BOUND STATES IN ONE AND TWO SPATIAL DIMENSIONS

K. Chadan  
*Laboratoire de Physique Théorique*  
*Université de Paris XI, Bâtiment 210, 91405 Orsay Cedex, France*

N. N. Khuri  
*Department of Physics, The Rockefeller University, New York, NY 10021, U.S.A.*

A. Martin  
*TH Division, CERN, Geneva, Switzerland and*  
*Laboratoire de Physique Théorique, F-74941 Annecy-le-Vieux, France*

Tai Tsun Wu  
*Gordon McKay Laboratory, Harvard University, Cambridge, MA 02138, U.S.A.*  
*and TH Division, CERN, Geneva, Switzerland*

Abstract  

In this paper we study the number of bound states for potentials in one and two spatial dimensions. We first show that in addition to the well-known fact that an arbitrarily weak attractive potential has a bound state, it is easy to construct examples where weak potentials have an infinite number of bound states. These examples have potentials which decrease at infinity faster than expected. Using somewhat stronger conditions, we derive explicit bounds on the number of bound states in one dimension, using known results for the three-dimensional zero angular momentum. A change of variables which allows us to go from the one-dimensional case to that of two dimensions results in a bound for the zero angular momentum case. Finally, we obtain a bound on the total number of bound states in two dimensions, first for the radial case and then, under stronger conditions, for the non-central case.

LPT Orsay 02-57  
CERN-TH/2002-128  
June 2002
I. Introduction

In recent years, it has become apparent that studying physics in two spatial dimensions is not just an academic exercise, especially for condensed matter physics where there are bound states due to impurities on the surface of a semiconductor or at a junction [1]. In addition, we have established a remarkable universality property for low energy scattering in two dimensions. Namely, excluding some well-defined and rare exceptional cases, the $m = 0$ phase shift for a radial potential behaves like $(\pi/2)(\ell nk)^{-1}$ as $k \to 0$ [2]. This result has been recently generalized to non-radial and even non-local potentials [3].

We believe that relatively little is known about bound states in one and two dimensions. For any dimension, including one and two, we know that if the potential is sufficiently smooth and sufficiently rapidly decreasing at large distances, there is a semi-classical asymptotic estimate of the number of bound states for a potential $gV, g \to \infty$, which was first established for the radial case in [4], then generalized in [5] to arbitrary dimensions.

However, concerning strict bounds on the number of bound states the situation is radically different for one and two dimensions from that in higher dimensions (including three dimensions). Lieb [6], Cwikel [7] and Rozenblum [8] have shown that for $n \geq 3$, $n$ being the number of spatial dimensions, there is a bound

$$N \leq B_n \int |V|^{n/2} \, d^n x ,$$

(1)

where $B_n$ is definitely larger, even for very large dimensions, contrary to earlier belief [9], than the semi-classical constant $C_n$ appearing in the asymptotic estimate [4]

$$N(g) \sim C_n \, g^{n/2} \int (V^-)^{n/2} \, d^n x , \quad g \to \infty , C_n = \frac{2^{-n} \pi^{-n/2}}{\Gamma(1 + \frac{n}{2})}$$

(2)

for a potential $gV$ where $-V^-$ is the negative part of the potential : $V = V^+ - V^-$, $V^\pm \geq 0$. For central potentials, $B_n/C_n \to 1$ for $n \to \infty$ [4]. Other proofs have been obtained [4], [11]. Furthermore, it is well known that for one and two dimensions a potential globally attractive, arbitrarily weak, such that

$$\int d^n x V(x) < 0 , \quad n = 1, 2$$

(3)
has a bound state. The proof is trivial for \( n = 1 \) by using a Gaussian trial function. For \( n = 2 \), there is a proof by Simon, for instance [11]. The simplest one is by Yang and De Llano \([12]\) who use a trial function \( \exp(-(r + r_0)^\alpha) \), \( \alpha \) sufficiently small.

However, this bound state has an incredibly small binding energy in absolute value, for a potential \( gV \), which behaves like \( \exp(-\frac{c}{g}) \) for small \( g \), as shown in Appendix I.

In addition to the above, we note that for the \( s \)-state \( (m = 0) \), and \( n = 2 \), there is an old bound on the number of bound states due to Newton [13] and Setô [14]. However, this bound is bilinear in \( V \) and does not behave like the semiclassical result for large \( g \).

It was noticed in Ref. [9] that the number of bound states in two dimensions is certainly larger than \(-\frac{1}{4} \int rV(r)dr\), in the central case.

In this paper we first find examples of potentials in one dimension for which the number of bound states is infinite. Using a transformation which is systematically studied, one can find more refined potentials for which the number of bound states is infinite.

This same transformation allows us also to find radial potentials in two dimensions for which the zero angular momentum bound states are infinite in number. Examples with non-radial potentials are also constructed. All these examples possess the property, \( \int d^2x|V(\vec{x})| < \infty \), and in addition \( \int d^2x|V(\vec{x})|\ln(2 + |\vec{x}|)^{1-\varepsilon} < \infty \).

In section III we find explicit bounds on the number of bound states in one dimension by using well-known bounds for the three dimensional radial case with zero angular momentum. In addition, using the above noted change of variables, we also obtain bounds on the number of zero angular momentum bound states in two dimensions.

Finally, in section IV, we get bounds on the total number of bound states in two dimensions. This bound has the property that it is linear in \( g \) for a potential \( gV \) and is thus similar to the semi-classical estimate.

In Appendix I we give upper and lower bounds on the ground state energy in two dimensions.

Next, in Appendix II, we present a system of transformations which first allow us to derive more and more refined examples of limit potentials with a finite or infinite number of bound states. Secondly, these transformations allow us to convert results obtained in a given dimension to results for another dimension for zero angular momentum.

In Appendix III we compare one of our two dimensional bounds with the Newton-Setô bound. Finally, in Appendix IV, we sketch the proof that bound states are on real
analytic Regge trajectories. [13]

A preliminary account of these results was presented at a workshop in Les Houches [13].

