On position, momentum and their correlation

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Position and momentum observables are considered and their correlation is studied for the simplest quantum system of a free particle moving in one dimension. The algebra and the eigenvalue problem for the correlation observable is presented and its possible relevance for the solution of the Pauli problem is analysed. The correlation provides a simple explanation of the shrinking and spreading of wave packets in an interpretation of quantum mechanics based in an ontology suggested by quantum field theory. Several properties and speculations concerning position-momentum correlations are mentioned.

Keywords: position momentum correlations, Pauli problem, interpretation of quantum mechanics, shrinking and spreading of wave packets

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I. INTRODUCTION

The relation between position and momentum in quantum mechanic has many features unexpected from the classical point of view. The simplest quantum system consisting in one structureless free particle moving in one dimensional space has position and momentum as unique relevant observables. Although this is the simplest system we may think of, it has sufficient weirdness to justify the claim that, from the point of view of the foundations of quantum mechanics, we can say that we do not completely understand it. In this work we will present several arguments supporting this claim and we will give a detailed analysis of the correlation observable

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II. QUANTUM MECHANIC RECIPE

According to standard quantum mechanics, in order to describe one free structureless particle moving in one dimension we define a (Rigged) Hilbert space $\mathcal{H}$ whose elements $\Psi \in \mathcal{H}$ represent the state that encodes all information about the system. Position and momentum are represented by hermitian operators $X$ and $P$ in $\mathcal{H}$ such that their associated bases, $\{\varphi_x\}$ and $\{\phi_p\}$, are mutually unbiased. Their internal product is $\langle \varphi_x, \phi_p \rangle = \frac{1}{\sqrt{2\pi}} \exp(\frac{i}{\hbar} px)$. $P$ and $X$ are, respectively, the generators of translations $X \to X + a I$ and impulsions $P \to X + g I$ and therefore (if the dimension of $\mathcal{H}$ is infinite) their commutator is $[X, P] = i \hbar$. The quantum mechanical prediction is that the measurement of position or momentum in an ensemble of systems in the state $\Psi$ will be distributed according to the densities $\rho(x)$ and $\varpi(p)$ given by

$$\rho(x) = |\langle \varphi_x, \Psi \rangle|^2,$$

$$\varpi(p) = |\langle \phi_p, \Psi \rangle|^2,$$

and similarly, the prediction for any observable $C(X, P)$ is given in terms of its eigenvectors $\{\eta_c\}$ by the density

$$\sigma(c) = |\langle \eta_c, \Psi \rangle|^2.$$

The state $\Psi$ of the system is determined after the measurement of any observable $A(X, P)$ with result $\alpha$ by the eigenvalue equation $A(X, P)\Psi = \alpha \Psi$ (for simplicity we assume pure states and nondegeneracy). The time evolution of the state, and therefore of any distribution, is controlled by a unitary operator $U_t$ such that $\Psi_t = U_t \Psi_0$ with $U_t = \exp(-\frac{i}{\hbar} t H)$ where $H$ is the hamiltonian ($P^2/2m$ in this case). That’s all.

III. EXISTENTIAL WEIGHT

Does the recipe above provide sufficient explanation for the complete understanding of the system? There are arguments to deny this. Although many physicists
would feel satisfied because every prediction can be tested in a laboratory measurement, there are questions concerning the reality of the system that do not refer to laboratory measurements but, for many of us, their answer is necessary for a complete understanding of the system. What is the nature of the distributions $\rho(x)$, $\varpi(p)$ and $\sigma(c)$? The standard answer that “they are probability distributions” is not really satisfactory\[1\]. Indeed, if position and momentum are random variables with their corresponding probability distributions given by $\rho(x)$ and $\varpi(p)$ then the well-established theory of random variables provides the probability distribution for any function $C(X, P)$ that turns out to be different from the quantum mechanical prediction given in Eq.\[3\]. Consequently $\rho(x)$ and $\varpi(p)$ are not the probability density functions of some stochastic process for a particle diffusing in space with some random velocity. Even though $\rho(x)$, $\varpi(p)$ and $\sigma(c)$ are related to frequency measurements, as if they were probabilities, strictly speaking they are not and it would be better to use another name to denote them. The term existential weight has been proposed\[1\] although the misnomer “probability distribution” appears to be irreversibly installed in quantum mechanics.

