EXISTENCE AND MULTIPLICITY OF SIGN-CHANGING SOLUTIONS FOR QUASILINEAR SCHRÖDINGER EQUATIONS WITH SUB-CUBIC NONLINEARITY

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Abstract. In this paper, we consider the quasilinear Schrödinger equation
\[-\Delta u + V(x)u - u\Delta(u^2) = g(u), \quad x \in \mathbb{R}^3,\]
where \(V\) and \(g\) are continuous functions. Without the coercive condition on \(V\) or the monotonicity condition on \(g\), we show that the problem above has a least energy sign-changing solution and infinitely many sign-changing solutions. Our results especially solve the problem above in the case where \(g(u) = |u|^{p-2}u\) \((2 < p < 4)\) and complete some recent related works on sign-changing solutions, in the sense that, in the literature only the case \(g(u) = |u|^{p-2}u\) \((p \geq 4)\) was considered. The main results in the present paper are obtained by a new perturbation approach and the method of invariant sets of descending flow. In addition, in some cases where the functional merely satisfies the Cerami condition, a deformation lemma under the Cerami condition is developed.

1. Introduction and main results

The present paper is devoted to studying the existence and multiplicity of sign-changing solutions of the quasilinear Schrödinger equation
\[-\Delta u + V(x)u - u\Delta(u^2) = g(u), \quad x \in \mathbb{R}^3,\]
where \(V \in C(\mathbb{R}^3, \mathbb{R})\) and \(g \in (\mathbb{R}, \mathbb{R})\). Problem (1.1) is referred to as the so-called Modified Nonlinear Schrödinger Equation (MNLS) and is related to solitary wave solutions of the equation
\[i\partial_t z = -\Delta z + W(x)z - m(|z|^2)z - \kappa\Delta \phi(|z|^2)\phi'(|z|^2)z,\]
where \(z : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}, W : \mathbb{R}^3 \to \mathbb{R}\) is a given potential, \(m, \phi : \mathbb{R}^+ \to \mathbb{R}\) are suitable functions, and \(\kappa \in \mathbb{R}\). The form of (1.2) has been derived as models of several physical phenomena corresponding to various types of \(\kappa\) and \(\phi\). For example, if \(\kappa = 0\), (1.2) turns out to be a semilinear Schrödinger equation, which has been widely investigated, we refer the readers to \([2, 21, 22]\). The case \(\phi(s) = s\), as a model of the time evolution of the
condensate wave function in super-fluid film, has been studied by Kurihara in [10]. While for \( \phi(s) = \sqrt{1 + s} \), the equations are the models of the self-channeling of a high-power ultra short laser in matter, see [4]. For more physical applications, we refer to [3,11,20] and references therein.

In the past decades, great progress has been made in studying sign-changing solutions of (1.1) and such problems have attracted a considerable researchers’ attention. In [15], Liu et al. considered the equation

\[-\Delta u + V(x)u - u\Delta (u^2) = |u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.3}\]

where \( N \geq 3, p \in [4, 22^*) \) with \( 2^* = \frac{2N}{N-2} \), and \( V(x) \in C(\mathbb{R}^N) \) satisfies

\((V') \) \( 0 < \inf_{\mathbb{R}^N} V(x) \leq \lim_{|x| \to +\infty} V(x) = V_\infty \), and \( V(x) \leq V_\infty - \frac{A}{1+|x|^m} \) for \( |x| \geq M \),

where \( A, M, m \) are positive constants. They proved that (1.3) has a least energy sign-changing solution by using an approximating sequence of problems on a Nehari manifold defined in an appropriate subset of \( H^1(\mathbb{R}^N) \). In [6], Deng et al. treated the equation (1.3) with \( p \in (4, 22^*) \) and showed that, for any given \( k \in \mathbb{N} \), there is a pair of sign-changing solutions with \( k \) nodes by using a minimization argument and an energy comparison method. Later, Deng et al. [7] extended the results in [6] to the critical growth case. Zhang et al. [25] showed the existence of infinitely many sign-changing solutions of the equation

\[-\Delta u + u - u\Delta (u^2) = a(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N, \tag{1.4}\]

where \( p \in (4, 22^*) \) and \( a(x) \) satisfies

\[ a(x) > 0, \quad a \in L^r(\mathbb{R}^N) \text{ with } r \geq 22^*/(22^* - p), \]

and the proof is based on the methods of perturbation and invariant sets of descending flow. Recently, Yang et al. [24] dealt with the equation with critical or supercritical growth

\[-\Delta u + V(x)u - u\Delta (u^2) = a(x)[g(u) + |u|^{p-2}u], \quad x \in \mathbb{R}^N, \]

where \( p \geq 22^* \). By assuming that \( a(x) > 0 \) a.e. in \( \mathbb{R}^N \), \( V \) satisfies one of the following conditions

\((V'_2) \) \( V(x) \geq V_0 > 0 \) for all \( x \in \mathbb{R}^N \), \( V(x) = V(|x|) \) and \( V \in L^\infty(\mathbb{R}^N) \),

\((V'_3) \) \( V(x) \geq V_0 > 0 \) for all \( x \in \mathbb{R}^N \), \( \lim_{|x| \to \infty} V(x) = +\infty \),

and \( g \in C(\mathbb{R}) \) is odd, subcritical at infinity, superlinear near zero and satisfies

\((g'_1) \) \( \frac{g(t)}{t^3} \) is nondecreasing in \( t \neq 0 \),

\((g'_2) \) \( g(t)t \geq \mu G(t) > 0, \quad t \neq 0 \), for some \( \mu \in (4, 22^*) \), where \( G(t) = \int_0^t g(s)ds \),

Yang et al. obtained a least energy sign-changing solution by means of the method of Nehari manifold, deformation arguments and \( L^\infty \)-estimates. Here, we also would like to mention a more general quasilinear equation than equation (1.1). Liu et al. [14] investigated the following quasilinear equation of the form

\[ \sum_{i,j=1}^{N} D_j (a_{ij}(x,u)D_i u) - \frac{1}{2} \sum_{i,j=1}^{N} D_s a_{ij}(x,u)D_i u D_j u + f(x,u) = 0, \tag{1.5} \]
where $D_i = \partial / \partial x_i$ and $D_s a_{ij}(x,s) = \partial / \partial s a_{ij}(x,s)$. Equation (1.1) can be regarded as a special case of equation (1.5) for $a_{ij}(x,u) = (1 + u^2) \delta_{ij}$. By introducing a $p$-Laplacian perturbation approach, the authors obtained, via the method of invariant sets of descending flow, infinitely many sign-changing solutions of equation (1.5) in a bounded domain $\Omega \subset \mathbb{R}^N$. We point out that in [14] the following assumption is imposed:

$$\lim_{s \to \infty} f(x,s)/s = +\infty$$ uniformly with respect to $x \in \bar{\Omega}$ and for some $q > 4$ and $c_0 \in \mathbb{R},$

$$\frac{1}{q} s f(x,s) - F(x,s) \geq -c_0, \quad \forall x \in \bar{\Omega}, s \in \mathbb{R},$$

where $F(x,s) = \int_0^s f(x,t)dt$. One can get that $\lim_{s \to \infty} f(x,s)/s^3 = +\infty$ uniformly with respect to $x \in \bar{\Omega}$.

Observe that the previous results on sign-changing solutions of (1.1) were focused on the case where $g(u) = |u|^{p-2}u$ with $p \geq 4$. A natural problem is whether or not equation (1.1) has sign-changing solutions if $p \in (2, 4)$. In addition, for the case $p = 4$, in [15] Liu et al. considered the existence of a least energy sign-changing solution under the condition $(V')$, another interesting problem is whether or not a least energy sign-changing solution or higher energy sign-changing solutions exist if $(V')$ is not satisfied? In this paper, we will fill these gaps and give affirmative answers. Firstly we state the result about sign-changing solutions of (1.1) with sub-cubic nonlinearity and give the assumptions on $V$ and $g$ as follows.

$(V_1)$ $V \in C(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, $V(x) = V(|x|)$ and $\inf_{x \in \mathbb{R}^3} V(x) > 0$.

$(g_1)$ $g \in C(\mathbb{R})$ and $g(t) = o(t)$ as $t \to 0$.

$(g_2)$ $\limsup_{|t| \to +\infty} \frac{|g(t)|}{|t|^p} < +\infty$ for some $p \in (2, 12)$.

$(g_3)$ There exists $4 \geq \mu > 2$ such that $g(t)t \geq \mu G(t) > 0$ for $t \neq 0$, where $G(t) = \int_0^t g(s)ds$.

As a consequence of $(g_2)$ and $(g_3)$, one has $2 < \mu \leq p < 12$. In order to deal with sub-cubic nonlinearity, we also introduce the following condition on $V$.

$(V_2)$ $V$ is weakly differentiable, $(\nabla V(x), x) \in L^\infty(\mathbb{R}^3) \cup L^{\frac{4}{3}}(\mathbb{R}^3)$, and

$$\frac{\mu - 2}{\mu} V(x) - (\nabla V(x), x) \geq 0 \quad \text{a.e. } x \in \mathbb{R}^3,$$

where $\mu$ is given in $(g_3)$ and $(\cdot, \cdot)$ is the usual inner product in $\mathbb{R}^3$.

Now we state our first result.

**Theorem 1.1.** Let $(V_1)$, $(V_2)$ and $(g_1)$-$(g_3)$ hold. Then problem (1.1) has a least energy sign-changing solution. If additionally $g$ is odd, then equation (1.1) admits infinitely many sign-changing solutions.

The main difficulties in the proof of Theorem 1.1 are three-fold.

First, a typical way in showing the existence and multiplicity of sign-changing solutions of (1.1) is to use the method of sign-changing Nehari manifold, which depends on the monotonicity condition

$$\frac{g(t)}{t^3}$$ is increasing in $(0, +\infty)$ and decreasing in $(-\infty, 0)$. (1.6)
Moreover, the Ambrosetti-Rabinowitz type condition: for some \( \mu > 4 \),
\[
g(t)t \geq \mu G(t) > 0, \quad t \neq 0
\]
is assumed to ensure the boundedness of Palais-Smale ((PS) for short) sequences.

In our paper, the nonlinearity is allowed to be the form of \( g(t) = |t|^p t \) with \( p \in (2, 4] \), which does not satisfy (1.6), let alone (1.7), so the standard variational methods cannot be used directly. Inspired by [17], we will use a perturbation method and the method of invariant sets of descending flow. In [17], Liu et al. considered the existence and multiplicity of sign-changing solutions for the Schrödinger-Poisson system
\[
\begin{cases}
-\Delta u + V(x)u + \phi u = |u|^{p-2}u, & \text{in } \mathbb{R}^3, \\
-\Delta \phi = u^2, & \text{in } \mathbb{R}^3,
\end{cases}
\]
where \( p \in (3, 4] \), and they overcame the difficulty brought by the term \(|u|^{p-2}u\) with \( p \in (3, 4] \) by adding a higher order nonlinear term and the coercive condition \((V'_{0})\).

Motivated by [17], we shall use a perturbation method by adding a higher order term \( \beta|u|^{r-2}u \) with \( \beta > 0 \) and \( r > 4 \). However, different from [17], we do not impose the coercive condition \((V'_{3})\), which plays an important role in showing the boundedness of (PS) sequence in [17]. So we shall add another perturbed term \( \lambda \left( \int_{\mathbb{R}^3} u^2 dx \right)^{\alpha}u \) with \( \lambda, \alpha > 0 \) to show the boundedness of (PS) sequences. We remark that the perturbation method is different from that in [25], where the authors added a coercive potential term and a 4-Laplacian operator, and obtained infinitely many sign-changing solutions of (1.4) with \( p \in (4, 22^*) \).

Second, we shall use the method of invariant sets of descending flow to solve the perturbed problem, and go back to the original problem via the Pohozaev equality as in [17]. However, in this process, there are a lot of additional difficulties caused by the change \( f \), which is used to transform a quasilinear problem into a semilinear one. Especially, the change \( f \) leads that it is not easy to find a similar auxiliary operator \( T_{\lambda, \beta} \) as in [17], which plays a crucial role in constructing invariants sets of descending flow. In addition, some verifications have to be involved in the change \( f \).

For instance, when showing the properties of the invariant sets of descending flow, we need to estimate \( \langle u - T_{\lambda, \beta}(u), f(u) \rangle \) rather than \( \langle u - T_{\lambda, \beta}(u), u \rangle \) as in [17]. When going back to the original problem, we have to combine a Pohozaev equality with the equality \( \langle \lambda \mu G(u), f(u) \rangle = 0 \) rather than \( \langle \lambda \mu G(u), u \rangle = 0 \), where \( \lambda \mu G \) is the associated functional of the perturbed problem. Hence, some new estimations and tricks are required.

Third, in the process in showing the multiplicity of sign-changing solutions, we will use the arguments of the existence part of Theorem 1.1 and some critical points theorem about multiple sign-changing solutions in [13]. Firstly, for the perturbed problem with two perturbation terms \( \lambda \left( \int_{\mathbb{R}^3} u^2 dx \right)^{\alpha}u \) and \( \beta|u|^{r-2}u \), we obtain infinitely many sign-changing solutions \( u_{j}^{\lambda, \beta} \) with the energy \( c_{j}^{\lambda, \beta} \), \( j = 1, 2, \ldots \), and \( c_{j}^{\lambda, \beta} \to +\infty \) as \( j \to +\infty \). Then by taking \( \lambda \to 0^+ \) and \( \beta \to 0^+ \), sign-changing solutions \( u_{j}^{\lambda, \beta} \) with the energy \( c_{j}^{\lambda, \beta} \) of the original problem are obtained. To illustrate that there are infinitely many sign-changing solutions, we need to show \( c_{j}^{\lambda, \beta} \to +\infty \) as \( j \to +\infty \). In [17], there is just one perturbation term \( \beta|u|^{r-2}u \), so it is easy to see that \( c_{j}^{\lambda, \beta} \geq c_{j}^{\lambda} \to +\infty \), where \( c_{j}^{\lambda, \beta}, j = 1, 2, \ldots \) are the energy of sign-changing solutions of the perturbed problem in [17]. In this paper, since
the minimax values \( c^j_{\lambda, \beta} \) have different monotonicity properties on the two perturbation terms, it seems difficult to show \( c^j_{\gamma} \to +\infty \) by \( c^j_{\lambda, \beta} \to +\infty \). As in [18, 19], by using an auxiliary functional, we obtain \( c^j_{\gamma} \to +\infty \) as \( j \to \infty \).

**Remark 1.1.** In this paper, we only consider the case \( 4 \geq \mu > 2 \) in \((g_3)\) since the case \( \mu > 4 \) can be treated without any perturbation. Actually, let \((V_1), \ (g_1), \ (g_2)\) and \((g_3)\) with \( \mu > 4 \) hold, then equation (1.1) admits a least energy sign-changing solution. If additionally \( g \) is odd, then equation (1.1) admits infinitely many sign-changing solutions.

