On construction of bounded sets not admitting a general type of Riesz spectrum

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Abstract
Despite the recent advances in the theory of exponential Riesz bases, it is yet unknown whether there exists a set \( S \subset \mathbb{R}^d \) which does not admit a Riesz spectrum, meaning that for every \( \Lambda \subset \mathbb{R}^d \) the set of exponentials \( e^{2\pi i \lambda \cdot x} \) with \( \lambda \in \Lambda \) is not a Riesz basis for \( L^2(S) \). As a meaningful step towards finding such a set, we construct a set \( S \subset [-\frac{1}{2}, \frac{1}{2}] \) which does not admit a Riesz spectrum containing a nonempty periodic set with period belonging in \( \alpha \mathbb{Q}_+ \) for any fixed constant \( \alpha > 0 \), where \( \mathbb{Q}_+ \) denotes the set of all positive rational numbers. In fact, we prove a slightly more general statement that the set \( S \) does not admit a Riesz spectrum containing arbitrarily long arithmetic progressions with a fixed common difference belonging in \( \alpha \mathbb{N} \). Moreover, we show that given any countable family of separated sets \( \Lambda_1, \Lambda_2, \ldots \subset \mathbb{R} \) with positive upper Beurling density, one can construct a set \( S \subset [-\frac{1}{2}, \frac{1}{2}] \) which does not admit the sets \( \Lambda_1, \Lambda_2, \ldots \) as Riesz spectrum. An interesting consequence of our results is the following statement. There is a set \( V \subset [-\frac{1}{2}, \frac{1}{2}] \) with arbitrarily small Lebesgue measure such that for any \( N \in \mathbb{N} \) and any proper subset \( I \) of \( \{0, \ldots, N-1\} \), the set of exponentials \( e^{2\pi ikx} \) with \( k \in \bigcup_{n \in I} (N\mathbb{Z}+n) \) is not a frame for \( L^2(V) \). The results are based on the proof technique of Olevskii and Ulanovskii in 2008.

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1. Introduction and Main Results

One of the fundamental research topics in Fourier analysis is the theory of exponential bases and frames. The elementary fact that \( \{e^{2\pi in \cdot x}\}_{n \in \mathbb{Z}^d} \) forms an orthogonal basis for \( L^2[-\frac{1}{2}, \frac{1}{2}]^d \), has far-reaching implications in many areas of mathematics and engineering. For instance, the celebrated Whittaker-Shannon-Kotel’nikov sampling theorem in sampling theory is an important consequence of this fact (see e.g., [19]).

As a natural generalization of the functions \( \{e^{2\pi in \cdot x}\}_{n \in \mathbb{Z}^d} \) in \( L^2[-\frac{1}{2}, \frac{1}{2}]^d \), one considers the set of exponentials \( E(\Lambda) := \{e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda\} \), where \( \Lambda \subset \mathbb{R}^d \) is a discrete set consisting of the pure frequency components of exponentials (thus...
called the frequency set or spectrum), in the Hilbert space $L^2(S)$ for a finite positive measure set $S \subset \mathbb{R}^d$. That is, for each $\lambda \in \Lambda$ the map $x \mapsto e^{2\pi i \lambda \cdot x}$ restricted to the set $S$ is considered as a function in $L^2(S)$. Characterizing the properties of $E(\Lambda)$ in the space $L^2(S)$, such as whether $E(\Lambda)$ forms an orthogonal/Riesz basis or a frame, has been an important problem in nonharmonic Fourier analysis. The problem has a close connection to the theory of entire functions of exponential type in complex analysis, through the celebrated work of Paley and Wiener [35]. For more details on this connection and for some historical background, we refer to the excellent book by Young [38]. Below we give a short overview of some known results on exponential bases and frames.

1.1. An overview of existing work on exponential bases and frames

**Exponential orthogonal bases.** For the case of orthogonal bases, Fuglede [13] posed a famous conjecture (also called the spectral set conjecture) which states that if $S \subset \mathbb{R}^d$ is a finite positive measure set, then there is an exponential orthogonal basis $E(\Lambda)$ (with $\Lambda \subset \mathbb{R}^d$) for $L^2(S)$ if and only if the set $S$ tiles $\mathbb{R}^d$ by translations along a discrete set $\Gamma \subset \mathbb{R}^d$ in the sense that

$$
\sum_{\gamma \in \Gamma} \chi_S(x + \gamma) = 1 \text{ for a.e. } x \in \mathbb{R}^d,
$$

where $\chi_S(x) = 1$ for $x \in S$, and 0 otherwise. The conjecture turned out to be false for $d \geq 3$ but is still open for $d = 1, 2$. There are many special cases where the conjecture is known to be true. For instance, the conjecture is true when $\Gamma$ is a lattice of $\mathbb{R}^d$, in which case the set $\Lambda \subset \mathbb{R}^d$ can be chosen to be the dual lattice of $\Gamma$ [13], and also when $S \subset \mathbb{R}^d$ is a convex set of finite positive measure for all $d \in \mathbb{N}$ [25]. In particular, it was shown in [20] that there is no exponential orthogonal basis for $L^2(S)$ when $S$ is the unit ball of $\mathbb{R}^d$ for $d \geq 2$, in contrast to the case $d = 1$ where the unit ball is simply $S = [-1,1]$ and $E(\frac{1}{2}\mathbb{Z})$ is an orthogonal basis for $L^2([-1,1])$. For more details of the recent progress on Fuglede’s conjecture, we refer to [25] and the reference therein.

**Exponential Riesz bases.** The relaxed case of Riesz bases is yet more challenging. Certainly, relaxing the condition of orthogonal bases to Riesz bases allows for potentially much more feasible sets $S \subset \mathbb{R}^d$. However, there are only several classes of sets $S \subset \mathbb{R}^d$ that are known to admit a Riesz spectrum, meaning that there exists an exponential Riesz basis for $L^2(S)$. For instance, the class of convex symmetric polygons in $\mathbb{R}^2$ [26], the class of sets that are finite unions of intervals in $\mathbb{R}^d$ [22, 23], and the class of certain symmetric convex polytopes in $\mathbb{R}^d$ for all $d \geq 1$ [9]. On the other hand, to the best of our knowledge, nobody was able to find a set $S \subset \mathbb{R}^d$ which does not admit a Riesz spectrum.

In search for an analogue of Fuglede’s conjecture for Riesz bases, Grepstad and Lev [15] considered the sets $S \subset \mathbb{R}^d$ that satisfy for some discrete set $\Gamma \subset \mathbb{R}^d$ and some $k \in \mathbb{N}$,

$$
\sum_{\gamma \in \Gamma} \chi_S(x + \gamma) = k \text{ for a.e. } x \in \mathbb{R}^d.
$$
have universal properties, namely the so-called universal measurable subset of $[-\frac{1}{2}, \frac{1}{2}]^d$ and thus a frame for $L^2([-\frac{1}{2}, \frac{1}{2}]^d)$. It was shown in [13] that if $S \subset \mathbb{R}^d$ is a bounded $k$-tile set with respect to a lattice $\Gamma \subset \mathbb{R}^d$ and has measure zero boundary, then the set $S$ admits a Riesz spectrum $\Lambda \subset \mathbb{R}^d$. Moreover in this case, the set $\Lambda$ can be chosen to be a union of $k$ translations of $\Gamma^*$ (referred to as a $(k, \Gamma^*)$-structured spectrum), where $\Gamma^* := (A^{-1})^{\ast} \mathbb{Z}^d$ is the dual lattice of $\Gamma = A \mathbb{Z}^d$ with $A \in \text{GL}(d, \mathbb{R})$. Later, Kolountzakis [21] gave a simpler proof of this result using elementary arguments and also eliminated the requirement of $S$ having measure zero boundary. The converse of the statement was proved by Agora et al. [1], thus establishing the equivalence: Given a lattice $\Gamma \subset \mathbb{R}^d$, a bounded set $S \subset \mathbb{R}^d$ is a $k$-tile with respect to $\Gamma$ if and only if it admits a $(k, \Gamma^*)$-structured Riesz spectrum. They also showed that the boundedness of $S$ is essential by constructing an unbounded 2-tile set $S \subset \mathbb{R}^d$ with respect to $\mathbb{Z}$ which does not admit a $(2, \mathbb{Z})$-structured Riesz spectrum. Nevertheless, for unbounded multi-tiles $S \subset \mathbb{R}^d$ with respect to a lattice $\Gamma$, Cabrelli and Carbajal [4] were able to provide a sufficient condition for $S$ to admit a structured Riesz spectrum. Recently, Cabrelli et al. [5] found a necessary and sufficient condition for a multi-tile $S \subset \mathbb{R}^d$ of finite positive measure to admit a structured Riesz spectrum, which is given in terms of the Bohr compactification of the tiling lattice $\Gamma$.

**Exponential frames.** Since frames allow for redundancy, it is relatively easier to obtain exponential frames than exponential Riesz bases. For instance, the set of exponentials $\{e^{2\pi in \cdot x}\}_{n \in \mathbb{Z}^d}$ is an orthonormal basis for $L^2([-\frac{1}{2}, \frac{1}{2}]^d)$ and thus a frame for $L^2(S)$ with frame bounds $A = B = 1$ whenever $S$ is a measurable subset of $[-\frac{1}{2}, \frac{1}{2}]^d$.

Nitzan et al. [31] proved that if $S \subset \mathbb{R}^d$ is a finite positive measure set, then there exists an exponential frame $E(\Lambda)$ (with $\Lambda \subset \mathbb{R}^d$) for $L^2(S)$ with frame bounds $c |S|$ and $C |S|$, where $0 < c < C < \infty$ are absolute constants. The proof is based on a lemma from Marcus et al. [27] which resolved the famous Kadison-Singer problem in the affirmative.

**Universality.** In [32, 33], Olevskii and Ulanovskii considered the interesting question of universality. They discovered some frequency sets $\Lambda \subset \mathbb{R}^d$ that have universal properties, namely the so-called universal uniqueness/sampling/interpolation sets $\Lambda \subset \mathbb{R}^d$ for Paley-Wiener spaces $PW(S)$ with all sets $S \subset \mathbb{R}^d$ in a certain class. In our notation, this corresponds to the set of exponentials $E(\Lambda)$ being a complete sequence/frame/Riesz sequence in $L^2(S)$ for all sets $S \subset \mathbb{R}^d$ in a certain class. For the convenience of readers, we include a short exposition on the relevant notions in Paley-Wiener spaces in [Appendix A].

It was shown that universal complete sets of exponentials exist, for instance, the set $E(\Lambda)$ with $\Lambda = \{\ldots, -6, -4, -2, 1, 3, 5, \ldots\}$ is complete in $L^2(S)$ for every measurable set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|S| \leq \frac{1}{2}$. Furthermore, any set $E(\Lambda)$ with $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}}$ satisfying $0 < |\lambda_n - n| \leq 1/2^n$ for all $n \in \mathbb{Z}$, is complete in $L^2(S)$ whenever $S \subset \mathbb{R}$ is a bounded measurable set with $|S| < 1$.

On the other hand, the existence of universal exponential frames and universal exponential Riesz sequences depend on the topological properties of $S$. As a
positive result, it was shown that there is a perturbation $\Lambda$ of $\mathbb{Z}$ such that $E(\Lambda)$ is a frame for $L^2(S)$ whenever $S \subset \mathbb{R}$ is a compact set with $|S| < 1$; a different construction of such a set $\Lambda \subset \mathbb{R}$ was given by Matei and Meyer \cite{MateiMeyer1, MateiMeyer2} based on the theory of quasicrystals. Similarly, there is a perturbation $\Lambda$ of $\mathbb{Z}$ such that $E(\Lambda)$ is a Riesz sequence in $L^2(S)$ whenever $S \subset \mathbb{R}$ is an open set with $|S| > 1$. However, in the negative side, it was shown that given any $0 < \epsilon < 2$ and a separated set $\Lambda \subset \mathbb{R}$ with $D^-(\Lambda) < 2$, there is a measurable set $S \subset [0, 2]$ with $|S| < \epsilon$ such that $E(\Lambda)$ is not a frame for $L^2(S)$, indicating that the compactness of $S$ in the aforementioned result cannot be dropped. Similarly, it was shown that given any $0 < \epsilon < 2$ and a separated set $\Lambda \subset \mathbb{R}$ with $D^+(\Lambda) > 0$, there is a measurable set $S \subset [0, 2]$ with $|S| > 2 - \epsilon$ such that $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$, indicating similarly that the restriction to open sets cannot be dropped.

For more details on the universality results, we refer to Lectures 6 and 7 in the excellent lecture book by Olevskii and Ulanovskii \cite{OlevskiiUlanovskii}.

### 1.2. Contribution of the paper

The current paper is motivated by the following open problem which was mentioned above.

