An effective criterion for algebraicity of rational normal surfaces

Pinaki Mondal
Weizmann Institute of Science
pinaki@math.toronto.edu

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Abstract

We give a novel and effective criterion for algebraicity of rational normal analytic surfaces constructed from resolving the singularity of an irreducible curve-germ on $\mathbb{P}^2$ and contracting the strict transform of a given line and all but the ‘last’ of the exceptional divisors. As a by-product we construct a new class of analytic non-algebraic rational normal surfaces which are ‘very close’ to being algebraic. These results are local reformulations of some results in [Mon11a] which sets up a correspondence between normal algebraic compactifications of $\mathbb{C}^2$ with one irreducible curve at infinity and algebraic curves in $\mathbb{C}^2$ with one place at infinity. This article is meant partly to be an exposition to [Mon11a] and we give a proof of the correspondence theorem of [Mon11a] in the ‘first non-trivial case’.

1 Introduction

In [Mon11a] we give an explicit criterion to determine when a normal analytic compactification of $\mathbb{C}^2$ with an irreducible curve at infinity of $X$ is algebraic. The geometric counterpart of this criterion is a correspondence between the following categories of objects:

\[
\begin{array}{c}
\text{normal algebraic compactifications of } \mathbb{C}^2 \\
\text{with one (irreducible) curve at infinity}
\end{array} \longleftrightarrow \begin{array}{c}
\text{algebraic curves in } \mathbb{C}^2 \\
\text{with one place at infinity}
\end{array}
\] (1)

(recall that ‘one place’ means only one branch which is analytically irreducible). In this article we reformulate the results in the local setting and describe some of the (hopefully interesting) consequences. We also give a proof of the results under a (greatly) simplifying assumption (which however applies to most of the examples we consider in this article). The paper consists of two parts which can be read more or less independently: the (rest of the) Introduction and Section 2 deals with the local setting, whereas in Sections 3 and 4 we give a complete statement of the main correspondence result in the global setting and give a proof under the simplifying assumption mentioned above.

1.1 Introduction to the problem

Fix a line $L \subseteq \mathbb{P}^2$ and let $\pi : Y \to \mathbb{P}^2$ be a birational morphism of non-singular algebraic surfaces. Fix an irreducible component $E^*$ of the exceptional divisor $E$ of $\pi$, and let $\tilde{E}$ be the union of the strict transform $\tilde{L}$ of $L$ with all components of $E$ except for $E^*$.

**Question 1.1.** When is $\tilde{E}$ contractible, i.e. when does there exist a proper surjective morphism $\tilde{\pi} : Y \to \tilde{Y}$ of normal analytic surfaces such that $\tilde{\pi}(\tilde{E})$ is a point in $\tilde{Y}$ and $\tilde{\pi}$ restricts to an isomorphism on $Y \setminus \tilde{E}$?

**Question 1.1’.** When is $\tilde{E}$ algebraically contractible, i.e. when does there exist $\tilde{Y}$ as in the preceding question such that $\tilde{Y}$ is also algebraic?

It follows from a criterion of Grauert [Gra62] that the answer to Question 1.1 is affirmative iff the matrix of intersection numbers of the irreducible components of $\tilde{E}$ is negative definite, or equivalently, as we showed in [Mon11b], iff the valuation corresponding to $E^*$ is positively skewed in the sense of [FJ07] as a valuation centered at infinity with respect to $\mathbb{P}^2 \setminus L$ (see Proposition 2.6 for an explicit version). In particular, the answer to Question 1.1 depends only on the configuration of the curves in $\tilde{E}$. The answer to Question 1.1’ however is more delicate, as the following example shows.
Example 1.2. Consider the set up of Question 1.1. Let \( O \subset \mathbb{P}^2 \) and \((u,v)\) be a system of affine coordinates at \( O \) (‘affine’ means that both \( u = 0 \) and \( v = 0 \) are lines on \( \mathbb{P}^2 \)) such that \( L = \{ u = 0 \} \). Let \( C_1 \) and \( C_2 \) be curve-germs at \( O \) defined respectively by \( f_1 := v^5 - u^3 \) and \( f_2 := (v - u^2)^5 - u^3 \). Note that \( C_j \)'s are isomorphic as curve-germs via the map \((u,v) \mapsto (u,v-u^2)\). For each \( i \), let \( Y_i \) be the surface constructed by resolving the singularity of \( C_i \) at \( O \) and then blowing up 8 more times the point of intersection of the strict transform of \( C_i \) with the exceptional divisor. Let \( E_i^* \) be the last exceptional divisor, and let \( E_i \) be the union of the strict transform \( L_i \) (on \( Y_i \)) of \( L \) and (the strict transforms of) all but the last of the exceptional divisors. It is straightforward to check that both \( E_i \) have the same dual graph (i.e. the graph whose vertices are the irreducible components of \( E_i \) and there is an edge between two vertices iff corresponding curves intersect) and are analytically contractible. Figure 1 depicts the dual graph of \( E_i \); note that we labelled the vertices according to the order of appearance of corresponding curves in the sequence of blow-ups. Below we list some other common properties of \( E_1 \) and \( E_2 \).

1. Removing (from the dual graph) the vertex corresponding to \( E_{11} \) turns it into the resolution graph of a rational singularity.
2. Removing the vertex corresponding to \( L \) turns it into the resolution graph of a sandwiched singularity (which is a special class of rational singularities - see \[Spi90\]).
3. The normal analytic surface \( \hat{Y}_i \) constructed from blowing down \( E_i \) has a trivial canonical sheaf and a unique singular point which is almost rational in the sense of \[Ném07\]. However, it turns out that \( \hat{Y}_1 \) is algebraic, but \( \hat{Y}_2 \) is not (see Example 2.8). In the algebraic case, it can also be shown that the image of \( E_i^* \) on \( \hat{Y}_1 \) is non-singular; we don’t know what happens in the non-algebraic case.

Note that Property 2 of the resolution graph of Figure 1 in fact holds true in general in the set up of Question 1.1 (so that the singularities on \( \hat{Y} \) resulting from contraction of \( \hat{E} \) are almost sandwiched). Indeed, removing the vertex corresponding to \( L \) and then adding a vertex corresponding to \( E^* \) produces the dual graph of \( E \) (using the notation of the set up of Question 1.1), which is simply the exceptional divisor of \( \pi \). Then Property 2 follows from the definition of sandwiched singularities \[Spi90\] Definition 1.9], namely a singularity is sandwiched iff the dual graph of its resolution is a part of the dual graph of the exceptional divisor of a morphism between non-singular surfaces.

We give two versions of the answer to Question 1.1: a geometric, but non-effective version (Theorem 2.1) as depicted in Figure 2 and an effective version; to avoid being redundant, we state (and prove) the effective version only for the simplest case (Theorem 2.7) and give the complete statement only for the global version (Theorem 3.3). The effective answer is especially useful to construct new classes of non-algebraic analytic (normal) rational surfaces - see also Remark 2.14 and Remark-Example 2.15. Since having only rational singularities implies algebraicity

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\[\text{Note: The examples in the existing literature (that we know of) of constructions (e.g. in \[Gra62\]) of non-algebraic normal Moishezon surfaces (such as}\]
corresponding to $E \nu := Palka$ for enlightening discussions. This exposition is in a large part motivated by their suggestions. I would like to thank Professor Peter Russell for the generosity with his time and very helpful remarks and Karol Palka for enlightening discussions. This exposition is in a large part motivated by their suggestions.

1.2 Acknowledgements

It is well known (and also illustrated by Example 1.2) that in general algebraicity can not be determined only from the dual graph of the exceptional divisor of the resolution of singularities. However, in the set up of Question 1.1 we can completely classify (in terms of two semigroup conditions) dual graphs of $E$ which correspond to only algebraic contractions, those which correspond to only non-algebraic contractions, and those which correspond to both types of contractions (Theorem 2.10).

The problem of determining algebraicity of a (compact) analytic surface (or more generally, variety) $Y$ has been extensively studied. [Gra62 Satz 2] gives a criterion in terms of the existence of a positive holomorphic line bundle on $Y$. On the other hand, a necessary requirement for $Y$ to be algebraic is that the transcendence degree over $C$ of the meromorphic function field of $Y$ should equal $\dim(Y)$, in which case $Y$ is said to be a Moishezon space. It was shown in [Art62] that a normal Moishezon surface with at most rational singularities is projective. There have been a number of other works which give criteria for analytic surfaces to be algebraic, see e.g. [MR75], [Bre77], [FJ99], [Sch00], [Bäd01], [Pal12]. However, all the criteria (for algebraicity) that appear in the literature are given in terms of cohomological or analytic invariants which are not suitable for examining the set-up of Question 1.1 Our `geometric criterion' (see also Remark 2.3) is stated in terms of the existence of a certain kind of divisors and has in a sense the same spirit as the criteria of [Sch00, Theorem 3.4] and [Pal12, Corollary 2.6]. Our `effective criterion' states that $\bar{Y}$ of Question 1.1 is algebraic iff a certain element of $C[x, x^{-1}, y]$ we compute from the input data of Question 1.1 is in fact an element of $C[x, y]$ (Theorem 3.3). To our knowledge this type of criterion for algebraicity does not exist in the literature - it would certainly be interesting to relate it to classical invariants.

Finally, we point out that if we identify (in the set up of Question 1.1) $\mathbb{P}^2 \setminus L$ with $C^2$, then the geometric criterion (Theorem 2.1) for algebraicity of $\bar{Y}$ of Question 1.1 is precisely the existence of a certain algebraic curve in $C^2$ with one place at infinity on $C^2$ (it is more explicit in the global version - Theorem 3.2). Moreover, Theorem 3.2 has an (almost immediate) translation in the terminology of valuative tree $\mathcal{P}[304]$ which we now describe. In the set up of Question 1.1 let $X := \mathbb{P}^2 \setminus L \cong C^2$ and $\nu$ be the divisorial valuation (see Definition 3.1) on $C(X)$ corresponding to $E^*$. Choose polynomial coordinates $(x, y)$ on $X$. Then the valuative tree at infinity $V_0$ on $C[x, y]$ is the space of all valuations $\mu$ on $C[x, y]$ such that $\min\{\mu(x), \mu(y)\} = -1$. It turns out that $V_0$ has the structure of a tree with root at $-\deg_{(x,y)}$ (C.07, Section 7.1), where $\deg_{(x,y)}$ is the usual degree in $(x, y)$-coordinates. Let $\nu := \nu/\max\{-\nu(x), -\nu(y)\}$ be the `normalized' image of $\nu$ in $V_0$.

**Theorem 1.3** (A corollary of Theorem 3.2). Assume $\bar{Y}$ of Question 1.1 exists. Then it is algebraic iff there is a tangent vector $\tau$ of $\nu$ on $V_0$ such that

1. $\tau$ is not represented by $-\deg$, and
2. $\tau$ is represented by a curve valuation corresponding to an algebraic curve with one place at infinity.