Next we further study the case \( \mu = 4 \) in \((g_3)\), some typical problems can be considered without the condition \((V_2)\). That is to say, we can deal with these problems directly without using the perturbation method. Now we give the assumptions on \( g \) and state the second result as follows.

\((g_4)\) \( G(t) > 0 \) for all \( t \neq 0 \) and \( \lim_{|t| \to +\infty} \frac{G(t)}{t^4} = +\infty \), where \( G(t) = \int_0^t g(s) \, ds \).

\((g_5)\) There exists \( \gamma \in [1, +\infty) \) such that

\[
G(s) \leq \gamma G(\tau), \quad \text{for all } 0 \leq s \leq \tau \text{ or } \tau \leq s \leq 0,
\]

where \( G(t) = \frac{1}{4} g(t) t - G(t) \) for any \( t \in \mathbb{R} \).

**Remark 1.2.** The condition \((g_3)\) was firstly introduced by Jeanjean in [9]. \((g_3)\) is weaker than (1.6) since \((g_5)\) holds with \( \gamma = 1 \) if (1.6) is satisfied. By \((g_5)\), it is easy to verify

\[
\frac{1}{4} g(t) t - G(t) = G(t) \geq 0, \quad \forall t \in \mathbb{R}.
\]  

(1.8)

An example of \( g \) satisfying \((g_1), \ (g_2), \ (g_4)\) and \((g_5)\) is \( g(t) = t^3 \log(1 + |t|) \) for any \( t \in \mathbb{R} \).

**Theorem 1.2.** Let \((V_1), \ (g_1), \ (g_2), \ (g_4)\) and \((g_5)\) hold. Then problem (1.1) has a least energy sign-changing solution. If additionally \( g \) is odd, then equation (1.1) admits infinitely many sign-changing solutions.

**Remark 1.3.** The proof of Theorem 1.2 is based on the method of invariant sets of descending flow in [17] without any perturbation. However, under the assumptions of Theorem 1.2, it seems difficult to verify the associated functional satisfies the \((PS)\) condition, which plays an important role in using the method of invariant sets of descending flow. In the present paper, we consider the Cerami ((Ce) for short) condition instead of the \((PS)\) condition. So we have to establish a deformation lemma under the (Ce) condition and show that the associated functional satisfies the (Ce) condition.

**Remark 1.4.** We would like to point out that, if the functional satisfies the (Ce) condition under suitable assumptions on \( V \) and \( g \), then arguing as in Theorem 1.2, we can show the existence and multiplicity of sign-changing solutions for equation (1.1). Moreover, we highlight that the method of invariant sets of descending flow under the (Ce) condition is also applicable for other related problems.

As a by-product, we finally state the result in the most typical case of \( \mu = 4 \) in \((g_3)\), i.e. \( g(u) = u^3 \), and consider

\[
- \Delta u + V(x) u - u \Delta (u^2) = u^3, \quad x \in \mathbb{R}^3.
\]

(1.9)

**Theorem 1.3.** Let \((V_1)\) hold. Then problem (1.9) has a least energy sign-changing solution. Moreover, equation (1.9) admits infinitely many sign-changing solutions.
Remark 1.5. In [15], under the condition (V1), Liu et al. considered the existence of a least energy sign-changing solution of (1.9) and there are few results about the multiplicity of sign-changing solutions. Hence, to a certain extent, Theorem 1.3 completes the study made in [15] on problem (1.9).

The paper is organized as follows. In Section 2, we give some preliminaries. In Sections 3, 4 and 5, we prove Theorems 1.1, 1.2 and 1.3 respectively.

2. Preliminaries

In this paper we use the following notations. \(\int_{\mathbb{R}^3} f(x)dx\) is represented by \(\int_{\mathbb{R}^3} f(x)\). For \(2 \leq p \leq \infty\), the norm in \(L^p(\mathbb{R}^3)\) is denoted by \(|\cdot|_p\). For any \(r > 0\) and \(x \in \mathbb{R}^3\), \(B_r(x)\) denotes the ball centered at \(x\) with the radius \(r\). The Hilbert space

\[ E := H^1_0(\mathbb{R}^3) = \{ u \in H^1(\mathbb{R}^3) : u(x) = u(|x|) \} \]

is the space endowed with the following inner product and norm

\[ (u, v) = \int_{\mathbb{R}^3} (\nabla u \nabla v + V(x)uv), \quad \|u\|^2 = \int_{\mathbb{R}^3} (|\nabla u|^2 + V(x)u^2). \]

Obviously, under (V1), the norm \(|\cdot|\) is an equivalent norm to the standard norm \(|\cdot|_0 = (|\nabla u|^2 + |u|^2)^{\frac{1}{2}}\) in \(E\), and so the embedding \(E \hookrightarrow L^p(\mathbb{R}^3)\) is compact for any \(p \in (2, 6)\).

Due to the quasilinear term \(-u\Delta(u^2)\), the functional of problem (1.1)

\[ J(u) = \frac{1}{2} \int_{\mathbb{R}^3} (1 + 2u^2)|\nabla u|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)u^2 - \int_{\mathbb{R}^3} G(u), \]

is not well defined in \(E\). As in [15], we make use of a suitable change, namely, the change of variables \(v = f^{-1}(u)\), and then we can choose \(E\) as the research space. The change \(f\) is defined by

\[ f'(t) = \frac{1}{(1 + 2f^2(t))^{\frac{1}{2}}} \quad \text{on} \ [0, +\infty), \]

\[ f(t) = -f(-t) \quad \text{on} \ (-\infty, 0]. \]

Below we state some properties of \(f\) given in [5, 8].

Lemma 2.1. (1) \(f\) is uniquely defined, \(C^\infty\) and invertible;
(2) \(|f'(t)| \leq 1\) for all \(t \in \mathbb{R}\);
(3) \(|f(t)| \leq |t|\) for all \(t \in \mathbb{R}\), and \(|f(t)| \leq 2^{\frac{1}{2}}|t|^{\frac{1}{2}}\) for all \(t \in \mathbb{R}\);
(4) \(\frac{f(t)}{t} \to 1\) as \(t \to 0\);
(5) \(\frac{|f(t)|}{|t|^\frac{1}{2}} \to 2^\frac{1}{4}\) as \(|t| \to +\infty\);
(6) \(\frac{f^2(t)}{2} \leq tf'(t)f(t) \leq f^2(t)\) for all \(t \in \mathbb{R}\);
(7) there exists a positive constant \(C\) such that \(|f(t)| \geq C|t|\) if \(|t| \leq 1\), and \(|f(t)| \geq C|t|^{\frac{1}{2}}\)
if \(|t| \geq 1\);
(8) \(|f(t)f'(t)| \leq \frac{1}{\sqrt{2}}\) for all \(t \in \mathbb{R}\).
In view of the properties of the change $f$, from $J$ we obtain the functional
\[
I(v) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)f^2(v) - \int_{\mathbb{R}^3} G(f(v)),
\]
which is well defined in $E$ and of $C^1$ under our hypotheses. Moreover, the critical points of $I$ are the weak solutions of the problem
\[
-\Delta v + V(x)f(v)f'(v) = g(f(v))f'(v), \quad x \in \mathbb{R}^3.
\]
By the one-to-one correspondence $f$, we just need to study equation (2.2). Denote
\[
\tilde{g}(x,u) = g(f(u))f'(u) - V(x)f(u)f'(u) + V(x)u, \quad \tilde{G}(x,u) = \int_0^u \tilde{g}(x,s)ds.
\]
One easily has the following properties of $\tilde{g}$.

**Lemma 2.2.** For $q = \max\{3, \frac{5}{2}\} \in (2, 6)$, there holds
1. $\tilde{g}(x,t) = o(t)$ uniformly in $x$ as $t \to 0$;
2. $\limsup_{|t| \to +\infty} \frac{\tilde{g}(x,t)}{|t|^q} < +\infty$ uniformly in $x$;
3. for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that $|\tilde{g}(x,t)| \leq \epsilon |t| + C_\epsilon |t|^{q-1}$ for any $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$.

The following lemma plays an important role in showing the (PS) sequence is bounded.

**Lemma 2.3.** Let $(V_1)$ hold. Then there exists $C > 0$ such that
\[
\|v\| \leq C(|\nabla v|_2 + |f(v)|_2^2 + |f(v)|_2) \leq C(1 + |\nabla v|_2 + |f(v)|_2^2), \quad \forall v \in E.
\]

**Proof:** For any $v \in E$, from Lemma 2.1 (7), the Hölder inequality and Young inequality it follows that
\[
\int_{\{v > 1\}} v^2 = \int_{\{|v| > 1\}} |v|^\frac{q}{2} |v|^\frac{q}{2} \leq C \int_{\{|v| > 1\}} |v|^\frac{q}{2} |f(v)|^\frac{q}{2}
\]
\[
\leq C \left( \int_{\{|v| > 1\}} |v|^{\frac{q}{5}} \right)^{\frac{5}{q}} \left( \int_{\{|v| > 1\}} |f(v)|^2 \right)^{\frac{q}{2}} \leq C \left( \frac{3}{5} |v|_6^2 + \frac{2}{5} |f(v)|_{12}^4 \right).
\]
Using Lemma 2.1 (7) again we get
\[
\int_{\{|v| \leq 1\}} v^2 \leq C \int_{\{|v| \leq 1\}} f^2(v).
\]
Then
\[
|v|_2 \leq C(|v|_6 + |f(v)|_2^2 + |f(v)|_2) \leq C(1 + |\nabla v|_2 + |f(v)|_2^2).
\]
Therefore, (2.4) yields. $\square$

In what follows, we recall the abstract critical point theorems developed by Liu et al. in [13], which will be used to show the existence and multiplicity of sign-changing solutions. Let $X$ be a Banach space, $J \in C^1(X, \mathbb{R})$ and $P, Q \subset X$ be open sets, $M = P \cap Q$, $\Sigma = \partial P \cap \partial Q$ and $W = P \cup Q$. For $c \in \mathbb{R}$, $K_c = \{x \in X : J(x) = c, \quad J'(x) = 0\}$ and $J_c = \{x \in X : J(x) \leq c\}$. 
Definition 2.1. ([13]) \( \{ P, Q \} \) is called an admissible family of invariant sets with respect to \( J \) at level \( c \) provided that the following deformation property holds: if \( K_c \setminus W = \emptyset \), then there exists \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0) \), there exists \( \eta \in C(X, X) \) satisfying

1. \( \eta(P) \subset P \), \( \eta(Q) \subset Q \); 
2. \( \eta|_{J^{c-\epsilon}} = id \); 
3. \( \eta(J^{c+\epsilon}\setminus W) \subset J^{c-\epsilon} \).

Theorem 2.1. ([13]) Assume that \( \{ P, Q \} \) is an admissible family of invariant sets with respect to \( J \) at any level \( c \geq c_* := \inf_{u \in \Sigma} J(u) \) and there exists a map \( \varphi_0 : \Delta \to X \) satisfying

1. \( \varphi_0(\partial_1 \Delta) \subset P \) and \( \varphi_0(\partial_2 \Delta) \subset Q \), 
2. \( \varphi_0(\partial_0 \Delta) \cap M = \emptyset \), 
3. \( \sup_{u \in \varphi_0(\partial_0 \Delta)} J(u) < c_* \),

where \( \Delta = \left\{ (t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 \leq 1 \right\} \), \( \partial_1 \Delta = \{0\} \times [0, 1] \), \( \partial_2 \Delta = [0, 1] \times \{0\} \) and \( \partial_0 \Delta = \left\{ (t_1, t_2) \in \mathbb{R}^2 : t_1, t_2 \geq 0, t_1 + t_2 = 1 \right\} \). Define

\[ c = \inf_{\varphi \in C(\Delta, E)} \sup_{u \in \varphi(\Delta) \setminus W} J(u) , \]

where \( \Gamma := \{ \varphi \in C(\Delta, E) : \varphi(\partial_1 \Delta) \subset P, \varphi(\partial_2 \Delta) \subset Q, \varphi|_{\partial_0 \Delta} = \varphi_0|_{\partial_0 \Delta} \} \). Then \( c \geq c_* \) and \( K_c \setminus W \neq \emptyset \).

If additionally assume \( G : X \to X \) is an isometric involution, i.e. \( G^2 = id \) and \( d(Gx, Gy) = d(x, y) \) for \( x, y \in X \). We assume \( J \) is \( G \)-invariant on \( X \) in the sense that \( J(Gx) = J(x) \) for any \( x \in X \). We also assume \( Q = GP \). A subset \( F \subset X \) is said to be symmetric if \( Gx \in F \) for any \( x \in F \). The genus of a closed symmetric subset \( F \) of \( X \setminus \{0\} \) is denoted by \( \gamma(F) \).

Definition 2.2. ([13]) \( P \) is called a \( G \)-admissible invariant set with respect to \( J \) at level \( c \), if the following deformation property holds: there exist \( \epsilon_0 > 0 \) and a symmetric open neighborhood \( N \) of \( K_c \setminus W \) with \( \gamma(N) < +\infty \), such that for \( \epsilon \in (0, \epsilon_0) \) there exists \( \eta \in C(X, X) \) satisfying

1. \( \eta(P) \subset P \), \( \eta(Q) \subset Q \); 
2. \( \eta \circ G = G \circ \eta \); 
3. \( \eta|_{J^{c-\epsilon}} = id \); 
4. \( \eta(J^{c+\epsilon}(N \cup W)) \subset J^{c-\epsilon} \).

Theorem 2.2. ([13]) Assume that \( P \) is a \( G \)-admissible invariant set with respect to \( J \) at level \( c \geq c_* := \inf_{u \in \Sigma} J(u) \) and for any \( n \in \mathbb{N} \), there exists a continuous map \( \varphi_n : B_n = \{ x \in \mathbb{R}^n : |x| \leq 1 \} \to X \) satisfying

1. \( \varphi_n(0) \subset M := P \cap Q \), \( \varphi_n(-t) = G \varphi_n(t) \) for \( t \in B_n \); 
2. \( \varphi_n(\partial B_n) \cap M = \emptyset \); 
3. \( \sup_{u \in Fix_G \cup \varphi_n(\partial B_n)} J(u) < c_* \), where \( Fix_G := \{ u \in X : Gu = u \} \).

For \( j \in \mathbb{N} \), define

\[ c_j = \inf_{B \in \Gamma_j} \sup_{u \in B \setminus W} J(u) , \]

where

\[ \Gamma_j := \{ B \mid B = \varphi(B_n \setminus Y) \text{ for some } \varphi \in G_n, n \geq j \}, \]

and open \( Y \subset B_n \) such that \( Y = Y \) and \( \gamma(Y) \leq n - j \).
Lemma 3.3. \( G_n := \{ \varphi | \varphi \in C(B_n, X), \varphi(-t) = G\varphi(t) \text{ for } t \in B_n, \varphi(0) \in M \text{ and } \varphi|_{\partial B_n} = \varphi_n|_{\partial B_n} \}. \) Then for \( j \geq 2, c_j \geq c_*, K_{c_j}\setminus W \neq \emptyset \) and \( c_j \to +\infty \) as \( j \to +\infty \).

