Nonuniversal entanglement level statistics in projection-driven quantum circuits

Lei Zhang,1 Justin A. Reyes,2 Stefanos Kourtis,1 Claudio Chamon,1 Eduardo R. Mucciolo,2 and Andrei E. Ruckenstein1

1Department of Physics, Boston University, Boston, MA 02215, USA
2Department of Physics, University of Central Florida, Orlando, FL 32816, USA

We study the level-spacing statistics in the entanglement spectrum of output states of random universal quantum circuits where qubits are subject to a finite probability of projection to the computational basis at each time step. We encounter two phase transitions with increasing projection rate: The first is the volume-to-area law transition observed in quantum circuits with projective measurements; The second separates the pure Poisson level statistics phase at large projective measurement rates from a regime of residual level repulsion in the entanglement spectrum within the area-law phase, characterized by non-universal level spacing statistics that interpolates between the Wigner-Dyson and Poisson distributions. By applying a tensor network contraction algorithm introduced in Ref. [1] to the circuit spacetime, we identify this second projective-measurement-driven transition as a percolation transition of entangled bonds. The same behavior is observed in both circuits of random two-qubit unitaries and circuits of universal gate sets, including the set implemented by Google in its Sycamore circuits.

I. INTRODUCTION

Closed quantum many-body systems undergoing unitary evolution generically reach a thermalized regime, exhibiting volume-law entanglement [2–6]. Exceptions to this scenario have drawn considerable attention, due to their relevance to experimentally controllable quantum systems. For example, many-body localization, which precludes thermalization and leads instead to area-law entanglement, has been the subject of extensive theoretical and experimental work [7–16]. Recently, failure to thermalize has also been reported in simulations of quantum circuits subjected to random measurement events that model coupling to a classical environment [17–24]. Interest in these models is fueled by the ongoing efforts to exploit noisy intermediate-scale quantum devices for tasks beyond the reach of classical computers.

The studies of Refs. [17–24] follow the time evolution of an initial product state of qubits arranged in a one-dimensional chain induced by local unitary gates randomly chosen from a volume-law entangling set (such as, e.g., the Clifford set), and subsequently measurement of each qubit with probability $p$. Intuitively, the non-unitary projective measurement operation effectively disentangle the state and, at sufficiently large measurement rate, results in a localization of the system in Hilbert space characterized by the volume-to-area law transition described in Refs. [17–24].

In this work, we study volume-law entangling unitary quantum circuits subjected to a different kind of disentangling perturbation, namely, projection operations that forcibly “reset” qubits to the computational basis, randomly inserted at a finite rate throughout the time evolution of the circuit. Furthermore, we employ the level spacing statistics of the entanglement spectrum [25], referred to hereafter as “the entanglement spectrum statistics” (ESS), as a finer measure of thermalization and entanglement [26–28]. Our computations are carried out by adopting the iterative tensor network contraction method introduced in Ref. [1] to the 1+1D spacetime of the circuits.

The main result of this paper is that the EES features three qualitatively different regimes separated by two transitions: the first transition displays the same phenomenology as discussed in the case of random projective measurements, namely, a volume-to-area law transition at a finite projection rate, $p = p_S$ [29]. In the context of the EES, this volume-to-area law transition separates the Wigner-Dyson entanglement spectrum statistics at low projection rate, $p < p_S$, from a finite regime within the area-law phase, $p_S < p < p_c$, in which the EES assumes a non-universal form that interpolates between the Wigner-Dyson and Poisson statistics (see Fig. 1). By resolving the entanglement bond dimensions of the 1+1D network spatially, we conclude that the transition from non-universal to Poisson statistics at $p_c$ is associated with the percolation of entangled bonds in the spacetime geometry of the circuit. We observe the same behavior, with two transitions, in 1+1D circuits comprised of either two-qubit random Haar unitaries or of gates drawn from universal sets, including that...
This paper is organized as follows. In Sec. II, we detail the construction of our quantum circuit and of the random one-qubit projection operators, and outline the method used to compute the ESS. We then map each of these circuits into a tensor network and describe the algorithm for contracting these networks in Sec. III. In Sec. IV, we show numerical results for the ESS in the thermalizing, non-universal, and Poisson phases and locate the transition point, \( p = p_c \), between the latter two, which we interpret as a two-dimensional percolation transition in the spacetime of the circuit in Sec. V. Sec. VI summarizes our conclusions.

