COMPLETION OF SKEW COMPLETABLE UNIMODULAR ROWS

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ABSTRACT. In this paper, we prove that if $R$ is a local ring of dimension $d \geq 3$, $d$ odd and $\frac{1}{(d-1)!} \in R$ then any skew completable unimodular row $v \in Um_d(R[X])$ is completable. It is also proved that skew completable unimodular rows of size $d \geq 3$ over a regular local ring of dimension $d$ are first row of a 2-stably elementary matrix.

Throughout this article we will assume $R$ to be a commutative noetherian ring with $1 \neq 0$.

1. INTRODUCTION

In 1955, J.P. Serre asked whether there were non-free projective modules over a polynomial extension $k[X_1,\ldots,X_n]$, over a field $k$. D. Quillen ([5]) and A.A. Suslin ([9]) settled this problem independently in early 1976; and is now known as the Quillen–Suslin theorem. Since every finitely generated projective module over $k[X_1,\ldots,X_n]$ is stably free, to determine whether projective modules are free, it is enough to determine that unimodular rows over $k[X_1,\ldots,X_n]$ are completable. Therefore, problem of completion of unimodular rows is a central problem in classical $K$-Theory.

In [11], R.G. Swan and J. Towber showed that if $(a^2,b,c) \in Um_3(R)$ then it can be completed to an invertible matrix over $R$. This result of Swan and Towber was generalised by Suslin in [10] who showed that if $(a_0^1,a_1,\ldots,a_r) \in Um_{r+1}(R)$ then it can be completed to an invertible matrix. In [7], Ravi Rao studied the problem of completion of unimodular rows over $R[X]$, where $R$ is a local ring. Ravi Rao showed that if $R$ is a local ring of dimension $d \geq 2$, $\frac{1}{d!} \in R$, then any unimodular row over $R[X]$ of length $d + 1$ can be mapped to a factorial row by elementary transformations. In [8], Ravi Rao proved that if $R$ is a local ring of dimension 3 with $2R = R$, then unimodular rows of length 3 are completable. In [2], Ravi Rao generalised his result with Anuradha Garge and proved that if $R$ is a
local ring of dimension 3 with $2R = R$ then any unimodular row of length 3 can be mapped to a factorial row via a two stably elementary matrix.

In this article, we generalise the result of Garge–Rao for skew completable unimodular rows. We prove:

**Theorem 1.1.** Let $R$ be a local ring of Krull dimension $d \geq 3$ with $d$ odd and $\frac{1}{(d-1)!} \in R$. Let $v = (v_0, v_1, \ldots, v_{d-1}) \in Um_d(R[X])$ be skew-completable unimodular row over $R[X]$. Then there exists $\rho \in SL_d(R[X]) \cap E_{d+2}(R[X])$ and an invertible alternating matrix $W \in SL_{d+1}(R[X])$ such that $v\rho = e_1K(W)$.

In the last section, we study the completion of skew completable unimodular rows over regular local rings. Since $SK_1(R[X])$ is trivial for a regular local ring $R$, we get the following result:

**Theorem 1.2.** Let $R$ be a regular local ring of Krull dimension $d \geq 3$ with $d$ odd and $\frac{1}{(d-1)!} \in R$. Let $v = (v_0, v_1, \ldots, v_{d-1}) \in Um_d(R[X])$ be skew-completable unimodular row over $R[X]$. Then there exists $\rho \in SL_d(R[X]) \cap E_{d+2}(R[X])$ such that $v = e_1\rho$.

2. Preliminary Remarks

A row $v = (a_0, a_1, \ldots, a_r) \in R^{r+1}$ is said to be unimodular if there is a $w = (b_0, b_1, \ldots, b_r) \in R^{r+1}$ with $\langle v, w \rangle = \sum_{i=0}^{r} a_i b_i = 1$ and $Um_{r+1}(R)$ will denote the set of unimodular rows (over $R$) of length $r+1$.

