BRAIDED FREE ORTHOGONAL QUANTUM GROUPS

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Abstract. We construct some braided quantum groups over the circle group. These are analogous to the free orthogonal quantum groups and generalise the braided quantum SU(2) groups for complex deformation parameter. We describe their irreducible representations and fusion rules and study when they are monoidally equivalent.

1. Introduction

The quantum SU(2) groups for a real deformation parameter \( q \) are archetypes for the theory of compact quantum groups. These are generalised in \cite{Meyer} to the case \( q \in \mathbb{C}^\times \). The underlying \( C^* \)-algebra \( B \) of \( SU_q(2) \) for \( q \in \mathbb{C}^\times \) is the universal \( C^* \)-algebra with two generators \( \alpha, \gamma \), subject to the relations \( \alpha^*\alpha + \gamma^*\gamma = 1 \), \( a\alpha^* + |q|^2\gamma^*\gamma = 1 \), \( \gamma^*\gamma = \gamma^*\gamma \), \( \alpha^*\gamma = \overline{\gamma}\alpha \) and \( \alpha\gamma^* = q\gamma^*\alpha \). The comultiplication no longer takes values in \( B \otimes B \), but in a certain twisted tensor product \( B \boxtimes B \). This twisted tensor product uses a parameter \( \zeta \in \mathbb{T} \) and the action of the circle group \( \mathbb{T} \) on \( B \) defined by \( \varphi_z(\alpha) = \alpha \) and \( \varphi_z(\gamma) = z\gamma \) for all \( z \in \mathbb{T} \).

For real \( q \), \( SU_q(2) \) is the universal quantum group with a representation \( S \) on \( \mathbb{C}^2 \) where a certain vector in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) is invariant under the tensor product representation \( S \boxtimes S \). Replacing \( \mathbb{C}^2 \) by \( \mathbb{C}^n \) gives the free orthogonal quantum groups by Van Daele and Wang \cite{VanDaeleWang}. The braided quantum group \( SU_q(2) \) for complex \( q \) has a similar universal property (see \cite{Meyer} Theorem 5.4). The difference is that \( \mathbb{C}^2 \) now carries a non-trivial representation of the circle group, which modifies the formula for \( S \boxtimes S \) and thus the condition for a vector to be invariant. In this article, we are going to construct braided compact quantum groups over the circle group \( \mathbb{T} \) that generalise both the free orthogonal quantum groups and the braided \( SU_q(2) \) of \cite{Meyer}.

The data to define them is a triple \((V, \pi, \omega)\) consisting of a finite-dimensional vector space \( V \) with a representation \( \pi \) of \( \mathbb{T} \) and a \( \mathbb{T} \)-homogeneous vector \( \omega \in V \otimes V \), subject to a condition that requires some notation. Let \( V_k \subseteq V \) for \( k \in \mathbb{Z} \) be the subspace of \( k \)-homogeneous elements of \( V \). Choose bases for each \( V_k \) so that a vector \( \omega \in V \otimes V \) corresponds to a matrix. Write this matrix as a block matrix \( \Omega = (\Omega_{ij}) \) according to the decomposition \( V = \bigoplus V_k \). Assume that \( \omega \) is \( d \)-homogeneous for \( d \in \mathbb{Z} \). Then \( \Omega_{ij} = 0 \) unless \( i + j = d \). To construct a braided quantum group, we need \( \Omega \) to satisfy the following condition: there is a non-zero scalar \( c \in \mathbb{C}^\times \) with

\[
\Omega_{i,d-i} \cdot \Omega_{d-i,i} = c \cdot \zeta^{d^2} \cdot 1 \quad \text{for all} \quad i \in \mathbb{Z},
\]

where \( 1 \) denotes the identity matrix of appropriate size and \( \zeta \) is the twisting parameter for the braided tensor product. Then the matrices \( \Omega_{i,d-i} \) are invertible for all \( i \in \mathbb{Z} \), and \( \tau/c = \zeta^{d^2} \) is needed for the equations for \( i \) and \( d-i \) to be compatible. Combining the blocks \( \Omega_{ij} \) in one matrix \( \Omega \), we may rewrite (1.1) more...
briefly as
\begin{equation}
\Omega\Omega = c \cdot \pi_d.
\end{equation}
Equation (1.1) is easy to solve for \(i \neq d/2\) by choosing arbitrary invertible matrices \(\Omega_{n,i,d-1}\) for \(i \in \mathbb{Z}\) with \(i < d/2\) and letting \(\Omega_{d-1,i,i} := c^{d-i}\Omega_{n,i,d-1}^{-1}\). If \(d\) is even and \(i = d/2\), then (1.1) becomes \(\Omega_{d/2,d/2} \cdot \Omega_{d/2,d/2} = c^d\), which can be solved because \(c \cdot \zeta^d/2\) is real if \(c^d = \zeta^d\). Thus there is a large parameter space for our new braided quantum groups. We now fix the deformation parameter \(\zeta\).

A braided compact quantum group over \(\mathbb{T}\) is a unital \(C^*\)-algebra \(B\) with a continuous \(\mathbb{T}\)-action \(\beta: B \to B \otimes C(\mathbb{T})\) and a coassociative, bisimplifiable comultiplication \(\Delta_B: B \to B \otimes \xi\). Here \(\otimes\) is a Rieffel deformation of the usual spatial tensor product \(B \otimes B\), where we use the canonical action of \(\mathbb{T}^2\) on \(B \otimes B\). We treat this using the monoidal structure on \(\mathbb{T}^\ast\)-algebras discussed in [4, Section 3]. A right representation of \(G\) on \((V, \pi)\) is a \(\mathbb{T}\)-invariant unitary \(u \in U(\text{End}(V) \otimes B)\) with \(\Delta_B(u) = (j_1 \otimes \text{id})(u) \cdot (j_2 \otimes \text{id})(u)\), where \(j_1, j_2: B \cong B \otimes \xi\) are the two canonical embeddings. There is a tensor product operation on representations of \(B\). Roughly speaking, the braided orthogonal quantum group \(A_o(V, \pi, \omega)\) is the universal \(\mathbb{T}\)-braided quantum group with a representation \(u\) on \((V, \pi)\) such that \(\omega \in V \otimes V\) is invariant for the representation \(u \circ \pi\). This invariance condition is equivalent to a linear relation among the matrix entries of \(u\) and \(u^\ast\). The condition on \(\omega\) in (1.1) ensures that this relation is equivalent to its adjoint, so that it does not produce linear relations among the matrix entries of \(u\). It also implies that the representation \(u\) is irreducible.

We now describe the braided free orthogonal quantum groups explicitly. Choose a basis \(e_1, \ldots, e_n\) for \(V\) such that \(\pi_x(e_i) = z^{d_i} e_i\) with \(d_1 \leq d_2 \leq \cdots \leq d_n\). Write \(\omega\) in this basis as \(\omega = \sum_{i,j=1}^n \omega_{ij} e_i \otimes e_j\). The \(C^\ast\)-algebra of the braided quantum group \(A_o(V, \pi, \omega)\) is the universal unital \(C^\ast\)-algebra with generators \(u_{ij}\) for \(1 \leq i, j \leq n\) subject to the relations
\begin{align}
\sum_{k=1}^n u_{ki}^* u_{kj} &= \delta_{i,j}, \\
\sum_{k=1}^n u_{ik} u_{jk}^* &= \delta_{i,j}, \\
\zeta^{d_i d_j} \sum_{k=1}^n \omega_{ik} u_{jk} &= \zeta^{d_i (d_j-d_i)} \sum_{k=1}^n \omega_{kj} u_{ki},
\end{align}
for \(1 \leq i, j \leq n\). Here \(u_{ij}\) is the \(i, j\)th coefficient of the canonical representation \(u \in M_n(A_o(V, \pi, \omega))\). The two relations (1.3) and (1.4) say that \(u\) is unitary. Equation (1.5) says that \(\omega\) is invariant. The action \(\beta\) of \(\mathbb{T}\) on \(A_o(V, \pi, \omega)\) and the comultiplication \(\Delta: A_o(V, \pi, \omega) \to A_o(V, \pi, \omega) \otimes_{\xi} A_o(V, \pi, \omega)\) are defined by
\begin{align}
\beta_x(u_{ik}) &= z^{d_k-d_i} u_{ik}, \\
\Delta(u_{ik}) &= \sum_{l=1}^n j_1(u_{il}) j_2(u_{lk}),
\end{align}
where \(j_1, j_2: A_o(V, \pi, \omega) \cong A_o(V, \pi, \omega) \otimes_{\xi} A_o(V, \pi, \omega)\) are the canonical morphisms to the braided tensor product. These are the unique action of \(\mathbb{T}\) and comultiplication for which \(u\) is a representation.

If \(V = \mathbb{C}^2\) and \(d_1 = 0, d_2 = 1\), the definition above gives the braided \(SU_q(2)\) groups of [3]. If \(d_i = 0\) for \(i = 1, \ldots, n\), then \(\beta\) is trivial and we get the usual free orthogonal quantum groups of [13]. As an interesting new case, let \(n = 2m\) be even and let \(d_i = 0\) for \(1 \leq i \leq m\) and \(d_i = 1\) for \(m < i \leq 2m\). Let \(d = 1\). Then a
d-homogeneous vector $\omega$ is equivalent to two $m \times m$-matrices $\Omega_0$, $\Omega_1$, which form the block matrix

$$
\begin{pmatrix}
0 & \Omega_1 \\
\Omega_0 & 0
\end{pmatrix}.
$$

Our construction requires two conditions, namely, $\overline{\Omega_0} \cdot \Omega_1 = c$ and $\overline{\Omega_1} \cdot \Omega_0 = c \zeta$. Thus $\Omega_0$ may be an arbitrary invertible $m \times m$-matrix, and $\Omega_1 = \frac{1}{c} \Omega_0^{-1}$, $\zeta = c/e$. If $m = 1$, this gives the braided $SU_q(2)$ groups of \cite{[4]}. If $c$ is real or, equivalently, if $\zeta = 1$, then it gives free orthogonal quantum groups in the usual sense.

Banica \cite{[1]} has described the irreducible representations of the free orthogonal quantum groups and their tensor products, showing that they behave like those of the Lie group $SU(2)$. His result was made more precise by Bichon, De Rijdt and Vaes \cite{[2]}, who established that any free orthogonal quantum group is monoidally equivalent to $SU_q(2)$ for a specific $q$. We prove analogues of these results for our braided free orthogonal quantum groups.

