Parabolic variational inequalities with generalized reflecting directions

Abstract: We study, in a Hilbert framework, some abstract parabolic variational inequalities, governed by reflecting subgradients with multiplicative perturbation, of the following type:

$$y'(t) + Ay(t) + \Theta(t, y(t)) \partial \varphi(y(t)) \ni f(t, y(t)), \quad y(0) = y_0, \quad t \in [0, T].$$

where $A$ is a linear self-adjoint operator, $\partial \varphi$ is the subdifferential operator of a proper lower semicontinuous convex function $\varphi$ defined on a suitable Hilbert space, and $\Theta$ is the perturbing term which acts on the set of reflecting directions, destroying the maximal monotony of the multivalued term. We provide the existence of a solution for the above Cauchy problem. Our evolution equation is accompanied by examples which aim to (systems of) PDEs with perturbed reflection.

Keywords: Evolution equations, Oblique reflection, PDEs

MSC: 47J35, 65J08, 35R15

1 Introduction

Multivalued deterministic and stochastic evolution equations featuring a transformed subdifferential operator are introduced and studied by Gassous, Răşcanu, Rotenstein [1] (for the convex framework) and Răşcanu, Rotenstein [2] (for the non-convex setup). The novelty of these type of equations is given by the presence of a time-state operator beside the subdifferential one, a couple which destroys the maximal monotony of the new term obtained. As a consequence, a new approach for the qualitative analysis of the equation is mandatory. In the finite-dimensional setting used in the mentioned papers, the study of the smooth problem is continued with the analysis of some generalized Skorohod problems, which lead to stochastic variational inequalities. The convergence of the approximating equations uses some arguments unavailable in the present infinite-dimensional setup. The aim is to obtain the existence and uniqueness of a solution when we translate the problem to infinite-dimensional spaces. Consistent results concerning parabolic variational inequalities, with or without a singular input, are produced by Lions, Sznitman [3], Barbu, Răşcanu [4], Răşcanu [5], Pardoux, Răşcanu [6] and the references within. Their papers cover also a wide horizon of applications and examples in the PDEs’ field. A new approach for dealing with multivalued differential equations is introduced in Răşcanu, Rotenstein [7]. The authors establish a one-to-one correspondence between the solutions of different type of variational inclusions and the solutions of some convex optimization problems. The Fitzpatrick function is the main ingredient which offers interesting perspectives for numerical approximations.

While our generalization is focused on transforming the reflection direction at the frontier of the domain into a new one, which is no longer normal at the frontier, different research directions aim at perturbing the Laplacian

*Corresponding Author: Eduard Rotenstein: Faculty of Mathematics, "Alexandru Ioan Cuza" University of Iaşi, 9 Carol I Blvd., 700506, Iaşi, România, E-mail: eduard.rotenstein@uaic.ro
operator (see Eidus [8], Barbu, Favini [9] or Altamore, Milella, Musceo [10]). Multiplicative perturbations of the Laplacian play an important role in the theory of wave propagation in nonhomogeneous media whose density is related to the perturbation coefficient.

The article is organized as follows. Section 2 presents the main notations and some basic assumptions on the spaces and the problem approached in the current study. Additional hypothesis on the terms of our evolution equation will be introduced in the following section, when we present also some particular problems. Section 3 proves the existence of a solution for a smooth multivalued problem with oblique reflection at the frontier of the domain and some applications to systems of PDEs are also provided.

2 Preliminaries, notations and basic assumptions

If \([a, b]\) is a real, closed interval and \(\mathbb{V}\) is a Banach space, then we denote by \(L^p(\mathbb{V})\), \(C([a, b]; \mathbb{V})\), \(BV([a, b]; \mathbb{V})\) and \(AC([a, b]; \mathbb{V})\) the usual spaces of \(p\)-integrable, continuous, with bounded variation, and, respectively, absolutely continuous \(\mathbb{V}\)-valued function on \([a, b]\). By \(W^{1,p}([a, b]; \mathbb{V})\) (and, in the same manner, we can define \(W^{2,p}([a, b]; \mathbb{V})\)) we shall denote the space of \(y \in L^p(a, b; \mathbb{V})\) such that \(y' \in L^p(a, b; \mathbb{V})\), where \(y'\) is the derivative in the sense of distributions. Equivalently, according to Barbu [11], we know that

\[
W^{1,p}([a, b]; \mathbb{V}) = \left\{ y \in AC([a, b]; \mathbb{V}) : \frac{dy}{dt} \in L^p(a, b; \mathbb{V}), y(t) = y(a) + \int_a^t \frac{dy}{dt}(s)ds, \forall t \in [a, b] \right\}.
\]

We denote by \(L(\mathbb{V})\) the Banach space of linear operators \(A : \mathbb{V} \to \mathbb{V}\), with the norm \(\|A\|_{L(\mathbb{V})} := \sup\{\|Ay\|_\mathbb{V} : \|y\|_\mathbb{V} = 1\}\).

