DYNAMICAL FRICTION IN A GASEOUS MEDIUM WITH A LARGE-SCALE MAGNETIC FIELD

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ABSTRACT

The dynamical friction force experienced by a massive gravitating body moving through a gaseous medium is modified by sufficiently strong large-scale magnetic fields. Using linear perturbation theory, we calculate the structure of the wake generated by, and the gravitational drag force on, a body traveling in a straight-line trajectory in a uniformly magnetized medium. The functional form of the drag force as a function of the Mach number \( \equiv V_0/c_s \), where \( V_0 \) is the velocity of the body and \( c_s \) is the sound speed, depends on the strength of the magnetic field and on the angle between the velocity of the perturber and the direction of the magnetic field. In particular, the peak value of the drag force is not near Mach number \( \sim 1 \) for a perturbing moving in a sufficiently magnetized medium. As a rule of thumb, we may state that for supersonic motion, magnetic fields act to suppress dynamical friction; for subsonic motion, they tend to enhance dynamical friction. For perturbers moving along the magnetic field lines, the drag force at some subsonic Mach numbers may be stronger than at supersonic velocities. We also mention the relevance of our findings to black hole coalescence in galactic nuclei.

Key words: black hole physics – hydrodynamics – ISM: general – waves

Online-only material: color figures

1. INTRODUCTION

An object moving in a background medium induces a gravitational wake. The asymmetry of the mass density distribution upstream and downstream from the perturber produces a drag on the body, which is often referred to as gravitational drag or dynamical friction (DF) force. A body in orbital motion may undergo a radial decay of its orbit owing to the loss of angular momentum by the negative torque caused by DF drag. Chandrasekhar (1943) derived the DF on a massive particle passing through a homogeneous and isotropic background of light stars. His formula is applied to estimate the merge timescale of satellite systems or to study the accretion history of galaxies. Bondi & Hoyle (1944) considered the problem of the mass accretion by a point mass \( M \) traveling at velocity \( V_0 \) in a collisional homogeneous medium of sound speed \( c_s \), in the limit where the perturber moves at supersonic velocities relative to the ambient gas (i.e., high Mach numbers). If the perturber is an accretor, streamlines with small impact parameter may become bound because of energy dissipation in shocks and can be accreted to the perturber. Hence, the force on the perturber consists of two parts: the gravitational drag and the momentum accretion force. The latter contribution may be decelerating or accelerating (Ruffert 1996). If the size of the perturber is larger than the Bondi–Hoyle accretion radius, defined as \( R_{\text{BH}} \equiv GM/[c_s^2(1 + M^2)] \) with \( M = V_0/c_s \), the density and velocity structure of the wake, at any Mach number, can be inferred analytically in linear theory because the body produces a small perturbation in the ambient gaseous medium at any location. The gravitational drag is inferred as the gravitational attraction between the perturber and its own wake (e.g., Dokuchaev 1964; Rephaeli & Salpeter 1980; Just & Kegel 1990; Ostriker 1999; Kim & Kim 2007; Sánchez-Salcedo 2009; Namouni 2010).

The studies of the gravitational drag in gaseous media have enjoyed widespread theoretical application, ranging from protoplanets to galaxy clusters. It seems to play a significant role in the growth of planetesimals (Hornung et al. 1985; Stewart & Wetherill 1988), the eccentricity excitation of planetary embryo orbits (Iida 1990; Namouni et al. 1996), the orbital decay of common-envelope binary stars (e.g., Taam & Sandquist 2000; Nordhaus & Blackman 2006; Ricker & Taam 2008; Maxted et al. 2009; Stahler 2010), the evolution of the orbits of planets around the more massive stars (Villaver & Livio 2009), the evolution of low-mass condensations in the cores of molecular clouds (Nejad-Asghar 2010), the mass segregation of massive stars in young clusters embedded in dense molecular cores (Chavarría et al. 2010), the orbital decay of kiloparsec-sized giant clumps in galaxies at high redshift (Immeli et al. 2004; Bournaud et al. 2007), or the heating of intracluster gas by supersonic galaxies (El-Zant et al. 2004; Kim et al. 2005; Kim 2007; Conroy & Ostriker 2008). Special work has been devoted to understanding the role of gaseous DF in the orbital decay of stars and supermassive black holes (SMBHs) as a result of hydrodynamic interactions with an accretion flow in galactic nuclei (Narayan 2000). Mergers of SMBH binaries may be accelerated on subparsec scales by angular momentum loss to surrounding gas (Armitage & Natarajan 2005). In particular, gaseous DF expedites the growth of SMBHs by mergers in colliding galaxies (Escala et al. 2004, 2005; Dotti et al. 2006; Mayer et al. 2007; Tanaka & Haiman 2009; Colpi & Dotti 2011).

Less developed is the corresponding theory of DF in a magnetized gaseous medium. As far as we know, the analytic estimate of the gravitational drag for a body moving on a rectilinear trajectory parallel to the uniform unperturbed magnetic field lines by Dokuchaev (1964) is the only work in this area. He concluded that the DF force on a supersonic body is reduced by a factor that depends on the ratio between the Alfvén speed and the sound speed. Since large-scale magnetic fields are ubiquitous in many astronomical systems such as molecular clouds (Tamura & Sato 1989; Goodman & Heiles 1994; Matthews & Wilson 2002; Heiles & Crutcher 2005) or galactic nuclei, it is important to understand how the DF force is affected by the presence of ordered large-scale magnetic fields. In fact, young stellar systems
and low-mass condensations orbiting in the potential of their birth clusters can interact with the surrounding dense and magnetized molecular interstellar medium during the dispersal of the cluster’s gas. In the Galactic center, structures associated with ordered magnetic fields, called arches and threads, are detected in radio continuum maps (Yusef-Zadeh et al. 1984). The magnetic field configuration of the Galactic center has been viewed as poloidal in the diffuse, interstellar (intercloud) medium and approximately parallel to the Galactic plane only in the dense molecular clouds (Nishiyama et al. 2010). On the scale of 400 pc, fields of 100 μG have been reported (Chuss et al. 2003; Crocker et al. 2010).

The importance of gaseous DF in the evolution and coalescence of a massive black hole binary is motivated by both observational and theoretical work that indicates the presence of large amounts of gas in the central region of merging galaxies. During the mergers of galaxies, the inflow of gas material toward their galactic centers driven by tidal torques associated with bar instabilities and shocks will sweep up and amplify the magnetic field in the central region (Callegari et al. 2009; Guedes et al. 2011). Observations of gas-rich interacting galaxies such as the ultraluminous infrared galaxies (ULIRGs) show that their central regions contain massive and dense clouds of molecular and atomic gas (Sanders & Mirabel 1996). ULIRGs are natural locations in which to expect very strong magnetic fields (Thompson et al. 2006, 2009; Robishaw et al. 2008).

In this paper, we study the DF in a gaseous medium on a body moving on a rectilinear orbit in a homogeneous, uniform magnetized cloud. This is the simplest idealized extension of the unmagnetized case and is the first step in understanding the role of ordered magnetic fields. Previous works have shown that although the formulae of the gaseous drag force in an unmagnetized gas medium were derived for rectilinear orbits in homogeneous and infinite media (Dokuchaev 1964; Rephaeli & Salpeter 1980; Just & Kegel 1990; Ostriker 1999; Sánchez-Salcedo & Brandenburg 1999; Kim & Kim 2009; Namouni 2010; Lee & Stahler 2011; Cantó et al. 2011), simple “local” extensions have been proven very successful in more realistic situations, e.g., smoothly decaying density backgrounds or when the perturber is moving on a circular orbit (Sánchez-Salcedo & Brandenburg 2001; Kim & Kim 2007; Kim et al. 2008; Kim 2010). As a useful starting point for understanding the role of a large-scale magnetic field, we also consider that the unperturbed medium is homogeneous and uniformly magnetized. A discussion on the DF in other initial force-free configurations will be given in a separate paper. Even in the simple case of a uniformly magnetized medium, the magnetic field produces qualitatively new phenomena.

The paper is organized as follows. In Section 2, we discuss the basic concepts on the ideal problem of a particle traveling at constant speed through a uniform gas, both in the purely hydrodynamic case and when the plasma is magnetized. In Section 3, we outline the linear derivation for calculating the steady-state density wake generated by an extended body moving along the magnetic fields, give an analytical solution of the problem, and compare it with previous work. The time-dependent linear perturbation theory is presented in Section 4. Section 5 describes the structure of the resulting wake and evaluates the DF force as a function of Mach number, for different angles between the direction of the perturber’s velocity and the direction of the magnetic field. In Section 6, we summarize our results and discuss their implications.

2. DYNAMICAL FRICITION IN GASEOUS MEDIA: BASIC FORMULAE

2.1. Unmagnetized Medium

Under the assumption of a steady state, Dokuchaev (1964), Ruderman & Spiegel (1971), and Rephaeli & Salpeter (1980) derived the drag force on a point mass $M$ moving at velocity $V_0$ on a straight-line trajectory through a uniform medium with unperturbed density $\rho_0$ and sound speed $c_s$. For subsonic perturbers ($\lambda \equiv V_0/c_s < 1$, where $\lambda$ is the Mach number), these authors found that the drag force is zero because of the front–back symmetry of the density distribution about the perturber. For the steady-state supersonic case, the drag force takes the form

$$\nabla F_{\text{DF}} = \frac{4\pi \rho_0(GM)^2}{V_0^2} \ln \lambda, \quad (1)$$

where $\lambda = r_{\text{max}}/r_{\text{min}}$, with $r_{\text{max}}$ and $r_{\text{min}}$ being the maximum and minimum radii of the effective gravitational interaction of a perturber with the gas. For extended perturbers $r_{\text{min}}$ is its characteristic size, whereas for point-like perturbers $r_{\text{min}}$ is of the order of the Bondi–Hoyle radius $R_{\text{BH}}$ (Cantó et al. 2011), as defined in Section 1.

Using a time-dependent analysis in the unmagnetized case, Ostriker (1999) found that (1) the force is not zero at the subsonic regime because, although a subsonic perturber generates a density distribution with contours of constant density corresponding to ellipsoids, there are always cutoff ones within the sonic sphere that exert a gravitational drag and (2) $r_{\text{max}}$ increases with time in the supersonic case. More specifically, she found that the Coulomb logarithm is given by

$$\ln \lambda = \frac{1}{2} \ln \left( \frac{1 + M}{1 - M} \right) - M, \quad (2)$$

for $M < 1$ and $t > r_{\text{min}}/(c_s - V_0)$, and

$$\ln \lambda = \frac{1}{2} \ln(1 - \lambda^{-2}) + \ln \left( \frac{V_0}{r_{\text{min}}} \right), \quad (3)$$

for $M > 1$ and $t > r_{\text{min}}/(V_0 - c_s)$. The perturber is assumed to be formed at $t = 0$. The transition from the subsonic to the supersonic regime is smooth without any divergence at a Mach number of unity (see Figure 3 in Ostriker 1999). Sánchez-Salcedo & Brandenburg (1999) tested numerically that Ostriker’s formula is very accurate for nonaccreting extended perturbers. In many astrophysical situations, one needs to assign a softening radius to the gravitational potential that in turn determines $r_{\text{min}}$ without any ambiguity. For a body described by a Plummer model with core radius $R_{\text{soft}} \gg R_{\text{BH}}$, Sánchez-Salcedo & Brandenburg (1999) found that $r_{\text{min}} \approx 2.25 R_{\text{soft}}$.

