On turbulence in hydrodynamic lubrication and in ground effect

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Abstract. The contribution deals with a self-consistent description of time-mean turbulent lubricant flow, i.e. flow through a wedge-shaped gap confined by counter-sliding solid surfaces. The methods adopted are matched asymptotic expansions, where the slenderness or aspect ratio of the gap and the accordingly defined Reynolds number represent the perturbation parameters. The limit considered is conveniently approached from the typical aerodynamical problem of wing–ground interference. This then ties in with appropriate inflow and outflow conditions, which here appear quite naturally rather than form a common uncertainty in lubrication theory. As a remarkable finding, a lifting force as a consequence of the resultant pressure distribution can only be maintained for fully developed turbulent flow provided its asymptotic structure flow differs distinctly to that known from other turbulent internal flows as pipe or channel flows or classical turbulent boundary layers. The basic analysis is carried out without resorting to a specific Reynolds shear stress closure. However, the resulting requirements for asymptotically correct turbulence models are discussed. The theoretical study is accompanied by a numerical study of the boundary layer equations governing the fully turbulent core flow. Finally, the impact of cavitation, a phenomenon highly relevant in lubrication theory, on the novel flow structure is addressed.

1. Introduction

Lubrication and boundary layer theory meet once the clearance of the lubrication gap under consideration is sufficiently wide or the impressed sliding motion sufficiently fast so that the effect of inertia on the flow through the gap cannot be neglected any longer, as it is the case in the classical description of lubricant flow. In general, the correct form of the inflow and outflow conditions and the associated (weak) elliptic character of the resulting boundary layer problem then pose a severe difficulty. At first, this is owing to the downstream growth of small perturbations triggered at the initial stage of the flow which finally lead to non-unique solutions and might render the original problem ill-posed. Secondly, it not least might reflect an uncertainty regarding the values of velocity and pressure at inlet and discharge side of the gap.

Both issues can be mastered in a rigorous manner by putting the problem formulation in a correct asymptotic framework, such that the slenderness of the gap serves as the principal perturbation parameter and the correct form of in-/outflow conditions is determined by matching as provided by the flow regions adjacent to inlet and outlet. That is, a sound global flow model
has to be established to some extent or the other. Here a classical situation encountered in aerodynamical research, namely an airfoil or blade entirely immersed in a Newtonian fluid and moving (steadily) close to a planar ground, thereby forming a relatively thin gap having an aspect ratio comparable to its angle of attack/camber/thickness, represents one particular problem, though at first sight patently not tied in with the typical problems addressed in lubrication science. If viscous forces in the gap can be neglected, the increase of the lifting force due to the proximity of the ground or interface with another medium is exploited efficiently in the design of special vehicles, as e.g. hydrofoils or ram wings in guideways. An exhaustive study of the related inviscid problems on the basis of thin-airfoil theory was put forward by Widnall & Barrows (1970a, b); here the inclusion of viscous forces and the wake flow by Purvis & Smith (2004) deserves special attention. For the extension of the theory to the generic case where the interplay of the gap contour and the smallness of the clearance prevents a linearisation about the uniform free stream flow, also in account of three-dimensional effects, see Tuck (1983). Finally, the aforementioned associated boundary layer limit was addressed in the pioneering work by Tuck & Bentwich (1983). More recently, Jones & Smith (2003) took it up in connection with the design of (racing) car undertrays and spoilers, where a gap shape producing a negative lift, i.e. a downforce rather than an upforce, proves crucial.

Jones & Smith (2003) also give estimates for the Reynolds numbers based on body lengths at play and hence for the penetration depth of viscous diffusion on the gap flow on the basis of flow conditions of engineering relevance. From this they conclude that the assumption of laminar flow, to our knowledge adopted in previous related investigations throughout, is presumably unrealistic when applied to the problem of car–ground interference. Eventually, they point out that and how the theory should be extended to the turbulent case, but by taking the traditional high-Reynolds-number scalings for wall-bounded turbulent flows in a rather ad-hoc manner. This serves as a motivation for the present contribution on turbulence in thin-gap flows from a more general and fundamental point of view, which we conveniently commence by revisiting the classical lubrication limit. The specific situation of a wing in ground effect addressed by the aforementioned studies shall exemplify the problem, in order to pin down inflow and outflow conditions in a self-consistent flow description. In contrast to the asymptotic theories of fully developed turbulent boundary layers driven by external flows, here the initially unknown gap pressure introduces a peculiar difficulty in the subtle task of how to scale the flow correctly.