Throughout this article we shall situate our work in the Gelfand triple framework. More precisely, we consider two real separable Hilbert spaces \(\mathbb{V}\) and \(\mathbb{H}\) such that \(\mathbb{V} \subset \mathbb{H} \cong \mathbb{H}^* \subset \mathbb{V}^*\), with continuous and dense embeddings, where \(\mathbb{V}^*\) denotes the dual of \(\mathbb{V}\). Moreover, assume that the inclusion \(\mathbb{V} \subset \mathbb{H}\) is a compact one. The norm from \(\mathbb{V}\) is given by \(\|\cdot\|\), the one from \(\mathbb{H}\) is \(\|\cdot\|_\mathbb{H}\) and \(\mathbb{V}^*\) is endowed with the norm \(\|\cdot\|_\mathbb{V}^*\). The scalar product of \(\mathbb{H}\) is \((\cdot, \cdot)\) and the duality pairing between \(\mathbb{V}\) and \(\mathbb{V}^*\) is given by \((\cdot, \cdot)\). Let \(\gamma_1, \gamma_2 > 0\) be some positive constants of boundedness, corresponding to the above inclusions \(\|y\|_\mathbb{V} \leq \gamma_1 \|y\|_\mathbb{H} \leq \gamma_2 \|y\|_\mathbb{V}^*, \forall y \in \mathbb{V}\).

**Remark 2.1.** For the situation when \(\mathbb{H}\) is a finite dimensional space, the problem was studied by Gassous, Rășcanu, Rotenstein [1], but with the essential assumption \(\text{int}(\text{Dom}(\varphi)) \neq \emptyset\). They considered even a more general differential inclusion, featuring a singular input which drives the equation. To analyze such a Skorohod problem in the infinite dimensional setting we should consider also a real separable Banach space \((\mathbb{X}, ||\cdot||_{\mathbb{X}})\), with its separable dual \((\mathbb{X}^*, ||\cdot||_{\mathbb{X}^*})\) such that \(\mathbb{X} \subset \mathbb{H} \subset \mathbb{X}^*\) and \(\mathbb{V} \cap \mathbb{X}\) is dense in \(\mathbb{V}\) and \(\mathbb{X}\) (for more details see Rășcanu [5]). In the infinite dimensional case, if \(\text{int}(\text{Dom}(\varphi)) \neq \emptyset\) then \(\mathbb{H} = \mathbb{X}\), but this assumption is too strong because, for example, even in the case of PDEs with barriers, this assumption is not satisfied. If \(\text{int}(\text{Dom}(\varphi)) = \emptyset\) the necessity of using the Banach space \(\mathbb{X}\) is mandatory. One can see this if we consider, for example, for a non-empty domain \(D \subset \mathbb{R}^k\), \(\mathbb{H} = L^2(D; \mathbb{R})\), \(K = \{u \in L^2(D; \mathbb{R}) : u \geq 0 \text{ a.e. } t \in [0, T]\}\) and \(\varphi = I_K^{1}\). It is obvious that \(\text{int}(\text{Dom}(\varphi))\) is empty and, therefore, one can take \(\mathbb{X} = C(D; \mathbb{R})\), which satisfies \(\mathbb{X} \subset \mathbb{H} \subset \mathbb{X}^*\).

We study the following type of multivalued evolution equation, driven by oblique reflected subgradients:

\[
\begin{cases}
y'(t) + Ay(t) + \Theta(t, y(t)) \partial \varphi(y(t)) \ni f(t, y(t)), \\
y(0) = y_0 \in \mathbb{H}, \ t \in [0, T],
\end{cases}
\]

where:

\[(A\varphi) : \varphi : \mathbb{H} \to (-\infty, +\infty]\text{ is a proper, convex, lower semicontinuous function.}\]

\(^1\text{We understand by } I_K \text{ the convexity indicator function of the closed } K, \text{ that is } I_K(y) = 0 \text{ if } y \in K \text{ and } I_K(y) = +\infty \text{ otherwise.}\)
(A_{A}) : A \in \mathcal{L}(\mathbb{V}, \mathbb{V}^{*}), A = A^{*} \text{ s.t., for some constants } \alpha_{0}, \alpha_{1} > 0, \forall y \in \mathbb{V}, \langle Ay, y \rangle \geq \alpha_{0} ||y||^{2} \text{ and } \langle Ay, \nabla \psi_{e}(y) \rangle \geq -\alpha_{1}(1 + ||\nabla \psi_{e}(y)||)(1 + ||y||), \forall y \in D(A_{H}). \quad (2)

Here \( D(A_{H}) := \{ v \in \mathbb{V} : Av \in \mathbb{H} \} \) and \( A_{H}v = Av, \forall v \in D(A_{H}) \).

