Gauss-Bonnet Quintessence: Background Evolution, Large Scale Structure and Cosmological Constraints

Tomi Koivisto*1 and David F. Mota1,2,3,4

1 Helsinki Institute of Physics, FIN-00014 Helsinki, Finland
2 Institute for Theoretical Physics, University of Heidelberg, 69120 Heidelberg, Germany
3 Institute of Theoretical Astrophysics, University of Oslo, Box 1029, 0315 Oslo, Norway
4 Perimeter Institute, Waterloo, Ontario N2L 2Y5, Canada

(Dated: August 22, 2018)

We investigate a string-inspired dark energy scenario featuring a scalar field with a coupling to the Gauss-Bonnet invariant. Such coupling can trigger the onset of late dark energy domination after a scaling matter era. The universe may then cross the phantom divide and perhaps also exit from the acceleration. We discuss extensively the cosmological and astrophysical implications of the coupled scalar field. Data from the Solar system, supernovae Ia, cosmic microwave background radiation, large scale structure and big bang nucleosynthesis is used to constrain the parameters of the model. A good Newtonian limit may require to fix the coupling. With all the data combined, there appears to be some tension with the nucleosynthesis bound, and the baryon oscillation scale seems to strongly disfavor the model. These possible problems might be overcome in more elaborate models. In addition, the validity of these constraints in the present context is not strictly established. Evolution of fluctuations in the scalar field and their impact to clustering of matter is studied in detail and more model-independently. Small scale limit is derived for the perturbations and their stability is addressed. A divergence is found and discussed. The general equations for scalar perturbations are also presented and solved numerically, confirming that the Gauss-Bonnet coupling can be compatible with the observed spectrum of cosmic microwave background radiation as well as the matter power spectrum inferred from large scale surveys.

PACS numbers: 98.80.-k,98.80.Jk
Keywords: Cosmology: Theory

I. INTRODUCTION

General relativity predicts singularities. Therefore, in spite of its being a highly successful as a classical theory of gravitation, its modification seems inevitable at high energy scales. The low energy action could then also feature corrections to the Einstein-Hilbert term from additional fields and curvature invariants. Among the possible corrections, a particular combination of the quadratic Riemann invariants, the Gauss-Bonnet term, is of special interest. It appears in extensions of gravitational actions, whether motivated by the form of most general of a scalar-tensor theory, uniqueness of the Lagrangian in higher dimensions or the leading order corrections from string theory. An interesting possibility then arises that the modifications could change the way the universe itself gravitates at large scales.

Indeed, the current cosmological observations cannot be explained by the standard model of particle physics and general relativity. Precise cosmological experiments have confirmed the standard Big Bang scenario of a flat universe undergoing an inflation in its earliest stages, where the perturbations are generated that eventually form into structure in matter. Most of this matter must be non-baryonic, dark matter. Even more curiously, the universe has presently entered into another period of acceleration. Such a result is inferred from observations of extra-galactic supernovae of type Ia (SNeIa) and is independently supported by the cosmic microwave background radiation (CMBR) and large scale structure data. It seems that some dark energy, with its negative pressure that speeds up the universal expansion, dominates the density of the universe. This concordance model agrees very well with the astrophysical data, but features inflation, dark matter and dark energy as phenomenological ingredients of undisclosed nature. Perhaps dark energy is the most enigmatic of the latter two, as at least there are reports of tentative discoveries of dark matter.

The problem of dark energy reduces to the questions 1) what is the culprit (perhaps in fundamental physics) for the acceleration, and 2) why did the speed-up begin just at the present stage of the cosmological evolution. In recent years dark energy and its cosmological and astrophysical signatures has been addressed in many papers considering both modifications of the energy-momentum tensor, with the inclusion of scalar fields or imperfect fluids and of gravitational physics at large scales. The latter approach has proven to be perhaps surprisingly difficult. Extending the action with curvature invariants rather generically results in fourth-order gravity, which is problematical because of the implied instabilities. An $f(R)$ extension, although not generically, can be stable. However, though some $f(R)$-type modifications might be

*tomikoiv@pcu.helsinki.fi
†d.mota@thphys.uni-heidelberg.de
compatible with the acceleration following a standard matter dominated phase\cite{19,20}, the simplest modifications as proposed this far seem not to be able to generate a viable cosmology\cite{21}. On the other hand, within the Palatini formulation, the corresponding extensions result in second order field equations, but it turns out that the observationally allowed modifications are then practically indistinguishable from a cosmological constant\cite{22,23,24,25}.

Scalar-tensor theories of gravity are interesting alternatives to the concordance model and seem to have potential to provide a linkage between the acceleration and fundamental physics\cite{26,27,28,29}. They have a desirable feature which is their quasi-linearity: the property that the highest derivatives of the metric appear in the field equations only linearly, so as to make the theory ghost free. Interestingly, there is a particular combination of the curvature squared terms with such behavior, known as the Gauss-Bonnet (GB) integrand. It is constructed from the metric as

$$\mathcal{R}_{GB}^2 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^\mu R^\nu + R^2. \quad (1)$$

As mentioned in the beginning, this term appears frequently in attempts at a quantum gravity, especially in stringy set-ups. In fact, all versions of string theory (except Type II) in 10 dimensions ($D = 10$) include this term as the leading order $\alpha'$ correction\cite{30,31}. In $D = 4$ the Gauss-Bonnet term is a topological invariant. It’s appearance alone in the action can then be neglected as a total divergence (though even then this topological term can have interesting role in view of the boundary terms and regularization of the $D = 4$ action in asymptotically AdS spacetimes\cite{32,33}). However, if coupled, its presence may lead to contributions to the field equations also in $D = 4$. The low energy string action typically features scalar fields with such couplings. Namely the moduli, associated to the internal geometry of the hidden dimensions, become non-minimally coupled to the curvature terms. Hence, in a general $D = 4$ scalar-tensor theory one might couple the scalar field non-minimally not only to the Ricci curvature $R$ but to the Gauss-Bonnet invariant $\mathcal{R}_{GB}^2$ as well, and in $D > 4$ it is necessary to include this term in order to preserve the uniqueness of the gravitational action.

Such a coupling could thus be useful in modeling both the early inflation and the late acceleration\cite{34,35,36,37,38}. Here we will concentrate on the post-inflationary epochs in such a universe, and more specifically, within the scenario we recently studied\cite{39}. There the scalar field lives in an exponential potential, and the coupling can likewise be exponential. This kind of model was originally proposed by Nojiri, Odintsov and Sasaki\cite{40}. More recently it was shown that if the coupling grows steeper than the potential decays with the field, acceleration can occur with a canonical scalar, after a transition from a scaling era\cite{41}. Then the interaction with the Gauss-Bonnet curvature term causes the universe to enter from a scaling matter era to an accelerating era. Existence of scaling solutions with more general parameterizations has then been taken under investigation\cite{42}, noting that requiring exact scaling is otherwise possible only with a tachyon field which is non-minimally coupled with matter. Other dark energy solutions in a broad variety of second order string cosmologies, taking into account both coupled and uncoupled matter, have been explored also previously\cite{42,43}, higher order terms have been also incorporated and a method to reconstruct the coupling and the potential has been found\cite{44}. If the scalar field is non-dynamical, the model is equivalent with a modified Gauss-Bonnet gravity, featuring a function of $\mathcal{R}_{GB}^2$ the action\cite{45,46,47,48}.

The coupling mechanism can also momentarily push the universe to a phantom era, until the scalar field potential begins to dominate. Therefore it is possible to get a very good accordance with the SNeIa data. Dark energy models other than of the Gauss-Bonnet form but featuring transition to phantom expansion and having possible origin in string theory have been considered also\cite{49,50}. In addition, one should clearly distinguish the models investigated here from those featuring a coupling to the Ricci scalar\cite{49}, since for them the so-called R-boost\cite{50} plays a role in the early universe, while the potential has to be shallow in order to have accelerating behaviour at the present. On the other hand, one can couple the self-tuning scaling field to matter\cite{51,52,53}, and thereby indeed set the acceleration ongoing and thereby perhaps alleviate the coincidence problem\cite{54,55}. However, implications for the large-scale structure formation seem to make this possibility problematical\cite{56,57}. Here we will not find the same, since modifications to the growth of matter perturbations are much less drastic as the scalar field remains minimally coupled to the matter sector.

In this paper, besides investigating the cosmological evolution of such models, we also derive quantitative constraints on the scenario of Gauss-Bonnet dark energy mentioned above. Our aim is to present in more detail this scenario and to generally uncover in more depth what are the possible effects of the Gauss-Bonnet combination on dark energy cosmologies, in order to assess whether they could be compatible with present days observations, and to estimate the amount of fine-tuning that this might require. To this end the the model in Ref.\cite{39} is subjected to further constraints. In addition to tightening the limits for this particular parameterization, we will discuss in more detail and generality the phenomenology of Gauss-Bonnet dark energy at cosmological, astrophysical and Solar system scales during the cosmological evolution. Especially, we consider the impact of the coupling to the CMBR and large scale structure. In view of the remarks above, this is a crucial aspect of a dark energy model with any non-minimal couplings or modifications of gravity. Since the class of models studied here features both and these can lead to completely different predictions for the expansion rate of the Universe than without the $\mathcal{R}_{GB}^2$ term, it is rather non-trivial to find that the
perturbation evolution in these models is in some sense only modestly altered. Namely, the shape of the matter power spectrum is retained during the presence of the Gauss-Bonnet gravity and the effects one expects to see in the large multipoles of the CMBR spectrum are typically small. Still, there are distinguishable features peculiar to these models that can lead to essential constraints for the model parameters.

The article is organized as follows. The cosmological dynamics for these models is described in Section III for both the background expansion and for linear perturbations. In Section IIII we discuss the implications and derive constraints of the Gauss-Bonnet coupling to nucleosynthesis, Solar system, large scale structure, CMBR and SNeIa. Section IV contains our conclusions. Some technical details are confined to the Appendixes.