2. Pathway to turbulence starting at classical lubrication theory

The classical long-standing theory of lubrication is essentially traced back to the following five assumptions: (i) a Newtonian lubricant filling (ii) a relatively slender gap separating the counter-sliding rigid smooth surfaces, i.e. with (iii) a surface roughness being insignificantly small compared to the gap height (lubrication in the fully hydrodynamic regime), (iv) the neglect of inertia terms in the Navier–Stokes equations, and (v) strictly laminar flow. Then the continuity equation can be integrated across the clearance subject to the kinematic boundary conditions provided the well-known Reynolds equation (in its most general form) governs the pressure distribution between the wetted surfaces.

2.1. Classical lubrication theory in a proper asymptotic framework

In this classical theory the streamwise velocity profile represents a superposition of Poiseuille and Couette flow. It is noteworthy that for impervious surfaces as considered in the following the special configuration of a constant gap height is well-understood, albeit in different context: in absence of a counter-sliding motion such a flow is commonly referred to as Hele–Shaw flow (i.e. pure Poiseuille flow in the case of plane flow), and in absence of an imposed pressure difference between the inlet and outlet boundaries of the clearance it reduces to (in general
three-dimensional unsteady) pure Couette flow.

As further assumptions adopted in the present study, the specific gap geometry, i.e. the distribution of its height, and hence the flow observed in a suitably adopted frame of reference are stationary, and mass forces due to the motion of the latter relative to an inertial frame of reference can be neglected. Then the basic prerequisites (ii) and (iv) are formally expressed as

\[(ii) \quad \epsilon = \frac{\tilde{H}}{\tilde{L}} \ll 1, \quad (iv) \quad Re = \frac{\tilde{U} \tilde{L}}{\tilde{\nu}} \ll 1/\epsilon^2, \quad (1)\]

where the slenderness parameter $\epsilon$ and the Reynolds number $Re$ are formed with a typical gap height $\tilde{H}$, a length $\tilde{L}$ characteristic of the lubricated region in the main direction of the sliding motion, a reference value $\tilde{U}$ of the sliding speed, and the (constant) kinematic viscosity $\tilde{\nu}$.

2.2. Extension of the classical theory for moderate to large Reynolds numbers

Interestingly, the relaxation of assumption (iv) has attracted little attention from theoreticians, despite the undeniable importance of convective effects in engineering applications as e.g. thrust bearings for high-speed rotors. In those cases $Re_\epsilon = \epsilon^2 Re$ can no longer be regarded as a small parameter but rather as a quantity of $O(1)$. On condition (v) the lubricant flow then is of boundary-layer type, where the pressure is not known in advance but adjusts as part of the solution of the underlying boundary layer equations, replacing the conventional Reynolds equation. Without doubt, a systematic treatment of such “internal” boundary layers for a most comprehensive variety of gap geometries and inlet and outlet conditions is rather involved as dealing with the three-dimensionality of the flow requires advanced numerical methods.

Here we restrict the analysis to incompressible lubricants of uniform density $\tilde{\rho}$ and plane gaps: then the resulting two-dimensional flow is essentially governed by the inflow and outlet conditions and the slope of the gap height, which fix the constant volume flux. Such flows have been envisaged in general by Tuck & Bentwich (1983) and more specifically e.g. by Jones & Smith (2003). It seems expedient to briefly discuss some interesting facets associated with such types of internal boundary layer problems, parametrised by $Re_\epsilon$ for a given gap geometry, as these are also highly relevant in the turbulent case considered subsequently.