(A_f) : \begin{cases} f : [0, T] \times \mathbb{H} \rightarrow \mathbb{H} \text{ is a Carathéodory function, i.e.:} \\
 f(t, \cdot) : [0, T] \rightarrow \mathbb{H} \text{ is measurable and } f(\cdot, \cdot) : \mathbb{H} \rightarrow \mathbb{H} \text{ is continuous,} \\
 \Theta : [0, T] \times \mathbb{H} \rightarrow \mathcal{L}(\mathbb{H}), \text{ such that} \\
 (t, y) \mapsto \Theta(t, y)h : [0, T] \times \mathbb{H} \rightarrow \mathbb{H} \text{ is a continuous function, for all } h \in \mathbb{H}.
\end{cases}

(A_{\Theta}) : \begin{cases} \theta_{0} > 0 \text{ such that } A_{\Theta} \theta_{0} = \mathbb{H}, \text{ and } \theta_{0} \in \mathbb{H} \text{ such that } (\theta_{0}, \theta_{0}) \in \partial C_{\mathbb{H}}. \end{cases}

We denoted by \( \partial C_{\mathbb{H}} \) the subdifferential operator of \( C_{\mathbb{H}} \), that is

\[ \partial C_{\mathbb{H}}(u) := \{ \tilde{u} \in \mathbb{H} : (\tilde{u}, v - u) + \varphi(u) \leq \varphi(v), \forall v \in \mathbb{H} \} \]

and \( \text{Dom}(\partial C_{\mathbb{H}}) := \{ u \in \mathbb{H} : \partial C_{\mathbb{H}}(u) \neq \emptyset \} \). We will use the notation \( (u, \tilde{u}) \in \partial C_{\mathbb{H}} \) in order to express that \( u \in \text{Dom}(\partial C_{\mathbb{H}}) \) and \( \tilde{u} \in \partial C_{\mathbb{H}}(u) \). Assume also that there exist \( u_{0} \in \mathbb{V} \) and \( \tilde{u}_{0} \in \mathbb{H} \) such that \( (u_{0}, \tilde{u}_{0}) \in \partial C_{\mathbb{H}} \). The element defined by the quantity \( \Theta(t, y) \tilde{y} \), with \( \tilde{y} \in \partial C_{\mathbb{H}}(y) \), will be called \textit{oblique subgradient}. Recall that \( \partial C_{\mathbb{H}}(\varphi_{e}) = \mathbf{e}^{-1}(I - (I + \varepsilon \partial C_{\mathbb{H}})^{-1}) \partial C_{\mathbb{H}}((I + \varepsilon \partial C_{\mathbb{H}})^{-1}) \) represents the classical Yosida approximation of \( \varphi \) and \( \partial C_{\mathbb{H}}(\varphi_{e}) = \nabla \psi_{e} \), where \( \psi_{e}(u) := \inf\{ 2\varepsilon^{-1} |v - u|^{2} + \varphi(v) : v \in \mathbb{H} \} \).

The above hypothesis assure that problem (1) is well posed. Some additional, specific hypothesis, which are necessary for the existence of a solution, will be added in Section 3.

### 3 Existence and uniqueness of a solution

Concerning the Cauchy problem (1), we first prove the existence of at least one solution. For its uniqueness we will renounce at the dependence on the state for the perturbing term \( \Theta \) and we consider some particular systems of PDEs. Assume \( (A_{\Theta}), (A_f) \) and \( (A_{\Theta}) \) still hold and we enhance them by adding the additional hypothesis:

\[ \begin{cases} (i) \forall (t, y) \in [0, T] \times \mathbb{H}, \Theta(t, y) : \mathbb{H} \rightarrow \mathbb{H} \text{ is a self-adjoint linear operator;} \\
(ii) \beta_{0} > 0 \text{ such that } \forall (t, y, h) \in [0, T] \times \mathbb{H} \times \mathbb{H}, \\
|f(t, y)| \leq \mu_{1}(t) + \mu_{2}(t) ||y||, \forall y \in \text{Dom}(\varphi), \text{ a.e. } t \quad (\text{for } \mu_{1}, \mu_{2} \in L^{2}(0, T; \mathbb{R}_{+})). \tag{H_{f}} \end{cases} \]

Without losing the generality, for convenience only, we can suppose that \( \varphi(y) \geq \varphi(0) = 0 \), which easily implies that \( 0 = \varphi_{e}(0) \leq \varphi_{e}(y) \), for every \( \varepsilon > 0 \) and \( y \in \mathbb{H} \). This is not an effective restriction since, using the pair \( (u_{0}, \tilde{u}_{0}) \in \partial C_{\mathbb{H}}, \tilde{u}_{0} \in \mathbb{V}, \tilde{u}_{0} \in \mathbb{H}, \) by the change of function \( \tilde{\varphi}(x) := \varphi(x + u_{0}) - \varphi(u_{0}) - \langle \tilde{u}_{0}, x \rangle \), with \( \partial \tilde{\varphi}(x) = \partial C_{\mathbb{H}}(x + u_{0}) - \tilde{u}_{0} \), one can transform our equation into a new one where this positivity condition is satisfied.

