A functional-type a posteriori error estimate for a linear Koiter’s shell

O Chistiakova
Department of Applied Mathematics, Peter the Great St. Petersburg Polytechnic University, 195251, Polytechnicheskaya st. 29, St. Petersburg, Russia
E-mail: chistiakova.olga@gmail.com

Abstract. In this paper a functional approach of a posteriori error estimation is applied to a shell deformation problem. A linear Koiter’s shell model is considered and a new a posteriori error estimate for this problem is derived. Presented estimate can be used for any conforming approximate solution of the problem including solutions computed by commercial solvers with a closed source code.

1. Introduction
A posteriori error estimation has been an area of interest both for scientists and engineers for more than three decades as there is a need for a tool for a reliable estimation of the accuracy of a given approximate solution while exact solution of most practically valuable problems is unknown. Moreover, a posteriori error estimation is used as a basis of adaptive mesh refinement algorithms. Research in this domain is active and there are several approaches that attempt to solve this non-trivial problem.

Classical approaches such as residual-based or postprocessing-based are perhaps most known from the literature [1]–[3] and already appeared in some commercial and open-source engineering software packages. However these methods can be reliably used only for Galerkin approximations. On the contrary, a functional approach [4]–[7] seems to be more complicated as it is based on functional analysis and variational calculus, but is able to control accuracy for all conforming approximations with a theoretically proven reliability.

Functional approach has already been applied to the problems of beam and plates bending [8] - [12] and even for some problems in Cosserat elasticity [13] - [14]. A posteriori estimates were derived and implemented. It was shown that theoretical properties of these estimates are justified by numerical experiments applied to the problems of beam and plates bending. In this paper we apply this approach to one of the widely used classical shell models.

2. Linear Koiter’s shell model
There are quite a few shell models that are widely used in mechanical engineering. Generally they can be classified into two types: classical models in which the rotation of particles of the shell is only related to the deformation of its middle surface and models, which take into account shear deformations.

One of the well-known classical shell models is a two-dimensional linear Koiter’s shell. This model is based on the following assumptions [15]:
• Stress inside the shell is planar
• Stresses parallel to middle surface may vary linearly across the thickness
• Every point on the normal to the middle surface remains after deformation on the normal to the deformed middle surface (part of the well-known Kirchhoff-Love assumptions)

We describe a model of the shell following a traditional notation which can be found, for example, in [15]. Let \( w \) be a two-dimensional open bounded set with a Lipschitz-continious boundary \( \gamma \). Let \((x^1, x^2)\) denote a point in \( \bar{w} \), \( \partial_a = \frac{\partial}{\partial x^a} \); \( \varphi(\bar{w}) \) describes the middle surface of the shell as an image of mapping \( \varphi : \bar{w} \to \mathbb{R}^3 \); \( \varphi \) is injective and two vectors \( a_\alpha \) are linearly independent at each point \((x^1, x^2)\) \( \in \bar{w} \).

\[
a_\alpha = \partial_\alpha \varphi = \partial_\alpha (\varphi^i e_i), \quad a^\alpha \cdot a_\beta = \delta^\alpha_\beta
\]

\[
a_3 = a^3 = \frac{a_1 \times a_2}{|a_1 \times a_2|}
\]

First fundamental form (metric tensor):

\[
a_{\alpha\beta} = a_{\beta\alpha} = a_\alpha \cdot a_\beta = \partial_\alpha \varphi^i \cdot \partial_\beta \varphi^i
\]

Second fundamental form (curvature tensor):

\[
b_{\alpha\beta} = a_3 \cdot \partial_\alpha a_\beta = -a_\alpha \cdot \partial_\alpha a_\beta
\]

Third fundamental form:

\[
c_{\alpha\beta} = c_{\beta\alpha} = b^\rho_\alpha b_\rho_\beta = b_{\alpha\rho} b_{\beta\rho}
\]

