Cohomology of Quotients in Real Symplectic Geometry

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Abstract

Given a Hamiltonian system \((M, \omega, G, \mu)\) where \((M, \omega)\) is a symplectic manifold, \(G\) is a compact connected Lie group acting on \((M, \omega)\) with moment map \(\mu : M \to \mathfrak{g}^*\), then one may construct the symplectic quotient \((M//G, \omega_{\text{red}})\) where \(M//G := \mu^{-1}(0)/G\). Kirwan used the norm-square of the moment map, \(|\mu|^2\), as a \(G\)-equivariant Morse function on \(M\) to derive formulas for the rational Betti numbers of \(M//G\).

A real Hamiltonian system \((M, \omega, G, \mu, \sigma, \phi)\) is a Hamiltonian system along with a pair of involutions \((\sigma : M \to M, \phi : G \to G)\) satisfying certain compatibility conditions. These imply that the fixed point set \(M^\sigma\) is a Lagrangian submanifold of \((M, \omega)\) and that \(M^\sigma//G^\phi := (\mu^{-1}(0) \cap M^\sigma)/G^\phi\) is a Lagrangian submanifold of \((M//G, \omega_{\text{red}})\). In this paper we prove analogues of Kirwan’s Theorems that can be used to calculate the \(\mathbb{Z}_2\)-Betti numbers of \(M^\sigma//G^\phi\). In particular, we prove (under appropriate hypotheses) that \(|\mu|^2\) restricts to a \(G^\phi\)-equivariantly perfect Morse-Kirwan function on \(M^\sigma\) over \(\mathbb{Z}_2\) coefficients, describe its critical set using explicit real Hamiltonian subsystems, prove equivariant formality for \(G^\phi\) acting on \(M^\sigma\), and combine these results to produce formulas for the \(\mathbb{Z}_2\)-Betti numbers of \(M^\sigma//G^\phi\).

1 Introduction

1.1 Motivation and Goals

A Hamiltonian system \(\mathcal{H} = (M, \omega, G, \mu)\) consists of a Lie group \(G\) with Lie algebra \(\mathfrak{g}\) acting on a symplectic manifold \((M, \omega)\) via a moment map \(\mu : M \to \mathfrak{g}^*\). We will always assume that \(M\) is connected, that \(G\) is both compact and connected and
identify $g \cong g^*$ using an invariant inner product. If $G$ acts freely on the zero level set $M_0 := \mu^{-1}(0)$ then we may construct the symplectic quotient $(M//G, \omega_{\text{red}})$, where $M//G := M_0/G$ is a smooth manifold with symplectic form $\omega_{\text{red}}$.

A real Hamiltonian system $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$ is a Hamiltonian system $(M, \omega, G, \mu)$ equipped with an anti-symplectic involution $\sigma : M \to M$ ($\sigma^2 = \text{Id}_M$ and $\sigma^* \omega = -\omega$) and a Lie group automorphism $\phi : G \to G$ of order two ($\phi^2 = \text{Id}_G$) satisfying certain compatibility conditions (see Definition 3). These imply that the real locus $M^\sigma := \{x \in M \mid \sigma(x) = x\}$ is a Lagrangian submanifold of $M$ and that the real subgroup $G^\phi := \{g \in G \mid \phi(g) = g\}$ restricts to an action on $M^\sigma$. If $G$ acts freely on $M_0$, then the real quotient $M^\sigma//G^\phi := M^\sigma_0/G^\phi$ embeds as a Lagrangian submanifold in the symplectic quotient $M//G$. The goal of this paper is to develop Morse theory techniques to calculate the $\mathbb{Z}_2$-Betti numbers of the real quotient $M^\sigma//G^\phi$ in analogy with Kirwan’s techniques from [16].

In the special case when $G$ is a torus and $\phi(g) = g^{-1}$, a real analogue of Kirwan’s equivariant perfection was proven by Goldin-Holm [10], and a real analogue of equivariant formality was proven by Biss-Guillemin-Holm [3], building on work of Duistermaat [7]. Our paper extends these results to non-abelian real Hamiltonian systems.

Equivariant perfection for non-abelian real Hamiltonian systems has been used in topological gauge theory by Liu-Schaffhauser [19] and by the first author [2] to study moduli spaces of real vector bundles over a real curve. Those papers use a real structure on the Atiyah-Bott [1] Yang-Mills Hamiltonian system.

1.2 Kirwan’s Theorems

Given an invariant inner product on the Lie algebra $g$, one can form the norm-square of the moment map

$$f = |\mu|^2 : M \to \mathbb{R}.$$ 

Suppose that $f$ is proper. Kirwan [16] showed that $f$ is minimally degenerate (or Morse-Kirwan) and used this to derive formulas for the equivariant cohomology. In particular, the critical set $C_f$ of $f$ is a disjoint union of $G$-invariant closed subsets $\{C_\beta \mid \beta \in \Lambda\}$ and the function $f$ is $G$-equivariantly perfect over the field of the rational numbers $\mathbb{Q}$; i.e.

$$P^G_t(M; \mathbb{Q}) = \sum_{\beta \in \Lambda} t^{2d_\beta} P^G_t(C_\beta; \mathbb{Q}),$$
where we define the equivariant Poincaré series relative to a field $\mathbb{F}$ to be the generating function
\[
P_t^G(M, t; \mathbb{F}) := \sum_{i=0}^{\infty} \dim(H^i_G(M; \mathbb{F})) t^i
\]
and $2d_\beta$ is the Morse index of $f$ along $C_\beta$ which is necessarily even. If we assume for simplicity that the Morse index is constant along $C_\beta$, then the index set $\Lambda \subset \mathfrak{g}$ is a discrete subset of the Lie algebra and $C_0 = M_0$ is the global minimum.

If additionally $M$ is compact, then Kirwan also proved that $M$ is $G$-equivariantly formal over the field of the rational numbers $\mathbb{Q}$; i.e., the Serre spectral sequence of the fibration $M \hookrightarrow M_G \to BG$, induced by the homotopy quotient space $M_G$, collapses at page two and thus
\[
H^*_G(M; \mathbb{Q}) \cong H^*(BG; \mathbb{Q}) \otimes_{\mathbb{Q}} H^*(M; \mathbb{Q}),
\]
where $BG$ is the classifying space of the Lie group $G$. Combining these results yields a formula for the rational equivariant Betti numbers of the zero level set $M_0 := \mu^{-1}(0)$:
\[
P_t^G(M_0; \mathbb{Q}) = P_t(M; \mathbb{Q})P_t(BG; \mathbb{Q}) - \sum_{\beta \neq 0} t^{2d_\beta} P_t^G(C_\beta; \mathbb{Q}),
\]
where $C_0 = M_0$. If $G$ acts freely on the zero level set $M_0$, then $H^*_G(M_0; \mathbb{Q}) = H^*(M/\!/G; \mathbb{Q})$, so (1) can be used to compute the Betti numbers of $M/\!/G$.

Kirwan showed that for each $\beta \in \Lambda$, there is a Hamiltonian subsystem $\mathcal{H}_\beta = (Z_\beta, \omega, G_\beta, \mu_\beta)$ in which $Z_\beta \subset M$ is a symplectic submanifold, $G_\beta \leq G$ is the stabilizer subgroup of $\beta$ and $C_\beta = G \times_{G_\beta} M_\beta$ where $M_\beta = \mu_\beta^{-1}(0)$. This implies $P_t^G(M_\beta; \mathbb{Q}) = P_t^G(C_\beta; \mathbb{Q})$ so we obtain the formula
\[
P_t^G(M_0; \mathbb{Q}) = P_t(M; \mathbb{Q})P_t(BG; \mathbb{Q}) - \sum_{\beta \neq 0} t^{2d_\beta} P_t^G(M_\beta; \mathbb{Q}),
\]
which is recursive in the dimension of $M_\beta$.

1.3 Summary of Results

Consider a real Hamiltonian system $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$ where $G$ is compact and connected and $f = |\mu|^2$ is proper. Consider the restricted function $f^\sigma := f|_{M^\sigma} : M^\sigma \to \mathbb{R}$. We construct a Morse stratification
\[
M^\sigma = \bigcup_{I \in \mathcal{I}} S^\sigma_I
\]
such that
(i) Each stratum \( S^\sigma_I \subseteq M^\sigma \) is a \( G^\sigma \)-invariant locally closed submanifold of constant codimension \( d_I \) that equivariantly deformation retracts onto its critical subset \( C^\sigma_I = C_{f^\sigma} \cap S^\sigma_I \).

(ii) There are real Hamiltonian subsystems \( \mathcal{RH}_I = (Z_I, \omega, G_I, \mu_I, \sigma, \phi) \) such that

\[
C^\sigma_I \cong G^\sigma \times_{G^\sigma_I} M_I
\]

where \( M_I := \mu_I^{-1}(0) \subseteq Z_I \).

