Mean Field Theory for Josephson Junction Arrays with Charge Frustration

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Abstract

Using the path integral approach, we provide an explicit derivation of the equation for the phase boundary for quantum Josephson junction arrays with offset charges and non-diagonal capacitance matrix. For the model with nearest neighbor capacitance matrix and uniform offset charge \( q/2e = 1/2 \), we determine, in the low critical temperature expansion, the most relevant contributions to the equation for the phase boundary. We explicitly construct the charge distributions on the lattice corresponding to the lowest energies. We find a reentrant behavior even with a short ranged interaction.

A merit of the path integral approach is that it allows to provide an elegant derivation of the Ginzburg-Landau free energy for a general model with charge frustration and non-diagonal capacitance matrix. The partition function factorizes as a product of a topological term, depending only on a set of integers, and a non-topological one, which is explicitly evaluated.

1 Introduction

Josephson junction arrays (JJA) and granular superconductors, namely systems of metallic grains embedded in an insulator, become superconducting in two steps. First, at the bulk critical temperature each grain develops a superconducting gap but the phases of the order parameter on different grains are uncorrelated. Then, at a lower temperature \( T_c \), the Cooper pair tunneling between grains gives rise to a long-range phase coherence and the system as a whole exhibits a phase transition to a superconducting state.

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The phase transition is governed by the competition between the Josephson tunneling, characterized by a Josephson coupling energy $E_J$, and the Coulomb interaction between Cooper pairs, described by a charging energy $E_C$. In classical junction arrays the Josephson coupling $E_J$ is dominant, the transition separates a superconducting low temperature phase from a normal high temperature phase. When $E_C$ is comparable to $E_J$ (small grains) charging effects give rise to a quantum dynamics. The grain capacitance is small, so that the energy cost of Cooper pair tunneling may be higher than the energy gained by the formation of a phase-coherent state. Zero point fluctuations of the phase may destroy the long range superconducting order even at zero temperature (see for example [1]).

Within the framework of the BCS theory, Efetov derived an effective quantum Hamiltonian in terms of the phases $\varphi_i$ of the superconducting order parameter at the grain $i$, and their conjugate variables $n_i$ number of Cooper pairs. Efetov’s Hamiltonian for the quantum phase model reads

$$H = \frac{1}{2} \sum_{ij} C_{ij}^{-1} Q_i Q_j - E_J \sum_{<ij>} \cos(\varphi_i - \varphi_j)$$

$$Q_i = 2e n_i \quad [\varphi_i, n_i] = i \delta_{ij},$$

where $Q_i$ is the excess of charge due to Cooper pairs (charge $2e$) on the site $i$ of a square lattice in D-space dimension and $C_{ij}$ is the capacitance matrix describing the electrostatic coupling between Cooper pairs. The diagonal elements of the inverse matrix $C_{ij}^{-1}$ provide the charging energy: $E_C = e^2 C_{ii}^{-1} / 2 \equiv e^2 / 2 C_0$, where $C_0$ is the self-capacitance.

The superconductor-insulator transition depends crucially on the spatial dimensionality $D$. For $D = 1$ there may exist also other phases. For $D = 2$ the system exhibits a richly structured phase diagram (see for example [7, 8]). In higher dimensions it is believed that the mean field theory analysis provides qualitatively correct results.

It is relevant to understand how the transition from insulator to superconductor depends on the relevant constitutive parameters - such as the capacitances of, and between, the junctions - as well as on external parameters such as the temperature, offset charges and external magnetic fields.

Much work has been done to study the phase diagram of quantum JJ A, in the $T/E_C-E_J/E_C$ plane [1]. The analysis uses mean field theory as well as the renormalization group approach. There is the claim that the phase diagram -under suitable circumstances- may exhibit a reentrant character with the superconducting phase existing between upper and lower critical temperature. In the influence of the Coulomb energy on the transition temperature was investigated for a model with a diagonal capacitance matrix. The effects of off-diagonal terms in the charging energy were investigated by several authors within the mean field theory approximation: while it is widely believed that the nearest neighbors interaction enhance the transition temperature $T_c$ by lowering the energy cost for a Cooper pair to tunnel from one neutral grain to another, there is
still some dispute on whether there is a reentrance or not for models with non-diagonal capacitance matrix [12, 20, 21].

It is relevant for physical applications to consider the effect of a background of external charges on the superconductor-insulator transition of a quantum JJA [7, 8, 22, 23]. Such an offset of charges might arise in physical systems as a result of charged impurities or by application of a gate voltage between the array and the ground. In the former case offset charges are distributed randomly on the lattice while in the latter case they play the role of a chemical potential for charges. They might be regarded as effective charges \( q_i \) on the sites of the lattice. When \( q_i \neq 2e \), the offset charges cannot be eliminated by Cooper pair tunneling.

Offset charges frustrate the attempts of the system to minimize the energy of the charge distribution of the ground state. Consequently the charging energy of any excitations is smaller compared to the unfrustrated case and superconductivity is enhanced. With offset charges the Hamiltonian (1) becomes

\[
H = \frac{1}{2} \sum_{ij} C_{ij}^{-1}(Q_i + q_i)(Q_j + q_j) - E_J \sum_{<ij>} \cos(\phi_i - \phi_j). 
\] (2)

In order to study the effect of charge frustration on the phase diagram of the system described by the Hamiltonian (2), it is our purpose to revisit the mean field theory of quantum JJA using the path-integral method. The approach uses the Hubbard-Stratonovich [24] representation for the partition function in terms of coarse-grained classical local variables \( \psi_i \) for which the effective action is computed [19]. We find a reentrant behavior for models with a nearest neighbor capacitance matrix and a uniform offset charge \( q_i = e \), even if the interaction among grains is short ranged. We find analytically the equation which determines the critical temperature as a function of \( E_J/E_C \). This allows us to analyze the low temperature limit of the theory and to find the regimes in which a reentrant behavior might be observed.

In section 2 we review the self consistent mean field theory approximation within the Hamiltonian formalism for quantum JJA. We study the eigenvalue equation of the mean field Hamiltonian with diagonal capacitance, and uniform offset charge \( q_i = e \) showing explicitly that at zero temperature there is superconductivity for all values of \( \alpha = zE_J/4E_C \).

In section 3 we use the coarse grained approach to compute the Ginzburg-Landau free energy for quantum JJA with charge frustration and a general Coulomb interaction matrix. The path integral providing the phase correlator needed to investigate the critical behavior of the system, is explicitly computed.

In section 4, from the Ginzburg-Landau free energy, we derive, within the mean field theory approximation, the analytical form of the critical line equation. The phase diagram is drawn in the diagonal case for a generic external charge distribution. We then analyze the low temperature limit of a system with nearest neighbor interaction matrix and find a reentrant behavior when a uniform background of external charges \( q_i = e \) is considered.
Section 5 is devoted to some concluding remarks. The appendices contain the derivation of some relevant formulas of the main text.