This correspondence between algebraicity of $\bar{Y}$ and existence of plane curves with one place at infinity is also evident in the comparison of the semigroup conditions. More precisely, it is possible to encode the input data for Question 1.1 in terms of a curve-germ $C$ at $O$ and a positive integer $r$ (see Subsection 2.2). Then we show that for a fixed $r$ and a fixed singularity type (of plane curve-germs), there is a curve-germ $C$ with the given singularity type such that the corresponding $\bar{Y}$ is algebraic, iff the sequence of virtual poles (Definition 2.9) satisfies a `semigroup condition'. On the other hand, it follows from a fundamental result (developed in AM73, Abh77, Abh78, SS94) of the theory of plane curves with one place at infinity that the same semigroup condition implies the existence of a plane algebraic curve $\bar{C}$ with one place at infinity with `almost' the given singularity type at infinity. Moreover, if the curve $\bar{C}$ exists, then the virtual poles are (up to a constant factor) precisely the generators of the semigroup of poles at the point at infinity of $\bar{C}$ - i.e. in this case virtual poles are real! We refer to Subsection 2.3 for details.

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involve contraction of non-rational curves from smooth surfaces. On the other hand, the non-algebraic surfaces emanating from negative answers to Question 1.1 come from contraction of rational trees.

We note however that there are numerical criteria (e.g. in [Art66]) applicable in our setting to determine if the singularities of the surface $\bar{Y}$ of Question 1.1 are rational - but in general (e.g. in Example 1.2) $\bar{Y}$ will have non-rational singularities, so that these tests do not apply.
2 Algebraicity in the local setting

2.1 Geometric criterion for algebraicity

We use here the notations and set-up of Question 1.1 and give the geometric answer.

Theorem 2.1 (Geometric criterion for algebraic contractibility). Assume $\tilde{E}$ is contractible, i.e. there exists $\tilde{Y}$ as in Question 1.1. Then $\tilde{Y}$ is algebraic iff there is a compact algebraic curve $C \subseteq Y \setminus E'$ such that $C$ has only one place at $E \cup L$, where $E'$ is the connected component of $E \cup L$ that contains $E''$ (see Figure 2).

Remark 2.2. The phrase ‘$C$ has only one place at $E \cup \tilde{L}$‘ (which is essentially the ‘essence’ of Theorem 2.1) means that $C$ intersects $E \cup \tilde{L}$ at only one point $P$ and $C$ is analytically irreducible at $P$. Identifying $Y \setminus (E \cup \tilde{L})$ with $\mathbb{C}^2$, this is equivalent to saying that $C \cap \mathbb{C}^2$ has one place at infinity. This observation sets up the correspondence and provides the equivalence between Theorem 2.1 and its ‘global’ incarnation (Theorem 3.2).

Remark 2.3. The non-trivial part of Theorem 2.1 is the condition ‘only one place at $E \cup \tilde{L}$‘ which is not hard to show. Indeed, here we sketch a proof. If $\tilde{Y}$ is algebraic, then there exists a compact algebraic curve $C \subseteq \tilde{Y}$ which does not pass through $P := \tilde{\pi}(\tilde{E}) \in \tilde{Y}$, which implies that $\tilde{\pi}^{-1}(C)$ does not intersect $\tilde{E} \supseteq E'$. For the opposite implication, consider the surface $Y'$ obtained from $Y$ by contracting all the components of $E$ other than $E'$ (the contraction is possible due to Grauert’s criterion). The singularities of $Y'$ are sandwicched, since there is a morphism $Y' \to \mathbb{P}^2$. Since sandwicched singularities are rational [Lip69] Proposition 1.2, a criterion of Artin [Art02] implies that $Y'$ is projective. Let $C$ be a closed algebraic curve on $Y'$ which does not intersect $E'$ and let $C'$ (resp. $L'$, $E''$) be the image of $C$ (resp. $\tilde{L}$, $E''$) on $Y'$. Then $C'$ is linearly equivalent (as a $\mathbb{Q}$-Cartier divisor) to $rL' + r_*E''$ for some $r, r_* \in \mathbb{Q}_{>0}$ and therefore a theorem of Zariski-Fujita [Laz04] Remark 2.1.32 implies that for some $m \geq 1$, the line-bundle $O_{Y'}(mC')$ is base-point free. Let $\tilde{Y}'$ be the image of the morphism defined by sections of $O_{Y'}(mC')$. Since $C'$ does not intersect $L'$, it follows that $L'$ maps to a point in $\tilde{Y}'$, and therefore $\tilde{Y} \cong \tilde{Y}'$. Consequently, $\tilde{Y}$ is projective, and in particular, algebraic.

2.2 Effective criterion for algebraicity (in a simple case)

In this subsection we state the effective version of Theorem 2.1 in the simplest case (Theorem 2.7). We start with a discussion of a way to encode the input data of Question 1.1 in terms of a germ of a curve (and a positive integer).

We continue to use the notations of Subsection 1.1. At first note that in the set up of Question 1.1 we may w.l.o.g. assume the following

1. $\pi$ is a sequence of blow-ups such that every blow-up (other than the first one) is centered at a point on the exceptional divisor of the preceding blow-up.
2. $E^*$ is the exceptional divisor of the last blow-up.

Now assume the above conditions are satisfied. Let

$\hat{C} := \text{an analytic curve germ at a generic point on } E^* \text{ which is transversal to } E^*$,
$C := \pi(\hat{C})$,

$r := \text{number of total blow-ups in } \pi \text{ } - \text{ the minimum number of blow-ups after which the strict transform of } C \text{ transversally intersects the union of the strict transform of } L \text{ and the exceptional divisor}$.

It is straightforward to see that $L, \hat{C}$ and $r$ uniquely determine $Y, E^*$ and $\tilde{E}$ via the following construction:

Construction of $Y, E^*$ and $\tilde{E}$ from $(L, C, r)$:

$Y := \text{the surface formed by at first constructing (via a sequence of blow-ups) the minimal resolution of the singularity of } C \cup L \text{ and then blowing up the point of intersection of the strict transform of } C \text{ and the exceptional divisor } r \text{ more times},$

$E^* := \text{the ‘last’ exceptional divisor, i.e. the exceptional divisor of the last of the sequence of blow-ups in the construction of } Y^*$,
Proposition 2.6. This is an immediate corollary of [Mon11b, Corollary 4.11 and Remark-Definition 4.13].

Proof. Indeed, ord $\phi$ is the polydromy order of $\phi$ (Definition 3.5) at $p$, which is defined by

$$\alpha_{L,C,r} := \text{intersection multiplicity at } O \text{ of } C \text{ and a curve-germ with Puiseux expansion}$$

$$v = [\phi(u)]_{<q_1+r} + \xi^* u^{(q_1+r)/p} + \text{h.o.t. for a generic } \xi^* \in \mathbb{C}$$

$$= p \left( (p \cdots p_1 - p_2 \cdots p_1) \frac{q_1}{p_1} + (p_2 \cdots p_1 - p_3 \cdots p_1) \frac{q_2}{p_1 p_2} + \cdots + (p_{l-1} p_1 - p_l) \frac{q_{l-1}}{p_1 \cdots p_{l-1}} + (p_l - 1) \frac{q_l}{p_1 \cdots p_l} \right) + q_l + r,$$

where $p := \text{polydromy order of } \phi$ (Definition 3.5).

Grauert’s criterion for contractibility translates (after some work) into the following in the set up of Question 2.4. This is an immediate corollary of [Mon11b, Corollary 4.11 and Remark-Definition 4.13].

Proposition 2.6. $\tilde{E}_{L,C,r}$ is contractible iff $\text{ord}_u(\phi) < 0$ and $\alpha_{L,C,r} < p^2$.

Now we give our criterion for algebraic contractibility in the case that $C$ has only one Puiseux pair, i.e. $l = 1$.  

Figure 3: Formulation of Question 1.1 in terms of $(L, C, r)$
Theorem 2.7 (Effective criterion for algebraic contractibility when \( l = 1 \)). Let \((L, C, r)\) be as in Question 2.4. Assume that the Puiseux expansion \( v = \phi(u) \) of \( C \) at \( O \) has only one Puiseux pair \((q, p)\). Let \( \omega \) be the weighted order on \( C(u, v) \) which gives weights \( p \) to \( u \) and \( q \) to \( v \). Let \( f(u, v) \) be the (unique) Weierstrass polynomial in \( v \) which defines \( C \) near \( O \). Define \( \tilde{f} \) to be the sum of all monomial terms of \( f \) which have \( \omega \)-value less than \( \alpha_{L,C,r} = pq + r \). Then \( \tilde{E}_{L,C,r} \) is algebraically contractible iff it is contractible and \( \deg_{(u,v)}(\tilde{f}) \leq p_1 \), where \( \deg_{(u,v)} \) is the usual degree in \((u,v)\)-coordinates.

We prove Theorem 2.7 in Subsection 4.2.

Example 2.8 (Continuation of Example 1.2 - see also Remark 2.14). Let \( L \) and \( C_1 \) and \( C_2 \) be as in Example 1.2. We consider Question 2.4 for \( C_1 \) and \( C_2 \) and \( r \geq 0 \) (Example 1.2 considered the case \( r = 8 \)). Figure 4 depicts the dual graph \( \tilde{E}_{L,C_1,r} \); in particular \( \tilde{E}_{L,C_1,r} \) is disconnected for \( r = 0 \).

![Dual graph of \( \tilde{E}_{L,C_1,r} \)](image)

Recall that \( C_i \)'s are defined by \( f_i = 0 \), with \( f_1 := v^5 - u^3 \) and \( f_2 := (v - u^2)^5 - u^3 \). It follows that the Puiseux expansions in \( u \) for each \( C_i \) has only one Puiseux pair, namely \((3, 5)\). Moreover, each \( f_i \) is a Weierstrass polynomial in \( v \), so that we can use Theorems 2.6 and 2.7 to determine contractibility and algebraic contractibility of \( \tilde{E}_{L,C_i,r} \).

Identity 2 implies that \( \alpha_{L,C_i,r} = pq + r = r + 15 \) for each \( i = 1, 2 \), and therefore Theorem 2.6 implies that \( \tilde{E}_{L,C_i,r} \)'s are contractible iff \( r < p^2 - pq = 10 \). We now determine if the contractions are algebraic. The weighted degree \( \omega \) Theorem 2.7 is the same for both \( i \)'s, and it corresponds to weights 5 for \( u \) and 3 for \( v \). The \( f \) of Theorem 2.7 (computed from \( f_i \)'s) are as follows:

\[
\tilde{f}_1 = \begin{cases} 
0 & \text{if } r = 0, \\
v^5 - u^3 & \text{if } r \geq 1.
\end{cases}
\]

\[
\tilde{f}_2 = \begin{cases} 
0 & \text{if } r = 0, \\
v^5 - u^3 & \text{if } 1 \leq r \leq 7, \\
v^5 - u^3 - 5v^4u^2 & \text{if } 8 \leq r \leq 9.
\end{cases}
\]

Theorem 2.7 therefore implies that \( \tilde{E}_{L,C_1,r} \) is algebraically contractible for all \( r < 10 \), but \( \tilde{E}_{L,C_2,r} \) is algebraically contractible only for \( r \leq 7 \). In particular, for \( r = 8, 9 \), the contraction of \( \tilde{E}_{L,C_2,r} \) produces a normal non-algebraic analytic surface.

2.3 The semigroup conditions on the sequence of virtual poles

In this subsection we define the sequence of ‘virtual poles’ corresponding to a curve-germ and state two ‘semigroup conditions’ on these sequences. For a given singularity type (and a given \( r \)), if the virtual poles satisfy the first semigroup condition, this implies the existence of a curve-germ \( C \) (with the prescribed singularity type) such that \( \tilde{E}_{L,C,r} \) is algebraically contractible. On the other hand, satisfying both semigroup conditions ensures that \( \tilde{E}_{L,C,r} \) are algebraically contractible for all curves \( C \) with the given singularity type. The first semigroup condition is precisely the classical semigroup condition satisfied by generators of the semigroup of poles of a plane curve with one place at infinity.

