tau}_s = \mu_1(X_s) - \mu_0(X_s) + D_s \frac{Y_s - \mu_1(X_s)}{e(X_s)} - (1 - D_s) \frac{Y_s - \mu_1(X_s)}{1 - e(X_s)}. \tag{2.3} \]
1/\( S \sum_{s=1}^S \hat{\tau}_s \) is the oracle version of the AIPW estimator. This oracle estimator is first-order equivalent to the feasible AIPW estimator where \( e(X_s) \), \( \mu_0(X_s) \), and \( \mu_1(X_s) \) are replaced by their estimators, for instance if one can form estimators of those quantities that converge at a faster rate than \( n^{1/4} \), a mild requirement (see Farrell, 2015). Under (2.2),
\[ E(\hat{\tau}_s | X_s) = \tau_s, \]
so Point 1 of Assumption 1 holds conditional on \( X_s \). Point 2 mechanically holds as units are assumed to be iid. If \( V(Y_s(1) | X_s) = V(Y_s(0) | X_s) = \sigma^2 \), then
\[ V(\hat{\tau}_s | X_s) = \sigma^2 \frac{1}{e(X_s)(1 - e(X_s))}, \]
so Point 3 of Assumption 1 also holds conditional on \( X_s \), with \( v_s = \frac{1}{e(X_s)(1 - e(X_s))} \). Finally, Point 4 requires that CATEs be bounded in absolute value by \( B\sigma \).

3 Minimax-linear estimator under homoscedasticity

For any \( 1 \times S \) deterministic vector \( w = (w_1, ..., w_S) \), let
\[ \hat{\tau}(w) = \sum_{s=1}^S w_s \hat{\tau}_s. \tag{3.1} \]
\( \hat{\tau}(w) \) is a linear combination of the estimators \( \hat{\tau}_s \). Lemma 3.1 gives its worst-case MSE.

**Lemma 3.1** (Worst-case MSE of \( \hat{\tau}(w) \))


If Assumption 1 holds, then for any \(1 \times S\) deterministic vector \(w = (w_1, ..., w_S)\)

\[
E \left( (\hat{\tau}(w) - \tau)^2 \right) \leq \text{MSE}(w) \equiv \sigma^2 \left( \sum_{s=1}^{S} w_s^2 v_s + B^2 \left( \sum_{s=1}^{S} |w_s - p_s| \right)^2 \right).
\]

The upper bound in the previous display is sharp: it is attained if \(\tau_s = \sigma B \left( 1\{w_s \geq p_s\} - 1\{w_s < p_s\} \right)\).

Without loss of generality, assume that

\[ p_1 v_1 \leq p_2 v_2 \leq ... \leq p_S v_S. \]

Let \(\bar{s} = \min\{s \in \{1, ..., S\} : \frac{1}{\sigma^2 + \sum_{s'=s}^{S} v_{s'}^{-1}} \sum_{s'=s}^{S} p_{s'} \leq p_s v_s\}\). \(\bar{s}\) is well defined, because \(\frac{1}{\sigma^2 + v_{\bar{s}}^{-1}} \sum_{s'=s}^{S} p_{s'} \leq p_s v_s\). For any \(h \in \{1, ..., S\}\), let \(w_h\) be such that

\[
w_{s,h} = p_s \text{ for all } s < h
\]

\[
w_{s,h} = \frac{1}{v_s \sigma^2 + \sum_{s'=h}^{S} v_{s'}^{-1}} \sum_{s'=h}^{S} p_{s'} \text{ for all } s \geq h.
\]

Finally, let

\[
w^* = \arg\min_{w \in \mathbb{R}^S} \text{MSE}(w).
\]

It follows from Lemma 3.1 that \(\hat{\tau}(w^*)\) is the minimax-linear estimator of \(\tau\).

**Theorem 3.1 (Minimax-linear estimator of \(\tau\), with bounded \(\tau_s\) and homoscedasticity)**

If Assumption 1 holds, then \(\hat{\tau}(w^*) = \hat{\tau}(w_{h^*})\), where \(h^* = \arg\min_{h \in \{\bar{s}, ..., S\}} \text{MSE}(w_h)\).

Theorem 3.1 shows that under Assumption 1 the minimax-linear estimator is a weighted sum of the \(\hat{\tau}_s\)'s, with positive weights that sum to less than 1. For a precisely estimated \(\hat{\tau}_s\) (one with a low value of \(p_s v_s\)), the optimal weight is just \(p_s\). On the other hand, for an imprecisely estimated \(\hat{\tau}_s\) (one with a high value of \(p_s v_s\)), the optimal weight is proportional to one over \(v_s\), the non-constant part of its variance. For an imprecisely estimated \(\hat{\tau}_s\), the optimal weight is also shrunk towards zero, where the amount of shrinkage depends on \(B\).

Operationally, to find the minimax-linear estimator, one just needs to compute \(\bar{s}\), and then evaluate \(\text{MSE}(w)\) at \(w_h\) for \(h \in \{\bar{s}, ..., S\}\). The following lemma shows that to compute \(\bar{s}\), one just needs to evaluate the inequalities \(\sum_{s'=s}^{S} \frac{1}{v_{s'}} \sum_{s'=s}^{S} p_{s'} \leq p_s v_s\) for \(s = S - 1, s = S - 2, \) etc., until one finds a first value where the inequality fails. \(\bar{s}\) equals that value plus one.

**Lemma 3.2**

\[
\frac{1}{\sum_{s'=s}^{S} v_{s'}} \sum_{s'=s}^{S} p_{s'} \leq p_s v_s \Rightarrow \frac{1}{\sum_{s'=s+1}^{S} v_{s'}} \sum_{s'=s+1}^{S} p_{s'} \leq p_{s+1} v_{s+1}.
\]
Finally, I give sufficient conditions under which the minimax-linear estimator under homoscedasticity still has lower worst-case MSE than the unbiased estimator, even if the outcome is actually heteroscedastic.

