THE SMOOTH CLASSIFICATION OF 4-DIMENSIONAL COMPLETE INTERSECTIONS

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Abstract. We prove the “Sullivan Conjecture” on the classification of 4-dimensional complete intersections up to diffeomorphism. Here an $n$-dimensional complete intersection is a smooth complex variety formed by the transverse intersection of $k$ hypersurfaces in $\mathbb{C}P^{n+k}$.

Previously Kreck and Traving proved the 4-dimensional Sullivan Conjecture when 64 divides the total degree (the product of the degrees of the defining hypersurfaces) and Fang and Klaus proved that the conjecture holds up to the action of the group of homotopy 8-spheres $\Theta_8 \cong \mathbb{Z}/2$.

Our proof involves several new ideas, including the use of the Hambleton-Madsen theory of degree-$d$ normal maps, which provide a fresh perspective on the Sullivan Conjecture in all dimensions. This leads to an unexpected connection between the Segal Conjecture for $S^1$ and the Sullivan Conjecture.

1. Introduction

1.1. Complete intersections and the Sullivan Conjecture. A complete intersection $X_n(d) \subset \mathbb{C}P^{n+k}$ is the transverse intersection of $k$ complex hypersurfaces of degrees $d = \{d_1, \ldots, d_k\}$. We regard $X_n(d)$ as an oriented smooth manifold and consider the problem of classifying complete intersections up to orientation preserving diffeomorphism. Hence throughout this paper, all manifolds are oriented and all diffeomorphisms and homeomorphisms are assumed to preserve orientations. By an observation of Thom, the diffeomorphism type of $X_n(d)$ depends only on the multidegree $d$.

The main conjecture organising the classification of complete intersections for $n \geq 3$ is the “Sullivan Conjecture”. The statement of the conjecture relies on the following fact (see Remark 2.6): There are integers $p_i(n, d)$ such that the Pontryagin classes of $X_n(d)$ satisfy $p_i(X_n(d)) = p_i(n, d)x^{n/2}$, where $x \in H^2(X_n(d))$ is the pullback of a generator of $H^2(\mathbb{C}P^{n+k})$. Let $d := d_1 \cdots d_k$ denote the total degree of $X_n(d)$, which is the product of the individual degrees.

Definition 1.1. The Sullivan data associated to the complete intersection $X_n(d)$ is the tuple

$$SD_n(d) := (d, (p_i(n, d))^{|n/2|}_i, \chi(X_n(d))) \in \mathbb{Z}^+ \times \mathbb{Z}^{|n/2|} \times \mathbb{Z},$$

which consists of the total degree $d$, the Pontryagin classes of $X_n(d)$ regarded as integers and the Euler-characteristic of $X_n(d)$. For a fixed $n$, each of these integers is a polynomial function of the individual degrees; see Section 2.1.

Conjecture 1.2 (The Sullivan Conjecture). Suppose that $X_n(d)$ and $X_n(d')$ are complete intersections with $SD_n(d) = SD_n(d')$. If $n \geq 3$, then $X_n(d)$ is diffeomorphic to $X_n(d')$.

The main result of this paper is that the Sullivan Conjecture holds in complex dimension 4.

Theorem 1.3. Suppose that $X_4(d)$ and $X_4(d')$ are complete intersections with $SD_4(d) = SD_4(d')$. Then $X_4(d)$ is diffeomorphic to $X_4(d')$.

1.2. Background and an application. We first list some existing results about the Sullivan Conjecture, its analogue in dimensions $n < 3$ and its converse.

When $n = 1$, $X_1(d)$ is an oriented surface and the classification is classical (in particular the Sullivan Conjecture holds but its converse does not).

When $n = 2$, $X_2(d)$ is a simply-connected smooth manifold and smooth classification results are currently out of reach. However, the topological classification can be deduced from results of Freedman [F]: Two complete intersections are homeomorphic if and only if they have the same Pontryagin class $p_1$ and the same
Euler-characteristic. The converse fails, because the total degree is not even a diffeomorphism invariant (e.g., $X_2(4)$, $X_2(3, 2)$ and $X_2(2, 2, 2)$ are all K3-surfaces.)

When $n \geq 3$, the converse of the Sullivan Conjecture holds, see Proposition 2.10.

If $n = 3$ the Sullivan Conjecture follows from classification theorems of Wall or Jupp [W1, 10].

If $n = 4$, Fang and Klaus [F-K] Remark 2 proved that the Sullivan Conjecture holds up to connected sum with homotopy 8-spheres:

**Theorem 1.14 (Fang and Klaus [F-K]).** Suppose that $X_4(d)$ and $X_4(d')$ are complete intersections with $SD_4(d) = SD_4(d')$. Then there is a homotopy 8-sphere $\Sigma$ such that $X_4(d')$ and $X_4(d)\Sigma$ are diffeomorphic.

If $5 \leq n \leq 7$, then Fang and Wang [F-W] proved that the Sullivan Conjecture holds up to homeomorphism.

For $n \geq 3$ Kreck and Traving proved the following general statement. Let $v_p(d)$ be the largest integer such that $p^{v_p(d)} | d$. If $SD_n(d) = SD_n(d')$ and $v_p(d) \geq \frac{2d+1}{2p+1} + 1$ for every prime $p$ with $p(p-1) \leq n+1$, then $X_n(d)$ and $X_n(d')$ are diffeomorphic [K] Theorem A. If $n = 4$, then the condition says that $64 | d$.

A motivation for the diffeomorphism classification of complete intersections is the result of Libgober and Wood [L-W] Corollary 8.3, which says that if $n \geq 3$ and diffeomorphic complete intersections have different multidegrees, then their complex structures lie in different connected components of the moduli space of complex structures on the underlying smooth manifold. Here and in general, multidegrees are regarded as equal if one can be obtained from the other by adding or removing 1s, because then the corresponding complete intersections have a common representative. Libgober and Wood used this result to show that for all odd $n \geq 3$ there are complete intersections having a complex moduli space with arbitrarily many connected components. Their proof relied on a counting argument, valid in all dimensions, which shows that the sets $\{d' \mid SD_n(d') = SD_n(d)\}$ of multidegrees with the same Sullivan data can be arbitrarily large.

In future work we give an effective algorithm for finding pairs of multidegrees with the same Sullivan data [C-N]. The Sullivan Conjecture then allows us to construct explicit examples of complete intersections in different components of the complex moduli space and we obtain the following application of Theorem 1.3.

**Example 1.5.** The complete intersections $X_4(3^{(150)}, 7^{(89)}, 9^{(65)}, 15, 25^{(130)})$ and $X_4(5^{(261)}, 21^{(89)}, 27^{(64)})$ (where $3^{(150)}$ stands for 150 copies of 3, etc.) are diffeomorphic by Theorem 1.3 and the formulae in Section 2.4. Hence the corresponding complex structures lie in different components of the complex moduli space.

1.3. The outline of the proof of Theorem 1.3

If $SD_4(d) = SD_4(d')$ then by Theorem 1.4 of Fang and Klaus there is a diffeomorphism $X_4(d) \to X_4(d')\Sigma$ for some homotopy sphere $\Sigma$. The group of homotopy 8-spheres, $\Theta_8 \cong \mathbb{Z}/2$, is known by Kervaire and Milnor [K-M] and so we let $\Sigma_{ex}$ denote the unique diffeomorphism class of the exotic 8-sphere and introduce the following terminology:

**Definition 1.6.**

- An 8-manifold $M$ is $\Theta$-**rigid** if $M\Sigma_{ex}^8$ is diffeomorphic to $M$.
- An 8-manifold $M$ is $\Theta$-**flexible** if $M\Sigma_{ex}^8$ is not diffeomorphic to $M$.
- A complete intersection $X_4(d)$ is strongly $\Theta$-**flexible** if $X_4(d)\Sigma_{ex}^8$ is not diffeomorphic to a complete intersection.

As our proof of Theorem 1.3 involves treating several cases separately, we shall say that the Sullivan Conjecture holds for a fixed complete intersection $X_4(d)$ if, for every $d'$, $SD_4(d) = SD_4(d')$ implies that $X_n(d')$ is diffeomorphic to $X_n(d)$. By Theorem 1.4 and Remark 2.11 the Sullivan Conjecture holds for $X_4(d)$ if and only if $X_4(d)$ is either $\Theta$-rigid or strongly $\Theta$-flexible. To prove the Sullivan Conjecture in dimension 4 we consider four cases, which are indexed by the Wu classes of $X_4(d)$ and parity of the total degree as follows:

| $v_2(X_4(d))$ | $v_4(X_4(d))$ | $d \mod 2$ | $\Theta$-rigidity | Treated in |
|---------------|---------------|-------------|------------------|-----------|
| 0             | -             | -           | Strongly $\Theta$-flexible | Theorem 1.7 |
| 1             | 0             | -           | $\Theta$-rigid | Theorem 1.12 |
| 1             | 1             | 0           | Unknown in general | Theorem 1.14(a) |
| 1             | 1             | 1           | Unknown in general | Theorem 1.14(b) |

Here $v_i(X_4(d)) \in H^i(X_4(d); \mathbb{Z}/2)$ is the $i^{th}$ Wu class of $X_n(d)$, which can be regarded as an element of $\mathbb{Z}/2$ by Remark 2.6 a “-” indicates the value of the invariant is not relevant in that case and in the cases when the $\Theta$-rigidity of $X_4(d)$ is unknown, we conjecture that it depends on $p_1(4, d) \mod 8$; see Conjecture 1.15.
Now we discuss the proof in each of the four cases.

For a spin complete intersection $X_d$ (equivalently, by Proposition 2.8 when $v_2(X_d) = 0$) we find a diffeomorphism invariant property of complete intersections not shared by $X_d$; see Section 3. Namely, if $S(X, \alpha)$ denotes the total space of the circle bundle over a space $X$ with first Chern class $\alpha$, then $S(X_n, \pm x)$ admits a framing making it a null-cobordant framed ($2n+1$)-manifold (for any $X_n$), whereas $S(X_d)_{\Sigma_{\infty}}$ does not (for a spin $X_d$). Hence we have the following (see Theorem 3.11).

**Theorem 1.7.** If $X_d$ is spin, then $X_d$ is strongly $\Theta$-flexible. In particular, the Sullivan Conjecture holds for $X_d$. 

**Remark 1.8.** For the 5-dimensional Sullivan Conjecture, the group of homotopy 10-spheres $\Theta_{10} \cong \mathbb{Z}/2 \times \mathbb{Z}/3$ will play a central role. We believe that “transfer” arguments similar to those we use in the 4-dimensional spin case will control the 5/3 factor of $\Theta_{10}$. The 5/3-factor of $\Theta_{10}$ is detected by the $\alpha$-invariant and Baraglia has recently computed the $\alpha$-invariant of spin complete intersections. We anticipate that these ideas will lead to a proof of the 5-dimensional Sullivan Conjecture in future work.

In the non-spin cases we apply Kreck’s modified surgery theory [K1]. Consider $B_n := CP^\infty \times BO(n+1)$ with the stable bundle $\xi_n(d) \times \gamma_{BO(n+1)}$ over it; see Definition 2.4 and Section 2.2. Recall from [K2] Section 8 that a normal $(n-1)$-smoothing in $(B_n, \xi_n(d) \times \gamma_{BO(n+1)})$ is a pair $(f, \tilde{f})$ where $f: M \to B_n$ is an $n$-equivalence from a closed smooth manifold $M$ and $f: M \to \xi_n(d) \times \gamma_{BO(n+1)}$ is a map of stable bundles from the normal bundle of $M$. Recall also that the normal $(n-1)$-type of $X_n$ is $(B_n, \xi_n(d) \times \gamma_{BO(n+1)})$, in particular $X_n$ admits a normal $(n-1)$-smoothing in $(B_n, \xi_n(d) \times \gamma_{BO(n+1)})$. In this setting [K1] Proposition 10 reduces the Sullivan Conjecture to a statement about bordism classes over $(B_n, \xi_n(d) \times \gamma_{BO(n+1)})$. For our purposes, it is useful to state an altered version of [K1] Proposition 10, which compares a complete intersection $X_n(d)$ to a somewhat more general closed 2n-manifold $X'$. The proof of Proposition 1.9 is identical to the proof of the sufficient condition of [K1] Proposition 10.

**Proposition 1.9.** Let $n \geq 3$, $X_n(d)$ be a complete intersection and $X'$ a closed 2n-manifold such that $\chi(X_n(d)) = \chi(X')$ and $X_n(d)$ and $X'$ admit bordant normal $(n-1)$-smoothings over $(B_n, \xi_n(d) \times \gamma_{BO(n+1)})$. If $d \neq \{1\}, \{2\}$ or $\{2,2\}$, then $X_n(d)$ and $X'$ are diffeomorphic.

**Remark 1.10.** In fact, the assumption that $d \neq \{2\}$ can be removed by applying [K1] Proposition 8 (i). We do not know the situation for $d = \{1\}, \{2\}$. However, for all three of these exceptional multidegrees $d$, it is elementary that $SD_n(d) = SD_n(d')$ implies $d = d'$ and so the Sullivan Conjecture holds for these complete intersections.

The main challenge when applying Proposition 1.9 is showing that the bordism condition holds; see the discussion in Section 2.2. Note that the bordism group of 8-manifolds over $(B_4, \xi_4(d) \times \gamma_{BO(5)})$ is canonically isomorphic to the twisted string bordism group $\Omega_8^{O(7)}(CP^{\infty}; \xi_4(d))$, since $BO(5) = BO(8) = B(O(7))$.

In the case of a non-spin complete intersection $X_d$ with $v_2(X_d) = 0$, we will use Proposition 1.9 to compare $X_d$ with $X' = X_d(\Sigma_{\infty})$. They both admit normal 3-smoothings over $(B_4, \xi_4(d) \times \gamma_{BO(8)})$ and the difference of their bordism classes is the image of $\Sigma_{\infty}$ under the canonical map $\Theta_8 \to \Omega_8^{O(7)}(CP^{\infty}; \xi_4(d))$. This map factors through $\text{Tors} \Omega_8^{O(7)}(CP^1; \xi_4(d)|_{CP^1})$ and we prove (see Lemma 4.2) the following

**Proposition 1.11.** If $X_d$ is non-spin, then $\text{Tors} \Omega_8^{O(7)}(CP^1; \xi_4(d)|_{CP^1}) \cong \mathbb{Z}/4$.

When $v_2(X_d) = 0$, we combine Proposition 1.11 with the computations of [F-K] Section 2.2] to show that the map $\Theta_8 \to \Omega_8^{O(7)}(CP^{\infty}; \xi_4(d))$ vanishes (Proposition 4.3), which gives (see Theorem 4.4) the following

**Theorem 1.12.** Suppose that $X_d$ is a non-spin complete intersection with $v_2(X_d) = 0$. If $d \neq \{2\}$ then $X_d$ is $\Theta$-rigid and the Sullivan Conjecture holds for $X_d$.

**Remark 1.13.** In fact $X_d(2,2)$ is $\Theta$-rigid too. This follows from from results in the second author’s PhD thesis but will not be proven here.

If $X_d$ is non-spin, $v_4(X_d) = 1$ and the total degree $d$ is even, then $16|d$ (see Remark 2.9). We add Proposition 1.11 to the Adams filtration argument of Kreck and Traving [K1] Section 8 and the calculations of [F-K] Section 2.4 to prove (see Proposition 4.6) Part (a) of the following theorem.
Theorem 1.14. Let $X_d(4)$ and $X_d(2')$ be non-spin complete intersections with $SD_d(4) = SD_d(2')$ and suppose that either
(a) $v_4(X_d(4)) \neq 0$ and the total degree $d$ is even, or
(b) the total degree $d$ is odd.
Then $X_d(4)$ and $X_d(2')$ admit bordant normal 3-smoothings over $(B_d(4), 4 \times \gamma_{BO}(8))$. Consequently, $X_d(4)$ and $X_d(2')$ are diffeomorphic and the Sullivan Conjecture holds for $X_d(d)$.

Note that the cases discussed so far (i.e. those prior to Theorem 1.14(b)), have a significant overlap with, but are not implied by, the theorem of Kreck and Traving. However, the case of odd total degree covered in Theorem 1.14(b) is completely new. Note also that the total degree can be odd only if $v_2(X_d(4)) \neq 0$ and $v_4(X_d(4)) \neq 0$; see Proposition 2.8.

To prove Theorem 1.14(b) we use the Hambleton-Madsen theory of degree-d normal maps [H-M]. A complete intersection $X_n(2')$ (with a canonical choice of normal data) represents an element in the set $N_d(CP^n)$ of normal bordism classes of degree-d normal maps over $CP^n$ and the Hambleton-Madsen theory gives a bijective normal invariant map

$$\eta: N_d(CP^n) \equiv [CP^n, (QS^0/O)d].$$

which is the usual normal invariant in the familiar case when $d = 1$ and $(QS^0/O)_d$ is in general a certain classifying space, which was identified by Brumfiel and Madsen [B-M]. We establish a relationship between certain “relative divisors” of a vector bundle and degree-d normal maps over the vector bundle (Lemma 5.14) and then use this to give a formula for the canonical degree-d normal invariant of $X_n(2)$ (Theorem 5.16).

