Algebraic Bethe ansatz for the one-dimensional Hubbard model with open boundaries

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Abstract

The one-dimensional Hubbard model with open boundary conditions is exactly solved by means of algebraic Bethe ansatz. The eigenvalue of the transfer matrix, the energy spectrum as well as the Bethe ansatz equations are obtained.

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1 Introduction

The one-dimensional (1D) Hubbard model has been one of the most fundamental and favorite integrable models in the non-perturbative quantum field theory. It exhibits on-site Coulomb interaction and correlated hopping which will possibly reveal a promising role in understanding the mystery of the high-Tc superconductivity. Since Lieb and Wu [1], in 1968, solved the 1D Hubbard model with periodic boundary conditions (BC) by coordinate Bethe ansatz, there has been a great deal of papers devoted to the study of the model. A remarkable step was done by Shastry [2] who showed that the Hamiltonian of the 1D Hubbard periodic chain commutes with a one-parameter family of transfer matrix of an equivalent coupled symmetric XY spin chain and who also gave a direct proof of the integrability of the model by presenting the quantum R-matrix. Later on, Wadati and coworkers [3, 4] further studied its integrability in terms of quantum inverse scattering method (QISM) [5, 6]. Very recently, Martins and Ramos [7] proposed a desirable way to solve the eigenvalue problem of the transfer matrix of the 1D Hubbard model with periodic BC by means of algebraic Bethe ansatz. Their approach provides us a unified way to solve a wide class of Hubbard-like models [8, 10] by algebraic Bethe ansatz.

On the other hand, in recent years, there has been much interest in the study of the quantum integrable systems with open BC, i.e. the systems on finite interval with independent boundary conditions on each end. Due to the presence of the boundary fields which lead to a pure back-scattering on each end of the quantum chain and the exhibition of the quantum group symmetry by special choice of the boundary parameters make the system possess rich physical properties [11, 24] in thermodynamical point of view. A systematic approach to handle the open BC for 1D integrable quantum chains was proposed by Sklyanin [12]. A further extension of Sklyanin’s formalism to deal with more general class of models associated with Lie (super) algebras was proposed by Mezincescu and Nepomechie [13]. We also remark that the coordinate Bethe ansatz for 1D Hubbard model with integrable boundary conditions was studied in [17]. By now, though there are several authors [14, 17, 16, 17] have studied the open BC for the 1D Hubbard model, the algebraic Bethe ansatz solution have not yet been achieved. Actually, the diagonalization of the transfer matrix which provide us with the spectrum of all conserved charges should be more essential in studying finite temperature properties of the integrable models [18, 19] than diagonalization of the underlying Hamiltonian. But as we know the reflection equations for the 1D Hubbard model are much more involved and the quantum R-matrix does not
have the additive property that make it difficult to built up the necessary commutation rules among the diagonal fields and creation fields. In this paper, we intend to generalize Sklyanin’s formalism to solve the 1D Hubbard model with open BC. The eigenvalue of the transfer matrix and Bethe ansatz equations for the model will be given. It will be found that the model exhibits a hidden XXX spin open chain which play a crucial role to solve the model.

This paper is organized as follows. In section 2, we shall recall the main results about open BC for the 1D Hubbard model in order to introduce the notations which shall be used in this paper. In section 3, we perform the algebraic Bethe ansatz approach for the model. In section 4, we formulate the nested algebraic Bethe ansatz for the hidden quantum spin open chain and present our main results. Section 5 is devoted to the conclusion.

2 The 1D Hubbard model with boundary fields

Let us consider the 1D Hubbard model with boundary fields determined by the Hamiltonian \([14, 15, 16]\)

\[
H = -\sum_{j=1}^{N-1} \sum_{s} (a_{j+1s}^\dagger a_{js} + a_{js}^\dagger a_{j+1s}) + U \sum_{j=1}^{N} (n_{j\uparrow} - \frac{1}{2})(n_{j\downarrow} - \frac{1}{2}) \tag{1}
\]

\[+ p_+(2n_{1\uparrow} - 1) + p_-(2n_{1\downarrow} - 1) + q_+(2n_{N\uparrow} - 1) + q_-(2n_{N\downarrow} - 1). \]

Here \(p_\pm\) and \(q_\pm\) are the free boundary parameters characterizing the boundary fields. The coupling \(U\) describes the on-site Coulomb interaction and \(a_{js}^\dagger\) and \(a_{js}\) are creation and annihilation operators with spins \((s = \uparrow \text{ or } \downarrow)\) at site \(j\) satisfying the anti-commutation relations

\[\{a_{js}, a_{j's'}^\dagger\} = \{a_{js}^\dagger, a_{j's'}\} = 0, \{a_{js}, a_{j's'}^\dagger\} = \delta_{jj'} \delta_{ss'}., \tag{2}\]

and \(n_{js} = a_{js}^\dagger a_{js}\) is the density operator. The Lax operator is given \([3, 4]\) by

\[
\mathcal{L}_j(u) = \begin{pmatrix}
-e^{h(u)}f_{j\uparrow}f_{j\downarrow} & -f_{j\uparrow}a_{j\downarrow} & ia_{j\uparrow}f_{j\downarrow} & ie^{h(u)}a_{j\uparrow}a_{j\downarrow} \\
-i f_{j\uparrow}a_{j\downarrow}^\dagger & e^{-h(u)}f_{j\uparrow}g_{j\downarrow} & e^{-h(u)}a_{j\uparrow}a_{j\downarrow}^\dagger & ia_{j\uparrow}g_{j\downarrow} \\
a_{j\uparrow}^\dagger f_{j\downarrow} & e^{-h(u)}a_{j\uparrow}a_{j\downarrow} & e^{-h(u)}g_{j\uparrow}f_{j\downarrow} & g_{j\uparrow}a_{j\downarrow} \\
-ie^{h(u)}a_{j\uparrow}a_{j\downarrow}^\dagger & a_{j\uparrow}^\dagger g_{j\downarrow} & ig_{j\uparrow}a_{j\downarrow}^\dagger & -e^{h(u)}g_{j\uparrow}a_{j\downarrow}
\end{pmatrix} \tag{3}
\]

