ON THE GREATEST COMMON DIVISOR OF INTEGER PARTS OF POLYNOMIALS

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Abstract. Motivated by a question of V. Bergelson and F. K. Richter (2017), we obtain asymptotic formulas for the number of relatively prime tuples composed of positive integers $n \leq N$ and integer parts of polynomials evaluated at $n$. The error terms in our formulas are of various strengths depending on the Diophantine properties of the leading coefficients of these polynomials.

1. Introduction

1.1. Motivation. Let $\lfloor t \rfloor$ and $\{t\}$ denote the integer and fractional parts of a real number $t$, respectively; thus, $t = \lfloor t \rfloor + \{t\}$ for all $t \in \mathbb{R}$.

Watson [13], answering a question of K. F. Roth, proved that for any given irrational number $\alpha$, the set of positive integers $n$ for which $\gcd(n, \lfloor \alpha n \rfloor) = 1$ has a natural density

$$\delta \left( \{n \in \mathbb{N} : \gcd(n, \lfloor \alpha n \rfloor) = 1 \} \right) = \frac{6}{\pi^2}. \quad (1.1)$$

In the same paper, Watson showed that a similar result holds for all rational numbers $\alpha$, albeit with a natural density that depends on $\alpha$ and differs from $6/\pi^2$. Shortly thereafter, Estermann [5] gave a different proof of a slight generalization of Watson’s theorem. Later, Erdős and Lorentz [4] gave sufficient conditions for a differentiable function $f : [1, \infty) \to \mathbb{R}$ to satisfy

$$\delta \left( \{n \in \mathbb{N} : \gcd(n, \lfloor f(n) \rfloor) = 1 \} \right) = \frac{6}{\pi^2}.$$ 

The problem of finding functions $f$ with this property has been studied by several authors; see [2] for a historical account of these results.

The present paper is inspired by a result of Bergelson and Richter [2], which asserts that the natural density

$$\delta(\{n \in \mathbb{N} : \gcd(n, \lfloor f_1(n) \rfloor, \ldots, \lfloor f_k(n) \rfloor) = 1 \}) = \frac{1}{\zeta(k+1)}$$

for any functions $f_1, \ldots, f_k$ belonging to a given Hardy field $\mathcal{H}$ and satisfying some mild conditions. Here, $\zeta(s)$ is the Riemann zeta function. At the end of their paper as a natural extension to Watson’s original result (1.1), Bergelson and Richter pose the following question (see [2, Question 1]):
**Question.** Let $\alpha_1, \ldots, \alpha_k$ be irrational numbers. Is it true that the natural density of the set

$$\{ n \in \mathbb{N} : \gcd(n, \lfloor \alpha_1 n \rfloor, \ldots, \lfloor \alpha_k n \rfloor) = 1 \}$$

exists and is equal to $1/\zeta(k+1)$?  

In this paper, we show this question has an affirmative answer whenever the numbers $\{\alpha_j\}$ satisfy some mild Diophantine conditions. For example, when all of the numbers $\{\alpha_j\}$ are of finite type, we establish an asymptotic formula with a strong bound on the error term. Our techniques also apply to certain classes of Liouville numbers (i.e., numbers of infinite type) but with somewhat weaker bounds on the error terms. We note that, in complete generality, the original question remains open.

To formulate our various results precisely, we first recall some standard notions from the theory of Diophantine approximations.

**1.2. Types of irrational numbers.** Let $[t]$ denote the distance from a real number $t$ to the nearest integer:

$$[t] := \min_{n \in \mathbb{Z}} |t - n| \quad (t \in \mathbb{R}).$$

For any irrational number $\alpha$, we define its type $\tau$ by the relation

$$\tau := \sup \left\{ \vartheta \in \mathbb{R} : \lim_{q \to \infty} q^{\vartheta} \left[ q \alpha \right] = 0 \right\}.$$

We say that $\alpha$ is of finite type when $\tau < \infty$. Using Dirichlet’s approximation theorem, one sees that $\tau \geq 1$ for every irrational $\alpha$. The celebrated theorems of Khinchin [7] and of Roth [9,10] assert that $\tau = 1$ for almost all real (in the sense of the Lebesgue measure) and all irrational algebraic numbers $\alpha$, respectively.

Similarly, for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we define its exponential type $\tau_*$ by

$$\tau_* := \sup \left\{ \vartheta \in \mathbb{R} : \lim_{q \to \infty} \exp(q^{\vartheta}) \left[ q \alpha \right] = 0 \right\}.$$

We say that $\alpha$ is of finite exponential type whenever $\tau_* < \infty$. Note that if $\alpha$ is of finite type $\tau$, then its exponential type $\tau_*$ is also finite, and one has $\tau_* \leq \tau$. The converse is not true in general.

**1.3. Statement of results.** Let $k \geq 1$ be a fixed integer. Given a sequence $\alpha := (\alpha_j)_{j=1}^k$ of irrational real numbers and a sequence $m := (m_j)_{j=1}^k$ of integers such that

$$1 = m_1 < m_2 < \cdots < m_k,$$

denote by $N_{\alpha,m}(x)$ the number of positive integers $n \leq x$ that satisfy

$$\gcd(n, \lfloor \alpha_1 n^{m_1} \rfloor, \lfloor \alpha_2 n^{m_2} \rfloor, \ldots, \lfloor \alpha_k n^{m_k} \rfloor) = 1.$$
Theorem 1.1. Let \( \alpha \) as in (1.4) be such that every \( \alpha_j \) is an irrational number of finite type not exceeding \( \tau \). Then the estimate
\[
N_{\alpha, m}(x) = \frac{x}{\zeta(k+1)} + O\left(x^{1-\gamma+o(1)}\right) \quad (x \to \infty),
\]
holds with
\[
\gamma := \begin{cases} (3\tau + 2)^{-1} & \text{if } k = 1, \\ \frac{1}{8} \min\{ (m_k \tau)^{-1}, (m_k^2 - m_k)^{-1} \} & \text{if } k \geq 2. \end{cases}
\]
A proof of Theorem 1.1 is given in §3. As alluded to above, for almost all vectors \( \alpha \) (in the sense of Lebesgue measure) one can use \( \tau := 1 \) in applications of Theorem 1.1; thus, for such vectors one can take
\[
\gamma = \frac{1}{8} (m_k^2 - m_k)^{-1}
\]
in all these cases.
We note that although we have optimized the general shape of the dependence on \( m_k \) and \( \tau \), the constant \( \frac{1}{8} \) in the above is certainly not optimal and, with a bit of tedious work, can be improved.

Our next result tests the limits of our approach as we consider abnormally well-approximable vectors \( \alpha \). More precisely, in some cases in which the terms of the sequence \( \alpha \) as in (1.4) are all irrational and of finite exponential type, we still manage to establish the expected asymptotic relation, albeit with a weaker error term.

