TWO-BUBBLE DYNAMICS FOR THRESHOLD SOLUTIONS TO THE WAVE MAPS EQUATION

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Abstract. We consider the energy-critical wave maps equation $\mathbb{R}^{1+2} \to S^2$ in the equivariant case, with equivariance degree $k \geq 2$. It is known that initial data of energy $< 8\pi k$ and topological degree zero leads to global solutions that scatter in both time directions. We consider the threshold case of energy $8\pi k$.

We prove that the solution is defined for all time and either scatters in both time directions, or converges to a superposition of two harmonic maps in one time direction and scatters in the other time direction. In the latter case, we describe the asymptotic behavior of the scales of the two harmonic maps.

The proof combines the classical concentration-compactness techniques of Kenig-Merle with a modulation analysis of interactions of two harmonic maps in the absence of excess radiation.

1. Introduction

This paper concerns energy critical wave maps $\Psi : (\mathbb{R}^{1+2}, \text{m}) \to (\mathcal{M}, \text{g})$, where $\text{m}$ is the Minkowski metric and $\mathcal{M}$ is a Riemannian manifold with a metric $\text{g}$. Wave maps arise in the physics literature as examples of nonlinear $\sigma$-models. A particularly interesting case is when the target manifold admits nontrivial finite energy stationary wave maps, or harmonic maps, as these give simple examples of topological (albeit unstable) solitons. Mathematically, wave maps simultaneously generalize the classical harmonic maps equation to Lorentzian domains as well as the free wave equation to manifold-valued maps.

Viewing $(\mathcal{M}, \text{g})$ as an isometrically embedded sub-manifold of Euclidean space $(\mathbb{R}^N, \langle , \rangle_{\mathbb{R}^N})$, a wave map is defined as a formal critical point of the Lagrangian action

$$L(\Psi) = \frac{1}{2} \int_{\mathbb{R}^{1+2}} \text{m}^{\alpha\beta} \langle \partial_\alpha \Psi, \partial_\beta \Psi \rangle_{\mathbb{R}^N} \, dx \, dt.$$ 

The Euler-Lagrange equations are given by

$$\Box \Psi \perp T_\Psi \mathcal{M},$$

which can be rewritten as

$$\Box \Psi = S(\Psi)(\partial_\psi \Psi, \partial_\psi \Psi), \quad (1.1)$$

where $S$ denotes the second fundamental form of the embedding $(\mathcal{M}, \text{g}) \hookrightarrow (\mathbb{R}^N, \langle , \rangle_{\mathbb{R}^N})$. The conserved energy is given by

$$E(\Psi, \partial_t \Psi)(t) = \frac{1}{2} \int_{\mathbb{R}^2} |\partial_t \Psi(t)|^2 + |\nabla \Psi(t)|^2 \, dx = \text{constant.} \quad (1.2)$$

Smooth finite energy initial data for $(1.1)$ consist of a pair $\Psi(0) = (\Psi_0, \Psi_1)$, where $\Psi_0(x) \in \mathcal{M} \subset \mathbb{R}^N$, $\Psi_1(x) \in T_{\Psi_0(x)} \mathcal{M}$, $\forall \, x \in \mathbb{R}^2$. 

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If the initial data are smooth with finite energy, then we can find a fixed vector \( \Psi_\infty \in \mathcal{M} \) so that
\[
\Psi_0(x) \to \Psi_\infty \quad \text{as} \quad |x| \to \infty.
\] (1.3)

We remark that wave maps on \( \mathbb{R}^{1+2} \) are called *energy critical* because the conserved energy and the equation are invariant under the same scaling: If \( \tilde{\Psi}(t) \) solves (1.1) then so does
\[
\tilde{\Psi}_\lambda(t, x) := (\Psi_\lambda(t, x), \partial_t \Psi_\lambda(t, x)) = (\Psi(t/\lambda, x/\lambda), \frac{1}{\lambda} \partial_t \Psi(t/\lambda, x/\lambda))
\] (1.4)
and it also holds that
\[
E(\tilde{\Psi}_\lambda) = E(\tilde{\Psi}).
\]

The geometry of the target manifold, and in particular the existence of non-constant finite energy harmonic maps \( \Psi : \mathbb{R}^2 \to \mathcal{M} \), plays a crucial role in determining the possible dynamics of solutions to the wave maps equation. Here we’ll focus on a special case when the target manifold is the 2-sphere, \( \mathcal{M} = S^2 \subset \mathbb{R}^3 \) with the round metric \( g \). One advantage is that here the harmonic maps are explicit: by a classical theorem of Eells and Wood [10] they are either holomorphic or anti-holomorphic with respect to the complex structure on \( S^2 \), and by (1.3) with the removable singularity theorem [15] they can thus be identified with the rational functions \( \rho : \mathbb{C}_\infty \to \mathbb{C}_\infty \). It follows that each harmonic map \( \mathbb{R}^2 \to S^2 \) has a topological degree given by the degree of the corresponding rational map.

In fact, the condition (1.3) allows us to assign a topological degree to each smooth finite energy wave map. Given data \((\Psi_0, \Psi_1)\), we can identify \( \Psi_0 \) with a map \( \tilde{\Psi}_0 : S^2 \to S^2 \) by assigning the point at \( \infty \) to the vector \( \Psi_\infty := \lim_{|x| \to \infty} \Psi(x) \).

Abusing notation slightly by writing \( \tilde{\Psi}_0 = \Psi_0 \), the degree of the map \( \Psi_0 \) is defined by
\[
\deg(\Psi_0) := \frac{1}{\text{Area}(S^2)} \int_{S^2} \Psi_0^*(\omega) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \Psi^*_0(\omega) \in \mathbb{Z}
\]
where \( \omega \) is the area element of \( S^2 \subset \mathbb{R}^3 \). The degree \( \deg(\Psi_0) \) is preserved by the smooth wave map flow on its maximal interval of existence \( I_{\text{max}} \), that is,
\[
\deg(\Psi_0) = \deg(\Psi(t)) \quad \forall t \in I_{\text{max}}.
\]
Importantly, the unique (up to symmetries) harmonic map \( Q_k \) of degree \( k \) minimizes the energy amongst all degree \( k \) wave maps, and in fact
\[
\mathcal{E}(Q_k) = 4\pi |\deg(Q_k)| = 4\pi |k|.
\]
As a holomorphic map \( \mathbb{C}_\infty \to \mathbb{C}_\infty \), \( Q_k \) is simply represented up to scaling, rotation, translation, and inversion by the rational function \( z \mapsto z^k \).

1.1. \( k \)-equivariant wave maps. To simplify the analysis we’ll take advantage of a symmetry reduction and study a restricted class of maps \( \Psi \) satisfying the equivariance relation \( \Psi \circ \rho^k = \rho^k \circ \Psi \) for all rotations \( \rho \in SO(2) \). We consider a subclass of such maps known as \( k \)-equivariant, or \( k \)-corotational, which correspond to equivariant maps that in local coordinates take the form
\[
\Psi(t, r, \theta) = (\psi(t, r), k\theta) \mapsto (\sin \psi \cos k\theta, \sin \psi \sin k\theta, \cos \psi) \in S^2 \subset \mathbb{R}^3,
\]
where $\psi$ is the colatitude measured from the north pole of the sphere and the metric on $S^2$ is given by $ds^2 = d\psi^2 + \sin^2 \psi \, d\omega^2$. The Euler-Lagrange equations \ref{eq:Euler-Lagrange} reduce to an equation for $\psi$ and we are led to the Cauchy problem:

$$\begin{cases}
\psi_{tt} - \frac{1}{r} \psi_{rr} - \frac{1}{r^2} \psi_r + k^2 \sin^2 \psi = 0, \\
(\psi(0), \partial_t \psi(0)) = (\psi_0, \psi_1).
\end{cases} \tag{1.5}$$

We'll often use the notation $\vec{\psi}(t)$ to denote the pair $\vec{\psi}(t,r) := (\psi(t,r), \psi_t(t,r))$ and we remark that the scaling \ref{eq:scaling} can be expressed as follows: If $\vec{\psi}(t,r)$ is a solution to \ref{eq:Cauchy} then so is $\vec{\psi}_\lambda(t,r) := (\psi(t/\lambda, r/\lambda), \frac{1}{\lambda} \psi_t(t/\lambda, r/\lambda))$ for each fixed $\lambda > 0$.

The conserved energy from \ref{eq:energy} takes the form

$$E(\vec{\psi}(t)) = 2\pi \frac{1}{2} \int_0^\infty \left( (\partial_t \psi(t,r))^2 + (\partial_r \psi(t,r))^2 + k^2 \sin^2 \psi(t,r) \right) r \, dr.$$ 

From the above it's clear that any $k$-equivariant data $\vec{\psi}(0,r)$ of finite energy must satisfy $\lim_{r \to 0} \psi(0,r) = m\pi$ and $\lim_{r \to \infty} \psi(0,r) = n\pi$ for some $m, n \in \mathbb{Z}$. Since the smooth wave map flow depends continuously on the initial data these integers are fixed over any time interval $t \in I$ on which the solution is defined. This splits the energy space into disjoint classes according to this topological condition and it is natural to consider the Cauchy problem \ref{eq:Cauchy} within a fixed class

$$\mathcal{H}_{m\pi,n\pi} := \{ (\psi_0, \psi_1) \mid E(\psi_0, \psi_1) < \infty \text{ and } \lim_{r \to 0} \psi_0(r) = m\pi, \lim_{r \to \infty} \psi_0(r) = n\pi \}.$$ 

We can restrict to $\mathcal{H}_{0,n\pi}$ and we'll denote these by $\mathcal{H}_{n\pi} := \mathcal{H}_{0,n\pi}$. We also define $\mathcal{H} = \bigcup_{n \in \mathbb{Z}} \mathcal{H}_{n\pi}$ to be the full energy space.

The equivariant reduction introduces a good deal of rigidity into the problem, but still allows us access to the family of harmonic maps. Indeed the degree $k$ harmonic map $Q_k$ corresponding to $z \mapsto z^k$ can be expressed uniquely (up to scaling) as the $|k|$-equivariant map

$$Q_k(r, \theta) = (Q_k(r), k\theta) \mapsto (\sin Q_k \cos k\theta, \sin Q_k \sin k\theta, \cos Q_k) \in S^2 \subset \mathbb{R}^3,$$

where $Q_k$ is the explicit finite energy stationary solution to \ref{eq:Cauchy} given by

$$Q_k(r) := 2 \arctan r^k.$$ 

Note that $Q_k(r)$ satisfies

$$Q_k(0) = 0, \quad \lim_{r \to \infty} Q_k(r) = \pi.$$ 

We often write $\vec{Q}_k := (Q_k, 0)$. We see that $E(\vec{Q}_k) = 4\pi k$, which is minimal amongst all $k$-equivariant maps in the energy class $\mathcal{H}_{n\pi}$; see Section \ref{sec:direct_argument} for a direct argument.

Here we consider $k$-equivariant maps $\vec{\psi} = (\psi_0, \psi_1)$ in the class $\mathcal{H}_0$, i.e., that satisfy

$$\lim_{r \to 0} \psi_0(r) = 0 \text{ and } \lim_{r \to \infty} \psi_0(r) = 0,$$
so that $\psi_0$ is the polar angle of a finite energy map $\Psi_0$ into $S^2$ with $\text{deg}(\Psi_0) = 0$.

Before stating our main results let us first motivate this restriction with a brief summary of recent developments.

1.2. **Threshold Theorems and Bubbling.** The energy critical wave maps equation (1.1) has been extensively studied over the past several decades; [4, 5, 27–33, 46, 47, 50, 51, 52, 54, 55]. In recent years the focus has centered on understanding the nonlinear dynamics of solutions with large energy. At the end of the last decade, the following remarkable **sub-threshold conjecture** was established [37, 48, 49, 53]:

Every wave map with energy less than that of the first nontrivial harmonic map is globally regular on $\mathbb{R}^{1+2}$ and scatters to a constant map. The role of the least energy harmonic map in the statement of the sub-threshold conjecture is based on fundamental work of Struwe [50], who showed that the smooth equivariant wave map flow can only develop a singularity by concentrating energy at the tip of a light cone by bubbling off at least one non-trivial finite energy harmonic map. In breakthrough works, Krieger, Schlag, Tataru [38], Rodnianski, Sterbenz [44], and Raphael, Rodnianski [43], constructed examples solutions of such blow-up by bubbling, with the latter two works yielding a stable blow-up regime; see also the recent stability analysis of Krieger [34] for type-II blow-ups solutions to the energy critical NLW, which suggests that the solutions from [38] should also exhibit stability properties.

The starting point for the present work is the following natural question: Can one give a satisfactory description of the possible dynamics for arbitrary initial data? In dispersive models such as (1.1) this is typically referred to as the **soliton resolution conjecture**, which states roughly that any smooth solution asymptotically decouples into weakly interacting (possibly concentrating) solitons plus free radiation. The wave maps equation (1.1) with $\mathcal{N} = S^2$ is an intriguing model in which to study this question: all stationary solutions (the harmonic maps) are known explicitly, the conserved topological degree of the solution introduces additional rigidity, and the equivariant reduction (1.5) greatly simplifies certain aspects of the analysis without destroying the essential mechanisms of truly nonlinear behavior, e.g., solitons, blow-up. There has been exciting recent progress in this direction for the general equation (1.1), see [12, 20]. Here we focus on the equivariant model (1.5) where more is known.

Our analysis is motivated by several results proved in the last few years; we will in fact use some of them explicitly. First, note that, by continuity, $\lim_{r \to 0} \psi(t, r)$ and $\lim_{r \to \infty} \psi(t, r)$ are independent of $t$. Hence scattering to a constant map is only possible if $\lim_{r \to 0} \psi_0(r) = \lim_{r \to \infty} \psi_0(r)$. We can assume without loss of generality that both these limits equal 0, i.e., the initial data $(\psi_0, \psi_1)$ is in $\mathcal{H}_0$. For such maps, the following refined threshold theorem was proved in [8].

**Theorem 1.1 (2E(\vec{Q}) Threshold Theorem).** [8, Theorem 1.1] For any smooth initial data $\vec{\psi}(0) \in \mathcal{H}_0$ with

$$\mathcal{E}(\vec{\psi}(0)) < 2\mathcal{E}(\vec{Q}_k) = 8\pi k,$$

there exists a unique global evolution $\vec{\psi} \in C^0(\mathbb{R}; \mathcal{H}_0)$. Moreover, $\vec{\psi}(t)$ scatters to zero in both time directions, i.e., there exist solutions $\vec{\varphi}_L^\pm$ to the linearized equation (2.2) such that

$$\vec{\psi}(t) = \vec{\varphi}_L^+(t) + o_{\mathcal{H}_0}(1) \text{ as } t \to \pm \infty.$$
The analogous result for the full model without symmetries was obtained by the second author and Oh in [39], as a consequence of the bubbling analysis in [49].

The heuristic reasoning behind the threshold \(2\mathcal{E}(\vec{Q})\) is as follows. The topological degree counts (with orientation) the number of times a map ‘wraps around’ \(S^2\). If a harmonic map of degree \(k\) bubbles off from a wave map \(\vec{\psi}(t)\), then, in order for \(\vec{\psi}(t)\) to satisfy \(\deg(\psi) = 0\), it must ‘unwrap’ precisely \(k\) times away from the bubble.

The minimum energy cost for wrapping and unwrapping is \(4\pi k\), which is also the energy of \(Q_k\). The total energy cost is at least \(8\pi k = 2\mathcal{E}(\vec{Q}_k)\).

Similar intuition motivated the works [8, 9], which established soliton resolution for energies that only allow for one concentrating bubble, namely for data in \(H_\pi\) with energy below \(3\mathcal{E}(Q)\). These works showed that for any such solution there exists a regular map \(\vec{\varphi} \in H_0\) (free radiation if the solution is global) and a continuous dynamical scale \(\lambda(t) \in [0, \infty)\) such that

\[
\vec{\psi}(t) = \vec{Q}_{\lambda(t)} + \vec{\varphi}(t) + o_{H_0}(1) \quad \text{as} \quad t \to T_+.
\] (1.6)

Cote [7] and later Jia, Kenig [24] extended the theory to handle arbitrary energies. It was shown that in this case the decomposition (1.6) holds with possibly many concentrating harmonic maps, but only along at least one sequence of times \(t_n \to T_+\). The proofs of [7–9, 24] rely heavily on concentration compactness techniques and were all inspired by the remarkable series of papers by Duyckaerts, Kenig, and Merle [13–16] on the focusing quintic nonlinear wave equation in 3 space dimensions.

We’ll discuss these latter works more below; see Remark 1.13.

**Theorem 1.2 (Sequential Decomposition).** [7, 24] Let \(\vec{\psi}(t) \in H_{n\pi}\) be a smooth solution to (1.5) on \([0, T_+)\). Then there exists a sequence of times \(t_n \to T_+\), an integer \(J \in \mathbb{N}\), a regular map \(\vec{\varphi} \in H_0\) (free radiation if the solution is global) and continuous dynamical scales \(\lambda(t) \in [0, \infty)\) such that

\[
\vec{\psi}(t_n) = \sum_{j=1}^J \iota_j \vec{Q}_{\lambda_{n,j}} + \vec{\varphi}(t) + o_{H_0}(1) \quad \text{as} \quad n \to \infty
\] (1.7)

In the case of finite time blow-up at least one scale \(\lambda_{n,1} \to 0\) as \(n \to \infty\) and \(\vec{\varphi}(t) \to \vec{\varphi}(1)\) is a regular map in \(H_0\) with \(\mathcal{E}(\vec{\varphi}(1)) = \mathcal{E}(\vec{\psi}) - J\mathcal{E}(\vec{Q})\). In the case of a global solution, \(\vec{\varphi}(t)\) can be taken to be a solution to the linear wave equation (2.2).

Note that above the signs \(\iota_j\) are required to match up so that

\[
\lim_{r \to \infty} \vec{\psi}(0, r) = n\pi = \lim_{r \to \infty} \sum_{j=1}^J \iota_j \vec{Q}_{\lambda_{n,j}}(r).
\]

**Remark 1.3.** A decomposition into bubbles for a sequence of times for the full non-equivariant model was obtained by Grinis [20] up to an error that vanishes in a weaker Besov-type norm. Duyckaerts, Jia, Kenig and Merle [12] proved that for energies slightly above \(\mathcal{E}(\vec{Q})\) a one-bubble decomposition holds for continuous time. The same authors obtained in [11] a sequential decomposition into bubbles in the case of the focusing energy critical power-type nonlinear wave equation (NLW).

Theorem 1.2 raises two natural questions:

- Are there any solutions to (1.5) with \(J \geq 2\) in (1.7), i.e., are there any solutions that form more than one bubble?
And, if so, does the decomposition \( \{1.7\} \) hold continuously in time, i.e., does soliton resolution hold for \( \{1.5\} \)?

In view of Theorems \([1.1]\) and \([1.2]\) it is natural to ask both questions at the minimal possible energy level where multiple bubble dynamics can occur, namely for solutions \( \vec{\psi}(t) \in \mathcal{H}_0 \) having threshold energy, that is such that

\[
\mathcal{E}(\vec{\psi}) = 2\mathcal{E}(\vec{Q}).
\]

In \([22]\) the first author obtained an affirmative answer to the first question, proving the following result.

**Theorem 1.4.** \([22]\) Theorem 2 Let \( k > 2 \). There exists a solution \( \vec{\psi} : (-\infty, T_0] \to \mathcal{H}_0 \) of \( \{1.5\} \) such that

\[
\lim_{t \to -\infty} \| \vec{\psi}(t) - (\vec{Q} + \vec{Q}_{\gamma_k|t|^{\frac{1}{2}}} ) \|_{\mathcal{H}_0} = 0,
\]

where \( \gamma_k > 0 \) is an explicit constant depending on \( k \).

**Remark 1.5.** Similar solutions could be obtained for \( k = 2 \) by the same method.

1.3. **Main result.** In this paper, we address the problem of classification of solutions at threshold energy level, in the spirit of the works of Duyckaerts and Merle \([17, 18]\). The major difficulty in the analysis is that in our case the threshold solutions contain two bubbles, which leads to significantly more complicated dynamics.

Let \( \vec{\psi}(t) : (T_-, T_+) \to \mathcal{H}_0 \) be a solution to \( \{1.5\} \) with \( \mathcal{E}(\vec{\psi}) = 2\mathcal{E}(\vec{Q}) \). We will say that \( \vec{\psi}(t) \) is a two-bubble in the forward time direction if there exist \( \iota \in \{1, -1\} \) and continuous functions \( \lambda(t), \mu(t) > 0 \) such that

\[
\lim_{t \to T_+} \| \psi(t) - \iota (Q_{\lambda(t)} - Q_{\mu(t)}), \vec{\psi}(t) \|_{\mathcal{H}_0} = 0, \quad \lambda(t) \ll \mu(t) \text{ as } t \to T_+.
\]

The notion of a two-bubble in the backward time direction is defined similarly. We prove the following result.

**Theorem 1.6** (Main Theorem). Fix any equivariance class \( k \geq 2 \). Let \( \vec{\psi}(t) : (T_-, T_+) \to \mathcal{H}_0 \) be a solution to \( \{1.5\} \) such that

\[
\mathcal{E}(\vec{\psi}) = 2\mathcal{E}(\vec{Q}) = 8\pi k.
\]

Then \( T_- = -\infty, T_+ = +\infty \) and one the following alternatives holds:

- \( \vec{\psi}(t) \) scatters in both time directions,
- \( \vec{\psi}(t) \) scatters in one time direction and is a two-bubble in the other time direction with the scales of the bubbles \( \lambda(t), \mu(t) \) satisfying
  \[
  \mu(t) \to \mu_0 \in (0, +\infty), \quad \lambda(t) \to 0.
  \]

**Remark 1.7.** As a by-product of the proof, we will determine the rate of decay of \( \lambda(t) \) in the two-bubble case. Suppose a two-bubble solution forms as \( t \to \infty \): If \( k \geq 3 \) there exists a constant \( C_k > 0 \) such that \( \frac{1}{C_k} \mu_0^k t^{-\frac{n-k}{2}} \leq \lambda(t) \leq C_k \mu_0^k t^{-\frac{n-k}{2}} \) for \( t \) large enough, see \([1.34]\). In the case \( k = 2 \) there exists a constant \( C > 0 \) such that we have \( \exp(-Ct) \leq \lambda(t) \leq \exp(-t/C) \) for \( t \) large enough, see \([1.33]\).

**Remark 1.8.** In particular, the two-bubble solutions from Theorem \( \{1.3\} \) scatter in forward time, which provides an example of an orbit connecting different types of dynamical behavior for positive and negative times.
Remark 1.9. Non-existence of solutions which form a pure two-bubble in both time directions is reminiscent of the work of Martel and Merle [40, 41]. This seems to be a typical feature of models which are not completely integrable.

One of the main points of our paper is an analysis of what we could call a collision of bubbles in the simplest possible case of threshold energy.

Remark 1.10. Recall that in [17] a complete classification at the threshold energy was obtained. It is tempting to believe that the solutions from Theorem 1.4 should play a similar role as the solution \(W^{-}\) from [17], in which case they would be unique non-dispersive solutions up to rescaling. This remains an open question.

Remark 1.11. We conjecture that for \(k = 1\) a similar result holds, but in the two-bubble case \(\lambda(t) \to 0\) in finite time. The slower decay of \(Q\) would be a source of additional technical difficulties in Section 3, but the general scheme could be applied without major changes.

Remark 1.12. Our method establishes the exact analog of Theorem 1.6 in the case of the equivariant Yang-Mills equation, by making the usual analogy between equivariant Yang-Mills and \(k = 2\)-equivariant wave maps, see for example [8, Appendix] for the analog of the Threshold Theorem and [22] for the analog of Theorem 1.4. There the harmonic map \(Q\) is replaced by the first instanton, the notion of topological degree is replaced by the second Chern number, and the threshold energy is exactly twice the energy of the first instanton.

Remark 1.13. The full soliton resolution conjecture was established for the radial solutions of focusing energy critical NLW by Duyckaerts, Kenig, and Merle in the landmark work [16]. This result is the only known case of a complete continuous-in-time classification for a model that is not completely integrable. The proof relies on a particularly strong form of the “channels of energy” method introduced by the same authors. However, proving channel of energy estimates in other settings is a delicate issue, and the strong form of these estimates used in [16] is known to fail for the linear wave equation in even dimensions, see [10].

Aside from [16], Theorem 1.6 is the only other classification result for a dispersive equation that holds for continuous times in the presence of more than one non-trivial elliptic profile. Upgrading sequential decompositions such as Theorem 1.2 or the one in [11] to hold for continuous times is regarded as a major open problem.

1.4. Structure of the proof. Inspired by the work of Duyckaerts and Merle [17], we merge the concentration-compactness techniques with a careful analysis of the modulation equations governing the evolution of the scales \(\lambda(t)\) and \(\mu(t)\). As mentioned above, the main difference with respect to [17] consists in the fact that our threshold solutions contain two bubbles, one of which is concentrating, whereas in [17] the modulation happens essentially around one stationary bubble. Thus our analysis requires substantially new technique. Our proof can be summarized as follows.

Step 1. If the solution does not scatter, then, by a special case of Theorem 1.2, it approaches a two-bubble configuration for a sequence of times.

Step 2. We divide the time axis into regions where the solution is close to a two-bubble configuration, which we can call the bad intervals \([a_m, b_m]\), and regions where it is not, which are the good intervals \([b_m, a_{m+1}]\).
Step 3. On a bad interval $[a_m, b_m]$, we decompose the solution as follows:

$$\vec{\psi}(t) = (-Q_{\mu(t)} + Q_{\lambda(t)} + g(t), \partial_t \psi(t)).$$

In order to specify the values of $\lambda(t)$ and $\mu(t)$, we use suitable orthogonality conditions, see Lemma 3.1. We consider $c_m \in (a_m, b_m)$ where the quantity $\lambda(t)/\mu(t)$ attains its global minimum on $[a_m, b_m]$ (we make sure that the minimum is not attained at one of the endpoints).

