LINEAR STABILITY OF EXACT SOLUTIONS FOR THE GENERALIZED KAUP-BOUSSINESQ EQUATION AND THEIR DYNAMICAL EVOLUTIONS

RUIZHI GONG
Laboratory of Mathematics and Complex Systems (Ministry of Education)
School of Mathematical Sciences, Beijing Normal University
Beijing 100875, China
School of Applied Science
Beijing Information Science and Technology University
Beijing 100192, China

YUREN SHI
College of Physics and Electronic Engineering
Northwest Normal University
Lanzhou 730070, China

DENG-SHAN WANG*
Laboratory of Mathematics and Complex Systems (Ministry of Education)
School of Mathematical Sciences, Beijing Normal University
Beijing 100875, China

(Communicated by Adrian Constantin)

Abstract. The integrability, classification of traveling wave solutions and stability of exact solutions for the generalized Kaup-Boussinesq equation are studied by prolongation structure technique and linear stability analysis. Firstly, it is proved that the generalized Kaup-Boussinesq equation is completely integrable in sense of having Lax pair. Secondly, the complete classification of exact traveling wave solutions of the generalized Kaup-Boussinesq equation are given and a family of exact solutions are proposed. Finally, the stability of these exact solutions are investigated by linear stability analysis and dynamical evolutions, and some stable traveling wave solutions are found. It is shown that the results of linear stability analysis are in excellent agreement with the results from dynamical evolutions.

1. Introduction. The rapid development of nonlinear science benefits from the significant role played by nonlinear phenomena in many research fields, which has aroused lots of attention. It is necessary to develop proper theory and method to find the exact solution and stability of nonlinear systems to reveal the exotic physical phenomena. In this sense, the soliton theory [13] provides efficient tools

2020 Mathematics Subject Classification. Primary: 35C07, 35Q51, 37C75; Secondary: 37K10.
Key words and phrases. Linear stability, generalized Kaup-Boussinesq equation, traveling wave solution, dynamical evolution.

This work was supported by the National Natural Science Foundation of China through grant Nos. 11971067 and 12065022, and the Fundamental Research Funds for the Central Universities under Grant No. 2020NTST22.

*Corresponding author: Deng-Shan Wang.
for studying integrable [1] and nonintegrable systems [32]. In particular, as an important branch of nonlinear science, soliton theory has important applications in nonlinear optics [14], fluid mechanics [16, 7, 8, 6], Bose-Einstein condensates [2] and plasma physics [24, 33, 31].

The Kaup-Boussinesq (KB) equation [22]

\[
\begin{align*}
    u_t + v_x + uu_x &= 0, \\
    v_t + (uv)_x - \frac{1}{4}u_{xxx} &= 0,
\end{align*}
\]

is an important integrable nonlinear evolution equation arising from the shallow water waves, which was first introduced by Kaup [22] in 1975 and then was re-derived by Ivanov through asymptotic expansion [17]. In terms of physical meaning, \( v \) is the local height of the water layer and \( u \) is a local mean flow velocity. Because of its excellent integrability, previous researchers have carried out a lot of outstanding work on the KB equation (1). For example, the Hamiltonian structure of the KB equation was given by Pavlov [25] in 2001. Kamchatnov et al. studied the asymptotic soliton train of KB equation by quasi-classical quantization method [21]. The spectral theory and the inverse scattering for the integrable Kaup-Boussinesq equation has been also developed [15, 26, 19, 23]. El, Grimshaw, Kamchatnov et al. have made many achievements in Whitham modulation theory on the KB equation [12, 11, 5]. Zhou et al. studied the traveling wave solution of the dual equation of KB system by using the bifurcation theory of planar dynamical system [35]. Dutykh and Ionescu-Kruse [10] found analytical multi-pulsed traveling wave solutions of the KB equation by means of the simple and geometrically intuitive phase space analysis. Recently, the exact solutions and conservation laws of multi KB system with fractional order are also considered [27].

In fact, there is a more general form of the KB equation (1), i.e., the generalized KB equation introduced by Ivanov and Lyons [18]

\[
\begin{align*}
    u_t + v_x + \left( \frac{3}{2} + \kappa \right)uu_x &= 0, \\
    v_t + (uv)_x + \frac{1}{4}\sigma u_{xxx} - \left( \frac{1}{2} + \kappa \right)uv_x - \kappa \left( \frac{1}{2} + \kappa \right)u^2u_x &= 0,
\end{align*}
\]

which can be reduced to the KB equation (1) for \( \sigma = -1 \) and \( \kappa = -\frac{1}{2} \). The generalized KB equation includes many special and interesting KB-type equations by choosing the real parameters \( \kappa \) and \( \sigma \). In particular, the positive and negative \( \sigma \) correspond to different kinds of KB-type equations.

The purpose of this paper is to investigate the integrability, classification of traveling wave solutions and stability of exact solutions for the generalized KB equation (2). Specifically, the prolongation structure technique is applied to explore the Lax integrability of the generalized KB equation (2). Taking the traveling wave transformation, the generalized KB equation (2) is reduced to a set of ordinary differential equation, then the exact traveling wave solutions are classified and formulated. The linear stability analysis of these exact solution is carried out by means of the perturbation expansion method. Furthermore, the results of linear stability analysis are compared with the results from dynamic evolutions to ensure the correctness of the theoretical analysis.

This paper is organized as follows: in Section 2, the generalized KB equation (2) is proved to be Lax integrable by prolongation structure technique. The complete classification of traveling wave solutions of the generalized KB equation are given
and many types of exact solutions are found explicitly in Section 3. In Section 4, the perturbation expansion of small parameters is used to analyze the stability of the exact solution of the generalized KB equation: firstly, the dispersion relation of the constant solution is formulated and is compared with the results from linear stability analysis, which shows excellent agreement with each other. This indicates that the results from linear stability analysis are reliable enough. Later, two exact solutions of the generalized KB equation (2) are selected to examine the linear stability, whose results were consistent with the results from dynamic evolutions. The paper is summarized in the Section 5.

2. Prolongation structure and integrability of the generalized KB equation. Integrability plays a key role in soliton theory and nonlinear evolution equations. The integrability of nonlinear equations not only reflects their abundant mathematical structures, but also closely relates to the existence of soliton solutions. However, there is no definitive definition of the integrability, and usually integration of differential equations refers to the integrability in a certain sense. The integrability discussed in this paper is Lax integrability, which can be checked by the theory of prolongation structure [28, 29, 30], one of the important applications of exterior differential form in the study of nonlinear wave equations. In what follows, the prolongation structure of the generalized KB equation (2) are investigated by using the prolongation technique, and the Lax pair of this equation is derived immediately.

