Numerical analysis of nonhomogeneous and nonprismatic members under generalised loadings

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Abstract. This research studied the problem of varying depth and varying elastic properties along flexural member lengths subjected to generalised loading, with such members considered to be nonhomogeneous and non-prismatic. The differential equation for classical thin beams was thus derived and the finite differences used in solving this equation. Fortran programs were written to solve the problem in finite differences, and three-dimensional elements were used to simulate or model the beams in finite element ABAQUS software. A parametric study of the influence of beam width, depth, load nature, and boundary conditions on deformations and internal forces was thus accomplished. The maximum variation in deformations seen between this study and previous studies was 8%. The deflection of the non-prismatic beam was reduced by 31% when the ratio of the beam moment of inertia at different sections increased by up to three times, while, the moment capacity and the right end support shear capacity were increased by 78% and 50%, respectively.

1. Introduction

Non-prismatic members are used in cases that require reduced weight and additional section efficiency. Non-prismatic beams are thus used in various building structures and bridges, with applications increasing depending on structural engineering technique improvement. Such members may include hybrid materials along their lengths, and in such cases the members are assumed to be nonhomogeneous.

Eisenbenger [1] derived the exact stiffness matrix for the non-prismatic beam in 1985 and solved various examples for beams with different loading and end conditions. In 1995 Al-Gahthi and Khan [2] derived the exact solution for non-prismatic flexural members with cross sections with linear and parabolic profiles. The obtained solution was derived in terms of the beam section variables and properties with different end conditions. Further research was then done to obtain the stiffness matrix for the non-prismatic beam element, based on direct integration of the governing differential equations by researchers such as Tena-Colunga [3] in 1996.

The issue of obtaining a closed form solution for members such as beams with variable sections along their span was investigated by Yavari et al. [4,5] and Yavari and Sarkani [6]. The Maculay theory has been used by Yavari et al. [5] in 2001 to simulate the singularities in the obtained solutions for both deep and thin beams, for example.

Biondi and Caddemi, [7] investigated the integration problem of the constant or uniform cross section Euler-Bernoulli beam governed by equations with discontinuities in 2005. The discontinuities in beams were simulated as bending stiffness singularities by using superimposition of compatible formulas onto a constant one-dimensional domain.
Girgin and Girgin [8] developed a procedure to derive the stiffness coefficient for static and dynamic stiffness with an equivalent nodal vector for analysis of non-uniform deep beam–columns with several factors in 2006, and in 2013, Al–Shareef [9] carried out a study on non-prismatic thin members using numerical methods under different loading cases and section distribution.

This paper is an attempt to analyse non-prismatic beams with different configurations of cross-section and different materials along their length by utilizing the method of finite differences and finite elements. The classical theory will thus be improved to take into account the non-homogeneity of material as well as the variable cross section. This problem is easily solved using finite elements: however, the finite difference solution for such problems, considering both non-prismatic and non-homogeneous members, is not yet available in the literature.

2. Finite difference method
The flexural member or beam can be discretised into steps (Δx) along its length (Figure 1). Here, (i) exemplifies the selected node and (n) the total number of nodes in the finite difference equation. In the finite difference model, the beam deflected curve is simulated by a straight line connecting the nodes.

\[
\frac{d^2}{dx^2} \left( E(x) I(x) \frac{dw}{dx} \right) + q(x) = 0
\]

(1)

The above equation is modified herein as follows:

\[
\frac{d^2}{dx^2} \left( E(x) I(x) \frac{dw}{dx} \right) + q(x) = 0
\]

(2)

where \( w=w(x) \) is the beam deflection, \( E(x) \) is the beam modulus of elasticity, \( I(x) \) is the beam moment of inertia, and \( q(x) \) is the beam distributed load.

Let

\[
\phi(x) = \frac{d^2w}{dx^2}
\]

(3)

Then

\[
\frac{d^2}{dx^2} \left( E(x) I(x) \phi(x) \right) + q(x) = 0
\]

(4)

and therefore

\[
\left( \frac{d^2}{dx^2} \right)^2 E(x) I(x) \phi(x) + 2 \left( \frac{dE}{dx} \frac{dI}{dx} \right) \phi(x) + \left( \frac{d^2E}{dx^2} \right) \phi(x) + q = 0
\]

(5)

The finite difference equations used here were thus written for an interior node \((i)\) based on differential equation (5) and computerised using Fortran.

