Equation of motion for relativistic compact binaries
with the strong field point particle limit:
Formulation, the first post-Newtonian and multipole terms

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We derive the equation of motion for the relativistic compact binaries
in the post-Newtonian approximation taking explicitly their strong internal
gravity into account. For this purpose we adopt the method of the point
particle limit where the equation of motion is expressed in terms of the sur-
face integrals. We examine carefully the behavior of the surface integrals in
the derivation. As a result, we obtain the Einstein-Infeld-Hoffman equation
of motion at the first post-Newtonian (1PN) order, and a part of the 2PN
order which depends on the quadrupole moments and the spins of component
stars. Hence, it is found that the equation of motion in the post-Newtonian
approximation is valid for the compact binaries by a suitable definition of the
mass, spin and quadrupole moment.

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I. INTRODUCTION

The motion of self gravitating systems and the associated gravitational waves have been among the major research interest in general relativity. Furthermore, understanding systems with strong internal gravity such as neutron star binaries are important also in astrophysics, since they are thought as a possible candidate for gamma ray bursts \[1\] as well as the most likely sources of gravitational waves. Laser interferometric detectors are under construction for detecting gravitational waves (LIGO \[2\], VIRGO \[3\], TAMA \[4\] and GEO600 \[5\]). It is expected that these will become new eyes looking into the universe and bring great impacts on astrophysics. For this to become realistic, reliable comparisons between theoretical predictions and the observations are definitely necessary. Namely we have to prepare sufficiently accurate theoretical templates for waveforms for likely sources including inspiralling neutron star binaries. This is possible only if we know their precise equation of motion \[1, 2\]. Since the Einstein equation is highly nonlinear system as is well-known, the use of some kind of approximation is unavoidable to describe a dynamical system such as a neutron stars binary.

Among various approximation schemes, the post-Newtonian approximation is most popular for treating nonlinear gravitational interactions \[10\], and the equation of motion has been derived successfully up to the 2.5PN order \[11-15\]. At higher orders, however, an expansion in power series seems to break down (e. g. \[16\]). Hence, it is very interesting to see what form the equation of motion takes at higher orders. In recent, 3PN, 3.5PN and higher orders are being tackled \[17, 18\]. Furthermore, some methods have been developed to treat the matter of binaries and internal gravity of them. For instance, in the post-Minkowskian approximation \[11, 12\], the self gravity causes divergent behavior at the position of a point source, so that regularization schemes have been used \[13, 14\]. Other method assumed that the neutron star is modeled by the spherical distribution of the perfect fluid \[20, 14\]. In these derivations where the standard post-Newtonian approximation is used, however, the weak field is assumed everywhere or everywhere except at the singular point. As a result, strictly speaking, the obtained equation of motion is applicable only to a object with weak internal
gravity or to a singular source whose relation to the real system is not clear. It is then necessary to extrapolate the equivalence principle to the extended objects with strong self gravity, in order to apply the post-Newtonian equation of motion derived from the standard approach to neutron star binaries. Since such an extrapolation is not trivial at all, it must be shown within the framework of general relativity if it works.

The equation of motion for bodies with strong self gravity has been studied by various authors; D’Eath developed an approach based on the asymptotic matching \cite{21}. However, this approach is based on a perturbation of black holes. Hence, there remain several problems to be solved in this scheme; how can we take into account the difference between neutron stars and black holes? How can we go to nonlinear perturbations? (Recall that the nonlinear perturbation is required for equal-mass binaries even at the 1 PN order.) Thorne and Hartle’s derivation is also using the linear order perturbation \cite{22}. However, there seem a great of difficulties in pushing their method forward to obtain a higher order equation of motion.

In the present paper, we take other approach, the so-called point particle limit in order to treat the strong self gravity where the size of the component star is reduced to a point keeping the strength of the internal gravity \cite{23,24}. The method has been used to derive only the equation of motion at the Newtonian order and the quadrupole radiation reaction at 2.5PN order for compact binaries. Here and henceforth, by the post-Newtonian approximation, we mean only the smallness of the ratio of the orbital velocity to the light velocity and the weakness of the gravitational field outside the compact objects, but not the weakness of the gravitational field inside these. As discussed above, we need to know higher post-Newtonian orders up to, say, 4th post-Newtonian order to have an accurate prediction of the emitted waveform \cite{25}. This is what we are aiming at. Namely, we wish to derive the equation of motion applicable to systems with strong internal gravity such as compact binaries up to a sufficiently higher post-Newtonian orders using the point particle limit. As the first step, here we derive the first and (partly) second post-Newtonian equation of motion for compact binaries taking fully strong internal gravity into account. For the sake of the completeness, we shall show some details of the approach which has been proposed by one of the present
Our main contributions in this paper are to derive the first post-Newtonian equation of motion, the lowest spin-orbit coupling force, the lowest quadrupole-orbit coupling force for relativistic compact binaries and a general momentum velocity relation which was not discussed in [24].

This paper is organized as follows. Section II presents a general framework for the derivation; the equation of motion expressed with surface integrals and the point particle limit. In section III, we review the Newtonian-like equation of motion for compact binaries to show our basic calculation. We derive the 1PN equation of motion in section IV, namely with some remarks on the definition of the mass and momentum in this paper and take account of effects of the quadrupole moment and the spin of compact objects in section V. Section VI is devoted to the conclusion. In appendix A we will give some comments on the derivation of the components of the metric. Some formulae are given in the appendix B.

Greek indices take from 0 to 3, and Latin indices from 1 to 3.

II. GENERAL FORMALISM

A. Newtonian scalings and point particle limit

It is expected that the orbital motion of neutron star binaries in the beginning of the inspiralling phase are governed mainly by Newtonian gravity so that it is natural to formulate our approach based on the post-Newtonian approximation. For our purpose it turns out convenient to introduce a Newtonian dynamical time as \[ \tau = \epsilon t. \] (2.1)

Then, the coordinate velocity is

\[ \frac{dx^i}{dt} = \epsilon \frac{dx^i}{d\tau}, \] (2.2)

so that the velocity scales as \( \epsilon \). Note that events with different \( \epsilon \) but at the same Newtonian dynamical time \( \tau \) are at (roughly) the same phase in the orbit. This motivates us to define
the post-Newtonian sequence by letting $\epsilon$ go to zero with $\tau$ staying constant. We call this limit, $\epsilon \to 0$, the Newtonian limit. In the binary whose mass and separation are denoted by $M$ and $L$ respectively, the balance of Newtonian forces is written as

$$\left| \frac{dx^i}{dt} \right|^2 \sim \frac{M}{L}.$$  \hspace{1cm} (2.3)

Hence, we find that $M$ scales as $\epsilon^2$ when we fix the separation $L$.

In order to treat strong internal gravity we introduce the point particle limit in the post-Newtonian sequence. Namely, denoting a size of a neutron star by $R$, we expect $M/R$ is of order of unity. To be so, $R$ must scale as $\epsilon^2$, which means that the density is of order $\epsilon^{-4}$. Hence, we define the body zones $B_A$ and the body zone coordinates as $B_A = \{ x^k | |x^k - z_A^k(\tau)| < \epsilon R_A \}$ for some $R_A$ and

$$\alpha_A^k = \epsilon^{-2}(x^k - z_A^k(\tau)),$$  \hspace{1cm} (2.4)

so that in the body zone coordinate, a size of a neutron star remains constant even as $\epsilon \to 0$. Here the underlined indices refer to the body zone coordinate and $z_A^k(\tau)$, $A = 1, 2$, are the near zone coordinates of the representative points of two bodies. There are three purposes to define the body zone and the body zone coordinate. First, it is useful to define physical quantities such as mass, spin with the body zone coordinate, since in this coordinate the body is nearly isolated. Second, as can be seen later, we define a gravitational force on the object $A$ as a momentum flow per unit time through the surface of the body zone of the object $A$. Thus we only need to know the metric at the boundary of the body zone where the assumption of weak gravity is valid. To deal with strong internal gravity, this is a great advantage since we do not have to care about the internal strong gravity. Third, the multipole moments of the body are reduced to the higher order terms relative to monopole terms thanks to the scaling of the body zone coordinate, as will be seen later. This is a nice feature because in the inspiralling phase it is a good approximation that compact objects are expressed as point particles with their multipoles as higher corrections.

