A new class of infinite rank $\mathbb{Z}$-graded Lie conformal algebras

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Abstract: In this paper, we construct a new class of infinite rank $\mathbb{Z}$-graded Lie conformal algebra, denoted by $CW(a,c)$. And $CW(a,c)$ contains the loop Virasoro Lie conformal algebra and a Block-type Lie conformal algebra. $CW(a,c)$ has a $\mathbb{C}[\partial]$-basis $\{L_\alpha | \alpha \in \mathbb{Z}\}$ and $\lambda$-brackets $[L_\alpha \lambda L_\beta] = ((a\alpha + c)\partial + (a(\alpha + \beta) + 2c)\lambda)L_{\alpha + \beta}$, where $\alpha, \beta \in \mathbb{Z}, a, c \in \mathbb{C}$. Then the associated Lie algebra $W(a,c)$ is studied, where $W(a,c)$ has a basis $\{L_\alpha, i | \alpha, \beta, i, j \in \mathbb{Z}\}$ over $\mathbb{C}$ and Lie brackets $[L_\alpha, i, L_{\beta, j}] = (a(\beta(i + 1) - \alpha(j + 1)) + c(i - j))L_{\alpha + \beta, i + j}$, where $\alpha, \beta, i, j \in \mathbb{Z}, a, c \in \mathbb{C}$. Clearly, we find that $W(a,c)$ is also a new class of infinite dimensional $\mathbb{Z}$-graded Lie algebras. In particular, the conformal derivations of $CW(a,c)$ are determined. Finally, rank one conformal modules over $CW(a,c)$ are classified.

Key words: Lie conformal algebra, Lie algebra, conformal derivation, conformal module

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1 Introduction

Lie conformal algebra encodes an axiomatic description of the operator product expansion of chiral fields in conformal field theory. Kac introduced the notion of conformal algebra in [9]. Conformal module is a basic tool for the construction of free field realization of infinite dimensional Lie (super)algebras in conformal field theory. In recent years, the structure theory, representation theory and cohomology theory of Lie conformal algebras have been extensively studied by many scholars. For example, a finite simple Lie conformal algebra was proved to be isomorphic to either the Virasoro conformal algebra or the current conformal algebra associated with a finite dimensional simple Lie algebra in [3]. Finite irreducible conformal modules over the Virasoro conformal algebra were determined in [2]. The cohomology theory of conformal algebras was developed in [1]. The low dimensional cohomologies of the infinite rank general Lie conformal algebras $g_{CN}$ with trivial coefficients were computed in [10]. Two new nonsimple conformal algebras associated with the Schrödinger-Virasoro Lie algebra and the extended Schrödinger-Virasoro Lie algebra were constructed in [11]. The Lie conformal algebra of a Block type was introduced and free intermediate series modules were classified in [6]. The loop Virasoro Lie conformal algebra and the loop Heisenberg-Virasoro Lie conformal algebra were studied in [4, 14]. In generally, we can start from a given Lie algebra to construct the related Lie conformal algebra. However, there is little about its converse course. We believe this article would play an energetic on the study how to obtain a new Lie algebra from the Lie conformal algebra.

Infinite rank Lie conformal algebras are important ingredients of Lie conformal algebras. In this paper, we begin from the definitions of Lie conformal algebras, and we obtain a new class of infinite rank $\mathbb{Z}$-graded Lie conformal algebra $CW(a,c)$. The Lie conformal algebra $CW(a,c)$, is

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constructed in Section 3. It has a $\mathbb{C}[\partial]$-basis $\{L_\alpha \mid \alpha \in \mathbb{Z}\}$ and $\lambda$-brackets

$$[L_\alpha, L_\beta] = \left( (a\alpha + c)\partial + (a(\alpha + \beta) + 2c)\lambda \right)L_{\alpha + \beta},$$

(1.1)

where $\alpha, \beta \in \mathbb{Z}, a, c \in \mathbb{C}$.

Obviously, we can see that $CW(a, c)$ contains the loop Virasoro Lie conformal algebra and a Block type Lie conformal algebra. In particular, $CW(0, 1)$ is actually isomorphic to the loop Virasoro Lie conformal algebra studied in [14]. And $CW(1, 0)$ is actually a Block type Lie conformal algebra studied in [6].

The associated Lie algebra of $CW(a, c)$, denoted by $W(a, c)$, is defined to be a Lie algebra with basis $\{L_\alpha, i \mid \alpha, i \in \mathbb{Z}\}$ and Lie brackets given by

$$[L_\alpha, i, L_\beta, j] = \left( a(\beta(i + 1) - \alpha(j + 1)) + c(i - j) \right)L_{\alpha + \beta, i + j},$$

where $\alpha, \beta, i, j \in \mathbb{Z}, a, c \in \mathbb{C}$.

This paper is organized as follows. In Section 2, some basic definitions of Lie conformal algebras are recalled. In Section 3, we construct a new class of infinite rank $\mathbb{Z}$-graded Lie conformal algebra, denoted by $CW(a, c)$. In Section 4, the associated Lie algebra of $CW(a, c)$ is studied. In Section 5, the conformal derivations of $CW(a, c)$ are determined. Finally, rank one conformal modules are classified in Section 6.

Throughout the paper, we denote by $\mathbb{C}, \mathbb{C}^*, \mathbb{Z}$ the sets of complex numbers, nonzero complex numbers, integers respectively.

## 2 Preliminaries

In this section, we recall some definitions related to Lie conformal algebras in [3, 7, 9].

A formal distribution (usually called a field by physicists) with coefficients in a complex vector space $U$ is a series of the following form:

$$a(z) = \sum_{i \in \mathbb{Z}} a(i)z^{-i-1},$$

where $z$ is an indeterminate and $a(i) \in U$. Denote by $U[[z, z^{-1}]]$ the space of formal distribution with coefficients in $U$. The space $U[[z, z^{-1}, w, w^{-1}]]$ is defined in a similar way. A formal distribution $a(z, w) \in U[[z, z^{-1}, w, w^{-1}]]$ is called local if $(z - w)^Na(z, w) = 0$ for some $N \in \mathbb{Z}^+$. Let $g$ be a Lie algebra. Two formal distributions $a(z), b(z) \in g[[z, z^{-1}]]$ are called pairwise local if $[a(z), b(w)]$ is local in $g[[z, z^{-1}, w, w^{-1}]]$.