II. QUANTUM CIRCUITS AND RANDOM MATRIX THEORY

We consider \( n \) qubits evolving in time \( t \) from an initial product state of the form

\[
|\Psi(t = 0)\rangle = |\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle ,
\]

where the single-qubit state for the \( j \)-th qubit is defined as \( |\psi_j\rangle = \cos(\theta_j/2) |0\rangle + \sin(\theta_j/2) e^{i\phi_j} |1\rangle \) with arbitrary angles \( \theta_j \) and \( \phi_j \). In what follows, the initial state evolves under the action of (i) random unitary gates, and (ii) single-qubit projection operators, randomly inserted after each gate with a finite probability \( p \). The state at time \( t \) is

\[
|\Psi(t)\rangle = M |\Psi(t = 0)\rangle = \sum_x \Psi_x(t) |x\rangle ,
\]

where \( |x\rangle = |x_1 x_2 \ldots x_n\rangle \) is a configuration in the computational basis with \( x_j = 0, 1 \) for \( j = 1, \ldots, n \), and \( M \) is a \( 2^n \times 2^n \) non-unitary matrix describing both the unitary evolution and the projection operations. The resulting circuit is illustrated in Fig. 2, where two-qubit gates are represented as blocks and projection operators as circles.

We choose the projection operator acting on the \( j \)-th qubit to take the form \( M_0 = I_1 \otimes I_2 \otimes \cdots \otimes [I_j] (|0\rangle \otimes \cdots \otimes I_n) \), where \( I_j \) is the identity operator on a single qubit. Although we have chosen to project to the \(|0\rangle\) instead of the \(|1\rangle\) state, this choice is immaterial in what follows. Projection operators are not norm-preserving, and hence the final state \( |\Psi(t)\rangle \) is not normalized by default. We normalize final states for consistency. Projection operators can be physically interpreted as randomly picking a qubit and resetting it to the computational basis. As will become evident below, the projector operators have a disentangling effect, similar to that of the addition of measurement operators in random Clifford or Haar-random circuits [17–24]. In particular, projectors also lead to a volume-to-area law transition as a function of nonunitary operator density.

Now, consider the pure state \( |\Psi\rangle = \sum_x \Psi_x |x\rangle \) after evolution with a quantum circuit as specified above, where \( x \) is the configuration of the qubits in the computational basis and \( \Psi_x \) is the coefficient for each \( x \). \( \Psi_x \) can be reshaped into a matrix \( \Psi_{A,B} \) by splitting the state into subsystems \( A \) and \( B \). In this way, \( |\Psi\rangle \) is expressed as

\[
|\Psi\rangle = \sum_{x,A,B} \Psi_{A,B} |x_A\rangle \otimes |x_B\rangle ,
\]

where \( x_A \) and \( x_B \) are the local configurations for subsystems \( A \) and \( B \), respectively. The entanglement spectrum can be obtained by a Schmidt decomposition [25]

\[
|\Psi\rangle = \sum_k \lambda_k |x_A^k\rangle \otimes |x_B^k\rangle ,
\]

which is equivalent to singular value decomposition (SVD) of matrix \( \Psi_{A,B} \) with singular values \( \lambda_k \).

