1. Introduction

We consider the following multi-marginal optimal transport (MOT) problem

$$\inf_{\gamma \in \Gamma(\rho)} \int_{X^N} c(x_1, \ldots, x_N) \, d\gamma(x_1, \ldots, x_N),$$

(1.1)

where \((X, d)\) is a Polish space and \(\Gamma(\rho)\) denotes the set of Borel probability measures in \(X^N\) having all \(N\) marginals equal to a Borel probability measure \(\rho\). We are interested in cost
functions of the type
\[ c(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} f(d(x_i, x_j)), \]
where \( f : [0, +\infty[ \to \mathbb{R} \) is a continuous, decreasing function, not necessarily bounded above or below. An interesting example of such cost is given by minus the logarithmic: \( f(d(x, y)) = -\log(d(x, y)) \).

Our aim is to study properties of the so-called Kantorovich formulation of \((1.1)\) for such costs
\[
\sup \left\{ N \int_X u \, d\rho \mid u \in L^1_{\rho}(X), \sum_{i=1}^N u(x_i) \leq c(x_1, \ldots, x_N) \text{ for } \rho^{\otimes N}\text{-a.e. } (x_1, \ldots, x_N) \right\}, \tag{1.2}
\]
where \( \rho^{\otimes N} \) denotes the product of \( N \) measures \( \rho \). Optimal Transport problems with logarithmic-type costs were first considered in the literature by W. Wang \cite{26} and W. Gangbo and V. Oliker \cite{14} motivated by the reflector problem. In this case, \( X = S^d \), \( N = 2 \) and the authors show the existence of optimal transport plans \( \gamma = (\text{Id}, T)\#\rho \) in \((1.1)\) concentrated on the graph of a map \( T : S^d \to S^d \). Generally, in the reflector problem, the marginals are not necessarily equal.

In the multi-marginal case, logarithmic-type costs appear in Density Functional Theory (DFT), in the so-called strictly correlated limit (SCE). In SCE-DFT, the multi-marginal optimal transport problem is interpreted as the equilibrium configuration of a distribution of \( N \) charges in \( (SCE) \), in the so-called strictly correlated limit \( (1.3) \). In particular, when the particles are confined in the plane \( \mathbb{R}^2 \), the natural model of electrostatic potential between two charges \( x_i \) and \( x_j \) is given by the logarithmic interaction. We present in subsection 1.2 a pedagogical example of a charged wire, where the logarithmic electrostatic potential appears naturally.

In the following, we give a brief overview on DFT-OT. For a complete presentation on the topic, we refer the reader to \cite{12} and the references therein.

1.1. A brief review on the literature in DFT-OT. The problem \((1.1)\) when \( X = \mathbb{R}^3 \) and \( c \) is the Coulomb cost \( (f(|x - y|) = 1/|x - y|) \) was introduced in 1999 by M. Seidl \cite{23}. By using arguments from physics, Seidl suggested that, at least in the case when \( \rho \) is radially symmetric, a minimizer \( \gamma \) in \((1.1)\) exists and is concentrated on the graph of a map \( T : \mathbb{R}^3 \to \mathbb{R}^3 \), \( T\#\rho = \rho \), and its iterates, i.e.
\[ \gamma = (\text{Id}, T, T^{(2)}, \ldots, T^{(N-1)})\#\rho, \]
where \( T^{(N)} = \text{Id} \) and \( T^{(i)} \) is the \( i \)-times composition of the map \( T \) with itself. In particular, via the map \( T \), the optimality condition in the Kantorovich formulation of \((1.2)\) with Coulomb cost reads
\[
\nabla u(x) = -\sum_{i=1}^N \frac{x - T^{(i)}(x)}{|x - T^{(i)}(x)|^3}, \tag{1.3}
\]
As pointed out in [23] (see also [3]), the constraint in (1.2),
\[ \sum_{i=1}^{N} u(x_i) \leq \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|}, \]
has a simple physical meaning: it is required that, at optimality, the allowed manifold of the full 3D configuration space is the minimum of the classical potential energy given by the Coulomb interaction. Also, the equation (1.3) means that if such an optimal map \( T \) exists, the Kantorovich potential \( u(x) \) must compensate the net force acting on the electron in \( x \), resulting from the repulsion of the other \( N - 1 \) electrons at positions \( T(i) \) [24].

In Density Functional Theory (DFT), the problem (1.1) can be seen as a sort of a semi-classical limit (dilute limit of DFT) of the Hohenberg-Kohn functional\(^1\) [17, 20, 21]. This was suggested in the physics literature by Gori-Giorgi, Seidl and Vignale [16] and, proved rigorously in 2017 by Cotar, Friesecke and Kläppelberg [7, 8].

For the Coulomb cost in the 2-marginal case \( (N = 2) \), the existence of a unique optimal transport plan in (1.1) of type \( \gamma = (\text{Id}, T)_{\sharp} \rho \) \( (N = 2) \) was obtained, independently, by Cotar, Friesecke and Kläppelberg [7] and by Buttazzo, De Pascale and Gori-Giorgi [3]. In the multi-marginal case \( (N > 2) \) on the real line \( (d = 1) \), Colombo, De Pascale and Di Marino [10] proved the existence of optimal transport plans \( \gamma = (\text{Id}, T, \ldots, T^{(N-1)})_{\sharp} \rho \) in (1.1) for Coulomb costs. In [11, 12, 24], the repulsive harmonic cost
\[ c_w(x_1, \ldots, x_N) = - \sum_{1 \leq i < j \leq N} |x_i - x_j|^2 \]
was studied: Friesecke \textit{et al} [13] have shown the existence of optimal transport plans supported in \((N - 1)d\)-dimensional sets; in [12], explicit examples of such higher dimensional optimal transport plans as well as an example of an optimal transport plan \( \gamma \) concentrated on the graphs of \( \text{Id}, T, \ldots, T^{(N-1)} \) for a nowhere continuous map \( T : [0, 1]^d \to [0, 1]^d \) are presented. In [15], we gave an example of a three-marginal harmonic repulsion case with absolutely continuous marginals in \( \mathbb{R}^n \) for which there is a unique optimal transport plan which is not induced by a map.

