Nonlinearity induced destruction of resonant tunneling in the Wannier-Stark problem

S. Wimberger\textsuperscript{1}, R. Mennella\textsuperscript{1}, O. Morsch\textsuperscript{1}, E. Arimondo\textsuperscript{1}, A.R. Kolovsky\textsuperscript{2,3}, and A. Buchleitner\textsuperscript{2}
\textsuperscript{1}Dipartimento di Fisica E. Fermi, CNR-INFM, Università di Pisa, Largo Pontecorvo 3, 56127 Pisa, Italy
\textsuperscript{2}Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Str. 38, 01187 Dresden, Germany
\textsuperscript{3}Kirensky Institute of Physics, 660036 Krasnoyarsk, Russia

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We present detailed numerical results on the dynamics of a Bose-Einstein condensate in a tilted periodic optical lattice over many Bloch periods. We show that an increasing atom-atom interaction systematically affects coherent tunneling, and eventually destroys the resonant tunneling peaks.

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Experiments with cold and ultracold atoms made it possible in the last decade to prepare and control the centre-of-mass motion of atoms with unprecedented precision. Many toy models of either many-body solid state physics \cite{1,2,3,4,5} or of simple Hamiltonian systems, whose complexity arises from an external driving force \cite{6}, were realized with the exceptional control offered by static or time-dependent optical potentials.

Particularly Bose-Einstein condensates (BEC) whose initial momentum spread can be adjusted in width and absolute position have proved to be an extremely helpful experimental tool \cite{3,4,7,8,9}. In addition, a BEC offers interesting new features originating from the intrinsic interactions between the atoms. Examples of such effects are new quantum phases \cite{10}, soliton-like motion \cite{11}, the occurrence of energetic or dynamical instabilities in condensates \cite{12,13,14}, or the decay and subsequent revival of Bloch oscillations (BO) \cite{15}.

We focus on the evolution of a BEC loaded into a one-dimensional lattice and subjected to an additional static or time-dependent optical potentials.

We present detailed numerical results on the dynamics of a Bose-Einstein condensate in a tilted periodic optical lattice over many Bloch periods. We show that an increasing atom-atom interaction systematically affects coherent tunneling, and eventually destroys the resonant tunneling peaks.

If we neglect interactions for a moment, our system will be described by the Hamiltonian

\begin{equation}
H = -\frac{\hbar^2}{2M} \frac{d^2}{dx^2} + V \sin^2 \left( \frac{\pi x}{F_L} \right) + F x .
\end{equation}

Here $d_L$ is the spatial period of the optical lattice with maximal amplitude $V$, and $M$ the atomic mass. Eq. (1) defines the well-known Wannier-Stark problem, which gives rise to BO with period $F_{\text{Bloch}} = h/d_L F$ ($h$ is Planck’s constant). If tunneling is small, we can view the system as moving at a constant speed in momentum space within the fundamental BZ. At the zone edge, most of the wave packet is reflected (giving rise to BO) while a small part can tunnel across the first band gap to the next higher-lying energy band and then escape quickly by successive tunneling events across the smaller (higher) band gaps. Landau-Zener theory predicts a decay rate \cite{16}

\begin{equation}
\Gamma(F) \propto F e^{-\frac{b}{F}},
\end{equation}

where $b$ is proportional to the square of the energy gaps. Eq. (2) is modified by RET which occurs when two Wannier-Stark levels in neighboring potential wells are coupled strongly due to their accidental degeneracy. The RET results in pronounced peaks in the tunneling rates, e.g., as a function of $1/F$ on top of the global exponential decay described by \cite{17,18}. In this paper we investigate the impact of the effective shift of the Wannier-Stark levels by a nonlinear interaction term.

For the linear problem \cite{11}, the decay rates have been measured previously in the regime of short life-times in the ground state band (of the order of 100 $\mu$s), where $\Gamma(F)$ is essentially smooth \cite{17}. Since RET is a coherent quantum effect, the peaks should be sensitively affected by the atom-atom interaction, which can be varied experimentally by changing either the density of the BEC or through the atom-atom scattering potential via a Feshbach resonance \cite{19}. Our results are a consequence of many sequential Landau-Zener events, and they show the destruction of a RET peak with increasing interaction strength, in a regime which is experimentally accessible.