The set of entanglement levels \( \lambda_k \) defines the entanglement spectrum (ES). The entanglement entropy is given by

\[
S = - \sum_k \lambda_k^2 \ln \lambda_k^2 .
\]

With the ES in descending order, \( \lambda_k > \lambda_{k+1} \), the ratio of adjacent gaps in the spectrum can be defined as

\[
r_k = \frac{\lambda_k - \lambda_{k+1}}{\lambda_k - \lambda_{k+1}} .
\]

For Haar-random states, the probability distribution for adjacent entanglement level ratios, which defines the ESS, follows Wigner-Dyson statistics from random matrix theory [30, 31] and fits well the surmise [32]

\[
P_{WD}(r) = \frac{1}{Z} \frac{(r + r^2)^\beta}{(1 + r + r^2)^{1 + 3\beta/2}}
\]

with \( Z = 4\pi/81\sqrt{3} \) and \( \beta = 2 \) for the Gaussian Unitary Ensemble (GUE) distribution. In contrast, the ESS for integrable systems takes the Poisson form

\[
P_{\text{Poisson}}(r) = \frac{1}{(1 + r)^2} .
\]

The most marked difference between the GUE and Poisson distributions is the level repulsion (\( P_{WD} \to 0 \) for \( r \to 0 \)) in the former and its absence (\( P_{\text{Poisson}} > 0 \) at \( r = 0 \)) in the latter.
II. QUANTUM CIRCUITS AS TENSOR NETWORKS

A. Tensor network mapping

In this section, we map the random quantum circuits introduced above and illustrated in Fig. 1 into a square-grid tensor network and detail the contraction algorithm we use to compute entanglement properties. The tensorial representation of the elements of the circuits is illustrated in Fig. 2.

Each two-qubit gate $g$ is expressed as a $4 \times 4$ unitary matrix $T_{(i_1i_2)(o_1o_2)}^g$, where $(i_1i_2)$ and $(o_1o_2)$ are combined indices corresponding to gate input and output qubit states, respectively. Each $4 \times 4$ matrix can be reshaped into a $2 \times 2 \times 2 \times 2$ tensor $T_{(i_1i_2)(o_1o_2)}^g$. Since we want to transform the circuit into a square lattice geometry, we regroup the indices of each tensor to reshape it to a matrix $T_{(i_1i_2)(i_2o_2)}^g$ and use a SVD to decompose it as

$$
T_{(i_1i_2)(i_2o_2)}^g = \sum_{m,m'} U_{(i_1o_1)m} V_{(i_2o_2)m'}^* \Lambda_{mm'}^f = \sum_{m,b,m'} U_{(i_1o_1)m} \Lambda_{mb} \Lambda_{bm'}^f V_{(i_2o_2)m'}^* = \sum_b T_{(i_1i_2)}^g b T_{(i_2o_2)}^g b,$$

where $U_{(i_1o_1)m}$ and $V_{(i_2o_2)m'}$ are unitary matrices and $\Lambda_{mm'}$ is a semi-positive diagonal matrix containing the singular values. The two new matrices $T_{(i_1i_2)}^g b = \sum_m U_{(i_1o_1)m} \Lambda_{mb}$ and $T_{(i_2o_2)}^g b = \sum_{m'} \Lambda_{bm'}^f V_{(i_2o_2)m'}^*$, with $b$ an index running over singular values, are then reshaped to tensors $T_{(i_1i_2)}^g b$ and $T_{(i_2o_2)}^g b$. To end up with a rectangular geometry, we add a fourth “dummy” index with dimension 1 to each tensor, connecting it with a neighboring tensor in the space dimension, as indicated by the faint vertical lines in Fig. 3a.

B. Contraction algorithm

To contract tensor networks, we employ a variant of the iterative compression-decimation (ICD) algorithm introduced in Ref. [1]. We perform iterations of alternating compression...
and \textit{decimation} steps until the width of the lattice is fully contracted. The compression step is a sweep over lattice bonds where we first contract the tensors at the ends of each visited bond and then perform a SVD to restore the structure of the lattice, in a way reminiscent of the density-matrix renormalization group algorithm [33]. This step becomes significant when the projector density in quantum circuits is increased, as we will discuss below. We use the tools developed in Ref. [1] to implement compression efficiently.

