What is geometry in quantum theory

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Abstract. In this scientific preface to the first issue of International Journal of Geometric Methods in Modern Physics¹, we briefly survey some peculiarities of geometric techniques in quantum models.

Contemporary quantum theory meets an explosion of different types of quantization. Some of them (geometric quantization, deformation quantization, noncommutative geometry, topological field theory etc.) speak the language of geometry, algebraic and differential topology. We do not pretend for any comprehensive analysis of these quantization techniques, but aims to formulate and illustrate their main peculiarities. As in any survey, a selection of topics has to be done, and we apologize in advance if some relevant works are omitted.

Geometry of classical mechanics and field theory is mainly differential geometry of finite-dimensional smooth manifolds, fiber bundles and Lie groups. The key point why geometry plays a prominent role in classical field theory lies in the fact that it enables one to deal with invariantly defined objects. Gauge theory has shown clearly that this is a basic physical principle. At first, a pseudo-Riemannian metric has been identified to a gravitational field in the framework of Einstein’s General Relativity. In 60-70th, one has observed that connections on a principal bundle provide the mathematical model of classical gauge potentials [1-3]. Furthermore, since the characteristic classes of principal bundles are expressed in terms of the gauge strengths, one can also describe the topological phenomena in classical gauge models [4]. Spontaneous symmetry breaking and Higgs fields have been explained in terms of reduced $G$-structures [5]. A gravitational field seen as a pseudo-Riemannian metric exemplifies such a Higgs field [6]. In a general setting, differential geometry of smooth fiber bundles gives the adequate mathematical formulation of classical field theory, where fields are represented by sections of fiber bundles and their dynamics is phrased in terms of jet manifolds [7]. Autonomous classical mechanics speaks the geometric language of symplectic and Poisson

¹Web: http://www.worldscinet.com/ijgmmp/ijgmmp.shtml
manifolds [8-10]. Nonrelativistic time-dependent mechanics can be formulated as a particular field theory on fiber bundles over \( \mathbb{R} \) [11].

At the same time, the standard mathematical language of quantum mechanics and perturbative field theory, except gravitation theory, has been long far from geometry. In the last twenty years, the incremental development of new physical ideas in quantum theory (including super- and BRST symmetries, geometric and deformation quantization, topological field theory, anomalies, noncommutativity, strings and branes) has called into play advanced geometric techniques, based on the deep interplay between algebra, geometry and topology.

In the framework of algebraic quantization, one associates to a classical system a certain (e.g., von Neumann, \( C^* \)- or \( Op^* \)-) algebra whose different representations are studied. Quantization techniques under discussion introduce something new. Namely, they can provide nonequivalent quantizations of a classical system corresponding to different values of some topological and differential invariants. For instance, a symplectic manifold \( X \) admits a set of nonequivalent star-products indexed by elements of the cohomology group \( H^2(X[[\lambda]]) \) [12, 13]. Thus, one may associate to a classical system different underlying quantum models. Of course, there is a question whether this ambiguity is of physical or only mathematical nature. From the mathematical viewpoint, one may propose that any quantization should be a functor between classical and quantum categories (e.g., some subcategory of Poisson manifolds on the classical side and a subcategory of \( C^* \)-algebras on the quantum side) [14]. From the physical point of view, dequantization becomes important.

There are several examples of \textit{sui generis} dequantizations. For instance, Berezin’s quantization [15] in fact is dequantization. One can also think of well-known Gelfand’s map as being dequantization of a commutative \( C^* \)-algebra \( \mathcal{A} \) by the algebra of continuous complex functions vanishing at infinity on the spectrum of \( \mathcal{A} \). This dequantization has been generalized to noncommutative unital \( C^* \)-algebras [16, 17]. The concept of the of the strict \( C^* \)-algebraic deformation quantization implies an appropriate dequantization when \( \hbar \to 0 \) [18, 19]. In Connes’ noncommutative geometry, dequantization of the spectral triple in the case of a commutative algebra \( C^\infty(X) \) is performed in order to restart the original differential geometry of a spin manifold \( X \) [20, 21]. One can also treat the method of an effective action in perturbative quantum field theory as an example how an underlying quantum model can contribute to the classical one [22, 23]. For instance, quantum models of inflationary cosmology are selected by very particular properties of observable (classical) Universe [24]. It also seems that the main criterion of selecting quantum gravitation theories is their appropriate ‘dequantization’ up to low energy (quantum and classical) physics [25].

Of course, our exposition is far from covering the whole scope of \textit{International Journal of Geometric Methods in Modern Physics}. For instance, we are not concerned here with quantization of geometric objects such as loop quantum gravity [25-28], strings and branes
It is also worth mentioning the theory of quantum information and computation which throws new lights on quantum physics [39-41]. Geometric techniques in this theory (e.g., quantum holonomy computation) are intensively developed [42, 43].

I.

Let us start from familiar differential geometry. There are the following reasons why this geometry contributes to quantum theory.

(i) Most of the quantum models come from quantization of the original classical systems and, therefore, inherits their differential geometric properties. First of all, this is the case of canonical quantization which replaces the Poisson bracket \( \{ f, f' \} \) of smooth functions with the bracket \( \hat{f}, \hat{f}' \) of Hermitian operators in a Hilbert space such that Dirac’s condition

\[
[\hat{f}, \hat{f}'] = -i\hbar \{ f, f' \}
\]

holds. Let us mention Berezin–Toeplitz quantization [15, 44, 45] and geometric quantization [10, 46-48] of symplectic, Poisson and Kähler manifolds. This is also the case of quantum gauge theory whose generating functional is expressed in the classical action of gauge fields containing topological terms and anomalies [49].

(ii) Many quantum systems are considered on a smooth manifold equipped with some background geometry. As a consequence, quantum operators are often represented by differential operators which act in a pre-Hilbert space of smooth functions. A familiar example is the Schrödinger equation. The Kontsevich deformation quantization is based on the quasi-isomorphism of the graded differential Lie algebra of multivector fields (endowed with the Schouten–Nijenhuis bracket and the zero differential) to that of polydifferential operators (provided with the Gerstenhaber bracket and the modified Hochschild differential) [50, 51].

