EXISTENCE FOR NONLINEAR FINITE DIMENSIONAL STOCHASTIC DIFFERENTIAL EQUATIONS OF SUBGRADIENT TYPE

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This paper is dedicated to Professor Jiongmin Yong on the occasion of his 60th birthday.

Abstract. One proves via variational techniques the existence and uniqueness of a strong solution to the stochastic differential equation
\[ dX(t) + \partial \varphi(t, X_t) dt \ni \sum_{i=1}^{N} \sigma_i(X_t) d\beta_i, \quad X(0) = x, \]
where \( \partial \varphi : \mathbb{R}^d \to \mathbb{R}^d \) is the subdifferential of a convex function \( \varphi : \mathbb{R}^d \to \mathbb{R} \), and \( \sigma_i \in L(\mathbb{R}^d, \mathbb{R}^d), 1 \leq d < \infty \).

1. Introduction. We consider here the stochastic differential equation with linear multiplicative noise
\[ dX(t) + A(t, X_t) dt \ni \sum_{i=1}^{N} \sigma_i(X_t) d\beta_i, \quad t \in (0, T), \]
\[ X(0) = x, \]
where \( A : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \), \( \sigma_i \in L(\mathbb{R}^d, \mathbb{R}^d), i = 1, \ldots, N; 1 \leq d < \infty \), and \( \{\beta_i\}_{i=1}^{N} \) is an independent system of Brownian motions in a probability space \( \{\Omega, \mathcal{F}, \mathbb{P}\} \) with the natural filtration \( (\mathcal{F}_t)_{t \geq 0} \).

The hypothesis below will be assumed in what follows.

(i) \( A(t, y) = \partial_y \varphi(t, y), \quad \forall t \in [0, T], \ y \in \mathbb{R}^d, \) where \( \varphi \in C([0, T] \times \mathbb{R}^d) \) and, for each \( t \in [0, T], \ y \to \varphi(t, y) \) is a convex function on \( \mathbb{R}^d \).

Here \( \partial_y \varphi(t, y) \), simply denoted \( \partial \varphi(t, y) \) in the following, is the subgradient (subdifferential) of \( \varphi(t, \cdot) \) in \( y \), that is,
\[ \partial \varphi(t, y) = \{ z \in \mathbb{R}^d; \ \varphi(t, y) \leq \varphi(t, \tilde{y}) + z \cdot (y - \tilde{y}), \ \forall \tilde{y} \in \mathbb{R}^d \}. \]

It should be said that the function \( y \to \varphi(t, y) \) is, in general, multivalued (this is the case if \( \varphi \) is not differentiable in \( y \)) and stochastic differential equations of the form (1) arise, for instance, in the case of discontinuous equations of the form
\[ dX + A_0(X) dt = \sum_{i=1}^{N} \sigma_i(X) d\beta_i, \quad t \in (0, T), \]

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where \( A_0 : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a monotone measurable function, that is, \((A_0 x - A_0 \bar{x}) \cdot (x - \bar{x}) \geq 0, \forall x, \bar{x} \in \mathbb{R}^d \) and \( A_0 = \nabla \varphi \). Consider the Filipov mapping associated with \( A_0 \), defined by

\[
A x = \bigcap_{\delta > 0} \bigcap_{m(G) = 0} \overline{\text{conv}}(A_0(B_\delta(x) \setminus G)), \quad \forall x \in \mathbb{R}^d,
\]

where \( B_\delta(x) = \{ y \in \mathbb{R}^d; |x - y| \leq \delta \} \) and \( m \) is the Lebesgue measure. Then \( A : \mathbb{R}^d \rightarrow 2^{\mathbb{R}^d} \) is maximal monotone (see [1], p. 46) and so it satisfies hypothesis (i). The solution to (1) with \( A \) given by (4) might be viewed as a generalized Filipov solution to equation (3).

By our knowledge, so far the existence of a strong solution to (1) under the general hypothesis (i) was open and the main result of this work, Theorem 2.2 below, addresses such a problem. We mention also the work [5] for a similar result in the infinite dimensional Hilbert spaces.

