Non-Abelian fields in AdS$_4$ spacetime: axially symmetric, composite configurations

Olga Kichakova$^1$, Jutta Kunz$^1$, Eugen Radu$^2$ and Yasha Shnir$^{1,3,4}$

$^1$Institut für Physik, Universität Oldenburg, Postfach 2503 D-26111 Oldenburg, Germany
$^2$Departamento de Fisica da Universidade de Aveiro and I3N, Campus de Santiago, 3810-183 Aveiro, Portugal
$^3$Department of Theoretical Physics and Astrophysics, BSU, Minsk, Belarus
$^4$BLTP, JINR, Dubna, Russia

September 8, 2014

Abstract

We construct new finite energy regular solutions in Einstein-Yang-Mills-SU(2) theory. They are static, axially symmetric and approach at infinity the anti-de Sitter spacetime background. These configurations are characterized by a pair of integers $(m,n)$, where $m$ is related to the polar angle and $n$ to the azimuthal angle, being related to the known flat space monopole-antimonopole chains and vortex rings. Generically, they describe composite configurations with several individual components, possessing a nonzero magnetic charge, even in the absence of a Higgs field. Such Yang-Mills configurations exist already in the probe limit, the AdS geometry supplying the attractive force needed to balance the repulsive force of Yang-Mills gauge interactions. The gravitating solutions are constructed by numerically solving the elliptic Einstein-DeTurck-Yang-Mills equations. The variation of the gravitational coupling constant $\alpha$ reveals the existence of two branches of gravitating solutions which bifurcate at some critical value of $\alpha$. The lower energy branch connects to the solutions in the global AdS spacetime, while the upper branch is linked to the generalized Bartnik-McKinnon solutions in asymptotically flat spacetime. Also, a spherically symmetric, closed form solution is found as a perturbation around the globally anti-de Sitter vacuum state.

1 Introduction

The study of solutions of the Yang-Mills (YM) theory in a curved spacetime geometry can be traced back at least to the early work [1]. Among other results, that study has given an exact solution of the YM equations in a fixed Schwarzschild black hole background. This shows that the non-trivial solutions to the full system of Einstein–Yang-Mills (EYM) equations are likely to exist, at least for large enough event horizon black hole radius.

Indeed, this has been confirmed ten years later, when several different authors have constructed asymptotically flat, black hole (BH) solutions within the framework of $d = 4$ SU(2) EYM theory [2]. Although these BHs were static and spherically symmetric, with vanishing YM charges, they were different from the Schwarzschild one and, therefore, not characterized exclusively by their total mass. Unfortunately, soon after their discovery, it has been shown that these solutions are unstable [3], [4]. However, despite this fact, they still present a challenge to the standard ‘no hair conjecture’ [5], [6].

These results have led to a revision of some of the basic concepts of BH physics based on the uniqueness and no-hair theorems. For example, the Israels theorem does not generalize to the non-Abelian (nA) case, since static EYM black holes with non-degenerate horizon turn out to be not necessarily spherically symmetric [7]. Moreover, in strong contrast to the Abelian case, the EYM hairy black holes in [2] do not trivialize in...
the limit of a vanishing horizon are reducing to horizonless, globally regular, particle-like configurations, originally found by Bartnik and McKinnon in Ref. [8].

As a result, the subject of particle-like and hairy BH solutions in EYM theory has become an active field of research, with many new results being reported each year. One interesting question addressed in this context is what happens if one drops the assumption of asymptotic flatness for the spacetime background. The case of the EYM system with a negative cosmological constant $\Lambda$ is of particular interest, the natural background of the theory corresponding to anti-de Sitter (AdS) spacetime. Solutions of various physical models in this geometry received recently much interest due to the conjectured anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence. This is a concrete realization of the holographic principle, which asserts that a consistent theory of quantum gravity in $(d-1)$-dimensions must have an alternate formulation in terms of a nongravitational theory in $(d-1)$-dimensions. Despite the fact that string theory in AdS space is still too complicated to be dealt with in detail, in many interesting cases, it is sufficient to consider the low energy limit of the superstring theory, namely, supergravity. However, the gauged supergravity theories generically contain YM fields (although most of the studies in the literature have been restricted to the case of Abelian matter content in the bulk), and thus the interest in the study of the EYM system with $\Lambda < 0$.

The first result on nA fields in a globally AdS $d$ geometry can be found again in Ref. [1], where a non-trivial solution of the YM equations is exhibited in closed form. This solution describes a globally regular, finite energy soliton, with a nonzero magnetic flux at infinity (despite the absence of a Higgs field), anticipating most of the basic properties of the (gravitating) nA fields in an AdS background.

The study of EYM solutions in a globally AdS $d$ background have started with Refs. [10], [11], where spherically symmetric BHs and solitons have been studied, again for the gauge group SU(2). As shown there, a variety of well known features of asymptotically flat self-gravitating nonabelian solutions are not shared by their AdS counterparts. Restricting for simplicity to purely magnetic configurations, one finds a continuum of particle-like and BH solutions describing nA monopoles with a non-integer magnetic charge (we recall that the asymptotically flat EYM configurations are magnetically neutral, forming a discrete sequence indexed by the node number of the magnetic gauge potential [2]). Moreover, perhaps most remarkable, some of the solutions are stable against spherically symmetric linear perturbations [12], [13]. Also, as already found in Ref. [1], the curved background geometry provided an extra attracting force, which makes possible the existence of finite mass, particle-like YM solutions already in the probe limit, i.e. in a fixed AdS spacetime. As discussed in [14], [15] the soliton and black hole solutions in [10], [11] possess interesting generalizations with higher gauge groups. A review of the EYM solutions in a globally AdS $d$ background can be found in Ref. [16].

We note that an even more intricate picture is found when studying EYM topological BHs [17]. For example, no globally regular particle-like limit of the solutions is found in this case. Moreover, as discovered in [18], [19], the planar hairy BHs describe gravity duals of $p$–wave superconductors. As a result, this type of EYM solutions enjoyed recently much interest. Also, the $d = 4$ configurations possess higher dimensional generalizations [20], with the $d = 5$ EYM planar black holes describing holographic $p$–wave superfluids [21]. However, these aspects are beyond the purposes of this paper, where we shall restrict ourselves to the case of the SU(2) gauge fields and the asymptotically AdS 4 spacetime. One should remark that, even in the case of the global AdS spacetime, the study of YM solutions is still far from being complete. For example, very few things are known about non-spherically symmetric EYM-AdS solutions. Non-Abelian solitons which are axially symmetric only have been studied in Ref. [22]. These configurations possess an azimuthal winding number $n > 1$ and describe (non-topological) monopoles localized at the origin, sharing all basic properties of the spherically symmetric counterparts (which have $n = 1$). Also, despite the generic presence of a net magnetic flux, they can be viewed as the natural AdS generalizations of the asymptotically flat EYM solitons in [21].

However, as discussed in [23], the EYM system with $\Lambda = 0$ possesses a variety of other globally regular

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1 In fact, rather curious, the EYM particle-like solutions have been discovered before their black hole generalizations.
2 A detailed review of the situation ten years after the discovery of the EYM solutions in [2], [8] can be found in [9].
3 However, note that the Ref. [1] has considered only the case of a positive cosmological constant and a slightly different coordinate system as compared to [2].
4 Their black hole generalizations have been considered in [23].
solutions, describing composite configurations, with several constituents. In the notation of \[25\], these solutions are characterized by a pair of positive integers \((m, n)\), where \(m\) is related to the polar angle and \(n\) to the azimuthal angle.

One should remark that the AdS solutions discussed so far in the literature cover the case \(m = 1\) only, that is, solutions with a single center. The main purpose of this work is to explicitly construct AdS composite configurations with \(m > 1\), looking for new features induced by the different asymptotic structure of the spacetime.

Although some common features are present, the results we find for \(\Lambda < 0\) are rather different from those valid in the asymptotically flat case. Perhaps the most prominent new result is the existence of multi-center solutions with a net magnetic flux at infinity, despite the absence of a Higgs field (such solutions are absent for \(\Lambda = 0\)). Similar to the spherically symmetric case \[1\], such nA configurations exist already in the probe limit (i.e. no backreaction). Moreover, when including the gravity effects, we establish the absence, for large enough values of \(|\Lambda|\), of configurations with a zero magnetic flux (which are the only ones existing in the asymptotically flat case).

This paper is structured as follows: in the next Section we introduce the gauge field ansatz and address the question of possible asymptotics for (purely magnetic) static, axially symmetric YM fields. In Section 3 we construct solutions with these asymptotics in the probe limit, i.e. for a fixed AdS background. Although being relatively simple, nevertheless this case appears to contain all the essential features of the gravitating solutions. The backreaction of the solutions on the spacetime geometry is considered in Section 4. There special attention is paid to a particular value of \(\Lambda\), for which the EYM system becomes a consistent truncation of the \(d = 4, \mathcal{N} = 4\) gauged supergravity \[28\]; therefore such solutions can be uplifted to \(d = 11\) dimensions \[29\]. When we abandon this restriction, the variation of the gravitational coupling constant \(\alpha\) (which is the ratio between the Planck length and AdS length scales) reveals the existence of two branches of gravitating solutions which bifurcate at some critical value \(\alpha_{cr}\). These branches interpolate between the solutions in the global AdS spacetime and, for small values of the coupling constant on the upper branches, the generalized Bartnik-McKinnon solutions in the asymptotically flat spacetime \[25\], confined in the interior region, and outer configurations in the global AdS spacetime.

We give our conclusions and remarks in the final Section. The Appendix A contains a derivation of an exact solution of the EYM equations with negative cosmological constant as a perturbation around the globally AdS vacuum state.

## 2 SU(2) Yang-Mills fields on AdS$_4$

### 2.1 The model

We consider the action of the SU(2) YM theory

\[
S = -\frac{1}{2} \int d^4x \sqrt{-g} \text{Tr}\{F_{\mu\nu}F^{\mu\nu}\}, \tag{1}
\]

with the field strength tensor

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu], \tag{2}
\]

and the gauge potential

\[
A_\mu = \frac{1}{2} \tau^a A^a_\mu, \tag{3}
\]

\(e\) being the gauge coupling constant. Also, \(\mu, \nu\) are space-time indices running from 1 to 4 and the gauge index \(a\) is running from 1 to 3.