**Assumption 2**

1. For all \( s \in \{1, ..., S\} \), there is a strictly positive unknown real number \( \sigma \), unknown real numbers \( (h_s)_1 \leq s \leq S \) such that \( h_s \geq 1 \) for all \( s \), and positive known real numbers \( (v_{0,s}, v_{1,s})_1 \leq s \leq S \), such that \( V(\hat{\tau}_s) = \sigma^2(v_{0,s} + h_s v_{1,s}) \).

2. For all \( s \in \{1, ..., S\} \), there is a strictly positive unknown real number \( \sigma \), an unknown real number \( h \) such that \( 0 \leq h \leq 1 \), and positive known real numbers \( (v_{0,s}, v_{1,s})_1 \leq s \leq S \), such that \( V(\hat{\tau}_s) = \sigma^2(v_{0,s} + hv_{1,s}) \).

**Corollary 3.2** (Ordering of \( \hat{\tau}(w^*) \)’s and \( \hat{\tau}(p) \)’s worst-case MSE with heteroscedasticity)

1. If Points 1, 2, and 4 of Assumption 1 and Point 1 of Assumption 2 hold, the worst-case MSE of \( \hat{\tau}(w^*) \) is lower than that of \( \hat{\tau}(p) \).

2. If Points 1, 2, and 4 of Assumption 1 and Point 2 of Assumption 2 hold, and if \( h \geq B^2 \left( \sum_{s=1}^{S} |w^*_s - p_s| \right)^2 - \sum_{s=1}^{S} ((p_s)^2 - (w^*_s)^2)v_{0,s} \), \( \sum_{s=1}^{S} ((p_s)^2 - (w^*_s)^2)v_{1,s} \), the worst-case MSE of \( \hat{\tau}(w^*) \) is lower than that of \( \hat{\tau}(p) \).

3.1 Application to SRCTs

**Computing the minimax-linear estimator and estimating its variance.** In an SRCT, the minimax-linear estimator in Theorem 3.1 is often feasible: given \( B \), the optimal weights \( w^* \) only depend on \( (n_{0,s}, n_{1,s}, p_s)_1 \leq s \leq S \), which are typically known quantities. Then, the weights \( w^* \) are not stochastic, and one can show that

\[
V(\hat{\tau}(w^*)) = \sigma^2 \sum_{s=1}^{S} (w^*_s)^2 \left( \sigma^2_{0,s}/n_{0,s} + \sigma^2_{1,s}/n_{1,s} \right),
\]

where \( \sigma^2_{0,s} \) and \( \sigma^2_{1,s} \) respectively denote the variances of the untreated and treated outcomes in stratum \( s \), and where the previous equality holds even if the outcome is heteroscedastic. The right hand side of the previous display can easily be estimated.

**In SRCTs, the minimax-linear estimator shares features with both the unbiased and strata-fixed-effects estimators.** Let \( p = (p_1, ..., p_S) \). The UATE is equal to \( \hat{\tau}(p) \). Let

\[
w_{fe} = \left( \frac{1}{n_{0,1}} + \frac{1}{n_{1,1}} \right)^{-1}, ..., \left( \frac{1}{n_{0,s}} + \frac{1}{n_{1,s}} \right)^{-1} \right),
\]

where \( n_{0,s} \) and \( n_{1,s} \) are the numbers of individuals in stratum \( s \) assigned to the untreated and treated groups, respectively.
It follows from, e.g., Angrist (1998), that the SFE is equal to \( \hat{\tau}(w_{fe}) \). Like \( \hat{\tau}(p) \), \( \hat{\tau}(w^*) \) assigns a weight equal to \( p_s \) to precisely estimated CATEs. Still, one always has \( w^*_S < p_S \), so \( \hat{\tau}(w^*) \) never coincides with \( \hat{\tau}(p) \). Like \( \hat{\tau}(w_{fe}) \), \( \hat{\tau}(w^*) \) assigns to imprecisely estimated \( \hat{\tau}_s \)’s weights proportional to one over their variance.

Interpreting Assumption 2 and Corollary 3.2 in an SRCT. In an SRCT, Point 1 of Assumption 2 assumes that the untreated outcome’s variance does not vary across strata, as is the case when in each stratum researchers standardize their outcome by its standard deviation among the stratum’s control group, and if in each stratum the variance of the treated outcome is larger than that of the untreated one. Then, \( \sigma^2 \) is the untreated outcome’s variance, \( h_s \) is the ratio of the treated and untreated outcomes’ variances in strata \( s \), \( v_{0,s} = 1/n_{0,s} \), and \( v_{1,s} = 1/n_{1,s} \). Then, Point 1 of Corollary 3.2 implies that the worst-case MSE of \( \hat{\tau}(w^*) \) is lower than that of \( \hat{\tau}(p) \), if the variance of the outcome with treatment is larger than that of the outcome without treatment. Intuitively, if \( \hat{\tau}(w^*) \) underestimates the variances of all the CATE estimators, then it does not shrink those estimators enough with respect to what an oracle estimator would do, but it still dominates the unbiased estimator that does not do any shrinkage. Point 2 of Assumption 2 instead assumes that the variance of the treated outcome is lower than that of the untreated one, and heteroscedasticity is constant across strata. Then, Point 2 of Corollary 3.2 shows that the worst-case MSE of \( \hat{\tau}(w^*) \) is still lower than that of \( \hat{\tau}(p) \) if the ratio of the treated and untreated outcomes’ variances is greater than a bound which only depends on the design and can be readily computed. In the first SRCT I revisit in Section 5, this lower bound is negative so the worst-case MSE of \( \hat{\tau}(w^*) \) is guaranteed to be lower than that of \( \hat{\tau}(p) \).

