A simple algorithm for extending the identities for quantum minors to the multiparametric case

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Abstract. For any homogeneous identity between $q$-minors, we provide an identity between $P,Q$-minors.

1. (128) Suppose we are given $n \times n$ matrices $P = (p_{ij})$ and $Q = (q_{ij})$ with invertible entries in the ground field $k$, for which there exist $q$ such that

\[ p_{ij}q_{ij} = q^2, q_{ij} = q_{ji}^{-1}, \quad i < j, \quad \text{and} \quad q_{ii} = p_{ii}, \quad \text{for all} \quad i. \tag{1} \]

Define an associative algebra $M(P, Q; k, n) := k \langle T_{ij} \rangle_{i,j=1,\ldots,n} / I$, where $I$ is the ideal spanned by the relations

\[ T_{ik}T_{kj} = q_{ij}T_{kj}T_{ik}, \quad i < j \]
\[ T_{ik}T_{jl} = p_{kl}T_{jl}T_{ik}, \quad k < l \]
\[ q_{ij}T_{kj}T_{il} = p_{kl}T_{il}T_{kj}, \quad i < j, k < l \]
\[ q_{ij}T_{kj}T_{il} - q_{ij}T_{il}^{-1}T_{kj}T_{ik} = (q_{ij} - p_{ij}^{-1})T_{kj}^{il}T_{ik}, \quad i < j, k < l \]

$M := M(P, Q; k, n)$ is a bialgebra with respect to the ”matrix” comultiplication which is the unique algebra homomorphism $\Delta : M \to M \otimes M$ extending the formulas $\Delta T_{ij} = \sum T_{ik}T_{kj}$ with counit $\epsilon T_{ij} = \delta_{ij}^i$ (Kronecker delta). The bialgebra $M$ is called the multiparametric quantum linear group.

Our conventions differ a bit from the cited references: we treat $p$-s and $q$-s symmetrically in the sense that if we interchange rows and columns of matrix $T$ and if we simultaneously interchange $P$ and $Q$, we obtain an isomorphic algebra. If $P = Q$ and $q_{ij} = q$ for $i < j$ and $q_{ij} = q^{-1}$ for $i > j$, then $M = M_q(k)$ (1-parametric quantized matrix bialgebra). In this paper, we will denote by $t_{ij}$ the generators for 1-parametric case.

2. (Labels.) It is convenient to consider that the row and column labels belong to some totally ordered set of labels, not necessarily the set $\{1, \ldots, n\}$. The main reason is that one often needs to treat some subsets of the set of labels and the corresponding submatrices of the matrix $T = (T_{ij})$.

3. The quantum $Q$-space $O(k^n_Q)$, quantum $P$-space $O(k^n_P)$, $q$- normalized $Q$-space $S_r(q, Q)$, dual $q$- normalized $Q$- space $S_l(q, Q)$, right $P$-exterior algebra
\( \Lambda_P \), and left \( Q \)-exterior algebra \( \Lambda_Q \), are the algebras defined by generators and relations as follows:

\[
\begin{align*}
\mathcal{O}(k^n) &= k\langle x^i, i = 1, \ldots, n \rangle / \langle x^i x^j - q_{ij} x^j x^i, i < j \rangle \\
\mathcal{O}(k^n_P) &= k\langle y_i, i = 1, \ldots, n \rangle / \langle y_i y_j - p_{ij} y_j y_i, i < j \rangle \\
S_r(q, Q) &= k\langle r_1, \ldots, r_n \rangle / \langle r_i r_j - q q_{ij}^{-1} r_j r_i, i < j \rangle \\
S_i(q, Q) &= k\langle l^1, \ldots, l^n \rangle / \langle l^i l^j - q q_{ij}^{-1} l^j l^i, i < j \rangle \\
\Lambda_P &= k\langle e^1, \ldots, e^n \rangle / \langle e^i e^j + p_{ij}^{-1} e^j e^i, i < j, (e^i)^2 \rangle \\
\Lambda_Q &= k\langle e_1, \ldots, e_n \rangle / \langle f_i f_j + q q_{ij}^{-1} f_j f_i, i < j, (f_i)^2 \rangle
\end{align*}
\]

