Card deals, lattice paths, abelian words and combinatorial identities

DAVID CALLAN
Dept. of Statistics, University of Wisconsin-Madison, 1300 University Ave, Madison, WI 53706
callan@stat.wisc.edu
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Abstract

We give combinatorial interpretations of several related identities associated with the names Barrucand, Strehl and Franel, including one for the Apéry numbers, $\sum_{k=0}^{n} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$. The combinatorial constructs employed are derangement-type card deals as introduced in a previous paper on Barrucand’s identity, labeled lattice paths and, following a comment of Jeffrey Shallit, abelian words over a 3-letter alphabet.

1 Introduction

The purpose of this paper is to give simple direct combinatorial interpretations of two identities of Strehl [1], for the Franel and Apéry numbers respectively,

$$\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{n},$$

and

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2,$$

and of the following curious sequence of identities involving powers of successively larger integers,

$$\sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} 2^k = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n} 3^k = \sum_{k=0}^{n} \binom{n}{k}^2 4^k = \sum_{k=0}^{n/2} \binom{n}{2k} \binom{2k}{k} 4^k 5^n - 2^k.$$

The first three of these expressions are equated in [2, Eqs. 34, 35], and all give sequence A084771 in OEIS.
The combinatorial constructs employed are (generalizations of) the derangement-type card deals introduced in a previous paper on Barrucand’s identity [3], the labeled lattice paths cited by Nour-Eddine Fahssi in A084771, and, following a comment of Jeffrey Shallit [4], abelian words over a 3-letter alphabet.

Section 2 reviews the card deals and abelian words/matrices. Section 3 presents a 1-to-1 correspondence between them and reinterprets Barrucand’s identity,

\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \tag{4}
\]

in terms of abelian matrices. Section 4 gives interpretations for (1) and Section 5 for (2). Section 6 presents three equinumerous combinatorial constructs involving lattice paths, card deals and matrices respectively, and Section 7 uses them to interpret (3).

## 2 Card deals and abelian words/matrices

A *Barrucand n-deal* [3] is formed as follows. Start with a deck of 3n cards, n each colored red, green and blue, in denominations 1 through n, choose an arbitrary subset of the denominations and deal all cards of the chosen denominations into three equal-size hands to players designated red, green and blue in such a way that no player receives a card of her own color. Let \( \mathcal{B}_n \) denote the set of Barrucand n-deals.

The left side of (4) counts \( \mathcal{B}_n \) by total number of cards, \( k \), in red’s hand and number of green cards, \( j \), in red’s hand: first, there are \( \binom{n}{k} \) ways to choose the denominations in the deal; next, \( j \) green cards in red’s hand implies both \( j \) blue cards in green’s hand and \( j \) red cards in blue’s hand, and these cards determine the deal. For each hand there are \( \binom{k}{j} \) ways to choose the determining cards, so \( \binom{k}{j}^3 \) choices in all. As shown in [3], the right side counts \( \mathcal{B}_n \) by number of distinct denominations, \( k \), in red’s hand; another approach to establishing this count is given below.

Serendipitously, on the day [3] was published, the editor emailed me that the counting sequence for \( \mathcal{B}_n \) also arose in his recently posted paper [5] counting abelian squares. An *abelian square* (over an alphabet) is a word of the form \( ww' \) where \( w' \) is a rearrangement of \( w \). Its *size* is the number of letters in \( w \) (= number of letters in \( w' \)). As easily seen, the number of abelian squares over a three-letter alphabet, say \( \{1, 2, 3\} \), of size \( n \) with \( n - k \) 1s in \( w \) is \( \binom{n}{k}^2 \binom{2k}{k} \) [5], the summand on the right in (4). This raises the questions of a bijection from \( \mathcal{B}_n \) to abelian squares over \( \{1, 2, 3\} \) and of an abelian squares interpretation
for the left side of (4). It is convenient to represent an abelian square $ww'$ of size $n$ as a $2 \times n$ matrix $(w)$, a so-called abelian matrix, so that we can refer to its columns.