Variation of (1) with respect to the gauge field \(A_\mu\) leads to the YM equations

\[
D_\mu F^{\mu\nu} \equiv \nabla_\mu F^{\mu\nu} + ie[A_\mu, F^{\mu\nu}] = 0, \tag{4}
\]

\(^{5}\text{Note that no such solutions exist with Abelian matter fields only, the closest approximation there being the Majumdar-Papapetrou \[26\], \[27\] extremal black holes in Einstein-Maxwell theory.}\)
while the variation with respect to the metric $g_{\mu\nu}$ yields the energy-momentum tensor of the YM fields

$$T_{\mu\nu} = 2\text{Tr}\left\{ F_{\alpha\beta} F_{\nu\beta} g^{\alpha\beta} - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right\}. \quad (5)$$

For the background metric, we shall consider the (covering-)AdS$_4$ spacetime, written in global coordinates as

$$ds^2 = \frac{dr^2}{N(r)} + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) - N(r) dt^2, \quad \text{with } N(r) = 1 + \frac{r^2}{\ell^2}, \quad (6)$$

where $(r, t)$ are the radial and time coordinates, respectively (with $0 \leq r < \infty$ and $-\infty < t < \infty$), while $\theta$ and $\varphi$ are angular coordinates with the usual range, parametrizing the two dimensional sphere $S^2$. Also, $\ell$ is the AdS length scale, which is fixed by the cosmological constant,

$$\Lambda = -\frac{3}{\ell^2}. \quad (7)$$

### 2.2 The axially symmetric YM Ansatz

In this work we shall restrict to purely magnetic YM configurations and employ a gauge field ansatz in the parametrization$^6$ originally proposed in [24]

$$A_\mu dx^\mu = \left( \frac{H_1}{r} dr + (1 - H_2) dt \right) \frac{u^{(n)}_r}{2e} - n \sin \theta \left( \frac{H_3 u^{(n)}_\theta}{2e} + (1 - H_4) \frac{u^{(n)}_{\varphi}}{2e} \right) d\varphi, \quad (8)$$

in terms of four gauge field functions $H_i$ which depend on $r$ and $\theta$ only. The SU(2) matrices $u^{(n)}_\alpha$ factorize the dependence on the azimuthal coordinate $\varphi$, with

$$u^{(n)}_r = \sin \theta (\cos n\varphi \tau_x + \sin n\varphi \tau_y) + \cos \theta \tau_z,$$

$$u^{(n)}_\theta = \cos \theta (\cos n\varphi \tau_x + \sin n\varphi \tau_y) - \sin \theta \tau_z,$$

$$u^{(n)}_{\varphi} = -\sin n\varphi \tau_x + \cos n\varphi \tau_y,$$

where $\tau_x, \tau_y, \tau_z$ are the Pauli matrices. The positive integer $n$ represents the azimuthal winding number of the solutions. For $n = 1$ and $H_1 = H_3 = 0$, $H_2 = H_4 = w(r)$ the usual spherically symmetric magnetic (singularity-free) YM ansatz is recovered.

This ansatz is axially symmetric in the sense that a rotation around the $z$–axis (with $z = r \cos \theta$) can be compensated by a suitable gauge transformation [32] [33]. However, note that the gauge transformation $U = \exp( i \Gamma(r, \theta) u^{(n)}_{\varphi} / 2)$ leaves the ansatz form-invariant [34]. Thus, to construct regular solutions we have to fix the gauge. The usual gauge condition [24] which is used also in this work is

$$r \partial_r H_1 - \partial_{\theta} H_2 = 0. \quad (9)$$

A straightforward computation leads to the following expression of the non-vanishing components of the SU(2) field strength tensor (with $F_{\mu\nu} = F^{(a)}_{\mu\nu} \frac{1}{2e} u^{(n)}_a$):

$$F^{(\varphi)}_{r\theta} = -\frac{1}{r} (H_{1,\theta} + r H_{2,r}), \quad (10)$$

$$F^{(r)}_{\varphi} = -\frac{n \sin \theta}{r} (r H_{3,r} - H_1 H_4), \quad F^{(\theta)}_{\varphi} = \frac{n \sin \theta}{r} (r H_{4,r} + H_1 H_3 + \cot \theta H_1),$$

$$F^{(r)}_{\theta \varphi} = -n \sin \theta (H_{3,\theta} - 1 + H_2 H_4 + \cot \theta H_3), \quad F^{(\theta)}_{\theta \varphi} = n \sin \theta (H_{4,\theta} - H_2 H_3 - \cot \theta (H_2 - H_4)).$$

$^6$This is in fact a suitable reparametrization of the axially symmetric YM ansatz introduced for the first time by Manton [30] and Rebbi and Rossi [31], which is better suited for numerical purposes. Also, note that [34] is a consistent truncation of the most general YM ansatz, which contains 12 potentials.
We are interested in singularity-free solutions of the YM equations (4) with a finite mass. For configurations in a fixed AdS background, the total mass $M$ is defined as the integral of the mass-energy density $\rho = -T_t^t$ over a $t = \text{const.}$ three-dimensional space, i.e.

$$M = -\int d^3 x \sqrt{-g} T_t^t = -2\pi \int_0^\infty dr \int_0^\pi d\theta \, r^2 \sin \theta \, T_t^t. \quad (11)$$

From (5) and (8), one finds

$$-T_t^t = \frac{1}{2e^{2r^2}} \left( NF_{r\theta}^2 + \frac{1}{\sin^2 \theta} \left( NF_{r\varphi}^2 + \frac{1}{r^2} F_{\theta\varphi}^2 \right) \right), \quad (12)$$

where we denote

$$F_{r\theta}^2 = (F_{r\theta}^{(s)})^2, \quad F_{r\varphi}^2 = (F_{r\varphi}^{(s)})^2 + (F_{r\varphi}^{(t)})^2, \quad F_{\theta\varphi}^2 = (F_{\theta\varphi}^{(s)})^2 + (F_{\theta\varphi}^{(t)})^2. \quad (13)$$

As known already from the study of spherically symmetric configurations [11], a generic feature of the generic YM solutions in an AdS$_4$ background is that they possess a nonvanishing magnetic flux through the sphere at infinity. A measure of this flux is provided by the magnetic charge, $Q_M$, which, in the absence of the Higgs field, does not have a meaning of a topological charge of the configuration, thus it is allowed to be non-integer. A possible gauge invariant definition of $Q_M$ which we shall employ in this work, is [35]

$$Q_M = \frac{1}{4\pi} \oint_{\infty} d\theta d\varphi \sqrt{(F_{\theta\varphi}^{(s)})^2 + (F_{\theta\varphi}^{(t)})^2}. \quad (14)$$

We have verified that in the spherically symmetric case, (14) agrees, up to a sign, with the magnetic charged expression in [11].

We close this part by noticing that the static axially symmetric YM configurations in a fixed AdS spacetime satisfy the virial identity [7]

$$\int_0^\infty \int_0^\pi dr \sin \theta \left( NF_{r\theta}^2 + \frac{1}{\sin^2 \theta} (NF_{r\varphi}^2 + \frac{1}{r^2} F_{\theta\varphi}^2) \right) = \int_0^\infty \int_0^\pi d\theta \sin \theta \frac{2\rho^2}{e^2} \left( F_{r\theta}^2 + \frac{1}{\sin^2 \theta} F_{r\varphi}^2 \right). \quad (15)$$

This makes clear that the AdS geometry supplies the attractive force needed to balance the repulsive force of Yang-Mills gauge interactions.

### 2.3 The issue of boundary conditions at infinity

#### 2.3.1 $\Lambda = 0$ flat space case

Let us start with the more familiar case of gauge fields in a Minkowski spacetime background. For $\Lambda = 0$, a systematic study of axially symmetric YM-Higgs configurations has revealed the existence of two possible types of asymptotics of the YM fields, describing different ground states of the model. These asymptotics are indexed by a number $m$, which is a positive integer.

**Odd-$m$ conditions**

For $m$ an odd number, $m = 1, 3, 5, \ldots$, the YM fields possess a solution with

$$H_1 = 0, \quad H_2 = 1 - m,$$

$$H_3 = \frac{\cos \theta}{\sin \theta} \left( \cos((m - 1)\theta) - 1 \right),$$

$$H_4 = -\frac{\cos \theta}{\sin \theta} \sin((m - 1)\theta), \quad (16)$$

As usual, this virial identity is found by considering the scale transformation $r \to \lambda r$ of the effective action $S_{\text{eff}}$ of the model (which essentially coincides with (11)), for a given set of boundary conditions. Then $S_{\text{eff}}$ must have a critical point at $\lambda = 1$, which results in the virial relation (15).
which describe an infinite energy embedded Abelian configuration with a singular origin. The configurations with these far field asymptotics carry a nonzero magnetic charge, \( Q_M = n \) (with \( Q_M \) computed according to (14)).

However, in a flat spacetime background, a magnetically charged configuration requires Higgs fields to exist. (or they are just embedded Abelian singular solutions). Indeed, as discussed in [36], the Yang-Mills-Higgs (YMH) system possesses regular, finite energy solutions whose gauge potentials approach (16) in the far field. The better known case are the \( m = 1 \), \( n \geq 1 \) self-dual magnetic monopoles (which are in fact the only YMH closed form solutions). For \( m > 1 \), they describe composite (non-self-dual) monopole-antimonopole configurations with a net magnetic charge. A systematic study of these solutions can be found in [36].

**Even-\( m \) conditions**

A different picture is found for \( m = 2, 4, \ldots \). The corresponding ground state YM solution reads

\[
H_1 = 0, \quad H_2 = 1 - m, \\
H_3 = \frac{\cos((m - 1)\theta) - \cos \theta}{\sin \theta}, \\
H_4 = -\frac{\sin((m - 1)\theta)}{\sin \theta}.
\]

(17)

Again, these asymptotics emerge from a systematic study of the axially symmetric YMH system [36]. The corresponding YMH solutions describe again (non-self dual) monopole-antimonopole chains. However, different from (16), the total magnetic charge vanishes in this case. \( Q_M = 0 \).

Moreover, as found in [25], in strong contrast to the odd–\( m \) case, these configurations survive in the limit of a vanishing Higgs field, provided that the gravity effects are included. For example, the well-known Bartnik-McKinnon EYM particle-like solutions [8] are recovered for \( m = 2, n = 1 \). The values \( m = 2, n > 1 \) lead to their axially symmetric generalizations in [24],[36].