The orthogonality conditions yield modulation equations for the evolution of $\lambda(t)$ and $\mu(t)$. From these equations we can deduce crucial information about the behavior of $\mu(t)$ and $\lambda(t)$ for $t \geq c_m$ and $t \leq c_m$. We refer to the beginning of Section 3 for a short description of the method. The main conclusion can be intuitively phrased as follows: $\mu(t)$ does not change much on a bad interval, whereas $\lambda(t)$ grows in a controlled way both for $t \geq c_m$ and $t \leq c_m$. The decisive point is that the bad interval $[a_m, b_m]$ can be long if $\lambda(c_m)/\mu(c_m)$ is small, but

$$\int_{a_m}^{b_m} \left( \frac{\lambda(t)}{\mu(t)} \right)^{\frac{1}{2}} dt \leq C_k, \quad C_k \text{ depending only on } k. \quad (1.8)$$

Note that the only information about the solution which is used in this process is the fact that $E(\vec{\psi}(t)) = 2E(Q)$, $\vec{\psi}(c_m)$ is close to a two-bubble configuration and

$$\frac{d}{dt} \bigg|_{t=c_m} (\lambda(t)/\mu(t)) = 0.$$

Step 4. Using concentration-compactness arguments and Theorem 1.1 we obtain that the solution has the compactness property on the union of the good intervals. Now the idea is to run a convexity argument based on a monotonicity formula between two times where $\vec{\psi}(t)$ is close to a two-bubble. It is as this stage that we reach a contradiction – if the solution has exited a neighborhood of two-bubble configurations during the interim, the total cost in terms of time derivative is too great to allow it to return. This is a type of no-return result and one can draw parallels here to the ignition and ejection lemmas from the work of Krieger, Nakanishi, Schlag [35, 36] concerning near ground-state dynamics for the energy critical NLW.

There can potentially be many good and bad intervals between the two times where $\vec{\psi}(t)$ is close to a two-bubble. It is well-known that one needs to use a cut-off in the monotonicity formula, which introduces an error in the estimates. On the good intervals, this error is controlled thanks to the compactness property. On the bad intervals, the bound (1.8) comes into play. More precisely, we obtain that the error on a bad interval is absorbed by positive terms obtained on intervals preceding and following the bad interval.

Step 5. Once the convergence to a two-bubble for continuous time is proved, we deduce easily from the modulation equations that the solution is global and $\mu(t) \to \mu_0 \in (0, +\infty)$. Scattering on at least one side follows easily from the previous analysis. Namely, if the solution is non-scattering in both time directions, then the time axis is divided into two bad regions near $\pm \infty$ and one good interval in between. We reach a contradiction by a similar (but simpler) argument as in Step 4.

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2. Preliminaries and technical lemmas

In this section we establish a few preliminary facts about solutions to (1.5) that will be required in our analysis. We first aggregate here some notation.

2.1. Notation. Given a radial function \( f : \mathbb{R}^d \to \mathbb{R} \) we’ll abuse notation and simply write \( f = f(r) \), where \( r = |x| \). We’ll also drop the factor \( 2\pi \) in our notation for the \( L^2 \) pairing of radial functions on \( \mathbb{R}^2 \)

\[
\langle f, g \rangle := \frac{1}{2\pi} \langle f, g \rangle_{L^2(\mathbb{R}^2)} = \int_0^\infty f(r)g(r) r \, dr
\]

Recall the definition of the space \( H_0 \):

\[
H_0 := \{ (\psi_0, \psi_1) \mid E(\psi_0, \psi_1) < \infty, \quad \lim_{r \to 0} \psi_0(r) = \lim_{r \to \infty} \psi_0(r) = 0 \}
\]

We define a norm \( \| \psi \|_{H^1} \) by

\[
\| \psi \|_{H^1}^2 := \int_0^\infty \left( \partial_r \psi(r)^2 + k^2 \frac{\psi(r)^2}{r^2} \right) r \, dr
\]

and for pairs \( \vec{\psi} = (\psi_0, \psi_1) \in H_0 \) we write

\[
\| \vec{\psi} \|_{H_0} := \| (\psi_0, \psi_1) \|_{H \times L^2}.
\]

The change of variables \( r \mapsto e^x \) gives us an identification between the radial functions \( H(\mathbb{R}^2) \) and \( H^1(\mathbb{R}) \), i.e., \( \psi_0(r) \in H \iff \psi_0(e^x) \in H^1(\mathbb{R}) \). In particular this means that

\[
\| \psi_0 \|_{L^\infty} \leq C \| \psi_0 \|_H
\]

Scaling invariance plays a key role in our analysis. Given a radial function \( \phi : \mathbb{R}^2 \to \mathbb{R} \) we denote the \( \dot{H}^1 \) and \( L^2 \) re-scalings as follows

\[
\phi_\lambda(r) = \phi(r/\lambda), \quad \dot{\phi}_\lambda(r) = \frac{1}{\lambda} \dot{\phi}(r/\lambda)
\]

The corresponding infinitesimal generators are given by

\[
\Lambda \phi := -\frac{\partial}{\partial \lambda} \bigg|_{\lambda=1} \phi_\lambda = r \partial_r \phi \quad (\dot{H}^1_{\text{rad}}(\mathbb{R}^2) \text{ scaling})
\]

\[
\Lambda_0 \phi := -\frac{\partial}{\partial \lambda} \bigg|_{\lambda=1} \phi_\lambda = (1 + \lambda r \partial_r) \phi \quad (L^2_{\text{rad}}(\mathbb{R}^2) \text{ scaling})
\]

2.2. Review of the Cauchy theory. For initial data \( (\psi_0, \psi_1) \) in the class \( H_0 \) the formulation of the Cauchy problem (1.5) can be modified to take into account the strong repulsive potential term in the nonlinearity:

\[
k^2 \sin(2\phi) = k^2 \frac{\sin(2\phi)}{r^2} + k^2 \frac{\sin(2\phi) - 2\phi}{r^2} \approx k^2 \frac{\sin(2\phi)}{r^2} + O(\phi^3)
\]

The presence of the potential \( \frac{k^2}{r^2} \) indicates that the linear wave equation,

\[
(\partial_t^2 - \Delta_{\mathbb{R}^2} + \frac{k^2}{r^2}) \psi = 0,
\]

of (1.5) has more dispersion than the 2d wave equation. In fact, it has the same dispersion as the free wave equation in dimension \( d = 2k + 2 \) as can be seen from
the following change of variables: given a radial function \( \phi \in H \), define \( v(r) \) by 
\[ \phi(r) = r^k v(r). \]
Then
\[ \frac{1}{r^k}(-\Delta r^2 + \frac{k^2}{r^2})\phi = -\Delta_{\mathbb{R}^{2k+2}} v, \quad \|\phi\|_H = \|v\|_{H^1(\mathbb{R}^{2k+2})}. \tag{2.3} \]
Thus one way of studying solutions \( \tilde{\psi}(t) \in \mathcal{H}_0 \) of Cauchy problem (1.5) is to define
\[ \tilde{\psi}(t) = (r^{-k}\psi(t), r^{-k}\psi_r(t)) \in \dot{H}^1 \times L^2(\mathbb{R}^{2k+2}) \]
and analyze the equivalent Cauchy problem for the radial nonlinear wave equation in \( \mathbb{R}^{1+(2k+2)} \) satisfied by \( \tilde{\psi}(t) \). Unfortunately, this route leads to unpleasant technicalities when \( k > 2 \) (spatial dimension = \( 2k + 2 > 6 \)) due to the high dimension and the particular structure of the nonlinearity.

There is a simpler approach that allows us to treat the scattering theory for the Cauchy problem (1.5) for all equivariance classes \( k \geq 1 \) in a unified fashion. The idea is to make use of some, but not all, of the extra dispersion in \( \mathcal{L}_0 \). Indeed, given a solution \( \psi(t) \) to (1.5) we define \( u \) by \( ru = \psi \) and obtain the following Cauchy problem for \( u \).

\[ u_{tt} - u_{rr} - \frac{3}{r} u_r + \frac{k^2 - 1}{r^2} u = k^2 ru - \sin(2ru) =: Z(ru)u^3 \tag{2.4} \]

where the function \( Z \) defined above is a clearly smooth, bounded, even function. The linear part of (2.4) is the radial wave equation in \( \mathbb{R}^{1+4} \) with a repulsive inverse square potential, namely

\[ v_{tt} - v_{rr} - \frac{3}{r} v_r + \frac{k^2 - 1}{r^2} v = 0. \tag{2.5} \]

For each \( k \geq 1 \), define the norm \( H_k \) for radially symmetric functions \( v \) on \( \mathbb{R}^4 \) by

\[ \|v\|_{H_k(\mathbb{R}^4)}^2 := \int_0^\infty \left[ (\partial_r v)^2 + \frac{(k^2 - 1)}{r^2} v^2 \right] r^3 dr \]

Solutions to (2.4) conserve the \( H_k \) norms. By Hardy’s inequality we have

\[ \|v\|_{H_k(\mathbb{R}^4)} \simeq \|v\|_{\dot{H}^1(\mathbb{R}^4)} \]

The mapping

\[ H_k \times L^2(\mathbb{R}^4) \ni (u_0, u_1) \mapsto (\psi_0, \psi_1) := (ru_0, ru_1) \in H \times L^2(\mathbb{R}^2) \]

satisfies

\[ \|(u_0, u_1)\|_{\dot{H}^1 \times L^2(\mathbb{R}^4)} \simeq \|(u_0, u_1)\|_{H_k \times L^2(\mathbb{R}^4)} = \|(\psi_0, \psi_1)\|_{H \times L^2(\mathbb{R}^2)} \tag{2.6} \]

Thus we can conclude that the Cauchy problem for (2.4) with initial data in \( \dot{H}^1 \times L^2(\mathbb{R}^4) \) is equivalent to the Cauchy problem for (1.5) for initial data \((\psi_0, \psi_1) \in \mathcal{H}_0\), allowing us to give a scattering criterion for solutions \( \psi(t) \in \mathcal{H}_0 \) to (1.5).

**Lemma 2.1.** Let \( \tilde{\psi}(0) = (\psi_0, \psi_1) \in \mathcal{H}_0 \). Then there exists a unique solution \( \tilde{\psi}(t) \in \mathcal{H}_0 \) to (1.5) defined on a maximal interval of existence \( I_{\text{max}}(\tilde{\psi}) := (-T_-(\tilde{\psi}), T_+(\tilde{\psi})) \) with the following properties: Define

\[ \tilde{u}(t,r) = (r^{-1}\psi(t,r), r^{-1}\psi_r(t,r)) \in \dot{H}^1 \times L^2(\mathbb{R}^4) \]
Then for any compact time interval \( J \subseteq [0,T] \) we have

\[
\|u\|_{L^6_t(J;L^6_x(\mathbb{R}^4))} \leq C(J) < \infty
\]

In addition, if

\[
\|u\|_{L^6_t(0,\infty;L^6_x(\mathbb{R}^4))} < \infty
\]

then \( T_+ = \infty \) and \( \tilde{\psi}(t) \) scatters \( t \to \infty \), i.e., there exists a solution \( \tilde{\phi}_L(t) \in \mathcal{H}_0 \) to (2.2) so that

\[
\|\tilde{\psi}(t) - \tilde{\phi}_L(t)\|_{\mathcal{H}_0} \to 0 \quad \text{as} \quad t \to \infty.
\]

Conversely, any solution \( \tilde{\psi}(t) \) that scatters as \( t \to \infty \) satisfies

\[
\|\psi/r\|_{L^6_tL^6_x((0,\infty) \times \mathbb{R}^4)} < \infty.
\]

The proof of Lemma 2.1 is standard consequence of Strichartz estimates for (2.5) and the equivalence of the Cauchy problems (1.5) and (2.4). In this case, we need Strichartz estimates for the radial wave equation in \( \mathbb{R}^{1+4} \) with a repulsive inverse square potential. For these we can cite the more general results of Planchon, Stalker, and Tahvildar-Zadeh [42]; see also [20, Lemma 2.18] which cover the non-radial case.

**Lemma 2.2 (Strichartz estimates).** [20, Corollary 3.9] Fix \( k \geq 1 \) and let \( \vec{v}(t) \) be a radial solution to the linear equation

\[
v_{tt} - v_{rr} - \frac{3}{r} v_r + \frac{k^2 - 1}{r^2} v = F(t,r), \quad \vec{v}(0) = (v_0, v_1) \in \dot{H}^1 \times L^2(\mathbb{R}^4)
\]

Then, for any time interval \( I \subset \mathbb{R} \) we have

\[
\|\vec{v}\|_{L^6_tL^6_x(I \times \mathbb{R}^4)} + \sup_{t \in I} \|\vec{v}(t)\|_{\dot{H}^1 \times L^2(\mathbb{R}^4)} \lesssim \|\vec{v}(0)\|_{\dot{H}^1 \times L^2(\mathbb{R}^4)} + \|F\|_{L^6_tL^6_x(I \times \mathbb{R}^4)}
\]

where the implicit constant above is independent of \( I \).

We’ll also explicitly require the following nonlinear perturbation lemma from [20]; see also [8, Lemma 2.18].

**Lemma 2.3 (Perturbation Lemma).** [20, Theorem 2.20] [8, Lemma 2.18] There are continuous functions \( \varepsilon_0, C_0 : (0,\infty) \to (0,\infty) \) such that the following holds: Let \( I \subset \mathbb{R} \) be an open interval, (possibly unbounded), \( \psi, \varphi \in C^0(I;H) \cap C^1(I;L^2) \) radial functions satisfying for some \( A > 0 \)

\[
\|\varphi\|_{L^\infty(I;H \times L^2(\mathbb{R}^2))} + \|\vec{\varphi}\|_{L^\infty(I;H \times L^2(\mathbb{R}^2))} + \|\varphi/r\|_{L^6(I;L^6_x(\mathbb{R}^4))} \leq A
\]

\[
\|eq(\psi/r)\|_{L^1_tL^2_x(I \times \mathbb{R}^4)} + \|eq(\vec{\varphi}/r)\|_{L^1_tL^2_x(I \times \mathbb{R}^4)} + \|w_0\|_{L^6_tL^6_x(I \times \mathbb{R}^4)} \leq \varepsilon \leq \varepsilon_0(A)
\]

where \( eq(\psi/r) := (\Box + \frac{k^2 + 1}{r^2})(\psi/r) + (\psi/r)^3Z(\psi) \) in the sense of distributions, and \( \tilde{w}_0(t) := S(t-t_0)(\vec{\psi} - \vec{\varphi})(t_0) \) with \( t_0 \in I \) arbitrary, but fixed and \( S \) denoting the linear wave evolution operator in \( \mathbb{R}^{1+4} \) (i.e., the propagator for (2.2)). Then,

\[
\|\vec{\psi} - \vec{\varphi} - \tilde{w}_0\|_{L^\infty(I;H \times L^2(\mathbb{R}^2))} + \|\frac{1}{r}(\psi - \varphi)\|_{L^6(I;L^6_x(\mathbb{R}^4))} \leq C_0(A)\varepsilon
\]

In particular, \( \|\psi/r\|_{L^6_tL^6_x(\mathbb{R}^4)} < \infty \).
2.3. Concentration Compactness. Another consequence of (2.6) and Lemma 2.2 is that we can translate the concentration compactness theory of Bahouri and Gérard to solutions to (2.2) and (1.5). We begin by stating the linear profile decompositions in the 4d setting for solutions to (2.5).

Lemma 2.4 (Linear 4d profile decomposition). [Main Theorem] Let \( k \geq 1 \) be fixed. Consider a sequence \( \tilde{u}_n = (u_{n,0}, u_{n,1}) \in H_k \times L^2(\mathbb{R}^4) \) which is bounded in the sense that \( \|\tilde{u}_n\|_{H_k \times L^2(\mathbb{R}^4)} \lesssim 1 \). Then, up to passing to a subsequence, there exists a sequence of solutions to (2.5), \( \tilde{V}_j^k \in H_k \times L^2(\mathbb{R}^4) \), sequences of times \( \{t_{n,j}\} \subset \mathbb{R} \), and scales \( \{\lambda_{n,j}\} \subset (0, \infty) \), and \( \tilde{w}_n \) defined by

\[
\tilde{u}_n(r) = \sum_{j=1}^{k} \left( \frac{1}{\lambda_{n,j}} V_j^k \left( \frac{-t_{n,j}}{\lambda_{n,j}}, \frac{r}{\lambda_{n,j}} \right) , \frac{1}{(\lambda_{n,j})^2} \partial_t V_j^k \left( \frac{-t_{n,j}}{\lambda_{n,j}}, \frac{r}{\lambda_{n,j}} \right) \right) + (w_{n,0}^k, w_{n,1}^k)(r)
\]

so that the following statements hold: Let \( w_{n,L}^k(t) \) denote the linear evolution of the data \( \tilde{w}_n^k \), i.e., solutions to (2.5). Then, for any \( j \leq k \),

\[
(\lambda_{n,j}^4 w_{n,L}^k(\lambda_{n,j} t_{n,j}, \lambda_{n,j}^2) , \lambda_{n,j}^2 w_{n,L}^k(\lambda_{n,j} t_{n,j}, \lambda_{n,j}^4)) \to 0 \quad \text{weakly in } H_k \times L^2(\mathbb{R}^4).
\]

In addition, for any \( j \neq k \) we have

\[
\frac{\lambda_{n,j}}{\lambda_{n,k}} + \frac{\lambda_{n,k}}{\lambda_{n,j}} + \frac{|t_{n,j} - t_{n,k}|}{\lambda_{n,j}} + \frac{|t_{n,j} - t_{n,k}|}{\lambda_{n,k}} \to \infty \quad \text{as } n \to \infty.
\]

Moreover, the errors \( \tilde{w}_n^k \) vanish asymptotically in the sense that

\[
\limsup_{n \to \infty} \|w_{n,L}^k\|_{L^1_t L^\infty_x \cap L^1_t L^2_x(\mathbb{R} \times \mathbb{R}^4)} \to 0 \quad \text{as } k \to \infty. \tag{2.7}
\]

Finally, we have almost-orthogonality of the \( H_k \times L^2 \) norms of the decomposition:

\[
\|\tilde{u}_n\|_{H_k \times L^2} = \sum_{1 \leq j \leq k} \|V_j^k(-t_{n,j}/\lambda_{n,j})\|_{H_k \times L^2}^2 + \|\tilde{w}_n^k\|_{H_k \times L^2}^2 + o_n(1) \quad \text{as } n \to \infty.
\]

Remark 2.5. The difference between Lemma 2.4 and the main theorem in [1] is that here we have phrased matters in terms of solutions to the 4d linear wave equation with a repulsive inverse square potential (2.5) (which conserve the \( H_k \times L^2 \) norm), as opposed to the free wave equation in 4d with data in \( H^1 \times L^2 \). However, a proof identical to the one in [1] can be used to establish Lemma 2.4. Alternatively, one can establish Lemma 2.4 by conjugating (2.5) to the free wave equation in dimension \( d = 2k + 2 \) via the map \( v(r) \mapsto r^{-k+1} v(r) = u \). This map induces an isometry \( H_k(\mathbb{R}^4) \to H^1(\mathbb{R}^{2k+2}) \); see (2.8). Then the usual Bahouri-Gérard profile decomposition in dimension \( d = 2k + 2 \) induces a profile decomposition in \( H_k \). Once must check that the errors \( w_{n,L}^j \) can be made to vanish as in (2.7), but this follows by combining the vanishing of \( r^{-k+1} w_{n,L}^j \) in appropriate \( \text{dim} = 2k+2 \) Strichartz norms with the Strauss estimate,

\[
\sup_{t \in \mathbb{R}, r > 0} |r w_{n,L}^j(t, r)| \lesssim \|w_{n,L}^j\|_{L^\infty_t H_k(\mathbb{R}^4)},
\]

and interpolation.
A direct consequence of Lemma 2.7 and (2.6) with the identifications
\[\psi_n(r) := ru_n(r), \quad \gamma_n^j(r) := rw_n^j,\]
\[\varphi^j_L(-t_{n,j}/\lambda_{n,j}, r/\lambda_{n,j}) := \frac{r}{\lambda_{n,j}}V^j_L(-t_{n,j}/\lambda_{n,j}, r/\lambda_{n,j}),\]
is the following profile decomposition for bounded sequences \(\tilde{\psi}_n \in \mathcal{H}_0\).

**Corollary 2.6 (Linear profile decomposition).** Consider a sequence \(\tilde{\psi}_n \in \mathcal{H}_0\) that is uniformly bounded in \(\mathcal{H}_0\). Then, up to passing to a subsequence, there exists a sequence of solutions \(\varphi^j_L \in \mathcal{H}_0\) to (2.2), sequences of times \(\{t_{n,j}\} \subset \mathbb{R}\), sequences of scales \(\{\lambda_{n,j}\} \subset (0, \infty),\) and errors \(\tilde{\gamma}_n^j\) defined by
\[\tilde{\psi}_n = \sum_{j=1}^{J} \left(\varphi^j_L(-t_{n,j}/\lambda_{n,j}, r/\lambda_{n,j}), \frac{1}{\lambda_{n,j}}\partial_r\varphi^j_L(-t_{n,j}/\lambda_{n,j}, r/\lambda_{n,j}) + (\gamma_n^j, \gamma_n^{j})\right)\]
so that the following statements hold: Let \(\gamma_n^j(t) \in \mathcal{H}_0\) denote the linear evolution, (i.e., solution to (2.2)) of the data \(\tilde{\gamma}_n^j \in \mathcal{H}_0\). Then, for any \(j \leq \ell, \)
\[(\gamma_n^j(\lambda_{n,j}t_{n,j}, \lambda_{n,j}r), \lambda_{n,j}\gamma_n^j(\lambda_{n,j}t_{n,j}, \lambda_{n,j}r) \to 0 \text{ weakly in } \mathcal{H}_0.\]
In addition, for any \(j \neq \ell\) we have
\[\frac{\lambda_{n,j}}{\lambda_{n,\ell}} + \frac{t_{n,j} - t_{n,\ell}}{\lambda_{n,j}} + \frac{|t_{n,j} - t_{n,\ell}|}{\lambda_{n,\ell}} \to \infty \text{ as } n \to \infty. \tag{2.8}\]
Moreover, the errors \(\tilde{\gamma}_n^j\) vanish asymptotically in the sense that
\[\lim_{n \to \infty} \sup_J \left\|\frac{1}{r} \tilde{\psi}_n^j\right\|_{L^\infty_t L^2_x \cap L^2_t L^6_x(\mathbb{R} \times \mathbb{R}^4)} \to 0 \text{ as } J \to \infty.\]
Finally, we have almost-orthogonality of the \(\mathcal{H}_0\) norms of the decomposition:
\[\|\tilde{\psi}_n\|_{\mathcal{H}_0} = \sum_{1 \leq j \leq J} \|\varphi^j_L(-t_{n,j}/\lambda_{n,j})\|^2_{\mathcal{H}_0} + \|\tilde{\gamma}_n^j\|^2_{\mathcal{H}_0} + o_n(1) \text{ as } n \to \infty.\]

Our applications of the concentration-compactness techniques developed by Kenig and Merle in [25,26] requires a “Pythagorean decomposition” of the nonlinear energy proved in [8].

**Lemma 2.7.** [8 Lemma 2.16] Let \(\tilde{\psi}_n \in \mathcal{H}_0\) be a bounded sequence with a linear profile decomposition as in Corollary 2.6. Then the following Pythagorean decomposition holds for the nonlinear energy of the sequence:
\[E(\tilde{\psi}_n) = \sum_{j=1}^{J} E(\varphi^j_L(-t_{n,j}/\lambda_{n,j})) + E(\tilde{\gamma}_n^j) + o_n(1) \text{ as } n \to \infty. \tag{2.9}\]

We will also require the following nonlinear profile decomposition analogous to [8 Proposition 2.17], or [13 Proposition 2.8]. We’ll use the following notation: Given a linear profile decomposition as in Corollary 2.6 with profiles \(\{\varphi^j_L\}\) and parameters \(\{t_{n,j}, \lambda_{n,j}\}\) we denote by \(\{\varphi^j\}\) the nonlinear profile associated to \(\{\varphi^j_L(-t_n^j/\lambda_n^j), \varphi^j_L(-t_n^j/\lambda_n^j)\}\), i.e., the unique solution to (1.5) so that for all \(-t_n^j/\lambda_n^j \in I_{\text{max}}(\varphi^j)\) we have
\[\lim_{n \to \infty} \|\varphi^j(-t_{n,j}/\lambda_{n,j}) - \varphi^j_L(-t_{n,j}/\lambda_{n,j})\|_{\mathcal{H}_0} = 0.\]
The existence of a non-linear profile is immediate from the local well-posedness theory for (1.5) in the case that \(-t_{n,j}/\lambda_{n,j} \to \tau_{\infty,j} \in \mathbb{R}\). If \(-t_{n,j}/\lambda_{n,j} \to \pm \infty\) then the existence of the nonlinear profile follows from the existence of wave operators for (1.5) and it follows that the maximal forward/backward time of existence \(T_{\pm}(\tilde{\varphi}) = \infty\). Each of these facts are now standard consequences of the Strichartz estimates in Lemma 2.2.