2 Self-consistent mean field theory in the Hamiltonian approach

Mean field theory for quantum JJA with diagonal capacitance matrix was first used by Symanek [9]. The approximation consists in replacing the Josephson coupling of the phase on a given island $i$ to its neighbors by an average coupling so that:

$$E_J \sum_{<ij>} \cos(\varphi_i - \varphi_j) = zE_J \langle \cos \varphi \rangle \sum_i \cos \varphi_i. \quad (3)$$

In (3) $z$ is the coordination number; it is assumed also that $\langle \cos \varphi \rangle$ does not depend on the island index $i$ and the choice $\langle \sin \varphi_i \rangle = 0$, which provides a real order parameter, has been made.

Within the mean field approximation the Hamiltonian (1) becomes

$$H_{mf} = \frac{1}{2} \sum_{ij} C_{ij}^{-1} Q_i Q_j - zE_J \langle \cos \varphi \rangle \sum_i \cos \varphi_i \quad (4)$$

and the order parameter $\langle \cos \varphi \rangle$ is evaluated self-consistently from (4). For a diagonal capacitance matrix ($C_{ij} = C_0 \delta_{ij}$) mean field theory computation are very simple since (4) describes on each site a quantum particle in a periodic potential $\cos \varphi_i$ [9].

The eigenfunction of the array is a product of eigenfunctions $\psi_n(\varphi)$ describing the individual islands and satisfying the Mathieu equation [25]

$$\left( -\frac{d^2}{d\varphi^2} - \frac{zE_J}{4EC} < \cos \varphi > \cos \varphi \right) \psi_n(\varphi) = \frac{E_n}{4EC} \psi_n(\varphi) \quad (5)$$

with periodic boundary conditions $\psi_n(\varphi) = \psi_n(\varphi + 2\pi)$.

It is well known that the Mathieu equation admits also antiperiodic solutions, $\psi_n(\varphi) = -\psi_n(\varphi + 2\pi)$ (see appendix A). If both periodic and antiperiodic solutions are used, the general solution of (5) does not have a definite periodicity and, consequently, the charges $n_i$ take continuous eigenvalues. Such continuous eigenvalues are expected to be relevant in the description of continuous flows of currents due for example to ohmic shunt resistances [26, 27]. Although the superposition of both periodic and antiperiodic solutions yields to a reentrant behavior even in the unfrustrated dissipationless diagonal model [10, 11, 29], this superposition is not allowed in describing physical situations in which the only excitations are Cooper pairs of charge $2e$ [1, 12, 7]. Thus the use of both periodic and antiperiodic solutions does not have physical significance in the models considered in this paper.

The mean field self-consistency condition gives

$$\langle \cos \varphi \rangle = \sum_n e^{-\beta E_n} \frac{\langle \cos \varphi \rangle |\psi_n\rangle \langle \psi_n|}{\sum_n e^{-\beta E_n}} \quad (6)$$

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with $\beta = 1/k_B T$. For high temperatures or low $E_J$ only the solution $< \cos \varphi > = 0$ exists and there is not superconductivity. For low temperatures or high $E_J$ $< \cos \varphi > \neq 0$ and the system as a whole behaves as a superconductor.

From (6) one gets the equation for the critical temperature [9]

$$\alpha = \frac{\sum_{n=-\infty}^{+\infty} e^{-\frac{4}{y} n^2}}{\sum_{n=-\infty}^{+\infty} \frac{1}{1-4n^2} e^{-\frac{4}{y} n^2}}$$  \hspace{1cm} (7)

with $y = k_B/T_c E_C$ and $\alpha = zE_J/4E_C$.

In fig.1 we plot $T_c$ as a function of $\alpha$.

![Figure 1: Phase diagram for the diagonal model without charge frustration.](image)

If one considers a diagonal capacitance matrix and uniform offset charges of magnitude $e$ on each site ($q_i/2e = 1/2$), the Hamiltonian reads

$$H_d = \frac{1}{2C_0} \sum_i (Q_i + q_i)(Q_i + q_i) - E_J \sum_{<ij>} \cos(\varphi_i - \varphi_j).$$  \hspace{1cm} (8)

Mean field theory of this model leads to a Schrödinger equation of the form

$$\left[ -\frac{d^2}{d\varphi^2} - 2i \frac{q}{2e} \frac{d}{d\varphi} + \left( \frac{q}{2e} \right)^2 - \alpha < \cos \varphi > \cos \varphi \right] \psi_n(\varphi) = \frac{E_n}{4E_C} \psi_n(\varphi).$$  \hspace{1cm} (9)

Redefining the phase of the wave function as

$$\psi_n(\varphi) = e^{-i\tilde{q}\varphi} \rho_n(\varphi)$$
(10) reduces to a Mathieu equation for $\rho_n(\phi)$

$$\frac{d^2 \rho_n}{d \phi^2} + \left(\frac{\lambda}{4} - \frac{v}{2} \cos \phi\right) \rho_n = 0$$

with $\lambda_n = E_n/E_C$ and $v = -z E_J < \cos \phi > /2E_C$. Equations (8),(9),(10) lead to the following modification of (7) [see Appendix A]:

$$\alpha = \frac{e^{-\frac{\phi}{2}} + \sum_{n=1}^{+\infty} e^{-\frac{\phi}{2}(n+\frac{1}{2})^2}}{\frac{1}{2} + \sum_{n=1}^{+\infty} \frac{1}{1 - 4(n+\frac{1}{2})^2} e^{-\frac{\phi}{2}(n+\frac{1}{2})^2}}$$

which - at variance with the unfrustrated model - exhibits superconductivity even for infinitesimal values of $\alpha$. This feature is shown in fig. 2 which also shows the absence of a reentrant behavior at low $T$.

![Figure 2: Phase diagram of the diagonal model with half-integer charge frustration.](image)

For frustrated models with non diagonal capacitance matrix, the self-consistent mean-field theory approximation becomes very cumbersome and one should resort to the more powerful functional approach. A reentrance at a low $T_c$ is expected at least when the interaction between grains is long-ranged [22, 18].


### 3 Ginzburg-Landau free energy

The partition function for the frustrated off-diagonal model is given by

$$ Z = \text{Tr}e^{-\beta H} = \sum_n \langle \psi_n | e^{-\beta H} | \psi_n \rangle $$  \hfill (12)

where $H$ is given in (2) and the sum is extended only to states of charge $2e$ and thus with definite periodicity.