The surgery argument of Proposition 1.9 also works if we have bordant representatives in $N_d(CP^n)$ (Lemma 5.15). This and the formula of Theorem 5.16 leads to a new perspective on the stable homotopy-theoretic input needed to prove the Sullivan Conjecture (see Theorem 5.17). This new perspective allows us to prove the 4-dimensional Sullivan Conjecture when the total degree is odd and we anticipate that it will lead to other new results in higher dimensions; e.g. see Remark 5.28.

Notice that Fang and Klaus (Theorem 1.4) reduced the 4-dimensional Sullivan Conjecture to a 2-local problem. When $d$ is odd, the work of Brumfiel and Madsen [B-M] shows that there is an equivalence of 2-localisations $[(QS^0/O)_d] 2^2 \simeq (G/O) 2^2$, where $G/O$ is the familiar classifying space from classical surgery theory [B-M]. We can then exploit Sullivan’s 2-local splitting (see [M-M Theorem 5.18]),

$$(G/O) 2^2 \simeq (BSO) 2^2 \times \ker(J)(2),$$

where $\ker(J)(2)$ is a 2-local space whose homotopy groups are certain large summands of the 2-primary component of the cokernel of the $J$-homomorphism (see [M-M Definition 5.16]). It follows that we have a sequence of maps

$$[CP^n, (QS^0/O)d] \rightarrow [CP^n, (QS^0/O)_d] 2^2 \rightarrow [CP^n, (G/O)(2)] \rightarrow [CP^n, (BSO)(2)] \times [CP^n, \ker(J)(2)].$$

The formula for the degree-d normal invariant of $X_n(2)$ shows that it is the restriction of a map $CP^n \rightarrow (QS^0/O) d$. Now the proof of a theorem of Feshbach [Fesh Theorem 6], which is based on the Segal Conjecture for the Lie group $S^3$, implies that any map $CP^n \rightarrow \ker(J)(2)$ is null-homotopic and this is enough to prove that the $[CP^n, \ker(J)(2)]$ factor of the 2-localised normal invariant is trivial (Lemma 5.25). The $[CP^n, (BSO)(2)]$ factor is controlled by the Sullivan data, hence in dimension 4 the degree-d normal invariant is completely determined by the Sullivan data (Theorem 5.26). The 4-dimensional Sullivan Conjecture for complete intersections with odd total degree follows (Theorem 5.27).

1.4. Inertia groups of 4-dimensional complete intersections. Recall that the inertia group of a closed connected $m$-manifold $M$ is the subgroup

$$I(M) := \{ \Sigma \in \Theta_m \mid M \text{ and } M \Sigma \text{ are diffeomorphic} \} \subseteq \Theta_m$$

of the group of homotopy $m$-spheres $\Theta_m$ [K-M]. For example, an 8-manifold $M$ is $\Theta$-rigid if and only if $I(M) = \Theta_8$. The results in Section 1.3 determine the inertia groups of a 4-dimensional complete intersection when $X_4(2)$ is spin, or when $X_4(2)$ is non-spin and $v_4(X_4(2)) = 0$. When $X_4(2)$ is non-spin and $v_4(X_4(2)) \neq 0$, we have $p_1(4, d) \equiv 3 \mod 4$ (see Proposition 2.8 and the calculations in Section 2.1) and we offer the third and fourth rows of the table in the following conjecture.
Conjecture 1.15. The inertia groups $I(X_4(d)) \subseteq \Theta_8 \cong \mathbb{Z}/2$ of 4-dimensional complete intersections are given by the table below:

| $v_2(X_4(d))$ | $v_4(X_4(d))$ | $p_1(4,d)$ mod 8 | $I(X_4(d))$ |
|---------------|----------------|-----------------|-------------|
| 0             | 1              | 0               | $\Theta_8$  |
| 1             | 0              | 3               | 0           |
| 1             | 1              | 7               | $\Theta_8$  |

Here a “−” indicates the value of the invariant is not relevant for $I(X_4(d))$ in that case.

Remark 1.16. The first line of the table follows from Theorem 1.12 and the case of given by the table below: Section 5 treats the case of even total degree. Finally, Section 6 is an appendix about Toda brackets and extensions, which is needed in the Θ-flexible non-spin cases, besides the bordism class in $\Omega_8^{O(7)}(\mathbb{C}P^n; \xi_4(d))$, we do not know of such a property.

The rest of this paper is organised as follows. Section 2 covers necessary preliminaries on Sullivan data and modified surgery. Section 3 treats the spin case. Section 4 treats the two non-spin cases whose solution relies on Proposition 1.11, which are the case with $v_4 = 0$ and even total degree (together comprising all non-spin complete intersections with even total degree). Section 5 treats the case of odd total degree. Finally, Section 6 is an appendix about Toda brackets and extensions, which is needed in Section 4 and specifically for the proof of Proposition 1.11.

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2. Preliminaries

Given a finite multiset $d = \{d_1, d_2, \ldots, d_k\}$ of positive integers, consider homogeneous polynomials $f_1, f_2, \ldots, f_k \in \mathbb{C}[x_0, x_1, \ldots, x_{n+k}]$ with these degrees. If the zero set $\{x \in \mathbb{C}P^{n+k} \mid f_i(x) = 0\}$ of $f_i$ is a smooth submanifold of $\mathbb{C}P^{n+k}$ for every $i$ and these submanifolds are transverse, then their intersection is a representative of the complete intersection $X_n(d)$. As Thom noted, any two representatives are diffeomorphic and $X_n(d)$ is defined as the common diffeomorphism type of its representatives. Moreover, if two representatives are identified via a diffeomorphism coming from Thom’s theorem, then their natural embeddings in $\mathbb{C}P^{n+k}$ are isotopic; hence $X_n(d)$ comes equipped with an embedding $i : X_n(d) \to \mathbb{C}P^{n+k}$, well-defined up to isotopy.

2.1. Computation of Sullivan data and the converse of the Sullivan Conjecture.

Definition 2.1. Let $x \in H^2(\mathbb{C}P^{n+k})$ denote the standard generator. The pullbacks of $x$ (by the standard embeddings) in $H^2(\mathbb{C}P^n)$ and $H^2(X_n(d))$ will also be denoted by $x$.

Definition 2.2. For a (complex) bundle $\xi$ and a positive integer $r$ let $r\xi = \xi \oplus \cdots \oplus \xi$ denote the $r$-fold Whitney-sum of $\xi$ with itself and let $-r\xi$ denote the stable bundle which is the inverse of $r\xi$. Let $\xi^r = \xi \otimes \cdots \otimes \xi$ be the $r$-fold tensor product (over $\mathbb{C}$) of $\xi$ with itself. For a tuple $\xi = (r_1, r_2, \ldots, r_k)$ let $\xi^{\xi} = \xi^{r_1} \otimes \cdots \otimes \xi^{r_k}$.

Definition 2.3. Let $\gamma$ be the conjugate of the tautological complex line bundle over $\mathbb{C}P^{\infty}$.

With this notation, the tautological bundle is $\bar{\gamma}$ and since $c_1(\bar{\gamma}) = -x$, we have $c_1(\gamma) = x$. It is well-known that the normal bundle of $\mathbb{C}P^n$ in $\mathbb{C}P^{n+1}$ is $\nu(\mathbb{C}P^n \to \mathbb{C}P^{n+1}) \cong \xi_{\mathbb{C}P^n}$ and that the stable normal bundle of $\mathbb{C}P^n$ is $\nu_{\mathbb{C}P^n} \cong -(n+1)\gamma_{\mathbb{C}P^n}$.

Definition 2.4. The stable vector bundle $\xi_n(d)$ over $\mathbb{C}P^{\infty}$ is defined to be $\xi_n(d) := -(n+k+1)\gamma \oplus \gamma^{d_1} \oplus \cdots \oplus \gamma^{d_k}$. 
Proposition 2.5. The stable normal bundle $\nu_{X_n(d)}$ of $X_n(d)$ is isomorphic to $i^*(\xi_n(d)|_{\mathbb{C}P^{n+k}})$. 

Remark 2.6. Since $\nu_{X_n(d)}$ is the pullback of a bundle over $\mathbb{C}P^n$, all of the characteristic classes of $X_n(d)$ lie in the subring $i^*(H^*(\mathbb{C}P^n)) \subseteq H^*(X_n(d))$, which is generated by $x \in H^2(X_n(d))$. In particular, $p_j(X_n(d)) \in \langle x^j \rangle \cong \mathbb{Z}$, $c_i(X_n(d)) \in \langle x^i \rangle \cong \mathbb{Z}$ and if $2j \leq n$, then $w_j(X_n(d))$, $v_2(X_n(d)) \in \langle x^j \rangle \cong \mathbb{Z}/2$, where $\varphi_2: H^*(X_n(d)) \to H^*(X_n(d); \mathbb{Z}/2)$ is reduction mod 2. (If $2j > n$ and $d$ is even, then $\varphi_2(x^j) = 0$.)

Proposition 2.5 allows us to compute the characteristic classes of $X_n(d)$ in terms of the degrees $d_1, \ldots, d_k$. Since $c(\gamma^r) = 1 + rx$, the total Chern class of $\xi_n(d)$ is $c(\xi_n(d)) = (1 + x)^{-\langle n+k+1 \rangle} \prod_{i=1}^{k} (1 + d_i x)$. The same formula holds for the normal bundle $\nu_{X_n(d)}$, because it is the pullback of $\xi_n(d)$. This implies that $c(X_n(d)) = (1 + x)^{\langle n+k+1 \rangle} \prod_{i=1}^{k} (1 + d_i x)^{-1}$. For the Pontryagin classes we have $p(\gamma^r) = 1 - r^2 x^2$, hence $p(X_n(d)) = (1 - x^2)^{\langle n+k+1 \rangle} \prod_{i=1}^{k} (1 - d_i^2 x^2)^{-1}$.

The Euler-characteristic of $X_n(d)$ can also be determined, namely

$$\chi(X_n(d)) = \langle c_n(X_n(d)), [X_n(d)] \rangle = \langle c_n(-\nu_{X_n(d)}), [X_n(d)] \rangle = \langle c_n(-i^*(\xi_n(d))), [X_n(d)] \rangle = \langle c_n(-\xi_n(d)), i_*(|[X_n(d)]|) \rangle,$$

where $i_*([X_n(d)]) \in H_{2n}(\mathbb{C}P^{n+k})$ is $d$ times the generator.

It will be useful to explicitly compute the Stiefel-Whitney classes $w_2$ and $w_4$ and Wu classes $v_2$ and $v_4$ of a 4-dimensional complete intersection $X_4(d)$.

Definition 2.7. For a multidegree $d$ let $p(d)$ denote the number of even degrees in $d$.

Proposition 2.8. The Stiefel-Whitney classes $w_2$ and $w_4$ of $\nu_{X_4(d)}$ and $X_4(d)$ and Wu classes $v_2$ and $v_4$ of $X_4(d)$ are determined by $p(d)$ mod 4 as follows (by Remark 2.6 these Stiefel-Whitney classes and Wu classes can be regarded as elements of $\mathbb{Z}/2$):

| $p(d)$ mod 4 | 0 | 1 | 2 | 3 |
|--------------|---|---|---|---|
| $w_2(\nu_{X_4(d)}) = w_2(X_4(d)) = v_2(X_4(d))$ | 1 | 0 | 1 | 0 |
| $w_4(\nu_{X_4(d)}) = v_4(X_4(d))$ | 1 | 1 | 0 | 0 |
| $w_4(X_4(d))$ | 0 | 1 | 1 | 0 |

Proof. The total Chern class of $\xi_n(d)$ is given by the following formula:

$$c(\xi_n(d)) = (1 + x)^{-\langle n+k+1 \rangle} \prod_{i=1}^{k} (1 + d_i x) = 1 + \left((n + 1) + \sum_{i=1}^{k} (d_i - 1)\right) x + \left(\begin{pmatrix} n + 2 \\ 2 \end{pmatrix} - (n + 2) \sum_{i=1}^{k} (d_i - 1) + \sum_{1 \leq i < j \leq k} (d_i - 1)(d_j - 1)\right) x^2 + \ldots .$$

We have $w_{2i} = \varphi_2(c_i)$. Therefore

$$w_2(\xi_n(d)) = \varphi_2 \left(\sum_{i=1}^{k} (d_i - 1)\right) x = \varphi_2(1 + p(d)) x.$$

We have the same formulas for the Stiefel-Whitney classes of $\nu_{X_4(d)}$, because $\nu_{X_4(d)}$ is the pullback of $\xi_4(d)$. Since $H^1(X_4(d); \mathbb{Z}/2) \cong H^1(X_4(d); \mathbb{Z}/2) \cong 0$, the Stiefel-Whitney classes $w_2(\xi_4(d))$ and $w_4(\xi_4(d))$ are determined by $w_2(\nu_{X_4(d)})$ and $w_4(\nu_{X_4(d)})$ via the Cartan-formula. We get that $w_2(X_4(d)) = w_2(\nu_{X_4(d)})$ and $w_4(X_4(d)) = w_2(\nu_{X_4(d)})^2 + w_4(\nu_{X_4(d)})$. By applying the Wu-formula we get that $v_2(X_4(d)) = w_2(X_4(d))$ and $v_4(X_4(d)) = w_2(X_4(d))^2 + w_4(X_4(d)) = w_4(\nu_{X_4(d)})$. 

\qed
Remark 2.9. Notice that if \( v_2(X_4(d)) \neq 0 \) and \( v_2(X_4(d')) \neq 0 \), then \( p(d) \) is divisible by 4. This means that either \( p(d) = 0 \), hence all degrees are odd and so the total degree is odd; or \( p(d) \geq 4 \), so there are at least 4 even degrees and then the total degree is divisible by 16.

The following proposition implies that the converse of the Sullivan Conjecture holds:

**Proposition 2.10.** Let \( n \geq 3 \) and let \( d \) and \( d' \) be two multidegrees. If there is a homotopy equivalence \( f: X_n(d) \to X_n(d') \) such that \( f^* (\nu_{X_n(d)}) \cong \nu_{X_n(d')} \) (eg. if \( f \) is a diffeomorphism), then \( SD_n(d) = SD_n(d') \).

**Proof.** If \( n \geq 3 \), then \( H^2(X_n(d)) \cong H^2(X_n(d')) \cong \mathbb{Z} \), so any homotopy equivalence \( X_n(d) \to X_n(d') \) preserves \( x \) up to sign. If \( f \) sends \( x \) to \(-x\), then we can replace it with another homotopy equivalence that preserves \( x \), by composing it with a self-diffeomorphism of \( X_n(d) \) (or \( X_n(d') \)) that changes the sign of \( x \). (Consider the conjugation map of the ambient \( CP^{n+k} \), it sends \( x \) to \(-x\).)

If a representative of \( X_n(d) \) is given by polynomials \( f_1, f_2, \ldots, f_k \), then its image is another representative of the same complete intersection, given by the conjugate polynomials \( \bar{f}_1, \bar{f}_2, \ldots, \bar{f}_k \). The embeddings of the two representatives are isotopic, using an isotopy we can get a self-diffeomorphism of either representative that changes the sign of \( x \). Since \( \langle X_n(d) \rangle, x^n = d \) and \( \langle X_n(d') \rangle, x^n = d' \), this means that \( d = d' \). The Euler-characteristic is a homotopy invariant. The Pontryagin classes are preserved by \( f \) because of the assumption on the normal bundles and since the elements \( x^{2i} \) are preserved, the Pontryagin classes are also invariant when regarded as elements of \( \mathbb{Z} \).

\[ \square \]

Remark 2.11. If \( \Sigma \in \Theta_{2n} \) is a homotopy sphere, then there is a homomorphism between \( X_n(d) \) and \( X_n(d) \sharp \Sigma \) which preserves normal bundles. Thus if \( n \geq 3 \) and \( X_n(d) \sharp \Sigma \) is diffeomorphic to a complete intersection \( X_n(d') \), then \( SD_n(d) = SD_n(d') \).

### 2.2. The setting for modified surgery

We recall the setup for the modified surgery arguments of Kreck [K] Section 8 and Fang and Klaus [FK].

By the Lefschetz Hyperplane Theorem, the inclusion \( i: X_n(d) \to CP^\infty \) is \( n \)-connected. It is covered by a bundle map \( i: \nu_{X_n(d)} \to \xi_n(d) \) (Proposition 2.5) and therefore \((i, i)\) is a normal \((n-1)\)-smoothing over \((CP^\infty, \xi_n(d))\).