In \( \S \) 6, we consider equivariant Thom-Gysin long exact sequence

\[
\cdots \to H^{*-d_I}_{G^\sigma}(S^\sigma_I; \mathbb{Z}_2) \xrightarrow{i_I} H^{*-d_I}_{G^\sigma}(\bigcup_{J \leq I} S^\sigma_J; \mathbb{Z}_2) \to H^{*-d_I}_{G^\sigma}(\bigcup_{J < I} S^\sigma_J; \mathbb{Z}_2) \to \cdots
\]

using an appropriate total order on \( \mathcal{I} \). We show that if

(i) \((G, \phi)\) has the free extension property (see \( \S \) 4.3), and

(ii) \( \mathcal{RH} \) is 2-primitive (see \( \S \) 6)

then the top equivariant Stiefel-Whitney class of the \( G^\sigma \)-equivariant normal bundle of each stratum \( S^\sigma_I \) is not a zero divisor. This forces \( i_I \) to be injective and breaks the long exact sequence into short exact sequences:

\[
0 \to H^{*-d_I}_{G^\sigma}(S^\sigma_I; \mathbb{Z}_2) \xrightarrow{i_I} H^{*-d_I}_{G^\sigma}(\bigcup_{J \leq I} S^\sigma_J; \mathbb{Z}_2) \to H^{*-d_I}_{G^\sigma}(\bigcup_{J < I} S^\sigma_J; \mathbb{Z}_2) \to 0. \tag{2}
\]

By induction, this implies that \( f^\sigma \) is equivariantly perfect over the field \( \mathbb{Z}_2 \); i.e.,

\[
\mathbf{P}_t^{G^\sigma}(M^\sigma) = \sum_I t^{d_I} \mathbf{P}_t^{G^\sigma}(S^\sigma_I) = \sum_I t^{d_I} \mathbf{P}_t^{G^\sigma}(M^\sigma_I)
\]

Rearranging yields a recursive formula

\[
\mathbf{P}_t^{G^\sigma}(M^\sigma_0) = \mathbf{P}_t^{G^\sigma}(M^\sigma) - \sum_{I \neq 0} t^{d_I} \mathbf{P}_t^{G^\sigma}(M^\sigma_I).
\]

and \( \mathbf{P}_t(M^\sigma \sslash G^\sigma) = \mathbf{P}_t^{G^\sigma}(M^\sigma_0) \) if the action of \( G \) on \( M_0 \) is free.

In \( \S \) 7 we prove that if \( M^\sigma \) is compact and the pair \((G, \phi)\) satisfies a toral free extension property (Definition \( \S \) 10), then \( M^\sigma \) is an equivariantly formal \( G^\sigma \)-space, meaning

\[
H^*_{G^\sigma}(M^\sigma; \mathbb{Z}_2) \cong H^*(BG^\phi; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(M^\sigma; \mathbb{Z}_2)
\]
and $P_t^{G^\phi}(M^\sigma) = P_t(M^\sigma)P_t(BG^\phi)$.

In §8 we apply our results in a number of examples.

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## 2 Real Symplectic Geometry

In this section we recall relevant notions in real symplectic geometry. We suggest Sjamaar [23] for a nice introduction to this topic.

### 2.1 Hamiltonian actions and symplectic quotients

**Definition 1.** Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that $G$ acts on a symplectic manifold $(M, \omega)$. A map $\mu : M \to \mathfrak{g}^*$ is called a **moment map** if the following are satisfied.

(i) For any $X \in \mathfrak{g}$, we have

$$d\mu^X = \iota_{X^\#} \omega,$$

where $\mu^X : M \to \mathbb{R}$ is defined $\mu^X(p) = \langle \mu(p), X \rangle$ and $X^\#$ is the vector field on $M$ generated by the action of $G$; i.e.,

$$X^\#(p) = \frac{d}{dt}\bigg|_{t=0} \left( \exp(tX).p \right).$$

(ii) $\mu$ is an equivariant map; i.e.,

$$\mu(g.p) = Ad^*_g \left( \mu(p) \right), \ \forall g \in G, \forall p \in M.$$

In this case, the tuple $(M, \omega, G, \mu)$ is called a **Hamiltonian system**.

Let $(M, \omega, G, \mu)$ be a Hamiltonian system where $G$ is a compact Lie group. Suppose that $M_0 = \mu^{-1}(0)$ is the zero level set of $\mu$ and $i : M_0 \hookrightarrow M$ is the inclusion map. Since $M_0$ is $G$-invariant, $G$ acts on $M_0$. Denote the orbit space $MG := M_0/G$ and let $q : M_0 \to MG$ the quotient map.
Proposition 1. If $M_0$ is nonempty and $G$ acts freely on $M_0$, then the orbit space $M//G$ is a smooth manifold with $\dim M//G = \dim M - 2\dim G$ equipped with a symplectic structure $\omega_{\text{red}}$ defined by $i^*\omega = q^*\omega_{\text{red}}$. The pair $(M//G, \omega_{\text{red}})$ is called the symplectic reduction or symplectic quotient of $(M, \omega, G, \mu)$.

Proof. See [6], Chapter 23.

2.2 Real Structures on Symplectic Manifolds

Definition 2. Let $(M, \omega)$ be a symplectic manifold. An anti-symplectic involution on $M$ is a diffeomorphism $\sigma : M \to M$ having the following properties.

(1) $\sigma^2 = \text{Id}.$

(2) $\sigma^*\omega = -\omega.$

We call $(M, \omega, \sigma)$ a real symplectic manifold. The fixed point set $M^\sigma$ is a Lagrangian submanifold of $M$, called a real Lagrangian (see [7]).

Example 1. Equip $\mathbb{C}^n$ with the standard symplectic structure $\omega = \sqrt{-1} \sum_{i=1}^n dz_i \wedge d\bar{z}_i$. The involution $\sigma : \mathbb{C}^n \to \mathbb{C}^n$ by $\sigma(z_1, \ldots, z_n) = (\bar{z}_1, \ldots, \bar{z}_n)$ which conjugates each coordinate is an anti-symplectic involution. The real locus is $\mathbb{R}^n$.

Example 2. The complex Grassmannian $\text{Gr}_k(\mathbb{C}^n)$ equipped with the standard symplectic form (see Example 10) admits a real structure induced by the complex conjugation involution on $\mathbb{C}^n$ described in Example 1. The real locus is the real Grassmanian $\text{Gr}_k(\mathbb{R}^n)$.

Example 3. Consider a smooth manifold $M$ and denote $(x, \xi)$ for the coordinates on its cotangent bundle $T^*M$. Let $\omega = \sum_1^n dx_i \wedge d\xi_i$ be the canonical symplectic form on $T^*M$. Suppose there exists a smooth automorphism $\sigma : M \to M$ of order two. This extends to an anti-symplectic involution $\sigma : T^*M \to T^*M$ as follows:

$$\sigma(x, \xi) = (\sigma(x), -\xi \circ d\sigma_x).$$

Definition 3. A real structure on a Hamiltonian system $(M, \omega, G, \mu)$ is a pair of smooth maps $\sigma : M \to M$ and $\phi : G \to G$ such that the following are satisfied.

(i) $\phi$ is a group involution; i.e., a group automorphism of order two.

(ii) $\sigma$ is an anti-symplectic involution.
(iii) $\sigma$ and $\phi$ satisfy the following compatibility conditions:

$$\sigma \circ g = \phi(g) \circ \sigma, \quad \forall g \in G, \mu \circ \sigma = -\phi \circ \mu. \quad (3)$$

Here, $\phi_* = d\phi : \mathfrak{g} \to \mathfrak{g}$.

We call $(\sigma, \phi)$ a real pair for $\mathcal{I}$ and the tuple $\mathcal{R}H = (M, \omega, G, \mu, \sigma, \phi)$ a real Hamiltonian system.

It follows from Definition 3 the real subgroup $G^\phi$ restricts to an action on the real locus $M^\sigma$ and on $M^\sigma_0 := \mu^{-1}(0) \cap M^\sigma$.

**Definition 4.** Let $\mathcal{R}H = (M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system. The orbit space $M^\sigma / G^\phi := M^\sigma_0 / G^\phi$ is called the real quotient.

The following is due to Foth [8].

**Proposition 2.** Let $\mathcal{R}H = (M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system for which $G$ compact and connected and suppose $G$ acts freely on $M_0 := \mu^{-1}(0)$. Then

- The anti-symplectic involution $\sigma : M \to M$ descends to an anti-symplectic involution $\sigma_{\text{red}} : M / G \to M / G$.

- The real quotient $M^\sigma / G^\phi$ embeds naturally in $M / G$ as a union of path components of the real locus $(M / G)^{\sigma_{\text{red}}}$.

The involution $\phi : G \to G$ induces an involution $\phi_* : \mathfrak{g} \to \mathfrak{g}$ determining an eigenspace decomposition

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-. \quad (4)$$

Clearly $\mathfrak{g}_+ = \text{Lie}(G^\phi)$ and the adjoint action by $G^\phi$ on $\mathfrak{g}$ preserves the decomposition. A consequence of (3) is that $\mu$ sends $M^\sigma$ to $\mathfrak{g}_-$.

**Example 4.** The standard action of $U(n + 1)$ on $\mathbb{C}P^n$ is Hamiltonian with respect to the Fubini-Study form $\omega_{FS}$ and moment map $\mu_{FS} : \mathbb{C}P^n \to \mathfrak{u}(n + 1)$ by

$$\mu_{FS}[z] = \frac{zz^*}{2\pi |z|^2}.$$
action by the complexification $G_C$ on $M$ and the Kempf-Ness Theorem states that the Hamiltonian quotient $M/G$ is isomorphic to a symplectic manifold with the GIT quotient $M/G_C$.

Suppose now that $M$ and $G$ are invariant under the standard complex conjugations $\sigma$ of $\mathbb{C}P^n$ and $\phi$ of $\text{GL}_n(\mathbb{C})$. Then $(M, \omega, G, \mu, \phi, \sigma)$ is a real Hamiltonian system and the quotient $M^\sigma/G^\phi$ corresponds to a totally real Lagrangian submanifold of $M/G_C$.

For any $\beta \in \mathfrak{g}$, let $O^-_\beta = \{ \text{Ad}_g \beta \mid g \in G^\phi \}$ and $O_\beta = \{ \text{Ad}_g \beta \mid g \in G \}$ be the orbits of $\beta$ with respect to the adjoint actions of $G^\phi$ and $G$, respectively.