At first, the ellipticity of the boundary layer problem due to the unknown pressure distribution demands particular attention. Widnall & Barrows (1970a) and Tuck & Bentwich (1983) account for it by prescribing the (uniform) inlet velocity in dependence of the inlet and the outlet pressure, as a consequence of the Kutta condition that has to satisfied at the trailing edge of the wing: in this specific context, it discards the possibility of break-away separation from the airfoil surface further upstream as this would contradict the assumption of globally attached flow. However, in any well-posed formulation of the problem a variety of different combinations of the inlet and outlet values of velocity and pressure may likely serve as upstream and downstream boundary conditions, but the last word is not yet spoken. Here we refer to the close resemblance of such internal flows to thin shear layers with free surfaces under the action of gravity, see e.g. Higuera (1994), and boundary layers in mixed convection, see e.g. Steinrück (1994). In those cases a well-posed formulation of the boundary layer problem is crucially associated with appropriate downstream conditions: these suppress the generation of eigensolutions of unbounded growth, which (under certain conditions) eventually enforces termination of the downstream integration in form of a singularity. Secondly, we note that classical lubrication theory predicts flow reversal if the pressure gradient is adverse (negative contribution of the Poiseuille flow) and the local (negative) ratio $r$ of the volume fluxes due to the Couette and the Poiseuille flow is greater than $-3$: then $(3+r)/2$ and $(3+r)/3$ denote the fractions of the gap height occupied by the reverse-flow zone attached at the fixed boundary and that of upstream flow, respectively. It is interesting how the prediction of reverse flow based on numerical boundary layer calculations, as carried out first by Tuck & Bentwich (1983) and more thoroughly by Jones & Smith (2003),
affects this criterion. To this end, a careful study by the present authors of the positions of separation and reattachment in dependence of $Re_\epsilon$ is under way.

In a next step, assumptions (iii) and (v) are reviewed critically. It is well-known that steady Couette and typical Poiseuille flows as pipe and channel flows become (globally) unstable when the values of the characteristic Reynolds numbers exceed critical thresholds of an order of magnitude of $10^4$. In the present context this Reynolds number is given by $\tilde{U}\tilde{H}/\tilde{\nu}$, equal to $Re_\epsilon/\epsilon$. Simultaneously, in engineering applications $\epsilon$ is typically of the order of $10^{-4}$, so that assuming $Re_\epsilon = O(1)$ in connection with strictly laminar flow seems questionable. The herewith indicated inclusion of the Reynolds shear stress in the boundary layer equations coins the notion of superlaminar flow (remaining in a state of laminar–turbulent transition). Eventually, for $Re_\epsilon \gg 1$ the contribution of the viscous stress can be neglected and the fully turbulent regime is reached. This situation has been tackled on a semi-empirical basis, see e.g. Fréne et al. (2006). Unfortunately, for developed turbulence this approach impedes a match of the bulk region with the (extremely thin) viscous wall layers adjacent to the surfaces, where both stresses are still of equal importance.

This inconsistency ties in with the presence of the inertia terms in leading order balancing the Reynolds stress gradient as it contradicts the classical notion of an asymptotically small streamwise velocity deficit with respect to the characteristic flow speed in the fully turbulent part of boundary layers or internal flows. However, here this classical description (suggested by Jones & Smith, 2003) would predict a predominantly inviscid “lubricant” flow prevailing in most of the gap and thus in general a breakdown of the load-bearing capacity. Therefore, in a generic formulation of the boundary-layer-type turbulent flow the aforementioned balance is retained, so that the velocity deficit is seen to be of $O(1)$.

3. Rigorous flow description: a novel distinguished limit

A rational treatment of the time-mean or, equivalently, Reynolds-averaged fully developed turbulent flow for $Re_\epsilon \gg 1$ exhibiting a large velocity defect gives rise to a multi-structured flow picture, in interesting analogy to that put forward by Melnik (1989). It typically employs

$$\sigma = 1/\ln Re$$

rather than $1/Re$ (as in the laminar case) as the second perturbation parameter besides $\epsilon$. It consists of the core (bulk) region comprising most of the gap, the two viscous layers adjacent to the walls with thicknesses $1/(\sigma Re)$, and two intermediate layers of widths $\epsilon^{3/2}$, separating the first from the latter (figure 2). In the wall layers the leading-order equilibrium between the total and the wall shear stresses leads to the commonly accepted universal form of the wall functions. The intermediate regions are reminiscent of the small velocity defect characteristic of wall-bounded turbulent shear flows, but here with respect to a slip velocity exerted by the core flow. In turn, they appear to be passive constant-stress or so-called buffer layers which modify the near-wall portion of the pronounced wake-type core flow so that the streamwise velocity component matches the celebrated logarithmic law of the wall holding on top of the viscous sublayers.