Introduce the transformation [29, 30]

\[ u_x = p, \quad p_x = q, \]

which reduces the generalized KB equation (2) to a set of first-order partial differential equations

\[ \begin{align*}
    u_t + v_x + \left( \frac{3}{2} + \kappa \right) up &= 0, \\
    v_t + pv + uv_x + \frac{\sigma}{4} q_x - \left( \frac{1}{2} + \kappa \right) uv_x - \kappa \left( \frac{1}{2} + \kappa \right) u^2 p &= 0.
\end{align*} \]

Define a set of differential 2-form \( I = \{ \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) on the manifold \( M \) with local coordinate \( \{ x, t, u, v, p, q \} \) as

\[ \begin{align*}
    \alpha_1 &= du \wedge dt - p dx \wedge dt, \\
    \alpha_2 &= q dx \wedge dt - dp \wedge dt, \\
    \alpha_3 &= dx \wedge du + dv \wedge dt + \left( \frac{3}{2} + \kappa \right) up dx \wedge dt, \\
    \alpha_4 &= dx \wedge dv + pv dx \wedge dt + udv \wedge dt + \frac{1}{4} \sigma dq \wedge dt - \left( \frac{1}{2} + \kappa \right) udv \wedge dt - \kappa \left( \frac{1}{2} + \kappa \right) u^2 p dx \wedge dt,
\end{align*} \]

then the first-order partial differential equations (3) are equivalent to the null values of the set of differential 2-form \( I \). Furthermore, introduce the differential 1-form system

\[ w^i = d\Phi^i - F^i(u, v, p, q, \Phi^i) dx - G^i(u, v, p, q, \Phi^i) dt, \quad i = 1, 2, \ldots, n, \]
where $F^i$ and $G^i$ are dependent on $\Phi^i$ linearly, i.e., $F^i = F_j^i \Phi^j, G^i = G_j^i \Phi^j$, and $n$ is a number to be determined. For simplicity, take $F_j^i = F$ and $G_j^i = G$.

Following the procedure of prolongation structure technique \cite{29, 30}, letting $\bar{\mathcal{I}} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4; w^i\}$ be closed ideal, yields

$$d w^i = \xi^i \wedge w^i + \sum_{j=1}^{4} f_j^i \alpha_j$$ \hspace{1cm} (6)

where $f_j^i$ are function of $(x, t)$ to be determined and $\xi^i$ are differential 1-form, which can be written as $\xi^i = a^i dx + b^i dt$, where $a^i$ and $b^i$ are function of $(x, t)$.

Inserting (4)-(5) into (6), collecting the coefficients of the basis of the differential 2-form on the manifold $M$ and removing the indexes on functions $F^i$ and $G^i$, a set of nonlinear partial differential equations about function $F$ and $G$ is obtained as follows

$$F_p = F_q = 0$$

$$G_q + \frac{1}{4} \sigma F_v = 0,$$

$$F_v(\frac{1}{2} - \kappa)u + F_u + G_v = 0,$$

$$pG_u + qG_p + \left( \frac{3}{2} + \kappa \right) u p F_u + (v - \kappa(\frac{1}{2} + \kappa)u^2) pF_v - [F, G] = 0,$$

where $[F, G] = FG - GF$.

The solution of the nonlinear partial differential equations in (7) can be given

$$F = X_1 + X_2 v + X_3 u + X_4 u^2,$$

$$G = -\frac{1}{4} \sigma q X_2 - X_3 v + (-2X_4 + \kappa X_2 - \frac{1}{2} X_4) v u + X_5 u^3 + X_6 u^2 + X_7 u + X_8 p + X_9,$$

where $X_i$ ($i = 1, 2, \ldots, 9$) are $n \times n$ matrices and satisfy the following commutation relations

$$[X_1, X_6] + [X_3, X_7] + [X_4, X_9] = 0, \left( \frac{1}{2} - \kappa \right) [X_1, X_2] + 2[X_1, X_4] - [X_2, X_4] = 0,$$

$$[X_4, X_8] + \kappa[X_2, X_8] = 3X_3 + 3X_4, [X_1, X_3] + [X_3, X_6] + [X_4, X_7] = 0, [X_4, X_5] = 0,$$

$$[X_2, X_8] = (\kappa + \frac{1}{2}) X_2 - 2X_4, [X_3, X_8] = (\kappa + \frac{3}{2}) X_3 + 2X_6, [X_2, X_5] = [X_2, X_4] = 0,$$

$$[X_2, X_9] = [X_1, X_4], [X_3, X_5] + [X_4, X_6] = 0, (\kappa - \frac{1}{2}) [X_2, X_3] = [X_2, X_6] - [X_3, X_4],$$

$$[X_3, X_9] + [X_1, X_7] = 0, X_8 + \frac{1}{4} \sigma [X_1, X_2] = 0, X_7 = [X_1, X_8], [X_1, X_9] = [X_2, X_3] = 0.$$

The set $L = \{X_i : i = 1, 2, \ldots, 9\}$ consists of the prolongation algebra \cite{29, 30}, which can be represented by embedding $L$ into the semisimple Lie algebra $sl(n, C)$. After some calculations, it is found that the case $n = 2$ is enough to represent the commutation relations and the $2 \times 2$ matrix representations of the prolongation algebra $L$ are

$$X_1 = \begin{pmatrix} 0 & 1 \\ \sigma \lambda^2 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 \\ -\sigma & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 0 \\ -\sigma \lambda & 0 \end{pmatrix},$$

$$X_4 = \begin{pmatrix} 0 & \sigma \kappa \\ -\frac{\sigma}{2} & 0 \end{pmatrix}, X_5 = \begin{pmatrix} 0 & 0 \\ \sigma \kappa & 0 \end{pmatrix}, X_6 = \begin{pmatrix} 0 & 0 \\ \sigma \lambda (\frac{\kappa + 1}{2}) & 0 \end{pmatrix},$$

$$X_7 = \begin{pmatrix} 0 & 0 \\ \sigma \lambda & 0 \end{pmatrix}, X_8 = \begin{pmatrix} 0 & 0 \\ \sigma \lambda & 0 \end{pmatrix}, X_9 = \begin{pmatrix} 0 & 0 \\ \sigma \lambda (\frac{\kappa + 1}{2}) & 0 \end{pmatrix}.$$
Thus we conclude that the generalized KB equation (2) is Lax integrable and its Lax pair is derived by vanishing the differential 1-form (5)
\[
\begin{align*}
\Phi_x &= F \Phi, \\
\Phi_t &= G \Phi.
\end{align*}
\] (9)
where \( \Phi = (\Psi, \Psi_x)^T \), and matrix-valued functions \( F \) and \( G \) are are
\[
F = \begin{pmatrix}
\sigma(\lambda^2 - \lambda u - \frac{\kappa}{2} u^2 - v) & 1 \\
\frac{1}{2} u_x & 0
\end{pmatrix},
\] (10)
\[
G = \begin{pmatrix}
\sigma(-\lambda - \frac{1}{2} u)(\lambda^2 - \lambda u - \frac{\kappa}{2} u^2 - v) + \frac{1}{4} u_{xx} & -\lambda - \frac{1}{2} u \\
\frac{1}{4} u_x & -\frac{1}{4} u_x
\end{pmatrix}.
\] (11)
Equations in (9) are the Lax pair in matrix form. In fact, one can formulate the operator form of the Lax pair. The parameter \( \sigma \) is a nontrivial index in the generalized KB equation (2), which determines the stability of this equation. When \( \sigma = 1 \), the generalized KB equation (2) is completely integrable and has Lax pair in operator form
\[
\begin{align*}
\Psi_{xx} &= (\lambda^2 - \lambda u - \frac{\kappa}{2} u^2 - v) \Psi, \\
\Psi_t &= (-\lambda - \frac{1}{2} u) \Psi_x + \frac{1}{4} u_x \Psi.
\end{align*}
\] (12)
Similarly, when \( \sigma = -1 \), it is also integrable and has Lax pair in operator form
\[
\begin{align*}
\Psi_{xx} &= (-\lambda^2 + \lambda u + \frac{\kappa}{2} u^2 + v) \Psi, \\
\Psi_t &= (-\lambda - \frac{1}{2} u) \Psi_x + \frac{1}{4} u_x \Psi.
\end{align*}
\] (13)
The Lax integrability of the generalized KB equation (2) indicates that it may have exact soliton solutions and other mathematical structures, which inspires us to explore the exotic nonlinear structures in this equation. One of the simplest exact solutions is the traveling wave solution, thus we will investigate the complete classification of traveling wave solutions for the generalized KB equation (2) firstly.