IV. ONTOLOGICAL INDEFINITENESS

Let us consider now one of these existential weights, say $\rho(x)$. If we make an experiment to detect the position of the particle in an ensemble of systems we will find that the eigenvalues of the position observable are distributed according to $\rho(x)$. Now, what is the nature of this distribution? Should we think that the particle has a definite position, the putative value\[2\], that can not be determined by quantum mechanics and the best that the theory can provide is the distribution $\rho(x)$ of the observed values? In this case $\rho(x)$ reflects our ignorance of the reality (gnoseological interpretation) and we can think that the experiment reveals a pre-existent value of the observable. On the contrary, we may think that the position observable doesn’t have a precise value and $\rho(x)$ represents this inherent indefiniteness in the observable (ontological interpretation). In this case we don’t have a pre-existent value for the observable and the experiment creates the observed result. The gnoseological interpretation appears easier to accept, however there are very strong arguments against it: the Bell-Kochen-Specker theorem\[3\,\text{-}\,4\] applied to the position observable\[5\] shows that the existence of context independent putative values for position is in contradic-
tion with quantum mechanics. There is however a stronger argument: the violation of Bells inequalities [6–8] imply that the existence of context independent putative values is in contradiction, not only with quantum mechanics, but also with empirical reality.

Context independence, means that the putative value should not depend on the value taken by other commuting observables, for instance, the position or momentum of another noninteracting particle located far away or any other commuting observable that the theorist may think of. Although context dependent putative values are not excluded, their existence appears suspicious and difficult to accept and therefore many experts in the foundations of quantum mechanics adopt the ontological interpretation of the distributions even though it is not a logical necessity.

V. POSITION AND MOMENTUM DEFINITION

If position and momentum of a particle do not assume exact values but are instead diffuse by nature, as the ontological interpretation of $\rho(x)$ and $\varpi(p)$ suggests, then we must review the intuitive understanding of the relation between position and momentum. Of course, in this diffuse position case we can not define the velocity as the time derivative of position because there is no such a position. Therefore the momentum definition of elementary mechanics $P = mv$ is not acceptable. Anyway, this definition is abandoned also in classical physics because it is incorrect for a charged particle in the presence of an external electromagnetic field (furthermore it results inadequate in special relativity). The standard way in quantum mechanics is to define momentum as the observable whose associated operator has the commutation relation $[X, P] = i\hbar$ with position and that can also be understood as generator of translations because we can prove that, for the observable whose operator $F(X)$ is a function of position (that can be expanded in a power series) we have

$$[F(X), P] = i\hbar \frac{dF(X)}{dX},$$

and therefore $U_a F(X) U_a = F(X + a)$, where $U_a = \exp(-\frac{i}{\hbar}aP)$. Similarly, the commutator with $X$ acts as a derivative with respect to $P$

$$[G(P), X] = -i\hbar \frac{dG(P)}{dP}.$$

(5)
These relations can be generalized, only in some cases, to functions of X and P taking partial derivatives in the right hand side. This is not always possible because the chain rule of derivatives becomes ambiguous due to the noncommutativity of X and P. The generalization should be used with extreme care only for functions like \( \sum a_{kr}X^kP^r \) where the partial derivatives are unambiguous. For instance, a careless application of Eq.(4) for \( F(X,P) = e^{-i\frac{2\hbar}{2}}(XP+PX) \) using the chain rule leads to the obviously wrong result: \( Pe^{-i\frac{2\hbar}{2}}(XP+PX) = 0. \)