**Lemma 2.8 (Nonlinear Profile Decomposition).** [8, Proposition 2.17] [13, Proposition 2.8] Let \(\tilde{\psi}_n(0) \in \mathcal{H}_0\) be a uniformly bounded sequence with a profile decomposition as in Corollary 2.6. Assume that the nonlinear profile \(\varphi^j\) associated to the linear profile \(\varphi^j_L\) has maximal forward time of existence \(T_+(\varphi^j_L)\). Let \(s_n \in (0, \infty)\) be any sequence such that for all \(j\) and for all \(n\),

\[
\frac{s_n - t_{n,j}}{\lambda_{n,j}} < T_+(\varphi^j_L), \quad \limsup_{n \to \infty} \|\varphi^j_L/r\|_{L^2_1([-\frac{t_{n,j}}{\lambda_{n,j}}, \frac{s_n - t_{n,j}}{\lambda_{n,j}}); L^2_2(\mathbb{R}^4))} < \infty.
\]

Let \(\tilde{\psi}_n(t)\) denote the solution of (1.5) with initial data \(\psi_n(0)\). Then for \(n\) large enough \(\tilde{\psi}_n(t)\) exists on the interval \((0, s_n)\) and satisfies,

\[
\limsup_{n \to \infty} \|\psi_n/r\|_{L^2_0((0, s_n); L^2_2(\mathbb{R}^4))} < \infty.
\]

Moreover, the following non-linear profile decomposition holds for all \(s \in [0, s_n)\),

\[
\tilde{\psi}_n(s, r) = \sum_{j=1}^{J} (\varphi^j \left( \frac{s - t_{n,j}}{\lambda_{n,j}}, \frac{r}{\lambda_{n,j}} \right), \frac{1}{\lambda_{n,j}} \partial_t \varphi^j \left( \frac{s - t_{n,j}}{\lambda_{n,j}}, \frac{r}{\lambda_{n,j}} \right) + \gamma^j_{n,L}(s, r) + \tilde{\theta}^j_{n}(s, r)
\]

with \(\gamma^j_{n,L}(t)\) as in (2.17) and

\[
\lim_{n \to \infty} \limsup_{n \to \infty} \left( \|\theta^j_{n}/r\|_{L^2_0((0, s_n); L^2_2(\mathbb{R}^4))} + \|\tilde{\theta}^j_{n}\|_{L^\infty((0, s_n); \mathcal{H}_0)} \right) = 0.
\]

The analogous statement holds for sequences \(s_n \in (-\infty, 0)\).

Our main application of these ideas can be summarized in the following compactness lemma.

**Lemma 2.9.** Let \(\tilde{\psi}(t) \in \mathcal{H}_0\) be a solution to (1.5) defined on its forward maximal interval of existence \([0, T_+(\tilde{\psi}))\). Suppose that \(\mathcal{E}(\tilde{\psi}) = 2\mathcal{E}(Q_k)\) and that \(\tilde{\psi}(t)\) does not scatter as \(t \to T_+(\tilde{\psi})\). Then the following holds: Suppose that \(t_n \to T_+\) is any sequence of times such that

\[
\sup_n \|\tilde{\psi}(t_n)\|_{\mathcal{H}_0} \leq C < \infty \quad (2.10)
\]

Then, up to passing to a subsequence of the \(t_n\), there exists scales \(\nu_n > 0\) and a nonzero \(\tilde{\varphi} \in \mathcal{H}_0\) such that

\[
\tilde{\psi}(t_{n})_{\nu_n} \to \tilde{\varphi} \in \mathcal{H}_0
\]

strongly in \(\mathcal{H}_0\) and \(\mathcal{E}(\tilde{\varphi}) = 2\mathcal{E}(Q_k)\). Moreover, the nonlinear evolution \(\varphi(s)\) of the data \(\tilde{\varphi}(0) = \tilde{\varphi}\) is non-scattering in both forwards and backwards time.

**Remark 2.10.** One consequence the main result, Theorem 1.6 is that the hypothesis of Lemma 2.9 are not satisfied by any solution! However, we’ll use Lemma 2.9 in the context of a contradiction argument in the proof of Proposition 4.1 in Section 4.

Since the proof of the lemma uses only standard facts about profile decompositions,
the local Cauchy theory, and the Threshold Theorem \[\text{1.1}\] we include it here in Section \[\text{2.3}\].

**Proof of Lemma \[\text{2.9}\]**. By \[\text{2.10}\] we can perform a linear profile decomposition as in Corollary \[\text{2.6}\] on \(\psi(t_n)\).

First we observe that there can only be one non-zero profile \(\bar{\varphi} = \varphi^1\) and that the errors \(\bar{\varphi}^{(J)}_{n,L}\) must vanish strongly \(\mathcal{H}_0\) as \(n \to \infty\). Indeed, if there were two non-trivial profiles, or if the errors did not vanish strongly in \(\mathcal{H}_0\), then \[\text{2.9}\] along with our hypothesis that \(\mathcal{E}(\bar{\varphi}) = 2\mathcal{E}(Q)\) imply that every nonzero profile must have energy < \(2\mathcal{E}(\bar{Q})\). Thus each non-zero nonlinear profile scatters in both directions by the Threshold Theorem \[\text{1.1}\].

Now assume that \(\nu_n\) and scales \(\nu_{n,1}\), and a single limiting profile \(\bar{\varphi} = (\varphi_0, \varphi_1)\) so that

\[
(\psi(t_n + \nu_n t_{n,1}, \nu_{n,1}), \nu_{n,1} \psi(t_n + \nu_n t_{n,1}, \nu_{n,1}) \to \bar{\varphi} \in \mathcal{H}_0 \quad \text{as} \quad n \to \infty
\]

Next we claim both \(-\frac{\nu_n}{\nu_{n,1}} \to \pm \infty\) are impossible and we can therefore assume without loss of generality that \(t_{n,1} = 0\) for all \(n\). To see this, first assume first \(-\frac{\nu_n}{\nu_{n,1}} \to +\infty\). Then \(\bar{\varphi}\) scatters in forward time and we can deduce that

\[
\|\varphi(r)\|_{L^2_t L^6_x([-\frac{\nu_n}{\nu_{n,1}} \infty) \times \mathbb{R}^4)} \to 0 \quad \text{as} \quad n \to \infty
\]

by the definition of the nonlinear profile. But then the Nonlinear Perturbation Lemma \[\text{2.3}\] implies that \(\bar{\psi}(t)\) must also scatter in forward time, which contradicts our initial assumptions on \(\psi(t)\).

Now assume that \(-\frac{\nu_n}{\nu_{n,1}} \to -\infty\). Then the nonlinear profile \(\bar{\varphi}(s)\) scatters in backwards time, and the Nonlinear Perturbation Lemma \[\text{2.3}\] implies that

\[
\|\psi(r)\|_{L^2_t L^6_x([0, t_n] \times \mathbb{R}^4)} = \|\varphi(r)\|_{L^2_t L^6_x([-\frac{\nu_n}{\nu_{n,1}} \infty) \times \mathbb{R}^4)} + o_n(1) \to 0,
\]

a contradiction. Thus we can assume that \(t_{n,1} \equiv 0\) and we simply write \(\nu_{n,1} = \nu_n\). At this point we’ve shown that up to passing to a subsequence in \(t_n\) we have

\[
\bar{\psi}(t_n) \xrightarrow{\mathcal{H}_0} \varphi \in \mathcal{H}_0, \quad \mathcal{E}(\bar{\varphi}) = 2\mathcal{E}(Q)
\]

We can now run a nearly identical argument to show that nonlinear evolution \(\bar{\varphi}(s) \in \mathcal{H}_0\) can not scatter in either time direction. To see this, first suppose that \(\bar{\varphi}\) scatters in forward time. Then, the Nonlinear Perturbation Lemma \[\text{2.3}\] implies that \(\bar{\psi}(t)\) must also scatter as \(t \to \infty\). If \(\bar{\varphi}(s)\) were to scatter as \(s \to -\infty\), then,

\[
\|\psi(r)\|_{L^2_t L^6_x([0, t_n] \times \mathbb{R}^4)} = \|\varphi(r)\|_{L^2_t L^6_x([-\frac{\nu_n}{\nu_{n,1}} \infty) \times \mathbb{R}^4)} + o_n(1) \leq C < \infty.
\]

Letting limit \(n \to \infty\), we see that \(\|\psi(r)\|_{L^2_t L^6_x([0, T_n] \times \mathbb{R}^4)} \leq C\), which again means that \(\bar{\psi}(t)\) scatters in forward time, a contradiction. Hence \(\bar{\varphi}(s)\) does not scatter in either direction. \(\square\)

### 2.4. The harmonic maps \(Q = Q_k\)

We record a few properties about the unique (up to scaling) \(k\)-equivariant harmonic map \(Q = Q_k(r) = 2 \arctan r^k\) and some consequences of the fact that each \(Q_k\) minimizes the energy functional amongst all \(k\)-equivariant maps.
First observe that $Q$ satisfies
\[ r \partial_r Q(r) = k \sin Q(r), \quad Q(0) = 0, \quad Q(\infty) = \pi. \]
Recall that $\mathcal{H}_\pi$ the set of all finite energy $k$-equivariant maps, with $\phi_0(0) = 0$ and $\phi_0(\infty) = \pi$,
\[ \mathcal{H}_\pi := \{ (\phi_0, \phi_1) \mid \mathcal{E}(\tilde{\phi}) < \infty, \quad \phi_0(0) = 0, \lim_{r \to \infty} \varphi_0(r) = \pi \} \]
The fact that $Q$ minimizes the energy in $\mathcal{H}_\pi$ can be easily seen from the following Bogomol'nyi factorization of the energy:
\[
\mathcal{E}(\varphi_0, \varphi_1) = \pi \| \varphi_1 \|^2_{L^2} + \pi \int_0^\infty \left( \partial_r \varphi_0 - k \frac{\sin \varphi_0}{r} \right)^2 r \, dr + 2\pi k \int_0^\infty \sin \varphi_0 \partial_r \varphi_0 \, dr \\
= \pi \| \varphi_1 \|^2_{L^2} + \pi \int_0^\infty \left( \partial_r \varphi_0 - k \frac{\sin \varphi_0}{r} \right)^2 r \, dr + 2\pi k \int_0^\varphi_0(\infty) \sin(\rho) \, d\rho \\
= \pi \| \varphi_1 \|^2_{L^2} + \pi \int_0^\infty \left( \partial_r \varphi_0 - k \frac{\sin \varphi_0}{r} \right)^2 r \, dr + 4\pi k
\]
Hence,
\[
\mathcal{E}(\varphi_0, \varphi_1) \geq \pi \| \varphi_1 \|^2_{L^2} + 4\pi k = \pi \| \varphi_1 \|^2_{L^2} + \mathcal{E}(Q_k, 0)
\]
where the inequality in the last line above is in fact strict if $\varphi_0 \neq Q_k$.
We define a functional on maps $\Phi : \mathbb{R}^2 \to \mathbb{S}^2$ of finite energy. Let $\omega_{\mathbb{S}^2}$ denote the volume form on $\mathbb{S}^2$. Given $\Omega \subset \mathbb{R}^2$ set
\[ G(\Phi, \Omega) := \int_{\Phi(\Omega)} \omega_{\mathbb{S}^2} = \int_{\Omega} \Phi^*(\omega_{\mathbb{S}^2}) \]
where $\Phi^*(\omega_{\mathbb{S}^2})$ denotes the pull-back. Given $k$-equivariant $\Phi$ with polar angle $\phi$, this reduces to
\[ G(\phi_0(\cdot)) := 2\pi \int_{\phi_0(0)}^{\phi_0(R)} k |\sin \rho| \, d\rho \]
Observe that for any $(\phi, 0)$ with $\mathcal{E}(\tilde{\phi}) < \infty$ and for any $R \in [0, \infty)$ we have
\[ |G(\phi_0(R))| = \left| 2\pi \int_{\phi_0(0)}^{\phi_0(R)} k |\sin \rho| \, d\rho \right| = 2\pi \int_0^R |k \sin(\phi_0(r))| \partial_r \phi_0(r) \, dr \leq \mathcal{E}_0^R(\phi_0, 0) \]
The same argument shows that
\[ |G(\phi_0(R))| \leq \mathcal{E}_0^R(\phi_0, 0) \]
On the other hand, since $Q$ satisfies $r \partial_r Q(r) = k \sin(Q)$, for any $0 \leq a \leq b < \infty$ we see that
\[ G(Q(b)) - G(Q(a)) = 2\pi \int_a^b |\sin(Q(r))| \, Q_r(r) \, dr = \mathcal{E}_0^b(Q, 0) \]
Letting $a \to 0$ and $b \to \infty$ we recover the fact that $\mathcal{E}(Q, 0) = G(\pi) = 4\pi k$.
We recall the following variational characterization of $Q$ in $\mathcal{H}_\pi$ from [6], which amounts to the coercivity of the energy functional near $Q$. 

Lemma 2.11. [6 Proposition 2.3] There exists a function \( c : [0, \infty) \rightarrow [0, \infty) \) such that \( c(\alpha) \rightarrow 0 \) as \( \alpha \rightarrow 0 \) and such that the following holds: Let \((\phi_0, 0) \in \mathcal{H}_\pi\).

Suppose

\[
\alpha := \mathcal{E}(\phi_0, 0) - \mathcal{E}(Q, 0) \geq 0
\]

Then for \( \lambda > 0 \) defined so that \( \mathcal{E}_0^\lambda(\phi_0, 0) = \mathcal{E}_0^1(Q) = \mathcal{E}(Q)/2 \), we have

\[
\|\phi_0 - Q\lambda\|_H \leq c(\alpha)
\]

Moreover, \( \alpha = 0 \) if and only if \( \phi_0(r) = Q(r/\lambda) \) for some \( \lambda > 0 \).

2.5. Threshold solutions near a 2-bubble configuration. The goal of this section is to relate the proximity of a map \( \vec{\phi} \in \mathcal{H}_0 \) to a 2-bubble configuration to the size of the \( \mathcal{H}_0 \)-norm of \( \vec{\phi} \). With this in mind we make the following definition.

Definition 2.12 (Proximity to a 2-bubble). Given a map \( \vec{\phi} = (\phi_0, \phi_1) \in \mathcal{H}_0 \) we define its proximity \( d(\vec{\phi}) \) to a pure 2-bubble by

\[
d(\vec{\phi}) := \inf_{\lambda, \mu > 0, i \in \{+1, -1\}} \left( \|\phi_0 - i(Q\lambda - Q\mu), \phi_1\|_{\mathcal{H}_0}^2 + (\lambda/\mu)^k \right)
\]

The proof of Theorem 1.6 will require a few technical lemmas concerning \( d \). We’ll state the lemmas first and postpone the proofs until the end of this section.

Lemma 2.13. Suppose that \( \vec{\phi} = (\phi_0, \phi_1) \in \mathcal{H}_0 \) is \( k \)-equivariant and satisfies,

\[
\mathcal{E}(\vec{\phi}) = 2\mathcal{E}(\vec{Q}_k).
\]

Then for each \( \beta > 0 \) there exists a \( \varepsilon > 0 \) such that

\[
d(\vec{\phi}) \geq \beta \implies \|\phi_0, \phi_1\|_{\mathcal{H}_0} \leq \varepsilon(\beta)
\]

Conversely, for each \( \varepsilon > 0 \) we can find \( \alpha = \alpha(A) \) such that

\[
d(\vec{\phi}) \leq \alpha(A) \implies \|\phi_0, \phi_1\|_{\mathcal{H}_0} \geq A
\]

Note that \( d \) is small when \( \vec{\phi} \) is close to either a bubble/anti-bubble \((i = + \) in the definition of \( d \)) or anti-bubble/bubble configuration \((i = -)\). The next lemma makes precise the intuitive notion that a map \( \vec{\phi} \) cannot be simultaneously close to both configurations. With this in mind we define

\[
d_\pm(\vec{\phi}) := \inf_{\lambda, \mu > 0} \left( \|\phi_0 \mp (Q\lambda - Q\mu), \phi_1\|_{\mathcal{H}_0}^2 + (\lambda/\mu)^k \right)
\]

Lemma 2.14. There exists \( \alpha_0 > 0 \) with the following property: Let \( \vec{\phi} \in \mathcal{H}_0 \). For all \( \alpha \leq \alpha_0 \),

\[
d_\pm(\vec{\phi}) \leq \alpha \implies d_\mp(\vec{\phi}) \geq \alpha_0
\]

We begin by proving Lemma 2.13.

Proof of Lemma 2.13. It suffices to consider \( \vec{\phi} \) of the form \( \vec{\phi} = (\phi, 0) \). First we prove (2.14). To see this we’ll first show that for each \( \beta > 0 \) there exists a constant \( \delta = \delta(\beta) \) so that for any \( \vec{\phi} \in \mathcal{H}_0 \) with \( \mathcal{E}(\vec{\phi}) = 2\mathcal{E}(Q) \) we have

\[
d(\vec{\phi}) \geq \beta \implies \|\phi\|_{L^\infty} \leq \pi - \delta(\beta),
\]
Suppose \([2.17]\) fails. Then we can find \(\beta > 0\), a sequence \(\tilde{\phi}_n = (\phi_n, 0) \in \mathcal{H}_0\) with \(\mathcal{E}(\tilde{\phi}_n) = 2\mathcal{E}(Q)\), and numbers \(r_n > 0\) so that
\[
d(\phi_n, 0) \geq \beta \quad \text{and} \quad |\phi_n(r_n) - \pi| = o_n(1) \quad \text{as} \quad n \to \infty \quad (2.18)
\]
Define scales \(\lambda_n\) and \(\mu_n\) by
\[
\mathcal{E}_0^{\lambda_n}(\tilde{\phi}_n) = \mathcal{E}(Q)/2, \quad \mathcal{E}_\mu^{\infty}(\tilde{\phi}_n) = \mathcal{E}(Q)/2 \quad (2.19)
\]
Then, by \((2.12)\) we see that for \(n\) large enough \(\lambda_n < r_n\) and \(\mu_n > r_n\). Now define \(\phi_{n,1}\) and \(\phi_{n,2}\) as follows
\[
\phi_{n,1}(r) = \begin{cases} 
\phi_n(r) & \text{if} \ 0 \leq r \leq r_n \\
\pi + \frac{\pi - \phi_n(r_n)}{r_n}(r - 2r_n) & \text{if} \ r \in [r_n, 2r_n] \\
\pi & \text{if} \ r \geq 2r_n
\end{cases}
\]
\[
\phi_{n,2}(r) = \begin{cases} 
\pi + \frac{\phi_n(r_n)}{r_n}r & \text{if} \ r \leq r_n \\
\phi_n(r) & \text{if} \ r \geq r_n
\end{cases}
\]
And define \(\eta_n(r)\) by
\[
\eta_n(r) := \phi_n(r) - \phi_{n,1}(r) - \phi_{n,2}(r) + \pi
\]
We claim that
\[
\mathcal{E}(\phi_{n,1}, 0) = \mathcal{E}(Q, 0) + o_n(1) \quad \text{as} \quad n \to \infty \quad (2.20)
\]
\[
\mathcal{E}(\phi_{n,2}, 0) = \mathcal{E}(Q, 0) + o_n(1) \quad \text{as} \quad n \to \infty \quad (2.21)
\]
First we prove \((2.20)\), \((2.21)\). Since \(\phi_n(r_n) \to \pi\) we have
\[
\mathcal{E}_0^{\infty}(\phi_{n,1}, 0) = \mathcal{E}_0^{\infty}(\phi_{n,1}, 0) \geq G(\phi_{n,1}(r_n)) \to G(\pi) = \mathcal{E}(Q, 0) \quad \text{as} \quad n \to \infty
\]
\[
\mathcal{E}_\infty^{\infty}(\phi_{n,2}, 0) = \mathcal{E}_\infty^{\infty}(\phi_{n,2}, 0) \geq G(\phi_{n,2}(r_n)) \to G(\pi) = \mathcal{E}(Q, 0) \quad \text{as} \quad n \to \infty
\]
From the above and the fact that \(\mathcal{E}(\tilde{\phi}) = 2\mathcal{E}(Q)\) we see that in fact
\[
\mathcal{E}_0^{\infty}(\phi_{n,1}, 0) = \mathcal{E}(Q, 0) + o_n(1) \quad \text{as} \quad t \to \infty
\]
\[
\mathcal{E}_\infty^{\infty}(\phi_{n,2}, 0) = \mathcal{E}(Q, 0) + o_n(1) \quad \text{as} \quad t \to \infty
\quad (2.22)
\]
Direct computations using the definitions of \(\phi_{n,1}, \phi_{n,2}\) then show that
\[
\mathcal{E}_0^{\infty}(\phi_{n,1}, 0) \leq (\pi - \phi_n(r_n))^2 \to 0 \quad \text{as} \quad n \to \infty
\]
\[
\mathcal{E}_\infty^{\infty}(\phi_{n,2}, 0) \leq (\pi - \phi_n(r_n))^2 \to 0 \quad \text{as} \quad n \to \infty
\quad (2.23)
\]
Combining \((2.22)\) and \((2.23)\) gives \((2.20)\) and \((2.21)\). By construction
\[
\eta_n(r) = \pi - \phi_{n,2}(r) \quad \text{if} \ r \leq r_n, \quad \eta_n(r) = \pi - \phi_{n,1}(r) \quad \text{if} \ r \geq r_n
\]
A direct computation using the above and the definitions of \(\phi_{n,1}, \phi_{n,2}\) on the relevant intervals then yields
\[
\|\eta_n\|^2_H \leq (\pi - \phi_n(r_n))^2 \to 0 \quad \text{as} \quad n \to \infty
\]
By \((2.20)\), \((2.21)\), and \(\lambda_n, \mu_n\) defined in \((2.19)\) we use Lemma \(2.11\) to find \(\eta_{n,1}, \eta_{n,2} \in H\) so that
\[
\phi_{n,1}(r) = Q\lambda_n + \eta_{n,1}(r), \quad \phi_{n,2}(r) = \pi - Q\mu_n - \eta_{n,2}(r)
\]
\[
\|\eta_{n,j}\|_H \to 0 \quad \text{as} \quad n \to \infty
\]
Proof of Lemma 2.14. If the conclusion fails we could find a sequence which completes the proof. □

Two sequences of scales \(A\) direct computation then shows, such that \(\lambda\).

Moreover, we must have \(j\) on \(\beta\) and thus \(d\) implies the estimate \(\lambda\).

Passing to subsequences if necessary, relabeling \(\beta\) times \(Q\).

Lastly, we prove (2.15). Suppose that \(d(\lambda) \leq \alpha\). Then we can find, say, \(\lambda_0, \mu_0\) such that

\[
\alpha \leq \| (\phi_0 - Q \lambda_0 + Q \mu_0, \phi_1) \|_{H_0} + (\lambda_0/\mu_0)^k \leq 2\alpha
\]

A direct computation then shows,

\[
\| \phi_n\|_H \geq \| Q \lambda_0 - Q \mu_0 \|_H - \| \phi_0 - Q \lambda_0 + Q \mu_0 \|_H
\]

\[
\geq \| \log(\lambda_0/\mu_0) \| - 2\alpha \to \infty \quad as \quad \alpha \to 0
\]

which completes the proof.

We next prove Lemma 2.14

Proof of Lemma 2.14. If the conclusion fails we could find a sequence \(\phi_n \in H\), and two sequences of scales \(\lambda_n^+, \mu_n^+, \lambda_n^-, \mu_n^-\) so that

\[
\| \phi_n - Q \lambda_n^+ + Q \mu_n^+ \|_H + \frac{\lambda_n^+}{\mu_n^+} \to 0 \quad as \quad n \to \infty
\]

\[
\| \phi_n + Q \lambda_n^- - Q \mu_n^- \|_H + \frac{\lambda_n^-}{\mu_n^-} \to 0 \quad as \quad n \to \infty
\]

It follows that

\[
0 \leq \| (\phi_n - Q \lambda_n^+ + Q \mu_n^+) - (\phi_n + Q \lambda_n^- - Q \mu_n^-) + (Q \lambda_n^+ - Q \mu_n^+ + Q \lambda_n^- - Q \mu_n^-) \|_H
\]

\[
\geq \| Q \lambda_n^+ - Q \mu_n^+ + Q \lambda_n^- - Q \mu_n^- \|_H - o_n(1) \quad as \quad n \to \infty
\]

Passing to subsequences if necessary, relabeling \(\pm\), or rescaling, we can assume that \(\lambda_n^+ \leq \lambda_n^-\) for all \(n\) and that one of the following three possibilities holds

\[
\frac{\lambda_n^+}{\mu_n^+} \to 0, \quad or \quad \frac{\lambda_n^-}{\mu_n^-} \to \infty, \quad or \quad \frac{\lambda_n^+}{\mu_n^+} \to 1 > 0, \quad as \quad n \to \infty
\]

Assume we are in the first situation. Then, we can choose \(n\) large enough so that

\[
Q \lambda_n^+(r) + Q \lambda_n^-(r) \geq \pi \quad \forall r \in [\lambda_n^-, 2\lambda_n^-]
\]

\[
Q \mu_n^+(r) + Q \mu_n^-(r) \leq \frac{\pi}{2} \quad \forall r \in [\lambda_n^-, 2\lambda_n^-]
\]
and thus
\[ \|Q_{\lambda_n^+} - Q_{\mu_n^+} + Q_{\lambda_n^-} - Q_{\mu_n^-}\|_H^2 \geq \frac{\pi^2}{4} \int_{\lambda_n^-}^{2\lambda_n^-} \frac{dr}{r} \geq \frac{\pi^2}{4} \]
for all \( n \) large enough, which is impossible by (2.25). Now suppose we are in the second case \( \frac{\lambda_n^+}{\mu_n^-} \to \infty \). This means that
\[ \lambda_n^+ \ll \mu_n^+ \ll \lambda_n^- \ll \mu_n^- \]
and so for large enough \( n \) we have
\[ (Q_{\lambda_n^+} - Q_{\mu_n^+} + Q_{\lambda_n^-} - Q_{\mu_n^-})(r) \geq \frac{\pi}{4}, \quad \forall r \in [\lambda_n^+, 2\lambda_n^+] \]
which similarly leads (2.25) into a contradiction. Finally, if \( \frac{\lambda_n^-}{\mu_n^+} \to 1 \) we have
\[ \|Q_{\lambda_n^+} - Q_{\mu_n^+} + Q_{\lambda_n^-} - Q_{\mu_n^-}\|_H \geq \|Q_{\lambda_n^+} - Q_{\mu_n^-}\|_H - o_n(1) \]
Then setting \( \varphi_n := Q_{\lambda_n^+} - Q_{\mu_n^-} \) we see that \( d((\varphi_n, 0)) \to 0 \) and hence the right-hand-side above is bounded below by a fixed constant by (2.13) in Lemma 2.13. This again leads to a contradiction in (2.25), which completes the proof. \( \square \)

2.6. Virial identity. In this section we record a nonlinear estimates related to a virial-type identity that will be used in the proof of Theorem 1.6.

