On AWGN Channels and Gaussian MACs with Variable-Length Feedback

Lan V. Truong and Vincent Y. F. Tan

Abstract

We characterize the information-theoretic limits of the additive white Gaussian noise (AWGN) channel and the Gaussian multiple access channel (MAC) when variable-length feedback is available at the encoder and a non-vanishing error probability is permitted. For the AWGN channel, we establish the $\varepsilon$-capacity (for $0 < \varepsilon < 1$) and show that it is larger than the corresponding $\varepsilon$-capacity when fixed-length feedback is available. Due to the continuous nature of the channel and the presence of expected power constraints, we need to develop new achievability and converse techniques. In addition, we show that a variable-length feedback with termination (VLFT) code outperforms a stop-feedback code in terms of the second-order asymptotic behavior. Finally, we extend our analyses to the Gaussian MAC with the two types of variable-length feedback where we establish the $\varepsilon$-capacity region. Due to the multi-terminal nature of the channel model, in the achievability part, we are faced with the need to bound the expected value of the maximum of several stopping times. To establish the bound, we develop new results based on work on renewal theory by Lai and Siegmund that may be of independent interest.

Index Terms

Additive white Gaussian noise channel, Gaussian multiple access channel, Variable-length feedback with termination, Stop feedback, Non-vanishing error probability, Second-order asymptotics

I. INTRODUCTION

A. Background

For fixed-length feedback communication, Shannon [1] showed that noiseless feedback does not increase the capacity of single-user memoryless channels. Despite this seemingly negative result, it is known that feedback significantly simplifies achievable coding schemes and the performance in terms of the error probability. For example, Schalkwijk and Kailath [2] proposed a simple coding scheme for the AWGN channel with (fixed-length) feedback based on the idea of refining the receiver’s knowledge of the initial noise in each transmission. The sender iteratively corrects each receiver’s error in estimating the previous transmission. The error probability for this scheme is known to decay (at least) doubly exponentially fast in the blocklength. Burnashev and Yamamoto [3] showed that even with noisy feedback, the reliability function of an AWGN channel improves (over the no feedback case). Ozarow [4] extended Schalkwijk and Kailath’s coding scheme [2] and showed that the capacity region of the Gaussian MAC is enlarged in the presence of feedback. These ideas are collectively known as posterior matching [5]. These ideas have also been extended by Truong, Fong and Tan [6] to the case where the error probability is not required to vanish with increasing blocklength.

It is also well known that feedback can increase the capacity channels with memory. Cover and Pombra [7] characterized the feedback capacity of non-stationary additive Gaussian noise channels with memory. Kim [8] found the capacity first-order autoregressive moving-average AWGN channel with feedback. For finite alphabet channels with memory and feedback, expressions of feedback capacity have been derived for the trapdoor channel [9] and the Ising channel [10]. It is also known that feedback also can increase the second-order coding rates of certain discrete memoryless channels (DMCs) [11].

An even greater advantage of feedback can be observed if one allows for the amount or length of the feedback signal to vary based on the quality of the channel output. Burnashev [12] demonstrated that the error exponent improves dramatically in this variable-length feedback setting. In fact, the error exponent of a DMC with variable-length feedback admits a particularly simple expression

$$E(R) = C_1 \left(1 - \frac{R}{C}\right)$$

for all rates $0 \leq R \leq C$, where $C$ is the capacity of the DMC and $C_1$ is the maximal relative entropy between the conditional output distributions. Yamamoto and Itoh [13] proposed a simple and conceptually important two-phase coding scheme that attains the optimal error exponent in (1). While the error exponent results in [12] and [13] are of paramount importance in feedback communications, in this paper, we focus on the scenario in which the error probability is non-vanishing.

For variable-length codes under the non-vanishing error probability formalism, Polansky, Poor and Verdú [14] provided non-asymptotic achievability and converse bounds for the maximal achievable coding rates. They also derived asymptotic

The authors are with the Department of Electrical and Computer Engineering, National University of Singapore (NUS). V. Y. F. Tan is also with the Department of Mathematics, NUS. Emails: lantruong@u.nus.edu; vtan@nus.edu.sg
The authors are supported by an NUS Young Investigator Award (R-263-000-B37-133) and a Singapore Ministry of Education (MOE) Tier 2 grant (R-263-000-B61-112).
expansions for the optimal code lengths of DMCs and showed dramatic improvements over the no feedback and the fixed-length feedback settings. In particular the channel dispersion vanishes, and so the backoffs from capacity at finite blocklengths are significantly reduced. Trillingsgaard and Popovski [15] generalized the results for DMCs in [14] to the discrete memoryless multiple access channel (DM-MAC). In it, they used ideas contained in Tan and Kosut [16] and MolavianJazi and Laneman [17] to analyze achievable second-order asymptotics for the DM-MAC. However, only achievability results were provided therein. It was also shown numerically in [15] that variable-length feedback outperforms fixed-length feedback. Achievability and converse bounds under variable-length full-feedback (VLF) and variable-length stop-feedback (VLFS) for the binary erasure channel (BEC) have recently been derived by Devassy et al. [18]. In addition, recently, Trillingsgaard et al. used ideas related to the compound channel [19] to study the 2-user [20] and $K$-user [21] common-message discrete memoryless broadcast channel with stop-feedback. However, the techniques used in both the achievability and converse parts in [18], [20] and [21] are difficult to extend to Gaussian channels. This is because the authors heavily exploit the fact that a relevant set of information densities for discrete channels can be bounded. This, together with Hoeffding’s inequality, allows the authors to control the expectation of the maximum of a set of stopping times to eventually upper bound the average transmission time. The relevant information density terms for Gaussian channels are not bounded. Hence, to study this important class of channels under variable-length feedback, we are required to develop new techniques.

In this paper, we characterize the information-theoretic limits of the AWGN channel and the Gaussian MAC when variable-length feedback is available at the encoder and a non-vanishing error probability is permitted. In particular, we circumvent the problem of the continuous nature of the input and output alphabets by deriving new bounds on the moments of the maximum of a set of stopping times. These techniques may be of independent interest in other problems.

B. Main Contributions

In this paper, we first propose a variable-length feedback model for the single-user AWGN channel. For AWGN channels, we carefully define the expected power constraint so that it is analogous to the definition in the fixed-length feedback setting. In the latter setting, the expected power constraint of a code with blocklength $N$ is defined to be

$$\sum_{n=1}^{N} \mathbb{E}[X_n^2] \leq NP,$$

where $X_n$ is the input to the channel at the $n$-th time slot and $P > 0$ is the admissible power. However, in the variable-length feedback setting, the analogue of $N$, usually denoted as $\tau$, is a stopping time. Hence, one needs to carefully think of how to define the expected power constraint so that we can utilize existing mathematical techniques for analyzing stopping times.

The second contribution of our paper is the derivation of achievability and converse bounds for the single-user AWGN channel with two forms of variable-length feedback—stop-feedback and variable-length feedback with termination (VLFT). These bounds are tight in the first-order terms and thus, we have successfully established the $\varepsilon$-capacity under the two different forms of variable-length feedback. In addition, we show that under the VLFT setting, we can achieve a better second-order coding rate (compared to the stop-feedback setting) due to the use of full-feedback available at the encoder. Note that in [14], to prove the achievability statements, the authors use a stop-feedback code for both settings. As such, they could not distinguish the difference between the performances of stop-feedback and VLFT codes in their achievability considerations. Furthermore, the authors in [14] exploit the discreteness of the DMC to bound a relevant information density term $i(X; Y)$. This is clearly inapplicable to the AWGN channel since its input and output alphabets are uncountable. To overcome this problem, we leverage ideas from Gut [22] to bound the asymptotics of the expected values of relevant stopping times. The converse proof also requires some new arguments. A direct application of the arguments in [14] does not seem to apply directly since, in contrast to the DMC, for the AWGN channel, the mutual information term $I(\hat{X}_n; \hat{Y}_n|V_n = 0)$ [14, Equation (82)] is not upper bounded by the capacity of the channel.

In our third and final contribution, we extend the abovementioned analyses to the Gaussian MAC under two different variable-length settings—stop-feedback and VLFT. We establish the $\varepsilon$-capacity region. We show that under the VLFT setting, we can achieve a larger $\varepsilon$-capacity region compared to the stop-feedback setting. Our achievability proof for the Gaussian MAC with stop-feedback uses some non-standard techniques. We find that Doob’s optional stopping theorem [23, Theorem 10.10], which was used in [14] for the single-user case, is not sufficient to bound the expected blocklength of the code. We need to develop new results based on work on renewal theory by Lai and Siegmund [24] to be able to bound the expected blocklength. The converse proof for the Gaussian MAC borrows some ideas from the weak converse proof in Ozarow’s analysis for the Gaussian MAC with fixed-length feedback [4]. However, our choice of parameters is slightly different. This is to account for the variable-length setting that we study.

C. Paper Organization

The rest of this paper is structured as follows: In Section II, we provide a precise problem setting for the AWGN channel, state the main results and provide self-contained proofs for them. We also explain the novelties of our arguments relative to existing works. In Section III, we do the same for the Gaussian MAC. Auxiliary technical results that are not essential to the main arguments are relegated to the appendices.
II. AWGN CHANNEL WITH VARIABLE-LENGTH FEEDBACK

A. Channel Model, Notation, and Definitions

1) Notation: We use $\log x$ to denote the natural logarithm so information units throughout are in nats. We also define $x^+ = \max(x, 0), x^- = \max(-x, 0)$. The Gaussian capacity and binary entropy functions are respectively defined as

$$C(x) := \frac{1}{2} \log(1 + x), \text{ and }$$

$$h_b(x) := -x \log x - (1 - x) \log(1 - x).$$

Random variables and information-theoretic quantities are standard and mainly follow the text by El Gamal and Kim [25]. We also use asymptotic notation such as $O(\cdot)$ in the standard manner: $f(n) = O(g(n))$ holds if and only if the implied constant $\limsup_{n \to \infty} |f(n)/g(n)| < \infty$.

2) Channel Model: We consider the standard AWGN channel model

$$Y = X + Z$$

where $X$ is the input to the channel, $Z$ is independent and identically distributed Gaussian noise with zero mean and unit variance, and $Y$ is the output of the channel. Thus, for a single-channel use, the channel from $X$ to $Y$ can be written as

$$\mathbb{P}(y|x) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2} (y - x)^2 \right).$$

3) Basic Definitions:

**Definition 1.** An $(M, N, P, \varepsilon)$ stop-feedback code for the AWGN channel, where $P, N$ are positive reals, $M$ is a positive integer, and $0 \leq \varepsilon \leq 1$, is defined by:

1) A space $\mathcal{U}$ and probability distributions $P_U$ on them, defining a random variable $U$ which is revealed to the encoder and the receiver before the start of transmission; i.e., $U$ acts as common randomness used to initialize the encoder and the decoder before the start of transmission.
2) A sequence of encoders $f_n : \mathcal{U} \times \{1, 2, \ldots, M\} \to \mathbb{R}, n \geq 1$, defining channel inputs

$$X_n = f_n(U, W),$$

where $W$ is equiprobable on $\{1, 2, \ldots, M\}$.
3) A sequence of decoders $g_n : \mathcal{U} \times \mathcal{Y}^n \to \{1, 2, \ldots, M\}$ (can be random) providing estimates of $W$ at various times $n$.
4) A non-negative integer-valued random variable $\tau$, a stopping time of the filtration $\{\mathcal{F}_n = \sigma(U, W^n)\}_{n=0}^\infty$, which satisfies

$$\mathbb{E}(\tau) \leq N.$$

5) The expected power constraint at the encoder

$$\sum_{n=1}^\infty \mathbb{E}[X_n^2] \leq \mathbb{E}(\tau) P.$$

The final decision $\hat{W}$ is computed at the stopping time $\tau$

$$\hat{W} = g_\tau(U, Y^\tau)$$

and must satisfy

$$\mathbb{P} [\hat{W} \neq W] \leq \varepsilon.$$

The fundamental limit of channel coding with stop-feedback is given by

$$M^{\ast}_{st}(N, P, \varepsilon) := \max \{ M : \exists \text{ an } (M, N, P, \varepsilon) \text{ stop-feedback code} \}.$$  

**Definition 2.** An $(M, N, P, \varepsilon)$ variable-length feedback with termination (VLFT) code for the AWGN channel, where $P, N$ are positive reals, $M$ is a positive integer, and $0 \leq \varepsilon \leq 1$, is defined as in Definition 1 except that $\tau$ is a stopping time of the filtration $\{\mathcal{F}_n = \sigma(U, W, Y^n)\}_{n=0}^\infty$ and

$$X_n = f_n(U, W, Y^{n-1}).$$

The fundamental limit of channel coding with VLFT feedback is given by

$$M^{\ast}_{st}(N, P, \varepsilon) := \max \{ M : \exists \text{ an } (M, N, P, \varepsilon) \text{ VLFT code} \}.$$  

The VLFT code in Definition 2 is similar to the VLFT code defined in Polyanskiy, Poor and Verdú [14, Definition 2], which includes as a special case the stop-feedback code in Definition 1. However, by making use of the full-feedback that is available at the encoder as in (13), a better achievability result can be obtained in general. Moreover, the converse results for two cases are also slightly different. Therefore, for greater clarity, we separately consider two nested classes of variable-length feedback codes—stop-feedback codes and VLFT codes.
B. Main Results

We now state our main results for the AWGN channel. The achievability proofs of Theorem 1 and 2 can be found in Sections II-C and II-D respectively. The converse proofs for both theorems follow from largely the same lines of arguments and can be found in Section II-E.

**Theorem 1.** For an AWGN channel with stop-feedback code and expected power constraint \( P \), we have for any \( 0 < \varepsilon < 1 \)

\[
\frac{NC(P)}{1 - \varepsilon} - \log N + O(1) \leq \log M^*_d(N, P, \varepsilon) \leq \frac{NC(P) + h_b(\varepsilon)}{1 - \varepsilon}.
\] (15)

**Theorem 2.** For an AWGN channel with VLFT code and expected power constraint \( P \), we have for any \( 0 < \varepsilon < 1 \)

\[
\frac{NC(P)}{1 - \varepsilon} - \log \log N + O(1) \leq \log M^*_v(N, P, \varepsilon) \leq \frac{NC(P) + (N + 1)h_b \left( \frac{1}{N + 1} \right) + h_b(\varepsilon)}{1 - \varepsilon} \leq \frac{NC(P) + \log(N + 1) + h_b(\varepsilon) + 1}{1 - \varepsilon}.
\] (16)

In particular, Theorems 1 and 2 imply that

**Corollary 1.** The \( \varepsilon \)-capacities are

\[
\liminf_{N \to \infty} \frac{1}{N} \log M^*_d(N, P, \varepsilon) = \frac{C(P)}{1 - \varepsilon} \quad \text{and}
\]

\[
\liminf_{N \to \infty} \frac{1}{N} \log M^*_v(N, P, \varepsilon) = \frac{C(P)}{1 - \varepsilon}.
\] (18)

Some remarks are now in order:

- Our achievability result in Theorem 2 is better than the corresponding one in Theorem 1 (in the second-order term). This shows that when full-feedback is available at the encoder (i.e., a VLFT code), a better achievability result can be obtained compared to when only stop-feedback is available. This suggests that a better achievability result compared to [14] may be proved for a DMC given a VLFT code.
- In the achievability proof in [14], Polyanskiy, Poor and Verdú utilize the fact that the relevant information density random variable \( i(X; Y) \) (induced by the capacity-achieving input distribution and the channel) is bounded when the channel is a DMC [14, Equation (107)]. However, this fact does not hold for the AWGN channel and so our achievability proof requires some novel elements. To the best of the authors’ knowledge, all previous works on variable-length feedback for systems with non-vanishing error probabilities [14, 18, 20, 21] involve channels with *discrete* alphabets.
- For the converse proofs of Theorems 1 and 2, although some of the ideas are inspired by [14], we need to augment the original arguments so that the proof is amenable for the AWGN channel. More specifically, in [14], the authors use the fact that the capacity of the DMC is \( \sup_{P_X} I(\hat{X}; \hat{Y}) : P_{\hat{X}}(T = 0) \) where \( T \) is a new symbol appended to the input and output alphabets of the DMC to form \( \hat{X} \) and \( \hat{Y} \) respectively and \( \hat{X} \in \hat{X} \) is the input random variable of the new DMC. However, for the AWGN channel with the expected power constraint in (9), it is not clear that this fact holds.
- In addition, we need to take into account the expected power constraint under the variable-length setting. This is rather different from the fixed-length setting studied in Truong, Fong and Tan [6] since the stopping time \( \tau \) is now a random variable. Also see the discussion of (2) in Section I-B.
- A strict improvement in the first-order coding rate compared to the fixed-length feedback setting can clearly be observed. The first-order term, i.e., the \( \varepsilon \)-capacity, under the fixed-length setting was shown in [6] to be \( C \left( \frac{P}{1 - \varepsilon} \right) \). However, the strict concavity of \( P \mapsto C(P) \) reveals that

\[
C \left( \frac{P}{1 - \varepsilon} \right) < C \left( \frac{P}{1 - \varepsilon} \right), \quad \forall P > 0, \varepsilon \in (0, 1).
\] (20)

This advantage is present because variable-length feedback codes are adaptive, i.e., the blocklength is adapted to the quality of the output sequence \( Y^\infty \).

C. Achievability Proof for Theorem 1

Before the proof, we commence with some technical results in Lemmas 1 and 2. The achievability part of Theorem 1 will be proved via a combination of Lemmas 3 and 4.

**Definition 3** (Strongly nonlattice [26]). We say that a distribution function \( F \) is strongly nonlattice if

\[
\liminf_{|t| \to \infty} |1 - f(t)| > 0,
\] (21)
where
\[ f(t) := \int_{-\infty}^{\infty} e^{itx} \, dF(x) \] (22)
is the characteristic function of F. This can be shown to be equivalent to Cramer’s condition (C):
\[ \limsup_{|t| \to \infty} |f(t)| < 1. \] (23)

The following lemma is adapted from Gut [22, Theorem 2.6].

**Lemma 1** (Asymptotics of Expected Values of Stopping Times). Let \( X_1, X_2, \ldots \) be i.i.d. random variables with positive mean \( \mu = \mathbb{E}[X_1] \), finite variance \( \sigma^2 = \text{Var}(X_1) \) and \( \mathbb{E}[X_1^+ < \infty \). Let \( S_n := X_1 + X_2 + \ldots + X_n \). For each \( b \geq 0 \) define
\[
\tau = \tau(b) = \inf\{n : S_n > b\},
\]
\[
\tau_+ = \tau(0) = \inf\{n : S_n > 0\}.
\]
Assume that \( X_1 \) has a distribution function \( F_{X_1} \) that is strongly non-lattice in the sense of Definition 3. Then as \( b \to \infty \),
\[
\mu \mathbb{E}(\tau) = b + \frac{\mathbb{E}(S_{\tau_+}^2)}{2\mathbb{E}(S_{\tau_+})} + o(1), \quad \text{as} \quad b \to \infty.
\] (26)

**Proof:** In the original statement of [22, Theorem 2.6], it is stated that
\[
\mathbb{E}(\tau) = \frac{b}{\mu} + \mathbb{E}(\tau_+) \frac{\mathbb{E}(S_{\tau_+}^2)}{2(\mathbb{E}(S_{\tau_+}))^2} + o(1).
\] (27)
Now we claim that (i) \( \mathbb{E}(\tau_+) < \infty \) and (ii) \( \mathbb{E}[|X_1|] < \infty \). Claim (i) follows by first noticing that the finiteness of \( \mu = \mathbb{E}[X_1] \) and \( \mathbb{E}[X_1^+ \) implies the same for \( \mathbb{E}[X_1] = \mathbb{E}[X_1^+] - \mathbb{E}[X_1] \). By [22, Theorem 2.1(a)] (with \( r = 1 \) in the statement therein), we have that \( \mathbb{E}(\tau_+) < \infty \). Claim (ii) follows because the finiteness of \( \mathbb{E}[X_1^+] \) and \( \mathbb{E}[X_1^+] \) implies the same for \( \mathbb{E}[|X_1|] = \mathbb{E}[X_1^+] + \mathbb{E}[X_1^+] \).

