BOOTSTRAPPING MAX STATISTICS IN HIGH
DIMENSIONS: NEAR-PARAMETRIC RATES UNDER
WEAK VARIANCE DECAY AND APPLICATION TO
FUNCTIONAL DATA ANALYSIS

BY MILES E. LOPES∗, ZHENHUA LIN† AND HANS-GEORG MÜLLER‡

University of California, Davis

In recent years, bootstrap methods have drawn attention for their
ability to approximate the laws of “max statistics” in high-dimensional
problems. A leading example of such a statistic is the coordinate-wise
maximum of a sample average of \( n \) random vectors in \( \mathbb{R}^p \). Existing re-
sults for this statistic show that the bootstrap can work when \( n \ll p \),
and rates of approximation (in Kolmogorov distance) have been ob-
tained with only logarithmic dependence in \( p \). Nevertheless, one of
the challenging aspects of this setting is that established rates tend
to scale like \( n^{-1/6} \) as a function of \( n \).

The main purpose of this paper is to demonstrate that improve-
ment in rate is possible when extra model structure is available.
Specifically, we show that if the coordinate-wise variances of the ob-
servations exhibit decay, then a nearly \( n^{-1/2} \) rate can be achieved,
\textit{independent of} \( p \). Furthermore, a surprising aspect of this dimension-
free rate is that it holds even when the decay is \textit{very weak}. As a
numerical illustration, we show how these ideas can be used in the
context of functional data analysis to construct simultaneous confi-
dence intervals for the Fourier coefficients of a mean function.

1. Introduction. One of the current challenges in theoretical statistics
is to understand when bootstrap methods work in high-dimensional
problems. In this direction, there has been a surge of recent interest in connection
with “max statistics” such as

\[
T = \max_{1 \leq j \leq p} S_{n,j},
\]

where \( S_{n,j} \) is the \( j \)th coordinate of the sum \( S_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mathbb{E}[X_i])\),
involving i.i.d. vectors \( X_1, \ldots, X_n \) in \( \mathbb{R}^p \).

∗Supported in part by NSF grant DMS 1613218
†Supported in part by NIH grant 5UG3OD023313-03
‡Supported in part by NSF grant DMS 1712864 and NIH grant 5UG3OD023313-03
MSC 2010 subject classifications: Primary 62G09, 62G15; secondary 62G05, 62G20.
Keywords and phrases: bootstrap, high-dimensional statistics, rate of convergence, functional data analysis, confidence region
This type of statistic has been a focal point in the literature for at least two reasons. First, it is an example of a statistic for which bootstrap methods can succeed in high dimensions under mild assumptions, which was established in several pathbreaking works (Arlot, Blanchard and Roquain, 2010a,b; Chernozhukov, Chetverikov and Kato, 2013, 2017). Second, the statistic $T$ is closely linked to several fundamental topics, such as suprema of empirical processes, nonparametric confidence regions, and multiple testing problems. Likewise, many applications of bootstrap methods for max statistics have ensued at a brisk pace in recent years (see, e.g., Chernozhukov, Chetverikov and Kato, 2014; Wasserman, Kolar and Rinaldo, 2014; Chen, Genovese and Wasserman, 2015; Chang, Yao and Zhou, 2017; Zhang and Cheng, 2017; Dezeure, Bühlmann and Zhang, 2017; Chen, 2018; Fan, Shao and Zhou, 2018; Belloni et al., 2018).

One of the favorable aspects of bootstrap approximation results for the distribution $L(T)$ is that rates have been established with only logarithmic dependence in $p$. For instance, the results in Chernozhukov, Chetverikov and Kato (2017) imply that under certain conditions, the Kolmogorov distance $d_K$ between $L(T)$ and its bootstrap counterpart $L(T^*|X)$ satisfies the bound

$$d_K(L(T), L(T^*|X)) \leq \frac{c \log(p)^a}{n^{1/6}}$$

with high probability, where $c, a > 0$ are constants not depending on $n$ or $p$, and $X$ denotes the matrix whose rows are $X_1, \ldots, X_n$. (In the following, $c$ will be often re-used to designate a positive constant, possibly with a different value at each occurrence.) Additional refinements of this result can be found in the same work, with regard to the choice of metric, or choice of bootstrap method. Also, recent progress in sharpening the exponent $a$ has been made by Deng and Zhang (2017). However, this mild dependence on $p$ is offset by the $n^{-1/6}$ dependence on $n$, which differs from the $n^{-1/2}$ rate in the multivariate Berry-Esseen theorem when $p \ll n$.

Currently, the general question of determining the best possible rates of bootstrap approximation in high dimensions is largely open. In particular, rates of the form (1.1) for $T$ and related statistics have been conjectured to be minimax optimal in the settings considered by Chernozhukov, Chetverikov and Kato (2017) and Chen (2018). Nevertheless, in finite-sample experiments, the performance of bootstrap methods for max statistics is often more encouraging than what might be expected from the $n^{-1/6}$ dependence on $n$ (see, e.g. Zhang and Cheng, 2017; Fan, Shao and Zhou, 2018; Belloni et al., 2018). This suggests that improved rates are possible in at least some situations.
The purpose of this paper is to quantify an instance of such improvement when additional model structure is available. Specifically, we consider the case when the coordinates of $X_1, \ldots, X_n$ have decaying variances. If we let $\sigma_j^2 = \text{var}(X_{1,j})$ for each $1 \leq j \leq p$, and write $\sigma_{(1)} \geq \cdots \geq \sigma_{(p)}$, then this condition may be formalized as

$$
\sigma_{(j)} \leq cj^{-\alpha} \quad \text{for all} \quad 1 \leq j \leq p, \tag{1.2}
$$

where $\alpha > 0$ is a parameter not depending on $n$ or $p$. (A complete set of assumptions is given in Section 2.) This type of condition arises in many contexts, and in Section 2 we discuss examples related to principal component analysis, sparse count data, and Fourier coefficients of functional data. Furthermore, this condition can be empirically verified in an approximate sense, due to the fact that the parameters $\sigma_1, \ldots, \sigma_p$ can be accurately estimated, even in high dimensions.

Within the setting of decaying variances, our main results show that bootstrap approximation of $\mathcal{L}(T)$ can be achieved at a nearly parametric rate. More precisely, for any fixed $\delta \in (0, 1/2)$, the bound

$$
d_K \left( \mathcal{L}(T), \mathcal{L}(T^* | X) \right) \leq c_{\delta, \alpha} n^{-1/2+\delta} \tag{1.3}
$$

holds with high probability, where $c_{\delta, \alpha} > 0$ is a number that depends only on $\delta$ and $\alpha$. Here, it is worth emphasizing a few basic aspects of this bound. First, it is non-asymptotic and does not depend on $p$. Second, the parameter $\alpha$ is allowed to be arbitrarily small, and in this sense, the decay condition (1.2) is very weak. Third, the result holds when $T^*$ is constructed using the standard multiplier bootstrap procedure (Chernozhukov, Chetverikov and Kato, 2013), and it is not necessary to use any auxiliary dimension reduction or variable selection.

With regard to previous bootstrap approximation results for $\mathcal{L}(T)$, it is important to clarify that our bound (1.3) does not conflict with the conjectured optimality of the rate $n^{-1/6}$. The reason is that the $n^{-1/6}$ rate has been established in settings where the values $\sigma_1, \ldots, \sigma_p$ are restricted from becoming too small. A basic version of such a requirement is that

$$
\min_{1 \leq j \leq p} \sigma_j \geq c. \tag{1.4}
$$

Hence, the conditions (1.2) and (1.4) are complementary. Also, it is interesting to observe that the two conditions “intersect” in the limit $\alpha \to 0^+$, suggesting a phase transition between the rates $n^{-1/6}$ and $n^{-1/2+\delta}$ at the “boundary” corresponding to $\alpha = 0$. 
Another important consideration that is related to the conditions (1.2) and (1.4) is the use of standardized variables. Namely, it is of special interest to approximate the distribution of the statistic

\[ T' = \max_{1 \leq j \leq p} \frac{S_{n,j}}{\sigma_j}, \]

which is equivalent to approximating \( \mathcal{L}(T) \) when each \( X_{i,j} \) is standardized to have variance 1. Given that standardization eliminates variance decay, it might seem that the rate \( n^{-1/2+\delta} \) has no bearing on approximating \( \mathcal{L}(T') \). However, it is still possible to take advantage of variance decay, by using the basic notion of “partial standardization”.

The idea of partial standardization is to slightly modify \( T' \) by using a fractional power of each \( \sigma_j \). Specifically, if we let \( \tau_n \in [0,1] \) be a free parameter, then we can consider the partially standardized statistic

\[ M = \max_{1 \leq j \leq p} \frac{S_{n,j}}{\sigma_j^{\tau_n}}, \]

which interpolates between \( T \) and \( T' \) as \( \tau_n \) ranges over \([0,1]\). This statistic has the following significant property: If \( X_1,\ldots,X_n \) satisfy the variance decay condition (1.2), and if \( \tau_n \) is chosen to be slightly less than 1, then our main results show that the rate \( n^{-1/2+\delta} \) holds for bootstrap approximations of \( \mathcal{L}(M) \). In fact, this effect occurs even when \( \tau_n \to 1 \) as \( n \to \infty \). Further details can be found in Section 3. Also note that our main theoretical results are formulated entirely in terms of \( M \), which covers the statistic \( T \) as a special case.

In practice, simultaneous confidence intervals derived from approximations to \( \mathcal{L}(M) \) are just as easy to use as those based on \( \mathcal{L}(T') \). Although there is a slight difference between the quantiles of \( M \) and \( T' \) when \( \tau_n < 1 \), the important point is that the quantiles of \( \mathcal{L}(M) \) may be preferred, since faster rates of bootstrap approximation are available. (See also Figure 1 in Section 4.) In this way, the statistic \( M \) offers a simple way to blend the utility of standardized variables with the beneficial effects of variance decay.

The remainder of the paper is organized as follows. In Section 2, we outline the problem setting, with a complete statement of the theoretical assumptions, as well as some motivating facts and examples. Our main results are given in Section 3, which consist of a Gaussian approximation result for \( \mathcal{L}(M) \) (Theorem 3.1), and a corresponding bootstrap approximation result (Theorem 3.2). To provide a numerical illustration of our results, in Section 4 we discuss a problem in functional data analysis, where the variance decay condition naturally arises. Specifically, we show how bootstrap
approximations to $\mathcal{L}(M)$ can be used to derive simultaneous confidence intervals for the Fourier coefficients of a mean function. Lastly, our conclusions are summarized in Section 5. Proofs are in the appendices as well as in the supplementary material. The organization of the proofs is described at the beginning of the appendices.

**Notation.** For any symmetric matrix $A \in \mathbb{R}^{d \times d}$, the ordered eigenvalues are denoted as $\lambda(A) = (\lambda_1(A), \ldots, \lambda_d(A))$, where $\lambda_1(A) \geq \cdots \geq \lambda_d(A)$. If $v \in \mathbb{R}^d$ is a fixed vector, and $r > 0$, we write $\|v\|_r = \left(\sum_{j=1}^d |v_j|^r\right)^{1/r}$. In a slight abuse of notation, we also write $\|\xi\|_{\psi_1} = \inf\{t > 0 \mid \mathbb{E}[\exp(|\xi|/t)] \leq 2\}$, and a random variable satisfying $\|\xi\|_{\psi_1} < \infty$ is said to be sub-exponential. If $\{a_n\}$ and $\{b_n\}$ are sequences of positive real numbers, then the relation $a_n \preceq b_n$ means that there is an absolute constant $c > 0$, and an integer $n_0 \geq 1$, such that $a_n \leq cb_n$ for all $n \geq n_0$. Also, define the abbreviations $a_n \vee b_n = \max\{a_n, b_n\}$ and $a_n \wedge b_n = \min\{a_n, b_n\}$. Lastly, when using symbols such as $c, c_\delta, c_\alpha, \delta$, etc., to refer to constants that do not depend on $n$ or $p$, we often allow their value to change from line to line in order to simplify presentation.

2. **Setting and preliminaries.** We consider a sequence of models indexed by $n$, with all parameters depending on $n$, except for those that are explicitly stated to be fixed.

**Assumption 2.1 (Data-generating model).**

(i). There is a vector $\mu = \mu(n) \in \mathbb{R}^p$ and positive definite matrix $\Sigma = \Sigma(n) \in \mathbb{R}^{p \times p}$, such that the observations $X_1, \ldots, X_n \in \mathbb{R}^p$ are generated as $X_i = \mu + \Sigma^{1/2}Z_i$ for each $1 \leq i \leq n$, where the vectors $Z_1, \ldots, Z_n \in \mathbb{R}^p$ are the rows of a matrix $Z \in \mathbb{R}^{n \times p}$ with i.i.d. entries.

(ii). The entries of $Z$ satisfy $\mathbb{E}[Z_{1,1}] = 0$, $\text{var}(Z_{1,1}) = 1$ and $\|Z_{1,1}\|_{\psi_1} \leq c_0$, where $c_0 > 0$ is an absolute constant.

(iii). There is an absolute constant $c_1 > 0$, such that the dimension $p = p(n)$ satisfies $p \geq c_1 n$.

**Remarks.** With regard to the dimension in part (iii), note that we allow the ratio $p/n$ to less than 1, or to diverge at an arbitrarily fast rate as $n \to \infty$. Meanwhile, the sub-exponential condition in part (ii) is similar to other tail conditions that have been used in previous works on bootstrap methods for max statistics (Chernozhukov, Chetverikov and Kato, 2013;
Deng and Zhang, 2017), and is considerably weaker than requiring $Z_{1,1}$ to be sub-Gaussian.

To state our next assumption, fix any $d \in \{1, \ldots, p\}$, and let $\mathcal{J}(d)$ denote a set of indices corresponding to the $d$ largest values among $\sigma_1, \ldots, \sigma_p$, i.e., $\{\sigma_1, \ldots, \sigma_{\sigma(d)}\} = \{\sigma_j \mid j \in \mathcal{J}(d)\}$. Next, define the quantity

$$
\rho_{\max}(d) = \max \left\{ \text{cor}(X_{1,j}, X_{1,j'}) \left| j, j' \in \mathcal{J}(d), \text{ and } j \neq j' \right. \right\}
$$

as the largest correlation among distinct variables indexed by $\mathcal{J}(d)$. Lastly, define the integer $\ell_n$ according to

$$
\ell_n = \left\lceil (1 \lor \log(n)^2) \wedge p \right\rceil,
$$

which occurs in the following conditions.