In this paper we adopt the initial value approach as in \[20,24\] and we choose two stationary uniformly rotating fluids with spatially compact support as the initial data for
the matter. Also, we assume that both objects rotate slowly, that is, the velocities of the spinning motion of the bodies scale as $\epsilon$. It is easy to incorporate rapidly rotating stars into our formalism. The result is that a coefficient $\epsilon^3$ appears instead of the coefficient $\epsilon^4$ in the equation (5.14). We note that, with this scaling, the spin-orbit coupling force appears at the 2PN order, which is different from the usual result [13, 22, 28]. The initial data of the stress-energy tensor $T_A^{\mu\nu}$ of the body $A$ is supposed to scale like

\begin{align}
T_A^{\tau\tau} &= \epsilon^2 T_A^{\tau\tau} \sim \epsilon^{-2}, \\
T_A^{\tau i} &= \epsilon^{-1} T_A^{\tau i} \sim \epsilon^{-4}, \\
T_A^{ij} &= \epsilon^{-4} T_A^{ij} \sim \epsilon^{-8}.
\end{align}

(2.5) (2.6) (2.7)

Note that as a slowly rotating fluid, the object is supported by the pressure. Also, note that the assumption that the densities of the fluids scale as $\epsilon^{-4}$ are applied to the components in the near zone coordinate, not in the body zone coordinate. The scalings of the stress energy tensor of bodies on the initial hypersurface can be found in the Table I.

Before ending up this subsection, we mention briefly the coordinate transformation from the near zone coordinate to the Fermi normal coordinate of the extended body which is needed to derive the spin-orbital coupling force in the same form as the previous works [13, 22, 28].

These transformation may be obtained order by order, and at the lowest order, we identify this transformation as the Galilei transformation, so that we have

\begin{align}
T_N^{\tau\tau} &= T_A^{\tau\tau}, \\
T_N^{\tau i} &= \epsilon^2 T_A^{\tau i} + T_A^{\tau\tau} v_A^i, \\
T_N^{ij} &= T_A^{\tau\tau} v_A^i v_A^j + 2\epsilon^2 v_A^i (T_A^{\tau\tau})^j + \epsilon^4 T_A^{ij}.
\end{align}

(2.8) (2.9) (2.10)

where the subscript N means that these are components of the stress energy tensor in the near zone coordinate. These equations are enough for obtaining the 1PN equation of motion.

It is difficult, however, to obtain the transformation even at the next order when we consider a body with strong internal gravity (for the case of weak internal gravity, see e.g.
Fortunately, it turns out in section V that we do not need to know an explicit form of the transformation to derive the lowest spin-orbital coupling force.

**B. A general form of the equation of motion**

As discussed above we need to know the metric at the boundary of the body zone where it is assumed that the metric deviates slightly from the flat metric. Thus we define the small deviation from the flat metric $\eta^{\mu\nu}$ as

$$\bar{h}^{\mu\nu} = \eta^{\mu\nu} - \sqrt{-g}g^{\mu\nu}, \tag{2.11}$$

where we denote the determinant of the metric $g_{\mu\nu}$ by $g$.

We choose the harmonic condition on the metric as

$$\bar{h}^{\mu\nu,\nu} = 0, \tag{2.12}$$

where the comma denotes the partial derivative. Then, the Einstein’s equation is rewritten as

$$\Box \bar{h}^{\mu\nu} = -16\pi \Lambda^{\mu\nu}, \tag{2.13}$$

where we defined

$$\Box = \eta^{\mu\nu} \partial_\mu \partial_\nu, \tag{2.14}$$

$$\Lambda^{\mu\nu} = \Theta^{\mu\nu} + \chi^{\mu\nu,\alpha\beta}, \tag{2.15}$$

$$\Theta^{\mu\nu} = (-g)(T^{\mu\nu} + l_{LL}^{\mu\nu}), \tag{2.16}$$

$$\chi^{\mu\nu,\alpha\beta} = \frac{1}{16\pi}(\bar{h}^{\alpha\nu}\bar{h}^{\beta\mu} - \bar{h}^{\alpha\beta}\bar{h}^{\mu\nu}). \tag{2.17}$$

Here, $T^{\mu\nu}$ and $l_{LL}^{\mu\nu}$ denote the stress-energy tensor and the Landau-Lifshitz pseudotensor, respectively. The conservation law is expressed as

$$\Lambda^{\mu\nu,\nu} = 0. \tag{2.18}$$

The reduced Einstein’s equation is solved formally as
\[ \bar{h}^{\mu\nu} = 4 \int_{C(\tau,x^k;\epsilon)} d^3y \Lambda_{N}^{\mu\nu}(\tau - \epsilon|\vec{x} - \vec{y}|, y^k; \epsilon) \frac{\Lambda_{N}(\tau - \epsilon|\vec{x} - \vec{y}|, y^k; \epsilon)}{|\vec{x} - \vec{y}|}, \]  

(2.19)

where \( C(\tau,x^k;\epsilon) \) means the past light cone emanating from the event \((\tau,x^k)\). To clarify that \( \Lambda^{\mu\nu} \) appearing in the Eq. (2.19) are components in the near zone coordinate \((\tau,x^k)\), we added a subscript \( N \) to \( \Lambda^{\mu\nu} \). This equation must be solved iteratively to obtain an explicit form of the metric. We have ignored the homogeneous solution, since it is irrelevant to the dynamics of the binary at the order considered in this paper [27].

Let us define the four momentum of the object \( A \) as the integral over the body zone \( B_A \) like

\[ P_\mu^A(\tau) = \epsilon^2 \int_{B_A} d^3\alpha \Lambda^{\tau\mu}_N(\tau, \alpha_A^k; \epsilon). \]  

(2.20)

Note that \( \chi^{\tau\nu\alpha\beta} \) does not play any role up to the 2PN order. We can understand this from the fact that the lowest order of the fields \( \bar{h}^{\mu\nu} \) is \( \epsilon^4 \), which is seen in the next section, and the volume integrals of \( \chi^{\tau\nu\alpha\beta}, \alpha_\beta \) can be transformed into the surface integrals.

It should be noted that \( P_\mu^A \) depends only on the time coordinate. We call the \( \mu = \tau \) and \( i \) component of \( P_\mu^A \) simply the energy and the momentum of the object \( A \) respectively. Note that we have not specified \( z_A^k(\tau) \) yet. We define the derivative which does not change the region \( B_A \) as

\[ \frac{D}{D\tau} = \frac{\partial}{\partial\tau} + v_A^k \frac{\partial}{\partial x_k}, \]  

(2.21)

where we defined \( v_A^k \) as

\[ v_A^k = \dot{z}_A^k(\tau), \]  

(2.22)

and the dot denotes a temporal derivative. Note that the differential operator \( D/D\tau \) differs essentially from the usual Lagrange derivative \( d/d\tau \). But since the body zones for each body remain unchanged (in the near zone coordinate sense), these two operators are effectively the same when they act on the integrals over the body zones.

Here, we define the near zone dipole moment of the object as
\[
D^i_A(\tau) = \epsilon^2 \int_{B_A} \, d^3 \alpha_A \Lambda^\tau_\alpha (\tau, \alpha^\lambda_A; \epsilon) \alpha^i_A. \tag{2.23}
\]

To define the center of the mass \( z^i_A(\tau) \) of the object A it is required that \( D^i_A \) takes a certain value. At the Newtonian and post-Newtonian order, we will require that \( D^i_A \) vanishes. But to derive the spin-orbit coupling force in the usual form \([13,22,28]\), care must be taken when we define the center of the mass as can be seen later. By taking the temporal derivative of \( D^i_A \), we obtain

\[
P^i_A = P_A^\tau v_A^i + Q_A^i + \epsilon^2 \frac{dD_A^i}{d\tau}, \tag{2.24}
\]

where we used the derivative defined by Eq. (2.21). Here, we defined \( Q_A^i \) as

\[
Q_A^i = \epsilon^{-4} \int_{\partial B_A} dS_k \Lambda_N^{ki} \{x^i - z_A^i(\tau)\} - \epsilon^{-4} v_A^i \int_{\partial B_A} dS_k \Lambda_N^{\tau i} \{x^i - z_A^i(\tau)\}, \tag{2.25}
\]

where \( \partial B_A \) denotes the surface of the sphere \( B_A \). Equation (2.24) is thought as the relation between the \textit{velocity} and \textit{momentum} if we make a certain choice of the near zone dipole moment \( D^i_A \), that is, we define the center of mass of the object A. It is worthwhile to mention that the velocity and the momentum are not proportional to each other, but \( Q_A^i \) appears as its correction to be considered. Actually, \( Q_A^i \) contributes to the equation of motion as shown in the section V.