3.2 Application to doubly-robust matching estimators.

Computing the minimax-linear estimator and estimating its variance. With doubly-robust unit-level matching estimators as building blocks, the minimax-linear estimator in Theorem 3.1 is not feasible. Then, the optimal weights \( w^* \) depend on the propensity-score \( e(X_s) \), which is typically unknown. One can still estimate this propensity score and the optimal weights, to form a feasible estimator \( \hat{\tau}(\hat{w}^*) \) proxying for \( \hat{\tau}(w^*) \). One
may then use the bootstrap to estimate the variance of \( \hat{\tau}(\hat{w}^*) \), making sure to re-estimate the propensity score in each bootstrap sample.

**Comparison with the AIPW estimator.** The (oracle) AIPW estimator is equal to \( \hat{\tau}(p) \). Like \( \hat{\tau}(p) \), \( \hat{\tau}(w^*) \) assigns weight \( 1/S \) to units such that \( e(X_s) \) is not too close to zero or one. But unlike \( \hat{\tau}(p) \), it downweights units such that \( e(X_s) \) is close to zero or one. How close to zero or one should \( e(X_s) \) be for a unit to be downweighted depends on \( B \) and on the structure of the data, just as the amount of downweighting. In the third application I revisit in Section 5, units with a value of \( e(X_s) \) below 0.03 or above 0.97 get downweighted, and the weight they receive in \( \hat{\tau}(w^*) \) is strictly positive but very close to zero.

**Comparison with the AIPW estimator with trimming.** Let \( S_{\text{trim}} = \# \{ s : e(X_s) > 0.1 \text{ or } e(X_s) > 0.9 \} \) denote the number of trimmed units under the rule of thumb of Crump et al. (2009). Let

\[
w_{\text{trim}} = \frac{1}{S - S_{\text{trim}}} 1 \{ e(X) \in [0.1, 0.9] \}.
\]

The AIPW estimator with trimming is \( \hat{\tau}(w_{\text{trim}}) \). There are three differences between \( \hat{\tau}(w^*) \) and \( \hat{\tau}(w_{\text{trim}}) \). First, \( \hat{\tau}(w_{\text{trim}}) \) trims units such that \( e(X_s) \) is close to zero or one. \( \hat{\tau}(w^*) \) downweights them, though in practice in the aforementioned application this downweighting is effectively close to trimming. Second, the two estimators do not trim/downweight the same units: in that application, \( \hat{\tau}(w^*) \) trims/downweights much fewer units than \( \hat{\tau}(w_{\text{trim}}) \). Third, the weights attached to \( \hat{\tau}(w_{\text{trim}}) \) sum to one, while those attached to \( \hat{\tau}(w^*) \) sum to less than one: \( \hat{\tau}(w_{\text{trim}}) \) rescales the weights attached to non-trimmed units, unlike \( \hat{\tau}(w^*) \).

### 4 Minimax-linear estimator without homoscedasticity

A result similar to Theorem 3.1 still holds without the homoscedasticity assumption, and under a modified version of Point 4 in Assumption 1:

**Assumption 3** For all \( s \in \{1, ..., S\} \): there is a strictly positive known real number \( B \) such that \( |\tau_s| \leq B \).

Without loss of generality, assume that

\[
p_1 V(\hat{\tau}_1) \leq p_2 V(\hat{\tau}_2) \leq ... \leq p_s V(\hat{\tau}_s).
\]

11
Let $s_2 = \min\{s \in \{1, \ldots, S\} : \frac{1}{B^2 + \sum_{s' = s}^{S} \frac{1}{V(\tau_{s'})}} \sum_{s' = s}^{S} p_{s'} \leq p_s V(\hat{\tau}_s)\}$. For any $h \in \{1, \ldots, S\}$, let $w_{h,2}$ be such that

\begin{align*}
    w_{s, h, 2} &= p_s \text{ for all } s < h \\
    w_{s, h, 2} &= \frac{1}{V(\hat{\tau}_s)} \frac{1}{B^2 + \sum_{s' = h}^{S} \frac{1}{V(\tau_{s'})}} \sum_{s' = h}^{S} p_{s'} \text{ for all } s \geq h.
\end{align*}

Finally, let

\[
    \text{MSE}_2(w) = \sum_{s=1}^{S} w_s^2 V(\hat{\tau}_s) + B^2 \left( \sum_{s=1}^{S} |w_s - p_s| \right)^2
\]

and

\[
    w^*_2 = \arg\min_{w \in \mathbb{R}^S} \text{MSE}_2(w).
\]

**Theorem 4.1 (Minimax-linear estimator of $\tau$, with bounded $\tau_s$)**

If Assumption 3 holds, then for any $1 \times S$ deterministic vector $w = (w_1, \ldots, w_S)$,

\[
    E \left( (\hat{\tau}(w) - \tau)^2 \right) \leq \text{MSE}_2(w).
\]

The upper bound in the previous display is sharp: it is attained if $\tau_s = B \left( 1\{w_s \geq p_s\} - 1\{w_s < p_s\} \right)$. 

$\hat{\tau}(w^*_2) = \hat{\tau}(w^*_{h,2})$, where $h^*_2 = \arg\min_{h \in \{s_2, \ldots, S\}} \text{MSE}_2(w_{h,2})$.

Theorem 4.1 shows that without the homoscedasticity assumption, the minimax-linear estimator is still a weighted sum of the $\hat{\tau}_s$, with positive weights that sum to less than 1, and where the most precisely estimated CATEs receive a weight equal to $p_s$, while the least precisely estimated CATEs receive a weight proportional to one over their variance.