For point-mass accretors, the friction force has been derived by Lee & Stahler (2011) in the subsonic regime and by Cantó et al. (2011) in the hypersonic limit.

2.2. Magnetized Medium

The presence of a small-scale magnetic field tangled at scales below $r_{\text{min}}$ will change the speed of sound. For isotropic compression of a random magnetic field, the effective sound speed is $(c_s^2 + \frac{4}{3} c_a^2)^{1/2}$ (e.g., Zweibel 2002), where $c_a$ is the Alfvén speed of the random small-scale component of the magnetic field. Therefore, in order to include the effect of a
small-scale magnetic field, one has to replace the sound speed by the effective sound speed in the definition of \( M \) in Equations (2) and (3).

The extension of the drag force formulae is by no means straightforward if the gaseous medium is permeated by a regular magnetic field. Dokuchaev (1964) derived the gravitational drag force in the steady state for perturbers moving along the lines of the unperturbed magnetic field. He found that the DF drag is

\[
F_{DF} = \left( 1 - \frac{c_A^2}{V_0^2} \right) 4\pi \rho_0 (GM)^2 \frac{1}{V_0^2} \ln \Lambda, \tag{4}
\]

at \( V_0 > (c_s^2 + c_A^2)^{1/2} \), where \( c_A \) is the Alfvén speed of the regular magnetic field, and it is zero for \( V_0 < (c_s^2 + c_A^2)^{1/2} \).

By comparing Equations (1) and (4), we see that the drag in a uniform magnetized background is never larger than in the unmagnetized case. According to Dokuchaev (1964), the gravitational drag on a body with velocity \( V_0 \) in a uniformly magnetized medium is equal to the drag on a body with velocity \( V_0'(1 - c_A^2/V_0^2)^{1/2} \) in an unmagnetized medium. Therefore, if one naively uses the nonmagnetic formulae (1)–(3) by replacing the sound speed \( c_s \) by the magnetosonic speed \( (c_s^2 + c_A^2)^{1/2} \), this would yield incorrect results. In the next section, we show, however, that the paper of Dokuchaev (1964) contains an error and it is not true that the drag force in the magnetized medium case is always smaller than in the unmagnetized case.

As Ostriker (1999) demonstrated in the field-free case, the steady-state result found by Dokuchaev that the net force is zero at \( V_0 < (c_s^2 + c_A^2)^{1/2} \), because of the front–back symmetry of the density perturbation in the medium, may be misleading. It is also unclear how \( \ln \Lambda \) varies in time for the magnetized supersonic case. Moreover, it left unexplored how the drag force depends on the angle between the velocity of the perturber and the direction of the magnetic field. Before addressing these questions, however, it is still worthwhile to find analytical solutions for the perturbed steady density and the resulting drag force in the simplest scenario in which the velocity of the perturber and the magnetic field are parallel. Such an exact treatment will allow us to gain insight into more complicated situations. This is done in the next section.

### 3. Axisymmetric Case: Velocity of the Perturber Parallel to the Direction of the Magnetic Field

We consider a gravitational perturber moving on a straight line at constant velocity in a medium with unperturbed density \( \rho_0 \) and thermal sound speed \( c_s \). In the absence of magnetic fields, the linearized equations of motion can be reduced to a nonhomogeneous wave equation for the relative perturbation \( \rho = \rho_0 + \rho' \) (e.g., Ruderman & Spiegel 1971; Ostriker 1999). Once a uniform magnetic field, \( B_0 \), parallel to the direction of the perturber’s velocity is included, Dokuchaev (1964) showed that the relative perturbation obeys an equation of fourth order in \( t \) and solved it using a double Fourier–Hankel transformation. As will become clear later, we prefer to describe the evolution of the system through wave equations because it facilitates the physical interpretation of the problem and because the contact with the analysis of Ostriker (1999) is easier. In addition, the extension of the equations for a case in which the angle between \( B_0 \) and the velocity of the perturber is arbitrary becomes straightforward in our approach.

#### 3.1. Perturbed Density Distribution

We study first the completely steady flow created by a mass on a constant-speed trajectory parallel to the lines of the unperturbed magnetic field \( B_0 = B_0 \hat{z} \). To do this, consider a particle at the origin of our coordinate system, surrounded by a gas whose velocity far from the particle is \( V_0 = -V_0 \hat{z} \), with \( V_0 > 0 \). We further assume that the gas evolves under flux-freezing conditions. Our analysis begins with the linearized magnetohydrodynamic (MHD) equations to describe the medium’s response to the perturber’s presence \( \rho = \rho_0 + \rho' \), \( V = V_0 + \nu' \), and \( B = B_0 + B' \), in a stationary state (\( \partial/\partial t = 0 \)):

\[
\rho_0 \nu' \cdot V_0 + \rho_0 \nabla \rho' = 0, \tag{5}
\]

\[
(V_0 \cdot \nabla)\nu' = \frac{c_A^2}{\rho_0} - \nabla \Phi + \frac{1}{4\pi \rho_0} (\nabla \times B') \times B_0, \tag{6}
\]

\[
\nabla \times (V_0 \times B') + \nabla \times (\nu' \times B_0) = 0, \tag{7}
\]

\[
\nabla \cdot B' = 0, \tag{8}
\]

where \( \Phi \) is the gravitational potential created by the perturber. The Poisson equation links the potential with the density profile of the perturber \( \rho_p \):

\[
\nabla^2 \Phi = 4\pi G \rho_p. \tag{9}
\]

The Lorentz force, which provides the magnetic back-reaction on the flow pattern, is given by

\[
(\nabla \times B') \times B_0 = B_0 \left[ \left( \frac{\partial B'_x}{\partial z} - \frac{\partial B'_y}{\partial x} \right) \hat{x} - \left( \frac{\partial B'_y}{\partial y} - \frac{\partial B'_x}{\partial z} \right) \hat{y} \right]. \tag{10}
\]

Hence, the divergence of the Lorentz force is

\[
\nabla \cdot \left( (\nabla \times B') \times B_0 \right) = B_0 \left( \frac{\partial^2 B'_x}{\partial z \partial z} - \frac{\partial^2 B'_y}{\partial x \partial z} - \frac{\partial^2 B'_y}{\partial y \partial y} + \frac{\partial^2 B'_x}{\partial y \partial z} \right) = -B_0 \nabla^2 B'_z. \tag{11}
\]

In the last equality we have used \( \nabla \cdot B' = 0 \). By substituting Equations (5) and (11) into the divergence of the equation of motion, we have

\[
-\frac{V_0^2}{\rho_0} \frac{\partial^2 \rho'}{\partial z^2} = -\frac{c_A^2}{\rho_0} \nabla^2 \rho' - \nabla^2 \Phi - \frac{B_0}{4\pi \rho_0} \nabla^2 B'_z. \tag{12}
\]

By comparing the second and third terms on the right-hand side of the above equation, we see that, formally, the magnetic back-reaction term \( \nabla^2 B'_z \) is mathematically equivalent to having an external potential term. However, while \( \Phi \) is known (Equation (9)), \( B'_z \) is coupled to the fluid motions through the flux-freezing Equation (7).
Next, we need an independent equation for $B_z'$ to close the system. This can be accomplished using the third component of the induction equation (7), which has the form

$$ B_0 \left( \nabla \cdot \mathbf{v}' - \frac{\partial \mathbf{v}'}{\partial z} \right) = V_0 \frac{\partial B_z'}{\partial z}, \quad (13) $$

Our strategy is to eliminate $\mathbf{v}'$ in Equation (13). The third component of the equation of motion (6) can be written as

$$ - V_0 \frac{\partial \mathbf{v}'}{\partial z} = \frac{c_s^2}{\rho_0} \frac{\partial \rho'}{\partial z} - \frac{\partial \Phi}{\partial z}. \quad (14) $$

This equation does not depend explicitly on the frozen-in magnetic field because the $z$-component of the Lorentz force vanishes in the linear approximation (see Equation (10)). Substituting Equations (5) and (14) into Equation (13), we obtain the desired equation

$$ B_0 \left( \frac{1}{\rho_0} \frac{\partial \rho'}{\partial z} - \frac{1}{\rho_0 M^2} \frac{\partial \rho'}{\partial z} - \frac{1}{V_0^2} \frac{\partial \Phi}{\partial z} \right) = \frac{\partial B_z'}{\partial z}, \quad (15) $$

where we recall that $M \equiv V_0/c_s$ is the (sonic) Mach number. Once again, the magnetic term $\partial B_z'/\partial z$ is formally identical to $\partial \Phi/\partial z$, but some caution should be used when interpreting it; the $z$-component of the Lorentz force is not $\partial B_z'/\partial z$ but zero.

Equations (12) and (15) constitute a system of two coupled linear differential equations for $\rho'$ and $B_z'$ that may be solved once we have chosen suitable boundary conditions. Defining the dimensionless perturbations $\alpha \equiv \rho'/\rho_0$ and $\beta \equiv B_z'/B_0$, the equations to solve are

$$ M^2 \frac{\partial^2 \alpha}{\partial z^2} = \nabla^2 \alpha + \frac{1}{c_s^2} \nabla^2 \Phi + \gamma^2 \nabla^2 \beta_z, \quad (16) $$

$$ (M^2 - 1) \frac{\partial \alpha}{\partial z} - \frac{1}{c_s^2} \frac{\partial \Phi}{\partial z} = M^2 \frac{\partial \beta_z}{\partial z}, \quad (17) $$

where $\gamma \equiv c_A/c_s$ and $c_A$ is the Alfvén speed in the unperturbed medium. In the limit of vanishing magnetic field, $\beta_z = \gamma = 0$ and Equation (16) reduces to that of the wake of a body in an unmagnetized medium (e.g., Sánchez-Salcedo 2009).

For a point-like perturber of mass $M$, an analytical solution can be derived for the density enhancement, velocity, and magnetic fields in the wake. In order to calculate the DF force exerted on the body, we only need the gas density enhancement in the wake, which is derived in Appendix A and is given by

$$ \alpha(R, z) = \frac{\lambda (1 - \eta) GM}{\xi c_s^2} \frac{1}{\sqrt{z^2 + R^2 \gamma^2}}, \quad (18) $$

where $R = \sqrt{x^2 + y^2}$ is the cylindrical radius and

$$ \eta = (c_A/V_0)^2 = (\gamma/M)^2, \quad (19) $$

$$ \xi = 1 + (1 - M^{-2}) \gamma^2 = 1 - \eta + \gamma^2, \quad (20) $$

$$ \gamma^2 = 1 - \frac{M^2}{\xi}, \quad (21) $$

and

$$ \lambda = \begin{cases} 1 & \text{if } M < M_{\text{crit}}, \\ 2 & \text{if } M_{\text{crit}} < M < \min(1, \gamma) \text{ and } z/R > |\gamma|; \\ 1 & \text{if } \min(1, \gamma) < M < \max(1, \gamma); \\ 2 & \text{if } M > \max(1, \gamma) \text{ and } z/R < -|\gamma|; \\ 0 & \text{otherwise.} \end{cases} \quad (22) $$

Here, the critical Mach number is defined as

$$ M_{\text{crit}} \equiv (1 + \gamma^{-2})^{-1/2}. \quad (23) $$

Because of the linear-theory assumption, Equation (18) is properly valid only for $(z^2 + \gamma^2 R^2)^{1/2} > (1 - \eta)GM/(\xi c_s^2)$. The nonmagnetic steady-state solution for density in the wake past a gravitating body is recovered when $\gamma = 0$.