Notice however that this definition of momentum relies on the formal aspects of the recipe of quantum mechanics and has lost the strong connection found in classical physics where momentum is related to matter in motion (as is suggested in \( P = mV \)). The lost connection between position and momentum observable in quantum mechanics results in that, even if we know the time evolution of position, that is, if we know \( \rho(x,t) \), we can not derive from it the momentum distribution \( \varpi(p) \). To prove this, consider

\[
\varpi(p) = \langle \Psi, \phi_p \rangle \langle \phi_p, \Psi \rangle = \langle \Psi_t, \phi_p \rangle \langle \phi_p, \Psi_t \rangle 
\]

\[
= \int dx \ dx' \langle \Psi_t, \varphi_{x'} \rangle \langle \varphi_{x'}, \phi_p \rangle \langle \phi_p, \varphi_x \rangle \langle \varphi_x, \Psi_t \rangle 
\]

\[
= \int dx \ dx' f(x, x') \langle \varphi_{x'}, \Psi_t \rangle \langle \Psi_t, \varphi_x \rangle ,
\]

where \( f(x, x') \) is a known function but of \( \langle \varphi_{x'}, \Psi_t \rangle \langle \Psi_t, \varphi_x \rangle \) we know only the “diagonal” terms given by \( \rho(x,t) \). Viceversa, in a similar way one can easily prove that the knowledge of the momentum distribution \( \varpi(p) \) and of an initial position distribution \( \rho(x) \) are not sufficient in order to calculate the time evolution of the position distribution \( \rho(x,t) \). The complete information on position and movement of the system encoded in the existential weights \( \rho(x,t) \) and \( \varpi(p) \) is not sufficient for a complete determination of the state of the system \( \Psi \). This fact, surprising from the classical point of view, was first recognized by Pauli\([9]\) and triggered an intense investigation on the necessary and sufficient information needed for an unambiguous state determination\([10–15]\).

The existence of Pauli partners states, that is, different states \( \psi \neq \phi \) having the same distributions for position \( \rho(x) \) and momentum \( \varpi(p) \), implies that, in general, there exists no functional \( C \) such that the distribution of the eigenvalues of some observable \( C(X, P) \) is given by \( C(\rho(x), \varpi(p)) \). Besides the knowledge of position and movement of the system we need some additional information concerning some
function of position and movement (the correlation perhaps) in order to determine the state of the system. What is the cause and physical meaning of this additional function that contributes to fix the state of the system? Is there something besides position and momentum in the ontology of the system that is described by such a function? is this some consequence of an unknown geometrical space-time structure? These are questions indicative of our lack of understanding of quantum mechanics even at the elementary level of this simplest physical system.

VI. POSITION-MOMENTUM CORRELATION OBSERVABLE

We can conclude from the arguments of the previous sections that, besides position and momentum, we need another observable to provide additional information on the system that may render possible the determination of the state of the system without the possible ambiguity of the Pauli partners.

One could first think that the best observable to provide the additional information should be an observable whose basis is unbiased to the bases of position and momentum. For instance, every linear combination \( \alpha X + \beta P \) has a basis unbiased to \( \{ \phi_x \} \) and \( \{ \phi_p \} \). This “unbiased” observable would bring information with the highest independence from position and momentum and therefore we may think that it is an optimum choice. However this is not true. The Pauli partners ambiguity is not avoided by this choice. One can prove that there is an infinite number of states having the same existential weight for position, momentum, and the third unbiased observable. This follows from the fact that the three bases associated to the three observables are mutually unbiased. However (if the Hilbert space is infinite dimensional) there exist a fourth, and infinite many, other bases unbiased to the previous three. Every element of these bases is an example of a state with identical (uniform) distributions for all three observables. That is, all the basis elements are Pauli partners. Therefore, if we hope to resolve the Pauli partners ambiguity, besides position and momentum we should include a third observable whose basis is not unbiased with position and momentum. One observable with this requirement is the correlation, defined as

\[
C = \frac{1}{2}(XP + PX) .
\]  

(9)