II. Examples where the number of bound states is infinite

We begin by using the well known result that in one dimension, and for the radial case in 2 and 3 dimensions, the number of negative energy bound states is equal to the number of nodes of the zero energy wave-function [14].

For any two potentials \( V_1(x) \leq 0, \) and \( V_2(x) \leq 0 \) in one dimension, one can easily show that, if \( V_1(x) > V_2(x) \), then for any interval \( a \leq x \leq b \), we have

\[
  n_2(a,b) \geq n_1(a,b) - 1 ;
\]

where \( n(a,b) \) is the number of nodes in the interval \((a,b)\). Thus if \( n_1(x, \infty) \) is infinite, \( n_2(x, \infty) \) is also infinite.

We write the zero energy one-dimensional Schrödinger equation for an attractive potential \( V = -\lambda/x^2 ; x > x_0 > 0, \lambda > 0, \)

\[
  \left( -\frac{d^2}{dx^2} - \frac{\lambda}{x^2} \right) \phi(x) = 0 .
\]

Because of the homogeneity of Eq. (3), \( \phi = x^s \), where \( s \) is given by the two roots \( s\pm \) of the equation

\[
  s(s - 1) = -\lambda ,
\]

or

\[
  s\pm = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \lambda} .
\]

For \( \lambda > 1/4 \), both \( s_+ \) and \( s_- \) are complex, and the solution \( \phi \) can be constructed by taking a linear combination of \( x^{s_+} \) and \( x^{s_-} \). We have

\[
  \phi(x) = \sqrt{x} \cos \left( \sqrt{\lambda - 1/4} \log x + \delta \right) .
\]

Obviously, this \( \phi \) has an infinite number of nodes for any \( X \leq x < \infty, X > 0 \).

We can now use the theorem summarized in Eq. (4), to get the following general result: the number of one dimensional bound states is infinite if there exists an \( X > 0 \)
such that

\[ x^2V(x) < L < -1/4, \quad \text{for } x > X, \]

and/or

\[ x^2V(x) < L < -1/4, \quad \text{for } x < -X. \] (8)

On the other hand, if \( V \) is bounded from below and if \( x^2V(x) > -1/4 \) for \( |x| > |X| \), then the number of bound states is finite.

Using the series of transformation described in Appendix II it is possible to approach the limiting case in a more refined way. For example: if

\[ V(x) < -\frac{1}{4x^2} - \frac{\mu_1}{4x^2(\ell n x)^2}, \quad x > X, \mu_1 > 1 \]

or

\[ V(x) < -\frac{1}{4x^2} - \frac{1}{4x^2(\ell n x)^2} \left[ 1 + \frac{\mu_2}{(\ell n x)^2} \right], \quad x > X, \mu_2 > 1 \] (9)

the number of bound states is \textit{infinite}. Notice that this is true for \( X \) arbitrarily large, i.e., in a way, \( V \) arbitrarily small.

These two examples are such that \( \int dx|V(x)|^{1/2} \to \infty \). This is not surprising since in the three dimensional radial case we have for a monotonic potential the Cohn-Calogero bound,

\[ n < \frac{2}{\pi} \int_0^\infty dr |V|^{1/2}. \] (10)

However we can have non-monotonic potentials such that the above integral converges but the number of bound states is infinite. For example one can set

\[ V = -\sum_{n=0}^{+\infty} \delta(x - 2n). \] (11)

For this potential \( \int |V|^{1/2} dx = 0 \) since the \( \delta \)-function can be effectively replaced by suitably chosen square wells of decreasing widths \( \varepsilon_n \) and depth \( \frac{1}{\varepsilon_n} \) with \( \Sigma \sqrt{\varepsilon_n} \) convergent, and \( \varepsilon_n \) arbitrarily small.

Next we consider the two-dimensional case. In this case we introduce a simple transformation which converts the one-dimensional zero energy Schrödinger equation to the \( m = 0, \) two-dimensional radial Schrödinger equation. In one dimension \( -\infty < x < +\infty \) we have,
\[
\left[ -\frac{d^2}{dx^2} + U(x) \right] \phi(x) = 0 .
\] (12)

Our change of variables is given by:

\[
x \equiv \ln \frac{r}{R} , \quad 0 \leq r < \infty ;
\]

\[
U(x) \equiv r^2 V(r) ;
\]

\[
\phi(x) = \psi(r) .
\] (13)

This transformation is a particular case of the Liouville transformation \cite{19}. Equation (12) now becomes

\[
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + V(r) \right) \phi(r) = 0 .
\] (14)

But this equation is precisely the \( m = 0 \) two-dimensional radial Schrödinger equation.

Using Eq. (8) we now see that for a radial potential \( V(r) \), the number of bound states is infinite if,

\[
r^2 \left( \ln \frac{r}{R} \right)^2 V(r) < L < -1/4 ; \quad r > R_0 > R ;
\] (15)

This time we see that the integral appearing in the semiclassical estimate, \( \int_0^\infty r |V(r)| dr \) is convergent and yet the number of bound states is infinite. Furthermore the integral \( \int_0^\infty r dr |V(r)||\ln(2 + r)|^{1-\varepsilon} \), is also convergent for \( \varepsilon > 0 \), and the integral can be made arbitrarily small by taking \( R_0 \) arbitrarily large.

Our limit potentials in the 2-dimensional case are given by

\[
V(r) = -\frac{\mu}{4} \frac{1}{r^2 \left( \ln \frac{r}{R} \right)^2} , \quad r \geq R_0 \geq 1 ;
\]

\[
V(r) = 0 , \quad r < R_0 ,
\] (16)

with \( \mu > 1 \).

