The infinitesimal multiplicities and orientations of the blow-up set of the Seiberg–Witten equation with multiple spinors

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Abstract

I construct multiplicies and orientations of tangent cones to any blow-up set $Z$ for the Seiberg–Witten equation with multiple spinors. This is used to prove that $Z$ determines a homology class, which is shown to be equal to the Poincaré dual of the first Chern class of the determinant line bundle. I also obtain a lower bound for the 1-dimensional Hausdorff measure of $Z$.

1 Introduction

Let $M$ be a closed oriented Riemannian three-manifold. Denote by $\mathcal{S}$ the spinor bundle associated with a fixed $\text{Spin}^c$–structure on $M$. Also fix a $U(1)$–bundle $\mathcal{L}$ over $M$, a positive integer $n \in \mathbb{N}$ and an $\text{SU}(n)$–bundle $E$ together with a connection $B$. Consider triples $(A, \Psi, \tau) \in \mathcal{A}(\mathcal{L}) \times \Gamma(\text{Hom}(E, \mathcal{S} \otimes \mathcal{L})) \times (0, \infty)$ consisting of a connection $A$ on $\mathcal{L}$, an $n$–tuple of twisted spinors $\Psi$, and a positive number $\tau$ satisfying the Seiberg–Witten equation with $n$ spinors:

$$\|\Psi\|_{L^2} = 1,$$
$$\partial_A \otimes B \Psi = 0, \quad \text{and}$$
$$\tau^2 F_A = \mu(\Psi).$$

(1)

A reader can find further details about (1) as well as a motivation for studying these equations in [HW15, Hay17]. The main result of [HW15] states in particular, that if $(A_k, \Psi_k, \tau_k)$ is an arbitrary sequence of solutions such that $\tau_k \to 0$, then there is a closed nowhere dense subset $Z \subset M$ and a subsequence, which converges to a limit $(A, \Psi, 0)$ on compact subsets of $M \setminus Z$ possibly after applying gauge transformations. Moreover, $(A, \Psi, 0)$ is a solution of (1) on $M \setminus Z$ and the function $|\Psi|$ has a continuous extension to $M$ such that $Z = |\Psi|^{-1}(0)$.

It follows from the proof of the above mentioned result that if a sequence $(A_k, \Psi_k, \tau_k)$ of solutions of (1) converges to $(A, \Psi, 0)$, then

$$\left\{ m \in M \mid \exists r_k \to 0 \text{ s.t. } r_k \int_{B_{r_k}(m)} |F_{A_k}|^2 \to \infty \right\} \subset Z,$$

(2)

where $B_r(m)$ is the geodesic ball of radius $r$ centered at $m$. This motivates the following.
Definition 3. A closed nowhere dense set $Z \subset M$ is called a blow-up set for the Seiberg–Witten equation with multiple spinors, if there is a solution $(A, \Psi, 0)$ of (1) defined over $M \setminus Z$ such that the following holds:

(i) $|\Psi|$ extends as a continuous function to all of $M$ and $Z = |\Psi|^{-1}(0)$;
(ii) $\int_{M \setminus Z} |\nabla A \Psi|^2 < \infty$.

As explained in Remark 33 below, (ii) holds automatically provided $(A, \Psi, 0)$ is a limit of the Seiberg–Witten monopoles with $\tau_k \to 0$, $\tau_k \neq 0$. Notice also that a result of [Tau14] implies that the Hausdorff dimension of $Z$ is at most one.

Denote by $\mathcal{L}/ \pm 1$ the bundle, whose fiber at $m \in M$ is the orbifold $\mathcal{L}_m/ \pm 1 \cong \mathbb{C}^2/ \pm 1$. Despite of the singularity, one can still make sense of the notion of harmonicity for sections of this bundle as follows. If $\psi$ is not identically zero, which is always assumed to be the case below, $\psi$ takes values in the non-singular stratum of $\mathcal{L}/ \pm 1$ over an open set; Therefore, in a neighborhood of any point of this open set $\psi$ admits a lift, which is a local section of $\mathcal{L}$. If all such lifts are harmonic, $\psi$ is said to be harmonic.

Coming back to (1), assume for a while that $n = 2$, which simplifies the upcoming discussion somewhat. By [Hay12] (see also [HW15, App. A]) the gauge-equivalence class $[A, \Psi, 0]$ of a solution of (1) on $M \setminus Z$ corresponds, roughly speaking, to a harmonic section of $\mathcal{L}/ \pm 1$ (Somewhat more precisely, the gauge-equivalence class $[A, \Psi, 0]$ corresponds to a $\mathbb{Z}/2\mathbb{Z}$ harmonic spinor in the sense of [Tau14, (1.3)], see [HW15, App. A]; slightly abusing notations I refer sometimes to harmonic sections of $\mathcal{L}/ \pm 1$ as $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors too). This correspondence is important for the intended applications of the Seiberg–Witten theory with multiple spinors, see for example [Hay17]. However, by interpreting the limit $[A, \Psi, 0]$ as a harmonic section of $\mathcal{L}/ \pm 1$ we loose information about the background $\text{Spin}^c$-structure (or, in our notations, about the line bundle $\mathcal{L}$). This raises naturally the question of how to recover this piece of information. Another question, which naturally appears in this context, is the following: Can any harmonic section of $\mathcal{L}/ \pm 1$ defined over a suitable open subset of $M$ appear as a limit of the Seiberg–Witten monopoles?

In this preprint I give an answer to these questions by showing that the blow-up set $Z$ can be equipped with a certain infinitesimal structure, which encodes in particular the missing piece of information about the background $\text{Spin}^c$-structure.

To be more precise, it follows from the results of [Tau14] that at each point $z \in Z$ there is a tangent cone $Z_z = \bigcup \ell_j$ consisting of finitely many rays. I show that each ray $\ell_j$ can be equipped with a weight $\theta^j \in \mathbb{Z}_{\geq 0}$ and an orientation provided $\theta^j \neq 0$. The collection of all weights and orientations is abbreviated as $(\theta, \text{or})$. For example, if $Z$ is smooth, $\theta$ is a locally constant function on $Z$, i.e., each connected component of $Z$ is equipped with a non-negative integer multiplicity. Moreover, the components with non-vanishing multiplicities are also oriented.

The main result of this preprint is the following theorem. A more precise version is stated as Theorem 36.

Theorem 4. Let $Z$ be a blow-up set for the Seiberg–Witten equation with two spinors. The triple $(Z, \theta, \text{or})$ determines a class $[Z, \theta, \text{or}] \in H_1(M, \mathbb{Z})$. This satisfies

$$[Z, \theta, \text{or}] = \text{PD}(c_1(L)),$$

where $L = \mathcal{L}^2$ is the determinant line bundle and $\text{PD}$ stays for the Poincaré dual.

I would like to stress that no extra assumptions on the Riemannian metric on $M$ or the regularity of $Z$ are required in Theorem 4. In particular, viewing $Z$ as a subset of $M$ only, the homology class of $Z$ may be ill defined.
An interpretation of Theorem 4 is that there are topological restrictions on blow-up sets for the Seiberg–Witten equation with a fixed Spin\(^c\)-structure. For example, if \(L\) is non-trivial, then \(Z\) cannot be empty. Although this follows immediately from Theorem 4, this statement can be proved directly by an elementary argument. Indeed, assume that for some non-trivial \(L\) there is a solution \((A, \Psi, 0)\) of (1) such that \(\Psi\) vanishes nowhere, i.e., \(Z = \emptyset\). The equation \(\mu(\Psi) = \Psi \Psi^* - \frac{1}{2}|\Psi|^2 = 0\) implies that \(\Psi\) is surjective, which in turn shows that \(\Psi\) is an isomorphisms provided \(n = 2\). Hence, we obtain \(\Lambda^2 E \cong \Lambda^2 (\mathcal{S} \otimes \mathcal{L}) \cong \mathcal{L}^2 = L\), which shows that \(L\) is trivial thus providing a contradiction. This argument shows in fact, that \(Z\) must be infinite if \(L\) is non-trivial. A generalization of this is stated in Theorem 34, which gives a lower bound for the 1-dimensional Hausdorff measure of \(Z\) in terms of \(c_1(L)\).

The proof of Theorem 4 is based on the following observation: Any blow-up set for the Seiberg–Witten equation with multiple spinors is a zero locus of some continuous section \(s\) of \(L\) (details can be found at the beginning of Section 3). This of course immediately implies the statement of Theorem 4 if \(Z\) is sufficiently regular, for example smooth. However, a priori \(Z\) does not need to be smooth.

The reader may wonder why should we care about infinitesimal weights and orientations, since, after all, one can imagine more conventional ways to keep track of information about the determinant line bundle. One reason is as follows: Since the blow-up set for the Seiberg–Witten equations is never empty provided the determinant line bundle is non-trivial, a natural question is which structure a blow-up set can be equipped with. By (2), it seems reasonable to expect \(Z\) to be the support of some sort of \(\delta\)-function, where components of \(Z\) may have different multiplicities and orientations. The approach utilized here provides one way to formalize this.