3 Bijection from Barrucand deals to abelian matrices

The following table describes a bijection from $B_n$, the set of Barrucand $n$-deals, to $2 \times n$ abelian matrices over $\{1, 2, 3\}$ by specifying the locations of the 9 possible distinct columns in the matrix ($R, G, B$ are short for red, green, blue respectively).

| matrix column | locations given by denominations that are . . . |
|---------------|-----------------------------------------------|
| $\frac{1}{1}$ | in $[n]$, not in deal                          |
| $\frac{1}{2}$ | in deal, not in red’s hand and not on R in blue’s hand |
| $\frac{1}{3}$ | not in red’s hand but do occur on R in blue’s hand |
| $\frac{2}{1}$ | in red’s hand on G and B and also occur on R in blue’s hand |
| $\frac{2}{2}$ | in red’s hand on G only and also occur on R in blue’s hand |
| $\frac{2}{3}$ | in red’s hand on B only and also occur on R in blue’s hand |
| $\frac{3}{1}$ | in red’s hand on G and B and don’t occur on R in blue’s hand |
| $\frac{3}{2}$ | in red’s hand on G only and don’t occur on R in blue’s hand |
| $\frac{3}{3}$ | in red’s hand on B only and don’t occur on R in blue’s hand |

Bijection from deals to matrices

Table 1

Note, for example, that the denominations not in red’s hand give the locations of 1s in the top row. It is straightforward to check that this mapping is a bijection as claimed and that its inverse is given by the following table.
| player | denominations given by | locations of . . . |
|--------|------------------------|--------------------|
|        | on . . . cards          |                    |
|        | G and B                | \(2, 3\)           |
|        | G only                 | \(2, 3\)           |
|        | B only                 | \(2, 3\)           |
| green  | B and R                | \(1, 3\)           |
|        | B only                 | \(1, 3\)           |
|        | R only                 | \(3, 3\)           |
| blue   | R and G                | \(1, 2\)           |
|        | R only                 | \(2, 2\)           |
|        | G only                 | \(1, 3\)           |

**Bijection from matrices to deals**

Table 2

For example, with \(n = 5\) and subscripts referring to card color, the deal for which red’s hand contains \(2_G, 2_B, 4_B, 5_G\), green’s hand contains \(1_B, 2_R, 4_R, 5_B\), and blue’s hand contains \(1_G, 1_R, 4_G, 5_R\) corresponds to the abelian matrix \(\begin{pmatrix} 1 & 3 & 1 & 3 & 2 \\ 3 & 1 & 1 & 3 & 2 \end{pmatrix}\).

Evidently, abelian matrices are somewhat more concise than Barrucand deals but, on the other hand, some statistics on \(B_n\) are more appealing than their counterparts for abelian matrices. For example,

\[
\begin{align*}
\text{# cards in red’s hand} & \leftrightarrow n - \text{# (1)} \\
\text{# distinct denominations in red’s hand} & \leftrightarrow \text{total # 2s and 3s in top row} \\
\text{# green cards in red’s hand} & \leftrightarrow \text{# columns (\(p\) \(q\)) with } p > 1 \text{ and } q < 3.
\end{align*}
\]

In particular, using these correspondences and the second paragraph of Section 2, the left side of Barrucand’s identity \((4)\) counts abelian matrices of size \(n\) over \(\{1, 2, 3\}\) by number, \(k\), of columns \((\begin{pmatrix} p \\ q \end{pmatrix}) \neq (\begin{pmatrix} 1 \\ 1 \end{pmatrix})\) and number, \(j\), of columns \((\begin{pmatrix} p \\ q \end{pmatrix})\) with \(p > 1\) and \(q < 3\). Summarizing these observations, we have the following alternative interpretation.

**Proposition 1.** For Barrucand’s identity \((4)\), the right side of counts abelian words \(ww'\) of length \(2n\) by number, \(n - k\), of 1s in \(w\) while the left side counts them by number of positions, \(n - k\), in which both \(w\) and \(w'\) have a 1.
A generalization of Barrucand’s identity (identity (37) in [2]),
\[
\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}^2 \binom{k}{j-a}^2 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k-a},
\]
(5)
can be treated similarly. Let \(A_{n,a}\) denote the set of \(2 \times n\) matrices with entries in \(\{1, 2, 3\}\), the same number of 1s in each row, and \(a\) more 3s in the top row than in the bottom row. For example, \(\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \end{pmatrix} \in A_{3,1}\), and \(a = 0\) gives abelian matrices. Then the two sides of (5) count \(A_{n,a}\) by the very same statistics as the two sides of (4) count abelian matrices.