3. Proof of Theorem 1.1

In this section, we will show Theorem 1.1 and assume (\( V_1 \)), (\( V_2 \)) and (\( g_1 \))-(\( g_3 \)) are satisfied. Since \( \mu \leq 4 \) in (\( g_3 \)), it is not easy to show the (PS) sequence is bounded. A perturbed problem is introduced to overcome this difficulty.

3.1. A perturbed problem. Set \( \alpha \in (0, \frac{\mu^2}{4\mu+2}) \) and fix \( \lambda, \beta \in (0,1) \) and \( r \in (\max\{4,p\}, 12) \), we consider the perturbed problem of (2.2) that

\[-\Delta u + V(x) f(u) f'(u) + \lambda f(u) |f(u)|^{2\alpha} f(u) f'(u) = g(f(u)) f'(u) + \beta |f(u)|^{r-2} f(u) f'(u). \tag{3.1}\]

The associated functional is

\[ I_{\lambda,\beta}(u) = I(u) + \frac{\lambda}{2(1+\alpha)} |f(u)|^{2(1+\alpha)} - \frac{\beta}{r} |f(u)|^r, \]

where \( I \) is given in (2.1). In order to go back to the original problem (2.2), we will make use of the Pohozaev type identity of (3.1), whose proof is standard and can be referred to [23, Lemma 2.6] for example.

Lemma 3.1. Let \( u \) be a critical point of \( I_{\lambda,\beta} \) for \( (\lambda, \beta) \in (0,1] \times (0,1] \), then

\[ \begin{align*}
\frac{1}{2} \| \nabla u \|^2 + \frac{1}{2} \int_{\mathbb{R}^3} \left[ 3V(x) + (\nabla V)(x, x) \right] f^2(u) \\
+ \frac{3\lambda}{2(1+\alpha)} |f(u)|^{2(1+\alpha)} - 3 \int_{\mathbb{R}^3} G(f(u)) - \frac{3\beta}{r} |f(u)|^r = 0.
\end{align*} \]

We now introduce an auxiliary operator \( T_{\lambda,\beta} \), which will be used to construct the descending flow for the functional \( I_{\lambda,\beta} \). As an application of Lax-Milgram theorem, for any \( u \in E \), there is a unique solution \( v = T_{\lambda,\beta}(u) \in E \) of the equation

\[-\Delta v + V(x) v + \lambda |f(u)|^{2\alpha} f(u) f'(u) = \tilde{g}(x, u), \tag{3.2}\]

where

\[ \tilde{g}(x, u) = g(f(u)) f'(u) + \beta |f(u)|^{r-2} f(u) f'(u) - V(x) f(u) f'(u) + V(x) u, \]

and \( \tilde{G}(x, u) := \int_0^u \tilde{g}(x, s) ds. \) Clearly, the three statements are equivalent: \( u \) is a solution of (3.1), \( u \) is a critical point of \( I_{\lambda,\beta} \), and \( u \) is a fixed point of \( T_{\lambda,\beta} \). Moreover, \( \tilde{g} \) has the following properties.

Lemma 3.2. (1) \( \tilde{g}(x, t) = o(t) \) uniformly in \( x \) as \( t \to 0 \);

(2) for any \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that

\[ |\tilde{g}(x, t)| \leq \epsilon |t| + C_\epsilon |t|^\frac{r-1}{2}, \forall (x, t) \in \mathbb{R}^3 \times \mathbb{R}. \tag{3.3}\]

Lemma 3.3. \( T_{\lambda,\beta} \) is continuous and compact.
\textbf{Proof:} We firstly show that $T_{\lambda, \beta}$ is continuous. Assume that $u_n \to u$ in $E$. Up to a subsequence, suppose that $u_n \to u$ in $L^s(\mathbb{R}^3)$ with $s \in (2, 6)$. Set $v_n = T_{\lambda, \beta}(u_n)$ and $v = T_{\lambda, \beta}(u)$, we have

$$-\Delta v_n + V(x)v_n + \lambda|f(u_n)|^{2\alpha} \frac{f(u_n)f'(u_n)}{u_n} v_n = g(x, u_n),$$

(3.4)

and

$$-\Delta v + V(x)v + \lambda|f(u)|^{2\alpha} \frac{f(u)f'(u)}{u} v = g(x, u).$$

(3.5)

Testing with $v_n$ in (3.4), by (3.3) and $r \in (4, 12)$ we get

$$\|v_n\|^2 + \lambda|f(u_n)|^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n} v_n^2 = \int_{\mathbb{R}^3} g(x, u_n)v_n$$

$$\leq \epsilon\|u_n\||v_n| + C\|u_n\|^{\frac{r}{2}}\|v_n\|.$$

(3.6)

Then $\{v_n\}$ is bounded in $E$. After passing to a subsequence, suppose $v_n \to v^* \in E$, $v_n \to v^*$ in $L^2_{\text{loc}}(\mathbb{R}^3)$ and $v_n \to v^*$ in $L^s(\mathbb{R}^3)$ with $s \in (2, 6)$. Using (3.4) it is easy to see that $v^*$ is a solution of (3.5) and then $v^* = v$ using the uniqueness. Moreover, testing with $v_n - v$ in (3.4) and (3.5) we obtain

$$\|v_n - v\|^2 + \lambda\mathcal{A}_1 - \lambda\mathcal{A}_2 = \int_{\mathbb{R}^3} (\tilde{g}(x, u_n) - \tilde{g}(x, u))(v_n - v),$$

(3.7)

where

$$\mathcal{A}_1 := |f(u_n)|^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n} v_n(v_n - v), \quad \mathcal{A}_2 := |f(u)|^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u} v(v_n - v).$$

Since $v_n \to v$ in $L^2_{\text{loc}}(\mathbb{R}^3)$, $|f(u_n)|_2$ is bounded and the fact that $|\frac{f(t)f'(t)}{t}| \leq 1$ for any $t \neq 0$, one easily has

$$|f(u_n)|^{2\alpha} \left| \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n} - \frac{f(u)f'(u)}{u} \right| v(v_n - v) \leq 2|f(u_n)|^{2\alpha} \int_{\mathbb{R}^3} |v(v_n - v)| = o_n(1),$$

$$\left( |f(u)|^{2\alpha} - |f(u_n)|^{2\alpha} \right) \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u} v(v_n - v) = o_n(1).$$

Then

$$\mathcal{A}_1 - \mathcal{A}_2 = |f(u_n)|^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n} (v_n - v)^2$$

$$+ |f(u_n)|^{2\alpha} \int_{\mathbb{R}^3} \left[ \frac{f(u_n)f'(u_n)}{u_n} - \frac{f(u)f'(u)}{u} \right] v(v_n - v)$$

$$+ \left( |f(u_n)|^{2\alpha} - |f(u)|^{2\alpha} \right) \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u} v(v_n - v)$$

$$= |f(u_n)|^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n} (v_n - v)^2 + o_n(1).$$

(3.8)

Next we show

$$\mathcal{B}_n := \int_{\mathbb{R}^3} (\tilde{g}(x, u_n) - \tilde{g}(x, u))(v_n - v) \to 0.$$

(3.9)
In fact, let \( \phi \in C_0^\infty(\mathbb{R}, [0, 1]) \) be such that \( \phi(t) = 1 \) for \( |t| \leq 1 \) and \( \phi(t) = 0 \) for \( |t| \geq 2 \). Setting
\[
\bar{g}_1(x, t) = \phi(t)\bar{g}_1(x, t), \quad \bar{g}_2(x, t) = \bar{g}(x, t) - \bar{g}_1(x, t).
\]

By (3.3), there exists \( C > 0 \) such that
\[
|\bar{g}_1(x, t)| \leq C|t|, \quad |\bar{g}_2(x, t)| \leq C|t|^\frac{7}{2}, \quad \text{for all } (x, t) \in \mathbb{R}^3 \times \mathbb{R}.
\]

Therefore
\[
B_n = \int_{\mathbb{R}^3} (\bar{g}_1(x, u_n) - \bar{g}_1(x, u))(v_n - v) + \int_{\mathbb{R}^3} (\bar{g}_2(x, u_n) - \bar{g}_2(x, u))(v_n - v)
\]
\[
\leq \epsilon \int_{\mathbb{R}^3} (|u_n| + |u|)|v_n - v| + |\bar{g}_2(x, u_n) - \bar{g}_2(x, u)||v_n - v| \leq C\epsilon + o_n(1).
\]

By the arbitrariness of \( \epsilon \), we know \( B_n \to 0 \) as \( n \to \infty \). Combining with (3.7), (3.8) and the fact that \( \frac{f(t)}{t^2} \geq 0 \) for any \( t \neq 0 \), we have \( v_n \to v \) in \( E \).

Below we show that \( T_{\lambda, \beta} \) is compact. Assume that \( u_n \to u \) in \( E \) and \( u_n \to u \) in \( L^s(\mathbb{R}^3) \) with \( s \in (2, 6) \). Set \( \lim_{n \to \infty} |f(u_n)|^{2n} = a \) and \( v_n = T_{\lambda, \beta}(u_n) \). As the above argument we infer \( \{v_n\} \) is bounded in \( E \). Up to a subsequence, suppose that \( v_n \to v_0 \) in \( E \), \( v_n \to v_0 \) in \( L^s(\mathbb{R}^3) \) with \( s \in (2, 6) \) and \( v_n(x) \to v_0(x) \) a.e. in \( \mathbb{R}^3 \). For any \( w \in C_0^\infty(\mathbb{R}^3) \), by the fact that \( |\frac{f(t)}{t^2}| \leq 1 \) for any \( t \neq 0 \), we know
\[
\int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n}(v_n - v_0)w = o_n(1),
\]
and using the mean value theorem we infer
\[
\int_{\mathbb{R}^3} \left[ \frac{f(u_n)f'(u_n)}{u_n} - \frac{f(u)f'(u)}{u} \right] v_0w = \int_{\mathbb{R}^3} \frac{f(\xi_n)f'(\xi_n)}{\xi_n}(u_n - u)v_0w
\]
\[
\leq \int_{\mathbb{R}^3} |(u_n - u)v_0w| \leq |u_n - u|_3|v_0|_3|w|_3 = o_n(1),
\]
where \( \xi_n = \theta u_n + (1 - \theta)u \) for some \( \theta \in (0, 1) \). Then there holds
\[
|f(u_n)|^{2n} \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n}v_nw - a \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u}v_0w
\]
\[
= a \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n}v_nw - a \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u}v_0w + o_n(1)
\]
\[
= a \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n}(v_n - v_0)w + a \int_{\mathbb{R}^3} \left[ \frac{f(u_n)f'(u_n)}{u_n} - \frac{f(u)f'(u)}{u} \right] v_0w = o_n(1).
\]

In view of (3.4) we know \( v_0 \) is a solution of the equation
\[
-\Delta v + V(x)v + \lambda a \frac{f(u)f'(u)}{u}v = \bar{g}(x, u).
\]

Then
\[
\|v_0\|^2 + \lambda a \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u}v_0^2 = \int_{\mathbb{R}^3} \bar{g}(x, u)v_0.
\]

(3.10)

In the same way as (3.9), we have
\[
\int_{\mathbb{R}^3} (\bar{g}(x, u_n) - \bar{g}(x, u))v_n = o_n(1).
\]
and so
\[
\int_{\mathbb{R}^3} (\bar{g}(x, u_n)v_n - \bar{g}(x, u)v_0) = o_n(1) + \int_{\mathbb{R}^3} \bar{g}(x, u)(v_n - v_0) = o_n(1). \tag{3.11}
\]

From Fatou lemma, the equality in (3.6) and (3.11) it follows that
\[
\|v_0\|^2 + \lambda a \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u} v_0^2 \leq \liminf_{n \to \infty} \left[ \|v_n\|^2 + \lambda |f(u_n)| \right] \frac{2a}{2} \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n} v_n^2
\]
\[
= \liminf_{n \to \infty} \int_{\mathbb{R}^3} \bar{g}(x, u_n)v_n = \int_{\mathbb{R}^3} \bar{g}(x, u)v_0.
\]

Using (3.10) we know \( v_n \to v_0 \) in \( E \). This ends the proof.

\[\square\]

**Lemma 3.4.** (1) \( \langle I_{\lambda,\beta}(u), u - T_{\lambda,\beta}(u) \rangle \geq \|u - T_{\lambda,\beta}(u)\|^2 \) for all \( u \in E \);
(2) \( \|I_{\lambda,\beta}(u)\| \leq \|u - T_{\lambda,\beta}(u)\| (1 + C_1\|u\|^{2a}) \) for all \( u \in E \), where \( C_1 > 0 \) is a positive constant independent on \( \lambda \) and \( \beta \).

**Proof:** (1) Since \( T_{\lambda,\beta}(u) \) is the solution of (3.2) and the fact that \( \frac{f'(t)f'(t)}{t} \geq 0 \) for any \( t \neq 0 \), we obtain
\[
\langle I_{\lambda,\beta}(u), u - T_{\lambda,\beta}(u) \rangle = \|u - T_{\lambda,\beta}(u)\|^2 + \lambda |f(u)| \frac{2a}{2} \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u} [u - T_{\lambda,\beta}(u)]^2
\]
\[
\geq \|u - T_{\lambda,\beta}(u)\|^2.
\]

(2) For any \( \varphi \in E \), by Lemma 2.1 (2) and (3) we infer
\[
\langle I_{\lambda,\beta}(u), \varphi \rangle = \langle u - T_{\lambda,\beta}(u), \varphi \rangle + \lambda |f(u)| \frac{2a}{2} \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u} (u - T_{\lambda,\beta}(u)) \varphi
\]
\[
\leq \|u - T_{\lambda,\beta}(u)\| \|\varphi\| + \lambda |u| \frac{2a}{2} \|u - T_{\lambda,\beta}(u)\| \|\varphi\|.
\]

Then \( \|I_{\lambda,\beta}(u)\| \leq \|u - T_{\lambda,\beta}(u)\| (1 + C_1\|u\|^{2a}) \). \[\square\]

**Lemma 3.5.** Fix \( (\lambda, \beta) \in (0, 1) \times (0, 1) \) and assume \( a < b \) and \( \tau > 0 \). If \( u \in E \), \( I_{\lambda,\beta}(u) \in [a, b] \) and \( \|I_{\lambda,\beta}(u)\| \geq \tau \), then there exists \( \delta > 0 \) (which depends on \( \lambda \) and \( \beta \)) such that \( \|u - T_{\lambda,\beta}(u)\| \geq \delta \).