For clarity, it is convenient to distinguish four intervals depending on the value of the Mach number of the body: $M < M_{\text{crit}}$ (case or interval I), $M_{\text{crit}} < M < \min(1, \gamma)$ (case or interval II), $\min(1, \gamma) < M < \max(1, \gamma)$ (case III), and $M > \max(1, \gamma)$ (interval IV). In cases I and III $\gamma^2$ is a positive number, whereas it is negative in cases II and IV. In the latter cases where $\gamma^2 < 0$, the density perturbation $\alpha$ vanishes at some spatial locations. In case IV, for instance, $\alpha$ outside the cone defined by the surface $z = -|\gamma| R$ is actually zero. Turning to Equation (15), we see that the magnetic perturbation $B_z'$ in these regions does not vanish but obeys the relation $\partial B_z'/\partial z = -(B_0/V_0^2)\partial \Phi/\partial z$. Now, from Equation (14), the axial component of the velocity satisfies a similar equation, $\partial v_z'/\partial z = (1/V_0)\partial \Phi/\partial z$. Using the fact that $\nabla \cdot \mathbf{v}' = 0$ in regions of constant density (see Equation (5)), it is simple to show that $B_z'$ is parallel to $\mathbf{v}'$ in regions where $\alpha = 0$, and thus the magnetic configuration is force-free in these zones.

Once $\rho/\rho_0$ is known, the gravitational drag can be computed; this is done in Section 3.4. Nevertheless, in order to gain more insight into the physics of the wake, we describe the morphology and structure of the steady-state wake in the next section.

### 3.2. Physical Interpretation

Consider first subsonic perturbers. In the limit $M \to 0$, we have $\gamma \to 1$, $\xi \to -1/M^2$, $(1 - \eta) \to -\gamma^2/M^2$, and $\lambda = 1$. Therefore, the density enhancement is $GM/(\xi c_s^2)$, which is $\gamma$-independent, and corresponds to the linearized solution of the hydrostatic envelope, $\rho/\rho_0 = \exp \left[ GM/(\xi c_s^2) \right]$, around a stationary perturber (e.g., Ostriker 1999). In this case, the magnetic field lines remain straight and the whole magnetic configuration is force-free.

The surfaces of constant density for subsonic perturbers may be either ellipsoids or hyperbolae, depending on the Mach number. At $M < M_{\text{crit}}$ it holds that $\gamma^2 > 1$ and the isodensity surfaces are ellipsoids elongated along the trajectory of the perturber with eccentricity $e = M/\sqrt{\xi}$. This is in sharp contrast to what happens without any magnetic field where the ellipses are elongated along $R$ for perturbers at any subsonic Mach number ($\gamma^2 < 1$). Therefore, the existence of ellipsoidal density isodectours elongated along $z$ is a clear signature of curved magnetic fields.

In the nonmagnetic subsonic case, $v'_z < 0$ at $z > 0$ and $v'_z > 0$ at any $z$, signifying that the incoming fluid is veering toward the perturber, but then turning away again once it passes the body. The result of $v'_z > 0$ implies that the gas is being dragged by the gravitational pushing of the body. In order to understand the morphology of the wake in a magnetized medium, in the
following we calculate \(v'_R\) and \(v'_z\) for a perturber with Mach number in the interval I.

From Equation (14) and using the result for \(\alpha\) in Equation (18), we find that

\[
V_0 \frac{\partial v'_z}{\partial z} = -\frac{\lambda(1-\eta)}{\xi} \frac{GMz}{(z^2 + R^2\gamma^2)^{3/2}} + \frac{GMz}{(z^2 + R^2\gamma^2)^{3/2}},
\]

which leads to

\[
v'_z = -\frac{GM}{V_0} \left[ \frac{1}{(z^2 + R^2\gamma^2)^{1/2}} - \frac{\lambda(1-\eta)}{\gamma^2}\xi(z^2 + R^2\gamma^2)^{1/2} \right].
\]

(24)

It is simple to show that \(v'_z > 0\) in case I, regardless of the value of \(\Upsilon\).

The radial component of the velocity can be found using Equation (5):

\[
\frac{\partial Rv'_R}{R \partial R} = -\frac{\partial v'_z}{\partial z} + \frac{\partial v_0}{\partial \alpha} = \frac{-V_0^2}{M^2} \left[ (1 - M^2) \frac{\partial \alpha}{\partial z} + \frac{1}{\xi} \frac{\partial \Phi}{\partial z} \right].
\]

(25)

Since we already know \(\alpha\) and \(\Phi\), this equation can be solved to obtain the radial velocity:

\[
v'_R = \frac{GMz}{V_0R} \left[ \frac{1}{(z^2 + R^2\gamma^2)^{1/2}} - \frac{\lambda(1-M^2)(1-\eta)}{\gamma^2\xi(z^2 + R^2\gamma^2)^{1/2}} \right].
\]

(26)

From the above equation, it follows that, in case I, \(v'_R > 0\) at the head of the perturber, regardless of the magnetic field strength. Thus, a parcel of fluid in the upstream region circulates around the perturber, reaching its maximum \(R\)-value at \(z = 0\) and then turning back again. Since frozen-in magnetic field lines are dragged by the gas, the upstream magnetic field lines are decompressed radially, resulting in arched magnetic field lines with negative \(B'_z\)-values. In fact, integration of Equation (15) gives

\[
B'_z = \frac{GM \xi}{V_0^2} \left[ \frac{1}{(z^2 + R^2\gamma^2)^{1/2}} - \frac{\lambda(1-M^2)(1-\eta)}{\gamma^2\xi(z^2 + R^2\gamma^2)^{1/2}} \right].
\]

(27)

This clearly states that \(B'_z < 0\) (here we assume \(B_0 > 0\)). Thus, an anticorrelation arises between density and \(B'_z\).

At Mach numbers close to \(M_{\text{crit}}\), that is, \(M = M_{\text{crit}} - \epsilon\) with \(\epsilon\) a very small positive number, the density profile, at not extremely large \(z\) distances, is

\[
\alpha(R, z) \simeq \frac{\gamma^{3/2}GM/c_s^2}{\sqrt{2}\epsilon(1 + \Upsilon'z)^{1/4}R}.
\]

(28)

Hence, the density enhancement is large and its \(z\)-gradient very small.

At subsonic Mach numbers in the interval \(M_{\text{crit}} < M < \min(1, \Upsilon)\), the surfaces of constant density exhibit no front–back symmetry. The isodensity contours correspond to hyperbolae in the \(z - R\) plane, as occurs for supersonic perturbers in the absence of magnetic fields, but now the density perturbation is null in the rear Mach cone and is nonvanishing in a modified Mach cone leading the perturber. The physical reason is as follows. The density perturbation \(\alpha\) is non-positive at any location because the streamlines diverge at the front cone (\(n > 1\) and \(\xi > 0\)). The quantity \(\alpha\) at the edges of the cone is minus infinity for a point mass. In the front cone, \(\partial v'_z/\partial z > 0\), meaning that the flow in that region is being accelerated by the inward net pressure force.

Across the edge of the modified Mach cone, there is a rapid rise in pressure and density, and the gas velocity quickly slows. In fact, the causality criterion used in Appendix A is tantamount to selecting the solution in which a rapid flow is slowed in a short distance, as occurs in shock waves. Indeed, in Section 5.1 we find numerically that the system adopts this solution.

In case III, the body moves at intermediate velocities, i.e., either in the range \(c_1 < V_0 < c_\Lambda\) (if \(\Upsilon > 1\)) or in the range \(c_\Lambda < V_0 < c_1\) (if \(\Upsilon < 1\)). It is only in this case that \(\gamma \) lies in the range \(0 < \gamma < 1\) and, therefore, the ellipsoids are flattened along \(z\). Remember that in the unmagnetized background, a subsonic perturber also produces elliptical density distributions flattened along \(z\) (e.g., Ostriker 1999); the latter is a particular case of \(c_\Lambda < V_0 < c_1\). For \(c_1 < V_0 < c_\Lambda\) (which requires that \(\Upsilon > 1\)), however, the density perturbation is negative (\(\alpha \leq 0\)). In this case, the thermal pressure decreases toward the body. Therefore, ahead of the perturber the gas is accelerated along the \(z\)-direction by the gravitational force plus the pressure gradient (it holds that \(\partial v'_z/\partial z > 0\) at \(z > 0\)). A negative \(\alpha\) is a consequence of the action of the pressure gradient plus the reduction of the radial convergence of the flow due to the presence of the ordered magnetic field that preferentially allows motions along \(z\), which lead to an accelerated flow falling toward the perturber. In the radial direction, the magnetic field lines are compressed at \(z > 0\) because \(v_R < 0\) upstream of the perturber.

From Equations (18) to (22) we see that if the motion of the perturber is both supersonic and super-Alfvénic, which corresponds to case IV, the density disturbance is confined to a rear cone defined by the condition \(z < -\gamma R\). In this regime, the surfaces of constant density within the wake correspond to similar hyperbolae in the \(z - R\) plane, at the rear of the body, with eccentricity \(e = M/\sqrt{\Upsilon}\). From Equation (20), it is simple to show that \(\xi > 1\) in this case. Therefore, given a sonic Mach number \(M\), the angular aperture of the cone is larger in the presence of magnetic fields. In analogy to the unmagnetized medium, one could define the effective speed of propagation of the disturbance as \(v_p = \sqrt{\Upsilon}c_s\) in case IV. At large Mach numbers (say, \(M \gg \sqrt{\Upsilon}\)), the cone is very narrow (\(e \gg 1\)) and, as expected, the propagation speed coincides with the magnetoacoustic velocity, \(v_p \simeq \sqrt{c_s^2 + B_0^2/c_s^2}\). Hence, at these large \(M\)-values, the stationary flow is similar to that in the nonmagnetic case but replacing the sound speed by the magnetosonic speed.

Now consider case IV but when the perturber moves at the same velocity as the effective speed of the disturbance, so that \(V_0 = v_p\) or, equivalently, \(M = \sqrt{\Upsilon}\). In the nonmagnetic case, this condition corresponds to \(\xi = 1\) and, therefore, the velocity of the body is in resonance with the sound speed in the medium. One could naively think that, at \(M = \sqrt{\Upsilon}\), the response of the medium is maximum because of the resonance between \(V_0\) and \(v_p\). This is not true for \(\Upsilon > 1\) because \(M = \sqrt{\Upsilon}\) implies \(M = \Upsilon(\text{using Equation (20)}, \eta = 1\) (from Equation (19)), and, thereby, \(\alpha = 0\). We learn that a mass moving at the Alfvén speed in a medium with \(\Upsilon > 1\) does not generate any density disturbance in the ambient gas because the velocity field of the stationary flow is divergence-free \((\nabla \cdot v' = 0)\).

