Schwarzschild and linear potentials in Mannheim’s model of conformal gravity

Peter R. Phillips
Department of Physics, Washington University, St. Louis, MO 63130

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ABSTRACT
We study the equations of conformal gravity, as given by Mannheim, in the weak field limit, so that a linear approximation is adequate. Specialising to static fields with spherical symmetry, we obtain a second-order equation for one of the metric functions. We obtain the Green function for this equation, and represent the metric function in the form of integrals over the source. Near a compact source such as the Sun the solution no longer has a form that is compatible with observations. We conclude that a solution of Mannheim type (a Schwarzschild term plus a linear potential of galactic scale) cannot exist for these field equations.

Key words: gravitation – cosmology: theory

1 INTRODUCTION
In this paper we will derive solutions, in the weak field limit, of the field equations of conformal gravity as given by Mannheim (2006, equation 186); (this paper will be referred to as PM from now on):

\[ 4\alpha g W^{\mu\nu} \equiv 4\alpha g \left[ W^{\mu\nu}_{(2)} - \frac{1}{3} W^{\mu\nu}_{(1)} \right] = T^{\mu\nu}. \]  

(1)

The tensor \( W^{\mu\nu} \) is derived by variation of the Weyl action, defined in PM (182). Its two separate parts, \( W^{\mu\nu}_{(1)} \) and \( W^{\mu\nu}_{(2)} \), are defined in PM (107) and (108); these definitions are repeated here:

\[ W^{\mu\nu}_{(1)} = 2g^{\mu\nu} (R^\alpha_{\nu})_{;\beta} - 2(R^\alpha_{\nu})^{\mu\nu} - 2R^\alpha_{\nu} R^{\mu\nu} + \frac{1}{2} g^{\mu\nu} (R^\alpha_{\nu})^2 \]  

\[ W^{\mu\nu}_{(2)} = \frac{1}{2} g^{\mu\nu} (R^\alpha_{\nu})_{;\beta} + R^{\mu\nu}_{;\beta} - R^{\mu\beta}_{;\nu} - R^{\nu\beta}_{;\mu} + \frac{1}{2} g^{\mu\nu} R_{;\beta} R^\alpha_{\nu} R^\beta_{\alpha}. \]  

(2)

\( \alpha g \) in (1) is a dimensionless coupling constant. (We adopt the notation of Weinberg (1972), with units such that \( c = \hbar = 1 \).)

The energy-momentum tensor, \( T^{\mu\nu} \), is derived from an action principle involving a scalar field, \( S \) (see PM (61)). Appropriate variation of this action yields \( T^{\mu\nu} \) as given in PM (64). In Mannheim’s model, the solutions of the field equations undergo a symmetry breaking transition (SBT) in the early Universe, with \( S \) becoming a constant, \( S_0 \). Making this change in PM (64) we obtain

\[ T^{\mu\nu} = -\frac{1}{6} S_0^2 \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha_{\alpha} \right) - g^{\mu\nu} \lambda S_0^4 + T_M^{\mu\nu}, \]  

where \( T_M^{\mu\nu} \) is the matter tensor, containing all the usual fermion and boson fields. From now on we will ignore the term \( g^{\mu\nu} \lambda S_0^4 \), because we are not concerned with the Hubble expansion.

We break from Mannheim’s development at this point. The factor 1/6 in (3) derives from the original, conformally invariant action. A SBT, however, will not in general preserve such relations, and we will instead write

\[ T^{\mu\nu} = -4\alpha g \eta \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha_{\alpha} \right) + 4\alpha g \xi T_M^{\mu\nu}, \]  

(4)

so that the field equations can be written

\[ W^{\mu\nu} + \eta \left( R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha_{\alpha} \right) = \xi T_M^{\mu\nu}. \]  

(5)

\( \xi \) is dimensionless, but \( \eta \) has dimension length\(^{-2} \), so its magnitude can be written \( |\eta| = 1/r_0^2 \), where \( r_0 \) divides lengths into two regimes, in one of which \( (r < r_0) \) \( W^{\mu\nu} \) is dominant, and in the other \( (r > r_0) \) the Einstein tensor, \( R^{\mu\nu} - g^{\mu\nu} R^\alpha_{\alpha}/2 \).

We will call these equations the Weyl-Einstein equations, or “W-E equations” for short. We will not try to justify these equations; Mannheim has written extensively in support of them. We are concerned only with some of their consequences.

In the important special case that \( \alpha g W^{\mu\nu} \) is negligible, or even identically zero, we obtain equations of Einstein form:

\[ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R^\alpha_{\alpha} = \frac{\xi}{\eta} T_M^{\mu\nu}. \]  

(6)

If \( \xi/\eta = -8\pi G_0 \), where \( G_0 \) is the usual Newtonian gravita-
tional constant, we regain the usual Einstein equations, as given, for example, in Weinberg (1972, equation 16.2.1).

In the opposite limit, \( \eta \to 0 \), we obtain
\[
W^\mu_\nu = \xi T^\mu_\nu,
\]
the Bach equations. Some solutions of these have been obtained by Fiedler & Schimming (1980).

We can take the trace of (5), to get
\[
R^\alpha_\alpha = -\frac{\xi}{\eta} T^\alpha_\alpha,
\]
which is, of course, the same as we would get from the Einstein equations since \( W^\mu_\nu \) is traceless.

From this Mannheim derives a traceless energy-momentum tensor, PM (65). We shall not use this, however, because it contains less information than the original tensor, and must be supplemented by the trace equation.

No exact solutions of the W-E equations seem to be available, except for the usual Schwarzschild solution, which satisfies both the Einstein and the Bach equations independently.