Using the conservation law, Eq. (2.18), we obtain

\[
\frac{d}{d\tau} P_A^\mu = -\epsilon^{-4} \int_{\partial B_A} dS_k \Lambda_N^{ki} + \epsilon^{-4} v_A^i \int_{\partial B_A} dS_k \Lambda_N^{\tau i}, \tag{2.26}
\]

where we used Eq. (2.21). This is an evolution equation for the four momentum \( P_A^\mu(\tau) \).

In order to obtain the equation of motion, an evolution equation for the momentum must be changed into an evolution equation for the velocity. Inserting Eq. (2.24) into the \( i \)-th component of Eq. (2.26), we obtain the general form of the equation of motion as

\[
P_A^\tau v_A^i = -\epsilon^{-4} \int_{\partial B_A} dS_k \Lambda_N^{ki} + \epsilon^{-4} v_A^i \int_{\partial B_A} dS_k \Lambda_N^{\tau i}
+ \epsilon^{-4} v_A^i \left( \int_{\partial B_A} dS_k \Lambda_N^{k\tau} - v_A^i \int_{\partial B_A} dS_k \Lambda_N^{\tau i} \right)
- \frac{d}{d\tau} Q_A^i - \epsilon^2 \frac{d^2 D_A^i}{d\tau^2}. \tag{2.27}
\]
The above set of our basic equations (2.25) and (2.27) consist only of the surface integrals when the representative point of the body A, \( z_A^i \), or equivalently, the value of the near zone dipole moment \( D_A^i \) is specified. Therefore, all we must do is to evaluate these surface integrals perturbatively. This is a great advantage of our formalism, since our evaluations can be done outside the strong self gravitational field.

### III. NEWTONIAN EQUATION OF MOTION

From the point of view of the point particle limit, the Newtonian order of the equation of motion for compact objects has been examined, where \( Q_A^i \) was not considered in \[24\]. However, since \( Q_A^i \) is newly introduced in this paper, we rederive the Newtonian equation of motion in this section. We also intend the completeness and to show our basic calculation.

At the Newtonian order, we use equations (2.8), (2.9) and (2.10). The near zone dipole moment becomes

\[
D_A^i(\tau) = \epsilon^2 \int_{B_A} d^3 \alpha_A \Lambda_A^\tau(\tau, \alpha_A; \epsilon) \alpha_A^i. \tag{3.1}
\]

So, we define the center of mass of the object A as \( z_A^i \), by requiring that the near zone dipole moment vanishes

\[
D_A^i(\tau) = 0. \tag{3.2}
\]

Then metric up to \( O(\epsilon^4) \) becomes \[23\]

\[
\bar{h}^{\tau\tau} = 4\epsilon^4 \sum_{A=1,2} \frac{m_A}{r_A} + O(\epsilon^5), \tag{3.3}
\]

\[
\bar{h}^{\tau i} = 4\epsilon^4 \sum_{A=1,2} \frac{J_A^i}{r_A} + 4\epsilon^4 \sum_{A=1,2} \frac{m_A v_A^i}{r_A} + O(\epsilon^5), \tag{3.4}
\]

\[
\bar{h}^{ij} = 4\epsilon^2 \sum_{A=1,2} \frac{Z_A^{ij}}{r_A^2} + 8\epsilon^4 \sum_{A=1,2} \frac{v_A^{(i} J_A^{j)}}{r_A} + 4\epsilon^4 \sum_{A=1,2} \frac{m_A v_A^i v_A^j}{r_A} + O(\epsilon^5), \tag{3.5}
\]

\[
+ \epsilon^4 \sum_{A=1,2} \frac{m_A^2}{r_A^4} \gamma_A^{ij} + \epsilon^4 \left\{ \frac{8m_1 m_2}{r_{12} S^2} \bar{n}_{12} \bar{n}_{12}^i \bar{n}_{12}^j \right\} - 8\epsilon^4 \left[ \delta^{ij} \delta_{kl} - \frac{1}{2} \delta^{ij} \delta_{kl} \right] \frac{m_1 m_2}{S^2} (\bar{n}_{12} - \bar{n}_1) (\bar{n}_{12} + \bar{n}_2) \bigl] + O(\epsilon^5),
\]
where \( r_A^i = x^i - z_A^i \), \( r_A = |\vec{r}_A| \), \( r_{12}^i = z_1^i - z_2^i \) and \( S = r_1 + r_2 + r_{12} \). We defined

\[
m_A = \lim_{\epsilon \to 0} \epsilon^2 \int_{B_A} d^3 \alpha_A \Lambda_{\tau \tau}^A(\tau, \alpha^k_A; \epsilon),
\]
(3.6)

\[
J_A^i = \lim_{\epsilon \to 0} \epsilon^4 \int_{B_A} d^3 \alpha_A \Lambda_{\tau i}^A(\tau, \alpha^k_A; \epsilon),
\]
(3.7)

\[
Z_A^{ij} = \lim_{\epsilon \to 0} \epsilon^8 \int_{B_A} d^3 \alpha_A \Lambda_{ij}^A(\tau, \alpha^k_A; \epsilon).
\]
(3.8)

The \( m_A \) is the ADM mass that the body \( A \) would have if the body \( A \) were isolated. By choosing the center of mass, \( J_A^i \) vanishes. Furthermore, we assume that the fluids of the two bodies are (quasi-) stationary so that \( Z_A^{ij} \) are of higher order, \( \epsilon^6 \) \[24\]. Note that at the Newtonian order, we have \( P_\tau^A = m_A \) by virtue of Eq. (2.8). We also note that the temporal derivative of the \( P_\tau^A \) vanishes at the Newtonian order from Eqs. (B1) and (B2). To obtain the Newtonian equation of motion, we need only the field \( \bar{h}^\tau\tau \) of order \( \epsilon^4 \).

Now, to evaluate the surface integrals we take

\[
\vec{r}_1 = \epsilon R_1 \vec{n}_1,
\]
(3.9)

\[
\vec{r}_2 = \vec{r}_{12} + \epsilon R_1 \vec{n}_1,
\]
(3.10)

\[
\vec{r}_{12} = \vec{z}_1 - \vec{z}_2,
\]
(3.11)

where \( \vec{n}_1 \) denotes the spatial unit vector emanating from \( \vec{z}_1 \) and \( \epsilon R_A \) is the radius of the sphere \( B_A \). The pseudotensor at this order is

\[
(4)[(-g)^{ik}_{LL}] = \frac{1}{64\pi} \left( (4)\bar{h}^{\tau\tau,i}_{(4)}(4)\bar{h}^{\tau\tau,k}_{(4)} - \frac{1}{2} \delta^{ik}_{(4)}(4)\bar{h}^{\tau\tau,l}_{(4)}(4)\bar{h}^{\tau\tau,l}_{(4)} \right),
\]
(3.12)

where \((n)\) means the order of \( \epsilon^n \). We evaluate

\[
\oint_{\partial B_1} dS_k(4)[(-g)^{ik}_{LL}] = \frac{1}{64\pi} \left( \delta^i_m \delta^k_m - \frac{1}{2} \delta^{ik} \delta^i_m \right) \sum_{A=1,2} \sum_{B=1,2} \oint_{\partial B_1} dS_k \frac{m_A m_B r^{i}_A r^{j}_B}{r_A^3 r_B^3},
\]

\[
= \frac{m_1 m_2}{r_{12}^3} n_{12}^i,
\]
(3.13)

where we defined \( n_{12}^i \) as

\[
\vec{n}_{12} = \frac{\vec{r}_{12}}{|\vec{r}_{12}|}.
\]
(3.14)
Hence, we obtain

$$\frac{dP_i^i}{d\tau} = -\frac{m_1 m_2}{r_{12}^2} n_{12}^i. \quad (3.15)$$

At the Newtonian order, we have $Q_A^i = 0$ from Eqs. (B1) and (B2), so we have the Newtonian momentum velocity relation, $P_A^i = m_A v_A^i$. Hence we obtain the Newtonian equation of motion for relativistic compact binaries.