**Theorem 1.8.** If $d$ is even, then the braided orthogonal quantum group $A_d(V, \pi, \omega)$ has irreducible representations $r_{(k,l)}$ for $k \in \mathbb{N}$, $l \in \mathbb{Z}$, such that any irreducible representation is unitarily equivalent to exactly one of these and $r_{(k,l)} = r_{(k,-l)}$ and

$$
\begin{align*}
\forall (a,b) \in \mathbb{N}^2 & \quad r_{(a,b)} \otimes r_{(m,k)} \cong r_{(a+m,b+k)} \oplus r_{(a+m-2,b+k)} \oplus r_{(a+m-4,b+k)} \oplus \cdots \oplus r_{(a-m,b+k)}.
\end{align*}
$$

If $d$ is odd, then a similar statement holds, but we only allow those representations where $a - b$ is even.

The fusion rules above are those of the group $SU(2) \times T$ in the even case, and those of the group $U(2)$ in the odd case (see \cite{[10] Theorem 6.3]). The extra circle occurs because representations of a braided quantum group are equivalent to representations of an associated quantum group, namely, its semidirect product with the quantum group over which it is braided (see \cite{[6] Theorem 3.4}). This is a C*-algebraic variant of the bosonisation of Radford and Majid (see \cite{[5][12]}). Therefore, we also change our notation from “semidirect product” to “bosonisation”. The classical case of bosonisation is a semidirect product of groups. The relevant classical example for us is that $U(2)$ is a semidirect product of $SU(2) \subseteq U(2)$ and the circle group $\mathbb{T}$, embedded into $U(2)$ through $\iota: \mathbb{T} \rightarrow U(2)$, $x \mapsto \text{diag}(x,1)$.

The bosonisation for $SU_q(2)$ is isomorphic to $U_q(2)$ for the quantum $U(2)$ groups of \cite{[17][18]}. If $q$ is real, then $U_q(2)$ is a special case of the groups $U_q(n)$, which go back to \cite{[11]}. If $q = \exp(i\theta)$ has absolute value 1, then $U_q(2)$ specialises to the $\theta$-deformation $U_0$ of $U(2)$ defined by Connes and Dubois-Violette \cite{[3]}.

In particular, when we view the usual $SU_q(2)$ for real $q$ as a braided quantum group over $\mathbb{T}$, we are implicitly dealing with the corresponding $U_q(2)$, which is its bosonisation. We describe the representation category of $A_d(V, \pi, \omega)$ using the representation category of an ordinary orthogonal quantum group. Using the monoidal equivalences proven in \cite{[2]}, this implies a monoidal equivalence between $A_d(V, \pi, \omega)$ and either $SU_q(2) \times T$ or $U_q(2)$ depending on the parity of $d$.

**Theorem 1.9.** There is a unique $q \in [-1,1] \setminus \{0\}$ such that the representation category of $A_d(V, \pi, \omega)$ is equivalent to the representation category of $SU_q(2) \times T$ if $d$ is even and to the representation category of $U_q(2)$ if $d$ is odd.

## 2. The universal property

Let $G = (B, \beta, \Delta)$ be a braided quantum group over $\mathbb{T}$. A representation of $G$ on $\mathcal{C}^*$ consists of a representation of $\mathbb{T}$ on $\mathcal{C}^*$ and a unitary $u \in \mathbb{M}_n(\mathbb{C}) \otimes B$ that is $\mathbb{T}$-invariant and satisfies a braided form of the corepresentation condition (see, for instance, \cite{[4][6]}). In particular, any representation of $\mathbb{T}$ and $u = 1$ form a “trivial” representation of $G$. Let $\mathcal{C}_\ell$ for $\ell \in \mathbb{Z}$ be $\mathbb{C}$ with the representation $z \mapsto z^\ell$ of $\mathbb{T}$ and
with the representation \( u = 1 \) of \( G \). In the following, we will identify \( M_n(\mathbb{C}) \otimes B \) with \( M_n(B) \) and write elements as \((b_{ij})_{1 \leq i, j \leq n}\) or \( \sum_{1 \leq i, j \leq n} e_{ij} \otimes b_{ij} \).

Let \( u = (u_{ij})_{1 \leq i, j \leq n} \in M_n(B) \) be a representation of \( G \) on \( V = \mathbb{C}^n \) with underlying representation \( \pi \) of \( T \). A \( d \)-homogeneous \( G \)-invariant vector is defined as a vector \( \omega \in V \otimes V \) such that the map \( C_d \to V \otimes V, c \mapsto c \cdot \omega \), is an intertwining operator for the representations of both \( T \) and \( G \). Let \( e_1, \ldots, e_n \) be an eigenbasis for \( \pi \), that is, there are \( d_1, \ldots, d_n \in \mathbb{Z} \) with \( \pi_z(e_i) = z^{d_i} e_i \) for \( i = 1, \ldots, n \). We order the basis so that \( d_1 \leq d_2 \leq \cdots \leq d_n \). Let \( \omega \in V \otimes V \) and write \( \omega = \sum_{1 \leq i, j \leq n} \pi_z(e_i, e_j) = \sum_{1 \leq i, j \leq n} u_{ij} \otimes e_i \otimes e_j \). The vector \( \omega \) is \( d \)-homogeneous – that is, the resulting map \( C_d \to V \otimes V \) is \( T \)-equivariant – if and only if \( d_i + d_j = d \) for all \( i, j \) with \( \omega_{i,j} \neq 0 \).

We assume this from now on.

**Proposition 2.1.** The matrix \((u_{ij})_{1 \leq i, j \leq n}\) satisfies the relations \([1.3],[1.4]\) if and only if it is a representation of \( G \) on \( V \) with underlying representation \( \pi \) of \( T \) and \( \omega \) is a \( d \)-homogeneous \( G \)-invariant vector in \( V \otimes V \).

**Proof.** A representation must be unitary, \( T \)-invariant and satisfy

\[
\Delta(u) = (j_1 \otimes \text{id})(u) \cdot (j_2 \otimes \text{id})(u).
\]

Being unitary says that \( u^* u = 1 \) and \( uu^* = 1 \), which translates to the relations \([1.3],[1.4]\) for the coefficients \( u_{ij} \). The action \( \Pi \) on \( M_n(\mathbb{C}) \) induced by \( \pi \) is \( \Pi_z(u_{ij}) = z^{d_i - d_j} u_{ij} \). Thus \( u = \sum e_{ij} \otimes u_{ij} \) is \( T \)-invariant if and only if the \( T \)-action \( \beta \) on \( B \) satisfies \( \beta_z(u_{ij}) = z^{d_i - d_j} u_{ij} \), which is \([1.6]\). Equation \([1.7]\) is equivalent to \( \Delta(u) = (j_1 \otimes \text{id})(u) \cdot (j_2 \otimes \text{id})(u) \). Thus these four equations say exactly that \( u \) is a representation of \( G \). It remains to show that the \( G \)-invariance of \( \omega \) is equivalent to \([1.5]\). We have already assumed that the map \( C_d \to V \otimes V, c \mapsto c \omega \), is \( T \)-equivariant. We must describe when this map is \( G \)-equivariant for the tensor product representation of \( G \) on \( V \otimes V \).

We represent \( B \) faithfully on a Hilbert space to explain the definition of the tensor product for representations of braided quantum groups. Let \( \mathcal{L} \) be a separable Hilbert space with a continuous representation \( \rho \) of \( T \) and let \( B \hookrightarrow \mathcal{B}(\mathcal{L}) \) be a faithful, \( T \)-equivariant representation. There is an orthonormal basis \((\lambda_m)_{m \in \mathbb{N}}\) for \( \mathcal{L} \) consisting of eigenvectors for the \( T \)-action, that is, \( \rho_z(\lambda_m) = z^{l_m} \lambda_m \) with some \( l_m \in \mathbb{N} \). Since the representation of \( B \) is \( T \)-equivariant,

\[
\rho_z(u_{ij} \lambda_m) = \beta_z(u_{ij}) \rho_z(\lambda_m) = z^{d_j - d_i + l_m} u_{ij} \lambda_m,
\]

\[
\rho_z(u_{ij}^* \lambda_m) = \beta_z(u_{ij}^*) \rho_z(\lambda_m) = z^{d_i - d_j + l_m} u_{ij}^* \lambda_m
\]

for all \( 1 \leq i, j \leq n \) and \( m \in \mathbb{N} \).

Fix \( \zeta \in T \) and define the \( \mathbb{R} \)-matrix on \( T \) by \( \mathbb{Z} \times \mathbb{Z} \ni (l, m) \mapsto \zeta^{l_m} \in T \). The associated braiding unitary \( \zeta^{X_L} : \mathcal{L} \otimes \mathcal{V} \to \mathcal{V} \otimes \mathcal{L} \) and its inverse \( \zeta^{-X_L} : \mathcal{V} \otimes \mathcal{L} \to \mathcal{L} \otimes \mathcal{V} \) act on the basis by

\[
\zeta^{X_L}(\lambda_m \otimes e_i) = \zeta^{d_j - l_m} e_i \otimes \lambda_m,
\]

\[
\zeta^{-X_L}(e_i \otimes \lambda_m) = \zeta^{d_i - l_m} \lambda_m \otimes e_i.
\]

The tensor product representation \( u \otimes u \) is the preimage in \( M_n(\mathcal{B}) \) of the unitary

\[
u_{23}\omega_{12} = (\zeta^{X_L}_{23} u_{12} \zeta^{X_L}_{23})^* \omega_{12} = \zeta^{X_L}_{23} u_{12}^* \zeta^{X_L}_{23} \omega_{12}.
\]
We use that \( u_{k,l}^* \lambda_m \) is T-homogeneous of degree \( d_k - d_l + l_m \) to compute
\[
\left( \zeta \lambda_{V,23} u_{12} \lambda_{V,23} \right)^n e_k \otimes e_j \otimes \lambda_m = \zeta^{-d_k+l_m} \lambda_{V,23} u_{12}^* \lambda_{V,23} (u_{k,l} \otimes \lambda_m) \otimes e_j
\]
\[
= \sum_{i=1}^{n} \zeta^{-d_k} \lambda_{V,23} (u_{ik}^* \lambda_m) \otimes e_j
\]
\[
= \sum_{i=1}^{n} \zeta^{-d_j l_m + d_j (d_k - d_l + l_m)} e_i \otimes e_j \otimes u_{ik}^*(\lambda_m)
\]
\[
= \sum_{i=1}^{n} \zeta^{d_j (d_k - d_l)} e_i \otimes e_j \otimes u_{ik}^*(\lambda_m).
\]
As a result, the invariance condition (2.4) becomes
\[
\sum_{i,j,k=1}^{n} e_i \otimes e_j \otimes u_{ik} \omega_{kj} = \sum_{i,j,k=1}^{n} e_i \otimes e_j \otimes u_{ik}^* \zeta^{d_j (d_k - d_l)} \omega_{kj}.
\]
Comparing the coefficient of each summand \( e_i \otimes e_j \) gives (1.5) because \( d_k + d_j = d \) whenever \( \omega_{kj} \neq 0 \) by the homogeneity of \( \omega \).