**Proof:** By Lemma 2.1 (6), for any \( u \in E \) we have
\[
\nabla \left( \frac{f(u)}{f'(u)} \right) = \frac{1 + 4f^2(u)}{1 + 2f^2(u)} \nabla u, \quad \left| \frac{f(u)}{f'(u)} \right| = \left| \frac{f(u)}{f'(u)} \right| \leq \frac{|f(u)|}{|f'(u)|} \leq 2|u|.
\]

Then
\[
\left\| \frac{f(u)}{f'(u)} \right\| \leq 2\|u\|. \tag{3.12}
\]
Taking $\gamma \in (4, r)$, we obtain
\[
I_{\lambda, \beta}(u) - \frac{1}{\gamma} \langle u - T_{\lambda, \beta}(u), f(u) \rangle
= I_{\lambda, \beta}(u) - \frac{1}{\gamma} \langle I_{\lambda, \beta}(u), f(u) \rangle + \frac{\lambda}{\gamma} \|f(u)\|_{2}^{2} \int_{\mathbb{R}^{3}} \frac{f^{2}(u)}{u} (u - T_{\lambda, \beta}(u))
= \int_{\mathbb{R}^{3}} \left( \frac{1}{2} - \frac{1}{\gamma(1 + 2f^{2}(u))} \right) |\nabla u|_{2}^{2} + \left( \frac{1}{2} - \frac{1}{\gamma} \right) \int_{\mathbb{R}^{3}} V(x) f^{2}(u)
+ \int_{\mathbb{R}^{3}} \left[ \frac{1}{\gamma} g(f(u)) f(u) - G(f(u)) \right] + \frac{(r - \gamma)\beta}{r\gamma} |f(u)|_{r}^{p}
+ \left( \frac{1}{2(1 + \alpha)} - \frac{1}{\gamma} \right) \lambda |f(u)|_{2}^{2 + (1 + \alpha)} + \frac{\lambda}{\gamma} |f(u)|_{2}^{2\alpha} \int_{\mathbb{R}^{3}} \frac{f^{2}(u)}{u} (u - T_{\lambda, \beta}(u)).
\]
Using (g1) and (g2), for any $\epsilon > 0$, there exists a constant $C_{\epsilon} > 0$ such that
\[
|g(t)| \leq \epsilon |t| + C_{\epsilon} |t|^{p - 1}, \quad \forall t \in \mathbb{R}. \tag{3.13}
\]
Then
\[
I_{\lambda, \beta}(u) - \frac{1}{\gamma} \langle u - T_{\lambda, \beta}(u), f(u) \rangle \geq \left( \frac{1}{2} - \frac{2 - \epsilon}{\gamma} \right) |\nabla u|_{2}^{2} + \int_{\mathbb{R}^{3}} V(x) f^{2}(u) - C_{\epsilon} |f(u)|_{p}^{p} + \frac{(r - \gamma)\beta}{r\gamma} |f(u)|_{r}^{p}
+ \left( \frac{1}{2(1 + \alpha)} - \frac{1}{\gamma} \right) \lambda |f(u)|_{2}^{2 + (1 + \alpha)} - \frac{\lambda}{\gamma} |f(u)|_{2}^{2\alpha} |u - T_{\lambda, \beta}(u)|_{2}.
\]
Choosing $\epsilon$ small enough, by (2.4) and (3.12) we deduce
\[
|\nabla u|_{2}^{2} + \int_{\mathbb{R}^{3}} V(x) f^{2}(u) + \lambda |f(u)|_{2}^{2 + (1 + \alpha)} + \beta |f(u)|_{r}^{p} - C_{0} |f(u)|_{p}^{p}
\leq C \left[ |I_{\lambda, \beta}(u)| + 2 |u - T_{\lambda, \beta}(u)| \left( |\nabla u|_{2} + 1 + |f(u)|_{2} \right)
+ |f(u)|_{2}^{2\alpha} (|\nabla u|_{2} + 1 + |f(u)|_{2} |u - T_{\lambda, \beta}(u)|) \right]. \tag{3.14}
\]
Now we assume on the contrary that there exists $\{u_{n}\} \subset E$ with $I_{\lambda, \beta}(u_{n}) \in [a, b]$ and $\|I_{\lambda, \beta}'(u_{n})\| \geq \alpha$ such that
\[
\|u_{n} - T_{\lambda, \beta}(u_{n})\| \to 0, \quad \text{as } n \to \infty.
\]
By (3.14) we get
\[
|\nabla u_{n}|_{2}^{2} + \int_{\mathbb{R}^{3}} V(x) f^{2}(u_{n}) + \lambda |f(u_{n})|_{2}^{2 + (1 + \alpha)} + \beta |f(u_{n})|_{r}^{p} - C_{0} |f(u_{n})|_{p}^{p}
\leq C \left[ C_{1} + 2 |u_{n} - T_{\lambda, \beta}(u_{n})| \left( |\nabla u_{n}|_{2} + 1 + |f(u_{n})|_{2} \right)
+ |f(u_{n})|_{2}^{2\alpha} + |\nabla u_{n}|_{2}^{2} + |f(u_{n})|_{2}^{2\alpha} + |f(u_{n})|_{2}^{2 + 2\alpha} |u_{n} - T_{\lambda, \beta}(u_{n})| \right]. \tag{3.15}
\]
In the same way as (3.7) in [18], for any $c, d > 0$ and $2 < p < r < 12$, there holds
\[
\inf_{u \in H^{1}(\mathbb{R}^{3})} \left( |f(u)|_{r}^{p} + c |f(u)|_{2}^{2\alpha + 2} - d |f(u)|_{p}^{p} \right) > -\infty. \tag{3.16}
\]
Therefore
\[ \frac{\lambda}{2} |f(u_n)|_2^{2(1+\alpha)} + \beta |f(u_n)|_r^r - C_0 |f(u_n)|_p^p \geq -C_2, \]
where \( C_2 > 0 \) is independent on \( n \). From (3.15) it follows that
\[
\begin{align*}
|\nabla u_n|^2_2 + \int_{\mathbb{R}^3} V(x)f^2(u_n) + \frac{\lambda}{2} |f(u_n)|_2^{2(1+\alpha)} \\
\leq C_2 + C \left[ C_1 + 2 \|u_n - T_{\lambda,\beta}(u_n)\| (|\nabla u_n|^2_2 + 1 + |f(u_n)|^2_2) \\
+ (|f(u_n)|^{4\alpha}_2 + |\nabla u_n|^2_2 + |f(u_n)|^{2+2\alpha}_2 + |f(u_n)|^{2+2\alpha}_2) \|u_n - T_{\lambda,\beta}(u_n)\| \right].
\end{align*}
\]
Since \( \alpha < 1 \) we deduce that \( \{ |\nabla u_n|^2_2 + |f(u_n)|^2_2 \} \) is bounded and so \( \{ u_n \} \) is bounded in \( E \) using (2.4). According to Lemma 3.4 (2) we know \( \|I^\prime_{\lambda,\beta}(u_n)\| \to 0 \) as \( n \to \infty \). This is impossible since \( \|I^\prime_{\lambda,\beta}(u_n)\| \geq \alpha \). The proof is complete. \( \square \)

3.2. Invariant subsets of descending flows. To look for sign-changing solutions, we define the positive and negative cones by
\[
P^+ := \{ u \in E : u \geq 0 \} \quad \text{and} \quad P^- := \{ u \in E : u \leq 0 \}
\]
respectively. For \( \epsilon > 0 \), set
\[
P^\epsilon_+ := \{ u \in E : \text{dist}(u, P^+) < \epsilon \} \quad \text{and} \quad P^\epsilon_- := \{ u \in E : \text{dist}(u, P^+) < \epsilon \}
\]
where \( \text{dist}(u, P^\pm) = \inf_{u \in P^\pm} \|u - v\| \). Clearly, \( P^\epsilon_- = -P^\epsilon_+ \). Let \( W = P^\epsilon_+ \cup P^\epsilon_- \). Then, \( W \) is an open and symmetric subset of \( \overline{E} \) and \( E \setminus W \) contains only sign-changing functions. In the following lemma, we will show that, for \( \epsilon \) small enough, all sign-changing solutions to (3.1) are contained in \( E \setminus W \).

**Lemma 3.6.** There is \( \epsilon_0 > 0 \) independent on \( \lambda \) and \( \beta \) such that for \( \epsilon \in (0, \epsilon_0) \),
(1) \( T_{\lambda,\beta}(\partial P^-) \subset P^-_\epsilon \) and every nontrivial solution \( u \in P^-_\epsilon \) is negative.
(2) \( T_{\lambda,\beta}(\partial P^+) \subset P^+_\epsilon \) and every nontrivial solution \( u \in P^+_\epsilon \) is positive.

**Proof:** We only prove (1) since the argument of (2) is similar. For \( u \in E \), define \( v := T_{\lambda,\beta}(u) \). Note that
\[
\frac{f(t)f'(t)}{t} \geq 0, \quad \text{for all } t \neq 0.
\]
Then using (3.3), for any \( \delta > 0 \), there exists \( C_\delta > 0 \) such that
\[
\begin{align*}
\text{dist}(v, P^-) \|v^+\| &\leq \|v^+\|^2 = \int_{\mathbb{R}^3} \bar{g}(x, u)u^+ - \lambda|f(u)|_2^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u)f'(u)}{u}vv^+ \\
&\leq \int_{\mathbb{R}^3} \bar{g}(x, u)u^+ \leq \int_{\mathbb{R}^3} (\delta|u^+| + C_\delta |u^+|^\frac{2}{\delta - 1})v^+ \\
&\leq C(\delta \text{dist}(u, P^-) + C_\delta \text{dist}(u, P^-)^\frac{2}{\delta - 1})\|v^+\|.
\end{align*}
\]
Thus
\[
\text{dist}(v, P^-) \leq C(\delta \text{dist}(u, P^-) + C_\delta \text{dist}(u, P^-)^\frac{2}{\delta - 1}).
\]
Choosing \( \delta \) small enough, there exists \( \epsilon_0 > 0 \) such that for \( \epsilon \in (0, \epsilon_0) \),
\[
\text{dist}(T_{\lambda,\beta}(u), P^-) \leq \frac{1}{2} \text{dist}(u, P^-), \quad \forall u \in P^-_\epsilon.
\]
So $T_{\lambda,\beta}(\partial P^-) \subset P^-$. If there exists $u \in P^-$ such that $T_{\lambda,\beta}(u) = u$, then $u \in P^-$. If $u \neq 0$, then the maximum principle implies that $u < 0$ in $\mathbb{R}^3$. □

Denote the set of fixed points of $T_{\lambda,\beta}$ by $K$. Since the operator $T_{\lambda,\beta}$ is not locally Lipschitz continuous, we need to construct a locally Lipschitz continuous operator $B_{\lambda,\beta}$ on $E_0 := E \setminus K$ which inherits its properties. Arguing as the proof of [1, Lemma 2.1], we have the following result.

Lemma 3.7. There exists a locally Lipschitz continuous operator $B_{\lambda,\beta} : E_0 \to E$ such that

1. $B_{\lambda,\beta}(\partial P^+) \subset P^+$ and $B_{\lambda,\beta}(\partial P^-) \subset P^-$ for $\epsilon \in (0, \epsilon_0)$;
2. $\frac{1}{2} \|u - B_{\lambda,\beta}(u)\| \leq \|u - T_{\lambda,\beta}(u)\| \leq 2 \|u - B_{\lambda,\beta}(u)\|$ for all $u \in E_0$;
3. $\langle I_{\lambda,\beta}'(u), u - B_{\lambda,\beta}(u) \rangle \geq \frac{1}{2} \|u - T_{\lambda,\beta}(u)\|^2$ for all $u \in E_0$;
4. if $g$ is odd, then $B_{\lambda,\beta}$ is odd.

In the following, we show that $I_{\lambda,\beta}$ satisfies the (PS) condition.

Lemma 3.8. For fixed $(\lambda, \beta) \in (0, 1) \times (0, 1)$, $I_{\lambda,\beta}$ satisfies the $(PS)_c$ condition with any $c \in \mathbb{R}$.

Proof: Assume that there exists $\{u_n\} \subset E$ satisfies

$$I_{\lambda,\beta}(u_n) \to c \text{ and } I'_{\lambda,\beta}(u_n) \to 0, \quad (3.17)$$

as $n \to \infty$. Set $\gamma \in (4, r)$, for any $\epsilon > 0$, by (3.13) there exists $C_\epsilon > 0$ such that

$$I_{\lambda,\beta}(u_n) - \frac{1}{\gamma} \left( I'_{\lambda,\beta}(u_n), \frac{f(u_n)}{f'(u_n)} \right) \leq \int_{\mathbb{R}^3} \left( \frac{1}{2} - \frac{1}{\gamma} + 4 \frac{f^2(u_n)}{f'(u_n)} \right) |\nabla u_n|^2 + \left( \frac{1}{2} - \frac{1}{\gamma} \right) \int_{\mathbb{R}^3} V(x) f^2(u_n) + \frac{(r - \gamma)\beta}{r\gamma} |f(u_n)|_r^r \times$$

$$+ \left( \frac{1}{2(1 + \alpha)} - \frac{1}{\gamma} \right) \lambda |f(u_n)|_2^{2(1 + \alpha)} + \int_{\mathbb{R}^3} \left[ \frac{1}{r\gamma} g(f(u_n)) f(u_n) - G(f(u_n)) \right] \geq \left( \frac{1}{2} - \frac{2}{\gamma} - \epsilon \right) \|\nabla u_n\|^2_{r} + \int_{\mathbb{R}^3} V(x) f^2(u_n) \right] + \left( \frac{1}{2(1 + \alpha)} - \frac{1}{\gamma} \right) \lambda |f(u_n)|_2^{2(1 + \alpha)}

$$

$$- C_1 |f(u_n)|_p^p + \frac{(r - \gamma)\beta}{r\gamma} |f(u_n)|_r^r.$$

Choosing small enough $\epsilon > 0$ we have

$$I_{\lambda,\beta}(u_n) - \frac{1}{\gamma} \left( I'_{\lambda,\beta}(u_n), \frac{f(u_n)}{f'(u_n)} \right) \geq \left( \frac{1}{4} - \frac{1}{\gamma} \right) \|\nabla u_n\|^2_{r} + \int_{\mathbb{R}^3} V(x) f^2(u_n) \right] + \left( \frac{1}{2(1 + \alpha)} - \frac{1}{\gamma} \right) \lambda |f(u_n)|_2^{2(1 + \alpha)}

$$

$$- C_1 |f(u_n)|_p^p + \frac{(r - \gamma)\beta}{r\gamma} |f(u_n)|_r^r.$$

Using $\alpha < 1$ and (3.16) we infer

$$|\nabla u_n|^2_{r} + \int_{\mathbb{R}^3} V(x) f^2(u_n) \leq C \left( C_2 + |I_{\lambda,\beta}(u_n)| + o_n(1) \|\frac{f(u_n)}{f'(u_n)}\| \right),$$
From (3.12) and (2.4) it follows that
\[
|\nabla u_n|^2 + \int_{\mathbb{R}^3} V(x) f^2(u_n) \leq C \left[ C_3 + o_n(1) \left( 1 + |\nabla u_n|^2 + |f(u_n)|^2 \right) \right].
\]
Then \(|\nabla u_n|^2 + |f(u_n)|^2\) is bounded and so \(\{u_n\}\) is bounded in \(E\) using (2.4) again.

