SEMIINFINITE COHOMOLOGY OF TATE LIE ALGEBRAS

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1. Introduction.

This note is a natural extension of the final part in [Ar] where a natural homological construction for graded Lie algebra semiinfinite cohomology and a natural explanation for the phenomenon of the critical cocycle was found. Here we propose a variant of the construction that works in the Tate Lie algebra case. Note that the standard complex for the computation of the Tate Lie algebra semiinfinite cohomology was written down by Beilinson and Drinfeld in [BD] and probably by some physicists. Still the construction was rather an indirect one and was not formulated in terms similar to the ones in [Ar]. In this note we spell out the construction of the complex in terms of some kind of quadratic-linear Koszul duality rather than Conformal Field Theory.

Let us say a few words about the contents of the note. In the second section we recall the notion of a differential graded Lie algebra with a curvature. We show that the standard Chevalley complex for computation of the Lie algebra cohomology can be modified in such a way that it still exists in this exotic case.

In the third section we recall a construction of an analogue of the Chevalley cohomological complex for a left module over a Lie algebroid $A$ over a commutative algebra $R$. Then we combine the setup from the previous section with the described one and obtain a picture consisting of a graded supercommutative algebra with a derivation, a graded (super) Lie algebroid over the algebra carrying an extension of the derivation of the basic algebra and finally an analogue of curvature that “corrects” the fact that both derivations do not satisfy the constraint $d^2 = 0$. We construct an analogue of the cohomological Chevalley complex with coefficients in a left CDG-module over the differential graded Lie algebroid with curvature (CDG Lie algebroid).

In the fourth section we show that the standard complex for semiinfinite cohomology in the Tate Lie algebra case is a particular example of the described situation. Namely for a Tate Lie algebra $g$ with a compact Lie subalgebra $b$ we consider the graded supercommutative algebra $\Lambda^\bullet((g/b)^*)$ and a graded Lie algebroid $b \otimes \Lambda^\bullet((g/b)^*)$ over it. We show that the components of the Lie bracket in $g$ provide the derivations and the curvature.

Finally we show that for a discrete module $M^g$ over the extension of $g$ with the help of the critical cocycle of $g$ the space $M^g \otimes \Lambda^\bullet((g/b)^*)$ carries a structure of a left CDG module over the above CDG Lie algebroid and that for this CDG-module the standard Chevalley complex from the third section coincides with the semiinfinite complex of $g$ with coefficients in $M^g$.

In order to simplify the exposition we never use the language of derived categories in the note but rather work with concrete complexes.

2. Toy example.

The material of this section is based on a partly unpublished construction of A.Polishchuk and L.Positselsky (still see [P]).

We begin with the case of a differential graded Lie superalgebra with a curvature. The standard complex appearing in the case seems to be more understandable.

Let $A = \bigoplus A_k$ be a graded Lie algebra and $d : A_k \to A_{k+1}$ be a derivation of $A$ of order 1. Note that at this point we do not put the constraint $d^2 = 0$. Instead we require an additional part of the data — an element $h \in A_2$ such that $d^2(a) = [h, a]$ for any $a \in A$ and $d(h) = 0$.

2.1. Definition: The data $(A, d, h)$ described above are called the differential graded Lie superalgebra with curvature or, for short, the CDG Lie algebra.
By definition a left (resp. a right) CDG-module over a CDG Lie algebra $A$ is a graded left (resp. right) module $M = \oplus M_k$ over the Lie algebra $A$ with the differential $d : M_k \to M_{k+1}$ satisfying the Leibnitz rule such that $d^2(m) = h \cdot m$ (resp. $d^2(m) = m \cdot h$) for any $m \in M$.

Denote the category of left (resp. right) CDG $A$-modules by $CDG \cdot A$-mod (resp. $CDG \cdot mod \cdot A$).

