String solutions in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ with $B$-field

Plamen Bozhilov

Institute for Nuclear Research and Nuclear Energy
Bulgarian Academy of Sciences
1784 Sofia, Bulgaria
bozhilov@inrne.bas.bg, bozhilov.p@gmail.com

Abstract

We consider strings living in $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ with nonzero $B$-field. By using specific ansatz for the string embedding, we obtain a class of solutions corresponding to strings moving in the whole ten dimensional space-time. For the $\text{AdS}_3$ subspace, these solutions are given in terms of incomplete elliptic integrals. For the two three-spheres, they are expressed in terms of Lauricella hypergeometric functions of many variables. The conserved charges, i.e. the string energy, spin and angular momenta, are also found.
1 Introduction

A very important development in the field of string theory has been achieved for the case of AdS/CFT duality \[1\] between strings and conformal field theories in various dimensions. The most developed case is the correspondence between strings living in \(AdS_5 \times S^5\) and \(\mathcal{N} = 4\) SYM in four dimensions. Another example is the duality between strings on \(AdS_4 \times CP^3\) background and \(\mathcal{N} = 6\) super Chern-Simons-matter theory in three space-time dimensions. The main achievements in the above examples are due to the discovery of integrable structures on both sides of the correspondence. Many other cases have been considered also \[2\].

An interesting area of research is the \(AdS_3/CFT_2\) duality \[3\]-\[28\], related to \(AdS_3 \times S^3 \times T^4\) and \(AdS_3 \times S^3 \times S^3 \times S^1\) string theory backgrounds where nontrivial two-form \(B\)-field appears. For a review, see e.g \[26\].

The classical string solutions and their semiclassical limits, corresponding to large conserved charges \[29\], play important role in checking and understanding the AdS/CFT correspondence. Here we obtain a class of solutions corresponding to strings moving in the whole ten dimensional space-time \(AdS_3 \times S^3 \times S^3 \times S^1\) with nonzero \(B\)-field.

The paper is organized as follows. In Sec.2 we describe the background. In Sec.3 we present our general approach to string dynamics in curved backgrounds with nonzero \(B\)-field. In Sec.4 we apply it to strings moving in \(AdS_3 \times S^3 \times S^3 \times S^1\) with \(B\)-field. In Sec.5 we obtain the conserved charges for the case under consideration. Sec.6 is devoted to our concluding remarks.

2 The background

The metric of \(AdS_3 \times S^3 \times S^3 \times S^1\) is

\[
ds^2 = ds^2_{AdS_3} + ds^2_{S^3_+} + ds^2_{S^3_-} + dw^2,
\]

where \(w\) is the coordinate along \(S^1\). As found in \[30\], the radii of \(AdS_3\) and of the two three-spheres satisfy the relation

\[
\frac{1}{R^2_{AdS_3}} = \frac{1}{R^2_+} + \frac{1}{R^2_-}.
\]
If we normalize the AdS$_3$ radius to one, (2.2) is solved by
\begin{equation}
\frac{1}{R^2_+} = \cos^2 \varphi, \quad \frac{1}{R^2_-} = \sin^2 \varphi.
\end{equation}

According to [27], the metric on AdS$_3$ and the two three-spheres can be written as
\begin{align}
\text{ds}^2_{AdS_3} &= -\left(\frac{1 + \frac{z_1^2 + z_2^2}{4}}{1 - \frac{z_1^2 + z_2^2}{4}}\right)^2 dt^2 + \left(\frac{1}{\frac{1}{1 - \frac{z_1^2 + z_2^2}{4}}}\right)^2 (dz_1^2 + dz_3^2), \\
\text{ds}^2_{S^3_+} &= \left(\frac{1 - \cos^2 \varphi \frac{y_3^2 + y_4^2}{4}}{1 + \cos^2 \varphi \frac{y_3^2 + y_4^2}{4}}\right)^2 (dy_3^2 + dy_4^2), \\
\text{ds}^2_{S^3_-} &= \left(\frac{1 - \sin^2 \varphi \frac{x_6^2 + x_7^2}{4}}{1 + \sin^2 \varphi \frac{x_6^2 + x_7^2}{4}}\right)^2 (dx_6^2 + dx_7^2).
\end{align}

The $B$-field in these coordinates is given by [27]
\begin{align}
B &= \frac{q}{\left(1 - \frac{z_1^2 + z_2^2}{4}\right)^2} (z_1 dz_2 - z_2 dz_1) \wedge dt \\
&\quad + \frac{q \cos \varphi}{\left(1 + \cos \varphi \frac{y_3^2 + y_4^2}{4}\right)^2} (y_3 dy_4 - y_4 dy_3) \wedge d\phi_5 \\
&\quad + \frac{q \sin \varphi}{\left(1 + \sin \varphi \frac{x_6^2 + x_7^2}{4}\right)^2} (x_6 dx_7 - x_7 dx_6) \wedge d\phi_8,
\end{align}

where the parameter $q$ is related to the quantized coefficient $k$ of the Wess-Zumino term by [27]
\begin{equation}
k = q \sqrt{\lambda}.
\end{equation}

For our purposes here, we introduce new background coordinates:
\begin{align}
z_1 &= 2 \tanh \frac{\rho}{2} \cos \phi, \quad z_2 = 2 \tanh \frac{\rho}{2} \sin \phi, \\
\phi_5 &= R_+ \phi_{2+} \\
y_3 &= R_+ w_1 = 2R_+ \tan \frac{\theta_+}{2} \cos \phi_{1+}, \\
y_4 &= R_+ w_2 = 2R_+ \tan \frac{\theta_+}{2} \sin \phi_{1+}, \\
\phi_8 &= R_- \phi_{2-} \\
x_6 &= R_- v_1 = 2R_- \tan \frac{\theta_-}{2} \cos \phi_{1-}, \\
x_7 &= R_- v_2 = 2R_- \tan \frac{\theta_-}{2} \sin \phi_{1-}.
\end{align}
As a consequence, the resulting description of the background becomes:

\[ ds_{_{\text{AdS}_3}}^2 = - \cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\phi^2, \]  
(2.12)

or \((\sinh^2 \rho = r^2)\)

\[ ds_{_{\text{AdS}_3}}^2 = -(1 + r^2) \, dt^2 + (1 + r^2)^{-1} dr^2 + r^2 d\phi^2 \]  
(2.13)

\[ ds_{_{S^3_+}}^2 = \frac{1}{\cos^2 \varphi} \left( d\theta_+^2 + \sin^2 \theta_+ d\phi_{1+}^2 + \cos^2 \theta_+ d\phi_{2+}^2 \right) \]  
(2.14)

\[ ds_{_{S^3_-}}^2 = \frac{1}{\sin^2 \varphi} \left( d\theta_-^2 + \sin^2 \theta_- d\phi_{1-}^2 + \cos^2 \theta_- d\phi_{2-}^2 \right) \]  
(2.15)

\[ ds_{_{S^1}}^2 = dw^2 = g_{ww} dw^2, \]

\[ B = qr^2 d\phi \wedge dt \]  
(2.16)

\[ B = qr^2 d\phi \wedge dt \]

\[ \equiv b_{\phi t} \, d\phi \wedge dt + b_{\phi_1 \phi_2} \, d\phi_{1+} \wedge d\phi_{2+} + b_{\phi_1 \phi_2} \, d\phi_{1-} \wedge d\phi_{2-}. \]

3 The approach

Here, we will use the Polyakov type action for the bosonic string in a \(D\)-dimensional curved space-time with metric tensor \(g_{MN}(x)\), interacting with a background 2-form gauge field \(b_{MN}(x)\) via Wess-Zumino term

\[ S^P = \int d^2 \xi L^P, \quad L^P = -\frac{1}{2} \left( T \sqrt{-\gamma} \gamma^{mn} G_{mn} - Q \epsilon^{mn} B_{mn} \right), \]

\[ \xi^m = (\xi^0, \xi^1) = (\tau, \sigma), \quad m, n = 0, 1, \]

where

\[ G_{mn} = \partial_m X^M \partial_n X^N g_{MN}, \quad B_{mn} = \partial_m X^M \partial_n X^N b_{MN}, \]

\[ (\partial_m = \partial / \partial \xi^m, \quad M, N = 0, 1, \ldots, D - 1), \]
are the fields induced on the string worldsheet, $\gamma$ is the determinant of the auxiliary worldsheet metric $\gamma_{mn}$, and $\gamma^{mn}$ is its inverse. The position of the string in the background space-time is given by $x^M = X^M(\xi^m)$, and $T = 1/2\pi\alpha'$, $Q$ are the string tension and charge, respectively. If we consider the action $S^P$ as a bosonic part of a supersymmetric one, we have to put $Q = \pm T$. In what follows, $Q = T$.