**Open Problem.** Is there a bounded/unbounded set $S \subset \mathbb{R}^d$ which does not admit a Riesz spectrum, meaning that for every $\Lambda \subset \mathbb{R}^d$ the set of exponentials \{e^{2\pi i \lambda \cdot x} : \lambda \in \Lambda\} is not a Riesz basis for $L^2(S)$?

We believe that the answer is positive, and in this paper we make a meaningful step towards finding such a set $S$. Adapting the proof technique of Olevskii and Ulanovskii \cite{OlevskiiUlanovskii}, we will construct a bounded subset of $\mathbb{R}$ which does not admit a certain general type of Riesz spectrum. As the proof technique of \cite{OlevskiiUlanovskii} works also in higher dimensions (see the end of Section 1 in \cite{OlevskiiUlanovskii}), our results can be extended to higher dimensions to obtain a bounded subset of $\mathbb{R}^d$ with similar properties. For simplicity of presentation, we will only consider the dimension one case ($d = 1$).

Let us point out that the stated problem has been deemed difficult by many researchers in the field, see for instance, \cite[Section 1]{OlevskiiUlanovskii}. To resolve the problem in full may require a far more advanced proof technique than the one used in this paper.

Before presenting our results, note that for any bounded set $S \subset \mathbb{R}$ there are some parameters $\sigma > 0$ and $a \in \mathbb{R}$ such that $\frac{1}{\sigma}S + a \subset [-\frac{1}{2}, \frac{1}{2}]$. It is therefore enough to restrict our attention to sets $S \subset [-\frac{1}{2}, \frac{1}{2}]$ (see Lemma \ref{lem:1} below). Also, recall that a set $S \subset \mathbb{R}$ is said to admit a Riesz spectrum $\Lambda \subset \mathbb{R}$ if the set $E(\Lambda)$ is a Riesz basis for $L^2(S)$.

Our first main result is as follows.

**Theorem 1.** Let $0 < \alpha \leq 1$ and $0 < \epsilon < 1$. There exists a measurable set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|S| > 1 - \epsilon$ satisfying the following property: if $\Lambda \subset \mathbb{R}$ contains arbitrarily long arithmetic progressions with a fixed common difference belonging
in $\alpha \mathbb{N}$, then $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$. Moreover, such a set can be constructed explicitly as

$$S = [-\frac{1}{2}, \frac{1}{2}] \setminus V \quad \text{with} \quad V = [-\frac{1}{2}, \frac{1}{2}] \cap \left( \bigcup_{\ell=1}^{\infty} \bigcup_{m=\mathbb{Z}} \left( \frac{m}{\ell^{1+\epsilon}} + \left[ -\frac{\ell}{2^{\ell^{1+\epsilon}}}, \frac{\ell}{2^{\ell^{1+\epsilon}}} \right] \right) \right).$$

(2)

It should be noted that the set $V \subset [-\frac{1}{2}, \frac{1}{2}]$ is a countable union of closed intervals, i.e., an $F_\sigma$ Borel set, which contains $\frac{1}{n} \mathbb{Q} \cap [-\frac{1}{2}, \frac{1}{2}]$. Yet, the set has small Lebesgue measure $|V| < \epsilon$ due to the exponentially decreasing length of the intervals. It is worth comparing the set $V$ with a fat Cantor set which is a closed, nowhere dense subset of $[-\frac{1}{2}, \frac{1}{2}]$ with positive measure containing uncountably many elements (see e.g., [11, 14]). In contrast to the fat Cantor sets, the set $V$ is not closed and has nonempty interior. Also, the set $V$ is dense in $[-\frac{1}{2}, \frac{1}{2}]$ because it contains $\frac{1}{n} \mathbb{Q} \cap [-\frac{1}{2}, \frac{1}{2}]$.

To illustrate the dense set $V \subset [-\frac{1}{2}, \frac{1}{2}]$, we truncate the infinite union in its expression to the finite union over $\ell = 1, \ldots, 10$. The corresponding sets for $\alpha = 1$ and $\epsilon = \frac{1}{10}, \frac{1}{2}, \frac{9}{10}$ are shown in Figure 1.

![Figure 1](image_url)

Figure 1: The characteristic function of the corresponding truncated set for $\alpha = 1$ and $\epsilon = \frac{1}{10}, \frac{1}{2}, \frac{9}{10}$ (from left to right).

To help the understanding of readers, we provide two sets $\Lambda \subset \mathbb{R}$, one which meets and the other which does not meet the condition stated in Theorem 1.

**Example.** (a) Let $M_1 < M_2 < \cdots$ be an increasing sequence in $\mathbb{N}$, and let $P \in \mathbb{N}$. Define the sequence $d_1 < d_2 < \cdots$ by $d_1 = 0$ and $d_k = 2 \sum_{n=1}^{k-1} M_n P$ for $k \geq 2$. Clearly, we have $d_{k+1} - d_k = 2M_k P$ for all $k \in \mathbb{N}$. Consider the set

$$\Lambda = \pm \bigcup_{k=1}^{\infty} \{d_k + P, d_k + 2P, \ldots, d_k + M_k P\} \subset \mathbb{Z},$$

where $\pm \Lambda_0 := \Lambda_0 \cup (-\Lambda_0)$ for any set $\Lambda_0 \subset \mathbb{R}$. This set contains arbitrarily long arithmetic progressions with common difference $P$, and has lower and upper

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1A set is called nowhere dense if its closure has empty interior.
Beurling density given by $D^-(\Lambda) = \frac{1}{\mathcal{H}^1}$ and $D^+(\Lambda) = \frac{1}{\mathcal{H}^1}$, respectively (see Section 2.3 for the definition of Beurling density).

(b) Let $N \in \mathbb{N}$ and let $\{\sigma_k\}_{k=1}^{\infty} \subset (0,1)$ be a sequence of distinct irrational numbers between 0 and 1. Consider the set

$$\Lambda = \pm \bigcup_{k=1}^{\infty} \left(\sigma_k + Nk + \{0 \cdot 100^k, 1 \cdot 100^k, \ldots, (k-1) \cdot 100^k\}\right) \subset \mathbb{R}$$

which has uniform Beurling density $D(\Lambda) = \frac{1}{N}$. For each $k \in \mathbb{N}$, the set $\Lambda$ contains exactly one arithmetic progression with common difference $100^k$ in the positive domain $(0, \infty)$, namely the arithmetic progression $\sigma_k + Nk, \sigma_k + Nk + 100^k, \ldots, \sigma_k + Nk + (k-1) \cdot 100^k$ of length $k$. Due to the ± mirror symmetry, the set $\Lambda$ has another such an arithmetic progression in the negative domain $(-\infty, 0)$. Note that all of these arithmetic progressions have integer-valued common difference and are distanced by some distinct irrational numbers, so none of them can be connected with another to form a longer arithmetic progression. Hence, there is no number $P \in \mathbb{N}$ for which the set $\Lambda$ contains arbitrarily long arithmetic progressions with common difference $P$. Such a set $\Lambda \subset \mathbb{R}$ is not covered by the class of frequency sets considered in Theorem 1.

Our second main result is the following.

**Theorem 2.** Let $0 < \epsilon < 1$ and let $\Lambda_1, \Lambda_2, \ldots \subset \mathbb{R}$ be a family of separated sets with $D^+(\Lambda_\ell) > 0$ for all $\ell \in \mathbb{N}$. One can construct a measurable set $S = S(\epsilon, \{A_\ell\}_{\ell=1}^{\infty}) \subset [-\frac{1}{2}; \frac{1}{2}]$ with $|S| > 1 - \epsilon$ such that $E(\Lambda_\ell)$ is not a Riesz sequence in $L^2(S)$ for all $\ell \in \mathbb{N}$.

Let us present some interesting implications of our main results.

By convention, a discrete set $\Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R}$ with $\lambda_n < \lambda_{n+1}$ is called periodic with period $t > 0$ (or $t$-periodic) if there is a number $N \in \mathbb{N}$ such that $\lambda_{n+N} - \lambda_n = t$ for all $n \in \mathbb{Z}$. Note that if $\Lambda \subset \mathbb{R}$ is a nonempty periodic set with period $\alpha \cdot \frac{P}{Q} \in \alpha\mathbb{Q}$, where $P, Q \in \mathbb{N}$ are coprime numbers, then it must contain a translated copy of $\alpha P\mathbb{Z}$, that is, $\alpha P\mathbb{Z} + d \subset \Lambda$ for some $d \in \mathbb{R}$. As a result, we have the following corollary of Theorem 1.

**Corollary 3.** For any $0 < \alpha < 1$ and $0 < \epsilon < 1$, let $S \subset [-\frac{1}{2}; \frac{1}{2}]$ be the set given by (2). Then for any nonempty periodic set $\Lambda \subset \mathbb{R}$ with period belonging in $\alpha\mathbb{Q}_+ = \alpha\mathbb{Q} \cap (0, \infty)$, the set $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$. Consequently, the set $S$ does not admit a Riesz spectrum containing a nonempty periodic set with period belonging in $\alpha\mathbb{Q}_+$.

It is worth noting that the class of nonempty periodic sets with rational period is uncountable, because of the flexibility in placement of elements in each period; hence, Corollary 3 cannot be deduced from Theorem 2.

As mentioned in Section 1.1, Agora et al. [11] constructed an unbounded 2-tile set $S \subset \mathbb{R}$ with respect to $\mathbb{Z}$ which does not admit a Riesz spectrum of the form $(\mathbb{Z} + \sigma_1) \cup (\mathbb{Z} + \sigma_2)$ with $\sigma_1, \sigma_2 \in \mathbb{R}$. By a dilation, one could easily generalize this example to an unbounded 2-tile set $W \subset \mathbb{R}$ with respect to $\frac{1}{n}\mathbb{Z}$ for any fixed
\( \alpha > 0 \), which does not admit a Riesz spectrum of the form \( (\alpha \mathbb{Z} + \sigma_1) \cup (\alpha \mathbb{Z} + \sigma_2) \) with \( \sigma_1, \sigma_2 \in \mathbb{R} \). Note that such a form of Riesz spectrum is \( \alpha \)-periodic and thus not admitted by our set \( S \) given by (2) with any \( 0 < \epsilon < 1 \). In fact, our set \( S \) has a much stronger property than \( W \), namely that \( S \) does not admit a periodic Riesz spectrum with period belonging in \( \alpha \mathbb{Q}_+ \) and moreover, the set \( S \) is bounded.

Since the set \( S \) is contained in \([-\frac{1}{2}, \frac{1}{2}]\), it is particularly interesting to consider the frequency sets consisting of integers \( \Omega \subset \mathbb{Z} \). Noting that a periodic subset of \( \mathbb{Z} \) is necessarily \( N \)-periodic for some \( N \in \mathbb{N} \), we immediately deduce the following result from Corollary 3.

**Corollary 4.** Let \( S \subset [-\frac{1}{2}, \frac{1}{2}] \) be the set given by (2) with \( \alpha = 1 \) and any \( 0 < \epsilon < 1 \). Then for any nonempty periodic set \( \Omega \subset \mathbb{Z} \), the set \( E(\Omega) \) is not a Riesz sequence in \( L^2(S) \).

Alternatively, one could construct such a set \( S \subset [-\frac{1}{2}, \frac{1}{2}] \) from Theorem 2 by observing that the family of all nonempty periodic integer sets is countable; indeed, the one and only nonempty 1-periodic integer set is \( \mathbb{Z} \), the nonempty 2-periodic integer sets are \( 2\mathbb{Z}, 2\mathbb{Z} + 1, \mathbb{Z} \), and so on.

Further, it is easy to deduce the following result from Corollary 4 and Proposition 7 below, by setting \( V := [-\frac{1}{2}, \frac{1}{2}] \setminus S \) and \( \Omega' := \mathbb{Z} \setminus \Omega \).

**Corollary 5.** Let \( S \subset [-\frac{1}{2}, \frac{1}{2}] \) be the set given by (2) with \( \alpha = 1 \) and any \( 0 < \epsilon < 1 \), and let \( V := [-\frac{1}{2}, \frac{1}{2}] \setminus S \). Then for any proper periodic subset \( \Omega' \subset \mathbb{Z} \), the set \( E(\Omega') \) is not a frame for \( L^2(V) \).

The significance of Corollary 5 is in the fact that for any \( N \in \mathbb{N} \) and any proper subset \( I \subset \{0, \ldots, N-1\} \), the set of exponentials \( E(\bigcup_{n \in I} (N\mathbb{Z} + n)) \) is not a frame for \( L^2(V) \) even though the set \( V \) has very small Lebesgue measure \( |V| < \epsilon \). Note that \( E(\mathbb{Z}) \) is a frame for \( L^2(V) \) with frame bounds \( A = B = 1 \), since it is an orthonormal basis for \( L^2[0,1] \).