It is also worth reminding perturbative quantum field theory in curved space. Firstly, the counter-terms in the energy-momentum tensor of quantum fields can contribute to the Einstein equations. This is the case of some models of inflationary cosmology, e.g., its seminal one taking into account the conformal anomaly [52]. Secondly, different counter-terms in an effective action can be computed [22, 23, 53]. In the case of Riemannian manifolds (with boundaries), the heat kernel technique is applied [54-56]. It is based on spectral properties of Laplace-type and Dirac-type operators on compact manifolds [57, 58].

(iii) In some quantum models, differential geometry is called into play as a technical tool. For instance, a suitable \( U(1) \)-principal connection is used in order to construct the operators \( \hat{f} \) in the framework of geometric quantization. Another example is Fedosov’s deformation quantization where a symplectic connection plays a similar role [59]. Note that this application has stimulated the study of symplectic connections [60].

(iv) Geometric constructions in quantum models often generalize the classical ones, and they are build in a similar way. For example, connections on principal superbundles [61],
graded principal bundles [62], and quantum principal bundles [63] are defined by means of the corresponding one-forms in the same manner as connections on smooth principal bundles with structure finite-dimensional Lie groups.

II.

In quantum models, one deals with infinite-dimensional smooth Banach and Hilbert manifolds and (locally trivial) Hilbert and C*-algebra bundles. The definition of smooth Banach (and Hilbert) manifolds follows that of finite-dimensional smooth manifolds in general, but infinite-dimensional Banach manifolds are not locally compact and paracompact [64, 65]. In particular, a Banach manifold admits the differentiable partition of unity iff its model space does. It is essential that Hilbert manifolds (but not, e.g., nuclear (Schwartz) manifolds) satisfy the inverse function theorem and, therefore, locally trivial Hilbert bundles are defined. However, they need not be bundles with a structure group.

(i) Infinite-dimensional Kähler manifolds provide an important example of Hilbert manifolds [66]. In particular, the projective Hilbert space of complex rays in a Hilbert space $E$ is such a Kähler manifold. This is the space the pure states of a C*-algebra $A$ associated to the same irreducible representation $\pi$ of $A$ in a Hilbert space $E$ [67]. Therefore, it plays a prominent role in many quantum models. For instance, it has been suggested to consider a loop in the projective Hilbert space, instead of a parameter space, in order to describe Berry’s phase [68, 69]. We have already mentioned the dequantization procedure which represents a unital C*-algebra by a Poisson algebra of complex smooth functions on a projective Hilbert space [16].

(ii) Gauge theory on a principal bundle $P \to X$ with a structure compact semisimple matrix Lie group $G$ over an oriented compact smooth manifold $X$ also provides an example of Hilbert manifolds. Namely, the Sobolev $k$-completions $\mathcal{E}_k$, $\overline{\mathcal{E}}_k$ and $\mathcal{E}_k^0$ of the gauge group, the effective gauge group, and the pointed gauge group, respectively, are infinite-dimensional Lie groups, while the Sobolev completion $\mathcal{C}_k$ of the space of principal connections and that $\overline{\mathcal{C}}_k$ of irreducible connections are a Hilbert manifold and its dense open subset, respectively [70]. If $k > \dim X/2 + 1$, the orbit space $\overline{\mathcal{M}}_k = \overline{\mathcal{C}}_k/\overline{\mathcal{E}}_{k+1}$ of principal connections is a smooth Hilbert manifold, and $\overline{\mathcal{C}}_k \to \overline{\mathcal{M}}_k$ is a principal bundle with the structure Lie group $\overline{\mathcal{E}}_{k+1}$. The quotient $\mathcal{C}_k/\mathcal{E}^0_{k+1}$ is also a smooth Hilbert manifold, whereas $\mathcal{M}_k = \mathcal{C}_k/\mathcal{E}_{k+1}$ falls into a countable union of Hilbert manifolds [71, 72]. For instance, if the principal bundle $\overline{\mathcal{C}}_k \to \overline{\mathcal{M}}_k$ is not trivial, one meets the well-known Gribov ambiguity [73]. It is also interesting that connections on the $G$-principal bundle $P \times \overline{\mathcal{C}} \to X \times \overline{\mathcal{C}}$ restart the BRST transformations of the geometric sector of the Donaldson theory [74] and those by Witten [75].

(iii) A Hilbert bundle over a smooth finite-dimensional manifold $X$ is a particular locally trivial continuous field of Hilbert spaces in [67]. Conversely, one can think of any locally trivial continuous fields of Hilbert spaces and C*-algebras as being topological fiber bundles.
Given a Hilbert space $E$, let $B \subset \mathcal{B}(E)$ be some $C^*$-algebra of bounded operators in $E$. The following fact reflects the nonequivalence of Schrödinger and Heisenberg quantum pictures. There is the obstruction to the existence of associated (topological) Hilbert and $C^*$-algebra bundles $\mathcal{E} \to X$ and $\mathcal{B} \to X$ with the typical fibers $E$ and $B$, respectively. Firstly, transition functions of $\mathcal{E}$ define those of $\mathcal{B}$, but the latter need not be continuous, unless $B$ is the algebra of compact operators in $E$. Secondly, transition functions of $\mathcal{B}$ need not give rise to transition functions of $\mathcal{E}$. This obstruction is characterized by the Dixmier–Douady class of $\mathcal{B}$ in the Čech cohomology group $H^3(X, \mathbb{Z})$. There is the similar obstruction to the $U(1)$-extension of structure groups of principal bundles [76, 77]. One also meets the Dixmier–Douady class as the obstruction to a bundle gerbe being trivial [78].

