IWASAWA THEORY OF FINE SELMER GROUPS OVER GLOBAL FIELDS

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Abstract. The $p^\infty$-fine Selmer group of an elliptic curve $E$ over a number field $F$ is a subgroup of the classical $p^\infty$-Selmer group of $E$ over $F$. Fine Selmer group is closely related to the 1st and 2nd Iwasawa cohomology groups. Coates-Sujatha observed that the structure of the fine Selmer group of $E$ over a $p$-adic Lie extension of a number field is intricately related to some deep questions in classical Iwasawa theory; for example, Iwasawa’s classical $\mu$-invariant vanishing conjecture. In this article, we study the properties of the $p^\infty$-fine Selmer group of an elliptic curve over certain $p$-adic Lie extensions of a number field. We also define and discuss $p^\infty$-fine Selmer group of an elliptic curve over function fields of characteristic $p$ and also of characteristic $\ell \neq p$. We relate our study with a conjecture of Jannsen.

Introduction

We fix an odd prime $p$ throughout. Let $E$ be an elliptic curve defined over a number field $F$ and let $S(E/F)$ be the $p^\infty$-Selmer group of $E$ over $F$ (Definition 1.4). The $p^\infty$-fine Selmer group $R(E/F)$ is a subgroup of $S(E/F)$ (Definition 1.4), obtained by putting stringent local condition at primes dividing $p$. Let us assume that $E/\mathbb{Q}$ has good, ordinary reduction at $p$ and $\mathbb{Q}_{\text{cyc}}$ be the cyclotomic $\mathbb{Z}_p$ extension of $\mathbb{Q}$. Let $u$ be the unique prime in $\mathbb{Q}_{\text{cyc}}$ dividing $p$ and $\mathbb{Q}_{\text{cyc},u}$ be the completion at $u$. It is known that $H^2(G_S(\mathbb{Q}_{\text{cyc}}), E_{p^\infty}) = 0$ [23]. Then the following exact sequence can be deduced from §1, equations (3) and (5):

$$0 \to H^1_{Iw}(\mathbb{Z}_p E/\mathbb{Q}_{\text{cyc}}) \to (E(\mathbb{Q}_{\text{cyc},u}) \otimes \mathbb{Q}_p/\mathbb{Z}_p)^\vee \to S(E/\mathbb{Q}_{\text{cyc}})^\vee \to R(E/\mathbb{Q}_{\text{cyc}})^\vee \to 0.$$ 

Using Kato’s Euler system [23] and the Perrin-Riou-Coleman map, from the above equation, we have an alternative formulation of the Iwasawa main conjecture involving $R(E/\mathbb{Q}_{\text{cyc}})^\vee$ (see [23, Theorem 16.6.2]). The fine Selmer group has been studied in Iwasawa theory throughout, under different names, like III$_1$, Sel$_0$ by Billot, Greenberg, Kurihara [25], Perrin-Riou [38] and others over the cyclotomic $\mathbb{Z}_p$ extensions of number fields. Coates-Sujatha [9] formally defined fine Selmer group of an elliptic curve and we largely follow their notation in this article. They observed an important relation between the structure of $R(E/F_{\text{cyc}})$ and Iwasawa’s $\mu = 0$ conjecture about the growth of the $p$-part of the ideal class group in the cyclotomic tower for $F_{\text{cyc}}$.

[9, Theorem 3.4] Let $E/F$ be an elliptic curve and $p$ be an odd prime such that $F(E_{p^\infty})/F$ is a pro-$p$ extension. Then $R(E/F_{\text{cyc}})^\vee$ is a finitely generated $\mathbb{Z}_p$ module if and only if Iwasawa’s $\mu = 0$ conjecture holds for $F_{\text{cyc}}$.

Motivated by this, [9] formulated the following conjecture:

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**Conjecture A.** For all elliptic curves $E$ over $F$, $R(E/F_{\text{cyc}})^{\vee}$ is a finitely generated $\mathbb{Z}_p$ module.

Following Conjecture A, the fine Selmer group over cyclotomic $\mathbb{Z}_p$ of a number field have been studied by various authors including [47, 30, 32], Jha-Sujatha and others.

**Main results of the article: part (a):** In this article, we initiate the study of fine Selmer group of an elliptic curve over $p$-adic Lie extensions of function fields of characteristic $p$ as well as $\ell \neq p$. In one of our main results in this article, we show that the analogues of Conjecture A are true for the $p^\infty$-fine Selmer groups over the function fields of characteristic $p$ (Theorem 3.7 and Corollary 3.15) and characteristic $\ell \neq p$ (Remark 2.7), respectively.

Coates-Sujatha also studied the structure of the fine Selmer over more general $p$-adic Lie extensions of a number field, instead of the cyclotomic $\mathbb{Z}_p$ extension, in the framework of so-called non-commutative Iwasawa theory [8]. Motivated by a conjecture of Greenberg [12, Conjecture 3.5], the following was predicted regarding the structure of the fine Selmer group over an admissible (see Definition 1.1) $p$-adic Lie extension of a number field:

**Conjecture B.** [9] Assume that the Conjecture A holds for $E$ over $F_{\text{cyc}}$. Let $F_\infty$ be an admissible $p$-adic Lie extension of $F$ such that $G = \text{Gal}(F_\infty/F)$ has dimension at least 2 as a $p$-adic Lie group. Then $R(E/F_\infty)^{\vee}$ is a pseudonull (defined in §1) $\mathbb{Z}_p[[G]]$-module.

Following Conjecture B, the properties of the fine Selmer group over $p$-adic Lie extensions of a number field have been investigated by various authors including [28, 29, 19], Lim and the third named authors of the article. In particular, Lei and Palvannan in [28] and [29], have studied the pseudonullity of fine Selmer groups of elliptic curves and Hida families over the $\mathbb{Z}_p^2$-extension of an imaginary quadratic field $K$, respectively. The work of Harchimori-Sharifi [14] is also related to the Conjecture B.

**Main results of the article: part (b):** In another main result in this article, we establish analogues of Conjecture B (Theorem 3.14) over the function fields of characteristic $p$. On the other hand, over the function fields of characteristic $\ell \neq p$, we give an explicit counterexample in Example 2 to show that the analogue of Conjecture B does not hold in that setting.

Another theme of this article is the study of the $G$-Euler characteristic of the fine Selmer group $R(E/F_\infty)$. Recall that the $\Gamma$-Euler characteristic of the Selmer group $S(E/Q_{\text{cyc}})$ encodes important arithmetic properties of the elliptic curve. Let $E/Q$ be an elliptic curve with good, ordinary reduction at $p$ and assume that $L_E(s)$, the Hasse-Weil $L$-function of $E/Q$ satisfies $L_E(1) \neq 0$. Then, under suitable hypotheses, $\chi(\Gamma, S(E/Q_{\text{cyc}})) := \frac{\#H^0(\Gamma, S(E/Q_{\text{cyc}}))}{\#H^1(\Gamma, S(E/Q_{\text{cyc}}))}$ is related to $L_E(1)/\Omega_E$ (cf. [11, Theorem 4.1]). Over an admissible $p$-adic Lie extension $F_\infty$ of $F$, the existence of the $G$-Euler characteristic of the dual Selmer group $S(E/F_\infty)$ and its relation with the special values of the $L$-function of $E$ has been established in various cases due to Coates-Howson, [10, 15] etc. On the other hand, Wuthrich [47] under certain conditions, has proven the existence of $\chi(\Gamma, R(E/F_{\text{cyc}}))$ and has given a formula to compute it. In this paper, we discuss the $G$-Euler characteristic of $R(E/F_\infty)$, where $F_\infty$ is certain admissible $p$-adic Lie extension of a
number field or a function field $F$. We would like to mention that it does not seem to be easy to prove the existence of the Euler characteristic even over a specific non-commutative $p$-adic Lie extension of a number field, like the false-Tate curve extension (see Remark 1.24).

**Main results of the article: part (c):** Our main result over number fields is Theorem 1.10. In this theorem, under suitable hypotheses and assuming Jannsen’s conjecture (Conjecture 1), we prove the existence of the $G$-Euler characteristic of $R(E/F_{\infty})$ over the false-Tate curve extension $F_{\infty}$ of $F$ (see §1). In fact, in §1, we also prove the existence of $\chi(G, R(E/F_{\infty}))$ without assuming Conjecture 1, as long as $\mathbb{Z}_p[[H]]$ corank of $R(E/F_{\infty})$ is at most 1 (see Propositions 1.13 and 1.14). We stress that Theorem 1.10 and Proposition 1.13 are valid irrespective of whether $E$ has ordinary or supersingular reduction above $p$. We also prove the existence of $\chi(G, R(E/F_{\infty}))$ of a function field $F$ of characteristic $\ell \neq p$; in this setting the analogue of Conjecture 1 has already proved by Jannsen. The existence of the $G$-Euler characteristic of $p^\infty$-fine Selmer group over $\mathbb{Z}_p$ extension a field characteristic of $p$, under appropriate condition, is established in Proposition 3.3.

In the number field case, for proving Theorem 1.10, we assume Jannsen’s Conjecture and make use of a result of Kato [22, Theorem 5.1] along with other Iwasawa theoretic techniques and the theorem holds whether $E$ has ordinary or supersingular reduction at the primes above $p$. For the other results in §1, we use the structure of modules over non-commutative Iwasawa algebras and the properties of elliptic curve. In §2, we notice that the image of the kummer map is zero, thus $p^\infty$-fine Selmer group over the function field of characteristic $\ell \neq p$ coincides with the $p^\infty$-Selmer group. Over the function field of characteristic $p$, we give emphasis to two special $\mathbb{Z}_p$ extensions; the arithmetic or the unramified $\mathbb{Z}_p$ extension and the geometric $\mathbb{Z}_p$ extension constructed from Carlitz module, a particular type of Drinfeld module (see §3.1). In addition, we also provide a modest evidence towards Conjecture B over a general $\mathbb{Z}_p$ extension in Corollary 3.9. In §3, our main tool is to compare the fine Selmer group of $E[p]$ with the corresponding fine Selmer groups of the group schemes $\mu_p$ and $\mathbb{Z}/p\mathbb{Z}$ and then make use of the classical results on the divisor class group over the function field. In the characteristic $p$ setting, we discuss the dependence of the fine Selmer group on the set $S$ (Remark 3.2) and also comment on the zero Selmer group (Remark 3.18).

The structure of the article is the following: §1 contains the results over the number fields. In particular, we discuss various cases in which, over the false-Tate curve extension $F_{\infty}/F$, $\chi(G, R(E/F_{\infty}))$ exists and Conjecture B holds. In §2, we consider results over the function fields of characteristic $\ell \neq p$ and we notice that in this setting, an analogue of Conjecture A is true but an analogue of Conjecture B is false. The results for $p^\infty$-fine Selmer group over function field of characteristic $p$ are contained in §3 and there we prove that an analogue of Conjecture A holds and moreover show that an analogue of Conjecture B holds over a certain class of $\mathbb{Z}_p^2$ extensions.
1. Results over number fields

Throughout the section 1, $E$ will be an elliptic curve defined over a number field $F$ with good reduction at all the primes of $F$ dividing $p$. (Note: Our Theorem 1.10 and Proposition 1.13 covers both the cases; $E$ has ordinary reduction or supersingular reduction at primes above $p$. However, from Proposition 1.14 onwards, we assume that $E$ has ordinary reduction at all primes above $p$.) Also, throughout §1, $S$ will denote a finite set of primes of $F$ containing the primes dividing $p$, the infinite primes of $F$ and the primes where $E$ has bad reduction.

Let $F_S$ be the maximal algebraic extension of $F$ unramified outside $S$. For a field $L$ with $F \subset L \subset F_S$, write $G_S(L) = \text{Gal}(F_S/L)$. For an abelian group $M$, $M[p^n]$ and $M(p)$ will respectively denote its $p^n$-torsion and $p$-primary torsion subgroup of $M$. For a compact or discrete $\mathbb{Z}_p$ module $M$, let $M^\vee := \text{Hom}_{\text{cont}}(M, \mathbb{Q}_p/\mathbb{Z}_p)$ denote its Pontryagin dual. Let $E[p^n]$ (respectively $E_{p^\infty}$) denote the Galois module $E(\mathbb{Q})[p^n]$ (respectively $E(\mathbb{Q})$). Let $T_p(E) = \lim_{\leftarrow n} E[p^n]$ be the Tate module associated to $E$. Let $F_{\text{cyc}}$ be the cyclotomic $\mathbb{Z}_p$ extension of a number field $F$ and set $\Gamma := \text{Gal}(F_{\text{cyc}}/F)$. For any profinite group $\mathcal{G}$, the Iwasawa algebra $\mathbb{Z}_p[[\mathcal{G}]]$ is defined by $\Lambda(\mathcal{G}) := \mathbb{Z}_p[[\mathcal{G}]] := \lim_U \mathbb{Z}_p[\mathcal{G}/U]$, where $U$ varies over open normal subgroups of $\mathcal{G}$ and the inverse limit is taken with respect to the natural projection maps.

**Definition 1.1.** A Galois extension $F_\infty/F$ is called an admissible $p$-adic Lie extension if $F_{\text{cyc}} \subset F_\infty$, at most finitely many primes of $F$ ramify in $F_\infty$ and $G := \text{Gal}(F_\infty/F)$ is a compact $p$-adic Lie group without any $p$-torsion element.