Equipped with these two claims and [27, Equation (13) in Section 12.5] (also see [28, Problem 22.8(a)]), Wald’s identity
\[
\mu \mathbb{E}(\tau) = \mu \mathbb{E}(S_{\tau_+})
\] (28)
applies. Uniting (27) and (28), we see that Lemma 1 follows. 

The following result can be regarded as a generalization of Wald’s equation (cf. [29]).

**Lemma 2.** Let \( \{X_n\}_{n=1}^{\infty} \) be an infinite sequence of real-valued random variables and let \( \tau \) be a non-negative integer-valued random variable. Assume that
\begin{itemize}
  \item \( \{X_n\}_{n=1}^{\infty} \) are all integrable (finite-mean) random variables,
  \item for all natural numbers \( n \)
  \[ \mathbb{E}[X_n 1\{\tau \geq n\}] = \mathbb{E}[X_n] \mathbb{P}(\tau \geq n) \] (29)
  \item the infinite series satisfies
  \[ \sum_{n=1}^{\infty} \mathbb{E}[|X_n| 1\{\tau \geq n\}] < \infty. \] (30)
\end{itemize}
\item \( \{X_n\}_{n=1}^{\infty} \) all have the same expectation, and
\item \( \tau \) has finite expectation.

Define
\[ S_\tau := \sum_{n=1}^{\tau} X_n. \] (31)
Then, we have
\[ \mathbb{E}[S_\tau] = \mathbb{E}[\tau] \mathbb{E}[X_1]. \] (32)

**Proof:** In Appendix A, we provide an elementary proof which uses only Lebesgue’s monotone and dominated convergence theorems.

**Lemma 3.** For the standard AWGN channel \( \mathbb{P}(y|x) \), there exists a sequence of \((M, N' + o(1), P', \frac{1}{N'})\) stop-feedback codes for any \( M \) satisfying
\[ 0 \leq \log M \leq N'C(P') - \log N' + O(1). \] (33)
Proof: The proof is partly based on [14]. However, we need some additional arguments for the proof to go through for the AWGN channel. For completeness, we provide the entire proof, including some overlap with [14].

First, we show that there exists an \((M, N’ + o(1), P', \frac{1}{N'})\) stop-feedback code with stopping time \(\tau^*\) such that (33) holds and

\[
\mathbb{E}\left[\sum_{n=1}^{\tau^*} X_n^2\right] \leq \mathbb{E}(\tau^*)P'.
\] (34)

To define such a code we need to specify \((U, f_n, g_n)\). First, we define a random variable \(U\) as follows:

\[
\mathcal{U} = \mathbb{R}^\infty \times \mathbb{R}^\infty \times \cdots \times \mathbb{R}^\infty, \quad M \text{ times},
\]

\[
\mathbb{P}_U = (\mathbb{P}_X)^\infty \times (\mathbb{P}_X)^\infty \times \cdots \times (\mathbb{P}_X)^\infty, \quad M \text{ times},
\] (35) (36)

where \(\mathbb{P}_X \sim \mathcal{N}(0, P')\). As [14], the cardinality of \(\mathcal{U}\) can be bounded by 3.

The realization of \(U\) defines \(M\) infinite dimensional vectors \(C_j \in \mathbb{R}^\infty, j = 1, 2, \ldots, M\) which are independently generated and distributed according to \((\mathbb{P}_X)^\infty\) on \(\mathbb{R}^\infty\). Our encoder and decoder will depend on \(U\) implicitly through \(C_j\). The encoding scheme consists of a sequence of encoders \(f_n\) that map a message \(j\) to an infinite sequence of inputs \(C_j \in \mathbb{R}^\infty\) without any regard to feedback

\[
X_n = f_n(w) = (C_w)_n,
\] (37)

where \((C_j)_n\) is the \(n\)-th coordinate of the vector \(C_j\).

At time instant \(n\), the decoder computes \(M\) information densities

\[
S_{j,n} := i(C_j(n); Y^n), \quad j = 1, 2, \ldots, M
\] (38)

where \(C_j(n)\) is the restriction of \(C_j\) to the first \(n\) symbols and

\[
i(C_j(n); Y^n) = \log \frac{\text{d}\mathbb{P}_{X^nY^n}}{\text{d}(\mathbb{P}_{X^n} \times \mathbb{P}_{Y^n})}(C_j(n), Y^n).
\] (39)

The decoder also defines \(M\) stopping times

\[
\tau_j := \inf\{n \geq 0 : S_{j,n} > \gamma\},
\]

for some \(\gamma\) to be determined later. The final decision is made by the decoder at the stopping time \(\tau^*\)

\[
\tau^* := \min_{j=1,2,\ldots,M} \tau_j.
\] (41)

This means that \(\tau^*\) is the time of the first \(\gamma\)-upcrossing among all \(S_{j,n}\) for \(j = 1, 2, \ldots, n\). The output of the decoder is

\[
g(Y^{\tau^*}) = \max\{j : \tau_j = \tau^*\}.
\] (42)

Let \(X^\infty, Y^\infty, \bar{X}^\infty\) be i.i.d. infinite dimensional vectors, each distributed according to

\[
\mathbb{P}_{XY\bar{X}}(x, y, \bar{x}) = \mathbb{P}_X(x)\mathbb{P}(y|x)\mathbb{P}_X(\bar{x}),
\]

where \(\mathbb{P}_X = \mathcal{N}(0, P')\). For this joint distribution, we consider the information density function \(i(x^n; y^n)\) defined as in (39), and hitting times

\[
\tau' := \inf\{n \geq 0 : i(X^n; Y^n) > \gamma\},
\]

\[
\bar{\tau}' := \inf\{n \geq 0 : i(\bar{X}^n; Y^n) > \gamma\}.
\] (44) (45)

Then, the average length of transmission satisfies

\[
\mathbb{E}[\tau^*] \leq \frac{1}{M} \sum_{j=1}^{M} \mathbb{E}[\tau_j|W = j]
\]

\[
= \mathbb{E}[\tau_1|W = 1]
\]

\[
= \mathbb{E}[\tau'].
\] (46) (47) (48)
Analogously, the average error probability of error satisfies

\[ P[g(Y^{\tau}) \neq W] \leq P[g(Y^{\tau}) \neq 1|W = 1] \]
\[ \leq P[\tau_1 \geq \tau^*|W = 1] \]
\[ \leq P\left[ \bigcup_{j=2}^{M} \{\tau_j \leq \tau_1\} \bigg| W = 1 \right] \]
\[ \leq (M - 1)P[\tau_2 \leq \tau_1|W = 1] \]
\[ = (M - 1)P(\tau' \leq \tau^*) \]
\[ \leq (M - 1)\exp(-\gamma). \] (54)

Here, (a) follows from [14, Equation (118)].

To show that

\[ \mathbb{E}(\tau^*) \leq N', \] (55)

we will show that

\[ \mathbb{E}(\tau') \leq N', \] (56)

by choosing \( \gamma \) appropriately and using Lemma 1 for the sequence of i.i.d. random variables \( i(X_1; Y_1), \ldots, i(X_n; Y_n) \). This non-standard step differs from the proof for the DMC case in [14] because the channel is not discrete. Hence, we need to leverage Lemma 1 appropriately to bound \( \mathbb{E}(\tau') \). We note that the following quantities are finite

\[ \mathbb{E}(i(X_1; Y_1)) = I(X_1; Y_1) = C(P') < \infty, \] (57)
\[ \text{Var}(i(X_1; Y_1)) = \frac{P'}{1 + P'} < \infty, \] (58)
\[ \mathbb{E}(i(X_1; Y_1)^+) \leq \mathbb{E}(i(X_1; Y_1)^+ + i(X_1; Y_1)^-) \]
\[ = \mathbb{E}(i(X_1; Y_1)) \] (59)
\[ \leq \sqrt{\mathbb{E}[i(X_1; Y_1)^2]} \] (60)
\[ = \sqrt{\text{Var}(i(X_1; Y_1)) + [\mathbb{E}(i(X_1; Y_1))]^2} \] (61)
\[ = \sqrt{\frac{P'}{1 + P'} + (C(P'))^2} \] (62)
\[ < \infty. \] (63)

In addition, by [30, pp. 207], Cramer’s condition (C) in Definition 3 is satisfied by those distributions having at least a continuous component in its Lebesgue decomposition. Since \( i(X_1; Y_1) \) is a continuous random variable, this condition is, of course, satisfied. Therefore, from Lemma 1 we obtain

\[ C(P')\mathbb{E}(\tau') = \gamma + \frac{ES^2_{\tau^+}}{2ES_{\tau^+}} + o(1), \quad \text{as} \quad \gamma \to \infty, \] (65)

where

\[ \tau_+ = \tau(0) = \inf\{n : i(X^n; Y^n) > 0\}. \] (66)

Now, we choose

\[ \gamma = C(P')N' - \frac{ES^2_{\tau^+}}{2ES_{\tau^+}} \] (67)

then, from (65) and (67) we see that the average length of the stopping time \( \tau' \) of the stop-feedback code satisfies

\[ C(P')\mathbb{E}(\tau') = C(P')N' + o(1), \] (68)

and hence,

\[ \mathbb{E}(\tau') \leq N' + o(1). \] (69)

Now, by choosing

\[ \log M = \gamma - \log N', \] (70)
we obtain from (54) that
\[ \epsilon' \leq \frac{1}{N'} \]  
(71)

Also by (67) and (70) and the fact that \( \frac{BS^2 + \epsilon S^2}{2BS_{\epsilon}} = O(1) \), the size of the code \( M \) satisfies (33).

Finally, we consider the expected power consumption of this coding scheme. We check all the conditions in Lemma 2 (with \( X_n^2 \) here playing the role of \( X_n \) in Lemma 2).

- We have
  \[ \mathbb{E}[X_n^2] = P', \]  
(72)
  so it follows that \( X_n^2 \) are integrable for all \( n \geq 1 \).

- Now, we see that \( 1\{\tau^* \geq n\} = 1 - 1\{\tau^* \leq n - 1\} \in \sigma(X_n^{-1}, Y_n^{-1}) \). Moreover, since the sequence \( X_1, X_2, \ldots \) is i.i.d., \( 1\{\tau^* \geq n\} \) is independent of \( X_n \). It follows that
  \[ \mathbb{E}[X_n^2 1\{\tau^* \geq n\}] = \mathbb{E}[X_n^2 1\{\tau^* \geq n\}] \]
  \[ = \mathbb{E}[X_n^2] \mathbb{P}(\tau^* \geq n). \]  
(73)
(74)

- the infinite series satisfies
  \[ \sum_{n=1}^{\infty} \mathbb{E}[X_n^2 1\{\tau^* \geq n\}] = \sum_{n=1}^{\infty} \mathbb{E}[X_n^2] \mathbb{P}(\tau^* \geq n) \]  
(75)
\[ \leq P' \sum_{n=1}^{\infty} \mathbb{P}(\tau^* \geq n) \]  
(76)
\[ = P' \mathbb{E}(\tau^*) \]  
(77)
\[ \leq P' \mathbb{P}(\tau') \]  
(78)
\[ \leq P' N' < \infty. \]  
(79)

- all random variables \( X_n^2 \) have the same expectation \( P' \).

- \( \mathbb{E}(\tau^*) \leq N' < \infty \).

Hence, by (32), the expected power constraints at the encoder satisfies
\[ \mathbb{E} \left[ \sum_{n=1}^{\tau^*} X_n^2 \right] = \mathbb{E}(\tau^*) \mathbb{E}[X_1^2] \]  
(80)
\[ = \mathbb{E}(\tau^*) P'. \]  
(81)

Finally, observe that if there exists an \((M, N', P', \frac{1}{N'})\) stop-feedback code and stopping time \( \tau^* \) such that (81) holds, then we easily see that there exists another \((M, N', P', \frac{1}{N'})\) stop-feedback code having the same stopping rule for \( \tau^* \), the same sequence of decoders \( \{g_n\}_{n=1}^{\infty} \) but with encoders defined as
\[ \tilde{X}_n := \begin{cases} X_n, & n \leq \tau^* \\ 0, & n > \tau^*. \end{cases} \]  
(82)

Now, for this new stop-feedback code, we have
\[ \sum_{n=1}^{\infty} \mathbb{E}[\tilde{X}_n^2] \overset{(a)}{=} \mathbb{E} \left[ \sum_{n=1}^{\tau^*} X_n^2 \right] \]  
(83)
\[ = \mathbb{E} \left[ \sum_{n=1}^{\tau^*} X_n^2 \right] \]  
(84)
\[ = \mathbb{E}(\tau^*) P'. \]  
(85)
where (a) follows from Lebesgue’s monotone convergence theorem. This completes the proof of Lemma 3.

**Lemma 4.** Given a standard AWGN channel \( \mathbb{P}(y|x) \), there exists a sequence of \((M, N, P, \epsilon)\) stop-feedback codes for any \( M \) satisfying
\[ 0 \leq \log M \leq \frac{NC(P)}{1 - \epsilon} - \log N + O(1). \]  
(86)

**Proof:** We use the power control ideas combined with Lemma 3. First, we show that there exists an \((M, N + o(1), P, \epsilon)\) stop-feedback code for any \( M \) satisfying (86).
• The decoder chooses numbers \( N', P' \) such that
\[
\frac{(N')^2(1 - \varepsilon)}{N' - 1} = N, \tag{87}
\]
\[
P' = P. \tag{88}
\]

• The decoder generates a Bernoulli random variable \( D \sim \text{Bern}(p) \), where
\[
p = \frac{N'\varepsilon - 1}{N' - 1}. \tag{89}
\]

• If \( D = 1 \), the decoder sends a stop-feedback signal (or a NACK) to the encoder via the feedback link. This means that \( \tau = 0 \).

• If \( D = 0 \), the encoder starts to send the intended message to the decoder using the \((M, N' + o(1), P', \frac{1}{N'})\) stop-feedback code mentioned in Lemma 3 and stops at time \( \tau' \). This means that \( \tau = \tau' \).

Note that with this coding scheme, the expected length of the combined scheme is
\[
\mathbb{E}(\tau) = \left(1 - \frac{N'\varepsilon - 1}{N' - 1}\right) \mathbb{E}(\tau') \tag{90}
\]
\[
= \frac{N'(1 - \varepsilon)}{N' - 1} \mathbb{E}(\tau') + o(1) \frac{N'(1 - \varepsilon)}{N' - 1} \tag{91}
\]
\[
\leq N + o(1) \frac{N'(1 - \varepsilon)}{N' - 1} \tag{92}
\]
\[
\leq N + o(1). \tag{93}
\]

The expected average error probability of the combined scheme is upper bounded by
\[
\frac{N'\varepsilon - 1}{N' - 1} + \left(1 - \frac{N'\varepsilon - 1}{N' - 1}\right) \frac{1}{N'} = \varepsilon. \tag{94}
\]

The average power consumption is
\[
\left(1 - \frac{N'\varepsilon - 1}{N' - 1}\right) \mathbb{E}(\tau') P' = \frac{N'(1 - \varepsilon)}{N' - 1} \mathbb{E}(\tau') P' \tag{95}
\]
\[
= \mathbb{E}(\tau) P', \tag{96}
\]
\[
= \mathbb{E}(\tau) P. \tag{97}
\]

Here, the last equality follows from (88).

By Lemma 3, we have
\[
\log M = N'C(P') - \log N' + O(1) \tag{98}
\]
\[
= \frac{NC(P)}{1 - \varepsilon} - \log N + O(1). \tag{99}
\]

Here, the last equality follows from (87) and (88).

Finally, we see that if there exists a \((M, N + o(1), P, \varepsilon)\) stop-feedback code for any \( M \) satisfying (86), then we can find a \((M, N, P, \varepsilon)\) stop-feedback code for any \( M \) satisfying (86) by replacing \( N \) by \( N - o(1) \). That concludes our proof. \( \blacksquare \)

D. Achievability Proof for Theorem 2

**Lemma 5.** Given a standard AWGN channel \( \mathbb{P}(y|x) \), there exists an \((M, N, P, \varepsilon)\) VLFT code for any \( M \) satisfying
\[
0 \leq \log M \leq \frac{NC(P)}{1 - \varepsilon} - \log N + O(1). \tag{100}
\]

First, we note from [2] that the Schalkwijk-Kailath (S-K) coding scheme with a fixed blocklength \( N' \) (a natural number), expected power constraints \( P' \), and number of messages \( M \) satisfying
\[
\log M = N'C(P') - \frac{1}{2} \log \left[ \frac{2}{P'} \log N' \right], \tag{101}
\]
has an average error probability upper bounded by
\[
\varepsilon' \leq \sqrt{\frac{2}{\pi}} \exp \left( - \frac{P'}{2} \cdot e^{2N'[C(P') - \log P']/N} \right) \tag{102}
\]
\[
= \sqrt{\frac{2}{\pi}} \frac{1}{N'} \leq \frac{1}{N'}. \tag{103}
\]
As such, we can construct a VLFT coding scheme with stopping time \( \tau \) as follows:

- The decoder chooses the largest natural number \( N' \) such that
  \[
  \frac{(N')^2(1 - \varepsilon)}{N' - 1} \leq N
  \]
  is satisfied and a positive number \( P' \) as in (88).
- The decoder generates a Bernoulli random variable \( D \sim \text{Bern}(p) \), where \( p \) is as in (89).
- If \( D = 1 \), the decoder sends a stop-signal (or a NACK) to the encoder via the feedback link. This means that \( \tau = 0 \).
- If \( D = 0 \), the encoder starts to send the intended message to the decoder using the S-K coding scheme mentioned above with parameters \((M, N', P', \frac{1}{N'M})\). The expected power \( P' = P \) and transmission stops at time \( \tau' = N' \). This means that \( \tau = N' \).

The average length of the proposed VLFT code is upper bounded by

\[
\mathbb{E}(\tau) = \left(1 - \frac{N'\varepsilon - 1}{N' - 1}\right) \mathbb{E}(\tau') = \left(1 - \frac{N'\varepsilon - 1}{N' - 1}\right) N' = \frac{(N')^2(1 - \varepsilon)}{N' - 1} = N.
\]

Similarly to the stop-feedback case, the error probability of the proposed VLFT code is upper bounded by (94). The expected power of the combined scheme also satisfies (95)–(97). It follows from (101), (87), and (88) that

\[
\log M = \frac{NC(P)}{1 - \varepsilon} - \log \log N + O(1),
\]

completing the proof of the achievability part of Theorem 2.