One should also mention that, as discussed in [36], the YM configuration (17) with \( m = 2k \) corresponds to a gauge transformed trivial solution,

\[
A_{\mu} = \frac{i}{e}(\partial_{\mu}U)U^\dagger.
\]

(18)

However, the \( m = 2k + 1 \) configurations (16) describe a gauge transformed charge-\( n \) Abelian multimonopole (\( H_i \equiv 0 \)):

\[
A_{\mu} = \frac{i}{e}(\partial_{\mu}U)U^\dagger + UA_{\mu}^{(0)}U^\dagger,
\]

(19)

where

\[
A_{\mu}^{(0)} = \frac{1}{2e}u_{\varphi}^{(n)} d\theta - n \sin \theta u_{\theta}^{(n)} d\varphi.
\]

(20)

Also,

\[
U = \exp\{-iku_{\varphi}^{(n)}\},
\]

(21)

for both \( m = 2k + 1 \) and \( m = 2k \).

**The relation with the gauge field parameterization in [36] and [25]**

Although relying on the same ansatz proposed by Manton [30] and Rebbi and Rossi [31], the Ref. [36] expresses the YM potentials in a slightly different SU(2) basis,

\[
A_{\mu} dx^\mu = \left( \frac{K_1}{r} dr + (1 - K_2) d\theta \right) \left( \frac{r_{\varphi}^{(n)}}{2e} - n \sin \theta \left( K_3 \frac{r_{\varphi}^{(n,m)}}{2e} + (1 - K_4) \frac{r_{\theta}^{(n,m)}}{2e} \right) d\varphi ,
\]

(22)

\[ \text{(Note that for any value of } m, \text{ the expression of the magnetic charge of the YMH solutions found by employing the Abelian 't Hooft tensor agrees with that from (14).)} \]
with the number \(m\) entering also the SU(2) matrices:

\[
\tau_r^{(n,m)} = \sin m\theta (\cos n\varphi \tau_x + \sin n\varphi \tau_y) + \cos m\theta \tau_z,
\]

\[
\tau_\theta^{(n,m)} = \cos m\theta (\cos n\varphi \tau_x + \sin n\varphi \tau_y) - \sin m\theta \tau_z,
\]

\[
\tau_{\varphi}^{(n)} = -\sin n\varphi \tau_x + \cos n\varphi \tau_y.
\]

A direct comparison with (8) implies \(H_1 = K_1,\ H_2 = K_2\), while \(H_3 = K_3 \cos((m - 1)\theta) - (1 - K_4) \sin((m - 1)\theta)\), \(1 - H_4 = K_3 \sin((m - 1)\theta) + (1 - K_4) \cos((m - 1)\theta)\).

Note also that the Ref. [25], dealing with pure EYM solutions, uses a version of (22) with \(m \to k\) and \(-1 - K_4 = K_4\) (and a set of boundary conditions at infinity resulting from [17], with \(m = 2k\)).

We would like to emphasize that the description of a given nA configuration in terms of \((\tau_a^{(n,m)}, K_i)\), or in terms of \((u_a^{(n)}, H_i)\) are equivalent. The choice in this work for \((u_a^{(n)}, H_i)\) has the advantage to simplify somehow the emerging general picture, providing a unified framework. Also, it leads to slightly better numerical results for the gravitating solutions.

### 2.3.2 YM far field asymptotics in AdS\(_4\) spacetime

As we know already from the study in [11], [10] of the spherically symmetric case, the \(r \to \infty\) asymptotics of the YM fields are less constrained for \(\Lambda < 0\). In the generic axially symmetric case, an obvious condition results from the requirement that \(T_\ell^i\), as given by [12], decays faster than \(1/r^3\) as \(r \to \infty\), as imposed by the assumption that the total mass \(M\) is finite. This implies

\[
F_{r\theta}^{(r)} \to \frac{1}{r^{3+\epsilon}}, \quad F_{r\varphi}^{(r)} \to \frac{1}{r^{3+\epsilon}}, \quad F_{\varphi \theta}^{(r)} \to F_{\varphi \theta}^{(\theta)}(\theta), \quad F_{\theta \varphi}^{(r)} \to G_{\theta \varphi}(\theta), \tag{23}
\]

with \(\epsilon > 0\) and \(F_{\varphi \theta}(\theta), G_{\theta \varphi}(\theta)\) two regular functions which vanish as \(\theta \to 0, \pi\) as imposed by the regularity of the configurations.

A general expansion of the YM potentials \(H_i\) in AdS spacetime reads (with \(i = 1, \ldots, 4\)):

\[
H_i = H_i^{(0)} + H_i^{(1)}/r + H_i^{(2)}/r^2 + \ldots, \tag{24}
\]

where the functions \(H_i^{(k)}\) depend on the angular coordinate \(\theta\) only. Once an expression is chosen for \(H_i^{(0)}\), \(H_i^{(1)}\) the functions \(H_i^{(k)}\) (with \(k \geq 2\)) are found by solving the YM equations in the far field, as a power series in \(1/r\). Note, however that the gauge condition (9) implies

\[
H_1^{(0)} = 0, \quad H_2^{(0)} = \text{const.}, \tag{25}
\]

while no obvious expressions are found for \(H_3^{(0)}, H_4^{(0)}\) (note that the regularity of the solutions implies \(H_2^{(0)}|_{\theta = 0, \pi} = H_1^{(0)}|_{\theta = 0, \pi}\)).

In what follows, among all possible sets, we shall restrict ourselves to those YM asymptotics which provide natural AdS generalizations of the flat spacetime boundary conditions [10], [17].

**Odd-\(m\) conditions**

One can prove that [10] is still a solution of the YM equations for \(\Lambda < 0\). However, in strong contrast to the (asymptotically) flat case, solutions with a non-zero magnetic charge exist even in the absence of a Higgs field and/or the gravity effects. This can easily be seen for the simplest case with \(m = 1, n = 1\), where the following spherically symmetric exact solution is known:

\[
H_1 = H_3 = 0, \quad H_2 = H_4 = w(r) = \frac{1}{\sqrt{1 + r^2}}, \tag{26}
\]

This is the exact solution found in [11], which describes a unit charge magnetic monopole with a finite mass \(M = 3\pi^2/2e^2\ell\). However, the results in [39] show the existence of (numerical) generalizations of this
solution with the same behaviour at the origin, \( w(0) = 1 \), and a relaxed set of boundary conditions at infinity, \( w(\infty) = w_0 \neq 0 \). As discussed in [22], these solutions admit axially symmetric generalizations with \((m = 1, n > 1)\), possessing many similar properties.

A systematic study of possible boundary conditions compatible with regularity and finite energy requirements led us to the following AdS natural generalizations of (16):

\[
H_1 = 0, \quad H_2 = 1 - m + w_0, \\
H_3 = \frac{\cos \theta}{\sin \theta} \left( \cos((m - 1)\theta) - 1 \right) + w_0 \sin((m - 1)\theta), \\
H_4 = -\frac{\cos \theta}{\sin \theta} \sin((m - 1)\theta) + w_0 \cos((m - 1)\theta),
\]

with \( w_0 \) a constant which is not fixed a priori, the flat space boundary conditions being recovered for \( w_0 = 0 \). For example, one takes

\[
H_1 = 0, \quad H_2 = w_0, \quad H_3 = 0, \quad H_4 = w_0, \quad \text{for } m = 1,
\]

and

\[
H_1 = 0, \quad H_2 = w_0 - 2, \quad H_3 = (w_0 - 1)\sin 2\theta, \quad H_4 = -2\cos^2 \theta + w_0 \cos 2\theta, \quad \text{for } m = 3.
\]

For the general asymptotics [27], one finds \( F_{\theta\varphi}^{(r)} \to (1 - w_0^2)n \sin \theta \cos((m - 1)\theta) \), \( F_{\theta\varphi}^{(\theta)} \to (1 - w_0^2)n \sin \theta \sin((m - 1)\theta) \), which implies

\[
Q_M = n|1 - w_0^2|.
\]

Therefore, the magnetic charge of the AdS configurations with the generic asymptotics [27] is no longer an integer, a feature which occurs already in the spherically symmetric case. However, one can see that magnetically neutral solutions are found for \( w_0 = \pm 1 \).

**Even-\( m \) conditions**

The situation is somehow different for even values of \( m \). It turns out that [17] provides the only possible set of boundary conditions compatible with the condition of a vanishing magnetic charge [10].

Of course, this does not exclude the existence of AdS deformations of (17). However, they would generically possess a nonzero magnetic charge. For example, we have studied in a systematic way \( m = 2k \) solutions with

\[
H_1 = 0, \quad H_2 = 1 - mw_0, \\
H_3 = \frac{\cos((m - 1)\theta) - \cos \theta}{\sin \theta}, \\
H_4 = 1 - w_0 - w_0 \frac{\sin((m - 1)\theta)}{\sin \theta},
\]

behaviour as \( r \to \infty \), such that the \( \Lambda = 0 \) conditions (17) are recovered for \( w_0 = 1 \). For example, one has

\[
H_1 = 0, \quad H_2 = 1 - 2w_0, \quad H_3 = 0, \quad H_4 = 1 - 2w_0, \quad \text{for } m = 2,
\]

---

9Note that [27] is not a solution of the YM equations, unless \( w_0 = 0 \). It describes instead the leading order behaviour of the YM potentials in the AdS spacetime, as given by \( H^{(0)} \).

10We did not find AdS generalizations of (17) with \( Q_M = 0 \), which would be similar to the odd-\( m \) conditions [27]. In our approach, we consider a deformation of (17) with:

\[
H_1^{(0)} = 0, \quad H_2^{(0)} = 1 - m - w_0, \quad H_3^{(0)} = \frac{\cos((m - 1)\theta) - \cos \theta}{\sin \theta} + w_0 S_3(\theta), \quad H_4^{(0)} = -\frac{\sin((m - 1)\theta)}{\sin \theta} + w_0 S_4(\theta),
\]

with \( w_0 \) a constant. The YM equations are supplemented with the zero-magnetic charge condition \( F_{\theta\varphi}^{(r)} \to 0, \quad F_{\theta\varphi}^{(\theta)} \to 0 \), as \( r \to \infty \). This results in two first order differential equations for \( S_3(\theta) \) and \( S_4(\theta) \), which have closed form solutions. However, the functions \( S_3(\theta) \) and \( S_4(\theta) \) fail to satisfy the regularity conditions on the symmetry axis. Thus we conclude that (17) are the only boundary conditions compatible with the assumption of a zero-magnetic flux at infinity.
and
\[ H_1 = 0, \quad H_2 = 1 - 4w_0, \quad H_3 = -2w_0 \sin 2\theta, \quad H_4 = 1 - 4w_0 \cos^2 \theta, \quad \text{for } m = 4. \tag{34} \]

This reveals an interesting feature: the lowest polar winding number solutions \( m = 1 \) and \( m = 2 \), are in fact identical, since (28) results from (33) via the identification \((1 - 2w_0)(m=2) \rightarrow w_0(m=1)\). This is related to the existence of a continuum of solutions (in terms of \( w_0 \)), which is a pure AdS property. The solutions with higher \( m \) satisfy different boundary conditions and are inequivalent\(^{11}\).