Assume the matrix \( \tilde{\omega} := (\omega_{ij} \zeta^{d_i d_j}) \) to be invertible. Multiplying both sides in (1.5) by \( (\tilde{\omega}^{-1})_{jl} \) and summing over \( j \) gives
\[
\sum_{j,k=1}^{n} \zeta^{d_j d_k} u_{ik} \omega_{kj} (\tilde{\omega}^{-1})_{jl} = \sum_{j,k=1}^{n} \zeta^{d_j d_k} u_{ik}^* \omega_{kj} (\tilde{\omega}^{-1})_{jl} = \sum_{k=1}^{n} u_{ki}^* \delta_{k,l} = u_{il}.
\]
Taking adjoints in \( B \) and substituting \((s,t)\) for \((j,k)\) then gives
\[
\sum_{j,k=1}^{n} \zeta^{-d_j d_k} \tilde{\omega}_{jl} (\tilde{\omega}^{-1})_{st} u_{st} = u_{li}.
\]
Now plug (2.5) for \( l = s, i = t \) into (2.6). This gives the following linear relation among the generators \( u_{j,k} \):
\[
u_{li} = \sum_{j,k,s,t=1}^{n} \zeta^{-d_j d_i + d_j d_i} \tilde{\omega}_{is} (\tilde{\omega}^{-1})_{sl} u_{js} \omega_{tk} (\tilde{\omega}^{-1})_{jst}.
\]
We want the generators \( u_{j,k} \) of our braided quantum group to be linearly independent. So the linear relation (2.7) should be trivial. Equivalently,
\[
\sum_{j,k,s,t=1}^{n} \zeta^{-d_j d_i + d_j d_i} \tilde{\omega}_{is} (\tilde{\omega}^{-1})_{sl} u_{js} \omega_{tk} (\tilde{\omega}^{-1})_{jst} = \delta_{j,l} \delta_{k,i}.
\]

Lemma 2.9. The relation (2.8) is equivalent to (1.1).

Proof. Since \( \tilde{\omega}_{ll} = \omega_{ll} \zeta^{-d_i d_i} \) and \( \tilde{\omega}_{lk} = \omega_{jk} \zeta^{d_j d_k} \), (2.8) is equivalent to
\[
\delta_{j,l} \delta_{k,i} = \sum_{s,t=1}^{n} \zeta^{-d_j d_i + d_j d_i} \tilde{\omega}_{is} (\tilde{\omega}^{-1})_{sl} u_{js} \omega_{tk} (\tilde{\omega}^{-1})_{jst}.
\]
Since \( \omega \) is of degree \( d \) in \( V \otimes V \), so is \( \tilde{\omega} \). Equivalently, the corresponding linear map \( \tilde{\omega} : V \to V \) maps elements of degree \( a \) to elements of degree \( d - a \). Then the inverse maps elements of degree \( d - a \) to elements of degree \( a \). Thus both \( \tilde{\omega} \) and \( \tilde{\omega}^{-1} \) are homogeneous of degree \( d \) in \( V \otimes V \). That is, \( \tilde{\omega}_{ab} = 0 \) and \( \tilde{\omega}_{ab}^{-1} = 0 \) unless \( d_a + d_b = d \). Therefore, if \( \tilde{\omega}_{j,k} (\tilde{\omega}^{-1})_{j,k} \neq 0 \), then \( d_i + d_j = d, d_s + d_i = d, d_t + d_k = d \) and \( d_j + d_s = d \). Equivalently, \( d_i = d_k, d_i = d_j, d_i = d - d_i \) and \( d_s = d - d_j \). Then the exponent of \( \zeta \) becomes
\[
-d_s d_i + d_i d_s + d_j d_t - d_i d_k = (d_j - d) d_i + d_j (d - d_i) = d(d_j - d_i).
\]
This does not involve the summation indices $s,t$, so we may put it on the other side of the equation. Thus (2.8) is equivalent to

\[(\zeta^{-d}d_{\delta_{j,k}}) \cdot (\zeta^{d}d_{\delta_{i,k}}) = \sum_{s,t=1}^{n} \omega_{st}(\bar{\omega}^{-1})_{is} \bar{\omega}_{tk} (\bar{\omega}^{-1})_{js} = (\bar{\omega}^{-1})_{ij} (\bar{\omega} \cdot \bar{\omega})_{ik}.
\]

This is an equality of two exterior tensor products of matrices with the entries $j,l$ and $i,k$, respectively. A matrix equality $X_1 \otimes Y_1 = X_2 \otimes Y_2$ with $X_1,Y_1 \neq 0$ holds if and only if $X_1 = c \cdot X_2$, $Y_2 = c \cdot Y_1$ with some non-zero scalar $c$. Thus we get a scalar $c$ with $(\bar{\omega} \cdot \bar{\omega})_{ik} = c \zeta^{dd_{\delta_{j,k}}}$ and $(\bar{\omega}^{-1})_{ij} = c^{-1} \zeta^{-dd_{\delta_{j,k}}}$. These two conditions are equivalent to each other by taking inverses.

Now we combine the entries $\omega_{ij}$ with $d_i + d_j = d$ into a block matrix $\Omega_{a,b}$ as in (1.1). Recall that $\Omega_{a,b} = 0$ unless $a + b = d$. The change from $\omega_{ij}$ to $\bar{\omega}_{ij}$ replaces $\Omega_{a,d-a}$ by $\bar{\Omega}_{a,d-a} := \zeta^{a(d-a)} \Omega_{a,d-a}$. So $(\bar{\omega} \cdot \bar{\omega})_{ik}$ is equivalent to $\zeta^{a(d-a)} \bar{\Omega}_{a,d-a} - \zeta^{a(d-a)} \Omega_{d-a,a} = c \cdot \zeta^{dd_{\delta_{j,k}}}$. Thus $\bar{\Omega}_{a,d-a}$ is an intertwiner for the representation $\Omega_{a,b}$ and $\bar{\Omega}_{a,d-a}$. We may replace the exponent $d_i$ with some non-zero scalar $1$, where 1 denotes the identity matrix of appropriate size. This is exactly (1.1).

Lemma 2.9 explains why we impose the condition (1.1) for our construction: otherwise, some of our generators $u_{ij}$ would be redundant. The following lemma was suggested by the anonymous referee. It interprets this relation in another way:

**Lemma 2.10.** Assume only that $\Omega$ is invertible. The representation $u$ is irreducible if and only if (1.1) holds.

**Proof.** The proof of Theorem 1.8 will show that $u$ is irreducible. It uses (1.1), which is a standing assumption for the construction of $A_o(V,\pi,\omega)$. Here we will only prove the converse. If $u$ is irreducible, any intertwiner of $u$ is a constant multiple of the identity matrix. We are going to rewrite (2.7) as saying that the transpose of $\overline{\Omega} \Omega \pi_{\zeta^{-d}}$ is an intertwiner of $u$.

We have seen in the proof of Lemma 2.9 that $\omega$ and $\bar{\omega}$ are homogeneous of degree $d$. Therefore, any non-zero summand on the right hand side of (2.7) has $d_i + d_j = d = d_i + d_k$ and $d_j + d_\ell = d$. Thus we may replace the exponent $d_i d_j - d_i d_\ell$ of $\zeta$ by $d_j (d - d_i) - d_\ell (d - d_j) = d d_j - d d_\ell$. We combine the entries $\omega_{ij}$ and $\bar{\omega}_{ij}$ to matrices $\Omega$ and $\bar{\Omega}$, and we let $u^\top$ be the transpose of $u$. We rewrite (2.7) as an equality of matrix products:

\[u^\top = \overline{\Omega} \Omega \pi_{\zeta^{-d}} u^\top \pi_{\zeta^{-d}} \overline{\Omega}^{-1}\Omega^{-1}.
\]

This is equivalent to $u^\top \overline{\Omega} \Omega \pi_{\zeta^{-d}} = \overline{\Omega} \Omega \pi_{\zeta^{-d}} u^\top$. Using that $\Omega$ is homogeneous, we rewrite the entries of $\overline{\Omega}$ as

\[(\overline{\Omega})_{ab} = \sum_{b=1}^{n} c^{-(d-d_\ell) d_\ell} d_\ell \omega_{a\ell} \omega_{b\ell} = \sum_{b=1}^{n} c^{-(d-d_\ell) d_\ell} \omega_{a\ell} \omega_{b\ell} = \sum_{b=1}^{n} \omega_{a\ell} \omega_{b\ell}.
\]

That is, $\overline{\Omega} = \Omega \pi_{\zeta^{-d}} u^\top = \overline{\Omega} \Omega \pi_{\zeta^{-d}} u^\top$. Taking transposes, we see that the transpose of $\overline{\Omega} \Omega \pi_{\zeta^{-d}}$ is an intertwiner for the representation $u$. Thus it is a scalar multiple of 1. Equivalently, $\overline{\Omega} \Omega \pi_{\zeta^{-d}} = c \cdot 1_n$ for some $c \in \mathbb{C}$. Since $\Omega$ is invertible, $c \in \mathbb{C}^\times$. This is equivalent to (1.2), which is equivalent to (1.1).

So far we have worked with a general braided quantum group $G$ to explain the defining equations (1.3)–(1.7) of $A_o(V,\pi,\omega)$. Now we show that these equations do define a braided quantum group.

Since the matrix $(u_{ij})_{1 \leq i,j \leq n} \in M_n(A_o(V,\pi,\omega))$ is unitary, $\|u_{ij}\| \leq 1$ for $1 \leq i,j \leq n$. Hence the universal $C^\ast$-algebra with the generators $u_{ij}$ and the relations (1.3)–(1.7) exists. To construct it, we start with the universal unital $^\ast$-algebra $A_o(V,\pi,\omega)$ with these generators and relations. Since any $C^\ast$-seminorm
on $A_0(V, \pi, \omega)$ satisfies $\|u_{ij}\| \leq 1$ for $1 \leq i, j \leq n$, there is a largest $C^*$-semigroup on $A_0(V, \pi, \omega)$. And $A_0(V, \pi, \omega)$ is the completion of $A_0(V, \pi, \omega)$ in this largest $C^*$-semigroup.

