The method of unitary clothing transformations in the theory of nucleon–nucleon scattering

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1 Introductory remarks and some recollections

We know that there are a number of high precision, boson-exchange models of the two nucleon force $V_{NN}$, such as Paris [1], Bonn [2], Nijmegen [3], Argonne [4], CD Bonn [5] potentials and a modern family of covariant one-boson-exchange (OBE) ones [6]. Note also successful treatments based on chiral effective field theory [7, 8], for a review see [9].

In this paper, we would like to draw attention to the first application of unitary clothing transformations (UCTs) [10, 11] in describing the nucleon–nucleon scattering. Recall that such transformations $W$, being aimed at the inclusion of the so-called cloud or persistent effects, make it possible the transition from the bare-particle representation (BPR) to the clothed-particle representation (CPR) in the Hilbert space $\mathcal{H}$ of meson–nucleon states. In this way, a large amount of virtual processes induced with the meson absorption/emission, the $NN$–pair annihilation/production and other cloud effects can be accumulated in the creation (destruction) operators $\alpha_c$ for the "clothed" (physical) mesons and nucleons. Such a bootstrap reflects the most significant distinction between the concepts of clothed and bare particles.

In the course of the clothing procedure all the generators of the Poincaré group get one and the same sparse structure on $\mathcal{H}$ [10]. Here we will focus upon one of them, viz., the total Hamiltonian

$$ H = H_F(\alpha) + H_I(\alpha) \equiv H(\alpha) $$

(1)

with

$$ H_I(\alpha) = V(\alpha) + \text{mass and vertex counterterms}, $$

(2)

where free part $H_F(\alpha) \sim \alpha^\dagger \alpha$ belongs to the class [1.1], if one uses the terminology adopted in [10], and inter-

Abstract The clothing procedure, put forward in quantum field theory (QFT) by Greenberg and Schweber, is applied for the description of nucleon–nucleon ($N – N$) scattering. We consider pseudoscalar ($\pi$ and $\eta$), vector ($\rho$ and $\omega$) and scalar ($\delta$ and $\sigma$) meson fields interacting with $\frac{1}{2}$ spin ($N$ and $\bar{N}$) fermion ones via the Yukawa-type couplings to introduce trial interactions between "bare" particles. The subsequent unitary clothing transformations (UCTs) are found to express the total Hamiltonian through new interaction operators that refer to particles with physical (observable) properties, the so-called clothed particles. In this work, we are focused upon the Hermitian and energy-independent operators for the clothed nucleons, being built up in the second order in the coupling constants. The corresponding analytic expressions in momentum space are compared with the separate meson contributions to the one–boson–exchange potentials in the meson theory of nuclear forces. In order to evaluate the $T$ matrix of the $N – N$ scattering we have used an equivalence theorem that enables us to operate in the clothed-particle representation (CPR) instead of the bare-particle representation (BPR) with its large amount of virtual processes. We have derived the Lippmann–Schwinger type equation for the CPR elements of the $T$–matrix for a given collision energy in the two–nucleon sector of the Hilbert space $\mathcal{H}$ of hadronic states.
action $V(\alpha)$ is a function of destruction (destructor) operators $a_\alpha^\dagger(a)$ in the BPR, i.e., referred to bare particles with physical masses \[\text{(11)},\] where they have been introduced via the mass-changing Bogoliubov–type UTs. To be more definite, let us consider fermions (nucleons and antinucleons) and bosons ($\pi^-, \eta^-, \rho^-, \omega$-mesons, etc.) interacting via the Yukawa-type couplings for scalar ($s$), pseudoscalar ($ps$) and vector ($v$) mesons. Then, as seen from Appendix A, $V(\alpha) = V_s + V_{ps} + V_v$ with

$$V_s = g_s \int dx \bar{\psi}(x)\gamma_5 \psi(x) \varphi_s(x)$$

$$V_{ps} = ig_{ps} \int dx \bar{\psi}(x) \gamma_\mu \psi(x) \varphi_{ps}(x)$$

$$V_v = \int dx \left\{ g_v \bar{\psi}(x) \gamma_\mu \psi(x) \varphi_v(x) + \frac{eF}{4m} \bar{\psi}(x) \sigma_{\mu\nu} \psi(x) \varphi_v^{\mu\nu}(x) \right\} + \int dx \left\{ \frac{g_2^2}{2m_v^2} \bar{\psi}(x) \gamma_0 \psi(x) \bar{\psi}(x) \gamma_0 \psi(x) + \frac{eF}{4m^2} \bar{\psi}(x) \sigma_{0\nu} \psi(x) \varphi_v(x) \sigma_{0\nu} \psi(x) \right\},$$

where $\varphi_v^{\mu\nu}(x) = \partial^\mu \varphi_v^{\nu}(x) - \partial^\nu \varphi_v^{\mu}(x)$ the tensor of the vector field included. The mass (vertex) counterterms are given by Eqs. (32)–(33) of Ref. \[\text{(11)},\] (the difference $V_0(\alpha) - V(\alpha)$ where a primary interaction $V_0(\alpha)$ is derived from $V(\alpha)$ replacing the "physical" coupling constants by "bare" ones).

The corresponding set $\alpha$ involves operators $a_\alpha^\dagger(a)$ for the bosons, $b_\alpha^\dagger(b)$ for the nucleons and $d_\alpha^\dagger(d)$ for the antinucleons. Following a common practice, they appear in the standard Fourier expansions of the boson fields $\varphi$ and the fermion field $\psi$, though for our purposes the use of such a representation is not compulsory (see, e.g., Chapter 3 of the monograph \[\text{(12)}\]). In any case we have the free pion and fermion parts

$$H_F(\alpha) = \int dk \omega_k a_\alpha^\dagger(k)a(k) + \int dp E_p \sum_\mu [b_\alpha^\dagger(p, \mu)b(p, \mu) + d_\alpha^\dagger(p, \mu)d(p, \mu)]$$

and the primary trilinear interaction

$$V(\alpha) \sim ab^\dagger b + ab d^\dagger d + a b^\dagger + d^\dagger + H.c.\, (7)$$

with the three-legs vertices. Here $\omega_k = \sqrt{m_\pi^2 + k^2}$ ($E_p = \sqrt{m^2 + p^2}$) represents the pion (nucleon) energy with physical mass $m_\pi(m)$ while $\mu$ denotes the fermion polarization index. It may be the particle spin projection onto the quantization axis (the particle momentum) for the so-called canonical (helicity–state) basis (see, e.g., Chapter 4 of Ref. \[\text{(14)}\]).

In the context, we have tried to draw parallels with that field–theoretic background which has been employed in boson-exchange models. First of all, we imply the approach by the Bonn group \[\text{(2)}\], where, following the idea by Schütte \[\text{(15)}\], the authors started from the total Hamiltonian (in our notations),

$$H = H_F(\alpha) + V(\alpha) \, (8)$$

with the boson-nucleon interaction

$$V(\alpha) \sim ab^\dagger b + H.c.\, (9)$$

For brevity, the contributions from other mesons are omitted.

Unlike Eq. (7) the antinucleonic degrees of freedom were disregarded in Ref. \[\text{(2)}\]. There (see also \[\text{(14)}\]) the transition matrix $T(z)$ (in the two–nucleon space) was considered in the framework of the three-dimensional perturbation theory when handling the integral equation,

$$T_{NN}(z) = V_{NN}(z) + V_{NN}(z)(z - H_N)^{-1}T_{NN}(z), \quad (10)$$

where the energy–dependent ”quasipotential” $V_{NN}(z)$ approximates a sum of all relevant non-iterative diagrams, $H_N$ the nucleon contribution to $H_F$. Such a potential has ”the unpleasant feature of being energy–dependent. This complicates applications to nuclear structure physics considerably” (quoted from p.40 in \[\text{(2)}\]). Therefore, further simplifications are welcome (see, e.g., Refs. \[\text{(2, 5)}\]).

Along with our derivation of a Lippmann–Schwinger (LS) equation for the T matrix of the $N-N$ scattering, we will demonstrate its solutions to be compared with those by the Bonn group.

2 Analytic expressions for the quasipotentials in momentum space

As shown in \[\text{(10)}\], after eliminating the so-called bad term\[\footnote{By definition, they prevent the bare vacuum $|\Omega\rangle(a|\Omega\rangle = b|\Omega\rangle = \ldots = 0$ and the bare one–particle states $|bare\rangle \equiv a_\alpha^\dagger|\Omega\rangle(a|bare\rangle = b_\alpha^\dagger|\Omega\rangle, \ldots)$ to be $H$ eigenstates.}\] from $V(\alpha)$ the primary Hamiltonian $H(\alpha)$ can be represented in the form,

$$H(\alpha) = K_F(\alpha_e) + K_I(\alpha_e) \equiv K(\alpha_e) \, (11)$$
The free part of the new decomposition is determined by

\[ K_F(\alpha_c) = \int dk \omega_k a_+^c(k) a_c(k) \]

\[ + \int dp E_p \sum_\mu \left[ b_+^c(p, \mu) b_c(p, \mu) + d_+^c(p, \mu) d_c(p, \mu) \right] \]

while \( K_I \) contains only interactions responsible for physical processes, these quasipotentials between the clothed particles, e.g.,

\[ K_i^{(2)}(\alpha_c) = K(NN \rightarrow NN) + K(\bar{N}N \rightarrow \bar{N}N) \]

\[ + K(N\bar{N} \rightarrow N\bar{N}) + K(bN \rightarrow bN) + K(b\bar{N} \rightarrow b\bar{N}) \]

\[ + K(bb' \rightarrow N\bar{N}) + K(N\bar{N} \rightarrow bb') \]

A key point of the clothing procedure developed in [10] is to fulfill the following requirements:

i) The physical vacuum (the \( H \) lowest eigenstate) must coincide with a new no-particle state \( \Omega \), i.e., the state that obeys the equations

\[ a_c(k) \langle \Omega | \rangle = b_c(p, \mu) \langle \Omega | \rangle = d_c(p, \mu) \langle \Omega | \rangle = 0, \quad \forall \, k, p, \mu \]

\[ \langle \Omega | \Omega \rangle = 1. \]

ii) New one-clothed-particle states \( |k\rangle_c \equiv a_+^c(k) |\Omega \rangle \) etc. are the eigenvectors both of \( K_F \) and \( K_i \),

\[ K(\alpha_c) |k\rangle_c = K_F(\alpha_c) |k\rangle_c = \omega_k |k\rangle_c \]

\[ K_i(\alpha_c) |k\rangle_c = 0 \]

iii) The spectrum of indices that enumerate the new operators must be the same as that for the bare ones.

iv) The new operators \( \alpha_c \) satisfy the same commutation rules as do their bare counterparts \( \alpha \), since the both sets are connected to each other via the similarity transformation

\[ \alpha_c = W^+ \alpha W, \]

with a unitary operator \( W \) to be obtained as in [10].

It is important to realize that operator \( K(\alpha_c) \) is the same Hamiltonian \( H(\alpha) \). Accordingly [10,11] the \( N-N \) interaction operator in the CPR has the following structure:

\[ K(NN \rightarrow NN) = \sum_b K_b(NN \rightarrow NN), \]

\[ K_b(NN \rightarrow NN) = \int dp_1 dp_2 \sum_\mu \left[ b_+^b(p_1, \mu) b_c(p_1, \mu) + d_+^b(p_1, \mu) d_c(p_1, \mu) \right] \]

\[ \times V_b(1', 2'; 1, 2) b_+^b(1') b_+^b(2') b_c(1) b_c(2) \]

where the symbol \( \sum_\mu \) denotes the summation over the nucleon spin projections, \( 1 = \{p_1, \mu_1\}, \) etc.