The equations of motion for $X^M$ following from $S^P$ are:

$$-g_{LK}[\partial_m (\sqrt{-\gamma}\gamma^{mn}\partial_n X^K) + \sqrt{-\gamma}\gamma^{mn}\Gamma^K_{MN}\partial_m X^M \partial_n X^N]$$

$$= \frac{1}{2}H_{LMN}\epsilon^{mn}\partial_m X^M \partial_n X^N, \quad (3.1)$$

where $(\partial_M = \partial/\partial x^M)$

$$\Gamma_{L,MN} = g_{LK}\Gamma^K_{MN} = \frac{1}{2}(\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN}),$$

$$H_{LMN} = \partial_L b_{MN} + \partial_M b_{NL} + \partial_N b_{LM},$$

are the components of the symmetric connection corresponding to the metric $g_{MN}$, and the field strength of the gauge field $b_{MN}$ respectively. The constraints are obtained by varying the action $S^P$ with respect to $\gamma_{mn}$:

$$\delta_{\gamma_{mn}} S^P = 0 \Rightarrow (\gamma^{kl}\gamma_{mn} - 2\gamma^{km}\gamma^{ln}) G_{mn} = 0. \quad (3.2)$$

Further on, we will use conformal gauge $\gamma^{mn} = \eta^{mn} = diag(-1, 1)$ in which the string Lagrangian, the Virasoro constraints and the equations of motion take the following form:

$$\mathcal{L} = \frac{T}{2} (G_{00} - G_{11} + 2B_{01}), \quad (3.3)$$

$$G_{00} + G_{11} = 0, \quad G_{01} = 0,$$

$$g_{LK} \left[ (\partial_0^2 - \partial_1^2) X^K + \Gamma^K_{MN} (\partial_0 X^M \partial_0 X^N - \partial_1 X^M \partial_1 X^N) \right] = H_{LMN}\partial_0 X^M \partial_1 X^N.$$

Now, we suppose that there exist some number of commuting Killing vector fields along part of $X^M$ coordinates and split $X^M$ into two parts

$$X^M = (X^\mu, X^a),$$

where $X^\mu$ are the isometric coordinates, while $X^a$ are the non-isometric ones. The existence of isometric coordinates leads to the following conditions on the background fields:

$$\partial_\mu g_{MN} = 0, \quad \partial_\mu b_{MN} = 0. \quad (3.4)$$
Then from the string action, we can compute the conserved charges

\[ Q_\mu = \int d\sigma \frac{\partial \mathcal{L}}{\partial (\partial_0 X^\mu)} \]  

(3.5)

under the above conditions.

Next, we introduce the following ansatz for the string embedding

\[ X^\mu(\tau, \sigma) = \Lambda^\mu \tau + \tilde{X}^\mu(\alpha \sigma + \beta \tau), \quad X^a(\tau, \sigma) = \tilde{X}^a(\alpha \sigma + \beta \tau), \]  

(3.6)

where \( \Lambda^\mu, \alpha, \beta \) are arbitrary parameters. Further on, we will use the notation \( \xi = \alpha \sigma + \beta \tau \).

Applying this ansatz, one can find that the equalities (3.3), (3.5) become

\[ \mathcal{L} = \frac{T}{2} \left[ - (\alpha^2 - \beta^2) g_{MN} \frac{d\tilde{X}^M}{d\xi} \frac{d\tilde{X}^N}{d\xi} + 2\Lambda^\mu (\beta g_{\mu N} + \alpha b_{\mu N}) \frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu \Lambda^\nu g_{\mu \nu} \right], \]  

(3.7)

\[ G_{00} + G_{11} = (\alpha^2 + \beta^2) g_{MN} \frac{d\tilde{X}^M}{d\xi} \frac{d\tilde{X}^N}{d\xi} + 2\beta \Lambda^\mu g_{\mu N} \frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu \Lambda^\nu g_{\mu \nu} = 0, \]  

(3.8)

\[ G_{01} = \alpha \beta g_{MN} \frac{d\tilde{X}^M}{d\xi} \frac{d\tilde{X}^N}{d\xi} + \alpha \Lambda^\mu g_{\mu N} \frac{d\tilde{X}^N}{d\xi} = 0, \]  

(3.9)

\[ -(\alpha^2 - \beta^2) \left[ g_{LK} \frac{d^2 \tilde{X}^K}{d\xi^2} + \Gamma_{L,MN} \frac{d\tilde{X}^M}{d\xi} \frac{d\tilde{X}^N}{d\xi} \right] + 2\beta \Lambda^\mu \Gamma_{L,\mu N} \frac{d\tilde{X}^N}{d\xi} + \Lambda^\mu \Lambda^\nu \Gamma_{L,\mu \nu} \]

\[ = \alpha \Lambda^\mu H_{L,\mu N} \frac{d\tilde{X}^N}{d\xi}, \]  

(3.10)

\[ Q_\mu = \frac{T}{\alpha} \int d\xi \left[ (\beta g_{\mu N} + \alpha b_{\mu N}) \frac{d\tilde{X}^N}{d\xi} + \Lambda^\nu g_{\mu \nu} \right]. \]  

(3.11)

Our next task is to try to solve the equations of motion (3.10) for the isometric coordinates, i.e. for \( L = \lambda \). Due to the conditions (3.4) imposed on the background fields, we obtain that

\[ \Gamma_{\lambda,ab} = \frac{1}{2} (\partial_a g_{b \lambda} + \partial_b g_{a \lambda}), \quad \Gamma_{\lambda,\mu a} = \frac{1}{2} \partial_a g_{\mu \lambda}, \quad \Gamma_{\lambda,\mu \nu} = 0, \]

\[ H_{\lambda a b} = \partial_a b_{b \lambda} + \partial_b b_{a \lambda}, \quad H_{\lambda \mu a} = \partial_a b_{\lambda \mu}, \quad H_{\lambda \mu \nu} = 0. \]
By using this, one can find the following first integrals for $\tilde{X}^\mu$:

$$
\frac{d\tilde{X}^\mu}{d\xi} = \frac{1}{\alpha^2 - \beta^2} \left[ g^{\mu\nu} (C_\nu - \alpha \Lambda^\rho b_{\nu\rho}) + \beta \Lambda^\mu \right] - g^{\mu\nu} g_{\nu a} \frac{d\tilde{X}^a}{d\xi},
$$

(3.12)

where $C_\nu$ are arbitrary integration constants. Therefore, according to our ansatz (3.6), the solutions for the string coordinates $X^\mu$ can be written as

$$
X^\mu(\tau, \sigma) = \Lambda^\mu \tau + \frac{1}{\alpha^2 - \beta^2} \int d\xi \left[ g^{\mu\nu} (C_\nu - \alpha \Lambda^\rho b_{\nu\rho}) + \beta \Lambda^\mu \right] - \int g^{\mu\nu} g_{\nu a} d\tilde{X}^a(\xi).
$$

(3.13)

Now, let us turn to the remaining equations of motion corresponding to $L = a$, where

$$
\begin{align*}
\Gamma_{a,\mu b} &= -\frac{1}{2} (\partial_a g_{b\mu} - \partial_b g_{a\mu}), \\
\Gamma_{a,\mu\nu} &= -\frac{1}{2} \partial_a g^{\mu\nu}, \\
H_{a\mu\nu} &= \partial_a b_{\mu\nu}, \\
H_{a\mu b} &= -\partial_a b_{b\mu} + \partial_b b_{a\mu}.
\end{align*}
$$

(3.15)

Taking this into account and replacing the first integrals for $\tilde{X}^\mu$ already found, one can write these equations in the form (prime is used for $d/d\xi$)

$$
(\alpha^2 - \beta^2) \left[ h_{ab} \ddot{X}^{b'} + \Gamma^h_{a,bc} \dot{X}^b \dot{X}^c \right] = 2 \partial_a [A_b] \ddot{X}^{b'} - \partial_a U,
$$

(3.14)

where

$$
\begin{align*}
h_{ab} &= g_{ab} - g_{a\mu} g^{\mu\nu} g_{\nu b}, \\
\Gamma^h_{a,bc} &= \frac{1}{2} (\partial_b h_{ca} + \partial_c h_{ba} - \partial_a h_{bc}), \\
A_a &= g_{a\mu} g^{\mu\nu} (C_\nu - \alpha \Lambda^\rho b_{\nu\rho}) + \alpha \Lambda^\mu b_{a\mu}, \\
U &= \frac{1}{\alpha^2 - \beta^2} \left[ (C_\mu - \alpha \Lambda^\rho b_{\mu\rho}) g^{\mu\nu} (C_\nu - \alpha \Lambda^\lambda b_{\nu\lambda}) + \alpha^2 \Lambda^\mu \Lambda^\nu g_{\mu\nu} \right].
\end{align*}
$$

(3.15-3.17)

The Virasoro constraints (3.8), (3.9) become:

$$
\frac{1}{2} (\alpha^2 - \beta^2) h_{ab} \dddot{X}^{a'} \dddot{X}^{b'} + U = 0, \quad \alpha \Lambda^\mu C_\mu = 0.
$$

(3.18)

