Optimal Stopping Problems with Regime Switching: A Viscosity Solution Method

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Abstract

We employ the viscosity solution technique to analyze optimal stopping problems with regime switching. Specifically, we obtain the viscosity property of value functions, the uniqueness of viscosity solutions, the regularity of value functions and the form of optimal stopping intervals. Finally, we provide an application of the results.

Key Words: dynamic programming; optimal stopping; regime switching; viscosity solution

1 Introduction

Regime-switching processes are appropriate candidates for describing the price of financial assets (Bae, Kim, & Mulvey, 2014; Chernova, Gallantb, Ghyselsb, & Tauchen, 2003) and the price of some commodities (Casassus, Collin-Dufresne, & Routledge, 2005). In addition, as Elias, Wahab, and Fang (2014) point out, regime-switching processes are also plausible choices of modelling the stochastic behavior of temperature. Last but not the least, regime-switching processes appear in real option pricing (Bollen, 1999). Thus it is reasonable to consider the optimal stopping problems in which underlying processes and payoff functions are modulated by Markov chains.

Many specific problems of optimal stopping with regime switching have been studied. Assuming that the stock price follows a geometric Brownian motion modulated by a two-state Markov chain, Guo (2001) provides an explicit closed solution for Russian options. Under the same assumption as that
of (Guo, 2001), Guo and Zhang derive an explicit closed solution for perpetual American options in (Guo & Zhang, 2004) and for optimal selling rules in (Guo & Zhang, 2005), respectively, and Buffington and Elliott (2002a) explore American options with finite maturity date. Eloe, Liu, Yatsuki, Yin, and Zhang (2008) develop optimal selling rules via using a regime-switching exponential Gaussian diffusion model. In a regime-switching Lévy model, Boyarchenko and Levendorskiĭ (2008) show a pricing procedure for perpetual American and real options which is efficient even though the number of states is large provided transition rates are not large with respect to riskless rates.

The method used in (e.g., Guo, 2001; Guo & Zhang, 2004, 2005) is to construct a solution to some equations by guessing a priori a strategy and then validate it by a verification argument. In this paper, we will employ the viscosity solution technique to determine the solution of optimal stopping problems with regime switching. First, we prove the value function is a viscosity solution of some variational inequalities. Second, we prove the uniqueness of viscosity solutions. Third, we show the regularity of the value function. Finally, we determine the form of optimal stopping intervals.

We outline the structure of this paper. In Section 2, we prove the viscosity property of the value function of optimal stopping problems with regime switching (Theorem 2.1), the uniqueness of viscosity solutions (Theorem 2.5 and Theorem 2.6), the regularity of the value function (Theorem 2.7) and the form of optimal stopping intervals (Theorem 2.11). In Section 3, we provide an application of the results obtained in Section 2. Some conclusions are drawn in Section 4.

2 Optimal Stopping Problems with Regime Switching

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be a filtered probability space with the filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions and \(\mathcal{F}_0\) being the completion of \(\{\emptyset, \Omega\}\).

Let \(X := (X(t), t \geq 0)\) be a time homogeneous Markov chain defined on \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) taking values in the standard orthogonal basis of \(\mathbb{R}^m\), \(I := \{e_1, e_2, \ldots, e_m\}\), whose rate matrix is \(A := (a_{ij})_{m \times m}\) with \(a_{ii} < 0\), for each \(i = 1, 2, \ldots, m\). Then as in (Buffington & Elliott, 2002b), we can show that

\[X(t) = X_0 + \int_0^t AX(s)ds + M(t),\]

where \(M := (M(t), t \geq 0)\) is a martingale with respect to the filtration generate
Let $B := (B(t), t \geq 0)$ be a one dimensional standard Brownian motion, which is independent of $X$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

Assume that the process $Y := (Y(t), t \geq 0)$ satisfies
\[
dY(t) = \alpha(X(t), Y(t))dt + \beta(X(t), Y(t))dB(t), \quad X(0) = e_i, \ Y(0) = y,
\]
where $\alpha : I \times \mathbb{R} \rightarrow \mathbb{R}$ and $\beta : I \times \mathbb{R} \rightarrow \mathbb{R}$ are two functions such that $\alpha(e_i, \cdot)$ and $\beta(e_i, \cdot)$ are Lipschitz continuous for each $i = 1, 2, \cdots, m$. We assume that $\beta(\cdot, \cdot) > 0$.

Let $T$ denote the set of all stopping times. For any $\tau \in T$, $e_i \in I$ and $y \in \mathbb{R}$, we define
\[
J^\tau(e_i, y) := \mathbb{E} \left[ \int_0^\tau \exp(-rt)f(X(t), Y(t))dt + \exp(-r\tau)g(X(\tau), Y(\tau)) \middle| X(0) = e_i, Y(0) = y \right],
\]
where $r$ is a real number, and $f : I \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : I \times \mathbb{R} \rightarrow \mathbb{R}$ are two functions such that $f(e_i, \cdot)$ and $g(e_i, \cdot)$ are Lipschitz continuous for each $i = 1, 2, \cdots, m$. We assume that $\exp(-r\tau)g(X(\tau), Y(\tau)) = 0$ on $\{\tau = \infty\}$.

Then the optimal stopping problem with regime switching is described as follows.

Find $V(e_i, y)$ and $\tau^* \in T$ such that $V(e_i, y) = \sup_{\tau \in T} J^\tau(e_i, y) = J^{\tau^*}(e_i, y).$ (1)

For each $i = 1, 2, \cdots, m$, setting $\alpha_i(\cdot) := \alpha(e_i, \cdot)$, $\beta_i(\cdot) := \beta(e_i, \cdot)$, $f_i(\cdot) := f(e_i, \cdot)$, $g_i(\cdot) := g(e_i, \cdot)$ and $V_i(\cdot) := V(e_i, \cdot)$, we have the following theorem.

**Theorem 2.1.** Assume that $r$ is large enough. Then for each fixed $i \in \{1, 2, \cdots, m\}$, $V_i$ is the unique viscosity solution with at most linear growth of the following variational inequality,
\[
\min\{rV_i - \mathcal{L}_iV_i - \sum_{q=1}^m a_{iq}V_q - f_i, V_i - g_i\} = 0 \text{ on } \mathbb{R},
\] (2)

where $\mathcal{L}$ is defined by
\[
\mathcal{L}_i\xi(y) := \frac{1}{2} \beta_i(y)^2 \frac{\partial^2 \xi(y)}{\partial y^2} + \alpha_i(y) \frac{\partial \xi(y)}{\partial y},
\]
for any $\xi \in C^2(\mathbb{R})$. 

Before proving the above theorem, let us first recall the definition of viscosity solutions.

The theory of viscosity solutions applies to certain partial differential equations of the form $F(x, u, Du, D^2u) = 0$ where $F : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times S(N) \to \mathbb{R}$ along with $S(N)$ is the set of all symmetric $N \times N$ matrices.

We require $F$ to satisfy the monotonicity condition

$$F(x, r, p, X) \leq F(x, s, p, Y), \text{ whenever } r \leq s \text{ and } Y \leq X. \quad (3)$$

Here $r, s \in \mathbb{R}, x, p \in \mathbb{R}^N, X, Y \in S(N)$ and $S(N)$ is equipped with its usual order.

It will be convenient to have the following notations.

$$USC(O) := \{\text{upper semicontinuous functions } u : O \to \mathbb{R}\},$$

$$LSC(O) := \{\text{lower semicontinuous functions } u : O \to \mathbb{R}\},$$

$$J^{2,+}_O u(x) := \{(D\varphi(x), D^2\varphi(x)) : \varphi \text{ is } C^2 \text{ and } u - \varphi \text{ has a maximum at } x\},$$

$$J^{2,-}_O u(x) := \{(D\varphi(x), D^2\varphi(x)) : \varphi \text{ is } C^2 \text{ and } u - \varphi \text{ has a minimum at } x\},$$

$$J^{2,+}_O u(x) := \{(p, X) \in \mathbb{R}^N \times S(N) : \text{there is a sequence } (x_n, p_n, X_n)_{n \in \mathbb{N}} \subset O \times J^{2,+}_O u(x_n) \text{ such that } (x_n, u(x_n), p_n, X_n) \text{ converges to } (x, u(x), p, X) \text{ as } n \to \infty\},$$

$$J^{2,-}_O u(x) := \{(p, X) \in \mathbb{R}^N \times S(N) : \text{there is a sequence } (x_n, p_n, X_n)_{n \in \mathbb{N}} \subset O \times J^{2,-}_O u(x_n) \text{ such that } (x_n, u(x_n), p_n, X_n) \text{ converges to } (x, u(x), p, X) \text{ as } n \to \infty\}.$$

**Definition 2.2.** (Crandall, Ishii, & Lions, 1992, Definition 2.2) Let $F$ satisfy (3) and $O \subset \mathbb{R}^N$. A viscosity subsolution of $F = 0$ (equivalently, a viscosity solution of $F \leq 0$) on $O$ is a function $u \in USC(O)$ such that

$$F(x, u(x), p, X) \leq 0, \text{ for all } x \in O \text{ and } (p, X) \in J^{2,+}_O u(x).$$

Similarly, a viscosity supersolution of $F = 0$ (equivalently, a viscosity solution of $F \geq 0$) on $O$ is a function $u \in LSC(O)$ such that

$$F(x, u(x), p, X) \geq 0, \text{ for all } x \in O \text{ and } (p, X) \in J^{2,-}_O u(x).$$

Finally, $u$ is a viscosity of $F = 0$ in $O$ if it is both a viscosity subsolution and a viscosity supersolution of $F = 0$ in $O$.

The following lemma is used in the proof of Theorem 2.1.
Lemma 2.3. There is a positive number $r_0$ such that $V_i$ is Lipschitz continuous for each $i = 1, 2, \ldots, m$ if $r > r_0$.