3.3. A Comparison with Dokuchaev (1964)

As already mentioned, Dokuchaev (1964) calculated, for the first time, the properties of the wake created by a star moving along the field lines, by treating it as a linear perturbation. His analysis started from the time-dependent linearized equations.
of MHD, including a source term $Q$ in the continuity equation, representing the gas replenishment by the star. Although he used the time-dependent equations, he tacitly assumed that the object’s gravitational field is active since $t = -\infty$, so that the wake is in a steady state. For the case without mass injected by the star ($Q = 0$), the physical standpoints used by Dokuchaev (1964) are exactly the same as those adopted in Section 3.1, except that he chose a reference frame in which the unperturbed background gas is at rest.

Dokuchaev (1964) found closed fourth-order differential equations for $\rho$ and the radial component $v_r$ of the velocity. Through Fourier–Hankel transformations, he was able to solve the differential equation for $\rho$. He found a similar expression for $\rho$ as that given in Equation (18) but he failed to separate correctly the different intervals for $\lambda$ (Equation (22)) and the intervals at which the isocontours are ellipsoids or hyperboloids. In particular, he claimed that the isocontours are ellipsoids at any Mach number below $(c_s^2 + c_A^2)\eta > 0$, which is misleading.

3.4. Gravitational Drag Force in the Axisymmetric Case

Once we have the gas density enhancement $\alpha(r)$ in the ambient medium, we can calculate the gravitational force exerted on the body by its own wake:

$$ F_{DF} = 2\pi GM\rho_0 \int \int dz \, dR \, R \alpha(r) \frac{z}{(z^2 + R^2)^{3/2}}. $$

(30)

The net drag is zero when the isodensity contours are ellipsoids, i.e., when $\gamma^2 > 0$, because the wake exhibits front–back symmetry. At values $\gamma^2 < 0$, however, the region of perturbed density is confined to a cone and the drag force is nonvanishing. Evaluating the integrals in spherical coordinates ($R = r \sin \theta$ and $z = r \cos \theta$), and using the variable $\mu$ defined as $\mu = \cos \theta$, the drag force can be expressed as

$$ F_{DF} = (1 - \eta) \frac{4\pi G M^2}{V_0^2} I \int_{\mu_{\min}}^{\mu_{\max}} \frac{d\mu}{r} \hat{z}, $$

(31)

where

$$ I = \frac{1}{2} \int_{-1}^{1} d\mu \sqrt{1 - \xi^{-1} - \mu^2 - \xi^{-1} \mu^2 M^2}. $$

(32)

As already said, the drag force is nonzero in cases II and IV, where $\gamma^2 < 0$ and thus $\xi > 0$. In case II, $\lambda = 2$ for all $\mu$ between $\mu_{\min} = (M^2 - \xi)^{1/2} / M$ and 1, so that $I = 1$. In case IV, $\lambda = 2$ for all $\mu$ between 1 and $\mu_{\max} = -(M^2 - \xi)^{1/2} / M$ and thus $I = -1$. Since $1 - \eta < 0$ in case II, we can write $(1 - \eta)I = |1 - \eta|$ and the resultant expression for the force is

$$ F_{DF} = \begin{cases} 
-|1 - \eta| F \ln \hat{\Lambda} \hat{z} & \text{if } \mathcal{M} > \max(1, \Upsilon) \\
0 & \text{or } \mathcal{M}_{\min} < \mathcal{M} < \min(1, \Upsilon);
\end{cases} $$

(33)

and $\ln \Lambda$ is the Coulomb logarithm. Note that the density diverges in the wake at Mach numbers close to $\mathcal{M}_{\min}$ because $\xi \to 0$. However, the drag force is finite because the opening angle of the cone becomes very narrow. Still, the drag force is nonzero in the intervals $0 < \mathcal{M} < 1$, so that the width of the interval $|1 - \eta| V_0^2$ in the formula for $\alpha$ increases when $\mathcal{M}$ decreases.

Figure 1 shows the DF force felt by the body at $t = 100 r_{\min} / c_s$, as a function of the Mach number and for different values of $\Upsilon$. In analogy to the unmagnetized case, we take $\Lambda = V_0 / r_{\min} = 100 \nu_0 / c_s$. Dokuchaev (1964) claimed that the drag force is nonzero only at $\mathcal{M} > (1 + \gamma^2)^{1/2}$ (see Section 2.2). This is incorrect. For instance, for $\Upsilon = 1$, the drag force is different from zero at $\mathcal{M} > \mathcal{M}_{\crit} = 0.70$. If the ratio between the Alfvén and sound speeds is $\gamma^2 = 2$, the DF force is nonvanishing in the intervals $0.816 < \mathcal{M} < 1$ and $\mathcal{M} > 1.41$. In fact, there exists always a subsonic velocity range at which the drag force is nonzero.

As long as $\gamma^2 \neq 0$, the DF force has two local maxima: one located at $\mathcal{M}_{\crit}$ and the other one at $\mathcal{M} = \max(1, \sqrt{2} \gamma)$. The drag force strength at $\mathcal{M}_{\crit}$ increases with $\Upsilon$, whereas the drag force at the second local maximum decreases with $\Upsilon$. As Figure 1 clearly shows, at low $\Upsilon$-values, the width of the interval with $F_{DF} \neq 0$ around $\mathcal{M}_{\crit}$ becomes very narrow. For instance, the width of that interval is only $4 \times 10^{-3}$ for $\Upsilon = 0.2$. Hence, the drag force at subsonic values is irrelevant for astrophysical purposes when the Alfvén speed is sufficiently small as compared with the sound speed. For $\Upsilon > 0.4$, the drag force at the local maximum $\mathcal{M}_{\crit}$ is always larger than the drag force at the other local maximum $\mathcal{M} = \max(1, \sqrt{2} \gamma)$. In the particular case of $\Upsilon = 1.41$, the drag force is a factor $> 6$ stronger in the interval $0.816 < \mathcal{M} < 1$ than it is at $\mathcal{M} > 1$. 

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
$\Upsilon$ & $\mathcal{M}_{\crit}$ & $F_{DF}$ \\
\hline
0.2 & 0.6 & 0.8 \\
0.4 & 0.8 & 1.2 \\
0.6 & 1.0 & 1.6 \\
0.8 & 1.2 & 2.0 \\
1.0 & 1.4 & 2.4 \\
1.4 & 1.8 & 3.2 \\
\hline
\end{tabular}
\caption{Values of $\mathcal{M}_{\crit}$ for different $\Upsilon$.}
\end{table}
4. TIME-DEPENDENT EQUATIONS

The steady-state analysis in the axisymmetric case predicts zero drag force at certain Mach numbers because the perturber is surrounded by complete ellipsoids that exert no net force. As Ostriker (1999) demonstrated in the field-free case, the time-dependent analysis in which the body is dropped suddenly at \( t = 0 \) allows one to capture the asymmetric density shells in the far field that exert a gravitational drag on the body. Another advantage of the time-dependent approach is that, contrary to what happens when assuming a steady state, the ambiguity in the definition of the maximum cutoff distance \( r_{\text{max}} \) is fixed.

Without loss of generality, it is convenient to use the gas frame of reference in which the ambient gas is initially at rest, the initial magnetic field is along the \( z \)-axis, and the body moves with velocity \( V_0, \dot{y} + V_0, \dot{z} \). The first-order continuity equation is

\[
\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0, \tag{35}
\]

the MHD Euler equation is

\[
\frac{\partial \mathbf{v}'}{\partial t} = -\frac{c_s^2}{\rho_0} \nabla \Phi + \frac{1}{4\pi \rho_0} (\nabla \times \mathbf{B}') \times \mathbf{B}_0, \tag{36}
\]

and the induction equation is

\[
\frac{\partial \mathbf{B}'}{\partial t} = \nabla \times (\mathbf{v}' \times \mathbf{B}_0). \tag{37}
\]

The initially uniform medium will respond to the gravitational pull of the body through the emission of fast and slow Alfvén waves and sound waves. In the following, we manipulate the above equations to obtain a closed system of two differential equations for \( \rho' \) and \( B'_z \) in analogy to the steady-state case.

Using Equation (11) in the divergence of Equation (36),

\[
\frac{\partial (\nabla \cdot \mathbf{v}')}{\partial t} = -\frac{c_s^2}{\rho_0} \nabla^2 \rho' - \nabla^2 \Phi - \frac{B_0}{4\pi \rho_0} \nabla^2 B'_z. \tag{38}
\]

By substituting Equation (35) into Equation (38), we obtain

\[
\frac{1}{\rho_0} \frac{\partial^2 \rho'}{\partial t^2} = \frac{c_s^2}{\rho_0} \nabla^2 \rho' + \nabla^2 \Phi + \frac{B_0}{4\pi \rho_0} \nabla^2 B'_z. \tag{39}
\]

In terms of \( \alpha \) and \( \beta_z \), it yields

\[
\frac{\partial^2 \alpha}{\partial t^2} = c_s^2 \nabla^2 \alpha + \nabla^2 \Phi + c_A^2 \nabla^2 \beta_z. \tag{40}
\]

Here, the magnetic effect on the density perturbation appears as an inhomogeneous term. We may recover the classical nonmagnetic equation for \( \alpha \) by taking \( c_A = 0 \).

On the other hand, the third component of the induction equation (Equation (37)) implies

\[
\frac{\partial B'_z}{\partial t} = -B_0 \left( \frac{\partial v'_x}{\partial x} + \frac{\partial v'_y}{\partial y} \right). \tag{41}
\]

Equations (35) and (41) give

\[
\frac{\partial B'_z}{\partial t} = B_0 \left( \frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} + \frac{\partial \rho'}{\partial z} \right). \tag{42}
\]

From the third component of the equation of motion (Equation (36))

\[
\frac{\partial v'_x}{\partial t} = \frac{c_s^2}{\rho_0} \frac{\partial \rho'}{\partial z} - \frac{\partial \Phi}{\partial z}, \tag{43}
\]

we know that

\[
\frac{\partial}{\partial t} \left( \frac{\partial v'_x}{\partial z} \right) = \frac{c_s^2}{\rho_0} \frac{\partial^2 \rho'}{\partial z^2} - \frac{\partial^2 \Phi}{\partial z^2}. \tag{44}
\]

Inserting Equation (44) into the temporal derivative of Equation (42), one finds

\[
\frac{1}{B_0} \frac{\partial^2 B'_z}{\partial t^2} = \frac{1}{\rho_0} \frac{\partial^2 \rho'}{\partial t^2} - \frac{c_s^2}{\rho_0} \frac{\partial^2 \rho'}{\partial z^2} - \frac{\partial^2 \Phi}{\partial z^2}. \tag{45}
\]

In dimensionless form

\[
\frac{\partial^2 \beta_z}{\partial t^2} = \frac{\partial^2 \alpha}{\partial t^2} - \frac{c_s^2}{\rho_0} \frac{\partial^2 \alpha}{\partial z^2} - \frac{\partial^2 \Phi}{\partial z^2}. \tag{46}
\]

Putting them together, Equations (40) and (46) can be written as

\[
\Box \alpha = \nabla^2 \tilde{\Phi} + \gamma^2 \nabla^2 \beta_z, \tag{47}
\]

\[
\Box_A \beta_z = \gamma^{-2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) (\alpha + \tilde{\Phi}), \tag{48}
\]

where \( \tilde{\Phi} \) is the gravitational potential in units of \( c_s^2 \) (i.e., \( \tilde{\Phi} = \Phi/c_s^2 \)) and we have used the Lorentz invariant D’Alembertian \( \Box \), defined as

\[
\Box \phi = \left( \frac{1}{c_s^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \phi. \tag{49}
\]

The first equation (Equation (47)) governs the evolution of the density in the presence of a gravitational potential and magnetic fields. The second equation (Equation (48)) describes the evolution of a frozen-in magnetic field when the gas is subject to pressure gradients and to an external gravitational potential. The inhomogeneous term in Equation (48) does not have \( z \)-derivatives because gas motions in that direction do not compress, stir, or stretch the background magnetic field. The resulting Equations (47) and (48) conform to a set of two coupled nonhomogeneous wave equations. For a point-mass perturber, it is simple to find \( \alpha \) in the Fourier–Laplace space, \( \alpha(k, \omega) \), but the inverse Fourier–Laplace integral cannot be given in a closed analytic form.