In the functional approach $Z$ reads

$$ Z = \int \prod_i D\varphi_i \exp \left\{ - \int_0^\beta d\tau L_E \left( \varphi_i(\tau), \frac{d\varphi_i}{d\tau}(\tau) \right) \right\} $$  \hfill (13)

where the Euclidean Lagrangian $L_E$ can be derived from

$$ L = \frac{1}{2} \left( \frac{\hbar}{2e} \right)^2 \sum_{ij} C_{ij} \frac{d\varphi_i}{dt} \frac{d\varphi_j}{dt} - \left( \frac{\hbar}{2e} \right) \sum_i \frac{d\varphi_i}{dt} q_i + E_J \sum_{i<j} \cos(\varphi_i - \varphi_j) $$  \hfill (14)

by replacing $it/\hbar \rightarrow \tau$. The path integral that one should compute is then given by:

$$ Z = \int \prod_i D\varphi_i \exp \left\{ \int_0^\beta d\tau \left[ -\frac{1}{2} \sum_{ij} C_{ij} \frac{\varphi_i}{2e} \frac{\varphi_j}{2e} + i \sum_i q_i \frac{\varphi_i}{2e} + \frac{E_J}{2} \sum_{ij} e^{i\varphi_i} \gamma_{ij} e^{-i\varphi_j} \right] \right\} $$  \hfill (15)

where $-\infty < \varphi_i < +\infty$, $\varphi_i(0) = \varphi_i(\beta) + 2\pi n_i$ and $\gamma_{ij} = 1$ if $i,j$ are nearest neighbors and equals zero otherwise. The integers $n_i$ appearing in this boundary conditions take into account the $2\pi$-periodicity of the states $\psi_n$ appearing in (12).

In order to derive the Ginzburg-Landau free energy for the order parameter, it is convenient to carry out the integration over the phase variables by means of the Hubbard-Stratonovich procedure [24]: using the identity

$$ e^{J+\Gamma J} = \frac{\det \Gamma^{-1}}{\pi^N} \int \prod_i D^2 \psi_i e^{-\psi^\dagger \Gamma^{-1} \psi - J^+ \psi - \psi^+ J} $$  \hfill (16)

the partition function may be rewritten as

$$ Z = \int \prod_i D\psi_i D\psi_i^\dagger e^{\int_0^\beta d\tau (\frac{1}{2} \sum_{ij} \psi_i^\dagger \gamma_{ij} \psi_j + S_{Eff}[\psi])} $$  \hfill (17)

where the effective action for the auxiliary Hubbard-Stratonovich field $\psi$, $S_{Eff}[\psi]$, is given by

$$ S_{Eff}[\psi] = -\log \left\{ \int \prod_i D\varphi_i \ exp \left\{ \int_0^\beta d\tau \left[ -\frac{1}{2} \sum_{ij} C_{ij} \frac{\varphi_i}{2e} \frac{\varphi_j}{2e} + \frac{E_J}{2} \sum_{ij} e^{i\varphi_i} \gamma_{ij} e^{-i\varphi_j} \right] \right\} \right\} $$


\[ +i \sum_i \left( q_i \frac{\psi_i}{2e} - \psi_i e^{i\varphi_i} - \psi_i^* e^{-i\varphi_i} \right) \right \}. \] (18)

The Hubbard-Stratonovich field \( \psi_i \) may be regarded as the order parameter for the insulator-superconductor phase transition since it turns out to be proportional to \( < e^{i\varphi_i} > \), as it can be easily seen from the classical equations of motion. From (18) the Ginzburg-Landau free-energy may be derived by integrating out the phase field \( \phi_i \).

Since the phase transition is second order \([28]\), close to the onset of superconductivity, the order parameter \( \psi_i \) is small. One may then expand the effective action up to the second order in \( \psi_i \), getting

\[ S_{Eff}[\psi] = S_{Eff}[0] + \int_0^\beta d\tau \int_0^\beta d\tau' G_{rs}(\tau, \tau') \psi_r(\tau) \psi_s^*(\tau') + \cdots \] (19)

where \( G_{rs} \) is the phase correlator

\[ G_{rs}(\tau, \tau') = \frac{\delta^2 S_{Eff}[\psi]}{\delta \psi_r(\tau) \delta \psi_s(\tau')} \bigg|_{\psi, \psi^* = 0} = \langle e^{i\varphi_r(\tau) - i\varphi_s(\tau')} \rangle_{0}. \] (20)

The partition function (17), can be written as

\[ Z = \int \prod_i d\psi_i d\psi_i^* e^{-F[\psi]} \] (21)

where \( F[\psi] \) is the Ginzburg-Landau free energy; due to (18,19), up to the second order in \( \psi_i \), one has

\[ F[\psi] = \int_0^\beta d\tau \int_0^\beta d\tau' \sum_{ij} \psi_i^*(\tau) \gamma_{ij}^{-1} \delta(\tau - \tau') - G_{ij}(\tau, \tau') \psi_j(\tau'). \] (22)

We shall now compute the phase correlator \( G_{rs} \) by evaluating the expectation value in (21) by means of the path integral over the phase variables \( \varphi_i(\tau) \). In performing this integration one should take into account that the field configurations satisfy

\[ \varphi_i(\beta) - \varphi_i(0) = 2\pi n_i. \] (23)

For this purpose it turns out very convenient to untwist the boundary conditions by decomposing the phase field in terms of a periodic field \( \phi_i(\tau) \) and a term linear in \( \tau \) which takes into account the boundary conditions \([23]\); namely, one sets

\[ \varphi_i(\tau) = \phi_i(\tau) + \frac{2\pi}{\beta} n_i \tau, \] (24)

with \( \phi_i(\beta) = \phi_i(0) \). Summing over all the phases \( \varphi_i(\tau) \) amounts then to integrate over the periodic field \( \phi_i \) and to sum over the integers \( n_i \). As a result the
phase correlator factorizes as the product of a topological term depending on the integers \( n_i \) and a non-topological one; namely

\[
G_{rs}(\tau; \tau') = \frac{\int D\phi_t e^{i\phi_t(\tau') - i\phi_t(\tau')} \exp\left\{ \int_0^\beta d\tau - \frac{1}{2} C_{ij} \frac{\partial^2 \phi_i}{\partial x_j} \right\}}{\int D\phi_t \exp\left\{ \int_0^\beta d\tau - \frac{1}{2} C_{ij} \frac{\partial^2 \phi_i}{\partial x_j} \right\}} \cdot \\
\sum_{[n]} e^{i\frac{2\pi}{\beta}(n_r \tau - n_s \tau')} e^{\left\{ \sum_{i} - \frac{e^2}{2\pi \tau} C_{ij} n_i n_j + \sum_{i} 2i\pi \frac{\beta}{2} n_i \right\}}. 
\]

(25)

After a lengthy computation, the first (non-topological) factor appearing in the l.h.s. of equation (25) has the following simple expression [see Appendix B]:

\[
\delta_{rs} \exp\left\{ -2e^2 C^{-1}_{rr} \left( |\tau - \tau'| - \frac{(\tau - \tau')^2}{\beta} \right) \right\}. 
\]

(26)

The sum over the integers in the topological factor in (25) is done by means of the Poisson resummation formula

\[
|\det G|^{-\frac{1}{2}} \sum_{[n]} e^{-\pi(n-a_i)G_{ij}(n-a_i)} = \sum_{[n]} e^{-\pi m_i (\tau - a_i)},
\]

Thus eq. (25) becomes

\[
G_{rs}(\tau; \tau') = \delta_{rs} e^{-2e^2 C^{-1}_{rr} |\tau - \tau'|} \cdot \\
\sum_{[n]} e^{\sum_{ij} 2e^2 C^{-1}_{ij} (n_i + \frac{\beta}{2})(n_j + \frac{\beta}{2}) - \sum_{ij} 4e^2 C^{-1}_{ij} (n_i + \frac{\beta}{2})(n_j + \frac{\beta}{2})(\tau - \tau')} \\
\sum_{[n]} e^{\sum_{ij} 2e^2 C^{-1}_{ij} (n_i + \frac{\beta}{2})(n_j + \frac{\beta}{2})}
\]

(27)

with \( n_i \) assuming all integer values and \( \sum_{[n]} \) being a sum over all the configurations.