(iv) There is a problem in the definition of a connection on $C^*$-algebra bundles which comes from the fact that a $C^*$-algebra (e.g., any commutative $C^*$-algebra) need not admit nonzero bounded derivations. An unbounded derivation of a $C^*$-algebra $A$ obeying certain conditions is an infinitesimal generator of a strongly (but not uniformly) continuous one-parameter group of automorphisms of $A$ [79]. Therefore, one may introduce a connection on a $C^*$-algebra bundle in terms of parallel transport curves and operators, but not their infinitesimal generators [80]. Moreover, a representation of $A$ does not imply necessarily a unitary representation of its strongly (not uniformly) continuous one-parameter group of automorphisms. In contrast, connections on a Hilbert bundle over a smooth manifold can be defined both as particular first order differential operators on the module of its sections [81] and a parallel displacement along paths lifted from the base [82].

(v) Instantwise geometric quantization of time-dependent mechanics is phrased in terms of Hilbert bundles over $\mathbb{R}$ [46, 83]. Holonomy operators in a Hilbert bundle with a structure finite-dimensional Lie group are well known to describe the non-Abelian geometric phase phenomena [84]. At present, holonomy operators in Hilbert bundles attract special attention in connection with quantum computation and control theory [42, 43, 85].

III.

Geometry in quantum systems speaks mainly the algebraic language of rings, modules and sheaves due to the fact that the basic ingredients in the differential calculus and differential geometry on smooth manifolds (except nonlinear differential operators) can be restarted in a pure algebraic way.

(i) Any smooth real manifold $X$ is homeomorphic to the real spectrum of the $\mathbb{R}$-ring $C^\infty(X)$ of smooth real functions on $X$ provided with the Gelfand topology [86, 87]. Furthermore, the sheaf $C^\infty_X$ of germs of $f \in C^\infty(X)$ on this topological space fixes a unique smooth manifold structure on $X$ such that it is the sheaf of smooth functions on $X$. The pair $(X, C^\infty_X)$ exemplifies a local-ringed space. A sheaf $\mathfrak{R}$ on a topological space $X$ is said to be a local-ringed space if its stalk $\mathfrak{R}_x$ at each point $x \in X$ is a local commutative ring [88]. One
can associate to any commutative ring $\mathcal{A}$ the particular local-ringed space, called an affine scheme, on the spectrum $\text{Spec} \mathcal{A}$ of $\mathcal{A}$ endowed with the Zariski topology [89]. Furthermore, one can assign the following algebraic variety to any commutative finitely generated $K$-ring $\mathcal{A}$ over an algebraically closed field $K$. Given a ring $K[x]$ of polynomials with coefficients in $K$, let us consider the epimorphism $\phi : K[x] \to \mathcal{A}$ defined by the equalities $\phi(x_i) = a_i$, where $a_i$ are generating elements of $\mathcal{A}$. Zeros of polynomials in $\text{Ker} \phi$ make up an algebraic variety $V$ whose coordinate ring $K^V$ is exactly $\mathcal{A}$. The subvarieties of $V$ constitute the system of closed sets of the Zariski topology on $V$ [90]. Every affine variety $V$ in turn yields the affine scheme $\text{Spec} K^V$ such that there is one-to-one correspondence between the points of $\text{Spec} K^V$ and the irreducible subvarieties of $V$. For instance, complex algebraic varieties have a structure of complex analytic manifolds.

(ii) Given a (connected) compact topological space $X$ and the ring $C^0(X)$ of continuous complex functions on $X$, the well-known Serre–Swan theorem [91] states that a $C^0(X)$-module is finitely generated projective iff it is isomorphic to the module of sections of some (topological) vector bundle over $X$. Moreover, this isomorphism is a categorial equivalence [92], and its variant takes place if $X$ is locally compact [93]. If $X$ is a compact smooth manifold, there is the similar isomorphism of a finitely generated projective $C^\infty(X)$-modules on $X$ to the modules of sections of some smooth vector bundle over $X$ [94], and this is also true if $X$ is not necessarily compact. A variant of the Serre–Swan theorem for Hilbert modules over noncommutative $C^*$-algebras holds [95].

(iii) Let $\mathcal{K}$ be a commutative ring, $\mathcal{A}$ a commutative $\mathcal{K}$-ring, and $P, Q$ some $\mathcal{A}$-modules. The $\mathcal{K}$-linear $Q$-valued differential operators on $P$ can be defined [87, 96, 97]. The representative objects of the functors $Q \to \text{Diff}_s(P, Q)$ are the jet modules $J^sP$ of $P$. Using the first order jet module $J^1P$, one also restarts the notion of a connection on a $\mathcal{A}$-module $P$ [81, 98]. Such a connection assigns to each derivation $\tau \in \partial\mathcal{A}$ of a $\mathcal{K}$-ring $\mathcal{A}$ a first order $P$-valued differential operator $\nabla_\tau$ on $P$ obeying the Leibniz rule

$$\nabla_\tau(ap) = \tau(a)p + a\nabla_\tau(p).$$

For instance, if $P$ is a $C^\infty(X)$-module of sections of a smooth vector bundle $Y \to X$, we come to the familiar notions of a linear differential operator on $Y$, the jets of sections of $Y \to X$ and a linear connection on $Y \to X$. Similarly, connections on local-ringed spaces are introduced [81]. In supergeometry, connections on graded modules over a graded commutative ring and graded local-ringed spaces are defined [61]. In noncommutative geometry, different definitions of a differential operator on modules over a noncommutative ring have been suggested [99-101]. Roughly speaking, the difficulty lies in the fact that, if $\partial$ is a derivation of a noncommutative $\mathcal{K}$-ring $\mathcal{A}$, the product $a\partial$, $a \in \mathcal{A}$, need not be so. There are also different definitions of a connection on modules over a noncommutative ring [102, 103].
Let \( K \) be a commutative ring, \( A \) a (commutative or noncommutative) \( K \)-ring, and \( Z(A) \) the center of \( A \). Derivations of \( A \) make up a Lie \( K \)-algebra \( \mathcal{D}A \). Let us consider the Chevalley–Eilenberg complex of \( K \)-multilinear morphisms of \( \mathcal{D}A \) to \( A \), seen as a \( \mathcal{D}A \)-module [10, 104]. Its subcomplex \( \mathcal{O}^*(\mathcal{D}A, d) \) of \( Z(A) \)-multilinear morphisms is a graded differential algebra, called the Chevalley–Eilenberg differential calculus over \( A \). It contains the universal differential calculus \( \mathcal{O}^*A \) generated by elements \( da, a \in A \). If \( A \) is the \( R \)-ring \( C^\infty(X) \) of smooth real functions on a smooth manifold \( X \), the module \( \mathcal{D}C^\infty(X) \) of its derivations is the Lie algebra of vector fields on \( X \) and the Chevalley–Eilenberg differential calculus over \( C^\infty(X) \) is exactly the algebra of exterior forms on a manifold \( X \) where the Chevalley–Eilenberg coboundary operator \( d \) coincides with the exterior differential, i.e., \( \mathcal{O}^*(\mathcal{D}C^\infty(X), d) \) is the familiar de Rham complex. In a general setting, one therefore can think of elements of the Chevalley–Eilenberg differential calculus \( \mathcal{O}^k(\mathcal{D}A, d) \) over an algebra \( A \) as being differential forms over \( A \). Similarly, the Chevalley–Eilenberg differential calculus over a graded commutative ring is constructed [104].