Suppose \(u_n \to u_0\) in \(E\) and \(u_n \to u_0\) in \(L^s(\mathbb{R}^3)\) with \(2 < s < 6\). It suffices to show that \(u_n \to u_0\) in \(E\) up to a subsequence. Indeed, arguing as (3.8) and (3.9) we get
\[
|f(u_n)|^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n} u_n(u_n - u_0) - |f(u_0)|^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u_0)f'(u_0)}{u_0} u_0(u_n - u_0)
\]
\[
= |f(u_n)|^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n} (u_n - u_0)^2 + o_n(1),
\]
and
\[
\int_{\mathbb{R}^3} (\bar{g}(x, u_n) - \bar{g}(x, u_0))(u_n - u_0) = o_n(1).
\]
Then
\[
\langle I_{\lambda, \beta}'(u_n) - I_{\lambda, \beta}'(u_0), u_n - u_0 \rangle
\]
\[
= |u_n - u_0|^2 + \lambda |f(u_n)|^{2\alpha} \int_{\mathbb{R}^3} \frac{f(u_n)f'(u_n)}{u_n} (u_n - u_0)^2 + o_n(1).
\]
On the other hand, by (3.17) we obtain \(\langle I_{\lambda, \beta}'(u_n) - I_{\lambda, \beta}'(u_0), u_n - u_0 \rangle = o_n(1)\). Therefore, \(u_n \to u_0\) in \(E\).

3.3. Existence of a sign-changing solution. In this subsection, we apply Theorem 2.1 to prove the existence of sign-changing solutions of the problem (3.1), and take \(X = E, P = P_0^+, Q = Q_0^+\) and \(J = I_{\lambda, \beta}\). We will prove that \(\{P_0^+, Q_0^+\}\) is an admissible family of invariant sets for the functional \(I_{\lambda, \beta}\) at any level \(c \in \mathbb{R}\). Indeed, if \(K_c \setminus W = \emptyset\), then \(K_c \subset W\). By Lemma 3.8, the functional \(I_{\lambda, \beta}\) satisfies the (PS)\(_c\) condition, and so \(K_c\) is compact. Then \(2\delta := \text{dist}(K_c, \partial W) > 0\).

Here we give a deformation lemma to the functional \(I_{\lambda, \beta}\) whose proof is almost the same as that of [17, Lemma 3.6].

**Lemma 3.9.** (Deformation lemma) If \(K_c \setminus W = \emptyset\), then there exists \(\epsilon_0 > 0\) such that, for \(0 < \epsilon < \epsilon' < \epsilon_0\), there exists a continuous map \(\sigma : [0, 1] \times E \to E\) satisfying
1. \(\sigma(0, u) = u\) for \(u \in E\);
2. \(\sigma(t, u) = u\) for \(t \in [0, 1], u \not\in I_{\lambda, \beta}^{-1}[c - \epsilon', c + \epsilon']\);
3. \(\sigma(1, I_{\lambda, \beta}^{-1}[c - \epsilon', c + \epsilon']) \subset I_c^{-\epsilon'}\);
4. \(\sigma(t, P_0^+) \subset P_0^+\) and \(\sigma(t, P_0^-) \subset P_0^-\) for \(t \in [0, 1]\).

**Lemma 3.10.** For any \(q \in [2, 6]\), there exists \(C > 0\) independent of \(\epsilon\) such that \(|u|_q \leq C\epsilon\) for \(u \in M = P_0^+ \cap P_0^-\).

**Proof:** For any fixed \(u \in M\), we have
\[
|u^\pm|_q = \inf_{v \in P^\pm} |u - v|_q \leq C \inf_{v \in P^\pm} \|u - v\| \leq C \text{dist}(u, P^\pm).
\]
Then \(|u|_q \leq C\epsilon\) for \(u \in M\). \(\Box\)

**Lemma 3.11.** If \(\epsilon > 0\) small enough, then \(I_{\lambda, \beta}(u) \geq \frac{\epsilon^2}{12}\) for \(u \in \Sigma = \partial P_0^+ \cap \partial P_0^-\), and so \(c_* := \inf_{u \in \Sigma} I_{\lambda, \beta}(u) \geq \frac{\epsilon^2}{12}\).
**Proof:** For \( u \in \partial P^+ \cap \partial P^- \), we have \( \|u^\pm\| \geq \text{dist}(u, P^\pm) = \epsilon \). In view of (3.3) and Lemma 3.10, for any \( \delta > 0 \), there exists \( C_\delta > 0 \) such that

\[
I_{\lambda, \beta}(u) \geq \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}^3} G(x, u) \geq \left( \frac{1}{2} - \delta \right)\|u\|^2 - C_\delta u_\frac{\epsilon}{2}.
\]

Choosing \( \delta \) small enough, we get

\[
I_{\lambda, \beta}(u) \geq \frac{1}{6}\|u\|^2 - C|u|_2 \epsilon^2 \geq \frac{1}{6}\epsilon^2 - C\epsilon^2 \geq \frac{\epsilon^2}{12},
\]

for \( \epsilon \) small enough. \( \square \)

**Proof of Theorem 1.1 (Existence part)** In what follows, we divide the proof into three steps.

**Step 1** In this step, we will apply Theorem 2.1 to look for a sign-changing solutions of (3.1) for any fixed \( \lambda, \beta \in (0, 1) \times (0, 1) \). By Lemma 3.9, we know \( \{P^+, P^-\} \) is an admissible family of invariant sets with respect to \( I_{\lambda, \beta} \) at any level \( c \in \mathbb{R} \). It suffices to verify assumptions (1)-(3) of Theorem 2.1.

Let \( v_1, v_2 \in C^\infty_0(\mathbb{R}^3) \backslash \{0\} \) be such that \( \text{supp}(v_1) \cap \text{supp}(v_2) = \emptyset \) and \( v_1 \leq 0, v_2 \geq 0 \). For \( (t, s) \in \Delta \), let

\[
\varphi_0(t, s)(\cdot) = R(tv_1(R^{-1}\cdot) + sv_2(R^{-1}\cdot)),
\]

where \( R > 0 \) will be determined later. Obviously, for \( t, s \in [0, 1] \), \( \varphi_0(0, s) = Rsv_2(R^{-1}\cdot) \in P^+ \) and \( \varphi_0(t, 0) = Rtv_1(R^{-1}\cdot) \in P^- \). Observe that \( \rho := \min\{|tv_1 + (1-t)v_2|_2 : 0 \leq t \leq 1\} > 0 \). Then \( |u|^2 \geq \rho^2 R^3 \) for \( u \in \varphi_0(\partial_0 \Delta) \). It follows from Lemma 3.10 that \( \varphi_0(\partial_0 \Delta) \cap M = \emptyset \). In view of Lemma 3.11, for small \( \epsilon \) we have \( c_\epsilon = \inf_{u \in \Sigma} I_{\lambda, \beta}(u) = \frac{\epsilon^2}{12} \) for any \( \lambda, \beta \in (0, 1) \times (0, 1) \). Below we show that \( \sup_{u \in \varphi_0(\partial_0 \Delta)} I_{\lambda, \beta}(u) < 0 \). Set \( u_t = \varphi_0(t, 1-t) \) for \( t \in [0, 1] \). A direct computation shows that

\[
|\nabla u_t|^2 = R^3 \int_{\mathbb{R}^3} \left( t^2 |\nabla v_1|^2 + (1-t)^2 |\nabla v_2|^2 \right);
\]

\[
|f(u_t)|_{2(1+\alpha)}^2 = R^{3(1+\alpha)} \left( \int_{\mathbb{R}^3} (|f(Rtv_1)|^2 + |f(R(1-t)v_2)|^2) \right)^{1+\alpha};
\]

\[
|f(u_t)|_2^2 = R^3 \int_{\mathbb{R}^3} (|f(Rtv_1)|^2 + |f(R(1-t)v_2)|^2);
\]

\[
|f(u_t)|_\mu^\mu = R^3 \int_{\mathbb{R}^3} (|f(Rtv_1)|^\mu + |f(R(1-t)v_2)|^\mu).
\]

Note that \( G(s) \geq C|s|^\mu - C_1 \) for any \( s \in \mathbb{R} \), we deduce

\[
I_{\lambda, \beta}(u_t) \leq \frac{R^3}{2} \int_{\mathbb{R}^3} \left( t^2 |\nabla v_1|^2 + (1-t)^2 |\nabla v_2|^2 \right) + \frac{\lambda R^{3(1+\alpha)}}{2(1+\alpha)} (|f(Rtv_1)|_{2(1+\alpha)}^2 + |f(R(1-t)v_2)|_{2(1+\alpha)}^2)
\]

\[
+ \int_{\text{supp}v_1} G(f(Rtv_1)) - \int_{\text{supp}v_2} G(f(R(1-t)v_2)) \leq \frac{R^3}{2} \int_{\mathbb{R}^3} \left( t^2 |\nabla v_1|^2 + (1-t)^2 |\nabla v_2|^2 \right) + \frac{R^{3(1+\alpha)}}{2(1+\alpha)} (|f(Rtv_1)|_{2(1+\alpha)}^2 + |f(R(1-t)v_2)|_{2(1+\alpha)}^2)
\]

\[
+ \int_{\text{supp}v_1} G(f(Rtv_1)) - \int_{\text{supp}v_2} G(f(R(1-t)v_2)) + C_1 R^3.
\]
Since \[ \frac{|f(s)|}{|s|^\frac{1}{2}} \rightarrow 2^+, \quad \text{as } |s| \rightarrow +\infty, \quad \text{and } \alpha < \frac{\mu - 2}{3\mu + 2} < \frac{1}{4}\left(\frac{\mu}{2} - 1\right), \]

we have \( I_{\lambda,\beta}(u_t) \rightarrow -\infty \) as \( R \rightarrow +\infty \) uniformly for \((\lambda, \beta) \in (0, 1) \times (0, 1)\). So we can choose large enough \( R > 0 \) independent on \( \lambda \) and \( \beta \) such that \( I_{\lambda,\beta}(u_t) < 0 \). Then

\[
\sup_{u \in \varphi_0(\partial \Delta)} I_{\lambda,\beta}(u) < 0 < \frac{\epsilon^2}{12} \leq c_* := \inf_{u \in \Sigma} I_{\lambda,\beta}(u), \quad \forall (\lambda, \beta) \in (0, 1) \times (0, 1).
\]

Applying Theorem 2.1 we know

\[
c_{\lambda,\beta} = \inf_{\varphi \in \Gamma} \sup_{u \in \varphi(\Delta) \setminus W} I_{\lambda,\beta}(u),
\]

is a critical value of \( I_{\lambda,\beta} \)

and

\[
c_{\lambda,\beta} \geq c_* \geq \frac{\epsilon^2}{12}, \quad (3.18)
\]

Therefore, there exists \( u_{\lambda,\beta} \in E \setminus (P^+ \cup P^-) \) such that \( I_{\lambda,\beta}(u_{\lambda,\beta}) = c_{\lambda,\beta} \) and \( I'_{\lambda,\beta}(u_{\lambda,\beta}) = 0 \) for fixed \((\lambda, \beta) \in (0, 1) \times (0, 1)\).

**Step 2.** Set \( \lambda \rightarrow 0 \) and \( \beta \rightarrow 0 \). In view of the definition of \( c_{\lambda,\beta} \), for any \((\lambda, \beta) \in (0, 1) \times (0, 1)\), there holds

\[
c_{\lambda,\beta} \leq C_R := \sup_{u \in \varphi_0(\Delta)} I_{1,0}(u) < +\infty, \quad (3.19)
\]

where \( C_R \) is independent on \((\lambda, \beta) \in (0, 1) \times (0, 1)\). Without loss of generality, we set \( \lambda = \beta \). Choosing a sequence \( \{\lambda_n\} \subset (0, 1) \) satisfying \( \lambda_n \rightarrow 0^+ \), we find a sequence of sign-changing critical points \( \{u_{\lambda_n}\} \) (still denoted by \( \{u_n\} \) for simplicity) of \( I_{\lambda_n,\beta_n} \) and \( I_{\lambda_n,\beta_n}(u_n) = c_{\lambda_n,\beta_n} \). Now we show that \( \{u_n\} \) is bounded in \( E \). Note that

\[
c_{\lambda_n,\beta_n} = \frac{1}{2}|\nabla u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)f^2(u_n) + \frac{\lambda_n}{2(1 + \alpha)}|f(u_n)|_2^{2(1 + \alpha)} - \int_{\mathbb{R}^3} G(f(u_n)) - \frac{\beta_n}{r}|f(u_n)|_r^r, \quad (3.20)
\]

and

\[
0 = \left\langle I'_{\lambda_n,\beta_n}(u_n), \frac{f(u_n)}{f'(u_n)} \right\rangle = \int_{\mathbb{R}^3} \left(1 + \frac{2f^2(u_n)}{1 + 2f^2(u_n)}\right)|\nabla u_n|^2 + \int_{\mathbb{R}^3} V(x)f^2(u_n) + \lambda_n|f(u_n)|_2^{2(1 + \alpha)} - \int_{\mathbb{R}^3} g(f(u_n))f(u_n) - \beta_n|f(u_n)|_r^r. \quad (3.21)
\]

Moreover, from Lemma 3.2 we have

\[
\frac{1}{2}|\nabla u_n|^2 + \frac{3}{2} \int_{\mathbb{R}^3} V(x)f^2(u_n) + \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x)f^2(u_n) + \frac{3\lambda_n}{2(1 + \alpha)}|f(u_n)|_2^{2(1 + \alpha)} - 3 \int_{\mathbb{R}^3} G(f(u_n)) - \frac{3\beta_n}{r}|f(u_n)|_r^r = 0. \quad (3.22)
\]
Multiplying (3.20), (3.21) and (3.22) by 4, $-\frac{1}{\mu}$ and $-1$ respectively and adding them up, we obtain

$$4c_{\lambda_n, \beta_n} = \int_{\mathbb{R}^3} \left[ \frac{3}{2} - \frac{1}{\mu} \left( 1 + \frac{2f^2(n)}{1 + 2f^2(n)} \right) \right] |\nabla u_n|^2 + \left( \frac{1}{2} - \frac{1}{\mu} \right) \int_{\mathbb{R}^3} V(x) f^2(u_n)$$

$$- \frac{1}{2} \int_{\mathbb{R}^3} (\nabla V(x), x) f^2(u_n) + \left[ \frac{1}{2(1 + \alpha)} - \frac{1}{\mu} \right] \lambda_n |f(u_n)|_2^{2(1+\alpha)}$$

$$+ \int_{\mathbb{R}^3} [-G(f(u_n)) + \frac{1}{\mu}g(f(u_n)) f(u_n)] + \left( \frac{1}{\mu} - \frac{1}{r} \right) \beta_n |f(u_n)|_r^r.$$

