Palatini-Born-Infeld Gravity and Black Hole Formation

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We consider the Palatini formulation of the Born-Infeld gravity. In the Palatini formulation, the propagating mode is only graviton, whose situation is different from that in the metric formulation. We discuss about the FRW cosmology and the black hole formation by using an effective potential. Especially we consider the condition that the bouncing could occur.

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I. INTRODUCTION

Motivated with the accelerating expansion of the present universe, we are considering many kinds of gravity theories beyond the Einstein gravity (for review, see [1]). In this paper, we consider about the Born-Infeld gravity [2] in the Palatini formulation [3] and cosmology has been considered [4]. In the metric formulation of the Born-Infeld gravity, theory includes a ghost in general and we need to tune the action so that the ghost does not appear [2]. In the Palatini formulation, however, as we explicitly show that there does not appear ghost [3] and only propagating mode is massless graviton. Even in the Palatini-Born-Infeld gravity, the Schwarzschild and Kerr black hole space-time are exact solutions. Then we calculate the entropy of the Schwarzschild black hole and we find the entropy is not changed from that in the Einstein gravity. We consider the FRW cosmology by including dust as a matter and show that the bouncing universe can be realized, whose behavior is, in some sense, similar to the loop quantum gravity [5–7].

We also consider the formation of the black hole by considering the collapse of the sphere of the dust. We should note that in the previous works, there were too strong constrains on the variables but we consider more general treatment in this paper. Because the pressure of the dust vanishes, we can regard the inside of the sphere as the FRW universe. Then by using the results in the FRW universe, we show that the small black hole could not be formed by the bouncing although the large black holes might be created.

In the next section, we show that the only propagating mode is massless graviton. In Section III, we calculate the entropy of the Schwarzschild black hole. In Section IV, we consider the FRW cosmology. After that in Section V, we investigate the formation of the black hole. The last section is devoted to the summary.

II. ABSENCE OF GHOST

In this section, we show that any ghost does not appear in the Palatini formulation of the Born-Infeld gravity [3, 4]. The action is given by

\[
S = \frac{1}{\kappa^2 b} \int d^4x \left\{ \sqrt{\left| \det (g_{\mu\nu} + b R_{\mu\nu}) \right|} - \sqrt{\left| \det (g_{\mu\nu}) \right|} \right\} + S_{\text{matter}} . \tag{1}
\]

Here \( S_{\text{matter}} \) is an action for the matters and \( R_{\mu\nu} \) is given by \( R_{\mu\nu} = -\Gamma^\rho_{\mu\lambda\nu} + \Gamma^\rho_{\mu\nu,\lambda} - \Gamma^\eta_{\mu\nu} \Gamma^\rho_{\eta\lambda} + \Gamma^\eta_{\mu\rho} \Gamma^\rho_{\eta\nu} \) and we regard the connection \( \Gamma^\rho_{\mu\nu} \) is a variable independent from the metric \( g_{\mu\nu} \). In the Palatini formulation, the propagating mode is only graviton, whose situation is different from that in the metric formulation [2].

In [2], the following action was investigated:

\[
S = \frac{1}{\kappa^2 b} \int d^4x \left\{ \sqrt{\left| \det (g_{\mu\nu} + b R_{\mu\nu} + c X_{\mu\nu}) \right|} - \sqrt{\left| \det (g_{\mu\nu}) \right|} \right\} . \tag{2}
\]

Here \( X_{\mu\nu} \) is rank two tensor, which is the sum of the products of curvatures and \( R_{\mu\nu} \) is defined by the metric tensor \( g_{\mu\nu} \) as in the Einstein gravity. In [2], Deser and Gibbons have shown that \( X_{\mu\nu} \) cannot be arbitrary but constrained for the consistency. It is well known that the model including higher derivative terms generates ghost in general. The ghost appears from the terms which is the sum of the products of two curvature like \( \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + \alpha_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \). What Deser and Gibbons have shown is that we can choose \( X_{\mu\nu} \) to avoid the ghost but \( X_{\mu\nu} \) is not uniquely determined. An important point is that if \( X_{\mu\nu} = 0 \), there always appears a ghost.