## 5 Illustrative applications

### 5.1 Behaghel et al. (2017)

The authors conducted an SRCT to estimate the effect of a boarding school for disadvantaged students in France. The boarding school’s pedagogy is similar to that of “No Excuse” charter schools in the US. It has capacity constraints at the gender $\times$ grade level, and in 2009 and 2010, the school had more applicants than seats in 14 gender $\times$ grade strata. In each stratum, seats were randomly offered to some applicants.\(^5\) Two years after the randomization, 363 applicants out of the 395 that participated in a lottery took a standardized

\(^5\)In this illustrative application, I do not take into account the fact that randomization followed a waitlist process, which generates complications orthogonal to the issues discussed in this paper. This explains why some numbers below do not exactly match the corresponding numbers in Behaghel et al. (2017), where the authors account for those complications.
maths test. Those students are the study sample, their score is the outcome. Treatment is defined as receiving an offer to enter the school. The treatment probability varies substantially across strata: it ranges from 0.17 to 0.93. With this treatment definition, \( \tau_s \) is the intention-to-treat effect of receiving an offer on students’ test scores two years after the lottery, in stratum \( s \). At that point, the first-stage effect of receiving an offer on the number of years spent in the school is equal to 1.34, so the \( \tau_s \)'s can be interpreted as effects of having spent 1.34 years in the boarding school. The first-stage effects may vary across strata, which would complicate this interpretation. However, I run a first-stage regression of attendance to the boarding school on a full set of strata fixed effects (FEs) interacted with the treatment, and cannot reject the null that the coefficients on the interactions of the offer and strata FEs are all equal to zero (p-value=0.25).

Based on the literature, \( 0.5\sigma \) is a plausible upper bound for the effect of spending one year in this boarding school. The paper studying the closest intervention is Curto & Fryer Jr (2014), who study a “No Excuse” charter boarding school in Washington DC. In their full sample, they find that one year spent in the school increases students’ math test scores by \( 0.23\sigma \). They also estimate CATEs in eight subgroups of students: males/females, students benefiting/not benefiting from the free lunch program, students in/not in special education, and students above/below the median at baseline. The estimated effects in those 8 subgroups are included between 0.04 and 0.36\( \sigma \), and in 7 of the 8 subgroups one can reject at the 90% level that the effect is greater than \( 0.5\sigma \), the only exception being the special education stratum that only has 30 students. Results from Angrist et al. (2010), Dobbie & Fryer Jr (2011), and Abdulkadiroğlu et al. (2011), three papers studying successful non-boarding “No Excuse” charter schools in New-York and Boston, also suggest that \( 0.5\sigma \) is a plausible upper bound. Together, these papers estimate 14 CATEs of spending one year in those schools on students’ math test scores. All estimates are included between 0.18 and 0.36\( \sigma \), and for 13 of the 14 CATEs, one can reject at the 90% level an effect greater than \( 0.5\sigma \). Accordingly, I let \( B = 1.34 \times 0.5 \). As a robustness check, I will also let \( B = 1.34 \times 0.6 \).

Results are shown in Table 1. \( \hat{\tau}(p) \) is 10% larger than \( \hat{\tau}(w_{fe}) \), and its standard error is 4% larger. \( \hat{\tau}(p) \) is marginally significant at the 5% level (t-stat=1.99), and \( \hat{\tau}(w_{fe}) \) is significant at the 10% level (t-stat=1.88). The difference between the two estimators is insignificant. \( \hat{\tau}(w^*) \) is very close to \( \hat{\tau}(p) \), but its standard error is 3% smaller, so it is the most significant of the three estimators (t-stat=2.05). The standard error of \( \hat{\tau}(w^*) \) is 1% larger than that of \( \hat{\tau}(w_{fe}) \). Assuming homoscedasticity, one can compute the worst-case MSE of the three estimators. That of \( \hat{\tau}(w^*) \) is 10% smaller than that of \( \hat{\tau}(p) \), and 76% smaller than that of \( \hat{\tau}(w_{fe}) \). As shown in Point 1 of Corollary 3.2 with an heteroscedastic outcome, \( \hat{\tau}(w^*) \)'s
worst-case MSE is still lower than $\hat{\tau}(p)$’s if the treated outcome’s variance is higher than that of the untreated outcome. This may be a plausible assumption in this context, as quantile treatment effects of this intervention are higher at the top than at the bottom of the distribution of test scores, thus suggesting that the intervention increases scores’ variance (see Figure 2 of Behaghel et al. [2017]). I still compute the lower bound in Point 2 of Corollary 3.2 and find that it is negative: under Point 2 of Assumption 2, the worst-case MSE of $\hat{\tau}(w^*)$ is guaranteed to be lower than that of $\hat{\tau}(p)$. The weight assigned by $\hat{\tau}(w^*)$ to the strata with the treatment probability furthest from $1/2$ is roughly half as large as the proportion this strata accounts for in the sample. The weights assigned by $\hat{\tau}(w^*)$ to all the other strata are equal to their shares in the population. Letting $B = 1.34 \times 0.6$ leaves $\hat{\tau}(w^*)$ and its standard error unchanged to the third digit.

Table 1: $\hat{\tau}(p)$, $\hat{\tau}(w_{fe})$, and $\hat{\tau}(w^*)$ in Behaghel et al. (2017)

|                      | $\hat{\tau}(p)$ | $\hat{\tau}(w_{fe})$ | $\hat{\tau}(w^*)$ |
|----------------------|------------------|------------------------|-------------------|
| Point estimate       | 0.267            | 0.243                  | 0.269             |
| Robust standard error| 0.135            | 0.129                  | 0.131             |
| Worst-case MSE       | 0.018            | 0.068                  | 0.016             |

Notes: This table shows $\hat{\tau}(p)$, $\hat{\tau}(w_{fe})$, and $\hat{\tau}(w^*)$ in Behaghel et al. (2017). $\hat{\tau}(w^*)$ is computed with $B = 1.34 \times 0.5$. The treatment is defined as being offered a seat in the boarding school. The outcome is students’ standardized maths test scores two years after the lottery. The table shows the point estimates, their robust standard errors, and their worst-case MSE computed assuming homoscedasticity and expressed in percentage points of the outcome’s variance.