We continue to use the notations of the set-up of Subsection 2.2; in particular, we assume that the Puiseux expansion for \( C \) is \( v = \phi(u) \) with Puiseux pairs (Definition 3.5) \((q_1, p_1), \ldots, (q_l, p_l)\) with \( l \geq 1 \). Define \( C_0 := L = \{u = 0\} \), and for each \( k, 1 \leq k \leq l \), let \( C_k \) be the curve-germ at \( O \) with the Puiseux expansion \( v = \phi_k(u) \), where \( \phi_k(u) \) is the Puiseux series (with finitely many terms) consisting of all the terms of \( \phi \) upto, but not including, the \( k \)-th characteristic exponent. Then it is a standard result (see e.g. [CA00, Lemma 5.8.1]) that \( m_k := (C, C_k) \), \( 0 \leq k \leq l \), are generators of the semigroup \( \{(C, D) \} \) of intersection numbers at \( O \), where \( D \) varies among analytic
curve-germs at \( O \) not containing \( C \). It follows from a straightforward computation that

\[
m_0 = p_1 \cdots p_l, \quad m_1 = q_1 p_2 \cdots p_l, \quad \text{and} \quad m_k = p_k \left( \frac{q_1}{p_1} + \frac{q_2}{p_1 \cdots p_{k-1}} \right), \quad 2 \leq k \leq l.
\]

**(Definition 2.9)** (Virtual poles). Let

\[
\tilde{l} := \begin{cases} 
    l - 1 & \text{if } r = 0, \\
    l & \text{if } r > 0.
\end{cases}
\]

The sequence of virtual poles at \( O \) on \( C \) are \( \tilde{m}_0, \ldots, \tilde{m}_\tilde{l} \) defined as

\[
\tilde{m}_0 := m_0, \quad \tilde{m}_1 := p_1 \cdots p_l - m_1, \quad \tilde{m}_k := p_1^2 \cdots p_{k-1}^2 p_k \cdots p_l - m_k, \quad 2 \leq k \leq \tilde{l}.
\]

The generic virtual pole at \( O \) is

\[
\tilde{m}_{\tilde{l}+1} := \begin{cases} 
    p_1^2 \cdots p_{\tilde{l}-1}^2 p_\tilde{l} - m_\tilde{l} = \frac{1}{p_\tilde{l}} \left( p^2 - \alpha_{L,C,r} \right) & \text{if } r = 0, \\
    p_1^2 \cdots p_{\tilde{l}-1}^2 p_\tilde{l}^2 - pm_\tilde{l} - r = p^2 - \alpha_{L,C,r} & \text{if } r > 0.
\end{cases}
\]

Fix \( k, 1 \leq k \leq \tilde{l} \). The semigroup conditions for \( k \) are:

\[
\begin{align*}
    p_k \tilde{m}_k & \in \mathbb{Z}_{\geq 0}(\tilde{m}_0, \ldots, \tilde{m}_{k-1}), \quad (\text{S1-k}) \\
    (\tilde{m}_{k+1}, p_k \tilde{m}_k) & \cap \mathbb{Z}(\tilde{m}_0, \ldots, \tilde{m}_k) = (\tilde{m}_{k+1}, p_k \tilde{m}_k) \cap \mathbb{Z}_{\geq 0}(\tilde{m}_0, \ldots, \tilde{m}_k), \quad (\text{S2-k})
\end{align*}
\]

where \( (\tilde{m}_{k+1}, p_k \tilde{m}_k) := \{ a \in \mathbb{R} : \tilde{m}_{k+1} < a < p_k \tilde{m}_k \} \) and \( \mathbb{Z}_{\geq 0}(\tilde{m}_0, \ldots, \tilde{m}_k) \) (respectively, \( \mathbb{Z}(\tilde{m}_0, \ldots, \tilde{m}_k) \)) denotes the semigroup (respectively, group) generated by linear combinations of \( \tilde{m}_0, \ldots, \tilde{m}_k \) with non-negative integer (respectively, integer) coefficients.

**Theorem 2.10.** Let \((q_1, p_1), \ldots, (q_l, p_l)\) be pairs of relatively prime positive integers with \( p_k \geq 2, 1 \leq k \leq l \), and \( r \) be a non-negative integer. Let \( \tilde{l} \) and \( \tilde{m}_0, \ldots, \tilde{m}_\tilde{l} \) be as in Definition 2.9. Assume \( \tilde{m}_{\tilde{l}+1} > 0 \) (so that \( \tilde{E}_{L,C,r} \) is contractible for every curve \( C \) with Puiseux pairs \((q_1, p_1), \ldots, (q_l, p_l)\)). Then

1. There exists a curve-germ \( C \) at \( O \) with Puiseux pairs \((q_1, p_1), \ldots, (q_l, p_l)\) (for its Puiseux expansion \( v = \phi(u) \)) such that \( \tilde{E}_{L,C,r} \) is algebraically contractible, iff the semigroup condition \( (\text{S1-k}) \) holds for all \( k, 1 \leq k \leq \tilde{l} \).

2. There exists a curve-germ \( C \) at \( O \) with Puiseux pairs \((q_1, p_1), \ldots, (q_l, p_l)\) (for its Puiseux expansion \( v = \phi(u) \)) such that \( \tilde{E}_{L,C,r} \) is not algebraically contractible, iff either \( (\text{S1-k}) \) or \( (\text{S2-k}) \) fails for some \( k, 1 \leq k \leq \tilde{l} \).

We prove the theorem in Section 4.2 assuming the general case of the ‘effective criterion’ (Theorem 3.3).

**Remark 2.11** (‘Explanation’ of the term ‘virtual poles’). Let all notations be as in Theorem 2.10. In the set up of Question 2.4 identify \( \mathbb{P}^2 \setminus L \) with \( \mathbb{C}^2 \), so that \((1/u, v/u)\) is a system of coordinates on \( \mathbb{C}^2 \). The terminology ‘virtual poles’ for \( \tilde{m}_0, \ldots, \tilde{m}_\tilde{l} \) is motivated by the last assertion of the following result which is a reformulation of a fundamental result of the theory of plane algebraic curves with one point at infinity.

**Theorem 2.12** ([AM73, Abh77, Abh78, SS94]). The semigroup condition \( (\text{S1-k}) \) is satisfied for all \( k, 1 \leq k \leq \tilde{l} \), iff there exists a curve \( C \) in \( \mathbb{C}^2 \) such that \( C \) has only one place at infinity and has a Puiseux expansion at the point at infinity with Puiseux pairs \((q_1, p_1), \ldots, (q_l, p_l)\). Moreover, if \( C \) exists, then \( \tilde{m}_0/\tilde{p}, \ldots, \tilde{m}_{\tilde{l}}/\tilde{p} \) are the generators of the semigroup of poles at infinity on \( C \), where

\[
\tilde{p} := \begin{cases} 
    p_l & \text{if } \tilde{l} = l - 1, \\
    1 & \text{if } \tilde{l} > l.
\end{cases}
\]
In the situation of [2.12] the numbers \( \hat{m}_k, 0 \leq k \leq \tilde{l} \), are usually denoted in the literature by \( \delta_k, 0 \leq k \leq \tilde{l} \), and are called the \( \delta \)-sequence of \( \widetilde{C} \).

For positive integers \( q, p \), and a curve-germ \( C \) at \( O \), we say that \( C \) is of \((q, p)\)-type with respect to \((u, v)\)-coordinates iff \( C \) has a Puiseux expansion \( v = \phi(u) \) such that \((q, p)\) is the only Puiseux pair of \( \phi \). The following result is a straightforward corollary of Theorem 2.10 and the fact (which is a special case of [Her70, Proposition 2.1]) that the greatest integer not belonging to \( \mathbb{Z}_{\geq 0}(p, p - q) \) is \( p(p - q) - p - (p - q) \).

**Corollary 2.13.** Let \( p, q \) be positive relatively prime integers and \( r \) be a non-negative integer.

1. Let \( C \) be a \((q, p)\)-type curve germ at \( O \) with respect to \((u, v)\)-coordinates. Then \( E_{L, C, r} \) is contractible iff \( r < p(p - q) \).

2. There is a \((q, p)\)-type curve germ \( C \) at \( O \) with respect to \((u, v)\)-coordinates such that \( E_{L, C, r} \) is contractible, but not algebraically contractible, iff \( 2p - q < r < p(p - q) \).

**Remark-Example 2.14.** In fact, if \( 2p - q < r < p(p - q) \), Theorem 2.10 gives an easy recipe to construct a curve \( C \) such that \( E_{L, C, r} \) is contractible, but not algebraically contractible; e.g. the curve given by \((v - f(u))^p = u^q \) would suffice for any polynomial \( f(u) \in \mathbb{C}[u] \) such that the coefficient of \( u^2 \) in \( f(u) \) is non-zero. In Examples 1.2 and 3.13 we considered the case \((q, p) = (3, 5) \) and \( f(u) = u^2 \).

**Remark-Example 2.15 (Dual graphs arising from only non-algebraic contractions).** Note that the ‘virtual poles’ of Theorem 2.10 depend only on the singularity type of \( C \cup L \), i.e. Puiseux pairs \((q_1, p_1), \ldots, (q_l, p_l) \) of the Puiseux expansion of the given curve \((u, v)\)-coordinates. If \((q_1, p_1), (q_2, p_2) \) are pairs of relatively prime positive integers such that \( p_1, p_2 \geq 2 \), \( q_1 < p_1 \) and

\[
q_2 = (p_1 - q_1)(p_2 - 1)(p_1 - 1) + p_1(p_2 + 1),
\]

then the ‘fact’ stated preceding Corollary 2.13 implies that the condition \((51-3)\) fails for \( k = 2 \) and therefore Theorem 2.10 implies that the dual graph for \( E_{L, C, r} \) for \( r = 1 \) and any curve \( C \) with Puiseux pairs \((q_1, p_1), (q_2, p_2) \) (for the Puiseux expansion in \( u \)) corresponds only to non-algebraic analytic contractions. Setting \((q_1, p_1) = (3, 5) \) and \( p_2 = 2 \) in equation \((5)\) gives \( q_2 = 23 \). Figure 5 depicts the dual graph of \( \widetilde{E}_{L, C, 1} \) for a curve with Puiseux pairs \{\((3, 5), (23, 2)\)\} (for its Puiseux expansion in \( u \)).

![Figure 5: A dual graph of \( \widetilde{E}_{L, C, r} \) which comes from only non-algebraic analytic contractions](image)

\[
\begin{array}{c}
\begin{array}{c}
-3 \\
\text{L}
\end{array}
\begin{array}{c}
-2 \\
\text{7 vertices of weight -2}
\end{array}
\begin{array}{c}
-2 \\
\text{-3}
\end{array}
\begin{array}{c}
-3 \\
\end{array}
\begin{array}{c}
-2 \\
\end{array}
\begin{array}{c}
2
\end{array}
\end{array}
\]

3. **The global incarnation of the question of algebraicity**

In the set up of Question 1.1, identifying \( \mathbb{P}^2 \setminus L \) with \( \mathbb{C}^2 \) and \( \widetilde{Y} \) with a compactification of \( \mathbb{C}^2 \) translates Question 1.1 to the following

**Question 3.1.** Let \( \widetilde{X} \) be a normal analytic compactification of \( X := \mathbb{C}^2 \) such that \( \widetilde{X} \setminus X \) is an irreducible curve. When is \( \widetilde{X} \) algebraic?