We now show that the action (1), where the curvature is given in terms of the connection which can be regarded as a variable independent of the metric, does not generate ghost. By the variations of the metric \( g_{\mu\nu} \) and the connection
\[ \Gamma^\lambda_{\mu\nu} \], we obtain the following equations, respectively,
\[ 0 = \sqrt{-P} (P^{-1})^{\mu\nu} - \sqrt{-g} g^{\mu\nu}, \tag{3} \]
\[ 0 = \nabla_\lambda \left( \sqrt{-P} (P^{-1})^{\mu\nu} \right) = 0. \tag{4} \]
Here \( P_{\mu\nu} \) is defined by
\[ P_{\mu\nu} \equiv g_{\mu\nu} + b R_{\mu\nu}. \tag{5} \]
It is clear that the Minkowski space-time is a solution of Eqs. (3) and (4).

Eqs. (3) and (4) tell that the connection is given by that in the Einstein gravity, which we show now. By multiplying Eq. (3) with \( \nabla_\lambda \) and using (4), we obtain the following equation,
\[ 0 = \nabla_\lambda \left( \sqrt{-g} g^{\mu\nu} \right), \tag{6} \]
which can be solved with respect of \( \Gamma^\lambda_{\mu\nu} \) as follows,
\[ \Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left( \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} \right), \tag{7} \]
which is identical with the expression in the Einstein gravity.

It is clear that the Minkowski space-time is the solution of Eqs. (3) and (4). We now consider the perturbation from the Minkowski background:
\[ g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}. \tag{8} \]
By using (8) and keeping only the terms linear to \( h_{\mu\nu} \), we obtain
\[ \Gamma^\lambda_{\mu\nu} = \frac{\epsilon}{2} \left( \partial_\mu h^\lambda_{\nu\lambda} + \partial_\nu h^\lambda_{\mu\lambda} - \partial^\lambda h_{\mu\nu} \right), \tag{9} \]
\[ R_{\mu\nu} = \frac{\epsilon}{2} \left( \partial_\nu \partial_\mu h^\lambda_{\lambda\lambda} - \partial_\lambda \partial_\mu h^\lambda_{\nu\lambda} - \partial_\lambda \partial_\nu h^\lambda_{\mu\lambda} + \partial^\lambda \partial_\lambda h_{\mu\nu} \right). \tag{10} \]
Then we obtain
\[ P_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu} + \frac{b\epsilon}{2} \left( \partial_\nu \partial_\mu h^\lambda_{\lambda\lambda} - \partial_\lambda \partial_\mu h^\lambda_{\nu\lambda} - \partial_\lambda \partial_\nu h^\lambda_{\mu\lambda} + \partial^\lambda \partial_\lambda h_{\mu\nu} \right), \]
and
\[ (P^{-1})^{\mu\nu} = \eta^{\mu\nu} - \epsilon h^{\mu\nu} - \frac{b\epsilon}{2} \left( \partial^\nu \partial^\mu h^\lambda_{\lambda\lambda} - \partial_\lambda \partial^\mu h^\lambda_{\nu\lambda} - \partial_\lambda \partial^\nu h^\lambda_{\mu\lambda} + \partial^\lambda \partial_\lambda h_{\mu\nu} \right), \tag{11} \]
\[ \sqrt{-\det P} = 1 + \frac{\epsilon}{2} \text{tr} \left( h^\mu_{\nu} + \frac{b\epsilon}{2} \left( \partial_\nu \partial^\mu h^\lambda_{\lambda\lambda} - \partial_\lambda \partial^\mu h^\lambda_{\nu\lambda} - \partial_\lambda \partial^\nu h^\lambda_{\mu\lambda} + \partial^\lambda \partial_\lambda h_{\mu\nu} \right) \right). \tag{12} \]
By substituting Eqs. (11) and (12) into (3) and keeping only linear terms, we find
\[ 0 = \frac{b\epsilon}{2} \left( \partial_\nu \partial_\mu h^\lambda_{\lambda\lambda} - \partial_\lambda \partial_\mu h^\lambda_{\nu\lambda} - \partial_\lambda \partial_\nu h^\lambda_{\mu\lambda} + \partial^\lambda \partial_\lambda h_{\mu\nu} \right), \tag{13} \]
which is identical with the equation for the graviton in the Einstein gravity. Therefore the only propagating mode is graviton and any other propagating mode like ghost does not appear, whose situation is different from that in the metric formulation.

III. BLACK HOLE ENTROPY

Eqs. (3) and (4) tells that the vacuum solutions of the Einstein gravity are also exact solution of the Palatini-Born-Infeld gravity. Especially the Schwarzschild and Kerr space-time are solutions. Then we may consider the entropy of the Schwarzschild black hole. The prescription to obtain the entropy is given in, for example, [2].