3. Finite element method
Finite elements (FE) are used to simulate a flexural member with varying depth along its span. In this case the beam is assumed to be non-prismatic. A beam in other finite element simulations is
considered nonhomogeneous when the beam material has varying elastic modulus along the beam span. Three-dimensional brick elements are utilized in this study for modelling such beams in ABAQUS software (C3D8R), and each node has three degrees of freedom (the vertical deformation \( w \) and the horizontal displacements \( u \) and \( v \)). The beam geometry, load, mesh and boundary conditions are shown in Figure 2.

![Finite element simulation](image)

**Figure 2.** Finite element simulation

4. Verification of the numerical method

4.1 Fixed ends beam with varying section

In this study, the problem of fixed ends non-prismatic beam under uniform load which was solved by Biondi and Caddemi in 2005 [7] is considered. The finite difference and finite element methods are thus used to analyse the same application. All information for the relevant beam is shown in Figure 3.

Figures 4, 5, and 6 show the deflection profile, bending moment, and shear force diagram along the beam length for the closed form solution offered by Biondi and Caddemi [7] and the present research. The agreement is acceptable, with the maximum differences between the finite difference and closed form solution being 3%, 3% and 7% for deflection, moment, and shear, respectively, while the maximum differences between the finite element and closed form solutions were 9%, 8%, and 6% for deflection, moment and shear respectively.

![Fixed ends beam with varying depth](image)

**Figure 3.** Fixed ends beam with varying depth

![Deflection curves for fixed ends beam with varying depth](image)

**Figure 4.** Deflection curves for fixed ends beam with varying depth
4.2 Simply supported non-prismatic beam

In this study, the problem of a simply supported ends non-prismatic beam under uniform load was considered, and finite difference and finite element methods were used to analyse the same application. All information for the relevant beam is shown in Figure 7.

Figures 8, 9 and 10 show the deflection profile, bending moment, and shear force diagram along the half of beam length for the present research finite difference and finite elements solutions. The agreement is acceptable. The maximum difference between the finite difference and finite elements solutions are 5%, 6% and 8% for deflection, moment, and shear respectively.

Figure 5. Bending moment curves for fixed ends beam with varying depth

Figure 6. Shear force curves for fixed ends beam with varying depth

Figure 7. Simple ends beam with varying depth

Figure 8. Deflection curves for simple ends beam with varying depth
4.3 Simply supported non-homogeneous prismatic beam

In this study, the problem of a simply supported ends non-homogeneous prismatic beam under uniform load was also considered. The prismatic beam has modulus of elasticity $E_1$ on the left side of the beam ($s_1$) and $E_2$ in other parts ($s_2$). Finite difference and finite element methods were used to analyse the same application. All information for this beam is shown in Figure 11.

Figures 12, 13 and 14 show the deflection profile, bending moment, and shear force diagram along half of the beam length for the finite difference and finite elements solutions. The agreement is acceptable. The maximum differences between the finite difference and finite elements solutions were 8%, 9% and 7% for deflection, moment, and shear respectively.
4.4 Simply supported cracked prismatic beam

In this study, the problem of simply supported ends prismatic cracked reinforced concrete beam under uniform load was also considered. Such a beam becomes non-prismatic when the moment exceeds the cracking moment \( M_{cr} = 51.8 \text{ kN m} \), according to ACI 318 code equations for estimating cracking stress as shown below:

\[
M_{cr} = \frac{f_{r}I_g}{\gamma}
\]

If the flexural stress is less than the modulus of rupture, about \( f_r = 0.62 \sqrt{f'_c} \), the full uncracked section provides rigidity, and the moment of inertia for the gross section \( I_g \) is thus available. The cracking moment is estimated based on section dimension: here \( f'_c = 25 \text{ MPa}, f_y = 420 \text{ MPa}, E_c = 23500 \text{ MPa}, E_s = 200000 \text{ MPa}, \) and the steel ratio is 0.003. The section effective moment of inertia can thus be calculated using the ACI 318 equation shown below

\[
I_e = \left( \frac{M_{cr}}{M_a} \right)^3 I_g + \left[ 1 - \left( \frac{M_{cr}}{M_a} \right)^3 \right] I_{cr} \leq I_g
\]

where \( M_a \) is the applied moment, and \( I_{cr} \) is the cracked section moment of inertia.