**Notation.** We denote by \( u \cdot v \) the scalar product of vectors \( u, v \in \mathbb{R}^d \) and by \( | \cdot |_d \) the corresponding Euclidean norm. By \( L(\mathbb{R}^d, \mathbb{R}^d) \) we denote the linear space of \( d \times d \) matrices and by \( AC([0, T]; \mathbb{R}^d) \) the space of absolutely continuous functions \( u : [0, T] \rightarrow \mathbb{R}^d \). Given a Banach space \( \mathcal{X} \), we denote by \( L^p(0, T; \mathcal{X}) \) the space of all Bochner \( p \)-integrable functions \( v : [0, T] \rightarrow \mathcal{X}, 1 \leq p < \infty \). \( C([0, T]; \mathcal{X}) \) is the space of continuous \( \mathcal{X} \)-valued functions on \([0, T]\) and denote by \( L^p(\Omega; C([0, T]; \mathcal{X})) \), \( 1 \leq p < \infty \), the space of \( (\mathcal{F}_t)_{t \geq 0} \)-adapted \( \mathcal{X} \)-valued continuous processes \( u : [0, T] \rightarrow \mathcal{X} \) such that

\[
\mathbb{E} \left[ \sup_{t \in [0, T]} \| u(t) \|_\mathcal{X}^p \right] < \infty.
\]

We shall use also the standard notation \( L^p((0, T) \times \Omega; \mathbb{R}^d) \) for the space of \( \mathbb{R}^d \)-valued, \( L^p \)-integrable functions on the set \((0, T) \times \Omega\) endowed with the measure \( dt \times d\mathbb{P} \). Given a convex and lower-semicontinuous function \( \psi : \mathbb{R}^d \rightarrow \overline{\mathbb{R}} = -\infty, +\infty \), we denote by \( \psi^* : \mathbb{R}^d \rightarrow \overline{\mathbb{R}} \) the conjugate function

\[
\psi^*(v) = \sup \{ v \cdot u - \psi(u); \ u \in \mathbb{R}^d \}.
\]

We note that (see, e.g., [7], p. 79)

\[
\psi(u) + \psi^*(v) - u \cdot v \geq 0, \quad \forall u, v \in \mathbb{R}^d,
\]

\[
\psi(u) + \psi^*(v) - u \cdot v = 0, \quad \text{iff} \ v \in \partial \psi(u).
\]

We refer also to [10] for basic results on stochastic analysis to be used later on in this work.

2. The main result.

**Definition 2.1.** By strong solution to (1), we mean an \((\mathcal{F}_t)_{t \geq 0}\)-adapted process \( X : [0, T] \rightarrow \mathbb{R}^d \) which is pathwise continuous and

\[
\mathbb{E} \int_0^T |X(t)|^2 dt < \infty,
\]

\[
X(t) = x - \int_0^t \eta(s)ds + \int_0^t \sum_{i=1}^N \sigma_i(X(s))d\beta_i(s), \quad \forall t \in [0, T], \ P\text{-a.s.,}
\]

\[
\eta \in L^1((0, T) \times \Omega; \mathbb{R}^n), \ t \rightarrow \eta(t) \text{ is } (\mathcal{F}_t)_{t \geq 0}\text{-adapted},
\]

\[
\eta(t, \omega) \in A(t, X(t, \omega)), \ dt \times d\mathbb{P}, \ a.e. \ (t, \omega) \in (0, T) \times \Omega.
\]
Let $\Gamma = \Gamma(t,\omega) \in L(\mathbb{R}^d, \mathbb{R}^d)$, $\forall (t, \omega) \in [0, T] \times \Omega$, be the solution to the stochastic differential equation in $L(\mathbb{R}^d, \mathbb{R}^d)$

\[
d\Gamma = \sum_{i=1}^{N} \sigma_i \Gamma d\beta_i, \forall t \in [0, T],
\]

\[
\Gamma(0) = I,
\]

where $I$ is the unity $d \times d$-matrix.

We have $\Gamma, \Gamma^{-1} \in L^2(\Omega; C([0, T]; L(\mathbb{R}^d, \mathbb{R}^d)))$.