For the technical reasons, instead of (4), we consider the following action,
\[ S = \frac{1}{\kappa^2 b} \int d^4 x \left\{ \sqrt{|\det (g_{\mu\nu} + b R_{\mu\nu})|} - \lambda \sqrt{|\det (g_{\mu\nu})|} \right\} + S_{\text{matter}}. \tag{14} \]
When $\lambda = 1$, the above action reduces to that in [1] and if $\lambda < 1$ (when $b > 0$), the Schwarzschild-anti-de Sitter space-time is an exact solution,

$$ds^2 = -e^{2\rho_0}dt^2 + e^{-2\rho_0}dr^2 + r^2 \sum_{i,j=1,2} \tilde{g}_{ij}dx_i dx_j, \quad e^{2\rho_0} = \frac{1}{r} \left(-\mu + r + \frac{r^3}{l^2}\right).$$  \hspace{1cm} (15)

Here $\tilde{g}_{ij}$ is the metric of two dimensional sphere and $M = \frac{\mu}{16\pi \kappa^2}$ is the mass of the black hole. The parameter $l$ is the length parameter of the anti-de Sitter space-time and given by

$$l^2 = \frac{3b}{1-\lambda}. \hspace{1cm} (16)$$

By Wick rotating the signature to the Euclidean one and substituting the solution (15), we now evaluate the action (15):

$$S = \frac{4\pi}{\kappa^2 b T} \int_{r_H}^{r_{\infty}} d\eta r^2 \left(-\frac{3b}{l^2} + \frac{9b^2}{l^4}\right). \hspace{1cm} (17)$$

Here $T$ is the Hawking temperature given by

$$T = \frac{3\mu - 2r_H}{4\pi r_H^2}, \hspace{1cm} (18)$$

and $r_H$ is the radius of the horizon given by

$$0 = -\mu + r_H + \frac{r_H^3}{l^2}. \hspace{1cm} (19)$$

We introduce $r_\infty$ in order to regularize the expression (17), which diverges when $r_\infty \to \infty$. The divergence can be renormalized by subtracting the contribution from the background without the black hole:

$$S_{\text{reg}} = \frac{4\pi}{\kappa^2 b T} \left(-\frac{3b}{l^2} + \frac{9b^2}{l^4}\right) \left(\int_{r_H}^{r_\infty} d\eta r^2 - e^{\phi(\mu=0) - \rho(\mu \neq 0)}\bigg|_{r=r_\infty} \int_0^{r_\infty} d\eta r^2\right). \hspace{1cm} (20)$$

The factor $e^{\phi(\mu=0) - \rho(\mu \neq 0)}$ is introduced so that the periodicity of the Euclidean time in the background coincides with the periodicity of the Euclidean black hole space-time. Then by taking the limit of $r_\infty \to \infty$, we find

$$S_{\text{reg}} = \frac{4\pi}{\kappa^2 b T} \left(-\frac{3b}{l^2} + \frac{9b^2}{l^4}\right) \left(-\frac{r_H^3}{3} - \frac{l^2 \mu}{6}\right). \hspace{1cm} (21)$$

We identify the free energy $F$ by $F = TS$. Then by using (19) etc., we find

$$F = \frac{4\pi}{\kappa^2 b} \left(-\frac{3b}{l^2} + \frac{9b^2}{l^4}\right) \left(-\frac{l^2 r_H}{6} - \frac{r_H^3}{2}\right), \quad T = \frac{1}{4\pi r_H^2} \left(r_H + \frac{3r_H^3}{l^2}\right). \hspace{1cm} (22)$$

By considering the limit $\lambda \to 1$, that is, $l \to \infty$, we find

$$F = \frac{2\pi r_H}{\kappa^2}, \quad T = \frac{1}{4\pi r_H^2}, \hspace{1cm} (23)$$

that is,

$$F = \frac{2}{\kappa^2 T}. \hspace{1cm} (24)$$

Therefore the entropy $S$ is given by

$$S = -\frac{\partial F}{\partial T} = \frac{8\pi^2 r_H^2}{\kappa^2} = \frac{2\pi A}{\kappa^2} = \frac{A}{4G}. \hspace{1cm} (25)$$

Here $A$ is the area of horizon $A = 4\pi r_H^2$ and $\kappa^2 = 8\pi G$. Then the entropy is not changed from that in the Einstein gravity. Especially, we should note that the entropy does not depend on the parameter $b$. 
IV. FRW UNIVERSE WITH DUST