Another reason comes from the conjectural relation between the Seiberg–Witten monopoles and \(G_2\) instantons [Hay17]. It seems plausible that if gauge–theoretic invariants of compact \(G_2\) manifolds in the sense of [DT98, DS11] exist, their construction should take into account not only honest \(G_2\) instantons, but also \(G_2\) instantons with singularities along one dimensional subsets (the author learned this from S. Donaldson). To the best of author’s knowledge, no direct evidence is known at present, however, by comparing with Donaldson–Thomas invariants for Calabi–Yau three-folds, it seems plausible that singular \(G_2\) instantons should play a rôle indeed. If this is true, it seems reasonable that the Seiberg–Witten monopoles with a non-empty blow-up set may be related to singular \(G_2\) instantons. If so, weights and orientations of \(Z\) are likely to encode information about the singularity of the corresponding \(G_2\) instanton. However, how much of this, if any, is true goes beyond the goals of this preprint.

Theorem 44 below generalizes Theorem 4 for any \(n \geq 2\). Notice, however, that an extra hypothesis, which becomes vacuous for \(n = 2\), appears in Theorem 44. Examples of solutions of (1) with \(\tau = 0\) and even \(n \geq 2\) satisfying this hypothesis are constructed in in Section 4. This also yields explicit examples of Fueter sections [HW15, App. A] with values in the moduli space \(M_{1,n}\) of framed centered charge one \(SU(n)\)-instantons over \(\mathbb{R}^4\).

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2 The zero locus of a continuous section of a line bundle

It is a classical fact that if a smooth section \(s\) of a line bundle \(L\) intersects the zero locus transversely, than \(s^{-1}(0)\) is a smooth oriented embedded submanifold whose homology class is
the Poincaré dual to the first Chern class of $L$. Even though a generic section does intersect the zero section transversely, in applications one can not always assume that a section at hand is in fact generic. This may happen for instance when $s$ satisfies some sort of PDE and perturbations are not readily available or do not fit the set-up.

In this section the structure and topology of $s^{-1}(0)$ is studied in the case when $s$ is assumed to be continuous only (and the base manifold $M$ is three-dimensional). In particular, no transversality arguments are available and $s^{-1}(0)$ is allowed to have singularities.

## 2.1 The case of an embedded graph

In this subsection I prove that if the zero locus $Z$ of a continuous section of a complex line bundle $L$ is an embedded graph, then edges can be equipped with weights and orientations resulting in a singular $1$-chain, whose homology class represents $\text{PD}(c_1(L))$. Although the material of this section is elementary, this provides a useful toy model for what follows in the next section.

Let $G$ be an (abstract) graph with a finite set of edges $E$.

### Definition 6 ([GL06, Def. 2.2]).

A flow on $G$ consists of a weight function $\theta: E \to \mathbb{Z}_{\geq 0}$ and orientations of edges with non-zero weights such that for each vertex $v$ we have

$$\sum_{e \text{ begins at } v} \theta(e) = \sum_{e \text{ ends at } v} \theta(e), \quad (7)$$

Notice that there is one minor difference between the above definition and the one of [GL06]; Namely, the weights in [GL06] are assumed to be positive whereas here $\theta$ is allowed to attain the zero value.

Observe that the set $\text{Flow}(G)$ of all flows on $G$ has a natural structure of an abelian group. Indeed, declare the sum of two flows $(\theta_1, or_1)$ and $(\theta_2, or_2)$ to be $(\theta, or)$ according to the following rule:

- If $e \in E$ has the same orientation with respect to $or_1$ and $or_2$, then this is also the orientation of $e$ with respect to $or$ and $\theta(e) = \theta_1(e) + \theta_2(e)$;
- If the orientations of $e$ with respect to $or_1$ and $or_2$ are opposite or one of the weights vanishes, declare $\theta(e) := |\theta_1(e) - \theta_2(e)|$; If $\theta(e) \neq 0$, declare the orientation of $e$ to be the one corresponding to the bigger weight.

Clearly, the inverse element is obtained by reversing the orientations of all edges.

Equivalently, we can also choose arbitrarily a reference orientation of all edges. Then a flow $(\theta, or)$ on $G$ can be conveniently interpreted as a map $\Theta: E \to \mathbb{Z}$, where the sign of $\Theta(e)$ encodes the difference between orientations of $e$ with respect to $or$ and the reference orientation. In particular, this shows that $\text{Flow}(G)$ is a subgroup of the free abelian group generated by $E$; Hence, $\text{Flow}(G)$ is also free and finitely generated.

### Definition 8.

A subset $Z$ of a manifold $M$ is called an embedded graph if there is a finite subset $V \subset Z$, whose elements are called vertices, such that $Z \setminus V$ consists of finitely many connected components; The closure of each component, which is called an edge, is a smooth embedded 1-dimensional submanifold, possibly with boundary.

Let $E$ denote the set of all edges of an embedded graph $Z$. Notice that by introducing extra vertices if necessary we can always assume that each edge contains at least one vertex (loops are allowed).
Let $Z$ be an embedded graph equipped with a flow. Assume for simplicity that the ambient manifold $M$ is compact. Clearly, $\sum_{e \in E} \theta(e)e$ is a singular 1-cycle in $M$. Denote

$$[Z, \theta, \text{or}] := \left[ \sum_{e \in E} \theta(e)e \right] \in H_1(M; \mathbb{Z}).$$

If confusion is unlikely to arise, we write simply $[Z] := [Z, \theta, \text{or}]$ for brevity. It should be pointed out that this notation does not suggest that the corresponding class depends on $Z$ as a set only.

The following proposition summarizes the above considerations.

**Proposition 9.** For each embedded graph $Z$ in a compact manifold $M$ there is a natural homomorphism

$$\Gamma: \text{Flow}(Z) \longrightarrow H_1(M; \mathbb{Z}), \quad (\theta, \text{or}) \mapsto [Z, \theta, \text{or}].$$

Let $s$ be a continuous section of a complex line bundle $L$ over a closed three-manifold $M$ such that the zero locus $Z := s^{-1}(0)$ is an embedded graph. Denote by $E$ the set of edges, which is finite. The section $s$ may be used to produce a flow on $Z$ as follows. Pick an edge $e \in E$ and a point $p$ on $e$ such that $p$ is not a vertex. Choose $\varepsilon > 0$ so small that $H_1(B_{\varepsilon}(p) \setminus e; \mathbb{Z}) \cong \mathbb{Z}$, where $B_{\varepsilon}(p)$ is a ball of radius $\varepsilon$ in a chart centered at $p$. Furthermore, choose an embedded circle $\gamma_e \subset B_{\varepsilon}(p) \setminus e$ such that $\gamma_e$ generates $H_1(B_{\varepsilon}(p) \setminus e; \mathbb{Z})$ for some choice of orientation on $\gamma_e$. Notice that at this point $\gamma_e$ is not assumed to be equipped with an orientation.

Furthermore, choose a local trivialization of $L$ over $B_{\varepsilon}(p)$ so that the restriction of $s$ to $\gamma_e$ can be thought of as a map $\gamma_e \rightarrow \mathbb{C}^*$. Declare

$$\theta(e) := |\deg(s: \gamma_e \rightarrow \mathbb{C}^*)| \in \mathbb{Z}_{\geq 0}, \quad (10)$$

where $\deg$ denotes the topological degree. Notice that this definition implicitly requires a choice of orientation of $\gamma_e$, however for the absolute value of the degree this choice is immaterial. If $\theta(e) \neq 0$, there is a unique orientation of $\gamma_e$ such that

$$\deg(s: \gamma_e \rightarrow \mathbb{C}^*) > 0. \quad (11)$$

Since the ambient manifold $M$ is oriented, the orientation of $\gamma_e$ yields a unique orientation of $e$.

Clearly, the map (10) as well as the orientations of edges with non-vanishing weights depends only on $s$ but not on the choices made in its definition.

**Proposition 12.** Let $M$ be an oriented three-manifold. Let $s$ be a continuous section of a complex line bundle $L \rightarrow M$ whose zero locus $Z := s^{-1}(0)$ is an embedded graph with the finite set of edges $E$. Then the following holds:

(i) $(\theta, \text{or})$ is a flow on $Z$, where $\theta$ is defined by (10) and the orientation of edges with non-zero weights is determined by (11);

(ii) If $M$ is also compact, then

$$[Z, \theta, \text{or}] = \text{PD}(c_1(L)),$$

where the right hand side of the equation denotes the Poincaré dual to the first Chern class of $L$.

**Proof.** Pick a vertex $v \in Z$, an open contractible neighborhood $U$ of $v$ such that $\Sigma := \partial U$ is a smoothly embedded surface in $M$, and a trivialization of $L$ over a neighborhood of the closure $\overline{U}$ (shrink $U$ if necessary). Without loss of generality we can also assume that $\Sigma$ intersects each
edge at most at one point and that this intersection is transverse. In particular, \( \Sigma \cap Z \) consists of finitely many points, say \( m_1, \ldots, m_k \). For each \( m_j \) choose a small embedded disc \( D_j \subset \Sigma \) containing \( m_j \); Clearly, these discs can be chosen so that there closures are disjoint. Denote also \( \gamma_j := \partial D_j \). Notice that \( \Sigma \) is oriented as the boundary of \( U \); This in turn induces an orientation of each \( D_j \) and, hence, also of \( \gamma_j \).

