SUPERCURRENT COUPLING IN THE FADDEEV-SKYRME MODEL

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Abstract. Motivated by the sigma model limit of multicomponent Ginzburg-Landau theory, a version of the Faddeev-Skyrme model is considered in which the scalar field is coupled dynamically to a one-form field called the supercurrent. This coupled model is investigated in the general setting where physical space is an oriented Riemannian manifold and the target space is a Kähler manifold. It is shown that supercurrent coupling destroys the topological stability enjoyed by the usual Faddeev-Skyrme model, so that there can be no globally stable knot solitons in this model. Nonetheless, local energy minimizers may still exist. The first variation formula is derived and used to construct three families of static solutions of the model, all on compact domains. In particular, a coupled version of the unit-charge hopfion on a three-sphere of arbitrary radius is found. The second variation formula is derived, and used to analyze the stability of some of these solutions. A family of stable solutions is identified, though these may exist only in spaces of even dimension. Finally, it is shown that, in contrast to the uncoupled model, the coupled unit hopfion on the three-sphere of radius $R$ is unstable for all $R$. This gives an explicit, exact example of supercurrent coupling destabilizing a stable solution of the uncoupled Faddeev-Skyrme model, and casts doubt on the conjecture of Babaev, Faddeev and Niemi that knot solitons should exist in the low-energy regime of two-component superconductors.

1. Introduction

The Faddeev-Skyrme model is a nonlinear scalar field theory which possesses so-called knot solitons, classified topologically by their Hopf degree. Motivated by the sigma model limit of two-component Ginzburg-Landau theory, Babaev, Faddeev and Niemi conjectured that such knot solitons may exist in the low energy regime of certain exotic superconductors [2]. However, the sigma model limit contains another dynamical field besides the usual (two-sphere valued) scalar field of the Faddeev-Skyrme model, a one-form field which may be interpreted physically as the supercurrent. In this paper we study this extended version of the Faddeev-Skyrme model in which dynamical coupling to the supercurrent is taken into account, concentrating primarily on the (analytically more accessible) case of compact physical domain. We will see that supercurrent coupling destroys the topological stability enjoyed by knot solitons, in that configurations of arbitrarily small energy can be found in every homotopy class. This should be contrasted with the usual Faddeev-Skyrme model where one has the Vakulenko-Kapitanski bound on $\mathbb{R}^3$ [16], or a linear energy bound on compact domains [13]. We will develop the variational calculus for the coupled model in a rather general geometric setting, and use our results to show explicitly and exactly that supercurrent coupling destabilizes the unit charge “hopfion” on the three-sphere of small radius.

Consider static two-component Ginzburg-Landau theory on physical space $M = \mathbb{R}^3$. This field theory models, among other things, certain exotic superconducting materials, including liquid metallic hydrogen, in which there are two different species of charge-carrying Cooper pairs [1]. It consists of two complex scalar fields $\psi_a : M \rightarrow \mathbb{C}$.
\( C, a = 1, 2, \) minimally coupled to a \( U(1) \) gauge connexion \( A \in \Omega^1(M) \), so that the total energy functional is

\[
E_{GL} = \frac{1}{2} \| d_A \psi_1 \|^2 + \frac{1}{2} \| d_A \psi_2 \|^2 + \frac{1}{2} \| dA \|^2 + \int_M V(\psi_1, \psi_2)
\]

where \( d_A \psi_a = d\psi - iA\psi \), \( \| \cdot \| \) denotes \( L^2 \) norm and \( V \) is a phenomenologically determined interaction potential whose details depend strongly on the precise physical context. To preserve gauge invariance, one must have \( V(\lambda \psi_1, \lambda \psi_2) = V(\psi_1, \psi_2) \) for all \( \lambda \in U(1) \). Babaev, Faddeev and Niemi [2] made the following interesting observation (the essential argument appeared somewhat earlier in a paper of Hindmarsh [5], and perhaps goes back further still). Define the total condensate density \( \rho = \sqrt{|\psi_1|^2 + |\psi_2|^2} \) and let \( \varphi = [\psi_1, \psi_2] : M \to \mathbb{C}P^1 \equiv S^2 \). Note that both these fields are gauge invariant and that the second field makes sense globally only if \( \psi_1, \psi_2 \) never vanish simultaneously, presumably a sensible assumption provided \( V(0, 0) \) is made sufficiently large. Further, let \( \rho^2 C \) be the total supercurrent of the condensates, that is,

\[
C = \frac{i}{2\rho^2} \sum_{a=1}^2 (\overline{\psi_a} d_A \psi_a - \psi_a d_A \overline{\psi_a}),
\]

which is also gauge invariant. Then the Ginzburg-Landau energy is precisely

\[
E_{GL} = \frac{1}{8} \| \rho d\varphi \|^2 + \frac{1}{2} \| dC \| + \frac{1}{2} d\varphi^* \omega \| + \frac{1}{2} \| d\rho \|^2 + \frac{1}{2} \| \rho C \|^2 + \int_M \hat{V}(\rho, \varphi)
\]

where \( \omega \) is the Kähler form on \( \mathbb{C}P^1 \) (equivalently, the area form on \( S^2 \)). The first two terms of this energy are strongly reminiscent of the Faddeev-Skyrme energy of a \( S^2 \)-valued field on \( M \). In fact, if \( \rho \) is constant and \( C = 0 \), \( E_{GL} \) reduces precisely to \( E_{FS} \), the Faddeev-Skyrme energy of \( \varphi \). This led Babaev, Faddeev and Niemi to suggest that the GL model, like its truncated version, possesses knot solitons, in which the field \( \varphi : \mathbb{R}^3 \to S^2 \) has nonzero Hopf degree.

The numerical evidence concerning this claim is a little mixed, but seems, on the whole, to be negative [8, 13, 6]. (For a comprehensive review of the current status of knot solitons in field theory, see [10].) In particular, a crucial role in destabilizing localized \( \varphi \) configurations of nonzero Hopf degree seems to be played by the coupling of \( \varphi \) and \( C \). That is, we can imagine reducing \( E_{GL} \) to \( E_{FS} \) by a two-step truncation. In the first step, we impose a sigma model limit on the \( \mathbb{C}^2 \) valued field \( (\psi_1, \psi_2) \), demanding that \( \rho^2 = |\psi_1|^2 + |\psi_2|^2 = \rho_0^2 = 1 \) everywhere, motivated by the choice \( V = \lambda(1 - |\psi_1|^2 - |\psi_2|^2)^2 \) in the limit of large \( \lambda \), for example. (There is no loss of generality in the choice \( \rho_0 = 1 \) since any other \( \rho_0 \) can be scaled away by a homothety of \( M \).) This yields a “halfway house” model, which one might call the supercurrent coupled Faddeev-Skyrme model (henceforth, the SCFS model),

\[
E(\varphi, C) = \frac{1}{8} \| d\varphi \|^2 + \frac{1}{2} \| dC \| + \frac{1}{2} d\varphi^* \omega \| + \frac{1}{2} \| C \|^2
\]

where, for simplicity, we have assumed the potential \( V(\psi_1, \psi_2) \) has \( U(2) \) symmetry (so is constant on surfaces of constant \( \rho \)). Experience with similar ungauged GL models [3] suggests that this truncation is unlikely to cause trouble (that is, if \( E \) has minimizers with nonzero Hopf degree, they probably survive the thawing of
the field $\rho$). The second step, where we set $C$ to zero, is more problematic. This amounts to ignoring the cross term $\frac{1}{2}(dC, \varphi^*\omega)$ in the second term of $E(\varphi, C)$, and there does not seem to be a strong justification for this.