Proof. The proof is similar to that of (Pham, 2009, p. 96, Lemma 5.2.1).

Now we provide a proof of Theorem 2.1.

Proof of Theorem 2.1. 1. Recall the dynamic programming principle (Pham, 2009, p. 97). For any $j \in \{1, 2, \ldots, m\}$, $y \in \mathbb{R}$ and $\theta \in T$, we have

$$V_j(y) = \sup_{\tau \in T} \mathbb{E} \left[ \int_0^{\theta \land \tau} \exp(-rt)f(X(t), Y(t))dt + \exp(-r\tau)g(X(\tau), Y(\tau))1_{\{\theta > \tau\}} + \exp(-r\theta)\phi_i(X(\theta), Y(\theta))1_{\{\theta \leq \tau\}} | X(0) = e_j, Y(0) = y \right].$$ \hspace{1cm} (4)

2. The viscosity supersolution property. Fix an $i \in \{1, 2, \ldots, m\}$. Suppose that $\phi_i \in C^2(\mathbb{R})$ and $V_i - \phi_i$ has a minimum at some point $y_i \in \mathbb{R}$ such that $V_i(y_i) = \phi_i(y_i)$. For any positive integer $k$, choose some functions $\phi_k^j \in C^2(\mathbb{R})$ such that $\max_{y \in \mathbb{R}} |\phi_k^j(y) - V_j(y)| < 1/k$, where $j \neq i$ (see, for example, (Zhang, 1996, p. 63, Lemma 1.3)). Set $\phi_k^i := \phi_i$ for convenience.

Define a stopping time $\theta_0 := \inf\{t : t > 0, X(t) \neq e_i, |Y(t) - y_i| > 1\}$. Then by the dynamic programming principle (4), for any positive number $\varepsilon$, we have, via taking $\tau = \theta_0 \land \varepsilon$,

$$V_i(y_i) \geq \mathbb{E} \left[ \int_0^{\theta_0 \land \varepsilon} \exp(-rt)\phi_i(X(t), Y(t))dt + \exp(-r(\theta_0 \land \varepsilon))\phi_i(X(\theta_0 \land \varepsilon), Y(\theta_0 \land \varepsilon)) \mid X(0) = e_i, Y(0) = y_i \right] \geq -\frac{1}{k} + \mathbb{E} \left[ \int_0^{\theta_0 \land \varepsilon} \exp(-rt)f(X(t), Y(t))dt + \exp(-r(\theta_0 \land \varepsilon)) \sum_{j=1}^m \phi_k^j(X(\theta_0 \land \varepsilon))X_j(\theta_0 \land \varepsilon) \mid X(0) = e_i, Y(0) = y_i \right].$$ \hspace{1cm} (5)
In addition, by Itô’s formula, we have

\[
\exp(-r(\theta_0 \wedge \varepsilon)) \sum_{j=1}^{m} \varphi_j^k(Y(\theta_0 \wedge \varepsilon))X_j(\theta_0 \wedge \varepsilon)
\]

\[= \sum_{j=1}^{m} \varphi_j^k(Y(0))X_j(0) + \sum_{j=1}^{m} \int_{0}^{\theta_0 \wedge \varepsilon} \exp(-rt) \left( \frac{1}{2} \beta(X(t), Y(t))^2 \frac{\partial^2 \varphi_j^k}{\partial y^2}(Y(t)) + \alpha(X(t), Y(t)) \frac{\partial \varphi_j^k}{\partial y}(Y(t)) - r\varphi_j^k(Y(t)) + \sum_{q=1}^{m} a_{jq} \varphi_q^k(Y(t)) \right) X_j(t) \, dt
\]

\[+ \sum_{j=1}^{m} \int_{0}^{\theta_0 \wedge \varepsilon} \exp(-rt) \beta(X(t), Y(t)) \frac{\partial \varphi_j^k}{\partial y}(Y(t))X_j(t) \, dB(t)
\]

\[+ \sum_{j=1}^{m} \int_{0}^{\theta_0 \wedge \varepsilon} \exp(-rt) \varphi_j^k(Y(t)) \, dM_j(t).
\]

By combining (5) and (6), it follows that

\[
\sum_{j=1}^{m} \mathbb{E} \left[ \frac{1}{\varepsilon} \int_{0}^{\theta_0 \wedge \varepsilon} \exp(-rt) \left( -f_j(Y(t)) + r\varphi_j^k(Y(t)) - \frac{1}{2} \beta(X(t), Y(t))^2 \frac{\partial^2 \varphi_j^k}{\partial y^2}(Y(t)) - \alpha(X(t), Y(t)) \frac{\partial \varphi_j^k}{\partial y}(Y(t)) - \sum_{q=1}^{m} a_{jq} \varphi_q^k(Y(t)) \right) X_j(t) \, dt \mid X(0) = e_i, Y(0) = y_i \right] \geq -\frac{1}{k\varepsilon},
\]

i.e., as \( X(t) = e_i \) for \( t \in (0, \theta_0) \),

\[
\mathbb{E} \left[ \frac{1}{\varepsilon} \int_{0}^{\theta_0 \wedge \varepsilon} \exp(-rt) \left( -f_i(Y(t)) + r\varphi_i(Y(t)) - \mathcal{L}_i \varphi_i(Y(t)) - \sum_{q=1}^{m} a_{qi} \varphi_q^k(Y(t)) \right) \, dt \mid X(0) = e_i, Y(0) = y_i \right] \geq -\frac{1}{k\varepsilon}.
\]

By sending \( k \to +\infty \), it follows that

\[
\mathbb{E} \left[ \frac{1}{\varepsilon} \int_{0}^{\theta_0 \wedge \varepsilon} \exp(-rt) \left( -f_i(Y(t)) + r\varphi_i(Y(t)) - \mathcal{L}_i \varphi_i(Y(t)) - \sum_{q=1}^{m} a_{qi} V_q(Y(t)) \right) \, dt \mid X(0) = e_i, Y(0) = y_i \right] \geq 0.
\]

Now taking limits in the above inequality as \( \varepsilon \to 0 \) and applying the mean
value theorem, we obtain

\[ rV_i(y_i) - \mathcal{L}_i \varphi_i(y_i) - \sum_{q=1}^{m} a_{qi} V_q(y_i) - f_i(y_i) \geq 0. \]  

(7)

By the definition of \( V_i \), we have

\[ V_i(y_i) - g_i(y_i) \geq 0. \]  

(8)

From (7) and (8), we see that \( V_i \) is a viscosity supersolution of the variational inequality (2).

3. The viscosity subsolution property. Fix an \( i \in \{1, 2, \cdots, m\} \). Suppose that \( \varphi_i \in C^2(\mathbb{R}) \) and \( V_i - \varphi_i \) has a maximum at some point \( \bar{y}_i \in \mathbb{R} \) such that \( V_i(\bar{y}_i) = \varphi_i(\bar{y}_i) \). We will prove that \( V_i \) is a viscosity subsolution of the variational inequality (2) by contradiction.

Suppose that

\[ rV_i(\bar{y}_i) - \mathcal{L}_i \varphi_i(\bar{y}_i) - \sum_{q=1}^{m} a_{qi} V_q(\bar{y}_i) - f_i(\bar{y}_i) > 0, \]

and

\[ V_i(\bar{y}_i) - g_i(\bar{y}_i) > 0. \]

Hence for any positive number \( \varepsilon \) small enough and \( k \) large enough, thanks to continuity of the functions \( \mathcal{L}_i \varphi_i(\cdot), f_i(\cdot) \) and \( V_j(\cdot) \), where \( j = 1, 2, \cdots, m \), there is a positive number \( \rho \) such that

\[ r\varphi_i(Y(t)) - \mathcal{L}_i \varphi_i(Y(t)) - \sum_{q=1}^{m} a_{qi} \varphi^h_q(Y(t)) - f_i(Y(t)) > \varepsilon \text{ for } 0 \leq t \leq \delta, \]  

(9)

and

\[ \varphi_i(Y(t)) - g_i(Y(t)) > \varepsilon \text{ for } 0 \leq t \leq \delta, \]  

(10)

where \( X(0) = e_i, Y(0) = \bar{y}_i \) and \( \delta := \inf\{t : t > 0, X(t) \neq e_i, Y(t) \notin (\bar{y}_i - \rho, \bar{y}_i + \rho)\} \).

For any stopping time \( \tau \in \mathcal{T} \), by applying Itô’s formula and noting (9) and
(10), we obtain
\[ V_i(\bar{y}_i) = \varphi_i(\bar{y}_i) \]
\[ = E \left[ \int_0^{\delta \wedge \tau} \exp(-rt) \left( r\varphi_i(Y(t)) - \mathcal{L}_i\varphi_i(Y(t)) - \sum_{q=1}^m a_{qi}\varphi_q^k(Y(t)) \right) dt \right. \]
\[ + \sum_{j=1}^m \exp(-r(\delta \wedge \tau))\varphi_j^k(Y(\delta \wedge \tau))X_j(\delta \wedge \tau) \bigg| X(0) = e_i, Y(0) = \bar{y}_i \bigg] \]
\[ \geq E \left[ \int_0^{\delta \wedge \tau} \exp(-rt)f(X(t), Y(t))dt + \exp(-r\tau)g(X(\tau), Y(\tau))1_{\{\delta \geq \tau\}} \right. \]
\[ + \exp(-r\delta)V(X(\delta), Y(\delta))1_{\{\delta \leq \tau\}} \bigg| X(0) = e_i, Y(0) = \bar{y}_i \bigg] \]
\[ + \varepsilon E \left[ \int_0^{\delta \wedge \tau} \exp(-rt)dt + \exp(-r\tau)1_{\{\delta > \tau\}} \bigg| X(0) = e_i, Y(0) = \bar{y}_i \bigg] \]
\[ - \frac{1}{k} \] \hspace{1cm} (11)

We will show in Lemma 2.4 ahead that there is a positive constant $C$ such that
\[ E \left[ \int_0^{\delta \wedge \tau} \exp(-rt)dt + \exp(-r\tau)1_{\{\delta \geq \tau\}} \bigg| X(0) = e_i, Y(0) = \bar{y}_i \bigg] \geq C. \]

Sending $k \to \infty$ and then taking supremum over $\tau \in T$ in (11), we find
\[ V_i(\bar{y}_i) \geq V_i(\bar{y}_i) + \varepsilon C, \]
which is a contradiction. Consequently, $V_i$ is a viscosity subsolution of the variational inequality (2).

