DESCARTES’ RULE OF SIGNS, CANONICAL SIGN PATTERNS
AND RIGID ORDERS OF MODULI

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Abstract. We consider real polynomials in one variable without vanishing coefficients and with all roots real and of distinct moduli. We show that the signs of the coefficients define the order of the moduli of the roots on the real positive half-line exactly when no four consecutive signs of coefficients equal (+, +, −, −), (−, −, +, +), (+, −, −, +) or (−, +, +, −).

Key words: real polynomial in one variable; hyperbolic polynomial; sign pattern; Descartes’ rule of signs

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1. Introduction

We consider real univariate polynomials. When all roots of such a polynomial are real, the polynomial is called hyperbolic. We are interested in the generic case, i.e. the case of hyperbolic polynomials without multiple roots and with all coefficients non-vanishing (hence without root at 0).

The present paper continues a recent activity in the research about real univariate polynomials and their roots. The classical Descartes’ rule of signs applied to a real degree \(d\) polynomial \(Q\) without vanishing coefficients states that the number \(pos\) of its positive roots is bounded by the number of sign changes \(c\) in the sequence \(A\) of its coefficients the difference \(c − pos\) being even. The same rule applied to the polynomial \(Q(−x)\) implies that the number \(neg\) of negative roots of \(Q\) is not larger than the number \(p\) of sign preservations in \(A\), \(c + p = d\), and the difference \(p − neg\) is also even. For a hyperbolic polynomial one has \(pos = c\) and \(neg = p\).

The problem for which couples \((pos, neg)\) satisfying these conditions can one find a real polynomial with prescribed signs of its coefficients seems to have been explicitly formulated for the first time in [2]. The first non-trivial result occurs for \(d = 4\), see [6]. The exhaustive answer to the question for \(d = 5\) and 6 is provided in [1]; for \(d = 7\) and 8 it is given in [4] and [7]. For \(4 \leq d \leq 8\), in all compatible but not realizable cases one has \(pos = 0\) or \(neg = 0\). For \(d \geq 9\), there are compatible and non-realizable couples with \(pos \geq 1\) and \(neg \geq 1\), see [8] and [3]. For a tropical analog of Descartes’ rule of signs a discussion can be found in [5]. Various problems concerning hyperbolic polynomials in one variable are exposed in [9].

Definition 1. A sign pattern of length \(d + 1\) is a sequence of \(d + 1\) signs plus and/or minus. The polynomial \(Q := \sum_{j=0}^{d} a_j x^j\), \(a_j \in \mathbb{R}^*\), is said to define the sign pattern \(\sigma(Q) := (\text{sgn}(a_d), \text{sgn}(a_{d−1}), \ldots, \text{sgn}(a_0))\). Most often we set \(a_d := 1\). In this case the first sign of \(\sigma(Q)\) is a plus.

A natural question to ask about a degree \(d\) generic hyperbolic polynomial is: Suppose that the moduli of all its \(d\) roots are distinct. When these moduli are
arranged on the real positive half-axis, at which positions can the moduli of its negative roots be? This question leads to the following formal definition:

**Definition 2.** An order of moduli of length $d$ is a string of $p$ letters $N$ and $c$ letters $P$, $c + p = d$, separated by signs of inequality <. The letters $N$ and $P$ indicate the relative positions of the moduli of the negative and positive roots of the hyperbolic polynomial on $\mathbb{R}_+$. For example, to say that the degree 7 hyperbolic polynomial $Q$ defines (or realizes) the order of moduli $N < N < P < N < P < N < P$ means that for the sign pattern $\sigma(Q)$, one has $c = 3$, $p = 4$, and that for the positive roots $\alpha_1 < \alpha_2 < \alpha_3$ and the negative roots $-\gamma_j$ of $Q$, one has $\gamma_1 < \gamma_2 < \alpha_1 < \gamma_3 < \alpha_2 < \gamma_4 < \alpha_3$.

A given order of moduli realizes a given sign pattern if there exists a hyperbolic polynomial which defines the given order of moduli and the given sign pattern.