The study of pseudonull modules play an important role in commutative Iwasawa theory. For a compact $p$-adic Lie group $G$ without $p$-torsion element, the notion of torsion modules and pseudonull modules over Iwasawa algebra $\mathbb{Z}_p[[G]]$ was generalised by Venjakob. If $M$ is a finitely generated $\mathbb{Z}_p[[G]]$-module, then $M$ is said to be a pseudonull (respectively torsion) $\mathbb{Z}_p[[G]]$ module if $\text{Ext}_{\mathbb{Z}_p[[G]]}^i(M, \mathbb{Z}_p[[G]]) = 0$ for $i = 0, 1$ (respectively $i = 0$). We have the following criterion for pseudonullity due to Venjakob:

**Theorem 1.2** (Venjakob). Let $F_\infty$ be an admissible $p$-adic Lie extension with Galois group $G$ and set $H := \text{Gal}(F_\infty/F_{\text{cyc}})$. Let $M$ be a $\mathbb{Z}_p[[G]]$ module, which is a finitely generated $\mathbb{Z}_p[[H]]$ module. Then $M$ is a pseudonull $\mathbb{Z}_p[[G]]$ module if and only if it is $\mathbb{Z}_p[[H]]$ torsion.

We also recall the following result:

**Theorem 1.3.** [16, Theorem 1.1] Let $G$ be a compact, pro-$p$, $p$-adic Lie group without any element of order $p$ and $M$ be a cofinitely generated discrete $\mathbb{Z}_p[[G]]$ module. Then $\text{corank}_{\mathbb{Z}_p[[G]]}(M) = \sum_{i \geq 0} (-1)^i \text{corank}_{\mathbb{Z}_p}(H^i(G, M))$. 

Let $F, S$ be as above and $L$ be a finite extension of $F$ with $L \subset F_S$. For each $v \in S$, set $J_v(E/L) := \prod_{w | v} H^1(L_w, E(L_w))(p)$ and $K^s_v(E/L) := \prod_{w | v} H^i(L_w, E_{p^\infty}(w))$, for $i = 0, 1$. Here $\overline{L}_w$ is the completion of $L$ at $w$. The $p^\infty$-Selmer and $p^\infty$-fine Selmer group of $E$ over $L$, denoted respectively as $S(E/L)$ and $R(E/L)$ are defined as:
Definition 1.4.
\[
S(E/L) := \ker(H^1(G_S(L), E_{p\infty}) \to \bigoplus_{v \in S} J_v(E/L))
\] (1)
\[
R(E/L) := \ker(H^1(G_S(L), E_{p\infty}) \to \bigoplus_{v \in S} K_v^1(E/L)),
\] (2)

In fact, \(S(E/L)\) and \(R(E/L)\) are independent of \(S\) as long as \(S\) contains all primes of bad reduction of \(E/K\). We have the following relation between \(R(E/L)\) and \(S(E/L)\) [9]
\[
0 \to R(E/L) \to S(E/L) \to \bigoplus_{w|p} (E(L_w) \otimes_{\mathbb{Q}_p} \mathbb{Z}_p)
\] (3)

The definitions of \(S(E/L)\) and \(R(E/L)\), over an infinite extension \(L\) of \(F\) contained in \(F_S\), extends as usual, by taking inductive limit over all finite subextensions of \(L\) containing \(F\). Moreover, using the Poitou-Tate exact sequence for \(E_{p\infty}\) over \(L\) and taking taking inductive limits over all such finite extensions \(F \subset L \subset \mathcal{L}\), [9, Equations (44), (45)] we get:
\[
0 \to H^i(L, E_{p\infty}) \to \bigoplus_{v \in S} K_v^i(E/L) \to R(E/L) \to 0,
\] (4)
\[
0 \to R(E/L) \to H^1(G_S(L), E_{p\infty}) \to \bigoplus_{v \in S} K_v^1(E/L) \to R(E/L) \to 0.
\] (5)

Here for \(i = 1, 2\), \(H^i_{Iw}(T_p E/L) := \varprojlim_{L} H^i(K_{S,L}(T_p E))\) is the Iwasawa cohomology of \(T_p E\) over \(L\) and the projective limit is taken with respect to the corestriction maps. Equations (4) and (5) explains the relation between the Iwasawa cohomologies and the fine Selmer group.

Let \(G\) be a compact \(p\)-adic Lie group without any element of order \(p\). Then, by results of Brumer and Lazard, the global dimension of \(\mathbb{Z}_p[[G]]\) is finite.

Definition 1.5. Let \(G\) be a compact \(p\)-adic Lie group without any \(p\)-torsion element and \(M\) be a discrete \(G\)-module. If \(H^i(G, M)\) is finite for all \(i \geq 0\), then we say that the \(G\)-Euler characteristic of \(M\) exists and it is defined as
\[
\chi(G, M) = \prod_{i \geq 0} (\#H^i(G, M))^{(-1)^i}.
\]

In the above setting, if \(G\) is commutative or more generally ‘finite by nilpotent’ [21, Remark 1.5], then it is known that \(\chi(G, M) \iff H^0(G, M)\) is finite.

Now we discuss the Euler characteristic of fine Selmer groups of elliptic curves over perhaps the simplest non-commutative admissible \(p\)-adic Lie extension, the so called false-Tate curve extension \(F_\infty\) of \(\mathbb{Q}(\mu_p)\). We prove the finiteness of the Euler characteristic of the fine Selmer group of an elliptic curve over the false-Tate curve extension in two different situations. For the first result, we explore the relation between the fine Selmer group and Conjecture 1 of Jannsen on the twist of \(\ell\)-adic cohomology of a motive. Using this together with a result of Kato [22], we prove the finiteness of the Euler characteristic of the fine Selmer group. For the second sets of results, we do not assume any conjecture but instead make some hypothesis on the corank of the Selmer group of \(E\) over \(F_{cyc}\).

Let \(\mu_{p^n}\) denote the group of \(p^n\)-th roots of unity in \(\mathbb{Q}\). Let \(F\) be a number field containing \(\mu_p\). Now, let \(m\) be a \(p\)-power free positive integer. Then \(F_\infty := \bigcup F(\mu_{p^n}, m^{1/p^n})\) is the false Tate curve extension over \(F\). The Galois group \(G := \text{Gal}(F_\infty/F) \cong \mathbb{Z}_p \times \mathbb{Z}_p\). Set \(H = \text{Gal}(F_\infty/F_{cyc}) \cong \mathbb{Z}_p\) and \(\Gamma = \text{Gal}(F_{cyc}/F) \cong \mathbb{Z}_p\).
\(G/H \cong \mathbb{Z}_p\). Let \(\chi : G_\mathbb{Q} \rightarrow \mathbb{Z}_p^\times\) be the \(p\)-adic cyclotomic character. Given a \(G_\mathbb{Q}\) module \(M\), we denote by \(M(k)\) the twist of \(M\) by \(\chi^k\). Define

\[R(E^k/F) := \text{Ker}(H^1(G_S(F), \mathcal{E}_{p^\infty}(k)) \rightarrow \bigoplus_{v \in S} H^1(F_v, \mathcal{E}_{p^\infty}(k))).\]

Next, we discuss the finiteness of \(H^0(G, R(E/F_\infty)) = R(E/F_\infty)^G\).

**Proposition 1.6.** Assume that \(R(E/F)\) is finite. Then \(R(E/F_\infty)^G\) is finite.

**Proof.** The proof follows easily from [15, Theorem 4.1 and Lemma 3.12].

**Proposition 1.7.** Assume that \(R(E/F)\) is finite. If \(R(E^k/F)\) is finite for every \(k \geq 1\), then \(\chi(G, R(E/F_\infty))\) is finite.

**Proof.** As \(R(E/F)\) is finite, applying a control theorem (Theorem 1.6), we get that \(R(E/F_\infty)^G\) is finite. Further, it is well known that the finiteness of \(R(E/F)\) implies that \(R(E/F_{\text{cyc}})^v\) is \(\mathbb{Z}_p[[\Gamma]]\) torsion and hence \(H^2(F_{\text{cyc}}, \mathcal{E}_{p^\infty}) = 0\). Consequently, \(H^2(F_\infty, \mathcal{E}_{p^\infty}) = 0\) as well and by [9, Lemma 3.1], we deduce that \(R(E/F_\infty)^v\) is a torsion \(\mathbb{Z}_p[[G]]\) module. Also note that \(G\) has \(p\)-cohomological dimension = 2. Thus, to show Euler characteristic is finite, it suffices to show \(R(E/F_\infty)^G\) is finite and \(H^2(G, R(E/F_\infty))\) is finite (by Theorem 1.3).

Next from the equation (4), it is easy to see that \(\chi(G, H^0_{\text{f}}(T_p E/F_\infty)^v)\) exists if and only if \(\chi(G, R(E/F_\infty))\) exists. From the same equation, we can also deduce that \(H^0_{\text{f}}(T_p E/F_\infty)\) is torsion over \(\mathbb{Z}_p[[\Gamma]]\) (respectively \((H^0_{\text{f}}(T_p E/F_\infty)^G)\) is finite) if and only if \(R(E/F_\infty)^v\) is torsion over \(\mathbb{Z}_p[[G]]\) (respectively \((R(E/F_\infty)^G)\) is finite). From these discussions, we are reduced to show \(H^2(G, R(E/F_\infty))\) is finite or equivalently \(H^2(G, H^0_{\text{f}}(T_p E/F_\infty)^v)\) is finite. As \(G \cong \mathbb{Z}_p \times \mathbb{Z}_p\), using Hochshild-Serre spectral sequence [36, Page-119] \(H^2(G, H^0_{\text{f}}(T_p E/F_\infty)^v) \cong H^1(\Gamma, H^1(H, H^0_{\text{f}}(T_p E/F_\infty)^v))\). Further, \(H^1(\Gamma, H^1(H, H^0_{\text{f}}(T_p E/F_\infty)^v))\) is finite if and only if \(H^0(\Gamma, H^1(H, H^0_{\text{f}}(T_p E/F_\infty)^v))\) is finite.

Thus we are further reduced to show \(H^0(\Gamma, H^1(H, H^0_{\text{f}}(T_p E/F_\infty)^v))\) is finite. By \([22, \text{Proposition 4.2}]\), we have a filtration of \(H^1(H, H^0_{\text{f}}(T_p E/F_\infty)^v)\) given by \(0 = S_0 \subset S_1 \subset \ldots \subset S_K = H^1(H, H^0_{\text{f}}(T_p E/F_\infty)^v)\) such that \(S_i/S_{i-1} \cong T(\chi^s)\), where \(T\) is a subquotient of \(H^0(H, H^1_{\text{f}}(T_p E/F_\infty)^v)\) and \(s \in \mathbb{N}\). Now by an argument similar to \([22, \text{Page 562, last paragraph}]\) which uses Nekovar’s spectral sequence [35, Proposition 8.4.8.3], we get that \(H^0(H, H^1_{\text{f}}(T_p E/F_\infty)^v) \cong H^0_{\text{f}}(T_p E/F_{\text{cyc}})^v\).

Then from the proof of \([22, \text{Theorem 5.1}]\), it suffices to show that for every \(\mathbb{Z}_p[[\Gamma]]\) subquotient \(T\) of \(H^0(H, H^1_{\text{f}}(T_p E/F_\infty)^v)\) and for every \(k \in \mathbb{N}\), \(H^0(\Gamma, T(k)^v)\) is finite if and only if \(\chi(\Gamma)\) does not divide the characteristic ideal of \(T(k)^v\). Notice that, by our assumption, \(H^0_{\text{f}}(T_p E/F_{\text{cyc}})(k)\) is \(\mathbb{Z}_p[[\Gamma]]\) torsion and hence \(T(k)^v\) is \(\mathbb{Z}_p[[\Gamma]]\) torsion. We fix a topological generator \(\chi \in \Gamma\). Now from the structure theorem of finitely generated torsion modules over \(\mathbb{Z}_p[[\Gamma]]\), \(H^0(\Gamma, T(k)^v)\) is finite if and only if \(\chi(k)^v\) does not divide the characteristic ideal of \(T(k)^v\). Note that \(T(k)^v\) is a subquotient of \(H^1_{\text{f}}(T_p E/F_{\text{cyc}})(k)\), whence it is enough to prove that \(\chi(k)^v\) does not divide characteristic ideal of \(H^1_{\text{f}}(T_p E/F_{\text{cyc}})(k)\) (also see the proof of \([22, \text{Theorem 5.1}]\) i.e., \(H^0(\Gamma, (H^1_{\text{f}}(T_p E/F_{\text{cyc}})(k))^v)\) is finite, for every \(k \in \mathbb{N}\).

The finiteness of \(H^0(\Gamma, (H^1_{\text{f}}(T_p E/F_{\text{cyc}})(k))^v)\) is equivalent to the finiteness of \(H^0(\Gamma, R(E/K_{\text{cyc}})(k))\), again by equation (4). As \(\chi\) is trivial on \(G_{\text{cyc}}\), we have \(H^0(\Gamma, R(E/K_{\text{cyc}})(k)) \cong H^0(\Gamma, R(E/F)(k))\). By a control theorem for \(K_{\text{cyc}}\)
over $K$, similar to the control theorem in [47], this is equivalent to $R(E_k/F)$ being finite for every $k \in \mathbb{N}$. Finally, we are done by the hypothesis that $R(E^k/F)$ is finite for every $k \geq 1$.

