A NOTE ON THE UPPER RADICALS OF SEMINEARRINGS

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Abstract

In this paper we work in the class of seminearrings. Hereditary properties inherited by the lower radical generated by a class $M$ have been considered in [2, 5, 6, 7, 9, 10, 12]. Here we consider the dual problem, namely strong properties which are inherited by the upper radical generated by a class $M$. 
1. Introduction and Preliminaries

V. G. Van Hoorn and B. Van Rootselaar [11] discussed general theory of seminearrings. The theory was further enriched by many authors (see [1, 3, 4, 13, 14]). The upper radicals were investigated by (see [2, 9, 12]) for radical classes of rings. Here we are interested in generalizing several results from [2, 5, 6, 7, 9, 10, 12] in the framework of seminearring, which is quite different from the ring theoretical approach discussed in [2, 5, 6, 7, 9, 10, 12]. Throughout this paper $N$ will denote an seminearrings and $\omega$ be the universal class of all seminearrings. An semi-ideal $I$ of $N$ is denoted by $I \leq N$. In the following we shall be working within the class of all seminearrings.

Consider non-associative seminearrings as general algebras $(N, +, .)$, where $(N, +)$ is a semigroup, $(N, .)$ is a groupoid, and only the one-side distributive law holds.

Lower radical classes for seminearrings can be constructed similar to the construction of lower radicals for rings (see [2, 5, 6, 7, 8, 9, 10, 12, 15, 16]). First we include necessary preliminary, let $\omega$ be the universal class of all seminearrings and $M$ be a sub-class of $\omega$ and let $M_0$ be the homomorphic closure of $M$ in $\omega$. For each $N \in \omega$, let $D_1(N)$ be the set of all semi-ideals of $N$. Inductively we define

$$D_{n+1}(N) = \{ I : I \text{ is an semi-ideal of some seminearring in } D_n(N) \}$$

Let $D(N) = \bigcup_{n \in N} D_n(N)$, $n = 1, 2, 3, \ldots$. By using rings theoretical approach discussed in [12], we have

$$\mathcal{L}M = \{ N \in \omega : D(N/I) \cap M_0 \neq 0, \text{for each proper semi-ideal } I \text{ of } N \},$$

is the Lee construction for lower radical determined by $M$, and $M \subseteq \mathcal{L}M$ (see also [8, 15, 16]).

First we give a construction for the upper radical, dual to the construction of [10] for the lower radical. From this the theorem on the inheritance of the left strong property is deduced.

We define the following classes from a given class $M$ of seminearrings:

$$IM = \{ N : N \text{ is a subsemi-ideal of some seminearring of } M \};$$
$TM = \{ N : N \text{ contains a semi-ideal } B \text{ such that } B \in M \text{ and } N/B \in M \}$;

$SM = \{ N : N \text{ contains a descending chain of semi-ideals } B_i \text{ such that } B_i = 0 \text{ and } N/B_i \in M \}$.

It is clear that $M$ is contained in $IM$ and $SM$ and that $M$ is contained in $TM$ if $0$ belongs to $M$. The class $M$ is semi-ideally closed if and only if $M = IM$. If $M$ is semi-ideally closed then it follows easily that $TM$ and $SM$ are also semi-ideally closed. For undefined terms of seminearrings we may refer (see [1, 3, 4, 9, 11, 12, 13, 14, 16]).

2. Upper Radicals

We extend the results of [2, 5, 6, 7, 9, 10, 12] by using the above construction of upper radical for seminearring which is indeed provides an excellent and different approach to handle the many results of [2, 5, 6, 7, 9, 10, 12] in the framework of seminearring.

Definition 2.1. If $\rho$ is a radical class of seminearring then it admits a semisimple class:

$$S\rho = \{ N\in \omega : \rho(N) = 0 \}.$$

The following theorems were proved by N. J. Divinsky [2] for rings. Here we generalize it for seminearring which can be obtained on the line of rings theoretical approach discussed in [2].

Theorem 2.2. For any radical property $N$, every semi-ideal of an $N$-semisimple is itself $N$-semisimple.

Proof. The proof of our Theorem 2.2 is very similar to the proof of [2].

Theorem 2.3. The class $M$ is the class of all $S$-semisimple seminearrings with respect to some radical property $S$ if and only if $M$ satisfies the following conditions:

1. Every non-zero semi-ideal of a seminearring of $M$ can be mapped homomorphically on to some non-zero seminearring of $M$.
2. If every non-zero semi-ideal of a seminearring $N$ can be mapped homomorphically onto some non-zero seminearring of $M$, then the seminearring
$N$ must be in $M$.

