QUANTITATIVE UNIQUENESS OF CONTINUATION RESULT RELATED TO HOPF’S LEMMA

MOURAD CHOULLI, FAOUZI TRIKI, AND QI XUE

Abstract. The classical Hopf’s lemma can be reformulated as uniqueness of continuation result. We aim in the present work to quantify this property. We show precisely that if a solution $u$ of a divergence form elliptic equation attains its maximum at a boundary point $x_0$ then both $L^1$-norms of $u - u(x_0)$ on the domain and on the boundary are bounded, up to a multiplicative constant, by the exterior normal derivative at $x_0$.

1. Introduction

Let $\Omega$ be a $C^{1,1}$ bounded domain of $\mathbb{R}^n$ ($n \geq 2$) with boundary $\Gamma$. All functions we consider are assumed to be real valued.

Fix $\kappa > 1$, $0 < \beta < 1$ and let $\Sigma$ be the set of functions $\sigma = (\sigma^{ij}) \in C^{1,\beta}(\Omega, \mathbb{R}^{n \times n})$ satisfying $\sigma^{ji} = \sigma^{ij}$, $1 \leq i, j \leq n$,

$$\kappa^{-1} |\xi|^2 \leq \sigma \xi \cdot \xi \quad \text{for each } \xi \in \mathbb{R}^n \quad \text{and} \quad ||\sigma||_{C^{1,\beta}(\Omega, \mathbb{R}^{n \times n})} \leq \kappa.$$ 

We associate to any $\sigma \in \Sigma$ the operator $L_\sigma$ acting as follows

$$L_\sigma u = -\text{div}(\sigma \nabla u), \quad u \in C^2(\Omega).$$

Define

$$\mathcal{J} = \{ u \in C^2(\Omega) \cap C^1(\overline{\Omega}); \ L_\sigma u = 0 \text{ for some } \sigma \in \Sigma \}$$

and set

$$M(u) = \{ x \in \overline{\Omega}; \ u(x) = \max_{\overline{\Omega}} u \}, \quad u \in \mathcal{J}.$$ 

When $x \in \Gamma$ we denote by $\nu(x)$ the unit normal vector to $\Gamma$ pointing outward $\Omega$. Let $u \in \mathcal{J}$ so that $\nabla u \neq 0$. According to the strong maximum principle $M(u) \subset \Gamma$ (e.g. [5, Theorem 3.5, page 35]) and by Hopf’s Lemma\(^1\), \( \partial_\nu u(x) = \nabla u(x) \cdot \nu(x) > 0 \) for any $x \in M(u)$ (e.g [5, Lemma 3.4, page 34]).

The first result of this kind goes back to the pioneering paper by Zaremba [11] and generalized later independently by Hopf [8] and Oleinik [9]. We refer to the nice historical survey in the introduction of [1] including results with less regularity on the domain and the coefficients of the operators under consideration.

Hopf’s lemma can be rephrased as uniqueness of continuation result: let $u \in \mathcal{J}$ and $x_0 \in M(u)$. If $\partial_\nu u(x_0) = 0$ then $u$ is identically equal to $u(x_0)$.

The following theorem quantify this uniqueness of continuation property.

2010 Mathematics Subject Classification. 35B50, 35C15, 35J08, 35J15.

Key words and phrases. Elliptic equation in divergence form, Hopf’s lemma, maximum principle, uniqueness of continuation, Green’s function, Poisson type kernel.

The authors are supported by the grant ANR-17-CE40-0029 of the French National Research Agency ANR (project MultiOnde).

\(^1\)Also called Hopf-Oleinik-Zaremba’s lemma or boundary point lemma.
Theorem 1.1. For any \( u \in \mathcal{S} \) and \( x_0 \in M(u) \) we have
\[
\|u(x_0) - u\|_{L^1(\Omega)} \leq C\|u(x_0) - u\|_{L^1(\Gamma)},
\]
\[
\|u(x_0) - u\|_{L^1(\Omega)} + \|u(x_0) - u\|_{L^1(\Gamma)} \leq C\partial_n u(x_0),
\]
where \( C = C(n, \Omega, \kappa, \beta) > 0 \) is a generic constant.

We used \( C^{1,\beta} \)-regularity of the coefficients of the operator \( L_\sigma \) only in the two-sided inequality (2.8) (which is contained in [7, main Theorem in page 105]). For all other results \( C^{0,1} \)-regularity of the coefficients of the operator \( L_\sigma \) is sufficient. It is not known presently whether the result of [7, main Theorem in page 105] can be extended to coefficients with \( C^{0,1} \)-regularity.