For our evaluations of the \( c \)-number matrices \( V_b \) we have employed some experience from Refs. [10,11] to get in the second order in the coupling constants (see Appendix A)

\[ V_b(1', 2'; 1, 2) = \frac{1}{(2\pi)^5} \frac{m^2}{\sqrt{E_{p_1} E_{p_2} E_{p_3} E_{p_4}}} \]

\[ \times \delta (p_1' + p_2' - p_1 - p_2) v_b(1', 2'; 1, 2), \]

\[ v_b(1', 2'; 1, 2) = -\frac{g_2^2}{2} \bar{u}(p_1') \gamma_5 u(p_1) \frac{1}{(p_1' - p_1)^4 - m_{ps}^2} \bar{u}(p_2') \gamma_5 u(p_2), \]

\[ v_{c,s}(1', 2'; 1, 2) = \frac{g_2^2}{2} \bar{u}(p_1') \gamma_5 u(p_1) \frac{1}{(p_1' - p_1)^4 - m_{ps}^2} \bar{u}(p_2') \gamma_5 u(p_2). \]
Such a feature of the UCTs method allows us to use the graphic language of the old-fashioned perturbation theory (OFPT) (see, e.g., Chapter 13 in Schweber’s book [17]) when addressing the graphs in Fig. 1. As noted in [11], the graphs within our approach should not be interpreted as the two time-ordered Feynman diagrams. Indeed, all events in the S picture used here are related to the same instant \( t = 0 \). Being aware of this, the line directions in Fig 1 are given with the sole scope to discriminate between nucleons and antinucleons. The latter will inevitably appear in higher orders in coupling constants and for other physical processes (e.g., the \( \pi N \) scattering) as it has been demonstrated in Ref. [11].

Further, for each boson included the corresponding relativistic and properly symmetrized \( N - N \) interaction, the kernel of integral equations for the \( N - N \) bound and scattering states, is determined by

\[
\langle b_i^\dagger(p_i')b_i^\dagger(p_i')\Omega| K_b(NN \to NN) | b_i^\dagger(p_i)b_i^\dagger(p_i)\Omega \rangle = V_b^{dir}(1',2';1,2) - V_b^{exc}(1',2';1,2) \tag{23}
\]

where we have separated the so-called direct

\[
V_b^{dir}(1',2';1,2) = -V_b(1',2';1,2) - V_b(2',1';2,1) \tag{24}
\]

and exchange

\[
V_b^{exc}(1',2';1,2) = V_b^{dir}(2',1';1,2) \tag{25}
\]

terms. For example, the one–pion–exchange contribution can be divided into the two parts:

\[
V_\pi^{dir}(1',2';1,2) = -\frac{g_\pi^2}{2\pi^2} \frac{m_\pi^2}{\sqrt{E_{p_1}E_{p_2}E_{p_1}E_{p_2}}} \times \delta(p_1 + p_2 - p_1 - p_2) \bar{u}(p_1')\gamma_5 u(p_1)\bar{u}(p_2')\gamma_5 u(p_2)
\]

\[
\times \frac{1}{2} \left\{ \frac{1}{(p_1 - p_1')^2 - m_\pi^2} + \frac{1}{(p_2 - p_2')^2 - m_\pi^2} \right\} \tag{26}
\]

and

\[
V_\pi^{exc}(1',2';1,2) = -\frac{g_\pi^2}{2\pi^2} \frac{m_\pi^2}{\sqrt{E_{p_1}E_{p_2}E_{p_1}E_{p_2}}} \times \delta(p_1 + p_2 - p_1 - p_2) \bar{u}(p_1')\gamma_5 u(p_2)\bar{u}(p_2')\gamma_5 u(p_1)
\]

\[
\times \frac{1}{2} \left\{ \frac{1}{(p_2 - p_1')^2 - m_\pi^2} + \frac{1}{(p_1 - p_2')^2 - m_\pi^2} \right\} \tag{27}
\]

for the direct contribution, and

\[
\frac{1}{2} \left\{ \frac{1}{(p_1 - p_1')^2 - m_\pi^2} + \frac{1}{(p_2 - p_2')^2 - m_\pi^2} \right\}
\]

for the exchange contribution to be depicted in Fig. 2, where the dashed lines correspond to the following Feynman-like "propagators":

\[
\frac{1}{2} \left\{ \frac{1}{(p_1 - p_1')^2 - m_\pi^2} + \frac{1}{(p_2 - p_2')^2 - m_\pi^2} \right\}
\]

on the left panel and

\[
\frac{1}{2} \left\{ \frac{1}{(p_2 - p_1')^2 - m_\pi^2} + \frac{1}{(p_1 - p_2')^2 - m_\pi^2} \right\}
\]

on the right panel. Other distinctive features of the result (23) have been discussed in [10][11]. Note also that expressions (26) – (27) determine the one–pion–exchange part of one–boson–exchange interaction derived via the Okubo transformation method in [18] (cf. [19][11]) taking into account the pion and heavier–meson exchanges.

3 The field–theoretic description of the elastic \( N-N \) scattering

3.1 The \( T \)–matrix in the CPR

Usually in nonrelativistic quantum mechanics (NQM) the LS equation for the \( T \) operator

\[
T(E + i0) = V_{NN} + V_{NN}(E + i0 - h_0)^{-1}T(E + i0) \tag{28}
\]

with a given kernel \( V_{NN} \) is a starting point in evaluating the \( N-N \) phase shifts. All operators in Eq. (28) (including the sum \( h_0 \) of the nucleon kinetic energies) act onto the subspace of two–nucleon states, and remain confined in this subspace (the particle number is conserved within nonrelativistic approach). The matrix elements \( \langle N'N'|T(E + i0)|NN \rangle \), form the corresponding \( T \)–matrix and, at the collision energy

\[
E = E_1 + E_2 = E_1' + E_2',
\]

the on–energy–shell elements can be expressed through the phase shifts and mixing parameters (see below).
In relativistic QFT the situation is completely different. Though one can formally introduce a field operator $T$ that meets the equation

$$T(E + i0) = H_I + H_I(E + i0 - H_F)^{-1}T(E + i0) \quad (29)$$

the field interaction $H_I$, as a rule, does not conserve the particle number, being the spring of particle creation and destruction. The feature makes the problem of finding the $N\rightarrow N$ scattering matrix much more complicated than in the framework of nonrelativistic approach, since now the $T$ matrix enters an infinite set of coupled integral equations.

Such a general field–theoretic consideration can be simplified with the help of an equivalence theorem [21, 20] according to which the $S$ matrix elements in the Dirac (D) picture, viz.,

$$S_{fj} = \langle \alpha | ... \Omega \rangle S(\alpha) | \alpha | ... \Omega \rangle \quad (30)$$

are equal to the corresponding elements

$$S_{fj}^c = \langle \alpha | ... \Omega \rangle S(\alpha_c) | \alpha | ... \Omega \rangle \quad (31)$$

of the $S$ matrix in the CPR. We say ”corresponding” keeping in mind the requirement (iii) of Sec. 2. The $S$ operators in Eqs. (20)–(21) are determined by the time evolution from the distant past to the distant future, respectively, for the two decompositions

$$H = H(\alpha) = H_F + H_I$$

and

$$H = K(\alpha_c) = K_F + K_I$$

Note that the equality $S_{fj} = S_{fj}^c$ in question becomes possible owing to certain isomorphism between the $\alpha_c$ algebra and the $\alpha$ algebra once the UCTs $W_D(t) = \exp(iK_FT)W \exp(-iK_FT)$ obey the condition

$$W_D(\pm \infty) = 1 \quad (32)$$

The $T$ operator in the CPR satisfies the equation

$$T_{cloth}(E + i0) = K_I + K_I(E + i0 - K_F)^{-1}T_{cloth}(E + i0) \quad (33)$$

and the matrix

$$T_{fj} = \langle f; c | T(E + i0) | i; b \rangle = \langle f; c | T_{cloth}(E + i0) | i; b \rangle = T_{fj}^c, \quad (34)$$

where $| b \rangle ( | b \rangle )$ are the $H_F (K_F)$ eigenvectors, may be evaluated relying upon properties of the new interaction $K_I(\alpha_c)$. The latter has nonzero matrix elements only between the clothed–particle (physical) states. Such a restriction helps us to facilitate the further consideration compared to the BPR with its large amount of virtual transitions (cf., our discussion in Sect.1).

If in Eq. (29) we approximate $K_I$ by $K_I^{(2)}$ (see Eq. (13)), then initial task of evaluating the BPR matrix elements $\langle N'N' | T(E + i0) | NN \rangle$ can be reduced to solving the equation

$$\langle 1', 2'|TN_N(E) | 1, 2 \rangle = \langle 1', 2'|K_{NN} | 1, 2 \rangle$$

$$\Rightarrow \langle 1', 2'|K_{NN}N(E + i0 - K_F)^{-1}TN_N(E) | 1, 2 \rangle \quad (35)$$

with $K_{NN} = K(NN \rightarrow NN)$.

Actually, let us employ the relation,

$$G_F(z)T_{cloth}(z) = G(z)K_I \quad (36)$$

with the two resolvents

$$G_F(z) = (z - K_F)^{-1} \quad G(z) = (z - K)^{-1}$$

In its turn, Eq. (35) can be rewritten as

$$iG_F(z)T_{cloth}(z) = \int_0^\infty d\tau e^{i(-K)\tau}K_I \quad (37)$$

To the approximation in question, where

$$K \simeq K_F + K_I^{(2)} \equiv K_2 + K_4 \quad (38)$$

with

$$K_2 = K_2(\text{ferm}) + K_2(\text{mes}),$$

$$K_2(\text{ferm}) \sim b_i^\dagger b_i + d_i^\dagger d_i, \quad K_2(\text{mes}) \sim a_i^\dagger a_i$$

and

$$K_4 \sim b_i^\dagger b_j^\dagger b_j b_i + d_i^\dagger d_j^\dagger d_j d_i + b_i^\dagger b_j^\dagger a_i a_j + a_i^\dagger a_j^\dagger a_j a_i + H.c.$$.

we have

$$K_I b_i^\dagger b_j^\dagger | \Omega \rangle \simeq K_4 b_i^\dagger b_j^\dagger | \Omega \rangle = K_2 N b_i^\dagger b_j^\dagger | \Omega \rangle,$$

$$K_4 N = b_i^\dagger b_j^\dagger b_i b_j + H.c.$$.

so

$$e^{i(z-K)\tau} K_I b_i^\dagger b_j^\dagger | \Omega \rangle \simeq e^{i(z-K+K_4)\tau} K_2 N b_i^\dagger b_j^\dagger | \Omega \rangle.$$

But

$$e^{-i(K_2 + K_4)\tau} K_4 N b_i^\dagger b_j^\dagger | \Omega \rangle = \left[ 1 - i(K_2 + K_4)\tau \right.$$

$$\left. + \frac{\tau^2}{2!} (K_2 + K_4)^2 \tau^2 + \ldots \right] K_4 N b_i^\dagger b_j^\dagger | \Omega \rangle$$

with

$$(K_2 + K_4)K_2 N b_i^\dagger b_j^\dagger | \Omega \rangle = (K_2 N + K_4 N)K_4 N b_i^\dagger b_j^\dagger | \Omega \rangle,$$
\[(K_2 + K_3)^2 K_2^N b_i^1 b_j^1 |\Omega\rangle = (K_2^N + K_3^N)^2 K_3 b_i^1 b_j^1 |\Omega\rangle\]

and so on. Here \(K_2^N \sim b_i^1 b_j^1\).

Thus, to the approximation \(38\)

\[G_F(z) T(z) b_i^1 b_j^1 |\Omega\rangle = (z - K_2^N - K_3^N)^{-1} K_3 b_i^1 b_j^1 |\Omega\rangle\]

or

\[T_{NN}(z)|1,2\rangle = K_3^N|1,2\rangle + K_2^N(z - K_2^N)^{-1} T_{NN}(z)|1,2\rangle.\]  

(39)

Sometimes it is convenient to use the notation \(|1,2\rangle = b_i^1 b_j^1 |\Omega\rangle\) for any two nucleon state. Equation \(35\) follows from \(39\) if we take into account the completeness condition

\[\sum_{NN} |NN\rangle \langle NN| = 2\]  

and put \(z = E + i\delta\). Here the symbol \(\sum_{NN}\) means the summation over nucleon polarizations and the integration over nucleon momenta.