IV.

As was mentioned above, homology and cohomology of spaces and algebraic structures often play a role of \textit{sui generis} hidden quantization parameters which can characterize nonequivalent quantizations.

(i) First of all, let us mention the abstract de Rham theorem [105] and, as its corollary, the homomorphism \( H^*(X, \mathbb{Z}) \rightarrow H^*(X) \) of the Čech cohomology of a smooth manifold \( X \) to the de Rham cohomology of exterior forms on \( X \). For instance, the Chern classes \( c_i \in H^{2i}(X, \mathbb{Z}) \) of a \( U(n) \)-principal bundle \( P \rightarrow X \) are represented by the de Rham cohomology classes of certain characteristic exterior forms \( \mathcal{P}_{2i}(F_A) \) on \( X \) expressed in the strength two-form \( F_A \) of a principal connection \( A \) on \( P \rightarrow X \) [4]. The Chern class \( c_2 \) of a complex line bundle plays a prominent role in many quantization schemes, e.g., geometric quantization. Conversely, given a principal bundle \( P \rightarrow X \) with a structure Lie group \( G \), the Weil homomorphism associates to any invariant polynomial \( I_k \) on the Lie algebra of \( G \) a closed exterior form \( \mathcal{P}_{2k}(F_A) \) on \( X \) whose de Rham cohomology class is independent of the choice of a connection \( A \) on \( P \). Furthermore, given another principal connection \( A' \) on \( P \), the global transgression formula

\[
\mathcal{P}_{2k}(F_A) - \mathcal{P}_{2k}(F_{A'}) = dS_{2k-1}(A, A')
\]

defines the secondary characteristic form \( S_{2k-1}(A, A') \). In particular, if \( \mathcal{P}_{2k} \) is the characteristic Chern form, then \( S_{2k-1}(A, A') \) is the familiar Chern–Simons \((2k - 1)\)-form utilized in many models of topological field theory [74, 106] and (gauge and BRST) anomalies [49].

(ii) The well-known index theorem establishes the equality of the index of an elliptic operator on a fiber bundle to its topological index expressed in terms of the characteristic
forms of the Chern character, Todd and Euler classes. Note that the classical index theorem deals with linear elliptic operators on compact manifolds. They are Fredholm operators. In order to generalize the index theorem to noncompact manifolds, one either imposes conditions sufficient to force operators to be the Fredholm ones or considers the operators which are no longer Fredholm, but their index can be interpreted as a real number by some kind of averaging procedure [107]. The index problem for nonlinear elliptic operators has also been discussed [108]. From the physical viewpoint, the index theorem has shown that topological invariants can be expressed in terms of functional analysis, e.g., functional integrals in field models [74, 109]. The following model is quite illustrative. Given two circles \( s, s' \in \mathbb{R}^3 \), their linking number obeys the well-known integral formula

\[
L(s, s') = (4\pi)^{-1} \int_s dx^i \int_{s'} dy^j \varepsilon_{ijk} \partial_k \left( \sum_r (x^r - y^r)^2 \right)^{-1/2}.
\]

This formula is generalized to compact homologically trivial surfaces of dimensions \( p \) and \( n - p - 1 \) in \( \mathbb{R}^n \). Let us now consider the topological field theory of a \( p \)-form \( B \) and a \((n - p - 1)\)-form \( A \) on a compact oriented \( n \)-dimensional manifold \( Z \) (without boundary). Its action is

\[
S(A, B) = \int_Z B \wedge dA.
\]

It is invariant under adding an exact form either to \( A \) or \( B \). Then the linking number of compact homologically trivial surfaces \( M \) and \( N \) of dimensions \( p \) and \( n - p - 1 \), respectively, can be given by the functional integral

\[
L(M, N) = i^{-1} \int \mu(A, B) \exp(iS(A, B))
\]

by the appropriate choice of the measure \( \mu(A, B) \) on the moduli space of fields \( A \) and \( B \) [110]. This is the simplest example of abelian and non-Abelian topological BF theories of antisymmetric tensor fields [111]. For instance, polynomial invariants of knots are obtained in three-dimensional versions of BF theory [112], where a knot \( K = \partial \Sigma_K \) is represented by a (classical) observable

\[
\gamma^{(\Sigma_K, K, x_0)} = \int_{\Sigma_K} B \wedge A + \frac{1}{2} \oint_{x < y \in K} [A(x)B(y) - B(x)A(y)].
\]

A non-Abelian BF theory also serves as a dual model for Yang–Mills theory [113].