4 Franel numbers, \(\sum_{k=0}^{n} \binom{n}{k}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}\)

A Franel n-deal is a Barrucand n-deal in which all the cards are dealt to the players. Let \(\mathcal{F}_n\) denote the set of Franel n-deals. As observed in Section 2, the left side of the identity for the Franel numbers counts \(\mathcal{F}_n\) by number, \(k\), of green cards in red’s hand. Translated to abelian matrices, the left side counts \(\mathcal{F}_n'\), the abelian matrices of size \(n\) over \(\{1, 2, 3\}\) with no \((1, 1)\) columns, by number, \(k\), of columns \((p, q)\) with \(p > 1\) and \(q < 3\).

As for the right side, let us count \(\mathcal{F}_n'\) by number, \(j\), of 1s in each row: \(\binom{n}{j}\) [place 1s in top row] \(\times\) \(\binom{n-j}{j}\) [place 1s in bottom row] \(\times\) \(\binom{2n-2j}{n-j}\) [choose \(n-j\) of the remaining \(2n-2j\) positions; place 2s in the chosen positions in the top row and fill out the top row with 3s; place 3s in the chosen positions in the bottom row and fill out the bottom row with 2s]. (The latter clever argument is due to Richmond and Shallit [5].) Thus, with \(k := n-j\), the number of abelian matrices in \(\mathcal{F}_n'\) with a total of \(k\) 2s and 3s in each row is \(\binom{n}{n-k} \binom{k}{n-k}^2 \binom{2k}{n}\). Translated back to card deals, the right side counts \(\mathcal{F}_n\) by number of distinct denominations in red’s hand.

5 Apéry numbers,
\[
\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sum_{j=0}^{k} \binom{k}{j}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2
\]

The counting sequence for this identity, A005259, cropped up in Roger Apéry’s celebrated proof of the irrationality of \(\zeta(3)\) [6] and the identity inspired a survey paper by Volker Strehl [2] in which he offers six different proofs including a combinatorial proof of a substantial generalization and, indeed, proves most of the other identities in this paper. Still, simple direct fully bijective proofs may be of interest.
Let $B_{n,k}$ denote the set of deals in $B_n$ with $k$ cards in red’s hand, equivalently, $k$ denominations in the deal. Thus $|B_{n,k}| = \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}^3$. To get the left side of Apéry (2), we need an additional factor of $\binom{n+k}{k}$ on the left side of Barrucand (4). This motivates us to consider a simple construction and define $A_{n,k}$ to be the set of pairs $(D, i)$ where $D \in B_{n,k}$ and $1 \leq i \leq \binom{n+k}{k}$. Thus $|A_{n,k}| = \binom{n+k}{k} |B_{n,k}| = \binom{n}{k} \binom{n+k}{n} \sum_{j=0}^{k} \binom{k}{j}^3$ and $A_n := \bigcup_{k=0}^{n} A_{n,k}$ is counted by the left side of Apéry.

**Proposition 2.** Just as for Barrucand, the right side of Apéry counts $A_n$ by number of distinct denominations in the red player’s hand in the associated deal.

The proof needs the identity
\[
\sum_{a \geq 0} \binom{k}{a} \binom{n-k}{a} \binom{n+k+a}{n} = \binom{n+k}{k} \binom{n+k}{n-k},
\]
proved combinatorially by George Andrews [7] in a more general form (see also [2, Eqs. (19) and (20)]). Applied to (6), his proof shows that the right side counts pairs $(K, L)$ where $K$ is a $k$-element subset of $[n+k]$ and $L$ is an $(n-k)$-element subset of $[n+k]$ while the left side counts these pairs by “intermingling coefficient” $a$: the number of elements in $L$ among the $k$ smallest elements of $K \cup L$.

A proof of Prop. 2 can now be devised following the analysis of $B_n$ in [3] but it is a little simpler to translate to abelian matrices and prove the following equivalent result.

**Proposition 3.** Let $A'_n$ denote the set of pairs $(A, i)$ with $A$ a $2 \times n$ abelian matrix over $\{1, 2, 3\}$ and $1 \leq i \leq \binom{n+j}{j}$ where $n-j$ is the number of $(\frac{1}{i})$ columns in $A$.