### 3.2 The \(R\)-matrix equation and its angular–momentum decomposition. The phase–shift relations

For practical applications, in order to get rid of some discomfort in handling the singularity of the resolvent \((E + i\delta - K_2^N)^{-1}\), one prefers to work with the \(R\)-matrix which is related to the \(T\)-matrix by the Heitler equation (see, e.g., Sect. 6 of Chapter V in the monograph \[22\]):

\[T(E) = R(E) - i\pi R(E)\delta(E - K_2^N) T(E)\]  

(41)

Thus, for \(R(E)\) in our case we obtain

\[R_{NN}(E)|1,2\rangle = K_{NN}|1,2\rangle + K_{NN} \frac{P}{E - K_2^N} R_{NN}(E)|1,2\rangle,\]  

(42)

where \(P\) denotes the principal value (p.v.) to be applied when the integration over the continuous spectrum of the operator

\[K_2^N = \sum_{\mu} \int d^3p E_p b_i^\mu (p,\mu) b_i^\mu (p,\mu)\]

is performed.

\[\text{After this, one can write}\]

\[\langle 1'2' | \tilde{R}(E) | 12\rangle = \langle 1'2' | \tilde{K}_{NN} | 12\rangle + \sum_{34} \langle 1'2' | \tilde{K}_{NN} | 34\rangle \frac{\langle 34 | \tilde{R}(E) | 12\rangle}{E - E_3 - E_4}\]  

(43)

with \(\tilde{R}(E) = R(E)/2\) and \(\tilde{K}_{NN} = K_{NN}/2\), where the operation \(\sum_{34}\) involves the p.v. integration.

Certainly, the integral equation \(43\) has much in common with the two–body \(R\)-matrix equation in NQM. It is true, unlike the latter, in our case the center–of–mass motion is not separated from the internal motion that is typical of relativistic theories of interacting particles. The kernel of Eq. \(43\) is

\[\langle 1'2' | \tilde{K}_{NN} | 12\rangle = \delta (p'_1 + p'_2 - p_1 - p_2) \langle 1'2' | \tilde{V} | 12\rangle \]

\[\equiv \delta (p'_1 + p'_2 - p_1 - p_2) \times \langle p'_1 p'_2 | \tilde{\tau}_1, p_2 p_2 | \tilde{\tau}_2 \rangle \]

that provides the total momentum conservation in every intermediate state. The subsequent calculations are essentially simplified in the center–of–mass system (c.m.s), in which we will employ the notations

\[|p_{\mu_1\mu_2, \tau_1\tau_2}\rangle = |p_{\mu_1\tau_1}, -p_{\mu_2\tau_2}\rangle,\]

\[|p'_1 p'_2, \tau'_1 \tau'_2\rangle = |p'_1 \tau'_1, -p'_2 \tau'_2\rangle\]

and

\[|q_{\mu_3\mu_4, \tau_3\tau_4}\rangle = |q_{\mu_3\tau_3}, -q_{\mu_4\tau_4}\rangle,\]  

respectively, for the initial, final and intermediate states. Here the quantum numbers \(\mu (\tau)\) are the individual spin (isospin) projections.

Using these notations Eq. \(43\) in the c.m.s. can be written as

\[\langle 1'2' | \tilde{V} | 12\rangle = \frac{1}{2(2\pi)^3} \frac{m^2}{E_p E_p'} \]

\[\times \sum_{b} [v_b^{dir}(1',2';1,2) - v_b^{exc}(1',2';2,1)]\]  

(44)

with

\[v_b^{dir}(1',2';1,2) = v_b(1',2';1,2) + v_b(2',1';2,1)\]

(46)
and
\[ v_b^{ex}(1',2',2,1) = v_b^{dir}(2',1';1,2), \]
where the separate boson contributions are determined by Eqs. (20–22) with \( p_1 = -p_2 \) and \( p'_1 = p'_2 = -p_2'. \)

Following a common practice we are interested in the angular momentum decomposition of Eq.(44) assuming a nonrelativistic analog of relativistic partial-wave expansions (see [23] and refs. therein) for two–particle states. For example, the clothed two–nucleon state (the so-called two–nucleon plane wave) can be represented as
\[ |p_\mu \mu_2, 7\tau_2\rangle = \sum \left( \frac{1}{2} \gamma_{\mu_1} \frac{1}{2} \gamma_2 |SM_S\rangle \right) \left( \frac{1}{2} \gamma_{\tau_1} \frac{1}{2} \gamma_{\tau_2} |TM_T\rangle \right), \]
where, as shown in Appendix B, \( J, S \) and \( T \) are, respectively, total angular momentum, spin and isospin of the \( NN \) pair. All necessary summations over dummy quantum numbers are implied.

Of course, when deriving expansions like (44) (see Appendix B) one needs to take into account a distinctive feature of any QFT associated with particle production and annihilation. The use of such expansions (see [23] and refs. therein) for two–particle states. For example, the clothed two–nucleon state (the so-called two–nucleon plane wave) can be represented as
\[ \langle p'|J'(l'S')M'_J,T'M'_T|V|pJ(lS)M_J,TM_T\rangle = \tilde{V}_{iJ}^{JST}(p',p)d_{J'J}d_{M'_J M_J}d_{S'S}d_{T'T}d_{M'_T M_T}, \]
so Eq.(44) reduces to the set of simple integral equations,
\[ \tilde{R}_{iJ}^{JST}(p',p) = \tilde{V}_{iJ}^{JST}(p',p) + \sum_{p''} P \int_0^\infty \frac{q^2 dq}{2(E_p - E_q)} \tilde{V}_{iJ}^{JST}(p',q)\tilde{R}_{iJ}^{JST}(q,p), \]
to be solved for each submatrix \( R_{JST} \) composed of the elements
\[ {R}_{iJ}^{JST}(p',p) = \tilde{R}_{iJ}^{JST}(p',p;2E_p), \]
where \( 2E_p = 2\sqrt{\mathbf{p}^2 + m^2} \) represents the collision energy in the c.m.s. As usually, we distinguish the on–energy–shell \( (p' = p) \) and half–off–energy–shell \( (p' \neq p) \) elements. Apparently, it is pertinent to stress that even if we are interested in the \( R \)–matrix on the energy shell one has inevitably to go out beyond it.

In addition, one should note that in view of the charge independence assumed in this work one has to solve two separate equations for isospin values \( T = 0 \) and \( T = 1 \). Regarding the \( T \)–dependence of the corresponding solutions in details one needs to keep in mind the typical factor \( [1 - (-1)^{l+S+T}] \) that is discussed in Appendix C (below Eq.(C.24)). It means such a selection of the partial \( R \)–equations, where the number \( l + S + T \) must be odd.

Properties (48) and (49) are resulted from rotational invariance, parity and isospin conservation in combination with the antisymmetry requirement for two–nucleon states. At the point, let us recall that the clothing procedure violates no one of these symmetries (details can be found in survey [10]). In particular, it implies that
\[ \| J_F(\alpha,\tau)K_{l}(\alpha) \| = \| J_{ferm},K(NN \rightarrow NN) = 0, \]
\[ \| P_F(\alpha,\tau)K_{l}(\alpha) \| = \| P_{ferm},K(NN \rightarrow NN) \| = 0 \]
so
\[ \| J_F(\alpha,\tau)S(\alpha) \| = \| P_F(\alpha,\tau)S(\alpha) \| = 0, \]
where \( J_F(\alpha,\tau) = J \) the operator of total angular momentum, \( P_F(\alpha,\tau) = P \) the operator of space inversion. As in Appendix B, we distinguish their fermionic parts \( J_{ferm} \) and \( P_{ferm} \) determined by Eqs. (48) and (49).

Further, apart from the transition with \( l = l' = J \) and \( l(l') = J \pm 1 \) all different ones are forbidden owing to the parity conservation. Therefore, for each \( JST \)–channel there are only six independent elements:
\[ R_{J=0}^{J'}(p',p), R_{J=1}^{J'}(p',p), R_{J=1T}^{J+1}(p',p), R_{J=1T}^{J-1}(p',p), \]
\[ R_{J=1T}^{J+1}(p',p), R_{J=1T}^{J-1}(p',p), R_{J=1T}^{J+1}(p',p). \]

The first two of them satisfy the equations:
for spin singlet
\[ R_{J=0}^{J'=0T}(p',p) = V_{J=0}^{J'=0T}(p',p) + \frac{1}{2} P \int_0^\infty \frac{q^2 dq}{E_p - E_q} V_{J=0}^{J'=0T}(p',q)R_{J=0}^{J'=0T}(q,p), \]
for spin triplet
\[ R_{J=0}^{J'=0T}(p',p) = \frac{1}{2} \int_0^\infty \frac{q^2 dq}{E_p - E_q} V_{J=0}^{J'=0T}(p',q)R_{J=0}^{J'=0T}(q,p), \]
\[ R_{J=1}^{J'=1T}(p',p) = V_{J=1}^{J'=1T}(p',p) + \frac{1}{2} P \int_0^\infty \frac{q^2 dq}{E_p - E_q} V_{J=1}^{J'=1T}(p',q)R_{J=1}^{J'=1T}(q,p), \]
for spin triplet
\[ R_{J=0}^{J'=0T}(p',p) = V_{J=0}^{J'=0T}(p',p) + \frac{1}{2} P \int_0^\infty \frac{q^2 dq}{E_p - E_q} V_{J=0}^{J'=0T}(p',q)R_{J=0}^{J'=0T}(q,p), \]
for spin singlet
\[ R_{J=1}^{J'=1T}(p',p) = V_{J=1}^{J'=1T}(p',p) + \frac{1}{2} P \int_0^\infty \frac{q^2 dq}{E_p - E_q} V_{J=1}^{J'=1T}(p',q)R_{J=1}^{J'=1T}(q,p), \]
for spin triplet
\[ R_{J=0}^{J'=0T}(p',p) = V_{J=0}^{J'=0T}(p',p) + \frac{1}{2} P \int_0^\infty \frac{q^2 dq}{E_p - E_q} V_{J=0}^{J'=0T}(p',q)R_{J=0}^{J'=0T}(q,p), \]
and for uncoupled spin triplet

\[ R_{jj}^{JS=IT}(p', p) = V_{jj}^{JS=IT}(p', p) + \frac{1}{2} \int_0^\infty \frac{q^2 dq}{E_p - E_q} V_{jj}^{JS=IT}(p', q) R_{jj}^{JS=IT}(q, p) \]

while for the rest one needs to solve the set of four coupled integral equations

\[ R_{j-1, j-1}^{JS=IT}(p', p) = V_{j-1, j-1}^{JS=IT}(p', p) + \frac{P}{2} \int \frac{q^2 dq}{E_p - E_q} [V_{j-1, j-1}^{JS=IT}(p', p) R_{j-1, j-1}^{JS=IT}(p', p) + R_{j-1, j-1}^{JS=IT}(p', p) R_{j-1, j-1}^{JS=IT}(q, p)] \]

\[ R_{j+1, j+1}^{JS=IT}(p', p) = V_{j+1, j+1}^{JS=IT}(p', p) + \frac{P}{2} \int \frac{q^2 dq}{E_p - E_q} [V_{j+1, j+1}^{JS=IT}(p', p) R_{j+1, j+1}^{JS=IT}(p', p) + R_{j+1, j+1}^{JS=IT}(p', p) R_{j+1, j+1}^{JS=IT}(q, p)] \]

\[ R_{j+1, j+1}^{JS=IT}(p', p) = V_{j+1, j+1}^{JS=IT}(p', p) + \frac{P}{2} \int \frac{q^2 dq}{E_p - E_q} [V_{j+1, j+1}^{JS=IT}(p', p) R_{j+1, j+1}^{JS=IT}(p', p) + R_{j+1, j+1}^{JS=IT}(p', p) R_{j+1, j+1}^{JS=IT}(q, p)] \]

\[ R_{j+1, j+1}^{JS=IT}(p', p) = V_{j+1, j+1}^{JS=IT}(p', p) + \frac{P}{2} \int \frac{q^2 dq}{E_p - E_q} [V_{j+1, j+1}^{JS=IT}(p', p) R_{j+1, j+1}^{JS=IT}(p', p) + R_{j+1, j+1}^{JS=IT}(p', p) R_{j+1, j+1}^{JS=IT}(q, p)] \]

One should stress that after omitting the factor \([1 - (-1)^{JST}]\) the matrix elements \(V_{ij}^{ST}\) in these equations are given by the direct parts of our energy independent quasipotentials, which are derived in Appendix C.

The Heitler equation (11) enables one to get the partial T–matrix elements

\[ T_{ij}^{JST}(p', p) = V_{ij}^{JST}(p', p) - i\pi \rho(p) \sum_{i'} R_{i'i}^{JST}(p', p) T_{i'i}^{JST}(p, p) \]

whence it follows the on-shell relationship:

\[ T_{ij}^{JST}(p) = R_{ij}^{JST}(p) - i\pi \rho(p) \sum_{i'} R_{i'i}^{JST}(p) T_{i'i}^{JST}(p) \]

where

\[ T(R)_{ij}^{JST}(p) \equiv T(R)_{ij}^{JST}(p, p; E = 2E_p) \]

and \(\rho(p) = pE_p/2\).