**Definition 2.1.** A formal distribution Lie algebra is a pair $(g, F)$, where $g$ is a Lie algebra and $F$ is a family of pairwise local formal distributions whose coefficients $F$ spans $g$. 

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Define the formal delta function to be
\[ \delta(z, w) = \sum_{i \in \mathbb{Z}} z^i w^{-i-1}. \]

The following proposition describes an equivalent condition for a formal distribution to be local.

**Proposition 2.2.** A formal distribution \( a(z, w) \in U[[z, z^{-1}, w, w^{-1}]] \) is local if and only if \( a(z, w) \) can be written as
\[
a(z, w) = \sum_{j \in \mathbb{Z}^+} c^j(w) \frac{\partial^j \delta(z, w)}{j!} \quad \text{(finite sum)} \quad \text{for some } c^j(w) \in U[[w, w^{-1}]].
\]

In this paper, we adopt the following definition of Lie conformal algebras using \( \lambda \)-brackets as in [9].

**Definition 2.3.** A Lie conformal algebra is a \( \mathbb{C}[\partial] \)-module \( A \) endowed with a linear map \( A \otimes A \to A[\lambda], \ a \otimes b \to [a, b]_\lambda \), called \( \lambda \)-bracket, where \( \lambda \) is an indeterminate and \( A[\lambda] = \mathbb{C}[\lambda] \otimes A \), subject to the following axioms:
\[
[\partial a, b] = -\lambda [a, b], \quad [a, \partial b] = (\partial + \lambda)[a, b]; \\
[a, b] = -[b - \lambda - \partial a]; \\
[a, [b, c]] = [[a, b]_\lambda + \mu c] + [b, [a, c]].
\]

For any local formal distribution \( a(z, w) \), the formal Fourier transform \( F_{z,w}^\lambda \) is defined as \( F_{z,w}^\lambda a(z, w) = \text{Res}_z e^{\lambda (z-w)} a(z, w) \). Suppose \((g, F)\) is a formal distribution Lie algebra.

Define
\[
[a(w), b(w)] = F_{z,w}^\lambda [a(z), b(w)],
\]
for any \( a(w), b(w) \in F \). One can easily check that this defines a \( \lambda \)-bracket satisfying equation (2.1)(where \( \partial = \partial_w \)). Given a formal distributions Lie algebra \((g, F)\), we may always include \( F \) in the minimal family \( F^c \) of pairwise local distributions which is closed under the derivative \( \partial \) and the \( \lambda \)-brackets. Then \( F^c \) is actually a Lie conformal algebra.

**Definition 2.4.** A conformal module \( M \) over a Lie conformal algebra \( A \) is a \( \mathbb{C}[\partial] \)-module endowed with a \( \lambda \)-action \( A \otimes M \to M[\lambda] \) such that
\[
(\partial a)_\lambda v = -\lambda a_\lambda v, \quad a_\lambda (\partial v) = (\partial + \lambda)a_\lambda v; \\
\]
\[
a_\lambda (b_\mu v) - b_\mu (a_\lambda v) = [a_\lambda b]_\lambda + \mu v.
\]

**Definition 2.5.** A Lie conformal algebra \( A \) is \( \mathbb{Z} \)-graded if \( A = \oplus_{\alpha \in \mathbb{Z}} A_\alpha \), where each \( A_\alpha \) is a \( \mathbb{C}[\partial] \)-submodule and \( [A_\alpha, A_\beta] \subset A_{\alpha + \beta}[\lambda] \) for any \( \alpha, \beta \in \mathbb{Z} \). Similarly, a conformal module \( V \) over \( A \) is \( \mathbb{Z} \)-graded if \( V = \oplus_{\alpha \in \mathbb{Z}} V_\alpha \), where each \( V_\alpha \) is a \( \mathbb{C}[\partial] \)-submodule and \( A_\alpha V_\beta \subset V_{\alpha + \beta}[\lambda] \) for any \( \alpha, \beta \in \mathbb{Z} \).
3 Infinite rank $\mathbb{Z}$-graded Lie Conformal algebras

In this section, we start from the definitions of Lie conformal algebras to construct a new class of infinite rank $\mathbb{Z}$-graded Lie conformal algebra.

Firstly, we consider the Lie conformal algebra, denoted by $\mathcal{CL}$. It has a $\mathbb{C}[\partial]$-basis $\{L_\alpha \mid \alpha \in \mathbb{Z}\}$ and with $\lambda$-brackets

$$[L_\alpha \lambda L_\beta] = (f(\alpha, \beta)\partial + g(\alpha, \beta)\lambda)L_{\alpha+\beta}, \tag{3.1}$$

where $\alpha, \beta \in \mathbb{Z}$, $f(\alpha, \beta), g(\alpha, \beta) \in \mathbb{C}[\alpha, \beta]$.

Therefore, we obtain that $\mathcal{CL}$ must satisfy the conformal sesquilinearity, conformal skew symmetry, and conformal Jacobi identity.

For the conformal skew symmetry,

$$[L_\beta \lambda L_\alpha] = -[L_\alpha - \lambda - \partial L_\beta],$$

then we can obtain that

$$f(\beta, \alpha)\partial + g(\beta, \alpha)\lambda = -(f(\alpha, \beta)\partial + g(\alpha, \beta))(-\lambda - \partial), \tag{3.2}$$

Comparing the coefficients of $\partial$ and $\lambda$ in equation (3.2), so we have

$$g(\alpha, \beta) = f(\alpha, \beta) + f(\beta, \alpha), \tag{3.3}$$

and

$$g(\alpha, \beta) = g(\beta, \alpha). \tag{3.4}$$

As it is easy to see, if satisfying the equality (3.3) then must be satisfying the equality (3.4), then we only consider the equality (3.3).