We consider the FRW space-time with flat spatial part:

\[ ds^2 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2 , \]  

(26)

and assume that the non-vanishing components of the connection is given by

\[ \Gamma^a_{i0} = A(t) , \quad \Gamma^a_{ij} = a(t)^2 B(t) \delta_{ij} , \quad \Gamma^a_{jt} = \Gamma^a_{ij} = C(t) \delta^a_j . \]  

(27)

In the Einstein gravity, we have \( A = 0 , \quad B = C = H \equiv \dot{a}/a . \) Then the Ricci tensors are given by

\[ R_{tt} = -3 \left( \dot{C} + C^2 - AC \right) , \quad R_{ij} = a^2 \left( \ddot{B} + 2HB + BC + BA \right) \delta_{ij} , \quad R_{ti} = R_{it} = 0 , \]  

(28)

and the matter is given by the dust whose pressure \( p \) vanishes and the energy density is denoted by \( \rho . \) Then we obtain the following equations:

\[ b \dot{r}^2 \rho = \left\{ 1 + b \left( \dot{B} + 2HB + BC + BA \right) \right\}^{3/2} \left\{ 1 + 3b \left( \dot{C} + C^2 - AC \right) \right\}^{-\frac{1}{2}} - 1 , \]  

(29)

\[ 0 = \left\{ 1 + b \left( \dot{B} + 2HB + BC + BA \right) \right\}^{1/2} \left\{ 1 + 3b \left( \dot{C} + C^2 - AC \right) \right\}^{\frac{1}{2}} - 1 , \]  

(30)

\[ \Gamma^i_{tt} = \frac{1}{2} \frac{d}{dt} \left\{ \ln \left\{ 1 + 3b \left( \dot{C} + C^2 - AC \right) \right\} \right\} , \]  

(31)

\[ \Gamma^i_{ij} = \frac{a(t)^2}{2} \left\{ 1 + 3b \left( \dot{C} + C^2 - AC \right) \right\}^{-1} \right. \right. \times \left\{ 2H + b \left\{ 4H \dot{B} + 4H^2 B + 2HBC + 2HBA + \dot{B} + 2\dot{H}B + \dot{B}(C + A) + B \left( \dot{C} + \dot{A} \right) \right\} \right\} \delta_{ij} , \]  

(32)

\[ \Gamma^i_{jt} = \Gamma^i_{ij} = \frac{1}{2} \frac{d}{dt} \left\{ \ln \left\{ a^2 + ba^2 \left( \dot{B} + 2HB + BC + BA \right) \right\} \right\} \delta^i_j . \]  

(33)

By using (27), (30), (31), and (33), we find

\[ C + A = H . \]  

(34)

We may delete \( A \) by using (34) and obtain

\[ b \dot{r}^2 \rho = \left\{ 1 + b \left( \dot{B} + 3HB \right) \right\}^{3/2} \left\{ 1 + 3b \left( \dot{C} + 2C^2 - CH \right) \right\}^{-\frac{1}{2}} - 1 , \]  

(35)

\[ 0 = \left\{ 1 + b \left( \dot{B} + 3HB \right) \right\}^{1/2} \left\{ 1 + 3b \left( \dot{C} + 2C^2 - CH \right) \right\}^{\frac{1}{2}} - 1 , \]  

(36)

\[ -C + H = \frac{1}{2} \frac{d}{dt} \left\{ \ln \left\{ 1 + 3b \left( \dot{C} + 2C^2 - CH \right) \right\} \right\} , \]  

(37)

\[ B = \left\{ 1 + 3b \left( \dot{C} + 2C^2 - CH \right) \right\}^{-1} \right. \right. \times \left\{ 2H + b \left\{ 5H \dot{B} + 6H^2 B + 3\dot{H}B + \dot{B} \right\} \right\} . \]  

(38)