Define first the notion of solution for Eq.(1).
3.1 Existence of a solution

Definition 3.1. A pair of functions \( y, k: [0, T] \to \mathbb{H} \) is a (strong) solution of the oblique reflected evolution equation (1) if

\[
(i) \ y \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}), \\
(ii) \ k \in C([0, T]; \mathbb{H}) \cap BV([0, T]; \mathbb{H}), k(0) = 0, \\
(iii) \ y(t) + \int_0^t Ay(s)ds + \int_0^t \Theta(s, y(s))dk(s) = y_0 + \int_0^t f(s, y(s))ds, \\
(iv) \ \int_0^t (z(r) - y(r), dk(r)) + \int_0^t \varphi(y(r))dr \leq \int_0^t \varphi(z(r))dr, \\
\forall 0 \leq s \leq t \leq T, \forall z \in C([0, T]; \mathbb{H}).
\]

Remark 3.2. By the notation \( dk(t) \in \partial \varphi(y(t))(dt) \) on \([0, T]\) we shall understand that \((y, k) \in C([0, T]; \mathbb{H}) \times BV([0, T]; \mathbb{H})\) and the condition (3-(iv)) is satisfied; consequently it follows that \( \varphi(y(\cdot)) \in L^1([0, T]) \) and \( y(t) \in \text{Dom}(\varphi), \forall t \in [0, T] \).

We provide now the main result of this section.

Theorem 3.3. Consider the hypothesis \((A_\varphi), (A_f), (A_{\Theta}), (H^1_d), (H^1_{\varphi})\) and \((H^1_{\varphi})\) is satisfied. Then, the evolution equation with oblique reflecting subgradients (1) admits at least one strong solution \((y, k)\), in the sense of Definition 3.1. Moreover, the feedback reflecting process \( k \) is an absolutely continuous one, that is there exists \( h \in L^2(0, T; \mathbb{H}) \) such that \( k(t) = \int_0^t h(s)ds \).

Proof. Let \( n \in \mathbb{N}^* \), with \( n \geq T \), consider \( \varepsilon = T/n \) and the extension \( f(s, x) = 0 \) for \( s \leq 0 \). Construct first the penalized problem, written, for all \( t \in [0, T] \) as:

\[
y_{\varepsilon}(t) + \int_0^t Ay_{\varepsilon}(s)ds + \int_0^t \Theta(s, y_{\varepsilon}(s))\nabla \varphi_{\varepsilon}(y_{\varepsilon}(s))ds = y_0 + \int_0^t f(s, y_{\varepsilon}(s))ds (4)
\]

where \( \pi_D(x) \) is the projection of \( x \) on the set \( D = \bar{\text{Dom}(\varphi)} \), uniquely defined by \( \pi_D(x) \in D \) and \( \text{dist}_D(x) = |x - \pi_D(x)| \). By convention, we set \( y_{\varepsilon}(t) = y_0 \in \mathbb{H} \), for \( t \leq 0 \).

According to a classical result from Lions [12] (see also Barbu [13], Theorem 1.14), Eq.(4) admits a unique solution \( y_{\varepsilon} \in C([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{V}) \). Since the function defined on each interval \([0, \varepsilon], [\varepsilon, 2\varepsilon], [2\varepsilon, 3\varepsilon], \ldots\) by

\[
g(t) := f(t - \varepsilon, \pi_D(y_{\varepsilon}(t - \varepsilon))) - \Theta(\varepsilon [t/\varepsilon], y_{\varepsilon}(\varepsilon [t/\varepsilon])) \nabla \varphi_{\varepsilon}(y_{\varepsilon}(t))
\]

belongs to \( L^2(0, T; \mathbb{H}) \) then, if we take the initial datum \( y_0 \in \mathbb{V} \subset \mathbb{H} \), it infers

\[
y_{\varepsilon} \in W^{1,2}([0, T]; \mathbb{H}) \cap L^2(0, T; \mathbb{H}) \cap C([0, T]; \mathbb{V}).
\]

One justifies this last statement by citing Barbu [13], Theorem 1.13. The presence of the oblique reflection brought by the term \( \Theta \nabla \varphi_{\varepsilon} \) does not permit the use of the technique introduced by Barbu & Răscanu in [4]. Theorem 3.1., even if we construct the space setup used there. Instead, we will use compactness results in order to obtain a solution for our equation. The regularity of the function \( y \mapsto |y|^2 + \varphi_{\varepsilon}(y) \) and the definition of the approximating sequence lead to

\[
|y_{\varepsilon}(t)|^2 + \varphi_{\varepsilon}(y_{\varepsilon}(t)) + \int_0^t (Ay_{\varepsilon}(s), 2y_{\varepsilon}(s))ds + \int_0^t (Ay_{\varepsilon}(s), \nabla \varphi_{\varepsilon}(y_{\varepsilon}(s)))ds \\
+ \int_0^t (\Theta(\varepsilon [s/\varepsilon], y_{\varepsilon}(\varepsilon [s/\varepsilon])) \nabla \varphi_{\varepsilon}(y_{\varepsilon}(s)), 2y_{\varepsilon}(s) + \nabla \varphi_{\varepsilon}(y_{\varepsilon}(s)))ds \\
= |y_0|^2 + \varphi_{\varepsilon}(y_0) + \int_0^t (f(s - \varepsilon, \pi_D(y_{\varepsilon}(s - \varepsilon))), 2y_{\varepsilon}(s) + \nabla \varphi_{\varepsilon}(y_{\varepsilon}(s)))ds.
\]
We first provide some useful estimates on the terms appearing in (5). Denote by $C$ a generic constant independent of $\varepsilon$ and, taking into consideration the hypothesis imposed on $\Theta$, $f$ and using standard inequalities of the form $xy \leq \frac{\lambda x^2}{2} + \frac{\gamma y^2}{2}$, with $\lambda > 0$, the following estimates hold:

