Dynamics Analysis of Anti-predator Model on Intermediate Predator With Ratio Dependent Functional Responses

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Abstract. This article discusses a predator prey model with anti-predator on intermediate predator using ratio dependent functional responses. Dynamical analysis performed on the model includes determination of equilibrium point, stability and simulation. Three kinds of equilibrium points have been discussed, namely the extinction of prey point, the extinction of intermediate predator point and the extinction of predator point are exists under certain conditions. It can be shown that the result of numerical simulations are in accordance with analitical results.

1. Introduction
In population dynamics, we need to build a mathematical model to understand some phenomena. Mathematical models of predation represents the interactions between species of animals predators and prey that live in the same environment. The prey predator model is expanded by considering the density of prey dependent growth and functional responses. Functional response is predatory predator rate by predator per capita. It is often used to describe the increasing population density of predators as well as the decrease in prey population density. Holling [4] has introduced functional responses that depend only on prey species, namely Holling types I, II, and III. The functional response of the dependent ratio shows the behavior by which the number of prey per predator or prey per predator ratio is replaced by prey density. Predator-prey model with dependent ratio was developed [2]. The functional response that relies only on prey population density is said to be prey-dependent, if it is dependent on the population density of prey and predator, it is called a predator-dependent response function. The predator-dependent model is more suitable for prey predator interactions in which predation involves the search process. One functional response that depends on the density of prey and predator is the ratio dependent functional response.

A study of prey predator interactions to avoid prey extinctions was introduced [1] taking prey behavior in defending against predation pressure called antipredator behavior. The anti-predator behavior that is a natural response of the prey, occurs when prey is threatened, one at the expense of certain body parts. For example lizards can release the tail while trying to save themselves from predators. Fish and insects have spines that prevent predators or birds from eating. This article was inspired model [9], who studied the stability and dynamical behaviour the model of three species consisting of a prey species and two predator species also consider the competition for...
their prey with Holling type II functional response that incorporates prey refuge. One of the more relevant behavioral traits affecting the dynamics of a predator-prey system is the use of refuge by prey. The use of refuge has been shown to increase the coexistence of predator-prey by preventing prey predators. The extinction of prey species can be avoided from predation in two ways. First, competition between two predators indirectly helps the prey species survive, and secondly, the most important part of the prey survival strategy is to consider antipredator behavior. Therefore, in this paper we consider the dynamic analysis of predator-prey models with ratio-dependent functional response. Finally a numerical simulation is taken to verify some of the results we obtained.

2. Materials and Methods

The classical model are mostly variation of Lotka (a physical chemist)-Volterra (a mathematician) model which is the product of chemistry, physics and mathematics. We simply should not equal or approximate all biological interactions by chemical. There is a growing explicit biological and physiological evinences in [2].

2.1. Basic Assumption

A simple ecosystem model, a predator prey interactions model with ratio-dependent functional responses in [6], where predation involves searching process. Especially when predator have to search for food (therefore have to share or compete for food). Generally, a ratio-dependent predator prey model takes the form

\[
\frac{dx}{dt} = r x \left(1 - \frac{x}{K}\right) - \frac{a x y}{m y + x}
\]

\[
\frac{dy}{dt} = \frac{\mu x y}{m y + x} - dy
\]

Based on prey predator model (1), we consider the impact of antipredator behaviour and propose the following model in [10].

\[
\frac{dx}{dt} = r_1 x \left(1 - \frac{x}{K}\right) - \frac{a x y}{m + x^2}
\]

\[
\frac{dy}{dt} = \frac{a x y}{m + x^2} - dy - \gamma x y
\]

System (2), considering of anti-predator behavior by substituting Holling type IV functional response, where \( \gamma \) is the rate of antipredator behaviour of prey to the predator spesies.