3. **Classification of the exact traveling wave solutions.** It is of great theoretical significance and has potential physical applications to explore more new types of exact solutions of nonlinear systems, especially in the form of solitary waves, which is helpful to the analysis and study of natural phenomena. There are several feasible ways to obtain the exact solution of the nonlinear partial differential equations. In this section, we focus on the traveling wave solutions of the generalized KB equation (2), which are explicit, exact solutions based on the traveling wave transformation. To be specific, the complete classification of the exact traveling wave solutions along with the corresponding traveling wave solutions in each class are given. Among these exact solutions, the soliton solutions and periodic wave solutions are physically interesting. In the analysis, we will get a first-order nonlinear ordinary differential equation \( \left( \frac{du}{d\xi} \right)^2 = Q(u) \), where \( Q(u) \) has four roots \( u_1, u_2, u_3, u_4 \). The classification mainly depend on the relationship among the four roots.
To derive the exact solution of generalized KB equation (2), take the traveling wave transformation of the form
\[ u(x, t) = u(\xi), \quad v(x, t) = v(\xi), \quad \xi = x - ct, \]
where \( c \) is the velocity of the traveling wave.

At this point, the generalized KB equation (2) is reduced to two nonlinear ordinary differential equations
\[
\begin{align*}
&cu' = v' + \left( \frac{3}{2} + \kappa \right) uu', \\
&cv' = u'v + v'u + \frac{1}{4} \sigma u''' - \left( \frac{1}{2} + \kappa \right) uv' - \kappa \left( \frac{1}{2} + \kappa \right) u^2 u',
\end{align*}
\]
where \( ' \) above means the derivative with respect to variable \( \xi \), i.e., \( u' = \frac{du}{d\xi} \) and \( u''' = \frac{d^3u}{d\xi^3} \). Integrating the equations in (15) once and taking the integration constant to be \( A \), yield
\[
\begin{align*}
v &= cu - \frac{1}{2} \left( \frac{3}{2} + \kappa \right) u^2 + A, \\
&cv = uv + \frac{1}{4} \sigma u'' - \int \left( \frac{1}{2} + \kappa \right) uv' d\xi - \frac{1}{3} \kappa \left( \frac{1}{2} + \kappa \right) u^3.
\end{align*}
\]
Substituting the first equation in (16) into the second equation, multiplying two sides of the obtained equation by the \( u' \) and integrating twice, we have
\[
\left( \frac{du}{d\xi} \right)^2 = \frac{1}{\sigma} \left[ u^4 - 4cu^3 + 4(c^2 - A)u^2 + 8(B + Ac)u + 8D \right],
\]
where \( B \) and \( D \) are integration constants. Set the right hand side of the equation (17) to be \( Q(u) \), which is called “Potential” function, i.e.,
\[
Q(u) = \frac{1}{\sigma} \left[ u^4 - 4cu^3 + 4(c^2 - A)u^2 + 8(B + Ac)u + 8D \right].
\]

For the convenience of the following discussion and without loss of generality, assume the function \( Q(u) \) in (18) has four roots \( u_1, u_2, u_3, u_4 \) satisfying \( u_4 \leq u_3 \leq u_2 \leq u_1 \). So rewrite the ordinary differential equation (17) as
\[
\left( \frac{du}{d\xi} \right)^2 = \frac{1}{\sigma}(u - u_1)(u - u_2)(u - u_3)(u - u_4).
\]

The sign of the parameter \( \sigma \) plays a key role in the ordinary differential equation (19). Figure 1 and Figure 2 show the “Potential” curve of function \( Q(u) \) for the four roots are \( u_4 < u_3 < u_2 < u_1 \) with \( \sigma > 0 \) and \( \sigma < 0 \), respectively. It is observed that the intervals \( (-\infty, u_4], [u_3, u_2], [u_1, +\infty) \) for variable \( u \) are available for parameter \( \sigma > 0 \), while only the intervals \( [u_4, u_3], [u_2, u_1] \) are available for parameter \( \sigma < 0 \) (see the gray regions in the figures). In other word, when the parameter \( \sigma > 0 \), the exact solution of ordinary differential equation (19) only exists on the intervals \( (-\infty, u_4], [u_3, u_2], [u_1, +\infty) \), however, the existence intervals of the solutions are \( [u_4, u_3] \) and \( [u_2, u_1] \) for the parameter \( \sigma < 0 \).

The function \( Q(u) \) is a polynomial function of degree four, so in view of the relationship between the four roots and the coefficients in equation (18), it is seen
that
\[
\begin{align*}
\quad & u_1 + u_2 + u_3 + u_4 = 4c, \\
\quad & u_1u_2 + u_1u_3 + u_1u_4 + u_2u_3 + u_3u_4 + u_2u_4 + u_3u_4 = 4(c^2 - A), \\
\quad & u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4 = -8(B + Ac), \\
\quad & u_1u_2u_3u_4 = 8D,
\end{align*}
\]
(20)
which establishes the relationship between the four roots \(u_1, u_2, u_3, u_4\) of function \(Q(u)\) and four constants \(A, B, c, D\).

In what follows, based on the existence intervals of the solutions, we consider the complete classification of the exact traveling wave solutions of the generalized KB equation (2) for parameter \(\sigma > 0\) and \(\sigma < 0\), respectively.

3.1. **Classification of the exact solutions for parameter \(\sigma > 0\).**

When the parameter \(\sigma > 0\), the solutions of the ordinary differential equation (19) may locate in the intervals \([u_3, u_2]\), \((−\infty, u_4]\) and \([u_1, +\infty)\), respectively. In what follows, without loss of generality we set the parameter \(\sigma = 1\).

Firstly, following the procedure of the Appendix A in Kamchatnov’s monograph [20] integrating the equation (19) on the interval \([u_3, u_2]\) yields the periodic wave solution expressed by the elliptic function as follows
\[
\begin{align*}
u & = u_3(u_2 - u_4) + u_4(u_3 - u_2)\text{sn}^2(W_1, m_1) \\
& = u_2 - u_4 + (u_3 - u_2)\text{sn}^2(W_1, m_1),
\end{align*}
\] (21)

**Figure 1.** The “Potential” curve of function \(Q(u)\) for the four roots are \(u_4 < u_3 < u_2 < u_1\) and \(\sigma > 0\).  

