Recognition of commutative algebra spectra through an idempotent quasiadjunction

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Abstract

In this article a recognition principle for $\infty$-loop pairs of spaces of connective commutative algebra spectra over connective commutative ring spectra is proved. This is done by generalizing the classical recognition principle for connective commutative ring spectra using relative operads. The machinery of idempotent quasiadjunctions is used to handle the model theoretical aspects of the proof.

1 Introduction

The category $\mathcal{S}_{\mathcal{P}}$ of sequential prespectra \cite{12} consists of sequences of spaces $\langle Y_N \rangle \in \prod_N \mathbf{Top}$ equipped with structural maps $\sigma_N^M : Y_M \wedge S^{N-M} \to Y_N$ for $M \leq N$ satisfying compatibility conditions. An $\Omega$-spectrum is a prespectrum whose adjoint structural maps $\tilde{\sigma}_N^M$ are weak equivalences, which by Brown representability represents (co)homology theories \cite{3}. Spectra are prespectra such that the $\tilde{\sigma}_N^M$ are homeomorphisms (see for instance \cite{5}). In this article we will work exclusively in the category of prespectra, so from now on we will simply refer to prespectra as spectra. From $\mathcal{S}_{\mathcal{P}}$ we can define via filtered colimits over the dual structural maps the $\infty$-loop spaces functor

$$\Omega^\infty : \mathcal{S}_{\mathcal{P}} \to \mathbf{Top}; \quad \Omega^\infty Y := \text{colim}_N Y_N S^N.$$

The $\infty$-loop spaces $\Omega^\infty Y$ are homotopy commutative $H$-spaces, but such description ignores a lot of information. In order to describe the algebraic structure completely we require an $E_\infty$-operad $\mathcal{E}$, a gadget used to describe topological spaces with operations that are associative and commutative up to coherent homotopy \cite{16}.

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For the category of finite sets a topological operad is a contravariant functor equipped with an abstract identity element and composition maps

\[ P : \mathcal{S}^{op} \to \text{Top}; \]

\[ \text{id} \in P_1, \quad \circ : P A \times \prod A P B^a \to P \Sigma A B^a \]

with \( P \emptyset = * \) satisfying invariance, associativity and unitary laws. We can interpret points in the underlying spaces as abstract multivariable functions with inputs indexed by the sets \( A \). This structures allows us to define via the coend construction \[13] the monad \( P : \text{Top}^* \to \text{Top}^* \), \( PX \) with

\[ \eta x := [\text{id}, x], \quad \mu[\alpha, \langle \beta^a, \langle x^a \rangle \rangle] := [\alpha \circ \langle \beta^a \rangle, \langle x^a \rangle]. \]

The category \( P[\text{Top}] \) of \( P \)-spaces consists of pointed spaces \( X \in \text{Top}_* \) equipped with maps \( \xi : PX \to X \) compatible with the monad maps, which we interpret as an instantiation of the abstract operations of \( P \).

An important family of operads are the embeddings operads \( \text{Emb}_N \) for \( N \in \mathbb{N} \) with

\[ \text{Emb}_N A := \left\{ \alpha = \langle \alpha_n \rangle \in (\mathbb{R}^N)^{\sqcup A \mathbb{R}^N} \mid \alpha \text{ an embedding} \right\}. \]

There are natural operad inclusions \( \text{End}_M \to \text{End}_N \) and we define \( \text{End}_\infty := \text{colim}_R \text{End}_N \). All \( N \)-loop spaces are naturally \( \text{End}_N \)-spaces with

\[ \alpha(\gamma^a) := \begin{cases} \tilde{u} & \gamma^a \alpha^a = \alpha \gamma, \\ \ast & \tilde{u} \notin \alpha \sqcup A \mathbb{R}^N \end{cases} \]

and these induce \( \text{End}_\infty \)-space structures on \( \infty \)-loop spaces.

An \( E_\infty \)-operad is an operad \( E \) with each underlying space \( E A \) a contractible free \( \mathcal{S}A \)-space. For the purpose of studying \( \infty \)-loop spaces we further require \( E_\infty \)-operads to be equipped with an operad map \( \psi : E \to \text{End}_\infty \). This allows us to define by pullback an \( E \)-algebra structure on \( \infty \)-loop spaces \( \Omega^\infty Y \), which induces a functor \( \Omega^\infty : \text{Sp} \to \mathcal{E}[\text{Top}] \). This functor is not a right adjoint since any abelian group \( G \) is an \( E \)-spaces and it can be shown that due to the strictness of the operations in \( G \) any \( E \)-map \( \varphi : G \to \Omega^\infty Y \) must be trivial, so no unit of adjunction can be constructed.

In May’s recognition theorem \[16\] the solution was to consider the resolution of \( E \)-spaces by the bar construction

\[ \overline{B} : \mathcal{E}[\text{Top}] \to \mathcal{E}[\text{Top}]; \quad \overline{B} X := B(E, E, X), \]

which comes equipped with a natural weak equivalence \( \eta : \overline{B} \Rightarrow Id. \)

The maps \( \psi : E \to \text{End}_\infty \) induce by pullback a suboperad filtration \( \mathcal{E}_N \) on \( E \). If each underlying space \( \mathcal{E}_N A \) is equivariantly homotopy equivalent to the
configuration space of $A$ elements in $\mathbb{R}^N$ then we can define the $\infty$-delooping functor

$$B^\infty : \mathcal{E}[\text{Top}] \to \text{Sp}; \quad B^\infty X := \langle B(\Sigma^N, E_N, X) \rangle;$$

such that there is a natural transformation $\eta : B \Rightarrow \Omega^\infty B^\infty$, with $\eta_X$ a weak equivalence if and only if $X$ is grouplike, meaning that $\pi_0 X$ is not only a monoid but also a group.

Dually there is no counit map. There is a spectrification functor\(^1\)

$$\tilde{\Omega} : \text{Sp} \to \text{Sp}; \quad \tilde{\Omega} Y := \langle \text{colim}_{M \leq N} \tilde{Y}_N \rangle,$$

where $\tilde{Y}$ is a certain inclusion prespectrum constructed from $Y$, such that we have natural inclusions $\epsilon^{'\prime} : \text{Id} \Rightarrow \tilde{\Omega}$ which are stable weak equivalences. This functor plays an important role in the construction of the stable model structures of spectra. There is a natural transformation $\epsilon : B^\infty \tilde{\Omega}^\infty \Rightarrow \tilde{\Omega}$ such that the equation $\Omega^\infty \epsilon_B \Omega^\infty \approx \Omega^\infty \epsilon^{'\prime} B^\infty \eta_X \approx \epsilon^{'\prime} B^\infty \eta_X$ holds in $\mathcal{E}[\text{Top}]$ and we have a homotopy equivalence $\epsilon_B = B^\infty \eta_X \approx \epsilon^{'\prime} B^\infty \eta_X$ in $\text{Sp}$.

Note the similarity of these equations to the ones for an adjunction. Indeed if $B, \tilde{\Omega}, \eta^{'\prime}$ and $\epsilon^{'\prime}$ were substituted by identities and both equations held strictly we would have an adjunction in the regular sense. In [25] I defined a generalization of Quillen adjunctions, called weak Quillen quasiadjunctions, that allowed for units and counits to exist up to functorial resolutions. I proved that weak Quillen quasiadjunctions still induce adjunctions of the homotopy categories, generalizing the analogous result for Quillen adjunctions. In the same vein I defined a generalization of Quillen idempotent (co)monads that induce left (right) Bousfield localizations of model structures, and through these we have a natural definition of idempotent quasiadjunctions which induce equivalences between the associated homotopy subcategories.

Adapting May’s original proof of the recognition principle in [16, 14] we can show that the weak Quillen quasiadjunction

$$(B^\infty \dashv \pi_\tilde{\Omega} \Omega^\infty) : \mathcal{E}[\text{Top}] \equiv \text{Sp}^N$$

is idempotent and induces an equivalence between the homotopy category of grouplike $\mathcal{E}$-spaces and the the homotopy category of connective spectra. Idempotent quasiadjunctions provide a model categorical axiomatization of the essential elements of May’s original proof, and it can be adapted to prove variations of the recognition principle. For instance the relative recognition principle for

\(^1\)The spectrification functor $\tilde{\Omega}$ is the left adjoint to the inclusion of the category of spectra in the sense used in [15] into the category of prespectra, hence the name.
\(\infty\)-loop pairs of spaces of spectra maps of degree 1 was proved using the above machinery and relative operads in [20].

In this article I show that the machinery of quasiadjunctions and relative operads are also compatible with actions by a natural relative version \(\mathcal{L}^-\) of the linear isometries operad \(\mathcal{L}\), introducing in particular a relative version of actions between operads which provides a natural definition of \(E_\infty\)-algebra spaces over \(E_\infty\)-ring spaces. The main theorem 4.5 is a recognition principle for \(\infty\)-loop pairs of spaces of commutative algebra spectra over commutative ring spectra. Explicitly it states that the homotopy category of algebralike \(E_\infty\)-algebras is equivalent to the homotopy category of connective commutative algebra spectra over connective commutative ring spectra. As in [5] this will require us to work on the more structured category \(\text{Mod}_S\) of \(S\)-modules, which is monoidal and so provides a convenient language to describe algebraic structures. In particular we will work with the coordinate-free spectra of [11] which substitutes the natural numbers \(\mathbb{N}\) by the set of finitely dimensional subspaces of some countably infinite dimensional inner product spaces such as \(\mathbb{R}^\infty\) as the indexing set of spectra. This result is a simple consequence of the intermediary theorems 4.2 and 4.3, which is a recognition principle for \(\infty\)-loop pairs of spaces of spectra maps.

1.1 Structure of the article

In section 2 we review the definition of weak Quillen quasiadjunctions, idempotent quasimonads and idempotent quasiadjunctions. Our main theorem will be a particular case of the fact that idempotent quasiadjunctions induce equivalences between the associated homotopy subcategories.

In section 3 we present the definition of \(E_\infty\)-algebras over \(E_\infty\)-rings through relative operads. A detailed description of relative sets and filtered rooted relative trees and operations on them will be required to construct bar resolutions and delooping spectra, as well as describe their algebraic structures. We then give a brief review on relative operads and \(E_\infty^-\)-operads and the bar resolution of \(E_\infty^-\)-algebras. Relative operad actions are then introduced which provides an account of distributivity laws between multiplicative and additive relative operad actions and is central in the definition of the category \((\delta^-, \mathcal{L}^-)\) of \(E_\infty^-\)-algebras. We also give a brief review of how the Quillen model structure on \((\delta^-, \mathcal{L}^-)\) is transferred from the one on \(\text{Top}^2\).

The main theorems are in section 4. We review the basics of coordinate-free spectra and the construction of the stable mixed model structure. The recognition principle for \(\infty\)-loop pairs of spaces of spectra maps is proved via an idempotent quasiadjunction in theorems 4.2 and 4.3 which imply the homotopy category of grouplike \(E_\infty^-\)-pairs is equivalent to the homotopy category of spectra maps between connective spectra. After a review of the basics of \(S\)-modules and commutative algebra spectra, including the construction of stable mixed model structures, the main theorem 4.5 is proved.

\(^2\)An \(E_\infty^-\)-algebra is algebralike if it admits additive inverses up to homotopy.
1.2 Notation and terminology

We assume the theory of model categories in [7, 8, 10], and the theory of monoids, their algebras and the bar construction in [16, Section 9]. In diagrams in a model category $\mathcal{T}$ the morphisms in the class of weak equivalences $W$ are denoted by arrows marked with a tilde $\sim$, the ones in the class of cofibrations $C$ by hooked arrows $\hookrightarrow$ and the ones in the class of fibrations $F$ by double headed arrows $\rightarrow$.

The functorial weak factorization systems are denoted by $(\text{Fat}_{C,F}, C \rightarrow, F \rightarrow)$ and $(\text{Fat}_{C,F}, C_\leq, F \rightarrow)$ such that a morphism $f \in \mathcal{T}(X,Y)$ is factored for instance as $X \xrightarrow{\text{cf}} \text{Fat}_{C,F}, f \xrightarrow{\text{fib}} Y$.

The notations $\mathcal{E} : \mathcal{T} \rightarrow \mathcal{T}$ and $\mathcal{col} : \mathcal{E} \Rightarrow \text{Id}$ are used for the cofibrant resolution functor and the associated natural trivial fibration, and the notations $\mathcal{F} : \mathcal{T} \rightarrow \mathcal{T}$ and $\mathcal{fib} : \text{Id} \Rightarrow \mathcal{F}$ are used for the fibrant resolution functor and the associated natural trivial cofibration. The homotopy category of $\mathcal{T}$ with objects the bifibrant objects of $\mathcal{T}$ and with morphisms between bifibrant objects $X$ and $Y$ the set $\mathcal{T}(X,Y)/_z$ of homotopy classes of maps [10, Section 1.2] is denoted by $\mathcal{H}o\mathcal{T}$.

The monoidal category $\text{Top}$ of compactly generated weakly Hausdorff spaces as presented in Strickland’s [23] admits two model structures, the cofibrantly generated Quillen model structure [17] with weak equivalences the weak homotopy equivalences ($q$-equivalences), fibrations of this model structure the Serre fibrations ($q$-fibrations), and cofibrations retracts of inclusions of well pointed relative CW-complexes ($q$-cofibrations), and the Hurewicz/Strøm model structure [24] with distinguished classes of maps the homotopy equivalences ($h$-equivalences), the Hurewicz fibrations ($h$-fibrations) and the Hurewicz cofibrations ($h$-cofibrations). As Cole proved in [4] we can mix these model structures into one with weak equivalences the $q$-equivalences, fibrations the $h$-fibrations and cofibrations the maps that can be factored as a $q$-cofibration followed by an $h$-equivalence. We use the notation $K \subset_{\text{cpct}} X$ to indicate $K$ is a compact subspace of $X$. We denote by $I$ the interval $[0, 1] \subset \mathbb{R}$. We denote by $\text{Top}_*$ the category of pointed spaces and for $X \in \text{Top}$ we denote by $X_* \in \text{Top}_*$ the pointed space obtained by adjoining a disjoint base point.