For generic hyperbolic polynomials, we give the exhaustive answer to the following question:

**Question 3.** When does the sign pattern $\sigma(Q)$ determine the order of moduli defined by the hyperbolic polynomial $Q$?

The answer is given in terms of the following definition:

**Definition 4.** We define after each sign pattern of length $d + 1$ its corresponding canonical order of moduli: the sign pattern is read from right to left and to each couple of opposite (resp. identical) consecutive signs one puts in correspondence the letter $P$ (resp. $N$). Example: for $d = 8$, to the sign pattern $(+, +, -, -, +, -, +, +, -)$ corresponds the canonical order of moduli $P < N < N < P < P < P < N < P < N$.

**Remark 5.** Each sign pattern is realizable by its canonical order of moduli, see Proposition 1 and its proof in [11]. A sign pattern is called canonical if it is realizable only by its canonical order of moduli. So to answer Question 3 means to say which sign patterns are canonical.

**Definition 6.** We call configuration any four consecutive components of a given sign pattern. For instance, the sign pattern from Definition 4 contains 7 configurations the first, second and last of which are $(+, +, -, -)$, $(+, -, -, +)$ and $(+, +, +, -)$. We give names to the following four configurations:

$A := (+, +, -, -)$, $B := (-, -, +, +)$, $C := (+, -, -, +)$, $D := (-, +, +, -)$.

It is clear that if the sign pattern $\sigma(Q(x))$ contains the configuration $A$ or $B$ (resp. $C$ or $D$), then in the same positions the sign pattern $\sigma(Q(-x))$ contains the configuration $C$ or $D$ (resp. $A$ or $B$).

Our main result is the following theorem:

**Theorem 7.** A sign pattern is canonical if and only if it contains none of configurations $A - D$.

**Remark 8.** (1) The “only if” part of the theorem is proved in [11] (see Theorem 2 therein), so we prove only the “if” part. The formulation of the result of [11] is not given in terms of the configurations $A - D$, but is equivalent to such a formulation.
In [10] it is shown that the following sign patterns with one or two sign changes are canonical:

- \((+,-,+,-, +, - ,\ldots, - , + )\)
- \((+,-,-,-, +, - ,\ldots, +)\)
- \((+,+,-, +,-, +, +,\ldots, + )\) and \((+,+,-, +,-, +,\ldots, +)\).

(2) Each of configurations \(A\) and \(B\) contains an isolated sign change, i.e. a sign change between two sign preservations, and vice versa for configurations \(C\) and \(D\). Thus one can reformulate Theorem 7 as follows: A sign pattern is canonical if and only if it contains no isolated sign change and no isolated sign preservation.

We prove Theorem 7 in Section 3. In Section 2 we discuss problems related to Question 3.

2. Some related problems

It is obvious that one can ask the inverse to Question 3:

**Question 9.** When does an order of moduli determine the sign pattern? In other words, do there exist orders of moduli such that each of them realizes a single sign pattern? (We call such orders of moduli rigid.)

The exhaustive answer to this question (given in [12], assuming that the leading coefficient of the generic hyperbolic polynomial is positive) looks like this:

These are the orders of moduli of the form \(\cdots < P < N < P < N < \cdots\), and the corresponding sign patterns equal

- \((+,+,-,+,+,+,-,\ldots)\) or \((+,-,-+,+,-,+,+,+,\ldots)\),
- and also the orders of moduli \(\cdots < N < N < N < \cdots\) and \(\cdots < P < P < P < \cdots\)

the corresponding sign patterns being

- \((+,+,+,-,+,+,+\ldots)\) and \((+,+,+,-,+,\ldots)\).

The latter two sign patterns are trivially canonical, because all roots are of the same sign and hence one cannot compare moduli of roots of opposite signs. For the former two sign patterns one observes that every configuration of theirs is of the form \(A, B, C\) or \(D\).

