A completely positive map associated with a positive map

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Abstract
We show that each positive map from $B(K)$ to $B(H)$ is a scalar multiple of a map of the form $Tr - \psi$ with $\psi$ completely positive. This is used to give necessary and sufficient conditions for maps to be $C$-positive for a large class of mapping cones; in particular we apply the results to $k$-positive maps.

Introduction
In [6] we studied several norms on positive maps from $B(K)$ into $B(H)$, where $K$ and $H$ are finite dimensional Hilbert spaces. These norms were very useful in the study of maps of the form $Tr - \lambda \psi$, where $Tr$ is the usual trace on $B(K), \lambda > 0,$ and $\psi$ a completely positive map of $B(K)$ into $B(H)$. In the present paper we shall see that every positive map is a positive scalar multiple of a map of the above form with $\lambda = 1$, hence the results in [6] are applicable to all positive maps. In particular they yield a simple criterion for some maps to be $k$-positive but not $k+1$-positive. As an illustration we give a new proof that the Choi map of $B(C^3)$ into itself is atomic, i.e. not the sum of a 2-positive and a 2-copositive map.

$C$-positive maps
Let $K$ and $H$ be finite dimensional Hilbert spaces. We denote by $B(B(K), B(H))$ (resp.$B(B(K), B(H))^+$) the linear (resp. positive linear) maps of $B(K)$ into $B(H)$. In the case $K = H$ we denote by $P(H) = B(B(H), B(H))^+$. Following [8] we say a closed cone $C_H \subset P(H)$ is a mapping cone if $\alpha \circ \phi \circ \beta \in C$ for all $\phi \in C$ and $\alpha, \beta \in CP$ - the completely positive maps in $P(H)$. A map $\phi$ in $B(B(K), B(H))$ defines a linear functional $\hat{\phi}$ on $B(K) \otimes B(H)$, identified with $B(K \otimes H)$ in the sequel, by $\hat{\phi}(a \otimes b) = Tr(\phi(a)b^t)$, where $Tr$ is the usual trace on $B(H)$ and $t$ denotes the transpose. Let $P(B(K), C)$ denote the closed cone

$$P(B(K), C) = \{ a \in B(K \otimes H) : \iota \otimes \alpha(a) \geq 0 \ \forall \ \alpha \in C \},$$
Theorem 1

A map \( \phi \) is said to be \( \mathcal{C} \)-positive if \( \phi \) is positive on \( P(B(K), \mathcal{C}) \). We denote by \( \mathcal{P}_C \) the cone of \( \mathcal{C} \)-positive maps.

If \( (e_{ij}) \) is a complete set of matrix units for \( B(K) \) then the Choi matrix for a map \( \phi \) is

\[
C_{\phi} = \sum e_{ij} \otimes \phi(e_{ij}) \in B(K \otimes H).
\]

By [10] and [11], the transpose \( C_{\phi}^t \) of \( C_{\phi} \) is the density operator for \( \tilde{\phi} \), and by [1] \( \phi \) is completely positive if and only if \( C_{\phi} \geq 0 \) if and only if \( \tilde{\phi} \geq 0 \) as a linear functional on \( B(K \otimes H) \). In the case \( \mathcal{C} = CP \), \( P(CP, B(K)) = B(K \otimes H)^+ \), so \( \phi \) is CP-positive if and only if \( \phi \) is completely positive.

If \( \mathcal{C}_1 \subset \mathcal{C}_2 \) are two mapping cones on \( B(H) \), then \( P(\mathcal{C}_1, \mathcal{C}_2) \) is a mapping cone on \( B(K, \mathcal{C}_1) \) if \( \imath \), all \( \alpha \in \mathcal{C}_1 \). Thus \( \phi \geq 0 \) on \( P(\mathcal{C}_1, \mathcal{C}_2) \) implies \( \phi \geq 0 \) on \( P(\mathcal{C}_1, \mathcal{C}_2) \), so \( \mathcal{P}_C \subset \mathcal{P}_{C_2} \).