(iii) Several important characteristics come from geometry and topology of moduli spaces. They are exemplified by Donaldson and Seiberg–Witten invariants. Given an \( SU(2) \)-principal bundle \( P \to X \) over a compact four-dimensional manifold \( X \), we have the \( SU(2) \)-principal bundle \( (P \times \mathbb{C})/\mathbb{S} \to X \times \mathbb{M} \), where the appropriate Sobolev completion is assumed. By
the K"unneth formula its second Chern class $c_2 \in H^4(X \times \mathcal{M})$ is decomposed into the terms $c_{i,4-i} \in H^i(X) \otimes H^{4-i}(\mathcal{M})$, $i = 0, \ldots, 4$, which provide the map $\times H_2(X; \mathbb{Z}) \to H^{2m}(\mathcal{M}, \mathbb{Q})$ via the cup product in $H^{2m}(\mathcal{M}, \mathbb{Q})$. This map yields an injection of the polynomial algebra on $H_{2m}(\mathcal{M}, \mathbb{Q})$. Evaluated on the homology class $[\mathcal{M}] \in H_*(\mathcal{M}, \mathbb{Q})$ of the module space $\mathcal{M}$ of irreducible instantons, these polynomials are the Donaldson polynomials [114]. Expressed in the strength of a connection on the irreducible orbit space $\mathcal{M}$, they are the differential Donaldson invariants of a four-dimensional manifold $X$. Similarly, let $X$ be an oriented compact four-dimensional Riemannian manifold provided with a spin$^c$ structure, $S \to X$ the corresponding positive spinor bundle, and $L \to X$ the associated complex line bundle. Let $\mathcal{G}$ be an appropriate Sobolev completion of the irreducible configuration space of connections on $L \to X$ and sections of $S \to X$, except the zero section. Given the Sobolev completions $\mathcal{G}$ and $\mathcal{G}^0$ of the $U(1)$-gauge group and its pointed subgroup, respectively, we have the $U(1)$-principal bundle $\mathcal{G}/\mathcal{G}^0 \to \mathcal{G}/\mathcal{G}$. The Seiberg–Witten invariant is defined as the cup-product $\cup c_1$ of the first Chern class of this bundle evaluated on the homology class the module space $\mathcal{M}_\phi$ of solutions of the perturbed Seiberg–Witten equations $(d = \dim \mathcal{M}_\phi/2)$ [115].

(iv) Geometric quantization of a symplectic manifold $(X, \Omega)$ is affected by the following ambiguity. Firstly, the equivalence classes of admissible connections on a prequantization bundle (whose curvature obeys the prequantization condition $R = i\Omega$) are indexed by the set of homomorphisms of the homotopy group $\pi_1(X)$ of $X$ to $U(1)$ [116, 117]. Secondly, there are nonequivalent bundles of half-forms over $X$ in general and, consequently, the nonequivalent quantization bundles [48]. This ambiguity leads to nonequivalent quantizations.

(v) The cohomology analysis gives a rather complete picture of deformation quantization of symplectic manifolds. Let $\mathcal{K}$ be a commutative ring and $\mathcal{K}[[\lambda]]$ the ring of formal series in a real parameter $\lambda$. Recall that, given an associative (resp. Lie) algebra $A$ over a commutative ring $\mathcal{K}$, its Gerstenhaber deformation [118] is an associative (resp. Lie) $\mathcal{K}[[\lambda]]$-algebra $\mathcal{A}$ such that $\mathcal{A}/\lambda \mathcal{A} \approx A$. The multiplication in $\mathcal{A}$ reads

$$a \star b = a \circ b + \sum_{r=1}^{\infty} \lambda^r C_r(a, b)$$

where $\circ$ is the original associative (resp. Lie) product and $C_r$ are 2-cochains of the Hochschild (resp. Chevalley–Eilenberg) complex of $A$. The obstruction to the existence of a deformation of $A$ lies in the third Hochschild (resp. Chevalley–Eilenberg) cohomology group. Let now $A = \mathbb{C}^\infty(X)$ be the ring of complex smooth functions on a smooth manifold $X$. One considers its associative deformations $\mathcal{A}$ where the cochains $C_r$ are bidifferential operators of finite order. The multidifferential cochains make up a subcomplex of the Hochschild complex of $A$, and its cohomology equals the space of multi-vector fields on $X$ [119]. If $\mathbb{C}^\infty(X)$ is provided with the standard Fréchet topology of compact convergence for all derivatives, one
can consider its continuous deformation. The corresponding subcomplex of the Hochschild complex of $A$ is proved to have the same cohomology as the differential one [120]. Let now $X$ be a symplectic manifold, and let $A = C^\infty(X)$ be the Poisson algebra. Since the Poisson bracket is a bidifferential operator of order $(1,1)$, one has studied the similar deformations of $A$ where the cochains $C_r$ are differential operators of order $(1,1)$ with no constant term. The cohomology of the corresponding subcomplex of the Chevalley–Eilenberg complex of $A$ equals the de Rham cohomology $H^*(X)$ of $X$ [121]. The equivalence classes of Poisson deformations of the Poisson bracket on a symplectic manifold $X$ are parametrized by $H^2(X)[[\lambda]]$. A star-product on a Poisson manifold is defined as an associative deformation of $C^\infty(X)$ such that $C_1(f, f') - C_1(f', f)$ is the Poisson bracket. The existence of a star product on an arbitrary symplectic manifold has been proved in [122], and this is true for any regular Poisson manifold [123, 124]. Moreover, any star-product on a symplectic manifold is equivalent to Fedosov’s one, and its equivalence classes are parametrized by $H^2(X)[[\lambda]]$ [12, 13].