The even-\( m \) solutions with the asymptotics (32) carry a magnetic charge
\[ Q_M = \frac{mn}{2} |(1 - w_0)w_0|. \tag{35} \]

Also, let us note that in contrast to the \( \Lambda = 0 \) case, both (27) and (32) correspond to intrinsic nA configurations (for \( w_0 \neq 0 \) and \( w_0 \neq 1 \), respectively).

3 The solutions in the probe limit

In this section we shall establish the existence of finite energy, regular YM configurations approaching at infinity the asymptotics (27) and (32), respectively. Different from the case of a (asymptotically) Minkowski spacetime background, such configurations exist already in the probe limit (i.e. when neglecting the backreaction of the matter fields on the spacetime geometry). This approximation greatly simplifies the problem but retains most of the interesting physics. For example, the probe limit was implemented recently to analyse properties of axially symmetric solutions of the SU(2) Yang-Mills-Higgs theory \(^{45}\).

The problem has an intrinsic length scale \( \ell \) (we recall \( \Lambda = -3/\ell^2 \)). Without any loss of generality, we fix \( \ell = 1 \), the total mass of solutions, as given by (11), being expressed in units of \( 4\pi/e^2 \). The numerical approach employed in this case is similar to those described in Section 4 for gravitating configurations and we shall not discussed it here\(^{12}\). We mention only that the solutions are found by directly solving the YM equations with a given set of boundary conditions.

3.1 The results

Fixing \( \Lambda = -3 \), the only input parameters of the problem are the integers \( (m, n) \) and the continuous constant \( w_0 \), which enters the boundary conditions at infinity (27), (32). We have studied a large number of configurations with \( m = 1, 3, 4, 5, 6, 7 \) and \( n = 1, \ldots, 10 \). Then we conjecture the existence of YM solutions in AdS\(_4\) background for any values\(^{13}\) of \( (m, n) \).

The YM solutions are found subject to the following boundary conditions:
\[ H_1|_{r=0} = H_3|_{r=0} = 0, \quad H_2|_{r=0} = H_4|_{r=0} = 1, \tag{36} \]

at the origin, and (27), (32) at infinity.

Also, for solutions with parity reflection symmetry (the only type we consider in this paper), the boundary conditions at \( \theta = 0, \pi/2 \) are
\[ H_1|_{\theta=0,\pi/2} = H_3|_{\theta=0,\pi/2} = 0, \quad \partial_\theta H_2|_{\theta=0,\pi/2} = \partial_\theta H_4|_{\theta=0,\pi/2} = 0, \tag{37} \]

(therefore we need to consider the solutions only in the region \( 0 \leq \theta \leq \pi/2 \)).

Regularity of the solutions on the symmetry axis imposes also
\[ H_2|_{\theta=0,\pi} = H_4|_{\theta=0,\pi}. \tag{38} \]

\(^{11}\)This can be seen by comparing gauge invariant quantities, e.g. the energy density.

\(^{12}\)Note that the typical numerical accuracy is better for pure YM solutions, the typical numerical error estimates being on the order of \( 10^{-5} \).

\(^{13}\)This contrasts with the \( \Lambda = 0 \) gravitating YM solutions in \cite{25}, which do not cover the whole \( (m, n) \) space.
a condition which is verified from the numerical output.

For given \((m, n)\), the solutions are found by varying the parameter \(w_0\) which enters the boundary conditions at infinity. As expected, we have found a continuum of solutions in terms of \(w_0\).

It is not easy to extract some general characteristic properties of the solutions, valid for every choice of the parameters \((w_0, m, n)\). However, we have found that the functions \(H_i\) present always a considerable angle-dependence, except for the \(m = 1\) case (there one finds usually a small angular dependence for the potentials \(H_2, H_4\)). We have also noticed that the angular dependence generally increases with \(Q_M\). The profiles of typical \(m = 1, 3, 4, 5\) solutions are given in Figures 1-4, both as 3D-plots (with \(\rho = r \sin \theta, z = r \cos \theta\)), and as a function of the radial coordinate for several different angles. There we show the gauge potentials \(H_i\) together with the mass-energy density as given by \(T_i\).

Also, we have found that for all sets \((m, n)\), in the absence of backreaction, the solutions exist for a single interval in \(w_0\), the mass of the solutions strongly increasing for large values of \(|w_0|\). One can see this in Figures 5, 6 for the spectrum of the \(m = 1, 3, 4, 5, 6\) solutions. There we plot the mass of the solutions in terms of the parameter \(w_0\) which enters the far field asymptotics \((27)\) and \((32)\) (we recall that \(w_0\) fixes the magnetic charge of solutions via \(Q_M = n|1 - w_0^2|\) for odd \(m\), and \(Q_M = \frac{m}{2}(1 - w_0)w_0|\) for even \(m\)). It is interesting to notice that the \(M(w_0)\) picture (as shown in Figures 5, 6), resembles the behaviour of non-gravitating Q-balls, whose frequency-mass diagram exhibits a similar pattern, see e.g. [39]. However, different from the frequency in that case, we could not derive analytical bounds for the parameter \(w_0\) which enters the boundary conditions for the YM potentials.

As seen in Figures 5, 6, one important difference between odd-\(m > 1\) and even-\(m\) solutions is that the former configurations are disconnected from the vacuum\(^{14}\). That is, solutions with \(M = 0\) are found for \(m = 2k\) only. This can be understood heuristically as follows. For any azimuthal winding number \(n \geq 1\), the solutions with an even \(m\) are found by varying the parameter \(w_0\) in \((32)\), starting with \(w_0 = 0\). However, in that case \(w_0 = 0\) results in a similar set of boundary conditions at \(r = 0\) and at infinity. The only solution we could find in this case corresponds to the trivial one, \(H_1 = H_3 = 0, H_2 = H_4 = 1\), with \(M = 0\). Then \(m = 2k\) solutions with a small value of \(|w_0|\) would be just deformations of this ground state and would possess a small mass. However, this does not hold for \(m = 2k + 1 > 1\), since any value of \(w_0\) is leading to a set of boundary conditions at infinity different from the the trivial solution \(H_1 = H_3 = 0, H_2 = H_4 = 1\) (as set by the boundary conditions at \(r = 0\)). As a result, the mass of the odd-\(m > 1\) solutions possesses always a nonzero minimal value.

An unexpected feature of the odd-\(m \geq 4\) solutions with large enough \(n\) is their non-uniqueness. That is, three different solutions are found for given \(m, n\) and some range of the parameter \(w_0\). This property is shown in Figure 6 for \(m = 4, 6\). To illustrate this behaviour, we also exhibit in Figure 7 the energy isosurfaces of the \(m = 6\) configurations at \(w_0 = 0.7\) and the fixed value of the energy density \(\epsilon = 2.1\).

One can understand this pattern as a manifestation of the composite structure of the \(m \geq 4\) configurations. Indeed, from Figure 7 we can clearly see that a typical \(m = 6\) solution consists of three constituents, each of them representing a \(m = 2\) soliton. Similarly, a \(m = 4\) configuration consists of two \(m = 2\) components. Since each of the \(m = 2\) components of the composite configuration possesses a magnetic dipole moment \([25]\), whose magnitude increases with \(n\), the energy of the dipole-dipole interaction between the components becomes a significant part of the total energy. However this interaction energy can be both repulsive and attractive, depending on the orientation of the dipoles. Thus the lowest branch corresponds to the aligned triplet of dipoles, when the forces are most attractive (Figure 7, right plot), and two other branches correspond to the higher mass solutions with two other possible orientations of the triplet of dipoles.

Also, one can observe that, for \(m = 2k\) and \(w_0 = 1\) in \((32)\), one finds zero magnetic charge configuration\(^{15}\) which are the direct AdS counterparts of the solutions in \([25]\).

---

\(^{14}\)Since the \(m = 1\) and \(m = 2\) configurations coincide, this feature occurs also for the \(m = 1\) case.

\(^{15}\)As noticed already, such solutions exist also for \(m = 2k + 1\) and \(w_0^2 = 1\).
Figure 1. The profiles of a typical $m = 1$ solution with $n = 2$, $u_0 = 0.1$. 
Figure 2. The profiles of a typical $m = 3$ solution with $n = 2$, $w_0 = 0.1$. 
Figure 3. The profiles of a typical $m = 4$ solution with $n = 5$, $\omega_0 = 1$. 
Figure 4. The profiles of a typical $m = 5$ solution with $n = 2, u_0 = 0.1$. 
Figure 5. The spectrum of solutions is shown for several odd-$m$ solutions.

Figure 6. The same as Figure 5 for even-$m$ solutions. One can notice the non-uniqueness of solutions for large enough values of the winding number $n$. 
Figure 7. The energy isosurfaces at the value of the energy density $\epsilon = -T_t = 2.1$ of the $m = 6, n = 6$ solutions are shown for $w_0 = 0.7$.

3.2 From AdS YM monopoles to flat space dynamical YM configurations

We close this section by remarking that the solutions discussed above may be relevant in yet another direction. It is well known that the coordinate transformation (see, e.g., Eq. (39)

$$r = \frac{2R\ell^2}{\ell^2 + T^2 - R^2}, \quad t = \ell \left( \arctan \left( \frac{R + T}{\ell} \right) - \arctan \left( \frac{R - T}{\ell} \right) \right),$$

(39)

puts the AdS metric (3) in a conformally flat form

$$ds^2 = \frac{4\ell^2}{(\ell^2 + T^2 - R^2)^2} (dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) - dT^2).$$

(40)

Then, due to the conformal invariance of the $d = 4$ YM system, any solution of the YM equations in an AdS background results in a solution in a Minkowski spacetime.