Theorem 1.2. Let \( \alpha \) as in (1.4) be such that every \( \alpha_j \) is an irrational number of finite exponential type not exceeding \( \tau_* \). Then the estimate
\[
N_{\alpha, m}(x) = \frac{x}{\zeta(k+1)} + O\left(x^{1-\gamma_*+o(1)}\right) \quad (x \to \infty),
\]
holds with
\[
\gamma_* := \begin{cases} \frac{1}{2} \left( \tau_*^{-1} + 1 \right) & \text{if } k = 1, \\ \frac{1 - (m_k^2 - m_k + 1)\tau_*}{(m_k^2 + 2)\tau_*} & \text{if } k \geq 2. \end{cases}
\]
A proof of Theorem 1.2 is given in §4.

Remark 1.3. Examining our proofs, one can immediately notice that without changing anything in the statements of Theorems 1.1 and 1.2, one can replace \( \alpha_j n^{m_j}, j = 2, \ldots, k \), in (1.2) with \( \alpha_j n^{m_j} + g_j(n) \) where \( g_j \in \mathbb{R}[X], \deg g_j < m_j \) (however we still have to keep \( \alpha_1 n \) in (1.2)).
Lemma 2.1. Suppose that \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) has finite type \( \tau \), and that \( \varpi \in (0, \tau^{-1}) \). If \( Q \) is large enough (depending on \( \alpha \) and \( \varpi \)), then there are integers \( a \) and \( q \) such that
\[
\left| \alpha - \frac{a}{q} \right| < \frac{1}{qQ}, \quad \gcd(a, q) = 1, \quad Q^\varpi < q \leq Q. \tag{2.1}
\]

Proof. By Dirichlet’s approximation theorem, there are coprime integers \( a \) and \( q \leq Q \) such that the first inequality of (2.1) holds. Then
\[
\lfloor \alpha q \rfloor \leq |\alpha q - a| < Q^{-1}.
\]
On the other hand, since \( \alpha \) is of type \( \tau < \varpi^{-1} \), we have
\[
q^{1/\varpi} \lfloor \alpha q \rfloor \geq 1
\]
if \( q \) is large enough. Combining these inequalities, the lemma follows. \( \square \)

We also use a similar result for irrational numbers of finite exponential type; the proof is nearly identical to that of Lemma 2.1.

Lemma 2.2. Suppose that \( \alpha \in \mathbb{R} \setminus \mathbb{Q} \) has finite exponential type \( \tau_* \), and that \( \varpi \in (0, \tau_*^{-1} - 1) \). If \( Q \) is sufficiently large (depending on \( \alpha \) and \( \varpi \)), then there are integers \( a \) and \( q \) such that
\[
\left| \alpha - \frac{a}{q} \right| < \frac{1}{qQ}, \quad \gcd(a, q) = 1, \quad (\log Q)^\varpi + 1 < q \leq Q.
\]

2.2. Discrepancy and the Koksma-Szüsz inequality. Let us consider the collection \( S \) consisting of all subsets \( S \) of \( \Omega := [0, 1)^k \) of the form
\[
S = \bigotimes_{1 \leq j \leq k} [a_j, b_j)
\]
with \( 0 \leq a_j < b_j \leq 1 \) for each \( j \). For any given sequence \( v := (v_n)_{n \geq 1} \) of vectors \( v_n \in \Omega \) and a fixed set \( S \in \mathcal{S} \), we denote
\[
A(v, S; N) := \left| \{ n \leq N : v_n \in S \} \right|.
\]
The (extreme) discrepancy is the quantity defined by
\[
\mathcal{D}(v; N) := \sup_{S \in \mathcal{S}} \left| \frac{A(v, S; N)}{N} - m(S) \right| \quad \text{with} \quad m(S) := \prod_{1 \leq j \leq k} (b_j - a_j).
\]
Note that if the vectors \( v_n \) in \( v \) are chosen uniformly at random from \( \Omega \) and independently for each \( n \), then \( m(S) \) (the Lebesgue measure of the subset \( S \subset \Omega \)) represents the proportion of the vectors \( v_n \) expected to lie in \( S \).

One of the basic tools used to study uniformity of distribution is the celebrated Koksma–Szüsz inequality [8, 12] (see also Drmota and Tichy [3, Theorem 1.21]), which links the discrepancy of a sequence of points to certain exponential sums. To formulate the result, let us recall the standard notation.
\[
e(t) := \exp(2\pi it) \quad (t \in \mathbb{R}).
\]
Also, identifying each real sequence $v := (v_j)_{j=1}^k$, with the vector $(v_1, \ldots, v_k) \in \mathbb{R}^k$, we denote the inner product of two such sequences $v$ and $w$ by
\[
\langle v, w \rangle := \sum_{j=1}^k v_j w_j.
\]
The Koksma–Szüsz inequality can be stated as follows.

**Lemma 2.3.** There is an absolute constant $C > 0$ with the following property. For any integer $H > 1$ and any sequence of points $v := (v_n)_{n \geq 1}$ in $\Omega$, we have
\[
\mathcal{D}(v; N) \leq C^k \left( \frac{1}{H} + \frac{1}{N} \sum_{0 < \|h\| \leq H} \frac{1}{r(h)} \left| \sum_{n=1}^N e((h, v_n)) \right| \right),
\]
where
\[
\|h\| := \max_j |h_j|, \quad r(h) := \prod_{j=1}^k \max(|h_j|, 1),
\]
and the sum is taken over all vectors $h = (h_1, \ldots, h_k) \in \mathbb{Z}^k$ with $0 < \|h\| \leq H$.

### 2.3. Bounds on Weyl sums
We use the following result of Shparlinski and Thuswaldner [11, Lemma 3.2].

**Lemma 2.4.** Let $m \geq 2$ be a fixed integer. Suppose that $\alpha \in [0, 1)$ satisfies
\[
\left| \alpha - \frac{a}{q} \right| < \frac{1}{q^2}
\]
with some coprime integers $a$ and $q \geq 1$. Then, for any integer $h \neq 0$, any polynomial $g(X) \in \mathbb{R}[X]$ of degree at most $m - 1$, and any integer $N$, we have
\[
\sum_{n=1}^N e(\alpha n^m + g(n)) \ll N^{1+o(1)} \Delta^{1/(m^2-m)} \quad (N \to \infty),
\]
and
\[
\sum_{n=1}^N e(\alpha n^m + g(n)) \ll N(\log N) \Delta^{1/(m^2-m+2)},
\]
where
\[
\Delta := q^{-1}|h| + N^{-1} + qN^{-m} + \gcd(q, h)N^{-m+1}.
\]

At first glance, the statement of Lemma 2.4 may appear to be different from [11, Lemma 3.2], which is formulated for Weyl sums with polynomials of the form $hf(X)$ with $f$ a real polynomial of degree $m \geq 2$. However, since the bound given in [11] depends only on the leading term of $f$, Lemma 2.4 is actually equivalent, for it corresponds to the choice $f(X) := \alpha X^m + h^{-1}g(X)$. We also remark (as in [11]) that the bound (2.2) has a smaller exponent of $\Delta$ than that of (2.3), hence the first bound is typically stronger. For very small $q$, however, the factor $N^{o(1)}$ can make (2.2) trivial, whereas (2.3) is nontrivial in the same situation.
For the case $m = 2$, we have a more precise statement, which follows from the Weyl differencing method; see [1, Equation (3.5)].