4. Uniqueness. The proof of uniqueness is the same as that of (Pham, 2009, pp. 98-99, Uniqueness property). Also refer to (Crandall et al., 1992, pp. 31-32). \(\square\)

Lemma 2.4. Using the notations introduced in the proof of Theorem 2.1, we have
\[ E \left[ \int_0^{\delta \wedge \tau} \exp(-rt)dt + \exp(-r\tau)1_{\{\delta > \tau\}} \bigg| X(0) = e_i, Y(0) = \bar{y}_i \bigg] \geq C \]
for some positive constant $C$.

Proof. 1. Let $Y_i$ be the solution of the following equation,
\[ dY_i(t) = \alpha_i(Y_i(t))dt + \beta_i(Y_i(t))dB(t), \quad Y_i(0) = \bar{y}_i. \]
Define \( T_{\bar{y}_i} := \inf\{ t : t > 0, |Y_i(t) - \bar{y}_i| = \rho \} \). Since \( \beta_i(\cdot) > 0 \), we have, by (Klebaner, 1998, p. 152, Corollary 6.12.2),

\[
\mathbb{P}(T_{\bar{y}_i} < +\infty | Y_i(0) = \bar{y}_i) > 0. \tag{12}
\]

2. Define \( T := \inf\{ t : t > 0, X(t) \neq e_i \} \). Thanks to the independence of \( X \) and \( B \), it follows that

\[
\mathbb{P}(\exists t \in (0, T), \text{ s.t. } |Y(t) - \bar{y}_i| = \rho |X(0) = e_i, Y(0) = \bar{y}_i) = \int_0^{+\infty} \mathbb{P}(\exists s \in (0, t), \text{ s.t. } |Y_i(s) - \bar{y}_i| = \rho |Y(0) = \bar{y}_i)(-a_i \exp(a_i t))dt,
\]

where we have used the fact that \( \mathbb{P}(T > t | X(0) = e_i) = \exp(a_i t) \).

The above equality and (12) yield

\[
\mathbb{P}(\exists t \in (0, T), \text{ s.t. } |Y(t) - \bar{y}_i| = \rho |X(0) = e_i, Y(0) = \bar{y}_i) > 0.
\]

Consequently, we obtain

\[
\mathbb{E} \left[ |Y(\delta) - \bar{y}_i|^2 | X(0) = e_i, Y(0) = \bar{y}_i \right] > 0. \tag{13}
\]

3. Set \( B_i := (\bar{y}_i - \rho, \bar{y}_i + \rho) \) and

\[
C_1 := \min \left\{ \left( r + \frac{2}{\rho} \sup_{y \in \partial B_i} |\alpha(e_i, y)| + \frac{1}{\rho^2} \sup_{y \in B_i} \beta(e_i, y)^2 \right)^{-1}, 1 \right\}.
\]

Define a function \( G : B_i \rightarrow \mathbb{R} \) by

\[
G(y) := C_1 \left( 1 - \frac{|y - \bar{y}_i|^2}{\rho^2} \right).
\]

Then we have

\[
\mathbb{E} \left[ \int_0^{\delta \wedge \tau} \exp(-rt)dt + \exp(-r\tau)1_{\{\delta > \tau\}} | X(0) = e_i, Y(0) = \bar{y}_i \right]
\]

\[
\geq \mathbb{E} \left[ \int_0^{\delta \wedge \tau} \exp(-rt) (rG(Y(t)) - \mathcal{L}_t G(Y(t))) dt 
\]

\[
+ \exp(-r\delta) G(Y(\delta)) 1_{\{\delta \leq \tau\}} | X(0) = e_i, Y(0) = \bar{y}_i \right]
\]

\[
= G(\bar{y}_i) - \mathbb{E} \left[ \exp(-r\delta) G(Y(\delta)) 1_{\{\delta \leq \tau\}} | X(0) = e_i, Y(0) = \bar{y}_i \right]
\]

\[
\geq C_1 \rho^{-2} \mathbb{E} \left[ |Y(\delta) - \bar{y}_i|^2 | X(0) = e_i, Y(0) = \bar{y}_i \right] =: C > 0,
\]

where we have used the facts \( rG(y) - \mathcal{L}_t G(y) \leq 1 \) and \( G(y) \leq 1 \) for the first inequality, Itô’s formula for the first equality, and (13) for the last inequality. The proof is complete.
We will prove that \( V \) is uniquely determined by the system (2). To do this, we introduce some notations.

Set \( \mu_i := \lim \sup_{x \to \infty} \frac{n_i(x)}{x} \) and \( \sigma_i := \lim \sup_{x \to \infty} \left| \frac{\beta_i(x)}{x} \right| \), \( i = 1, 2, \cdots, m \). Then we define the functions \( \omega_i(\lambda) := r - a_{ii} - \mu_i\lambda - \frac{1}{2} \sigma_i^2 \lambda (\lambda - 1) \), \( i = 1, 2, \cdots, m \), and a matrix \( H(\lambda) := (h_{ij}(\lambda))_{m \times m} \), where \( h_{ii} := \omega_i \) and \( h_{ij} := -a_{ij} \) for \( i \neq j \).

We make the following hypothesis.

(H1) There are positive numbers \( b_i \)'s such that \( \sum_{i=1}^{m} b_i h_{ij}(\lambda) \geq 0 \) for some constant \( \lambda > 1 \), \( j = 1, 2, \cdots, m \).

**Theorem 2.5.** Assume that (H1) holds. Let \( \{U_i\}_{i=1}^{m} \) be a family of functions defined on \( \mathbb{R} \) with at most linear growth such that for each fixed \( i \), \( U_i \) is a viscosity solution of

\[
\min \{ rU_i - L_i U_i - \sum_{q=1}^{m} a_{qi} U_q - f_i, U_i - g_i \} = 0 \text{ on } \mathbb{R}.
\]

Then \( U_i = V_i \) for \( i = 1, 2, \cdots, m \).

**Proof.** 1. Since \( U_i \) and \( V_i \), \( i = 1, 2, \cdots, m \), grow at most linearly, we have

\[
\lim_{x \to \infty} (U_i(x) - V_i(x) - b_i|x|^{\lambda}) = -\infty, \quad i = 1, 2, \cdots, m.
\]

2. Thanks to (H1), there is a positive constant \( K \) such that, for each \( i = 1, 2, \cdots, m \),

\[
\min \{ r\tilde{\psi}_i - L_i \tilde{\psi}_i - \sum_{q=1}^{m} a_{qi} \tilde{\psi}_q, \tilde{\psi}_i - g_i \} \geq 0 \text{ on } \mathbb{R} \setminus [-K, K],
\]

where \( \tilde{\psi}_i(x) := b_i|x|^{\lambda} \).

Choose some functions \( \bar{\psi}_i \in C^2(\mathbb{R}) \) satisfying \( \bar{\psi}_i|_{\mathbb{R} \setminus [-K,K]} = \tilde{\psi}_i \), \( i = 1, 2, \cdots, m \). Then it follows that

\[
\min \{ r\psi_i - L_i \psi_i - \sum_{q=1}^{m} a_{qi} \psi_q, \psi_i - g_i \} \geq 0 \text{ on } \mathbb{R},
\]

where \( \psi_i(x) := C + \bar{\psi}_i(x), \quad i = 1, 2, \cdots, m \), for some constant \( C \) large enough.

3. Define \( V_i^\varepsilon := V_i + \varepsilon \psi_i \) for \( \varepsilon \in (0,1) \) and \( i = 1, 2, \cdots, m \). Then \( V_i^\varepsilon \) is a viscosity supersolution of

\[
\min \{ rV_i^\varepsilon - L_i V_i^\varepsilon - \sum_{q=1}^{m} a_{qi} V_q^\varepsilon - f_i, V_i^\varepsilon - g_i \} = 0 \text{ on } \mathbb{R}.
\]
4. We prove \( U_i \leq V_i^\varepsilon, \ i = 1, 2, \cdots, m, \) by contradiction. Otherwise, we have by Step 1

\[
M := \max_{i \in \{1, \cdots, m\}} \sup_{x \in \mathbb{R}} \{U_i(x) - V_i^\varepsilon(x)\} = \max_{x \in \mathbb{R}} \{U_k(x) - V_k^\varepsilon(x)\} = U_k(\hat{x}) - V_k^\varepsilon(\hat{x}) > 0
\]

for some \( k \in \{1, 2, \cdots, m\} \) and \( \hat{x} \in \mathbb{R} \).