2.2. Construction of the standard complex. For $M^• \in CDG \cdot A$-mod consider the bigraded vector space $C^{••}(A, M^•)$ as follows: $C^{••}(A, M^•) = \Hom(\Lambda^{••}(A), M^•)$, here the first grading comes from the number of wedges in the exterior product and

$$C^{•k}(A, M^•) = \bigoplus_{p+q=k} \Hom((\Lambda^p(A))_q, M^q).$$

Consider the two differentials on the bigraded vector space. The first one of the grading $(1, 0)$ is the usual Chevalley differential:

$$(d_1 f)(a_1 \wedge \ldots \wedge a_m) = \sum_i (-1)^i f(a_1 \wedge \ldots \wedge da_i \wedge \ldots \wedge a_m) + d_{M^•} f(a_1 \wedge \ldots \wedge a_m)$$

$$+ \sum_i (-1)^i a_i^{M^•} f(a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge a_m)$$

$$+ \sum_{i<j} (-1)^{i+j} f([a_i, a_j] \wedge a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge \hat{a}_j \wedge \ldots \wedge a_m).$$

The second differential of the grading $(-1, 2)$ is given by the formula that uses the curvature $h$:

$$(d_2 f)(a_1 \wedge \ldots \wedge a_{m-1}) = f(h \wedge a_1 \wedge \ldots \wedge a_{m-1}).$$

Consider the total grading on the bigraded space.

2.2.1. Lemma: The differential $d = d_1 + d_2$ satisfies $d^2 = 0$.

Proof. The corresponding calculation repeats the one of Polishchuk and Positselsky. □

3. Standard complex for a CDG Lie algebroid.

3.1. Chevalley complex of a Lie algebroid. Let $R$ be a commutative algebra, and let $A$ be a Lie algebroid over $R$, i.e. $A$ is a $R$-module carrying a Lie algebra structure over the base field and acting on $R$ by derivations such that $[a, rb] = a(r)b + r[a, b]$ for any $a, b \in A$ and $r \in R$.

By definition a $A$-module is a $R$-module $M$ with the Lie action of $A$ satisfying the constraint $a \cdot (rm) = r(a \cdot m) + (a(r)) \cdot m$ for any $a \in A$, $r \in R$ and $m \in M$.

3.1.1. For a left $A$-module $M$ consider the graded vector space $C^•(A, M)$ as follows:

$$C^•(A, M) = \bigoplus_k C^k(A, M), \quad C^k(A, M) = \Hom_R(\Lambda^k_R(A), M)$$

We endow the graded vector space with the differential as follows:

$$(df)(a_1 \wedge \ldots \wedge a_m) = + \sum_i (-1)^i a_i^M f(a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge a_m)$$

$$+ \sum_{i<j} (-1)^{i+j} f([a_i, a_j] \wedge a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge \hat{a}_j \wedge \ldots \wedge a_m).$$

Lemma:

(i) The differential in the complex is well defined.
(ii) The differential satisfies $d^2 = 0$.

Proof. (i) Let us perform a calculation showing that the differential $d : C^k \to C^{k+1}$ is well defined for $k = 1$, the general case is quite similar. Note that

$$df(ra_1 \wedge a_2) = ra_1 f(a_2) - a_2 f(ra_1) - f([ra_1, a_2]) = ra_1 f(a_2) - a_2 f(a_1) + -rf([a_1, a_2]) + a_2 r f(a_1)$$

$$= ra_1 f(a_2) - ra_2 f(a_1) + -rf([a_1, a_2]) = df(a_1 \wedge ra_2).$$

(ii) The calculation does not differ from the usual Lie algebra case. □

3.1.2. Remark: Note that there is no such standard complex for a right $A$-module $M$.

The constructed complex is called the (cohomological) Chevalley complex of the Lie algebroid $A$ with coefficients in the left $A$-module $M$. 

3.2. CDG-algebroids. Now let $R = \bigoplus_k R_k$ be a graded supercommutative algebra and let $A = \bigoplus A_k$ be a graded super Lie algebroid over $R$. Suppose also there are a super derivation $d_R$ on $R$ of degree 1 and a super derivation $d_A$ of the Lie superalgebra $A$ also of degree 1 satisfying Leibniz rule with respect one to another. Again we put the constraint $d^2 = 0$ on neither of the derivations. Instead of that we fix the choice of an element $h \in A_2$ such that $d_A^2(a) = [h, a]$ and $d^2_R(r) = h(r)$.