2. Preliminaries

2.1. Sequences in separable Hilbert spaces

**Definition.** A sequence \( \{f_n\}_{n \in \mathbb{Z}} \) in a separable Hilbert space \( \mathcal{H} \) is called

- a **Bessel sequence** in \( \mathcal{H} \) (with a Bessel bound \( B \)) if there is a constant \( B > 0 \) such that
  \[
  \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2 \leq B \|f\|^2 \quad \text{for all } f \in \mathcal{H};
  \]

- a **frame** for \( \mathcal{H} \) (with frame bounds \( A \) and \( B \)) if there are constants \( 0 < A \leq B < \infty \) such that
  \[
  A \|f\|^2 \leq \sum_{n \in \mathbb{Z}} |\langle f, f_n \rangle|^2 \leq B \|f\|^2 \quad \text{for all } f \in \mathcal{H};
  \]
• a Riesz sequence in $\mathcal{H}$ (with Riesz bounds $A$ and $B$) if there are constants $0 < A \leq B < \infty$ such that

$$A \|c\|_{\ell_2}^2 \leq \left\| \sum_{n \in \mathbb{Z}} c_n f_n \right\|_{\ell_2}^2 \leq B \|c\|_{\ell_2}^2$$

for all $c = \{c_n\}_{n \in \mathbb{Z}} \in \ell_2(\mathbb{Z})$;

• a Riesz basis for $\mathcal{H}$ (with Riesz bounds $A$ and $B$) if it is a complete Riesz sequence in $\mathcal{H}$ (with Riesz bounds $A$ and $B$);

• an orthogonal basis for $\mathcal{H}$ if it is a complete sequence of nonzero elements in $\mathcal{H}$ such that $\langle f_m, f_n \rangle = 0$ whenever $m \neq n$.

• an orthonormal basis for $\mathcal{H}$ if it is complete and $\langle f_m, f_n \rangle = \delta_{m,n}$ whenever $m \neq n$.

The associated bounds $A$ and $B$ are said to be optimal if they are the tightest constants satisfying the respective inequality.

In general, an orthonormal basis is a Riesz basis with Riesz bounds $A = B = 1$, but an orthogonal basis is not necessarily norm-bounded below and thus generally not a Riesz basis (for instance, consider the sequence $\{e_n\}_{n=1}^\infty$ where $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for $\mathcal{H}$). Nevertheless, exponential functions have constant norm in $L^2(S)$ for any finite measure set $S \subset \mathbb{R}^d$, namely $\|e^{2\pi i \lambda \cdot \cdot}\|_{L^2(S)} = \sqrt{|S|}$ for all $\lambda \in \mathbb{R}^d$. Thus, an exponential orthogonal basis is simply an exponential orthonormal basis scaled by a common multiplicative factor.

**Proposition 6.** Let $\mathcal{H}$ be a separable Hilbert space.

(a) [7, Corollary 3.7.2] Every subfamily of a Riesz basis is a Riesz sequence with the same bounds (the optimal bounds may be tighter).

(b) [18, Corollary 8.24] If $\{f_n\}_{n \in \mathbb{Z}}$ is a Bessel sequence in $\mathcal{H}$ with Bessel bound $B$, then $\|f_i\|^2 \leq B$ for all $i \in I$. If $\{f_n\}_{n \in \mathbb{Z}}$ is a Riesz sequence in $\mathcal{H}$ with bounds $0 < A \leq B < \infty$, then $A \leq \|f_i\|^2 \leq B$ for all $i \in I$.

(c) [7, Lemma 3.6.9, Theorems 3.6.6, 5.4.1, and 7.1.1] (or see [18, Theorems 7.13, 8.27, and 8.32]) Let $\{e_n\}_{n \in \mathbb{Z}}$ be an orthonormal basis for $\mathcal{H}$ and let $\{f_n\}_{n \in \mathbb{Z}} \subset \mathcal{H}$. The following are equivalent.

• $\{f_n\}_{n \in \mathbb{Z}}$ is a Riesz basis for $\mathcal{H}$.

• $\{f_n\}_{n \in \mathbb{Z}}$ is an exact frame (i.e., a frame that ceases to be a frame whenever a single element is removed) for $\mathcal{H}$.

• $\{f_n\}_{n \in \mathbb{Z}}$ is an unconditional basis of $\mathcal{H}$ with $0 < \inf_{n \in \mathbb{Z}} \|f_n\| \leq \sup_{n \in \mathbb{Z}} \|f_n\| < \infty$.

• There is a bijective bounded operator $T : \mathcal{H} \to \mathcal{H}$ such that $Te_n = f_n$ for all $n \in \mathbb{Z}$.

Moreover in this case, the optimal frame bounds coincide with the optimal Riesz bounds.
Proposition 7 (Proposition 5.4 in [2]). Let \( \{e_n\}_{n \in \mathbb{Z}} \) be an orthonormal basis of a separable Hilbert space \( \mathcal{H} \). Let \( P : \mathcal{H} \to \mathcal{M} \) be the orthogonal projection from \( \mathcal{H} \) onto a closed subspace \( \mathcal{M} \). Let \( J \subset \mathbb{Z} \) and \( 0 < \alpha < 1 \). The following are equivalent.

(i) \( \{P e_n\}_{n \in J} \subset \mathcal{M} \) is a Bessel sequence with optimal bound \( 1 - \alpha \). (Note that \( \{P e_n\}_{n \in J} \) is always a Bessel sequence with bound 1.)

(ii) \( \{P e_n\}_{n \in J} \subset \mathcal{M} \) is a frame for \( \mathcal{M} \) with optimal lower bound \( \alpha \) and upper bound 1 (not necessarily optimal).

(iii) \( \{(I - P)e_n\}_{n \in J} \subset \mathcal{M} \) is a Riesz sequence with optimal lower bound \( \alpha \) and upper bound 1 (not necessarily optimal).

2.2. Exponential systems

As already introduced in Section 1, we define the exponential system \( E(\Lambda) = \{e^{2\pi i \lambda \cdot (\cdot)} : \lambda \in \Lambda\} \) for a discrete set \( \Lambda \subset \mathbb{R}^d \) (called a frequency set or a spectrum).

Lemma 8. Assume that \( E(\Lambda) \) is a Riesz basis for \( L^2(S) \) with optimal bounds \( 0 < A \leq B < \infty \), where \( \Lambda \subset \mathbb{R}^d \) is a discrete set and \( S \subset \mathbb{R}^d \) is a measurable set.

(a) For any \( a \in \mathbb{R}^d \), \( E(\Lambda) \) is a Riesz basis for \( L^2(S + a) \) with bounds \( A \) and \( B \).

(b) For any \( b \in \mathbb{R}^d \), \( E(\Lambda + b) \) is a Riesz basis for \( L^2(S) \) with bounds \( A \) and \( B \).

(c) For any \( \sigma > 0 \), \( \sqrt{\sigma} E(\sigma \Lambda) \) is a Riesz basis for \( L^2(\frac{1}{\sigma} S) \) with bounds \( \frac{A}{\sigma} \) and \( \frac{B}{\sigma} \).

A proof of Lemma 8 is given in Appendix B.

Remark 9. Lemma 8 remains valid if the term “Riesz basis” is replaced with one of the following: “Riesz sequence”, “frame”, and “frame sequence” (and also “Bessel sequence” in which case the lower bound is simply neglected).

Theorem 10 (The Paley-Wiener stability theorem [35]). Let \( V \subset \mathbb{R} \) be a bounded set of positive measure and \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) be a sequence of real numbers such that \( E(\Lambda) \) is a Riesz basis for \( L^2(V) \) (resp. a frame for \( L^2(V) \), a Riesz sequence in \( L^2(V) \)). There exists a constant \( \theta = \theta(\Lambda, V) > 0 \) such that whenever \( \Lambda' = \{\lambda'_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) satisfies

\[ |\lambda'_n - \lambda_n| \leq \theta, \quad n \in \mathbb{Z}, \]

the set of exponentials \( E(\Lambda') \) is a Riesz basis for \( L^2(V) \) (resp. a frame for \( L^2(V) \), a Riesz sequence in \( L^2(V) \)).

For a proof of Theorem 10, we refer to [38, p. 160] for the case where \( V \) is a single interval, and [22, Section 2.3] for the general case. It is worth noting that the constant \( \theta = \theta(\Lambda, V) \) depends on the Riesz bounds of the Riesz basis \( E(\Lambda) \) for \( L^2(V) \), which are determined once \( \Lambda \) and \( V \) are given. Also, it is pointed out in [22, Section 2.3, Remark 2] that the theorem holds also for frames and Riesz sequences.
2.3. Density of frequency sets

The lower and upper (Beurling) density of a discrete set \( \Lambda \subset \mathbb{R}^d \) is defined respectively by (see e.g., [17])

\[
D^- (\Lambda) = \lim_{r \to \infty} \inf_{x \in \mathbb{R}^d} \frac{|\Lambda \cap [x, x+r]|}{r}
\]
and

\[
D^+ (\Lambda) = \lim_{r \to \infty} \sup_{x \in \mathbb{R}^d} \frac{|\Lambda \cap [x, x+r]|}{r}.
\]

If \( D^- (\Lambda) = D^+ (\Lambda) \), we say that \( \Lambda \) has uniform (Beurling) density \( D (\Lambda) := D^- (\Lambda) = D^+ (\Lambda) \). A discrete set \( \Lambda \subset \mathbb{R}^d \) is called separated (or uniformly discrete) if its separation constant \( \Delta (\Lambda) := \inf \{ |\lambda - \lambda'| : \lambda \neq \lambda' \in \Lambda \} \) is positive.

The following proposition is considered folklore. The corresponding statements for Gabor systems of \( L^2 (\mathbb{R}^d) \) are well-known (see [8, Theorem 1.1] and also [16, Lemma 2.2]) and the following proposition can be proved similarly.

**Proposition 11.** Let \( \Lambda \subset \mathbb{R}^d \) be a discrete set and let \( S \subset \mathbb{R}^d \) be a finite positive measure set which is not necessarily bounded.

(i) If \( E (\Lambda) \) is a Bessel sequence in \( L^2 (S) \), then \( D^+ (\Lambda) < \infty \).

(ii) If \( E (\Lambda) \) is a Riesz sequence in \( L^2 (S) \), then \( \Lambda \) is separated, i.e., \( \Delta (\Lambda) > 0 \).

A proof of Proposition 11 is given in Appendix B.

**Theorem 12** ([24, 30]). Let \( \Lambda \subset \mathbb{R}^d \) be a discrete set and let \( S \subset \mathbb{R}^d \) be a finite positive measure set.

(i) If \( E (\Lambda) \) is a frame for \( L^2 (S) \), then \( |S| \leq D^- (\Lambda) \leq D^+ (\Lambda) < \infty \).

(ii) If \( E (\Lambda) \) is a Riesz sequence in \( L^2 (S) \), then \( \Lambda \) is separated and \( D^+ (\Lambda) \leq |S| \).

**Corollary 13.** Let \( \Lambda \subset \mathbb{R}^d \) be a discrete set and let \( S \subset \mathbb{R}^d \) be a finite positive measure set. If \( E (\Lambda) \) is a Riesz basis for \( L^2 (S) \), then \( \Lambda \) is separated and has uniform Beurling density \( D (\Lambda) = |S| \).

3. A result of Olevskii and Ulanovskii

As our main results (Theorems 1 and 2) hinge on the proof technique of Olevskii and Ulanovskii [33], we will briefly review the relevant result in [33].

**Theorem 14** (Theorem 4 in [33]). Let \( 0 < \epsilon < 1 \) and let \( \Lambda \subset \mathbb{R} \) be a separated set with \( D^+ (\Lambda) > 0 \). One can construct a measurable set \( S = S (\epsilon, \Lambda) \subset [-\frac{1}{2}, \frac{1}{2}] \) with \( |S| > 1 - \epsilon \) such that \( E (\Lambda) \) is not a Riesz sequence in \( L^2 (S) \).