In this section we give complete statements of geometric and algebraic (which is also effective!) answers to Question 3.1 and in Section 4 we present a proof of these statements under an additional simplifying condition.

### 3.1 Geometric answer

Let \( X := \mathbb{C}^2 \) and \( X^0 := \mathbb{P}^2 \supset X \). Let \( \widetilde{X} \) be a normal analytic compactification of \( X \) such that \( \widetilde{X} \setminus X \) is an irreducible curve and \( \sigma' : X^0 \to \widetilde{X} \) be the bimeromorphic map induced by identification of \( X \). Let \( S' \) be the (finite) set of points of indeterminacies of \( \sigma' \).

**Theorem 3.2.** Assume \( \sigma' \) is not an isomorphism, so that \( \sigma' \) maps \( L_{\infty} \setminus S' \) to a point \( P_{\infty} \in C_{\infty} \). Then \( \widetilde{X} \) is algebraic iff there is an algebraic curve \( C \subseteq X \) with one place at infinity such that \( \widetilde{C}^X \cap P_{\infty} = \emptyset \), where \( \widetilde{C}^X \) is the closure of \( C \) in \( \widetilde{X} \).
3.2 Algebraic answer

As in the preceding subsection, let \( \bar{X} \) be a normal analytic compactification of \( X := \mathbb{C}^2 \) such that \( C_\infty := \bar{X} \setminus X \) is an irreducible curve. Let \( \nu : \mathbb{C}(X) \setminus \{0\} \to \mathbb{Z} \) be the order of vanishing along \( C_\infty \). Then \( \nu \) is a divisorial valuation on \( \mathbb{C}(X) \) (see Definition 3.4) which is centered at infinity (i.e. there are \( f \in \mathbb{C}[x,y] \setminus \{0\} \) such that \( \nu(f) < 0 \). We study \( \bar{X} \) via studying \( \nu \), or more precisely \( \delta := -\nu \), which we call a semidegree (Definition 4.4). In Subsection 3.4 below we associate with \( \delta \) a (finite) sequence of elements of \( \mathbb{C}[x,x^{-1},y] \) which we call key forms (which are analogues of key polynomials \([\text{Mac36}]\) associated to \( \nu \)). The algebraic formulation of our result is then:

**Theorem 3.3.** \( \bar{X} \) is algebraic iff all the key forms associated to \( \delta \) are polynomials iff the last key form associated to \( \delta \) is a polynomial.

Below we recall the notion of key polynomials associated to valuations and then define key forms for semidegrees.

3.3 Puiseux series and Key Polynomials corresponding to valuations

**Definition 3.4** (Divisorial valuations). Let \( u, v \) be polynomial coordinates on \( X' \cong \mathbb{C}^2 \). A discrete valuation on \( \mathbb{C}(u,v) \) is a map \( \nu : \mathbb{C}(u,v) \setminus \{0\} \to \mathbb{Z} \), such that for all \( f, g \in \mathbb{C}(u,v) \setminus \{0\} \),

1. \( \nu(f + g) \geq \min\{\nu(f), \nu(g)\} \),
2. \( \nu(fg) = \nu(f) + \nu(g) \).

Let \( \bar{X}' \) be an algebraic compactification of \( X' \). A discrete valuation \( \nu \) on \( \mathbb{C}(u,v) \) is called divisorial iff there exists a normal algebraic surface \( Y \) equipped with a birational morphism \( \sigma : Y \to \bar{X}' \) and a curve \( C_\nu \) on \( Y \) such that for all non-zero \( f \in \mathbb{C}[x,y] \), \( \nu(f) \) is the order of vanishing of \( \sigma^*(f) \) along \( C_\nu \). The center of \( \nu \) on \( \bar{X}' \) is \( \sigma(C_\nu) \).

Let \( u, v \) be as in Definition 3.4 and \( \nu \) be a divisorial valuation on \( \mathbb{C}(u,v) \) with \( \nu(u) > 0 \) and \( \nu(v) > 0 \). We recall two of the standard ways of representing a valuation: by a Puiseux series and by key polynomials \([\text{Mac36}]\).

**Definition 3.5** (Puiseux series). Recall that the ring of Puiseux series in \( u \) is

\[ \mathbb{C}\{\{u\}\} := \bigcup_{p=1}^{\infty} \mathbb{C}[\{u^{1/p}\}] = \left\{ \sum_{k=0}^{\infty} a_k u^{k/p} : p \in \mathbb{Z}, \ p \geq 1 \right\}. \]

Let \( \phi \in \mathbb{C}\{\{u\}\} \). The **polydromy order** \([\text{CA00}]\) Chapter 1 of \( \phi \) is the smallest positive integer \( p \) such that \( \phi \in \mathbb{C}\{\{u^{1/p}\}\} \).

For any \( r \in \mathbb{Q} \), let us denote by \( [\phi]_{<r} \) (resp. \( [\phi]_{\leq r} \)) sum of all terms of \( \phi \) with order less than (resp. less than or equal to) \( r \). Then the **Puiseux pairs** of \( \phi \) are the unique sequence of pairs of relatively prime positive integers \((q_1, p_1), \ldots, (q_k, p_k)\) such that the polydromy order of \( \phi \) is \( p_1 \cdots p_k \), and for all \( j, 1 \leq j \leq k \),

1. \( p_j \geq 2 \),
2. \( [\phi]_{< \frac{q_j}{p_j}} \in \mathbb{C}[u^{\frac{1}{p_j-1}}] \) (where we set \( p_0 := 1 \)), and
3. \( [\phi]_{\leq \frac{q_j}{p_j}} \notin \mathbb{C}[u^{\frac{1}{p_j-1}}] \).

**Proposition 3.6** (Valuation via Puiseux series: cf. \([\text{FJ04}]\) Proposition 4.1). There exists a Puiseux polynomial (i.e. a Puiseux series with finitely many terms) \( \phi_\nu \in \mathbb{C}\{\{u\}\} \) and a rational number \( r_\nu \) such that for all \( f \in \mathbb{C}[u,v] \),

\[ \nu(f) = \nu(u) \text{ord}_u \left( f(u,v)\big|_{v=\phi_\nu(u)+\xi u^{r_\nu}} \right), \tag{6} \]

where \( \xi \) is an indeterminate.

**Definition 3.7.** If \( \phi_\nu \) and \( r_\nu \) are as in Proposition 3.6 we say that \( \tilde{\phi}_\nu(u, \xi) := \phi_\nu(x) + \xi u^{r_\nu} \) is the **generic Puiseux series** associated to \( \nu \).
**Definition 3.8 (Key Polynomials of \cite{Mac36} after \cite{FJ04} Chapter 2).** Let \( \nu \) be as above. A sequence of polynomials \( U_0, U_1, \ldots, U_k \in \mathbb{C}[u, v] \) is called the sequence of *key polynomials* for \( \nu \) if the following properties are satisfied:

1. Let \( \omega_j := \nu(U_j), \) \( 0 \leq j \leq k. \) Then
   \[
   \omega_{j+1} > n_j \omega_j = \sum_{i=0}^{j-1} m_{j,i} \omega_i \text{ for } 1 \leq j < k,
   \]
   where \( n_j \in \mathbb{Z}_{>0} \) and \( m_{j,i} \in \mathbb{Z}_{\geq 0} \) satisfy
   \[
   n_j = \min\{l \in \mathbb{Z}_{>0}; l \omega_j \in \mathbb{Z} \omega_0 + \cdots + \mathbb{Z} \omega_{j-1}\} \text{ for } 1 \leq j < k, \text{ and}
   0 \leq m_{j,i} < n_i \text{ for } 1 \leq i < j < k.
   \]
2. For \( 1 \leq j < k, \) there exists \( \theta_j \in \mathbb{C}^* \) such that
   \[
   U_{j+1} = U_j^{n_j} - \theta_j U_0^{m_{j,0}} \cdots U_j^{m_{j,j-1}}.
   \]
3. Let \( u_0, \ldots, u_k \) be indeterminates and \( \omega \) be the *weighted order* on \( \mathbb{C}[u_0, \ldots, u_k] \) corresponding to weights \( \omega_j \) for \( u_j, 0 \leq j \leq k \) (i.e., the value of \( \omega \) on a polynomial is the smallest ‘weight’ of its monomials). Then for every polynomial \( f \in \mathbb{C}[u, v], \)
   \[
   \nu(f) = \max\{\omega(F) : F \in \mathbb{C}[u_0, \ldots, u_k], F(U_0, \ldots, U_k) = f\}.
   \]

**Theorem 3.9 (\cite{FJ04} Theorem 2.29).** There is a unique and finite sequence of key polynomials for \( \nu. \)

**Example 3.10.** If \( \nu \) is the multiplicity valuation at the origin, then the generic Puiseux series corresponding to \( \nu \) is \( \tilde{\nu}_\nu = \xi u \) and the key polynomials are \( u, v. \)

**Example 3.11.** If \( \nu \) is the weighted order in \((u, v)\)-coordinates corresponding to weights \( p \) for \( u \) and \( q \) for \( v \) with \( p, q \) positive integers, then \( \tilde{\nu}_\nu = \xi^{q/p} \) and the key polynomials are again \( u, v. \)

**Example 3.12.** Let \( C \) be a singular irreducible analytic curve-germ at the origin with Puiseux expansion \( v = \phi(u). \) Pick any positive integer \( r. \) Construct the minimal resolution of singularity of \( C \) (at \( O \)) and then blow up \( r \) more times the point where the strict transform of \( C \) intersects the exceptional divisor. Let \( E \) be the last exceptional divisor constructed via this process and \( \nu \) be the valuation corresponding to \( E. \) Then the generic Puiseux series corresponding to \( \nu \) is
   \[
   \tilde{\nu}_\nu = [\phi(u)]_{\leq (q+r)}/p + \xi u^{(q+r)}/p, \quad \text{where}
   p = \text{the smallest positive integer such that } \phi \in \mathbb{C}[[u^{1/p}]],
   \]
   \[
   q/p = \text{the last Puiseux exponent of } \phi,
   [\phi(u)]_{\leq (q+r)}/p = \text{sum of all terms of } \phi(u) \text{ with order less than } (q+r)/p.
   \]

**Example 3.13.** Let \( C_1 \) and \( C_2 \) be the curves from Example 2.8. We apply the construction of Example 3.12 to \( C_1 \) and \( C_2. \) The Puiseux expansion for \( C_1 \) and \( C_2 \) at the origin are respectively given by: \( v = u^{3/5} \) and \( v = u^{3/5} + u^2. \) It follows that the generic Puiseux series for the valuation of Example 3.12 applied to \( C_1 \)’s are:
   \[
   \tilde{\nu}_{\nu_1} = \begin{cases} 
   \xi u^{3/5} & \text{if } r = 0, \\
   u^{3/5} + \xi u^{(3+r)/5} & \text{if } r \geq 1.
   \end{cases}
   \quad \tilde{\nu}_{\nu_2} = \begin{cases} 
   \xi u^{3/5} & \text{if } r = 0, \\
   u^{3/5} + \xi u^{(3+r)/5} & \text{if } 1 \leq r \leq 7, \\
   u^{3/5} + u^2 + \xi u^{(3+r)/5} & \text{if } 8 \leq r.
   \end{cases}
   \]