Therefore, the equality (3.1) becomes

$$[L_\alpha \lambda L_\beta] = (f(\alpha, \beta)\partial + f(\alpha, \beta + \gamma)\lambda)L_{\alpha+\beta}. \tag{3.5}$$

In fact, we have known that the conformal Jacobi identity,

$$[L_\alpha \lambda [L_\beta \mu L_\gamma]] = [[L_\alpha \lambda L_\beta] \lambda + \mu] L_\gamma + [L_\beta \mu [L_\alpha \lambda L_\gamma]]. \tag{3.6}$$

According to (3.5), and comparing the coefficients of $\partial^2$, $\partial$, $\partial \lambda$, $\partial \mu$, $\lambda^2$, $\lambda$, $\mu^2$ and $\mu$ in equation (3.6) separately, so we obtain

$$f(\alpha, \gamma)f(\beta, \alpha + \gamma) = f(\beta, \gamma)f(\alpha, \beta + \gamma), \tag{3.7}$$

and

$$f(\beta, \alpha)(f(\alpha + \beta, \gamma) + f(\gamma, \alpha + \beta)) = f(\beta, \gamma)(f(\alpha, \beta + \gamma) + f(\beta + \gamma, \alpha)). \tag{3.8}$$
**Theorem 3.1.** If $f(\alpha, \beta) \in \mathbb{C}[\alpha, \beta]$ and $f(\alpha, \beta)$ satisfies (3.7), (3.8), for any $\alpha, \beta, \gamma \in \mathbb{Z}$, then $f(\alpha, \beta) = a\alpha + c$, where $a, c \in \mathbb{C}$.

**Proof.** Firstly, we consider two special cases: $f(\alpha, \beta) \in \mathbb{C}[\alpha]$ or $f(\alpha, \beta) \in \mathbb{C}[\beta]$.

If $f(\alpha, \beta) \in \mathbb{C}[\beta]$, then we can assume

$$f(\alpha, \beta) = \sum_{i=0} a_{i0} \alpha^i \text{ (finite sum), where } a_{i0} \in \mathbb{C}, i \in \mathbb{Z}. \quad (3.9)$$

Due to the finity of $f(\alpha, \beta)$, we may suppose that the highest degree of $\alpha$ is $m(>1)$ and $a_{m0} \neq 0$.

Substituting (3.9) into (3.8), we have

$$\left( \sum_{i=0} a_{i0} \beta^i \right) \left( \sum_{i=0} a_{i0} (\alpha + \beta)^i + \sum_{i=0} a_{i0} \gamma^i \right) = \left( \sum_{i=0} a_{i0} \beta^i \right) \left( \sum_{i=0} a_{i0} \alpha^i + \sum_{i=0} a_{i0} (\beta + \gamma)^i \right). \quad (3.10)$$

Because the equality (3.10) must remain valid for any $\alpha, \beta, \gamma \in \mathbb{Z}$, then the coefficients of $\alpha \beta^{m-1}$ must be equal.

Due to $m > 1$, then the left of (3.10) must have $\alpha \beta^{m-1}$ (its coefficient is not equal to zero), but the right doesn’t have, which is contradictory. So $m \leq 1$.

And if $f(\alpha, \beta) \in \mathbb{C}[\beta]$, then we can assume

$$f(\alpha, \beta) = \sum_{j=0} a_{0j} \beta^j \text{ (finite sum), where } a_{0j} \in \mathbb{C}, j \in \mathbb{Z}. \quad (3.11)$$

Due to the finity of $f(\alpha, \beta)$, we may suppose that the highest degree of $\beta$ is $n(>0)$ and $a_{0n} \neq 0$.

Substituting (3.11) into (3.7), then we have

$$\sum_{j=0} a_{0j} \gamma^j + \sum_{j=0} a_{0j} (\alpha + \gamma)^j = \sum_{j=0} a_{0j} \gamma^j + \sum_{j=0} a_{0j} (\beta + \gamma)^j. \quad (3.12)$$

Because the equality (3.12) must be always valid for any $\alpha, \beta, \gamma \in \mathbb{Z}$, so $n = 0$. Therefore, we have $f(\alpha, \beta) = a_{00}$, where $a_{00} \in \mathbb{C}$.

Then we consider the general case, assuming

$$f(\alpha, \beta) = \sum_{i,j=0} a_{ij} \alpha^i \beta^j \text{ (finite sum), where } a_{ij} \in \mathbb{C}, i, j \in \mathbb{Z}. \quad (3.13)$$

We could suppose the highest degree of $\alpha$ is $m'$, and the degree of corresponding $\beta$ is $m''$. Similarly, the highest degree of $\beta$ is $n'$, and the degree of corresponding $\alpha$ is $n''$. Obviously, we can assume $a_{m'm''} \neq 0, a_{n'n''} \neq 0$.

Substituting (3.13) into (3.7), comparing the coefficients of the highest of $\alpha$, then we have

$$a_{m'm''} \alpha^{m'} \gamma^{n''} + a_{n'n''} \beta^{n'} \gamma^{n''} = f(\beta, \gamma) a_{m'm''} \alpha^{m'} (\beta + \gamma)^{n''}, \quad (3.14)$$
Because of $d'_{m'm''} \neq 0$ and $\forall \alpha \in \mathbb{Z}$, then we have $n' = 0$.

That is easy to say that

$$f(\alpha, \beta) = \sum_{i=0} a'_{i0} \alpha^i.$$  \hspace{1cm} (3.15)

Then $f(\alpha, \beta) \in \mathbb{C}[\alpha]$, obviously so we have $f(\alpha, \beta) = a\alpha + c$, where $a, c \in \mathbb{C}$.

Obviously, we can now see that

$$g(\alpha, \beta) = a(\alpha + \beta) + 2c, g(\alpha, \beta) = a(\alpha + \beta) + 2c.$$  \hspace{1cm} \square

According to Theorem 3.1, and there is nothing to prove.