In order to gain physical insight, consider first a two-dimensional example. If \( \Phi = \Phi(x, y) \), that is, if the perturber is an infinite cylinder with a certain radial density profile \( \rho_p = \rho_p(R) \), then \( \beta_z = \alpha \) because of the flux-freezing condition, and \( \alpha \) satisfies a simple wave equation with magnetoeacoustic speed:

\[
\left( \frac{1}{c_s^2 + c_A^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right) \alpha = \frac{1}{1 + \gamma^2} \nabla^2 \Phi. \tag{50}
\]

The physical reason is that motions are always perpendicular to the frozen-in magnetic field lines. Magnetohydrodynamical equilibrium is reached within the magnetosonic cylinder. At a later stage, Parker instabilities can develop (Sánchez-Salcedo & Santillán 2011).

In the purely hydrodynamical problem, the equation governing the evolution of \( \alpha \) is \( \Box \alpha = \nabla^2 \tilde{\Phi} \). If the perturber is a point source, we have \( \Box \alpha = (4\pi G M/c_s^2) \delta(x - V_{0,z} t) \delta(y - V_{0,y} t) \).
δ(z − V₀,t)H(t). Hence, the density remains unperturbed outside the causal region for sound waves (see Ostriker 1999). In a magnetized medium, however, the situation is different because Equation (48) for the perturbed magnetic field β has a source term \((∂²Φ/∂x² + ∂²Φ/∂y²)\) that does not vanish even outside the causal region for magnetosonic waves.

In the next section, we solve the coupled wave equations numerically. To do so, the perturber gravitational potential will be modeled by a smooth core Plummer potential:

$$\Phi(r, t) = -\frac{GMt(t)}{\sqrt{x^2 + (y - V₀,y)t^2 + (z - V₀,z)t^2 + R_{soft}^2}}$$

where \(R_{soft}\) is the softening radius and \(t(t)\) is a Heaviside step function. Stellar and globular clusters can be accurately described by Plummer potentials. These types of models were also used in Sánchez-Salcedo & Brandenburg (1999, 2001), Kim & Kim (2009), and Kim (2010) to study DF.

5. RESULTS

The coupled inhomogeneous wave equations (47) and (48) were solved using a finite difference scheme in a uniform grid. The scheme is second order in space and third order in time. The temporal algorithm was described in Sánchez-Salcedo & Brandenburg (2001). Calculations start with a uniform background density and magnetic field, and the body is initially placed at the origin of the coordinate system with velocity \(V₀,x + V₀,y\). For the axisymmetric case, which occurs when \(V₀,y = 0\), the calculations were carried out on a two-dimensional \((R, z)\)-plane in cylindrical symmetry. In the general three-dimensional case \((V₀,y \neq 0)\), the variables \(α, β\), and \(γ\) are symmetric about the plane \(z = 0\). Hence, we considered a finite domain with \(x \in [0, L_{x, max}]\) and used symmetric boundary conditions at \(x = 0\) and outflow boundary conditions in the other five caps of the computational domain. However, the size of the domain was taken large enough to ensure that the perturbed density and magnetic field do not reach the boundaries.

As a test of the algorithm, we studied the convergence of homogeneous wave modes by perturbing a uniform background medium. We further tested convergence of our models for several resolutions and found that four zones per \(R_{soft}\) suffice to have converged results.

We take \(R_{soft}, c_s\), and \(1_{cross} = R_{soft}/c_s\) as the units of length, velocity, and time, respectively. A model can thus be specified with four dimensionless parameters: \(GM/(c_s^2 R_{soft})\), \(M, Ψ, \text{ and } Θ = \arctan(V₀,y/V₀,z)\). Here, \(Θ\) is the angle between \(V₀\) and \(B₀\). With fixed \(M, Ψ, \text{ and } Θ\), the variables \(α, β\), and \(γ\) depend linearly on \(GM/(c_s^2 R_{soft})\). Hence, in our calculations we always take \(GM/(c_s^2 R_{soft}) = 0.01/3\) and explore how the density and the magnetic field in the wake depend on the other three parameters.

5.1. Axisymmetric Case

We start run models with the magnetic field terms switched off and compare the density enhancement and the gravitational drag with previous linear calculations in Ostriker (1999) and Sánchez-Salcedo & Brandenburg (1999). We found excellent agreement, backing up our numerical model. In the following, we present results for a body moving along the field lines of the unperturbed magnetic field, which corresponds to \(Θ = 0\).

The simplest scenario occurs when the gravitational perturber is dropped at \(t = 0\) at rest \((V₀ = 0)\). As discussed at the beginning of Section 3.2, the steady-state density distribution is identical to that without any magnetic field. However, the initial relaxation stage and the far density distribution are sensitive to magnetic effects. In fact, while the problem has spherical symmetry at any time in the purely hydrodynamical case, this symmetry is broken in a magnetized medium because the magnetic field dictates a preferential direction. Figure 2 shows maps of density and \(B_r\) for a case with \(Y = 1.41\). We see that the density distribution in the vicinity of the body is indeed spherically symmetric and the magnetic field takes essentially its initial value, implying that this part has reached hydrostatic equilibrium. However, there are two symmetric underdense regions along the \(z\)-axis in the outer parts. Physically, the origin of them is that the magnetic field reduces the flow convergence toward the symmetry axis because magnetic forces mainly affect the radial component of the velocity \(v_R\). This loss of radial convergence produces a wave with a negative density enhancement \((β_r > 0)\) because of the compression of the magnetic field lines in the radial direction.

Figure 3 shows snapshots of the density at the \((R, z)\)-plane for \(Y = 1.41\) and two subsonic velocities \((M = 0.75\) and 0.9). In the time-dependent analysis, both cases present a region of negative density enhancement \((α < 0)\) at the head of the perturber. As \(M\) increases from 0 to 0.9, the underdense region at the head of the body remains and gets deeper, while the underdensity in the downstream region becomes less pronounced. This is simply a consequence of the Doppler effect: gradients become steeper upstream (remember that this also occurs in the purely hydrodynamic case). By comparing the density at two times (the central and the right panels of Figure 3), we see that the evolution of the density looks self-similar.

At \(M = 0.75\), the steady-state analysis predicts a null drag force because of the front–back symmetry in the wake (see the upper left panel of Figure 3). In the finite-time case, complete ellipsoids are visible only in the vicinity of the body. For instance, at \(t = 173.6_{\text{cross}}\) (with \(Y = 1.41\) and \(M = 0.75\)), the isodensity contours are no longer ellipsoids at distances beyond \(\approx 8R_{soft}\) from the body’s center. For comparison, in the absence of magnetic fields \((Y = 0)\) and at \(M = 0.75\), ellipsoids within a radius of \(43R_{soft}\) are complete at \(t = 173.6_{\text{cross}}\). This difference is a consequence of the coupling between \(α\) and \(β\). At \(z = 136R_{soft}\) and \(R = 0\) (i.e., in the symmetry axis), a steep front separates the fluid into two regions: one with \(α > 0\) and \(β_r < 0\), and another with \(α < 0\) and \(β_r > 0\). This front leads to a deceleration of the gas, which expands radially.

At \(M = 0.9\), a cone of negative density enhancement is located at the head of the perturber (see Figure 3), as predicted in Section 3.1, and a region of positive density enhancement is located at the rear. Because of the back-reaction of the magnetic field, the isodensity contours of the tail are not incomplete ellipsoids at all. Note that the body is at the apex of the cone, whereas the overdensity is detached from the perturber. It is important to remark that, according to the analysis in Section 3.4 and Figure 1, the maximum drag force for \(Y = 1.41\) occurs at a Mach number close to 0.9.

Figure 4 displays density maps for \(Y = 1.41\) and two supersonic Mach numbers: \(M = 1.2\) (sub-Alfvénic perturber) and \(M = 1.4\) (trans-Alfvénic perturber). In the first case, the steady-state analysis predicts ellipsoidal isocontours. In the time-dependent case, however, ellipsoids are incomplete far enough away upstream from the perturber and, again, an overdensity wave at the rear of the perturber moves away from.
the body. When the perturber moves at $M = 1.4$, a tenuous ellipsoidal underdense envelope is still visible.

The overdensity behind the body has the shape of a tulip. This tulip-shaped overdensity also appears at the rear of the body at $M = 1.4$ and 1.7 (see Figures 4 and 5). A feature of the tulip-shaped overdensity wave is that it has a negative magnetic enhancement. The tulip-shaped overdensity is a consequence of our initial conditions and, as expected, is detached from the body. In order to illustrate the birth of the tulip-shaped wave, Figure 6 shows the density and magnetic perturbations along the symmetry axis for $M = 1.7$, at two early times. Initially, $\beta_z$ increases as a result of the compression of magnetic field lines (see the panel at $t = 2.83t_{cross}$). At the far edge of the tail, $z \approx -3R_{soft}$, an underdense region with positive $\beta_z$ appears. Later on (see the profiles at $t = 6.47t_{cross}$), the overdensity loses gravitational support and expands behind the body, decreasing the magnetic field strength, until the magnetic pressure plus the magnetic tension provides sufficient radial confinement to the tulip-shaped structure, allowing it to remain over long times.

In Figure 5, it is simple to identify the modified Mach cone dragged by its point by the perturber. At $t = 64.7t_{cross}$, the Mach cone at the rear of the body is well defined. Clearly, the timescale for the development of the Mach cone at the rear for $M = 1.7$ is shorter than the timescale to form the upstream Mach front at $M = 0.9$. In fact, Figure 3 shows that the cone for $M = 0.9$ is not well developed at $t = 57.7t_{cross}$.

Figure 7 shows the gravitational DF drag as a function of time for $M = 0.55, 0.75, 0.9, 1.1, 1.4,$ and 2. For $\Upsilon = 1$ and $M = 0.55$, the drag force clearly saturates in $\sim 30t_{cross}$. In the remaining cases the drag force increases with time. However, the drag force on a perturber moving in a medium with $\Upsilon = 1.41$ saturates for $M = 0.55, 0.75, 1.1,$ and 1.4. This means that the DF force saturates to a constant value at those Mach numbers for which the steady-state analysis predicts a null drag force.