**Lemma 2.5.** For any integer $h \neq 0$, any linear polynomial $g(X) \in \mathbb{R}[X]$, and any integer $N$, we have

$$\left| \sum_{n=1}^{N} e(h\alpha n^2 + g(n)) \right|^2 \ll \sum_{v=1}^{N} \min \left( N, \frac{1}{\lfloor 2hv\alpha \rfloor} \right),$$

where $\lfloor \cdot \rfloor$ is defined by (1.3).

We also need a version of Lemma 2.4 to handle the case $m = 1$, i.e., the case of linear sums.

**Lemma 2.6.** Suppose that $\alpha \in [0, 1)$ satisfies

$$\left| \alpha - \frac{a}{q} \right| < \frac{1}{q^2}$$

with some coprime integers $a$ and $q \geq 1$. Then, for any integer $h \neq 0$, we have

$$\sum_{n=1}^{N} e(h\alpha n) \ll N\Delta,$$

where $\Delta := q^{-1}|h| + qN^{-1}$.

**Proof.** For $|h| \geq \frac{1}{2}q$ the bound is trivial, thus we can assume $|h| < \frac{1}{2}q$.

Using the well known inequality (see, e.g., [6, Equation (8.6)])[1]

$$\left| \sum_{n=1}^{N} e(h\alpha n) \right| \leq \min \left( N, \frac{1}{2\lfloor h\alpha \rfloor} \right).$$

Since $\gcd(a, q) = 1$ and $0 < |h| < \frac{1}{2}q$, the ratio $ah/q$ is a non-integer rational number whose denominator (when it is expressed in reduced form) is at most $q$; therefore, $\lfloor ah/q \rfloor \geq q^{-1}$. Since $|h/q^2| \leq (2q)^{-1}$, we conclude that $\lfloor h\alpha \rfloor \geq (2q)^{-1}$, and the lemma follows. \(\square\)

### 2.4. Bounds on some reciprocal sums.

The following well-known result is used in conjunction with Lemma 2.5; see, e.g., [1, Lemma 3.2].

**Lemma 2.7.** Suppose that $\alpha \in [0, 1)$ satisfies

$$\left| \alpha - \frac{a}{q} \right| < \frac{1}{q^2}$$

with some coprime integers $a$ and $q \geq 1$. For any real integer $K, N \geq 1$, we have

$$\sum_{\nu=1}^{K} \min \left( N, \frac{1}{\nu\alpha} \right) \ll (N + q \log q)(K/q + 1).$$
2.5. Some elementary calculus. We also need the following straightforward statements.

**Lemma 2.8.** For any real numbers \( u > 0 \) and \( v \geq 1 \), the sequence
\[
\left( \frac{u}{v^m - 1} \right)^{1/(m^2 - m)} \quad \text{with} \quad m = 2, 3, 4, \ldots
\]
is nondecreasing if \( u \leq v \).

**Lemma 2.9.** For any real numbers \( u > 0 \) and \( v \geq 1 \), the sequence
\[
\left( \frac{v^m}{u} \right)^{1/(m^2 - m + 2)} \quad \text{with} \quad m = 2, 3, 4, \ldots, M
\]
is nondecreasing if \( v \leq u^{1/M} \).

3. Proof of Theorem 1.1

3.1. Preliminary transformation and plan of the proof. Our approach is based on the following equivalence, which is easily verified:

\[
[t] \equiv 0 \mod d \quad \iff \quad \{t/d\} \in [0, d^{-1}) \quad (t \in \mathbb{R}, \ d \in \mathbb{N}). \quad (3.1)
\]

We begin our estimation of \( N_{\alpha, m}(x) \) by applying a familiar inclusion-exclusion argument, using the Möbius function to detect the coprimality condition (1.5):

\[
N_{\alpha, m}(x) = \sum_{n \leq x} \sum_{d|\gcd(n, \lfloor \alpha_1 n^m_1 \rfloor, \ldots, \lfloor \alpha_k n^m_k \rfloor)} \mu(d) = \sum_{d \leq x} \mu(d) \sum_{n \leq x/d} 1.
\]

Using the criterion (3.1) it follows that

\[
N_{\alpha, m}(x) = \sum_{d \leq x} \mu(d) \cdot \left| \left\{ n \leq x/d : \nu_{d, n} \in \Omega_d \right\} \right|, \quad (3.2)
\]

where \( \Omega_d \) is used to denote the subset \([0, d^{-1})^k\) of \( \mathbb{R}^k \), and \( \nu_d := (\nu_{d, n})_{n \geq 1} \) is the sequence of vectors in \([0, 1)^k\) given by

\[
\nu_{d, n} := \left( \left\{ \alpha_1 d^{m_1 - 1} n^{m_1} \right\}, \ldots, \left\{ \alpha_k d^{m_k - 1} n^{m_k} \right\} \right).
\]

The strength of our estimate for \( N_{\alpha, m}(x) \) via (3.2) depends to a large extent on the Diophantine properties of the sequence \( \alpha := (\alpha_j)_{j=1}^k \).

The plan of the proof is as follows:

1. For a real parameter \( D \in (1, x] \), we split the sum in (3.2) into two sums, one varying over large \( d \) (i.e., \( d > D \)), the other over small \( d \) (i.e., \( d \leq D \)).
2. For the sum over large \( d \), we obtain only an upper bound using the trivial bound

\[
\sum_{n \leq x/d} 1 \leq \sum_{n \leq x/d} \sum_{\left[ \alpha_j d^{m_j} n^{m_j} \right] \equiv 0 \mod d \forall j} 1
\]

(which holds since \( m_1 = 1 \)) along with some ideas from [13].
3. For the sum over small \( d \), we require an asymptotic formula. To derive such a formula, we relate the conditions \( \left[ \alpha_j d^{m_j} n^{m_j} \right] \equiv 0 \mod d \) for all \( j \) to a certain uniformity of distribution problem, where we can then apply modern bounds on Weyl sums.
3.2. Large $d$. First, we consider the “tail” contribution to (3.2) coming from integers $d > D$, where $D$ is a real parameter to be specified later. We follow some ideas of Watson [13].