IV. THE FIRST POST-NEWTONIAN EQUATION OF MOTION

In this section, we shall derive the 1PN equation of motion. At this order $T^\tau_\tau$ is transformed as

$$T^\tau_\tau = (\Gamma_A)^2 T_A^\tau + 2\varepsilon A^i T_A^i T_A^\tau + O(\varepsilon^6), \quad (4.1)$$

where $\Gamma_A$ is a transformation coefficient whose explicit expression is irrelevant for the purpose in this paper. The second term of the above equation does not contribute to the near zone dipole moment when we require that $J_A^i$ vanishes, so the condition $D_A^i = 0$ defines the same center of the mass as in the Newtonian case. However, we must take into account this transformation when we consider the spin-orbit coupling force as seen in the next section.

As for $Q_A^i$, this term appears from the 0.5PN order and depends on the size of the body zone. However up to the 0.5PN order, $Q_A^i$ does not affect the equation of motion. We will show this explicitly in the following two subsections.

A. The boundary dependent terms

Before deriving the 1PN equation of motion, we present some comments on the mass and momentum defined in this paper, which are relevant to the cancellation between terms depending on the size of the body zone; our definition depends on the size of the body zone since the pseudotensor has non-compact support. The boundary of the body zone is given
by hand so that the mass must depend on the size of the body zone. Thus, we may split the mass as

\[ m_A = \bar{m}_A + \tilde{m}_A, \tag{4.2} \]

where \( \bar{m}_A \) does not depend on the boundary size but \( \tilde{m}_A \) does. In order to perform this splitting, we use the fact that the integral of \( \Lambda_{N}^{\tau\tau} \) over the whole space \( W \) does not depend on the size of the body zone, say \( \epsilon R_1 \). In fact, from the relevant expression of the pseudotensor, the equations (B3) in the appendix B, we obtain at the lowest order

\[ \epsilon^2 \int_{W/B_1} d^3 \alpha \epsilon^6(0) \Lambda_{N}^{\tau\tau} = \epsilon^2 \int_{W/B_1} d^3 y(0) \Lambda_{N}^{\tau\tau} \]
\[ = -\epsilon^2 \frac{7}{8} \int_{\epsilon R_1}^{\infty} y_1^2 dy_1 \oint d\Omega_1 \left( \frac{m_1}{|y_1|} \right)_k \left( \frac{m_1}{|y_1|} \right)_k \]
\[ + \text{(terms independent of } R_1) \]
\[ = -\epsilon^2 \frac{7}{2} \frac{m_2^2}{\epsilon R_1} + \text{(terms independent of } R_1), \]

where \( W/B_1 \) means the spatial domain obtained by subtracting \( B_1 \) from \( W \). Since the total energy must not depend on the size of the body zone, we find

\[ \tilde{m}_A = \epsilon^2 \frac{7}{2} \frac{m_2^2}{\epsilon R_A} + O(\epsilon^2). \tag{4.3} \]

In the similar manner, from Eq. (B5) we split \( P_A^i \) into \( P_A^i = \bar{P}_A^i + \tilde{P}_A^i \) and obtain

\[ \tilde{P}_A^i = \epsilon^2 \frac{11}{3} \frac{m_2^2 v_A^i}{\epsilon R_A} + O(\epsilon^2). \tag{4.4} \]

Furthermore, we evaluate each integral in Eq. (2.25) as

\[ \epsilon^{-4} \int_{\partial B_1} dS_k(0) \Lambda_{N}^{\tau\tau} \left\{ x^i - z_1^i(\tau) \right\} = -\epsilon^2 \frac{7 m_1^2}{6 \epsilon R_1}, \tag{4.5} \]
\[ \epsilon^{-4} \int_{\partial B_1} dS_k(0) \Lambda_{N}^{\tau k} \left\{ x^i - z_1^i(\tau) \right\} = -\epsilon^2 \frac{m_2^2 v_1^i}{\epsilon R_1}. \tag{4.6} \]

Hence, we obtain \( \tilde{Q}_A^i = O(\epsilon^4) \) and

\[ \tilde{Q}_A^i = \epsilon^2 \frac{m_2^2 v_A^i}{6 \epsilon R_A} + O(\epsilon^2), \tag{4.7} \]
where we split $Q^i_A$ like $m_A$. Thus we have

$$\bar{P}^i_A = \bar{m}_A v^i_A + O(\epsilon^2). \quad (4.8)$$

This is the momentum velocity relation independent of the size of the boundary. Note that this relation is valid up to the 0.5PN order, since $Q^i_A$ and all the terms which depend on the size of the body zone are evaluated with the 1PN pseudotensor. There are other terms which depend on the size of the body zone. These come from the integrals which are formally of the 2PN and 2.5PN order (e.g. $(8) \Lambda^\tau_N$ in Eq. (4.3) ) and, at first sight, contribute to the 1PN equation of motion. We will discuss this issue in the forthcoming paper where we will derive the 2.5PN equation of motion.

Since the boundary-dependent term of the mass is of order $\epsilon$, when we pay attention only to the Newtonian order, we can rewrite the Newtonian equation of motion as

$$\bar{m}_1 \frac{dv^i_1}{d\tau} = -\bar{m}_1 \bar{m}_2 \frac{r^3_{12}}{r^2_{12}} n^i_{12}. \quad (4.9)$$

B. The first post-Newtonian equation of motion

In this subsection we derive the 1PN equation of motion paying attention to the cancellation between terms dependent on the size of the body zone.

First we consider the temporal component of the four momentum at the 1PN order. From Eqs. (B4), (B5), (3.3) and (3.4), we obtain

$$\frac{dP^\tau_1}{d\tau} = -\epsilon^{-4} \int_{\partial B_1} dS_k \epsilon^{(6)} \Lambda^k_N + \epsilon^{-4} v^k_1 \int_{\partial B_1} dS_k \epsilon^{(6)} \Lambda^\tau_N$$

$$= -\epsilon^2 \bar{m}_1 \bar{m}_2 \frac{r^3_{12}}{r^2_{12}} \left(4(\bar{r}_{12} \cdot \bar{v}_1) - 3(\bar{r}_{12} \cdot \bar{v}_2)\right), \quad (4.10)$$

where we used the following integral formulae

$$\frac{1}{16\pi} \int_{\partial B_1} dS_{m(4)} \bar{h}^{\tau\tau,i}_{(4)} \bar{h}^{\tau\tau,j}_{(4)} = \frac{4 \bar{m}_1 \bar{m}_2 \delta^i_{m} r^j_{12}}{3 r^3_{12}} + \frac{4 \bar{m}_1 \bar{m}_2 \delta^j_{m} r^i_{12}}{3 r^3_{12}}, \quad (4.11)$$

$$\frac{1}{16\pi} \int_{\partial B_1} dS_{m(4)} \bar{h}^{\tau\tau,i}_{(4)} \bar{h}^{\tau\tau,k}_{(4)} = \frac{4 \bar{m}_1 \bar{m}_2 \delta^i_{m} r^j_{12}}{3 r^3_{12}} + \frac{4 \bar{m}_1 \bar{m}_2 \delta^j_{m} r^k_{12}}{3 r^3_{12}}. \quad (4.12)$$
Using the Newtonian equation of motion, this is rewritten as
\[
\frac{dP_1^r}{d\tau} = \epsilon^2 \bar{m}_1 \frac{d}{d\tau} \left( \frac{1}{2} v_1^2 + \frac{3\bar{m}_2}{r_{12}} \right). \tag{4.13}
\]

Hence, we obtain
\[
P_1^r = m_1 \left[ 1 + \epsilon^2 \left( \frac{1}{2} v_1^2 + \frac{3\bar{m}_2}{r_{12}} \right) \right] + O(\epsilon^3), \tag{4.14}
\]
where we renormalized the integration constant into \( m_1 \) and replaced \( \bar{m}_1 \) by \( m_1 \) at the \( \epsilon^2 \) order because the difference is of order \( \epsilon \) higher. We split \( P_1^r \) into the boundary independent part \( \bar{P}_1^r \) and the boundary dependent part \( \tilde{P}_1^r \). Up to the 0.5PN order \( \tilde{P}_1^r = \tilde{m}_A \).