Next we discuss Jannsen’s Conjecture [17, Question 2, Page-349]. Jannsen formulates this as “Question 2” in his article but as we show in Remark 1.9, in our setting of Tate twists of an elliptic curve, [17, Question 2, Page-349] is equivalent to the Conjecture 1 stated by Jannsen [17, page 317]. Also note that Bellaiche [2, Proposition 5.1] states Question 2 of Jannsen as a conjecture.

Recall that certain finite set $S$ of primes of $F$ has been chosen in the beginning of this section. For a Galois module $V$, set $V^* := \text{Hom}(V, \mathbb{Q}_p)$.

Conjecture 1. [17, Question 2] Let $V$ be a p-adic representation coming from geometry of $G_S(F)$ which is pure of weight $w \neq -1$ let $W = V^*(1)$. Then the natural map $H^1(G_S(F), W) \to \prod_{v \in S} H^1(G_v, W)$ is injective.

Proposition 1.8. [2, Proposition 5.1] Let $V$ be a p-adic geometric representation of $G_S(F)$ and let $W = V^*(1)$. The following are equivalent:

(i) The natural map $H^1(G_S(F), W) \to \prod_{v \in S} H^1(G_v, W)$ is injective.

(ii) $\dim H^1(G_S(F), V) = |F : \mathbb{Q}| \dim V + \dim H^0(G_S(F), V) - \dim H^0(G_S(F), V^*(1)) + \sum_{v \in S, \text{finite}} \dim H^0(G_v, V^*(1)) - \sum_{\infty \cap S, \text{finite}} \dim H^0(G_v, V).

(iii) $\dim H^2(G_S(F), V) = \sum_{v \in S, \text{finite}} \dim H^0(G_v, V^*(1)) - \dim H^0(G_S(F), V^*(1))$

Moreover, in (ii) and (iii), the LHS is never less that the RHS. □

Remark 1.9. Let $k$ be a positive integer. By [17, Theorem 5(a)], the statement (iii) in Proposition 1.8 for $V = V_p E^*(1-k)$ is equivalent to $H^2(G_S(F), V_p E^*(1-k)) = 0$. On the other hand, [17, Conjecture 1, page 317] states that with $k$ as above, $H^2(G_S(F), V_p E^*(1-k)) = 0$. Thus the statement in Proposition 1.8(i) for $V_p E^*(1-k)$ with $k \geq 1$ is equivalent to the [17, Conjecture 1, page 317].

Assuming Conjecture 1, the main result of §1 follows from Proposition 1.7:

Theorem 1.10. Let $E/F$ be an elliptic curve with good reduction at all the primes of $F$ dividing $p$. Assume that $R(E/F)$ is finite and Jannsen’s Conjecture 1 is true. Then, $\chi(G, R(E/F_\infty))$ is finite.

Proof. By Proposition 1.7, it suffices to show that $R(E^k/F)$ is finite for every $k \geq 1$. For each $k \geq 1$, recall that the pure weight of $V_p E^*(1-k)$ is $2k - 1$ (see [2, Page 8]). As $2k - 1 \neq -1$, by Conjecture 1, we get that $H^1(G_S(F), V_p E(k)) \xrightarrow{\phi_k} \prod_{v \in S} H^1(G_v, V_p E(k))$ is injective. It follows that $R(E^k/F) = \ker (H^1(G_S(F), E_p^*(k))) \to \prod_{v \in S} H^1(G_v, E_p^*(k))$ is finite. Indeed, $\ker(\phi_k)$ surjects onto the divisible part of $R(E^k/F)$. This completes the proof. □

Remark 1.11. Jannsen has shown [17, Theorem 4] that the analogue of his Conjecture 1 holds true for the function field of characteristic $\ell \neq p$. Using [17, Theorem 4], we prove the existence of $\chi(G, R(E/F_\infty))$ for the false Tate extension in the function field of characteristic $\ell \neq p$ setting, in Theorem 2.5.

We continue to discuss $\chi(G, R(E/F_\infty))$ under different sets of hypotheses. Recall that $H^1(H, S(E/F_\infty)) = 0$ [15].
Lemma 1.12. Assume that $R(E/F)$ is finite. Then $\chi(G, R(E/F_{\infty}))$ exists if and only if $H^1(H, R(E/F_{\infty}))^\Gamma$ is finite.

Proof. Note that $G$ has $p$-cohomological dimension 2 and by Proposition 1.6 $H^0(G, R(E/F_{\infty}))$ is finite. Hence to show $\chi(G, R(E/F_{\infty}))$ is finite it suffices to show that $H^1(G, R(E/F_{\infty}))$ is finite (by Theorem 1.3). Now from the Hochschild-Serre spectral sequence, we have

$$0 \to H^1(\Gamma, R(E/F_{\infty})) \to H^1(G, R(E/F_{\infty})) \to H^1(H, R(E/F_{\infty}))^\Gamma \to 0 \quad (6)$$

Since $\Gamma = G/H \cong \mathbb{Z}_p$, the finiteness of $H^1(\Gamma, R(E/F_{\infty}))^\Gamma$ is equivalent to the finiteness of $H^0(G, R(E/F_{\infty}))$. Thus from (6), we get that $H^1(G, R(E/F_{\infty}))$ is finite if and only if $H^1(H, R(E/F_{\infty}))^\Gamma$ is finite. \hfill \Box

Proposition 1.13. Let $R(E/F)$ be finite. Also assume that $\text{corank}_{\mathbb{Z}_p}(R(E/F_{\infty})^H) = \text{corank}_{\mathbb{Z}_p}(R(E/F_{\infty})^H) = r$ (say) with $r \leq 1$. Then $\chi(G, R(E/F_{\infty}))$ exists and $\chi(G, R(E/F_{\infty})) = 1$.

Proof. As $R(E/F_{\infty})^H$ is a co-finitely generated $\mathbb{Z}_p$ module, it follows that by Nakayama lemma that $R(E/F_{\infty})$ is a co-finitely generated $\mathbb{Z}_p[[H]]$ module. Consider the exact sequence of $\mathbb{Z}_p[[\Gamma]]$ modules:

$$0 \to \ker(g) \to (R(E/F_{\infty})^\vee)^H \to (R(E/F_{\infty})^\vee)^H \to \text{coker}(g) \to 0 \quad (7)$$

where $g$ is the composition of the natural maps $(R(E/F_{\infty})^\vee)^H \to (R(E/F_{\infty})^\vee)^H \to \text{coker}(g)$. We claim that the map $g$ is a pseudo-isomorphism of $\mathbb{Z}_p[[\Gamma]]$ modules. Assume the claim for a moment. Then from (7), we deduce that the characteristic ideal of $(R(E/F_{\infty})^\vee)^H$ and $(R(E/F_{\infty})^\vee)^H$ as $\mathbb{Z}_p[[\Gamma]]$ modules are the same. By Proposition 1.6, $(R(E/F_{\infty})^\vee)^H$ is finite and hence $(R(E/F_{\infty})^\vee)^H \cong (H^1(H, R(E/F_{\infty}))^\Gamma)^\vee$ is finite as well. Thus $\chi(G, R(E/F_{\infty}))$ exists by Lemma 1.12. Moreover, by our claim the Akashi series ([8, Page-177]) of $R(E/F_{\infty})$, $\text{Ak}^H_{\mathbb{Z}_p}(R(E/F_{\infty})) = \frac{\text{char}_{\mathbb{Z}_p}[[H]] H_0(H, R(E/F_{\infty})^\vee)}{\text{char}_{\mathbb{Z}_p}[[H]] H_0(H, R(E/F_{\infty})^\vee)} = 1$. Consequently, $\chi(G, R(E/F_{\infty})) = 1$ follows from [21, Lemma 3.6].

For the rest of the proof, we establish the claim: the kernel and the cokernel of $g$ in (7) are finite. We only need to consider the case when $\text{corank}_{\mathbb{Z}_p}(R(E/F_{\infty})^H) = \text{corank}_{\mathbb{Z}_p}(R(E/F_{\infty})^H) = 1$. Further, it is enough to show $\ker(g)$ is finite in order to deduce that $\text{coker}(g)$ is finite. Identifying $\mathbb{Z}_p[[H]] \cong \mathbb{Z}_p[[X]]$, by the structure theorem of $\mathbb{Z}_p[[X]]$ modules, there is a $\mathbb{Z}_p[[X]]$ module homomorphism

$$R(E/F_{\infty})^\vee \to \bigoplus_{i=1}^s \frac{\mathbb{Z}_p[[X]]}{P_i^{n_k}}$$

with finite kernel and cokernel, where $P_i$'s are height one primes in $\mathbb{Z}_p[[X]]$. Notice that $\text{rank}_{\mathbb{Z}_p}\left(\bigoplus_{i=1}^s \frac{\mathbb{Z}_p[[X]]}{P_i^{n_k}}\right) = \text{rank}_{\mathbb{Z}_p}\left(\bigoplus_{i=1}^s \frac{\mathbb{Z}_p[[X]]}{P_i^{n_k}}\right) = 1$. Therefore, there exists $1 \leq k \leq s$ such that $P_k = (X)$, $n_k = 1$ and $P_i \neq (X)$, for $i \neq k$. Consider the commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \to & \ker(f_1) & \to & (R(E/F_{\infty})^\vee)^H & \to & \bigoplus_{i=1}^s \frac{\mathbb{Z}_p[[X]]}{P_i^{n_k}} & \to & 0 \\
& & \downarrow{g_1} & & \downarrow{g} & & \downarrow{\bigoplus_{i=1}^s g_2} & & \\
0 & \to & \ker(f_2) & \to & (R(E/F_{\infty})^\vee)^H & \to & \bigoplus_{i=1}^s \frac{\mathbb{Z}_p[[X]]}{P_i^{n_k}} & \to & 0
\end{array}
$$

\hfill (9)
For an element \( x \in \frac{\mathbb{Z}_p[[X]]}{F^\circ_i}[X] \), \( g_{2,i}(x) \) denotes the residue class of \( x \) in \( \frac{\mathbb{Z}_p[[X]]}{F^\circ_i}[X] \). The map \( g_i \) is induced by the restriction of \( g \). The map \( g_{2,k} \) is an isomorphism and for \( i \neq k \), \( \frac{\mathbb{Z}_p[[X]]}{F^\circ_i}[X] = 0 \). Thus, \( \ker( \bigoplus_{i=1}^{s} g_{2,i} ) = 0 \). Also, from (8), we know that \( \ker(f_2) \) is finite. Consequently from (9), we deduce that \( \ker(g) \) is finite. This establishes the claim.

From now onwards, we assume that \( E/F \) has (good) ordinary reduction at all the primes of \( F \) dividing \( p \). We recall the following set of primes of \( F_{\text{cyc}} \) associated to \( E \) from [15].

\[
P_1 := \{ u \text{ is a prime in } F_{\text{cyc}} : u \nmid p \text{ and } u \mid m \},
\]

\[
P_2 := \{ u \in P_0 : E/F_{\text{cyc}} \text{ has split multiplicative reduction at } u \} \text{ and } m_1 := \#P_1,
\]

\[
P_2 := \{ u \in P_0 : E \text{ has good reduction at } u \text{ and } E(F_{\text{cyc,}}.p) \neq 0 \} \text{ and } m_2 := \#P_2.
\]

**Proposition 1.14.** Assume that \( R(E/F) \) is finite and \( S(E/F_\infty)^\vee \) is a finitely generated \( \mathbb{Z}_p[[H]] \) module of rank 1. Then \( \chi(G, R(E/F_\infty)) \) exists.

**Proof.** Let \( f \) be the natural inclusion map \( R(E/F_\infty) \xrightarrow{f} S(E/F_\infty) \). It is shown in [15, proof of Theorem 5.3] that \( H^1(H, S(E/F_\infty)) = 0 \). Hence \( H^1(H, coker(f)) = 0 \) and we also have:

\[
0 \to R(E/F_\infty)^H \xrightarrow{f^H} S(E/F_\infty)^H \to (coker(f))^H \to H^1(H, R(E/F_\infty)) \to 0
\]

Now by [15, Theorem 3.1], \( S(E/F_\infty)^\vee \) is a finitely generated \( \mathbb{Z}_p[[H]] \) module and there is an injective homomorphism \( S(E/F_\infty)^\vee \to \mathbb{Z}_p[[H]]^{\lambda + m_1 + 2m_2} \) with finite cokernel, where \( \lambda = \text{corank}_{\mathbb{Z}_p}(S(E/F_{\text{cyc}})) \). By our hypothesis that \( \text{corank}_{\mathbb{Z}_p[[H]]}(S(E/F_\infty)) = 1 \), we get that \( m_2 = 0 \) and \( \lambda \in \{0, 1\} \).

First, we consider the case \( \lambda = 0 \). Notice that, \( R(E/F_\infty) \) being a subgroup of \( S(E/F_\infty) \), \( \text{corank}_{\mathbb{Z}_p[[H]]}(R(E/F_\infty)) \leq 1 \).