Thus for any degree \(d \geq 3\), there exist exactly two non-trivial rigid orders of moduli and corresponding sign patterns. The number of canonical sign patterns, on the contrary, increases with \(d\) and tends to infinity. To see this it suffices to notice that canonical are, in particular, all sign patterns of the form \(([s_1],-[s_2],-[s_3],-\ldots)\), where \([s_j]\) denotes a sequence of \(s_j\) consecutive signs plus, \(s_j \geq 3\).

3. Proof of Theorem 7

In the proof of Theorem 7 we use also generalized orders of moduli, i.e. orders of moduli in which one of the signs of inequality \(<\) in a given order of moduli \(S\) is replaced by the sign of equality \(=\) with the obvious meaning that the moduli of the corresponding two roots are equal. We say that the thus obtained generalized order of moduli is adjacent to the order of moduli \(S\).

Consider a sign pattern \(\sigma^b\) of length \(d+1\) and containing none of configurations \(A - D\). Then there exists a degree \(d\) hyperbolic polynomial \(U\) realizing the sign pattern \(\sigma^b\) with the canonical order of moduli \(R\) defined by \(\sigma^b\) (see [11, Proposition 1]).
Denote by \( \Omega(\sigma^\triangle) \) the set of degree \( d \) monic generic hyperbolic polynomials whose coefficients define the sign pattern \( \sigma^\triangle \). Hence the set \( \Omega(\sigma^\triangle) \) is open and contractible (see \[13\] Theorem 2).

Suppose that the set \( \Omega(\sigma^\triangle) \) contains a polynomial \( V \) which defines an order of moduli different from \( \mathcal{R} \). Then one can connect the polynomials \( U \) and \( V \) by a continuous path \( \gamma \subset \Omega(\sigma^\triangle) \). The set \( \Omega(\sigma^\triangle) \) being open, one can choose \( \gamma \) such that for each point of it, in the corresponding polynomial there is at most one equality between the moduli of a negative and of a positive root. Hence there exists a polynomial \( W^\dagger \in \gamma \) defining a generalized order of moduli adjacent to \( \mathcal{R} \). Hence \( W^\dagger \) is of the form \((x^2 - \alpha^2)W_\diamond\), where \( \alpha > 0 \) and \( W_\diamond \) is a degree \( d - 2 \) monic hyperbolic polynomial. One can make the change of variables \( x \mapsto \alpha x \) after which the polynomial \( W := W^\dagger/\alpha^2 \) takes the form

\[
(3.1) \quad W = (x^2 - 1)W_\ast, \quad W_\ast := x^{d-2} + u_1x^{d-3} + \cdots + u_{d-3}x + u_{d-2}, \quad u_j \in \mathbb{R}.
\]

Notice that the coefficients \( u_j \) are not indexed in the same way as the coefficients \( a_j \) above.

We want to prove that if the sign pattern \( \sigma^\triangle \) contains none of configurations \( A - D \), then such a polynomial \( W \) does not exist. Hence \( V \) also does not exist which proves Theorem \([7]\).

**Remark 10.** One can make two non-restrictive assumptions on \( W \):

1) One can assume that \( u_1 > 0 \). Indeed, as \( W = x^d + u_1x^{d-1} + \cdots \), one has \( u_1 \neq 0 \). If \( u_1 < 0 \), then one can make the change \( x \mapsto -x \) which preserves the factor \( x^2 - 1 \). This means that to prove Theorem \([7]\) it suffices to consider only half of the possible cases, the ones of sign patterns beginning with two pluses.

2) One can assume that \( u_j \neq 0, 2 \leq j \leq d - 2 \). Indeed, if some of the coefficients \( u_j \) equals 0, then for its nearby values the coefficients \( a_j \) retain the same signs, so the sign pattern of \( W \) remains the same.

**Notation 11.** For a given sign pattern \( \sigma^\triangle \), we denote by \( c(\sigma^\triangle) \) the number of its sign changes. For each sign pattern \( \sigma(W_\ast) \) of length \( d - 1 \) beginning with two pluses, we denote by \( S_d(\sigma(W_\ast)) \) the set of possible sign patterns \( \sigma(W) \). By \( T_d(\sigma(W_\ast)) \subseteq S_d(\sigma(W_\ast)) \) we denote its subset of sign patterns \( \sigma(W) \) such that \( c(\sigma(W)) = c(\sigma(W_\ast)) + 1 \).