Now we turn to the equation of motion. The relevant expressions for the pseudotensor and the metric components are given in the appendix B. The surface integrals appearing here are as follows [32];
\[
\oint_{\partial B_1} dS_{k(4)}(-g t_{LL}^{ik}) = \frac{P_1^r P_2^r}{r_{12}^2} n_{12}^i, \tag{4.15}
\]
\[
\oint_{\partial B_1} dS_{k(6)}(-g t_{LL}^{ik}) = \frac{\bar{m}_1 \bar{m}_2}{r_{12}^2} n_{12}^i \left( -\frac{3}{2} (\bar{v}_2 \cdot \bar{n}_{12})^2 + \frac{3}{2} v_2^2 - \frac{8}{3} (\bar{v}_1 \cdot \bar{v}_2) - \frac{8\bar{m}_1}{r_{12}} - \frac{4\bar{m}_2}{r_{12}} \right) - \frac{\tilde{m}_1 \tilde{m}_2}{r_{12}^2} (\bar{v}_2 \cdot \bar{n}_{12}) v_1^i + \frac{\bar{m}_1 \bar{m}_2}{r_{12}^2} \left( \frac{8}{3} (\bar{v}_1 \cdot \bar{n}_{12}) - 3(\bar{v}_2 \cdot \bar{n}_{12}) \right) v_1^i - \frac{\epsilon^2 \tilde{m}_1 \tilde{m}_2}{3 R_{12}^2} n_{12}^i, \tag{4.16}
\]
\[
v_1^k \oint_{\partial B_1} dS_{k(6)}(-g t_{LL}^{ir}) = \frac{\bar{m}_1 \bar{m}_2}{r_{12}^2} n_{12}^i \left( \frac{4}{3} (\bar{v}_1 \cdot \bar{v}_2) - v_1^2 \right) - \frac{\bar{m}_1 \bar{m}_2}{r_{12}^2} (\bar{v}_2 \cdot \bar{n}_{12}) v_1^i - \frac{4\bar{m}_1 \bar{m}_2}{3 r_{12}^2} (\bar{n}_{12} \cdot \bar{v}_1) v_1^i. \tag{4.17}
\]

We note that the last term in Eq. (4.16) precisely cancels out the time derivative of \( \tilde{P}_A^i \), Eq. (4.4) (or equivalently, \( \tilde{m}_A v_A^i + Q_A^i \)) when we use the Newtonian equation of motion (4.9).

Inserting these results and Eq. (4.14) into Eq. (2.27) and using Eq. (4.8), we obtain
\[
\bar{m}_1 \frac{d v_1^i}{d\tau} = \frac{-\bar{m}_1 \bar{m}_2}{r_{12}^2} n_{12}^i + \epsilon^2 \bar{m}_1 \bar{m}_2 \left[ n_{12}^i \left( -v_1^2 - 2v_2^2 + \frac{3}{2} (\bar{v}_2 \cdot \bar{n}_{12})^2 + 4(\bar{v}_1 \cdot \bar{v}_2) + \frac{5\bar{m}_1}{r_{12}} + \frac{4\bar{m}_2}{r_{12}} \right) + V^i \left( 4(\bar{v}_1 \cdot \bar{n}_{12}) - 3(\bar{v}_2 \cdot \bar{n}_{12}) \right) \right], \tag{4.18}
\]
where we defined the relative velocity as
Thus, it has been shown that the terms dependent on the size of the body zone do not affect the 1 PN equation of motion, since parts proportional to $R_A^{-1}$ cancel out totally.

The equation (4.18) takes the same form as those in previous works [19,11]. Hence, the applicability of the 1PN equation of motion has been extended to compact objects.

V. THE EQUATION OF MOTION WITH EFFECTS OF THE QUADRUPOLE MOMENT AND THE SPIN

Now we consider the forces associated with the spins and quadrupole moments of the component stars in the equation of motion which appear at the 2PN order in our ordering.

First, we must derive the metric components which depend on the spin and the quadrupole moment. The calculations are straightforward except for the spin part of the field $\bar{h}_{\tau\tau}$. From the equation (4.1) and the requirement that $J_A^i$ vanishes, we can calculate this term as

$$\bar{h}_{\tau\tau} = 4\epsilon^6 \sum_{A=1,2} \int_{B_A} d^3\alpha_A \frac{\Lambda^\tau_\tau_A (\tau - \epsilon |\vec{r}_A - \epsilon^2 \vec{\alpha}_A|, \vec{\alpha}_A; \epsilon)}{|\vec{r}_A - \epsilon^2 \vec{\alpha}_A|} + 4 \int_{C/B} d^3y \frac{\Lambda^\tau_\tau_A (\tau - \epsilon |\vec{x} - \vec{y}|, \vec{y}; \epsilon)}{|\vec{x} - \vec{y}|}$$

$$= 4\epsilon^4 \sum_{A=1,2} \left[ \frac{1}{r_A} \int_{B_A} d^3\alpha_A \epsilon^2 \Lambda^\tau_\tau_A (\tau, \vec{\alpha}_A; \epsilon) + \frac{\epsilon^2 r_A^i}{r_A^j} \int_{B_A} d^3\alpha_A \epsilon^2 \alpha^i_A \Lambda^\tau_\tau_A (\tau, \vec{\alpha}_A; \epsilon) \right]$$

$$+ \text{(terms irrelevant to the spin and higher order terms than 2PN)}$$

$$= 4\epsilon^4 \sum_{A=1,2} \left[ \frac{D^\tau_A}{r_A} + \epsilon^2 \frac{r_A^i}{r_A^j} (\tilde{d}^i_A + \epsilon^2 M^i_A v_A^k) \right]$$

$$+ \text{(terms irrelevant to the spin and higher order terms than 2PN),}$$

where in the last equality, we used the equation (2.19) in [24]. We defined $\tilde{d}^i_A$ and the spin tensor $M^{ij}_A$ of the body $A$ as

$$\tilde{d}^i_A = \lim_{\epsilon \to 0} \epsilon^2 \int_{B_A} d^3\alpha_A \Gamma^2_A \alpha^i_A \Lambda^\tau_\tau_A,$$

$$M^{ij}_A = \lim_{\epsilon \to 0} 2\epsilon^4 \int_{B_A} d^3\alpha_A \alpha^i_A \alpha^j_A \Lambda^\tau_\tau_A.$$

Recalling that $\Lambda^\tau_\tau_A$ is a component of a tensor in the Fermi normal coordinate, we define the center of mass by requiring that $\tilde{d}^i_A$, instead of $D^i_A$, vanishes.
It is important to note that the above choice of the center of the mass leads us to the following velocity-momentum relation;

\[ P^i_1 = P^r_1 v^i_1 + Q^i_1 - \epsilon^4 m_2 M^{ij}_1 n^j_{12} / r^2_{12}. \]  

(5.4)

Here we used the fact that the temporal derivative of the spin \( M^{ij}_A \) is of order \( \epsilon^2 \).

The required metric components are expressed as

\[
(8) \bar{h}_{\tau \tau} = 6 \sum_{A=1,2} \frac{r^k_A r^l_A}{r^A} \hat{I}^{kl}_A + 4 \sum_{A=1,2} \frac{r^k_A}{r^A} M^{ki}_A v^j_A,
\]

\[
(6) \bar{h}_{\tau i} = 2 \sum_{A=1,2} \frac{r^k_A}{r^A} M^{ki}_A,
\]

\[
(6) \bar{h}_{ij} = 4 \sum_{A=1,2} \frac{r^k_A}{r^A} M^{k(i,j)}_A,
\]

where \( \hat{I}^{ij}_A \) denotes the tracefree tensor for the reduced quadrupole moment of the body \( A \) defined as

\[
\hat{I}^{ij}_A = \lim_{\epsilon \to 0} \epsilon^2 \int_{B_A} d^3 \alpha_A \Lambda^{ij}_A (\alpha^i_A \alpha^j_A - \delta^{ij} |\alpha_A|^2).
\]

(5.5)

The spin vector \( S^i_A \) is related with \( M^{ij}_A \) as

\[
S^i_A = \frac{1}{2} \epsilon_{ijk} M^{jk}_A,
\]

(5.6)

where \( \epsilon_{ijk} \) means the spatially unit antisymmetric symbol.