This motivates us to study the SCFS model (1.4) in detail, and compare its properties with those of the usual Faddeev-Skyrme model (henceforth, FS model). Since generalization costs no extra effort, and working in a natural geometric context often reveals structure otherwise hidden, we will study both the SCFS and FS models in the general case where $\varphi : M \rightarrow N$, $M$ being an oriented Riemannian manifold and $N$ a Kähler manifold. We will still call $C$ the supercurrent, and interpret $dC + \frac{1}{2}\varphi^*\omega$ as the electromagnetic field two-form (that is, magnetic field, if $\dim M = 3$). Indeed, in the case $N = \mathbb{C}P^{k-1}$, with the Fubini-Study metric of unit holomorphic sectional curvature, this model is precisely the naive sigma model limit of $k$-component Ginzburg-Landau theory [5], which gives it an immediate physical interpretation.

In section 2 we compare the best known topological lower energy bounds for the FS and SCFS models in the case where $M$ has dimension 3 and $N = S^2$. Whereas the FS energy is bounded below by some power of the Hopf degree ($Q^2$ for $M = \mathbb{R}^3$, $Q$ for $M^3$ compact), we will find that the SCFS energy can be arbitrarily small in every (algebraically inessential) homotopy class. This is strikingly different behaviour, and it follows that no nontrivial global minimizer of the SCFS energy can exist in this case. One can still hope for local minimizers, however, and to find these one must solve the variational problem for the SCFS energy. In section 3, we compute the first variation formula for $E(\varphi, C)$, that is, the field equations that a pair $(\varphi, C)$ must satisfy in order to be a critical point of $E$. For a given $\varphi : M \rightarrow N$, we will show there is at most one $C \in \Omega^1(M)$ so that $(\varphi, C)$ is a critical point of $E$. We shall call a critical point $(\varphi, C)$ embedded if $\varphi$ is a critical point of $E_{FS}$, and refer to $(\varphi, C)$ as an embedding of $\varphi$. Clearly an embedding of $\varphi$, if it exists, is unique. We shall construct three families of embedded critical points, all of which are submersive (that is, $d\varphi_x$ is surjective at each $x \in M$). To analyze the stability of a critical point, one must consider the second variation formula for $E$, to which we turn in section 4. We compute an explicit formula for the Hessian operator associated with a critical point, and use this formula to show that the critical point $\varphi = Id : N \rightarrow N, C = 0$ is stable for every Kähler manifold $N$.

Ideally, one would like to apply the first and second variation formulae in the case of direct physical interest, namely $M = \mathbb{R}^3, N = S^2$, but this case of the SCFS model is (like the FS model) analytically intractable. It would be interesting, but very challenging, and beyond the scope of the present work, to conduct a large-scale numerical analysis of this problem. In lieu of hard numerics, we will consider the case nearest to $M = \mathbb{R}^3$ where exact results can be obtained, namely $M = S^3_R$, the sphere of radius $R > 0$. Here, as for $M = \mathbb{R}^3$, configurations are labelled homotopically by their Hopf degree $Q$, but (unlike on $\mathbb{R}^3$) an explicit solution of the FS model in the $Q = 1$ class is known, namely the Hopf fibration $\varphi : S^3 \rightarrow S^2$[17]. This solution is known to be stable for $0 < R \leq 2$[12], so it can be thought of as the spherical analogue of the unit “hopfion” at least on small spheres. In section 4 we will show that this hopfion has a (necessarily unique) embedding in the SCFS model. The associated supercurrent is homogeneous and tangent to the fibres of the
Hopf map. We go on to construct the Hessian operator for this embedded hopfion explicitly, and show that it has a negative eigenvalue of total multiplicity 4, for all $R > 0$. This, then, provides an explicit, exact example (albeit on a compact domain) of the process whereby supercurrent coupling destabilizes a previously stable critical point of the Faddeev-Skyrme energy. Clearly this does not conclusively rule out the existence of stable knot solitons in the SCFS model, but it is one more piece of evidence in favour of scepticism.

2. Energy bounds

We first setup some notation and conventions, following [12]. Let $(M, g)$ be a Riemannian manifold and $(N, h, J)$ be a Kähler manifold, with Kähler form $\omega(X, Y) = h(JX, Y)$. Let $\langle \cdot, \cdot \rangle$ denote $L^2$ inner product, $\| \cdot \|_{L^2}$ norm, $\Omega^k(M)$ the space of smooth $k$-forms on $M$, $\Gamma(E)$ the space of smooth sections of vector bundle $E$, $\delta : \Omega^k \to \Omega^{k-1}$ the coderivative $L^2$ adjoint to $d$, and $\sharp = b^{-1}$ the musical isomorphisms induced by the metric $g$ on $M$. Given $\varphi : M \to N$, $\varphi^{-1}TN$ will denote the vector bundle over $M$ whose fibre above $x \in M$ is $T_{\varphi(x)}N$. The symbol $\nabla$ will denote the Levi-Civita connexion on $TM$, $TN$ or its canonical extension to $T^*M \otimes \varphi^{-1}TN$ (which being clear from context). The pullback of the Levi-Civita connexion on $TN$ to $\varphi^{-1}TN$ will be denoted $\nabla^\varphi$. All maps will be assumed smooth. We shall frequently need to refer to specific terms in the energy functional (1.4), and so define, for any $\varphi : M \to N$ and $C \in \Omega(M)$,

$$E(\varphi, C) = \frac{1}{8} \|d\varphi\|^2 + \frac{1}{2} \|dC + \frac{1}{2} \varphi^* \omega\|^2 + \frac{1}{2} \|C\|^2$$

$$E_1(\varphi) = \frac{1}{2} \|d\varphi\|^2$$

$$E_2(\varphi) = \frac{1}{2} \|\varphi^* \omega\|^2$$

$$E_3(C) = \frac{1}{2} \|dC\|^2 + \frac{1}{2} \|C\|^2$$

$$E_4(\varphi, C) = \frac{1}{2} \langle dC, \varphi^* \omega \rangle$$

$$E_{FS}(\varphi) = E_1(\varphi) + E_2(\varphi),$$

so that $E = \frac{1}{4}(E_1 + E_2) + E_3 + E_4$.

For the rest of this section, $N = S^2$ and $M$ has dimension 3. We begin by contrasting the energy bounds for $E$ and $E_{FS}$ in the case $M = \mathbb{R}^3$. We impose the usual boundary condition on $\varphi$ ($\varphi(x) \to (0, 0, 1)$ as $|x| \to \infty$) so that field configurations are homotopic if and only if they have the same Hopf degree $Q \in \mathbb{Z}$. Recall that

$$Q = \frac{1}{(4\pi)^2} \int_M A \wedge dA$$

(2.1)

where $A \in \Omega^1(M)$ is chosen so that $dA = \varphi^* \omega$ (note that $A$ certainly exists since $\varphi^* \omega$ is closed and $H^2(M) = 0$). Then, for $E_{FS}$ we have the well-known Vakulenko-Kapitanski bound:
Theorem 2.1 (Vakulenko-Kapitanski [16]). There exists a constant \( c > 0 \) such that \( E_{FS}(\varphi) \geq c|Q(\varphi)|^{\frac{4}{3}} \) for all \( \varphi \).

The power \( \frac{4}{3} \) is believed to be sharp, and is certainly consistent with numerics. The optimal constant \( c \) is not known. In contrast to this energy growth with \( |Q| \), we will now see that the infimum of \( E(\varphi, C) \) is zero in every homotopy class. This fact is already known in the physics literature, at least informally, in the multicomponent Ginzburg-Landau setting [9, 10]. The point is that, for a given \( \varphi \), we can choose \( dC \) to exactly cancel \( \frac{1}{2} \varphi^* \omega \) in \( E \), so that \( (\varphi, C) \) has no stability against Derrick scaling.

Proposition 2.2. \( \inf \{ E(\varphi, C) : Q(\varphi) = n \} = 0 \) for all \( n \in \mathbb{Z} \).