To prove Theorem 1.1 we modify the proof of Hopf's lemma itself, we make use the integral representation
\[
(u(x) - u(x_0) - \int_\Gamma K_\sigma(x, y)(u(x) - u(x_0))dS(y), \quad x \in \Omega,
\]
where \( K_\sigma \) is Poisson type kernel associated to the operator \( L_\sigma \). The proof is completed by showing beforehand two-sided inequality for \( K_\sigma \) involving the weakly singular kernel \( \text{dist}(x, \Gamma)|x - y|^{-n} \).

Theorem 1.1 confirms numerical testing we obtained before. The details of these numerical testing are given in Appendix B.

The rest of this text is organized as follows. In Section 2 we collect some properties of the Green function associated to the operator \( L_\sigma \) and establish two-sided inequality for the Poisson type kernel \( K_\sigma \). We give in Section 3 the proof of Theorem 1.1. We also added two appendices. Appendix A contains the proof of a regularity result we used in Section 2. While Appendix B is devoted to numerical testing.

2. Preliminaries

Define, where \( \kappa > 1 \), \( \Sigma_0 \) as the set of functions \( \sigma = (\sigma^{ij}) \in C^{0,1}(\overline{\Omega}, \mathbb{R}^{n \times n}) \) satisfying \( \sigma^{ij} = \sigma^{ji} \), \( 1 \leq i, j \leq n \),
\[
\kappa^{-1}|\xi|^2 \leq \sigma \xi \cdot \xi \quad \text{for each } \xi \in \mathbb{R}^n \quad \text{and} \quad \|\sigma\|_{C^{0,1}(\overline{\Omega}, \mathbb{R}^{n \times n})} \leq \kappa.
\]

It is worth noticing that according to Rademacher's theorem (e.g. [4, Theorem 2 in page 81]) \( C^{0,1}(\overline{\Omega}, \mathbb{R}^{n \times n}) \) is continuously embedded in \( W^{1,\infty}(\overline{\Omega}, \mathbb{R}^{n \times n}) \).

We associate to \( \sigma \in \Sigma_0 \) the symmetric bounded and coercive bilinear form
\[
a_\sigma(u, v) = (\sigma \nabla u \cdot \nabla v), \quad u, v \in H_0^1(\Omega),
\]
where \( (\cdot | \cdot)_2 \) is the usual scalar product on \( L^2(\Omega) \).

In this section we prove the following result, where
\[
\mathcal{U} = \{(x, y) \in \Omega \times \Omega; \ x \neq y\}.
\]

Theorem 2.1. Let \( \sigma \in \Sigma_0 \). Then
1. There exists a unique \( G_\sigma \in L^1(\Omega \times \Omega) \cap C^1(\mathcal{U}) \) satisfying
\[
a_\sigma(G_\sigma(\cdot, y), v) = v(y), \quad v \in C_0^\infty(\Omega), \quad y \in \Omega,
\]
and \( G_\sigma(x, y) = G_\sigma(y, x), \ (x, y) \in \Omega \times \Omega \).
2. Let \( \omega \subset \omega_0 \subset \Omega \). Then \( G_\sigma(x, \cdot) \in C^{1,\alpha}(\overline{\Omega} \setminus \omega_0) \) for each \( x \in \Omega \), where \( 0 \leq \alpha = \alpha(n) < 1 \) is a constant. We have in addition
\[
\|G_\sigma(x, \cdot)\|_{C^{1,\alpha}(\overline{\Omega} \setminus \omega_0)} \leq c \quad \text{for each } x \in \omega
\].
(2.2) \[ \| G_\sigma(x_1, \cdot) - G_\sigma(x_2, \cdot) \|_{C^{1,\alpha}((\Omega \setminus \omega))} \leq c|x_1 - x_2| \quad \text{for each } x_1, x_2 \in \omega, \]

where \( c = c(n, \Omega, \kappa, \omega, \omega_0) > 0 \) is a generic constant.

Prior to proving this theorem we state a regularity result of the solution of the Dirichlet BVP associated to \( L_\sigma \).