We begin with a virial-type identity for solutions to (1.5). In what follows we fix a smooth radial cut-off function \( \chi \in C^\infty_{\text{rad}}(\mathbb{R}^2) \), so that, writing \( \chi(r) = 1 \) if \( r \leq 1 \) and \( \chi(r) = 0 \) if \( r \geq 3 \) and \( |\chi'(r)| \leq 1 \) \( \forall r \geq 0 \) For each \( R > 0 \) we then define
\[ \chi_R(r) := \chi(r/R) \]

Lemma 2.15. Let \( \tilde{\psi}(t) \) be a solution to (1.5) on a time interval \( I \). Then for any time \( t \in I \) and \( R > 0 \) fixed we have
\[ \frac{d}{dt} \left( \psi_t | r \partial_r \psi \right)_{L^2} (t) = - \int_0^{\infty} \psi_t^2(t, r) r \, dr + \Omega_R(\tilde{\psi}(t)) \]
where
\begin{equation}
\Omega_R(\tilde{\psi}(t)) := \int_0^{\infty} \psi_t^2(t)(1 - \chi_R) r \, dr - \frac{1}{2} \int_0^{\infty} \left( \psi_t^2(t) + \psi_r^2(t) - k^2 \sin^2 \psi(t) \right) \frac{r}{R} \chi'(r/R) r \, dr \tag{2.26} \end{equation}

satisfies
\[ \left| \Omega_R(\tilde{\psi}(t)) \right| \lesssim \int_R^{\infty} \psi_t^2(t, r) r \, dr \, dt + \int_R^{\infty} \left| \psi_r^2 - k^2 \sin^2 \psi \right| r \, dr \, dt \]
\[ \lesssim \mathcal{E}_R(\tilde{\psi}(t)) \]
Proof. By direct calculation, using (1.5) we have
\[
\frac{d}{dt} \langle \psi_1 | \chi_R r \partial_r \psi_1 \rangle_{L^2} (t) = - \int_0^\infty \psi_1^2 (t) r \, dr + \int_0^\infty \psi_1^2 (t) (1 - \chi_R) r \, dr
\]
\[
- \frac{1}{2} \int_0^\infty \left( \psi_1^2 (t) + \psi_1^2 (t) - k^2 \frac{\sin^2 \psi (t)}{r^2} \right) \frac{r}{R} \chi' (r/R) r \, dr
\]

We show below how the quantities appearing on the right hand side of the virial identity can be estimated in terms of \( d(\vec{\phi}) \) in the vicinity of a two-bubble.

Lemma 2.16. There exists a number \( C_0 > 0 \) depending only on \( k \) such that for all \( \vec{\phi} = (\phi_0, \phi_1) \in H_0 \) with \( \mathcal{E}(\vec{\phi}) = 2\mathcal{E}(Q) \) and all \( R > 0 \) there holds
\[
| \langle \phi_1, \chi_R r \partial_r \phi_0 \rangle | \leq C_0 R \sqrt{d(\vec{\phi})},
\]
(2.27)
\[
\Omega_R(\vec{\phi}) \leq C_0 \sqrt{d(\vec{\phi})}.
\]
(2.28)

Proof. By Cauchy-Schwarz, we get
\[
| \langle \phi_1, \chi_R r \partial_r \phi_0 \rangle | \lesssim R \| \phi_1 \|_{L^2} \| \partial_r \phi_0 \|_{L^2}.
\]
(2.29)
We have \( \| \partial_r (Q_\lambda - Q_\mu) \|_{L^2} \lesssim 1 \) for all \( \lambda \) and \( \mu \), hence by the triangle inequality \( \| \partial_r \phi_0 \|_{L^2} \lesssim 1 + \sqrt{d(\vec{\phi})} \). If \( d(\vec{\phi}) \leq 1 \), then we obtain \( \| \partial_r \phi_0 \|_{L^2} \lesssim 1 \). If \( d(\vec{\phi}) \geq 1 \), then Lemma 2.13 gives \( \| \phi_0 \|_{H_0} \lesssim 1 \), in particular again \( \| \partial_r \phi_0 \|_{L^2} \lesssim 1 \). Thus (2.29) yields (2.27).

To prove (2.28), we write
\[
| \Omega_R(\vec{\phi}) | \lesssim \int_0^{+\infty} \phi_1^2 r \, dr + \left| (\partial_r \phi_0)^2 - k^2 \frac{\sin^2 \phi_0}{r^2} \right| r \, dr.
\]
Again, the conclusion is clear if \( d(\vec{\phi}) \geq 1 \), we can assume \( d(\vec{\phi}) \leq 1 \). Find \( \lambda, \mu > 0 \) such that, say,
\[
\left( \frac{\lambda}{\mu} \right)^k \leq 2d(\vec{\phi}) \quad \text{and} \quad \| (\phi_0 - Q_\lambda + Q_\mu) \|_{H_0} \leq 2d(\vec{\phi})
\]
By the above it suffices to show that for \( g := \phi_0 - Q_\lambda + Q_\mu \) we have
\[
\int_0^{+\infty} \left| (\partial_r \phi_0)^2 - k^2 \frac{\sin^2 \phi_0}{r^2} \right| r \, dr \lesssim \left( \frac{\lambda}{\mu} \right)^{k/2} + \| g \|_H
\]
Using trigonometric identities we expand
\[
\sin^2 (Q_\lambda - Q_\mu + g) = \sin^2 Q_\lambda + \sin^2 Q_\mu - \frac{1}{2} \sin 2Q_\lambda \sin 2Q_\mu + 2 \sin^2 Q_\lambda \sin^2 Q_\mu + \frac{1}{2} \sin 2g \sin 2(Q_\lambda - Q_\mu) + \sin^2 g \cos 2(Q_\lambda - Q_\mu)
\]
Then, since \( \Lambda Q_\lambda := r \partial_r Q_\lambda = k \sin Q_\lambda \) we have
\[
\int_0^\infty \left| (r \partial_r \phi_0)^2 - k^2 \sin^2 \phi_0 \right| \frac{dr}{r} \lesssim \int_0^\infty |\Lambda Q_\lambda \Lambda Q_\mu| + |\Lambda Q_\lambda r \partial_r g| + |\Lambda Q_\mu r \partial_r g| \frac{dr}{r}
\]
\[
+ \int_0^\infty |g \Lambda Q_\lambda| + |g \Lambda Q_\mu r| + |r \partial_r g|^2 + |g|^2 \left| \right| \frac{dr}{r}
\]
To estimate the first term above we see that setting $\sigma = \lambda/\mu$ we have
\[
\int_0^\infty |\Lambda Q_\lambda \Lambda Q_\mu| \frac{dr}{r} = k^2 \int_0^\infty \frac{(r/\lambda)^k (r/\mu)^k}{(1 + (r/\lambda)^{2k})(1 + (r/\mu)^{2k})} \frac{dr}{r} = \sigma^k \int_0^\infty r^{2k-1} \frac{dr}{(\sigma^{2k} + r^{2k})(1 + r^{2k})} \lesssim \sigma^k \|\log \sigma\| \lesssim \left(\frac{\lambda}{\mu}\right) \frac{1}{\mu}.
\]
The remaining terms can be controlled by $\|g\|_H$ by Cauchy-Schwarz. \hfill $\square$

3. The modulation method: analysis of 2-bubble collisions

In this section we give a careful analysis of the modulation equations that govern the evolution of 2-bubble configurations. The intuition is that the less concentrated bubble does not change its scale and influences the dynamics of the more concentrated bubble. We will quantify this influence.

3.1. Modulation Equations. We consider solutions $\tilde{\psi}(t)$ to (1.5) that are close to a 2-bubble configuration on a time interval $J$ in the sense that $d(\tilde{\psi}(t))$, defined in (2.13), is small for all $t \in J$. Recall that $d(\tilde{\psi}(t))$ is the smaller of the numbers $d_+(\tilde{\psi}(t))$ and $d_-(\tilde{\psi}(t))$ defined in (2.10).

Linearizing (1.5) about $Q_\Lambda$ leads to the Schrödinger operator
\[
\mathcal{L}_\lambda := -\partial_r^2 - \frac{1}{r} \partial_r + k^2 \cos 2Q_\lambda \frac{2Q_\lambda}{r^2}
\]
We write $\mathcal{L} := \mathcal{L}_1$. Recall from (2.11) that $\Lambda = r \partial_r$ is the infinitesimal generator of dilations in $\dot{H}^1(\mathbb{R}^2)$. One can check that $\Lambda Q$ is a zero energy eigenfunction for $\mathcal{L}$, i.e.,
\[
\mathcal{L}\Lambda Q = 0, \quad \text{and} \quad \Lambda Q \in L^2_{\text{rad}}(\mathbb{R}^2).
\]
When $k = 1$, $\mathcal{L}\Lambda Q = 0$ still holds but in this case $\Lambda Q \notin L^2$ due to slow decay as $r \to \infty$ and is 0 is referred to as a threshold resonance.

In fact, $\Lambda Q$ spans the kernel of $\mathcal{L}$. This can be seen using the following well known factorization of $\mathcal{L}$,
\[
\mathcal{L} = A^* A \quad \text{where} \quad A^* = \partial_r + \frac{1 + k \cos(Q)}{r}, \quad A = -\partial_r + \frac{k \cos(Q)}{r}
\]
(3.1) together with the observation that $A(\Lambda Q) = 0$; we note that (3.1) is a consequence of the Bogomol’nyi factorization (2.11); see [43, 44] for more.

The fact that $\mathcal{L}_\lambda \Lambda Q_\lambda = 0$ will play an important role in the modulation estimates.

We fix a radial cutoff $\chi \in C_0^\infty(\mathbb{R}^2)$ such that $\chi(r) = 1$ if $r \leq 1$, $\text{supp}(\chi) \subset B(0, 2)$. We also fix a radial function $\mathcal{Z} \in C_0^\infty(\mathbb{R}^2)$ so that
\[
\int_0^\infty \mathcal{Z}(r) \cdot \Lambda Q(r) r \, dr > 0, \quad \left|\frac{\mathcal{Z}(r)}{r^k}\right| \lesssim 1, \quad \forall r \leq 1.
\]

Lemma 3.1 (Modulation Lemma). There exist $\eta_0 > 0$ and $C > 0$ with the following property: Let $J \subset \mathbb{R}$ be a time interval, $\tilde{\psi}(t)$ a solution to (1.5) defined on $J$, and assume that
\[
d_+(\tilde{\psi}(t)) \leq \eta_0 \quad \forall t \in J.
\]
Then, there exist unique $C^1(J)$ functions $\lambda(t), \mu(t)$ so that, defining $g(t) \in H$ by
\[ g(t) := \psi(t) - Q_{\lambda(t)} + Q_{\mu(t)} \] (3.2)
we have, for each $t \in J$,
\[ \langle \chi_{\mu(t)} A Q_{\lambda(t)} | g(t) \rangle = 0 \] (3.3)
\[ \langle Z_{\mu(t)} | g(t) \rangle = 0 \] (3.4)
\[ d_+ (\tilde{\psi}(t)) \leq \| (g(t), \psi(t)) \|_{H^2 \times L^2} + (\lambda(t)/\mu(t))^k \leq C d_+ (\tilde{\psi}(t)) \] (3.5)
Moreover,
\[ \| (g(t), \psi(t)) \|_{H^0} \leq C \left( \frac{\lambda(t)}{\mu(t)} \right)^{\frac{1}{2}} \] (3.6)
and hence
\[ d_+ (\tilde{\psi}(t)) \approx \left( \frac{\lambda(t)}{\mu(t)} \right)^k \] (3.7)

Proof. The proof follows by standard techniques that we outline below; we refer the reader to [21, Lemma 3.3] for a detailed proof of a similar statement.

We begin by showing that for each $t \in J$ there exist unique $\lambda(t), \mu(t)$, and $g(t)$ that satisfy (3.2), (3.5) and the orthogonality conditions (3.3) (3.4) using an argument based on the implicit function theorem. That $\lambda(t)$ and $\mu(t)$ are actually $C^1(J)$ is then proved via a standard ODE argument, which we postpone until Remark 3.13 in Section 3.2.

To establish the former statement let $\tilde{\varphi} \in H_0$ be such that $d_+ (\tilde{\varphi}) \leq \eta_0$. This means we can find $\lambda_0, \mu_0 > 0$ such that for $g_0 \in H$ defined by
\[ g_0 := \varphi_0 - (Q_{\lambda_0} - Q_{\mu_0}) \]
we have
\[ \| (g_0, \varphi_1) \|_{H^2 \times L^2} + \left( \frac{\lambda_0}{\mu_0} \right)^k \leq 2 \eta_0 \]

Define the mapping $F : H \times (0, \infty) \times (0, \infty) \to H$ by
\[ F(g, \lambda, \mu) := g - (Q_{\lambda} - Q_{\mu}) + (Q_{\lambda_0} - Q_{\mu_0}) \]
and note that $F(0, \lambda_0, \mu_0) = 0$. Next define a mapping $G : H \times (0, \infty) \times (0, \infty) \to \mathbb{R}^2$ by
\[ G(g, \lambda, \mu) := \left( \langle \chi_{\mu} A Q_{\lambda} | F(g, \lambda, \mu) \rangle, \langle Z_{\mu} | F(g, \lambda, \mu) \rangle \right) \]
For $g \in H$ we have
\[ \langle \chi_{\mu} A Q_{\lambda} | g \rangle \leq \| AQ \|_{L^2} \| \chi_{\mu} g \|_{L^2} \lesssim_{\mu} \| g \|_H, \quad \langle Z_{\mu} | g \rangle \lesssim_{\mu} \| g \|_H \]
which ensures that the mapping $G$ is well-defined and continuous. Taking the $\lambda, \mu$ derivatives of $G$ and restricting to $(g, \lambda, \mu) = (0, \lambda_0, \mu_0)$ yields
\[ D_{\lambda, \mu} G |_{(g=0, \lambda=\lambda_0, \mu=\mu_0)} = \left( \begin{array}{c} \langle \chi_{\mu_0} A Q_{\lambda_0} | A Q_{\lambda_0} \rangle - \langle \chi_{\mu_0} A Q_{\lambda_0} | A Q_{\mu_0} \rangle \\ \langle Z_{\mu_0} | A Q_{\lambda_0} \rangle - \langle Z_{\mu_0} | A Q_{\mu_0} \rangle \end{array} \right) \]
The diagonal terms in the matrix above are size \(O(1)\) and the off-diagonal terms satisfy
\[
\left\langle \chi_{\mu_0} \Lambda Q_{\lambda_0} \mid \Lambda Q_{\mu_0} \right\rangle = O((\lambda_0/\mu_0)^{k-1}), \quad \left\langle Z_{\mu_0} \mid \Lambda Q_{\lambda_0} \right\rangle = O((\lambda_0/\mu_0)^{k-1})
\]
Hence for \(k \geq 2\) the matrix is invertible as long as \((\lambda_0/\mu_0)^{k-1} \leq 2\eta_0\) is small enough.

An argument based on the implicit function theorem yields the following: There exists \(\eta_0 > 0\) small enough, an open neighborhood \(U \ni (\lambda_0, \mu_0)\) in \(\mathbb{R}^2\) and mapping
\[\varsigma : B_H(0, \eta_0) \to U\]
with \(\varsigma(g) = (\lambda, \mu)\) so that for all \((g, \lambda, \mu) \in B_H(0, \eta_0) \times U\) we have
\[G(g, \lambda, \mu) \equiv 0 \iff (\lambda, \mu) = \varsigma(g)\]

Finally, we observe that if we let \(\tilde{\varsigma}_0\) be as above, and define \((\tilde{\lambda}, \tilde{\mu})\) and \(\tilde{g} \in H\) by
\[(\tilde{\lambda}, \tilde{\mu}) := \varsigma(\tilde{\varsigma}_0), \quad \tilde{g} := F(\tilde{\varsigma}_0, \tilde{\lambda}, \tilde{\mu})\]
we see that \(\tilde{\phi}_0 = Q_\lambda - Q_\mu + g\) and moreover that
\[\left\langle \chi_{\mu} \Lambda Q_{\tilde{\lambda}} \mid \tilde{g} \right\rangle = 0 \quad \text{and} \quad \left\langle Z_{\tilde{\mu}} \mid \tilde{g} \right\rangle = 0\]

Shrinking \(\eta_0\) further we can also ensure that \((3.5)\) holds as well.

It remains to establish the estimate \((3.6)\). This follows by expanding the nonlinear energy. We’ll make use of trigonometric identities here for simplicity but note that the following computation relies only on the fact that the nonlinearity in \((1.5)\) is smooth, that \(Q\) is a solution, and on the orthogonality conditions \((3.3)\) \((3.4)\).

\[
\frac{2}{\pi} \mathcal{E}(Q) = \frac{1}{\pi} \mathcal{E}(\tilde{\phi}) = \frac{2}{\pi} \mathcal{E}(Q) + \int_0^\infty g_r^2 r \, dr + \int_0^\infty \psi_r^2 r \, dr
- 2 \int_0^\infty \partial_r Q_\lambda \partial_r Q_\mu \, r \, dr + 2 \int_0^\infty \partial_r Q_\lambda g_r \, r \, dr - 2 \int_0^\infty \partial_r Q_\mu g_r \, r \, dr
+ k^2 \int_0^\infty \frac{\sin^2(Q_\lambda - Q_\mu + g)}{r^2} r \, dr - k^2 \int_0^\infty \frac{(\sin^2 Q_\lambda + \sin^2 Q_\mu)}{r^2} r \, dr
\]

We expand the nonlinear terms on the last line using trigonometric identities
\[
\sin^2(Q_\lambda - Q_\mu + g) = \sin^2(Q_\lambda - Q_\mu) + \frac{1}{2} \sin 2g \sin 2(Q_\lambda - Q_\mu)
+ \sin^2 g \cos 2(Q_\lambda - Q_\mu)
+ g \sin 2(2Q_\lambda - Q_\mu) + g^2 \cos 2(Q_\lambda - Q_\mu) + O(|g|^3)
\]
which further reduces to
\[
= \sin^2 Q_\lambda + \sin^2 Q_\mu + g^2 \cos 2(Q_\lambda - Q_\mu) + 2 \sin^2 Q_\lambda \sin^2 Q_\mu
+ g \sin 2Q_\lambda - g \sin 2Q_\mu - \sin 2Q_\lambda Q_\mu + \frac{1}{2} \sin 2Q_\lambda [2Q_\mu - \sin 2Q_\mu]
- g(2\sin 2Q_\lambda \sin^2 Q_\mu - 2 \sin 2Q_\mu \sin^2 Q_\lambda) + O(|g|^3)
\]

Next we observe that the first three terms in the second line of \((3.9)\) will give exact cancelations with the terms in the second line of \((3.8)\). Indeed, using the identity
\[
\frac{1}{r} \partial_r (r \partial_r Q_\lambda) = k^2 \frac{\sin 2Q_\lambda}{2r^2}
\]
we integrate by parts to obtain
\[ k^2 \int_0^\infty \frac{\sin 2Q}{r^2} \sin^2 Q \mu g r \, dr = 2 \int_0^\infty \frac{1}{r} \partial_r (r \partial_r Q_\lambda) Q_\mu g r \, dr = -2 \int_0^\infty \partial_r Q_\lambda \partial_r Q_\mu g r \, dr \]
\[ k^2 \int_0^\infty \frac{\sin 2Q}{r^2} g r \, dr = 2 \int_0^\infty \frac{1}{r} \partial_r (r \partial_r Q_\lambda) g r \, dr = -2 \int_0^\infty \partial_r Q_\lambda g r \, dr \]
\[ k^2 \int_0^\infty \frac{\sin 2Q}{r^2} g r \, dr = 2 \int_0^\infty \frac{1}{r} \partial_r (r \partial_r Q_\mu) g r \, dr = -2 \int_0^\infty \partial_r Q_\mu g r \, dr \]

We can use the same identity to integrate by parts the terms arising from the rest of (3.9).

\[ -2k^2 \int_0^\infty \frac{\sin 2Q}{2r^2} \sin^2 Q_\mu g r \, dr = -4 \int_0^\infty \frac{1}{r} \partial_r (r \partial_r Q_\lambda) \sin^2 Q_\mu g r \, dr \]
\[ = 4 \int_0^\infty \partial_r Q_\lambda \partial_r Q_\mu \sin 2Q_\mu g r \, dr + 4 \int_0^\infty \partial_r Q_\lambda \sin^2 Q_\mu g r \, dr \]

and the same for the symmetric term in \( \lambda, \mu \). Finally, we have

\[ k^2 \int_0^\infty \frac{\sin 2Q}{2r^2} [2Q_\mu - \sin 2Q_\mu] g r \, dr = \int_0^\infty \frac{1}{r} \partial_r (r \partial_r Q_\lambda) [2Q_\mu - \sin 2Q_\mu] g r \, dr \]
\[ = -4 \int_0^\infty \partial_r Q_\lambda \partial_r [2Q_\mu - \sin 2Q_\mu] g r \, dr = -4 \int_0^\infty \partial_r Q_\lambda \partial_r Q_\mu g r \, dr \]

Therefore, using the above, (3.9), the identity \( \Lambda Q = k \sin Q \), and the fact that \( E(\psi) = 2E(Q) \), we can deduce from (3.8) that

\[
\int_0^\infty \psi_t^2 r \, dr + \int_0^\infty g_r^2 r \, dr + k^2 \int_0^\infty \frac{\cos 2(Q_\lambda - Q_\mu)}{r^2} g^2 r \, dr \\
= \frac{4}{k^2} \int_0^\infty \Lambda Q_\lambda (\Lambda Q_\mu)^3 \frac{dr}{r} - \frac{2}{k^2} \int_0^\infty (\Lambda Q_\lambda)^2 (\Lambda Q_\mu)^2 \frac{dr}{r} \\
- 4 \int_0^\infty \Lambda Q_\lambda \Lambda Q_\mu \sin 2Q_\mu g \frac{dr}{r} - \frac{4}{k^2} \int_0^\infty \Lambda Q_\lambda (\Lambda Q_\mu)^2 (rg_r) \frac{dr}{r} \\
+ 4 \int_0^\infty \Lambda Q_\lambda \Lambda Q_\mu \sin 2Q_\lambda g \frac{dr}{r} + \frac{4}{k^2} \int_0^\infty \Lambda Q_\lambda (\Lambda Q_\mu)^2 (rg_r) \frac{dr}{r} \\
+ O \left( \int_0^\infty |g|^3 \frac{dr}{r} \right)
\]

Next, we estimate each of the terms on the right-hand-side of (3.10). Denote \( \sigma := \lambda/\mu \). We claim that the first term on the right-hand-side of (3.10) gives the leading order, i.e., we claim that

\[ \frac{4}{k^2} \int_0^\infty \Lambda Q_\lambda (\Lambda Q_\mu)^3 \frac{dr}{r} = 16k^2 \sigma^k (1 + O(\sigma^{-k})) \quad (3.11) \]

We compute, using the identity \( k \sin Q_\lambda = \Lambda Q_\lambda \), and setting \( \sigma = \lambda/\mu \),

\[ \frac{4}{k^2} \int_0^\infty \Lambda Q_\lambda (\Lambda Q_\mu)^3 \frac{dr}{r} = \frac{4}{k^2} \int_0^\infty \Lambda Q_\sigma (\Lambda Q)^3 \frac{dr}{r} \]
\[ = 64k^2 \sigma^k \int_0^\infty (\sigma^{-2k} + r^{2k})(1 + r^{2k})^{\sigma^{-1}} \, dr \quad (3.12) \]
First we estimate the contribution of the integral on the interval $[0, \sigma]$. Since we can assume that $\sigma \ll 1$, we have

\[
\int_0^\sigma \frac{r^{4k-1}}{(\sigma^{2k} + r^{2k})(1 + r^{2k})^3} \, dr \simeq \sigma^{-2k} \int_0^\sigma r^{4k-1} \, dr \simeq \sigma^{2k}
\]

Next, we estimate the integral on $[\sigma, \infty]$. For $\sigma < r$ we have

\[
\frac{1}{\sigma^{2k} + r^{2k}} = \frac{1}{r^{2k}} + \left( \frac{1}{\sigma^{2k} + r^{2k}} - \frac{1}{r^{2k}} \right)
\]

\[
= \frac{1}{r^{2k}} + \frac{1}{r^{2k}} \left( \frac{1}{1 - (\sigma/r)^{2k}} - 1 \right)
\]

\[
= \frac{1}{r^{2k}} + \frac{1}{r^{2k}} \left( - (\sigma/r)^{2k} + O((\sigma/r)^{4k}) \right)
\]

Hence,

\[
\int_\sigma^\infty \frac{r^{4k-1}}{(\sigma^{2k} + r^{2k})(1 + r^{2k})^3} \, dr = \int_0^\infty \frac{r^{2k-1}}{(1 + r^{2k})^3} \, dr - \int_0^\sigma \frac{r^{2k-1}}{(1 + r^{2k})^3} \, dr + O(\sigma^{2k})
\]

\[
= \int_0^\infty \frac{r^{2k-1}}{(1 + r^{2k})^3} \, dr + O(\sigma^{2k})
\]

\[
= \frac{1}{4k} + O(\sigma^{2k})
\]

where the integral on the second to last line can be computed explicitly by contour integration. Inserting the above into the last line of (3.12) yields (3.11).