The group of elementary matrices is a subgroup of $GL_{r+1}(R)$, denoted by $E_{r+1}(R)$, and is generated by the matrices of the form $E_{ij}(\lambda) = I_{r+1} + \lambda e_{ij}$, where $\lambda \in R$, $i \neq j$, $1 \leq i, j \leq r+1$, $e_{ij} \in M_{r+1}(R)$ whose $ij$th entry is 1 and all other entries are zero. The elementary linear group $E_{r+1}(R)$ acts on the rows of length $r+1$ by right multiplication. Moreover, this action takes unimodular rows to unimodular rows: $\frac{Um_{r+1}(R)}{E_{r+1}(R)}$ will denote set of orbits of this action; and we shall denote by $[v]$ the equivalence class of a row $v$ under this equivalence relation.

2.1. The elementary symplectic Witt group $W_E(R)$. If $\alpha \in M_r(R), \beta \in M_s(R)$ are matrices then $\alpha \perp \beta$ denotes the matrix $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in M_{r+s}(R)$. $\psi_1$ will denote $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in E_2(\mathbb{Z})$, and $\psi_r$ is inductively defined by $\psi_r = \psi_{r-1} \perp \psi_1 \in E_{2r}(\mathbb{Z})$, for $r \geq 2$.

A skew-symmetric matrix whose diagonal elements are zero is called an alternating matrix. If $\phi \in M_{2r}(R)$ is alternating then $\det(\phi) = (\text{pf}(\phi))^2$ where pf is a polynomial (called the Pfaffian) in the matrix elements with coefficients $\pm 1$. 

Note that we need to fix a sign in the choice of pf; so we insist pf(ψr) = 1 for all r. For any α ∈ M2r(R) and any alternating matrix φ ∈ M2r(R) we have pf(αtφα) = pf(φ)det(α). For alternating matrices φ, ψ it is easy to check that pf(ψ ⊥ ψ) = (pf(φ))(pf(ψ)).

Two matrices α ∈ M2r(R), β ∈ M2r(R) are said to be equivalent (w.r.t. E(R)) if there exists a matrix ε ∈ SL2(r+s+l)(R) \cap E(R), such that α − ψs+l = εt(β ⊥ ψr+l)ε, for some l. Denote this by α ⪅ β. Thus ⪅ is an equivalence relation; denote by [α] the orbit of α under this relation.

It is easy to see ([12] p. 945) that ⊥ induces the structure of an abelian group on the set of all equivalence classes of alternating matrices with pfaffian 1; this group is called elementary symplectic Witt group and is denoted by WE(R).

2.2. W. Van der Kallen’s group structure on Um d+1(R)/Ed+1(R).

Definition 2.1. Essential dimension: Let R be a ring whose maximal spectrum Max(R) is a finite union of subsets Vi, where each Vi, when endowed with the (topology induced from the) Zariski topology is a space of Krull dimension d. We shall say R is essentially of dimension d in such a case.

For instance, a ring of Krull dimension d is obviously essentially of dimension ≤ d; a local ring of dimension d is essentially of dimension 0; whereas a polynomial extension R[X] of a local ring R of dimension d ≥ 1 has dimension d + 1 but is essentially of dimension d as Max(R[X]) = Max(R/(a)[X]) ∪ Max(Ra[X]) for any non-zero divisor a ∈ R.

In ([4] Theorem 3.6]), W. van der Kallen derives an abelian group structure on Um d+1(R)/Ed+1(R) when R is essentially of dimension d, for all d ≥ 2. Let * denote the group multiplication henceforth. He also proved in ([4] Theorem 3.16(iv)], that the first row map is a group homomorphism

\[ SL_{d+1}(R) \rightarrow \frac{Um_{d+1}(R)}{Ed+1(R)} \]

when R is essentially of dimension d, for all d ≥ 2.

