Three-qubit dynamics of entanglement in the magnetic field

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Abstract

A closed system of the equations for the local Bloch vectors and spin correlation functions is obtained by decomplexification of the Liouville-von Neumann equation for three magnetic qubits with the exchange interaction, that takes place in an arbitrary time-dependent external magnetic field. The numerical comparative analysis of entanglement is carried out depending on initial conditions and the magnetic field modulation. The present study may be useful for analysis of interference experiments and in the field of quantum computing.

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INTRODUCTION

The phenomenon of entanglement (inseparability) is one of the properties of quantum systems. A quantum entanglement is the key source of such applications as superdense coding, teleportation, quantum calculations researched by the quantum information theory \[1, 2, 3\]. The most known and natural way creating the entanglement is the global unitary evolution caused by the interaction between subsystems, which is controllable by external fields. The aim of this work has been to make a comparative analysis of the measures of entanglement in a three-qubit system located in a magnetic field, because up to now contrary to two-qubit system, there has been no unified measure entanglement for three qubits.

THE MODEL HAMILTONIAN

The Hamiltonian of three coupled by exchange interaction magnetic qubits $e, p, n$ (particles with spin 1/2) placed in an external variable magnetic field $H = (H_1, H_2, H_3)$ looks like

$$\hat{H} = h^e_i s^e_i + h^p_i s^p_i + h^n_i s^n_i + 2J^{ep} s^e_i s^p_i + 2J^{en} s^e_i s^n_i + 2J^{pn} s^p_i s^n_i,$$

where $h^e_i, h^p_i, h^n_i$ are the Cartesian components of the external magnetic field multiplied by gyromagnetic ratio of the corresponding qubits; $s^e_i = \frac{1}{2} \sigma_0 \otimes \sigma_0 \otimes \sigma_i$, $s^p_i = \frac{1}{2} \sigma_0 \otimes \sigma_i \otimes \sigma_0$, $s^n_i = \frac{1}{2} \sigma_0 \otimes \sigma_0 \otimes \sigma_i$ is the matrix representation of spin operators of magnetic qubits; the Pauli matrices are equal to $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; $\otimes$ is the symbol of direct product \[4\]; $J^{ep}, J^{en}, J^{pn}$ are the constants of isotropic exchange interaction between qubits; the summation over $e, p, n$ is absent.

DECOMPLEXIFICATION OF THE LIOUVille-VON NEUMANN EQUATION

The Liouville-von Neumann equation for the density matrix $\rho$, describing the dynamics of a three-qubit system, looks like

$$i\partial_t \rho = [\hat{H}, \rho], \rho(t = 0) = \rho_0.$$  \hspace{1cm} (2)

Let us present the solution of the equation (2) as

$$\rho = \frac{1}{8} R_{\alpha\beta\gamma} \sigma_\alpha \otimes \sigma_\beta \otimes \sigma_\gamma, \rho^+ = \rho, R_{000} = 1, S \rho \rho = 1.$$  \hspace{1cm} (3)

Hereinafter summation is taken over the repeating Greek indices from zero up to four, and over Latin indices from one up to three. The three coherence vectors (the Bloch vectors) widely used in the magnetic resonance theory, are written as

$$R_{000} = S \rho \rho \sigma_i \otimes \sigma_0 \otimes \sigma_0, \hspace{1cm} (4a)$$

$$R_{00i} = S \rho \rho \sigma_0 \otimes \sigma_i \otimes \sigma_0, \hspace{1cm} (4b)$$

$$R_{0ii} = S \rho \rho \sigma_0 \otimes \sigma_0 \otimes \sigma_i. \hspace{1cm} (4c)$$
They characterize the local properties of individual qubits, whereas the following tensors

\[
R_{kq0} = Spp\sigma_k \otimes \sigma_q \otimes \sigma_0, \quad (5a)
\]

\[
R_{k0q} = Spp\sigma_k \otimes \sigma_0 \otimes \sigma_q, \quad (5b)
\]

\[
R_{0kq} = Spp\sigma_0 \otimes \sigma_k \otimes \sigma_q, \quad (5c)
\]

\[
R_{kql} = Spp\sigma_k \otimes \sigma_q \otimes \sigma_l \quad (5d)
\]

describe the spin correlations.