where

\[
f_{js} = \sin u - (\sin u - i \cos u)n_{js}, \quad g_{js} = \cos u - (\cos u + i \sin u)n_{js}.
\]
With the grading $P(1) = P(4) = 0$, $P(2) = P(3) = 1$ and the constraint condition
\[
\frac{\sinh 2h(u)}{\sin 2u} = \frac{U}{4},
\]
the Lax operator satisfies the graded Yang-Baxter algebra
\[
\mathcal{R}_{12}(u, v) \frac{1}{\mathcal{T}}(u) \frac{2}{\mathcal{T}}(v) = \frac{2}{\mathcal{T}}(v) \frac{1}{\mathcal{T}}(u) \mathcal{R}_{12}(u, v),
\]
where the monodromy matrix $\mathcal{T}(u)$ is defined by
\[
\mathcal{T}(u) = \mathcal{L}_N(u) \cdots \mathcal{L}_1(u),
\]
and
\[
\frac{1}{\mathcal{T}}(u) = \mathcal{T}(u) \otimes_s I; \quad \frac{2}{\mathcal{T}}(u) = I \otimes_s \mathcal{T}(u).
\]
here $\otimes_s$ is the super direct product:
\[
[A \otimes_s B]_{\alpha \beta \gamma \delta} = (-1)^{[P(\alpha)+P(\gamma)]P(\beta)} A_{\alpha \gamma} B_{\beta \delta}.
\]
For our convenience in practical calculation, we display the associated quantum $\mathcal{R}_{12}(u, v)$-matrix in Appendix. One may show that $\mathcal{R}_{12}(u, v)$ enjoys the following graded reflection equations:
\[
\mathcal{R}_{12}(u, v) \frac{1}{\mathcal{K}}(u) \mathcal{R}_{21}(v, -u) \frac{2}{\mathcal{K}}(v) = \frac{1}{\mathcal{K}}(v) \mathcal{R}_{12}(u, -v) \frac{1}{\mathcal{K}}(u) \mathcal{R}_{21}((-v, -u)),
\]
which ensure the integrability of the model provided that
\[
\begin{pmatrix}
K_1(u) & 0 & 0 & 0 \\
0 & K_2(u) & 0 & 0 \\
0 & 0 & K_3(u) & 0 \\
0 & 0 & 0 & K_4(u)
\end{pmatrix},
\]
where $p_+ = p_- = \xi_-/2$. 

3
\[
K1_-(u) = \lambda_- (e^{-h(u)} \cos u - e^{h(u)} \xi_- \sin u)(e^{h(u)} \cos u - e^{-h(u)} \xi_- \sin u),
\]
\[
K2_-(u) = \lambda_- (e^{-h(u)} \cos u + e^{h(u)} \xi_- \sin u)(e^{h(u)} \cos u - e^{-h(u)} \xi_- \sin u),
\]
\[
K3_-(u) = \lambda_- (e^{-h(u)} \cos u + e^{h(u)} \xi_- \sin u)(e^{-h(u)} \cos u - e^{h(u)} \xi_- \sin u),
\]
\[
K4_-(u) = \lambda_- (e^{h(u)} \cos u - e^{-h(u)} \xi_- \sin u)(e^{-h(u)} \xi_- \cos u + e^{h(u)} \xi_- \sin u),
\]
\begin{equation}
(11)
\end{equation}

and \(q_+ = q_- = \xi_+/2\)

\[
\]
\[
K1_+(u) = \lambda_+ (e^{-h(u)} \xi_+ \cos u + e^{h(u)} \sin u)(e^{h(u)} \xi_+ \cos u + e^{-h(u)} \sin u),
\]
\[
K2_+(u) = \lambda_+ (e^{h(u)} \xi_+ \cos u - e^{-h(u)} \sin u)(e^{h(u)} \xi_+ \cos u - e^{-h(u)} \sin u),
\]
\[
K3_+(u) = \lambda_+ (e^{h(u)} \xi_+ \cos u + e^{-h(u)} \sin u)(e^{h(u)} \xi_+ \cos u - e^{-h(u)} \sin u),
\]
\[
K4_+(u) = \lambda_+ (e^{h(u)} \xi_+ \cos u - e^{-h(u)} \sin u)(e^{-h(u)} \xi_+ \cos u + e^{h(u)} \sin u).
\]
\begin{equation}
(12)
\end{equation}

Here \(\lambda_\pm\) and \(\xi_\pm\) are arbitrary constants describing boundary effects. \(\widetilde{St}\) stands for the inverse of the supertransposition in the space \(a\). The supertransposition is defined by

\[
(A_{ij})^{\widetilde{St}} = (-1)^{|P(i)+1|P(j)} A_{ji}.
\]

We would like to remark that Zhou \([14]\) first time gave a class of boundary \(K_\mp\)-matrices equivalent to \((11)\) and \((12)\) in terms of QISM. Consequently, using Lax pair formulation, the author \([15]\) presented two class of boundary \(K_\pm\)-matrices leading to four possible boundary terms in the 1D Hubbard open chain Hamiltonian. While, Shiroishi and Wadati \([16]\) studied the open BC for the model in terms of the graded version of QISM and also presented two class of the solutions to the graded RE. The second solution to the graded RE \((8)\) and \((9)\) permits the boundary fields with \(p_+ = -p_-\) and \(q_+ = -q_-\) corresponding to magnetic boundary fields (see ref. \([15, 16]\) ). In this paper, we restrict to study the chemical boundary fields \((11)\) and \((12)\) basing on the consideration that this kind of boundary conditions will bring us a simple boundary \(K\)-matrix for the hidden XXX open chain. To other kinds of boundary conditions, of course, we may treat them in a similar way. It is found that the Hamiltonian \((10)\) is related to the double-row monodromy matrix

\[
\tau(u) = \text{Str}_0 K_+(u) T(u) K_-(u) T^{-1}(-u)
\]

\begin{equation}
(13)
\end{equation}
in the following way

\[ \tau(u) = c_1 u + c_2 u^2 + c_3 (H + \text{const.}) u^3 + \cdots \]  

(14)

where \( c_i, i = 1, \cdots, 4 \), are some scalar functions of boundary parameters. \( \text{Str}_0 \) denotes the supertrace carried out in auxiliary space \( v_0 \).