In addition we can also solve the Schrödinger equation exactly for the class,

\[
V(r) = \begin{cases} 
0 , & r < R , \text{with } R > 1 ; \\
-\frac{g}{r^2 (\ln r)^\alpha} , & r > R , \quad 1 < \alpha < 2
\end{cases}
\] (17)
with $g > 0$. The solution is given by

$$
\begin{align*}
\psi(r) &= a + b \ln r ; \quad r < R ; \\
\psi(r) &= (\log r)^{1/2} \left[ AJ_\nu \left( 2\nu \sqrt{g} (\log r)^{1/2\nu} \right) + BY_\nu \left( 2\nu \sqrt{g} (\log r)^{1/2\nu} \right) \right] , \quad r \geq R .
\end{align*}
$$

where $\nu \equiv (2 - \alpha)^{-1}$, and $J_\nu$ and $Y_\nu$ are Bessel functions. This last solution has an infinite number of nodes for $r > R$ and hence the potential (17) has an infinite number of bound states, and this is true for arbitrarily small $g$.

A completely different approach to get infinitely many bound states abandons radial symmetry and considers scattering by circular “delta shell” potentials in the plane. Indeed a very simple example where $\int V d^2x$ is finite, arbitrarily small, and where one sees that has a bound state has been invented by Richard [20]. It is a delta shell potential:

$$
V = -g \delta(r - 1) .
$$

Here $\int d^2x V = -2\pi g$ is finite. The zero-energy Schrödinger equation

$$
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + V \right) \psi = 0
$$

has a solution, finite at the origin, which is

$$
\begin{align*}
\psi &= 1 \quad \text{for } r < 1 \\
\psi &= 1 - g \ln r \quad \text{for } r \geq 1 .
\end{align*}
$$

Hence the zero-energy radial solution has a node at

$$
r_0 = \exp \frac{1}{g} ,
$$

and therefore this potential has a bound state for arbitrarily small $g$.

If, in addition, we now impose a Dirichlet boundary condition at $r = \exp \left( \frac{1}{g} \right)$ and set $\psi$ to be identically zero for $r > \exp \left( \frac{1}{g} \right)$, i.e., physically, having an infinitely repulsive wall, we will still have a solution with a node at $r = \exp \left( \frac{1}{g} \right)$, and hence a zero-energy bound state.

Take now a sequence of potentials
\[ V_n = -g_n \delta(|\vec{x} - \vec{x}_n| - 1), \]  
\[ g_n > 0, \text{ such that } \Sigma g_n \text{ converges, } \vec{x}_n \text{ on the positive } x \text{ axis. For simplicity, } g_n \text{ will be chosen a decreasing sequence. It is always possible to choose the } \vec{x}_n \text{'s in such a way that the disks} \]

\[ |\vec{x} - \vec{x}_n| \leq \exp \frac{1}{g_n} = r_n \]

do not overlap.

The number of bound states of \( V = \sum_{n=0}^{n_0} V_n \) is certainly larger than \( n_0 \), the result one gets when one imposes Dirichlet boundary conditions on the border of each disk (this strategy was already used in Ref. \[5\]). Letting \( n_0 \) go to infinity, we see that we have infinitely many bound states, and yet the integral \( \int |V| d^2 x = 2\pi \Sigma g_n \) is finite and can be arbitrarily small.

We can, however, do better than that, i.e., try to build an example in which

\[ \int |V| [\ell n(2 + |\vec{x}|)]^\alpha \ d^2 x \]

is finite, where \( \alpha \) is to be determined. We take the centres of the circles on a line, and since the \( g_n \)'s are decreasing, we have

\[ |\vec{x}_n| + r_n < (2n + 1) \exp \frac{1}{g_n}, \]

and hence

\[ \int |V_n| [\ell n(2 + |\vec{x}|)]^\alpha \ d^2 x < g_n \ell n \left[ 2 + (2n + 1) \exp \frac{1}{g_n} \right]^\alpha. \]

However,

\[ \ell n \left( 2 + (2n + 1) \exp \frac{1}{g_n} \right) < \frac{\ell n 3}{\ell n 2} \left[ \ell n(2n + 1) + \frac{1}{g_n} \right], \]

and hence

\[ \sum_{n=1}^{\infty} \int |V_n| [\ell n(2 + |\vec{x}|)]^\alpha \ d^2 x < 2\pi \left( \frac{\ell n 3}{\ell n 2} \right)^\alpha \left[ \Sigma g_n \left[ \ell n(2n + 1) + \frac{1}{g_n} \right]^\alpha \right]. \]

Since we want the series on the right-hand side to converge, \( \alpha \) is chosen to be less than 1.

With the choice
\[ g_n = g_0 \exp(-\lambda n), \]

this series will converge for any \( \alpha < 1. \)
III. Bounds on the number of bound states in one and two dimensions

We start by considering the one dimensional case, and write always, in obvious notations,

\[ V = V^+ - V^- \]

where \( V^+ \) and \( V^- \) are both \( \geq 0 \).

The zero-energy one-dimensional Schrödinger equation is

\[
\left(-\frac{d^2}{dx^2} + V(x)\right) \psi(x) = 0, \quad x \in (-\infty, +\infty). \tag{24}
\]

Except for the fact that one is restricted to the half line, the above equation is the same as the reduced \( \ell = 0 \), 3-dimensional Schrödinger equation

\[
\left(-\frac{d^2}{dr^2} + V(r)\right) u(r) = 0, \quad r \in [0, \infty). \tag{25}
\]

Now if, in the one dimensional case \( V(x) \) has \( N \) bound states, then \( \psi(x) \) has \( N \) nodes, \( x_p, p = 1, \cdots, N \). Let \( k \) be such that

\[ x_k < 0 < x_{k+1}, \]

then the 3-dimensional potential, \( V_1(r) = V(x) \) with \( r \equiv x - x_{k+1} \), has \( (N - k - 1) \ell = 0 \) bound states. Also the potential, \( V_2(r) \equiv V(x) \) with \( r = -(x - x_k) \) has \( k \) bound states with \( \ell = 0 \). Hence any three dimensional bound gives a one dimensional bound.