One finds by discarding both the prerequisites (iv) and (v) that the bounds of validity (1) of classical lubrication theory are violated in favour of the new characterisation

$$\epsilon \ll 1, \quad \text{(fully developed turbulence)} \quad \kappa K = \sqrt{\epsilon} \ln Re = O(1).$$

Here $\kappa$ denotes the von Kármán constant and $K$ represents a similarity parameter. It describes a distinguished (least-degenerate) limit that introduces a formal coupling between $\epsilon$ and $Re$. Also, it clarifies why the possibility of fully developed turbulence is inconsistent with the assumption that $Re_\epsilon = O(1)$, in agreement with the notion of superlaminar flow proposed above.
For the specific flow configuration we refer to figure 1. The stationary gap geometry is formed by two sheets, one sliding with the constant speed $\tilde{U}$ and the other at rest. Let $x$, $y$, $h$, $u$, $v$, $\tau$ and $p$ represent normal coordinates attached to the particular frame of reference along and perpendicular, respectively, to the wetted surface of the sliding sheet, the local gap height, the components of the flow velocity in $x$-(streamwise) and $y$-direction, the Reynolds shear stress, and the pressure. Here $x$ is made non-dimensional by $L$, $y$ and $h$ by $\tilde{H}$, the flow velocity by $\tilde{U}$, and $\tau$ and $p$ by $\rho \tilde{U}^2$. In the core region the flow quantities are appropriately expanded in the form $[u, v/\epsilon, \tau/\epsilon] \sim [U, V, T](x, y; K) + \cdots$ and $p \sim P(x; K) + \cdots$ as $\epsilon \to 0$. We then arrive at the full boundary layers equations for turbulent flow, subject to the conditions of matching with the intermediate layers:

$$
\partial_y U + \partial_x V = 0, \quad U \partial_y U + V \partial_y U = -dP/dx + \partial_y T,
$$

$$
y = 0: \quad V = 0, \quad K \sqrt{|T|} \text{sgn}(T) = U - 1, \quad y = h(x): \quad V = U \partial_x h/dx, \quad K \sqrt{|T|} \text{sgn}(T) = U.
$$

The relationships between the wall shear stress $T$ and the slip velocity relative to the motion of the respective sliding and fixed sheets in (5) provide the skin-friction law. Equations (4), (5) have to be supplemented with appropriate inflow and outflow conditions, where we recall the impact of the latter in the context of laminar flow as addressed above. In order to complete the problem formulation, we now state these conditions explicitly for the specific problem of an airfoil in effect of ground, the first having a chord length $1$ and the latter identified with a planar sliding sheet. They remain unchanged in the turbulent case as found by matching the gap flow with the predominantly inviscid flow in the square regions I and II, where ($x/\epsilon, y$) $= O(1)$ and $[(x - 1)/\epsilon, y]$ $= O(1)$ and which encompass the leading edge ($x = 0$) and the trailing edge ($x = 1$), respectively (see figure 1):

$$
x \to 0+: \quad U \to \lambda(K) \leq 1, \quad P \to P(0; K) = (1 - \lambda^2) / 2 \geq 0, \quad x \to 1-: \quad P \to 0.
$$

Here the flow speed $\lambda$ at inlet represents an eigenvalue of the resulting elliptic problem (4)–(6), parametrised by $K$. That is, in the thin-gap limit considered the Kutta condition $P = 0$ satisfied at the trailing edge causes a jump of the pressure from its value at free-stream ($P = 0$) to $P(0)$ at the entrance of the gap region and thus promotes a net lifting force acting on the airfoil. This must be be determined from the numerical solution of (4)–(6), which can be obtained once this problem is closed by an asymptotically correct shear stress model, i.e. one consistent with the skin-friction law and the multi-layered structure of the flow (cf. Melnik, 1989; Scheichl & Kluwick, 2007). This issue is the main topic of the current research activities.