3 Algebraic Bethe ansatz approach

According to the algebraic Bethe ansatz, let us first choose the standard ferromagnetic pseudovacuum state \( |0\rangle_i \):

\[ |0\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \otimes_s \begin{pmatrix} 1 \\ 0 \end{pmatrix}_i \]  

(15)

as a highest vector, which corresponds to the doubly occupied state. Following the notation introduced in [9], we define the monodromy matrix \( \mathcal{T}(u) \) as

\[
\mathcal{T}(u) = \begin{pmatrix}
B(u) & B_1(u) & B_2(u) & F(u) \\
C_1(u) & A_{11}(u) & A_{12}(u) & E_1(u) \\
C_2(u) & A_{21}(u) & A_{22}(u) & E_2(u) \\
C_3(u) & C_{4}(u) & C_{5}(u) & D(u)
\end{pmatrix}
\]  

(16)

and

\[
\mathcal{T}^{-1}(-u) = \begin{pmatrix}
\bar{C}_1(u) & \bar{A}_{11}(u) & \bar{A}_{12}(u) & \bar{E}_1(u) \\
\bar{C}_2(u) & \bar{A}_{21}(u) & \bar{A}_{22}(u) & \bar{E}_2(u) \\
\bar{C}_3(u) & \bar{C}_{4}(u) & \bar{C}_{5}(u) & \bar{D}(u)
\end{pmatrix}
\]  

(17)

\[
\mathcal{T}_-(u) = \mathcal{T}(u) K_-(u) \mathcal{T}^{-1}(-u)
\]

\[
= \begin{pmatrix}
\tilde{B}(u) & \tilde{B}_1(u) & \tilde{B}_2(u) & \tilde{F}(u) \\
\tilde{C}_1(u) & \tilde{A}_{11}(u) & \tilde{A}_{12}(u) & \tilde{E}_1(u) \\
\tilde{C}_2(u) & \tilde{A}_{21}(u) & \tilde{A}_{22}(u) & \tilde{E}_2(u) \\
\tilde{C}_3(u) & \tilde{C}_{4}(u) & \tilde{C}_{5}(u) & \tilde{D}(u)
\end{pmatrix}
\]  

(18)

(19)

It is not difficult to show that \( \mathcal{T}_-(u) \) also satisfies the RE [8]. Acting \( \mathcal{T}(u) \) and \( \mathcal{T}^{-1}(-u) \) on the pseudovacuum state

\[ |0\rangle = \otimes_{i=1}^{N} |0\rangle_i, \]  

(20)
we have the following properties (upon a common factor):

\[
B(u)|0\rangle = \widetilde{B}(u)|0\rangle = \left\{ \frac{\cos u}{\sin u} e^{2h(u)} \right\}^N |0\rangle,
\]

\[
D(u)|0\rangle = \widetilde{D}(u)|0\rangle = \left\{ \frac{\sin u}{\cos u} e^{2h(u)} \right\}^N |0\rangle,
\]

\[
A_{aa}(u)|0\rangle = \widetilde{A}_{aa}(u)|0\rangle = |0\rangle, \ a = 1, 2,
\]

\[
A_{ab}(u)|0\rangle = \widetilde{A}_{ab}(u)|0\rangle = 0, \ a \neq b = 1, 2,
\]

\[
B_a(u)|0\rangle \neq 0, \ \widetilde{B}_a(u)|0\rangle \neq 0, \ a = 1, 2,
\]

\[
E_a(u)|0\rangle \neq 0, \ \widetilde{E}_a(u)|0\rangle \neq 0, \ a = 1, 2,
\]

\[
F(u)|0\rangle \neq 0, \ \widetilde{F}(u)|0\rangle \neq 0,
\]

\[
C_i(u)|0\rangle = \widetilde{C}_i(u)|0\rangle = 0, \ i = 1, \cdots, 5.
\]

Using the properties [21], and the Yang-Baxter algebra

\[
\mathcal{T}^{2^{-1}} (-u) \mathcal{R}_{12}(u, -u) \mathcal{T} (u) = \mathcal{T} (u) \mathcal{R}_{12}(u, -u) \mathcal{T}^{2^{-1}} (-u), \tag{22}
\]

and after some algebra, one can obtain

\[
\widetilde{B}(u)|0\rangle = W_1^- (u) B(u) \widetilde{B}(u)|0\rangle, \tag{23}
\]

\[
\widetilde{A}_{aa}(u)|0\rangle = \left\{ \frac{\rho_3(u, -u)}{\rho_1(u, -u)} B(u) \widetilde{B}(u) + W_2^- (u) A_{aa}(u) \widetilde{A}_{aa}(u) \right\}|0\rangle, \tag{24}
\]

\[
\widetilde{D}(u)|0\rangle = \left\{ \frac{1}{\rho_4(u, -u)} (K2_-(u) - \frac{\rho_2(u, -u)}{\rho_1(u, -u)} \sum_{a=1}^2 A_{aa}(u) \widetilde{A}_{aa}(u))
\right.
\]

\[
+ \left. \frac{\rho_3(u, -u)}{\rho_1(u, -u)} B(u) \widetilde{B}(u) + W_3^- (u) D(u) \widetilde{D}(u) \right\} |0\rangle, \tag{25}
\]

\[
\widetilde{B}_a(u)|0\rangle \neq 0, \ \widetilde{E}_a(u)|0\rangle \neq 0, \ a = 1, 2,
\]

\[
\widetilde{A}_{ab}(u)|0\rangle = 0, \ a \neq b = 1, 2, \ \widetilde{F}(u) \neq 0,
\]

\[
\widetilde{C}_i(u)|0\rangle = 0, \ i = 1, \cdots, 5, \tag{28}
\]

where

\[
W_1^- (u) = 1,
\]

\[
W_2^- (u) = - \frac{(e^{-2h(u)} + e^{2h(u)}) \sin u \cos u (\xi e^{h(u)} \cos u - e^{-h(u)} \sin u)}{(e^{2h(u)} \cos^2 u - e^{-2h(u)} \sin^2 u)(\xi e^{-h(u)} \sin u - e^{h(u)} \cos u)}.
\]

\[6\]
\[
W_4^-(u) = \frac{(e^{-2h(u)} + e^{2h(u)}) \sin u \cos u \sin 2u}{\cos 2u(e^{-2h(u)} \cos^2 u - e^{2h(u)} \sin^2 u)} \times \frac{(e^{-h(u)} \xi_+ \cos u - e^{h(u)} \sin u)(e^{h(u)} \xi_+ \cos u - e^{-h(u)} \sin u)}{(e^{-h(u)} \cos u - e^{h(u)} \sin u)(e^{h(u)} \cos u - e^{-h(u)} \sin u)}.
\]