II. COSMOLOGICAL DYNAMICS

Before writing down the cosmological equations, let us briefly discuss the model and its motivation. A peculiar property of the string effective action is the presence of scalar fields and couplings which are field dependent, and thus in principle space-time dependent. The scalar fields are moduli associated to geometrical properties of compactified extra dimensions. Thus one could consider multiple scalar fields, representing the dilaton and various other moduli, but here we stick to one field \( \phi \). The reason is that the dilaton couples to the Ricci curvature, and this would lead to variations of the Newton’s constant. When going from the string frame to the Einstein frame, the coupling to \( R \) is transformed into a non-minimal interaction with matter. This in turn would lead to violations of the equivalence principle. Thus, in general, also the matter Lagrangian \( L_m \) could be non-minimally coupled to the scalar field. There are tight constraints from observations to such effects [58] (see however, [59, 60]). However, it is (usually) only the dilaton field which in heterotic string theory acquires the geometrical coupling to \( R \) and thus enters in the conformal factor and the matter Lagrangian in the Einstein frame[61]. If the conformal factor is not (nearly) trivial, it can evolve so, for instance due to the so called least coupling principle [62]. In the present article we will make the usual assumption in cosmology, that possible non-minimal couplings to the matter sector are negligible, since our specific purpose here is to study the novel features resulting from the Gauss-Bonnet curvature interaction. Thus we identify \( \phi \) as a run-away modulus without direct matter couplings. Then the equivalence principle is automatically satisfied, though gravitational dynamics are modified due to the modulus-dependent loop corrections.

Thus we can write the action for the system to consider as

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa^2} - \frac{\gamma}{2} (\nabla \phi)^2 - V(\phi) - f(\phi)R_{GB}^2 + L_m \right],
\]

here \( \kappa = (8\pi G)^{-1/2}, \) \( G \) being the Newton’s constant. The \( \gamma \) could be a function of \( \phi \) as well, and there could be additional kinetic terms. In the following we will mostly consider canonic scalar field and thus set \( \gamma = 1 \) (which can be achieved by a redefinition of the field for any constant \( \gamma > 0 \)). \( V(\phi) \) is the field potential which could result from nonperturbative effects. In four dimensions the GB term makes no contribution if \( f(\phi) = \text{const.} \). However, it is natural to consider \( f(\phi) \) to be dynamical. This follows, for example, from the one-loop corrected string effective action [61, 62], where the function \( f(\phi) = \sigma - \xi(\phi) \): the coupling \( \sigma \) may be related to string coupling \( g_s \) via \( \sigma \sim 1/g_s^2 \), and the numerical coefficient \( \xi \) typically depends on the massless spectrum of every particular model [64]. It thus seems that the dilaton might couple already at the string tree-level next to leading order expansion in the inverse string tension \( \alpha' \), and the modulus functions \( \xi(\phi) \) are non-trivial at the one-loop order.

In the numerical examples we will adopt an exponential form for the potential and the coupling,

\[
V(\phi) = V_0 e^{-\lambda \phi/\sqrt{\gamma}}, \quad f = f_0 e^{\alpha \phi/\sqrt{\gamma}}.
\]

On one hand, the nonperturbative effects from instantons or gaugino condensation typically result in an exponential potential. On the other hand, there is also phenomenological motivation for this, since, besides its being a simple choice, an exponential potentials allows to consider scaling solutions in cosmology. An exponential field-dependence for coupling is a reasonable assumption in supergravity actions. For massless dilaton one in fact has \( f(\phi) = \sum C_i e^{(n-1)e^\phi} \) [65]. For another instance, a known example from heterotic string theory [64] yields for the modulus coupling, in terms of the Dedekind function \( \eta \), the form \( f(\phi) \sim \log [2e^{\alpha e^\phi} \eta^4 (ie^{\alpha e^\phi})] \), which behaves like Eq. (3) to a good approximation. Again, it is a simple choice for \( f \), introducing but one extra parameter \( \alpha \). Such minimalism is practical when considering a phenomenological model of dark energy, though from a particle physics point of view one might expect that corrections to the Eqs. [3] come into play, in particular when the Gauss-Bonnet term begins participate in cosmological dynamics in the late time in our model.

In the presence of the coupling \( f \), the field equations can be written as

\[
G_{\mu\nu} = \kappa^2 \left[ T^m_{\mu\nu} + T^\phi_{\mu\nu} + T^f_{\mu\nu} \right]
\]

where the two first terms in the right hand side are the energy-momentum tensors for matter and the scalar field, respectively. The curvature corrections resulting from
taking into account the Gauss-Bonnet contribution involve only terms proportional to derivatives of $f(\phi)$. A partial integration of the coupled term in action (2) gives a vanishing boundary term but in addition, because of the field dependent interaction, a contribution involving the integral of $R^2_{GB}$ which is of the first order by construction. Explicitly, one obtains

$$T_{\mu\nu}^I = -8 \left[ f_{\alpha\beta} R_{\mu}{}^\alpha{}_{\nu} \right] + \Delta f R_{\mu\nu} - 2 f_{\alpha(\mu} R^\alpha_{\nu)} + \frac{1}{2} f_{\mu\nu} - \frac{4}{\kappa} \left[ 2 f_{\alpha\beta} R^{\alpha\beta} - \Delta f R \right] g_{\mu\nu}. \quad (5)$$

The equation of motion for the scalar field reads

$$\gamma \Delta \phi - V'(\phi) - f'(\phi) R_{GB}^2 = 0. \quad (6)$$

Thus the field lives in an effective potential given by $V(\phi) + f(\phi) R_{GB}^2$. Matter, since minimally coupled, is conserved as usually $\nabla^\mu T^\mu_{\mu'} = 0$. The general covariant expression for curvature corrections Eq. (5) is rather complicated combination of the derivatives of the coupling and their contractions with the Riemann and Ricci objects, but due to the geometric properties of these contributions, they vanish in constant curvature spacetimes, and also otherwise can assume rather tractable forms, as we will see in the following.

**A. Background**

We consider a flat, homogeneous and isotropic background universe with scale factor $a(t)$ in the Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t) \delta_{ij} dx^i dx^j. \quad (7)$$

The action (2) then yields the Friedmann equation as the $t-t$ component of Eq. (1),

$$\frac{3}{\kappa^2} H^2 = \frac{\gamma}{2} \dot{\phi}^2 + V(\phi) + \rho_m + 24 H^2 f'(\phi) \dot{\phi}, \quad (8)$$

where an overdot denotes derivative with respect to the cosmic time, $H \equiv \dot{a}/a$ is the Hubble rate and $\rho_m$ represents the matter component. The Klein-Gordon equation Eq. (3) reads

$$\gamma \ddot{\phi} + V'(\phi) + f'(\phi) R_{GB}^2 = 0, \quad (9)$$

where the Gauss-Bonnet invariant is $R^2_{GB} = 24 H^2 (H + \dot{H})$. It will be convenient to work with the dimensionless variables defined as

$$\Omega_m \equiv \frac{\kappa^2 \rho_m}{3 H^2}, \quad x \equiv \frac{\kappa}{\sqrt{2}} \frac{\dot{\phi}}{H}, \quad y \equiv \kappa^2 \frac{V(\phi)}{H^2},$$

$$\mu \equiv 8 \kappa^2 \dot{\phi} H f'(\phi), \quad \epsilon \equiv \frac{\dot{H}}{H^2}. \quad (10)$$

In addition it is useful to define, in analogy with $\Omega_m$, the relative contributions from the scalar field and the Gauss-Bonnet correction as

$$\Omega_\phi \equiv \frac{\kappa^2 \rho_\phi}{3 H^2} = \frac{\gamma x^2 + y}{3}, \quad \Omega_f \equiv 1 - \Omega_m - \Omega_\phi = \mu. \quad (11)$$

Then

$$w_{eff} = w_m \Omega_m + w_\phi \Omega_\phi + w_f \Omega_f = -\frac{2}{3} \epsilon - 1 \quad (12)$$

is the total effective equation of state in the sense that the Universe expands as if dominated by a fluid with this relation between its pressure and energy. One notes that the Gauss-Bonnet term can be written as $R^2_{GB} = -12 H^2 (3 w_{eff} + 1)$, and is thus negative if and only if the scale factor is decelerating.

With the aim to understand the behaviour of the background cosmology for the model we perform a dynamical system analysis. In terms of the dimensionless variables (10), the complete dynamical system is then given by (11),

$$0 \equiv -3 + \gamma x^2 + y + 3 \mu + 3 \Omega_m,$$

$$0 \equiv 2 \epsilon + 3 + \gamma x^2 - y - \mu' - (\epsilon + 2) \mu + 3 w_m \Omega_m,$$

$$0 \equiv 2 \gamma [xx' + x^2 (\epsilon + 3)] + y' + 2 y e + 3 \mu (\epsilon + 1),$$

$$0 \equiv 2 x x' + y' + 3 \mu' + 3 \Omega_m',$$

$$\Omega_m' = -3 (1 + w_m) \Omega_m - 2 \epsilon \Omega_m,$$

$$\mu' = (x' + 2 x + 2 \epsilon + 2 \alpha x) \mu,$$

$$y' = - (\lambda x + 2 \epsilon) y, \quad (13)$$

where prime means a derivative with respect to $\log(a)$ and $w_m$ is the equation of state of the background. The first five equations hold generally for an action like (2), whereas the two last equations encode the information about the specific model (3). For simplicity, we set from now on $\gamma = 1$, so the scalar field kinetic term is canonical.

We find several fixed points for the system (13), characterized by $x' = y' = \mu' = 0$:

- **A:** $(x, y, \mu) = (0, 0, 0)$. This is an unstable point which would correspond to domination of the background fluid with $w_{eff} = w_m$ and completely vanishing contribution of the scalar field.

- **B:** $(x, y, \mu) = (\pm \sqrt{3}, 0, 0)$. In this case, the kinetic energy of the scalar field dominates. Then $w_{eff} = 1$. The inflation phase is not stable, since the kinetic energy will always redshift away faster than the other contributions to the energy density. This solution with $\phi > 0$ is a saddle point (it has one positive eigenvalue, implying that the solution attracts from some direction while repelling from some other), whenever $\alpha < 2 \sqrt{3}$ or $\lambda, \alpha \geq 2 \sqrt{3}$.

- **C:** $(x, y, \mu) = (0, 3, 0)$. In this fixed point, the potential of the scalar field dominates. Thus this is a de Sitter solution with $w_{eff} = -1$. In the absence of the coupling, all other contributions to the energy density have larger effective equations of state, and the solution is stable. With the coupling, stability condition is simply $\alpha \geq \lambda$.

- **D:** $(x, y, \mu) = (\lambda/2, 3 - \lambda^2/4, 0)$. This fixed point corresponds to scalar field domination with $w_{eff} = \lambda^2/6 - 1$. It does not exist when $\lambda > 2 \sqrt{3}$ and is
thus irrelevant to us here. When $\alpha = \lambda$, the Gauss-Bonnet term can reduce $w_{eff}$ of this solution $E$.