**Proposition 2.11.** There is a unique unital $^*$-homomorphism $\beta: A_0(V, \pi, \omega) \to C(\mathbb{T}, A_0(V, \pi, \omega))$ satisfying (1.6). It defines a continuous $\mathbb{T}$-action $\beta$ on $A_0(V, \pi, \omega)$.

*Proof.* The formula $\beta_z(u_{ij}) = z^{d_j-d_i}u_{ij}$ clearly defines a $\mathbb{T}$-action on the free $^*$-algebra generated by $u_{i,j}$. We must show that the relations that define $A_0(V, \pi, \omega)$ are homogeneous. The relations (1.3) and (1.4) are homogeneous because

$$\beta_z(u^*_{ik}u_{kj}) = z^{d_j-d_i}u^*_{ik}u_{kj}, \quad \beta_z(u_{ik}u^*_{kj}) = z^{d_j-d_i}u_{ik}u^*_{kj}$$

for all $1 \leq i, j, k \leq n$. The relation (1.5) is homogeneous because $\omega_{i,k} = 0$ for $d_i + d_k \neq d_i$, so that

$$\beta_z(\zeta^{d_i}d_j \omega_{ik}u_{j,k}) = z^{d_k-d_i} \cdot \zeta^{d_i}d_j \omega_{ik}u_{j,k} = z^{d_k-d_j}d_i \omega_{ik}u_{j,k},$$

$$\beta_z(\zeta^{d_i}d_j \omega_{kj}u^*_{k,i}) = z^{d_k-d_i} \cdot \zeta^{d_i}d_j \omega_{kj}u^*_{k,i} = z^{d_k-d_j}d_i \omega_{kj}u^*_{k,i}$$

for all $i, j, k$. \hfill $\square$

**Proposition 2.12.** There is a unique unital $^*$-homomorphism $\Delta: A_0(V, \pi, \omega) \to A_0(V, \pi, \omega) \otimes A_0(V, \pi, \omega)$ satisfying (1.7). It is $\mathbb{T}$-equivariant, coassociative, and satisfies the Podleś condition.

*Proof.* We abbreviate $B := A_0(V, \pi, \omega)$. Recall the faithful representation $B \hookrightarrow B(\mathcal{L})$ from the proof of Proposition 2.1. Define the braiding unitaries $\xi^\mathcal{L}_V$ and $\epsilon^\mathcal{L}_V$ on $\mathcal{L} \otimes \mathcal{L}$ by

(2.13) $\xi^\mathcal{L}_V(\lambda_\alpha \otimes \lambda_\beta) := \zeta^{l_\alpha} \lambda_\alpha \otimes \lambda_\beta$, \quad $\epsilon^\mathcal{L}_V(\lambda_\alpha \otimes \lambda_\beta) := \zeta^{-l_\alpha} \lambda_\beta \otimes \lambda_\alpha$.

There is a faithful representation $B \otimes B \hookrightarrow B(\mathcal{L} \otimes \mathcal{L})$, such that the inclusions $j_1, j_2: B \hookrightarrow B \otimes B$ of the tensor factors become $j_1(x) = x_1$ and $j_2(x) = \xi^\mathcal{L}_V x_1 \epsilon^\mathcal{L}_V$ in leg numbering notation. Let $U := u_{12} \xi^\mathcal{L}_V u_{23} \xi^\mathcal{L}_V u_{34} \in B(\mathcal{L} \otimes \mathcal{L})$. A $^*$-homomorphism $\Delta: B \to B \otimes B$ satisfying (1.7) exists if and only if the matrix coefficients of $U$ satisfy the relations (1.3)–(1.5) that define $A_0(V, \pi, \omega)$. Equations (1.3) and (1.4) hold because $U$ is unitary. Define

$$U \otimes U := (\xi^\mathcal{L}_V)^{234} U_{123} V^\mathcal{L} \otimes \mathcal{L}^{234} U_{234} \in B(V \otimes V \otimes \mathcal{L} \otimes \mathcal{L})$$

as in (2.3); here

(2.14) $\xi^\mathcal{L}_V(\lambda_\alpha \otimes \lambda_\beta \otimes e_1) = \zeta^{l_\alpha} \lambda_\alpha \otimes \lambda_\beta \otimes e_1$, \quad $\epsilon^\mathcal{L}_V = (\xi^\mathcal{L}_V)^*.

The proof of Proposition 2.1 implies $\epsilon^\mathcal{L}_V u_{12} \xi^\mathcal{L}_V u_{23} \xi^\mathcal{L}_V u_{34} = (u \otimes u) \omega_{12} = \omega_{12}$ and shows that the coefficients of $U$ satisfy (1.3) if and only if $(U \otimes U) \omega_{12} = \omega_{12}$. Now we compute

$$(U \otimes U) \omega_{12} = \xi^\mathcal{L}_V u_{12} \xi^\mathcal{L}_V u_{23} \xi^\mathcal{L}_V u_{34} \xi^\mathcal{L}_V \epsilon^\mathcal{L}_V \epsilon^\mathcal{L}_V = (u \otimes u) \omega_{12} = \omega_{12}.$$  

We are going to prove $(U \otimes U) \omega_{12} = \omega_{12}$ using $(u \otimes u) \omega_{12} = \omega_{12}$ and properties of the braiding unitaries. This computation is more readable when written in pictures. Using the definition (2.3) of $u \otimes u$, we rewrite $(u \otimes u) \omega_{12} = \omega_{12}$ through the diagram

(2.15)
The following diagrams illustrate the proof that $(U ⊠ U)ω_{12} = ω_{12}$.

This finishes the construction of $\Delta : A_o(V, π, ω) → A_o(V, π, ω) ⊠ A_o(V, π, ω)$.

A direct computation shows that $\Delta$ is $\mathbb{T}$-equivariant for the $\mathbb{T}$-action $β$ on $A_o(V, π, ω)$ and the induced $\mathbb{T}$-action on $A_o(V, π, ω) ⊠ A_o(V, π, ω)$. It is coassociative because both $(\Delta ⊠ \text{id}) ◦ \Delta$ and $(\text{id} ⊠ \Delta) ◦ \Delta$ map $u$ to $j_1(u) ⊠ j_2(u) ⊠ j_3(u)$.

The Podleś condition for $B := A_o(V, π, ω)$ follows by the same argument as in [4, Section 4]. Let $S := \{b ∈ B : j_1(b) ∈ \Delta(B)j_2(B)\}$. The set $S$ contains $u_{ij}$ and $u_{ij}^*$ for all $1 ≤ i, j ≤ n$ because $j_{j_1} = \sum_{k=1}^n \Delta(u_{ij})j_2(u_{kj})$. Let $x, y ∈ S$ be homogeneous of degree $k$ and $l$, respectively. Then $j_1(x)j_2(y) = \zeta^{ij}j_2(y)j_1(x)$. This implies $j_1(x)j_2(B) = j_2(B)j_1(x)$ and $j_1(B)j_2(y) = j_2(y)j_1(B)$ as in [3, Proposition 3.1]. So

$$j_1(x · y) = j_1(x)j_1(y) ∈ \Delta(B)j_2(B)j_1(y) = \Delta(B)j_1(y)j_2(B) \subseteq \Delta(B)\Delta(B)j_2(B) = \Delta(B)j_2(B).$$

Thus $x · y ∈ S$. Therefore, all the monomials in $u_{ij}$ and $u_{ij}^*$ belong to $S$. Then $S$ is dense in $B$. This gives one of the Podleś conditions: $B ⊠ B = j_1(B)j_2(B) \subseteq \Delta(B)j_2(B)j_2(B) = \Delta(B)j_2(B)$. The other Podleś condition is shown similarly by proving that $R := \{x ∈ B : j_2(x) ∈ j_1(B)Δ(B)\}$ is dense in $B$.

Propositions 2.11 and 2.12 show that $A_o(V, π, ω)$ with the $\mathbb{T}$-action $β$ and the comultiplication $\Delta$ is a braided compact quantum group over $\mathbb{T}$. By construction, $u ∈ M_\mathbb{T}(A_o(V, π, ω))$ and the underlying representation $π$ of $\mathbb{T}$ form a representation on $V ∼= \mathbb{C}^n$. Up to isomorphism, $A_o(V, π, ω)$ does not depend on the choice of the basis $e_1, \ldots, e_n$ in $V$. To make this clearer, we now treat $u$ as a representation in $B(V) ⊗ A_o(V, π, ω)$.

\textbf{Theorem 2.16.} The braided quantum group $A_o(V, π, ω)$ is the universal one with a representation on $(V, π)$ for which $ω ∈ V ⊗ V$ is $d$-homogeneous and $G$-invariant. That is, if $G = (B, β, Δ)$ is another braided quantum group and $U ∈ B(V) ⊗ B$ is a representation of $G$ on $(V, π)$ such that $ω ∈ V ⊗ V$ is $d$-homogeneous and $G$-invariant, then there is a unique $\mathbb{T}$-equivariant Hopf $*$-homomorphism $A_o(V, π, ω) → B$ mapping $u ⊠ u → U$.

\textbf{Proof.} Choose any orthonormal basis $f_1, \ldots, f_n$ of $T$-eigenvectors in $V$ as above and use it to turn $U$ into a matrix $(U_{ij})_{1 ≤ i, j ≤ n} ∈ M_n(B)$. Proposition 2.11 gives a unique unital $*$-homomorphism $A_o(V, π, ω) → B$ that maps $u_{ij} → U_{ij}$ and that is $\mathbb{T}$-equivariant and compatible with comultiplications.