In turn, with the help of a standard definition of the on–shell S–matrix elements

\[ S_{jj}^{JST}(p) \equiv \exp[2i\delta^J_{ST}] = 1 - 2\pi i\rho(p) T_{jj}^{JST}(p) \]

the R–matrix elements for the uncoupled states can be expressed through the phase shifts \(\delta^J_{ST}\):

\[ -\pi \rho(p) R_{jj}^{JST}(p) = \tan \delta^J_{ST} \]

Usually the isospin label is suppressed to write simply \(\delta^J\). Of course, these quantities depend either on the incoming momentum \(p\) or the laboratory energy \(E_{lab}\).

We choose the second alternative.

For the coupled states the on–shell R–matrix elements are conventionally parameterized in terms of the phase shifts \(\delta^J_{\pm}\) and the mixing parameters \(\varepsilon_J\):

\[ -\pi \rho(p) \left( \begin{array}{c} R_{j+1, j-1}^{J1} \quad R_{j-1, j+1}^{J1} \\ R_{j-1, j+1}^{J1} \quad R_{j-1, j+1}^{J1} \end{array} \right) \left( \begin{array}{c} \cos \varepsilon_J \quad -\sin \varepsilon_J \\ \sin \varepsilon_J \quad \cos \varepsilon_J \end{array} \right) \left( \begin{array}{c} \cos \delta^J_{\pm} \quad \sin \delta^J_{\pm} \\ \sin \delta^J_{\pm} \quad -\cos \delta^J_{\pm} \end{array} \right) \]

This a parametrization was put forward in [23]. From Eq. (22) it follows that

\[ \tan \delta^J_{\pm} = -\frac{1}{2}\pi \rho(p) \left( \begin{array}{c} R_{j+1, j-1}^{J1} + R_{j-1, j+1}^{J1} \\ \frac{R_{j+1, j+1}^{J1} + R_{j-1, j+1}^{J1}}{\cos \varepsilon_J} \end{array} \right) \]

and

\[ \tan 2\varepsilon_J = \frac{R_{j+1, j+1}^{J1} + R_{j-1, j-1}^{J1}}{\cos \varepsilon_J} - R_{j-1, j+1}^{J1} \]

Here in \(R_{ij}^{J1}(p)\) the isospin index and the argument \(p\) are suppressed too.

However, in the next section our calculations are shown for the so–called bar convention introduced in [29]. These are related to the Blatt–Biedenharn phase shifts by

\[ \delta^J_+ + \delta^J_- = \delta^J_+ + \delta^J_-, \quad \sin(\delta^J_+ - \delta^J_-) = \tan 2\varepsilon_J, \quad \tan 2\varepsilon_J = \frac{\delta^J_+ - \delta^J_-}{\sin 2\varepsilon_J} \]

(cf. Eq. (132) from [2]). It will allow us to compare our results directly with those by the Bonn group (in particular, from the survey [13]).
4 Results of numerical calculations and their discussion

Available experience of solving integral equations similar to Eqs. (55–57) shows that it is convenient to employ the so-called matrix inversion method (MIM) [30, 31] (more sophisticated methods are discussed in the monograph [32]). In the course of our numerical calculations we have improved a code [33] based upon the MIM and successfully applied for the treatment of the final-state interaction in studies [34], [35] of the deuteron breakup by electrons and protons in the GeV region.

Since we deal with the relativistic dispersion law for the particle energies, the well known subtraction procedure within the MIM in our case leads to equations

\[ R_{i i}^{JST}(p', p) = V_{i i}^{JST}(p', p) + \frac{1}{2} \sum_{\nu} \int \frac{dq}{p^2 - q^2} \left( g^2 (E_p + E_q) V_{i i}^{JST}(p', q) R_{i i}^{JST}(q, p) - 2p^2 E_p V_{i i}^{JST}(p', p) R_{i i}^{JST}(p, p) \right). \]  

(66)

To facilitate comparison with some derivations and calculations from Refs. [2, 13], we introduce the notation

\[
(p'\mu_1\mu_2|v_{0}^{UCT}|p\mu_1\mu_2) = -F_0^2(p', p)[v_0(1', 2', 1, 2) + v_0(2', 1', 1, 2)]
\]

for the regularized UCT quasipotentials in the c.m.s.

(see Appendix C). As in Ref.[2], we put

\[
F_b(p', p) = \left[ \frac{A_0^2 - m_0^2}{A_0^2 - (p' - p)^2} \right]^{n_b} \equiv F_b[(p' - p)^2]
\]

Doingso, we have

\[
(p'\mu_1\mu_2|v_{0}^{UCT}|p\mu_1\mu_2) = g_0^2 \bar{u}(p')u(p) \frac{F_0^2[(p' - p)^2]}{2m_0}\bar{u}(-p')u(-p),
\]

(67)

and

\[
(p'\mu_1\mu_2|v_{ps}^{UCT}|p\mu_1\mu_2) = -g_{ps}^2 \bar{u}(p')\gamma_5 u(p) \frac{F_{ps}^2[(p' - p)^2]}{2m_{ps}}\bar{u}(-p')\gamma_5 u(-p)
\]

(68)

Table 1 The best-fit parameters for the two models. The third (fourth) column taken from Table A.1 [33] (obtained by solving Eqs. (66) with a least squares fitting [36, 37] to OBEP values in Table 2). All masses are in MeV, and $n_0 = 1$ except for $n_\rho = n_\omega = 2$.

| Meson | Potential B | UCT |
|-------|-------------|-----|
| $\pi$ | 14.4        | 14.5 | 138.03 |
| $\rho$ | 1700        | 2200 | 1538.13 |
| $\omega$ | 138.03 | 138.03 | 548.8 |
| $\delta$ | 1850 | 1450 | 717.72 |
| $\sigma$, $T = 0$ | 18.3773 | 19.4434 | 2035.59 |
| $\sigma$, $T = 1$ | 8.9437 | 10.8292 | 568.86 |

At first sight, such a regularization can be achieved via a simple substitution $g_0 \rightarrow g_0 F_b(p', p)$ with some cutoff functions $F_b(p', p)$ depending on the 4-momenta $p'$ and $p$. 
Table 2: Neutron–proton phase shifts (in degrees) for various laboratory energies (in MeV). The OBEP(OBEP*)–rows taken from Table 5.2 [13] (calculated by solving Eqs.(71) with the model parameters from the third column in Table 1). The UCT(UCT*)–rows calculated by solving Eqs.(66) with the parameters from the third (fourth) column in Table 1. As in [2], we have used the bar convention [29] for the phase parameters.

| State | Potential | 25   | 50   | 100  | 150  | 200  | 300  |
|-------|-----------|------|------|------|------|------|------|
| $^1S_0$ | OBEP      | 50.72| 39.98| 25.19| 14.38| 5.66 | -8.18|
|       | OBEP*     | 50.71| 39.98| 25.19| 14.37| 5.66 | -8.18|
|       | UCT       | 66.79| 53.01| 36.50| 25.27| 16.54| 3.12 |
|       | UCT*      | 50.03| 39.77| 25.55| 15.20| 6.92 | -6.07|
| $^1P_1$ | OBEP      | -7.21| -11.15| -16.31| -20.21| -23.47| -28.70|
|       | OBEP*     | -7.17| -11.15| -16.32| -20.21| -23.48| -28.71|
|       | UCT       | -7.40| -11.70| -17.73| -22.63| -26.98| -34.54|
|       | UCT*      | -7.15| -10.95| -15.62| -19.90| -21.49| -25.41|
| $^1D_2$ | OBEP      | 0.68 | 1.58 | 3.34 | 4.94 | 6.21 | 7.49 |
|       | OBEP*     | 0.68 | 1.58 | 3.34 | 4.94 | 6.21 | 7.49 |
|       | UCT       | 0.68 | 1.56 | 3.22 | 4.68 | 5.77 | 6.68 |
|       | UCT*      | 0.68 | 1.59 | 3.40 | 5.10 | 6.52 | 8.20 |
| $^3P_0$ | OBEP      | 9.34 | 12.24| 9.80 | 4.57 | -1.02| -11.48|
|       | OBEP*     | 9.34 | 12.24| 9.80 | 4.57 | -1.02| -11.48|
|       | UCT       | 9.48 | 12.53| 10.32| 5.27 | -0.15| -10.27|
|       | UCT*      | 9.30 | 12.16| 9.81 | 4.73 | -0.68| -10.76|
| $^3P_1$ | OBEP      | -5.33| -8.77| -13.47| -17.18| -20.49| -26.38|
|       | OBEP*     | -5.33| -8.77| -13.47| -17.18| -20.48| -26.38|
|       | UCT       | -5.27| -8.62| -13.09| -16.56| -19.63| -25.06|
|       | UCT*      | -5.28| -8.58| -12.85| -16.06| -18.86| -23.79|
| $^3D_2$ | OBEP      | 3.88 | 9.29 | 17.67| 22.57| 24.94| 25.36|
|       | OBEP*     | 3.89 | 9.29 | 17.67| 22.57| 24.94| 25.36|
|       | UCT       | 3.86 | 9.15 | 17.12| 21.51| 23.47| 23.48|
|       | UCT*      | 3.89 | 9.25 | 17.31| 21.77| 23.75| 23.61|
| $^3S_1$ | OBEP      | 80.32| 62.16| 41.99| 28.94| 19.04| 4.07 |
|       | OBEP*     | 80.31| 62.15| 41.98| 28.93| 19.03| 4.06 |
|       | UCT       | 92.30| 72.71| 51.44| 38.10| 28.20| 13.70|
|       | UCT*      | 79.60| 61.53| 41.57| 28.75| 19.08| 4.60 |
| $^3D_1$ | OBEP      | -2.99| -6.86| -12.98| -17.28| -20.28| -23.72|
|       | OBEP*     | -2.99| -6.87| -12.99| -17.28| -20.29| -23.72|
|       | UCT       | -2.74| -6.43| -12.36| -16.54| -19.47| -22.78|
|       | UCT*      | -3.00| -6.90| -13.12| -17.66| -21.11| -26.03|
| $^1\epsilon_1$ | OBEP | 1.76 | 2.00 | 2.24 | 2.58 | 3.03 | 4.03 |
|       | OBEP*     | 1.76 | 2.00 | 2.24 | 2.58 | 3.03 | 4.03 |
|       | UCT       | 0.02 | -0.12| -0.17| 0.04 | 0.41 | 1.40 |
|       | UCT*      | 1.80 | 2.01 | 2.19 | 2.50 | 2.90 | 3.83 |
| $^3P_2$ | OBEP      | 2.62 | 6.14 | 11.73| 14.99| 16.65| 17.40|
|       | OBEP*     | 2.62 | 6.14 | 11.73| 14.99| 16.65| 17.39|
|       | UCT       | 2.80 | 6.61 | 12.71| 16.28| 18.10| 18.91|
|       | UCT*      | 2.57 | 6.00 | 11.32| 14.18| 15.37| 15.07|
| $^3F_2$ | OBEP      | 0.11 | 0.34 | 0.77 | 1.04 | 1.10 | 0.52 |
|       | OBEP*     | 0.11 | 0.34 | 0.77 | 1.04 | 1.10 | 0.52 |
|       | UCT       | 0.11 | 0.34 | 0.77 | 1.05 | 1.13 | 0.64 |
|       | UCT*      | 0.11 | 0.34 | 0.75 | 1.00 | 1.03 | 0.41 |
| $^1\epsilon_2$ | OBEP | -0.86 | -1.82 | -2.84 | -3.05 | -2.85 | -2.02 |
|       | OBEP*     | -0.86 | -1.82 | -2.84 | -3.05 | -2.85 | -2.02 |
|       | UCT       | -0.87 | -1.83 | -2.82 | -2.99 | -2.75 | -1.88 |
|       | UCT*      | -0.86 | -1.83 | -2.84 | -3.05 | -2.89 | -2.18 |
Fig. 3  Neutron-proton phase parameters for the uncoupled partial waves, plotted versus the nucleon kinetic energy in the lab. system. Dashed[solid] curves calculated with Potential B parameters (Table 1) by solving Eqs. (66)[(71)]. Dotted represent the solutions of Eqs. (66) with UCT parameters (Table 1). The rhombs show original OBEP results (see Table 2).

However, a principal moment is to satisfy the requirement (86) for the Hamiltonian invariant under space inversion, time reversal and charge conjugation. A constructive consideration of the issue is given in Appendix C.