In the subcase, when \( \text{corank}_{\mathbb{Z}_p[[H]]}(R(E/F_\infty)) = 0 \), by Theorem 1.3, we get that \( \text{corank}_{\mathbb{Z}_p}(R(E/F_\infty)^H) = \text{corank}_{\mathbb{Z}_p}(R(E/F_\infty)_{\text{cyc}}) \). Again applying Theorem 1.3, we obtain that \( \text{corank}_{\mathbb{Z}_p}(S(E/F_{\text{cyc}})) = 0 \), via a control theorem for \( F_{\text{cyc}}/F \), we deduce that \( S(E/F) \) is finite. Now applying [15, Theorem 4.1], \( \chi(G, S(E/F_\infty)) = \chi(G, R(E/F_\infty)) \) is finite.

Now, we consider the subcase where \( \lambda = 1 \) and \( \text{corank}_{\mathbb{Z}_p[[H]]}(R(E/F_\infty)) = 1 \). Now by our hypothesis, \( \text{corank}_{\mathbb{Z}_p[[H]]}(S(E/F_\infty)) = 1 \). Hence \( \text{corank}_{\mathbb{Z}_p[[H]]}(\text{coker}(f)) = 0 \) and by Theorem 1.2, \( \text{coker}(f) \) is a \( \mathbb{Z}_p[[G]] \) pseudonull module. Further, by [15, Remark 3.2], the maximal pseudonull submodule of \( S(E/F_\infty)^\vee \) is 0. Thus \( \text{coker}(f) = 0 \) and \( S(E/F_\infty) = R(E/F_\infty) \). As \( \lambda = \text{corank}_{\mathbb{Z}_p}(S(E/F_{\text{cyc}})) = 0 \), via a control theorem for \( F_{\text{cyc}}/F \), we deduce that \( S(E/F) \) is finite. Now applying [15, Theorem 4.1], \( \chi(G, S(E/F_\infty)) = \chi(G, R(E/F_\infty)) \) is finite.

Now, we consider the second case, where \( \lambda = 1 \). In this case, \( m_1 = m_2 = 0 \) and \( E \) has good reduction at primes \( v \mid p \) of \( F \). Then, by a control theorem similar to [19, Lemma 2], we get that \( \text{corank}_{\mathbb{Z}_p}(R(E/F_{\text{cyc}})) = \text{corank}_{\mathbb{Z}_p}(R(E/F_\infty)^H) \). Also, given \( \lambda = \text{corank}_{\mathbb{Z}_p}(S(E/F_{\text{cyc}})) \) = 1, \( \text{corank}_{\mathbb{Z}_p}(R(E/F_{\text{cyc}})) \) is atmost 1.

If \( \text{corank}_{\mathbb{Z}_p}(R(E/F_{\text{cyc}})) = 0 \), then (by Theorem 1.3) \( \text{corank}_{\mathbb{Z}_p}(R(E/F_\infty)^H) = \text{corank}_{\mathbb{Z}_p}(R(E/F_\infty)_{\text{cyc}})^H = 0 \). Once again, by applying Proposition 1.13, we deduce that \( \chi(G, R(E/F_\infty)) \) exists. On the other hand, if \( \text{corank}_{\mathbb{Z}_p}(R(E/F_{\text{cyc}})) = 1 \), then by Theorem 1.3 \( \text{corank}_{\mathbb{Z}_p}(R(E/F_\infty)^H) \) could be 0 or \( \text{corank}_{\mathbb{Z}_p}(R(E/F_\infty)_{\text{cyc}})^H \) could be 1. In the former case, \( \chi(G, R(E/F_\infty)) \)
exists by Lemma 1.12 and in the later case, the existence of \( \chi(G,R(E/F_\infty)) \) follows from Proposition 1.13. This completes the proof of this theorem. \( \square \)

We give an example where all the hypotheses of Proposition 1.14 are satisfied:

**Example 1.** Let \( p = 3 \), \( F = \mathbb{Q}(\mu_3) \) and \( F_\infty = \mathbb{Q}(\mu_3^{1/3},1/3^{\infty}) \). Consider the following elliptic curve \( E \) with LMFDB label 306.a2:

\[
y^2 + xy = x^3 - x^2 - 927x + 11097
\]  

(11)

By discussions on [11, Pages-129, 130], it follows that \( \text{rank}(E(F)) \geq 1 \), \( E \) has good, ordinary reduction at the prime of \( K \) dividing \( p \), \( \mu \)-invariant of \( S(E/F_\text{cyc})^\vee \) vanishes and \( \text{corank}_{\mathbb{Z}_p}(S(E/F_\text{cyc})) = 1 \). By [15, Theorem 3.1], we deduce that \( \text{corank}_{\mathbb{Z}_p[[H]]}(S(E/F_\infty)) = 1 \). So, all the conditions of Proposition 1.14 are satisfied.

**Remark 1.15.** Among the various cases discussed in the proof of Theorem 1.14, we have considered the situations where \( \text{corank}_{\mathbb{Z}_p[[H]]}(R(E/F_\infty)) = 1 \). Assume that Conjecture B holds. Then these situations cannot occur.

**Corollary 1.16.** Let us keep the hypotheses of Proposition 1.14. Further, assume that Conjecture B holds. Then, by Proposition 1.13, \( \chi(G,R(E/F_\infty)) = 1 \).

**Remark 1.17.** Let \( G \) be a commutative compact \( p \)-adic Lie group without any element of order \( p \). Then it is well known that for a finitely generated pseudonull \( \mathbb{Z}_p[[G]] \) module \( M \) with well-defined Euler characteristic, \( \chi(G,M) = 1 \) holds. In general, there are examples [10, Example 3] of non-commutative compact \( p \)-adic Lie group \( G \) and finitely generated pseudonull \( \mathbb{Z}_p[[G]] \) modules \( M \) such that \( \chi(G,M) \neq 1 \). However, in the special case where \( G \cong \mathbb{Z}_p \times \mathbb{Z}_p \) and \( M = R(E/F_\infty)^\vee \), we see in Corollary 1.16 that the pseudonullity of \( R(E/F_\infty)^\vee \) implies \( \chi(G,R(E/F_\infty)) = 1 \).

The following variant of Proposition 1.13 can be proved easily.

**Lemma 1.18.** Let \( m_1 = m_2 = 0 \) and assume that \( R(E/F_\infty) \) is finite. Then \( R(E/F_\infty) \) is a pseudonull \( \mathbb{Z}_p[[G]] \) module and \( \chi(G,R(E/F_\infty)) \) exists. \( \square \)

Next, we discuss \( \chi(G,R(E/F_\infty)) \) when \( \text{corank}_{\mathbb{Z}_p[[H]]}(S(E/F_\infty)) = 2 \):

**Proposition 1.19.** Suppose \( \text{rank}_{\mathbb{Z}_p}(E(F)) > 0 \). Also assume that \( R(E/F) \) is finite, \( \mu \)-invariant of \( S(E/F_\text{cyc})^\vee \) vanishes, \( \text{corank}_{\mathbb{Z}_p[[H]]}(S(E/F_\infty)) = 2 \) and \( m_1 = m_2 = 0 \). Then \( \chi(G,R(E/F_\infty)) \) exists.

**Proof.** Since \( m_1 = m_2 = 0 \), we get from [15, Theorem 3.1] that \( \text{corank}_{\mathbb{Z}_p}(S(E/F_\text{cyc})) = 2 \). Then by applying [9, Corollary 4.4 and Proposition 4.9], we further deduce that \( \text{corank}_{\mathbb{Z}_p}(R(E/F_\text{cyc})) \leq 1 \) and \( \text{corank}_{\mathbb{Z}_p[[H]]}(R(E/F_\infty)) \leq 1 \).

Now, the result can be deduced following the proof of Proposition 1.14. \( \square \)

We also briefly mention the commutative case with \( G \cong \mathbb{Z}_p^{d+1} \).

**Proposition 1.20.** Let \( F \) be a totally real field of degree \( d \) over \( \mathbb{Q} \) and \( K \) be a CM field which is a quadratic extension of \( F \). Let \( K_\infty \) be a Galois extension of \( K \), such that \( G := \text{Gal}(K_\infty/K) \cong \mathbb{Z}_p^{d+1} \). Let \( E \) be an elliptic curve over \( K \). Then \( \chi(G,R(E/K_\infty)) \) exists if \( R(E/K) \) is finite.

**Proof.** As \( G \) is commutative, it suffices to show \( R(E/K_\infty)^G \) is finite. Finiteness of \( R(E/K_\infty)^G \) follows from [20, Example 1.9 and the proof of Theorem 1]. \( \square \)
Remark 1.21. The Euler characteristic of the fine Selmer group of an elliptic curve over the false Tate-curve extension has been discussed in [13]. Assume that $R(E/F)$ is finite, then it follows from [13, Theorem 3.1], that $\chi(G, R(E/F))$ exists if and only if the kernel of the natural map $\phi_{F_{\infty}/F}$ (see [9, equation 71]),

$$H^1_{lw}(T_p E/F)_{G} \overset{\phi_{F_{\infty}/F}}{\longrightarrow} H^1_{lw}(T_p E/F) = H^1(F/S, T_p E)$$

is finite.

However, applying Nekovar’s spectral sequence we actually get that [35, Proposition 8.4.8.3] $\ker(H^1_{lw}(T_p E/F)_{G} \longrightarrow H^1(F/S, T_p E)) = H_2(G, H^3_{lw}(T_p E/F_{\infty})).$

By Poitou-Tate exact sequence (4), we have seen that the finiteness of $H^2(G, H^3_{lw}(T_p E/F_{\infty}))$ is equivalent to the finiteness of $H^2(G, R(E/F_{\infty})).$

Thus, following the criterion of [13, Theorem 3.1], establishing the finiteness of $H_2(G, H^3_{lw}(T_p E/F_{\infty}))$ seems as difficult as showing the existence of $\chi(G, R(E/F_{\infty})).$

We now discuss a criterion for the existence of $\chi(G, R(E/F_{\infty}))$ in terms of the map $\phi_{F_{\infty}/F_{cyc}} : H^1_{lw}(T_p E/F_{\infty})_{H} \rightarrow H^1_{lw}(T_p E/F_{cyc}).$

Proposition 1.22. Assume that $m_1 = m_2 = 0$, $R(E/F)$ is finite and $S(E/F_{\infty})^\vee$ is a finitely generated $\mathbb{Z}_p[[H]]$ module. Furthermore assume that $\text{rank}_{\mathbb{Z}_p}(\text{coker}(\phi_{F_{\infty}/F_{cyc}})) = 0.$ Then $\chi(G, R(E/F_{\infty}))$ exists.

Proof. By Lemma 1.12, it suffices to show that $H^1(H, R(E/F_{\infty})) = 0.$ Using (10), it further reduces to show that $\text{corank}_{\mathbb{Z}_p}(\text{coker}(f)^H) = r - s$, where $r = \text{corank}_{\mathbb{Z}_p}S(E/F_{\infty})^H$ and $s = \text{corank}_{\mathbb{Z}_p}R(E/F_{\infty})^H.$ As $H^1(H, S(E/F_{\infty})) = 0$ [15, proof of Theorem 5.3], we observe that $r = \text{corank}_{\mathbb{Z}_p[[H]]}S(E/F_{\infty})$ and it follows from (10) that $\text{corank}_{\mathbb{Z}_p}(\text{coker}(f)^H) = \text{corank}_{\mathbb{Z}_p[[H]]}(\text{coker}(f)).$

As $m_1 = m_2 = 0$, using a control theorem for $R(E/F_{cyc}) \rightarrow R(E/F_{\infty})^H,$ we have $\text{corank}_{\mathbb{Z}_p}R(E/F_{\infty})^H = \text{corank}_{\mathbb{Z}_p}R(E/F_{cyc}).$ Finally, we use the hypothesis $\text{rank}_{\mathbb{Z}_p}(\phi_{F_{\infty}/F_{cyc}}) = 0$ and deduce from [9, Theorem 4.11] that $\text{corank}_{\mathbb{Z}_p}R(E/F_{cyc}) = \text{corank}_{\mathbb{Z}_p[[H]]}R(E/F_{\infty}).$ Therefore, $\text{corank}_{\mathbb{Z}_p}(\text{coker}(f)^H) = r - s.$ \hfill $\square$

Remark 1.23. We keep the hypotheses and setting of Proposition 1.22. Then from [9, Theorem 4.11], it follows that if we assume Conjecture B holds, then the following conditions are equivalent:

(i) $\text{rank}_{\mathbb{Z}_p}(\phi_{F_{\infty}/F_{cyc}}) = 0.$

(ii) $R(E/F_{cyc})$ is finite.

Remark 1.24. In [31], the Euler characteristic of the fine Selmer group of an abelian variety is discussed and contains the following corollary.

[31, Corollary 3.5] Let $A$ be an abelian variety over $K$. Let $T$ be the Tate module of the dual of the abelian variety $A^\vee$ of $A$. Let $K_{\infty}$ be a $p$-adic Lie extension of a number field $K$ with Galois group $G$. Suppose that there is a finite family of closed normal subgroups $G_i (0 \leq i \leq r)$ of $G$ such that $1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_i = G, G_i/G_{i-1} \cong \mathbb{Z}_p$, for every $i.$ Then if $Y(T/K)_{lw}$ is finite, the $G$-Euler characteristic of $Y(T/K_{\infty})$ is defined. \hfill $\square$

The following result of Kato was used crucially in proving [31, Corollary 3.5].