Then $\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ counts $A'_n$ by total number, $k$, of 2s and 3s in the top row.

**Proof** Suppose $(A, i) \in A'_n$ has $k$ 2s and 3s, hence $n-k$ 1s, in the top row. Now count by number of $(\frac{1}{i})$ columns, say $n-k-a$. Thus we have $\binom{n+k+a}{k+a}$ choices for the second member of the pair $(A, i)$ and choices for $A$ as follows: place 1s in top row $\lfloor \binom{n-k}{n-k} \rfloor$ choices], locate $(\frac{1}{i})$ columns $\lfloor \frac{n-k}{n-k-a} \rfloor$ choices], place a 1s in the bottom row not below 1s in the top row $\lfloor \binom{k}{k} \rfloor$ choices], place 2s and 3s $\lfloor \binom{2k}{k} \rfloor$ choices, as explained in Section 4. All told, the number of choices for $(A, i)$ is
\[
\left(\frac{n}{k}\right) \left(\frac{2k}{k}\right) \sum_{a \geq 0} \binom{n-k}{a} \binom{k}{a} \binom{n+k+a}{n} = \left(\frac{n}{k}\right) \left(\frac{2k}{k}\right) \binom{n+k}{k} \binom{n+k}{n-k} = \left(\frac{n}{k}\right)^2 \binom{n+k}{k}^2,
\]
using (6) at the first equality. $\square$
6 Combinatorial constructs for (3)

A Delannoy path is a lattice path of upsteps \( U = (1, 1) \), downsteps \( D = (1, -1) \), and flatsteps \( F = (1, 0) \) with an equal number of Us and Ds. The line joining its endpoints, necessarily horizontal, is ground level. Each upstep in a Delannoy path has a matching downstep (and conversely): given an upstep above ground level (resp. below ground level), travel directly east (resp. west) until you encounter a downstep.

![matching step pairs in a Delannoy path](image)

Thus the slanted steps (\( U \) and \( D \)) in a Delannoy path are partitioned into matching pairs of opposite-slope steps.

A Hanna \( n \)-path is a Delannoy path with \( n \) labeled steps: each slanted step gets one of two labels (colors), say 1 or 2, and each flat step gets one of five labels, say 1, 2, 3, 4 or 5. As observed by Nour-Eddine Fahssi, Hanna \( n \)-paths are counted by A084771.

A Hanna \( n \)-deal is formed in the same way as a Barrucand deal except that the hands need not all be of equal size: if there are \( j \) denominations in the deal, only red’s hand is required to contain its fair share of \( j \) cards and the remaining \( 2j \) cards are split arbitrarily between the green and blue players.

A Hanna \( n \)-matrix is a \( 2 \times n \) matrix with entries in \( \{1, 2, 3\} \) and the same number of 1s in each row.

Hanna \( n \)-matrices, \( n \)-deals, and \( n \)-paths are equinumerous: the mapping in Table 1 of Section 3 (with a larger domain) is a bijection from the matrices to the deals, and there is a simple bijection from the matrices to the paths: transform each column in turn (subscripts denote step labels) according to the following table.

| matrix column | 1 1 1 2 2 2 3 3 3 |
|---------------|---------------------|
| labeled step  | \( F_1 \) \( U_1 \) \( U_2 \) \( D_1 \) \( F_2 \) \( F_3 \) \( D_2 \) \( F_4 \) \( F_5 \) |

In the next section, we use these constructs to give a combinatorial interpretation of the identities (3).
7 Combinatorial interpretations for (3)

The summand in the first expression in (3), \( \binom{n}{k} \binom{2k}{k} 2^k \), is the number of Hanna \( n \)-deals with \( k \) cards in red’s hand. To see this, expand \( 2^k \) as \( \sum_{j=0}^{k} \binom{k}{j} \). Then the resulting summand, \( \binom{n}{k} \binom{2k}{k} \), is the number of Hanna \( n \)-deals with \( k \) cards in red’s hand and \( j \) red cards in blue’s hand: choose denominations in the deal \( \binom{n}{k} \) choices, choose red denominations in blue’s hand \( \binom{k}{j} \) choices and the remaining red cards are forced into green’s hand, select red’s hand from the green and blue cards \( \binom{2k}{k} \) choices and the remaining green and blue cards are forced into the hand of opposite color.