Let \( \gamma_{BO} \) denote the universal stable vector bundle over \( BO \) and \( \gamma_{BO(\ell)} \), its pullback to \( BO(\ell) \), the \((\ell-1)\)-connected cover of \( BO \). Let \( \gamma_{n} := CP^\infty \times BO(n+1) \), then \((i, i)\) can be regarded as a normal map over \((\gamma_{n}, \xi_n(d) \times \gamma_{BO(n+1)}) \) and it is still \(n\)-connected. Moreover, the map \( \gamma_{n} \to BO \) inducing \( \xi_n(d) \times \gamma_{BO(n+1)} \) from \( \gamma_{BO} \) is \(n\)-co-connected, therefore \((\gamma_{n}, \xi_n(d) \times \gamma_{BO(n+1)}) \) is the normal \((n-1)\)-type of \( X_n(d) \). When \( n = 4 \), we have that \( BO(\ell) = BO(8) = B\text{String} \) by Bott periodicity and thus \((i, i)\) represents an element in the bordism group of closed 8-manifolds with normal maps to \((\gamma_{4}, \xi_4(d) \times \gamma_{BO(8)}) \).

We denote this bordism group by \( \Omega_{8}^{\gamma_4}(\gamma_{4}, \xi_4(d) \times \gamma_{BO(8)}) \) and it is canonically isomorphic to the twisted string bordism group \( \Omega_{8}^{\gamma_4}(CP^\infty, \xi_4(d)) \).

First we will want to apply Proposition 1.9 when \( X' = X_4(d) \sharp \Sigma_{ex}^{8} \). There is a canonical homoeomorphism \( h: X_4(d) \sharp \Sigma_{ex}^{8} \to X_4(d) \) and since homoeomorphies are stably parallelisable [K-M] Theorem 3.1), \( h \) is covered by a bundle map \( \tilde{h} \) of stable normal bundles. Then \((i \circ h, i \circ \tilde{h})\) is also a normal \(3\)-smoothing over \((\gamma_{4}, \xi_4(d) \times \gamma_{BO(8)}) \) and in the bordism group \( \Omega_{8}^{\gamma_4}(CP^\infty, \xi_4(d)) \) it represents \((i, i) + \Sigma_{ex}^{8} \), where \( \Sigma_{ex}^{8} \) is the image of \( \Sigma_{ex}^{8} \) under the canonical homoeomorphism \( \Theta_{8} \to \Omega_{8}^{\gamma_4}(CP^\infty, \xi_4(d)) \). So to apply Proposition 1.9 in this setting we need to show that this homoeomorphism is trivial and we do this in the non-spin case with \( v_1(X_4(d)) = 0 \); see Proposition 4.4.

Now suppose that \( X_4(d') \) is another complete intersection with an analogous normal \(3\)-smoothing \((i', \tilde{i}')\) over \((CP^\infty, \xi_4(d'))\). If the Pontryagin classes of \( X_4(d) \) and \( X_4(d') \) agree, in particular if \( SD_4(d) = SD_4(d') \), then the Pontryagin classes \( p_2 \) of \( \xi_4(d) \) and \( \xi_4(d') \) also agree. By results of Sanderson [S], this implies that \( \xi_4(d') \big|_{CP^4} \cong \xi_4(d) \big|_{CP^4} \). Thus \( \xi_4(d') \bigcap \xi_4(d) \) is trivial over \( CP^4 \), so it has an \( O(7) \)-structure. Therefore \( Id_{CP^8} \) has a lift \( g: CP^\infty \to B3 \) which induces \( \xi_4(d') \big|_{CP^4} \) from \( \xi_4(d) \big|_{CP^4} \times \gamma_{BO(8)} \). Hence if \( \bar{g}: \xi_4(d') \to \xi_4(d) \times \gamma_{BO(8)} \) is a bundle map over \( g \), then \((g \circ i', \bar{g} \circ \tilde{i}')\) is a normal \(3\)-smoothing of \( X_4(d') \) over \((B4, \xi_4(d) \times \gamma_{BO(8)}) \).

If \( SD_4(d) = SD_4(d') \), then the discussion in the paragraph above shows that \( X_4(d) \) and \( X_4(d') \) admit normal \(3\)-smoothings over \((B4, \xi_4(d) \times \gamma_{BO(8)}) \) and \( \chi(X_4(d)) = \chi(X_4(d')) \), therefore to apply Proposition 1.9 it is enough to prove that these normal \(3\)-smoothings represent the same bordism class in \( \Omega_{8}^{\gamma_4}(CP^\infty, \xi_4(d)) \).

Fang and Klaus obtained Theorem 1.4 by showing that the difference of these bordism classes is in the image
of the canonical homomorphism \( \Theta_k \to \Omega_k^{(7)}(\mathbb{C}P^\infty; \xi(d)) \). In the non-spin cases with \( v_4(X(d)) \neq 0 \), we are able to show in Sections 3.3 and 5 that the bordism classes agree.

### 3. THE SPIN CASE

In this section we prove that the Sullivan-conjecture holds for 4-dimensional spin complete intersections.

**Definition 3.1.** For a smooth manifold \( X \) and a cohomology class \( \alpha \in H^2(X) \) let \( E(X, \alpha) \) denote the total space of the complex line bundle over \( X \) with first Chern class \( \alpha \). Let \( D(X, \alpha) \) denote its disc bundle and \( S(X, \alpha) \) denote its sphere bundle.

Recall that \( x \in H^2(X(d)) \) is the pullback of the standard generator of \( H^2(\mathbb{C}P^\infty) \). First we will prove that for every complete intersection \( X_n(d) \) the total space \( S(X_n(d), x) \) admits a framing which is framed null-cobordant; see Theorem 3.4.

Recall that (a representative of) the complete intersection \( X_{n+1}(d) \subset \mathbb{C}P^{n+k+1} \) is the set of common zeros of some homogeneous polynomials \( f_1, f_2, \ldots, f_k \in \mathbb{C}[x_0, x_1, \ldots, x_{n+k+1}] \). If \( f_{k+1} \in \mathbb{C}[x_0, x_1, \ldots, x_{n+k+1}] \) is linear and its zero set \( L \) is transverse to \( X_{n+1}(d) \), then \( X_n(d) = X_{n+1}(d) \cap L \).

**Proposition 3.2.** \( X_{n+1}(d) \setminus X_n(d) \) is stably parallelisable.

**Proof.** We have the following commutative diagram of embeddings:

\[
\begin{array}{ccc}
\mathbb{C}P^{n+k+1} \setminus L & \to & \mathbb{C}P^{n+k+1} \\
i & & \downarrow i \\
X_{n+1}(d) \setminus X_n(d) & \to & X_{n+1}(d)
\end{array}
\]

So \( \nu_{X_{n+1}(d) \setminus X_n(d)} \cong \nu_{X_{n+1}(d)}|_{X_{n+1}(d) \setminus X_n(d)} \cong \nu^*(\xi_{n+1}(d))|_{X_{n+1}(d) \setminus X_n(d)} \cong \nu^*(\xi_{n+1}(d)|_{\mathbb{C}P^{n+k+1} \setminus L}) \) and this is trivial, because \( \mathbb{C}P^{n+k+1} \setminus L \) is contractible (recall that \( L \) is a hyperplane).

**Proposition 3.3.** \( \nu(X_n(d) \to X_{n+1}(d)) \cong \nu^*(\gamma) \).

**Proof.** Since \( L \) is transverse to \( X_{n+1}(d) \) and \( X_n(d) = X_{n+1}(d) \cap L \), the normal bundle \( \nu(X_n(d) \to X_{n+1}(d)) \) is the restriction of \( \nu(L \to \mathbb{C}P^{n+k+1}) \), hence \( \nu(X_n(d) \to X_{n+1}(d)) \cong \nu^*(\nu(L \to \mathbb{C}P^{n+k+1})) \cong \nu^*(\gamma) \).

**Theorem 3.4.** For any complete intersection \( X_n(d) \) the \( S^1 \)-bundle \( S(X_n(d), x) \) has a framing \( F_0 \) such that \( [S(X_n(d), x), F_0] = 0 \in \Omega_{2n+1}^f \).

**Proof.** Let \( U \) be a tubular neighbourhood of \( X_n(d) \) in \( X_{n+1}(d) \). By Proposition 3.3 it is diffeomorphic to the disc bundle of \( \nu^*(\gamma) \), whose first Chern class is \( x \), therefore \( \partial U \approx S(X_n(d), x) \). Its complement, \( X_{n+1}(d) \setminus \text{int } U \) is a codimension-0 submanifold in \( X_{n+1}(d) \setminus X_n(d) \). The latter is stably parallelisable by Proposition 3.2 so \( X_{n+1}(d) \setminus \text{int } U \) is stably parallelisable too. If we choose \( F_0 \) to be the restriction of a framing of \( X_{n+1}(d) \setminus \text{int } U \) to the boundary \( \partial(X_{n+1}(d) \setminus \text{int } U) \approx \partial U \approx S(X_n(d), x) \), then \( (S(X_n(d), x), F_0) \) is framed null-cobordant.

The goal of the rest of this section is to prove that \( S(X_4(d), \Sigma^8 \epsilon_x, x) \) is not framed nullcobordant (with any framing) if \( X_4(d) \) is spin, see Theorem 3.10. First we show that, when an \( m \)-manifold \( X \) is replaced by \( X_2 \Sigma \) for a homotopy \( m \)-sphere \( \Sigma \), the framed cobordism class of \( S(X, \alpha) \) changes by \( \Sigma \times S^1 \) (with a certain choice of framings), see Lemma 3.5. In Lemma 3.7 we give a formula to compute the framing of the \( S^1 \) component. By applying this formula we prove that if \( X_4(d) \) is spin, then \( S(X_4(d), \Sigma^8 \epsilon_x, x) \) has a framing such that it is not framed nullcobordant (Theorem 3.4). Finally we show that we cannot make the framed cobordism class vanish by changing the framing.

**Lemma 3.5.** Let \( m \geq 3 \). For every \( m \)-manifold \( X \), cohomology class \( \alpha \in H^2(X) \) and framing \( F_0 \) of \( S(X, \alpha) \) and every \( \Sigma \in \Theta_m \) and framing \( F_1 \) of \( \Sigma \) there exists a framing \( F_2 \) of \( S^1 \) and a framing \( F \) of \( S(X_2 \Sigma, \alpha) \) such that

\[
[S(X, \alpha), F_0] + [\Sigma \times S^1, F_1 \times F_2] = [S(X_2 \Sigma, \alpha), F] \in \Omega_{m+1}^f.
\]
Proof. Fix an embedding $D^m \to X$ where the connected sum is done. There is a homotopically unique homeomorphism between $X$ and $X \sharp \Sigma$ that is identical on $X \setminus \text{int } D^m$, so there is a canonical isomorphism $H^2(X) \cong H^2(X \sharp \Sigma)$, so $\alpha$ can be regarded as an element of $H^2(X \sharp \Sigma)$, so $S(X \sharp \Sigma, \alpha)$ makes sense. The homomorphisms $H^2(X) \to H^2(X \setminus \text{int } D^m) \to H^2(X \sharp \Sigma)$ are injective (isomorphisms if $m \geq 4$), therefore $S(X \sharp \Sigma, \alpha)$ is (the total space of) the unique $S^1$-bundle over $X \sharp \Sigma$ whose restriction to $X \setminus \text{int } D^m$ is isomorphic to that of $S(X, \alpha)$.

Let $W = (S(X, \alpha) \cup \Sigma \times S^1) \times I$ and $f: D^m \times S^1 \times \partial I \to (S(X, \alpha) \cup \Sigma \times S^1) \times \{1\}$ is the disjoint union of the (homotopically unique) local trivialisation $f_0: D^m \times S^1 \times \{0\} \to S(X, \alpha) \times \{1\}$ of $S(X, \alpha)$ over the fixed $D^m$ and the product map $f_1: D^m \times S^1 \times \{1\} \to \Sigma \times S^1 \times \{1\}$, where $D^m \to \Sigma$ is the embedding used to construct the connected sum $X \sharp \Sigma$. Then $\partial W = \partial \cdot W \cup \partial_0 W$, where $\partial \cdot W = (S(X, \alpha) \cup \Sigma \times S^1) \times \{0\}$ and $\partial_0 W = (S(X, \alpha) \setminus (\text{int } D^m \times S^1)) \cup ((\Sigma \setminus \text{int } D^m) \times S^1) \times \{1\} \cup I \times S^{m-1} \times S^1 \times I$. Thus $\partial_0 W$ is an $S^1$-bundle over $(X \setminus \text{int } D^m) \cup S^{m-1} \times I \cup ((\Sigma \setminus \text{int } D^m) \approx X \sharp \Sigma$ and it coincides with $S(X, \alpha)$ over $X \setminus \text{int } D^m$, therefore $\partial_0 W \approx S(X \sharp \Sigma, \alpha)$.

The inclusion $S(X, \alpha) \times \{0\} \to S(X, \alpha) \times I$ is a homotopy equivalence, covered by a bundle map between the stable normal bundles, therefore the framing $F_0$ can be extended to a framing of $S(X, \alpha) \times I \cup f_0 D^m \times S^1 \times I$. The restriction of this framing to $D^m \times S^1 \times \{1\}$ is $E_m = F_2$, where $E_m$ is the homotopically unique framing of $D^m$ and $F_2$ is some framing of $S^1$ (because every framing of $D^m \times S^1$ is of this form). Similarly, we can take the framing $F_1 \times F_2$ of $\Sigma \times S^1$ and extend it to $\Sigma \times S^1 \times I$. The restriction of this framing to $D^m \times S^1 \times \{1\}$ is again $E_m = F_2$ (up to homotopy), therefore the framings of $S(X, \alpha) \times I \cup f_0 D^m \times S^1 \times I$ and $\Sigma \times S^1 \times I$ together determine a framing of $W$. Let $F$ denote its restriction to $\partial_0 W \approx S(X \sharp \Sigma, \alpha)$. Then $W$ is a framed cobordism between the framed manifolds $(S(X, \alpha), F_0) \cup (\Sigma \times S^1, F_1 \times F_2)$ and $(S(X \sharp \Sigma, \alpha), F)$.

Remark 3.6. Notice that the choice of $F_2$ was determined by $F_0$ alone, so we could have rearranged the quantifiers to get a stronger statement: for every $F_0$ there exists an $F_2$ such that for every $\Sigma$ and $F_1$ there exists an $F$ such that the equality holds.

Lemma 3.7. Suppose that, in addition to the assumptions of Lemma 3.5, there is an $[a] \in \pi_2(X)$ such that $h([a], \alpha) = 1$, where $h: \pi_2(X) \to H_2(X)$ is the Hurewicz-homomorphism. Then for any such $[a] \in \pi_2(X)$ and the framing $F_2$ constructed in the proof of Lemma 3.5 we have

$$[S^1, F_2] = \langle h([a]), w_2(X) \rangle + 1,$$

where both sides are regarded as elements of $Z/2$ (using that $\Omega_2^1 \cong Z/2$).

Proof. Fix a local trivialisation $f_0: D^m \times S^1 \to S(X, \alpha)$, as in the proof of Lemma 3.5. The framing $F_2$ is defined by the property that the restriction of $F_0$ to $f_0(D^m \times S^1)$ is $E_m \times F_2$ (throughout this proof we will identify the framings of $f_0(D^m \times S^1)$ with the framings of $D^m \times S^1$ via the derivative of $f_0$). First we will give another characterisation of $E_m \times F_2$.

If $\partial_0 \pi_2(X) \to \pi_1(S^1) \cong \mathbb{Z}$ denotes the boundary map in the homotopy long exact sequence of the fibration $S^1 \to S(X, \alpha) \to X$, then $\partial([a]) = \langle h([a]), \alpha \rangle$ (because $\alpha$ is the first Chern class of $S(X, \alpha)$). Moreover, $\partial_0$ is the composition of the isomorphism $\pi_2(X) \cong \pi_2(S(X, \alpha), S^1)$ and the boundary map $\pi_2(S(X, \alpha), S^1) \to \pi_1(S^1)$. Therefore for any $[a]$ with $\langle h([a]), \alpha \rangle = 1$ there is a map $\tilde{a}: D^2 \to S(X, \alpha)$ (well-defined up to homotopy) such that $\tilde{a}_{|S^1}$ is the inclusion of a fibre and $[\tilde{a}, \tilde{a}_{|S^1}] \in \pi_2(S(X, \alpha), S^1)$. We can lift $\tilde{a}$ to a map $\tilde{a}: D^2 \to S(X, \alpha) \times \mathbb{R}^+_0$ such that $\tilde{a}$ is an embedding, it is transverse to $S(X, \alpha) \times \{0\}$, $\tilde{a}^{-1}(S(X, \alpha) \times \{0\}) = S^1$, $\tilde{a}_{|S^1}: S^1 \to S(X, \alpha) \times \{0\}$ is the inclusion of a fibre and $[\tilde{a}, \tilde{a}_{|S^1}] \in \pi_2(S(X, \alpha) \times \mathbb{R}^+_0, S^1 \times \mathbb{R}^+_0) \cong \pi_2(S(X, \alpha), S^1)$.