Denote by \( \gamma_0 \) the bounded trace operator from \( H^1(\Omega) \) onto \( H^{1/2}(\Gamma) \) defined by

\[ \gamma_0 u = u_{|\Gamma}, \quad u \in C^\infty(\overline{\Omega}) \]

Let \( \sigma \in \Sigma_0, f \in L^\infty(\Omega) \) and consider the BVP

(2.3) \[ \begin{cases} -\text{div}(\sigma \nabla u) = f & \text{in } \Omega, \\ \gamma_0 u = 0. \end{cases} \]

Then Lax-Milgram’s lemma allows us to conclude that the BVP (2.3) has a unique variational solution \( u = u_\sigma(f) \in H^1_0(\Omega) \):

(2.4) \[ a_\sigma(u, v) = (f|v)_2 \quad \text{for each } v \in H^1_0(\Omega). \]

**Theorem 2.2.** For any \( \sigma \in \Sigma_0 \) and \( f \in L^\infty(\Omega) \), \( u_\sigma(f) \in H^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega}) \), for some \( 0 < \alpha = \alpha(n) < 1 \). Furthermore the following estimate holds

(2.5) \[ \| u_\sigma(f) \|_{H^2(\Omega) \cap C^{1,\alpha}(\overline{\Omega})} \leq C \| f \|_{L^\infty(\Omega)}, \]

where \( C = C(n, \Omega, \kappa) > 0 \) is a constant.

We give the proof Theorem 2.2 in Appendix A.

**Proof of Theorem 2.1.** (1) is contained in [3, Proposition 24 in page 625 and Proposition 26 in page 629].

(2) Fix \( \omega \in \omega_0 \subseteq \Omega \) and \( \phi \in C^\infty_0(\mathbb{R}^n \setminus \omega) \) satisfying \( \phi = 1 \) in a neighborhood of \( \overline{\Omega} \setminus \omega_0 \). Then it is not difficult to check that \( \phi G_\sigma(x, \cdot), x \in \overline{\Omega}, \) is the solution of the BVP (2.3) with

\[ f = 2\sigma \nabla \phi \cdot \nabla G_\sigma(x, \cdot) + \sigma G_\sigma(x, \cdot) \Delta \phi. \]

In light of (1) \( f \in C(\Omega) \) and according to [6, Inequalities (i) and (iv) of Theorem 3.3 in page 305] we have \( f \in L^\infty(\Omega) \). The expected result follows by applying Theorem 2.2. Furthermore from [6, Inequality (1.8) of Theorem 1.1 in page 305] and (2.5) we get

\[ \| G_\sigma(x, \cdot) \|_{C^{1,\alpha}(\overline{\Omega \setminus \omega_0})} \leq c, \quad x \in \overline{\Omega}. \]

Here and henceforward \( c = c(n, \Omega, \kappa, \omega_0, \kappa) \).

Similarly, simple calculations show that \( \phi|G_\sigma(x_1, \cdot) - G_\sigma(x_2, \cdot)|, x_1, x_2 \in \overline{\Omega}, \) is the solution of the BVP (2.3) when

\[ f = 2\sigma \nabla \phi \cdot \nabla [G_\sigma(x_1, \cdot) - G_\sigma(x_2, \cdot)] + \sigma [G_\sigma(x_1, \cdot) - G_\sigma(x_2, \cdot)] \Delta \phi. \]

In light of [6, Inequality (vi) of Theorem 3.3 in page 333] we get by applying the mean value theorem that

\[ \| f \|_{L^\infty(\Omega)} \leq c|x_1 - x_2|. \]

This inequality together with Theorem 2.2 yield

\[ \| G_\sigma(x_1, \cdot) - G_\sigma(x_2, \cdot) \|_{C^{1,\alpha}(\overline{\Omega \setminus \omega_0})} \leq c|x_1 - x_2|, \quad x_1, x_2 \in \overline{\Omega}. \]

The proof is then complete. □
We denote hereafter by $G$ the function $G_\sigma$ when $\sigma$ is identically equal to 1. As usual the Poisson kernel is given by

$$K(x, y) = -\partial_{\nu(y)} G(x, y), \quad x \in \Omega, \ y \in \Gamma.$$ 

Note that according to Theorem 2.1 $K \in C(\Omega \times \Gamma)$.

From [12, Lemma 1 in page 21] we have the following two-sided inequality

$$(2.6) \quad \kappa^{-1} \frac{\dist(x, \Gamma)}{|x - y|^n} \leq K(x, y) \leq \kappa \frac{\dist(x, \Gamma)}{|x - y|^n}, \quad x \in \Omega, \ y \in \Gamma.$$ 

where $\kappa = \kappa(n, \Omega) > 1$ is a constant.