The pseudotensor needed in this section is expressed as the equations (B8), (B9) and (B10) in the appendix B.

**A. Quadrupole-orbital couplings**

First we calculate the contribution from quadrupole moments. The surface integrals that we need to evaluate are only

\[
\oint_{\partial B_i} dS_{k(8)} (-gt^{ik}_{LL}) = -\frac{3}{2r^3_{12}} \left( \tilde{m}_1 \hat{I}^{kl}_2 + \tilde{m}_2 \hat{I}^{kl}_1 \right) \left( 2 \epsilon^{il} n^k_{12} - 5 n^i_{12} n^k_{12} n^l_{12} \right).
\]

(5.7)

From Eq. (2.27), we obtain immediately the result:
\[ m_1 \frac{dv_i}{d\tau} = \epsilon^4 \frac{3}{2r_{12}^4} \left( \bar{m}_1 \dot{I}_2^k + \bar{m}_2 \dot{I}_1^k \right) \left( 2\delta^{il}n_{12}^k - 5n_{12}^i n_{12}^k n_{12}^l \right). \] (5.8)

Here we omitted the Newtonian and the 1PN force and the spin-orbit coupling force. Note that we use \( \bar{m}_A \) instead of \( P^r_A \) in the left hand side of the equation (2.27) here since the difference between \( P^r_A \) and \( \bar{m}_A \) does not affect the quadrupole-orbital coupling. This is because from the equations (B8) and (B9) the equation of evolution for time component \( P^r_A \), Eq. (2.26), does not contain the quadrupole moment at the 2PN order, hence \( P^r_A \) does not contain the quadrupole moment.

This equation, (5.8), has the same form as the Newtonian equation of motion for the so-called tidal force. Therefore, it has been shown that the above equation is valid even for the compact objects.

**B. spin-orbital couplings**

Next we turn our attention to the spin-orbital coupling in the equation of motion. The relevant surface integrals are as follows:

\[ \oint_{\partial B_1} dS_{k(8)} \left( -g_{LL}^{\tau i} \right) = -\epsilon^3 \frac{2\bar{m}_1}{3r_{12}^3} M_{1i}^k - \frac{2\bar{m}_1}{3r_{12}^3} \left( M_{2i}^k \Delta^{lk} + M_{2l}^k \Delta^{li} \right) \]

\[ -\frac{2\bar{m}_2}{15r_{12}^3} \left( M_{1i}^k \Delta^{lk} + 3M_{1l}^k \Delta^{li} \right), \] (5.9)

\[ \oint_{\partial B_1} dS_{k(8)} \left( -g_{LL}^{ki} \right) = -\epsilon^3 \frac{2\bar{m}_1}{3r_{12}^3} M_{1i}^k v_1^k + \frac{\bar{m}_1}{r_{12}^3} \left( \frac{4}{3} v_1^k - 2v_2^k \right) \left( M_{2i}^k \Delta^{lk} + M_{2l}^k \Delta^{li} \right) \]

\[ + \frac{\bar{m}_2}{r_{12}^3} \left[ v_1^k \left( \frac{6}{5} M_{1i}^k \Delta^{lk} + \frac{8}{5} M_{1l}^k \Delta^{li} \right) \right. \]

\[ + v_2^k \left( -\frac{4}{3} M_{1i}^k \Delta^{lk} - 2M_{1l}^k \Delta^{li} \right), \] (5.10)

where we defined

\[ \Delta^{ij} = \delta^{ij} - 3n_{12}^i n_{12}^j. \] (5.11)

We also need to evaluate the surface integrals in \( Q_1^i \) which are

\[ \oint_{\partial B_1} dS_{k(8)} \left( -g_{LL}^{\tau k} \right) \left\{ x^i - z_1^i(\tau) \right\} = \frac{2\bar{m}_2}{3r_{12}^3} M_{1i}^k n_{12}^k. \] (5.12)
Then we obtain

\[ Q_i^j = \epsilon^4 \frac{2\bar{m}_2}{3r_{12}^2} M_{1i}^k \bar{m}_n^k. \]  

(5.13)

Note that the surface integral of \( (8) \left( -g_{LL}^\tau \right) \) does not depend on the spins of the bodies.

Inserting the above results into Eq. (2.27) and recalling the velocity-momentum relation (5.4), we obtain

\[
\bar{m}_1 \frac{dv_i^j}{d\tau} = -\epsilon^4 \frac{V^k}{r_{12}^3} \left[ \left( 2\bar{m}_1 M_{2i}^k + \bar{m}_2 M_{1i}^k \right) \Delta^{ik} + 2 \left( \bar{m}_1 M_{2l}^k + \bar{m}_2 M_{1l}^k \right) \Delta^{lk} \right],
\]

(5.14)

where we omitted irrelevant terms to the spin-orbit coupling force. Note that the difference between \( P_A^r \) and \( \bar{m}_A \) has no influence on the spin-orbital coupling because of the same reason as in the quadrupole-orbital coupling.

Here, it is worthwhile to mention that the terms proportional to \( R_{1}^{-3} \) cancel out totally in the equation of motion as we expect. This equation is exactly the same as the equation in [13, 22].

VI. CONCLUSION

Based on the point particle limit, we derived the equation of motion not only at the first post-Newtonian (1PN) order but also with effects of the quadrupole moment and the spin of compact objects at the 2PN order. It is worthwhile to mention why these effects are of the 2PN order, though they are usually of the Newtonian and the 1PN order. In the point particle limit, the ratio of the size of compact objects to the separation of the binary is taken as an expansion parameter, which is equal to \( \epsilon^2 \). Hence, the quadrupole moment and the spin become of higher orders in the ratio of the radius to the separation. Also, we assume that the velocity of the spinning motion of the body scales as \( \epsilon \). Therefore, their effects appear at the 2PN order.

As for the spin-orbit coupling, we must carefully define the center of the mass of the object to have the same expression as the previous works [13, 22, 28]. It is well-known that
the form of the spin-orbit coupling force depends on the definition of the center of the mass of the object [28].

We emphasize that the definitions of the mass, spin, quadrupole moment of objects include the effect of the strong self gravity explicitly (for the case of the weak internal gravity, e.g. [29,30]). Also we note that we did not need any regularization schemes. As a result, we have shown that the post-Newtonian equation of motion can be applied to compact binaries with strong internal gravity when we use the suitably defined mass, spin and quadrupole moment up to the order studied in this paper. We are pushing the present scheme ahead to complete the 2nd and 2.5 post-Newtonian equation of motion for the strong self-gravitating objects which will be investigated in the near future. It is also very interesting to see if the present scheme works for the 3rd post-Newtonian order. We hope to tackle this problem also in the near future.

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APPENDIX A: COMMENTS ON THE DERIVATION OF THE METRIC

The mass and four momentum are defined as the volume integral of $\Lambda^{\mu\nu}$ in the body zone so that these depend on the size of the body zone. On the other hand the integrated form of the reduced Einstein equation (2.19) has no information of the size of the body zone, so we expect that the metric components do not depend on it. We here show that the field $\tilde{h}^{\tau\tau}$ is really boundary-independent up to the 0.5PN order.
From Eq. (2.19), we have up to the 1PN order

$$\bar{h}^{\tau\tau} = 4\epsilon^4 \sum_{A=1,2} \frac{P_A^\tau}{r_A} + 4\epsilon^6 \int_{C/B} d^3y \frac{\Lambda_N^{\tau\tau}(\tau, y^k)}{|\vec{x} - \vec{y}|} + O(\epsilon^7). \quad (A1)$$

We can evaluate explicitly the boundary-dependent term arising from the non-compact source, i.e. the last term in the above equation.

