Tsirelson’s bound and supersymmetric entangled states

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In order to see whether superqubits are more nonlocal than ordinary qubits, we construct a class of two-superqubit entangled states as a nonlocal resource in the CHSH game. Since super Hilbert space amplitudes are Grassmann numbers, the result depends on how we extract real probabilities and we examine three choices of map: (1) DeWitt (2) Trigonometric (3) Modified Rogers. In cases (1) and (2) the winning probability reaches the Tsirelson bound $p_{\text{win}} = \cos^2 \pi/8 \approx 0.8536$ of standard quantum mechanics. Case (3) crosses Tsirelson’s bound with $p_{\text{win}} \approx 0.9265$. Although all states used in the game involve probabilities lying between 0 and 1, case (3) permits other changes of basis inducing negative transition probabilities.

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In providing nonlocal resources, quantum mechanics fundamentally distinguishes itself from the classical world. Perhaps the best known example of a nonlocal resource is the EPR-pair, introduced by Einstein, Podolsky and Rosen [1]. Bell famously showed that this quantum state can be used to violate an inequality required to be satisfied by any local hidden variable model [2]. It was later realised by Tsirelson that not only does the EPR-pair violate the Bell inequality, it violates it maximally [3]: there is no quantum system that can do better. In particular, Tsirelson used a variant of the Bell inequality introduced by Clauser, Horne, Shimony and Holt (CHSH) [5]. The CHSH inequality can be nicely rephrased as a limit on the probability of winning a nonlocal game [6], which we describe below. In these terms the CHSH inequality is given by $p_{\text{win}} \leq 3/4$, where $p_{\text{win}}$ denotes the probability of winning the game. On the other hand Tsirelson’s bound corresponds to $p_{\text{win}} \leq \cos^2 \pi/8 \approx 0.8536$.

Why does quantum mechanics stop at $p_{\text{win}} = \cos^2 \pi/8$? In an effort to address this very question Popescu and Rohrlich [4] asked whether one could envision a theory which beats $p_{\text{win}} = \cos^2 \pi/8$, i.e. is more nonlocal than quantum mechanics, without violating other established principles of physics, in particular, the no-signalling condition imposed by special relativity. They answered this question in the positive: there are hypothetical resources [4], known as nonlocal-boxes or PR-boxes, that have $p_{\text{win}} = 1$ without violating the no-signalling condition. Thus far, however, they are purely mathematical constructs with no links to existing physical theories.

In this paper we ask whether superqubits can cross Tsirelson’s bound. Superqubits are a supersymmetric extension of qubits, introduced in [7], where the even (commuting) computational basis vectors $|0\rangle$ and $|1\rangle$ are augmented by an odd (anticommuting) basis vector $|\cdot\rangle$. They transform as a triplet under the orthosymplectic group $\text{UOSp}(1|2)$ (the “compact real form” of $\text{OSp}(1|2)$), which is the minimal supersymmetric extension of the SU(2) group of local unitaries. The $\text{osp}(1|2)$ algebra has been extensively studied in the past [11–14] in the context of Lie superalgebras [9, 17, 18]. In general, the main physical motivation for studying supersymmetry comes from high energy physics where it is a leading candidate for physics beyond Standard Model. However, supersymmetry also appears in several proposals from condensed matter physics. In particular, the orthosymplectic algebra $\text{osp}(1|2)$ plays a key role in [15].

We begin by recalling the usual CHSH game [5]. It is a so-called nonlocal game [6, 24] with three players: a referee who competes with Alice and Bob. The referee chooses with probability 1/4 two bits $i \in \{0, 1\}$ and $j \in \{0, 1\}$ and sends one of them to Alice and Bob such they are not aware of one another’s bit value. Alice and Bob send a bit of communication (denoted $a$ and $b$) back to the referee. The conditions for Alice and Bob to win the game are captured in the following table:

| $ij$ | $a \oplus b$ |
|------|--------------|
| 00   | 0            |
| 01   | 0            |
| 10   | 0            |
| 11   | 1            |

Alice and Bob cannot communicate during the game but they may establish their strategy beforehand. Tsirelson’s bound is achieved if they share a maximally entangled state such as $\Psi = (|00\rangle + |11\rangle)/\sqrt{2}$ accompanied by an agreed measurement strategy. The PR-box always wins - it is simply a black-box that when fed $i$ by Alice and $j$ by Bob spits out $a, b$ such that the table is satisfied.

