Economics Mapping to the Renormalization Group
Scaling of Stock Markets

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We make an attempt to map a simple economically motivated model for
the price evolution [J. Phys. A: Gen. Math 33, 3637 (2000)] to the phe-
nomenological renormalization group scaling of stock markets. This mapping
gives insight into the critical exponents and the renormalization group pre-
dictions for the log-periodic oscillations preceding some stock market crashes
from the perspective of non-linear changes in ‘the level of stocks’.

I. INTRODUCTION

Several papers have appeared in recent years showing increasing evidence that at least
some market crashes are often anticipated by a power law behaviour of the stock market
index which fluctuates with oscillations that are periodic in the logarithm of the time to
-crash (see, for example, [1-8]). From these observations, it has been argued that there is a
close relation between the stock market crashes and the renormalization group (RG) theory
[1]. Precursory logarithm- (log-)periodic patterns can also emerge from percolation models
by applying the cluster concept to groups of investors acting collectively [7,8].

Though the RG approach has shown to model remarkably well the stock market time
evolution and to predict the existence of a large price crash, a possible universality for
the real exponents quantifying the observed behaviour in the market prices -which would
allow to define a crash- has not been yet established [2]. Unlike in the case of systems
in thermodynamic equilibrium, there is no known underlying Hamiltonian from which RG
critical exponents could be deduced. In this paper we propose a simplified dynamics for the
price evolution and made an attempt to map this dynamics to the RG predictions. We show
how the simplest, non-linear economics model proposed by the Author [9] may be mapped
into the non-linear RG scaling of stock markets in order to understand the critical exponents
in terms of relevant economics variables such as the demand and supply of a commodity (or
product). As an illustrative example, we apply the mapping to the NY Standard & Poor
(S&P)500 index crash of Oct. 1987.

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II. NON-LINEAR RENORMALIZATION GROUP GENERALIZATION

In analogy with RG theory, it is assumed that the temporal variation of the stock market index \( I(t) \) is related to future events at \( t' \) by the transformations [1,2]

\[
x' = \phi(x) ,
\]

\[
F(x) \equiv I(t_c) - I(t) = g(x) + \frac{1}{\mu} F(\phi(x)) ,
\]

where \( x = t_c - t \). \( \phi \) is called the RG flow map and \( \mu \) is a constant describing the scaling of \( I \) upon the rescaling of \( t \) in Eq.(1). \( F = 0 \) at the critical point \( t_c \) (the time of a large crash), \( g(x) \) represents the non-singular part of the function \( F(x) \), which is assumed to be continuous, and \( \phi(x) \) is assumed to be differentiable.

An extension of these results to a more general RG approach begins by considering that the solution of the RG Eq.(2), in conjunction with Eq.(1) and the linear approximation \( \phi(x) = \hat{\lambda}x \), can be rewritten as

\[
\frac{dF(x)}{d\log x} = \alpha F(x) ,
\]

This defines limiting power laws as \( t \to t_c \). Eq.(3) is then extended to include corrections to the power law with log-periodicity by introducing the amplitude \( B \) and phase \( \psi \) of \( F(x) = Be^{i\psi(x)} \). The symmetry law used is the phase shift that should keep the observable constant under a change of units [2]. This leads to postulate the following Landau expansion [10]:

\[
\frac{dF(x)}{d\log x} = (\alpha + i\omega)F(x) + (\eta + i\kappa)|F(x)|^2F(x) + O(F^5) .
\]

where \( \alpha > 0, \omega, \eta \) and \( \kappa \) are real coefficients and \( O(F^5) \) represents higher order terms to be neglected.

In terms of the amplitude and phase of \( F \), Eq.(4) yields

\[
\frac{\partial B}{\partial \log x} = \alpha B + \eta B^3 + \cdots ; \quad \frac{\partial \psi}{\partial \log x} = \omega + \kappa B^2 + \cdots
\]

whose solutions are found to be

\[
B^2 = \left( \frac{x}{x_0} \right)^{2\alpha} \frac{B^2_{\infty}}{1 + \left( \frac{x}{x_0} \right)^{2\alpha}} ,
\]
ψ = ω \log\left(\frac{x}{x_0}\right) + \frac{\Delta \omega}{2\alpha} \log\left(1 + \left(\frac{x}{x_0}\right)^{2\alpha}\right), \quad (7)