- $E: (x, y, \mu) = (\frac{4}{3} (1 + w_m), \frac{2}{3} (1 - w_m^2), 0)$. This is the well-known scaling solution, where a scalar field with exponential potential mimics exactly the background equation of state $w_m$. This fixed point is a stable spiral when $\lambda > \alpha, \sqrt{6}(1 + w_m)$. It is, however, a saddle point when $\lambda, \alpha < \sqrt{6}(1 + w)$ or $\alpha \geq \lambda, \sqrt{6}(1 + w_m)$. Also this solution exists in a modified form in the special case $\alpha = \lambda$.

- $F: (x, y, \mu) = (\frac{4}{3} (1 + w_m), 0, \frac{18}{(1 + w_m)(1 + w_m^2)}).$ This is an a new scaling solution, in which field has run to large values and the potential can be neglected. Then $\Omega_m$ is nonzero and the effective equation of state in this case is again just $w_m$. The condition for the existence of this fixed point is that either $w_m > 7/3$ or $\alpha^2 = 3(1 + w_m)^2(7 - 3w_m)/(1 + 3w_m)$. Then it might be a saddle point, but it is generically unstable. Thus this possibility of the kinetic energy together with the Gauss-Bonnet contribution scaling like matter seems not to be useful for cosmological applications, as it will not be reached by dynamical means. Even if one sets the field to this solution as an initial condition during radiation domination, the Gauss-Bonnet contribution will drop away when transition to matter domination occurs.

- $G$: There exists also a solution where only the Gauss-Bonnet term and the kinetic term of the scalar field survive while $y = \Omega_m = 0$ (see Appendix A). This solution is complicated and of no cosmological interest to us here.

From this we can see that the standard tracking behaviour of exponential quintessence is available whenever the coupling term is negligible. This tracker solution has been shown to exist for extremely wide range of initial conditions. Unfortunately, while in the tracking regime, the scalar field equation of state equals exactly the background $w$, and thus this solution cannot account for the accelerating universe.

If, however, the coupling becomes significant at late times, the situation will change. Since for the fixed point $E$ we have $H^2 \sim \rho \sim a^{-3(1+w_m)}$, the last term in the Friedmann equation $E$ scales like $\rho_f \equiv H^2 f(\phi) \sim a^{-3(1+w_m)}(2-\alpha/\lambda)$. This follows from the tracking behaviour of the scalar field; since $\phi' = 3(1+w_m)/\lambda$, we have that $\phi = \phi_0 + 3(1+w_m) log(a)/\lambda$, which implies $f'(\phi) \sim a^{3\alpha(1+w_m)}$. Thus we find that the effective energy density due to the presence Gauss-Bonnet term, $\rho_f$, dilutes slower than the energy density due to ordinary matter, $\rho_m$, if and only if $\alpha > \lambda$. We have confirmed this simple result by numerically integrating our system (see Appendix A for details). Then we found that as the coupling begins to affect the evolution of the field, it will always be passed to the fixed point $F$ from the saddle point $E$ as depicted in FIG.1. Hence the universe is approaching a de Sitter phase and an acceleration occurs as observations indicate.

Furthermore, during the transient epoch between matter and scalar field domination, the universe can enter into a transient phantom stage (with $w_{eff} < 1$), which is possible due to the $\rho_f$ term in the Friedmann equation. Even if $w_{eff} > -1$, a result $w_{eff} < -\Omega_0$ would imply effective phantom dark energy when $\Omega_0$ is interpreted as due to an uncoupled dark energy component. We plot two typical examples of such evolution in FIG.2 with $\lambda = 4$ or $\lambda = 8$ and $\alpha = 20$. Note that $\Omega_m + \Omega_\phi > 1$ is possible, since the (effective) Gauss-Bonnet energy density $\rho_f$ can be negative when the field momentarily rolls backwards. This can occur since, as noted previously, when the universe begins to accelerate the Gauss-Bonnet term flips its sign, and might overturn the slope of the effective potential.

Having now found the background expansion in the model, we can check what parameter values are needed to produce it. We see from FIG.2 that all the variables $x, y$ and $\mu$ are roughly of the order of one today. Then we have that

$$\mu_0 = 8\kappa^2 f(\phi_0)\dot{\phi}_0 H_0 = 8\kappa^4 H_0^2 x_0^2 f_0 e^{\alpha \phi_0}/\sqrt{7} \sim 10^{-120} e^{\alpha \phi_0}/\sqrt{7} \sim 1,$$

if we set $f_0$ of order one as expected in string theory and recall that $(\kappa^2 H)^2 \approx 10^{122}$. Note that though $f(\phi_0)$ has to be very large, because of the exponential amplification of $f(\phi)$ we can here have avoid introducing huge numbers into the Lagrangian $E$. Then it is seen that the present value of the scalar field is about $\kappa \phi_0 \sim 390/\alpha$. It seems that typically our models the field has run to perhaps a couple of dozen Planck masses. Then the scale of the
According to the very rough estimates (14) and (15), say $\lambda_0 \sim 10^{10}$ GeV. Nevertheless, this suggests that the scales of the model could be obtained from more fundamental physics.

### B. Linear Perturbations

The line-element Eq. (17) generalizes in the perturbed FRW spacetime to

$$ds^2 = - (1 + 2\varphi_0) dt^2 - 2a(t)\beta_{ij} dt dx^i + a^2(t) \left[ \delta_{ij} (1 + 2\varphi + \psi_{ij}) + 2\varphi \beta_{ij} \right] dx^i dx^j.$$  

We consider variables in the Fourier space. The transformation is simple since at linear order each $k$-mode evolves independently. We do not consider vector perturbations since their evolution is unmodified [67], and we will only briefly comment on tensor perturbations. We characterize the scalar perturbations in a general gauge by the four variables $\varphi_A, \beta, \varphi, \psi$. Some of these degrees of freedom are due to arbitrariness in separating the background from the perturbations. One can deal with these gauge degrees of freedom by noting that the homogeneity and isotropy of the background space implies invariance of all physical quantities under purely spatial gauge transformations [68]. Therefore one can trade $\beta$ and $\psi$ to the shear perturbation

$$\chi \equiv a\dot{\beta} + a^2\dot{\psi}.$$  

Since both $\beta$ and $\psi$ vary under spatial gauge transformation, they appear only through the spatially invariant linear combination $\chi$ in all relevant equations. In addition, one can define the perturbed expansion scalar

$$\kappa \equiv 3(H\varphi_A - \varphi) + \frac{k^2}{a^2}\chi.$$  

Using these variables in the so called gauge-ready formalism is useful in studying generalized gravity [22, 67, 69]. The general equations for scalar perturbations are given in Appendix B. From here on we set $(8\pi G)^{-1/2}$ to unity to avoid confusion since it was also denoted by $\kappa$.

#### 1. Small scales

Because the Gauss-Bonnet interaction becomes here dynamically important at late times, let us for now neglect the contribution from radiation and consider the matter dominated and subsequent epochs, when $w_m = -1, \Lambda_m = \Pi_m = 0$. Dropping the subscripts $m$, the continuity equations [18] and [19] then read

$$\delta = \frac{k^2}{a^2} v + \kappa - 3H\varphi_A, \quad \dot{v} = -Hv + \frac{1}{a^2} \varphi_A.$$  

We choose to work in a synchronous gauge, defined by $\varphi_A = 0$. One can then see from Eq. [19] that it is convenient and legitimate to set $v = 0$ to fix the remaining gauge mode. Then Eq. [19] tells that $\kappa = \delta$. Thus
the large-scale limit of the Raychaudhuri equation (B5),

\[ \frac{\dot{\rho}^2}{k^2} = (1 - 8H\dot{f})\dot{\chi} + (8H\ddot{f} - 4H^2\dot{f} + 8H\dot{f})\chi - 8\ddot{f} - 2H\dot{f})\varphi - 4(\dot{H} + H^2)\mathcal{f}^2\delta \phi. \]  

(23)

With some algebra, it is now possible to deduce the evolution equation for the matter overdensity \( \delta \). One arrives at

\[ \dot{\delta} + 2H\delta = 4\pi G_s \rho \delta. \]  

(24)

The effective gravitational constant \( G_s \), seen by the matter inhomogeneities depends rather non-trivially on the evolution of the background quantities, and in terms of our dimensionless variables defined in Eq. (10) it can be written as

\[ G_s = 4 \frac{x^4 + \mu^2(1 + \epsilon^2) + x^2[2(1 + \epsilon)(\mu - 1) + y]}{x^2[4 + \mu(5\mu - 8)] - \mu^2[6(1 + \epsilon)(\mu - 1) + y]} \]  

(25)

We plot \( G_s/G \) in the upper panel of Fig. 3 for selection of parameter values. It is clear that at high redshifts, when the scalar field and so coupling term is negligible, all the models reduce to the standard value of \( G \).

When the coupling can be neglected \( (\mathcal{f}', \mathcal{f}'' \approx 0) \), \( G_s \) reduces to unity. This is understandable since standard uncoupled quintessence does not cluster at small scales, and the evolution equation for linear growth stays unmodified. On the other hand, in the slow-roll limit where \( \phi \) and \( \phi' \) can be neglected we have

\[ G_s = \frac{1 + 32f^2(-2H^2 + \rho)^2}{1 + 32H^2f^2(3H^2 - 2\rho)}. \]  

(26)

In the case that the coupling is subdominant, so that \( H^2\mathcal{f}', H^2f'' \approx \epsilon \), one has

\[ G_s = 1 - 8 \left[ \dot{\phi}f' + \phi(\dot{\phi}f'' - 2f'H) \right] + 32f'\left[ \bar{\phi}^4 f' - 4\bar{\phi}\dot{\phi}f' \right] 

\[ - 4\bar{\phi}^3 f''H + f'(H^2 - \rho)'^2 + 2\bar{\phi}^2 f'(2H^2 + \rho) \]  

(27)

where the first and second square brackets respectively embrace corrections of the order \( \mathcal{O}(\epsilon) \) and \( \mathcal{O}(\epsilon^2) \).