**Figure 2.** The “Potential” curve of function \(Q(u)\) for the four roots are \(u_4 < u_3 < u_2 < u_1\) and \(\sigma < 0\).
Figure 3. The Jacobi elliptic function periodic wave solution of the generalized KB equation (2) at $t = 0$ based on the traveling wave solution (21): (a) the $u$-component and (b) the $v$-component. Here the parameters are chosen to be $u_1 = 1, u_2 = 0, u_3 = -1, u_4 = -2, \kappa = -1, \xi_0 = 0, \sigma = 1$.

where $W_1 = (\xi - \xi_0)\sqrt{(u_4 - u_2)(u_3 - u_1)}/2$, and $m_1$ is the modulus of the Jacobian elliptic $sn$ function

$$m_1 = \frac{(u_3 - u_2)(u_4 - u_1)}{(u_4 - u_2)(u_3 - u_1)}, \quad 0 \leq m_1 \leq 1.$$  

Figure 3 shows the spatial distributions of the periodic wave solution to the generalized KB equation (2) in term of the traveling wave solution (21), where the parameters are $u_1 = 1, u_2 = 0, u_3 = -1, u_4 = -2, \kappa = -1, \xi_0 = 0$.

When the roots of the function $Q(u)$ in equation (18) satisfy $u_1 = u_2$, we have $m_1 = 1$, then the periodic wave solution (21) can degenerate into the solitary wave solution

$$u = \frac{u_3(u_2 - u_4) + u_4(u_4 - u_2)\tanh^2(W_{11})}{u_2 - u_4 + (u_3 - u_2)\tanh^2(W_{11})},$$

where $W_{11} = (\xi - \xi_0)\sqrt{(u_2 - u_3)(u_2 - u_4)}/2$. It is obvious that there is singular point in this exact solution, so one only calls it solitary wave solution but not soliton solution. Moreover, although it may have singularity, one can choose special parameters $u_2, u_3, u_4$ to avoid this situation because of the boundary of the function $\tanh^2(W_{11})$, i.e. $0 \leq \tanh^2(W_{11}) \leq 1$. Figure 4 displays the spatial distributions of the generalized KB equation (2) based on the solitary wave solution (22) under parameters $u_1 = u_2 = 2, u_3 = 0, u_4 = -2, \kappa = -1, \xi_0 = 0$. It is observed that the $u$-component demonstrates an anti-bell solitary wave and the $v$-component corresponds to a two-hump type solitary wave, which are found in the generalized KB equation (2) for the first time. In fact, solitary wave solution (22) may have singularities for some parameters, however, the parameters in Figure 4 are chosen to avoid the singularities.

Moreover, the periodic wave solution (21) degenerates into trivial constant solutions $u = u_3$ and $u = u_4$ for the roots of the function (18) satisfying $u_2 = u_3$ and $u_3 = u_4$, respectively. If the roots of the function (18) have one triple real root, there are two possible cases, one is $u_1 = u_2 = u_3 \neq u_4$ and the other is $u_1 \neq u_2 = u_3 = u_4$. On this case, the periodic wave solution (21) can also degenerate into constant solutions. What’s more, trivial constant solutions can also be obtained in the case the roots of the function (18) have two double real roots (for
Figure 4. The solitary wave solution of the generalized KB equation (2) at \( t = 0 \) based on the traveling wave solution (22): (a) the \( u \)-component of anti-bell type and (b) the \( v \)-component of two-hump type, where the parameters here are chosen to be \( u_1 = u_2 = 2, u_3 = 0, u_4 = -2, \kappa = -1, \xi_0 = 0 \).

Figure 5. The solitary wave solution of the generalized KB equation (2) at \( t = 0 \) based on the traveling wave solution (24): (a) the \( u \)-component of kink type and (b) the \( v \)-component of two-hump and anti-bell type. Here the parameters are chosen to be \( u_1 = u_2 = 8, u_3 = 0, u_4 = -8, \kappa = -1, \xi_0 = 0 \). (c) The “Potential” curve \( Q(u) \) in case of the roots \( u_1 = u_2 \).

Example, \( u_1 = u_2, u_3 = u_4 \) or only one quadruple real root, i.e., \( u_1 = u_2 = u_3 = u_4 \). These are trivial cases, so it makes sense to take the focus off the analysis.

Secondly, when the solution of the ordinary differential equation (19) is located on the interval \((-\infty, u_4]\), the other periodic wave solution in form of Jacobi elliptic
function are derived

\[ u = \frac{u_3(u_1 - u_4)\text{sn}^2(W_1, m_1) - u_4(u_1 - u_3)}{u_3 - u_1 + (u_1 - u_4)\text{sn}^2(W_1, m_1)}, \tag{23} \]

which can also be degenerated into the solitary wave form in case of \( u_1 = u_2 \), that is

\[ u = \frac{u_3(u_2 - u_4)\text{tanh}^2(W_{11}) - u_4(u_2 - u_3)}{u_3 - u_2 + (u_2 - u_4)\text{tanh}^2(W_{11})}, \tag{24} \]

where again we have \( W_{11} = (\xi - \xi_0)\sqrt{(u_2 - u_3)(u_2 - u_4)}/2 \). Figure 5 demonstrates the spatial distributions of the solitary wave solution of the equation (2) based on the solution (24) at time \( t = 0 \). It is seen that the \( u \)-component demonstrates a kink type solitary wave while the \( v \)-component corresponds to a two-hump and anti-bell type solitary wave. Figure 5(c) shows the shape of the “Potential” curve \( Q(u) \) in case of the roots \( u_1 = u_2 \).

Moreover, there are other degenerating cases for the periodic wave solution (23). It degenerates into the trivial constant solution \( u = u_4 \) for \( u_3 = u_4 \). For \( u_2 = u_3 \), it degenerates into the periodic wave solution in form of trigonometric function as

\[ u = \frac{u_3(u_1 - u_4)\sin^2(W_{12}) - u_4(u_1 - u_3)}{u_3 - u_1 + (u_1 - u_4)\sin^2(W_{12})}, \tag{25} \]

where \( W_{12} = (\xi - \xi_0)\sqrt{(u_1 - u_3)(u_3 - u_4)}/2 \).

Interestingly, an algebraic solitary wave solution is found for \( u \in (-\infty, u_4) \) when the roots of the “Potential” curve \( Q(u) \) satisfy \( u_1 = u_2 = u_3 \neq u_4 \), which can be expressed explicitly of the form

\[ u = \frac{u_3(u_3 - u_4)^2\xi^2 - 4u_4}{(u_3 - u_4)^2\xi^2 - 4}. \tag{26} \]

It is not difficult to see that this algebraic solitary wave solution has singularity at point \( \xi^2 = 4/(u_3 - u_4)^2 \), but one can choose special values of \( u_3 \) and \( u_4 \) to avoid the singularity on certain interval. The graphs of the algebraic solitary wave solution for the generalized KB equation (2) is displayed in Figure 6(a)-(b), which demonstrates novel spatial structures in the generalized KB equation (2). In addition, Figure 6(c) shows the shape of the “Potential” curve \( Q(u) \) in case of the roots satisfying \( u_1 = u_2 = u_3 \neq u_4 \).

