Fluctuations of the order parameter of a mesoscopic Floquet condensate

Bettina Gertjerenken and Martin Holthaus
Institut für Physik, Carl von Ossietzky Universität, D-26111 Oldenburg, Germany
(Dated: October 23, 2014)

We suggest that nonequilibrium Bose-Einstein condensates may occur in time-periodically driven interacting Bose gases. Employing the model of a periodically forced bosonic Josephson junction, we demonstrate that resonance-induced ground state-like many-particle Floquet states possess an almost perfect degree of coherence, as corresponding to a mesoscopically occupied, explicitly time-dependent single-particle orbital. In marked contrast to the customary time-independent Bose-Einstein condensates, the order parameter of such systems is destroyed by violent fluctuations when the particle number becomes too large, signaling the non-existence of a proper mean field limit.

PACS numbers: 03.75.Kk, 03.75.Lm, 03.65.Sq, 05.45.Mt

I. NONEQUILIBRIUM CONDENSATES

In the wake of traditional textbook teaching, Bose-Einstein condensation usually is associated with thermal equilibrium: At sufficiently low temperatures a Bose gas "condenses" into the lowest single-particle state \( |\psi_{\text{min}}(t)\rangle \). In the present paper we take a theoretical step towards the exploration of nonequilibrium condensates \[ \text{(1)} \]

The possible existence of such nonequilibrium condensates is reflected in the fundamental Penrose-Onsager criterion \[ \text{(2)} \] for Bose-Einstein condensation in a system of \( N \) repulsively interacting Bose particles, where \( N \) is large: This criterion does neither require thermal equilibrium nor even steady states \[ \text{(3)} \]. Instead, it takes recourse to the one-particle reduced density matrix

\[
\varrho(\mathbf{r}, \mathbf{r}'; t) = \langle \Psi_N(t) | \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}') | \Psi_N(t) \rangle ,
\]

where \( |\Psi_N(t)\rangle \) denotes the state of the \( N \)-Boson system at time \( t \), and \( \hat{\psi}^\dagger(\mathbf{r}) \) and \( \hat{\psi}(\mathbf{r}) \) are the usual creation and annihilation operators, obeing the Bose commutation relation \( \left[ \hat{\psi}(\mathbf{r}), \hat{\psi}^\dagger(\mathbf{r}') \right] = \delta(\mathbf{r} - \mathbf{r}') \). Considered as a matrix with indices \( \mathbf{r} \) and \( \mathbf{r}' \), its diagonal elements \( \varrho(\mathbf{r}, \mathbf{r}; t) \) provide the particle density of the system at the position \( \mathbf{r} \). Because at each moment this matrix is Hermitian, it can be decomposed in terms of a complete set of orthonormal single-particle functions \( \chi_j(\mathbf{r}; t) \) with eigenvalues \( n_j(t) \), such that

\[
\varrho(\mathbf{r}, \mathbf{r}'; t) = \sum_j n_j(t) \chi_j(\mathbf{r}, t) \chi_j^\dagger(\mathbf{r}', t) .
\]

According to Penrose and Onsager one has a simple Bose-Einstein condensate when the largest eigenvalue \( n_{\text{max}}(t) \) is on the order of \( N \), all others being of order 1; the corresponding eigenfunction \( \chi_{\text{max}}(\mathbf{r}; t) \) then is the condensate wave function \[ \text{(4)} \]. In the most favorable case where \( n_{\text{max}}(t) = N \), the density matrix \[ \text{(5)} \] reduces to a projector, times \( N \), onto the \( N \)-fold occupied single-particle orbital \( \chi_{\text{max}}(\mathbf{r}; t) \). As a matter of principle, this orbital can have an arbitrarily strong time-dependence.

Here we suggest that a particular type of nonequilibrium condensate may become experimentally accessible when an interacting Bose gas is subjected to a resonant time-periodic force. In general, when a quantum system evolves according to a Hamiltonian \( H(t) = H(t + T) \) which depends periodically on time with period \( T \), and remains bounded, the Floquet theorem asserts that there exists a complete set of solutions to the time-dependent Schrödinger equation which possess the particular form

\[
|\psi_j(t)\rangle = |u_j(t)\rangle \exp(-i\varepsilon_j t/\hbar),
\]

where the Floquet functions \( |u_j(t)\rangle = |u_j(t + T)\rangle \) inherit the imposed periodicity in time, and the quantities \( \varepsilon_j \) which determine the growth rates of the accompanying phases are known as quasienergies \[ \text{(6)} \]. Each solution to the time-dependent Schrödinger equation can be expanded in this Floquet-state basis with constant coefficients, implying that one can describe, e.g., a time-periodically driven ideal Bose gas by means of single-particle Floquet orbitals which carry constant occupation numbers \[ \text{(7)} \]. In particular, it makes sense to introduce the notion of a macroscopically occupied Floquet state.