Let $U$ be a tubular neighbourhood of $\tilde{a}(D^2)$ in $S(X, \alpha) \times \mathbb{R}^+_0$, we can assume that $U \cap S(X, \alpha) \times \{0\} = f_0(D^m \times S^1)$. (Note that $U$ is the total space of a $D^m$-bundle over $D^2$, so it has a homotopically unique trivialisation $D^m \times D^2 \to U$, but the restriction of this trivialisation to $S^1$ may differ from $f_0$, the difference is given by an element of $\pi_1(SO_m) \cong \mathbb{Z}/2$. The framing $F_0$ can be extended to a framing of $S(X, \alpha) \times \mathbb{R}^+_0$ and then restricted to a framing of $U$. As mentioned above, if we further restrict this framing to $f_0(D^m \times S^1)$, we get $E_m \times F_2$. Since $U$ is contractible, it has a homotopically unique framing, so this means that $E_m \times F_2$ is the restriction of the homotopically unique framing of $U$.}
The local trivialisation $f_0$ is the restriction of a local trivialisation $\tilde{f}_0: D^m \times D^2 \to D(X, \alpha)$. The homotopically unique framing of $\tilde{f}_0(D^m \times D^2)$ is $E_m \times E_2$, its restriction to $f_0(D^m \times S^1)$ is $E_m \times (E_2|_{S^1})$. Since $(S^1, E_2|_{S^1})$ is the framed boundary of $(D^2, E_2)$, $[S^1, E_2|_{S^1}] = 0 \in \Omega_1^F \cong \mathbb{Z}/2$. So if $g \in \pi_1(SO) \cong \mathbb{Z}/2$ denotes the difference of the framings $E_2$ and $E_2|_{S^1}$ of $S^1$, then $[S^1, F_2] = g \in \mathbb{Z}/2$.

We have $D(X, \alpha) \cup_{(X, \alpha)} S(X, \alpha) \times \mathbb{R}^+ \cong E(X, \alpha)$ (in each fibre $D^2 \cup_{S^1} S^3 \times \mathbb{R}^+ \cong \mathbb{R}^3$) and $f_0(D^m \times D^2) \cup U$ is a tubular neighbourhood of $f_0([0] \times D^2) \cup \partial(D^2) \cong S^2$ in $E(X, \alpha)$. As a $D^m$-bundle over $S^2$ it is classified by an element of $\pi_2(BSO_m) \cong \pi_2(BSO) \cong \pi_1(SO)$ this element corresponds to $g$ (because it is the difference of the framings $E_m \times (E_2|_{S^1})$ and $E_m \times F_2$, which are the restrictions of the unique framings of $f_0(D^m \times D^2)$ and $U$ respectively).

So we need to determine the normal bundle of the embedding $S^2 \to E(X, \alpha)$ as an element of $\pi_2(BSO)$. Since $S^2$ is stably parallelisable, it is the same as the restriction of the tangent bundle $TE(X, \alpha)$ to $S^2$. The embedding $S^2 \to E(X, \alpha)$ is homotopic to its projection to the zero-section $(X)$. Since $f_0([0] \times D^2)$ is a fiber of $D(X, \alpha)$, its projection to $X$ is one point. The map $\bar{a}: D^2 \to S(X, \alpha) \times \mathbb{R}^+$ is a lift of $a; D^2 \to S(X, \alpha)$, which is a lift of a map $a: S^2 \to X$ representing $[a] \in \pi_2(X)$. Therefore the composition of the embedding $S^2 \to E(X, \alpha)$ and the projection to $X$ is $a$. The restriction of $TE(X, \alpha)$ to $X$ is $E(X, \alpha) \oplus TX$. So the bundle we are interested in is the pullback of $E(X, \alpha) \oplus TX$ by $a$.

The second Stiefel-Whitney class detects $\pi_2(BSO)$, so
\[
g = ([S^2], w_2(a^*(E(X, \alpha) \oplus TX))) = \langle a_*([S^2]), w_2(E(X, \alpha) \oplus TX) \rangle =
\]
\[
= \langle h([a]), w_2(E(X, \alpha)) + w_2(TX) \rangle = \langle h([a]), g_2(a) \rangle + \langle h([a]), w_2(X) \rangle = 1 + \langle h([a]), w_2(X) \rangle,
\]
where $g_2: H^2(X) \to H^2(X; \mathbb{Z}/2)$ denotes reduction mod 2 and we used that $E(X, \alpha)$ is a complex line bundle, so $w_1(E(X, \alpha)) = 0$ and $w_2(E(X, \alpha)) = g_2(c_1(E(X, \alpha))) = g_2(a)$ and that $\langle h([a]), \alpha \rangle = 1$.

We already saw that $g$ corresponds to $[S^2, F_2]$, so the statement follows.

**Proposition 3.8.** If $F_2$ is the (homotopically unique) framing of $S^1$ such that $[S^1, F_2]$ is the non-trivial element in $\Omega_1^{F} \cong \mathbb{Z}/2$ and $F_1$ is any framing of $\Sigma^8_{ex}$, then $[\Sigma^8_{ex} \times S^1, F_1 \times F_2] \neq 0 \in \Omega_2^{F}$. Moreover, $[\Sigma^8_{ex} \times S^1, F_1 \times F_2]$ is not contained in the image of the $J$-homomorphism $J_2: \pi_9(SO) \to \Omega_2^{F}$.

**Proof.** Recall that $\Omega_2^{F} \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\text{Im} J_3 \cong \mathbb{Z}/2$. It follows from Kervaire-Milnor [K-M] that $\Sigma^8_{ex}$ can represent one of the two elements of $\Omega_2^{F}$ \setminus \text{Im} J_3 (depending on framing). These elements are denoted by $\varepsilon$ and $\varepsilon + \eta \sigma$ in Toda [T]. Then $[\Sigma^8_{ex} \times S^1, F_1 \times F_2]$ is either $\eta \sigma$ or $\eta \sigma + \eta^2 \sigma$, neither of these is in $\text{Im} J_3$ (in particular neither of them is 0). \hfill $\Box$

**Theorem 3.9.** If $X_4(d)$ is spin, then there is a framing $F$ such that $[S(X_4(d)) \Sigma^8_{ex}, x], F] \neq 0 \in \Omega_2^{F}$.

**Proof.** Let $F_0$ be a framing of $S(X_4(d), x)$ such that $[S(X_4(d), x), F_0] = 0$ (see Theorem 3.4). Let $F_1$ be any framing of $\Sigma^8_{ex}$. By Lemma 3.5 there are framings $F_2$ and $F$ such that $[S(X_4(d)) \Sigma^8_{ex}, F] = [S(X_4(d), x), F_0] + [\Sigma^8_{ex} \times S^1, F_1 \times F_2] = [\Sigma^8_{ex} \times S^1, F_1 \times F_2]$. Since $x$ is a generator of $H^2(X_4(d))$, there is a generator $[a]$ of $\pi_2(S(X_4(d))$ such that $\langle h([a]), x \rangle = 1$, so we can apply Lemma 3.7 and since $X_4(d)$ is spin, we get that $[S^1, F_2] = 1$. By Proposition 3.8 $[\Sigma^8_{ex} \times S^1, F_1 \times F_2] \neq 0$ and this implies that $[S(X_4(d)) \Sigma^8_{ex}, x], F] \neq 0$. \hfill $\Box$

**Theorem 3.10.** If $X_4(d)$ is spin, then for every framing $F$, $[S(X_4(d)) \Sigma^8_{ex}, x], F] \neq 0 \in \Omega_2^{F}$.

**Proof.** First we show that $S(X_4(d)) \Sigma^8_{ex}$ is $3$-connected. It follows from the Lefschetz hyperplane theorem that the embedding $X_4(d) \to \mathbb{C}P^{2n+k}$ is $4$-connected. Therefore we have $\pi_1(X_4(d)) \cong \pi_1(X_4(d)) \cong 0$ and $\pi_2(X_4(d)) \cong \mathbb{Z}$. From the homotopy long exact sequence of the fibration $S^1 \to S(X_4(d), x) \to X_4(d)$, we obtain that $S(X_4(d), x)$ is $3$-connected. Since $S(X_4(d)) \Sigma^8_{ex}$ is homeomorphic to $S(X_4(d), x)$, it is also a $3$-connected $9$-manifold. This implies that $S(X_4(d)) \Sigma^8_{ex}$ is homotopy equivalent to a CW-complex with cells only in dimensions 0, 4, 5 and 9.

Any two framings of $S(X_4(d)) \Sigma^8_{ex}$ differ by a map $S(X_4(d)) \Sigma^8_{ex}, x) \to SO$. Since $\pi_4(SO) \cong \pi_5(SO) \cong 0$, this difference is in fact an element of $\pi_9(SO)$. The effect on the framed cobordism class is given by the J-homomorphism $J_9: \pi_9(SO) \to \Omega_9^{F}$. Therefore the set of cobordism classes in $\Omega_9^{F}$ represented by $S(X_4(d)) \Sigma^8_{ex}$ (with any framing) is a coset of $\text{Im} J_9$. By Proposition 3.8 and the proof of Theorem 3.9 this coset has an element which is not in $\text{Im} J_9$, therefore it is not the trivial coset. So it does not contain 0, therefore 0 $\in \Omega_9^{F}$ is not represented by $S(X_4(d)) \Sigma^8_{ex}$, x with any framing. \hfill $\Box$
Now we can conclude that 4-dimensional spin complete intersections are strongly $\Theta$-flexible:

**Theorem 3.11.** If $X_4(d)$ is spin, then $X_4(d)^g\Sigma^8_{ex}$ is not diffeomorphic to a complete intersection.

**Proof.** Suppose that $X_4(d)^g\Sigma^8_{ex}$ is diffeomorphic to some complete intersection $X_4(d')$. The diffeomorphism induces an isomorphism between $H^2(X_4(d'))$ and $H^2(X_4(d)^g\Sigma^8_{ex})$. We may assume that the generator $x \in H^2(X_4(d'))$ goes into the generator of $H^2(X_4(d)^g\Sigma^8_{ex})$ corresponding to $x$ under the isomorphism $H^2(X_4(d)^g\Sigma^8_{ex}) \cong H^2(X_4(d)'|\text{int } D^4) \cong H^2(X_4(d))$ (see the proof of Lemma 3.5). This is because $X_4(d')$ has a self-diffeomorphism which sends $x$ to $-x$ (see the proof of Proposition 2.10). This implies that $S(X_4(d)^g\Sigma^8_{ex}, x)$ is diffeomorphic to $S(X_4(d'), x)$.

By Theorem 3.4 $S(X_4(d'), x)$ has a framing $F_0$ such that $S(X_4(d'), x, F_0)$ is framed nullcobordant, but by Theorem 3.10 $S(X_4(d)^g\Sigma^8_{ex}, x)$ does not have such a framing, so they are not diffeomorphic. This contradiction shows that $X_4(d)^g\Sigma^8_{ex}$ is not diffeomorphic to any complete intersection $X_4(d')$. \hfill $\square$

### 4. The non-spin cases with even total degree

In this section we prove Theorem 1.12 and Theorem 1.14(a). Both of these results rely on the computation $\text{Tors } \Omega^8_{\Theta}(\mathbb{C}P^1; \xi) \cong \mathbb{Z}/4$ of Lemma 4.2 below, where $\xi$ denotes the (unique up to isomorphism) non-trivial stable bundle over $\mathbb{C}P^1 = S^2$.

#### 4.1. The computation of $\text{Tors } \Omega^8_{\Theta}(\mathbb{C}P^1; \xi)$

We first establish the necessary background to state and prove Lemma 4.1.

Since $\xi$ is the non-trivial stable bundle over $\mathbb{C}P^1$, the Thom spectrum of $\xi$ is given by

$$\text{Th}(\xi) \cong C_\eta := S^0 \cup_\eta D^2,$$

where $\eta: S^1 \to S^0$ generates the 1-stem $\pi^1_\eta = \mathbb{Z}/2$. If $MO(8)$ denotes the Thom-spectrum of $BO(8) = B(\Omega(7))$, then the Pontryagin-Thom isomorphism for $\Omega^8_{\Theta}(\mathbb{C}P^1; \xi)$ is the isomorphism

$$\text{PT}: \Omega^8_{\Theta}(\mathbb{C}P^1; \xi) \to \pi_8(MO(8) \wedge \text{Th}(\xi)) = \pi_8(MO(8) \wedge C_\eta).$$

From the cobordism sequence $S_0 \to C_\eta \to S^2$, we see that there is a long exact sequence

$$\ldots \to \pi_7(MO(8)) \xrightarrow{\eta} \pi_8(MO(8)) \to \pi_8(MO(8) \wedge C_\eta) \to \pi_6(MO(8)) \xrightarrow{\eta} \pi_7(MO(8)) \to \ldots.$$  \hfill (1)

The String bordism groups $\pi_*(MO(8)) \cong \Omega^8_{\Theta}$ have been computed in low dimensions [G] and we shall need the following lemma, which is easily deduced from results in [G].

**Lemma 4.1.** The natural map $\Omega^8_{\Theta} \to \Omega^8_{\Theta}$ satisfies:

1. $\Omega^8_{\Theta} \to \Omega^8_{\Theta}$ is an isomorphism;
2. $\Omega^8_{\Theta} = 0$;
3. $\Omega^8_{\Theta} \to \Omega^8_{\Theta}$ is a surjection onto $\text{Tors } \Omega^8_{\Theta} \cong \mathbb{Z}/2$.

Moreover the kernel of the natural map $\Omega^8_{\Theta} \to \Omega^8_{\Theta}$ is the image of $J$-homomorphism $J_5: \pi_4(O) \to \pi_4^s = \Omega^8_{\Theta}$ and $\Omega^8_{\Theta} \cong \mathbb{Z}/2 \oplus \mathbb{Z}$, where the $\mathbb{Z}$ summand is detected by the signature homomorphism. \hfill $\square$

From the exact sequence and Lemma 4.1, we deduce that there is a short exact sequence

$$0 \to \Omega^8_{\Theta} \to \Omega^8_{\Theta}(\mathbb{C}P^1; \xi) \to \Omega^8_{\Theta} \to 0.$$  \hfill (2)

Noting that $\Omega^8_{\Theta} \cong \Omega^8_{\Theta} \cong \mathbb{Z}/2$ is detected by the Arf-invariant, it is easy to see that the homomorphism $\Omega^8_{\Theta}(\mathbb{C}P^1; \xi) \to \Omega^8_{\Theta}$ can be identified with the codimension-2 Arf-invariant

$$A_{\mathbb{C}P^1}: \Omega^8_{\Theta}(\mathbb{C}P^1; \xi) \to \mathbb{Z}/2,$$

which is defined by making a normal map $(g, \tilde{g}): M \to S^2$ transverse to a point $* \in S^2$ and taking the Arf invariant of the resulting 6-manifold $g^{-1}(\ast)$, which is canonically framed.

**Lemma 4.2.** There is a non-split short exact sequence of abelian groups

$$0 \to \Theta_8 \to \text{Tors } \Omega^8_{\Theta}(\mathbb{C}P^1; \xi) \xrightarrow{\Lambda_{\mathbb{C}P^1}} \mathbb{Z}/2 \to 0.$$  \hfill In particular $\text{Tors } \Omega^8_{\Theta}(\mathbb{C}P^1; \xi) \cong \mathbb{Z}/4$. 