5.2 Blattman & Annan (2016)

After the second Liberian civil war, some ex-fighters started engaging in illegal activities, and working abroad as mercenaries. The authors conducted an SRCT to estimate the effect of an agricultural training on their employment and on their social networks. By improving their labor market opportunities, the program hoped to reduce their interest in illegal and mercenary activities, and to sever their relationships with other ex-combatants. To allocate the treatment, the authors divided applicants into 67 strata, according to the training site they applied for, their former military rank, and their community, and made treatment offers within each stratum. Treatment is defined as receiving a program offer,

\[^6\]Here again, I do not take into account the fact that randomization followed a waitlist process, which generates complications orthogonal to the issues discussed in this paper.
so the $\tau_s$s are intention-to-treat effects. The first-stage effect of receiving an offer on the probability to join the program is 0.75. To test if first-stage effects vary across strata, I run a first-stage regression with a full set of strata FEs interacted with the treatment offer, and cannot reject the null that the coefficients on the interactions of the offer and strata FEs are all equal to zero (p-value=0.19). The treatment probability varies substantially across strata: it ranges from 0.33 to 0.89.

Blattman & Annan (2016) estimate the training’s effect on 62 outcomes, that are either applicants’ answers to survey questions, or standardized scores averaging their answers to several related questions. To preserve space, I follow de Chaisemartin & Behaghel (2020) and only consider 10 main outcomes (see de Chaisemartin & Behaghel (2020) for the rules used to make that selection). For all outcomes, $0.7\times$ the outcome’s standard deviation is more than twice larger, in absolute value, than the program’s average LATE across strata (see Table 3 of de Chaisemartin & Behaghel (2020) for those LATE estimates). Thus, $0.7\sigma$ seems like a plausible upper bound for the program’s LATE in every stratum. Accordingly, I let $B = 0.75 \times 0.7$. As a robustness check, I will also let $B = 0.75 \times 0.8$.

For each outcome, Table 2 below shows $\hat{\tau}(p)$, $\hat{\tau}(w_{fe})$, and $\hat{\tau}(w^*)$. Standard errors are shown between parentheses. Under the homoscedasticity assumption, estimators’ worst-case MSEs only depend on the design and do not vary across outcomes, so I show them once, at the bottom of the table. For five outcomes out of ten, the difference between $\hat{\tau}(p)$ and $\hat{\tau}(w_{fe})$ is not completely trivial, with $\hat{\tau}(p)/\hat{\tau}(w_{fe})$ either lower than 0.85 or larger than 1.15. For the “Interest in mercenary work” and “Relations with ex-combatants” outcomes, the two estimators are significantly different, at the 5 and 10% level respectively (t-stat=-2.12 and -1.83). On average across the 10 outcomes, the standard deviation of $\hat{\tau}(p)$ is 5% larger than that of $\hat{\tau}(w_{fe})$. Its worst-case MSE is 57% smaller. $\hat{\tau}(w^*)$ is at most 11% away from $\hat{\tau}(w_{fe})$, while there is an outcome for which it is 42% away from $\hat{\tau}(w_{fe})$. For the “Hours of illegal work” and “Relations with ex-combatants” outcomes, $\hat{\tau}(w^*)$ and $\hat{\tau}(w_{fe})$ are significantly different, at the 10 and 1% level respectively (t-stat=1.82 and -3.00). For the “Relations with ex-combatants ” and “Social network quality” outcomes, $\hat{\tau}(w^*)$ and $\hat{\tau}(p)$ are significantly different at the 10 and 5% level respectively (t-stat=1.86 and 2.13). On average across the 10 outcomes, the standard deviation of $\hat{\tau}(w^*)$ is 3% lower than that of $\hat{\tau}(p)$, and 2% larger than that of $\hat{\tau}(w_{fe})$. Its worst-case MSE is 3% smaller than that of $\hat{\tau}(w_{fe})$. Those numbers do not change much with $B = 0.75 \times 0.8$. 


Table 2: $\hat{\tau}(p)$, $\hat{\tau}(w_{fe})$, and $\hat{\tau}(w^*)$ in Blattman & Annan (2016)

|                                | $\hat{\tau}(p)$ (s.e.) | $\hat{\tau}(w_{fe})$ (s.e.) | $\hat{\tau}(w^*)$ (s.e.) |
|--------------------------------|--------------------------|-------------------------------|---------------------------|
| Works in agriculture           | 0.115 (0.029)            | 0.122 (0.030)                 | 0.119 (0.029)             |
| Hours illegal work             | -2.488 (1.403)           | -2.396 (1.351)                | -2.155 (1.314)            |
| Hours farming work             | 2.441 (1.299)            | 2.656 (1.285)                 | 2.406 (1.254)             |
| Income index                   | 0.104 (0.062)            | 0.094 (0.065)                 | 0.096 (0.061)             |
| Interest mercenary work        | -0.239 (0.092)           | -0.284 (0.116)                | -0.271 (0.115)            |
| Relations ex-combatants        | 0.086 (0.058)            | 0.054 (0.059)                 | 0.050 (0.057)             |
| Relations ex-commanders        | -0.131 (0.062)           | -0.168 (0.068)                | -0.152 (0.067)            |
| Social network quality         | 0.049 (0.059)            | 0.054 (0.066)                 | 0.049 (0.065)             |
| Social support                 | 0.129 (0.064)            | 0.110 (0.064)                 | 0.121 (0.063)             |
| Relationships families         | 0.109 (0.063)            | 0.126 (0.063)                 | 0.120 (0.061)             |
| Worst-case MSE under homoscedasticity | 0.0047                  | 0.0109                       | 0.0045                    |

Notes: This table shows $\hat{\tau}(p)$, $\hat{\tau}(w_{fe})$, and $\hat{\tau}(w^*)$ in Blattman & Annan (2016). $\hat{\tau}(w^*)$ is computed with $B = 0.75 \times 0.7$. The treatment is defined as being offered a program seat. The table shows the point estimates and their robust standard errors between parentheses. Under the homoscedasticity assumption, estimators’ worst-case MSEs only depend on the design and do not vary across outcomes, so I show them once, at the bottom of the table.