In any case, we can here make the important conclusion from Eq. (24) that the shape of the matter power spectrum is the same as for \( \Lambda \)CDM cosmology. The superhorizon scales, where our approximation breaks down and the density perturbation becomes gauge dependent, are not efficiently probed by the present large scale structure surveys. As we have shown that at subhorizon scales the growth rate of structure is the same at all scales, which is the also case when the acceleration is driven by vacuum energy, we cannot distinguish between these two very different scenarios by comparing the shape of matter power spectrum (assuming of course that the primordial spectrum is the same in both of the cases). Therefore the primary constraints arising from the matter power spectrum could be deduced from the overall normalization. Note that these conclusions are model independent, the only assumption being that the scalar field is not very massive. We did not make specific assumptions on the coupling or the potential or assume that \( \Omega_m \) dominates in order to deduce these results. The constraints from large scale structure are still of course relevant, since it is possible that for some models the (scale-independent) growth rate is heavily modified and the normalization of the power spectrum can be used to exclude such cases. On the other hand, the perturbations at large scales will affect the integrated Sachs-Wolfe effect of the CMBR, which might constrain models although the cosmic variance weakens the significance of the low CMBR multipoles.
have to fix several other parameters besides the $\Omega$ model presented in the previous subsection. Then we the full matter power and CMBR spectra for the exam-

deviation is visible.

For simplicity, we set the scalar spectral index $n_S$ to one and the optical depth $\tau$ to zero and keep the present Hup-

ble constant $h$ fixed to 0.7. The normalization we use for each model is such that the first peak in the CMBR

spectrum matches with WMAP observations. For com-

parison, we plot also the concordance $\Lambda$CDM model with its parameters set according to the best-fit obtained by

combining the WMAP and SDSS data [3]. We have checked that its value is now independent of scale in the matter dom-

inated era, when $k \gtrsim 0.02 h^{-1}$. Note that there

$$ F_{MD} = 1 + \frac{5}{4} \left( 1 - \sqrt{1 - \frac{24}{25} \Omega_\phi} \right), $$

which is the small scale solution to Eq. (24) when the

Gauss-Bonnet terms can be neglected. The subsequent

evolution of this quantity is plotted for various parameter

choices in the lower panel of FIG. 3 showing that this evolution is indeed reproduced to high accuracy by the

simple second order differential equation Eq. (24). Dis-

crepancy with the numerical solutions to the full lin-

earized equations appear only for some extreme cases.

The effects on the CMBR and matter power spectra of varying the model parameters is shown in FIGS. (4)

- (9). Note that though we include the error bars in the

figure, they only roughly indicate how the observations

constrain the models, since likelihoods should be com-

puted using the window functions that depend on each

model. The CMBR error bars are from the three year

WMAP data [3] and the error bars for the matter power

spectra are provided by SDSS [3]. A complete likelihood

analysis taking into account all these data would require

exploring a vast parameter space, which is beyond the

scope of the present study. Here we are rather interested in study how the model qualitatively differs from the concordance cosmology when the inhomogeneous evolution is considered. In the next Section we will see that in fact it is enough to impose tight constraints on the model parameters considering only the background expansion.

FIG. (4) seems to indicate that the model favours mat-

ter densities about $\Omega_m^0 \sim 0.4$. It seems clear that we cannot do without dark matter within this framework. Low matter densities result in large relative contribution from the Gauss-Bonnet term (this means very negative effective equation of state, as shown before), and leads to, in addition to a modification of the peak structure in CMB, a large ISW effect. Requiring viable normalization for both the CMBR and the matter power spectrum can

2 In numerical calculations we use a modified version of the public CAMB code [72], see http://camb.info/. Our numerical implementa-

tion is discussed in the Appendixes.
also significantly constrain the allowed matter density. In FIGs. 4 and 5 we show how the cosmological predictions are changed when the slope of the potential or of the coupling are varied. The main imprint from different potential slopes $\lambda$ seems to be in the normalization. For low values of $\lambda$, there is significant contribution of the scalar field during the scaling matter era. This slows down the rate of growth of matter inhomogeneities, see Eq.(28). Hence the fact that there is less structure nowadays than for larger $\lambda$ is not a consequence of the Gauss-Bonnet modification, but rather an effect of the presence of the scalar field in the earlier scaling era. Finally, in FIG. 6 we see that the strength of the coupling $\alpha$ might be more difficult to deduce from these data. With steep coupling slopes, the scalar field domination takes place more rapidly and with more negative $w_{eff}$, which can somewhat amplify the ISW effect. The contrary happens for smaller $\alpha$.

The effective gravitational constant Eq.(25) can in principle diverge. Indeed, we have found that in the model studied here $G_*$ typically diverges in the future, see FIG. 3. For low matter densities or large $\alpha$ this can happen even before $a = 1$, as happens in an example plotted in FIG. 3. Such a case would clearly be ruled out. However, in the model parameter combinations that would lead to this divergence of perturbations before the present day, though may fit the SNeIa or other individual data, are not usually compatible with the combined constraints (these will be presented in the next section).

Matter perturbations will at least for a while grow explosively, but this is different from the Big Rip singularity in phantom models where the background energy density will approach infinity in finite time. Note also that the perturbation singularity does not correspond to the crossing of phantom divide, where $1 + w_{eff}$ changes its sign. In some phantom fluid dark energy models crossing this divide might seem problematical, since fluid perturbation
Previously, instabilities have been found to occur in Gauss-Bonnet pre-big bang cosmology. Such have been also recently studied in the context of ghost conditions in Gauss-Bonnet cosmologies. Though as well known, when expanded about de Sitter spacetimes, ghost modes do not appear in Gauss-Bonnet gravity, it has been pointed out that this does not necessarily hold in the FRW background since such is characterized by non-constant curvature. Quantum field theoretical consistency requires absence of ghosts, which means that a function $T$ multiplying the time derivatives of a variable to quantized should be positive. If a function $S$ multiplying the spatial gradient is negative, a related instability might occur when the sound speed $c^2_S = S/T$ is not positive definite. In fact, when an imaginary propagation speed appears in an action for a canonical field, one would expect already the solutions of its classical evolution equation to exhibit divergent behaviour. This is what we found here. It can be shown that the canonical action for the potential $\Phi$ in the case $\rho_m = 0$ features an effective propagation speed.

\[
 c^2_S = \frac{-x^2[4 + \mu(5\mu - 8)] + \mu^2[6(1 + \epsilon)(\mu - 1) + \epsilon]}{(\mu - 1)[3\mu^2 - 4(\mu - 1)x^2]} \quad (29)
\]

When this exceeds one, the speed is superluminal and causality could be violated; when it is negative, the stability might be lost. Since Eq. (29) corresponds to propagation of the scalar field in vacuum, the same $c^2_S$ would be found for any other gauge-invariant variable as there is only one scalar degree of freedom. However, in dark energy cosmologies featuring the Gauss-Bonnet term one should take into account both matter and the corrections to Einstein gravity. This does not complicate the analysis of the tensor mode, since it decouples from other fluids (as long as they are isotropic), and one can just include the background solution (for which matter has been taken in account) into an expression for the tensor $c^2_T$ (which is the same as without matter). However, scalar modes are non-trivially coupled. In the presence of matter, Eq. (29) does not describe the evolution of $\Phi$. It might still describe the evolution of $\delta\phi$ in some suitable gauge at a limit where the impact of matter perturbations on the field fluctuations can be neglected, but a priori it is not clear that such a limit of cosmological equations exists. For the models considered here, one notes by comparing Eqs. (29) and (25), that the linearized matter perturbations diverge precisely in the points where the propagation speed squared $c^2_S$ changes its sign. Our interpretation is that the fluctuation of the field $\phi$ becomes unstable when Eq. (29) dictates, and since it interacts (via gravitational potentials) with the fluctuation in $\rho_m$, also the latter signals instability at the same instant. Therefore the stability of the linear scalar modes is now indeed determined by a canonical action for any gauge-invariant scalar perturbation variable in the vacuum.
III. CONSTRAINTS

A. Cosmological and Astrophysical Constraints

Armed with all the equations describing the cosmological dynamics, we can now derive the constraints arising from astrophysical and cosmological observations. In this section we will consider kinematic tests related to the background expansion of the Universe: the SNeIa luminosity distance-redshift relationship, the CMBR shift parameter and the baryon oscillation length scale.

1. Supernovae Ia

The luminosity distance in a flat space is defined as

\[ D_L(z) = (1 + z) \int_0^z \frac{dz'}{H(z')} \]  \hspace{1em} (30)

The distance modulus probed by SNeIa observations is then given by

\[ \mathcal{M} = m - M = 5 \log_{10} \left( \frac{D_L(z)}{10 \text{pc}} \right) \]  \hspace{1em} (31)

We use the "Gold" sample of 157 SNeIa [1]. There the observed magnitude \( m \), its error and the redshift is given for each supernova. The absolute magnitude \( M \) is unknown, and its effect is degenerated with \( H_0 \) entering the formula [31] from Eq. (30). We marginalize over \( H_0 \) (or equivalently, \( \mathcal{M} \)), by integrating over the likelihood as

\[ \chi^2 = \int \chi^2(H_0) e^{-\chi^2(H_0)/2} dH_0/\int e^{-\chi^2(H_0)/2} dH_0. \]

We marginalize similarly over the model parameter \( \alpha \).

For comparison, we perform the likelihood calculations also with the Supernova Legacy Survey (SNLS) data [2] using then a bit different method. For each of the 115 supernovae in the SNLS set (labeled here with the index \( i \)), the distance modulus has been given as a function of the stretch factor \( s_i \), color factor \( c_i \) and the apparent magnitude \( m_i^* \) as

\[ \mathcal{M}_i(M, a_1, a_2) = m_i^* - M + a_1(s_i - 1) - a_2 c_i. \]  \hspace{1em} (32)

We will treat the parameters \( M \), \( a_1 \) and \( a_2 \) like cosmological parameters and find their values which maximize the likelihood. For individual supernovae, we keep the parameters \( m_i^* \), \( s_i \) and \( c_i \) in their best-fit values. The uncertainty in \( \mathcal{M}_i \) would in principle depend on \( a_1 \) and \( a_2 \), but should be kept fixed while optimizing these global parameters [2]. Thus we compute

\[ \chi^2 = \sum_{i=1}^{115} \frac{\mathcal{M}_i(M, a_1, a_2) - 5 \log_{10} \left[ \frac{D_L(z_i, a_1, a_2, \Omega_m) 10^{18}}{10 \text{pc}} \right]}{\sigma_{int}^2} + \frac{\sigma_{int}^2}{\sigma^2_{int}}. \]  \hspace{1em} (33)

The quantities with lower index \( i \) we get directly from the data, for the parameters \( M \), \( a_1 \), \( a_2 \) and \( \beta \) we use their best-fit values, and for the intrinsic dispersion we use \( \sigma_{int} = 0.15 \) for the nearby and \( \sigma_{int} = 0.12 \) for the distant supernovae. Similar procedure was used in Ref. [79]. We have checked that modifications of the scheme change the results only very slightly. The SNLS data set can give little bit tighter contours than GOLD set for the model at hand. The contours are shown in dotted lines in FIG. [7]

In principle when considering the supernovae observations in extended gravity theories, one should take into account the possible effects of modified gravity on the evolution of the supernovae [50]. Here we however assume that the SNeIa can be treated as standard candles. If variations in the scalar field \( \phi \) can be neglected at the scales relevant to supernova evolution, it is determined by general relativity alone, though corrections from \( R_{G\beta} \) might be crucial at cosmological scales and have to be taken into account in the luminosity-distance relation [50]. This assumption needs to be strictly justified; we return to a related discussion in Section III B 2.