We denote by $\mathcal{T}^-$ the category of morphisms $f : X_d \rightarrow X_c$ in $\mathcal{T}$ as objects and commutative squares as morphisms. For notational convenience we denote elements of categories of pairs $\mathcal{T}^2$ as $X = (X_d, X_c)$, and we will consider relative operads colored on the set $\{d, c\}$, with $d$ being the “domain” color and $c$ the “codomain” color.

Let $\mathcal{F}$ denote the topological category of finite or countably infinite dimensional real inner product spaces and linear isometries, with the topology defined as the colimit of the finite dimensional sub-spaces. This category is monoidal under direct sums. For $U \in \mathcal{F}$ we denote by $\mathcal{A}_U$ the set of finite dimensional subspaces of $U$, partially ordered by inclusion, and for $U \in \mathcal{A}_U$ we define $\mathcal{A}_U := \{V \in \mathcal{A}_U \mid U \subseteq V\}$. For $U = \mathbb{R}^\infty$ we simply write $\mathcal{A} := \mathcal{A}_{\mathbb{R}^\infty}$. For $\langle f_a \rangle \in \mathcal{F}(\oplus A U^a, V)$ and $\langle u^a \rangle \in \oplus A U^a$ we use the Einstein summation convention $f_a u^a := \sum_A f_a u^a$. For $U \in \mathcal{F}$ and $U \subset U$ any subspace we use the
notation $U^\perp := \{ \vec{v} \in U \mid \forall \vec{u} \in U : \vec{v} \cdot \vec{u} = 0 \}$ for the orthogonal complement.  
For any $\vec{u} \in U$ and $U \subseteq U$ we use the notation $\vec{u}_U$ to denote the projection of $\vec{u}$ on $U$. For $\vec{v} \in V$ and $f \in \mathcal{F}(U, V)$ we use the notation $\vec{v}_f := \vec{v}_fU \in V$ for the projection of $\vec{v}$ onto the image of $f$ and $\vec{u}^f := f^{-1} \vec{u} \in U$. For all $U \in \mathcal{S}_U$ let $S^U$ be the one point compactification of $U$ obtained by adding a point $\infty$ at infinity and for $(U, V) \in \Sigma_{\mathcal{S}}\mathcal{A}_U$, let $V - U := U \cap V^\perp$.

We will make extensive use of mapping spaces $Y^X$ and will express their elements as $x \mapsto \Phi$ for some expression $\Phi$ which may use the variable $x$. For $X$ a set (or space) equipped with an equivalence relation $\sim$ we will denote the equivalence classes of $x \in X$ using square brackets $[x] \in X / \sim$.

We denote by $\text{Set}$ the category of sets and functions, by $\mathcal{S}^\text{fin}$ the subcategory of finite sets and injections and by $\mathcal{S}$ the subcategory of finite sets and bijections. We will use the notation $m$ for the sets $\{1, \ldots, m\}$, with $\emptyset = \emptyset$.

Given a class $A$ and a family of classes $(B^a)$ indexed by $A$ the dependent sum $\Sigma_A B^a$ is the class of pairs $(a, b)$ with $a \in A$ and $b \in B^a$ and the dependent product $\Pi_A B^a$ is the class of sequences $(b^a)$ indexed on $A$ with $b^a \in B^a$ for each $a \in A$, or equivalently it is the class of sections of the natural surjection $\Sigma_A B^a \to A$.

We denote by $\text{POS}et$ the category of ordered sets and monotone functions and $\Delta$ the full subcategory on $\langle m \rangle = \{ 0 < \cdots < m \}$ for $m \in \mathbb{N}$. This category is generated by the coface injections $\partial_i : \langle m - 1 \rangle \to \langle m \rangle$, with $i \notin \partial_i \langle m - 1 \rangle$, and codegeneracy surjections $\delta_i : \langle m + 1 \rangle \to \langle m \rangle$, with $\delta_i = \delta_i(i + 1)$, for all $i \in \langle m \rangle$.

Consider the cosimplicial space of partitions of the interval $\text{Part}^\Delta \in \text{Top}^\Delta$ with

$\partial_i \cdot t := \begin{cases} 
(0, \ j = 0) & \text{if } i = 0 \\
(t^i, \ j > 0) & \text{if } 0 < i < m, \\
(t^i, \ j < i) & \text{if } 0 < i < m, \\
(t^i, \ j < m - 1) & \text{if } i = m \\
(1, \ j = m - 1) & \text{if } i = m 
\end{cases}$

with $\text{Part}^{\langle m \rangle}$ topologized as a subspace of $I^{\langle m - 1 \rangle}$. For each $\langle t^a \rangle \in \Pi_A \text{Part}^{\langle m^a \rangle}$ there is a unique

$\chi_A t^a \in \text{Part}^{\langle \Sigma_A m^a \rangle}$

obtained by ordering the elements $t^{a,i}$ for $a \in A$ and $i \in \langle m^a - 1 \rangle$. For each $a \in A$ and $\langle \alpha^a \rangle \in \Pi_A \Delta^{\langle m^a \rangle}$ we can define

$\delta^a \in \Delta^{\langle \Sigma_A m^a \rangle}, \langle m^a \rangle}$, $\delta^a : \begin{cases} 
\min(j \mid (\chi_A t^a)^i \leq t^{a,j}), & (\chi_A t^a)^i \leq t^{a,m^a-1} \\
\chi_A t^a)^i > t^{a,m^a-1} & (\chi_A t^a)^i > t^{a,m^a-1} 
\end{cases}$

$6$
such that $\delta^a \cdot \a A t^a = t^a$.

For any simplicial space $X^- \in \text{Top}^\Delta_{op}$ its geometric realization $|X^-|$ is defined via the coend construction [13] as

$$|X^-| := \int^\Delta X^{(m)} \times \text{Part}^{(m)}.$$  

The reason we consider the geometric realization via the partitions cosimplicial space instead of the usual homeomorphic cosimplicial space of topological simplexes is that this choice simplifies the algorithm in [16 Theorem 11.5].

2 Idempotent quasiadjunctions

2.1 Weak Quillen quasiadjunction

The following definition introduced in [25] is a generalization of Quillen adjunctions between model categories. The basic idea is that to construct the unit and counit natural transformations of an adjunction between the homotopy categories it suffices to construct a unit natural span and counit natural cospan at the model categories level, plus some natural compatibility conditions with the model structures.

**Definition 2.1.** Let $\mathcal{T}$ and $\mathcal{A}$ be model categories. A weak Quillen quasiadjunction, or just quasiadjunction, between $\mathcal{T}$ and $\mathcal{A}$, denoted by

$$(S \dashv \eta, \varepsilon : \Lambda) : \mathcal{T} \rightleftarrows \mathcal{A},$$

is a quadruple of functors

$$\begin{array}{cc}
\varepsilon & \downarrow \gamma \\
\mathcal{T} & \mathcal{A} \\
\Lambda & \uparrow \delta
\end{array}$$

with $S$ the left quasiadjoint and $\Lambda$ the right quasiadjoint, equipped with a natural span in $\mathcal{T}$ and a natural cospan in $\mathcal{A}$

$$\begin{array}{cc}
\text{Id}_\mathcal{T} & \xrightarrow{\eta^l} \mathcal{C} & \xrightarrow{\eta} \Lambda S \\
S \Lambda & \xleftarrow{\varepsilon} \mathcal{F} & \xleftarrow{\varepsilon^l} \text{Id}_\mathcal{A}
\end{array}$$

such that

(i) $S$ is left derivable;

(ii) $\Lambda$ is right derivable;

(iii) $\mathcal{C}$ and $\mathcal{F}$ preserve cofibrant and fibrant objects;

(iv) $\eta^l$ and $\varepsilon^l$ are natural weak equivalences;

(v) If $X \in \mathcal{T}$ is cofibrant then $\varepsilon_S X S \eta_X \approx \varepsilon_S X S \eta_X^l$;
(vi) If $Y \in \mathcal{A}$ is fibrant then $\Lambda \delta_Y \eta_{\Lambda Y} = \Lambda \delta_Y \eta_{\Lambda Y}^f$.

$$
\begin{align*}
S \xi X &\xrightarrow{S \eta_X} S \xi S X &\xrightarrow{S \xi S X} \xi S X \\
\xi S X &\xrightarrow{\xi s_X} \xi \xi S X &\xrightarrow{\xi \xi \xi S X} \xi \xi \xi S X \\
\xi X &\xrightarrow{\xi s_X} \xi \xi X &\xrightarrow{\xi \xi \xi X} \xi \xi \xi X
\end{align*}
$$

\[\xi \Lambda Y \xrightarrow{\eta_{\Lambda Y}} \Lambda \xi \Lambda Y \xrightarrow{\Lambda \delta_Y} \Lambda \xi \Lambda Y \]

\[\xi \Lambda Y \xrightarrow{\eta_{\Lambda Y}^f} \Lambda \xi \Lambda Y \xrightarrow{\Lambda \delta_Y^f} \Lambda \xi \Lambda Y \]

\[\xi \Lambda Y \xrightarrow{\eta_{\Lambda Y}} \Lambda \xi \Lambda Y \xrightarrow{\Lambda \delta_Y} \Lambda \xi \Lambda Y \]

Theorem 2.2 ([25, Theorem 2.1.2]). A quasiadjunction induces an adjunction

$$
(LS \dashv \mathcal{R} \Lambda) : \text{Ho} \mathcal{T} \Rightarrow \text{Ho} \mathcal{A};
$$

\[\begin{array}{c}
\text{Id}_{\text{Ho} \mathcal{T}} \\
\text{L} S \mathcal{R} \Lambda \\
\text{R} \mathcal{F} \Lambda \text{S}
\end{array} \xrightarrow{[(\text{co} \mathcal{F} \Lambda \mathcal{S})^{-1}]} \xrightarrow{[(\Lambda \mathcal{R} \mathcal{S} \mathcal{F})^{-1}]} \xrightarrow{\text{Id}_{\text{Ho} \mathcal{A}}} \]

between the homotopy categories.

2.2 Idempotent quasi(co)monads

The following generalization of idempotent Quillen monads [2] was also introduced following the same principle of only requiring the existence of a unit natural span, and they also induce Bousfield localizations.

Definition 2.3. Let $\mathcal{T}$ be a right proper model category with distinguished subclasses of morphisms $(W, C, F)$. A Quillen idempotent quasimonad on $\mathcal{T}$, or simply an idempotent quasimonad, is a pair of endofunctors $Q, C : \mathcal{T} \to \mathcal{T}$ equipped with a natural span

$$
\begin{array}{c}
\text{Id}_{\mathcal{T}} \\
\text{L} S \mathcal{R} \Lambda \\
\text{R} \mathcal{F} \Lambda \text{S}
\end{array} \xrightarrow{[\mathcal{F} \Lambda \mathcal{S} \text{Id}_{\mathcal{T}}]} \xrightarrow{[\mathcal{S} \text{Id}_{\mathcal{T}} \mathcal{F}]} \xrightarrow{\text{Id}_{\mathcal{A}}} \]

such that:

(i) $Q$ preserves weak equivalences;

(ii) $Q \eta$ and $\eta Q$ are natural weak equivalences;

(iii) If $f \in T(X, B), p \in F(E, B)$ and $\eta_E, \eta_B, Qf \in W$ then $Q(f^* p) \in W$;

$$
\begin{array}{c}
X \times_B E \xrightarrow{f^* p} E \xleftarrow{\eta_B} E \xrightarrow{\eta_E} E \xrightarrow{Q(f^* p)} Q(X \times_B E) \\
X \xrightarrow{j} E \xleftarrow{\eta_B} E \xrightarrow{\eta_E} E \xrightarrow{Q(j)} Q(X)
\end{array}
$$

(iv) $\eta'$ is a natural weak equivalence;
An idempotent quasimonad induces a left Bousfield localization

\[ \mathcal{T}_Q = (T; W_Q := Q^{-1}W, C_Q := C, F_Q := \{ p \in F \mid (2.1) \text{ a homotopy pullback} \}) \]

The resulting homotopy category is the reflective subcategory

\[ \mathcal{H}oT_Q := \{ X \in \mathcal{H}oT \mid (i_X : X \rightarrow X \sqcup X QX) \in W \} \]

of \( Q \)-fibrant objects.

The above definition can be dualized and the resulting idempotent quaşicomonads induce right Bousfield localizations and associated coreflective homotopy subcategories.

### 2.3 Idempotent quasiadjunctions

A quasiadjunction \((S \dashv \varrho, \varLambda) : \mathcal{T} \leftrightarrows \mathcal{A}\) induces the following natural span on \(\mathcal{T}\) and natural cospan on \(\mathcal{A}\):

\[
\begin{array}{cccccc}
1_{\mathcal{T}} & \overset{\text{cofib}_{\varrho}}{\leftarrow} & \mathcal{C} & \overset{\varLambda\varrho_{\text{fib}}}{\rightarrow} & \Lambda \mathcal{S} \mathcal{C} & \overset{}{\downarrow} & \mathcal{F} \mathcal{S} & \overset{\varrho_{\text{fib}}}{\rightarrow} & 1_{\mathcal{A}} \\
\end{array}
\]

**Definition 2.5.** An idempotent quasiadjunction is a quasiadjunction such that the induced span and cospan are respectively an idempotent quasimonad and an idempotent quaşicomonad.

**Theorem 2.6** ([25 Theorem 2.3.8]). An idempotent quasiadjunction \((S \dashv \varrho, \varLambda) : \mathcal{T} \leftrightarrows \mathcal{A}\) induces an equivalence between the associated (co)reflective homotopy subcategories.
3 $E_{\infty}^{-}$-algebras

3.1 Relative sets and filtered rooted relative trees

Relative operads are abstract operations with entries indexed by relative sets. We now give the basic definitions and constructions on these colored sets. We will also require filtered rooted relative trees in the construction of the bar resolutions and delooping spectra, and we provide here the relevant definitions and constructions.