Replacing in equations (67)-(69)

\[ F_b^2 [(p' - p)^2] \left\{ (p' - p)^2 - m_b^2 \right\}^{-1} \]

by

\[ -F_b^2 [-(p' - p)^2] \left\{ (p' - p)^2 + m_b^2 \right\}^{-1} \]

and neglecting the tensor-tensor term

\[ \frac{f^2}{4m^2} (E_{p'} - E_p)^2 \bar{u}(p') [\gamma_0 \gamma_\nu - g_{\nu\nu}] u(p) \]

\[ \times \bar{u}(-p') [\gamma_0 \gamma_\nu - g^\nu_\nu] u(-p) \]  (70)
in (69), we obtain approximate expressions that with the common factor
\[(2\pi)^{-3}m^2 / E'_p E_p\]
instead of
\[(2\pi)^{-3}m / \sqrt{E'_p E_p}\]
are equivalent to Eqs.(E.21)–(E.23) from [2]. Such an equivalence becomes coincidence if in our formulae instead of the canonical two-nucleon basis \(|p_{\mu_1\mu_2}\rangle\) one uses the helicity basis as in [2].

In parallel, we have considered the set of equations

\[
B_{R_{ij}^{IST}}(p', p) = B_{V_{ij}^{IST}}(p', p)
\]

\[
+ m \sum_{\nu'} \int_0^\infty \frac{dq}{p'^2 - q^2} \{ q^2 B_{V_{ij}^{IST}}(p', q) B_{R_{ij}^{IST}}(q, p) - p'^2 B_{V_{ij}^{IST}}(p', p) B_{R_{ij}^{IST}}(p, p) \}, \quad (71)
\]
Fig. 5  Half–off–shell $R$–matrices for uncoupled waves at laboratory energy equal to 150 MeV($p_0$=265 MeV). Other notations as in Fig.3

where the superscript $B$ refers to the partial matrix elements of the potential $B$ determined in [13] with the just mentioned interchange of the bases. It is important to note that Eqs. (71) can be obtained from Eqs. (66) ignoring some relativistic effects. In particular, it means that the covariant OBE propagators

$$\frac{1}{(p' - p)^2 - m_b^2} = \frac{1}{(E_{p'} - E_p)^2 - (p' - p)^2 - m_b^2}$$  \hspace{1cm} (72)

are replaced by their nonrelativistic counterparts

$$\frac{1}{(p' - p)^2 + m_b^2}$$  \hspace{1cm} (73)

Such an approximation,\footnote{Sometimes associated with ignoring the so-called meson retardation (see, e.g., Appendix E from [12] and a discussion therein)} is a key point that gives rise to the potential $B$ from [13].
However, the transition from (72) to (73) being valid on the energy shell

\[ E_{p'} = \frac{1}{2} E_{\text{cms}}, \]

cannot be \textit{a priori} justified when finding the \( R \) matrix even on the shell. Thus our calculations of the \( R \) matrices that meet the equations (66) and (71) are twofold. On the one hand, we will check reliability of our numerical procedure (in particular, its code). On the other hand, we would like to show similarities and discrepancies between our results and those by the Bonn group both on the energy shell and beyond it. These results are depicted in Figs. 3–8 and collected in Table 2.

As seen in Figs. 3–4, the most appreciable distinctions between the UCT and OBEP curves take place for the phase shifts with the lowest \( l \)–values. As the orbital angular momentum increases the difference between the solid and dashed curves decreases. Such features may be explained if one takes into account that...
the approximations under consideration affect mainly high–momentum components of the UCT quasipotentials (their behavior at "small" distances). With the $l$–increase the influence of small distances is suppressed by the centrifugal barrier repulsion.

Of course, it would be more instructive to compare the corresponding half–off–energy–shell $R$–matrices (see definition (51)). Their $p'$–dependencies shown in Fig.5–6 are necessary to know when calculating the $\psi^{(\pm)}$ scattering states for a two–nucleon system. Such states may be expressed through the partial–wave functions $\varphi_{l'j}^{(a)}$, $a = J, S, T$ that have the asymptotic of standing waves (see, for example, [34]). Within the MIM every $\varphi$ can be represented as

$$\varphi_{l'j}^{(a)}(p) = \sum_{j=1}^{N+1} B_{l'j}^{(a)}(j) \delta(p - p_j)/p_j^2,$$

where the coefficients $B_{l'j}^{(a)}(j)$ are the solutions of the set of linear algebraic equations approximately equiva-
lent to the integral equation for the corresponding $R$–matrix; $N$ is the dimension of the set, $p_j$ are the grid points associated with the Gaussian nodes on the interval $[-1, 1]$, $p_{N+1} = p_0$ (details can be found in [34, 35]). Meanwhile, our computations presented here have been done with $N = 32$ (we do not talk about tests with other $N$-values to get the results stable with respect to $N$).

Completing the comparison of the UCT and Bonn results we display in Figs. 7–8 some interactions in relevant partial states. Appreciable distinctions between dotted and solid curves in these figures mean that the UCT and Bonn parameters from Table 1, ensuring a fair treatment of such on–energy–shell quantities as the phase shifts, may be inadequate in constructing model nucleon–nucleon potentials. It seems to be especially prominent in case of the $^3S_1 - ^3D_1$ potentials responsible for the formation of the tensor part of nuclear forces.

Fig. 8 The same as in Fig.7 for the coupled waves.
In addition, one should emphasize that hitherto we have explored the OBEP and UCT $R$–matrices in the c.m.s., where the both approaches yield the most close results. It is not the case in those situations when the c.m.s. cannot be referred to everywhere (e.g., in the reactions $NN \rightarrow \gamma NN$ and $\gamma d \rightarrow pd$). In this respect our studies of the differences between UCT and OBE approaches are under way.

5 Summary, Conclusions and Prospects

The present work has been made to develop a consistent field-theoretical approach in the theory of nucleon–nucleon scattering. It has been shown that the method of UCT’s, based upon the notion of clothed particles, is proved to be appropriate in achieving this purpose.

Starting from a primary Lagrangian for interacting meson and nucleon fields, we come to the corresponding Hamiltonian whose interaction part $K_I$ consists of new relativistic interactions responsible for physical (not virtual) processes in the system of the bosons ($\pi$, $\eta$, $\rho$, $\omega$, $\delta$, and $\sigma$–mesons) and the nucleon. Proceeding with the CPR we have confined ourselves to constructing the four-legs interaction operators $K_I^{(2)}$ in the two-nucleon sector of the Hilbert space of hadronic states. The corresponding quasipotentials (these essentially non-local objects) for binary processes $NN \rightarrow NN$, $NN \rightarrow N\bar{N}$, $NN \rightarrow NN$, etc. are Hermitian and energy–independent. It makes them attractive for various applications in nuclear physics. They embody the off–shell effects in a natural way without addressing to any off–shell extrapolations of the $S$–matrix for the $NN$ scattering.

Using the unitary equivalence of the CPR to the BPR, we have shown in the approximation $K_I = K_I^{(2)}$ the extremely complicated scattering problem in QFT can be reduced to the three –dimensional $LS$–type equation for the $T$–matrix in momentum space. The equation kernel is given by the clothed two-nucleon interaction of the class [2.2]. A conversion becomes possible owing to the property of $K_I^{(2)}$ to leave the two–nucleon sector and its separate subsectors to be invariant.

Special attention has been paid to the elimination of auxiliary field components. We encounter such a necessity for interacting vector and fermion fields when in accordance with the canonical formalism the interaction Hamiltonian density embodies not only a scalar contribution but nonscalar terms too. It has proved (at least, for the primary $pN$ and $\omega N$ couplings) that the UCT method allows us to remove such noncovariant terms directly in the Hamiltonian. To what extent this result will take place in higher orders in coupling constants it will be a subject of further explorations.

Being concerned with constructing the two–nucleon states from $\mathcal{H}$ and their angular–momentum decomposition we have not used the so–called separable ansatz, where every such state is a direct product of the corresponding one–nucleon (particle) states. The clothed two–nucleon partial waves have been built up as common eigenstates of the field total angular–momentum generator and its polarization (fermionic) part expressed through the clothed creation/destruction operators and their derivatives in momentum space.

We have not tried to attain a global treatment of modern precision data. But a fair agreement with the earlier analysis by the Bonn group makes sure that our approach may be useful for a more advanced analysis. In the context, to have a more convincing argumentation one needs to do at least the following. First, show the low–energy scattering parameters and the deuteron wave function calculated within the UCT method. Second, consider triple commutators $[R, [R, [R, V_0]]]$ to extract the two–boson–two–nucleon interaction operators of the same class [2.2] in the fourth order in the coupling constants. Third, extend our approach for describing the $NN$ scattering above the pion production threshold. All the things are in progress.

As a whole, the persistent clouds of virtual particles are no longer explicitly contained in the CPR, and their influence is included in the properties of the clothed particles (these quasiparticles of the UCT method). In addition, we would like to stress that the problem of the mass and vertex renormalizations is intimately interwoven with constructing the interactions between the clothed nucleons. The renormalized quantities are calculated step by step in the course of the clothing procedure unlike some approaches, where they are introduced by ”hands”.

6 Acknowledgements

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A Model Lagrangians and Hamiltonians within Canonical Formalism. Transition to Clothed Particle Representation

We will focused upon three boson fields (pseudoscalar, scalar and vector) that are coupled with the nucleon by means of the often employed interaction Lagrangians,

$$\mathcal{L}_s = -g_s \bar{\psi} \gamma_5 \psi \Phi_s,$$  \hspace{1cm} (75)
\[ \mathcal{L}_{ps} = -ig^0_{ps} \bar{\psi} \gamma_0 \psi \phi_{ps}, \]  
\[ \mathcal{L}_\gamma = -g^0 \bar{\psi} \gamma_\mu \phi^\mu - \frac{\eta_0}{4m} \bar{\psi} \sigma_{\mu\nu} \phi^\mu \phi^\nu, \]  
(76)

where \( g^0_{ps} \) and \( g^0 \) are coupling constants, \( \eta_0 \) is a cutoff factor, and \( \phi^\mu \) is the field. As in Refs. (19, 22), throughout this paper we use the definitions and notations of (22, 23) and (19). In addition, following Ref. (19), we shall distinguish via upper (lower) case letters between the various fields and operators (\( \Psi \) and \( \Phi \) vs \( \psi \) and \( \varphi \), respectively, for fermions and bosons).

Of course, we could incorporate the so-called pseudovector (pv) coupling
\[ \mathcal{L}_{pv} = -f^0_{ps} \bar{\psi} \gamma_5 \gamma^\mu \psi \partial_\mu \phi_{ps}, \]  
(78)

Since in this paper all used model interactions are suggested to be modified by introducing some cutoff factors, it has no matter that couplings (76) and (78) with derivatives are nonrenormalizable (cf. an instructive discussion of this subject in Subsect. 3.4 of the survey (13). In the context, starting from couplings (76)–(78) with "bare" constants \( g^0_{ps} \), \( g^0 \), \( f^0 \), and \( \eta_0 \), we have tried to reproduce some results obtained in Refs. [2]. However, the paper does not explain how we restrict to these Lagrangian densities. Recall also that for isospin 1 bosons one needs to write \( \tau \Phi \) instead of \( \Phi \), where \( \tau \) is the Pauli vector in isospin space.