For each positive integer \( n \in \mathbb{N} \), we define a function \( \Phi_n \) by

\[
\Phi_n(x, y) := U_k(x) - V_k^\varepsilon(y) - \frac{n}{2}(x - y)^2.
\]

In virtue of Step 1, we take \( \mathcal{O} := (-R, R) \) with \( U_k(x) - V_k^\varepsilon(x) < 0 \) for all \( |x| \geq R \). Then \( \Phi_n \) has a maximum point \((x_n, y_n)\) on \( \mathcal{O} \). By the similar argument to the step 1 of the proof of (Pham, 2009, p. 77, Theorem 4.4.4), we have, up to a subsequence,

\[
\lim_{n \to \infty} \Phi_n(x_n, y_n) = M, \quad \lim_{n \to \infty} n(x_n - y_n)^2 = 0 \quad \text{and} \quad \lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = \hat{x}.
\]

Therefore, for \( n \) large enough, \((x_n, y_n)\) is a local maximum point of \( \Phi_n \) relative to \( \mathcal{O} \). Consequently, according to (Crandall et al., 1992, p. 17, p. 19), there are two constants \( a \) and \( b \) with \( a \leq b \) such that

\[
(n(x_n - y_n), a) \in J_{\mathcal{O}}^\mathbb{R} U_k(x_n), \quad (n(x_n - y_n), b) \in J_{\mathcal{O}}^\mathbb{R} V_k(x_n)
\]

and

\[
a\beta_k(x_n)^2 - b\beta_k(y_n)^2 \leq 3n(\beta_k(x_n) - \beta_k(y_n))^2.
\]

Then, in light of the viscosity property of \( U_k \) and \( V_k^\varepsilon \), it follows that

\[
\min \left\{ \alpha_k(x_n)(n(x_n - y_n)) - \frac{a}{2}\beta_k(x_n)^2 - \sum_{q=1}^{m} a_{qk} U_q(x_n) - f_k(x_n), U_k(x_n) - g_k(x_n) \right\} \leq 0
\]

and

\[
\min \left\{ \alpha_k(y_n)(n(x_n - y_n)) - \frac{b}{2}\beta_k(y_n)^2 - \sum_{q=1}^{m} a_{qk} V_q^\varepsilon(y_n) - f_k(y_n), V_k^\varepsilon(y_n) - g_k(y_n) \right\} \geq 0.
\]

**Case 1** \( U_k(x_n) - g_k(x_n) > 0 \) for all \( n \) large enough.
Subtracting (18) from (17) yields:

\[ r(U_k(x_n) - V_k^\varepsilon(y_n)) \leq n(\alpha_k(x_n) - \alpha_k(y_n))(x_n - y_n) + \frac{a}{2}\beta_k(x_n)^2 - \frac{b}{2}\beta_k(y_n)^2 + \sum_{q=1}^{m} a_{qk}(U_q(x_n) - V_q^\varepsilon(y_n)) + f_k(x_n) - f_k(y_n) \]

for all \( n \) large enough.

Note (15), (16) and the Lipschitz continuity of the functions \( \alpha_k, \beta_k \) and \( f_k \). By taking limits in the above inequality as \( n \to +\infty \), it follows that

\[ r(U_k(\hat{x}) - V_k^\varepsilon(\hat{x})) \leq \sum_{q=1}^{m} a_{qk}(U_q(\hat{x}) - V_q^\varepsilon(\hat{x})). \]

Therefore, in light of (14) and \( \sum_{q=1}^{m} a_{qk} = 0 \), we have \( rM \leq 0 \), which is a contradiction.

**Case 2** \( U_k(x_n) - g_k(x_n) \leq 0 \) frequently.

In this case, the following inequality

\[ U_k(x_n) - V_k^\varepsilon(y_n) \leq g_k(x_n) - g_k(y_n) \]

holds frequently.

Then, using the continuity of the function \( g_k \), we get \( M \leq 0 \), which is a contradiction.

In conclusion, we have proved \( U_i \leq V_i^\varepsilon, i = 1, 2, \cdots, m \).

5. Taking limits in \( U_i \leq V_i^\varepsilon, i = 1, 2, \cdots, m, \) as \( \varepsilon \to 0 \), we have \( U_i \leq V_i, i = 1, 2, \cdots, m. \)

6. Similarly, \( U_i \geq V_i, i = 1, 2, \cdots, m. \) This and Step 5 imply \( U_i = V_i, i = 1, 2, \cdots, m. \)

**Theorem 2.6.** Assume that

\[ r > \max \left\{ \limsup_{x \to \infty} \frac{\alpha_1(x)}{x}, \cdots, \limsup_{x \to \infty} \frac{\alpha_m(x)}{x} \right\}. \tag{19} \]

Let \( U_i, i = 1, 2, \cdots, m \) be a family of functions defined on \( \mathbb{R} \) with at most linear growth such that for each \( i \), \( U_i \) is a viscosity solution of

\[ \min\{rU_i - L_iU_i - \sum_{q=1}^{m} a_{qi}U_q - f_i, U_i - g_i\} = 0 \text{ on } \mathbb{R}. \]

Then \( U_i = V_i \) for \( i = 1, 2, \cdots, m. \)
Proof. 1. Since $U_i$ and $V_i$, $i = 1, 2, \cdots, m$, grow at most linearly, there is a positive constant $C_1$ such that
\[ \lim_{x \to \infty} \{ U_i(x) - V_i(x) - C_1|x| \} = -\infty, \ i = 1, 2, \cdots, m. \]

2. Thanks to (19), there are two positive constants $C_2$ and $K$ such that, for each $i = 1, 2, \cdots, m$,
\[ \min \{ r \tilde{\psi} - L_i \tilde{\psi}, \tilde{\psi} - g_i \} \geq 0 \text{ on } \mathbb{R} \setminus [-K, K], \]
where $\tilde{\psi}(x) := C_2|x|$ with $C_2 > C_1$.

Choose a function $\psi \in C^2(\mathbb{R})$ satisfying $\psi |_{\mathbb{R} \setminus [-K, K]} = \tilde{\psi}$. Then it follows that
\[ \min \{ r\psi - L_i \psi, \psi - g_i \} \geq 0 \text{ on } \mathbb{R}, \]
where $\psi(x) := C + \psi(x)$ for some constant $C$ large enough.

3. Define $V_i^\varepsilon := V_i + \varepsilon \psi$ for $\varepsilon \in (0, 1)$ and $i = 1, 2, \cdots, m$. Then $V_i^\varepsilon$ is a viscosity supersolution of
\[ \min \{ rV_i^\varepsilon - L_i V_i^\varepsilon - \sum_{q=1}^m a_{qj}V_q^\varepsilon - f_i, V_i^\varepsilon - g_i \} = 0 \text{ on } \mathbb{R}. \]

4. The remaining is to repeat Step 4 through Step 6 of the proof of Theorem 2.5.

Let us introduce some notations as follows.
\[
\mathcal{C}_i := \{ y : V_i(y) > g_i(y) \}, \\
\mathcal{S}_i := \{ y : V_i(y) = g_i(y) \}, \\
\mathcal{D}_i := \{ y : rg_i(y) - L_i g_i(y) - \sum_{q=1}^m a_{qj}g_q(y) - f_i(y) \geq 0 \}, \]
and
\[ \hat{V}_i(y) := \mathbb{E} \left[ \int_0^\infty \exp(-rt)f(X(t), Y(t))dt \middle| X(0) = e_i, Y(0) = y \right], \]
for each $i = 1, 2, \cdots, m$.

We consider the regularity of $V$ in the following theorem.

**Theorem 2.7.** The function $V_i$ is $C^2$ continuous on $\mathcal{C}_i$ and $C^1$ continuous on $\partial \mathcal{C}_i$, $i = 1, 2, \cdots, m$. 

Proof. It follows from $\beta(\cdot,\cdot) > 0$ that $\mathcal{L}_i$’s are locally elliptic. Then, thanks to Theorem 2.1, the proof is completed in the same way as that of (Pham, 2009, p. 100, Lemma 5.2.2, Proposition 5.2.1).

**Theorem 2.8.** The stopping time $\tau^* := \inf \{ t > 0 : (X(t), Y(t)) \in \bigcup_{i=1}^{m} e_i \times S_i \}$ is an optimal stopping time of the problem (1).

**Proof.** We refer to (Pham, 2009, pp. 101-102) for the proof.

Theorem 2.9 states some properties about $S_i$’s.

**Theorem 2.9.** Assume that (H1) holds. Then the following conclusions are true.

1. If $\bigcup_{i=1}^{m} S_i = \emptyset$, then $\hat{V}_i \geq g_i$ for each $i = 1, 2, \cdots, m$.
2. If $\hat{V}_i \geq g_i$ for each $i = 1, 2, \cdots, m$, then $V_i = \hat{V}_i$.
3. If $g_i$ is $C^2$ continuous for some $i \in \{1, 2, \cdots, m\}$, then $S_i$ is included in $D_i$.

**Proof.** By virtue of Theorem 2.5, (1) and (2) are proved in a similar way to that of (Pham, 2009, p. 102, Lemma 5.2.3); for the proof of (3), we apply Theorem 2.5 and refer to (Pham, 2009, p. 102, Lemma 5.2.4).

Now we study the forms of $S_i$’s. To do this, we make the following assumption.

(H2) The process $Y$ takes values in $(0, +\infty)$ and $|\alpha_i(y)| + |\beta_i(y)| \leq C|y|$ for some constant $C$, $i = 1, 2, \cdots, m$.

We will use the fact that the matrix $rI_{m \times m} - (a_{ij})_{m \times m}$ is invertible. This follows from that $rI_{m \times m} - (a_{ij})_{m \times m}$ is a strictly diagonally dominant matrix, since $r > 0$ and $\sum_{i=1}^{m} a_{ij} = 0$ for all $j = 1, 2, \cdots, m$. Let $(b_{ij})_{m \times m}$ denote the inverse of $rI_{m \times m} - (a_{ij})_{m \times m}$.