Proof. Each degree class contains \( \varphi \) with \( \varphi = (0, 0, 1) \) outside some closed ball \( B \). For this \( \varphi \), there exists \( C \in \Omega^1(\mathbb{R}^3) \) such that \( dC = -\frac{1}{2} \varphi^* \omega \) (since \( H^2(\mathbb{R}^3) = 0 \)). Since \( d\varphi_x = 0 \) for all \( x \notin B \), \( \varphi^* \omega \) vanishes on \( \mathbb{R}^3 \setminus B \). Without loss of generality, we may assume that \( C \) itself vanishes on \( \mathbb{R}^3 \setminus B \). (Assume \( C \) does not vanish on \( \mathbb{R}^3 \setminus B \). \( C \) is closed on \( \mathbb{R}^3 \setminus B \) and hence there exists \( f : \mathbb{R}^3 \setminus B \to \mathbb{R} \) such that \( C = df \), since \( H^1(\mathbb{R}^3 \setminus B) = 0 \). We may smoothly extend \( f \) to a function on the whole of \( \mathbb{R}^3 \). Let \( C' = C - df \). Then \( dC' = dC = -\frac{1}{2} \varphi^* \omega \) and \( C' \) vanishes on \( \mathbb{R}^3 \setminus B \).) For each \( \lambda > 0 \) let \( D_\lambda : \mathbb{R}^3 \to \mathbb{R}^3 \) be the dilation map \( D_\lambda(x) = \lambda x \). Clearly \( \varphi \circ D_\lambda \) is homotopic to \( \varphi \) and, by the usual Derrick argument [4],

\[
E(\varphi_\lambda, C_\lambda) = \frac{1}{8\lambda} \| d\varphi \|^2 + \frac{1}{2\lambda} \| C \|^2 \to 0
\]

as \( \lambda \to \infty \). \( \square \)

We turn now to the case where \( M \) is a compact oriented 3-manifold. The homotopy classification of maps \( M \to S^2 \) was completed by Pontryagin.

Theorem 2.3 (Pontryagin [9]). Let \( M \) be a compact, connected, oriented 3-manifold, and \( \mu \) be a generator of \( H^2(S^2; \mathbb{Z}) \cong \mathbb{Z} \) (for example, \( \mu = \omega/4\pi \) in the de Rham model). The homotopy classes of based maps \( \varphi : M \to S^2 \) fall into disjoint families labelled by \( [\varphi^* \mu] \in H^2(M; \mathbb{Z}) \). Within the family of classes with fixed \( [\varphi^* \mu] \), the classes are labelled by elements of \( H^3(M; \mathbb{Z})/(2[\varphi^* \mu] \sim H^1(M; \mathbb{Z})) \).

If \( H^2(M; \mathbb{Z}) \neq 0 \), therefore, maps \( M \to S^2 \) are not classified homotopically by a single integer. We will be interested primarily in the case \( M = S^3 \), where this complication does not arise. However, even on general \( M \), there is a family of maps which are classified homotopically by a single integer, the algebraically inessential maps, that is, those for which \( [\varphi^* \mu] = 0 \). These maps fall into homotopy classes labelled by \( H^3(M; \mathbb{Z}) \cong \mathbb{Z} \), and one may identify this integer homotopy invariant with the Hopf degree of \( \varphi \), defined as in (2.1). From the standpoint of topological solitons, algebraically inessential maps are the most interesting, since all other maps have the property that regular preimages \( \varphi^{-1}(p) \subset M \) are nontrivial in \( H_1(M) \) (one may think of such preimages as being Poincaré dual to \( [\varphi^* \mu] \in H^2(M; \mathbb{Z}) \)), so they are not, in a topological sense, spatially localized (they are “tied” to some nontrivial 1-cycle in \( M \)).

So, on a compact domain, provided we restrict to the case of algebraically inessential maps (no restriction if \( H^2(M; \mathbb{Z}) = 0 \)), configurations \( \varphi : M \to S^2 \) are still classified by their Hopf degree \( Q(\varphi) \). An interesting fact, which does not seem to
have been appreciated previously, is that the Faddeev-Skyrme energy $E_{FS}$ grows at least linearly with $|Q|$ in this case, in contrast to the $|Q|^\frac{3}{2}$ growth found on $\mathbb{R}^3$.

The essential proof has appeared previously for the case $M = S^3$ [13], but adapts readily to the case of general $M$.

**Theorem 2.4** (Speight-Svensson). *Let $M$ be a compact, connected, oriented Riemannian 3-manifold and $\varphi : M \to S^2$ be a smooth, algebraically inessential map. Then

$$E_{FS}(\varphi) > E_2(\varphi) \geq \frac{\sqrt{\lambda_1}}{32\pi^2} |Q(\varphi)|$$

where $\lambda_1 > 0$ is the first nonzero eigenvalue of the Laplacian restricted to coexact one-forms on $M$.*

**Proof.** Since $\varphi$ is algebraically inessential, $\varphi^*\omega$ is exact, and there exists $A \in \Omega^1(M)$ such that $dA = \varphi^*\omega$. By Hodge decomposition, we may assume, without loss of generality that $A$ is coexact. (Consider the Hodge decomposition of $A$, $A = A_h + dA_0 + \delta A_2$, where $A_h$ is a harmonic one-form, $A_0$ is a zero-form and $A_2$ a two-form. Then $\varphi^*\omega = dA = 0 + 0 + d\delta A_2$.) We may also assume that $Q(\varphi) > 0$. (The bound is trivial if $Q = 0$, and if $Q(\varphi) < 0$ then $Q(\varphi \circ \rho) = -Q(\varphi)$ where $\rho : M \to M$ is an orientation reversing diffeomorphism.) Then

$$E_2(\varphi) = \frac{1}{2}\|\varphi^*\omega\|^4 = \frac{1}{2} \langle dA, dA \rangle = \frac{1}{2} \langle A, \Delta A \rangle \geq \frac{1}{2} \lambda_1 \|A\|^2.$$

But

$$(4\pi)^2 Q(\varphi) = \langle A, *dA \rangle \leq \|A\| \|dA\| = \|A\| \sqrt{2E_2(\varphi)}.$$ 

The result follows immediately. $\square$

Note that this bound follows from a straightforward estimate of just the Skyrme term $E_2$ in $E_{FS}$, using only the Cauchy-Schwartz inequality and ellipticity of $\Delta$. By contrast, the Vakulenko-Kapitanski bound involves a subtle trade-off between $E_1$ and $E_2$, and uses rather more advanced estimates from functional analysis. Note also that under a homothety $g \mapsto R^2 g$, the spectrum of $\Delta$ scales as $\lambda \mapsto \lambda/R^2$, so our bound becomes trivial in the limit where $M$ attains infinite volume (for example $M = S^3_R$, the three-sphere of radius $R$, in the limit $R \to \infty$). It is an interesting and (apparently) open question whether the power $|Q|^1$ in this bound is optimal.

Once again, the issue of primary interest in this paper is the effect that supercurrent coupling has on this energy bound. In fact, as for the model on $\mathbb{R}^3$, the bound disappears entirely in the SCFS model.

**Proposition 2.5.** *Let $M$ be a compact, connected, oriented 3-manifold. Then for each $n \in \mathbb{Z}$, $\inf E(\varphi, C) = 0$, where the infimum is taken over all smooth, algebraically inessential maps $\varphi : M \to S^2$ with $Q(\varphi) = n$, and all $C \in \Omega^1(M)$.*

**Proof.** Each algebraically inessential homotopy class contains maps which are constant outside some (arbitrarily small) closed ball. The argument of the proof of Proposition [2.2] can be applied to such maps. $\square$

So there can be no global minimizer of $E(\varphi, C)$ in any nontrivial algebraically inessential homotopy class. It does not follow that the SCFS model can have no
stable static solutions, however, since local minima may exist. To seek them, we require the first variation formula for $E$.

3. The first variation

In this section and the next, we revert to the general setting in which $M$ is an oriented Riemannian manifold and $N$ is a Kähler manifold.