The sequence of key polynomials for \( \nu_1 \) and \( \nu_2 \) for \( 0 \leq r < 10 \) are as follows:

- key polynomials for \( \nu_1 \) = \( \{u, v\} \) if \( r = 0, \)
- key polynomials for \( \nu_1 \) = \( \{u, v, \nu^5 - u^3\} \) if \( r \geq 1. \)
- key polynomials for \( \nu_2 \) = \( \{u, v\} \) if \( r = 0, \)
- key polynomials for \( \nu_2 \) = \( \{u, v, v^5 - u^3\} \) if \( 1 \leq r \leq 7, \)
- key polynomials for \( \nu_2 \) = \( \{u, v, v^5 - u^3, v^5 - u^3 - 5u^4 u^2\} \) if \( 8 \leq r \leq 9. \)

In particular, note that for \( r \geq 1 \) the last key polynomials are precisely the \( \tilde{f}_i \)’s of Example 2.8. This is in fact the key observation for the proof of Theorem 2.7 using Theorem 3.3.
3.4 Degree-wise Puiseux series and Key Forms corresponding to semidegrees

Let $X \cong \mathbb{C}^2$ with coordinates $(x, y)$ and let $\delta$ be a divisorial semidegree (i.e. $\nu := -\delta$ is a divisorial valuation) on $\mathbb{C}[x, y]$ such that $\delta(x) > 0$. Let $u := 1/x$ and $v := y/x^k$ for some $k$ such that $\delta(y) < k\delta(x)$. Then $\nu(u) > 0$ and $\nu(v) > 0$. Applying Proposition 3.14 to $\nu$ and then translating in terms of $(x, y)$-coordinates yields Proposition 3.14 below. Recall that the field $\mathbb{C}(\langle x \rangle)$ of Laurent series in $x$ is the field of fractions of the formal power series ring $\mathbb{C}[x]$. The field of degree-wise Puiseux series in $x$

\[ \mathbb{C}(\langle x \rangle) := \mathbb{C}(\langle x^{-1/p} \rangle) = \left\{ \sum_{j \leq k} a_j x^{j/p} : k, p \in \mathbb{Z}, p \geq 1 \right\}. \]

**Proposition 3.14** ([Mon11b Theorem 1.2]). There exists a degree-wise Puiseux polynomial (i.e. a degree-wise Puiseux series with finitely many terms) $\phi_\delta \in \mathbb{C}(\langle x \rangle)$ and a rational number $r_\delta < \text{ord}_x(\phi_\delta)$ such that for every polynomial $f \in \mathbb{C}[x, y]$,

\[ \delta(f) = \delta(x) \deg_x (f(x, y) |_{y = \phi_\delta(x) + \xi x^{r_\delta}}), \]

(7)

where $\xi$ is an indeterminate.

**Definition 3.15.** If $\phi_\delta$ and $r_\delta$ are as in Proposition 3.14, we say that $\tilde{\phi}_\delta(x, \xi) := \phi_\delta(x) + \xi x^{r_\delta}$ is the generic degree-wise Puiseux series associated to $\delta$.

We will need the following geometric interpretation of degree-wise Puiseux series: assume that $\bar{X}$ is a normal analytic compactification of $X$ with an irreducible curve $C_\infty$ at infinity and $\delta$ is precisely the order of pole along $C_\infty$. Let $X^0 \cong \mathbb{P}^2$ be the compactification of $X$ induced by the map $(x, y) \mapsto [1 : x : y]$, $\sigma : \bar{X} \rightarrow X^0$ be the natural bimeromorphic map, and $S$ (resp. $S'$) be the finite set of points of indeterminacy of $\sigma$ (resp. $\sigma^{-1}$). Assume that $\sigma$ maps $C_\infty \setminus S$ to a point $O \in L_\infty := X^0 \setminus X$. It then follows that $\sigma^{-1}$ maps $L_\infty \setminus S'$ to a point $P_\infty \in C_\infty$.

**Proposition 3.16** ([Mon11b Proposition 4.2]). Let $\tilde{\phi}_\delta(x, \xi)$ be the generic degree-wise Puiseux series associated to $\delta$ and $\gamma$ be an (analytically) irreducible curve-germ at $O$ (on $X^0$) which is distinct from the germ of $L_\infty$. Then the strict transform of $\gamma$ on $\bar{X}$ intersects $C_\infty \setminus \{P_\infty\}$ iff $\gamma \cap X$ (i.e. the finite part of $\gamma$) has a parametrization of the form

\[ t \mapsto (t, \tilde{\phi}_\delta(t, \xi)|_{\xi = c} + \text{l.o.t.}) \quad \text{for } |t| \gg 0 \]

(8)

for some $c \in \mathbb{C}$, where l.o.t. means ‘lower order terms’ (in $t$).

Now we adapt the notion of key polynomials to the case of semidegrees. The main difference from the case of valuations is that these may not be polynomials (hence the word ‘form’ instead of ‘polynomial’) - see Example 3.20 and Remark 3.21.

**Definition 3.17** (Key Forms). Let $\delta$ be as above. A sequence of elements $f_0, f_1, \ldots, f_k \in \mathbb{C}[x, x^{-1}, y]$ is called the sequence of key forms for $\delta$ if the following properties are satisfied:

P0. $f_0 = x$, $f_1 = y$.

P1. Let $\omega_j := \delta(f_j)$, $0 \leq j \leq k$. Then

\[ \omega_{j+1} < n_j \omega_j = \sum_{i=0}^{j-1} m_{j,i} \omega_i \quad \text{for } 1 \leq j < k, \]

where

(a) $n_j = \min\{l \in \mathbb{Z}_{>0} : l \omega_j \in \mathbb{Z} \omega_0 + \cdots + \mathbb{Z} \omega_{j-1}\}$ for $1 \leq j < k$,

(b) $m_{j,i}$’s are integers such that $0 \leq m_{j,i} < n_i$ for $1 \leq i < j < k$ (in particular, $m_{j,0}$’s are allowed to be negative).

P2. For $1 \leq j < k$, there exists $\theta_j \in \mathbb{C}^*$ such that

\[ f_{j+1} = f_j^{m_{j+1}} \theta_j f_j^{m_{j+0}} \cdots f_j^{m_{j-1}}. \]

*We use the word ‘form’ in particular, because the key forms have a property analogous to ‘weighted homogeneous forms’ for weighted degrees. Indeed, for each key form $f_j$ of $\delta$ there is a semidegree $\delta_j$ which is an approximation of $\delta$ such that $f_j$ is the leading form of an element in $\mathbb{C}[x, x^{-1}, y]$ with respect to $\delta_j$. 

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P3. Let \( y_0, \ldots, y_k \) be indeterminates and \( \omega \) be the weighted degree on \( \mathbb{C}[y_0, \ldots, y_k] \) corresponding to weights \( \omega_j \) for \( y_j, \ 0 \leq j \leq k \) (i.e. the value of \( \eta \) on a polynomial is the maximum ‘weight’ of its monomials). Then for every polynomial \( f \in \mathbb{C}[x, y] \),
\[
\nu(f) = \min\{\eta(F) : F \in \mathbb{C}[y_0, \ldots, y_k], F(f_0, \ldots, f_k) = f\}.
\]

**Theorem 3.18.** There is a unique and finite sequence of key forms for \( X \) at infinity on an algebraic compactification of \( C_\delta \).

**Definition 4.1.**

4.1.1 Degree-like functions and compactifications

4.1.2 Background

series). We use in the proof: the process of compactifications via degree-wise Puiseux series (the latter being just a reformulation of the factorization in terms of Puiseux series).

**Example 3.19.** If \( \delta \) is a weighted degree in \((x, y)\)-coordinates corresponding to weights \( p \) for \( x \) and \( q \) for \( y \) with \( p, q \) positive integers, then the generic degree-wise Puiseux series corresponding to \( \delta \) is \( \tilde{\phi}_\delta = \xi^{q/p} \) and the key polynomials are \( f_0 = x \) and \( f_1 = y \).

**Example 3.20.** Set \( u := 1/x \) and \( v := y/x \). Let \( \nu_1 \) and \( \nu_2 \) be valuations from Example 3.13 and set \( \delta_i := -\nu_i, \ 1 \leq i \leq 2 \). It follows from the computations of Example 3.13 shows that the generic degree-wise Puiseux series for the valuation of Example 3.12 applied to \( C_i \)'s are:

\[
\tilde{\phi}_{\nu_1} = \begin{cases} 
\xi x^{2/5} & \text{if } r = 0, \\
x^{2/5} + \xi x^{(2-r)/5} & \text{if } r \geq 1.
\end{cases}
\]

\[
\tilde{\phi}_{\nu_2} = \begin{cases} 
\xi x^{2/5} & \text{if } r = 0, \\
x^{2/5} + \xi x^{(2-r)/5} & \text{if } 1 \leq r \leq 7, \\
x^{2/5} + x^{2} + \xi x^{(2-r)/5} & \text{if } 8 \leq r.
\end{cases}
\]

The sequence of key polynomials for \( \delta_1 \) and \( \delta_2 \) for \( 0 \leq r < 10 \) are as follows:

| Key polynomials for \( \nu_1 \) | Key polynomials for \( \nu_2 \) |
|---------------------------------|---------------------------------|
| \( x, y \) if \( r = 0 \) | \( x, y \) if \( r = 0 \) |
| \( x, y, y^5 - x^2 \) if \( r \geq 1 \) | \( x, y, y^5 - x^2 \) if \( 1 \leq r \leq 7 \) |
| \( x, y, y^5 - x^2, y^5 - x^2 - 5y^4x^{-1} \) if \( 8 \leq r \leq 9 \) | \( x, y, y^5 - x^2, y^5 - x^2 - 5y^4x^{-1} \) if \( 8 \leq r \leq 9 \) |

In particular, for \( 8 \leq r \leq 9 \), the last key polynomial for \( \delta_2 \) is not a polynomial. On the other hand, recall (from Example 2.8) that \( \tilde{E}_{L, C_2, r} \) is contractible for these values of \( r \), which implies that \( \delta_2 \) is positive on \( \mathbb{C}[x, y] \setminus \{0\} \).

**Remark 3.21.** As Example 3.20 illustrates, even if \( \delta \) is positive on \( \mathbb{C}[x, y] \setminus \{0\} \), some of the key forms may not be polynomials. This is precisely the reason of the difficulty of the global case and the ‘content’ of the algebraicity criteria of this article is the statement that this does not happen if \( \delta \) is the semidegree corresponding to the curve at infinity on an algebraic compactification of \( \mathbb{C}^2 \) for which the curve that infinity is irreducible.

### 4 Proof of the results in the case of one Puiseux pair

Let \( \tilde{X} \) be a normal analytic compactification of \( X := \mathbb{C}^2 \) with \( C_\infty := \tilde{X} \setminus X \) irreducible and let \( \delta \) be the semidegree on \( \mathbb{C}(x, y) \) corresponding to \( C_\infty \). In this section we give a proof of Theorems 3.2 and 3.3 under the additional assumption that the generic degree-wise Puiseux series for \( \delta \) has at most one Puiseux pair. In the local setting this gives a complete proof of Theorem 2.7. We also give a proof of Theorem 2.10. At first we briefly recall some notions we use in the proof: the process of compactifications via degree-like functions and the factorization of polynomials in terms of degree-wise Puiseux series (the latter being just a reformulation of the factorization in terms of Puiseux series).