**Lemma 2.10.** Assume that (H2) holds. Then

$$V_i(0) := V_i(0^+) = \max \left\{ \sum_{i=1}^{m} b_{qi} f_q(0), g_i(0) \right\}, \quad i = 1, 2, \cdots, m.$$ 

**Proof.** Theorem 2.1 implies that $V_i(0)$ is a solution of

$$\min \{ rV_i(0) - \sum_{q=1}^{m} a_{qi} V_q(0) - f_i(0), V_i(0) - g_i(0) \} = 0 \text{ on } \mathbb{R}.$$
Consequently, \( V_i(0) = \max \{ \sum_{i=1}^{m} b_i q_i f_q(0), g_i(0) \} \). This completes the proof. □

**Theorem 2.11.** Assume that (H1) and (H2) hold, \( S_i \)'s are nonempty connected sets, and \( g_i \)'s are \( C^2 \) continuous. Then the following are true.

1. If \( D_i = [a_i, +\infty) \) for some positive constant \( a_i \), \( i = 1, 2, \ldots, m \), then \( S_i = [\inf S_i, +\infty) \).

2. If \( g_i(0) > \sum_{i=1}^{m} b_i q_i f_q(0) \) and \( D_i = (0, b_i] \) for some positive constant \( b_i \), \( i = 1, 2, \ldots, m \), then \( S_i = (0, \sup S_i] \).

**Proof.** Set \( a_i^* := \inf S_i \) in Case (1) and \( b_i^* := \sup S_i \) in Case (2). Then we define \( A_i := [a_i^*, +\infty) \) in Case (1) and \( A_i := (0, b_i^*] \) in Case (2). We will prove \( S_i = A_i \).

Note the following facts,

(i) \( g_i \)'s grow at most linearly;

(ii) \( g_i(a_i^*) = V_i(a_i^*) \) or \( g_i(b_i^*) = V_i(b_i^*) \) by the continuity of \( g_i \) and \( V_i \);

(iii) in light of Lemma 2.10, \( g_i(0) = V_i(0) \) if \( g_i(0) > \sum_{i=1}^{m} b_i q_i f_q(0) \) and \( D_i = (0, b_i] \).

Moreover, thanks to \( A_i \in D_i \) by Lemma 2.9, \( g_i \) solves the inequality

\[
rg_i - \mathcal{L}_i g_i - \sum_{q=1}^{m} a_{iq} g_q - f_i \geq 0 \text{ on } A_i.
\]

Thus, by Theorem 2.5, it follows that \( g_i = V_i \) on \( A_i \). The proof is complete. □

3 **An Application**

A firm is extracting a kind of natural resource (oil, gas, etc.). We will determine what time is optimal to stop the extraction.

Let \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) be a filtered probability space. We assume that \( \{\mathcal{F}_t\}_{t \geq 0} \) satisfies the usual conditions and \( \mathcal{F}_0 \) is the completion of \( \{\emptyset, \Omega\} \).

Let \( X := (X(t), t \geq 0) \) be a time homogeneous Markov chain defined on \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) taking values in the standard orthogonal basis of \( \mathbb{R}^2 \), \( I := \{e_1, e_2\} \), with a rate matrix \( A := \begin{pmatrix} -\lambda_1 & \lambda_2 \\ \lambda_1 & -\lambda_2 \end{pmatrix} \) for some positive constants \( \lambda_1, \lambda_2 \).
Let $B(t)$ be a one dimensional standard Brownian motion, which is independent of $X$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$.

We assume that the price process $P$ of the resource satisfies

$$dP(t) = \mu(X(t))P(t)dB(t) + \sigma(X(t))P(t)dB(t) \text{ and } P(0) > 0,$$

where $\mu(e_1) := \mu_1$, $\mu(e_2) := \mu_2$, $\sigma(e_1) := \sigma_1 > 0$ and $\sigma(e_2) := \sigma_2 > 0$. We will assume that $\mu_1 \leq \mu_2$.

Applying Itô’s formula, we deduce that the solution of Equation (20) is

$$P(t) = P(0) \exp \left[ \int_0^t \left( \mu(X(s)) - \frac{1}{2} \sigma(X(s))^2 \right) ds + \int_0^t \sigma(X(s))dB(s) \right]. \quad (21)$$

To answer the question— what time is optimal to stop the extraction, we will solve the following optimal problem,

$$J_i(p) := \sup_{\tau \in \mathcal{T}} \mathbb{E} \left[ \int_0^\tau \exp(-rt)(P(t) - C)dt \right] - \exp(-r\tau)K | X(0) = e_i, P(0) = p], \quad (22)$$

where $\mathcal{T}$ is the collection of all stopping times, $r$ is the discount rate, $C$ is the running cost rate with $C > 0$, and $K$ is the cost at which the firm stops the extraction.

**Lemma 3.1.** (Guo & Zhang, 2005, Lemma 1) we have

$$\mathbb{E} \left[ \exp \left( \int_0^t \mu(X(s))ds \right) | X(0) = e_i \right] = \frac{x_2 - \mu_1}{x_2 - x_1} \exp(x_1t) + \frac{\mu_1 - x_1}{x_2 - x_1} \exp(x_2t),$$

where $x_1$ and $x_2$ are the solutions of

$$x^2 + (\lambda_1 - \mu_1 + \lambda_2 - \mu_2)x + (\lambda_1 - \mu_1)(\lambda_2 - \mu_2) - \lambda_1\lambda_2 = 0$$

with $x_1 < x_2$.

**Remark 3.2.** If $\mu_1 < \mu_2$, then $x_1 < \mu_1 < x_2 < \mu_2$; if $\mu_1 = \mu_2$, then $x_1 = \mu_1 - \lambda_1 - \lambda_2$ and $x_2 = \mu_1$.

**Theorem 3.3.** Assume $r \leq x_2$. Then the optimal stopping time $\tau^*$ is given by $\tau^* = +\infty$ a.s., and the function $J_i$ in (22) is given by $J_i(p) = +\infty$, $i = 1, 2$. 

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Proof. By (21) and (22), we have, for fixed \( T > 0 \),

\[
J_i(p) \geq p \int_0^T \mathbb{E} \left[ \exp(-rt) \exp \left( \int_0^t \mu(X(s)) - \frac{1}{2} \sigma(X(s))^2 \right) ds \right. \\
+ \int_0^t \sigma(X(s)) dB(s) \left| X(0) = e_i \right. \] dt \\
- r^{-1}(1 - \exp(-rT))C - \exp(-rT)K.
\]

Note that

\[
\left( \exp \left( \int_0^t \sigma(X(s)) dB(s) - \frac{1}{2} \int_0^t \sigma(X(s))^2 ds \right) , t \geq 0 \right)
\]
is a martingale. Then in light of the independence of \( X \) and \( B \), we get

\[
J_i(p) \geq p \int_0^T \exp(-rt) \mathbb{E} \left[ \exp \left( \int_0^t \mu(X(s)) \right) ds \right| X(0) = e_i \] dt \\
- r^{-1}(1 - \exp(-rT))C - \exp(-rT)K \\
= p \left( \frac{x_2 - \mu_i}{x_2 - x_1} \int_0^T \exp((x_1 - r)t) dt + \frac{\mu_i - x_1}{x_2 - x_1} \int_0^T \exp((x_2 - r)t) dt \right) \\
- r^{-1}(1 - \exp(-rT))C - \exp(-rT)K,
\]

where we have used Lemma 3.1.

Sending \( T \to +\infty \) in the above inequality, we obtain \( J_i(p) = +\infty \) since \( r \leq x_2 \).

\[ \square \]

Corollary 3.4. If \( r \leq \mu_1 \), then the conclusions in Theorem 3.3 hold.

Proof. The proof can be completed by the fact that \( r \leq \mu_1 \) implies \( r \leq x_2 \).

In the following discussion, we consider the case \( r > x_2 \).

Lemma 3.5. (1) (Guo, 2001, Remark 2.1) The following equation

\[
w_1(z)w_2(z) - \lambda_1 \lambda_2 = 0,
\]

where \( w_i(z) := r + \lambda_i - \mu_i z - \frac{\sigma^2}{2} z (z - 1) \), has four distinct solutions \( z_i \)'s satisfying \( z_1 < z_2 < 0 < z_3 < z_4 \) with \( w_1(z_3) > 0 \).

(2) Assume that \( r > x_2 \). Then \( z_3 > 1 \) and there are positive constants \( b_i \)'s fulfilling \( b_1 w_1(z_3) - \lambda_1 b_2 = 0 \) and \( b_2 w_2(z_3) - \lambda_2 b_1 = 0 \). See Figure 1.
Figure 1: Solutions of $w_1(z)w_2(z) - \lambda_1\lambda_2 = 0$.

Proof. 1. Set $Q(z) := w_1(z)w_2(z) - \lambda_1\lambda_2$. Note that the equation $w_1(z) = 0$ has two solutions, say, $z^1$ and $z^2$ with $z^1 < z^2$. Then $Q(z^i) < 0$, where $i = 1, 2$. In addition, we have $Q(0) > 0$ and $\lim_{z \to \infty} Q(z) = +\infty$. Therefore, by the intermediate value theorem, it follows that the equation $w_1(z)w_2(z) - \lambda_1\lambda_2 = 0$ has four distinct solutions $z^i$’s satisfying $z^1 < z^2 < 0 < z^3 < z^4$. Furthermore, $w_1(z^3) > 0$, since $z^1 < z^3 < z^2$.

2. Since $r > x_2$, we have

$$r^2 + (\lambda_1 - \mu_1 + \lambda_2 - \mu_2)r + (\lambda_1 - \mu_1)(\lambda_2 - \mu_2) - \lambda_1\lambda_2 > 0 \quad (23)$$

and

$$2r + \lambda_1 - \mu_1 + \lambda_2 - \mu_2 > 0. \quad (24)$$

Then it follows from (23) that $Q(1) > 0$ and $(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) > 0$. Combining (24) and $(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) > 0$, we have $r + \lambda_1 - \mu_1 > 0$ and $r + \lambda_2 - \mu_2 > 0$.