In Figure 8, we plot the drag force at $t = 40t_{cross}$ and $200t_{cross}$ for different values of $\Upsilon$, together with the predicted force with $r_{min} = 2.25R_{soft}$. We see that for $\Upsilon = 0$, there is a perfect agreement between the Ostriker formula and the inferred values, confirming the result $r_{min} = 2.25R_{soft}$ reported in Sánchez-Salcedo & Brandenburg (1999). The drag force formula given in Equation (34), with $r_{min} = 2.25R_{soft}$, overestimates the drag force in a neighborhood of $M_{crit}$, where the drag force as a
The Astrophysical Journal, 745:135 (19pp), 2012 February 1

Sánchez-Salcedo

Figure 3. Color map of the density $\rho/\rho_0$, in the $(R,z)$-plane, for the cylindrical case with $M = 0.75$ (upper panels) and $M = 0.9$ (lower panels), in a natural logarithmic scale. Both models have $\Upsilon = 1.41$, implying $M_{\text{crit}} = 0.816$. Therefore, $M = 0.75$ falls into interval I, while $M = 0.9$ lies in interval II (see Section 3.1). For easy comparison, the density map in the steady state for a perturber seated at $R = z = 0$ is shown in the left panels. The central and right panels display the density in the wake at two snapshots, when the perturber is dropped suddenly at $t = 0$ at the origin of the coordinate system. The time of the snapshots is given in the lower right-hand corner of each panel in units of $t_{\text{cross}}$.

(A color version of this figure is available in the online journal.)

function of $M$ becomes very cuspy. As expected, the steady-state formula is more accurate at long timescales, except when it predicts a zero net force. Roughly speaking, we may say that, for the axisymmetric case, the gravitational drag in a magnetized medium is always smaller than or equal to the drag force in the unmagnetized case for supersonic perturbers ($M > 1$), whereas the drag is always larger or equal in a magnetized medium at subsonic perturbers ($M < 1$).

5.2. Magnetic Field Perpendicular to the Direction of Motion of the Perturber

In Section 5.1, we have focused on the case where the angle formed between the velocity of the perturber and the magnetic field, $\Theta$, was equal to $0$. In such a situation, the problem has axial symmetry and it is possible to find the analytical solution in the steady state. However, it is by no means clear how the gravitational drag force depends on $\Theta$. A visual comparison of the density wake structures for $\Theta = 0, \pi/4$, and $\pi/2$ in Figures 3 and 9 would lead us to think that the resulting wake at $\Theta = \pi/4$ is more similar to the case with $\Theta = \pi/2$ than to $\Theta = 0$. In particular, we would like to stress the remarkably different structure of the wake for $\Theta = 0$ (Figure 3) and $\Theta = \pi/2$ (Figure 9) at $M = 0.9$.

In this section, we discuss in detail the extreme case where the perturber moves perpendicular to the field lines. In Appendix B, the perturbed density is given in Fourier space. At subsonic Mach numbers and $\Theta = \pi/2$, the perturber is surrounded by an ellipsoid-like envelope but also presents a tail with positive and negative $\alpha$-values separated by a sharp front (see Figure 9 for $\Upsilon = 1.41$). Approaching the upstream axis of motion of the body, the $y$-axis, the plowing up of field lines increases the total pressure. At low Mach numbers, say, $M = 0.3$, underdense regions are now formed in the direction of the ambient field lines, which are lagged behind the body (remember that regions with negative $\alpha$ appear along the field lines; see Figure 2). At $M = 0.9$, even if the motion is subsonic and sub-Alfvénic, a magnetic bow wave with sharp edges and opening angle $\arctan(c_\lambda/V_0)$ is apparent in Figure 9. Note that when we say that it is sub-Alfvénic, we only mean $c_\lambda/V_0 < 1$. However, some caution should be used when interpreting this ratio because
the velocities are oriented in different directions. Since the velocity of the perturber is always orthogonal to the ambient direction of propagation of Alfvén waves, the Alfvén speed in the direction of motion of the perturber is zero and, thus, the body is always infinitely super-Alfvénic in the direction of motion. The morphology of the wake is the result of a competition between the gravitational focusing of gas by the perturber and the drainage of gas along magnetic field lines. We should warn here that the wake is not axisymmetric and thus the density map in the $(x, y)$-plane is different from the map in the $(y, z)$-plane.

When the perturber travels faster than the magnetosonic velocity $c_s(1 + \Upsilon^2)^{1/2}$, a magnetosonic Mach cone is formed at the rear of the perturber; the entire perturbed density distribution lags the perturber. In the $(y, z)$-plane, the perturber creates two magnetic bow waves: the Alfvénic wave with opening angle
\[ \arctan \left( \frac{c_A}{V_0} \right) \]
and the magnetosonic wave with opening angle
\[ \arctan \left[ \frac{c_A^2 + c_s^2}{V_0} \right]^{1/2} \]

The gravitational drag force is the result of the contribution of all the parcels in the domain, and it is not possible to estimate its value just by comparing the density structure by eye. Figure 10 shows the gravitational drag as a function of the perturber’s Mach number for different values of $\Upsilon$, together with the gravitational drag in the unmagnetized case using Ostriker’s formula. All the points at Mach number larger than 0.7 lie on or below Ostriker’s curve, implying that at $M > 0.7$ the drag force is equal to or smaller than it is in the unmagnetized medium. At $\Upsilon \lesssim 0.5$, the effect of including the magnetic field on the drag force is rather small. Interestingly, at $\Upsilon \gtrsim 0.5$, the strength of the drag for Mach numbers $> (1.7 + \Upsilon^2)^{1/2}$ is the same as it is in the unmagnetized case.

Figure 4. Same as Figure 3 but for $M = 1.2$ (upper panels), which falls into interval III, and $M = 1.4$ (lower panels). Again $\Upsilon = 1.41$.

(A color version of this figure is available in the online journal.)
For $\Upsilon = 1.41$, the drag force shows a plateau between $M = 0.6$ and 1.4 (see Figure 10). In general, the drag force is remarkably suppressed at Mach numbers around $\sim 1$ in the magnetized case as compared with the unmagnetized case, as long as $\Upsilon > 0.5$ (see, for instance, the drag at $0.9 < M < 1.4$ for $\Upsilon = 1$). In fact, the temporal evolution of the drag force is given in Figure 11 for $\Upsilon = 1$. The DF force on perturbers moving at Mach numbers $0.75 \leq M \leq 1.2$ saturates asymptotically to a constant value. Hence, at $0.75 \leq M < 1$, the drag forces reach a steady-state value whether the medium is magnetized or not. However, we know that the unmagnetized drag force increases logarithmically in time for supersonic perturbers. This implies that the drag force may be suppressed by one order of magnitude at $1 < M < 1.2$ in the magnetized as compared with the unmagnetized medium because the magnetized drag force saturates to a constant value. On the other hand, at $M < 0.75$ (again $\Upsilon = 1$) there is no indication that the drag force saturates, at least up to $t = 40t_{cross}$, implying that the DF force in a magnetized medium may be larger by a factor of a few than the drag in the unmagnetized case.

In summary, when the magnetic fields are relevant, that is, for $\Upsilon > 0.5$, we distinguish three ranges. At high Mach numbers (i.e., $M > (1.7 + \Upsilon^2)^{1/2}$), the drag is the same as in the unmagnetized case. At intermediate Mach numbers...
Figure 7. Temporal evolution of the gravitational drag force in the axisymmetric model, at six different Mach numbers between 0.55 and 2, for $\Upsilon = 1$ (left panel) and $\Upsilon = 1.41$ (right panel). The number on each curve is the Mach number $M$.

Figure 8. Gravitational drag force for the time-dependent axisymmetric models against the Mach number at $t = 40t_{\text{cross}}$ (left panel) and at $t = 200t_{\text{cross}}$ (right panel). Symbols correspond to the numerical models. The solid curves plot Ostriker’s formula, while the remaining curves draw the drag force as predicted by Equation (33), adopting $r_{\text{min}} = 2.25R_{\text{soft}}$ and $r_{\text{max}} = V_0t$. 

13
Figure 9. Snapshots of the density map along a cutoff through the \((y, z)\)-plane at \(x = 0\), in natural logarithmic scale, for \(\Theta = 45^\circ\) (upper panels) and \(\Theta = 90^\circ\) (lower panels), at three different Mach numbers. The Mach number \(M\) is indicated at the right corner in each panel. The perturber was dropped at \(t = 0\) and moves on a rectilinear orbit in the \((y, z)\)-plane. In all panels, \(\Upsilon = 1.41\).

(A color version of this figure is available in the online journal.)

Figure 10. Gravitational DF force as a function of Mach number for different \(\Upsilon\)-values at \(t = 40t_{\text{cross}}\). The perturber moves perpendicular to the magnetic field lines (i.e., \(\Theta = 90^\circ\)). The solid curve plots Ostriker’s formula, which was derived for unmagnetized media, with \(r_{\text{min}} = 2.25R_{\text{soft}}\) and \(r_{\text{max}} = V_0t\). To make the plot readable, the symbols at \(M = 0.3, 0.55, 0.75,\) and 2 have been slightly shifted in the horizontal direction.

Figure 11. Temporal evolution of the gravitational drag force for \(\Upsilon = 1\). The perturber moves in a direction perpendicular to the magnetic field lines. The numbers given at each curve represent the Mach number.

\((1 < M \leq M_{\text{crit}}),\) the drag is highly suppressed. Finally, at low Mach numbers, the drag force is stronger in the MHD case than in a purely hydrodynamical medium.

5.3. Intermediate Angle between Perturber’s Velocity and Magnetic Field \(\Theta = 45^\circ\)

The upper panels of Figure 9 exhibit the complex morphology of the wake when \(\Theta = \pi/4\). As is obvious from these
panels, the gravitational drag will have two components: one parallel to \( V_0 \), which produces the drag and loss of kinetic energy by the perturber, and one component perpendicular to \( V_0 \), which would change the direction of \( V_0 \). Given the symmetry of the problem, both components lie in the \((y, z)\)-plane. We start our discussion by considering the gravitational drag force.

For those simulations presented in Figure 12 with \( \Theta = \pi/4 \), the drag force saturates within \( t = 40\zeta_{\text{cross}} \) only for the model with \( M = 0.9 \) and \( \Upsilon = 0.71 \). The maximum of the drag force occurs at Mach numbers near \( \approx 1 \). At intermediate Mach numbers, say, at 1.2–1.6 for \( \Upsilon = 1.41 \), the drag force may decrease by a factor of 2–3 as compared with the force without magnetic fields. At high Mach numbers, \( M > M_{\text{crit}} \), the drag force is slightly suppressed as compared with the unmagnetized case, but this reduction is more modest than for \( \Theta = 0^\circ \). Once again, at low Mach numbers (\( M < 0.75 \)), the drag force is stronger than in the unmagnetized case.