Next we observe that all of the remaining terms on the right-hand-side of (3.10) are $o(\sigma^k)$. Indeed, a similar computation to the one performed above yields,

\[
\int_0^\infty (\Lambda Q_\lambda)^2 (\Lambda Q_\mu)^2 \frac{dr}{r} \lesssim \sigma^{2k} |\log \sigma|
\]

Moreover since $\|g\|_{L^\infty} \lesssim \|g\|_H \lesssim \sigma^{\frac{k}{2}}$ we have

\[
\int_0^\infty |g|^3 \frac{dr}{r} \lesssim \|g\|_H^3 \lesssim \sigma^{\frac{3k}{2}}
\]

And the remaining terms in (3.10) can be controlled by a combination of these last two estimates together with Cauchy-Schwarz. Therefore as long as $\eta_0$ is small enough and since $\sigma = \lambda/\mu \lesssim \eta_0$ we have

\[
\int_0^\infty \psi_t r \, dr + \int_0^\infty g_t^2 r \, dr + k^2 \int_0^\infty \frac{\cos 2(Q_\lambda - Q_\mu)}{r^2} g^2 r \, dr = 16k \sigma^k - O(\sigma^{\frac{3k}{2}} |\log \sigma|)
\]

To complete the proof of (3.6) we claim the following coercivity statement: there exists a uniform constant $c = c(M) > 0$ so that

\[
\int_0^\infty g_t^2 r \, dr + k^2 \int_0^\infty \frac{\cos 2(Q_\lambda - Q_\mu)}{r^2} g^2 r \, dr \geq c(M) \|g\|_H^2
\]

for all $g \in H$ such that (3.3) (3.4) hold and such that $\|g\|_H$ is small enough. This is a standard consequence of the orthogonality conditions (3.3), (3.4) and the smallness of $\lambda/\mu$, $\|g\|_H$ and we refer the reader to [22] Lemma 5.4 for a detailed proof. □
3.2. Dynamical control of the modulation parameters. In this section we obtain precise control of the evolution of the modulation parameters \( \lambda(t), \mu(t) \) on any time interval \( J \) on which \( d_+(\tilde{\psi}(t)) \) is small. We’ll show that any solution \( \tilde{\psi}(t) \) that lies within a small enough \( \epsilon \)-neighborhood of a 2-bubble at some time \( t_0 \), must be ejected from this \( \epsilon \)-neighborhood in at least one time direction.

This ejection happens by a defocalisation of the more concentrated bubble \( Q_\lambda \) until its scale becomes comparable with the less concentrated bubble \( Q_\mu \) (which does not change in the process). The influence of the bubble \( Q_\mu \) on the evolution of \( Q_\lambda \) is reflected in the time derivative of the function \( b(t) \) defined in (3.24) below. Indeed, the main term of \( b'(t) \) is given precisely by the interaction between the two bubbles, see (3.29). The main term of \( b(t) \) is related to \( \lambda'(t) \). Hence \( b'(t) \) is related to \( \lambda''(t) \) so that the interaction influences the acceleration, as it should be expected.

In this subsection we define a truncated virial functional and state some estimates related to it. The same functional was used crucially in the two-bubble construction by the first author in [22]. For the proofs of the following statements we refer the reader to [22, Lemma 4.6] and [22, Lemma 5.5].

Lemma 3.2. [22, Lemma 4.6] For each \( c, R > 0 \) there exists a function \( q(r) = q_{c,R}(r) \in C^{3,1}((0, +\infty)) \) with the following properties:

(P1) \( q(r) = \frac{1}{2} r^2 \) for \( r \leq R \),

(P2) there exists \( R = \tilde{R}(R, c) > R \) such that \( q(r) \equiv \text{const} \) for \( r \geq \tilde{R} \),

(P3) \( |q'(r)| \lesssim r \) and \( |q''(r)| \lesssim 1 \) for all \( r > 0 \), with constants independent of \( c, R \),

(P4) \( q''(r) \geq -c \) and \( \frac{1}{r} q'(r) \geq -c \), for all \( r > 0 \),

(P5) \( \left( \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right)^2 q(r) \leq c \cdot r^{-2} \), for all \( r > 0 \),

(P6) \( |r \left( \frac{q(r)}{r} \right)'| \leq c \), for all \( r > 0 \).

For each \( \lambda > 0 \) we define the operators \( A(\lambda) \) and \( A_0(\lambda) \) as follows:

\[
[A(\lambda)g](r) := q \left( \frac{r}{\lambda} \right) \cdot \partial_r g(r), \quad (3.13)
\]

\[
[A_0(\lambda)g](r) := \left( \frac{1}{2\lambda} q'' \left( \frac{r}{\lambda} \right) + \frac{1}{2r} q' \left( \frac{r}{\lambda} \right) \right) g(r) + q \left( \frac{r}{\lambda} \right) \cdot \partial_r g(r). \quad (3.14)
\]

Note the similarity between \( A \) and \( \Lambda \) and between \( A_0 \) and \( \Lambda_0 \). For technical reasons we introduce the space

\[
X := \{ g \in H, \left\| \frac{2}{r} \partial_r g \right\|_H \}
\]

Let \( f(\rho) := \frac{1}{2} \rho \sin 2\rho \) denote the nonlinearity in (1.4).

Lemma 3.3. [22, Lemma 5.5] Let \( c_0 > 0 \) be arbitrary. There exists \( c > 0 \) small enough and \( R, \tilde{R} > 0 \) large enough in Lemma 3.2 so that the operators \( A(\lambda) \) and \( A_0(\lambda) \) defined in (3.13) and (3.14) have the following properties:

- the families \( \{ A(\lambda) : \lambda > 0 \}, \ldots \) are bounded in \( \mathscr{L}(H; L^2) \), with the bound depending only on the choice of the function \( q(r) \),
• For all \( \lambda > 0 \) and \( g_1, g_2 \in X \) there holds
\[
\left| \left( A(\lambda)g_1 \mid \frac{1}{r^2}(f(g_1 + g_2) - f(g_1) - f'(g_1)g_2) \right) \right|
+ \left| \left( A(\lambda)g_2 \mid \frac{1}{r^2}(f(g_1 + g_2) - f(g_1) - k^2g_2) \right) \right|
\leq \frac{c_0}{\lambda}((\|g_1\|_H^2 + 1)\|g_2\|_H^2 + \|g_2\|_H^2), \tag{3.15}
\]

For all \( g \in X \) we have
\[
\left\langle A_0(\lambda)g \left( \frac{1}{r^2} \frac{1}{r^2} - \frac{k^2}{r^2} \right) g \right\rangle \leq \frac{c_0}{\lambda}g\|g\|_H^2 - \frac{1}{\lambda} \int_0^{R\lambda} (\partial_r g)^2 + \frac{k^2}{r^2}g^2 \, rdr \tag{3.16}
\]

Moreover,
\[
\|A_0\Lambda Q_\lambda - A_0(\lambda)\Lambda Q_\lambda\|_{L^2} \leq c_0, \tag{3.17}
\]
\[
\|\Lambda Q_\lambda - A(\lambda)Q_\lambda\|_{L^\infty} \leq \frac{c_0}{\lambda}, \tag{3.18}
\]
and, for any \( g \in H, \lambda, \mu > 0 \) with \( \lambda/\mu \ll 1 \),
\[
\left| \int_0^{+\infty} \frac{1}{2} \left( q''(r^2) + \frac{\lambda}{r}q'(r^2) \right) \frac{1}{r^2} f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - k^2g \, g \, rdr \right|
- \int_0^{+\infty} \frac{1}{r^2} (f(Q_\lambda) - k^2)g^2 \, rdr \leq c_0(||g||_H^2 + (\lambda/\mu)^k). \tag{3.19}
\]

Remark 3.4. The conditions \( g, g_1, g_2 \in X \) is required only to ensure that the left-hand-side of (3.15) and (3.16) are well defined, but do not appear on the right-hand-side of the estimates. Note also that in (3.15), (3.16) and (3.19) we have extracted the linear part of \( f \).

We are now ready to state the main modulation estimates. Proofs are given in Section 4.3. Our first estimate is a consequence of the orthogonality conditions (3.3) and (3.4).

**Proposition 3.5** (Modulation Control Part 1). Let \( \eta_0 > 0 \) be as in Lemma 3.1 and let \( J \subset \mathbb{R} \) be a time interval, and let \( \psi(t) \) be a solution to (1.5) on \( J \) such that
\[
d(\psi(t)) \leq \eta_0 \quad \forall t \in J.
\]

Let \( \lambda(t), \mu(t) \) be given by Lemma 3.2. Then the following estimates hold for \( t \in J \):
\[
|\lambda'(t)| \leq \lambda(t)^{1/2}/\mu(t)^{1/2}, \tag{3.20}
\]
\[
|\mu'(t)| \leq \lambda(t)^{1/2}/\mu(t)^{1/2}. \tag{3.21}
\]

The control we obtain on \( \lambda(t), \mu(t) \) above is not sufficient for our purposes. In particular, we’d like to show that the ratio \( \lambda(t)/\mu(t) \) grows in a controlled fashion away from any small enough local minimum value. For this purpose we introduce a virial-type correction \( b(t) \) to \( \lambda(t) \). The idea of modifying a modulation parameter by a virial term was used in [23].

Given scaling parameters \( \lambda(t), \mu(t) \) we write
\[
g(t) := \psi(t) - Q_{\lambda(t)} + Q_{\mu(t)}
\]
\[
\dot{g}(t) := \dot{\psi}(t)
\]

\[
\dot{g}(t) := \dot{\psi}(t)
\]
so that the vector \( \vec{g} := (g, \dot{g}) \) satisfies the system of equations

\[
\begin{align*}
\partial_t g &= \dot{g} + \lambda' \Lambda Q_{\lambda} - \mu' \Lambda Q_{\mu} \\
\partial_t \dot{g} &= \partial_r^2 g + \frac{1}{r} \partial_r g - \frac{1}{r^2} \left( f(Q_{\lambda} - Q_{\mu} + g) - f(Q_{\lambda}) + f(Q_{\mu}) \right)
\end{align*}
\] (3.22) (3.23)

We then define the auxiliary function \( b(t) \) by

\[
b(t) := -\left\langle \Lambda Q_{\lambda(t)} | \dot{g}(t) \right\rangle - \left\langle \dot{g}(t) | A_0(\lambda(t)) g(t) \right\rangle
\] (3.24)

We’ll show below that we can think of \( b(t) \) as a subtle monotonic correction to the derivative \( \lambda'(t) \).

Before stating the estimates satisfied by \( b(t) \) we record the following numbers, which can be computed using contour integration:

\[
\|\Lambda Q\|_2^2 = \frac{2\pi i \text{Res}[(\Lambda Q(z))^2 z; \omega_k]}{1 - \omega_k^2}, \quad \omega_k := \exp(2\pi i/4k)
\]

which means that

\[
\|\Lambda Q\|_2^2 = \frac{2\pi}{\sin(\pi/k)} =: \kappa = \kappa(k) > 0
\]

We will also use fact that

\[
\int_0^\infty (\Lambda Q(r))^3 r^{k-1} dr = 2k^2
\] (3.25)

**Proposition 3.6** (Modulation Control Part 2). Assume the same hypothesis as in Proposition 3.5. Let \( \delta > 0 \) be arbitrary and let \( \eta_0 \) be as in Lemma 3.1. Let \( b(t) \) be as in (3.24). Then, there exists \( \eta_1 = \eta_1(\delta) < \eta_0 \) such that if \( d(\vec{\psi}(t)) \leq \eta_1 \) for all \( t \in J \) we have

\[
|\kappa \lambda'(t) - b(t)| \leq \delta \lambda(t)^{\frac{3}{2}} / \mu(t)^{\frac{1}{2}}.
\] (3.26)

\[
|b(t)| \leq 4 \sqrt{\kappa k (\lambda(t)/\mu(t))^{\frac{3}{2}}} + \delta \lambda^{\frac{3}{2}}(t)/\mu(t)^{\frac{3}{2}}
\] (3.27)

In addition the derivative \( b'(t) \) satisfies

\[
|b'(t)| \lesssim \frac{\lambda(t)^{k-1}}{\mu(t)^k},
\] (3.28)

\[
b'(t) \geq 8 k^2 \frac{\lambda^{k-1}(t)}{\mu^k(t)} - \delta \lambda^{k-1}(t)/\mu^k(t)
\] (3.29)

where all unspecified constants depend only on \( k \).

We’ll deduce the following consequence of Proposition 3.6 and Proposition 3.6.

**Proposition 3.7.** Let \( C > 0 \). There exist \( \epsilon, \epsilon_0 > 0 \), depending only on \( C \) and \( k \), such that the following holds. Let \( \vec{\psi}(t) : [T_0, T_+] \to \mathcal{H}_0 \) be a solution of (1.5). Assume that \( t_0 \in [T_0, T_+] \) is such that \( d(\vec{\psi}(t_0)) \leq \epsilon \) and \( \frac{d}{dt}(\lambda(t)/\mu(t)) |_{t=t_0} \geq 0 \).
Then there exist $t_1$ and $t_2$, $T_0 \leq t_0 \leq t_1 \leq t_2 < T_+$, such that
\begin{equation}
\mathbf{d}(\bar{\psi}(t)) \geq 2\epsilon, \quad \text{for } t \in [t_1, t_2],
\end{equation}
\begin{equation}
\mathbf{d}(\bar{\psi}(t)) \leq \frac{1}{4} \epsilon_0, \quad \text{for } t \in [t_0, t_1],
\end{equation}
\begin{equation}
\mathbf{d}(\bar{\psi}(t_2)) \geq 2\epsilon_0,
\end{equation}
\begin{equation}
\int_{t_1}^{t_2} \|\partial_\nu \psi(t)\|_{L^2}^2 dt \geq C \int_{t_0}^{t_1} \sqrt{\mathbf{d}(\bar{\psi}(t))} dt
\end{equation}

Analogous statements hold with times $t_2 \leq t_1 \leq t_0$ if $\frac{d}{dt}(\lambda(t)/\mu(t))|_{t=t_0} \leq 0$.

**Remark 3.8.** We will take $\epsilon < \eta_0$, so that $\mathbf{d}(\bar{\psi}(t_0)) \leq \epsilon$ implies that the modulation parameters $\lambda(t)$ and $\mu(t)$ are well-defined $C^1$ functions in a neighborhood of $t = t_0$.

We also note that it follows from the proof that $\epsilon_0$ can be taken as small as we wish.

**Remark 3.9.** We will actually deduce (3.34) from the following stronger statement. There exist $\epsilon_0, C_k$ depending only on $k$ such that for any $\epsilon > 0$ small enough, Proposition 3.7 holds and additionally
\begin{equation}
\int_{t_0}^{t_1} \sqrt{\mathbf{d}(\bar{\psi}(t))} dt \leq C_k \epsilon^\frac{3}{2},
\end{equation}
\begin{equation}
\int_{t_1}^{t_2} \|\partial_\nu \psi(t)\|_{L^2}^2 dt \geq \frac{1}{C_k}.
\end{equation}

### 3.3. Proofs of the Modulation Estimates

We first assume the conclusions of Propositions 3.5 and 3.6 and prove Proposition 3.7. We record here a few useful formulae:

\begin{align}
\Lambda Q := r \partial_r Q &= k \sin Q = \frac{2k r^k}{1 + r^{2k}} \\
\Lambda^2 Q &= \frac{k^2}{2} \sin 2Q = 2k^2 r^k \left( \frac{1 - r^{2k}}{(1 + r^{2k})^2} \right) \\
\Lambda^3 Q &= 2k^3 r^k \left( \frac{1 + r^{2k} - 5r^{4k} - r^{6k}}{(1 + r^{2k})^4} \right) \\
\Lambda_0 \Lambda Q &= (r \partial_r + 1) (r \partial_r Q) = 2 \Lambda Q + r^2 \partial_r^2 Q
\end{align}

**Proof of Proposition 3.7.** From (3.6) it follows that there exists $\epsilon_1 > 0$ such that if $\lambda(t)/\mu(t) \leq \epsilon_1$, the modulation estimates hold in a neighborhood of time $t$. If needed, we will assume that $\epsilon_1$ is sufficiently small, but depending only on $k$. Let $t_2$ be the first time $t_2 \geq t_0$ such that $\lambda(t_2)/\mu(t_2) = \epsilon_1$ (if there is no such time, we set $t_2 = T_+$). By the estimate (3.7) in Lemma 3.1 there exists $\epsilon_0$ sufficiently small such that $\lambda(t_2)/\mu(t_2) = \epsilon_1$ implies (3.32). The number $\epsilon > 0$ will be chosen later in the proof and should be thought of as being much smaller than $(\epsilon_1)^\frac{3}{2}$, whereas we can think of $\epsilon_0$ as comparable to $(\epsilon_1)^\frac{1}{2}$.

Without loss of generality we can assume that $\mu(t_0) = 1$. Let $t_3 \leq t_2$ be the last time such that $\mu(t) \in [\frac{3}{2}, 2]$ for all $t \in [t_0, t_3]$. If there is no such final time we set $t_3 = t_2$. (Later we will see that we can always take $t_3 = t_2$ as long as $\epsilon_1 > 0$ is small enough.)
For \( t \in [t_0, t_3] \), from (3.29) we obtain
\[
b'(t) \geq \frac{k^2}{2k-1} \lambda(t)^{k-1}.
\]
We also obtain from (3.26)
\[
\lambda'(t) \geq \frac{1}{\kappa} b(t) - \sqrt{\frac{k}{2k-1}} \lambda(t)^{\frac{k}{2}}.
\]
Let \( \kappa_1 := \sqrt{\frac{k}{2}} \) and consider \( \xi(t) := b(t) + \kappa_1 \lambda(t)^{\frac{k}{2}} \). Using the two inequalities above we obtain
\[
\xi'(t) \geq \frac{k^2}{2k-1} \lambda(t)^{k-1} + \kappa_1 \frac{k}{2} \lambda(t)^{\frac{k}{2}-1} \left( \frac{1}{\kappa} b(t) - \sqrt{\frac{k}{2k-1}} \lambda(t)^{\frac{k}{2}} \right)
\]
\[
= \frac{\kappa_1 k}{2\kappa} \lambda(t)^{\frac{k}{2}-1} b(t) + \left( \frac{k^2}{2k-1} - \frac{\kappa_1 k}{2\kappa} \sqrt{\frac{k}{2k-1}} \right) \lambda(t)^{k-1}
\]
\[
= k \sqrt{\frac{k}{\kappa_{2k+1}}} \lambda(t)^{\frac{k}{2}-1} b(t) + \frac{k^2}{2\kappa} \lambda(t)^{k-1}
\]
\[
= k \sqrt{\frac{k}{\kappa_{2k+1}}} \lambda(t)^{\frac{k}{2}-1} \xi(t).
\]
It is easy to compute that (3.27) yields \( |b(t)| \leq \kappa_1 2^{k+3} \lambda(t)^{\frac{k}{2}} \), so we have
\[
\xi(t) \leq \kappa_1 2^{k+4} \lambda(t)^{\frac{k}{2}}
\]
and (3.40) leads to
\[
\xi'(t) \geq \kappa_2 \xi(t)^{\frac{2k-2}{k}},
\]
for some constant \( \kappa_2 > 0 \) depending only on \( k \).

Let \( \xi_1(t) := b(t) + \frac{\kappa_1}{\kappa} \lambda(t)^{\frac{k}{2}} = \frac{1}{\kappa} b(t) + \frac{\kappa_1}{\kappa} \xi(t) \). Since \( b'(t) \geq 0 \), we have
\[
\xi_1'(t) \geq \frac{1}{2} \xi'(t) \geq \frac{k}{2} \sqrt{\frac{k}{\kappa_{2k+1}}} \lambda(t)^{\frac{k}{2}-1} \xi(t) \geq \frac{k}{2} \sqrt{\frac{k}{\kappa_{2k+1}}} \lambda(t)^{\frac{k}{2}-1} \xi_1(t).
\]
Since \( \mu(t_0) = 1 \), we have \( 0 \leq \frac{\kappa_1}{\kappa} \lambda(t)/\mu(t) \big|_{t=t_0} = \lambda'(t_0) - \lambda(t_0) \mu'(t_0) \), so (3.21) implies that \( \kappa \lambda'(t_0) \geq -\frac{\kappa_1}{\kappa} \lambda(t_0)^{\frac{k}{2}} \) as long as \( \epsilon \) is taken small enough. Now (3.26) gives \( b(t_0) \geq -\frac{\kappa_1}{\kappa} \lambda(t_0)^{\frac{k}{2}} \), so \( \xi_1(t_0) > 0 \) and (3.43) yields \( \xi_1(t) > 0 \) for all \( t \in [t_0, t_3] \). Thus
\[
\xi(t) \geq \frac{\kappa_1}{2} \lambda(t)^{\frac{k}{2}}, \quad \text{for } t \in [t_0, t_3].
\]
In particular, (3.42) implies that \( \xi(t) \) is strictly increasing on \( [t_0, t_3] \) and by (3.41) we see that \( \lambda(t) \) is far from 0 on \( [t_0, t_3] \).

Bound (3.41) implies that there exists a constant \( \kappa_3 \) depending only on \( k \) such that \( \xi(t) \geq \kappa_3 \sqrt{\epsilon} \) forces \( d(\vec{\psi}(t)) \geq 2 \epsilon \). Let \( t_1 \in [t_0, t_3] \) be the last time such that \( \xi(t_1) = \kappa_3 \sqrt{\epsilon} \) (set \( t_1 = t_3 \) if no such time exists). Then by (3.44) we have
\[
\lambda(t)^{\frac{k}{2}} \leq \frac{2\kappa_3}{\kappa_1} \sqrt{\epsilon} \quad \text{for } t \in [t_0, t_1],
\]
which yields (3.31) if \( \epsilon \) is small enough.

Case \( k = 2 \). In this case (3.42) reads
\[
\xi'(t) \geq \kappa_2 \xi(t).
\]
Integrating between \( t \) and \( t_3 \) we get \( \xi(t) \leq e^{\kappa_2(t-t_3)}\xi(t_3) \). Thus (3.41) and (3.41) yield
\[
\lambda(t) \leq \kappa_3 e^{\kappa_2(t-t_3)}\lambda(t_3) \leq 2\kappa_3 e^{\kappa_2(t-t_3)}\epsilon_1,
\]
with a constant \( \kappa_3 \) depending only on \( k \). Thus integrating (3.41) and using \( \mu(t) = 1 \) we get \( \mu(t_3) \in [2/3, 3/2] \) if \( \epsilon_1 \) is small enough, which implies that \( t_3 = t_2 \). Also, suppose that there is no \( t_2 \) \( \geq t_0 \) such that \( \lambda(t_2)/\mu(t_2) = \epsilon_1 \). Then, since \( \lambda(t) \) is far from 0, by known arguments, see for instance [22, Corollary A.4], the solution is global and (3.42) implies that \( \xi(t) \) is unbounded. Thus \( \lambda(t) \) is also unbounded, which is a contradiction. We infer that there must be \( t_2 < T_+ \) such that \( \lambda(t_2)/\mu(t_2) = \epsilon_1 \), which implies (3.32) by choosing \( \epsilon_0 \) comparable to \( (\epsilon_1)^{1/k} \).

We have \( |\lambda(t)| \lesssim |\lambda(t)| \), see (3.20), hence there exists a constant \( \kappa_4 \) depending only on \( k = 2 \) such that \( \lambda(t) \geq \frac{1}{k} \epsilon_1 \) for \( t \in [t_2 - \kappa_4, t_2] \). Thus (3.39) yields
\[
b(t) - b(t_0) \geq \kappa_5(t - (t_2 - \kappa_4))\epsilon_1, \quad \text{for} \ t \in [t_2 - \kappa_4, t_2],
\]
with \( \kappa_5 \) depending only on \( k = 2 \). Thus, if \( \epsilon \) is small enough, we get \( b(t) \geq \kappa_6\epsilon_1 \) for \( t \in [t_2 - \frac{1}{2}k_4, t_2] \). Note that \( \kappa_6 \) is independent of \( \epsilon_1 \). From the definition of \( b(t) \) and the Cauchy-Schwarz inequality we can deduce, if \( \epsilon_1 \) is small enough, that \( \|\dot{g}(t)\|_{L^2} \geq \kappa_7\epsilon_1 \) for \( t \in [t_2 - \frac{1}{2}k_4, t_2] \), which leads to
\[
\int_{t_2 - \frac{1}{2}k_4}^{t_2} \|\dot{g}(t)\|_{L^2}^2 \, dt \geq \kappa_8\epsilon_1^2. \tag{3.46}
\]
Integrating (3.45) between \( t \) and \( t_1 \) and using (3.41), (3.41) and the definition of \( t_1 \) we obtain
\[
\lambda(t) \leq \kappa_9 e^{\kappa_2(t-t_1)}\sqrt{\epsilon}, \quad \text{for} \ t \in [t_0, t_1],
\]
with a constant \( \kappa_9 > 0 \) depending only on \( k \). Thus
\[
\int_{t_0}^{t_1} \sqrt{d(\psi(t))} \, dt \leq \kappa_{10}\sqrt{\epsilon}.
\]
Comparing this bound with (3.46) and choosing \( \epsilon \) small enough, we get (3.33).