#### 4.1 Background

4.1.1 Degree-like functions and compactifications

**Definition 4.1.** Let \( X \) be an irreducible affine variety over an algebraically closed field \( \mathbb{K} \). A map \( \delta : \mathbb{K}[X] \setminus \{0\} \to \mathbb{Z} \) is called a degree-like function if

1. \( \delta(f + g) \leq \max\{\delta(f), \delta(g)\} \) for all \( f, g \in \mathbb{K}[X] \), with \( < \) in the preceding inequality implying \( \delta(f) = \delta(g) \).
2. \( \delta(fg) \leq \delta(f) + \delta(g) \) for all \( f, g \in \mathbb{K}[X] \).

Every degree-like function \( \delta \) on \( \mathbb{K}[X] \) defines an ascending filtration \( \mathcal{F}^\delta := \{F^\delta_d\}_{d \geq 0} \) on \( \mathbb{K}[X] \), where \( F^\delta_d := \{f \in \mathbb{K}[X] : \delta(f) \leq d\} \). Define

\[
\mathbb{K}[X]^\delta := \bigoplus_{d \geq 0} F^\delta_d, \quad \text{gr } \mathbb{K}[X]^\delta := \bigoplus_{d \geq 0} F^\delta_d/F^\delta_{d-1}.
\]
Remark 4.2. For every $f \in \mathbb{K}[X]$, there are infinitely many ‘copies’ of $f$ in $\mathbb{K}[X]^\delta$, namely the copy of $f$ in $F^\delta_d$ for each $d \geq \delta(f)$; we denote the copy of $f$ in $F^\delta_d$ by $(f)_d$. If $t$ is a new indeterminate, then
\[ \mathbb{K}[X]^\delta \cong \sum_{d \geq 0} F^\delta_d t^d, \]
via the isomorphism $(f)_d \mapsto ft^d$. Note that $t$ corresponds to $(1)_1$ under this isomorphism.

We say that $\delta$ is \textit{finitely-generated} if $\mathbb{K}[X]^\delta$ is a finitely generated algebra over $\mathbb{K}$ and that $\delta$ is \textit{projective} if in addition $F^\delta_0 = \mathbb{K}$. The motivation for the terminology comes from the following straightforward

Proposition 4.3 ([Mon10 Proposition 2.5]). If $\delta$ is a projective degree-like function, then $X^\delta := \text{Proj} \mathbb{K}[X]^\delta$ is a projective compactification of $X$. The hypersurface at infinity $\hat{X}_\infty^\delta := X^\delta \setminus X$ is the zero set of the $\mathbb{Q}$-Cartier divisor defined by $(1)_1$ and is isomorphic to $\text{Proj} \mathbb{K}[X]^\delta$. Conversely, if $\hat{X}$ is any projective compactification of $X$ such that $\hat{X} \setminus X$ is the support of an effective ample divisor, then there is a projective degree-like function $\delta$ on $\mathbb{K}[X]$ such that $X^\delta \cong \hat{X}$.

Definition 4.4. A degree-like function $\delta$ is called a \textit{semidegree} if it always satisfies property 2 with an equality, and $\delta$ is called a \textit{subdegree} if it is the maximum of finitely many semidegrees. As we have already seen in Section 3, a semidegree is the negative of a \textit{discrete valuation}.

Theorem 4.5 (cf. [Mon10 Theorem 4.1]). Let $\delta$ be a finitely generated degree-like function on the coordinate ring of an irreducible affine variety $X$. Let $I$ be the ideal of $\mathbb{K}[X]^\delta$ generated by $(1)_1$. Then

1. $\delta$ is a semidegree (resp. subdegree) iff $I$ is a prime (resp. radical) ideal.
2. If $\delta$ is a subdegree, then it has a unique minimal presentation as the maximum of finitely many semidegrees.
3. The non-zero semidegrees in the minimal presentation of $\delta$ are (up to integer multiples) precisely the orders of pole along the irreducible components of the hypersurface at infinity.

4.1.2 Factorization in terms of degree-wise Puiseux series

Given a degree-wise Puiseux series $\psi$ in $x$, the \textit{polydromy order} of $\psi$ is the smallest positive integer $p$ such that the exponents of all terms in $\psi$ are of the form $q/p$, $q \in \mathbb{Z}$. Let $\psi = \sum_{q \leq q_0} a_q x^{q/p}$, where $p$ is the polydromy order of $\psi$. Then the \textit{conjugates} of $\psi$ are $\psi_j := \sum_{q \leq q_0} a_q \zeta^j x^{q/p}$, $1 \leq j \leq p$, where $\zeta$ is a primitive $p$-th root of unity. The usual factorization of polynomials in terms of Puiseux series implies the following

Theorem 4.6. Let $f \in \mathbb{C}[x,y]$. Then there are unique (up to conjugacy) degree-wise Puiseux series $\psi_1, \ldots, \psi_k$ and a unique non-negative integer $m$ such that
\[ f = x^m \prod_{i=1}^k \prod_{\text{conjugate of } \psi_i} (y - \psi_{ij}(x)) \]

4.2 Idea of the proof

Definition 4.7. Let $X := \mathbb{C}^2$ with coordinates $(x,y)$. Let $\phi(x)$ be a degree-wise Puiseux series in $x$ and $C \subseteq X$ be an analytic curve. We say that $(x, \phi(x))$ is a \textit{parametrization of a branch of $C$ at infinity} iff there is a branch of $C$ with a parametrization of the form $t \mapsto (t, \phi(t))$ for $|t| \gg 0$.

Let $\hat{X}$ be a normal analytic compactification of $X$ with $C_\infty := \hat{X} \setminus X$ irreducible and let $\delta$ be the semidegree on $\mathbb{C}(x,y)$ corresponding to $C_\infty$. Let $\phi_\delta(x, \xi)$ be the generic degree-wise Puiseux series for $\delta$. The following is the key Proposition for the proof.

Proposition 4.8. Let $f_0, \ldots, f_k$ be the key forms associated to $\delta$.

1. If $f_0, \ldots, f_k$ are all polynomials, then $\hat{X}$ is isomorphic to the closure of the image of $X$ in the weighted projective variety $\mathbb{P}^{k+1}(1, \delta(f_0), \ldots, \delta(f_k))$ under the mapping $(x,y) \mapsto [1 : f_0 : \cdots : f_k]$. 

2. If $f_k$ is a polynomial then $C_k := V(f_k) \subseteq X$ is a curve with one place at infinity and its unique branch at infinity has a parametrization of the form (1) (from Proposition 3.16).

3. If there exists $j$, $0 \leq j \leq k$, such that $f_j$ is not a polynomial, then there does not exist any polynomial $f \in \mathbb{C}[x,y]$ such that every branch of $V(f) \subseteq X$ at infinity has a parametrization of the form (3).

Below we use Proposition 4.8 to prove Theorems 2.7, 2.10, 3.2 and 3.3. In the next subsection we prove Proposition 4.8 under the additional assumption that $\phi_\delta(x, \xi)$ has at most one Puiseux pair.

**Remark 4.9.** Assertions 1 and 2 of Proposition 4.8 are more or less straightforward to see. The hard part in our proof of assertion 3 is to keep track of all the ‘cancellations’. However, if $\phi_\delta(x, \xi)$ has at most one Puiseux pair, then the problem is much simpler and the proof is much shorter.

*Proof of Theorem 3.3.* Note that assertions 2 and 3 of Proposition 4.8 imply that the last key form of $\delta$ is a polynomial iff all the key forms of $\delta$ are polynomials. Moreover, assertion 1 shows that the latter (and hence both) of the equivalent properties of the preceding sentence imply that $X$ is algebraic. Therefore it only remains to show that if $X$ is algebraic then all the key forms of $\delta$ are polynomials. So assume that $X$ is algebraic. Let $X^0 \cong \mathbb{P}^2$ be the compactification of $X$ induced by the map $x, y \mapsto [1 : x : y]$, $\sigma : X \to X^0$ be the natural bimeromorphic map, and $S$ (resp. $S'$) be the finite set of points of indeterminacy of $\sigma$ (resp. $\sigma^{-1}$). We have two cases to consider:

**Case 1:** $\sigma(C_\infty \setminus S)$ is dense in $L_\infty := \bar{X}^0 \setminus X$. In this case it follows from basic geometry of bimeromorphic maps that $\sigma$ must be an isomorphism. In particular, this implies that $\delta$ is precisely the usual degree in $(x, y)$-coordinates, i.e. $\phi_\delta(x, \xi) = \xi x$. The theorem then follows from Example 3.19.

**Case 2:** $\sigma(C_\infty \setminus S)$ is a point $O \in L_\infty$. In this case we are in the situation of Proposition 3.16. In particular, $\sigma^{-1}(L_\infty \setminus S')$ is a point $P_\infty \in C_\infty$. Since $\bar{X}$ is algebraic, it follows that there is an algebraic curve $C \subseteq X$ such that the closure of $C$ in $\bar{X}$ does not intersect $P_\infty$. Proposition 3.16 then implies that every branch of $C$ at infinity has a parametrization of the form (3). Then assertion 3 of Proposition 4.8 implies that all the key forms of $\delta$ are polynomials, as required.

*Proof of Theorem 3.2.* We continue to use the notation of the proof of Theorem 3.3. Note that $\sigma'$ of Theorem 3.2 is precisely $\sigma^{-1}$. At first assume $\bar{X}$ is algebraic. Since the last key form $f_k$ is a polynomial (which follows from Theorem 3.3), assertion 2 of Proposition 4.8 and Proposition 3.16 imply that $C := V(f_k) \subseteq X$ satisfies the requirement of Theorem 3.2 and completes the proof of ($\Rightarrow$) direction of Theorem 3.2.

Now we assume that $\bar{X}$ is not algebraic. Then Theorem 3.3 implies that one of the key polynomials is not a polynomial. It then follows from assertion 3 of Proposition 4.8 and Proposition 3.16 that $P_\infty$ lies on the closure in $\bar{X}$ of all algebraic curves in $X$, which completes the proof of ($\Leftarrow$) direction of Theorem 3.2 as required.

*Proof of Theorem 2.7.* Recall that $L = \{u = 0\}$. Let the Puiseux expansion for $C$ at $O := (0,0)$ be

$$v = a_0 u^{q/p} + a_1 u^{(q+1)/p} + \ldots$$

Let $\tilde{f}$ be as in Theorem 2.7. Then it is straightforward to see that

$$\tilde{f} = \begin{cases} 
0 & \text{if } r = 0, \\
\text{a monic polynomial in } v \text{ of degree } p & \text{otherwise.}
\end{cases}$$

Let $\nu$ be the divisorial valuation on $\mathbb{C}(u, v)$ corresponding to $E_{L,C,r}^*$ (i.e. the last exceptional divisor in the set up of Question 2.4). Then the generic Puiseux series (Definition 3.1) corresponding to $\nu$ is

$$\phi_\nu(u, \xi) = \begin{cases} 
\xi u^{q/p} & \text{if } r = 0, \\
\xi u^{q/p} + \ldots + a_{r-1} u^{(q+r-1)/p} + \xi u^{(q+r)/p} & \text{otherwise.}
\end{cases}$$

(8)

(this is a special case of Example 3.12). If $r = 0$, then the key polynomials for $\nu$ are $U_0 = u$ and $U_1 = v$. For $r \geq 1$, the sequence continues with $U_2 = v^p - a_0 u^q$ and so on, with

$$U_j = U_{j-1} - a_0 u^q$$

for $j \geq 3$. 