With these preliminaries at hand we have

\[
\sum_{j=1}^{k} \deg \left( s : \gamma_j \to \mathbb{C}^* \right) = \deg(L|_{\Sigma}) = 0, \tag{13}
\]

where the last equality holds by the triviality of \( L \) over \( \Sigma \).

Furthermore, notice that if an edge \( e \) begins at \( v \) and intersects \( \Sigma \) at some \( m_j \) we have \( \theta(e) = + \deg \left( s : \gamma_j \to \mathbb{C}^* \right) \), whereas if \( e \) ends at \( v \) we have \( \theta(e) = - \deg \left( s : \gamma_j \to \mathbb{C}^* \right) \). Hence, (13) shows that \( (\theta, \text{or}) \) is a flow, thus proving (i).

To see (ii), let \( s_1 \in C^\infty(M; L) \) be a perturbation of \( s \) intersecting the zero section transversely. In particular, \( s_1^{-1}(0) \) is a smooth closed oriented curve representing \( \text{PD}(c_1(L)) \). This is schematically shown on Figure 1. Notice that each “blue” connected component must be equipped with weight 1.

![Figure 1: A graph (in black) and its perturbation (in blue); Numbers near black edges represent their weights; Dashed lines represent boundaries of balls to be collapsed.](image1)

![Figure 2: The graph obtained by a perturbation and collapse of balls.](image2)

Without loss of generality we can assume that \( s_1^{-1}(0) \) is contained in an arbitrarily small neighborhood of \( Z \). By “collapsing” suitable disjoint balls centered at the vertices of \( Z \), we obtain a new graph \( G \) shown on Figure 2. The graph \( G \) can be though of as being obtained from \( Z \) by replacing each edge \( e \) by a number of “parallel” edges all equipped with an orientation and weighted by 1.

For any \( e \in E(Z) \) denote by \( E_e(G) \) the set of all edges in \( G \) connecting the same vertices as \( e \). We have

\[
[Z] = \left[ \sum_{e \in E(Z)} \theta(e) e \right] = \left[ \sum_{e \in E(Z)} \sum_{e' \in E_e(G)} e' \right] = \left[ \sum_{e' \in E(G)} e' \right] = [s_1^{-1}(0)] = \text{PD}(c_1(L)).
\]
This finishes the proof of this proposition. 

**Remark 14.** Let \( Z \subset M \) be an embedded graph. Choose a neighborhood \( V \) of \( Z \) such that \( V \) retracts on \( Z \) and the boundary \( \partial V \) is a smoothly embedded surface. One can think of \( V \) as a thickening of \( Z \). The long exact sequence of the pair \((V, \partial V)\) yields

\[
H_1(Z) \cong H_1(\partial V)/H_2(V, \partial V) \cong H^1(\partial V)/H^1(V).
\]

Since \( \text{Flow}(Z) \cong H_1(Z) \), the rightmost space is isomorphic to \( \text{Flow}(Z) \), which is intuitively clear, since its elements can be seen as “assigning weights” to elements of \( \text{Im}(H_2(V, \partial V) \to H_1(\partial V)) \).

### 2.2 The case of a graph-like set

Let \( M \) be a closed oriented three-manifold. It is convenient to fix a Riemannian metric \( g \) on \( M \) and a positive number \( r_0 \) which is smaller than the injectivity radius of \( g \). For any point \( m \in M \) denote by \( \exp_m : T_m M \to M \) the exponential map and \( \exp_{m, \lambda}(v) := \exp_m(\lambda v) \), where \( \lambda > 0 \). These maps are defined on the balls centered at the origin and of radius \( r_0 \) and \( \lambda^{-1} r_0 \) respectively. If the point \( m \) is clear from the context we will write simply \( \exp \) and \( \exp_{\lambda} \) respectively. Denote also by \( d : M \times M \to \mathbb{R}_{\geq 0} \) the distance function corresponding to \( g \).

Following [Tau14], we say that \( Z_s \) is a rescaling limit of \( Z \) at \( z \), if there is a sequence of positive numbers \( \lambda_i \to 0 \) with the following property: For any \( \varepsilon > 0 \) there is \( I_\varepsilon > 0 \) such that for all \( i \geq I_\varepsilon \) we have:

\[
\text{(a) } \exp_{\lambda_i}^{-1}(Z) \cap B_{\varepsilon^{-1}}(0) \subset U_{\varepsilon}(Z_s).
\]

Here \( B_{\varepsilon^{-1}}(0) \) is the ball of radius \( \varepsilon^{-1} \) centered at the origin and \( U_{\varepsilon}(Z_s) \) is the \( \varepsilon \)-neighborhood of \( Z_s \).

**Definition 15.** A set \( Z \) is said to be *locally graph-like*, if the following holds:

1. At each \( z \in Z \) there is a rescaling limit, which is a cone consisting of finitely many rays;
2. At all but at most countably many points of \( Z \) there is a rescaling limit, which is a line.

In what follows for a locally graph-like set we consider only those rescaling limits, which are cones consisting of finitely many rays.

Let \( \Sigma \subset M \) be a compact embedded oriented surface. We say that \( \Sigma \) intersects \( Z \) transversally, if at each point \( z \in Z \cap \Sigma \) there is a rescaling limit \( Z_s \), which is a line, such that \( T_z \Sigma \) and \( Z_s \) are transverse. If \( Z_s \) is equipped with a weight \( \theta_s \) and an orientation, we define \( I(Z_s, T_z \Sigma) := \varepsilon \cdot \theta_s \in \mathbb{Z} \), where \( \varepsilon \) is \(+1\) or \(-1\) depending on the orientation of \( Z_s \oplus T_z \Sigma \).

**Definition 16.** A flow \((\theta, \sigma)\) on a locally graph-like set \( Z \) is a collection of flows on each rescaling limit \( Z_{z,s} \) at each point \( z \in Z \) such that the following holds: Let \( \Sigma \) be any compact embedded oriented surface contained in an open contractible subset of \( M \) and intersecting \( Z \) transversally at each point. Then there is \( \delta > 0 \) such that for each finite covering of \( \Sigma \cap Z \subset \Sigma \) by disjoint discs \( D_r(z_j) \subset \Sigma \) with \( r \leq \delta \) and \( z_j \in Z \cap \Sigma \) we have

\[
\sum_j I(Z_{z_j,s}, T_{z_j} \Sigma) = 0
\]

for any choice of rescaling limits \( Z_{z,s} \), which is transverse to \( T_{z} \Sigma \) at any \( z_j \in Z \cap \Sigma \).
Remark 17. If in the setting of the previous definition $\Sigma$ intersects $Z$ at a finite number of points, the condition is that the “intersection number” vanishes, i.e.,

$$\sum_{z \in Z \cap \Sigma} I(Z_z, T_z \Sigma) = 0.$$ 

However, I do not wish to exclude a priori the case when the intersection contains infinitely many points.

Let $Z$ be a locally graph-like set equipped with a flow. We say that $z$ is a regular point of $Z$, if there is a neighborhood $U \subset M$ of $z$ such that $Z \cap U$ is a $C^1$-embedded submanifold of $U$. Let $Z_{\text{reg}}$ denote the set of all regular points. Clearly, at a regular point we have $Z_s = T_z Z_{\text{reg}}$, which is equipped with a unique weight $\theta_s$ and an orientation, provided $\theta_s \neq 0$. In other words, over $Z_{\text{reg}}$ we can regard $\theta$ as a locally constant function with values in $\mathbb{Z}_{\geq 0}$, i.e., $\theta$ attaches an integer multiplicity to each connected component of $Z_{\text{reg}}$. Moreover, each connected component with $\theta \neq 0$ is equipped with an orientation. With this in mind it is easy to see that in the case $Z$ is an embedded graph, the above definition yields a flow in the sense of Definition 6.

Clearly, the set $\text{Flow}(Z)$ of all flows on a given $Z$ has a natural structure of an abelian group. I show below that there is a natural homomorphism $\text{Flow}(Z) \to H_1(M; \mathbb{Z})$. However, before doing this let me construct some examples.

Assume that a locally graph-like set $Z$ is the zero locus of a continuous section of a complex line bundle $L \to M$. Let $s_* : T_z M \to \mathbb{C}^*$ be a continuous map such that $s_*^{-1}(0) = Z_s$.

Definition 18. A pair $(Z_s, s_*)$ is said to be a rescaling limit of $(Z, s)$ at $z \in Z$, if there is a sequence of positive numbers $\lambda_i \to 0$ with the following property: For any $\varepsilon > 0$ there is $I_* > 0$ such that for all $i \geq I_*$ in addition to (a) we have:

(b) There is a sequence of trivializations of $\exp_{\lambda_i}^* L$ such that the $C^0$-norm of $\exp_{\lambda_i}^* s - s_*$ over $B_{\varepsilon^{-1}}(0) \setminus U_{\varepsilon}(Z_s)$ is less than $\varepsilon$.

Since $Z_s$ is a cone consisting of finitely many rays, by Proposition 12, (i) (applied to $s_*$ in place of $s$) we obtain an infinitesimal flow $(\theta_*, or)$, i.e., a flow on $Z_s$. The collection of all these infinitesimal flows is abbreviated as $(\theta, or)$ and we say that $(\theta, or)$ is induced by $s$.