2. CMBR Peak Locations

In addition, we compute the CMBR (cosmic microwave background radiation) shift parameter [51] given by

\[ \mathcal{R} = \sqrt{\Omega_m H_0} \int_0^{z_{dec}} \frac{dz}{H(z)}. \]  \hspace{1em} (34)

where \( z_{dec} \) is the redshift at decoupling. This number \( \mathcal{R} \) captures the correspondence between the angular diameter distance to last scattering and the relation of the angular scale of the acoustic peaks to the physical scale of the sound horizon. Its value is expected to be very weakly model-dependent, and it can be inferred rather accurately from the latest data [82]. We use here the parameter values \( \mathcal{R} = 1.70 \pm 0.03 \) and \( z_{dec} = 1088 \pm 1 \). The resulting contours are plotted with dashed lines in FIG. [9]

3. Baryon Oscillations

The imprint of primordial baryon-photon oscillations in the matter power spectrum is related to the dimensionless quantity \( A \) in the following way

\[ A = \sqrt{\Omega_m H_0^2} \left( \frac{n_S}{0.98} \right)^{1.2} \left[ \frac{1}{z_1} \int_{z_1}^{z_i} \frac{dz}{H(z)} \right]^2. \]  \hspace{1em} (35)

The physical length scale associated with the oscillations is set by the sound horizon at recombination, which can be estimated from the CMBR data [3]. Measuring the apparent size of the oscillations in a galaxy survey allows one to determine the angular diameter distance at the survey redshift. Together with the angular size of the CMBR sound horizon, the baryon oscillation size can then be a powerful probe of the properties and evolution
of the universe. From the large scale structure data one can infer that $A = 0.469 \pm 0.017$ and $z_1 = 0.35$, assuming the $\Lambda$CDM model. For the scalar spectral index we use the best-fit WMAP value 0.95.\(^{[3]}\) The resulting constraints are plotted with solid lines in FIG. \((9)\)

### 4. Combined constraints

The combined constraints arising from the CMBR shift parameter, baryon oscillations and the SNeIa are presented in the shaded contours in FIG. \((7)\). There the dependence on the parameter $\alpha$ is not shown, but we have included also likelihood slices on three constant-$\Omega_m$ slices of the parameter space in FIG. \((8)\), to explicitly show that the value of $\alpha$ is not so tightly constrained, as the contours on the $\alpha$-$\lambda$ planes have are nearly horizontal planes. The best-fit values for the $\chi^2$ are given in Table \((4)\). Note that we have reported the $\chi^2$ per effective degree of freedom, $\chi^2_{dof}$, thus taking into account that the coupled model has more free parameters than the $\Lambda$CDM model. The fit with CMBR and SNeIa is as good as with the concordance model, but when the constraint from $A$ is included the fit gets worse. This conclusion is the same for both the Gold and the SNLS data. The latter data gives slightly worse fit to the model than to $\Lambda$CDM; when $R$-constraint is included the goodness of fit is slightly better than for $\Lambda$CDM; and when the $A$-constraint is added the goodness of fit is significantly worse, difference of $\chi^2$ being almost $\Delta \chi^2 = 9.5$. Taken at face value, the model is ruled out by the combining the three kinematical tests. The reason why the baryon oscillation data seems to be in tension with the other data for this model is that it fixes the present matter density rather tightly to the $\Lambda$CDM region, $\Omega_m \approx 0.27$. As a matter of fact in a flat universe this constraint is equivalent to requiring that the matter density is $\Omega_m = 0.273 + 0.123(1 + w_0) \pm 0.025$, where $w_0$ is an averaged equation state between the present and $z_1$\(^{[83]}\). On the other hand, because the Gauss-Bonnet coupled field produces a very negative $w_{eff}$ today, the other data sets we included prefer higher matter densities than in the standard $\Lambda$CDM model.

However, on one hand it is not clear how model-independent probe of the background expansion the baryon oscillation constraint in its present form is. Here the the non-negligible contribution of the scalar field to the energy density during matter domination probably affects the dependence of $A$ on $\Omega_m$. In addition, the Gauss-Bonnet modification of gravity might have such effects on the nonlinear evolution of galaxy distributions that shift the scale at which the oscillation appear us today. Finally, in general in dark energy models with rapidly varying equations of state the validity of the approximation of lumping the observations to the single redshift $z_1$ has not been established.\(^{[79]}\) For such reasons one has to be careful before ruling out alternative cosmologies based on the baryon oscillation constraint.

On the other hand, as shown in Fig. \((8)\) the baryon oscillations and the CMBR acoustic scale alone or together do not limit the allowed parameter space very strictly. The best fit to these observations is $\chi^2 \ll 1$, and they are compatible with lower matter densities than with the SNeIa data included. As mentioned previously, it is possible that the supernova constraints do not apply in their present form to the model, and hence it is interesting to note that when they are discarded, the model matches excellently with the other observations. It is then also consistent with high values of $\lambda$, and the tension with nucleosynthesis bound, to be discussed next, disappears.

### 5. Nucleosynthesis

The nucleosynthesis bound restricts the allowed amount of dark energy in the early universe. If the ex-
universe.

\( R \) also constrains arising from SNeIa and the solid contours include those arising from SNeIa, the solid contours include also \( R \). In the left panels, we have in addition included the \( \lambda \). There at the solid lines one has \( G'_\star = 0 \) at the present. In the upper pair of figures \( \Omega_m = 0.3 \), in the middle panels \( \Omega_m = 0.35 \) and in the lowest two figures \( \Omega_m = 0.4 \). One notes that the background expansion does not very sensitively probe \( \alpha \), and that requiring \( G'_\star = 0 \) restricts very strictly the parameter combinations.

\[ \chi^2_{\text{dof}} \]

\[ \Omega_m \]

\[ \lambda \]

| Data set   | ΛCDM     | \( R^2_{CO \phi} \) model |
|------------|-----------|---------------------------|
| SNeIa      | 1.142 0.31   ±0.04 | 1.140 0.22±0.32 ±0.11 | 0.3±4.3 ±2.2 |
| SNeIa+R    | 1.144 0.28±0.05 | 1.136 0.45±0.02 | 0.4±1.5 |
| SNeIa+R+\( \Delta \) | 1.137 0.28±0.02 | 1.205 0.34±0.02 | 5.7±0.5 |

\[ \text{TABLE I: The best-fit values for } \Lambda \text{CDM model compared with fits of the coupled scalar field model for some parameter values when } \alpha \text{ is marginalized over. The error limits correspond to change of unity in the } \chi^2. \text{ The degrees of freedom in the first row are } 157 - d, \text{ in the second } 158 - d \text{ and in the third } 159 - d, \text{ where } d = 1 \text{ for the } \Lambda \text{CDM and } d = 2 \text{ for the } G\phi \text{ models.} \]

pansion rate is not due to radiation alone, the prediction for the abundance of the light elements produced during nucleosynthesis will be modified. This places constraints on \( \Omega_\phi \) at \( \alpha \sim 10^{-10} \). As tight constraints as \( \Omega_\phi < 0.045 \) have been reported \[84\], but more a conservative limit is \( \Omega_\phi < 0.2 \) \[3\]. Here one should keep in mind that the observationally preferred value of \( \omega_m h^2 \) might turn out to be different than in the concordance models (as it seems to be with \( \omega_m h^2 \)). Nevertheless, if the scalar field is tracking during nucleosynthesis, the more conservative bound translates into \( \lambda > 6.3 \). While with potential slopes corresponding to such values, a good fit can still be achieved, the best fit values are typically at smaller \( \lambda \) if the SNeIa data is taken into account. Therefore one might want to consider relaxing the nucleosynthesis constraint.

There are several ways to do that. The simplest option is to assume that the scalar field has not reached the tracker solution yet, but begins to approach it during or after the nucleosynthesis. Then one just considers large enough initial values for the field, so that the potential term in Eq. (9) is small compared to the Hubble drag at very early times. This could be the natural outcome of inflation \[5\]. If one however insists that the field must have entered the tracking phase well before nucleosynthesis, it would be necessary to modify the early dynamics of the field. If the slope of the potential is steeper for small values of the field, the energy density residing in the quintessence field is subdominant until it reaches the shallower region, and choosing the parameters suitably, this could happen after a sufficient amount of light elements is produced. It might be reasonable to consider \( \lambda \) as a function of the scale factor \[3\]. Alternatively, an additional field dependent features in the coupling \( f \) could be used to hold the field subdominant enough in very early universe.

### B. Solar System Constraints

Time variation of the gravitational constant is tightly constrained by observations. These observations include various tests of the force of gravity within the Solar system and laboratories, and indicate that \( (dG_\star/df)/G_\star \) is
less than about $10^{-12}$ per year, where $G_*$ is the effective gravitational constant [58]. This bounds translates into

$$\frac{G'_*}{G_*} \lesssim 0.01.$$  \hfill (36)

To derive the variation of this constant for the coupled Gauss-Bonnet gravity, we follow the approach of Ref. [86] where cosmological perturbation equations were considered at their Newtonian limit, i.e. assuming small scales and small velocities the Poisson equation was derived, and the effective strength of gravitational coupling was read from the resulting expression, which relates the gradient of the gravitational potential to the perturbations in matter density. We will first compute results ensuing from this approach and then discuss the Post-Newtonian parameterization.