Let \( \mathcal{C} \) be a mapping cone on \( B(H) \). Let \( \mathcal{P}_C^\circ \) denote the dual cone of \( \mathcal{P}_C \) defined as

\[
\mathcal{P}_C^\circ = \{ \phi \in B(B(K), B(H)) : Tr(C_{\phi} C_\psi) \geq 0 \quad \forall \psi \in \mathcal{P}_C \}.
\]

Thus if \( \mathcal{C}_1 \subset \mathcal{C}_2 \) then \( \mathcal{P}_{C_1}^\circ \supset \mathcal{P}_{C_2}^\circ \). In the particular case when \( \mathcal{C} \supset CP \) we thus get \( \mathcal{P}_C^\circ \supset \mathcal{P}_{C_{CP}}^\circ = CP(K, H) \) - the completely positive maps of \( B(K) \) into \( B(H) \).

Following [6], \( \mathcal{C} \) defines a norm on \( B(B(K), B(H)) \) by

\[
\| \phi \|_C = \sup \{ | Tr(C_{\phi} C_\psi) | : \psi \in \mathcal{P}_C^\circ, Tr(C_\psi) = 1 \}.
\]

In the special case when \( \mathcal{C} \supset CP \) it follows from the above that

\[
\| \phi \|_C = \sup \{ |\rho(C_{\phi})| \}
\]

where the sup is taken over all states \( \rho \) on \( B(K \otimes H) \) with density operator \( C_\psi \) with \( \psi \in \mathcal{P}_C^\circ \). Let \( \phi \in B(B(K), B(H)) \) be a self-adjoint map, i.e. \( \phi(a) \) is self-adjoint for a self-adjoint \( a \). Then \( C_{\phi} \) is a self-adjoint operator, so is a difference \( C_+ - C_- \) of two positive operators with orthogonal supports. Let \( c \geq 0 \) be the smallest positive number such that \( c1 \geq C_{\phi} \). Then \( c = \| C_{\phi}^+ \| \). Hence, if \( c \neq 0 \) there exists a map \( \phi_{CP} \in B(B(K), B(H)) \) such that the Choi matrix for \( \phi_{CP} \) equals \( 1 - c^{-1} C_{\phi} \), which is a positive operator. Thus, if we let \( Tr \) denote the map \( x \mapsto Tr(x)1, \phi_{CP} \) is completely positive, and \( c^{-1} \phi = Tr - \phi_{CP} \), since \( C_{Tr} = 1 \), as is easily shown. Combining the above discussion with [6], Prop. 2, we thus have.

**Theorem 1** Let \( \phi \) be a self-adjoint map of \( B(K) \) into \( B(H) \). Then

(i) There exists a completely positive map \( \phi_{CP} \in B(B(K), B(H)) \) such that

\[
\| C_{\phi}^+ \|^{-1} \phi = Tr - \phi_{CP}.
\]

(ii) If \( \mathcal{C} \) is a mapping cone on \( B(H) \) containing \( CP \) then \( \phi \) is \( \mathcal{C} \)-positive if and only if

\[
1 \geq \| \phi \|_C = \sup \{ \rho(C_{\phi_{CP}}) \}.
\]
where the sup is taken over all states $\rho$ on $B(K \otimes H)$ with density operator $C_\psi$, with $\psi \in \mathcal{P}_C^n$.

Note that we did not need to take the absolute value of $\rho(C_{\phi,cp})$ because $C_{\phi,cp} \geq 0$ and $\psi \in \mathcal{P}_C^n \subset CP$.

We next spell out the theorem for some well known mapping cones. Recall that a map $\phi$ is decomposable if $\phi = \phi_1 + \phi_2$ with $\phi_1$ completely positive and $\phi_2$ copositive, i.e. $\phi_2 = t \circ \psi$ with $\psi$ completely positive. Also recall that a state $\rho$ on $B(K \otimes H)$ is a PPT-state if $\rho \circ (\iota \otimes t)$ is also a state.

**Corollary 2** Let $\phi \in B(B(K), B(H))$ be a self-adjoint map. Then we have.

(i) $\phi$ is positive if and only if $\rho(C_{\phi,cp}) \leq 1$ for all separable states $\rho$ on $B(K \otimes H)$.