We also notice that the operators \(\tilde{B}_a(u), \tilde{E}_a(u)\) and \(\tilde{F}(u)\) are still creation fields, otherwise, \(\hat{C}_i(u)\) are the annihilation fields. Via the transformations

\[
\tilde{A}'_{aa}(u) = \tilde{A}_{aa}(u) - \frac{\rho_2(u,-u)}{\rho_1(u,-u)} \tilde{B}(u),
\]
\[
D'(u) = D(u) - \frac{\rho_3(u,-u)}{\rho_1(u,-u)} \tilde{B}(u) - \frac{1}{\rho_4(u,-u)} \sum_{a=1}^{2} A'_{aa}(u),
\]

we may express the transfer matrix \([13]\) in the following way

\[
\tau(u) = \text{Str}_0 K_+(u) T_-(u) = W_1^+(u) \tilde{B}(u) + W_2^+(u) \sum_{a=1}^{2} \tilde{A}'_{aa}(u) + W_4^+(u) \tilde{D}'(u),
\]

where

\[
W_1^+(u) = \frac{(e^{-2h(u)} + e^{2h(u)}) \sin u \cos u \sin 2u}{\cos 2u(e^{-2h(u)} \cos^2 u - e^{2h(u)} \sin^2 u)} f(u) \times \frac{(e^{-h(u)} \xi_+ \sin u - e^{h(u)} \cos u)(e^{h(u)} \xi_+ \sin u - e^{-h(u)} \cos u)}{(e^{-h(u)} \xi_+ \cos u + e^{h(u)} \sin u)(e^{h(u)} \xi_+ \cos u + e^{-h(u)} \sin u)},
\]
\[
W_2^+(u) = \frac{(e^{-2h(u)} + e^{2h(u)}) \sin u \cos u}{(e^{-2h(u)} \cos^2 u - e^{2h(u)} \sin^2 u)} f(u) \times \frac{(e^{h(u)} \xi_+ \cos u - e^{-h(u)} \sin u)(e^{-h(u)} \xi_+ \cos u - e^{h(u)} \sin u)}{(e^{-h(u)} \xi_+ \cos u + e^{h(u)} \sin u)(e^{h(u)} \xi_+ \cos u + e^{-h(u)} \sin u)},
\]
\[
W_4^+ = \frac{(e^{-h(u)} \xi_+ \cos u - e^{h(u)} \sin u)(e^{h(u)} \xi_+ \cos u - e^{-h(u)} \sin u)}{(e^{-h(u)} \xi_+ \cos u + e^{h(u)} \sin u)(e^{h(u)} \xi_+ \cos u + e^{-h(u)} \sin u)} f(u)
\]

with

\[
f(u) = e^{-2Nh(u)} \cos^{2N} u \sin^{2N} u K_1^-(u) K_1^+(u).
\]

Now we proceed the key step to built up the necessary commutation relations between the diagonal fields and the creation fields respectively. From the RE \([8]\) and definition \([14]\), after many steps of substitution, we can get the following important commutation relations:

\[
\tilde{B}(u) \tilde{B}_a(v) = \frac{\rho_1(v,u) \rho_{10}(u,-v)}{\rho_1(v,-u) \rho_{10}(-u,-v)} \tilde{B}_a(v) \tilde{B}(u) + u.t.,
\]

\[
\tilde{B}(u) \tilde{F}(u) = \frac{e^{2h(u)} - e^{-2h(u)}}{e^{-h(u)} \cos u - e^{h(u)} \sin u} \tilde{F}(u) \tilde{B}(u) + u.t.,
\]

\[
\tilde{B}(u) \tilde{E}_a(u) = \frac{\rho_{10}(u,\xi)}{\rho_{10}(u,\xi)} \tilde{E}_a(u) \tilde{B}(u) + u.t.,
\]

\[
\tilde{F}(u) \tilde{E}_a(u) = \frac{\rho_{10}(u,\xi)}{\rho_{10}(u,\xi)} \tilde{E}_a(u) \tilde{F}(u) + u.t.,
\]

\[
\tilde{B}_a(u) \tilde{F}(u) = \frac{e^{2h(u)} - e^{-2h(u)}}{e^{-h(u)} \cos u - e^{h(u)} \sin u} \tilde{F}(u) \tilde{B}_a(u) + u.t.,
\]

\[
\tilde{B}_a(u) \tilde{E}_a(u) = \frac{\rho_{10}(u,\xi)}{\rho_{10}(u,\xi)} \tilde{E}_a(u) \tilde{B}_a(u) + u.t.,
\]

\[
\tilde{F}(u) \tilde{E}_a(u) = \frac{\rho_{10}(u,\xi)}{\rho_{10}(u,\xi)} \tilde{E}_a(u) \tilde{F}(u) + u.t.,
\]
\[ \tilde{D}'(u) \tilde{D}_a(v) = -\frac{\rho_7(u,-v)\rho_9(-v,-u)}{\rho_9(u,-v)\rho_8(u,v)} \tilde{D}_a(v) \tilde{D}'(u) + u.t., \] (40)

\[ \tilde{A}_ab(u) \tilde{D}_a(v) = -\frac{\rho_4(-v,-u)\rho_1(u,v)}{\rho_1(u,-v)\rho_8(u,v)} \tilde{A}_ch(u,-v) \tilde{\rho}_{ch}^{ih} (-v,-u) \tilde{D}_c(v) \tilde{A}_a^i(u) + u.t., \] (41)

\[ \tilde{B}_a(u) \otimes \tilde{B}_b(v) = \frac{\rho_{10}(u,-v)\rho_4(-v,-u)}{\rho_1(u,v)\rho_8(u,v)} \left\{ \tilde{B}_c(v) \otimes \tilde{B}_d(u) \right\} \frac{\rho_6(u,-v)}{\rho_{10}(u,-v)} F(v) \tilde{\xi} (I \otimes \tilde{A}(u)) \tilde{\tilde{\rho}} (-v,-u) + \frac{\rho_6(v,-u)}{\rho_{10}(v,-u)} F(u) \tilde{\xi} (I \otimes \tilde{A}(v)) + \frac{\rho_8(v,-u)\rho_6(-v,-u)}{\rho_{10}(v,-u)\rho_8(-v,-u)} [\tilde{F}(v) \tilde{B}(u) - \tilde{F}(u) \tilde{B}(v)] \tilde{\tilde{\xi}}, \] (42)