1. Newtonian limit

Consider subhorizon scales at a limit where the time derivatives of the perturbations are much smaller than their gradients. In the Newtonian gauge we have $\varphi = -\Phi$ and $\varphi_A = \Psi$. Then $\kappa = 3H\Psi$. This gauge is also called zero-shear or longitudinal for the reason that there $\chi = 0$. The ADM energy constraint Eq. (B2) is then, at this limit

$$\frac{a^2 \rho \delta}{k^2} = -2(1 - 8H^2)\Phi - 8H^2 \delta f.$$  \hfill (37)

The shear equation [14] reveals there is effective anisotropic stress in the sense that $\Phi \neq -\Psi$:

$$(1 - 8H^2)\Psi + (1 - 8\dot{f})\Phi = -8(\dot{H} + H^2)\delta f.$$  \hfill (38)

To obtain the effective Poisson equation here, we need to eliminate the scalar field perturbation from the above two equations. The fluctuation $\delta \phi$ is given by the Klein-Gordon equation [16],

$$\delta \phi = 8f(2(\dot{H} + H^2)\Phi - H^2 \Psi).$$  \hfill (39)

From the previous three equations, it is then straightforward to obtain the Poisson equation,

$$\frac{k^2 H^2}{a^2} \Psi = 8\pi G_\star \rho \delta.$$  \hfill (40)

We find that the effective gravitational constant here coincides with the expression [25]. This is in spite that the Newtonian limit is considered at a static configuration of the gravitational sources. This takes the approximation we used in Section (IIB) further, since there we derived the small scale limit of the dynamical equations. We can understand that the two $G_\star$ agree since the time derivatives of say $\Psi$ are expected to be proportional to $\dot{\Psi} \sim H\Psi$. The gradient terms are more important at small scales and hence determine there the main contribution to the gravitational coupling of evolving structures.

Though not obvious from the formula (25), it equals one when the coupling goes to zero. Generally, when the coupling is significant, i.e. $\mu$ is of order one, then one expects the $G'_*/G_*$ to be of roughly of order one as well. This is confirmed by numerical evaluation of the time variation of $G_\star$ for the present model considered here. As claimed in Ref. [86], one has to assume "an accidental cancellation" to satisfy the bound Eq. (36) in the presence of significant Gauss-Bonnet contribution to the energy density. This means that we have to fine-tune the coupling $\alpha$ in order to eliminate the time variation of the effective gravitational coupling. Then the Newtonian limit in general exhibits time-varying $G_\star$, but at the present this $G_\star$ appears to us to be a constant. In practice, the requirement of well-behaved Newtonian limit sets the magnitude of the coupling $\alpha$. Such a limit does not exist for all $\lambda$ and $\Omega_m$. When it is found and it requires $\alpha$ to be fixed with the accuracy of about 0.01, see FIGs. 3 for actual constraints and FIG 10 for the variation as function of $\alpha$.

In Table 11 we report the $\chi^2_{dof}$, computed under the condition $G_\star = 0$, for some parameter choices for our model. Even with the coupling fixed in to yield $G_\star = 0$, the coupled Gauss-Bonnet quintessence fits both the SNeIa and the CMBR shift parameter data as well as, or slightly better than the $\Lambda$CDM concordance model. However, again this is only when the baryon oscillation constraint is not taken into account: with the $\Lambda$ combined with all the other data sets including the Solar system constraints the model is in blatant contradiction with the data.

3 It is possible to find an expression for the $G'_*/G_*$ in terms of $x$, $y$ and $\mu$, but it is somewhat lengthy.
2. Schwarzschild metric

The rather tight bounds we get could be loosened if one takes into account that cosmological variations of $G$ and other gauge-couplings might be different from the ones we measure on Earth or within our Solar system \[^{57, 58, 59, 60}\]. The model is set up in such a way that the corrections to the Einstein gravity will affect the overall expansion of the Universe, while the Solar system is clearly subcosmological in scale and nonlinear in nature. Thus the local values of the scalar field and the potentials could be something else than at cosmological scales where the linear perturbation equations of Appendix Eq. (14) are expected to apply. If for example, the scalar field happened to be nearly frozen at our neighbourhood we would observe $G_s = G$ though at larger scales the linear perturbation would evolve according to Eq. (24) where possibly $G_s \neq 0$.

Conventionally the constraints ensuing from Solar system experiments are imposed on the deviations from the Schwarzschild solution written in form of the Post-Newtonian parameters (PPN). Here we computed the Newtonian limit of the cosmological perturbation equations in the FRW background to derive the constraints, and there is reason to doubt that results ensuing within the PPN formalism would not be equivalent. Therefore our considerations of the Newtonian limit in the previous subsection must be regarded as preliminary. More detailed study of the Solar system constraints is left for future. Since the $R^2_{GB}$ is quadratic, its value for the Schwarzschild metric goes like $1/r^6$. However, due to the dynamics of the field, this does not necessarily mean that the curvature correction would dominate at small scales. Previously local constraints for a scalar-Gauss Bonnet coupling have been investigated in the case that the potential can be neglected \[^{61, 62}\]. Even then all the effects of the scalar field can be negligible for any Solar system experiment provided $f''(\phi_0)$ is negative. One also notes that due to specific properties of $R^2_{GB}$, if the coupling $f(\phi_0)$ happens to be in its minimum at present all the corrections trivially disappear. With these remarks in mind, it seems easy to adjust the function $f(\phi)$ in such a way that the local constraints would be satisfied (even if our preliminary analysis would turn out to be inadequate and setting $\alpha$ to a certain value would not guarantee the viability of the model). Such an adjustment would apparently ruin the potential of the model to alleviate coincidence problem, since the slope of the coupling function $f(\phi)$ would change just today. However, it might be possible to associate the change of the slope with the triggering of the acceleration, since it is then that the curvature corrections become dynamically important and one could expect higher order modifications to enter in the play. We hope to address these issues more quantitatively elsewhere.

| Data set     | $\Lambda$CDM | $R^2_{GB}$ model |
|--------------|---------------|-----------------|
|              | $\chi^2_{dof}$ | $\chi^2_{dof}$ | $\Omega_m$ | $\Omega_m$ | $\lambda$ | $\alpha$ |
| SNeIa        | 1.142         | 0.314           | 1.146      | 0.42       | 5.1        | 32.3      |
| SNeIa+R      | 1.144         | 0.277           | 1.141      | 0.44       | 5.2        | 33.8      |

**TABLE II**: The best-fit values for $\Lambda$CDM model compared with fits of the coupled scalar field model for some parameter values when the coupling $\alpha$ is set in order to fix the present time variation of $G_s$ to zero. The degrees of freedom in the first row are $157 - d$, in the second $158 - d$ and in the third $159 - d$, where $d = 1$ for the $\Lambda$CDM and $d = 3$ for the $R^2_{GB}$ models.

IV. CONCLUSIONS

We studied cosmological phenomenology of dark energy based upon a low-energy effective string theory action featuring a compactification modulus and taking into account the leading order curvature corrections. One is then lead to consider a generalized scalar-tensor theory which includes a coupling to the Gauss-Bonnet invariant. Specifically, we investigated the evolution of perturbations. There the simple closed form equation (21) for the linear matter inhomogeneities, together with its effective gravitational constant (25) could be highlighted as one of the main results of the paper, since it enabled to find several new model-independent results of perturbation evolution in Gauss-Bonnet dark energy, these including the following.

- The evolution of matter perturbations is scale-invariant at small scales in the presence of the Gauss-Bonnet term, and thus the shape of the matter power spectrum is retained.
- The growth rate of matter perturbations, which is easily extracted from Eq. (21), can be compatible with observations even in the presence of significant contribution from the Gauss-Bonnet interaction.
- The stability of perturbations can be read off from the expression for the effective gravitational constant. A divergence is possible.

This gravitational constant cannot be deduced by matching Schwarzschild and FRW-type metrics for subhorizon evolution of spherical matter overdensities, as the Jebsen-Birkhoff theorem is not in general respected in these models. The Newtonian limit, as discussed in Section III B, does feature this effective strength of gravity, which might be used to efficiently constrain the coupling. However, the relevance of this limit to the experiments within Solar system is an open question to be addressed more carefully in forthcoming studies.

In addition to such general considerations mentioned above, we chose a particular model and scrutinized its predictions for cosmology. Specifically, we characterized the potential and the coupling by exponential functions, as with the Nojiri-Odintsov-Sasaki modulus \[^{41}\]. This parameterization is appealing in its minimality, as within
it one can describe Gauss-Bonnet dark energy with just two extra degrees of freedom compared to the $\Lambda$CDM model. Moreover, the exponential forms are well motivated on fundamental grounds, and allow dark energy solutions without introducing unnatural scales into the Lagrangian. Many of the previous cosmological studies of this type of low-energy string theory have been concerned with asymptotic solutions and confined to cases where the Gauss-Bonnet term is subdominant, and thus only introducing small deviations to standard quintessence scenarios. Here allowed a crucial role for the Gauss-Bonnet term and examined its consequences in detail. Such approach also proved to be very useful, as it revealed several possibilities of cosmological evolution present already in this simple model, these including the following.

- The Gauss-Bonnet coupling can act to switch the decelerating expansion into acceleration after a scaling matter era, thus perhaps alleviating the coincidence problem.

- The curvature interaction may momentarily push the universe into a phantom era, which will then not lead into a Big Rip singularity in the future. Hence the model can explain superacceleration.

- However, the linear perturbations might diverge in the future. This might terminate the de Sitter phase, possibly helping to make contact with string theory.

Let us briefly comment each of these possibilities.

The field is drawn into the scaling attractor virtually regardless of initial conditions, and at late times acceleration can be onset provided simply $\alpha > \lambda$. This can be seen as a possible way towards solution of the coincidence problem, though it is occasionally considered that in a preferable solution the future evolution of the Universe would be characterized by a constant ratio of $\Omega_\phi$ and $\Omega_m$, different from $1$. However, a transition from a scaling matter era to an accelerating scaling era has been shown to be impossible for a very general class of scalar field models \cite{32}. Another remark is that the question of why dark energy is beginning its domination today is now translated into the question that why the coupling comes into play just recently. The same holds for any other coupling mechanism proposed so far, but it is still fair to say that such mechanism can provide an approach to tackle the coincidence problem. The relevant dynamics are not always obvious when only the asymptotic behaviour is considered, and therefore it was only very recently \cite{9, 11} that one would have found that the simple model considered here can (without flipping the field to a phantom, i.e. putting $\gamma = -1$ in the action \cite{2}) feature a transition from a matter dominated to an accelerating phase.