where \( B^2_\infty = \alpha/|\eta| \), \( \Delta \omega = B^2_\infty \kappa \) and \( x_0 \) is an arbitrary coefficient characterizing the time scale. These general forms lead to the following solutions of the non-linear RG Eq.(4):

\[ I(\tau) = \sum A_i \left( \frac{\tau_c - \tau}{\Delta t} \right)^{\alpha} \times \left[ 1 + A_3 \cos\left( \omega \log\left(\frac{\tau_c - \tau}{\Delta t}\right) + \frac{\Delta \omega}{2\alpha} \log\left(1 + \left(\frac{\tau_c - \tau}{\Delta t}\right)^{2\alpha}\right) \right) \right], \quad (8)\]

where \( \tau = t/\phi \), \( \Delta t = x_0 \) and \( A_i = 1, 2, 3 \) are linear variables.

### III. NON-LINEAR ECONOMICS MODEL

Within our economics model \[4\], only one stock of the commodity is assumed and the market is considered competitive so it self-organizes to determine the behaviour of the asset price \( p \). We derive a dynamical price equation which results from the prevailing market conditions in terms of the excess demand function \( E(p) = D(p) - Q(p) \), where \( D \) and \( Q \) are the demand and supply functions, respectively. In our description, asterisks \( (*) \) denote quantities in equilibrium and all variables are dimensionless.

In a competitive market the rate of price increase usually is a functional of \( E(p) \) such that \( dp/dt \equiv f[E(p)] \). Considering that in general a commodity can be stored, then stocks of the commodity build up when the flow of output exceeds the flow of demand and vice-versa. The rate at which ‘the level of stocks’ \( S \) changes can be approximated as \( dS/dt = Q(p) - D(p) \). Thus a price adjustment relation that takes into account deviations of the stock level \( S \) above certain optimal level \( S_o \) (to meet any demand reasonably quickly) is given by

\[
\frac{dp}{dt} = -\gamma \frac{dS}{dt} + \lambda (S_o - S), \quad (9)
\]

where \( \gamma \) (\( i.e. \), the inverse of excess demand required to move prices by one unity \[12\]) and \( \lambda \) are positive factors. For \( \lambda > 0 \), prices increase when stock levels are low and raise when they are high (with respect to \( S_o \)). When \( \lambda = 0 \), the price adjusts at a rate proportional to the rate at which stocks are either raising or running down.

For all asset prices \( p(t) \), non-linear forms for the quantities \( D \) demanded and \( Q \) supplied are postulated such that
\[ D(p) = d^* + d_o[1 - \frac{\delta^2}{2!}(p - p^*)^2 + \ldots] (p - p^*) \]
\[ Q(p) = q^* + q_o[1 - \frac{\delta^2}{2!}(p - p^*)^2 + \ldots] (p - p^*) \]

where \( d_o, q_o \) and \( d^* = D(p^*), q^* = Q(p^*) \) are arbitrary coefficients (related to material costs, wage rate, etc), \( p^* = p(t^*) \) is an equilibrium price and \( \delta < 0 \) is our order parameter as discussed in [9]. Expansion terms \( O(5) \) are here neglected.

Considering the simplest, complete economics model as in [11], we assume that \( S_o \) depends linearly on the demand; e.g., \( S_o = \ell_o + \ell D \), with \( \ell_o \) a constant and \( \ell \) satisfying
\[ \ell = \frac{\gamma \beta_o}{\lambda d_o} \]

where \( \beta_o \equiv q_o - d_o \). We have shown that only this condition can lead to solutions of the dynamical price equation in real space [4]. Therefore, in equilibrium (where \( \frac{dp}{dt}|_{p^*} = 0 \) and \( \frac{dS}{dt}|_{S^*} = 0 \), so that demand equals supply and \( S = S_o \)), we obtain \( d^* - q^* = 0 \) and \( S^* = \ell_o + \ell (d^* + d_o p^*) \).

After some algebra, the second derivative of the price adjustment Eq.(9) of one commodity can be approximated as
\[ \frac{d^2 p}{dt^2} \approx -\lambda \beta_o (p - p^*) + \frac{\delta^2 \lambda \beta_o}{2} (p - p^*)^3 \]

For \( \delta \neq 0 \) and \( [\lambda \beta_o, \delta^2 \lambda \beta_o/2] > 0 \), it has the well-known kink solutions
\[ p(t) = p^* + \frac{\sqrt{2}}{\delta} \tanh \left( \frac{\lambda \beta_o}{2} (t - t^*) \right) \]
such that \( \beta_o \) is positive. As in a competitive market economy the demand for a commodity fall when its price increases, then it is reasonable to assume \( d_o < 0 \) in Eq.(10). And as the price raises, the supply usually also increases; hence in general one also assumes \( q_o > 0 \). These conditions yield \( \beta_o > 0 \) as required and also \( d_o \ell > 0 \).