In this paper we will seek a solution of the W-E equations in the limit of weak fields. This should be adequate for studies of galactic rotation and gravitational lensing, and may give us insight into what a more complete solution would look like. We will be particularly interested in the Solar System (SS), for which the most incisive observations exist. We are trying to construct a theory that is similar to Mannheim’s, so we will choose values of parameters that seem likely to bring this about.

We can locate the present paper within the context provided by several papers about conformal gravity, both critical and supportive, that have appeared in recent years. Mannheim (2007) has responded to the critique of Flanagan (2006); this debate will not concern us here. Several papers, (Edery & Paranjape 1998; Walker 1994; Yoon 2013), have shown that a linear potential of Mannhein type is incompatible with observations; in the present paper we show that a linear potential is not a consequence of the the field equations of conformal gravity, so these critiques are not needed.

Gegenberg et al. (2017) have used fourth-order gravity to try to understand inflation in the early Universe and accelerated expansion at late times (see also Gegenberg et al. 2016; Gegenberg & Seahra 2018). This work deals with the Universe in the large, and is not directly connected with the present paper, which considers objects of the scale of galaxies or the Solar System.

We note also recent work on the elimination of ghosts in fourth-order theories (Bender & Mannheim 2008a,b,c); the presence of ghosts had previously been a major obstacle to the development of such theories. These papers deal with the quantum mechanical aspects of fourth-order theories; the present treatment is purely classical.

2 STATIC FIELDS WITH SPHERICAL SYMMETRY

We now specialize further, to static fields with spherical symmetry. Like Fiedler and Schimming, but apparently independent of them, Mannheim & Kazanas (1989) addressed the problem of the solution of the Bach equations under these conditions. They found that in addition to the usual \( 1/r \) term of the Schwarzschild solution there was a term \( \gamma r \). Mannheim has used this linear potential to obtain a fit to the rotation curves of galaxies; for a recent paper, see Mannheim & O’Brien (2012). However, the relevant field equations in the Mannheim model are not the Bach equations, but the W-E equations, for which a linear potential is not a solution. It is therefore not clear what use a linear potential can be in such studies, except as an approximation over a limited range.

3 FIELD EQUATIONS IN THE LINEAR APPROXIMATION

The most general form for a static metric with spherical symmetry is given in Weinberg (1972, equation 8.1.6):
\[
d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2).
\]

For weak fields we write \( A(r) = 1 + a(r) \) and \( B(r) = 1 + b(r) \), where \( a(r) \) and \( b(r) \) are assumed small compared to unity, so that only terms linear in \( a(r) \) and \( b(r) \) need be considered.

We will be considering a source such as the Sun, with density \( \rho(r) \) and pressure \( p(r) \). Within such a source, the pressure terms in the field equations are much smaller than the density terms and will normally be omitted. The trace equation then becomes, with primes denoting differentiation with respect to \( r \):
\[
\eta R^\alpha_\alpha = -\xi T^\alpha_\alpha = \xi \rho(r) \tag{10}
\]
\[
-\frac{\eta}{2\pi^2} \left( 4a - 4rb' + 4ra' - 2r^2b'' \right) = \xi \rho(r) \tag{11}
\]
\[
2(r a')' - (r^2 b')' = -\frac{2\pi \rho(r)}{\eta}. \tag{12}
\]

We assume the density is a smooth, monotonically decreasing function, so that \( a'(r) \) and \( b'(r) \) are both zero at \( r = 0 \). Then we can integrate out from the origin to \( r \) to get
\[
2ra - r^2 b' = -\frac{\xi}{4\pi \eta} \int_0^r 4\pi u^2 \rho(u) \, du = -\frac{\xi}{4\pi \eta} m_c(r), \tag{13}
\]
where \( m_c(r) \) is the enclosed mass out to \( r \).

The \( r, r \) component of the W-E equations gives
\[
-3b'' - 2r^2 b'' - r^2 a'' + 2rb' + 2a + 3r^2 \eta (-rb' + a) = 0. \tag{14}
\]
Before going further, we can check that the Schwarzschild solution is a possible solution of (14) and the trace equation, (11). This solution is characterised by \( A(r) = 1/B(r) = 1 + \beta/r \), i.e. \( a(r) = -b(r) = \beta/r \); we will call this relation the Schwarzschild condition. 2 Substituting these expressions into (14) and (11), we can verify that the equations are satisfied.

1 For the geometrical calculations we have used GRTEXTORII followed by a MAPLE script to extract the linear terms.
2 A few years ago this writer speculated (Phillips 2015) that a second solution of the W-E equations might also satisfy these Schwarzschild conditions. The present paper suggests this idea is mistaken.
In the limit $\alpha \rightarrow \infty$ ($\eta \rightarrow 0$), the Weyl tensor is everywhere dominant. The trace equations are irrelevant, and (14) admits the solution
\[
a(r) = -b(r) = \gamma r,
\]
the Mannheim linear potential.

These are not, however, the only possibilities, and we will now derive a different form for $a(r)$ and $b(r)$. The solution we will obtain does not, of course, guarantee that a corresponding solution exists for the full nonlinear W-E equations. But it does provide a limiting form, for weak fields, of such a solution, if it exists.