Starting with the well known Bargmann [21] bound for angular momentum \( \ell \), we write

\[
N(\ell) < \frac{1}{2\ell + 1} \int_0^\infty r V^-(r) \, dr. \tag{26}
\]

Using \( \ell = 0 \), we get, for the one dimensional case :

\[
N(1D) - 1 < \int_{-\infty}^{x_\kappa} |x - x_n| V^-(x) \, dx + \int_{x_{\kappa+1}}^{\infty} |x - x_{\kappa+1}| V^-(x) \, dx,
\]

and hence

\[
N(1D) < 1 + \int_{-\infty}^{+\infty} |x| V^-(x) \, dx. \tag{27}
\]

Similarly, we can use the bound obtained by one of us [22] in the radial three-dimensional case :
\[ N(3D, \ell = 0) < \left[ \int_{0}^{\infty} r^2 V^{-}(r) \, dr \int_{0}^{\infty} V^{-}(r) \, dr \right]^{1/4} \]  

(28)

to get, in the one-dimensional case, after some manipulations:

\[ N(1D) < 1 + \sqrt{2} \left[ \int_{-\infty}^{+\infty} x^2 V^{-}(x) \, dx \int_{-\infty}^{+\infty} V^{-}(x) \, dx \right]^{1/4} \]  

(29)

which behaves like \( \sqrt{g} \) if \( V = gV \), like the semi-classical estimate.

Now to get bounds in two dimensions for the \( m = 0 \) case is very simple. The change of variables given in Eq. (13) allows us to go from Eq. (27) to a bound for the 2D case:

\[ N(2D, m = 0) < 1 + \int_{0}^{\infty} r |\ln r| V^{-}(r) \, dr . \]  

(30)

In this bound \( R \) is arbitrary. We can minimize with respect to \( R \). \( R_{\text{min}} \) is given by

\[
\int_{0}^{R_{\text{min}}} x |V(x)| \, dx = \int_{R_{\text{min}}}^{\infty} x |V(x)| \, dx .
\]  

(31)

The bound (30) with \( R = R_{\text{min}} \) should be compared with the bound previously obtained by Newton \([13]\) and Setô \([14]\) which is

\[ N(m = 0) < 1 + \frac{1}{2} \int_{0}^{\infty} \frac{r \, dr \, r' \, dr' V^{-}(r) V^{-}(r') |\ln \left( \frac{r}{r'} \right)|}{\int_{0}^{\infty} r \, dr \, V^{-}(r)} \]  

(32)

It turns out that

\[ J < I(R_{\text{min}}) < 2J . \]  

(33)

This is demonstrated in Appendix III. So the Newton-Setô bound is slightly better but has a more complex structure. Both bounds are “optimal” in the sense that multiplying factors in them cannot be improved. This is because the Bargmann bound is itself known to be optimal.

Applying the same change of variable in equation (13) and (29) gives

\[ N(m = 0, 2D) < 1 + \sqrt{2} \left[ \int_{0}^{\infty} (\ell r^2 r \, dr \, V^{-}(r) \int_{0}^{\infty} r \, dr \, V(r) \right] . \]  

(34)

For large coupling this behaves like \( \sqrt{g} \) for a potential \( gV \). The integrals appearing in Eq. (34) are those which were required to converge in our original paper on low energy...
scattering in 2 dimensions.

IV. A bound on the total number of bound states in two dimensions

In this section, we study the total number of bound states in two dimensions, mostly for a rotationally symmetrical potential. The bound for this rotationally symmetrical case gives also some information for the general case, as discussed near the end of this section.

For the radial case, the easiest thing to do is to notice that the radial reduced equation (11) can be viewed as a radial three-dimensional equation with non-integer angular momentum $\ell = m - 1/2$. Therefore the Bargmann bound [18] is valid:

$$N_m < \frac{1}{2m} \int_0^\infty r V^- (r) \, dr .$$

To get the total number of bound states, we must remember that for $m \neq 0$ we have a multiplicity 2 and for $m = 0$ multiplicity 1. Hence

$$N_{\text{total}} < N_0 + \sum_{m=1}^{m=2} \frac{1}{m} \int r V^- (r) \, dr$$

where $N_0$ is for instance given by (30).

However, the logarithm is spurious. This has already happened in the past, for instance in the three-dimensional bound obtained by Glaser, Grosse, Martin and Thirring [21].

To show this, we use a technique due to Glaser, Grosse and Martin [9], in which the counting of bound states for a radial potential is reduced to the calculation of a bound on the moment of the eigenvalues of a one-dimensional problem.

The reduced radial Schrödinger equation for bound states

$$\left[ -\frac{d^2}{dr^2} + \frac{m^2 - 1/4}{r^2} + V(r) - E_i(m) \right] u_i(r) = 0 ,$$

where $i$ designates the number of nodes of the solution ($i$-th eigenfunction starting from the ground state designated by $i = 0$), has been generalized by Regge [23] to non-integer and even complex angular momentum. What can be shown, under the weak condition
\[ \int r|V(r)|dr < \infty , \quad (38) \]
is that each \( E_i(m) \), \( i = 0, 1, \ldots \) is the restriction to \( m \) integer (physical) of a real analytic, monotonically increasing function of \( m \), \( 0 < m < m_i \), where \( m_i \) is such that \( E_i(m_i) = 0 \). That \( m_i \) exist follows from the Bargmann bound and condition (38). (Notice that \( m_0 > m_1 > \cdots \)). This is what is called a “Regge trajectory”. Different trajectories with different \( m_i \)’s do not intersect, due to general Sturm-Liouville theory. In Appendix IV, we sketch the proof of these statements.