Lemma 19. Let $s$ be a continuous section of a complex line bundle $L$. Assume the zero locus $Z = s^{-1}(0)$ is locally graph-like and at each point $z \in Z$ there is a rescaling limit $(Z_s, s_*)$ of $(Z, s)$. The collection $(\theta, or)$ induced by $s$ is a flow on $Z$.

Proof. Let $\Sigma \subset M$ be a compact embedded oriented surface in $M$ contained in an open contractible set. Trivialize $L$ over $\Sigma$ so that $s$ can be thought of as a map $\Sigma \to \mathbb{C}$. Cover $\Sigma \cap Z$ by a finite collection of disjoints discs $D_r(z_j)$. If $r$ is sufficiently small we can assume $\theta_s(z_j) = |\deg(s : \partial D_r(z_j) \to \mathbb{C}^*)|$. Hence,

$$\sum_j \epsilon(z_j) \theta_s(z_j) = \sum_j \deg(s : \partial D(z_j) \to \mathbb{C}^*) = 0. \quad \Box$$

The rest of this subsection is devoted to the proof of the following result.

Proposition 20. Let $Z$ be a compact locally graph-like set.

(i) There is a natural homomorphism $\Gamma : \text{Flow}(Z) \to H_1(M; \mathbb{Z})$, $(\theta, or) \mapsto [Z, \theta, or]$; 
(ii) If $(\theta, or)$ is induced by a section of a line bundle $L$, then $[Z, \theta, or] = \text{PD}(c_1(L))$. 

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Moreover, $(i)$ and $Q$ a smooth function $\chi(U)$.

Assume that at $Z$.

The proof consists of three steps.

**Proof.** The proof consists of three steps.

**Step 1.** Assume that at $z \in Z$ there is a rescaling limit $(Z_\ast, s_\ast)$ such that $Z_\ast$ is a line. Then for any neighborhood $U$ of $z$ in $M$ there is a neighborhood $Q \subset U$ and a smooth map $f_U \simeq id$ with the following properties:

(i) $f_U(Z) \cap Q$ is a smooth embedded curve;

(ii) $f_U(z) = z$ and $f_U$ is the identity map on the complement of $U$;

(iii) The restriction of $f_U$ to $U \setminus Q$ is a diffeomorphism onto its image;

(iv) $f_U(Z) = s_U^{-1}(0)$ for some $s_U \in C^0(M; L)$.

Moreover, $f_U$ induces a natural homomorphism $\text{Flow}(Z) \to \text{Flow}(f_U(Z))$.

Without loss of generality, we can assume that $U$ is a coordinate chart centered at $z$. Let $(x_1, x_2, x_3)$ be local coordinates on $U$ such that $Z_\ast$ is tangent to the $x_1$–axis. It follows from the definition of the rescaling limit that there is a cube $Q := (-\lambda, \lambda)^3 \subset U$ such that $Z \cap Q \subset C := (-\lambda, \lambda) \times D_{\lambda/2}$, where $D_{\lambda/2} \subset \mathbb{R}^2$ denotes the disc of radius $\lambda/2$ centered at the origin. Choose a smooth function $\chi$, which vanishes on $C$ and equals 1 on the complement of a cube containing $Q$. Then $f_U(x_1, x_2, x_3) = (x_1, \chi \cdot x_2, \chi \cdot x_3)$ is homotopic to the identity map and clearly satisfies (i) and (ii). Property (iii) can be checked directly for a special choice $\chi(x) = \chi_1(x_1)\chi_2(x_2^2 + x_3^2)$, where $\chi_1$ and $\chi_2$ are suitable functions. To see (iv), notice that fixing a trivialization of $L$ over $U$, the equation $\chi \cdot s = s_U \circ f_U$ determines uniquely $s_U$, which has the required property.

Furthermore, to define the homomorphism $(f_U)_*: \text{Flow}(Z) \to \text{Flow}(f_U(Z))$ it is enough to consider the points at $Z \cap Q$. If $f_U(Z) \cap Q$ is not connected, then by the construction there is a disc $D \subset Q$ such that $\partial D \subset Q \setminus (-\lambda, \lambda) \times D_{\lambda/2}$ and $D \cap Z = \emptyset$. It is then easy to see that $(f_U)_*$ must vanish for all $z \in Z \cap Q$. If $f_U(Z) \cap Q$ is connected, then $(f_U)_*$ is uniquely specified by $(f_U)_* = id$ at the point $z$.

**Step 2.** Let $B$ be a ball in $M$ centered at $v \in Z$ such that $\partial B \cap Z = \{z_1, \ldots, z_n\}$, $Z$ is smooth in a neighborhood of each $z_i$, and the intersection of $\partial B$ and $Z$ is transverse. Then there is a smooth map $f_B \simeq id_M$ such that

(i) $f_B(Z) \cap B$ is an embedded graph equipped with a flow such that $v$ is an $n$-valent vertex at which (7) is satisfied. Moreover, each $z_i$ is connected with $v$ by a unique edge;

(ii) $f_B$ is the identity map on the complement of $B$;

(iii) $f_B(Z) = s_B^{-1}(0)$ for some $s_B \in C^0(M; L)$;
(iv) There is a natural homomorphism \((f_B)_*: \text{Flow}(Z) \rightarrow \text{Flow}(f_B(Z))\).

Without loss of generality we can assume that the radius of \(B\) equals 1. Choose \(\varepsilon > 0\) so small that \(Z \cap B \setminus B_{1-2\varepsilon}(v)\) consists of \(n\) smooth connected curves. Choose also a smooth monotone function \(\chi: [0, 1] \rightarrow [0, 1]\) such that

\[
\chi(t) = \begin{cases} 
0 & \text{if } t \in [0, 1 - 2\varepsilon], \\
1 & \text{if } t \in [1 - \varepsilon, 1].
\end{cases}
\]

Define

\[
f_B(x) := \begin{cases} 
\chi(|x|)x & \text{if } x \in B, \\
x & \text{otherwise}.
\end{cases}
\]

Clearly, \(f_B(Z) \cap B\) is an embedded graph, which inherits a weight function and orientation from \(Z \cap B \setminus B_{1-\varepsilon}\). By Definition 16, we have \(\sum \theta(z_i)\varepsilon(z_i) = 0\), which immediately implies that (7) holds at \(v\). Part (iii) is proved just like the corresponding statement in Step 1. Finally, the last part follows, since the restriction of \(f_B\) to \(B \setminus B_{1-\varepsilon}\) is a diffeomorphism onto its image.

**Step 3. I prove the statement of this lemma.**

Let \(z\) be an arbitrary point in \(Z\) and let \(Z_s\) be a rescaling limit of \(Z\) such that \(Z_s\) consists of finitely many rays. By the definition of the rescaling limit, for any \(\varepsilon > 0\) we can find some \(\lambda > 0\) such that \(\exp_{\lambda Z}(Z) \cap B_1(0) \subset U_\varepsilon(Z_s)\). Choose \(r \in \left[\frac{1}{2}, 1\right]\) such that \(\partial B_{\lambda r}(z) \cap Z\) contains only points admitting a line as a rescaling limit. The existence of \(r\) follows from the observation that there are uncountably many choices for \(r\), however \(Z\) contains at most countably many points which do not admit a line as a rescaling limit. Denote by \(B(z)\) the chosen ball and by \(U(z) \subset B(z)\) the corresponding open neighborhood of a cone containing \(Z \cap B(z)\).

Since \(Z\) is compact, there is a finite collection of balls as above \(\{B_i = B(z_i) \mid 1 \leq i \leq I\}\) covering \(Z\). Pick one of these balls, say \(B_1\). For each \(z \in Z \cap \partial B_1\) choose a ball \(B'(z)\) such that \(\text{diam } B'(z) \leq \frac{1}{4} \text{diam } B_1\) and denote by \(Q(z)\) the open subset supplied by Step 1. Choose a finite collection \(\{(B'_k, Q_k) \mid 1 \leq k \leq K\}\) such that \(\{Q_k\} \text{ covers } Z \cap \partial B_1\). Without loss of generality we can assume in addition that the following holds: If for some \(i \in \{1, \ldots, I\}\) we have \(B'_k \cap U_i \neq \emptyset\), then \(B'_k \subset U_i\). Notice also, that for each \(B'_k\) there is a geodesic segment through the center of \(B'_k\) such that \(Z \cap B'_k\) is contained in a neighborhood of this geodesic segment.

Apply Step 1 consecutively to \(B'_1, \ldots, B'_K\) to obtain a map \(f' \simeq id_M\) such that \(f'(Z)\) is a smooth embedded submanifold in a neighborhood of \(\partial B_1\). Notice that the choice of the balls \(B'_k\) ensures that \(\{U_i \mid i = 2, 3, \ldots, I\}\) covers \(f'(Z) \setminus B_1\). If the intersection \(\partial B_1 \cap f'(Z)\) is not transverse, we can decrease slightly the radius of \(B_1\) to get rid of the non-transverse intersection points. This can be done so that \(\{U_i \mid i = 2, 3, \ldots, I\}\) still covers the part of \(f'(Z)\) which is not contained in the new \(B_1\).