\textbf{Corollary 2.17.} Up to isomorphism, the braided quantum group $A_o(V, π, ω)$ does not depend on the choice of the basis $e_1, \ldots, e_n$ used to build it. Even more, a
\[ T\text{-equivariant unitary operator } \varphi: (V_1, \pi_1) \to (V_2, \pi_2) \text{ with } \varphi(\omega_1) = \omega_2 \text{ induces an}
\]
isomorphism of braided quantum groups \( A_o(V_1, \pi_1, \omega_1) \cong A_o(V_2, \pi_2, \omega_2). \)

**Proof.** The universal property in Theorem 2.16 does not mention the basis \( e_1, \ldots, e_n \). Hence the braided quantum groups built from different orthonormal bases of eigenvectors for the \( T \)-action \( \pi \) have the same universal property. This implies that they are isomorphic as braided quantum groups. More generally, let \( \varphi: (V_1, \pi_1) \to (V_2, \pi_2) \) be a \( T \)-equivariant unitary operator with \( \varphi(\omega_1) = \omega_2. \) Let \( G = (B, \beta, \Delta) \) be another braided quantum group and let \( U_1 \in B(V_1) \otimes B \) be a representation of \( G \) on \((V_1, \pi_1)\) such that \( \omega_1 \in V_1 \otimes V_1 \) is \( d \)-homogeneous and \( G \)-invariant. Then \( U_2 := (\varphi \otimes 1)U_1(\varphi \otimes 1)^* \in B(V_2) \otimes B \) is a representation of \( G \) on \((V_2, \pi_2)\) such that \( \omega_2 \in V_2 \otimes V_2 \) is \( d \)-homogeneous and \( G \)-invariant. Conversely, any representation \( U_2 \) of \( G \) on \((V_2, \pi_2)\) such that \( \omega_2 \in V_2 \otimes V_2 \) is \( d \)-homogeneous and \( G \)-invariant gives a representation \( U_1 := (\varphi \otimes 1)^*U_2(\varphi \otimes 1) \in B(V_1) \otimes B \) on \((V_1, \pi_1)\) such that \( \omega_1 \in V_1 \otimes V_1 \) is \( d \)-homogeneous and \( G \)-invariant. By Theorem 2.16, this bijection on certain classes of representations induces a bijection between \( T \)-equivariant Hopf \(^*\)-homomorphisms from \( A_o(V_1, \pi_1, \omega_1) \) and \( A_o(V_2, \pi_2, \omega_2) \) to \( G \). By the Yoneda Lemma, this bijection comes from an isomorphism of braided quantum groups \( A_o(V_1, \pi_1, \omega_1) \cong A_o(V_2, \pi_2, \omega_2). \)

Next we describe some isomorphisms of our braided quantum groups that shift the homogeneity degree \( d \) of \( \omega \). Let \((V, \pi)\) be a representation of \( T \) on a finite-dimensional vector space and let \( s \in \mathbb{Z} \). Then \( \pi'_s := z^{-s} \pi z \) is another representation of \( T \) on \( V \). Let \( d_1, \ldots, d_n \in \mathbb{Z} \) and let \( e_1, \ldots, e_n \in V \) be an orthonormal basis of \( V \) such that \( e_i(e_j) = z^{d_i} \delta_{ij} \) for all \( i \in \mathbb{T} \). Then \( \pi'_s(e_i) = z^{d'_i} e_i \) with \( d'_i := d_i - s \), that is, the degrees \( d_i \) are shifted.

**Lemma 2.18.** Let
\[ \omega' = \sum_{i,j=1}^n \zeta^{-s}d'_i \omega_{ij}e_i \otimes e_j. \]
Then there is a \( T \)-equivariant Hopf \(^*\)-isomorphism \( A_o(V, \pi, \omega) \cong A_o(V, \pi', \omega') \) which maps the generator \( u_{ij} \) of \( A_o(V, \pi, \omega) \) to \( u_{ij} \in A_o(V, \pi', \omega') \).

**Proof.** When we replace \( \pi, \omega \) by \( \pi', \omega' \), then we replace \( \omega_{ik} \) by \( \zeta^{-sd_k} \omega_{ik} \), \( d_i \) by \( d'_i = d_i - s \), and \( d \) by \( d' = d - 2s \) because \( \omega' \) has degree \( d - 2s \) for the representation \( \pi' \). The block matrix \( \Omega \) in the introduction becomes \( \Omega'_{ij} = \zeta^{-s} \cdot \Omega_{ij} \). Equation (1.3) for \( \Omega \) implies
\[ \Omega'_{i,d-1} \cdot \Omega'_{d-1,i} = \zeta^{s(d-1)} \cdot \Omega_{i,d-1} \cdot \Omega_{d-1,i} = \zeta^{sd - 2si} \cdot \Omega_{i,d-1} \cdot \Omega_{d-1,i} = \zeta^{s d' - s} \cdot \Omega_{i,d-1} \cdot \Omega_{d-1,i} \]
with \( d' := \zeta^{sd} d \). Thus \( A_o(V, \pi', \omega') \) is defined.

The effect of our substitutions on (1.3) is that the two sides \( \zeta^{sd} d \sum_{k=1}^n \omega_{ik} u_{jk} \) and \( \zeta^{-s} \cdot \sum_{k=1}^n \omega_{ik} u_{jk}^* \) are multiplied by \( \zeta^{-s} \cdot \sum_{k=1}^n \omega_{ik} u_{jk} \) and \( \zeta^{-s} \cdot \sum_{k=1}^n \omega_{ik} u_{jk}^* \), respectively; here we use that \( d_j + d_k = d \) if \( \omega_{jk} \neq 0 \). Since the two factors are the same and do not involve the summation index \( k \) after simplification, the relation (1.3) holds for \( (\pi', \omega') \) if and only if it holds for \( (\pi, \omega) \).

The same is obvious for the other two relations (1.3) and (1.4). Hence the generators \( u_{ij} \) of \( A_o(V, \pi, \omega) \) and \( A_o(V, \pi', \omega') \) are subjected to the same relations. The formulas for the \( T \)-action and the comultiplication in (1.6) and (1.7) also remain the same because \( d'_i - d'_j = d_i - d_j \) for all \( 1 \leq i, j \leq n \). So the canonical map \( A_o(V, \pi, \omega) \to A_o(V, \pi', \omega') \) is a \( T \)-equivariant Hopf \(^*\)-isomorphism.

The formula for \( \omega' \) in Lemma 2.18 may be understood using the universal property in Theorem 2.10. Let \( G = (B, \beta, \Delta) \) be another braided quantum group. The \( T \)-equivariant Hopf \(^*\)-isomorphism in Lemma 2.18 means that \( U \in B(V) \otimes B \)
is a representation of $G$ on $(V, \pi)$ for which $\omega \in V \otimes V$ is $d$-homogeneous and $G$-invariant if and only if $U \in B(V) \otimes B$ is a representation of $G$ on $(V, \pi')$ for which $\omega \in V \otimes V$ is $d'$-homogeneous and $G$-invariant. A $d$-homogeneous invariant vector corresponds to an intertwining operator $\omega : C_d \to V \otimes V$. And such an intertwining operator is equivalent to an intertwining operator

$$C_{d-2s} \cong C_{-s} \otimes C_d \otimes C_{-s} \xrightarrow{1 \otimes \omega \otimes 1} C_{-s} \otimes V \otimes V \otimes C_{-s} \xrightarrow{V \chi^{C_{-s}}} (C_{-s} \otimes V) \otimes (C_{-s} \otimes V).$$

Now $C_{-s} \otimes V$ is $V$ with the shifted representation $\pi'$ and the same $U$, exactly as needed. The braiding operator $V \chi^{C_{-s}}$ maps the basis vector $e_j \otimes 1$ of $V \otimes C_{-s}$ to $\zeta^{-sd} \cdot 1 \otimes e_j$. This is the origin of the factors $\zeta^{-sdj}$ in the definition of $\omega'$.

The isomorphism in Lemma 2.18 allows to reduce the case of even $d$ to $d = 0$, up to a canonical $T$-equivariant Hopf $^*$-isomorphism. We shall use this simplification later to describe the representation category of $A_o(V, \pi, \omega)$.

The case $d = 1$ is particularly important because it occurs for $SU_q(2)$. We can also reduce this case to $d = 0$ if we replace the circle $T$ by a double cover.

Namely, we may compose the representation $\pi$ with the endomorphism $z \mapsto z^2$ of $T$. We may also use any other endomorphism, but only this two-fold covering map will be important later. Replacing $\pi$ by $\pi'' = \pi_z$ replaces $d$ by $2d$ for $1 \leq i \leq n$. The $d$-homogeneous vector $\omega$ for $\pi$ is $d''$-homogeneous for $\pi''$ with $d'' := 2d$. In order to keep the braiding unitaries the same, we replace $\zeta$ by a fourth root $\zeta'' = \sqrt[4]{\zeta}$. We leave the vector $\omega'' = \omega$ unchanged. Then the relations (1.3) for $(V, \pi, \omega)$ and $(V, \pi'', \omega)$ are the same. And (1.7) is also the same. But the action $\beta''$ on $A_o(V, \pi'', \omega)$ defined in (1.6) becomes $\beta'' = \beta_z$.

Thus there is a Hopf $^*$-isomorphism $\varphi : A_o(V, \pi, \omega) \xrightarrow{\sim} A_o(V, \pi'', \omega)$ that maps the generator $u_{ij}$ of $A_o(V, \pi, \omega)$ to $u_{ij} \in A_o(V, \pi'', \omega)$. But $\varphi$ is not $T$-equivariant. Instead, it satisfies $\beta'' \circ \varphi = \varphi \circ \beta_z$ for all $z \in T^2$. Thus the braided quantum groups $A_o(V, \pi, \omega)$ and $A_o(V, \pi'', \omega)$ are different. They have different representations. Namely, representations of $A_o(V, \pi, \omega)$ are equivalent to those representations of $A_o(V, \pi'', \omega)$ where the underlying representation of $T$ factors through the map $T \to T, z \mapsto z^2$.

Since $d'' = 2d$ is even, Lemma 2.18 for $s = d$ identifies $A_o(V, \pi'', \omega)$ with $A_o(V, \pi', \omega')$ for $\pi'' = z^{-d'} \pi'' = z^{-d} \pi_z$, $\zeta'' = \sqrt[4]{\zeta}$, and $\omega_{ij}' = (\zeta')^{-dd} \omega_{ij} = \zeta^{-ddj}/2 \omega_{ij}$. This gives a braided free orthogonal quantum group with $d'' = 0$.

3. The bosonisation

The bosonisation of a braided quantum group is an ordinary quantum group together with an idempotent quantum group homomorphism (“projection”). Conversely, any quantum group with projection is the bosonisation of a braided quantum group (see [3]). In the compact case, the bosonisation is constructed in [8, Corollary 6.5], where it is called “semidirect product”. When we specialise this construction to the braided quantum group $A_o(V, \pi, \omega)$, we get the following:

**Theorem 3.1.** Let $(C, \Delta_C)$ be the bosonisation associated to the braided quantum group $A_o(V, \pi, \omega)$. Then $C$ is isomorphic to the universal unital $C^*$-algebra generated by elements $z$ and $u_{ij}$ for $1 \leq i, j \leq n$ subject to the relations $z z^* = 1 = z^* z$, \[1.3\] and the commutation relations $z u_{ij} = \zeta^{d_i-d_j} u_{ij} z$ for all $1 \leq i, j \leq n$. The comultiplication $\Delta_C$ is given by

$$\Delta_C(z) = z \otimes z, \quad \Delta_C(u_{ik}) = \sum_{l=1}^{n} u_{il} \otimes z^{d_i-d_l} u_{lk}.$$
Thus the unitaries with the identity function. This generates $C$.