5.3 Connors et al. (1996)

In this section, I revisit Connors et al. (1996), a famous example of a matching study with limited overlap, that was also revisited by Crump et al. (2009) and Rothe (2017). Connors et al. (1996) study the impact of right heart catheterization (RHC) on patient mortality. RHC is a diagnostic procedure used for critically ill patients. The data contain information on 5,735 patients. For each patient, I observe the treatment status $D_s$, defined as RHC being applied within 24 hours of admission, the outcome $Y_s$, an indicator for survival at 30 days, and 71 covariates deemed related to the decision to perform the RHC by a panel of experts. Using a propensity score matching approach, the authors concluded that RHC causes a substantial increase in patient mortality.

I follow Hirano & Imbens (2001) and Crump et al. (2009), and estimate the propensity score $e(X_s)$ using a logistic regression that includes all the covariates. In the treatment and control groups, the support of the estimated propensity scores is nearly the entire unit interval, so the AIPW estimator may be affected by limited overlap. As the outcome is binary, I also use logistic regressions including all covariates to estimate $\mu_0(X_s)$ and $\mu_1(X_s)$. 
Column (1) of Table 3 shows \( \hat{\tau}(p) \), the feasible AIPW estimator where \( e(X_s) \), \( \mu_0(X_s) \), and \( \mu_1(X_s) \) are replaced by their estimators. Column (2) shows \( \hat{\tau}(\hat{w}_{\text{trim}}) \), the AIPW estimator after trimming patients with propensity scores outside of the \([0.1, 0.9]\) interval, as suggested by [Crump et al. 2009]. This leads to trimming 1,008 patients, namely 18% of the sample. Finally, Column (3) shows the minimax-linear estimator \( \hat{\tau}(\hat{w}^*) \), with \( B = 1/3 \). As survival’s standard deviation is around 0.48, \( 1/3 \sigma \approx 0.16 \), a large effect for a binary outcome, around 2.5 times larger than \( \hat{\tau}(p) \), the estimated ATE. In this data and with that value of \( B \), the minimax-linear estimator assigns a weight close to 0 to 159 patients, namely 3% of the sample, with a propensity score outside of the \([0.03, 0.97]\) interval. Estimators’ standard errors are computed using the bootstrap, drawing samples of patients without replacement from the original sample, and reestimating \( e(X_s) \), \( \mu_0(X_s) \), and \( \mu_1(X_s) \) in each sample. The three estimators are close and insignificantly different from each other. The standard errors of \( \hat{\tau}(\hat{w}_{\text{trim}}) \) and \( \hat{\tau}(\hat{w}^*) \) are respectively 17% and 10% smaller than that of \( \hat{\tau}(p) \). \( \hat{\tau}(\hat{w}^*) \) achieves 56% of the precision gains achieved by \( \hat{\tau}(\hat{w}_{\text{trim}}) \), trimming six times fewer observations, without changing the target parameter, but at the expense of assuming a bound on the CATEs. Finally, the worst-case MSES of \( \hat{\tau}(p) \) and \( \hat{\tau}(\hat{w}_{\text{trim}}) \) are respectively 1.14 and 10.82 times larger than that of \( \hat{\tau}(\hat{w}^*) \). As a robustness check, I recompute \( \hat{\tau}(\hat{w}^*) \) and its standard error with \( B = 0.5 \). Doing so leaves the point estimate and its standard error unchanged to the third digit.

|                  | \( \hat{\tau}(p) \) | \( \hat{\tau}(\hat{w}_{\text{trim}}) \) | \( \hat{\tau}(\hat{w}^*) \) |
|------------------|---------------------|-----------------------------|---------------------|
| Point estimate   | -0.064              | -0.069                      | -0.067             |
| Robust standard error | 0.018              | 0.015                       | 0.016              |
| Worst-case MSE/Worst-case MSE \( \hat{\tau}(\hat{w}^*) \) | 1.142              | 10.822                      | 1                  |
| N                | 5,735               | 4,727                       | 5,735              |

Notes: Column (1) of Table 3 shows \( \hat{\tau}(p) \), the augmented inverse propensity weighting (AIPW) estimator of [Robins et al. 1994], computed on the data of [Connors et al. 1996]. Column (2) shows \( \hat{\tau}(\hat{w}_{\text{trim}}) \), the AIPW estimator after trimming patients with propensity scores outside of the \([0.1, 0.9]\) interval, as suggested by [Crump et al. 2009]. Finally, Column (3) shows the minimax-linear estimator \( \hat{\tau}(\hat{w}^*) \), with \( B = 1/3 \). The treatment is defined as having received right heart catheterization. The outcome is an indicator for patients’ survival at 30 days. The 71 covariates present in the data are used to estimate the propensity score and the conditional means of the untreated and treated outcomes, using logistic regressions. The table shows the point estimates, their bootstrapped standard errors, and their worst-case MSE computed assuming homoscedasticity and normalized by the worst-case MSE of \( \hat{\tau}(\hat{w}^*) \).
6 Conclusion