\[
\begin{align*}
(\Theta (s, y_{\varepsilon}(s - \varepsilon)) \nabla \varphi_e (y_{\varepsilon}(s))) \cdot \nabla \varphi_e (y_{\varepsilon}(s)) & \geq \beta_0 |\nabla \varphi_e (y_{\varepsilon}(s))|^2 ; \\
(\Theta (s, y_{\varepsilon}(s - \varepsilon)) \nabla \varphi_e (y_{\varepsilon}(s)), 2y_{\varepsilon}(s)) & \geq -2\beta_1 |y_{\varepsilon}(s)| |\nabla \varphi_e (y_{\varepsilon}(s))| \\
& \geq -4\frac{\beta_1^2}{\beta_0} \sup_{r \leq s} |y_{\varepsilon}(r)|^2 - \frac{1}{4}\beta_0 |\nabla \varphi_e (y_{\varepsilon}(s))|^2 ; \\
(2y_{\varepsilon}(s) + \nabla \varphi_e (y_{\varepsilon}(s)), f(s - \varepsilon, \pi_{D}(y_{\varepsilon}(s - \varepsilon)))) & \leq \frac{1}{8}\beta_0 |2y_{\varepsilon}(s) + \nabla \varphi_e (y_{\varepsilon}(s))|^2 + \frac{2}{\beta_0} |f(s - \varepsilon, \pi_{D}(y_{\varepsilon}(s - \varepsilon)))|^2 \\
& \leq \frac{\beta_0}{4} |\nabla \varphi_e (y_{\varepsilon}(s))|^2 + \beta_0 |y_{\varepsilon}(s)|^2 + 4\frac{\beta_0}{\beta_0} \mu_2^2(s - \varepsilon) + 4\frac{\beta_0}{\beta_0} \mu_2^2(s - \varepsilon) \leq |\pi_{D}(y_{\varepsilon}(s - \varepsilon)))|^2 .
\end{align*}
\]

Introduce the above estimates, as well as (2), into (5), we write the obtained inequality for

\[
\int_0^T \sup_{t \in [0,T]} |y_{\varepsilon}(t)|^2 + \frac{\beta_0}{4} |\nabla \varphi_e (y_{\varepsilon}(t))|^2 + \beta_0 |y_{\varepsilon}(t)|^2 + \mu_1(s - \varepsilon) |y_{\varepsilon}(s)| ds \leq C_T .
\]

The Gronwall inequality assures the existence of a positive constant $C_T = C_T(\alpha_0, \alpha_1, \beta_0, \beta_1, \mu_1, \mu_2)$, independent of $\varepsilon$, such that

\[
\sup_{t \in [0, T]} |y_{\varepsilon}(t)|^2 + \frac{\beta_0}{4} |\nabla \varphi_e (y_{\varepsilon}(t))|^2 + \beta_0 |y_{\varepsilon}(t)|^2 + \mu_1(s - \varepsilon) |y_{\varepsilon}(s)| ds \leq C_T .
\]

We shall prove now that the sequence $(y_{\varepsilon})_{\varepsilon \in (0,1)}$ is a equi-integrable one in $L^2(0,T;\mathbb{Y}^*)$, that is, we must show

\[
\lim_{\varepsilon \to 0} \int_0^{T-\tau} ||y_{\varepsilon}(t + \tau) - y_{\varepsilon}(t)||_{\mathbb{Y}^*}^2 dt = 0 .
\]

Since

\[
y_{\varepsilon}(t + \tau) - y_{\varepsilon}(t) = \int_t^{t+\tau} [-Ay_{\varepsilon}(s) - \Theta (\varepsilon [s/\varepsilon], y_{\varepsilon} (\varepsilon [s/\varepsilon]))] \nabla \varphi_e (y_{\varepsilon}(s))) + f(s - \varepsilon, \pi_{D}(y_{\varepsilon}(s - \varepsilon))) ds,
\]

we obtain

\[
||y_{\varepsilon}(t + \tau) - y_{\varepsilon}(t)||_{\mathbb{Y}^*} \leq \int_t^{t+\tau} \left( [||A||_{L^2(\mathbb{Y}^*)} ||y_{\varepsilon}(s)|| + \beta_1 ||\nabla \varphi_e (y_{\varepsilon}(s))|| + \mu_1(s - \varepsilon) \right) ds + \int_t^{t+\tau} \mu_2(s - \varepsilon) |y_{\varepsilon}(s - \varepsilon)| ds,
\]

which implies, by standard computations and using the estimates (6),

\[
||y_{\varepsilon}(t + \tau) - y_{\varepsilon}(t)||_{\mathbb{Y}^*}^2 \leq 3\tau ||A||_{L^2(\mathbb{Y}^*)}^{2} \int_t^{t+\tau} ||y_{\varepsilon}(s)||^2 ds
\]