Proof. There is natural forgetful map $F : \Omega^e_8(CP^1; \xi) \to \Omega^e_8(CP^1; \xi)$ and the exact sequence of (2) forms part of the following commutative diagram:

$$
\begin{array}{ccc}
\Omega^e_7 & \xrightarrow{\eta_*} & \Omega^e_8 \\
\downarrow & & \downarrow F \\
\Omega^e_8(CP^1; \xi) & \xrightarrow{A^e_{CP^1}} & \Omega^e_9 \\
\end{array}
$$

Here $A^e_{CP^1} : \Omega^e_8(CP^1; \xi) \to \mathbb{Z}/2$ is a codimension-2 Arf-invariant, which is defined analogously to the codimension 2 Arf-invariant on $\Omega^e_8(CP^1; \xi)$. We shall first compute $\Omega^e_8(CP^1; \xi)$ and we do this via the Pontryagin-Thom isomorphism

$$
\Omega^e_8(CP^1; \xi) \cong \pi^*_8(C_\eta).
$$

The cofibration $S_0 \to C_\eta \to S^2$ sequence leads to a long exact sequence

$$
\ldots \to \pi^*_7 \xrightarrow{\eta_*} \pi^*_8 \to \pi^*_8(C_\eta) \to \pi^*_6 \xrightarrow{\eta_*} \pi^*_7 \to \ldots
$$

From Toda’s calculations [1], we have $\pi^*_8 \cong \mathbb{Z}/2(\nu^2)$, $\pi^*_7 \cong \mathbb{Z}/240(\sigma)$, $\pi^*_8 \cong \mathbb{Z}/2(\eta \sigma) \oplus \mathbb{Z}/2(\nu)$, where $\nu \in \pi^*_2$. It follows that $\eta_* : \pi^*_6 \to \pi^*_7$ is the zero map and that there is a short exact sequence

$$
0 \to \mathbb{Z}/2[\nu] \to \pi^*_8(C_\eta) \to \mathbb{Z}/2 \to 0,
$$

which is determined by the Toda bracket

$$
(\eta, \nu^2, 2) \in \pi^*_8.
$$

By [1] Proposition 3.4 ii], there is a Jacobi identity for Toda brackets,

$$
0 \in \angle \{\eta, \nu^2, 2\} + \langle 2, \eta, \nu^2 \rangle + \langle \eta^2, 2, \eta \rangle,
$$

where we have ignored signs since all the Toda brackets consist of elements of order two or one. Now by [1] Proposition 1.2, $\langle 2, \eta, \nu^2 \rangle \subseteq \langle 2, \eta, \nu \rangle \nu$. Since $\langle 2, \eta, \nu \rangle \subseteq \pi^*_8 = \{0\}$, $\langle 2, \eta, \nu \rangle \nu = \{0\}$ and so $\langle 2, \eta, \nu^2 \rangle = \{0\}$. By [1] p. 189), $\langle \nu^2, 2, \eta \rangle = \{\epsilon, \epsilon + \eta \sigma\}$. It follows that $\langle \eta, \nu^2, 2 \rangle = \{\epsilon, \epsilon + \eta \sigma\}$ is non-trivial and maps to the generator $[\epsilon] \in \pi^*_8$. Applying Lemma [1] we deduce that the extension (3) is non-trivial and hence is isomorphic to the extension

$$
0 \to \mathbb{Z}/2 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0.
$$

The above shows that $\Omega^e_8(CP^1; \xi) \cong \mathbb{Z}/4$.

To complete the proof, we note that the image of the $J$-homomorphism $J_8 : \pi^*_8(SO) \to \pi^*_8$ is precisely the subgroup generated by $\eta \sigma$ [1]. Hence we may identify $\mathbb{Z}/2[\nu] = \text{coker}(J_8) = \Theta_8$ and the lemma follows. □

4.2. The non-spin case with $v_4(X_4(\xi)) = 0$. Let $X_4(\xi)$ be a non-spin complete intersection with $v_4(X_4(\xi)) = 0$. We will prove that $X_4(\xi)$ is $\Theta$-rigid. As explained in Section 2.2 it is enough to show that the canonical homomorphism $i_* : \Theta_8 \cong \text{Tors} \Omega^e_8(\xi) \to \Omega^e_8(\xi_4(\xi))$ is trivial. We will exploit the fact that $i_*$ factors through the group $\text{Tors} \Omega^e_8(\xi_4(\xi))$. Proposition 4.3. Let $\xi$ be a stable bundle over $CP^\infty$ such that $w_2(\xi) \neq 0$ and $w_4(\xi) = 0$, then the natural map $\Theta_8 \to \Omega^e_8(\xi)$ is trivial.

Proof. By [F-K] Section 2.2 we have $\Omega^e_8(\xi) \cong \mathbb{Z}$ From the exactness of the sequence

$$
\ldots \to \xi^e(\xi) \xrightarrow{i_*} \xi^e(\xi) \to \xi^e(\xi) \to \ldots
$$

we deduce that $i_*$ is surjective onto the torsion subgroup of $\Omega^e_8(\xi)$. The signature defines non-trivial homomorphisms $\Omega^e_8(\xi) \to \mathbb{Z}$ and $\Omega^e_8(\xi) \to \mathbb{Z}$ which commute with $i_*$. By Lemma 4.1 $\Omega^e_8(\xi) \cong \mathbb{Z} \oplus \mathbb{Z}/2$ and so $i_*$ is rationally injective. Therefore its restriction to $\text{Tors} \Omega^e_8(\xi)$ is surjective onto $\text{Tors} \Omega^e_8(\xi)$. Therefore if $\Omega^e_8(\xi)$ has a non-trivial torsion element, then it has order 2.
Let \( i_{\mathbb{C}P^1_*} : \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^1; \xi|_{\mathbb{C}P^1}) \to \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi) \) denote the homomorphism induced by the inclusion \( \mathbb{C}P^1 \to \mathbb{C}P^\infty \). By Lemma 4.2 we have \( \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^1; \xi|_{\mathbb{C}P^1}) \approx \mathbb{Z}/4 \) and if \( a \) denotes a generator, then \( \Sigma_{ex}^a \) represents \( 2a \). Therefore \( i_*(\Sigma_{ex}^a) = i_{\mathbb{C}P^1_*}(2a) = 2i_{\mathbb{C}P^1_*}(a) = 0 \).

**Theorem 4.4.** If \( X_4(d) \) is a non-spin complete intersection with \( v_4(X_4(d)) = 0 \) and \( d \neq \{2, 2\} \), then \( X_4(d) \) and \( X_4(d)\Sigma_{ex}^a \) are diffeomorphic.

**Proof.** In this case \( w_2(\xi_4(d)) \neq 0 \) and \( w_4(\xi_4(d)) = 0 \) (see Proposition 2.8), so by Proposition 4.3 the canonical homomorphism \( i_* : \Theta_8 \to \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi_4(d)) \) is trivial. Therefore \( X_4(d) \) and \( X_4(d)\Sigma_{ex}^a \) admit bordant normal 3-smoothings over \( (B_4, \xi_4(d) \times \gamma_{BO(8)}) \). By Proposition 1.9, \( X_4(d) \) and \( X_4(d)\Sigma_{ex}^a \) are diffeomorphic. □

4.3. The non-spin case with \( v_4(X_4(d)) = 0 \) and even total degree. Now we suppose that \( v_4(X_4(d)) = 0 \) and the total degree \( d \) is even. By Remark 2.9 this implies that \( v_2(d) \geq 4 \). We will apply the Adams filtration argument of Kreck and Traving [K5, Section 8], for this we need the following improved vanishing result:

**Lemma 4.5.** Let \( \xi \) be a stable bundle over \( \mathbb{C}P^\infty \) such that \( w_2(\xi) \neq 0 \), \( w_4(\xi) \neq 0 \) and the homomorphism \( i_* : \Theta_8 \to \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi) \) is injective. Then the only element of \( \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi) \) with Adams filtration 4 or higher is the trivial element.

**Proof.** Consider the exact sequence

\[
\cdots \to \Omega_8^{O(7)} \xrightarrow{i_*} \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi) \to \Omega_8^{O(7)}(\mathbb{C}P^\infty, \ast; \xi) \to \cdots
\]

Fang and Klaus [F-K, Section 2.4] proved that \( \Omega_8^{O(7)}(\mathbb{C}P^\infty, \ast; \xi) \approx \mathbb{Z} \oplus \mathbb{Z}/2 \), where the \( \mathbb{Z}/2 \) is detected by the co-dimension 2 Arf invariant. Hence we have the following commutative diagram between exact sequences:

\[
\begin{array}{ccc}
\Theta_8 & \xrightarrow{i_*} & \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^1; \xi|_{\mathbb{C}P^1}) \\
\downarrow & & \downarrow \text{Ad} \\\n\text{Tors} \Omega_8^{O(7)} & \xrightarrow{i_*} & \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi)
\end{array}
\]

The top sequence is short exact by Lemma 4.2. The bottom sequence is also short exact (the surjectivity of \( A \) follows from the commutativity of the diagram and the injectivity of \( i_* \) was assumed). It follows that \( i_{\mathbb{C}P^1_*} : \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^1; \xi|_{\mathbb{C}P^1}) \to \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi) \) is an isomorphism. Now we choose a generator \( a \in \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^1; \xi|_{\mathbb{C}P^1}) \approx \mathbb{Z}/4 \). Then \( i_{\mathbb{C}P^1_*}(a) \in \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi) \) is also a generator and since it is detected by the co-dimension 2 Arf invariant, it has Adams filtration 2. It follows that \( i_*(\Sigma_{ex}^a) = 2i_{\mathbb{C}P^1_*}(a) \) has Adams filtration 3 and so the only element of \( \text{Tors} \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi) \) with Adams filtration 4 or higher is the trivial element. □

**Proposition 4.6.** Let \( X_4(d) \) and \( X_4(d') \) be non-spin complete intersections with \( SD_n(d) = SD_n(d') \) and \( v_4(X_4(d)) = 0 \). If the total degree \( d \) is even, then \( X_4(d) \) and \( X_4(d') \) are diffeomorphic.

**Proof.** By Theorem 4.4 there is a homotopy sphere \( \Sigma \in \Theta_8 \) such that \( X_4(d) \approx X_4(d')\Sigma \). By Proposition 2.8 we have \( w_2(\xi_4(d)) \neq 0 \) and \( w_4(\xi_4(d)) \neq 0 \). Again we consider the natural map \( i_* : \Theta_8 \to \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi_4(d)) \). If it is zero then \( X_4(d') \) and \( X_4(d')\Sigma \) are diffeomorphic by the same argument as in the proof of Theorem 4.4. Hence \( X_4(d) \) and \( X_4(d') \) are diffeomorphic.

Now suppose that \( i_* : \Theta_8 \to \Omega_8^{O(7)}(\mathbb{C}P^\infty; \xi_4(d)) \) is non-zero (hence injective). The arguments of Kreck and Traving [K5, Section 8] show that \( X_4(d) \) and \( X_4(d') \) admit normal 3-smoothings over \( (B_4, \xi_4(d) \times \gamma_{BO(8)}) \) whose bordism classes differ by a torsion element of Adams filtration \( \nu_2(d) \) or higher. Since \( d \) is even, we have \( \nu_2(d) \geq 4 \) (Remark 2.9), so by Lemma 4.3 any such torsion element is trivial. Hence \( X_4(d) \) and \( X_4(d') \) admit bordant normal 3-smoothings over \( (B_4, \xi_4(d) \times \gamma_{BO(8)}) \) and so by Proposition 1.9, \( X_4(d) \) and \( X_4(d') \) are diffeomorphic. □
5. The case of odd total degree

It remains then to consider the case where the total degree \( d \) is odd. Note that in general, this case is not \( \Theta \)-rigid as the following theorem of Kasilingam shows.

**Theorem 5.1** (Kasilingam \([Ka, Remark 2.6(1)]\)). \( CP^4 \) is not diffeomorphic to \( CP^4 \# \Sigma_{ex}^8 \).

To prove the Sullivan Conjecture for \( X_4(d) \) when \( d \) is odd, we find a new way to compare normal bordism classes for \( X_4(d) \) and \( X_4(d') \). This is a difficult problem and its solution is one of the main achievements of this paper. In particular, we believe that introducing the Hambleton-Madsen \([HM]\) theory of degree-\( d \) normal invariants will provide a new perspective on the Sullivan Conjecture in all dimensions.

5.1. Degree-\( r \) normal maps and their normal invariants. In this section we review the surgery classification of bordism classes of degree-\( r \) normal maps for a non-negative integer \( r \). Our treatment follows Hambleton and Madsen \([HM]\) but we choose to work with stable normal bundles in the source of normal maps, as opposed to stable tangent bundles. Also for simplicity, we only formulate the statements in the special case when the target space of a degree-\( r \) normal map is a closed smooth connected oriented \( m \)-manifold \( P \).

**Definition 5.2.** Let \( M \) and \( P \) be closed smooth oriented \( m \)-manifolds and assume that \( P \) is connected. A degree-\( r \) normal map \( (f, \bar{f}): M \to P \) is a map of stable vector bundles

\[
\begin{aligned}
\nu_M & \xrightarrow{f} \xi \\
M & \xrightarrow{f} P \\
\end{aligned}
\]

from the stable normal bundle of \( M \) to some stable vector bundle over \( P \) such that \( f: M \to P \) has degree \( r \).

When \( r = 1 \), then \( \xi \) is a vector bundle reduction of the Spivak normal fibration of \( P \), but in general this only holds away from \( r \). Normal bordism of degree-\( r \) normal maps is defined analogously to normal bordism of degree-1 normal maps \([W2, Proposition 10.2]\): the normal maps \( (f, \bar{f}): (M, \nu_M) \to (P, \xi) \) and \( (f', \bar{f}'): (M', \nu_{M'}) \to (P, \xi') \) are normally bordant, if there is an isomorphism \( \alpha: \xi' \to \xi \) and a bordism between \( (f, \bar{f}) \) and \( (f', \alpha \circ f') \) over \( (P, \xi) \).

**Definition 5.3.** We denote the set of normal bordism classes of degree-\( r \) normal maps to \( P \) by \( \mathcal{N}_r(P) \).

Let \( \Omega^m_{\text{fr}}(P; \xi) \) denote the subset of \( \Omega^m_m(P; \xi) \) consisting of bordism classes whose representatives have degree \( r \). The group of stable bundle automorphisms of \( \xi \), \( \text{Aut}(\xi) \), acts on \( \Omega^m_{\text{fr}}(P; \xi) \), by post-composition. Let \( \mathcal{N}_r(P; \xi) \subseteq \mathcal{N}_r(P) \) denote the subset of normal bordism classes that are representable by normal maps to \( (P, \xi) \). Then we have a canonical bijection

\[
\mathcal{N}_r(P; \xi) \equiv \Omega^m_{\text{fr}}(P; \xi)/\text{Aut}(\xi).
\]

Moreover,

\[
\mathcal{N}_r(P) = \bigsqcup_{[\xi]} \mathcal{N}_r(P; \xi),
\]

where we take the union over the isomorphism classes of stable bundles over \( P \) which admit degree-\( r \) normal maps. To distinguish degree-\( r \) normal bordism classes from usual bordism classes we use the following

**Notation 5.4.** We shall denote the bordism class of \( (f, \bar{f}) \) in \( \Omega^m_{\text{fr}}(P; \xi) \) by \([f, \bar{f}]_\xi \) and in \( \mathcal{N}_r(P) \) by \([f, \bar{f}] \).

Hambleton and Madsen \([HM]\) showed that as in the degree-1 case, the computation of \( \mathcal{N}_r(P) \) proceeds via fibrewise degree-\( r \) maps between vector bundles. Recall that for a bundle \( \xi \), the total space is denoted by \( E\xi \), the disc bundle is \( D\xi \), the sphere bundle is \( S\xi \) and the projection is \( \pi_\xi \). For a space \( Y \) with oriented vector bundles \( \zeta \) and \( \theta \) of the same rank over \( Y \), we consider fibrewise maps

\[
\begin{aligned}
S\zeta & \xrightarrow{t} S\theta \\
Y & \xrightarrow{\pi_\zeta} \pi_\theta \\
\end{aligned}
\]
between the associated sphere bundles, where the restriction of $t$ to each fibre has degree $r$. Given a fibrewise degree-$r_1$ map $t_1: S\zeta_1 \to S\theta_1$ and a fibrewise degree-$r_2$ map $t_2: S\zeta_2 \to S\theta_2$, their fibrewise join is a fibrewise degree-$r_1+r_2$ map $t_1 + t_2: (S\zeta_1 \oplus S\zeta_2) \to (S\theta_1 \oplus S\theta_2)$ between the spheres bundles of the Whitney sums of the original bundles. An isomorphism between two fibrewise degree-$r$ maps, $t_i: S\zeta_i \to S\theta_i$, $i = 0, 1$, is a pair of vector bundle isomorphisms $g: \eta_0 \to \eta_1$ and $h: \theta_0 \to \theta_1$ such that the following diagram commutes up to stable fibre homotopy over $Y$:

\[
\begin{array}{c}
S\zeta_0 \\
\downarrow t_0 \\
S\theta_0 \\
\downarrow g \\
S\zeta_1 \\
\downarrow t_1 \\
S\theta_1
\end{array}
\]

where $Sg$ and $Sh$ are the induced maps of sphere bundles.

**Definition 5.5.** Two fibrewise degree-$r$ maps are equivalent if they become isomorphic after fibrewise join with the restriction of a vector bundle isomorphism and we define

\[
F_r(Y) := \{ t: S\zeta \to S\theta \}/ \sim
\]

to be the set of equivalence classes of fibrewise degree-$r$ maps of vector bundles over $Y$. The equivalence class of $t$ is denoted by $[t]$.

For the case when $P$ is a smooth manifold (which we have assumed for simplicity), the following theorem is an equivalent formulation of a foundational result of Hambleton and Madsen on degree-$r$ normal maps.