By means of a Euclidean-time Fourier transform, the fields \( \psi_i \) are written as

\[
\psi_i(\tau) = \frac{1}{\beta} \sum_{\mu} \psi_i(\omega_\mu) e^{i\omega_\mu \tau},
\]

where \( \omega_\mu \) are the Matsubara frequencies. As a consequence, the phase correlator \( G_{ij} \) can be expressed as

\[
G_{ij}(\tau; \tau') = \frac{1}{\beta} \sum_{\mu \mu'} G_{ij}(\omega_\mu; \omega_{\mu'}) e^{i\omega_\mu \tau} e^{i\omega_{\mu'} \tau'}.
\]

(28)

From (27) one can show that \( G_{rs}(\omega_\mu; \omega_{\mu}') \) is diagonal in the Matsubara frequencies and can be written as

\[
G_{rs}(\omega_\mu; \omega_{\mu}') = G_r(\omega_\mu) \cdot \delta_{rs} \cdot \delta(\omega_\mu + \omega_{\mu'})
\]

(29)
with
\[ G_r(\omega_\mu) = \frac{1}{2E_C} \sum_{[n_i]} e^{-\frac{1}{2} \frac{U_{ij}}{U_{00}} (n_i + \frac{q_i}{2}) (n_j + \frac{q_j}{2})} \cdot \frac{1}{Z_0}. \] (30)

In (31) \( Z_0 \) is given by
\[ Z_0 = \sum_{[n_i]} e^{-\frac{1}{2} \sum_{[n_i]} U_{ij} (n_i + \frac{q_i}{2 e}) (n_j + \frac{q_j}{2 e})}. \]

with \( U_{ij} = C_{ij}^{-1} \), \( E_C = e^2 C_{rr}^{-1} / 2 \) and \( y = k_B T_c / E_C \). In terms of Matsubara frequencies the Ginzburg-Landau free energy (22) becomes

\[ F[\psi] = \frac{1}{\beta} \sum_{\mu ij} \psi_i^*(\omega_\mu) \left[ \frac{2}{E_j} \gamma_{ij}^{-1} - G_i(\omega_\mu) \delta_{ij} \right] \psi_j(\omega_\mu). \] (31)

This is our starting point for any analysis of the phase boundary between the insulating and the superconducting phases in JJA with arbitrary capacitance matrix and with charge frustration.

4 Mean field theory analysis

In the following we shall derive the equation determining the phase boundary in the plane \((\alpha, K_B T_c / E_C)\), in mean field theory and for a system with arbitrary capacitance matrix and a uniform distribution of off-set charges. For this purpose it is convenient to expand the fields \( \psi_i(\omega_\mu) \) and \( G_i(\omega_\mu) \) in terms of the vectors of the reciprocal lattice \( q \). One has
\[ \psi_i(\omega_\mu) = \frac{1}{N} \sum_q \psi_q(\omega_\mu) e^{i q \cdot \mathbf{i}} \] (32)
\[ G_i(\omega_\mu) = \frac{1}{N} \sum_q G_q(\omega_\mu) e^{i q \cdot \mathbf{i}}. \] (33)

Moreover
\[ \gamma_{ij}^{-1} = \frac{1}{N} \sum_q \gamma_q^{-1} e^{i q \cdot (i-j)}. \] (34)

where \( \gamma_q^{-1} \) is the inverse of the Fourier transform of the Josephson coupling strength \( \gamma_{ij} \) which equals 1 for \( i, j \) nearest neighbors and 0 otherwise. As a consequence
\[ \gamma_q^{-1} = \frac{1}{\sum_p e^{-i q \cdot p}} \]

where \( p \) is a vector connecting two nearest neighbors sites. Expanding in \( q \) one gets
\[ \gamma_q^{-1} = \frac{1}{z} + \frac{q^2 a^2}{2 z^2} + \cdots \] (35)
where $a$ is the lattice spacing and $z$ the coordination number. The first term in (35) provides the mean field theory approximation which, as expected, is exact in the limit of large coordination number.

The Ginzburg-Landau free energy (31), reads

$$F[\psi] = \frac{1}{\beta N} \sum_{\mu q q'} \psi_q(\omega_\mu)^* \left[ \gamma_{q q'}^{-1} \delta_{q q'} - \frac{G_{q-q'}(\omega_\mu)}{N} \right] \psi_{q'}(\omega_\mu) \simeq$$

Using (35) and keeping only terms of zero-th order in $\omega_\mu$ and $q$ one obtains the mean field theory approximation to the coefficient of the quadratic term of $F$

$$\simeq \frac{1}{\beta N} \sum_{q \mu} \left[ \frac{2}{E_j z} - G_0(0) + \cdots \right] |\psi_q(\omega_\mu)|^2. \quad (36)$$

The equation for the phase boundary line then reads as

$$1 = z \frac{E_j}{2} G_0(0) \quad (37)$$

with

$$G_0(0) = \frac{1}{N} \sum_r G_r(0). \quad (38)$$

Equation (37) determines the relation between $T_c$ and $\alpha$ at the phase boundary.

For a uniform distribution of offset charges eq.(37) simplifies further since in (38) $G_r$ does not depend on $r$. As a consequence, the phase boundary equation becomes

$$1 = \alpha \cdot \sum_{[n_i]} e^{-\frac{1}{2} \sum \mu \nu \omega_{00} (n_i+\frac{q}{2}) (n_j+\frac{q}{2})} \cdot \frac{1}{Z_0} \cdot \frac{1}{Z_0} \cdot (39)$$

with

$$\alpha = \frac{z E_j}{4 E_c} \quad \text{and} \quad Z_0 = \sum_{[n_i]} e^{-\frac{1}{2} \sum \mu \nu \omega_{00} (n_i+\frac{q}{2}) (n_j+\frac{q}{2})}.$$

In the following we shall derive the physical implications of (39) in a variety of models describing JJA.

4.1 Self-charging model

For a diagonal capacitance matrix, $U_{ij} = \delta_{ij} U_0$, one singles out only the self-interaction of plaquettes. This case was already analyzed in section 2 within the approach of self-consistent mean field theory. As a check of the path integral approach we shall show that one is able to reproduce the same results from eq.(33).