In addition, Cristoffel symbols of the middle surface of the shell are denoted as \( \Gamma^\rho_{\alpha\beta} \):

\[
\Gamma^\rho_{\alpha\beta} = \Gamma^\rho_{\beta\alpha} = a^\gamma \cdot \partial_\beta a_\alpha
\]

Finally, we denote the thickness of the shell as \( 2t > 0 \), middle surface of the shell as \( S = \varphi(\bar{w}) \) and Lame constants as \( \lambda > 0 \) and \( \mu > 0 \).

\[
dS = \sqrt{a} dx^1 dx^2, \quad \text{where} \quad a = det(a_{\alpha\beta}) = det(a_\alpha \cdot a_\beta) = det(\partial_\alpha \varphi^i \cdot \partial_\beta \varphi^i)
\]

Thus, a shell is defined as an elastic body which configuration before deformation is a closure of a following set:

\[
\Omega = \{(\varphi(x^1, x^2) + x^3 a_3(x^1, x^2)) \in \mathbb{R}^3; (x^1, x^2) \in w, |x^3| < t\}
\]

Here we assume the shell to be hard clamped on a boundary part \( \gamma_0 \subset \gamma \) while the other part \( \gamma_1 = \gamma \setminus \gamma_0 \) is left free.

\[
\Gamma_0 = \{(\varphi(x^1, x^2) + x^3 a_3(x^1, x^2)) \in \mathbb{R}^3; (x^1, x^2) \in \gamma_0, |x^3| < t\}
\]

where \( \gamma_0 \) denotes a measurable subset of boundary of \( w \) and may be also subjected to surface forces along the remaining part of its lateral surface:

\[
\Gamma_1 = \{(\varphi(x^1, x^2) + x^3 a_3(x^1, x^2)) \in \mathbb{R}^3; (x^1, x^2) \in \gamma_1, |x^3| < t\}
\]

Given all the notation above we can now give a variational formulation of the linear Koiter’s shell bending problem: find three covariant components \( \varsigma_i : \bar{w} \to \mathbb{R} \) of the displacement \( \varsigma_i a^i \) of the points of the middle surface \( S = \varphi(\bar{w}) \) \( \varsigma_i : \bar{w} \to \mathbb{R} \) so that they would satisfy the following:
\[ \varsigma \in V(w) = \{ \eta = ((\eta_\alpha), \eta_3) \in H^1(w) \times H^2(w) ; \eta_\iota = \partial_\iota \eta_3 = 0 \text{ on } \gamma_0 \} \]

and

\[ B(\varsigma, \eta) = L(\eta) \quad \forall \eta \in V(w), \]

where

\[ B(\varsigma, \eta) = \int_w \left\{ t A\gamma_{\rho\sigma}(\varsigma) \gamma_{\alpha\beta}(\eta) + \frac{\ell^3}{3} A\Upsilon_{\rho\sigma}(\varsigma) \Upsilon_{\alpha\beta}(\eta) \right\} \sqrt{\text{ad}w,} \]

\[ A = a^{\alpha\beta\rho\sigma} = \frac{4\lambda\mu}{\lambda + 2\mu} a^{\alpha\beta} a^{\rho\sigma} + 2\mu (a^{\alpha\rho} a^{\beta\sigma} + a^{\alpha\sigma} a^{\beta\rho}), \]

\[ L(\eta) = \int_w p \cdot \eta \sqrt{\text{ad}w} + \int_{\gamma_1} q \cdot \eta d\gamma + \int_{\gamma_1} m^\alpha (\partial_\alpha \eta_3 + b^{\alpha}_\sigma \eta_\sigma) d\gamma \]

\[ L \text{ is a linear form that describes body forces and surface forces applied to the shell in its interior, upper and lower faces; vector fields } p : w \to \mathbb{R}^3, q : \gamma_1 \to \mathbb{R}^3 \text{ and } m^\alpha : \gamma_1 \to \mathbb{R}^2 \text{ are determined through appropriate integration across the thickness of the shell. First term of } B(\varsigma, \eta) \text{ is a so-called bending part, second is a membrane part. } \gamma_{\alpha\beta} \text{ and } \Upsilon_{\alpha\beta} \text{ are covariant components of the linearized strain and change of curvature (related to the arbitrary displacement } \eta \text{ of the surface } S) \text{ tensors. Note that both tensors are symmetrical:} \]