\[
+ 3\tau \beta_1^2 \int_t^{t+\tau} ||\nabla \varphi_e (y_{\varepsilon}(s)))||^2 ds + 3 \left( \int_{t-\varepsilon}^{t+\tau-\varepsilon} \mu_1(s) + \mu_2(s) |y_{\varepsilon}(s)| ds \right)^2
\]

\[
\leq C_1 \tau \left( \int_t^{t+\tau} ||y_{\varepsilon}(s)||^2 + ||\nabla \varphi_e (y_{\varepsilon}(s)))||^2 ds \right) + \tau \int_{t-\varepsilon}^{t+\tau-\varepsilon} \mu_1(s) ds + \int_{t-\varepsilon}^{t+\tau-\varepsilon} \mu_2(s) ds \int_{t-\varepsilon}^{t+\tau-\varepsilon} |y_{\varepsilon}(s)|^2 ds
\]

\[
\leq C_2 \tau + m_{\varepsilon}(\tau) \int_0^T ||y_{\varepsilon}(s)||^2 ds,
\]

We denote by $C_1, C_2, \ldots$ some generic constants which vary from one line to another, but which remain independent of $\varepsilon$. 

\[\Box\]
where \( \tilde{\mu} : [0, T] \to \mathbb{R}, \tilde{\mu}(t) := \int_0^t \mu^2(s)ds \) is a uniformly continuous function, for which \( m_{\tilde{\mu}}(\tau) := \sup_{|t-s| \leq \tau} |\tilde{\mu}(t) - \tilde{\mu}(s)| \) represents the modulus of continuity of \( \tilde{\mu} \). Therefore,

\[
||y_e(t + \tau) - y_e(t)||_{\mathbb{H}}^2 \leq C_3(\tau + m_{\tilde{\mu}}(\tau))
\]

and (7) easily follows.

Consequently, since the sequence \( (y_{\varepsilon_n})_{\varepsilon \in (0,1)} \) is bounded in \( L^2(0, T; \mathbb{V}) \), equi-integrable in \( L^2(0, T; \mathbb{V}^*) \) and the inclusion \( \mathbb{V} \subset \mathbb{E} \) is a compact one, it follows, due to a compacticity result of Aubin [14], that \( (y_{\varepsilon_n})_{\varepsilon \in (0,1)} \) is relatively compact in \( L^2(0, T; \mathbb{H}) \) and, as a consequence, there exist a subsequence \( \varepsilon_n \to 0 \) such that, as \( n \to \infty \),

\[
y_{\varepsilon_n} \rightharpoonup y \quad \text{in} \quad L^2(0, T; \mathbb{H}) \quad \text{and} \quad a.e. \quad \text{on} \quad [0, T].
\]

Using (6), there exist a subsequence of \( \varepsilon_n \) (we denote it also with \( \varepsilon_n \)) and \( h \in L^2(0, T; \mathbb{H}) \) such that, as \( n \to \infty \),

\[
y_{\varepsilon_n} \to y, \quad \text{weakly in} \quad L^2(0, T; \mathbb{V}), \quad Ay_{\varepsilon_n} \to Ay, \quad \text{weakly* in} \quad L^2(0, T; \mathbb{V}) \quad \text{and} \quad
\]

\[
\nabla \psi_{\varepsilon_n}(y_{\varepsilon_n}) \to h, \quad \text{weakly} \quad \text{in} \quad L^2(0, T; \mathbb{H}).
\]

One can now pass to limit in (4) and we obtain that the pair \((y, k)\), with \( k(t) := \int_0^t h(s)ds \), verifies

\[
y(t) + \int_0^t Ay(s)ds + \int_0^t \Theta(s, y(s))dk(s) = y_0 + \int_0^t f(s, y(s))ds.
\]

It also follows that \( y \in W^{1,2}(0, T; \mathbb{V}^*) \). In order to conclude that \((y, k)\) is a solution of Eq.(1) we must prove that \( dk(t) = \partial \varphi(y(t))(dt) \). The lower semicontinuity property of \( \varphi \) yields, a.e. \( t \in [0, T] \),

\[
\varphi(y(t)) \leq \liminf_{n \to +\infty} \varphi_{\varepsilon_n}(y_{\varepsilon_n}(t) - \varepsilon_n \nabla \psi_{\varepsilon_n}(y_{\varepsilon_n}(t))) = \liminf_{n \to +\infty} \varphi_{\varepsilon_n}(y_{\varepsilon_n}(t)) \leq C_T.
\]

Since \( \nabla \psi_{\varepsilon_n}(y_{\varepsilon_n}(t)) \) is \( \partial \varphi(y_{\varepsilon_n}(t) - \varepsilon \nabla \psi_{\varepsilon_n}(y_{\varepsilon_n}(t))) \), then, for all \( z \in C([0, T]; \mathbb{H}) \) and all \( 0 \leq s \leq t \leq T \),

\[
\int_s^t \varphi_{\varepsilon_n}(y_{\varepsilon_n}(t) - \varepsilon \nabla \psi_{\varepsilon_n}(y_{\varepsilon_n}(t)))dr + \int_s^t \varphi_{\varepsilon_n}(y_{\varepsilon_n}(t) - \varepsilon \nabla \psi_{\varepsilon_n}(y_{\varepsilon_n}(t)))dr \leq \int_s^t \varphi(z(r))dr.
\]