Note that the domain in [12, Lemma 1 in page 21] is assumed to be of class $C^2$ in order to guarantee the uniform interior sphere property but we know that this property is in fact satisfied by $C^{1,1}$-domains (see Section 3).

In the sequel we use the following notation, where $\sigma \in \Sigma_0$,

$$K_\sigma(x, y) = -\partial_{\nu(y)} G_\sigma(x, y), \quad x \in \Omega, \ y \in \Gamma.$$ 

Here again $K_\sigma \in C(\Omega \times \Gamma)$ by Theorem 2.1.

**Proposition 2.1.** For all $\sigma \in \Sigma$ we have

$$(2.7) \quad \kappa^{-1} \frac{\dist(x, \Gamma)}{|x - y|^n} \leq K_\sigma(x, y) \leq \kappa \frac{\dist(x, \Gamma)}{|x - y|^n}, \quad x \in \Omega, \ y \in \Gamma.$$ 

where $\kappa = \kappa(n, \Omega, \kappa) > 1$ is a constant.

**Proof.** In this proof $\kappa = \kappa(n, \Omega, \kappa) > 1$ is a generic constant. Let $\sigma \in \Sigma$. Then the following two sided inequality is contained in [7, main Theorem in page 105]

$$(2.8) \quad -\kappa^{-1} G(x, y) \leq -G_\sigma(x, y) \leq -\kappa G(x, y), \quad x, y \in \Omega.$$ 

Fix $x \in \Omega$ and $y_0 \in \Gamma$. Then for sufficiently small $t$ we have from (2.8)

$$\kappa^{-1} \frac{G(x, y_0 - t\nu(y_0)) + G(x, y_0)}{t} \leq \frac{-G_\sigma(x, y_0 - t\nu(y_0)) + G_\sigma(x, y_0)}{t} \leq \kappa \frac{G(x, y_0 - t\nu(y_0)) + G(x, y_0)}{t}, \quad t > 0,$$ 

where we used that $G(x, y_0) = G_\sigma(x, y_0) = 0$. Passing to the limit when $t$ goes to zero (observe that $G(x, \cdot)$ and $G_\sigma(x, \cdot)$ are $C^1$ up to the boundary) we find

$$-\kappa^{-1} \partial_{\nu(x)} G(x, y_0) \leq -\partial_{\nu(x)} G_\sigma(x, y_0) \leq -\kappa \partial_{\nu(x)} G(x, y_0)$$ 

or equivalently

$$\kappa^{-1} K(x, y_0) \leq K_\sigma(x, y_0) \leq \kappa K(x, y_0).$$

We obtain the expected inequality by using (2.6). \hfill \Box

3. **Proof of Theorem 1.1**

In the sequel we use the fact that $\Omega$, which is $C^{1,1}$, admits the uniform interior sphere condition (e.g. [2, Theorem 1.0.9, page 7]). This means that there exists $r > 0$ so that for any $x \in \Gamma$, there exists $\hat{x} \in \Omega$ with the property that $B(\hat{x}, r) \subset \Omega$ and $\partial B(\hat{x}, r) \cap \Gamma = \{x\}$.

Let $u \in \mathcal{F}$ so that $\nabla u \neq 0$ and pick $x_0 \in M(u)$. Note that if $u$ is constant then (1.2) holds obviously.
Take $\bar{x}_0 \in \Omega$ with the property that $B(\bar{x}_0, r) \subset \Omega$ and $\partial B(\bar{x}_0, r) \cap \Gamma = \{x_0\}$. As in the proof of Hopf’s Lemma, we introduce the function, where $0 < \rho < r$ is arbitrary fixed,
\[ u(x) = e^{-\lambda|x-\bar{x}_0|^2} - e^{-\lambda r^2} \text{ in } \omega = \{x \in \mathbb{R}^n; \rho < |x-\bar{x}_0| < r\}, \]
where the constant $\lambda > 0$ is to determined hereafter.

Straightforward computations show
\[ L_\alpha v \geq (4\lambda^2 \rho^2 \kappa^{-1} - c\lambda)e^{-\lambda r^2} \text{ in } \omega, \]
where $c = c(\kappa, r)$ is a constant. We fix then $\lambda = \lambda(\kappa, \rho, r)$ sufficiently large in such a way that $L_\alpha v \geq 0$.