The proof of Theorem 14 relies on a technical lemma (Lemma 15 below) which is based on the celebrated Szemerédi’s theorem [36] asserting that any
integer set \( \Omega \subset \mathbb{Z} \) with positive upper Beurling density \( D^+ (\Omega) > 0 \) contains at least one arithmetic progression of length \( M \) for all \( M \in \mathbb{N} \). Here, an arithmetic progression of length \( M \) means a sequence of the form

\[
d, d+P, d+2P, \ldots, d+(M-1)P \quad \text{with} \quad d \in \mathbb{Z} \quad \text{and} \quad P \in \mathbb{N}.
\]

As a side remark, we mention that the common difference \( P \in \mathbb{N} \) of the arithmetic progression resulting from Szemerédi’s theorem, can be restricted to a fairly sparse subset of positive integers \( C \subset \mathbb{N} \). For instance, one can ensure that \( P \) is a multiple of any prescribed number \( L \in \mathbb{N} \), by passing to a subset of \( \Omega \) that is contained in \( L\mathbb{Z} + u \) for some \( u \in \{0, 1, \ldots, L-1\} \) and has positive upper Beurling density. This allows us to take \( C = L\mathbb{N} \) which clearly satisfies \( D^+(C) = 1/L \). Further, one can even choose \( C = \{q, 2q, 3q, \ldots\} \) for any \( q \in \mathbb{N} \), which satisfies

\[
D^+(C) = \begin{cases} 1 & \text{if } q = 1, \\ 0 & \text{if } q > 1. \end{cases}
\]

More generally, one may choose \( C = \{p(n) : n \in \mathbb{N}\} \) for any polynomial \( p \) with rational coefficients such that \( p(0) = 0 \) and \( p(n) \in \mathbb{Z} \) for \( n \in \mathbb{Z} \backslash \{0\} \) (see [3] p.733). On the other hand, it was shown in [10, Theorem 7] that \( C \subset \mathbb{N} \) cannot be a lacunary sequence, i.e., a sequence \( \{a_n\}_{n=1}^{\infty} \) satisfying \( \liminf_{n \to \infty} a_{n+1}/a_n > 1 \) (for instance, \( \{2^n : n = 0, 1, 2, \ldots\} \)). Note that the aforementioned set \( C = \{p(n) : n \in \mathbb{N}\} \) can be sparse but not lacunary since \( \lim_{n \to \infty} p(n+1)/p(n) = 1 \) for any polynomial \( p \). We refer to [FLW16, Section 2] for a short review on the possible choice of (deterministic) sets \( C \subset \mathbb{N} \), and also for the situation where \( C \) is chosen randomly.

**Lemma 15** (Lemma 5.1 in [33]). Let \( \Lambda \subset \mathbb{R} \) be a separated set with \( D^+(\Lambda) > 0 \). For any \( M \in \mathbb{N} \) and \( \delta > 0 \), there exist constants \( c = c(M, \delta, \Lambda) \in \mathbb{N} \), \( d = d(M, \delta, \Lambda) \in \mathbb{R} \), and an increasing sequence \( s(-M) < s(-M+1) < \ldots < s(M) \) in \( \Lambda \) such that

\[
|s(j) - cj - d| \leq \delta \quad \text{for} \quad j = -M, \ldots, M.
\]

Moreover, the constant \( c = c(M, \delta, \Lambda) \in \mathbb{N} \) can be chosen to be a multiple of any prescribed number \( L \in \mathbb{N} \).

As Lemma 15 will be used in the proof of Theorem 2, we include a short proof of Lemma 15 in Appendix B for self-containedness of the paper.

---

2When restricted to subsets of integers \( \Lambda \subset \mathbb{Z} \), the upper Beurling density is equal to the so-called upper Banach density which is defined as \( \limsup_{n \to \infty} \sup_{x \in \mathbb{Z}} |\Lambda \cap \{n+1, n+2, \ldots, n+r\}|/r \). In the literature, Szemerédi’s theorem is often stated for sets \( \Lambda \subset \mathbb{N} \) with positive upper natural density (upper asymptotic density) \( \limsup_{n \to \infty} |\Lambda \cap \{1, 2, \ldots, r\}|/r > 0 \). It should be noted that the statements of Szemerédi’s theorem with different types of density are equivalent, but the proofs are not easily converted from one density type to the other.
4. Proof of Theorem 1

Before proving Theorem 1, we note that Theorem 2 is an extension of Theorem [4] from a single set $A \subset \mathbb{R}$ to a countable family of sets $A_1, A_2, \ldots \subset \mathbb{R}$. We will first consider a particular choice of sets $A_1 = \alpha \mathbb{Z}$, $A_2 = 2\alpha \mathbb{Z}$, $A_3 = 3\alpha \mathbb{Z}$, \ldots for any fixed $0 < \alpha \leq 1$, from which a desired set for Theorem 1 will be acquired.

**Proposition 16.** Let $0 < \alpha \leq 1$ and $A_1 = \alpha \mathbb{Z}$, $A_2 = 2\alpha \mathbb{Z}$, $A_3 = 3\alpha \mathbb{Z}$, \ldots, that is, $A_\ell = \ell \alpha \mathbb{Z}$ for $\ell \in \mathbb{N}$. Given any $0 < \epsilon < 1$, one can construct a measurable set $S \subset \left[ -\frac{1}{2}, \frac{1}{2} \right]$ with $|S| > 1 - \epsilon$ such that $E(\Lambda_\ell)$ is not a Riesz sequence in $L^2(S)$ for all $\ell \in \mathbb{N}$.

**Proof.** Fix any $0 < \epsilon < 1$ and choose an integer $R > \frac{1}{1 - \epsilon}$ so that $0 < \epsilon < \frac{R - 1}{R}$. We claim that for each $0 < \eta < \frac{R - 1}{R}$ there exists a set $V_\eta \subset \left[ -\frac{1}{2}, \frac{1}{2} \right]$ with $|V_\eta| < \eta$ satisfying the following property: for each $\ell \in \mathbb{N}$ there is a finitely supported sequence $b^{(\eta, \ell)}_j = \{b^{(\eta, \ell)}_j\}_{j \in \mathbb{Z}}$ satisfying

$$\int_{\left[-\frac{1}{2}, \frac{1}{2}\right] \setminus V_\eta} \left| \sum_{j \in \mathbb{Z}} b^{(\eta, \ell)}_j e^{2\pi i x j} \right|^2 \, dx \leq R^{\frac{1}{2\pi}} \sum_{j \in \mathbb{Z}} |b^{(\eta, \ell)}_j|^2. \tag{4}$$

If this claim is proved, one could take $V := \cup_{\eta=1}^\infty V_\eta$ and $S := \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus V$. Indeed, we have $|V| \leq \sum_{\eta=1}^\infty |V_\eta| < \sum_{\eta=1}^\infty \frac{\epsilon}{\eta} = \epsilon$, so that $|S| > 1 - \epsilon$. Also, it holds for any $k, \ell \in \mathbb{N},$

$$\int_S \left| \sum_{j \in \mathbb{Z}} b^{(\epsilon, 2^k, \ell)}_j e^{2\pi i x j} \right|^2 \, dx \leq \int_{\left[-\frac{1}{2}, \frac{1}{2}\right] \setminus V_{\epsilon/2^k}} \left| \sum_{j \in \mathbb{Z}} b^{(\epsilon, 2^k, \ell)}_j e^{2\pi i x j} \right|^2 \, dx$$

$$\leq R^{\frac{1}{2\pi}} \sum_{j \in \mathbb{Z}} |b^{(\epsilon, 2^k, \ell)}_j|^2. \tag{5}$$

By fixing any $\ell \in \mathbb{N}$ and letting $k \to \infty$, we conclude that $E(\ell \alpha \mathbb{Z})$ is not a Riesz sequence in $L^2(S)$.

To prove the claim (5), fix any $0 < \eta < \frac{R - 1}{R}$. For each $\ell \in \mathbb{N}$, let $\tilde{a}^{(\eta/2^\ell)} := \{\tilde{a}^{(\eta/2^\ell)}_j\}_{j \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$ be the sequence given by

$$\tilde{a}^{(\eta/2^\ell)}_j := \begin{cases} \sqrt{\frac{2^{2^{\ell+1}}}{2^{2^{\ell+1}}}} & \text{if } j = 0, \\ \sqrt{\frac{2^{2^{\ell+1}}}{\eta^2}} \sin \left( \frac{\pi j \eta}{2^{2^{\ell+1}}} \right) & \text{if } j \neq 0, \end{cases} \tag{5.5}$$

which is the Fourier coefficient of the 1-periodic function

$$\tilde{\nu}^{(\eta/2^\ell)}(x) := \begin{cases} \sqrt{\frac{2^{2^{\ell+1}}}{\eta^2}} & \text{for } x \in \left[ -\frac{\eta}{4^{2^{\ell+1}}}, \frac{\eta}{4^{2^{\ell+1}}} \right], \\ 0 & \text{for } x \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus \left[ -\frac{\eta}{4^{2^{\ell+1}}}, \frac{\eta}{4^{2^{\ell+1}}} \right], \end{cases}$$

that is, $\tilde{\nu}^{(\eta/2^\ell)}(x) = \sum_{j \in \mathbb{Z}} \tilde{a}^{(\eta/2^\ell)}_j e^{2\pi i j x}$ for a.e. $x \in \left[ -\frac{1}{2}, \frac{1}{2} \right]$. Note that $\|\tilde{a}^{(\eta/2^\ell)}\|_{\ell^2} = \|\tilde{\nu}^{(\eta/2^\ell)}\|_{L^2(\left[-\frac{1}{2}, \frac{1}{2}\right])} = 1$. Choose a number $M := M(\eta/2^\ell) \in \mathbb{N}$ satisfying

$$\sum_{|j| > M} \left| \tilde{a}^{(\eta/2^\ell)}_j \right|^2 < \frac{1}{\alpha} \cdot \left( \frac{\eta}{2^\ell} \right) = \frac{\eta}{2^\ell},$$
so that
\[
\sum_{j=-M}^{M} |\tilde{a}_j^{(\eta_0/2^\ell)}|^2 > 1 - \frac{2}{2^\ell} \geq 1 - \eta > \frac{1}{M}.
\] (7)

Now, the set \(\Lambda_\ell\) comes into play. We write \(\Lambda_\ell = \{s_\ell(j) : j \in \mathbb{Z}\}\) with \(s_\ell(j) := \ell \alpha j\) for all \(j \in \mathbb{Z}\).

For \(x \in [-\frac{1}{2}, \frac{1}{2}]\), we define
\[
\tilde{f}_{\eta_0/2^\ell, \Lambda_\ell}(x) := \sum_{j=-M}^{M} \tilde{a}_j^{(\eta_0/2^\ell)} e^{2\pi i s_\ell(j)x}
\] (8)

and observe that
\[
\tilde{f}_{\eta_0/2^\ell, \Lambda_\ell}(x) - \tilde{p}_{\eta_0/2^\ell}(\ell \alpha x) = - \sum_{|j|>M} \tilde{a}_j^{(\eta_0/2^\ell)} e^{2\pi i \alpha jx}.
\] (9)

Setting \(V_{\eta}(\ell) := [-\frac{1}{2}, \frac{1}{2}] \cap \text{supp} \tilde{p}_{\eta_0/2^\ell}(\ell \alpha x)\) for \(\ell \in \mathbb{N}\), we obtain
\[
\int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_{\eta}(\ell)} \left| \tilde{f}_{\eta_0/2^\ell, \Lambda_\ell}(x) \right|^2 dx 
\leq \int_{-1/2}^{1/2} \left| \sum_{|j|>M} \tilde{a}_j^{(\eta_0/2^\ell)} e^{2\pi i \alpha jx} \right|^2 dx = \sum_{|j|>M} |\tilde{a}_j^{(\eta_0/2^\ell)}|^2
\]
\[
< \frac{2}{2^\ell} \sum_{j=-M}^{M} |\tilde{a}_j^{(\eta_0/2^\ell)}|^2.
\]

Note from (6) that \(\text{supp} \tilde{p}_{\eta_0/2^\ell}(\ell \alpha x) = \frac{1}{\ell \alpha} \bigcup_{m \in \mathbb{Z}} (m + [-\frac{\eta_0}{4^\ell \cdot 2^\ell}, \frac{\eta_0}{4^\ell \cdot 2^\ell}]) = \bigcup_{m \in \mathbb{Z}} (m + [-\frac{\eta}{4^\ell}, \frac{\eta}{4^\ell}])\) which implies \(|V_{\eta}(\ell)| < \frac{2}{2^\ell}\). Indeed, the set \([-\frac{1}{2}, \frac{1}{2}] \cap \bigcup_{m \in \mathbb{Z}} (m + [-\frac{\eta}{4^\ell}, \frac{\eta}{4^\ell}])\) is of length \(\frac{1}{2^\ell}\), and the dilated set \(\frac{1}{\ell} \bigcup_{m \in \mathbb{Z}} (m + [-\frac{\eta}{4^\ell}, \frac{\eta}{4^\ell}])\) restricted to \([-\frac{1}{2}, \frac{1}{2}]\) has Lebesgue measure \(\frac{\eta}{2^\ell}\) as well, so the set \(V_{\eta}(\ell) = [-\frac{1}{2}, \frac{1}{2}] \cap \bigcup_{m \in \mathbb{Z}} (m + [-\frac{\eta}{4^\ell}, \frac{\eta}{4^\ell}])\) with \(0 < \alpha \leq 1\) has Lebesgue measure at most \(\frac{\eta}{2^\ell}\) which is strictly less than \(\frac{\eta}{2^\ell}\).