**Case** \( k \geq 3 \). Most of the argument can be repeated without major changes. We can rewrite (3.42) as
\[
(\xi(t)^{\frac{k}{2} - 1})' \leq -\frac{(k - 2)\kappa_2}{k}.
\]
Integrating and using (3.41), (3.41) we obtain
\[
\lambda(t)^{\frac{k}{2}} \leq \kappa_3(\lambda(t_3)^{\frac{k}{2}} + (t_3 - t))^{\frac{1}{2k}}. \tag{3.47}
\]
Thus
\[
\int_{t_0}^{t_3} \lambda(t)^{\frac{k}{2}} \, dt \leq \kappa_3 \int_{\lambda(t_3)}^{+\infty} \tau^{\frac{1}{2k}} \lambda(t)^{\frac{k-1}{2}} \, d\tau \leq \frac{k - 2}{2\kappa_3}\lambda(t_3) \leq (k - 2)\kappa_3\epsilon_1.
\]
As in the case \( k = 2 \), we can deduce that \( t_3 = t_2 \) and that \( \lambda(t_2)/\mu(t_2) = \epsilon_1 \), which implies (3.32).

The proof of (3.46) applies without significant changes and yields
\[
\int_{t_2 - \frac{1}{2}k_4}^{t_2} \|\dot{g}(t)\|_{L^2}^2 \, dt \geq \kappa_8\epsilon_1^{2k-2}. \tag{3.48}
\]
The proof of (3.47) yields
\[ \lambda(t)(t) \leq \kappa_9 \left( \epsilon \frac{\mu(t)}{\mu(t_0)} + (t_1 - t) \right) \frac{1}{\sqrt{\mu(t)}} \quad \text{for } t \in [t_0, t_1]. \]

After integrating, this implies
\[ \int_{t_0}^{t_1} \sqrt{d(\psi(t))} \, dt \leq \kappa_{10} \epsilon \frac{1}{\mu(t_0)}. \]

Comparing this bound with (3.48) and choosing \( \epsilon \) small enough, we get (3.33). □

Proof of Proposition 3.5. Let \( t_0 \in J \) be any point in \( J \). By rescaling \( \psi(t) \mapsto \psi(t)(\mu(t_0)^{-1}) \) we can assume without loss of generality that \( \mu(t_0) = 1 \). We can also assume that
\[ \frac{1}{2} \leq \mu(t) \leq 2 \]
for all \( t \in J \) (we work in a small neighborhood of \( t_0 \)).

We begin by differentiating the orthogonality conditions (3.3) and (3.4) to derive a linear system for \((\lambda', \mu')\). Differentiating (3.3) gives
\[ 0 = \frac{d}{dt} \left( \chi_\mu \Lambda Q_\lambda | g \right) = \left( \partial_t (\chi_\mu) \Lambda Q_\lambda | g \right) + \left( \chi_\mu \partial_t (\Lambda Q_\lambda) | g \right) + \left( \chi_\mu \Lambda Q_\lambda | \partial_t g \right) \]
\[ = -\frac{\mu'}{\mu} \left( \Lambda \chi_\mu \Lambda Q_\lambda | g \right) - \frac{\lambda'}{\lambda} \left( \chi_\mu [\Lambda_0 \Lambda Q_\mu] | g \right) \]
\[ + \left( \chi_\mu \Lambda Q_\lambda | \dot{g} \right) + \lambda' \left( \chi_\mu \Lambda Q_\lambda | \Lambda Q_\mu \right) - \mu' \left( \chi_\mu \Lambda Q_\lambda | \Lambda Q_\mu \right) \]

The above yields
\[ - \left( \chi_\mu \Lambda Q_\lambda | \dot{g} \right) = \lambda' \left( \left( \chi_\mu \Lambda Q_\lambda | \Lambda Q_\lambda \right) - \left( \frac{1}{\lambda} \chi_\mu [\Lambda_0 \Lambda Q_\mu] | g \right) \right) \]
\[ - \mu' \left( \left( \chi_\mu \Lambda Q_\lambda | \Lambda Q_\mu \right) + \left( \Lambda \chi_\mu \Lambda Q_\lambda | g \right) \right) \]

Differentiating (3.4) yields
\[ 0 = \frac{d}{dt} \left( Z_\mu | g \right) = -\mu' \left( \frac{1}{\mu} [\Lambda_0 Z_\mu] | g \right) + \left( Z_\mu | \partial_t g \right) \]

Plugging in (3.22) above and rearranging we have
\[ - \left( Z_\mu | \dot{g} \right) = \lambda' \left( Z_\mu | \Lambda Q_\lambda \right) + \mu' \left( - \left( Z_\mu | \Lambda Q_\mu \right) - \left( \frac{1}{\mu} [\Lambda_0 Z_\mu] | g \right) \right) \]

We then arrive at the following linear system for \((\lambda', \mu')\),
\[ \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \lambda' \\ \mu' \end{pmatrix} = \begin{pmatrix} - \left( \chi_\mu \Lambda Q_\lambda | \dot{g} \right) \\ - \left( Z_\mu | \dot{g} \right) \end{pmatrix} \]
where

\[ M_{11} := \langle \chi_\mu \Lambda Q_\Delta | \Lambda Q_\Delta \rangle - \left\langle \frac{1}{\lambda} \chi_\mu [\Lambda_0 \Lambda Q_\Delta] | g \right\rangle \]
\[ M_{12} := - \left\langle \chi_\mu \Lambda Q_\Delta | \Lambda Q_\mu \right\rangle - \left\langle \Lambda \chi_\mu \Lambda Q_\Delta | g \right\rangle \]  
\[ M_{21} := \left\langle Z_\mu | \Lambda Q_\lambda \right\rangle \]
\[ M_{22} := - \left\langle Z_\mu | \Lambda Q_\mu \right\rangle - \left\langle \frac{1}{\mu} \Lambda_0 Z_\mu | g \right\rangle \]  

(3.49)

Our first goal is to show that \( M = (M_{ij}) \) is diagonally dominant with coefficients of size \( \simeq 1 \) on the diagonal. This will allow us to invert \( M \) and estimate \( \lambda', \mu' \). We estimate each of the terms in the matrix \( M \) beginning with \( M_{11} \).

**Claim 3.10.** \( M_{11} \) satisfies the estimate

\[ |M_{11} - \kappa| \lesssim \begin{cases} 
\lambda \sqrt{\log \lambda} & \text{if } k = 2 \\
\lambda^{\frac{2}{k-1}} & \text{if } k \geq 3
\end{cases} \]

(3.50)

where \( \kappa = \kappa(k) = \frac{2\pi}{\sin(\pi/k)} \).

To see this note that

\[ M_{11} = \langle \Lambda Q_\Delta | \Lambda Q_\Delta \rangle - \left\langle (1 - \chi_\mu) \Lambda Q_\Delta | \Lambda Q_\Delta \right\rangle - \left\langle \frac{1}{\lambda} \chi_\mu [\Lambda_0 \Lambda Q_\Delta] | g \right\rangle \]  

(3.51)

For the first term we have

\[ \langle \Lambda Q_\Delta | \Lambda Q_\Delta \rangle = \kappa \]

Since \( \lambda = \lambda(t) \ll 1 \) on \( J \) the second term is estimated by

\[ |\left\langle (1 - \chi_\mu) \Lambda Q_\Delta | \Lambda Q_\Delta \right\rangle| \lesssim \int_{1/2}^{\infty} (\Lambda Q_\Delta)^2 r dr = \int_{\frac{1}{2}}^{\infty} (\Lambda Q)^2 r dr \lesssim \int_{\frac{1}{2}}^{\infty} r^{-2k+1} dr \lesssim \lambda^{2k-2} \]

(3.52)

Finally we estimate the last term on the right-hand-side of (3.51). First observe that by (3.33) and (3.38)

\[ \left\langle \frac{1}{\lambda} \chi_\mu [\Lambda_0 \Lambda Q_\Delta] | g \right\rangle = \left\langle \frac{1}{\lambda} \chi_\mu (2\Lambda Q_\Delta + [r^2 \partial_r Q]_\Delta) | g \right\rangle = \left\langle \frac{1}{\lambda} \chi_\mu [r^2 \partial_r Q]_\Delta | g \right\rangle \]

We then can integrate by parts,

\[ \left\langle \frac{1}{\lambda} \chi_\mu [r^2 \partial_r Q]_\Delta | g \right\rangle = - \int_0^{\infty} \frac{1}{\lambda^3} Q_r(r/\lambda) \partial_r (r^3 \chi_\mu(r)) g(r) dr \]
\[ = -3 \int_0^{\infty} \frac{1}{\lambda^3} Q_r(r/\lambda)^2 \chi_\mu(r) g(r) dr - \int_0^{\infty} \frac{1}{\lambda^3} Q_r(r/\lambda) r^3 \chi_\mu'(r) g(r) dr \]
\[ - \int_0^{\infty} \frac{1}{\lambda^3} Q_r(r/\lambda) r^3 \chi_\mu(r) g_r(r) dr \]
We estimate each of the three terms on the right above as follows. First, using (3.6) we have

\[ \left| \int_0^\infty \frac{1}{\lambda^3} Q_r(r/\lambda) r^2 \chi_\mu(r) g(r) \, dr \right| \lesssim \left( \int_0^{2/\lambda} Q_r^2(r) \, dr \right)^{\frac{1}{2}} \| r^{-1} g \|_2 \]

\[ = \left( 1 + \int_1^{2/\lambda} r^{-2k+3} \, dr \right)^{\frac{1}{2}} \| r^{-1} g \|_2 \]

\[ \lesssim \begin{cases} \lambda \sqrt{\log \lambda} & \text{if } k = 2, \\ \lambda^{\frac{k}{2}} & \text{if } k \geq 3 \end{cases} \]

Arguing similarly we note that the other two terms satisfy the same estimate. Putting the above together we arrive at the estimate

\[ |M_{11} - \kappa| \lesssim \begin{cases} \lambda \sqrt{\log \lambda} & \text{if } k = 2, \\ \lambda^{\frac{k}{2}} & \text{if } k \geq 3 \end{cases} \]

which is precisely (3.50). Next we estimate \( M_{12} \) and \( M_{21} \).

**Claim 3.11.** We have

\[ |M_{12}| \lesssim \lambda^{\frac{k}{2}}, \quad |M_{21}| \lesssim \lambda^{k-1} \quad (3.53) \]

We prove that the first term of \( M_{12} \) is \( O(\lambda^{k-1}) \). Indeed, since we are assuming \( \frac{1}{2} \leq \mu \leq 2 \), we have

\[ |M_{12}| \lesssim \int_0^4 \frac{r}{\lambda^3} Q_r(r/\lambda) \frac{1}{\mu \mu} Q_r(r/\mu) \, r \, dr \]

\[ \lesssim \frac{1}{\lambda} \int_0^4 \frac{r}{1 + (r/\lambda)^{2k}} \frac{r^{2k+1}}{1 + r^{2k}} \, dr \]

\[ \lesssim \lambda^{k-1} \int_0^\lambda \frac{r^{2k+1}}{\lambda^{2k} + r^{2k}} \frac{1}{1 + r^{2k}} \, dr + \lambda^{k-1} \int_\lambda^4 \frac{r^{2k+1}}{\lambda^{2k} + r^{2k}} \frac{1}{1 + r^{2k}} \, dr \]

To estimate the first integral on the right above on the interval \([0, \lambda]\) we note that since \( \lambda \ll 1 \) we have

\[ \lambda^{k-1} \int_0^\lambda \frac{r^{2k+1}}{\lambda^{2k} + r^{2k}} \frac{1}{1 + r^{2k}} \, dr \lesssim \lambda^{k-1} \lambda^{-2k} \int_0^\lambda r^{2k+1} \, dr \lesssim \lambda^{k+1} \]

On the interval \([\lambda, 4]\) we write

\[ \frac{1}{\lambda^{2k} + r^{2k}} = \frac{1}{\lambda^{2k}} + \left( \frac{1}{\lambda^{2k} + r^{2k}} - \frac{1}{r^{2k}} \right) \]

\[ = \frac{1}{r^{2k}} + \frac{1}{\lambda^{2k}} \left( \frac{1}{1 + (\lambda/r)^{2k}} - 1 \right) \]

\[ = \frac{1}{r^{2k}} + \frac{1}{\lambda^{2k}} \left( -\lambda^{2k} r^{-2k} + O(\lambda^{4k} r^{-4k}) \right) \]
This yields,
\[
\lambda^{k-1} \int_\lambda^4 \frac{r^{2k+1}}{\lambda^{2k} + r^{2k}} \frac{1}{1 + r^{2k}} \, dr = \lambda^{k-1} \int_0^\infty \frac{r}{1 + r^{2k}} \, dr \\
- \lambda^{k-1} \int_0^\lambda \frac{r}{1 + r^{2k}} \, dr + O(\lambda^{k+1}) \\
= C\lambda^{k-1} + O(\lambda^{k+1})
\]
A similar argument yields the corresponding estimate for $|M_{21}|$.

Now we estimate the second term of $M_{12}$. By Cauchy-Schwarz, we have
\[
\left| \left\langle \Lambda \chi_\mu \Lambda Q_\Delta | g \right\rangle \right|^2 \leq \|r\Lambda Q_\Delta\|_{L^2(r \leq 4)}^2 \|\frac{\partial}{\partial r}\|_{L^2}^2 \lesssim \lambda^2 \|g\|_{H^2}^2 \int_0^{4/\lambda} (r + 1)^{-k+1} r \, dr,
\]
so the bound follows (with a margin) from the fact that $\|g\|_{H} \lesssim \lambda^2$ and that $k \geq 2$.

Lastly we estimate $M_{22}$.

**Claim 3.12.** Define $\beta := \langle Z | \Lambda Q \rangle$ and note that $\beta > 0$ is a fixed positive number. Then
\[
|M_{22} + \beta| \lesssim \lambda^\frac{k}{2}
\]
To see this, note that by the definition of $M_{22}$ and the facts that $\mu \simeq 1$ and $Z \in C_0^\infty$ we have
\[
|M_{22} + \beta| \leq \left| \left\langle \frac{1}{\mu} [\Lambda_0 Z]_\mu | g \right\rangle \right| \lesssim \|g\|_{H^2} \lesssim \lambda^\frac{k}{2}
\]
where the last inequality above follows from (3.6).

The last three claims imply that the matrix $M = (M_{ij})$ is diagonally dominant with coefficients on the diagonal of size $\simeq 1$. We solve for $(\lambda', \mu')$, by inverting $M$,
\[
\begin{pmatrix}
\lambda' \\
\mu'
\end{pmatrix} = \frac{1}{\det M} \begin{pmatrix}
-M_{22} \left\langle \chi_\mu \Lambda Q_\Delta | \dot{g} \right\rangle + M_{12} \left\langle Z_\mu | \dot{g} \right\rangle \\
M_{21} \left\langle \chi_\mu \Lambda Q_\Delta | \dot{g} \right\rangle - M_{11} \left\langle Z_\mu | \dot{g} \right\rangle
\end{pmatrix}
\]
Now note that by (3.6) we have
\[
\left| \left\langle \chi_\mu \Lambda Q_\Delta | \dot{g} \right\rangle \right| \lesssim \|\dot{g}\|_{L^2} \lesssim \lambda^\frac{k}{2}
\]
\[
\left| \left\langle Z_\mu | \dot{g} \right\rangle \right| \lesssim \|\dot{g}\|_{L^2} \lesssim \lambda^\frac{k}{2}
\]
(3.54)

Our estimates for the coefficients of $M$ imply that
\[
\det M = M_{11} M_{22} + O(\lambda^{\frac{k}{2}-1})
\]
\[
\frac{1}{\det M} = \frac{1}{M_{11} M_{22}} + O(\lambda^{\frac{k}{2}-1})
\]

Using the above we now write $\lambda'$ as follows,
\[
\lambda' = \left( \frac{1}{M_{11} M_{22}} + O(\lambda^{\frac{k}{2}-1}) \right) \left( -M_{22} \left\langle \chi_\mu \Lambda Q_\Delta | \dot{g} \right\rangle + M_{12} \left\langle Z_\mu | \dot{g} \right\rangle \right)
\]
and thus using (3.53) and (3.54) we obtain the estimates
\[
\left| \lambda' - \frac{1}{M_{11}} (\langle \chi_\mu \Lambda Q_\Delta | \dot{g} \rangle) \right| \lesssim |M_{12}| \left| \left\langle Z_\mu | \dot{g} \right\rangle \right| + O(\lambda^{2k-1}) \lesssim \lambda^k
\]
Then, recalling (3.50), we deduce that
\[ |\kappa \lambda(t) + \langle \chi_{\mu} A Q_{\Delta} | \dot{g} \rangle| \lesssim \begin{cases} \lambda^2 \sqrt{\log \lambda} & \text{if } k = 2 \\
 \lambda^k & \text{if } k \geq 3 \end{cases} \] (3.55)
With (3.54) we see that then,
\[ |\lambda| \lesssim \lambda^k \]
Similarly, for \( \mu' \) we have
\[ \mu' = \left( \frac{1}{M_{11} M_{22}} + O(\lambda^{\frac{k}{2}})^{-1} \right) \left( M_{21} \langle \chi_{\mu} A Q_{\Delta} | \dot{g} \rangle - M_{11} \langle Z_{\mu} | \dot{g} \rangle \right) \]
and hence,
\[ |\mu'| \lesssim \lambda^k \]
which proves (3.21) and completes the proof of Proposition 3.6. \( \square \)

**Remark 3.13.** We remark here that \( \lambda(t), \mu(t) \) obtained in the proof of Lemma 3.1 can be easily seen to be \( C^1 \) functions. Recall the ODE
\[ \left( \begin{array}{cc} M_{11} & M_{12} \\
 M_{21} & M_{22} \end{array} \right) \left( \begin{array}{c} \lambda' \\
 \mu' \end{array} \right) = \left( -\langle \chi_{\mu} A Q_{\Delta} | \psi_t(t) \rangle \\
 -\langle Z_{\mu} | \psi_t(t) \rangle \right), \]
(3.56)
obtained by formally differentiating the orthogonality conditions 3.3 and 3.4; the coefficients \( M_{ij} \) are given explicitly in 3.39. For any \( t_0 \in J \) the smallness of \( \lambda(t_0)/\mu(t_0) \) guarantees the existence of a unique \( C^1 \) solution \((\hat{\lambda}(t), \hat{\mu}(t))\) with initial data \((\lambda(t_0), \mu(t_0))\) in a neighborhood of \( t_0 \). Because of how the system 3.50 was derived, \( \hat{\lambda}(t), \hat{\mu}(t) \) and \( g(t) := \psi(t) - (Q_{\hat{\lambda}(t)} + Q_{\hat{\mu}(t)}) \) satisfy 3.3 3.4 and 3.5 in a small enough neighborhood of \( t_0 \). Since the \( \lambda(t), \mu(t) \) obtained by the implicit value theorem are unique with these properties we have \( \lambda(t) = \hat{\lambda}(t) \) and \( \mu(t) = \hat{\mu}(t) \) proving that \( \lambda(t), \mu(t) \) are indeed \( C^1 \).

**Proof of Proposition 3.6.** We first prove 3.20. As in the proof of Proposition 3.5 we can assume that \( \frac{1}{2} \leq \mu(t) \leq 2 \) below.

Recall that
\[ b(t) := -\langle A Q_{\Delta} | \dot{g} \rangle - \langle \dot{g} | A_0(\lambda) g \rangle \]
By (3.6) and the fact that \( A_0 : H \rightarrow L^2 \) is bounded independently of \( \lambda \) – see Lemma 3.3 – we have
\[ \langle \dot{g} | A_0(\lambda) g \rangle \lesssim \| g \|_{L^2} \| A_0(\lambda) g \|_{L^2} \lesssim \| (g, \dot{g}) \|_{H_0}^2 \lesssim \lambda^k \] (3.57)
Together with (3.55) this means that
\[ |\kappa \lambda - b| \lesssim |\kappa \lambda' + \langle \chi_{\mu} A Q_{\Delta} | \dot{g} \rangle| + \| (1 - \chi_{\mu}) A Q_{\Delta} | \dot{g} \rangle \| + \| \dot{g} | A_0(\lambda) g \| \]
\[ \lesssim \begin{cases} \lambda^2 |\log \lambda| & \text{if } k = 2 \\
 \lambda^k & \text{if } k \geq 3 \end{cases} \]
where we have used 3.55, 3.52, 3.6, and 3.57 in the last line above. This completes the proof of (3.20).

Arguing as above we have
\[ |b(t)| \leq \| A Q_{\Delta} \|_{LL^2} \| \psi_t \|_{L^2} - O(\lambda^k) = \sqrt{\kappa} \| \psi_t \|_{L^2} - O(\lambda^k) \]
From the expansion of the nonlinear energy in the proof of (3.6) and our assumption that \( \mu(t) \approx 1 \) we see that

\[
\| \psi_t(t) \|^2 \leq 16k(\lambda/\mu)^k + o(\lambda^k)
\]

Plugging this in above yields (3.27).

Finally, we begin the delicate proof of (3.29); we note that (3.28) will also be a consequence of this analysis. Differentiating \( b(t) \) gives, and recalling the formulæ (3.22), (3.23) we have

\[
b'(t) = \frac{\lambda'}{\lambda} \left( \langle \Lambda_0Q_\Delta | \dot{g} \rangle - \langle \Lambda_0Q_\Delta | \partial_t \dot{g} \rangle - \langle \partial_t \dot{g} | \Lambda_0(\lambda)g \rangle \right)
\]

\[
= \frac{\lambda'}{\lambda} \left( \langle \Lambda_0Q_\Delta | \dot{g} \rangle - \langle \Lambda_0Q_\Delta | \partial_t \dot{g} \rangle - \langle \partial_t \dot{g} | \Lambda_0(\lambda)g \rangle \right)
\]

\[
- \left( \langle \partial_t \dot{g} | \Lambda_0(\lambda)g \rangle - \langle \Lambda_0Q_\Delta | \dot{g} \rangle \right)
\]

\[
- \left( \langle \partial_t \dot{g} | \Lambda_0(\lambda)g \rangle - \langle \Lambda_0Q_\Delta | \dot{g} \rangle \right)
\]

\[
+ \mu' \langle \dot{g} | \Lambda_0(\lambda)\Lambda_0Q_\mu \rangle
\]

Let us first identify terms above that we’ve already established to be \( \ll \lambda^{k-1} \) and discard them. First note that since \( (\lambda \partial_\lambda \Lambda_0(\lambda)) : H \to L^2 \) is bounded, and since we’ve already shown \( |\lambda| \lesssim \lambda^k \) we have

\[
\frac{\lambda'}{\lambda} \langle \dot{g} | (\lambda \partial_\lambda \Lambda_0(\lambda)) \rangle \lesssim \| \dot{g} \|_H^2 \lambda^k \lesssim \lambda^{k-1}
\]

Then we note that

\[
\langle \dot{g} | \Lambda_0(\lambda) \rangle \dot{g} = 0,
\]

which can be shown directly by integration by parts. Next, using the fact that \( \mu \approx 1 \), along with the boundedness of \( \Lambda_0(\lambda) : H \to L^2 \) we have

\[
\mu' \langle \dot{g} | \Lambda_0(\lambda)\Lambda_0Q_\mu \rangle = \frac{\mu'}{\mu} \langle \dot{g} | \Lambda_0(\lambda)\Lambda_0Q_\mu \rangle
\]

\[
\lesssim |\mu'| \| \dot{g} \|_H \| \Lambda_0Q_\mu \|_H \lesssim |\mu'| \| \dot{g} \|_{L^2} \lesssim \lambda^k \ll \lambda^{k-1}
\]

Next, the combination of the first and sixth terms on the right-hand-side of (3.58) can be estimated using (3.17),

\[
\left| \frac{\lambda'}{\lambda} \langle \Lambda_0Q_\Delta | \dot{g} \rangle - \lambda' \langle \dot{g} | \Lambda_0(\lambda)\Lambda_0Q_\Delta \rangle \right|
\]

\[
\lesssim \lambda^{k-1} \langle \Lambda_0Q_\Delta - \Lambda_0(\lambda)\Lambda_0Q_\Delta | \dot{g} \rangle \| \dot{g} \|_{L^2}
\]

\[
\lesssim c_0 \lambda^{k-1} \lambda^k \ll \lambda^{k-1}
\]

where in the last line above we rely on our ability to take \( c_0 \) as small as we like in the estimate (3.17) from Lemma 3.3.
Thus we’ve show that up to terms of order $\ll \lambda^{k-1}$, which can be absorbed into the error, we have
\begin{equation}
 b'(t) \simeq -\left\langle \Lambda Q_{\Delta} | \partial^2_r g + \frac{1}{r} \partial_t g - \frac{1}{r^2} f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu) \right\rangle \\
- \left\langle \partial^2_r g + \frac{1}{r} \partial_t g - \frac{1}{r^2} f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu) | A_0(\lambda)g \right\rangle 
\end{equation}
(3.59)

Next, rescaling the equation $\mathcal{L} \Lambda Q = 0$, we see that
\[ \mathcal{L}_\lambda \Lambda Q_{\Delta} := (-\partial_{rr} - \frac{1}{r} \partial_r + \frac{f'(Q_\lambda)}{r^2}) \Lambda Q_{\Delta} = 0 \]
And since $\mathcal{L}_\lambda$ is symmetric we have
\[ \left\langle \Lambda Q_{\Delta} | \partial^2_r g + \frac{1}{r} \partial_t g \right\rangle = \left\langle \Lambda Q_{\Delta} | \frac{f'(Q_\lambda)}{r^2} g \right\rangle \]
We thus rewrite (3.59) as
\begin{align*}
 b'(t) & \simeq \left\langle \Lambda Q_{\Delta} | \frac{1}{r^2} \left( f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu) \right) \right\rangle \\
 & \quad - \left\langle \partial^2_r g + \frac{1}{r} \partial_t g - \frac{1}{r^2} \left( f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu) \right) | A_0(\lambda)g \right\rangle 
\end{align*}
where the symbol $\simeq$ above means “up to terms of order $\ll \lambda^{k-1}$”. Adding and subtracting we have
\begin{align*}
 b'(t) & \simeq \left\langle \Lambda Q_{\Delta} | \frac{1}{r^2} \left( f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu) \right) \right\rangle \\
 & \quad + \left\langle \Lambda Q_{\Delta} | \frac{1}{r^2} \left( f'(Q_\lambda - Q_\mu) - f'(Q_\lambda) \right) g \right\rangle \\
 & \quad + \left\langle \Lambda Q_{\Delta} | \frac{1}{r^2} \left( f(Q_\lambda - Q_\mu + g) - f(Q_\lambda - Q_\mu) - f'(Q_\lambda - Q_\mu) g \right) \right\rangle \\
 & \quad - \left\langle \partial^2_r g + \frac{1}{r} \partial_t g - \frac{1}{r^2} \left( f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu) \right) | A_0(\lambda)g \right\rangle 
\end{align*}
(3.60)
(3.61)
(3.62)

Let’s begin by estimating the first term on the right-hand-side above, which we’ll show contributes the leading order:

Claim 3.14.
\[ \left\langle \Lambda Q_{\Delta} | \frac{1}{r^2} \left( f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu) \right) \right\rangle \simeq 8k^2 \lambda^{k-1} \frac{\mu^k}{\mu^k} \]
where again $\simeq$ means “up to terms of order $\ll \lambda^{k-1}$.”