**Theorem 3.2.** Let $CL$ have a $\mathbb{C}[\partial]$-basis $\{L_\alpha | \alpha \in \mathbb{Z}\}$ and $\lambda$-brackets

$$[L_\alpha L_\beta] = (f(\alpha, \beta)\partial + g(\alpha, \beta)\lambda)L_{\alpha + \beta}, \text{ where } f(\alpha, \beta), g(\alpha, \beta) \in \mathbb{C}[x, y], \text{ and } x, y \in \mathbb{Z}.$$  

If $CL$ could be a Lie conformal algebra, then

$$[L_\alpha L_\beta] = (a\alpha + c)\partial + (a(\alpha + \beta) + 2c)\lambda L_{\alpha + \beta}, \text{ where } \alpha, \beta \in \mathbb{Z}, a, c \in \mathbb{C}.$$  \hspace{1cm} (3.16)

This class Lie conformal algebra is denoted by $CW(a, c)$. We simply denote $CW = CW(a, c)$.

Obviously, we can see that $CW$ is a $\mathbb{Z}$-graded Lie conformal algebra, and $CW = \oplus_{\alpha \in \mathbb{Z}} CW_\alpha$, where $CW_\alpha = \mathbb{C}[\partial]L_\alpha$.

Hence we shall restrict our attention to some examples which are useful for the development of the infinite rank Lie conformal algebras.

**Example 3.3.** In (3.16), let $a = 0, c = 1$, then the equality (3.16) becomes

$$[L_\alpha L_\beta] = (\partial + 2\lambda)L_{\alpha + \beta}.$$  \hspace{1cm} (3.17)

We see that $CW(0, 1)$ is actually isomorphic to the loop Virasoro Lie conformal algebra studied in [14].

**Example 3.4.** In (3.16), let $a = 1, c = 0$, then the equality (3.16) becomes

$$[L_\alpha L_\beta] = \left(\alpha\partial + (\alpha + \beta)\lambda\right)L_{\alpha + \beta}. \hspace{1cm} (3.18)$$

It is clear that $CW(1, 0)$ is the Lie conformal algebra of a Block type Lie algebra. And the $\mathbb{Z}$-graded free intermediate series modules of $CW(1, 0)$ were studied in [6].

**Example 3.5.** In (3.16), let $a = 1, c = 1$, then the equality (3.16) becomes

$$[L_\alpha L_\beta] = \left((\alpha + 1)\partial + (\alpha + \beta + 2)\lambda\right)L_{\alpha + \beta}. \hspace{1cm} (3.19)$$

We obtain that $CW(1, 1)$ is a new type Lie conformal algebra. We can still consider the $\mathbb{Z}$-graded free intermediate series modules of $CW(1, 1)$, and it may be useful and interesting.
4 The Lie algebra $W(a, c)$

In this section, we start from the Lie conformal algebra $CW(a, c)$ to construct $CW(a, c)$ via formal distribution Lie algebra, then we obtain the associated Lie algebra $W(a, c)$ with basis $\{L_{\alpha,i} | \alpha, i \in \mathbb{Z}\}$ and Lie brackets given by

$$[L_{\alpha,i}, L_{\beta,j}] = f(\alpha, \beta, i, j)L_{\alpha+\beta,i+j},$$

(4.1)

where $\alpha, \beta, i, j \in \mathbb{Z}$, $f(\alpha, \beta, i, j) \in \mathbb{C}$.

**Proposition 4.1.** We have

$$[L_{\alpha}(z), L_{\beta}(w)] = (a\alpha + c)\partial_w L_{\alpha+\beta}(w)\delta(z,w) + (a(\alpha + \beta) + 2c)L_{\alpha+\beta}(w)\partial_w \delta(z,w),$$

where $\alpha, \beta \in \mathbb{Z}$, $a, c \in \mathbb{C}$.

**Proof.** On the one hand, using (3.16), we can obtain

$$[L_{\alpha}(w), L_{\beta}(w)] = \left((a\alpha + c)\partial + (a(\alpha + \beta) + 2c)\lambda\right)L_{\alpha+\beta}(w)$$

$$= (a\alpha + c)\partial_w L_{\alpha+\beta}(w) + (a(\alpha + \beta) + 2c)\lambda L_{\alpha+\beta}(w)$$

$$= (a\alpha + c)\partial_w L_{\alpha+\beta}(w)\text{Res}_z\delta(z,w)$$

$$+ (a(\alpha + \beta) + 2c)L_{\alpha+\beta}(w)\text{Res}_z\left(\partial_w \delta(z,w) + \lambda(z-w)\partial_w \delta(z,w)\right)$$

$$= (a\alpha + c)\partial_w L_{\alpha+\beta}(w)\text{Res}_z \sum_{j \in \mathbb{Z}^+} \frac{\lambda^k(z-w)^k\delta(z,w)}{k!}$$

$$+ (a(\alpha + \beta) + 2c)L_{\alpha+\beta}(w)\text{Res}_z \sum_{j \in \mathbb{Z}^+} \frac{\lambda^k(z-w)^k\partial_w \delta(z,w)}{k!}$$

$$= \text{Res}_z e^{\lambda(z-w)}\left((a\alpha + c)\partial_w L_{\alpha+\beta}(w)\delta(z,w) + (a(\alpha + \beta) + 2c)L_{\alpha+\beta}(w)\partial_w \delta(z,w)\right).$$

On the other hand, we know that

$$[L_{\alpha}(w), L_{\beta}(w)] = \text{Res}_z e^{\lambda(z-w)}[L_{\alpha}(z), L_{\beta}(w)].$$

So we have

$$[L_{\alpha}(z), L_{\beta}(w)] = (a\alpha + c)\partial_w L_{\alpha+\beta}(w)\delta(z,w) + (a(\alpha + \beta) + 2c)L_{\alpha+\beta}(w)\partial_w \delta(z,w).$$

□

For any $\alpha \in \mathbb{Z}$, let $L_{\alpha}(z) = \sum_{i \in \mathbb{Z}} L_{\alpha,i} z^{-i-1}$, where $x \in \mathbb{Z}$. Let $F = \{L_{\alpha}(z) | \alpha \in \mathbb{Z}\}$ be the set of $W(a, c)$-valued formal distributions.