Let $\text{Set}_{(d,c)}$ be the category of relative sets composed of sets equipped with a coloring on the colors $\langle d,c \rangle$, ie the class of objects

$\{ (A,c) \in \Sigma_{\text{Set}} \Sigma_{\mathbb{A} \uplus A, \{d,c\}} \ | \ \epsilon A = d \implies \forall a \in A : ca = d \},$

with $(A,c)$ usually being denoted simply as $A$ or explicitly as a set of elements in brackets with coloring given by subscripts, eg $\{1_d, 2_d, 3_c, 4_d, 5_c\}$. The morphisms sets are

$\text{Set}_{(d,c)}(A, A') := \{(\sigma \in \text{Set}\langle d, A' \rangle | ca = c \implies \epsilon'\sigma a = c), \epsilon A = d \text{ or } \epsilon' A' = c \}
\begin{cases}
\emptyset, & \epsilon A = c \text{ and } \epsilon' A' = d
\end{cases}$

For $\star \in \{d,c\}$ we denote by $\text{Set}_{\star} \subset \text{Set}_{(d,c)}$ the full subcategory of relative sets $A$ such that $\epsilon A = \star$.

Given $((A,c), ((B^a, c^a))) \in \Sigma_{\text{Set}_{(d,c)}} \Sigma_A \text{Set}_{ca}$ we have the dependent sum

$(\Sigma_A B^a, \Sigma_A c^a) \in \text{Set}_{(d,c)}, \Sigma_A \epsilon^a(\Sigma_A B^a) = \epsilon A, \Sigma_A c^a(a, b) = c^a b.$

For $\sigma \in \text{Set}_{(d,c)}(A, A')$ let

$\sigma(B^a) \in \text{Set}_{(d,c)}(\Sigma_A B^a, \Sigma_A B^a); \sigma(B^a)(a', b) := (\sigma a', b)$

and for $\tau^a \in \Sigma_A \text{Set}_{(a)}(B^a, B^a)$ let

$\Sigma_A \tau^a \in \text{Set}_{(d,c)}(\Sigma_A B^a, \Sigma_A B^a); \Sigma_A \tau^a(a', b) := (a', \tau^a b).$

We also have the dependent product

$\Pi_A B^a \in \text{Set}_{(d,c)}, \Pi_A \epsilon^a \Pi_A B^a = \epsilon A, \Pi_A \epsilon^a(\langle b^a \rangle) = \begin{cases} d, & \forall a \in A : c^a b^a = d; \\ c, & \exists a \in A : c^a b^a = c. \end{cases}$

For $\sigma \in \text{Set}_{(d,c)}(A, A')$ let

$\sigma(B^a) \in \text{Set}_{(d,c)}(\Pi_A B^a, \Pi_A B^a); \sigma(B^a)(b^a) := \langle b^a \rangle$

and for $\tau^a \in \Sigma_A \text{Set}_{(a)}(B^a, B^a)$ let

$\Pi_A \tau^a \in \text{Set}_{(d,c)}(\Pi_A B^a, \Pi_A B^a); \Pi_A \tau^a(\langle b^a \rangle) := \langle \tau^a b^a \rangle.$
For every \( \langle b^a \rangle \in \Pi_A B^a \) we can form a new relative set \( A_{\langle b^a \rangle} \) composed of the pairs \( (a, b^a) \) with coloring \( \xi_{\langle b^a \rangle} A_{\langle b^a \rangle} = \Pi_A \xi^a(b^a) \) and \( \xi_{\langle b^a \rangle}(a, b^a) = \xi^a b^a \). This relative set is naturally equipped with \( \pi_{\langle b^a \rangle} \in \text{Set}_{(d,c)}(A_{\langle b^a \rangle}, A) \) with \( \pi_{\langle b^a \rangle}(a, b^a) = a \). Let

\[
\nu \in \text{Set}_{(d,c)}(\Pi_A \Sigma_{B^a} C_{a,b}, \Sigma_{\Pi_A B^a \Pi_A (\xi^a b^a)}); \quad \nu(\langle b^a, c^a \rangle) := (\langle b^a, \xi^a \rangle).
\]

This is a key element in distributivity properties.

Let \( S^{\text{inj}}_{(d,c)} \subset \text{Set}_{(d,c)} \) be the subcategory of \( \text{Set}_{(d,c)} \) composed of the finite relative sets and the injective functions that preserve coloring, i.e.

\[
S^{\text{inj}}_{(d,c)} (A, A') = \begin{cases} 
\{ \sigma \in \text{Set}_{(d,c)}(A, A') \mid \sigma \text{ is injective}, \, \xi^a \sigma a = \xi^a c, \, \xi^a A = \xi^a A' \} \\
\emptyset,
\end{cases}
\]

Let \( S_{(d,c)} \subset S^{\text{inj}}_{(d,c)} \) be the subcategory with the same objects and bijections that preserve coloring as morphisms. For \( \star \in \{d, c\} \) we denote by \( S^{\text{inj}}_{(d,c)} \) and \( S_{(d,c)} \) the full subcategories of \( S^{\text{inj}}_{(d,c)} \) and \( S_{(d,c)} \) respectively composed of relative sets \( A \) such that \( \xi^a A = \star \). Define also the subcategory \( S_{(d,c)} \subset \text{Set}_{(d,c)} \) with objects the finite relative sets and with morphisms the bijections (that don’t necessarily preserve coloring). Note that \( S_{(d,c)} \) is a subcategory of \( S_{(d,c)} \).

Many spaces of interest are built via the two sided bar construction for monads induced by operads, which can be described using filtered rooted relative trees.

**Definition 3.1.** The simplicial category \( T_{(d,c)} \in \text{Cat}^{\Delta^{op}} \) of filtered rooted relative trees has as objects quintuples

\[
T = (\langle V^i \rangle, \langle E^i \rangle, \langle s_i \rangle, \langle t_i \rangle, e) \in T_{(d,c)} \langle m \rangle
\]

composed of

- A sequence of nonempty finite sets \( \langle V^i \rangle \in S^{(m-1)} \). We also set \( V^{-1} := \{v^r\} \), and call \( v^r \) the root vertex of \( T \). We set \( V := \bigcup_{(m-1)} V^i \) and \( V_* := V^{-1} \cup V \). For \( v \in V \) we will denote by \( |v| \in \langle m - 1 \rangle \) the element such that \( v \in V^{|v|} \).

- A sequence of finite sets \( \langle E^i \rangle \in S^{(m)} \). We also set \( E^{-1} := \{e^r\} \), and call \( e^r \) the root edge of \( T \). We set \( E := E^{-1} \cup \bigcup_{(m)} E^i \). The edges in \( E^m \) are called the leaves of \( T \).

- A sequence of bijections \( \langle s_i \rangle \in \Pi_{(m-1)} S(E^i, V^i) \), called the start of the edges. We sometimes omit the subscript and write simply \( se := s_i e \). Note that the leaves don’t have a source.

- A sequence of functions \( \langle t_i \rangle \in \Pi_{(m)} S(E^i, V^{i-1}) \), the target of the edges. We sometimes omit the subscript and write simply \( te := t_i e \). Note that the root edge doesn’t have a target.
- A function \( c \in \text{Set}\{E, \{d, c\}\} \), the coloring of the edges, such that if \( te' = se \) and \( cc' = d \) then \( cc' = d \).

We sometimes just write \( T = \langle (V^i), \langle E^i \rangle \rangle \) and leave the mappings implicit. Morphisms \( \sigma \in \prod_{\langle m \rangle} \langle T, T' \rangle \) are pairs of sequences of bijections

\[
\langle (\sigma_V^i), (\sigma_E^i) \rangle \in \prod_{\langle m-1 \rangle} S^{\text{ini}}(V^i, V'^i, i) \times \prod_{\langle m \rangle} S^{\text{ini}}(E^i, E'^i, i)
\]

that commute with the structural functions.

The simplicial structural functors are defined on objects as:

\[
T \cdot \partial_j := \langle V^{\partial_j}, E^{\partial_j}, s^{\partial_j} \rangle \left( \begin{array}{l}
t_{\partial_j, j}, \quad j \neq i \\
t_{\partial_j, i-1}, \quad j = i
\end{array} \right), \quad c \cdot \partial_i, \\
T \cdot \delta_i := \langle V^{\delta_i}, E^{\delta_i}, s^{\delta_i} \rangle \left( \begin{array}{l}
t_{\delta_i, j}, \quad j \neq i + 1 \\
s_i, \quad j = i + 1
\end{array} \right), \quad c \cdot \delta_i
\]

with the coloring maps induced naturally from the ones in \( T \).

Define also \( \mathcal{T}^0_{\{d, c\}} \in \text{Cat}^{\Delta^{op}} \) as the full simplicial subcategory of relative trees such that \( |V^1| = 1 \). Define also the simplicial full subcategories \( \mathcal{T}_\star \in \mathcal{T}_{\{d, c\}} \) for \( \star \in \{d, c\} \) of the trees such that \( cc' = \star \). We similarly define the simplicial full subcategories \( \mathcal{T}^0_\star \in \mathcal{T}^0_{\{d, c\}} \).

Note that any \( T \in \mathcal{T}_{\{d, c\}} \langle m \rangle \) has a natural partial order structure on the union of the set of vertices and edges induced by the start and target maps such that \( e^r \) is the unique minimal element. For each \( e \in E^1 \) let \( T_{e} \in \mathcal{T}^0_{\{d, c\}} \langle m \rangle \) be the sub-tree composed of the root vertex, root edge and the vertices and edges greater than \( e \).

For all \( T \in \mathcal{T}_{\{d, c\}} \langle m \rangle \) and \( v \in V \) define the relative set

\[
in v := \{ e \in E \mid te = v \} \in \Sigma_{\{d, c\}}.
\]

Note that \( \Sigma_{\{d, c\}} \) is isomorphic to \( \mathcal{T}_{\{d, c\}} \langle 0 \rangle \).

We have natural dependent sums and dependent products of filtered rooted trees of a fixed height \( T^a \in \Pi_4 \mathcal{T}_{\{d, c\}} \langle m \rangle \) defined as

\[
\Sigma_A T^a := \langle (\Sigma_A V^a, i), (\Sigma_A E^a, i), (\Sigma_A s^a), (\Sigma_A t^a), (\Sigma_A c^a) \rangle, \\
\Pi_A T^a := \langle (\Pi_A V^a, i), (\Pi_A E^a, i), (\Pi_A s^a), (\Pi_A t^a), (\Pi_A c^a) \rangle.
\]

We also have for all

\[
T = \langle (V^i), (E^i) \rangle \in \mathcal{T}_{\{d, c\}} \langle m \rangle \text{ and } S^c = \langle \langle W^c, i \rangle, (F^c, i) \rangle \in \Pi_{E^m} T_{ce} \langle n \rangle
\]

the grafting

\[
T \circ S^c := \left( \begin{array}{l}
V^i, \quad i < m \\
\sum_{E^m} W^c, \quad i = m \\
E^i, \quad i \geq m
\end{array} \right), \quad \left( \begin{array}{l}
E^i, \quad i < m \\
\sum_{E^m} F^c, \quad i = m \\
E^i, \quad i \geq m
\end{array} \right)
\]

in \( \mathcal{T}_{\{d, c\}} \langle m + n + 1 \rangle \), with the obvious start, target and coloring maps.
3.2 Relative operads

We now give a brief review of relative operads, a kind of colored operad introduced by Voronov in [26].

**Definition 3.2.** The category of $S_{\{d,c\}}$-spaces is the contravariant functor category $\text{Top}_{S_{\{d,c\}}}$. A topological relative operad is an $S_{\{d,c\}}$-space $P \in \text{Top}_{S_{\{d,c\}}}$ equipped with elements $\text{id}_* \in P\{1,\}*$ for $* \in \{d,c\}$ and structural maps

$$\langle \circ_{A}, \langle B_a \rangle \rangle \in \Pi_A \Sigma_{A} S_{\{d,c\}} \times \Pi_A P B_a, P \Sigma_{A} B_a$$

such that $P \varnothing_* = *$ for $* \in \{d,c\}$ and, using the notation

$$\alpha \langle \beta^a \rangle := \circ_{A_*}(\alpha, \langle \beta^a \rangle)$$

the following equations are satisfied:

$$\alpha\langle \beta^a \rangle \langle \gamma^a \rangle = \alpha\langle \beta^a \rangle \langle \gamma^a \rangle;$$

$$\text{id}_c \alpha = \alpha = \alpha \langle \text{id}_{c*} \rangle;$$

$$\alpha \cdot \sigma(\beta^a) = \alpha(\beta^a \cdot \sigma(B^a));$$

$$\alpha(\beta^a \cdot \tau^a) = \alpha(\beta^a) \cdot \Sigma_{A} \tau^a.$$

Operad morphisms are natural transformations that preserve the unit and compositions, and we denote the category of topological relative operad as $\text{Op}_{\{d,c\}}[\text{Top}]$.

For $X = ((X_d, e_d), (X_c, e_c)) \in \text{Top}_{\{d,c\}}$ define

$$\Pi_X : S_{\{d,c\}} \rightarrow \text{Top}; \Pi_A X := \Pi_A X_{ca}, \sigma \cdot \langle x^a \rangle := \begin{cases} e_{ca}, & a \notin \text{Im } \sigma; \\ x^{a'}, & a' \in \text{Im } \sigma. \end{cases}$$

---

Figure 1: A filtered rooted relative tree in $T_c(3)$ with wiggled edges representing “domain” edges and straight edges “codomain” edges. The leaves are the only edges that are labeled.
The underlying functor of a unital relative operad \( \mathcal{P} \) can be extended to a functor on \( S_{(d,c)}^{\text{inj,op}} \). For \( \sigma \in \mathcal{S}_{(d,c)}(A,A') \) the right action \( \cdot \sigma \in \text{Top}(\mathcal{P} A', \mathcal{P} A) \) is defined as

\[
\alpha \cdot \sigma := \alpha \left\{ \begin{array}{ll}
\star \circ \alpha', & a' \notin \text{Im} \sigma;
\text{id}_{\alpha'}, & a' \in \text{Im} \sigma.
\end{array} \right.
\]

These morphisms are the degenerations of the relative operad.