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It then follows from the construction of $\tilde{f}$ and the defining properties (and uniqueness) of key polynomials that $\tilde{f}$ is precisely the last key polynomial $U_k$ of $\nu$.

Now identify $X := \mathbb{P}^2 \setminus L$ with $\mathbb{C}^2$ with coordinates $(x, y) := (1/u, v/u)$. Then $\tilde{E}_{L,C,r}$ is algebraically contractible iff the compactification $\tilde{X}$ of $X$ corresponding to the semidegree $\delta := -\nu$ is algebraic. There are two cases to consider:

**Case 1:** $\tilde{f} = 0$. This corresponds to the case that $r = 0$. Then (8) implies that $\delta$ is precisely the weighted degree corresponding to weights $p$ for $x$ and $p - q$ for $y$. It follows that $X$ is the weighted projective space $\mathbb{P}^2(1, p, p - q)$ and therefore $\tilde{E}_{L,C,r}$ is algebraically contractible, as required.

**Case 2:** $\tilde{f} \neq 0$. This means $r \geq 1$ and $\tilde{f}$ is the last key polynomial $U_k$ of $\nu$. It is straightforward to see (e.g. using the uniqueness of key forms) that the last key form of $\delta$ is precisely $x^pU_k(y/x, 1/x)$, and the latter is a polynomial if $\deg((u,v))(U_k) \leq p$. Theorem 2.7 now follows from Theorem 3.3.

**Proof of Theorem 2.10** We use the notations of Theorem 2.10 and Question 2.4. Set

$$
\tilde{q}_j := \begin{cases} 
p_1 \cdots p_j - q_j & \text{for } 1 \leq j \leq \bar{l}, 
p_1 \cdots p_1 - q_1 - r & \text{for } j = \bar{l} + 1.
\end{cases}
$$

Consider a generic degree-wise Puiseux series of the form

$$
\tilde{\phi}_\delta := \left( \alpha_1 x^{\tilde{q}_1} + \sum_{j=1}^{k_1} \alpha_{1j} x^{\tilde{q}_{1j}} \right) + \left( \alpha_2 x^{\tilde{q}_2} + \sum_{j=1}^{k_2} \alpha_{2j} x^{\tilde{q}_{2j}} \right) + \cdots + \left( \alpha_l x^{\tilde{q}_l} + \sum_{j=1}^{k_l} \alpha_{lj} x^{\tilde{q}_{lj}} \right) + \xi x^{\tilde{q}_{l+1}}
$$

where $a_1, \ldots, a_l, a_{ij}$'s belong to $\mathbb{C}$. As in the proof of Theorem 2.7, identify $X := \mathbb{P}^2 \setminus L$ with $\mathbb{C}^2$ with coordinates $(x, y) := (1/u, v/u)$. Recall that we assume in Theorem 2.10 that $\tilde{E}_{L,C,r}$ is contractible for every curve $C$ with Puiseux pairs $(q_1, p_1), \ldots, (q_l, p_l)$. This is equivalent to saying that for all choices of $a_i$'s and $a_{ij}$'s, the semidegree $\delta$ corresponding to $\phi_\delta$ is algebraic. There are two cases to consider:

1. There exist $a_i$'s and $a_{ij}$'s such that $\tilde{X}_\delta$ is algebraic, iff the semigroup condition $(S1-k)$ holds for all $k, 1 \leq k \leq \bar{l}$.
2. There exist $a_i$'s and $a_{ij}$'s such that $\tilde{X}_\delta$ is not algebraic iff either $(S1-k)$ or $(S2-k)$ fails for some $k, 1 \leq k \leq \bar{l}$.

At first we prove $(\Leftarrow)$ implication of Statement 1. So assume that the semigroup condition $(S1-k)$ holds for all $k, 1 \leq k \leq \bar{l}$. Let $\alpha_0$ corresponds to the choice $a_1 = \cdots = a_l = 1$ and $a_{ij} = 0$ for all $i, j$. It suffices to show that $\tilde{X}_\alpha$ is algebraic. Indeed, it follows from semigroup conditions $(S1-k)$ that for all $k, 1 \leq k \leq \bar{l}$,

$$p_k \bar{m}_k = \sum_{j=0}^{k-1} \beta_{k,j} \bar{m}_j$$

for non-negative integers $\beta_{k,0}, \ldots, \beta_{k,k-1}$. It is then straightforward to compute that the key forms of $\delta$ are $f_0, \ldots, f_{\bar{l}+1}$, with $f_0 := x$, $f_1 := y$, and

$$f_{k+1} = f_k^{p_k} - c_k \prod_{j=0}^{k-1} f_j^{\beta_{k,j}}, \quad c_k \in \mathbb{C}, \quad 1 \leq k \leq \bar{l}.
$$

In particular, each key form is a polynomial, and therefore Theorem 3.3 implies that $\tilde{X}_\alpha$ is algebraic, as required.

Statement 2 and the $(\Rightarrow)$ implication of Statement 1 follow from the properties of key forms of $\delta_\alpha$ listed in the following Claim. The Claim follows from an induction on $\bar{l}$ via a straightforward (but a bit messy) computation and we omit the proof.

**Claim.** Let $\delta_\alpha$ be the semidegree defined as above and $f_0, \ldots, f_s$ be the key polynomials of $\delta_\alpha$. Pick the subsequence $f_{i_1}, f_{i_2}, \ldots$ of $f_j$'s consisting of all $f_{i_k}$ such that $n_{i_k} > 1$ (where $n_{i_k}$ is as in Property $\mathcal{P}$ of Definition 3.7). Then
1. There are precisely $\tilde{l}$ of these $f_{jk}$’s.

2. $j_l < s$.

3. $\delta(f_{jk}) = \tilde{m}_k/\tilde{p}$ and $n_{jk} = p_k$, $1 \leq k \leq \tilde{l}$, where

$$\tilde{p} := \begin{cases} p_l & \text{if } \tilde{l} = l - 1, \\ 1 & \text{if } \tilde{l} > l. \end{cases}$$

4. Define $j_0 := 0$, i.e. $f_{j_0} = x$ and $\delta(f_{j_0}) = \tilde{m}_0$. Then for each $k$, $1 \leq k \leq \tilde{l}$,

$$f_{j_{k+1}} = f_{j_k}^{p_k} - c_k \prod_{i=0}^{k-1} f_{j_i}^{\delta_k,i} \text{ for some } c_k \in \mathbb{C}^*.$$

5. Define $j_{l+1} := s$. Then $\delta(f_{j_{l+1}}) = \delta(f_s) = \tilde{m}_{l+1}$.

6. Fix $k$, $0 \leq k \leq \tilde{l}$. For every $i$ such that $j_k < i < j_{k+1}$,

$$\delta(f_i) \in M_k := (\tilde{m}_{k+1}, p_k \tilde{m}_k) \cap \mathbb{Z}(\tilde{m}_0, \ldots, \tilde{m}_k)$$

and on conversely, for every $m \in M_k$, there is a choice of $\tilde{a}$ such that there is $i$ with $j_k < i < j_{k+1}$ and $\delta(f_i) = m$. \qed

### 4.3 Proof of Proposition 4.8 in the case of at most one Puiseux pair

In this subsection we prove Proposition 4.8 under the assumption that the generic degree-wise Puiseux series $\tilde{\phi}_s(x, \xi)$ for $\delta$ has at most one Puiseux pair, i.e. it has one of the following two forms:

Case 1: $\tilde{\phi}_s(x, \xi) = h(x) + \xi x^r$ for some $h(x) \in \mathbb{C}[x, x^{-1}]$ and $r \in \mathbb{Q}$, or

Case 2: $\tilde{\phi}_s(x, \xi) = h(x) + a_0 x^{q/p} + a_1 x^{(q-1)/p} + \cdots + a_{s-1} x^{(q-s+1)/p} + \xi x^{(q-s)/p}$ for some $h(x) \in \mathbb{C}[x, x^{-1}]$ and $p, q, s \in \mathbb{Z}$ such that $p \geq 2$, $s \geq 1$, $q \neq 0$ and $p, q$ are co-prime.

Assume we are in Case 1. We claim that $h(x) \in \mathbb{C}[x]$. Indeed, otherwise we have $h(x) = h_0(x) + x^{-1}h_1(x)$ for some $h_0 \in \mathbb{C}[x]$ and $h_1 \neq 0 \in \mathbb{C}[x^{-1}]$, and it would follow from \[7\] that $\delta(y - h_0(x)) < 0$, which is impossible, since $\delta$ takes positive value on all non-constant polynomials. Similarly, we must have $r > 0$. Let $h(x) = \sum_{i=1}^m b_i x^{d_i}$ with $d_1 > \cdots > d_m > r$. It follows that the key forms of $\delta$ are precisely, $x, y, y - b_1 x^{d_1}, y - b_1 x^{d_1} - b_2 x^{d_2}, \ldots, y - b_1 x^{d_1} - \cdots - b_m x^{d_m}$. Since the ‘last’ key form is $y - h(x)$ and all key forms are polynomials, assertions 2 and 3 of Proposition 4.8 are automatically satisfied. For assertion 1, we set $y' := y - h(x)$ and observe that $\delta$ is a weighted degree in $(x, y')$-coordinates corresponding to weights $p$ for $x$ and $q$ for $y'$, where $r = q/p$ with $p, q$ co-prime. It follows that $X$ is precisely the weighted projective space $\mathbb{P}^2(1, p, q)$ with the embedding $X \hookrightarrow \bar{X}$ given by $(x, y') \mapsto [1 : x : y']$. Assertion 4 of Proposition 4.8 now follows in a straightforward way.

Now assume Case 2 holds. It follows as in Case 1 that $h(x) \in \mathbb{C}[x]$ and $q > 0$. Moreover, letting $h(x) = \sum_{i=1}^m a_i x^{d_i}$ with $d_1 > \cdots > d_m > q/p$, we have that the first $m + 2$ key forms of $\delta$ are $x, y, y - b_1 x^{d_1}, \ldots, y - h(x)$. Since these are already polynomials, it follows from the definition of key forms that in order to prove Proposition 4.8 w.r.o.g. we may apply the change of coordinates $(x, y) \mapsto (x, y - h(x))$ and assume that

$$\tilde{\phi}_s(x, \xi) = a_0 x^{q/p} + a_1 x^{(q-1)/p} + \cdots + a_{s-1} x^{(q-s+1)/p} + \xi x^{(q-s)/p}.$$ 

We now compute the key forms of $\omega$. Define

$$\Phi_s(x, y) := \prod_{j=1}^p \left(y - a_0 \xi_j x^{q/p} - a_1 \xi_j x^{(q-1)/p} - \cdots - a_{s-1} \xi_j x^{(q-s+1)/p}\right),$$
where ζ is a primitive p-th root of unity. In other words, Φδ is the unique monic polynomial in y with coefficients in \( \mathbb{C}[x,x^{-1}] \) whose roots are conjugates of \( a_0 y^{q/p} + a_1 x^{(q-1)/p} + \cdots + a_{s-1} x^{(q-s+1)/p} \). Let \( d_\delta := \delta(\Phi_\delta) \). Then

\[
d_\delta = \delta(x) \text{ord}_x \left( \Phi_\delta(x,y) \right)_{y=\bar{\phi}_\delta(x,\xi)} = (p-1)q + q - s = pq - s.
\]

Let \( \omega \) be the weighted degree on \( \mathbb{C}(x,y) \) which gives weight \( p \) to \( x \) and \( q \) to \( y \). Note that

\[
\Phi_\delta = y^p - a_0^p x^q - \sum_j g_j
\]

where each \( g_j \)'s are monomial terms of the form \( c_j x^{\alpha_j} y^{\beta_j} \) for some \( c_j \in \mathbb{C} \) and integers \( \alpha_j, \beta_j \) such that \( 0 \leq \beta_j < p \). Order the \( g_j \)'s so that \( \omega(g_1) \geq \omega(g_2) \geq \cdots \geq \omega(g_m) = d_\delta \geq \omega(g_{m+1}) \geq \cdots \).