As already said, the component of the force perpendicular to the velocity of the perturber, \( F_{\text{perp}} \), will tend to induce a change in the direction of the velocity (note that we force the body to move along a straight line). We use the following sign convention for \( F_{\text{perp}} \). For an angle \( \Theta \) in the interval \( 0 \leq \Theta \leq \pi/2 \), \( F_{\text{perp}} > 0 \) will mean that \( \Theta > 0 \), in our convention. In Figure 12, \( F_{\text{perp}} \) is shown as a function of Mach number. The magnitude of \( F_{\text{perp}} \) may be comparable to the drag force. For instance, at \( M = 1.2 \), the perpendicular force is only a factor of two smaller for \( \Upsilon = 1.41 \) and a factor of three for \( \Upsilon = 1 \). Given a certain supersonic velocity, \( F_{\text{perp}} \) increases with \( \Upsilon \), while \( F_{\text{DF}} \) shows the opposite behavior. For supersonic motions with angle \( \Theta = \pi/4 \), \( F_{\text{perp}} \) is always positive and increases monotonically in time (see Figure 13). This implies that \( F_{\text{perp}} \) will tend to redirect the perturber’s velocity to a higher \( \Theta \). For the cases shown in Figure 12, the perpendicular force saturates in the run of the calculation (\( t = 40\zeta_{\text{cross}} \)) only in two cases: for \( M = 0.5 \) and \( \Upsilon = 0.71 \), and for \( M = 0.75 \) and \( \Upsilon = 1.41 \) (the latter case is shown in Figure 13). In some cases with subsonic Mach numbers, the perpendicular component of the force is initially positive, achieves a maximum, and then starts a linear decline up to negative values (see Figure 13).

![Figure 12](image-url)

**Figure 12.** Components of the gravitational DF force parallel (left panel) and perpendicular (right panel) to the direction \( V_0 \) (at \( t = 40\zeta_{\text{cross}} \)) vs. \( M \), for three different values of \( \Upsilon \) (\( \Upsilon = 0.71 \), asterisks; \( \Upsilon = 1 \), crosses; \( \Upsilon = 1.41 \), diamonds). The angle between the perturber velocity and the magnetic field is 45°. The key to symbols is the same as in Figure 10.

5.4. **Dependence of the Drag Force on \( \Theta \)**

In Figure 14, we plot the drag force as a function of \( \Theta \), for \( \Upsilon = 1 \) and 1.41. The dependence of \( F_{\text{DF}} \) on \( \Theta \) is not always monotonic. The strongest variation of \( F_{\text{DF}} \) with \( \Theta \) occurs for \( M = 0.9 \). For this Mach number, the drag force may decrease by a factor of 2–3 from \( \Theta = 0 \) to 30°. For \( M = 1.2 \), the drag force may change up to a factor of two depending on the \( \Theta \)-value. For \( M = 0.3 \) and 1.4, the drag force depends gently on the angle.

In many astrophysical scenarios, the perturber will be subject to an external gravitational potential and will describe a nonrectilinear orbit. Sánchez-Salcedo & Brandenburg (2001) numerically treated the orbital decay of a perturber in orbit around an unmagnetized gaseous sphere. They found that the “local approximation,” that is, estimating the drag force at the present location of the perturber as if the medium were homogeneous but taking appropriately the Coulomb logarithm, is very successful. Consider now a perturber on a circular orbit in...
a magnetized medium. If the orbit lies in a plane perpendicular to the magnetic field, the attack angle $\Theta$ is always $\pi/2$. In the local approximation, the maximum drag for $\Theta = \pi/2$ and $\Upsilon = 0$ occurs at $\mathcal{M} \approx 1$ and at $\mathcal{M} \approx 1.7$ for $\Upsilon = 1$. However, if the plane of the circular orbit is parallel to the direction of the initial magnetic field, $\Theta$ will change periodically in time as $\Theta = \Theta_0 + \Omega_{\phi} t$. Therefore, if the local approximation is valid, one can estimate the mean drag force over a rotation period, which is approximately equivalent to taking the mean value of $F_{\text{DF}}$ over $\Theta$. In particular, for $\Upsilon = 1$, the $\Theta$-average drag force is maximum at $\mathcal{M} \approx 1.4$. This example illustrates how $F_{\text{DF}}$ may depend on $\Upsilon$ and on the inclination of the orbit with respect to the magnetic field lines. A more detailed analysis of the drag force on a body on a circular orbit will be given elsewhere.

6. SUMMARY AND DISCUSSION

Understanding the nature of the DF force experienced by a gravitational object that moves against a mass density background is of great importance for describing the evolution of gravitational systems. In this work, we investigated the DF on a body moving in rectilinear trajectory through a gaseous medium with a magnetic field uniform on the scales considered. In linear theory, the problem is largely characterized by three dimensionless parameters: $\mathcal{M}$, which is defined as the ratio of the particle velocity to the sound speed of the uniform gas, $\Upsilon$, defined as the ratio between the Alfvén and sound speeds, and $\Theta$, the angle between the magnetic field direction and the particle velocity. We find that magnetic effects may alter the drag force, especially for $\Upsilon > 0.5$, because the magnetic field affects the flow velocity field in the perpendicular direction of the ambient field lines, and thereby the morphology of the wake. Note that the plasma beta, defined as the ratio of gas to magnetic pressure, is $2/\Upsilon^2$ for an isothermal system.

There are two major differences between the magnetized and unmagnetized case. One conceptual difference is that, while gravitational focusing in an unmagnetized medium always generates a positive density enhancement, this is not the case in a magnetized medium (see, e.g., Figures 2, 3, and 9). A second result is that the peak value of the drag force is not near $\mathcal{M} = 1$ for a mass moving in a magnetized medium. In fact, the sharp peak of $F_{\text{DF}}$ at $\mathcal{M} = 1$ found in the $\Upsilon = 0$ case is no longer present in a magnetized medium with $\Upsilon > 0.5$. For instance, for a perturber in perpendicular motion to the field lines ($\Theta = \pi/2$) in a medium with $\Upsilon = 1.41$, the drag force is essentially constant from $\mathcal{M} = 0.5$ to 1.4 and its maximum is located around $\mathcal{M} = 2$ (see Figure 10). The flat plateau in the drag force between $\mathcal{M} = 0.5$ and 1.4 is partly because of the extra rigidity of the magnetic field in the $x$ and $y$ directions.

For a body traveling along the field lines, i.e., $\Theta = 0$, the steady-state problem can be treated analytically. We focus first on this case. For $\Upsilon \neq 0$, the drag force presents two local maxima (see Figure 1): one is located in the subsonic branch ($\mathcal{M}_{\text{crit}}$), and the other peak value is in the supersonic branch ($\mathcal{M} = \max(1, \sqrt{2}\Upsilon)$). When the velocity of the perturber is supersonic and super-Alfvénic (and $\Theta = 0$), the DF force in a magnetized medium is weaker than in the unmagnetized case by a factor of $(1 - \eta)$, with $\eta = (c_s/V_0)^2$. The physical reason is that the medium becomes more rigid in the radial direction and, hence, the opening aperture of the modified Mach cone is the same as that in an unmagnetized medium with effective sound speed $(c_s^2 + c_{\perp}^2)^{1/2}$, but the density enhancement is smaller by a factor of $(1 - \eta)$. By contrast, the drag force for subsonic velocities is stronger if the medium is uniformly magnetized. For $\Theta = 0$, an underdense region is formed upstream because of the gas channeling along the direction of the magnetic field, following the path of less resistance. The steady-state theory predicts that the gravitational drag on a body with $\Theta = 0$ vanishes at Mach numbers in the following two ranges: (1) at $\mathcal{M} < \mathcal{M}_{\text{crit}}$ and (2) at min($1, \Upsilon$) < $\mathcal{M}$ < max($1, \Upsilon$). However, using time-dependent analysis we find that the DF force asymptotically approaches a nonzero steady-state value at these Mach numbers. For $\Upsilon > 0.4$ (still $\Theta = 0$), the drag force is maximized for perturbers moving at a Mach number close to $\mathcal{M}_{\text{crit}}$ (Figure 8). At Mach numbers around $\mathcal{M}_{\text{crit}}$, the density enhancement is large but negative in a cone in front of the body. At those Mach numbers, the DF may be even more efficient than in the stellar case. For example, for a medium with $\Upsilon = 1.41$, the drag force peaks between $\mathcal{M} = 0.6$ and 1.1. As a consequence of the stronger DF force, subsonic massive objects in an orbit elongated along the magnetic field lines in a constant-density core of a nonsingular gaseous sphere will suffer an orbital decay faster if the medium is pervaded by a large-scale magnetic field.

We have also explored the $\Theta$-dependence of the DF drag. For Mach numbers around $\mathcal{M}_{\text{crit}}$, the drag force exhibits the strongest variations with $\Theta$ (see Figure 14). For magnetized media with $\Upsilon \geq 1$ and regardless of the exact value of $\Theta$, we find that (1) the drag force for subsonic perturbers is higher by a factor of 1.5–2 than it is for the unmagnetized DF drag and (2) for supersonic perturbers ($\mathcal{M} > 1$), the magnetized

![Figure 14. Dependence of the drag force on the angle $\Theta$ for $\Upsilon = 1$ (upper panel) and $\Upsilon = 1.41$ (lower panel). The numbers given at each curve represent the Mach number $\mathcal{M}$. The drag force was computed at $t = 40t_{\text{cross}}$.](image-url)
drag force is always weaker than the unmagnetized drag force. At intermediate Mach numbers, $1.1 \leq \mathcal{M} \leq 1.4$, the drag force is a factor of 2–3 weaker than it is in the absence of magnetic fields.\footnote{This factor may be larger at later times because the magnetized drag force saturates, whereas it increases logarithmically in time in the unmagnetized case. See Figure 8 for an evolved stage.} At high Mach numbers, $\mathcal{M} > (1.7 + \Upsilon^2)^{1/2}$, the suppression of the drag force is more important at small values of $\Theta$ (Figure 14). At these high Mach numbers and for an angle of $\Theta = \pi/2$, the drag forces are similar with and without magnetic fields (Figure 10).

As a consequence, supersonic massive objects may make their way more slowly to the center of the system if the medium is pervaded by a large-scale magnetic field. As a model problem, consider a singular isothermal spherical cloud threaded by a uniform magnetic field and a small-scale random magnetic field with Alfvén speed $\cA$ everywhere constant. The density profile of the cloud is given by $\rho(r) = (c_s^2 + c_d^2)/(2\pi G r^2)$, where $c_s$ is the isothermal sound speed. The circular speed is $V_0 = \sqrt{2c_s^2 + c_d^2}$. Since the effective sound speed is $\sqrt{\gamma c_s^2 + 2c_d^2}/3$, the Mach number of a body on a quasi-circular orbit is

$$\mathcal{M} = \left(\frac{2c_s^2 + c_d^2}{\gamma c_s^2 + (2/3)c_d^2}\right)^{1/2}, \quad (52)$$

which varies from 1.1 to 1.4 depending on the value of $c_s$ and whether the perturbations are isothermal or adiabatic.\footnote{In the nonmagnetic simulations of the orbital decay of a single black hole due to gaseous DF in Escala et al. (2004), the velocity of the black hole is initially supersonic ($\mathcal{M} = 1.4$) and remains barely supersonic through most of the simulation.} If the uniform magnetic field component has a $\Upsilon$-value between 1 and 1.4, the time for the perturber’s orbit to decay will be a factor of 2–5 larger than the corresponding decay time for $\Upsilon = 0$. Our results demonstrate that, in the presence of ordered magnetic fields with $\Upsilon > 0.7$, the role of the magnetic field on the drag force should be taken into account to have accurate estimates of the timescales of orbital decay via gravitational DF.