\section*{E. Converse Proofs of Theorems 1 and 2}

\begin{lemma}
Given a standard AWGN channel \( \mathbb{P}(y|x) \), we have for any \( 0 < \varepsilon < 1 \)
\[
\log M^*_a(N, P, \varepsilon) \leq \frac{NC(P) + hb(\varepsilon)}{1 - \varepsilon},
\]
\[
\log M^*_i(N, P, \varepsilon) \leq \frac{NC(P) + (N + 1)hb \left(\frac{1}{N+1}\right) + hb(\varepsilon)}{1 - \varepsilon} \leq \frac{NC(P) + \log(N + 1) + hb(\varepsilon) + 1}{1 - \varepsilon}.
\]
\end{lemma}

\textbf{Proof:} Initially, we assume that the code is deterministic and \( |\mathcal{U}| = 1 \). Now, consider a triplet \((f_n, g_n, \tau)\) defining a given code. For a VLFT code, \( \tau \) is a stopping time of the filtration \( \{\sigma(W, Y^n)\}_{n=0}^\infty \). Now, we append a special symbol \( T \notin \mathbb{R} \) to the input and output alphabets \((\mathbb{R}, \mathbb{R})\) and create an extended channel as follows

\[
\hat{Y} = \begin{cases} 
\hat{X} + N'(0,1), & \hat{X} \in \mathbb{R} \\
\hat{X}, & \hat{X} = T
\end{cases}.
\]

Next, we convert the given VLFT code \((f_n, g_n, \tau)\) to a new code \((\hat{f}_n, \hat{g}_n, \hat{\tau})\) for the extended channel as follows: The new encoder is

\[
\hat{X}_n = \hat{f}_n(W) = \begin{cases} 
 f_n(W, \hat{Y}^{n-1}), & \tau \geq n \\
 T, & \tau < n
\end{cases}.
\]

the new stopping time is

\[
\hat{\tau} = \tau + 1 = \inf\{n : \hat{Y}_n = T\}.
\]

and the new decoder is

\[
\hat{g}_n(\hat{Y}^n) = \begin{cases} 
g_n(\hat{Y}^n), & \hat{\tau} > n \\
g_n(\hat{Y}^{\hat{\tau}-1}), & \hat{\tau} \leq n
\end{cases}.
\]

By using the same arguments as in \cite{14}, we arrive at the following bound (cf. \cite[Equation (68)]{14})

\[
(1 - \varepsilon) \log M \leq I(W; \hat{Y}^\infty) + hb(\varepsilon).
\]
Now, we note that

\[
I(W; \hat{Y}^\infty) = \sum_{n=1}^\infty I(W; \hat{Y}_n | \hat{Y}^{n-1}). \tag{117}
\]

To bound \( I(W; \hat{Y}_n | \hat{Y}^{n-1}) \), we use a similar method as that in [14]. However, we need to employ new techniques to take into account the expected power constraints for the AWGN channel. These steps are non-standard.

Define

\[
V_n := 1\{\hat{r} \leq n\} \in \sigma(\hat{Y}^n), \tag{118}
\]

then we have

\[
I(W; \hat{Y}_n | \hat{Y}^{n-1}) = I(W; \hat{Y}_n V_n | \hat{Y}^{n-1}) \tag{119}
\]

\[
= I(W; V_n | \hat{Y}^{n-1}) + I(W; \hat{Y}_n | V_n, \hat{Y}^{n-1}) \tag{120}
\]

\[
\leq H(V_n | \hat{Y}^{n-1}) + I(W; \hat{Y}_n | V_n, \hat{Y}^{n-1}) \tag{121}
\]

\[
\leq H(V_n | \hat{Y}^{n-1}) + \mathbb{P}(V_n = 0)I(\hat{X}_n; \hat{Y}_n | V_n = 0, \hat{Y}^{n-1}) + \mathbb{P}(V_n = 1)I(\hat{X}_n; \hat{Y}_n | V_n = 1, \hat{Y}^{n-1}) \tag{122}
\]

\[
\leq H(V_n | \hat{Y}^{n-1}) + \mathbb{P}(V_n = 0)I(\hat{X}_n; \hat{Y}_n | V_n = 0, \hat{Y}^{n-1}) \tag{123}
\]

\[
\leq H(V_n | \hat{Y}^{n-1}) + \mathbb{P}(V_n = 0)I(\hat{X}_n; \hat{Y}_n | V_n = 0, \hat{Y}^{n-1}) \tag{124}
\]

\[
\leq H(V_n | \hat{Y}^{n-1}) + \mathbb{P}(V_n = 0)I(\hat{X}_n; \hat{Y}_n | V_n = 0, \hat{Y}^{n-1}) \tag{125}
\]

\[
= H(V_n | \hat{Y}^{n-1}) + \mathbb{P}(V_n = 0)\left[h(Y_n | V_n = 0, Y^{n-1}) - h(Y_n | X_n, V_n = 0, Y^{n-1})\right] \tag{126}
\]

\[
= H(V_n | \hat{Y}^{n-1}) + \mathbb{P}(V_n = 0)\left[h(Y_n | V_n = 0, Y^{n-1}) - h(X_n + Z_n | X_n, V_n = 0, Y^{n-1})\right] \tag{127}
\]

\[
\leq H(V_n | \hat{Y}^{n-1}) + \mathbb{P}(V_n = 0)\left[h(Y_n | V_n = 0, Y^{n-1}) - h(Z_n)\right] \tag{128}
\]

\[
\leq H(V_n | \hat{Y}^{n-1}) + \mathbb{P}(V_n = 0)\left[h(Y_n | V_n = 0) - h(Z_n)\right] \tag{129}
\]

\[
\leq H(V_n | \hat{Y}^{n-1}) + \mathbb{P}(V_n = 0)\left[\frac{1}{2} \log \left(2\pi e (\mathbb{E}[X_n^2 | V_n = 0] + 1)\right) - \frac{1}{2} \log(2\pi e)\right] \tag{130}
\]

\[
= H(V_n | \hat{Y}^{n-1}) + \frac{1}{2} \mathbb{P}(V_n = 0) \log \left(1 + \mathbb{E}[X_n^2 | V_n = 0]\right). \tag{131}
\]

Here, (a) follows from the fact that \( V_n \) is a discrete random variable taking values in \( \{0, 1\} \), (b) follows from the Markov chain \( W - \hat{X}_n - \hat{Y}_n \) which holds almost surely when conditioned on \( \hat{Y}^{n-1} \), (c) follows from the fact that \( \hat{X}_n = \hat{Y}_n = T \) when \( V_n = 1 \) or \( n \geq \hat{r} \), (d) follows from the fact that for \( V_n = 0 \) or \( \tau \geq n \) we have \( \hat{Y}_k = Y_k, \hat{X}_k = X_k, 1 \leq k \leq n \), (e) follows from the fact that \( Z_n \) is independent of \( W, X_n, Y^{n-1} \) and \( V_n = 1 \{\hat{r} \leq n\} = 1 \{\tau \leq n - 1\} \) is a function of \( (W, Y^{n-1}) \), (f) follows from the fact that \( \mathbb{E}[Y_n^2 | V_n = 0] = \mathbb{E}[X_n^2 | V_n = 0] + 1 \).

From (116), (117) and (131) and the fact that \( I(W; \hat{Y}_n | Y^{n-1}) \geq 0 \), we obtain

\[
(1 - \varepsilon) \log M \leq \sum_{n=1}^\infty \left( H(V_n | \hat{Y}^{n-1}) + \frac{1}{2} \mathbb{P}(V_n = 0) \log \left(1 + \mathbb{E}[X_n^2 | V_n = 0]\right)\right) + h_b(\varepsilon). \tag{132}
\]

Now, we observe that

\[
\sum_{n=1}^\infty \mathbb{P}(V_n = 0) = \sum_{n=1}^\infty \mathbb{P}(\tau \geq n) \tag{133}
\]

\[
= \sum_{n=1}^\infty \mathbb{P}(\tau \geq n) \tag{134}
\]

\[
= \mathbb{E}(\tau). \tag{135}
\]

It follows that

\[
\sum_{n=1}^\infty \frac{\mathbb{P}(V_n = 0)}{\mathbb{E}(\tau)} = 1, \tag{136}
\]
so \( \left\{ \frac{P(V_n = 0)}{E(\tau)} \right\}_{n=1}^{\infty} \) is a probability distribution. Moreover, since the function \( f(x) = \log(1 + x) \) is concave, we have that

\[
\sum_{n=1}^{\infty} \frac{1}{2} P(V_n = 0) \log \left( 1 + \mathbb{E}[X_n^2 | V_n = 0] \right) = \sum_{n=1}^{\infty} \frac{E(\tau) P(V_n = 0)}{2 E(\tau)} \log \left( 1 + \mathbb{E}[X_n^2 | V_n = 0] \right)
\]

\[
\leq \frac{E(\tau)}{2} \log \left( 1 + \sum_{n=1}^{\infty} \frac{P(V_n = 0)}{E(\tau)} \mathbb{E}[X_n^2 | V_n = 0] \right)
\]

\[
\leq \frac{N}{2} \log \left( 1 + \frac{1}{E(\tau)} \sum_{n=1}^{\infty} \mathbb{E}[X_n^2] \right)
\]

\[
\leq \frac{N}{2} \log \left( 1 + \frac{P E(\tau)}{E(\tau)} \right)
\]

\[
= \frac{N}{2} \log (1 + P).
\]

Here, (a) follows from the fact that

\[
\mathbb{E}[X_n^2] = P(V_n = 0) \mathbb{E}[X_n^2 | V_n = 0] + P(V_n = 1) \mathbb{E}[X_n^2 | V_n = 1]
\]

\[
\geq P(V_n = 0) \mathbb{E}[X_n^2 | V_n = 0],
\]

and (b) follows from the power constraint of the stop-feedback code in (9).

Using the same arguments as [14, Equation (90)] we obtain

\[
\sum_{n=1}^{\infty} H(V_n | \hat{Y}^{n-1}) = H(\tau)
\]

\[
\leq (N + 1) h_b \left( \frac{1}{N + 1} \right).
\]

It follows from (132), (142), and (146) that

\[
(1 - \varepsilon) \log M \leq (N + 1) h_b \left( \frac{1}{N + 1} \right) + N C(P) + h_b(\varepsilon)
\]

\[
\leq N C(P) + \log(N + 1) + 1 + h_b(\varepsilon).
\]

For stop-feedback codes, we use same steps as above except that we set \( \hat{\tau} = \tau = \inf \{ n : Y_n = T \} - 1 \) and \( V_n = 1 \{ \hat{\tau} \leq n - 1 \} \), which is a function of \( \hat{Y}^{n-1} \). It follows that

\[
H(V_n | \hat{Y}^{n-1}) = 0 \ \forall n \geq 1,
\]

hence from (132) and (142) we obtain

\[
(1 - \varepsilon) \log M \leq N C(P) + h_b(\varepsilon).
\]

For the case \( |U| > 1 \), using the same arguments as [14], we have almost surely

\[
(1 - P(W \neq W|U)) \log M \leq C(P) E(\tau|U) + H_{\sigma(U)}(\tau) + h_b(P|W \neq \hat{W}|U)).
\]

Taking the expectation with respect to \( U \) on both sides of (151) and applying Jensen’s inequality to the concave function \( h_b(\cdot) \), we obtain (109)–(111).

**III. GAUSSIAN MAC CHANNEL WITH VARIABLE-LENGTH FEEDBACK**

**A. Channel Model and Definitions**

1) **Channel Model:** The channel model is given by

\[
Y = X_1 + X_2 + Z,
\]

where \( X_1 \) and \( X_2 \) represent the inputs to the channel, \( Z \sim \mathcal{N}(0, 1) \) is additive Gaussian noise with zero mean and unit variance, and \( Y \) is the output of the channel. Thus, the channel from \( (X_1, X_2) \) to \( Y \) can be written as

\[
P(y|x_1, x_2) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{1}{2}(y - x_1 - x_2)^2 \right).
\]
2) Basic Definitions:

**Definition 4.** An \((M_1, M_2, N, P_1, P_2, \varepsilon)\) stop-feedback code for the Gaussian MAC \(\mathbb{P}(y|x_1, x_2)\), where \(N, P_1, P_2\) are positive real, \(M_1, M_2\) are positive integers, and \(0 \leq \varepsilon \leq 1\), is defined by:

1) Two spaces \(U_1, U_2\) and probability distributions \(P_{U_1}, P_{U_2}\) on them, defining independent random variables \(U_j, j = 1, 2\) each of which is revealed to transmitter \(j = 1, 2\) and the receiver before the start of transmission; i.e., \((U_1, U_2)\) acts as common randomness used to initialize the encoders and the decoder before the start of transmission.

2) Two sequences of encoders \(f_n^{(1)} : U_1 \times \{1, 2, \ldots, M_1\} \rightarrow \mathbb{R}, n \geq 1\) and \(f_n^{(2)} : U_2 \times \{1, 2, \ldots, M_2\} \rightarrow \mathbb{R}, n \geq 1\), defining channel inputs

\[
X_{jn} = f_n^{(j)}(U_j, W_j), \quad j = 1, 2 
\]

where \(W_j\) is equiprobable on the message set \(\{1, 2, \ldots, M_j\}\) for \(j = 1, 2\).

3) A sequence of decoders \(g_n : \mathcal{U}_1 \times \mathcal{U}_2 \times \mathbb{R} \rightarrow \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\}\) (can be random) providing estimates \((W_1, W_2)\) at various times \(n\).

4) A non-negative integer-valued random variable \(\tau\), a stopping time of the filtration \(\{\sigma(U_1, U_2, Y^n)\}\) for which satisfies

\[
E(\tau) \leq N. 
\]

5) The expected power constraints at the encoders

\[
\sum_{n=1}^{\infty} E[X_{jn}^2] \leq E(\tau)P_j, \quad j = 1, 2. 
\]

The final decision \((\hat{W}_1, \hat{W}_2)\) is computed at time \(\tau\)

\[
(\hat{W}_1, \hat{W}_2) = g_\tau(U_1, U_2, Y^\tau) 
\]

and must satisfy

\[
\mathbb{P}[(\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)] \leq \varepsilon. 
\]

**Definition 5.** An \((M_1, M_2, N, P_1, P_2, \varepsilon)\) variable-length feedback with termination code (VLFT), where \(N, P_1, P_2\) are positive real, \(M_1, M_2\) are positive integers, and \(0 \leq \varepsilon \leq 1\), is defined as Definition 4 except that \(\tau\) is a stopping time of the filtration \(\{\sigma(U_1, U_2, W_1, W_2, Y^n)\}\) and

\[
X_{jn} = f_n^{(j)}(U_j, W_j, Y^{n-1}), \quad j = 1, 2. 
\]

B. Main Results

We now state our main results for the Gaussian MAC under various forms of variable-length feedback. The achievability parts of Theorems 3 and 4 are provided in Sections III-C and III-D respectively. The converse parts of Theorems 3 and 4 are provided in Sections III-E and III-F respectively.

**Theorem 3.** For the Gaussian MAC \(\mathbb{P}(y|x_1, x_2)\), there exists a sequence of \((M_1, M_2, N, P_1, P_2, \varepsilon)\) stop-feedback codes for any \((M_1, M_2)\) satisfying

\[
0 \leq \log M_j \leq \left(\frac{N}{1-\varepsilon} - A\sqrt{\frac{N}{1-\varepsilon}}\right) C(P_j) - \log N + O(1), \quad j = 1, 2 
\]

\[
0 \leq \log M_1M_2 \leq \left(\frac{N}{1-\varepsilon} - A\sqrt{\frac{N}{1-\varepsilon}}\right) C(P_1 + P_2) - \log N + O(1). 
\]

where \(A \geq 0\) is a constant given as

\[
A := \min_{(i,j,k) \in \text{perm}[3]} \frac{1}{2}\left(\sqrt{2(L_i + L_j) + \sqrt{4L_k}} + \frac{1}{4}\left(\sqrt{2(L_i + L_j) + \sqrt{2(L_j + L_k)}}\right)^2\right). 
\]

Here, we define \(\text{perm}[3]\) as the set of all permutations of the tuple \((1, 2, 3)\) and

\[
L_j := \frac{4P_1}{(1 + P_j)[\log(1 + P_j)]^2}, \quad j = 1, 2 \quad (163)
\]

\[
L_3 := \frac{4(P_1 + P_2)}{(1 + P_1 + P_2)[\log(1 + P_1 + P_2)]^2}. 
\]
Conversely, given any \( (M_1, M_2, N, P_1, P_2, \varepsilon) \) stop-feedback code, the following inequalities hold

\[
0 \leq \log M_j \leq \frac{NC(P_j) + h_b(\varepsilon)}{1 - \varepsilon}, \quad j = 1, 2
\]  \hspace{1cm} (165)

\[
0 \leq \log M_1 M_2 \leq \frac{NC(P_1 + P_2) + h_b(\varepsilon)}{1 - \varepsilon}.
\]  \hspace{1cm} (166)

**Theorem 4.** Given a Gaussian MAC, for any \( \rho \in [0, 1] \), there exist a sequence of \( (M_1, M_2, N, P_1, P_2, \varepsilon) \) VLFT-feedback codes for any \( M_1, M_2 \) satisfying

\[
0 \leq \log M_j \leq \left( \frac{N}{1 - \varepsilon} \right) C(P_j (1 - \rho^2)) - \log \log N + O(1), \quad j = 1, 2
\]  \hspace{1cm} (167)

\[
0 \leq \log M_1 M_2 \leq \left( \frac{N}{1 - \varepsilon} \right) C(P_1 + P_2 + 2\rho\sqrt{P_1 P_2}) - \log \log N + O(1).
\]  \hspace{1cm} (168)

Conversely, for any \( (M_1, M_2, N, P_1, P_2, \varepsilon) \)-VLFT feedback code for the Gaussian MAC, then the following inequalities hold for some \( \rho \in [0, 1] \)

\[
0 \leq \log M_j \leq \frac{NC(P_j (1 - \rho^2)) + (N + 1) h_b \left( \frac{1}{N + 1} \right) + h_b(\varepsilon)}{1 - \varepsilon},
\]  \hspace{1cm} (169)

\[
\leq \frac{NC(P_j (1 - \rho^2)) + \log(N + 1) + h_b(\varepsilon) + 1}{1 - \varepsilon}, \quad j = 1, 2
\]  \hspace{1cm} (170)

\[
0 \leq \log M_1 M_2 \leq \frac{NC(P_1 + P_2 + 2\rho\sqrt{P_1 P_2}) + (N + 1) h_b \left( \frac{1}{N + 1} \right) + h_b(\varepsilon)}{1 - \varepsilon},
\]  \hspace{1cm} (171)

\[
\leq \frac{NC(P_1 + P_2 + 2\rho\sqrt{P_1 P_2}) + \log(N + 1) + h_b(\varepsilon) + 1}{1 - \varepsilon}.
\]  \hspace{1cm} (172)

We define the \( \varepsilon \)-capacity region of a Gaussian MAC under the stop-feedback (resp. VLFT) formalisms \( C_{sf}(P_1, P_2, \varepsilon) \) (resp. \( C_{v}(P_1, P_2, \varepsilon) \)) to be the closure of the set of all rate pairs \( (R_1, R_2) \) such that there exists a sequence of \( (M_1, M_2, N, P_1, P_2, \varepsilon) \) stop-feedback codes (resp. VLFT codes) such that

\[
\lim \inf_{N \to \infty} \frac{1}{N} \log \log M_j \geq R_j, \quad j = 1, 2
\]  \hspace{1cm} (173)

and also that (158) holds. Theorems 3 and 4 imply the following corollary.