**Theorem 2.2.** Assume that hypothesis (i) holds. Then, for each $x \in \mathbb{R}^d$, there is a unique strong solution $X$ to (1). Moreover, one has $\Gamma^{-1}(X) \in AC([0, T]; \mathbb{R}^d)$, $\mathbb{P}$-a.s.

**Example 1.** Consider the second order stochastic differential equation

\[
\ddot{X} + f(t, \dot{X}) + \omega^2 \dot{X} = (\sigma_0 \dot{X} + \sigma_1 \dot{X}) \dot{\beta}, \ t \in (0, T),
\]

\[
X(0) = x_0, \ \dot{X}(0) = x_1,
\]

or, more exactly,

\[
dX + A_0(t, X) dt = \sigma(X) d\beta,
\]

\[
X(0) = x_0,
\]

where $\beta$ is a Brownian motion and

\[
X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \quad A_0(t, X) = \begin{bmatrix} X_2 \\ f(t, X_2) - \omega^2 X_1 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & 0 \\ \sigma_0 & \sigma_1 \end{bmatrix}.
\]

Here $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous in $t$ and monotonically nondecreasing in the second argument with a discontinuity in $r = r_0$. By filling the jump in $r_0$, i.e., by replacing the function $f$ by

\[
\bar{f}(t) = f(r) \text{ for } r \neq r_0, \quad \bar{f}(r_0) = [f(r_0 - 0), f(r_0 + 0)]
\]

and substituting in (17) $A_0$ by the corresponding mapping $A$ defined by (4), we rewrite the above system under the form (1), where $A$ satisfies hypothesis (i). (By redefining the scalar product of $\mathbb{R}^1 \times \mathbb{R}^1$, we see that $A$ is of gradient type.)

Equation (15) describes the motion of a particle subject to a linear deterministic forcing $\omega^2 \dot{X}$ and to a nonlinear, time-dependent discontinuous friction force $f(t, \dot{X})$.

3. **Proof of Theorem 2.2.** Without no loss of generality, we may prove the theorem for the modified equation

\[
dX + A(t, X) dt + \lambda X dt = \sum_{i=1}^{N} \sigma_i(X) d\beta_i,
\]

\[
X(0) = x,
\]

where $\lambda > 0$ is arbitrary. Indeed, equation (1) can be reduced to (18) by the substitution $X \to e^{\lambda t} X$ and by replacing $A(t, X)$ by $e^{-\lambda t} A(t, e^{\lambda t} X)$. We see that hypothesis (i) is invariant to such a substitution.

Consider now the transformation

\[
X = \Gamma y,
\]
where $\Gamma \in L^2(\Omega;C([0, T]; L(R^d, R^d)))$ is the solution to (12). Then, by Itô’s formula, we obtain for $y = \Gamma^{-1}(X)$ the random differential equation
\begin{equation}
\frac{dy}{dt} (t) + \Gamma^{-1}(t)A(t, \Gamma(t)y(t)) + \lambda y(t) \ni 0, \ t \in (0, T),
\end{equation}
y(0) = x.
Equivalently,
\begin{equation}
\frac{dy}{dt} (t) + \Gamma^{-1}(t)\partial\psi(t, \Gamma(t)y(t)) \ni 0, \ t \in (0, T),
y(0) = x,
\end{equation}
where
\begin{equation}
\psi(t, r) = \varphi(t, r) + \frac{\lambda}{2} |r|^2, \ \forall t \in [0, T], \ r \in R^d.
\end{equation}

Since the mapping $y \rightarrow \Gamma^{-1}A(t, \Gamma(t)y))$ is not monotone, the existence for (20) does not follow by general existence results for the deterministic Cauchy problem in $R^d$. We have, however,

**Lemma 3.1.** Under hypothesis (i), equation (21) has a unique $\mathcal{F}_t$-adapted solution $y^*$ which satisfies
\begin{equation}
y^* \in L^2((0, T) \times \Omega; R^d), \end{equation}
\begin{equation}
y^* \in AC([0, T]; R^n), \ \mathbb{P}\text{-a.s.}
\end{equation}

**Proof.** We shall associate to (21) the following optimal control problem
\begin{equation}
\text{Minimize} \quad I(y, z) = E\int_0^T (\psi(t, \Gamma(t)y(t)) + \psi^*(t, z(t))dt + \frac{1}{2} E|\Gamma(T)(y(T))|^2
\end{equation}
such that $(y, z) \in U$.