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1 In the previous works, the FRW metric was assumed for \( P_{\mu \nu} \) in (1):

\[ ds^2 = \sum_{\mu, \nu = 0}^4 = -dt^2 + a(t)^2 \sum_{i=1,2,3} (dx^i)^2 , \]  

and the connection \( \Gamma^\lambda_{\mu \nu} \) is given by \( P_{\mu \nu} \):

\[ \Gamma^\lambda_{\mu \nu} = \frac{1}{2} \left( P^{-1} \right)^{\lambda \rho} (\partial^\rho P_{\mu \nu} + \partial^\mu P_{\rho \nu} - \partial^\nu P_{\mu \rho}) , \]  

which, however, reduces the degrees of freedom in \( \Gamma^\lambda_{\mu \nu} \) so that \( A = 0 \) and \( B = C . \)
When \( b < 0 \), Eq. (29) or (35) tells that there is a maximum \( \rho_{\text{max}} \):

\[
\rho_{\text{max}} = \frac{1}{b\kappa^2}.
\]

(39)

When we consider the shrinking universe where \( H < 0 \), does the energy density \( \rho \) go to the maximum \( \rho_{\text{max}} \) asymptotically or bounces at \( \rho = \rho_{\text{max}} \)? If the energy density \( \rho \) goes to the maximum \( \rho_{\text{max}} \) asymptotically, there should be a static solution where \( H = 0 \) and \( B \) and \( C \) are constant. If we assume that \( H = 0 \) and \( B \) and \( C \) are constant, however, Eqs. (37) and (38) tells \( B = C = 0 \) and therefore Eq. (35) tells \( \rho = 0 \), which contradicts with the assumption. Therefore the energy density \( \rho \) does not go to the maximum \( \rho_{\text{max}} \) asymptotically.

Furthermore by using (35) and (36), we obtain

\[
bn^2 \rho = \left\{ 1 + b \left( \dot{B} + 3HB \right) \right\}^2 - 1.
\]

(40)

Then when \( \rho = \rho_{\text{max}} \), we find \( 1 + b \left( \dot{B} + 3HB \right) = 0 \), which tell that \( \dot{C} + 2C^2 - CH \) diverges positively, due to Eq. (35) and therefore \( R_{tt} \) diverge negatively, \( R_{tt} = -3 \left( \dot{C} + 2C^2 - CH \right) \to -\infty \). Here we used (38). Therefore \( \rho \) could not reach \( \rho_{\text{max}} \).

We now delete \( B \) and \( C \) in (35), (36), (37), (38) and obtain a single equation with respect to the scale factor \( a \). We now assume

\[
\rho = \rho_0 a^{-3}.
\]

(41)

which can be obtained from the conservation law

\[
\dot{\rho} + 3H (\rho + p) = 0.
\]

(42)

Then by combining (36) and (38), we obtain

\[
4a^4 B = \frac{d}{dt} \left[ a^4 \left\{ 1 + b \left( \dot{B} + 3HB \right) \right\}^2 \right].
\]

(43)

Furthermore by combining (40) and (43), we find

\[
B = H + \frac{bn^2}{4} H \rho.
\]

(44)

On the other hand, Eqs. (35) and (36) give

\[
bn^2 \rho = \left\{ 1 + 3b \left( \dot{C} + 2C^2 - CH \right) \right\}^{-2} - 1.
\]

(45)

By using (37) and (35), we obtain

\[
C = H - \frac{3}{4} bn^2 H \rho \left( 1 + bn^2 \rho \right)^{-1}.
\]

(46)

A single equation with respect to the scale factor \( a \) can be obtained by deleting \( B \) in (40) by using (44) as follows,

\[
bn^2 \rho = \left\{ 1 + b \left( \dot{H} + 3H^2 + \frac{bn^2}{2} \dot{H} \rho \right) \right\}^2 - 1.
\]

(47)

Because \( H = \dot{a}/a \) and \( \dot{H} = \ddot{a}/a - \dot{a}^2/a^2 \), by using (41) Eq. (47) can be rewritten as

\[
bn^2 \rho_0 a^{-3} = \left\{ 1 + b \left( \frac{\ddot{a}}{a} + 2 \left( \frac{\dot{a}}{a} \right)^2 + \frac{bn^2}{4} \left( \frac{\dot{a}}{a} \right)^2 \right) \rho_0 a^{-3} \right\}^2 - 1,
\]

(48)

which is a single equation with respect to the scale factor \( a \). If we use the e-foldings \( N \) defined by \( a = e^N \), we obtain

\[
bn^2 \rho_0 e^{-3N} = \left\{ 1 + b \left( \ddot{N} + 3\dot{N}^2 + \frac{bn^2}{4} \dot{N} \rho_0 e^{-3N} \right) \right\}^2 - 1,
\]