**Lemma 1.** Let $G$ be a compact connected Lie group and $\phi \in \text{Aut}(G)$ of order two. For every $G$-orbit $O_\beta \subseteq \mathfrak{g}$ the intersection $O_\beta \cap \mathfrak{g}_-$ is a union of finitely many $G^\phi$-orbits.

**Proof.** Clearly $O_\beta \cap \mathfrak{g}_-$ is a union of $G^\phi$-orbits. We only need to show that the number of orbits is finite.

Observe that $\mathfrak{g}_-$ is the fixed point set of the linear automorphism $-\phi^*$ of $\mathfrak{g}$, which sends $G$-orbits to $G$-orbits. If $O_\beta \cap \mathfrak{g}_-$ is non-empty, then $-\phi^*$ sends $O_\beta$ to itself and $O_\beta \cap \mathfrak{g}_- = O_\beta^{-\phi^*}$ is the fixed point set of an order two automorphism. Therefore $O_\beta \cap \mathfrak{g}_-$ is union of submanifolds. Given $\alpha \in O_\beta \cap \mathfrak{g}_-$ we claim that

$$T_\alpha O^-_\beta = (T_\alpha O_\beta) \cap \mathfrak{g}_-. \quad (5)$$

Clearly $T_\alpha O^-_\beta \subseteq (T_\alpha O_\beta) \cap \mathfrak{g}_-$. Conversely, suppose that $\xi \in (T_\alpha O_\beta) \cap \mathfrak{g}_-$. Then $\xi = [X, \alpha]$, for some $X \in \mathfrak{g}$. Decompose $X = X_+ + X_-$ into eigenvectors, so $[X_+, \alpha] \in \mathfrak{g}_+^*$ and $[X_-, \alpha] \in \mathfrak{g}_-^*$. Since $\xi \in \mathfrak{g}_-$, it follows that

$$\xi = [X_+, \alpha] + [X_-, \alpha] = [X_+, \alpha] \in T_\alpha O^-_\beta.$$

Since both $O^-_\beta$ and $O_\beta \cap \mathfrak{g}_-$ are manifolds, (5) implies that $O^-_{\beta_i}$ is an open subset of $O_\beta \cap \mathfrak{g}_-$. Since $O_\beta \cap \mathfrak{g}_-$ is compact, it must be covered by a finite number of them, completing the proof.

### 3 Morse stratification for Real Hamiltonian systems

#### 3.1 Morse Stratification for Hamiltonian systems

Let $\mathcal{H} = (M, \omega, G, \mu)$ be a Hamiltonian system in which $G$ is compact and connected and $M$ is connected of dimension $2n$. Fix an Ad-invariant inner product on the Lie
algebra \( g \) which identifies \( g \cong g^* \) and fix a \( G \)-invariant Riemannian metric on \( M \) compatible with \( \omega \). Define \( f : M \to \mathbb{R} \) by

\[
f(p) = |\mu(p)|^2, \quad \forall p \in M,
\]

and assume that \( f \) is proper. This ensures that the negative gradient flow determined by the vector field \(-\nabla f\) exists for all positive time.

For any \( \beta \in g \), the component map \( \mu^\beta : M \to \mathbb{R} \) is a (not necessarily proper) Morse-Bott function whose critical set \( C_{\mu^\beta} := \{ x \in M \mid d\mu^\beta_x = 0 \} \) is a union of symplectic submanifolds with even Morse indices. Let \( Z_\beta \) be the union of path components of \( C_{\mu^\beta} \) on which \( \mu^\beta \) takes the value \( |\beta|^2 \); i.e.

\[
Z_\beta := C_{\mu^\beta} \cap \left[ (\mu^\beta)^{-1}(|\beta|^2) \right].
\]

Decompose

\[
Z_\beta = \coprod_{m=0}^n Z_{\beta,m}
\]

where \( Z_{\beta,m} \) is the set of points with Morse index \( 2m \). Denote the isotropy group \( G_\beta = \{ g \in G \mid \text{Ad}_g \beta = \beta \} \) with Lie algebra \( g_\beta \). Then each \( Z_{\beta,m} \) is \( G_\beta \)-invariant and

\[
\mathcal{H}_{\beta,m} = (Z_{\beta,m}, \omega, G_\beta, \mu_{\beta,m})
\]

is a Hamiltonian system with moment map \( \mu_{\beta,m} = \text{Pr}_\beta \circ \mu|_{Z_{\beta,m}} - \beta \) where \( \text{Pr}_\beta : g \to g_\beta \) is orthogonal projection. The following are due to Kirwan [16] (except K2 which is an improvement of Kirwan’s result due to Duistermaat-Lerman [18]).

K1 The critical set \( C_f \) of \( f \) is a finite union of disjoint, \( G \)-invariant, compact (possibly disconnected) subsets

\[
C_f = \coprod_{\beta,m} C_{\beta,m},
\]

where \( \beta \in \mathfrak{t}_+ \) are elements of the positive Weyl chamber and \( f(C_{\beta,m}) = \mathcal{O}_\beta \) is the (co)adjoint orbit of \( \beta \) and \( m \in \{0,1,\ldots,\dim(M)/2\} \).

K2 \( f \) determines a \( G \)-invariant Morse stratification into locally closed submanifolds

\[
M = \bigcup_{\beta,m} S_{\beta,m}
\]
where $S_{\beta,m}$ deformation retracts onto $C_{\beta,m}$ under the negative gradient flow by $f$. Partially order the indices by

$$(\beta_1, m_1) < (\beta_2, m_2) \Leftrightarrow |\beta_1| < |\beta_2|.$$ 

The closure of a stratum $S_{\beta,m}$ in $M$ satisfies

$$\overline{S_{\beta,m}} \subseteq \bigcup_{(\gamma, k) \geq (\beta, m)} S_{\gamma,k}$$

K3 Given $p \in C_{\beta,m}$, the tangent space $T_p S_{\beta,m}$ is a symplectic vector subspace of $T_p M$. The codimension of $S_{\beta,m}$ is even and equal to

$$2d(\beta, m) := 2m - \dim G + \dim G_{\beta}.$$ 

K4 For any $\beta$ and $m$, the Hamiltonian subsystem $H_{\beta,m} = (Z_{\beta,m}, \omega, G_{\beta}, \mu_{\beta,m})$ satisfies

$$\mu_{\beta,m}^{-1}(0) = Z_{\beta,m} \cap \mu^{-1}(\beta)$$

and

$$C_{\beta,m} = G \cdot \mu_{\beta,m}^{-1}(0) \cong G \times G_{\beta} \mu_{\beta,m}^{-1}(0).$$

### 3.2 Morse Stratification for real Hamiltonian systems

Consider a real Hamiltonian system $RH = (M, \omega, G, \mu, \sigma, \phi)$ where $G$ is compact and connected and $M$ is connected. Choose invariant compatible metrics on $M$ and $g$ so that $f = |\mu|^2$ is $\sigma$-invariant and suppose that $f$ is proper. Let

$$f^\sigma : M^\sigma \rightarrow \mathbb{R}$$

be the restriction of $f$ to the real locus $M^\sigma$. This implies that

$$\nabla f = \sigma_*(\nabla f) \quad (8)$$

and in particular that along $M^\sigma$, we have equality $\nabla f = \nabla f^\sigma$. Therefore the negative gradient flow on $M$ preserves $M^\sigma$. 

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Proposition 3. Let $\mathcal{RH}$ and $f^\sigma$ be as above. Then

(i) The critical set of $f^\sigma$ is $C_{f^\sigma} = C_f \cap M^\sigma$ and thus

$$C_{f^\sigma} = \bigsqcup_{\beta,m} C_{\beta,m}^\sigma,$$

where $C_{\beta,m}^\sigma = C_{\beta,m} \cap M^\sigma$ are $G^\phi$-invariant closed subsets of $M^\sigma$ on each of which $f^\sigma$ takes a constant value.

(ii) We have a stratification into locally closed submanifolds

$$M^\sigma = \bigcup_{\beta,m} S_{\beta,m}^\sigma,$$

where $S_{\beta,m}^\sigma = S_{\beta,m} \cap M^\sigma$ and $S_{\beta,m}^\sigma$ deformation retracts onto $C_{\beta,m}^\sigma$ via the negative gradient flow.

(iii) The closure of $S_{\beta,m}^\sigma$ in $M^\sigma$ satisfies

$$\overline{S_{\beta,m}^\sigma} \subseteq \bigcup_{(\gamma,k) \geq (\beta,m)} S_{\gamma,k}^\sigma.$$

(iv) The codimension of $S_{\beta,m}^\sigma$ in $M^\sigma$ is half the codimension of $S_{\beta,m}$ in $M$.

Proof. Both (i) and (ii) follows easily from Kirwan’s Theorems K1, K2 combined with (8).

The closure of $S_{\beta,m}^\sigma$ satisfies

$$\overline{S_{\beta,m}^\sigma} \subseteq \overline{S_{\beta,m}} \cap M^\sigma \subseteq \left( \bigcup_{(\gamma,k) \geq (\beta,m)} S_{\gamma,k} \right) \cap M^\sigma = \bigcup_{(\gamma,k) \geq (\beta,m)} S_{\gamma,k}^\sigma,$$

proving (iii).

To prove (iv) recall that at $p \in C_{\beta,m}^\sigma$ the tangent space $T_p S_{\beta,m}^\sigma$ is a symplectic vector subspace of $T_p M$. Because $\sigma$ is anti-symplectic it follows that $T_p S_{\beta,m}^\sigma$ is Lagrangian in $T_p S_{\beta,m}$, hence half dimensional. Since $M^\sigma$ has half the dimension of $M$ the result follows. ■
As we saw in Kirwan’s result K4, for each $\beta$ and $m$, there exists a Hamiltonian subsystem $H_{\beta,m} = (Z_{\beta,m},\omega,G_\beta,\mu_{\beta,m})$. It is easy to see that $(\sigma,\phi)$ also restricts to a real pair on this system.

