Gravitational Lensing Effects of Fermion-Fermion Stars: I. Strong Field Case

Ke-Jian Jin\textsuperscript{1,3}, Yuan-Zhong Zhang\textsuperscript{2,1*,4}, and Zong-Hong Zhu\textsuperscript{5,1}

\textsuperscript{1}Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing, China
\textsuperscript{2}CCAST (World Laboratory), P.O. Box 8730, Beijing, China
\textsuperscript{3}Department of Physics, Northern Jiaotong University, Beijing, China
\textsuperscript{4}The State Key Lab. of Scientific and Engineering Computing, Chinese Academy of Sciences
\textsuperscript{5}National Astronomical Observatories and Beijing Astronomical Observatory, Chinese Academy of Sciences, Beijing, China

We investigate a two-component model for gravitational lenses, i.e., the fermion-fermion star as a dark matter self-gravitating system made from two kinds of fermions with different masses. We calculate the deflection angles varying from arcseconds to even degrees. There is one Einstein ring. In particular, we find three radial critical curves for radial magnifications and four or five images of a point source. These are different from the case of the one-component model such as the fermion stars and boson stars. This is due to the fermion-fermion star being a two-component concentric sphere. Our results suggest that any possible observations of the number of images more than 3 could imply a polytropic distribution of the mass inside the lens in the universe.

PSCA numbers: 98.62.Sb, 95.35.+d, 04.40.-b

It is suggested that most of the matter in the universe may be dark. Several types of dark matter distribution, such as local dark matter, galaxy dark matter, cluster dark matter and background dark matter, exist in the universe [1]. The dark matter may be consist of bosons or/and fermions [2]. Many authors have studied the dark matter self-gravitating systems, e.g., fermion stars [3], boson stars [4], boson-fermion stars [5] and fermion-fermion stars [6]. The dark matter stars could be formed by ejecting part of the dark matter, carrying out the excess kinetic energy [7]. This may also be a mechanism on the formation of such dark matter stars, though a finite temperature situation is still needed to be studied.

It is assumed that the only coupling of the dark matter stars to ordinary matter and radiation is gravitational. So the stars would be transparent which allows the light to pass through them. General relativity predicts the deflection of light in gravitational fields, which is the foundation of compact objects as gravitational lenses. The basic theory of gravitational lensing was developed by Liebes and others [8]. The first example of gravitational lensing, twin images QSO 0957+561 A,B separated by 5.7 arcseconds at the same redshift $z_s = 1.405$ and $\text{mag} \approx 17$, was discovered in 1979 [9]. In 1988 Hewitt et al. [10] observed the first Einstein ring MG1131+0456 at redshift $z_s = 1.13$. Schwarzschild gravitational lensing in the week gravitational field region is well-known [11]. Recently,
gravitational lensing effects in strong gravitational field regions of black holes, neutron stars and boson stars were discussed [12]. However, all the stars have a smooth mass distribution. In the present paper, we give a two-component model for gravitational lenses. We calculate gravitational lensing effect for the relativistic fermion-fermion stars (for the case of weak gravitational fields, see [13]). We find that there are typically four or five images of a point source, being different from the cases of both the fermion stars and boson stars. This is due to the mass distribution inside both the boson stars and fermion stars being smooth, while the fermion-fermion star is a two-component concentric sphere (a polytropic mass distribution). We also get one tangential critical curve (Einstein ring) for tangential magnification and three radial critical curves for radial magnification.

Consider a two-component model consisting of two types of fermions with different masses $m_1$ and $m_2$. Each of the two types is assumed to be a Fermi gas. For the system, Einstein’s field equations (in G=c=1 units) read

$$ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi \left[ T_{(1)\mu\nu} + T_{(2)\mu\nu} \right]. \quad (1) $$

The assumption of the only gravitational interaction inside the system implies the covariant conservation equations,

$$ T_{(i)\mu\nu} = 0, \quad (i = 1, 2), \quad (2) $$

where

$$ T_{(i)\mu\nu} = \text{diag}(-\rho_i, p_i, p_i, p_i) \quad (3) $$

is the energy–momentum tensor for the $i$-th Fermi gas. What is more, the equations of state (in the parametric form) will be

$$ \rho_i = K_i (\text{sh} t_i - t_i), \quad (4) $$

$$ p_i = \frac{1}{3} K_i \left( \text{sh} t_i - 8\text{sh} \frac{1}{2} t_i + 3t_i \right), \quad (5) $$

where

$$ K_i \equiv \pi m_i^4 / 4\hbar^3. \quad (6) $$

Let

$$ K_2 = \frac{1}{4\pi}, \quad (7) $$

then the unit of length is

$$ \xi = \frac{1}{\pi} \left( \frac{\hbar}{m_2 c} \right)^{3/2} \frac{c}{(m_2 G)^{1/2}} = \frac{392}{[m_2 (\text{eV})]^{1/2}} \text{ kpc}. \quad (8) $$