\[ \gamma_{\alpha\beta} = \frac{1}{2} (\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) + \Gamma^\rho_{\alpha\beta} \eta_\rho - b_{\alpha\beta} \eta_3 \]

\[ \Upsilon_{\alpha\beta} = \partial_\alpha \beta \eta_3 - \Gamma^\rho_{\alpha\beta} \partial_\rho \eta_3 + b_{\beta}^\rho (\partial_\alpha \eta_\rho + \Gamma^\sigma_{\rho\alpha} \eta_\sigma) + b^\rho_\alpha (\partial_\beta \eta_\rho + \Gamma^\sigma_{\rho\beta} \eta_\sigma) + \]"

\[ + (\partial_\alpha b^\rho_\beta + \Gamma^\sigma_{\alpha\beta} b^\rho_\sigma)(\eta_\rho - c_{\alpha\beta} \eta_3) \]

For the simplification of the following formulae from now on we consider the shell to be hard clamped on all its boundary (\( \gamma = \gamma_0 \)). In this case \( L(\eta) \) can be rewritten as:

\[ L(\eta) = \int_w p \cdot \eta \sqrt{\text{ad}w} \]

It is proved [16] that if \( \gamma_0 > 0 \), bilinear form \( B(\eta, \eta) \) is \( V(w) \)-elliptic and there exists such a constant \( \beta > 0 \) that \( \| B(\eta, \eta) \| \geq \| \eta \|^2_{H^1(w) \times H^2(w)} \) for all \( \eta \in V(w) \):

\[ \| \eta \|^2_{H^1(w) \times H^2(w)} = \sum_\alpha \| \eta_\alpha \|^2_{1,w} + \| \eta_3 \|^2_{2,w} \]

As a consequence, an existence theorem was proved in [15]: if the linear form \( L \) is continuous with respect to the norm \( \| \cdot \|_{H^1(w) \times H^2(w)} \), linear Koiter’s shell model has one and the only one solution.
3. A posteriori error estimation

The energy functional for a linear Koiter’s shell model is as follows:

\[ J(\varsigma) = \frac{1}{2} B(\varsigma, \varsigma) - L(\varsigma) \]  

(20)

Our goal is to find an estimate for an energy error norm which is defined as a doubled difference of the energy functionals on the approximate and exact solutions of the problem:

\[ \| e \|_2^2 = 2(J(\tilde{\varsigma}) - J(\varsigma)) = B(e, e), \]  

(21)

where error \( e = \varsigma - \tilde{\varsigma} \).

As it is done in the functional approach of a posteriori error estimation, let us introduce a pair of free variables, which in our case are symmetric tensors \((\kappa, \phi) \in H^1_{\text{sym}}(w) \times H^2_{\text{sym}}(w)\).

Using a technique of integration by parts we get

\[ \int_w \kappa \gamma_{\alpha\beta}(\eta) \sqrt{\text{ad}}w + \int_w \Psi_1(\kappa) \cdot \eta \sqrt{\text{ad}}w = 0 \]

\[ \int_w \phi \Upsilon_{\alpha\beta}(\eta) \sqrt{\text{ad}}w + \int_w \Psi_2(\phi) \cdot \eta \sqrt{\text{ad}}w = 0, \]  

(22)

where

\[ \Psi_1(\kappa) = \text{Div} \kappa - \Gamma^\rho_{\alpha\beta} \kappa_\rho + b_{\alpha\beta} \kappa_3 \]  