Finally, let us write down the expressions for the conserved charges (3.11)

$$
Q_\mu = \frac{T}{\alpha^2 - \beta^2} \int d\xi \left[ \frac{\beta}{\alpha} C_\mu + \alpha \Lambda^\nu g_{\mu\nu} + b_{\mu\nu} g^{\mu\rho} (C_\rho - \alpha \Lambda^\lambda b_{\rho\lambda}) \\
+ (\alpha^2 - \beta^2) (b_{\mu a} - b_{\nu\mu} g^{\nu\rho} g_{\rho a}) \dddot{X}^{a'} \right].
$$

(3.19)
4 String solutions in $AdS_3 \times S^3 \times S^3 \times S^1$ with $B$-field

In accordance with our notations

\[
\begin{align*}
\mu &= (t, \phi, \phi_{1+}, \phi_{2+}, \phi_{1-}, \phi_{2-}, w), \quad a = (r, \theta_{+}, \theta_{-}), \\
g_{\mu\nu} &= (g_{tt}, g_{\phi\phi}, g_{\phi_{1+}\phi_{1+}}, g_{\phi_{2+}\phi_{2+}}, g_{\phi_{1-}\phi_{1-}}, g_{\phi_{2-}\phi_{2-}}, g_{ww}), \\
g_{ab} &= (g_{rr}, g_{\theta_{+}\theta_{+}}, g_{\theta_{-}\theta_{-}}), \\
g_{\alpha\mu} &= 0, \quad h_{ab} = g_{ab}, \\
b_{\mu\nu} &= (b_{\phi t}, b_{\phi_{1+}\phi_{2+}}, b_{\phi_{1-}\phi_{2-}}), \quad b_{\alpha\nu} = 0, \\
A_a &= 0, \quad (4.1)
\end{align*}
\]

where

\[
\begin{align*}
g_{tt} &= -(1 + r^2), \quad g_{rr} = (1 + r^2)^{-1}, \quad g_{\phi\phi} = r^2, \\
g_{\theta_{+}\theta_{+}} &= \frac{1}{\cos^2 \varphi}, \quad g_{\phi_{1+}\phi_{1+}} = \frac{1}{\cos^2 \varphi} \sin^2 \theta_{+}, \quad g_{\phi_{2+}\phi_{2+}} = \frac{1}{\cos^2 \varphi} \cos^2 \theta_{+}, \\
g_{\theta_{-}\theta_{-}} &= \frac{1}{\sin^2 \varphi}, \quad g_{\phi_{1-}\phi_{1-}} = \frac{1}{\sin^2 \varphi} \sin^2 \theta_{-}, \quad g_{\phi_{2-}\phi_{2-}} = \frac{1}{\sin^2 \varphi} \cos^2 \theta_{-}, \\
g_{ww} &= 1, \\
b_{\phi t} &= -qr^2, \\
b_{\phi_{1+}\phi_{2+}} &= \frac{q \sin^2 \theta_{+}}{\cos^2 \varphi \left( \cos^2 \frac{\theta_{+}}{2} + \frac{\sin^2 \frac{\theta_{+}}{2}}{\cos \varphi} \right)^2}, \\
b_{\phi_{1-}\phi_{2-}} &= \frac{q \sin^2 \theta_{-}}{\sin^2 \varphi \left( \cos^2 \frac{\theta_{-}}{2} + \frac{\sin^2 \frac{\theta_{-}}{2}}{\sin \varphi} \right)^2}. \quad (4.2)
\end{align*}
\]

The effective scalar potential (3.17) can be computed to be

\[
U = \sum_{i=1}^{4} U_i, \quad (4.3)
\]

where
\[ U_1(r) = \frac{1}{2} \left( \frac{(C_t + \alpha \Lambda \phi r^2)^2}{1 + r^2} + \frac{(C_\phi - \alpha \Lambda t q^2 r)^2}{r^2} \right) - \alpha^2 \left[ (\Lambda^t)^2 (1 + r^2) - (\Lambda^\phi)^2 r^2 \right], \quad (4.4) \]

\[ U_2(\theta_+) = \frac{1}{2} \frac{\sin^2 \theta_+}{\cos^2 \varphi \left( \cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \theta_+}{\cos \varphi} \right)^2} \left( C_{\phi_1} - \frac{\alpha \Lambda \phi_{2+} q \sin^2 \theta_+}{\cos \varphi} \right)^2 + \left( C_{\phi_2} + \frac{\alpha \Lambda \phi_{2+} q \sin^2 \theta_+}{\cos \varphi} \right)^2 \cos^2 \varphi \left( \cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \theta_+}{\cos \varphi} \right)^2 \right) \quad (4.5) \]

\[ U_3(\theta_-) = \frac{1}{2} \frac{\sin^2 \theta_-}{\cos^2 \varphi \left( \cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \theta_-}{\sin \varphi} \right)^2} \left( C_{\phi_1} - \frac{\alpha \Lambda \phi_{2-} q \sin^2 \theta_-}{\sin \varphi} \right)^2 + \left( C_{\phi_2} + \frac{\alpha \Lambda \phi_{2-} q \sin^2 \theta_-}{\sin \varphi} \right)^2 \sin^2 \varphi \left( \cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \theta_-}{\sin \varphi} \right)^2 \right) \quad (4.6) \]

\[ U_4 = C_w^2 + (\Lambda^w)^2 = \text{const.} \quad (4.7) \]

By using that in the case under consideration the metric \( g_{ab} \) is diagonal, one can prove that the equations of motion (3.14) possess the following first integrals

\[ \frac{d\tilde{X}^a}{d\xi} = \sqrt{\frac{C_a - 2U_a}{(\alpha^2 - \beta^2)g_{aa}}}. \quad (4.8) \]
where \( \tilde{X}^a = (r, \theta_+, \theta_-) \), \( C_a = (C_r, C_{\theta_+}, C_{\theta_-}) \) are arbitrary integration constants and \( U_a = (U_1(r), U_2(\theta_+), U_3(\theta_-)) \).

The replacement of (4.8) into (3.18) reduces the first Virasoro constraint to the following equality

\[
C_r + C_{\theta_+} + C_{\theta_-} = 0. \tag{4.9}
\]

Thus, the Virasoro constraints are simplified to relations between the integration constants and embedding parameters on this type of string solutions.

### 4.1 Solutions in \( AdS_3 \)

For the \( AdS_3 \) subspace, (4.8) gives

\[
d\xi = \frac{dr}{\sqrt{(C_r - 2U_1(r))(1 + r^2)}}. \tag{4.10}
\]

By using the expression (4.4) for \( U_1(r) \) and introducing the variable \( y = r^2 \), one obtains

\[
d\xi = \frac{\alpha^2 - \beta^2}{2\alpha \sqrt{1 - q^2} \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right]} \frac{dy}{\sqrt{(y_p - y)(y - y_m)(y - y_n)}}, \tag{4.10}
\]

where

\[
y_p > y > y_m \geq 0, \quad y_n < 0,
\]

and \( y_p, y_m, y_n \) satisfy the equalities

\[
y_p + y_m + y_n = \frac{1}{\alpha^2(1 - q^2) \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right]} \left[ C_r(\alpha^2 - \beta^2) - \alpha \left( \alpha (\Lambda^\phi)^2 - 2\alpha (\Lambda^t)^2 - 2q (C_\phi \Lambda^t + C_t \Lambda^\phi) + q^2 \alpha (\Lambda^t)^2 \right) \right], \tag{4.11}
\]

\[
y_p y_m + y_p y_n + y_m y_n = -\frac{1}{\alpha^2(1 - q^2) \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right]} \left[ C_r(\alpha^2 - \beta^2) + C_t^2 - C_\phi^2 + \alpha^2 (\Lambda^t)^2 + 2q \alpha C_\phi \Lambda^t \right],
\]

\[
y_p y_m y_n = -\frac{C_\phi^2}{\alpha^2(1 - q^2) \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right]}.
\]
Integrating (4.10) and inverting
\[\xi(y) = \frac{\alpha^2 - \beta^2}{\alpha \sqrt{(1 - q^2) \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right] (y_p - y_n)}} F\left( \text{arcsin} \frac{y_p - y}{y_p - y_m}, \frac{y_p - y_n}{y_p - y_m} \right)\]
to \(y(\xi)\), one finds the following solution
\[y(\xi) = (y_p - y_n) \text{DN}^2 \left[ \frac{\alpha \sqrt{(1 - q^2) \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right] (y_p - y_n)}}{\alpha^2 - \beta^2}, \xi, \frac{y_p - y_m}{y_p - y_n} \right] + y_n, \quad (4.12)\]
where \(F\) is the incomplete elliptic integral of first kind and \(\text{DN}\) is one of the Jacobi elliptic functions.