Here $U$ is the set of all $(\mathcal{F}_t)_{t \geq 0}$-adapted processes $(y, z) \in L^2((0, T) \times \Omega; R^d) \times L^1((0, T) \times \Omega; R^d)$ such that $y(T) \in L^2(\Omega; R^d)$ and $\mathbb{P}$-a.s.
\begin{equation}
\frac{dy}{dt} + \Gamma^{-1}z = 0, \ \text{a.e.} \ t \in (0, T),
y(0) = x.
\end{equation}

We are going to prove that, for $\lambda$ sufficiently large, there is a solution $(y^*, z^*)$ to (25) and that
\begin{equation}
z^* \in \partial\psi(t, \Gamma y^*), \ \text{a.e.} \ (0, T) \times \Omega,
\end{equation}
which implies, therefore, that $y^*$ is a solution to equation (21).

To prove the existence in (25), we consider a sequence $\{(y_n, z_n)\} \subset U$ such that
\begin{equation}
\inf\{I(y, z); (y, z) \in U\} \leq I(y_n, z_n) \leq \inf\{I(y, z); (y, z) \in U\} + \frac{1}{n}.
\end{equation}

Since $\psi \in C([0, T] \times R^d)$ and is convex in the second variable, it follows that (see, e.g., [7], p. 80)
\begin{equation}
\lim_{|r| \to \infty} \frac{\psi^*(t, r)}{|r|} = +\infty \text{ uniformly on } [0, T].
\end{equation}

Taking into account that, by (28),
\begin{equation}
\mathbb{E}\int_0^T \psi^*(t, z_n(t))dt \leq C < \infty, \ \forall n \in \mathbb{N},
\end{equation}
it follows by (29) that the sequence

\[ \left\{ \int_G |z_n(t, \omega)| dt \ dP; \ G \subset (0, T) \times \Omega, \ dt \times dP \ \text{measurable} \right\} \]

is equi-absolutely continuous, that is, for each \( \varepsilon > 0 \), there is \( \delta(\varepsilon) > 0 \) such that

\[ \int_G |z_n(t, \omega)| dt \ dP \leq \varepsilon \text{ if } m(G) \leq \delta. \]

(Here \( m \) is the measure \( dt \times dP \) on \( (0, T) \times \Omega \).) Then, by the Dunford–Pettis theorem, the sequence \( \{z_n\} \) is weakly compact in \( L^1((0, T) \times \Omega; \mathbb{R}^d) \) and so, on a subsequence again denoted \( \{n\} \to \infty \), we have

\[ z_n \to z^* \text{ weakly in } L^1((0, T) \times \Omega; \mathbb{R}^d). \] (30)

Moreover, by (25) and (28), it follows that

\[ E \int_0^T |\Gamma y_n|^2 dt + E|y_n(T)|^2 \leq C, \ \forall n \in \mathbb{N}, \]

where \( C \) is independent of \( n \). Hence, selecting further a subsequence of \( \{n\} \), we have by (12)

\[ y_n \to y^*, \ \Gamma y_n \to \Gamma y^* \text{ weakly in } L^2((0, T) \times \Omega; \mathbb{R}^d), \] (31)

\[ y_n(T) \to \zeta \text{ weakly in } L^2(\Omega). \] (32)

On the other hand, by (30), (31), it follows via Fatou’s lemma (or, more exactly, by the weak-lower semicontinuity property of convex integrands)

\[ \liminf_{n \to \infty} E \int_0^T \psi(t, \Gamma(t)y_n(t)) dt \geq E \int_0^T \psi(t, \Gamma(t)y^*(t)) dt, \] (33)

\[ \liminf_{n \to \infty} E \int_0^T \psi^*(t, z_n(t)) dt \geq E \int_0^T \psi^*(t, z^*(t)) dt. \] (34)