**Proposition 4.** Choose $G^\phi$-orbits representatives $\beta_1,\ldots,\beta_k \in O_\beta \cap g_-$ (these are finite by Lemma 4). Then for each $m$ we have a natural $G^\phi$-equivariantly diffeomorphism

$$C_{\beta,m}^\sigma = \prod_{i=1}^k G^\phi \times G_{\beta_i}^\phi \mu_{\beta_i,m}^{-1}(0)^\sigma.$$ 

**Proof.** Since $\mu(C_{\beta,m}^\sigma) \subseteq O_\beta \cap g_- = \bigsqcup_{i=1}^k O^-_{\beta_i}$ is a disconnected union, we obtain

$$C_{\beta,m}^\sigma = \prod_{i=1}^k [C_{\beta,m}^\sigma \cap \mu^{-1}(O^-_{\beta_i})].$$

By K4, we have a $G$-diffeomorphism

$$C_{\beta,m} = C_{\beta_1,m} \cong G \times_{G_{\beta_1}} \mu_{\beta_1,m}^{-1}(0).$$

The subset $C_{\beta_1,m}^\sigma$ corresponds to those equivalence classes of pairs $(g,x) \in G \times \mu_{\beta_1,m}^{-1}(0)$ for which there exists $h \in G_{\beta_1}$ such that $(gh^{-1},hx) = (\phi(g),\sigma(x))$.

The subset $C_{\beta_1,m}^\sigma \cap \mu^{-1}(O^-_{\beta_1})$ corresponds to those equivalence classes containing a representative $(g,x)$ for which $g = \phi(g)$. Such a class is fixed by $\sigma$ if and only if $(gh^{-1},hx) = (\phi(g),\sigma(x)) = (g,\sigma(x))$, so $(g,x) \in G^\phi \times \mu_{\beta_1,m}^{-1}(0)$. Finally, if $g \in G^\phi$ then $gh^{-1} \in G^\phi$ if and only if $h \in G^\phi$. We conclude that

$$C_{\beta,m}^\sigma \cap \mu^{-1}(O^-_{\beta_1}) \cong G^\phi \times_{G_{\beta_1}} \mu_{\beta_1,m}^{-1}(0)^\sigma.$$ 

$\blacksquare$

## 4 The Free Extension Property

### 4.1 Elementary abelian 2-subgroups

**Definition 5.** Let $G$ be a compact Lie group. An **elementary abelian 2-subgroup** is a subgroup $E \leq G$ which is isomorphic to $(\mathbb{Z}_2)^n$ for some $n \geq 0$. We say such an $E$ is **maximal** if it is not contained in a larger elementary abelian $p$-subgroup.

**Example 5.** The $n$-torus $T^n = U(1)^n$ contains a unique maximal abelian 2-subgroup, $E(n) = \{ \lambda \in T \mid \lambda^2 = 1 \} = O(1)^n \cong (\mathbb{Z}_2)^n$. 

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Example 6. The diagonal matrix group

\[ D(n) = \left\{ \begin{pmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_n \end{pmatrix} \mid \varepsilon_i \in \{\pm 1\} \right\} \cong (\mathbb{Z}_2)^n \]

is the unique maximal elementary abelian 2-group up to conjugation in both O(n) and U(n).

Example 7. The matrix group

\[ SD(n) = \left\{ \begin{pmatrix} \varepsilon_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \varepsilon_n \end{pmatrix} \in D(n) \mid \prod_{i=1}^n \varepsilon_i = 1 \right\} \cong (\mathbb{Z}_2)^{n-1} \]

is the unique maximal elementary abelian 2-group up to conjugation in both SO(n) and SU(n).

In general, a compact connected Lie group G may contain more than one conjugacy class of maximal elementary abelian 2-groups. We refer to Griess [9] for a classification.

4.2 Free extensions and spectral sequences

Definition 6. Let \( X \) be a left \( G \)-space and \( EG \to BG \) be a fixed universal \( G \)-bundle. The twisted product space \( X_G = EG \times_G X \) is called the homotopy quotient or the Borel construction of \( X \) with respect to the fixed universal bundle. Given a commutative ring with unity \( R \) define the equivariant cohomology

\[ H^*_G(X; R) = H^*(X_G; R). \]

Let \( i : K \hookrightarrow G \) be an inclusion of compact Lie groups, and suppose \( G \) acts on a connected manifold \( X \). Then the homotopy quotients fit into the natural commutative diagram:

\[
\begin{array}{ccc}
G/K & \xrightarrow{jx} & X_K & \xrightarrow{iX} & X_G \\
\downarrow & & \downarrow & & \downarrow \\
G/K & \xrightarrow{j} & BK & \xrightarrow{Bi} & BG
\end{array}
\]
Proposition 5. Let $G$, $K$, and $X$ be as above and let $F$ be a field. If $j^* : H^*(BK; F) \to H^*(G/K)$ is surjective then we have an isomorphism

$$H_K^*(X; F) \cong H^*(G/K; F) \otimes_K H_G^*(X; F)$$

as graded $H^*_G(X; F)$-modules. In particular, $i_X^* : H^*_G(X; F) \to H^*_K(X; F)$ is injective.

Proof. If $j^*$ is surjective, then $j_X^* : H_K^*(X; F) \to H^*(G/K; F)$ must also be surjective by applying cohomology to the commutative diagram (9). The result now follows by the Leray-Hirsch Theorem.

In the following proposition, we give a criterion for the surjectivity of the morphism $j^*$ in the fibration $G/K \to BK \to BG$. Recall that a ring homomorphism $\varphi : R \to S$ between commutative rings with unity is called a free extension if $S$ is a free $R$-module with respect to the module structure induced by $\varphi$.

Proposition 6. Let $K \leq G$ be a pair of compact Lie groups and $F$ be a field. Then the following are equivalent

(i) $j^* : H^*(BK; F) \to H^*(G/K)$ is surjective.

(ii) $Bi^* : H^*(BG; F) \to H^*(BK; F)$ is a free extension and the action of $\pi_1(BG)$ on $H^*(G/K; F)$ is trivial.

Proof. That (i) implies (ii) is proven in ([21] Theorem 4.4). Conversely, suppose $Bi^* : H^*(BG; F) \to H^*(BK; F)$ is a free extension and the action of $\pi_1(BG)$ on $H^*(G/K; F)$ is trivial. The Eilenberg-Moore spectral sequence of this fibration converges strongly to $H^*(G/K; F)$ and its second page has the following form:

$$E_2^{*,*} \cong F \otimes_{H^*(BG; F)} H^*(BK; F).$$

The free extension condition forces this spectral sequence to collapse to the zeroth column $E_2^{0,*}$ which implies that

$$H^*(G/K; F) \cong F \otimes_{H^*(BG; F)} H^*(BK; F).$$

We can write

$$H^*(BK; F) \cong H^*(BG; F) \otimes_F F \otimes_{H^*(BG; F)} H^*(BK; F) \cong H^*(BG; F) \otimes_F H^*(G/K; F).$$

That is, the Serre spectral sequence of the fibration collapses at page two and by the Leray-Hirsch theorem, $j^* : H^*(BK; F) \to H^*(G/K; F)$ is surjective.
Example 8. Let $T := U(1)^n$ be an $n$-torus and $E(n) := O(1)^n$ be its unique maximal elementary abelian 2-subgroup. Consider the fibration $T/E(n) \xrightarrow{j} BE(n) \xrightarrow{Bi} BT$. The cohomology of classifying spaces $BT$ and $BE(n)$ are

$$H^*(BT; \mathbb{Z}_2) = \mathbb{Z}_2[c_1, \ldots, c_n]$$

and

$$H^*(BE(n); \mathbb{Z}_2) = \mathbb{Z}_2[x_1, \ldots, x_n]$$

where $c_i$ is the Chern class of degree $i$, $x_i$ has degree one and the induced morphism $Bi^*: H^*(BT; \mathbb{Z}_2) \to H^*(BE(n); \mathbb{Z}_2)$ sends $c_i$ to $x_i^2$ (see [21], Theorem 5.11). This shows that the set

$$\{x_1^{m_1} \ldots x_n^{m_n} \mid m_k \in \{0, 1\}\}$$

is a basis for $H^*(BE(n); \mathbb{Z}_2)$ as a $H^*(BT; \mathbb{Z}_2)$-module. Therefore, $Bi^*$ is a free extension. Since $T$ is connected, $\pi_1(BT)$ is trivial and by Proposition 6 the induced morphism $j^*: H^*(BE(n); \mathbb{Z}_2) \to H^*(T/E(n); \mathbb{Z}_2)$ is surjective.

Example 9. Let $D(n) \subseteq O(n)$ and $SD(n) \subseteq SO(n)$ be the subgroups of diagonal matrices respectively. Consider two fibrations $O(n)/D(n) \xrightarrow{j} BD(n) \xrightarrow{Bi} BO(n)$ and $SO(n)/SD(n) \xrightarrow{j_0} BSD(n) \xrightarrow{Bi_0} BSO(n)$. It is known by Theorem 5.9 in [21] that the induced morphisms

$$j^*: H^*(BD(n); \mathbb{Z}_2) \to H^*(O(n)/D(n); \mathbb{Z}_2)$$

and

$$j_0^*: H^*(BSD(n); \mathbb{Z}_2) \to H^*(SO(n)/D(n); \mathbb{Z}_2)$$

are both surjective.