Correspondingly the unit of mass is

$$ \eta = \frac{c^2 a}{G} = 8.18 \times 10^{18} \left[ m_2 (\text{eV}) \right]^2 \text{ M}_\odot. \quad (9) $$
For the polytropic sphere model, the general metric takes the form
\[ ds^2 = e^\nu dt^2 - e^\mu dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \] (10)
with \( \nu \) and \( \mu \) being functions of the radial distance \( r \) from the center of the star. Finally, we get the basic equations as follows:
\[
\frac{dt_i}{dr} = -4 \frac{M + 4\pi r^3 p}{r(r - 2M)} \text{cth} \frac{1}{4} t_i, \quad \frac{dM_i}{dr} = k_i r^2 (\text{sh} t_i - t_i),
\] (11)
and
\[
e^\mu = (1 - 2M/r)^{-1}, \quad \frac{de^\nu}{dr} = \frac{M + 4\pi r^3 p}{r(r - 2M)}.
\] (12)
where
\[
M = M_1 + M_2, \quad p = p_1 + p_2,
\] (13)
with the boundary conditions:
\[
t_i(r = 0) = t_{i0} > 0, \quad M_i(r = 0) = 0,
\] (14)
and
\[
t_i(r = R_i) = 0
\] (15)
where \( R_i \) is the radius of the \( i \)-th fermion sphere.

A light ray would be deflected by gravitational field, and then the deflection angle of which is given by [14]
\[
\hat{\alpha}(r_0) = \Delta \phi(r_0) - \pi,
\] (16)
with
\[
\Delta \phi(r_0) = 2 \int_{r_0}^{\infty} \frac{e^{\mu/2}}{\sqrt{(r^4/b^2)} e^{-\nu} - r^2} dr.
\] (17)
where \( r_0 \) is the closest distance between the light ray and the center of the gravitational force, and the impact parameter \( b \) is defined by
\[
b = r_0 \text{exp}[-\nu(r_0)/2].
\] (18)

For the case of the light ray deflected by a fermion-fermion star, the equation (17) becomes
\[
\Delta \phi(r_0) = 2 \int_{r_0}^{R} \frac{e^{\mu/2}}{\sqrt{(r^4/b^2)} e^{-\nu} - r^2} dr + 2 \int_{R}^{\infty} \frac{1}{\sqrt{(r^4/b^2) - r^2 [1 - 2M(R)/r]}} dr.
\] (19)
where \( R \) is the radius of the star.

An observer \( O \) and a point source \( S \) are assumed to be located in an asymptotically flat spacetime far away from a fermion-fermion star (sa a lens \( L \)). Let \( D_{ol} \) denote the distance from the observer to the center of the lens, \( D_{ls} \) the distance between the lens and the source, and \( D_{os} \) the distance between the observer
and the source. An image position is specified by the angle $\theta$ between OL and the tangent to the null geodesic at the observer. $\beta$ stands for the true angular position of the source. The lens equation may be expressed as [12]

$$\sin(\theta - \beta) = \frac{D_{ls}}{D_{os}} \sin \hat{\alpha}. \quad (20)$$

From the geometry of the lens we have

$$\sin \theta = \frac{b}{D_{ol}} = \frac{r_0}{D_{ol}} e^{-\nu(r_0)/2}. \quad (21)$$

The magnification of images is given by

$$\mu = \left( \frac{\sin \beta \, d\beta}{\sin \theta \, d\theta} \right)^{-1}. \quad (22)$$

The tangential and radial critical curves (TCC and RCC, respectively) follow from the singularities of the tangential

$$\mu_t \equiv \left( \frac{\sin \beta}{\sin \theta} \right)^{-1} \quad (23)$$

and the radial magnification

$$\mu_r \equiv \left( \frac{d\beta}{d\theta} \right)^{-1}. \quad (24)$$

Then we have from (20)

$$\mu_r^{-1} = \frac{d\beta}{d\theta} = 1 - \frac{D_{ls}}{D_{os}} \cos \hat{\alpha} \frac{d\hat{\alpha}}{dr_0} \, dr_0 \, d\theta \quad (25)$$

where $\cos(\theta - \beta)$ can be found from (20), $dr_0/d\theta$ can be got from the derivative of (21) with respect to $r_0$,

$$\frac{dr_0}{d\theta} = \frac{2D_{ol} e^{\nu(r_0)/2} \sqrt{1 - \frac{r_0^2}{D_{ol}^2} e^{-\mu(r_0)}}}{2 - r_0 \frac{dr(r_0)}{dr_0}}, \quad (26)$$

and $d\hat{\alpha}/dr_0$ can be calculated by parametric differential of (16) and (17) with respect to $r_0$.

