Effects of free will and massive opinion in majority rule model

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Abstract

We study the effects of free will and massive opinion of multi-agents in a majority rule model wherein the competition of the two types of opinions is taken into account. To address this issue, we consider two specific models (model I and model II) involving different opinion-updating dynamics. During the opinion-updating process, the agents either interact with their neighbors under a majority rule with probability $1 - q$, or make their own decisions with free will (model I) or according to the massive opinion (model II) with probability $q$. We investigate the difference of the average numbers of the two opinions as a function of $q$ in the steady state. We find that the location of the order-disorder phase transition point may be shifted according to the involved dynamics, giving rise to either smooth or harsh conditions to achieve an ordered state. For the practical case with a finite population size, we conclude that there always exists a threshold for $q$ below which a full consensus phase emerges. Our analytical estimations are in good agreement with simulation results.

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For a long time in history simple statistical models are used to study complex biological, social, and geological phenomena [1, 2, 3]. Recently, a great deal of efforts are devoted to mathematical modelling of collective behaviors of individuals, particularly opinion spreading and formation, by using for example two-state interacting spin models [4, 5, 6, 7, 8, 9]. In this context, the common formulation assumes that the agents are located at the nodes of a network, and are endowed with two states, i.e., spin up and spin down, which mimic human attitudes or decisions: like/dislike, agree/disagree, accept/reject, etc. The interactions are assumed to take place only between linked nodes, where the links represent the relations between individuals such as friendship, coauthorship, neighborhood, etc. Through interactions, the agents can update their opinions by ways of convincement [5], imitation [8], following local majority [4, 6, 7], and so on. Generally, the main concern of these models is whether there arises an order-disorder phase transition of the system dynamics [4, 5] and, if yes, how the time needed to attain consensus depends on the system size [6], the initial configuration [5, 6, 9], or the topology of the underlying network [7, 9].

Very recently, some extensions of the majority rule model (MRM) [4, 6] were proposed and investigated within different scenarios [10, 11, 12]. In Ref. [10], Lambiotte studied a variant of the MRM on heterogeneous networks, namely dichotomous networks, which are composed of two kinds of nodes characterized by their distinct link degrees, $k_1$ and $k_2$. It was found that the degree heterogeneity (characterized by the ratio $\gamma = k_1/k_2$) affects the location of the order-disorder phase transition point and that the system exhibits non-equipartition of average opinion between the two kinds of nodes. The effects of community structure on opinion spreading and formation in the framework of MRM was also considered by Lambiotte and coworkers [11]. Motivated by the fact that a social system is inhomogeneous in many aspects of its inherent nature, Guan et al. introduced two types of agents in the MRM where one type of agents have less ability to persuade the other [12]. It was shown that, as the inhomogeneous effect is strengthened, the location of phase transition point is shifted along the direction where the ordered state is more difficult to be realized.

In the present paper, we continue the research in line of Ref. [12] to study the effects of free will and massive opinion of multi-agents in the MRM. The motivation comes from the following observations. In some cases, our opinions are strongly influenced by our social surroundings (the opinions of our friends, colleagues and neighbors), e.g., the fact that a majority of our acquaintances are smokers will easily convince ourselves to smoke; the fact
that a large number of our friends have MySpace will likely urge us to apply for one account. This phenomenon can be modelled by implementing a local majority rule when the agents update their opinions. In other cases, however, we make decisions in a way less dependent on the others, such as the type of coffee we like to drink, the fashion of clothes we like to wear, etc., where our taste, or “free will”, dominates. In some other cases, not only the local majority opinion but also the global one would affect much of our behavior. Therefore, it is important to investigate how the free will and massive opinion of multi-agents influence the processes of their opinion spreading and formation.