Lemma 2.2. Let R be essentially of dimension d ≥ 2, and let Cd+1(R) denote the set of all completable (d + 1)-rows in Um d+1(R). Then,

- The map σ → [e1σ], where e1 = (1, 0, ..., 0) ∈ Um d+1(R), is a group homomorphism \[ SL_{d+1}(R) \rightarrow \frac{Um_{d+1}(R)}{Ed+1(R)} \].
- Cd+1(R) / Ed+1(R) is a subgroup of Um d+1(R) / Ed+1(R).

Proof: First follows from ([4] Theorem 3.16(iv]). Since v ∈ Cd+1(R) can be completed to a matrix of determinant one, Cd+1(R) / Ed+1(R) is the image of SL d+1(R) under the above mentioned homomorphism; whence is a subgroup of Um d+1(R) / Ed+1(R).
Proposition 2.3. Let $R$ be a local ring of dimension $d$, $d \geq 3$ and \( \frac{1}{(d-1)!} \in R \). Let \( v = (v_0, \ldots, v_d) \in Um_{d+1}(R[X]). \) Then $v$ is completable if and only if $v^{(d-1)} = (v_0^{(d-1)}, v_1, \ldots, v_d)$ is completable.

Proof: In view of ([7, Remark 1.4.3]), we may assume that $R$ is a reduced ring. By ([7, Lemma 1.3.1, Example 1.5.3]),
\[
[v^{(d-1)}] = [v] \ast [v] \ast \ldots \ast [v], \quad (d - 1) \text{ times}
\]
in \( \frac{Um_{d+1}(R[X])}{E_{d+1}(R[X])} \). By Lemma 2.2, $v$ is completable implies $v^{(d-1)}$ is also completable.

Conversely, let $v^{(d-1)}$ be completable. By ([7, Proposition 1.4.4]),
\[
v \sim (w_0, w_1, \ldots, w_{d-1}, c)
\]
with $c \in R$ a non-zero-divisor. Since $\dim(R/(c)) = d - 1$ and \( \frac{1}{(d-1)!} \in R \), by ([7, Corollary 2.3]),
\[
(\tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_{d-1}) \in e_1 SL_d(R/(c)[X]).
\]

By ([3, Proposition 1.2, Chapter 5]), $(w_0, w_1, \ldots, w_{d-1}, c^d)$ is completable. Thus,
- \( (v_0, v_1, \ldots, v_{d-1}, v_d^d) \sim (w_0, w_1, \ldots, w_{d-1}, c^d) \) by ([14, Theorem 1]),
- \( [v]^n = [(v_0, v_1, \ldots, v_{d-1}, v_d^n)] \) for all $n$ by ([7, Lemma 1.3.1]).

Thus \( [v]^d = [(w_0, w_1, \ldots, w_{d-1}, c^d)] \in \frac{C_{d+1}(R[X])}{E_{d+1}(R[X])} \) and by hypothesis \( [v]^{d-1} = [v^{(d-1)}] \in \frac{C_{d+1}(R[X])}{E_{d+1}(R[X])} \). Therefore by Lemma 2.2, $v$ is completable. \( \square \)

3. Krusemeyer’s completion of the square of a skew completable row

Definition 3.1. A row \( v \in Um_{2r-1}(R) \) is said to be skew completable if there is an invertible alternating matrix \( V \in GL_{2r}(R) \) with \( e_1 V = (0, v) \).

First we note an example of skew completable unimodular row which is not completable.

Example 3.2 (Kaplansky). Let \( A = \frac{\mathbb{R}[x_0, x_1, x_2]}{(x_0 + x_1 + x_2 - 1)} \) and \( v = (x_0, x_1, x_2) \in Um_3(A) \).
In view of ([12 Section 5]), every unimodular row of length 3 is skew completable. Thus \( v = (x_0, x_1, x_2) \) is skew completable. Next we will show that $v$ is not completable.