Finally, define \(V_\eta := \bigcup_{\ell=1}^{\infty} V_{\eta}(\ell)\) which clearly satisfies \(|V_\eta| \leq \sum_{\ell=1}^{\infty} |V_{\eta}(\ell)| < \sum_{\ell=1}^{\infty} \frac{\eta}{2^\ell} = \eta\). Then for each \(\ell \in \mathbb{N}\),
\[
\int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_\eta} \left| \sum_{j=-M}^{M} \tilde{a}_j^{(\eta_0/2^\ell)} e^{2\pi i \alpha jx} \right|^2 dx = \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_\eta} \left| \tilde{f}_{\eta_0/2^\ell, \Lambda_\ell}(x) \right|^2 dx
\]
\[
\leq \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_\eta} \left| \tilde{f}_{\eta_0/2^\ell, \Lambda_\ell}(x) \right|^2 dx < R \frac{2}{2^\ell} \sum_{j=-M}^{M} |\tilde{a}_j^{(\eta_0/2^\ell)}|^2
\]
which establishes the claim (4). This completes the proof. \(\square\)
Proof of Theorem 1. Let $\Lambda$ in Remark 17.

In the proof above, the set $S$ is constructed as follows. Given any $0 < \epsilon < 1$, choose an integer $R > \frac{1}{\epsilon^2}$ so that $0 < \epsilon < \frac{R-1}{R}$. The set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ is then given by $S := [-\frac{1}{2}, \frac{1}{2}] \setminus V$ with $V := \bigcup_{k=1}^{\infty} V_{\epsilon/2^k}$, where

$$V_{\eta} := \bigcup_{k=1}^{\infty} V_{\eta}^{(f)}$$

and

$$V_{\eta}^{(f)} := \left[-\frac{1}{2}, \frac{1}{2}\right] \cap \operatorname{supp} \hat{p}_{\eta/2^f}(\ell x)$$

$$= \left[-\frac{1}{2}, \frac{1}{2}\right] \cap \left( \bigcup_{m \in \mathbb{Z}} \left( m + \left[ \frac{m}{4^f}, \frac{m}{4^f} \right] \right) \right)$$

$$= \left[-\frac{1}{2}, \frac{1}{2}\right] \cap \left( \bigcup_{m \in \mathbb{Z}} \left( \left[ \frac{m}{4^f}, \frac{m}{4^f} \right] \right) \right)$$

for any $0 < \eta < \frac{R-1}{R}$ and $f \in \mathbb{N}$.

In short,

$$S := [-\frac{1}{2}, \frac{1}{2}] \setminus V$$

with

$$V := \bigcup_{k=1}^{\infty} V_{\epsilon/2^k} = \bigcup_{k=1}^{\infty} V_{\epsilon/2^k}^{(f)}$$

$$= \left[-\frac{1}{2}, \frac{1}{2}\right] \cap \left( \bigcup_{m \in \mathbb{Z}} \left( m + \left[ \frac{m}{4^f}, \frac{m}{4^f} \right] \right) \right)$$

$$= \left[-\frac{1}{2}, \frac{1}{2}\right] \cap \left( \bigcup_{m \in \mathbb{Z}} \left( \left[ \frac{m}{4^f}, \frac{m}{4^f} \right] \right) \right)$$

where the set $V$ satisfies $|V| < \epsilon$ and thus $|S| > 1 - \epsilon$. See Figure 1 for an illustration of the set $V$.

We are now ready to prove Theorem 1. Note that the set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ stated in Theorem 1 is exactly the resulting set of Proposition 16 which is described in Remark 17.

Proof of Theorem 1. Let $\Lambda \subset \mathbb{R}$ be a set containing arbitrarily long arithmetic progressions with a fixed common difference $P\alpha$ for some $P \in \mathbb{N}$. To prove that $E(\Lambda)$ is not a Riesz sequence in $L^2(S)$, it suffices to show that the set $V_{\eta} \subset [-\frac{1}{2}, \frac{1}{2}]$ given by (10) for $0 < \eta < \frac{R-1}{R}$ and $f \in \mathbb{N}$ (with a fixed integer $R > \frac{1}{\epsilon^2}$), satisfies the following property: there is a finitely supported sequence $b^{(\eta, \Lambda)}(\lambda) = \{b^{(\eta, \Lambda)}_{\lambda}\}_{\lambda \in \Lambda}$ with

$$\int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_0} \left| \sum_{\lambda \in \Lambda} b^{(\eta, \Lambda)}_{\lambda} e^{2\pi i \lambda x} \right|^2 dx \leq \frac{R}{2\pi} \sum_{\lambda \in \Lambda} |b^{(\eta, \Lambda)}_{\lambda}|^2.$$  \hspace{1cm} (11)

Indeed, since $S := [-\frac{1}{2}, \frac{1}{2}] \setminus \bigcup_{k=1}^{\infty} V_{\epsilon/2^k}$ (see Remark 17), it then holds for any $k \in \mathbb{N},$

$$\int_{S} \left| \sum_{\lambda \in \Lambda} b^{(\epsilon/2^k, \Lambda)}_{\lambda} e^{2\pi i \lambda x} \right|^2 dx \leq \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus S} \left| \sum_{\lambda \in \Lambda} b^{(\epsilon/2^k, \Lambda)}_{\lambda} e^{2\pi i \lambda x} \right|^2 dx \leq \frac{R}{2\pi} \sum_{\lambda \in \Lambda} |b^{(\epsilon/2^k, \Lambda)}_{\lambda}|^2$$  \hspace{1cm} (11)
which implies that \( E(\Lambda) \) is not a Riesz sequence in \( L^2(S) \).

To prove the claim (11), consider the sequence \( \tilde{a}^{(\eta\alpha/2^\nu)} = \{ \tilde{a}_j^{(\eta\alpha/2^\nu)} \}_{j \in \mathbb{Z}} \in \ell_2(\mathbb{Z}) \), the function \( \tilde{p}_\eta/2^\nu \), and the number \( \tilde{M} = \tilde{M}(\eta\alpha/2^\nu) \in \mathbb{N} \) taken respectively from \([4]-[7]\) with \( \ell = P \). By the assumption, the set \( \Lambda \subset \mathbb{R} \) contains an arithmetic progression of length \( 2\tilde{M}+1 \) with common difference \( P\alpha \), which can be expressed as

\[
s_\Lambda(j) := P\alpha j + d, \quad j = -\tilde{M}, \ldots, \tilde{M},
\]

for some \( d \in \mathbb{Z} \). Similarly as in \([8]\) and \([9]\), we define

\[
\tilde{f}_\Lambda(x) := \sum_{j=-\tilde{M}}^{\tilde{M}} \tilde{a}_j^{(\eta\alpha/2^\nu)} e^{2\pi i s_\Lambda(j)x} \quad \text{for } x \in \mathbb{R}
\]

and observe that

\[
\tilde{f}_\Lambda(x) - \tilde{p}_{\eta\alpha/2^\nu}(P\alpha x) e^{2\pi i dx} = \sum_{|j|>\tilde{M}} |\tilde{a}_j^{(\eta\alpha/2^\nu)}| e^{2\pi i (P\alpha j + dx)} \quad \text{for all } x \in \mathbb{R}.
\]

Recalling that \( V_\eta^{(P)} := [-\frac{1}{2}, \frac{1}{2}] \cap \text{supp } \tilde{p}_{\eta\alpha/2^\nu}(P\alpha x) \) (see Remark 17), we have

\[
\int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_\eta^{(P)}} |\tilde{f}_\Lambda(x)|^2 dx 
\leq \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_\eta^{(P)}} \left| \sum_{|j|>\tilde{M}} \tilde{a}_j^{(\eta\alpha/2^\nu)} e^{2\pi i (P\alpha j + dx)} \right|^2 dx 
\leq \frac{n}{2^\nu} < R \frac{n}{2^\nu}
\]

where the inequality \([7]\) for \( \ell = P \) is used in the last step. Since \( V_\eta := \bigcup_{\ell = 1}^\infty V_\eta^{(\ell)} \), we have

\[
\int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_\eta} \left| \sum_{j=-\tilde{M}}^{\tilde{M}} \tilde{a}_j^{(\eta\alpha/2^\nu)} e^{2\pi i s_\Lambda(j)x} \right|^2 dx = \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_\eta} |\tilde{f}_\Lambda(x)|^2 dx 
\leq \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_\eta^{(P)}} |\tilde{f}_\Lambda(x)|^2 dx < R \frac{n}{2^\nu}
\]

which establishes the claim (11).

5. Proof of Theorem 2

We will now prove Theorem 2 which generalizes Proposition 16 from \( \Lambda_1 = \alpha \mathbb{Z}, \Lambda_2 = 2\alpha \mathbb{Z}, \Lambda_3 = 3\alpha \mathbb{Z}, \ldots \) to arbitrary separated sets \( \Lambda_1, \Lambda_2, \ldots \subset \mathbb{R} \) with positive upper Beurling density. The proof is similar to the proof of Proposition 16 but since an arbitrary separated set is in general non-periodic we need the additional step of extracting an approximate arithmetic progression from each set \( \Lambda_\ell \) with the help of Lemma 15.
**Proof of Theorem 2**  Fix any $0 < \epsilon < 1$ and choose an integer $R > \frac{1}{\epsilon}$ so that $0 < \epsilon < \frac{R^{-1}}{R}$. We claim that for each $0 < \eta < \frac{R^{-1}}{R}$ there exists a set $V_\eta \subset [-\frac{1}{2}, \frac{1}{2}]$ with $|V_\eta| < \eta$ satisfying the following property: for each $\ell \in \mathbb{N}$ there is a finitely supported sequence $b(\eta, \Lambda_\ell) = \{b^\ell\}_{\lambda \in \Lambda_\ell}$ with

$$\int_{[-\frac{1}{2}, \frac{1}{2}]} \left| \sum_{\lambda \in \Lambda_\ell} b^\ell \lambda e^{2\pi i \lambda x} \right|^2 \, dx \leq R \eta^2 \sum_{\lambda \in \Lambda_\ell} |b^\ell \lambda|^2. \quad (12)$$

To prove the claim (12), fix any $0 < \eta < \frac{R^{-1}}{R}$. For each $\ell \in \mathbb{N}$, let $a(\eta/2\ell) = \{a_j(\eta/2\ell)\}_{j \in \mathbb{Z}}$ be an $\ell_1$-sequence with unit $\ell_2$-norm $\|a(\eta/2\ell)\|_{\ell_2} = 1$ such that

$$p_{\eta/2\ell}(x) := \sum_{j \in \mathbb{Z}} a_j(\eta/2\ell) e^{2\pi i j x}$$

satisfies $p_{\eta/2\ell}(x) = 0$ for $\frac{\eta}{4\ell^2} \leq |x| \leq \frac{1}{2}$. \quad (13)

Since the sequence $a(\eta/2\ell) \in \ell_1(\mathbb{Z})$ is not finitely supported, there is a number $M = M(\eta/2\ell) \in \mathbb{N}$ with $0 < \sum_{|j| > M} |a_j(\eta/2\ell)| < \frac{\eta}{R}$. Note that since $|a_j(\eta/2\ell)| \leq \|a(\eta/2\ell)\|_{\ell_2} = 1$ for all $j \in \mathbb{Z}$, we have

$$\sum_{|j| > M} |a_j(\eta/2\ell)|^2 = \sum_{|j| > M} |a_j(\eta/2\ell)| < \frac{\eta}{R},$$

so that

$$\sum_{j = -M}^{M} |a_j(\eta/2\ell)|^2 > 1 - \frac{\eta}{R} \geq 1 - \eta > \frac{1}{R}. \quad (14)$$

We then choose a small parameter $0 < \delta = \delta(\eta/2\ell) < 1$ satisfying

$$\sin(\pi \delta/2) < \frac{\eta/2\ell}{2 \sum_{j = -M}^{M} |a_j(\eta/2\ell)|},$$

so that $\sum_{j = -M}^{M} |a_j(\eta/2\ell)| \cdot |e^{i\pi \delta} - 1| = \sum_{j = -M}^{M} |a_j(\eta/2\ell)| \cdot 2 \sin(\pi \delta/2) < \frac{\eta}{4\ell}$. Note that all the terms up to this point depend only on the parameters $\eta$ and $\ell$, in fact, only on the value $\eta/2\ell$.