We calculated numerically the angle $\beta$ as a function of the angle $\theta$, and the tangential and radial magnifications. In the present paper, we consider the maximal fermion-fermion star. The relevant parameters of the star are: the mass ratio for the two kinds of fermions $m_2/m_1 = 5$, the maximal central gravitational redshift $z_c = 1.22$, the total mass $M = 1.73\eta$, and the radius $R = 16.6\xi$; By using the definitions (9) and (8) for $\eta$ and $\xi$, the total mass of the maximal star is $M = 1.42 \times 10^{19}M_\odot/[m_2(eV)]^2$; The radius is $R = 6.51\text{Mpc}/[m_2(eV)]^2$; For
example, in case of \( m_2 = 10\text{eV} \) and then \( m_1 = 2\text{eV} \), \( M = 1.42 \times 10^{17}\text{M}_\odot \) and \( R = 65.1 \times \text{kpc}; \) for \( m_2 = 10\text{GeV} \) and then \( m_1 = 2\text{GeV} \), \( M = 0.14\text{M}_\odot \) and \( R = 2.0\text{km} \).

The angular position \( \beta \) of the point source, the tangential magnification \( \mu_t \), and the radial magnification \( \mu_r \) are plotted against the angular position \( \theta \) of the image, which being shown in figures 1–4.

In Fig. 1 (it is assumed that \( D_{ls}/D_{os} = 1/2 \) and \( D_{ol} = 5 \times 10^5\xi \)), the continuous curve denotes the plot of the source position angle \( \beta \) against the image position angle \( \theta \), and the lines of \( \beta = \text{constants} \) are given by the dashed straight lines. The intersections between the lines (with \( \beta = \text{constants} \)) and the continuous curve indicate the angular positions of the images and their numbers from 1 to 5, of which the ones with \( \theta \neq 0 \) and \( \beta = 0 \) present the Einstein ring that corresponds to the tangential critical curve (TCC) coming from the singularity of \( \mu_t \) in Fig. 2. The continuous curve with \( \beta > 0 \) or \( \beta < 0 \) shows two peaks following from the two-component concentric sphere structure of the fermion-fermion star, so that the numbers of the images may be 4 or 5. The maximal deflection angle \( \hat{\alpha}_{\text{max}} \) corresponds to the maximum (28 degrees) of \( \beta \) at about \( \theta = 0.25 \text{ arcseconds} \), and then the maximal reduced deflection angle \( |\alpha| = |\theta - \beta| \simeq 28^\circ \). From (20) and because of \( \theta \ll \beta \) and \( D_{ls}/D_{os} = 1/2 \), we have the maximal deflection angle \( \hat{\alpha}_{\text{max}} \simeq \sin^{-1}(2\sin 28^\circ) \simeq 70^\circ \).

In Fig. 2 the tangential magnification \( \mu_t \) is plotted as a function of the image position \( \theta \). The singularity in the magnification \( \mu_t \) shows the angular position of the Einstein ring (TCC), which is at about \( \theta = 542 \text{ arcseconds} \).

Figure 3 gives the radial magnification \( \mu_r \) as a function of \( \theta \). There are three singularities in the magnification \( \mu_r \) corresponding to three extreme values (i.e., two maxima and a minimum) on the the continuous curve of \( \beta(\theta) > 0 \) in Fig. 1, which indicate the angular positions of the three radial critical curves (RCCs) at about \( \theta = 0.254, 0.883, 3.811 \text{ arcseconds} \), respectively.

Comparing to the one-component models such as the boson stars [12] and the fermion stars, the results from figures 1 and 3 give us the difference: There may be four or five images of the point source; At the same time, there are three radial critical curves. This is due to that the mass distribution for the fermion-fermion stars are generally not smooth but has two-component structure. Our results suggest that any possible observations of the number of images more than 3 could imply a polytropic distribution of the mass inside the lens in the universe.