Besides providing a possibility to solve the Pauli problem, this observable is in-
teresting in itself and therefore we will now study its properties. The commutation relations with \( X \) and \( P \) are
\[
[X, C] = i\hbar X \quad \text{and} \quad [P, C] = -i\hbar P ,
\]
and by induction we can prove that
\[
[X^n, C] = i\hbar^n X^n \quad \text{and} \quad [P^n, C] = -i\hbar^n P^n .
\]
From these Eqs. (11) and using Leibniz rule for the commutator of products we get
\[
[X^r P^s, C] = i\hbar(r - s)X^r P^s \quad \text{and} \quad [P^s X^r, C] = i\hbar(r - s)P^s X^r ,
\]
and therefore, for any hermitian operator of the type \( D_{rs} = X^r P^s + P^s X^r \) we have
\[
[D_{rs}, C] = i\hbar(r - s)D_{rs} .
\]
Notice that all these commutation relations above are of the type \([B, C] = ikB\) where \( k \) is a real constant. This relation will be relevant later when we study the eigenvectors of \( C \).

One interesting thing is to try to determine an operator \( A(X, P) \) that has, with the correlation operator \( C(X, P) \), the commutation relation equal to the corresponding commutator of position and momentum, that is, \([A, C] = i\hbar\). Physically, we are looking for an observable that could act as a generator of correlations. One can prove that such an operator \( A(X, P) \) can not be expanded in a power series \( \sum a_{kr} X^k P^r \) (notice that any power series can be brought to this “normal order” with all powers of \( X \) at the left of all powers of \( P \)). In order to prove this we use the commutation relations in Eq. (12) and we can see that there exists no choice of the coefficients \( a_{kr} \) that satisfy the commutation relation \([A, C] = i\hbar\).

The eigenvectors of \( A(X, P) \) (if they exist) and \( C(X, P) \) would build two mutually unbiased bases and these two observables could be chosen as a pair of canonical conjugate coordinates for the description of the system. This choice is related to the canonical transformation of classical mechanics where the coordinates \((x, p)\) are transformed to \( a(x, p) \) and \( c(x, p) \) in a way to preserve the Poisson brackets, that is \( \{x, p\} = \{a, c\} = 1 \). If we take \( c(x, p) = xp \), then the conjugate coordinate is \( a(x, p) = \frac{1}{2} \ln(x) \). Following this suggestion we can see that the operator
\[
A = \frac{1}{2}(\ln X - \ln P) ,
\]
at least \textit{formally}, has the wanted commutation relation. To prove this we use Eqs.(4) and (5) in order to obtain \([\ln X, P] = i\hbar/X\) and \([\ln P, X] = -i\hbar/P\). With more mathematical rigour it is not clear that such an operator exists. Furthermore, the physical meaning of an observable such as \(\ln X\), undefined for negative values of position, is unclear, leaving alone what would be the mysterious physical procedure to measure \(A\). The question of the existence of a generator of correlations is open.

We come now to the question of the existence of the eigenvectors of the correlation operator \(C\), that is, to determine the basis \(\{\eta_c\}\) and the real numbers \(c\) such that

\[
C\eta_c = c\eta_c .
\]

(15)

Strictly speaking, the correlation operator does not have eigenvectors \textit{in} the Hilbert space. The reason for this, is that this operator \(C\), as it also happens with position and momentum operators, is unbound as can be proven from its definition in Eq.(9). If we assume the existence of the eigenvectors \(\{\eta_c\}\) we can arrive at several contradictions. One of them arises from the commutation relations of the type \([B, C] = ikB\) shown in Eqs.(10-13). This commutation relation implies that the operator \(B\) acts as a “shift” operator for the eigenvectors of \(C\). In fact, it can be easily shown that if \(\eta_c\) is an eigenvector of \(C\) with eigenvalue \(c\), then \(B\eta_c\) is also an eigenvector corresponding to the eigenvalue \((c-ik)\); but then the \textit{hermitian} operator \(C\) could have \textit{complex} eigenvalues reaching a contradiction.