The comultiplication

$$
\Delta : C \otimes C \to C \otimes C
$$

using the $T$-homogeneous basis $(\lambda_m)$ of $L$. The underlying $C^*$-algebra of the bosonisation is the braided tensor product

$$
C := C(T) \boxtimes A_o(V,v,\omega)
$$

Let $z \in C(T) \subseteq B(L^2\mathbb{T})$ denote the unitary operator of pointwise multiplication with the identity function. This generates $C(T)$. Since $A_o(V,v,\omega)$ is generated by the matrix coefficients $u_{ij}$ of $u$, the $C^*$-algebra $C$ is generated by the operators $j_1(z) := z \otimes 1$ and $j_2(u_{ij}) := \xi(z) \cdot (u_{ij} \otimes 1) \cdot \xi^-(T \otimes L^2)^{z \otimes \lambda_m}$, which acts on $L^2(T) \otimes L$ by

$$
j_2(u_{ij})(z^{k} \otimes \lambda_m) = \xi^{-k} \cdot \xi(z) \cdot (u_{ij} \otimes 1) \cdot \xi(z^{k} \otimes \lambda_m).
$$

This implies the commutation relation

$$
j_1(z) j_2(u_{ij}) j_1(z^*) = \xi^{-d_{ij}} \cdot j_2(u_{ij}) = j_2(\beta_{-1} u_{ij}).
$$

Thus the unitaries $j_1(z^n)$ for $n \in \mathbb{Z}$ and the representation $j_2$ of $A_o(V,v,\omega)$ form a covariant representation for the $\mathbb{Z}$-action on $A_o(V,v,\omega)$ generated by the automorphism $\beta_{-1}$. This covariant representation is unitarily equivalent to the regular representation that defines the reduced crossed product. So

$$
C = C(T) \boxtimes A_o(V,v,\omega) \cong A_o(V,v,\omega) \rtimes \mathbb{Z}
$$

for the $\mathbb{Z}$-action on $A_o(V,v,\omega)$ generated by $\beta_{-1}$. This crossed product $C^*$-algebra is also the universal unital $C^*$-algebra generated by elements $(u_{ij})_{1 \leq i,j \leq n}$ and $z$ subject to the relations (2.23)–(2.25), the relation $zz^* = 1 = z^*z$ saying that $z$ is unitary, and the commutation relation

$$
u u_{ij} = \xi^{-d_{ij}} u_{ij} z.
$$

The comultiplication $\Delta_C : C \to C \otimes C$ is defined in [8] as the composite of

$$\text{id}_A \boxtimes \Delta : A \boxtimes A \to A \boxtimes A \boxtimes A \boxtimes A
$$

and the unique $^*$-homomorphism $\psi : A \boxtimes B \boxtimes B \to (A \boxtimes B) \otimes (A \boxtimes B)$ with

$$
\psi(j_1(a)) = (j_1 \otimes j_1) \Delta_A(a), \quad \psi(j_2(b)) = (j_2 \otimes j_1) \beta(b), \quad \psi(j_3(b)) = 1 \otimes j_2(b)
$$

for all $a \in A$, $b \in B$. Here $j_l$ denotes the embedding of the $l$th factor in a braided tensor product, and $\beta : B \to B \otimes A$ is the $T$-action on $B$. Using $\Delta_A(z) = z \otimes z$
and \([1,7]\), we compute

\[
\Delta_C(z) = z \otimes z,
\]

\[
\Delta_C(u_{ik}) = \psi \left( \sum_{l=1}^{n} j_2(u_{il}) j_3(u_{ik}) \right) = \sum_{l=1}^{n} (j_2 \otimes j_1)(u_{il} \otimes z^{d_l-d_i}) \cdot (1 \otimes j_2(u_{ik}))
\]

\[
= \sum_{l=1}^{n} j_2(u_{il}) \otimes j_1(z^{d_l-d_i}) j_2(u_{ik}).
\]

This gives the formulas for \( \Delta_C \) in the theorem when we drop the inclusion maps \( j_1 \) and \( j_2 \) from our notation as above. \( \square \)

A sanity check for the formulas in Theorem 3.1 is that \( \Delta_C \) is coassociative:

\[
(id \otimes \Delta_C) \Delta_C(z) = (\Delta_C \otimes id) \Delta_C(z) = z \otimes z \otimes z,
\]

\[
(id \otimes \Delta_C) \Delta_C(u_{ik}) = (\Delta_C \otimes id) \Delta_C(u_{ik}) = \sum_{l,m=1}^{n} u_{il} \otimes z^{d_l-d_i} u_{im} \otimes z^{d_m-d_i} u_{mk}.
\]

It can also be checked directly that \( \Delta_C \) is well defined, that is, the elements \( \Delta_C(z) \) and \( \Delta_C(u_{ik}) \) of \( C \otimes C \) verify the relations in the universal property of \( C \).

Representations of the braided quantum group \( A_o(V, \pi, \omega) \) are equivalent to representations of the bosonisation by [6, Theorem 3.4]. Here a representation of \( A_o(V, \pi, \omega) \) on a Hilbert space \( \mathcal{H} \) is a pair \((\mathcal{S}, \mathcal{U})\), where \( \mathcal{S} \in \mathcal{U}(K(\mathcal{H}) \otimes C(\mathcal{T})) \subseteq \mathcal{U}(\mathcal{H} \otimes \mathcal{L}(\mathcal{T})) \) is a representation of \( \mathcal{T} \) and \( \mathcal{U} \in \mathcal{U}(K(\mathcal{H}) \otimes A_o(V, \pi, \omega)) \subseteq \mathcal{U}(\mathcal{H} \otimes \mathcal{L}) \) is a representation of \( A_o(V, \pi, \omega) \). By [6, Theorem 3.4], the corresponding representation of \( C \subseteq \mathcal{B}(\mathcal{L}(\mathcal{T}) \otimes \mathcal{L}) \) is

\[
\mathcal{S}_{12}(\mathcal{S} \otimes 1_{L^2(\mathcal{T})}) = \mathcal{S}_{12} \cdot \mathcal{L}^{L^2(\mathcal{T})}_{23} \cdot \mathcal{U}_{12} \cdot \mathcal{L}^{L^2(\mathcal{T})}_{23} \in \mathcal{U}(\mathcal{H} \otimes L^2(\mathcal{T}) \otimes \mathcal{L}).
\]

In particular, the fundamental representation of \( A_o(V, \pi, \omega) \) on \( V \) is given by the pair \((\mathcal{S}V, \mathcal{U}V)\) with

\[
\mathcal{S}V(e_i \otimes z^k) = e_i \otimes z^{d_i+k}, \quad \mathcal{U}V(e_i \otimes \lambda_m) = \sum_{k=1}^{n} e_k \otimes u_{ki} \lambda_m.
\]

The corresponding representation \( t \) of \( C \) is given by the formula

\[
t(e_i \otimes z^k \otimes \lambda_m) = \mathcal{S}_{12} \mathcal{L}^{L^2(\mathcal{T})}_{23} \mathcal{U}_{12}(e_i \otimes \lambda_m \otimes z^k) = \sum_{h=1}^{n} \zeta^{- kl} \mathcal{S}_{12} \mathcal{L}^{L^2(\mathcal{T})}_{23} (e_h \otimes u_{hi} \lambda_m \otimes z^k)
\]

\[
= \sum_{h=1}^{n} \zeta^{- kl} \mathcal{L}^{L^2(\mathcal{T})}_{23} (e_h \otimes z^{d_h} u_{hi} \lambda_m) = \sum_{h=1}^{n} \zeta^{k(d_h-d_i)} (e_h \otimes z^{d_h} u_{hi} \lambda_m).
\]

This and [9,2] show that \( t \in \mathcal{M}_n(C) \) has matrix entries \( t_{hi} = j_1(z^{d_h}) j_2(u_{hi}) \) or briefly

\[
t_{hi} = z^{d_h} u_{hi}.
\]

A sanity check is that \( t \) is a representation of \( C \). Indeed, we compute that \( \Delta_C(t) = t_{13} t_{13} \) is the matrix over \( C \otimes C \) with \( i, k \)th entry

\[
\sum_{m=1}^{n} z^{d_i u_{im}} \otimes z^{d_m u_{mk}}.
\]

Since \( t_{ij} = z^{d_i} u_{ij} \) and \( z^{-d_i} t_{ij} = u_{ij} \), we may also describe \( C \) as the universal unital \( C^* \)-algebra generated by \( z \) and \( t_{ij} \) for \( 1 \leq i, j \leq n \). These are subject to
the following relations. First, $z$ and the matrix $t = (t_{ij})$ are unitary. Secondly, $zt_{ij} = z^{d_{ij}}t_{ij}z$ for $1 \leq i, j \leq n$. And third, (1.5) becomes

$$
\zeta^{-d_{ij}} \sum_{k=1}^{n} \omega_{ik} z^{-d_{ij}} t_{jk} = \zeta^{ij} (\zeta^{ij}) \sum_{k=1}^{n} \omega_{kj} (z^{-d_{ij} t_{ki}})^* = \zeta^{ij} (\zeta^{ij}) \sum_{k=1}^{n} \omega_{kj} z^{d_{ij}} t_{ki}^{*}
$$

because $\omega_{kj} = 0$ unless $d_{k} + d_{j} = d$ and $t_{ki}^{*} z^{-d_{ij}} = \zeta^{ij} (\zeta^{ij}) z^{d_{ij} t_{ki}}$. Cancelling $z^{-d_{ij}}$ and multiplying with $\zeta^{ij}$, this becomes

$$
(3.3) \sum_{k=1}^{n} t_{jk} (\zeta^{ik} \omega_{ik}) = z^{d} \sum_{k=1}^{n} (\zeta^{ik} \omega_{jk}) t_{ki}^{*}.
$$

Since $(C, \Delta_C)$ is a bosonisation of a braided quantum group, it comes with quantum group morphisms $(C, \Delta_C) \mapsto (A, \Delta_A)$ whose composite makes $(C, \Delta_C)$ a $C^*$-quantum group with projection (see [9]). We also describe these quantum group morphisms explicitly:

**Proposition 3.4.** The quantum group morphisms $(C, \Delta_C) \mapsto (A, \Delta_A)$ are given by the Hopf $^*$-homomorphisms $\iota: A \to C$ and $\pi: C \to A$ defined by $\iota(z) := z$, $\pi(z) := z$ and $\iota(\iota_{ij}) := d_{ij}$.