3.2.1. Definition: The data $(A, R, d_A, d_R, h)$ are called the differential graded Lie algebroid with curvature or, for short, a CDG Lie algebroid.

The notion of a left (resp. right) CDG-module over a CDG-algebroid $A$ of the previous definitions and we do not spell it out explicitly. The category of left (resp. right) CDG-modules over a CDG-algebroid $A$ is denoted by $\text{CDG-} A\text{-mod}$ (resp. $\text{CDG- mod-} A$).

3.3. Standard complex for a CDG Lie algebroid. Now we sort of put together definitions of the standard complexes given in \[2.2\] and \[3.1.1\]. For $M^\bullet \in \text{CDG-} A\text{-mod}$ consider the bigraded vector space $C^{\bullet\bullet}(A, M^\bullet)$ as follows: $C^{\bullet\bullet}(A, M^\bullet) = \text{Hom}_R(\Lambda_R \idot(A), M^\bullet)$, here the first grading comes from the number of wedges in the exterior product and

\[ C^{\bullet\bullet}(A, M^\bullet) = \text{Hom}_R(\Lambda_R \idot(A), M^\bullet) \]

in the graded Hom sense. Consider the two differentials on the bigraded vector space. The first one of the grading (1,0) is the usual Chevalley differential like in \[3.1.1\]:

\[ (d_1 f)(a_1 \wedge \ldots \wedge a_m) = \sum_i (-1)^i f(a_1 \wedge \ldots \wedge da_i \wedge \ldots \wedge a_m) + d_{M^\bullet} f(a_1 \wedge \ldots \wedge a_m) \]

\[ + \sum_i (-1)^i a_i^{M^\bullet} f(a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge a_m) + \sum_{i<j} (-1)^{i+j} f([a_i, a_j] \wedge a_1 \wedge \ldots \wedge \hat{a}_i \wedge \ldots \wedge \hat{a}_j \wedge \ldots \wedge a_m). \]

The second differential of the grading (−1,2) is again given by the formula that uses the curvature $h$:

\[ (d_2 f)(a_1 \wedge \ldots \wedge a_{m-1}) = f(h \wedge a_1 \wedge \ldots \wedge a_{m-1}). \]

Consider the total grading on the bigraded space.

3.3.1. Lemma:

(i) The differential $d_1$ is well defined and its square equals zero.

(ii) The differential $d_2$ is well defined and its square equals zero.

(iii) The differential $d = d_1 + d_2$ satisfies $d^2 = 0$.

Proof. (i) Follows from Lemma \[3.1.1\]. (ii) Follows from the obvious fact that $h \wedge h = 0 \in \Lambda^2_R(\idot(A))$. (iii) Repeats the proof of Lemma \[2.2.1\]. \hfill \Box

The obtained complex is called the (cohomological) Chevalley complex of the CDG Lie algebroid $A$ with coefficients in the left CDG $A$-module $M$.

4. Semiinfinite cohomology via CDG Lie algebroids.

In this section we show that the standard complex for the computation of semiinfinite cohomology of a discrete module over a Tate Lie algebra coincides with the Chevalley complex of the form \[3.3.1\] for a certain CDG Lie algebroid and a certain left module over it.

4.1. Tate Lie algebras. Recall that a Tate space is a complete topological vector space having a base of neighbourhoods of 0 consisting of commensurable vector spaces (i.e., $\dim U_1 / (U_1 \cap U_2) < \infty$ for any $U_1$ and $U_2$ from this base).

Recall also that a $c$-lattice in a Tate space $V$ is an open bounded subspace and a $d$-lattice $L \subset V$ is a discrete subspace such that there exists a $c$-lattice $P$ with $L + P = V$. Note that the quotient of a Tate space by a $c$-lattice (resp. by a $d$-lattice) is discrete (resp. compact) in its natural topology. It is known that there is a natural duality on the category of Tate spaces and that $V^{**} \cong V$ for any Tate space $V$.