In the diagonal case eq.(39) becomes

$$1 = \alpha \left( \sum_n e^{-\frac{1}{2} (n+\frac{q}{2})^2} \frac{1}{1-4(n+\frac{q}{2})^2} \right) \cdot \frac{1}{Z_0} \cdot (40)$$
Since \( n \) is an integer \( [40] \) is invariant under the shift \( \frac{q}{2e} \to \frac{q}{2e} + 1 \). For \( q = 0 \) eq.\( [40] \) reduces to \( [7] \). From figure 1 one readily sees that there is no superconductivity for \( \alpha < 1 \). Due to the periodicity of \( [40] \) this holds for any integer \( q \). For \( q/2e \) equal to 1/2 one gets equation \( [11] \). From figure 2 one sees that superconductivity is attained for all the values of \( \alpha \), since the superconducting order parameter at zero temperature is different from zero.

For the self-charging model the system exhibits superconductivity for all the values of \( \alpha \) also if the distribution of offset charges is such that integer and half-integer charges coexist on the lattice. If one denotes by \( f_0 \) the fraction of integer charges and by \( f_{1/2} = 1 - f_0 \) the fraction of half-integer charges, eq.\( [40] \) implies that

\[
\alpha = \left( f_0 \frac{\sum_n e^{-\frac{4}{\pi}n^2 \frac{1}{1-4m^2}}}{\sum_m e^{-\frac{4}{\pi}m^2}} + f_{1/2} \frac{\sum_n e^{-\frac{4}{\pi}(n+1/2)^2 \frac{1}{1-4(n+1/2)^2}}}{\sum_m e^{-\frac{4}{\pi}(m+1/2)^2}} \right)^{-1}.
\]

In fig.3 we plot \( T_c \) as a function of \( \alpha \) for several values of \( f_0 \). As expected superconductivity is enhanced as \( f_{1/2} \) increases.

### 4.2 Models with non-diagonal capacitance matrix

In \( [20] \) Fishman and Stroud, using a low temperature expansion, determined \( T_c \) as a function of \( \alpha \) for models with non diagonal interaction matrix without
considering the effect of offset charges. They did not find signs of normal state reentrance for nearest neighbor interaction matrix models in which only the diagonal interaction matrix element $U_{00}$ and the nearest neighbor interaction matrix element $U_{0p} = \theta U_{00}$ are nonzero. This can be seen from the expansion of the critical line eq.(39) for $q = 0$ and small critical temperatures:

$$\alpha = 1 + \left[ \frac{8}{3} + 2z(1 - \frac{1}{1 - 4\theta^2}) \right] e^{-\frac{4}{\theta}} + \ldots$$

Reentrant behavior is possible [12] for $\theta > \theta_c = \frac{1}{\sqrt{4 + 3z}}$ when the coefficient of the exponential $e^{-4/\theta}$ is negative; in fact, the phase boundary line $\alpha = \alpha(T_c)$ first bends to the left due to the negative coefficient of $e^{-4/\theta}$ and finally, when the critical temperature is high enough, bends to the right, favoring the insulating phase.

As evidenced by Fishman and Stroud [20], the regime of physical interest is $\theta < \frac{1}{z}$; namely, when the capacitance matrix is invertible. Reentrance is possible only in one dimension ($\theta_c = 1/\sqrt{10} < 1/z = 1/2$); in higher dimensions reentrance occurs only when the electrostatic interaction is long ranged [20].

If there are half-integer offset charges on the sites of a square lattice, our analysis shows that the equation for the critical line is

$$\alpha = \frac{\sum_{[n_i]} e^{-\frac{4}{\theta} E_{0i}} (n_i + \frac{1}{2})(n_j + \frac{1}{2})}{\sum_{[n_i]} e^{-\frac{4}{\theta} E_{0i}} (n_i + \frac{1}{2})(n_j + \frac{1}{2})}$$

In eq.(41) appears the expression

$$E_{[n_i]} = \sum_{ij} \frac{U_{ij}}{U_{00}} (n_i + \frac{1}{2})(n_j + \frac{1}{2})$$

which is the electrostatic energy of a generic charge distribution on the lattice.

Denoting with $n_{0i}$ and $n_{1i}$ the charge distributions of the two lowest lying energy states and with $E_0$ and $E_1$ the corresponding energies, the low temperature expansion of eq.(41) yields

$$\alpha = \frac{\sum_{[n_{0i}]} e^{-\frac{4}{\theta} E_{0i}} + \sum_{[n_{1i}]} e^{-\frac{4}{\theta} E_{1i}} + \cdots}{\sum_{[n_{0i}]} e^{-\frac{4}{\theta} E_{0i}} (n_i + \frac{1}{2})^2 + \sum_{[n_{1i}]} e^{-\frac{4}{\theta} E_{1i}} (n_i + \frac{1}{2})^2 + \cdots}.$$ (43)

Independently on the explicit form of $U_{ij}$, $E_{[ni]}$ reaches its minimum value when $(n_i + \frac{1}{2}) = \pm \frac{1}{2}(-1)^{i_1 + i_2 + \cdots + i_D}$ with $i_j$ ($j = 1, \ldots, D$) the components of the lattice position vector $i$ in units of the lattice spacing. This charge configuration is exhibited in figure 4. For models with nearest-neighbor interaction, i.e. $U_{ij} = \delta_{ij} + \theta \sum_p \delta_{i+p,j}$ with $\sum_p$ denoting summation over nearest neighbors, the charge
Figure 4: ground state.

\[
\begin{array}{ccccccc}
- & + & - & + & - & + & - \\
+ & - & + & - & + & - & - \\
- & + & - & + & - & + & - \\
+ & - & + & - & + & - & - \\
- & + & - & + & - & + & - \\
\end{array}
\]

Figure 5: first excited state.

\[
\begin{array}{ccccccc}
- & + & - & + & - & + & - \\
+ & - & + & - & + & - & - \\
- & + & - & + & - & + & - \\
+ & - & + & - & + & - & - \\
- & + & - & + & - & + & - \\
\end{array}
\]

The configuration corresponding to the first excited state is given in Fig. 5. The energy of the charge distribution of Fig. 5 is

\[ E[n^1_i] = E[n^0_i] + z\theta, \]

where \( E[n^0_i] \), the ground state energy, is given by

\[ \sum_i \left( 1 - \frac{1}{1 - (1 - z\theta)^2} \right) \cdot e^{-\frac{4z\theta}{\pi}} + \cdots \]

Reentrant behavior at low temperature occurs when the coefficient of the exponential is negative, namely when

\[ a_1 \equiv \left( 1 - \frac{1 - (1 - z\theta)^2}{1 - (1 - (z - 2)\theta)^2} \right) < 0. \]

In Appendix C we compute also the coefficients \( a_2 \) and \( a_3 \) of the higher order exponentials in the expansion (44). In Fig. 6 we plot the coefficients \( a_1, a_2 \) and \( a_3 \) as a function of \( \theta \) for \( z = 6 \), i.e. for a 3-D array on a square lattice. One sees that the inequality (45) can be satisfied for values of \( \theta \) consistent with the physical constraint \( \theta < 1/z = 1/6 \).