**Assumption 2.2 (Structural assumptions).**

(i). There is a parameter $\alpha > 0$ not depending on $n$, and absolute constants $c, c' > 0$, such that

$$(2.1) \quad \sigma_{(j)} \leq c j^{-\alpha} \quad \text{for all} \quad 1 \leq j \leq p$$

and

$$(2.2) \quad \sigma_{(j)} \geq c' j^{-\alpha} \quad \text{for all} \quad 1 \leq j \leq \ell_n.$$  

(ii). There is an absolute constant $\epsilon_0 \in (0, 1)$ such that

$$(2.3) \quad \rho_{\max}(\ell_n) \leq 1 - \epsilon_0.$$

**Remarks.** These assumptions are approximately checkable in practice, since the parameters $\sigma_j$ and $\rho_{\max}(\ell_n)$ can be estimated at nearly parametric rates, even in high dimensions (see Lemmas D.6 and D.7). When considering the size of the decay parameter $\alpha$, note that if $\Sigma$ is viewed as a covariance operator acting on a Hilbert space, then the condition $\alpha > 2$ essentially corresponds to the case of a trace-class operator — a property that is typically assumed in functional data analysis (Hsing and Eubank, 2015). From this perspective, the condition $\alpha > 0$ is very weak, and allows the trace of $\Sigma$ to diverge as $p \to \infty$.

The conditions (2.2) and (2.3) are also mild in the sense that they only apply to a small index set of size $\ell_n \lesssim \log(n)^2$. Furthermore, the correlation structure for the variables outside of $\mathcal{J}(\ell_n)$ is completely unrestricted. Lastly, the condition (2.3) can actually be relaxed so that $\rho_{\max}(\ell_n)$ is allowed to approach 1 at a certain rate as $n \to \infty$, but we do not pursue such refinements for simplicity.
2.1. Examples of correlation structures. Some examples of correlation matrices satisfying the condition (2.3) are given below.

- **Autoregressive:** $R_{i,j} = \rho_0^{|i-j|}$, for some $\rho_0 \in (0, 1)$.

- **Algebraic decay:** $R_{i,j} = 1\{i = j\} + \frac{1\{i \neq j\}}{4|i - j|^\gamma}$, for some $\gamma \geq 2$.

- **Compound symmetry:** $R = \epsilon_0 I_p + (1 - \epsilon_0) 1_p 1_p^\top$, for some $\epsilon_0 \in (0, 1)$.

- **Banded:** $R_{i,j} = \left(1 - \frac{|i-j|}{c_0}\right)_+$, for some $c_0 \geq 1$.

Based on these examples, it is also straightforward to construct covariance matrices $\Sigma$ that jointly satisfy (2.1), (2.2), and (2.3). Specifically, let $\sigma_1, \ldots, \sigma_p$ be any sequence of positive numbers satisfying (2.1) and (2.2) and let $R \in \mathbb{R}^{p \times p}$ be one of the matrices above. Then, a suitable covariance matrix can be obtained by conjugating $R$ with $\text{diag}(\sigma_1, \ldots, \sigma_p)$.

2.2. Examples of variance decay. To provide additional context for the decay condition (2.1), we describe some general situations where it arises.

- **Principal components analysis (PCA).** The broad applicability of PCA rests on the fact that many types of data have an underlying covariance matrix with weakly sparse eigenvalues. Roughly speaking, this means that most of the eigenvalues of $\Sigma$ are negligible in comparison to the top few. Similar to the condition (1.2), this situation is commonly modeled with the decay condition

$$
\lambda_j(\Sigma) \leq c_j^{-\gamma},
$$

for some parameter $\gamma > 0$ (see, e.g., Johnstone and Lu, 2009), where $\lambda_1(\Sigma) \geq \cdots \geq \lambda_p(\Sigma)$ are the sorted eigenvalues of $\Sigma$. Whenever this holds, it can be shown that the variance decay condition (1.2) must hold for some associated parameter $\alpha > 0$, and this is done in Proposition 2.1 below. So, in a qualitative sense, this indicates that if a dataset is amenable to PCA, then it is also likely to fall within the scope of our setting.

- **Sparse count data.** Consider a multinomial model based on $p$ cells and $n$ trials, parameterized by a vector of cell frequencies $\pi = (\pi_1, \ldots, \pi_p)$. 
The case when the vector $\mathbf{\pi}$ is approximately sparse often occurs in the analysis of contingency tables (see, e.g. Cressie and Read, 1984; Zelterman, 1987; Plunkett and Park, 2017). If the $i$th trial is represented as a vector $X_i \in \mathbb{R}^p$ in the set of standard basis vectors $\{e_1, \ldots, e_p\}$, then $\text{var}(X_{i,j}) = \pi_j(1 - \pi_j) \leq \pi_j$. Therefore, a weak sparsity condition on $\mathbf{\pi}$ conforms naturally with the variance decay condition (1.2). Similar considerations also apply to multivariate models with sparse Poisson marginals. Namely, if each observation $X_i \in \mathbb{R}^p$ has Poisson marginals and a weakly sparse mean vector $\mathbb{E}[X_i]$, then the basic fact $\text{var}(X_{i,j}) = \mathbb{E}[X_{i,j}]$ leads to variance decay.

- Fourier coefficients of functional data. Let $Y_1, \ldots, Y_n$ be an i.i.d. sample of functional data, taking values in a separable Hilbert space $\mathcal{H}$. In addition, suppose that the covariance operator $\mathcal{C} = \text{cov}(Y_1)$ is trace-class and satisfies an eigenvalue decay condition of the form (2.4) — which is common in functional data analysis (see, e.g., Cai and Hall, 2006). Lastly, for each $i = 1, \ldots, n$, let $X_i \in \mathbb{R}^p$ denote the first $p$ generalized Fourier coefficients of $Y_i$ with respect to some fixed orthonormal basis $\{\psi_j\}$ of $\mathcal{H}$. That is, $X_i = (\langle Y_i, \psi_1 \rangle, \ldots, \langle Y_i, \psi_p \rangle)$.

Under the above conditions, it can be shown that no matter which basis $\{\psi_j\}$ is chosen, the vectors $X_1, \ldots, X_n$ always satisfy the variance decay condition (1.2). (This follows from Proposition 2.1 below.) In Section 4, we explore some consequences of this condition as it relates to simultaneous confidence intervals for the Fourier coefficients of the mean function $\mathbb{E}[Y_1]$.

To conclude this section, we state a proposition that was used in the examples above. In essence, this basic result shows that decay among $\lambda_1(\Sigma), \ldots, \lambda_p(\Sigma)$ requires at least some decay among $\sigma_1, \ldots, \sigma_p$. As a matter of notation, if $v \in \mathbb{R}^p$ is a fixed vector, and $r > 0$, then the weak-$\ell_r$ quasi-norm is given by $\|v\|_w\ell_r = \max_{1 \leq j \leq p} j^{1/r} |v|(j)$, where $|v|(1) \geq \cdots \geq |v|(p)$ are the sorted absolute entries of $v$.

**Proposition 2.1.** Fix two numbers $s \geq 1$, and $r \in (0, s)$. Then, there is a constant $c_{r,s}$ depending only on $r$ and $s$, such that for any positive semidefinite matrix $A \in \mathbb{R}^{p \times p}$, we have

$$\|\text{diag}(A)\|_{w\ell_s} \leq c_{r,s} \|\lambda(A)\|_{w\ell_r}.$$ 

In particular, if $A = \Sigma$, and if there is a constant $c_0 > 0$ such that the inequality

$$\lambda_j(\Sigma) \leq c_0 j^{-1/r}$$

holds for all $j$.

holds for all $1 \leq j \leq p$, then the inequality
\[ \sigma^2_{(j)} \leq c_0 c_{r,s} j^{-1/s} \]
holds for all $1 \leq j \leq p$.

The proof is given in Appendix A, and follows essentially from the Schur-Horn majorization theorem, as well as inequalities relating $\| \cdot \|_r$ and $\| \cdot \|_{w_ℓ_r}$.

3. Main results. In this section, we present our main results on Gaussian approximation and bootstrap approximation.

3.1. Gaussian approximation. Let $\tilde{X}_1, \ldots, \tilde{X}_n$ be independent random vectors drawn from $N(\mu, \Sigma)$, and let
\[ \tilde{S}_n = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\tilde{X}_i - \mu). \]
The Gaussian counterpart of the partially standardized statistic $M$ (1.5) is defined as
\[ \tilde{M} = \max_{1 \leq j \leq p} \frac{\tilde{S}_{n,j}}{\sigma_j \tau_n}. \]

Our first theorem shows that in the presence of variance decay, the distribution $\mathcal{L}(\tilde{M})$ can approximate $\mathcal{L}(M)$ at a nearly parametric rate in Kolmogorov distance. Recall that for any random variables $U$ and $V$, this distance is given by
\[ d_K(\mathcal{L}(U), \mathcal{L}(V)) = \sup_{t \in \mathbb{R}} |\mathbb{P}(U \leq t) - \mathbb{P}(V \leq t)|. \]

**Theorem 3.1 (Gaussian approximation).** Fix any number $\delta \in (0, 1/2)$, and suppose that Assumptions 2.1 and 2.2 hold. In addition, suppose that $\tau_n \in [0, 1)$ with $(1 - \tau_n) / \sqrt{\log(n)} \geq 1$. Then, there is a constant $c_{α, δ} > 0$ depending only on $α$ and $δ$, such that
\[ d_K(\mathcal{L}(M), \mathcal{L}(\tilde{M})) \leq c_{α, δ} n^{-\frac{1}{2} + δ}. \]

**Remarks.** As a basic observation, note that the result handles the ordinary max statistic $T$ as a special case with $τ_n = 0$. In addition, it is especially notable that the rate does not depend on the dimension $p$, or the variance decay parameter $α$ (provided that it is positive). In this sense, the result shows that even a small amount of structure can have a substantial impact of Gaussian approximation, in relation to existing $n^{-1/6}$ rates that hold when $α = 0$. Lastly, the lower bound on $1 - τ_n$ is needed, because if $τ_n$ quickly approaches 1 as $n \to ∞$, then the variances $\text{var}(S_{n,j} / \sigma_j^{τ_n})$ will also quickly approach 1, eliminating the beneficial effect of variance decay.
3.2. Multiplier bootstrap approximation. In order to define the multiplier bootstrap counterpart of \( \tilde{M} \), first let \( X_1^*, \ldots, X_n^* \) be independent vectors drawn from \( N(0, \tilde{\Sigma}_n) \), where we define

\[
\tilde{\Sigma}_n = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})(X_i - \bar{X})^\top,
\]

and \( \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i \). Noting that each \( X_i^* \) has mean 0, we let \( S_n^* = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i^* \), and define the associated max statistic as

\[
M^* = \max_{1 \leq j \leq p} S_{n,j}^*/\tilde{\sigma}_{j}^{\tau_n},
\]

where \( (\tilde{\sigma}_1^2, \ldots, \tilde{\sigma}_p^2) = \text{diag}(\tilde{\Sigma}_n) \). In the exceptional case when \( \tilde{\sigma}_j = 0 \) for some \( j \), the expression \( S_{n,j}^*/\tilde{\sigma}_j \) is understood to be 0. This convention is natural, because if \( \tilde{\sigma}_j = 0 \), then \( S_{n,j}^* = 0 \) almost surely.

Remarks. The above description of \( M^* \) differs from some previous works insofar as we have suppressed the role of “multiplier variables”, and have defined \( S_n^* \) in terms of direct samples from \( N(0, \tilde{\Sigma}_n) \). From a mathematical standpoint, this is equivalent to the multiplier formulation (Chernozhukov, Chetverikov and Kato, 2013), where \( S_n^* = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^* (X_i - \bar{X}) \) and \( \xi_1^*, \ldots, \xi_n^* \) are independent \( N(0, 1) \) random variables, conditionally on \( X \).

**Theorem 3.2** (Bootstrap approximation). Suppose the conditions of Theorem 3.1 hold, with the same choice of \( \delta \in (0, 1/2) \). Then, there is an absolute constant \( c > 0 \), and a constant \( c_{\delta, \alpha} > 0 \) depending only \( \delta \) and \( \alpha \), such that the event

\[
d_K(\mathcal{L}(\tilde{M}), \mathcal{L}(M^*|X)) \leq c_{\delta, \alpha} n^{-\frac{1}{2}+\delta}
\]

occurs with probability at least \( 1 - \xi_n \).

Remarks on proofs. At a high level, the proofs of Theorems 3.1 and 3.2 are based on the following observation. When the variance decay condition holds, there is a relatively small subset of \( \{1, \ldots, p\} \) that is likely to contain the maximizing index for \( M \). In other words, if \( J_{\text{max}} \in \{1, \ldots, p\} \) denotes a random index satisfying \( M = S_n,J_{\text{max}}/\sigma_{J_{\text{max}}}^{\tau_n} \), then the “effective range” of \( J_{\text{max}} \) is fairly small. Although this situation is quite intuitive when the decay parameter \( \alpha \) is large, what is more surprising is that the effect persists even for small values of \( \alpha \).

Once the maximizing index \( J_{\text{max}} \) has been localized to a small set, it becomes possible to use tools that are specialized to the regime where \( p \ll n \).
For example, Bentkus’ multivariate Berry-Esseen theorem (Bentkus, 2003) (cf. Lemma E.1) is especially important in this regard. Another technical aspect of the proofs worth mentioning is that they make essential use of the sharp constants in Rosenthal’s inequality, as established in (Johnson, Schechtman and Zinn, 1985) (Lemma E.4).

4. Numerical illustration with functional data. Due to advances in technology and data collection, functional data have become ubiquitous in the past two decades, and statistical methods for their analysis have received growing interest. General references and surveys may be found in Ramsay and Silverman (2005); Ferraty and Vieu (2006); Horváth and Kokoszka (2012); Hsing and Eubank (2015); Wang, Chiou and Müller (2016).

The purpose of this section is to present an illustration, showing how the partially standardized statistic \( M \) and the bootstrap can be used to do inference on functional data. More specifically, we consider a one-sample test for a mean function, which proceeds by constructing simultaneous confidence intervals (SCI) for its Fourier coefficients. With regard to our theoretical results, this is a natural problem for illustration, because the Fourier coefficients of functional data typically satisfy the variance decay condition (1.2) — as explained in the third example of Section 2.2. Additional background on mean testing for functional data may be found in Benko, Härdle and Kneip (2009); Degras (2011); Cao, Yang and Todem (2012); Horváth, Kokoszka and Reeder (2013); Zheng, Yang and Härdle (2014); Choi and Reimherr (2018); Zhang et al. (2018), as well as references therein.

4.1. Testing the mean function. To set the stage, let \( \mathcal{H} \) be a separable Hilbert space of functions, and let \( Y \in \mathcal{H} \) be a random function with mean \( \mathbb{E}[Y] = \mu \). Given a sample \( Y_1, \ldots, Y_n \) of i.i.d. realizations of \( Y \), a basic goal is to test

\[
H_0 : \mu = \mu^\circ \quad \text{versus} \quad H_1 : \mu \neq \mu^\circ,
\]

where \( \mu^\circ \) is a fixed element in \( \mathcal{H} \).