Since $m_1 = 1$, we have the trivial bound:

$$
\{n \leq x/d : \nu_{d,n} \in \Omega_d\} \leq T(x, d),
$$

where

$$
T(x, d) := \{n \leq x : n \equiv [\alpha_1 n] \equiv 0 \mod d\},
$$

and observe that the congruence $[\alpha_1 dn] \equiv 0 \mod d$ is equivalent to the fact that $[\alpha_1 dn] = dm$ with some integer $m$, hence

$$
\alpha_1 n = m + \{\alpha_1 dn\}/d.
$$

For any fixed $\alpha \in (0, \tau^{-1})$, Lemma 2.1 shows that for all large $Q$ (depending on $\alpha_1$ and $\alpha$) there are integers $a$ and $q$ such that

$$
|\alpha_1 - a/q| < 1/qQ, \quad \gcd(a, q) = 1, \quad Q^\alpha < q \leq Q. \quad (3.3)
$$

It is convenient to assume that $Q \leq x^{O(1)}$. This condition is not restrictive as it holds for our choice of parameters at the optimization stage. Using (3.3) and the fact that $n \leq x/d$, the inequality

$$
|an/q - m| \leq |\alpha_1 n - m| + n|\alpha_1 - a/q| < 1/d + x/dQ \quad (3.4)
$$

holds with some coprime integers $a$ and $q$ such that $Q^\alpha < q \leq Q$.

If both inequalities

$$
d \geq 2q \quad \text{and} \quad d \geq \frac{2x}{Q} \quad (3.5)
$$

hold, then (3.4) implies that $an/q = m \in \mathbb{N}$, hence $q \mid n$. In this case, there are at most $x/(dq)$ such positive integers $n \leq x/d$, and so

$$
T(x, d) \ll \frac{x}{dq} < \frac{x}{dQ^\alpha}. \quad (3.6)
$$

On the other hand, if either inequality in (3.5) fails, then

$$
d \ll q + \frac{x}{Q}.
$$

In this case, (3.4) implies

$$
an = mq + O(q/d + x/(dQ)).
$$

Since $\gcd(a, q) = 1$, it follows that $n$ belongs to one of $O(q/d + x/(dQ))$ distinct residue classes modulo $q$. Since each residue class modulo $q$ contains no more
than \(O \left( x/(dq) + 1 \right) \) positive integers \(n \leq x/d\), we get that
\[
T(x, d) \ll (q/d + x/(dQ)) \left( x/(dq) + 1 \right)
\]
\[
= x/d^2 + x^2/(d^2 qQ) + q/d + x/(dQ)
\]
\[
\ll x/d^2 + x^2/(d^2 Q^{1+\omega}) + Q/d + x/(dQ),
\]
where we have used (3.13) in the last step. This implies the slightly weaker bound
\[
T(x, d) \ll x/d^2 + x^2/(d^2 Q^{1+\omega}) + Q/d + x/(dQ),
\]
which we also use to replace (3.6) in the previous case. Optimizing the choice \(Q\) in (3.7), leads to
\[
T(x, d) \ll x d^{-2} + x^{c_1} d^{-1} + x^{2c_2} d^{-1-c_2},
\]
where
\[
c_1 := (1 + \omega)^{-1} \quad \text{and} \quad c_2 := (2 + \omega)^{-1}.
\]
Summing over \(d > D\), we find that
\[
\sum_{D < d \leq x} \mu(d) \cdot \left| \{ n \leq x/d : \mathbf{v}_{d,n} \in \Omega_d \} \right| \ll \sum_{D < d \leq x} T(x, d) \ll E_0,
\]
where
\[
E_0 := x D^{-1} + x^{c_1} \log x + x^{2c_2} D^{-c_2}.
\]

3.3. Small \(d\). Next, we consider the contribution to (3.2) from integers \(d \leq D\). Using the definitions of \(\Omega_d\) and the fact that \(m(\Omega_d) = d^{-k}\), we have
\[
\left| \{ n \leq x/d : \mathbf{v}_{d,n} \in \Omega_d \} \right| = A(\mathbf{v}_d, \Omega_d; x/d) = x d^{-k-1} + O(xd^{-1} \mathcal{D}_d),
\]
where \(\mathcal{D}_d\) is shorthand for the discrepancy \(\mathcal{D}(\mathbf{v}_d; x/d)\). By Lemma 2.3, for any positive real parameter \(H \leq x\) we have
\[
\mathcal{D}_d \ll \frac{1}{H} + \frac{d}{x} \sum_{0 < \| h \| \leq H} \frac{1}{r(h)} \left| \sum_{n \leq x/d} e(h_k d^{m_k-1} \alpha_k n^{m_k} + \cdots + h_d d^{m_d-1} \alpha_1 n^{m_1}) \right|,
\]
where the outer sum runs over all \(h = (h_1, \ldots, h_k) \in \mathbb{Z}^k\) with \(0 < \| h \| \leq H\). For each \(j = 1, \ldots, k\), let \(\mathcal{H}_j\) be the set of such vectors \(h = (h_1, \ldots, h_k)\) with \(h_j \neq 0\) and \(h_{j+1} = \cdots = h_k = 0\). Then
\[
\mathcal{D}_d \ll \frac{1}{H} + \sum_{j=1}^k \mathcal{D}_{d,j},
\]
where
\[
\mathcal{D}_{d,j} := \left( x/d \right)^{-1} \sum_{h \in \mathcal{H}_j} \frac{1}{r(h)} |S_j(d, h)|
\]
and
\[
S_j(d, h) := \sum_{n \leq x/d} e(h_j d^{m_j-1} \alpha_j n^{m_j} + \cdots + h_d d^{m_d-1} \alpha_1 n^{m_1}).
\]
As in §3.2 we fix \( \varpi \in (0, \tau^{-1}) \). For each \( j = 1, \ldots, k \), Lemma 2.1 shows that for all large \( Q \) (depending on \( \alpha_j \) and \( \varpi \)) there are integers \( a \) and \( q \) such that

\[
|\alpha_j - \frac{a}{q}| < \frac{1}{qQ}, \quad \gcd(a, q) = 1, \quad Q^\varpi < q \leq Q. \tag{3.13}
\]

As before, we assume that \( Q \leq x^{\mathcal{O}(1)} \).

We now turn to the problem of bounding \( D_{d,j} \) for any given \( j \). Because the different bounds on Weyl sums given in §2.3 vary in strength, we examine several cases according to whether \( m_j = 1, m_j = 2, \) or \( m_j \geq 3 \).