Now, the set $\Lambda_\ell$ comes into play. Applying Lemma 15 to the set $\Lambda_\ell$ with the parameters $M$ and $\delta$ chosen above, we deduce that there exist constants $c = c(\eta/2\ell, \Lambda_\ell) \in \mathbb{N}$ and $d = d(\eta/2\ell, \Lambda_\ell) \in \mathbb{R}$, and an increasing sequence $s_{\eta/2\ell, \Lambda_\ell}(-M) < s_{\eta/2\ell, \Lambda_\ell}(-M + 1) < \ldots < s_{\eta/2\ell, \Lambda_\ell}(M)$ in $\Lambda_\ell$ satisfying

$$|s_{\eta/2\ell, \Lambda_\ell}(j) - cj - d| \leq \delta \quad \text{for } j = -M, \ldots, M.$$  

For $x \in [-\frac{1}{2}, \frac{1}{2}]$, we define

$$f_{\eta/2\ell, \Lambda_\ell}(x) := \sum_{j = -M}^{M} a_j(\eta/2\ell) \exp(2\pi i s_{\eta/2\ell, \Lambda_\ell}(j)x).$$

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and observe that
\[ |f_{n/2^j, \Lambda}(x) - p_{n/2^j}(c_x) e^{2\pi i dx}| \]
\[ \leq \left| \sum_{j=-M}^{M} a_j^{(n/2^j)} \left( \exp \left( 2\pi i s_{n/2^j, \Lambda}(j)x \right) - e^{2\pi i (c_j + d)x} \right) \right| + \left| \sum_{|j|>M} a_j^{(n/2^j)} e^{2\pi i (c_j + d)x} \right| \]
\[ \leq \sum_{j=-M}^{M} |a_j^{(n/2^j)}| \cdot \left| \exp \left( 2\pi i (s_{n/2^j, \Lambda}(j) - c_j - d)x \right) - 1 \right| + \sum_{|j|>M} |a_j^{(n/2^j)}| \]
\[ < \frac{\eta}{2^2} + \frac{\eta}{2^2} = \frac{\eta}{2^{2+2}} \leq \eta. \]

Setting \( V_{n=(1)} := [-\frac{1}{2}, \frac{1}{2}] \cap \text{supp } p_{n/2^j}(c(n/2^j, \Lambda_x)x), \) we have
\[ \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_{n=(1)}} |f_{n/2^j, \Lambda}(x)|^2 \, dx \leq \eta^2 \leq R \eta^2 \sum_{j=-M}^{M} |a_j^{(n/2^j)}|^2. \]

Similarly as in the proof of Proposition \ref{prop16} we have \( |V_{n=(1)}| < \frac{\eta}{2^2} \) and therefore the set \( V_{n} := \cup_{\ell=1}^{\infty} V_{n=(\ell)} \) satisfies \( |V_{n}| < \eta. \) It then holds for each \( \ell \in \mathbb{N}, \)
\[ \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_{n}} \sum_{j=-M}^{M} a_j^{(n/2^j)} \exp \left( 2\pi i s_{n/2^j, \Lambda}(j)x \right)^2 \, dx = \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_{n}} |f_{n/2^j, \Lambda}(x)|^2 \, dx \]
\[ \leq \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_{n}} |f_{n/2^j, \Lambda}(x)|^2 \, dx < R \eta^2 \sum_{j=-M}^{M} |a_j^{(n/2^j)}|^2 \]
which proves the claim \ref{claim12}.

Finally, based on the established claim \ref{claim12} we define \( V := \cup_{k=1}^{\infty} V_{\epsilon/2^k} \) and \( S := [-\frac{1}{2}, \frac{1}{2}] \setminus V. \) Clearly, we have \( |V| \leq \sum_{k=1}^{\infty} |V_{\epsilon/2^k}| < \sum_{k=1}^{\infty} \frac{\epsilon}{2^k} = \epsilon, \) so that \( |S| > 1 - \epsilon. \) Also, it holds for any \( k, \ell \in \mathbb{N}, \)
\[ \int_{S} \left| \sum_{\lambda \in \Lambda_k} b_{\lambda}^{(\epsilon/2^k, \Lambda)} e^{2\pi i \lambda x} \right|^2 \, dx \leq \int_{[-\frac{1}{2}, \frac{1}{2}] \setminus V_{\epsilon/2^k}} \left| \sum_{\lambda \in \Lambda_k} b_{\lambda}^{(\epsilon/2^k, \Lambda)} e^{2\pi i \lambda x} \right|^2 \, dx \]
\[ \leq R \left( \frac{\epsilon}{2^k} \right)^2 \sum_{\lambda \in \Lambda_k} |b_{\lambda}^{(\epsilon/2^k, \Lambda)}|^2. \]

By fixing any \( \ell \in \mathbb{N} \) and letting \( k \to \infty, \) we conclude that \( E(\Lambda_k) \) is not a Riesz sequence in \( L^2(S). \)

\[ \square \]

**Remark 18** (The construction of \( S \) for arbitrary separated sets \( \Lambda_1, \Lambda_2, \ldots \subset \mathbb{R} \)).

In the proof above, the set \( S \) for arbitrary separated sets \( \Lambda_1, \Lambda_2, \ldots \subset \mathbb{R} \) is constructed as follows. Given any \( 0 < \epsilon < 1, \) choose an integer \( R > \frac{1}{1-\epsilon} \) so that \( 0 < \epsilon < \frac{R-1}{R}. \) The set \( S \subset [-\frac{1}{2}, \frac{1}{2}] \) is then given by \( S := [-\frac{1}{2}, \frac{1}{2}] \setminus V \) with
\[ V := \bigcup_{k=1}^{\infty} V_{\varepsilon/2^k}, \text{ where} \]

\[ V_0 := \bigcup_{k=1}^{\infty} V_{\eta/2^k}^{(k)} \quad \text{and} \]

\[ V_{\eta/2^k}^{(k)} := \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \text{supp} p_{\eta/2^k} \left( c(\eta/2^k, \Lambda_k) x \right) = \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \left( \frac{1}{c(\eta/2^k, \Lambda_k)} \text{supp} p_{\eta/2^k} \right) \]

\[ \subset \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \left( \bigcup_{m \in \mathbb{Z}} \left( \frac{m}{c(\eta/2^k, \Lambda_k)} + \left[ -\frac{\eta}{4c(\eta/2^k, \Lambda_k)}, \frac{\eta}{4c(\eta/2^k, \Lambda_k)} \right] \right) \right) \]

for any \( 0 < \eta < \frac{B-1}{M} \) and \( k \in \mathbb{N} \).

Here, \( c(\eta/2^k, \Lambda_k) \) is a positive integer depending on the value \( \eta/2^k \) and the set \( \Lambda_k \). In short,

\[ S := \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus V \quad \text{with} \]

\[ V := \bigcup_{k=1}^{\infty} V_{\varepsilon/2^k} = \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} V_{\eta/2^k}^{(k)} \]

\[ = \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \left( \bigcup_{k=1}^{\infty} \bigcup_{\ell=1}^{\infty} \frac{1}{c(\varepsilon/2^{k+\ell}, \Lambda_\ell)} \text{supp} p_{\varepsilon/2^{k+\ell}} \right) \]

\[ \subset \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \left( \bigcup_{k=1}^{\infty} \bigcup_{m \in \mathbb{Z}} \left( \frac{m}{c(\varepsilon/2^{k+\ell}, \Lambda_\ell)} + \left[ -\frac{\varepsilon}{4c(\varepsilon/2^{k+\ell}, \Lambda_\ell)}, \frac{\varepsilon}{4c(\varepsilon/2^{k+\ell}, \Lambda_\ell)} \right] \right) \right) \].

Recall that we were able to eliminate the union \( \bigcup_{k=1}^{\infty} \) in the expression of \( V \) in Remark\[17\], because for any fixed \( \ell \in \mathbb{N} \) the sets \( \left[ \frac{1}{2^{k+\ell}}, \frac{1}{2^{k+\ell}}, \frac{1}{2^{k+\ell}} \right] \), \( k = 1, 2, \ldots \)

are decreasingly nested. Unfortunately, the trick cannot be applied here even if \( \text{supp} p_{\varepsilon/2^{k+\ell}} \cap \left[ -\frac{1}{2}, \frac{1}{2} \right] = \left[ -\frac{\varepsilon}{2^{k+\ell}}, \frac{\varepsilon}{2^{k+\ell}} \right] \) for all \( k, \ell \in \mathbb{N} \) and the numbers \( c(\varepsilon/2^{k+\ell}, \Lambda_\ell), k = 1, 2, \ldots \) increase by factors of positive integers (exploiting the ‘moreover’ part of Lemma\[15\]) for \( \ell \in \mathbb{N} \) fixed, in which case the sets \( \left[ -\frac{\varepsilon}{4c(\varepsilon/2^{k+\ell}, \Lambda_\ell)}, \frac{\varepsilon}{4c(\varepsilon/2^{k+\ell}, \Lambda_\ell)} \right], k = 1, 2, \ldots \) are decreasingly nested for \( \ell \in \mathbb{N} \) fixed. This is because the period \( \frac{1}{c(\varepsilon/2^{k+\ell}, \Lambda_\ell)} \) of the periodization involved with the union \( \bigcup_{m \in \mathbb{Z}} \), depends also on \( k \).

6. Remarks

Let us discuss some obstacles in extending our results (Theorems\[1\] and \[2\]) to the class of arbitrary separated sets \( \Lambda \subset \mathbb{R} \) with positive upper Beurling density. Our result relies on the proof technique of Olevskii and Ulanovskii\[33\] which is based on the celebrated Szemerédi’s theorem\[36\] stating that

any integer set \( \Omega \subset \mathbb{Z} \) with positive upper Beurling density \( D^+(\Omega) > 0 \) contains at least one arithmetic progression of length \( M \) for all \( M \in \mathbb{N} \).

If it were even true that for any integer set \( \Omega \subset \mathbb{Z} \) with \( D^+(\Omega) > 0 \),

there exists a number \( P \in \mathbb{N} \) such that \( \Omega \) contains arbitrarily long arithmetic progressions with common difference \( P \),

(*)
then Theorem 1 would imply a stronger result:

\[ S \subset [-\frac{1}{2}, \frac{1}{2}] \] be the set given by \( \alpha \) with \( \alpha = 1 \) and any \( 0 < \epsilon < 1 \).

If \( \Lambda \subset \mathbb{R} \) is a separated set with \( D^+(\Lambda) > 0 \), then \( E(\Lambda) \) is not a Riesz sequence in \( L^2(S) \).

To see this, suppose to the contrary that \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) is a separated set with \( D^+(\Lambda) > 0 \) such that \( E(\Lambda) \) is a Riesz sequence in \( L^2(S) \). Then according to Theorem 10 there is a constant \( \theta = \theta(\Lambda, S) > 0 \) such that \( E(\Lambda') \) is a Riesz sequence in \( L^2(S) \) whenever \( \Lambda' = \{\lambda_n'\}_{n \in \mathbb{Z}} \subset \mathbb{R} \) satisfies \( |\lambda_n' - \lambda_n| \leq \theta \) for all \( n \in \mathbb{Z} \). This allows for a replacement of the set \( \Lambda \subset \mathbb{R} \) with its perturbation \( \Lambda' \subset \frac{1}{N} \mathbb{Z} \) for some large \( N \in \mathbb{N} \). Certainly, the set \( N\Lambda' \subset \mathbb{Z} \) satisfies \( D^+(N\Lambda') > 0 \), and thus \( \Omega \) would imply that there is a number \( P \in \mathbb{N} \) such that \( N\Lambda' \) contains arbitrarily long arithmetic progressions with common difference \( P \). In turn, the set \( \Lambda' \) would also contain arbitrarily long arithmetic progressions with common difference \( P \), and therefore \( E(\Lambda') \) would not be a Riesz sequence in \( L^2(S) \) by Theorem 1 yielding a contradiction.

Unfortunately, as shown in the following example, there exist some sets \( \Omega \subset \mathbb{Z} \) with \( D^+(\Omega) > 0 \) which do not satisfy \( \Omega \). Hence, the improvement of Theorem 1 to \( (15) \) does not work as we wished.

**Example.** As mentioned in [6] Section 1], very little is known about the integer sets that satisfy \( \Omega \), i.e., the integer sets containing arbitrarily long arithmetic progressions with fixed common difference. Motivated by a discussion in [6], we will now provide a set \( \Omega \subset \mathbb{Z} \) with positive uniform Beurling density which does not satisfy \( \Omega \).

Consider the set \( \Omega := \pm\Omega_0 = \Omega_0 \cup (-\Omega_0) \) where \( \Omega_0 := \{1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, \ldots\} \) is the set of square-free integers, i.e., the integers that are not divisible by \( n^2 \) for \( n \in \mathbb{N} \) prime. It is well-known (see e.g., [17]) that \( \lim_{r \to \infty} \frac{|\Omega_0 \cap \{1, 2, \ldots, r\}|}{r} \approx 0.6079 \), which implies \( D(\Omega) = \frac{6}{\pi^2} \). Note that if \( \Omega \) contains a long arithmetic progression, then either \( -\Omega_0 \) or \( \Omega_0 \) contains at least half portion of that arithmetic progression. By symmetry, this implies that if \( \Omega \) holds for \( \Omega_0 \), then it holds also for \( -\Omega_0 \). Thus, to prove that \( \Omega \) does not hold for \( \Omega_0 \), it will be enough to show that \( \Omega \) does not hold for \( \Omega_0 \).