We will now transform (14) and the trace equation to get a second-order equation in $a(r)$ only. Differentiating (11):
\[
-4a'' - 2ra'' + 4r b'' + r^2 b''' + 2b' = \frac{2r^2 c(r)}{\eta} + \frac{r^2 c'(r)}{\eta}.
\]
Combining this with (14) we can eliminate $b'''(r)$:

\[
-3r^2 a'' - 2ra'' + 2a + 2r^2 b'' + 4r b' + 3r^2 \eta (-rb' + a) = \frac{2r^2 c(r)}{\eta} + \frac{r^2 c'(r)}{\eta}.
\]

We can now use (11) and (13) to eliminate all terms involving $b(r)$, to arrive at

\[
a'' + \left(2 - \frac{2}{r^2} + \eta \right) a = -\frac{6}{4\pi} m(r) = \frac{r^2 c'(r)}{3\eta} \equiv -\mathcal{H}.
\]

Associated with this equation is the homogeneous equation

\[
a'' + \left(2 - \frac{2}{r^2} + \eta \right) a = 0.
\]

The appearance of the derivative of the density on the right side of (18) reminds us that although this equation is second order, it originates in the third-order equation (14), and therefore will probe more intimately into the density distribution than we are familiar with in conventional general relativity. Indeed, as we will see in sections 10–12, it is precisely this term, through the associated integrals $\mathcal{H}_2$ and $\mathcal{J}_2$, that causes us the most trouble in practice.

At this point we choose $\alpha \eta < 0$, and therefore $\eta < 0$. This will ensure that we deal with modified Bessel functions, which have a particularly simple form. Also, because in this paper we are looking for solutions analogous to Mannheim’s linear potential, we will assume, tentatively, that the length $r_0 = 1/k$ is of galactic scale, intermediate between the scale of the SS and truly cosmological scales.

We will develop the solution of (18) as an integral over the source density, using a Green function constructed from the related homogeneous equation, which now can be written

\[
a'' - \left(\nu^2 - \frac{1}{r^2} + k^2 \right) a = 0,
\]
with $\nu = 3/2$ and $k^2 = -\eta > 0$. We use the notation of Abramowitz & Stegun (1970) (AS in what follows). For the solution of (20), AS, equation 9.1.49, and the paragraph preceding 9.6.41, gives $a(r) = r^3/2 \mathcal{E}_3/2(kr)$, where $\mathcal{E}_\nu$ stands for $I_\nu$ or $K_\nu$. We can express these solutions in terms of spherical Bessel functions. Setting $kr = z$, and using AS, equations 10.2.13 and 10.2.17, we define:

\[
a_1(z) \equiv \frac{2}{\pi} z^{1/2} I_{1/2}(z) = \left(\frac{\sinh z}{z} + \cosh z\right),
\]
\[
a_K(z) \equiv \frac{2}{\pi} z^{1/2} K_{1/2}(z) = \left(1 + \frac{1}{z}\right) e^{-z},
\]
\[
W[a_1(z), a_K(z)] = a_1(z) \frac{da_K(z)}{dz} - \frac{da_1(z)}{dz} a_K(z) = -1 \quad \text{(Wronskian)}.
\]

Associated with these metric functions are $b_1(z)$ and $b_K(z)$, obtained by integrating the trace equation (13):

\[
b_1(z) = \frac{2 \sinh z}{z},
\]
\[
b_K(z) = -\frac{2e^{-z}}{z}.
\]

If we can find a Particular Integral (PI) of our equation (18), the general solution is the PI plus a Complementary Function (CF) that is a solution of the homogeneous equation (19). This construction is useful only if a PI can actually be found, but this turns out to be the case for our problem.

### 4 THE GREEN FUNCTION

We will consider two basic forms for our Green function, $G_1(y, z)$ and $G_2(y, z)$, where $y$ and $z$ are both positive, $y$ refers to the source point, and $z$ to the field point. We will use a range of radii from $r_{\min}$ to $r_{\max}$, and anticipate that we may be able to let $r_{\min}$ tend to zero, if all metric functions are regular at the origin. But we will have to be cautious about letting $r_{\max}$ tend to infinity if we have to deal with a potential that rises indefinitely, like Mannheim’s linear potential.

For a compact source such as the Sun, our main concern in this paper, $G_1$ has the form (see Arfken & Weber 1995):

\[
G_1(y, z) = \begin{cases} a_1(y) a_K(z) / k, & r_{\min} \leq y < z, \\ a_K(y) a_1(z) / k, & z < y < r_{\max}. \end{cases}
\]

$a(r)$ is then given as the integral:

\[
a(r) = \int_{r_{\min}}^{r_{\max}} G_1(kr, kt) H(t) \, dt.
\]

Our second Green function is

\[
G_2(y, z) = \begin{cases} -a_K(y) a_1(z) / k, & r_{\min} \leq y < z, \\ -a_1(y) a_K(z) / k, & z < y < r_{\max}. \end{cases}
\]

with $a(r)$ now given by

\[
a(r) = \int_{r_{\min}}^{r_{\max}} G_2(kr, kt) H(t) \, dt.
\]

More generally, we can consider a combination of $G_1$ and $G_2$.
\[ G(y, z) = PG_1(y, z) + (1 - P)G_2(y, z), \]  
\[ \text{where } P \text{ can be chosen to satisfy the constraints of problem.} \]

The condition that the metric functions be regular at the origin is easily met, and the conditions at \( r_{\text{max}} \) are what we are trying to discover; the important constraint is that in the SS the metric functions satisfy what we will call the Schwarzschild condition, to be discussed more fully later, in section 9.

5 THE SOURCE

In this paper we will be mainly concerned with the SS, so our source is the Sun. However, when dealing with any compact system with a long-range potential we have to worry about contributions from distant matter. Let us call this the “embedding problem”. For galaxies, Mannheim has proposed a solution (see Mannheim 2006, section 9.3), and there is a long tradition stemming from the paper of Einstein & Strauss (1945) (the “Einstein vacuole”).