This testing problem can be naturally formulated in terms of SCI, as follows. Let \( \{\psi_j\} \) denote any orthonormal basis for \( \mathcal{H} \). Also, let \( \{u_j\} \) and \( \{u_j^\circ\} \) respectively denote the generalized Fourier coefficients of \( \mu \) and \( \mu^\circ \) with respect to \( \{\psi_j\} \), so that

\[
\mu = \sum_{j=1}^{\infty} u_j \psi_j \quad \text{and} \quad \mu^\circ = \sum_{j=1}^{\infty} u_j^\circ \psi_j.
\]

Then, the null hypothesis is equivalent to \( u_j = u_j^\circ \) for all \( j \geq 1 \). To test this condition, one can construct a confidence interval \( \tilde{I}_j \) for each \( u_j \), and
reject the null if \( u_j^0 \not\in \hat{I}_j \) for at least one \( j \geq 1 \). In practice, due to infinite dimensionality, one will choose a sufficiently large integer \( p \), and reject the null if \( u_j^0 \not\in \hat{I}_j \) for at least one \( j = 1, \ldots, p \).

Recently, this general strategy was pursued by Choi and Reimherr (2018), hereafter CR, who developed a test for the problem (4.1) based on a hyper-rectangular confidence region for \( (u_1, \ldots, u_p) \) — which is equivalent to constructing SCI. In the CR approach, the basis is taken to be the eigenfunctions \( \{\psi_{C,j}\} \) of the covariance operator \( C = \text{cov}(Y) \), and \( p \) is chosen as the number of eigenfunctions \( \psi_{C,1}, \ldots, \psi_{C,p} \) required to explain a certain fraction (say 99%) of variance in the data. However, since \( C \) is unknown, the these functions must be estimated.

When \( p \) is large, estimating the eigenfunctions \( \psi_{C,1}, \ldots, \psi_{C,p} \) is a well-known challenge in functional data analysis. For instance, if the sample paths of \( Y_1, \ldots, Y_n \) are not sufficiently smooth, then a large number \( p \) may be needed to explain most of the variance. Another example occurs when \( H_1 \) holds, but \( \mu \) and \( \mu^0 \) are not well separated. If this is the case, then a large choice of \( p \) may be needed in order to distinguish \( (u_1, \ldots, u_p) \) and \( (u_1^0, \ldots, u_p^0) \). In our illustration below, we consider an alternative approach to constructing SCI that leverages the fact that the bootstrap method in Section 3 can accommodate large values of \( p \).

4.2. Applying the bootstrap. Let \( \{\psi_j\} \) be any pre-specified orthonormal basis for \( \mathcal{H} \). For instance, when \( \mathcal{H} = L^2[0,1] \), a commonly considered option is to let \( \{\psi_j\} \) be the standard Fourier basis. Letting \( Y_1, \ldots, Y_n \) be as before, define a sample of vectors \( X_1, \ldots, X_n \) in \( \mathbb{R}^p \) according to

\[
X_i = (\langle Y_i, \psi_1 \rangle, \ldots, \langle Y_i, \psi_p \rangle),
\]

and note that \( \mathbb{E}[X_1] = (u_1, \ldots, u_p) \). For simplicity, we retain the other notation associated with \( X_1, \ldots, X_n \) in previous sections, so that \( S_{n,j} = n^{-1/2} \sum_{i=1}^n (X_{i,j} - u_j) \), and likewise for other quantities. In addition, for any \( \tau_n \in [0,1] \), let

\[
L = \min_{1 \leq j \leq p} S_{n,j}/\sigma_j^{\tau_n} \quad \text{and} \quad M = \max_{1 \leq j \leq p} S_{n,j}/\sigma_j^{\tau_n}.
\]

For a given significance level \( \varrho \in (0,1) \), the quantiles of \( L \) and \( M \) are denoted \( q_L(\varrho) \) and \( q_M(\varrho) \). This implies the following event occurs with probability at least \( 1 - \varrho \),

\[
\bigcap_{j=1}^p \left\{ \frac{q_L(\varrho/2)\sigma_j^{\tau_n}}{\sqrt{n}} \leq \bar{X}_j - u_j \leq \frac{q_M(1-\varrho/2)\sigma_j^{\tau_n}}{\sqrt{n}} \right\},
\]

(4.2)
which leads to theoretical SCI for \((u_1, \ldots, u_p)\).

We now apply the bootstrap from Section 3.2 to estimate \(q_L(g/2)\) and \(q_M(1 - g/2)\). Specifically, if we generate \(B \geq 1\) independent samples of \(M^*\) as in (3.4), then we define \(\hat{q}_M(1 - g/2)\) to be the empirical \(1 - g/2\) quantile of the \(B\) samples, and similarly for \(\hat{q}_L(g/2)\). In turn, the bootstrap SCI are defined by

\[
\hat{I}_j = \left[ \bar{X}_j + \frac{\hat{q}_L(g/2)\hat{\sigma}_{j\tau n}}{\sqrt{n}}, \bar{X}_j + \frac{\hat{q}_M(1-g/2)\hat{\sigma}_{j\tau n}}{\sqrt{n}} \right]
\]

for each \(j = 1, \ldots, p\).

It remains to select a value for \(\tau_n\), which can be done with the following simple rule. For each value of \(\tau_n\) in a set of candidates \(T = \{0, 0.1, \ldots, 0.9, 1\}\), we construct the associated intervals \(\hat{I}_1, \ldots, \hat{I}_p\) as in (4.3). Then, we choose the value \(\tau_n \in T\) for which the average width \(\frac{1}{p} \sum_{j=1}^p |\hat{I}_j|\) is the smallest. (Note that \(|[a, b]| = b - a\).

In Figure 1, we illustrate the influence of \(\tau_n\) on the shape of SCI. There are two main points to notice: (1) The intervals change very gradually as a function of \(\tau_n\), which shows that partial standardization is a mild adjustment to ordinary standardization. (2) The choice of \(\tau_n\) involves a tradeoff, which controls the “allocation of power” among the \(p\) intervals. When \(\tau_n\) is close
to 1, the intervals are wider for the top coefficients (small \( j \)), and narrower for the bottom coefficients (large \( j \)). However, as \( \tau_n \) decreases from 1, the widths of the intervals gradually become more uniform, and the intervals for the top coefficients become narrower. Hence, if the vectors \((u_1, \ldots, u_p)\) and \((u_1', \ldots, u_p')\) differ in the top coefficients, then choosing a smaller value of \( \tau_n \) may lead to a gain in power. One last interesting point to mention is that in the simulations below, the selection rule of “minimizing the average width” typically selected values of \( \tau_n \) around 0.8, and hence strictly less than 1.

4.3. Simulation settings. To study the numerical performance of the SCI described above, we generated i.i.d. samples from a Gaussian process on \([0, 1]\), with population mean function

\[
\mu_{\omega, \rho, \theta}(t) = (1 + \rho) \cdot (\exp[-\{g_\omega(t) + 2\}] + \exp[-\{g_\omega(t) - 2\})] + \theta
\]

indexed by parameters \((\omega, \rho, \theta)\), where \(g_\omega(t) := 8h_\omega(t) - 4\), and \(h_\omega(t)\) denotes the Beta distribution function with shape parameters \((2 + \omega, 2)\). This family of functions was considered in Chen and Müller (2012). To interpret the parameters, note that \(\omega\) determines the shape of the mean function (see Figure 2), whereas \(\rho\) and \(\theta\) are scaling and shift parameters. In terms of these parameters, the null hypothesis corresponds to \(\mu = \mu^\circ := \mu_{0,0,0}\).

The population covariance function was taken to be the Matern function

\[
C(s, t) = \frac{(\sqrt{2\nu}|t-s|)\nu}{16|t-s|^{\nu-1}} K_\nu(\sqrt{2\nu}|t-s|),
\]

which was considered in CR, with \(K_\nu\) being a modified Bessel function of the second kind. We set \(\nu = 0.1\), which results in relatively rough sample paths,
as illustrated in Figure 3. To understand how this covariance structure relates to Assumption 2.2, we can numerically verify that Assumption 2.2(i) is satisfied with \( c = 0.153 \) and \( \alpha = 0.69 \) (see Figure 3). In addition, Assumption 2.2(ii) is satisfied with \( \rho_{\text{max}}(\ell_n) \leq 3 \times 10^{-15} \) when \( n = 50 \) (\( \ell_n = 16 \)), as well as \( \rho_{\text{max}}(\ell_n) \leq 0.027 \) when \( n = 200 \) (\( \ell_n = 29 \)).

![Figure 3](image)

Fig 3. Left: A sample of the functional data \( Y_1, \ldots, Y_n \) in the simulation study. Right: The ordered values \( \sigma_{(j)} = \sqrt{\text{var}(X_{1:j})} \) are represented by dots, which are well approximated by the decay profile \( 0.153j^{-0.69} \) (solid line).

When implementing the bootstrap in Section 4.2, we always used the first \( p = 100 \) functions from the standard Fourier basis on \([0,1]\). (In principle, an even larger value \( p \) could have been selected, but we chose \( p = 100 \) to limit computation time.) Meanwhile, we implemented the CR method using its accompanying R package \texttt{fregion} (Choi and Reimherr, 2016) under default settings, which typically estimated the first \( p \approx 50 \) eigenfunctions of the covariance operator \( \mathcal{C} \).

Results on type I error. The nominal significance level was set to 0.05 in all simulations. To assess the actual type I error, we carried out 5,000 simulations under the null hypothesis, for both of the cases \( n = 50 \) and \( n = 200 \). When \( n = 50 \), the type I error was 6.7% for the bootstrap method, and 1.6% for CR. When \( n = 200 \), the results were 5.7% for the bootstrap method, and 2.6% for CR. So, in these cases, the bootstrap respects the nominal significance level relatively well.

Results on power. To consider power, we varied each of the parameters \( \omega \), \( \rho \) and \( \theta \), one at a time, while keeping the other two at their baseline value of zero. In each parameter setting, we carried out 1,000 simulations with
sample size $n = 50$. The results are summarized in Figure 4, showing that the bootstrap achieves relative gains in power — especially when $H_0$ and $H_1$ differ in shape ($\omega$) or scale ($\rho$). Indeed, it seems that using a large number of basis functions can help to catch small differences in shape or scale near the endpoints of the domain $[0,1]$ (see also Figure 2).

![Figure 4](image-url)

**Fig 4.** Empirical power for the partially standardized bootstrap method (solid) and the CR method (dotted) Left: Empirical power for varying shape parameters $\omega$ while $\rho = \theta = 0$. Middle: Empirical power for varying scale parameters $\rho$ while $\omega = \theta = 0$. Right: Empirical power for varying shift parameters $\theta$ while $\omega = \rho = 0$.

### 5. Conclusions.

The main conclusion to draw from our work is that a modest amount of variance decay in a high-dimensional model can substantially enhance rates of bootstrap approximation for max statistics. In particular, there are three aspects of this type of model structure that are worth emphasizing. First, the variance decay condition (1.2) is very weak, in the sense that the decay parameter $\alpha > 0$ is allowed to be arbitrarily small. Second, the condition is approximately checkable in practice, since the variances $\sigma_1^2, \ldots, \sigma_p^2$ can be accurately estimated when $n \ll p$. Third, this type of structure arises naturally in a variety of contexts, such as in applications of PCA, as well as in the analysis of sparse count data, and functional data.

Beyond our main theoretical focus on rates of approximation, we have also shown that the technique of partial standardization leads to favorable numerical results at moderate sample sizes. Specifically, this was illustrated with an example from functional data analysis, where the inherent variance decay of Fourier coefficients can be leveraged. Finally, we note that this application to functional data is just one of many possible illustrations, and the adaptation of these ideas to other situations may provide some opportunities for future work.

**Appendices.**
Organization of appendices. In Appendix A we prove Proposition 2.1, and in Appendices B and C we prove Theorems 3.1 and 3.2 respectively. These proofs rely on numerous technical lemmas, which are stated and proved in Appendix D of the supplementary material. Lastly, in Appendix E of the supplementary material, we provide statements of background results that are used in the proofs.

General remarks and notation. It will simplify some of the proofs to make use of the fact that the metric $d_K$ is always bounded by 1, and therefore, it is sufficient to show that Theorems 3.1 and 3.2 hold for all large values of $n$. (This is because a constant $c_{\alpha,\delta}$ can be chosen large enough so that $c_{\alpha,\delta} n^{-\frac{1}{2} + \delta} \geq 1$ for finitely many values of $n$.)

To fix some notation that will be used throughout the appendices, let $d \in \{1, \ldots, p\}$, and define a generalized version of $M$ as

$$M_d = \max_{j \in J(d)} S_{n,j}/\sigma_j^{\tau_n}.$$ 

In particular, note that the statistic $M$ defined in equation (1.5) is the same as $M_p$. Similarly, the Gaussian and bootstrap versions of $M_d$, denoted $\tilde{M}_d$ and $M_d^*$, are defined as

$$\tilde{M}_d = \max_{j \in J(d)} \tilde{S}_{n,j}/\sigma_j^{\tau_n},$$

and

$$M_d^* = \max_{j \in J(d)} S_{n,j}^{\ast}/\hat{\sigma}_j^{\tau_n}.$$ 

In addition, define the parameter

(5.1) $$\beta_n = \alpha(1 - \tau_n),$$

as well as the integer

(5.2) $$k_n = k_n(\delta) = \left\lceil \left( n^{\frac{\delta}{4(\beta_n + 3)}} \vee \ell_n \right) \wedge p \right\rceil,$$

where $\delta > 0$ is the value fixed in Theorems 3.1 and 3.2. This integer always satisfies $1 \leq \ell_n \leq k_n \leq p$. Lastly, we will often use the fact that if a random variable satisfies $\|\xi\|_{\psi_1} \leq c$ for some absolute constant $c$, then there is another absolute constant $c' > 0$, such that $\|\xi\|_r \leq c'r$ for all $r \geq 1$ (Vershynin, 2018, Proposition 2.7.1).
APPENDIX A: PROOF OF PROPOSITION 2.1

**Proof.** It is a standard fact that for any \( s \geq 1 \), the \( \ell_s \) norm dominates \( \ell_w \) norm, and so \( \| \text{diag}(\Sigma) \|_{\ell_w} \leq \| \text{diag}(\Sigma) \|_s \). Next, since \( \Sigma \) is symmetric, the Schur-Horn Theorem implies that the vector \( \text{diag}(\Sigma) \) is majorized by \( \lambda(\Sigma) \) (Marshall, Olkin and Arnold, 2011, p.300). Furthermore, when \( s \geq 1 \), the function \( \| \cdot \|_s \) is Schur-convex on \( \mathbb{R}^p \), which means that if \( u \in \mathbb{R}^p \) is majorized by \( v \in \mathbb{R}^p \), then \( \| u \|_s \leq \| v \|_s \) (Marshall, Olkin and Arnold, 2011, p.138). Hence,

\[
\| \text{diag}(\Sigma) \|_{\ell_w} \leq \| \lambda(\Sigma) \|_s.
\]