Passing now to \( \liminf_{n \to +\infty} \) in the above inequality we obtain that \( dk(t) = \partial \varphi(y(t))(dt) \), a.e. on \([0, T]\), and the proof is now complete.

\[\square\]

Remark 3.4. The previous result still holds if we consider in Eq. (1), instead of A, a family \( \{A_t\}_{t \in [0, T]} \) of linear operators on \( \mathbb{H} \) (instead of \( \mathbb{V} \)) which satisfies, along with assumption \((A_A)\): \( \sup_{t \in [0, T]} \|A_t\|_{L^2(\mathbb{H})} < +\infty \) and, for every \( y \in L^2(0, T; \mathbb{V}) \), the function \( t \to A_t y(t) \) is strongly \( \mathbb{V}^* \)-measurable on \([0, T]\).

### 3.2 Parabolic variational inequalities. Uniqueness of the solution

In order to prove the uniqueness of a solution we restrict the study of the general problem (1) and analyze a scenario given by the consideration of a particular system of multivalued PDEs. For doing this, let \( D \subset \mathbb{R}^d \) be a domain with a smooth frontier (for example, of class \( C^2 \)), \( \mathbb{H} = L^2(D; \mathbb{R}^k) \equiv (L^2(D))^k \), \( \mathbb{V} = H^1_0(D; \mathbb{R}^k) \equiv (H^1_0(D))^k \) and consider \( \Theta : [0, T] \to \mathbb{R}^{k \times k}, \Theta(t) := diag(\Theta_1(t), ..., \Theta_k(t)) \), with \( \Theta \in C^1([0, T]; \mathbb{R}) \). Assume also that \( 0 < c \leq \Theta_i(t) \leq C, \left| \frac{\partial \Theta_i(t)}{\partial \theta} \right| \leq C \), for some positive constants \( c, C \) and for all \( t \in [0, T], i = 1, ..., k \). It is obvious that, for every \( \tau \), the inverse matrix \( [\Theta(t)]^{-1} \) has the same properties as \( \Theta(t) \). For a convex, proper, l.s.c. function \( \varphi : \mathbb{R}^k \to (-\infty, +\infty] \), consider the semilinear system of multivalued parabolic PDEs:

\[
\begin{cases}
\frac{du_i(x, t)}{dt} - \Delta u_i(x, t) + \Theta(t) \partial \varphi(u(x, t)) \ni f(t, u(x, t)), & x \in Q := D \times (0, T), i = 1, ..., k, \\
u(x, 0) = u_0(x), & x \in D, \\
u(x, t) = 0, & x \in \partial D \times (0, T).
\end{cases}
\]

where \( u = (u_1, ..., u_k) : D \times [0, T] \to \mathbb{R}^k \) and \( u_0 \in H^1_0(D; \mathbb{R}^k), f \in L^2(D \times (0, T); \mathbb{R}^k) \).
Theorem 3.5. Under the setup introduced above, there exists a unique solution for problem (8).

Proof. The existence of a solution is provided by Theorem 3.3. We focus now on its uniqueness. For doing this, assume that \( u \) and \( \tilde{u} \) are two solutions for (8) and define \( v : [0, T] \to \mathbb{R} \),

\[
v(t) := \left[ \Theta(t) \right]^{-1} (u(r) - \tilde{u}(r)), u(r) - \tilde{u}(r) = \left[ \Theta(r) \right]^{-1/2} (u(r) - \tilde{u}(r)) \right]^2.
\]

The assumptions on \( \Theta \) assure the existence of some positive constants \( c_1 \) and \( c_2 \) such that

\[
c_1 |u(r) - \tilde{u}(r)|^2 \leq v(r) \leq c_2 |u(r) - \tilde{u}(r)|^2.
\] (9)

For all \( 0 \leq s \leq t \), we have

\[
v(t) = v(s) + \int_s^t dv(r) = v(s) + \int_s^t ((d \left[ \Theta(r) \right]^{-1})(u(r) - \tilde{u}(r)), u(r) - \tilde{u}(r))dr
\]

\[
- 2 \int_s^t (u(r) - \tilde{u}(r)), h(r) - \tilde{h}(r))dr + 2 \int_s^t \left[ \Theta(r) \right]^{-1} (u(r) - \tilde{u}(r)), \Delta(u(r) - \tilde{u}(r)) \right] dr
\]

\[
+ 2 \int_s^t \left[ \Theta(r) \right]^{-1} (u(r) - \tilde{u}(r)), f(r, u(r)) - \tilde{f}(r, \tilde{u}(r)) \right] dr
\]

\[
\leq v(s) + C \int_s^t |u(r) - \tilde{u}(r)|^2 dr - 2 \int_s^t \left[ \Theta(r) \right]^{-1} (u(r) - \tilde{u}(r)), (-\Delta)(u(r) - \tilde{u}(r)) \right] dr
\]

Since

\[
\left[ \Theta(t) \right]^{-1} (u(t) - \tilde{u}(t)), (-\Delta)(u(t) - \tilde{u}(t)) \right] = \sum_{i=1}^k \int_\mathbb{D} \left[ \Theta(t) \right]^{-1} \left( \nabla(u_i(t) - \tilde{u}_i(t)) \right) \geq c \sum_{i=1}^k \left| \nabla(u_i(t) - \tilde{u}_i(t)) \right|^2
\]

\[
\geq c \left| |u(t) - \tilde{u}(t)| \right|,
\]

the inequality (9) leads to \( v(t) \leq v(s) + C \int_s^t v(r)dr \), for all \( 0 \leq s \leq t \). Apply now the Gronwall’s inequality and the uniqueness of the solution for (8) follows.