We note after Manin (e.g. \[7, 8\]) the simple fact that \( \mathcal{O}(k^n_Q), \Lambda_Q \) are right and \( \mathcal{O}(k^n_P), \Lambda_P \) are left comodule algebras over \( \mathcal{M} \) via coactions \( \rho, \rho_\Lambda, \rho', \rho'_\Lambda \) which are the unique coactions which are algebra maps and extend formulas \( \rho(x^j) = \sum_j x^j \otimes T^j_i, \rho_\Lambda(e^j) = \sum_j e^j \otimes T^j_i, \rho'(y_j) = \sum_i T^j_i \otimes y_i \) and \( \rho'_\Lambda(f_j) = \sum_i T^j_i \otimes f_i \).

4. Tuples of labels will be called multilabels. Let \( L = (l_1, \ldots, l_r), K = (k_1, \ldots, k_s) \). The concatenation will be denoted by juxtaposition: \( KL = (l_1, \ldots, l_r, k_1, \ldots, k_s) \). Usually the multilabels will be (ascendingly) ordered to start with and \( \hat{L} \) denotes the ordered complement of a submultilabel \( L \) (usually in \( \{1, \ldots, n\} \)). By placing the multilabel within the colons, we will denote its ascendingly ordered version. For example, if \( K \) and \( L \) are ordered, then \( KL \) is not necessarily ordered, because some labels in \( L \) may be smaller than some labels in \( K \). However \( :KL:\) is the multilabel obtained from \( KL \) by permuting the labels until they are ascendingly ordered. We identify the notation for a single label \( j \) and the multilabel \( (j) \), and in this vein \( \hat{j} = (\hat{j}) \) is the same as multilabel \( (1, 2, \ldots, j - 1, j + 1, \ldots, n) \).

We use obvious exponent notation: \( r^j := r^{j_1} \cdots r^{j_n} \) and alike.

5. Algebras \([2]\) satisfy the obvious normal basis ("PBW type") theorems: fix an order on generators, then the monomials ordered compatibly with this order form a vector space basis ("PBW basis"), for example \( x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} \) in the case of \( \mathcal{O}(k^n) \); however, no higher exponents than 1 appear in the bases for \( \Lambda_P \) and \( \Lambda_Q \) because \( e_i^2 = 0 \).

Passing from arbitrary words in generators to the basis elements clearly reduces to reordering the generators, and accumulating the proportionality constants. Let us introduce the notation for those rearrangements factors which will be needed below. The following are the defining properties of
coefficient functions \( \epsilon_P, \epsilon_Q, \zeta_r, \zeta_l \) from sets of multilabels to \( k \):\[
\begin{align*}
\epsilon_J & := \epsilon_P(J) e_J, \quad r_J := \zeta_r(J) r_J, \\
f_J & := \epsilon_Q(J) f_J, \quad l_J := \zeta_l(J) l_J,
\end{align*}
\]
where \( \epsilon_P(J) \), \( \epsilon_Q(J) \) are defined only when \( J \) has no repeated labels inside, but \( \zeta_r \) and \( \zeta_l \) are defined even for multilabels with repetition. The following is obvious:
\[
\zeta_r(k_1, \ldots, k_s) = \prod_{i < j, k_i > k_j} q^{-1} q_{k_i k_j}
\]
\[
\zeta_l(k_1, \ldots, k_s) = \prod_{i < j, k_i > k_j} q q_{k_i k_j} = \zeta_r^{-1}(k_1, \ldots, k_s)
\]
Clearly, if \( J \) is ascendingly ordered multilabel without repetitions, then
\[
\begin{align*}
\epsilon_P(\sigma J) &= (-q)^{-l(\sigma)} \zeta_r(\sigma J) = \prod_{k < l, \sigma(k) > \sigma(l)} (-p_{j(\sigma(k)) j(l)}), \\
\epsilon_Q(\sigma J) &= (-q)^{l(\sigma)} \zeta_l(\sigma J) = \prod_{k < l, \sigma(k) > \sigma(l)} (-q_{j(\sigma(k)) j(l)}),
\end{align*}
\]
(recall that \( (q_{ij})^{-1} = q_{ji} \)).