In light of the strong maximum principle $\max_{|x-x_0|=\rho} u < u(x_0)$. Let then
\[ \epsilon = \frac{u(x_0) - \max_{|x-x_0|=\rho} u}{e^{-\lambda \rho^2} - e^{-\lambda r^2}} \]
and
\[ w(x) = u(x) - u(x_0) + \epsilon v(x) \quad x \in \omega. \]

Our choice of $\epsilon$ guarantees that $w \leq 0$ on $\partial \omega$. As $L_\alpha w = \epsilon L_\alpha v \geq 0$ in $\Omega$ we derive from the weak maximum principle that $w \leq 0$ in $\mathbb{R}$ (e.g. [5, Theorem 3.1, page 32]). But $w(x_0) = 0$ which means that $w$ achieves its maximum at $x_0$. In consequence $\partial_v w(x_0) \geq 0$ and hence
\[ \partial_v u(x_0) \geq -\epsilon \partial_v v(x_0) = 2r\epsilon\lambda e^{-\lambda r^2}. \]

In particular we have
\[ \partial_v u(x_0) \geq \gamma \min_{|x-x_0|=\rho} (u(x_0) - u(x)), \] 
where
\[ \gamma = \frac{2r\lambda e^{-\lambda r^2}}{e^{-\lambda \rho^2} - e^{-\lambda r^2}}. \]

On the other hand, according to [3, formula (8.95) in page 628] we know that
\[ u(x_0) - u(x) = \int_\Gamma K_\rho(x, y)(u(x_0) - u(y))dS(y), \quad x \in \Omega, \]
which in light of the lower bound in (2.7) yields
\[ \min_{|x-x_0|=\rho} (u(x_0) - u(x)) \geq C \int_\Gamma (u(x_0) - u(y))dS(y). \]

Here and until the rest of this proof $C = C(n, \Omega, \kappa) > 0$ is a generic constant.

We obtain by putting together inequalities (3.1) and (3.3)
\[ \int_\Gamma (u(x_0) - u(y))dS(y) \leq C \partial_v u(x_0). \]

Let $d$ be the diameter of $\Omega$. Then
\[ \int_\Omega \frac{\text{dist}(x, \Gamma)}{|x-y|^{n-1}}dx \leq \int_\Omega \frac{dx}{|x-y|^{n-1}} \leq |S^{n-1}|d \quad \text{for all } y \in \Gamma. \]

This and the upper bound in (2.7) show that $K_\rho \in L^\infty(\Gamma, L^1(\Omega))$ with
\[ \|K_\rho\|_{L^\infty(\Gamma, L^1(\Omega))} \leq C. \]
Using again (3.2) we get by applying Fubini’s theorem
\[ \int_{\Omega} (u(x_0) - u(x)) dx = \int_{\Gamma} (u(x_0) - u(y)) dS(y) \int_{\Omega} K_\sigma(x, y) dx. \]
Therefore (3.5) gives
\[ \int_{\Omega} (u(x_0) - u(x)) dx \leq C \int_{\Gamma} (u(x_0) - u(y)) dS(y), \]
which combined with (3.4) implies
\[ \|u(x_0) - u\|_{L^1(\Omega)} \leq C \partial \nu u(x_0). \]
Hence
\[ \|u(x_0) - u\|_{L^1(\Omega)} + \|u(x_0) - u\|_{L^1(\Gamma)} \leq C \partial \nu u(x_0), \]
as expected.

**Appendix A. Proof of Theorem 2.2**

In all this proof \( C = C(n, \Omega, \kappa) > 0 \) is a generic constant.

We denote Poincaré’s constant of \( \Omega \) by \( p_2 \):
\[ \|w\|_{L^2(\Omega)} \leq p \|\nabla w\|_{L^2(\Omega)} \]
for each \( w \in H^1_0(\Omega) \).

Let \( \sigma \in \Sigma_0, f \in L^\infty(\Omega) \) and \( u = u_\sigma(f) \). We get in a straightforward manner by taking \( v = u \) in (2.4)
\[ \|\nabla u\|^2_{L^2(\Omega)} \leq \kappa \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}. \]
This inequality together with Poincaré’s inequality give
\[ \|\nabla u\|_{L^2(\Omega)} \leq p \kappa \|f\|_{L^2(\Omega)}. \]

We can then apply [10, Theorem 8.53 in page 326] and its proof in order to conclude that \( u \in H^2(\Omega) \) and
\[ \|u\|_{H^2(\Omega)} \leq C \|f\|_{L^2(\Omega)}. \]

We now discuss \( C^{1,\alpha} \) regularity of \( u \). To this purpose we shall use repeatedly [5, Theorem 9.14 in page 240 and Theorem 9.15 in page 241] concerning \( W^{2,p} \) elliptic regularity and the corresponding \( W^{2,p} \) a priori estimate.