**Proof.** The proof of our Theorem 2.3 is very similar to the proof of [2].

**Theorem 2.4.** A non-empty class of seminearrings $M$ is the semisimple class with respect to some radical if and only if $M = IM = TM = SM$.

**Proof.** By Theorem 2.2 a semisimple class is semi-ideally closed. Also if $B$ and $N/B$ are semisimple and $\rho(N)$ is the radical of $N$, $(\rho(N) + B)/B$ is semisimple, being an semi-ideal of $N/B$ and is also radical being isomorphic to $\rho(N)/(\rho(N) \cap B^*)$, (where $B^*$ is a $k$-semi-ideal generated by $B$ (see [8, 15, 16]). Hence $\rho(N) \subseteq B$. As $B$ is semisimple we have $\rho(N) = 0$. Therefore $N$ is semisimple. If $B_i$ is a family of semi-ideals of $N$ such that $N/B_i$ is semisimple and $\cap B_i = 0$, then as above, $\rho(N) \subseteq B_i$ and so $\rho(N) \subseteq \cap B_i$. Therefore $N$ is semisimple. It follows that a semisimple class $M$ satisfies $M = IM = TM = SM$.

Conversely let $M$ be a class of seminearrings such that $M = IM = TM = SM$. We show that $M$ is a semisimple class by verifying the conditions (1) and (2) of Theorem 2.3. Since $M = IM$ condition (1) of Theorem 2.3 is clear. Now let $N$ be a seminearring such that every non-zero semi-ideal of $N$ can be mapped onto some non-zero seminearring of $M$. To complete the proof we must show that $N$ is in $M$. Consider the family of proper semi-ideals $G$ of $N$ such that $N/G \in M$ and using Zorn’s Lemma we see that there is a semi-ideal $B$ minimal in this family. If $B = 0$, we are finished. If not there is a semi-ideal $B$ minimal in the family of proper semi-ideals of $B$ whose quotients belong to $M$. $J$ is not a semi-ideal of $N$. Since $M = TM$ would then imply $N/J \in M$, contradicting the minimality of $B$. Hence either $NJ$ or $JN$. We may assume without loss of generality that there exists $n \in N$ with $nJ$. Consider $(nJ + J)/J$. We have $nJB \subseteq nJ$ and $BnJ \subseteq BJ \subseteq J$. Hence $(nJ + J)/J$ in an semi-ideal of $B/J$. Since $B/J \in M$ and $IM = M$ we have $(nJ + J)/J \in M$. Consider the mapping from $J$ to $(nJ + J)/J$ given by $\eta(x) = nx + J$, $\eta(xy) = nxy + J \subseteq NJJ + J \subseteq BJ + J \subseteq J$; $\eta(x)\eta(y) = nxy + J \subseteq NJNJ + J \subseteq BJ + J \subseteq J$. Therefore $\eta$ is an epimorphism. Let $K$ be the kernel of $\eta$, i.e. $K = \{x \in J : nx \in J\}$. Let $x \in K$, $b \in B$; then $nxb \subseteq BJ \subseteq J$. Hence $K$ is an semi-ideal of $B$. However we have $B/J \in M$ and

$$J/K \cong (nJ + J)/J \in M$$
Since $TM = M$ it follows that $B/K \varepsilon M$. This contradicts the minimality of $J$. Therefore $N \varepsilon M$ and $M$ is a semisimple class.

Now we can give the upper radical construction. Let $M$ be a non-empty class of seminearrings. Let $M_1 = IM$. We define $M_\alpha$ inductively on ordinals $\alpha > 1$ as follows. If $\alpha$ is not a limit ordinal $M_\alpha = TM_{\alpha - 1}$. If $\alpha$ is a limit ordinal $M_\alpha = S(\bigcup_{\beta < \alpha} M_\beta)$. Finally we set $\bar{M} = \bigcup M_\alpha$.

**Theorem 2.5.** For any non-empty class $M$ of seminearrings, $\bar{M}$ is the smallest semisimple class containing $M$.