We also showed that during this transition, the dynamics of the coupling can cause a phantom expansion. Again, this phenomenon is unaccessible when only asymptotic behaviour or vacuum is taken into account. Several authors have shown that phantom dark energy does not necessarily lead a Big Rip singularity in the presence of the Gauss-Bonnet term. Here we found that this term can in fact ally with a canonical scalar field into an effective phantom energy. This is clearly a more appealing way to produce $w_{\text{eff}} < -1$, since thereby the phantom era can take place without the introduction strong energy condition violating components, while the Big Rip singularity is still avoided.

However, one of the peculiarities of this class of models is that their linear perturbations can diverge, even while the background solution is not singular. We argued that such a possibility does not necessarily indicate an inconsistency of the model. For the specific model considered here, the divergence typically awaits in the future when the evolution would begin to asymptote to the de Sitter expansion. The dynamics is such that the (linearized) inhomogeneities grow explosively as the $s_{\text{SC}} < 0$ limit is approached. Hence the linear approximation perhaps together with the whole FRW description breaks down. It is tempting to believe that the de Sitter phase will not then be reached but the inhomogeneities take over. This would enable to reconcile the accelerating universe with string theory in these models in yet another way. Since if the acceleration is transient one can consistently define the set of observables in string theories\cite{74}. Their present S-matrix formulations seem to be in odds with an eternally accelerating universe, like in the concordance model with the cosmological constant. Would the universe however somehow pass through $c_s^2 = 0$, one would have to invoke higher-order curvature corrections or modifications of the exponential parameterization to escape the possible singularity.

We calculated the implications of Gauss-Bonnet dark energy to various different cosmological and astrophysical observations and matched them with several data sets. The general conclusion we arrive at is that the quadratic curvature coupling has interesting and non-trivial signatures while being in a good agreement with a large amount of data. We checked how the Gauss-Bonnet interaction affects the evolution of scalar field fluctuations and how these in turn impact the large scale structure and CMBR observations. Using combined datasets related to the CMBR shift parameter, baryon oscillation scale, supernovae Ia luminosity distance and variations of the effective gravitational constant in the Solar system, we derived confidence limits on the parameters of the model. The study presented not only shows the requirements to be imposed on models of Gauss-Bonnet dark energy for them to exhibit completely realistic cosmology, but also quantifies the degree to which these requirements can be within the most minimalistic set-ups. Due to its simplicity, matching with observations, capability to shed light on the coincidence problem, possibility incorporate phantom (and perhaps also transient) acceleration, and not least its theoretical motivation, the Gauss-Bonnet quintessence seems to be a promising alternative to a cosmological constant.
We end by mentioning some issues yet to be pursued. Among these are the nature of the divergence and the Solar System limit in these models. Also, we found that there is some contrast with the amount of early quintessence and the nucleosynthesis bound and with the present matter density and the baryon oscillation scale. Whether these enforce one to resort to more elaborate models depends on several assumptions, like the validity of the baryon oscillation constraint and the initial conditions of the scalar field after inflation. One caveat is also that these contrasts can disappear if the Gauss-Bonnet modification affects the intrinsic evolution of supernovae and thus changes the effective luminosity-distance relationships of these objects. This does render some of our conclusions indecisive, but in an interesting way.

Among these are the nature of the divergence and the potential, the initial condition for $\mu$ makes difference is for $\Omega_m$. The fractional matter density $\Omega_m$ as a function of the coupling and the potential, the initial condition for $\mu$ is determined by the required amount of matter today $\Omega_m^0$. The fractional matter density $\Omega_m$ as a function of $\epsilon$, Eq. (A4). One might be concerned that the system (A1)-(A3) is not yet algebraically solved.

\[ x' = \frac{-2\epsilon(-3 + x^2 + 6\mu) + 3x^2(1 + w) + 3(1 + w)(-3 + 3\mu + y) + x(3\alpha\mu - \lambda y)}{2x^2 + 3\mu} x, \tag{A1} \]
\[ y' = -(2\epsilon + x\lambda)y, \tag{A2} \]
\[ \mu' = \frac{2\alpha x^3 + 2\epsilon(3 + x^2 - 3\mu) - 3x^2(1 + w) + x\lambda y - 3(1 + w)(-3 + 3\mu + y)}{2x^2 + 3\mu} \mu, \tag{A3} \]

where the slow-roll parameter $\epsilon$ is given by
\[ \epsilon = \frac{2\alpha x^3\mu - 3\mu^2 + 2x^4(-1 + w) + 2x^2[\mu(-1 + 3w) + (1 + w)(-3 + y)] + \lambda x\mu y}{4x^2(-1 + \mu) - 3\mu^2}. \tag{A4} \]

Here $w$ is given by $w = \Omega_H^0/(3(\Omega_\gamma^0 + \alpha\Omega_m^0))$. Since the field is attracted to the scaling solution $\Sigma$ regardless of the initial values for $x$ and $y$, the only initial condition that makes difference is for $\mu$. However, given the exponents of the coupling and the potential, the initial condition for $\mu$ is determined by the required amount of matter today $\Omega_m^0$. The fractional matter density $\Omega_m$ as a function of $\log(a)$ is given by $\Omega_m = (3 - x^2 - 3\mu - y)/3$. Since on the other hand $\Omega_m = \Omega_m^0 a^{-3(1+w)/H^2}$, we can readily obtain the Hubble parameter as function of redshift from the solution for $x$, $y$ and $\mu$. One consistency check that we have made is the comparison of the numerical derivative of the Hubble parameter with the algebraic solution for $\epsilon$, Eq. (A4). One might be concerned that the system (A1)-(A3) is not yet algebraically solved.

\[ x' = \frac{-2\epsilon(-3 + x^2 + 6\mu) + 3x^2(1 + w) + 3(1 + w)(-3 + 3\mu + y) + x(3\alpha\mu - \lambda y)}{2x^2 + 3\mu} x, \tag{A1} \]
\[ y' = -(2\epsilon + x\lambda)y, \tag{A2} \]
\[ \mu' = \frac{2\alpha x^3 + 2\epsilon(3 + x^2 - 3\mu) - 3x^2(1 + w) + x\lambda y - 3(1 + w)(-3 + 3\mu + y)}{2x^2 + 3\mu} \mu, \tag{A3} \]

where the slow-roll parameter $\epsilon$ is given by
\[ \epsilon = \frac{2\alpha x^3\mu - 3\mu^2 + 2x^4(-1 + w) + 2x^2[\mu(-1 + 3w) + (1 + w)(-3 + y)] + \lambda x\mu y}{4x^2(-1 + \mu) - 3\mu^2}. \tag{A4} \]

In the appendices we will consider some technical details relevant in solving the cosmological equations. The first appendix concerns only the kinematic evolution model specified by Eq. (3). The equations in the second appendix are for perturbations in general for an action $\gamma$. In fact, when one sets $\gamma = 0$, they describe also modified Gauss-Bonnet models featuring a function $f(R^2_{GB})$ in their action, since there exists a simple mapping between the theories.

\[ x' = \frac{-2\epsilon(-3 + x^2 + 6\mu) + 3x^2(1 + w) + 3(1 + w)(-3 + 3\mu + y) + x(3\alpha\mu - \lambda y)}{2x^2 + 3\mu} x, \tag{A1} \]
\[ y' = -(2\epsilon + x\lambda)y, \tag{A2} \]
\[ \mu' = \frac{2\alpha x^3 + 2\epsilon(3 + x^2 - 3\mu) - 3x^2(1 + w) + x\lambda y - 3(1 + w)(-3 + 3\mu + y)}{2x^2 + 3\mu} \mu, \tag{A3} \]

where the slow-roll parameter $\epsilon$ is given by
\[ \epsilon = \frac{2\alpha x^3\mu - 3\mu^2 + 2x^4(-1 + w) + 2x^2[\mu(-1 + 3w) + (1 + w)(-3 + y)] + \lambda x\mu y}{4x^2(-1 + \mu) - 3\mu^2}. \tag{A4} \]
The equation of motion for the scalar field is
\[ r = \frac{\mu}{3\alpha}, \] (A5)
and
\[ \mu = -\left[ 27\lambda^4 + 3r k + 100k^2 + 20(500 + 3r + 50k) \right. \]
\[ \left. + 3\alpha^2(320 + 26k + 5k^2) \right]/(9k^2) \] (A6)
where \( r = a(2000 - 264\alpha^2 + 9\alpha^4)^{1/2} \) and \( k = (1000 - 54\alpha^2 + 9r)^{1/3} \).

**APPENDIX B: LINEARLY PERTURBED EQUATIONS**

In this appendix we consider the scalar perturbation equations for coupled Gauss-Bonnet gravity in a general gauge. These equations are consistent with the ones presented in Refs. [67, 69].

The metric is defined as in Eq. (10). Recall that the variables satisfy the constraints
\[ \chi \equiv a^2 + a^2 \dot{\psi}, \quad \kappa \equiv 3(\dot{H} \varphi_A - \dot{\psi}) + \frac{k^2}{a^2} \chi. \] (B1)

Due to general structure of covariantly modified gravity [66], the last three conservation equations are deducible from the set of field equations (B2)-(B5) when the matter content is known.