The least obvious statistic for the sums in (3) is the one for the second sum. Actually, it is a sum of two statistics. On Hanna \( n \)-paths, define the statistic \( X \) to be the number of matching pairs of slanted steps not both labeled 1, and define \( Y \) to be the number of flat steps whose label exceeds 2. Then the summand in the second expression in (3), \( \binom{n}{k} \binom{2n-k}{n} 3^k \), is the number of Hanna \( n \)-paths for which \( X + Y = k \). This is an immediate consequence of the following two propositions.

**Proposition 4.** The number of Hanna \( n \)-paths with \( X = i \) and \( Y = j \) is

\[
\binom{n}{j} \binom{n-j}{i} \binom{2n-2i-2j}{n-j} 3^{i+j}.
\]

**Proposition 5.**

\[
\sum_{i,j : i+j=k} \binom{n}{j} \binom{n-j}{i} \binom{2n-2i-2j}{n-j} 3^{i+j} = \binom{n}{k} \binom{2n-k}{n} 3^k.
\]

**Proof of Prop. (4)** To form a Hanna \( n \)-path with \( X = i \) and \( Y = j \), choose locations in the path for flat steps whose label exceeds 2 \( \binom{n}{i} \) choices, label these flat steps \( 3^j \) choices, choose locations for the up steps in matching pairs whose members are not both labeled 1 \( \binom{n-j}{i} \) choices, assign labels to these pairs \( 3^i \) choices, since each \( U-D \) pair may be labeled 1-2, 2-1, or 2-2. Now consider the steps in the \( n-i-j \) locations not yet filled (including the down steps in the matching pairs). These steps form a path of \( U \)s, \( D \)s, and \( F \)s of length \( n-i-j \) with \( i \) more \( D \)s than \( U \)s. The labels on the slanted steps in this path are already determined and the flat steps are bicolored (labeled 1 or 2). Expanding the path via the transformation rules \( U \rightarrow UU \), \( D \rightarrow DD \), \( F_1 \rightarrow UD \), \( F_2 \rightarrow DU \) (subscript denotes label), it becomes a path of \( U \)s and \( D \)s of length \( 2n-2i-2j \) with \( n-2i-j \) \( U \)s and \( n-j \) \( D \)s. There are \( \binom{2n-2i-2j}{n-j} \) such paths, and the expansion is reversible. Thus all factors in the expression of Prop. (4) have been accounted for. \( \square \)
Proof of Prop. (5)

\[
\sum_{i+j=k} \binom{n}{j} \binom{n-j}{i} \left(\binom{2n-2i-2j}{n-j} \right)^{3i+j} = \sum_{j} \binom{n}{j} \binom{n-j}{k-j} \left(\binom{2n-2k}{n-j} \right)^{3k} = \sum_{j} \binom{n}{k} \binom{k}{j} \left(\binom{2n-k}{n-j} \right)^{3k} = \binom{n}{k} \left(\binom{2n-k}{n} \right)^{3k},
\]

using the Chu-Vandermonde identity at the last equality.

The summand in the third expression in (3), \(\binom{n}{k}^2 4^k\), is the number of \(2 \times n\) Hanna \(n\)-matrices with \(n - k\) 1s in each row: place the 1s \([\binom{n}{n-k}^2 = \binom{n}{k}^2\] choices\) and then fill the remaining \(2k\) entries with 2s and 3s arbitrarily \([2^{2k}\] choices\). Equivalently, it counts Hanna \(n\)-deals by number, \(n - k\), of denominations appearing in red’s hand. (Alternative interpretations of the other expressions in (3) are left to the reader.)

The summand in the fourth expression in (3), \(\binom{n}{2k} \binom{2k}{k} 4^k 5^{n-2k}\), is the number of Hanna \(n\)-paths with \(k\) upsteps: choose locations for the slanted steps \([\binom{n}{2k} \] choices\], insert \(U\)s and \(D\)s into these locations \([\binom{2k}{k} \] choices\], label the slanted steps \([2^{2k} \] choices\], and lastly, label the flatsteps \([5^{n-2k} \] choices\).

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