The decimation step coarsens the lattice at the expense of increasing bond dimensions. As illustrated in Fig. 4 the tensor network coarse-graining is performed in time direction, so that every two columns of tensors are contracted into one. At the end of decimation, we get a tensor chain representing the final state.

IV. ENTANGLEMENT SPECTRUM STATISTICS AND PHASE TRANSITIONS

We now use the formulations and tools previously discussed to perform numerical investigations of the ESS as a function of projector density, \( p \), in random quantum circuits satisfying the geometry given in Fig. 2. For each circuit realization, the initial state is of the form of Eq. (1), where all \( \theta_j \) and \( \phi_j \) are selected uniformly at random. For comparative purposes, two separate gate sets were used to construct the random circuits. The first case is composed of two-qubit gates selected from the Haar-random measure, while the second consists of gates uniformly selected from the universal gate set \( U_{\text{sys}} = \{ \sqrt{X}, \sqrt{Y}, \sqrt{W}, f\text{Sim} \} \) [34]. A single-qubit projector is applied with probability \( p \) to each qubit after every gate and before the final time step. We calculate the ESS of many random realizations and bin the spectra for the same \( p \) to obtain the ESS. Our results are summarized in Fig. 5.

We begin by characterizing the quantum chaotic and integrable regimes for these circuits with Wigner-Dyson and Poisson distribution ESS, respectively, and then proceed to describe a previously unforeseen intermediate regime. For \( p < p_S \), the ESS follows the GUE distribution over the entire range of \( r \), as seen in Fig. 5a,d. This corresponds to a highly entangled final state (i.e. volume-law state), indicating that the system has settled into the quantum chaotic regime. On the other hand, for \( p > p_c \), with \( p_c \approx 0.41 \), the ESS follows the Poisson distribution, which indicates that the system is integrable — see Fig. 5c,f. The intuition for this change in the ESS as a function of \( p \) is that as the frequency of projectors is increased, the system becomes frozen in local states, therefore failing to entangle [17–24]. The small deviations from the expected exact Poisson distribution are due to our choice to avoid placing projectors in the final time step of simulations. This choice prevents us from potentially reducing the number of qubits in the system at the final time step. This effect disappears in the thermodynamic limit, which is investigated below by analysis using finite size scaling.

The primary result in our work is the discovery of an intermediate regime between \( p_S < p < p_c \), where the ESS smoothly transitions from the GUE distribution to the Poisson. We call this phase the \textit{residual repulsion phase}. As shown in Fig. 5b,e, the strict level repulsion emblematic of the chaotic regime disappears, i.e. \( P(r) \) becomes nonzero for \( r \to 0 \), and is replaced by a distribution that is between the two regimes, having a maxima at a non-zero value of \( r \).

For the quantum circuits taken from the set \( U_{\text{sys}} \), this transitional phase exists within a shifted window of \( p \) values, specifically \( 0.15 < p < 0.3 \). We infer that this quicker transition occurs as a result of the particular universal gate set which we have selected. The argument is as follows: from \( U_{\text{sys}} \), the gate responsible for introducing entanglement into the system is the \text{fSim} gate, which is an \text{iSWAP} gate concatenated with a controlled \( Z \), having an internal block structure as \( 1 \times 1, 2 \times 2, 1 \times 1 \). The entanglement created by this gate is not as robust as that introduced by a random Haar unitary gate, having no predefined symmetries or structure. The entanglement arising from this structured gate is therefore more susceptible to the presence of projective measurements. Similar evidence for this argument is also found when considering a separate universal set \( U = \text{CNOT, T, Hadamard} \). For this gate set, the entangling gate (\text{CNOT}) has an internal structure of \( 2 \times 2, 2 \times 2 \), and only encodes a bit flip operation. In this case, we found that the residual repulsion phase exists only within the narrow window \( 0.01 < p < 0.03 \). In light of these narrower transition windows for the gate sets \( U_{\text{sys}} \) and \( U \), we choose to focus our attentions on the Haar-random circuits when investigating the thermodynamic limit of the ESS.