[31, Proposition 2.3] (22, Proposition 4.2) Let $G$ be a compact pro-$p$ $p$-adic Lie group without $p$-torsion. Let $N$ be a closed normal subgroup of $G$ such that $G/N$ has no $p$-torsion. Suppose that there is a finite family of closed normal subgroups $N_i$ $(0 \leq i \leq r)$
when is abelian. Hence, \( G \) is a torsion \( R[[G/N]] \)-module.

\( H \)

Remark 3.5, \[ \text{seems to have some gap for the false Tate curve extension.} \]

\[ \text{Taking } G = N \text{ in [31, Proposition 2.3], the author obtains [31, Corollary 3.5] via [31, Theorem 3.4].} \]

However, the assumption \( G = N \) forces the action of \( G \) on \( N_i/N_{i-1} \) by inner automorphism \( \chi_i \) to be trivial for all \( i \). Note that, this in particular implies that \( G \) is abelian. Hence, [31, Proposition 2.3] do not apply for \( G = N \) in the case when \( G \) is not abelian, in particular for the false Tate curve extension case with \( G \cong \mathbb{Z} \times \mathbb{Z}_p \). Thus, the proof of [31, Theorem 3.4] and hence [31, Corollary 3.5] seems to have some gap for the false Tate curve extension.

2. Fine Selmer group over function fields of characteristic \( \ell \neq p \)

We choose and fix a rational prime \( \ell \) distinct from \( p \) and take \( r \in \mathbb{N} \) such that the finite field \( \mathbb{F} = \mathbb{F}_\ell \) contains \( \mu_p \). Set \( K = \mathbb{F}(t) \). Let \( \mathbb{F}^{(p)} \) be the unique (unramified) \( \mathbb{Z}_p \) extension of \( \mathbb{F} \) contained in \( \mathbb{F}_\ell \). Note that \( \mathbb{F}^{(p)}(t) = K(\mu_p^{\infty}) \) and we denote \( \mathbb{F}^{(p)}(t) \) by \( K_{\text{cyc}} \). Further, we choose any non-constant polynomial \( q(t) \) in \( K \) and put \( K_\infty := \cup \mathbb{F}^{(p)}(t)(q(t)^{1/p^n}) \). Then \( G := \text{Gal}(K_\infty/K) \cong \mathbb{Z}_p \times \mathbb{Z}_p \) [46, Remark 3.5], \( H := \text{Gal}(K_\infty/K_{\text{cyc}}) \cong \mathbb{Z}_p \) and \( \Gamma = G/H \cong \text{Gal}(K_{\text{cyc}}/K) \cong \mathbb{Z}_p \). \( K_\infty \) is an analogue of the false-Tate extension of number fields. Throughout section 2, \( E \) will be a non-isotrivial elliptic curve defined over \( K \), i.e., \( j(E) \notin \mathbb{F} \) and \( S \) will be a finite set of primes of \( K \) containing the primes of bad reduction for \( E \) and the primes which ramify over \( K_\infty \). In this section, we discuss \( p^{\infty} \)-fine Selmer groups of \( E \) over \( K_\infty \).

For a finite extension \( L/K \) and each \( v \in S \), put \( K^1_v(E/L) := \prod_{w|v} H^1(L_w, E_{p^{\infty}}) \).

We define the \( p^{\infty} \)-fine Selmer group \( R(E/L) \) of \( E \) over \( L \) as:

\[
R(E/L) := \ker(H^1(G_S(L), E_{p^{\infty}}) \rightarrow \bigoplus_{v \in S} K^1_v(E/L)).
\]

As before, the definition of \( R(E/L) \) extends to an infinite extension \( \mathcal{L}/K \) by taking inductive limit over finite subextensions.

**Remark 2.1.** Recall that the classical \( p^{\infty} \)-Selmer group of \( E \) over \( L \) is defined as:

\[
S(E/L) := \text{Ker}(H^1(G_S(L), E_{p^{\infty}}) \rightarrow \bigoplus_{w|S} H^1(L_w, E_{p^{\infty}})/\text{im}(\kappa_w)),
\]

where \( \kappa_w : E(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(L_w, E_{p^{\infty}}) \) is the local Kummer map. Moreover, in this function field setting with \( \ell \neq p \), \( \text{im}(\kappa_w) = 0 \) [4, Prop. 3.3].

Thus, we see that the classical Selmer group \( S(E/L) \) is the smallest possible Selmer group and in fact we have \( S(E/L) = R(E/L) \).

Note that the definition of \( R(E/L) = S(E/L) \) is independent of the choice of \( S \) as long as \( S \) contains all the primes of bad reduction of \( E/L \) [45, Page 119].
Remark 2.2. We will discuss the Euler characteristic \( \chi(G, R(E/K_\infty)) \) over the extension \( K_\infty/K \). In fact, as explained in Remark 1.11, we will prove the existence of \( \chi(G, R(E/K_\infty)) \) using [17, Theorem 4]. Note that, for the cyclotomic case, the existence of \( \chi(\Gamma, S(E/K_{\text{cyc}})) \) is discussed under a variety of hypotheses in [37] (see [37, Theorems 3.7 and 3.9]).

Let \( \chi \) be the \( p \)-adic cyclotomic character. Given a \( G_K \) module \( M \), we denote by \( M(k) \) the twist of \( M \) by \( \chi^k \). Define \( R(E^k/K) \) by replacing \( E_p^\infty \) by \( E_p^\infty(k) \) in the equation (13). We begin with the following lemma.

Lemma 2.3. Let \( K_\infty/K \) be the extension as defined above in \( \S 2 \). Let \( E/K \) be a non-isotrivial elliptic curve with good or split multiplicative reductions at all primes of \( K \). If \( R(E/K) \) is finite, then \( H^0(G, R(E/K_\infty)) \) is finite.

Proof. Note that it is enough to establish a control theorem by showing that the kernel and the cokernel of the natural map \( R(E/K) \rightarrow R(E/K_\infty)^G \) are finite. By the snake lemma, it suffices to show that \( H^i(G, E_p^\infty(K_\infty)) \) is finite for \( i = 1, 2 \) and \( H^1(G_v, E_p^\infty(K_\infty,w)) \) is finite for every place \( w \) of \( K \) with \( w | v \), \( v \in S \). The finiteness of these local and global terms follows from [5, Corollary 4.9, Lemmas 3.3 and 4.2]. \( \square \)

Proposition 2.4. Let us keep the setting and hypotheses of Lemma 2.3. If \( R(E^k/K) \) is finite for every \( k \geq 1 \), then \( \chi(G, R(E/K_\infty)) \) is finite.

Proof. By Lemma 2.3, we deduce that \( H^0(G, R(E/K_\infty)) \) is finite. Hence, to show that \( \chi(G, R(E/K_\infty)) \) exists, it suffices to show that \( H^2(G, R(E/K_\infty)) \) is finite (Theorem 1.3). Note that the equation (4) continue to hold in this function field setting (\( \ell \neq p \)) for the extension \( K_\infty \). Then using [5, Lemmas 4.2 and 3.3] in (4), we obtain that \( H^2(G, R(E/K_\infty)) \) is finite if and only if \( H^2(G, H^1_{Iw}(T_p, E/K_\infty)^{\chi}(k)) \) is finite. Now, using the techniques of Kato [22, Theorem 5.1] as in the proof of Proposition 1.7, it is enough to show that \( H^0(\Gamma, R(E/K_{\text{cyc}})^{\chi}(k)) \cong H^0(\Gamma, R(E^k/K_{\text{cyc}})) \) is finite for every \( k \geq 1 \). By a control theorem similar to [37, Theorem 3.9], the finiteness of \( H^0(\Gamma, R(E^k/K_{\text{cyc}})) \) is equivalent to the finiteness of \( R(E^k/K) \). \( \square \)

Theorem 2.5. Let \( K_\infty/K \) be the false Tate curve extension as described above. Let \( E/K \) be a non-isotrivial elliptic curve with good or split multiplicative reductions at all primes of \( K \). If \( R(E/K) \) is finite, then \( \chi(G, R(E/K_\infty)) \) is finite.

Proof. By Proposition 2.4, it suffices to show that \( R(E^k/K) \) is finite for every \( k \geq 1 \). For each \( k \), set \( V_k = V_p E(k) \). Then the pure weight of \( V_k = -2k - 1 \) (see [2, Page-8] and [17, Page-356]). From [17, Theorem 4], for each \( k \geq 1 \), the map \( \phi_k : H^1(G_S(K), V_k) \rightarrow \prod_{s \in S} H^1(K_s, V_k) \) is injective. As the kernel of \( \phi_k \) surjects onto the divisible part of \( R(E^k/K) \), it follows that \( R(E^k/K) \) is finite. \( \square \)

Remark 2.6. Comparison of Theorem 2.5 with the work of [45]. The existence of \( \chi(G, R(E/L)) \) for some compact \( p \)-adic Lie extension \( L \) of \( K \) has been proved in [45], under certain set of hypotheses. In particular, for the false Tate curve extension \( K_\infty/K \), it is shown in [45, Theorem 1.2] that \( \chi(G, S(E/K_\infty)) \) exists under the following assumptions:

(i) \( E \) has either good or split multiplicative reduction at every prime of \( K \),
(ii) \( R(E/K) \) is finite,
(iii) $H^2(G_S(K∞), E_p∞) = 0$,
(iv) $\chi(G, E_p∞(K∞))$ exists,
(v) for every $v \mid S$, $\chi(G_v, E_p∞(K∞,v))$ exists, where $G_v$ is the decomposition subgroup of $G$ at $v$, and
(vi) the map $H^1(G_S(K∞), E_p∞) \to \bigoplus_{v \mid S} H^1(K∞,v, E_p∞)$ is surjective.

In Theorem 2.5, under the assumptions (i), (ii) and $j(E) \notin \mathbb{F}$, we prove the existence of $\chi(G, R(E/K∞))$. Our proof is different from the proof of [45, Theorem 1.2] and uses the results of Kato [22] and Jannsen [17]. Moreover, we do not need the assumptions (iii)-(vi) in our proof.

2.1. Analogue of Conjecture A. In this function field setting ($\ell \neq p$), the analogue of Conjecture A, i.e., $R(E/K_{cyc})^\dagger$ is a finitely generated $\mathbb{Z}_p$ module, is known and we briefly explain it here. Note that for an elliptic curve $E/K$, $R(E/K_{cyc})^\dagger$ is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$ module [46, Theorem 4.24].

Let $K$ be a function field of transcendence degree 1 over a finite field of characteristic $\ell$ and let $U$ be an open dense subset of $C$, the proper smooth curve with function field $K$. For any algebraic extension $L/K$, Witte [46] defined the $U$-Selmer group, $Sel_U(L, E_p∞)$ as follows:

$$Sel_U(L, E_p∞) := \ker(H^1(L, E_p∞) \to \bigoplus_{v \in U_p L} K^1_v(E/L)),$$

where $v \in U_p L$ varies over the closed points of normalisation of $U$ in $L$. Further the Iwasawa $\mu$-invariant of $Sel_U(K_{cyc}, E_p∞)$ vanishes [46, Corollary 4.38]. Since $R(E/K_{cyc}) \subset Sel_U(L, E_p∞)$, Conjecture A is true for $R(E/K_{cyc})^\dagger$ as well.

Remark 2.7. Alternatively, $R(E/K_{cyc})^\dagger$ is a finitely generated $\mathbb{Z}_p$ module can be shown using the relation between the ideal class group and $R(E/K_{cyc})$. Let $E/K$ be a non-isotrivial elliptic curve with good or split multiplicative reduction at all places of $K$. Let $L = K(E[p])$ and let $L \subset L_cyc$ be such that $[L_cyc : L] = p^n$. Let $Cl(F)$ denote the divisor class group of $F$, where $F$ is a finite extension of $K$. Then, using a proof similar to [32, Theorem 5.5] and [39, Proposition 11.16], we see that $R(E/L_{cyc})^\dagger$ is a finitely generated $\mathbb{Z}_p$ module and consequently the same holds for $R(E/K_{cyc})^\dagger$ as well.

2.2. Analogue of Conjecture B. Since $R(E/K_{cyc})$ is a finitely generated $\mathbb{Z}_p$ module, it is easy to see that the kernel and the cokernel of the map $R(E/K_{cyc}) \to R(E/K_{cyc})^\dagger$ is a finitely generated $\mathbb{Z}_p$ module. Thus, we arrive at the following:

Proposition 2.8. Let $K_{cyc}/K$ be as above with $Gal(K_{cyc}/K) \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Let $E/K$ be such that $j(E) \notin \mathbb{F}$ and it has good or split multiplicative reductions at all primes of $K$. Then $R(E/K_{cyc})^\dagger$ is a finitely generated $\mathbb{Z}_p[[H]]$ module and in particular, it is $\mathbb{Z}_p[[H]]$-torsion.

At this point, it is natural to ask if the analogue of Conjecture B is true in this function field setting ($\ell \neq p$) and in particular, for $K_{cyc}$, whether $R(E/K_{cyc})^\dagger$ is $\mathbb{Z}_p[[H]]$ torsion? We will give an explicit counterexample (Example 2) to this question using the following result of [6].

Proposition 2.9. [6, Proposition 4.3] Let $E, K$ and $S$ be as defined in §2. Let $G = Gal(L/K)$ be a compact $p$-adic Lie group without elements of order $p$ and of dimension $\geq 2$. If $H^2(G_S(L), E_p∞) = 0$ and $cd_p(G_v) = 2$ for every $v \in S$, then $R(E/L)^\dagger$ has no non-trivial pseudonull submodule.
Thus to use the above result for $\mathcal{L} = K_\infty$, we need to establish the vanishing of $H^2(G_S(K_\infty), E_p^\infty)$.