But we have to devise a reasonable embedding for the SS. We should not simply assume a compact mass at the center (the Sun) surrounded by vacuum. The SS is embedded in a galaxy (the Milky Way), with a mean background center (the Sun) surrounded by vacuum. We should not simply assume a compact mass at the center (the Sun) surrounded by vacuum. The SS is embedded in a galaxy (the Milky Way), with a mean background center (the Sun) surrounded by vacuum. So it is reasonable to suppose, at least in the linear approximation, that what best characterises the source is the difference between \( \rho_b \) and the actual density in the SS. For simplicity let us suppose the SS formed by the collapse of material enclosed within a sphere of radius \( r_v \) (\( r_\star \) for vacuole), while material at a larger radius is unaffected. \( r_v \), of course, is much larger than \( r_\star \), the radius of the Sun. Then we have

\[ M_\odot = 4\pi \int_{r_v}^{r_\star} r^2 \rho(r) \, dr = \frac{4\pi \rho_b r_v^3}{3}. \]  
\[ \text{In the range } r_\star < r < r_v \text{ we must use a constant negative density, } -\rho_b. \]

6 NUMBERS

We give here some numbers for the SS and other objects; this is mainly for orientation, since we will not make much actual use of the numbers:

- background density, \( \rho_b = 1.5 \times 10^{-23} \text{ g cm}^{-3} \)
- mean solar density, \( \rho_s = 1.4 \text{ g cm}^{-3} \)
- radius of the Sun, \( r_\star = 7 \times 10^{10} \text{ cm} \)
- radius of Neptune’s orbit, \( r_N = 4.5 \times 10^{14} \text{ cm} \)
- radius of the vacuole, \( r_v = 3.6 \times 10^{18} \text{ cm} \)
- size of the Galaxy, \( r_0 = 10^{23} \text{ cm} \).

We will take \( r_{\text{max}} \) to be \( 100r_0 = 10^{25} \text{ cm} \). The two regions of interest are

- \( \mathcal{R}_1, r_\star < r < r_v \): this region includes the SS.
- \( \mathcal{R}_2, r_v < r < r_{\text{max}} \): this region is outside all sources.

For the sake of completeness, we will retain terms involving \( \rho_b \). We note here, however, that in the SS the effect of these terms is very small; for example, at \( r = r_N \), the related acceleration is about seven orders of magnitude smaller than that associated with the Pioneer anomaly (Turyshev et al. 2012), which is itself near the limit of detectability.

7 THE METRIC IN REGION \( \mathcal{R}_1 \), GENERAL CONSIDERATIONS

In this region, we have

\[ m_e(r) = M_\odot - \frac{4\pi \rho_b r^3}{3} \quad \text{and} \quad \rho' = 0, \]

and a suitable PI for (18) is

\[ a_{\text{PI}}(r) = \frac{\xi M_\odot}{4\pi k^2 r} - \frac{\xi \rho_b r^2}{3k^2}. \]  
\[ \text{Since } r_0 \text{ is assumed to be of galactic scale, } kr \text{ is much less than unity for } r \text{ within the SS.} \]

Associated with this metric function is \( b_{\text{PI}}(r) \), obtained by integrating the trace equation (13):

\[ b_{\text{PI}}(r) = -\frac{\xi M_\odot}{4\pi k^2 r} - \frac{\xi \rho_b r^2}{6k^2} + \mathcal{E}, \]

where \( \mathcal{E} \) is an integration constant chosen to ensure \( b_{\text{PI}}(r_0) = 0 \):

\[ \mathcal{E} = \frac{\xi M_\odot}{4\pi k^2 r_v} + \frac{\xi \rho_b r_v^2}{6k^2}. \]  
\[ \text{In equations (34) and (35) the most important terms, for } r \text{ within the SS, are the first ones. If we neglect the others, we see that these first terms give us just the Schwarzschild solution.} \]

We will write the CF as

\[ \text{CF} = B_1 a_1 (kr) + B_K a_K (kr). \]  
\[ \text{We can verify that our original third-order equation, (14), is solved not only by the PI pair, } a_{\text{PI}}(r), b_{\text{PI}}(r), \text{ but also by the CF pairs } a_1(r), b_1(r) \text{ and } a_K(r), b_K(r). \]

Our Green function will determine the constants \( B_1 \) and \( B_K \). It will be constructed to satisfy certain conditions, the most important of which is discussed in section 9.

8 THE METRIC IN REGION \( \mathcal{R}_2 \), GENERAL CONSIDERATIONS

In this region, \( m_e(r) = \rho' = 0 \), so the PI is also zero. The general behaviour of the PI across the boundary at \( r = r_v \) is shown in figure 1. The graph, with its discontinuity in slope, is reminiscent of the potential in problems of electrostatics.

The CF, however, being the solution of a homogeneous equation, is independent of sources, and will show no discontinuity of any kind across the boundary.

9 THE SCHWARZSCHILD CONDITION

Observations tell us that to high accuracy the metric in the SS is of Schwarzschild form, according to which \( A(r) = 1/B(r) \), or, for weak fields, \( a(r) = -b(r) \). This we will call the Schwarzschild condition. A solution that is pure PI represents the usual Schwarzschild solution, so we have to ask what CF we can add and still preserve the Schwarzschild condition.
This figure illustrates the behavior of the PI’s, using (upper curve) $a_{\pi} = 1/r - r^2$ for $r < 1$, and $a_{\pi} = 0$ for $r > 1$, and (lower curve) $b_{\pi} = -1/r - r^2/2 + 3/2$ for $r < 1$, and $b_{\pi} = 0$ for $r > 1$.