Finally, if \( r \in (0, s) \), then for any \( v \in \mathbb{R}^p \), the inequality

\[
\| v \|_s \leq \left( \zeta(s/r) \right)^{1/s} \| v \|_{\ell_r}.
\]

holds, where \( \zeta(x) := \sum_{j=1}^{\infty} j^{-x} \) for \( x > 1 \). This bound may be derived as in (Johnstone, 2017, p.257),

\[
\| v \|_s = \sum_{j=1}^{p} |v_j|^s \leq \sum_{j=1}^{p} \left( \| v \|_{\ell_r} j^{-1/r} \right)^s \leq \zeta(s/r) \cdot \| v \|_{\ell_r}^s,
\]

which completes the proof. \( \square \)

APPENDIX B: PROOF OF THEOREM 3.1

**Proof.** Consider the inequality

\[
d_K(\mathcal{L}(M_p), \mathcal{L}(\tilde{M}_p)) \leq I_n + II_n + III_n,
\]

where we define

\[
I_n = d_K\left( \mathcal{L}(M_p), \mathcal{L}(M_{kn}) \right) \quad (B.2)
\]

\[
II_n = d_K\left( \mathcal{L}(M_{kn}), \mathcal{L}(\tilde{M}_{kn}) \right) \quad (B.3)
\]

\[
III_n = d_K\left( \mathcal{L}(\tilde{M}_{kn}), \mathcal{L}(\tilde{M}_p) \right) \quad (B.4)
\]

Below, we show that the term \( \Pi_n \) is at most of order \( n^{-\frac{1}{2}+\delta} \) in Proposition B.1. Later on, we establish a corresponding result for \( I_n \) and \( III_n \) in Proposition B.2. Taken together, these results complete the proof of Theorem 3.1. \( \square \)
Proposition B.1. Suppose the conditions of Theorem 3.1 hold, with the same choice of $\delta \in (0, 1/2)$. Then, there is a constant $c_\delta > 0$ depending only on $\delta$ such that

\[(B.5) \quad \Pi_n \leq c_\delta n^{-\frac{1}{2} + \delta}.\]

Proof. For ease of notation, we will write $k_0 = k$ below. Let $\Pi_k \in \mathbb{R}^{k \times p}$ denote the projection onto the coordinates indexed by $\mathcal{J}(k)$. This means that if $\mathcal{J}(k)$ is enumerated as $(\sigma_{j_1}, \ldots, \sigma_{j_k}) = (\sigma_{(1)}, \ldots, \sigma_{(k)})$, and if $l \in \{1, \ldots, k\}$, then the $l$th row of $\Pi_k$ is the standard basis vector $e_{j_l} \in \mathbb{R}^p$.

Next, define the diagonal matrix $D_k = \text{diag}(\sigma_{(1)}, \ldots, \sigma_{(k)})$. It follows that

\[(B.6) \quad M_k = \max_{1 \leq j \leq k} e_j^\top D_k^{-\tau_n} \Pi_k S_n, \]

In light of this relation, we will deal with the covariance matrix of the random vector $D_k^{-\tau_n} \Pi_k S_n$, which is

\[(B.6) \quad \mathcal{G}_k := D_k^{-\tau_n} \Pi_k \Sigma \Pi_k^\top D_k^{-\tau_n}. \]

Also, let $\breve{Z} \in \mathbb{R}^k$ denote the random vector with zero mean and identity covariance matrix such that

\[D_k^{-\tau_n} \Pi_k S_n = \mathcal{G}_k^{1/2} \breve{Z}.\]

It is simple to check that any fixed $t \in \mathbb{R}$, there is a Borel-measurable convex set $\mathcal{A}_t \subset \mathbb{R}^k$ such that $P(M_k \leq t) = P(\breve{Z} \in \mathcal{A}_t)$. By the same reasoning, we also have $P(\breve{M}_k \leq t) = \gamma_k(\mathcal{A}_t)$, where $\gamma_k$ is the standard Gaussian distribution on $\mathbb{R}^k$. Therefore, the quantity $\Pi_n$ satisfies the bound

\[(B.7) \quad \Pi_n \leq \sup_{\mathcal{A} \in \mathcal{A}} \left| P(\breve{Z} \in \mathcal{A}) - \gamma_k(\mathcal{A}) \right|, \]

where $\mathcal{A}$ denotes the collection of all Borel-measurable convex subsets of $\mathbb{R}^k$.

We now apply Theorem 1.1 of Bentkus (2003) (Lemma E.1), to handle the supremum above. First observe that

\[(B.8) \quad \breve{Z} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} J_k Z_i, \]

where $J_k \in \mathbb{R}^{k \times p}$ is a deterministic matrix given by

\[J_k := \mathcal{G}_k^{-1/2} D_k^{-\tau_n} \Pi_k \Sigma^{1/2}. \]
The representation of $\tilde{Z}$ in (B.8) satisfies the conditions of (Bentkus, 2003, Theorem 1.1.), since the terms $J_k Z_1, \ldots, J_k Z_n$ are i.i.d. with zero mean and identity covariance matrix. Therefore,$$
P_n \lesssim k^{1/4} \cdot E[\|J_k Z_1\|_2^3] \cdot n^{-1/2}.$$

It remains to bound the factor $E[\|J_k Z_1\|_2^3]$. By Lyapunov’s inequality,$$
E[\|Z_1\|_2^3] \leq E[\|Z_1\|_4^2]^{3/4},
$$
and furthermore,$$
E[\|J_k Z_1\|_4^2] = E[(Z_1^\top J_k^\top J_k Z_1)^2] = E[(Z_1^\top J_k^\top J_k Z_1 - \text{tr}(J_k^\top J_k))^2] \lesssim k^2,$$
where the last step follows from Lemma E.6, as well as the fact that $J_k^\top J_k$ is idempotent with rank $k$. Altogether, we have $E[\|J_k Z_1\|_2^3] \lesssim k^{6/4}$, and hence$$
P_n \lesssim k^{7/4} n^{-1/2} \leq c_\delta n^{-1/2 + \delta},$$
as needed. \hfill \Box

**Proposition B.2.** Assume that the conditions of Theorem 3.1 hold, with the same choice of $\delta \in (0, 1/2)$. Then, there is a constant $c_{\alpha, \delta} > 0$ depending only on $\alpha$ and $\delta$ such that
\begin{equation}
I_n \leq c_{\alpha, \delta} n^{-1/2 + \delta} \quad \text{and} \quad III_n \leq c_{\alpha, \delta} n^{-1/2 + \delta}.
\end{equation}

**Proof.** We only prove the bound for $I_n$, since the same argument applies to $III_n$. It is simple to check that for any fixed real number $t$,
$$
\left| P\left( \max_{1 \leq j \leq p} \frac{S_n,j}{\sigma_n^j} \leq t \right) - P\left( \max_{j \in J(k_n)} \frac{S_n,j}{\sigma_n^j} \leq t \right) \right| = P(A(t) \cap B(t)),
$$
where we define the events
\begin{equation}
A(t) = \left\{ \max_{j \in J(k_n)} \frac{S_n,j}{\sigma_n^j} \leq t \right\} \quad \text{and} \quad B(t) = \left\{ \max_{j \in J(k_n)^c} \frac{S_n,j}{\sigma_n^j} \leq t \right\},
\end{equation}
and $J(k_n)^c$ denotes the complement of $J(k_n)$ in $\{1, \ldots, p\}$. Also, for any pair of real numbers $t_{1,n}$ and $t_{2,n}$ satisfying $t_{1,n} \leq t_{2,n}$, it is straightforward to check that the following inclusion holds for all $t \in \mathbb{R}$,
\begin{equation}
A(t) \cap B(t) \subset A(t_{2,n}) \cup B(t_{1,n}).
\end{equation}
Applying a union bound, and then taking the supremum over $t \in \mathbb{R}$, we obtain
$$I_n \leq \mathbb{P}(A(t_{2,n})) + \mathbb{P}(B(t_{1,n})).$$

The remainder of the proof consists in selecting $t_{1,n}$ and $t_{2,n}$ so that $t_{1,n} \leq t_{2,n}$ and that the probabilities $\mathbb{P}(A(t_{2,n}))$ and $\mathbb{P}(B(t_{1,n}))$ are sufficiently small. Below, Lemma B.1 shows that if $t_{1,n}$ and $t_{2,n}$ are chosen as

$$t_{1,n} = \kappa_{\alpha} \cdot k_{-n}^{\beta_n} \cdot \log(n) \quad (B.13)$$

$$t_{2,n} = \kappa \cdot \ell_{-n}^{\beta_n} \cdot \sqrt{(1 - \rho_{\max}(\ell_n)) \cdot \log(\ell_n)}, \quad (B.14)$$

for certain constants $\kappa_{\alpha}$ and $\kappa$, then $\mathbb{P}(A(t_{2,n}))$ and $\mathbb{P}(B(t_{1,n}))$ are at most of order $n^{-\frac{1}{2} + \delta}$. Furthermore, it follows from Assumption 2.2, as well as the condition $(1 - \tau_n)\sqrt{\log(n)} \gtrsim 1$, that the inequality $t_{1,n} \leq t_{2,n}$ holds for all $n \geq n_0(\alpha, \delta)$, where $n_0(\alpha, \delta)$ is an integer that depends only on $\alpha$ and $\delta$.

(Note that the constant $c_{\alpha,\delta}$ in the bounds (B.10) can be chosen so that $c_{\alpha,\delta} \geq n_{0}(\alpha, \delta)^{1/2 - \delta}$, ensuring the bounds are valid when $n \leq n_0(\alpha, \delta)$.)

**Lemma B.1.** Suppose the conditions of Theorem 3.1 hold, with the same choice of $\delta \in (0, 1/2)$. Then, there are positive constants $\kappa_{\alpha}$ and $\kappa$ that can be selected in equations (B.13) and (B.14) so that

(a) $\mathbb{P}(A(t_{2,n})) \leq c_{\delta} n^{-\frac{1}{2} + \delta},$

and

(b) $\mathbb{P}(B(t_{1,n})) \leq n^{-1},$

where $c_{\delta} > 0$ is a constant depending only on $\delta$.

**Proof of Lemma B.1 part (a).** Due to Proposition B.1 and the fact that $\mathcal{J}(\ell_n) \subset \mathcal{J}(k_n)$, we have

$$\mathbb{P}(A(t_{2,n})) \leq \mathbb{P}\left( \max_{j \in \mathcal{J}(k_n)} \frac{\hat{S}_{n,j}}{\sigma_j^{\tau_n}} \leq t_{2,n} \right) + I_{1,n} \quad (B.15)$$

$$\leq \mathbb{P}\left( \max_{j \in \mathcal{J}(k_n)} \frac{\hat{S}_{n,j}}{\sigma_j^{\tau_n}} \leq t_{2,n} \right) + c_{\delta} n^{-\frac{1}{2} + \delta}. $$

Since $\hat{S}_n$ is a Gaussian vector, we may use Slepian’s lemma (Lemma E.3) to derive an upper bound for the first term in the last line. (In fact, we will show it is of order $n^{-1/2}.$)
Observe that for any distinct indices \( j, j' \in J(\ell_n) \), the vector \( S_n \) satisfies

\[
E \left[ \left( \frac{S_{n,j}}{\tau_n} - \frac{S_{n,j'}}{\tau_n} \right)^2 \right] = \left( \sigma_j^{1-\tau} - \sigma_j'^{1-\tau_n} \right)^2 + 2 \sigma_j^{1-\tau} \sigma_j'^{1-\tau_n} (1 - \rho_j j')
\]

\( (B.16) \)

\[\geq 2 \sigma_j^{1-\tau} \sigma_j'^{1-\tau_n} (1 - \rho_j j') \]

\[\geq \ell_n^{-2\beta_n} (1 - \rho_{\max}(\ell_n)).\]

Based on this lower bound, if we let \( \xi_1, \ldots, \xi_{\ell_n} \) be independent \( N(0, 1) \) random variables, and put

\[c_n = \kappa_0 \cdot \ell_n^{-\beta_n} \cdot \sqrt{1 - \rho_{\max}(\ell_n)}\]

for a sufficiently small absolute constant \( \kappa_0 > 0 \), then for any distinct indices \( j, j' \in J(\ell_n) \), we have

\[
E \left[ \left( \frac{S_{n,j}}{\tau_n} - \frac{S_{n,j'}}{\tau_n} \right)^2 \right] \geq 2 c_n^2 = E \left[ (c_n \xi_j - c_n \xi_j')^2 \right].
\]

Consequently, Slepian’s lemma implies

\[
P \left( \max_{j \in J(\ell_n)} \frac{S_{n,j}}{\tau_n} \leq t_{2,n} \right) \leq P \left( \max_{j \in J(\ell_n)} c_n \xi_j \leq t_{2,n} \right)
\]

\( (B.18) \)

\[
= \Phi \left( \frac{t_{2,n}}{c_n} \right) \ell_n,
\]

where \( \Phi \) is the standard normal distribution function. In turn, we will use an elementary numerical inequality,

\[
\Phi \left( \frac{1}{2} \sqrt{\log \left( \frac{x}{2} \right)} \right) \leq 1 - \frac{1}{x},
\]

which can be verified to hold when \( x \geq 5/2 \). Now consider the choice

\[t_{2,n} = \frac{c_n}{2} \sqrt{\log(\sqrt{\ell_n})},\]

and let \( 2 \sqrt{\ell_n} \) play the role of \( x \). Then, for sufficiently large \( n \) we have \( 2 \sqrt{\ell_n} \geq 5/2 \), and so

\[
P \left( \max_{j \in J(\ell_n)} \frac{S_{n,j}}{\tau_n} \leq t_{2,n} \right) \leq \left( 1 - \frac{1}{2 \sqrt{\ell_n}} \right) \ell_n
\]

\( (B.20) \)

\[\leq \exp \left( -\frac{1}{2} \sqrt{\ell_n} \right)
\]

\[\lesssim n^{-\frac{1}{2}},\]

which completes the proof.
Proof of Lemma B.1 part (b). Define the random variable

\[ V = \max_{j \in \mathcal{J}(k_n)^c} \frac{S_{n,j}}{\sigma_j^\tau_n}, \]

and the parameter

\[ q = \max \{ \frac{2}{\beta_n}, \log(n), 3 \}. \]

For ease of notation, we omit the dependence of \( V \) and \( q \) on \( n \). Clearly, for any \( t > 0 \), we have the tail bound

\[ \mathbb{P}(V \geq t) \leq \frac{\|V\|^q}{t^q}, \tag{B.21} \]

and furthermore

\[ \|V\|^q = \mathbb{E}\left( \left| \max_{j \in \mathcal{J}(k_n)^c} \frac{S_{n,j}}{\sigma_j^\tau_n} \right|^q \right) \leq \sum_{j \in \mathcal{J}(k_n)^c} \sigma_j^{q(1-\tau_n)} \mathbb{E}\left[ \left| \frac{1}{\sigma_j} S_{n,j} \right|^q \right]. \tag{B.22} \]