The turbulent flow governed by (4)–(6) matches the superlaminar flow regime when $K \ll 1$. The case $K \gg 1$, i.e. $\epsilon \gg \sigma^2$ with $\sigma$ defined in (2), either results in a further internal splitting.
of the core layer near the walls and thus in a flow structure known from turbulent boundary layers prone to internal separation (cf. Scheichl & Kluwick, 2007) or degenerates to the classical small-deficit structure, proposed by Jones & Smith (2003) even as the generic flow picture. This here implies $K = O(1)$, whereas (3) is superseded by the distinguished limit

$$\epsilon \ll 1, \quad \epsilon \ln Re = O(1) \quad (7)$$

under the small-defect assumption. It then is characterised by the absence of the intermediate layers and typically core-layer subexpansions of the form $U \sim U_0(x; \epsilon)[1 - \sigma F_1(x, y) + O(\sigma^2)], T \sim \sigma T_1(x, y) + O(\sigma^2)$, where $U_0 \sim h(1)/h(x) + \epsilon U_1(x) + O(\epsilon^3)$ is governed by inviscid-flow theory (cf. Tuck, 1983), so that $\lambda(\infty) = h(1)/h(0)$. This finally yields

$$p \sim (1 - [h(1)/h(x)]^2)/2 + \epsilon P_1(x) + O(\epsilon^2, \sigma) \quad (8)$$

However, a comprehensive analysis of the case $K \gg 1$ is one goal of the present investigations. At this stage a conceptional difficulty should be noted which arises when commonly employed shear stress closures are considered from an asymptotic viewpoint: typical model constants have to be identified with $\epsilon$ in the case (3) (cf. Melnik, 1989; Scheichl & Kluwick, 2007) but taken to be finite for arbitrarily high values of $Re$ if (7) applies. Also, it is instructive to demonstrate that the double expansion (8) provides a match with the potential-flow regime obtained when the distance $\epsilon$ from ground to airfoil becomes large while the latter remains thin: see the Appendix.

4. Further perspectives

The a priori unknown scaling of the gap pressure points to the alternative of a leading-order balance between pressure and shear stress gradient and, as in laminar lubrication, hence to a fully developed internal flow, as e.g. pipe flow. Under this assumption the effect of inertia is negligibly small, so that the momentum equation in (4) can be integrated once to give $y dP/dx \sim T - T|_{y=0}$, where $P$ and $T$ have to be rescaled accordingly. This limit was brought forward by Jones & Smith (2003) on account of a likewise thin gap, i.e. for $K \ll 1$. Under their assumption of a small velocity defect perturbing the almost inviscid core flow, the latter would inevitably exhibit a pronounced variation with $x$ due to the non-parallel walls but represent a higher-order effect. Since this contradicts the proposed flow structure, the present authors at least cast doubt on it. In view of the contrasting picture of a large velocity deficit, the study of a fully developed internal turbulent flow one-sidedly driven by a belt reported in the textbook by Schlichting & Gersten (2003, pp. 544–549) should be mentioned. However, here the neglect of the intermediate layers severely hampers a correct match of the fully turbulent flow with that in the viscous wall layers. Consequently, as long as a self-consistent asymptotic splitting of the flow in an inertia-free limit is still lacking, its existence can hardly be accepted. Still more endeavour concerning the flow scaling seems required.

Moreover, two phenomena (and their interplay) are of specific interest: (a) the occurrence of flow reversal, so that both the slip velocity and the wall shear stress in (5) change sign close to the positions of separation and reattachment which renders the proposed skin-friction locally invalid; (b) the (in applications undesired) onset of cavitation of an initially liquid lubricant when the value of $P$ falls below $P_c (< 0)$, referring to the vapour pressure, so that the lubricant undergoes a phase change. Both events require an extension of the flow description presented here, and efforts in this direction yield first encouraging results. Here the presence of a minimum gap height $h_{\text{min}}$, as indicated in figures 1 and 2, thus both a convergent and divergent part of the duct, has a central impact on the flow picture. This topic is also of vital importance for strongly compressible flows, particularly in view of the transition between the sub- and supersonic regime as the duct acts as a Laval nozzle for inviscid flow. Another aspect to be noted here concerns
the inclusion of surface roughness, which may become important when \( h_{\text{min}} \) is comparable to the average asperity height. Here the flow configuration is inevitably non-stationary on the relevant micro scale.