Now we are going to find the solutions for the isometric coordinates \(t(\xi)\) and \(\phi(\xi)\). In accordance with (3.12), the first integrals for \(\bar{X}^t\) and \(\bar{X}^\phi\) can be computed to be given by
\[
\frac{d\bar{X}^t}{d\xi} = \frac{1}{\alpha^2 - \beta^2} \left[ \beta \Lambda^t - q \alpha \Lambda^\phi - \frac{C_t - q \alpha \Lambda^\phi}{1 + y} \right],
\]
\[
\frac{d\bar{X}^\phi}{d\xi} = \frac{1}{\alpha^2 - \beta^2} \left( \beta \Lambda^\phi - q \alpha \Lambda^t + \frac{C_\phi}{y} \right).
\]

Integrating and using (4.10), we obtain
\[t(\tau, \xi) = \Lambda^t \tau + \frac{1}{\alpha \sqrt{(1 - q^2) \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right] (y_p - y_n)}} \left[ (\beta \Lambda^t - q \alpha \Lambda^\phi) F\left( \text{arcsin} \frac{y_p - y}{y_p - y_m}, \frac{y_p - y_n}{y_p - y_m} \right) \right.\]
\[- \frac{C_t - q \alpha \Lambda^\phi}{1 + y_p} \Pi\left( \text{arcsin} \frac{y_p - y}{y_p - y_m}, \frac{y_p - y_m}{1 + y_p}, \frac{y_p - y_n}{y_p - y_m} \right) \left. \right], \quad (4.13)\]
\[\phi(\tau, \xi) = \Lambda^\phi \tau + \frac{1}{\alpha \sqrt{(1 - q^2) \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right] (y_p - y_n)}} \left[ (\beta \Lambda^\phi - q \alpha \Lambda^t) F\left( \text{arcsin} \frac{y_p - y}{y_p - y_m}, \frac{y_p - y_n}{y_p - y_m} \right) \right.\]
\[+ \frac{C_\phi}{y_p} \Pi\left( \text{arcsin} \frac{y_p - y}{y_p - y_m}, \frac{y_p - y_m}{y_p - y_n} \right) \left. \right], \quad (4.14)\]
where \(\Pi\) is the incomplete elliptic integral of third kind.
4.2 Solutions on the two $S^3$

It is clear from (2.14) and (2.15) that the solutions on the two three-spheres $S^3_+$ and $S^3_-$ can be obtained from each other by the exchanges $\sin \varphi \leftrightarrow \cos \varphi$ and $+ \leftrightarrow -$ in the subscripts of the coordinates, the corresponding integration constants, and in the superscripts of the embedding parameters. That is why we are going to present here the string solutions for one of the spheres only, say $S^3_-$. 

The non-isometric coordinate on $S^3_-$ is $\theta_-$ for which the first integral (4.8) reads

$$\frac{d\theta_-}{d\xi} = \sqrt{\frac{\sin^2 \varphi}{\alpha^2 - \beta^2}} \left[ C_{\theta_-} - 2U_3(\theta_-) \right].$$

(4.15)

$U_3(\theta_-)$ is given in (4.6).

Now we introduce the variable

$$\gamma = \cos^2 \frac{\theta_-}{2}.$$

This allows us to rewrite (4.15) in the following form

$$d\xi = \frac{\alpha^2 - \beta^2}{\sin \varphi} (-a_{10})^{-1/2} (1 - (1 - \sin \varphi)\gamma)^2 (2\gamma - 1)$$

$$\left[ (\gamma_1 - \gamma)(\gamma - \gamma_2)(\gamma - \gamma_3)(\gamma - \gamma_4)(\gamma - \gamma_5)(\gamma - \gamma_6) \right]^{-1/2} d\gamma, \quad \gamma_1 = \gamma_{\text{max}}.$$

(4.16)

The corresponding computations are given in an Appendix. Next, we integrate

$$\xi = \frac{\alpha^2 - \beta^2}{\sin \varphi} (-a_{10})^{-1/2} \int_{\gamma_{\text{max}}}^{\gamma_{\text{max}} - u} (1 - (1 - \sin \varphi)u)^2 (2u - 1)$$

$$\left[ (\gamma_{\text{max}} - u)(u - \gamma_2)(u - \gamma_3)(u - \gamma_4)(u - \gamma_5)(u - \gamma_6) \right]^{-1/2} du,$$

and introduce new integration variable

$$\delta = \frac{\gamma_{\text{max}} - u}{\gamma_{\text{max}} - \gamma}.$$
Then (4.17) becomes

$$\xi(\gamma) = \frac{\alpha^2 - \beta^2}{\sin \varphi} (a_{10})^{-1/2}(1 - \gamma_{\text{max}}(1 - \sin \varphi))^2(2\gamma_{\text{max}} - 1)(\gamma_{\text{max}} - \gamma)^{1/2} \quad (4.18)$$

$$\left[\prod_{i=2}^{10}(\gamma_{\text{max}} - \gamma_i)\right]^{-1/2} \int_0^1 \delta^{-1/2} \left(1 + \frac{(\gamma_{\text{max}} - \gamma)(1 - \sin \varphi)}{(1 - \gamma_{\text{max}}(1 - \sin \varphi))} \right)^2$$

$$\left(1 - \frac{2(\gamma_{\text{max}} - \gamma)}{2\gamma_{\text{max}} - 1}\right) \prod_{i=2}^{10} \left(1 - \frac{\gamma_{\text{max}} - \gamma}{\gamma_{\text{max}} - \gamma_i}\right)^{-1/2} d\delta.$$  

Comparing the integral in (4.18) with the integral representation of the Lauricella hypergeometric functions of many variables $F_D^{(a)}$ [31]

$$F_D^{(a)}(a; b_1, \ldots, b_n; c; z_1, \ldots, z_n) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c - a)} \int_0^1 \delta^{a-1}(1 - \delta)^{c-a-1}(1 - z_1\delta)^{-b_1} \ldots (1 - z_n\delta)^{-b_n} d\delta,$$

where $Re(a) > 0, \quad Re(c - a) > 0,$

one finds

$$\xi(\gamma) = 2\frac{\alpha^2 - \beta^2}{\sin \varphi} (a_{10})^{-1/2}(1 - \gamma_{\text{max}}(1 - \sin \varphi))^2(2\gamma_{\text{max}} - 1)(\gamma_{\text{max}} - \gamma)^{1/2} \quad (4.20)$$

$$\left[\prod_{i=2}^{10}(\gamma_{\text{max}} - \gamma_i)\right]^{-1/2} F_D^{(11)}(1/2; b_1, \ldots, b_{11}; 3/2; z_1, \ldots, z_{11}),$$

where

$$b_1 = -2, \quad z_1 = -\frac{(\gamma_{\text{max}} - \gamma)(1 - \sin \varphi)}{1 - \gamma_{\text{max}}(1 - \sin \varphi)},$$

$$b_2 = -1, \quad z_2 = 2\frac{(\gamma_{\text{max}} - \gamma)}{2\gamma_{\text{max}} - 1},$$

$$b_k = 1/2, \quad z_k = \frac{\gamma_{\text{max}} - \gamma}{\gamma_{\text{max}} - \gamma_{k-1}}, \quad k = 3, \ldots, 11.$$  

This is our final result for $\xi(\gamma)$. Unfortunately this solution is not invertible, so we can not write down $\gamma(\xi)$.

Now, let us proceed with obtaining the solutions for the isometric coordinates on $S^3$. From (3.12) one finds the following first integrals:

$$\frac{d\tilde{X}^{\phi_1-}}{d\xi} = \frac{1}{\alpha^2 - \beta^2} \left\{ \beta\Lambda^{\phi_1-} + \sin^2 \varphi \left[ C_{\phi_1-} \sin^2 \theta_- - \frac{\alpha\Lambda^{\phi_2-}}{\sin^2 \varphi \left( \cos^2 \frac{\varphi}{2} + \frac{\sin^2 \varphi}{\sin \varphi} \right)^2} \right] \right\}, \quad (4.22)$$
\[
\frac{d\tilde{X}_{\phi_2-}}{d\xi} = \frac{1}{\alpha^2 - \beta^2} \left\{ \beta \Lambda_{\phi_2-} + \frac{\sin^2 \varphi}{\cos^2 \theta_-} \left[ C_{\phi_2-} + \frac{q \alpha \Lambda_{\phi_1-} \sin^2 \theta_-}{\sin^2 \varphi \left( \cos^2 \theta_- + \frac{\sin^2 \theta_-}{\sin \varphi} \right)^2} \right] \right\}. \quad (4.23)
\]