1.2. Logarithmic Electrostatic potential: Charged wire. Consider a uniformly charged (infinitely thin) wire on the \( z \)-axis:
\[ \mathcal{W} := \{ x = (x, y, z) \in \mathbb{R}^3 : |z| < \delta \}, \quad 0 < \delta \ll 1. \]
Suppose that the wire has a charge density \( \rho(x) \). The resulting electric field is defined by
\[ E(x) = \frac{1}{4\pi \varepsilon_0} \int_{\mathbb{R}^3} \frac{x - s}{|x - s|^3} \rho(s) \, ds, \]
where \( \varepsilon_0 > 0 \) is a constant (permittivity of the free space). Due to \textit{Maxwell’s first equation} (or \textit{Gauss’ law} of electrostatics) the scalar field \( \rho : \mathbb{R}^3 \to \mathbb{R} \) and the vector field \( E(x) \) are related by
\[ \nabla \cdot E(x) = \frac{1}{\varepsilon_0} \rho(x). \]
\(^1\)Also known as the Levy-Lieb functional.
We define the total amount of charge $Q_\Omega$ in a cylinder $\Omega = \Omega_{R,H} \subset \mathbb{R}^3$ of radius $R > 0$ and height $H$, which has the wire as its axis of symmetry:

$$Q_\Omega = \int_{\Omega} \rho(s) \, ds = \epsilon_0 \int_{\Omega} \nabla \cdot E(x) \, dx = \epsilon_0 \int_{\partial\Omega} E(a) \cdot da,$$  

(1.4)

where the second equality is obtained using the Gauss’ theorem. Due to symmetry, the magnitude $|E(x)|$ of the electric field depends only on the Euclidean distance $s = d(x,w) = d(x,W)$ of a point $x$ from the wire, $|E(x)| = E(s)$, i.e $E(x) = (E(s) \cos \theta, E(s) \sin \theta, 0)$. Moreover, at each point $w$ on the lateral surface of this cylinder, the vector $E(w)$ is normal to the surface and has everywhere the same magnitude $|E(w)| = E(R)$.

Therefore, if $\rho(x) = \rho > 0$ is constant inside the cylinder, the flux integral and the total amount of charge in the cylinder $\Omega_{R,H}$ in (1.4) read

$$\frac{1}{\epsilon_0} \pi H = (2\pi R)H \cdot E(R),$$

and therefore,

$$E(R) = \frac{1}{2\pi \epsilon_0 R}.$$

Let us write $E(s) = 1/(2\pi \epsilon_0 s)$. Since $E(s) = -V'(s)$, the corresponding electrostatic potential $V(s)$ is of logarithmic form

$$V(s) = -\frac{1}{2\pi \epsilon_0} \log \frac{s}{s_0}, \quad s_0 > 0.$$

1.3. **Kantorovich duality.** The duality (1.2) and the existence of a maximizer in (1.2) was shown by Kellerer [18] in the case there exist $L^1(X)$-functions $h_1, \ldots, h_N$ and a constant $C$ such that

$$C \leq c(x_1, \ldots, x_N) \leq h(x_1) + \cdots + h(x_N).$$

More recently, De Pascale [9] and Buttazzo, Champion and De Pascale [2] extended the duality theory for a class of repulsive cost functions $c: \mathbb{R}^{dN} \to \mathbb{R} \cup \{+\infty\}$ which are bounded from below, allowing, for instance, the inclusion of the Coulomb $(s = 1)$ and Riesz cost functions $(1 \leq s \leq d)$

$$c(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} \frac{1}{|x_i - x_j|^s}.$$

The main contribution of this paper is to extend the duality theory for logarithmic costs. Some of our proofs are based on arguments present in [2]. One ingredient to tackle the problem of costs that are not bounded from below is to consider, for $R \in ]0, \infty[$, the truncated cost functions

$$c_R(x_1, \ldots, x_N) := \sum_{1 \leq i < j \leq N} \max\{f(R), f(d(x_i, x_j))\}, \text{ for all } (x_1, \ldots, x_N) \in X^N,$$  

(1.5)

and related total cost $C_R$, and collection $\mathcal{F}_R$ of functions for the dual problem:

$$C_R(\gamma) := \int_{X^N} c_R(x_1, \ldots, x_N) \, d\gamma(x_1, \ldots, x_N), \text{ for each } \gamma \in \Gamma(\rho),$$

and

$$\mathcal{F}_R := \left\{ u \in L^1(X) \mid u(x_1) + \cdots + u(x_N) \leq c_R(x_1, \ldots, x_N) \text{ for } \rho^{\otimes(N)}\text{-a.e. } (x_1, \ldots, x_N) \right\}.$$

In this paper, we will deal with the unbounded costs via the $\Gamma$-limit of their truncations.
1.4. **Organization of the paper.** This paper is divided as follows: in Section 2 we present the general setting and introduce briefly some properties of Γ-convergence. In Section 3 we discuss the existence of a minimizer in (1.1) by assuming that the marginals $\rho$ satisfy, with respect to the function $f$ that appears in our cost $c$, a condition analogous to the common assumption of the marginal measures having finite second moments (see condition (B) in Section 3).

In Section 4, we extend the duality results of [18, 9, 2] for a class of unbounded cost functions (Theorem 4.1) and in Section 5 we obtain regularity results of Kantorovich potentials (Theorem 5.2) as well as continuity of the cost functional as a function of the marginal $\rho$.

Finally, in Section 6 we give some applications of our results: we note the existence of optimal plans in (1.1), for log-type costs, which are concentrated on maps when $X = \mathbb{R}$, and we prove the existence of an optimal transport map for the logarithmic cost when $N = 2$.