**Theorem 5.6 (H-M Theorem 2.2).** There is a bijection $T: F_r(P) \to \mathcal{N}_r(P)$.

The map $T$ can be described as follows. If $g: S\zeta \to S\theta$ represents an element $[g] \in F_r(P)$, then we can extend it to a fibre-preserving map $f: D\zeta \to D\theta$ that is transverse to the zero-section $P \subset D\theta$. We set $M := f^{-1}(P)$ and $f_M := f|_M$. Since $g$ has degree $r$, the map $f_M: M \to P$ has degree $r$ too. The map $f$ determines a bundle map $f_0: \nu(M \to D\zeta) \to \nu(P \to D\theta) \cong \theta$ over $f_M$. We have

\[
\nu_M \cong \nu(M \to D\zeta) \oplus \nu(D\zeta) \cong \nu(M \to D\zeta) \oplus (\pi|_M)^* (\nu_P \oplus \eta) = \nu(M \to D\zeta) \oplus f_M^*(\nu_P \oplus \zeta).
\]

By adding the canonical map $f_M^*(\nu_P \oplus \zeta) \to \nu_P \oplus \zeta$ to $f_0$, we get a bundle map

\[
f_M^*: \nu_M \to \theta \oplus \nu_P \oplus \zeta
\]

over $f_M$. Then $T([g]) = [f_M, f_M]$. In order to apply Theorem 5.6 we need to be able to compute $F_r(P)$. The assignment $Y \mapsto F_r(Y)$ is a homotopy functor from the category of spaces to the category of sets. By Brown representability, this functor (restricted to CW-complexes) is represented by a classifying space.

**Definition 5.7.** The classifying space of the functor $F_r$ is denoted by

\[
(\text{QS}^0/O)_r,
\]

and the canonical bijection between $F_r(Y)$ and $[Y, (\text{QS}^0/O)_r]$ will be denoted by

\[
\text{Br}: F_r(Y) \to [Y, (\text{QS}^0/O)_r].
\]

In the case $r = 1$, we may identify $(\text{QS}^0/O)_1 = G/O$, where $G/O$ is the homotopy fibre of the canonical map $BO \to BG$, the forgetful map from the classifying space of stable vector bundles to the classifying space of stable spherical fibrations.

The equivalence $\text{Br}$ and Theorem 5.6 combine to give the following important

**Definition 5.8.** Let $\eta: \mathcal{N}_r(P) \to [P, (\text{QS}^0/O)_r]$ denote the composition $\text{Br} \circ T^{-1}$. For a degree-$r$ normal map $(f, \tilde{f}): M \to P$, the homotopy class $\eta([f, \tilde{f}]) \in [P, (\text{QS}^0/O)_r]$ is called the normal invariant of $(f, \tilde{f})$.

An important example of a fibrewise degree-$r$ map is when $\zeta$ has real rank 2 and we regard $\zeta$ as a complex line bundle over $Y$. Setting $\theta := \zeta^r$ to be the $r$-fold complex tensor product of $\zeta$ with itself, we have the canonical degree-$r$ map

\[
t_r(\zeta): S\zeta \to S\zeta^r, \quad v \mapsto v^r = v \oplus v \oplus \ldots \oplus v.
\]

For the classification of complete intersections, the universal examples of such maps, where $Y = \mathbb{C}P^n$ or $\mathbb{C}P^\infty$ and $\zeta = \gamma|_{\mathbb{C}P^n}$ or $\gamma$ will play a central role.
**Definition 5.9.** For a k-tuple of integers $\underline{r} = (r_1, \ldots, r_k)$ with $r = r_1 r_2 \ldots r_k$ set
\[
\eta_n(\underline{r}) := Br([t_1(\gamma)|_{CP^n} \ast \ldots \ast t_k(\gamma)|_{CP^n}]) \in [CP^n, (QS^0/O)_r]
\]
and
\[
\eta_\infty(\underline{r}) := Br([t_1(\gamma) \ast \ldots \ast t_k(\gamma)]) \in [CP^\infty, (QS^0/O)_r].
\]

The notation in Definition 5.9 is designed to match Theorem 5.16 which states that the complete intersection $X_n(d)$ admits a degree-d normal map $(f_n(d), \bar{f}_n(d)) : X_n(d) \to CP^n$ such that $\eta((f_n(d), \bar{f}_n(d))) = \eta_n(d)$.

5.2. The space $(QS^0/O)_r$ and the action of $\Theta_m^r$ on $N_r(P)$. In order to apply Theorem 5.6 we will need to make computations with the set of normal invariants $[P,(QS^0/O)_r]$ and for this we need information about the space $(QS^0/O)_r$. When $r = 1$, the space $(QS^0/O)_1 = G/O$ has been extensively studied. In general, there is a map $\psi : G/O \to (QS^0/O)_r$, which classifies taking fibrewise join with the trivial degree-$r$ map and Brumfiel and Madsen prove that $\psi$ is a homotopy equivalence when $r$ is inverted:

**Proposition 5.10** ([B-M] Proposition 4.6]). The map $\psi : G/O \to (QS^0/O)_r$ induces a homotopy equivalence $(G/O)[1/r] \simeq (QS^0/O)_r[1/r]$.

Brumfiel and Madsen prove [B-M] Proposition 4.6] by observing that there is a fibration sequence
\[
QS^0_0 \to (QS^0/O)_r \to BSO,
\]
where the map to $(QS^0/O)_r \to BSO$ classifies taking the formal difference of the source and target vector bundles of a fibrewise degree-$r$ map and $QS^0_0$ is the space of stable degree-$r$ self maps of the sphere, which classifies fibrewise degree-$r$ self-maps of trivialised vector bundles. The map $\psi$ fits into a map of fibration sequences from the fibration sequence of (4) when $r = 1$ to the general fibration sequence. For our purposes, we will only need the fact that there is a canonical homotopy equivalence $QS^0_0 \to QS^0_1$, given by taking the loop sum with a fixed map of degree $r$ and we let
\[
\iota_r : QS^0_0 \to (QS^0/O)_r
\]
be the composition of the equivalence $QS^0_0 \to QS^0_1$ with the inclusion of fibre of (4).

We conclude this subsection by considering the effect of addition of homotopy spheres in the sources of normal maps. A choice of stable framing of a homotopy $m$-sphere $\Sigma$ defines a degree zero normal map $(f_\Sigma, \bar{f}_\Sigma) : \Sigma^m \to S^m$, where $f_\Sigma$ is the constant map. It also defines an element $[\Sigma, \bar{f}_\Sigma]$ in $m$-dimensional framed bordism, $\Omega_m^b$.

We define the framed normal invariant of $(f_\Sigma, \bar{f}_\Sigma)$,
\[
\eta^b([f_\Sigma, \bar{f}_\Sigma]) \in \pi_m(QS^0_0) \cong \Omega_m^b,
\]
to be the homotopy class which maps to $[\Sigma, \bar{f}_\Sigma]$ under the Pontryagin-Thom isomorphism. Given a general degree-$r$ normal map $(f, \bar{f}) : M \to P$, we can assume that it is constant over a small $m$-disc $D^m \subset M$ and by taking connected sum in the source and extending with the constant map, we obtain a degree-$r$ normal map
\[
(f \# f_\Sigma, \bar{f} \# \bar{f}_\Sigma) : M \# \Sigma^m \to P.
\]
If $c : P \to S^m$ is the degree one collapse map obtained by collapsing around a small $m$-disc containing $f(D^m)$, then we have the induced map $c^* : \pi_m([QS^0/O)_r] \to [P, (QS^0/O)_r]$. Pinching off the top cell of $P$ gives the pinch map $p : P \to P \vee S^m$, which induces the action
\[
[P, (QS^0/O)_r] \times \pi_m([QS^0/O)_r] \xrightarrow{\gamma} [P, (QS^0/O)_r], \quad ([\varphi], [\psi]) \mapsto ([\varphi \vee \psi] \circ p).
\]
Tracing through the definition of the Hambleton-Madsen normal invariant in the discussion following Theorem 5.6, we see that the normal invariant of $(f \# f_\Sigma, \bar{f} \# \bar{f}_\Sigma)$ is given by the following formula
\[
\eta((f \# f_\Sigma, \bar{f} \# \bar{f}_\Sigma)) = \eta([f, \bar{f}]) + c^*(\iota_r(\eta^b([f_\Sigma, \bar{f}_\Sigma]))) \in [P, (QS^0/O)_r].
\]

5.3. Relative divisors. In this subsection we reformulate the description of the bijection $T : \mathcal{F}_r(P) \to \mathcal{N}_r(P)$ from Theorem 5.6 in terms of sections and divisors.

Suppose that $\theta$ is a rank-$k$ smooth vector bundle over a smooth manifold $V$ with boundary $\partial V$ and $s_\theta : \partial V \to S\theta|_{\partial V}$ is a section of $S\theta|_{\partial V}$ (hence a nowhere zero section of $E\theta|_{\partial V}$).
**Definition 5.11.** If \( s: V \to E\bar{\theta} \) is a smooth section of \( \bar{\theta} \), proper near \( \partial V \), which extends \( s_\theta \) and which is transverse to the zero-section, \( s_0 \), then we call
\[
Z(s) := s(V) \cap s_0(V) \subset s_0(V) \cong V
\]
a divisor of \( \bar{\theta} \) relative to \( s_\theta \).

**Remark 5.12.** The normal bundle of the embedding \( Z(s) \hookrightarrow V \) is given by \( \nu(Z(s) \to V) \cong \bar{\theta}|_{Z(s)} \), hence \( \nu_{Z(s)} \cong (\bar{\theta} \oplus \nu_V)|_{Z(s)} \). Since the fibre of \( E\bar{\theta} \) is contractible, \( s_\theta \) can always be extended to a (transverse) section \( s \) and the extension is unique up to homotopy (rel \( \partial V \)). This also implies that the normal bordism class of the normal map
\[
\begin{array}{c}
\nu_{Z(s)} \\
\downarrow \\
Z(s) \\
\downarrow \\
V
\end{array}
\]
is independent of the choice of \( s \) (and it only depends on the homotopy class of \( s_\theta \) as a nowhere zero section).

Suppose in addition that \( V = D\bar{\theta} \) itself is the disc bundle of a rank-\( k \) smooth vector bundle \( \zeta \) over a closed smooth manifold \( P \). Let \( \theta = \bar{\theta}|_P \) be the restriction of \( \bar{\theta} \), then \( \theta \) can be identified with \( (\pi|_{\mathcal{D}\zeta})^*(\theta) \).

**Definition 5.13.** (a) For a fibre-preserving map \( g: S\zeta \to S\theta \) we can define a section \( s_g: S\zeta \to (\pi|_{\mathcal{D}\zeta})^*(S\theta) \subseteq S\zeta \times S\theta \) of the pull-back sphere bundle \( (\pi|_{\mathcal{D}\zeta})^*(S\theta) \) by the formula \( s_g(x) = (x, g(x)) \). The assignment \( g \mapsto s_g \) is a bijection between fibre-preserving maps \( S\zeta \to S\theta \) and sections of \( (\pi|_{\mathcal{D}\zeta})^*(S\theta) \).

(b) For a fibre-preserving map \( f: D\zeta \to D\theta \) we can define a section \( s_f: D\zeta \to (\pi|_{\mathcal{D}\zeta})^*(D\theta) \subseteq D\zeta \times D\theta \) by \( s_f(x) = (x, f(x)) \). The assignment \( f \mapsto s_f \) is a bijection between fibre-preserving maps \( D\zeta \to D\theta \) and sections of \( (\pi|_{\mathcal{D}\zeta})^*(D\theta) \). Moreover, \( f \) is transverse to the zero-section of \( D\theta \) if and only if \( s_f \) is transverse to the zero-section of \( (\pi|_{\mathcal{D}\zeta})^*(D\theta) \).

These two bijections are compatible in the sense that if \( g \) is the restriction of some \( f \), then \( s_g \) is the restriction of \( s_f \).

Let \( g: S\zeta \to S\theta \) be a fibrewise degree-\( r \) map and \( s_g \) the corresponding section. There exists a section \( s: D\zeta \to (\pi|_{\mathcal{D}\zeta})^*(D\theta) \) that extends \( s_g \) and is transverse to the zero-section. Let \( p = \pi|_{Z(s)}: Z(s) \to P \).

**Lemma 5.14.** The map \( p: Z(s) \to P \) has degree \( r \) and it is covered by a bundle map \( \bar{p}: \nu_{Z(s)} \to \theta \oplus \nu_P \oplus \zeta \) such that
\[
T([g]) = [p, \bar{p}] \in \mathcal{N}_r(P).
\]

**Proof.** Using the bijection from Definition 5.13(b) there is a fibre-preserving map \( f: D\zeta \to D\theta \) such that \( s = s_f \). This \( f \) extends \( g \) and is transverse to the zero-section, so it satisfies the conditions in the definition of \( T \) (see below Theorem 5.6). The manifold \( M = f^{-1}(P) \) is then equal to \( Z(s) \) and \( f|_M = p \) (and it has degree \( r \)). We can choose \( p = f|_M \) and then \( T([g]) = [p, \bar{p}] \).

### 5.4. The canonical-degree-\( d \) normal invariant of a complete intersection.
Consider a complete intersection \( X_n(d) \). By cellular approximation the canonical embedding \( i: X_n(d) \to \mathbb{C}P^{n+k} \) is homotopic to a map \( f_n(d): X_n(d) \to \mathbb{C}P^n \) and since \( \mathbb{C}P^{n+k} \) has no \( (2n+1) \)-cells, \( f_n(d) \) is well-defined up to homotopy. Since \( i_*([X_n(d)]) \) is \( d \) times the preferred generator of \( H_{2n}(\mathbb{C}P^n) \), \( f_n(d) \) is a degree-\( d \) map.

The main result of this section is the computation of the normal invariant of a certain degree-\( d \) normal map covering \( f_n(d) \) in Theorem 5.16 below. The importance of this calculation comes from the next lemma (which can be regarded as a variation of [Kr, Proposition 10]) and its application, Theorem 5.17.

**Lemma 5.15.** Let \( X_n(d) \) and \( X_n(d') \) be complete intersections with \( \chi(X_n(d)) = \chi(X_n(d')) \) and the same total degree \( d \). Suppose that there are degree-\( d \) normal maps \( (f, f'): (X_n(d), \nu_{X_n(d)}) \to (\mathbb{C}P^n, \xi_0(d)|_{\mathbb{C}P^n}) \) and \( (f', f''): (X_n(d'), \nu_{X_n(d')}) \to (\mathbb{C}P^n, \xi_0(d')|_{\mathbb{C}P^n}) \) such that
\[
[f, f'] = [f', f''] \in \mathcal{N}_d(\mathbb{C}P^n).
\]

If \( n \geq 3 \) then \( X_n(d) \) and \( X_n(d') \) are diffeomorphic.
Proof. Let $\xi = \xi_n(d)|_{CP^n}$, and recall (see Notation 5.4) that $[g, \bar{g}]_d \in \Omega^d_{2n}(CP^n; \xi)d$ denotes the element represented by a degree-$d$ normal map $[g, \bar{g}]$ and the image of $[g, \bar{g}]_d$ in $N_d(CP^n)$ is $[g, \bar{g}]$. By definition, the condition $[f, f'] = [f', f]$ means that there is an isomorphism $\alpha: \xi_n(f')|_{CP^n} \to \xi_n(f)|_{CP^n} = \xi$ (which in particular implies that $SD_n(d) = SD_n(f')$) such that $\ [f, f]_d = [f', \alpha \circ f']_d \in \Omega^d_{2n}(CP^n; \xi)_d$.

Now consider the composition

$$\Omega^d_{2n}(CP^n; \xi_n(d)|_{CP^n}) \to \Omega^d_{2n}(CP^n; \xi_n(d)|_{CP^n}) \to \Omega^d_{2n}(CP^n; \xi_n(d)) \to \Omega^d_{2n}(CP^n; \xi_n(d)).$$

We see that $X_n(d)$ and $X_n(d')$ admit normal $(n-1)$-smoothings over $(B_n; \xi_n(d) \times \gamma_{BO(n+1)}$ and if $d \neq \{1\}, \{2\}$ or $\{2, 2\}$, then the lemma follows from Proposition 1.9. If $d = \{1\}, \{2\}$ or $\{2, 2\}$, then $d' = d$ (because $SD_n(d) = SD_n(d')$).

**Theorem 5.16.** There is a bundle map $\tilde{f}_n(d): \nu_{X_n(d)} \to \xi_n(d)|_{CP^n}$ over $f_n(d)$ such that $\eta([\tilde{f}_n(d), f_n(d)]) = \eta_n(d) \in [CP^n, (QS^d/O)d]$. (For the definition of $\eta_n(d)$ see Definition 5.9.)

An immediate consequence of Theorem 5.16 is that the fact that $\eta$ is a bijection and Lemma 5.15 is the following

**Theorem 5.17.** Let $X_n(d)$ and $X_n(d')$ be complete intersections with the same total degree $d$ and the same Euler-characteristic. If $n \geq 3$ and $\eta_n(d) = \eta_n(d') \in [CP^n, (QS^d/O)d]$, then $X_n(d)$ and $X_n(d')$ are diffeomorphic.