(vi) Let us also mention BRST cohomology, called into play in order to describe constrained symplectic systems [125-127]. Let $(Z, \Omega)$ be a symplectic manifold endowed with a Hamiltonian action of a Lie group $G$, $\hat{J}$ the corresponding momentum mapping of $Z$ to the Lie coalgebra $g^*$ of $G$, and $N = \hat{J}^{-1}(0)$ a regular constraint surface. The classical BRST complex is defined as the bicomplex

$$B^{n,m} = \Lambda^n g^* \otimes \Lambda^m g \otimes C^\infty(Z),$$

where the $n$- and $m$-gradings are the ghost and antighost degrees, respectively. The differential $\delta : B^{*,*} \rightarrow B^{*+1,*}$ is the coboundary operator of the Chevalley–Eilenberg cohomology of $g$ with coefficients in the $g$-module $g \otimes C^\infty(Z)$, while $\partial : B^{*,*} \rightarrow B^{*,*-1}$ is the Koszul boundary operator. The total differential $d = \delta + 2\partial$ is the classical BRST operator. The algebra $B$ is provided with the super Poisson bracket $\{ , \}$, and there exists an element $\Theta$ of $B$, called the BRST charge, such that $d = \{ \Theta, . \}$. The BRST cohomology is defined as the cohomology of $d$. The BRST complex has been built for constrained Poisson systems [128] and time-dependent Hamiltonian systems with Lagrangian constraints [129] as an extension of the Koszul–Tate complex of constraints through introduction of ghosts. Quantum BRST cohomology has been studied in the framework of geometric [130] and deformation [131] quantization. In field theory, BRST transformations are the particular generalized supersymmetry transformations of first order (depending on jets of ghosts) whose infinitesimal generator (the BRST operator) is nilpotent. In the framework of the Lagrangian field-antifield formalism on a base space $X$ [132, 133], one considers the bicomplex of horizontal (local in the terminology of [132]) exterior forms on the infinite order jet space of fields, ghosts and anti-fields with respect to the BRST operator $s$ and the horizontal differential $d_H$. In particular, it is essential for applications that relative ($s$ modulo $d_H$) cohomology
of this bicomplex in the maximal form-degree \( n = \dim X \) are related to the cohomology of the total BRST operator \( \mathbf{s} + d_H \) via the descent equation. Stated for a contractible base space \( X \), this relation can be extended to an arbitrary manifold \( X \) due to the fact that the cohomology of \( d_H \) equals the de Rham cohomology of \( X \) [134].

\textbf{V.}

Contemporary quantum models appeal to a number of new algebraic structures and the associated geometric techniques.

(i) For instance, SUSY models deal with graded manifolds and different types of supermanifolds, namely, \( H^\infty, G^\infty, GH^\infty \), \( G \)-supermanifolds over (finite) Grassmann algebras, \( R^\infty \)- and \( R \)-supermanifolds over Arens–Michael algebras of Grassmann origin and the corresponding types of DeWitt supermanifolds [61, 135, 136]. Their geometries are phrased in terms of graded local-ringed spaces. Note that one usually considers supervector bundles over \( G \)-supermanifolds. Firstly, the category of these supervector bundles is equivalent to the category of locally free sheaves of finite rank (in contrast, e.g., with \( GH^\infty \)-supermanifolds). Secondly, derivations of the structure sheaf of a \( G \)-supermanifold constitute a locally free sheaf (this is not the case, e.g., of \( G^\infty \)-supermanifolds). Moreover, this sheaf is again a structure sheaf of some \( G \)-superbundle (in contrast with graded manifolds). At the same time, most of the quantum models uses graded manifolds. They are not supermanifolds, though there is the correspondence between graded manifolds and DeWitt \( H^\infty \)-supermanifolds. By virtue of the well-known Batchelor theorem, the structure ring of any graded manifold with a body manifold \( Z \) is isomorphic to the graded ring \( \mathcal{A}_E \) of sections of some exterior bundle \( \wedge E \rightarrow Z \). In physical models, this isomorphism holds fixed from the beginning as a rule and, in fact, by geometry of a graded manifold is meant the geometry of the graded ring \( \mathcal{A}_E \). For instance, the familiar differential calculus in graded exterior forms is the graded Chevalley–Eilenberg differential calculus over such a ring.

(ii) Noncommutative geometry is mainly developed as a generalization of the calculus in commutative rings of smooth functions [20, 103, 137]. In a general setting, any noncommutative \( \mathcal{K} \)-ring \( \mathcal{A} \) over a commutative ring \( \mathcal{K} \) can be called into play. One can consider the above mentioned Chevalley–Eilenberg differential calculus \( \mathcal{O}^* \mathcal{A} \) over \( \mathcal{A} \), differential operators and connections on \( \mathcal{A} \)-modules (but not their jets). If the derivation \( \mathcal{K} \)-module \( \mathfrak{d} \mathcal{A} \) is a finite projective module with respect to the center of \( \mathcal{A} \), one can treat the triple \( (\mathcal{A}, \mathfrak{d} \mathcal{A}, \mathcal{O}^* \mathcal{A}) \) as a noncommutative space. For instance, this is the case of the matrix geometry, where \( \mathcal{A} \) is the algebra of finite matrices, and of the quantum phase space, where \( \mathcal{A} \) is a finite-dimensional CCR algebra. Noncommutative field theory also can be treated in this manner [138, 139], though the bracket of space coordinates \( [x^\mu, x^\nu] = i\theta^{\mu\nu} \) in this theory is also restarted from Moyal’s star product \( x^\mu \star x^\nu [138, 140] \). A different linear coordinate product \( [x^\mu, x^\nu] = ic_{\lambda}^{\mu\nu} x^\lambda \) comes from Connes’ noncommutative geometry [141]. In Connes’ noncommutative geometry,
the more deep analogy to the case of commutative smooth function rings leads to the notion of a spectral triple \((\mathcal{A},E,D)\) [20, 142]. It is given by an involutive subalgebra \(\mathcal{A} \subset B(E)\) of bounded operators on a Hilbert space \(E\) and an (unbounded) self-adjoint operator \(D\) in \(E\) such that the resolvent \((D - \lambda)^{-1}, \lambda \in \mathbb{C} \setminus \mathbb{R}\) is a compact operator and \([D,\mathcal{A}] \subset B(E)\). Furthermore, one assigns to elements \(\phi = a_0da_1 \cdots da_k\) of the universal differential calculus \((\mathcal{O}^*\mathcal{A},d)\) over \(\mathcal{A}\) the operators \(\pi(\phi) = a_0[D,a_1] \cdots [D,a_k]\) in \(E\). This however fails to be a representation of the differential algebra \(\mathcal{O}^*\mathcal{A}\) because \(\pi(\phi) = 0\) does not imply \(\pi(d\phi) = 0\).