(49)
or

\[ \dot{N} = -\frac{3N^2}{1 + \frac{b\kappa^2\rho_0}{4} e^{-3N}} \cdot \frac{1 - \sqrt{1 + b\kappa^2\rho_0 e^{-3N}}}{b \left(1 + \frac{b\kappa^2\rho_0}{4} e^{-3N}\right)}. \]  

(50)

By an analogy with the Newton equation in the classical mechanics, the first term in the r.h.s. may be regarded as a drag force and the second term could be a force by a potential, which we denote by \( F(N) \) as

\[ F(N) \equiv -\frac{1 - \sqrt{1 + b\kappa^2\rho_0 e^{-3N}}}{b \left(1 + \frac{b\kappa^2\rho_0}{4} e^{-3N}\right)}. \]  

(51)

We should note that the potential force \( F(N) \) is positive, which does not depend on the sign of the parameter \( b \) and therefore the force act so that the e-foldings \( N \) increases. Then even if the universe is shrinking, it turns to expand. Since

\[ F'(N) \equiv \frac{3\kappa^2\rho_0 e^{-3N} \left(1 - \frac{b\kappa^2\rho_0}{2} e^{-3N} + \sqrt{1 + b\kappa^2\rho_0 e^{-3N}}\right)}{4 \left(1 + \frac{b\kappa^2\rho_0}{4} e^{-3N}\right)^2 \sqrt{1 + b\kappa^2\rho_0 e^{-3N}}}. \]  

(52)

Therefore when \( b > 0 \), there is a maximum \( F(N) = 2/3b \) at \( b\kappa^2\rho_0 e^{-3N} = 8 \) and when \( b < 0 \), we find \( F'(N) < 0 \) and therefore there is a maximum \( F(N) = -4/3b \) at \( b\kappa^2\rho_0 e^{-3N} = 1 \). We should also note that \( F(N) \to 0 \) when \( N \to +\infty \) independently to the sign of \( b \) and if \( b > 0 \), \( F(N) \to 0 \) when \( N \to -\infty \).

Eq. (50) can be further rewritten in the following form:

\[ 0 = \frac{d^2}{dt^2} \left( \frac{e^{3N}}{3} + \frac{b\kappa^2\rho_0}{4} N \right) + \frac{e^{3N} \left(1 - \sqrt{1 + b\kappa^2\rho_0 e^{-3N}}\right)}{b}, \]  

(53)

which tells that there is a conserved quantity \( E \),

\[ E = \frac{1}{2} \left\{ \frac{d}{dt} \left( \frac{e^{3N}}{3} + \frac{b\kappa^2\rho_0}{4} N \right) \right\}^2 + \int_{N_0}^{N} \frac{e^{3N} \left(1 - \sqrt{1 + b\kappa^2\rho_0 e^{-3N}}\right) \left(e^{3N} + \frac{b\kappa^2\rho_0}{4}\right)}{b} dN, \]  

(54)

which corresponds to the total energy in the classical mechanics. Here \( V(N) \) is given by

\[ V(N) = \frac{e^{6N}}{6b} \left(1 + \frac{1}{2} b\kappa^2\rho_0 e^{-3N}\right) \left(1 - \sqrt{1 + b\kappa^2\rho_0 e^{-3N}}\right) - \frac{b\kappa^2\rho_0}{12b} e^{3N} \sqrt{1 + b\kappa^2\rho_0 e^{-3N}}. \]  

(55)

When \( N \) is positive and large, \( V(N) \) behaves as

\[ V(N) \sim -\frac{\kappa^2\rho_0 e^{3N}}{6}. \]  

(56)

In case \( b > 0 \), when \( N \) is negative and large, we find

\[ V(N) \sim -\left(\frac{b\kappa^2\rho_0}{12b}\right)^{\frac{3}{2}} \frac{e^{3N}}{6}. \]  

(57)

On the other hand, in case \( b < 0 \), there is a maximum in \( V(N) \) when \( 1 + b\kappa^2\rho_0 e^{-3N} = 0 \):

\[ V(N) = V_{\text{max}} \equiv \frac{(b\kappa^2\rho_0)^2}{12b} < 0. \]  

(58)

We now assume that the universe may have started from \( N \to +\infty \) and after that they have started to shrink. Then from the above results, by the analogy with the classical mechanics, we find the followings:
• In case $b > 0$, if $E < 0$, the shrinking of the universe will stop and turn to expand. On the other hand if $E > 0$, the universe will continue to shrink and the scale factor $a$ vanishes in the infinite future.