Noting that \( q > 2 \), we may apply Rosenthal’s inequality (Lemma D.4) to obtain \( \| \frac{1}{\sigma_j} S_{n,j} \|_q \leq c_\alpha q \) for some constant \( c_\alpha > 0 \), and so

\[ \|V\|^q \leq (c_\alpha q)^q \sum_{j \in \mathcal{J}(k_n)^c} \sigma_j^{q(1-\tau_n)} \]

\[ \lesssim (c_\alpha q)^q \sum_{j = k_n+1}^p j^{-q\beta_n} \]

\[ \leq (c_\alpha q)^q \int_{k_n}^p x^{-q\beta_n} dx \]

\[ \leq \frac{(c_\alpha q)^q}{q\beta_n - 1} k_n^{-q\beta_n + 1}, \tag{B.23} \]

where we recall \( \beta_n = \alpha(1 - \tau_n) \), and note that \( q\beta_n > 1 \), which holds by the definition of \( q \). Hence, if we put \( C_n := \frac{c_\alpha}{(q\beta_n - 1)^{1/q}} \cdot k_n^{1/q} \), then

\[ \|V\|_q \leq C_n \cdot q \cdot k_n^{-\beta_n}. \]

It is simple to check that \( C_n \leq \kappa_\alpha \) for some constant \( \kappa_\alpha > 0 \) depending only on \( \alpha \). Also, the assumption \((1 - \tau_n)\sqrt{\log(n)} \gtrsim 1\) implies

\[ q \lesssim \frac{1}{\alpha \wedge 1} \log(n). \]
Therefore, the inequality (B.21) with $t = e\|V\|_q$ gives

$$\mathbb{P}\left(V \geq \kappa_\alpha \cdot \log(n) \cdot k_n^{-\beta} \right) \leq e^{-q} \leq \frac{1}{n},$$

for some constant $\kappa_\alpha > 0$ depending only on $\alpha$, as needed. \qed

**APPENDIX C: PROOF OF THEOREM 3.2**

**Proof.** Consider the inequality

$$(C.1) \quad d_K(\mathcal{L}(\tilde{M}_p), \mathcal{L}(M^*_p|X)) \leq I'_n + II'_n(X) + III'_n(X),$$

where we define

$$(C.2) \quad I'_n = d_K(\mathcal{L}(\tilde{M}_p), \mathcal{L}(\tilde{M}_{k_n}))$$

$$(C.3) \quad II'_n(X) = d_K(\mathcal{L}(\tilde{M}_{k_n}), \mathcal{L}(M^*_{k_n}|X))$$

$$(C.4) \quad III'_n(X) = d_K(\mathcal{L}(M^*_{k_n}|X), \mathcal{L}(M^*_p|X)).$$

Note that $I'_n$ is deterministic, whereas $II'_n(X)$ and $III'_n(X)$ are random variables depending on $X$. The remainder of the proof consists in showing that each of these terms are at most of order $n^{-\frac{1}{2}+\delta}$, with probability at least $1 - \frac{c}{n}$. The terms $II'_n$ and $III'_n$ are handled in Sections C.2 and C.1 respectively. The first term $I'_n$ requires no further work, due to Proposition B.2 (since $I'_n$ is equal to $III_n$, defined in equation (B.4)).

**C.1. Handling the term $III'_n(X)$.** The proof of Proposition B.2 can be partially re-used to show that for any fixed realization of $X$, and any real numbers $t'_{1,n} \leq t'_{2,n}$, the following bound holds

$$(C.5) \quad III'_n(X) \leq \mathbb{P}(A'(t'_{2,n})|X) + \mathbb{P}(B'(t'_{1,n})|X),$$

where we define the following events for any $t \in \mathbb{R}$,

$$(C.6) \quad A'(t) = \left\{ \max_{j \in J(k_n)} S^*_{n,j}/\hat{\sigma}^*_j \leq t \right\} \quad \text{and} \quad B'(t) = \left\{ \max_{j \in J(k_n)^c} S^*_{n,j}/\hat{\sigma}^*_j > t \right\}.$$

Below, Lemma C.1 ensures that $t'_{1,n}$ and $t'_{2,n}$ can be chosen so that the random variables $\mathbb{P}(B'(t'_{1,n})|X)$ and $\mathbb{P}(A'(t'_{2,n})|X)$ are at most of order $n^{-\frac{1}{2}}$, with probability at least $1 - \frac{c}{n}$. Also, it is straightforward to check that under Assumption 2.2, the choices of $t'_{1,n}$ and $t'_{2,n}$ given in Lemma C.1 satisfy $t'_{1,n} \leq t'_{2,n}$ when $n$ is sufficiently large.
Lemma C.1. Suppose the conditions of Theorem 3.1 hold, with the same choice of \( \delta \in (0, 1/2) \). Then, there are positive constants \( \kappa'_\alpha, \kappa', c \) for which the following statement is true:

If \( t'_{1,n} \) and \( t'_{2,n} \) are chosen as

\[
t'_{1,n} = \kappa'_\alpha \cdot \log(n)^{3/2} \cdot \kappa_n^{-\beta_n}
\]

and

\[
t'_{2,n} = \kappa' \cdot \ell_n^{-\beta_n} \cdot \sqrt{\left(1 - \rho_{\max}(\ell_n) - \epsilon_n\right)_+} \cdot \log(\sqrt{\ell_n}),
\]

with \( \epsilon_n = c \cdot n^{-1/2} \cdot \log(n)^3 \), then the events

(a) \( \mathbb{P}(A'(t'_{2,n})|X) \leq c n^{-\frac{1}{2}} \)

and

(b) \( \mathbb{P}(B'(t'_{1,n})|X) \leq n^{-1} \)

each hold with probability at least \( 1 - \frac{c}{n} \).

Proof of Lemma C.1 part (a). Similarly to Lemma B.1 part (a), the proof is based on Slepian’s lemma (Lemma E.3). For any \( j, j' \in J(\ell_n) \), let \( \hat{\rho}_{j,j'} \) denote the sample correlation associated with the \( (j, j') \) entry of \( \hat{\Sigma}_n \), and define the sample version of \( \rho_{\max}(\ell_n) \) as

\[
\hat{\rho}_{\max}(\ell_n) = \max \left\{ \hat{\rho}_{j,j'} \mid j, j' \in J(\ell_n), \text{ and } j \neq j' \right\}.
\]

The argument in the proof of Lemma B.1 part (a) may be repeated to show there is an absolute constant \( \kappa' > 0 \), such that if we define

\[
\hat{t}_{2,n} = \kappa' \left( \min_{j \in J(\ell_n)} \frac{\hat{\sigma}_j^{(1-\tau_n)}}{\hat{\sigma}_j} \right) \cdot \sqrt{\left(1 - \hat{\rho}_{\max}(\ell_n)\right)_+} \cdot \log(\sqrt{\ell_n}),
\]

then the following bound holds for any realization of \( X \),

\[
\mathbb{P}\left( \max_{j \in J(\ell_n)} S_{n,j} / \hat{\sigma}_j^{\tau_n} \leq \hat{t}_{2,n} \mid X \right) \leq c n^{-\frac{1}{2}},
\]

for some absolute constant \( c > 0 \). The only remaining task is to select a deterministic value \( t'_{2,n} \) that satisfies \( \hat{t}_{2,n} \geq t'_{2,n} \) with high probability. For this purpose, Lemmas D.6 and D.7 imply there is a sufficiently large absolute constant \( c > 0 \) such that events

\[
\min_{j \in J(\ell_n)} \hat{\sigma}_j^{(1-\tau_n)} \geq \frac{1}{2} \ell_n^{-\beta_n},
\]
and
\[
(\text{C.12}) \quad 1 - \tilde{\rho}_{\text{max}}(\ell_{n}) \geq \left( 1 - \rho_{\text{max}}(\ell_{n}) - \frac{c\log(n)^{3}}{\sqrt{n}} \right) +
\]
each hold with probability at least \( 1 - \frac{c}{n} \). Consequently, if \( t'_{2,n} \) is selected as in equation (C.8), then the event
\[
(\text{C.13}) \quad \mathbb{P}\left( \max_{j \in \mathcal{J}(\ell_{n})} S_{n,j}^{*}/\hat{\sigma}_{j}^{\tau_{n}} \leq t'_{2,n} \middle| X \right) \leq cn^{-\frac{1}{2}}
\]
holds with probability at least \( 1 - \frac{c}{n} \), which completes the proof.

**Proof of Lemma C.1 part (b).** Define the random variable
\[
(\text{C.14}) \quad V^{*} := \max_{j \in \mathcal{J}(k_{n})} S_{n,j}^{*}/\hat{\sigma}_{j}^{\tau_{n}},
\]
and as in the proof of Lemma B.1(b), let
\[
q = \max\left\{ \frac{2}{\beta_{n}}, \log(n), 3 \right\}.
\]
The idea of the proof is to construct a function \( b(\cdot) \) such that the following bound holds for every realization of \( X \),
\[
\left( \mathbb{E}[|V^{*}|^{q} | X] \right)^{1/q} \leq b(X),
\]
and then Chebyshev’s inequality implies
\[
\mathbb{P}\left( V^{*} \geq eb(X) \middle| X \right) \leq e^{-q} \leq \frac{1}{n}.
\]
In turn, we will derive a constant \( b_{n} \) such that the event \( \{ b(X) \leq b_{n} \} \) holds with high probability, which implies that the event
\[
\mathbb{P}\left( V^{*} \geq eb_{n} | X \right) \leq \frac{1}{n},
\]
holds with high probability, and this will give the statement of the lemma by setting \( t'_{1,n} = eb_{n} \).

To construct the function \( b(\cdot) \), observe that the initial portion of the proof of Lemma B.1(b) shows that for any realization of \( X \),
\[
(\text{C.15}) \quad \mathbb{E}[|V^{*}|^{q} | X] \leq \sum_{j \in \mathcal{J}(k_{n})^{c}} \hat{\sigma}_{j}^{q(1-\tau_{n})} \mathbb{E}[|\frac{1}{\hat{\sigma}_{j}} S_{n,j}^{*}|^{q} | X].
\]
Next, Lemma D.4 ensures that for every $j \in \{1, \ldots, p\}$, the event

$$
\mathbb{E} \left[ \left| \frac{1}{\sigma_j} S_{n,j}^{*} \right| \right] \leq (c_{\alpha} q)^q,
$$

holds with probability 1. Consequently, if we let $s = q(1 - \tau_n)$ and consider the random variable

(C.16) 
$$
\hat{s} := \left( \sum_{j \in \mathcal{F}(k_n)} \tilde{\sigma}_j^s \right)^{1/2},
$$

as well as

$$
b(X) := c_{\alpha} \cdot q \cdot \hat{s}^{(1 - \tau_n)},
$$

then we obtain the bound

(C.17) 
$$
\left( \mathbb{E} \left[ |V^*|^q \right] \right)^{1/q} \leq b(X),
$$

with probability 1. To proceed, Lemma D.2 implies

(C.18) 
$$
\mathbb{P} \left( b(X) \geq q \cdot \frac{(c_{\alpha} \sqrt{q})^{1 - \tau_n}}{(q \beta_n - 1)^{1/q}} \cdot k_n^{-\beta_n + 1/q} \right) \leq e^{-q} \leq \frac{1}{n},
$$

for some constant $c_{\alpha} > 0$ depending on $\alpha$. By weakening this tail bound slightly, it can be simplified to

(C.19) 
$$
\mathbb{P} \left( b(X) \geq C_{\alpha}' \cdot q^{3/2} \cdot k_n^{-\beta_n} \right) \leq \frac{1}{n},
$$

where $C_{\alpha}' := \frac{c_{\alpha} k_n^{1/q}}{(q \beta_n - 1)^{1/q}}$, and we recall $\beta_n = \alpha(1 - \tau_n)$. To simplify further, it can be checked that $C_{\alpha}' \leq c_{\alpha}$ for some possibly larger constant $c_{\alpha}$. Finally, the assumption that $(1 - \tau_n)\sqrt{\log(n)} \geq 1$ gives $q \lesssim \frac{\log(n)}{\alpha}$, and it follows that there is a constant $\kappa_{\alpha}' > 0$ depending only on $\alpha$ such that

$$
b_n := \kappa_{\alpha}' \cdot \log(n)^{3/2} \cdot k_n^{-\beta_n},
$$

then

(C.20) 
$$
\mathbb{P} \left( b(X) \geq b_n \right) \leq \frac{1}{n},
$$

which completes the proof. \qed


C.2. Handling the term $\Pi'_n(X)$.

**Proposition C.1.** Suppose the conditions of Theorem 3.1 hold, with the same choice of $\delta \in (0, 1/2)$. Then, there is a constant $c_{\delta, \alpha} > 0$ depending only on $\delta$ and $\alpha$, as well as an absolute constant $c > 0$, such that the event

\begin{equation}
\Pi'_n(X) \leq c_{\delta, \alpha} n^{-\frac{1}{2} + \delta}
\end{equation}

holds with probability at least $1 - \frac{c}{n}$.

**Proof.** Define the random variable

\begin{equation}
\tilde{M}^*_k := \max_{j \in J(k_n)} S_{n,j}^* / \sigma_{\tau n},
\end{equation}

which differs from $M^*_k$, since $\sigma_{\tau n}$ is used in place of $\hat{\sigma}_{\tau n}$. Consider the triangle inequality

\begin{equation}
\Pi'_n(X) \leq d_K\left( \mathcal{L}(\tilde{M}_n), \mathcal{L}(\tilde{M}_n^{*} | X) \right) + d_K\left( \mathcal{L}(\tilde{M}_n^{*} | X), \mathcal{L}(M_k^{*} | X) \right).
\end{equation}

The two terms on the right will be bounded separately. With regard to the first term, note that $\tilde{M}_n$ and $\tilde{M}_n^{*}$ are the coordinate-wise maxima of Gaussian vectors drawn from $N(0, \mathcal{S}_k)$ and $N(0, \mathcal{S}_k)$ respectively, where

\begin{equation}
\mathcal{S}_k = D_{\kappa^{-\tau n}} \Pi_k, \Sigma^\top_k D_{\kappa^{-\tau n}},
\end{equation}

and

\begin{equation}
\mathcal{S}_k = D_{\kappa^{-\tau n}} \Pi_k \hat{\Sigma}_n \Pi_k^\top D_{\kappa^{-\tau n}},
\end{equation}

where the projection matrix $\Pi_k \in \mathbb{R}^{k_n \times p}$ is defined in the proof of Proposition B.1. Next, let $I_k$ be the $k_n \times k_n$ identity matrix. Lemma D.3 ensures that if the event

\begin{equation}
\left\| \mathcal{S}_k^{-1/2} \mathcal{S}_k \mathcal{S}_k^{-1/2} - I_k \right\|_{op} \leq \epsilon
\end{equation}

holds for some $\epsilon > 0$, then the event

\begin{equation}
d_K\left( \mathcal{L}(\tilde{M}_n), \mathcal{L}(\tilde{M}_n^{*} | X) \right) \leq c \cdot \sqrt{k_n} \cdot \epsilon
\end{equation}

also holds for some absolute constant $c > 0$. Furthermore, Lemma D.8 shows that if $\epsilon = c \cdot n^{-1/2} \cdot k_n^{3/2} \cdot \log(n)^4$, then the event (C.26) holds with probability at least $1 - \frac{c}{n}$. So, given that

\[ n^{-\frac{1}{2}} k_n^{3/2} \log(n)^4 \leq c_{\delta, \alpha} n^{-\frac{1}{2} + \delta} \]
for some constant $c_δ$, the first term in the bound (C.23) requires no further consideration.