**Example 12.** Suppose that \( d = 5 \) and \( \sigma(W_\ast) = (+, +, +) \). The coefficients of \( W \) equal

\[
1, \quad u_1, \quad u_2 - 1, \quad u_3 - u_1, \quad -u_2, \quad -u_3.
\]

One has \( u_1 > 0, -u_2 < 0 \) and \(-u_3 < 0 \). A priori the coefficients \( u_2 - 1 \) and \( u_3 - u_1 \) can have any sign. Therefore the set \( S_5(\sigma(W_\ast)) \) consists of the four sign patterns of the form \((+, +, \pm, \pm, -,-)\). As \( c(\sigma(W_\ast)) = 0 \), for the sign patterns of \( T_5(\sigma(W_\ast)) \) one has \( c(\sigma(W)) = 1 \), so

\[
T_5(\sigma(W_\ast)) = \{(+, +, +, +, -,-)\}, \quad (+, +, +, +,-,-), \quad (+, +,-,-,-, -,-)
\]

and \( S_5(\sigma(W_\ast)) \setminus T_5(\sigma(W_\ast)) = \{(+, +, +, +,-,-)\} \). We consider only polynomials \( W \) with all coefficients non-vanishing, therefore the possibilities to have \( u_2 - 1 = 0 \) and/or \( u_3 - u_1 = 0 \) are not taken into account.
More generally, suppose that \( d \geq 3 \). If \( u_j > 0, 1 \leq j \leq d - 2 \) (see (3.1), then \( c(\sigma(W_*)) = 0 \). The first two and the last two coefficients of \( W \) equal 1 > 0, \( u_1 > 0 \) and \( -u_{d-3} < 0, -u_{d-2} < 0 \) respectively. If \( \sigma(W) \in T_d(\sigma(W_*)) \) (so \( c(\sigma(W)) = 1 \)), then \( \sigma(W) \) consists of \( m \geq 2 \) signs plus followed by \( d + 1 - m \geq 2 \) signs minus. Hence this sign pattern contains a configuration \( A \).

**Remark 13.** The sets \( S_d(\sigma(W_*)) \) and \( T_d(\sigma(W_*)) \) can be defined for any real monic univariate, not necessarily hyperbolic, polynomial \( W_* \). In this case the polynomial \( W \) is also not necessarily hyperbolic. When the polynomials \( W_* \) and \( W \) are hyperbolic, they have \( c(\sigma(W_*)) \) and \( c(\sigma(W_*)) + 1 \) positive and \( d - 2 - c(\sigma(W_*)) \) and \( d - 1 - c(\sigma(W_*)) \) negative roots respectively, see the second paragraph of the Introduction.

Theorem 7 results from the following proposition:

**Proposition 14.** For any real monic univariate polynomial \( W_* \) as in (3.1), every sign pattern of its corresponding set \( T_d(\sigma(W_*)) \) contains at least one of configurations \( A - D \).

Hence if the sign pattern of the generic hyperbolic polynomial \( W \) contains none of configurations \( A - D \), then \( W \) is not representable in the form (3.1).

We give first examples of polynomials \( W \) in some of which we use a different notation. These examples will be used in the proof of Proposition 14.

**Example 15.** Set \( d := 3 \). Consider the polynomial \((x^2-1)(x+a) = x^3 + ax^2 - x - a, a \in \mathbb{R}^* \). For \( a > 0 \) (resp. for \( a < 0 \)), it defines the sign pattern \((+, +, -, -)\) (resp. \((+,-,-,+)\)) which is configuration \( A \) (resp. configuration \( C \)).

**Example 16.** Set \( d := 4 \). Consider for \( a, b \in \mathbb{R}^*, a > 0 \), the polynomial

\[(x^2 - 1)(x^2 + ax + b) = x^4 + ax^3 + (b-1)x^2 - ax - b .\]

For \( b > 0 \), it defines one of the sign patterns \((+, +, +, -)\) both of which contain configuration \( A \). For \( b < 0 \), it defines the sign pattern \((+,-,-,+)\) containing configurations \( A \) and \( C \).