We use a fully 3D Gross-Pitaevskii equation (GPE) \cite{20} to describe the temporal evolution of a BEC which
is subject to realistic potentials:

\[ i \hbar \frac{\partial}{\partial t} \psi(\vec{r}, t) = \left[ -\frac{e^2}{2M} \nabla^2 + \frac{1}{2} \left( \omega^2 x^2 + \omega^2 r^2 \right) + V \sin^2 \left( \frac{\pi}{d_L} \right) + F x + gN |\psi(\vec{r}, t)|^2 \right] \psi(\vec{r}, t). \]  

(3)

\( \psi(\vec{r}, t) \) represents the condensate wave function, and the frequencies \( \omega_x \) and \( \omega_r \) characterize the longitudinal and transverse harmonic confinement (here with cylindrical symmetry: \( \rho = \sqrt{y^2 + z^2} \)). We fixed \( d_L = 1.56 \mu m \) and \( V/E_R = 5 \) for our computations, with the recoil energy \( E_R = p_R^2/2M \) for \( p_R = h\pi/d_L \), and the recoil period \( T_R = h/E_R \). The above values for \( d_L \) and \( V \) were realized in the experiments reported in [3, 5] based on two laser beams propagating at an angle different from \( \pi \). In Eq. (3), the nonlinear coupling constant is given by \( g = 4\pi R^2 a_s/M \), where \( a_s \) is the s-wave scattering length and \( N \) the number of atoms in the BEC [15, 21]. The dimensionless nonlinearity \( C = gn_{t}/(8E_R) \) is computed from the peak density of the initial state of the condensate, with \( C = 0.027 \ldots 0.31 \) for the experimentally investigated range of [3], and with \( C = 0.5 \) reached in [21]. Here we focus on \( C > 0 \), but report briefly also on attractive interactions with \( C < 0 \). The latter case leads to a fundamentally different behavior of the system because the collapse of the condensate introduces an additional time scale, which for experimentally relevant parameters is of the order of 10 msec [18, 22] (slightly longer than \( T_{\text{Bloch}} = 1.8 \ldots 3.0 \) msec here).

The GPE (3) is numerically integrated using finite difference propagation, adapted by a predictor-corrector estimate to reliably evaluate the nonlinear interaction [19]. Since our system is essentially the problem of a constantly accelerated particle for the part of the wave function which has tunnelled out of the first BZ already, one must be careful with the application of absorbing boundary conditions or complex coordinate methods [23, 24]. To avoid any spurious effects due to the fast spreading, we use a large numerical basis. In this way, we fully cover the 3D expansion of the entire wave packet, including its tunnelled tail, without the use of non-Hermitian potentials. The initial state propagated by (3) is the relaxed condensate wave function, adiabatically loaded into the confining potential given by the harmonic trap and the optical lattice (with \( F = 0 \)). Approximate analytic forms of the relaxed state are found, e.g., in [22], but we used an imaginary time propagation to reliably compute the initial state for \( C > 0 \).

The linear decay rates for non-interacting atoms in the optical lattice are computed from the spectrum of the 1D Wannier-Stark problem of Eq. (1) using, e.g., the method of [16]. Those linear rates are plotted in Fig. 1. The maxima in the rates occur when \( Fd_L/m \) (with \( m \) integer) is close to the difference between the first two energy bands (averaged over the BZ) of the \( F = 0 \) problem [16]. The actual peaks are slightly shifted with respect to the above estimate (marked by arrows in the inset of Fig. 1), owing to a field-induced level shift [16].

Experimentally, the most easily measurable quantity is the momentum distribution of the BEC obtained from a free expansion after the evolution inside the lattice. From the momentum distribution we determine the survival probability by projection of the evolved state \( \psi(\vec{r}, t) \) onto the support of the initial state

\[ P_{\text{sur}}(t) = \int_{-p_c}^{p_c} dp_x \left( \int dp_y dp_z |\psi(\vec{p}, t)|^2 \right), \]  

(4)

where \( p_c \geq 3p_R \) is a good choice since three momentum peaks are initially significantly populated, corresponding to \(-2p_R, 0, 2p_R\) [3, 23].