To proceed, we first introduce our model of opinion dynamics. The population is composed of \( N \) agents located on a fully connected network. This simple setting allows us to look for a mean-field analytical solution for the problem. Initially, two types of opinions, \( S_1 \) and \( S_2 \), are assigned to the agents with an equal probability. The densities of populations holding opinions \( S_1 \) and \( S_2 \) are denoted by \( \rho_1 \) and \( \rho_2 \), respectively. We investigate two types of models (model I and model II) involving two different opinion-updating dynamics. At each time step, one agent is randomly selected to update its opinion state. Following Refs. [10, 11, 12], two types of processes may take place: (i) With probability \((1 - q)\), the selected agent interacts with two neighboring agents randomly selected from its neighborhood, and the three agents adopt the opinion of the local majority. The magnitude of \((1 - q)\) characterizes the frequency of confrontation, or the strength of aggressiveness [3]. This step is compulsory for both models I and II. (ii) With probability \( q \), the selected agent picks an opinion by using one of the following two different strategies.

In model I, we assume that where there is no neighboring-interaction happening, an agent updates its opinion state according to its own will. In particular, the selected agent may change its current opinion to the opposite one with a free will whose strength is weighed by \( \alpha \in [0, 1] \). Smaller values of \( \alpha \) implies stronger confidence for the agent to stick to its current opinion. In model II, we assume that during updating, the selected agent has a tendency to adopt the global massive opinion with a probability proportional to its corresponding population density in the system:

\[
P(S_1) = \frac{\rho_1^\beta}{\rho_1^\beta + \rho_2^\beta},
\]

where \( \beta \) characterizes the strength of the massive opinion. Larger values of \( \beta \) correspond to stronger tendency to become the majority. Hereafter, we always denote the global massive
opinion by $S_1$. When there exist only neighboring interactions, i.e., in the case of $q = 0$, it is already known that the system asymptotically reaches global consensus where all agents share the same opinion [6]. In the following, we investigate how the phase diagram of the system varies when the free will and massive opinion of agents are taken into account in the case of $q \neq 0$. In this case, the quantity $q$ serves as the control parameter in the dynamical process over the network.

For model I, it is easy to write down the following mean-field rate equation for the system:

$$A_{t+1} = A_t + (1 - q)W + q[\alpha(1 - a) - \alpha a], \quad (1)$$

where $A_t$ is the average number of agents with opinion $S_1$ at time $t$, $a = A_t/N$ is the corresponding average proportion of agents with this opinion, and $W$ is the total contribution to the evolution of $A_t$ due to neighboring interactions. The term proportional to $(1 - q)$ accounts for local majorities and the last term, for the flips with free will. The probability for two agents with opinions $S_1(S_2)$ and one with $S_2(S_1)$ to be selected is $3a^2(1 - a)$ and $3a(1 - a)^2$, respectively, so that

$$W = 3a^2(1 - a) - 3a(1 - a)^2 = -3a(1 - 3a + 2a^2). \quad (2)$$

So, the evolution equation for $A_t$ becomes

$$A_{t+1} = A_t + (1 - q)[-3a(1 - 3a + 2a^2)] + q\alpha(1 - 2a). \quad (3)$$

Due to the existence of symmetry for the two opinions, it is easy to see that $a = 1/2$ is always a trivial stationary solution of the above equation. In order to find nontrivial solutions of the system, we use the method proposed in [10]. Instead of considering directly the quantity $A_t$, we rewrite Eq. (3) as follows and consider the quantities $\Delta = A - N/2$ and $\delta = a - 1/2$:

$$\Delta_{t+1} = \Delta_t + \frac{\delta}{2} [3 - (3 + 4\alpha)q - 12(1 - q)\delta^2]. \quad (4)$$

We can easily verify that the symmetric solution $a = 1/2$ ceases to be stable when $q < 3/(3 + 4\alpha)$, and in that case the system reaches the following asymmetric solutions:

$$\rho_{1,2} = a_{1,2} = \frac{1}{2} \pm \frac{\sqrt{3 - (3 + 4\alpha)q}}{12(1 - q)}. \quad (5)$$
For convenience, we define the order parameter of the system by the difference between the average density of the majority and that of the minority, in the steady state, as follows:

$$\rho_1 - \rho_2 = \sqrt{\frac{3 - (3 + 4\alpha)q}{3(1 - q)}}. \quad (6)$$

Thus, the system undergoes an order-disorder transition at the critical point

$$q_c(\alpha) = \frac{3}{3 + 4\alpha}. \quad (7)$$

Below this value, one type of opinion dominates the other, or an ordered collective phenomenon emerges due to the interactions between neighboring agents. According to Eq. (6), the location of the order-disorder phase transition point depends strongly on the value of $\alpha$, i.e., the strength of the free will of the agents to change their current opinion states. The stronger the inclination (to make a change) is, the more difficult an ordered collective behavior emerges. Evidently, model I is equivalent to the MRM considered in [10] with $\alpha = 0.5$, and in that case, $q_c = 3/5$.

To test the above analysis, we perform computer simulations on model I and compare the simulation results with the analytical estimations in Fig. 1. Simulations were carried out for a population of $N = 5 \times 10^3$ agents located on the sites of a fully connected network. We

FIG. 1: The order parameter $\rho_1 - \rho_2$ versus the control parameter $q$ for several values of $\alpha$. The solid lines are the analytical solutions obtained by calculating Eq. (6). The symbols are the simulation results obtained from averaging over twenty independent experiments within a population of size $N = 5 \times 10^3$.  

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FIG. 2: The order parameter $\rho_1 - \rho_2$ versus the control parameter $q$ for several values of $\beta$. The solid lines are the analytical solutions obtained by calculating Eq. (11). The symbols are the simulation results obtained from averaging over twenty independent experiments within a population of size $N = 5 \times 10^3$.

study the order parameter $\rho_1 - \rho_2$ as a function of the control parameter $q$. Initially, the two opinions $S_1$ and $S_2$ are randomly distributed among the agents with equal probability $1/2$. After evolution, the system reaches a dynamic equilibrium state where the densities of the opinions fluctuate stably. In our simulations, one Monte Carlo step is accomplished after all agents have updated their opinions. The simulation results were obtained by averaging over the last $10^4$ Monte Carlo steps out of $10^5$. From Fig. 1 we can see that the simulation results are in very good agreement with the analytical solutions. The small differences between the simulation results and the analytical predictions are due to the finite-size effect.

Now, we consider model II, for which the mean-field rate equation for $A_t$ is given by

$$A_{t+1} = A_t + (1 - q)W + q\frac{(1 - a)a^\beta - a(1 - a)^\beta}{a^\beta + (1 - a)^\beta},$$

(8)

where the formulation of $W$ is the same as that in model I. As assumed above, the global majority opinion has a greater attraction than the minority one, therefore we restrict our attention to the region of $\beta \in [0, 1]$. The last term in Eq. (8) accounts for the flips influenced by the massive opinion, i.e., the global majority opinion of the population. Since the average density of the majority opinion $a$ is less than unity (except for the full consensus case where $a = 1$), we can approximately solve Eq. (8) by using of the following truncated
Taylor expansions:

\[(1 - a)^\beta = 1 - \beta a + \mathcal{O}(a^2),\]
\[a^\beta = 1 - \beta(1 - a) + \mathcal{O}(a^2).\]  

(9)

Substituting (2) and (9) into (8), with some algebraic calculations, we get

\[A_{t+1} = A_t + (1 - q)[-3a(1 - 3a + 2a^2)] + q\gamma(1 - 2a),\]  

(10)

where \(\gamma = (1 - \beta)/(2 - \beta)\).

The remaining part is the same as what has been done for model I. We hence omit the lengthy calculations and simply write out the final result for the order parameter of model II:

\[\rho_1 - \rho_2 = \sqrt{\frac{3 - (3 + 4\gamma)q}{3(1 - q)}} = \sqrt{\frac{3 - [3 + 4(1 - \beta)/(2 - \beta)]q}{3(1 - q)}}.\]  

(11)

Thus, in model II, the system undergoes an order-disorder transition at the critical point

\[q_c(\beta) = \frac{3(2 - \beta)}{3(2 - \beta) + 4(1 - \beta)} = \frac{6 - 3\beta}{10 - 7\beta}.\]  