From the Eqs. (B4) and (3.3),

$$\int_{C/B} d^3y \frac{\Lambda_N^{\tau\tau}(\tau, y^k)}{|\vec{x} - \vec{y}|} = -\frac{7}{8\pi} \sum_{A=1,2} \int_{C/B} d^3y \frac{m_A^2}{|\vec{y}_A|^4 |\vec{x} - \vec{y}|} + (\text{irrelevant terms}). \quad (A2)$$

Then it is easily verified that

$$\int_{C/B} d^3y \frac{1}{|\vec{y}_1|^4 |\vec{x} - \vec{y}|} = \int_{\epsilon R_1}^{r_1} d^3y \frac{1}{|\vec{y}_1|^4 |\vec{r}_1 - \vec{y}_1|} + \int_{r_1}^{\infty} d^3y \frac{1}{|\vec{y}_1|^4 |\vec{r}_1 - \vec{y}_1|} + (\text{higher order terms})
= \frac{4\pi}{r_1} \left( \frac{1}{\epsilon R_1} - \frac{1}{2r_1} \right) + (\text{higher order terms}). \quad (A3)$$

Inserting the above result into Eq. (A1) we obtain

$$\bar{h}^{\tau\tau} = 4\epsilon^4 \sum_{A} \frac{1}{r_A} \left( P_A^\tau - \epsilon^2 \frac{7m_A^2}{2\epsilon R_A} \right) + O(\epsilon^6). \quad (A4)$$

Let us split $P_A^\tau$ into $\tilde{P}_A^\tau$ and $\bar{P}_A^\tau$ as $P_A^\tau = \tilde{P}_A^\tau + \bar{P}_A^\tau$ where $\tilde{P}_A^\tau$ depends on the size of the boundary while the $\bar{P}_A^\tau$ does not. Then from the equation (1.14) we have relations like $P_A^\tau = m_A$, $\tilde{P}_A^\tau = \tilde{m}_A$ and $\bar{P}_A^\tau = \bar{m}_A$ up to the 0.5PN order. In section IV we obtained the following relation

$$\bar{m}_A = \epsilon^2 \frac{7m_A^2}{2\epsilon R_A}. \quad (A5)$$

Thus it is shown that up to the 0.5PN order the boundary-dependent term in $P_A^\tau$, i.e. $\tilde{P}_A^\tau$ cancels out the term arising from the non-compact source. We can write the metric up to the 0.5PN order as

$$\bar{h}^{\tau\tau} = 4\epsilon^4 \sum_{A} \frac{\bar{P}_A^\tau}{r_A} + O(\epsilon^6). \quad (A5)$$
APPENDIX B: THE METRIC AND THE PSEUDOTENSOR EXPANDED IN $\epsilon$

For convenience, we collect some formulae here. First, we show the components of the Landau-Lifshitz pseudotensor expanded in $\epsilon$ up to an order relevant in this paper. The lowest order in the pseudotensor is $\epsilon^4$ since the lowest order of the near zone field $\bar{h}^{\mu \nu}$ is $\epsilon^4$ as shown later. The components at the fifth order are identically zero, while the components at the seventh order vanish because $(5)\bar{h}^{\mu \nu}$ and $(7)\bar{h}^{\tau \tau}$ depend only on time $\tau$ [24]. The required orders of the gravitational field and the Landau-Lifshitz pseudotensor to obtain a certain order equation of motion are summarized in the Tables II and III.

$(\ )$ and $[\ ]$ mean the symmetrization and the antisymmetrization respectively.

$O(\epsilon^4)$

\[
\begin{align*}
(4)[-16\pi g^{i\tau}_{LL}t^{\tau\tau}] &= 0, \quad (B1) \\
(4)[-16\pi g^{i\tau}_{LL}t^{i\tau}] &= 0, \quad (B2) \\
(4)[-16\pi g^{ij}_{LL}t^{\tau\tau}] &= \frac{1}{4} \left(\begin{array}{c} (4)\bar{h}^{\tau\tau,(i)}(4)\bar{h}^{\tau\tau,j} - \frac{1}{2} \delta^{ij}(4)\bar{h}^{\tau\tau,k}\bar{h}^{\tau\tau,k}\end{array}\right). \quad (B3)
\end{align*}
\]

$O(\epsilon^6)$

\[
\begin{align*}
(6)[-16\pi g^{i\tau}_{LL}t^{i\tau}] &= -\frac{7}{8}\frac{1}{(4)}\bar{h}^{\tau\tau,k}(4)\bar{h}^{\tau\tau,k}, \quad (B4) \\
(6)[-16\pi g^{i\tau}_{LL}t^{i\tau}] &= 2\frac{1}{(4)}\bar{h}^{\tau\tau,k}(4)\bar{h}^{\tau\tau,i} - 3\frac{3}{4}\frac{1}{(4)}\bar{h}^{\tau\tau,k}(4)\bar{h}^{\tau\tau,i}, \quad (B5) \\
(6)[-16\pi g^{ij}_{LL}t^{\tau\tau}] &= 4\frac{1}{(4)}\bar{h}^{\tau\tau,(i)}(4)\bar{h}^{\tau\tau,j} - 2\frac{1}{(4)}\bar{h}^{\tau\tau,i}(4)\bar{h}^{\tau\tau,k} - \frac{1}{2}\frac{1}{(4)}\bar{h}^{\tau\tau,(i)}(6)\bar{h}^{\tau\tau,j} + \frac{1}{2}\frac{1}{(4)}\bar{h}^{\tau\tau,i}(4)\bar{h}^{\tau\tau,k} - 1\frac{1}{2}\frac{1}{(4)}\bar{h}^{\tau\tau,(i)}(4)\bar{h}^{\tau\tau,j} \\
&+ \delta^{ij} \left[ -\frac{3}{8}\frac{1}{(4)}\bar{h}^{\tau\tau,k}(4)\bar{h}^{\tau\tau,l} + (4)\bar{h}^{\tau\tau,k}(4)\bar{h}^{\tau\tau,l} \right. \\
&\left. + (4)\bar{h}^{\tau\tau,k}(4)\bar{h}^{\tau\tau,l} + (4)\bar{h}^{\tau\tau,k}(4)\bar{h}^{\tau\tau,l} \right] + (7)\bar{h}^{\tau\tau,k}(4)\bar{h}^{\tau\tau,k}. \quad (B6)
\end{align*}
\]

$O(\epsilon^7)$

\[
\begin{align*}
(7)[-16\pi g^{i\tau}_{LL}t^{i\tau}] &= 0, \\
(7)[-16\pi g^{i\tau}_{LL}t^{i\tau}] &= 0, \\
(7)[-16\pi g^{ij}_{LL}t^{\tau\tau}] &= \frac{1}{2} \left[ (4)\bar{h}^{\tau\tau,(i)}(5)\bar{h}^{\tau\tau,k} + (4)\bar{h}^{\tau\tau,(i)}(7)\bar{h}^{\tau\tau,j}\right].
\end{align*}
\]
and partly in the section V and appendix A. We do not show the field of order $O(\epsilon^5)$ up to the order relevant to this paper [32].

Next, we show here the field $\bar{h}_{ij}$ expanded in $\epsilon$. The derivations can be seen in [24], [17], [15] and partly in the section V and appendix A. We do not show the field of order $O(\epsilon^5)$ and $O(\epsilon^7)$, since these depend only on time [24] and hence have no effect on the equation of motion up to the order relevant to this paper [32].