**Lemma 3.1.** With the notation as above, we have for each \( j \):

\[
D_{d,j} \ll \begin{cases} 
  x^{c_1-1}d^{1-c_1}H^{c_1}(\log H)^{1-c_1} & \text{if } m_j = 1, \\
  x^{c_1-1}d^{1-c_1/2} + x^{-1/2}d & \text{if } m_j = 2, \\
  x^{o(1)}\left( \frac{H^{c_1}d^{(m_j-1)\alpha_1} \lambda_j}{(x^{(m_j-1)(1-\alpha_1)})} + \left( \frac{d}{x} \right)^{\lambda_j} \right) & \text{if } m_j \geq 3,
\end{cases}
\]

where

\[
c_1 := (1 + \varpi)^{-1} \quad \text{and} \quad \lambda_j := (m_j^2 - m_j)^{-1}. \tag{3.14}
\]

**Proof.** First, suppose that \( m_j = 1 \). For each vector \( \mathbf{h} = (h_1, 0, \ldots, 0) \in \mathcal{H}_1 \) we apply Lemma 2.6 with

\[
N := \lfloor x/d \rfloor \quad \text{and} \quad h := h_1,
\]

deriving the bound

\[
S_1(d, \mathbf{h}) = \sum_{n \leq x/d} e(h_1\alpha_1 n) \ll \frac{x|h_1|}{dq} + q < \frac{x|h_1|}{dQ^\varpi} + Q,
\]

where we used (3.13) in the second step. By (3.12) we have

\[
\mathcal{D}_{d,1} = \frac{d}{x} \sum_{\mathbf{h} \in \mathcal{H}_1} \frac{1}{|h_1|} \left( \frac{x|h_1|}{dQ^\varpi} + Q \right) \ll \frac{H}{Q^\varpi} + \frac{dQ \log H}{x}.
\]

Taking

\[
Q := \left( \frac{xH}{d \log H} \right)^{c_1},
\]

we obtain the desired bound for \( \mathcal{D}_{d,1} \).

Next, suppose that \( m_j = 2 \). For any vector \( \mathbf{h} = (h_1, h_2, 0, \ldots, 0) \in \mathcal{H}_2 \) we apply Lemma 2.5 with \( N := \lfloor x/d \rfloor \) and \( h := h_2d \), deriving the bound

\[
S_2(d, \mathbf{h}) \ll \left( \sum_{v=1}^{N} \min \left( N, \frac{1}{\lfloor 2h_2dv \varpi_2 \rfloor} \right) \right)^{1/2},
\]
hence by (3.12) (and symmetry) we have

\[ \mathcal{D}_{d, 2} \ll \frac{d}{x} \sum_{h_1 \in H} \sum_{0 < h_2 \leq H} \frac{1}{\max(h_1, 1)} \left( \sum_{v=1}^{N} \min \left( N, \frac{1}{[2h_2d\alpha_2]} \right) \right)^{1/2} \]

\[ \ll \frac{d \log H}{x} \sum_{0 < h \leq H} \frac{1}{h} \left( \sum_{v=1}^{N} \min \left( N, \frac{1}{[2hd\alpha_2]} \right) \right)^{1/2}. \]

Splitting the summation range over \( h \) into \( O(\log H) \) dyadic intervals of the form \( R < h \leq 2R \) with \( \frac{1}{2} \leq R \ll H \), it suffices to bound each term

\[ \Sigma_R := \frac{d \log H}{xR} \sum_{R < h \leq 2R} \left( \sum_{v=1}^{N} \min \left( N, \frac{1}{[2hd\alpha_2]} \right) \right)^{1/2}. \]  

(3.15)

By the Cauchy inequality,

\[ \sum_{R < h \leq 2R} \left( \sum_{v=1}^{N} \min \left( N, \frac{1}{[2hd\alpha_2]} \right) \right)^{1/2} \leq R^{1/2} \Xi^R_{1/2}, \]  

(3.16)

where

\[ \Xi_R := \sum_{R < h \leq 2R} \sum_{v=1}^{N} \min \left( N, \frac{1}{[2hd\alpha_2]} \right). \]

Collecting together products \( 2hdv \) with the same value \( \nu := 2hdv \), and using a well-known bound on the divisor function (see, e.g., [6, Equation (1.81)]), we have

\[ \Xi_R \leq (dNR)^{o(1)} \sum_{1 \leq \nu \leq 4dNR} \min \left( N, \frac{1}{[\nu\alpha_2]} \right). \]

Finally, using Lemma 2.7 and (3.13), we conclude that

\[ \Xi_R \leq x^{o(1)} (N + q \log q) (dNR/q + 1) \]

\[ \leq x^{o(1)} (dN^2R/q + dNR + q) \]

\[ \leq x^{o(1)} \left( \frac{dN^2R}{Q^\omega} + dNR + Q \right), \]

which together with (3.15) and (3.16) implies

\[ \Sigma_R \leq x^{-1+o(1)} d \left( \frac{dN^2}{Q^\omega} + dN + QR^{-1} \right)^{1/2} \leq x^{-1+o(1)} d \left( \frac{x^2}{dQ^\omega} + x + Q \right)^{1/2}. \]

The bound is optimized with the choice \( Q := x^{2c_1}d^{c_1} \), and we get that

\[ \Sigma_R \leq x^{o(1)} \left( x^{c_1-1} d^{1-c_1/2} + x^{-1/2} d \right). \]

Summing over all possibilities for \( R \), we finish the proof in this case.

Finally, suppose that \( m_j \geq 3 \). For each vector \( \mathbf{h} \in \mathcal{H}_j \) we use the bound (2.2) from Lemma 2.4 with \( N := \lfloor x/d \rfloor \) and \( h := h_jd^{m_j-1} \); taking into account (3.13), we find that the bound

\[ S_j(d, \mathbf{h}) \ll N^{1+o(1)} \left( \frac{|h_jd^{m_j-1}|}{Q^\omega} + \frac{1}{N} + \frac{Q}{N^{m_j-1}} \right)^{\lambda_j}. \]
holds with \( \lambda_j := (m_j^2 - m_j)^{-1} \) as in (3.14). To optimize the bound, we choose 
\( Q := (h_j d^{m_j-1} N^{m_j-1})^{c_1} \), which leads to
\[
S_j(d, h) \ll N^{1+o(1)} \left( \frac{|h_j| c_1 d^{(m_j-1)c_1}}{N^{(m_j-1)(1-c_1)}} + \frac{1}{N} \right)^{\lambda_j}.
\]
Recalling (3.12) and noting that (with any fixed \( C > 0 \))
\[
\sum_{h \in H_j} \frac{1}{r(h)} \ll (\log H) \quad \text{and} \quad \sum_{h \in H_j} \frac{|h_j|^C}{r(h)} \ll H^C (\log H)^{j-1}, \tag{3.17}
\]
we derive the bound
\[
\mathcal{D}_{d,j} \leq x^{o(1)} \left( \frac{H^{c_1} d^{(m_j-1)c_1}}{N^{(m_j-1)(1-c_1)}} \right)^{\lambda_j} + \left( \frac{d}{x} \right)^{\lambda_j}.
\]
which concludes the proof.