Letting \( n \to \infty \) in the equation

\[ \frac{dy_n}{dt} + \Gamma^{-1} z_n = 0, \ \text{a.e. } t \in (0, T), \]

\[ y_n(0) = x, \]

we see that

\[ \frac{dy^*}{dt} + \Gamma^{-1} z^* = 0, \ \text{a.e. } t \in (0, T), \]

\[ y^*(0) = x, \]

and that \( y^* \in AC([0, T]; \mathbb{R}^d) \), \( \mathbb{P}\)-a.s. It is also clear that the processes \( y^* \) and \( z^* \) are \((\mathcal{F}_t)_{t \geq 0}\)-adapted. Hence \( (y^*, z^*) \in \mathcal{U} \). Moreover, it follows by (35)–(36) that \( y_n, y^* \in L^1(\Omega; C([0, T]; \mathbb{R}^d)) \) and so, by (32), it follows that \( \zeta = y^*(T) \) (which is well defined \( \mathbb{P}\)-a.e. on \( \Omega \)). Also, by the weak lower-semicontinuity of the function \( u \to E|u|^2 \), we infer by (31) that

\[ \liminf_{n \to \infty} E|\Gamma(T)(y_n(T))|^2 \geq E|\Gamma(T)(y^*(T))|^2. \] (37)

By (33), (34), (37), it follows that

\[ I(y^*, z^*) = \inf \{ I(y, z); (y, z) \in \mathcal{U} \} \]

and so \( (y^*, z^*) \) is a solution to the minimization problem (25), as claimed. It is also clear that \( (y^*, z^*) \) is unique.
To prove (27), we shall write the Euler–Lagrange optimality conditions in the optimal control problem (25). Namely, we have
\[
\frac{dy^*}{dt} = -\Gamma^{-1}z^*, \quad \text{a.e. } t \in (0, T),
\]
\[
\frac{dp^*}{dt} \in \Gamma^* \partial \psi(t, \Gamma y^*), \quad \text{a.e. } t \in (0, T),
\]
\[
y^*(0) = x, \quad p(T) = -\Gamma^*(T)\Gamma(T)(y^*(T)),
\]
\[
z^*(t) \in \partial \psi(t, -\Gamma^*(t)^{-1}p(t)), \quad t \in (0, T). \tag{39}
\]
To prove that \((y^*, z^*)\) satisfies (38)–(39), we shall use an argument from [2], [7]. Namely, we consider the approximating optimal control problem
\[
\begin{align*}
\text{Minimize} & \quad I_\varepsilon(y, z) = \mathbb{E} \int_0^T \left( \psi_\varepsilon(t, \Gamma(t)y(t)) + \frac{1}{2} |z(t) - z^*(t)|^2 + \psi^*(t, z(t)) \right) dt \\
& \quad + \frac{1}{2} \mathbb{E} |\Gamma(T)(y(T))|^2 
\end{align*}
\tag{25}_\varepsilon
\]
subject to \((y, z) \in \mathcal{U} \). Here
\[
\psi_\varepsilon(t, u) = \inf \left\{ \frac{1}{2\varepsilon} |u - v|^2 + \psi(t, v) \colon v \in \mathbb{R}^d \right\}.
\]
Since the function \(u \mapsto \psi_\varepsilon(t, u)\) is differentiable, the unique optimal pair \((y_\varepsilon, z_\varepsilon)\) to 
\((25)_\varepsilon\) is given by the Euler–Lagrange system
\[
\begin{align*}
\frac{dy_\varepsilon}{dt} &= -(\Gamma^*)^{-1}z_\varepsilon, \quad \text{a.e. } t \in (0, T), \\
\frac{dp_\varepsilon}{dt} &= \Gamma^* \partial \psi_\varepsilon(t, \Gamma y_\varepsilon), \quad \text{a.e. } t \in (0, T), \\
y_\varepsilon(0) &= x, \quad p_\varepsilon(T) = -\Gamma^*(T)\Gamma(T)(y_\varepsilon(T)), \\
-\Gamma^* p_\varepsilon &\in \partial \psi_\varepsilon(t, z_\varepsilon) + z_\varepsilon - z^*, \quad \text{a.e. } t \in (0, T). \tag{41}
\end{align*}
\]
We have
\[
\begin{align*}
I_\varepsilon(y_\varepsilon, z_\varepsilon) &\leq \mathbb{E} \int_0^T \left( \psi_\varepsilon(t, \Gamma(t)y_\varepsilon(t)) + \psi^*(t, z_\varepsilon(t)) \right) dt + \frac{1}{2} \mathbb{E} |\Gamma(T)(y_\varepsilon(T))|^2 \\
&\leq I(y^*, z^*), \quad \forall \varepsilon > 0.
\end{align*}
\tag{42}
\]
On a subsequence, we have, for \(\varepsilon \to 0\),
\[
y_\varepsilon \to \tilde{y}, \quad z_\varepsilon \to \tilde{z} \quad \text{weakly in } L^2((0, T) \times \Omega; \mathbb{R}^d),
\]
and by the weak lower-semicontinuity of convex integrands, we infer that
\[
\liminf_{\varepsilon \to 0} \left\{ \mathbb{E} \int_0^T \left( \psi_\varepsilon(t, \Gamma(t)y_\varepsilon(t)) + \psi^*(t, z_\varepsilon(t)) \right) dt + \frac{1}{2} \mathbb{E} |\Gamma(T)(y_\varepsilon(T))|^2 \right\} \geq I(\tilde{y}, \tilde{z}).
\]
Then, by (42), we see that \(y^* = \tilde{y}, \quad z^* = \tilde{z}\) and that \(z_\varepsilon \to z^*\) strongly in \(L^2((0, T) \times \Omega; \mathbb{R}^d)\) as \(\varepsilon \to 0\). Then \(y_\varepsilon \to y^*\) strongly in \(L^2((0, T) \times \Omega; \mathbb{R}^d)\).