A relative operad \( \mathcal{P} \) induces a monad \(( \mathcal{P}; \eta, \mu )\) on \( \text{Top}_*^2 \) with

\[
PX_* := \int \mathcal{S}(\mathcal{P} A \times \Pi A X);
\]

\[
\eta_* x := [\text{id}_*, x], \quad \mu_*(\alpha, ([\beta^a, \langle x^{a,b} \rangle]) := [\alpha(\beta^a), \langle x^{a,b} \rangle].
\]

**Definition 3.3.** Let \( \mathcal{P} \) be a relative operad. A \( \mathcal{P} \)-space is a \( \mathcal{P} \)-algebra, i.e. a pair of pointed spaces \( X \in \text{Top}_*^2 \) equipped with structural maps

\[
\langle \theta_A \rangle \in \Pi \mathcal{S}_{(d,c)} \text{Top}(\mathcal{P} A \times \Pi A X, X e A),
\]

satisfying, using the notation \( \alpha(x^a) = \theta_A(\alpha, \langle x^a \rangle) \), the following equations:

\[
\alpha(\beta^a \langle x^{a,b} \rangle) = \alpha \circ (\beta^a) \langle x^{a,b} \rangle;
\]

\[
\text{id}_* x = x;
\]

\[
\alpha \cdot \sigma(x^a) = \alpha(\sigma \cdot x^a).
\]

The category of \( \mathcal{P} \)-spaces is denoted \( \mathcal{P} [\text{Top}] \).

The following are the relative operads relevant to the main result.

The terminal relative operad is \( \text{Com}^\sim \) with underlying \( \mathcal{S}_{(d,c)} \)-space given by \( \text{Com}^\sim(A) := \ast \). The \( \mathcal{S}_{(d,c)} \) right actions, units and compositions are the unique terminal maps. The \( \text{Com}^\sim \)-spaces are pairs \((M_d, M_e)\) of topological commutative monoids equipped with a continuous homomorphism \( \iota : M_d \to M_e \) induced by the unique element in \( \text{Com}^\sim(\{1\}) \).

For \( U \in \mathcal{A} \) let the relative operad of \( U \)-embeddings \( \text{Emb}_U \) be

\[
\text{Emb}_U A := \{ \alpha = \langle \alpha_a \rangle \in U A | \langle \alpha_a \rangle \text{ is an embedding} \};
\]

\[
\langle \alpha_a \rangle \cdot \sigma := \langle \alpha_{a'\sigma} \rangle, \quad \text{id}_U := \text{id}_U, \quad \alpha(\beta^a) := \langle \alpha_a \beta^a \rangle
\]

and degenerations deleting embeddings.

For \( U \in \mathcal{A} \) the loop space map functors image has natural \( \text{Emb}_U \)-pairs structure, giving us the functor

\[
\Omega^U_2 : \text{Top}_*^2 \to \text{Emb}_U[\text{Top}] : \Omega^U_2(\iota : Y_d \to Y_e) := \langle Y_d, Y_e \rangle;
\]

\[
\alpha(\gamma^a) := \left\{ \begin{array}{ll}
\gamma^a \alpha^a \iota^a, & \gamma \alpha = \gamma \alpha A.
\end{array} \right.
\]

(3.1)

For \((U,V) \in \Sigma_{\mathcal{A} \mathcal{U}}\) we have natural inclusion of relative operads

\[
\iota^U_{V} : \text{Emb}_U \to \text{Emb}_V; \quad \iota^U_{V} \alpha := \langle \nu \mapsto \nu_{V-U} + \alpha_{V-U} \rangle
\]

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and we define $\text{Emb}^c := \text{colim}_{\mathcal{A}} \text{Emb}^c_U$.

The embeddings operad contains embeddings of configuration spaces, and these embeddings are relevant to the definition of $E^c$-operads we give here. For each $U \in \mathcal{A}$ define the configurations $S_{(d,e)}$-space

$$\text{Conf}_U : S_{(d,e)}^{op} \rightarrow \text{Top}; \text{Conf}_U A := \{\vec{x} = \{x_a\} \in U^A \mid a \neq a' \implies \vec{x}_a \neq \vec{x}_{a'}\}.$$  

Note that $\text{Conf}_U$ is $m$-cofibrant, since it is $h$-equivalent to the underlying space of the Fulton-MacPherson operads which are $q$-cofibrant [18, 9]. We can define the $S_{(d,e)}$-space maps

$$\chi_U : \text{Conf}_U \Rightarrow \text{Emb}^c_U; \chi_U \vec{x} := \left(\vec{u} \mapsto \vec{x}_a + \frac{\min_{a' \neq a} \|\vec{x}_{a'} - \vec{x}_a\| \vec{u}}{\min_{a' \neq a} \|\vec{x}_{a'} - \vec{x}_a\| + 2\|\vec{u}\|}\right).$$

**Definition 3.4.** An $E^c_{\infty}$-operad is an operad

$$E^c \in \mathcal{Op}_{(d,e)}[\text{Top}]$$

equipped with a relative operad map

$$\Psi \in \mathcal{Op}_{(d,e)}[\text{Top}](E^c, \text{Emb}^c_{\infty})$$

and, for the induced $\mathcal{A}$-filtration $E^c_U := \Psi^{-1}\text{Emb}^c_U$, a $S_{(d,e)}$-space homotopy equivalence

$$\Phi_U \mathcal{Top}^{op}_{(d,e)}(\text{Conf}_U, E_U^c)$$

for each $U \in \mathcal{A}$ such that $\Psi |_U \Phi_U = \chi_U$.

By this definition the $E^c_U$ are $m$-cofibrant as $S_{(d,e)}$-spaces and $E^c$ is contractible and free. One of the main examples of $E_{\infty}$-operads we will consider is the Steiner relative operad, composed of paths of embeddings [22].

For all $U \in \mathcal{A}$ define the relative operad $\mathcal{H}^c_U$ as

$$\mathcal{H}^c_U A := \left\{ \alpha = \{\alpha_a\} \in U^{I \times U} \mid \begin{array}{ll} \forall a \in A, t \in I : (\vec{u} \mapsto \alpha_a(t, \vec{u})) \in \text{Emb}^c_U \{a\}; \\
\forall a \in A, t \in I, \forall \vec{u}, \vec{v} \in U : \\
\|\alpha_a(t, \vec{u}) - \alpha_a(t, \vec{v})\| \leq \|\vec{u} - \vec{v}\|; \\
\forall a \in A, \vec{u} \in U : \alpha_a(1, \vec{u}) = \vec{u}; \\
\{\vec{u} \mapsto \alpha_a(0, \vec{u})\} \in \text{Emb}^c_U A. \end{array} \right\}$$

$$(\alpha, \cdot) \ast (\alpha_{\sigma a}), \text{id}_* := ((t, \vec{u}) \mapsto \vec{u}), \alpha(\beta) := ((t, \vec{u}) \mapsto \alpha_a(t, \beta(t, \vec{u})))$$

and degenerations deleting paths of embeddings.

We have natural inclusions $\iota_U^V : \mathcal{H}^c_U \Rightarrow \mathcal{H}^c_V$ for all $(U, V) \in \Sigma_{\mathcal{A}} A_U$ with

$$\iota_U^V \alpha := \{(t, \vec{v}) \mapsto \vec{v}_V - U + \alpha_a(t, \vec{u}_U)\}$$

and we define $\mathcal{H}^c_{\infty} := \text{colim}_{\mathcal{A}} \mathcal{H}^c_U$.  

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The $E_{\infty}^-$-structural transformations are
\[
\begin{align*}
\Psi : \mathcal{H}^- &\Rightarrow \text{Emb}^-; \quad \Psi_U \alpha := (u \mapsto \alpha_u(0, \tilde{u})); \\
\Phi_U : \text{Conf}^U &\Rightarrow \mathcal{H}^U; \quad \Phi_U x := ((t, \tilde{u}) \mapsto (1 - t)((\chi_U x) a \tilde{u}) + t \tilde{u}).
\end{align*}
\]
The homotopy inverses of the $\Phi_U$ are
\[
\begin{align*}
\bar{\Phi}_U : \mathcal{H}^U &\Rightarrow \text{Conf}^U; \quad \bar{\Phi}_U \alpha = (\alpha_a(0, 0)).
\end{align*}
\]
See [22] for the construction of the homotopies.

### 3.3 Bar resolution

For the construction of the quasiadjunctions in our main theorems we will require the bar resolution of $E^-$-pairs. Recall from [16] Construction 9.6 that for a monad $(C, \eta, \mu)$ in the category $\mathcal{T}$, a $C$-functor $(F, \lambda)$ in the category $\mathcal{A}$ and a $C$-algebra $(X, \xi)$ the two sided bar construction $B_-(F, C, X) \in \mathcal{A}^\Delta$ with

\[
B_{(m)}(F, C, X) := FC^m X; \quad \delta_i := FC^i \xi_{C^m-1}, \quad \partial_i := \begin{cases} 
\lambda_{C^m}, & i = 0; \\
FC^{i-1} \mu_{C^{m-1}}, & 0 < i < m; \\
FC^{m-1} \xi, & i = m.
\end{cases}
\]

In particular for a relative operad $\mathcal{P}$ and $C = P = F$ we have a natural isomorphism

\[
B_{(m)}(P, P, X) \cong \int^{T_{(m)}} \Pi_{\mathcal{V}} \mathcal{P} \mathcal{m} \times \Pi_{E^m} X_{cc},
\]

\[
\begin{align*}
[a^r, (\alpha^u), (x^r)] &\cdot \delta_i := \\
\begin{cases} 
[a^r (\alpha^u, a^u), (x^r)] & i = 0; \\
[a^r, \begin{cases} \alpha^r & |v| = i - 1; \\
\alpha^u & |v| \neq i - 1
\end{cases}, (x^r)] & 0 < i < m; \\
[a^r, \begin{cases} \alpha^r & |v| = 1; \\
\alpha^u & |v| \neq 1
\end{cases}, (x^r)] & i = m.
\end{cases}
\end{align*}
\]

The $E^-$-pair structural maps in each dimension are

\[
\alpha([\beta^{a,r}, (\beta^{a,u}), (x^{a,e})]_{T^m}) := [\alpha(\beta^{a,r}), (\beta^{a,u}), (x^{a,e})]_{\Sigma A T^m}
\]  

(3.2)

The bar resolution of $E^-$-pairs is then the geometric realization of this simplicial $E^-$-pair functor

\[
\bar{B}_2 : E^-[\text{Top}] \to E^-[\text{Top}], \quad \bar{B}_2 X_\bullet := \left|B_-(E^-, E^-, X)_\bullet\right|.
\]

By the above isomorphism we can intuitively think of points in $\bar{B}_2 X$ as equivalence classes of filtered rooted relative trees with vertices decorated with
elements of $\mathcal{E}^-$, leaves decorated with elements of $X$ and we associate an ordered partition with the filtration of the inner vertices.

It is not the case in general that the geometric realization of a simplicial $C$-algebra for a topological monad $C$ is a $C$-algebra. This is however the case when the monad is the one induced by an operad. The structural maps are induced by the algorithm described in [16, Theorem 11.5]. For a sequence of elements with representatives of distinct dimensions we can systematically determine equivalent representatives of the same dimension, and then apply the formula 3.2, so that the $\mathcal{E}^-$-pair structural maps of $B_2X$ are defined by the formula

$$\alpha \langle \beta^a_r, \langle \beta^a_v, \langle x^a_e \rangle \rangle \rangle = \langle \beta^a_r, \langle \beta^a_v, \langle x^a_e \rangle \rangle \rangle \in \Pi \Sigma_{\text{Top}} \mathcal{G} \times \Pi \mathcal{P} B^a, \mathcal{P} \Pi A B^a \rangle$$

which is illustrated in figure 2.

This functor can be equipped with the natural transformation

$$\eta : B_2 \Rightarrow \text{Id}, \quad \eta \left[ \left[ (\alpha^r, \langle \alpha^v, \langle x^e \rangle \rangle) \right] \right] := \circ T \alpha^r \circ \alpha^v (x^e).$$

where $\circ T \alpha^r$ is the composition of all the $\alpha^r$, including $\alpha^r$, induced by the operadic composition and the structure of $T$.

### 3.4 Relative operad action

Operad actions encode distributive laws between operations defined by operads [15, Definition VI.1.6]. The following definition is a relative version of this notion.

**Definition 3.5.** A relative operad pair is a pair of relative operads $\langle \mathcal{P}, \mathcal{G} \rangle$ equipped with an extension of $\mathcal{G}$ to $\mathcal{S}^\text{op}_{(d,c)}$ and structural maps

$$\langle \kappa \rangle \in \Pi \Sigma_{\text{Top}} \mathcal{G} \times \Pi \mathcal{P} B^a, \mathcal{P} \Pi A B^a \rangle$$
the action of \( G \) on \( P \), such that, using the notations

\[
f \times \langle \alpha^a \rangle := \langle f, \langle \alpha^a \rangle \rangle, \quad f \times \langle \beta^{a,b} \rangle := f \cdot \pi_{(b^a)} \times \langle \beta^{a,b} \rangle
\]

the following equations are satisfied:

\[
f \times \langle g^a \rangle \times \langle \alpha^{a,b} \rangle = f \circ \langle g^a \rangle \times \langle \alpha^{a,b} \rangle; \quad (3.5)
\]

\[
f \times \langle \alpha^a \circ \langle \beta^{a,b} \rangle \rangle = f \times \langle \alpha^a \circ (f \times \langle \beta^{a,b} \rangle) \rangle \cdot \nu; \quad (3.6)
\]

\[
\text{id}_{x_A} \times \alpha = \alpha; \quad (3.7)
\]

\[
f \times \langle \text{id}_{x_A} \rangle = \text{id}_{x_A}; \quad (3.8)
\]

\[
f \times \sigma \times \langle \alpha^{a} \rangle = f \times \langle \alpha^{a-1} \rangle \cdot \sigma(B^a); \quad (3.9)
\]

\[
f \times \langle \alpha^a \cdot \tau^a \rangle = f \times \langle \alpha^a \rangle \cdot \Pi_A \tau^a. \quad (3.10)
\]

We refer to the operad \( P \) as the additive relative operad and \( G \) as the multiplicative relative operad of the pair.