where

\[ \tilde{\xi} = (0, 1, -1, 0), \]

\[ r(u,-v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(u,-v) & b(u,-v) & 0 \\ 0 & b(u,-v) & a(u,-v) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \] (43)

\[ \tilde{r} (-v,-u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \tilde{a} (-v,-u) & \tilde{b} (-v,-u) & 0 \\ 0 & \tilde{b} (-v,-u) & \tilde{a} (-v,-u) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \] (44)

with the weights

\[ a (u,-v) = \frac{\rho_3(u,-v)\rho_4(u,-v) - 1}{\rho_4(u,-v)\rho_8(u,v)}, \quad b (u,-v) = 1 - a (u,-v), \] (45)

\[ \tilde{a} (-v,-u) = \frac{\rho_5(-v,-u)\rho_6(-v,-u) + \rho_5^2(-v,-u)}{\rho_4(-v,-u)\rho_8(-v,-u)}, \quad \tilde{b} (-v,-u) = 1 - \tilde{a} (-v,-u). \] (46)

In the commutation relations (43)-(44), we had to omit all unwanted terms because they take a big space to display. It turns out that the auxiliary matrices \( r(u,-v) \) and \( \tilde{r} (-v,-u) \) are nothing but the rational \( R \)-matrices of isotropic six-vertex model. The structure of the auxiliary matrices is very important to solve the Hubbard-like \([10, 21]\) models with open BC that exhibit a similar structure of the auxiliary matrices like Eq. (13) and (14). If performing the parameterization introduced in \([7]\),

\[ \tilde{x} = \frac{\sin x}{\cos x} e^{-2h(x)} - \frac{\cos x}{\sin x} e^{2h(x)}, \quad x = u, v, \] (47)
one may also find

$$a(u, -v) = \frac{U}{\check{u} + \check{v} + U}, \quad b(u, -v) = \frac{\check{u} + \check{v}}{\check{u} + \check{v} + U},$$

(48)

$$\check{a}(-v, -u) = \frac{U}{\check{u} - \check{v} + U}, \quad \check{b}(-v, -u) = \frac{\check{u} - \check{v}}{\check{u} - \check{v} + U}.$$  

(49)

In view of the commutation relation (42), the creation operators $\check{B}_a, \check{E}_a$ do not interwine. So it is reasonable that the eigenvectors of the transfer matrices are generated only by the creation operators $B_a(u)$ and $F(u)$ or $E_a(u)$ and $F(u)$. Unfortunately, it seems to be very difficult to construct the explicit form of the multi-particle vector even in the case of the Hubbard periodic chain [7]. But it does have a similar recursive relation as that for the Hubbard periodic chain. Here we prefer the $n$-particle vector in a formal form, namely

$$| \Phi_n(v_1, \cdots, v_n) \rangle = \Phi_n(v_1, \cdots, v_n) F^{a_1, \cdots, a_n} |0\rangle.$$  

(50)

Where the $n$-particle vector $\Phi_n(v_1, \cdots, v_n)$ may be given by a recursive relation

$$\Phi_n(v_1, \cdots, v_n) = \check{B}_a(v_1) \otimes \Phi_{n-1}(v_2, \cdots, v_n)$$

$$+ \sum_{j=2}^{n} [\xi \otimes \check{F}(v_1)] \Phi_{n-2}(v_2, \cdots, v_{j-1}, v_{j+1}, \cdots, v_n) \check{B}(v_j) g^{(n)}_{j-1}(v_1, \cdots, v_n)$$

$$- \sum_{j=2}^{n} [\xi \otimes \check{F}(v_1)] \Phi_{n-2}(v_2, \cdots, v_{j-1}, v_{j+1}, \cdots, v_n) (I \otimes \hat{A}(u)) h^{(n)}_{j-1}(v_1, \cdots, v_n).$$

(51)

From the commutation relation (42), we can conclude that $\Phi_n(v_1, \cdots, v_n)$ also satisfies the symmetry relation

$$\Phi_n(v_1, \cdots, v_j, v_{j+1}, \cdots, v_n) = \frac{\rho_{10}(v_j, -v_{j+1}) \rho_4(-v_{j+1}, -v_j)}{\rho_4(v_j, v_{j+1}) \rho_{10}(v_{j+1}, -v_j)} \times$$

$$\Phi_n(v_1, \cdots, v_j, v_{j+1}, \cdots, v_n), \quad \check{r}(-v_{j+1}, -v_j)$$

(52)

basing on the following relation:

$$\frac{\rho_4(-v_{j+1}, -v_j)}{\rho_4(v_j, v_{j+1})} \xi . \check{r}(-v_{j+1}, -v_j) = \frac{\rho_8(v_{j+1}, -v_j) \rho_6(-v_{j+1}, -v_j) \rho_8(-v_j, -v_{j+1})}{\rho_8(v_j, v_{j+1}) \rho_6(-v_j, -v_{j+1})} \xi .$$

(53)

This symmetry giving a restriction to the functions $h^{(n)}_{j-1}(v_1, \cdots, v_n)$ and $g^{(n)}_{j-1}(v_1, \cdots, v_n)$ is very useful to deduce the coefficients and simplify the unwanted terms in the eigenvalue of the transfer matrix. In fact, after performing three-particle scattering, the explicit form of these coefficients can be fixed. But
checking three-particle scattering is indeed a extremely tough work, we had to leave the coefficients to be determined. Explicitly, we display the two-particle state

\[
\Phi_2(v_1, v_2) = \hat{B}_{a_1}(v_1) \otimes \hat{B}_{a_2}(v_2) - \frac{\rho_6(v_2, -v_1)}{\rho_{10}(v_2, -v_1)} F(v_1) \xi (I \otimes \hat{A}(v_2)) + \frac{\rho_8(v_2, -v_1)\rho_6(-v_2, -v_1)}{\rho_{10}(v_2, -v_1)\rho_8(-v_2, -v_1)} \tilde{F}(v_1) \tilde{B}(v_2). \xi .
\]  

(54)