The number of bound states on a given trajectory, with \( m \geq 1 \), will be \( \lfloor m_i \rfloor \), where \( \lfloor x \rfloor \) is the integer part of \( x \). Each of those bound states with \( m \neq 0 \) has a multiplicity 2. So the total number of bound states with \( m \neq 0 \) is

\[ 2 \sum_{i, \lfloor m_i \rfloor \geq 1} \lfloor m_i \rfloor . \]

On the other hand, by using the change of variables (13) already employed in sections II and III the zero-energy reduced Schrödinger equation

\[ \left( -\frac{d^2}{dr^2} + \frac{m^2 - 1/4}{r^2} + V(r) \right) u(r) = 0 \quad (39) \]

becomes

\[ \left( -\frac{d^2}{dz^2} + U(x) \right) \phi(x) = -(m^2 - 1/4) \phi(x) \quad (40) \]

The eigenvalues of (39) are just the \( m_i^2 - 1/4, m_i \) defined previously. The sum \( \sum [m_i] \) is very similar to the sum of moments of power 1/2 of the eigenvalues of (38):

\[ \sum_{\lfloor m_i \rfloor \geq 1} [m_i] < \frac{2}{\sqrt{3}} \sum (m_i^2 - 1/4)^{1/2} \quad (41) \]

It happens that this moment satisfies a bound proposed by Lieb and Thirring [24]

\[ \sum |e_i|^{1/2} < L_{1/2,1} \int_{-\infty}^{+\infty} dx U^-(x) = L_{1/2,1} \int_0^{+\infty} r V^-(r) dr \quad (42) \]

where the \( e_i \)'s are the eigenvalues of the one-dimensional Schrödinger equation with a potential \( U \). \( L_{1/2,1} \) has been shown to be finite by Weidel [25] and less than 1.005. More
recently Hundertmark, Lieb and Thomas \cite{26} have found the optimal value for $L_{1/2,1}$, namely $1/2$:

$$\sum |e_i|^{1/2} < \frac{1}{2} \int_{-\infty}^{+\infty} U^-(x) \, dx$$

which is obtained in the one-bound-state case with a delta function potential.

Therefore, using (30), (41) and (43) we get a bound on the total number of bound states in two space dimensions for a central potential

$$N < 1 + \int_0^\infty r V^-(r) \left| \ln \left( \frac{R}{r} \right) \right| \, dr$$

$$+ \frac{2}{\sqrt{3}} \int_0^\infty r V^-(r) \, dr .$$

We notice that for a potential $gV$ the bound is linear in $g$, similar to the semiclassical estimate for large $g$. It is probably almost optimal, in the sense that it is optimal for $m = 0$ and that for $m \neq 0$ the only foreseeable improvement is to remove the multiplicative factor $2/\sqrt{3}$.

It is trivial, but not very elegant, to obtain also a bound on the total number of bound states for a non-central potential. Let

$$B(r) = \sup_{0<\theta<2\pi} V^-(r,\theta) .$$

Then replacing $V(r)$ by $B(r)$ in (44) we get a bound on the total number of bound states in a non-radial potential because of the monotonicity of the bound-state energies with respect to the potential.

For a potential with a single singular point the replacement of $V^-$ by $B(r)$ is not too bad. However, if $V$ has several singular points the replacement will be catastrophic since $B$ will be infinite on successive circles corresponding to these singular points. It is certainly desirable to find a better bound.

Our conjecture is

$$N < 1 + 2 \int \frac{d^2 x}{2\pi} V_R(|x|) \ell n^{-} \left( \frac{|x|}{R} \right)$$

$$+ \int \frac{d^2 x}{2\pi} V^-(x) \ell n \left( \frac{|x|}{R} \right) + \frac{2}{\sqrt{3}} \int \frac{d^2 x}{2\pi} V^-(x) ,$$

(45)
where \( V_R(|x|) \) is the decreasing rearrangement of \( V^-(x) \) (see footnote in Appendix I). The reasons for which we propose this are

(i) for a central decreasing potential \( (46) \) coincides with \( (44) \);

(ii) for a central potential not necessarily decreasing, the r.h.s. of \( (46) \) is larger than the r.h.s. of \( (44) \);

(iii) if we take a shifted central with a centre outside the origin, the first and the last integrals in \( (46) \) are, of course, invariant. The second integral, because of the harmonic properties of \( \ell n r \) in 2 dimensions, is larger than the one corresponding to a central potential centred at the origin.

Proving \( (46) \) or something similar might be rather difficult but, seeing what has been achieved for higher dimensions, not impossible.

Notice that the integrals in \( (44) \) and \( (46) \) will certainly converge under the conditions of Ref. [1], and we can announce that they do converge in Ref. [2] also.

**Acknowledgements**

We should like to thank J. M. Richard and W. Thirring for crucial information. Two of us, N.N.K and T.T.W., are grateful to the CERN Theory Division for its kind hospitality. This work was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-91ER40651, Task B, and under Grant No. DE-FG02-84ER40158.

*Note added:* Dr. P. Blanchard drew our attention to a paper by A. Laptev [28] in which he finds a bound on the number of bound states for a potential \( b|x|^{-2} - |V(|x|)| \), which is

\[
N < \frac{A(b)}{4\pi} \int |V(x)|d^2x,
\]

when \( A(b) \to \infty \) for \( b \to 0 \). With methods developed in the present paper, using the Bargmann bound for the \( m = 0 \) contribution and (42) for the rest, we get

\[
A(b) < \frac{1}{\sqrt{b}} + \frac{4}{\sqrt{3}}.
\]
Appendix I

Upper and lower bounds on the ground state energy in two dimensions

We use the Schrödinger equation in integral form, for a potential $gV$:

$$
\psi(x) = -\frac{g}{2\pi} \int K_0(\kappa |x - y|) V(y) \psi(y) d^2 y ,
$$

for an energy $E = -\kappa^2$.