2. **Preliminaries**

2.1. **General assumptions.** Let $(X, d)$ a Polish space. We consider a Borel probability measure $\rho \in \mathcal{P}(X)$ having small concentration, meaning

$$\lim_{r \to 0} \sup_{x \in X} \rho(B(x, r)) < \frac{1}{N(N-1)^2}. \quad (A)$$

We denote by $(x_1, \ldots, x_N)$ points in $X^N$, so $x_i \in X$ for each $i$. If we do not otherwise specify, each quantification with respect to $i$ or $i, j$ is from 1 to $N$. For a fixed $N \geq 1$, we assume that the cost $c: X^N \to \mathbb{R} \cup \{+\infty\}$ is of the form

$$c(x_1, \ldots, x_N) = \sum_{1 \leq i \neq j \leq N} f(d(x_i, x_j)), \quad \text{for all } (x_1, \ldots, x_N) \in X^N; \quad (2.1)$$

where $f: [0, \infty) \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions

$$f|_{[0,\infty[} \text{ is continuous and decreasing, and} \quad (F1)$$

$$\lim_{t \to 0^+} f(t) = +\infty. \quad (F2)$$

Let us denote for a fixed $R > 0$, for all $t > 0$

$$f_R(t) = \begin{cases} f(t) & \text{if } t < R \\ f(R) & \text{otherwise} \end{cases}$$

$$f_R^{-1}(t) = \inf \{s \mid f_R(s) = t\};$$

of course, if $f$ is not strictly decreasing, the inverse function $f^{-1}$ is not well defined, but still the left-inverse of $f$ can be defined as above.

We denote the set of couplings or transport plans having $N$ marginals equal to $\rho$ by

$$\Gamma(\rho) = \{ \gamma \in \mathcal{P}(X^N) \mid \text{pr}^i_\gamma = \rho \text{ for all } i \},$$

where pr$^i$ is the projection on the $i$-th coordinate

$$\text{pr}^i(x_1, \ldots, x_i, \ldots, x_N) = x_i, \quad \text{for all } (x_1, \ldots, x_i, \ldots, x_N) \in X^N.$$
this is the transportation cost related to $\gamma$.

We want to study the dual problem, so we set

$$
\mathcal{F} := \left\{ u \in L^1(\rho) \, \bigg| \, u(x_1) + \cdots + u(x_N) \leq c(x_1, \ldots, x_N) \text{ for } \rho^{\otimes N}-\text{a.e. } (x_1, \ldots, x_N) \right\}
$$

and

$$
D: L^1(\rho) \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}, \quad D(u) = N \int_X u \, d\rho \text{ for all } u \in L^1(\rho).
$$

Here one should note that, in the definition of $\mathcal{F}$ and also in future considerations, we identify the elements of $L^1(\rho)$ with their representatives unless otherwise stated. That is why the constraint

$$
u(x_1) + \cdots + u(x_N) \leq c(x_1, \ldots, x_N)
$$

is required to hold only for $\rho^{\otimes N}$-almost-every $(x_1, \ldots, x_N)$. Also, we do not allow the representatives to get the value $+\infty$. This we may do without loss of generality, since $L^1$-functions are finite almost everywhere.

We aim at showing that

$$
\min_{\gamma \in \Gamma(\rho)} C(\gamma) = \max_{u \in \mathcal{F}} D(u).
$$

In order to guarantee the existence of a minimizer on the left-hand side of (2.2), we also assume that there exist a point $o \in X$ and a radius $r_0 > 0$ such that

$$
\int_{X \setminus B(o, r_0)} f(2d(x, o)) \, d\rho(x) > -\infty.
$$

This is a similar assumption than requiring, in the case of quadratic cost, that the marginal measures have finite second moments.

Notice that even when $X = \mathbb{R}^d$ the cost function $c$ in (2.1) does not fall in the class of functions considered by Buttazzo, Champion and de Pascale [2], since it may not be bounded from below. However, by suitably truncating the cost $c$, the truncated functions $c_R$ are bounded from below for each $R$ and, modulo translation, fall into the category of functions considered in [2].

2.2. $\Gamma$-convergence. We briefly outline the relevant definitions and properties of $\Gamma$ and $\Gamma^+$-convergences. The former is a type of convergence of functionals adjusted to minimal value problems and the latter to maximal value problems. For a thorough presentation of $\Gamma$-convergence, we refer the reader to Braides’ book [1].

**Definition 2.1** ($\Gamma$-convergence and $\Gamma^+$-convergence). Let $(S, d)$ be a metric space. We say that a sequence $(F_n)_{n \in \mathbb{N}}$ of functions $F_n: S \rightarrow \mathbb{R}$ $\Gamma$-converges to a function $F: S \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ and denote $F_n \rightharpoonup \Gamma F$ if for all $y \in S$ the following two conditions hold:

For all sequences $(y_n)_{n \in \mathbb{N}}$ that converge to $y$ we have

$$
\liminf_n F_n(y_n) \geq F(y), \text{ and } \quad (I)
$$

there exists a sequence $(y_n)_{n \in \mathbb{N}}$ converging to $y$ such that

$$
\limsup_n F_n(y_n) \leq F(y). \quad (II)
$$

Correspondingly, we say that a sequence $(D_n)_{n \in \mathbb{N}}$ of functions $D_n: S \rightarrow \mathbb{R}$, $\Gamma^+$-convergence to a function $D: S \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ and denote $D_n \rightharpoonup \Gamma^+ D$ if for all $u \in S$ the following two
conditions hold:

For any sequence \((u_n)_{n \in \mathbb{N}}\) converging to \(u\) we have
\[
\limsup_{n} D_n(u_n) \leq D(u), \quad \text{and} \quad (I+)
\]
there exists a sequence \((u_n)_{n \in \mathbb{N}}\) converging to \(u\) such that
\[
\limsup_{n} D_n(u_n) \leq D(u). \quad (II+)
\]

In order to be able to take advantage of these notions, the underlying space \(S\) must satisfy some compactness properties with respect to the minima/maxima of the functionals of interest. The following definition takes care of this.

**Definition 2.2.** Let \((S, d)\) be a metric space. We say that a sequence \((F_n)_{n \in \mathbb{N}}\) of functions \(F_n: S \to \mathbb{R} \cup \{-\infty, +\infty\}\) is equi-mildly coercive on \(S\) if there exists a compact and non-empty subset \(K\) of \(S\) such that for all \(n \in \mathbb{N}\) we have
\[
\inf_{y \in S} F_n(x) = \inf_{y \in K} F_n(y).
\]
Analogously, we say that a sequence \((D_n)_{n \in \mathbb{N}}\) of functions \(D_n: S \to \mathbb{R} \cup \{-\infty, +\infty\}\) is equi-mildly \(+\)-coercive on \(S\) if there exists a compact and non-empty subset \(K\) of \(S\) such that for all \(n \in \mathbb{N}\) we have
\[
\sup_{u \in S} D_n(u) = \sup_{u \in K} D_n(u).
\]

**Theorem 2.3.** [Theorem 1.21] Let \((S, d)\) be a metric space. Let \((F_n)_{n \in \mathbb{N}}\) be an equi-mildly coercive sequence of functions \(F_n: S \to \mathbb{R} \cup \{-\infty, +\infty\}\) that \(\Gamma\)-converges to some function \(F: S \to \mathbb{R} \cup \{-\infty, +\infty\}\). Then there exists a minimum \(y \in S\) of \(F\) and the sequence \((\inf_{y \in S} F_n(y))_{n \in \mathbb{N}}\) converges to \(\min_{y \in S} F(y)\). In addition, if \((y_n)_{n \in \mathbb{N}}\) is a sequence of elements of \(S\) such that
\[
\lim_{n} F_n(y_n) = \liminf_{n} F_n(y),
\]
then every limit of a subsequence of \((y_n)_{n \in \mathbb{N}}\) is a minimizer of \(F\).