A normalized superqubit $\psi$ [7, 8] may be regarded as an element of the projective space $S^{2^{12}} = \text{UOSp}(1|2)/\text{U}(0|1)$ known as the supersphere [11–15], where $\text{U}(0|1)$ is the even Grassmann generalization of the $\text{U}(1)$ Lie group. The definition implies that the supersphere is nothing...
other than a super version of the Bloch sphere $S^2 = \text{SU}(2)/\text{U}(1)$. An arbitrary superqubit can be generated by a general group action

$$Z(\eta, \alpha, \beta) = S(\eta)U(\alpha, \beta)$$

where $\alpha, \beta$ are even Grassmann supernumbers satisfying $\alpha\alpha^\# + \beta\beta^# = 1$ and $\eta$ is an unconstrained odd supernumber. Supernumbers are elements of a complex Grassmann algebra which is a complex vector space equipped with an alternating product. A Grassmann algebra is isomorphic to an exterior algebra. The hash operator denotes the graded involution which is a generalized complex conjugation. For even supernumbers $X^\# = X$ holds for the Grassmann-valued transition probability function between two superqubits $\varphi$ and $\psi$ as

$$p_{\varphi}(\varphi, \psi) = \langle \varphi | \psi \rangle (\langle \varphi | \psi \rangle)^\#.$$  

The rationale behind this definition is clear: for ordinary (non-Grassmann) states we recover the usual Born rule. Furthermore, for any superqubit $\psi$ we find that $\sum_{m=1}^3 p_G(m, \psi) = 1$ where $m = \{0, 1, \bullet\}$. Hence all superqubits are normalized to one. Adopting the rule of one Grassmann generator $\theta$ per superqubit the single superqubit outcome probabilities in the computational basis are given by

\begin{align}
  p_{\varphi_0} &= \alpha\bar{\alpha}(1 - r^2\theta\theta^\#) \quad (4a) \\
  p_{\varphi_1} &= \beta\bar{\beta}(1 - r^2\theta\theta^\#) \quad (4b) \\
  p_{\varphi_\bullet} &= r^2\theta\theta^\#, \quad (4c)
\end{align}

where we set $\eta = 2r\theta$ in Eq. (2) ($r \in \mathbb{R}$). The parameters $\alpha, \beta$ are in general even Grassmann numbers but we regard them as complex for simplicity. We need a map to convert the Grassmann-valued probabilities $p_{\varphi_m}$ into a real-valued probabilities, $p_m$, which respects the UOSp$(1|2)$ symmetry. Here, we consider three alternatives. The first map, introduced by DeWitt [20], is to simply ignore the Grassmann part of the probability. It boils down to ordinary quantum mechanics: $p_0 = \alpha\bar{\alpha}, p_1 = \beta\bar{\beta}$ and $p_\bullet = 0$ where the bar denotes complex conjugation. The second possibility is the trigonometric map: $r^2\theta\theta^\# \mapsto \cos^2 r$. Hence $p_0 = \alpha\bar{\alpha}\sin^2 r, p_1 = \beta\bar{\beta}\sin^2 r$ and $p_\bullet = \cos^2 r$. Both of these maps have the virtue that the individual probabilities lie between zero and one. The third map we consider is defined by the rule $r^2\theta\theta^\# \mapsto r^2$, where we fix the orientation of the Grassmann generators to be $\theta\theta^\#$. Hence $p_0 = \alpha\bar{\alpha}(1 - r^2), p_1 = \beta\bar{\beta}(1 - r^2)$ and $p_\bullet = r^2$. We refer to this map as the modified Rogers map since, in spite of their similarity, the Rogers construction [21, 22] does not respect the UOSp$(1|2)$ symmetry. For a $2^{2^n}$-dimensional Grassmann algebra generated by $\{\theta, \theta^\#\}_{i=1}^{2^n}$ the definition of the modified Rogers map can be generalized as follows

$$|\tau\rangle_R^\# \equiv \prod_{i=1}^{2^n} e^{i\theta_i d\theta_i} \tau \, d^2\theta_i,$$  

where $\tau$ is an arbitrary supernumber, $d^2\theta = \prod_{i=1}^{2^n} d\theta_i d\theta_i^\#$ and the orientation is fixed by $\prod_{i=1}^{2^n} \theta_i \theta_i^\#$. The map is linear and respects the UOSp$(1|2)$ symmetry. However, although the probabilities sum to one, they are not always guaranteed individually to lie between zero and one.