In Fig. 7 we plot \( T_c \) versus \( \alpha \) for \( \theta = 0.05 \) and \( z = 6 \). In this plot we keep into account also the next two orders of (44) with coefficients \( a_2 \) and \( a_3 \). The resulting diagram exhibits reentrance in the insulating phase even for models with nearest neighbors interaction.

In Fig. 8 we plot \( \alpha_0 = \alpha(T_c = 0) \) as a function of \( \theta \) for \( q \) integer and \( q \) half-integer and for \( z = 6 \). The plot shows that half-integer offset charges always favor superconductivity and that -at variance with the self-charging model- for non-diagonal interaction matrix there is always a range of \( \alpha \) in which the system
Figure 6: Expansion coefficients of $\alpha$ as a function of nearest neighbor interaction $\theta$.

Figure 7: Phase diagram for small critical temperatures with $z = 6$ and $\theta = 0.05$. 
Figure 8: Broadening of the superconducting phase at $T = 0$ with $z = 6$ and nearest neighbor interaction.

is in the insulating phase. The plot also shows that for $q/2e = 1/2$ and $T = 0$ the size of the superconducting region in the phase diagram depends on $\theta$.

5 Discussion

In this paper, using the path integral approach, we provided an explicit derivation of the equation for the phase boundary for quantum Josephson junction arrays with offset charges and non-diagonal capacitance matrix.

For the model with nearest neighbor capacitance matrix and uniform offset charge $q = 1/2$ (in units of $2e$), using a procedure developed in [20], we were able to determine, in the low temperature expansion, the most relevant contributions to the equation for the phase boundary. For this purpose we explicitly constructed the charge distributions on the lattice corresponding to the lowest energies.

Confirming the results of the numerical analysis of ref. [8], we found a reentrant behavior even with a short ranged interaction. Our analysis extends the results found in [20] to the situation in which offset charges are present and provides a physical picture of the states contributing to the reentrant behavior.

For a model with diagonal capacitance matrix our analysis confirms the absence of reentrant behavior for the physical situation where the phase variable is $2\pi$-periodic. The diagonal model with offset charge $q = 1/2$ exhibits
superconductivity for all the values of $\alpha = zE_J/4EC$, since in this case the superconducting order parameter is different from zero at zero temperature; this is evidenced by eq.(53) in Appendix A. An offset charge $q = 1/2$ tends to decrease the charging energy and thus favors the superconducting behavior even for small Josephson energy $E_J$.

A merit of the path integral approach, used in this paper, is that it allows to follow at each stage of the analysis the effects of the $2\pi$-periodicity of the phase variable. In fact, one can untwist this periodicity by introducing a set of integers, so that the partition function factorizes as a product of a topological term, depending only on this set of integers, and a non-topological one explicitly evaluated in Appendix B. The Poisson resummation formula for the topological part of the partition function turns out very useful for the derivation of the low critical temperature expansion.

It would be interesting to investigate the superconducting-insulating behavior in quantum JJA in lower dimensional models, where mean field theory is not expected to provide accurate results. For $D = 1$ there is evidence [6] for a new phase separating the superconducting and the insulating phase. The analysis of the phase diagram for this case should be carried out with different methods such as the renormalization group [15, 16].

A Derivation of the self consistency equation

With a uniform charge frustration $q$ the pertinent Mathieu equation is given by

$$\left[ -\frac{d^2}{d\varphi^2} - 2i \frac{q}{2e} \frac{d}{d\varphi} + \left( \frac{q}{2e} \right)^2 - \alpha <\cos \varphi> \cos \varphi \right] \psi_n(\varphi) = \frac{E_n}{4EC} \psi_n(\varphi) \quad (46)$$

Upon defining

$$\psi_n(\varphi) = e^{-i\frac{q}{2e}\varphi} \rho_n(\varphi) \quad (47)$$

eq.(46) becomes

$$\frac{d^2}{d\varphi^2} \rho_n + \left( \frac{\lambda}{4} - \frac{v}{2} \cos \varphi \right) \rho_n = 0 \quad (48)$$

with $\lambda_n = E_n/EC$ and $v = -zE_J <\cos \varphi>/2EC$. Eq.(48) yields the canonical form of the Mathieu equations

$$\frac{d^2}{dx^2} y + (\lambda - 2v \cos 2x) y = 0 \quad (49)$$

if one puts $\varphi = 2x$ and $\psi_n = y$. 

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The Mathieu equation has the well known periodic solutions [25]:

\[
\begin{align*}
ce_{2n}(x, v) & \quad \text{even solutions with period } \pi \\
se_{2n+2}(x, v) & \quad \text{odd solutions with period } \pi \\
\end{align*}
\]

with eigenvalues \(a_{2n}(v)\) and

\[
\begin{align*}
ce_{2n+1}(x, v) & \quad \text{even solutions with period } 2\pi \\
se_{2n+1}(x, v) & \quad \text{odd solutions with period } 2\pi \\
\end{align*}
\]

with eigenvalues \(b_{2n+1}(v)\)

If \(q/2\pi\) is integer, the periodic boundary conditions \(\psi_n(\varphi = 0) = \psi_n(\varphi = 2\pi)\) singles out only the \(2\pi\)-periodic Mathieu eigenfunctions \(ce_{2n}, se_{2n}\). With these eigenfunctions one may derive (7) \([\text{10}]\). If \(q/2\pi\) is half-integer, the periodic boundary conditions together with (48) single out the \(\pi\)-anti-periodic Mathieu eigenfunctions (i.e. \(\rho_n\) is anti-periodic of \(2\pi\) and periodic of \(4\pi\)). These are the Mathieu eigenfunctions \(ce_{2n+1}\) and \(se_{2n+1}\).