Introducing the variable \(\gamma\) and using (4.16), one arrives at
\[
\tilde{X}_{\phi_1-} \equiv \tilde{\phi}_1- = 2 \sin \varphi (-a_{10})^{-1/2} (1 - \gamma_{\max} (1 - \sin \varphi))^2 (2 \gamma_{\max} - 1) (\gamma_{\max} - \gamma)^{1/2} \quad (4.24)
\]
\[
\left[ \prod_{i=2}^{10} (\gamma_{\max} - \gamma_i) \right]^{-1/2} \left[ \frac{\beta \Lambda_{\phi_1-}}{\sin^2 \varphi} F_D^{(11)} (1/2; b_1, \ldots, b_{11}; 3/2; z_1, \ldots, z_{11}) \right]
+ \frac{C_{\phi_1-}}{4 (1 - \gamma_{\max}) \gamma_{\max}} F_D^{(13)} (1/2; b_1, \ldots, b_{11}, b_{12}, b_{13}; 3/2; z_1, \ldots, z_{11}, z_{12}, z_{13})
- q \alpha \Lambda_{\phi_2-} (1 - \gamma_{\max} (1 - \sin \varphi))^{-2} F_D^{(10)} (1/2; b_2, \ldots, b_{11}; 3/2; z_2, \ldots, z_{11}) \right],
\]

\[
\tilde{X}_{\phi_2-} \equiv \tilde{\phi}_2- = 2 \sin \varphi (-a_{10})^{-1/2} (1 - \gamma_{\max} (1 - \sin \varphi))^2 (2 \gamma_{\max} - 1) (\gamma_{\max} - \gamma)^{1/2} \quad (4.25)
\]
\[
\left[ \prod_{i=2}^{10} (\gamma_{\max} - \gamma_i) \right]^{-1/2} \left[ \frac{\beta \Lambda_{\phi_2-}}{\sin^2 \varphi} F_D^{(11)} (1/2; b_1, \ldots, b_{11}; 3/2; z_1, \ldots, z_{11}) \right]
+ \frac{C_{\phi_2-}}{(2 \gamma_{\max} - 1)^2} F_D^{(12)} (1/2; b_1, \ldots, b_{11}, c_{12}; 3/2; z_1, \ldots, z_{11}, y_{12})
+ 4 q \alpha \Lambda_{\phi_1-} (2 \gamma_{\max} - 1)^{-2} \gamma_{\max} (1 - \gamma_{\max}) (1 - \gamma_{\max} (1 - \sin \varphi))^{-2} \times 
F_D^{(15)} (1/2; b_1, \ldots, b_{11}, c_{12}, c_{13}, c_{14}, c_{15}; 3/2; z_1, \ldots, z_{11}, y_{12}, z_{12}, z_{13}, y_{15}) \right],
\]

where
\[
b_{12} = 1, \quad z_{12} = \frac{\gamma_{\max} - \gamma}{1 - \gamma_{\max}}, \quad (4.26)
\]
\[
b_{13} = 1, \quad z_{13} = \frac{\gamma_{\max} - \gamma}{\gamma_{\max}},
\]
\[
c_{12} = 2, \quad y_{12} = 2 \frac{\gamma_{\max} - \gamma}{2 \gamma_{\max} - 1},
\]
\[
c_{13} = -1, \quad c_{14} = -1,
\]
\[
c_{15} = 2, \quad y_{15} = - \frac{(\gamma_{\max} - \gamma)(1 - \sin \varphi)}{1 - \gamma_{\max} (1 - \sin \varphi)}.
\]

Therefore, according to (3.13), the solutions for the isometric coordinates on \(S^3\) are given by
\[
\phi_{1-}(\tau, \xi) = \Lambda_{\phi_1-} \tau + \tilde{\phi}_1-(\xi), \quad \phi_{2-}(\tau, \xi) = \Lambda_{\phi_2-} \tau + \tilde{\phi}_2-(\xi).
\]
5 Conserved charges

The expressions for the conserved charges corresponding to the isometric coordinates can be found from (3.19) to be

\[ Q_t \equiv -E_s = \frac{T}{\alpha^2 - \beta^2} \int \left[ \frac{\beta}{\alpha} C_t - \alpha \Lambda^t (1 + r^2) - q \left( C_\phi - q \alpha \Lambda^t r^2 \right) \right] d\xi, \quad (5.1) \]

\[ Q_\phi \equiv S = \frac{T}{\alpha^2 - \beta^2} \int \left[ \frac{\beta}{\alpha} C_\phi - \alpha \Lambda^{\phi} r^2 - q \frac{r^2}{1 + r^2} \left( C_t + q \alpha \Lambda^t r^2 \right) \right] d\xi, \quad (5.2) \]

\[ Q_{\phi_1+} \equiv J_{1+} = \frac{T}{\alpha^2 - \beta^2} \int \left[ \frac{\beta}{\alpha} C_{\phi_1+} + \frac{\alpha \Lambda^{\phi_1+}}{\cos^2 \varphi} \sin^2 \theta_+ \right. \]
\[ + \frac{q \sin^2 \theta_+}{\cos^2 \theta_+ \left( \cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \left( \frac{q \alpha \Lambda^{\phi_1+} \sin^2 \theta_+}{\cos^2 \phi \left( \cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \right) \] \[ \left. \left. - q \left( \cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right) \right) \right] d\xi, \quad (5.3) \]

\[ Q_{\phi_2+} \equiv J_{2+} = \frac{T}{\alpha^2 - \beta^2} \int \left[ \frac{\beta}{\alpha} C_{\phi_2+} + \frac{\alpha \Lambda^{\phi_2+}}{\cos^2 \varphi} \cos^2 \theta_+ \right. \]
\[ - \frac{q}{\left( \cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \left( \frac{q \alpha \Lambda^{\phi_2+} \sin^2 \theta_+}{\cos^2 \phi \left( \cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right)^2} \right) \] \[ \left. \left. - q \left( \cos^2 \frac{\theta_+}{2} + \frac{\sin^2 \frac{\theta_+}{2}}{\cos \varphi} \right) \right) \right] d\xi, \quad (5.4) \]

\[ Q_{\phi_1-} \equiv J_{1-} = \frac{T}{\alpha^2 - \beta^2} \int \left[ \frac{\beta}{\alpha} C_{\phi_1-} + \frac{\alpha \Lambda^{\phi_1-}}{\sin^2 \varphi} \sin^2 \theta_- \right. \]
\[ + \frac{q \sin^2 \theta_-}{\cos^2 \theta_- \left( \cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \left( \frac{q \alpha \Lambda^{\phi_1-} \sin^2 \theta_-}{\sin^2 \phi \left( \cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2} \right) \] \[ \left. \left. \cos^2 \theta_- \left( \cos^2 \frac{\theta_-}{2} + \frac{\sin^2 \frac{\theta_-}{2}}{\sin \varphi} \right)^2 \right) \right] d\xi, \quad (5.5) \]
\[ Q_{\phi_2-} \equiv J_{2-} = \frac{T}{\alpha^2 - \beta^2} \int \left[ \frac{\beta}{\alpha} C_{\phi_2-} + \frac{\alpha \Lambda_{\phi_2-}}{\sin^2 \varphi} \cos^2 \theta_- \right] \] (5.6)
\[ - \frac{q}{\left( \cos^2 \frac{\vartheta_-}{2} + \frac{\sin^2 \varphi}{2} \right)^2} \left( C_{\phi_1-} - \frac{q \alpha \Lambda_{\phi_2-} \sin^2 \theta_-}{\sin^2 \varphi \left( \cos^2 \frac{\vartheta_-}{2} + \frac{\sin^2 \varphi}{2} \right)^2} \right) \] d\xi.

Here we introduced the following notations: \( E_s \) is the string energy, \( S \) is the spin of the string in \( AdS_3 \), \( J_{1+}, J_{2+}, J_{1-}, J_{2-} \), are the angular momenta on the two three spheres \( S^3_\pm \).

In order to compute \( E_s \) and \( S \), we introduce the variable \( y = r^2 \) and use the expression (4.10) for \( d\xi \) in \( AdS_3 \). This leads to the following results

\[ E_s = \frac{2T}{\sqrt{(1-q^2) \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right] (y_p - y_n)}} \] (5.7)
\[ \left[ \left( \Lambda^t - \frac{\beta}{\alpha^2} C_t + \frac{C_\phi}{\alpha} \right) K \left( 1 - \frac{y_m - y_n}{y_p - y_n} \right) \right. + \]
\[ (1-q^2) \Lambda^t \left( y_n E \left( 1 - \frac{y_m - y_n}{y_p - y_n} \right) \right) \bigg], \]
\[ S = \frac{2T}{\sqrt{(1-q^2) \left[ (\Lambda^\phi)^2 - (\Lambda^t)^2 \right] (y_p - y_n)}} \] (5.8)
\[ \left[ \left( \frac{\beta}{\alpha^2} C_\phi - \frac{C_t}{\alpha} + \Lambda^\phi q^2 \right) K \left( 1 - \frac{y_m - y_n}{y_p - y_n} \right) \right. + \]
\[ (1-q^2) \Lambda^\phi \left( y_n E \left( 1 - \frac{y_m - y_n}{y_p - y_n} \right) \right) \bigg] + \frac{q \xi}{1 + y_p} \Pi \left( \frac{y_p - y_m}{1 + y_p}, 1 - \frac{y_m - y_n}{y_p - y_n} \right), \]

where \( K, E \) and \( \Pi \) are the complete elliptic integrals of first, second and third kind.