4.3 The Free Extension Property for Involutive Lie Groups

An **involutive Lie group** is a pair $(G, \phi)$ in which $G$ is a Lie group and $\phi: G \to G$ is an automorphism of order 2. The subgroup of invariant elements is denoted by $G^\phi = \{g \in G \mid \phi(g) = g\}$. Decompose the Lie algebra into $\phi$-eigenspaces $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$. Given $\beta \in \mathfrak{g}_-$, consider

$$G^\phi_\beta := \{g \in G^\phi \mid \text{Ad}_g \beta = \beta\}.$$

**Definition 7.** We say the involutive Lie group $(G, \phi)$ has the **free extension property** (FEP) if for all $\beta \in \mathfrak{g}_-$ and every maximal abelian 2-subgroup $D_\beta \leq G^\phi_\beta$, the induced morphism $H^*(BD_\beta; \mathbb{Z}_2) \to H^*(G^\phi_\beta/D_\beta; \mathbb{Z}_2)$ is surjective.
Remark 1. The terminology “free extension” comes from Proposition 6.

Remark 2. An involutive Lie group \((G, \phi)\) has FEP if and only if \((G^\phi_{\beta}, \text{Id})\) has FEP for every \(\beta \in g_-\).

Proposition 7. Let \(G\) be a compact connected Lie group. If the integral cohomology \(H^*(G; \mathbb{Z})\) contains no 2-torsion, then the pair \((G, \text{Id})\) has the free extension property.

Proof. Let \(D\) be a maximal elementary abelian 2-subgroup of \(G\). By a theorem of Borel (see [14], Theorem 1.1), since \(H^*(G; \mathbb{Z})\) has no 2-torsion, there exists a maximal torus \(T\) in \(G\) such that \(D \leq T\). The morphism \(j : G/D \to BD\) fits into a commuting diagram

\[
\begin{array}{ccc}
T/D & \xrightarrow{k} & G/D \\
\downarrow & & \downarrow j \\
T/D & \xrightarrow{i} & BD \\
\end{array}
\]

which is a pullback of \(T/D\) fibre bundles. By Example 8 we know that \(i^*\) is surjective, from which it follows that \(k^*\) is surjective. Therefore by Leray Hirsch we have \(H^*(G/D; \mathbb{Z}_2) \cong H^*(T/D; \mathbb{Z}_2) \otimes H^*(G/T; \mathbb{Z}_2) \cong H^*(T/D; \mathbb{Z}_2) \otimes H^*(BT; \mathbb{Z}_2)\) and these isomorphisms identify \(j^*\) with \(\text{Id}_{H^*(T/D; \mathbb{Z}_2)} \otimes (j')^*\). Since \((j')^*\) is known to be surjective by Theorem 8.3 in [20], we conclude that \(j^*\) is surjective. \(\blacksquare\)

Proposition 8. The following pairs

\[
(U(n), \text{Id}), (SU(n), \text{Id}), (Sp(n), \text{Id}), (O(n), \text{Id}), (SO(n), \text{Id})
\]

have the free extension property.

Proof. Since the integral cohomology of compact connected Lie groups \(U(n), SU(n)\) and \(Sp(n)\) have no 2-torsion (see [21], Corollary 3.11), Proposition 7 follows that \((U(n), \text{Id}), (SU(n), \text{Id}), (Sp(n), \text{Id})\) have free extension property. The result for \((O(n), \text{Id})\) and \((SO(n), \text{Id})\) was established in Example 9. \(\square\)

Proposition 9. If \((G, \phi)\) and \((H, \psi)\) have the free extension property, then \((G \times H, \phi \times \psi)\) also has the free extension property.

Proof. Given \(\beta = \beta_1 + \beta_2 \in (g \oplus h)_- = g_- \oplus h_-\) then

\[
(G \times H)^{\phi \times \psi}_\beta = G^{\phi}_{\beta_1} \times H^{\psi}_{\beta_2}.
\]

A maximal abelian 2-subgroup in \(G^{\phi}_{\beta_1} \times H^{\psi}_{\beta_2}\) must be a product \(D_{\beta_1} \times D_{\beta_2}\) of a maximal abelian 2-subgroups of the factors. Applying the Kunneth Theorem to \(G^{\phi}_{\beta_1} / D_{\beta_1} \times H^{\psi}_{\beta_2} / D_{\beta_2} \to BD_{\beta_1} \times BD_{\beta_2}\) completes the proof. \(\blacksquare\)
Proposition 10. Both \((U(n), \phi)\) and \((SU(n), \phi)\) have the free extension property, where \(\phi(A) = \overline{A}\) is entry-wise complex conjugation.

Proof. We start with the case \(U(n)\). We have \(U(n)^\phi = O(n)\) and \(\sqrt{-1}\beta\) is a real symmetric matrix for every \(\beta \in u(n)_-\). By the Spectral Theorem, \(\beta\) is orthogonally diagonalizable with imaginary eigenvalues. Therefore

\[ O(n)^{\beta} \cong O(k_1) \times \cdots \times O(k_p). \]

where \(k_1, \ldots, k_p\) are the multiplicities of the eigenvalues of \(\beta\), so \(k_1 + \ldots, + k_p = n\). The result now follows from Propositions 8, 9 and Remark 2.

In the case \(SU(n)\) we obtain similarly \(SU(n)^\phi = SO(n)\) and

\[ SO(n)^{\beta} \cong SO(n) \cap (O(k_1) \times \cdots \times O(k_p)). \]

Given a maximal elementary abelian 2-group \(SD \leq SO(n)^{\beta}\), there exists a maximal elementary abelian 2-group \(D \subseteq O(n)^{\beta}\) such that \(SD = D \cap SO(n)^{\beta}\). We obtain an equality of quotients

\[ F := O(n)^{\beta}/D = SO(n)^{\beta}/SD, \]

and a pullback of fibre bundles

\[
\begin{array}{ccc}
SO(n)^{\beta}/SD & \rightarrow & O(n)^{\beta}/D \\
j \downarrow & & \downarrow j' \\
BSD & \rightarrow & BD \\
\downarrow & & \downarrow \downarrow \\
BSO(n)^{\beta} & \rightarrow & BO(n)^{\beta}
\end{array}
\]

(10)

so surjectivity of \(j^*\) follows from the surjectivity of \(j'^*\) by commutativity. 

Proposition 11. Let \(G = U(n) \times U(n)\) and \(\phi : G \rightarrow G\) be defined by \(\phi(A, B) = (\overline{B}, \overline{A})\). Then \((G, \phi)\) satisfies the free extension property.

Proof. We have \(G^\phi = \{(A, \overline{A}) \mid A \in U(n)\} \cong U(n)\) and \(g_- = \{(X, -\overline{X}) \mid X \in u(n)\}\).

Given \(\beta = (X, -\overline{X}) \in g_-\), observe that \(AXA^{-1} = X\) if and only if \(\overline{A}(-\overline{X})\overline{A}^{-1} = -\overline{X}\). Therefore \(G^\phi_\beta\) is isomorphic to the centralizer of \(X\) in \(U(n)\). Since \(X\) is skew-Hermitian, the Spectral Theorem implies \(X\) is orthogonally diagonalizable, so

\[ G^\phi_\beta \cong U(k_1) \times \cdots \times U(k_p). \] (11)
for some natural numbers \(k_1, \ldots, k_p\) with \(k_1 + \cdots + k_p = n\). The proof now follows from Propositions 8, 9 and Remark 2.

**Proposition 12.** Consider the projective unitary group \(G = \text{PU}(4n + 2)\) for \(n \geq 1\) and \(\phi : G \to G\) defined by \(\phi(A) = \overline{A}\). Then \((G, \phi)\) does not satisfy the free extension property.

**Proof.** A necessary condition for \((G, \phi)\) to have the free extension property is that \(H^*(BG^n \phi, \mathbb{Z}_2)\) be an integral domain for all \(\beta \in \mathfrak{g}_-\), for otherwise it can't possibly inject into the polynomial ring \(H^*(BD\beta; \mathbb{Z}_2)\). The real subgroup is \(\text{PU}(4n + 2)\), and the ring \(H^*(B\text{PO}(4n + 2); \mathbb{Z}_2)\) contains zero divisors (see [17] Proposition 5.8).

### 5 An Atiyah-Bott Lemma

In this section we prove a version of Atiyah-Bott Lemma (Proposition 13.4 [1]) for \(\mathbb{Z}_2\)-cohomology.

**Definition 8.** Let \(E \to X\) be a \(G\)-equivariant real vector bundle where \(G\) is a compact Lie group. We say \(E\) is 2-primitive if there exists an elementary abelian 2-subgroup \(D_0\) of \(G\) that acts trivially on \(X\) and fixes no nonzero vectors in \(E\).

**Proposition 13 (Atiyah-Bott Lemma for \(\mathbb{Z}_2\)-cohomology).** Let \(G\) be a compact Lie group and \(\pi : E \to X\) be a \(G\)-equivariant vector bundle of rank \(m\) over a connected manifold \(X\). If \((G, \text{Id})\) has the free extension property and the \(G\)-vector bundle is 2-primitive, then the equivariant top Stiefel-Whitney class \(w^G_m(E) := w_m(E_G)\) is not a zero-divisor in \(H^*_G(X; \mathbb{Z}_2)\).

**Proof.** By the assumptions, the vector bundle \(E \to X\) is 2-primitive, so there exists an elementary abelian 2-subgroup \(D_0\) of \(G\) that acts trivially on \(X\) and fixes no nonzero vectors in \(E\). Choose a maximal elementary abelian 2-subgroup \(D\) containing \(D_0\). The inclusion map \(i : D \hookrightarrow G\) induces the following commutative diagram of homotopy quotients:

\[
\begin{array}{ccc}
X_D & \xrightarrow{i_D} & X_G \\
\downarrow q_D & & \downarrow q_G \\
BD & \xrightarrow{B \pi} & BG
\end{array}
\]

By functoriality of Stiefel-Whitney classes,

\[
w^D_m(E) = i_D^*(w^G_m(E)).
\]
Since $G$ has the free extension property, Proposition 5 implies that $i^*_D$ is injective. Thus, it suffices to show $w_{m}^D(E)$ is not a zero divisor in $H^*_D(X; \mathbb{Z}_2)$.