Suppose to the contrary that \( \Omega \) holds for \( \Omega_0 \). Then there is a number \( P \in \mathbb{N} \) such that \( \Omega_0 \) contains an arithmetic progression of length \( Q^2 \) with common difference \( P \), where \( Q \in \mathbb{N} \) is any prime number greater than \( P \), that is,

\[ d, d+P, d+2P, \ldots, d+(Q^2-1)P \in \Omega_0 \quad \text{for some} \quad d \in \mathbb{N}. \]

Since \( Q \) is prime and \( P < Q \), we have \( \gcd(P, Q^2) = 1 \) which implies that all the numbers \( d+jP \) for \( j = 0, \ldots, Q^2-1 \) have distinct residues modulo \( Q^2 \). In particular, there is a number \( d+jP \in \Omega_0 \) which is divisible by \( Q^2 \), contradicting the choice of \( \Omega_0 \). Hence, the property \( \Omega \) does not hold for \( \Omega_0 \) and thus neither for \( \Omega \).

**Remark 19.** It is possible to slightly improve the set \( S \subset [-\frac{1}{2}, \frac{1}{2}] \) appearing in the statement of Theorem 1 (see Remark 17 for the construction of \( S \)). Instead
of the set $S$ in Theorem 1 consider the set $S := \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus V$ with $V := \cup_{k=1}^{\infty} V_{\epsilon/2^k}$, where

$$V_{\eta} := \cup_{\ell=1}^{\infty} V_{\ell}^{(\ell)}$$
and

$$V_{\eta}^{(\ell)} := \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \left( \cup_{r=1}^{2^\ell} \text{supp} \tilde{p}_{\eta/4^\ell} \left( c_{r}^{(\ell)} \alpha x \right) \right)$$

$$= \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \left( \cup_{r=1}^{2^\ell} \left( \mathbb{Z} + \left[ -\frac{n}{4^\ell \eta}, \frac{n}{4^\ell \eta} \right] \right) \right)$$

for any $0 < \eta < \frac{1}{\ell+1}$ and $\ell \in \mathbb{N}$.

Here, $c_{r}^{(\ell)} = \{ c_r^{(\ell)} \}_{r=1}^{2^\ell} \in \mathbb{N}^{2^\ell}$ is any $\mathbb{N}$-valued vector of size $2^\ell$ (for instance, in the light of the set $C$ discussed before Lemma 15, one may choose $c_r^{(\ell)} = r$ or $c_r^{(\ell)} = r^{1000}$ for $r = 1, 2, 3, \ldots, 2^\ell$), and $\tilde{p}_{\eta/4^\ell}$ is the 1-periodic function given by

$$\tilde{p}_{\eta/4^\ell}(x) = \begin{cases} \sqrt{\frac{2^\ell}{\eta^2}} & \text{for } x \in \left[ -\frac{n}{4^\ell \eta}, \frac{n}{4^\ell \eta} \right], \\ 0 & \text{for } x \in \left[ -\frac{1}{2}, \frac{1}{2} \right] \setminus \left[ -\frac{n}{4^\ell \eta}, \frac{n}{4^\ell \eta} \right], \end{cases}$$

which is consistent with the notation of $\tilde{p}_{\eta/2^\ell}(x)$ in (10). In short,

$$V := \cup_{k=1}^{\infty} V_{\epsilon/2^k} = \cup_{k=1}^{\infty} \cup_{\ell=1}^{\infty} V_{\ell}^{(\ell)}$$

$$= \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \left( \cup_{k=1}^{\infty} \cup_{\ell=1}^{\infty} \cup_{r=1}^{2^\ell} \mathbb{Z} + \left[ -\frac{n}{4^\ell \eta}, \frac{n}{4^\ell \eta} \right] \right)$$

Note that for each $\ell \in \mathbb{N}$, we have $\left| \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \text{supp} \tilde{p}_{\eta/4^\ell} \left( c_r^{(\ell)} \alpha x \right) \right| < \frac{n}{4^\ell}$ for all $r = 1, 2, 3, \ldots, 2^\ell$, regardless of the choice of $c_{r}^{(\ell)} = \{ c_r^{(\ell)} \}_{r=1}^{2^\ell} \in \mathbb{N}^{2^\ell}$, which then implies $|V_{\eta}^{(\ell)}| < 2^\ell \cdot \frac{n}{4^\ell} = \frac{n}{2^\ell}$. In turn, we have $|V_{\eta}| \leq \sum_{\ell=1}^{\infty} |V_{\eta}^{(\ell)}| < \sum_{\ell=1}^{\infty} \frac{n}{\eta 2^{\ell}} = \eta$ and consequently, $|V| \leq \sum_{k=1}^{\infty} |V_{\epsilon/2^k}| < \sum_{k=1}^{\infty} \frac{\epsilon}{\eta} = \epsilon$ and $|S| > 1 - \epsilon$.

An inspection of the proof of Proposition 10 and Theorem 1 shows that the original set $V_{\eta}^{(\ell)} := \left[ -\frac{1}{2}, \frac{1}{2} \right] \cap \text{supp} \tilde{p}_{\eta/2^\ell} \left( \ell \alpha x \right)$ defined in (10) can accommodate all the arithmetic progressions with common difference $\ell \alpha$ through the function $\tilde{p}_{\eta/2^\ell}(\ell \alpha x)$ whose dilation factor is $\ell \alpha$. Defining $V_{\eta}^{(\ell)}$ as in (16), on the other hand, allows for a multiple choice of common difference parameter $\ell \alpha$ with $P \in \{ c_r^{(\ell)} \}_{r=1}^{\infty} : r = 1, 2, 3, \ldots, 2^\ell \}$, which is to be used as the dilation factor associated with the function $\tilde{p}_{\eta/4^\ell}$. Accordingly, the new set $V_{\eta}^{(\ell)}$ can accommodate all the arithmetic progressions with common difference $\ell \alpha$ for $r = 1, 2, 3, \ldots, 2^\ell$, through the function $\tilde{p}_{\eta/4^\ell}(c_r^{(\ell)} \alpha x)$.

However, such flexibility is yet too weak for generalizing Theorem 1 to the class of arbitrary separated sets $\Lambda \subset \mathbb{R}$ with positive upper Beurling density. Indeed, to adapt the proof technique of Theorem 1 to an arbitrary separated
set $\Lambda \subset \mathbb{R}$, one needs to extract from $\Lambda$ an arithmetic progression (resp. an approximate arithmetic progression in the sense of [3] in Lemma [15]) of length $2\tilde{M}+1$ with common difference $P\alpha$ for some $P \in \{c_r^{(f)} : r = 1, 2, 3, \ldots, 2^f\}$, where $\tilde{M} = \hat{M}(\eta/4^f) \in \mathbb{N}$ is a large number chosen similarly as in [7]. However, setting $\alpha = 1$ for simplicity, we note that Szemerédi’s theorem (resp. Lemma [15]) only guarantees the existence of an arithmetic progression (resp. an approximate arithmetic progression) of length $2\tilde{M}+1$ in $\Lambda$, where the common difference $P \in \mathbb{N}$ of the progression can be arbitrarily large. While the flexibility in choosing the set $\{c_r^{(f)} : r = 1, 2, 3, \ldots, 2^f\}$ is certainly advantageous, there is no guarantee that the parameter $P$ will be in this set. Hence, even for the improved set $S \subset [-\frac{1}{2}, \frac{1}{2}]$ given by (16), the general case of arbitrary separated sets $\Lambda \subset \mathbb{R}$ is still out of reach.

Note that the issue of $P \in \mathbb{N}$ being potentially very large is easily avoided when $\Lambda$ is assumed to have arbitrarily long arithmetic progressions with a fixed common difference $P$ (with $\alpha = 1$ chosen for simplicity), which has led to our first main result Theorem [11].

Appendix A. Related notions in Paley-Wiener spaces

The Fourier transform[3] is defined densely on $L^2(\mathbb{R}^d)$ by

$$\mathcal{F}(f) := \hat{f}(\omega) = \int f(x) e^{2\pi i x \cdot \omega} \, dx \quad \text{for} \quad f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d).$$

It is easily seen that $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is a unitary operator satisfying $\mathcal{F}^2 = I$, where $I : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is the reflection operator defined by $If(x) = f(-x)$, and thus $\mathcal{F}^4 = \text{Id}_{L^2(\mathbb{R}^d)}$. The Paley-Wiener space over a measurable set $S \subset \mathbb{R}^d$ is defined by

$$PW(S) := \{f \in L^2(\mathbb{R}^d) : \text{supp} \hat{f} \subset S\} = \mathcal{F}^{-1}[L^2(S)]$$

equipped with the norm $\|f\|_{PW(S)} := \|f\|_{L^2(\mathbb{R}^d)} = \|\hat{f}\|_{L^2(S)}$, where $L^2(S)$ is embedded into $L^2(\mathbb{R}^d)$ by the trivial extension. Denoting the Fourier transform of $f \in PW(S)$ by $F \in L^2(S)$, we see that for almost all $x \in \mathbb{R}^d$,

$$f(x) = (\mathcal{F}^{-1}F)(x) = \int_S F(\omega) e^{-2\pi i x \cdot \omega} \, d\omega = \langle F, e^{2\pi i x \cdot \omega} \rangle_{L^2(S)} \quad (A.1)$$

Moreover, if the set $S \subset \mathbb{R}^d$ has finite measure, then $f$ is continuous and thus (A.1) holds for all $x \in \mathbb{R}^d$.

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[3] This is a nonstandard but equivalent definition of the Fourier transform which has no negative sign in the exponent. This definition is employed only to justify the relation (A.1). Alternatively, as in [32, 34, 35] one could use the standard definition of the Fourier transform which has negative sign in the exponent, and define the Paley-Wiener space $PW(S)$ to be the space of Fourier transforms of $L^2(S)$. 

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Definition. Let $S \subset \mathbb{R}^d$ be a measurable set. A discrete set $\Lambda \subset \mathbb{R}^d$ is called

- a uniqueness set (a set of uniqueness) for $PW(S)$ if the only function $f \in PW(S)$ satisfying $f(\lambda) = 0$ for all $\lambda \in \Lambda$ is the trivial function $f = 0$;

- a sampling set (a set of sampling) for $PW(S)$ if there are constants $0 < A \leq B < \infty$ such that
  \[
  A \|f\|_{PW(S)}^2 \leq \sum_{\lambda \in \Lambda} |f(\lambda)|^2 \leq B \|f\|_{PW(S)}^2
  \]
  for all $f \in PW(S)$;

- an interpolating set (a set of interpolation) for $PW(S)$ if for each $c = \{c_\lambda\}_{\lambda \in \Lambda} \in \ell^2(\Lambda)$ there exists a function $f \in PW(S)$ satisfying $f(\lambda) = c_\lambda$ for all $\lambda \in \Lambda$.

It follows immediately from (A.1) that

- $\Lambda$ is a uniqueness set for $PW(S)$ if and only if $E(\Lambda)$ is complete in $L^2(S)$;

- $\Lambda$ is a sampling set for $PW(S)$ if and only if $E(\Lambda)$ is a frame for $L^2(S)$.

Also, we have the following characterization of interpolation sets for $PW(S)$ (see [38, p.129, Theorem 3]):

- $\Lambda$ is an interpolating set for $PW(S)$ if and only if there is a constant $A > 0$ such that
  \[
  A \|c\|_{\ell^2}^2 \leq \left\| \sum_{n \in \mathbb{Z}} c_n e^{2\pi i \lambda \cdot n} \right\|_{L^2(S)}^2
  \]
  for all $c = \{c_\lambda\}_{\lambda \in \Lambda} \in \ell^2(\Lambda)$,

  meaning that the lower Riesz inequality of $E(\Lambda)$ for $L^2(S)$ holds.

Combining with the Bessel inequality (which corresponds to the upper Riesz inequality), we obtain a more convenient statement:

- If $E(\Lambda)$ is a Bessel sequence in $L^2(S)$, then $\Lambda$ is an interpolating set for $PW(S)$ if and only if $E(\Lambda)$ is a Riesz sequence in $L^2(S)$.

In fact, this statement can be proved by elementary functional analytic arguments. Indeed, if $E(\Lambda)$ is Bessel, i.e., if the synthesis operator $T : \ell^2(\Lambda) \to L^2(S)$ defined by $T(\{c_\lambda\}_{\lambda \in \Lambda}) = \sum_{\lambda \in \Lambda} c_\lambda e^{2\pi i \lambda \cdot \cdot}$ is a bounded linear operator (equivalently, the analysis operator $T^* : L^2(S) \to \ell^2(\Lambda)$ defined by $T^* F = \{\langle F, e^{2\pi i \lambda \cdot \cdot} \rangle_{L^2(S)}\}_{\lambda \in \Lambda}$ is a bounded linear operator), then $T$ is bounded below (that is, the lower Riesz inequality holds) if and only if $T$ is injective and has closed range, if and only if $T^*$ has dense and closed range, i.e., $T^*$ is surjective, which means that $E(\Lambda)$ is an interpolating set for $PW(S)$ by (A.1).