For the case of the axially symmetric configurations discussed above, these configurations are described by a YM ansatz with a supplementary electric potential as compared to (8):

$$A_{\mu} dx^\mu = \left( H_1(R, \theta, T) \frac{R}{R} dR + (1 - H_2(R, \theta, T)) d\theta \right) \frac{u_2^{(n)}}{2e}$$

$$- n \sin \theta \left( H_3(R, \theta, T) \frac{u_3^{(n)}}{2e} + (1 - H_4(R, \theta, T)) \frac{u_4^{(n)}}{2e} \right) d\phi + V(R, \theta, T) \frac{u_2^{(n)}}{2e} dT,$$

(41)

with the same functions $H_2, H_3, H_4$ and

$$H_1(R, \theta, T) = H_1 \frac{\ell^2 + T^2 + R^2}{\ell^2 + T^2 - R^2}, \quad V(R, \theta, T) = -\frac{2H_1}{\ell^2 + T^2 - R^2}.$$  

(42)

One can see the gauge potentials acquire a dependence on the time coordinate $T$, via the transformation (39).

Due to their numerical nature, the study of the physical properties of the corresponding Minkowski spacetime YM solutions is a difficult task, which is beyond the purposes of this work. Here we mention only

---

\textsuperscript{16} One can consider as well YM solutions in a fixed Einstein universe background, see e.g. [47].
that the AdS exact solution \(26\) corresponds to the flat spacetime meron configuration \([48]\), written in a special gauge \([1]\). Thus we conjecture that at least the \(m = 1, n > 1\) AdS solutions are likely to be relevant to the subject of multi-merons.

4 Including the backreaction: Einstein–Yang-Mills-\(\Lambda\) solutions

Now we shall address the question on how the YM solutions discussed in the previous Section would deform the spacetime geometry, by including the backreaction and solving the full set of EYM equations.

4.1 The EYM action and field equations

We consider the Einstein-Yang-Mills action with a cosmological term:

\[
I_{\text{bulk}} = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G} (R - 2\Lambda) - \text{Tr}\{\frac{1}{2} F_{\mu\nu} F^{\mu\nu}\} \right].
\] (43)

When including the backreaction, apart from the YM equations (4), one solves also the Einstein equations,

\[
E_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + 8\pi G T_{\mu\nu} = 0.
\] (44)

This model has two length scales: the Planck one, \(L_{Pl} = \sqrt{\frac{4\pi G}{e}}\), and the cosmological one, \(\ell\), which is fixed by the cosmological constant. These two length scales define the dimensionless ratio

\[
\alpha = \frac{\sqrt{4\pi G}}{e\ell}.
\] (45)

4.2 EYM-\(\Lambda\) system as a truncation of the \(N = 4\) gauged supergravity

For the generic EYM case, the value of \(\alpha\) is not fixed but an input parameter of the theory. However, among all values of the cosmological constant, there is a case of special interest. This corresponds to a consistent truncation of the \(N = 4\) SO(4) gauged supergravity in \(d = 4\) [28]. Note that this supergravity model can be viewed as a reduction of \(d = 11\) supergravity on \(S^7\) [29]. Thus these particular EYM-\(\Lambda\) solutions have a higher-dimensional interpretation.

The bosonic sector of \(N = 4\) SO(4) gauged supergravity contains two SU(2) fields \(F_{\mu\nu}\) and \(\tilde{F}_{\mu\nu}\), a dilaton \(\phi\) and an axion \(\chi\). In the conventions of [28], the Lagrangian density of the model reads

\[
\mathcal{L} = -\frac{1}{2} \partial_{\mu}\phi \partial^{\mu}\phi - \frac{1}{2} e^{2\phi} \partial_{\mu}\chi \partial^{\mu}\chi - V(\phi, \chi) - \frac{1}{2} e^{-\phi} F_{\mu\nu}^{a} F^{a\mu\nu} - \frac{1}{2} e^{\phi} \tilde{F}_{\mu\nu}^{a} \tilde{F}^{a\mu\nu} - \frac{1}{2} e^{\phi} \epsilon_{\mu\nu\rho\sigma} F^{a\mu\nu} F^{a\rho\sigma} + \frac{1}{2} e^{-\phi} \epsilon_{\mu\nu\rho\sigma} \tilde{F}^{a\mu\nu} \tilde{F}^{a\rho\sigma},
\] (46)

where the potential \(V(\phi, \chi)\) is

\[
V(\phi, \chi) = -2 e^2 (4 + 2 \cosh \phi + \chi^2 e^\phi).
\] (47)

It is easy to verify that

\[
\chi = \phi = 0 \quad \text{and} \quad \tilde{F}_{\mu\nu} = F_{\mu\nu},
\] (48)

is a consistent truncation of the general model [46]. As a result, we end up with the EYM-\(\Lambda\) Lagrangian

\[
\mathcal{L} = R + 12 e^2 - F_{\mu\nu}^{a} F^{a\mu\nu}
\] (49)
Working in conventions with a length scale set by $\ell$ (i.e., taking in the numerics $\Lambda = -3$), it follows that the generic EYM-$\Lambda$ Lagrangian reduces to \[ \alpha = \sqrt{2}, \]
(note also that $e = 1/\sqrt{2}$). As a result, all EYM solutions with the above particular ratio between Planck and cosmological length scales extremize also the action of the $d = 11$ supergravity. Then, by using the formulas in \[ \text{[28]} \]
with two equal gauge fields, $A^a = A^\alpha$, one can uplift any such $d = 4$ configuration to eleven dimensions. For example, the corresponding $d = 11$ metric ansatz reads
$ds^2_{11} = ds^2_4 + 4d\xi^2 + \cos^2 \xi \sum_a (\Theta^a - A^\mu_\nu dx^\mu)^2 + \sin^2 \xi \sum_a (\tilde{\Theta}^a - A^\mu_\nu dx^\mu)^2,$
noindent where $ds^2_4$ is the four dimensional line element, and $(\Theta^a, \tilde{\Theta}^a)$ are SU(2) right invariant one forms on two 3-spheres $S^3, \tilde{S}^3$. The corresponding expression of the $d = 11$ matter fields can be found e.g. in \[ \text{[29]}, \]
and we shall not display them here.

Also, note that, however, the solutions discussed in this work are generically not supersymmetric. The interesting task of constructing supersymmetric configurations would require a different approach\[ \text{[17]} \].

4.3 Solving the gravity field equations: the Einstein-De Turck approach

4.3.1 The metric ansatz

The previous work \[ \text{[22]} \] has solved the EYM field equations by using a standard approach as originally proposed in \[ \text{[24]} \] for asymptotically flat solutions. There one employs a metric ansatz with three unknown functions, $f_i \ (i = 1, 2, 3)$ and
$ds^2 = f_1(r, \theta) \left( \frac{dr^2}{N(r)} + r^2 d\theta^2 \right) + f_2(r, \theta) r^2 \sin^2 \theta d\varphi^2 - f_0(r, \theta) N(r) dt^2,$
noindent (we recall $N(r) = 1 + r^2 \ell^2$). The line-element \[ \text{[52]} \] is inspired by the one in \[ \text{[24]} \] and uses a quasi-conformal choice of the gauge for the $(r, \theta)$-part of the metric (i.e. $g_{r\theta} = 0, g_{\theta\theta}/g_{rr} = r^2 N(r)$). In this approach, one solves only a part of the full set of Einstein equations \[ \text{[44]}, \]
namely
$E^r_r + E^\theta_\theta = 0, \quad E^\varphi_\varphi = 0, \quad E^t_t = 0.$

The remaining equations, $E^r_\theta - E^\theta_\theta$ and $E^0_\varphi$ provide two constraints, which were used to test the numerical accuracy of the results.

Our choice in this work was to construct the solutions by employing the Einstein-De Turck approach. This approach has been proposed in \[ \text{[40], [41]} \] (see also \[ \text{[42]} \] for a review), and been employed recently in the study of various asymptotically AdS configurations\[ \text{[18]} \]. This scheme has the advantage of not fixing apriori a metric gauge, and leads to an overall better quality of the numerical results.

In this approach one solves the so called Einstein-DeTurck (EDT) equations
$R_{\mu\nu} - \nabla_{(\mu} \xi_{\nu)} = \Lambda g_{\mu\nu} + 2\alpha^2 T_{\mu\nu},$
instead of \[ \text{[44]}, \]
where $\xi^\mu$ is a vector defined as
$\xi^\mu = g^{\nu\rho} (\Gamma^\mu_{\nu\rho} - \tilde{\Gamma}^\mu_{\nu\rho}),$

and $\Gamma^\mu_{\nu\rho}$ is the Levi-Civita connection associated to the spacetime metric $g$ that one wants to determine. Also, a reference metric $\tilde{g}$ is introduced, with $\tilde{\Gamma}_{\nu\rho}$ the corresponding Levi-Civita connection. Solutions to

\[ \text{[17]} \] To our knowledge, this task has not been considered yet in the literature. The supersymmetric solutions in \[ \text{[38]} \] have been found for a different truncation of the $N = 4 \ SO(4)$ gauged supergravity model than \[ \text{[18]} \].

\[ \text{[18]} \] Note, however, that only configurations in a Poincaré patch of AdS (and possibly with Abelian matter fields), have been considered so far in the literature.
solve the Einstein equations iff $\xi^\mu \equiv 0$ everywhere on $\mathcal{M}$. To achieve this, we shall impose boundary conditions which are compatible with $\xi^\mu = 0$ on the boundary of the domain of integration. Then, this should imply $\xi^\mu \equiv 0$ everywhere, a condition which is verified from the numerical output.