Similarly, let \((D_n)_{n \in \mathbb{N}}\) be an equi-mildly \(\Gamma^+\)-coercive sequence of functions \(D_n: S \to \mathbb{R} \cup \{-\infty, +\infty\}\) that \(\Gamma^+\)-converges to some function \(D: S \to \mathbb{R} \cup \{-\infty, +\infty\}\). Then there exists a maximum \(u \in S\) of \(D\) and the sequence \((\sup_{u \in S} D_n(u))_{n}\) converges to \(\max_{u \in S} D(u)\). In addition, if \((u_n)_{n \in \mathbb{N}}\) is a sequence of elements of \(S\) such that
\[
\lim_{n} D_n(u_n) = \limsup_{n} D_n(u),
\]
then every limit of a subsequence of \((u_n)_{n \in \mathbb{N}}\) is a maximizer of \(D\).

3. **Monge-Kantorovich problem**

First, we prove the existence of a minimizer for the Monge-Kantorovich problem \((\text{I.I})\) in our framework. Notice that the conditions (A) and (B) guarantee that the cost has a finite value.

**Proposition 3.1.** Let \((X, d)\) be a Polish space. Suppose that \(\rho \in \mathcal{P}(X)\) satisfies (A) and (B), and \(c: X^N \to \mathbb{R} \cup \{+\infty\}\) is a cost function
\[
c(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} f(d(x_i, x_j)), \quad \text{for all} \ (x_1, \ldots, x_N) \in X^N,
\]
where \( f : [0, \infty] \to \mathbb{R} \) satisfies (F1) and (F2). Then, the following minimum is achieved

\[
\min_{\gamma \in \Gamma(\rho)} \int_{X^N} c(x_1, \ldots, x_N) \, d\gamma(x_1, \ldots, x_N).
\]

**Proof.** The proof follows standard arguments. From [18] we know that \( \Gamma(\rho) \) is compact. Therefore, it suffices to prove the lower semicontinuity of the cost \( C(\gamma) \). For this, it suffices (see [25, Theorem 4.3]) to find an upper semicontinuous function \( h \) such that

1. \( h \in L^1_c(X^N) \) for all \( \gamma \in \Gamma(\rho) \),
2. \( c \geq h \), and
3. \( \int_{X^N} h \, d\gamma = \int_{X^N} h \, d\gamma' \) for all \( \gamma, \gamma' \in \Gamma(\rho) \).

Let us define \( g : [0, \infty] \to \mathbb{R} \) by

\[
g(r) = \begin{cases} 
  f(r_0) & \text{if } r < r_0 \\
  f(r) & \text{if } r \geq r_0
\end{cases},
\]

and set \( h : X^N \to \mathbb{R} \)

\[
h(x_1, \ldots, x_N) = \frac{1}{2} \sum_{1 \leq i < j \leq N} \left( g(2d(x_i, o)) + g(2d(x_j, o)) \right).
\]

As a finite sum of continuous functions, \( h \) is continuous and thus trivially upper semicontinuous. In addition, for any \( \gamma \in \Gamma(\rho) \) we have

\[
\int_{X^N} h \, d\gamma = \frac{1}{2} \sum_{1 \leq i < j \leq N} \int_{X^N} (g(2d(x_i, o)) + g(2d(x_j, o))) \, d\gamma
\]

\[
= \frac{1}{2} N(N - 1) \int_X g(2d(x_i, o)) \, d\rho(x)
\]

\[
= \frac{1}{2} N(N - 1) \left( \int_{B(o, r_0)} f(2r_0) \, d\rho(x) + \int_{X \setminus B(o, r_0)} f(2d(x, o)) \, d\rho(x) \right).
\]

Therefore, due to Assumption (B) condition \( 3.1 \) holds. Similarly, condition \( 3.3 \) follows by

\[
\int_{X^N} h \, d\gamma' = \frac{1}{2} \sum_{1 \leq i < j \leq N} \int_X (g(2d(x_i, o)) + g(2d(x_j, o))) \, d\rho = \int_{X^N} h \, d\gamma.
\]
Finally, to prove condition (3.2), we fix \((x_1, \ldots, x_N) \in X^N\) and by (F1) we have that
\[
c(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} f(d(x_i, x_j)) \geq \sum_{1 \leq i < j \leq N} g(d(x_i, x_j)) \\
\geq \sum_{1 \leq i < j \leq N} g(d(x_i, o) + d(x_j, o)) \\
\geq \sum_{1 \leq i < j \leq N} g(2 \max\{d(x_i, o), d(x_j, o)\}) \\
= \sum_{1 \leq i < j \leq N} \min\{g(2d(x_i, o)), g(2d(x_j, o))\} \\
= \frac{1}{2} \sum_{1 \leq i < j \leq N} (g(2d(x_i, o)) + g(2d(x_j, o)) - |g(2d(x_i, o)) - g(2d(x_j, o))|) \\
\geq \frac{1}{2} \sum_{1 \leq i < j \leq N} (g(2d(x_i, o)) + g(2d(x_j, o)) - 0) = h(x_1, \ldots, x_N).
\]
This concludes the proof. \(\square\)

For \(\alpha > 0\) we define the set \(D_\alpha\) as
\[
D_\alpha := \{(x_1, \ldots, x_N) \in X^N \mid \text{there exist } i, j \text{ such that } d(x_i, x_j) < \alpha\}.
\]
The next theorem states that for any measure \(\rho\) there exists \(\overline{\alpha} > 0\) for which the support of any optimal plan is concentrated away from the set \(D_\overline{\alpha}\).