The Kullback-Leibler (KL) divergence, defined as

\[
D_{KL}(P(x)||Q(x)) = \sum_x P(x) \ln \left( \frac{P(x)}{Q(x)} \right),
\]

provides a measure of distance between two distributions \( P(x) \) and \( Q(x) \). If we calculate the \( D_{KL} \) between numerically calculated ESS distributions in the residual repulsion phase and the Poisson distribution, there should exist a point \( p \) where \( D_{KL} \) goes to zero, indicating that the numerical distributions have definitively become Poisson. However, as previously mentioned, the numeric distributions will necessarily deviate from the Poisson distribution for small-\( n \) circuits, such that the \( D_{KL} \) takes on a nonzero even in the Poisson phase. We therefore instead calculate the quantity

\[
\Delta D_{KL} = D_{KL}(P_{\text{final}}||P_{\text{Poisson}}) - D_{KL}(P_{\text{2layer}}||P_{\text{Poisson}}),
\]

where \( P_{\text{final}} \) is the calculated final state ESS distribution and \( P_{\text{2layer}} \) is the ESS distribution obtained by evolving random initial \( n \)-qubit product states with a single time step of Haar-random two-qubit gates. This quantity is positive in the residual repulsion phase but vanishes in the Poisson phase, and can hence be used as an indicator of the transition between the two phases with varying \( p \).

In Fig. 6 we locate the transition point \( p_c \) by use of two distinct figures of merit. The first one is the aforementioned \( \Delta D_{KL} \), shown in Fig. 6a. In the inset of Fig. 6b, we plot \( \Delta D_{KL} \) as a function of \( 1/n \) and use the data for various \( n \) to extrapolate linearly to \( n \to \infty \). For small \( p_c \), \( \Delta D_{KL} \) extrapolates to a positive value at infinite size. The slope of the finite-
size scaling curve increases with increasing $p$ and at $p_c \approx 0.41$ the curves start intersecting the $\Delta D_{KL} = 0$ axis at finite $n$, indicating a transition to the Poisson phase. Additionally, in Fig. 6c we show how the position of the maximum of $P_{\text{final}}(r)$ changes with $p$. For the Poisson distribution the maximum is at $r = 0$, whereas the GUE distribution has its maximum $P(r)$ at $r = (\sqrt{5} - 1)/2 \approx 0.618$. Fig. 6c shows that the maximum of $P_{\text{final}}(r)$ decreases from $r \approx 0.618$, reaching zero close to $p_c$.

V. ESS TRANSITION AT $p_c$ AND BOND PERCOLATION IN SPACETIME

We propose that the transition at $p_c$ can be explained as bond percolation in a square lattice [35]. Percolation theory, however, predicts a transition at $p = 0.5$ and not at the observed $p_c \approx 0.41$. The reason for this discrepancy is that a single projector may affect multiple bonds.

To illustrate this, we consider the more direct mapping from circuit to tensor network shown in Fig. 7a, which implements only the first step of Fig. 3a. Without projectors, this yields a rotated square lattice with uniform bond dimension 2. Adding a projection operator to a bond reduces the dimension of that bond to 1. Naively, one would expect that percolation occurs when a giant component of bonds with a projector forms, which for the square lattice would happen at density $p = 0.5$. However, the projection of a qubit to the computational basis has a disentangling effect also in its vicinity in spacetime and not only at the particular point of insertion of a projector. This effect is resolved by the compression step of our algorithm. In the example of Fig. 7a, after the compression step, all bonds have a disentangling effect also in its vicinity in spacetime and not only at the particular point of insertion of a projector. This effect is resolved by the compression step of our algorithm. In the example of Fig. 7a, after the compression step, all bonds indicated by a green line are also reduced to dimension 1. This reasoning suggests a modified percolation threshold based on the density of dimension-1 bonds after a compression sweep, which we call the effective projection rate.