**Proposition 3.1.** Let $\phi_t$ be a variation of $\phi_0 = \phi : M \to N$ and $C_t$ a variation of $C_0 = C \in \Omega^1(M)$. Let $X = \partial_t \phi|_{t=0} \in \Gamma(\phi^{-1}TN)$ and $Y = \partial_t C|_{t=0} \in \Omega^1(M)$, both assumed to be of compact support (if $M$ is noncompact). Then the corresponding first variation of $E$ is

$$
\left. \frac{dE(\phi_t, C_t)}{dt} \right|_{t=0} = \langle X, -\frac{1}{4}(\tau(\phi) + Jd\phi^\sharp\delta(\phi^*\omega + 2dC)) \rangle + \langle Y, \delta(dC + \frac{1}{2}\phi^*\omega) + C \rangle
$$

where $\tau(\phi)$ is the tension field of the mapping $\phi$, that is, $\tau(\phi) = \text{tr} \nabla d\phi$.

**Proof.** Using the notation introduced in Section 2 above, one has $E(\phi_t, C_t) = \frac{1}{4}(E_1(t) + E_2(t)) + E_3(t) + E_4(t)$. It is well known [14, p. 131] that

(3.1) $\dot{E}_1(0) = -\langle X, \tau(\phi) \rangle$,

and it was shown in [12] that

(3.2) $\dot{E}_2(0) = -\langle X, Jd\phi^\sharp\delta(\phi^*\omega) \rangle$.

Turning to $E_3$, one sees that

(3.3) $\dot{E}_3(0) = \langle dY, dC \rangle + \langle Y, C \rangle = \langle Y, \delta dC + C \rangle$.

Finally, by the homotopy lemma (see, for example, [12]),

(3.4) $\dot{E}_4(0) = \frac{1}{2}\langle dY, \phi^*\omega \rangle + \frac{1}{2}\langle dC, d(\phi^*\iota_X\omega) \rangle = \frac{1}{2}\langle Y, \delta(\phi^*\omega) \rangle + \frac{1}{2}\langle \delta dC, \phi^*\iota_X\omega \rangle$.

Now, for any $\eta \in \Omega^1(M)$,

(3.5) $g(\eta, \phi^*\iota_X\omega) = (\phi^*\iota_X\omega)(\sharp\eta) = \omega(X, d\phi^\sharp\eta) = -h(X, Jd\phi^\sharp\eta)$.

Hence

(3.6) $\langle \eta, \phi^*\iota_X\omega \rangle = \int_M g(\eta, \phi^*\iota_X\omega) = -\langle X, Jd\phi^\sharp\eta \rangle$.

Applying this in the case $\eta = \delta dC$, we have, from (3.4)

(3.7) $\dot{E}_4(0) = \frac{1}{2}\langle Y, \delta(\phi^*\omega) \rangle - \frac{1}{2}\langle X, Jd\phi^\sharp\delta dC \rangle$.

The result immediately follows.

The first variation formula, or field equations, follow immediately from this:

**Corollary 3.2.** $(\phi, C)$ is a critical point of $E$ if and only if

(3.8) $\delta(dC + \frac{1}{2}\phi^*\omega) + C = 0$
(3.9) $\tau(\phi) - 2Jd\phi^\sharp C = 0$

where, once again, $\tau(\phi) = \text{tr} \nabla d\phi$. 

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Proof. For all variations \( X, Y \) in Proposition 3.1 we must have \( \dot{E}(0) = 0 \). Hence, by the fundamental lemma of the calculus of variations,

\[ \delta(dC + \frac{1}{2} \phi^* \omega) + C = 0, \]

\[ -\frac{1}{4}(\tau(\phi) + Jd\phi \delta(\phi^* \omega + 2dC)) = 0. \]

Substituting equation (3.10) into (3.11) we obtain the pair claimed. \( \square \)

**Corollary 3.3.** Let \( \phi : M \to N \). If there exists \( C \in \Omega^1(M) \) such that \((\phi, C)\) is a critical point of \( E \), then \( C \) is unique.

**Proof.** Assume that \( C' \in \Omega^1(M) \) also renders \((\phi, C')\) a critical point. Then by equation (3.8), \( C'' = C - C' \) satisfies \( \delta dC'' + C'' = 0 \), whence \( 0 = (C'', \delta dC'' + C'') = \|dC''\|^2 + \|C''\|^2 \). Hence \( C'' = 0 \). \( \square \)

**Definition 3.4.** A critical point \((\phi, C)\) of \( E \) will be called an embedding of \( \phi \) if \( \phi \) is a critical point of \( E_{FS} \). By Corollary 3.3, if an embedding of \( \phi \) exists, it is unique.

It was shown in [12] that \( \phi \) is a critical point of \( E_2 \) if and only if \( \delta \phi^* \omega \in \ker d\phi \) everywhere. If \( \phi \) is also harmonic, so \( \tau(\phi) = 0 \) (hence a critical point of \( E_{FS} \)), it is natural to seek an embedding of \( \phi \) with \( C = \mu \delta \phi^* \omega \), where \( \mu \) is a constant. The point is that equation (3.9) is satisfied automatically in this case, and we are left to check the somewhat simpler equation (3.8). We will apply this idea in the next three examples.

**Example 3.5.** Under what circumstances is \((\phi, 0)\) a critical point of \( E \)? From Corollary 3.2 if and only if \( \phi \) is harmonic \((\tau(\phi) = 0)\) and coclosed \((\delta \phi^* \omega = 0)\). Such a map is separately a critical point of \( E_1 \) and \( E_2 \), and hence is a (very special) critical point of \( E_{FS} \). A trivial example is the identity map \( \text{Id} : N \to N \). This can be easily extended to projection on a Riemannian product \( \phi = \pi : N \times K \to N \). So \( \text{Id} : S^2 \to S^2 \), \( \pi : S^2 \times S^1 \to S^2 \) and \( \pi : S^2 \times \mathbb{R} \to S^2 \) embed trivially (i.e. with \( C = 0 \)) in the coupled model. A less trivial family can be adapted from [13]. Let \( M \) be the space of full flags in \( \mathbb{C}^k \) and \( N \) be the Grassmannian of \( l \)-planes in \( \mathbb{C}^k \). Then one has a natural projection \( \pi_1 : M \to N \) which maps each flag to its \( l \)-dimensional entry. This map is holomorphic and hence harmonic (with respect to the usual \( \text{SU}(k) \) homogeneous metrics on \( M \) and \( N \)), and is coclosed, and hence embeds trivially in the coupled model. In particular, this gives a family of examples \( \pi_1 : M \to \mathbb{C}P^{k-1} \).

**Example 3.6.** Given the physical origin of \( E(\phi, C) \) in multicomponent Ginzburg-Landau theory, it is natural to consider the general Hopf fibration from \( M = S^{2n+1} \subset \mathbb{C}^{n+1} \) to \( N = \mathbb{C}P^n \), given by \( \pi : (z_1, z_2, \ldots, z_{n+1}) \mapsto [z_1, z_2, \ldots, z_{n+1}] \). This map is harmonic, and is known [12] to be a critical point of \( E_2 \), and hence is a critical point of \( E_{FS} \). We will now show that it can be embedded in the coupled model, but in contrast to Example 3.3 the embedding has \( C \neq 0 \).