Finally, consider the solution of the ordinary differential equation (19) in the infinite interval \([u_1, \infty)\). In the similar way, the periodic wave solution in term of Jacobi elliptic function is also obtained for the four different roots of the “Potential” curve \( Q(u) \), that is

\[ u = \frac{u_2(u_1 - u_4)\text{sn}^2(W_1, m_1) - u_1(u_2 - u_4)}{u_4 - u_2 + (u_1 - u_4)\text{sn}^2(W_1, m_1)}, \tag{27} \]

which can also be degenerated into the periodic wave solution in form of trigonometric function for case of \( u_2 = u_3 \), whose form is very similar to the trigonometric function in the infinite interval \((-\infty, u_4)\), which is

\[ u = \frac{u_3(u_1 - u_4)\sin^2(W_{12}) - u_1(u_3 - u_4)}{u_4 - u_3 + (u_1 - u_4)\sin^2(W_{12})}. \tag{28} \]
Moreover, in the special case $u_3 = u_4$, the periodic wave solution (27) can degenerate into the solitary wave solution of the form

$$u = \frac{u_2(u_1 - u_4)\tanh^2(W_{13}) - u_1(u_2 - u_4)}{u_4 - u_2 + (u_1 - u_4)\tanh^2(W_{13})},$$

(29)

where $W_{13} = (\xi - \xi_0)\sqrt{(u_1 - u_4)(u_2 - u_4)}/2$. The shapes of this solitary wave solution is similar to the solution in (24), which along with the corresponding “Potential” curve $Q(u)$ are shown in Figure 7 for parameters $u_1 = 8, u_2 = 0, u_3 = u_4 = -8, \kappa = -1, \xi_0 = 0$.

It is interesting to observe that the solitary wave solution (29) can further degenerate into the algebraic solitary wave solution for the limit $u_2 \to u_4$ with $u_1 \neq u_4$, that is

$$u = \frac{u_4(u_1 - u_4)^2\xi^2 - 4u_1}{(u_1 - u_4)^2\xi^2 - 4},$$

(30)

which also has singularity at point $\xi^2 = 4/(u_1 - u_4)^2$. The spatial structure along with the corresponding shape of the “Potential” curve $Q(u)$ are shown in Figure 8 with parameters $u_1 = 4, u_2 = u_3 = u_4 = -2, \kappa = -1, \xi_0 = 0$.

Still there are some other degeneration cases for the periodic wave solution (27). For example, it degenerates into the trivial constant solution $u = u_2$ for $u_1 = u_2$. The cases of both two double real roots and one quadruple real roots for the “Potential” curve $Q(u)$ also correspond to trivial constant solutions, which will not

---

**Figure 6.** The algebraic solitary wave solution of the generalized KB equation (2) at $t = 0$ based on the algebraic solitary wave solution (26): (a) the $u$-component of kink type and (b) the $v$-component of two-hump and anti-bell type, where the parameters are chosen to be $u_1 = u_2 = u_3 = 5, u_4 = -10, \kappa = -1, \xi_0 = 0$. (c) The “Potential” curve $Q(u)$ in case of the roots $u_1 = u_2 = u_3 \neq u_4$. 

---

**Figure 7.** The shapes of this solitary wave solution is similar to the solution in (24), which along with the corresponding “Potential” curve $Q(u)$ are shown for parameters $u_1 = 8, u_2 = 0, u_3 = u_4 = -8, \kappa = -1, \xi_0 = 0$. 

---

**Figure 8.** The spatial structure along with the corresponding shape of the “Potential” curve $Q(u)$ are shown with parameters $u_1 = 4, u_2 = u_3 = u_4 = -2, \kappa = -1, \xi_0 = 0$. 

---

**Figure 9.** The “Potential” curve $Q(u)$ also correspond to trivial constant solutions, which will not...
be discussed here in details. Examining the exact traveling wave solutions above, it is easy to find that most of them have singularities for some parameters, which is caused by the parameter $\sigma > 0$ in the generalized KB equation (2). In case of the parameter $\sigma < 0$, it will be seen in the following subsection that all the exact traveling wave solutions to the generalized KB equation (2) doesn’t have singularities, which is of particular interest for the parameter $\sigma < 0$. Similar to the “good” Boussinesq equation [4, 3] and “bad” Boussinesq equation [34, 9], we call the equation (2) to be the “good” generalized KB equation for the parameter $\sigma < 0$ and the “bad” generalized KB equation for the parameter $\sigma > 0$.

### 3.2. Classification of the exact solutions for parameter $\sigma < 0$.

In the case of the parameter $\sigma < 0$, Figure 2 indicates that the solutions of the ordinary differential equation (19) may locate in the intervals $[u_2, u_1]$ and $[u_4, u_3]$, respectively. In this subsection, we constrain the parameter $\sigma = -1$ for simplicity.

Firstly, consider the exact solution of the equation (19) on the interval $[u_2, u_1]$. When the “Potential” curve $Q(u)$ has four mutually distinct real roots shown in Figure 2, integrating the ordinary differential equation (19) yields the periodic wave solution in term of the Jacobi elliptic function as

$$u = \frac{u_1(u_2 - u_4) + u_4(u_1 - u_2)sn^2(W_2, m_2)}{u_2 - u_4 + (u_1 - u_2)sn^2(W_2, m_2)}, \quad (31)$$
where $W_2 = (\xi - \xi_0) \sqrt{(u_1 - u_3)(u_2 - u_4)} / 2$ and the modulus of the Jacobian elliptic sn function is $m_2 = \frac{(u_1 - u_2)(u_3 - u_4)}{(u_1 - u_3)(u_2 - u_4)}$, $0 \leq m_2 \leq 1$.

When the roots of the "Potential" curve $Q(u)$ satisfy $u_1 \neq u_2$ and $u_3 = u_4$, the periodic wave solution (31) can degenerate into the following periodic wave solution in term of the trigonometric function

$$u = \frac{u_1(u_2 - u_4) + u_4(u_1 - u_2)\sin^2(W_{21})}{u_2 - u_4 + (u_1 - u_2)\sin^2(W_{21})},$$

(32)

where $W_{21} = \frac{1}{2}(\xi - \xi_0) \sqrt{(u_1 - u_3)(u_2 - u_4)}$. For clarity, this periodic wave solution of the generalized KB equation (2) at time $t = 0$ is displayed Figure 9 under the parameters $u_1 = 4, u_2 = 2, u_3 = u_4 = -2, \kappa = -1, \xi_0 = 0$, where it is seen that the periods of the $u$-component and the $v$-component are different.