(ii) $\phi$ is decomposable if and only if $\rho(C_{\phi,cp}) \leq 1$ for all PPT-states $\rho$ on $B(K \otimes H)$.

(iii) $\phi$ is completely positive if and only if $\rho(C_{\phi,cp}) \leq 1$ for all states $\rho$ on $B(K \otimes H)$.

**Proof.** (i) That $\phi$ is positive is the same as saying that $\phi$ is $P(H)$-positive. Since the dual cone of $P(H)$ is the cone of separable states (i) follows.

(ii) A state $\rho$ is PPT if and only if its density operator is of the form $C_\psi$ with $\psi$ a map which is both positive copositive, see e.g. [10], Prop.4. But the dual of those maps is the cone of decomposable maps, see e.g. [7]. Thus (ii) follows from the theorem.

(iii) This follows since the dual cone of the completely positive maps is the cone of completely positive maps, and that the density operator for a state is positive, hence the corresponding map $\psi$ is completely positive.

**k-positive maps**

A map $\phi \in B(B(K), B(H))$ is said to be k-positive if $\phi \otimes t \in B(B(K \otimes L), B(H \otimes L))^+$ whenever $L$ is a k-dimensional Hilbert space. The k-positive maps in $P(H)$ form a mapping cone $P_k$ containing $CP$. Denote by $P_k(K, H)$ the cone of k-positive maps in $B(B(K), B(H))$. Then we have,

**Lemma 3** With the above notation we have $\mathcal{P}_{P_k} = P_k(K, H)$.

**Proof.** We have $P_k^o = SP_k$, the k-superpositive maps in $P(H)$, which is the mapping cone generated by maps of the form $AdV$ defined by $AdV(a) = VaV^*$, where $V \in B(H), rank V \leq k$, see e.g. [7]. By [11] the dual cone of $\mathcal{P}_{P_k}$ is given by

$$\mathcal{P}_{P_k}^o = \{ \phi \in B(B(K), B(H)) : AdV \circ \phi \in CP(K, H) \ \forall V \in B(H), \ rank V \leq k \}.$$  

By [5], Theorem 3, or [9], Theorem 2, it follows that $\mathcal{P}_{P_k}^o = P_k(K, H)$. By [8], Theorem 3.6, $\mathcal{P}_{P_k}$ is generated by maps of the form $\alpha \circ \beta$ with $\alpha \in P_k, \beta \in SP_k$.
Therefore, for function lemma. density operators $C$ the assumption that $\Re(1 - Ay, y) = 0$, since $\Re(1 - Ay, y) = 0$. Since $\Re(1 - Ay, y) = 0$, it follows that $\mathcal{P}_k = \mathcal{P}_k^0 = \mathcal{P}_k(K, H)$, completing the proof of the lemma.

It follows from the above description of $\mathcal{P}_k^0$ that the states with density operators $C, \psi \in \mathcal{P}_k^0$, are the same as the vector states generated by vectors in the Schmidt class $S(k)$, i.e. the vectors $y = \sum_{i=1}^k x_i \otimes y_i, x_i \in K, y_i \in H$, where the $x_i$ and $y_i$ are not necessarily all $\neq 0$.

**Theorem 4** Let $\phi \in B(B(K), B(H))^+$. Then we have.

(i) $\phi$ is $k$-positive if and only if $\sup_{x \in S(k), \|x\|=1} (C_{\phi,x}, x) \leq 1$.

(ii) Suppose $k < \min(\dim K, \dim H)$, and that there exists a unit vector $y = \sum_{i=1}^k x_i \otimes y_i \in S(k)$ such that $y \perp C_\phi y \notin X \otimes Y$, where $X = \text{span}(x_i), Y = \text{span}(y_i)$. Then $\phi$ is not $k+1$-positive.

In order to prove the theorem we first prove a lemma.

**Lemma 5** Let $A$ be a self-adjoint operator in $B(K \otimes H)$. Suppose $y = \sum_{i=1}^k x_i \otimes y_i$ satisfies $(Ay, y) = 1$, and $Ay \notin X \otimes Y$ with $X, Y$ as in Theorem 4. Then there exist a unit product vector $x \perp X \otimes Y$ and $s \in (0, 1)$ such that $(A(sx + (1 - s^2)^{1/2})y), sx + (1 - s^2)^{1/2}y) > 1$.