IV. THE MAPPING

We show next how a mapping can be established in order to identify the real, phenomenological RG critical exponents \( \alpha \) and \( \eta \) in terms of our non-linear economics model variables. From a comparison between Eqs.(5) and (12) we identify the following relation between the expansion terms
\[ \alpha B \rightleftharpoons -\delta \beta_o (p - p^*) \] .

(14)

It is from this simple mapping that we can make an attempt to understand RG modelling of stock markets and from it to analyse and predict financial crashes in analogy to critical points studied in physics with log-periodic correction to scaling [2].

If for \( t \to t^* \) we approximate \( \sinh\left( \sqrt{\frac{\lambda \beta_o}{2}} (t - t^*) \right) = \frac{\sqrt{\frac{\lambda \beta_o}{2}} (t - t^*)}{\sqrt{1 + \frac{\lambda \beta_o}{2} (t - t^*)^2}} \),

then using the mapping in Eq.(14) and the solutions for \( B \) and \( p \) given by Eqs.(6) and (13), respectively, we obtain

\[ \alpha \rightleftharpoons \lim_{t \to t_c} \log\left( \frac{t^* - t}{\sqrt{2/\lambda \beta_o}} \right) \log(\frac{x/x_0}{\sqrt{1 + \frac{\lambda \beta_o}{2} (t - t^*)^2}}) \to 1 \]

(15)

\[ |\eta| \rightleftharpoons \frac{\delta^2}{2\lambda^2 \beta_o^2} \]

(16)

such that, as before \( x = t_c - t \) and \( x_o = \Delta t \), and \( \Delta t \to \sqrt{2/\lambda \beta_o} \). The above \( \alpha \) is consistent with the definition of critical exponents [10].

Using these mappings for \( \alpha \to 1 \) and \( \eta \), it is straightforward to show that they also relate the second series expansion terms between Eqs.(3) and (12), namely: \( \eta B^3 \rightleftharpoons \frac{\delta^2 \lambda \beta_o}{2} (p - p^*)^3 \) provided that \( \delta < 0 \). Hence, in terms of our non-linear economics model variables, we find the following extended solutions in analogy with the non-linear RG framework:

\[ I(t) = A_1 + A_2 \frac{(t^* - t)}{\sqrt{1 + \left( \frac{t^* - t}{\Delta t} \right)^2}} \times \]

\[ \left[ 1 + A_3 \cos\left( \omega \log\left( \frac{t^* - t}{\sqrt{2/\lambda \beta_o}} \right) + \frac{\Delta \omega}{2} \log\left( 1 + \left( \frac{t^* - t}{\Delta t} \right)^2 \right) \right) \right] \]

(17)

with \( \tau \rightleftharpoons t, \tau_c \rightleftharpoons t^*, \Delta \omega \rightleftharpoons 2(\lambda \beta_o/\delta)^2 \kappa, \) and \( \Delta \tau \rightleftharpoons \sqrt{2/\lambda \beta_o} \).

V. DISCUSSION

As an example, we apply next the present mapping to the S&P500 index. In Fig. 1 we show the fit of Eq.(17) to the time dependence of the logarithm of the NY S&P 500 index from Jan 1985 to the October 1987 crash. The parameters used in this illustrative curve (full line) are: \( A_1 = 5.79, A_2 = -0.32, A_3 = 0.059, \omega = 6.47, \Delta t = 2.29, \Delta \omega = 15.42 \) and
\( t^* = 87.70 \) decimal years (with \( rms = 0.02 \)). The parameter values used for the best fit of Eq.\((8)\) (dotted lines in the figure) are those found in \([2]\).

Similarly to the non-linear RG scaling results, we see that the general trend of the S&P 500 data is also reproduced by the mapping of our economics model in the limit \( t \to t^* \) so that \( \alpha \to 1 \) as deduced from Eq.\((16)\). This is to be expected due to the oscillations that are periodic in the logarithm of the time to crash appearing in both Eqs.\((8)\) and \((17)\).