**Corollary 2.** Let \( 0 < \varepsilon < 1 \). The \( \varepsilon \)-capacity region \( C_{sf}(P_1, P_2, \varepsilon) \) is the set of all \( (R_1, R_2) \in \mathbb{R}^2_+ \) satisfying

\[
R_j \leq \frac{C(P_j)}{1 - \varepsilon}, \quad j = 1, 2
\]  \hspace{1cm} (174)

\[
R_1 + R_2 \leq \frac{C(P_1 + P_2)}{1 - \varepsilon}.
\]  \hspace{1cm} (175)

Similarly, the \( \varepsilon \)-capacity region \( C_{v}(P_1, P_2, \varepsilon) \) is the set of all \( (R_1, R_2) \in \mathbb{R}^2_+ \) satisfying

\[
R_j \leq \frac{C(P_j (1 - \rho^2))}{1 - \varepsilon}, \quad j = 1, 2
\]  \hspace{1cm} (176)

\[
R_1 + R_2 \leq \frac{C(P_1 + P_2 + 2\rho\sqrt{P_1 P_2})}{1 - \varepsilon}.
\]  \hspace{1cm} (177)

for some \( \rho \in [0, 1] \).

Some remarks are now in order:

- Trillingsgaard and Popovski [15] generalized the single-user variable-length results for the DMC in Polyanskiy, Poor and Verdù [14] to the DM-MAC. In it, they used ideas contained in Tan and Kosut [16] and MolavianJazi and Laneman [17] to analyze achievable second-order asymptotics for the DM-MAC with variable-length feedback. However, Trillingsgaard and Popovski [15] could not analytically bound the expectation of the maximum of several relevant stopping times \( \mathbb{E}(\max_{k \leq T_k} \tau_k) \) and they also could not prove a matching converse. Instead, they provided numerical results to show that stop-feedback can increase the first-order coding rate compared to the fixed-length feedback setting.

- Inequalities (176)–(177) in Corollary 2 describe the \( \varepsilon \)-capacity region using VLFT codes. This region is easily seen to be larger than the corresponding \( \varepsilon \)-capacity region for fixed-length feedback codes studied by Truong, Fong and Tan [6]. This is a consequence of the simple inequality in (20). In our proof of achievability part for Theorem 3, we provide a simple method to upper bound the important quantity \( \mathbb{E}(\max_{k \leq T_k} \tau_k) \) for the Gaussian MAC, leading to tight first-order terms in Corollary 2. These non-standard arguments are mainly contained in Lemma 8 to follow. We are unable to do the
same for the DM-MAC due to a technical requirement of Lemma 7 (to follow) that requires relevant information density random variables to be nonlattice.

- The $\varepsilon$-capacity region $C_1(P_1, P_2, \varepsilon)$ is strictly larger than $C_{sf}(P_1, P_2, \varepsilon)$, which clearly illustrates the fact that full-feedback at encoders can help to enlarge the (first-order) $\varepsilon$-capacity region compared to the case where only stop-feedback is available. Contrast this to the point-to-point scenario where it is shown in Corollary 1 that the $\varepsilon$-capacities are the same under both settings but only the achievable second-order term using VLFT codes is superior to that using stop-feedback codes (cf. the lower bounds in (15) and (16)). That $C_1(P_1, P_2, \varepsilon)$ is strictly larger than $C_{sf}(P_1, P_2, \varepsilon)$ is completely analogous to the fact that fixed-length full-feedback enlarges the capacity region of the Gaussian MAC (cf. Ozarow [4]).
- For the converse proof for the VLFT formalism in Theorem 4, we borrow some ideas from Ozarow’s weak converse proof for the Gaussian MAC with fixed-length full-feedback [4]. However, our parameter settings and the manipulations of the resultant bounds are slightly different from Ozarow. See (382) and (383) in Lemma 13.

C. Achievability Proof for Theorem 3

To show the achievability result for Theorem 3 in (160) and (161), we first introduce some lemmas. The achievability result for Theorem 3 follows from a combination of Lemmas 9 and 10 to follow. The first technical lemma we state is a result from Lai and Siegmund [24, Theorem 5].

**Lemma 7** (Asymptotics of Variance of Stopping Times). Let $X_1, X_2, \ldots$ be independent identically distributed random variables with positive mean $\mu$ and finite variance $\sigma^2$ and $\mathbb{E}(X_i^+) < \infty$. Let $S_n := X_1 + X_2 + \ldots + X_n$. For each $b \geq 0$ define $\tau$ and $\tau_+$ as in (24) and (25). Under the condition that $X_1$ has a distribution function that is strongly nonlattice (refer to Definition 3), as $b \to \infty$,

$$\text{Var}(\tau) = \mu^{-3}\sigma^2b + \mu^{-2}K + o(1),$$

(178)

where $K$ is a constant that does not depend on $b$ and given by

$$K = \frac{\sigma^2\mathbb{E}S_{\tau_+}^2}{2\mu\mathbb{E}S_{\tau_+}} + \frac{3}{4} \left( \frac{\mathbb{E}S_{\tau_+}^2}{\mathbb{E}S_{\tau_+}} \right)^2 - \frac{2}{3} \left( \frac{\mathbb{E}S_{\tau_+}^3}{\mathbb{E}S_{\tau_+}} \right) - \frac{\mathbb{E}S_{\tau_+}^2}{\mathbb{E}S_{\tau_+}} \mathbb{E}\{\min S_n\} - 2 \int_0^\infty \mathbb{E}\{S_{\tau(x)} - x\} \mathbb{P}\{\min S_n \leq -x\} \, dx.$$  

(179)

The second important technical lemma will eventually allow us to control the asymptotic behavior of the expectation of the maximum of several stopping times.

**Lemma 8** (Asymptotics of the Expectation of the Maximum of Sequences of Random Variables). Let $\{(X_{1N}, X_{2N}, X_{3N})\}_{N \geq 1}$ be three sequences of random variables satisfying

$$\mathbb{E}[X_{jN}] = N - A\sqrt{N} - D - B_j + o(1), \quad j = 1, 2, 3.$$  

(180)

for some constants $A \geq 0$, $D \geq 0$ and $B_1, B_2, B_3 \in \mathbb{R}$ and

$$\text{Var}(X_{jN}) \leq L_jN + S_j + o(1), \quad j = 1, 2, 3.$$  

(181)

for some constants $L_1 > 0$, $L_2 > 0$, $L_3 > 0$ and $S_1, S_2, S_3 \in \mathbb{R}$. Here $o(1)$ denotes an arbitrary sequence that vanishes as $N \to \infty$. Then for $A$ as given in (162), and

$$(i_0, j_0, k_0) := \arg \min_{(i, j, k) \in \text{perm}[3]} \frac{1}{2} \sqrt{2(L_i + L_j)} + \frac{1}{4} \left( \sqrt{2(L_i + L_k)} + \sqrt{2(L_j + L_k)} \right),$$

(182)

$$D := \frac{1}{4} (B_{i_0} + B_{j_0} + 2B_{k_0}) + \frac{1}{2} \left( \sqrt{2|S_{i_0} + S_{j_0}| + (B_{i_0} - B_{j_0})^2} \right.$$

$$\left. + \frac{1}{4} \left( \sqrt{2|S_{i_0} + S_{k_0}| + (B_{i_0} - B_{k_0})^2} + \sqrt{2|S_{j_0} + S_{k_0}| + (B_{j_0} - B_{k_0})^2} \right), \right.$$  

(183)

we have

$$\mathbb{E}(\max\{X_{1N}, X_{2N}, X_{3N}\}) \leq N + o(1).$$  

(184)

**Proof:** The proof of this result is deferred to Appendix B.

**Lemma 9.** Consider a standard Gaussian MAC $\mathbb{P}(y|x_1, x_2)$ with expected power constraints $P_1, P_2$. For any $N' > 0$, and $(M_1, M_2)$ satisfying

$$0 \leq \log M_j \leq (N' - A\sqrt{N'}) C(P_j) - \log N' + O(1), \quad j = 1, 2,$$

(185)

$$0 \leq \log M_1 M_2 \leq (N' - A\sqrt{N'}) C(P_1 + P_2) - \log N' + O(1),$$

(186)
we can find an \((M_1, M_2, N' + o(1), \frac{1}{M'})\) stop-feedback code with \(A\) defined as in (162).

**Proof:** Part of the proof is based on [14] and [15]. First, we show that there exists an \((M_1, M_2, N' + o(1), P_1, P_2, \frac{1}{M'})\) stop-feedback code with stopping time \(\tau^*\), where
\[
\mathbb{E}(\tau^*) \leq N' + o(1),
\]
the sizes of the message sets \(M_1, M_2\) satisfy (185) and (186), and finally,
\[
\mathbb{E}\left[ \sum_{n=1}^{\tau^*} X_{jn}^2 \right] = \mathbb{E}(\tau^*) P_j, \quad j = 1, 2.
\]
Indeed, to define this code, we need to specify \((U_1, U_2)\). We define two random variables \(U_1, U_2\) with alphabets and distributions
\[
U_j := \mathbb{R}^\infty \times \mathbb{R}^\infty \times \cdots \times \mathbb{R}^\infty, \quad (M_j \text{ times})
\]
\[
\mathbb{P}_{U_j} := (\mathbb{P}_{X_j})^\infty \times (\mathbb{P}_{X_j})^\infty \times \cdots \times (\mathbb{P}_{X_j})^\infty, \quad (M_j \text{ times}),
\]
where \(j = 1, 2\) and \(\mathbb{P}_{X_j} \sim \mathcal{N}(0, P_j)\).

We generate the codebook as follows. For a realization of \(U_1\), we generate \(M_1\) i.i.d. infinite dimensional vectors \(\{C_j^{(1)}\}\) from the distribution \(\mathbb{P}_{X_j} \sim \mathcal{N}(0, P_1)\). Similarly, for each realization of \(U_2\), we generate \(M_2\) i.i.d. infinite dimensional vectors \(\{C_k^{(2)}\}\) from the distribution \(\mathbb{P}_{X_j} \sim \mathcal{N}(0, P_2)\). The encoder and decoder will depend on \(U_1, U_2\) implicitly through \(\{C_j^{(1)}\}, \{C_k^{(2)}\}\).

The encoding scheme 1 consists of sequences of encoders \(f_n^{(1)}\) that maps messages \(j \in \{1, 2, \ldots, M_1\}\) to an infinite sequence of input \(C_j^{(1)} \in \mathbb{R}^\infty\). The encoding scheme 2 consists of sequences of encoders \(f_n^{(2)}\) that maps messages \(k \in \{1, 2, \ldots, M_2\}\) to an infinite sequence of input \(C_k^{(2)} \in \mathbb{R}^\infty\). The mappings are without regard to feedback,
\[
X_{jn} = f_n^{(j)}(w_j) := C_{w_{jn}}^{(j)}, \quad j = 1, 2
\]
where \(C_{w_{1,n}}^{(1)}\) and \(C_{w_{2,n}}^{(2)}\) are respectively the \(n\)-th coordinates of the vectors \(C_{w_{1,n}}^{(1)}\) and \(C_{w_{2,n}}^{(2)}\).

At the time \(n\), the decoder computes the following (conditional) information densities:
\[
S_{j,k}^{(1,n)} := i(C_j^{(1)}(n); Y^n | C_k^{(2)}(n)),
\]
\[
S_{j,k}^{(2,n)} := i(C_k^{(2)}(n); Y^n | C_j^{(1)}(n)),
\]
\[
S_{j,k}^{(3,n)} := i(C_j^{(1)}(n), C_k^{(2)}(n); Y^n),
\]
for all \((j, k) \in \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\}\), where
\[
i(C_j^{(1)}(n); Y^n | C_k^{(2)}(n)) := \log \frac{d\mathbb{P}_{X_j^{(1)}}^{Y^{(n)}} | X_j^{(2)}}{d\mathbb{P}_{X_j^{(1)}}^{Y^{(n)}} | X_j^{(2)} \times \mathbb{P}_{Y^n} | X_j^{(2)}} (C_j^{(1)}(n), C_k^{(2)}(n), Y^n),
\]
\[
i(C_k^{(2)}(n); Y^n | C_j^{(1)}(n)) := \log \frac{d\mathbb{P}_{X_k^{(2)}}^{Y^{(n)}} | X_k^{(1)}}{d\mathbb{P}_{X_k^{(2)}}^{Y^{(n)}} | X_k^{(1)} \times \mathbb{P}_{Y^n} | X_k^{(1)}} (C_j^{(1)}(n), C_k^{(2)}(n), Y^n),
\]
\[
i(C_j^{(1)}(n), C_k^{(2)}(n); Y^n) := \log \frac{d\mathbb{P}_{X_j^{(1)}}X_k^{(2)}Y^n}{d\mathbb{P}_{X_j^{(1)}}X_k^{(2)} \times \mathbb{P}_{Y^n} | X_j^{(1)}X_k^{(2)}} (C_j^{(1)}(n), C_k^{(2)}(n), Y^n).
\]

For a triple of positive real numbers \((\gamma_1, \gamma_2, \gamma_3)\) to be chosen later, the decoder also defines a number of stopping times as follows:
\[
\tau_{j,k}^{(1)} := \inf \{n \geq 0 : i(C_j^{(1)}(n); Y^n | C_k^{(2)}(n)) > \gamma_1\},
\]
\[
\tau_{j,k}^{(2)} := \inf \{n \geq 0 : i(C_k^{(2)}(n); Y^n | C_j^{(1)}(n)) > \gamma_2\},
\]
\[
\tau_{j,k}^{(3)} := \inf \{n \geq 0 : i(C_j^{(1)}(n), C_k^{(2)}(n); Y^n) > \gamma_3\},
\]
\[
\tau_{j,k,l} := \max \{\tau_{j,k}^{(1)}, \tau_{j,k}^{(2)}, \tau_{j,k}^{(3)}\}.
\]

The final decision is made by the decoder at the stopping time
\[
\tau^* := \min_{j,k,l} \tau_{j,k,l},
\]
The output of the decoder is given by
\[
g(Y^{\tau^*}) = \max \{(j, k, l) : \tau_{j,k,l} = \tau^*\},
\]
where the maximum is in lexicographic order. Note from (202) that we always can choose a smallest tuple since there exists at least one tuple \((j, k, l)\) such that \(\tau_{j,k,l} = \tau^*\).
Analogously, the average probability of error satisfies

\[ \mathbb{P}(\bar{X}_1, \bar{X}_2, Y_1, Y_2, (x_1, x_2, y, \bar{x}_1, \bar{x}_2)) = \mathbb{P}(X_1, (x_1)\mathbb{P}(X_2)\mathbb{P}(y|x_1,x_2)\mathbb{P}(X_1, (\bar{x}_1, \bar{x}_2)), \tag{204} \]

where \( \mathbb{P}(X_1) \sim \mathcal{N}(0, P_1), \mathbb{P}(X_2) \sim \mathcal{N}(0, P_2) \) and \( \mathbb{P}(y|x_1,x_2) \) is the law of the Gaussian MAC.

For each finite \( n \), define three random information density variables \( i(X_1^n; Y^n | X_2^n), i(X_2^n; Y^n | X_1^n), \) and \( i(X_1^n, X_2^n; Y^n) \) similarly to (195)–(197), and hitting times

\[ \tau^{(1)} := \inf\{ n \geq 0 : i(X_1^n; Y^n | X_2^n) > \gamma_1 \}, \tag{205} \]

\[ \tau^{(2)} := \inf\{ n \geq 0 : i(X_2^n; Y^n | X_1^n) > \gamma_1 \}, \tag{206} \]

\[ \bar{\tau} := \inf\{ n \geq 0 : i(\bar{X}_1^n; Y^n | \bar{X}_2^n) > \gamma_1 \}, \tag{207} \]

\[ \tau^{(2)} := \inf\{ n \geq 0 : i(X_2^n; Y^n | X_1^n) > \gamma_2 \}, \tag{208} \]

\[ \tau^{(3)} := \inf\{ n \geq 0 : i(X_1^n, X_2^n; Y^n) > \gamma_3 \}, \tag{209} \]

\[ \tau^{(3)} := \inf\{ n \geq 0 : i(\bar{X}_1^n, \bar{X}_2^n; Y^n) > \gamma_3 \}, \tag{210} \]

\[ \tau := \max\{ \tau^{(1)}, \tau^{(2)}, \tau^{(3)} \}. \tag{211} \]

Then, it follows that the average length of the transmission satisfies

\[ \mathbb{E}(\tau^*) = \frac{1}{M_1M_2} \sum_{j,k} \mathbb{E}(\tau^* | W_1 = j, W_2 = k) \tag{215} \]

\[ = \mathbb{E}(\tau^* | W_1 = 1, W_2 = 1) \tag{216} \]

\[ \leq \mathbb{E}(\max\{ \tau^{(1)}, \tau^{(2)}, \tau^{(3)} \} | W_1 = 1, W_2 = 1) \tag{217} \]

\[ = \mathbb{E}(\max\{ \tau^{(1)}, \tau^{(2)}, \tau^{(3)} \}), \tag{218} \]

\[ = \mathbb{E}(\tau). \tag{219} \]

Analogously, the average probability of error satisfies

\[ \mathbb{P}(g(Y^*) \neq (W_1, W_2)) \leq \frac{1}{M_1M_2} \sum_{w_1, w_2} \mathbb{P}(g(Y^*) \neq (w_1, w_2) | W_1 = w_1, W_2 = w_2) \tag{220} \]

\[ \leq \mathbb{P}(g(Y^*) \neq (1, 1) | W_1 = 1, W_2 = 1) \tag{221} \]

\[ \leq \mathbb{P}(\tau_{1,1} \geq \tau^* | W_1 = 1, W_2 = 1) \tag{222} \]

\[ \leq \mathbb{P} \left( \bigcup_{(j,k) \neq (1,1)} \{ \tau_{1,1} \geq \tau_{j,k} \} \left| W_1 = 1, W_2 = 1 \right. \right) \tag{223} \]

\[ \leq \sum_{(j,k) \neq (1,1)} \mathbb{P}(\tau_{1,1} \geq \max\{ \tau_{j,k}^{(1)}, \tau_{j,k}^{(2)}, \tau_{j,k}^{(3)} \} | W_1 = 1, W_2 = 1) \tag{224} \]

\[ = \sum_{j=1, k \neq 1} \mathbb{P}(\tau_{1,1} \geq \max\{ \tau_{j,k}^{(1)}, \tau_{j,k}^{(2)}, \tau_{j,k}^{(3)} \} | W_1 = 1, W_2 = 1) \tag{225} \]

\[ + \sum_{j \neq 1, k = 1} \mathbb{P}(\tau_{1,1} \geq \max\{ \tau_{j,k}^{(1)}, \tau_{j,k}^{(2)}, \tau_{j,k}^{(3)} \} | W_1 = 1, W_2 = 1) \tag{226} \]

Observe that

\[ \sum_{j \neq 1, k \neq 1} \mathbb{P}(\tau_{1,1} \geq \max\{ \tau_{j,k}^{(1)}, \tau_{j,k}^{(2)}, \tau_{j,k}^{(3)} \} | W_1 = 1, W_2 = 1) \leq \mathbb{P}(\tau_{1,1} \geq \tau_{j,k}^{(3)} | W_1 = 1, W_2 = 1) \tag{227} \]

\[ = (M_1 - 1)(M_2 - 1) \mathbb{P}(\tau \geq \tau_{j,k}^{(3)}). \tag{228} \]
We also have

\[
\sum_{j=1, k \neq 1} \mathbb{P}(\tau_{1,1} \geq \max\{\tau_{j,k}^{(1)}, \tau_{j,k}^{(2)}, \tau_{j,k}^{(3)}\}|W_1 = 1, W_2 = 1) \\
\leq \sum_{j=1, k \neq 1} \mathbb{P}(\tau_{1,k}^{(2)}|W_1 = 1, W_2 = 1) \\
= (M_2 - 1)\mathbb{P}(\tau \geq \bar{\tau}^{(2)}).
\]