On the other hand, since \(\partial \psi^* = (\partial \psi)^{-1}\), we see that the map \(\partial \psi^*\) is continuous in \(z\) (in fact, it is Lipschitz) and so, by (40), we infer that \(p_\varepsilon \to p, \quad \Gamma p_\varepsilon \to \Gamma p\) strongly in \(L^2((0, T) \times \Omega; \mathbb{R}^d)\) for \(\varepsilon \to 0\). Then, letting \(\varepsilon\) go to zero in (40)–(41), we get (38)–(39), as claimed.
The sufficiency of \((38)–(39)\) for optimality follows easily by integrating on \((0, T)\) the obvious inequalities
\[
\psi(t, \Gamma^*) \leq \psi(t, \Gamma y) + \partial \psi(t, \Gamma y) \cdot \Gamma(y^* - y), \quad \forall y \in \mathbb{R}^d,
\]
\[
\psi^*(t, z^*) \leq \psi(t, z) + \partial \psi^*(t, z^*) \cdot (z^* - z), \quad \forall z \in \mathbb{R}^d.
\]
In order to prove \((27)\), by \((39)\) it suffices to check that
\[
\Gamma y^* = - (\Gamma^*)^{-1} p, \quad dt \times d\mathbb{P}, \quad \text{a.e. in} \ (0, T) \times \Omega. \tag{43}
\]
To this end, we set \(Y = \Gamma y^*\), \(Z = - (\Gamma^*)^{-1} p\) and note that, by \((12)\), we have, via Itô’s formula,
\[
d(\Gamma^*)^{-1} = - \sum_{j=1}^N \sigma_j^*(\Gamma^*)^{-1} d\beta_j \quad \text{in} \ (0, T).
\]
(Here \(\Gamma^*\) is the adjoint of \(\Gamma\) and \(\sigma_j^*\) is the adjoint of \(\sigma_j\).) Then, substituting in \((38)–(39)\), we get by Itô’s formula that
\[
dY + \partial \psi(t, Z) dt \ni \sum_{j=1}^N \sigma_j Y d\beta_j, \quad t \in (0, T),
\]
\[
dZ + \partial \psi(t, Y) dt \ni \sum_{j=1}^N \sigma_j^* Z d\beta_j, \quad t \in (0, T), \tag{44}
\]
\[Y(0) = x, \quad Z(T) = Y(T).\]
Since \((Y, Z)\) is a solution to \((44)\), it suffices to show that \((44)\) has at most one solution \((Y, Z)\). If \((Y_i, Z_i), i = 1, 2\), are two solutions to \((44)\), we have
\[
d(Y_1 - Y_2) + (\partial \psi(t, Z_1) - \partial \psi(t, Z_2)) dt \ni \sum_{j=1}^N \sigma_j (Y_1 - Y_2) d\beta_1,
\]
\[
d(Z_1 - Z_2) + (\partial \psi(t, Y_1) - \partial \psi(t, Y_2)) dt \ni \sum_{j=1}^N \sigma_j^* (Z_1 - Z_2) d\beta_j,
\]
\[(Y_1 - Y_2)(T) = (Z_1 - Z_2)(T).
\]
By Itô’s formula, we get
\[
\mathbb{E}(Y_1(t) - Y_2(t)) \cdot (Z_1(t) - Z_2(t)) + \lambda \mathbb{E} \int_0^t ((Y_1 - Y_2)(s))^2 + ((Z_1 - Z_2)(s))^2) ds
\]
\[= \mathbb{E} \int_0^t \sum_{j=1}^N \sigma_j (Y_1 - Y_2) \cdot \sigma_j^* (Z_1 - Z_2) ds, \quad \forall t \in (0, T),
\]
and so, for \(\lambda\) sufficiently large, we get that \(Y_1 \equiv Y_2, Z_1 \equiv Z_2\), as claimed.