The length of the generalized Bloch vector \( b \) is conserved under unitary evolution.

\[
b = \sqrt{R_{\alpha\beta\gamma}^2}. \quad (6)
\]

The Liouville-von Neumann equation accepts the real form in terms of the functions \( R_{\alpha\beta\gamma} \) as closed system of 63 differential equations for the local Bloch vectors and spin correlation functions

\[
\partial_t R_{q00} = \varepsilon_{ilq}h_i^e R_{l00} + \varepsilon_{mlq}(J^{ep} R_{lm0} + J^{en} R_{l0m}), \quad (7a)
\]

\[
\partial_t R_{0q0} = \varepsilon_{ilq}h_i^p R_{l0l} + \varepsilon_{mlq}(J^{ep} R_{ml0} + J^{en} R_{mlm}), \quad (7b)
\]

\[
\partial_t R_{00q} = \varepsilon_{ilq}h_i^n R_{l0l} + \varepsilon_{imq}(J^{en} R_{lom} + J^{pm} R_{l0m}), \quad (7c)
\]

\[
\partial_t R_{qk0} = \varepsilon_{ilq}h_i^e R_{lkl} + \varepsilon_{imk}h_i^p R_{qlm} + J^{en}\varepsilon_{kmq}(R_{m00} - R_{0m0}) + J^{en}\varepsilon_{lmq} R_{mlk} + J^{pm}\varepsilon_{lmk} R_{qml}, \quad (7d)
\]

\[
\partial_t R_{q0k} = \varepsilon_{ilq}h_i^p R_{l0k} + \varepsilon_{imk}h_i^n R_{qlm} + J^{en}\varepsilon_{qmk}(R_{00m} - R_{00m}) + J^{en}\varepsilon_{lmq} R_{lmk} + J^{pm}\varepsilon_{lkm} R_{qlm}, \quad (7e)
\]

\[
\partial_t R_{0qk} = \varepsilon_{ilq}h_i^n R_{l0k} + \varepsilon_{imk}h_i^p R_{qlm} + J^{en}\varepsilon_{qmk}(R_{00m} - R_{00m}) + J^{en}\varepsilon_{lmq} R_{lmk} + J^{pm}\varepsilon_{lkm} R_{qlm}, \quad (7f)
\]

\[
\partial_t R_{qkl} = \varepsilon_{imq}h_i^e R_{mlk} + \varepsilon_{imk}h_i^p R_{qm0} + \varepsilon_{iml}h_i^n R_{qkm} + J^{ep}\varepsilon_{kmq}(R_{m0l} - R_{0ml}) + J^{en}\varepsilon_{qml}(R_{0km} - R_{km0}) + J^{pm}\varepsilon_{kmq}(R_{q0m} - R_{q0m}). \quad (7g)
\]

for the set of initial conditions.

In system (7), assuming, for example, \( J^{en} = 0 \), \( J^{pm} = 0 \), we get the closed system of equations for the description of two-qubit dynamics

\[
\partial_t R_{q0} = \varepsilon_{ilq}h_i^e R_{l0l} + \varepsilon_{mlq} J^{ep} R_{lm}, \quad (8a)
\]

\[
\partial_t R_{0q} = \varepsilon_{ilq}h_i^p R_{l0l} + \varepsilon_{mlq} J^{ep} R_{ml}, \quad (8b)
\]
\[ \partial_t R_{jk} = \varepsilon_{ilk} h^i_l R_{lk} + \varepsilon_{imk} h^p_k R_{qm} + J^{ep} \varepsilon_{kmq} (R_{m0} - R_{0m}), \]  

(8c)

where \( R_{q0} = S p \sigma_q \otimes \sigma_0, \ R_{0q} = S p \sigma_0 \otimes \sigma_q, \ R_{kq} = S p \sigma_k \otimes \sigma_q. \)

The concrete calculations will be carried out for the following initial conditions.

The fully separable state (S) is

\[ | S > = | 111 >, \]  

(9)

the biseparable state (BS) is \[ | BS > = \frac{1}{\sqrt{2}}(| 001 > + | 010 >), \]  

(10)

the Greenberger-Horne-Zeilinger maximally entangled state (GHZ) is

\[ | GHZ > = \frac{1}{\sqrt{2}}(| 000 > + | 111 >), \]  

(11)

the Werner entangled state (W) is

\[ | W > = \frac{1}{\sqrt{3}}(| 001 > + | 010 > + | 100 >), \]  

(12)

the mixed state (Mix) \[ \rho_0 = x | GHZ > < GHZ | + \frac{1 - x}{2} (| W > < W | + | V > < V |), \]  

(13)

where

\[ | V > = \frac{1}{\sqrt{3}}(| 110 > + | 101 > + | 011 >), \]  

\[ 1/3 < x \leq 1. \]  

(14)

The length of the generalized Bloch vector (6) for pure states (9-12) is equal to \( \sqrt{7} \), and for the mixed state (13) at \( x = \frac{2}{3} \) it is equal to \( \sqrt{3} \).