\begin{equation}
O(\epsilon^8)
\end{equation}

\begin{equation}
(8)[-16\pi g t_{\tau\tau}^{ij}] = -\frac{7}{4}(4)\bar{h}_{\tau\tau,k}^{ij}(6)\bar{h}_{\tau\tau,k}^{ij}
\end{equation}

\begin{equation}
- \frac{3}{8}(4)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij} + (4)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij}
\end{equation}

\begin{equation}
+ \frac{1}{4}(4)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij}
\end{equation}

\begin{equation}
+ \frac{7}{8}(4)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij},
\end{equation}

\begin{equation}
(8)[-16\pi g t_{\tau\tau}^{ij}] = 2(4)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij} + (4)\bar{h}_{\tau\tau,l}^{ij}(4)\bar{h}_{\tau\tau,k}^{ij}
\end{equation}

\begin{equation}
- \frac{1}{4}(4)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij}
\end{equation}

\begin{equation}
+ 2(6)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij} - \frac{3}{4}(6)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij}
\end{equation}

\begin{equation}
+ 2(6)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij} - \frac{3}{4}(6)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij}
\end{equation}

\begin{equation}
+ 2(4)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij} + \frac{3}{4}(4)\bar{h}_{\tau\tau,l}^{ij}(4)\bar{h}_{\tau\tau,k}^{ij}
\end{equation}

\begin{equation}
- \frac{1}{4}(4)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij} + \frac{1}{8}(4)\bar{h}_{\tau\tau,l}^{ij}(4)\bar{h}_{\tau\tau,k}^{ij},
\end{equation}

\begin{equation}
(8)[-16\pi g t_{\tau\tau}^{ij}] = -2(4)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij} + \frac{1}{2}(4)\bar{h}_{\tau\tau,l}^{ij}(6)\bar{h}_{\tau\tau,k}^{ij}
\end{equation}

\begin{equation}
+ 2(6)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij} - 2(6)\bar{h}_{\tau\tau,k}^{ij}(4)\bar{h}_{\tau\tau,l}^{ij} + 2(4)\bar{h}_{\tau\tau,l}^{ij}(6)\bar{h}_{\tau\tau,k}^{ij} - 2(6)\bar{h}_{\tau\tau,l}^{ij}(4)\bar{h}_{\tau\tau,k}^{ij}
\end{equation}

\begin{equation}
+ \frac{1}{2}(4)\bar{h}_{\tau\tau,l}^{ij}(4)\bar{h}_{\tau\tau,k}^{ij}
\end{equation}

\begin{equation}
+ \delta^{ij}\left[ - \frac{3}{4}(4)\bar{h}_{\tau\tau,k}^{ij}(6)\bar{h}_{\tau\tau,l}^{ij} + 2(4)\bar{h}_{\tau\tau,k}^{ij}(6)\bar{h}_{\tau\tau,l}^{ij}
\end{equation}

\begin{equation}
+ (4)\bar{h}_{\tau\tau,l}^{ij}(6)\bar{h}_{\tau\tau,k}^{ij} - \frac{1}{4}(4)\bar{h}_{\tau\tau,k}^{ij}(6)\bar{h}_{\tau\tau,l}^{ij}
\end{equation}

\begin{equation}
- \frac{1}{4}(4)\bar{h}_{\tau\tau,k}^{ij}(6)\bar{h}_{\tau\tau,l}^{ij}
\end{equation}

\begin{equation}
+ (\text{Terms independent of the spin and quadrupole moment}).
\end{equation}
\( \bar{h}^{\tau\tau} = \epsilon^4(4)\bar{h}^{\tau\tau} + \epsilon^6(6)\bar{h}^{\tau\tau} + \epsilon^7(7)\bar{h}^{\tau\tau} + \epsilon^8(8)\bar{h}^{\tau\tau} + O(\epsilon^9), \)  
(B11)

\( (4)\bar{h}^{\tau\tau} = 4 \sum_{A=1,2} \frac{P^\tau_A}{r_A}, \)  
(B12)

\( (6)\bar{h}^{\tau\tau} = -2 \sum_{A=1,2} \frac{m_A}{r_A} \{ (\vec{v}_A \cdot \vec{v}_A)^2 - v_A^2 \} + 2 \frac{\bar{m}_1 \bar{m}_2}{r_{12}^2} \bar{n}_{12} \cdot (\bar{n}_1 - \bar{n}_2) \)  
+ 7 \sum_{A=1,2} \frac{\bar{m}_A^2}{r_A^2} + 14 \frac{\bar{m}_1 \bar{m}_2}{r_{12} r_{12}} \sum A=1,2 \frac{1}{r_A}, \)  
(B13)

\( (8)\bar{h}^{\tau\tau} = 6 \sum_{A=1,2} \frac{r_A^k r_A^l}{r_A^3} \bar{k}^{kl} + 4 \sum_{A=1,2} \frac{r_A^k}{r_A^3} M_A^{kl} v_A^l \)  
+ (Terms independent of the spin and quadrupole moment).  
(B14)

\( \bar{h}^{\tau i} = \epsilon^4(4)\bar{h}^{\tau i} + \epsilon^6(6)\bar{h}^{\tau i} + O(\epsilon^7), \)  
(B15)

\( (4)\bar{h}^{\tau i} = 4 \sum_{A=1,2} \frac{P^\tau_A v_A^i}{r_A}, \)  
(B16)

\( (6)\bar{h}^{\tau i} = 2 \sum_{A=1,2} \frac{r_A^k}{r_A^3} M_A^{ki} \)  
+ (Terms independent of the spin and quadrupole moment).  
(B17)

\( \bar{h}^{ij} = \epsilon^4(4)\bar{h}^{ij} + \epsilon^5(5)\bar{h}^{ij} + \epsilon^6(6)\bar{h}^{ij} + O(\epsilon^7), \)  
(B18)

\( (4)\bar{h}^{ij} = 4 \sum_{A=1,2} \frac{P^\tau_A v_A^i v_A^j}{r_A} \)  
+ \sum_{A=1,2} \frac{\bar{m}_A^2}{r_A^4} r_A^i r_A^j - \frac{8 \bar{m}_1 \bar{m}_2}{r_{12} S^2} n_{12}^i n_{12}^j \)  
- 8 \left[ \delta^i_k \delta^j_l - \frac{1}{2} \delta^i_l \delta^j_k \right] \frac{\bar{m}_1 \bar{m}_2}{S^2} (\bar{n}_{12} - \bar{n}_1)^k (\bar{n}_{12} + \bar{n}_2)^l, \)  
(B19)

\( (6)\bar{h}^{ij} = 4 \sum_{A=1,2} \frac{r_A^k}{r_A^3} M_A^{k(i} v_A^{j)} \)  
+ (Terms independent of the spin and quadrupole moment).  
(B20)

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[32] Mathematically, equations (1.13) and (B12) are wrong about the ordering. But it was easier to calculate with these expressions than those in a correct ordering, so we adopted the former ones.
| Near zone | Near zone | Body zone |
|-----------|-----------|-----------|
| $(t, x^i)$ | $(\tau, x^i)$ | $(\tau, \alpha^2)$ |

\[
T_{tt}^{\tau\tau} \sim \epsilon^{-4} \quad T_{tt}^{\tau i} \sim \epsilon^{-2} \quad T_{tt}^{\tau i} \sim \epsilon^{-2}
\]
\[
T_{\alpha}^{\tau\tau} \sim \epsilon^{-3} \quad T_{\alpha}^{\tau i} \sim \epsilon^{-4} \quad T_{\alpha}^{\tau i} \sim \epsilon^{-4}
\]
\[
T_{ij}^{\tau\tau} \sim \epsilon^{-4} \quad T_{ij}^{\tau i} \sim \epsilon^{-4} \quad T_{ij}^{\tau i} \sim \epsilon^{-8}
\]

**TABLE I.** Scalings of the stress energy tensor $T_{A}^{\mu\nu}$ of matters on the initial hypersurface in various coordinates.

| Newtonian order | 1 PN order | 2 PN order |
|----------------|------------|------------|
| $\tilde{h}^{\tau\tau}$ | $\epsilon^4$ | $\epsilon^6$ | $\epsilon^8$ |
| $\tilde{h}^{\tau i}$ | $-$ | $\epsilon^4$ | $\epsilon^6$ |
| $\tilde{h}^{ij}$ | $-$ | $\epsilon^4$ | $\epsilon^6$ |

**TABLE II.** Required order of the gravitational field to obtain an equation of motion (EOM) up to a certain order. For example, to obtain the Newtonian EOM, we have to calculate $\tilde{h}^{\tau\tau}$ up to $\epsilon^4$.

| Newtonian order | 1 PN order | 2 PN order |
|----------------|------------|------------|
| $t_{\tau\tau}^{LL}$ | $-$ | $\epsilon^6$ | $\epsilon^8$ |
| $t_{\tau i}^{LL}$ | $-$ | $\epsilon^6$ | $\epsilon^8$ |
| $t_{ij}^{LL}$ | $\epsilon^4$ | $\epsilon^6$ | $\epsilon^8$ |

**TABLE III.** Required order of the Landau-Lifshitz pseudotensor to obtain an equation of motion (EOM) up to a certain order. For example, to obtain the Newtonian EOM, we have to calculate $t_{ij}^{LL}$ up to $\epsilon^4$. 

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