In view of Proposition (4.1), it is sufficient to prove the following statement.
Proposition 4.2. We have

\[ [L_{\alpha,i}, L_{\beta,j}] = (a(\beta(i+1) - \alpha(j+1)) + c(i-j))L_{\alpha+\beta,i+j}, \]

where \( \alpha, \beta, i, j \in \mathbb{Z}, a, c \in \mathbb{C}. \)

Proof. On the one hand, using Proposition (4.1), we can obtain

\[
[\L_{\alpha}(z), \L_{\beta}(w)] = (a\alpha + c)\partial_{w}L_{\alpha+\beta}(w)\delta(z,w) + (a(\alpha + \beta) + 2c)L_{\alpha+\beta}(w)\partial_{w}\delta(z,w)
\]

\[
= -(a\alpha + c)(\sum_{k \in \mathbb{Z}} (k+1)L_{\alpha+\beta,k+x}w^{-k-2})(\sum_{i \in \mathbb{Z}} z^{-i-1}w^{i})
\]

\[
+ (a(\alpha + \beta) + 2c)(\sum_{k \in \mathbb{Z}} L_{\alpha+\beta,k+x}w^{-k-1})(\sum_{i \in \mathbb{Z}} iz^{-i-1}w^{i-1})
\]

\[
= -(a\alpha + c)(\sum_{i,j \in \mathbb{Z}} (i+j)L_{\alpha+\beta,i+j+x-1}z^{-i-1}w^{-j-1})
\]

\[
+ (a(\alpha + \beta) + 2c)(\sum_{i,j \in \mathbb{Z}} iL_{\alpha+\beta,i+j+x-1}z^{-i-1}w^{-j-1})
\]

\[
= \sum_{i,j \in \mathbb{Z}} \left( a(\beta i - \alpha j) + c(i-j) \right) L_{\alpha+\beta,i+j+x-1}z^{-i-1}w^{-j-1}.
\]

On the other hand, we have

\[
[\L_{\alpha}(z), \L_{\beta}(w)] = \left[ \sum_{i \in \mathbb{Z}} L_{\alpha,i+x}z^{-i-1}, \sum_{j \in \mathbb{Z}} L_{\beta,j+x}w^{-j-1} \right]
\]

\[
= \sum_{i,j \in \mathbb{Z}} [L_{\alpha,i+x}, L_{\beta,j+x}]z^{-i-1}w^{-j-1}.
\]

It is not difficult to see that

\[
[\L_{\alpha,i+x}, \L_{\beta,j+x}] = (a(\beta i - \alpha j) + c(i-j))L_{\alpha+\beta,i+j+x-1}.
\]

Furthermore, by (4.1), we have to take \( x = -1. \)

\[ \square \]

Proposition 4.3. Let \( W(a,c) \) be an algebra with basis \( \{ L_{\alpha,i} \mid \alpha, i \in \mathbb{Z} \} \), where \( a, c \in \mathbb{C} \). Then \( W(a,c) \) is a Lie algebra with Lie brackets defined as in Proposition (4.2).

Proof. For the skew symmetry, we have

\[
[L_{\alpha,i}, L_{\beta,j}] = \left( a(\beta(i+1) - \alpha(j+1)) + c(i-j) \right) L_{\alpha+\beta,i+j}
\]

\[
= -\left( a(\alpha(j+1) - \beta(i+1)) + c(j-i) \right) L_{\alpha+\beta,i+j} = -[L_{\beta,j}, L_{\alpha,i}].
\]

Finally, we can also check the Jacobi equality easily.

\[
[L_{\alpha,i}, [L_{\beta,j}, L_{\gamma,k}]] = [[L_{\alpha,i}, L_{\beta,j}], L_{\gamma,k}] + [L_{\beta,j}, [L_{\alpha,i}, L_{\gamma,k}]].
\]
We simply denote $W = W(a, c)$. It is easy to say that the Lie algebra $W$ is $\mathbb{Z}$-graded
\[
W = \bigoplus_{a \in \mathbb{Z}} W_{\alpha}, \quad W_{\alpha} = \text{span}\{L_{\alpha,i} | i \in \mathbb{Z}\} \text{ for } \alpha \in \mathbb{Z}.
\]

Furthermore, we find that $W$ has many Lie algebras which we are familiar with. For example, $W(0, 1)$ is actually isomorphic to the centerless loop-Witt Lie algebra studied in [13]. And $CW(1, 0)$ is actually a Block type Lie algebra studied in [15]. Moreover, $W(0, 1)$ is a new type Lie algebra. Its structure and representation theories could also be studied.

5 Conformal derivations of $CW(a, c)$

Suppose $L$ is a Lie conformal algebra. A linear map $\phi_\lambda : L \to L[\lambda]$ is called a conformal derivation if the following equalities hold:
\[
\phi_\lambda (\partial v) = (\partial + \lambda) \phi_\lambda (v), \quad \phi_\lambda ([a, b]) = ([\phi_\lambda a, b] + [a, \phi_\lambda b]).
\]

We often write $\phi$ instead of $\phi_\lambda$ for simplicity.