Since $\alpha < \frac{\mu - 2}{\mu + 2} < \frac{\mu}{2} - 1$, it follows from (V2), (g3) and (3.19) that

$$4C_R \geq \left( \frac{3}{2} - \frac{2}{\mu} \right) |\nabla u_n|^2,$$

which implies that there exists $C > 0$ independent of $\lambda, \beta$ such that

$$|\nabla u_n|^2 < C.$$  

Moreover, in view of (3.19) and (3.20) we infer that for small $\delta > 0$, there exists $C_\delta > 0$ such that

$$C_R \geq \frac{1}{2} \int_{\mathbb{R}^3} V(x) f^2(u_n) - \int_{\mathbb{R}^3} G(f(u_n)) - \frac{\beta_n}{r} |f(u_n)|_r^r$$

$$\geq \frac{1 - \delta}{2} \int_{\mathbb{R}^3} V(x) f^2(u_n) - C_\delta \int_{\mathbb{R}^3} |u_n|^6 - \frac{1}{r} |f(u_n)|_r^r.$$  

Observe that $r \in (4, 12)$, we may assume that $r = 2\tau + 12(1 - \tau)$, $\tau \in (0, 1)$. Then for the above $\delta$, there exists $C_\delta > 0$ such that

$$|f(u_n)|_r^r \leq |f(u_n)|_2^{2\tau} |f(u_n)|_2^{12(1-\tau)} \leq |f(u_n)|_2^{2\tau} |u_n|_6^{6(1-\tau)}$$

$$\leq \delta \int_{\mathbb{R}^3} V(x) f^2(u_n) + C_\delta |u_n|^6.$$  

According to (3.24) and (3.25) we have

$$C_R \geq \frac{1 - 2\delta}{2} \int_{\mathbb{R}^3} V(x) f^2(u_n) - (C_\delta + C_\delta) \int_{\mathbb{R}^3} |u_n|^6.$$

Using (3.23) and the arbitrariness of $\delta$, we get $|f(u_n)|_2^2$ is bounded. By (2.4) we know $\{u_n\}$ is bounded in $E$. From (3.18) it follows that

$$\lim_{n \to \infty} I(u_n) = \lim_{n \to \infty} \left( f_{\lambda_n, \beta_n}(u_n) - \frac{\lambda_n}{2(1 + \alpha)} |f(u_n)|_2^{2(1+\alpha)} + \frac{\beta_n}{r} |f(u_n)|_r^r \right)$$

$$= \lim_{n \to \infty} c_{\lambda_n, \beta_n} := c^* \geq \frac{\epsilon^2}{12}.$$  

Moreover, for any $\psi \in C_0^\infty(\mathbb{R}^3)$,

$$\lim_{n \to \infty} \langle I'(u_n), \psi \rangle = \lim_{n \to \infty} \left( I_{\lambda_n, \beta_n}'(u_n), \psi \right) - \lambda_n |f(u_n)|_2^{2\alpha} \int_{\mathbb{R}^3} f(u_n) f'(u_n) \psi$$

$$+ \beta_n \int_{\mathbb{R}^3} |f(u_n)|_r^{r-2} f(u_n) f'(u_n) \psi = 0.$$
Then \( \{u_n\} \) is a bounded (PS) sequence for \( I \) at level \( c^* \). Hence, we may assume that \( u_n \to u^* \) in \( E \) and \( u_n \to u^* \) in \( L^s(\mathbb{R}^3) \) with \( s \in (2, 6) \). By Lemma 2.2 (3), arguing as (3.9) we have

\[
\int_{\mathbb{R}^3} (\bar{g}(x, u_n) - \bar{g}(x, u^*)) (u_n - u^*) = o_n(1).
\]

Then

\[
\langle I'(u_n) - I'(u^*), u_n - u^* \rangle = \|u_n - u^*\|^2 + o_n(1).
\]

On the other hand, since \( I'(u_n) \to 0 \) we get \( \langle I'(u_n) - I'(u^*), u_n - u^* \rangle = o_n(1) \). Consequently

\[
u_n \to u^* \text{ in } E.
\] (3.27)

Then \( I'(u^*) = 0 \) and \( I(u^*) = c^* \). From the fact that \( u_n \in E \setminus (P_+^+ \cup P_-^-) \) we infer \( u^* \in E \setminus (P_+^+ \cup P_-^-) \) and so \( u^* \) is a sign-changing solution of problem (2.2).

**Step 3.** Define

\[ c := \inf_{u \in \mathcal{N}} I(u), \quad \mathcal{N} := \{u \in E : I'(u) = 0, u^\pm \neq 0\}. \]

Based on Step 2, we see that \( \mathcal{N} \neq \emptyset \) and \( c \leq c^* \), where \( c^* \) is given in (3.26). By the definition of \( c \), there exists \( \{u_n\} \subset E \) such that \( I(u_n) \to c \) and \( I'(u_n) = 0 \). Arguing as (3.27), we infer that there exists \( u \neq 0 \) such that \( u_n \to u \) in \( E \), \( I(u) = c \) and \( I'(u) = 0 \). Moreover, note that \( \langle I'(u_n), u_n^\pm \rangle = 0 \). From Lemma 2.2 (3) we infer that for any \( \delta > 0 \), there exists \( C_\delta > 0 \) such that

\[
|\nabla u_n^\pm|_2^2 + \int_{\mathbb{R}^3} V(x)(u_n^\pm)^2 = \int_{\mathbb{R}^3} \bar{g}(x, u_n^\pm) u_n^\pm \leq \delta \|u_n^\pm\|^2 + C_\delta \|u_n^\pm\|^q.
\]

Then \( \|u_n^\pm\| \geq \delta > 0 \) and \( \|u^\pm\| \geq \delta^{\frac{1}{2}} \). Consequently, \( u \) is a least energy sign-changing solution of problem (2.2). The proof is complete. \( \square \)

### 3.4. Multiplicity of sign-changing solutions

In this subsection, \( g \) is assumed to be odd, and so \( I_{\lambda, \beta} \) is even. We shall apply Theorem 2.2 to obtain infinitely many sign-changing solutions and set \( X = E, G = -id, J = I_{\lambda, \beta} \) and \( P = P_r^+ \). Then \( M = P_+^+ \cap P_-^- \), \( \Sigma = \partial P_+^+ \cap \partial P_-^- \), and \( W = P_+^+ \cup P_-^- \). Since \( K_c \) is compact, there exists a symmetric open neighborhood \( N \) of \( K_c \setminus W \) such that \( \gamma(\overline{N}) < +\infty \).

**Lemma 3.12.** There exists \( \epsilon_0 > 0 \) such that, for \( 0 < \epsilon < \epsilon' < \epsilon_0 \), there exists a continuous map \( \sigma : [0, 1] \times E \to E \) satisfying

1. \( \sigma(0, u) = u \) for \( u \in E \);
2. \( \sigma(t, u) = u \) for \( t \in [0, 1], u \not\in I^{-1}[c - \epsilon', c + \epsilon'] \);
3. \( \sigma(1, I^{c+\epsilon} \setminus (N \cup W)) \subset I^{c-\epsilon} \);
4. \( \sigma(t, P_+^+) \subset P_+^+ \) and \( \sigma(t, P_-^-) \subset P_-^- \) for \( t \in [0, 1] \);
5. \( \sigma(t, -u) = -\sigma(t, u) \) for \( (t, u) \in [0, 1] \times E \).

**Proof:** The arguments of (1)-(4) are similar to those of Lemma 3.9. Concerning (5), since \( I_{\lambda, \beta} \) is even and the operator \( B_{\lambda, \beta} \) in Lemma 3.7 is odd, we infer \( \sigma \) is odd in \( u \). \( \square \)

**Proof of Theorem 1.1** (Multiplicity part) We complete the proof in two steps.

**Step 1.** Since \( g \) is odd, in view of Lemma 3.12 we know \( P_r^+ \) is a \( G \)-admissible invariant set with respect to \( I_{\lambda, \beta} \) at level \( c \). In order to apply Theorem 2.2, we are now constructing
For any \( n \in \mathbb{N} \), let \( \{v_i\}_{i=1}^n \subset C^\infty_0(\mathbb{R}^3) \setminus \{0\} \) be such that \( \text{supp}(v_i) \cap \text{supp}(v_j) = \emptyset \) for \( i \neq j \). Define \( \varphi_n \in C(B_n, E) \) as
\[
\varphi_n(t) = R_n \sum_{i=1}^n t_i v_i(R_n^{-1} \cdot t), \quad t = (t_1, t_2, \ldots, t_n) \in B_n,
\]
where \( R_n > 0 \) will be determined later. Obviously, \( \varphi_n(0) = 0 \in P_\varepsilon^+ \cap P_\varepsilon^- \) and \( \varphi_n(-t) = -\varphi_n(t) \) for \( t \in B_n \). Observe that
\[
\rho_n = \min \{ |t_1 v_1 + t_2 v_2 + \ldots + t_n v_n|_2 : \sum_{i=1}^n t_i^2 = 1 \} > 0,
\]
then \( |u_i|^2 \geq \rho_n^2 R_n^2 \) for \( u_i \in \varphi_n(\partial B_n) \) and it follows from Lemma 3.10 that \( \varphi_n(\partial B_n) \cap (P_\varepsilon^+ \cap P_\varepsilon^-) = \emptyset \). Similar to the proof of Theorem 1.1 (existence part), for large enough \( R_n > 0 \) independent on \( \lambda \) and \( \beta \) we also have
\[
\sup_{u \in \varphi_n(\partial B_n)} I_{\lambda, \beta}(u) < 0 < \inf_{u \in \Sigma} I_{\lambda, \beta}(u).
\]
For any \( j \in \mathbb{N} \) and \( (\lambda, \beta) \in (0, 1] \times (0, 1] \), we define
\[
c_{\lambda, \beta}^j = \inf_{B \in \Gamma_j} \sup_{u \in B \setminus W} I_{\lambda, \beta}(u),
\]
where \( W = P_\varepsilon^+ \cup P_\varepsilon^- \) and \( \Gamma_j \) is defined in Theorem 2.2. Applying Theorem 2.2 and Lemma 3.11, for any \( (\lambda, \beta) \in (0, 1] \times (0, 1] \) and \( j \geq 2 \),
\[
\frac{\varepsilon^2}{12} \leq \inf_{u \in \Sigma} I_{\lambda, \beta}(u) \leq c_{\lambda, \beta}^j \rightarrow +\infty, \quad as \quad j \rightarrow +\infty,
\]
and there exists \( \{u_{\lambda, \beta}^j\} \subset E \setminus W \) such that \( I_{\lambda, \beta}(u_{\lambda, \beta}^j) = c_{\lambda, \beta}^j \) and \( I'_{\lambda, \beta}(u_{\lambda, \beta}^j) = 0 \).

**Step 2.** Using similar arguments as those in Theorem 1.1 (existence part), for any fixed \( j \geq 2 \), \( \{u_{\lambda, \beta}^j\}_{\lambda, \beta \in (0, 1]} \) is bounded in \( E \). Namely, there exists \( C > 0 \) independent of \( \lambda, \beta \) such that \( \|u_{\lambda, \beta}^j\| \leq C \). Without loss of generality, we assume \( u_{\lambda, \beta}^j \rightarrow u^j \) in \( E \) as \( \lambda, \beta \rightarrow 0 \). By (3.28) we have
\[
\frac{\varepsilon^2}{12} \leq c_{\lambda, \beta}^j \leq c_{R_n} := \sup_{u \in \phi_n(B_n)} I_{1,0}(u),
\]
where \( c_{R_n} \) is independent of \( \lambda, \beta \). Assume that \( c_{\lambda, \beta}^j \rightarrow c^*_j \) as \( \lambda, \beta \rightarrow 0 \). Then as the proof of Theorem 1.1 (existence part), we can prove that \( u_{\lambda, \beta}^j \rightarrow u^j \) in \( E \) as \( \lambda, \beta \rightarrow 0^+ \) and \( u^j \in E \setminus W \) satisfying \( I'(u^j) = 0 \) and \( I(u^j) = c^*_j \). Below we claim that \( c^*_j \rightarrow +\infty \) as \( j \rightarrow \infty \). Indeed, by (g1) and (g2) we infer
\[
I_{\lambda, \beta}(u) \geq \frac{1}{2} |\nabla u|^2_2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) f^2(u) - \int_{\mathbb{R}^3} G(f(u)) - \frac{1}{r} |f(u)|^r_\varepsilon
\]
\[
\geq \frac{1}{2} |\nabla u|^2_2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) f^2(u) - \int_{\mathbb{R}^3} \left( \inf_{\mathbb{R}^3} V \frac{4}{4} f^2(u) + \frac{C}{r} |f(u)|^r_\varepsilon \right) - \frac{1}{r} |f(u)|^r_\varepsilon
\]
\[
\geq \frac{1}{2} |\nabla u|^2_2 + \frac{1}{4} \int_{\mathbb{R}^3} V(x) f^2(u) - \frac{1+C}{r} |f(u)|^r_\varepsilon := \hat{I}(u),
\]
where \( C > 0 \) depends on \( \inf_{\mathbb{R}^3} V \). Since \( r \in (4, 12) \), it is easy to verify that the (PS) condition of \( \hat{I} \) holds. Then replacing \( I_{\lambda, \beta} \) by \( \hat{I} \), the arguments of \( I_{\lambda, \beta} \) are still valid with
some suitable modifications. In particular, the assumptions of Theorem 2.2 are satisfied. So we can define

\[ d^j := \inf_{B \in \Gamma_j} \sup_{u \in B \cap W} \bar{I}(u), \]

where \( W = P^+_\epsilon \cup P^-_\epsilon \) and \( \Gamma_j \) is defined in Theorem 2.2. Moreover, Theorem 2.2 implies that \( d^j \to +\infty \) as \( j \to +\infty \). Combining (3.29) and the definitions of \( c^j_{\lambda, \beta} \) and \( d^j \), we have \( c^j_{\lambda, \beta} \geq d^j \). Taking \( \lambda, \beta \to 0^+ \) we get \( c^j \geq d^j \to +\infty \) as \( j \to +\infty \). Hence, problem (2.2) has infinitely many sign-changing solutions. This ends the proof. \( \square \)

4. Proof of Theorem 1.2

In this section, we are devoted to showing Theorem 1.2 and assume \((V)_1, (g)_1, (g)_2, (g)_4\) and \((g)_5\) are satisfied. As said in Remark 1.3, we shall prove Theorem 1.2 without using the perturbation method. However, under the assumptions of Theorem 1.2, we will need to show that the functional satisfies the \((Ce)\) condition and establish the deformation lemma under the \((Ce)\) condition.

4.1. Properties of operator \( A \). We firstly introduce an auxiliary operator \( A \), which will be used to construct the descending flow for the functional \( \bar{I} \) given in (2.1). As an application of Lax-Milgram theorem, for any \( u \in E \), there is a unique solution \( v = A(u) \in E \) of the equation

\[ -\Delta v + V(x)v = \tilde{g}(x, u), \]

where \( \tilde{g}(x, u) \) is given in (2.3). Arguing as Lemma 3.3, we have that \( A \) is continuous and compact.