We verify this numerically. Fig. 7b shows the effective projection rate as a function of the density of projectors $p$. Dashed
VI. CONCLUSION

The work presented here explores projection-driven quantum circuits from the perspective of the ESS of the output state. Our results uncover three distinct behaviors of the entanglement spectrum with increasing the rate, $p$, of projection of qubits to the computational basis. The first regime, $0 < p < p_S$, displays volume-law entanglement entropy and Wigner-Dyson statistics of the ESS. At $p = p_S$ the systems undergoes a volume-to-area-law transition similar to that studied in Refs. [17–24]. The principal result of this paper is that the ESS of the area law phase emerging at $p = p_S$ is non-universal and interpolates between Wigner-Dyson and Poisson statistics, with the region of residual level repulsion extending up to a second transition, $p = p_c > p_S$, beyond which the ESS of the system is Poisson. Our tensor network algorithm, which resolves entanglement by monitoring the distribution of bond dimensions across the 1+1D spacetime of the circuit, allows us to identify the transition between non-universal and Poisson statistics as a percolation transition of entangled bonds and to locate the corresponding critical value $p = p_c$ via finite size scaling. We note that, unlike previous results in Refs. [22, 23] obtained in the limit of large local Hilbert space dimension that associate the volume-to-area law transition to percolation in a classical Potts model, here we find that the ESS transition from non-universal to Poisson statistics is due to percolation of entangled bonds in the circuit spacetime itself.

This work leaves open the question of the origin of the intermediate regime with non-universal statistics of the entanglement spectrum. The nature of level statistics is determined by the details of the interactions between eigenvalues which can induce complex non-universal level statistics, including a Griffiths-like phase observed in studies of MBL [36, 37]. More detailed work is needed to elucidate the non-universal range correlations, this reflects the fact that the system becomes integrable at $p > p_c$. It should be noted that the compression algorithm is exact to machine precision and hence accuracy does not factor appreciably into this reasoning.
regime in the ESS identified in this paper.

ACKNOWLEDGMENTS

J.A.R and E.R.M. acknowledge partial financial support from NSF grant No. CCF-1844434. L.Z., C.C., and A.E.R acknowledge partial financial support from NSF grant No. CCF-1844190. Numerical calculations were performed on the Boston University Shared Computing Cluster, which is administered by Boston University Research Computing Services.