3. In this step, we prove $z_3 > 1$.

Case 1 $\mu_i + \sigma_i^2/2 > 0$, where $i = 1, 2$.

In this case, we have

$$Q'(1) = -(r + \lambda_1 - \mu_1)(\mu_2 + \sigma_2^2/2) - (r + \lambda_2 - \mu_2)(\mu_1 + \sigma_1^2/2) < 0.$$  

This and $Q(1) > 0$ imply $z_3 > 1$.

Case 2 $\mu_i + \sigma_i^2/2 < 0$, where $i = 1, 2$.

In this case, we have

$$Q''(1) = 3\sigma_1^2\sigma_2^2 + 3(\sigma_1^2\mu_2 + \sigma_2^2\mu_1) < 0.$$  

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This and \( Q(1) > 0 \) imply \( z_3 > 1 \).

**Case 3** \( (\mu_1 + \sigma^2_1/2)(\mu_2 + \sigma^2_2/2) \leq 0 \).

In this case, we have

\[
Q''(1) = -\sigma^2_1(r + \lambda_2 - \mu_2) + 2(\mu_1 + \sigma^2_1)(\mu_2 + \sigma^2_2) - \sigma^2_1(r + \lambda_2 - \mu_2) < 0.
\]

This and \( Q(1) > 0 \) imply \( z_3 > 1 \).

4. Take \( b_1 = 1 \) and \( b_2 = w_1(z_3)/\lambda_1 \). Then we have \( b_1 w_1(z_3) - \lambda_1 b_2 = 0 \) and \( b_2 w_2(z_3) - \lambda_2 b_1 = 0 \).

**Lemma 3.6.** Assume that \( r > x_2 \). Then \( J_i \) is a Lipschitz continuous function, \( i = 1, 2 \). In addition, \( S_i := \{ p : p \in (0, +\infty) \) and \( J_i(p) = -K \} \) is connected, \( i = 1, 2 \).

**Proof.** 1. By (21) and (22), we have

\[
|J_i(p_1) - J_i(p_2)| \leq |p_1 - p_2| \int_0^{+\infty} \mathbb{E} \left[ \exp(-rt) \exp \left( \int_0^t \mu(X(s)) - \frac{1}{2} \sigma(X(s))^2 ds \right) \right] dt.
\]

Note that

\[
\left( \exp \left( \int_0^t \sigma(X(s)) dB(s) - \frac{1}{2} \int_0^t \sigma(X(s))^2 ds \right) , t \geq 0 \right)
\]

is a martingale. Then in light of the independence of \( X \) and \( B \), we get

\[
|J_i(p_1) - J_i(p_2)| \leq |p_1 - p_2| \int_0^{+\infty} \mathbb{E} \left[ \exp \left( \int_0^t \mu(X(s)) ds \right) |X(0) = e_i| \right] dt
\]

where we have used Lemma 3.1 and \( r > x_2 \) for the equality.

2. We prove that \( S_i := \{ p : p \in (0, +\infty) \) and \( J_i(p) = -K \} \) is connected by contradiction. Suppose that \( a, b \in S_i \) but there is a point \( c \in (a, b) \) with \( c \notin S_i \) (i.e., \( J_i(c) > -K \)).

Noting that \( J_i \) is convex, we have

\[
J_i(c) \leq \frac{b - c}{b - a} J_i(a) + \frac{c - a}{b - a} J_i(b) = -K,
\]

which contradicts \( J_i(c) > -K \). \( \square \)
Theorem 3.7. Assume $r > x_2$. Then $\{J_1, J_2\}$ is the unique solution with at most linear growth of the following variational inequalities

$$
\begin{align*}
\min \{ (r + \lambda_1)J_1(p) - \mu_1 p J'_1(p) - \frac{1}{2} \sigma_1^2 p^2 J''_1(p) - \lambda_1 J_2(p) - p + C, J_1(p) + K \} &= 0 \\
\min \{ (r + \lambda_2)J_2(p) - \mu_2 p J'_2(p) - \frac{1}{2} \sigma_2^2 p^2 J''_2(p) - \lambda_2 J_1(p) - p + C, J_2(p) + K \} &= 0
\end{align*}
$$
on $(0, +\infty)$ in viscosity sense.

Proof. By Lemma 3.5 and Lemma 3.6, the result is a straight corollary of Theorem 2.5. \qed

Theorem 3.8. Assume that $r > x_2$ and $C \leq rK$. Then the optimal stopping time $\tau^*$ is given by $\tau^* = +\infty$ a.s.; furthermore, the functions $J_i$'s are given by

$$
J_1(p) = k_2 p - r^{-1} C, \quad J_2(p) = k_1 p - r^{-1} C,
$$

where

$$
k_i = \frac{r + \lambda_1 + \lambda_2 - \mu_i}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1 \lambda_2}, \quad i = 1, 2.
$$

Proof. Note that, for $i = 1, 2$,

$$
S_i \subset \{ p : 0 < p \leq C - rK \}
$$

by Theorem 2.9. Thus we have $S_i = \emptyset$.

Therefore, it follows from Theorem 3.7 that

$$
J_1(p) = \frac{(r + \lambda_1 + \lambda_2 - \mu_2)p}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1 \lambda_2} - \frac{C}{r}
$$

and

$$
J_2(p) = \frac{(r + \lambda_1 + \lambda_2 - \mu_1)p}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1 \lambda_2} - \frac{C}{r}.
$$

The proof is complete. \qed

Corollary 3.9. If $r > \mu_2$ and $C \leq rK$, then the conclusions in Theorem 3.8 hold.

Proof. The proof can be completed by the fact that $r > \mu_2$ implies $r > x_2$. \qed

Recall that $z_1$ and $z_2$ are the solutions of $w_1(z)w_2(z) - \lambda_1 \lambda_2 = 0$ with $z_1 < z_2 < 0$ (see Lemma 3.5).
Theorem 3.10. Assume that \( r > x_2 \) and \( C > rK \). Then one and only one of the following holds.

(1) The equation

\[
\begin{bmatrix}
p_{1}^{-y_1} & 0 \\
0 & p_{1}^{-y_2}
\end{bmatrix}
\begin{bmatrix}
a_{11} p_{1} + b_{11} \\
a_{21} p_{1} + b_{21}
\end{bmatrix} =
\begin{bmatrix}
p_{2}^{-y_1} & 0 \\
0 & p_{2}^{-y_2}
\end{bmatrix}
\begin{bmatrix}
a_{12} p_{2} + b_{12} \\
a_{22} p_{2} + b_{22}
\end{bmatrix}
\]

has a solution \( \{p_1^*, p_2^*\} \) with \( p_1^* < p_2^* \), where \( y_1 \) and \( y_2 \) are the two solutions of the following quadratic equation

\[
r + \lambda_1 - \mu_1 y - \frac{1}{2} \sigma_1^2 y(y - 1) = 0
\]

with \( y_1 < y_2 \),

\[
\begin{bmatrix}
a_{11} \\
a_{21}
\end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix}
\frac{-1}{r + \lambda_1 - \mu_1} \\
\frac{1}{r + \lambda_1 - \mu_1}
\end{bmatrix},
\quad
\begin{bmatrix}
b_{11} \\
b_{12}
\end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix}
\frac{C - rK}{r + \lambda_1} \\
0
\end{bmatrix},
\quad
\begin{bmatrix}
a_{12} \\
a_{22}
\end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix}
k_2 - \frac{1}{r + \lambda_1 - \mu_1} \\
\frac{1}{r + \lambda_1 - \mu_1}
\end{bmatrix} \right)
\begin{bmatrix}
\lambda_1 \\
z_1
\end{bmatrix}
\begin{bmatrix}
w_1(z_1) & w_1(z_2) \\
w_1(z_1)z_1 & w_1(z_2)z_2
\end{bmatrix}^{-1}
\begin{bmatrix}
k_1
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
b_{12} \\
b_{22}
\end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix}
\frac{\lambda_1 K + C}{r + \lambda_1} - \frac{C}{r} \\
0
\end{bmatrix}
\right)
\begin{bmatrix}
\lambda_1 \\
z_1
\end{bmatrix}
\begin{bmatrix}
w_1(z_1) & w_1(z_2) \\
w_1(z_1)z_1 & w_1(z_2)z_2
\end{bmatrix}^{-1}
\begin{bmatrix}
k_1
\end{bmatrix}.
\]