Suppose to the contrary that $v = e_1 \sigma$ for some $\sigma \in SL_3(A)$. Let $\sigma = (\sigma_{ij})$. We can think $\sigma_{ij}$’s as a function on $S^2$. Let us define tangent vector field
\[
\phi : S^2 \longrightarrow \mathbb{R}^3
\]
\[
w \longmapsto ((\sigma_{21}^{-1})^t(w), \sigma_{22}^{-1}t(w), \sigma_{23}^{-1}t(w)).
\]
As \( \sigma_{ij} \)'s are polynomials, \( \phi \) is a differential function. Since \( (\sigma^{-1})^t \in SL_3(A) \), \( \phi \) is a nonvanishing continuous tangent vector field on \( S^2 \) which is a contradiction to Hairy ball theorem. Thus \( v \) is not completable.

**Theorem 3.3.** (M. Krusemeyer) ([5] Theorem 2.1) Let \( R \) be a commutative ring and \( v = (v_1, \ldots, v_n) \) be skew completable. Let \( V \) be a skew completion of \( v \), then 
\[(v_1^2, v_2, \ldots, v_n) \text{ is completable.} \]

**Notation 3.4.** In the above theorem we will denote \( K(V) \in SL_n(R) \) to be a completion of \((v_1^2, v_2, \ldots, v_n)\) for a skew completable unimodular row \( v = (v_1, \ldots, v_n) \) and its skew completion \( V \).

**Remark 3.5.** M. Krusemeyer’s proof in ([5] Theorem 2.1), shows that \( V \in (1 \perp K(V))E_{n+1}(R) \).

**Lemma 3.6.** Let \( R \) be a commutative ring and \( v = (v_1, \ldots, v_n) \in Um_n(R) \) be skew completable to \( V \). Then \( [e_1 K(V)] = [e_1 K(V)^t] \).

**Proof:** By Remark 3.5, \( V \in (1 \perp K(V))E_{n+1}(R) \). Since \(-I_{2k} \in E_{2k}(R)\), we have \( V \in V^t E_{n+1}(R) \). Therefore \((1 \perp K(V))^t \in (1 \perp K(V))E_{n+1}(R)\). Since stably \( K(V) \) and \( K(V)^t \) are in same elementary class, therefore in view of ([13] Lemma 10), we have \([e_1 K(V)] = [e_1 K(V)^t] \). \( \square \)

**Lemma 3.7.** Let \( R \) be a local ring with \( 1/2 \in R \) and let \( V \) be an invertible alternating matrix of Pfaffian \( 1 \). Let \( e_1 V = (0, v_1, \ldots, v_{2r-1}) \). Then \([V^{2n}] = [W]\), with \( e_1 W = (0, v_1^{2n}, \ldots, v_{2r-1})\).

**Proof:** We will prove it by induction on \( n \). For \( n = 1 \), by ([2] Corollary 4.3), \( W_E(R[X]) \hookrightarrow SK_1(R[X]) \) is injective, we have
\[ V \perp V \equiv V^2 \equiv V^t \psi_r V \equiv SK_1(1 \perp K(V))^t \psi_r (1 \perp K(V)). \]

Therefore \([V^2] = [U]\) with \( e_1 U = (0, v_1^2, \ldots, v_{2r-1})\). Now assume that result is true for all \( k \leq n - 1 \) and let \([W_1] = [V^{2^{n-1}}]\) with \( e_1 W_1 = (0, v_1^{2^{n-1}}, \ldots, v_{2r-1})\). Then by lemma ([2] Corollary 4.3), \( W_E(R[X]) \hookrightarrow SK_1(R[X]) \) is injective, we have
\[ W_1 \perp W_1 \equiv W_1^2 \equiv W_1^t \psi_r W_1 \equiv (1 \perp K(W_1))^t \psi_r (1 \perp K(W_1)). \]

Therefore \([V^{2n}] = [W]\) with \( e_1 W = (0, v_1^{2n}, \ldots, v_{2r-1})\). \( \square \)