To find the final expressions for the angular momenta, we now introduce the variable \( \gamma \) (\( \gamma = \cos^2 \frac{\theta_-}{2} \) for \( S^3_+ \) and \( \gamma = \cos^2 \frac{\theta_-}{2} \) for \( S^3_- \)). Here we present the results for \( J_{1-} \) and \( J_{2-} \)
only. By using (4.10) we derive:

\[
J_{1-} = \frac{2\pi T}{\sin \varphi} (-a_{10})^{-1/2} (1 - \gamma_{max} (1 - \sin \varphi))^{2} (2 \gamma_{max} - 1) (\gamma_{max} - \gamma_{min})^{1/2} \] (5.9)

\[
\left[ \prod_{i=2}^{10} (\gamma_{max} - \gamma_i) \right]^{-1/2} \times \frac{\beta}{\alpha} C_{\phi_{1-}} F_{D}^{(10)}(1/2; b_1, \ldots, b_1, b_{k-1}, b_{k+1}, \ldots, b_{11}; 1; Z_1, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_{11}) + \frac{4 \alpha \Lambda_{\phi_{1-}}}{\sin^2 \varphi} \gamma_{max} (1 - \gamma_{max}) \times F_{D}^{(12)}(1/2; b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_{11}, b_{12}, b_{13}; 1; Z_1, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_{11}, Z_{12}, Z_{13}) + 4 q \sin^2 \varphi \gamma_{max} (1 - \gamma_{max}) \left( \frac{C_{\phi_{2-}}}{(2 \gamma_{max} - 1)^{2} (1 - \gamma_{max} (1 - \sin \varphi))^{2}} \right) \times F_{D}^{(11)}(1/2; b_2, \ldots, b_{k-1}, b_{k+1}, \ldots, b_{11}, b_{12}, b_{13}; 1; Z_2, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_{11}, Z_{12}, Z_{13}) + \frac{4 q \alpha \Lambda_{\phi_{1-}}}{(2 \gamma_{max} - 1)^{2} (1 - \gamma_{max} (1 - \sin \varphi))^{4}} \times F_{D}^{(12)}(1/2; 2, 2, 1, b_3, \ldots, b_{k-1}, b_{k+1}, \ldots, b_{11}, -2, -2; 1; Z_1, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_{11}, Z_{12}, Z_{13}) \right],
\]

\[
J_{2-} = \frac{2\pi T}{\sin \varphi} (-a_{10})^{-1/2} (1 - \gamma_{max} (1 - \sin \varphi))^{2} (2 \gamma_{max} - 1) (\gamma_{max} - \gamma_{min})^{1/2} \] (5.10)

\[
\left[ \prod_{i=2}^{10} (\gamma_{max} - \gamma_i) \right]^{-1/2} \times \frac{\beta}{\alpha} C_{\phi_{2-}} F_{D}^{(10)}(1/2; b_1, \ldots, b_1, b_{k-1}, b_{k+1}, \ldots, b_{11}; 1; Z_1, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_{11}) + \frac{\alpha \Lambda_{\phi_{2-}}}{\sin^2 \varphi} (2 \gamma_{max} - 1)^{2} \times F_{D}^{(10)}(1/2; b_1, -3, b_3, \ldots, b_{k-1}, b_{k+1}, \ldots, b_{11}; 1; Z_1, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_{11}) - q \sin^2 \varphi C_{\phi_{1-}} (1 - \gamma_{max} (1 - \sin \varphi)) \times F_{D}^{(9)}(1/2; b_2, \ldots, b_{k-1}, b_{k+1}, \ldots, b_{11}; 1; Z_2, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_{11}) + 4 q^2 \alpha \Lambda_{\phi_{2-}} \sin^2 \varphi \gamma_{max} (1 - \gamma_{max}) (1 - \gamma_{max} (1 - \sin \varphi))^{-4} \times F_{D}^{(13)}(1/2; 2, -1, b_3, \ldots, b_{k-1}, b_{k+1}, \ldots, b_{11}, -1, -1; 1; Z_1, \ldots, Z_{k-1}, Z_{k+1}, \ldots, Z_{11}, Z_{12}, Z_{13}) \right],
\]

where $Z_k$ are related to the previous $z_k$ by the change $\gamma \rightarrow \gamma_{min}$.

In writing (5.9), (5.10), we used the following property of the hypergeometric functions.
\[ F_{D}^{(n)}(a; b_1, \ldots, b_n; c; z_1, \ldots, z_{k-1}, 1, z_{k+1}, \ldots, z_n) = \frac{\Gamma(c)\Gamma(c-a-b_k)}{\Gamma(c-a)\Gamma(c-b_k)} \times (5.11) \]

\[ F_{D}^{(n-1)}(a; b_1, \ldots, b_{k-1}, b_{k+1}, \ldots, b_n; c - b_k; z_1, \ldots, z_{k-1}, z_{k+1}, \ldots, z_n). \]

It follows from the integral representation (4.19). We needed to use this property in order to take into account that for some \( k \)

\[ Z_k = \frac{\gamma_{\text{max}} - \gamma_{\text{min}}}{\gamma_{\text{max}} - \gamma_{k-1}} = 1, \]

i.e. \( \gamma_{k-1} = \gamma_{\text{min}} \geq 0. \)

### 6 Concluding remarks

Here we considered strings living in \( AdS_3 \times S^3 \times S^3 \times S^1 \) with nonzero 2-form \( B \)-field. By using specific ansatz for the string embedding, we obtained a class of solutions corresponding to strings moving in the whole ten dimensional space-time. For the \( AdS_3 \) subspace, these solutions are given in terms of incomplete elliptic integrals as expected. For the two three-spheres, they are expressed in terms of Lauricella hypergeometric functions of many variables. The same is true for the corresponding conserved angular momenta related to the isometries of the three-spheres. This is in contrast with the case of \( AdS_3 \times S^3 \times T^4 \) background with \( B \)-field, where the solutions for the string coordinates on \( S^3 \) are given in terms of incomplete elliptic integrals \([25, 28]\). The complications here arise because of the specific form of the \( B \)-field for this supergravity solution (see \([2,16]\)).

### Acknowledgements

This work is partially supported by the NSF grant DFNI T02/6.

### Appendix

Here we explain how (4.16) is derived. First we represent \( d\xi \) as

\[ d\xi = \frac{\alpha^2 - \beta^2}{\sin \varphi} (1 - (1 - \sin \varphi)\gamma)^2 (2\gamma - 1) \sum_{i=0}^{10} a_i \gamma^i d\gamma, \]
where

\[ a_0 = \frac{1}{4}C_{\phi_1}^2 \sin^2 \varphi, \]

\[ a_1 = C_{\theta_-} (\alpha^2 - \beta^2) - \alpha^2 (\Lambda_{\phi^2-})^2 \csc^2 \varphi \]
\[ - \sin^2 \varphi \left[ (C_{\phi_1})^2 \sin \varphi + (C_{\phi_2})^2 - 2C_{\phi_1} (C_{\phi_1} + q\alpha \Lambda_{\phi^2-}) \right], \]

\[ a_2 = -9C_{\theta_-} (\alpha^2 - \beta^2) - 4\alpha^2 (\Lambda_{\phi^2-})^2 \csc \varphi + \alpha^2 \left[ 13 (\Lambda_{\phi^2-})^2 - 4 (\Lambda_{\phi^1-})^2 \right] \csc^2 \varphi \]
\[ + \frac{1}{2} \sin \varphi \left( 8C_{\theta_-} (\alpha^2 - \beta^2) - \sin \varphi \left( 13 (C_{\phi_1})^2 - 2C_{\phi_2} - 5C_{\phi_2} - 8q\alpha \Lambda_{\phi^1-} \right) \right) \]
\[ + 28q\alpha C_{\phi_1} \Lambda_{\phi^2-} + 8q^2 \alpha^2 (\Lambda_{\phi^2-})^2 + \sin \varphi \left( 8 (C_{\phi_2})^2 \right) \]
\[ - 2C_{\phi_1} \left( 7C_{\phi_1} + 4q\alpha \Lambda_{\phi^2-} \right) + 3 \left( (C_{\phi_1})^2 \sin \varphi \right) \right), \]

\[ a_3 = 34C_{\theta_-} (\alpha^2 - \beta^2) - 6\alpha^2 (\Lambda_{\phi^2-})^2 - 16\alpha^2 \left( (\Lambda_{\phi^1-})^2 - 3 (\Lambda_{\phi^2-})^2 \right) \csc \varphi \]
\[ + 2\alpha^2 \left( 20 (\Lambda_{\phi^1-})^2 - 37 (\Lambda_{\phi^2-})^2 \right) \csc^2 \varphi + \sin \varphi \times \]
\[ - 32C_{\theta_-} (\alpha^2 - \beta^2) + \sin \varphi \left( 11 (C_{\phi_1})^2 + 38q\alpha C_{\phi_1} \Lambda_{\phi^2-} \right) \]
\[ + 2 \left( 3C_{\theta_-} (\alpha^2 - \beta^2) - 5 (C_{\phi_2})^2 + 16q\alpha C_{\phi_2} \Lambda_{\phi^1-} \right) \]
\[ - 4q^2 \alpha^2 \left( 2 (\Lambda_{\phi^1-})^2 - 3 (\Lambda_{\phi^2-})^2 \right) \right) \right) \]
\[ - \sin \varphi \left( 19 (C_{\phi_1})^2 \right) \]
\[ - 16C_{\phi_2} \left( C_{\phi_2} - 4q\alpha \Lambda_{\phi^1-} \right) + 24q\alpha C_{\phi_1} \Lambda_{\phi^2-} + \sin \varphi \times \]
\[ \left( 6 (C_{\phi_2})^2 - 9 (C_{\phi_1})^2 + (C_{\phi_1})^2 \sin \varphi - 2q\alpha C_{\phi_1} \Lambda_{\phi^2-} \right) \right) \]