Making $B_1$ non-zero is allowed, providing it is not too large, because $a_I(z) \approx z^2/3$ for small $z$. (See section 12 for a discussion of $B_1$). But with $B_K$ we have to be careful, because $a_K(z) \approx 1/z$, similar to the PI. So let us examine a solution of the form

$$a(r) = B_1 + B_K a_K (kr) + \frac{\xi M_\odot}{4 \pi k^2 r} \left[ 1 + \frac{1}{kr} \right] e^{-kr}$$

for small $kr$. We can derive $b(r)$ from (13). For points in the SS,

$$b'(r) = \frac{2a}{r} + \frac{\xi M_\odot}{2 \pi \eta r^2} + \frac{\xi M_\odot}{4 \pi k^2 r^2} + B_K \left( \frac{2}{kr^2} - \frac{1}{kr} \right)$$

We note that the terms derived from the PI satisfy the Schwarzschild condition, as expected, but the terms involving $B_K$ do not. Our Green function must therefore be constructed to minimise $|B_K|$.

10 THE METRIC IN REGION $\mathcal{R}_1$, USING $G_1$

We will define our Green function over the region $r_{\text{min}} < r < r_e$, i.e. we set $r_{\text{max}} = r_e$. In practice, we will be able to let $r_{\text{min}} \to 0$ in the end, so this region is larger than region $\mathcal{R}_1$, because it includes $0 < r < r_e$, the interior of the Sun. Note that the range of $G_1$ does not include the step in density at $r_e$; let us use the notation $r_{\text{min}}$ to emphasise this point.

In the body of this paper we assume the Weyl limit within the SS, $kr \ll 1$. But in appendix A we check the Green function $G_1$ using the Einstein limit, $kr \gg 1$, where we know the solution.

Our metric function in region $\mathcal{R}_1$, $a(r)$, can be written as four integrals:

$$a(r) = a_<(r) + a_>(r) ,$$

where

$$a_<(r) = \frac{a_K (kr)}{kr} [H_1 (r) + H_2 (r)]$$

$$H_1 (r) = \int_{r_{\text{min}}}^{r} a_1 (kt) \left[ \frac{\xi}{4 \pi t} m_e (t) \right] dt$$

$$H_2 (r) = \int_{r_{\text{min}}}^{r} a_t (kt) \left[ \frac{\xi \rho (t)}{3 \eta} \right] dt$$

and

$$a_>(r) = \frac{a_K (kr)}{kr} [H_3 (r) + H_4 (r)]$$

$$H_3 (r) = \int_{r}^{r_e} a_K (kt) \left[ \frac{\xi}{4 \pi t} m_e (t) \right] dt$$

$$H_4 (r) = \int_{r}^{r_e} a_t (kt) \left[ \frac{\xi \rho (t)}{3 \eta} \right] dt .$$

The last of our four integrals, $H_4$, is clearly zero. The first, $H_1$, will be split by writing

$$H_1 = \frac{\xi}{4 \pi} \int_{kr}^{kr_e} \frac{a_I (z)}{z} \left( \frac{M_\odot - 4 \pi \rho_0 z^3}{3 k^3} \right) dz + H_{\text{sun}} ,$$

where

$$H_{\text{sun}} = \frac{\xi}{4 \pi} \int_{kr_{\text{min}}}^{r_e} \frac{a_I (kt)}{t} m_e (t) dt$$

depends on the density distribution within the Sun, as modelled in the Appendix.

The integrals in (45) are straightforward:

$$H_1 = \frac{\xi}{4 \pi} \int_{kr}^{kr_e} \frac{a_I (z)}{z} \left( M_\odot - 4 \pi \rho_0 z^3 \right) dz + H_{\text{sun}} ,$$

where

$$H_{\text{sun}} = \frac{\xi}{4 \pi} \int_{kr_{\text{min}}}^{r_e} \frac{a_I (kt)}{t} m_e (t) dt$$

It will be convenient in what follows to divide $H_1$ further, according to the limits on $I_1 (z)$:

$$H_1 = H_5 + H_6 + H_{\text{sun}} ,$$

where

$$H_5 = \frac{\xi}{4 \pi} \int_{kr}^{kr_e} \frac{a_K (kr)}{kr} \left( \frac{M_\odot - 4 \pi \rho_0 z^3}{3 k^3} \right) dz .$$

For $H_2$ we set $a_I (kt)$ equal to its limiting value for small $kt$, namely $k^2 t^2 / 3$:

$$H_2 = \frac{\xi k^2}{9 \eta} \int_{r_{\text{min}}}^{r} t^3 \rho (t) dt$$

$$= \frac{\xi}{9} \left[ t^3 \rho (t) \right]_{r_{\text{min}}}^{r} - \frac{\xi k^2}{12 \pi} \left[ \frac{1}{r_{\text{min}}} - \int_{r_{\text{min}}}^{r} 3 t^2 \rho (t) dt \right]$$

Finally, $H_3$ evaluates to:

$$H_3 = \frac{\xi}{4 \pi} \int_{kr}^{kr_e} \frac{a_K (z)}{z} \left( \frac{M_\odot - 4 \pi \rho_0 z^3}{3 k^3} \right) dz .$$

As with $H_1$, the integrals in (51) are straightforward, and lead to

$$H_3 = \frac{\xi}{4 \pi} \int_{kr}^{kr_e} \frac{a_K (z)}{z} \left( \frac{M_\odot - 4 \pi \rho_0 z^3}{3 k^3} \right) dz .$$
We divide $H_3$ according to the limits on the integral:

$$H_3 = H_\mu + H_\nu , \quad \text{where}$$

$$H_\mu = -\frac{\xi}{4\pi} H_3(\xi = kr)$$

$$H_\nu = \frac{\xi}{4\pi} H_3(\xi = kr \rightarrow 0) .$$

We note that $H_\mu$ and $H_\nu$ are functions of $r$, and together produce the PI of (34).