(229)

Similarly, we have

\[
\sum_{j \neq 1, k = 1} \mathbb{P}(\tau_{1,1} \geq \max\{\tau_{j,k}^{(1)}, \tau_{j,k}^{(2)}, \tau_{j,k}^{(3)}\}|W_1 = 1, W_2 = 1) \\
\leq (M_1 - 1)\mathbb{P}(\tau \geq \bar{\tau}^{(1)}).
\]

(231)

It follows that

\[
\mathbb{P}(g(Y^+) \neq (W_1, W_2)) \leq (M_1 - 1)(M_2 - 1)\mathbb{P}(\tau \geq \bar{\tau}^{(3)}) \\
+ (M_1 - 1)\mathbb{P}(\tau \geq \bar{\tau}^{(1)}) \\
+ (M_2 - 1)\mathbb{P}(\tau \geq \bar{\tau}^{(2)}).
\]

(232)

We define random walks \(S^{(1)}_n := i(X_1^n; Y^n|X_2^n), S^{(2)}_n := i(X_2^n; Y^n|X_1^n), \) and \(S^{(3)}_n := i(X_1^n, X_2^n; Y^n). \) Now, observe that the following statistics are all finite:

\[
\mu_1 = \mathbb{E}[i(X_1; Y|X_2)] = I(X_1; Y|X_2) = C(P_1),
\]

(233)

\[
\mu_2 = \mathbb{E}[i(X_2; Y|X_1)] = I(X_2; Y|X_1) = C(P_2),
\]

(234)

\[
\mu_3 = \mathbb{E}[i(X_1, X_2; Y)] = I(X_1, X_2; Y) = C(P_1 + P_2),
\]

(235)

\[
\sigma_1^2 = \text{Var}(i(X_1; Y|X_2)) = \frac{P_1}{1 + P_1},
\]

(236)

\[
\sigma_2^2 = \text{Var}(i(X_2; Y|X_1)) = \frac{P_2}{1 + P_2},
\]

(237)

\[
\sigma_3^2 = \text{Var}(i(X_1, X_2; Y)) = \frac{P_1 + P_2}{1 + P_1 + P_2},
\]

(238)

\[
\mathbb{E}[i(X_1; Y|X_2)^+] \leq \mathbb{E}[i(X_1; Y|X_2)^+ + i(X_1; Y|X_2)^-] \\
= \mathbb{E}[|i(X_1; Y|X_2)|] \\
\leq \sqrt{\mathbb{E}[i(X_1; Y|X_2)]^2} \\
= \sqrt{\mu_1^2 + \sigma_1^2} < \infty,
\]

(241)

\[
\mathbb{E}[i(X_2; Y|X_1)^+] \leq \sqrt{\mu_2^2 + \sigma_2^2} < \infty,
\]

(243)

\[
\mathbb{E}[i(X_1, X_2; Y)^+] \leq \sqrt{\mu_3^2 + \sigma_3^2} < \infty.
\]

(244)

Moreover, by [30, pp. 207], Cramer’s condition (C) in Definition 3 (see (23)) is satisfied by those distributions having at least a continuous component in its Lebesgue decomposition. Since \(i(X_1, Y|X_2), i(X_2; Y|X_1), \) and \(i(X_1, X_2; Y) \) are all continuous random variables, their distribution functions are strongly nonlattice. Hence, it follows from Lemma 1 that

\[
I(X_1; Y|X_2)\mathbb{E}(\tau^{(1)}) = \gamma_1 + \frac{\mathbb{E}S_{S_1}^{(1)}}{2\mathbb{E}S_{S_1}^{(1)}} + o(1), \quad \text{as } \gamma_1 \to \infty,
\]

(245)

\[
I(X_2; Y|X_1)\mathbb{E}(\tau^{(2)}) = \gamma_2 + \frac{\mathbb{E}S_{S_2}^{(2)}}{2\mathbb{E}S_{S_2}^{(2)}} + o(1), \quad \text{as } \gamma_2 \to \infty,
\]

(246)

\[
I(X_1, X_2; Y)\mathbb{E}(\tau^{(3)}) = \gamma_3 + \frac{\mathbb{E}S_{S_3}^{(3)}}{2\mathbb{E}S_{S_3}^{(3)}} + o(1), \quad \text{as } \gamma_3 \to \infty.
\]

(247)

In addition, from Lemma 7 we also have for \(j = 1, 2, 3 \) that

\[
\text{Var}(\tau^{(j)}) = \mu_j^{-3} \sigma_j^2 \gamma_j + \mu_j^{-2} K_1 + o(1), \quad \text{as } \gamma_j \to \infty,
\]

(248)
where
\[
K_j = \frac{\sigma_j^2 \mathbb{E}S^2_{\tau_j}^{(0)}}{2 \mu_1 \mathbb{E}S_{\tau_j}^{(0)}} + 3 \left( \frac{\mathbb{E}S^2_{\tau_j}^{(0)}}{\mathbb{E}S_{\tau_j}^{(0)}} \right)^{\frac{2}{3}} - 2 \left( \frac{\mathbb{E}S^3_{\tau_j}^{(0)}}{\mathbb{E}S_{\tau_j}^{(0)}} \right) - \left( \frac{\mathbb{E}S^2_{\tau_j}^{(0)}}{\mathbb{E}S_{\tau_j}^{(0)}} \right) \mathbb{E}\{\min_{n \geq 0} S_n^{(0)} \}
- 2 \int_0^\infty \mathbb{E}\{S_{\tau_j}^{(0)}(x) - x\} \mathbb{P}\{\min_{n \geq 0} S_n^{(0)} \leq -x\} \, dx, \quad j = 1, 2, 3
\] (249)
are constants (i.e. $K_1, K_2, K_3 = O(1)$), and
\[
\tau_j^{(0)}(x) = \inf\{n \geq 0 : S_n^{(0)} > x\}, \quad j = 1, 2, 3.
\] (250)

Now, for any positive real number $N'$, choose
\[
\gamma_1 = I(X_1; Y|X_2)(N' - A \sqrt{N'} - D) - \frac{\mathbb{E}S^2_{\tau_j}^{(1)}}{2 \mathbb{E}S_{\tau_j}^{(1)}},
\] (251)
\[
\gamma_2 = I(X_2; Y|X_1)(N' - A \sqrt{N'} - D) - \frac{\mathbb{E}S^2_{\tau_j}^{(2)}}{2 \mathbb{E}S_{\tau_j}^{(2)}},
\] (252)
\[
\gamma_3 = I(X_1, X_2; Y)(N' - A \sqrt{N'} - D) - \frac{\mathbb{E}S^2_{\tau_j}^{(3)}}{2 \mathbb{E}S_{\tau_j}^{(3)}},
\] (253)
and a pair $(M_1, M_2)$ satisfying
\[
0 \leq \log M_j \leq \gamma_j - \log(3N'), \quad j = 1, 2,
\] (254)
\[
0 \leq \log M_1M_2 \leq \gamma_3 - \log(3N'),
\] (255)
for some $A \geq 0, D \geq 0$ to be determined later. These choices of $M_1$ and $M_2$ and the fact that $\mathbb{E}S^2_{\tau_j}^{(i)}/(2 \mathbb{E}S_{\tau_j}^{(i)}) = O(1)$ for $j = 1, 2, 3$ show that (185) and (186) are satisfied.

Then, combining these choices with (245)–(247) we obtain
\[
\mathbb{E}[\tau_j^{(0)}] = N' - A \sqrt{N'} - D - B_j + o(1) \quad j = 1, 2, 3,
\] (256)
where
\[
B_j = -\frac{\mathbb{E}S^2_{\tau_j}^{(0)}}{\mathbb{E}S_{\tau_j}^{(0)}} \left[ \frac{1}{\log (1 + P_j)} \right], \quad j = 1, 2
\] (257)
\[
B_3 = -\frac{\mathbb{E}S^2_{\tau_j}^{(3)}}{\mathbb{E}S_{\tau_j}^{(3)}} \left[ \frac{1}{\log (1 + P_1 + P_2)} \right],
\] (258)
are constants. Besides, by using the facts that $A \geq 0$ and $D \geq 0$, we also have
\[
\mathbb{V}ar(\tau_j^{(0)}) = L_j(N' - A \sqrt{N'} - D) + S_j + o(1) \leq L_j N' + S_j + o(1),
\] (259)
where the constants $L_1, L_2, L_3$ and $S_1, S_2, S_3$ are defined according to Lemma 7. Specifically,
\[
L_j = \left( \frac{\sigma_j}{\mu_j} \right)^2 = (163), \quad j = 1, 2, 3
\] (260)
\[
S_j = \mu_j^{-2} K_j - \sigma_j^2 \mu_j^{-3} \frac{\mathbb{E}S^2_{\tau_j}^{(0)}}{2 \mathbb{E}S_{\tau_j}^{(0)}}, \quad j = 1, 2, 3.
\] (261)

Then, it follows from Lemma 8 that
\[
\mathbb{E}(\tau^*) \leq \mathbb{E}[\max\{\tau^{(1)}, \tau^{(2)}, \tau^{(3)}\}] \leq N' + o(1), \quad \text{as} \quad N' \to \infty.
\] (262)
if we choose $A$ as (162), and $D$ sufficiently large.

Moreover, from (256) we have
\[
\mathbb{P}(\tau_j^{(0)} < \infty) = 1, \quad j = 1, 2, 3.
\] (263)
Then, applying a change of measure, we see from (204) that for any measurable function $f$,

\[
E[f(\bar{X}_1^n, X_2^n, Y^n)] = E\left[ f(X_1^n, X_2^n, Y^n) \exp\left(-S_n^{(1)}\right) \right],
\]

(264)

\[
E[f(\bar{X}_1^n, \bar{X}_2^n, Y^n)] = E\left[ f(X_1^n, X_2^n, Y^n) \exp\left(-S_n^{(2)}\right) \right],
\]

(265)

\[
E[f(\bar{X}_1^n, X_2^n, Y^n)] = E\left[ f(X_1^n, X_2^n, Y^n) \exp\left(-S_n^{(3)}\right) \right].
\]

(266)

Observe that

\[
1\{\tau(j) \leq n\} \in \sigma(X_1^n, X_2^n, Y^n), \quad j = 1, 2, 3
\]

(267)

\[
1\{\tau \leq n\} \in \sigma(X_1^n, X_2^n, Y^n),
\]

(268)

\[
1\{\bar{\tau}(1) \leq n\} \in \sigma(\bar{X}_1^n, X_2^n, Y^n),
\]

(269)

\[
1\{\bar{\tau}(2) \leq n\} \in \sigma(X_1^n, \bar{X}_2^n, Y^n),
\]

(270)

\[
1\{\bar{\tau}(3) \leq n\} \in \sigma(\bar{X}_1^n, \bar{X}_2^n, Y^n).
\]

(271)

Then, it follows the same arguments as [14], we have

\[
P(\bar{\tau}(3) \leq \tau) \leq P(\bar{\tau}(3) < \infty)
\]

(272)

\[
= \lim_{n \to \infty} P(\bar{\tau}(3) < n)
\]

(273)

\[
\overset{(a)}{=} \lim_{n \to \infty} P\left(\exp\left(-S_n^{(3)}\right) 1\{\tau < n\}\right)
\]

(274)

\[
\overset{(b)}{=} \lim_{n \to \infty} P\left(\exp\left(-S_n^{(3)}\right) 1\{\tau^{(3)} < n\}\right)
\]

(275)

\[
\overset{(c)}{=} \exp(-\gamma_3),
\]

(276)

\[
P(\bar{\tau}(j) \leq \tau) \leq P(\bar{\tau}(j) < \infty) \leq \exp(-\gamma_j), \quad j = 1, 2.
\]

(277)

Here, (a) follows from the fact that $1\{\bar{\tau}(3) \leq n\} \in \sigma(\bar{X}_1^n, X_2^n, Y^n), 1\{\tau \leq n\} \in \sigma(X_1^n, X_2^n, Y^n)$ and the change of measure formula (266), (b) follows from the fact that $\tau = \max\{\tau^{(1)}, \tau^{(2)}, \tau^{(3)}\} \geq \tau^{(3)}$, so $1\{\tau \leq n\} \leq 1\{\tau^{(3)} \leq n\}$, (c) follows from the fact that $1\{\tau \leq n\} \in \sigma(X_1^n, X_2^n, Y^n)$ and Doob’s optional stopping theorem [23, Theorem 10.10].

Now, from the bound on the error probability in (232), the bounds on the individual probabilities in (276) and (278), the choices of $M_1$ and $M_2$ in (254) and (255), we see that the average error probability of the stop-feedback code satisfies

\[
\varepsilon' \leq (M_1 - 1)(M_2 - 1) \exp(-\gamma_3)
\]

(278)

\[
+ (M_1 - 1) \exp(-\gamma_1)
\]

(279)

\[
+ (M_2 - 1) \exp(-\gamma_2)
\]

(280)

\[
\leq \frac{1}{N^2},
\]

Finally, we check all the conditions of Lemma 2 (with $X_{2n}^2$ for $j = 1, 2$ here playing the role of $X_n$ in Lemma 2).

- We have

\[
E[X_{2n}^2] = P_j, \quad j = 1, 2,
\]

(281)

so it follows that $X_{1n}^2$ and $X_{2n}^2$ are integrable for all $n \geq 1$.

- Now, we see that $1\{\tau^* \geq n\} = 1 - 1\{\tau^* \leq n - 1\} \in \sigma(X_1^{n-1}, X_2^n, Y^{n-1})$. Moreover, since the sequence $\{X_{1n}\}_{n \geq 1}$ as well as the sequence $\{X_{2n}\}_{n \geq 1}$ are i.i.d. generated, hence we have $1\{\tau^* \geq n\}$ is independent of $X_{1n}$ and $X_{2n}$. It follows that

\[
E[X_{2n}^2 1\{\tau^* \geq n\}] = E[X_{2n}^2]E[1\{\tau^* \geq n\}]
\]

(282)

\[
= E[X_{2n}^2]P(\tau^* \geq n), \quad j = 1, 2
\]

(283)
• For each $j = 1, 2$, the infinite series satisfies
\[
\sum_{n=1}^{\infty} \mathbb{E}[X_{jn}^2 \{ \tau^* \geq n \}] = \sum_{n=1}^{\infty} \mathbb{E}[X_{jn}^2] \mathbb{P}(\tau^* \geq n) 
\leq P_j \sum_{n=1}^{\infty} \mathbb{P}(\tau^* \geq n)
= P_j \mathbb{E}(\tau^*)
\leq P_j N < \infty
\] (284)

• For each $j = 1, 2$, all random variables $X_{jn}^2, n \geq 1$ have the same expectation $P_j$.

• $E(\tau^*) \leq N < \infty$.

Hence, by Lemma 2, the expected power constraints at the encoders satisfy
\[
E \left( \sum_{n=1}^{\tau^*} X_{jn}^2 \right) = E(\tau^*)E[X_{n1}^2]
= E(\tau^*)P_j, \quad j = 1, 2.
\] (288)

This means that we have shown there exists an $(M_1, M_2, N' + o(1), \frac{1}{M_1})$ stop-feedback code with stopping time $\tau^*$ such that (188) holds.

Now, by keeping the same stopping rule and decoding algorithm of the aforementioned $(M_1, M_2, N' + o(1), \frac{1}{M_1})$ stop-feedback code and setting
\[
\hat{X}_{jn} := \begin{cases} X_{jn}, & n \leq \tau^* \\ 0, & n > \tau^* \end{cases}, \quad j = 1, 2
\] (290)
we have a new $(M_1, M_2, N' + o(1), \frac{1}{M_1})$ stop-feedback code satisfying:
\[
\sum_{n=1}^{\infty} \mathbb{E}[\hat{X}_{jn}^2] \overset{(a)}{=} \mathbb{E} \left[ \sum_{n=1}^{\tau^*} X_{jn}^2 \right] 
= \mathbb{E} \left[ \sum_{n=1}^{\tau^*} X_{jn}^2 \right]
= \mathbb{E}(\tau^*)P_j, \quad j = 1, 2.
\] (291)

Here, (a) follows from the monotone convergence theorem. That concludes our proof.

**Lemma 10.** For the Gaussian MAC $P(y|x_1, x_2)$, there exists an $(M_1, M_2, N, P_1, P_2, \varepsilon)$ stop-feedback code for $M_1, M_2$ satisfying
\[
0 \leq \log M_j \leq \left( \frac{N}{1 - \varepsilon} - A\sqrt{\frac{N}{1 - \varepsilon}} \right) C(P_j) - \log N + O(1), \quad j = 1, 2
\] (292)
\[
0 \leq \log M_1 M_2 \leq \left( \frac{N}{1 - \varepsilon} - A\sqrt{\frac{N}{1 - \varepsilon}} \right) C(P_1 + P_2) - \log N + O(1).
\] (293)

Here, $A$ is defined in (162).

**Proof:** We propose a stop-feedback coding scheme as follows:

• The decoder chooses a number $N', P'_1, P'_2$ such that
\[
\frac{(N')^2(1 - \varepsilon)}{N' - 1} = N, \quad P'_j = P_j, \quad j = 1, 2.
\] (294)

• The decoder generates a Bernoulli random variable $D \sim \text{Bern}(p)$, where $p$ is defined in (89).

• If $D = 1$, the decoder sends a stop-feedback (or a NACK) to the encoder via the feedback link. This means that $\tau = 0$.

• If $D = 0$, the encoder starts to send the intended message to the decoder using the stop-feedback $(M_1, M_2, N' + o(1), P'_1, P'_2, \frac{1}{M_1})$ mentioned in the Lemma 9 for the Gaussian MAC with common messages with expected power $P'_1 = P_1$ and $P'_2 = P_2$ and stops at time $\tau'$. This means that $\tau = \tau'$. 

Proof: We propose a stop-feedback coding scheme as follows:

• The decoder chooses a number $N', P'_1, P'_2$ such that
\[
\frac{(N')^2(1 - \varepsilon)}{N' - 1} = N, \quad P'_j = P_j, \quad j = 1, 2.
\] (295)

• The decoder generates a Bernoulli random variable $D \sim \text{Bern}(p)$, where $p$ is defined in (89).

• If $D = 1$, the decoder sends a stop-feedback (or a NACK) to the encoder via the feedback link. This means that $\tau = 0$.

• If $D = 0$, the encoder starts to send the intended message to the decoder using the stop-feedback $(M_1, M_2, N' + o(1), P'_1, P'_2, \frac{1}{M_1})$ mentioned in the Lemma 9 for the Gaussian MAC with common messages with expected power $P'_1 = P_1$ and $P'_2 = P_2$ and stops at time $\tau'$. This means that $\tau = \tau'$. 

Proof: We propose a stop-feedback coding scheme as follows:

• The decoder chooses a number $N', P'_1, P'_2$ such that
\[
\frac{(N')^2(1 - \varepsilon)}{N' - 1} = N, \quad P'_j = P_j, \quad j = 1, 2.
\] (296)

• The decoder generates a Bernoulli random variable $D \sim \text{Bern}(p)$, where $p$ is defined in (89).