This concludes the proof of Lemma 3.1. \(\square\)

Now, by Lemma 3.1, we infer that \(X\) given by \((19)\) is a strong solution to equation \((18)\). Since the solution \(y\) to \((20)\) is unique and \((18)\) is equivalent to \((20)\) through \((19)\), we conclude that \((18)\) (and, consequently, \((1)\)) has a unique solution \(X\).

Remark 1. A nice feature of the variational approach used above is that it reduces the nonlinear stochastic differential equation \((1)\) to a convex optimization problem which can be studied and, eventually, approximated in the framework of convex analysis. A similar method was used in the author works [3]–[6], [8] on the existence theory of nonlinear stochastic partial differential equations.
It should be mentioned also that there is a large number of works devoted to multivalued equations of the form (1), formulated as stochastic variational inequalities and a notable recent contribution is the recent work [9] of R. Buckdahn et al. The results of the present work do not cover entirely that of [9] which refers to stochastic variational inequalities on the nonconvex domain $O \subset \mathbb{R}^d$. However, the case where $O$ is closed, convex and with nonempty interior can be treated as in Theorem 2.2 by redefining the function $\varphi$ as $\tilde{\varphi}(t,y) = \varphi(y)$ for $y \in O$, $\tilde{\varphi}(t,y) = +\infty$ for $y \notin O$. Since the new function $\varphi$ has the domain $D(\varphi(t))$ with nonempty interior, one can proceed by reducing as above the existence problem to a convex minimization problem.

We note also that Theorem 2.2 remains true for the mappings $A : [0,T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}^d$, which are progressively measurable, i.e., for every $t \in [0,T]$, the function $y \to A(t,y) = \partial \varphi(t,y)$ restricted to $[0,t] \times \mathbb{R}^d \times \Omega$ is $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{F}_t$-measurable and
\[
\sup\{|\varphi(\cdot, y)|_{L^\infty([0,T] \times \Omega)} : |y| \leq \rho\} < \infty, \; \forall \rho > 0.
\]
The proof is exactly the same and will be omitted.

It should be also said that the extension of Theorem 2.2 to the general mapping $A : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, which are continuous in $t$ and maximal monotone with respect to $y$, remains open.

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