• In case $b > 0$, if $E < V_{\text{max}}$, the shrinking of the universe will stop and turn to expand. On the other hand if $E > V_{\text{max}}$, the universe will reach the singular point at $1 + b\kappa^2 \rho_0 e^{-3N} = 0$.

In order to estimate $E$, we now solve (50) by assuming $N \gg 1$. Then (50) can be rewritten as

$$\dot{N} + 3N^2 - \frac{\kappa^2 \rho_0 e^{-3N}}{2} = \frac{3}{4} b\kappa^2 \rho_0 e^{-3N} N^2 - \left(\frac{b\kappa^2 \rho_0}{4b}\right)^2 e^{-6N} + \mathcal{O} \left( b^2 \right).$$

Then in the limit $b \to 0$, we find

$$N = \frac{2}{3} \ln \left| \frac{t}{t_0} \right|, \quad t_0^2 \equiv \frac{4}{3\kappa^2 \rho_0}.$$

Then for the finite $b$, by writing $N = \frac{2}{3} \ln \frac{t}{t_0} + \delta N$ and by using (50), we find

$$\delta \ddot{N} + 4 \frac{\dot{t}}{t} \delta \dot{N} + \frac{1}{t} \delta N = \mathcal{O} \left( b^2 \right) + \mathcal{O} \left( (\delta N)^2 \right).$$

Then we find

$$\delta N = C_+ \left| t \right|^{-\frac{3}{2}} + C_- \left| t \right|^{-\frac{3}{2}} + \mathcal{O} \left( b^2 \right).$$

Here $C_{\pm}$ are arbitrary constants. Because the first and the second terms do not depend on $b$, we may put $C_+ = 0$. If we keep $C_-$, we find $E$ diverges and therefore physically not acceptable. Even if keep $C_-$, this term does not contribute to $E$. Then for the large $N$, by using the expression of $E$ in (54) with (55), we find

$$E = -\frac{\left( b\kappa^2 \rho_0 \right)^2}{16b}.$$ 

Therefore when $b > 0$, the shrinking of the universe will always stop and turn to expand, that is, we obtain the bouncing universe. On the other hand, when $b < 0$, the shrinking universe always reaches the singular point at $1 + b\kappa^2 \rho_0 e^{-3N} = 0$.

When $b > 0$, we may estimate $N$ when the shrinking universe turns to expand and therefore $V(N) = E$. When $b\kappa^2 \rho_0 \gg 1$, by using the expression of $V(N)$ in (50) and $E$ in (68), we find

$$e^{3N} \sim \frac{3}{8} b\kappa^2 \rho_0.$$ 

On the other hand, when $b\kappa^2 \rho_0 \ll 1$, by using (67), we find

$$e^{3N} \sim \frac{9}{64} b\kappa^2 \rho_0.$$ 

We should note that Eq. (54) can be identified with the first FRW equation because $H = \dot{N}$ and rewritten as

$$\frac{3}{\kappa^2} H^2 = \frac{6}{\kappa^2} e^{-6N} \left( 1 + \frac{b\kappa^2 \rho_0}{4} e^{-3N} \right)^2 \left( E - V(N) \right).$$

For large $N$, the r.h.s. in (66) can be expanded as a power series with respect to $e^{-3N}$ and we find

$$\frac{3}{\kappa^2} H^2 = \rho \left( 1 - \frac{\rho}{\rho_1} \right) + \mathcal{O} \left( e^{-9N} \right), \quad \rho = \rho_0 e^{-3N}, \quad \rho_1 \equiv \frac{2}{b\kappa^2}.$$ 

The above structure is similar to the loop quantum cosmology although the critical energy density $\rho_c$ is not given by $\rho_1$ but by using (63) or (67), we find

$$\rho_c = \begin{cases} \frac{8}{9\kappa^2} & \text{when } b\kappa^2 \rho_0 \gg 1 \\ \frac{8}{9\kappa^2} & \text{when } b\kappa^2 \rho_0 \ll 1 \end{cases}.$$ 

Therefore the obtained behavior of the bouncing is similar to that in the loop quantum gravity, there is quantitative difference.
V. BLACK HOLE FORMATION BY THE COLLAPSE OF DUST

Now we consider if black hole can be formed by the collapse of dust. We now assume there is a spherically symmetric and uniform ball made of dust and consider the collapse of ball. This assumption is valid because the pressure of the dust vanishes nor the density of ball cannot be uniform because the pressure should vanish at the boundary between the ball and bulk, which is assumed to be vacuum. Inside the ball, the space-time can be regarded with the shrinking FRW universe as in the last section. The results in the previous section tell that there could be a bouncing. If the radius of the ball at the bouncing is larger than the Schwarzschild radius, the black hole cannot be formed.