• If $D = 1$, the decoder sends a stop-feedback (or a NACK) to the encoder via the feedback link. This means that $\tau = 0$.

• If $D = 0$, the encoder starts to send the intended message to the decoder using the stop-feedback $(M_1, M_2, N' + o(1), P'_1, P'_2, \frac{1}{M_1})$ mentioned in the Lemma 9 for the Gaussian MAC with common messages with expected power $P'_1 = P_1$ and $P'_2 = P_2$ and stops at time $\tau'$. This means that $\tau = \tau'$. 

Proof: We propose a stop-feedback coding scheme as follows:

• The decoder chooses a number $N', P'_1, P'_2$ such that
\[
\frac{(N')^2(1 - \varepsilon)}{N' - 1} = N, \quad P'_j = P_j, \quad j = 1, 2.
\] (297)
It follows that the error probability of the proposed stop-feedback coding scheme is upper bound by
\[ \frac{N'\varepsilon - 1}{N' - 1} + \left( 1 - \frac{N'\varepsilon - 1}{N' - 1} \right) \frac{1}{N'} = \varepsilon. \] (298)

In addition, the average length of the proposed stop-feedback coding scheme is less than or equal to
\[ \left( 1 - \frac{N'\varepsilon - 1}{N' - 1} \right) \mathbb{E}(\tau') \leq \left( 1 - \frac{N'\varepsilon - 1}{N' - 1} \right) N' + o(1) \left( 1 - \frac{N'\varepsilon - 1}{N' - 1} \right) \]
\[ = \frac{(N')^2(1-\varepsilon)}{N' - 1} + o(1) \] (299)
\[ = N + o(1). \] (300)

From (296) and (297), the expected powers of the combined scheme satisfy
\[ \left( 1 - \frac{N'\varepsilon - 1}{N' - 1} \right) \mathbb{E}(\tau') P'_j = \mathbb{E}(\tau) P'_j \] (302)
\[ = \mathbb{E}(\tau) P'_j, \quad j = 1, 2. \] (303)

Therefore, combining this code construction with Lemma 9, we see that there exists an \((M_1, M_2, N + o(1), P_1, P_2, \varepsilon)\) stop-feedback code where
\[ 0 \leq \log M_j \leq \left( \frac{N}{1 - \varepsilon} - A \sqrt{\frac{N}{1 - \varepsilon}} - D + o(1) \right) C(P_j) - \log \left( \frac{3N}{1 - \varepsilon} \right) + O(1), \quad j = 1, 2 \] (304)

Now, it is easy to see that if there exists an \((M_1, M_2, N + o(1), P_1, P_2, \varepsilon)\) stop-feedback code by setting the expected length equal to \(N - o(1)\). Note that this change of the expected length does not affect the asymptotic approximation of the code rates.

That concludes our proof of the achievability part of Theorem 3.

D. Achievability Proof for Theorem 4

Lemma 11. Given a Gaussian MAC, for any \(\rho \in [0, 1]\), there exist an \((M_1, M_2, N, P_1, P_2, \varepsilon)\) VLFT-feedback code for any \(M_1, M_2\) satisfying
\[ 0 \leq \log M_j \leq \left( \frac{N}{1 - \varepsilon} \right) \mathbb{C} \left( P_j \left(1 - \rho^2\right) \right) - \log \log N + O(1), \quad j = 1, 2 \] (306)
\[ 0 \leq \log M_1 M_2 \leq \left( \frac{N}{1 - \varepsilon} \right) \mathbb{C} \left( P_1 + P_2 + 2\rho \sqrt{P_1 P_2} \right) - \log \log N + O(1). \] (307)

Proof: Consider Ozarow’s coding scheme (for the Gaussian MAC with fixed-length feedback) [4] with fixed blocklength \(N'\) (a natural number), expected powers bounded by \(P'_1\) and \(P'_2\), and message sizes \(M_1\) and \(M_2\) satisfying
\[ \log M_j = N'\mathbb{C} \left( P'_j \left(1 - \rho^2\right) \right) - \frac{1}{2} \log[v \log N'], \quad j = 1, 2 \] (308)
where \(\rho \in [0, 1]\), and \(v\) is some constant which does not depend on \(N'\). Then, it is easy to see from [6] that Ozarow’s scheme results in an error probability
\[ \varepsilon' \leq \frac{2}{(N')^2} \leq \frac{1}{N'}, \] (309)
for \(N' \geq 2\) (cf. [6, Equation (121)]).

Therefore, we can construct our VLFT coding scheme as follows:
- The decoder chooses the largest natural number \(N'\) such that (104) is satisfied and positive numbers \(P'_1, P'_2\) as in (297).
- The decoder generates a Bernoulli random variable \(D \sim \text{Bern}(p)\), where \(p\) is defined in (89).
- If \(D = 1\), the decoders send a stop-feedback (or a NACK) to the encoder via the feedback link. This means that \(\tau = 0\).
- If \(D = 0\), the encoder starts to send the intended message to the decoder using the Ozarow’s coding scheme with parameters \((M_1, M_2, N', P'_1, P'_2, \frac{N'}{N})\) mentioned above for the Gaussian MAC with expected powers \(P'_1 = P_1\) and \(P'_2 = P_2\) and stops at time \(\tau'\). This means that \(\tau = N' + 1\).

Similarly to the stop-feedback case, it follows that the error probability of the proposed VLFT coding scheme is upper bound by (298). The expected powers of the combined scheme satisfy (302) and (303). In addition, the average length of the proposed VLFT coding scheme can be computed in the same way as we did for the single-user AWGN channel in (105)–(107). Hence it is no larger than \(N\). Consequently, the achievability part of Theorem 4 follows from (308), (296), and (297).
E. Converse Proof for Theorem 3

Lemma 12. Given a Gaussian MAC $P(y|x_1, x_2)$, $0 \leq \varepsilon \leq 1 - \max\left\{\frac{1}{M_1}, \frac{1}{M_2}\right\}$, any $(M_1, M_2, N, P_1, P_2, \varepsilon)$ stop-feedback code satisfies the following inequalities:

\[0 \leq \log M_j \leq \frac{NC(P_j) + h_b(\varepsilon)}{1 - \varepsilon}, \quad j = 1, 2\]  
\[0 \leq \log M_1M_2 \leq \frac{NC(P_1 + P_2) + h_b(\varepsilon)}{1 - \varepsilon}.\]  

**Proof:** First, we consider the case $|U_1| = |U_2| = 1$. For the stop-feedback formalism, $\tau$ is a stopping time of the filtration $\{\sigma(Y^n)\}_{n=0}^{\infty}$. As in [14], we can convert any given code $(f^{(1)}_n, f^{(2)}_n, g_n, \tau)$ to an equivalent code $(\hat{f}^{(1)}_n, \hat{f}^{(2)}_n, \hat{g}_n, \tau)$ as follows. We add a special symbol $T \notin \mathbb{R}$ to input alphabets and output alphabet to form new input-output alphabets $\mathbb{R} \cup \{T\}, \mathbb{R} \cup \{T\}, \mathbb{R} \cup \{T\}$ and then assign:

\[P_{\hat{Y}|X_1, X_2}(\hat{y} | \hat{x}_1, \hat{x}_2) = \begin{cases} P(y | \hat{x}_1, \hat{x}_2), & \hat{x}_1 \neq T, \hat{x}_2 \neq T \\ 1 \{y = T\}, & \hat{x}_1 = \hat{x}_2 = T \end{cases}\]  
\[\hat{X}_n = \hat{f}^{(j)}_n(W_j) := \begin{cases} f^{(j)}_n(W_j), & n \leq \tau \\ T, & n > \tau, \quad j = 1, 2. \end{cases}\]  

Define

\[\hat{\tau} = \tau = \inf\{n : \hat{Y}_n = T\} - 1\]  
and

\[\hat{g}_n(\hat{Y}^n) := \begin{cases} g_n(Y^n), & n < \hat{\tau} \\ g_n(Y^{\hat{\tau}-1}), & n \geq \hat{\tau}. \end{cases}\]

Let $\phi : \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\} \times \hat{Y}^\infty \to \{0, 1\}$ be defined as

\[\phi = 1\{g_{\hat{\tau}-1}(Y^{\hat{\tau}-1}) = (W_1, W_2)\}.\]  
Then, the following expressions hold for all $(w_1, w_2) \in \{1, 2, \ldots, M_1\} \times \{1, 2, \ldots, M_2\}$:

\[P_{W_1W_2Y^\infty}[\phi = 1] \geq 1 - \varepsilon,\]  
\[P_{W_1W_2} \times P_{Y^\infty}[\phi = 1] = \frac{1}{M_1M_2},\]  
\[P_{W_1Y^\infty|W_2=w_2}[\phi = 1] \geq 1 - \varepsilon,\]  
\[P_{W_1} \times P_{Y^\infty|W_2=w_2}[\phi = 1] = \frac{1}{M_1},\]  
\[P_{W_2} \times P_{Y^\infty|W_1=w_1}[\phi = 1] \geq 1 - \varepsilon,\]  
\[P_{W_2} \times P_{Y^\infty|W_1=w_1}[\phi = 1] = \frac{1}{M_2}.\]

Then for $1 - \varepsilon \geq \max\left\{\frac{1}{M_1}, \frac{1}{M_2}\right\}$, by the data-processing inequality for the relative entropy, we have

\[D(P_{W_1W_2Y^\infty} \| P_{W_1W_2} \times P_{Y^\infty}) \geq d \left(1 - \varepsilon \right) \frac{1}{M_1M_2},\]  
\[D(P_{W_1Y^\infty|W_2=w_2} \| P_{W_1} \times P_{Y^\infty|W_2=w_2}) \geq d \left(1 - \varepsilon \right) \frac{1}{M_1},\]  
\[D(P_{W_2Y^\infty|W_1=w_1} \| P_{W_2} \times P_{Y^\infty|W_1=w_1}) \geq d \left(1 - \varepsilon \right) \frac{1}{M_2},\]

where $d(p \| q) := p \log \frac{p}{q} + (1 - p) \log \frac{1 - p}{1 - q}$ is the binary divergence. Then, we obtain

\[(1 - \varepsilon) \log M_1M_2 \leq I(W_1W_2; \hat{Y}^\infty) + h_b(\varepsilon),\]  
\[(1 - \varepsilon) \log M_1 \leq I(W_1; Y^\infty | W_2 = w_2) + h_b(\varepsilon),\]  
\[(1 - \varepsilon) \log M_2 \leq I(W_2; Y^\infty | W_1 = w_1) + h_b(\varepsilon).\]
By taking expectations of (327) and (328), we obtain

\[(1 - \varepsilon) \log M_1 \leq I(W_1; \hat{Y}_n) + h_b(\varepsilon), \tag{329}\]
\[(1 - \varepsilon) \log M_2 \leq I(W_2; \hat{Y}_n) + h_b(\varepsilon). \tag{330}\]

Define

\[\Psi_n := 1\{\hat{n} \leq n - 1\} \in \sigma(\hat{Y}_n^{-1}). \tag{331}\]

Then, by Lemma 14 in Appendix C, we have

\[I(W_1W_2; \hat{Y}_n) \leq \sum_{n=1}^{\infty} \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log [1 + (X_{1n} + X_{2n})^2 | \Psi_n = 0], \tag{332}\]
\[I(W_1; \hat{Y}_n|W_2) \leq \sum_{n=1}^{\infty} \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log (1 + \mathbb{E}[X_{1n}^2 | \Psi_n = 0]), \tag{333}\]
\[I(W_2; \hat{Y}_n|W_1) \leq \sum_{n=1}^{\infty} \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log (1 + \mathbb{E}[X_{2n}^2 | \Psi_n = 0]). \tag{334}\]

Now, we observe that

\[\sum_{n=1}^{\infty} \mathbb{P}(\Psi_n = 0) = \sum_{n=1}^{\infty} \mathbb{P}(\tau \geq n) = \mathbb{E}(\tau), \tag{335}\]
\[= \mathbb{E}(\tau). \tag{336}\]

It follows that

\[\sum_{n=1}^{\infty} \frac{\mathbb{P}(\Psi_n = 0)}{\mathbb{E}(\tau)} = 1, \tag{337}\]

so \(\{\frac{\mathbb{P}(\Psi_n = 0)}{\mathbb{E}(\tau)}\}_{n=1}^{\infty}\) is a probability distribution. Moreover, since the function \(f(x) = \log(1 + x)\) is concave, we have from (326) and (332) that

\[(1 - \varepsilon) \log M_1 M_2 \leq \frac{1}{2} \mathbb{E}(\tau) \log \left(1 + \sum_{n=1}^{\infty} \frac{\mathbb{P}(\Psi_n = 0)}{\mathbb{E}(\tau)} \mathbb{E}[(X_{1n} + X_{2n})^2 | \Psi_n = 0] \right) + h_b(\varepsilon) \leq \frac{N}{2} \log \left(1 + \frac{1}{\mathbb{E}(\tau)} \sum_{n=1}^{\infty} \mathbb{P}(\Psi_n = 0) \mathbb{E}[(X_{1n} + X_{2n})^2 | \Psi_n = 0] \right) + h_b(\varepsilon) \tag{338}\]
\[\leq \frac{N}{2} \log \left(1 + \frac{1}{\mathbb{E}(\tau)} \sum_{n=1}^{\infty} \mathbb{E}[(X_{1n} + X_{2n})^2] \right) + h_b(\varepsilon) \tag{339}\]
\[= \frac{N}{2} \log \left(1 + \frac{1}{\mathbb{E}(\tau)} \sum_{n=1}^{\infty} \mathbb{E}[X_{1n}^2] + \mathbb{E}[X_{2n}^2] + 2\mathbb{E}[X_{1n}X_{2n}] \right) + h_b(\varepsilon) \tag{340}\]
\[\leq \frac{N}{2} \log \left(1 + \frac{P_1\mathbb{E}(\tau) + P_2\mathbb{E}(\tau)}{\mathbb{E}(\tau)} \right) + h_b(\varepsilon) \tag{341}\]
\[= \frac{N}{2} \log (1 + P_1 + P_2) + O(1). \tag{342}\]

Here, (a) follows from the fact that

\[\mathbb{E}[(X_{1n} + X_{2n})^2] = \mathbb{P}(\Psi_n = 0)\mathbb{E}[(X_{1n} + X_{2n})^2 | \Psi_n = 0] + \mathbb{P}(\Psi_n = 1)\mathbb{E}[(X_{1n} + X_{2n})^2 | \Psi_n = 1] \tag{343}\]
\[\geq \mathbb{P}(\Psi_n = 0)\mathbb{E}[(X_{1n} + X_{2n})^2 | \Psi_n = 0], \tag{344}\]

and (b) follows from the power constraints of the stop-feedback code and the fact that \(X_{1n} = f_n^{(1)}(W_1)\) is independent of \(X_{2n} = f_n^{(2)}(W_2)\).
Similarly, we have from (329) and (333) that

\[(1 - \varepsilon) \log M_1 \leq \frac{1}{2} \mathbb{E}(\tau) \log \left(1 + \sum_{n=1}^{\infty} \frac{\mathbb{P}(\Psi_n = 0)}{\mathbb{E}(\tau)} \mathbb{E}[X^2_{1n} | \Psi_n = 0]\right) + h_b(\varepsilon) \tag{346}\]

\[\leq \frac{N}{2} \log \left(1 + \frac{1}{\mathbb{E}(\tau)} \sum_{n=1}^{\infty} \mathbb{P}(\Psi_n = 0) \mathbb{E}[X^2_{1n} | \Psi_n = 0]\right) + h_b(\varepsilon) \tag{347}\]

\[\leq \frac{N}{2} \log \left(1 + \frac{1}{\mathbb{E}(\tau)} \sum_{n=1}^{\infty} \mathbb{E}[X^2_{1n}]\right) + h_b(\varepsilon) \tag{348}\]

\[\leq \frac{N}{2} \log \left(1 + \frac{1}{\mathbb{E}(\tau)} \mathbb{P}(\Psi_n = 0) \mathbb{E}[X^2_{1n} | Y_n = 0]\right) + h_b(\varepsilon) \tag{349}\]

\[= \frac{N}{2} \log (1 + P_1) + O(1). \tag{350}\]

Here, inequalities (a) and (b) follow from the same reasonings as those in the previous set of inequalities in (338)–(343).

For the case \(|\mathcal{U}_1| \geq 1, |\mathcal{U}_2| \geq 1\), with the above arguments and \(\mathcal{F}_n = \sigma(U_1, U_2, Y^n)\), the following expressions hold almost surely:

\[(1 - \mathbb{P}([\hat{W}_1, \hat{W}_2] \neq (W_1, W_2)|U_1, U_2)) \log M_1 M_2 \leq \frac{1}{2} \log(1 + P_1 + P_2) + h_b((\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)|U_1, U_2), \tag{351}\]

\[(1 - \mathbb{P}([\hat{W}_1, \hat{W}_2] \neq (W_1, W_2)|U_1, U_2)) \log M_j \leq \frac{1}{2} \log(1 + P_j) + h_b((\hat{W}_1, \hat{W}_2) \neq (W_1, W_2)|U_1, U_2), \quad j = 1, 2. \tag{352}\]

By taking the expectation with respect to \((U_1, U_2)\) on both sides of (351)–(352) and applying Jensen inequality’s for the binary entropy terms, we obtain (310)–(311). This confirms the converse proof of Theorem 3.

\[\square\]

**F. Converse Proof for Theorem 4**

**Lemma 13.** Given a Gaussian MAC \(\mathbb{P}(y|x_1, x_2), 0 \leq \varepsilon \leq 1 - \max\left\{\frac{1}{M_1}, \frac{1}{M_2}\right\}\), any \((M_1, M_2, N, P_1, P_2, \varepsilon)\) VLFT code satisfies the following inequalities for some \(\rho \in [0, 1]\):

\[0 \leq \log M_j \leq \frac{\mathbb{N}(\rho, 1) + \log(N + 1) + h_b(\varepsilon)}{1 - \varepsilon}, \quad j = 1, 2 \tag{353}\]

\[\log M_1 M_2 \leq \frac{\mathbb{N}(\rho, 1) + \log(N + 1) + h_b(\varepsilon)}{1 - \varepsilon}. \tag{354}\]

**Proof:** Similarly to the converse proof for Gaussian MAC with a stop-feedback code, we first consider the case in which \(|\mathcal{U}_1| = |\mathcal{U}_2| = 1\). Since the receiver decides the transmitted messages based only on \(Y^\tau\) (not dependent on the channel outputs that are received later), we can convert any given code \((f^{(1)}_n, f^{(2)}_n, g_n, \tau)\) to an equivalent code \((\hat{f}^{(1)}_n, \hat{f}^{(2)}_n, \hat{g}_n, \tau)\) as follows. We add a special symbol \(T \not\in \mathbb{R}\) to the input and output alphabets to form new alphabets \(\mathbb{R} \cup \{T\}, \mathbb{R} \cup \{T\}, \mathbb{R} \cup \{T\}\) and then define \(\mathbb{P}_{Y|\hat{X}_1, \hat{X}_2}\) as in (312), \(\hat{X}_jn\) as in (313), \(\hat{g}_n(\hat{Y}^n)\) as in (315) and

\[\hat{\tau} = \tau + 1 = \inf\{n : \hat{Y}_n = T\}. \tag{357}\]

Define

\[\Psi_n := 1\{\hat{\tau} \leq n\} \in \sigma(\hat{Y}^n). \tag{358}\]