Note that \( \pi \) is a submersion with one dimensional fibres. At each \( z \in M \subset \mathbb{C}^{n+1} \) we have an orthogonal decomposition \( T_z M = V_z \oplus H_z \) where \( V_z = \ker d\pi_z \) is the vertical space, tangent to the fibres, and \( H_z = V_z^\perp \) is the horizontal space, its orthogonal complement. It is convenient to give \( \dot{N} \) the Fubini-Study metric of
constant holomorphic sectional curvature 4 (in the case \( n = 1 \), this corresponds to giving the target two-sphere radius \( \frac{1}{2} \)) since \( \pi \) is then a Riemannian submersion, that is \( d\pi_z : T_{\pi(z)}N \to \mathcal{H}_z \) is an isometry for all \( z \). Clearly, \( \mathcal{H}_z \) is spanned by the unit vector field \( V(z) = iz \) (which generates the \( U(1) \) action \( z \mapsto e^{it}z \) on \( M \subset \mathbb{C}^{n+1} \)). Since \( \pi \) is a critical point of \( E_2 \), we know that \( d\pi^\# \delta(\pi^*\omega) = 0 \), and hence \( \delta(\pi^*\omega) = f^\flat V \) for some \( f : M \to \mathbb{R} \). By homogeneity of the map, \( f \) must be constant, and a short calculation based (for example) at \( z = (1, 0, \ldots, 0) \) shows that \( f = n \). Another short calculation shows that \( \pi^*\omega = d^\flat V \). Substituting into (3.8) and (3.9) one sees that \( (\pi, \lambda^\flat V) \) is a critical point of \( E \) if (and only if) \( \lambda = \frac{n^2}{2n+2} \).

Note that Example 3.6 includes the standard Hopf fibration \( S^3 \to S^2 \), which may be thought of as the unit charge Hopf “soliton” on \( S^3 \) (the special case \( n = 1 \)).

We may generalize the basic Hopf fibration in a different direction by thinking of it as the coset projection \( SU(2) \to SU(2)/S(U(1) \times U(1)) \) onto a Kähler symmetric space:

**Example 3.7.** Let \( G \) be a compact, connected, simple Lie group and \( K \) be a compact subgroup of \( G \) such that \( G/K \) is an irreducible Hermitian symmetric space of compact type. Denote by \( \mathfrak{g}, \mathfrak{k} \) the Lie algebras of \( G, K \). Then there is an \( \text{Ad}_K \)-invariant subspace \( \mathfrak{p} \) of \( \mathfrak{g} \) such that \( [\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k} \) and

\[
\mathfrak{g} = \mathfrak{k} + \mathfrak{p},
\]

this decomposition being orthogonal with respect to the Killing form. We give \( G \) the bi-invariant metric coinciding with (minus) the Killing form at \( e \), and \( G/K \) the metric which makes the homogeneous projection

\[
\varphi : G \to G/K, \quad g \mapsto g \cdot o
\]
a Riemannian submersion (here \( o \) denotes the identity coset in \( G/K \)). Denote by \((\cdot, \cdot)\) minus the Killing form on \( \mathfrak{g} \). The almost complex structure on \( G/K \) (with respect to which the homogeneous metric is Kähler) coincides with the adjoint action of an element in the centre of \( \mathfrak{k} \). In a slight abuse of notation, we shall denote this element \( J \). By left translation, we may identify vector fields on \( G \) with \( \mathfrak{g} \)-valued functions on \( G \) and sections of \( \varphi^{-1}T(G/K) \) with \( \mathfrak{p} \)-valued functions on \( G \). The connexions on \( TG \) and \( \varphi^{-1}T(G/K) \) are then

\[
\nabla_X Y = dY(X) + \frac{1}{2}[X, Y] \quad (X, Y \in C^\infty(G, \mathfrak{g})),
\]

\[
\nabla^\varphi_X Y = dY(X) + [X, Y]_p \quad (X \in C^\infty(G, \mathfrak{g}), \, Y \in C^\infty(G, \mathfrak{p})),
\]

respectively.

The map \( \varphi \) is well known to be harmonic, and was shown to be a critical point of \( E_2 \) in [12]. In fact

\[
\sharp \delta(\varphi^*\omega) = -\frac{\lambda}{2} J
\]

where \( \lambda \) is the eigenvalue of the Casimir operator associated with the adjoint representation of \( G \), that is,

\[
\mathfrak{g} \to \mathfrak{g}, \quad X \mapsto -\sum_{k=1}^m [e_k, [e_k, X]]
\]
where \( \{e_1, \ldots, e_m\} \) is an orthonormal basis for \( g \). Hence, for any vector fields \( X, Y \) on \( G \),
\[
\begin{align*}
d(\delta \varphi^* \omega)(X,Y) &= -\frac{\lambda}{2} \{ X(J,Y) - Y(J,X) - (J, [X,Y]) \} \\
&= -\frac{\lambda}{2} \{ (\nabla_X J,Y) - (\nabla_Y J,X) \} \\
&= -\frac{\lambda}{2} \{ (dJ(X),Y) - (dJ(Y),X) + \frac{1}{2}([X,J],Y) - \frac{1}{2}([Y,J],X) \} \\
&= -\frac{\lambda}{2} \{ 0 - 0 - \frac{1}{2}(\text{ad}_J X,Y) + \frac{1}{2}(\text{ad}_J Y,X) \} \\
&= \frac{\lambda}{2} \varphi^* \omega(X,Y),
\end{align*}
\]
where we have used the fact that \( J \) is a constant mapping \( M \to g \) (so \( dJ = 0 \)).

Hence
\[
(3.16)\quad d\delta \varphi^* \omega = \frac{\lambda}{2} \varphi^* \omega.
\]

We now seek an embedding of \( \varphi \) with \( C = \mu \delta \varphi^* \omega \). As remarked previously, \( (3.9) \) is satisfied automatically, and, in light of \( (3.16) \), we see that \( (3.8) \) is satisfied if and only if \( \mu = -(\lambda + 2)^{-1} \) (which always exists since the Casimir is a positive operator). In particular, let \( G = SU(n+1) \) and \( K = S(U(1) \times U(n)) \). Then this gives an embedding of the projection \( \varphi : SU(n+1) \to G/K \equiv \mathbb{C}P^n \). Note that the vertical space of the fibration \( \varphi \) has dimension \( n^2 \) in this case, rather than 1, as in Example 3.6. From the point of view of satisfying equation \( (3.9) \), it looks like choosing \( C \) parallel to \( \delta \varphi^* \omega \) is needlessly restrictive (\( C \) need only be restricted to the \( n^2 \)-dimensional vertical space). However, by Corollary 3.3, we know that no alternative embedding of \( \varphi \) exists, despite the apparent extra freedom.

4. Second variation: the Hessian

Let \( (\varphi, C) \) be a critical point of \( E \) and \( (\varphi_{s,t}, C_{s,t}) \) be a two-parameter variation of \( (\varphi, C) = (\varphi_{0,0}, C_{0,0}) \). The Hessian of \( E \) at \( (\varphi, C) \) is
\[
H_{(\varphi, C)}((X,Y), (\hat{X}, \hat{Y})) = \frac{\partial^2 E(\varphi_{s,t}, C_{s,t})}{\partial s \partial t} \bigg|_{s=t=0}
\]
where
\[
\begin{align*}
X &= \partial_s \varphi_{s,t} \big|_{s=t=0}, \quad \hat{X} = \partial_t \varphi_{s,t} \big|_{s=t=0} \in \Gamma(\varphi^{-1}TN), \\
Y &= \partial_s C_{s,t} \big|_{s=t=0}, \quad \hat{Y} = \partial_t C_{s,t} \big|_{s=t=0} \in \Omega^1(M).
\end{align*}
\]
Let \( \mathcal{E} \) denote the vector bundle \( \varphi^{-1}TN \oplus T^*M \) over \( M \). Then \( H_{(\varphi, C)} \) is a symmetric bilinear form on \( \Gamma(\mathcal{E}) \). The critical point \( (\varphi, C) \) is stable if the associated quadratic form is non-negative, that is,
\[
H_{(\varphi, C)}((X,Y), (X,Y)) \geq 0 \quad \text{for all } (X,Y) \in \Gamma(\mathcal{E}).
\]
Otherwise, \( (\varphi, C) \) is unstable. The index of an unstable critical point is the dimension of the largest subspace of \( \Gamma(\mathcal{E}) \) on which the quadratic form is negative. We
Recall that $X$ is a section of $\varphi^{-1}TN$ and $Y$ is a one-form on $M$.