We assume the ball of dust with radius \( R \) at \( N = N_0 \). We choose \( N_0 \) to be large enough. Then the total mass \( M \) is given by

\[
M = \frac{4\pi}{3} R^3 \rho_0 e^{-3N_0} .
\]

We now consider the case that \( b > 0 \). First we assume

\[
bn^2 \rho_0 = \frac{3bn^2 M e^{3N_0}}{4\pi R^3} \gg 1 .
\]

Then by using (64), we find \( N = N_b \) at the bouncing is given by

\[
e^{3N_b} \sim \frac{9bn^2 M e^{3N_0}}{32\pi R^3} ,
\]

which give the radius \( R_b \) at the bouncing by

\[
R_b^3 = R^3 e^{3(N_b-N_0)} = \frac{9bn^2 M}{32\pi} .
\]

On the other hand, the Schwarzschild radius \( R_s \) is given by

\[
R_s = \frac{\kappa^2 M}{4\pi} .
\]

Then we find

\[
\frac{R_b^3}{R_s^3} = \frac{18\pi b}{\kappa^4 M^2} .
\]

Therefore large black hole, where \( M^2 \gg \frac{b}{\kappa^4} \), can be formed because \( R_b \ll R_s \) and therefore the bouncing can occur after the formation of the horizon.

Instead of (70), we may also consider the case

\[
bn^2 \rho_0 = \frac{3bn^2 M e^{3N_0}}{4\pi R^3} \ll 1 .
\]

Then by using (65), we find that the bouncing occurs when

\[
e^{3N} \sim e^{3N_b} \sim \frac{27bn^2 M e^{3N_0}}{256\pi R^3} ,
\]

and the radius \( \bar{R}_b \) at the bouncing is given by

\[
\bar{R}_b^3 = R^3 e^{3(N_b-N_0)} = \frac{27bn^2 M}{256\pi} ,
\]

and we obtain

\[
\frac{\bar{R}_b^3}{R_s^3} = \frac{27\pi b}{4\kappa^4 M^2} .
\]

Therefore small black hole, where \( M^2 \ll \frac{b}{\kappa^4} \), cannot be formed because \( R_b \gg R_s \).
We now consider the case that $b < 0$. In this case, there is a maximum in the energy density $\rho$ given by (39). We now assume that the black hole is formed by the collapse of the star made of the dust with radius $r$. Then the energy density $\rho$ is given by

$$\rho = \hat{\rho}_0 r^{-3}. \quad (79)$$

Here $\hat{\rho}_0$ is a constant. Then the mass $M$ and the Schwarzschild radius $R_s$ of the star is given by

$$M = \frac{4\pi}{3} \rho r^3 = \frac{4\pi}{3} \hat{\rho}_0, \quad R_s = \frac{\kappa^2 M}{4\pi} = \frac{\kappa^2 \hat{\rho}_0}{3}. \quad (80)$$

Then Eq. (80) tells that the minimum of $r$ is given by

$$r_{\text{min}} = (-3bR_s)^{\frac{1}{3}}. \quad (81)$$

The black hole cannot be formed if $r_{\text{min}} > R_s$, that is

$$R_s^2 < -3b. \quad (82)$$

Therefore small black holes may be prohibited if $b < 0$ but large ones are not prohibited. This result may tell that the creation of the primordial black holes might be prohibited.

VI. SUMMARY

We have shown that the Born-Infeld gravity in the Palatini formulation has several interesting properties, especially, the only propagating mode is massless graviton and no ghost appears. We investigated the entropy of the Schwarzschild black hole and we have shown that the entropy is identical with the entropy in the Einstein gravity. We also investigated the FRW cosmology where the matter is dust and when $b > 0$, there occurs the bouncing. The cosmology in the Palatini-Born-Infeld gravity has been investigated in several papers, but in the most of the previous works, the connections are assumed to be given by regarding $P_{\mu\nu}$ with the metric of the FRW universe but this requirement is too strong and we considered more general case. By applying the results in the FRW universe, we also investigated the collapse of the sphere of dust and the black hole formation. Then we have shown that although the large black hole might be formed but the small black holes are prohibited to be formed.

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