In addition, the optimal stopping time \( \tau^* \) is given by

\[
\tau^* = \inf\{t : t > 0, (X(t), P(t)) \in (e_1) \times (0, p_1^*) \cup (e_2) \times (0, p_2^*)\},
\]

and the \( J_i \)'s are given by

\[
J_1(p) = \begin{cases}
-\frac{K}{p} & \text{on } (0, p_1^*) \\
\frac{\lambda_1 K + C}{r + \lambda_1} + A_1 p + A_2 p^2 & \text{on } (p_1^*, p_2^*) \\
k_2 p - r^{-1} C + B_1 p + B_2 p^2 & \text{on } (p_2^*, +\infty)
\end{cases}
\]

and

\[
J_2(p) = \begin{cases}
-\frac{K}{p} & \text{on } (0, p_1^*) \\
k_1 p - r^{-1} C + \lambda_1^{-1} w_1(z_1) B_1 p + \lambda_1^{-1} w_1(z_2) B_2 p^2 & \text{on } (p_1^*, +\infty),
\end{cases}
\]
respectively. Here
\[
\begin{bmatrix}
A_1 \\
A_2
\end{bmatrix} = \begin{bmatrix}
p_1^{-y_1} & 0 \\
0 & p_1^{-y_2}
\end{bmatrix}
\begin{bmatrix}
a_{11}p_1^* + b_{11} \\
a_{21}p_2^* + b_{21}
\end{bmatrix},
\]
\[
\begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} = \lambda_1 \begin{bmatrix}
p_2^{-z_1} & 0 \\
0 & p_2^{-z_2}
\end{bmatrix}
\begin{bmatrix}
w_1(z_1) & w_1(z_2) \\
w_1(z_1)z_1 & w_1(z_2)z_2
\end{bmatrix}^{-1}
\begin{bmatrix}
-k_1p_2^* + C-rK \\
-k_1p_2^* \lambda
\end{bmatrix}.
\]
(2) The equation
\[
\begin{bmatrix}
p_2^{-\varphi_1} & 0 \\
0 & p_2^{-\varphi_2}
\end{bmatrix}
\begin{bmatrix}
\varphi_{11}p_2 + \varphi_{11} \\
\varphi_{21}p_2 + \varphi_{21}
\end{bmatrix} = \begin{bmatrix}
p_1^{-\varphi_1} & 0 \\
0 & p_1^{-\varphi_2}
\end{bmatrix}
\begin{bmatrix}
\varphi_{12}p_2 + \varphi_{12} \\
\varphi_{22}p_2 + \varphi_{22}
\end{bmatrix}
\]
has a solution \(\{\varphi_1, \varphi_2\}\) with \(\varphi_1 > \varphi_2\), where \(\varphi_1\) and \(\varphi_2\) are the two solutions of the following quadratic equation
\[
r + \lambda_2 - \mu_2y - \frac{1}{2}\sigma_2^2y(y - 1) = 0
\]
with \(\varphi_1 < \varphi_2\),
\[
\begin{bmatrix}
\varphi_{11} \\
\varphi_{21}
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
\varphi_1 & \varphi_2
\end{bmatrix}^{-1}\begin{bmatrix}
\frac{-1}{r + \lambda_2 - \mu_2} \\
\frac{-1}{\varphi_1 + \lambda_2 - \mu_2}
\end{bmatrix},
\begin{bmatrix}
\varphi_{12} \\
\varphi_{22}
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
\varphi_1 & \varphi_2
\end{bmatrix}^{-1}\begin{bmatrix}
k_1 - \frac{1}{\varphi_1 + \lambda_2 - \mu_2} \\
k_1 - \frac{1}{r + \lambda_2 - \mu_2}
\end{bmatrix} - \lambda_2 \begin{bmatrix}
1 & 1 \\
z_1 & z_2
\end{bmatrix}
\begin{bmatrix}
w_2(z_1) & w_2(z_2) \\
w_2(z_1)z_1 & w_2(z_2)z_2
\end{bmatrix}^{-1}
\begin{bmatrix}
k_2 \\
k_2
\end{bmatrix},
\]
and
\[
\begin{bmatrix}
\varphi_{12} \\
\varphi_{22}
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
\varphi_1 & \varphi_2
\end{bmatrix}^{-1}\begin{bmatrix}
\frac{\lambda_2K + C}{r + \lambda_2} - \frac{C}{r} \\
0
\end{bmatrix}
+ \lambda_2 \begin{bmatrix}
1 & 1 \\
z_1 & z_2
\end{bmatrix}
\begin{bmatrix}
w_2(z_1) & w_2(z_2) \\
w_2(z_1)z_1 & w_2(z_2)z_2
\end{bmatrix}^{-1}\begin{bmatrix}
\frac{C-rK}{r} \\
0
\end{bmatrix}.
\]
In addition, the optimal stopping time \(\tau^*\) is given by
\[
\tau^* = \inf\{t : t > 0, (X(t), P(t)) \in \{e_1\} \times (0, \varphi_1) \cup \{e_2\} \times (0, \varphi_2)\},
\]
and the \(J_i\)'s are given by
\[
J_i(p) = \begin{cases} 
-K & \text{on } (0, \varphi_1] \\
-k_2p - r^{-1}C + \lambda_2^{-1}w_2(z_1)B_1p^{z_1} + \lambda_2^{-1}w_2(z_2)B_2p^{z_2} & \text{on } (\varphi_1, +\infty)
\end{cases}
\]
respectively. Here

\[
\begin{bmatrix}
\tilde{A}_1 \\
\tilde{A}_2
\end{bmatrix} = \begin{bmatrix}
\tilde{p}_2^{-y_1} & 0 \\
0 & \tilde{p}_2^{-y_2}
\end{bmatrix}\begin{bmatrix}
\tilde{a}_{11} \tilde{p}_1 + \tilde{b}_{11} \\
\tilde{a}_{21} \tilde{p}_1 + \tilde{b}_{21}
\end{bmatrix},
\]

\[
\begin{bmatrix}
\tilde{B}_1 \\
\tilde{B}_2
\end{bmatrix} = \lambda_2 \begin{bmatrix}
\tilde{p}_2^{-z_1} & 0 \\
0 & \tilde{p}_2^{-z_2}
\end{bmatrix}\begin{bmatrix}
w_2(z_1) & w_2(z_2) \\
w_2(z_1)z_1 & w_2(z_2)z_2
\end{bmatrix}^{-1}\begin{bmatrix}
-k_2 \tilde{p}_1 + C-rK \\
-k_2 \tilde{p}_1
\end{bmatrix}.
\]

(3) The equation

\[
\begin{bmatrix}
\tilde{a}_{11}p + \tilde{b}_{11} \\
\tilde{a}_{21}p + \tilde{b}_{21}
\end{bmatrix} = \begin{bmatrix}
\tilde{a}_{12}p + \tilde{b}_{12} \\
\tilde{a}_{22}p + \tilde{b}_{22}
\end{bmatrix}
\]

has a positive solution \(p^*\), where

\[
\begin{bmatrix}
\tilde{a}_{11} \\
\tilde{a}_{12}
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
z_1 & z_2
\end{bmatrix}^{-1}\begin{bmatrix}
-k_2 \\
-k_2
\end{bmatrix}, \quad \begin{bmatrix}
\tilde{b}_{11} \\
\tilde{b}_{12}
\end{bmatrix} = \begin{bmatrix}
1 & 1 \\
z_1 & z_2
\end{bmatrix}^{-1}\begin{bmatrix}
C-rK \\
0
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
\tilde{a}_{12} \\
\tilde{a}_{22}
\end{bmatrix} = \lambda_1 \begin{bmatrix}
w_1(z_1) & w_1(z_2) \\
w_1(z_1)z_1 & w_1(z_2)z_2
\end{bmatrix}^{-1}\begin{bmatrix}
-k_1 \\
-k_1
\end{bmatrix}, \\
\begin{bmatrix}
\tilde{b}_{21} \\
\tilde{b}_{22}
\end{bmatrix} = \lambda_1 \begin{bmatrix}
w_1(z_1) & w_1(z_2) \\
w_1(z_1)z_1 & w_1(z_2)z_2
\end{bmatrix}^{-1}\begin{bmatrix}
C-rK \\
0
\end{bmatrix}.
\]

In addition, the optimal stopping time \(\tau^*\) is given by

\[
\tau^* = \inf \{ t : t > 0, P(t) \in (0, p^*) \},
\]

and the \(J_i\)'s are given by

\[
J_1(p) = \begin{cases}
-K & \text{on } (0, p^*], \\
k_2 p - r^{-1}C + \tilde{B}_1 p^{-z_1} + \tilde{B}_2 p^{-z_2} & \text{on } (p^*, +\infty)
\end{cases}
\]

and

\[
J_2(p) = \begin{cases}
-K & \text{on } (0, p^*], \\
k_1 p - r^{-1}C + \lambda_1^{-1} w_1(z_1) \tilde{B}_1 p^{-z_1} + \lambda_1^{-1} w_1(z_2) \tilde{B}_2 p^{-z_2} & \text{on } (p^*, +\infty),
\end{cases}
\]

respectively. Here

\[
\begin{bmatrix}
\tilde{B}_1 \\
\tilde{B}_2
\end{bmatrix} = \begin{bmatrix}
p^{*-z_1} & 0 \\
0 & p^{*-z_2}
\end{bmatrix}\begin{bmatrix}
\tilde{a}_{11}p^* + \tilde{b}_{11} \\
\tilde{a}_{21}p^* + \tilde{b}_{21}
\end{bmatrix}.
\]
Proof. 1. Since \(C > rK\), it follows from Lemma 2.10 that \(J_i(0^+) = -K\) and \(S_i \neq \emptyset\). In addition, by Lemma 2.9, we have \(S_i \subset (0, C - rK]\). Therefore, by Theorem 2.11 and Lemma 3.6, it follows that \(S_i = (0, p_1^*]\) for some positive number \(p_1^*\).

2. Case 1 \(p_1^* < p_2^*\). In this case, by Theorem 3.7, we have

\[
\begin{aligned}
J_1(p) &= -K \\
J_2(p) &= -K
\end{aligned}
\]

on \((0, p_1^*]\),

\[
\begin{aligned}
(r + \lambda_1)J_1(p) - \mu_1 p J_1'(p) - \frac{1}{2} \sigma_1^2 p^2 J_1''(p) - \lambda_1 J_2(p) - p + C &= 0 \\
J_2(p) &= -K
\end{aligned}
\]

(25)
on \((p_1^*, p_2^*],\) and

\[
\begin{aligned}
(r + \lambda_1)J_1(p) - \mu_1 p J_1'(p) - \frac{1}{2} \sigma_2^2 p^2 J_1''(p) - \lambda_1 J_2(p) - p + C &= 0 \\
(r + \lambda_2)J_2(p) - \mu_2 p J_2'(p) - \frac{1}{2} \sigma_2^2 p^2 J_2''(p) - \lambda_2 J_1(p) - p + C &= 0
\end{aligned}
\]

(26)
on \((p_2^*, +\infty)\).