The above model has been modified model in [9] 2 predator-1 prey with Holling type II response

\[
\frac{dx}{dt} = r x \left(1 - \frac{x}{K}\right) - \frac{a_{1} x y}{1 + a_{1} x} - \frac{a_{2} x z}{1 + a_{2} z}
\]

\[
\frac{dy}{dt} = \frac{a_{1} x y}{1 + a_{1} x} - \mu_{1} y - \sigma_{1} y z
\]

\[
\frac{dz}{dt} = \frac{a_{2} x y}{1 + a_{2} x} + -\mu_{2} z - \sigma_{2} y z
\]

Where \( x \) is the prey, \( y \) and \( z \) are the population size of the first predator spesies or intermediate predator, and second predator spesies or top predator, respectively at any time \( t \); \( r \) is the prey growth rate parameter, and \( K \) is the environmental carrying capacity of the prey spesies; \( \mu_{1} \) and \( \mu_{2} \) are the predator death rates, \( \sigma_{1} \) and \( \sigma_{2} \) are the rates at which the growth rate of the intermediate predator is annihilated by the top predator and vice versa. \( a \) is predator grazing rate; \( b \) prey depend on carring capacity for predator and \( b \) is measure the quality of food for predator. It is assumed that all parameters are positif. Observing the importance of ratio-dependent predator-prey model explained in the previous section. The analysis stability of solution of modified in [9] with ratio dependent functional response. Linearization critical points of system to find informations about the stability of
critical points. Moreover, for the case that linearization is not easy to be implemented to analyze the stability of critical point, we used numerical simulation to analyze.

2.2. Mathematical model
Mathematical model that represents the interaction between 2 predator species and 1 prey species will be constructed based on several assumptions.

\[
\frac{dx}{dt} = rx \left(1 - \frac{x}{K}\right) - \frac{axy}{x + y} - \frac{\beta xz}{x + z} = f_1 \\
\frac{dy}{dt} = \frac{\delta xy}{x + y} - \mu y - \gamma xy = f_2 \\
\frac{dz}{dt} = \frac{\tau zx}{x + z} - \rho z = f_3
\]

(4)

By taking the right hand side of (4) equal zero, we obtain the following critical points, where \(x, y\) and \(z\) have same meaning as \(x(t), y(t)\) and \(z(t)\), respectively. Observing the importance of ratio-dependent predator-prey model explained in the previous section.

2.2.1. Determination of the equilibrium point. Equilibrium points obtained from the equilibrium solutions of system, when population of prey, intermediate predator and top predator growth rates are zero. Equilibrium point illustrates a constant solution of system. Let

\[
0 = rx \left(1 - \frac{x}{K}\right) - \frac{axy}{x + y} - \frac{\beta xz}{x + z} \\
0 = \frac{\delta xy}{x + y} - \mu y - \gamma xy \\
0 = \frac{\tau zx}{x + z} - \rho z
\]

System (4) possesses have four equilibrium points that appear for any choice of positive value of parameters.

1. The predator extinction point \(T_1 = (K, 0, 0)\)
2. The intermediate predator extinction point \(T_2 = \left( \frac{K(\beta - \beta r - \gamma r)}{\tau p}, 0, \frac{K(\alpha - \alpha r - \beta r)}{\tau p}\right)\)
3. The extinction predator point \(T_3 = (X^*, Y^*, 0) = \left( \frac{K(\alpha - \alpha r - \beta r)\gamma (\beta - \beta r)}{\beta - \beta r - \gamma r}, \frac{K(\alpha - \alpha r - \beta r)\gamma (\beta - \beta r)}{\beta - \beta r - \gamma r}, 0\right)\)
4. The survival point of the three populations (prey, intermediate predator and top predator) \(T_4 = (X^{**}, Y^{**}, Z^{**})\)

2.2.2. The equilibrium point. Stability of the equilibrium point is observed by the Jacobian matrix of the system. Stability properties shows a prediction of the equilibrium point which may occur or not. If the equilibrium point is stable, then all of the system solution with different initial values will be converge to it, and vice versa.