**Theorem 3.2.** Let \((X, d), \rho, f, c\) as in the Proposition 3.1 and let \(\gamma\) be a minimizer of
\[
C(\rho) = \min_{\gamma \in \Gamma(\rho)} \int_{X^N} c(x_1, \ldots, x_N) d\gamma(x_1, \ldots, x_N).
\]
Let us fix \(0 < \beta < 1\) such that
\[
\sup_{x \in X} \rho(B(x, \beta)) < \frac{1}{N(N-1)^2}.
\]
Then, we have for all
\[
\alpha < f^{-1}\left(\frac{N^2(N-1)}{2} f(\beta)\right) \tag{3.4}
\]
the inclusion
\[
\text{spt}(\gamma) \subset X^N \setminus D_\alpha. \tag{3.5}
\]
**Proof.** The proof presented in [2] also works here. The fact that optimal plans stay out of the diagonal reflect the properties of the cost close to the singularity, not to the tail. \(\square\)

We recall that for all \(R > 0\), the truncated costs \(c_R\) and \(C_R\)
\[
c_R(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} \max\{f(R), f(d(x_i, x_j))\} \text{ for all } (x_1, \ldots, x_N) \in X^N,
\]
\[
C_R(\gamma) = \int_{X^N} c_R(x_1, \ldots, x_N) d\gamma(x_1, \ldots, x_N) \text{ for each } \gamma \in \Gamma(\rho).
\]
Using these we define the functionals \( K_R, K : \mathcal{P}(X^N) \to \mathbb{R} \cup \{+\infty\} \),

\[
K_R(\gamma) := \begin{cases} 
C_R(\gamma) & \text{if } \gamma \in \Gamma(\rho) \\
+\infty & \text{otherwise}
\end{cases},
\]

\[
K(\gamma) := \begin{cases} 
C(\gamma) & \text{if } \gamma \in \Gamma(\rho) \\
+\infty & \text{otherwise}
\end{cases}.
\]

An approximation result of convergence of minimizers of the truncated costs \((K_R)_{R \in \mathbb{N}}\) is given by the following proposition.

**Proposition 3.3.** The sequence of functionals \((K_R)_{R \in \mathbb{N}}\) is equicoercive and \(\Gamma\)-converges to \(K\) with respect to the weak convergence of measures.

**Proof.** First we notice that the equicoerciviness of \((K_R)_{R \in \mathbb{N}}\) follows from the fact that \(\Gamma(\rho)\) is weakly compact [18]. We then fix \(\gamma \in \mathcal{P}(X^N)\) and show that for all sequences \((\gamma_R)_{R \in \mathbb{N}}\) such that \(\gamma_R \rightharpoonup \gamma\) we have

\[
\liminf_{R \to \infty} K_R(\gamma_R) \geq K(\gamma), \quad \text{(3.6)}
\]

there exists a sequence \((\gamma_R)_{R \in \mathbb{N}}\) such that \(\gamma_R \rightharpoonup \gamma\) and

\[
\limsup_{R \to \infty} K_R(\gamma_R) \leq K(\gamma). \quad \text{(3.7)}
\]

Fix a sequence \((\gamma_R)_{R \in \mathbb{N}}\) in \(\mathcal{P}(X^N)\) such that \(\gamma_R \rightharpoonup \gamma\). By going to a subsequence we may assume that \(\liminf_{R \to \infty} K_R(\gamma_R) = \lim_{R \to \infty} K_R(\gamma_R)\). Thus, we may also suppose that \(K_R(\gamma_R) < \infty\) for all \(R \in \mathbb{N}\), since otherwise \((3.6)\) would trivially hold. Consequently, we have that \(\gamma_R \in \Gamma(\rho)\) for all \(R \in \mathbb{N}\) and thus also \(\gamma \in \Gamma(\rho)\) by compactness of \(\Gamma(\rho)\), see [18]. Now, by monotonicity of the integral and lower semi-continuity of \(K(\gamma)\) we get

\[
\liminf_{R \to \infty} K_R(\gamma_R) \geq \liminf_{R \to \infty} K(\gamma_R) \geq K(\gamma),
\]

so \((3.6)\) is satisfied. Finally, the condition \((3.7)\) is satisfied by the constant sequence \(\gamma_R = \gamma\) for all \(R \in \mathbb{N}\). \(\square\)

### 3.1. Symmetric probability measures

We remark that the Monge-Kantorovich problem \((1.1)\) can be restricted to symmetric transport plans.

**Definition 3.4** (Symmetric measures). A measure \(\gamma \in \mathcal{P}(X^N)\) is symmetric if

\[
\int_{X^N} \phi(x_1, \ldots, x_N) \, d\gamma = \int_{X^N} \phi(\sigma(x_1, \ldots, x_N)) \, d\gamma, \quad \text{for all } \phi \in \mathcal{C}(X^N)
\]

and for all permutations \(\sigma\) of \(N\) symbols. We denote by \(\Gamma^{sym}(\rho)\), the space of all \(\gamma \in \Gamma(\rho)\) which are symmetric.

**Proposition 3.5.** Let \((X,d)\) be a Polish space. Suppose \(\rho \in \mathcal{P}(X)\) such that \((A)\) and \((B)\) hold and \(c : X^N \to \mathbb{R} \cup \{+\infty\}\) is a continuous cost function. Then,

\[
\min_{\gamma \in \Gamma(\rho)} \int_{X^N} c(x_1, \ldots, x_N) \, d\gamma = \min_{\gamma \in \Gamma^{sym}(\rho)} \int_{X^N} c(x_1, \ldots, x_N) \, d\gamma. \quad \text{(3.8)}
\]
Duality Theory for log-type cost functions

4. Duality Theory for log-type cost functions

The following theorem extends Kantorovich duality for our class of cost functions.