**Proposition 2.10.** Let $E/K$ be such that $j(E) \notin \mathbb{F}$. Then $H^2(G_S(K_{\text{cyc}}), E_p^\infty) = 0$ and consequently, $H^2(G_S(K_\infty), E_p^\infty)$ also vanishes.

**Proof.** As $H \cong \mathbb{Z}_p$, using Hochschild-Serre spectral sequence $H^2(G_S(K_{\text{cyc}}), E_p^\infty) = 0$ implies $H^2(G_S(K_\infty), E_p^\infty) = 0$. As $R(E/K_{\text{cyc}})^s$ is a finitely generated $\mathbb{Z}_p$ module, the vanishing of $H^2(G_S(K_{\text{cyc}}), E_p^\infty)$ can be deduced form Jannsen’s spectral sequence [18] (see [9, Lemma 3.1] for the details).

Now, we present a counterexample to an analogue of Conjecture B.

**Example 2.** Set $K = \mathbb{F}_5(t)$ and let $E$ be the elliptic curve defined over $K$ given by the Weierstrass equation:

$$y^2 + xy = x^3 - t^6.$$ 

Here the discriminant of $E = t^6(1 + 3t^6) = 3t^6(t^2 - 2)(t^2 + 2t + 3)(t^2 + 3t + 3)$. Let $p \neq 5$ be an odd prime and take $q(t) = t(t^2 - 2)(t^2 + 2t + 3)(t^2 + 3t + 3)$. Now, define $K_\infty = \bigcup_{n \geq 0} \mathbb{F}_5^{(p)}(t)(q(t)^{1/p^n})$. The primes of $K$ that ramify in $K_\infty$ are given by the irreducible divisors of $q(t)$ in $K$ and also the infinite prime. In [44, §2.2 and §2.3], it was shown that the bad primes for $E$ are the irreducible divisors of $q(t)$ in $K$ and $E$ has split multiplicative reduction at all these bad primes. Hence, we can take $S$ = {irreducible divisors of $q(t)$ in $K$, the infinite prime}. By [5, Lemma 4.2], we get that $cd_p(G_v) = 2$ for every $v \in S$. Also by Proposition 2.10, we have $H^2(G_S(K_\infty), E_p^\infty) = 0$. Hence, all the hypotheses of Proposition 2.9 are satisfied. Using [44, Theorem 1.5], we also obtain that rank$_\mathbb{Z}(E(K)) \geq 2$ and hence corank$_\mathbb{Z}_p(R(E/K))^G \geq 2$. As $j(E) \notin \mathbb{F}_5$, from Lemma 2.3, we deduce that corank$_\mathbb{Z}_p(R(E/K_{\infty})^G) \geq 2$. This implies that $R(E/K_{\infty}) \neq 0$ (Theorem 1.3).

Hence, $R(E/K_{\infty})^G$ is a torsion $\mathbb{Z}_p[[G]]$ module, which is finitely generated over $\mathbb{Z}_p[[H]]$ and is not a pseudo null $\mathbb{Z}_p[[G]]$ module.

### 3. Fine Selmer group over function fields of characteristic $p$

Let $\mathbb{F}$ be a finite field of order $p^r$, for some $r \in \mathbb{N}$ and $K$ denote the function field $\mathbb{F}(t)$. In this section, we study the $p^\infty$-fine Selmer groups of elliptic curves over $p$-adic Lie extensions of $K$. In fact, for simplicity, we restrict ourselves to the case when $\text{Gal}(K_\infty/K) \cong \mathbb{Z}_p^d$. Although, we believe analogue of Conjecture B should be investigated over non-commutative $p$-adic Lie extensions of function fields of characteristic $p$ also. We crucially use the properties of the arithmetic and the geometric $\mathbb{Z}_p$ extension of $K$ and hence discuss an analogue of conjecture B for their composite $\mathbb{Z}_p^2$ extension and prove it under suitable hypotheses.

Set $C_K := \mathbb{P}_K^1$. Throughout this section, $E$ will denote an elliptic curve over $K$ and $U$ will be a dense open subset of $C_K$ such that $E/K$ has good reductions at every place of $U$. Let $\mathcal{E}$ denote the Néron model of $E$ over $C_K$. Let $\Sigma_K$ be the set of all the places of $K$ and $S$ be the set of places of $K$ outside $U$ i.e., the places of $C_K \setminus U$. In particular, throughout §3, $S$ will be a finite set of places of $K$ containing the set of all primes of bad reduction of $E/K$. Let $L$ be a finite extension of $K$ inside $K_S$, the maximal algebraic extension of $K$ unramified...
outside $S$. Let $v$ be any prime of $K$ and $w$ denote a prime of $L$. Define

$$J^v(E/L) := \prod_{w|v} \frac{H^1_{fl}(L_w, E_{p^\infty})}{im(\kappa_w)} \quad \text{and} \quad K^v(E/L) := \prod_{w|v} H^1_{fl}(L_w, E_{p^\infty}).$$

(15)

Here $H^1_{fl}(-, -)$ denote the flat cohomology [34, Chapters II, III] and $\kappa_w : E(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to H^1_{fl}(L_w, E_{p^\infty})$ is induced by the Kummer map. Recall the following definition of the Selmer group from [24]:

**Definition 3.1.** [24, Prop. 2.4] With $\Sigma_K, S$ and $K \subset L \subset K_S$ as above, define

$$S(E/L) := \ker \left( H^1_{fl}(L, E_{p^\infty}) \to \bigoplus_{v \in \Sigma_K} J^v(E/L) \right).$$

(16)

Analogous to the definition of the fine Selmer group over the number field (1.4) and the function field of characteristic $\neq p$ (13), we define the $S$-fine Selmer group as:

$$R^S(E/L) := \ker \left( H^1_{fl}(L, E_{p^\infty}) \to \bigoplus_{v \in S} K^v(E/L) \bigoplus_{v \in \Sigma_K \setminus S} J^v(E/L) \right) \cong \ker \left( S(E/L) \to \bigoplus_{w|v, v \in S} E(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p \right).$$

(17)

Note that in the number field case for all places $w \nmid p$ of $L$ and in the function field case of char $\neq p$, for all primes $w$ of $L$, $E(L_w) \otimes \mathbb{Q}_p/\mathbb{Z}_p = 0$.

For an infinite algebraic extension $\mathcal{L}$ of $K$, the above definitions extends, as usual, by taking inductive limit over finite subextensions of $\mathcal{L}$ over $K$.

**Remark 3.2** (Dependence on $S$). With $E, S, K$ as above, recall the following equivalent definition [24] of the Selmer group:

$$S(E/K) := \ker \left( H^1_{fl}(U, E_{p^\infty}) \to \bigoplus_{v \in S} J^v(E/K) \right).$$

(18)

Using definitions 17 and 18, we have

$$R^S(E/K) := \ker \left( H^1_{fl}(U, E_{p^\infty}) \to \bigoplus_{v \in S} K^v(E/K) \right).$$

(19)

In fact, in [24, Proposition 2.4], the authors showed that the two definitions (16 and 18) of $S(E/K)$ are equivalent. The key ingredient in the proof is the following exact sequence [33, Chapter 3, §7]:

$$0 \to H^1_{fl}(U, E_{p^\infty}) \to H^1_{fl}(K, E_{p^\infty}) \to \bigoplus_{v \in U} H^1_{fl}(K_v, E_{p^\infty}) / H^1_{fl}(O_v, E_{p^\infty}),$$

(20)

where $O_v$ is the valuation ring of $K_v$ and $H^1_{fl}(O_v, E_{p^\infty}) \cong E(K_v) \otimes \mathbb{Q}_p/\mathbb{Z}_p$. In particular, it shows that the definition of $S(E/K)$ in (18) is independent of $S$. However in the definition of $R^S(E/K)$, for $v \in S$, we have a different local term given by $K^v(E/K)$.

For any $p$-adic Lie extension $L_{\infty}/K$ where $G = \text{Gal}(L_{\infty}/K)$ is a compact $p$-adic Lie group without any $p$-torsion, $\mathbb{Z}_p[G]$ is left and right Noetherian. For any two finite sets $S_1 \subset S_2$ of primes of $K$ containing the primes of bad reduction of $E$, we have $R^{S_2}(E/L_{\infty}) \subset R^{S_1}(E/L_{\infty})$. Thus, for a sufficiently large $S$, using the Noetherianess of $\mathbb{Z}_p[G]$ and the module $R^{S_1}(E/L_{\infty})^\vee$, we see that $R^S(E/L_{\infty})$ is independent of $S$. However, we cannot determine this set $S$ explicitly.

Now we discuss Euler characteristic of $R^S(E/L_{\infty})$, where $\text{Gal}(L_{\infty}/K) \cong \mathbb{Z}_p^d$. 16
Proposition 3.3. Let $K \subset L_\infty \subset K_S$ be a Galois extension over $K$ such that $\text{Gal}(L_\infty/K) \cong \mathbb{Z}_p^d$. Let $E/K$ be an elliptic curve and $S$ be as defined before. Then $\chi(G, R^S(E/L_\infty))$ exists if $R^S(E/K)$ is finite.

Proof. Since $G$ is commutative, it suffices to show that $R^S(E/L_\infty)^G$ is finite. Note that $E_{p^{\infty}}(K)$ is finite. Also, for each $v \in S$, $E_{p^{\infty}}(K_v)$ is finite ([4, Theorem 4.12, §2.1.2] and [43, 2.5.1]). By our assumption, $R^S(E/K)$ is finite. Notice that the local terms appearing in the definitions of $R^S(E/K)$ and $S(E/K)$ outside $S$, are the same. Now using a control theorem for $R^S(E/K) \to R^S(E/L_\infty)^G$, similar to the control theorem for $S(E/K) \to S(E/L_\infty)^G$ [4, Theorem 4.4], we get that $R^S(E/L_\infty)^G$ is finite. \qed

3.1. Analogues of Conjecture A. Next, we will discuss analogues of Conjecture A in this setting. Note that the only $p^{\infty}$-root of unity in $\mathbb{F}_p$ is 1. We discuss analogues of Conjecture A for two specific $\mathbb{Z}_p$ extensions of $K$, widely discussed in the literature; namely, the arithmetic $\mathbb{Z}_p$ extension $K_\infty$ and the geometric $\mathbb{Z}_p$ extension $K'_\infty$. Let $\mathbb{F}_p^{(p)}$ be the unique subfield of $\mathbb{F}_p$ such that $\text{Gal}(\mathbb{F}_p^{(p)}/\mathbb{F}_p) \cong \mathbb{Z}_p$. Set $K_\infty := K_{\mathbb{F}_p^{(p)}}$. Notice that $K_\infty/K$ is unramified everywhere. On the other hand, let $K'_\infty$ be the geometric $\mathbb{Z}_p$ extension of $K$ constructed using Carlitz module, a particular type of Drinfeld module ([39, Chapter 12]). The extension $K'_\infty$ of $K$ is ramified at exactly one prime $v$ [5, Remark 4.6] of $K$ and $v$ is totally ramified in $K'_\infty/K$.

We will show that, under suitable hypotheses, the analogues of Conjecture A are true over $K_\infty$ (Theorem 3.7) and also $K'_\infty$ (Corollary 3.15) i.e. $R^S(E/K_\infty)^\vee$ and $R^S(E/K'_\infty)^\vee$ are finitely generated $\mathbb{Z}_p$ modules. One of our main tool in proving this is to explore the relation between the fine Selmer group of $E[p]$ with the corresponding divisor class group of extensions of function field. For the rest of this section 3, we assume that $E/K$ is an ordinary elliptic curve.

Definition 3.4. Let $E, K, S$ be as before and $L \subset K_S$ be a finite extension of $K$. For $* \in \{\mu_{p^{\infty}}, \mathbb{Q}_p/\mathbb{Z}_p, \mu_p, \mathbb{Z}/p\mathbb{Z}\}$, set $K_1^*(L) := \bigoplus_{w|v} H^1_{fl}(L, *)$ and $J_1^*(L) := \bigoplus_{w|v} H^1_{fl}(L, *)/H^1_{fl}(O_w, *)$. We define the groups $S^*(L)$ and $R^S*(L)$ as follows:

$$S^*(L) := \ker(H^1_{fl}(L, *)) \to J_1^*(L). \quad (21)$$

$$R^S*(L) := \ker(H^1_{fl}(L, *)) \to \bigoplus_{v \in \Sigma_K} K_1^*(L) \bigoplus_{v \in \Sigma_K \setminus S} J_1^*(L). \quad (22)$$

Similarly, the group $R^S(E[p]/L)$ of $E[p]$ over $L$ is defined by:

$$R^S(E[p]/L) := \ker(H^1_{fl}(L, E[p])) \to \bigoplus_{v \in \Sigma_K} K_1^*(E[p]/L) \bigoplus_{v \in \Sigma_K \setminus S} J_1^*(E[p]/L), \quad (23)$$

where $K_1^*(E[p]/L) := \bigoplus_{w|v} H_{fl}^1(L_w, E[p])$ and $J_1^*(E[p]/L) := \bigoplus_{w|v} H_{fl}^1(L_w, E[p])/H_{fl}^1(O_w, E[p]).$

At first, note that by [27, Theorem 1.7], $S(E/K_\infty)^\vee$ (and hence $R^S(E/K_\infty)^\vee$) is a finitely generated torsion $\mathbb{Z}_p[[\Gamma]]$ module.