It can be easily verified that for any $x \in L$, the map $\text{ad}_x$, defined by $(\text{ad}_x)_y = [x, y]$ for $y \in L$, is a conformal derivation of $L$. All conformal derivations of this kind are called inner conformal derivations. Denote by $\text{CDer}(CW)$ and $\text{CInn}(CW)$ the vector spaces of all conformal derivations and inner conformal derivations of $CW$, respectively. Assume $D \in \text{CDer}(CW)$. Define $D^\beta (L_{\alpha}) = \pi_{\alpha + \beta} D(L_{\alpha})$ for any $\alpha \in \mathbb{Z}$, where in general $\pi_{\beta}$ is the natural projection from
\[
\mathbb{C}[\lambda] \otimes CW \cong \bigoplus_{\gamma \in \mathbb{Z}} \mathbb{C}[\partial, \lambda] L_{\gamma}.
\]
onto $\mathbb{C}[\partial, \lambda] L_{\alpha}$. Then $D^\beta$ is a conformal derivation and $D = \sum_{\beta \in \mathbb{Z}} D^\beta$ in the sense that for any $x \in CW$ only finitely many $D^\beta(x) \neq 0$. Let $(\text{CDer}CW)^d$ be the space of conformal derivations of degree $d$, i.e.,
\[
(\text{CDer}(CW))^d = \{D \in \text{CDer}(CW) | D_{\alpha}(CW_{\alpha}) \subset CW_{\alpha+d}[\lambda]\}.
\]

Firstly, by [14], we have the following result.

**Theorem 5.1.** If $c \in \mathbb{C}^*$, then $\text{CDer}(CW(0, c)) = \text{CInn}(CW(0, c))$. □

**Theorem 5.2.** If $a \in \mathbb{C}^*$, then $\text{CDer}(CW(a, 0)) = \mathbb{C}D^0$.

**Proof.** Assume $D^\beta_{\lambda}(L_{\alpha}) = f_{\alpha}(\partial, \lambda)L_{\alpha+\beta}$, where $f_{\alpha}(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$. Applying $D^\beta_{\lambda}$ to $[L_0, \mu L_{\alpha}] = a\alpha \mu L_{\alpha}$, one has
\[
f_0(-\lambda - \mu, \lambda)(\beta \partial + (\beta + \alpha)(\lambda + \mu)) + f_\alpha(\partial + \mu, \lambda)(\alpha + \beta)\mu = \alpha \mu f_\alpha(\partial, \lambda). \quad (5.1)
\]
Setting $\mu = 0$ in (5.1), one gets
\[ f_0(-\lambda, \lambda)(\beta \partial + (\beta + \alpha)\lambda) = 0. \]  
(5.2)

For any $\alpha \in \mathbb{Z}$, the equality (5.2) remains valid, then $f_0(-\lambda, \lambda) = 0$.

Therefore, (5.1) becomes
\[(\alpha + \beta)f_a(\partial + \mu, \lambda) = \alpha f_a(\partial, \lambda). \]

If $\beta \neq 0$, then we have $f_a(\partial, \lambda) = 0$. Therefore, $D^\beta(\lambda) = 0$.

If $\beta = 0$, then we can obtain $f_a(\partial, \lambda) = h_a(\lambda)(\alpha \neq 0)$, where $h_a(\lambda) \in \mathbb{C}[\lambda]$. Applying $D^\beta$ to $[L_\mu L_\gamma] = a(\alpha \partial + (\alpha + \gamma)\mu)L_{\alpha + \gamma}$, one has
\[ f_a(-\lambda - \mu, \lambda)(\alpha \partial + (\alpha + \gamma)(\lambda + \mu)) + f_\gamma(\partial + \mu, \lambda)(\alpha \partial + (\alpha + \gamma)\mu) \]
\[ = f_{a + \gamma}(\partial, \lambda)(\alpha(\partial + \lambda) + (\alpha + \gamma)\mu). \]  
(5.3)

We consider that when $\alpha, \gamma, \alpha + \gamma$ are not equal to zero, comparing the coefficients of $\partial$ and $\lambda$ in equation (5.3), so we have $h_a(\lambda) = b\alpha(\alpha \neq 0)$, where $b \in \mathbb{C}$.

By (5.3), this is not hard to prove that $f_0(\partial, \lambda) = 0$. Therefore, $D^\beta(\lambda) = b\alpha L_\alpha$, where $b \in \mathbb{C}$.

Suppose $D^\beta(\lambda) = \alpha L_\alpha$.  \(\square\)

**Theorem 5.3.** If $a \in \mathbb{C}^*$, $c \in \mathbb{C}^*$ and $a^{-1}c \notin \mathbb{Z}$, then $C\text{Der}(CW(a, c)) = C\text{Inn}(CW(a, c))$.

**Proof.** Assume $D^\beta(\lambda) = f_a(\partial, \lambda)L_{\alpha + \beta}$, where $f_a(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$. Applying $D^\beta$ to $[L_\mu L_\alpha] = (c \partial + (a\alpha + 2c)\mu)L_\alpha$, one has
\[ f_0(-\lambda - \mu, \lambda)((a\beta + c)\partial + (\lambda + \mu)(a(\beta + \alpha) + 2c)) + f_a(\partial + \mu, \lambda)(c \partial + (a(\alpha + \beta) + 2c)\mu) \]
\[ = (c(\partial + \lambda) + (a\alpha + 2c)\mu)f_a(\partial, \lambda). \]  
(5.4)

Setting $\mu = 0$ in (5.4), one gets
\[ c\lambda f_a(\partial, \lambda) = f_0(-\lambda, \lambda)((a\beta + c)\partial + \lambda(a(\beta + \alpha) + 2c)) \]
\[ = \frac{f_0(\lambda, -\lambda)}{\lambda}. \]  
(5.5)

Because $a^{-1}c \notin \mathbb{Z}$, then $\lambda$ is a factor of $f_0(-\lambda, \lambda)$ in the polynomial ring $\mathbb{C}[\partial, \lambda]$. Setting $g(\lambda) = -\frac{f_0(\lambda, -\lambda)}{\lambda}$, we have $D^\beta = \text{ad}_{g(\partial)L_{\beta}}$.