In constructing the Hamiltonians with Lagrangian densities (75)–(77) as a departure point, we have first used the equations of motion for the \( H \) fields and the so-called Legendre transformation (from \( L \) to \( H \)) to express the total Hamiltonian \( H \) in terms of the independent canonical variables and their conjugates. Then, passing to the \( L \) picture (interaction representation) the Hamiltonian has been split into a physically satisfactory free-field part \( H_0 \) and an interaction \( V \). Int. al., since the component \( \phi^0 \) has no canonical conjugate, we have resorted to a trick (by Eq. (7.5.22) from (12)) to introduce a proper component \( \varphi^0 \) in the Dirac picture. As a result, we arrive to the interaction Hamiltonian densities:

\[ \mathcal{H}_s(x) = g^0 \bar{\psi}(x) \gamma^\mu \psi(x) \varphi^\mu(x), \]  
(79)

\[ \mathcal{H}_{ps}(x) = ig^0_{ps} \bar{\psi}(x) \gamma_0 \psi(x) \varphi_{ps}(x), \]  
(80)

\[ \mathcal{H}_\gamma(x) = \mathcal{H}_{\text{cov}}(x) + \mathcal{H}_{\text{ncov}}, \]  
(81)

\[ \mathcal{H}_{\text{cov}}(x) = g^0 \bar{\psi}(x) \gamma^\mu \varphi^\mu(x) + \frac{\eta_0}{4m} \bar{\psi}(x) \sigma_{\mu\nu} \varphi^\mu \varphi^\nu(x), \]  
(82)

and

\[ \mathcal{H}_{\text{ncov}}(x) = \frac{g^0}{2m^2} \bar{\psi}(x) \gamma_0 \psi(x) \varphi_0(x) \]  
\[ + \frac{\eta_0}{4m^2} \bar{\psi}(x) \sigma_{0\nu} \psi(x) \varphi_0(x). \]  
(83)

It is implied that the total Hamiltonian of interest \( H = H_0 + V \) consists of the sum \( H_0 = H_{0,f} + \sum_b H_{0,b} \), where \( H_{0,b} \) describes the free-fermion Hamiltonian, \( H_{0,f} \) the free-boson contribution, and the space integral
\[ V = \int dO(x) \]  
(84)
of Eq. (90) depend on the clothed-particle creation/destruction operators \( \alpha_e = W^{(1)^*} \alpha W^{(1)} \), for example, involved in the Fourier expansions

\[
\psi(x) = (2\pi)^{-3/2} \int dp \frac{e^{ip \cdot x}}{E_p} \sum_{\mu} \left[ u(p_\mu) b_\mu(p) \right] + v(-p_\mu) d_\mu^*(p) \exp(ip \cdot x),
\]

\[
\varphi_s^c(x) = (2\pi)^{-3/2} \int \frac{dk}{2\omega_k} \sum_s \left[ \epsilon^s(k, s) a_s^c(k, s) + \epsilon^s(-k, s) a_s^c(-k, s) \right] \exp(ik \cdot x),
\]

where the \( \epsilon^s(k, s) \) for \( s = +1, 0, -1 \) are three independent vectors, being transverse \( k_\mu \epsilon^s(k, s) = 0 \), and normalized so that

\[
\sum_s \epsilon_\mu(k, s) \epsilon^s_\mu(k, s) = -g_{\mu\nu} + k_\mu k_\nu/m^2.
\]

In this paper we do not intend to derive all interactions between the clothed mesons and nucleons, allowed by formula (90). Our aim is more humble, viz., to find in the r.h.s. of Eq. (90) terms of the type (85), responsible for the N–N interaction. Along the guideline the commutators \( \frac{1}{2}[H^{(1)}, V^{(1)}] \) and \( \frac{1}{2}[H^{(2)}, V^{(1)}] \) generate the scalar- and pseudoscalar–meson contributions \( K_v(NN \to NN) \) with coefficients (95)–(97). In case of the vector mesons we encounter an interplay between the commutator \( \frac{1}{2}[H^{(1)}, V^{(1)}] \) and the integral (81).

To show it explicitly, let us write

\[
V^{(1)}_v = \frac{1}{(2\pi)^3/2} \int \frac{dk dp' dp}{\sqrt{2m E_{p'} E_p}} \delta(p' - p - k) \left\{ v_\mu, v_\nu - \frac{f_v}{2m} (\gamma_\mu k^\nu + \gamma_\nu k^\mu) \right\} u(p_\mu) \\
\times b_1^v(p_\mu) b_v(p) a_v(k, s) + \text{H.c.}, \tag{92}
\]

\[
K^{(1)}_v = \frac{1}{(2\pi)^3/2} \int \frac{dk dp' dp}{\sqrt{2m E_{p'} E_p}} \delta(p' - p - k) \left\{ v_\mu, v_\nu - \frac{f_v}{2m} (\gamma_\mu k^\nu + \gamma_\nu k^\mu) \right\} u(p_\mu) \\
\times b_1^v(p_\mu) b_v(p) a_v(k, s) - \text{H.c.}, \tag{93}
\]

retaining only those parts of \( V^{(1)} \) and \( R^{(1)} \), which are necessary for deriving \( K_v(NN \to NN) \). After a simple algebra we find

\[
\frac{1}{2} \left[ R^{(1)}, V^{(1)} \right]_v (NN \to NN) = K_v(NN \to NN) + K_{cont}(NN \to NN)
\]

with

\[
K_v(NN \to NN) = \frac{1}{2(2\pi)^3} \sum_{\mu} \int dp'_1 dp'_2 dp_1 dp_2 \\
\times \frac{m^2}{\sqrt{E_{p'_1} E_{p'_2} E_{p_1} E_{p_2}}} \left\{ \delta(p'_1 + p'_2 - p_1 - p_2) \\
\times \bar{u}(p_\mu) \left\{ \left( g_{\nu, \gamma_{p_\nu}} - \frac{f_v}{2m} \gamma_{\mu, k^\nu} \right) u(p_\mu) \right\} \\
\times b_1^v(p_\mu) b_v(p) a_v(k, s) + \text{H.c.}, \right.
\]

\[
K_{cont}(NN \to NN) = \frac{1}{2(2\pi)^3} \sum_{\mu} \int dp'_1 dp'_2 dp_1 dp_2 \\
\times \frac{m^2}{\sqrt{E_{p'_1} E_{p'_2} E_{p_1} E_{p_2}}} \left\{ \delta(p'_1 + p'_2 - p_1 - p_2) \\
\times \bar{u}(p_\mu) \left\{ \left( g_{\nu, \gamma_{p_\nu}} - \frac{f_v}{2m} \gamma_{\mu, k^\nu} \right) u(p_\mu) \right\} \\
\times b_1^v(p_\mu) b_v(p) a_v(k, s) + \text{H.c.}, \right. \tag{92}
\]

\[
\frac{1}{4} \left[ R^{(1)}, V^{(1)} \right]_v (NN \to NN) = K_v(NN \to NN) + K_{cont}(NN \to NN)
\]

B Partial Wave Expansion of Two–Particles States in the CPR

We will show how one can proceed without the separable ansatz

\[
| p_1 \mu_1 ; p_2 \mu_2 \rangle = | p_1 \mu_1 \rangle | p_2 \mu_2 \rangle
\]

often exploited in relativistic quantum mechanics (RQM) (see, e.g., (22) and (23)) in getting expansions similar to Eq. (47). Unlike this, our consideration with particle creation/destruction as a milestone, where the clothed two–nucleon state is given by

\[
| p_1 \mu_1 ; p_2 \mu_2 \rangle = b_1^v(p_1 \mu_1) b_v(p_2 \mu_2) \Omega \tag{94}
\]

\[
5 \text{ Sometimes it is convenient to handle the operators } b_1^v(p_1 \mu_1) \text{ and their adjoint ones } b_v(p_\mu) \text{ that meet covariant relations } b_1^v(p_\mu') b_v(p_\mu) = \rho \delta(p' - p) \delta_{\mu' \mu}.
\]
By definition, it belongs to the two-nucleon sector of \( \mathcal{N} \) being the 
\( K_F \) eigenstate with energy \( E = p_1^2 \mu + p_2^2 \mu \). Moreover, it is assumed 
that vector \( |\Omega\rangle \), being the single clothed no-particle state, has the property
\[
U_F (A,a) |\Omega\rangle = 2 |\Omega\rangle = \ldots = |\Omega\rangle
\]
(95)
\[
\forall A \in L_+ \text{ and arbitrary spacetime shifts } a = (a^0, a) \text{ to be invariant with regard to the Poincaré group } H \text{, space innversion and other symmetries. Here } L_+ \text{ is the homogeneous (proper) orthoschronous Lorentz group. In turn, every } U_F (A,a) \text{ is expressed through the clothed free-particle generators of spacetime translations}
\]
\[
P_F^\mu (\alpha_c) = \{ K_F (\alpha_c), P_F (\alpha_c) \},
\]
space rotations
\[
J_F (\alpha_c) = \{ M_F^2 (\alpha_c), M_F^3 (\alpha_c), M_F^4 (\alpha_c) \}
\]
and the Lorentz boosts
\[
B_F (\alpha_c) = \{ M_F^0 (\alpha_c), M_F^1 (\alpha_c), M_F^2 (\alpha_c), M_F^3 (\alpha_c) \}
\]
and the Lorentz boosts
\[
U_F (A,a) = \exp \left[ ia_\mu P_F^\mu (\alpha_c) + \frac{i}{2} \omega_{\mu \nu} M_F^\mu (\alpha_c) \right],
\]
(96)
where the antisymmetric tensor \( \omega_{\mu \nu} = -\omega_{\nu \mu} \) with \( (\mu, \nu) = 0, 1, 2, 3 \) has six independent components.

Now, let us work out the following transformation property:
\[
U_F (A,a) b_\mu (p) U_F^{-1} (A,a) = e^{i A p} u_\mu (A p) \Delta_{\mu \nu}^{(1/2)} (W (A,p))
\]
(97)
with the \( \Delta_{\mu \nu}^{(1/2)} \) function whose argument is the Wigner rotation
\( W (A,p) \). The latter has the property \( W (R,p) = R \) for any three-
dimensional rotation \( R \). In other words, from Eq. \( 27 \) it follows that
under such a rotation, when \( U_F (A,a) = U_F (R,0) \equiv U_F (R) \), one has
\[
U_F (R) b_\mu (p) U_F^{-1} (R) = b_\mu (R p) \Delta_{\mu \nu}^{(1/2)} (R)
\]
(98)
In addition, we need to have an analytic expression for the operator
\( J_F \) to be expressed in terms of \( b_\mu \) and \( c_\mu \). To do it we recur to the well known result:
\[
J_F = \int d^3 x \psi (x) \left[ -i \mathbf{x} \frac{\partial}{\partial \mathbf{x}} + \frac{1}{2} \Sigma \right] \psi (x)
\]
(99)
with \( \Sigma = \begin{pmatrix} \sigma & 0 \\ 0 & -\sigma \end{pmatrix} \), where \( \sigma \) the Pauli vector.

After modest effort one can see that
\[
J_F = L^I \text{term} + S^I \text{term},
\]
(100)
where \( L^I \text{term} \) (\( S^I \text{term} \)) the orbital (spin) momentum of the fermion field, that are given by
\[
L^I = \frac{i}{2} \sum_{\mu} \int d^3 p \mathbf{p} \times \left[ \frac{\partial b_\mu (p)}{\partial p} \frac{\partial c_\mu (p)}{\partial p} - \frac{\partial b_\mu (p)}{\partial p} \frac{\partial c_\mu (p)}{\partial p} \right]
\]
(101)
\[
S^I = \frac{1}{2} \sum_{\mu \nu} \int d^3 p \left[ \left\{ u^\dagger (p, \mu) \sigma \nu \text{\Sigma} u (p, \mu) \right\} - \partial \frac{\text{\partial \sigma \nu \text{\Sigma} u (p, \mu)}}{\partial p} \right]
\]
and
\[
S^I = \frac{1}{2} \sum_{\mu \nu} \int d^3 p \left[ \left\{ u^\dagger (p, \mu) \sigma \nu \text{\Sigma} u (p, \mu) \right\} - \partial \frac{\text{\partial \sigma \nu \text{\Sigma} u (p, \mu)}}{\partial p} \right]
\]
(102)
since
\[
\left\{ u^\dagger (p, \mu) \frac{\partial u (p, \mu)}{\partial p} - \frac{\partial u^\dagger (p, \mu)}{\partial p} u (p, \mu) \right\}
\]
\[
\left\{ u^\dagger (p, \mu) \frac{\partial u (p, \mu)}{\partial p} - \frac{\partial u^\dagger (p, \mu)}{\partial p} u (p, \mu) \right\}
\]
(103)
\[
|p_{\mu_1} \mu_2\rangle = |p_{\mu_1} \rangle - |p_{\mu_2}\rangle
\]
(104)
\[
|p \text{SM}_\mu \rangle = \left( \frac{1}{2} |\frac{1}{2} \text{SM}_\mu \rangle \right) |p \text{SM}_\mu \rangle,
\]
so
\[
|p |\mu_1 \mu_2\rangle = \left( \frac{1}{2} |\frac{1}{2} \text{SM}_\mu \rangle \right) |p \text{SM}_\mu \rangle
\]
(105)
7 Of course, what follows can be extended to an arbitrary frame.
8 For a moment, the isospin quantum numbers are suppressed. We will come back to the point later.
Second, one introduces

$$|pJ(IS)M_J⟩ = \int dp Y_{lm_1}(\hat{p}) |pSM_s⟩ (lm_3SM_3 | JM_J⟩ \tag{106}$$

or, reversely,

$$|pSM_s⟩ = \langle lm_3SM_3 | JM_J⟩ Y_{lm_1}(\hat{p}) |pJ(IS)M_J⟩ \tag{107}$$

with the unit vector $\hat{p} = p/p$.