First we shall get an algebraic lower bound. Then $V$ can be replaced by $-V^-$, the attractive part of the potential. We have:

$$
|\psi(x)| < \frac{g}{2\pi} \int K_0(\kappa |x - y|) V^-(y) d^2 y \sup |\psi|
$$

Since $K_0(t)$ is a decreasing function of $t$ and given the rearrangement inequality,

$$
\int A B \, d^2 x < \int A_R \, B_R \, d^2 x
$$

where $A$ and $B$ are positive, going to zero at infinity, and $A_R$ and $B_R$ are their decreasing circular rearrangements, we have

$$
|\psi(x)| < \frac{g}{2\pi} \int K_0(\kappa |y|) V_R(|y|) d^2 y \sup |\psi|
$$

where $V_R$ is the rearrangement of $V^-$. Hence, if we take the supremum of the left-hand side over $x$, we can divide by $\sup |\psi|$ and obtain

$$
1 < \frac{g}{2\pi} \int K_0(\kappa |y|) V_R(|y|) d^2 y .
$$

From the property

$$
K_0(ab) < \ell n^+ \left( \frac{1}{a} \right) + K_0(b) ,
$$

where $\ell n^+(t) = \ell n t$ for $t > 1$, $= 0$ for $t < 1$, which is proved at the end of this Appendix, we get

\[A_R\] is a decreasing function of $|x|$ such that $\forall t$, $\mu(A_R > t) = \mu(A > t)$, where $\mu$ is the Lesbègue measure.
\[ K_0(\kappa) > \frac{1}{g} \left( 1 - \frac{g}{2\pi} \int \frac{\ell n^+ \left( \frac{1}{y_\ell} \right) V_R(y) d^2 y}{\frac{1}{2\pi} \int V^-(y) d^2 y} \right) = X. \] (I.5)

As long as \( X \) is positive, this gives a lower bound on \( K_0(\kappa) \) and hence an upper bound on \( \kappa \) and an upper bound on \( \kappa^2 \), the absolute value of the binding energy.

If \( X > K_0(1) = 0.42, \cdots \), we can again use the inequality (I.4) and get

\[ \kappa^2 < \exp \left( \frac{1}{g} \left( 1 - \frac{1}{2\pi} \int \ell n^+ \left( \frac{1}{y_\ell} \right) V_R(y) d^2 y \right) \right), \] (I.6)

which demonstrates that the absolute value of the binding energy is bounded by \( \exp -C/g \), \( C > 0 \) for \( g \to 0 \).

In the special case of a purely attractive potential we can get an inequality going in the opposite direction. We start again from (I.1) and use the fact that the ground-state wave function is positive. We have

\[ \psi(x) > \frac{g}{2\pi} \int_{|y|<R} K_0(\kappa|x-y|)|V(y)|d^2 y \times \text{Inf}_{|y|<R} |\psi(y)| \]

and, taking also \( |x| < R \), and using the fact that \( K_0 \) is decreasing:

\[ \text{Inf}|\psi(y)||x|<R > \frac{g}{2\pi} K_0(2\kappa R) \int_{|y|<R} |V(y)|d^2 y \text{Inf}|\psi(y)||x|<R. \] (I.7)

However, \( \text{Inf}|\psi(y)||x|<R \) cannot vanish in the ground state and hence we can divide (I.7) by \( \text{Inf}|\psi(x)| \). From

\[ K_0(t) > \ell n \frac{1}{t} + \ell n 2 - \gamma, \] (I.8)

when \( \gamma \) is the Euler constant = 0.577 ... we get

\[ \kappa^2 > \frac{e^{-2\gamma}}{R^2} \exp \frac{2}{g \int_{|x|<R} |V(\kappa)|d^2 x} \] (I.9)

which goes in the opposite direction to (I.6), but again has the form \( \exp -C/g \) for small \( g \).

Both upper and lower bounds on \( \kappa^2 \) have the same qualitative behaviour for small \( g \). The lower bound on \( \kappa^2 \) can be optimized with respect to \( R \). Of course we cannot do that for a potential which is not strictly attractive but only globally attractive. Nevertheless, we believe that the same qualitative result will hold.
In a recent paper [27] Nieto has given an explicit example in which he shows that the binding energy in absolute value is incredibly small. A square well with unit radius and strength 0.1 in natural units produces a bound state with energy $-10^{-18}$.

Finally we give a proof of (I.4) and (I.8): consider the quantity

$$Z = K_0(x) - \ln \left( \frac{x_0}{x} \right),$$
$$Z' = -K_1(x) + \frac{1}{x}.$$

From

$$K_1(x) = \int_1^\infty \frac{tdt}{\sqrt{t^2 - 1}} \exp(-tx) < \int_1^\infty \frac{tdt}{\sqrt{t^2 - 1}} \exp(-x\sqrt{t^2 - 1}),$$

we get $K_1(x) < \frac{1}{x}$, and hence

$$Z' > 0.$$ 

So, for $x < x_0$ $Z(x) < Z(x_0) = K_0(x_0)$, which proves (I.4). On the other hand, we have $\lim_{x \to 0} Z(x) = \ln 2 - \gamma$, and so

$$K_0(x) > \ln 2 - \gamma + \ln \left( \frac{1}{x} \right).$$
Appendix II

Transformations of the Schrödinger equation from one to two dimensions, the converse, limit potentials, and generalization

In section II we presented a transformation of the one dimensional zero energy Schrödinger equation,

\[
\left( -\frac{d^2}{dx^2} + U(x) \right) \phi(\kappa) = 0, \quad x \in (-\infty, +\infty);
\] (II.1)

into the two dimensional, zero angular momentum, Schrödinger equation,

\[
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + V(r) \right) \phi(r) = 0, \quad r \in [0, \infty).
\] (II.2)

The transformation is given by:

\[
\begin{align*}
    x & \equiv \ln \left( \frac{r}{R} \right); \quad x \in (-\infty, +\infty); \quad r \in [0, \infty); \\
    U(x) & \equiv r^2 V(r); \quad x \geq 0; \\
    \phi(x) & \equiv \psi(r); \quad x \geq 0.
\end{align*}
\] (II.3)