**Example 17.** Set \( d := 5 \). Consider for \( a, b, r \in \mathbb{R}^*, a > 0 \), the polynomial

\[W := (x^2 - 1)(x^3 + ax^2 + bx + r) = x^5 + ax^4 + (b-1)x^3 + (r-a)x^2 - bx - r .\]

For \( b > 0 \) and \( r < 0 \), it defines one of the sign patterns \((+, +, +, -)\); for \( b < 0 \) and \( r > 0 \), it defines one of the sign patterns \((+, +, +, +, -)\); for \( b < 0 \) and \( r < 0 \), it defines the sign pattern \((+, +, +, -)\). Each of these sign patterns contains at least one of configurations \( A - D \).

For \( b > 0 \) and \( r > 0 \), one obtains the sign patterns \((+, +, +, +, -)\) of which only \((+, +, +, +, -)\) contains neither of configurations \( A - D \). However this sign pattern has three sign changes, so it does not belong to the set \( T_5(\sigma(W_*)) \) with \( \sigma(W_*) = (+, +, +, +, +) \), see Example 12.

**Example 18.** Set \( d := 6 \). Consider for \( a, b, r, g \in \mathbb{R}^*, a > 0 \), the polynomial

\[(x^2 - 1)(x^4 + ax^3 + bx^2 + rx + g) = x^6 + ax^5 + (b-1)x^4 + (r-a)x^3 + (g-b)x^2 - rx - g .\]
We give a list of cases in each of which either the sign pattern \( c(\sigma(W)) \) thus obtained contains at least one of configurations \( A-D \) or the condition \( c(\sigma(W)) = c(\sigma(W_\ast)) + 1 \) is violated (this takes place in the sign pattern \( (+,+,\ldots,+,\ldots,+) \) of case 7):

1. \( b < 0, r > 0, g > 0: (+,+,\ldots,+,\ldots,-) \);
2. \( b < 0, r > 0, g < 0: (+,+,\ldots,\pm,\ldots,+) \);
3. \( b < 0, r < 0, g > 0: (+,+,\ldots,-,\ldots,+) \);
4. \( b < 0, r < 0, g < 0: (+,+,\ldots,\pm,\ldots,+) \);
5. \( b > 0, r < 0, g > 0: (+,+,\ldots,\pm,\ldots,-) \);
6. \( b > 0, r < 0, g < 0: (+,+,\ldots,\pm,\ldots,+) \);
7. \( b > 0, r > 0, g < 0: (+,+,\ldots,\pm,\ldots,-) \).

The only case not included in this list is \( b > 0, r > 0, g > 0 \). It is covered by Example 12.

**Proof of Proposition 14.** We prove Proposition 14 by induction on \( d \). The induction base are the cases \( 3 \leq d \leq 6 \) considered in Examples 6, 8.

We compare the coefficients of the polynomial \( W \) for two consecutive degrees, \( d \) and \( d + 1 \), \( d \geq 6 \). We denote these polynomials by \( W_d \) and \( W_{d+1} \) and their corresponding polynomials \( W_\ast \) (see (3.1)) by \( W_{d,\ast} \) and \( W_{d+1,\ast} \). The coefficients of \( W_d \) and \( W_{d+1} \) are:

\[
(3.2) \quad 1 \quad u_1 \quad u_2 - 1 \quad \ldots \quad u_{d-3} - u_{d-5} \quad u_{d-2} - u_{d-4} \quad -u_{d-3} \quad -u_{d-2} \quad
\]

\[
1 \quad u_1 \quad u_2 - 1 \quad \ldots \quad u_{d-3} - u_{d-5} \quad u_{d-2} - u_{d-4} \quad u_{d-1} - u_{d-3} \quad -u_{d-2} \quad -u_{d-1} .
\]

**Definition 19.** We say that a sign pattern \( \sigma^\ast \) of length \( d + 1 \) belongs to the class \( M_d(m) \), \( 1 \leq m \leq d-2 \), if it contains one of configurations \( A-D \) in positions \( m \), \( m + 1 \), \( m + 2 \), \( m + 3 \). In particular, if \( \sigma^\ast \in M_d(d-4) \), then the following coefficients of \( \sigma^\ast \) form one of configurations \( A-D \):

\( u_{d-5} - u_{d-7} \), \( u_{d-4} - u_{d-6} \), \( u_{d-3} - u_{d-5} \) and \( u_{d-2} - u_{d-4} \).