Apply Step 2 to obtain a map \(f_1 \simeq id_M\) such that \(f_1(Z) \cap B_1\) is an embedded graph equipped with a flow. Repeating this procedure consecutively for all balls \(B_i\) we obtain a map \(f \simeq id_M\) such that \(G := f(Z)\) is an embedded graph. Moreover, we obtain \(f_* : \text{Flow}(Z) \rightarrow \text{Flow}(G)\) by composing corresponding homomorphisms at each step of the construction; Also, if \(Z\) is the zero locus of a continuous section, so is \(G\), since this property is preserved by each step of the construction. Part (iii) can be seen by tracing through the above proof and shrinking the corresponding neighborhoods chosen above if necessary. Finally, the last part is immediate from the construction.
Remark 22. The proof of Lemma 21 shows that the following holds: The map \( f \) in fact can be chosen so that its restriction to a neighborhood \( V \) of \( Z \) is a homotopy equivalence between \( V \) and \( G = f(Z) \). The only minor modification in the proof is needed at Step 2, namely an extra collapse of a neighborhood of \( Z \) in \( B \setminus B_{1-2\varepsilon}(v) \).

Clearly, we can assume that the boundary of \( V \) is a smoothly embedded surface. To see this, it is enough to pick a non-negative smooth function \( \phi \) such that \( \phi^{-1}(0) = Z \) and shrink \( V \) to \( \phi^{-1}([0,\varepsilon)) \), where \( \varepsilon > 0 \) is a sufficiently small regular value of \( \varphi \).

Remark 23. The fact that \( Z \) can be mapped onto an embedded graph by a map homotopic to the identity map can be proved in a less technical manner. Namely, one can choose a handle decomposition of \( M \) and first “push” a suitable subset of each 3-handle to its boundary so that \( Z \) will be mapped to the union of 0-, 1-, and 2-handles. Using the same sort of arguments one can push \( Z \) further to a 1-skeleton of \( M \). This argument requires \( Z \) to be closed of 2-dimensional Hausdorff measure zero only, however the resulting map will not satisfy (iv) of Lemma 21, since the construction requires collapses of large subsets of \( M \).

Proof of Proposition 20. Fix a neighborhood \( V \) and a graph \( G \) as in Remark 22 (for some map \( M \to M \) supplied by Lemma 21). Observe that the long exact sequence of the pair \((\overline{V}, \partial \overline{V})\) yields
\[
0 \to H_2(\overline{V}, \partial \overline{V}) \to H_1(\partial \overline{V}) \to H_1(\overline{V}) \to 0,
\]
where we used \( H_2(\overline{V}) \cong 0 \) and \( H_1(\partial \overline{V}) \cong H^2(\overline{V}) \cong 0 \). Using \( H_1(\overline{V}) \cong H_1(G) \), we obtain
\[
H_1(G) \cong H_1(\partial \overline{V})/H_2(\overline{V}, \partial \overline{V}) \cong H^1(\partial \overline{V})/H^1(\overline{V}),
\]
cf. Remark 14.

Furthermore, pick any map \( f \) supplied by Lemma 21 such that \( f \) is the identity map outside of \( V \). Denote by \( G_f := f(Z) \) the corresponding embedded graph. Let also \( U \) be an open set containing \( Z \) such that \( U \) is homotopy equivalent to \( G_f \). Define
\[
\Gamma(Z, \theta, \text{or}) := [G_f, f_*(\theta, \text{or})],
\]
where the brackets on the right hand side denote the homology class of an embedded graph as in Proposition 9. The commutativity of the diagram
\[
\begin{array}{ccc}
0 & \longrightarrow & H_2(U, \partial U) \\
& & \downarrow \\
& & H_1(M)
\end{array}
\]
\[
\begin{array}{ccc}
0 & \longrightarrow & H_2(\overline{V}, \partial \overline{V}) \\
& & \downarrow \\
& & H_1(\overline{V})
\end{array}
\]
together with the discussion above show that \( \Gamma \) does not depend on the choice of \( f \).

Part (ii) of this proposition is obtained by combining Proposition 12, (ii) and Lemma 21, (iii).

\[ \square \]

2.3 A lower bound for the Hausdorff measure of a locally graph-like set

Denote by \( \mathcal{H}^1 \) the 1-dimensional Hausdorff measure induced by the Riemannian metric on \( M \).
Proposition 24. Let $Z$ be a compact locally graph-like set equipped with a flow $(\theta, or)$ such that $a := [Z, \theta, or] \neq 0$. There is a positive constant $C(a, g)$ depending only on $a$ and the Riemannian metric $g$ such that

$$\mathcal{H}^1(Z) \geq C(a, g).$$

Proof. For a subset $S \subset M$ denote by $\mathcal{H}^1_\delta(S)$ the $\delta$-approximation of $\mathcal{H}^1(S)$. For any $a \in H_1(M; \mathbb{Z})$ define

$$C_\delta(a, g) := \inf \{ \mathcal{H}^1_\delta(G) \mid G \text{ embedded graph, } \exists (\theta, or) \in \text{Flow}(G), [G, \theta, or] = a \}.$$ 

Notice that by passing to the limit $\delta \to 0$ we obtain

$$C(a, g) = C_0(a, g) := \inf \{ \ell(G) \mid G \text{ embedded graph, } \exists (\theta, or) \in \text{Flow}(G), [G, \theta, or] = a \},$$

where $\ell(G) := \sum_{e \in E} \text{length}(e)$ is the total length of $G$.

Observe that $C(a, g) > 0$ provided $a \neq 0$. Indeed, if $C(a, g) = 0$, the class $a$ could be represented by an embedded graph, whose connected components are contained in (small) balls, which contradicts $a \neq 0$.

Furthermore, let $\{V_i \mid i \in I\}$ be an arbitrary countable covering of $Z$ by open sets such that $\text{diam } V_i < \delta$ for all $i \in I$. There is $\sigma > 0$ such that

$$U_\sigma(Z) := \{ m \in M \mid d(m, Z) < \sigma \} \subset V := \bigcup_{i \in I} V_i.$$ 

By Lemma 21, there is an embedded graph $G$ equipped with a flow $(\theta_1, or_1)$ such that $G$ is contained in $U_\sigma(Z)$ and $[G, \theta_1, or_1] = a$. Since the collection $\{V_i\}$ covers $G$, we have

$$\sum_{i \in I} \text{diam } V_i \geq \mathcal{H}^1_\delta(G) \geq C_\delta(a, g).$$

Hence, for all $\delta > 0$ we have $\mathcal{H}^1_\delta(Z) \geq C_\delta(a, g)$, which in turn implies (35). \hfill \Box

2.4 The case of a rectifiable set

Let $Z$ be the zero locus of $s \in C^0(M; L)$ as in the previous sections. Here, however, we assume the following:

(A) $Z$ is locally graph-like;

(B) For $\mathcal{H}^1$–almost all $z \in Z$ there is $\bar{r} > 0$ such that $H_1(B_{\bar{r}}(z) \setminus Z; \mathbb{Z}) \cong \mathbb{Z}$ for all $r \in (0, \bar{r})$;

(C) $Z$ is a (countably) rectifiable subset of $M$ and $\mathcal{H}^1(Z) < \infty$.

Denote by $Z^\times$ the set of all those points $z \in Z$ such that at least one of the following conditions hold:

- $H_1(B_{r_k}(z) \setminus Z; \mathbb{Z}) \not\cong \mathbb{Z}$ for some sequence $r_k \to 0$;
- no line is a rescaling limit of $Z$ at $z$.

Notice that $Z^\times$ is of $\mathcal{H}^1$-measure zero.

Since $Z$ is rectifiable, we have

$$Z = \bigcup_{j=0}^{\infty} Z_j,$$

(26)
where $\mathcal{H}^1(Z_0) = 0$ and each $Z_j$, $j \geq 1$ is an embedded $C^1$-curve. Notice that without loss of generality we can assume that (26) is a disjoint union [Sim83, 11.7].

If $z \in Z_j$, then $T_zZ_j$ must be contained in any rescaling limit of $Z$ at $z$. Hence, if $Z$ admits a line as a rescaling limit at $z$, then this line must be $T_zZ_j$. If in addition $z \notin Z^\times$, then for any $r \ll 1$ there is a circle $\gamma_z \subset B_r(z) \setminus Z$ generating $H_1(B_r(z) \setminus Z)$. Clearly, the multiplicity (or weight)

$$\theta(z) := |\deg(s : \gamma_z \to \mathbb{C}^*)|$$

does not depend on the choice of $\gamma_z$. Moreover, if $\theta(z) \neq 0$, we can orient $T_zZ$ just like in the case of an embedded graph.

Thus, we obtain the multiplicity function $\theta : Z \setminus (Z_0 \cup Z^\times) \to \mathbb{Z}_{\geq 0}$, which is locally constant. This in turn determines an orientation of

$$\tilde{Z} = \{z \in Z \setminus (Z_0 \cup Z^\times) | \theta \neq 0\},$$

which can be interpreted as a continuous vector field $\xi$ on $\tilde{Z}$ such that $|\xi(z)| = 1$ for all $z \in \tilde{Z}$.

**Proposition 27.** If (A)–(C) holds, then $(\tilde{Z}, \theta, \xi)$ is an integer multiplicity current without boundary.