The finite difference and finite element methods were used to analyse the same application. The beam was simulated and simplified as elastic material concrete in tension and compression without reinforcement in order to use the differential equation or classical theory. All information for the beam is shown in Figure 15.

Figures 16, 17 and 18 show the deflection profile, bending moment and shear force diagram along half of the beam length for the finite difference and finite elements solutions. The agreement was acceptable. The maximum differences between the finite difference and finite elements solutions were 4%, 7% and 4% for deflection, moment, and shear respectively. The cracking phenomenon changed the behaviour of the beam from prismatic to non-prismatic showing higher deflection and bending moments and shear force to reduction in the moment of inertia.
5. Parametric study

A member with properties as displayed in Figure 19 was used for parametric study using finite differences to show case the capability of this simple method in terms of simulating such problems, which is a main goal for this research. The following parameters were thus considered:

\[ q = 33 \text{ kN/m}, S_1 = 0.62 \text{ m}, S_2 = 4.76 \text{ m}, S_3 = 0.62 \text{ m}, b = 0.3 \text{ m}, h_1 = 0.6 \text{ m}, \\
\quad h_2 = 0.48 \text{ m}, E = 23.5 \times 10^6 \text{ kN/m}^2, \text{ beam length 6m} \]

**Figure (15):** Simple ends beam with varying depth

**Figure (16):** Deflection curves for simple ends beam with varying depth

**Figure 17.** Bending moment curves for simple ends beam with varying depth

**Figure 18.** Shear force curves for simple ends beam with varying depth

\[ q = 20 \text{ kN/m}, P = 80 \text{ kN}, S_1 = 2 \text{ m}, S_2 = 2 \text{ m}, b = 0.3 \text{ m}, h_1 = 0.3 \text{ m}, h_2 = \text{varies from 0.3 to 0.43m,} \\
\quad E_1 = 20 \times 10^6 \text{ kN/m}^2, E_2 = \text{varies from } 10 \times 10^6 \text{ to } 30 \times 10^6 \text{ kN/m}^2. \]
5.1 Beam moment of inertia ratio ($I_2/I_1$)

The height of the beam ($h_1=0.3$ m or $I_1=6.75 \times 10^{-4}$ m$^4$) was kept constant, while the depth ($h_2$) was increased. Figure 20 reveals that the deflection is reduced when the ratio of the beam moment of inertia at section 2 to section 1 ($I_2/I_1$) increases because the beam’s flexural or bending rigidity increases. If the ratio of $I_2/I_1$ is increased from 1 to 3, the mid span deflection is decreased by 31%. Figure 21 reveals that the moment capacity is increased when the ratio of moment of inertia at section 2 to section 1 ($I_2/I_1$) increases because the beam’s flexural or bending rigidity increases. If the ratio of $I_2/I_1$ is increased from 1 to 3 the mid span moment is increased by 78%. Figure 22 shows that the right end support shear capacity was increased by 50% with an increase in the ratio of moment of inertia at section 2 to section 1 ($I_2/I_1$). The stress resistance was increased with an increase in the moment of inertia of the beam section, and thus moment and shear resistance also increased.

![Figure 20](image1.png)
**Figure 20.** Effect of the ratio of moment of inertia ($I_2/I_1$) on deflection for beam under uniform load.

![Figure 21](image2.png)
**Figure 21.** Effect of the ratio of moment of inertia ($I_2/I_1$) on moment for beam under uniform load.

![Figure 22](image3.png)
**Figure 22.** Effect of the ratio of moment of inertia ($I_2/I_1$) on right end support shear for beam under uniform load.

5.2 Beam modulus of elasticity ratio ($E_2/E_1$)
The beam is assumed prismatic with \( h_1 = h_2 = 0.3 \text{ m} \) or \( I_1 = I_2 = 6.75 \times 10^{-4} \text{ m}^4 \) while the value of modulus of elasticity (\( E_2 \)) along \( s_2 \) is made variable from 0.5 to 1.5 of \( E_1 \). This represents the beam having two materials along its length. Figure 23 reveals that the deflection is reduced when the ratio of the beam modulus of elasticity at section 2 to section 1 (\( E_2/E_1 \)) increases as the beam’s flexural or bending rigidity increases. If the ratio of \( E_2/E_1 \) increases from 0.5 to 1.5, the mid span deflection is decreased by 42%. Figure 24 reveals that the moment capacity is increased when the ratio of modulus of elasticity at section 2 to section 1 (\( E_2/E_1 \)) increases because the beam’s flexural or bending rigidity increases. If the ratio of (\( E_2/E_1 \)) is increased from 0.5 to 1.5, the mid span moment is increased by 48%. Figure 25 shows that the right end support shear capacity is increased by 24% with an increase in the ratio of modulus of elasticity at section 2 to section 1 (\( E_2/E_1 \)).