Recall also that by definition a Tate Lie algebra is a Tate vector space equipped with a Lie algebra structure continuous in the Tate topology.
4.2. Construction of the CDG Lie algebroid. Let \( \mathfrak{g} \) be a Tate Lie algebra with a subalgebra \( \mathfrak{b} \subset \mathfrak{g} \) that is a \( \mathfrak{c} \)-lattice In particular the space \( \mathfrak{c} := \mathfrak{g}/\mathfrak{b} \) is discrete and its dual space is compact. Choose a section of the projection \( \mathfrak{g} \to \mathfrak{c} \). Thus we fix a noncanonical decomposition \( \mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c} \) and the Lie algebra structure on \( \mathfrak{g} \) is provided by the following collection of maps:

\[
\mu_{2} : \mathfrak{b} \wedge \mathfrak{b} \to \mathfrak{b}, \quad \mu_{1} : \mathfrak{b} \otimes \mathfrak{c} \to \mathfrak{c}, \quad \mu_{c} : \mathfrak{c} \wedge \mathfrak{c} \to \mathfrak{c}, \quad \mu_{2} : \mathfrak{b} \otimes \mathfrak{c} \to \mathfrak{b}, \quad h^* : \mathfrak{c} \wedge \mathfrak{c} \to \mathfrak{b}.
\]

The existence of \( h^* \) means exactly that \( \mathfrak{c} \) is not a Lie subalgebra in \( \mathfrak{g} \). The construction below is parallel to the one in [Ar].

4.2.1. First consider the graded supercommutative algebra \( \Lambda^{*}(\mathfrak{c}^*) \) or rather its Tate completion denoted in the same way. So as a vector space it is Tate dual to the discrete coalgebra \( \Lambda^{*}(\mathfrak{c}) \). Consider the derivation \( d_{R} \) on \( \Lambda^{*}(\mathfrak{c}^*) \) of degree one generated by the map dual to \( \mu_{c} \) and extended to the whole algebra by Leibnitz rule and by continuity.

Next consider the graded Lie algebra \( \mathfrak{b} \otimes \Lambda^{*}(\mathfrak{c}^*) \) where the tensor product is understood in the completed sense so that the whole space is Tate dual to the discrete space \( \mathfrak{b}^* \otimes \Lambda^{*}(\mathfrak{c}) \). The commutator map in the above Lie algebra is generated by the one on \( \mathfrak{b} \) and by the action of \( \mathfrak{b} \) on the space \( \mathfrak{c}^* = (\mathfrak{g}/\mathfrak{b})^* \).

Denote the graded Lie algebra (resp. the graded supercommutative algebra) above by \( A \) (resp. by \( R \)). Evidently \( A \) is a \( R \)-module, moreover there is a natural adjoint action of \( A \) on \( R \).

**Lemma:** \( A \) is a graded Lie algebroid over the graded supercommutative algebra \( R \).

4.2.2. Construction of the CDG Lie algebroid structure on \( (A, R) \). We extend the derivation of \( R \) constructed above to the derivation of \( A \). Namely consider the map \( \mathfrak{b} \to \mathfrak{b} \otimes \mathfrak{c}^* \) dual to \( \mu_{2} \). Denote the map by \( d_{\mathfrak{b}} \). Extend the sum of \( d_{\mathfrak{b}} \) and \( d_{R} \) to the whole Lie algebra \( A \) by Leibnitz rule and by continuity. Denote the obtained derivation by \( d_{A} \). Finally consider the element \( h \in A_{2} = \mathfrak{b} \otimes \Lambda^{2}(\mathfrak{c}^*) \) corresponding to the component \( h^* \) of the bracket in \( \mathfrak{g} \).

4.2.3. **Proposition:** The data \( (A, R, d_{A}, d_{R}, h) \) form a CDG Lie algebroid.

4.3. Construction of the standard semiinfinite complex. First note that for any compact \( \mathfrak{g} \)-module \( M \) the (completed) tensor product \( M \otimes \Lambda^{*}(\mathfrak{c}^*) \) has a natural structure of a left CDG-module over the CDG algebroid \( (A, \ldots) \) constructed in the previous subsection.

4.3.1. **Construction of the right CDG-module.** Now fix a discrete \( \mathfrak{g} \)-module \( M \) and consider the graded space \( M \otimes \Lambda^{*}(\mathfrak{c}) \).