Therefore, we have $D = \sum_{\beta \in \mathbb{Z}}D^\beta = \sum_{\beta \in \mathbb{Z}}\text{ad}_{h^\beta(\partial)L_{\beta}}$ for some $h^\beta(\partial) \in \mathbb{C}[\partial]$. If $h^\beta(\partial) \neq 0$ for infinite many $\beta$’s, then $D_\lambda(L_0) = \sum_{\beta \in \mathbb{Z}}D^\beta(L_0) = \sum_{\beta \in \mathbb{Z}}[h^\beta(\partial)L_{\beta}L_0] = \sum_{\beta \in \mathbb{Z}}((a\beta + c)\partial + (a\beta + 2c)\lambda)h^\beta(-\lambda)L_{\beta}$ is an infinite sum, a contradiction to the definition of derivations.

Thus $D = \sum_{\beta \in \mathbb{Z}}\text{ad}_{h^\beta(\partial)L_{\beta}} = \text{ad}_h$ is a finite sum, where $h = \sum_{\beta \in \mathbb{Z}}h^\beta(\partial)L_{\beta} \in CW(a, c)$, i.e, $D \in C\text{Inn}(CW(a, c))$.  \(\square\)
Theorem 5.4. If $a \in \mathbb{C}^*$, $c \in \mathbb{C}^*$ and $a^{-1}c \in \mathbb{Z}$, then $CDer(CW(a,c)) = Cl_{\text{Vir}}(CW(a,c)) + \mathbb{C}D^d$, where $d = -a^{-1}c$.

Proof. Assume $D^\beta_\lambda(L_\alpha) = f_\alpha(\partial, \lambda)L_{\alpha+\beta}$, where $f_\alpha(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$. From (5.5), we know that if $\beta \neq -a^{-1}c$, then $\lambda$ is a factor of $f_0(-\lambda, \lambda)$ in the polynomial ring $\mathbb{C}[\partial, \lambda]$. Setting $g(\lambda) = -\frac{f_0(\lambda - \cdot \lambda)}{\lambda}$, we have $D^\beta = ad_{g(\partial)L_\beta}$.

For convenience, we simply denote $d = -a^{-1}c$, $f_0(-\lambda, \lambda) = \lambda l'(\lambda) + e$, where $l'(\lambda) \in \mathbb{C}[\lambda]$, $e \in \mathbb{C}$.

If $\beta = d$, then we can rephrase (5.5) as

$$f_\alpha(\partial, \lambda) = f_0(-\lambda, \lambda)(1 - d^{-1}\alpha).$$

Therefore, $f_\alpha(\partial, \lambda) = l_\alpha(\lambda) = \lambda l'(\lambda)(1 - d^{-1}\alpha) + e(1 - d^{-1}\alpha)$, for any $\alpha \in \mathbb{Z}$.

Replacing $D^d_\lambda = D^d_\lambda - ad_{d^{-1}f(\partial)L_d}$, one has $D^d_\lambda(L_\alpha) = e(1 - d^{-1}\alpha)L_{\alpha+d}$, where $e \in \mathbb{C}$.

Suppose $D^d(L_\alpha) = (1 - d^{-1}\alpha)L_{\alpha+d}$.

Therefore, we have $D = \sum_{\beta \in \mathbb{Z}} D^\beta = \sum_{\beta \in \mathbb{Z}} ad_{h_\beta(\partial)L_\beta} + eD^d$ for some $h_\beta(\partial) \in \mathbb{C}[\partial]$ and $e \in \mathbb{C}$. If $h_\beta(\partial) \neq 0$ for infinite many $\beta$’s, then $D^d(L_0) = \sum_{\beta \in \mathbb{Z}} D^\beta(L_0) = \sum_{\beta \in \mathbb{Z}} [h_\beta(\partial)L_{\beta}L_0] + eD^d(L_0) = \sum_{\beta \in \mathbb{Z}} ((a\beta + c)\partial + (a\beta + 2c)\lambda)h_\beta(-\lambda)L_{\beta} + eL_d$ is an infinite sum, a contradiction to the definition of derivations.

Thus $D = \sum_{\beta \in \mathbb{Z}} ad_{h_\beta(\partial)L_\beta} + eD^d = ad_h + eD^d$ is a finite sum, where $h = \sum_{\beta \in \mathbb{Z}} h_\beta(\partial)L_{\beta} \in \mathbb{C}L$, i.e., $D \in Cl_{\text{Vir}}(CW(a,c))$. \hfill $\square$

6 Rank one conformal modules over $CW(a,c)$

Suppose $M$ is a free conformal module of rank one over $CW$. We may write $M = \mathbb{C}[\partial]v$ and assume $L_{\alpha, \lambda}v = f_\alpha(\partial, \lambda)v$, where $f_\alpha(\partial, \lambda) \in \mathbb{C}[\partial, \lambda]$. We will compute the coefficients $f_\alpha(\partial, \lambda)$ in the rest of this section.

For the Virasoro conformal algebra $CVir$, it is well known that all the free nontrivial $CVir$-modules of rank one over $\mathbb{C}[\partial]$ are the following ones $(a', b' \in \mathbb{C})$,

$$M'_{a', b'} = \mathbb{C}[\partial]v, \quad L_{\lambda}v = (\partial + a'\lambda + b')v.$$ 

The module $M'_{a', b'}$ is irreducible if and only if $a' \neq 0$. The module $M'_{0, b'}$ contains a unique nontrivial submodule $(\partial + b')M'_{0, b'}$ isomorphic to $M'_{1, b'}$. It was proved that the modules $M'_{a', b'}$ with $a' \neq 0$ exhaust all finite irreducible nontrivial $CVir$-modules in [2].

Firstly, by [14], we have the following result.
Theorem 6.1. If $c \in \mathbb{C}^*$, then a nontrivial free conformal module of rank one over $CW(0,c)$ is isomorphic to $M_{d',b',c',d'}$ for some $d',b',d' \in \mathbb{C}, c' \in \mathbb{C}^*$, where $M_{d',b',c',d'} = \mathbb{C}[\partial]v$ and $\lambda$-actions are given by

$$L_{\alpha \lambda} v = c^i(\partial + a' \lambda + b')v.$$  

Furthermore, $M_{d',b',c',d'}$ is irreducible if and only if $a' \neq 0$. □

By [5], we have the following result.