This enables us to prove that since a potential, \(U(x)\), given by

\[
\begin{align*}
    U(x) &= 0; \quad x < X, \\
    U(x) &= -\frac{\mu}{4x^2}; \quad \mu > 1, \quad x \geq X,
\end{align*}
\] (II.4)

has infinitely many bound states in one dimension, the potential

\[
\begin{align*}
    V(r) &= 0; \quad r < R_0; \\
    V(r) &= -\frac{\mu}{r^2 \left( \ln \frac{r}{R_0} \right)^2}; \quad r \geq R_0 > R; \quad \mu > 1;
\end{align*}
\] (II.5)

will also have infinitely many bound states in two dimensions for the \(m = 0\), radial case.
This procedure can be continued further. We can re-transform (II.1) to make it look like a two dimensional equation by defining \( \chi(x) \) as

\[
\phi(x) \equiv x^{1/2} \chi(x) .
\] (II.6)

The \( \kappa \) satisfies the equation

\[
\left( -\frac{d^2}{dx^2} - \frac{1}{x} \frac{d}{dx} + W(x) \right) \chi(x) = 0 ,
\]

with

\[
W(x) = U(x) - \frac{1}{4x^2} .
\] (II.7)

Relabelling \( x \) as \( r \) we have

\[
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + W(r) \right) \chi(r) = 0 .
\] (II.8)

This last equation is for \( r \geq 0 \) exactly the two dimensional radial equation.

From the chain,

\[
V(r) \to U(x) \to W(r) ,
\]

we obtain,

\[
W(r) = -\frac{1}{4r^2(\ln r)^2} + \frac{1}{r^2} V(\ln r) .
\] (II.9)

Thus if for \( x > x_0 \) we set

\[
U(x) = -\frac{\mu}{4x^2} ,
\]

or

\[
V(r) = -\frac{\mu}{4r^2(\ln r)^2} ,
\]

we get

\[
W(r) = -\frac{1}{4r^2(\ln r)^2} - \frac{\mu}{4r^2(\ln r)^2(\ln \ln r)^2} ,
\]

with \( r > R_0 > 0 \).
This potential has infinitely many bound states if \( \mu > 1 \). Our procedure can be repeatedly iterated producing potentials which are closer to the limit, and with wave functions which can be expressed explicitly in terms of elementary functions.

Finally we stress that this procedure is not restricted to the connection between one dimension and two dimensions, and the construction of limit potentials in one or two dimensions. It also applies in \( N \) dimensions.

In \( N \)-dimensions the radial Schrödinger equation becomes

\[
\left( -\frac{d^2}{dr^2} - \frac{N-1}{r} \frac{d}{dr} + V(r) \right) \psi(r) = 0 .
\]

We set

\[
\psi(r) = r^{1-\frac{N}{2}} \tilde{\psi}(r) ;
\]

and obtain

\[
\left( -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \frac{(1-\frac{N}{2})^2}{r^2} + V(r) \right) \tilde{\psi}(r) = 0 .
\]

We can define \( \tilde{V}(r) \equiv V(r) + \left(1 - \frac{N}{2}\right)^2/r^2 \), and hence again obtain the 2-D form.

The conclusion is, using (II.5), that in \( N \) dimensions, the potential

\[
V(r) = -\frac{(N-2)^2}{4r^2} - \frac{\mu}{r^2(\ln r)^2} ; \quad r > R > 1 ;
\]

\[
= 0 \quad ; \quad r \leq R ; \quad (\text{II.11})
\]

has infinitely many bound states if \( \mu > 1 \), and a finite number if \( \mu < 1 \).

This procedure can be further iterated to get more refined results.
Appendix III

Comparison of our bound on the number of $m = 0$ bound states and of the Newton-Setô bound

We wish to compare our bound

\[
N(m = 0) < 1 + I(R)
\]
\[
I(R) = \int_0^\infty r dr V^-(r) \ln \left( \frac{R}{r} \right)
\]  \hspace{1cm} (III.1)

and $I(R_{\text{min}})$ given by

\[
\int_0^{R_{\text{min}}} r V^-(r) dr = \int_{R_{\text{min}}}^\infty r V^-(r) dr ,
\]  \hspace{1cm} (III.2)

with the Newton-Setô bound:

\[
N(m = 0) < 1 + J,
\]

where

\[
J = \frac{1}{2} \int r dr V^- V^- \ln \left( \frac{r}{r'} \right) V^-(r) V^-(r') \left[ V^-(r') V^- - V^-(r) V^-(r) \right] .
\]  \hspace{1cm} (III.3)

$J$ can be rewritten as

\[
J = \frac{1}{2} \int r dr V^- I^-(r) .
\]  \hspace{1cm} (III.4)

Hence, from the mean value theorem

\[
J \geq \frac{1}{2} I(R_{\text{min}}) .
\]  \hspace{1cm} (III.5)

On the other hand, taking into account (III.2), one has, with $R > R_{\text{min}}$,

\[
I(R) = I(R_{\text{min}}) + 2 \int_{R_{\text{min}}}^R r dr V^- \ln \left( \frac{R}{r} \right) .
\]  \hspace{1cm} (III.6)

One gets
\[ I(R) < I(R_{\text{min}}) + 2\ell n \left( \frac{R}{R_{\text{min}}} \right) \int_{R_{\text{min}}}^{\infty} r \, dr \, V^-(r) \]
\[ = I(R_{\text{min}}) + \ell n \left( \frac{R}{R_{\text{min}}} \right) \int_{0}^{\infty} r \, dr \, V^-(r) . \]

The case \( R < R_{\text{min}} \) can be treated in the same way and one gets

\[ I(R) < I(R_{\text{min}}) + \left| \ell n \left( \frac{R}{R_{\text{min}}} \right) \right| \int_{0}^{\infty} r \, dr \, V^-(r) . \]  

(III.7)

Inserting in (III.4) leads to

\[ J < I(R_{\text{min}}) . \]  

(III.8)
Appendix IV

Regge trajectories for bound states

What follows here is somewhat implicit in the work of Regge [23]. We give here some details for the sake of completeness.