**Proof.** We interpret a representation of $(C, \Delta_C)$ as a braided representation $(S, U)$ of $A_\iota(V, \pi, \omega)$. Then the functor induced by $\iota$ maps a representation $S$ of $A = C(\mathbb{T})$ (that is, a representation of $\mathbb{T}$) to the representation $(S, 1)$ of $A_\iota(V, \pi, \omega)$ on the same Hilbert space. And the functor induced by $\pi$ maps a representation $(S, U)$ of $A_\iota(V, \pi, \omega)$ to the underlying representation $S$ of $C(\mathbb{T})$ on the same Hilbert space. By [6] Proposition 2.15, a functor between the representation categories of two $C^*$-quantum groups (defined by manageable multiplicative unitaries) is equivalent to a quantum group morphism as defined in [7]. And it is checked in [6] Proposition 3.5 that the quantum group morphisms that correspond to the functors above are the ones defined in the study of $C^*$-quantum groups with projection in [9].

It is easy to check that there are unique $^*$-homomorphisms $\iota$ and $\pi$ given by the formulas in the statement of the proposition and that these are Hopf $^*$-homomorphisms. These Hopf $^*$-homomorphisms induce functors between the representation categories of $(C, \Delta_C)$ and $(A, \Delta_A)$ that do not change the underlying Hilbert space. A computation shows that the functors on the representation categories corresponding to $\iota$ and $\pi$ map $S$ to $(S, 1)$ and $(S, U)$ to $S$, respectively. □

4. The representation category

We are going to prove Theorem 1.8 In the previous section, we have described the bosonisation $(C, \Delta_C)$ and translated representations of the braided quantum group $A_\iota(V, \pi, \omega)$ to representations of $(C, \Delta_C)$. In particular, $z \in C$ and $t \in M_n(C)$ are representations of $(C, \Delta_C)$. The representation $z$ corresponds to the trivial representation of $A_\iota(V, \pi, \omega)$ on the 1-dimensional vector space on which $\mathbb{T}$ acts by the identity character $\mathbb{T} \to U(1)$, $z \mapsto z$. The representation $t$ corresponds to the fundamental representation of $A_\iota(V, \pi, \omega)$ on $V$.

**Corollary 4.1.** The $C^*$-algebra $C$ is generated by the coefficients of the direct sum representation $z \oplus t$. And $(C, z \oplus t)$ is a compact matrix quantum group.

**Proof.** The coefficients of $z \oplus t$ are $z$ and the coefficients $t_{ij}$ of $t$. We have seen in the last section that these elements generate $C$ as a $C^*$-algebra. The representation $z$ is a character, so that $z^* = z^{-1}$. Equation (3.3) is equivalent to an equality of matrices $t \cdot F = z^d \cdot F \cdot t$ for $t = (t_{ij})_{1 \leq i, j \leq n}$, $\overline{t} = (t_{ij}^*)_{1 \leq i, j \leq n}$ and $F = (F_{ij})$ with

$$
F_{ij} := \zeta^{ij d_{ij}} \omega_{ji}.
$$
The elements \( t_{ij} \) are linearly independent for all \( 1 \leq i, j \leq n \). Equation (1.2) implies that \( F \) is invertible. Since \( t \) and \( z \) are unitary, it follows that \( \tilde{t} = F^{-1} z d t \cdot F \) is invertible. Thus \( \pi \circ \tilde{t} \) is invertible as well. And this means that \((C, t \oplus z)\) is a compact matrix quantum group as defined in [15]. □

The representation theory of compact matrix quantum groups was studied by Woronowicz in [15]. It follows that any irreducible representation of \((C, \Delta_C)\) is contained in a tensor product of several copies of \( z, t, \pi \) and \( \tilde{t} \). The equation \( t \cdot F = z^d \cdot F \cdot \tilde{t} = F \cdot z^{-d} \cdot t \) says that \( F \) is an intertwiner from \( z^d \cdot \tilde{t} \) to \( t \). Equivalently, \( F \) is an intertwiner from \( \tilde{t} \) to \( z^{-d} t \). Hence the tensor factor \( \tilde{t} \) is redundant. The commutation relations \( z t_{ij} = \zeta^{d_i - d_j} t_{ij} z \) say that \( \pi \zeta \), the diagonal matrix with entries \( \zeta^d \), is an intertwiner from \( t \oplus z \) to \( z \oplus t \). Hence \( t \oplus z^k \cong z^k \oplus t \) for all \( k \in \mathbb{Z} \).

As a result, any irreducible representation of \((C, \Delta_C)\) is a direct summand in \( z^k \oplus t \) for some \( k \in \mathbb{Z}, l \in \mathbb{N} \).

Now we are going to relate the quantum group \((C, \Delta_C)\) to the usual free orthogonal quantum groups \( A_\alpha(F) \) of Wang and van Daele [13,14]. Recall that \( A_\alpha(F) \) for an invertible matrix \( F \in GL_n(\mathbb{C}) \) is the universal \( C^* \)-algebra with generators \( x_{ij} \) subject to the relations

\[
\sum_{k=1}^n x_{ki}^* x_{kj} = \delta_{i,j}, \tag{4.3}
\]

\[
\sum_{k=1}^n x_{ik} x_{jk}^* = \delta_{i,j}, \tag{4.4}
\]

\[
\sum_{k=1}^n x_{ik} F_{kj} = \sum_{k=1}^n F_{ik} x_{kj}^* \quad \text{for } 1 \leq i, j \leq n. \tag{4.5}
\]

Equivalently, \( x = (x_{ij})_{1 \leq i,j \leq n} \in M_n(A_\alpha(F)) \) is unitary and satisfies \( x \cdot F = F \cdot x \).

The comultiplication on \( A_\alpha(F) \) is the unique \(*\)-homomorphism \( \Delta_{A_\alpha(F)} \) with

\[
\Delta_{A_\alpha(F)}(x_{ij}) = \sum_{k=1}^n x_{ik} \otimes x_{kj}. \tag{4.4}
\]

Now we assume for some time that \( d = 0 \). Then (4.2) simplifies to \( F_{ij} = \omega_{ij} \) or \( F = \Omega \). Equation (1.2) says that \( F F^\dagger = c \cdot 1 \) because \( d = 0 \). So \( A_\alpha(F) \) is defined and the results of Banica [11] apply.

By assumption, \( F_{ij} = \omega_{ij} \) vanishes unless \( d_i + d_j = 0 \). Therefore, multiplying each \( x_{ij} \) by \( \zeta^{d_i - d_j} \) preserves the relations (4.3)–(4.5) that define \( A_\alpha(F) \). Then there is an automorphism \( \alpha \) of the \( C^* \)-algebra \( A_\alpha(F) \) with \( \alpha(x_{ij}) := \zeta^{d_i - d_j} x_{ij} \) for \( 1 \leq i, j \leq n \). This is even a Hopf \(*\)-homomorphism, that is, \( \Delta_{A_\alpha(F)} \circ \alpha = (\alpha \otimes \alpha) \circ \Delta_{A_\alpha(F)} \). There is also a unique unital Hopf \(*\)-homomorphism \( \varphi: A_\alpha(F) \to C \) mapping \( x_{ij} \mapsto t_{ij} \) for \( 1 \leq i, j \leq n \). This follows from the universal property of \( A_\alpha(F) \) because \( t \in M_n(\mathbb{C}) \) is an intertwiner from \( \tilde{t} \) to \( t \). The following proposition implies that \( \varphi \) is injective:

**Proposition 4.6.** Let \( d = 0 \). The Hopf \(*\)-homomorphism \( \varphi: A_\alpha(F) \to C \) extends to an isomorphism \( A_\alpha(F) \rtimes \mathbb{Z} \cong C \).

**Proof.** The crossed product \( A_\alpha(F) \rtimes \mathbb{Z} \) is generated by the generators \( x_{ij} \) of \( A_\alpha(F) \) and an extra unitary \( z \), subject to the relations (4.3)–(4.5) defining \( A_\alpha(F) \), the unitarity conditions \( z^* z = z z^* = 1 \), and the relation \( z x_{ij} z^* = \alpha(x_{ij}) = \zeta^{d_i - d_j} x_{ij} \) for \( 1 \leq i, j \leq n \). When we replace \( x_{ij} \) by \( t_{ij} \), this gives a presentation of \( C \) by the computations in Section 3 using \( d = 0 \). □

Using the Hopf \(*\)-homomorphism \( \varphi \), a representation \( g \in M_n(A_\alpha(F)) \) induces a representation in \( \varphi_*(g) \in M_n(C) \). A linear map \( \psi: \mathbb{C}^n \to \mathbb{C}^n \) is an intertwiner
for $g$ if and only if it is an intertwiner for $\varphi_*(g)$ because $\varphi$ is injective. So $\varphi_*$ is a fully faithful functor from the representation category of $A_o(F)$ to that of $C$. It is also a strict tensor functor because $\varphi_*(g_1 \oplus g_2) = \varphi_*(g_1) \oplus \varphi_*(g_2)$ and $\varphi_*(\varepsilon) = \varepsilon$ for the trivial representation $\varepsilon \equiv 1 \in A_o(F)$. The representation category of $A_o(F)$ is described by Banica [1]: he finds that $A_o(F)$ has irreducible representations $r_k$ for $k \in \mathbb{N}$ such that $\tau_k \cong r_k$ for all $k \in \mathbb{N}$ and

$$r_k \otimes r_n \cong r_{|k-n|} \oplus r_{|k-n|+2} \oplus \cdots \oplus r_{k+s-2} \oplus r_{k+s}.$$ 

These are the same fusion rules as for the group $SU(2)$. Using the fully faithful strict tensor functor $\varphi_*$, we get irreducible representations $\varphi_*(r_k)$ of $(C, \Delta_C)$ with the same fusion rules.

**Lemma 4.7.** Let $d = 0$. Let $g_1, g_2$ be representations of $A_o(F)$ and let $a, b \in \mathbb{Z}$. If $a \neq b$, then there are no intertwiners between the representations $z^a \varphi_*(g_1)$ and $z^b \varphi_*(g_2)$. If $a = b$, then the two maps

$$\text{Mor}(g_1, g_2) \cong \text{Mor}(z^a \varphi_*(g_1), z^b \varphi_*(g_2)),$$

are equal and invertible.