In quantum mechanics, however, we need the eigenvectors of unbound operators like \(X\), \(P\) or \(C\) because they represent possible states of the system. There are two standard ways out of this difficulty. One of them is to assume for the Hilbert space, not the squared integrable \textit{functions} but instead, \textit{distributions} that include also non squared integrable functions, like Dirac’s delta “function” and \(e^{ikx}\). These are precisely the eigenvectors of \(X\) and \(P\). The other way, mathematically more elegant, is to consider the Gel’fand triplet \(\mathcal{H}^0 \subseteq \mathcal{H} \subseteq \mathcal{H}'\) that amounts to an extension of the Hilbert space \(\mathcal{H}\) towards the so-called \textit{Rigged} Hilbert space \(\mathcal{H}'\) that includes the desired eigenvectors\[17\].

We have then the three bases \(\{\varphi_x\}\), \(\{\phi_p\}\) and \(\{\eta_c\}\) associated to \(X\), \(P\) and \(C\). The transformation between the first two, already known, is given by \(\langle \varphi_x, \phi_p \rangle = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{ipx}{\hbar}\right)\) and we must now determine \(\eta_c(x) = \langle \varphi_x, \eta_c \rangle\) and \(\eta_c(p) = \langle \phi_p, \eta_c \rangle\), that is, the eigenvectors of \(C\) in position and momentum representation. For this, we can write the eigenvalue Eq.(15) in the position or momentum representation and solve
it to find the associated eigenfunctions. That is, we must solve
\[ x \frac{d\eta_c(x)}{dx} = \left( i \frac{c}{\hbar} - \frac{1}{2} \right) \eta_c(x) \]  
(16)
\[ p \frac{d\eta_c(p)}{dp} = \left( -i \frac{c}{\hbar} - \frac{1}{2} \right) \eta_c(p) \]  
(17)

(To avoid confusion, notice that we are using the same letter, \( \eta \), to denote different functions \( \eta_c(x) \) and \( \eta_c(p) \) that are actually related by Fourier transformation).

The correlation operator is invariant under the (unitary) parity transformation \( P \) that changes \( X \rightarrow -X \) and \( P \rightarrow -P \). That is, \( [C, P] = 0 \) and this implies that the eigenvalues are also invariant. That is, if \( \eta_c \) is an eigenvector, then, \( P \eta_c \) is also an eigenvector with the same eigenvalue and therefore the correlation eigenvalues are twofold degenerate because the parity operator has two eigenvectors, even (\( \text{gerade} \)) \( \eta^g_c \) or odd (\( \text{ungerade} \)) \( \eta^u_c \). These two eigenvectors are orthogonal because
\[ \langle \eta^g_c, \eta^u_c \rangle = \langle \eta^g_c, P^2 \eta^u_c \rangle = \langle P^2 \eta^g_c, \eta^u_c \rangle = \langle \eta^g_c, -\eta^g_c \rangle = - \langle \eta^g_c, \eta^u_c \rangle. \]

The explicit treatment of the above equation in the position representation provides both degenerate solutions:
\[ \eta^g_c(x) = \frac{1}{2\sqrt{\hbar \pi}} |x|^{-\frac{1}{2}+i\frac{c}{\hbar}} e^{i\frac{c}{\hbar} \ln|x|}, \]  
(18)
\[ \eta^u_c(x) = \frac{\text{sign}(x)}{2\sqrt{\hbar \pi}} |x|^{-\frac{1}{2}+i\frac{c}{\hbar}} e^{i\frac{c}{\hbar} \ln|x|}, \]  
(19)
normalized such that \( \langle \eta^k_c, \eta^{k'}_c \rangle = \delta_{k,k'} \delta(c-c') \). The momentum representation of the eigenfunctions can be obtained in the same way, that is, solving Eq.(17), or by taking the Fourier transform of Eqs.(18,19) or, most easily, by noticing that the operator \( C \) in the momentum representation is obtained from the position representation by replacing \( x \rightarrow p \) and taking the complex conjugate. Therefore, if \( \eta_c(x) \) is an eigenfunction in the position representation, then \( \eta^*_c(p) \) is the corresponding eigenfunction in the momentum representation. These eigenfunctions have the interesting property that their Fourier transformation is equal to their complex conjugate.