**Theorem 4.1.** Let \((X, d)\) be a Polish space. Suppose \(\rho \in \mathcal{P}(X)\) such that (A) and (B) hold and \(c: X^N \to \mathbb{R} \cup \{+\infty\}\) is a cost function

\[
c(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} f(d(x_i, x_j)), \quad \text{for all } (x_1, \ldots, x_N) \in X^N,
\]

where \(f: [0, +\infty] \to \mathbb{R} \cup \{+\infty\}\) is a function satisfying (F1) and (F2). Then, the duality holds:

\[
\min_{\gamma \in \Gamma(\rho)} \int_{X^N} c \, d\gamma = \max_{u \in L^1_p(X)} \left\{ N \int_X u(x) \, d\rho(x) : \sum_{i=1}^N u(x_i) \leq c(x_1, \ldots, x_N) \rho^{\otimes (N)} - \text{a.e.} \right\}. \tag{4.1}
\]

**Proof.** Due to Proposition 3.1 the minimum on the left-hand side is realized. By using the monotonicity of integral and the fact that \(\gamma \in \Gamma(\rho)\), we easily get

\[
\min_{\gamma \in \Gamma(\rho)} C(\gamma) \geq \sup_{u \in \mathcal{F}} D(u).
\]

Hence, we need to show that

\[
\min_{\gamma \in \Gamma(\gamma)} C(\gamma) \leq \sup_{u \in \mathcal{F}} D(u) \tag{4.2}
\]

and that a maximizer for \(\max_{u \in \mathcal{F}} D(u)\) exists.

Towards this goal, let us fix a minimizer \(\gamma\) of \(C\). It now suffices to show that there exists a function \(u \in \mathcal{F}\) such that

\[
C(\gamma) \leq D(u).
\]

For each \(L > 0\), let us denote \(\gamma_L = \gamma|_{B(o,L)^N}\), and by \(\gamma_L^p\) the normalized versions of \(\gamma_L\). Notice that because of Assumption (B), \(\gamma_L \neq 0\) for large enough \(L > 0\). Let us denote the marginals of \(\gamma_L^p\) by \(\rho_L\).

Now, \(\gamma_L^p\) is optimal also for all \(C_R\) with \(R \geq 2L\), since \(C = C_R\) for all couplings of \(\rho_L\). Let \((u_R)\) be a sequence of Kantorovich potentials, each corresponding to \(\gamma_R^{p/2}\) with the cost \(c_R\) and the marginals \(\rho_R^{p/2}\). By [2, Lemma 3.3], we may assume that for all \(R\) and all \(x_1 \in X\) we have the representation

\[
u_R(x_1) = \inf \left\{ \sum_{i=1}^N c_R(x_1, x_2, \ldots, x_N) - \sum_{j=2}^N u_R(x_j) \mid (x_2, \ldots, x_N) \in X^{N-1} \right\}. \tag{4.3}
\]

Let us fix \(R_0 > 0\) such that \(\gamma_{R_0/2} \neq 0\), and a point \((\bar{x}_1, \ldots, \bar{x}_N) \in \text{spt}(\gamma_{R_0/2})\).
We may then assume that for all $R \geq R_0$, we have
\[ u_R(x_i) = \frac{1}{N} c_R(x_1, \ldots, x_N) = \frac{1}{N} c(x_1, \ldots, x_N) \text{ for all } i, \]
since $(\overline{x}_1, \ldots, \overline{x}_N) \in \text{spt}(\gamma_{R_0}/2) \subset \text{spt}(\gamma_R/2)$.

Now we have, for all $R \geq R_0$ and for all $x = (x_1, \ldots, x_N) \in X^N$, by $(4.3)$ and Theorem 3.2 for some $\alpha > 0$ the estimate
\[ u_R(x_1) \leq c_R(x_1, x_2, \ldots, x_N) - \frac{N-1}{N} c_R(x_1, \ldots, x_N) \]
\[ \leq \frac{N(N-1)}{2} f(\alpha/2) - \frac{N-1}{N} c_R(x_1, \ldots, x_N) =: M, \]
since by the fact that $(\overline{x}_1, \ldots, \overline{x}_2) \in X^N \setminus D_\alpha$, we may assume (by changing $\overline{x}_1$ with some other $x_i$), that $d(x_1, x_j) \geq \frac{\alpha}{2}$ for all $j \in \{2, \ldots, N\}$.

For the lower bound, we use again the representation $(4.3)$ and the upper bound that we just obtained. For all $x = (x_1, \ldots, x_N) \in \text{spt}(\gamma_R^P)$, when $R \geq 2L$, we have
\[ u_R(x_1) = \sum_{1 \leq i < j \leq N} f(d(x_i, x_j)) - \sum_{j=2}^N u_R(x_j) \]
\[ \geq \frac{N(N-1)}{2} f(2L) - (N-1)M. \]

What we have shown is that for each $L$ the sequence $(u_R)$ is bounded on $\text{spt}\rho_L$ when $R \geq 2L$. So, we may in each set $\text{spt}(\rho_L)$ define $u$ as the weak limit of $u_R$ along some subsequence, and finally define $u$ in the whole space by a diagonal argument. Now, assuming that we have that $u \in \mathcal{F}$, by the definition of $\gamma_L$, and by the weak convergence we get
\[ C(\gamma) = \lim_{R \to \infty} C(\gamma_R^P) = \lim_{R \to \infty} D_R(u_R) = D(u). \]
Thus, it remains to show that $u \in \mathcal{F}$. Supposing this is not the case, there exists a Borel set $A \subseteq X^N$ such that $\rho^{\otimes(N)}(A) > 0$ and
\[ u(x_1) + \cdots + u(x_N) > c(x_1, \ldots, x_N) \text{ for all } (x_1, \ldots, x_N) \in A. \]  
(4.4)
By going into a subset of $A$ if necessary, we may assume that $A \subset (\text{spt}\rho_L \cap B(0,L))^N$ for some $L > 0$. Now, by Mazur’s lemma, there is a sequence $(\tilde{u}_R)$ of convex combinations of $(u_R)_{R \geq 2L}$ strongly converging to $u$ in $L^1(\rho)$. Since, $c_R = c$ on $A$ for all $R \geq 2L$, we have
\[ \tilde{u}_R(x_1) + \cdots + \tilde{u}_R(x_N) \leq c(x_1, \ldots, x_N) \text{ for all } (x_1, \ldots, x_N) \in A, \]  
(4.5)
for all $R \geq 2L$, as the inequality is preserved under convex combinations.