**Proof of Theorem 5.16.** Let $f^0: D(k|_{CP^n}) \to D(\gamma^d|_{CP^n})$ denote the Whitney-sum of the tensor power maps $D(\gamma^d|_{CP^n}) \to D(\gamma^d|_{CP^n})$ and let $g^0: S(k|_{CP^n}) \to S(\gamma^d|_{CP^n})$ be its restriction to the sphere bundle. Hence, in the notation of Definition 5.9, $g^0 = d_0(\gamma|_{CP^n}) \ast \ldots \ast d_0(\gamma|_{CP^n})$ so $Br([g^0]) = \eta_n(d)$ and we must prove the following: There is a map of stable vector bundles $f_n(d): \nu_{X_n(d)} \to \xi_n(d)|_{CP^n}$ over $f_n(d)$ such that $T([g^0]) = [f_n(d), f_n(d)] \in N_n(d(CP^n))$.

First we describe a way of constructing the representative of the complete intersection $X_n(d)$ in an arbitrarily small neighbourhood of the subspace $CP^n \subset CP^{n+k}$. Let $[x_0, x_1, \ldots, x_{n+k}]$ be homogeneous coordinates on the ambient $CP^{n+k}$. Let $p_i^0(\bar{z}) = x_{n+i}^d$ for $i = 1, 2, \ldots, k$, where $\bar{z} = (x_0, x_1, \ldots, x_{n+k})$. Then $\{[\bar{z}] \in CP^{n+k} | p_1^0(\bar{z}) = p_2^0(\bar{z}) = \ldots = p_k^0(\bar{z}) = 0\} = \{[\bar{z}] \in CP^{n+k} \mid x_{n+1} = x_{n+2} = \ldots = x_{n+k} = 0\} \subset CP^n$. Note that if $d_1 > 1$, then $p_1^0$ is singular at its zeroes, so $CP^n$ is not a representative of $X_n(d)$ unless $d = \{1, 1, \ldots, 1\}$. However, by applying an arbitrarily small perturbation to the $p_i^0$ we can obtain new polynomials $p_i$ such that

$$\{[\bar{z}] \in CP^{n+k} \mid p_1(\bar{z}) = p_2(\bar{z}) = \ldots = p_k(\bar{z}) = 0\} = X_n(d)$$

is a complete intersection and it is contained in the interior of a closed tubular neighbourhood $U$ of $CP^n$ (we will fix a $U$ in Lemma 5.20 below).

By Lemma 5.19 the polynomials $p_i^0$ and $p_i$ define sections of $\gamma^d|_{CP^{n+k}}$. Therefore the tuples $(p_1^0, p_2^0, \ldots, p_k^0)$ and $(p_1, p_2, \ldots, p_k)$ define some sections $s^0$ and $s$ of $\gamma^d|_{CP^{n+k}}$ (so the zero sets of $s^0$ and $s$ are $CP^n$ and $X_n(d)$ respectively). Then we can assume that there is a homotopy between $s^0$ and $s$ that is non-zero on $CP^{n+k} \setminus \text{int} U \times I$. In particular, the restrictions of $s^0$ and $s$ are homotopic as non-zero sections over $\partial U$.

The normal bundle of $CP^n$ in $CP^{n+k}$ is $k|_{CP^n}$, so $U$ can be identified with $D(k|_{CP^n})$. Moreover, the projection $\pi_U: U \to CP^n$ of $U$ is a deformation retraction, hence the bundle $\pi_U^* (\gamma^d|_{CP^n})$ is isomorphic to $\gamma^d|_{U}$. We will fix an identification and an isomorphism in Lemma 5.20. With these identifications, the sections $s^0$, and $s|U$ correspond to fibre-preserving maps $D(k|_{CP^n}) \to D(\gamma^d|_{CP^n})$ under the bijection of Definition 5.13. Let $f: D(k|_{CP^n}) \to D(\gamma^d|_{CP^n})$ be the map such that $s_f = s|U$. In Lemma 5.20 we prove that $s_f$ is $s|U$. Let $g: S(k|_{CP^n}) \to S(\gamma^d|_{CP^n})$ be the restriction of $f$ (we can assume that it has values in the sphere bundle, because $f_s|_{k|_{CP^n}}$ is nowhere zero, because $s|_{\partial U}$ is nowhere zero), then $s_g = s|_{\partial U}$. The restriction of $f^0$ is $g^0$, so $s^0 = s^0|_{\partial U}$. Since $s_f^0 = s^0|_{\partial U}$ and $s_g = s|_{\partial U}$ are homotopic as non-zero sections, $g^0$ and $g$ are fibre homotopic. The bijection $T$ is well-defined on fibre homotopy classes, so $T([g^0]) = T([g])$. 


By construction $X_n(d) = Z(s)$. Since $\pi_U$ is homotopic to the identity, we have $f_n(d) = \pi_U|_{X_n(d)}$ (up to homotopy). By Lemma 5.14 there is a bundle map $f_n(d) : \nu_{X_n(d)} \to \gamma^d \oplus -(n+1)^d \otimes k^\gamma_{|CP^n}$ such that $T([g]) = [f_n(d), f_n(d)] \in \mathcal{N}_d(CP^n)$. Therefore $T([g]) = [f_n(d), f_n(d)]$.

**Remark 5.18.** There is a canonical bundle map $\nu_{X_n(d)} \to \xi_n(d)|_{CP^{n+k}}$ over $i$ (and in fact this is how Proposition 2.5 was proved), because there is a canonical isomorphism $\nu_{X_n(d)} \cong \nu(X_n(d) \to CP^{n+k}) \otimes \nu_{CP^{n+k}}$ and the normal bundle of a degree-$r$ hypersurface in $CP^{n+k}$ is canonically isomorphic to the restriction of $\gamma^r$ (this follows from Lemma 5.19 below). By following the definitions, we can see that the bundle map $f_n(d)$ constructed in the proof of Theorem 5.16 is equal to this canonical map (up to homotopy).

We used the following two lemmas:

**Lemma 5.19.** A homogeneous polynomial $q$ of degree $r$ in variables $x_0, x_1, \ldots, x_m$ determines a section of the bundle $\gamma^r|_{CP^n}$.

This statement is of course well-known, but we give a proof to establish the notation for the next lemma. 

**Proof.** If $r = 1$, then the assignment $[x_0, x_1, \ldots, x_m] \mapsto [x_0, x_1, \ldots, x_m, q(x_0, x_1, \ldots, x_m)]$ is a well-defined map $CP^n \to CP^{n+1} \setminus \{0, \ldots, 0, 1\}$. Since the map $CP^{n+1} \setminus \{0, \ldots, 0, 1\} \to CP^1 : [x_0, x_1, \ldots, x_m+1] \mapsto [x_0, x_1, \ldots, x_m]$ can be identified with the projection of the normal bundle of $CP^n$ in $CP^{n+1}$, which is isomorphic to $\gamma^r_{|CP^n}$, we get that $q$ determines a section of $\gamma^r_{|CP^n}$. So every linear monomial $x_i$ determines a section of $\gamma^r_{|CP^n}$. If we have sections $s_1, s_2, \ldots, s_r$ of some vector bundle $s_1s_2\ldots s_r$ is a section of the symmetric power $Sym^r(\xi)$ and if $\xi$ is a line bundle, then $Sym^r(\xi) = \xi^r$. Therefore every degree-$r$ monomial and hence every degree-$r$ homogeneous polynomial determines a section of $\gamma^r_{|CP^n}$.

**Lemma 5.20.** We can identify $D(k\gamma|CP^n)$ with a tubular neighbourhood $U$ of $CP^n$ in $CP^{n+k}$ and the bundle $\pi_U^*(\gamma^d|_{CP^n})$ with $\gamma^d|U$ such that after these identifications the section $s_0$ corresponding to $f^0$ under the bijection of Definition 5.13 is equal to $s^0|U$.

**Proof.** First we will introduce “coordinates” on the total space of $\gamma^r|_{CP^n}$, then we will define $U$ and describe the necessary identifications, then we will show that $s_0$ (regarded as a section of $\gamma^d|_{CP^n}$) is equal to $s^0|U$.

By Lemma 5.19 a pair $([a], q)$ (where $[a] = [a_0, a_1, \ldots, a_m] \in CP^n$ and $q$ is a homogeneous polynomial of degree $r$ in variables $x_0, x_1, \ldots, x_m$) determines a point in $E(\gamma^r|_{CP^n})$ (namely, the value of the section determined by $q$ over the point $[a]$). Every point in $E(\gamma^r|_{CP^n})$ can be described by such a pair and two pairs, $([a], q)$ and $([a], q')$, determine the same point if and only if $q([a]) = q'([a])$. Similarly, if $q_i$ is a homogeneous polynomial of degree $d_i$, then a pair $([a], (q_1, q_2, \ldots, q_k))$ determines a point in $E(\gamma^d|_{CP^n})$.

To simplify notation we will use the abbreviations $a = (a_0, a_1, \ldots, a_m) \in C^{n+1} \setminus \{0\}$, $b = (b_1, b_2, \ldots, b_k) \in C^k$ and $z = (z_0, c_1, \ldots, c_{n+k}) \in C^{n+k+1} \setminus \{0\}$. Also, $q_i$ and $r_i$ will always denote some homogeneous polynomials in variables $x_0, x_1, \ldots, x_m$ such that $q_i$ has degree $d_i$ and $r_i$ is linear.

The map $([a], (r_1, r_2, \ldots, r_k)) \mapsto ([a], r_1([a]), r_2([a]), \ldots, r_k([a]))$ is a homeomorphism between $E(k\gamma|_{CP^n})$ and an open tubular neighbourhood of $CP^n$ in $CP^{n+k}$ (which is diffeomorphic to $CP^{n+k} \setminus CP^{k-1}$). We use $U$ to be the image of the disc bundle $D(k\gamma|_{CP^n})$ under this map. Then this map identifies $D(k\gamma|_{CP^n})$ with $U$.

Points of the subspace $E(\pi_U^*(\gamma^d|_{CP^n})) \subset U \times E(\gamma^d|_{CP^n})$ are of the form $([a], ([a], (q_1, q_2, \ldots, q_k)))$. The map $([a], ([a], (q_1, q_2, \ldots, q_k))) \mapsto ([a], (q_1([a]), q_2([a]), \ldots, q_k([a]))) \in E(\gamma^d|_{CP^{n+k}})$ (where $q_i$ is equal to $q_i$, but it is regarded as a polynomial in the variables $x_0, x_1, \ldots, x_{n+k}$) is an isomorphism between the bundles $\pi_U^*(\gamma^d|_{CP^n})$ and $\gamma^d|U$.

By definition the section $s^0$ is $([a], ([a], (p^0_0, p^0_1, \ldots, p^0_k)))$.

The map $f^0$ is given by the formula $([a], (r_1, r_2, \ldots, r_k)) \mapsto ([a], (r^*_1, r^*_2, \ldots, r^*_k))$. After identifying $D(k\gamma|_{CP^n})$ with $U$ the formula becomes $([a], ([a], (r^*_1, r^*_2, \ldots, r^*_k)))$, where $r_i$ is chosen such that $r_1([a]) = b$. Therefore $s_f([a], ([a], (r^*_1, r^*_2, \ldots, r^*_k)))$ is a section of $([a], ([a], (p^0_0, p^0_1, \ldots, p^0_k)))$. We have $r^*_1([a], b) = r^*_1([a], b) = b^*_1 = s^0([a], b)$, so $([a], ([a], (p^0_0, p^0_1, \ldots, p^0_k))) = ([a], ([a], (p^0_0, p^0_1, \ldots, p^0_k)))$, therefore $s_f = s^0|U$. 


We conclude this section with a discussion of the bundle data \( f_n(d) \) in the canonical normal invariant of \( X_n(d) \). Although the degree-\( d \) normal map \((f_n(d), f_n(d)): X_n(d) \rightarrow \mathbb{C}P^n\) is canonically constructed, so far we have not been able to characterise its homotopy class amongst all such degree-\( d \) normal maps. In particular, if there is a diffeomorphism \( h: X_n(d) \rightarrow X_{n}(d') \), then up to homotopy it induces a unique bundle map \( h: \nu_{X_{n}(d)} \rightarrow \nu_{X_{n}(d')} \) covering \( h \) and in general we do not know whether \( f_n(d') \circ h \) and \( f_n(d) \) are homotopic stable bundle maps. In this paper, we shall only need to address this question when \( n = 4 \) and \( X_4(d) \) is non-spin. In this case, the problem is solved via the following

**Lemma 5.21.** Let \( X \) be a closed, connected non-spin 8-manifold which is homotopy equivalent to a CW-complex with only even dimensional cells, \( \xi \) a stable vector bundle over \( X \) and \( \bar{g}: \xi \rightarrow \xi \) an orientation preserving stable bundle automorphism. Then \( \bar{g} \) is homotopic to the identity.

**Proof.** By standard K-theoretic arguments (given for automorphisms of stable spherical fibrations in [Br, Lemma I.4.6]), it is sufficient to prove that \([X, \mathbb{S}O] = 0\). This follows from a calculation with the Atiyah-Hirzebruch spectral sequence, where the only non-zero term on the 0-line is \( H^8(X; \pi_8(\mathbb{S}O)) = \mathbb{Z}/2 \) and this term is killed by the \( d^2 \)-differential since \( X \) is non-spin. We leave the details to the reader. \( \square \)

**Corollary 5.22.** Let \((f_0, f_0), (f_1, f_1): (X_4(d), \nu_{X_4(d)}) \rightarrow (\mathbb{C}P^4, \xi_4(d)|_{\mathbb{C}P^4})\) be a pair normal maps from a non-spin complete intersection \( X_4(d) \) such that \( f_0(x) = f_0(x) \). Then \( (f_0, f_0) \) and \((f_1, f_1)\) are homotopic.

**Proof.** It follows from the assumption \( f_0(x) = f_0(x) \) that \( f_0 \) and \( f_1 \) are homotopic as maps into \( \mathbb{C}P^\infty \). By cellular approximation they are also homotopic as maps into \( \mathbb{C}P^4 \), so we may assume that \( f_0 = f_1 \).

Then the bundle maps \( f_0, f_1: \nu_{X_4(d)} \rightarrow \xi_4(d)|_{\mathbb{C}P^4} \) differ by pre-composition with a bundle automorphism \( \bar{g}: \nu_{X_4(d)} \rightarrow hX_4(d) \). By Lemma 5.21 \( \bar{g} \) is homotopic to the identity and so \( f_0 \) and \( f_1 \) are homotopic. \( \square \)

5.5. The Sullivan Conjecture in the case of odd total degree. By Theorem 5.10 there is a degree-\( d \) normal map \((f_n(d), f_n(d)): X_n(d) \rightarrow \mathbb{C}P^n\) such that

\[ \eta(f_n(d), f_n(d)) = \eta(d) \in [\mathbb{C}P^n, (QS^0/O)d]. \]

Now \( \eta(d) \) is the restriction of \( \eta_{\infty}(d) : \mathbb{C}P^\infty \rightarrow (QS^0/O)d \) and this allows us to apply results of Feshbach on the Segal Conjecture to compare the normal invariants \( \eta(d) \) and \( \eta(d) \). Combined with Theorem 1.4 of Fung and Klaus, this will be sufficient to prove the Sullivan Conjecture when \( n = 4 \) and \( d \) is odd.

When the total degree \( d \) is odd, localising at the prime 2 will prove to be an effective strategy. For a space \( Z \) and a prime \( p \) we shall write \( Z(p) \) (and even \((Z)p\) where necessary) for the \( p \)-localisation of \( Z \) and if \( \mu \in [Y, Z] \) is a homotopy class of maps from some other space \( Y \) to \( Z \), we write \( \mu(p) \) for the corresponding homotopy class in \([Y, Z(p)]\).

We write \( A(p) \) for the \( p \)-localisation of an abelian group \( A \).

**Lemma 5.23.** Let \( X_4(d) \) and \( X_4(d') \) be 4-dimensional complete intersections with \( SD_4(d) = SD_4(d') \). If \( \eta_4(d)(d) = \eta_4(d')(d) \in [\mathbb{C}P^4, ((QS^0/O)d)(d)] \), then \( \eta_4(d) = \eta_4(d') \).