To deal with the second term in (C.23), we proceed by considering the general inequality
\begin{equation}
\tag{C.28}
\mathbb{P}_K(\mathcal{L}(\xi), \mathcal{L}(\zeta)) \leq \sup_{t \in \mathbb{R}} \mathbb{P}(|\zeta - t| \leq r) + \mathbb{P}(|\xi - \zeta| > r),
\end{equation}

which holds for any random variables $\xi$ and $\zeta$, and any real number $r > 0$ (cf. Chernozhukov, Chetverikov and Kato (2016, Lemma 2.1)). Specifically, we will let $\mathcal{L}(\hat{M}^*_k | X)$ play the role of $\mathcal{L}(\xi)$, and let $\mathcal{L}(M^*_k | X)$ play the role of $\mathcal{L}(\zeta)$. In other words, we need to establish an anti-concentration inequality for $\hat{M}^*_k$, as well as a coupling inequality for $M^*_k$ and $\hat{M}^*_k$, conditionally on $X$.

To establish the coupling inequality, if we put
\begin{equation}
\tag{C.29}
r_n = c_\alpha \cdot n^{-1/2} \cdot \log(n)^{9/2},
\end{equation}

for a suitable constant $c_\alpha$ depending only on $\alpha$, then Lemma D.9 shows that the event
\begin{equation}
\tag{C.30}
\mathbb{P}\left(\left|\hat{M}^*_k - M^*_k\right| > r_n \left| X\right.\right) \leq \frac{\xi}{n}
\end{equation}

holds with probability at least $1 - \frac{\xi}{n}$.

Lastly, the anti-concentration inequality can be derived from Nazarov’s inequality (Lemma E.2), since $M^*_k$ is obtained from a Gaussian vector, conditionally on $X$. Indeed, Nazarov’s inequality implies that the event
\begin{equation}
\tag{C.31}
\sup_{t \in \mathbb{R}} \mathbb{P}\left(\left|M^*_k - t\right| \leq r_n \left| X\right.\right) \leq c \cdot \frac{r_n}{\hat{\sigma}_k} \cdot \sqrt{\log(k_n)}
\end{equation}

holds with probability 1, where we put $\hat{\sigma}_k := \min_{j \in J(k_n)} \hat{\sigma}_j$. Meanwhile, Lemma D.6 implies that the event
\begin{equation}
\tag{C.32}
\frac{1}{\hat{\sigma}_k^{1-r_n}} \leq c \frac{k^\beta n}{\hat{\sigma}_k} \frac{1}{n}
\end{equation}

holds with probability at least $1 - \frac{\xi}{n}$. Combining the last few steps, we conclude that the following bound holds with probability at least $1 - \frac{\xi}{n}$,
\begin{equation}
\tag{C.33}
\sup_{t \in \mathbb{R}} \mathbb{P}\left(\left|M^*_k - t\right| \leq r_n \left| X\right.\right) \leq c_\alpha \cdot n^{-1/2} \cdot k^\beta n \cdot \log(n)^{7/2} \cdot \sqrt{\log(k_n)}
\leq c_\delta, \alpha n^{-\frac{1}{2} + \delta},
\end{equation}

as needed. □
LEMMA D.1. Suppose the conditions of Theorem 3.1 hold, with the same choice of $\delta \in (0, 1/2)$, and let $q = \max\{\frac{2}{3\alpha}, \log(n), 3\}$. Then, there is a constant $c_\alpha > 0$ depending only on $\alpha$ such that for any fixed $j \in \{1, \ldots, p\}$, we have

\begin{equation}
\|\hat{\sigma}_j\|_q \leq c_\alpha \cdot \sigma_j \cdot \sqrt{q}.
\end{equation}

PROOF. Define the vector $u := \frac{1}{\sigma_j} \Sigma^{1/2} e_j \in \mathbb{R}^p$, which satisfies $\|u\|_2 = 1$. Observe that

\begin{equation}
\frac{1}{\sigma_j} \|\hat{\sigma}_j\|_q = \left\| \left( \frac{1}{n} \sum_{i=1}^{n} (u^T Z_i)^2 - (u^T \bar{Z})^2 \right)^{1/2} \right\|_q
\end{equation}

\begin{equation}
\leq \left\| \left( \frac{1}{n} \sum_{i=1}^{n} (u^T Z_i)^2 \right)^{1/2} \right\|_q
\end{equation}

\begin{equation}
= \left\| \frac{1}{n} \sum_{i=1}^{n} (u^T Z_i)^2 \right\|^{1/2}_{q/2}.
\end{equation}

Since the random variables $(u^T Z_1)^2, \ldots, (u^T Z_n)^2$ are i.i.d. and non-negative, part (i) of Rosenthal’s inequality in Lemma E.4 implies the $L^{q/2}$ norm in the last line satisfies

\begin{equation}
\left\| \frac{1}{n} \sum_{i=1}^{n} (u^T Z_i)^2 \right\|_{q/2} \leq c \cdot q \cdot \max \left\{ \|u^T Z_1\|_2^2, n^{-1+2/q} \|u^T Z_1\|_q^2 \right\},
\end{equation}

for an absolute constant $c > 0$. For the first term inside the maximum, observe that since $\|u\|_2 = 1$ and the entries of $Z_1$ are i.i.d. with mean 0 and variance 1, we have $\|u^T Z_1\|_2^2 = 1$. To handle the second term inside the maximum, we may view $u^T Z_1 = \sum_{l=1}^{p} u_l Z_{1,l}$ as a sum of independent random variables with mean 0 and $\|u_l Z_{1,l}\|_q \leq c \cdot |u_l| \cdot q$ for all $1 \leq l \leq p$. Hence, part (ii) of Lemma E.4 implies

\begin{equation}
\|u^T Z_1\|_q \lesssim q \cdot \max \left\{ 1, q \|u\|_q \right\} \leq q^2,
\end{equation}

where we have used $\|u\|_q \leq \|u\|_2 = 1$. Combining the last few steps, and noticing the square root on the $L^{q/2}$ norm in the last line of (D.2), we obtain

\begin{equation}
\frac{1}{\sigma_j} \|\hat{\sigma}_j\|_q \lesssim \sqrt{q} \cdot \max \left\{ 1, n^{-1/2+1/q} q^2 \right\},
\end{equation}

and this implies the statement of the lemma, since the quantity $n^{-1/2+1/q} q^2$ is bounded by a constant depending only on $\alpha$. $\square$
Lemma D.2. Let \( q = \max\{ \frac{2}{n}, \log(n), 3 \} \), and \( s = q(1 - \tau_n) \). Suppose the conditions of Theorem 3.1 hold with the same choice of \( \delta \in (0, 1/2) \). Consider the random variables \( \hat{s} \) and \( \hat{t} \) defined by

\[
\hat{s} = \left( \sum_{j \in \mathcal{J}(k_n)^c} \hat{\sigma}_j^s \right)^{1/s} \quad \text{and} \quad \hat{t} = \left( \sum_{j \in \mathcal{J}(k_n)} \hat{\sigma}_j^s \right)^{1/s}.
\]

Then, there is a constant \( c_\alpha > 0 \) depending only on \( \alpha \) such that

\[
\mathbb{P}\left( \hat{s} \geq \frac{c_\alpha \sqrt{q}}{(s\alpha - 1)^{1/s}} \cdot k_n^{-\alpha + 1/s} \right) \leq e^{-q},
\]

and

\[
\mathbb{P}\left( \hat{t} \geq \frac{c_\alpha \sqrt{q}}{(s\alpha - 1)^{1/s}} \right) \leq e^{-q}.
\]

Proof. In light of the Chebyshev inequality \( \mathbb{P}(\hat{s} \geq e\|\hat{s}\|_q) \leq e^{-q} \), it suffices to bound \( \|\hat{s}\|_q \) (and similarly for \( \hat{t} \)). We proceed by direct calculation,

\[
\|\hat{s}\|_q = \left\| \sum_{j \in \mathcal{J}(k_n)^c} \hat{\sigma}_j^s \right\|_q^{1/s} \leq \left( \sum_{j \in \mathcal{J}(k_n)^c} \|\hat{\sigma}_j^s\|_q \right)^{1/s} \quad \text{(triangle inequality for } \|\cdot\|_{q/s}, \text{ with } q/s \geq 1) \]

\[
= \left( \sum_{j \in \mathcal{J}(k_n)} \|\hat{\sigma}_j^s\|_q \right)^{1/s} \leq c_\alpha \cdot \sqrt{q} \cdot \left( \sum_{j \in \mathcal{J}(k_n)^c} \sigma_j^s \right)^{1/s} \quad \text{(Lemma D.1)}
\]

\[
\leq c_\alpha \cdot \sqrt{q} \cdot \left( \int_{k_n}^{p} x^{-s\alpha} \, dx \right)^{1/s} \leq c_\alpha \cdot \sqrt{q} \cdot \frac{k_n^{-\alpha + 1/s}}{(s\alpha - 1)^{1/s}},
\]

and in the last step we have used the fact that \( s\alpha > 1 \), which holds since \( q \) is defined to satisfy \( q\beta_n > 1 \). The calculation for \( \hat{t} \) is essentially the same, except that it involves the integral \( \int_{1}^{kn+1} x^{-s\alpha} \, dx \). \( \square \)
Lemma D.3. Let $A$ and $B$ be positive definite matrices in $\mathbb{R}^{d \times d}$, and let $U \sim N(0,A)$ and $V \sim N(0,B)$. Also, suppose there is a constant $\epsilon > 0$ such that $\|A^{-1/2}BA^{-1/2} - I_d\|_{op} \leq \epsilon$. Then, there is an absolute constant $c > 0$ such that

\[(D.8) \sup_{t \in \mathbb{R}} \left| \mathbb{P}\left( \max_{1 \leq j \leq d} U_j \leq t \right) - \mathbb{P}\left( \max_{1 \leq j \leq d} V_j \leq t \right) \right| \leq c\sqrt{d} \epsilon.\]

Proof. We may assume that $\sqrt{d} \epsilon \leq 1/2$, for otherwise the claim trivially holds with $c = 2$. Observe that the event $\{\max_{1 \leq j \leq d} U_j \leq t\}$ is equivalent to the vector $U$ lying in a certain Borel set, and so the left hand side of (D.8) is upper-bounded by the total variation distance between $L(U)$ and $L(V)$, which in turn, is upper-bounded by the Hellinger distance $d_H(L(U), L(V))$ (Gibbs and Su, 2002). Since $U$ and $V$ are centered Gaussian vectors, the following exact formula for the squared Hellinger distance is available (Pardo, 2005, p.51),

\[(D.9) \frac{1}{2} d_H(L(U), L(V))^2 = 1 - \frac{\det(A)^{1/4} \det(B)^{1/4}}{\det\left(\frac{1}{2}(A + B)\right)^{1/2}}.\]

Considering the basic identity,

\[\frac{1}{2}(A + B) = \frac{1}{2} A^{1/2}(I_d + A^{-1/2}BA^{-1/2})A^{1/2},\]

the squared Hellinger distance may be written as

\[(D.10) \frac{1}{2} d_H(L(U), L(V))^2 = 1 - \frac{\det(A^{-1/2}BA^{-1/2})^{1/4}}{\det(\frac{1}{2}(I_d + A^{-1/2}BA^{-1/2}))^{1/2}}.\]

Now let $C = A^{-1/2}BA^{-1/2}$, and for any $t \in [0,1]$, consider the function

\[g(t) = -\exp\left( t \log\left( \frac{\det(C)^{1/4}}{\det(\frac{1}{2}(I_d + C))^{1/2}} \right) \right),\]

so that

\[(D.11) \frac{1}{2} d_H(L(U), L(V))^2 = \int_0^1 g'(t) dt.\]

It suffices to derive an upper bound on $|g'(t)|$. To proceed, put $\eta_j := \lambda_j(C) - 1$, and note that $\max_j |\eta_j| \leq \epsilon \leq 1/2$ by assumption. In turn, by using the basic inequality

\[|\log(x+1) - x| \leq \frac{x^2}{1+x},\]
for any $x \in (-1, \infty)$, we obtain
\[
\left| \log \left( \frac{\det(C)^{1/4}}{\det(\frac{1}{2}(I_d + C))^{1/2}} \right) \right| \leq \sum_{j=1}^{d} \frac{1}{4} \left| \log \left( \frac{\lambda_j(C)}{\{\frac{1}{4}(1+\lambda_j(C))\}^2} \right) \right|
\]
(D.12)
\[
= \sum_{j=1}^{d} \frac{1}{4} \left| \log \left( \frac{-(1/4)n_j^2}{(1+(1/2)n_j)^2} + 1 \right) \right|
\]
\[
\leq c_0 d \epsilon^2,
\]
where $c_0 > 0$ is an absolute constant. It follows that for all $t \in [0, 1]$,
\[
|g'(t)| \leq e^{c_0 d \epsilon^2} c_0 d \epsilon^2,
\]
and so using the equation (D.11), in conjunction with $\sqrt{d} \epsilon \leq 1/2$, we have
(D.13)
\[
\frac{1}{2} d_H(\mathcal{L}(U), \mathcal{L}(V))^2 \leq e^{c_0/4} c_0 d \epsilon^2,
\]
which implies the stated bound. \hfill \square

**Lemma D.4.** Suppose the conditions of Theorem 3.1 hold, with the same choice of $\delta \in (0, 1/2)$, and let $q = \max\{\frac{2}{\delta_0}, \log(n), 3\}$. Then, there is a constant $c_\alpha > 0$ depending only on $\alpha$ such that for any $j \in \{1, \ldots, p\}$,
(D.14)
\[
\|\frac{1}{\sigma_j} S_{n,j}\|_q \leq c_\alpha q.
\]
In addition, the following event holds with probability 1,
(D.15)
\[
\left( \mathbb{E} \left[ \|\frac{1}{\sigma_j} S_{n,j}^*\|^q | X \right] \right)^{1/q} \leq c_\alpha q.
\]

**Proof.** We only prove the first bound, since the second one can be obtained by repeating the same argument, conditionally on $X$. Since $q > 2$, Lemma E.4 gives
(D.16)
\[
\|\frac{1}{\sigma_j} S_{n,j}\|_q \leq q \cdot \max\left\{ \|\frac{1}{\sigma_j} S_{n,j}\|_2, n^{-1/2+1/q}\|\frac{1}{\sigma_j} X_{1,j}\|_q \right\}.
\]
Clearly,
(D.17)
\[
\|\frac{1}{\sigma_j} S_{n,j}\|_2^2 = \text{var}(\frac{1}{\sigma_j} S_{n,j}) = 1.
\]
Furthermore, if we define the vector $u := \frac{1}{\sigma_j} \sum^{1/2} e_j$ in $\mathbb{R}^p$, which satisfies $\|u\|_2 = 1$, then

$$\|\frac{1}{\sigma_j} X_{1,j}\|_q = \|u^\top Z_1\|_q$$

(D.18)

$$= \left\| \sum_{l=1}^{p} u_l Z_{1,l} \right\|_q$$

$$\leq c \cdot q^2,$$

where the last step follows easily from Lemma E.4, and the fact that Assumption 2.1 ensures $\|Z_{1,1}\|_q \lesssim q$. Applying the work above to the bound (D.16) gives

(D.19)

$$\|\frac{1}{\sigma_j} S_{n,j}\|_q \lesssim q \cdot \max \left\{ 1, n^{-1/2+1/q} q^2 \right\}.$$

Finally, since $q \leq c_\alpha \log(n)$ for a constant $c_\alpha > 0$ depending only on $\alpha$, the quantity $n^{-1/2+1/q} q^2$ is bounded by a constant depending only on $\alpha$. 