For \( X = ((X_d, 0_d, 1_d), (X_c, 0_c, 1_c)) \in \text{Top}_2^0 \) define

\[
X^{\wedge -} : S^{(d,c)}_{(d,c)} \to \text{Top}; \quad X^{\wedge A} := \wedge_A X_{x_0}, \quad \sigma \cdot [x^a] := \begin{cases} 1_{\text{cat}}, & a' \notin \text{Im} \sigma; \\ x^{a-1}_{a'}, & a' \in \text{Im} \sigma. \end{cases}
\]

with the zeros as base points for the wedge products. We can then define the monad \( (G_0; \eta, \mu) \) on \( \text{Top}_2^0 \) with

\[
G_0X_* := \int_{S_{(d,c)}}^\times G A_* \wedge X^{\wedge A};
\]

\[
\eta_*x := [\text{id}_x, x], \quad \mu_*[f, [[g^a, [x^{a,b}]]]] := [f(g^a), [x^{a,b}]].
\]

**Definition 3.6.** A \( G_0 \)-space is a \( G_0 \)-algebra, i.e. a pair of \( S^0 \)-spaces \( X \in \text{Top}_2^0 \) equipped with a structural map \( \chi : G_0X \to X \) satisfying, using the notation

\[
f[x^a] = \chi_A[f, [x^a]]
\]

similar equations as in [3.3] and also that 0 is an absorbing element, i.e

\[
\exists a \in A : x^a = 0_{x^a} \Longrightarrow f[x^a] = 0_{x^a}.
\]

The category of \( G_0 \)-spaces is denoted \( G_0[\text{Top}] \).

If \( G \) acts on \( P \) then the functor \( P \) induces a monad on \( G_0[\text{Top}] \).

**Definition 3.7.** Let \((G, P)\) be a relative operad pair. A \((G, P)\)-space is a \( P \)-algebra in \( G_0[\text{Top}] \). Equivalently a \((G, P)\)-space is a pair of \( S^0 \)-spaces \( X \in \text{Top}_2^0 \) equipped with a \( G_0 \)-space structure and a \( P \)-space structure with neutral elements the zeros such that

\[
f[\alpha^a(x^{a,b})] = f \times \langle \alpha^a \rangle (f[x^{a,b}]).
\]

The category of \((P, G)\)-spaces is denoted \((P, G)[\text{Top}] \).
There is a natural operad pair structure on \((Com^{-}, Com^{-})\). Set the notation 
\(\sum_{A} \in Com^{-}(A)\) for the additive copy of \(Com^{-}\) and \(\prod_{A} \in Com^{-}(A)\) for the multiplicative copy of \(Com^{-}\). Then in a \((Com^{-}, Com^{-})\)-space the distributivity equations and the equality of the additive and multiplicative homomorphisms

\[
\prod_{A} \sum_{B^{n}} \alpha_{a,b} = \sum_{\prod_{A} B^{n}} \prod_{A(\alpha_{b})} \alpha_{a,b}
\]

\[
\phi_{+}, x = \prod_{\{1, 2, \cdots, n\}_{c}} \langle \phi_{+}, 1_{c} \rangle
\]

\[
= \prod_{\{1, 2, \cdots, n\}_{c}} \frac{\phi_{+}}{\phi_{+}, \text{id}_{1}} \prod_{\{1, 2, \cdots, n\}_{c}} \langle x, 1_{c} \rangle
\]

\[
= \phi_{+}x
\]

hold. This means that \((Com^{-}, Com^{-})[\text{Top}]\) is isomorphic to the category of topological commutative semi-algebras over commutative semi-rings.

The main example of multiplicative relative operad we will consider is the relative linear isometries operad \(\mathcal{L}^{-}\) with

\[
\mathcal{L}^{-} \text{A} := \mathcal{I}(\Phi_{A} R^{\infty}, R^{\infty});
\]

\[
f \cdot \sigma := \langle f_{\sigma_{a}} \rangle, \quad \text{id} := id_{\mathbb{R}^{\infty}}, \quad f \circ \langle a_{b} \rangle := \langle f_{a_{b}} \rangle.
\]

The classical action of the linear isometries operad on the Steiner operad induces an action on the relative versions. The extension of \(\mathcal{L}^{-}\) to \(S^{op}_{(d < c)}\) is given by identity maps and the action maps are given by the formula

\[
f \otimes \langle \alpha_{a} \rangle := [(t, f_{a_{b}}) \mapsto f_{a_{b}}(\alpha_{b} = (t, u_{a})].
\]

**Definition 3.8.** The category of \(E_{\infty}^{-}\)-algebras is \((\mathcal{E}^{-}, \mathcal{L}^{-})[\text{Top}]\) for an \(E_{\infty}^{-}\)-operad \(\mathcal{E}^{-}\) equipped with an action by \(\mathcal{L}^{-}\).

Although we give this general definition we note that there is no known non-trivial example of an \(E_{\infty}\) operad equipped with an \(\mathcal{L}\)-action other then the Steiner operad \(\mathcal{K}_{\infty}\). Having a \(q\)-cofibrant, not just mixed \(\Sigma\)-cofibrant example would be interesting and useful, but since we can work in the mixed model structure of spectra it is not necessary.

The images of \(B_{2}X\) are also \(\mathcal{L}^{-}\)-pairs with structural maps defined as

\[
f \otimes \langle \alpha_{a} \rangle, \langle f \otimes \langle \alpha_{a} \rangle, \langle \alpha_{a} \rangle \rangle_{T^{a}, d^{a}} :=
\]

\[
\left[ f \otimes \langle \alpha_{a} \rangle, \langle f \otimes \langle \alpha_{a} \rangle \rangle_{T^{a}, d^{a}} \right].
\]

which is illustrated in figure [3]

---

[3] Semi-algebras and semi-rings are like algebras and rings without the assumption that additive inverses exist, ie we have an additive commutative monoid instead of an additive abelian group.
Figure 3: $\mathcal{L}^\rightarrow$-structure of $B_2X$
3.5 Model structure of $E_{\infty}$-algebras

The model structure of $E_{\infty}$-pairs is transferred from the $q$-model structure of $\text{Top}_\ast^2$ by the adjunction

$$(E^{-} L^{-} \dashv U) : \text{Top}_\ast^2 \cong (\mathcal{E}^{-}, \mathcal{L}^{-})[\text{Top}],$$

so the weak equivalences and fibrations are respectively the maps that are $q$-equivalences and $q$-fibrations as topological space maps [17]. All objects are fibrant and cofibrant algebras are retracts of cellular $E_{\infty}$-algebras, with cells the $E^{-} L^{-}$-images of the cells in $\text{Top}_\ast^2$.

4 Recognition of algebra spectra

4.1 Coordinate-free spectra

We give a brief review of coordinate-free spectra [11] and give some examples.

Let $U \in \mathcal{F}$ be countably infinite dimensional (In the context of coordinate-free spectra we refer to $U$ as a universe). The topological category $\mathbf{Sp}_U$ of coordinate-free $U$-spectra is composed of the class of objects

$$\{Y = (\langle Y_U \rangle, \langle \sigma_V \rangle) \in \Sigma_{\Pi \sigma \text{Top}_{\ast}^2 \Pi \Sigma_{\sigma \text{Top}_{\ast}^2} \text{Top}_\ast^2 (Y_U \wedge S^{Y-U}, Y_V) | \sigma_U y, \vec{0} = y, \sigma_W [\sigma_V y, \vec{v}], \vec{w}] = \sigma_U y, \vec{v} + \vec{w}]\}$$

and the morphisms spaces $\mathbf{Sp}_U(Y, Z)$ defined as

$$\{f = \langle f_U \rangle \in \Pi \sigma \text{Top}_\ast^2 (Y_U, Z_U) | \sigma_V f_U y, \vec{v}, \bar{v} = f_V \sigma_V y, \vec{v} \}.$$

We are particularly interested here in the case $U = R^\infty$ and in this case we use the notation $\mathbf{Sp} := \mathbf{Sp}_{R^\infty}$.

Example 4.1. Interesting coordinate-free spectra to keep in mind are the following, with details similar to the equivalent symmetric examples in [20, Section I.2]:

- For each $p \in \mathbb{Z}$ the $p$-sphere spectrum is defined as
  $$S^p := \begin{cases} \langle S^{U-R^{-p}}, \sigma_U^U \hat{u}, \vec{v} \rangle := \hat{u} + \hat{v}_{U-R^{-p}}, & p < 0 \\ \langle S^U, \sigma_U^U \hat{u}, \vec{v} \rangle := \hat{u} + \hat{v}, & p = 0 \\ \langle S^{U-R^p}, \sigma_U^U ([\hat{u}, \vec{v}], \vec{w}) := (\hat{u} + \hat{v}, \vec{w}), & p > 0 \end{cases}$$

  We use the notation $S := S^0$.

- For each $G \in \text{AbGrp}$ define the Eilenberg-MacLane spectrum
  $$HG := (G \otimes F[S^U])_\ast; \quad \sigma_V^U [g_a \otimes \vec{u}^a, \vec{v}] := g_a \otimes \vec{u}^a + \vec{v}.$$
where \( F[S^U] \) denotes the quotient of the free abelian group generated by the points of the \( U \)-sphere by the subgroup generated by \( \infty \), and as in the Einstein convention \( g_a \otimes \bar{u}^2 \) indicates a finite sum of elements. Note that \( g \otimes \infty = 0 \).

- For each \( U \in \mathcal{A} \) let \( O_U \) be the orthogonal group of isometric automorphisms of \( U \). The total space \( EO_U \) of the universal principal \( O_U \)-bundle is the geometric realization of the simplicial space \( O_U \in \text{Top}^{\Delta^{op}} \) with

\[
\langle f \rangle \cdot \partial_i := \begin{cases}
  f^j, & j < i - 1 \\
  f^j f^{j+1}, & j = i - 1 \\
  f^{j+1}, & j > i - 1
\end{cases}, \quad \langle f \rangle \cdot \delta_i := \begin{cases}
  f^j, & j < i \\
  \text{id}, & j = i \\
  f^{j-1}, & j > i
\end{cases}.
\]

The \( U \)-spheres admit a left \( O_U \)-action by evaluation \( f \cdot \bar{u} := f \bar{u} \) and \( EO_U \) admits the right \( O_U \)-action

\[
\langle \langle g \rangle, t \rangle \cdot f := \begin{cases}
  g^i, & i < m \\
  g^i, & i = m
\end{cases}, t \]

For \( (U, V) \in \Sigma_{\mathcal{A}_{\mathcal{A}U}} \) we have a natural inclusion

\[
\iota^U_V : O_U \to O_V, \quad \iota^U_V f \bar{v} := \bar{v}_{V-U} + f \bar{v}_U.
\]

We can define the Thom spectrum as

\[
MO := \langle EO_{U,+} \wedge_{O_U} S^U \rangle; \quad \sigma^U_V [[\langle f \rangle, t, \bar{u}], \bar{v}] := [\langle \iota^U_V f \rangle, t, \bar{u} + \bar{v}].
\]

An \( \Omega \)-spectrum is a spectrum \( Y \in \text{Sp} \) such that the adjoint structural maps \( \sigma^U_V \in \text{Top}_*(Y_U, Y_V^{S^{V-U}}) \) are \( q \)-equivalences.

The stable homotopy groups of spectra are defined as \( \pi^S_p Y := \pi_0 \text{Sp}(S^p, Y) \).

If \( Y \) is an \( \Omega \)-spectrum then

\[
\pi^S_p Y \equiv \begin{cases}
  \pi_0 Y_{[p]}, & p < 0 \\
  \pi_p Y_0, & p \geq 0
\end{cases}
\]

Spectra maps that induce isomorphisms of the stable homotopy groups are called \textit{stable weak equivalences}, and spectra \( Y \in \text{Sp} \) with \( \pi^S_p Y \) trivial for \( p < 0 \) are called \textit{connective}.

We base space functor

\[
\Lambda^\infty : \text{Sp} \to \text{Top}_* ; \quad \Lambda^\infty Y := Y_0
\]

which is right adjoint to the suspension spectrum functor

\[
\Sigma^\infty : \text{Top}_* \to \text{Sp}; \quad \Sigma^\infty X := \{ X \wedge S^U \}; \quad \sigma^U_V [[x, \bar{u}], \bar{v}] := [x, \bar{u} + \bar{v}].
\]

with unit and counit of the adjunction

\[
\eta x := [x, 0]; \quad \epsilon_U [y, \bar{u}] := \sigma^0_U [y, \bar{u}].
\]

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4.2 Stable mixed model structure of spectra

For any spectrum \( Y \in \mathcal{Sp} \) the cylinder spectrum is defined as \( Y \wedge I_+ := \{ Y \wedge I_+ \} \) and the cone spectrum is \( CY := Y \wedge I_+/\{y,1\}[-y,1] \).

In the strict Quillen model structure on \( \mathcal{Sp} \) a morphism \( f \in \mathcal{Sp}(X,Y) \) is a weak equivalence if each \( f_U \) is a \( q \)-equivalence, a fibration if it is a Serre fibration, ie if it has the homotopy lifting property with respect to all cylinder inclusions in \( \mathcal{Sp} \), and a cofibration if it is a retract of a relative cell-spectrum, with cells given by cones of sphere spectra and domain of the attaching maps the boundary sphere spectra [5, Section VII.4]. This is a cofibrantly generated model structure with factorization systems induced by the small object argument. The weak equivalences, fibrations and cofibrations of this model structure are referred to as \( q \)-equivalences, \( q \)-fibrations and \( q \)-cofibrations respectively.

Homotopy equivalences in \( \mathcal{Sp} \) are spectra maps that admit an inverse up to homotopy, with homotopies defined via the cylinder spectra in the usual way. In the strict Hurewicz/Strøm model structure \( f \) is a weak equivalence if it is a homotopy equivalence, a fibration if it is a Hurewicz fibration, ie if it has the homotopy lifting property with respect to all cylinder inclusions \( in_0 \in \mathcal{Sp}(CS^q,CS^q \wedge I_+) \) for all \( q \in \mathbb{Z} \), and a cofibration if it has the left lifting property against trivial Hurewicz fibrations.