The equilibrium point \(T_1 = (K, 0, 0)\) represent the predator population extinction at steady state. Prey lives without predation by predators. Because the element is positive, the equilibrium point always exists. \(T_3 = (x^*, y^*, 0)\) describe the extinction of the top predator population at steady state, Meanwhile, the prey alive with a density of \(X^*\), and intermediate predator \(Y^*\). In contrast to all the equilibrium point, \(E_4(x^{**}, y^{**}, z^{**})\) shows that both the predator (intermediate predator and top predator) and the prey can survive at steady state.

3. Result and Discussion
Based on the above discussion, for the existence of the positive equilibria of system (4), we have the following some preliminary results.
Theorem 1. The system (4) always has a predator extinction equilibrium $T_1 = (K, 0, 0)$, and has only one unique positive equilibrium $T_3$.

3.1. Stability Analysis
In this section, stability of the equilibrium point is observed by the Jacobian matrix of the system. Stability properties shows a prediction of the equilibrium point which may occur or not. If the equilibrium point is stable, then all of the system solution with different initial values will be converge to it, and vice versa.

The Jacobian matrix of system (4) at equilibrium point $T_1 = (x_1, y_1, z_1)$ is

$$J = \begin{bmatrix}
\frac{\partial f_1(x, y, z)}{dx} & \frac{\partial f_1(x, y, z)}{dy} & \frac{\partial f_1(x, y, z)}{dz} \\
\frac{\partial f_2(x, y, z)}{dx} & \frac{\partial f_2(x, y, z)}{dy} & \frac{\partial f_2(x, y, z)}{dz} \\
\frac{\partial f_3(x, y, z)}{dx} & \frac{\partial f_3(x, y, z)}{dy} & \frac{\partial f_3(x, y, z)}{dz}
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix},$$

The local stability of all equilibria of system (4), first the Jacobian matrix on the system (4) is calculated as

$$j(x, y, z) = \begin{bmatrix}
\alpha y - \frac{\beta x y}{x + y} + \frac{\beta x z}{x + y} & \beta x & \frac{\beta x}{x + y} \\
\frac{\delta x}{x + y} \sigma z - \gamma y & \frac{\delta x}{x + y} \sigma z - \mu - \gamma x & 0 \\
\frac{\delta x}{x + y} \sigma z - \mu - \gamma x & \alpha y - \frac{\beta x y}{x + y} + \frac{\beta x z}{x + y}
\end{bmatrix}$$

The Jacobian matrix of system (4) at $T_1 = (K, 0, 0)$ is

$$j(k, 0, 0) = \begin{bmatrix}
r - k & 0 & -\frac{\beta k}{k} \\
0 & \frac{\delta k}{k} - \mu - \gamma k & 0 \\
0 & 0 & \frac{\sigma k}{k} - \rho
\end{bmatrix}.$$

3.2. Numerical Simulations
System (4) is a nonlinear system, so the analytical solution of system (4) is not easy to determine. Therefore, numerical simulation can be performed to investigate the system behavior. Numerical solution of the system can give an overview of population densities in the long term. To demonstrate the numerical solution of predator-prey model.

To confirm the results of the analysis given in the previous section, a numerical simulation is performed so that the behavior of the system solution (4) can be graphically depicted. This numerical simulation is done by the fourth-order Runge-Kutta method using the Matlab program. In the simulation shows the variation of predation prey level by predator.

The phase portraits of boundary equilibria are shown in figure 1. If system (4) has a positive equilibrium in $T_3 = (x^*, y^*, 0)$, then it shows the existence and stability of the equilibrium point.
For some value of these parameters and suitable initial populations, the prey, intermediate predator and top predator can coexist at a positive equilibrium. From the perspective of biology interpretation, antipredator on intermediate predator species have effects of predation is significant on destruction of ecological system.

4. Conclusions

In this paper, we have studied numerically antipredator model with ratio dependent functional response. It is found that the model has equilibrium points, namely the extinction of both predator point \( T_1 \), the intermediate predator extinction point \( T_2 \), the top predator extinction point \( T_3 \). Based on numerical simulations, there are two possible stable equilibrium points that are \( T_1 \) and \( T_3 \).

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