$H_2$, $H_5$, $H_8$ and $H_{sun}$ are constants, so we will write the CF for this Green function as

$$CF = C_1 a_1(kr) + C_K a_K(kr) , \quad \text{where}$$

$$C_1 = \frac{1}{k} H_8$$

$$C_K = \frac{1}{k} (H_2 + H_5 + H_{sun}) .$$

We can omit the term $H_{sun}$, which is smaller by a factor of order $k^2 r^2 \approx 1$ than the other two terms (see appendix C). This gives:

$$C_K = \frac{1}{k} (H_2 + H_5) = -\frac{\xi M_\odot}{6\pi k} .$$

For the metric function $a(r)$ in the SS we can omit the small contribution from $C_1$ and the term in $\rho_0$ in the PI:

$$a(r) = \frac{\xi M_\odot}{4\pi k^2 r} - \frac{\xi M_\odot}{6\pi k} \left( 1 + \frac{1}{kr} \right) e^{-kr} \approx \frac{\xi M_\odot}{12\pi k^2 r} , \quad \text{for} \quad kr \ll 1 .$$

If we ignore terms in $\rho_0$ for the moment, and simply use (56) in (13), we can integrate to get

$$b(r) = -\frac{\xi M_\odot}{4\pi k^2 r} + \frac{\xi M_\odot}{3\pi k^2 r} e^{-kr} ,$$

which reduces, in the SS, to

$$b(r) \approx \frac{\xi M_\odot}{12\pi k^2 r} . \quad \text{(59)}$$

$a(r)$, from (56), and $b(r)$, from (58), are plotted in figure 2.

$C_K$ is non-zero, so $G_1$ by itself is not an acceptable Green function; it does not lead to the Schwarzschild condition in the SS, $b(r) = -a(r)$. Instead we have $b(r) \approx a(r)$.

### 11 THE METRIC IN REGION $\mathcal{R}_1$, USING $G_2$

Here we recapitulate the previous section, with appropriate changes in the integrals.

Our metric function in region $\mathcal{R}_1$, $a(r)$, can be written as four integrals:

$$a(r) = a_<(r) + a_>(r) , \quad \text{(60)}$$

where

$$a_<(r) = -\frac{a_1(kr)}{k} [\mathcal{J}_1(r) + \mathcal{J}_2(r)]$$

$$\mathcal{J}_1(r) = \int_{r_{min}}^{r_1} a_K(kt) \left[ \frac{\xi}{4\pi t} m_e(t) \right] dt \quad \text{(61)}$$

$$\mathcal{J}_2(r) = \int_{r_{min}}^{r_1} a_K(kt) \left[ \frac{\epsilon \rho_t(t)}{3\eta} \right] dt , \quad \text{(62)}$$

and

$$a_>(r) = -\frac{a_1(kr)}{k} [\mathcal{J}_3(r) + \mathcal{J}_4(r)]$$

$$\mathcal{J}_3(r) = \int_{r_{min}}^{r_1} a_1(kt) \left[ \frac{\xi}{4\pi t} m_e(t) \right] dt \quad \text{(63)}$$

$$\mathcal{J}_4(r) = \int_{r_{min}}^{r_1} a_1(kt) \left[ \frac{\epsilon \rho_t(t)}{3\eta} \right] dt . \quad \text{(64)}$$

The last of our four integrals, $\mathcal{J}_4$, is clearly zero. The first, $\mathcal{J}_1$, will be split by writing:

$$\mathcal{J}_1 = \frac{\xi}{4\pi} \int_{kr_0}^{kr_1} a_K(z) \left( M_\odot - \frac{4\pi \rho_0 z^3}{3k^3} \right) dz + J_{sun} . \quad \text{(65)}$$

where

$$J_{sun} \approx \frac{\xi}{4\pi} \int_{r_{min}}^{r_1} a_K(kt) \frac{\epsilon \rho_t(t)}{t} dt \quad \text{(66)}$$

depends on the density distribution within the Sun, as modelled in the Appendix.

The integrals in (65) are straightforward, and result in

$$\mathcal{J}_1 = \frac{\xi}{4\pi} \int_{kr_0}^{kr_1} a_1(z) \left( M_\odot - \frac{4\pi \rho_0 z^3}{3k^3} \right) dz + J_{sun}$$

$$\mathcal{J}_1(z) = -M_\odot \frac{e^{-z}}{z} + \frac{4\pi \rho_0}{3k^3} \left( z^2 + 3z + 3 \right) e^{-z} . \quad \text{(67)}$$

It will be convenient in what follows to divide $\mathcal{J}_1$ further, according to the limits on $\mathcal{J}_1(z)$:

$$\mathcal{J}_1 = \mathcal{J}_5 + \mathcal{J}_6 + J_{sun} , \quad \text{where}$$

$$\mathcal{J}_5 = -\frac{\xi}{4\pi} \mathcal{J}_1(kr_0) \quad \text{(68)}$$

$$\mathcal{J}_6 = \frac{\xi}{4\pi} \mathcal{J}_1(kr_1) . \quad \text{(69)}$$

For $\mathcal{J}_2$ we set $a_K(kt)$ equal to its limiting value for small $kt$, namely $1/(kt)$:

$$\mathcal{J}_2 = \frac{\xi}{3\eta} \int_{r_{min}}^{r_1} \left( \frac{1}{kt} \right) t \rho_t dt \quad \text{→} \quad \int_{r_{min}}^{r_1} -\frac{\xi}{3\eta k} \rho(0) . \quad \text{(70)}$$