The weak factorization system can be constructed through (co)monads as described in [1]. For any \( Y \in \mathcal{Sp} \) let the spectrum of Moore paths in \( Y \) be

\[
MY := \{ \Sigma_{[0,\infty)}\{Y^{[0,\infty]}_U \mid s \geq t \implies \gamma s = \gamma t \}\},
\]

\[
\sigma^U_[(t, \gamma), \bar{v}] := (t, r \mapsto \sigma^U_[(\gamma r, \bar{v})]).
\]

The factorization systems are then defined as

\[
\Gamma f := X \times_Y MY; \quad C_t f x := (x, 0, r \mapsto f x), \quad F f(x, t, \gamma) := \gamma t.
\]

\[
E f := \Gamma f \wedge [0, \infty] \cup_{\Gamma f} Y; \quad C f x := (x, 0, r \mapsto f x, 0), \quad F f(x, t, \gamma, s) := \gamma s.
\]

The weak equivalences, fibrations and cofibrations of this model structure are referred to as \( h \)-equivalences, \( h \)-fibrations and \( h \)-cofibrations respectively. We then equip \( \mathcal{Sp} \) with the mixed model structure as described in [3, Prop 3.6].

Since the point of spectra is to study stabilization phenomena we are actually interested in inverting the stable weak homotopy equivalences. The stable model structure with stable weak homotopy equivalences as weak equivalences is obtained from the strict model structure by the process of Bousfield localization through the following idempotent monad [2, 19]. For every spectrum \( Y \in \mathcal{Sp} \) we can functorially define an inclusion spectrum \( \overline{\Omega} \) equipped with a quotient map \( Y \to \overline{Y} \), so we may think of points in \( \overline{Y} \) as equivalence classes of points in \( Y \) (see [11, Ap1] for a detailed construction). If \( Y \) is already an inclusion spectrum then \( \overline{Y} = Y \). We may then define the \textit{spectrification functor}

\[
\overline{\Omega} : \mathcal{Sp} \to \mathcal{Sp}; \quad \overline{\Omega} Y := (\text{colim}_{\sigma f} \overline{Y}^U) \quad \sigma^U_[(\gamma, \bar{v})] := (\bar{v} \mapsto \gamma(\bar{v} + \bar{w}))
\]

\footnote{Inclusion spectra are those with adjoint structural maps \( \sigma \) all inclusions.}
induced by the adjoint structural maps $\tilde{\sigma}$ and with the formula for the structural maps determined for a choice of representative $\gamma$ with domain $V \in \mathcal{A}_W$. This is a Quillen idempotent monad with structural natural map

$$\epsilon' : Id \Rightarrow \tilde{\Omega}; \quad (\epsilon'_{U,y}) := [v \mapsto \sigma_U^V[y,v]] \quad (4.1)$$

The stable model structure on spectra $\text{Sp}^{-}$ has as weak equivalences the stable weak equivalences and stable fibrations are $p \in \text{Sp}(E,B)$ composed of indexwise Hurewicz fibrations such that the maps

$$(\sigma_U^V, p_U) : E_U \rightarrow E_U^{S^{-V}} \times_{B_U^{S^{-V}}} B_U$$

are $q$-equivalences. The fibrant spectra are the $\Omega$-spectra, and the cofibrant spectra are those homotopy equivalent to retracts of $q$-cofibrant spectra. With the induced stable model structure the adjunction $(\Sigma^\infty \dashv \Lambda^\infty)$ is a Quillen adjunction.

The morphisms category $\text{Sp}^{-}$ admits a projective stable model structure with $(f_d, f_c) \in \text{Sp}^{-} : (1 : Y_d \rightarrow Y_c, i) : Z_d \rightarrow Z_c$ a weak equivalence or fibration if $f_c$ and $f_d$ are both stable weak equivalences or stable fibrations respectively, and it is a cofibration if both $f_d$ and $(f_c, i) : Y_c \vee Y_d \rightarrow Z_c$ are stable cofibrations.

### 4.3 Recognition of $\infty$-loop maps

We can now prove the recognition principle for $\infty$-loop pairs of spaces of spectra maps. The base pair of spaces functor is

$$\Lambda^\infty_2 : \text{Sp}^{-} \rightarrow \text{Top}^2, \quad \Lambda^\infty_2 i := (Y_{d,0}, Y_{c,0})$$

and the relative suspension functor is

$$\Sigma^\infty_\alpha : \text{Top}^2 \rightarrow \text{Sp}^{-}, \quad \Sigma^\infty_\alpha X := \Sigma^\infty(\text{in}_d : X_d \rightarrow X_d \vee X_c).$$

We have a Quillen adjunction

$$(\Sigma^\infty_\alpha \dashv \Lambda^\infty_2) : \text{Top}^2 \Rightarrow \text{Sp}^{-}; \quad \eta_\alpha x := [x, \tilde{\Omega}], \quad \epsilon_\alpha[y, u] := \begin{cases} \sigma_U^V[y, u], & \epsilon y = \star \\ \sigma_U^V[y, u], & \epsilon y \neq \star \end{cases}$$

The spectrification functor $\tilde{\Omega}$ induces

$$\tilde{\Omega}_\alpha : \text{Sp}^{-} \rightarrow \text{Sp}^{-}; \quad \tilde{\Omega}_\alpha i := (\tilde{\Omega}i : \tilde{\Omega}Y_d \rightarrow \tilde{\Omega}Y_c).$$

The $\infty$-loop pair of spaces functor is defined as

$$\Omega^\infty_2 : \text{Sp}^{-} \rightarrow \mathcal{E}^{-}[\text{Top}]; \quad \Omega^\infty_2 i := \Lambda^\infty_2 \tilde{\Omega}_\alpha i$$

with structural maps induced by the formula $[3.1]$ by taking representatives of the $\gamma^\alpha$ with a common domain.
This functor is not a right adjoint, but it is a weak Quillen right quasiadjoin.
The left quasiadjoint functor is defined as follows: We have simplicial pointed
maps $B_\times (\Sigma^\times U, E^\times_U, X) \in (\text{Top}_\times)^\Delta$ with

$$B_\times (\Sigma^\times U, E^\times_U, X) \cong \{ \int_{\mathcal{E}^-}[\Pi_V \mathcal{E}^- \in \mathcal{V} \times \mathcal{I} \mathcal{E}^- X_{\mathcal{E}^-} \wedge S^U, \mathcal{V}] \}
$$

The left quasiadjoint functor is defined as follows: We have simplicial pointed
maps $B_\times (\Sigma^\times U, E^\times_U, X) \in (\text{Top}_\times)^\Delta$ with

$$B_\times (\Sigma^\times U, E^\times_U, X)_\bullet \cong \{ \int_{\mathcal{E}^-}[\Pi_V \mathcal{E}^- \in \mathcal{V} \times \mathcal{I} \mathcal{E}^- X_{\mathcal{E}^-} \wedge S^U, \mathcal{V}] \}
$$

Define the relative $\infty$-delooping functor as

$$B_\times^\infty : \mathcal{E}^- \mathcal{[T} \mathcal{op}] \to \mathcal{S} \mathcal{p}^-; \quad B_\times^\infty X_\bullet := \{ \{ B_\times (\Sigma^\times U, E^\times_U, X)_\bullet \} \},$$

$$\sigma^\infty_\mathcal{V}[[\{ \mathfrak{a}^\infty_\mathcal{V} \}, (x^\mathfrak{e}_\mathcal{V})], (u_T, t), (v)] := [[\{ \mathfrak{a}_\mathcal{V}^\infty \}, (x^\mathfrak{e}_\mathcal{V})], (u_T, t), (v)].$$

Points in $B_\times^\infty X_{\bullet, U}$ are equivalence classes of decorated filtered rooted relative
trees as in the description of the bar resolution $\overline{B}_2 X$, except the root vertex is
decorated with a vector in $U$ and the relative operad points decorating the inner
vertices must be contained in the suboperad $\mathcal{E}^\times_U$ of the $\mathfrak{a}^\infty_\mathcal{V}$-filtration of $\mathcal{E}^-$.  

**Theorem 4.2.** For $\mathcal{E}^-$ an $E_\infty$-operad there is an idempotent quasiadjunction

$$(B_\times^\infty \dashv \pi^0_{2, \Omega} \Omega_\infty^\infty : \mathcal{E}^- \mathcal{[T} \mathcal{op}] \Rightarrow \mathcal{S} \mathcal{p}^-)$$

**Proof:** The unit span and cospan has $\eta$ the natural weak equivalence [3,4]
$\epsilon'$ induced by the idempotent monad transformation [4,1] and $\eta$ and $\epsilon$ are defined by the following formulas:

$$\eta: \overline{B} \Rightarrow \Omega^\infty \mathcal{B}_\times^\infty, \quad \epsilon: B^\infty \Omega^\infty \Rightarrow \overline{\Omega}_\times,$$

$$\eta_*[[\mathfrak{a}^\mathfrak{r}_\mathcal{V}, (x^\mathfrak{e}_\mathcal{V})], (t)] : \{ \overline{u} \mapsto \{ \{ (x^\mathfrak{e}_\mathcal{V}), (x^\mathfrak{e}_\mathcal{V}), (x^\mathfrak{e}_\mathcal{V}), \mathfrak{r}^{-1}_\mathcal{V} \mathfrak{u} \}_{\mathcal{V}}, t, \mathfrak{r} \in \mathfrak{r}^\mathfrak{r}_\mathcal{V} \}, \mathfrak{r} \in \mathfrak{r}^\mathfrak{r}_\mathcal{V} \Rightarrow \mathfrak{r}^\infty \cup E^0 \mathcal{U} \},$$

$$\epsilon_* \mathfrak{v}[[\gamma^\mathfrak{r}_\mathcal{V}, (x^\mathfrak{e}_\mathcal{V})], (u_T, t)] : \{ \mathfrak{v} \mapsto \sigma \alpha^\mathfrak{r}_\mathcal{V} \gamma^\mathfrak{r}_\mathcal{V} (\mathfrak{u} + \mathfrak{v}) \}.$$

We verify that the conditions for definition [2,1] are satisfied.
(i): By the assumptions on $\mathcal{E}^-$ and [2,4 Prop. 3.2.3] the functor $B^\infty$ is left
derivable.
(ii): Trivially $\Omega^\infty$ preserves fibrant objects. Since $\Omega^\infty = L^\infty \overline{\Omega}_\times$ and stable
weak equivalences are by definition maps whose images under $\overline{\Omega}_\times$ are strict weak
equivalences we have that $\Omega^\infty$ preserves weak equivalences.

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Figure 4: Representative $U$-loop of $\eta_c[[u^r, \langle v^r \rangle, (x^r)]_T, t]$

Figure 5: Representative $V$-loop of $\epsilon_{c,U}[[\langle v^r \rangle, \langle \gamma^r \rangle, \bar{u}]_T, t]$
(iii): The functor $\overline{B}_2$ preserves cofibrant objects by [25 Prop. 3.2.3] and trivially preserves fibrant objects. The functor $\Omega_*$ preserves cofibrant objects by the results in [6, Sec. 5.3] and the fact that we are using the mixed stable model structure on spectra, and it trivially preserves fibrant objects since it is the fibrant replacement functor of the stable model structure.

(iv): As a map of topological spaces $\eta$ is a realization of a simplicial strong deformation retract, so it is itself a strong deformation retract of topological spaces and therefore in particular a $q$-equivalence [16 theorems 9.10, 9.11 and 11.10]. The map $\epsilon'$ is a weak equivalence by the definition of the stable model structure.

(v): The natural homotopy $H$ which gives the homotopy commutativity in $\text{Sp}^{-}$

$\epsilon_{B_2^\infty X} B_2^\infty \eta_X [\{[\alpha^v, (\beta^e, r), (\beta^c, w), (x^c, f)]_{S^e}, s^e\}, [\tilde{u}], T, r]$

$= \left[\tilde{v} \mapsto \left[\left[\left[\alpha^v, \text{id}_{S^e, w}, \tilde{u} + \tilde{v} \right], \tilde{u}, T = (S^e, \delta^e)\right], \Phi t\right]\right]$,

where

$\Phi t := (1 - t)(\alpha^m \cdot \gamma^e, r^t) + t(\alpha^m \cdot \gamma^e, s^e)_t$,

with the conditions in the formula similar to the ones in [3,3]

(vi): In $\text{E}^-[\text{Top}]$ we have strict commutativity

$\Omega_2^\infty \eta^{\infty}_2 [\{\alpha^v, (\alpha^r, \gamma^e)\}, [\tilde{v}], t] = [\tilde{v} \mapsto \pi_\alpha^v (\gamma^e, \tilde{v})]$

$= \Omega_2^\infty \epsilon^{\infty}_2 [\{\alpha^r, (\alpha^v, \gamma^e)\}, [\tilde{v}], t]]$.

**Theorem 4.3.** The quasiadjunction in theorem 4.2 is idempotent and induces an equivalence

$[\text{LB}_2^\infty \rightleftarrows \Omega_2^\infty] : \text{HoE}^-[\text{Top}]_{\text{grp}} \simeq \text{HoSp}^{-}_{\text{conn}}$.

between the homotopy categories of grouplike $E^-$-pairs and maps between connective spectra.
Proof: In $\mathcal{E}^-[\text{Top}]$ the conditions for definition [2.3] are satisfied and the resulting reflective homotopy subcategory is composed of the grouplike $\mathcal{E}^-$-pairs:

(i) As we have seen $\eta^\prime$ is a natural weak equivalence and by definition cof is a natural trivial fibration, so $\text{cof}\eta^\prime$ is a weak equivalence.

(ii) Since $\Omega_2^\omega$ preserves weak equivalence between fibrant objects and $B^\omega_\omega$ preserves weak equivalences between cofibrant objects we have that $\Omega_2^\omega \tilde{\Omega} B^\omega_\omega \mathcal{C}$ preserves weak equivalences.

(iii) The natural transformation $\eta$ is a natural group completion, since it is a realization of a simplicial group completion map (see [16, Theorems 2.7, 9.10 and 9.11], and [14, Theorem 2.2]), and the images of $\Omega_2^\omega \tilde{\Omega} B^\omega_\omega \mathcal{C}$ are grouplike, therefore $\Omega_2^\omega \tilde{\Omega} B^\omega_\omega \mathcal{C}$ is a weak equivalence. By naturality $\Omega_2^\omega \tilde{\Omega} B^\omega_\omega \mathcal{C}$ is also a group completion, and since the domain and codomain are grouplike this is a natural weak equivalence.