**Proposition 4.1** (Smith, [11]). Let $\varphi : M \to N$ be a critical point of $E_1 = \frac{1}{2}\|d\varphi\|^2$. Then the Hessian of $E_1$ at $\varphi$ is $\langle X, \mathcal{J}_\varphi Y \rangle$ where the Jacobi operator is

$$\mathcal{J}_\varphi Y = \overline{\nabla}_\varphi Y - \mathcal{R}_\varphi Y.$$

The terms of $\mathcal{J}_\varphi$ are defined as follows. Let $\{e_1, \ldots, e_m\}$ be a local orthonormal frame on $M$ and $R$ be the curvature of $\nabla$ on $N$. Then the rough Laplacian is

$$\overline{\nabla}_\varphi Y = \sum_{k=1}^m (\nabla_{e_k} Y - \nabla_{e_k} e_k Y)$$

and

$$\mathcal{R}_\varphi Y = \sum_{k=1}^m R(Y, d\varphi E_k)d\varphi E_k.$$

**Proposition 4.2** (Speight-Svensson, [12]). Let $\varphi : M \to N$ be a critical point of $E_2 = \frac{1}{2}\|\varphi^*\omega\|^2$. Then the Hessian of $E_2$ at $\varphi$ is $\langle X, \mathcal{L}_\varphi Y \rangle$ where the symplectic Jacobi operator is

$$\mathcal{L}_\varphi Y = -J \left( \nabla_{Z_\varphi} Y + d\varphi(\sharp \delta d\varphi^*t_Y \omega) \right) \quad \text{and} \quad Z_\varphi = \sharp \delta \varphi^* \omega.$$

**Proposition 4.3.** Let $(\varphi, C)$ be a critical point of $E$. Then the Hessian operator for $E$ at $(\varphi, C)$ is

$$\mathcal{H}_{(\varphi, C)} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \frac{1}{4}(\mathcal{J}_\varphi X + \mathcal{L}_\varphi X) - \frac{1}{2} J\nabla_{\sharp \delta C} \varphi X - \frac{1}{2} Jd\varphi \sharp \delta dY \\ \delta d(Y + \frac{1}{2} \varphi^* t_X \omega) + Y \end{bmatrix}.$$
\[ \frac{1}{4} \langle X, (\mathcal{J}_\varphi + \mathcal{L}_\varphi) \tilde{X} \rangle + \langle Y, \delta d \tilde{Y} + \tilde{Y} \rangle + \frac{1}{2} \langle X, \delta d (\varphi^* t_X \omega) \rangle - \frac{1}{2} \langle X, J \delta t (d \varphi_{0,t} \delta d C_{0,t}) \rangle_{|_{t=0}} \]

by Propositions 4.1 and 4.2 and the Homotopy Lemma. Let \((\varphi_t, C_t) = (\varphi_{0,t}, C_{0,t})\), and \(F : (-\varepsilon, \varepsilon) \times M \to N\) be the total map \(F(t, x) = \varphi_t(x)\). Then

\[ \partial_t (d \varphi_t \delta d C_t)_{|_{t=0}} = \left( \nabla^F_{\partial/\partial t} dF \right)_{|_{t=0}} \delta d C + d \varphi_t \delta d (\partial_t C)_{|_{t=0}} \]

\[ = \left( \nabla^F_{\delta \delta d C} dF \right)_{|_{t=0}} \partial/\partial t + d \varphi_t \delta d \tilde{Y} \]

\[ = \nabla^F_{\delta \delta d C} \tilde{X} + d \varphi_t \delta d \tilde{Y} \]

since the Levi-Civita connexion on \(N\) is torsionless, and \((-\varepsilon, \varepsilon) \times M\) has the product metric. The result immediately follows. \(\square\)

**Remark 4.4.** A nontrivial check on this formula is that \(\mathcal{H}\) should be self-adjoint with respect to the \(L^2\) inner product on \(\Gamma(E)\). This is clear, provided that the operators

\[ \mathcal{A} : \Omega^1(M) \to \Gamma(\varphi^{-1}TN) \quad \mathcal{A} : Y \mapsto -J d \varphi^* \delta d Y \]

\[ \mathcal{B} : \Gamma(\varphi^{-1}TN) \to \Omega^1(M) \quad \mathcal{B} : X \mapsto \delta d (\varphi^* t_X \omega) \]

are an adjoint pair, that is, \(\mathcal{B}^* = \mathcal{A}\). Let us check this:

\[ \langle Y, \mathcal{B} X \rangle = \langle \delta d Y, \varphi^* t_X \omega \rangle = \int_M \varphi^* t_X \omega (\delta d Y) = \int_M \omega (X, d \varphi^* \delta d Y) = \langle J X, d \varphi^* \delta d Y \rangle = \langle X, \mathcal{A} Y \rangle \]

as required.

The formula for the total Hessian operator is, admittedly, rather complicated. However, it is not so complicated as to be unusable, as we shall now see.

**Corollary 4.5.** \((\text{Id}, 0)\) is a stable critical point of \(E\), where \(\text{Id} : N \to N\) is the identity map on any Kähler manifold.

**Proof.** Certainly \((\text{Id}, 0)\) is a critical point (Example 3.3). It is known [14, p. 172] that

\[ \langle X, \mathcal{J}_\varphi X \rangle \geq 0 \]

for any holomorphic map \(\varphi\) between Kähler manifolds, and hence, in particular, for \(\varphi = \text{Id}\). In the case of \(\varphi = \text{Id}\), \(\varphi^* \omega = \omega\), which is coclosed, and one has a canonical identification of \(\varphi^{-1}TN\) with \(TN = TN\), so that \(\varphi^* (t_X \omega) = b J X\), and one finds that \(\langle X, \mathcal{L}_{\text{Id}} X \rangle = \|db J X\|^2 [12]\). Hence

\[ \langle (X, Y), \mathcal{H}(X, Y) \rangle = \frac{1}{4} \langle (X, J \text{Id} X) + \frac{1}{4} \|db J X\|^2 + \|d Y\|^2 + \|d Y\|^2 + \langle d Y, db J X \rangle \geq \|db J \frac{X}{2}\|^2 + 2 \langle d Y, db J \frac{X}{2} \rangle + \|d Y\|^2 \geq 0 \]

\(\square\)
As a less trivial application of the second variation formula, we can use it to show that the embedded Hopf fibration $S^3 \to S^2$ is unstable. This is the subject of the next section.

5. Stability of the Embedded Hopf Map $S^3 \to S^2$

In this section we will analyze in detail the Hessian operator for the embedded Hopf fibration $S^3_R \to S^2$, where $S^3_R$ is the 3-sphere of radius $R$ and $S^2$ is the unit 2-sphere. It is convenient to identify the domain with $G = SU(2)$, and the codomain with $G/K$, where $K = \{ \text{diag}(\lambda, \lambda) : \lambda \in \mathbb{U}(1) \}$, so that $\varphi : G \to G/K$ is the coset projection, and use the machinery outlined in Example 3.7. Let $\vartheta_1, \vartheta_2, \vartheta_3$ be the usual basis of left-invariant vector fields on $G$ (coinciding at $e$ with $\frac{i}{2} \tau_a$, $a = 1, 2, 3$, where $\tau_a$ are the Pauli spin matrices). Note that $\mathfrak{g} = \{ \vartheta_3 \} \mathbb{R}$ and $\mathfrak{p} = \{ \vartheta_1, \vartheta_2 \} \mathbb{R}$, where $\{ \cdot \} \mathbb{R}$ denotes linear span. The almost complex structure on $G/K$ coincides with the adjoint action of $J = \vartheta_3$ on $\mathfrak{p}$, and the pullback of the Kähler form $\omega$ to $G$ is $\varphi^* \omega = -\sigma_1 \wedge \sigma_2$, where $\{ \sigma_a \}$ is the coframe dual to $\{ \vartheta_a \}$.