Third, substituting (107) into the r.h.s. of Eq. (105) we arrive to the desired expansion,

$$|pμ1μ2⟩ = \left(1 - \frac{1}{2}μ1\frac{1}{2}μ2\right) |SM_s⟩ (lm_3SM_3 | JM_J⟩ Y_{lm_1}(\hat{p}) |pJ(IS)M_J⟩ \tag{108}$$

From the physical viewpoint it is important to know that

$$J^2 |pJ(IS)M_J⟩ = J(J + 1) |pJ(IS)M_J⟩ \tag{109}$$

$$F^3 |pJ(IS)M_J⟩ = M_J |pJ(IS)M_J⟩ \tag{109}$$

and

$$S^2_{\text{ferm}} |pSM_s⟩ = S(S + 1) |pSM_s⟩ \tag{110}$$

$$S^3_{\text{ferm}} |pSM_s⟩ = M_S |pSM_s⟩ \tag{110}$$

A simple way of deriving these relations is to use the transformation

$$U_F(R)pSM_s⟩ = |R⟩pSM_s⟩D_{M_S′M_S}(R) \tag{111}$$

for infinitesimal rotations. With the help of (111) it is easily seen that

$$S^2_{\text{ferm}} |pJ(IS)M_J⟩ = S(S + 1) |pJ(IS)M_J⟩ \tag{113}$$

Thus we have built up in the CPR the common eigenvectors of operators $J^2$ and $S^2_{\text{ferm}}$. Probably, one should note that $S$ is not any eigenvalue of the so-called invariant (or internal) spin operator $I = (0, 1)$ introduced by M. Shirokov. While the latter involves internal orbital motion and polarization contributions, the quantum number $S$, whose values are regulated by Clebsch–Gordan coefficient in definition (113), characterizes rather the total spin of the two–nucleon system. Coming to the end, we allow ourselves to write the parity operator of the fermion (nucleon) field in the CPR (cf. Eq.15.93 from (125)):

$$P_{\text{ferm}} = \exp[\pi P_{\text{ferm}}],$$

$$\pi P_{\text{ferm}} = -\frac{π}{2} \int d^3p \left[ b^*_\perp(pμ)b_\perp(pμ) - b^*_\parallel(pμ)b_\parallel(-pμ) + d^*_\perp(pμ)d_\perp(pμ) + d^*_\parallel(pμ)d_\parallel(-pμ) \right] \tag{114}$$

with the relations

$$P_{\text{ferm}}b_\perp(pμ)P_{\text{ferm}}^{-1} = b_\perp(-pμ),$$

$$P_{\text{ferm}}d^*_\perp(pμ)P_{\text{ferm}}^{-1} = -d^*_\perp(-pμ)$$

that extend the rules (97) for the clothed particle operators to such an improper transformation as the space inversion.

C The Regularized Quasipotentials and their Angular–Momentum Decomposition

Trying to overcome ultraviolet divergences inherent in solving equation (50), we will regularize their driving terms by introducing some cutoff factors. It can be achieved if instead of Eq. (19) one assumes

$$V_{\text{reg}}^{\text{reg}}(p′_1μ′_1, p′_2μ′_2; pμ1, pμ2) = \frac{δ(p′_1 + p′_2 - p_1 - p_2)}{\sqrt{P₀₁P₀₂P₀₁P₀₂}} V_{\text{reg}}^{\text{reg}}(p′_1μ′_1, p′_2μ′_2; pμ1, pμ2) \tag{115}$$

omitting for the moment isospin indices, so

$$K_{\text{reg}}^{\text{reg}}(NN → NN) → K_{\text{reg}}^{\text{reg}}(NN → NN) \tag{115}$$

where the old ones $v_\text{reg}$ are given by

$$v_\text{reg}^{\text{reg}}(p′_1μ′_1, p′_2μ′_2; pμ1, pμ2) = b_\perp(p′_1μ′_1)b_\perp(p′_2μ′_2)b_\parallel(pμ1)b_\parallel(pμ2) \tag{116}$$

Here the new (regularized) coefficients $v_\text{reg}^{\text{reg}}$ are obtained by

$$v_\text{reg}^{\text{reg}}(p′_1μ′_1, p′_2μ′_2; pμ1, pμ2) \tag{117}$$

where the old ones $v_\text{reg}$ are determined by Eq. (113)–(124) and empirical cutoff functions $F_R$ should not violate the known symmetries of interactions. In particular, if one writes

$$K_{\text{reg}}^{\text{reg}}(NN → NN) = \int K_{\text{reg}}^{\text{reg}}(x)dx \tag{118}$$

with

$$K_{\text{reg}}^{\text{reg}}(x) = \frac{1}{(2\pi)^3} \int \frac{dp₁}{P₀₁} \frac{dp₂}{P₀₂} \frac{dp₃}{p₀₁p₀₂} \sum_μ v_{\text{reg}}^{\text{reg}}(p₁^′μ₁^′, p₂^′μ₂^′; pμ1, pμ2) b_\perp(p₁^′μ₁^′)b_\perp(p₂^′μ₂^′)b_\parallel(pμ1)b_\parallel(pμ2) \tag{119}$$

then the RI implies the property of the operator

$$K_{\text{reg}}^{\text{reg}}(x) = \exp(iK_FT)K_{\text{reg}}^{\text{reg}}(x) \exp(-iK_FT) \tag{120}$$

to be a scalar, viz.,

$$U_F(A, λ)K_{\text{reg}}^{\text{reg}}(x)U_F^{-1}(A, λ) = K_{\text{reg}}^{\text{reg}}(Ax + λ) \tag{121}$$

But accordingly (127) it imposes the following restrictions

$$D^{(\frac{1}{2})}_μ(λ, pμ) D^{(\frac{1}{2})}_μ(λ, pμ) \tag{122}$$

$$\times D^{(\frac{1}{2})}_μ(λ, pμ) D^{(\frac{1}{2})}_μ(λ, pμ) \tag{122}$$

$$\times v_\text{reg}^{\text{reg}}(p₁^′μ₁^′, p₂^′μ₂^′; pμ1, pμ2) = v_\text{reg}^{\text{reg}}(Ap₁^′μ₁^′, Ap₂^′μ₂^′; pμ1, pμ2) \tag{122}$$

to the coefficients $v_\text{reg}^{\text{reg}}$. In this connection, before going on, one needs to verify that the old (non–regularized) ones satisfy relation (122) themselves. It can be done with the help of the property

$$S(A)pμ1 = D^{(1/2)}_{μ′}(W(A, pμ))ν(Apμ′) \tag{122}$$

We do not consider the functions depending on nucleon polarizations.
where \( S(A) = \exp \left[ -\frac{i}{\hbar} \int dt \omega_{\mu} \sigma^{\mu
u} \right] \) the matrix of the nonunitary representation \( A \rightarrow S(A) \) in the space of spinor indices. Indeed, recalling more the relations
\[
S(A)^{-1} \gamma^a S(A) = \gamma^a A^\mu_{\mu'},
\]
one can easily seen that the quantities
\[
u_a(p_1',p_2',p_1,p_2) = \nu_a(p_1',p_2',p_2',p_1) = \nu_a(1',2',1,2)
\]
by Eqs.\((20)-(22)\) obey Eq.\((22)\). Also, let us remind (see, e.g., [22]) that for a Lorentz boost \( A = L(\nu) \) with the velocity \( \nu \), the Wigner transformation \( W(A,p) \) is the rotation about the \( \nu \times p \) direction by an angle \( \psi \), which can be represented as
\[
m \left( 1 + \gamma \frac{p_2 - p_1}{m} \right) \tan \frac{\psi}{2} = \gamma |\nu \times p_1|,
\]
where \( p_1^2 \) is the zeroth component of the nucleon momentum \( p^* = (p_1^2,\nu^*) = L(\nu)p \) in the moving frame and \( \gamma \) the corresponding Lorentz factor. As noted, \( W(R,p) = R \) for a pure rotation of \( \nu \).

Now, keeping in mind the relation \((17)\) we need to deal with a Lorentz–invariant cutoff,
\[
F_b(p_1',p_2',p_1,p_2) = F_b(Ap_1',Ap_2;Ap_1,Ap_2)
\]
(123)

In our model regularization,
\[
\begin{split}
V_{reg}^{\nu}(p_1',p_1) & = \frac{\delta (p_1' + p_2' - p_1 - p_2)}{\sqrt{p_1 p_2 p_1 p_2}} \\
F_b(p_1',p_2',p_1,p_2) & = \langle p_1 - p_1' | p_2 - p_2' \rangle^2
\end{split}
\]
(124)

Further, assuming that \( F_b \) depends on the two invariants \( (p_1 - p_1')^2 \) and \( (p_2 - p_2')^2 \),
\[
F_b(p_1',p_2',p_1,p_2) = f_b ((p_1 - p_1')^2, (p_2 - p_2')^2)
\]
we have on the momentum shell,
\[
(p_1 - p_1')^2 = (E_1 - E_1')^2 = (p_1^2 - p_1'^2)
\]
\[
(p_2 - p_2')^2 = (E_2 - E_2')^2 = (p_2^2 - p_2'^2)
\]
so in the c.m.s. one has to deal with the function \( f_b((p - p')^2) \)
\[(E^2 - E_2'^2) = (E^2 - E_2') - (p_2 - p_2') \]
Of course, a similar regularization could be implemented if the vertices
\[
\tilde{u}(p^*p^'|F_b(k)\nu(p\mu)) \quad \text{(see, e.g., Eq.\((92)\))}
\]
would be at the beginning modified by introducing cutoff factors \( g_0^b(p',p) \) with the bosons and fermions on their mass shells \( k_0^2 = m_0^2, k'^2 = m^2 \) and the momentum conservation \( k = p' - p \). Neglecting a possible dependence of such factors on particle polarizations, we preserve the subscript ++
keeping in mind a more symmetrical consideration, where some factors \( g_0^b(p',p) \) could be introduced with the proper energy–sign labels \( \mp \) and \( \epsilon \) for separate three–legs contributions to a given boson–fermion Yukawa–type interaction. Then, the corresponding regularized quasipotentials would be contained factors
\[
F_b(p_1',p_2',p_1,p_2) = \int d\nu_{p_1'}(p_1') g_{p_1++}(p_2',p_2)
\]
(125)
in the quadratic terms.

As noted below Eq.\((66)\) our choice of the phenomenological cutoff \( F_b(p' - p') \) has been prompted by those investigations [2] of the Bonn group. Of course, recent developments in studying the structure of meson–baryon interaction vertices are of great interest. Among certain achievements in this area we find a microscopic derivation of the strong \( \pi NN \) and \( \pi N A \) FFs within the quark model developed by the Graz group (see, e.g., [10]). Being free of any phenomenological input (fit parameters)
the corresponding prediction for the FF \( G_{eNN} \) (e.g., in an analytic form put forward in [10]) could be employed as the bare factor \( g_{p_1++}(p',p) \). The latter has the property \( g_{p_1++}(Ap',Ap) = g_{p_1++}(p',p) \), i.e., it should be dependent upon the Lorentz scalar \( p'p \) or \( (p' - p')^2 \) to fulfill condition \((66)\). Other vertex cutoffs \( g_{p_1-+}^{--} \) and \( g_{p_1-+}^{++} \) are also desirable to be introduced on the same physical footing.

By the way, along with properties \( g_{p_1++}(Ap',Ap) = g_{p_1++}(p',p) \) condition \((66)\) means that the crossed cutoffs \( g_{p_1-+}(p',p) \) and \( g_{p_1-+}(p',p) \) should be some functions of the four–product \( p'^2 = p'p \). In turn, other symmetries, mentioned below Eq.\((69)\), yield the following links \( g_{p_1-+}(p',p) = g_{p_1-+}(p,0) = g_{p_1-+}(p',p) \) for the real functions. All the things are extremely important for constructing a relativistic nonlocal QFT, where the boosts operators are determined as elements of the Lie algebra of the Poincaré group. It will be presented somewhere else.