We are now in a position to bound \( \mathcal{D}_d \) and to estimate the overall contribution to (3.2) from integers \( d \leq D \). We consider the cases \( k = 1 \) and \( k \geq 2 \) separately.

**Case 1:** \( (k = 1) \). In this case, \( m_k = 1 \). By (3.11) and Lemma 3.1 we have
\[
\mathcal{D}_d \ll H^{-1} + x^{c_1-1} d^{1-c_1} H^{c_1} (\log H)^{1-c_1}.
\]
The bound is optimized with the choice \( H := (x/(d \log x))^{1-2c_2} \), where
\[
c_2 := (2 + \omega)^{-1},
\]
and with this choice, we get that
\[
\mathcal{D}_d \ll \left( \frac{d \log x}{x} \right)^{1-2c_2}.
\]
Using this result in (3.10) and summing over all \( d \leq D \), it follows that
\[
\sum_{d \leq D} \mu(d) \cdot |\{ n \leq x/d : v_{d,n} \in \Omega_d \}| = \frac{x}{\zeta(2)} + O(E_1), \tag{3.18}
\]
where
\[
E_1 := xD^{-1} + x^{2c_2} (\log x)^{1-2c_2} D^{1-2c_2} \tag{3.19}
\]

**Case 2:** \( (k \geq 2) \). By (3.11), and putting together all available estimates from Lemma 3.1, we obtain the bound
\[
\mathcal{D}_d \ll \left( H^{-1} + x^{c_1-1} d^{1-c_1} H^{c_1} + x^{c_1-1} d^{1-c_1/2} + x^{-1/2} d + \sum_{j=2}^{k} \mathcal{D}_{d,j} \right) x^{o(1)},
\]
where for \( 2 \leq j \leq k \) we have
\[
\mathcal{D}_{d,j} \leq x^{o(1)} \left( \left( \frac{H^{c_1} d^{(m_j-1)c_1}}{x^{(m_j-1)(1-c_1)}} \right)^{\lambda_j} + \left( \frac{d}{x} \right)^{\lambda_j} \right).
\]
Clearly, the term \((d/x)^{\lambda_j}\) increases with the parameter \(j\). On the other hand, applying Lemma 2.8 with \(u := H^c\) and \(v := x^{1-c_1}d^{-c_1}\), we see that the first term does not decrease with \(j\) provided that \(u \leq v\), or equivalently, when
\[
H \leq x^{(1-c_1)/c_1}d^{-1} = x^{\varpi d^{-1}}.
\]
Assuming this condition is met, we derive the bound
\[
\mathcal{D}_d \ll \left( \frac{1}{H} + \frac{d^{1-c_1}H^{c_1}}{x^{1-c_1}} + \frac{d^{1-c_1/2}}{x^{1-c_1}} + \frac{d}{x^{1/2}} + \frac{H^{c_1}d^{c_1}}{x^{c_1}} + \frac{d^{\lambda}}{x^{\lambda}} \right) x^{o(1)},
\]
where
\[
m := m_k, \quad \lambda := (m^2 - m)^{-1}, \tag{3.20}
\]
and
\[
c_3 := \lambda c_1, \quad c_4 := \lambda(m - 1)c_1, \quad c_5 := \lambda(m - 1)(1 - c_1). \tag{3.21}
\]

It is now easy to see that there exists a choice of the parameter \(H \in [0, x^{\varpi d^{-1}}]\), for which
\[
\frac{1}{H} + \frac{d^{1-c_1}H^{c_1}}{x^{1-c_1}} + \frac{H^{c_1}d^{c_1}}{x^{c_1}} \ll \frac{d}{x^{\varpi}} + \frac{d^{(1-c_1)/(c_1+1)}}{x^{(1-c_1)/(c_1+1)}} + \frac{d^{c_4/(c_4+1)}}{x^{c_4/(c_4+1)}}.
\]
Hence
\[
\mathcal{D}_d \ll \left( \frac{d}{x^{\varpi}} + \frac{d^{\lambda}}{x^{\lambda}} + \frac{d}{x^{1/2}} + \frac{d^{1-c_1/2}}{x^{1-c_1}} + \frac{d^{(1-c_1)/(c_1+1)}}{x^{(1-c_1)/(c_1+1)}} + \frac{d^{c_4/(c_4+1)}}{x^{c_4/(c_4+1)}} \right) x^{o(1)}.
\]

Using this result in (3.10) and summing over all \(d \leq D\), we have
\[
\sum_{d \leq D} \mu(d) \cdot \left\{ n \leq x/d : (n, d, n) \in \Omega_d \right\} = \frac{x}{\zeta(k+1)} + O(E_2), \tag{3.22}
\]
where
\[
E_2 := \frac{x}{D^k} + x^{1+o(1)} \left( \frac{D}{x^{\varpi}} + \frac{D^{\lambda}}{x^{\lambda}} + \frac{D}{x^{1/2}} + \frac{D^{1-c_1/2}}{x^{1-c_1}} + \frac{D^{(1-c_1)/(c_1+1)}}{x^{(1-c_1)/(c_1+1)}} + \frac{D^{c_4/(c_4+1)}}{x^{c_4/(c_4+1)}} \right). \tag{3.23}
\]

3.4. Final optimizations. When \(m_k = 1\), that is, in Case 1, we combine (3.8), (3.9), (3.18), and (3.19), obtaining an overall error
\[
E := E_0 + E_1 \ll x^{o(1)} \left( xD^{-1} + x^{c_1} + x^{2c_2}D^{1-c_2} \right).
\]

Recalling the definitions of \(c_1\) and \(c_2\), and taking \(D := x^\varpi/(3+2\varpi)\), one has
\[
E \ll x^{o(1)} \left( x^{(3+\varpi)/(3+2\varpi)} + x^{1/(1+\varpi)} \right) \quad (x \to \infty).
\]

Since
\[
\frac{3+\varpi}{3+2\varpi} > \frac{1}{1+\varpi} \quad (0 < \varpi < 1),
\]
the second term can be dropped, and thus
\[
E \ll x^{(3+\varpi)/(3+2\varpi)+o(1)} \quad (x \to \infty).
\]