Using the same approach as the proof of converse for the Gaussian MAC with a stop-feedback code, we obtain from (491) and (502) in Appendix C but retaining the term \(H(\Psi_n|\hat{Y}^{n-1})\) that

\[I(W_1, W_2; \hat{Y}_n|\hat{Y}^{n-1}) \leq H(\Psi_n|\hat{Y}^{n-1}) + \frac{1}{2}\mathbb{P}(\Psi_n = 0) \log[1 + (X_{1n} + X_{2n})^2] + h_b(\varepsilon) \tag{359}\]

\[I(W_1; \hat{Y}_n|\hat{Y}^{n-1}W_2) \leq H(\Psi_n|\hat{Y}^{n-1}) + \mathbb{P}(\Psi_n = 0) I(X_{1n}; Y_n|\Psi_n = 0, \hat{Y}^{n-1}, X_{2n}, W_2), \tag{360}\]

\[I(W_2; \hat{Y}_n|\hat{Y}^{n-1}W_1) \leq H(\Psi_n|\hat{Y}^{n-1}) + \mathbb{P}(\Psi_n = 0) I(X_{2n}; Y_n|\Psi_n = 0, \hat{Y}^{n-1}, X_{1n}, W_1). \tag{361}\]
Note that \( \{\hat{\tau} \leq n - 1\} \) for the stop-feedback case (cf. Lemma 12) is equivalent to \( \{\hat{\tau} \leq n\} \) for the VLFT case we consider here. Observe that

\[
I(X_{1n}; Y_n | \Psi_n = 0, Y^{n-1}, X_{2n}, W_2) = h(Y_n | \Psi_n = 0, Y^{n-1}, X_{2n}, W_2) - h(Y_n | \Psi_n = 0, Y^{n-1}, X_{1n}, X_{2n}, W_2)
\]

From here on, we mimic Ozarow’s weak converse proof for the Gaussian MAC with fixed-length feedback [4] but with some changes in the parameter settings. First define

\[
\sigma^2_{jn} := \Var[X_{jn} | \Psi_n = 0], \quad j = 1, 2
\]

\[
\lambda_n := \Cov[X_{1n}, X_{2n} | \Psi_n = 0].
\]

Using the same approach as [4], we can show that

\[
h(X_{1n} + Z_n | \Psi_n = 0, X_{2n}) \leq \frac{1}{2} \log \left[ 2 \pi e \sigma^2_{1n} \left( 1 - \frac{\lambda_n^2}{\sigma^2_{1n} \sigma^2_{2n}} \right) + 2 \pi e \right].
\]

Therefore, we obtain

\[
I(W_1, W_2; \hat{Y}_n | \hat{Y}^{n-1}) \leq H(\Psi_n | \hat{Y}^{n-1}) + \frac{1}{2} P(\Psi_n = 0) \log \left[ 1 + \sigma^2_{1n} + \sigma^2_{2n} + 2 \lambda_n \right],
\]

\[
I(W_1; \hat{Y}_n | \hat{Y}^{n-1} W_2) \leq H(\Psi_n | \hat{Y}^{n-1}) + \frac{1}{2} P(\Psi_n = 0) \log \left[ 1 + \sigma^2_{1n} \left( 1 - \frac{\lambda_n^2}{\sigma^2_{1n} \sigma^2_{2n}} \right) \right].
\]

Similarly, we have

\[
I(W_2; \hat{Y}_n | \hat{Y}^{n-1} W_1) \leq H(\Psi_n | \hat{Y}^{n-1}) + \frac{1}{2} P(\Psi_n = 0) \log \left[ 1 + \sigma^2_{2n} \left( 1 - \frac{\lambda_n^2}{\sigma^2_{1n} \sigma^2_{2n}} \right) \right].
\]

It follows from (326), (329), and (330) and the above considerations that

\[
(1 - \varepsilon) \log M_1 M_2 \leq \sum_{n=1}^{\infty} H(\Psi_n | \hat{Y}^{n-1}) + \sum_{n=1}^{\infty} \frac{1}{2} P(\Psi_n = 0) \log \left[ 1 + \sigma^2_{1n} + \sigma^2_{2n} + 2 \lambda_n \right] + h_b(\varepsilon),
\]

\[
(1 - \varepsilon) \log M_1 \leq \sum_{n=1}^{\infty} H(\Psi_n | \hat{Y}^{n-1}) + \sum_{n=1}^{\infty} \frac{1}{2} P(\Psi_n = 0) \log \left[ 1 + \sigma^2_{1n} \left( 1 - \frac{\lambda_n^2}{\sigma^2_{1n} \sigma^2_{2n}} \right) \right] + h_b(\varepsilon),
\]

\[
(1 - \varepsilon) \log M_2 \leq \sum_{n=1}^{\infty} H(\Psi_n | \hat{Y}^{n-1}) + \sum_{n=1}^{\infty} \frac{1}{2} P(\Psi_n = 0) \log \left[ 1 + \sigma^2_{2n} \left( 1 - \frac{\lambda_n^2}{\sigma^2_{1n} \sigma^2_{2n}} \right) \right] + h_b(\varepsilon).
\]

Note that by [14, Equation (90)] and similarly to the analogous steps for the AWGN channel in (146)–(148), we have

\[
\sum_{n=1}^{\infty} H(\Psi_n | \hat{Y}^{n-1}) = H(\tau) \leq (N + 1) h_b \left( \frac{1}{N + 1} \right) \leq \log(N + 1) + 1.
\]

Moreover, since we have

\[
\sum_{n=1}^{\infty} P(\Psi_n = 0) = \sum_{n=1}^{\infty} P(\hat{\tau} > n)
\]

\[
= \sum_{n=1}^{\infty} P(\tau \geq n)
\]

\[
= \mathbb{E}(\tau),
\]
it follows that (337) holds so \( \left( \frac{P(\Psi_n = 0)}{2^\tau} \right)_{n=1}^\infty \) is a valid probability distribution. As in Ozarow’s weak converse proof for the Gaussian MAC with fixed-length feedback [4], the right-hand-sides of (375), (376), and (377) can be readily shown to be jointly concave in \((\sigma_{1n}^2, \sigma_{2n}^2, \lambda_n)\). Thus, we can use Jensen’s inequality to upper bound them.

More specifically, we set

\[
G_j^2 = \sum_{n=1}^\infty \frac{P(\Psi_n = 0)}{\mathbb{E}(\tau)} \sigma_{jn}^2, \quad j = 1, 2
\]

(382)

\[
\rho = \frac{1}{G_1 G_2} \sum_{n=1}^\infty \frac{P(\Psi_n = 0)}{\mathbb{E}(\tau)} \lambda_n.
\]

(383)

Note that, we have

\[
G_1^2 = \sum_{n=1}^\infty \frac{P(\Psi_n = 0)}{\mathbb{E}(\tau)} \sigma_{1n}^2,
\]

(384)

\[
\leq \sum_{n=1}^\infty \frac{P(\Psi_n = 0)}{\mathbb{E}(\tau)} \mathbb{E}(X_{1n}^2 | \Psi_n = 0)
\]

(385)

\[
\leq \sum_{n=1}^\infty \left[ \frac{P(\Psi_n = 0)}{\mathbb{E}(\tau)} \mathbb{E}(X_{1n}^2 | \Psi_n = 0) + \frac{P(\Psi_n = 1)}{\mathbb{E}(\tau)} \mathbb{E}(X_{1n}^2 | \Psi_n = 1) \right]
\]

(386)

\[
= \sum_{n=1}^\infty \frac{\mathbb{E}[X_{1n}^2]}{\mathbb{E}(\tau)}
\]

(387)

\[
\leq P_1.
\]

(388)

The last step follows from the expected power constraints in (156). Similarly, we have

\[
G_2^2 \leq P_2.
\]

(389)

Moreover, we also have \(|\lambda_n| \leq \sigma_{1n} \sigma_{2n}\) and so from (383) and the Schwarz inequality,

\[
|\rho|^2 \leq \left( \sum_{n=1}^\infty \frac{1}{G_1 G_2} \frac{P(\Psi_n = 0)}{\mathbb{E}(\tau)} \sigma_{1n} \sigma_{2n} \right)^2
\]

(390)

\[
\leq \left( \sum_{n=1}^\infty \frac{P(\Psi_n = 0)}{\mathbb{E}(\tau)} \sigma_{1n}^2 G_1^2 \right) \left( \sum_{n=1}^\infty \frac{P(\Psi_n = 0)}{\mathbb{E}(\tau)} \sigma_{2n}^2 G_2^2 \right)
\]

(391)

\[
= 1.
\]

(392)

Then, we have

\[
(1 - \varepsilon) \log M_1 M_2 \leq (N + 1) h_b \left( \frac{1}{N + 1} \right) + \frac{\mathbb{E}(\tau)}{2} \log \left[ 1 + G_1^2 + G_2^2 + 2 \rho G_1 G_2 \right]
\]

(393)

\[
\leq (N + 1) h_b \left( \frac{1}{N + 1} \right) + \frac{N}{2} \log \left[ 1 + G_1^2 + G_2^2 + 2 \rho G_1 G_2 \right]
\]

(394)

\[
\leq (N + 1) h_b \left( \frac{1}{N + 1} \right) + \frac{N}{2} \log \left[ 1 + P_1 + P_2 + 2|\rho| \sqrt{P_1 P_2} \right]
\]

(395)

and

\[
(1 - \varepsilon) \log M_j \leq (N + 1) h_b \left( \frac{1}{N + 1} \right) + \frac{\mathbb{E}(\tau)}{2} \log \left[ 1 + G_j^2 \left( 1 - \frac{(G_1 G_2 \rho)^2}{G_1^2 G_2^2} \right) \right]
\]

(396)

\[
= (N + 1) h_b \left( \frac{1}{N + 1} \right) + \frac{\mathbb{E}(\tau)}{2} \log \left[ 1 + G_j^2 (1 - \rho^2) \right],
\]

(397)

\[
\leq (N + 1) h_b \left( \frac{1}{N + 1} \right) + \frac{N}{2} \log \left[ 1 + G_j^2 (1 - \rho^2) \right],
\]

(398)

\[
\leq (N + 1) h_b \left( \frac{1}{N + 1} \right) + \frac{N}{2} \log \left[ 1 + P_j^2 (1 - \rho^2) \right], \quad j = 1, 2.
\]

(399)

This completes the converse proof of Theorem 4.
For completeness, we provide a proof of Lemma 2.

Proof: Define

\[ T_\tau := \sum_{n=1}^{\tau} E(X_n). \]  (400)

All we need to show is that \( S_\tau \) and \( T_\tau \) are integrable and that

\[ E(S_\tau) = E(T_\tau). \]  (401)

First, we show that \( S_\tau \) and \( T_\tau \) are integrable, i.e. \( E(|S_\tau|) < \infty \) and \( E(|T_\tau|) < \infty \). Indeed, define

\[ S_i = \sum_{n=1}^{i} X_n, \]  (402)

\[ T_i = \sum_{n=1}^{i} E(X_n), \]  (403)

for each \( i \in \mathbb{N} \). Since \( \tau \) takes values in \( \mathbb{N} \) and \( S_0 = 0, T_0 = 0 \), it follows that

\[ |S_\tau| = \sum_{i=1}^{\infty} |S_i| 1\{\tau = i\}, \]  (404)

\[ |T_\tau| = \sum_{i=1}^{\infty} |T_i| 1\{\tau = i\}. \]  (405)

Observe that

\[ E(|S_\tau|) = \sum_{i=1}^{\infty} E(|S_i| 1\{\tau = i\}) \]  (406)

\[ \overset{(a)}{=} \sum_{i=1}^{\infty} E \left( \sum_{n=1}^{i} |X_n| 1\{\tau = i\} \right) \]  (407)

\[ = \sum_{i=1}^{\infty} \sum_{n=1}^{i} E(|X_n| 1\{\tau = i\}) \]  (408)

\[ \overset{(b)}{=} \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} E(|X_n| 1\{\tau = i\}) \]  (409)

\[ \overset{(c)}{=} \sum_{n=1}^{\infty} E \left( |X_n| \sum_{i=n}^{\infty} 1\{\tau = i\} \right) \]  (410)

\[ = \sum_{n=1}^{\infty} E \left( |X_n| \sum_{i=n}^{\infty} 1\{\tau \geq n\} \right) \]  (411)

\[ \overset{(d)}{<} \infty. \]  (412)

Here, (a) follows from (402) and the triangle inequality, (b) follows from Tonelli’s theorem, (c) follows from the monotone convergence theorem, (d) follows from (30).
Similarly, we have

\[ \mathbb{E}(|T_\tau|) = \sum_{i=1}^{\infty} \mathbb{E}(|T_i| \mathbb{1}\{\tau = i\}) \]

\[ = \sum_{i=1}^{\infty} |T_i| \mathbb{E}(\mathbb{1}\{\tau = i\}) \]

\[ = \sum_{i=1}^{\infty} |T_i| \mathbb{P}(\tau = i) \]

\[ \leq (a) \sum_{i=1}^{\infty} \sum_{n=1}^{i} |\mathbb{E}(X_n)| \mathbb{P}(\tau = i) \]

\[ = (b) \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} |\mathbb{E}(X_n)| \mathbb{P}(\tau = i) \]

\[ = \sum_{n=1}^{\infty} |\mathbb{E}(X_n)| \sum_{i=n}^{\infty} \mathbb{P}(\tau = i) \]

\[ = \sum_{n=1}^{\infty} |\mathbb{E}(X_n)| \mathbb{P}(\tau \geq n) \]

\[ = \sum_{n=1}^{\infty} |\mathbb{E}(X_n)| \mathbb{P}(\tau \geq n) | \]

\[ \leq (c) \sum_{n=1}^{\infty} |\mathbb{E}(X_n)\mathbb{1}\{\tau \geq n\}) | \]

\[ \leq (d) \sum_{n=1}^{\infty} \mathbb{E}(|X_n| \mathbb{1}\{\tau \geq n\}) \]

\[ < \infty. \]

Here, (a) follows from (403) and the triangle inequality, (b) follows from Tonelli’s theorem, (c) follows from (29), (d) follows from (30). Now, we show that

\[ \mathbb{E}(S_\tau) = \mathbb{E}(T_\tau). \]

Indeed, we have

\[ \mathbb{E}(S_\tau) = \mathbb{E}\left( \sum_{i=1}^{\infty} S_i \mathbb{1}\{\tau = i\} \right) \]

\[ = (a) \sum_{i=1}^{\infty} \mathbb{E}[S_i \mathbb{1}\{\tau = i\}] \]

\[ = \sum_{i=1}^{\infty} \mathbb{E}\left[ \sum_{n=1}^{i} X_n \mathbb{1}\{\tau = i\} \right] \]

\[ = \sum_{i=1}^{\infty} \sum_{n=1}^{i} \mathbb{E}[X_n \mathbb{1}\{\tau = i\}] \]

\[ = (c) \sum_{n=1}^{\infty} \sum_{i=n}^{\infty} \mathbb{E}[X_n \mathbb{1}\{\tau = i\}] \]

\[ = (d) \sum_{n=1}^{\infty} \mathbb{E}\left[ \sum_{i=n}^{\infty} X_n \mathbb{1}\{\tau = i\} \right] \]

\[ = \sum_{n=1}^{\infty} \mathbb{E}\left[ \sum_{i=n}^{\infty} 1 \mathbb{1}\{\tau = i\} \right] \]

\[ = \sum_{n=1}^{\infty} \mathbb{E}[X_n \mathbb{1}\{\tau \geq n\}] \]
Since, we have

Here, (a) follows from the dominated convergence theorem (dominated by the integrable random variable $|S_n|$), (b) follows from (402), (c) follows from Fubini’s Theorem noting from (408) that $\sum_{n=1}^{\infty} \sum_{i=n}^{\infty} \mathbb{E}[|X_n^i|] < \infty$, (d) follows from the dominated convergence theorem (dominated by the integrable random variable $X_n$), (e) follows from (29), (f) follows from Fubini’s Theorem and the bound in (407), (g) follows from the dominated convergence theorem (dominated by integrable random variable $|T_\tau|$). This completes the proof.

**APPENDIX B**

**PROOF OF LEMMA 8**

*Proof:* First, observe that

$$\mathbb{E}[(X_{1N} + X_{2N})^2] + \mathbb{E}[(X_{1N} - X_{2N})^2] = 2[\mathbb{E}[X_{1N}^2] + \mathbb{E}[X_{2N}^2]]$$

$$= 2[\text{Var}(X_{1N}) + \mathbb{E}X_{1N}^2 + \text{Var}(X_{2N}) + \mathbb{E}X_{2N}^2]$$

$$\leq 2[L_1N + S_1 + o(1) + (N - A\sqrt{\frac{1}{N}} - D - B_1 + o(1))^2$$

$$+ L_2N + S_2 + o(1) + (N - A\sqrt{\frac{1}{N}} - D - B_2 + o(1))^2]$$

$$= 2[L_1N + S_1 + o(1) + L_2N + S_2 + o(1)] + (B_1 - B_2 + o(1))^2$$

$$\leq 2[L_1 + L_2]N + 2(S_1 + S_2) + (B_1 - B_2)^2 + o(1)$$

$$\leq 2[L_1 + L_2]N + 2|S_1 + S_2| + (B_1 - B_2)^2 + o(1).$$

Therefore, we have

$$(\mathbb{E}|X_{1N} - X_{2N}|^2 \leq \mathbb{E}[(X_{1N} - X_{2N})^2]$$

$$\leq 2[L_1 + L_2]N + 2|S_1 + S_2| + (B_1 - B_2)^2 + o(1).$$

$$\leq 2[L_1 + L_2]N + 2|S_1 + S_2| + (B_1 - B_2)^2 + o(1).$$
By using the fact that \((a + b)^{1/2} \leq a^{1/2} + b^{1/2}\) for nonnegative \(a, b\), it follows that
\[ \mathbb{E}|X_1 - X_2| \leq \sqrt{2(L_1 + L_2)\sqrt{N} + \sqrt{2}|S_1 + S_2| + (B_1 - B_2)^2} + o(1). \] (454)
Similarly, we have
\[ \mathbb{E}|X_i - X_j| \leq \sqrt{2(L_i + L_j)\sqrt{N} + \sqrt{2}|S_i + S_j| + (B_i - B_j)^2} + o(1). \] (455)
for any \((i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}\).
Now, we note that
\[ \max\{X_i, X_j\} = \frac{1}{2}[X_i + X_j + |X_i - X_j|] \] (456)
for any \((i, j) \in \{1, 2, 3\} \times \{1, 2, 3\}\).
Therefore, we have
\[ \max\{X_1, X_2, X_3\} = \max\{\max\{X_1, X_2\}, X_3\} \] (457)
\[ = \frac{1}{2}\max\{X_1 + X_2 + |X_1 - X_2|, 2X_3\} \] (458)
\[ = \frac{1}{4}\left[ X_1 + X_2 + |X_1 - X_2| + 2X_3 
+ |(X_1 + X_2 + |X_1 - X_2|) - 2X_3| \right] \] (459)
\[ = \frac{1}{4}\left[ (X_1 + X_2 + 2X_3) + |X_1 - X_2| 
+ |(X_1 - 3) + (X_2 - 3) + |X_1 - X_2| \right] \] (460)
\[ \leq \frac{1}{4}\left[ (X_1 + X_2 + 2X_3) + 2|X_1 - X_2| 
+ |X_1 - X_3| + |X_2 - X_3| \right]. \] (461)
It follows that
\[ \mathbb{E}[\max\{X_1, X_2, X_3\}] \leq \frac{1}{4}\mathbb{E}[X_1 + X_2 + 2X_3] 
+ \frac{1}{4}\mathbb{E}[2|X_1 - X_2| + |X_1 - X_3| + |X_2 - X_3|] \] (462)
\[ = \frac{1}{4}\left( \mathbb{E}[X_1] + \mathbb{E}[X_2] + 2\mathbb{E}[X_3] \right) 
+ \frac{1}{4}\left( \mathbb{E}[X_1 - X_2] \right) \] (463)
\[ = N - A\sqrt{N} - D - \frac{1}{4}(B_1 + B_2 + 2B_3) + o(1) 
+ \frac{1}{4}\left[ \sqrt{2(L_1 + L_2)\sqrt{N} + \sqrt{2}|S_1 + S_2| + (B_1 - B_2)^2} + o(1) \right] \] (464)
\[ = N - \sqrt{N}\left[ A - \frac{1}{2}(\sqrt{2(L_1 + L_2)}) 
- \frac{1}{4}(\sqrt{2(L_1 + L_3)} + \sqrt{2(L_2 + L_3)}) \right] - D - \frac{1}{4}(B_1 + B_2 + 2B_3) 
+ \frac{1}{2}\left( \sqrt{2|S_1 + S_2| + (B_1 - B_2)^2} \right) 
+ \frac{1}{4}\left( \sqrt{2|S_1 + S_3| + (B_1 - B_3)^2} + \sqrt{2|S_2 + S_3| + (B_2 - B_3)^2} \right) + o(1). \] (465)
Now, if we choose
\[ A = \frac{1}{2}\left( \sqrt{2(L_1 + L_2)} \right), \] (466)
\[ D = -\frac{1}{4}(B_1 + B_2 + 2B_3) + \frac{1}{2}\left( \sqrt{2|S_1 + S_2| + (B_1 - B_2)^2} \right) 
+ \frac{1}{4}\left( \sqrt{2|S_1 + S_3| + (B_1 - B_3)^2} + \sqrt{2|S_2 + S_3| + (B_2 - B_3)^2} \right), \] (467)
then we will have

\[ \mathbb{E}[\max\{X_{1N}, X_{2N}, X_{3N}\}] \leq N + o(1). \quad (468) \]

Note that since the role of \( X_{1N}, X_{2N}, X_{3N} \) are the same in the \( \max\{X_{1N}, X_{2N}, X_{3N}\} \), hence by the above approximation procedure, the smallest values of \( A \) that we can choose is given by (162). The proof of Lemma 8 is now complete.