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**APPENDIX A**

**FOURIER TRANSFORMATION: AXISYMMETRIC CASE**

The three-dimensional Fourier transform of a perturbed variable $f(r)$ is given by

$$\hat{f}(k) = \frac{1}{(2\pi)^{3/2}} \int \! f(r) e^{-ik \cdot r} \, d^3r. \quad (A1)$$

In the Fourier space, Equations (16) and (17) are transformed into

$$\mathcal{M}^2 k_z^2 \hat{\alpha} = k_z^2 \hat{\alpha} - \frac{4\pi G}{c_s^2} \hat{\rho}_p + \Upsilon^2 k_z^2 \hat{\beta}_z, \quad (A2)$$

$$i(\mathcal{M}^2 - 1) k_z \hat{\alpha} = i k_z \hat{\Phi} + i \mathcal{M}^2 k_z \hat{\beta}_z, \quad (A3)$$

where $\rho_p$ is the mass density of the perturber, thus $\nabla \cdot \Phi = 4\pi G \rho_p$. In order to have an equation for $\hat{\alpha}$, we will eliminate $\hat{\beta}_z$. From Equation (A3), we have

$$\hat{\beta}_z = \frac{1}{M^2} \left[ (\mathcal{M}^2 - 1) \hat{\alpha} - \frac{\hat{\Phi}}{c_s^2} \right], \quad (A4)$$

and substituting into Equation (A2), we find

$$\{\mathcal{M}^2 k_z^2 - k_z^2[1 + (1 - \mathcal{M}^{-2})\Upsilon^2]\} \hat{\alpha} = \frac{4\pi G}{c_s^2} \left( \frac{\Upsilon^2}{\mathcal{M}^2} - 1 \right) \hat{\rho}_p. \quad (A5)$$

In the absence of magnetic fields ($\Upsilon = 0$), the above equation reduces to

$$\{\mathcal{M}^2 k_z^2 - k_z^2\} \hat{\alpha} = \frac{4\pi G}{c_s^2} \hat{\rho}_p, \quad (A6)$$

and the standard steady-state equation for the wake past a gravitating body is recovered. At velocities much larger than the Alfvén speed, $\mathcal{M} \gg c_A/c_s = \Upsilon$, Equation (A5) is simplified to

$$\left[1 + (1 - \mathcal{M}^{-2}) \Upsilon^2 \right] \hat{\alpha} = \frac{-4\pi G}{(1 + \Upsilon^2)c_s^2} \hat{\rho}_p. \quad (A7)$$

By comparing the above equation with Equation (A6), we see that the response of the medium in this case is indistinguishable from that of an unmagnetized medium with sound speed $(c_s^2 + c_d^2)(k_z^2) = c_d(1 + \Upsilon^2)^{3/2}$.

It is interesting to note that when $V_0 = c_A \neq 0$, the right-hand side of Equation (A5) vanishes and thereby the solution is $\alpha = 0$, implying that the steady-state configuration satisfies $\nabla \cdot \vec{v} = 0$. Obviously, the drag force is exactly zero in this configuration.

There exist two situations where the differential equation (A5) is not well posed (1) at $\mathcal{M} = 1$ and (2) at the critical Mach number, $\mathcal{M}_{\text{crit}}$, satisfying that

$$1 + (1 - \mathcal{M}_{\text{crit}}^{-2}) \Upsilon^2 = 0. \quad (A8)$$

Thus,

$$\mathcal{M}_{\text{crit}} \equiv (1 + \Upsilon^{-2})^{-1/2}. \quad (A9)$$

It is clear that $\mathcal{M}_{\text{crit}} < 1$. If the dynamics is dominated by the magnetic field, i.e., when $\Upsilon \gg 1$, then $\mathcal{M}_{\text{crit}} \rightarrow 1$. If not specified, we consider $\mathcal{M} \neq 1$ and $\mathcal{M} \neq \mathcal{M}_{\text{crit}}$ throughout this section.

We now calculate the solution of Equation (A5) when the perturber is a point mass $M$, so that $\rho_p = M/(2\pi)^{3/2}$, which corresponds to the Fourier transformation of $\rho_p(r) = M\delta(r)$, to obtain the Green function. Using the convolution theorem, it is possible to evaluate $\alpha$ for any general distribution $\rho_p$. Hence, we solve for

$$\alpha(r) = \frac{(1 - \eta)GM}{2\pi^2 c_s^2} \int \frac{1}{\xi k^2 - M^2 k_z^2} e^{i k \cdot r} \, d^3k, \quad (A10)$$

where

$$\eta = \left( \frac{\Upsilon}{M} \right)^2 = \left( \frac{c_A}{V_0} \right)^2 \quad (A11)$$

and

$$\xi = 1 + (1 - \mathcal{M}^{-2})\Upsilon^2. \quad (A12)$$

The integral (A10) along $k_z$ is evaluated by transforming to the complex plane. It is convenient to define $\gamma^2 \equiv 1 - \mathcal{M}^2/\xi$. If $\mathcal{M}$ stands either in the range $0 < \mathcal{M} < \mathcal{M}_{\text{crit}}$ or in the range $\min(1, \Upsilon) < \mathcal{M} < \max(1, \Upsilon)$, then $\gamma^2 > 0$ and thus none of
the poles lies on the real axis. Hence, we may use Cauchy’s residue theorem to obtain
\[ \alpha(r) = \frac{(1 - \eta)GM}{\xi c_s^2} \frac{1}{\sqrt{z^2 + R^2/\gamma^2}}. \]  
(A13)

For Mach numbers in any of the two ranges, \( M_{\text{crit}} < M < \min(1, \gamma) \) and \( M > \max(1, \gamma) \), the integrand has poles on the real axis. Hence, we make “indentations” in the contour at the position of the poles. We consider first Mach numbers larger than \( \max(1, \gamma) \). Then, for \( z > 0 \), we close the contour at \( +i\infty \), leaving the poles outside the contour to preserve causality, whereas for \( z < 0 \), we consider a domain containing the lower half-plane, that is, where \( \text{Im}(k_z) < 0 \), and the contour slightly above the real axis, so that the two poles lie inside the contour. More specifically, for \( z < 0 \), the integration over \( k_z \) can be evaluated as
\[
\int_{-\infty}^{\infty} e^{ik_zz}dk_z = \int_{-\infty}^{\infty} e^{ik_zz}dk_z = -\frac{2\pi}{\gamma_lk_{\|}} \sin(k_{\|z}/\gamma_l), \quad (A14)
\]
where \( \gamma_l^2 = -\gamma^2 = \xi^{-1} M^2 - 1 \) and \( k_{\|}^2 = k_x^2 + k_y^2 \). The integration over \( k_x \) and \( k_y \) can be carried out in polar coordinates \( (k_x = k_{\phi} \cos \phi \text{ and } k_y = k_{\phi} \sin \phi) \):
\[
-\frac{2\pi}{\gamma_l} \int_0^{2\pi} \sin(k_{\|z}/\gamma_l) e^{ik_{\phi} \cos \phi} d\phi dk_R
\]
\[
-\frac{4\pi}{\gamma_l} \int_0^{2\pi} \sin(k_{\|z}/\gamma_l) J_0(k_{\|} R) dk_R
\]
\[
= \begin{cases} \frac{4\pi}{\gamma_l} & \text{if } z < -\gamma_l R; \\ 0 & \text{otherwise}. \end{cases}
\]  
(A15)

Here, \( J_0 \) is the cylindrical Bessel function of order zero. At Mach numbers in the interval \( M_{\text{crit}} < M < \min(1, \gamma) \), causality is preserved at \( z > 0 \) if we consider a domain containing the upper half-plane and the two poles lie inside the contour. Putting it all together, the solution is
\[ \alpha(r) = \frac{\lambda(1 - \eta)GM}{\xi c_s^2} \frac{1}{\sqrt{z^2 + R^2/\gamma^2}}. \]  
(A16)

where
\[
\lambda = \begin{cases} 2 & \text{if } M > \max(1, \gamma) \text{ and } z/R < -|\gamma|; \\ 2 & \text{if } M_{\text{crit}} < M < \min(1, \gamma) \text{ and } z/R > |\gamma|; \\ 1 & \text{if } M < M_{\text{crit}} \text{ or if } \min(1, \gamma) < M < \max(1, \gamma); \\ 0 & \text{otherwise}. \end{cases}
\]

This result can be compared with that in Dokuchaev (1964) by noting that he used the variable \( A \) to denote the combination \( (1 - \eta)/\xi \). Dokuchaev (1964) found the same functional form for \( \alpha \) but failed to divide correctly the cases according to the Mach number. In the absence of a background magnetic field, which corresponds to \( \xi = 1 \) and \( \eta = 0 \), we recover the classical form derived by previous authors.

**APPENDIX B**

**FOURIER TRANSFORMATION: THE PERTURBER’S VELOCITY PERPENDICULAR TO THE MAGNETIC FIELD**

We consider a gravitational perturber moving at constant velocity \( V_0 \hat{y} \) in an unperturbed medium with density \( \rho_0 \), sound speed \( c_s \), and a magnetic field \( B_0 \hat{z} \). Therefore, the velocity of the perturber and the magnetic field are perpendicular. We are interested in the stationary wake formed behind the body. To do this, we assume that the perturber is at rest and feels a wind with velocity \( -V_0 \hat{y} \) at infinity. Following the same approach as in Appendix A, the two coupled governing equations in this geometry are given by
\[
M^2 \frac{\partial^2 \alpha}{\partial y^2} = \nabla^2 \alpha + \frac{1}{c_s^2} \nabla \cdot \nu \nabla^2 \beta_z, \quad (B1)
\]
\[
M^2 \frac{\partial^2 \alpha}{\partial z^2} = \frac{\partial^2 \alpha}{\partial z^2} + \frac{1}{c_s^2} \frac{\partial^2 \beta_z}{\partial y^2} + M^2 \frac{\partial^2 \beta_z}{\partial y^2}. \quad (B2)
\]
In the Fourier space, these equations are transformed into
\[
- M^2 k_y^2 \hat{\alpha} = -k_y^2 \hat{\alpha} + \frac{4\pi G}{c_s^2} \hat{\rho}_p - \nabla^2 k_y^2 \hat{\beta}_z, \quad (B3)
\]
\[
\frac{M^2 k_y^2 \hat{\alpha}}{\beta_z} = \frac{k_y^2 \hat{\alpha}}{\beta_z} + \frac{1}{c_s^2} \frac{\partial^2 \beta_z}{\partial y^2} + M^2 \frac{\partial^2 \beta_z}{\partial y^2}. \quad (B4)
\]
In order to have an equation for \( \hat{\alpha} \), we eliminate \( \hat{\beta}_z \) using
\[
\hat{\beta}_z = \frac{1}{\gamma^2 k_y^2} \left[ \frac{4\pi G}{c_s^2} \hat{\rho}_p + (M^2 k_y^2 - k_y^2) \hat{\alpha} \right], \quad (B5)
\]
and we obtain the solution of the density disturbance in Fourier space:
\[
\frac{k_y^2 \hat{\alpha}}{\beta_z} = \frac{4\pi G}{c_s^2} \left[ \frac{\nabla^2}{M^2 k_y^2} \hat{\rho}_p \right]. \quad (B6)
\]
Of course, the structure of \( \alpha \) in this case is different than in the axisymmetric case (Equation (A5)). The inverse Fourier transform cannot be derived analytically, but since the above equation is well posed for any \( M \)-value if \( \gamma \neq 0 \), we expect the drag force to be a continuous function of \( M \).

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