\[ a_4 = -70C_{\theta_-} (\alpha^2 - \beta^2) - 6\alpha^2 \left( 4 (\Lambda_{\phi^1-})^2 - 11 (\Lambda_{\phi^2-})^2 \right) \]
\[ + 8\alpha^2 \left( 18 (\Lambda_{\phi^1-})^2 - 31 (\Lambda_{\phi^2-})^2 \right) \csc \varphi - 2\alpha^2 \left( 86 (\Lambda_{\phi^1-})^2 - 121 (\Lambda_{\phi^2-})^2 \right) \csc^2 \varphi \]
\[ + \frac{1}{4} \sin \varphi \left( 416C_{\theta_-} (\alpha^2 - \beta^2) - 16\alpha^2 (\Lambda_{\phi^2-})^2 + \sin \varphi \times \right) \]
\[ - 41 (C_{\phi_1})^2 - 200q\alpha C_{\phi_1} \Lambda_{\phi^2-} - 8 \left( 21C_{\theta_-} (\alpha^2 - \beta^2) - 5 (C_{\phi_2})^2 \right) \]
\[ + 4 (C_{\phi_1})^2 - 4 \left( C_{\theta_-} (\alpha^2 - \beta^2) - 6C_{\phi_2} \left( C_{\phi_2} - 2q\alpha \Lambda_{\phi^1-} \right) \right) \]
\[ + 4 (C_{\phi_1})^2 - 4 \left( C_{\theta_-} (\alpha^2 - \beta^2) - 6C_{\phi_2} \left( C_{\phi_2} - 2q\alpha \Lambda_{\phi^1-} \right) \right) \]
\[ + 52q\alpha C_{\phi_1} \Lambda_{\phi^2-} + \sin \varphi \left( 78 (C_{\phi_1})^2 - 8C_{\phi_2} \left( 9C_{\phi_2} - 4q\alpha \Lambda_{\phi^1-} \right) \right) \]
\[ + 40q\alpha C_{\phi_1} \Lambda_{\phi^2-} + \sin \varphi \left( (C_{\phi_1})^2 \sin \varphi - 20 (C_{\phi_1})^2 + 16 (C_{\phi_2})^2 \right) \right),\]

18
$$a_5 = 85C_{\theta_\alpha}(\alpha^2 - \beta^2) + 6\alpha^2 \left(32 (\Lambda^{\phi_1})^2 - 51 (\Lambda^{\phi_2})^2\right)$$

$$-16\alpha^2 \left(34 (\Lambda^{\phi_1})^2 - 45 (\Lambda^{\phi_2})^2\right) \csc \varphi + \alpha^2 \left(416 (\Lambda^{\phi_1})^2 - 501 (\Lambda^{\phi_2})^2\right) \csc^2 \varphi$$

$$+ \sin \varphi \left(-8 \left(22C_{\theta_\alpha}(\alpha^2 - \beta^2) + \alpha^2 \left(2 (\Lambda^{\phi_1})^2 - 5 (\Lambda^{\phi_2})^2\right)\right)\right)$$

$$+ \sin \varphi \left(5 (C_{\phi_1})^2 + 114C_{\theta_\alpha}(\alpha^2 - \beta^2) - (5C_{\phi_2} - 12\alpha\lambda\Lambda^{\phi_1}) (C_{\phi_2} - 4q\alpha\Lambda^{\phi_1}) + 32\alpha C_{\phi_1}\Lambda^{\phi_2} - (1 - 48q^2) \alpha^2 (\Lambda^{\phi_2})^2 + \sin \varphi \left(-8 \left(3C_{\theta_\alpha}(\alpha^2 - \beta^2) - 2 (C_{\phi_1} + 3q\alpha\Lambda^{\phi_2})\right)\right)$$

$$+ \sin \varphi \left(C_{\theta_\alpha}(\alpha^2 - \beta^2) - 2C_{\phi_2} (9C_{\phi_2} - 8q\alpha\Lambda^{\phi_1})\right) + 2C_{\phi_1} (9C_{\phi_2} + 8q\alpha\Lambda^{\phi_2})$$

$$+ \frac{1}{2} \left((C_{\phi_1})^2 - (C_{\phi_2})^2\right) (1 - \cos 2\varphi - 16 \sin \varphi)),$$

$$a_6 = -61C_{\theta_\alpha}(\alpha^2 - \beta^2) - 6\alpha^2 \left(104 (\Lambda^{\phi_1})^2 - 129((\Lambda^{\phi_2})^2\right)$$

$$+ 4\alpha^2 \left(280 (\Lambda^{\phi_1})^2 - 321 (\Lambda^{\phi_2})^2\right) \csc \varphi$$

$$- \alpha^2 \left(620 (\Lambda^{\phi_1})^2 - 681 (\Lambda^{\phi_2})^2\right) \csc^2 \varphi + \sin \varphi \times$$

$$\left(4 \left(41C_{\theta_\alpha}(\alpha^2 - \beta^2) + \alpha^2 \left(28 (\Lambda^{\phi_1})^2 - 41 (\Lambda^{\phi_2})^2\right)\right) + \sin \varphi \times$$

$$\left(-150C_{\theta_\alpha}(\alpha^2 - \beta^2) + (C_{\phi_2} + 2(1 - 2q)\alpha\Lambda^{\phi_1}) (C_{\phi_2} - 2(1 + 2q)\alpha\Lambda^{\phi_1}) - (C_{\phi_1} - (3 - 4q)\alpha\Lambda^{\phi_2}) (C_{\phi_1} + (3 + 4q)\alpha\Lambda^{\phi_2}) + \sin \varphi \times$$

$$\left(4 \left(13C_{\theta_\alpha}(\alpha^2 - \beta^2) + (C_{\phi_1})^2 - (C_{\phi_2})^2 + 4\alpha \left(C_{\phi_1}\Lambda^{\phi_2} + C_{\phi_2}\Lambda^{\phi_1}\right) - \sin \varphi \left(5C_{\theta_\alpha}(\alpha^2 - \beta^2) + 6 \left((C_{\phi_1})^2 - (C_{\phi_2})^2\right)\right)$$

$$+ 8q\alpha \left(C_{\phi_1}\Lambda^{\phi_2} + C_{\phi_2}\Lambda^{\phi_1}\right) + \frac{1}{2} \left((C_{\phi_1})^2 - (C_{\phi_2})^2\right) \times$$

$$(1 - \cos 2\varphi - 8 \sin \varphi))\right).$$

$$a_7 = -8(1 - \sin \varphi) \left(-3C_{\theta_\alpha}(\alpha^2 - \beta^2) - 5\alpha^2 \left(7 (\Lambda^{\phi_1})^2 - 8 (\Lambda^{\phi_2})^2\right)$$

$$+ 97\alpha^2 \left(97 (\Lambda^{\phi_1})^2 - 104 (\Lambda^{\phi_2})^2\right) \csc \varphi - \alpha^2 \left(73 (\Lambda^{\phi_1})^2 - 76 (\Lambda^{\phi_2})^2\right) \csc^2 \varphi$$

$$+ \sin \varphi \left(7C_{\theta_\alpha}(\alpha^2 - \beta^2) + \alpha^2 \left(3 (\Lambda^{\phi_1})^2 - 4 (\Lambda^{\phi_2})^2\right) + \frac{1}{2}C_{\theta_\alpha}(\alpha^2 - \beta^2) (1 - \cos 2\varphi - 10 \sin \varphi)\right).$$
\[
a_8 = -4 (1 - \sin \varphi)^2 \left( C_{\theta_-} (\alpha^2 - \beta^2) + \alpha^2 \left( 13 (\Lambda_{1\phi}^1)^2 - 14 (\Lambda_{1\phi}^2)^2 \right) \right. \\
-2 \alpha^2 \left( 37 (\Lambda_{1\phi}^1)^2 - 38 (\Lambda_{1\phi}^2)^2 \right) \csc \varphi \\
+ \alpha^2 \left( 85 (\Lambda_{1\phi}^1)^2 - 86 (\Lambda_{1\phi}^2)^2 \right) \csc^2 \varphi \\
\left. + \frac{1}{2} C_{\theta_-} (\alpha^2 - \beta^2) (1 - \cos 2\varphi - 4 \sin \varphi) \right),
\]

\[
a_9 = 16 \alpha^2 \left( (\Lambda_{1\phi}^1)^2 - (\Lambda_{1\phi}^2)^2 \right) \csc^2 \varphi (1 - \sin \varphi)^3 (7 - 3 \sin \varphi),
\]

\[
a_{10} = -16 \alpha^2 \left( (\Lambda_{1\phi}^1)^2 - (\Lambda_{1\phi}^2)^2 \right) \csc^2 \varphi (1 - \sin \varphi)^4.
\]