Recent experiments with Bose-Einstein condensates in optical lattices subjected to strong time-periodic forcing already have demonstrated dynamic localization \[ \text{(8)} \], coherent control of the superfluid-to-Mott insulator transition \[ \text{(9)} \], giant Bloch oscillations \[ \text{(10)} \], frustrated classical magnetism \[ \text{(11)} \], controlled correlated tunneling \[ \text{(12)} \], artificial tunable gauge fields \[ \text{(13)} \], and effective ferromagnetic domains \[ \text{(14)} \]. Without claiming completeness of this list, these experiments testify that a macroscopic matter wave persists in the presence of strong time-periodic forcing.

II. APPEARANCE OF NEW GROUND STATE

For our theoretical considerations we employ the model of a periodically driven bosonic Josephson junction, which can be realized, for instance, with Bose-Einstein condensates in optical double-well potentials \[ \text{(15)} \]. The
juncture itself is described by the Lipkin-Meshkov-Glick Hamiltonian \[ H_0 = -\frac{\hbar \Omega}{2} \left( a_1 a_2^\dagger + a_2 a_1^\dagger \right) + \hbar \kappa \left( a_1^\dagger a_1 a_1 + a_2^\dagger a_2 a_2 \right) \],

where the operators \( a_j^\dagger \) and \( a_j \) create and annihilate, respectively, a Bose particle in the \( j \)th well \((j = 1, 2)\), obeying the commutation relation \( [a_j, a_k^\dagger] = \delta_{jk} \). Moreover, \( \hbar \Omega \) is the single-particle tunneling splitting, and \( 2\hbar \kappa \) quantifies the repulsion energy of each pair of bosons occupying the same well. This Hamiltonian \[ \text{23} \] had originally been devised for testing many-body approximation schemes \[ \text{23} \]; its paradigmatic importance as a nontrivial, but well tractable model for interacting Bose gases is almost exactly on resonance. Note that a bona fide order parameter, namely, of a single-particle orbital which is occupied by almost all of the particles when the system \[ \text{3} \] is in its ground state, and which thus constitutes the macroscopic wave function. It is well known that the exact ground state of the Hamiltonian \[ \text{4} \] coincides with an exact coherent state only when \( \hbar \kappa = 0 \). However, the difference between the exact ground state \( |0\rangle \) and an exactly coherent state here becomes insignificant when approaching the mean field limit, when \( \hbar \kappa \) vanishes proportionally to \( 1/N \).

We now extend this analysis to the driven junction \[ \text{4} \]. Here we focus on resonant driving, \( \text{i.e.} \), we choose the frequency \( \omega \) such that \( \hbar \omega \) equals the spacing \( E_r = E_r+1 \) of the unperturbed energy eigenvalues \( E_j \) of the junction \[ \text{3} \] at a particular state label \( j = r \). Figure 2 (a) shows the exact quasienergies of the system for \( N = 100 \) particles, scaled interaction strength \( N \kappa/\Omega = 0.95 \), and scaled driving frequency \( \omega/\Omega = 1.62 \). This implies \( r = 8 \), so that the unperturbed \( N \)-particle energy eigenstates \[ \text{8} \] and \[ \text{9} \] are almost exactly on resonance. Note that a Floquet state can be factorized according to

\[
\langle u_j(t) \rangle \exp(-i\varepsilon_j t/\hbar) = \langle u_j(t) e^{im\omega t} \rangle \exp(-i[\varepsilon_j + m\hbar \omega] t/\hbar)
\]

with an arbitrary positive or negative integer \( m \), so that the Floquet function \( |u_j(t) e^{im\omega t}| \) remains \( T \)-periodic, with \( T = 2\pi/\omega \). This means, loosely speaking, that “the quasienergies are defined only up to an integer multiple of \( \hbar \omega \).” More precisely, the quasienergy of a Floquet state labeled by \( j \) has to be regarded as an infinite set of representatives \( \varepsilon_j + m\hbar \omega \) spaced by \( \hbar \omega \), implying that each Brillouin zone of the quasienergy spectrum of width \( \hbar \omega \) contains precisely one representative of each state.

The Brillouin zone of quasienergies displayed in Fig. 2 (a) features a regular fan of almost equidistant
The vicinity of the state \( |r\rangle \) singled out by the condition \( \hbar \omega = E_{r+1} - E_r \), the dynamics of the driven \( N \)-particle system can be mapped to that of an effective quasiparticle, named “floton”, which moves in a cosine potential well without external driving, such that the energies of this quasiparticle yield the quasienergies of the near-resonant Floquet states \[^{33, 34}\].

\[
\varepsilon_k = E_r + \frac{1}{8} \frac{E''_r}{\hbar \omega} \alpha_k(q) \mod \hbar \omega ,
\]

where \( E''_r \) denotes the formal (discrete) second derivative of the unperturbed eigenvalues \( E_j \) with respect to the state label \( j \), evaluated at the resonant state \( j = r \), and \( \alpha_k(q) \) is a characteristic value of the Mathieu equation. Using the notation of Ref. \[^{35}\], one has \( \alpha_k(q) = \alpha_k(q) \) for quantum numbers \( k = 0, 2, 4 \ldots \) labeling the even eigenstates of the floton quasiparticle, while \( \alpha_k(q) = b_k(q) \) for \( k = 1, 3, 5, \ldots \). The Mathieu parameter \( q \) is proportional to the driving amplitude,

\[
q = \frac{2}{E''_r/(\hbar \omega)} \frac{2\mu_1}{\omega} \varepsilon(\langle r|a_1^\dagger a_1 - a_2^\dagger a_2|r \rangle - 1) .
\]