I consider estimation of the average treatment effect (ATE), in a population composed of \( S \) groups or units, when one has unbiased estimators of each group’s conditional average treatment effect (CATE). These conditions are met in SRCTs and in matching studies. I assume that each CATE is bounded in absolute value by \( B \) standard deviations of the outcome, for some known \( B \). This restriction may be appealing: outcomes are often standardized in applied work, so researchers can use available literature to determine a plausible value for \( B \). I derive, across all linear combinations of the CATEs’ estimators, the minimax estimator of the ATE. In two SRCTs where the treatment probability varies substantially across strata, my estimator is very close to the unbiased estimator. It sometimes differs from the strata-fixed effects estimator, and it has twice lower worst-case MSE on average across eleven outcomes. Thus, in SRCTs where the treatment probability varies a lot across strata, I recommend using either the unbiased estimator or my minimax estimator. In a matching study with limited overlap, my estimator achieves 56% of the precision gains of a commonly-used trimming estimator, and has an 11 times smaller worst-case MSE. Thus, I recommend using the minimax estimator in matching studies with limited overlap.
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Proofs

Proof of Lemma 3.1

\[
E \left( (\tilde{\tau}(\mathbf{w}) - \tau)^2 \right) = V(\tilde{\tau}(\mathbf{w})) + E \left( (\tilde{\tau}(\mathbf{w})) - \tau \right)^2 \\
= \sum_{s=1}^{S} w_s^2 V(\tilde{\tau}_s) + \left( \sum_{s=1}^{S} (w_s - p_s) \tau_s \right)^2 \\
= \sigma^2 \sum_{s=1}^{S} w_s^2 v_s + \left( \sum_{s=1}^{S} (w_s - p_s) \tau_s \right)^2 \\
\leq \sigma^2 \sum_{s=1}^{S} w_s^2 v_s + \left( \sum_{s=1}^{S} |w_s - p_s| \tau_s \right)^2 \\
\leq \sigma^2 \left( \sum_{s=1}^{S} w_s^2 v_s + B^2 \left( \sum_{s=1}^{S} |w_s - p_s| \right)^2 \right).
\]

The first equality follows from the fact that an estimator’s MSE is the sum of its variance and squared bias. The second equality follows from the fact \( \mathbf{w} \) is deterministic, from Equations (3.1) and (2.1), and from Point 1 of Assumption 1. The third equality follows from Point 3 of Assumption 1. The first inequality follows from the fact that for any real number \( a \), \( a^2 = |a|^2 \), from the triangle inequality, and from the fact that \( x \mapsto x^2 \) is increasing on \( \mathbb{R}^+ \). The second inequality follows from Point 4 of Assumption 1. The sharpness of the upper bound follows from plugging \( \tau_s = \sigma B \left( 1 \{ w_s \geq p_s \} - 1 \{ w_s < p_s \} \right) \) into the second equality in the previous display.

Proof of Theorem 3.1

First, assume that \( \mathbf{w}^* \) has at least one coordinate that is strictly larger than the corresponding coordinate of \( (p_1, \ldots, p_S) \). Without loss of generality, assume that \( w_1^* > p_1 \). One has \( \text{MSE}(\mathbf{w}^*) > \text{MSE}(p_1, w_2^*, \ldots, w_S^*) \), a contradiction. Therefore, each coordinate of \( \mathbf{w}^* \) is at most as large as the corresponding coordinate of \( (p_1, \ldots, p_S) \). Accordingly, finding the minimax-linear estimator is equivalent to minimizing \( \text{MSE}(\mathbf{w}) \) with respect to \( \mathbf{w} \), across all \( \mathbf{w} = (w_1, \ldots, w_S) \) such that \( w_s \leq p_s \) for all \( s \in \{1, \ldots, S\} \).

If \( w_s \leq p_s \) for all \( s \in \{1, \ldots, S\} \),

\[
\text{MSE}(\mathbf{w}) = \sigma^2 \left( \sum_{s=1}^{S} w_s^2 v_s + B^2 \left( \sum_{s=1}^{S} (p_s - w_s) \right)^2 \right).
\]
Therefore, $w^*$ is the minimizer of

$$\sum_{s=1}^{S} w_s^2 v_s + B^2 \left( \sum_{s=1}^{S} (p_s - w_s) \right)^2,$$

subject to

$$w_s - p_s \leq 0$$

for all $s$.

The objective function is convex, and the inequality constraints are continuously differentiable and concave. Therefore, the necessary conditions for optimality are also sufficient.

The Lagrangian of this problem is

$$L(w, \mu) = \sum_{s=1}^{S} w_s^2 v_s + B^2 \left( \sum_{s=1}^{S} (p_s - w_s) \right)^2 + \sum_{s=1}^{S} 2\mu_s (w_s - p_s).$$

The Karush-Kuhn-Tucker necessary conditions for optimality are

$$w_s^* v_s - B^2 \left( 1 - \sum_{s'=1}^{S} w_{s'}^* \right) + \mu_s = 0$$

$$w_s^* \leq p_s$$

$$\mu_s \geq 0$$

$$\mu_s (w_s^* - p_s) = 0.$$

Those conditions are equivalent to

$$w_s^* = \min \left( \frac{1}{v_s} B^2 \left( 1 - \sum_{s'=1}^{S} w_{s'}^* \right), p_s \right)$$

$$\mu_s = \max \left( 0, B^2 \left( 1 - \sum_{s'=1}^{S} w_{s'}^* \right) - p_s v_s \right).$$

One has that

$$\frac{1}{v_s} B^2 \left( 1 - \sum_{s'=1}^{S} w_{s'}^* \right) < p_s$$

$$\Leftrightarrow B^2 \left( 1 - \sum_{s'=1}^{S} w_{s'}^* \right) < p_s v_s.$$

Together with (6.2), the previous display implies that

$$w_s^* < p_s \Rightarrow w_{s+1}^* < p_{s+1}.$$  (6.3)

Let $s^* = \min\{s \in \{1, \ldots, S\} : w_s^* < p_s\}$, with the convention that $s^* = S + 1$ if the set is empty. It follows from Equations (6.2) and (6.3) that

$$w_s^* = p_s$$

for all $s < s^*$

$$w_s^* = \frac{1}{v_s} B^2 \left( 1 - \sum_{s'=1}^{S} w_{s'}^* \right)$$

for all $s \geq s^*$.