Finally, to be more specific, issue (b) is associated with the occurrence of a thermodynamically stable homogeneous two-phase mixture. Under the assumption of isothermal flow, the accordingly modified boundary layer equations (4) then read

\[
\partial_x(\rho U) + \rho \partial_y V = 0, \quad \rho (U \partial_x U + V \partial_y U) = -dP/dx + \partial_y(\rho T). \tag{9}
\]

Herein overbars indicates Favre- rather than conventional Reynolds-averaging (entailing a difference in the definitions of \( \overline{\rho} \) and \( T \)). The density \( \rho(x) \) is non-dimensional with \( \bar{\rho} \), so that \( \rho \equiv 1 \) in the fully liquid and \( 1 > \rho > 0 \) in the two-phase regime. Since \( P \equiv P_c \) in the latter, there \( \rho \) rather than \( P \) is part of the solution and accounts for the ellipticity of the boundary layer problem. Finally, for sufficiently low values of \( \rho \) even full vaporisation has to be considered and hence an equation of state \( P = P(\rho) \) to be supplied.

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Appendix. Ground effect on potential flow past a thin airfoil

Without loss of generality, we take \( h(0) = 1 \). Then \( y = 1 \pm (\alpha/\epsilon)h_\pm(x) \) with \( h_+(0) = h_-(0) = 0 \) and \( h_+(1) = -h_-(1) \) describes the upper and the lower contour of the airfoil over planar ground and exposed to the oncoming unperturbed free stream, given by \( y = 0 \) and \([u, v] = [1, 0]\), respectively. The small positive parameter \( \alpha \) measures the aspect ratio of the airfoil, comprising slenderness/camber/angle of attack, in the \((x, \epsilon y)\)-plane. As long as the ratio \( \epsilon \) of the distance ground to \( \alpha \) is sufficiently large, classical potential-flow theory predicts a pressure disturbance

\[
p \sim \alpha p_1(x, y; \epsilon) + O(\alpha^2) \quad \text{(A.1)}
\]

and \([1-u, v] \sim \alpha[p_1, v_1](x, y; \epsilon) + O(\alpha^2) \) for \( y \geq 0 \). One conveniently determines \( p_1 \), \( v_1 \) by continuation into the half-plane \( y < 0 \) and exploitation of symmetry with respect to \( y = 0 \).

Let \( p_{1\pm}(x; \epsilon) = \lim_{y \rightarrow \pm 1} p_1 \) refer to the pressure perturbations at the upper/lower surface of the airfoil. Application of Cauchy’s integral formula to the holomorphic function \( p_1 + i v_1 \) of \( z = x + i \epsilon y \) and deformation of a simply-connected path of integration in the \( z \)-plane to the adjacent line segments \( y = 1 \pm 0, 0 \leq x \leq 1 \), their mirror images with respect to \( y = 0 \), and a circle with infinite radius (figure A1) yields

\[
p_1 = \frac{1}{2\pi i} \int_0^1 \left\{ \epsilon(p_{1+} - p_{1-})(s; \epsilon) \left[ \frac{y-1}{(x-s)^2 + \epsilon^2(y-1)^2} - \frac{y+1}{(x-s)^2 + \epsilon^2(y+1)^2} \right] - (h_+ + h_-)(s) \left[ \frac{x-s}{(x-s)^2 + \epsilon^2(y+1)^2} + \frac{x-s}{(x-s)^2 + \epsilon^2(y-1)^2} \right] \right\} ds. \quad \text{(A.2)}
\]

An analogous expression is found for \( v_1 \), with \( v_1(x, 0; \epsilon) = 0 \) and \( h'_\pm(x) = \pm \lim_{y \rightarrow 1} \pm v_1 \). The base pressure perturbation \( \alpha p_1(x, 0; \epsilon) \) is obtained directly from (A.2). However, \( p_{1+} \) and \( p_{1-} \)
are determined by solving two Fredholm integral equations of the second kind for the quantities
\( P_\pm = p_+ \pm p_- \), where we accordingly define \( H_\pm = h_+ \pm h_- \):

\[
\pi P_+(x; \epsilon) = \int_0^1 \frac{H_+'(s)}{s-x} \, ds - \int_0^1 \frac{(x-s)H_+'(s) + 2\epsilon P_-(s; \epsilon)}{(x-s)^2 + 4\epsilon^2} \, ds, \quad (A.3)
\]

\[
\pi H_+'(x) = \int_0^1 \frac{P_-(s; \epsilon)}{x-s} \, ds - \int_0^1 \frac{(x-s)P_-(s; \epsilon) - 2\epsilon H_+'(s)}{(x-s)^2 + 4\epsilon^2} \, ds. \quad (A.4)
\]