Finally we want to say that the function \( \beta(\theta) \) depends upon the values of \( D_{ls}/D_{os} \) and \( D_{ol} \), \( \beta \) being smaller as \( D_{ol} \) larger. In figure 4 we give, as an example, the source position angle \( \beta \) as a function of the image position \( \theta \) for the case of \( D_{ol} = 1.5 \times 10^{11}\xi \) and \( D_{ls}/D_{os} = 1/2 \), where the values of \( \beta \) and \( \theta \) are of the same order in arcseconds.

This work was supported partially by The National Natural Science Foundation of China under Grants Nos. 19745008 and 19835040.
[1] See, e.g., M.S. Turner, in the Proceedings of the Second International Workshop on Particle Physics and the Early Universe, Asilomar, USA, Nov. 15–20, 1998.

[2] L. Roszkowski, in the Proceedings (see [1]); [hep-ph/9903467].

[3] M.A. Markov, Phys. Lett. 10 122 (1964); J.G. Gao and R. Ruffini, Acta Astrophys. Sin. 1 (1981) 19; C.R. Ching, T.H. Ho and Y.Z. Zhang, Commun. in Theor. Phys. (China) 2 1145 (1983).

[4] M. Colpi, S.L. Shapiro and I. Wasserman, Phys. Rev. Lett. 57 2485 (1986).

[5] A.B. Henriques, A.R. Liddle and R.G. Moorhouse, Phys. Lett. B 233 99 (1989); Nucl. Phys. B337 737 (1990).

[6] Y.Z. Zhang and K.J. Jin, Phys. Lett. A128 309 (1988); K.J. Jin and Y.Z. Zhang, Phys. Lett. A142 79 (1989).

[7] E. Seidel and W.M. Suen, Phys. Rev. lett. 72 2516 (1994).

[8] Jr.S. Leibes, Phys. Rev. B133 835 (1964); S. Refsdal and J. Surdej, MN-RAS 128 295 (1964); R.R. Bourassa and R. Kantowski, APJ 195 13 (1975); for review see: R. Narayan and M. Bartelmann, Lectures on gravitational lensing, [astro-ph/9606001].

[9] D. Walsh, R.F. Carswell, and R.J. Weymann, Nature 279 381 (1979).

[10] J.N. Hewitt, et al., Nature 333 537 (1988).

[11] P. Schneider, J. Ehlers, and E.E. Falco, 1992, Gravitational lenses, Springer Verlag, Berlin.

[12] K.S. Virbhadra, D. Narasimha, and S.M. Chitre, Role of the scalar field in gravitational lensing, [astro-ph/9801174]. M.P. Dabrowski and F. Schunck, Boson stars as gravitational lenses, [astro-ph/9807039]. K.S. Virbhadra and G.F.R. Ellis, Schwarzschild black hole lensing, [astro-ph/9904193].

[13] Z.H. Zhu, K.J. Jin, and Y.Z. Zhang, Gravitational lensing effects of fermion-fermion stars: II. weak field case, submitted.

[14] S. Weinberg, 1972, Gravitation and cosmology: principles and applications of the general theory of relativity, John Wiely & Sons, NY.
FIG. 1. Gravitational lensing for the maximal fermion-fermion star. The true angular position $\beta$ of the point source is plotted as a function of the image position $\theta$ with $(D_{ls}/D_{os}) = 1/2$ and $D_{ol} = 10^5 \xi$, which is given by the continuous curve. The values of $\theta$ corresponding to the interactions between the dashed lines $\beta = \text{constants}$ and the continuous curve indicate the image positions; It is shown that the number of the images may be from 1 to 5. The interactions of the straight line $\beta = 0$ with the curve present the Einstein ring at $|\theta| = 542$ arcsecs. Note that the curve with $\beta > 0$ or $\beta < 0$ shows two peaks which come from the two-component structure of the fermion-fermion star.

FIG. 2. Corresponding to the curve in Fig. 1, the tangential magnification $\mu_t$ is plotted as a function of the image position $\theta$.

FIG. 3. The radial magnification $\mu_r$ as a function of $\theta$. Because of three extreme values (i.e., two maxima and a minimum) for the the continuous curve with $\beta(\theta) > 0$ in Fig. 1, there are three singularities in the magnification $\mu_r$.

FIG. 4. Gravitational lensing for the maximal fermion-fermion star in the case of $(D_{ls}/D_{os}) = 1/2$ and $D_{ol} = 1.5 \times 10^{11} \xi$. The angular positions of the point source and images are all of the same order in arcseconds.
Fig. 1
Fig. 3
Fig. 4