VII. PAULI PARTNERS AMBIGUITY

We can now analyse the possibility to resolve the Pauli partner ambiguity by means of the correlation operator \( C \). Let us recall that two different states \( \Psi \) and \( \Phi \) (that is, \( |\langle \Psi, \Phi \rangle| \leq 1 \)) are Pauli partners if they have equal position and
momentum existential weights $\rho(x)$ and $\varpi(p)$. That is, $|\langle \varphi_x, \Psi \rangle| = |\langle \varphi_x, \Phi \rangle|$ and $|\langle \phi_p, \Psi \rangle| = |\langle \phi_p, \Phi \rangle|$. This means that the states in position representation $\langle \varphi_x, \Psi \rangle$ and $\langle \varphi_x, \Phi \rangle$ differ at most by a phase $e^{i\alpha(x)}$. This condition in the abstract Hilbert space is that there exists an (hermitian) operator function of position $\alpha(X)$ such that

$$\Psi = e^{i\alpha(X)} \Phi,$$

and from the equality of the momentum existential weight it follows that there exist an operator function $\beta(P)$ such that

$$\Psi = e^{i\beta(P)} \Phi.$$  

The Pauli partners $\Psi$ and $\Phi$ are therefore eigenstates, or fix points, of two unitary operators,

$$e^{i\alpha(X)} e^{-i\beta(P)} \Psi = \Psi$$

$$e^{-i\alpha(X)} e^{i\beta(P)} \Phi = \Phi.$$  

To avoid misunderstanding it must be clear that the two operators $e^{i\alpha(X)}$ and $e^{i\beta(P)}$ are different operators but for the pair $(\Psi, \Phi)$ they have the same effect, that is, $\Phi \rightarrow \Psi$. For any other Hilbert space element they produce different results; clearly, we can not represent these operators by $\Psi \langle \Phi, \cdot \rangle$. The main difficulty in dealing with Pauli partners is that we know that they exist, but we do not have a complete characterization of them. Therefore we can not give explicit expressions for the functions $\alpha(X)$ and $\beta(P)$ and, for instance, we don’t know the commutator $[\alpha(X), \beta(P)]$.

In a numerical survey\cite{15} done with an iterative algorithm for the determination of states in finite dimensional Hilbert spaces\cite{15}, large number of Pauli partners were found and in all cases the partners were differentiated by the correlation operator. With this numerical result one could jump to the conjecture that the correlation observable always resolves the Pauli partner ambiguity. However this conclusion could be wrong because the existence of a set of Pauli partners with null measure is not excluded in a numerical survey and therefore we can never be sure to have analysed all Pauli partners. Furthermore the survey involves only low dimensional Hilbert spaces and there is no guaranty that the same is true in infinite dimensions.

For an analytical treatment of the possibility to resolve the Pauli partner ambiguity by means of the correlation observable, we should calculate and compare the
two distributions $\sigma_\Psi(c) = |\langle \eta^c, \Psi \rangle|^2 + |\langle \eta^c, \Psi \rangle|^2$ and $\sigma_\Phi(c) = |\langle \eta^c, \Phi \rangle|^2 + |\langle \eta^c, \Phi \rangle|^2$. A simpler approach, motivated by the numerical survey, is to look at the expectation values of these distributions and compare $\langle \Psi, C\Psi \rangle$ with $\langle \Phi, C\Phi \rangle$. This is however also affected by the same difficulties mentioned before and the question whether the additional information provided by the correlation observable is sufficient in order to solve the Pauli ambiguity is still open.

VIII. CORRELATION IN THE QFT INTERPRETATION OF QM

Position-momentum correlations have a simple explanation in an interpretation of quantum mechanics (QM) suggested by quantum field theory (QFT). In this interpretation, consistent with the ontological choice for the indeterminacies mentioned in fourth section of this work, we can view the “probability cloud” as a permanent creation, propagation and annihilation of virtual particles in an indefinite number making up the quantum field associated to some particle type. We can think that the virtual particles are the components of the field that have objective but ephemeral existence with position and momentum. In this view, the Feynman graphs are not only mathematical terms of a perturbation expansion but represent real excitations of the quantum field.