**Lemma D.5.** Let $Z_1, \ldots, Z_n \in \mathbb{R}^p$ be as in Assumption 2.1, and let $Q_{kn} \in \mathbb{R}^{p \times k_n}$ be a fixed matrix with orthonormal columns. Also let $I_{kn}$ denote the identity matrix of size $k_n \times k_n$, and let

(D.20)

$$W_n = \frac{1}{n} \sum_{i=1}^{n} (Z_i - \bar{Z})(Z_i - \bar{Z})^\top,$$

where $\bar{Z} = \frac{1}{n} \sum_{i=1}^{n} Z_i$. Then, there is an absolute constant $c > 0$ such that the event

(D.21)

$$\|Q_{kn}^\top W_n Q_{kn} - I_{kn}\|_{\op} \leq c \cdot n^{-1/2} \cdot k_n^{3/2} \cdot \log(n)^4$$

holds with probability at least $1 - \frac{c}{n}$.

**Proof.** Let $\epsilon \in (0,1/2)$, and let $\mathcal{N}$ be an $\epsilon$-net (with respect to the $\ell_2$-norm) for the unit $\ell_2$-sphere in $\mathbb{R}^{k_n}$. It is well known that $\mathcal{N}$ can be chosen so that $\text{card}(\mathcal{N}) \leq (3/\epsilon)^{k_n}$, and the inequality

$$\|Q_{kn}^\top W_n Q_{kn} - I_{kn}\|_{\op} \leq \frac{1}{1-2\epsilon} \cdot \max_{u \in \mathcal{N}} \left| u^\top \left( Q_{kn}^\top W_n Q_{kn} - I_{kn} \right) u \right|,$$

holds with probability 1 (Vershynin, 2012, Lemmas 5.2 and 5.4). Let $\xi \in \mathbb{R}^{pn \times 1}$ be a column vector obtained by concatenating $Z_1, \ldots, Z_n$, and for a
fixed vector $u \in \mathcal{N}$, let $A(u)$ be a $pn \times pn$ block-diagonal matrix with $n$ copies of the $p \times p$ matrix $B(u) := \frac{1}{n}Q_{kn} uu^T Q_{kn}^T$ along the diagonal. Noting that $\text{tr}(A(u)) = 1$, a bit of algebra gives

$$u^T (Q_{kn}^T W_n Q_{kn} - I_{kn}) u = \left( \xi^T A(u) \xi - \text{tr}(A(u)) \right) - n \bar{Z}^T B(u) \bar{Z}. \tag{D.22}$$

The remainder of the proof involves showing that both terms on the right side are close to 0 with high probability, and then taking a union bound over $u \in \mathcal{N}$.

First, we deal with the term $\xi^T A(u) \xi - \text{tr}(A(u))$. Note that $\|A(u)\|_F = \|A(u)\|_{op} = 1/n$. If we put $\epsilon_n := C \cdot n^{-1/2} \cdot k_n^{3/2} \cdot \log(n)^2$ for a suitable absolute constant $C > 0$, it follows from Lemma E.5 that there are absolute constants $c_0, c_1 > 0$ such that the quadratic form $\xi^T A(u) \xi$ satisfies the following concentration inequality

$$P\left( \left| \xi^T A(u) \xi - \text{tr}(A(u)) \right| \geq \epsilon_n \right) \leq c_0 \exp\left\{ - c_1 \cdot C \cdot k_n \cdot \log(n) \right\}. \tag{D.23}$$

Second, we deal with the term $n \bar{Z}^T B(u) \bar{Z}$. Define the vector $\zeta = \sqrt{n} \bar{Z}$, so that

$$n \bar{Z}^T B(u) \bar{Z} = \zeta^T B(u) \zeta. \tag{D.24}$$

Note that $\zeta$ has i.i.d. entries with mean 0, variance 1, and

$$\|\zeta_1\|_{\psi_1} \leq c$$

for an absolute constant $c \in (0, \infty)$, which follows from a standard facts about sub-exponential variables (Vershynin, 2018, Proposition 2.7.1, Theorem 2.8.1). Since $\text{tr}(B(u)) = \|B(u)\|_F = \|B(u)\|_{op} = 1/n$, we can apply Lemma E.5 to obtain

$$P\left( \left| \zeta^T B(u) \zeta \right| \geq \frac{1}{n} + \epsilon_n \right) \leq c_0 \exp\left\{ - c_1 \cdot C \cdot k_n \cdot \log(n) \right\}. \tag{D.25}$$

(This bound could be improved, but it is not necessary to use a tighter bound, and so we leave it in a form that matches (D.23) for simplicity.)

If $\mathcal{E}_n$ denotes the event (D.21), then a union bound over $u \in \mathcal{N}$ gives

$$P(\mathcal{E}_n) \geq 1 - 2c_0 \exp\left\{ - c_1 \cdot C \cdot k_n \cdot \log(n) - k_n \cdot \log(3/\epsilon_n) \right\}.$$

This implies the stated result as long as $C$ is chosen sufficiently large. \qed
Lemma D.6. Suppose the conditions of Theorem 3.1 hold, with the same choice of $\delta \in (0, 1/2)$. Fix any two distinct indices $j, j' \in \{1, \ldots, p\}$. Then, there is an absolute constant $c > 0$, such that the event

\[(D.25) \quad |\hat{\rho}_{j,j'} - \rho_{j,j'}| \leq c \frac{\log(n)^3}{\sqrt{n}} \]

holds with probability at least $1 - \frac{c}{n}$. Furthermore, the events

\[(D.26) \quad |\hat{\rho}_{\max}(\ell_n) - \rho_{\max}(\ell_n)| \leq c \frac{\log(n)^3}{\sqrt{n}} \]

and

\[(D.27) \quad \min_{j \in J(k_n)} \hat{\sigma}_{1 - \tau_n} \geq \left( \min_{j \in J(k_n)} \sigma_{1 - \tau_n} \right) \left(1 - c \frac{\log(n)^3}{\sqrt{n}}\right)_+ \]

each hold with probability at least $1 - \frac{c}{n}$.

Proof. The result is essentially a direct consequence of Lemma D.7 below. Note that choosing $j = j'$ in (D.28) leads to a concentration inequality for $|\hat{\sigma}_j/\sigma_j - 1|$. Likewise, this can be combined with the bound for $|\hat{\Sigma}_{j,j'}/(\sigma_j \sigma_{j'}) - \rho_{j,j'}|$ in order to control $|\hat{\rho}_{j,j'} - \rho_{j,j'}|$. These details are elementary, but tedious, and so are omitted.

Lemma D.7. Suppose that Assumption 2.1 holds, and fix any two (possibly equal) indices $j, j' \in \{1, \ldots, p\}$. Then, there is an absolute constant $c > 0$, such that the event

\[(D.28) \quad \left|\frac{\hat{\Sigma}_{j,j'}}{\sigma_j \sigma_{j'}} - \rho_{j,j'}\right| \leq c \cdot n^{-1/2} \cdot \log(n)^3 \]

holds with probability at least $1 - \frac{c}{n^2}$.

Remark. The event in the lemma has been formulated to hold with probability at least $1 - \frac{c}{n^2}$, rather than $1 - \frac{c}{n}$, in order to accommodate a simple union bound for proving inequality (D.26) in Lemma D.6.

Proof. Consider the $\ell_2$-unit vectors $u = \sum^{1/2} e_j/\sigma_j$ and $v = \sum^{1/2} e_{j'}/\sigma_{j'}$ in $\mathbb{R}^p$. Letting $W_n$ be as defined in (D.20), observe that

\[(D.29) \quad \frac{\hat{\Sigma}_{j,j'}}{\sigma_j \sigma_{j'}} - \rho_{j,j'} = u^\top (W_n - I_p)v.\]
Next, define the block-diagonal matrix $A(u, v) \in \mathbb{R}^{pn \times pn}$ with $n$ copies of the $p \times p$ matrix $B(u, v) := \frac{1}{2n} (uv^\top + vu^\top)$ along the diagonal, which satisfies
\begin{align}
\text{tr}(A(u, v)) &= \rho_{j,j'} \\
\|A(u, v)\|_F &\leq 1/\sqrt{n} \\
\|A(u, v)\|_{op} &\leq 1/n.
\end{align}

(D.30)

If we let let $\xi \in \mathbb{R}^{np \times 1}$ be the vector obtained by concatenating $Z_1, \ldots, Z_n$, then some algebra gives
\begin{equation}
(D.31) \quad u^\top (W_n - I_p) v = \left( \xi^\top A(u, v) \xi - \text{tr}(A(u, v)) \right) - n \bar{Z}^\top B(u, v) \bar{Z}.
\end{equation}

If we put
\begin{equation}
\epsilon_n := \frac{c\log(n)^3}{\sqrt{n}}
\end{equation}

for a sufficiently large absolute constant $c > 0$, then Lemma E.5 implies there are absolute constants $c_0, c_1 > 0$ such that
\begin{equation}
(D.32) \quad \mathbb{P}\left( \left| \xi^\top A(u, v) \xi - \text{tr}(A(u, v)) \right| \geq \epsilon_n \right) \leq c_0 \exp\left\{ - \left( \frac{c_1\epsilon_n^2}{\log(n)} \right)^{1/3} \right\}.
\end{equation}

As in the proof of Lemma D.5, let $\zeta = \sqrt{n}\bar{Z}$, which has i.i.d entries with mean 0, variance 1, and $\|\zeta_1\|_{\psi_1} \leq c$, for some absolute constant $c > 0$. Also note that
\begin{align}
n \bar{Z}^\top B(u, v) \bar{Z} &= \zeta^\top B(u, v) \zeta,
\end{align}

and that $|\text{tr}(B(u, v))|$, $\|B(u, v)\|_F$, and $\|B(u, v)\|_{op}$ are all at most $1/n$. Consequently, Lemma E.5 gives
\begin{equation}
(D.33) \quad \mathbb{P}\left( \left| \zeta^\top B(u, v) \zeta \right| \geq \frac{1}{n} + \epsilon_n \right) \leq c_0 \exp\left\{ - c_1 (\epsilon_n n)^{1/3} \right\}.
\end{equation}

Combining (D.32) and (D.33) with a union bound leads to the stated result.

Remark. Recall the following definitions from equations (C.24) and (C.25),
\begin{align}
\mathcal{S}_k = D_k^{-\tau_n} \Pi_k \Sigma_k \Pi_k^\top D_k^{-\tau_n},
\end{align}

and
\begin{align}
\mathcal{S}_k = D_k^{-\tau_n} \Pi_k \hat{\Sigma}_n \Pi_k^\top D_k^{-\tau_n}.
\end{align}
Lemma D.8. Suppose the conditions of Theorem 3.1 hold, with the same choice of $\delta \in (0, 1/2)$. Then, there is an absolute constant $c > 0$ such that the event

\[(D.34) \quad \left\| \tilde{\Sigma}^{-1/2} \tilde{\Sigma} \tilde{\Sigma}^{-1/2} - I_{kn} \right\|_{op} \leq c \cdot n^{-1/2} \cdot k_{n}^{3/2} \cdot \log(n)^{4} \]

holds with probability at least $1 - \frac{c}{n}$.

Proof. Let $U_{kn} \Lambda_{kn} V_{kn}^{T}$ be a thin s.v.d. for the matrix $\Sigma_{kn}^{1/2} \Pi_{kn}^{T} D_{kn}^{-\tau_{n}}$. This means that the matrix $U_{kn} \in \mathbb{R}^{p \times kn}$ has orthonormal columns, the matrix $\Lambda_{kn} \in \mathbb{R}^{kn \times kn}$ is diagonal and positive definite, and the matrix $V_{kn} \in \mathbb{R}^{kn \times kn}$ is orthogonal. Furthermore, if we let $W_{n}$ be as defined in (D.20) then it follows that

\[(D.35) \quad \tilde{\Sigma}^{-1/2} \tilde{\Sigma} \tilde{\Sigma}^{-1/2} = V_{kn} U_{kn}^{T} W_{n} U_{kn} V_{kn}^{T}. \]

Since the matrix $U_{kn} V_{kn}^{T}$ has orthonormal columns, the proof is completed by applying Lemma D.5.

Lemma D.9. Suppose the conditions of Theorem 3.1 hold, with the same choice of $\delta \in (0, 1/2)$. Then, there is an absolute constant $c > 0$ such that the event (C.30) holds with probability at least $1 - \frac{c}{n}$.

Proof. Let $(a_{1}, \ldots, a_{kn})$ and $(b_{1}, \ldots, b_{kn})$ be real vectors, and note the basic fact

\[ \left| \max_{1 \leq j \leq kn} a_{j} - \max_{1 \leq j \leq kn} b_{j} \right| \leq \max_{1 \leq j \leq kn} |a_{j} - b_{j}|. \]

From this, it is simple to derive the inequality

\[(D.36) \quad \left| \hat{M}_{kn}^{*} - M_{kn}^{*} \right| \leq \max_{j \in J(kn)} \left| \left( \frac{\hat{\sigma}_{j}}{\sigma_{j}} \right)^{\tau_{n}} - 1 \right| \cdot \max_{j \in J(kn)} \left| S_{n,j}^{*} / \hat{\sigma}_{j}^{\tau_{n}} \right|. \]

To handle the first factor on the right side, it follows from Lemma D.7 that the event

\[(D.37) \quad \max_{j \in J(kn)} \left| \left( \frac{\hat{\sigma}_{j}}{\sigma_{j}} \right)^{\tau_{n}} - 1 \right| \leq c \cdot n^{-1/2} \cdot \log(n)^{3} \]

holds with probability at least $1 - \frac{c}{n}$ for some absolute constant $c > 0$. To proceed, consider the random variable

\[(D.38) \quad U^{*} := \max_{j \in J(kn)} |S_{n,j}^{*} / \hat{\sigma}_{j}^{\tau_{n}}|. \]
It suffices to show there is an absolute constant $c > 0$, and a constant $c_\alpha > 0$ depending only on $\alpha$, such that the event
\begin{equation}
\mathbb{P}(U^* \geq c_\alpha \log(n)^{3/2} \mid |X|) \leq \frac{1}{n}
\end{equation}
holds with probability at least $1 - \frac{\varepsilon}{n}$. Using Chebyshev’s inequality with $q = \max\{\frac{2}{3n}, \log(n), 3\}$ gives
\begin{equation}
\mathbb{P}(U^* \geq e (\mathbb{E}[|U^*|^q \mid X])^{1/q} \mid |X|) \leq e^{-q}.
\end{equation}
Likewise, if the event
\begin{equation}
(\mathbb{E}[|U^*|^q \mid X])^{1/q} \leq c_\alpha \log(n)^{3/2}
\end{equation}
holds for some constant $c_\alpha > 0$, then the event (D.39) also holds. For this purpose, the argument in the proof of Lemma C.1(b) can be essentially repeated to show that the event (D.40) holds with probability at least $1 - \frac{\varepsilon}{n}$. The main detail to notice when repeating the argument is that $U^*$ involves a maximum over $J(k_n)$, whereas the argument for Lemma C.1(b) involves a maximum over $J(k_n)^c$. This distinction can be easily handled by using the bound (D.7) in Lemma D.2.