6. (Gradings.) Let \( \mathcal{I} \) be the set of labels of generators of \( S_r = S_r(q, Q) \) (they label rows!). Let \( \mathcal{J} \) be the set of labels of generators of \( S_l = S_l(q, Q) \) (they label columns!). Both sets are bijective to \{1, \ldots, n\}. Thus the free Abelian group \( \mathbb{Z}[\mathcal{I}] \) is isomorphic to \( \mathbb{Z}^n \) (and could be naturally identified with the weight lattice for \( SL_n \)). Now we assign \( \mathbb{Z}[\mathcal{I}] - \mathbb{Z}[\mathcal{J}] \) bigrading to algebras \( S_l(q, Q), S_r(q, Q) \) and \( \mathcal{M}_q(k) \). If \( i_1, \ldots, i_n, j_1, \ldots, j_n \) are the elements of \( \mathcal{I} \) and \( \mathcal{J} \), then a bidegree is a formal sum of the form \( a_1 i_1 + \ldots + a_n i_n + b_1 j_1 + \ldots + b_n j_n \), e.g. \(-i_3 + 2i_4 + j_3 \), and we may separate the \( \mathcal{I} \) and \( \mathcal{J} \) grading with comma for clarity, e.g. \((-i_3 + 2i_4, j_3) \equiv -i_3 + 2i_4 + j_3 \). We assign the bidegree \((-i, 0)\) to the generator \( r_i \) of \( S_l \) (notice the negative sign!) \( \mathbb{Z}[\mathcal{I}^*] \)-degree to zero and similarly the dual prescription \((0, -j)\) to \( \mathcal{M}_q(k) \). We also assign the bidegree \((+i, +j)\) to each generator \( t_i^j \) of the 1-parametric algebra \( \mathcal{M}_q(k) \). The defining ideals are bihomogeneous hence we extend this prescription multiplicatively to a bigrading on the algebras \( S_l(q, Q), S_r(q, Q) \) and \( \mathcal{M}_q(k) \).

7. Notation. Consider the tensor product of bigraded algebras
\[
\widehat{\mathcal{M}} := S_l(q, Q) \otimes \mathcal{M}_q(k) \otimes S_r(q, Q).
\]
8. Lemma. Any (bi)homogenous element in a tensor product of (bi)graded algebras is a sum of tensor products of (bi)homogenous elements in tensor factors. If one chooses a set of homogeneous generators in each tensor factor than the summands can be chosen as tensor products of monomials in those generators.

9. Lemma. \( l^i \otimes t^j_i \otimes r_i \) generate the subalgebra of all elements of bidegree \((0,0)\) in \( \widehat{M} \).

Proof. If \( J = (j_1, \ldots, j_s) \) is some ordered \( s \)-tuple of labels (repetitions of labels possible) denote \( l^J = l^{j_1} \cdots l^{j_s} \) and we adopt obvious extension of this multilabel notation for \( r \)-s and \( t \)-s. It is clear that any tensor product of monomials which is of bidegree \((0,0)\) is of the form \( l^t \otimes t^l_j \otimes r_{\sigma J} \) where \( \sigma \) and \( \tau \) are permutations on \( |I| = |J| \) letters. Then by lemma 8 it is enough to show that any such tensor product \( l^{t_x} \otimes t^l_x \otimes r_{\sigma J} \) may be written as sum of products of the form \( l^i \otimes t^i_j \otimes r_i \). But \( l^J \) and \( l^{t_x} \) are proportional in \( S_l \), and similarly \( r_I \) and \( r_{\sigma I} \) are proportional in \( S_r \). Hence, up to accounting for a scalar factor, we may assume that \( \sigma \) and \( \tau \) are trivial. But then the expression is manifestly the product of elements of the required form.