Since, near the critical temperature \(T_c\), the order parameter \(\langle \cos \varphi \rangle\) and \(v\) are small, apart from the phase factor \(e^{-i\varphi/2}\) (important only for the periodicity), to first order in \(v\), eq.\((46)\) has the solutions

\[
\psi_1^\pm = \frac{1}{\sqrt{\pi}} \left( \cos \frac{\varphi}{2} - \frac{q}{8} \cos \frac{3\varphi}{2} \right)
\]

\[
\psi_0^\pm = \frac{1}{\sqrt{\pi}} \left( \sin \frac{\varphi}{2} - \frac{q}{8} \sin \frac{3\varphi}{2} \right)
\]

\[
\psi_{2n+1}^\pm = \frac{1}{\sqrt{\pi}} \left( \cos \left(\frac{2n+1}{2}\right)\varphi - v \left[ \frac{\cos \left(\frac{2n+3}{2}\right)\varphi}{4(2n+2)} - \frac{\cos \left(\frac{2n-1}{2}\right)\varphi}{8n} \right] \right)
\]

\[
\left( n = 1, 2, \ldots \right)
\]

with the corresponding eigenvalues given by

\[
\begin{align*}
E_1^+ &= EC(1 + q) \\
E_0^+ &= EC(1 - q) \\
E_1^- &= E_0^- = EC(2n + 1)^2 \\
E_{2n+1}^- &= E_{2n+1}^+ = EC(2n + 1)^2 \\
\end{align*}
\]

(51)

The expectation values of the superconducting order parameter on the eigenfunctions \((50)\) are given by

\[
\langle \psi_n | \cos \varphi | \psi_n \rangle = \int_0^{2\pi} d\varphi \cos \varphi \left| \psi_n(\varphi) \right|^2 .
\]

(52)
Using (50), to the first order in \(v\) one gets

\[
\begin{aligned}
< \psi^e_1 | \cos \varphi | \psi^e_1 > &= \frac{1}{2} - \frac{v}{8} \\
< \psi^e_2 | \cos \varphi | \psi^e_2 > &= \frac{1}{2} - \frac{v}{8} \\
< \psi^o_{2n+1} | \cos \varphi | \psi^o_{2n+1} > &= \frac{v}{8(n+1)} \\
< \psi^o_{2n+1} | \cos \varphi | \psi^o_{2n+1} > &= \frac{v}{8(n+1)} \\
(n = 1, 2, \ldots)
\end{aligned}
\]  

(53)

Inserting (51) and (53) in (6) and keeping only the terms proportional to \(v \sim < \cos \varphi >\), one finds

\[
1 = \alpha \left( \frac{\left( \frac{2}{9} + \frac{1}{2} \right) e^{-\frac{\beta}{3}} - \sum_{n=1}^{\infty} \frac{e^{-\frac{\beta}{3}(2n+1)^2}}{2n(n+1)}}{2 e^{-\frac{\beta}{3}} + 2 \sum_{n=1}^{\infty} e^{-\frac{\beta}{3}(2n+1)^2}} \right);
\]

(54)

namely eq.(11).

### B The phase correlator

In this appendix we want to elucidate the computation of the correlator defined in equation (25). For this purpose one should compute the path integral

\[
\int D\phi e^{i\int d\phi (\frac{1}{2} C_{ij} \phi_i \phi_j - \frac{1}{2} C_{ij} \phi_i \phi_j)}
\]

(55)

Fourier transforming \(\phi_i(\tau)\) according to

\[
\phi_i(\tau) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \phi_{i,m} e^{i\omega_m \tau}
\]

(56)

with \(0 \leq \tau \leq \beta\) and \(\omega_m = \frac{2\pi}{\beta} m\), the numerator of (55) becomes

\[
\int \prod_i d\phi_{i,0} \prod_{n=1}^{\infty} d\phi_{i,n} d\phi_{i,n}^* \exp \left\{ - \frac{1}{4 e^2 \beta} \sum_{i,j} \sum_{n=1}^{+\infty} C_{ij} \omega_n^2 \phi_{i,n} \phi_{j,n}^* + \frac{i}{\beta} \sum_{n=1}^{+\infty} (\phi_{r,n} e^{i\omega_n \tau} - \phi_{s,n}^* e^{-i\omega_n \tau}) + c.c. + \frac{i}{\beta} (\phi_{r,0} - \phi_{s,0}) \right\}.
\]

(57)

Upon integrating over the components \(\phi_{r,0}, \phi_{s,0}\) one gets a factor \(\delta_{rs}\)

\[
\left( \prod_{i \neq r, s} \int_{-\infty}^{\infty} d\phi_{i,0} \right) \left( \int_{-\infty}^{\infty} d\phi_{r,0} \int_{-\infty}^{\infty} d\phi_{s,0} e^{i\frac{\phi_{r,0} - \phi_{s,0}}{2}} \right) = \delta_{rs} \cdot K
\]

(58)
where \( K \) is an irrelevant divergent constant which cancels against the denominator. Using (58), (57) becomes

\[
K \delta_{rs} \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} d\phi_{in} d\phi_{i,n}^* \exp \left( -\frac{1}{4e^2\beta} \sum_{ij} C_{ij} \omega_n^2 \phi_{i,n} \phi_{j,n}^* + \sum_i \phi_{i,n} \delta_{ri} \left( e^{i\omega_n \tau} - e^{i\omega_n \tau'} \right) - \sum_i \phi_{i,n}^* \delta_{ri} \left( e^{-i\omega_n \tau} - e^{-i\omega_n \tau'} \right) \right).
\]

The multiple Gaussian integral may be easily computed to give, up to an irrelevant constant which cancels against the denominator,

\[
\delta_{rs} \prod_{i=1}^{\infty} \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} d\phi_{in} d\phi_{i,n}^* \exp \left\{ \sum_{ij} \frac{4e^2\beta C_{ij}}{\omega_n^2} i \delta_{ri} \left( e^{i\omega_n \tau} - e^{i\omega_n \tau'} \right) \right\} = \delta_{rs} \exp \left\{ \frac{8e^2\beta C_{rr}^{-1}}{\beta} \sum_{n=1}^{\infty} \frac{1 - \cos \omega_n (\tau - \tau')}{\omega_n^2} \right\} = \delta_{rs} \exp \left\{ -\frac{2e^2\beta C_{rr}^{-1}}{\beta} \left| \tau - \tau' \right| - \frac{(\tau - \tau')^2}{\beta} \right\}
\]

where \(-\beta \leq \tau - \tau' \leq \beta\). In the last step, the identity

\[
|x| - \frac{x^2}{\beta} = \sum_{n=1}^{\infty} \left( \frac{4}{\beta \omega_n^2} - \frac{4 \cos \omega_n x}{\beta \omega_n^2} \right) \quad -\beta \leq x \leq \beta
\]

has been used. This completes the proof of (26).

### C Low \( T_c \) expansion

In this appendix we derive equation (44) and compute the next two orders whose coefficients are plotted in fig.6. Using the notation \((-1)^i = (-1)^{i_1 + \cdots + i_D}\), the ground state charge configuration \(n_0^i\) can be written as

\[
(n_0^i + \frac{1}{2}) = \frac{1}{2} (-1)^i.
\]

The first excited states read

\[
(n_0^i + \frac{1}{2}) = n_1^i (1 - 2\delta_{ri}) ;
\]

where the apex \(1_r\) means that this first excited state is obtained from the ground state by flipping the sign of the charge at the site \(r\). Higher excitations may
be obtained from the ground state by flipping the sign of two charges at sites $r$ and $s$ and can be represented as
\[
(n_1^{2rs} + \frac{1}{2}) = n_1^0(1 - 2\delta_{ri} - 2\delta_{si})
\]
The energy shifts are given by
\[
\Delta^1 = E^1 - E^0 = \sum_{i \neq r} U_{ir}(-1)^{r+i+1}
\]
and
\[
\Delta^{2rs} = E[n_1^{2rs}] - E^0 = 2\Delta^1 + 2(-1)^{r-s}U_{rs}.
\]
Note that, whereas the energy $E^1$ of the charge configurations $n_1^i$ does not depend on $r$, $E[n_1^{2rs}]$ depends on the relative position $r - s$ of the charges whose sign has been flipped.