Let us imagine then virtual components of the field created at a location at “the right” of the one dimensional distribution $\rho(x)$, that is with a positive value for the observable $X - \langle X \rangle$. If these components are moving with momentum smaller than the mean value, that is, with negative value for $P - \langle P \rangle$ the relative motion will be towards the center and the distribution will shrink. Similarly, the components created at the left and moving to the right have the two offsets $X - \langle X \rangle$ and $P - \langle P \rangle$ with different sign, that is, their (symmetrized) product is negative.

For simplicity, let us assume that in this state we have $\langle X \rangle = \langle P \rangle = 0$ (the general state is obtained with the translation and impulsion operator). Therefore the product of the two offsets in position and momentum is precisely the correlation observable and the previous argument means that if the correlation is negative the space distribution shrinks. We can prove this with rigour: let us calculate the time derivative of the width of the distribution $\Delta^2 x = \langle X^2 \rangle$. In the Heisenberg picture,
assuming a nonrelativistic hamiltonian $H = P^2/2m$, we have

\[ \frac{dX^2}{dt} = -\frac{i}{\hbar} [X^2, H] = \frac{-i}{2\hbar m} [X^2, P^2] = \frac{1}{m} (XP + PX) = \frac{2}{m} C. \quad (24) \]

Taking expectation values we conclude that states with negative correlation shrink and states with positive correlation expand, as expected from the heuristic argument given above.

The momentum distribution for a free particle is time independent and if the state is shrinking, that is, with negative correlation, we are approaching the limit imposed by Heisenberg indeterminacy principle. This principle will not be violated because the correlation will not remain always negative: at some time it will become positive and the state will begin to expand. In fact, we can prove that the correlation is always increasing in time:

\[ \frac{dC}{dt} = -\frac{i}{\hbar} [C, H] = -\frac{i}{4\hbar m} [XP + PX, P^2] = \frac{1}{m} P^2 = 2H, \quad (25) \]

and this is a nonnegative operator. If a state is shrinking, at some later time it will be spreading. Gaussian states of this sort have been reported in a very comprehensive paper.

It is interesting to notice that the fact that the correlation (like entropy in thermodynamics) is always increasing can be used to define a quantum mechanical arrow of time without recourse to the state collapse that is one of the most mysterious features of quantum mechanics.

**IX. FINAL COMMENTS**

In this work we have seen that position and momentum observables in quantum mechanics are more subtle than their corresponding classical variables. In particular it is interesting to notice that these are the unique relevant observables for the simplest quantum system of a free particle, but they are *not sufficient* to fix the state: in this quantum system there must be something else that has to be specified for an unambiguous determination of its behaviour. It is suggested that the correlation between position and momentum can play this role.

We have presented several features of the correlation that becomes an intuitive explanation in an interpretation of quantum mechanics where the virtual particles
acquire real, but ephemeral, existence. It is remarkable that the sign of the correlation expectation value controls the shrinking and spreading of a wave packet. The correlation for a free particle is always increasing and therefore in the long term states expand. In this context, it is easy to prove that there is an important class of states—the coherent states—with vanishing expectation value for the correlation (but there are other states, not coherent, that also have zero expectation value for the correlation).

Another place where the correlation appears, not shown in this work but related with the comment above, is in the improved version of Heisenbergs uncertainty principle, derived by Schrödinger [19, 20], where an extra term involving the anti-commutator (that is, the correlation) besides the commutator contribution limit the uncertainty product.

Finally let us conclude with some speculative comments. We usually try to understand quantum mechanics as an extension of classical mechanics with new concepts beyond classical physics. So, to the energy of an oscillator we must add the zero point energy $\omega \hbar /2$ and to the orbital angular momentum we include the intrinsic spin $\hbar /2$ that have an essential quantum origin. With the correlation something similar happens: if we use the position-momentum commutator to write it as $C = PX + i\hbar /2$, we see that an essential quantum contribution $\hbar /2$ is added to the classical correlation. If we are ever to have a different paradigm to explain quantum mechanics, it will have to bring some rationale for the zero point energy, the zero point angular momentum and the zero point correlation.

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