Letting \(\varpi\) approach \(\tau^{-1}\), Theorem 1.1 follows in this case.
When \( m_k \geq 2 \), that is, in Case 2, we use (3.8), (3.9), (3.22), and (3.23), observing also that the term \( x^{c_1} \log x \) from (3.9) can be discarded since it is dominated by the term \( D^{1-c_1/2}x^{c_1} \) in (3.23). Hence, the overall error becomes

\[
E := E_0 + E_2 \ll x^{1+o(1)} \left( \frac{1}{D} + \frac{1}{x^{1-2c_2}D^{c_2}} + \frac{D^\lambda}{x^{\lambda}} + \frac{D}{x^{1/2}} \right. \\
\left. + \frac{D^{1-c_1/2}}{x^{1-c_1}} + \frac{D^{(1-c_1)/(c_1+1)}}{x^{(1-c_1)/(c_1+1)}} + \frac{D^{c_4/(c_3+1)}}{x^{c_5/(c_3+1)}} \right). 
\]

(3.24)

We now choose

\[
D := x^{\varpi/8}. 
\]

Since

\[
D = x^{\varpi/8} < x^{1/8} \quad \text{and} \quad 1 - c_1 \in \left( \frac{1}{2}, \varpi \right),
\]

we have the bound

\[
\frac{1}{D} + \frac{D}{x^{\varpi}} + \frac{D^\lambda}{x^{\lambda}} + \frac{D}{x^{1/2}} \ll x^{-\varpi/8} + x^{-7\varpi/8} + x^{-7\lambda/8} + x^{-3/8},
\]

which we rewrite crudely in the form

\[
\frac{1}{D} + \frac{D}{x^{\varpi}} + \frac{D^\lambda}{x^{\lambda}} + \frac{D}{x^{1/2}} \ll x^{-\frac{1}{8}\min\{\varpi, \lambda\}}. 
\]

(3.25)

Next, using

\[
1 - 2c_2 = 1 - \frac{2}{2 + \varpi} = \frac{\varpi}{2 + \varpi} \geq \frac{\varpi}{3},
\]

we estimate

\[
\frac{1}{x^{1-2c_2}D^{c_2}} \ll x^{-\varpi/3}.
\]

Similarly, we have

\[
\frac{D^{1-c_1/2}}{x^{1-c_1}} \ll \frac{D}{x^{1/2}} \ll x^{-3\varpi/8}
\]

and

\[
\frac{D^{1/(c_1+1)}}{x^{(1-c_1)/(c_1+1)}} \leq \left( \frac{D}{x^{1-c_1}} \right)^{1/2} \ll x^{-3\varpi/16}.
\]

Finally, recalling (3.20) and (3.21), we see that for \( m \geq 2 \)

\[
c_3 < 1, \quad c_4 \leq \lambda m = \frac{1}{m - 1} \leq \frac{2}{m}, \quad c_5 = \frac{1 - c_1}{m} = \frac{\varpi}{m(1 + \varpi)} \geq \frac{\varpi}{2m},
\]

and therefore

\[
\frac{D^{c_4/(c_3+1)}}{x^{c_5/(c_3+1)}} \ll \left( \frac{D^{c_4}}{x^{c_5}} \right)^{1/2} \ll \left( \frac{D^{2/m}}{x^{\varpi/(2m)}} \right)^{1/2} = x^{-\varpi/(8m)}.
\]

(3.26)

Collecting the bounds (3.25)–(3.26) into (3.24), we find that

\[
E \ll x^{1-\frac{1}{8}\min\{\varpi/m, \lambda\} + o(1)} \quad (x \to \infty).
\]

Letting \( \varpi \) approach \( \tau^{-1} \), Theorem 1.1 follows in this case, and we are done.
4. Proof of Theorem 1.2

4.1. Plan of the proof. We follow a plan similar to the one outlined in §3.1. We proceed as in the proof of Theorem 1.1, however we now use Lemma 2.2 instead of Lemma 2.1 and the bound (2.3) instead of (2.2). We continue to use the notation introduced earlier, and since our arguments are essentially the same, we focus only on the needed adjustments.

Note that we can assume \( \tau_* < (m_k^2 - m_k + 1)^{-1} \) since the statement of the theorem is trivial otherwise.

4.2. Large \( d \). Let \( \varpi \in (0, \tau_*^{-1} - 1) \) be fixed in what follows. As before, we start by considering the contribution to (3.2) coming from integers \( d > D \), where \( D \) is a real parameter to be specified below.

Lemma 2.2 shows that for all large \( Q \) (depending on \( \alpha \) and \( \varpi \)) there are integers \( a \) and \( q \) such that
\[
|\alpha_j - a/q| < \frac{1}{qQ}, \quad \gcd(a, q) = 1, \quad (\log Q)^{\varpi + 1} < q \leq Q.
\]

Using this result in place of (3.13), the argument of §3.2 yields the bound
\[
T(x, d) \ll x/d^2 + x^2/(d^2 Q(\log Q)^{\varpi + 1}) + Q/d + x/(d(\log Q)^{\varpi + 1})
\]
instead of (3.7). Taking \( Q := x/(\log x)^{\varpi + 1} \), it follows that
\[
T(x, d) \ll x/d^2 + x/(d(\log x)^{\varpi + 1}).
\]

and therefore
\[
\sum_{D < d \leq x} \mu(d) \cdot |\{n \leq x/d : \nu_{d,n} \in \Omega_d\}| \ll E_0 := xD^\lambda \ll xD^{-1} + x(\log x)^{-\varpi}.
\]

Below, we use this bound in place of (3.8) and (3.9).

4.3. Small \( d \). Next, we consider the contribution to (3.2) from integers \( d \leq D \).

As before, Lemma 2.2 shows that for all large \( Q \) (depending only on \( \alpha \) and \( \varpi \)) and every \( j = 1, \ldots, k \), there are integers \( a \) and \( q \) such that
\[
|\alpha_j - a/q| < \frac{1}{qQ}, \quad \gcd(a, q) = 1, \quad (\log Q)^{\varpi + 1} < q \leq Q.
\]

**Lemma 4.1.** In the notation of §3.3, we have for each \( j \):
\[
G_{d,j} \ll \begin{cases} H(\log x)^{-\varpi - 1} & \text{if } m_j = 1, \\ H\lambda_j (\log H)^{j-1} d^{(m_j-1)\lambda_j} (\log x)^{1-(\varpi + 1)\lambda_j} & \text{if } m_j \geq 2, \end{cases}
\]

where
\[
\lambda_j := (m_j^2 - m_j + 2)^{-1}.
\]

**Proof.** First, suppose that \( m_j = 1 \). For each vector \( h = (h_1, 0, \ldots, 0) \in \mathcal{H}_1 \) we apply Lemma 2.6 with \( N := \lfloor x/d \rfloor \) and \( h := h_1 \), deriving the bound
\[
S_1(d, h) = \sum_{n \leq x/d} e(h_1|\alpha_1 n|) \ll \frac{x|h_1|}{d} + q < \frac{x|h_1|}{d(\log Q)^{\varpi + 1}} + Q,
\]
where we have used (4.2) in the second step. By (3.12) we have
\[ D_{d,1} = \frac{d}{x} \sum_{h \in H_i} \frac{1}{|h_1|} \left( \frac{x|h_1|}{(d)(\log Q)^{\omega+1}} + Q \right) \ll \frac{H}{(\log Q)^{\omega+1}} + \frac{dQ \log H}{x}. \]

Taking \( Q := xH/(d(\log x)^{\omega+2}) \) we obtain the bound for \( D_{d,1} \) stated in the lemma.