Note that \(r + \lambda_1 - \mu_1 > 0\) by Step 2 of the proof of Lemma 3.5. We have by (25)

\[
J_1(p) = \frac{p}{r + \lambda_1 - \mu_1} - \frac{\lambda_1 K + C}{r + \lambda_1} + A_1 p^{\rho_1} + A_2 p^{\rho_2} \text{ on } (p_1^*, p_2^*],
\]

where \(A_1\) and \(A_2\) are two constants, and \(y_1 \) and \(y_2\) are the two solutions of the following quadratic equation

\[
r + \lambda_1 - \mu_1 y - \frac{1}{2} \sigma_1^2 y(y - 1) = 0
\]

with \(y_1 < y_2\).

Since \(J_1\) and \(J_2\) are Lipschitz continuous, we have by (26)

\[
\begin{aligned}
J_1(p) &= \frac{(r + \lambda_1 + \lambda_2 - \mu_2)p}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1 \lambda_2} - \frac{C}{r} + B_1 p^{\gamma_1} + B_2 p^{\gamma_2} \\
J_2(p) &= \frac{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1 \lambda_2}{r} - \frac{\lambda_2}{r} + \lambda_1^{-1} w_1(z_1) B_1 p^{\gamma_1} + \lambda_1^{-1} w_1(z_2) B_2 p^{\gamma_2}
\end{aligned}
\]

on \((p_2^*, +\infty)\), where \(B_1\) and \(B_2\) are some constants, and \(z_1\) and \(z_2\) are the solutions introduced in Lemma 3.5.
Therefore, by $C^1$ continuity of $J_1$ and $J_2$, we have

$$\begin{align*}
\begin{cases}
\frac{p_1^*}{r + \lambda_1 - \mu_1} - \frac{\lambda_1 K + C}{r + \lambda_1} + A_1 p_1^{y_1} + A_2 p_1^{y_2} = -K \\
\frac{1}{r + \lambda_1 - \mu_1} + A_1 y_1 p_1^{y_1 - 1} + A_2 y_2 p_1^{y_2 - 1} = 0
\end{cases}
\end{align*} \tag{27}$$

and

$$\begin{align*}
\begin{cases}
\frac{p_2^*}{r + \lambda_1 - \mu_1} - \frac{\lambda_1 K + C}{r + \lambda_1} + A_1 p_2^{y_1} + A_2 p_2^{y_2} \\
\frac{1}{r + \lambda_1 - \mu_1} + A_1 y_1 p_2^{y_1 - 1} + A_2 y_2 p_2^{y_2 - 1} = C
\end{cases}
\end{align*} \tag{28}$$

and

$$\begin{align*}
\begin{cases}
\frac{(r + \lambda_1 + \lambda_2 - \mu_1) p_2^*}{r + \lambda_1 - \mu_1} - \frac{C}{r + \lambda_1} \\
\frac{(r + \lambda_1 + \lambda_2 - \mu_1) (r + \lambda_2 - \mu_2)}{r + \lambda_1 - \mu_1} - \frac{C}{r + \lambda_1} + \lambda_1^{-1} w_1(z_1) B_1 p_2^{x_2} + \lambda_1^{-1} w_1(z_2) B_2 p_2^{x_2} = -K
\end{cases}
\end{align*} \tag{29}$$

By solving $A_1$ and $A_2$ from (27) and solving $A_1$ and $A_2$ from (28) and (29), we get

$$\begin{bmatrix}
p_1^{y_1} & 0 \\
0 & p_2^{y_2}
\end{bmatrix}
\begin{bmatrix}
a_{11} p_1^* + b_{11} \\
0 & a_{21} p_1^* + b_{21}
\end{bmatrix}
= 
\begin{bmatrix}
p_2^{y_1} & 0 \\
0 & p_2^{y_2}
\end{bmatrix}
\begin{bmatrix}
a_{12} p_2^* + b_{12} \\
0 & a_{22} p_2^* + b_{22}
\end{bmatrix}.$$  

Case 2 $p_1^* > p_2^*$. This case is similar to Case 1.

Case 3 $p_1^* = p_2^* =: p^*$. By Theorem 3.7, we have

$$\begin{align*}
\begin{cases}
J_1(p) = -K \\
J_2(p) = -K
\end{cases}
\end{align*}$$
on $(0, p^*]$, and

$$\begin{align*}
\begin{cases}
(r + \lambda_1) J_1(p) - \mu_1 p J_1'(p) - \frac{1}{2} \sigma_1^2 p^2 J_1''(p) - \lambda_1 J_2(p) - p + C = 0 \\
(r + \lambda_2) J_2(p) - \mu_2 p J_2'(p) - \frac{1}{2} \sigma_2^2 p^2 J_2''(p) - \lambda_2 J_1(p) - p + C = 0
\end{cases}
\end{align*} \tag{30}$$
on $(p^*, +\infty)$.

Since $J_1$ and $J_2$ are Lipschitz continuous, we have by (30)

$$\begin{align*}
J_1(p) &= \frac{(r + \lambda_1 + \lambda_2 - \mu_2) p}{r + \lambda_1 - \mu_1} - \frac{C}{r + \lambda_1} + B_1 p^{x_1} + B_2 p^{x_2} \\
J_2(p) &= \frac{(r + \lambda_1 + \lambda_2 - \mu_1) p}{r + \lambda_1 - \mu_1} - \frac{C}{r + \lambda_1} + A_1 p^{x_1} + A_2 p^{x_2}
\end{align*}$$
on \((p^*, +\infty)\), where \(\tilde{B}_1\) and \(\tilde{B}_2\) are some constants, and \(z_1\) and \(z_2\) are the solutions introduced in Lemma 3.5.

Therefore, by \(C^1\) continuity of \(J_1\) and \(J_2\), we have

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{(r + \lambda_1 + \lambda_2 - \mu_2)p^*}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2} - \frac{C}{r} + \tilde{B}_1p^{z_1} + \tilde{B}_2p^{z_2} = -K \\
\frac{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2} + \tilde{B}_1z_1p^{*z_1} + \tilde{B}_2z_2p^{*z_2} = 0,
\end{array} \right. \\
\end{align*}
\]

and

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{(r + \lambda_1 + \lambda_2 - \mu_1)p^*}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2} - \frac{C}{r} + \tilde{B}_1p^{z_1} + \tilde{B}_2p^{z_2} = -K \\
\frac{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2}{(r + \lambda_1 - \mu_1)(r + \lambda_2 - \mu_2) - \lambda_1\lambda_2} + \tilde{B}_1z_1p^{*z_1} + \tilde{B}_2z_2p^{*z_2} = 0.
\end{array} \right. \\
\end{align*}
\]

By solving \(\tilde{B}_1\) and \(\tilde{B}_2\) from (31) and solving \(\tilde{B}_1\) and \(\tilde{B}_2\) from (32), we get

\[
\begin{bmatrix}
\tilde{a}_{11}p^* + \tilde{b}_{11} \\
\tilde{a}_{21}p^* + \tilde{b}_{21}
\end{bmatrix} = \begin{bmatrix}
\tilde{a}_{12}p^* + \tilde{b}_{12} \\
\tilde{a}_{22}p^* + \tilde{b}_{22}
\end{bmatrix}.
\]

The proof is complete. \(\square\)

**Corollary 3.11.** If \(r > \mu_2\) and \(C > rK\), then the conclusions in Theorem 3.10 hold.

**Proof.** The proof can be completed by the fact that \(r > \mu_2\) implies \(r > x_2\). \(\square\)

**Example 3.12.** Take \(\mu_1 = 0.01\), \(\mu_2 = 0.10\), \(\sigma_1 = \sigma_2 = 0.25\), \(r = 0.08\), \(\lambda_1 = \lambda_2 = 0.05\), \(C = 20\), \(K = 5\). Then we have \(r = 0.10 > 0.0723 = x_2\) and \(C = 20 > 0.40 = rK\). Thus we apply Theorem 3.10, and find that (2) of Theorem 3.10 gives us \((p_1^*, p_2^*) = (2.08, 1.04)\) and the optimal stopping time \(\tau^* = \inf\{t : t > 0, (X(t), P(t)) \in \{e_1\} \times (0, 0.08] \cup \{e_2\} \times (0, 1.04]\}\).

It is interesting to compare regime-switching cases with no regime-switching cases. If there is no regime switching and the price \(P\) satisfies

\[dP(t) = 0.01P(t)dt + 0.25P(t)dB(t),\]

the optimal stopping time is \(\inf\{t : t > 0, P(t) \in (0, 12.73]\}\). However, if the price \(P\) satisfies

\[dP(t) = 0.10P(t)dt + 0.25P(t)dB(t),\]

26
the firm should never stop the extraction since \( r = 0.08 < 0.10 \).

In summary, the firm may stop the extraction even though it should never stop the extraction in one of regimes.

4 Conclusions

Regime-switching processes are introduced to describe the price of financial assets and commodities (e.g., Casassus et al., 2005; Chernova et al., 2003), the stochastic behavior of temperature (Elias et al., 2014), and so on. Under the assumption that underlying processes are modeled by some regime-switching processes, Guo and Zhang (2004) derive an explicit closed solution for perpetual American options, Bae et al. (2014) investigate dynamic asset allocation among diverse financial markets, and Elias et al. (2014) valuate temperature-based weather options. These motivate us to study in a uniform way the optimal stopping problems in which underlying processes and payoff functions are modulated by Markov chains.

In this paper, we employ the viscosity solution technique to analyze optimal stopping problems with regime switching. Specifically, we first prove the value function is a viscosity solution of some variational inequalities; next, we obtain the uniqueness of the viscosity solution of the variational inequalities; then, we study the regularity of the value function and the form of optimal stopping intervals.

In Section 3, we provide an application of the results obtained in Section 2. At the end of the paper, a numerical example is demonstrated. From the example, we come to a conclusion that a firm may stop a project even though it should never stop the project in one of regimes.

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