**Proof.** By Theorem 1.4 there is a homotopy 8-sphere \( \Sigma \) and a diffeomorphism \( h: X_4(d) \cong X_4(d') \Sigma \). We may assume that \( h \) preserves the cohomology class \( x \) (see the proof of Proposition 2.10) and hence the maps \( f_3(d) \) and \( f_3(d') \circ h \) are homotopic. Let \( h: \nu_{X_4(d)} \rightarrow \nu_{X_4(d') \Sigma} \) be the stable bundle map covering \( h \) which is uniquely determined up to homotopy by the derivative of \( h \). As explained in Section 2.2, since \( SD_4(d) = SD_4(d') \) there is a stable bundle isomorphism \( \alpha: \xi_4(d')(d') \rightarrow \xi_4(d')(d') \). Choose a framing \( f_2 \) of \( \Sigma \) as in Section 5.2, then we have the following diagram of stable bundle maps, which commutes by Corollary 5.22.

\[
\begin{align*}
\nu_{X_4(d)} & \xrightarrow{f_3(d)} \xi_4(d)|_{\mathbb{C}P^4} \\
& \xrightarrow{h} \\
\nu_{X_4(d') \Sigma} & \xrightarrow{f_3(d') \circ f_2} \xi_4(d')|_{\mathbb{C}P^4}
\end{align*}
\]

It follows that the degree-\( d \) normal maps

\[
(f_3(d), f_3(d)): X_4(d) \rightarrow \mathbb{C}P^4 \quad \text{and} \quad (f_3(d') \circ f_2, \alpha \circ (f_3(d') \circ f_2)): X_4(d') \rightarrow \mathbb{C}P^4
\]
are normally bordant and so \( \eta([f_4(d), f_4(d)]') = \eta([f_4(d), f_4(d), s_f, c] \circ \eta([f_4(d), f_4(d[D)]) = \eta([f_4(d), f_4(d), s_f, c] \circ \eta([f_4(d), f_4(d[D)]) \). Now by Theorem 5.16 we have \( \eta_4(d) = \eta([f_4(d), f_4(d)]') \) and \( \eta_4(d') = \eta([f_4(d'), f_4(d')]) \) and Equation (5) applies to give 
\[
\eta_4(d) = \eta_4(d') + c^*(i_d, \eta([f_4(d), f_4(d)])),
\]
where \( c : CP^4 \to S^8 \) is the degree-1 collapse map and \( \eta_4([f_4(d), f_4(d)]') = \pi_4(QS^n_0) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2 \). Since the normal invariants \( \eta_4(d) \) and \( \eta_4(d') \) differ by the action of a 2-torsion element of \( \pi_4(QS^n_0/O) \) and we assumed that \( \eta_4(d), \eta_4(d') \) \( \in [CP^4, ([Q^n_0/O]_d)] \), it follows that \( \eta_4(d) = \eta_4(d') \).  

From the Brunfied-Madsen equivalence \( (G/O)[1/d] \simeq ([Q^n_0/O]_d[1/d]) \) of Proposition 5.10 we deduce the existence of a homotopy equivalence 
\[
\chi : ([Q^n_0/O]_d) \simeq (G/O).
\]
Moreover, by Sullivan’s 2-primary splitting theorem for \( G/O \) [M-M Theorem 5.18], there is a homotopy equivalence 
\[
\phi : (G/O) \simeq (BSO) \times \text{coker}(J),
\]
where the space \( \text{coker}(J) \) is defined in [M-M Definition 5.16]. From the splitting of \( (G/O) \) in (6) we obtain a projection map 
\[
\pi : (G/O) \to \text{coker}(J).
\]

The result of Theorem 5.24 is contained in Feshbach’s proof of [Fe1, Theorem 6], where the arguments rely on work of Feshbach and Ravenel’s on the Segal Conjecture [522 [R].

**Theorem 5.24 (c.f. [Fe1, Proof of Theorem 6]).** For any prime \( p \), \( [CP^\infty, \text{coker}(J)]_{(p)} \nsimeq 0 \).

**Proof.** The proof of [Fe1, Theorem 6] states that the stable cohomotopy group \( \pi_0^s(CP^\infty) \) is trivial. Now \( \pi_0^s(CP^\infty) = \pi_0^s(QS^0) \) and since \( QS^0_0 \) is connected with homotopy groups \( \pi_i(QS^0_0) = \pi_i^s \) which are finite, there is a weak equivalence \( QS^0_0 \sim \prod p \pi(QS^0_0) \) from \( QS^0_0 \) to the product of its \( p \)-localisations, taken over all primes \( p \). Now by Sullivan’s splitting of \( QS^0_1 \simeq QS^0_0, [M-M \text{ Theorem 5.18}], (Q^n_0)_{(p)} \simeq \text{im}_p \times \text{coker}_p \) for a certain \( p \)-local space \( \text{im}_p \). Therefore 
\[
0 \nsimeq [CP^\infty, QS^0_0] \simeq \prod_p ([CP^\infty, \text{im}_p] \times [CP^\infty, \text{coker}_p])
\]
and the theorem follows.

As a consequence of Theorem 5.24 we have

**Lemma 5.25.** If \( d \) is odd then \( \pi_4(\eta_4(d)) = 0 \in [CP^n, ([Q^n_0/O]_d)] \) for all \( n \).

**Proof.** Let \( i : CP^n \to CP^\infty \) be the inclusion and consider the following commutative diagram:
\[
\begin{array}{ccc}
[CP^\infty, ([Q^n_0/O]_d)] & \longrightarrow & [CP^\infty, ([Q^n_0/O]_d)] \\
\eta_4 \downarrow & & \eta_4 \downarrow \\
[CP^n, ([Q^n_0/O]_d)] & \longrightarrow & [CP^n, ([Q^n_0/O]_d)]
\end{array}
\]
Now \( \eta_4(d) = i^*(\eta_4(d)) \) by Definition 5.9 and \( [CP^\infty, \text{coker}(J)]_{(p)} \nsimeq 0 \) by Theorem 5.24 so the lemma follows from the commutativity of the diagram.

**Theorem 5.26.** Let \( X_4(d) \) and \( X_4(d') \) be complete intersections with \( SD_4(d) = SD_4(d') \), where \( d = d' \) is odd. Then \( \eta_4(d) = \eta_4(d') \in [CP^4, ([Q^n_0/O]_d)] \).

The Sullivan Conjecture for \( n = 4 \) and odd total degree follows directly from Theorems 5.17 and 5.26.

**Theorem 5.27.** Let \( X_4(d) \) and \( X_4(d') \) be complete intersections with \( SD_4(d) = SD_4(d') \) and odd total degree. Then \( X_4(d) \) is diffeomorphic to \( X_4(d') \).
Proof of Theorem 5.26. By Lemma 5.23 it is enough to prove that \( \eta_4(d) = \eta_4(d^\prime) \in [\mathbb{C}P^4, ((Q^S/\gamma)_{d})_{(2)}] \).

The equivalence
\[
\phi \circ \chi : (Q^S/\gamma)_{d})_{(2)} \simeq (G/O)_{(2)} \simeq (BSO)_{(2)} \times \text{coker}(J)_{(2)},
\]
leads to the bijection
\[
\phi_* \circ \chi_* : [\mathbb{C}P^4, ((Q^S/\gamma)_{d})_{(2)}] \equiv [\mathbb{C}P^4, (BSO)_{(2)}] \times [\mathbb{C}P^4, \text{coker}(J)_{(2)}].
\]

Since \( \chi \) is a homotopy equivalence, it suffices to show that \( \chi_* (\eta_4(d)) = \chi_* (\eta_4(d^\prime)) \) and to simplify the notation we set
\[
\hat{\eta}(d) := \chi_* (\eta_4(d)) \quad \text{and} \quad \hat{\eta}(d^\prime) := \chi_* (\eta_4(d^\prime)).
\]

Lemma 5.25 states that
\[
\pi_*(\hat{\eta}(d)) = 0 = \pi_*(\hat{\eta}(d^\prime)) \in [\mathbb{C}P^4, \text{coker}(J)_{(2)}]
\]
and it remains to consider the \([\mathbb{C}P^4, (BSO)_{(2)}] \) factor. Let
\[
\rho : (G/O)_{(2)} \rightarrow (BSO)_{(2)} \quad \text{and} \quad \sigma : (BSO)_{(2)} \rightarrow (G/O)_{(2)}
\]
be respectively the projection and inclusion defined by the Sullivan splitting of \((G/O)_{(2)}\) in [\ref{M-M}], \(\sigma\) is a solution to the Adams Conjecture, which means there is a commutative diagram
\[
\begin{array}{ccc}
(G/O)_{(2)} & \xrightarrow{\sigma} & (BSO)_{(2)} \\
\downarrow{\iota} & & \downarrow{(\psi^3 - \text{Id})} \\
(BSO)_{(2)} & \xrightarrow{\sigma \circ \iota} & (BSO)_{(2)}
\end{array}
\]

where \(\psi^3\) is the map induced by the third power Adams operation; see [M-M], 5.13 & Theorem 5.18. Now by \([\ref{M-M}]\) we have that \(\hat{\eta}(d) = (\sigma \circ \rho)_* (\hat{\eta}(d))\) and \(\hat{\eta}(d^\prime) = (\sigma \circ \rho)_* (\hat{\eta}(d^\prime))\). Let \(\iota : ((Q^S/\gamma)_{d})_{(2)} \rightarrow (BSO)_{(2)}\) and \(\iota : (G/O)_{(2)} \rightarrow (BSO)_{(2)}\) denote the canonical maps and consider the following commutative diagram:
\[
\begin{array}{ccc}
(Q^S/\gamma)_{d})_{(2)} & \xrightarrow{\iota} & (G/O)_{(2)} \\
\downarrow{\chi} & & \downarrow{\sigma} \\
(BSO)_{(2)} & \xrightarrow{\iota \circ \sigma} & (BSO)_{(2)}
\end{array}
\]

The assumption that \(SD_4(d) = SD_4(d^\prime)\) ensures \(\iota_* (\eta_4(d)) = \iota_* (\eta_4(d^\prime))\) and so \(\iota_* (\hat{\eta}(d)) = \iota_* (\hat{\eta}(d^\prime))\).

Applying \([\ref{M-M}]\) we obtain
\[
(\iota \circ \sigma)_*(\rho_* (\hat{\eta}(d))) = (\iota \circ \sigma)_*(\rho_* (\hat{\eta}(d^\prime))).
\]

Since \(\iota \circ \sigma : (BSO)_{(2)} \rightarrow (BSO)_{(2)}\) equals \(\psi^3 - \text{Id}\), it follows that \(\iota \circ \sigma\) is a rational equivalence. Since the total rational Pontryagin class
\[
p : [\mathbb{C}P^4, BSO] \rightarrow H^4([\mathbb{C}P^4, \mathbb{Q}])
\]
is injective, it follows that \((\iota \circ \sigma)_* : [\mathbb{C}P^4, (BSO)_{(2)}] \rightarrow [\mathbb{C}P^4, (BSO)_{(2)}]\) is injective. Now by \([\ref{M-M}]\) we have that \(\rho_* (\hat{\eta}(d)) = \rho_* (\hat{\eta}(d^\prime))\) and combining this with \([\ref{M-M}]\) we have that \(\hat{\eta}(d) = \hat{\eta}(d^\prime)\) as required.

Remark 5.28. The arguments of this section can be generalised to prove the Sullivan Conjecture “prime to the total degree”. We take this up in future work \([\ref{C-N}]\).

6. Appendix: Extensions and Toda Brackets

The aim of this Appendix is to prove Lemma 6.1 which concerns the role of Toda brackets in computing extensions for stable homotopy groups of two cell complexes. Lemma 6.1 is presumably well-known, but we did not find a proof for it in the literature so far.

Let \(f : S^k \rightarrow S^0\) be a stable map and let \(C_f := S^0 \cup D^{k+1}\) be the mapping cone of \(f\). The reduced stable homotopy groups of \(C_f\) lie in the following fragment of the long exact Puppe sequence:
\[
\cdots \rightarrow \pi^{s}_{j-k} \xrightarrow{f_*} \pi^{s}_{j} \xrightarrow{i_*} \pi^{s}_{j}(C_f) \xrightarrow{q_*} \pi^{s}_{j-k-1} \rightarrow \cdots
\]
Here \( f_\ast, i_\ast \) and \( c_\ast \) are respectively the homomorphisms induced by composition with \( f \), the inclusion \( i: S^i \subset C_f \), the collapse map \( c: C_f \to S^{i+k+1} \). We shall be interested in describing the extension
\[
0 \to \text{im}(i_\ast) \to \pi_j^i(C_f) \to \text{im}(c_\ast) \to 0. \tag{9}
\]
To describe the extension \( 0 \), we take an element \( g \in \pi_{j-k-1}^i(C_f) \) of order \( a \) for some positive integer \( a \), which lifts to \( \bar{g} \in \pi_j^i(C_f) \). Then \( a\bar{g} \in \text{im}(i_\ast) = \text{coker}(f_\ast) \). The element \( a\bar{g} \in \pi_{i+j}(S^i) \) will of course depend on the choice of \( \bar{g} \) in general.

To describe \( a\bar{g} \) we consider the sequence of stable maps
\[
S^{i-1} \xrightarrow{L} S^{i-k-1} \xrightarrow{\bar{S}} S^0 \xrightarrow{a} S^0,
\]
Since \( g \circ f \) and \( a \circ g \) are both null-homotopic, the Toda bracket
\[
\langle (a, g, f) \rangle \subseteq \pi_j^i\]
is defined. Representatives for the elements of \( \langle a, g, f \rangle \) are defined as unions
\[\langle a \circ H_1 \rangle \cup (C(f) \circ H_2) : C(S^{i-1}) \cup C(S^{i-1}) \to S^0,\]
where \( H_1 \) is a null-homotopy of \( g \circ f \) and \( H_2 \) is a null-homotopy of \( a \circ g \) and \( C(...) \) denotes the cone of a space or a map. The indeterminacy of \( \langle a, g, f \rangle \) arises from the choice of null-homotopies \( H_1 \) and \( H_2 \) and is given by
\[I(\langle a, g, f \rangle) = f_\ast(\pi_{j-k-1}^i) + a\pi_j^i \subseteq \pi_j^i.\]
We now relate the restriction of the extension \( 0 \) to the cyclic subgroup \( \langle \bar{g} \rangle \subseteq \pi_{j-k-1}^i \) generated by \( g \) to the Toda bracket \( \langle a, f, g \rangle \).

**Lemma 6.1.** Suppose that \( g \in \pi_{j-k-1}^i \) has order \( a \) and that \( \bar{g} : S^i \to C_f \) is a stable map such that \( c \circ \bar{g} = g \). Then
\[a\bar{g} \in \text{im}(i_\ast) \subseteq \pi_j^i(C_f).
\]
In particular, the extension
\[0 \to \text{im}(i_\ast) \to (c_\ast)^{-1}(\langle \bar{g} \rangle) \to \langle g \rangle \to 0\]
is trivial if and only if \( 0 \in \langle a, g, f \rangle \).

**Proof.** Given \( \bar{H}_1 : C(S^{i-1}) \to S^0 \), a null-homotopy of \( g \circ f : S^{i-1} \to S^0 \), we define a choice of \( \bar{g} \in \pi_j^i(C_f) \) by
\[\bar{g} = H_1 \cup C(\bar{g}) : C(S^{i-1})_1 \cup C(S^{i-1})_2 \to S^0 \cup D^{k+1},\]
where we take a representative \( g : S^i \to S^k \). Since we are in the stable category, there is an \( a \)-fold fold map \( a\bar{C}_f : (C_f, S^0) \to (C_f, S^0) \), which extends \( a : S^0 \to S^0 \) and we have \( a\bar{g} = a\bar{C}_f \circ \bar{g} \). On the first copy of \( C(S^{i-1}) \) we have \( (a\bar{C}_f \circ \bar{g})|_{C(S^{i-1})} = a \circ \bar{H}_1 \). On the second copy of \( C(S^{i-1}) \), the map \( (a\bar{C}_f \circ \bar{g})|_{C(S^{i-1})} \) defines the zero element of \( \pi_j^i(C_f, S^0) \cong \pi_{j-1}^i(S^0) \). It follows that \( (a\bar{C}_f \circ \bar{g})|_{C(S^{i-1})} \) is homotopic rel. \( S^{i-1} \) to \( H_2 \circ C(f) \), where \( H_2 : C(S^{i-k-1}) \to S^0 \) is a null-homotopy of \( a\bar{g} \). It follows that \( a\bar{g} = a\bar{C}_f \circ \bar{g} \) is homotopic to \( i \circ (a \circ \bar{H}_1 \cup H_2 \circ C(f)) \) and so \( a\bar{g} \in \text{im}(\langle a, g, f \rangle) \) as required.

Finally, the extension
\[0 \to \text{im}(i_\ast) \to (c_\ast)^{-1}(\langle \bar{g} \rangle) \to \langle g \rangle \to 0\]
is trivial if and only if there is \( \bar{g} \in \pi_j^i(C_f) \) such that \( a\bar{g} = 0 \). Given such a \( \bar{g} \), then \( 0 \in \text{im}(i_\ast) \) by the previous paragraph and so \( \langle a, g, f \rangle \)-contain an element of \( \ker(i_\ast) = f_\ast(\pi_{j-k-1}^i) \). Hence \( \langle a, g, f \rangle \cap I(\langle a, g, f \rangle) \neq 0 \) and so \( 0 \in \langle a, g, f \rangle \). Conversely, if \( 0 \in \langle a, g, f \rangle \) if and only if \( \langle a, g, f \rangle = f_\ast(\pi_{j-k-1}^i) + a\pi_j^i \) and then \( a\bar{g} \in \text{im}(i_\ast) \) by \( a\bar{g} = a\pi_j^i \). Hence we can modify our choice of \( \bar{g} \) to achieve \( a\bar{g} = 0 \).

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