Defining
\[
R^0 = \frac{1}{1 - 4[\sum_j U_{0j}(n_1^j + 1/2)]^2},
\]
\[
R^1_r = \frac{1}{1 - 4[\sum_j U_{0j}(n_1^j + 1/2)]^2}
\]
and
\[
R^{2rs} = \frac{1}{1 - 4[\sum_j U_{0j}(n_1^{2rs} + 1/2)]^2},
\]
one may expand eq. (41) for small critical temperatures ($y \propto T_c \to 0$), according to
\[
\alpha = \frac{1 + \sum_r e^{-\frac{2}{y}\Delta^1} + \sum_{r \neq s} e^{-\frac{2}{y}\Delta^{2rs}} + \cdots}{R^0 + \sum_r R^{1r} e^{-\frac{2}{y}\Delta^1} + \sum_{r \neq s} R^{2rs} e^{-\frac{2}{y}\Delta^{2rs}} + \cdots} = \frac{1}{R^0}\left[1 + \sum_r (1 - \frac{R^{1r}}{R^0}) e^{-\frac{2}{y}\Delta^1} + \sum_{r \neq s} (1 - \frac{R^{2rs}}{R^0}) e^{-\frac{2}{y}\Delta^{2rs}} + \cdots\right]
\]
where $\sum_{r \neq s}$ indicates a summation over pairs of different sites $r, s$, where each pair is counted only once.

For a nearest neighbor interaction $U_{0j} = \delta_{0j} + \theta \sum_p \delta_{jp}$ (where $p$ denotes the vector connecting two neighboring sites) one has
\[
\Delta^1 = z\theta,
\]
\[
\Delta^{2rs} = \begin{cases} 
2(z - 1)\theta & r - s = p \\
2z\theta & r - s \neq p 
\end{cases}
\]
The condition for the reentrant behavior is

\[ z < 1 - \frac{(1 - z\theta)^2}{2} \]

Substituting these relations in (60), one obtains the expansion for small temperatures of the critical line equation, up to the first four orders

\[ \alpha = \left(1 - (1 - z\theta)^2\right) \times \left(1 + a_1 e^{-\frac{1}{z\theta}} + a_2 e^{-\frac{2}{z\theta}(z-1)\theta} + a_3 e^{-\frac{3}{z\theta}}\right). \]  

(61)

\( a_1 \) is given in [47], \( a_2 \) is equal to

\[ (z-1)z\left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 - (2z-2)\theta)^2}\right) + z\left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 + (2z-2)\theta)^2}\right) \]

and \( a_3 \) is given by

\[ \left(1 - \frac{(1 - (1 - z\theta)^2}{1 - (1 + z\theta)^2}\right)^2 - \left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 + z\theta)^2}\right) + z(z-1) \cdot \left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 - (2z-2)\theta)^2} - 1\right) + \]

\[ + z \cdot \frac{1 - (1 - z\theta)^2}{1 - (1 - (2z-2)\theta)^2} \left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 - (2z-2)\theta)^2} - 1\right) + \frac{1 - (1 - z\theta)^2}{1 - (1 + z\theta)^2} \left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 - (2z-2)\theta)^2} - 1\right) + \]

\[ + z \cdot \frac{1 - (1 - z\theta)^2}{1 - (1 + z\theta)^2} \left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 + z\theta)^2} - 1\right) + \frac{1 - (1 - z\theta)^2}{2} \cdot \left(1 - \frac{1 - (1 - z\theta)^2}{1 - (1 - (2z-4)\theta)^2}\right). \]

The condition for the reentrant behavior is \( a_1 < 0 \). In fig.6 we plot the coefficients \( a_1, a_2, a_3 \) as a function of \( \theta \). In fig.7 we plot the critical equation (61) with \( \theta = 0.05 \) and \( z = 6 \).

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**References**

[1] E. Simanek, Inhomogeneous Superconductors, Oxford University Press, (1994).
[2] B. D. Josephson, Phys. Lett. 1,7 (1962).

[3] P.W. Anderson, in Lectures on the Many Body Problem, edited by E.R. Caianello (Academic, New York, 1964), p.113.

[4] B. Abeles, Phys. Rev. B 15, 2828 (1977).

[5] K. B. Efetov, Zh. Eksp. Teor. Fiz. 78, 2017 (1980) [Sov. Phys. JETP 51, 1015 (1980)].

[6] A. I. Larkin and L. I. Glazman, cond-mat/9705169.

[7] C. Bruder, R. Fazio, A. Kampf, A. van Otterlo and G. Schön, Phys. Scri. 42, 159 (1992).

[8] A. van Otterlo, K. H. Wagenblast, R. Fazio and G. Schön, Phys. Rev. B 48, 3316 (1993).

[9] E. Simànek, Solid State Commun. 31, 419 (1979).

[10] E. Simànek, Phys. Rev. B 22, 459 (1980).

[11] E. Simánek, Phys. Rev. B 23, 5762 (1981).

[12] P. Fazekas, Z. Phys. B, 45, 215 (1982).

[13] J. G. Kissner and U. Eckern, Z. Phys. B91, 155 (1993).

[14] M. V. Simkin, Phys. Rev. B 44, 7074 (1991).

[15] E. Granato and M. A. Continentino, Phys. Rev. B 48, 15 977 (1993).

[16] C. Rojas and J. V. José, cond-mat/9610051.

[17] P. Fazekas, B. Muhlschlegel and M. Schroter, Z. Phys. B57, 193 (1984).

[18] R.S. Fishman and Stroud, Phys. Rev. B 35, 1676 (1987).

[19] S. Doniach, Phys. Rev. B 24, 5063 (1981).

[20] R.S. Fishman and Stroud, Phys. Rev. B 37, 1499 (1988).

[21] R. S. Fishman, Phys. Rev. B 42 1985 (1990).

[22] E. Roddick and D. Stroud, Phys. Rev. B 48, 16 600 (1993).

[23] G. Luciano, U. Eckern and J.G. Kissner, Europhys. Lett. 32, 8 (1995).

[24] J. Hubbard, Phys. Rev. Lett. 3, 77 (1959); R. L. Stratonovich, Sov. Phys. Dokl. 2, 416 (1958).
[25] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1964.

[26] G. Schon and A. D. Zaikin, Physica, 152 B, 203 (1988).

[27] K. K. Likharev and A. B. Zorin, J. Low. Temp. Phys. 59, 347 (1985).

[28] M. V. Simkin, cond-mat/9607001.

[29] E. Simanek, Phys. Rev. B 32 500 (1985).