In above expressions, \( F^{a_1, \cdots, a_n} \) are the coefficients of arbitrary linear combination of the vectors reflect the "spin" degrees of freedom with \( a_i = 1, 2 \) and \( \xi \) plays the role of forbidding two spin up or two spin down at same site. \( \tilde{F}(u) \) creates a local hole pair with opposite spins. Acting the diagonal fields on the vector \((50)\), we way phenomenologically get

\[
\begin{align*}
\hat{B}(u) | \Phi_n(v_1, \cdots, v_n) \rangle &= \hat{B}(u) \prod_{i=1}^{n} \frac{\rho_1(v_i, u)\rho_{10}(u, -v_i)}{\rho_1(v_i, -u)\rho_{10}(-u, -v_i)} | \Phi_n(v_1, \cdots, v_n) \rangle + u.t., \\
\hat{D}(u) | \Phi_n(v_1, \cdots, v_n) \rangle &= \hat{D}(u) \prod_{i=1}^{n} \frac{\rho_7(u, -v_i)\rho_9(-v_i, -u)}{\rho_9(u, -v_i)\rho_8(u, v_i)} | \Phi_n(v_1, \cdots, v_n) \rangle + u.t., \\
\hat{A}_{a a}(u) | \Phi_n(v_1, \cdots, v_n) \rangle &= \hat{A}_{a a}(u) \prod_{i=1}^{n} \frac{\rho_4(-v_i, -u)\rho_{10}(u, -v_i)}{\rho_1(u, -v_i)\rho_9(u, v_i)} r_{12}(\tilde{u} + \tilde{v})^{e_1 a}_{h_{1} h_{1}} \times \\
&\quad r_{12}(\tilde{u} - \tilde{v})^{i_{1} h_{1}}_{a_{1} h_{1} i_{1}} r_{12}(\tilde{u} + \tilde{v})^{e_2 g_1}_{i_{2} g_2} r_{12}(\tilde{u} - \tilde{v})^{i_{2} b_2}_{i_{2} b_2} \cdots \\
&\quad r_{12}(\tilde{u} + \tilde{v})^{e_n g_n - 1}_{h_n g_n} r_{12}(\tilde{u} - \tilde{v})^{i_n h_n}_{i_{n-1} i_{n-1}} | \Phi_n(v_1, \cdots, v_n) \rangle + u.t.
\end{align*}
\]  

(57)

It follows that

\[
\begin{align*}
\tau(u) | \Phi_n(v_1, \cdots, v_n) \rangle &= \left\{ W_{1}^{+}(u) \hat{B}(u) \prod_{i=1}^{n} \frac{\rho_1(v_i, u)\rho_{10}(u, -v_i)}{\rho_1(v_i, -u)\rho_{10}(-u, -v_i)} \\
&\quad + W_{2}^{+}(u) \hat{D}'(u) \prod_{i=1}^{n} \frac{\rho_7(u, -v_i)\rho_9(-v_i, -u)}{\rho_9(u, -v_i)\rho_8(u, v_i)} \\
&\quad + W_{2}^{+}(u) \hat{A}_{a a}'(u) \prod_{i=1}^{n} \frac{\rho_4(-v_i, -u)\rho_{10}(u, -v_i)}{\rho_1(u, -v_i)\rho_9(u, v_i)} \Lambda^{(1)}(\tilde{u}, \{ \tilde{v}_i \}) \right\} | \Phi_n(v_1, \cdots, v_n) \rangle
\end{align*}
\]  

(58)

provided that

\[
\frac{W_{1}^{+}(u) \hat{B}(u)}{W_{2}^{+}(u) \hat{A}_{1}(u)} |_{u = v_i = \Lambda^{(1)}(\tilde{u} = \tilde{v}_i, \{ \tilde{v}_i \})} = 1, \cdots n.
\]  

(59)

Here \( r_{12}(u) = P r(u) \) and \( \Lambda^{(1)}(\tilde{u}, \{ \tilde{v}_i \}) \) is the eigenvalue of the nested transfer matrix \((57)\), i.e.

\[
\tau^{(1)}(\tilde{u}, \{ \tilde{v}_i \}) F^{e_1 \cdots e_n} = \Lambda^{(1)}(\tilde{u}, \{ \tilde{v}_i \}) F^{e_1 \cdots e_n},
\]  

(60)
where
\[ \tau^{(1)}(\tilde{u}, \{\tilde{v}_i\}) = \text{Tr}_0 T^{(1)}(\tilde{u})T^{(1)^{-1}}(-\tilde{u}). \] (61)

The nested monodromy matrices \( T^{(1)}(\tilde{u}) \) and \( T^{(1)^{-1}}(-\tilde{u}) \) read
\[
T^{(1)}(\tilde{u}) = r_{12}(\tilde{u} + \tilde{v}_1)c_{12} \cdots r_{12}(\tilde{u} + \tilde{v}_n)c_{n}\delta_{n-1},
\]
(62)
\[
T^{(1)^{-1}}(-\tilde{u}) = r_{12}(\tilde{u} - \tilde{v}_1)c_{12} \cdots r_{12}(\tilde{u} - \tilde{v}_1)c_{12}\delta_{1}. \] (63)

So far, the eigenvalue problem of the 1D Hubbard model with boundaries reduces to solve the nested auxiliary transfer matrix \( \Theta \) which corresponds to an isotropic six-vertex model with open boundary conditions.

4 The nested Bethe ansatz

In this section, we proceed the diagonalization of the auxiliary transfer matrix \( \Theta \). Following Sklyanin’s formalism \([12]\), performing the nested Bethe ansatz have not been a difficult problem yet.