(12)

![FIG. 3: The order parameter $\rho_1 - \rho_2$ versus the control parameter $q$ for different system sizes $N$. The parameter $\beta = 0.9$ is fixed. The arrows point to the locations of phase transition obtained by simulations.](image)
For $\beta = 0$, our model reduces to the MRM studied in [10], so it is not surprising to recover the known result of $q_c = 3/5$. While for other positive values of $\beta$ less than unity, we obtain a threshold of $q$ (at which the order-disorder transition takes place) greater than $3/5$. That is, if the information of global massive opinion is accessible to the agents who have tendency to becoming the majority, it is easier for the whole system to attain an ordered state.

The simulation results of the order parameter $\rho_1 - \rho_2$ as a function of the control parameter $q$ for several values of $\beta$ are shown in Fig. 2. The analytical solutions of Eq. (11) are also plotted there for comparison. Once again, they well match with each other. The differences between the simulation and analytical results come from two aspects: one is the finite-size effect, and the other is the approximation of Eq. (9). From Fig. 2 we also notice that for any large enough value of $\beta$, there arises another phase transition where a full consensus state emerges as the parameter $q$ goes below some critical value $q'_c$, i.e., one of the two opinions dominates the whole system for any $q < q'_c$.

We argue that the fact that the considered system is of finite size contributes to the phase transition. In fact, reviewing Eq. (8), we find that $A_{t+1}$ will be going to $N$ iff $q$ satisfies the following condition:

$$q < f(a) = \frac{3a(1-a)(2a-1)}{3a(1-a)(2a-1) + \frac{(1-a)a^\beta-a(1-a)^\beta}{a^\beta+(1-a)^\beta}}. \quad (13)$$

Note that $a$ is the average density of the majority, $a \in [0.5, 1]$. In this region of $a$, it is easy to verify that $f(a)$ is a monotonically decreasing function for positive $\beta$. Thus, as long as $q$ is smaller than the minimal value $f_{min}$ of $f(a)$, the whole system will evolve to a full consensus state.

From an ecological point of view, the number of species of a population must be kept above a minimum level to prevent going to extinction due to random fluctuations of environmental conditions. Inspired by this viewpoint, for the present MRM, we judge that the number of the minority should not be smaller than 2, below which they are doomed to extinct (since two agents of the same opinion is guaranteed to convince a neighbor), but above which they have a chance (though very small) to survive. Thus, a reasonable estimation is $f_{min} = f(1-2/N)$. In Fig. 3 we present simulation results for three systems with different sizes, $N = 5 \times 10^3$, $10^4$, and $2 \times 10^4$, respectively. We can see that the critical value of $q'_c$ is decreasing with the increase of the system size. For $\beta = 0.9$, the simulations yielded that $q'_c = 0.73, 0.70, \text{ and } 0.66$ for $N = 5 \times 10^3, 10^4, \text{ and } 2 \times 10^4$, respectively, which are in good accordance with the
estimations of \( f(1 - 2/N) = 0.717, 0.692, \) and \( 0.665 \) for the three corresponding values of \( N \).

To sum up, we have investigated the effects of free will and massive opinion of multi-agents in a network based on the majority rule model, which is a simple yet useful statistical model for studying the emergence of collective behaviors. Two types of models have been considered. In model I, the agents have a free will to either adopt the opposite opinion or remain its current opinion, whereas in model II, besides the local majority, the agents also have a tendency to becoming the global majority when there are no neighboring interactions among them. We found that the location of the order-disorder phase transition point strongly depends on the involved dynamics, which may give rise to either smooth or harsh conditions to achieve an ordered state. In addition, for finite system sizes, we found that there exists a threshold below which a full consensus can be attained if the agents have an inclination to becoming the global majority. Our analytical estimations are in good agreement with simulation results. In the present work, however, we only studied the majority rule model over fully connected networks. To model the real world in a more accurate fashion, future research may take into account the coevolution of the majority process and the underlying interaction pattern, i.e., from the viewpoint of adaptive and co-evolutionary networks [14].

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