**APPENDIX E: BACKGROUND RESULTS**

The following result is a multivariate version of the Berry-Esseen theorem due to Bentkus (2003).

**Lemma E.1** (Bentkus’ multivariate Berry-Esseen theorem). Let $V_1, \ldots, V_n$ be i.i.d. random vectors $\mathbb{R}^d$, with zero mean, and identity covariance matrix. Furthermore, let $\gamma_d$ denote the standard Gaussian distribution on $\mathbb{R}^d$, and let $\mathcal{A}$ denote the collection of all Borel-measurable convex subsets of $\mathbb{R}^d$. Then, there is an absolute constant $c > 0$ such that
\begin{equation}
\sup_{\mathcal{A} \in \mathcal{A}} \left| \mathbb{P}\left( \frac{1}{\sqrt{n}} (V_1 + \cdots + V_n) \in \mathcal{A} \right) - \gamma_d(\mathcal{A}) \right| \leq \frac{c \cdot d^{1/4} \cdot \mathbb{E}[\|V_1\|^2]}{n^{1/2}}.
\end{equation}

The following is a version of Nazarov’s inequality (Nazarov, 2003; Klivans, O’Donnell and Servedio, 2008), as formulated in (Chernozhukov, Chetverikov and Kato, 2016, Lemma 4.3).

**Lemma E.2** (Nazarov’s inequality). Let $(\xi_1, \ldots, \xi_m)$ be a multivariate normal random vector, and define $\sigma^2 = \min_{1 \leq j \leq m} \text{var}(\xi_j)$. Then, for any $r > 0$,
\begin{equation}
\sup_{t \in \mathbb{R}} \mathbb{P}\left( \max_{1 \leq j \leq m} |\xi_j - t| \leq r \right) \leq \frac{2r}{\sigma} \cdot (\sqrt{2 \log(m)} + 2).
\end{equation}
The following version of Slepian’s lemma follows from (van der Vaart and Wellner, 1996, Proposition A.2.6).

**Lemma E.3 (Slepian’s lemma).** Let \((\xi_1, \ldots, \xi_m)\) and \((\zeta_1, \ldots, \zeta_m)\) be zero-mean multivariate normal random vectors, and suppose that the following inequality holds for all \(j, j' \in \{1, \ldots, m\}\),

\[
\mathbb{E}[(\xi_j - \xi_{j'})^2] \leq \mathbb{E}[(\zeta_j - \zeta_{j'})^2].
\]

Then, for any \(t \geq 0\),

\[
\mathbb{P}\left( \max_{1 \leq j \leq m} \zeta_j \leq t \right) \leq \mathbb{P}\left( \max_{1 \leq j \leq m} \xi_j \leq t \right).
\]

The following inequalities are due to Johnson, Schechtman and Zinn (1985).

**Lemma E.4 (Rosenthal’s inequality with best constants).** Fix \(r \geq 1\) and put \(\text{Log}(r) := \max\{\log(r), 1\}\). Let \(\xi_1, \ldots, \xi_m\) be independent random variables satisfying \(\mathbb{E}[|\xi_j|^r] < \infty\) for all \(1 \leq j \leq m\). Then, there is an absolute constant \(c > 0\) such that the following two statements are true.

(i). When the random variables \(\xi_1, \ldots, \xi_m\) all non-negative,

\[
(E.3) \quad \| \sum_{j=1}^m \xi_j \|_r \leq c \cdot \frac{r}{\text{Log}(r)} \cdot \max \left\{ \| \sum_{j=1}^m \xi_j \|_1, \left( \sum_{j=1}^m \| \xi_j \|_r \right)^{1/r} \right\}.
\]

(ii). When \(r > 2\) and the random variables \(\xi_1, \ldots, \xi_m\) all have mean 0,

\[
(E.4) \quad \| \sum_{j=1}^m \xi_j \|_r \leq c \cdot \frac{r}{\text{Log}(r)} \cdot \max \left\{ \| \sum_{j=1}^m \xi_j \|_2, \left( \sum_{j=1}^m \| \xi_j \|_r \right)^{1/r} \right\}.
\]

**Remark.** The non-negative case is handled in (Johnson, Schechtman and Zinn, 1985, Theorem 2.5). With regard to the mean 0 case, the statement above differs slightly from (Johnson, Schechtman and Zinn, 1985, Theorem 4.1), which requires symmetric random variables, but the remark on page 247 of that paper explains why the variables \(\xi_1, \ldots, \xi_m\) need not be symmetric as long as they have mean 0.

The following result is an extension of the Hanson-Wright inequality (Rudelson and Vershynin, 2013) for quadratic forms involving sub-exponential random variables, due to (Vu and Wang, 2015, Theorem 1.5).
Lemma E.5. Let $\xi = (\xi_1, \ldots, \xi_m)$ be a random vector whose entries are independent with mean 0, and variance 1. Suppose there is an absolute constant $c > 0$ such that $\max_{1 \leq j \leq m} \| \xi_j \|_{\psi_1} \leq c$. Also, let $A \in \mathbb{R}^{m \times m}$ be a non-random matrix, and let $\epsilon_m > 0$ satisfy
\[
\epsilon_m \geq C \cdot (\| A_m \|_F + \log(m) \| A_m \|_{\text{op}}) \cdot \log(m)^3,
\]
for an absolute constant $C > 0$. Then, there are absolute constants $c_0, c_1 > 0$ such that
\[
\mathbb{P} \left( \| \xi^\top A \xi - \text{tr}(A) \| \geq \epsilon_m \right) \leq c_0 \exp \left(-c_1 \min \left\{ \left( \frac{\epsilon_m^2}{\| A \|_F^2 \log(m)} \right)^{1/3}, \left( \frac{\epsilon_m}{\| A \|_{\text{op}}} \right)^{1/3} \right\} \right).
\]

The following result on the moments of quadratic forms may be found in (Bai and Silverstein, 2010, Lemma B.26).

Lemma E.6. Let $A \in \mathbb{R}^{m \times m}$ be a non-random matrix, and let $\xi \in \mathbb{R}^m$ be a random vector having independent entries with mean 0 and variance 1. Also, let $r \in [1, \infty)$, and suppose that for each $s \in [1, 2r]$, there is a constant $c_s \in (0, \infty)$ such that $\max_{1 \leq j \leq m} \mathbb{E}[|\xi_j|^s] \leq c_s$. Then,
\[
\mathbb{E} \left[ \| \xi^\top A \xi - \text{tr}(A) \|^r \right] \leq C_r \left( c_4^{r/2} \| A \|_F^r + c_2r \text{ tr} \left( (A^\top A)^{r/2} \right) \right),
\]
where $C_r > 0$ is a number depending only on $r$.

REFERENCES

Arlot, S., Blanchard, G. and Roquain, E. (2010a). Some nonasymptotic results on resampling in high dimension, I: Confidence regions. The Annals of Statistics 38 51–82.
Arlot, S., Blanchard, G. and Roquain, E. (2010b). Some nonasymptotic results on resampling in high dimension, II: Multiple tests. The Annals of Statistics 38 83–99.
Bai, Z. and Silverstein, J. W. (2010). Spectral analysis of large dimensional random matrices 20. Springer.
Belloni, A., Chernozhukov, V., Chetverikov, D., Hansen, C. and Kato, K. (2018). High-dimensional econometrics and regularized GMM. arXiv:1806.01888.
Benko, M., Härdle, W. and Kneip, A. (2009). Common functional principal components. The Annals of Statistics 37 1–34.
Bentkus, V. (2003). On the dependence of the Berry–Esseen bound on dimension. Journal of Statistical Planning and Inference 113 385–402.
Cai, T. T. and Hall, P. (2006). Prediction in functional linear regression. The Annals of Statistics 34 2159–2179.
Cao, G., Yang, L. and Todem, D. (2012). Simultaneous inference for the mean function based on dense functional data. Journal of Nonparametric Statistics 24 359–377.
Chang, J., Yao, Q. and Zhou, W. (2017). Testing for high-dimensional white noise using maximum cross-correlations. Biometrika 104 111–127.
Chen, X. (2018). Gaussian and bootstrap approximations for high-dimensional U-statistics and their applications. The Annals of Statistics (to appear).
Chen, Y. C., Genovese, C. R. and Wasserman, L. (2015). Asymptotic theory for density ridges. *The Annals of Statistics* 43 1896–1928.

Chen, D. and Müller, H. G. (2012). Nonlinear manifold representations for functional data. *The Annals of Statistics* 40 1-29.

Chernozhukov, V., Chetverikov, D. and Kato, K. (2013). Gaussian approximations and multiplier bootstrap for maxima of sums of high-dimensional random vectors. *The Annals of Statistics* 41 2786–2819.

Chernozhukov, V., Chetverikov, D. and Kato, K. (2014). Anti-concentration and honest, adaptive confidence bands. *The Annals of Statistics* 42 1787–1818.

Chernozhukov, V., Chetverikov, D. and Kato, K. (2016). Empirical and multiplier bootstraps for suprema of empirical processes of increasing complexity, and related Gaussian couplings. *Stochastic Processes and their Applications* 126 3632–3651.

Chernozhukov, V., Chetverikov, D. and Kato, K. (2017). Central limit theorems and bootstrap in high dimensions. *The Annals of Probability* 45 2309–2352.

Choi, H. and Reimherr, M. (2016). R Package ‘fregion’. https://github.com/hpchoi/fregion.

Choi, H. and Reimherr, M. (2018). A geometric approach to confidence regions and bands for functional parameters. *Journal of Royal Statistical Society: Series B (Statistical Methodology)* 80 239–260.

Cressie, N. A. C. and Read, T. R. C. (1984). Multinomial goodness-of-fit tests. *Journal of the Royal Statistical Society: Series B* 46 440–464.

Degras, D. A. (2011). Simultaneous confidence bands for nonparametric regression with functional data. *Statistica Sinica* 11735–1765.

Deng, H. and Zhang, C. H. (2017). Beyond Gaussian Approximation: Bootstrap for Maxima of Sums of Independent Random Vectors. arXiv:1705.09528.

Dezeure, R., Bühlmann, P. and Zhang, C.-H. (2017). High-dimensional simultaneous inference with the bootstrap (with discussion). *Test* 26 685–719.

Fan, J., Shao, Q.-M. and Zhou, W.-X. (2018). Are discoveries spurious? Distributions of maximum spurious correlations and their applications. *The Annals of Statistics* 46 989–1017.

Ferraty, F. and Vieu, P. (2006). *Nonparametric Functional Data Analysis: Theory and Practice*. Springer.

Gibbs, A. L. and Su, F. E. (2002). On choosing and bounding probability metrics. *International statistical review* 70 419–435.

Horváth, L. and Kokoszka, P. (2012). *Inference for Functional Data with Applications*. Springer.

Horváth, L., Kokoszka, P. and Reeder, R. (2013). Estimation of the mean of functional time series and a two-sample problem. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)* 75 103–122.

Hsing, T. and Eubank, R. (2015). *Theoretical Foundations of Functional Data Analysis, with an Introduction to Linear Operators*. Wiley.

Johnson, W. B., Schecthman, G. and Zinn, J. (1985). Best constants in moment inequalities for linear combinations of independent and exchangeable random variables. *The Annals of Probability* 234–253.

Johnstone, I. M. (2017). Gaussian estimation: Sequence and wavelet models. http://statweb.stanford.edu/ imj/GE_08 ninety_17.pdf.

Johnstone, I. M. and Lu, A. Y. (2009). On consistency and sparsity for principal components analysis in high dimensions. *Journal of the American Statistical Association* 104 682–693.

Klivans, A. R., O’Donnell, R. and Servedio, R. A. (2008). Learning geometric con-
cepts via Gaussian surface area. In Foundations of Computer Science, 2008. FOCS’08. 541–550.

Marshall, A. W., Olkin, I. and Arnold, B. C. (2011). Inequalities: Theory of Majorization and Its Applications. Springer.

Nazarov, F. (2003). On the maximal perimeter of a convex set in $\mathbb{R}^n$ with respect to a Gaussian measure. In Geometric Aspects of Functional Analysis Springer.

Pardo, L. (2005). Statistical Inference Based on Divergence Measures. CRC press.

Plunkett, A. and Park, J. (2017). Two-Sample Test for Sparse High Dimensional Multinomial Distributions. arXiv:1711.05524.

Ramsay, J. O. and Silverman, B. W. (2005). Functional Data Analysis. Springer.

Rudelson, M. and Vershynin, R. (2013). Hanson-Wright inequality and sub-gaussian concentration. Electronic Communications in Probability 18 9 pp.

Van der Vaart, A. W. and Wellner, J. A. (1996). Weak Convergence and Empirical Processes. Springer.

Vershynin, R. (2012). Introduction to the non-asymptotic analysis of random matrices. In Compressed sensing: theory and applications (Y. C. Eldar and G. Kutyniok, eds.) Cambridge University Press.

Vershynin, R. (2018). High Dimensional Probability. Cambridge University Press.

Vu, V. and Wang, K. (2015). Random weighted projections, random quadratic forms and random eigenvectors. Random Structures & Algorithms 47 792–821.

Wang, J.-L., Chiou, J.-M. and Müller, H.-G. (2016). Review of functional data analysis. Annual Review of Statistics and Its Application 3 257–295.

Wasserman, L., Kolar, M. and Rinaldo, A. (2014). Berry-Esseen bounds for estimating undirected graphs. Electronic Journal of Statistics 8 1188–1224.

Zelterman, D. (1987). Goodness-of-fit tests for large sparse multinomial distributions. Journal of the American Statistical Association 82 624–629.

Zhang, X. and Cheng, G. (2017). Simultaneous inference for high-dimensional linear models. Journal of the American Statistical Association 112 757–768.

Zhang, J.-T., Cheng, M.-Y., Wu, H.-T. and Zhou, B. (2018). A new test for functional one-way ANOVA with applications to ischemic heart screening. Computational Statistics & Data Analysis.

Zheng, S., Yang, L. and Härdle, W. K. (2014). A smooth simultaneous confidence corridor for the mean of sparse functional data. Journal of the American Statistical Association 109 661–673.