Let us denote
\[ l := \int_A (u(x_1) + \cdots + u(x_N) - c(x_1, \ldots, x_N)) \, d\rho^{\otimes(N)}. \]
Due to $(4.4)$ we have $l > 0$. Because $\tilde{u}_R \to u$ strongly, there exists $R_1 \geq 2L$ such that
\[ \int_A \sum_{i=1}^N |\tilde{u}_R(x_i) - u(x_i)| \, d\rho^{\otimes(N)} < \frac{l}{2} \text{ for all } R \geq R_1. \]  
(4.6)
Then we have for all $R > R_1$
\[
\int_A \left( \sum_{i=1}^N \tilde{u}_R(x_i) - c(x_1, \ldots, x_N) \right) d\rho^{\otimes(N)} = \int_A \sum_{i=1}^N (\tilde{u}_R(x_i) - u(x_i)) d\rho^{\otimes(N)} + \int_A \sum_{i=1}^N u(x_i) - c(x_1, \ldots, x_N) d\rho^{\otimes(N)} > l - \frac{l}{2} = \frac{l}{2} > 0,
\]
contradicting (4.5).

5. Properties of the Kantorovich potentials

Let $C(\gamma)$ be as before
\[
C(\gamma) = \int_X \sum_{1 \leq i < j \leq N} f(d(x_i, x_j)) d\gamma.
\]
We denote by $C^R(\gamma)$ the truncation of a cost $C(\gamma)$ from above\footnote{Notice that we have used the notation $C_R$ to correspond to the cost truncated from below.},
\[
C^R(\gamma) = \int_{X^N} c^R(x_1, \ldots, x_N) d\gamma, \quad \text{for all } \gamma \in \mathcal{P}(X^N),
\]
where we have denoted by $c^R$ the corresponding truncation of $c$,
\[
c^R(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} \min\{R, f(d(x_i, x_j))\}.
\]

Proposition 5.1. Let $\rho \in \mathcal{P}(X)$ satisfy the assumptions (A) and (B). Fix $\beta > 0$ such that
\[
\sup_{x \in X} \rho(B(x, \beta)) < \frac{1}{N(N-1)^2}.
\]
Then, for any $\alpha < f^{-1} \left( \frac{N^2(N-1)}{2} f(\beta) \right)$ and for all optimal $\gamma \in \Gamma(\rho)$ associated to $C(\gamma)$, we have
\[
C(\gamma) \leq \frac{N^3(N-1)^2}{4} f(\beta) \quad \text{and} \quad C(\gamma) = C^{f(\alpha)}(\gamma). \tag{5.1}
\]
Moreover, for the same $\alpha$, any Kantorovich potential $u_\alpha$ for $C^{f(\alpha)}$ is also a Kantorovich potential for $C$.

Proof. For each
\[
\alpha < f^{-1} \left( \frac{N^2(N-1)}{2} f(\beta) \right),
\]
we know by Theorem 3.2 that the support of $\gamma$ can intersect at most the boundary of $D_\alpha$. Therefore, since $f$ is decreasing, we have for all $(x_1, \ldots, x_N) \in \text{spt}(\gamma)$ the estimate
\[
c(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} f(d(x_i, x_j)) \leq \frac{N(N-1)}{2} f(\alpha).
\]
Thus, since $\gamma$ is a probability measure, we have
\[ C(\gamma) \leq \int_{X^N} \frac{N(N - 1)}{2} f(\alpha) \, d\gamma = \frac{N(N - 1)}{2} f(\alpha). \]
Taking $\alpha \to f^{-1}\left(\frac{N^2(N - 1)}{2} f(\beta)\right)$, we then get
\[ C(\gamma) \leq \frac{N(N - 1)}{2} f\left(f^{-1}\left(\frac{N^2(N - 1)}{2} f(\beta)\right)\right) = \frac{N(N - 1)}{2} \cdot \frac{N^2(N - 1)}{2} f(\beta) = \frac{N^3(N - 1)^2}{4} f(\beta), \]
which gives the left-hand side in (5.1). Let us then fix an optimal plan $\gamma$ for the cost $C^{f(\alpha)}$. Then $\text{spt}(\gamma) \in X^N \setminus D_\alpha$, so $c = c^{f(\alpha)}$ on $\text{spt}(\gamma)$. Thus,
\[ C(\gamma) \leq \int_{X^N} c \, d\gamma = \int_{X^N} c^{f(\alpha)} \, d\gamma = C^{f(\alpha)}(\gamma). \]
The opposite inequality is simply due to the monotonicity of the integral. It remains to prove the last part of the statement. We fix a Kantorovich potential $u_\alpha$ for $C^{f(\alpha)}$. It satisfies, for $\rho^{\otimes(N)}$-almost every $(x_1, \ldots, x_N) \in X^N$ the estimate
\[ u_\alpha(x_1) + \cdots + u_\alpha(x_N) \leq c^{f(\alpha)}(x_1, \ldots, x_N) \leq c(x_1, \ldots, x_N). \]
Hence, $u_\alpha$ is also a Kantorovich potential for the cost function $c$ and, moreover,
\[ \int_X u(x) \, d\rho(x) = \min_{\gamma \in \Gamma(\rho)} C(\gamma) = \min_{\gamma \in \Gamma(\rho)} C^{f(\alpha)}(\gamma) = N \int_X u_\alpha(x) \, d\rho(x). \]
This concludes the proof. \hfill \square

**Theorem 5.2.** Let $(X, d)$ be a Polish space. Suppose $\rho \in \mathcal{P}(X)$ such that (A) and (B) hold and $c : X^N \to \mathbb{R} \cup \{+\infty\}$ is a cost function
\[ c(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} f(d(x_i, x_j)), \quad \text{for all } (x_1, \ldots, x_N) \in X^N, \]
where $f : [0, +\infty] \to \mathbb{R} \cup \{+\infty\}$ is a function satisfying (F1) and (F2).

Let $\beta > 0$ be such that
\[ \sup_{x \in X} \rho(B(x, \beta)) < \frac{1}{N(N - 1)^2}. \]
Assume additionally that, for some $\alpha < f^{-1}\left(\frac{N^2(N - 1)}{2} f(\beta)\right)$, the restriction $f|_{[\alpha, +\infty[}$ is Lipschitz. Then, there exists a Kantorovich potential $w$ in (1.1) that is Lipschitz.

The following lemma is useful for proving Theorem 5.2. The proof follows in the same way as the proof of [2, Lemma 3.3].