The appropriate quotient \(\mathcal{O}^*\mathcal{A} \ni \phi \to [\phi] \in \mathcal{O}^*_D\) together with the differential \(\delta[\phi] = [d\phi]\) overcomes this difficulty, though \(\pi([\phi])\) is not an operator in \(E\). The \((\mathcal{O}^*_D,\delta)\) is called the Connes–de Rham complex. Note that other variants of spectral data, besides a spectral manifold \(X\) exemplifies Connes’ commutative geometry [20, 21].

Spectral triples have been studied for noncommutative tori, the Moyal deformations of \(\mathbb{R}^n\), noncommutative spheres 2-, 3- and 4-spheres [144, 145], and quantum Heisenberg manifolds [146]. It is essential that, since the universal differential calculus over a unital commutative algebra \(A\) is a direct summand (not as a differential algebra) of the Hochschild homology of \(A\) [147], the differential calculus \(\mathcal{O}^*\mathcal{A}\) over \(A\) in Connes’ noncommutative geometry is generated by the Hochschild cycles of \(A\) so that the representation \(\pi\) of Hochschild boundaries \(b\) vanishes.

(iii) Formalism of groupoids provides the above mentioned categorial \(C^*\)-algebraic deformation quantization of some class of Poisson manifolds [14, 148]. A groupoid is a small category whose morphisms are invertible [149, 150]. For instance, given an action of a group \(G\) on a set \(X\) on the right, the product \(\mathfrak{G} = X \times G\) is brought into the action groupoid, where a pair \(((x,g),(x',g'))\) is composable iff \(x' = xg\), the inversion \((x,g)^{-1} := (xg,g^{-1})\), the partial multiplication \((x,g)(x',g') := (x,g'x)\), the range \(r((x,g)) := (x,1_G)\), and the domain \(l((x,g)) := (xg,1_G)\). The unit space \(\mathfrak{G}^0 = r(\mathfrak{G}) = l(\mathfrak{G})\) of this groupoid is naturally identified to \(X\). Any group bundle \(Y \to X\) (e.g., a vector bundle) is a groupoid whose elements make up composable pairs iff they belong to the same fiber, and whose unit space is the set of unit elements of fibers of \(Y \to X\). Let \(\mathfrak{A} \to \mathfrak{G}^0\) be an Abelian group bundle over the unit space \(\mathfrak{G}^0\) of a groupoid \(\mathfrak{G}\). The pair \((\mathfrak{G},\mathfrak{A})\) together with a homomorphism \(\mathfrak{G} \to \text{Iso} \mathfrak{A}\) is called the \(\mathfrak{G}\)-module bundle. One can associate to any \(\mathfrak{G}\)-module bundle a cochain complex \(C^*(\mathfrak{G},\mathfrak{A})\). Let \(\mathfrak{A}\) be a \(\mathfrak{G}\)-module bundle in groups \(U(1)\). The key point is that, similarly to the case of a locally compact group [67], one can associate a \(C^*\)-algebra \(A_{\mathfrak{G},\sigma}\) to any locally compact groupoid \(\mathfrak{G}\) provided with a Haar system by means of the choice of a two-cocycle \(\sigma \in C^2(\mathfrak{G},\mathfrak{A})\) [149]. The algebras \(A_{\mathfrak{G},\sigma}\) and \(A_{\mathfrak{G},\sigma'}\) are isomorphic if \(\sigma\) and \(\sigma'\) are cohomology equivalent. If \(\mathfrak{G}\) is an \(r\)-discrete groupoid, any measure \(\lambda\) of total mass 1 on its unit space \(\mathfrak{G}^0\) induces a state of the \(C^*\)-algebra \(A_{\mathfrak{G},\sigma}\). Moreover, it is the KMS state if a measure \(\lambda\) satisfies a certain cohomological condition. A Lie groupoid is a groupoid for which \(\mathfrak{G}\) and
\(G^0\) are smooth manifolds, the inversion and partial multiplication are smooth, while \(r\) and \(l\) are fibered manifolds. Since a Lie groupoid admits a Haar system, one can assign to it a \(C^*\)-algebra \(A_{\mathcal{G}}\). This assignment is functorial if certain classes of morphisms of Lie groupoids and \(C^*\)-algebras (isomorphism classes of regular bibundles and those of Hilbert bimodules, respectively) are considered [148]. A Lie groupoid is called symplectic if it is a symplectic manifold \((\mathcal{G}, \Omega)\) such that the multiplication relation \((x, y) \mapsto (xy, x, y)\) is a Lagrangian submanifold of the symplectic manifold \((\mathcal{G} \times \mathcal{G} \times \mathcal{G}, \Omega \oplus \Omega \oplus \Omega)\) [151]. A Poisson manifold \(P\) is called integrable if there exists a symplectic groupoid \(\mathcal{G}(P)\) over \(P\). It is unique up to an isomorphism. Integrable Poisson manifolds subject to a certain class morphisms (isomorphism classes of regular dual pairs) make up a suitable category \(Ps\) [148]. Since the groupoid \(\mathcal{G}(P)\) is \(l\) and \(l\)-simple connected, one considers the category \(LGc\) of Lie groupoids possessing this property. Any Lie groupoid yields an associated Lie algebroid \(L(\mathcal{G})\) which is the restriction to \(\mathcal{G}^0\) of the vertical tangent bundle of the fibration \(r : \mathcal{G} \to \mathcal{G}^0\) [150]. Sections of \(L(\mathcal{G})\) make up a real Lie algebra compatible with the anchor map \(L(\mathcal{G}) \to T\mathcal{G}^0\). The key point is that, similarly to the dual of a Lie algebra, the dual \(L^*(\mathcal{G})\) of \(L(\mathcal{G})\) is a Poisson manifold in a canonical way. Then the assignment \(\mathcal{G} \mapsto L^*(\mathcal{G})\) is a functor \(LGc \to Ps\) [148]. Let \(LPs\) denote its image. One can show that \(Q : L^*(\mathcal{G}) \mapsto A_{\mathcal{G}}\) is a functor from the category \(LPs\) to the above mentioned category of \(C^*\)-algebras [14]. It is a desired functorial quantization. This functor is equivariant under the Morita equivalence of Poisson manifolds in \(LGc\) [152] and that of \(C^*\)-algebras [153]. This property fails if the functor \(Q\) is extended to the category \(Ps\). In this case, the natural codomain of \(Q\) is the category \(KK\) whose objects are separable \(C^*\)-algebras and morphisms are Kasparov’s KK-groups [154]. Furthermore, the functorial quantization \(L^*(\mathcal{G}) \mapsto A_{\mathcal{G}}\) is amplified into the above mentioned strict quantization of \(A_{\mathcal{G}}\) by an appropriate continuous field of \(C^*\)-algebras over \(\mathbb{R}\) [14]. Connes’ tangent groupoid provides an example of such strict quantization [19, 155].