Lemma 3.5. $\mu(R^S(E/K_\infty)^\vee) = 0$ if and only if $R^S(E[p]/K_\infty)$ is finite.
Proof. It is easy to see that the kernel and the cokernel of the natural map $R^S(E[p]/K_\infty) \to R^S(E/K_\infty)[p]$ are finite. The result follows from this. \hfill \Box

**Proposition 3.6.** Define $K_\infty^p := \overline{K}_p^\infty$ and recall $E/K$ is ordinary. Then $R^S(E[p]/K_\infty^p)$ is finite.

Proof. Consider the connected-étale sequence (see, for example, [26, §3.2])

$$0 \to E[p] \to E \to \pi_0(E[p]) \to 0 \quad (24)$$

where $E[p]$ and $\pi_0(E[p])$ are Cartier dual to each other. As $E/K$ is ordinary, we know that $\pi_0(E[p]) \cong \mathbb{Z}/p\mathbb{Z}$ and $E[p] \cong \mu_p$, where $\mu_p$ is the Cartier dual to $\mathbb{Z}/p\mathbb{Z}$. Hence, we have the following exact sequences,

$$H^1_{fl}(K_\infty^p, \mu_p) \to H^1_{fl}(K_\infty^p, E[p]) \to H^1_{fl}(K_\infty^p, \mathbb{Z}/p\mathbb{Z}). \quad (25)$$

Also for each prime $w$ of $K_\infty$,

$$H^1_{fl}(K_{w}^p, \mu_p) \to H^1_{fl}(K_{w}^p, E[p]) \to H^1_{fl}(K_{w}^p, \mathbb{Z}/p\mathbb{Z}). \quad (26)$$

By [42, §1.4], for $w$ dividing $v$ of $K$, $v \not\in S$, we have the following exact sequence:

$$0 \to \mathcal{E}(O_w)[p] \to \mathcal{E}(O_w) \to \pi_0(\mathcal{E}(O_w)) \to 0, \quad (27)$$

where $\mathcal{E}(O_w)[p] \cong \mu_p$ and $\pi_0(\mathcal{E}(O_w)) \cong \mathbb{Z}/p\mathbb{Z}$.

From the definition in (23), we have the following commutative diagram of complexes, which is not necessarily exact:

$$0 \to R^S((\mathbb{Z}/p\mathbb{Z})/K_\infty^p) \to H^1_{fl}(K_\infty^p, \mathbb{Z}/p\mathbb{Z}) \to \prod_{w | v, v \in S} H^1_{fl}(K_{w}^p, \mathbb{Z}/p\mathbb{Z}) \prod_{w | v, v \in \bar{S}} H^1_{fl}(K_{w}^p, \mathbb{Z}/p\mathbb{Z})$$

$$0 \to R^S(E[p]/K_\infty^p) \to H^1_{fl}(K_\infty^p, E[p]) \to \prod_{w | v, v \in S} H^1_{fl}(K_{w}^p, E[p]) \prod_{w | v, v \in \bar{S}} H^1_{fl}(K_{w}^p, E[p])$$

$$0 \to R^S(\mu_p/K_\infty^p) \to H^1_{fl}(K_\infty^p, \mu_p) \to \prod_{w | v, v \in S} H^1_{fl}(K_{w}^p, \mu_p) \prod_{w | v, v \in \bar{S}} H^1_{fl}(K_{w}^p, \mu_p)$$

(28)

Note that $R^S(\mu_p/K_\infty^p) \cong S'(\mu_p/K_\infty^p)$ and $R^S(\mathbb{Z}/p\mathbb{Z}, K_\infty^p) \cong S'((\mathbb{Z}/p\mathbb{Z})/K_\infty^p)$, where $S'(-/K_\infty^p)$ is defined in (21). By the proof of [27, Lemma 3.4.1], we deduce that $S'((\mathbb{Z}/p\mathbb{Z})/K_\infty^p)$ is finite and $S'((\mathbb{Z}/p\mathbb{Z})/K_\infty^p) \cong \text{Cl}(K_\infty^p)[p]$, the $p$-part of the divisor class group, which is also finite [39, Proposition 11.16]. (Notice that for $* \in \{\mu_p, \mathbb{Z}/p\mathbb{Z}\}$, $S'(*/K_\infty^p)$ is being denoted by $H^1_{fl}(C_{K_\infty^p}, *)$ in [27].) Hence, $R^S(\mu_p/K_\infty^p)$ and $R^S(\mathbb{Z}/p\mathbb{Z}, K_\infty^p)$ are finite. Now, using [33, §III.7], we get that for $w | v, v \not\in S$, $\ker(\gamma_w) = 0$. Also, for $w | v, v \in S$, $\ker(\gamma_w)$ is finite. This implies that $\ker(\gamma)$ is finite. By applying a snake lemma to the lower complex in (28), we obtain a map from $\ker(\gamma) \to \text{coker}(\beta)$, such that $\text{coker}(\theta)$ injects into the finite group $R^S((\mathbb{Z}/p\mathbb{Z})/K_\infty^p)$. Therefore, $R^S(E[p]/K_\infty^p)$ is finite. \hfill \Box

As $G = \text{Gal}(K_\infty^p/K_\infty) \cong \prod_{l \neq p} \mathbb{Z}_l$, we get $R^S(E[p]/K_\infty^p)^G \cong R^S(E[p]/K_\infty)$. Thus using Lemma 3.5 and Proposition 3.6, we deduce the following theorem:

**Theorem 3.7.** Let $E/K$ be an ordinary elliptic curve and $S$ be as defined in §3. Then, $R^S(E/K_\infty)^Y$ is a finitely generated $\mathbb{Z}_p$ module. \hfill \Box
3.2. Pseudonullity. Now we discuss an analogue of Conjecture B over the extension $F_\infty := K_\infty K'_\infty$ of $K$. Put $G=\text{Gal}(F_\infty/K) \cong \mathbb{Z}_p^2$ and $H:=\text{Gal}(F_\infty/K_\infty) \cong \mathbb{Z}_p$. We will show, under suitable assumption $R^S(E/F_\infty)^\vee$ is a pseudonull $\mathbb{Z}_p[[G]]$ module. We begin by collecting some evidence towards this. At first, in Proposition 3.8, we show that the corank of $R^S(E/K)$ is strictly less than the corank of $S(E/K)$, whenever $E(K)$ is infinite.

**Proposition 3.8.** Let $E/K$ be an elliptic curve with $\text{rank}_\mathbb{Z}(E(K)) \geq 1$. Assume that there exists $v \in S$, where $E$ has good ordinary reduction or split multiplicative reduction. Then, $\text{corank}_{\mathbb{Z}_p}(R^S(E/K)) < \text{corank}_{\mathbb{Z}_p}(S(E/K))$.

**Proof.** We generalise the proof of [9, Lemma 4.1] from the number field case to the function field (of char. $p$) case. Recall the following short exact sequence,

$$0 \longrightarrow R^S(E/K) \longrightarrow S(E/K) \stackrel{v \in S}{\longrightarrow} \bigoplus_{v \in S} E(K_v) \otimes \mathbb{Q}_p / \mathbb{Z}_p. \quad (29)$$

So, it is enough to prove that $\text{image}(\bigoplus_{v \in S} r_v)$ is infinite. Note that $B = E(K) \otimes \otimes_{v \in S} \mathbb{Q}_p / \mathbb{Z}_p \hookrightarrow S(E/K)$. Since $E(K)$ is finitely generated, $B \cong (\mathbb{Q}_p / \mathbb{Z}_p)^{\text{rank}(E(K))}$ is divisible. Thus it is enough to show that $r_v(B) \neq 0$, for some $v \in S$. On the contrary, let us assume that $r_v(B) = 0$ for every $v \in S$ and let $R$ be a point of infinite order in $E(K)$. Then for every $k \geq 1$, $R \otimes p^{-k} = 0$ in $E(K_v) \otimes \mathbb{Q}_p / \mathbb{Z}_p$.

Let us consider the case where $E$ has a split multiplicative reduction at $v \in S$. Let $E_0(K_v)$ be the set of points of $E(K_v)$ which has non-singular reduction at the residue field of $K_v$. We can choose an integer $n \neq 0$ such that $Q = nR$ is a point on $E_0(K_v)$. Then, for every $k \geq 1$, $Q \otimes p^{-k} = 0$ in $E_0(K_v) \otimes \mathbb{Q}_p / \mathbb{Z}_p$. Note that $E_0(K_v) \cong O_v^\ast$ [4, §2.1.2], where $O_v$ is the ring of integers of $K_v$. This is a contradiction since $O_v^\ast \cong \mathbb{Z}_p^d \oplus \mathbb{Z}/(p^r - 1)\mathbb{Z}$, where $p^r = \#F$.

Next, let us assume that $v$ is a prime of good ordinary reduction. We can choose an integer $n \neq 0$ such that $Q = nR$ is a point on $E_v(m_v)^\vee$, the formal group of $E$ at $v$. Note that $E_v(m_v)^\vee$ is a torsion free $\mathbb{Z}_p$ module [43, Lemma 2.5.1]. Being a finite index subgroup of the maximal pro-$p$ subgroup of $E(K_v)$, $E_v(m_v)^\vee$ is a pro-$p$ group [43, page 4436]. By [40, Theorem 4.3.4], we know that a torsion free pro-$p$ group is a free $\mathbb{Z}_p$ module. Therefore, $E_v(m_v)^\vee$ is a free $\mathbb{Z}_p$ module and it cannot have an infinite $p$-divisible element. □

From Proposition 3.8, we get an evidence for Conjecture B over $\mathbb{Z}_p^d$-extension.

**Corollary 3.9.** Let us keep the hypotheses of Proposition 3.8 and assume that $j(E) \notin \mathbb{F}$. Let $K \subset L_\infty \subset K_S$ be such that $\text{Gal}(L_\infty/K) \cong \mathbb{Z}_p^d$. Then $S(E/L_\infty) \neq R^S(E/L_\infty)$.

**Proof.** We use Proposition 3.8 and using a control theorem, proceed in a similar way, as in the proof of [9, Proposition 4.3]. The details are omitted. □

For an algebraic extension $L/K$, let $Cl(L)$ denote the divisor class group of $L$.

**Proposition 3.10.** Recall $F_\infty = K_\infty K'_\infty$ and $H = \text{Gal}(F_\infty/K'_\infty) \cong \mathbb{Z}_p$. Then, $Cl(F_\infty)[p^\infty]^\vee$ is a finitely generated torsion $\mathbb{Z}_p[[H]]$ module.

**Proof.** By Nakayama lemma, it suffices to show that $(Cl(F_\infty)[p^\infty])^H$ is finite.
Using the commutative diagram in [27, Page-38], we obtain the following commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & \left( \lim_m \frac{(\mathbb{F}(p))^\times}{(\mathbb{F}(p))^\times \times \mathbb{F}^n} \right)^H & \rightarrow & S'(\mu_{p^n}/F_\infty)^H & \rightarrow & (Cl(F_\infty)[p^n])^H \\
\uparrow & & \uparrow \alpha & & \uparrow \beta & & \uparrow \gamma \\
0 & \rightarrow & \lim_m \frac{\mathbb{F}^x}{\mathbb{F}^x} & \rightarrow & S'(\mu_{p^n}/K'_\infty) & \rightarrow & Cl(K'_\infty)[p^n] \\
\end{array}
\]

(30)

Note that \(\frac{(\mathbb{F}(p))^\times}{(\mathbb{F}(p))^\times \times \mathbb{F}^n} = 0\) for all \(m\). Therefore, \(\beta\) is an isomorphism. Further, by [1, Proposition 2], \(Cl(K'_\infty)[p^n]\) is finite. We claim that \(\text{coker}(\alpha)\) is finite. Then using a snake lemma in diagram 30, \(\ker(\gamma)\) is finite and the lemma follows. We now establish the claim. Consider the commutative diagram:

\[
\begin{array}{cccccc}
0 & \rightarrow & S'(\mu_{p^n}/F_\infty)^H & \rightarrow & H_1^4(F_\infty, \mu_{p^n})^H & \rightarrow & \left( \oplus_{w,v,\in \Sigma_K} \frac{H_1^4(F_{K_\infty,w,\mu_{p^n}})}{\text{Gal}(\mathbb{F}_{K_\infty,w,\mu_{p^n}})} \right)^H \\
\uparrow \alpha & & \uparrow \delta & & \uparrow \gamma \\
0 & \rightarrow & S'(\mu_{p^n}/K'_\infty) & \rightarrow & H_1^4(K'_\infty, \mu_{p^n}) & \rightarrow & \left( \oplus_{w,v,\in \Sigma_K} \frac{H_1^4(K'_\infty,w,\mu_{p^n})}{\text{Gal}(\mathbb{F}_{K_\infty,w,\mu_{p^n}})} \right) \\
\end{array}
\]

(31)