4. **Completion of skew-completable unimodular rows of length \( d \)**

In this section, we prove that if \( R \) is a local ring of dimension \( d \geq 3 \), \( d \) odd and \( \frac{1}{(d-1)!} \in R \) then any skew completable unimodular row \( v \in Um_d(R[X]) \) is completable.
Proposition 4.1. Let \( R \) be a local ring of dimension \( d \geq 3 \) with \( d \) odd and \( \frac{1}{(d-1)!} \in R \). Let \( V \in SL_{d+1}(R[X]) \) be an alternating matrix with Pfaffian 1. Then \( [V] = [(1 \perp K(W)^t)\psi_{d+1}(1 \perp K(W))] \) for some \( [W] \in W_E(R[X]) \). Consequently, there is a 1-stably elementary matrix \( \gamma \in SL_{d+1}(R[X]) \) such that

\[
V = \gamma^t(1 \perp K(W)^t)\psi_{d+1}(1 \perp K(W))\gamma.
\]

**Proof**: By \( ([8] \text{ Proposition 2.4.1}) \), \([V] = [W_1]^2\) for some \( W_1 \in W_E(R[X]) \). By \( ([9] \text{ Theorem 2.6}) \), \( Um_r(R[X]) = e_1 E_r(R[X]) \) for \( r \geq d + 2 \), so on applying \( ([12] \text{ Lemma 5.3 and Lemma 5.5}) \), a few times, if necessary, we can find an alternating matrix \( W \in SL_{d+1}(R[X]) \) such that \([W_1] = [W]\). Therefore \([V] = [W]^2\). Now, we have

\[
W \perp W \equiv W^t \psi_{d+1} W \equiv (1 \perp K(W)^t)\psi_{d+1}(1 \perp K(W)).
\]

Since in view of \( ([2] \text{ Corollary 4.3}) \), \( W_E(R[X]) \hookrightarrow SK_1(R[X]) \) is injective. Thus \([W]^2 = [(1 \perp K(W)^t)\psi_{d+1}(1 \perp K(W))]\). Therefore, \([V] = [(1 \perp K(W)^t)\psi_{d+1}(1 \perp K(W))]\). The last statement follows by applying \( ([12] \text{ Lemma 5.5 and Lemma 5.6}) \).

\[\square\]

Theorem 4.2. Let \( R \) be a local ring of Krull dimension \( d \geq 3 \) with \( d \) odd and \( \frac{1}{(d-1)!} \in R \). Let \( v = (v_0, v_1, \ldots, v_{d-1}) \in Um_d(R[X]) \) be skew-completable unimodular row over \( R[X] \). Then there exists \( \rho \in SL_d(R[X]) \cap E_{d+2}(R[X]) \) and an invertible alternating matrix \( W \in SL_{d+1}(R[X]) \) such that

\[
v\rho = e_1 K(W).
\]

**Proof**: Let \( V \in SL_{d+1}(R[X]) \) be an invertible alternating matrix of Pfaffian 1 which is a skew completion of \( v = (v_0, v_1, \ldots, v_{d-1}) \). By Proposition 4.1 there exists an alternating matrix \( W \in SL_{d+1}(R[X]) \) of Pfaffian 1 such that

\[
[V] = [(1 \perp K(W)^t)\psi_{d+1}(1 \perp K(W))].
\]

Therefore there exists \( \gamma \in SL_{d+1}(R[X]) \cap E_{d+2}(R[X]) \) such that

\[
\gamma^tV\gamma = (1 \perp K(W))^t\psi_{d+1}(1 \perp K(W)).
\]

In view of \( ([2] \text{ Corollary 5.17}) \), \( e_1\gamma \) can be completed to an elementary matrix. Thus there exists \( \varepsilon \in E_{d+1}(R[X]) \), and \( \rho_1 \in SL_d(R[X]) \cap E_{d+2}(R[X]) \) such that

\[
\varepsilon^t(1 \perp \rho_1)^tV(1 \perp \rho_1)\varepsilon = (1 \perp K(W))^t\psi_{d+1}(1 \perp K(W)).
\]