**Lemma 4.1.** (1) \( \langle I'(u), u - A(u) \rangle \geq \|u - A(u)\|^2 \) for all \( u \in E \);

(2) \( \|I'(u)\| \leq \|u - A(u)\| \) for all \( u \in E \);

(3) for \( a < b \) and \( \alpha > 0 \), there exists \( \beta > 0 \) such that \( \|u - A(u)\| \geq \beta \) if \( u \in E \), \( I(u) \in [a, b] \) and \( \|I'(u)\| \geq \alpha \).

**Proof:** (1) For any \( u \in E \), we have

\[ \langle I'(u), u - A(u) \rangle = \langle I'(u) - I'(A(u)), u - A(u) \rangle = \|u - A(u)\|^2. \]

(2) For any \( u, \varphi \in E \), we get

\[ \langle I'(u), \varphi \rangle = \langle I'(u) - I'(A(u)), \varphi \rangle = \langle u - A(u), \varphi \rangle \leq \|u - A(u)\| \|\varphi\|. \]

Then \( \|I'(u)\| \leq \|u - A(u)\| \).

(3) From (2) it follows that the conclusion (3) holds true. \( \square \)

4.2. Invariant subsets of descending flows. The notations \( P^+, P^-, P^+_\epsilon, P^-_\epsilon \) and \( W \) in Section 3.2 are still valid. By Lemma 2.2 (3), arguing as Lemma 3.6 with small modifications, we have:

**Lemma 4.2.** There is \( \epsilon_0 > 0 \) such that, for \( \epsilon \in (0, \epsilon_0) \),

(1) \( A(\partial P^-_\epsilon) \subset P^-_\epsilon \) and every nontrivial solution \( u \in P^-_\epsilon \) is negative.

(2) \( A(\partial P^+_\epsilon) \subset P^+_\epsilon \) and every nontrivial solution \( u \in P^+_\epsilon \) is positive.

Denote the set of fixed points of \( A \) by \( K \). Similar to Lemma 3.7, we have the following result.
Lemma 4.3. There exists a locally Lipschitz continuous operator $B : E_0 = E \setminus K \to E$ such that

(1) $B(\partial P^\perp_\epsilon) \subset P^\perp_\epsilon$ and $B(\partial P^-) \subset P^- \epsilon$ for $\epsilon \in (0, \epsilon_0)$;
(2) $\frac{1}{2}\|u - B(u)\| \leq \|u - A(u)\| \leq 2\|u - B(u)\|$ for all $u \in E_0$;
(3) $\langle I'(u), u - B(u) \rangle \geq \frac{1}{2}\|u - A(u)\|^2$ for all $u \in E_0$;
(4) if $g$ is odd, then $B$ is odd.

Below we show that, $I$ satisfies the (Ce) condition. Although the proof is inspired by [9], some crucial modifications are needed to overcome the difficulties caused by the change $f$.

Lemma 4.4. For any $c \in \mathbb{R}$, $I$ satisfies the (Ce)$_c$ condition.

Proof: Let $\{u_n\} \subset E$ be a (Ce)$_c$ sequence of $I$, i.e.

$$I(u_n) \to c \text{ and } \|u_n\|(1 + \|I'(u_n)\|) \to 0. \quad (4.2)$$

Firstly, we show that $\|u_n\|$ is bounded. Argue by contradiction we may assume that $\|u_n\| \to +\infty$. Setting $v_n := \frac{u_n}{\|u_n\|}$. Up to a subsequence, we suppose that $v_n \to v$ in $E$, $v_n \to v$ in $L^s(\mathbb{R}^3)$ with $2 < s < 6$ and $v_n(x) \to v(x)$ a.e. in $\mathbb{R}^3$. Below we consider two cases that $v = 0$ and $v \neq 0$ separately, and show that there will be a contradiction in both cases.

Case 1: $v = 0$.

In this case, let $t_n \in [0, 1]$ such that $I(t_n u_n) = \max_{t \in [0, 1]} I(t u_n)$. For any positive constant $M$, we have $2M^\frac{1}{s}\|u_n\|^{-1} \in (0, 1)$ for large $n$. Denote $\tilde{v}_n = 2M^\frac{1}{s}v_n$. From Lemma 2.2 (3) and Lebesgue dominated convergence theorem it follows that $\int_{\mathbb{R}^3} \tilde{G}(x, \tilde{v}_n) = o_n(1)$. Then for $n$ large we get

$$I(t_n u_n) \geq I(\tilde{v}_n) = \frac{1}{2}\|\tilde{v}_n\|^2 - \int_{\mathbb{R}^3} \tilde{G}(x, \tilde{v}_n) \geq M.$$  

Hence

$$\lim_{n \to \infty} I(t_n u_n) = +\infty. \quad (4.3)$$

On the other hand, note that $I(0) = 0$ and $I(u_n) \to c$, we have $t_n \in (0, 1)$ and then $\frac{d}{dt}I(t u_n)|_{t=t_n} = 0$. Denote

$$\bar{f}(t) := f^2(t) - f(t)f'(t)t, \quad \bar{f}(t) := f(t)f'(t)t - \frac{f^2(t)}{2}.$$  

Clearly, using Lemma 2.1 (6) we have $\bar{f}(t), \bar{f}(t) \geq 0$. We claim that

$$\bar{f}(t) \text{ and } \bar{f}(t) \text{ are nondecreasing in } (0, +\infty) \text{ and nonincreasing in } (-\infty, 0). \quad (4.4)$$

Indeed, by Lemma 2.1 (6) and (8), for any $t > 0$ we deduce

$$\bar{f}'(t) = f(t)f'(t) - [f'(t)]^2 t + 2f^2(t)[f'(t)]^4 t \geq 2f^2(t)[f'(t)]^4 t \geq 0,$$

and

$$\bar{f}'(t) = t[f'(t)]^2 + f(t)f''(t)t = t[f'(t)]^2[1 - 2f^2(t)(f'(t))^2] \geq 0.$$
Note that \( \bar{f} \) and \( \bar{\bar{f}} \) are even, so (4.4) holds true. Using (4.4) and \((g_5)\), we infer
\[
I(t_n u_n) = I(t_n u_n) - \frac{1}{2} \langle I'(t_n u_n), t_n u_n \rangle
\]
\[
= \frac{1}{2} \int_{\mathbb{R}^3} V(x) \left[ f^2(t_n u_n) - f(t_n u_n) f'(t_n u_n) t_n u_n \right] + \int_{\mathbb{R}^3} \left[ \frac{1}{2} g(f(t_n u_n)) f'(t_n u_n) t_n u_n - G(f(t_n u_n)) \right]
+ \int_{\mathbb{R}^3} \left[ \frac{1}{2} g(f(t_n u_n)) f(t_n u_n) - 2G(f(t_n u_n)) \right] \frac{f'(t_n u_n) t_n u_n}{f(t_n u_n)}.
\tag{4.5}
\]
By Remark 1.2, \( \frac{G(s)}{f^2(t)} \) is nondecreasing for \( s > 0 \) and nonincreasing for \( s < 0 \). It follows that \( \frac{G(s)}{f^2(t)} \leq \frac{G(f(t))}{f^2(s)} \) if \( |t| \leq |s| \) and then
\[
\int_{\mathbb{R}^3} \frac{G(f(t_n u_n))}{f^2(t_n u_n)} \bar{f}(t_n u_n) \leq \int_{\mathbb{R}^3} \frac{G(f(u_n))}{f^2(u_n)} \bar{f}(u_n).
\]
By Lemma 2.1, \( \frac{f'(t)}{f(t)} \geq \frac{1}{2} \) and \( \frac{f'(t)}{f(t)} \leq 1 \) for any \( t \neq 0 \). This yields that
\[
\frac{f'(t) t}{f(t)} \leq \frac{f'(s) s}{f(s)}, \text{ for any } t, s \neq 0.
\]
So we have
\[
\frac{f'(t_n u_n) t_n u_n}{f(t_n u_n)} \leq 2 \frac{f'(u_n) u_n}{f(u_n)}, \text{ a.e. in } \mathbb{R}^3.
\]
Thanks to \((g_5)\) and the monotonicity of \( \bar{f} \), by (4.5) we get
\[
I(t_n u_n) \leq \frac{1}{2} \int_{\mathbb{R}^3} V(x) \bar{f}(u_n) + 2 \int_{\mathbb{R}^3} \frac{G(f(u_n))}{f^2(u_n)} \bar{f}(u_n)
+ 2\gamma \int_{\mathbb{R}^3} \left[ \frac{1}{2} g(f(u_n)) f(u_n) - 2G(f(u_n)) \right] \frac{f'(u_n) u_n}{f(u_n)}
\leq 2\gamma[I(u_n) - \frac{1}{2} \langle I'(u_n), u_n \rangle].
\]
In view of (4.2) we know \( I(t_n u_n) \leq 2\gamma[c + o_n(1)] \), contradicting (4.3).

**Case 2:** \( v \neq 0 \).
Set \( \Omega = \{ x \in \mathbb{R}^3 : v(x) \neq 0 \} \). Then \( |u_n(x)| \to \infty \) for \( x \in \Omega \). From \((g_4)\), Lemma 2.1 (5) and Fatou lemma it follows that
\[
\liminf_{n \to \infty} \int_{\mathbb{R}^3} \frac{G(f(u_n))}{\|u_n\|^2} \geq \liminf_{n \to \infty} \int_{\Omega} \frac{G(f(u_n))}{u_n^2} v_n^2
= \liminf_{n \to \infty} \int_{\Omega} \frac{G(f(u_n))}{f^4(u_n)} \frac{f^4(u_n)}{u_n^2} v_n^2 = +\infty
\]
Hence
\[
\frac{I(u_n)}{\|u_n\|^2} \leq \frac{1}{2} - \int_{\mathbb{R}^3} \frac{G(f(u_n))}{\|u_n\|^2} \to -\infty.
\]
However, by (4.2) and \( \|u_n\| \to +\infty \), we know \( \frac{I(u_n)}{\|u_n\|^{\frac{n}{p}}} \to 0 \). This is a contradiction.

According to the above discussion, \( \{u_n\} \) is bounded in \( E \) and we assume that \( u_n \to u \) in \( E \). Then as the argument of (3.27) we obtain \( u_n \to u \) in \( E \) after passing to a subsequence. This ends the proof. \( \Box \)

### 4.3. Existence of a sign-changing solution.

To apply Theorem 2.1, we take \( X = E \), \( P = P^+_\epsilon \), \( Q = Q^-_\epsilon \) and \( J = I \). In the process of applying Theorem 2.1, a deformation lemma of the functional \( I \) corresponding to [17, Lemma 3.6] is crucial. However, the proof of [17, Lemma 3.6] depends on the fact that the functional satisfies the (PS) condition and here the functional \( I \) merely satisfies the (Ce) condition, so we have to provide a new proof.

In the following, we firstly recall the local solvability and global existence theorems of solutions for initial value problems in [26], see also [12]. Let \( (X_0, \| \cdot \|_{X_0}) \) be a real Banach space, suppose that \( \varphi : [0, +\infty) \times X_0 \to X_0 \) is continuous and for all \( x_0 \in X_0 \) and all \( \alpha > 0 \), there exist \( R > 0 \) and \( L = L(x_0, \alpha, R) > 0 \) such that

\[
\| \varphi(t,x) - \varphi(t,y) \|_{X_0} \leq L \| x - y \|_{X_0}, \quad \forall x, y \in B(x_0, R), t \in [0, \alpha].
\]  

(4.6)

Consider the following initial value problem

\[
\begin{cases}
\frac{dx}{dt} = \varphi(t,x), \\
x(0) = x_0 \in X_0.
\end{cases}
\]  

(4.7)

**Lemma 4.5.** ([26, Theorem 5.1]) Suppose that \( \varphi \) satisfies the assumption (4.6). Then there exists \( \beta > 0 \) such that the problem (4.7) has a unique solution \( x(t) \) in \( [0, \beta] \) which continuously depends on \( x_0 \). More generally, if \( \| \varphi(x,t) - \varphi(y,t) \|_{X_0} \leq L \| x - y \|_{X_0} \), then

\[
\| x(t) - y(t) \|_{X_0} \leq L e^{Lt} \| x_0 - y_0 \|_{X_0}, \quad \forall x, y \in X_0, t \in [0, \beta],
\]

where \( x(t) \) and \( y(t) \) are the solutions of (4.7) with initial values \( x_0 \) and \( y_0 \) respectively.

**Lemma 4.6.** ([26, Theorem 5.3]) Suppose that \( \varphi \) satisfies the assumption (4.6). If there exist \( a, b > 0 \) such that

\[
\| \varphi(t,x) \|_{X_0} \leq a + b \| x \|_{X_0}, \quad \forall (t,x) \in [0, +\infty) \times X_0 \mapsto X_0,
\]

then the unique local solution of (4.7) can be extended as a global solution for \( t \in [0, +\infty) \).

Now we are ready to state the deformation lemma under the (Ce) condition and give the proof.

**Lemma 4.7.** (Deformation lemma) If \( K_\epsilon \setminus \partial W = \emptyset \), then there exists \( \epsilon_0 > 0 \) such that, for \( 0 < \epsilon < \epsilon' < \epsilon_0 \), there exists a continuous map \( \sigma : [0, 1] \times E \to E \) satisfying

1. \( \sigma(0, u) = u \) for \( u \in E \);
2. \( \sigma(t, u) = u \) for \( t \in [0, 1] \), \( u \notin I^{-1}[c - \epsilon', c + \epsilon'] \);
3. \( \sigma(1, I^{c+\epsilon}_-W) \subset I^{c-\epsilon}_- \);
4. \( \sigma(t, P^+_\epsilon) \subset P^+_\epsilon \) and \( \sigma(t, P^-_\epsilon) \subset P^-_\epsilon \) for \( t \in [0, 1] \).

**Proof:** Since \( K_\epsilon \setminus \partial W = \emptyset \), we have \( K_\epsilon \subset W \). By Lemma 4.4, \( I \) satisfies the (Ce)_\epsilon condition, and so \( K_\epsilon \) is compact. Then \( 2\delta := \text{dist}(K_\epsilon, \partial W) > 0 \). For any \( D \subset E \) and \( a > 0 \), let \( N_a(D) := \{ u \in E : \text{dist}(u, D) < a \} \). Then \( N_a(D) \subset W \). Moreover, since \( I \) satisfies the (Ce)_\epsilon condition, there exist \( \epsilon_0, \alpha > 0 \) such that

\[
(1 + \| u \|) \| I'(u) \| \geq \alpha \quad \text{for } u \in I^{-1}([c - \epsilon_0, c + \epsilon_0]) \setminus N_{\frac{\delta}{4}}(K_\epsilon).
\]
By Lemmas 4.1 (2) and Lemma 4.3 (2), there exists $\beta > 0$ such that

$$
(1 + \|u\|)\|u - B(u)\| \geq \beta \text{ for } u \in I^{-1}([c - \epsilon_0, c + \epsilon_0]) \setminus N_\delta(K_c). \quad (4.8)
$$

Without loss of generality, assume that $\epsilon_0 < \frac{\beta \delta}{32(\sup_u K_c + 2)}$. Let

$$
h(u) = \frac{u - B(u)}{\|u - B(u)\|^2} \quad \text{for } u \in E_0 = E \setminus K,
$$

and take a cut-off function $\zeta : E \to [0, 1]$, which is locally Lipschitz continuous, such that

$$
\zeta(u) = \begin{cases}
0, & \text{if } u \not\in I^{-1}[c - \epsilon', c + \epsilon'] \text{ or } u \in N_\delta(K_c), \\
1, & \text{if } u \in I^{-1}[c - \epsilon, c + \epsilon] \text{ and } u \not\in N_\delta(K_c).
\end{cases}
$$

By Lemma 4.3, $\zeta(\cdot)h(\cdot)$ is locally Lipschitz continuous on $E$.