**Lemma:** The graded space \( M \otimes \Lambda^{*}(\mathfrak{c}) \) has a natural structure of the right CDG-module over the CDG Lie algebroid \( (A, \ldots) \).

4.3.2. Here we come to the crucial point explaining the phenomenon of the critical cocycle in the semiinfinite cohomology of Tate Lie algebras. What we would like to do is to consider the standard complex of the CDG Lie algebroid \( (A, R, \ldots) \) with coefficients in \( M \otimes \Lambda^{*}(\mathfrak{c}) \). Yet as noted in 3.1.2 there is no naive way to do it. Somehow we have to make \( M \otimes \Lambda^{*}(\mathfrak{c}) \) into a left CDG-module over our CDG Lie algebroid.

Happily the answer how to do this is known at least in the graded Lie algebra case. The idea works for Tate Lie algebra semiinfinite cohomology as well.

4.3.3. **Critical cocycle of \( \mathfrak{g} \).** It is known that the Lie algebra of continuous endomorphisms of \( \mathfrak{g} \) (and of any other Tate vector space as well) has a remarkable class in \( H^{2} \) and the choice of the decomposition \( \mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c} \) fixes its representative in the space of cocycles called the critical 2-cocycle and denoted by \( \omega_{0} \). The adjoint action of \( \mathfrak{g} \) on itself provides the inverse image of the class denoted by \( \omega_{0}^{\mathfrak{g}} \) and called the critical 2-cocycle of \( \mathfrak{g} \).

Denote by \( \mathfrak{g}^{\mathfrak{c}} \) the central extension of \( \mathfrak{g} \) with the help of this cocycle. Note that \( \mathfrak{g}^{\mathfrak{c}} \) is a Tate Lie algebra with a fixed vector space decomposition \( \mathfrak{g} = \mathfrak{b} \oplus \mathfrak{c} \oplus \mathfrak{cK} \), where \( \mathfrak{K} \) denotes the central element.
4.3.4. Consider the CDG Lie algebroid $(\tilde{A}^\sharp, R^\sharp, \ldots)$ obtained from the pair $b \oplus CK \subset g^\sharp$ by our main construction. Note that the algebra $R^\sharp \rightarrow R = \Lambda^\bullet(c^\ast)$ and that the element $K$ is central in $\tilde{A}^\sharp$. Denote by $A^\sharp$ the quotient of $\tilde{A}^\sharp$ by $K - 1$.

**Proposition:** The CDG Lie algebroid $(A^\sharp, R^\sharp, d_A^\sharp, d_R^\sharp, h^\sharp)$ is isomorphic to the the CDG Lie algebroid $(A, R, d_A, d_R, h)^{opp}$.

**Remark:** In particular for any discrete $g^\sharp$-module $M^\sharp$ such that the central element $K$ acts on it by 1 the right $(A^\sharp, \ldots)$ CDG-module $M^\sharp \otimes \Lambda^\bullet(c)$ becomes a left CDG-module over the CDG Lie algebroid $(A, \ldots)$.

4.3.5. **Definition:** For a discrete $g^\sharp$-module $M^\sharp$ such that the central element $K$ acts on it by 1 the complex

$$C^\bullet(A, M^\sharp \otimes \Lambda(c)) = \text{Hom}_R(\Lambda^\bullet_R(A), M^\sharp \otimes \Lambda^\bullet(c))$$

is called the standard semiinfinite complex with coefficients in $M^\sharp$ and is denoted by $C^{\infty+\bullet}(g, M^\sharp)$.

4.3.6. **Theorem:**

(i) For a discrete $g^\sharp$-module $M^\sharp$ as above the graded vector space $C^{\infty+\bullet}(g, M^\sharp)$ is isomorphic to the graded vector space $\Lambda^\bullet(b^\ast) \otimes \Lambda^\bullet(g/b) \otimes M^\sharp$.

(ii) The cohomology of the complex $C^{\infty+\bullet}(g, M^\sharp)$ coincides with the semiinfinite cohomology of the Tate Lie algebra $g$ with coefficients in the discrete module $M^\sharp$ defined in [BD].

**References**

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