In this approach, we use a metric ansatz with two additional functions as compared to (52),
\[
d s^2 = f_1(r, \theta) \frac{d r^2}{N(r)} + S_1(r, \theta) (r d \theta + S_2(r, \theta) d r)^2 + f_2(r, \theta) r^2 \sin^2 \theta d \phi^2 - f_0(r, \theta) N(r) d t^2.
\]
The obvious reference metric is AdS$_4$ spacetime with
\[
S_1 = f_1 = f_2 = f_0 = 1, \quad S_2 = 0.
\]

### 4.3.2 The boundary conditions

While the boundary conditions for the gauge potentials $H_i$ are similar to those used in the probe limit (and we shall not give them again here), the choice of the boundary conditions for the metric functions require special care. These boundary conditions are found by constructing an approximate form of the solutions on the boundary of the domain of integration compatible with the requirement $\xi^\mu = 0$. For example, as $r \to 0$, the solution reads
\[
f_1(r, \theta) = \frac{(c_1^2 - 1)c_2}{c_1 + \cos 2\theta} + f_{12}(\theta) r^2 + \ldots, \quad f_2(r, \theta) = c_2(1 + c_1) + f_{22}(\theta) r^2 + \ldots, \quad f_0(r, \theta) = f_{00} + f_{02}(\theta) r^2 + \ldots,
\]

where $c_1, c_2, f_{00}$ are parameters which result from the numerics. This implies the following boundary conditions, which are imposed in the numerics:
\[
\frac{\partial_r f_1}{\partial r} \bigg|_{r=0} = \frac{\partial_r f_2}{\partial r} \bigg|_{r=0} = \frac{\partial_r f_0}{\partial r} \bigg|_{r=0} = \frac{\partial_r S_1}{\partial r} \bigg|_{r=0} = \frac{\partial_r S_2}{\partial r} \bigg|_{r=0} = 0.
\]

The far field behaviour of the solutions is
\[
f_0 = 1 + \frac{f_{03}(\theta)}{r^3} + O(1/r^4), \quad f_1 = O(1/r^4), \quad f_2 = 1 + \frac{f_{23}(\theta)}{r^3} + O(1/r^4), \quad S_1 = 1 + \frac{s_{13}(\theta)}{r^3} + O(1/r^4), \quad S_2 = O(1/r^5),
\]
with $f_{03}(\theta), f_{23}(\theta), s_{13}(\theta)$ functions fixed by the numerics. These functions satisfy the relations
\[
f_{03} + f_{23} + s_{13} = 0, \quad \cos \theta (s_{13} - f_{23}) = \sin \theta \frac{d}{d \theta} (f_{03} + f_{23} - s_{13}).
\]
Thus we impose
\[
f_1 \bigg|_{r=\infty} = f_2 \bigg|_{r=\infty} = f_0 \bigg|_{r=\infty} = S_1 \bigg|_{r=\infty} = 1, \quad S_2 \bigg|_{r=\infty} = 0.
\]

For $\theta \to 0$ one finds
\[
f_i = f_i^0(r) + f_i^2(r) \theta^2 + \ldots, \quad (i = 1, 2, 3),
\]
\[
S_1 = S_1^0(r) + S_1^2(r) \theta^2 + \ldots, \quad S_2 = S_2^1(r) \theta + \ldots,
\]
(a similar expression holds for $\theta \to \pi$). Then the boundary conditions at $\theta = 0$ are
\[
\frac{\partial_\theta f_1}{\partial \theta} \bigg|_{\theta=0} = \frac{\partial_\theta f_2}{\partial \theta} \bigg|_{\theta=0} = \frac{\partial_\theta f_0}{\partial \theta} \bigg|_{\theta=0} = \frac{\partial_\theta S_1}{\partial \theta} \bigg|_{\theta=0} = \frac{\partial_\theta S_2}{\partial \theta} \bigg|_{\theta=0} = 0.
\]
Moreover, we shall assume again that the solutions are symmetric w.r.t. a reflection in the equatorial plane, which implies
\[
\frac{\partial_\theta f_1}{\partial \theta = \pi/2} = \frac{\partial_\theta f_2}{\partial \theta = \pi/2} = \frac{\partial_\theta f_0}{\partial \theta = \pi/2} = \frac{\partial_\theta S_1}{\partial \theta = \pi/2} = \frac{\partial_\theta S_2}{\partial \theta = \pi/2} = 0.
\]
4.3.3 The holographic stress tensor and the mass of the solutions

Our solutions should describe some equilibrium states in the dual CFT, defined in an Einstein universe in \(d = 3\) dimensions, with a line element \(ds^2 = \ell^2(d\theta^2 + \sin^2 \theta \, d\bar{\theta}^2 - dt^2)\) (note that in the absence of a horizon, they do not possess an intrinsic temperature \(T\)). The SU(2) gauge symmetry of the bulk action corresponds to a global SU(2) symmetry in the dual field theory. Also, for the considered boundary conditions, it is natural to work in a grand canonical ensemble, the central quantity being the grand potential \(W\). In AdS/CFT, we identify \(W\) with \(T\) times the on-shell bulk action (in Euclidean signature). We thus analytically continue to Euclidean signature and compactify the time direction with \((\text{an arbitrary})\ \beta = 1/T\). Note that since our solutions are static, the analytical continuation \(t \rightarrow i\tau\) has no effects at the level of the equations of motion, and the solutions discussed above extremize also the Euclidean action.

As usual, the \(I_{\text{bulk}}\) action [43] is supplemented with a Gibbons-Hawking boundary term \(I_{\text{GH}}\) [43], and a boundary counterterm \(I_{\text{ct}}\) [44]. Both \(I_{\text{bulk}}\) and \(I_{\text{GH}}\) exhibit divergences, which are canceled by \(I_{\text{ct}}\). Following the standard prescription, we introduce a hypersurface \(r = r_0\) with some large but finite \(r_0\). Then we ultimately remove the regulator by taking \(r_0 \to \infty\). The leading terms in the asymptotic form of the metric functions have been already computed above, Rel. (58). Then, by using the equations of motion (supposing a fixed point with \(\xi^\mu = 0\)), and the Killing identity \(\nabla^\mu \nabla_\nu K^K = R^K_{\mu
u}K^K\), for the Killing vector \(K^K = \delta^K_t\), one arrives at the final expression for the grand potential \(W\):

\[
W = \frac{3}{8G\ell^2} \int_0^\pi d\theta (f_{23}(\theta) + s_{13}(\theta)) \sin \theta .
\] (60)

Within the same approach, one can construct a divergence-free boundary stress tensor according to the prescription in [44], by defining

\[
T_{ab} = \frac{2}{\sqrt{-h}} \frac{\delta I}{\delta h^{ab}} = \frac{1}{8\pi G} (K_{ab} - K h_{ab} - \frac{2}{\ell} h_{ab} + \ell E_{ab}),
\]

where \(E_{ab}\) is the Einstein tensor of the induced boundary metric \(h_{ab}\), and \(K_{ab}\) is the extrinsic curvature.

One finds in this way the following large-\(r\) expressions of the nonvanishing components of the boundary stress tensor:

\[
T_{\theta\theta} = -\frac{3}{16\pi G\ell} \left( \frac{f_{03}(\theta) + f_{23}(\theta)}{r^3} \right) + \ldots, \quad T_{\phi\phi} = \frac{3}{16\pi G\ell} \left( \frac{f_{23}(\theta)}{r^3} \right) + \ldots, \quad T_{tt} = \frac{3}{16\pi G\ell} \left( \frac{f_{03}(\theta)}{r^3} \right) + \ldots.
\] (62)

Note that this is a traceless tensor, which is a result expected from the AdS/CFT correspondence, since even-dimensional bulk theories are dual to odd-dimensional CFTs that have a vanishing trace anomaly.

From \(T_{ab}\), one can define a mass of the solutions as the conserved charge associated with time translation symmetry of the induced boundary metric \(h\) [44]. A straightforward computation leads to the following expression:

\[
M = \frac{3}{8G\ell^2} \int_0^\pi d\theta (f_{23}(\theta) + s_{13}(\theta)) \sin \theta,
\]

which, as expected in the absence of a horizon, equals the grand potential \(W\).

4.4 Remarks on the numerics

In the numerical construction of the solutions in this work we have used a slight generalization of the techniques employed in [24], [25] for asymptotically flat EYM regular configurations.

Our numerical scheme can be summarized as follows. In a first step, one chooses a suitable combination of the Einstein-De Turck–YM equations, such that the differential equations for the metric and gauge functions are diagonal in the second derivatives \(\partial^2_{\theta\theta}\) with respect to \(r\) (one should remark that the Einstein-De Turck...
This branch survives when including the backreaction, in which case one finds, however, a smaller size of
construct solutions strongly decreases.

There, as noticed above, one single branch of non-gravitating solutions is found, centered around
\( w = \frac{x}{1 - x} \),

(63)

with \( 0 \leq x \leq 1 \), such that we do not introduce a cutoff value \( r_{max} \). For the derivatives, this leads to the
following substitutions at the level of the equations

\[
\begin{align*}
  rF_r \rightarrow x(1 - x)F_x, & \quad rF_{r\theta} \rightarrow x(1 - x)F_{x\theta}, & \quad r^2F_{rr} \rightarrow x^2((1 - x)^2F_{xx} - 2(1 - x)F_x) \\
\end{align*}
\]

for any function \( F = (H; f_0; f_1, f_2, S_1, S_2) \). In the next step, the equations are discretized on a \((x, \theta)\) grid
with \( N_x \times N_\theta \) points. Usually, the grid spacing in the \( x \)-direction is non-uniform, while the values of the
grid points in the angular direction are given by \( \theta_k = (k - 1)\pi/(2(N_\theta - 1)) \). Typical grids have sizes around
150 \times 30 points, a limit imposed by the computational constraints.

The resulting system is solved iteratively until convergence is achieved. All numerical calculations have
been performed by using the professional package CADSOL, which uses a Newton-Raphson finite difference
method with an arbitrary grid and arbitrary consistency order (a detailed description of this package is given in [50]).
This code solves a given system of nonlinear partial differential equations subject to a set of boundary
conditions on a rectangular domain. Apart from some initial guess for the solution, CADSOL requires also
the Jacobian matrices for the equations \( w.r.t. \) the unknown functions and their first and second derivatives,
and the boundary conditions. This software package provides also error estimates for each function, which
allows to judge the quality of the computed solution. The typical numerical error for the solutions reported
in this work is estimated to be of the order of \( 10^{-4} \) (also, the order of the difference formulae was 6).

Another independent test of the numerics is provided by the functions \((\xi^1, \xi^2)\) as defined by [55], together
with their norm. Also, to monitor the numerical errors and test the convergence of our code, we have
computed \( R \), the Ricci scalar (we recall that we set \( \ell = 1 \) which implies \( R = -12 \)). The results presented
below have typically \( |\xi|^2 < 10^{-8} \), while the maximal value for \(|1 + R/12|\) was around \( 10^{-4} \) (note that the
numerical errors are maximal close to \( r = 0 \) and much smaller for large \( r \)). Also, as discussed below, the
numerical accuracy deteriorates towards the limits of the \( w_0 \)-interval, and the \( m = 3 \) configurations have
typically a lower accuracy.