Using the relation: \(H_1^4(\text{Spec}(R), \mu_{p^n}) = R^\times/(R^\times)^{p^n}\), for a local ring \(R\) and the Hochschild-Serre spectral sequence, we obtain that \(\ker(\gamma) = 0\). As \(H \cong \mathbb{Z}_p\), \(\text{coker}(\delta) = 0\). Then \(\text{coker}(\alpha) = 0\) follows from the diagram (31). \(\square\)

**Lemma 3.11.** \(R^S(\mathbb{Z}/p\mathbb{Z})/F_\infty)\) is finite.

**Proof.** We have \(F_\infty = \bigcup_{n,m} F_{n,m}\), where \(\text{Gal}(F_{n,m}/F) \cong \mathbb{Z}/p^n\mathbb{Z} \times \mathbb{Z}/p^m\mathbb{Z}\). Using [41, Theorem 27.6], we observe that \(R^S(\mathbb{Z}/p\mathbb{Z})/F_{n,m}) \cong \text{Hom}(G_{\alpha}^{ab}(F_{n,m})(p), \mathbb{Z}/p\mathbb{Z}) \rightarrow \text{Hom}(G_{\alpha}^{ab}(F_{n,m})(p), \mathbb{Z}/p\mathbb{Z})\), where \(G_{\alpha}^{ab}(F_{n,m})(p)\) is the Galois group of the maximal abelian everywhere unramified pro-\(n\) extension of \(F_{n,m}\). Also,

\[
\text{Hom}(G_{\alpha}^{ab}(F_{n,m})(p), \mathbb{Z}/p\mathbb{Z}) \cong \text{Hom}(\text{Cl}(F_{n,m}) \otimes \mathbb{Z}/p\mathbb{Z}) \cong \text{Hom}(\text{Cl}(F_{n,m})/p, \mathbb{Z}/p\mathbb{Z}).
\]

By applying a control theorem, similar to the proof of Proposition 3.10, we get that the kernel and the cokernel of the map \(\text{Cl}(F_{n,m})[p] \rightarrow \text{Cl}(F_\infty)[p]\) are finite and bounded independently of \(m\) and \(n\). As a result, \(\text{Cl}(F_{n,m})[p]\) is finite and bounded independently of \(m\) and \(n\). Moreover, as \(\text{Cl}(F_{n,m})(p)\) is finite, the same is true for \(\text{Cl}(F_{n,m})/p\). Thus, \(R^S((\mathbb{Z}/p\mathbb{Z})/F_{n,m})\) is finite and bounded independently of \(n\) and \(m\). Hence, we conclude that \(R^S((\mathbb{Z}/p\mathbb{Z})/F_\infty) \cong \lim_n R^S((\mathbb{Z}/p\mathbb{Z})/F_{n,m})\) is finite. \(\square\)

**Lemma 3.12.** \(S'(\mu_{p^n}/F_\infty)\) is finite. It follows that \(R^S(\mu_{p^n}/F_\infty)\) is also finite.

**Proof.** We have an exact sequence:

\[
0 \rightarrow \left( \mathbb{F}(p)^\times / \mathbb{F}(p)^\times \times \mathbb{F}^n \right)^p \rightarrow S'(\mu_{p^n}/F_\infty) \rightarrow \text{Cl}(F_\infty)[p] \rightarrow 0.
\]
Note that \( \frac{(p^n)^*}{(p^n)} = 0 \), hence \( S'(\mu p/F_\infty) \cong Cl(F_\infty)[p] \). We have \( K'_\infty = \bigcup_{n=1}^\infty K'_n \), where \( K \subset K'_n \subset K'_\infty \) with \( \text{Gal}(K'/K) \cong \mathbb{Z}/p^n\mathbb{Z} \). Set \( K_{n,\infty} = K'_n K_\infty \) and \( G_n = \text{Gal}(K_{n,\infty}/K_{n,\infty}) \) with \( K_{n,\infty} = K_n \mathbb{F}_p(t) \). As \( K_n \) is prime to \( p \), following a standard diagram chase using the definition of \( S'(\mu p/\infty) \), we get \( S'(\mu p/\infty) \cong S'(\mu p/K_{n,\infty}) \). Similarly, we obtain that \( Cl(K_{n,\infty})[p^\infty] \cong Cl(K_{n,\infty})[p^\infty] \mathbb{F}_p \). Let \( C_{K_{n,\infty}} \) be a proper smooth geometrically connected curve which is the model of the function field \( K_{n,\infty} \). On the other hand, [36, Proposition 10.1.1], we observe that \( \text{Cl}(K_{n,\infty})[p^\infty] \cong (\mathbb{Q}/p\mathbb{Z})^\infty \), where \( 0 \leq r_n \leq \text{genus}(C_{K_{n,\infty}}) \). Therefore \( Cl(F_\infty)[p^\infty] \cong \lim_{\rightarrow} Cl(K_{n,\infty})[p^\infty] \) is \( p \)-divisible. By Proposition 3.10, \( Cl(F_\infty)[p^\infty] \mathbb{F}_p \) is also a finitely generated torsion \( \mathbb{Z}_p[[H]] \cong \mathbb{Z}_p[[G]] \) module. Consequently, \( S'(\mu p/F_\infty) \cong Cl(F_\infty)[p] \) is finite. As \( R^S(\mu p/F_\infty) \rightarrow S'(\mu p/F_\infty) \) (see Definition 3.4), \( R^S(\mu p/F_\infty) \) is finite as well. □

**Proposition 3.13.** Recall \( H = \text{Gal}(F_\infty/K_\infty) \). Then \( R^S(E/F_\infty)^\vee \) is a finitely generated \( \mathbb{Z}_p[[H]] \) module and in particular, it is \( \mathbb{Z}_p[[G]] \)-torsion.

**Proof.** By Theorem 3.7, \( R^S(E/K_\infty)^\vee \) is a finitely generated \( \mathbb{Z}_p \) module. It is easy to see that the kernel and the cokernel of the map \( (R^S(E/F_\infty)^\vee)_H \rightarrow R^S(E/K_\infty)^\vee \) are finitely generated \( \mathbb{Z}_p \)-modules. Hence by Nakayama lemma, \( R^S(E/F_\infty)^\vee \) is a finitely generated \( \mathbb{Z}_p[[H]] \) module. □

**Theorem 3.14.** Let \( F_\infty = K_\infty K'_\infty \) be the \( \mathbb{Z}_p^2 \) extension of \( K \), defined in §3.2 and let \( v_r \) be the unique prime of \( K \) that ramifies in \( K'_\infty \). Assume \( E/K \) be an ordinary elliptic curve which has good reduction outside \( \{v_r\} \) and \( S \) be as specified before. Then \( R^S(E/F_\infty)^\vee \) is a pseudonull \( \mathbb{Z}_p[[G]] \) module.

**Proof.** Put \( S' := \{v_r\} \). Using ordinarity of \( E/K \), from (24), we obtain a complex (not necessarily exact):

\[
R^S(\mu p/F_\infty) \rightarrow R^S(E[p]/F_\infty) \rightarrow R^S((\mathbb{Z}/p\mathbb{Z})/F_\infty).
\]

Then, from the definition of \( R^S(\_ / F_\infty) \), there is a commutative diagram (not necessarily exact):

\[
\begin{array}{cccc}
0 & \rightarrow & R^S((\mathbb{Z}/p\mathbb{Z})/F_\infty) & \rightarrow & H^1_f(F_\infty, \mathbb{Z}/p\mathbb{Z}) & \rightarrow & \prod_{w|v_r} H^1_H(\mathbb{Z}/p\mathbb{Z}) & \rightarrow & \prod_{w|v_r} H^1_H(F_\infty, w, \mathbb{Z}/p\mathbb{Z}) \\
& & & & & & & \uparrow & \\
0 & \rightarrow & R^S(E[p]/F_\infty) & \rightarrow & H^1_f(F_\infty, E[p]) & \rightarrow & \prod_{w|v_r} H^1_H(E[p]) & \rightarrow & \prod_{w|v_r} H^1_H(F_\infty, w, E[p]) \\
& & & & & & & \uparrow & \\
0 & \rightarrow & R^S(\mu p/F_\infty) & \rightarrow & H^1_f(F_\infty, \mu_p) & \rightarrow & \prod_{w|v_r} H^1_H(H_1)(\mu_p) & \rightarrow & \prod_{w|v_r} H^1_H(F_\infty, w, \mu_p) \\
\end{array}
\]

(32)

Here \( w \) denote a place of \( F_\infty \). Note that \( R^S(\mu p/F_\infty) \) and \( R^S((\mathbb{Z}/p\mathbb{Z})/F_\infty) \) are finite by Lemmas 3.12 and 3.11, respectively. Note that there are only finitely many primes of \( F_\infty \) lying above \( v_r \) in \( F'_\infty \). Now, from the proof of Proposition 3.6, we get that \( \ker(\gamma) \) is finite. Again using a diagram chase, as in the proof of Proposition 3.6, we get that \( R^S(E[p]/F_\infty) \) is finite.
Next, it is easy to see that the kernel and the cokernel of the natural map $R^S(E[p]/F_\infty) \to R^S(E/F_\infty)[p]$ are finite. Thus $R^S(E/F_\infty)^\vee/(pR^S(E/F_\infty)^\vee)$ is finite. Then, by Nakayama lemma and Proposition 3.13, $R^S(E/F_\infty)^\vee$ is a finitely generated torsion $\mathbb{Z}_p[[H]]$ module and hence $\mathbb{Z}_p[[G]]$ pseudonull (Theorem 1.2).

Now for a finite set $S$ of primes of $K$ containing $S' = \{v_t\}$, such that $E/K$ has good reduction outside $v_t$, $R^S(E/F_\infty) \to R^S(E/F_\infty)$. Consequently, $R^S(E/F_\infty)^\vee$ is a pseudonull $\mathbb{Z}_p[[G]]$ module. $$\square$$

**Corollary 3.15.** Let us keep the setting and hypotheses of Theorem 3.14. Then $R^S(E/K_\infty')^\vee$ is a finitely generated $\mathbb{Z}_p$ module.

Proof. Again, put $S' = \{v_t\}$. Applying a control theorem $R^S(E[p]/K_\infty') \to R^S(E[p]/F_\infty)^{\text{Gal}(F_\infty/K_\infty')}$ and using Theorem 3.14, we deduce that $R^S(E[p]/K_\infty')$ is finite. Similarly, the kernel and the cokernel of the natural map $R^S(E[p]/K_\infty') \to R^S(E/K_\infty')[p]$ are finite and thus $R^S(E/K_\infty')^\vee/(pR^S(E/K_\infty')^\vee)$ is finite. Then by Nakayama lemma, $R^S(E/K_\infty')^\vee$ is a finitely generated $\mathbb{Z}_p$ module. Finally, for a general $S$, the argument extends as in the proof of Theorem 3.14. $$\square$$

**Remark 3.16.** Let $K = \mathbb{F}(t)$ be a function field of char $p > 2$. Let $K_\infty'$ be the $\mathbb{Z}_p$ extension, constructed using Carlitz module that is ramified only at the infinite prime $(1/t)$. Set $F_\infty = K_\infty K_\infty'$. Let $E/K$ given by the Weierstrass equation:

$$y^2 = x^3 + (1/t^n)x,$$

where $n > 0$. Then $E$ has bad reduction only at the infinite prime $(1/t)$ of $K$. Hence, these examples of $E/K$ satisfies the assumptions of Theorem 3.14.

**Zero Selmer group:** In this function field of characteristic $p$ setting, we discuss ‘$p^\infty$-zero Selmer group’ or ‘ $\text{III}(E_{p^\infty}/L)$’ of an elliptic curve $E$ defined over an algebraic extension $L/K$ and compare it with $R^S(E_{p^\infty}/L)$.

**Definition 3.17.** Define the ‘$p^\infty$-zero Selmer group’ of $E$ over a finite extension $L/K$ as:

$$R_0(E/L) := \ker (H^1_{\text{pr}}(L, E_{p^\infty}) \to \prod_{w \in \Sigma_L} K^1_w(E/L)).$$

Let $K \subset L_\infty \subset K_S$ be an infinite extension. We define $R_0(E/L_\infty) := \lim_{K \subset F \subset L_\infty} R_0(E/F)$, where $F$ varies over finite extensions of $K$.

In the following remark, we discuss various properties of $R_0(E/L)$.

**Remark 3.18.** (i) Recall $S$ is a finite set of primes of $K$ containing all the primes of bad reduction of $E/K$. Then for any algebraic extension $K \subset L_\infty \subset K_S$, clearly $R_0(E/L_\infty) \to R^S(E/L_\infty)$, for any such $S$.

Hence, under the setting of Theorem 3.7 (respectively Corollary 3.15), we can deduce from the same theorem (resp. corollary) that $R_0(E/K_\infty)^\vee$ (respectively $R_0(E/K_\infty')^\vee$) is a finitely generated $\mathbb{Z}_p$ module. Similarly, from Theorem 3.14, we obtain that $R_0(E/F_\infty)^\vee$ is a pseudo-null $\mathbb{Z}_p[[G]]$ module.

(ii) For a finite extension $L/K$, the $p^n$-zero Selmer group is defined as:

$$R_0(E[p^n]/L) := \ker (H^1_{\text{pr}}(L, E[p^n]) \to \prod_{w \in \Sigma_L} H^1(L_w, E[p^n])).$$

In fact, $R_0(E[p^n]/L) = 0$ in most cases (see [3, Proposition 6.1] for a precise statement).
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