**Proof.** We only need to show that the boundary of $(\tilde{Z}, \theta, \xi)$ is empty, i.e.,

$$\int_{\tilde{Z}} \theta(df, \xi) \, d\mathcal{H}^1 = 0 \quad (28)$$

for any smooth function $f$ on $M$. Since the left hand side of (28) is linear in $f$, it is enough to prove (28) for those functions, whose support is contained in a contractible subset.

Thus, let $U \subset M$ be contractible and $\operatorname{supp} f \subset U$. Denote $f_Z = f|_Z$. Let $Jf_Z$ denote the Jacobian of $f_Z$ (in the sense of the geometric measure theory). Then we have

$$\int_{\tilde{Z}} \theta(df, \xi) \, d\mathcal{H}^1 = \int_{\tilde{Z}} \theta \operatorname{sign}(\langle df, \xi \rangle) Jf_Z \, d\mathcal{H}^1 = \int \left( \sum_{z \in f^{-1}(t) \cap \tilde{Z}} \theta(z) \varepsilon(z) \right) d\mathcal{L}^1(t), \quad (29)$$

Here the first equality follows from the definition of the Jacobian and the second one follows from the area formula.

Notice that if $t$ is a regular value of $f$, then $\Sigma_t = f^{-1}(t) \subset U$ is smooth and contained in a contractible set, namely $U$. Also, almost any $t \in \mathbb{R}$ is a regular value of both $f$ and $f_Z$ and $f(Z_0 \cup Z^\times)$ is of measure zero. Hence, an argument used in the proof of Lemma 19 shows that the right hand side of (29) vanishes for almost all $t$. \hfill $\square$

**Remark 30.** As we will see below, in the case of the blow-up set for the Seiberg–Witten equation Conditions (A) and (C) are known to hold true, whereas (B) requires further studies. Clearly, it is also possible to replace (B) by other conditions, but we will not go into the details here.

### 3 The infinitesimal structure of the blow-up set for the Seiberg–Witten monopoles with multiple spinors

In this section we prove Theorem 36, which is a somewhat more precise version of Theorem 4, as well as its generalization for the case of $n \geq 3$ spinors. Also, we obtain a lower bound for the
1-dimensional Hausdorff measure of blow-up sets for the Seiberg–Witten equations with two spinors, see Theorem 34.

As already mentioned in the introduction, a blow-up set for the Seiberg–Witten equations with two spinors is the zero locus of a continuous section. Indeed, let \((A, \Psi, 0)\) be a solution of (1) over \(M \setminus Z\) with \(n = 2\). By [HW15, Thm. 1.5] (see also Prop. 0.1 of the Erratum) \(A\) is flat with the holonomy in \(\mathbb{Z}/2\mathbb{Z}\), in particular the holonomy of the induced connection on \(L \to M \setminus Z\) is trivial. Let \(s_0\) be a parallel section of \(L\) over \(M \setminus Z\). Then

\[
s := |\Psi| \cdot s_0
\]

is a continuous section of \(L\) defined on all of \(M\) such that \(Z = s^{-1}(0)\).

**Lemma 32.**

(i) A blow-up set \(Z\) for the Seiberg–Witten equations with \(n\) spinors is a compact locally graph-like set;

(ii) If \(n = 2\), the pair \((Z, s)\) admits a rescaling limit at each point \(z \in Z\), where \(s\) is given by (31).

**Proof.** Notice first that it is enough to prove (i) for \(n = 2\). Indeed, if \(n > 2\) and \(Z\) is a blow-up set in the sense of Definition 3, then \(Z\) is also a blow-up set for \(n = 2\) by Proposition 0.1 of [HW15].

Thus, assume that \(Z\) is a blow-up set for the Seiberg–Witten equation with two spinors and let \((A, \Psi, 0)\) be a corresponding solution of (1). Then the projection \(\psi\) of \(\Psi\) is a \(\mathbb{Z}/2\mathbb{Z}\)-harmonic spinor [HW15, Prop A.1]; Moreover, we have the pointwise equality \(|\Psi| = |\psi|\) as well as the estimate \(\int_{M \setminus Z} |\nabla \psi|^2 < \infty\), which follows from Definition 3, (ii). By [Tau14, Prop. 4.1] applied to the constant sequence \(z_i = z\) and arbitrary \(\lambda_i \to 0\) we obtain that there is a rescaling limit \((Z_s, \psi_s)\) of \((Z, \psi)\) at any point \(z \in Z\) in the sense described by [Tau14, Prop. 4.1]. In particular, \(Z_s\) is a cone consisting of finitely many rays [Tau14, Lemma 5.4] and the sequence \(\exp_{\lambda_i}^* |\Psi| = |\Psi| \circ \exp_{\lambda_i}\) converges to \(|\psi_s|\) in \(C^0_{\text{loc}}(T_z M \setminus Z_s)\). Moreover, by [Tau14, Lemmas 6.1 and 6.3] \(Z_s\) is a line for all but at most countably many points of \(Z_s\). In particular, \(Z_s\) is a locally graph-like set; Clearly, \(Z_s\) is also compact.

Furthermore, pick a smooth trivialization \(\sigma\) of \(L\) in a neighborhood of \(z\). Interpret \(\exp_{\lambda_i}^* A\) as a sequence of flat connections on the product bundle, where \(L\) is trivialized by \(\sigma_i := \sigma \circ \exp_{\lambda_i}\). Hence, the sequence \(\exp_{\lambda_i}^* A\) has a subsequence, which converges in \(C^\infty_{\text{loc}}(T_z M \setminus Z_s)\) to some flat connection \(A_s\). In particular, a subsequence of \(\exp_{\lambda_i}^* s_0\), which is considered as a section of the product bundle over \(T_z M \setminus Z_s\), converges in \(C^0_{\text{loc}}(T_z M \setminus Z_s)\) to a parallel section of \(A_s\). Hence, \(\exp_{\lambda_i}^* s\) also has a subsequence, which converges to some \(s_s\) in \(C^0_{\text{loc}}(T_z M \setminus Z_s)\). The pointwise equality \(|s_s| = |\psi_s|\) implies that \(s_s\) vanishes precisely on \(Z_s\). This shows that \(s_s\) is a rescaling limit of \(s\). \(\square\)

**Remark 33.** Observe that for any solution \((A, \Psi, \tau)\) of (1) with \(\tau \neq 0\) by the Weitzenböck formula we have

\[
\int_M |\nabla A\Psi|^2 \leq \int_M |\nabla^A \Psi|^2 + \tau^{-2} \int_M |\mu(\Psi)|^2 = -\frac{1}{4} \int_M \text{scal}_g |\Psi|^2 \leq C,
\]

where \(C = \max\{-s/4, 0\} \geq 0\) and \(\text{scal}_g\) denotes the scalar curvature of the background Riemannian metric \(g\) on \(M\). hence, any solution \((A, \Psi, 0)\) of (1) arising as a limit of some sequence \((A_k, \Psi_k, \tau_k)\) over \(M \setminus Z\) with \(\tau_k \to 0\) satisfies Condition (ii) of Definition 3.
**Theorem 34.** Let $Z$ be a blow-up set for the Seiberg–Witten equations with two spinors corresponding to the determinant line bundle $L$ with $a = \text{PD}(c_1(L)) \neq 0$.

(i) There is a positive constant $C(a, g)$ depending only on $a$ and the Riemannian metric $g$ such that

$$\mathcal{H}^1(Z) \geq C(a, g).$$

(ii) The Hausdorff dimension of $Z$ equals $1$.

**Proof.** Part (i) follows from Lemma 32 and Proposition 24. Moreover, (35) shows in particular that the Hausdorff dimension $\dim_H Z$ of $Z$ is at least $1$. Combining this with $\dim_H Z \leq 1$ [Tau14, Thm.1.3], yields $\dim_H Z = 1$. □

The following theorem follows directly from Proposition 20 and Lemma 32.

**Theorem 36.** Let $Z$ be a blow-up set for the Seiberg–Witten equations with two spinors. For any solution $(A, \Psi, 0)$ of (1) over $M \setminus Z$ define $s \in C^0(M; L)$ by (31). Let $(\theta, \text{or})$ be a flow on $Z$ induced by $s$. Then

$$[Z, \theta, \text{or}] = \text{PD}(c_1(L)),$$

where $[Z, \theta, \text{or}] = \Gamma(Z, \theta, \text{or})$, $L = \mathcal{L}^2$ is the determinant line bundle, and PD stays for the Poincaré dual. □

Theorem 4 implies that there are restrictions for $\mathbb{Z}/2\mathbb{Z}$ harmonic spinors, which can be lifted to a solution of (1) with $\tau = 0$. Namely, let $Z$ be an arbitrary locally graph-like subset of $M$. By applying a map $f \simeq \text{id}_M$ if necessary, we can assume that $Z$ is an embedded graph. Denote

$$\Lambda = \Lambda(Z) := \text{Im}(\Gamma: \text{Flow}(Z) \to H_1(M; \mathbb{Z})).$$

For instance, if $Z$ is a smooth connected oriented curve, then $\Lambda(Z) = \mathbb{Z}[Z]$.