Choose a complementary elementary abelian subgroup $D_1 \subset D$ such that $D = D_0 \times D_1$ where $D_1 = (\mathbb{Z}_2)^q$ and $p + q = n$. The action of $D_0$ on $X$ is trivial, so we have a homeomorphism

$$X_D \cong BD_0 \times X_{D_1}.$$ 

By the Kunneth formula,

$$H^*_D(X; \mathbb{Z}_2) \cong H^*(BD_0; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*_{D_1}(X; \mathbb{Z}_2).$$

This formula makes $H^*_D(X; \mathbb{Z}_2)$ into a bigraded ring. Decompose

$$w_{m}^D(E) = \alpha_0 \otimes 1 + \alpha'.$$

where $\alpha_0 \in H^m(BD_0; \mathbb{Z}_2)$ and $\alpha' \in \bigoplus_{i=1}^m H^{m-i}(BD_0; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^i_{D_1}(X; \mathbb{Z}_2)$. Because $H^*(BD_0; \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \ldots, x_p]$ is an integral domain, in order to prove $w_{m}^D(E)$ is not a zero divisor, it suffices to show that $\alpha_0$ is non-zero.

Choose a point $x \in X$. The inclusion maps $i_x : \{x\} \hookrightarrow X$ and $i_0 : D_0 \hookrightarrow D$ induce the following pullback diagram:

$$
\begin{array}{ccc}
(E_x)_{D_0} & \longrightarrow & E_{D_0} \\
\downarrow & & \downarrow \\
BD_0 & \longrightarrow & X_{D_0}
\end{array}
\quad (13)

$$

where $E_x$ is the fiber over $x$. Since $\alpha_0$ is the component of $w_{m}^D(E)$ in $H^*(BD_0; \mathbb{Z}_2)$, it follows from Diagram 13 that

$$\alpha_0 = (i_{D_0} \circ Bi_x)^* \left( w_{m}^D(E) \right).$$

Thus, $\alpha_0$ is the top Stiefel-Whitney class of the vector bundle $(E_x)_{D_0} \rightarrow BD_0$. Since $D_0$ fixes $X$, we see that $E_x$ is a representation of $D_0 \cong \mathbb{Z}_2^p$, which therefore decomposes into a direct sum of 1-dimensional $D_0$-representations $E_x^i$, determining a decomposition into line bundles

$$(E_x)_{D_0} = (E_x^1)_{D_0} \oplus \cdots \oplus (E_x^m)_{D_0}.$$ 

The Whitney sum formula yields

$$\alpha_0 = w_{m}^{D_0}(E_x) = \prod_{i=1}^m w_{1}^{D_0}(E_x^i).$$
Because $E$ is 2-primitive, each representation $E^i_x$ is non-trivial, hence the line bundles $(E^i_x)_{D_0}$ are nontrivial and

$$w_1^{D_0}(E^i_x) 
eq 0, \forall i = 1, \ldots, m.$$  

It follows from (5) that $\alpha_0 \neq 0$. This completes the proof. ■

6 Real Equivariant Perfection

Let $\mathcal{RH} = (M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system where $G$ is compact and connected and $f$ is proper. By Proposition 4 there exists a collection of real Hamiltonian subsystems $\mathcal{RH}_I = (Z_I, \omega, G_I, \mu_I, \sigma, \phi)$ indexed by $I := \{I = (\beta, m)\}$ such that the critical set of $f^\sigma$ is

$$C_{f^\sigma} = \coprod_I G^\phi \times_{G^\phi_I} M^\sigma_I.$$  \hspace{1cm} (14)

where $M^\sigma_I := \mu^{-1}_I(0) \cap Z^\sigma_I$.

Definition 9. A real Hamiltonian system $\mathcal{RH}$ is called 2-primitive if for every generated subsystem $\mathcal{RH}_I$, the negative normal bundle of $M^\sigma_I$ is a 2-primitive $G^\phi_I$-bundle (see Definition 8).

Theorem 14 (Real Equivariant Perfection). Let $(M, \omega, G, \mu, \sigma, \phi)$ be a 2-primitive real Hamiltonian system where $G$ is compact and connected and $f$ is proper. If $(G, \phi)$ has the free extension property, then $f^\sigma$ is equivariantly perfect over $\mathbb{Z}_2$. That is

$$P^G_{t^\phi} (M^\sigma; \mathbb{Z}_2) = \sum_I I_d^I P^G_{t^\phi} (C^\sigma_I; \mathbb{Z}_2),$$  \hspace{1cm} (15)

where $C^\sigma_I$ are critical subsets of $f^\sigma$ and $d_I$ is the Morse index of $f^\sigma$ along $C^\sigma_I$.

Proof. By Proposition 3 the critical set of $f^\sigma$ decomposes into a collection of disjoint closed $G^\phi$-invariant subsets $C^\sigma_I$ and there is a smooth invariant Morse stratification such that each stratum $S^\sigma_I$ has a constant codimension $d_I$ and deformation retracts onto the corresponding critical subset $C^\sigma_I$. Consider the generated real Hamiltonian subsystems $(Z_I, \omega, G_I, \mu_I, \sigma, \phi)$. By Proposition 4 we have

$$C^\sigma_I \cong G^\phi \times_{G_I} M^\sigma_I.$$  \hspace{1cm} (16)

Extend the partial ordering of Proposition 3(iii) to a total ordering on the index set $I$ satisfying the condition that $S^\sigma_I \subseteq \cup_{J \geq I} S^\sigma_J$. Define a topological filtration of
Let \( M^\sigma \) by \( S^\sigma_{\leq I} := \bigcup_{j \leq I} S^\sigma_j \) and let \( S^\sigma_{< I} := \bigcup_{j < I} S^\sigma_j \). Let \( \nu_I \) be the normal bundle of \( S^\sigma_{< I} \) in \( S^\sigma_{\leq I} \) which we identify with a tubular neighborhood, and let \( \nu^*_I = \nu_I \setminus S^\sigma_I \). Consider the commutative diagram

\[
\begin{array}{cccccccc}
\cdots \rightarrow H^*_G(S^\sigma_{\leq I}; Z_2) & \xrightarrow{\Phi_I} & H^*_G(S^\sigma_{< I}; Z_2) & \rightarrow & H^*_G(S^\sigma_{< I}; Z_2) & \rightarrow & \cdots \\
\psi_1 \equiv & & & & & & \\
\cdots \rightarrow H^*_G(\nu_I, \nu^*_I; Z_2) & \rightarrow & H^*_G(\nu_I; Z_2) & \rightarrow & H^*_G(\nu^*_I; Z_2) & \rightarrow & \cdots \\
\psi_2 \equiv & & & & & & \\
\psi_3 \equiv & & & & & & \\
H^*_G(\nu_I; Z_2) & \rightarrow & H^*_G(\nu^*_I; Z_2) & \rightarrow & H^*_G(\nu^*_I; Z_2) & \rightarrow & \cdots \\
\psi_4 \equiv & & & & & & \\
H^*_{G^\phi}(S^\sigma_I; Z_2) & \rightarrow & H^*_{G^\phi}(S^\sigma_I; Z_2) & \rightarrow & H^*_{G^\phi}(S^\sigma_I; Z_2) & \rightarrow & \cdots \\
\end{array}
\tag{17}
\]

In this diagram, the first and second rows are long exact sequences for the pairs \((S^\sigma_{\leq I}, S^\sigma_{< I})\) and \((\nu_I, \nu^*_I)\) respectively, \(\psi_1\) is an excision isomorphism, \(\psi_2\) is the Thom isomorphism and \(\psi_3\) is homotopy equivalence isomorphism. The morphism \(\psi_4\) is defined by taking cup product with the top Stiefel-Whitney class \(w^G_{m^\phi}(\nu_I)\) which plays the role of the “mod 2 Euler class”. Commutativity implies that the map \(\Phi_I\) is injective if \(w^G_{m^\phi}(\nu_I)\) is not a zero divisor.

Since \(S^\sigma_I\) deformation retracts onto \(C^\sigma_I \cong G^\phi \times C^\phi_I (\mu^{-1}_I(0)^\sigma)\), it follows that

\[
H^*_G(S^\sigma_I; Z_2) \cong H^*_G(M^\sigma_I; Z_2). \tag{18}
\]

The negative normal bundle \(\xi_I\) of \(\mu^{-1}_I(0)^\sigma\) is just the restriction of \(\nu_I\), so (18) sends \(w^G_{m^\phi}(\nu_I)\) to \(w^G_{m^\phi}(\xi_I)\).

By the assumption, \(\xi_I\) is 2-primitive with respect to the \(G^\phi_I\)-action so by the real Atiyah-Bott Lemma, \(w^G_{m^\phi}(\xi_I)\) is not a zero divisor and neither is \(w^G_{m^\phi}(\nu_I)\). Applying to (17) we get short exact sequences

\[
0 \rightarrow H^*_G(S^\sigma_{\leq I}; Z_2) \rightarrow H^*_G(S^\sigma_{< I}; Z_2) \rightarrow H^*_G(S^\sigma_{< I}; Z_2) \rightarrow 0
\]

yielding identities

\[
P^G_t(S^\sigma_{\leq I}) = P^G_t(S^\sigma_{< I}) + t^{d_I} P^G_t(S^\sigma_I)
\]

for all \(I\). Applying induction yields

\[
P^G_t(M^\sigma) = \sum_I t^{d_I} P^G_t(S^\sigma_I).
\]

Combine with \(P^G_t(S^\sigma_I) = P^G_t(C^\sigma_I) = P^G_t(M^\sigma_I)\) to complete the proof. \(\blacksquare\)

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Under the hypotheses of Theorem 14 we get a real version of Kirwan Surjectivity:

**Corollary 14.1.** The natural map \( \kappa_R : H^*_{G^\phi}(M^\sigma; \mathbb{Z}_2) \to H^*_{G^\phi}(M^\sigma_0; \mathbb{Z}_2) \) is surjective.