Let us then consider first the case \( n = 2 \). In that case since \( H^1(\Omega) \) is continuously embedded in \( C(\overline{\Omega}) \) we derive that
\[ L_\sigma u = f \in L^p(\Omega), \quad 1 < p < \infty. \]
Whence \( u \in W^{2,p}(\Omega) \) and
\[ \|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}. \]
This and (A.2) imply
\[ \|u\|_{W^{2,p}(\Omega)} \leq C \|f\|_{L^p(\Omega)}. \]

For \( 0 < s < 1 \), we can apply the preceding result with \( p_s = 2/(1-s) \). Noting that \( W^{2,p_s}(\Omega) \) is continuously embedded in \( C^{1,s}(\overline{\Omega}) \) we deduce that \( u \in C^{1,s}(\overline{\Omega}) \) and from (A.3) we have
\[ \|u\|_{C^{1,s}(\overline{\Omega})} \leq C \|f\|_{L^\infty(\Omega)}. \]
Similarly when $n = 3$, using that $H^1(\Omega)$ is continuously embedded in $L^6(\Omega)$, we obtain that $u \in W^{2,6}(\Omega)$. But $W^{2,6}(\Omega)$ is continuously embedded in $C^{1,1/2}(\bar{\Omega})$. Therefore $u \in C^{1,1/2}(\bar{\Omega})$ and the following estimate holds

$$
\|u\|_{C^{1,1/2}(\bar{\Omega})} \leq C\|f\|_{L^\infty(\Omega)}.
$$

Assume that $n \geq 4$ and let $k_n \geq 1$ be the smallest integer $k$ so that $k \geq n/2 - 1$. Define

$$
p_k = \frac{2n}{n - 2}, \quad p_{k-1} = \frac{np_k - 1}{n - p_{k-1}}, \quad 2 \leq k \leq k_n.
$$

It is then not hard to check that

$$
p_k = \frac{2n}{n - 2k}, \quad 0 \leq k \leq k_n.
$$

We proceed as before. First we use that $H^1(\Omega)$ is continuously imbedded in $L^{p_1}(\Omega)$ to derive that $u \in W^{p_1,1}(\Omega)$ and

$$
\|u\|_{W^{p_1,1}(\Omega)} \leq C\|f\|_{L^{p_1}(\Omega)}.
$$

We use then that $W^{1,p_1}(\Omega)$ is continuously embedded in $L^{p_2}(\Omega)$ and we repeat the preceding argument in order to obtain that $u \in W^{p_2,1}(\Omega)$ and

$$
\|u\|_{W^{p_2,1}(\Omega)} \leq C\|f\|_{L^{p_2}(\Omega)}.
$$

By induction in $k$ we get at the end that $u \in W^{p_n,1}(\Omega)$ and

$$
\|u\|_{W^{p_n,1}(\Omega)} \leq C\|f\|_{L^{p_n}(\Omega)}.
$$

When $n = 2m + 1$, $m \geq 1$ then $k_n = m$. In that case as $W^{2,p_n}(\Omega)$ is continuously embedded in $C^{1,1/2}(\Omega)$ we obtain that $u \in C^{1,1/2}(\Omega)$ and

$$
\|u\|_{C^{1,1/2}(\bar{\Omega})} \leq C\|f\|_{L^\infty(\Omega)}.
$$

If $n = 2m$, $m \geq 2$, we have $p_{k_n} = n$. In particular $u \in W^{2,n-1/2}(\Omega)$ and

$$
\|u\|_{W^{2,n-1/2}(\Omega)} \leq C\|f\|_{L^{n-1/2}(\Omega)}.
$$

Using that $W^{2,n-1/2}(\Omega)$ is continuously embedded in $L^{n(2n-1)}(\Omega)$ we get that $u \in W^{2,n(2n-1)}(\Omega)$ and