Unlike Example 3.7 we will not give $G$ the metric coinciding at $e$ with minus the Killing form, nor $G/K$ the metric which renders $\varphi$ a Riemannian submersion. Rather, $G$ and $G/K$ are given the round metrics of radius $R$ and 1, respectively. Making use of the canonical identifications $TG \equiv G \times \mathfrak{g}$ and $\varphi^* (TG/K) \equiv G \times \mathfrak{p}$, this amounts to declaring $\vartheta_a \in TG$, $a = 1, 2, 3$, to be orthogonal vector fields of length $\frac{R}{2}$, and $\vartheta_1, \vartheta_2$ to be orthonormal sections of $T(G/K)$. It follows that $\delta \varphi^* \omega = -\frac{1}{R^2} \sigma_3$, and that the pair $(\varphi, C)$ satisfies (3.8), (3.9) if and only if

\begin{equation}
C = \frac{2}{4 + R^2} \sigma_3.
\end{equation}

The pair $(\varphi, C)$ is the analogue for the supercurrent coupled Faddeev-Skyrme model on $S^3_R$ of the unit charge Hopf soliton on $\mathbb{R}^3$. We note in passing that the magnetic field of this coupled “hopfion” is tangent to the fibres of $\varphi$,

\begin{equation}
B = \mathbb{I} \ast (dC + \frac{1}{2} \varphi^* \omega) = -\frac{4}{R(4 + R^2)} \vartheta_3.
\end{equation}

Making use of the identifications $\varphi^{-1} T(G/K) \equiv G \times \mathfrak{p}$ and $T^* G \equiv G \times \mathfrak{g}^*$, we can identify any smooth section of $E$ with a quintuple of smooth functions $f_i : G \to \mathbb{R}$, $i = 1, 2, \ldots, 5$, by

\begin{equation}
(X, Y) = (f_1 \vartheta_1 + f_2 \vartheta_2, f_3 \sigma_1 + f_4 \sigma_2 + f_5 \sigma_3).
\end{equation}

If we similarly identify the section $\mathcal{H}(X, Y)$ with a mapping $G \to \mathbb{R}^5$, the Hessian operator for $E$ at $(\varphi, C)$ is represented by a $5 \times 5$ matrix of differential operators acting on real-valued functions on $G$. Similarly, the Jacobi and symplectic Jacobi operators $\mathcal{J}, \mathcal{L} : \Gamma(\varphi^{-1} T^* N) \to \Gamma(\varphi^{-1} T^* N)$ are represented by $2 \times 2$ matrices of differential operators. We shall denote the matrix representing $\mathcal{H}$ by $(\mathcal{H})$. Similarly, the $2 \times 2$ matrix of differential operators representing $\mathcal{J}$ will be denoted $(\mathcal{J})$, and so on. From Proposition 4.3, we see that $(\mathcal{H})$ has the block structure

\begin{equation}
(\mathcal{H}) = \begin{pmatrix}
\frac{1}{4} (\mathcal{J}) + \frac{1}{4} (\mathcal{L}) & + (\mathcal{C}) & \frac{1}{2} (\mathcal{A}) \\
\frac{1}{2} (\mathcal{B}) & (\delta \mathcal{d} + \mathbb{I}_3)
\end{pmatrix},
\end{equation}
where \( \mathcal{A}, \mathcal{B} \) were defined in Remark 4.4, and

\[
\mathcal{C} : \Gamma(\varphi^{-1}TN) \to \Gamma(\varphi^{-1}TN) \quad \mathcal{C} : X \mapsto -\frac{1}{2} J \nabla^2_{\varphi^*} X.
\]

We seek an explicit formula for \((\mathcal{H})\). The \((\mathcal{J})\) and \((\mathcal{L})\) parts of this formula are already known \([15, 12]\), in principle. These papers each equip \( S^3 \) with a fixed radius \((R = 2\sqrt{2})\) for \(\mathcal{J}\) in \([15]\) and \(R = 2\) for \(\mathcal{L}\) in \([12]\), however, so to make use of their results we must determine how \(\mathcal{J}\) and \(\mathcal{L}\) scale under homotheties of \(M\).

For completeness, we shall simultaneously consider their scaling under homotheties of \(N\) also.

**Proposition 5.1.** Let \(\varphi : (M^n, g) \to (N^n, h)\) be harmonic with Jacobi operator \(\mathcal{J}\).

If \(\tilde{g} = R_1^2 g, \tilde{h} = R_2^2 h\), where \(R_1, R_2 > 0\) are constants, then the Jacobi operator of \(\varphi\) as a harmonic map \((M, \tilde{g}) \to (N, \tilde{h})\) is

\[
\mathcal{J} = R_1^{-2} \mathcal{J}.
\]

Similarly, let \(\varphi : (M^n, g) \to (N^n, h)\) be a critical point of \(E_2\) with symplectic Jacobi operator \(\mathcal{L}\). Then the symplectic Jacobi operator of \(\varphi\) as a critical point \((M, \tilde{g}) \to (N, \tilde{h})\) is

\[
\mathcal{L} = R_1^{-4} R_2^2 \mathcal{L}.
\]

**Proof.** Let \(\varphi_{s,t} : M \to N\) be a smooth two-parameter variation of \(\varphi = \varphi_{0,0}\), with \(\partial_s \varphi_{s,t} \big|_{s=t=0} = X, \partial_t \varphi_{s,t} \big|_{s=t=0} = Y \in \Gamma(\varphi^{-1}TN)\), and note that the Hodge isomorphism \(\Omega^p(M) \to \Omega^{m-p}(M)\) scales as \(\varpi = R_1^{m-2p}\) under the homothety \(g \to \tilde{g}\). We have, in obvious notation,

\[
\tilde{E}_1(\varphi_{s,t}) = R_1^{m-2} R_2 E_1(\varphi_{s,t})
\]

\[
\Rightarrow \quad \frac{\partial^2 \tilde{E}_1(\varphi_{s,t})}{\partial s \partial t} \bigg|_{s=t=0} = R_1^{m-2} R_2^2 \int_M h(X, \mathcal{J} Y) \ast 1 = \int_M \tilde{h}(X, R_1^{-2} \mathcal{J} Y) \ast 1.
\]

Similarly, since \(\varpi = R_2^2 \omega\),

\[
\tilde{E}_2(\varphi_{s,t}) = \int_M \varphi_{s,t}^* \varpi \land \varpi \ast 1 = R_1^{m-4} R_2^4 E_2(\varphi_{s,t})
\]

\[
\Rightarrow \quad \frac{\partial^2 \tilde{E}_2(\varphi_{s,t})}{\partial s \partial t} \bigg|_{s=t=0} = R_1^{m-4} R_2^4 \int_M h(X, \mathcal{L} Y) \ast 1 = \int_M \tilde{h}(X, R_1^{-4} R_2^2 \mathcal{L} Y) \ast 1.
\]

\[\square\]

It follows from the results of \([15]\) and Proposition 5.1 that, for the Hopf fibration \(S^3_R \to S^2\),

\[
(\mathcal{J}) = \frac{4}{R^2} \left( \begin{array}{c} \vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2 \\ 2\vartheta_3 \end{array} \right) \left( \begin{array}{c} 2\vartheta_3 \\ \vartheta_1^2 + \vartheta_2^2 + \vartheta_3^2 \end{array} \right)
\]