Rearranging (15) yields

\[
P^G_{\phi t}(M^\sigma_0) = P^G_{\phi t}(M^\sigma) - \sum_{I \neq 0} t^{|I|} P^G_{\phi I}(M^\sigma_I).
\]  

(19)

If \( G \) acts freely on \( M_0 \) then \( P^G_{\phi t}(M^\sigma_0) = P^G_{t}(M^\sigma / / G^\phi) \).

## 7 Real Equivariant Formality

A \( G \)-space \( X \) is called *equivariantly formal* with respect to a field \( \mathbb{F} \) if the Serre spectral sequence of the fibration \( X \rightarrow X_G \rightarrow BG \) collapses at the \( E_2 \)-term; i.e.

\[
H^*_G(X; \mathbb{F}) = H^*(BG; \mathbb{F}) \otimes_{\mathbb{F}} H^*(X; \mathbb{F}).
\]  

(20)

This is equivalent to the Poincaré series satisfying

\[
P^G_{t}(X; \mathbb{F}) = P_t(BG; \mathbb{F})P_t(X; \mathbb{F}).
\]  

(21)

Kirwan proved that a Hamiltonian system \( (M, \omega, G, \mu) \) is equivariantly formal with respect to \( \mathbb{Q} \) if both \( G \) and \( M \) are compact and connected. For a real abelian Hamiltonian system, Biss-Guillemin-Holm (see [3], Theorem B) proved the following result.

**Proposition 15** (Biss-Guillemin-Holm). Let \( (M, \omega, T, \mu, \sigma, \phi) \) be a real Hamiltonian system in which \( M \) is a compact connected manifold, \( T \) is a torus and \( \phi(g) = g^{-1} \) for any \( g \in T \). If \( M^\sigma \) is the real locus and \( T^\phi \) is the real subgroup, then

\[
H^*_{T^\phi}(M^\sigma; \mathbb{Z}_2) \cong H^*(BT^\phi; \mathbb{Z}_2) \otimes_{\mathbb{Z}_2} H^*(M^\sigma; \mathbb{Z}_2)
\]  

(22)

as graded modules over \( H^*(BT^\phi; \mathbb{Z}_2) \).

We use this result to prove a real version of equivariant formality for nonabelian real Hamiltonian systems. First we have the following definition.

**Definition 10.** We say that a compact connected involutive Lie group \((G, \phi)\) has the **toral free extension property** if the following hold.

1. The pair \((G, \phi)\) has the free extension property.
2. There exists a maximal torus $T \leq G$ such that $\phi(g) = g^{-1}$ for all $g \in T$ and $T^\phi$ is a maximal elementary abelian 2-subgroup of $G^\phi$.

Both $(U(n), \phi)$ and $(SU(n), \phi)$ where $\phi(A) = \overline{A}$ satisfy the toral free extension party. However $(U(n) \times U(n), \phi)$ with $\phi(A, B) = (\overline{B}, \overline{A})$ does not.

**Theorem 16 (Real Equivariant Formality).** Let $(M, \omega, G, \mu, \sigma, \phi)$ be a real Hamiltonian system where both $G$ and $M$ are compact and connected. If $(G, \phi)$ has the toral free extension property, then the real locus $M^\sigma$ is $G^\phi$-equivariantly formal with respect to the field $\mathbb{Z}_2$.

**Proof.** Since $(G, \phi)$ has the toral free extension property, there exists a maximal torus $T \leq G$ such that $\phi(g) = g^{-1}$ for any $g \in T$. Let $\mu_T : M \to t^*$ be the composition of $\mu$ with the orthogonal projection $g \to t^*$. Then $(M, \omega, T, \mu_T, \sigma, \phi_T)$ is a real Hamiltonian system such that $\phi_T : T \to T$ is the inversion map. By the Biss-Guillemin-Holm theorem, $M^\sigma$ is $T^\phi$-equivariantly formal yielding

$$P_t(M^\sigma_{T^\phi}) = P_t(M^\sigma)P_t(BT^\phi). \quad (23)$$

Consider the following commutative diagram:

```
\begin{array}{ccc}
M^\sigma & \xrightarrow{i_1} & M^\sigma_{T^\phi} \\
\downarrow i_2 & & \downarrow j_1 \\
M^\sigma & \xrightarrow{j_2} & M^\sigma_{G^\phi} \\
\downarrow j_1 & & \downarrow \phi_T \\
G^\phi/T^\phi & \xrightarrow{j_2} & B(G^\phi) \\
\downarrow \phi_T & & \downarrow B(G^\phi) \\
BT^\phi & \xrightarrow{j_2} & BG^\phi
\end{array}
```

Since $(G, \phi)$ has the free extension property and $T^\phi$ is a maximal elementary abelian 2-subgroup of $G^\phi$, the induced maps $j_1^*$ and $j_2^*$ are surjective. By the Leray-Hirsch Theorem we deduce

$$P_t(BT^\phi) = P_t(BG^\phi)P_t(G^\phi/T^\phi),$$

and

$$P_t(M^\sigma_{T^\phi}) = P_t(M^\sigma_{G^\phi})P_t(G^\phi/T^\phi).$$

Combining these with $(23)$, we get

$$P_t(M^\sigma_{G^\phi})P_t(G^\phi/T^\phi) = P_t(M^\sigma)P_t(BG^\phi)P_t(G^\phi/T^\phi),$$

which implies that $P_t(M^\sigma_{G^\phi}) = P_t(M^\sigma)P_t(BG^\phi)$ completing the proof. \hfill \blacksquare
8 Examples

Example 10. Let $M = \text{Mat}_{n \times k}(\mathbb{C}) \cong \mathbb{C}^{nk}$ be the set of $n \times k$-matrices with the standard symplectic form (see Example 1). The action of $U(k)$ by right multiplication is Hamiltonian with moment map that sends a $n \times k$ matrix $A$ to $\mu(A) = \sqrt{-1/2}(A^T A - I_{nk})$. Observe that $\mu^{-1}(0)$ equals the set $n \times k$-matrices whose columns form an orthonormal frame in $\mathbb{C}^n$, therefore $U(k)$ acts freely on $\mu^{-1}(0)$ and

$$\mu^{-1}(0)/U(k) = \text{Gr}_k(\mathbb{C}^n).$$

This $U(k)$-action extends to a $\text{GL}_k(\mathbb{C})$-action by right multiplication and $\mu^{-1}(0)/U(k)$ corresponds to a GIT quotient by the Kempf-Ness Theorem. This implies that minimum Morse stratum $S_0$ is equal to the union of $\text{GL}_k(\mathbb{C})$-orbits intersecting $\mu^{-1}(0)$ for which it follows that if $A$ has full rank $k$ then it is a critical point of $f$ if and only if it lies in $\mu^{-1}(0)$.

Suppose that $A$ is critical point of $f$ and has rank $r < k$. Then after acting by some element of $U(k)$ if necessary, we can assume that the first $r$ columns of $A$ are linearly independent and the remaining columns are zero. Effectively then, we can regard $A$ as an element of $\text{Hom}(\mathbb{C}^r, \mathbb{C}^n)$ and deduce that $A$ is a critical point for $f$ if and only if the first $r$ columns of $A$ are orthonormal. The corresponding critical value is the adjoint orbit containing the diagonal matrix $\beta$ with $r$ entries equal to 0 and the rest equal to $\sqrt{-1/2}$. The corresponding Morse stratum is the set of all matrices of rank $r$. Therefore the Morse stratification agrees with the decomposition $M = S_0 \cup S_1 \cup \cdots \cup S_k$ where $S_i$ is the set of matrices of rank $k-i$. It is straightforward to calculate that the dimension of $S_i$ is $2(k-i)i + 2(k-i)(n) = 2(k-i)(n+i)$ and therefore has codimension $2kn - 2(k-i)(n+i) = 2i(n-k+i)$. The Hamiltonian subsystem corresponding to $S_i$ is isomorphic to $(\text{Mat}_{n \times (k-i)}(\mathbb{C}), \omega, U(k-i) \times U(i), \mu_i)$ which looks like a lower rank version of $M$ times an extra trivial action by $U(i)$. Applying perfection yields the formula

$$P^U(k)(\mu^{-1}(0); \mathbb{Q}) = P_t(\text{BU}(n); \mathbb{Q}) - \sum_{i=1}^k t^{2i(n-k+i)}P^U(k-i)(\mu_{i}^{-1}(0); \mathbb{Q})P_t(\text{BU}(i); \mathbb{Q}).$$

Inputting known values determines recursion relation for the Poincaré polynomials of the complex Grassmannians.

$$P_t(\text{Gr}_k(\mathbb{C}^n); \mathbb{Q}) = \prod_{p=1}^k \frac{1}{1 - t^{2p}} - \sum_{i=1}^k P_t(\text{Gr}_{k-i}(\mathbb{C}^n); \mathbb{Q}) \prod_{p=1}^i \frac{t^{2(n-k+i)}}{1 - t^{2p}}.$$
This system admits a real structure where $\sigma$ and $\phi$ are the standard conjugations. We get $M^\sigma = \text{Mat}_{n \times k}(\mathbb{R})$ and $U(k)^\phi = O(k)$. Running through the details produces an analogous recursive formula

$$P_t(\text{Gr}_k(\mathbb{R}^n); \mathbb{Z}_2) = \prod_{p=1}^{k} \frac{1}{1 - t^p} - \sum_{i=1}^{k} P_t(\text{Gr}_{k-i}(\mathbb{R}^n); \mathbb{Z}_2) \prod_{p=1}^{i} \frac{t^{(n-k+i)}}{1 - t^p}.$$  