(iv) This condition holds since fibrations are preserved by pullbacks, fibrations induce long exact sequences of homotopy groups and for a fibration $p : E \to B$ and a map $f : X \to B$ the fibers of the pullback $f^*p : X \times_B E \to X$ are homeomorphic to the fibers of $p$.

(v) Pushouts in $\mathcal{E}^-[\text{Top}]$ by a cofibration whose domain is $m$-cofibrant in $\text{Top}^*$ is a retract of a transfinite composition of pushouts by $m$-cofibrations in $\text{Top}_*$ (see [21, I.4]), hence this condition holds since $\text{Top}_*$ with the mixed model structure is left proper and the underlying functor of $\mathcal{E}^-$ is an $m$-cofibrant $S_{(d,c)}$-space.

By the characterization of fibrations in the resulting Bousfield localization in [25, Prop. 2.3.6] the fibrations are the group completions and fibrant objects are the grouplike $\mathcal{E}^-$-pairs.

The dual conditions for definition [2.3] are also satisfied in $\mathcal{S}^-$ and the resulting coreflective homotopy subcategory is composed of the maps between connective spectra. Note that conditions (i), (ii) and (iii) are self dual.

(i) By definition of the stable model structure $\epsilon$ is a natural stable weak equivalence and by definition fib is a natural trivial cofibration, so $\eta^\prime_{\text{fib}}$ is a weak equivalence.

(ii) That $B^\omega_\omega \tilde{\Omega} \Omega^\omega_2 \mathcal{F}$ preserves weak equivalences follows by the same argument for $\Omega^\omega_2 \tilde{\Omega} B^\omega_\omega \mathcal{C}$.

(iii) We have that $\eta_{\Omega^\omega_2}$ is a natural weak equivalence, and since $\Omega^\omega_2 \epsilon_{\Omega^\omega_2} = \Omega^\omega_2 \epsilon_{\tilde{\Omega} \Omega^\omega_2}$ and $\Omega^\omega_2 \epsilon_{\tilde{\Omega} \Omega^\omega_2}$ is a natural weak equivalence by the 2-out-of-3 property $\Omega^\omega_2 \epsilon$ is a natural weak equivalence. Since the images of $\tilde{\Omega}$ are $\Omega$-spectra by the formula for stable homotopy groups of $\Omega$-spectra we have that $\epsilon$ induces isomorphisms on the non-negative stable homotopy groups, and is therefore a stable weak equivalence on the maps between connective spectra. The images of $B^\omega_\omega$ are connective by [16, 11.12] and [14, A5]. Therefore $\epsilon_{B^\omega_\omega \tilde{\Omega} \Omega^\omega_2 \mathcal{F}}$ is a natural weak equivalence. By naturality $B^\omega_\omega \tilde{\Omega} \Omega^\omega_2 \mathcal{F} \epsilon$ also induces isomorphisms on the non-negative stable homotopy groups and so is also a natural weak equivalence.

(iv) This condition holds since cofibrations are preserved by pullbacks, spectra cofibrations induce long exact sequences of stable homotopy groups and for any cofibration $i : A \hookrightarrow X$ and map $f : A \to Y$ the cofiber of the pushout
\( f \circ i : Y \to X \sqcup_A Y \) are homeomorphic to the cofibers of \( i \).

(v) The stable model structure of spectra is right proper so the dual of (v) holds.

By the dual of the characterization in [25 Prop. 2.3.6] the cofibrant objects are the spectra maps such that

\[
\Gamma((\varepsilon B^\infty_{\Omega^m} \circ \delta i) \times \Omega^m i) \to i
\]

are weak equivalences, which is equivalent to \( \iota \) being a map of connective spectra.\[■\]

## 4.4 \textit{S}-modules and commutative algebra spectra

In order to define a monoidal category of spectra, so that we get natural definitions of spectral algebraic structures, we need to work on the more structured category of sphere modules \( \text{Mod}_S \)\[5\]. As a first step consider for \( A \in S \) the external smash product functor

\[
\wedge_A : \Pi_A \text{Sp} \to \text{Sp}_{\wedge_A R^1}, \quad \wedge_A(Y^m) := \langle \wedge_A Y^m \rangle,
\]

\[
\sigma_{\langle V^m \rangle}[[\nu_A], \langle \nu_A \rangle] := \langle \sigma_{\nu_A}[[\nu_A], \langle \nu_A \rangle] \rangle.
\]

The change of universe in this product is formally problematic, and the following construction is used to internalize the smash product in \( \text{Sp} \). For \( K \subset \text{cpct} \) define the monotone functions

\[
\mu \in \text{POSet}(\mathcal{A}_{\wedge_A R^1}, \mathcal{A}), \quad \mu\langle U^m \rangle := \sum_{K \subset \text{cpct}} f(U^m)
\]

\[
\nu \in \text{POSet}(\mathcal{A}, \mathcal{A}_{\wedge_A R^1}), \quad \nu U := \cap_K f^{-1} U
\]

which satisfy

\[
\mu \nu U \subset U, \quad \nu \mu \nu U = \nu U, \quad \langle U^m \rangle \subset \nu \mu \langle U^m \rangle, \quad \mu\langle U^m \rangle = \mu \nu \mu \langle U^m \rangle.
\]

For all \( \langle (U^m) \rangle, V \rangle \in \Sigma_{\mathcal{A}_{\wedge_A R^1}, \mathcal{A}_{\mu(U^m)}} \) we have the associated Thom complex

\[
TK_V^{\langle U^m \rangle} := \Sigma_K S^{V - f(U^m)} / [f, \infty) - \infty, \infty) \in \text{Top}^*.
\]

where \( \Sigma_K S^{V - f(U^m)} \) is topologized as a subspace of \( K \times S^V \), with the equivalence class \( [f, \infty] \) as base point. We will use the notation \( \vec{v} f := [f, \vec{v}] \in TK_V^{\langle U^m \rangle} \).

The twisted half-smash product is defined as

\[
\mathcal{L} A \ltimes - : \text{Sp}_{\wedge_A R^1} \to \text{Sp}; \quad \mathcal{L} A \ltimes Z := \left\{ \text{Colim}_{K \subset \text{cpct}} TK_{U}^{\nu U} \wedge Z_{\nu U} \right\},
\]

\[
\sigma_{\nu U}[[f, \vec{v}], z, \nu] := \left[ \sigma_{\nu U}[[\nu, \nu], \langle \nu, \nu \rangle] \right].
\]

We define the monad \( (\mathcal{L}; \eta, \mu) \) on \( \text{Sp} \) with

\[
\mathcal{L} Y := \mathcal{L} \ker Y; \quad \eta y := [\delta, y], \quad \mu y := \left[ \begin{array}{c} \nu f, \nu f y \end{array} \right] := \left[ \begin{array}{c} \nu f, \nu f y \end{array} \right] = \left[ \begin{array}{c} \nu f, \nu f y \end{array} \right].
\]
We refer to the $L$-algebras as $L$-spectra and for $(Y, y) \in L[Sp]$ we use the notation $\tilde{u}_y := \eta[f, y]$. The sphere spectrum $S$, the Eilenberg-Maclane spectra $HG$ and the Thom spectrum $MO$ in example 4.1, as well as the suspensions $\Sigma^\infty X$ of $L$-spaces, deloopings $B^\infty X$ of $E_\infty$-rings and the spectrifications $\tilde{\Omega}Y$ of $L$-spectra, are all $L$-spectra with structural morphisms given respectively by:

| $L$-spectrum | Structural Morphism |
|---------------|---------------------|
| $S$           | $\tilde{u} + f\tilde{v}$ |
| $HG$          | $g_a \otimes \tilde{u} + f \tilde{v}$ |
| $MO$          | $[(f U, f g^i f^{-1}), t, \tilde{u} + f \tilde{v}]$ |
| $\Sigma^\infty X$ | $[f y, \tilde{u} + f \tilde{v}]$ |
| $B^\infty X$  | $[(f \times \alpha^a), \langle f x^c \rangle, \tilde{u} + f \tilde{v}], t]$ |
| $\tilde{\Omega}Y$ | $[f \tilde{v} \mapsto \tilde{u}_y \in (f^{-1} \tilde{u}_V + \tilde{v})]$ |

The $A$-indexed smash product is

$$\wedge_{\mathcal{L} A} : \Pi A L[Sp] \to L[Sp]$$

$$\wedge_{\mathcal{L} A} \langle Y^a \rangle := \left\{ \mathcal{L} A \times \tilde{\mathcal{L}} A Y^a U / \phi_{\eta}^{\tilde{y}^a} \right\}$$

with structural maps induced by the ones for the twisted smash product. In order to make explicit the parallel between the smash product of spectra with the tensor product of abelian groups we will use the notation

$$\otimes_{\mathcal{L} A} [y^a] := [[\tilde{u}_f, [y^a]]] \in \wedge_{\mathcal{L} A} \langle Y^a \rangle,$$

so that the $L$ structural maps are given by the formula

$$\tilde{u}_f \otimes_{\mathcal{L} A} [y^a] := \otimes_f \tilde{u} + f \tilde{v} \tilde{y}^a.$$  

For $A = 2$ this defines an associative and symmetric smash product

$$Y^1 \wedge_{\mathcal{L} A} \wedge_{\mathcal{L} B} \langle Y^a \rangle := \wedge_{\mathcal{L} 2} \langle Y^1, Y^2 \rangle.$$  

Associativity follows from the fact that the maps

$$\wedge_{\mathcal{L} A} (\wedge_{\mathcal{L} B} \langle Y^a \rangle) \to \wedge_{\mathcal{L} \Sigma A B^a} \langle Y^{ab} \rangle; \quad \tilde{u}_f \otimes_{\mathcal{L} B} \tilde{v}^a \tilde{y}^a \mapsto \tilde{u}_f + \tilde{v} \tilde{y}^a$$

are isomorphisms [5] Theorems I.5.4, I.5.5 and I.5.6]. In particular when the $B^a$ are a constant set $B$ we have a natural isomorphism

$$\Phi_{A, B} : \wedge_{\mathcal{L} A} (\wedge_{\mathcal{L} B} \langle Y^a \rangle) \to \wedge_{\mathcal{L} B} (\wedge_{\mathcal{L} A} \langle Y^a \rangle)$$

and we set the notation

$$\check{\otimes}_{\mathcal{L} B} [\tilde{u}^a \tilde{y}^{ab}] := \Phi_{A, B} \otimes_{\mathcal{L} B} \check{\otimes} [\tilde{u}^a \tilde{y}^{ab}].$$
Symmetry is given by the natural isomorphism

\[ \tau_{Y^1, Y^2} : Y^1 \land Y^2 \xrightarrow{\cong} Y^2 \land Y^1, \quad \tau_{(f_1, f_2)} : [y^1, y^2] \mapsto \tau_{(f_2, f_1)}[y^2, y^1]. \]

For all \( Z \in L[Sp] \) set the notation \( \Sigma^Z := - \land Z : L[Sp] \to L[Sp] \).

This smash product almost has a unit given by the sphere spectrum \( S \), in that there are natural weak equivalences

\[ \rho_Y : \Sigma^S Y \xrightarrow{\sim} Y, \quad \phi^f[y, v] \mapsto \sigma^U \gamma \gamma^f[y, v, f] \varepsilon. \]

Unfortunately \( \rho \) is not in general a natural isomorphism, though it is a natural weak equivalence. The category of \( S \)-modules is the full subcategory

\[ \text{Mod}_S := \{ Y \in L[Sp] \mid \rho_Y \text{ is an isomorphism} \}. \]

With the same smash product and unit \( S \) the category \( \text{Mod}_S \) is a symmetric monoidal category.

From the nontrivial fact that \( L A / \land 1 \) has a single equivalence class \( [5] \) Theorems I.8.1 and Section XI.2) the sphere spectrum \( S \), the Eilenberg-Maclane spectra \( HG \) and the Thom spectrum \( MO \) in \( \langle 4 \rangle \) as well as \( \Sigma^S Y \) for \( Y \in L[Sp] \) and \( \tilde{\Omega} Y \) for \( Y \in \text{Mod}_S \), are all \( S \)-modules with inverse maps given respectively by:

| \( S \) | \( \phi^0[f_1^{-1} \tilde{u}, \tilde{v}] \) |
| --- | --- |
| \( HG \) | \( \phi^0[g_a \otimes f_1^{-1} \tilde{u}^a, \tilde{v}] \) |
| \( MO \) | \( [\langle f_1^{-1} \tilde{y}, f_1 \rangle, \langle t, f_1^{-1} \tilde{u} \rangle, \tilde{0}] \) |
| \( \Sigma^S Y \) | \( \phi^{\Sigma^S Y}(f_1, g_2, g_3)[y, g_2^{-1} f_2 \tilde{v}, \tilde{0}] \) |
| \( \tilde{\Omega} Y \) | \( \tilde{v} \mapsto \rho^{-1} \gamma \tilde{v} \) |

where in the first three lines \( f \in L \) is any linear isometry such that \( U \subset f_1 \mathbb{R}^\infty \), in the fourth \( \langle g_2, g_3 \rangle \in L \) are chosen such that \( \langle f_1, g_2, g_3 \rangle \in L \) and \( f_2 \nu U^1 \subset g_2 R^\infty \).