10. Theorem. Suppose (1) holds. Let \( t^i_j \) and \( T^i_j \) denote the generators of \( q \)-deformed and \( P, Q \)-deformed quantum matrix algebras respectively. Then

(i) the rule

\[
\iota_{q,Q} : T^i_j \mapsto l^i \otimes t^i_j \otimes r_i
\]

extends to a unique algebra homomorphism \( \iota_{q,Q} : \mathcal{M}(P, Q; k) \to \widehat{M} \).

(ii) This homomorphism is injective and its image is the subalgebra of all elements of \((0,0)\)-bidegree in \( \widehat{M} \).

(iii) Similarly, rescaling \( e^j \) by \( l^i \) produces the relations in \( \Lambda_P \) from the relations in \( \Lambda_q \).

Proof. (i) One needs to show that \( \iota_{q,Q} \) sends the ideal of relations (in free algebra on \( T \)-s) to zero. For example, omitting the tensor product notation, we calculate, for \( i < j \) and \( k < l \),

\[
\iota(q_{kl} T^i_l T^j_k) = q_{kl} t^i_l r_i \cdot t^j_k r_j
\]

\[
= q_{kl} (t^i_l t^j_k)(r_i r_j)
\]

\[
= q_{kl} (qq^{-1}_{kl} t^i_l t^j_k)(q q^{-1}_{ij} r_j r_i)
\]

\[
= q^2 q_{ij}^{-1} l^i j^j k^k l^l i^i r_j r_i
\]

The other cases are left to the reader.
(ii) For injectivity one can use e.g. the normal basis theorem for the quantum matrix algebras: monomials of the form \((T_1^{1 \alpha_11}) \alpha_11 \cdots (T_1^{1 \alpha_{nn}}) \alpha_{nn}\) make a basis of \(\mathcal{M}(P, Q; k)\). It is clear that the images are linearly independent because the middle tensor factors of the images are such (by the normal basis theorem for 1-parametric case) and the other two tensor factors are nonzero. The description of the image of \(\iota_{q,Q}\) follows from [9].

(iii) Easy.

11. Remarks. This isomorphism will be very useful for our purpose. Essentially this proposition is a mechanism essentially equivalent to the cocycle-twisting of [1]. Namely, in both approaches, the difference between the algebra relations for \(t_i^j\)-s and for \(T_i^j\)-s is reflected in rescaling factors for each monomial, which may be expressed in terms of a bicharacter and depends only on the bidegree of the monomial.

However, there is an important difference in using our isomorphism \(\iota\) from the usage of twisting in [1]. Namely, it is shown in [1] that the correspondence \(t_i^j \mapsto T_i^j\) which they use, extends multiplicatively on monomials to an isomorphism of vector spaces, and even of coalgebras; whereas it does not respect the algebra structure. On the other hand, our map \(\iota\) is a monomorphism of algebras, as stated above, but it does not respect the coalgebra structure!

12. (Quantum minors.) If \(J\) and \(K\) are ascendingly ordered row and column multilabels of the same cardinality \(m\) without repetitions, then the corresponding quantum minor \(D^K_L\) is the element of \(\mathcal{M}\) satisfying

\[
D^K_L = \sum_{\sigma \in \Sigma(m)} \epsilon_P(\sigma K) T_{i_{k_1} \cdots i_{k_m}}^{k_1 \cdots k_m} = \sum_{\sigma \in \Sigma(m)} \epsilon_P(\sigma K) \epsilon_P^{-1}(\tau L) T_{l_{k_1} \cdots l_{k_m}}^{k_1 \cdots k_m},
\]

\[
D^K_L = \sum_{\sigma \in \Sigma(m)} \epsilon_Q(\sigma L) T_{l_{i_1} \cdots l_{i_m}}^{i_1 \cdots i_m} = \sum_{\sigma \in \Sigma(m)} \epsilon_Q(\sigma L) \epsilon_Q^{-1}(\tau K) T_{l_{k_1} \cdots l_{k_m}}^{k_1 \cdots k_m},
\]

where \(\tau \in \Sigma(m)\) is a fixed permutation.