The solution of this recursion is the well known formula $P_t(\text{Gr}_k(\mathbb{R}^n)) = \prod_{p=1}^{n} \frac{1 - t^p}{1 - t^{2p}}$. Observe that even without solving the recursion, we can deduce immediately that

$$P_t(\text{Gr}_k(\mathbb{R}^n); \mathbb{Z}_2) = P_t^{1/2}(\text{Gr}_k(\mathbb{C}^n); Q).$$  

Example 11. Kirwan considers the diagonal action of $U(n)$ on a product of Grassmannians $M = \text{Gr}_{l_1}(\mathbb{C}^n) \times \cdots \times \text{Gr}_{l_r}(\mathbb{C}^n)$. This action is Hamiltonian with moment map

$$\mu(V_1, \ldots, V_r) = \sqrt{-1} \left( \sum_{j=1}^{r} \text{Pr}_{V_j} - \left( \frac{\sum_{j=1}^{r} l_j}{n} \right) \text{Id}_n \right),$$

where $\text{Pr}_{V_j}$ is the orthogonal projection on $V_j$. The scalar matrices $U(1) \leq U(n)$ act trivially on $M$ determining a Hamiltonian action by $PU(n) = U(n)/U(1)$ on $M$. If $\gcd(n, l_1 + \cdots + l_r) = 1$, then $PU(n)$ acts freely on $M_0$ so $M/\!/U(n) = M/\!/PU(n)$ is a manifold and

$$P_t^{U(n)}(M_0; Q) = P_t(BU(1); Q)P_t^{PU(n)}(M_0; Q) = \frac{1}{1 - t^2} P_t(M/\!/U(n); Q).$$  

We have a real pair $(\phi, \sigma)$ where $\phi$ and $\sigma$ are standard complex conjugations. Therefore $U(n)^\phi = O(n)$ and

$$M^\sigma = \text{Gr}_{l_1}(\mathbb{R}^n) \times \cdots \times \text{Gr}_{l_r}(\mathbb{R}^n).$$

If $\gcd(n, l_1 + \cdots + l_r) = 1$, then $M/\!/O(n) = M/\!/PO(n)$ is a Lagrangian submanifold of $M/\!/U(n) = M/\!/PU(n)$.

Proposition 17. We have equality

$$P_t^{O(n)}(M_0^\sigma; \mathbb{Z}_2) = P_t^{U(n)}(M_0; Q).$$

Proof. Following the general scheme, there exist a collection of Hamiltonian subsystems $(Z_I, \omega_I, U(n)_I, \mu_I)$ such that

$$P_t^{U(n)}(M_0; Q) = P_t(M; Q)P_t(BU(n); Q) - \sum_{I \neq 0} t^{2\omega_I} P_t^{U(n)_I}(M_I; Q),$$

25
where \( M_t := \mu^{-1}(0) \). It is more convenient to refine the index set described in §3.2 to one whose strata are connected.

Identify the Lie algebra of the maximal torus \( t \cong \mathbb{R}^n \) in the standard way, with positive Weyl chamber \( t_+ = \{ (x_1, \ldots, x_n) | x_1 \geq \cdots \geq x_n \geq 0 \} \). The index set \( \mathcal{I} \) is a finite subset of \( t_+ \times \text{Mat}_{s,r}(\mathbb{Z}) \) of pairs \((\beta, l)\) for which

\[
\beta = \left( \begin{array}{ccc}
\frac{k_1}{m_1} & \cdots & \frac{k_1}{m_1} \\
\frac{k_s}{m_s} & \cdots & \frac{k_s}{m_s}
\end{array} \right)
\]

where \( m_i, k_i \in \mathbb{Z}, m_i > 0, k_i \geq 0, m_1 + \cdots + m_s = n, \) and \( k_1 + \cdots + k_s = \sum_{j=1}^r l_j \), \( \frac{k_1}{m_1} > \cdots > \frac{k_s}{m_s} \) and \( l \) is a matrix of non-negative integers \( l = (l_{i,j}) \in \text{Mat}_{s,r}(\mathbb{Z}) \) such that \( \sum_{i=1}^s l_{i,j} = l_j \) and \( \sum_{j=1}^r l_{i,j} = k_i \). Then

\[
Z_{\beta,l} \cong \prod_{i=1}^s \prod_{j=1}^r \text{Gr}_{l_{i,j}}(\mathbb{C}^{m_i})
\]

and the Hamiltonian subsystem \((Z_{\beta,l}, \omega_{\beta,l}, U(n)_{\beta}, \mu_{\beta,l})\) is a product of Hamiltonian systems of the type (25) but with rank \( m_i \) less than \( n \). Consequently

\[
P^U(n)_t(M_0; \mathbb{Q}) = P_t(M; \mathbb{Q}) P_t(BU(n); \mathbb{Q}) - \sum_{(\beta,l) \neq 0} t^{2d_{\beta,l}} \prod_{i=1}^s P^{U(m_i)}_t(M_{\beta,l}; \mathbb{Q})
\]  

(29)

which can be calculated recursively in the rank \( n \). The base of the recursion occurs with \( n = 1 \) in which case \( \text{Gr}_0(\mathbb{C}^1) = \text{Gr}_1(\mathbb{C}^1) \) is a point and \( P^U(1)_t(\text{point}) = P_t(BU(1)) = (1 - t^2)^{-1} \).

The calculation for the real quotient proceeds analogously. The real Hamiltonian subsystems satisfy

\[
Z^\sigma_{\beta,l} \cong \prod_{i=1}^s \prod_{j=1}^r \text{Gr}_{l_{i,j}}(\mathbb{R}^{m_i})
\]

and are products of systems of the form (27) but of rank \( m_i < n \). The action of \( U(n)_{\beta} \) on the tangent spaces \( T_x M_{\beta,l} \) are \( \mathbb{Z} \)-primitive as observed by Kirwan (Remark 16.11 [16]), so the system is 2-primitive. Applying Theorem 14 we obtain the recursive formula

\[
P^O(n)_t(M_0^\sigma; \mathbb{Z}_2) = P_t(M^\sigma; \mathbb{Z}_2) P_t(BO(n); \mathbb{Z}_2) - \sum_{(\beta,l) \neq 0} t^{d_{\beta,l}} \prod_{i=1}^s P^O(m_i)_t(M_{\beta,l}^\sigma; \mathbb{Z}_2).
\]  

(30)

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The inputs to the recursions (29) and (30) satisfy
\[ P_t(M^\sigma; \mathbb{Z}_2) = P_{t/2}(M; \mathbb{Q}), \]
\[ P_t(BO(n); \mathbb{Z}_2) = P_{t/2}(BU(n); \mathbb{Q}), \]
\[ P_t(BO(1); \mathbb{Z}_2) = P_{t/2}(BU(1); \mathbb{Q}). \]

We conclude that
\[ P_{tO(n)}(M^\sigma_0; \mathbb{Z}_2) = P_{t/2}(M; \mathbb{Q}). \]

\[ \blacksquare \]

**Corollary 17.1.** If \( \gcd(n, l_1 + \cdots + l_r) = 1 \) and \( n \) is odd, then
\[ P_t(M^\sigma // O(n); \mathbb{Z}_2) = P_{t/2}(M // U(n); \mathbb{Q}). \] (31)

**Proof.** Since \( n \) is odd, the short exact sequence \( 1 \to O(1) \to O(n) \to PO(n) \to 1 \) splits, yielding an isomorphism \( O(n) \cong O(1) \times PO(n) \), sending \( SO(n) \cong PO(n) \). Since \( O(1) \) acts trivially and \( PO(n) \) acts freely, we get
\[ P_{tO(n)}(M^\sigma_0; \mathbb{Z}_2) = P_{tPO(n)}(M^\sigma_0)P_t(BO(1)) = \frac{1}{1 - t} P_t(M^\sigma_0/PO(n)) = \frac{1}{1 - t} P_t(M^\sigma // O(n)). \]

Compare with (26). \[ \blacksquare \]

The phenomenon revealed in (24) and (31) is a familiar one in real symplectic geometry (see for example [12]). The next example shows that (31) does not always hold for real quotients.

**Example 12.** Consider \((\mathbb{C}P^1, \omega_{FS}, U(1), \mu_1, \sigma_{\mathbb{C}P^1}, \phi)\) a special case of Example 11 and let \((M, \omega_M, \sigma_M)\) be any real symplectic manifold. Define a new real Hamiltonian system \((\mathbb{C}P^1 \times M, \omega, U(1), \mu, \sigma, \phi)\) where \( \omega := \omega_{FS} + \omega_M \) is the product symplectic form, \( \sigma := \sigma_{\mathbb{C}P^1} \times \sigma_M \), and \( \mu(x, y) = \mu_1(x) \). The preimage \( \mu^{-1}(0) = S^1 \times M \) where \( S^1 \) is the equator in \( \mathbb{C}P^1 \) so the symplectic quotient is \((\mathbb{C}P^1 \times M) // U(1) \cong M \) and the real quotient is \((\mathbb{C}P^1 \times M)^\sigma // O(1) \cong M^{\sigma M} \). Thus whenever \( M \) has non-trivial \( \mathbb{Q} \)-Betti numbers in odd degree we have \( P_t(M^{\sigma M}; \mathbb{Z}_2) \neq P_{t/2}(M; \mathbb{Q}) \) and (31) is not satisfied.

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