The coefficient \( u_{d-2} - u_{d-4} \) is the rightmost of the coefficients which are the same for \( W_d \) and \( W_{d+1} \), see (3.2).

**Remark 20.** When passing from \( W_d \) to \( W_{d+1} \), one deduces from (3.2) that:

i) Changes occur only among the coefficients at the right end. Namely, the coefficient \( -u_{d-3} \) becomes \( u_{d-1} - u_{d-3} \) and the last coefficient \( -u_{d-1} \) is added.

ii) If the sign pattern \( \sigma(W_d) \) is in the class \( M_d(m_0) \) for some \( 1 \leq m_0 \leq d-4 \), then \( \sigma(W_{d+1}) \) is in the class \( M_{d+1}(m_0) \), see Definition 19.

By induction hypothesis the sign pattern \( \sigma(W_d) \) belongs to at least one of the classes \( M_d(m) \). If \( m \leq d-4 \), by ii) of Remark 20 one has \( \sigma(W_{d+1}) \in M_{d+1}(m) \). Therefore we need to consider only the cases

1) \( \sigma(W_d) \in M_d(d-2) \) and II) \( M_d(d-2) \neq \sigma(W_d) \in M_d(d-3) \).

**Case I.** Suppose first that \( \text{sgn}(u_{d-1} - u_{d-3}) = \text{sgn}(-u_{d-3}) \). This is true when, but not only when \( \text{sgn}(u_{d-1}) = -\text{sgn}(u_{d-3}) \). Hence all signs of the sign pattern
σ(W_{d+1}) except the last one coincide with the corresponding signs of the sign pattern σ(W_d), so σ(W_{d+1}) ∈ M_{d+1}(d − 2).

Suppose that sgn(u_{d−1} − u_{d−3}) = −sgn(−u_{d−3}). Hence sgn(u_{d−1}) = sgn(u_{d−3}).

We list the possible last four signs of σ(W_d) in the first line and the corresponding last five signs of σ(W_{d+1}) in the second line:

\( (+,+,−,−) \quad (−,−,+,+) \quad (+,−,+,−) \quad (−,+,+,−) \)

\( (+,+,+,−,−) \quad (−,−,−,+,+) \quad (+,−,−,+,-) \quad (−,−,+,-,+). \)

In all four cases the sign pattern σ(W_{d+1}) contains one of configurations A − D in its last four positions.

**Case II.** We list the last five signs of σ(W_d) in the first line and the corresponding last six signs of σ(W_{d+1}) in the second line:

\( (+,+,−,−,−) \quad (−,+,+,+,+) \quad (+,−,−,+,+), \quad (−,+,+,−,+). \)

\( (+,+,−,+,−,−) \quad (−,−,−,+,+), \quad (+,−,−,+,,+). \)

In the first two cases one has sgn(u_{d−1}) = sgn(u_{d−3}), so c(W_{d,*}) = c(W_{d+1,*}). Therefore one should have

\[ c(W_{d+1,*}) + 1 = c(W_{d+1}) = c(W_d) = c(W_{d,*}) + 1. \]

However one has \( c(W_{d+1}) = c(W_d) + 2. \)

In the last two cases the equality sgn(u_{d−1}) = −sgn(u_{d−3}) implies \( c(W_{d+1,*}) = c(W_{d,*}) + 1, \) so one should have \( c(W_{d+1}) = c(W_d) + 1, \) but one has \( c(W_{d+1}) = c(W_d) − 1. \) This contradiction proves the proposition. \( \square \)

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