When the roots $u_2 = u_3$, the periodic wave solution (31) degenerates into the soliton solution of the form

$$u = \frac{u_1(u_3 - u_4) + u_4(u_1 - u_3)\tanh^2(W_{22})}{u_3 - u_4 + (u_1 - u_3)\tanh^2(W_{22})},$$

(33)

where $W_{22} = \frac{1}{2}(\xi - \xi_0) \sqrt{(u_1 - u_3)(u_3 - u_4)}$. This is a soliton solution without singularities, which is different from the solitary wave solutions (22), (24) and (29) of the "bad" generalized KB equation (2) with parameter $\sigma > 0$. Moreover, the
interaction between two such solitons is elastic. Figure 10 shows that spatial distributions of the soliton solution to the generalized KB equation (2) with \( \sigma = -1 \) at time \( t = 0 \) based on the traveling wave solution (33). It is observed that the \( u \)-component is a bell type bright soliton and the \( v \)-component is two-hump and anti-bell type bright soliton with nonvanishing boundary, which to the best of our knowledge is new nonlinear structures in the generalized KB equation (2).

Similar to the algebraic solitary wave solution above, the soliton solution (33) can further degenerate into the algebraic soliton solution as follows for the roots of “Potential” curve \( Q(u) \) satisfying \( u_2 = u_3 = u_4 \neq u_1 \), that is

\[
 u = \frac{u_4(u_1 - u_4)^2\xi^2 + 4u_1}{(u_1 - u_4)^2\xi^2 + 4},
\]

which is a rational solution that doesn’t have singularities at any points. Figure 11 demonstrates the spatial structure of the algebraic soliton solution to the generalized KB equation (2) with \( \sigma = -1 \) at time \( t = 0 \) based on the traveling wave solution (34). It is observed that this algebraic soliton is very similar to the soliton shown in Figure 10 except the altitudes of the two humps in the \( v \)-component.

In the same way, there are also many other trivial degenerations for the periodic wave solution (31). For example, when the roots satisfy \( u_1 = u_2 \) and \( u_3 \neq u_4 \), the periodic wave solution (31) is reduced to the constant solution \( u = u_2 \), while it degenerates into the constant solution \( u = u_3 \) for \( u_1 = u_2 = u_3 \neq u_4 \).
Secondly, consider the exact solution of the ordinary differential equation (19) on the interval \([u_4, u_3]\), which is very similar to case of \(u \in [u_2, u_1]\) above and one can also derive the periodic wave solutions in term of Jacobi elliptic function and trigonometric function, soliton solution, algebraic soliton solution and trivial constant solutions to the “good” generalized KB equation (2) with parameter \(\sigma < 0\). For simplicity, we don’t discuss them here in detail, but only list the expressions of these exact traveling wave solutions in Table 1 as follows. It is remarked that in Table 1, we have taken the parameter \(\sigma = -1\) and denoted that

\[
W_2 = \frac{1}{2}(\xi - \xi_0)\sqrt{(u_1 - u_3)(u_2 - u_4)},
\]

\[
W_{23} = \frac{1}{2}(\xi - \xi_0)\sqrt{(u_2 - u_3)(u_2 - u_4)},
\]

\[
W_{24} = \frac{1}{2}(\xi - \xi_0)\sqrt{(u_1 - u_3)(u_3 - u_4)}.
\]

| Classification of exact solutions on the interval \([u_4, u_3]\) |
|---|---|---|---|
| Distribution of the roots | The roots | The Exact Solutions | Properties of the solution |
| \(u_4\) | \(u_3\) | \(u_2\) | \(u_1\) |

Figure 10. The soliton solution of the generalized KB equation (2) at \(t = 0\) based on the traveling wave solution (33): (a) the \(u\)-component is a bell type bright soliton and (b) the \(v\)-component is two-hump and anti-bell type bright soliton with nonvanishing boundary, where the parameters are chosen to be \(u_1 = 4, u_2 = u_3 = 0, u_4 = -2, \kappa = -1, \xi_0 = 0\). (c) The “Potential” curve \(Q(u)\) in case of the roots \(u_2 = u_3\).
Four different real roots

\[ u_1, u_2, u_3, u_4 \]

\[ u = \frac{u_4(u_1-u_3)+u_1(u_3-u_4)\sin^2(W_2m)u_2-u_3+(u_3-u_4)\sin^2(W_2m_2)}{u_2-u_3+(u_3-u_4)\sin^2(W_2m_2)} \]

Periodic solution

Two simple real roots and one double real root

\[ u_1 = u_2, u_3, u_4 \]

\[ u = \frac{u_4(u_2-u_3)+u_2(u_3-u_4)\sin^2(W_2m)}{u_2-u_3+(u_3-u_4)\sin^2(W_2m_2)} \]

Periodic solution

\[ u_1, u_2 = u_3, u_4 \]

\[ u = \frac{u_4(u_1-u_3)+u_1(u_3-u_4)\tanh^2(W_2m_2)}{u_2-u_3+(u_3-u_4)\tanh^2(W_2m_2)} \]

Anti-bell soliton solution

\[ u_1, u_2, u_3 = u_4 \]

\[ u = u_4 \]

Constant solution

Table 1: Classification of the solutions on interval \([u_4, u_3]\)

4. Linear stability analysis and dynamical evolution. This section focuses on the linear stability and dynamical evolutions of the exact solutions to the generalized KB equation (2) by the standard perturbation expansion. To verify the correctness of the linear stability analysis, the dispersion relation and perturbation expansion for the constant solutions are deduced to observe the consistence, which displays excellent agreement. In the case of soliton solutions and periodic solutions, the dispersion relationship is complicated, and the results from stability analysis can be compared with the dynamical evolutions.

Firstly, we formulate the eigenvalue problem associated with the linear stability analysis by taking the general form of the perturbation expansion

\[ u = u_0(\xi) + \varepsilon e^{\lambda t}u_1(\xi) \],

\[ v = v_0(\xi) + \varepsilon e^{\lambda t}v_1(\xi) \],

where \(u_0(\xi)\) and \(v_0(\xi)\) are exact traveling wave solutions of the generalized KB equation (2) proposed in Section 3, \(\lambda\) is the eigenvalue parameter, \(\varepsilon\) is a small positive parameter, and \(u_1(\xi), v_1(\xi)\) are the eigenvectors.

The eigenvalue problem is derived by substituting (38) into (2) and linearizing (2), which is of the form

\[ \mathbf{L} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} \],

where the operator matrix \(\mathbf{L}\) is

\[ \mathbf{L} = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \],

and the elements of \(\mathbf{L}\) are represented as
Figure 11. The algebraic soliton solution of the generalized KB equation (2) at $t = 0$ based on the algebraic solitary wave solution (34): (a) the $u$-component is a bell type bright soliton and (b) the $v$-component is a two-hump and anti-bell type bright soliton, where the parameters are chosen to be $u_1 = 4, u_2 = u_3 = u_4 = -2, \kappa = -1, \xi_0 = 0$. (c) The “Potential” curve $Q(u)$ in case of the roots $u_2 = u_3 = u_4 \neq u_1$.