Consider the following initial value problem

$$
\begin{cases}
\frac{d\tau}{dt} = -\zeta(\tau)h(\tau), \\
\tau(0, u) = u.
\end{cases}
(4.9)
$$

For any $u \in E$, by (4.8) we have

$$
\|\zeta(u)h(u)\| \leq \frac{1}{\|u - B(u)\|} \leq \frac{1 + \|u\|}{\beta}.
$$

From Lemmas 4.5 and 4.6 it follows that the problem (4.9) admits a unique solution $\tau(\cdot, u) \in C(\mathbb{R}^+, E)$.

Define $\sigma(t, u) = \tau(16et, u)$. Obviously, the conclusions (1) and (2) hold true. It suffices to check (3) and (4). To verify (3), let $u \in I^{c+\epsilon}\setminus W$. By Lemma 4.3, $I(\tau(t, u))$ is decreasing in $t \geq 0$. If there exists $t_0 \in [0, 16\epsilon]$ such that $I(\tau(t_0, u)) < c - \epsilon$, then $I(\sigma(1, u)) = I(\tau(16\epsilon, u)) < c - \epsilon$. Otherwise, for any $t \in [0, 16\epsilon]$, there holds $I(\tau(16\epsilon, u)) \geq c - \epsilon$. Then $\tau(t, u) \in I^{-1}[c - \epsilon, c + \epsilon]$ for any $t \in [0, 16\epsilon]$. We claim that for any $t \in [0, 16\epsilon]$, $\tau(t, u) \not\in N_\delta(K_c)$. Otherwise, there exists $t_1 \in [0, 16\epsilon]$ such that $\tau(t_1, u) \in N_\delta(K_c)$. Observe that $u \not\in N_\delta(K_c)$. Then

there is $0 < t_2 < t_1$ such that $\tau(t_2, u) \in \partial N_\delta(K_c)$.

Hence

$$
\frac{\delta}{2} \leq \|\tau(t_2, u) - \tau(t_1, u)\| \leq \int_{t_2}^{t_1} \|\tau'(s, u)\| ds
\leq \int_{t_2}^{t_1} \frac{1}{\|\tau(s, u) - B(\tau(s, u))\|} ds
\leq \int_{t_2}^{t_1} \frac{1 + \|\tau(s, u)\|}{\beta} ds
\leq \frac{1 + (\sup_u K_c + \delta)}{\beta} \int_{t_2}^{t_1} ds \leq \frac{16\epsilon}{\beta} (\sup_u K_c + 2),
$$
which contradicts the fact that $\epsilon < \epsilon_0 < \frac{\beta_\delta}{32([\sup_u K_e]+2)}$. So $\zeta(t,u)) \equiv 1$ for $t \in [0,16\epsilon]$. From Lemma 4.3 (2) and (3) it follows that
\[
I(\tau(16\epsilon,u)) = I(u) - \int_0^{16\epsilon} \langle I'(\tau(s,u)), h(\tau(s,u)) \rangle \, ds \\
= I(u) - \int_0^{16\epsilon} \frac{\langle I'(\tau(s,u)), \tau(s,u) - B(\tau(s,u)) \rangle}{\|\tau(s,u) - B(\tau(s,u))\|^2} \, ds \\
\leq I(u) - \int_0^{16\epsilon} \frac{1}{8} \, ds \leq c + \epsilon - 2\epsilon = c - \epsilon.
\]
Thus, the conclusion (3) yields. Finally, (4) is a consequence of Lemma 4.3 (1), see [16] for a detailed proof. \qed

Arguing as Lemma 3.11 and using Lemma 2.2 (3) we have the following result.

**Lemma 4.8.** If $\epsilon > 0$ small enough, then $I(u) \geq \frac{\epsilon^2}{12}$ for $u \in \Sigma = \partial P_e^+ \cap \partial P_e^-$, that is $c_* \geq \frac{\epsilon^2}{12}$.

**Proof of Theorem 1.2 (Existence part)** We will apply Theorem 2.1 to look for a sign-changing solution of (2.2). By Lemma 4.5, we know $\{P_e^+, P_e^\pm\}$ is an admissible family of invariant sets with respect to $I$ at any level $c \in \mathbb{R}$. It suffices to verify assumptions (1)-(3) of Theorem 2.1. Let $v_1, v_2 \in C^\infty_0(\mathbb{R}^3)\setminus\{0\}$ be such that $\text{supp}(v_1) \cap \text{supp}(v_2) = \emptyset$ and $v_1 \leq 0$, $v_2 \geq 0$. For $(t,s) \in \Delta$, let $\varphi_0(t,s) = R(tv_1 + sv_2)$, where the constant $R > 0$ will be determined later. Obviously, for $t,s \in [0,1]$, $\varphi_0(0,s) = Rsv_2 \in P_e^+$ and $\varphi_0(t,0) = Rt v_1 \in P_e^-$. Set $\rho = \min\{|tv_1 + (1-t)v_2|_2 : 0 \leq t \leq 1\} > 0$. Then $|u|_2 \geq \rho R$ for $u \in \varphi_0(\partial_0 \Delta)$. It follows from Lemma 3.10 that $\varphi_0(\partial_0 \Delta) \cap M = \emptyset$.

For any $u \in \varphi_0(\partial_0 \Delta)$, we have
\[
I(u) = \frac{R^2 \rho^2}{2} |\nabla v_1|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) f^2(Rtv_1) - \int_{\text{supp} v_1} G(f(Rtv_1)) \\
+ \frac{R^2 (1-t)^2}{2} |\nabla v_2|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x) f^2(R(1-t)v_2) - \int_{\text{supp} v_2} G(f(R(1-t)v_2)).
\]
By $(g_4)$ and Lemma 2.1 (5) we deduce
\[
\frac{f^2(s)}{s^2} \to 0 \quad \text{and} \quad \frac{G(f(s))}{s^2} = \frac{G(f(s))}{f^4(s)} \frac{f^4(s)}{s^2} \to +\infty, \quad \text{as} \quad |s| \to +\infty.
\]
we have $I(u) < 0$ for large $R > 0$. In view of Lemma 4.8 we have $c_* \geq \frac{\epsilon^2}{12}$ for small $\epsilon$. Hence $\sup_{u \in \varphi_0(\partial_0 \Delta)} I(u) < 0 < c_\ast$. Applying Theorem 2.1 we know (2.2) admits a sign-changing solution $u_0 \in E \setminus (P_e^+ \cup P_e^-)$. Below arguing as the proof of Theorem 1.1 (Existence part), we show (2.2) has a least energy sign-changing solution. \qed

4.4. **Multiplicity of sign-changing solutions.** In order to apply Theorem 2.2, we set $X = E$, $G = -id$, $J = I$ and $P = P_e^\ast$. Then $M = P_e^\ast \cap P_e^\ast$, $\Sigma = \partial P_e^\ast \cap \partial P_e^\ast$, and $W = P_e^+ \cup P_e^-$. In this subsection, $g$ is assumed to be odd, and so $I$ is even. Since $K_e$ is compact, there exists a symmetric open neighborhood $N$ of $N_e \setminus W$ such that $\gamma(N) < +\infty$. The counterpart of Lemma 3.12 under the (Ce) condition is as follows.


Lemma 4.9. There exists $\epsilon_0 > 0$ such that, for $0 < \epsilon < \epsilon' < \epsilon_0$, there exists a continuous map $\sigma : [0, 1] \times E \to E$ satisfying

1. $\sigma(0, u) = u$ for $u \in E$;
2. $\sigma(t, u) = u$ for $t \in [0, 1]$, $u \notin I^{-1}[c - \epsilon', c + \epsilon']$;
3. $\sigma(1, I_{c+\epsilon'}(N \cup W)) \subset I_{c-\epsilon'}$;
4. $\sigma(t, P_{c+\epsilon'}^+) \subset P_{c+\epsilon'}^+$ and $\sigma(t, P_{c-\epsilon'}^-) \subset P_{c-\epsilon'}^-$ for $t \in [0, 1]$;
5. $\sigma(t, -u) = -\sigma(t, u)$ for $(t, u) \in [0, 1] \times E$.

Proof: The proof of (1)-(4) are similar to those of Lemma 4.7. Regarding (5), since $I$ is even and the operator $B$ is odd in Lemma 4.3, we infer $\sigma$ is odd in $u$. □

Proof of Theorem 1.2 (Multiplicity part) In view of Lemma 4.9 we know $P_{c+\epsilon'}^+$ is a $G$—admissible invariant set with respect to $I$ at level $c \in \mathbb{R}$. To apply Theorem 2.2, we are now constructing $\phi_n$. For any $n \in \mathbb{N}$, choose $\{v_i\}_{i=1}^n \subset C_0^\infty(\mathbb{R}^3) \setminus \{0\}$ such that $\text{supp}(v_i) \cap \text{supp}(v_j) = \emptyset$ for $i \neq j$. Define $\varphi_n \in C(B_n, E)$ as

$$\varphi_n(t) = R_n \sum_{i=1}^n t_i v_i, \quad t = (t_1, t_2, ..., t_n) \in B_n,$$

where $R_n > 0$. For $R_n$ large enough, it is easy to check that all the assumptions of Theorem 2.2 are satisfied. Therefore, (2.2) admits infinitely many sign-changing solutions. □

5. Proof of Theorem 1.3

In this section, we consider equation (1.9). By the change $f$, (1.9) is changed into

$$-\Delta u + V(x)f(u)f'(u) = f^3(u)f'(u). \quad (5.1)$$

In this case, the associated functional $I$ merely satisfies the (Ce) condition. So we use the deformation lemmas as in Sections 4.3 and 4.4 to show the existence and multiplicity of sign-changing solutions of (5.1). However, in this case, the condition

$$\lim_{|t| \to +\infty} \frac{G(t)}{t^4} = +\infty, \quad (5.2)$$

is not satisfied, then it is necessary to make suitable modifications in the arguments of Lemma 4.4 and Theorem 1.2. And the other results and their proof are similar to those in Section 4, so they will not be written out.

Lemma 5.1. For any $c \in \mathbb{R}$, $I$ satisfies the (Ce)$_c$ condition.

Proof: Assume that $\{u_n\} \subset E$ satisfies $I(u_n) \to c$ and $(1 + \|u_n\|)\|I'(u_n)\| \to 0$. Let $w_n = \frac{f(u_n)}{I'(u_n)}$. By (3.12) we deduce

$$c + o_n(1) = I(u_n) - \frac{1}{4}\langle I'(u_n), u_n \rangle \geq \frac{1}{4} \int_{\mathbb{R}^3} (f'(u_n))^2 |\nabla u_n|^2 + \frac{1}{4} \int_{\mathbb{R}^3} V(x)f^2(u_n)$$

$$= \frac{1}{4} \int_{\mathbb{R}^3} |\nabla f(u_n)|^2 + \frac{1}{4} \int_{\mathbb{R}^3} V(x)f^2(u_n).$$

Therefore, $\{f(u_n)\}$ is bounded in $E$ and then bounded in $L^s(\mathbb{R}^3)$ with $2 \leq s \leq 6$. Note that

$$c + o_n(1) = I(u_n) = \frac{1}{2} |\nabla u_n|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)f^2(u_n) - \frac{1}{4} |f(u_n)|^4.$$
So \(|\nabla u_n|^2\) is bounded and \(\{u_n\}\) is bounded in \(E\) using (2.4). In the same way as (3.27) we have \(u_n \to u\) in \(E\). Thus \(I\) satisfies the (Ce) condition. 

**Proof of Theorem 1.3** We take similar arguments of Theorem 1.2, and without the condition (5.2), we make some modification as follows. In applying Theorem 2.1, we need to choose \(v_1, v_2 \in C_0^\infty(\mathbb{R}^3)\) to be such that

\[
|\nabla v_i|^2 < |v_i|^2, \quad i = 1, 2, \quad \text{supp}(v_1) \cap \text{supp}(v_2) = \emptyset, \quad \text{and} \quad v_1 \leq 0, v_2 \geq 0.
\]

For \((t, s) \in \Delta\), let \(\varphi_0(t, s) = R(tv_1 + sv_2)\), where \(R > 0\) will be determined later. For any \(u \in \varphi_0(\partial_0 \Delta)\), we have

\[
I(u) = \frac{R^2t^2}{2}|\nabla v_1|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)f^2(Rtv_1) - \frac{1}{4} \int_{\mathbb{R}^3} |f(Rtv_1)|^4
\]

\[+ \frac{R^2(1-t)^2}{2}|\nabla v_2|^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(x)f^2(R(1-t)v_2) - \frac{1}{4} \int_{\mathbb{R}^3} |f(R(1-t)v_2)|^4.
\]

In view of Lemma 2.1 (5) we deduce

\[
\frac{f^2(s)}{s^2} \to 0 \quad \text{and} \quad \frac{|f(s)|^4}{s^2} \to 2, \quad \text{as} \quad |s| \to +\infty.
\]

Combining with the fact that \(|\nabla v_i|^2 < |v_i|^2\), we have \(I(u) < 0\) for large \(R > 0\). Then arguing as Theorem 1.2 (existence part), we infer that (5.1) has a least energy sign-changing solution.

In applying Theorem 2.2, as above we need to construct a different \(\phi_n\) from that in the proof of Theorem 1.2 (multiplicity part). For any \(n \in \mathbb{N}\), let \(\{v_i\}^n_{i=1} \subset C_0^\infty(\mathbb{R}^3)\) be such that

\[
|\nabla v_i|^2 < |v_i|^2, \quad \text{for all} \quad i = 1, 2, \ldots, n, \quad \text{and} \quad \text{supp}(v_i) \cap \text{supp}(v_j) = \emptyset \quad \text{for} \quad i \neq j.
\]

Define \(\varphi_n \in C(B_n, E)\) as \(\varphi_n(t) = R_n\sum_{i=1}^{n} t_iv_i\), where \(t = (t_1, t_2, \ldots, t_n) \in B_n\) and \(R_n > 0\). For \(R_n\) large enough, as above one easily check that all the assumptions of Theorem 2.2 are satisfied. Hence, equation (5.1) has infinitely many sign-changing solutions. 

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