The functor \( \Sigma^S := - \land S \) is the right adjoint of the inclusion of \( \text{Mod}_S \) in \( L[Sp] \). The functor \( \Sigma^S \) is also a left adjoint, with right adjoint induced by a closed structure on \( L[Sp] \) given by an \( L \)-mappings functor \( F \). Details of this construction can be found in \( [5] \) Section I.7, but we give an overview to establish notation. The twisted half-smash product \( L A \land - \) admits a right adjoint, the twisted function spectrum functor

\[ F[\land A, -] : \text{Sp} \to \text{Sp}_{\land A \land \mathbb{R}^\infty}, F[\land (A), Y] := \left\langle \lim_{K \in \text{spectral}} Y^T K_{\mu(U^0)} \right\rangle, \]

\[ \phi^0(V^a, \langle \tilde{u}, \tilde{v} \rangle) := \left\langle \tilde{u} \mapsto \phi^0(V^a, \langle \tilde{u} + f_a \tilde{v} \rangle) \right\rangle. \]
For $U^1 \in \mathcal{A}$ we also have a shift functor

$$-[U^1] : \text{Sp}_{\text{R}^\otimes} \to \text{Sp} : Y[U^1] = \{Y_{U^1, U^2}, \sigma_{U^1, U^2}[y, \tilde{v}] := \sigma_{U^1, U^2}[y, (\tilde{0}, \tilde{v})] \}.$$ 

If $Y \in \text{L}[\text{Sp}]$ then $F[L^2, Y][U^1] \in \text{L}[\text{Sp}]$ with structural map

$$\tilde{u}_f \varphi := \langle \tilde{v}, \varphi_{(g_1, g_2)f} \rangle.$$ 

Finally, we can now define

$$F_S(-, -) : \text{L}[\text{Sp}]^\otimes \times \text{L}[\text{Sp}] \to \text{L}[\text{Sp}]:$$

$$F_S(Z, Y) := \begin{cases} \phi \in \text{L}[\text{Sp}](Z, F[L^2, Y][U^1]) & \begin{cases} \tilde{u}_f(\varphi_{f \varphi}) = \tilde{u}_f(\varphi_{(g_1, g_2)f}) \\ \varphi_f = \varphi_{(g_1, g_2)f} \end{cases} \end{cases},$$

$$\sigma_{U^1, U^2}^{\tilde{v}, \tilde{u}} : (z \mapsto \sigma_{U^1, U^2}^{\tilde{v}, \tilde{u}}[\varphi_{(g_1, g_2)f}]), \quad \tilde{u}_f \varphi := \langle [z, \tilde{v}], \varphi_{(g_1, g_2)f} \rangle.$$ 

The functor $F_S := F_S(S, -) : \text{Mod}_S \to \text{L}[\text{Sp}]$ is right adjoint to $\Sigma^S$.

The monoidal structure of $\text{S}$-modules provides a natural definition of ring spectra, module spectra and algebra spectra.

**Definition 4.4.** A commutative ring spectrum $R$ is a commutative monoid in $\text{Mod}_S$, ie an $S$-module equipped with a *multiplication* map $\mu : R \otimes R \to R$ and a *unit* map $\eta : S \to R$ satisfying natural associative, unit and commutative laws. The category of commutative ring spectra is denoted $\text{CRingSp}$. For $R \in \text{CRingSp}$ an $R$-module $M$ is a module over $R$, ie an $S$-module equipped with an *action* $\lambda : R \otimes M \to M$, satisfying natural associative and unit laws. The category of $R$-modules is denoted as $\text{Mod}_R$.

The category of $R$-modules admits a monoidal structure with associative and symmetric tensor product the coequalizer

$$M \otimes_R N := \text{Coeq}(M \otimes_R R \otimes_R N \rightrightarrows M \otimes_R N)$$

and unit $R$. The category of $R$-modules is denoted $\text{Mod}_R$.

A commutative $R$-algebra is a commutative monoid in $(\text{Mod}_R, \otimes_R, R)$, and the category of commutative $R$-algebra is denoted $\text{CAlg}_R$.

The category of commutative algebra spectra is defined as

$$\text{CAlgSp} := \Sigma_{\text{CRingSp}} \text{CAlg}_R.$$ 

As in the classical set theoretical setting there is a natural isomorphism $\text{CAlgSp} \equiv \text{CRingSp}^{-\otimes}$ [VII.1]. Alternatively we have a monad $(P^\otimes ; \eta, \mu)$ on $\text{L}[\text{Sp}]^\otimes$ with

$$P^\otimes Y := \int_{\mathcal{A}} \text{Sp} \{Y_{\alpha} \} :$$

$$\eta_{\alpha} y := [\Theta_{id} \tilde{v}], \quad \mu_{\alpha} [\Theta_{f} \Theta_{g^\otimes} [y_{a, b}]] := [\Theta_{(f, g_1^\otimes)} _{f, g_2^\otimes} [y_{a, b}]]$$
which restricts to a monad on \( \text{Mod}^2 \). The objects of \( \mathbb{P}^{-}[\text{L}[\text{Sp}]] \) behave like algebra spectra over ring spectra except they have units only up to weak equivalence and are referred to as \( E_{\infty} \)-algebra spectra, similarly to how algebras in \( \text{L}[\text{Sp}] \) over the nonrelative version \( \mathbb{P} \) of this monad are called \( E_{\infty} \)-ring spectra. By the same argument as in [5, Prop. II.4.5] we have an isomorphism

\[
\mathbb{P}^{-}[\text{Mod}^2] \cong \text{CAlgSp}.
\]  

For \( R = ((R_d, R_u); \eta, \mu) \in \text{CAlgSp} \) and \( \Phi^\mu_f[r^a] \in \wedge_{\mathcal{P}} \langle R_{ca} \rangle \) we will use the notation

\[
\prod_f \Phi^\mu_f[r^a] := \mu(\Phi^\mu_f[r^a]).
\]

The sphere spectrum \( S \), the Eilenberg-MacLane spectrum of a commutative ring \( HR \), the Thom spectrum \( MO \), suspensions \( \Sigma^\infty X \) of \( \mathcal{P} \)-spaces, deloopings \( B^\infty X \) of \( E_{\infty} \)-rings, the \( S \)-module \( \Sigma^R \) associated to an \( E_{\infty} \)-ring spectrum \( R \) in \( \mathbb{P}[\text{L}[\text{Sp}]] \) and spectrifications \( \Omega R \) of ring spectra \( R \) are all commutative ring spectra with

\[
\begin{array}{|c|c|c|}
\hline
\text{S} & \eta \overline{u} & \prod_f \Phi^\mu_f[r^a] \\
\hline
HR & 1_R \otimes \overline{u} & \prod_f \langle \overline{u}_a \rangle \otimes \overline{u} + f_a \overline{v}^{2,a,b} \\
MO & [\text{id}, \varnothing, \overline{u}] & \prod_f \langle f_a \overline{v} \rangle \otimes \overline{u} + f_a \overline{v}^{2,a} \\
\Sigma^\infty X & [1_X, \overline{u}] & \prod_f \langle f_a \rangle \otimes \overline{u} + f_a \overline{v}^{2,a} \\
B^\infty X & \langle [\varnothing, 1_X, \overline{u}], \varnothing \rangle & \prod_f \langle f_a \rangle \otimes \overline{u} + f_a \overline{v}^{2,a} \\
\Sigma^R & \Phi^\mu_f[1_Y, \overline{u}] & \prod_f \Phi^\mu_f[r^a] \otimes \overline{u} + f_a \overline{v}^{2,a} \\
\tilde{\Omega} R & \overline{v} \mapsto \sigma^\mu_f[\eta \overline{u}, \overline{v}] & \prod_f \langle f_a \overline{v} \rangle \otimes \langle f_a \overline{v} \rangle \prod_f \langle f_a \overline{v} \rangle \otimes \overline{u} + f_a \overline{v}^{2,a} \\
\hline
\end{array}
\]

with the implicit conditions in the fifth line as in the formula [3.11]

There is a natural isomorphism \( \text{CAlgSp} \cong \text{CRingSp} \), which is analogous to the isomorphism between commutative rings and commutative \( \mathbb{Z} \)-algebras. Moreover \( (MO, HR) \in \text{CAlgSp} \) with

\[
\prod_f \Phi^\mu_f[\langle \overline{v}^{1,i}, \overline{v} \rangle, \overline{v}^{2,b}] := \overline{v}^{2,a} \otimes \overline{u} + f_1 \overline{v}^{1} + f_2 \overline{v}^{2,b}.
\]

### 4.5 Stable mixed model structure of commutative algebra spectra

The stable mixed model structure of \( \text{Mod}_S \) is right transferred from the one in \( \text{Sp} \) by the adjunction

\[
(\Sigma^S \text{L} \dashv F^S) : \text{Sp} \cong \text{Mod}_S
\]

as described in [14, 13, 5], so that weak equivalences and fibrations in \( \text{Mod}_S \) are those maps whose underlying spectrum mapping are \( q \)-equivalences and
Let $h$-fibrations respectively. The Hurewicz/Strøm factorization systems are constructed as in $\mathbf{Sp}$ with the $S$-module structures of $\Gamma f$ and $E f$ defined point-wise. The mixed model structure of $\mathbf{CAlgSp}$ is right transferred from the one in $\mathbf{Mod}_S$ by the adjunction

$$(P^{-} \rightarrow U): \mathbf{Mod}_S \Rightarrow \mathbf{CAlgSp}.$$ 

The Quillen model structure is transferred due to the fact that $\mathbf{CAlgSp}$ has continuous coequalizers and satisfies the “Cofibration Hypothesis” as in [3, Theorem VII.4.7]. The Hurewicz/Strøm model structure is transferred since we can define an algebra structure on $\mathbf{CAlgSp}$. As in $\mathbf{Sp}$ we have that $(\Gamma; C, F)$ forms an algebraic weak factorization system in $\mathbf{CAlgSp}$. On the other hand there doesn’t seem to be any natural algebra structure on $E f$ such that the $h$-cofibration/trivial $h$-fibration factorization $(E; C, F_t)$ in $\mathbf{Sp}$ induces a factorization in $\mathbf{CAlgSp}$. We do have an $h$-cofibration/$h$-equivalence factorization

$$(\eta u : (\eta X \tilde{u}, 0, r \mapsto \eta Y \tilde{u}) ; \prod f_t[(x^a, t^a, \gamma^a)] := (\prod f_t x^a, \max_A t^a, r \mapsto \prod f_t \gamma^a r)$$

As in $\mathbf{Sp}$ we have that $(\Gamma; C_t, F)$ is an algebraic weak factorization system in $\mathbf{CAlgSp}$. The Hurewicz/Strøm model structure is transferred since we can define an algebraic weak factorization system in $\mathbf{CAlgSp}$. We define the functors

$$\mathbf{Sp}_\infty \rightarrow \mathbf{CAlgSp},$$

and the fact that $C_t f$ has the left lifting property against $h$-fibrations in $\mathbf{Mod}_S$ induces the left lifting property against $h$-fibrations in $\mathbf{CAlgSp}$ on $\mathbf{in}_X$. The map $(f_t, \eta u, id)$ is an $h$-equivalence, but it is not necessarily an $h$-fibration. Applying $(\Gamma; C_t, F)$ then gives us the $h$-cofibration/trivial $h$-fibration factorization

$$X' \xrightarrow{\mathbf{in}_X} X \wedge_{\mathbf{P}_X} P(\Gamma f \wedge [0, \infty], \mathbf{P}_{\mathbf{G}_f}) Y \xrightarrow{(\eta f_t, \eta v, id)} Y$$

which determines the Hurewicz/Strøm, and therefore also the mixed, model structure on $\mathbf{CAlgSp}$.

### 4.6 Recognition of algebra spectra

Let $\mathcal{E}_-^-$ be an $E_\infty$-operad equipped with an $\mathcal{L}_-^-$-action. The functors $F^S$ and $\Sigma^S$ induces objectwise adjoint functors $F_\Sigma^S$ and $\Sigma^S$ on the morphism categories. We can then define the functors

$$\Omega_2^{\infty, S}: \mathbf{CAlgSp} \rightarrow (\mathcal{L}_-^- \mathcal{E}_-^-)[\mathbf{Top}], \quad \Omega_2^{\infty, S} R := \Omega_2^{\infty, S} F_\Sigma^S \eta;$$

$$f[\phi^a] := \left[\begin{array}{c} u^1 \mapsto (u^2, \delta^2) \mapsto \prod f_t \phi^a[u^1, u^2]^\delta \end{array}\right]$$

and

$$B_\Sigma^{\infty, S} : (\mathcal{L}_-^- \mathcal{E}_-^-)[\mathbf{Top}] \rightarrow \mathbf{AlgSp}; \quad B_\Sigma^{\infty, S} X := \Sigma_X B_\Sigma^S X.$$
Theorem 4.5. There is an idempotent quasiadjunction
\[(B^S, \infty \rightarrow \mathcal{H}_0^\circ, \mathcal{L}^-)[\text{Top}] \cong \text{CAlgSp}\]
that induces an equivalence of homotopy categories
\[(L B^S, \infty \rightarrow \mathbb{R} \Omega_2^{\infty, S}) : \mathcal{H}(\mathcal{H}_0^\circ, \mathcal{L}^-)[\text{Top}]_{\text{alg}} \cong \mathcal{H} \text{CAlgSp}_{\text{con}}.\]

Proof: The natural weak equivalences $\eta'$ and $\epsilon'$ are defined as in the proof of theorem 4.2. The other natural transformations of the unit span and counit cospan are
\[
\eta_\mu = \left[\tilde{u}^1 \mapsto \left\{\begin{array}{l}
(\tilde{u}^2, \tilde{v}) \mapsto \bigotimes_j \prod_i \left[\left(\alpha^v \cdot \langle x^e \rangle, \alpha_{\tilde{u}^i}^{-1} \tilde{u}^1 \right)_{T_{x^e}, t}, \tilde{u}^2\right]\right]\right.\]
\[
\epsilon_{\star, U} \bigotimes_j \left[\left(\alpha^v \cdot \langle x^e \rangle, \alpha_{\tilde{u}^i}^{-1} \tilde{v}^1 \right)_{T, t}, \tilde{v}^2\right] : \left[\tilde{w} \mapsto \circ \alpha^v \cdot \langle x^e \rangle \left[\tilde{v}^1 + \tilde{w}^f_1, \tilde{v}^2 + \tilde{w}^f_2 \right]_{r+\tilde{w}^r_f}\right].
\]
with the conditions in the first formula as in the proof of theorem 4.2 and with the domain in the last formula any $W \in \mathcal{A}_{U+f_1}$ with $V$ a common domain of representatives of the loops $\phi^e$.

That these maps satisfy the conditions for an idempotent quasiadjunction follows from the fact that $(\Sigma^S_\infty \rightarrow F^S_\infty) : P^- \mathbb{L}^2 [\text{Sp}^2]) \cong \text{CAlgSp}$ is a Quillen equivalence and the same argument as for 4.2 and 4.3.

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