Claim 4.10. The key forms of \( \delta \) are \( x, y, y^p - a_0^p x^q, y^p - a_0^p x^q - g_1, \ldots, y^p - a_0^p x^q - \sum_{j=1}^m g_j \).

Proof. Let \( h_0 := y^p - a_0^p x^q \) and \( h_j := y^p - a_0^p x^q - \sum_{i=1}^j g_i \) for \( 1 \leq j \leq m \). It follows from the definition of \( \Phi_\delta \) that

\[
h_m|_{y=\bar{\phi}_\delta(x,\xi)} = (c_\xi - c_\beta') x^{(pq-s)/p} + \text{l.o.t.} \quad (11)
\]

for some \( c, c' \in \mathbb{C} \), \( c \neq 0 \), which implies that \( \delta(h_m) = d_\delta \). A straightforward backward induction then proves that \( \delta(h_m) < \delta(h_{m-1}) < \cdots < \delta(h_0) \). The uniqueness of key forms then imply that \( x, y, h_0, \ldots, h_m \) are key forms for \( \delta \). Moreover, note that the leading term of the right hand side of identity (11) contains the indeterminate \( \xi \), which implies that for any \( n \geq 1 \), the value of \( \delta(h_m)^n \) can not be reduced via adding any polynomial in \( x, x^{-1}, y, h_0, \ldots, h_{m-1} \). In particular, \( h_m \) is the ‘last’ key form for \( \delta \).

Proof of assertion 1 of Proposition 4.8. Assume that \( h_j \in \mathbb{C}[x,y] \) for each \( j \), \( 0 \leq j \leq k \). Let \( \omega_j := \omega(h_j) \) for \( 0 \leq j \leq m \). Let \( \mathbb{W}_P \) be the weighted projective space \( \mathbb{P}(1,p,q,0,\ldots,\omega_m) \) with weighted homogeneous coordinates \( \left[ z : x : y : y_0 : \cdots : y_m \right] \). We have to show that \( \tilde{X} \) is isomorphic to the closure in \( \mathbb{W}_P \) of the image of \( X \) under the embedding \( (x,y) \mapsto [1 : x : y : h_0 : \cdots : h_m] \). Let \( R \) be the homogeneous coordinate ring of \( \mathbb{W}_P \), i.e., \( R := \mathbb{C}[z,x,y,y_0,\ldots,y_m] \) with the grading on \( R \) given by the weights \( 1 \) for \( z \), \( p \) for \( x \), \( q \) for \( y \), and \( \omega_j \) for \( y_j \), \( 0 \leq j \leq m \). Note that the closure \( Z \) of \( X \) in \( \mathbb{W}_P \) is naturally isomorphic to \( \text{Proj} R/J \) for some homogeneous ideal \( J \) of \( R \). Consequently, we have to show that \( \text{Proj} R/J \cong \text{Proj} \mathbb{C}[x,y]^{\delta} \), where the latter ring is the graded ring corresponding to \( \delta \) as in Subsubsection 4.1.1. For this it suffices to show that the graded \( \mathbb{C} \)-algebra homomorphism \( \mathbb{C}[z,x,y,y_0,\ldots,y_m] \rightarrow \mathbb{C}[x,y]^{\delta} \) which maps \( z \mapsto (1)_1, x \mapsto (x)_p, y \mapsto (y)_q \) and \( y_j \mapsto (y_j)_{\omega_j} \) is in fact a surjection. But the latter is an immediate consequence of Property (13) of key forms. This completes the proof of assertion 1 of Proposition 4.8.

Proof of assertion 2 of Proposition 4.8. Note that \( f_k \) of Proposition 4.8 is precisely \( h_m \). Observe that

1. \( h_m \) is a monic polynomial in \( y \) of degree \( p \).
2. Since \( h_m \) is a polynomial by assumption, it is also a monic polynomial in \( x \) of degree \( q \).
3. \( \omega(h_m) = pq \).

Since \( p \) and \( q \) are relatively prime, these observations imply that \( h_m \) has one place at infinity and there is a degree-wise Puiseux series \( \psi(x) \) with polydromy order \( p \) such that

\[
h_m = \prod_{j=1}^p (y - \psi_j(x)), \quad (12)
\]

\( \psi_j \)'s are the conjugates of \( \psi \). Identities (12) and (11) then imply that there must be \( j, 1 \leq j \leq p \), such that

\[
\psi_j(x) = \bar{\phi}_\delta(x,\xi)|_{\xi=c''} + \text{l.o.t.}
\]

for some \( c'' \in \mathbb{C} \). This proves that the curve of \( h_m \) has a parametrization at infinity of the form (12), as required.

Proof of assertion 3 of Proposition 4.8. We prove it by contradiction. So assume there exists \( j, 1 \leq j \leq m \), such that \( \alpha_j < 0 \) and that there exists a polynomial \( f \in \mathbb{C}[x,y] \) such that all of its branches at infinity has a parametrization of the form

\[
t \mapsto (t, a_0 t^{k/p} + a_1 t^{(q-1)/p} + \cdots + a_{s-1} t^{(q-s+1)/p} + ct^{(q-s)/p} + \text{l.o.t.})
\]
for some \( c \in \mathbb{C} \) (where \( c \) depends on the branch). Let us write \( \phi(x) := a_0 x^{a/p} + a_1 x^{(q-1)/p} + \cdots + a_{s-1} x^{(q-s+1)/p} \). Then it follows from Theorem 4.6 that there exist degree-wise Puiseux series \( \psi_1, \ldots, \psi_l \) in \( x \) such that \( f \) has a factorization of the form

\[
f = \prod_{i=1}^{l} \prod_{\text{is a conjugate of } \psi_i} (y - \psi_{ij}(x))
\]

where for each \( i, 1 \leq i \leq l \),

\[
\psi_i(x) = \phi(x) + c_i x^{(q-s)/p} + \text{l.o.t.}
\]

for some \( c_i \in \mathbb{C} \). Pick \( i, 1 \leq i \leq l \), and let

\[
\Psi_i := \prod_j (y - \psi_{ij}(x)).
\]

It follows from (13) that \( p \) divides the polydromy order \( p_i \) (which is also the number of conjugates) of \( \psi_i \), and for each conjugate \( \psi_{ij} \) of \( \psi_i \), there is a conjugate \( \phi_{ij} \) of \( \phi \) such that

\[
\psi_{ij}(x) = \phi_{ij}(x) + c_{ij} x^{(q-s)/p} + \text{l.o.t.}
\]

for some \( c_{ij} \in \mathbb{C} \). It follows that \( \Psi_i \) can be expressed in the following form:

\[
\Psi_i = \prod_{j=1}^{p_i/p} \left( y - \phi_{ij}(x) - \tilde{\psi}_{ij}(x) \right), \quad \text{where } \deg_x \left( \tilde{\psi}_{ij}(x) \right) \leq (q-s)/p,
\]

where

\[
\Psi_{ij} := \prod_{j=1}^{p_i/p} (y - \phi_{ij}(x) - \tilde{\psi}_{ij}(x)), \quad \text{for each } j, 1 \leq j \leq p_i/p.
\]

Fix a \( j, 1 \leq j \leq p_i/p \). Let \( \omega \) be the weighted degree defined preceding Claim 4.10. Note that \( \omega \) extends to a weighted degree on \( \mathbb{C} \langle \langle x \rangle \rangle[y] \) (corresponding to weight \( p \) for \( x \) and \( q \) for \( y \)), where \( \mathbb{C} \langle \langle x \rangle \rangle \) is the field of degree-wise Puiseux series in \( x \). Then it follows that

\[
\Psi_{ij} = \prod_{j=1}^{p_i/p} (y - \phi_{ij}(x)) + H_{ij}(x, y) \quad \text{for some } H_{ij} \in \mathbb{C} \langle \langle x \rangle \rangle[y], \quad \omega(H_{ij}) \leq pq - s,
\]

\[
= \Phi_\delta(x, y) + H_{ij}(x, y)
\]

\[
= y^p - a_0^\delta x^q - \sum_{k=1}^{m} g_k + \tilde{H}_{ij}(x, y) \quad \text{for some } \tilde{H}_{ij} \in \mathbb{C} \langle \langle x \rangle \rangle[y], \quad \omega(\tilde{H}_{ij}) \leq pq - s.
\]

Now we prepare for the contradiction. Pick the smallest integer \( k_0 \) such that \( \alpha_{k_0} < 0 \) and let \( W_0 := \omega(g_{k_0}) > pq - s \). Collecting all terms of \( \Psi_{ij} \) with \( \omega \) value less than \( W_0 \) yields:

\[
\Psi_{ij} = y^p - a_0^\delta x^q - \sum_{k=1}^{k_0} g_k + G_{ij}(x, y) \quad \text{for some } G_{ij} \in \mathbb{C} \langle \langle x \rangle \rangle[y], \quad \omega(G_{ij}) < W_0.
\]

In particular, note that \( \Psi_{ij} - G_{ij} \) is independent of \( i, j \). Now

\[
f = \prod_{i,j} \Psi_{ij} = \prod_{i,j} \left( y^p - a_0^\delta x^q - \sum_{k=1}^{k_0} g_k + G_{ij}(x, y) \right).
\]

Let \( M \) be the total number of factors in the product of right hand side, and for each \( W \in \mathbb{Q} \), let \( f_W \) be the sum of monomials that appear (after multiplying out all the factors) in the right hand side with \( \omega \)-value equal to \( W \). Then for all \( W > W_1 := (M - 1)pq + W_0 \), \( f_W \) is a polynomial in \( x \) and \( y \). Moreover, \( f_{W_1} = f - M(y^p - a_0^\delta x^q)^{M-1} g_{k_0} \) for a polynomial \( f \) in \( x \) and \( y \). Let \( f' := f - \sum_{W > W_1} f_W - f \). Then it follows that \( f' \) is a polynomial with \( \omega(f) = W_1 \), but the leading weighted homogeneous form (with respect to \( \omega \)) of \( f' \) is \( -M(y^p - a_0^\delta x^q)^{M-1} g_{k_0} \), which is not a polynomial. This gives the desired contradiction and completes the proof of assertion 3 of Proposition 4.8.
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