Figure 2 shows the initial population in momentum space (inset in (a)) as compared with the population after 10 BO periods, for both the linear and the nonlinear case. The increase of \( C > 0 \) has two effects: firstly, it enhances the tunneling for the first few crossings of the BZ. Secondly, it scrambles the out-coupled part of the wave function (see Fig. 2 and its complement in Fig. 4 below), as previously observed in [2, 3, 5]. The change in the momentum distributions after various Landau-Zener events is a manifestation of the intrinsic instability of the nonlinear GPE dynamics [4, 12]. Instead of studying the details of the distributions shown in Fig. 2 we will focus on the temporal decay of the survival probability in the following. Figure 3 presents \( P_{\text{sur}}(t) \), which for the linear case has an exponential form (apart from the \( t \to 0 \) limit [24]):

\[ P_{\text{sur}}(t) \sim e^{-\Gamma t}, \]  

(5)
with the characteristic exponent $\Gamma$. The temporal behavior of $P_{\text{sur}}$ depends significantly on $C$. For $C = \pm 0.31$, we observe clear deviations from a purely exponential decay, as present for small $C$. A repulsive nonlinearity initially enhances the tunneling more than after about five crossings of the BZ (see fits to data in Fig. 3). This deviation from the mono-exponential behavior means that the tunneling events occurring at different integer multiples of the Bloch period are correlated by the presence of the nonlinearity. Since the remaining density becomes smaller, the impact of the nonlinearity becomes less. The result is that the rate $\Gamma$ is defined only locally in time, and its value systematically decreases as time increases.

An attractive interaction can stabilize the system at the RET peak, which is shown for $C = -0.31$ in Fig. 3(b). For optimal comparison, we chose the same initial state (for $C = +0.31$) which then was evolved for $F \neq 0$ with $C = -0.31$. Such a scenario could be realized by a sudden change of the sign of the scattering length through the tunneling events occurring at different integer multiples of the Bloch period and then compare the ratio of the linear and the nonlinear area (denoted by $A_0$ and $A_C$). This rough estimate $\Gamma_C \approx \Gamma_0 A_0/A_C$ agrees within 25% with the rate extracted from the fits to the data of Fig. 3. The estimate could be improved if we knew the analytic form of the function $f(t)$, and it breaks down for large $C$, when the periodic oscillations in $P_{\text{rec}}$ are destroyed.

Having introduced two methods to extract the tunneling rates, we scan the parameter $F$ across a RET peak of the globally exponential curve $\Gamma(1/F)$ (see Fig. 4). The scanned range in $F$ corresponds to values of lattice accelerations between 0.99 ms$^{-2}$ and 1.65 ms$^{-2}$, which are standard in experiments.

A repulsive nonlinearity particularly affects the wings of the peak and, for small $C$, much less the peak maximum. The global increase of $\Gamma$ with increasing $C$ is qualitatively predicted in Fig. 2, with enhanced single Landau-Zener crossing probabilities induced by the effective reduction of the energy gap due to the nonlinearity. The left and right-most points in Fig. 4 are in the regime where an amended version of (2) indeed applies, and here $\Gamma/F$ is approximately proportional to $C$. However, near the peak, the rates do not follow a simple scaling.
Wannier-Stark problem. We verified that letting \( \omega_x \) tend to zero for the evolution with \( F \neq 0 \), or applying a small finite \( \omega_x \) gives the same results for the BO cycles studied here. Furthermore, for \( 0 < C \lesssim 0.05 \), using the renormalized nonlinearity of \( \mathcal{P} \) we observed that a 1D version of Eq. (4) reproduces well the 3D data. If \( |C| \) is larger, the nonlinearity couples the longitudinal and transverse degrees of freedom, which affects the dynamics of a real BEC in a non-trivial way \[21\]. The 1D computations are feasible up to 100 Bloch periods, and this would allow one to extract the tunneling rates more reliably. The effect of the nonlinearity is, however, hardly visible for \( 0 < C < 0.05 \), and quantitative predictions for a broad range of \( C \) relied on 3D computations.

To summarize, we observed and quantified the deformation and destruction of the RET peaks due to interactions in a BEC in an accelerated optical lattice. Our results complement ongoing studies of interaction-induced processes such as dynamical instabilities or the decay and subsequent revival of BO. In the regime of small nonlinearity, where dynamical instabilities are not fully developed, the survival and recurrence probabilities experience an exponential decay modified by the condensate nonlinearity. The temporal decay of these observables remains a useful indicator also for large nonlinearity, even if the resonant structure in the tunneling rate is washed out.

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