The important feature here is the appearance of a new quantum number \( k \): The resonant state \( |r\rangle \) turns into the floton ground state \( k = 0 \); the neighboring states of the unperturbed junction \[^{36}\] are transformed into its excitations \( k > 0 \). In Fig. 4 we depict the degree of coherence \( \eta \) for the exact near-resonant Floquet states, computed numerically, with floton quantum numbers \( k = 0, \ldots , 4 \). The similarity to the previous Fig. 1 is striking: Indeed the “resonant ground state” \( k = 0 \) is an almost coherent state, in the sense that it corresponds to an \( N \)-fold occupied, periodically time-dependent single-particle orbital. Thus, here we encounter an example of Floquet engineering: The driving is not employed primarily to excite the system, but rather to create a new effective Hamiltonian \[^{36}\], describing the floton quasiparticle, and providing a new ground state into which the actual particles can condense. This Floquet condensate constitutes a collective mode of response to the drive which remains perfectly coherent in the course of time.

**III. ORDER PARAMETER FLUCTUATIONS**

However, there is a fundamental difference between such Floquet condensates and the customary, time-independent Bose-Einstein condensates which shows up if one tries to recover the mean field regime: In Fig. 4 we show the maximum degree of coherence \( \eta_{\text{max}} \), taken over all Floquet states of the driven Josephson junction \[^{36}\] with \( \omega/\Omega = 1.62 \), vs. the scaled driving strength; again the interaction strength is adjusted such that \( N\kappa/\Omega = 0.95 \). In panel (a) we take \( N = 100 \): Here we observe extended intervals where \( \eta_{\text{max}} = 1 \) with high accuracy, caused by the floton state \( k = 0 \), and large fluctuations occurring when \( 2\mu_1/\omega \approx 0.9 \). The interval magnified in the inset is scanned again in panel (b), but now with \( N = 500 \); here additional small fluctuations appear. Iterating this procedure, the interval framed in the inset of panel (b) is evaluated in panel (c) with \( N = 1000 \); here

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**FIG. 2:** (a) One Brillouin zone of exact quasienergies for the driven bosonic Josephson junction \[^{36}\] with \( N = 100 \) particles, scaled interaction strength \( N\kappa/\Omega = 0.95 \), and scaled driving frequency \( \omega/\Omega = 1.62 \), for low scaled driving amplitudes \( 2\mu_1/\omega \). The fan of almost equidistant lines is well described by the Mathieu approximation \[^{36}\]. (b) Part of the quasienergy spectrum for \( N = 500 \), and higher driving amplitudes. Observe the scales!

**FIG. 3:** (Color online) Degree of coherence \( \eta \) for the near-resonant Floquet states with Mathieu quantum numbers \( k = 0, \ldots , 4 \) (top to bottom) of the driven bosonic Josephson junction \[^{36}\] with \( N\kappa/\Omega = 0.95 \) kept fixed, \( \omega/\Omega = 1.62 \), and \( 2\mu_1/\omega = 0.3 \), vs. particle number \( N \).
FIG. 4: Maximum degree of coherence $\eta_{\text{max}}$ of all $N$-particle Floquet states for $N\kappa/\Omega = 0.95$ and $\omega/\Omega = 1.62$. (a) $N = 100$; the inset delimits the interval of driving strengths inspected in the following panel. (b) $N = 500$; again the inset marks the interval investigated in the following panel. (c) $N = 1000$. (d) $N = 2000$. Observe the change of scale in comparison to (c), and the shift of the baseline.

The fluctuations become more violent. In panel (d), where $N = 2000$, even the baseline of the fluctuations is shifted downward. These results indicate that the size of a resonant Floquet condensate remains restricted to mesoscopically large particle numbers, while its order parameter would be destroyed for high $N$ by large fluctuations.

IV. CONCLUSIONS

Since the appearance of resonances is a generic feature of driven nonlinear quantum systems, we anticipate that the findings reported in this work are not restricted to our particular model (4). Thus, we may summarize our main results as follows: (i) Resonantly driven Bose gases allow the formation of nonequilibrium Bose-Einstein condensates, with the resonance-induced effective ground state corresponding to a mesoscopically occupied, periodically time-dependent single-particle orbital; (ii) the coherence of such condensates is destroyed when the particle number becomes large, a mean field limit cannot be reached. This non-existence of a proper mean field limit is closely related to the absence of an adiabatic limit in periodically driven quantum systems [37].
tectable through large fluctuations of the system’s coherence in a series of measurements in which the particle number varies slightly from shot to shot.

Acknowledgments

We acknowledge support from the Deutsche Forschungsgemeinschaft (DFG) through grant No. HO 1771/6-2. The computations were performed on the HPC cluster HERO, located at the University of Oldenburg and funded by the DFG through its Major Research Instrumentation Programme (INST 184/108-1 FUGG), and by the Ministry of Science and Culture (MWK) of the Lower Saxony State.

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