13. Proposition. Let \(\iota\) be the monomorphism [4]. Then \(\iota(D^K_L) = l^j d^K_j r_K\) where \(d^K_j\) denotes the quantum minor in 1-parametric case.

Proof.\\

\[
\iota(D^K_j) = \sum_{\sigma \in \Sigma(m)} (-q)^{l(\sigma)} \zeta_r(\sigma K) \iota(T_{j_1}^{k_{\sigma(1)}} \cdots i_{j_m}^{k_{\sigma(m)}}) \iota(T_{j_1}^{k_{\sigma(1)}} \cdots i_{j_m}^{k_{\sigma(m)}}) = \sum_{\sigma \in \Sigma(m)} (-q)^{l(\sigma)} \zeta_r(\sigma K) i_{j_1}^{k_{\sigma(1)}} \cdots i_{j_m}^{k_{\sigma(m)}} \otimes r_{k_1} \cdots r_{k_m} = l^j D^K_j r_K.
\]
This proposition reminds but is different to the statement of Lemma 5 in [1] which asserts that the twisting considered as an identity map but changing its algebra structure, interchanges the quantum determinants. Notice that the $D^K_L$ and $d^K_L$ on the two sides are given by different formulas in terms of generators $T^k_l$ and $t^k_l$ respectively.

14. Now one needs to see what happens when one considers monomials in $D$-s, for example $\iota(D^K_LD^M_N)$. By the same, method, one gets $(l^Ll^M)(d^K_Ld^M_N)(r_Kr_M)$. One knows that the relations in $M$ are homogeneous in the sense that the total row multilabel and column multilabel are the same up to the ordering. Every relation is a sum of homogeneous. Now if we take different monomials $\prod d^*_s = d^K_{L_1} \cdots d^K_{L_m}$ in usual quantum minors then in order to make them manifestly in the image of $\iota$ on some monomial in multiparametric quantum minors, we need to homogenize expression by multiplying it by $l^S$ and $r^V$ where $S$ and $V$ are the ascendingly ordered column and row total multilabel of $\prod d^*_s$, that is $S =: K_1 \cdots K_m$; and $V =: L_1 \cdots L_m$. Thus the $S$ and $V$ are the same for all monomials in the identity (this is more or less the definition of a homogeneous identity). Then we reorder the multilabels in $l$ and in $r$ separately to get the same ordering, but this involves introducing inverse of $\zeta_l$ and $\zeta_r$ corresponding to the ordering on the column and row multilabels seperately. For a homogeneous identity the multiplier $l^S$ and $r^V$ will be the same, however the reordering factors will be clearly different. Thus we get, in terms of $D$-s the same identity up to different homogeneous factors in front of different monomials will be $\zeta_l^{-1}(L_1L_2 \cdots L_m)\zeta_r^{-1}(K_1K_2 \cdots K_m)$ where $K_i$-s is the row multilabel of $i$-th row and $L_i$ of $i$-th column.

**Theorem.** This procedure induces the 1-1 correspondence between the quantum minor identities for the 1-parametric and the minor identities for multiparametric minors.

Notice that despite the fact that $\iota$ is not an algebra homomorphism, essentially $\iota$ and extracting the proprotionality constants from reorderings of $r$-s and $l$-s do the job. If one would use the original twisting of [1] one has there a coalgebra map, hence it sends identities to something what are not, hence it is not clear how to directly use it for the same result.

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