**Proposition 37.** Let $(\psi, Z)$ be a $\mathbb{Z}/2$-harmonic spinor. If $\text{PD}(c_1(L)) \not\in \Lambda(Z)$, then $(\psi, Z)$ cannot appear as the limit of a sequence of the Seiberg–Witten monopoles with two spinors for any Spin$^c$-structure, whose determinant line bundle is $L$. □

**Example 1.** To obtain an example of a $\mathbb{Z}/2$-harmonic spinor with a non-trivial subgroup $\Lambda$, consider a harmonic spinor $\psi$ on a Riemann surface $\Sigma$ with a non-empty zero locus, which is necessarily a finite collection of points. Viewing $\psi$ as a harmonic spinor on $M = \Sigma \times S^1$ equipped with the product metric, we obtain that the corresponding zero locus consists of finitely many copies of $\{pt\} \times S^1$, which implies that $\Lambda(\psi) = H_1(S^1) \subset H_1(\Sigma) \oplus H_1(S^1) = H_1(M)$.

In particular, this yields the following: Choose any Spin$^c$-structure, whose determinant line bundle is not the pull-back of a line bundle on $\Sigma$. Then $\psi$ cannot appear as the limit of a sequence of the Seiberg–Witten monopoles on $\Sigma \times S^1$ with two spinors for this choice of a Spin$^c$-structure.

Let us turn to the general case, i.e., $n \geq 2$. Notice first that $E$ admits a topological trivialization, since the structure group of $E$ is $\text{SU}(n)$ and the base manifold is three-dimensional. It is convenient to pick such a trivialization thus identifying $E$ with the product bundle $\mathbb{C}^n$. Notice that $E$ may be equipped with a nontrivial background connection $B$, however this will not have any significance for the upcoming discussion.
Recall that the quadratic map $\mu$ appearing in (1) is obtained from the following algebraic map

$$\mu: \text{Hom}(\mathbb{C}^n, \mathbb{C}^2) \to i\mathfrak{su}(2), \quad \mu(B) = BB^* - \frac{1}{2}|B|^2,$$

which is denoted by the same letter. The $U(n)$-action $A \cdot B = BA^*$ on the domain of $\mu$ combined with the action of $\mathbb{R}_{>0}$ by dilations yields a transitive action on $\mu^{-1}(0) \setminus \{0\}$. Observing that the stabilizer of a point, say the projection onto the first two components, is $U(n-2)$, we obtain $\mu^{-1}(0) \setminus \{0\} \cong \mathbb{R}_{>0} \times \frac{U(n)}{U(n-2)}$.

Furthermore, letting $U(1)$ act as the center of $U(n)$, we obtain a diffeomorphism

$$\tilde{M}_{1,n} = \mu^{-1}(0) \setminus \{0\}/U(1) \cong \mathbb{R}_{>0} \times \frac{U(n)}{U(1) \times U(n-2)}.$$

This yields a projection

$$\zeta: \tilde{M}_{1,n} \to \text{Gr}_{n-2}(\mathbb{C}^n), \quad (38)$$

which in fact represents $\tilde{M}_{1,n}$ as the total space of a fiber bundle over $\text{Gr}_{n-2}(\mathbb{C}^n)$ with the fiber $\mathbb{C}^2 \setminus 0/\pm 1$. Notice also that by viewing $\mathbb{C}^2$ as the tautological $SU(2)$-representation, we obtain an action of $SU(2)$ on $\tilde{M}_{1,n}$. Moreover, the induced action on $\text{Gr}_{n-2}(\mathbb{C}^n)$ is trivial.

Denote

$$\mathcal{M} := SU(\hat{\mathcal{U}}) \times SU(2) \tilde{M}_{1,n} \to M. \quad (39)$$

To each $\mathcal{I} \in \Gamma(\mathcal{M})$ defined on a subset of $M$, say $M \setminus Z$, with the help of (38) we can associate a map $\Phi_0: M \setminus Z \to \text{Gr}_{n-2}(\mathbb{C}^n)$, namely $\Phi_0 = \zeta \circ \mathcal{I}$. Assume $\mathcal{I}$ admits a lift $\Psi$, i.e., $\Psi$ is a section of $\text{Hom}(\mathbb{C}^n, \hat{\mathcal{U}} \otimes \mathcal{L})$ which vanishes nowhere on $M \setminus Z$ and satisfies $\mu \circ \Psi = 0, \pi \circ \Psi = \mathcal{I}$, where $\pi: \mu^{-1}(0) \setminus \{0\} \to \tilde{M}_{1,n}$ is the natural projection. Under these circumstances the map $\Phi_0$ can be described as follows. The equation $\mu(\Psi) = 0$ implies that for each $m \in M \setminus Z$ the homomorphism $\Psi_m$ is surjective and therefore

$$\Phi_0: M \setminus Z \to \text{Gr}_{n-2}(\mathbb{C}^n), \quad \Phi_0(m) = \ker \Psi_m \quad (40)$$

is well-defined and coincides with $\zeta \circ \mathcal{I}$. Notice that over $M \setminus Z$ we have the short exact sequence $0 \to \Phi_0^* S \to \mathbb{C}^n \to \mathcal{S} \to 0$, which implies $L|_{M \setminus Z} \cong \Phi_0^* \det S^\vee$, where $S$ denotes the tautological bundle over $\text{Gr}_{n-2}(\mathbb{C}^n)$.

**Remark 41.** $\tilde{M}_{1,n}$ can be identified with the framed moduli space of centered charge one $SU(n)$-instantons on $\mathbb{R}^4$, see for example [DK90, Sect. 3.3]. Then $\mathcal{M}$ can be thought of as the bundle whose fiber at $m$ is $M_{1,n}(\hat{\mathcal{U}})_m$.

**Lemma 42.** Let $(A, \Psi, 0)$ be a solution of (1) over $M \setminus Z$ such that $|\Psi|$ has a continuous extension to $M$ and $Z = |\Psi|^{-1}(0)$. Then the following holds:

(i) There is a neighbourhood $U$ of $Z$ such that $L$ is trivial over $U$;

(ii) For any closed oriented surface $\Sigma \subset U \setminus Z$ we have $\langle \Phi_0[\Sigma], c_1(\det S^\vee) \rangle = 0$.

**Proof.** By Lemma 32 (i), $Z$ is a compact locally graph-like subset of $M$. Hence, there is $f \simeq id_M$ such that $f(Z)$ is an embedded graph. Then $L$ must be trivial over a regular neighborhood $V$ of $f(Z)$. Therefore, $f^* L \cong L$ is trivial over $U := f^{-1}(V)$. This proves (i). Part (ii) follows immediately from (i) taking into account that $L|_{M \setminus Z} \cong \Phi^* \det S^\vee$. \qed

Recall [HW15, App. A] that each solution $(A, \Psi, 0)$ of (1) determines a Fueter section of $\mathcal{M}$ on the complement of the zero locus of $\Psi$. Given a pair $(\mathcal{I}, Z)$, where $\mathcal{I}$ is a Fueter section of $\mathcal{M}$ defined on the complement of $Z \subset M$, one can ask whether $(\mathcal{I}, Z)$ can be obtained from a solution of (1) with $\tau = 0$. This situation is considered in the following corollary.
Corollary 43. Let $\mathcal{I}$ be a Fueter section of $\mathfrak{M}$ over $M \setminus Z$, where $Z = \bigsqcup Z_i$ is a link in $M$ and each $Z_i \cong S^1$. Let $U_i$ be a tubular neighborhood of $Z_i$. Denote $a_i := \deg (\Phi_0^* \det S|_{\partial U_i})$. If $a(\mathcal{I}, Z) := \sum_i a_i^2 \neq 0$, then $\mathcal{I}$ cannot be lifted to a solution of (1) over $M \setminus Z$ with $\tau = 0$. □

Given a link $Z$ in $M$, this corollary clearly yields restrictions on homotopy classes of sections of $\mathfrak{M}|_{M \setminus Z}$ representable by solutions of (1). At present, to the best of author’s knowledge, the only known examples of Fueter sections with values in $\tilde{M}_{1,n}$ are those representable by solutions of (1), see Section 4 below. In particular, the question whether $a(\mathcal{I}, Z)$ is a non-trivial obstruction for the existence of a solution of (1) with $\tau = 0$ representing $(\mathcal{I}, Z)$ remains open.

In Theorem 44 below we assume that $\Phi_0$ can be extended to all of $M$ as a continuous map. In particular, $\Phi_0^* \det S'$ is trivial in a neighborhood of $Z$ so that obstructions based on Lemma 42 must vanish. In particular, for $n = 2$ we have $\text{Gr}_0(\mathbb{C}^2) = \{pt\}$ so that the existence of $\Phi$ becomes trivial.

Notice that by [HW15, Prop. 0.1] for any solution $(A, \Psi, 0)$ on $M \setminus Z$ we can construct a solution of the Seiberg–Witten equation with two spinors, say $(\hat{A}, \hat{\Psi}, 0)$, on $M \setminus Z$. More precisely, $\hat{\Psi}$ is a homomorphism from a trivial rank two bundle into $\mathfrak{S} \otimes \mathcal{L} \otimes \mathcal{K}^{1/2}$, where $\mathcal{K} = \det(\ker \Psi)^\perp \cong \Phi_0^* \det S$. Denote by $(\theta, or)$ the flow on $Z$ associated with $(\hat{A}, \hat{\Psi}, 0)$ as in Theorem 36.