**Proof.** We have already seen that the braiding unitary $\pi_\psi$ is a unitary intertwiner between $z \otimes t = z \otimes \varphi_*(x)$ and $t \otimes z = \varphi_*(x) \otimes z$. Hence the appropriate braiding unitary is a unitary intertwiner $z \otimes \varphi_*(x^{\mathfrak{D}^n}) \cong \varphi_*(x^{\mathfrak{D}^n}) \otimes z$. Then the braiding unitary is a unitary intertwiner $z \otimes \varphi_*(g) \cong \varphi_*(g) \otimes z$ for any representation $g$ of $A_o(F)$ because $g$ is a direct sum of direct summands in the tensor powers $x^{\mathfrak{D}^n}$ for $n \in \mathbb{N}$. This also implies $z^a \otimes \varphi_*(g) \cong \varphi_*(g) \otimes z^a$ for all $a \in \mathbb{Z}$ and all representations $g$ of $A_o(F)$.

The matrix coefficients of $z^a \varphi_*(g_1)$ belong to $A_o(F) \cdot z^a \subseteq A_o(F) \times \mathbb{Z} \cong C$. If $a \neq b$, then the subspaces $A_o(F) \cdot z^a$ and $A_o(F) \cdot z^b$ of $A_o(F) \times \mathbb{Z}$ are linearly independent. This implies that there cannot be any non-zero intertwiners between $z^a \varphi_*(g_1)$ and $z^b \varphi_*(g_2)$.

Now assume $a = b$. We have already seen that there are the same intertwiners $g_1 \rightarrow g_2$ and $\varphi_*(g_1) \rightarrow \varphi_*(g_2)$. Tensoring with the identity on $z^a$ on the left gives an intertwiner $z^a \otimes \varphi_*(g_1) \rightarrow z^a \otimes \varphi_*(g_2)$. This map between intertwiner spaces is invertible because tensoring with the identity on $z^{-a}$ gives an inverse map, using $z^{-a} \otimes z^a = \varepsilon$, the trivial representation. Similarly, tensoring on the right with the identity on $z^a$ gives an intertwiner $\varphi_*(g_1) \otimes z^a \rightarrow \varphi_*(g_2) \otimes z^a$. Conjugating the latter with the braiding unitaries $z^a \otimes \varphi_*(g_1) \cong \varphi_*(g_1) \otimes z^a$ for $i = 1, 2$ gives another intertwiner $z^a \otimes \varphi_*(g_1) \rightarrow z^a \otimes \varphi_*(g_2)$. We claim that these two intertwiners are equal. Since any representation is contained in a direct sum of copies of $x^{\mathfrak{D}^{m_i}}$, it is enough to show this in case $g_i = x^{\mathfrak{D}^{m_i}}$ for $i = 1, 2$ for some $m_1, m_2 \in \mathbb{N}$. Banica shows that any intertwiner between the representations $x^{\mathfrak{D}^{m_i}}$ is a $*$-polynomial in the special intertwiners $x^{\mathfrak{D}^{m_i}} \rightarrow x^{\mathfrak{D}^{m_i+2}}$ that add the invariant vector in two consecutive entries. Therefore, it is enough to prove the claim for these special intertwiners. In this case, the result follows because the invariant vector $\omega$ is $T$-invariant.

The lemma implies, in particular, that the representations $z^a \varphi_*(r_k)$ of $(C, \Delta_C)$ for $a \in \mathbb{Z}$, $k \in \mathbb{N}$ are all irreducible and distinct. And any irreducible representation is one of them because it is contained in $z^a \varphi_*(r_k)$ for some $a \in \mathbb{Z}$, $l \in \mathbb{N}$. As a result, any irreducible representation of $(C, \Delta_C)$ is unitarily equivalent to $z^a \varphi_*(r_k)$ for a unique $a \in \mathbb{Z}$, $k \in \mathbb{N}$.
These irreducible representations satisfy the fusion rules
\[ z^k \varrho r_l \cong \varrho \otimes z^{-k} \cong z^{-k} \varrho r_l, \]
\[ (z^b \varrho r_a) \varrho (z^k \varrho r_m) \cong z^{b+k} \varrho (r_{|a-m|} \otimes r_{|a-m|+2} \otimes \cdots \otimes r_{a+m-2} \otimes r_{a+m}). \]
These are the rules asserted in Theorem 1.8. Thus we have completed the proof of that theorem in the case \( d = 0 \).

Next, we describe the representation category \( \text{Rep}(C, \Delta_C) \) of \( (C, \Delta_C) \) for \( d = 0 \) as a monoidal category. Let \( \mathcal{R}_a \) be the full subcategory of all representations of \( (C, \Delta_C) \) of the form \( z^a \varrho \varphi \varphi (\varrho) \) for a representation \( \varrho \) of \( A_o(F) \). Lemma 4.7 implies that there are no non-zero intertwiners between representations in \( \mathcal{R}_a \) and \( \mathcal{R}_b \) for \( a \neq b \) and that the intertwiners between two representations in \( \mathcal{R}_a \) are the same as for the corresponding representations of \( A_o(F) \). Thus we get an equivalence of categories \( \text{Rep}(C, \Delta_C) \cong \text{Rep}(A_o(F)) \) and isomorphisms of categories \( \mathcal{R}_a \cong \text{Rep}(A_o(F)) \) for all \( a \in \mathbb{Z} \). The isomorphism \( \mathcal{R}_0 \cong \text{Rep}(A_o(F)) \) is \( \varphi \varphi \), which is a strict tensor functor and thus an isomorphism of monoidal categories. The tensor product of two representations in \( \mathcal{R}_a \) and \( \mathcal{R}_b \) belongs to \( \mathcal{R}_{a+b} \), and
\[ (z^a \varrho \varphi \varphi (\varrho_1)) \varrho (z^b \varrho \varphi \varphi (\varrho_2)) \cong z^{a+b} \varrho \varphi \varphi (\varrho_1 \varrho \varphi \varphi (\varrho_2)) \]
where the intertwiner is the braiding unitary \( \varphi \varphi (\varrho_1) \varrho \varphi \varphi (\varrho_2) \). The naturality of the braiding unitary says that it commutes with \( S \otimes T \) if \( S \in \mathcal{B}(C_{x,l}) \) and \( T \in \mathcal{B}(\varphi \varphi (\varrho_1)) \) are intertwiners or, more generally, \( \mathcal{T} \)-equivariant. Therefore, when we identify \( \mathcal{R}_a \cong \text{Rep}(A_o(F)) \) as above for all \( a \in \mathbb{Z} \), the tensor product functor on \( \text{Rep}(C, \Delta_C) \) restricted to a functor \( \mathcal{R}_a \times \mathcal{R}_b \to \mathcal{R}_{a+b} \) becomes the usual tensor product functor on \( \text{Rep}(A_o(F)) \). The associators \( (\varrho_1 \varrho \varphi \varphi (\varrho_2) \varrho \varphi \varphi (\varrho_3) \varrho \varphi \varphi (\varrho_4)) \) and the unit transformations \( \varepsilon \varrho \varphi \varphi \varphi \varphi (\varepsilon) \) also restrict to the usual ones. So the representation category of \( (C, \Delta_C) \) is equivalent to the representation category of \( A_o(F) \otimes \mathcal{C}(\mathbb{T}) \). In other words, \( (C, \Delta_C) \) is monoidally equivalent to \( A_o(F) \otimes \mathcal{C}(\mathbb{T}) \) (still if \( d = 0 \)). It is already known that \( A_o(F) \) is monoidally equivalent to \( \text{SU}_q(2) \) for a unique \( q \in [-1, 1] \setminus \{0\} \) (see [2]). Hence \( (C, \Delta_C) \) is monoidally equivalent to \( \text{SU}_q(2) \otimes \mathcal{C}(\mathbb{T}) \) for the same \( q \). This finishes the proof of Theorem 1.9 in case \( d = 0 \).

Remark 4.8. The equivalence of categories above does not preserve the functor to the underlying Hilbert spaces because the intertwiner \( z \varrho \varphi t \otimes t \varrho \varphi z \) is not just the flip, but a non-trivial braiding operator. Hence Woronowicz’s Tannaka–Krein Theorem for compact quantum groups from [16] does not apply and it does not follow that the quantum group \( (C, \Delta_C) \) would be isomorphic to \( \text{SU}_q(2) \otimes \mathcal{C}(\mathbb{T}) \).

Finally, we remove the assumption \( d = 0 \). First, if \( d \) is even, then we use the \( \mathcal{T} \)-equivariant Hopf *-isomorphism of Lemma 2.18 with the shift \( s = d/2 \) to reduce to the case \( d = 0 \). Let \( \pi^t = z^{-d/2} \pi \) and \( F_{ji} = \omega_{ij}^t = \zeta^{-dd_{ij}/2} \omega_{ij} \) for \( 1 \leq i, j \leq n \). Then \( A_o(V, \pi, \omega) \cong A_o(V, \pi^t, \omega^t) \) as a braided quantum group, and hence the bosonisations are also isomorphic. Thus the description of the irreducible representations and the monoidal equivalence to \( A_o(F) \otimes \mathcal{C}(\mathbb{T}) \) or to \( \text{SU}_q(2) \otimes \mathcal{C}(\mathbb{T}) \) for a unique \( q \in [-1, 1] \setminus \{0\} \) carry over to all \( A_o(V, \pi, \omega) \) with even \( d \).

If \( d \) is odd, then we pass to a double cover of \( \mathbb{T} \) as in Section 2. More precisely, we replace the representation \( \pi \) of \( \mathbb{T} \) by \( \pi'' := \pi \zeta^z \), take the fourth root of \( \zeta \), and the same vector \( \omega \). The fundamental representation \( t \) of \( A_o(V, \pi, \omega) \) corresponds to the representation \( (z^a)^{-d} \varrho \varphi t^a = (z^a)^{-d} \varrho \varphi r_1 \) of \( A_o(V, \pi^a, \omega) \), and the character \( z \) on \( A_o(V, \pi, \omega) \) corresponds to the character \( (z^a)^2 \). Since any representation of \( A_o(V, \pi, \omega) \) is contained in a tensor power of the two fundamental representations \( z \) and \( t \), we see that the irreducible representations of \( A_o(V, \pi'', \omega) \) that come from irreducible representations of \( A_o(V, \pi, \omega) \) are \( z^a \varrho \varphi \varphi (r_b) \) for which \( b - a \) is even. The representation category of \( A_o(V, \pi, \omega) \) is the full monoidal subcategory of
Rep(\(A_d(V, \pi'', \omega)\)) whose objects are isomorphic to direct sums of these particular irreducible representations. This allows to carry over the results proven for \(d = 0\) to the case of odd \(d\) as well. This gives Theorems 1.8 and 1.9 in complete generality; Rep(\(A_0(V, \pi, \omega)\)) and Rep\(U_q(2)\) correspond to the same subcategory of representations of \(SU_q(2) \times C(T)\) because our construction gives the same subcategory for all \(A_d(V, \pi, \omega)\) with odd \(d\), including \(SU_q(2)\).

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