In addition, let us address the matrix elements \( \langle p' | j_b(0) | p \rangle \) that are closely connected with the FFs in question (see, e.g., [13]) since we handle the corresponding baryon current density \( J_b(x) \) at \( x = 0 \) sandwiched between physical (clothed) one–nucleon states. Such matrix elements might be evaluated in terms of the cutoffs and other physical inputs using some idea from [11] (cf. the clothed particle representation of a current operator therein).

From the constructive point of view, it means the use of definitions \( j_b = b_h(p unix) / \Omega \) and \( j_b' = b_h(\nu' | p' | \Omega) \), adopted at the beginning of Appendix B, and the similarity transformation
\[
j_b(0) = e^{R(\alpha_0)} j_b^* (0)e^{-R(\alpha_0)} = j_b^* (0) + \frac{1}{2!} \left[ R(\alpha_0), j_b^* (0) \right] + \frac{1}{3!} \left[ R(\alpha_0), R(\alpha_0), j_b^* (0) \right] + \cdots ,
\]
where \( j_b^* (0) \) is the same current density but expressed through the clothed operators. The nonperturbative expansion in the commutators gives an opportunity for a systematic evaluation of corrections to matrix elements
\[
\langle \nu | J_b j_b^* (0) | \Omega \rangle = (\Omega_0 | J_b(0) b_h^* | \Omega_0).
\]

Some simplifications originate from the well–known fact that similar expectations of the commutators that involve odd number of meson operators are equal to zero.

In general, one can elaborate a recursive procedure of calculations, like that by Kharlov–Padova group, for manipulations with the multiple commutators \( [V]^{n} (n = 2,3, \cdots ) \) (see [11]). Doing so, one can find corrections to formula \((125)\) obtained from the commutator \( [V]^{1} = [R,V] \). This work is in progress.

After this prelude we note that the partial–wave matrix elements of interest are defined by
\[

\]
\[ V_{dir}^b(p', p; \rho, \sigma) = -\frac{1}{2(2\pi)^3} \frac{m^2}{E_p E_p'} \left( \frac{1}{2} \bar{\mu}_2^\rho \mu_2^\sigma | SM_{S_0}^b \right) \left( \frac{1}{2} \mu_1^\rho \bar{\mu}_1^\sigma | SM_{S_0}^b \right) \nu_{v, dir}^{(exc)}(p', \mu_2; \rho, \sigma), \]

(128)

where we have employed formula (45) and property (48).

In turn, the matrices \( v_{v, dir}^{(p', \mu_2)}(p', \rho, \sigma) \) can be represented as

\[ v_{v, dir}^{(p', \mu_2)}(p', \rho, \sigma) = \Gamma_\delta(p', \rho, \sigma; \mu_2) D_\delta(p', \rho, \sigma), \]

(129)

\[ D_\delta(p', \rho, \sigma) = \frac{F_\delta^2 \left( |p' - \rho|^2 \right)}{|p' - \rho|^2 - m_\delta^2}. \]

(130)

Recall that

\[ v_{v, exc}^{(p', \mu_2)}(p', \rho, \sigma) = v_{v, dir}^{(p', \mu_2)} - \mu \text{ a linear functional of } \sigma, \]

so

\[ V_{v, exc}(p', \rho, \sigma; \mu_2) = (-1)^{S_2-1} v_{v, dir}^{(p', \rho, \sigma; \mu_2)}, \]

(131)

Further, taking into account the completeness of the matrices \( \sigma \) in \( 2 \times 2 \) space, the non-regularized vertices from Eqs. (20, 24) can be written as

\[ \Gamma_\delta(p', \rho, \sigma; \mu) = A_\delta(p', \rho, \sigma; \mu) + a \text{ a linear functional of } \sigma. \]

Now, to get the matrix \( (p' SM_{S'}^b) \hat{V}_6(p SM_{S}^b) \) we could do all summations in formula (125) over \( \mu \) projections directly. However, we prefer the following way putting formally

\[ \langle \mu_1 | \sigma | \mu_1 \rangle = \langle \mu_1 | \sigma(1) | \mu_1 \rangle \text{ and } \langle \mu_2 | \sigma | \mu_2 \rangle = \langle \mu_2 | \sigma(2) | \mu_2 \rangle \]

to obtain

\[ V_{v, dir}^{(p', \rho, \sigma; \mu_2)}(p' SM_{S'}^b) | SM_{S}^b \rangle = \langle SM_{S'}^b | G_6(p', \rho, \sigma; \mu_1) | SM_{S}^b \rangle \]

(132)

with the \( S^2 \) eigenvalue equations,

\[ \hat{S}^2 | SM_{S}^b \rangle = S(S + 1) | SM_{S}^b \rangle \]

where \( \hat{S} \) is a \( 1 \times 2 \) matrix with the help of the relations

\[ \hat{\sigma}(1) \cdot \hat{n} \hat{\sigma}(2) \cdot \hat{n} = 2 \left( \hat{S} \cdot \hat{n} \right)^2 - \hat{n}^2 \]

\[ \hat{\sigma}(1) \times \hat{n} \hat{\sigma}(2) \times \hat{n} = \hat{\sigma}(1) \cdot \hat{\sigma}(2) \hat{n}^2 - \hat{\sigma}(1) \cdot \hat{\sigma}(2) \cdot \hat{n} \]

for any vector \( \hat{n} \).

As a result, we find with the models (20, 24)

\[ G_{\rho} = \frac{g_\rho^2}{2(2\pi)^3} CD_{\rho}(p', p) \left\{ U_1^2 + V_3^2 - 2(V_3 S_i U_1) + (V_3 S_i S_j) \right\}, \]

(133)

\[ G_{\rho} = \frac{g_\rho^2}{2(2\pi)^3} CD_{\rho}(p', p) \left\{ V_1^2 - 2(V_3 S_i U_1) \right\}, \]

(134)

\[ G_{\nu} = G_{\nu}^{\nu} + c_{\nu}^\nu + G_{\nu}^f, \]

(135)

\[ G_{\nu}^{\nu} = \frac{g_\nu^2}{2(2\pi)^3} CD_{\nu}(p', p) G_1, \]

(136)

\[ G_{\nu}^f = \frac{1}{2(2\pi)^3} \frac{f_{\rho} g_\rho}{2m_{\rho}} CD_{\nu}(p', p) \left\{ 4mG_1 - 2 \left( E_{\rho'} + E_p \right) G_2 + G_3 + (E_{\rho'} - E_p) G_4 \right\}, \]

(137)

where \( f_{\rho} \) and \( g_\rho \) are the isospin and spin factors, respectively.

Further, taking into account the completeness of the matrices \( \sigma \) in \( 2 \times 2 \) space, the non-regularized vertices from Eqs. (20, 24) can be written as

\[ \Gamma_\delta(p', \rho, \sigma; \mu) = A_\delta(p', \rho, \sigma; \mu) + a \text{ a linear functional of } \sigma. \]

Now, to get the matrix \( (p' SM_{S'}^b) \hat{V}_6(p SM_{S}^b) \) we could do all summations in formula (125) over \( \mu \) projections directly. However, we prefer the following way putting formally

\[ \langle \mu_1 | \sigma | \mu_1 \rangle = \langle \mu_1 | \sigma(1) | \mu_1 \rangle \text{ and } \langle \mu_2 | \sigma | \mu_2 \rangle = \langle \mu_2 | \sigma(2) | \mu_2 \rangle \]

to obtain

\[ V_{v, dir}^{(p', \rho, \sigma; \mu_2)}(p' SM_{S'}^b) | SM_{S}^b \rangle = \langle SM_{S'}^b | G_6(p', \rho, \sigma; \mu_1) | SM_{S}^b \rangle \]

(132)

with the \( S^2 \) eigenvalue equations,

\[ \hat{S}^2 | SM_{S}^b \rangle = S(S + 1) | SM_{S}^b \rangle \]

where \( \hat{S} \) is a \( 1 \times 2 \) matrix with the help of the relations

\[ \hat{\sigma}(1) \cdot \hat{n} \hat{\sigma}(2) \cdot \hat{n} = 2 \left( \hat{S} \cdot \hat{n} \right)^2 - \hat{n}^2 \]

\[ \hat{\sigma}(1) \times \hat{n} \hat{\sigma}(2) \times \hat{n} = \hat{\sigma}(1) \cdot \hat{\sigma}(2) \hat{n}^2 - \hat{\sigma}(1) \cdot \hat{\sigma}(2) \cdot \hat{n} \]

for any vector \( \hat{n} \).

As a result, we find with the models (20, 24)

\[ G_{\rho} = \frac{g_\rho^2}{2(2\pi)^3} CD_{\rho}(p', p) \left\{ U_1^2 + V_3^2 - 2(V_3 S_i U_1) + (V_3 S_i S_j) \right\}, \]

(133)

\[ G_{\rho} = \frac{g_\rho^2}{2(2\pi)^3} CD_{\rho}(p', p) \left\{ V_1^2 - 2(V_3 S_i U_1) \right\}, \]

(134)

\[ G_{\nu} = G_{\nu}^{\nu} + c_{\nu}^\nu + G_{\nu}^f, \]

(135)

\[ G_{\nu}^{\nu} = \frac{g_\nu^2}{2(2\pi)^3} CD_{\nu}(p', p) G_1, \]

(136)

\[ G_{\nu}^f = \frac{1}{2(2\pi)^3} \frac{f_{\rho} g_\rho}{2m_{\rho}} CD_{\nu}(p', p) \left\{ 4mG_1 - 2 \left( E_{\rho'} + E_p \right) G_2 + G_3 + (E_{\rho'} - E_p) G_4 \right\}, \]

(137)

where \( f_{\rho} \) and \( g_\rho \) are the isospin and spin factors, respectively.
The operators $G_n(p', p; S)$ depend on the scalars $(p' \cdot p)$, $(S')^2$, $(S)^2$, $(S' \cdot S)$, $(S \cdot S')$, so all we need is to calculate the following integrals:

$$b_{UL}^{J_{2S}}(p', p) = \int d\phi \int d\phi' \langle Y_{J_{2S}}(p')|D_b(p', p) f(p', p; S)|Y_{J_{2S}}(p)\rangle,$$

where $f(p', p; S)$ is a polynomial of these scalars.

After a lengthy calculation we arrive to our working formulae. In particular, in case of the tensor–tensor interaction in the $\rho$–exchange channel we have for uncoupled waves:

$$\rho^{\mu, UT}V_{JJ}(p', p) = \frac{f_0}{4\pi} \left[ \left\{ \left[ p^2p'^2 + (p^2 + p'^2)(E'E - 2m^2) + 6m^2(E'E - m^2) \right] s\tilde{Q}_J(p', p) 
- pp'(p^2 + p'^2 + 4m^2)p\tilde{Q}_{J}^{(1)}(p', p) - p'^2p\tilde{Q}_{J}^{(2)}(p', p)
- (E' - E)^2 \left[ (E'E - 2m^2)p\tilde{Q}_J(p', p) - pp'(p\tilde{Q}_{J}^{(1)}(p', p) \right] \right\} \right].$$

(141)

with

$$\rho^{\mu, UT}V_{JJ}^{(1)}(p', p) = \frac{1}{2J + 1} \left\{ J\tilde{Q}_{J-1}(p', p) + (J + 1)s\tilde{Q}_{J+1}(p', p) \right\},$$

(143)

and

$$\rho^{\mu, UT}V_{JJ}^{(2)}(p', p) = \frac{1}{2J + 1} \left\{ (J + 1)s\tilde{Q}_{J-1}(p', p) + J\tilde{Q}_{J+1}(p', p) \right\},$$

(144)

In these formulae

$$b_{\tilde{Q}_n}(p', p) = 2\pi \int_{-1}^{1} d(cos \theta) P_n(cos \theta) D_b(p', p),$$

where $P_n(cos \theta)$ is the Legendre polynomial. Using the Neumann integral representation for the Legendre function of second kind $Q_n(x)$ one can write for any $n$:

$$Q_n^2(p', p) = 2\pi \int_{-1}^{1} d(cos \theta) P_n(cos \theta) \left[ \frac{A^2 - m^2}{(p' - p)^2 - A^2} \right]^{2n},$$

$$= \frac{1}{2p'p} \sum_{m=0}^{2n-1} \frac{1}{m!} \frac{d^m}{(y - x)^m} Q_v(y),$$

where $x = p'^2 + p^2 + m^2 - (E' - E)^2$, $y = p'^2 + p^2 + A^2 - (E' - E)^2$ and $Q_{n>0}(p', p) \equiv 0$.

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