**MEASURES OF GLOBAL ENTANGLEMENT IN THE THREE-QUBIT SYSTEM**

We shall define the measures of entanglement of three qubits \[ \] on solutions of the system (7), by introducing the tensors of two-particle entanglement:

\[ m_{ij0} = R_{ij0} - R_{i00} R_{0j0}, \]  

(15a)

\[ m_{i0j} = R_{i0j} - R_{i00} R_{00j}, \]  

(15b)

\[ m_{0ij} = R_{0ij} - R_{0i0} R_{00j}. \]  

(15c)

The tensors (15) are equal to zero, if the two-particle correlation functions (5a, 5b, 5c) are factorized in terms of the Bloch vectors (4). With the help of these tensors we shall define the measure of two-particle entanglement in two-qubit system as

\[ m = m_{ij}^2, \]  

(16)
where $m_{ij} = R_{ij} - R_{i0}R_{0j}$. The three-particle entanglement tensor $m_{ijk}$ is obtained by subtraction from three-particle spin correlation function $R_{ijk}$ (5d) of all products of the Bloch vectors and the tensors of lower order

$$m_{ijk} = R_{ijk} - R_{i00}m_{0jk} - R_{0j0}m_{ij0} - R_{00k}m_{ij0} - R_{i00}R_{0j0}R_{00k}. \quad (17)$$

This decomposition can be considered as a kind of cluster expansion in statistical mechanics. In terms of these tensors the measure of three-particle entanglement becomes

$$m_{SM} = m_{ijk}^2. \quad (18)$$

This measure is equal to zero when the tensor $R_{ijk}$ can be expressed through two-particle entanglements (15) and the local Bloch vectors. It is also applicable for pure as well as for mixed states.

The concurrence depends on all six reduced density matrices, and for pure states it can be represented as

$$C_3 = \frac{1}{\sqrt{2}} \sqrt{6 - \frac{9}{4} + R_{i00}^2 + R_{0j0}^2 + R_{00k}^2 + \frac{1}{4}(R_{m00}^2 + R_{m0n}^2 + R_{0mn}^2)]. \quad (19)$$

The measures $m_{SM}$ and $C_3$ are not normalized to 1.

The measure of global entanglement in the form is expressed through the reduced matrices of individual qubits $\rho_s = S\rho_{pn}\rho$, $\rho_p = S\rho_{en}\rho$, $\rho_n = S\rho_{ep}\rho$ according to the formula

$$m_B = 1 - \frac{R_{i00}^2 + R_{0j0}^2 + R_{00k}^2}{3}. \quad (20)$$

For the initial GHZ state of three qubits the three-tangle measure, expressed through the Cayley hyperdeterminant, is equal to

$$m_K = 4\rho_{11}\rho_{88}, \quad (21)$$

where the diagonal density matrix elements determine population probabilities

$$\rho_{11} = \frac{1}{8}(R_{300} + R_{030} + R_{003} + R_{330} + R_{303} + R_{033} + R_{333} + 1), \quad (22)$$

$$\rho_{88} = \frac{1}{8}(-R_{300} - R_{030} - R_{003} + R_{330} + R_{303} + R_{033} - R_{333} + 1). \quad (23)$$

The measure of global entanglement is not equal to zero for fully inseparable pure states and is determined by the expression

$$m_L = \sqrt[3]{(1 - R_{i00}^2)(1 - R_{0j0}^2)(1 - R_{00k}^2)}. \quad (24)$$

This measure is equal to one for initial GHZ state and to 8/9 for the W state, and it takes the same values for the $m_B$ measure.
NUMERICAL RESULTS

Let us consider two cases, when the external magnetic fields depend on the dimensionless time $\tau = \omega t$:
the resonant (relative to $e$ qubit) circularly polarized field (R) is

$$
H = -\left( \frac{\omega_1}{\omega} \cos \tau, \frac{-\omega_1}{\omega} \sin \tau, \frac{\omega_0}{\omega} \right),
$$

(25)

and the nonresonant circularly polarized field (NR) is

$$
H = -\left( \frac{\omega_1}{\omega} \cos \tau, \frac{\omega_1}{\omega} \sin \tau, \frac{\omega_0}{\omega} \right),
$$

(26)

where $\omega$ is the frequency of the external field, $\omega_1$ and $\omega_0$ are the dimensionless amplitudes of transverse and longitudinal field correspondingly. Let us assume, that the fields, operating on the qubits $e$, $p$ and $n$, are equal to $h^e = H$, $h^p = 2H$ and $h^n = 4H$, respectively.

To perform the numerical simulation, we have chosen the following parameters in units of $2\pi \times 100$ MHz, $\omega = \omega_0 = 1$. It corresponds to the longitudinal field $H_3 = 2.3487$ T for the proton resonance of $e$ qubit. The exchange constants are equal to $J^{ep} = -0.2; J^{en} = -0.1; J^{pn} = -0.3$ and $\omega_1 = 0.3$ in the same units. In figures 1 and 2 the solid line corresponds to the NR field, and dashed line - to the R field.