It is easy to check that the \( r_{12}(u) \)-matrix satisfies the Yang-Baxter algebra
\[
r_{12}(u_1 - u_2) \frac{1}{T^{(1)}(u_1, \{v_i\})} T^{(1)}(u_2, \{\tilde{v}_i\}) = \frac{2}{T^{(1)}(\tilde{u}_2, \{\tilde{v}_i\})} T^{(1)}(\tilde{u}_1, \{\tilde{v}_i\}) T^{(1)}(u_1 - u_2), \] (64)

and the reflection equations
\[
r_{12}(\tilde{u}_1 - \tilde{u}_2) K^{(1)}_{-}(\tilde{u}_1) r_{12}(\tilde{u}_1 + \tilde{u}_2) K^{(1)}_{-}(\tilde{u}_2) = K^{(1)}_{-}(\tilde{u}_2) r_{12}(\tilde{u}_1 + \tilde{u}_2) K^{(1)}_{-}(\tilde{u}_1) r_{12}(\tilde{u}_1 - \tilde{u}_2), \] (65)
\[
r_{12}(\tilde{u}_2 - \tilde{u}_1) K^{(1)}_{+}(\tilde{u}_1) r_{12}(-\tilde{u}_1 - \tilde{u}_2 - 2U) K^{(1)}_{+}(\tilde{u}_2) = K^{(1)}_{+}(\tilde{u}_2) r_{12}(-\tilde{u}_1 - \tilde{u}_2 - 2U) K^{(1)}_{+}(\tilde{u}_1) r_{12}(\tilde{u}_2 - \tilde{u}_1). \] (66)

For our case, the \( K^{(1)}_{\pm}(u) = I \). Let us define the nested monodromy matrix
\[
\tilde{T}^{(1)}(\tilde{u}) = T^{(1)}(\tilde{u})T^{(1)^{-1}}(-\tilde{u}) = \left( \begin{array}{cc} \tilde{A}^{(1)}(\tilde{u}) & \tilde{B}^{(1)}(\tilde{u}) \\ \tilde{C}^{(1)}(\tilde{u}) & \tilde{D}^{(1)}(\tilde{u}) \end{array} \right) \] (67)

which also satisfies the RE \([63]\). Using the main ingredients \([64]\) describing the open BC compatible with the integrability of the model and following all steps solving XXZ open chain in \([12]\), one
can present the following results:

\[ \Lambda^{(1)}(\tilde{u}, \{\tilde{u}_1 \cdots \tilde{u}_M\}; \{\tilde{v}_i\}) = \Phi^{(1)}(\tilde{u}_l; \{\tilde{v}_i\}) = \left\{ \begin{array}{l} \frac{2(\tilde{u} + U)}{2\tilde{u} + U} \prod_{l=1}^{M} \frac{(\tilde{u} + \tilde{u}_l)(\tilde{u} - \tilde{u}_l - U)}{(\tilde{u} - \tilde{u}_l)(\tilde{u} + \tilde{u}_l + U)} \\
\frac{2\tilde{u}}{2\tilde{u} + U} \prod_{i=1}^{n} b(\tilde{u} + \tilde{v}_i) b(\tilde{u} - \tilde{v}_i) \prod_{l=1}^{M} \frac{(\tilde{u} + \tilde{u}_l + 2U)(\tilde{u} - \tilde{u}_l + U)}{(\tilde{u} - \tilde{u}_l)(\tilde{u} + \tilde{u}_l + U)} \end{array} \right\} \Phi^{(1)}(\tilde{u}_l; \{\tilde{v}_i\}) \]  

(68)

provided that

\[ \prod_{i=1}^{n} \frac{(\tilde{u}_j + \tilde{v}_i + U)(\tilde{u}_j - \tilde{v}_i + U)}{(\tilde{u}_j + \tilde{v}_i)(\tilde{u}_j - \tilde{v}_i)} = \prod_{l=1}^{M} \frac{(\tilde{u}_j + \tilde{u}_l + 2U)(\tilde{u}_j - \tilde{u}_l + U)}{(\tilde{u}_j + \tilde{u}_l)(\tilde{u}_j - \tilde{u}_l - U)}, \]  

(69)

which indeed ensure the cancellation of all unwanted terms in \[68\). Here the "spin" part of the multi-particle states is given by

\[ | \Phi^{(1)}(\tilde{u}_l; \{\tilde{v}_i\}) \rangle = B^{(1)}(\tilde{u}_1) \cdots B^{(1)}(\tilde{u}_M)|0\rangle, \]  

(70)

where \( M \) is the number of holes with spin down, \( n \) is the total number of the holes.

Finally, if we adopt the variables \( z_\pm(v_i) \) used in \[7\), i.e.

\[ z_-(v_i) = \frac{\cos v_i}{\sin v_i} e^{2h(v_i)}, \quad z_+(v_i) = \frac{\sin v_i}{\cos v_i} e^{2h(v_i)}, \]  

(71)

and make a shift \( \tilde{u}_j = \tilde{\lambda}_j - U/2 \), the eigenvalue of the transfer matrix \[13\] is given as

\[ \tau(u) | \Phi_n(v_1, \cdots, v_n) \rangle = \left\{ \begin{array}{l} W_1^+(u) W_1^-(u) [z_-(u)]^{2N} \prod_{i=1}^{n} \frac{\sin^2 u(1 + z_-(v_i)/z_+(u))(1 + 1/z_-(v_i)z_+(u))}{\cos^2 u(1 - z_-(v_i)/z_-(u))(1 - 1/z_-(v_i)z_-(u))} \\
+ W_4^+(u) W_4^- (u) [z_+(u)]^{2N} \prod_{i=1}^{n} \frac{\sin^2 u(1 + z_-(v_i)z_+(u))(1 + z_-(u)/z_-(v_i))}{\cos^2 u(1 - z_+(v_i)z_+(u))(1 - z_+(u)/z_+(v_i))} \\
+ W_2^+(u) W_2^- (u) \frac{2(\tilde{u} + U)}{2\tilde{u} + U} \prod_{i=1}^{n} \frac{\sin^2 u(1 + z_-(v_i)/z_+(u))(1 + z_-(v_i)z_+(u))}{\cos^2 u(1 - z_-(v_i)/z_-(u))(1 - z_-(v_i)z_-(u))} \times \prod_{l=1}^{M} \frac{(\tilde{u} + \tilde{\lambda}_l - U/2)(\tilde{u} - \tilde{\lambda}_l - U/2)}{(\tilde{u} - \tilde{\lambda}_l + U/2)(\tilde{u} + \tilde{\lambda}_l + U/2)} \\
+ W_2^+(u) W_2^- (u) \frac{2\tilde{u}}{2\tilde{u} + U} \prod_{i=1}^{n} \frac{\sin^2 u(1 + z_-(v_i)z_+(u))(1 + z_+(u)/z_-(v_i))}{\cos^2 u(1 - z_-(v_i)z_+(u))(1 - z_+(u)/z_-(v_i))} \times \prod_{l=1}^{M} \frac{(\tilde{u} + \tilde{\lambda}_l + 3U/2)(\tilde{u} - \tilde{\lambda}_l + 3U/2)}{(\tilde{u} - \tilde{\lambda}_l + U/2)(\tilde{u} + \tilde{\lambda}_l + U/2)} \end{array} \right\} \Phi_n(v_1, \cdots, v_n), \]  