$$
\|u\|_{W^{2,n(2n-1)}(\Omega)} \leq C\|f\|_{L^{n-1/2}(\Omega)}.
$$

We finally obtain $u \in C^{1,\alpha}(\bar{\Omega})$, with $\alpha = (2n - 2)/(2n - 1)$, and

$$
\|u\|_{C^{1,\alpha}(\bar{\Omega})} \leq C\|f\|_{L^\infty(\Omega)}.
$$

**Appendix B. Numerical testing**

We limit our numerical testing to (sufficiently smooth) isotropic $\sigma$ with $n = 2$, $\Omega = B(0, 1)$ and the following sequence of boundary conditions:

$$
h_k(x_1, x_2) = \left(\frac{x_2 + 3}{4}\right)^{1/k}, \quad n \in \mathbb{N}, \text{ and } |(x_1, x_2)| = 1.
$$

It is not hard to check that $(h_k)$ converges uniformly on $\mathbb{S}^1$ to the constant function equal to 1.

Denote by $u_k$ the solution of the BVP

$$
\begin{cases}
-\text{div}(\sigma \nabla u_k) = 0 & \text{in } \Omega, \\
\gamma_0 u_k = h_k.
\end{cases}
$$
Figure 1. Different choices of the coefficient $\sigma$. 

According to the maximum principle $u_k$ attains its maximum at $x_0 = (0, 1)$.

We considered four different choices of the coefficient $\sigma$ in Fig.1: linear, Gaussian, oscillating and realistic. We restrict the highest and lowest values to 4 and 1 respectively.

We observe in Fig.2 that $\|h(x_0) - h\|_{L^1(\Gamma)}$ converges to 0 linearly as $\partial_\nu u(x_0) \to 0$.

References

[1] D. E. Apushkinskaya and A. I. Nazarov, A counterexample to the Hopf-Oleinik lemma (elliptic case), Analysis PDE, 9 (2) (2016) 439-458.
[2] S. Barb, Topics in geometric analysis with applications to partial differential equations, Thesis (Ph.D.)-University of Missouri - Columbia, 2009, 238 pp.
[3] R. Dautray, Robert and J.-L. Lions, Mathematical analysis and numerical methods for science and technology, Vol. 1, Physical origins and classical methods. With the collaboration of Ph. Bénilan, M. Cessenat, A. Gervat, A. Kavenoky and H. Lanchon. Translated from the French by Ian N. Sneddon. With a preface by Jean Teillac. Springer-Verlag, Berlin, 1990. xviii+695 pp.
[4] L. C. Evans and R. F. Gariepy, Measure theory and fine properties of functions, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992. viii+268 pp.
[5] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer, Berlin, 1998.
[6] M. Grüter and K.-O. Widman, The Green function for uniformly elliptic equations, Manuscripta Math. 37 (3) (1982), 303-342.
[7] H. Hueber and M. Sieveking, Uniform bounds for quotients of Green functions on $C^{1,1}$-domains. Ann. Inst. Fourier (Grenoble) 32 (1) (1982), 105-117.
[8] E. Hopf, A remark on linear elliptic differential equations of second order, Proc. American Math. Soc. 3 (5) (1952), 791-793.
[9] O. A. Oleinik, On properties of solutions of certain boundary problems for equations of elliptic type, Matematicheskii Sbornik, 72 (3) (1952) 695-702.
[10] M. Renardy and R.C. Rogers, An introduction to partial differential equations, Texts in Applied Mathematics 13, Springer-Verlag, New York, 1993. xiv+428 pp.
[11] S. Zaremba, Sur un problème mixte relatif à l’équation de Laplace, Bull. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. Ser. A (1910), 313-344.
[12] Z. X. Zhao, Uniform boundedness of conditional gauge and Schrödinger equations, Comm. Math. Phys. 93 (1) (1984), 9-31.

Université de Lorraine, 34 cours Léopold, 54052 Nancy cedex, France
Email address: mourad.choulli@univ-lorraine.fr

Laboratoire Jean Kuntzmann, UMR CNRS 5224, Université Grenoble-Alpes, 700 Avenue Centrale, 38401 Saint-Martin-d’Hères, France
Email address: faouzi.triki@univ-grenoble-alpes.fr

Laboratoire Jean Kuntzmann, UMR CNRS 5224, Université Grenoble-Alpes, 700 Avenue Centrale, 38401 Saint-Martin-d’Hères, France
Email address: Qi.Xue@univ-grenoble-alpes.fr