Since we want the variable \( \gamma = \cos^2 \frac{\theta}{2} \) to have maximum, the coefficient \( a_{10} \) must be negative, i.e.
\[
(\Lambda_{1\phi}^1)^2 > (\Lambda_{1\phi}^2)^2.
\]

Taking this into account, we rewrite \( \sum_{i=0}^{10} a_i \gamma^i \) as
\[
\sum_{i=0}^{10} a_i \gamma^i = -a_{10} \left( -\gamma^{10} - \frac{1}{a_{10}} \sum_{j=0}^{9} a_j \gamma^j \right) = -a_{10} \left( -\gamma^{10} - \sum_{j=0}^{9} B_j \gamma^j \right)
\]

\[
= -a_{10} (\gamma_1 - \gamma) \prod_{k=2}^{10} (\gamma - \gamma_k),
\]

where \( \gamma_1 = \gamma_{\text{max}} \) and

\[
-B_0 = -\prod_{k=1}^{10} \gamma_k, \ldots, -B_9 = \sum_{k=1}^{10} \gamma_k.
\]

References

[1] J. M. Maldacena, “The large N limit of superconformal field theories and supergravity”, Adv. Theor. Math. Phys. 2, 231 (1998) [arXiv:hep-th/9711200];
S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “Gauge theory correlators from non-critical string theory”, Phys. Lett. B428, 105 (1998) [arXiv:hep-th/9802109];
E. Witten, “Anti-de Sitter space and holography”, Adv. Theor. Math. Phys. 2, 253 (1998) [arXiv:hep-th/9802150].

20
[2] N. Beisert et al., “Review of AdS/CFT Integrability: An Overview”, Lett. Math. Phys. 99 3 (2012), arXiv:1012.3982v5[hep-th].

[3] A. Babichenko, B. Stefanski, K. Zarembo, “Integrability and the AdS(3)/CFT(2) correspondence”, JHEP 1003 (2010) 058, [arXiv:hep-th/0912.1723v3].

[4] O. Ohlsson Sax, B. Stefanski Jr, “Integrability, spin-chains and the AdS3/CFT2 correspondence”, JHEP 1108 (2011) 029 [arXiv:hep-th/1106.2558v2].

[5] Ingo Kirsch, Tim Wirtz, “Worldsheet operator product expansions and p-point functions in AdS3/CFT2”, JHEP 1110 (2011) 049 [arXiv:hep-th/arXiv:1106.5876v2].

[6] Nitin Rughoonauth, Per Sundin, Linus Wulff, “Near BMN dynamics of the AdS(3) x S(3) x S(3) x S(1) superstring”, JHEP 1207 (2012) 159 [arXiv:hep-th/1204.4742v3].

[7] Per Sundin, Linus Wulff, “Classical integrability and quantum aspects of the AdS(3) x S(3) x S(3) x S(1) superstring”, JHEP 10 (2012) 109, arXiv:1207.5531 [hep-th]

[8] Alessandra Cagnazzo, Konstantin Zarembo, “B-field in AdS(3)/CFT(2) Correspondence and Integrability”, JHEP 1211 (2012) 133, [arXiv:hep-th/1209.4049v2].

[9] Olof Ohlsson Sax, Bogdan Stefanski jr, Alessandro Torrielli, “On the massless modes of the AdS3/CFT2 integrable systems”, JHEP 1303 (2013) 109, [arXiv:hep-th/1211.1952v2].

[10] Changrim Ahn, Diego Bombardelli, “Exact S-matrices for AdS3/CFT2”, [arXiv:hep-th/1211.4512].

[11] Riccardo Borsato, Olof Ohlsson Sax, Alessandro Sfondrini, “A dynamic su(1|1)^2 S-matrix for AdS3/CFT2”, JHEP 1304 (2013) 113 [arXiv:hep-th/1211.5119v3].

[12] M. Beccaria, F. Levkovich-Maslyuk, G. Macorini, A. A. Tseytlin, “Quantum corrections to spinning superstrings in AdS3 x S^3 x M^4: determining the dressing phase”, JHEP 1304 (2013) 006, [arXiv:hep-th/1211.6090v3].

[13] Riccardo Borsato, Olof Ohlsson Sax, Alessandro Sfondrini,“All-loop Bethe ansatz equations for AdS3/CFT2”, JHEP 1304 (2013) 116, [arXiv:hep-th/1212.0505v3].

[14] Matteo Beccaria, Guido Macorini, “Quantum corrections to short folded superstring in AdS3 x S^3 x M^4”, JHEP 1303 (2013) 040, [arXiv:hep-th/1212.5672v2].
Per Sundin, Linus Wulff, “Worldsheet scattering in AdS(3)/CFT(2)”, JHEP 1307 (2013) 007 [arXiv:hep-th/1302.5349v2].

B. Hoare, A. A. Tseytlin, “On string theory on $AdS_3 \times S^3 \times T^4$ with mixed 3-form flux: tree-level S-matrix”, Nucl.Phys. B873 (2013) 682-727, [arXiv:hep-th/1303.1037v4].

Riccardo Borsato, Olof Ohlsson Sax, Alessandro Sfondrini, Bogdan Stefanski, Alessandro Torrielli, “The all-loop integrable spin-chain for strings on $AdS_3 \times S^3 \times T^4$: the massive sector”, JHEP 1308 (2013) 043, [arXiv:hep-th/1303.5995v2].

B. Hoare, A. A. Tseytlin, “Massive S-matrix of $AdS_3 \times S^3 \times T^4$ superstring theory with mixed 3-form flux”, Nucl.Phys. B873 (2013) 395-418, [arXiv:hep-th/1304.4099v3].

Oluf Tang Engelund, Ryan W. McKeown, Radu Roiban, “Generalized unitarity and the worldsheet S matrix in $AdS_n \times S^n \times M(10 - 2n)$”, JHEP 1308 (2013) 023, [arXiv:hep-th/1304.4281v1].

Riccardo Borsato, Olof Ohlsson Sax, Alessandro Sfondrini, Bogdan Stefanski, Alessandro Torrielli, “Dressing phases of AdS3/CFT2”, Phys.Rev. D88 (2013) 066004, [arXiv:hep-th/1306.2512v2].

Michael C. Abbott, “The $AdS_3 \times S^3 \times S^3 \times S^1$ Hernandez-Lopez Phases: a Semiclassical Derivation”, J. Phys. A46 (2013) 445401 [arXiv:hep-th/1306.5106v2].

Per Sundin, Linus Wulff, “The low energy limit of the $AdS(3) \times S(3) \times M(4)$ spinning string”, JHEP 1310 (2013) 111 [arXiv:hep-th/1306.6918v1].

B. Hoare, A. Stepanchuk, A.A. Tseytlin, “Giant magnon solution and dispersion relation in string theory in $AdS_3 \times S^3 \times T^4$ with mixed flux”, [arXiv:hep-th/1311.1794].

Thomas Lloyd, Bogdan Stefaski jr, “AdS 3 /CFT 2, finite-gap equations and massless modes”, [arXiv:1312.3268] [hep-th]

Changrim Ahn, Plamen Bozhilov, “String solutions in $AdS_3 \times S^3 \times T^4$ with NS-NS B-field”, Phys.Rev. D90 (2014) 6, 066010, [arXiv:1404.7644] [hep-th]

A. Sfondrini, “Towards integrability for AdS3/CFT2”, [arXiv:hep-th/1406.2971].

Riccardo Borsato, Olof Ohlsson Sax, Alessandro Sfondrini, Bogdan Stefanski Jr, “The $AdS_3 \times S^3 \times S^3 \times S^1$ worldsheet S matrix”, J.Phys. A48 (2015) 41, 415401, [arXiv:1506.00218] [hep-th]
[28] Aritra Banerjee, Kamal L. Panigrahi, Manoranjan Samal, “A note on oscillating strings in $AdS_3 \times S^3$ with mixed three-form fluxes”, JHEP 11(2015)133, arXiv:1508.03430 [hep-th]

[29] S. S. Gubser, I. R. Klebanov, A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence”, Nucl. Phys. B636 (2002) 99-114, arXiv:hep-th/0204051

[30] Jerome P. Gauntlett, Robert C. Myers, Paul K. Townsend, “Supersymmetry of Rotating Branes”, Phys.Rev. D59 (1999) 025001, arXiv:hep-th/9809065

[31] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, “Integrals and series, vol.3: More special functions”, NY, Gordon and Breach (1990).