Similarly, from the calculation in \([12]\) and Proposition 5.1, one sees that

\[
(\mathcal{L}) = \frac{16}{R^4} \left( \begin{array}{c} -\vartheta_1^2 - \vartheta_2^2 \\ -\vartheta_1 \vartheta_2 \end{array} \right) \left( \begin{array}{c} -\vartheta_1 \vartheta_2 \\ -\vartheta_1^2 - \vartheta_2^2 \end{array} \right).
\]
To complete the computation of the top-left $2 \times 2$ block of $(\mathcal{H})$, we need the matrix representing $\mathcal{C}$. Now

\begin{equation}
\hat{\delta} \delta \mathcal{C} = \frac{32}{R^4(4 + R^2)} \vartheta_3
\end{equation}

and, from equation (3.13),

\begin{equation}
\nabla^2_{\vartheta_3} (f_1 \vartheta_1 + f_2 \vartheta_2) = \vartheta_3 (f_1) \vartheta_1 + \vartheta_3 (f_2) \vartheta_2 - f_1 \vartheta_2 + f_2 \vartheta_1.
\end{equation}

Hence, since $J \vartheta_1 = [\vartheta_3, \vartheta_1] = -\vartheta_2$,

\begin{equation}
(\mathcal{C}) = \frac{16}{R^4(4 + R^2)} \begin{pmatrix}
1 & -\vartheta_3 \\
\vartheta_3 & 1
\end{pmatrix}.
\end{equation}

A short calculation of $\delta d (f \sigma_a)$ using $d \sigma_1 = \sigma_2 \wedge \sigma_3$, $* \sigma_1 \wedge \sigma_2 = \frac{2}{R} \sigma_3$, and cyclic permutations, shows that

\begin{equation}
(\delta d) = \frac{4}{R^2} \begin{pmatrix}
1 - \vartheta_2^2 - \vartheta_3^2 & \vartheta_2 \vartheta_3 - 2 \vartheta_3 & \vartheta_3 \vartheta_1 + 2 \vartheta_2 \\
\vartheta_1 \vartheta_2 + 2 \vartheta_3 & 1 - \vartheta_1^2 - \vartheta_3^2 & \vartheta_3 \vartheta_2 - 2 \vartheta_1 \\
\vartheta_1 \vartheta_3 - 2 \vartheta_2 & \vartheta_2 \vartheta_3 + 2 \vartheta_1 & 1 - \vartheta_1^2 - \vartheta_2^2
\end{pmatrix}.
\end{equation}

It remains to compute the top-right block $(\mathcal{A})$ and the bottom-left block $(\mathcal{B})$ of $(\mathcal{H})$. These form an $L^2$ adjoint pair, but care must be taken when using this fact, since our basis for $\mathcal{E}$ is not orthonormal. Recall that $d \varphi$ coincides with orthogonal projection $\mathfrak{g} \to \mathfrak{p}$, so, making use of the formula (5.11) for $(\delta d)$ above, one finds,

\begin{equation}
(\mathcal{A}) = -\frac{1}{2} \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix} \frac{4}{R^2} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} (\delta d)
\end{equation}

\begin{equation}
= \frac{8}{R^4} \begin{pmatrix}
-\vartheta_1 \vartheta_2 - 2 \vartheta_3 & \vartheta_1^2 + \vartheta_3^2 & \vartheta_3 \vartheta_2 + 2 \vartheta_1 \\
-\vartheta_2^2 - \vartheta_3^2 & \vartheta_2 \vartheta_1 - 2 \vartheta_3 & \vartheta_3 \vartheta_1 + 2 \vartheta_2
\end{pmatrix}.
\end{equation}

Finally, the map $\Lambda : \Gamma(\varphi^{-1}TN) \to \Omega^1 (M)$, $\Lambda : X \mapsto \varphi^* t_X \omega$ has matrix representative

\begin{equation}
(\Lambda) = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix},
\end{equation}

which, together with (5.11), yields

\begin{equation}
(\mathcal{B}) = \frac{1}{2} (\delta d)(\Lambda) = \frac{2}{R^2} \begin{pmatrix}
-\vartheta_2 \vartheta_1 + 2 \vartheta_3 & -\vartheta_2^2 - \vartheta_3^2 \\
\vartheta_1^2 + \vartheta_3^2 & \vartheta_1 \vartheta_2 + 2 \vartheta_3 \\
-\vartheta_2 \vartheta_3 + 2 \vartheta_1 & \vartheta_1 \vartheta_3 - 2 \vartheta_2
\end{pmatrix}
\end{equation}

Assembling (5.6), (5.7), (5.10), (5.11), (5.12), (5.14) with (5.4) we obtain the explicit formula for $(\mathcal{H})$, which we will not reproduce here.

We now have an expression for $\mathcal{H}$ as a matrix of differentiable operators acting on functions on $G$. To proceed further, we must choose a basis for $L^2 (G, \mathbb{C})$, and for this purpose we appeal to the Peter-Weyl theorem [7] p. 17. According to this, the matrix elements of the finite-dimensional irreducible unitary representations of $G$ form an orthonormal basis for $L^2 (G, \mathbb{C})$. Now $\mathcal{H}$ commutes with the obvious action of $G$ on $\Gamma(\mathcal{E})$. Hence, for a given fixed irreducible representation of $G$, the finite dimensional subspace of $\Gamma(\mathcal{E})$ in which each of $f_1, f_2, \ldots, f_5 : G \to \mathbb{C}$ lies in the span of the matrix elements of that representation is invariant under
\( \mathcal{H} \). So we can break down the (infinite dimensional) eigenvalue problem for \( \mathcal{H} \) into an infinite sequence of finite dimensional eigenvalue problems, indexed by the irreducible unitary representations of \( G \). In this case, \( G = \text{SU}(2) \), whose irreducible unitary representations are indexed by a non-negative integer \( n \), called the weight (loosely speaking, twice the "spin" associated with the representation). For fixed \( n \), we can construct \((n + 1) \times (n + 1)\) antihermitian matrices representing \( \vartheta_1, \vartheta_2, \vartheta_3 \), and hence construct a \((5n + 5) \times (5n + 5)\) matrix \( \mathcal{H}^{(n)} \) representing \( \mathcal{H} \) acting on the weight \( n \) invariant subspace of \( \Gamma(\mathcal{E}) \). The machinery for dealing with the general \( n \) case is developed in [12]. However, for our purposes, we will need only the fundamental representation \( n = 1 \). In this case, we simply replace \( \vartheta_a \) by \( \frac{i}{2} \tau_a \) in \( (\mathcal{H}) \) to obtain a \( 10 \times 10 \) complex matrix \( \mathcal{H}^{(1)} \), each of whose entries is a rational function of \( R \), the radius of \( M \). The eigenvalue problem for \( \mathcal{H}^{(1)} \) can be solved exactly, by Maple, for example. One finds that \( \mathcal{H}^{(1)} \) has an eigenvalue of multiplicity two, namely

\[
\lambda(R) = \frac{p(R) - \sqrt{p(R)^2 + 3840R^4 + 1536R^6 + 208R^8 + 16R^{10}}}{8R^4(4 + R^2)},
\]

where \( p(R) = 48 + 144R^2 + 51R^4 + 4R^6 \).

Clearly \( \lambda(R) < 0 \) for all \( R > 0 \). Since our basis vectors for \( L^2(G, \mathbb{C}) \) are matrix elements, they are indexed by an ordered pair of indices, taking values in \( \{1, \ldots, n + 1\} \). It follows that an eigenvalue of \( \mathcal{H}^{(n)} \) of multiplicity \( m \) is an eigenvalue of \( \mathcal{H} \) (restricted to the weight \( n \) invariant subspace of \( \Gamma(\mathcal{E}) \)) of multiplicity \( (n + 1)m \). Hence, the index (dimension of the sum of the negative eigenspaces) of \( \mathcal{H} \) is at least \( 2 \times (1 + 1) = 4 \). The eigenvalue problem for \( \mathcal{H}^{(n)} \) for \( 1 \leq n \leq 10 \) has been solved numerically for various \( R \), no further negative eigenvalues being found. It seems likely, therefore, that the index of \( \mathcal{H} \) is exactly 4. In any case, the \( n = 1 \) calculation outlined above establishes rigorously that the embedded Hopf fibration is an unstable critical point of \( E \), for all choices of the radius \( R \) of \( S^3_R \).

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