Figure 12. The distribution of spectral points for the linear eigenvalue problem (42) with parameter $\sigma = 1$ (a) and $\sigma = -1$ (b), respectively.
Secondly, examine the dispersion relation of the trivial constant solution and compare the result with the linear stability analysis. Obviously, arbitrary constants \( u_0 \) and \( v_0 \) are the trivial constant solution of the generalized KB equation (2). Consider the perturbation of the trivial constant solution
\[
\begin{align*}
  u &= u_0 + \varepsilon u_1 e^{i(kx+\omega t)}, \\
  v &= v_0 + \varepsilon v_1 e^{i(kx+\omega t)}.
\end{align*}
\]
(40)

Substituting the perturbation (40) into equation (2) and recalling the compatibility condition \( v_{xt} = v_{tx} \), the dispersion relation between wave number \( k \) and frequency \( \omega \) can be formulated as
\[
\omega^2 = -\frac{1}{4} \sigma k^4 + \left[ v_0 + \left( \frac{1}{2} - \kappa \right) \frac{u_0 v_1}{u_1} - \kappa \left( \frac{1}{2} + \kappa \right) u_0^2 \right] k^2 - \left( \frac{3}{2} + \kappa \right) u_0 k \omega.
\]
(41)

It is found that in a large extent, the sign of the parameter \( \sigma \) affects the dispersion relation. In order to clarify the dispersion relation, we illustrate the dispersion relation by choosing special values of the constants \( u_0, v_0 \) and \( \kappa \), which are listed in Table 2.

It is observed from Case A in Table 2 that the parameter \( \sigma < 0 \) indicates that the generalized KB equation (2) is well-posed and the trivial zero solution is stable,
while the parameter $\sigma > 0$ shows that it is ill-posed. For the Case B, the trivial constant solution $(u, v) = (0, v_0)$ is stable for the parameter $\sigma < 0$, while instability may emerge for the parameter $\sigma > 0$. In order to better illustrate this case, we analyze the linear stability of the constant solution $(u, v) = (0, v_0)$, where the linear eigenvalue problem is formulated below and the distributions of the spectrum are demonstrated in Figure 12

\[ \hat{L} \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \]

(42)

where the operator matrix $\hat{L}$ is

\[ \hat{L} = \begin{pmatrix} \hat{L}_{11} & \hat{L}_{12} \\ \hat{L}_{21} & \hat{L}_{22} \end{pmatrix}, \]

with $\hat{L}_{11} = c \frac{\partial}{\partial \xi}, \hat{L}_{12} = -\frac{\partial}{\partial \xi}, \hat{L}_{21} = -v_0 \frac{\partial}{\partial \xi} - \frac{1}{4} \sigma \frac{\partial^3}{\partial \xi^3}$ and $\hat{L}_{22} = c \frac{\partial}{\partial \xi}$.

The numerical simulations of the linear eigenvalue problem (42) are carried out for different choice of the parameter $\sigma$. It can be clearly observed from Fig. 12(a) that for the parameter $\sigma = 1$ there are lots of spectral point with $\text{Re}(\lambda) > 0$, which lead to bad instability (see the perturbation expansion (38)); for the parameter $\sigma = -1$, the real part of the maximum spectral $\lambda$ is a even small number near 0, which is about $10^{-13}$, which displays the linear stability of the constant solution $(u, v) = (0, v_0)$. So we have illustrated that results of linear stability analysis are consistent with the result of dispersion relations, which is further verified that the perturbation expansion method is reliable for stability analysis.

Thirdly, investigate the linear stability and dynamical evolutions of the exact traveling wave solutions to the generalized KB equation (2) proposed in Section 3. For simplicity, we only take the soliton solution (33) and algebraic soliton solution (34) as examples, which are the exact solutions for the parameter $\sigma = -1$.

For the soliton solution (33), solve the linear eigenvalue problem (39) and it is observed from Figure 13 that all the eigenvalue spectrums locate on the imaginary axis, and the real parts and the imaginary parts of the eigenvector $(u_1, v_1)^T$ are localized, which indicates the reliability of the results from linear stability analysis. So we conclude that the soliton solution (33) is linearly stable. In order to determine the stability of the soliton solution (33), the dynamical evolutions of this solution are done by directly simulating the generalized KB equation (2), which is shown in Figure 14. It is seen that after a period of evolution (time $t = 30$), the solutions of $u$-component and $v$-component do not oscillate, and they just drift steadily along the direction of the characteristic velocity, which verifies the stability of the soliton solution (33) again from another perspective.
Figure 14. (a1)-(b1) The initial profiles (at time $t = 0$) and (a2)-(b2) time evolutions (at time $t = 30$) of the soliton solution (33), where the parameters are $u_1 = 4, u_2 = u_3 = 0, u_4 = -2, \kappa = -1, \xi_0 = 0$ and the 0.2% Gaussian white noise is added.

For the algebraic soliton solution (34), solve the linear eigenvalue problem (39) and it is seen from Figure 15 that some eigenvalue spectrums have positive real parts, which indicates that the solution (34) is linearly unstable. In order to check the validity of the linear stability analysis and exclude the eigenvalue spectral points caused by systematic error, the graphs of the real parts and the imaginary parts of the eigenvector $(u_1, v_1)^T$ are given and it is found that they are still localized, which also shows the reliability of the linear stability analysis. Moreover, from the perspective of dynamical evolutions, the algebraic soliton solution (34) does show great instability. Figure 16 displays that during the time evolutions the strong oscillation occurs in very short time, where the shapes of the solution change greatly and the amplitudes fluctuates greatly. Figure 16(a1)-(b1) show the initial profiles of the algebraic soliton solution (34), where the $u$-component has one peak while the $v$-component has three peaks. At time $t = 0.2$, the $u$-component keeps it shape and amplitude, but the $v$-component oscillates strongly. But at time $t = 0.4$, both $u$ and $v$-component break down immediately with large oscillations. Obviously, the results
of dynamical evolutions agree well with the results of linear stability analysis, which
double check the instability of the algebraic soliton solution (34).

5. Conclusion. In conclusion, this work investigates the integrability, classification of the exact traveling wave solutions and the linear stability of the exact solutions to the generalized Kaup-Boussinesq equation analytically and numerically. The Lax integrability of this equation is determined by the standard prolongation technique, which indicates that it may have exact soliton solutions. Then the complete classification of the exact traveling wave solutions and the spacial structures of these solutions are proposed according to the distribution of roots of the “Potential” curve $Q(u)$ for the cases $\sigma > 0$ and $\sigma < 0$, respectively. It is found that the exact traveling wave solutions may exist singularities for the parameter $\sigma > 0$ and never appear singularities for $\sigma < 0$. Finally, the linear stability of the exact traveling wave solution is analyzed by perturbation expansion. In order to ensure the reliability of the method, the dispersion relation of the constant solution is compared with the results of the linear stability analysis, and the results are proved to be consistent with each other. In particular, this method is used to explore the linear stability analysis of two special exact solutions, whose dispersion relationships are hard to get. Thus the dynamical evolutions are carried out to examine the validity of the theoretical analysis and the results show that they match well with each other.