Proof. Let’s prove the claim. Recall that the nonlinearity $f(\rho)$ is given by
\[ f(\rho) := \frac{k^2}{2} \sin(2\rho), \quad f'(\rho) = k^2 \cos(2\rho) \]
Indeed, writing $\sigma$ we show that the leading order contribution comes from the second term above.

Changing variables again we have

$$\sin 2\sigma k$$

Using trigonometric identities we can write

We show that the leading order contribution comes from the second term above. Indeed, writing $\sigma = \lambda/\mu$ and changing variables we have

$$\langle \Lambda Q_\lambda | \frac{1}{r^2}(\Lambda Q_\lambda)^2 \sin 2Q_\mu \rangle = \frac{1}{\lambda} \int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma(r))^3 \sin 2Q(r) \frac{dr}{r}$$

$$= \frac{1}{\lambda} \int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma(r))^3 \sin 2Q(r) \frac{dr}{r} + \frac{1}{\lambda} \int_0^{\infty} (\Lambda Q_\sigma(r))^3 \sin 2Q(r) \frac{dr}{r}$$

Since $\sigma = \lambda/\mu \lesssim 1$, on the interval $[0, \sqrt{\sigma}]$ we write

$$\sin 2Q = \frac{2}{k^2} 2k^2 r^k \frac{1 - r^{2k}}{(1 + r^{2k})^2} = 4r^k + O(r^{3k})$$

Changing variables again we have

$$\frac{1}{\lambda} \int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma(r))^3 4r^k \frac{dr}{r} = 4 \frac{\sigma^k}{\lambda} \int_0^{\sqrt{\sigma}} (\Lambda Q(r))^3 r^{k-1} dr$$

$$= 4 \frac{\sigma^k}{\lambda} \int_0^{\infty} (\Lambda Q(r))^3 r^{k-1} dr - 4 \frac{\sigma^k}{\lambda} \int_0^{\sqrt{\sigma}} (\Lambda Q(r))^3 r^{k-1} dr$$

$$= 8\lambda^k \frac{\lambda^{k-1}}{\mu^k} + O(\lambda^{-1}\sigma^{2k})$$

where we used (3.25) in the last line above. Moreover, using (3.65) we estimate

$$\frac{1}{\lambda} \int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma(r))^3 (\sin 2Q - 4r^k) \frac{dr}{r} \lesssim \frac{1}{\lambda} \int_0^{\sqrt{\sigma}} (\Lambda Q_\sigma(r))^3 r^{3k-1} dr$$

$$= \frac{\sigma^{3k}}{\lambda} \int_0^{\sqrt{\sigma}} (\Lambda Q(r))^3 r^{3k-1} dr \lesssim \frac{\sigma^{3k} |\log \sigma|}{\lambda}$$

Finally, we estimate the second term on the last line (3.64) by

$$\left| \frac{1}{\lambda} \int_0^{\infty} (\Lambda Q_\sigma(r))^3 \sin 2Q(r) \frac{dr}{r} \right| \lesssim \frac{1}{\lambda} \int_0^{\infty} (\Lambda Q_\sigma(r))^3 \frac{dr}{r} = \frac{1}{\lambda} \int_0^{\sqrt{\sigma}} (\Lambda Q(r))^3 \frac{dr}{r}$$

$$\lesssim \frac{1}{\lambda} \int_{\sqrt{\sigma}}^{\infty} r^{-3k-1} dr \lesssim \frac{\sigma^{2k}}{\lambda}$$

Next we estimate the contribution of the first term in (3.63). Recalling that

$$\sin 2Q_\lambda = \frac{\partial}{\partial r} \Lambda Q_\lambda = \frac{\partial}{\partial r} \partial_r (\Lambda Q_\lambda)$$

we integrate by parts to obtain

$$- \left\langle \Lambda Q_\lambda | \frac{1}{r^2} \sin 2Q_\lambda (\Lambda Q_\mu) \right\rangle = \frac{2}{k^2} \lambda \left\langle (\Lambda Q_\lambda)^2 | \frac{1}{r^2} \Lambda Q_\mu \Lambda^2 Q_\mu \right\rangle$$

$$= \frac{2}{k^2} \lambda \left\langle (\Lambda Q_\sigma)^2 | \frac{1}{r^2} \Lambda Q^2 \right\rangle$$
where \( \sigma = \lambda/\mu \) as before. We first estimate the last line above on the interval \([0, \sigma]\) and \([0, 1]\) using (3.35) (3.36) to obtain the bound
\[
|\Lambda Q(r)\Lambda^2 Q(r)| \lesssim r^{2k}
\]
which gives
\[
\frac{2}{k^2} \int_0^\sigma (\Lambda Q_\sigma)^2 \Lambda Q_\sigma^2 \frac{dr}{r} \lesssim \frac{1}{\lambda} \int_0^\sigma \Lambda Q_\sigma^2 r^{2k-1} dr = \frac{\sigma^{2k}}{\lambda} \int_0^1 \Lambda Q^2 r^{2k-1} dr \lesssim \frac{\sigma^{2k}}{\lambda}
\]
and
\[
\frac{2}{k^2} \int_\sigma^1 (\Lambda Q_\sigma)^2 \Lambda Q_\sigma^2 \frac{dr}{r} \lesssim \frac{\sigma^{2k}}{\lambda} \int_\sigma^1 r^{4k-1} dr \lesssim \frac{\sigma^{2k} |\log \sigma|}{\lambda}
\]
On the interval \([1, \infty]\) we use the formulas (3.35) (3.36) to estimate
\[
|\Lambda Q(r)\Lambda^2 Q(r)| \lesssim \frac{r^{4k}}{(1 + r^{2k/3})}
\]
which means
\[
\frac{2}{k^2} \int_\sigma^\infty (\Lambda Q_\sigma)^2 \Lambda Q_\sigma^2 \Lambda^2 Q_\sigma \frac{dr}{r} \lesssim \frac{\sigma^{2k}}{\lambda} \int_\sigma^\infty r^{6k-1} \left(\frac{\sigma^{2k} + r^{2k}}{(1 + r^{2k/3})^2}\right) dr \lesssim \frac{\sigma^{2k}}{\lambda} \int_0^\infty r^{2k-1} (1 + r^{2k/3})^3 dr \lesssim \frac{\sigma^{2k}}{\lambda}
\]
Putting this all together, we’ve shown that
\[
\langle \Lambda Q_\lambda | \left( f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu) \right) \rangle = 8k^2 \frac{\lambda^{k-1}}{\mu^k} + O\left(\frac{\lambda^{k-1}}{\mu^k} \left(\frac{\lambda}{\mu}\right)^{\frac{3}{2}}\right)
\]
which is precisely Claim 3.14. \(\square\)

Next, we claim the second term (3.60) in our expansion of \(b'(t)\) satisfies
\[
\langle \Lambda Q_\lambda | \frac{1}{r} (f'((Q_\lambda - Q_\mu) - f'(Q_\lambda)) \rangle = o(\lambda^{k-1})
\]
and can thus be absorbed into the error. First note that the we have
\[
f'(Q_\lambda - Q_\mu) - f'(Q_\lambda) = k^2 \sin 2Q_\lambda \sin 2Q_\mu - 2k^2 \cos 2Q_\lambda \sin^2 Q_\mu
\]
\[
= \frac{4}{k^2} \Lambda^2 Q_\lambda \Lambda^2 Q_\mu - (\Lambda Q_\mu)^2 \cos 2Q_\lambda
\]
For the contribution from the first term above we integrate by parts, change variables, and use the explicit formulæ (3.35) (3.36) (3.37) to estimate
\[
\left| \langle \Lambda Q_\lambda | \frac{4}{k^2} \Lambda^2 Q_\lambda \Lambda^2 Q_\mu \rangle \right| \lesssim \frac{1}{\lambda} \left| \int_0^\infty (\Lambda Q_\lambda)^2 \Lambda^3 Q_\mu g \frac{dr}{r} \right| + \left| \int_0^\infty (\Lambda Q_\lambda)^2 \Lambda^2 Q_\mu r \partial_r g \frac{dr}{r} \right|
\]
\[
\lesssim \frac{1}{\lambda} \|g\|_H \left[ \left( \int_0^\infty (\Lambda Q_\lambda)^4 (\Lambda^3 Q_\mu)^2 \frac{dr}{r} \right)^{\frac{1}{2}} + \left( \int_0^\infty (\Lambda Q_\lambda)^4 (\Lambda^2 Q_\mu)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right] = o(\sigma^k/\lambda)
\]
where \(\sigma = \lambda/\mu\) as before. For the second term we write
\[
\left| \langle \Lambda Q_\lambda | \frac{1}{r^2} (\Lambda Q_\mu)^2 \cos 2Q_\lambda g \rangle \right| \lesssim \|g\|_H \left( \int_0^\infty (\Lambda Q_\lambda)^2 (\Lambda Q_\mu)^4 \frac{dr}{r} \right)^{\frac{1}{2}} \lesssim \frac{1}{\lambda} \sigma^{\frac{3}{2}} \sigma^k |\log \sigma|^{\frac{1}{2}} = o(\sigma^k/\lambda)
which finishes the proof of (3.66).

Finally, we consider the last two terms (3.61, 3.62). We will reorganize these terms in anticipation of applications of Lemma 3.3. First we rewrite (3.61) as follows:

\begin{align*}
\left\langle \Lambda Q_\lambda \right| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda)g) \right\rangle & = - \left\langle A(\lambda)g \left| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - k^2 g) \right\rangle \right.
\left. + \left\langle A(\lambda)g \left| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - k^2 g) \right\rangle \right. \\
+ \left\langle A(\lambda)Q_\lambda \left| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda)g) \right\rangle \right. + \left\langle A(\lambda)Q_\lambda \left| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda)g) \right\rangle \right. \\
\end{align*}

The second two terms on the right-hand-side can be controlled by setting \( g_1 = Q_\lambda - Q_\mu \) and and \( g_2 = g \) in (3.65):

\begin{align*}
& \left| \left\langle A(\lambda)g \left| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - k^2 g) \right\rangle \right. \\
+ \left\langle A(\lambda)Q_\lambda \left| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda)g) \right\rangle \right. \right| \lesssim c_0 \lambda^{k-1}
\end{align*}

Using the pointwise bound

\begin{align*}
|f(Q_\lambda - Q_\mu + g) - f(Q_\lambda - Q_\mu) - f'(Q_\lambda - Q_\mu)g| = \frac{k^2}{2} |\sin(2Q_\lambda - 2Q_\mu)| |\cos 2g - 1| + \cos(2Q_\lambda - 2Q_\mu)| \sin 2g - 2g| \lesssim |g|^2
\end{align*}

the last line of the expansion of (3.61) can be controlled as follows

\begin{align*}
& \left| \left\langle \Lambda Q_\lambda - A(\lambda)Q_\lambda \left| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda)g) \right\rangle \right. \right| \\
& \lesssim \|\Lambda Q_\lambda - A(\lambda)Q_\lambda\|_{L^\infty} \|g\|_H^2 \leq C_0 \lambda^{k-1} \ll \lambda^{k-1}
\end{align*}

In the last line we used (3.18) and the fact that \( c_0 \) can be taken small independently of \( \lambda \) in Lemma 3.3.

Thus, up to terms of order \( \ll \lambda^{k-1} \) we have

\begin{align*}
\left\langle \Lambda Q_\lambda \right| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda)g) \right\rangle \\
\simeq - \left\langle A(\lambda)g \left| \frac{1}{r^2} (f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - k^2 g) \right\rangle \right. \right. (3.67)
\end{align*}
We now transform the last line (3.62) adding and subtracting terms as before. Using (3.10) we have

\[-\left\langle \partial_t^2 g + \frac{1}{r} \partial_r g - \frac{1}{r^2} (f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu)) \mid A_0(\lambda) g \right\rangle\]

\[= - \left\langle A_0(\lambda) g \mid \partial_t^2 g + \frac{1}{r} \partial_r g - \frac{k^2}{r^2} g \right\rangle\]

\[+ \left\langle A_0(\lambda) g \mid \frac{1}{r^2} \left( f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu) \right) \right\rangle\]

\[+ \left\langle A_0(\lambda) g \mid \frac{1}{r^2} \left( f(Q_\lambda - Q_\mu + g) - f(Q_\lambda - Q_\mu) - k^2 g \right) \right\rangle \tag{3.68}\]

where \( R \) is as in Lemma 3.3. Note that from (3.68) we have the pointwise inequality

\[|f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu)| \lesssim (\Lambda Q_\lambda)^2 (\Lambda Q_\mu) + \Lambda Q_\lambda (\Lambda Q_\mu)^2\]

Since \( \|A_0(\lambda) g\|_{L^2} \lesssim \|g\|_H \), and since \( A_0(\lambda) g \) is supported on a ball of radius \( R\lambda \), the term on the second to last line above can be estimated as follows,

\[\left| \left\langle A_0(\lambda) g \mid \frac{1}{r^2} \left( f(Q_\lambda - Q_\mu) - f(Q_\lambda) + f(Q_\mu) \right) \right\rangle \right|\]

\[\lesssim \|g\|_H \left[ \left( \int_0^{R_\sigma} r^{-2} (\Lambda Q_\sigma)^4 (\Lambda Q_\lambda)^2 \frac{dr}{r} \right)^{\frac{1}{2}} + \left( \int_0^{R_\sigma} r^{-2} (\Lambda Q)^4 (\Lambda Q_\mu)^2 \frac{dr}{r} \right)^{\frac{1}{2}} \right] \]

\[\lesssim \sigma^{\frac{1}{2}} \sigma^k \ll \lambda^{k-1}\]

where \( \sigma = \lambda/\mu \) as usual and \( \mu \simeq 1 \).

Therefore, up to terms of order \( \ll \lambda^k \), we can put together (3.67) and (3.68) can estimate the combination of (3.61) and (3.62) from below by

\[\left\langle \Lambda Q_\lambda \mid \frac{1}{r^2} \left( f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - f'(-Q_\mu + Q_\lambda) g \right) \right\rangle\]

\[-\left\langle \partial_t^2 g + \frac{1}{r} \partial_r g - \frac{1}{r^2} (f(Q_\lambda - Q_\mu + g) - f(Q_\lambda) + f(Q_\mu)) \mid A_0(\lambda) g \right\rangle\]

\[\geq \frac{1}{\lambda} \int_0^{R_\lambda} \left( (\partial_r g)^2 + \frac{k^2}{r^2} g^2 \right) rdr \]

\[+ \left\langle A_0(\lambda) - A(\lambda) \mid \frac{1}{r^2} \left( f(-Q_\mu + Q_\lambda + g) - f(-Q_\mu + Q_\lambda) - k^2 g \right) \right\rangle \tag{3.69}\]
Based on monotonicity formulas, which is also the scheme that we adopt here. Critical NLW. Such results are usually obtained by means of a convexity argument in their study of the dynamics near the ground state stationary solution for the energy configuration, can never approach a two-bubble again. Such a result is similar in nature to the no-return lemma proved by Krieger, Nakanishi and Schlag in their study of the dynamics near the ground state stationary solution for the energy critical NLW. Such results are usually obtained by means of a convexity argument based on monotonicity formulas, which is also the scheme that we adopt here.

An analogous statement holds if \( \psi(t) \) does not scatter in backwards time. □

4. Dynamics of Non-Scattering Threshold Solutions

4.1. Overall scheme. In this section we prove the Main Theorem. We deduce it from the following proposition, whose proof will be split into several lemmas.

**Proposition 4.1.** Let \( \psi(t) : (T_-, T_+) \to \mathcal{H}_0 \) be a solution to (1.5) with \( E(\psi) = 2E(\tilde{Q}) \) which does not scatter in forward time. Then

\[
\lim_{t \to T_+} d(\psi(t)) = 0. \tag{4.1}
\]

To begin with, note the following special case of Theorem 1.2.

**Proposition 4.2.** Let \( \tilde{\psi}(t) : (T_-, T_+) \to \mathcal{H}_0 \) be a solution to (1.5) with \( E(\tilde{\psi}) = 2E(\tilde{Q}) \) which does not scatter in forward time. Then

\[
\liminf_{t \to T_+} d(\tilde{\psi}(t)) = 0.
\]

An analogous statement holds if \( \tilde{\psi}(t) \) does not scatter in backwards time. □

Let us summarize the main idea of the proof of Proposition 4.1. We know from Proposition 4.2 that (4.1) holds for a sequence of times. Thus, in order to obtain (4.1), we should prove that \( \tilde{\psi}(t) \), after exiting a small neighborhood of a two-bubble configuration, can never approach a two-bubble again. Such a result is similar in nature to the no-return lemma proved by Krieger, Nakanishi and Schlag in their study of the dynamics near the ground state stationary solution for the energy critical NLW. Such results are usually obtained by means of a convexity argument based on monotonicity formulas, which is also the scheme that we adopt here.

Until the end of this section, \( \psi(t) \) always denotes a solution to (1.5). \( \psi(t) : (T_-, T_+) \to \mathcal{H}_0 \), such that \( E(\psi) = 2E(\tilde{Q}) \) and \( \psi(t) \) does not scatter in forward time. \( \tilde{\psi}(t) \), after exiting a small neighborhood of a two-bubble configuration, can never approach a two-bubble again. Such a result is similar in nature to the no-return lemma proved by Krieger, Nakanishi and Schlag in their study of the dynamics near the ground state stationary solution for the energy critical NLW. Such results are usually obtained by means of a convexity argument based on monotonicity formulas, which is also the scheme that we adopt here.
time. Let $T_- < \tau_1 \leq \tau_2 < T_+$. Integrating the virial identity from Lemma 2.15 for
\[ T \in [\tau_1, \tau_2] \] yields
\[
\int_{\tau_1}^{\tau_2} \| \partial_t \psi(t) \|_{L^2}^2 \, dt \leq \| \langle \partial_t \psi | \chi_R r \partial_r \psi \rangle (\tau_1) \| + \| \langle \partial_t \psi | \chi_R r \partial_r \psi \rangle (\tau_2) \|
+ \int_{\tau_1}^{\tau_2} \Omega_R(\bar{\psi}(t)) \, dt
\]
where $\Omega_R(\bar{\psi}(t))$ is defined in (2.26). Note that for any $R > 0$ we can use Lemma 2.16 to bound the first two terms on the right-hand-side above and obtain
\[
\int_{\tau_1}^{\tau_2} \| \partial_t \psi(t) \|_{L^2}^2 \, dt \leq C_0 \left( R \sqrt{d(\bar{\psi}(\tau_1))} + R \sqrt{d(\bar{\psi}(\tau_2))} \right) + \int_{\tau_1}^{\tau_2} \Omega_R(\bar{\psi}(t)) \, dt. \tag{4.2}
\]

Our goal is to show that with a good choice of $R$, $\tau_1$ and $\tau_2$ the right hand side can be absorbed into the left hand side. As mentioned in the Introduction, we use different arguments depending whether $d(\bar{\psi}(t))$ is small or not.

4.2. Splitting of the time axis. We would like to divide the time axis into good intervals where $d(\bar{\psi}(t))$ is large and bad intervals where it is small. We begin with a preliminary splitting, which will then need to be refined.

Claim 4.3. Suppose that (4.1) fails. Then for any $\epsilon_0 > 0$ sufficiently small there exist sequences $p_n, q_n$ such that
\[ T_- < p_0 < q_0 < p_1 < q_1 < \cdots < p_{n-1} < q_{n-1} < p_n < q_n < \cdots \]
such that the following holds for all $n \in \{0, 1, 2, 3, \ldots\}$:
\begin{align*}
\forall t \in [p_n, q_n] : d(\bar{\psi}(t)) &\leq \epsilon_0, \tag{4.3} \\
\forall t \in [q_n, p_{n+1}] : d(\bar{\psi}(t)) &\geq \frac{1}{2} \epsilon_0, \tag{4.4} \\
\lim_{n \to +\infty} p_n = \lim_{n \to +\infty} q_n &= T_+. \tag{4.5}
\end{align*}

Proof. Suppose that (4.1) fails and let $\epsilon_0$ be any number such that
\[
0 < \epsilon_0 < \min(\limsup_{t \to T_+} d(\bar{\psi}(t)), \eta_1) \tag{4.6}
\]
(recall that $d(\bar{\psi}(t_0)) < \eta_1$ guarantees that the modulation estimates hold for $t$ in some neighborhood of $t_0$). Let $T_0 \in (T_-, T_+)$ be such that $d(\bar{\psi}(T_0)) > \epsilon_0$. We set
\[
p_0 := \sup \left\{ t : d(\bar{\psi}(\tau)) \geq \frac{1}{2} \epsilon_0, \forall \tau \in [T_0, t] \right\}.
\]
Proposition 4.1 implies that $p_0 < T_+$ and $d(\bar{\psi}(p_0)) = \frac{1}{2} \epsilon_0$. Then we define inductively for $n \geq 1$:
\[
q_{n-1} := \sup \left\{ t : d(\bar{\psi}(\tau)) \leq \epsilon_0, \forall \tau \in [p_{n-1}, t] \right\},
\]
\[
p_n := \sup \left\{ t : d(\bar{\psi}(\tau)) \geq \frac{1}{2} \epsilon_0, \forall \tau \in [q_{n-1}, t] \right\}.
\]
By a simple inductive argument using (4.6) and Proposition 4.2 we can show that for \( n \in \{1, 2, \ldots\} \) there holds

\[
\begin{align*}
    p_{n-1} &< q_{n-1} < T_+,
    
    q_{n-1} &< p_n < T_+,
    
    d(\vec{\psi}(p_n)) &= \frac{1}{2} \epsilon_0, \\
    d(\vec{\psi}(q_n)) &= \epsilon_0.
\end{align*}
\]

Bounds (4.7) and (4.8) follow directly from the definitions of \( p_n \) and \( q_n \). Suppose that (4.5) does not hold. Then, by monotonicity, \( \lim_{n \to +\infty} p_n = \lim_{n \to +\infty} q_n = T_1 < T_+ \).

By the local well-posedness \( d(\vec{\psi}(t)) \) has a limit as \( t \to T_1 \), which is in contradiction with (4.7) and (4.8). \( \square \)

Claim 4.4. Let \( \epsilon > 0 \). There exist \( \lambda_0, \epsilon' > 0 \) having the following property. Assume that \( d(\vec{\psi}(t)) < \eta_1 \), with \( \eta_1 \) as in Proposition 3.6 and let \( \lambda(t), \mu(t) \) be the modulation parameters given by Lemma 3.1. Then

\[
\begin{align*}
    \frac{\lambda(t)}{\mu(t)} &\geq \lambda_0 \Rightarrow d(\vec{\psi}(t)) > \epsilon', \\
    \frac{\lambda(t)}{\mu(t)} &\leq \lambda_0 \Rightarrow d(\vec{\psi}(t)) < \epsilon.
\end{align*}
\]

Remark 4.5. Note that \( \epsilon' < \epsilon \).

Proof. Lemma 3.1 yields \( d(\vec{\psi}(t)) \leq (C^2 + 1)(\lambda(t)/\mu(t))^k \), so we get (4.10) with any \( \lambda_0 < (\epsilon/(C^2 + 1))^k \).

In order to prove (4.9), we notice that from Lemma 3.1 we get \( d(\vec{\psi}(t)) \geq \frac{1}{k}(\lambda(t)/\mu(t))^k \), hence it suffices to take \( \epsilon' < \frac{1}{k} \lambda_0^k \). \( \square \)

Lemma 4.6. Suppose that (4.1) fails. Then there exist \( \epsilon, \epsilon' > 0 \) and a splitting of the time axis

\[
T_- < a_1 < c_1 < b_1 < \cdots < a_m < c_m < b_m < a_{m+1} < \cdots
\]

such that the following holds for all \( m \in \{2, 3, 4, \ldots\} \):

\[
\begin{align*}
    &\forall t \in [b_m, a_{m+1}]: d(\vec{\psi}(t)) \geq \epsilon', \\
    &\exists t \in [b_m, a_{m+1}]: d(\vec{\psi}(t)) \geq 2\epsilon, \\
    &d(\vec{\psi}(a_m)) = d(\vec{\psi}(b_m)) = \epsilon, \\
    &C_0 \int_{b_m}^{a_{m+1}} \sqrt{d(\vec{\psi}(t))} \, dt \leq \frac{1}{10} \int_{b_m}^{a_{m+1}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt, \\
    &C_0 \int_{c_m}^{b_m} \sqrt{d(\vec{\psi}(t))} \, dt \leq \frac{1}{10} \int_{b_m}^{a_{m+1}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt
\end{align*}
\]

and

\[
\lim_{m \to +\infty} d(\vec{\psi}(c_m)) = 0.
\]
Thus the meaning of times $t$. We set $d$.