(72)
provided that
\[
\zeta(v_i, \xi_+ \xi_-)[z_-(v_i)]^{2N} = \prod_{i=1}^{M} \frac{(\tilde{\nu}_i + \tilde{\lambda}_l - U/2)(\tilde{\nu}_i - \tilde{\lambda}_l - U/2)}{(\tilde{\nu}_i - \tilde{\lambda}_l + U/2)(\tilde{\nu}_i + \tilde{\lambda}_l + U/2)} \tag{73}
\]
\[
\prod_{i=1}^{n} \frac{(\tilde{\lambda}_j + \tilde{v}_i + U/2)(\tilde{\lambda}_j - \tilde{v}_i + U/2)}{(\tilde{\lambda}_j + \tilde{v}_i - U/2)(\tilde{\lambda}_j - \tilde{v}_i - U/2)} = \prod_{l=1}^{M} \frac{(\tilde{\lambda}_j + \tilde{\lambda}_l + U)(\tilde{\lambda}_j - \tilde{\lambda}_l + U)}{(\tilde{\lambda}_j + \tilde{\lambda}_l - U)(\tilde{\lambda}_j - \tilde{\lambda}_l - U)}, \tag{74}
\]
where
\[
\zeta(u, \xi_\pm) = \frac{e^{-h(u)} \xi_\pm \sin u - e^{h(u)} \cos u}{e^{h(u)} \xi_\pm \cos u - e^{-h(u)} \sin u}. \tag{75}
\]
If we express the variable \(z_-(u_i)\) in terms of the momenta \(k_i\) (hole) by \(z_-(u_i) = e^{2k_i}\), from the relation (14), the energy is given by
\[
E_n = \xi_- + \xi_+ - (N/2 - n)U - \sum_{i=1}^{n} 4 \cos k_i. \tag{76}
\]
Equations (72)-(76) constitute our main results of this paper. It is found that the boundary fields are indeed nontrivial to the ground state properties and the boundary energy of the model.

5 Conclusion

We have formulated the algebraic Bethe ansatz solution for the 1D Hubbard model with open boundaries. Bethe ansatz equations, the eigenvalue of the transfer matrix and energy spectrum have also been given. Comparing our results with coordinate Bethe ansatz solution [17], the Bethe ansatz equations (73) and (74) coincide with ones obtained in [17], apart from this, we presented explicitly the eigenvalue of the transfer matrix and conjectured the main structure of the \(n\)-particle eigenvectors. The results obtained provides us with a start point to study the thermodynamical properties of the model [6, 18]. Especially, the proposed way is available to formulate the algebraic Bethe ansatz for other extented Hubbard models with open BC, such as 1D Bariev open chain [26, 10], \(U_q[Osp(2|2)]\) electronic system [21, 2], etc. We also notice that if we add the chemical potential term \(\nu \sum_{j=1}^{N} \sum_{s}(n_{js} - 1/2)\) to the Hamiltonian (11), the integrability of the model require the associated quantum \(R\)-matrix [22], which does not have crossing unitarity, should satisfy new RE. But the new class of the boundary \(K_\pm\)-matrices [23] shall not change the Bethe ansatz equations (73) and (74). Nevertheless, if we add
the Kondo impurities \[ J \sum_{ss'} a_s^{\dagger} \sigma_{ss'} a_{s'} \cdot S \] to each boundaries, the model is also integrable with a certain boundary \( K_{\pm} \)-matrices which lead to new Bethe ansatz equations. We hope following this paper we shall present a class of integrable Kondo impurities for the 1D Hubbard model in near future.
Appendix

We display the quantum $\mathcal{R}(u,v)$-matrix of the 1D Hubbard model below

\[
\begin{pmatrix}
\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -i\rho_{10} & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -i\rho_{10} & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_8 & 0 & 0 & i\rho_6 & 0 & -i\rho_6 & 0 & 0 & \rho_3 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_2 & 0 & 0 & i\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\rho_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & i\rho_6 & 0 & 0 & -\rho_7 & 0 & 0 & -\rho_5 & 0 & 0 & -i\rho_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_9 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & i\rho_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i\rho_6 & 0 & 0 & -\rho_5 & 0 & 0 & -\rho_7 & 0 & 0 & i\rho_6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\rho_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \rho_3 & 0 & 0 & -i\rho_6 & 0 & 0 & i\rho_6 & 0 & 0 & \rho_8 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & 0 & 0 & 0 & \rho_2 & 0 & 0 & -i\rho_{10} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_1 \\
\end{pmatrix},
\]

with the Boltzmann weights

\[
\begin{align*}
\rho_1 &= (\cos u \cos v e^l + \sin v \sin u e^{-l})\rho_2, \\
\rho_4 &= (\cos u \cos v e^{-l} + \sin v \sin u e^l)\rho_2, \\
\rho_9 &= (\sin u \cos v e^{-l} - \sin v \cos v e^l)\rho_2, \\
\rho_{10} &= (\sin u \cos v e^l - \sin v \cos u e^{-l})\rho_2, \\
\rho_3 &= \frac{(\cos u \cos v e^l - \sin v \sin u e^{-l})}{\cos^2 u - \sin^2 v}\rho_2, \\
\rho_5 &= \frac{(\cos u \cos v e^{-l} - \sin v \sin u e^l)}{\cos^2 u - \sin^2 v}\rho_2, \\
\rho_6 &= \frac{e^{-h}(\cos u \sin u e^l - \sin v \cos v e^{-l})}{\cos^2 u - \sin^2 v}\rho_2,
\end{align*}
\]

and

\[
\begin{align*}
\rho_8 &= \rho_1 - \rho_3; \quad \rho_7 = \rho_4 - \rho_5, \\
l &= h(u) - h(v), \quad h = h(u) + h(v)
\end{align*}
\]
which enjoy the following identities:

\[ \rho_4 \rho_1 + \rho_9 \rho_{10} = 1, \rho_1 \rho_5 + \rho_3 \rho_4 = 2, \]

\[ \rho_6^2 = \rho_3 \rho_5 - 1, \rho_6^2 = \rho_9 \rho_{10} + \rho_7 \rho_8. \]

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