The $S$ state. The measure $m_{SM}$ is not sensitive to the NR field (amplitude of fluctuations is less than 0.001), but it shows a resonant behavior in the R field (Fig.1). The measures $C_3, m_B, m_L$ display irregular fluctuations with a low amplitude in the NR field and with a high amplitude in the R field. Fig. 2 shows the dynamics of measure $m_L$. It can be seen that the behavior of measures $m_{SM}$ and $m_L$ is similarly.

The $BS$ state. The measure $m_{SM}$ has a periodic character in the NR field and shows irregular fluctuations with a high amplitude in the R field. The amplitude of fluctuations in the R field is 3 times larger than in the NR field. The entanglement measures $C_3, m_B, m_L$ display irregular fluctuations with a high amplitude, and they are also little different for the R and NR fields. The measure $C_3$ is shown in Fig. 2.

The $GHZ$ state. The measure $m_{SM}$ is well distinct for R and NR fields. The measures $C_3, m_B, m_L$ exhibit low-amplitude fluctuations in the NR field. In the R field the entanglement is less than in the NR field at all times, but the occurring fluctuations have a high amplitude. Fig. 2 shows the dynamics of measure $m_K$.

The $W$ state. The measure $m_{SM}$ has an oscillatory character at times of $0 < \tau < 15$ and also it is also practically indiscernible for R and NR fields. For $\tau > 20$ the amplitude of fluctuations in the R field is 3 times higher than in the NR field (it is not shown in Fig. 1). The entanglement measures $C_3, m_B, m_L$ display a qualitatively similar behavior in both the R and NR field. Fig. 2 shows the dynamics of measure $m_B$.

The $Mix$ state. It can be seen from Fig. 1, that the measure $m_{SM}$ insignificantly changes in comparison with the initial value in the NR field, while the R field appreciably reduces the entanglement.

As a matter of fact, the solution of equation (2) for a constant external field looks like

$$
\rho = \exp(-i\hat{H}t)\rho_0\exp(i\hat{H}t).
$$

(27)

Evidently, if $[\rho_0, \hat{H}] = 0$, then $\rho = \rho_0$ for all $t$. Thus the value of an arbitrary function dependent on solutions of equation (2) will be equal to the initial value. In other words, the
density matrix will be fixed for this Hamiltonian. The influence of time-dependent magnetic field is insignificant for $\omega_0 >> \omega_1$, therefore all entanglement measures are constant and equal to the initial entanglement.

In the model considered, a decrease in the transverse field amplitude leads to the decrease in the amplitudes of fluctuations for all entanglement measures, regardless of the initial conditions.

We shall consider the influence of qubits $e, p$ as two fluctuators on the $n$ qubit dynamics. From system (7c) it follows for $J^{en} = 0$, $J^{pn} = 0$, that the spin flip probability of the free qubit $n$ from an initial state $R_{001}(0) = 0$, $R_{002}(0) = 0$, $R_{003}(0) = 1$ is equal to

$$P = \frac{1 - R_{003}}{2}. \quad (28)$$

Fig. 3 shows the spin flip probability of $n$ qubit in the resonant field and the dependence of this probability on the field deformation caused by the presence of qubits $e, p$ in the initial S state. It can be seen that the Rabi oscillations transform to beats. This is in qualitative accordance with the results of work [14].
CONCLUSION

The closed system of equations for the local Bloch vectors and spin correlation functions for three magnetic qubits with the exchange interaction, that takes place in any time-dependent external magnetic field has been derived. The numerical comparative analysis of entanglement measures has been made, depending on initial conditions and the magnetic field modulation.

From the experimental point of view, the measure $m_K$ is more acceptable as it is expressed through the populations $\rho_{11}, \rho_{88}$. The measures $C_3$ and $m_{SM}$ for GHZ conditions which also show sharp difference for R and NR fields but they depend in a complicated manner on all elements of the density matrix $\rho$.

The present study will permit one to bring closer the theoretical results [7, 8, 10, 11, 12, 13] to the possibility of experimental corroboration.

The proposed approach can be realized without any specific difficulties for the four-qubit system the detailed dynamics of which is described by 255 equations with the length of the generalized Bloch vector for pure states $b = \sqrt{15}$.  

Figure 2: Entanglement dynamics of the measures $m_L$, $C_3$, $m_K$, $m_B$ versus initial conditions
Figure 3: The influence of $e$, $p$ qubits as fluctuators (a thick line) on spin flip probability of the free $n$ qubit $n$ (a thin line) in the $R$ field.

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