**Lemma 5.3.** Let $u$ be a Kantorovich potential for the problem (1.2), i.e. a maximizer of the problem (1.2). Then there exists a Kantorovich potential $\tilde{u}$ such that $\tilde{u} \geq u$ which satisfies the representation
\[ \tilde{u}(x) = \inf \left\{ c(x, x_2, \ldots, x_N) - \sum_{i \geq 2} \tilde{u}(x_i) : x_j \in X \text{ for all } j \right\}. \quad (5.2) \]
Under the same assumptions as in Theorem 5.2, let Proposition 5.4.

By [2, Theorem 3.9], the above result holds for the singular costs \( C_R \) which are bounded from below. Therefore, it suffices to show that for each \( \epsilon > 0 \) there exists \( R \in \mathbb{N} \) such that

\[
|C(\rho_n) - C_R(\rho_n)| < \epsilon
\]

for all \( n \in \mathbb{N} \). Since the inequality \( C \leq C_R \) always holds, it suffices to show that \( C_R(\rho_n) - C(\rho_n) < \epsilon \) for \( R \) large enough. In order to obtain this, we take a minimizer \( \gamma_n \) for \( C \) with marginals \( \rho_n \) (given by Proposition 3.1) and estimate, assuming \( f(R/2) \leq 0 \) by taking \( R \) large enough and \( \gamma_n \in \Gamma^{sym}(\rho_n) \) by Proposition 3.5.

\[
C_R(\rho_n) - C(\rho_n) \leq \int_{X^N} (C_R - C) \, d\gamma_n
\]

\[
\leq -N(N - 1) \int_{d(x_1, x_2) \geq R} f(d(x_1, x_2)) \, d\gamma_n
\]

\[
\leq -N(N - 1) \int_{d(x_1, x_2) \geq R} f(\max\{2d(x_1, o), 2d(x_2, o)\}) \, d\gamma_n
\]

\[
\leq -2N(N - 1) \int_{d(x, o) \geq \frac{R}{2}} f(2d(x, o)) \, d\rho_n < \epsilon,
\]

for large enough \( R \) by assumption (5.3).
6. Monge Problem for log-type costs

Regarding the existence of Monge-type minimizers in (1.1), the first positive result for repulsive type costs is shown in [10], where in dimension $d = 1$, $X = \mathbb{R}$, M. Colombo, L. De Pascale and S. Di Marino prove that, for an absolutely continuous measure, a symmetric optimal plan $\gamma$ is always induced by a cyclical optimal map $T$. One important ingredient of that proof relied on the fact that for symmetric cost functions (1.1) can be restricted for a class of symmetric transport plans (see Definition 3.3 and Proposition 3.5).

**Theorem 6.1** (Colombo, De Pascale and Di Marino, [10]). Let $\mu \in \mathcal{P}(\mathbb{R})$ be an absolutely continuous probability measure and $f : \mathbb{R} \to \mathbb{R}$ strictly convex, bounded from below and non-increasing function. Then there exists a unique optimal symmetric plan $\gamma \in \Gamma^{sym}(\mu)$ that solves

$$\min_{\gamma \in \Gamma^{sym}(\mu)} \int_{\mathbb{R}^N} \sum_{1 \leq i < j \leq N} f(|x_j - x_i|) \, d\gamma.$$ 

Moreover, this plan is induced by an optimal cyclical map $T$, that is, $\gamma^{sym} = \frac{1}{N!} \sum_{\sigma \in S_N} \sigma \gamma_T$, where $\gamma_T = (Id, T, T(2), \ldots, T^{(N-1)})\#\mu$. An explicit optimal cyclical map is

$$T(x) = \begin{cases} F^{-1}_\mu(F_\mu(x) + 1/N) & \text{if } F_\mu(x) \leq (N - 1)/N \\ F^{-1}_\mu(F_\mu(x) + 1 - 1/N) & \text{otherwise.} \end{cases}$$

Here $F_\mu(x) = \mu(-\infty, x]$ is the distribution function of $\mu$, and $F^{-1}_\mu$ is its lower semicontinuous left inverse.

We remark that, due to Theorem 2.3, the above Theorem 6.1 also holds for unbounded cost functions satisfying $(F1)$ and $(F2)$ and under the additional assumption $(B)$ on the absolutely continuous measure $\mu$. This can be seen for instance by taking a minimizer for the unbounded cost and observing that its restriction to a bounded set is also a minimizer of a truncated for and thus of the form given by Theorem 6.1.

6.1. Log-type cost ($N = 2$). Here we consider $X = \mathbb{R}^d$ with $d \geq 1$.

**Theorem 6.2.** Let $\rho \in \mathcal{P}(\mathbb{R}^d)$ be a probability measure such that (A) and (B) hold. Then there exists a unique optimal plan $\gamma_{O} \in \Gamma(\rho, \rho)$ for the problem

$$\min_{\gamma \in \Gamma(\rho, \rho)} \int_{\mathbb{R}^d \times \mathbb{R}^d} -\log(|x_1 - x_2|) \, d\gamma(x_1, x_2).$$

(6.1)

Moreover, this plan is induced by an optimal map $T$, that is, $\gamma = (Id, T)\#\rho$, and $T(x) = x - \frac{\nabla u}{|\nabla u|^2} \rho$-almost everywhere, where $u$ is a Lipschitz maximizer for the dual problem (1.2).

**Proof.** Let us consider $\gamma$ a minimizer for the problem (6.1) and $u$ a maximizer of the dual problem, which is Lipschitz by Theorem 5.2. Then, $F(x_1, x_2) = u(x_1) + u(x_2) + \log(|x_1 - x_2|) \leq 0,$

for $\rho \otimes \rho$-almost every $(x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d$. Moreover, $F = 0$ $\gamma$-almost everywhere. But then $F$ has a maximum on the support of $\gamma$ and so $\nabla F = 0$ in this set; in particular we have that $\nabla u(x_1) = \frac{(x_1 - x_2)}{|x_1 - x_2|^2}$ on the support of $\gamma$. By solving this equation for $x_2$, we have

$$x_2 = x_1 - \frac{\nabla u(x_1)}{|\nabla u(x_1)|^2}, \quad \gamma - \text{almost everywhere},$$

which implies $\gamma = (Id, T)\#\mu$ as we wanted to show. □
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