By \( ([1] \text{ Corollary 4.5}) \), there exists \( \varepsilon_1 \in E_d(R[X]) \) such that

\[
(1 \perp \varepsilon_1)^t(1 \perp \rho_1)^tV(1 \perp \rho_1)(1 \perp \varepsilon_1) = (1 \perp K(W))^t\psi_{d+1}(1 \perp K(W)).
\]

Now we set \( \rho = \rho_1\varepsilon_1 \). Thus \( v\rho = e_1 K(W) \). Hence \( v \) is completable. \[\square\]
5. Completion of unimodular 3-vectors

(8 Theorem 3.1), Ravi A. Rao proved that for a local ring of dimension 3, every \( v \in Um_3(R[X]) \) is completable. We get stronger results than Theorem 4.2 when we work with a local ring \( R \) of dimension 3. We reprove Anuradha Garge and Ravi Rao’s result in ([2] Corollary 5.18).

**Proposition 5.1.** Let \( R \) be a local ring of dimension 3 with \( \frac{1}{2k} \in R \) and let \( V \in SL_4(R[X]) \) be an alternating matrix of Pfaffian 1. Then \( [V] = [V^*] \) in \( W_E(R[X]) \) with \( e_1 V^* = (0, a^{2k}, b, c) \), and \( V^* \in SL_4(R[X]) \). Consequently, there is a stably elementary \( \gamma \in SL_4(R[X]) \) such that \( V = \gamma^t V^* \gamma \).

**Proof:** By ([8] Proposition 2.4.1), \( [V] = [W_1]^{2k} \) for some \( W_1 \in W_E(R[X]) \). By ([9] Theorem 2.6), \( Um_{r}(R[X]) = e_1 E_{r}(R[X]) \) for \( r \geq 5 \), so on applying ([12] Lemma 5.3 and Lemma 5.5), a few times, if necessary, we can find an alternating matrix \( V^* \in SL_4(R[X]) \) such that \( [W_1] = [V_1^*] \). Therefore \( [V] = [V^*]^{2k} \). Let \( [V^*]^{2k} = [V]^* \), thus \( [V] = [V^*] \). By ([2] Lemma 4.8), \( e_1 V^* = (0, a^{2k}, b, c) \). The last statement follows by applying ([12] Lemma 5.3 and Lemma 5.5). \( \square \)

**Theorem 5.2.** Let \( R \) be a local ring of Krull dimension 3 with \( \frac{1}{2k} \in R \). Let \( v = (v_0, v_1, v_2) \in Um_3(R[X]) \). Then there exists \( \rho \in SL_3(R[X]) \cap E_5(R[X]) \) such that

\[
v \rho = (a^{2k}, b, c) \text{ for some } (a, b, c) \in Um_3(R[X]).\]

**Proof:** Choose \( w = (w_0, w_1, w_2) \) such that \( \Sigma_{i=0}^2 v_i w_i = 1 \), and consider the alternating matrix \( V \) with Pfaffian 1 given by

\[
V = \begin{bmatrix}
0 & v_1 & v_2 \\
-v_0 & 0 & w_2 & -w_1 \\
-v_1 & -w_2 & 0 & w_0 \\
-v_2 & w_1 & -w_0 & 0
\end{bmatrix} \in SL_4(R[X]).
\]

By Proposition 5.1, there exists an alternating matrix \( V^* \in SL_4(R[X]) \), with \( e_1 V^* = (0, a^{2k}, b, c) \), of Pfaffian 1 such that

\[
[V] = [V^*].
\]

Therefore there exists \( \gamma \in SL_4(R[X]) \cap E_5(R[X]) \) such that

\[
\gamma^t V \gamma = V^*.
\]