Theorem 44. Let $Z$ be a blow-up set for the Seiberg–Witten equations with multiple spinors. Let $(A, \Psi, 0)$ be a corresponding solution of (1) over $M \setminus Z$. Assume the associated map $\Phi_0$ given by (40) admits a continuous extension $\Phi : M \to \text{Gr}_{n-2}(\mathbb{C}^n)$. Then $(A, \Psi, 0)$ induces a flow $(\theta, or)$ on $Z$ such that

$$[Z, \theta, or] = \text{PD}(c_1(L)) + \text{PD}(c_1(\Phi^* \det S)).$$

**Proof.** The hypothesis of this theorem implies that $\Phi^* \det S$ is an extension of $\mathcal{K}$ to $M$; Then, arguing just like at the beginning of this section, we can show that $Z$ is the zero locus of a continuous section $s$ of $\mathcal{L}^2 \otimes \Phi^* \det S \cong L \otimes \Phi^* \det S$; Moreover, similar arguments to the ones used in the proof of Lemma 32 (ii) show that $(Z, s)$ has rescaling limits at each point. The proof of this theorem is finished by appealing to Proposition 20. □

**Remark 45.** Even though a trivialization of $E$ simplifies somewhat the discussion above, it may be also of interest to outline necessary modifications in the case when $E$ is not trivialized. For example, this may be of interest in the four-dimensional setting.

Thus, denote $Q = \text{SU}(E) \times \text{SU}(\mathcal{S})$ the principal $\mathcal{G} := \text{SU}(n) \times \text{SU}(2)$–bundle. Notice that $\text{SU}(n)$ acts on $\tilde{M}_{1,n}$ by the change of frame and we can define a “twisted version” of (39) by

$$\mathfrak{M} := Q \times_G \tilde{M}_{1,n} \to M.$$ 

The map $\zeta$ is $\text{SU}(n)$–equivariant, which implies that we have a projection

$$\mathfrak{M} \to \text{Gr}_{n-2}(E),$$

where $\text{Gr}_{n-2}(E)$ is the bundle over $M$ with the fiber $\text{Gr}_{n-2}(E_m)$ at $m \in M$. The total space of $\text{Gr}_{n-2}(E)$ is equipped with the “tautological vector bundle” $S$, whose pull-back to $\text{Gr}_{n-2}(E_m)$ is the tautological vector bundle of this Grassman manifold. Under these circumstances, $\Phi_0$ is interpreted as a section of $\text{Gr}_{n-2}(E) \to M$. The rest goes through essentially without any modifications.
4 A construction of Fueter sections with values in $\hat{M}_{1,2n}$

Recall that $\mathcal{S}$ denotes the spinor bundle associated with a fixed Spin-structure on $M$. Notice that $\mathcal{S}$ is equipped with a quaternionic structure, i.e., a complex antilinear automorphism $J$ s.t. $J^2 = -\text{id}$.

**Proposition 46.** Let $\psi_1, \ldots, \psi_n$ be harmonic spinors. Denote $\Psi = (\psi_1, J\psi_1, \ldots, \psi_n, J\psi_n)$. Then $(\Psi, 0, 0)$ is a solution of (1) for the following data: $\mathcal{L} = \mathbb{C}$ and $\vartheta$ denotes the product connection, $E = \mathbb{C}^n$ and $B$ is also the product connection.

**Proof.** We only need to show that $\mu \circ \Psi = 0$. It is in turn enough to prove this for $n = 1$. Indeed, if $n = 1$, by the definition of $\mu$ we have

$$\mu(\psi, J\psi) = \langle J\psi, \cdot \rangle J\psi + \langle \psi, \cdot \rangle \psi - |\psi|^2.$$  \hfill (47)

Pick a point $m \in M$ and assume without loss of generality that $\psi(m) \neq 0$. Then $(\psi(m), J\psi(m))$ is a complex basis of the fiber $\mathcal{S}_m$ and any $\varphi \in \mathcal{S}_m$ can be represented as

$$\varphi = \frac{1}{|\psi(m)|^2} (\langle J\psi(m), \varphi \rangle J\psi(m) + \langle \psi(m), \varphi \rangle \psi(m)).$$

Substituting this in (47), we obtain $\mu(\psi, J\psi) = 0$. \hfill $\square$

Combining Proposition 46 with [HW15, Prop. A.1], we obtain the following result.

**Corollary 48.** Any $n$-tuple of harmonic spinors $(\psi_1, \ldots, \psi_n)$ defines a Fueter–section $\mathcal{I}$ with values in $\hat{M}_{1,2n}$ away from the common zero locus $Z := \psi_{1}^{-1}(0) \cap \cdots \cap \psi_{n}^{-1}(0)$.

**Remark 49.** Recall that $\hat{M}_{1,2}$ is isometric to $\mathbb{C}^2 \setminus 0/\pm 1$ with its flat metric. Hence, $\hat{M} = \hat{M}_{1,2}$ can be identified with $\mathcal{S} \setminus 0/\pm 1$, where 0 denotes the zero-section. Tracing through the construction, one can see that the Fueter section of Corollary 48 for $n = 1$ is the $\mathbb{Z}/2$-harmonic spinor obtained by projecting a classical harmonic spinor.

In the remaining part of this section we construct examples of solutions of (1) with $\tau = 0$ satisfying the hypotheses of Theorem 44. To this end, pick a harmonic spinor $\psi$ and assume that its zero locus is a smooth curve. Choosing suitable coordinates around any fixed $m \in \psi^{-1}(0)$, we can think of $\psi$ as a map $\mathbb{R}^3 \to \mathbb{C}^2$ such that $\psi^{-1}(0)$ is the $x_3$-axis. Assume in addition that the metric on $\mathbb{R}^3$ is the product metric with respect to the decomposition $\mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}^1$.

It follows from the harmonicity of $\psi$ that along the $x_3$-axis $\psi$ has an expansion of the form $\psi = \psi^{(N)}(x_3)w^N + o(|w|^N)$, where $w = x_1 + ix_2$ and $\psi^{(N)}(0) \neq 0$ with $1 \leq N < \infty$.

Let $(\psi_1, \ldots, \psi_n)$ be an $n$-tuple of harmonic spinors such that $Z := \psi_{1}^{-1}(0) \cap \cdots \cap \psi_{n}^{-1}(0)$ is a smooth curve. By the above consideration, write $\psi_j = \psi_j^{(N_j)}(x_3)w^{N_j} + o(|w|^{N_j})$ and assume without loss of generality that $N_1 = \min\{N_j\}$. Since

$$\Phi_0(w, x_3) = \left\{ y \in \mathbb{C}^n \mid \sum_{j=1}^n (y_{2j-1}\psi_j(w, x_3) + y_{2j}J\psi_j(w, x_3)) = 0 \right\}$$

and $(\psi_1^{(N_1)}(0), J\psi_1^{(N_1)}(0))$ is a basis of $\mathbb{C}^2$, the map $\Phi_0$ extends across the $x_3$-axis in a neighborhood of the origin.

Concrete examples can be constructed on $M = \Sigma \times S^1$ equipped with the product metric starting from a pair of harmonic spinors on $\Sigma$ with a common zero point.
5 Concluding remarks

As we have already seen in Lemma 32, a blow-up set \( Z \) for the Seiberg–Witten equation is locally graph-like. After the preliminary version of this preprint has been published, B. Zhang [Zha17] proved that the zero locus of a \( \mathbb{Z}/2 \) harmonic spinor is rectifiable (in dimension four). By [Tau14], \( Z \) contains an open everywhere dense subset \( I \), which is locally a subset of a Lipschitz graph. In particular, this implies that \( H_1(B_r(z) \setminus Z; \mathbb{Z}) \) is either trivial or isomorphic to \( \mathbb{Z} \) for all \( z \in I \) provided \( r \) is sufficiently small. Hence, the construction of Section 2.4 yields a multiplicity function \( \theta \) on \( I \) and an orientation on the subset where \( \theta \) does not vanish. This motivates the following.

Conjecture 50. Any blow-up set for the Seiberg–Witten equations with two spinors in dimension three admits a structure of an integer multiplicity rectifiable current, whose homology class is the Poincaré dual to the first Chern class of the determinant line bundle.

Let \( Z \) and \((A, \Psi, 0)\) be as in the setting of Definition 3. As we have already noticed at the beginning of Section 3, \( A \) is flat provided \( n = 2 \). Hence, if \((A, \Psi, 0)\) appears as the limit of a sequence \((A_k, \Psi_k, \tau_k)\) of the Seiberg–Witten monopoles with two spinors, then the corresponding sequence of curvatures \( F_{A_k} \) converges to some sort of \( \delta \)-function supported on \( Z \). More precisely, it seems likely that \( Z \) admits a structure of an integer multiplicity rectifiable current, such that the sequence \( F_{A_k} \) converges to this current. While the construction described in this preprint in general does not produce a multiplicity function and an orientation with this property (since we may change a solution \((A, \Psi, 0)\) by a gauge transformation over \( M \setminus Z \) and in general the resulting structure depends on this gauge transformation), the author believes that the choice of gauge may be fixed to obtain the desired property and intends to study this question further.

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