Finally, $\mathcal{J}_3$ evaluates to:

$$\mathcal{J}_3 = \frac{\xi}{4\pi} \int_{kr_0}^{kr_1} a_1(z) \left( M_\odot - \frac{4\pi \rho_0 z^3}{3k^3} \right) dz . \quad \text{(71)}$$
As with \( J_1 \), the integrals in (51) are straightforward, and lead to
\[
\mathcal{J}_s = \frac{\xi}{4\pi} \mathcal{J}_3(z) \Big|_{z=kr}^{z=kr-}\]
\[
\mathcal{J}_3(z) = \frac{M_\odot}{z} \sinh \frac{z}{2} - \frac{4\pi \rho_s}{3k^3} \left( \frac{z}{2} \sinh \frac{z}{2} - 3z \cosh \frac{z}{2} + 3 \sinh \frac{z}{2} \right). \tag{72}
\]

We divide \( \mathcal{J}_3 \) according to the limits on the integral:
\[
\mathcal{J}_3 = \mathcal{J}_r + \mathcal{J}_G, \quad \text{where}
\]
\[
\mathcal{J}_r = -\frac{\xi}{4\pi} \mathcal{J}_3(z = kr)
\]
\[
\mathcal{J}_G = \frac{\xi}{4\pi} \mathcal{J}_3(z = kr -)). \approx \frac{\xi M_\odot}{4\pi}. \tag{73}
\]

We note that \( \mathcal{J}_r \) and \( \mathcal{J}_G \) are functions of \( r \), and together produce the PI of (34).
\( \mathcal{J}_r, \mathcal{J}_G, \mathcal{J}_G \) and \( \mathcal{J}_G \) are constants, and (as for \( \mathcal{H}_\odot \)) \( \mathcal{J}_\odot \) is negligible in comparison to the others. So we will write the CF for this Green function as
\[
\mathcal{C}F = D_1 a_1(kr) + D_2 a_2(kr), \quad \text{where}
\]
\[
D_1 = -\frac{1}{k} (\mathcal{J}_r + \mathcal{J}_G)
\]
\[
D_2 = \frac{1}{4\pi k}. \quad \text{(74)}
\]

\( D_2 \) is non-zero, so \( G_2 \) by itself is not an acceptable Green function; it does not lead to the Schwarzschild condition in the SS.

### 12 THE METRIC IN REGION \( R_1 \), USING A LINEAR COMBINATION OF \( G_1 \) AND \( G_2 \)

Both \( c_K \) and \( D_K \) are non-zero, so we have to construct a linear combination of \( G_1 \) and \( G_2 \), \( G = PG_1 + (1-P)G_2 \). This preserves the PI, and for the coefficient \( B_K \) in the CF we get
\[
B_K = PC_K + (1-P)D_K
\]
\[
= -\frac{\xi M_\odot}{12\pi k^3} [2P + 3(1-P)]
\]
\[
= 0 \quad \text{if} \quad P = 3, \tag{75}
\]
so that the coefficient \( B_I \) is given by
\[
B_I = 3C_I - 2D_I
\]
\[
= \frac{3C}{k} (J_2 + J_6)
\]
\[
= \frac{3\xi}{4\pi k} \left( -\frac{M_\odot}{kr_\odot} \frac{4\pi \rho_s}{k^3} \right)
\]
\[
+ \frac{k}{2} \left[ \frac{\xi \rho_s(0)}{3k^3} - \frac{\xi M_\odot}{4\pi k^3} - \frac{4\pi \rho_s}{k^3} \right]. \tag{76}
\]

On the right side of this equation we need retain only the dominant term, the one derived from \( J_2 \):
\[
B_I = -\frac{2\xi \rho_s(0)}{3k^3}, \tag{77}
\]
so our CF is
\[
\mathcal{C}F = -\frac{2\xi \rho_s(0)}{3k^3} a_l(kr)
\]
\[
\approx -\frac{2\xi \rho_s(0)r^2}{9\eta}. \tag{78}
\]

Within the SS. Combining this with the PI, we obtain for the metric function \( a(r) \) within the SS:
\[
a(r) \approx \frac{\xi M_\odot}{4\pi \eta^2} \frac{2\xi \rho_s(0)r^2}{9\eta}, \tag{79}
\]
where we have omitted the small term in \( \rho_s \).

The second term on the right side of (79) will produce the long-range potential analogous to the linear potential of Mannheim. But we can see already that it is unacceptably large. With our assumption that the density is a maximum at the origin, and is monotonically decreasing, \( \rho(0)r^2 \) is of order \( M_\odot \). So this second term is of order \( \xi M_\odot / (\eta^2 r^2) \), and the magnitude of the second term exceeds that of the first, Newtonian, term by a factor of about \( (r/r_s)^3 \). Taking \( r \) to be the radius of the Earth’s orbit, this is \( 1500^3 \approx 3 \times 10^8 \).

Mannheim (2006) takes the coefficients of the Schwarzschild and the linear potentials to be independent, and determined from observation. Our Green function approach, on the other hand, shows these coefficients are connected, so that once we know the Schwarzschild coefficient (Mannheim’s \( b^s \)) we know not only the PI but also the CF, depending on our choice of Green function. This CF, moreover, turns out to be in conflict with observations of the Sun System.