(23)

\[ \Psi_2(\phi) = (b^\rho_{\beta} \Gamma^\rho_{\alpha\beta} + b^\rho_{\alpha} \Gamma^\rho_{\rho\beta}) \kappa_\rho - (\partial_\alpha b^\rho_{\beta} + \Gamma^\rho_{\alpha\sigma} b^\sigma_{\beta} - \Gamma^\rho_{\alpha\beta} b^\sigma_{\rho}) \kappa_\sigma + c_{\alpha\beta} \kappa_3 + \partial_\alpha (b^\rho_{\beta} \kappa_\rho) + \partial_\beta (b^\rho_{\alpha} \kappa_\rho) - \partial_\rho (\Gamma^\rho_{\alpha\beta} \kappa_3) + \partial_{\alpha\beta} \kappa_3. \]  

(24)

Now we add these equalities to the energy norm of the error and then regroup the terms:

\[ \| e \|_2^2 = \int_w t \gamma_{\rho\sigma}(\kappa) \gamma_{\alpha\beta}(e) \sqrt{\text{ad}}w + \int_w \frac{t^3}{3} A \gamma_{\rho\sigma}(e) \Upsilon_{\alpha\beta}(e) \sqrt{\text{ad}}w = \]

\[ = \int_w p \cdot e \sqrt{\text{ad}}w - \int_w t \gamma_{\rho\sigma}(\tilde{\varsigma}) \gamma_{\alpha\beta}(e) \sqrt{\text{ad}}w - \int_w \frac{t^3}{3} A \gamma_{\rho\sigma}(\tilde{\varsigma}) \Upsilon_{\alpha\beta}(e) \sqrt{\text{ad}}w = \]

\[ = \int_w t (\varsigma - A \gamma_{\rho\sigma}(\tilde{\varsigma})) \gamma_{\alpha\beta}(e) \sqrt{\text{ad}}w + \int_w \frac{t^3}{3} (\varphi - A \gamma_{\rho\sigma}(\tilde{\varsigma})) \Upsilon_{\alpha\beta}(e) \sqrt{\text{ad}}w + \]

\[ + \int_w \left( p + t \Psi_1(\varsigma) + \frac{t^3}{3} \Psi_2(\varphi) \right) \cdot e \sqrt{\text{ad}}w \]  

(25)

The last term is estimated with a Cauchy-Schwartz inequality:

\[ \int_w \left( p + t \Psi_1(\varsigma) + \frac{t^3}{3} \Psi_2(\varphi) \right) \cdot e \sqrt{\text{ad}}w \leq \| e \|_w \left\| p + t \Psi_1(\varsigma) + \frac{t^3}{3} \Psi_2(\varphi) \right\|_w \]  

(26)

And the same inequality is used for estimation of the sum of the first two terms:
\[
\int_{\omega} \left( t \left( \kappa - A \gamma_{\rho\sigma} (\zeta) \right) \gamma_{\alpha\beta} (e) \right) \sqrt{\alpha} dw + \int_{\omega} \frac{t^3}{3} \left( \varphi - A Y_{\rho\sigma} (\zeta) \right) Y_{\alpha\beta} (e) \sqrt{\alpha} dw \leq \\
\leq \left\| e \right\| \left( \int_{\omega} \left( t \left( \kappa - A \gamma_{\rho\sigma} (\zeta) \right) \left( \kappa - A \gamma_{\alpha\beta} (\zeta) \right) \right) \sqrt{\alpha} dw \right)^{1/2} + \\
+ \left\| e \right\| \left( \int_{\omega} \frac{t^3}{3} \left( \varphi - A Y_{\rho\sigma} (\zeta) \right) \left( \varphi - A Y_{\alpha\beta} (\zeta) \right) \sqrt{\alpha} dw \right)^{1/2},
\]

Thus, a following estimate is derived:

\[
\left\| e \right\| \leq M_1 (\zeta, \kappa, \varphi) + M_2 (\kappa, \varphi),
\]

where

\[
M_1 (\zeta, \kappa, \varphi) = \left( \int_{\omega} \left( t \left( \kappa - A \gamma_{\rho\sigma} (\zeta) \right) \left( \kappa - A \gamma_{\alpha\beta} (\zeta) \right) \right) \sqrt{\alpha} dw \right)^{1/2} + \\
+ \left( \int_{\omega} \frac{t^3}{3} \left( \varphi - A Y_{\rho\sigma} (\zeta) \right) \left( \varphi - A Y_{\alpha\beta} (\zeta) \right) \sqrt{\alpha} dw \right)^{1/2},
\]

\[
M_2 (\kappa, \varphi) = c_b \left\| p + t \Psi_1 (\kappa) + \frac{t^3}{3} \Psi_2 (\varphi) \right\|_w,
\]

and \( c_b \) — an auxiliary constant, which does not depend on the parameters of the mesh.