**Proof.** Since $Ay \notin X \otimes Y$ there exists a product vector $x \perp X \otimes Y$ such that $\Re(x, Ay) > 0$. Let $s \in (-1, 1)$ and $t = (1 - s^2)^{1/2}$, and let $f$ denote the function

$$f(s) = (A(sx + ty), st + ty) = s^2(Ax, x) + t^2(Ay, y) + 2st\Re(Ax, y).$$

Since $(Ay, y) = 1$ we get

$$f'(0) = 2(1 - s^2)^{1/2}\Re(Ax, y) > 0.$$ 

Therefore, for $s > 0$ and near 0 we have $(A(sx+ty), st+ty) > f(0) = 1$, proving the lemma.

**Proof of Theorem 4.**

(i) is a direct consequence of Theorem 1, since, as noted in the proof of Lemma 3, the vector states $\omega_x$ with $x \in S(k)$ generate the set of states with density operators $C_\psi$ with $\psi \in \mathcal{P}_k^0$.

(ii) By Theorem 1 $C_{\omega_x,^p} = 1 - \parallel C_\phi^+ \parallel^{-1} C_\phi$, so that $(C_{\omega_x,^p}, y) = 1$, using the assumption that $C_\phi y \perp y$. Furthermore $C_{\omega_x,^p} = y - \parallel C_\phi^+ \parallel^{-1} C_\phi y$. Since $C_\phi y \notin X \otimes Y$ $C_{\omega_x,^p} y \notin X \otimes Y$. Thus by Lemma 5 there exist a unit product vector $x \in X \otimes Y$ and $s, t = (1 - s^2)^{1/2} > 0$ such that $(C_{\omega_x} (sx+ty), sx+ty) > 1$. Since
$sx + ty$ is a unit vector in $S(k+1)$, $\phi$ is not $k+1$-positive by part (i), completing the proof of the theorem.

**Example** We illustrate the above results by an application to the Choi map $\phi \in B(B(C^3), B(C^3))$ defined by

$$\phi((x_{ij})) = \begin{bmatrix} x_{11} + x_{33} & -x_{12} & -x_{13} \\ -x_{21} & x_{11} + x_{22} & -x_{23} \\ -x_{31} & -x_{32} & x_{22} + x_{33} \end{bmatrix}$$

We have $C_{t\circ \phi} = (t \otimes t)C_\phi$. So if $y = x \otimes x$ with $x = 3^{-1/2}(1, 1, 1) \in C^3$, then $(C_\phi y, y) = (C_{t\circ \phi} y, y) = 0$, and $C_\phi y \neq 0 \neq C_{t\circ \phi} y$. Hence, by Theorem 4, neither $\phi$ nor $t \circ \phi$ is 2-positive, i.e. $\phi$ is neither 2-positive nor 2-copositive. Since $\phi$ is an extremal positive map of $B(C^3)$ into itself by [2], $\phi$ cannot be the sum of a 2-positive and a 2-copositive map, hence $\phi$ is atomic, a result first proved by Tanahashi and Tomiyama [12], and then extended to more general maps by others, see [3] for references.

$\phi$ can also be shown to be a positive map by a straightforward argument using Corollary 2.

It should be remarked that the Choi map $\phi$ also yields an example of a PPT-state on $B(C^3) \otimes B(C^3)$ which is not separable. Indeed, in [9] we gave an example of a positive matrix in $A$ in $B(C^3) \otimes B(C^3)$ such that its partial transpose $t \otimes \iota(A)$ is also positive, and that $\phi \otimes \iota(A)$ is not positive. Then $A$ cannot be of the form $\sum A_i \otimes B_i$ with $A_i$ and $B_i$ positive, hence the state $\rho(x) = Tr(A)^{-1} Tr(Ax)$ is PPT but not separable. An example of a PPT state on $B(C^3) \otimes B(C^3)$ which is not separable was later exhibited by P. Horodecki [4].

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