Theorem 6.2. If $a \in \mathbb{C}^*$, then all free conformal $CW(a,0)$-modules of rank 1 over $\mathbb{C}[\partial]$ are trival. □

We can obtain the following result easily.

Lemma 6.3. For any $\alpha, \beta \in \mathbb{Z}$, the following equality holds:

$$((a\beta + c)\lambda - (a\alpha + c)\mu))f_{\alpha + \beta}(\partial, \lambda + \mu) = f_{\beta}(\partial + \lambda, \mu)f_{\alpha}(\partial, \lambda) - f_{\alpha}(\partial + \mu, \lambda)f_{\beta}(\partial, \mu). \quad (6.1)$$

Proof. A direct computation shows that

$$[L_{\alpha \lambda} L_{\beta \mu}]_{\lambda + \mu} v = ((a\beta + c)\lambda - (a\alpha + c)\mu))f_{\alpha + \beta}(\partial, \lambda + \mu)v,$$

$$L_{\alpha \lambda} (L_{\beta \mu} v) = L_{\alpha \lambda}(f_{\beta}(\partial, \mu)v) = f_{\beta}(\partial + \lambda, \mu)L_{\alpha \lambda} v = f_{\beta}(\partial + \lambda, \mu)f_{\alpha}(\partial, \lambda)v,$$

and

$$L_{\beta \mu} (L_{\alpha \lambda} v) = f_{\alpha}(\partial + \mu, \lambda)f_{\beta}(\partial, \mu)v.$$ □

Lemma 6.4. We have $f_0(\partial, \lambda) = c \partial + d \lambda + e$, where $d, e \in \mathbb{C}$.

Proof. Setting $\alpha = \beta = 0$ in (6.1), then we have

$$c(\lambda - \mu)f_0(\partial, \lambda + \mu) = f_0(\partial + \lambda, \mu)f_0(\partial, \lambda) - f_0(\partial + \mu, \lambda)f_0(\partial, \mu). \quad (6.2)$$

Comparing the degree of $\lambda$, so we have $f_0(\partial, \lambda) = \partial h(\lambda) + l(\lambda)$, where $h(\lambda), l(\lambda) \in \mathbb{C}[\lambda]$.

By (6.4), and comparing the degree of $\partial, \lambda$ and $\mu$, we can obtain $h(\lambda) = c, l(\lambda) = d \lambda + e$, where $d, e \in \mathbb{C}$. □

Lemma 6.5. We have $f_{\beta}(\partial, \lambda) = 0$ for any $\beta \in \mathbb{Z}$.

Proof. Setting $\alpha = 0$ in (6.1), then we have

$$((a\beta + c)\lambda - c\mu)f_{\beta}(\partial, \lambda + \mu) = f_{\beta}(\partial + \lambda, \mu)f_0(\partial, \lambda) - f_0(\partial + \mu, \lambda)f_{\beta}(\partial, \mu). \quad (6.3)$$

By Lemma (6.4), then

$$((a\beta + c)\lambda - c\mu)f_{\beta}(\partial, \lambda + \mu) = f_{\beta}(\partial + \lambda, \mu)(c \partial + d \lambda + e) - f_{\beta}(\partial, \mu)(c \partial + c \mu + d \lambda + e). \quad (6.4)$$
If \( a^{-1}c \notin \mathbb{Z} \), then \( a\beta + c \neq 0 \). Taking \( \mu = c^{-1}(a\beta + c)\lambda \), according to (6.4), we can obtain
\[
f_\beta(\partial + \lambda, c^{-1}(a\beta + c)\lambda)(c\partial + d\lambda + e) = f_\beta(\partial, c^{-1}(a\beta + c)\lambda)(c\partial + (a\beta + c)\lambda + d\lambda + e).
\]
Comparing the degree of \( \partial \) and \( \lambda \), then we have \( f_\beta(\partial, \lambda) = 0 \).

If \( a^{-1}c \in \mathbb{Z} \) and \( \beta \neq a^{-1}c \), then \( a\beta + c \neq 0 \). Taking \( \mu = c^{-1}(a\beta + c)\lambda \), similarly we have \( f_\beta(\partial, \lambda) = 0 \) (\( \beta \neq a^{-1}c \)).

Then we consider \( a^{-1}c \in \mathbb{Z} \) and \( \beta = \alpha^{-1}c \), and we have
\[
-c\mu f_\beta(\partial, \lambda + \mu) = f_\beta(\partial + \lambda, \mu)(c\partial + d\lambda + e) - f_\beta(\partial, \mu)(c\partial + c\mu + d\lambda + e). \tag{6.5}
\]
Setting \( \mu = 0 \) in (6.5), then we have
\[
f_\beta(\partial + \lambda, 0)(c\partial + d\lambda + e) = f_\beta(\partial, 0)(c\partial + d\lambda + e).
\]
So \( f_\beta(\partial + \lambda, 0) = f_\beta(\partial, 0) \), then we have \( f_\beta(\partial, 0) = A \), where \( A \in \mathbb{C} \).

Setting \( \mu = -\lambda \) in (6.5), then we have
\[
cA\lambda = f_\beta(\partial + \lambda, -\lambda)(c\partial + d\lambda + e) - f_\beta(\partial, -\lambda)(c\partial - c\lambda + d\lambda + e). \tag{6.6}
\]
Comparing the degree of \( \partial \), then we have \( f_\beta(\partial, \lambda) = f_\beta(\lambda) \). And comparing the degree of \( \lambda \) in (6.5), then \( f_\beta(\partial, \lambda) = A \).

According to (6.1), we can easily see that \( f_\beta(\partial, \lambda) = A = 0 \).

Therefore, we have the following proposition.

**Theorem 6.6.** If \( a \in \mathbb{C}^* \) and \( c \in \mathbb{C}^* \), then all free conformal \( CW(a,c) \)-modules of rank 1 over \( \mathbb{C}[\partial] \) are trivial. \( \square \)

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