Note that this estimate is unconditionally reliable (which means that true error is never underestimated) and can be computed using only approximate solution and geometry and material parameters of the shell.

It may be more convenient to use this estimate without square roots by applying a Cauchy inequality with parameter \( \beta \):

\[
\left\| e \right\|^2 \leq (1 + \beta) M_1^2 (\zeta, \kappa, \varphi) + \left( 1 + \frac{1}{\beta} \right) M_2^2 (\kappa, \varphi).
\]

4. Conclusion

A new functional-type a posteriori error estimate was presented for a problem of a linear Koiter’s shell deformation. The estimate is reliable and does not depend on the type of approximation used to solve the problem, which makes it potentially useful for engineering practice.

Acknowledgments

Work is supported by the Grant of the President of the Russian Federation MD-1071.2017.1.

References

[1] Babuška I and Rheinboldt W 1978 Error estimates for adaptive finite element computations SIAM J. Numer. Anal. 15 736–54
[2] Babuška I 1981 A-posteriori error estimates for the finite element solutions for one-dimensional problems SIAM J. Numer. Anal. 18 565–89
[3] Ainsworth M and Oden J 2000 A Posteriori Error Estimation in Finite Element Analysis (Chichester: Wiley)
[4] Neittaanmäki P and Repin S 2004 Reliable Methods for Computer Simulation – Error Control and A Posteriori Estimates (Amsterdam: Elsevier)
[5] Repin S 2008 A Posteriori Estimates for Partial Differential Equations vol 4 (Berlin: Radon Series on Computational and Applied Mathematics, de Gruyter)
[6] Mali O, Neittaanmäki P and Repin S 2014 Accuracy Verification Methods: Theory and Algorithms vol 32 Computational Methods in Applied Sciences (Berlin: Springer)
[7] Repin S and Valdman J 2018 Error identities for variational problems with obstacles ZAMM Z. Angew. Math. Mech. 98 635–58
[8] Frolov M 2010 Functional a posteriori error estimates for Euler-Bernoulli beam bending problem (in Russian) SPb Phys. Math. J. 104 81–4
[9] Frolov M 2010 Functional a posteriori error estimates for certain models of plates and beams Russ. J. Numer. Anal. Math. Model. 25 117–29
[10] Mali O 2011 Analysis of errors caused by incomplete knowledge of material data in mathematical models of elastic media. Ph.D. thesis. Jyväskylä Studies in Computing 132
[11] Frolov M and Chistiakova O 2016 A new functional a posteriori error estimate for problems of bending of Timoshenko beams Lobachevskii J. Math. 37 534–40
[12] Frolov M and Chistiakova O 2019 Adaptive algorithm based on functional-type a posteriori error estimate for Reissner-Mindlin plates Advanced Finite Element Methods with Applications. Selected papers from the 30th Chemnitz Finite Element Symposium, LNCSE 128
[13] Churilova M and Frolov M 2017 Comparison of adaptive algorithms for solving plane problems of classical and Cosserat elasticity Materials Physics and Mechanics 32 370–82
[14] Churilova M and Frolov M 2019 A posteriori error estimates for linear problems in Cosserat elasticity J. Phys.: Conf. Series 1158 22–32
[15] Bernadou M, Ciarlet P and Miara B 1994 Existence theorems for two-dimensional linear shell theories J. Elast. 34 111–38
[16] Ciarlet P and Miara B 1992 On the ellipticity of linear shell models Z. Angew. Math. Phys. 43 243–53