Within the framework constructed in this section we present some particular cases of semilinear systems of multivalued parabolic PDEs.

Examples. Let \( K \) be a non-empty closed convex set from \( H \) and \( L = \{ L(t) \}_{t \in \mathbb{R}} \), \( L(t) : H \to H \) a strongly continuous group of linear operators, with \( B : D(B) \subset H \to H \) being its infinitesimal generator. Suppose that \( L \) satisfies the hypothesis imposed on \( \Theta \) along the previous section and consider the convex, l.s.c. function \( \varphi \) which appears in (8) to be the convexity indicator of the time-dependent convex set \( L(t)K \). The system (8) now becomes:

\[
\begin{align*}
\frac{d u_i(x,t)}{dt} - \Delta u_i(x,t) + \Theta(t) \partial L(t)K(u(x,t)) & \ni f(t, u(x,t)) \text{ on } Q, i = 1, k, u(x,0) = u_0(x), \text{ on } D, (10)
\end{align*}
\]

(a) Let \( \Theta = I_{k \times k} \in \mathbb{R}^{k \times k} \) and, for every \( t \), \( L(t) = l(t) \cdot I_{2\mathbb{R}} \), with \( l : \mathbb{R} \to \mathbb{R}^+ \) a continuous invertible function. By a suitable change of variable, we can reduce the problem to a reflected PVI with a time-independent convex domain. For example, for the situation of a contracting, time-dependent set characterized by the decreasing function \( l(t) = e^{-\lambda t}, \lambda > 0 \), we have

\[
\partial I_{e^{-\lambda t}K} \left( u \right) = \left\{ \left. \tilde{u} \in H : (\tilde{u}, z - u) \leq 0, \forall z \in e^{-\lambda t}K \right\} \right.
\]

\[
= \left\{ \left. \tilde{u} \in H : (e^{-\lambda t} \tilde{u}, w - e^{\lambda t} u) \leq 0, \forall w \in K \right\} \right. = \partial I_K(e^{\lambda t}u).
\]
Denote \( v(x,t) := e^{\lambda t} u(x,t) \) and we obtain that \( u \) is a solution for the constructed problem iff \( v \) is a solution for the following system of PDEs

\[
\begin{align*}
\frac{dv_i}{dt} - \Delta v_i + \partial I_K(v) \ni \lambda v + e^{\lambda t} f(t,e^{-\lambda t} v), \quad &\text{on} \ Q, \ i = 1,k, \\
v(x,0) = u_0(x), \quad &\text{on} \ D, \\
v(x,t) = 0, \quad &\text{on} \ \partial D \times (0,T).
\end{align*}
\]

According to Barbu, Răşcanu [4], the transformed problem admits a unique solution.

(b) If \( \Theta \) is no longer the identity matrix or/and \( L \) is the group of linear operators defined at the beginning of the Example, the situation from point (a) changes because we have

\[
L(t)\partial I_{L(t)K}(u) = L(t) \{ \hat{u} \in \mathbb{H} : (L(t)\hat{u}, z - L(-t)u) \leq 0, \forall z \in K \}
\]

\[
= \partial I_K(L(-t)u).
\]

Denoting again \( v(x,t) := [L(t)]^{-1} u(x,t) = L(-t)u(x,t) \), we deduce that \( v \) must be a solution of the following problem:

\[
\begin{align*}
\frac{dv_i}{dt} - \Delta v_i + \Theta(t)L(-2t)\partial I_K(v) \ni \tilde{f}(t,v), \quad &\text{on} \ Q, \ i = 1,k, \\
v(x,0) = u_0(x), \quad &\text{on} \ D, \\
v(x,t) = 0, \quad &\text{on} \ \Gamma \times (0,T),
\end{align*}
\]

where \( \tilde{f}(t,v) = L(-t)(f(t,L(t)v) - BL(t)v) \), since \( \frac{d}{dt}(L(t)v) = BL(t)v \). Apply now Theorem 3.3, with \( \mathbb{H} = L^2(D), \forall \mathbb{H} = H^1_0(D), A = -\Delta \) and the existence of a solution \( (v,k) \) for (11) easily follows. Its uniqueness is assured by Theorem 3.5.

Acknowledgement: The author would like to express his sincere gratitude to the anonymous referees for their valuable comments and suggestions, which have resulted in considerable improvement of the results and presentation of this article.

The work for this paper was supported by funds from the project POSDRU/159/1.5/S/137750, Programe doctorale și postdoctorale-suport pentru creșterea competitivității cercetării în domeniul Științelor exacte.

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