Next, suppose that \( m_j \geq 2 \). For each vector \( h \in H_j \) we use the bound (2.3) of Lemma 2.4 with \( N := \lfloor x/d \rfloor \) and \( h := h_jd^{m_j-1} \), taking into account (4.2) and using the crude inequality \( \gcd(q,h) \leq Q \), we find that the bound
\[ S_j(d,h) \ll \frac{x \log x}{d} \left( \frac{|h_j|d^{m_j-1}}{(\log x)^{\omega+1}} + \frac{dQ \log H}{xH} \right)^{\lambda_j} \]
holds with \( \lambda_j \) as in (4.3). To optimize, we choose
\[ Q := \frac{|h_j|x^{m_j-1}}{(\log x)^{\omega+1}}, \]
which leads to the bound
\[ S_j(d,h) \ll \frac{x \log x}{d} \left( \frac{|h_j|d^{m_j-1}}{(\log x)^{\omega+1}} + \frac{d}{x} \right)^{\lambda_j} \ll \frac{x \log x}{d} \left( \frac{|h_j|d^{m_j-1}}{(\log x)^{\omega+1}} \right)^{\lambda_j} = \frac{x|h_j|^{\lambda_j}d^{(m_j-1)\lambda_j-1}}{(\log x)^{\omega+1}\lambda_j-1}. \]

Using (3.12) and (3.17), we derive the bound for \( D_{d,j} \) stated in the lemma. \( \square \)

We now bound \( D_d \) and estimate the overall contribution to (3.2) coming from integers \( d \leq D \), considering separately the cases \( k = 1 \) and \( k \geq 2 \).

**Case 1:** \((k = 1)\). In this case \( m_j = m_1 = 1 \). By (3.11) and Lemma 4.1 we have
\[ D_d \ll H^{-1} + H(\log x)^{-\omega-1}. \]
The bound is optimized with the choice \( H := (\log x)^{(\omega+1)/2} \), which gives
\[ D_d \ll (\log x)^{-(\omega+1)/2}. \]
Using this result in (3.10) and summing over all \( d \leq D \), it follows that
\[ \sum_{d \leq D} \mu(d) \cdot \left\{ n \leq x/d : \nu_{d,n} \in \Omega_d \right\} = \frac{x}{\zeta(2)} + O(E_1), \quad (4.4) \]
where
\[ E_1 := xD^{-1} + x(\log x)^{-(\omega+1)/2} \log D. \quad (4.5) \]

**Case 2:** \((k \geq 2)\). By (3.11) and Lemma 4.1 we have
\[ D_d \ll \frac{1}{H} + \frac{H}{(\log x)^{\omega+1}} + (\log x)(\log H)^{k-1} \sum_{j=2}^{k} \left( \frac{Hd^{m_j-1}}{(\log x)^{\omega+1}} \right)^{\lambda_j}, \]
where \( \lambda_j := (m_j^2 - m_j + 2)^{-1} \). Using Lemma 2.9, we see that the terms in the above sum are nondecreasing as \( j \) increases provided that
\[ d^{m_k-1} \leq \frac{(\log x)^{\omega+1}}{H}. \quad (4.6) \]
Assuming this for the moment, we have
\[ \mathcal{D}_d \ll \frac{1}{H} + \frac{H}{(\log x)^{\omega+1}} + \frac{H^\lambda (\log H)^{k-1} d^{(m-1)\lambda}}{(\log x)^{(\omega+1)\lambda-1}}, \] (4.7)
where
\[ m := m_k \quad \text{and} \quad \lambda := (m^2 - m + 2)^{-1}. \]
Clearly, (4.6) implies \( H \leq (\log x)^{\omega+1} \), so we can drop the second term in the bound (4.7) since it is always dominated by the third term. We set
\[ H := \left( \frac{(\log x)^{\omega+1} \lambda - 1}{d^{(m-1)\lambda}} \right)^{1/(\lambda+1)} \]
to balance the two remaining terms in (4.7). Note that
\[ d^{(m-1)\lambda} H^\lambda = \frac{(\log x)^{\omega+1} \lambda}{H \log x} \leq (\log x)^{\omega+1} \lambda, \]
and therefore the condition (4.6) is met. Putting everything together, we get that
\[ \mathcal{D}_d \ll (\log x)^{o(1)} \left( \frac{d^{(m-1)\lambda}}{(\log x)^{(\omega+1)\lambda-1}} \right)^{1/(\lambda+1)} \quad (x \to \infty). \]
Inserting this bound into (3.10) and summing over \( d \leq D \), we find that
\[ \sum_{d \leq D} \mu(d) \cdot |\{ n \leq x/d : \nu_{d,n} \in \Omega_d \}| = \frac{x}{\zeta(k+1)} + O(E_2), \] (4.8)
where
\[ E_2 = xD^{-k} + x(\log x)^{o(1)} \left( \frac{D^{(m-1)\lambda}}{(\log x)^{(\omega+1)\lambda-1}} \right)^{1/(\lambda+1)} \quad (x \to \infty). \] (4.9)

4.4. Final optimizations. In Case 1, we combine (4.1), (4.4), and (4.5), which yields an overall error
\[ E := E_0 + E_1 \ll xD^{-1} + x(\log x)^{-\omega} + x(\log x)^{-(\omega+1)/2} \log D \]
Choosing \( D := (\log x)^{\vartheta} \) in this case, we have
\[ E \ll x(\log x)^{-\omega} + x(\log x)^{-(\omega+1)/2 + o(1)} \quad (x \to \infty). \]
Letting \( \omega \) approach \( \tau_*^{-1} \), we finish the proof of Theorem 1.2 in this case.
In Case 2, we combine (4.1), (4.8), and (4.9), which yields an overall error
\[ E := E_0 + E_2 \ll xD^{-1} + x(\log x)^{-\omega} + x(\log x)^{o(1)} \left( \frac{D^{(m-1)\lambda}}{(\log x)^{(\omega+1)\lambda-1}} \right)^{1/(\lambda+1)} \]
as \( x \to \infty \). We balance this bound by taking
\[ D := (\log x)^{\vartheta}, \quad \vartheta := \frac{\omega + 1}{m\lambda}, \]
and with this choice, we can drop the middle term \( x(\log x)^{-\omega} \) since \( \vartheta < \omega \).
Letting \( \omega \) approach \( \tau_*^{-1} \), we finish the proof of Theorem 1.2.
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