4.5 The solutions

In the numerics, we have studied in a systematic way the solutions which are relevant for the case of the
\( \mathcal{N} = 4 \) SO(4) gauged supergravity model. However, smaller values of \( \alpha \) have been considered as well (see
the last part of this Section) to clarify the relation between the YM solutions in the fixed AdS background
and the generalized Bartnik-McKinnon solutions in the asymptotically flat space [25].

For the case of main interest, \( \alpha = \sqrt{2} \), we have been able to establish the existence of gravitating
generalizations of some of the branches discussed in Section 3. There we have studied systematically solutions
with \( m = 1, 4 \) and \( n = 1, 2, 3, 4, 5 \) as well as \( m = 3, n = 1 \). A number of configurations with \( m = 6, n = 1 \)
have been also constructed.

Since the profiles of the gauge potentials are rather similar to those found in the probe limit, we shall
not display them here. The profiles of the metric functions of two typical \( m = 1, 4 \) solutions are shown in
Figure 8.

For all considered cases, the solutions always exist for a single \( w_0 \)-interval only (see Figures 9, 10), with
\( w_0^{\min} < w_0 < w_0^{\max} \) (we recall that the parameter \( w_0 \) enters the boundary conditions at infinity, fixing the
magnetic charge). However, when including the backreaction, the allowed range of \( w_0 \) for which we could
construct solutions strongly decreases.

This can be understood in analogy with the simpler \( m = 1, n = 1 \) spherically symmetric case [11, 39].
There, as noticed above, one single branch of non-gravitating solutions is found, centered around \( w_0 = 1 \).
This branch survives when including the backreaction, in which case one finds, however, a smaller size of
the \( w_0 \)-interval, as compared to the probe limit [26]. Moreover, the size of this interval decreases with \( |\Lambda| \).

\[20\text{Note that only nodeless solutions } (H_2 = H_4 = w(r) > 0) \text{ are found for large enough } |\Lambda|.\]
Figure 8. The profiles of the metric functions for typical $\alpha = \sqrt{2}$, $m = 1, 4$ solutions with $n = 4$, $w_0 = 0.75$ and $n = 3$, $w_0 = -0.175$, respectively.
Figure 9. The spectrum of $m = 1, 4$ magnetically charged solutions of the $\mathcal{N} = 4$ $SO(4)$ gauged supergravity model.

Figure 10. Left: The spectrum of $m = 3$, $n = 1$ magnetically charged gravitating solutions with several values of the coupling constant $\alpha$. Right: The value of the metric function $f_0 = -g_{tt}$ at $r = 0$ is shown for the same configurations.

As $w_0 \to w_0^{\text{min}}$, an extremal black hole is approached, with a finite mass and horizon area, while the solutions diverge as $w_0 \to w_0^{\text{max}}$ [13].

However, as found in [39], new $w_0$-intervals emerge as $|\Lambda|$ becomes smaller and the shape of the branches has approximate self-similarity. When the parameter $\Lambda$ approaches zero, an already-existing $w_0$-interval of monopole solutions collapses to a single point in the moduli space, $w_0 = \pm 1$ being the only allowed value[21]. This implies the existence of a fractal structure in the moduli space of the solutions [39].

The same pattern is likely to exist for other values of $(m, n)$ (although the study of the solutions is much more difficult in this case). To clarify this behaviour in the odd-$m$ case ($m > 1$), we plot in Figure 9 (left) the spectrum of solutions for $(m = 3, n = 1)$ configurations with several values of $\alpha$. One can see that the $w_0$-interval decreases with increasing $\alpha$. At the same time, the numerical construction of the solutions becomes more and more intricate. This can be partially understood from Figure 10 (right), where we show the value of the metric function $g_{tt}$ at $r = 0$ (which is an invariant quantity).

[21] Viewed from this perspective, the fundamental AdS branch of solution reduces to the one-node Bartnik-McKinnon solution.
Figure 11. Surfaces of constant energy density close to the maximum value of $-T^t_t$ are shown for typical $m = 3$ (left) and $m = 4$ (right) gravitating solutions.

One can see that, for any allowed $w_0$ and $\alpha = \sqrt{2}$, the values of $f_0(0)$ are very small; at the same time, the values at $r = 0$ of the functions $f_1$, $S_1$ (not shown there) are always very large.

This behaviour makes the numerical construction of the $m = 3$ solution within the $\mathcal{N} = 4$ SO(4) gauged supergravity model very difficult, since the gravity effects are always very strong in this case. In particular, we could not clarify the issue of limiting solutions at the ends of the $w_0$-interval.

However, the numerical construction of $m = 4$ configurations has proven to be much simpler. Since the boundary value $w_0 = 0$ corresponds to the AdS background case, these solutions possess a weak gravity regime (for small enough $|w_0|$). However, the AdS background becomes increasingly deformed as we increase $|w_0|$, with larger values of $M$.

As a new feature, we notice that for both $m = 3$ and $m = 4$, the decrease of the size of the $w_0$-interval results in the disappearance, for $\alpha = \sqrt{2}$, of the solutions with a zero net magnetic charge, $Q_M = 0$. Therefore, no counterparts of the even-$m$ solutions in [25] exist for the case of the $\mathcal{N} = 4$ SO(4) gauged supergravity model. Moreover, the same applies also for $m = 1$, in which case, the branches of solitons cease to exist before approaching $w_0 = -1$ (note that $w_0 = 1$ there corresponds to the vacuum case $M = 0$). We conclude that all solutions of the $\mathcal{N} = 4$ SO(4) gauged supergravity model with excited nA fields, possess a net magnetic charge.

However, this is a $\Lambda$–dependent feature, and the magnetically neutral solutions are recovered for large enough values of $\ell$ (see the discussion below for the case $m = 4$).

Also, we have found that for $m = 1, 3$ the energy of the solutions is located in a small region around the origin. The energy density typically exhibits a torus-like structure, with a maximum being located in the equatorial plane. (This becomes apparent by considering surfaces of constant energy density of, e.g., half the respective maximal value of the energy density.) Moreover, a double-tori structure appears for large enough values of $n$. However, this changes for higher values of $m$, in which case the maximum of the energy density is typically located on the symmetry axis. Thus these are composite configurations with a dumbbell-like structure. Such configurations have two distinct components only, located symmetrically with respect to the equatorial plane, and kept in balance due to the repulsive stresses of the YM fields. These features are illustrated in Figure 11, where we plot surfaces of constant energy density for $m = 1, 4$ gravitating solutions.

\footnote{Note that the numerical accuracy of the reported $m = 3$ gravitating solutions with $\alpha = \sqrt{2}$ is much lower as compared to all other solutions in this work. At the same time, the $m = 3$ solutions with small enough $\alpha$ could be constructed with very good accuracy.}
An interesting pattern is observed, once we abandon restrictions on the value of the parameters of the model imposed by the reduction of the $\mathcal{N} = 4$ SO(4) gauged supergravity. In particular, we consider the dependence of the solutions on the value of the coupling constant $\alpha^{(45)}$. Furthermore, we may restrict our consideration to the most interesting case $w_0 = 0$ (for $m = 2k + 1$) and $w_0 = 1$ (for $m = 2k$), in which case the boundary conditions are similar to those valid in the asymptotically flat space.

Note that the dimensionless coupling constant $\alpha^{(45)}$ vanishes if (i) the Newton constant $G \to 0$, or, (ii) $\ell \to \infty$ (or, equivalently, $\Lambda \to 0$). In the former case one recovers the YM solutions in a fixed globally AdS background, while the latter case corresponds to the strongly coupled gravitating configurations in the asymptotically flat spacetime discussed in [25].

We have found numerical evidence that, when gravity is coupled to the YM model in the globally AdS spacetime, a weakly gravitating solution emerges from a particular $(m,n)$ configuration which we discussed in the Section 3. For these solitons the values of the metric functions at small $\alpha$ are not very much different from those corresponding to the background metric, see (56). However, as the coupling constant $\alpha$ increases, the background is more and more deformed. At the same time, $\alpha$ cannot be arbitrarily large, the configurations approaching a critical solution at some maximal value, $\alpha_{cr}$ (which depends on $(m,n)$, see Figure 13). There a backbending in $\alpha$ is observed, with the occurrence of the second branch of solutions. With decreasing $\alpha$, the solutions smoothly reach the corresponding (generalized) Bartnik-McKinnon solutions in [8, 52, 25], in the limit $\alpha \to 0$. Along this branch, the mass of the $(m,n)$ configurations is higher than the mass of the corresponding solution on the branch linked to the global AdS space. Also, the gravitational interaction remains strong and the metric function $f_0$ at the origin approaches some very small but finite value, as seen in Figure 13.

We illustrate this pattern with a particular example of the $(m = 4, n = 1)$ configurations presented in Figure 12 (left). There we exhibit the evolution of the value of the mass and of the metric function $f(0)$ at the origin for the $w_0 = 1$ solutions along both branches. As seen in Figure 12 (right), a similar pattern is expected to hold for other values of $w_0$ (note also that the maximal (critical) value of $\alpha$ decreases with $w_0$).

Now let us consider the matter and metric functions for small values of the rescaled coupling constant $\alpha$. In Figure 13, top panel, we exhibit the gauge field function $H_2$ and the metric function $f_0$ for $n = 1$ configurations with $m = 3, 4, 5, 6$, on the upper branch at $\alpha = 0.05$. 

\textbf{Figure 12.} Left: The values of the mass and the metric function $f_0(0)$ of the $(m = 4, n = 1)$ configurations with zero magnetic charge ($w_0 = 1$) are plotted as functions of $\alpha$. Right: The value of the metric function $f_0(0)$ of the $(m = 4, n = 1)$ configurations is plotted as a function of $\alpha$ for several values of the parameter $w_0$. 

\textbf{4.6 EYM-Λ configurations: from probe limit to generalized Bartnik-McKinnon solutions}
Figure 13. Top plots: The profiles of the metric function $f_0$ and the gauge function $H_2$ of the $n = 1, m = 3, 4, 5, 6$, configurations on the upper branches are shown at $\alpha = 0.05$.

Bottom plot: The values of the metric function $f_0(0)$ of the $n = 1$ configurations with $(m = 4, 6, w_0 = 1)$ and $(m = 3, 5, w_0 = 0)$ are plotted as functions of $\alpha$.