In view of ([2] Corollary 5.17), \( e_1 \gamma \) can be completed to an elementary matrix. Thus there exists \( \varepsilon \in E_4(R[X]) \), and \( \rho_1 \in SL_3(R[X]) \cap E_5(R[X]) \) such that

\[
e_1^t (1 \perp \rho_1)^t V (1 \perp \rho_1) \varepsilon = V^*.
\]
By ([1] Corollary 4.5), there exists \( \varepsilon_1 \in E_3(R[X]) \) such that
\[
(1 \perp \varepsilon_1)^t(1 \perp \rho_1)^tV(1 \perp \rho_1)(1 \perp \varepsilon_1) = V^*.
\]
Now we set \( \rho = \rho_1\varepsilon_1 \). Thus \( v\rho = (a^{2k}, b, c) \) for some \( (a, b, c) \in Um_3(R[X]) \). \( \square \)

6. COMPLETION OVER REGULAR LOCAL RINGS

In this section, we prove that skew completable unimodular rows of size \( d \geq 3 \) over a regular local ring of dimension \( d \) are first row of a 2-stably elementary matrix.

We note a result of Rao and Garge in ([2] Corollary 4.3]).

**Lemma 6.1.** Let \( R \) be a local ring with \( 2R = R \). Then the natural map
\[
W_E(R[X]) \rightarrow SK_1(R[X])
\]
is injective.

**Corollary 6.2.** Let \( R \) be a regular local ring with \( 2R = R \). Then the Witt group \( W_E(R[X]) \) is trivial.

**Proof:** Since \( R \) is a regular local ring, \( SK_1(R[X]) = 0 \). Thus the result follows in view of Lemma 6.1. \( \square \)

**Lemma 6.3.** Let \( R \) be a regular local ring of Krull dimension \( d \geq 3 \) with \( d \) odd and \( \frac{1}{(d-1)!} \in R \). Let \( v = (v_0, v_1, \ldots, v_{d-1}) \in Um_d(R[X]) \) be skew-completable unimodular row over \( R[X] \). Then there exists \( \rho \in SL_d(R[X]) \cap E_{d+2}(R[X]) \) such that \( v = e_1\rho \).

**Proof:** Let \( V \) be a skew completion of \( v \). In view of Corollary 6.2, \( W_E(R[X]) = 0 \), we have \( [V] = [\psi_{d+1}] \). Thus upon applying ([12] Lemma 5.5 and Lemma 5.6]), there exists \( \gamma \in SL_{d+1}(R[X]) \cap E_{d+2}(R[X]) \) such that \( \gamma^tV\gamma = \psi_{d+1} \).

By ([2] Corollary 5.17]), \( e_1\gamma \) can be completed to an elementary matrix. Thus there exists \( \varepsilon \in E_{d+2}(R[X]) \), and \( \rho_1 \in SL_d(R[X]) \cap E_{d+2}(R[X]) \) such that
\[
\varepsilon^t(1 \perp \rho_1)^tV(1 \perp \rho_1)\varepsilon = \psi_{d+1}.
\]
By ([1] Corollary 4.5]), there exists \( \varepsilon_1 \in E_d(R[X]) \) such that
\[
(1 \perp \varepsilon_1)^t(1 \perp \rho_1)^tV(1 \perp \rho_1)(1 \perp \varepsilon_1) = \psi_{d+1}.
\]
Now we set \( \rho = (\rho_1\varepsilon_1)^{-1} \). Thus we have \( v = e_1\rho \). \( \square \)

**Corollary 6.4.** Let \( R \) be a regular local ring of Krull dimension 3 with \( \frac{1}{2} \in R \). Let \( v = (v_0, v_1, v_2) \in Um_3(R[X]) \). Then there exists \( \rho \in SL_3(R[X]) \cap E_5(R[X]) \) such that \( v = e_1\rho \).

**Proof:** Since every \( v \in Um_3(R[X]) \) is skew completable, thus the result follows in view of Lemma 6.3. \( \square \)
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