We are ready to play the superqubit CHSH game in each of the three cases, being careful in case (3) to avoid measurement outcomes with negative probabilities. A superqubit is formally a three-level system and so we have to adapt the CHSH game to this situation. We set the rules such that Alice announces $a = 0$ ($a = 1$) to the referee if her local measurement projects to $|0\rangle$ ($\langle 1 |$ or $|\bullet\rangle$). The same holds for the value of $b$ transmitted from Bob’s laboratory. The rest of the rules are identical to the usual CHSH game. Therefore the winning Grassmann-valued probability reads

$$p_G(\Gamma_{AB}) = \frac{1}{4} \left[ \sum_{i,j \in \{00, 01, 10\}} (p_{G^{(ij)}} + p_{G^{(ij)}_\bullet} + p_{G^{(ij)}_\bullet} + p_{G^{(ij)}_\bullet} + p_{G^{(ij)}_{01}} + p_{G^{(ij)}_{01}} + p_{G^{(ij)}_{10}} + p_{G^{(ij)}_{10}} + p_{G^{(ij)}_{0\bullet}} + p_{G^{(ij)}_{0\bullet}}) \right], \quad (6)$$
where subscripts $A$ and $B$ refer to Alice and Bob and where

$$p_{\text{win}}^{(ij)} = (m_{A^n B}|\Gamma_{A, i, j, B})\left((m_{A^n B}|\Gamma_{A, i, j, B})^\# \right)$$

is the Grassmann-valued probability function introduced in Eq. (4). $\Gamma_{A, i, j, B} = (Z_{iA} \otimes Z_{jB})\Gamma_{AB}$ is a locally superunitary rotated superqubit bipartite entangled state $\Gamma_{AB}$ where

$$Z_{iA} \otimes Z_{jB} = S(2r_i \theta_A)U(\alpha_i, \beta_i)\otimes S(2s_j \theta_B)U(\gamma_j, \delta_j). \tag{8}$$

Here, the Grassmann odd elements $\theta_A, \theta_B$ and their conjugates are the generators of the Grassmann algebra of order $2n = 4$ and $r_i, s_j \in \mathbb{R}$. The shared entangled state we use reads

$$\Gamma_{AB} = \left(1 + \frac{1}{2} X + \frac{3}{8} X^2 \right) \times \left(\frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) + \frac{p}{\sqrt{2}} \theta_A |1\rangle + \frac{q}{\sqrt{2}} \theta_B |0\rangle \right)$$

where $p, q \in \mathbb{R}$ and $X = -\frac{p^2}{2} \theta_A^2 - \frac{q^2}{2} \theta_B^2$. Note that this state reduces to the Bell state in the absence of the Grassmann numbers ($p = q = 0$). We define

$$p_{\text{win}} = \max_{\alpha_i, \beta_i, \gamma_j, \delta_j} p_{\text{win}}(\Gamma_{AB}) \tag{10}$$

s.t. $0 \leq p_{\text{win}}^{(ij)} \leq 1$, $\forall i, j, m, n$

where $p_{\text{win}}(\Gamma_{AB})$ comes from Eq. (6) after the modified Rogers map has been applied. The optimization procedure has to be performed numerically [27]. The winning parameters are

$$r_0 \simeq 0.7476, \quad s_0 \simeq 0.6329, \quad r_1 \simeq 0, \quad s_1 \simeq 0.6329$$

$$\alpha_0 \simeq -\pi/2, \quad \alpha_1 \simeq \pi/4, \quad \beta_0 \simeq \pi/4, \quad \beta_1 \simeq 3\pi/4$$

$$p \simeq 0.7476, \quad q \simeq -1.0949$$

This corresponds to $p_{\text{win}} \simeq 0.9265$. Note that this is a mere lower bound since the optimization problem is non-convex. We appreciate that this exceeds the triviality of communication complexity bound $p_{\text{win}} \leq (3 + \sqrt{6})/6 \simeq 0.908$ [23]. However, a case can be made for imposing $|r_i| \leq 1/2, |s_j| \leq 1/2$, which is computationally equivalent to ‘compactifying’ $S$ in UOSP$(1|2)$ appearing in (8). In this case we obtain a reduced $p_{\text{win}} = 0.8647$.

In this work we have proposed a model based on supersymmetry which provides bipartite entangled states more nonlocal than those allowed by quantum mechanics. Violating Tsirelson’s bound was always destined to involve paying a price and it remains to be seen whether the existence of negative transition probabilities is too high price to pay (even though we did not invoke them in the CHSH game). In fact, one might ask whether it is this feature alone, with or without supersymmetry, that is responsible for exceeding the bound. Indeed, it has been shown that the space of no-signalling models is equivalent to the space of local hidden-variable models with extended probabilities that marginalize to yield standard non-negative probabilities [28]. However, extended probabilities alone do not seem sufficient. Evidence for this is provided by the example of a ‘qutrit’ with non-compact local unitary group $SU(2,1)$, rather than $SU(3)$, which, like the superqubit, has an $SU(2)$ subgroup. The probabilities sum to one but, owing to the indefinite metric, are again not always guaranteed individually to lie between zero and one. Unlike the superqubit, however, there are no rotations, aside from the trivial $SU(2)$ subgroup, that do not lead to negative transition probabilities. Excluding such rotations, therefore, means that our ‘qutrists’ cannot exceed the bound.

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