5.3 Ends Boundary Conditions

The beam is assumed non-prismatic with \( b=0.3 \text{m}, h_1=0.3 \), \( h_2=0.378 \text{ m} \) or \( I_2/I_1=2 \), while the modulus of elasticity is assumed constant (\( E_1=E_2= 20 \times 10^6 \text{ kN/m}^2 \)). The beam is under a uniformly distributed load with both end boundary conditions are assumed to be fixed or simply supported. Figure 26 shows that
the mid span deflection decreases by 80% due to end condition alteration from simply supported to fixed. Figures 27 and 28 reveal that the moment is reduced by 42% and the right end support shear enlarged by 10% due to end condition alteration for varying depth beams.

Figure 26. Effect of boundary conditions on beam deflection

Figure 27. Effect of boundary conditions on beam moment

Figure 28. Effect of boundary conditions on beam right support shear

5.4 Loading Type

The beam is assumed non-prismatic with b=0.3m, h₁=0.3, h₂=0.378 m or I₂/I₁=2, while the modulus of elasticity is assumed constant (E₁=E₂= 20x10⁶ kN/m²). The beam has simply supported ends and the loading is assumed to change from distributed with 20 kN/m intensity to a point load at mid span of 80 kN. Figure 29 shows that the mid span deflection increases by 100% when the load changed from uniform to point load. Figures 30 and 31 reveal that the moment is increased by 35% and the right end support shear is enlarged by 34% due to the change in load type.
Figure 29. Effect of loading type on beam deflection

Figure 30. Effect of loading type on beam moment

Figure 31. Effect of loading type on beam right support shear

6. Conclusions
The main conclusions obtained from this study are set out below:
1. The results obtained from the closed form and numerical solutions revealed good agreement with maximum percentage differences of 7% and 9% for finite differences and finite elements respectively.
2. The problem of simply supported end non-homogeneous prismatic beams under uniform load was studied. It was found that such beams behave in similar ways to non-prismatic beams. The agreement between the finite differences and the finite elements forms used were acceptable. The maximum differences between the finite difference and finite elements figures were 8%, 9% and 7% for deflection, moment, and shear respectively. For non-prismatic members, the difference between the two numerical methods increased with an increase in the ratio of moment of inertia of both beams section ($I_2/I_1$).
3. The problem of simply supported end prismatic cracked reinforced concrete beams under uniform load was studied. Such beams become non-prismatic when the moment exceeds the cracking moment (51.8 kN m) as calculated using the ACI 318 equation. In this case, the effective moment of inertia is used for the cracked beam section along $s_2$ calculated again using the ACI 318 code equation. The agreement between the finite differences and the finite elements figures were acceptable. The
maximum differences between the finite difference and finite elements results were 4%, 7%, and 4% for deflection, moment and shear respectively.

4. The deflection of the non-prismatic beam was reduced by 31% when the ratio of the beam moment of inertia at section 2 to section 1 (I2/I1) increased from 1 to 3. The moment capacity and the right end support shear capacity were increased by 90% and 50% respectively when the ratio of moment of inertia at section 2 to section 1 (I2/I1) increased due to the beam bending rigidity increasing.

5. The deflection of the prismatic beam was reduced by 42% when the ratio of the beam modulus of elasticity at section 2 to section 1 (E2/E1) increased from 0.5 to 1.5. The moment capacity and the right end support shear capacity were increased by 48% and 24% respectively, when the ratio of the beam modulus of elasticity at section 2 to section 1 (E2/E1) increased due to the beam bending rigidity increasing.

6. The mid span deflection for the non-prismatic beam decreased by 80% due to end condition alteration from simply supported to fixed, while, the moment was reduced by 42% and the right end support shear enlarged by 10% due to end condition alteration for varying depth beams.

7. The mid span deflection increased by 100% when the load changed from uniform to point load. The moment was also increased by 35% and the right end support shear enlarged by 34% due to the change in load type.

7. References

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