**Proof.** It is clear that if $M$ is contained in semisimple class so also are $IM$, $TM$, and $SM$. Therefore $\bar{M}$ is contained in every semisimple class containing $M$. It remains to show that $\bar{M}$ is a semisimple class. It is clear that $IM_\alpha = M_\alpha$ for all $\alpha$ and hence that $IM = \bar{M}$. Let $N \varepsilon TM$. Then $N$ contains an semi-ideal $B$ with $B$ and $N/B$ in $\bar{M}$. Therefore there exists an ordinal $\alpha$ with both $B$ and $N/B \varepsilon M_\alpha$. Hence $N \varepsilon TM_\alpha = M_{\alpha + 1}$. Therefore $M = TM$. Let $N \varepsilon SM$. Let $B_i$ be the descending chain of semi-ideals of $N$ with $\cap B_i = 0$ and $N/B_i \varepsilon \bar{M}$. Then $N/B_i \varepsilon M_{\alpha_i}$, for some $\alpha_i$. Since the indices $I$ form a set there exists a limit ordinal $\alpha$ with $\alpha_i$ for all $i$. Then $N/B_i \varepsilon \bigcup M_\beta$ and so $N \varepsilon S(\bigcup M_\beta) = M_\alpha$. Therefore $A \varepsilon \bar{M}$ and $\bar{M} = SM$. It follows from Theorem 2.4 that $\bar{M}$ is a semisimple class of seminearrings.

**Definition 2.6.** A radical $\rho$ is said to be left strong, if every radical left semi-ideal of a seminearring $N$ is contained in the radical of $N$. Equivalently semisimple seminearrings contain no non-zero radical left semi-ideal of every seminearring in $M$ has a non-zero image in $IM$. Clearly a radical is strong if and only if its semisimple class is left strong.

**Theorem 2.7.** If a non-empty class $M$ of seminearrings is left strong then the upper radical generated by $M$ is left strong.

**Proof.** Let $\rho$ denote the upper radical generated by $M$. By Theorem 2.5 the semisimple class of the radical $\rho$ is $\bar{M}$. We need to show that if $K$ is a non-zero left semi-ideal of a seminearring $N$ in $\bar{M}$ then $K$ has a non-zero image in $\bar{M}$. Let $N \varepsilon M_\alpha$. The proof is by transfinite induction on $\alpha$. First suppose that $N \varepsilon M_1 = IM$. Then $N$ is a subsemi-ideal of a seminearring in $\bar{M}$. Let $N = N_1 \subseteq N_2 \subseteq \ldots \subseteq N_n$, where $N_n \varepsilon M$ and $N_i$ is an semi-ideal of $N_{i+1}$. We prove this case by induction on $n$. If $n = 1$ the required result
holds. \( K + N_2 K \) is a left semi-ideal of \( N_2 \). By the inductive assumption
\( K + N_2 K \) has a non-zero image \( (K + N_2 K)/J \) in \( \bar{M} \). If \( K \subseteq J \) then
\[
K/(K \cap J^*) \cong (K + J)/J
\]
(\( J^* \) is a k-semi-ideal generated by \( J \) (see [8, 15, 16])) is a non-zero
semi-ideal of \( (K + N_2 K)/J \) and so is in \( \bar{M} \). If \( K \subseteq J \) then there exist \( b \in N_2 \)
with \( bK \); then \( (J + bK)/J \) is an semi-ideal of \( (K + N_2 K)/J \) and so is in \( \bar{M} \).
As before, the mapping \( \eta \) from \( K \) to \( (J + bK)/J \) given by \( \eta(x) = bx + J \) is
an epimorphism and so \( K \) has a non-zero image in \( M \) as required. Now let
\( N \in M_\alpha \) and assume that the result has been proved for ordinals less then \( \alpha \). If \( \alpha \) in not a limit ordinal then \( N \) contains an semi-ideal \( B \) and \( N/B \) in
\( M_\alpha \). If \( K \subseteq B \) the required result holds. Otherwise
\[
K/(K \cap B^*) \cong (K + B)/B,
\]
(\( B^* \) is a k-semi-ideal generated by \( B \) (see [8, 15, 16])) which is a non-zero left semi-ideal of \( N/B \). Again the required result follows. If \( \alpha \) is
a limit ordinal then \( N \) contains a descending chain of semi-ideals \( B_i \) with
\( \cap B_i = 0 \) and \( N/B_i \in M_\alpha \), \( \alpha_i < \alpha \). For some \( i \), \( KB_i \). Then \( (K + B_i)/B_i \) is a
non-zero left semi-ideal of \( N/B_i \). Thus in all cases \( K \) has a non-zero image
in \( M \). Therefore the upper radical generated by \( M \) is left strong.

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A note on the upper radicals of seminearrings

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