Declaration of interests. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.
Figure 16. (a1)-(b1) The initial profiles, (a2)-(b2) time evolutions (at time $t = 0.2$) and (a3)-(b3) time evolutions (at time $t = 0.4$) of the algebraic soliton solution (34), where the parameters are $u_1 = 4, u_2 = u_3 = u_4 = -2, \kappa = -1, \xi_0 = 0$ and the 0.2% Gaussian White noise is added.
REFERENCES

[1] M. J. Ablowitz and P. A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering, University Press, Cambridge, 1991.

[2] R. Balakrishnan and I. I. Satija, Solitons in Bose-Einstein condensates, Pramana J. Phys., 77 (2011), 929–947.

[3] C. Charlier and J. Lenells, The “good” Boussinesq equation: A Riemann-Hilbert approach, Indiana Univ. Math. J., to appear 2021, arXiv:2003.02777, 48 pp.

[4] C. Charlier, J. Lenells and D. S. Wang, The “good” Boussinesq equation: Long-time asymptotics, Analysis & PDE, to appear 2021, arXiv:2003.04789, 34 pp.

[5] T. Congy, S. K. Ivanov, A. M. Kamchatnov and N. Pavloff, Evolution of initial discontinuities in the Riemann problem for the Kaup-Boussinesq equation with positive dispersion, Chaos, 27 (2017), Paper No. 083107, 12 pp.

[6] A. Constantin, Dispersion relations for periodic traveling water waves in flows with discontinuous vorticity, Comm. Pure Appl. Anal., 11 (2012), 1397–1406.

[7] A. Constantin and W. Strauss, Pressure beneath a Stokes wave, Comm. Pure Appl. Math., 63 (2010), 533–557.

[8] A. Constantin and W. Strauss, Periodic traveling gravity water waves with discontinuous vorticity, Arch. Rational Mech. Anal., 202 (2011), 133–175.

[9] P. Deift, C. Tomei and E. Trubowitz, Inverse scattering and the Boussinesq equation, Comm. Pure Appl. Math., 35 (1982), 567–628.

[10] D. Dutykh and D. Ionescu-Kruse, Effects of vorticity on the travelling waves of some shallow water two-component systems, Discrete Contin. Dyn. Syst., 39 (2019), 5521–5541.

[11] G. A. El, R. H. J. Grimshaw and A. M. Kamchatnov, Wave breaking and the generation of undular bores in an integrable shallow water system, Studies Appl. Math., 114 (2005), 395–411.

[12] G. A. El, R. H. J. Grimshaw and M. V. Pavlov, Integrable shallow-water equations and undular bores, Studies Appl. Math., 106 (2001), 157–186.

[13] C. S. Gardner, J. M. Greene, M. D. Kruskal and R. M. Miura, Method for solving the Korteweg-de Vries equation, Phys. Rev. Lett., 19 (1967), 1095–1097.

[14] V. S. Gerdjikov, E. V. Doktorov and J. Yang, Adiabatic interaction of N ultrashort solitons: Universality of the complex Toda chain model, Phys. Rev. E, 64 (2001), Paper No. 056617, 15 pp.

[15] J. Haberlin and T. Lyons, Solitons of shallow-water models from energy-dependent spectral problems, Eur. Phys. J. Plus, 133 (2018), 16.

[16] M. A. Helal, Soliton solution of some nonlinear partial differential equations and its applications in fluid mechanics, Chaos, Solitons and Fractals, 13 (2002), 1917–1929.

[17] R. Ivanov, Two-component integrable systems modelling shallow water waves: The constant vorticity case, Wave Motion, 46 (2009), 389–396.

[18] R. Ivanov and T. Lyons, Integrable models for shallow water with energy dependent spectral problems, J. Nonlinear Math. Phys., 19 (2012), Paper No. 1240008, 17 pp.

[19] M. Jaulent and C. Jean, The inverse s-wave scattering problem for a class of potentials depending on energy, Comm. Math. Phys., 28 (1972), 177–220.

[20] A. M. Kamchatnov, Nonlinear Periodic Waves and Their Modulations: An Introductory Course, World Scientific Publishing, 2000.

[21] A. M. Kamchatnov, R. A. Kraenkel and B. A. Umarov, Asymptotic soliton train solutions of Kaup-Boussinesq equations, Wave Motion, 38 (2003), 355–365.

[22] D. J. Kaup, A higher-order water-wave equation and the method for solving it, Progr. Theoret. Phys., 54 (1975), 396–408.

[23] V. B. Matveev and M. I. Yavor, Almost periodic N-soliton solutions of the nonlinear hydrodynamic Kaup equation, Ann. Inst. H. Poincar’e Sect. A, 31 (1979), 25–41.

[24] K. Nishinary, K. Abe and J. Satsuma, A new-type of soliton behavior in a two dimensional plasma system, J. Phys. Soc. Japan, 62 (1993), 2021–2029.

[25] M. Pavlov, Integrable systems and metrics of constant curvature, J. Nonlinear Math. Phys., 9 (2002), 173–191.

[26] D. H. Sattinger and J. Szmigielski, A Riemann-Hilbert problem for an energy dependent Schrödinger operator, Inverse Problems, 12 (1996), 1003–1025.

[27] K. Singla and M. Rana, Exact solutions and conservation laws of multi Kaup-Boussinesq system with fractional order, Analysis Math. Phys., 11 (2021), Paper No. 30, 15 pp.
[28] H. D. Wahlquist and F. B. Estabrook, Prolongation structures of nonlinear evolution equations, *J. Math. Phys.*, 16 (1975), 1–7.

[29] D.-S. Wang, Integrability of the coupled KdV equations derived from two-layer fluids: Prolongation structures and Miura transformations, *Nonlinear Anal.*, 73 (2010), 270–281.

[30] D.-S. Wang and J. Liu, Integrability aspects of some two-component KdV systems, *Appl. Math. Lett.*, 79 (2018), 211–219.

[31] B. Wang, Z. Zhang and B. Li, Soliton molecules and some hybrid solutions for the nonlinear Schrödinger equation, *Chin. Phys. Lett.*, 37 (2020), 030501.

[32] J. Yang, *Nonlinear Waves in Integrable and Non-Integrable Systems*, Society for Industrial and Applied Mathematics, 2010.

[33] Y. S. Zhang and J. S. He, Bound-state soliton solutions of the nonlinear Schrödinger equation and their asymmetric decompositions, *Chin. Phys. Lett.*, 36 (2019), 030201.

[34] V. E. Zakharov, On stochastization of one-dimensional chains of nonlinear oscillators, *Sov. Phys.-JETP*, 38 (1974), 108-110; translated from *Zh. Eksp. Teor. Fiz.*, 65 (1973), 219–225.

[35] J. Zhou, L. Tian and X. Fan, Solitary-wave solutions to a dual equation of the Kaup-Boussinesq system, *Nonlinear Anal.: Real World Appl.*, 11 (2010), 3229–3235.

Received August 2021; revised January 2022; early access March 2022.

E-mail address: rz_gong@163.com
E-mail address: syr317@126.com
E-mail address: dswang@bnu.edu.cn