[1] Z.-C. Yang, S. Kourtis, C. Chamon, E. R. Mucciolo, and A. E. Ruckenstein, Phys. Rev. E 97, 033303 (2018).
[2] J. M. Deutsch, Phys. Rev. A. 43, 2046 (1991).
[3] P. Calabrese and J. Cardy, Journal of Statistical Mechanics: Theory and Experiment 2005, 04010 (2005).
[4] M. Rigol, V. Dunjko, and M. Olshanii, Nature 452, 854 (2008).
[5] H. Kim and D. A. Huse, Phys. Rev. Lett. 111, 127205 (2013).
[6] M. Mezei and D. Stanford, Journal of High Energy Physics 2017, 65 (2017).
[7] B. L. Altshuler, Y. Gefen, A. Kamenev, and L. S. Levitov, Phys. Rev. Lett. 78, 2803 (1997).
[8] V. Oganesyan and D. A. Huse, Phys. Rev. B 75, 155111 (2007).
[9] A. Pal and D. A. Huse, Phys. Rev. B 82, 174411 (2010).
[10] J. H. Bardarson, F. Pollmann, and J. E. Moore, Phys. Rev. Lett. 109, 017202 (2012).
[11] T. Grover, arXiv:1405.1471 (2014), arXiv:1405.1471.
[12] S. S. Kondov, W. R. McGehee, W. Xu, and B. DeMarco, Phys. Rev. Lett. 114, 083002 (2015).
[13] M. Schreiber, S. S. Hodgman, P. Bordia, H. P. Lüschen, M. H. Fischer, R. Vosk, E. Altman, U. Schneider, and I. Bloch, Science 349, 842 (2015).
[14] J.-y. Choi, S. Hild, J. Zeiher, P. Schauß, A. Rubio-Abadal, T. Yefsah, V. Khemani, D. A. Huse, I. Bloch, and C. Gross, Science 352, 1547 (2016).
[15] K. X. Wei, C. Ramanathan, and P. Cappellaro, Phys. Rev. Lett. 120, 070501 (2018).
[16] J. Smith, A. Lee, P. Richerme, B. Neyenhuis, P. W. Hess, P. Hauke, M. Heyl, D. A. Huse, and C. Monroe, Nature Physics 12, 907 (2016).
[17] A. Cao, X.and Tilloy and A. D. Luca, arXiv:1804.04638 (2018), arXiv:1804.04638.
[18] A. Chan, R. M. Nandkishore, M. Pretko, and G. Smith, arXiv:1808.05949 (2018), arXiv:1808.05949.
[19] B. Skinner, J. Ruhman, and A. Nahum, arXiv:1808.05953 (2018), arXiv:1808.05953.
[20] Y. Li, X. Chen, and M. P. A. Fisher, arXiv:1901.08092 (2019), arXiv:1901.08092.
[21] Y. Li, X. Chen, and M. P. A. Fisher, Phys. Rev. B 98, 205136 (2018).
[22] Y. Bao, S. Choi, and E. Altman, arXiv:1908.04305 (2019), arXiv:1908.04305.
[23] C.-M. Jian, Y.-Z. You, and A. W. W. Ludwig, arXiv:1908.08051 (2019), arXiv:1908.08051.
[24] M. J. Gullans and D. A. Huse, arXiv:1910.00020 (2019), arXiv:1910.00020.
[25] H. Li and F. D. M. Haldane, Phys. Rev. Lett. 101, 010504 (2007).
[26] C. Chamon, A. Hamma, and E. R. Mucciolo, Phys. Rev. Lett. 112, 240501 (2014).
[27] S. D. Geraedts, R. Nandkishore, and N. Regnault, Phys. Rev. B 93, 174202 (2016).
[28] Z.-C. Yang, A. Hamma, S. M. Giampaolo, E. R. Mucciolo, and C. Chamon, Phys. Rev. B 96, 020408 (2017).
[29] Because of the sizeable bond dimensions encountered for large systems when p is near the first transition at $p_s$, we did not study directly the scaling of the entanglement entropy with system size around that point. Instead, we studied the first order mutual information between nearest neighbor spins, and reproduced results similar to those in Fig. 19 of Ref. [20] for that same quantity, at the same location that they find the volume-to-area transition.
[30] Z.-C. Yang, C. Chamon, A. Hamma, and E. R. Mucciolo, Phys. Rev. Lett. 115, 267206 (2015).
[31] V. A. Marchenko and L. A. Pastur, Mathematics of the USSR-Sbornik 1, 457 (1967).
[32] Y. Y. Atas, E. Bogomolny, O. Giraud, and G. Roux, Phys. Rev. Lett. 110, 084101 (2013).
[33] U. Schollwöck, Ann. Phys. (N. Y.) 326, 96 (2011).
[34] F. Arute, K. Arya, and R. t. Babbush, Nature 574, 505 (2019).
[35] H. Kesten, Commun. Math. Phys. 74, 41 (1980).
[36] S. Gopalakrishnan, K. Agarwal, E. A. Demler, and M. Knap, Phys. Rev. B 93, 134206 (2016).
[37] M. Serbyn and J. E. Moore, Phys. Rev. B 93, 041424 (2016).