The Kutta condition, \( P_-(1; \epsilon) = 0 \), fixes the intrinsic non-uniqueness of the solution \( P_\pm \) of (A.3) and (A.4). The ground effect is accounted for by the rightmost terms in these equations.

Essential conclusions are: (A) even the case \( \alpha = O(1) \), i.e. of an angle attack of \( O(1) \) and/or relatively thick profiles when the thin-airfoil approximation (A.1) fails, is principally amenable to the above approach; (B) in the distinguished limit \( [p_{1\infty}, v_{1\infty}](x, Y) = \lim_{e \to 1}\{p_1, v_1\}(x, y, \epsilon) \) with \( y = \epsilon(y-1) = O(1) \) the classical picture of a thin airfoil in unbounded domain (expressed in terms of the new coordinates \( x, Y \)) is recovered; (C) in general, even a symmetric profile \( (H_\pm \equiv 0) \) feels increasingly positive values of \( -P_- \) and hence the lift coefficient \( c_p \), as \( c_p \sim -\alpha \int_0^1 P_-(x; \epsilon) \, dx \), for a decreasing ground distance \( \epsilon \); (D) the two-parameter expansion by Widnall & Barrows (1970a) for a thin airfoil close to ground is recovered from (A.3) and (A.4) in the limit

\[
\epsilon \to 0: \quad p_{1+} \sim \frac{1}{\pi} \int_0^1 \frac{H_+'(s)}{s-x} \, ds + O(\epsilon), \quad p_{1-} \sim \frac{h_-(1) - h_-(x)}{\epsilon} + p_{1+} + O(\epsilon). \quad (A.5)
\]

That is, the distinguished limit \( [p_{10}, v_{10}](x, Y) = \lim_{e \to 0}[p_1, v_1]\{x, y, \epsilon\} \) with \( Y = O(1) \) found from (A.2) can be interpreted as a flow perturbation due to a weak local distortion of the ground given by \( Y = \alpha H_+(x) \) \( 0 \leq x \leq 1 \) or the thickness of the airfoil. Simultaneously, \( c_p \) increases by a factor of \( O(1/\epsilon) \) as we infer from (A.1) together with (A.2) subject to (A.5) for the pressure increase in the gap, i.e. where

\[
0 \leq y \leq 1: \quad p \sim (\alpha/\epsilon)[h_-(1) - h_-(x)] + \alpha p_{1+} + O(\alpha \epsilon). \quad (A.6)
\]
Finally, thin-airfoil theory ceases to be valid in the generic distinguished gap limit $\alpha/\epsilon = O(1)$, i.e. when $\epsilon$ is reduced such that the height and the streamwise variation of the gap formed by the airfoil are of comparable magnitude. Then the gap pressure is governed by (8). This can be restated as an expansion in terms of $\alpha$ in that limit as $h(x) = h(0) - (\alpha/\epsilon) h_\infty(x)$, and (A.6) is also obtained by re-expanding (8) for $\alpha/\epsilon$ being small. As an appealing aspect of issue (D), it then appears that (A.1) and (8) or, equivalently, the thin-airfoil approximation and the nonlinear regime of the inviscid gap flow have a common overlap. A further noteworthy topic associated with the transition between both flow regimes, still not fully resolved, concerns the downstream shift of the stagnation point at $x = x_s$, say, for a parabolic leading edge of a slender airfoil: for $\alpha \ll 1$ one finds that $x_s = O(\alpha^2)$ when $\alpha/\epsilon \gg 1$ (cf. Van Dyke, 1975, pp. 62–68) but $x_s = O(\alpha)$ (region I in figure 1) when $\alpha/\epsilon = O(1)$, where $x_s$ essentially depends on the value of $\lambda$ in (6) (Tuck, 1983).

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