The energy constraint (\( G^0_i \) component of the field equation) is
\[ \left( 1 - 8H \dot{f} \right) \frac{k^2}{a^2} \varphi + \left( \frac{1}{2} \gamma \dot{\phi}^2 + 12H^3 \dot{f} \right) \varphi_A - \left( H - 12H^2 \dot{f} \right) \kappa = \frac{1}{2} \left( \delta \rho_m + \gamma \dot{\phi} \delta \phi + V' \delta \phi \right) + 4H^2 \left( 3H \delta \dot{f} + \frac{k^2}{a^2} \delta \dot{f} \right) \] (B2)
and the momentum constraint (\( G^i_i \) component) is
\[ \left( 1 - 8H \dot{f} \right) \left( \kappa - \frac{k^2}{a^2} \chi \right) - 12H^2 \delta \dot{f} \varphi_A - 3 \left[ a (\rho_m + p_m) v_m + \gamma \dot{\phi} \delta \phi \right] - 12H^2 \left( \delta \dot{f} - H \delta f \right). \] (B3)

The shear propagation equation (\( G^j_j - \frac{1}{4} G^{ii} \) component) reads
\[ \left( 1 - 8H \dot{f} \right) \chi + \left( H - 8 \left( \dot{H} + H \dot{f} + H^2 \dot{f} \right) \right) \chi - \left( 1 - 8H \dot{f} \right) \varphi_A - \left( 1 - 8H \dot{f} \right) \varphi = \Pi_m - 8 \left( \dot{H} + H^2 \right) \delta f. \] (B4)

The Raychaudhuri equation (\( G^k_k - G^0_0 \) component) is now given by
\[ \left( 1 - 8H \dot{f} \right) \dot{\kappa} + \left( 2H - 8H \dot{f} - 8\dot{H} \dot{f} - 12H^2 \dot{f} \right) \kappa - 12H^2 \dot{f} \varphi_A \]
\[ + \left( 3H - \frac{k^2}{a^2} + 2\gamma \dot{\phi}^2 - 24H^2 \ddot{f} - 48H \dot{H} \ddot{f} + 12H^3 \dddot{f} + 8H f \frac{k^2}{a^2} \right) \varphi_A + 8 \left( \dddot{f} - H \ddot{f} \right) \frac{k^2}{a^2} \varphi \]
\[ = \frac{1}{2} \left( \delta \rho_m + 3 \delta p_m \right) + 2 \gamma \dot{\phi} \ddot{\phi} - V' \ddot{\phi} - 4 \left[ 3H^2 \delta \dot{f} + \left( 2H + H^2 \right) \left( 3H \delta \dot{f} + \frac{k^2}{a^2} \delta \dot{f} \right) \right]. \] (B5)

The previous three equations can be considered as evolution equations for \( \varphi, \chi \) and \( \kappa \), respectively. For instance, the Raychaudhuri equation might be written as \( \dot{\kappa} = \text{sources}/(1 - 8H \dot{f}) \). Also the other equations imply a divergence of perturbations in the case that \( 8H \dot{f} = 1 \). In the models considered here, \( 8H \dot{f} < 1 \) is always satisfied; this condition in fact is necessary for absence of tensor ghost modes [78]. Similar considerations apply for the factor \( (1 - 8\dot{f}) \) appearing in some of the equations. One might understand this by noting that the first condition for stability of scalar modes is automatically satisfied given the conditions for stability of the tensor modes [78]. The second condition for stability of scalar modes, which is not always guaranteed in models considered here, is discussed extensively in Section II.B.3.

The equation of motion for the scalar field is
\[ \gamma \left( \dddot{\phi} + 3H \ddot{\phi} \right) + \left( \frac{k^2}{a^2} + V'' \right) \ddot{\phi} = \gamma \dddot{\phi} (\varphi_A + \kappa) + \gamma \left( 2\ddot{\phi} + 3H \ddot{\phi} \right) \varphi_A - f'' R^2_{GB} \delta \phi - f' \delta R^2_{GB}. \] (B6)
where
\[ \delta R_{GB}^2 = 4H^2\delta R - 16\dot{H} \left( H\kappa - \frac{k^2}{a^2}\phi \right), \quad \delta R = 2 \left[ -\ddot{\kappa} - 4H\kappa + \left( \frac{k^2}{a^2} - 3\dot{H} \right) \varphi_A + 2\frac{k^2}{a^2}\phi \right]. \]  
(B7)

The continuity equations for matter are
\[ \dot{\delta}_m = 3H \left( w_m - \frac{\delta}{c_m} \right) \delta_m \left( 1 + w_m \right) \left( \frac{k^2}{a^2} \right) \varphi_A \right), \quad \dot{\varphi}_m = \left( 3c_m^2 - 1 \right) H c_m \frac{\varphi_m}{a} \left( \frac{\delta}{1 + w_m} - \frac{2k^2}{3(1 + w_m)a^2}\rho_m \right), \]  
(B8) \[ \dot{\varphi}_m = \left( 3c_m^2 - 1 \right) H c_m \frac{\varphi_m}{a} \left( \frac{\delta}{1 + w_m} - \frac{2k^2}{3(1 + w_m)a^2}\rho_m \right), \]  
(B9)

To implement the model in the CAMB code, we have included there an integration of the background equations described in the Appendix [A]. The solution is then interpolated to obtain it at arbitrary points. The perturbation equations are modified to be consistent with the set of equations listed above, as written in the synchronous gauge (with the CAMB variables slightly different from those $h$, $\eta$ and $\chi$ we discussed in [11]). Modifications have to be taken into account in the evolution equations for perturbations to obtain the matter power spectrum, and in the evaluation of their line-of-sight integral to obtain in addition the CMBR spectrum.

[1] A. G. Riess et al. (Supernova Search Team), Astrophys. J. 607, 665 (2004), astro-ph/0402512.
[2] P. Astier et al., Astron. Astrophys. 447, 31 (2006), astro-ph/0501047.
[3] D. N. Spergel et al. (2006), astro-ph/0603449.
[4] M. Tegmark et al. (SDSS), Phys. Rev. D69, 103501 (2004), astro-ph/0310723.
[5] E. J. Copeland, M. Sami, and S. Tsujikawa (2006), hep-th/0603057.
[6] G. Bertone, D. Hooper, and J. Silk, Phys. Rept. 405, 279 (2005), hep-ph/0404175.
[7] R. R. Caldwell, Phys. Lett. B545, 23 (2002), astro-ph/9908168.
[8] A. W. Brookfield, C. van de Bruck, D. F. Mota, and D. Tocchini-Valentini, Phys. Rev. D73, 083515 (2006), astro-ph/0512367.
[9] O. Bertolami and D. F. Mota, Phys. Lett. B455, 96 (1999), gr-qc/9811087.
[10] D. F. Mota and C. van de Bruck, Astron. Astrophys. 421, 71 (2004), astro-ph/0401504.
[11] T. Koivisto, H. Kurki-Suonio, and F. Ravndal, Phys. Rev. D71, 064027 (2005), astro-ph/0409163.
[12] T. Koivisto and D. F. Mota, Phys. Rev. D73, 083502 (2006), astro-ph/0512135.
[13] S. Capozziello, Int. J. Mod. Phys. D11, 483 (2002), gr-qc/0210033.
[14] S. Nojiri and S. D. Odintsov (2006), hep-th/0601213.
[15] N. Straumann, Mod. Phys. Lett. A21, 1083 (2006), hep-ph/0604231.
[16] A. Borowiec, W. Godlowski, and M. Szydlowski (2006), astro-ph/0607639.
[17] R. P. Woodard (2006), astro-ph/0601672.
[18] V. Faraoni (2006), astro-ph/0610734.
[19] S. Nojiri and S. D. Odintsov, Phys. Rev. D74, 086005 (2006), hep-th/0608008.
[20] S. Nojiri and S. D. Odintsov (2006), hep-th/0610164.
[21] L. Amendola, D. Polarski, and S. Tsujikawa (2006), astro-ph/0307033.
[22] T. Koivisto and H. Kurki-Suonio, Class. Quant. Grav. 23, 2355 (2006), astro-ph/0509422.
[23] M. Amarzguioui, O. Elgaroy, D. F. Mota, and T. Multamaki (2005), astro-ph/0510519.
[24] T. Koivisto, Phys. Rev. D73, 083517 (2006), astro-ph/0602031.
[25] B. Li, K. C. Chan, and M. C. Chu (2006), astro-ph/0610794.
[26] E. Elizalde, S. Nojiri, and S. D. Odintsov, Phys. Rev. D70, 043539 (2004), hep-th/0405034.
[27] B. Boisseau, G. Esposito-Farese, D. Polarski, and A. A. Starobinsky, Phys. Rev. Lett. 85, 2236 (2000), gr-qc/0001066.
[28] A. Albrecht, C. P. Burgess, F. Ravndal, and C. Skordis, Phys. Rev. D65, 123507 (2002), astro-ph/0107573.
[29] K. Kainulainen and D. Sunhede, Phys. Rev. D73, 083510 (2006), astro-ph/0412609.
[30] J. Callan, Curtis G., E. J. Martinec, M. J. Perry, and D. Friedan, Nucl. Phys. B262, 593 (1985).
[41] S. Tsujikawa and M. Sami (2006), hep-th/0608178.

[42] M. Sami, A. Toporensky, P. V. Tretjakov, and S. Tsujikawa, Phys. Lett. B619, 193 (2005), hep-th/0504154.

[43] G. Calcagni, S. Tsujikawa, and M. Sami, Class. Quant. Grav. 22, 3977 (2005), hep-th/0505193.

[44] S. Nojiri, S. D. Odintsov, and M. Sami (2006), hep-th/0605039.

[45] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini (2006), hep-th/0510183.

[46] M. Sami, A. Toporensky, P. V. Tretjakov, and S. Tsujikawa, Phys. Lett. B619, 193 (2005), hep-th/0504154.

[47] G. Calcagni, S. Tsujikawa, and M. Sami, Class. Quant. Grav. 22, 3977 (2005), hep-th/0505193.

[48] S. Nojiri, S. D. Odintsov, and M. Sami (2006), hep-th/0605039.

[49] G. Cognola, E. Elizalde, S. Nojiri, S. D. Odintsov, and S. Zerbini (2006), hep-th/0510183.

[50] M. Sami, A. Toporensky, P. V. Tretjakov, and S. Tsujikawa, Phys. Lett. B619, 193 (2005), hep-th/0504154.

[51] G. Calcagni, S. Tsujikawa, and M. Sami, Class. Quant. Grav. 22, 3977 (2005), hep-th/0505193.

[52] S. Nojiri, S. D. Odintsov, and O. G. Gorbunova, J. Phys. A39, 6627 (2006), hep-th/0510183.

[53] I. Y. Aref’eva, A. S. Koshelev, and S. Y. Vernov, Phys. Rev. D72, 046017 (2005), astro-ph/0507067.

[54] I. Y. Aref’eva and A. S. Koshelev (2006), hep-th/0605085.

[55] C. Baccigalupi, S. Matarrese, and F. Perrotta, Phys. Rev. D62, 123510 (2000), astro-ph/0005543.

[56] V. Pettorino, C. Baccigalupi, and F. Perrotta, JCAP 0512, 003 (2005), astro-ph/0508519.

[57] I. Y. Aref’eva, A. S. Koshelev, and S. Y. Vernov, Phys. Rev. D72, 046017 (2005), astro-ph/0507067.

[58] I. Y. Aref’eva and A. S. Koshelev (2006), hep-th/0605085.