(iv) Hopf algebras and, in particular, quantum groups make a contribution to many quantum theories [63, 156, 157]. At the same time, the development of differential calculus and differential geometry over these algebras has met difficulties. Given a (complex or real) Hopf algebra \(H = (H, m, \Delta, \varepsilon, S)\), one introduces the first order differential calculus (henceforth FODC) \((\mathcal{O}^1, d)\) over \(H\) as for a noncommutative algebra. It is said to be left-covariant if \(\mathcal{O}^1\) possesses the structure of a left \(H\)-comodule \(\Delta_l : \mathcal{O}^1 \to H \otimes \mathcal{O}^1\) such that \(\Delta_l(ab) = \Delta(a)(\text{Id} \otimes d)\Delta(b), a, b \in H\) [156]. By virtue of Woronowicz’s theorem [158], left-covariant FODCs are classified by right ideals

\[\mathcal{R} = \{x \in \text{Ker } \varepsilon : S(x_1)dx_2 = 0\}\]

of \(H\) contained in the kernel of its counit. The linear subspace

\[T = \{t \in H^* : t(1) = 0, t(\mathcal{R}) = 0\}\]
of the dual $H^*$ is the quantum (enveloping) Lie algebra (quantum tangent space [159]) associated to the left-covariant FODC $(\mathcal{O}^1, d)$ (see [100] for a general construction of the enveloping algebra for a noncommutative FODC). A problem lies in the definition of vector fields as a sum $a^i u_i$ of invariant vector fields $u_i(a) = a_1 t_i(a_2)$ [160] because they satisfy the deformed Leibniz rule deduced from the formula

$$t_i(ab) = t_i(a)\varepsilon(b) + \sum_i f_{ij}(a)t_j(b),$$

where $(t_i)$ is a basis for $T$ and $f_{ij}$ are functionals on $H$. One can model the vector fields obeying such a Leibniz rule by the so called Cartan pairs [101]. These are elements $u$ of the right $H$-dual $\mathcal{O}_R$ of $\mathcal{O}^1$ together with the morphisms $\hat{u}: H \ni a \mapsto u(da) \in H$ which obey the relations

$$\hat{(bu)}(a) = b\hat{u}(a), \quad \hat{u}(ba) = \hat{u}(b)a + \hat{(ub)}(a).$$

Note that the standard FODC given by the kernel of the product $A \otimes A \to A$ in a noncommutative algebra $A$ is not applied to a commutative algebra because it does not provide the equality $adb = (db)a$, $a, b \in A$. The similar commutative FODC deals with a certain quotient $A \otimes A/\mu^2$ [97]. Another problem of geometry of Hopf algebras is the notion of a quantum principal bundle [63, 161, 162]. In the case of Lie groups, there are two equivalent definitions of a smooth principal bundle, which is both a set of trivial bundles glued together by means of transition functions and a bundle provided with the canonical action of a structure group on the right. In the case of quantum groups, these two notions of a principal bundle are not matched, unless the base is a smooth manifold [163, 164]. The first definition of a quantum principal bundle repeats the classical one and makes use of the notion of a trivial quantum bundle, a covering of a quantum space (e.g., by a family of nonintersecting closed ideals), and its reconstruction from local pieces [165] which, however, is not always possible [166]. The second definition of a quantum principal bundle is algebraic [63, 167]. Let $H$ be a bialgebra and $P$ a right $H$-comodule algebra with respect to the coaction $\beta: P \to P \otimes H$. Let $M = \{p \in P : \beta(p) = p \otimes 1_H\}$ be its invariant subalgebra. The triple $(P, H, \beta)$ is called a quantum principal bundle if the map

$$\text{ver}: P \otimes_M P \ni (p \otimes_M q) \to p\beta(q) \in P \otimes H$$

is a linear isomorphisms. This condition, called the Hopf-Galois condition, is a key point of this algebraic definition of a quantum principal bundle. By some reasons, one can think of it as being a $sui generis$ local trivialization. In particular, let $\pi: P \to H$ be a Hopf-algebra surjection and $\beta := (1\text{d} \otimes \pi)\Delta$. If the product map $m: \text{Ker} \varepsilon|_M \otimes P \to \text{Ker} \pi$ is a surjection, then $(P, H, \beta)$ is a quantum principal bundle [63]. Many examples of a quantum principal bundle come from this fact.
(v) Finally, one of the main point of Tamarkin’s proof of the formality theorem in deformation quantization is that, for any algebra $\mathcal{A}$ over a field characteristic zero, its cohomological Hochschild complex and its Hochschild cohomology are algebras for the same operad. This observation has been the starting point of 'operad renaissance' [168, 169]. Monoidal categories provide numerous examples of algebras for operads. Furthermore, homotopy monoidal categories lead to the notion of a homotopy monoidal algebra for an operad. In a general setting, one considers homotopy algebras and weakened algebraic structures where, e.g., a product operation is associative up to homotopy [170]. Their well-known examples are $A_\infty$-spaces and $A_\infty$-algebras [171]. At the same time, the formality theorem is also applied to quantization of several algebro-geometric structures such as algebraic varieties [50, 172].

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