Clearly, there is an internal region where the gravitational interaction is strong and the YM configuration agrees well with the first member of spherically symmetric Bartnik-McKinnon family of solutions (in which case $H_1 = H_3 = 0, H_2 = H_4 = w(r)$, with $w(r)$ interpolating monotonically between 1 and $-1$). In the outer region the metric functions rapidly approach their trivial values and the remaining $(m - 2, 1)$ configuration approaches to the corresponding solution in the fixed AdS spacetime, which we considered above.

On the other hand, in the scaled coordinate $r \to r/\alpha$, the inner region expands all the way up to infinity and contains the generalized Bartnik-McKinnon solution with finite rescaled mass $\hat{M} = \alpha M$. Thus, the outer AdS configuration is shifted up to infinity.

Note that this type of behavior is almost identical to the picture which holds for self-gravitating composite solitons in the EYMH theory [53, 54]. In the latter case the gravitational coupling constant dependence reveals a similar two-branch structure, however the branch lower in energy emerges from the corresponding composite YMH solitons in the flat space [50, 57]. At some maximal value of the coupling, this branch bifurcates with the upper branch which, once again, extends all the way back to the limit of vanishing coupling constant where it approaches the corresponding generalized Bartnik-McKinnon solution. The difference to
the EYM solitons in AdS spacetime is that in the EYMH model the evolution along the upper branch is related with the decrease of the expectation value of the Higgs field while in the former case it is related with the decrease of $|\Lambda|$.

5 Further remarks. Conclusions

The purpose of this work was to extend the spectrum of known nA solutions in an AdS background by reporting a whole new class of solitons indexed by two integers $(m, n)$ and a continuous parameter $w_0$, which fixes the magnetic charge of the solutions. The main motivation for this task comes from the observation that the EYM action (43) enters the $d = 4$ gauged supergravities as the basic building block and one can expect the basic features of its solutions to be generic.

Our solutions are akin to the asymptotically flat static, axially symmetric EYM configurations studied in [25]. However, we have found that new features occur due to the different asymptotic structure of the spacetime. The most prominent one is perhaps the existence in this case of a set of non-gravitating YM solutions which, for $\Lambda = 0$, are forbidden in the absence of a Higgs field (the odd-$m$ configurations, in the conventions of this work). Also, the generic solutions possess a nonvanishing magnetic flux on the sphere at infinity, being interpreted as non-topological monopoles.

New features occur also when including the backreaction of the YM fields on the spacetime geometry. A rather unexpected result there is the absence, for large enough values of $|\Lambda|$, of magnetically neutral configurations. In particular, all solutions we have found for the case of main interest, corresponding to a consistent truncation of $\mathcal{N} = 4$ SO(4) gauged supergravity, carry a non-zero magnetic charge. Another interesting result is the existence of balanced, regular composite configurations, with several distinct components. To our knowledge, this is the first example in the literature of AdS$_4$ solutions with this property.

One question we did not address concerns the stability of the new solutions. However, it is known that some $(m = 1, n = 1)$ EYM solitons are stable, and there is all reason to believe, that at least some of the new $(m, n)$ configurations are also stable.

There are various possible extensions of the solutions discussed in this work. First, our preliminary results indicate the existence of static axially symmetric black hole solutions with nA hair. These solutions are found within the same YM ansatz [8], and the same boundary conditions at infinity, by using again the EDT approach. However, all configurations we have constructed so far possess a single horizon of spherical topology. Finding asymptotically AdS$_4$ multi-black hole solutions remains a challenge.

Another possible direction would be to generalize these configurations by including an electric potential for the YM connections. These axially symmetric solitons would carry automatically a nonzero electric charge and a non-vanishing angular momentum density.

Finally, perhaps the most interesting task would be to generalize the solutions in this work for a Poincaré patch of the AdS spacetime. In this case, all known solutions are just counterparts of the simplest $(m = 1, n = 1)$ spherically symmetric configurations in a globally AdS background. We expect at least the $m = 1$ configurations to possess planar generalizations with an azimuthal winding number $n > 1$.

We hope to report elsewhere on these problems.

Acknowledgements.– We acknowledge support by the DFG Research Training Group 1620 Models of Gravity. E.R. gratefully acknowledges support from the FCT-IF programme and Ya.S. support by the A. von Humboldt Foundation in the framework of the Institutes Linkage Programm.

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23 It is interesting to note that the AdS$_4$ generalization of the Israel-Khan multi-black hole solution [51] has not been found yet (however, these configurations would possess conical singularities for $\Lambda < 0$ as well). Moreover, the Majumdar-Papapetrou [20], [27] Einstein-Maxwell extremal black holes are also not known for a global AdS background (interestingly, such solutions exist for $\Lambda > 0$ [50], [58]).

24 In principle, one cannot exclude apriori that e.g. a composite EYM configuration with two solitons located symmetrically on the $z$-axis would lead to a balanced dihole, when adding a small black hole at the center of each soliton.
A  The spherically symmetric EYM-Λ system: an exact solution

The AdS generalizations of the Λ = 0 Bartnik-McKinnon configurations have been constructed in [11] by Bjoraker and Hosotani. In such a study, it is convenient to employ Schwarzschild-like coordinates, with a metric ansatz:

\[ ds^2 = \frac{dr^2}{N(r)} + r^2(d\theta^2 + \sin^2 \theta \, d\varphi^2) - N(r)\sigma^2(r)dt^2 \]  

where

\[ N(r) = 1 - \frac{2m(r)}{r} - \frac{\Lambda}{3}r^2, \]

\[ m(r) \] corresponding to the local mass-energy density. The corresponding Yang-Mills ansatz is a truncation of [8], with \( n = 1 \) and \( H_1 = H_3 = 0, \) \( H_2 = H_4 = w(r). \)

Then the EYM equations take the simple form

\[
\begin{align*}
\frac{m'}{r} &= \alpha^2(\omega^2 N + \frac{(w^2 - 1)^2}{2r^2}), \\
\frac{w''}{N} + \frac{\sigma'}{\sigma}w' + \frac{w(1 - w^2)}{r^2N} &= 0, \\
\sigma' &= 2\alpha^2\sigma\omega r.
\end{align*}
\]

The solutions with a regular origin have the following behaviour at \( r = 0 \)

\[ w(r) = 1 - br^2 + O(r^4), \quad m(r) = 2a^2b^2r^3 + O(r^5), \quad \sigma(r) = \sigma_0(1 + 4ab^2r^2) + O(r^4), \]  

where \( b, \sigma_0 \) are real constants. The corresponding expansion at large \( r \) reads

\[ m(r) = M + \frac{\alpha^2}{6r}(2\Lambda w_1^2 - 3(1 - w_0^2)^2) + \ldots, \quad w(r) = w_0 + \frac{w_1}{r} + \frac{3w_0(1 - w_0^2)}{2\Lambda r^2} + \ldots, \quad \sigma(r) = 1 - \frac{\alpha^2 w_1^2}{2r^4} + \ldots, \]  

where \( w_0, M \) and \( w_1 \) are constants determined by numerical calculations. \( M \) corresponds to the ADM mass of the solutions, while \( w_0 \) determines the value of the magnetic charge, \( Q_m = |1 - w_0^2| \).

The numerical solutions are found by varying the parameter \( b \) which enters the expansion at the origin [3]. Note that \( b \) takes both positive and negative values, with \( b = 0 \) corresponding to the ground (vacuum) state

\[ w(r) = 1, \quad m(r) = 0, \quad \sigma(r) = 1. \]  

In this way a continuum of monopole configurations is found, which are regular in the entire space. In particular, one finds solutions also for small \( b \), which are arbitrarily close to the vacuum state (i.e., \( w(r) \) close to one everywhere).

This suggests to construct the configurations as a power series expansion around the vacuum state [5],

\[ w(r) = 1 + \sum_{k \geq 1} e^k w_k(r), \quad m(r) = \sum_{k \geq 1} e^k m_k(r), \quad \sigma(r) = 1 + \sum_{k \geq 1} e^k \sigma_k(r), \]

with \( e \) a small parameter. To our best knowledge, although very simple, this approach did not appear in the literature.

To second order in \( e \), the solution of the EYM equations reads

\[
\begin{align*}
w_1(r) &= 1 - \frac{\ell}{r} \arctan \frac{r}{\ell}, & w_2(r) &= 3\frac{\ell}{4r} \arctan \frac{r}{\ell} \left( \frac{r}{\ell} + \frac{\ell}{r} \right) \arctan \frac{r}{\ell} - 1, \\
m_1(r) &= 0, & m_2(r) &= \frac{\alpha^2}{r} \left( 1 - \frac{\ell}{r} \arctan \frac{r}{\ell} \right) \left( \frac{r}{\ell} + \frac{\ell}{r} \right) \arctan \frac{r}{\ell} - 1, \\
\sigma_1(r) &= 0, & \sigma_2(r) &= \frac{\alpha^2}{2\ell r^2} \left[ 3\pi^2 - \ell^2 (1 + \frac{2}{1 + \frac{\ell^2}{r^2}}) - \ell^4 r^4 \arctan \frac{r}{\ell} \left( -2r + \frac{6r^3}{\ell^3} + (1 + 3\frac{r^4}{\ell^4}) \arctan \frac{r}{\ell} \right) \right].
\end{align*}
\]
Figure 14. The spectrum of spherically symmetric solutions as given by the numerics and by the analytical estimates.

Although one can extend the solution to higher orders, it was not possible to identify a general pattern\textsuperscript{25} the expressions for the functions becoming increasingly complicated with increasing $k$.

In Figure 14 we plot the spectrum of $\mathcal{N} = 4$ SO(4) solutions as found by solving numerically the EYM equations together with the analytical estimates found from (6). One can see that, for small values of $Q_M$, (6) provides a reasonable approximation of the full numerical solutions. One may expect similar closed form solutions to exist for other values of $(n, m)$ as well.

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\textsuperscript{25}Note that the parameter $\epsilon$ can be related to $b$ entering the small-$r$ expansion (3), and also to $w_0$ which enters the asymptotics (4). To second order, one finds $b = \frac{\epsilon^2 + 3\epsilon}{6\epsilon^2}$ and $w_0 = 1 + \epsilon + \epsilon^2 \frac{3\epsilon^2}{16}$, respectively.
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