**APPENDIX C**

**BOUNDS ON MUTUAL INFORMATION QUANTITIES FOR THE GAUSSIAN MAC**

We now prove the following lemma:

**Lemma 14.** For any stop-feedback code for the Gaussian MAC as in Definition 4 and its equivalent form as in (312)–(315) for the case \( |k_1| = |k_2| = 1 \), define

\[ \Psi_n := 1\{\hat{\tau} \leq n - 1\} \in \sigma(\hat{Y}^{n-1}). \quad (469) \]

Then the following inequalities hold:

\[ I(W_1 W_2; \hat{Y}^{\infty}) \leq \sum_{n=1}^{\infty} \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log(1 + \mathbb{E}[(X_{1n} + X_{2n})^2 | \Psi_n = 0]), \quad (470) \]

\[ I(W_1; \hat{Y}^{\infty} | W_2) \leq \sum_{n=1}^{\infty} \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log(1 + \mathbb{E}[X_{2n}^2 | \Psi_n = 0]), \quad (471) \]

\[ I(W_2; \hat{Y}^{\infty} | W_1) \leq \sum_{n=1}^{\infty} \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log(1 + \mathbb{E}[X_{2n}^2 | \Psi_n = 0]). \quad (472) \]

**Proof:** To show inequality (470), we observe that

\[ I(W_1 W_2; \hat{Y}^{\infty}) = \sum_{n=1}^{\infty} I(W_1, W_2; \hat{Y}_n | \hat{Y}^{n-1}). \quad (473) \]

Consider,

\[ I(W_1, W_2; \hat{Y}_n | \hat{Y}^{n-1}) \]

\[ = I(W_1, W_2; \hat{Y}_n, \Psi_n | \hat{Y}^{n-1}) \]

\[ = I(W_1, W_2; \hat{Y}_n | \hat{Y}^{n-1}) + I(W_1, W_2; \hat{Y}_n | \Psi_n, \hat{Y}^{n-1}) \quad (474) \]

\[ \leq H(\hat{Y}^{n-1}) + I(W_1, W_2; \hat{Y}_n | \Psi_n, \hat{Y}^{n-1}) \quad (475) \]

\[ = \mathbb{P}(\Psi_n = 0) I(W_1, W_2; \hat{Y}_n | \Psi_n = 0, \hat{Y}^{n-1}) + \mathbb{P}(\Psi_n = 1) I(W_1, W_2; \hat{Y}_n | \Psi_n = 1, \hat{Y}^{n-1}) \quad (476) \]

\[ \leq \mathbb{P}(\Psi_n = 0) I(\hat{X}_{1n}, \hat{X}_{2n}; \hat{Y}_n | \Psi_n = 0, \hat{Y}^{n-1}) \quad (477) \]

\[ = \mathbb{P}(\Psi_n = 0) \mathbb{E}[h(Y_n | \Psi_n = 0, \hat{Y}^{n-1}) - \mathbb{E}[h(Y_n | \Psi_n = 0, \hat{Y}^{n-1})] \quad (478) \]

\[ \leq \mathbb{P}(\Psi_n = 0) \mathbb{E}[h(Y_n | \Psi_n = 0) - h(Y_n | \Psi_n = 0, \hat{Y}^{n-1})] \quad (479) \]

\[ \mathbb{P}(\Psi_n = 0) \mathbb{E}[h(Y_n | \Psi_n = 0, \hat{Y}^{n-1}) - h(X_{1n} + X_{2n} + Z_n | X_{1n}, X_{2n}, \Psi_n = 0, \hat{Y}^{n-1})] \quad (480) \]

\[ \mathbb{P}(\Psi_n = 0) \mathbb{E}[h(Z_n | X_{1n}, X_{2n}, \Psi_n = 0, \hat{Y}^{n-1})] \quad (481) \]

\[ \mathbb{P}(\Psi_n = 0) \mathbb{E}[h(Z_n)] \quad (482) \]

\[ \leq \mathbb{P}(\Psi_n = 0) \left( \frac{1}{2} \log[2\pi e \mathbb{E}[(Y_n^2 | \Psi_n = 0)] - \frac{1}{2} \log[2\pi e] \right) \quad (483) \]

\[ = \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log[\mathbb{E}(Y_n^2 | \Psi_n = 0)] \quad (484) \]

\[ = \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log[\mathbb{E}(X_{1n} + X_{2n} + Z_n)^2 | \Psi_n = 0] \quad (485) \]

\[ = \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log[\mathbb{E}((X_{1n} + X_{2n})^2 | \Psi_n = 0) + \mathbb{E}(X_{1n} Z_n | \Psi_n = 0) + \mathbb{E}(X_{2n} Z_n | \Psi_n = 0) + \mathbb{E}(Z_n^2 | \Psi_n = 0)] \quad (486) \]

\[ = \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log[1 + \mathbb{E}((X_{1n} + X_{2n})^2 | \Psi_n = 0)]. \quad (487) \]
Here, (a) follows from the fact that \( \Psi_n \) is a discrete random variable taking values in \( \{0, 1\} \), (b) follows from the fact that \( \Psi_n \in \sigma(\hat{Y}^{n-1}) \), (c) follows from the fact that given \( \Psi_n = 1 \) or \( n \geq \hat{r} + 1 \) we always have \( \hat{Y}_n = T \), (d) follows from the fact that given \( \Psi_n = 0 \) or \( \tau \geq n \) we have \( X_{1n} = X_{1n}, \hat{X}_{2n} = X_{2n}, \hat{Y}_n = Y_n \), (e) follows from the fact that \( \Psi_n = 1 \{ \tau \leq n - 1 \} = 1 \{ \tau \leq n - 1 \} \) is a function of \( \sigma(\hat{Y}^{n-1}) \), \( X_{1n} = f_1^{(1)}(W_1) \), \( X_{2n} = f_2^{(1)}(W_2) \) and \( Z_n \) is independent of \( \hat{Y}^{n-1}, W_1, W_2 \), (f) follows from the maximal differential entropy formula, (g) follows from the fact that \( \Psi_n \) is a function of \( \hat{Y}^{n-1} \) and \( Z_n \) is independent of \( X_{1n}, X_{2n}, \hat{Y}^{n-1} \).

It follows that

\[
I(W_1W_2; \hat{Y}^{\infty}) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{P}(\Psi_n = 0) \log[1 + (X_{1n} + X_{2n})^2|\Psi_n = 0].
\]  

(492)

Now, we have

\[
I(W_1; \hat{Y}^{\infty}|W_2) = \sum_{n=1}^{\infty} I(W_1; \hat{Y}_n|W_2 \hat{Y}_n^{n-1}).
\]  

(493)

Using similar arguments, we have

\[
I(W_1; \hat{Y}_{1n}|\hat{Y}^{n-1}W_2)
\]

\[
\leq I(W_1; \hat{Y}_{1n}|\Psi_n \hat{Y}^{n-1}, W_2)
\]

\[
= I(W_1; \hat{Y}_{1n}|\hat{Y}^{n-1}W_2) + I(W_1; \hat{Y}_{1n}|\hat{Y}^{n-1}, W_2, \Psi_n)
\]

\[
\leq H(\hat{Y}_{1n}|\hat{Y}^{n-1}) + I(W_1; \hat{Y}_{1n}|\hat{Y}^{n-1}, W_2, \Psi_n)
\]

\[
= I(W_1; \hat{Y}_{1n}|\hat{Y}^{n-1}, W_2, \Psi_n)
\]

\[
= \mathbb{P}(\Psi_n = 0) I(W_1; \hat{Y}_{1n}|\Psi_n = 0, \hat{Y}^{n-1}, W_2)
\]

\[
+ \mathbb{P}(\Psi_n = 1) I(W_1; \hat{Y}_{1n}|\Psi_n = 1, \hat{Y}^{n-1}, W_2)
\]

\[
= \mathbb{P}(\Psi_n = 0) I(W_1; \hat{Y}_{1n}|\Psi_n = 0, \hat{Y}^{n-1}, W_2)
\]

\[
= \mathbb{P}(\Psi_n = 0) I(W_1; \hat{Y}_{1n}|\Psi_n = 0, \hat{Y}^{n-1}, \hat{X}_{2n}, W_2)
\]

\[
\leq \mathbb{P}(\Psi_n = 0) I(\hat{X}_{1n}; \hat{Y}_{1n}|\Psi_n = 0, \hat{Y}^{n-1}, \hat{X}_{2n}, W_2)
\]

\[
\leq \mathbb{P}(\Psi_n = 0) I(\hat{X}_{1n}; \Psi_n = 0, \hat{Y}^{n-1}, \hat{X}_{2n}, W_2)
\]

\[
\leq \mathbb{P}(\Psi_n = 0) [h(Y_n|\Psi_n = 0, Y^{n-1}, X_{2n}, W_2) - h(Y_n|\Psi_n = 0, Y^{n-1}X_{1n}, X_{2n}, W_2)]
\]

\[
\leq \mathbb{P}(\Psi_n = 0) [h(Y_n|\Psi_n = 0, X_{2n}) - h(X_{1n} + X_{2n} + Z_n|\Psi_n = 0, Y^{n-1}, X_{1n}, X_{2n}, W_2)]
\]

\[
= \mathbb{P}(\Psi_n = 0) [h(Y_n|\Psi_n = 0, X_{2n}) - h(Z_n|\Psi_n = 0, Y^{n-1}, X_{1n}, X_{2n}, W_2)]
\]

\[
= \mathbb{P}(\Psi_n = 0) [h(Y_n|\Psi_n = 0, X_{2n}) - h(Z_n)]
\]

\[
= \mathbb{P}(\Psi_n = 0) [h(Y_n|\Psi_n = 0, X_{2n}) - \frac{1}{2} \log(2\pi e)]
\]

\[
= \mathbb{P}(\Psi_n = 0) [h(X_{1n} + Z_n|\Psi_n = 0, X_{2n}) - \frac{1}{2} \log(2\pi e)]
\]

\[
= \mathbb{P}(\Psi_n = 0) [h(X_{1n} + Z_n|\Psi_n = 0, X_{2n}) - \frac{1}{2} \log(2\pi e)]
\]

\[
\leq \mathbb{P}(\Psi_n = 0) [h(X_{1n} + Z_n|\Psi_n = 0) - \frac{1}{2} \log(2\pi e)]
\]

\[
\leq \mathbb{P}(\Psi_n = 0) \left[ \frac{1}{2} \log(2\pi e) \mathbb{E}[(X_{1n} + Z_n)^2|\Psi_n = 0] - \frac{1}{2} \log(2\pi e) \right]
\]

\[
= \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log(1 + \mathbb{E}[X_{1n}^2|\Psi_n = 0])
\]

\[
= \frac{1}{2} \mathbb{P}(\Psi_n = 0) \log(1 + \mathbb{E}[X_{2n}^2|\Psi_n = 0]).
\]

(512)

Here, (a) follows from the fact that given \( \Psi_n = 0 \) or \( \tau \geq n \) we have \( \hat{X}_{1n} = X_{1n}, \hat{X}_{2n} = X_{2n}, \hat{Y}_n = Y_n \). It follows that

\[
I(W_1; \hat{Y}^{\infty}|W_2) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{P}(\Psi_n = 0) \log(1 + \mathbb{E}[X_{1n}^2|\Psi_n = 0]).
\]

(514)

Similarly, we also have

\[
I(W_2; \hat{Y}^{\infty}|W_1) \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \mathbb{P}(\Psi_n = 0) \log(1 + \mathbb{E}[X_{2n}^2|\Psi_n = 0]).
\]

(515)
This completes the proof of Lemma 14.

REFERENCES

[1] C. E. Shannon. The zero error capacity of a noisy channel. *IRE Trans. on Inform. Th.*, 2(3):8–19, 1956.
[2] J. Schalkwijk and T. Kailath. A coding scheme for additive noise channels with feedback—Part I: No bandwith constraint. *IEEE Trans. on Inform. Th.*, 12(2):172–182, 1966.
[3] M. Burnashev and H. Yamamoto. On using feedback in a Gaussian channel. *Problems of Information Transmission*, 50(3):19–34, 2014.
[4] L. H. Ozarow. The capacity of the white Gaussian multiple access channel with feedback. *IEEE Trans. on Inform. Th.*, 30(4):623–629, 1984.
[5] O. Shayevitz and M. Feder. Optimal feedback communication via posterior matching. *IEEE Trans. on Inform. Th.*, 57(3):1186–1222, 2011.
[6] L. V. Truong, S. L. Fong, and V. Y. F. Tan. On Gaussian channels with feedback under expected power constraints and with non-vanishing error probabilities. *Information Theory (cs.IT)*, 2015. arXiv:1512.05088 [cs.IT].
[7] T. Cover and S. Pombra. Gaussian feedback capacity. *IEEE Trans. on Inform. Th.*, 35(1):37–43, 1989.
[8] Y. H. Kim. Feedback capacity of stationary Gaussian channels. *IEEE Trans. on Inform. Th.*, 56(1):57–85, 2010.
[9] H. Permuter, P. Cuff, B. Van Roy, and T. Weissman. Capacity of the trapdoor channel with feedback. *IEEE Trans. on Inform. Th.*, 54(7):3150–3165, 2008.
[10] O. Elischo and H. Permuter. Capacity and coding for the Ising channel with feedback. *IEEE Trans. on Inform. Th.*, 60(9):5138–5149, 2014.
[11] Y. Altuğ and A. B. Wagner. Feedback can improve the second-order coding performance in discrete memoryless channels. In *Proc. of Intl. Symp. on Inform. Th.*, pages 2361–2365, Honolulu, HI, 2014.
[12] M. V. Burnashev. Data transmission over a discrete channel with feedback. Random transmission time. *Problems of Information Transmission*, 12(4):10–30, 1976.
[13] H. Yamamoto and K. Itoh. Asymptotic performance of a modified Schalkwijk-Barron scheme for channels with noiseless feedback. *IEEE Trans. on Inform. Th.*, 25(6):729–733, 1979.
[14] Y. Polyanskiy, H. V. Poor, and S. Verdú. Feedback in the non-asymptotic regime. *IEEE Trans. on Inform. Th.*, 57(8):4903–4925, 2011.
[15] K. F. Trillingsgaard and P. Popovski. Variable-length coding for short packets over a multiple access channel with feedback. In *Proc. 11th Intl. Symp. on Wireless Communications Systems*, pages 796–800, Barcelona, Spain, 2014.
[16] V. Y. F. Tan and O. Kosut. On the dispersions of three network information theory problems. *IEEE Trans. on Inform. Th.*, 60(2):881–903, 2014.
[17] E. MolavianJazi and J. N. Laneman. Simpler achievable rate regions for multiaccess with finite blocklength. In *Proc. of Intl. Symp. on Inform. Th.*, pages 36–40, Cambridge, MA, 2012.
[18] R. Devassy, G. Durisi, B. Lindqvist, W. Yang, and M. Dalai. Nonasymptotic coding-rate bounds for binary erasure channels with feedback. *Information Theory (cs.IT)*, 2016. arXiv:1607.06837 [cs.IT].
[19] Y. Polyanskiy. Dispersion of compound channels. In *Proc. of Allerton Conference*, Monticello, IL, 2013.
[20] K. F. Trillingsgaard, W. Yang, G. Durisi, and P. Popovski. Broadcasting a common message with variable-length stop-feedback codes. In *Proc. of Intl. Symp. on Inform. Th.*, pages 2505–2509, Hong Kong, China, 2015.
[21] K. F. Trillingsgaard, W. Yang, G. Durisi, and P. Popovski. Variable-length coding with stop-feedback for the common-message broadcast channel in the nonasymptotic regime. *Information Theory (cs.IT)*, 2016. arXiv:1607.03519 [cs.IT].
[22] A. Gut. On the moments and limit distributions of some first passage times. *The Annals of Probability*, 2(2):277–308, 1974.
[23] D. Williams. *Probabilities with Martingales*. Cambridge Univ. Press, 1991.
[24] T. L. Lai and D. Siegmund. A nonlinear renewal theory with applications to sequential analysis II. *The Annals of Statistics*, 7(1):60–76, 1979.
[25] A. El Gamal and Y.-H. Kim. *Network Information Theory*. Cambridge University Press, Cambridge, U.K., 2012.
[26] C. Stone. On characteristic functions and renewal theory. *Trans. Amer. Math. Soc.*, 120(2):327–342, 1965.
[27] G. R. Grimmett and D. R. Stirzaker. *Probability and Random Processes*. Oxford Science Publications, 2nd edition, 1992.
[28] P. Billingsley. *Probability and Measure*. Wiley-Interscience, 3rd edition, 1995.
[29] F. Thomas Bruss and J. B. Robertson. “Wald’s Lemma” for sums of order statistics of i.i.d. random variables. *Advances in Applied Probability*, 23(3):812–823, 1991.
[30] R. N. Bhattacharya and R. R. Rao. *Normal Approximation and Asymptotic Expansions*. Wiley, New Jersey, United States, 1976.