![Graph](image)

**FIG. 1.** Time dependence of the logarithm of the NY S&P 500 index from Jan 1985 to the October 1987 crash. The full line curve is the fit of Eq.\((17)\) for a year scale of 2.29 years and the dotted lines curves the fit of Eq.\((8)\) with \( \Delta t = 11 \) years.

The modulation of the logarithm frequency does not changes significatively when one approaches the critical point \( t_c \leftrightarrow t^* \). On relatively short time scales spikes are not accounted by both equations due to the complex self-organizing phenomena in stock markets other than the one analysed here. The log-periodic structure found prior to crashes implies the existence of a hierarchy of time scales \([3]\). The choice of the parameters \( A_{i=1,2,3} = 1 \) is still empirical in both cases.

Our mapping may give insight into the nature of market crashes from the new perspective of the demand \( D \) and supply \( Q \) of a commodity. Our non-linear expressions for \( D \) and \( Q \) of Eq.\((10)\) are plotted in Fig. 2 and are justified as follows: when \( |\delta p| << 1 \), these functions display similar behaviour to the (commonly used) linear \( p \)-dependence for \( D \) and
$Q$. Even more important, they depict the fact that as price falls, the quantity demanded for a commodity can increase in agreement with one of the basic principles of economy. On the other hand, our choice for $Q$ (with $q_o > 0$) also follows the typical behaviour observed in a competitive market (where no individual producer can set his own desired price). That is, the higher the price, the higher the profit, then the higher the supply (see [9] for a more extensive discussion).

![Graph](image)

**FIG. 2.** Non-linear forms for the demand and supply functions of Eq.(10) with $q_o > 0$, $d_o < 0$, $|\delta| = 0.8$ (full lines) and $\delta = 0$ (dotted lines).

We have related the critical exponents $\alpha$ and $\eta$ to the relevant variables of our non-linear economics model based on observed laws for $D$ and $Q$ [9]. As $t \to t^*$ we have identified $\alpha \to 1$ whereas the $\eta$ exponent given in Eq.(16) is found to depend on the order parameter $\delta < 0$, relating $D$ and $Q$ of a commodity as in Eq.(10) in conjunction with the economics model factors $\lambda$ and $\beta_o$ under the limiting constraints of Eq.(15).

Since our $\Delta t$ coefficient also depends on $\lambda$ and $\beta_o$, these factors drive the observed effects for $t/t_c \gg 1$, where there is a saturation of the function $I(t)$, and for $t \to t_c$, where the log-frequency shifts from $\omega + \Delta \omega$ to $\frac{\omega}{2\pi}$. In economics terms, these features are directly related to the temporal adjustments of ‘the level of stocks’ $S$ as given in Eq.(11).

The concept of a certain optimal level of stock is well-known in economics theory about stocks [11]. Planning ahead to have suitable ‘level of stocks’ is essential. If production had to be stopped every time a company ran out of raw materials, the time wasted would cost a fortune.
Indeed stocks are held for a variety of reasons. There may be stocks of raw materials ready for production, stocks of work-in-progress (e.g., production parts) or stocks of finished goods. Whichever they are it is vital for a company to control \textit{the level of stocks} very carefully. Too little and they may run into production problems, but too much and they have tied up money unnecessarily. Low \textit{level of stocks} -say 10\%- would certainly be adequate if production levels could be maintained during the years. Usually \textit{the level of stocks} needs to be adjusted as the marketing year progresses. Stocks are considered to be current assets because they can be converted into cash reasonably quickly. On the other hand, producers can also carry some stocks surplus as a way to speculate on prices.

\section*{VI. CONCLUSIONS}

We have shown that the present economics mapping to the RG scaling of stock markets reasonably allows to reproduce the S&P 500 index trends in the vicinity of the time of crash and also to predict the existence of a crash as in the RG model due to corrections to the power law with log-periodicity but such that $\alpha \to 1$. The main point of this work follows that of RG model. That is, the underlying cause of the crash must be searched years before by looking at the progressive accelerating ascent of the market price. However, our formalism differs in that we have a shorter year scale of about $\Delta t = 2.29$ years compared to the fitting of $\Delta t = 11$ years as reported in \cite{2} in which these cooperative phenomena are progressively being constructed. In this period of time, one should also look for the appearence of non-linearities in the behaviour of the demand and supply functions (or, alternatively, in \textit{the level of stocks}) of the commodities prior to the crash as recognized here.

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