To find bound state energies $E = -\kappa^2$ for a given $m$ (real > 0), but not necessarily integer, we must find a solution of

$$\left[ -\frac{d^2}{dr^2} + \frac{m^2 - 1/4}{r^2} + V(r) + \kappa^2 \right] u = 0 , \quad (IV.1)$$

such that $u \to 0$ for $r \to 0$ and $r \to \infty$. For general $m$ and $\kappa$, $Re m > 0$, $Re \kappa > 0$, if

$$\int r|V(r)|dr < \infty , \quad (IV.2)$$

(IV.1) has in general two independent solutions $y$ and $z$ such that

$$y \sim r^m , \quad r \to 0$$
$$z \sim \exp(-\kappa r) \quad , \quad r \to \infty . \quad (IV.3)$$

It is then shown that both $y(m, \kappa; r)$ and $z(m, \kappa; r)$ are analytic in $m$ and $\kappa$ in \( \{Re m > 0 \otimes Re \kappa > 0\} \). The Wronskian of $y$ and $z$ is given by

$$W(y, z) \equiv yz' - y'z = F(m, \kappa) ,$$

where $F$ is analytic in the same domain. The bound state energies are given by

$$F(m, \kappa) = 0 . \quad (IV.4)$$

This defines the bound state energies as implicit functions of $m$. If $F(\bar{m}_i, \bar{\kappa}_i) = 0$, $\bar{m}_i$ and $\bar{\kappa}_i > 0$, and $(\partial/\partial \kappa)^p F = 0$, $p = 1, 2 \cdots , q - 1$, and $(\partial/\partial \kappa)^q F \neq 0$ at that point, we have $q$ different solutions in the neighbourhood of $\bar{m}_i$, $\bar{\kappa}_i$. However, this is impossible for $q \geq 2$ because there cannot be any degeneracy as a general consequence of Sturm-Liouville theory. Hence, $\kappa$ is analytic in $m$ in the neighbourhood of $\bar{m}_i$, $\bar{\kappa}_i$, and $\kappa_i$ is a real analytic
function of $m$ for $0 < m < m_i$, where $m_i$ is such that $E_i(m_i) = 0$. In addition, $\kappa_i$ is a decreasing function of $m$ since, from the Feynman-Hellmann theorem

$$\frac{dE_i}{dm} = 2m \int \frac{u_i^2}{r^2} \, dr . \quad \text{(IV.5)}$$

Let us remark here that the condition (IV.2) is certainly too strong. It is needed to ensure that $y$ and $z$ have the properties given by (IV.3). But if $V$ has strong repulsive singularities, one could approach it by $V_M, V_M = V$ if $V < M$, $V_M = M$ if $V \geq M$, and use a limiting procedure.
References

[1] F. Bassani, T. Martin, private communications.

[2] K. Chadan, N. N. Khuri, A. Martin and T. T. Wu, *Phys. Rev.* **D58** (1998) 025014.

[3] N. N. Khuri, A. Martin, P. Sabatier and T. T. Wu, in preparation.

[4] K. Chadan, *Nuovo Cimento* **58A** (1968) 191.

[5] A. Martin, *Helv. Phys. Acta* **45** (1972) 140.
    H. Tamura, *Proc. Japan. Acad.* **50** (1974) 19.

[6] E. Lieb, *Bull. Amer. Math. Soc.* **82** (1978) 751 and *Proc. A.M.S., Symp. Pure Math.* **36** (1980) 241.

[7] M. Cwikel, *Trans. AMS* **224** (1977) 93.

[8] G. V. Rozenbljum, *Dokl. AN SSSR* **202 NS** (1972) 1012 ; *Izv. VUZOV Mathematika* **N1** (1978) 75.

[9] H. Grosse, V. Glaser and A. Martin, *Comm. Math. Phys.* **59** (1978) 197.

[10] P. Li and S. T. Yau, *Comm. Math. Phys.* **88** (1983) 309 ;
    Ph. Blanchard, J. Stubbe and J. Rezende, *Lett. Math. Phys.* **14** (1987) 215 ;
    J. G. Conlon, *Rocky Mountain J. Math.* **15** (1985) 117.

[11] B. Simon, *Ann. Phys.* **97** (1976) 279.

[12] K. Yang and M. de Llano, *Am. J. Phys.* **57** (1989) 85.

[13] R. G. Newton, *J. Math. Phys.* **3** (1962) 867.

[14] N. Setô, *Publ. RIMS*, Kyoto University (1979) 429.

[15] T. Regge, *Nuovo Cimento* **14** (1959) 951.

[16] N. N. Khuri, A. Martin and T.T. Wu, *Few-Body Systems* **31** (2002) 83-89.

[17] R. Courant and D. Hilbert, Methods of Mathematical Physics, vol. I, Interscience, New York (1953) p. 454.
[18] F. Calogero, *Comm. Math. Phys.* 1 (1965) 80.
    H. E. Cohn, *J. London Math. Soc.* 40 (1965) 523.

[19] See for instance, E. Hille, *Lectures on Ordinary Differential Equations*, Addison-Wesley (1969), p. 340.

[20] J. M. Richard, private communication.

[21] V. Bargmann, *Proc. Nat. Acad. Sci. USA* 38 (1952) 96.

[22] A. Martin, *Comm. Math. Phys.* 55 (1977) 293.

[23] V. Glaser, H. Grosse, A. Martin and W. Thirring, in “Studies in Mathematical Physics”, essays in honour of V. Bargmann, E. Lieb, B. Simon, A. Wightman eds., Princeton University Press, Princeton (1978), p. 169.

[24] E. H. Lieb and W. Thirring, Ref. [22], p. 269.

[25] T. Weidel, *Comm. Math. Phys.* 178 (1996) 135.

[26] D. Hundertmark, E. H. Lieb and L. E. Thomas, *Adv. Theor. Phys.* 2 (1998) 719.

[27] M. Nieto, *Phys. Letters* 293 (2002) 10.

[28] A. Laptev, *Functional Analysis and its Applications* 34 (2000) 305.