The statement above is often useful because $E(\Lambda)$ is necessarily a Bessel sequence in $L^2(S)$ whenever $\Lambda \subset \mathbb{R}^d$ is separated and $S \subset \mathbb{R}^d$ is bounded [38, p.135, Theorem 4]. Note that $\Lambda \subset \mathbb{R}^d$ is necessarily separated if $E(\Lambda)$ is a Riesz sequence in $L^2(S)$ (see Proposition 11).
Appendix B. Proof of some auxiliary results

Proof of Lemma [8] To prove (a), note that for any $a \in \mathbb{R}^d$,

$$T_{-a}[E(\Lambda)] = \{e^{2\pi i \lambda (\cdot + a)} : \lambda \in \Lambda\} = \{e^{2\pi i \lambda a} e^{2\pi i \lambda (\cdot)} : \lambda \in \Lambda\}. $$

Since the phase factor $e^{2\pi i \lambda a} \in \mathbb{C}$ for $\lambda \in \Lambda$ does not affect the Riesz basis property and Riesz bounds, it follows that $T_{-a}[E(\Lambda)]$ is a Riesz basis for $L^2(S)$ with optimal bounds $A$ and $B$. Consequently, $E(\Lambda)$ is a Riesz basis for $L^2(S+a)$ with bounds $A$ and $B$.

For (b) and (c), note that the modulation $F(x) \mapsto e^{2\pi i b x} F(x)$ is a unitary operator on $L^2(S)$ and that the dilation $F(x) \mapsto \sqrt{\sigma} F(\sigma x)$ is also a unitary operator from $L^2(S)$ onto $L^2(\frac{1}{\sigma} S)$. It is easily seen from Proposition [6] that if $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a unitary operators between two Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, and if $\{f_n\}_{n \in \mathbb{Z}}$ is a Riesz basis for $\mathcal{H}_1$, then $\{U f_n\}_{n \in \mathbb{Z}}$ is a Riesz basis for $\mathcal{H}_2$.

The parts (b) and (c) follow immediately from this statement.

Proof of Proposition [11] For simplicity, we will only consider the case $d = 1$. (i) Assume that $D^+(\Lambda) = \infty$. This means that there is a real-valued sequence $1 \leq r_1 < r_2 < \cdots \rightarrow \infty$ such that

$$\sup_{r \in \mathbb{R}} |\Lambda \cap [x, x+r_n]| > n \text{ for all } n \in \mathbb{N}. $$

Then for each $n \in \mathbb{N}$ there exists some $x_n \in \mathbb{R}$ satisfying

$$|\Lambda \cap [x_n, x_n+r_n]| \geq n. $$

For each $k \in \mathbb{N}$, we partition the interval $[x_n, x_n+r_n]$ into $k$ subintervals of equal length $\frac{r_n}{k}$, namely the intervals $[x_n, x_n+\frac{r_n}{k}], \ldots, [x_n+(k-1)\frac{r_n}{k}, x_n+r_n]$. Then at least one of the subintervals, which we denote by $I_{n,k}$, must satisfy

$$\frac{|\Lambda \cap I_{n,k}|}{|I_{n,k}|} \geq n, \quad (B.1)$$

where $|I_{n,k}| = \frac{r_n}{k}$. Letting $k \rightarrow \infty$, we see that

$$\limsup_{r \rightarrow 0} \frac{\sup_{x \in \mathbb{R}} |\Lambda \cap [x, x+r]|}{r} = \infty. $$

Define the function $g : \mathbb{R} \rightarrow \mathbb{C}$ by $g(x) = \frac{1}{\sqrt{|S|}} \chi_S(x)$ for $x \in \mathbb{R}$. Then $\|g\|_{L^2(\mathbb{R})} = \|g\|_{L^2(S)} = 1$ and $\hat{g}(0) = \int_S g(x) dx = \sqrt{|S|}$. Since $g \in L^1(\mathbb{R})$, its Fourier transform $\hat{g}$ is continuous on $\mathbb{R}$ and therefore exists $0 \leq \delta \leq \frac{1}{2}$ such that $|\hat{g}(\omega)| \geq \sqrt{|S|}/2$ for all $\omega \in [-\frac{\delta}{2}, \frac{\delta}{2}]$. For each $n \in \mathbb{N}$, we set $k_n := \lceil \frac{r_n}{\delta} \rceil \geq 2$, so that $k_n - 1 < \frac{r_n}{\delta} \leq k_n$ and thus $\frac{\delta}{2} < \frac{\delta}{k_n} \leq \frac{r_n}{k_n} \leq \delta$. It then follows from (B.1) that

$$|\Lambda \cap I_{n,k_n}| \geq n \cdot |I_{n,k_n}| \geq n \cdot \frac{r_n}{k_n} > n \cdot \frac{\delta}{2}. $$

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For each \( n \in \mathbb{N} \), we denote the center of the interval \( I_{n,k_n} \) by \( c_n \in \mathbb{R} \) and let \( f_n \in L^2(S) \) be defined by \( f_n(x) := e^{2\pi i c_n x} g(x) \) for \( x \in S \). Then

\[
\sum_{\lambda \in \Lambda} \left| \langle f_n, e^{2\pi i \lambda} \rangle_{L^2(S)} \right|^2 \geq \sum_{\lambda \in \Lambda \cap I_{n,k_n}} \left| \langle g, e^{2\pi i (\lambda-c_n)} \rangle_{L^2(S)} \right|^2
= \sum_{\lambda \in \Lambda \cap I_{n,k_n}} |\hat{g}(c_n - \lambda)|^2 \geq (n \cdot \delta) \cdot \frac{|S|}{4}, \tag{B.2}
\]

where used that \( c_n - \lambda \in [-\frac{\delta}{2}, \frac{\delta}{2}] \) for all \( \lambda \in \Lambda \cap I_{n,k_n} \), since \( I_{n,k_n} \) is an interval of length \( \frac{2\delta}{n} \leq \delta \). While \( \|f_n\|_{L^2(S)} = \|g\|_{L^2(S)} = 1 \) for all \( n \in \mathbb{N} \), the right hand side of (B.2) tends to infinity as \( n \to \infty \). Hence, we conclude that \( E(\Lambda) \) is not a Bessel sequence in \( L^2(S) \) if \( D^+(\Lambda) = \infty \).

(ii) Suppose to the contrary that \( E(\Lambda) \) is a Riesz sequence in \( L^2(S) \) with Riesz bounds \( A \) and \( B \), but the set \( \Lambda \subset \mathbb{R} \) is not separated. Then there are two sequences \( \{\lambda_n\}_{n=1}^{\infty} \) and \( \{\lambda'_n\}_{n=1}^{\infty} \) in \( \Lambda \) such that \( |\lambda_n - \lambda'_n| \to 0 \) as \( n \to \infty \). Note that \( S \subset \mathbb{R} \) is a finite measure set, and for each \( x \in S \) we have

\[
|e^{2\pi i \lambda_n x} - e^{2\pi i \lambda'_n x}| \leq 2 \quad \text{and} \quad e^{2\pi i \lambda_n x} - e^{2\pi i \lambda'_n x} \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus, we have

\[
\lim_{n \to \infty} \int_S |e^{2\pi i \lambda_n x} - e^{2\pi i \lambda'_n x}|^2 \, dx = 0
\]

by the dominated convergence theorem. For \( \lambda \in \Lambda \), let \( \delta_\lambda \in E(\Lambda) \) be the Kronecker delta sequence supported at \( \lambda \), that is, \( \delta_\lambda(\lambda') = 1 \) if \( \lambda' = \lambda \), and 0 otherwise. Then since \( E(\Lambda) \) is a Riesz sequence in \( L^2(S) \), we have

\[
2 = \|\delta_{\lambda_n} - \delta_{\lambda'_n}\|_{L^2(S)}^2 \leq \frac{2}{\mathcal{A}} \|e^{2\pi i \lambda_n} - e^{2\pi i \lambda'_n}\|_{L^2(S)}^2 = \frac{2}{\mathcal{A}} \int_S |e^{2\pi i \lambda_n x} - e^{2\pi i \lambda'_n x}|^2 \, dx \to 0,
\]

yielding a contradiction. \( \square \)

**Proof of Lemma 15** Let \( \Lambda = \{\lambda_n\}_{n \in \mathbb{Z}} \) with \( \lambda_n < \lambda_{n+1} \) for all \( n \), and fix any \( \delta > 0 \). Choose a sufficiently large number \( N \in \mathbb{N} \) so that \( \frac{1}{N} < \tau := \min\{\Delta(\Lambda), 2\delta\} \), where \( \Delta(\Lambda) := \inf\{|\lambda - \lambda'| : \lambda \neq \lambda' \in \Lambda\} \) is the separation constant of \( \Lambda \) (see Section 2.3). Consider the perturbation \( \tilde{\Lambda} \subset \frac{1}{N} \mathbb{Z} \) of \( \Lambda \), obtained by rounding each element of \( \Lambda \) to the nearest point in \( \frac{1}{N} \mathbb{Z} \) (if \( \lambda \in \Lambda \) is exactly the midpoint of \( \frac{k+1}{N} \) and \( \frac{k}{N} \), then we choose \( \frac{k+1}{N} \)). Since \( \Delta(\Lambda) > \frac{1}{N} \), all elements in \( \Lambda \) are rounded to distinct points in \( \frac{1}{N} \mathbb{Z} \), i.e., the set \( \tilde{\Lambda} = \{\tilde{\lambda}_n\}_{n \in \mathbb{Z}} \subset \frac{1}{N} \mathbb{Z} \) has no repeated elements. Clearly, there is a 1:1 correspondence between \( \lambda_n \) and \( \tilde{\lambda}_n \), and we have \( |\lambda_n - \tilde{\lambda}_n| \leq \frac{1}{N} < \frac{\delta}{2} \leq \delta \) for all \( n \in \mathbb{Z} \).

We claim that for any \( M \in \mathbb{N} \) there exist constants \( c \in \mathbb{N} \), \( d \in \frac{1}{N} \mathbb{Z} \), and an increasing sequence \( \tilde{s}(-M) < \tilde{s}(-M+1) < \ldots < \tilde{s}(M) \) in \( \tilde{\Lambda} \subset \frac{1}{N} \mathbb{Z} \) satisfying

\[
\tilde{s}(j) = cj + d \quad \text{for} \quad j = -M, \ldots, M.
\]

Once this claim is proved, it follows that the sequence \( \{s(j)\}_{j=-M}^{M} \subset \Lambda \) corresponding to \( \{\tilde{s}(j)\}_{j=-M}^{M} \subset \tilde{\Lambda} \) satisfies the condition (9) as desired.

To prove the claim, consider the partition of \( N \tilde{\Lambda} (\subset \mathbb{Z}) \) based on residue modulo \( N \), that is, consider the sets \( N \Lambda \cap N \mathbb{Z}, N \Lambda \cap (N \mathbb{Z} + 1), \ldots, N \Lambda \cap (N \mathbb{Z} + N-1) \).
Since $D^+(\tilde{\Lambda}) = D^+(\Lambda) > 0$, at least one of these $N$ sets must have positive upper density, i.e., $D^+(N\tilde{\Lambda} \cap (NZ+u)) > 0$ for some $u \in \{0, \ldots, N - 1\}$. Then Szemerédi’s theorem implies that for any $M \in \mathbb{N}$ the set $N\tilde{\Lambda} \cap (NZ+u)$ contains an arithmetic progression of length $2M+1$, that is, \( \{c_0j + d_0 : j = -M, \ldots, M\} \subset N\tilde{\Lambda} \cap (NZ+u) \) for some $c_0 \in \mathbb{N}$ and $d_0 \in \mathbb{Z}$. This means that there is an increasing sequence $\tilde{s}(-M) < \tilde{s}(-M+1) < \ldots < \tilde{s}(M)$ in $\tilde{\Lambda}$ satisfying

\[N\tilde{s}(j) = c_0j + d_0 \quad \text{for} \quad j = -M, \ldots, M.\]

Since the numbers $c_0j + d_0$, $j = -M, \ldots, M$ are in $NZ+u$, it is clear that $c_0 \in NN$ and $d_0 \in NZ+u$. Thus, setting $c := \frac{1}{N}c_0 \in \mathbb{N}$ and $d := \frac{1}{N}d_0 \in \mathbb{Z} + \frac{u}{N} \subset \frac{1}{N}\mathbb{Z}$, we have $\tilde{s}(j) = cj + d$ for $j = -M, \ldots, M$, as claimed.

Finally, one can easily force the constant $c \in \mathbb{N}$ to be a multiple of any prescribed number $L \in \mathbb{N}$. This is achieved by considering the partition of $N\tilde{\Lambda} (\subset Z)$ based on residue modulo $LN$, instead of modulo $N$.

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