(6.4)
\[ (6.4) \] implies that
\[
\sum_{s=s^*}^{S} w_s^* = \frac{B^2 \sum_{s=s^*}^{S} \frac{1}{v_s} S}{1 + B^2 \sum_{s=s^*}^{S} \frac{1}{v_s}} \sum_{s=s^*}^{S} p_s.
\]
Plugging this equation into \[ (6.4) \] yields
\[
w_s^* = p_s \text{ for all } s < s^*
\]
\[
w_s^* = \frac{1}{v_s} \frac{1}{B^2} + \sum_{s'=s^*}^{S} \frac{1}{v_{s'}} \sum_{s'=s^*}^{S} p_{s'} \text{ for all } s \geq s^*.
\]
(6.5)

Finally, assume that \( s^* < \mathfrak{s} \). Then, \( w_s^* > p_s^* \), a contradiction. Some algebra shows that
\[
\text{MSE}(p) - \text{MSE}(p_1, \ldots, p_{S-1}, \frac{1}{v_S}, \frac{1}{v_S}) = p_S^2 v_S - \left( p_S^2 \left( \frac{1}{v_S} \frac{1}{B^2} + \frac{1}{v_S} \right)^2 \right) v_S + B^2 p_S^2 \left( \frac{1}{B^2} \frac{1}{v_S} + \frac{1}{B^2} \right)^2 
\]
\[
= \frac{p_S^2}{\left( \frac{1}{B^2} + \frac{1}{v_S} \right)^2} \left( \frac{v_S}{B^2} + \frac{1}{B^2} \right) > 0.
\]
Therefore,
\[
s^* \in \{s, \ldots, S\}
\]
(6.6)
The result follows from Equations \[ (6.5) \] and \[ (6.6) \].

**Proof of Lemma 3.2**

Assume that
\[
\frac{1}{\sum_{s'=s}^{S} \frac{1}{v_{s'}}} \sum_{s'=s}^{S} p_{s'} \leq p_s v_s.
\]
Then,
\[
p_{s+1} v_{s+1} \sum_{s'=s}^{S} \frac{1}{v_{s'}}
\]
\[
= p_{s+1} v_{s+1} \sum_{s'=s}^{S} \frac{1}{v_{s'}} - p_{s+1} \frac{v_{s+1}}{v_s}
\]
\[
= p_{s} v_{s} \sum_{s'=s}^{S} \frac{1}{v_{s'}} + (p_{s+1} v_{s+1} - p_{s} v_{s}) \sum_{s'=s}^{S} \frac{1}{v_{s'}} - p_{s+1} \frac{v_{s+1}}{v_s}
\]
\[
\geq p_{s} v_{s} \sum_{s'=s}^{S} \frac{1}{v_{s'}} + p_{s+1} \frac{v_{s+1}}{v_s} - p_{s} v_{s} - p_{s+1} \frac{v_{s+1}}{v_s}
\]
\[
\geq \sum_{s'=s}^{S} p_{s'}.
\]

24
6.1 Proof of Corollary 3.2

Proof of Point 1
Assume that Point 4 of Assumption 1 holds. If Point 1 of Assumption 2 were to hold with \( h_s = 1 \) for all \( s \), then Point 3 of Assumption 1 would hold with \( v_s = v_{0,s} + v_{1,s} \), and \( \hat{\tau}(w^*) \) would be minimax-linear. Accordingly, its worst-case MSE under that DGP has to be lower than that of \( \hat{\tau}(p) \), which implies that

\[
\sigma^2 B^2 \left( \sum_{s=1}^{S} |w_s^* - p_s| \right)^2 \leq \sigma^2 \sum_{s=1}^{S} ((p_s)^2 - (w_s^*)^2)(v_{0,s} + v_{1,s}). \tag{6.7}
\]

As for all \( s \), \( v_{1,s} \geq 0 \), \( (p_s)^2 - (w_s^*)^2 \geq 0 \), and \( h_s \geq 1 \) under Point 1 of Assumption 2

\[
\sigma^2 \sum_{s=1}^{S} ((p_s)^2 - (w_s^*)^2)(v_{0,s} + v_{1,s}) \leq \sigma^2 \sum_{s=1}^{S} ((p_s)^2 - (w_s^*)^2)(v_{0,s} + h_s v_{1,s}). \tag{6.8}
\]

Combining (6.7) and (6.8) and rearranging proves the result.

Proof of Point 2
If Points 2 and 4 of Assumption 1 and Point 2 of Assumption 2 hold, the worst-case MSEs of \( \hat{\tau}(w^*) \) and \( \hat{\tau}(p) \) are respectively equal to

\[
\sigma^2 \left( \sum_{s=1}^{S} (w_s^*)^2(v_{0,s} + h v_{1,s}) + B^2 \left( \sum_{s=1}^{S} |w_s^* - p_s| \right)^2 \right)
\]

and

\[
\sigma^2 \sum_{s=1}^{S} (p_s)^2(v_{0,s} + h v_{1,s}).
\]

Taking the difference between the two preceding displays, setting that difference lower than 0 and rearranging yields the result.

Proof of Theorem 4.1
That \( E \left( (\hat{\tau}(w) - \tau)^2 \right) \leq \text{MSE}_2(w) \) follows from the same steps as the proof of Lemma 3.1

That \( \text{MSE}_2(w) \) is minimized at \( w_{h^2} \) follows from the same steps as the proof of Theorem 3.1.