By (3.30) we have

$$
\inf_{t \in [p_m, q_m]} d(\tilde{\psi}(t)) \leq \epsilon'.
$$

(4.16)

Recall that the modulation parameters $\lambda(t)$, $\mu(t)$ are well defined on $[p_m, q_m]$. Let $c_m \in [p_m, q_m]$ be such that

$$
\lambda(c_m)/\mu(c_m) = \inf_{t \in [p_m, q_m]} \lambda(t)/\mu(t).
$$

Claim 4.3 and (4.16) imply that $\lambda(c_m)/\mu(c_m) < \lambda_0$, which implies again by Claim 4.3 that $d(\tilde{\psi}(c_m)) < \epsilon < 1/10 \epsilon_0$. Hence $c_m \in (p_m, q_m)$ and

$$
\frac{d}{dt}\bigg|_{t=c_m} \left(\frac{\lambda(t)}{\mu(t)}\right) = 0.
$$

We will use Proposition 3.7 with various $t_0$, in forward and backward direction. Thus the meaning of $t_0$, $t_1$ and $t_2$ will change depending on the context.

Using Proposition 3.7 with $t_0 = c_m$ in the backward time direction we obtain times $t_1 \leq c_m$ and $t_2 \leq t_1$. Note that (3.31) and (4.1) imply that $t_1 \in (p_m, c_m]$. We set

$$
a_m := \sup\{t \geq t_1 : d(\tilde{\psi}(t)) \geq \epsilon, \forall \tau \in [t_1, t]\}.
$$

By (3.30) we have $d(\tilde{\psi}(t_1)) > \epsilon$. Since $d(\tilde{\psi}(c_m)) < \epsilon$, we have $a_m \in (p_m, c_m)$ and $d(\tilde{\psi}(a_m)) = \epsilon$.

Denote $\sigma_m := t_2$. Then (3.30) yields $d(\tilde{\psi}(t)) \geq \epsilon$ for $t \in [\sigma_m, t_1]$. By definition of $a_m$ we also have $d(\tilde{\psi}(t)) \geq \epsilon$ for $t \in [t_1, a_m]$, hence

$$
d(\tilde{\psi}(t)) \geq \epsilon, \forall t \in [\sigma_m, a_m].
$$

(4.17)

Bound (3.32) together with (3.3) yields $\sigma_m < p_m$, so (4.17) implies that

$$
d(\tilde{\psi}(t)) \geq \epsilon, \forall t \in [p_m, a_m].
$$

(4.18)

Finally, (3.33) yields

$$
\int_{\sigma_m}^{a_m} \|\partial_t \tilde{\psi}(t)\|_{L^2}^2 dt \geq C \int_{a_m}^{c_m} \sqrt{d(\tilde{\psi}(t))} dt.
$$

(4.19)

Now using Proposition 3.7 with $t_0 = c_m$ in the forward time direction we obtain times $t_1 \geq c_m$ and $t_2 \geq t_1$. Note that (3.31) and (4.1) imply that $t_1 \in [c_m, q_m]$. We set

$$
b_m := \inf\{t \leq t_1 : d(\tilde{\psi}(t)) \geq \epsilon, \forall \tau \in [t, t_1]\}.
$$
Lemma 4.8. There exists a continuous function \( \nu : I \to (0, +\infty) \) such that the set 
\[ \mathcal{K} := \{ \tilde{\psi}(t)_{1/\nu(t)} \mid t \in I \} \subset \mathcal{H}_0 \]
is pre-compact in \( \mathcal{H}_0 \).

Proof. We will first prove that for any sequence \( \{ t_n \} \subset I \) there exists a subsequence (still denoted by \( t_n \)) and a sequence of scales \( \tau_n \), so that
\[ \tilde{\psi}(t_n)_{1/\tau_n} \to \tilde{\varphi} \in \mathcal{H}_0 \] (4.24)
for some \( \tilde{\varphi} \in \mathcal{H}_0 \).
We observe that by Lemma 2.13 and (4.23) we have the uniform bound
\[ \| \vec{\psi}(t) \|_{H_0} \leq C(t') \quad \forall t \in I, \]
which means in particular that
\[ \| \vec{\psi}(t_n) \|_{H_0} \leq C(t') < \infty. \]
Thus (4.24) follows from Lemma 2.9.

We are now ready to construct the function \( \nu(t) \). For each \( t \in I \) let \( \nu(t) \) be the unique number such that
\[
\int_0^\infty e^{-r} \left( (\partial_t \psi_{1/\nu(t)}(t,r))^2 + (\partial_r \psi_{1/\nu(t)}(t,r))^2 + k^2 \frac{(\psi_{1/\nu(t)}(t,r))^2}{r^2} \right) \, r \, dr
= \frac{1}{2} \| \vec{\psi}(t) \|_{H_0}
\]
(the function \( e^{-r} \) could be replaced by any continuous strictly decreasing function whose value is 1 for \( r = 0 \) and tending to 0 as \( r \to +\infty \)). We see that \( \nu(t) \) is a continuous function.

Suppose that \( \vec{\psi}(t_{1/\nu(t)}) \) is not pre-compact in \( H_0 \). Thus there exists a sequence \( \vec{\psi}(t_{n_{1/\nu(n_n)}}) \) which has no convergent subsequence. But we know (by assumption) that there exist a subsequence (still denoted \( t_n \)) and numbers \( \nu_n \) such that \( \vec{\psi}(t_{n_{1/\nu_n}}) \) converges in \( H_0 \) to some \( \vec{\varphi} = (\varphi_0, \varphi_1) \). This implies that
\[
\int_0^\infty e^{-r} \left( (\partial_t \psi_{1/\nu_n}(t_n,r))^2 + (\partial_r \psi_{1/\nu_n}(t_n,r))^2 + k^2 \frac{(\psi_{1/\nu_n}(t_n,r))^2}{r^2} \right) \, r \, dr
\]
converges to
\[
\int_0^\infty e^{-r} \left( \varphi_1(r)^2 + (\partial_r \varphi_0(r))^2 + k^2 \frac{\varphi_0(r)^2}{r^2} \right) \, r \, dr \in (0, \| \vec{\varphi} \|_{H_0}).
\]
Since \( \| \vec{\psi}(t_n) \|_{H_0} \) converges to \( \| \vec{\varphi} \|_{H_0} \), we deduce that \( \nu_n/\nu(t_n) \) is bounded. This implies that \( \nu_n/\nu(t_n) \) has a convergent subsequence, hence \( \vec{\psi}(t_{n_{1/\nu(t_n)}}) \) has a convergent subsequence, so we have a contradiction. This completes the proof. \( \square \)

For \( m \in \{1, 2, 3, \ldots \} \) we define
\[ \nu_m := |I_m| = a_m - b_{m-1}. \]

**Lemma 4.9.** There exists \( C_1 > 0 \) such that for all \( m \geq 1 \) and all \( t \in I_m \) there holds
\[ \frac{1}{C_1} \nu(t) \leq \nu_m \leq C_1 \nu(t). \]

**Remark 4.10.** The last lemma tells us that \( \nu(t) \) is comparable to \( \nu_m \) for \( t \in I_m \). In particular, the set
\[
K_1 := \bigcup_{m \geq 1} \{ \vec{\psi}(t_{1/\nu_m}) \mid t \in I_m \}
\]
is pre-compact in \( H_0 \).
Proof of Lemma 4.9. Suppose that there exists a sequence \( m_\ell \) and times \( t_\ell \in I_{m_\ell} \) such that
\[
\lim_{\ell \to +\infty} \frac{\nu_{m_\ell}}{\nu(t_\ell)} = 0. \tag{4.26}
\]

Let \( \tilde{\psi}(s) \) be the solution of (1.5) with initial data \( \psi(0) = \tilde{\psi}(0) = \tilde{\varphi}(0) \) (where \( s_0 > 0 \)). By the standard Cauchy theory, for sufficiently large \( \ell \) the solution \( \tilde{\psi}(s) \) is defined for \( s \in [-s_0, s_0] \) and \( \tilde{\psi}(s) \to \varphi(s) \) in \( \mathcal{H}_0 \), uniformly for \( s \in [-s_0, s_0] \).

Let \( t'_\ell \in I_{m_\ell} \) be any sequence. Let \( s_\ell = \frac{t'_\ell - t_\ell}{\nu(t_\ell)} \). Then (4.26) implies that
\[
\lim_{\ell \to +\infty} s_\ell = 0.
\]
Thus \( s_\ell \) can be treated similarly.

Now suppose that there exist a sequence \( m_\ell \) and times \( t_\ell \in I_{m_\ell} \) such that
\[
\lim_{\ell \to +\infty} \frac{\nu(t_\ell)}{\nu_{m_\ell}} = 0.
\]
Without loss of generality we can assume that
\[
\lim_{\ell \to +\infty} \frac{\nu(t_\ell)}{m_\ell - t_\ell} = 0
\]
(the case \( \nu(t_\ell)/(t_\ell - b_{m_\ell-1}) \to 0 \) can be treated similarly).

Again, let \( \tilde{\psi}(s) \) be the solution of (1.5) with initial data \( \psi(0) = \tilde{\psi}(0) = \tilde{\varphi}(0) \), and let \( \tilde{\varphi} \in \mathcal{H}_0 \). Let \( \varphi(s) : (-\infty, T_+\varphi(0)) \to \mathcal{H}_0 \) be the solution of (1.5) with initial data \( \varphi(0) = \tilde{\varphi} \). By Lemma 2.9 we know that \( \varphi(s) \) is non-scattering in both time directions and satisfies
\[
E(\varphi) = E(\bar{\psi}) = 2E(\bar{Q}).
\]
Thus Proposition 4.2 implies that there exists \( \sigma \in [0, T_+\varphi(0)) \) such that \( d(\varphi(\sigma)) \leq \frac{1}{2} \epsilon' \). By Cauchy theory, for \( \ell \) large enough \( \tilde{\psi}(s) \) is defined for \( s \in [0, \sigma] \) and \( \tilde{\psi}(\sigma) \to \phi(\sigma) \) in \( \mathcal{H}_0 \), in particular \( d(\tilde{\psi}(\sigma)) \to d(\phi(\sigma)) \leq \frac{1}{2} \epsilon' \).

Let \( t'_\ell := t_\ell + \nu(t_\ell) \sigma \). Then \( \tilde{\psi}(t'_\ell) = \tilde{\psi}(\sigma) \nu(t_\ell) \), so we have
\[
\lim_{\ell \to +\infty} d(\tilde{\psi}(t'_\ell)) = \lim_{\ell \to +\infty} d(\tilde{\psi}(\sigma)) \leq \frac{1}{2} \epsilon'.
\]
However, (4.20) implies that for \( \ell \) large enough there holds \( t_\ell \leq t'_\ell \leq a_{m_\ell} \), thus (4.11) yields \( d(\tilde{\psi}(t'_\ell)) \geq \epsilon' \). The contradiction finishes the proof. \( \square \)
Lemma 4.11. There exists $\delta_1 > 0$ such that for all $m$ there holds
\[
\int_{I_m} \|\partial_t \psi(t)\|^2 \, dt \geq \delta_1^2 \nu_m.
\]

Proof. Let $t_m := \frac{1}{2}(b_{m-1} + a_m)$ and $s_1 := \frac{1}{2c_{1}}$. Then Lemma 4.9 yields
\[
b_{m-1} \leq t_m - \nu_m s_1 \leq t_m + \nu_m s_1 \leq a_m.
\]
(4.27)

We consider the following sequence of solutions of (1.5):
\[
\tilde{\psi}_m(s) := \tilde{\psi}(t_m + \nu_m s)^{1/\nu_m} \quad \text{for } s \in [-s_1, s_1].
\]
Then (4.27) implies that
\[
\int_{I_m} \Omega_{\nu_m R_{1}}(\tilde{\psi}(t_m)_{1/\nu_m}) \, dt \leq \delta_1^2 10 C_0 R_0.
\]
(4.29)

Suppose that the conclusion fails. Then there exists a sequence $m_1, m_2, \ldots$ such that
\[
\lim_{t \to +\infty} \int_{-s_1}^{s_1} \|\partial_s \tilde{\psi}_{m_1}(s)\|^2 \, ds = 0.
\]
(4.28)

After extraction of a subsequence, $\tilde{\psi}_{m_1}(0) \to \tilde{\varphi}_0 \in H_0$. Let $\varphi(s)$ be the solution of (1.3) such that $\varphi(0) = \tilde{\varphi}_0$. Then by the standard Cauchy theory $\tilde{\psi}_{m_1}(s) \to \varphi(s)$ in $H_0$, uniformly for $s \in [-s_1, s_1]$. In particular, (4.28) yields
\[
\int_{-s_1}^{s_1} \|\partial_s \varphi(s)\|^2 \, ds = 0,
\]
so the limiting wave map $\varphi(s)$ must be time-independent. Hence $\varphi(s) \in H$ is a harmonic map. But then $\varphi(s) \equiv 0$ since the constant map is the unique harmonic map with topological degree 0. However, we also have $E(\varphi) = 2E(\tilde{Q}) > 0$, which gives a contradiction. \square

Lemma 4.12. There exists $R_0 > 0$ such that if $R_1 \geq R_0$, then for all $m \in \{2, 3, \ldots\}$ there holds
\[
\int_{I_m} \Omega_{\nu_m R_1}(\tilde{\psi}(t)) \, dt \leq \frac{\delta_1^2}{10} \nu_m.
\]

Proof. With a change of variables, it suffices to prove that for all $t \in I$ there holds
\[
\Omega_{R_1}(\tilde{\psi}(t)_{1/\nu_m}) \leq \frac{\delta_1^2}{10}.
\]
This is a standard consequence of the pre-compactness of the set $K_1$ defined in (4.26). \square

4.4. Conclusions.

Proof of Proposition 4.1. Choose $1 \leq m_1 < m_2$ such that
\[
\sqrt{d(\tilde{\psi}(c_{m_1}))} + \sqrt{d(\tilde{\psi}(c_{m_2}))} \leq \frac{\delta_1^4}{10 C_0 R_0}.
\]
(4.29)

This is possible thanks to (4.15). Let
\[
R := R_0 \max_{m_1 \leq m \leq m_2} \nu_m.
\]
Inequalities (4.13) and (2.28) yield
\[
\frac{1}{5} \int_{c_{m1}}^{c_{m2}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \geq \sum_{m=m_1+1}^{m_2} \frac{1}{5} \int_{b_{m-1}}^{a_m} \|\partial_t \psi(t)\|_{L^2}^2 \, dt
\]
\[
\geq \sum_{m=m_1+1}^{m_2} 2C_0 \int_{a_m}^{c_m} \sqrt{\mathbf{d}(\psi(t))} \, dt
\]
\[
\geq 2 \sum_{m=m_1+1}^{m_2} \int_{a_m}^{b_m} \Omega_R(\psi(t)) \, dt.
\]
Similarly, (4.14) and (2.28) yield
\[
\frac{1}{5} \int_{c_{m1}}^{c_{m2}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \geq \sum_{m=m_1+1}^{m_2-1} \frac{1}{5} \int_{b_{m-1}}^{a_{m+1}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt
\]
\[
\geq \sum_{m=m_1+1}^{m_2-1} 2C_0 \int_{c_m}^{a_m} \sqrt{\mathbf{d}(\psi(t))} \, dt
\]
\[
\geq 2 \sum_{m=m_1+1}^{m_2-1} \int_{c_m}^{b_m} \Omega_R(\psi(t)) \, dt.
\]
Next, from Lemma 4.11 we have
\[
\frac{1}{5} \int_{c_{m1}}^{c_{m2}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \geq \sum_{m=m_1+1}^{m_2} \frac{1}{5} \int_{b_{m-1}}^{a_m} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \geq \frac{\delta^2}{5} \sum_{m=m_1+1}^{m_2} \nu_m. \tag{4.30}
\]
By the definition of $R$, for each $m \in \{m_1 + 1, m_1 + 2, \ldots, m_2\}$ we have $R = R_1 \nu_m$ with $R_1 \geq R_0$. Thus Lemma 4.12 gives
\[
\frac{1}{5} \int_{c_{m1}}^{c_{m2}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \geq 2 \sum_{m=m_1+1}^{m_2} \int_{b_{m-1}}^{a_m} \Omega_R(\psi(t)) \, dt.
\]
Finally, (4.30) and (4.29) imply
\[
\frac{1}{5} \int_{c_{m1}}^{c_{m2}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \geq \frac{\delta^2}{5} \max_{m_1 < m \leq m_2} \nu_m
\]
\[
\geq 2C_0 R \left( \sqrt{\mathbf{d}(\psi(c_{m_1}))} + \sqrt{\mathbf{d}(\psi(c_{m_2}))} \right).
\]
Summing the four inequalities above and using (1.24) for $(\tau_1, \tau_2) = (c_{m_1}, c_{m_2})$ we get
\[
\frac{4}{5} \int_{c_{m1}}^{c_{m2}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt \geq 2 \int_{c_{m1}}^{c_{m2}} \|\partial_t \psi(t)\|_{L^2}^2 \, dt.
\]
This contradiction finishes the proof.

Proof of the Main Theorem 1.6

**Step 1.** Let $\epsilon > 0$ be such that Proposition 3.7 holds with some $\epsilon_0 \geq 10\epsilon$. Define
\[
T_1 := \sup \{t : \exists \epsilon' \geq t \text{ such that } \mathbf{d}(\psi(t')) \geq \epsilon'\}.
\]
By Proposition 4.1 we have $T_1 < T_+$, and we know that the modulation parameters $\lambda(t)$ and $\mu(t)$ are well-defined for $t \in [T_1, T_+]$. Assume without loss of generality that $\mu(T_1) = 1$. 
There exists a sequence $\tau_n \to T_+$ such that
\[
\left. \frac{d}{dt} \right|_{t=\tau_n} \left( \frac{\lambda(t)}{\mu(t)} \right) \leq 0.
\]

For any such $t_0 = \tau_n$ we are in the setting of Proposition 3.17 in the backward time direction, so we obtain times $t_1 \leq \tau_n$ and $t_2 \leq t_1$. By the definition of $T_1$ and \(3.20\) we have $t_1 \leq T_1$, so the proof of Proposition 3.17 yields in particular $\frac{1}{2} \leq \mu(t) \leq 2$ for $t \in [t_1, \tau_n]$, thus $\frac{1}{2} \leq \mu(t) \leq 2$ for $t \in [T_1, T_+)$.

Hence, Proposition 3.17 implies that $\int_{t_1}^{T_1} \sqrt{d(\bar{\psi}(t))} \, dt$ is bounded as $\tau_n \to T_+$. Thus \(3.21\) implies that $\int_{t_1}^{T_1} |\mu'(t)| \, dt < +\infty$, hence $\mu(t)$ converges to some $\mu_0 \in [\frac{1}{2}, 2]$. Eventually rescaling again, we can assume that $\mu_0 = 1$.

As in the proof of Proposition 3.17, we consider $\xi(t) := b(t) + \kappa_1 \lambda(t)^{\frac{2}{k}}$ and we find that it is strictly decreasing on $[T_1, \tau_n]$ and satisfies
\[
\xi'(t) \leq -\kappa_2 \xi(t)^{\frac{2k-2}{k}}.
\]

Hence $\xi(t)$ is strictly decreasing on $[T_1, T_+]$ and satisfies \(4.31\) for $t \in [T_1, T_+]$. From the modulation equations we also obtain
\[
\xi'(t) \geq -\kappa_3 \xi(t)^{\frac{2k-2}{k}}.
\]
for some $\kappa_3$ depending only on $k$. Indeed, \(3.20\) and \(3.28\) yield
\[
|\xi'(t)| \lesssim |\beta'(t)| + |\lambda'(t)| |\lambda(t)|^{\frac{k-2}{k}} \lesssim \lambda^{k-1} \lesssim \xi(t)^{\frac{2k-2}{k}},
\]
Since $\lim_{t \to T_1} \xi(t) = 0$ and $\frac{2k-2}{k} \geq 1$, \(4.32\) implies that $T_+ = +\infty$.

showing that the sign $\iota$ is constant is standard. Lemma 2.14 implies that $d_+(\bar{\psi}(t)) \leq \epsilon$ for $t \in [T_1, +\infty)$ or $d_-(\bar{\psi}(t)) \leq \epsilon$ for $t \in [T_1, +\infty)$. Indeed, suppose that $t_1, t_2 \geq T_1$, $t_1 \leq t_2$ are such that $d_+(\bar{\psi}(t_1)) \leq \epsilon$ and $d_-(\bar{\psi}(t_2)) \leq \epsilon$. Without loss of generality we can assume that $\epsilon < \frac{1}{2} \alpha_0$, where $\alpha_0$ is the constant from Lemma 2.14. Then Lemma 2.14 yields $d_+(\bar{\psi}(t_2)) \geq \alpha_0$, hence there exists $t_0 \in [t_1, t_2]$ such that $d_+(\bar{\psi}(t_0)) = \frac{1}{2} \alpha_0 > \epsilon$. But Lemma 2.14 gives that also $d_-(\bar{\psi}(t_0)) \geq \alpha_0 > \epsilon$, which contradicts the choice of $T_1$.

**Step 2.** We now deduce the rate of decay of $\lambda(t)$ as $t \to +\infty$. Bounds \(3.41\) and \(3.44\) imply that $\xi(t)$ is comparable to $\lambda(t)^{\frac{2}{k}}$. Rewrite \(4.31\) and \(4.32\) as follows:
\[
-\kappa_3 \leq \frac{\xi'(t)}{\xi(t)} \leq -\kappa_2, \quad \kappa_2, \kappa_3 > 0,
\]

In the case $k = 2$, after integrating and possibly changing the values of the constants, we obtain
\[
e^{-\kappa_3 t} \leq \xi(t) \leq e^{-\kappa_2 t},
\]
which implies that there exists a constant $C$ such that
\[
e^{-Ct} \leq \lambda(t) \leq e^{-\frac{1}{2}t} \quad \text{as } t \to +\infty.
\]
(recall that we rescale the solution so that $\mu_0 = \lim_{t \to +\infty} \mu(t) = 1$).

Similarly, for $k > 2$ we obtain
\[
\frac{1}{C} e^{-\frac{k-2}{k} t} \leq \lambda(t) \leq C t^{-\frac{k-2}{k}} \quad \text{as } t \to +\infty,
\]
with a constant depending on $k$. 

**Step 3.** Suppose that $\psi$ does not scatter in either time direction. Take any $\delta > 0$. Bounds (4.33) and (4.34) imply that

$$\int_{-\infty}^{+\infty} \sqrt{d(\psi(t))} \, dt < +\infty. \quad (4.35)$$

Indeed, it suffices to consider the behavior as $t \to \pm \infty$. In this situation we have well-defined modulation parameters $\lambda(t)$, $\mu(t)$ and (3.6) together with the fact that $\mu(t) \to \mu_0 > 0$ imply that $d(\psi(t)) \lesssim \lambda(t)^k$, so (4.35) follows from time integrability of $\lambda(t)^k$.

Thus (2.28) implies that there exist $T_1$, $T_2$ such that for all $R > 0$

$$\int_{-\infty}^{T_1} \Omega_R(\psi(t)) \, dt \leq \frac{1}{3} \delta,$$

$$\int_{T_2}^{+\infty} \Omega_R(\psi(t)) \, dt \leq \frac{1}{3} \delta.$$

Since $[T_1, T_2]$ is a finite time interval, there exists $R > 0$ such that

$$\int_{T_1}^{T_2} \Omega_R(\psi(t)) \, dt \leq \frac{1}{3} \delta.$$

But $d(\psi(t)) \to 0$ as $t \to \pm \infty$, hence (4.2) yields

$$\int_{-\infty}^{+\infty} \| \partial_t \psi(t) \|_{L^2}^2 \, dt \leq \delta.$$

This would imply that $\psi(t)$ is a constant in time solution (because $\delta$ was any strictly positive number), which is impossible. \hfill \Box

**Remark 4.13.** Suppose that $\psi(t)$ does not scatter in the forward time direction. From the modulation equations we know that

$$b'(t) \leq -\kappa_6 \lambda(t)^{k-1}, \quad \kappa_6 > 0,$$

so integration yields

$$b(t) \geq \kappa_7 \int_t^{\infty} \lambda(s)^{k-1} \, ds.$$

But, as noticed in Step 1. above, we also have $|\lambda(t)^{k/2}'| \lesssim \lambda(t)^{k-1}$, which yields $\lambda(t)^{k/2} \lesssim \int_t^{\infty} \lambda(s)^{k-1} \, ds$, so we obtain

$$b(t) \geq \kappa_8 \lambda(t)^{k/2} \Rightarrow b(t)^2 \geq \kappa_9 d(\psi(t)),$$

which in turn implies

$$\left| \left\langle \frac{1}{\lambda(t)} \Lambda Q \lambda(t), \partial_t \psi \right\rangle \right|^2 \geq c d(\psi(t)), \quad \text{as } t \to +\infty,$$

where $c > 0$ is a constant depending only on $k$. Thus the projection of the time derivative of the solution constitutes at least a fixed fraction of the total distance from a two-bubble. In fact, if we were more precise in our computations, we could probably obtain that this projection is the leading term of the error.
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