### 13 CAN THE W-E EQUATIONS REPRESENT REALITY?

At the beginning of this paper we pointed out that a linear potential, used by Mannheim & O’Brien (2012) in their studies of galactic rotation, was not a solution of the W-E equations, which are the relevant field equations for Mannheim’s model. We then began to search for a solution of the W-E equations that might include a term that approximates a linear potential. Specialising to weak, static fields with spherical symmetry, and choosing the critical length \( r_0 \) to be of galactic scale, we used a Green function approach to construct the solution of the linearised W-E equations for a compact source such as the Sun (assumed to have a density that is a monotonically decreasing function of radius). This solution is in conflict with observations of the Solar System; either it does not have the required Schwarzschild form, or it has an unacceptably large contribution from the long-range function \( a_l(kr) \).

This does not mean, however, that the W-E equations are useless. We should simply discard our initial assumption, that \( r_0 = 1/k \) is of galactic scale. Indeed, it would be surprising if a SBT resulted in so large a value of \( r_0 \). More likely would seem to be a value of order of the size of elementary particles, \( 10^{-15} \) m or less. In this case the Einstein equations would be adequate at all scales accessible to experiment, and the Schwarzschild solution would be appropriate for the Solar System. We have seen in appendix A that our Green function correctly identifies the Schwarzschild solution in this limit.

The W-E equations could still have important theoretical applications, however, because at the highest energies we expect the SBT to be reversed, so that we recover the original conformal form in which all coupling constants are dimensionless. The theory is then potentially renormalisable.
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APPENDIX A: CHECKING THE GREEN FUNCTION IN THE EINSTEIN LIMIT

In the Einstein limit, $|\xi|$ and $|\eta|$ both become large, but in such a way that their ratio stays the same. For $r$ in the SS, $kr \gg 1$. Our metric function, $a(r)$, can be written in terms of the Green function $G_1$:

$$a(r) = a_c(r) + a_s(r)$$

$$a_c(r) = \frac{a_K(kr)}{k} \int_{r_{min}}^{r} a_I(kt)$$

$$\times \left[ \frac{\xi}{4\pi t} m_c(t) + \frac{t\xi\rho'(t)}{3\eta} \right] \, dt$$

$$a_s(r) = \frac{a_I(kr)}{k} \int_{r_{min}}^{r} a_K(kt)$$

In the large $z$ limit,

$$a_c(z) \to \frac{e^z}{2}$$

$$a_s(z) \to e^{-z},$$

so $a_c(r)$ and $a_s(r)$ can be written

$$a_c(r) = \int_{r_{min}}^{r} e^{k(r-t)} \left[ \frac{\xi}{4\pi t} m_c(t) + \frac{t\xi\rho'(t)}{3\eta} \right] \, dt$$

$$a_s(r) = \int_{r}^{r_{min}} e^{k(r-t)} \left[ \frac{\xi}{4\pi t} m_c(t) + \frac{t\xi\rho'(t)}{3\eta} \right] \, dt.$$  \hspace{1cm} (A1)

The exponentials are sharply peaked around $t = r$, so we can write

$$a_c(r) = \frac{1}{2k^2} \left[ \frac{\xi}{4\pi r} m_c(r) + \frac{r\xi\rho'(r)}{3\eta} \right]$$

$$a_s(r) = \frac{\xi M_\odot}{4\pi k^2 r} - \frac{\xi \rho s r^3}{3k^2},$$  \hspace{1cm} (A4)

in agreement with the PI of (34). There is no CF in this approximation, and the Schwarzschild condition is satisfied.

APPENDIX B: MODELLING THE DENSITY OF THE SUN

We will model the density of the Sun as a Gaussian:

$$\rho(r) = \rho_s \exp \left( -\frac{s_s r^2}{r_s^2} \right). \hspace{1cm} (B1)$$

Requiring that this density decreases by a factor of $10^4$ as we go from $r = 0$ to $r = r_s$ gives us $s_s = 9.2$.

The enclosed mass out to radius $r$ can be shown to be

$$m_e(r) = \frac{4\pi \rho_s r_s^2 E(z)}{s_s^{3/2}} \hspace{1cm} \text{where}$$

$$E(z) = -\frac{\sqrt{2}}{2} e^{-z} + \frac{\sqrt{\pi}}{4} \text{erf}(z).$$  \hspace{1cm} (B2)

The total mass, $M_\odot$, is given by

$$M_\odot = \frac{4\pi \rho_s r_s^2}{s_s^{3/2}} \int_{0}^{\sqrt{\pi}} z^2 e^{-z^2} \, dz$$

$$\approx 0.2 \rho_s r_s^2.$$  \hspace{1cm} (B3)
APPENDIX C: ESTIMATING THE MAGNITUDE OF $\mathcal{H}_{\text{sun}}$

\[
\mathcal{H}_{\text{sun}} = \frac{\xi}{4\pi} \int_{r_{\text{min}}}^{r_s} \frac{a_l(kt)}{t} m_e(t) \, dt
\]
\[
\approx \frac{\xi k^2}{4\pi} \int_{r_{\text{min}}}^{r_s} \frac{1}{3} m_e(t) \, dt. \quad (C1)
\]

We can let $r_{\text{min}} \to 0$, and take $m_e(r)$ from the previous section:

\[
\mathcal{H}_{\text{sun}} \approx \frac{\xi k^2 r_s^2 \rho_s}{3s^{3/2}} \int_0^{r_s} t E(z) \, dt \quad (C2)
\]

with $z = \sqrt{s t/r_s}$.

We are concerned now only with orders of magnitude. The integral in the previous equation is $O(r_s^2)$, so

\[
\mathcal{H}_{\text{sun}} \text{ is of order } \xi M_\odot k^2 r_s^2. \quad (C3)
\]

The presence of the very small factor $k^2 r_s^2$ shows that $\mathcal{H}_{\text{sun}}$ is negligible in comparison with quantities such as $\mathcal{H}_2 = \xi M_\odot/(12\pi)$.

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