A HARDY INEQUALITY FOR ULTRASPHERICAL EXPANSIONS
WITH AN APPLICATION TO THE SPHERE

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ABSTRACT. We prove a Hardy inequality for ultraspherical expansions by using a proper ground state representation. From this result we deduce some uncertainty principles for this kind of expansions. Our result also implies a Hardy inequality on spheres with a potential having a double singularity.

1. INTRODUCTION AND MAIN RESULT

For \( d \geq 3 \), the classical Hardy inequality states that
\[
\frac{(d - 2)^2}{4} \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^2} \, dx \leq \int_{\mathbb{R}^d} |\nabla u(x)|^2 \, dx.
\]
Due to its applicability, there is an extensive literature about the topic (see the references in [16]) covering many extensions of this estimate in several and different directions. We are interested in one involving the fractional powers of the Laplacian.

We can rewrite (1) as
\[
\frac{(d - 2)^2}{4} \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^2} \, dx \leq \int_{\mathbb{R}^d} u(x)(-\Delta u(x)) \, dx
\]
and, taking the fractional Laplacian \((\Delta)^\sigma\) defined by \((\Delta)^\sigma u = |\cdot|^{2\sigma} \hat{u}\), a natural extension is the inequality
\[
C_{\sigma,d} \int_{\mathbb{R}^d} \frac{u^2(x)}{|x|^{2\sigma}} \, dx \leq \int_{\mathbb{R}^d} u(x)(-\Delta)^\sigma u(x) \, dx,
\]
for which the sharp constant \(C_{\sigma,d}\) is well known (see [3, 20]).

From (2), we deduce the positivity (in a distributional sense) of the operator
\[
(-\Delta)^\sigma - \frac{C_{\sigma,d}}{|\cdot|^{2\sigma}}.
\]
Our target is to provide a Hardy inequality like (2) related to ultraspherical expansions and apply it to prove the positivity of certain operator on the sphere with a potential having singularities in both poles of the sphere.

Let \(C_n^\lambda(x)\) be the ultraspherical polynomial of degree \(n\) and order \(\lambda > -1/2\). We consider \(c_n^\lambda(x) = d^{-1}C_n^\lambda(x)\) with
\[
d_n^\lambda = \int_{-1}^{1} (C_n^\lambda(x))^2 \, d\mu_\lambda(x), \quad d\mu_\lambda(x) = (1 - x^2)^{\lambda-1/2} \, dx.
\]

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The sequence of polynomials \( \{c_\lambda^n\}_{n \geq 0} \) forms an orthonormal basis of the space \( L^2_\lambda := L^2((-1,1),d\mu_\lambda) \). For each \( c_\lambda^n \), it holds that \( \mathcal{L}_\lambda c_\lambda^n = -(n + \lambda)^2 c_\lambda^n \), where

\[
\mathcal{L}_\lambda = (1 - x^2) \frac{d^2}{dx^2} - (2\lambda + 1)x \frac{d}{dx} - \lambda^2.
\]

The ultraspherical expansion of each appropriate function \( f \) defined in \((-1,1)\) is given by

\[
f \mapsto \sum_{n=0}^{\infty} a_\lambda^n(f) c_\lambda^n,
\]
where \( a_\lambda^n(f) \) is the \( n \)-th Fourier coefficient of \( f \) respect to \( \{c_\lambda^n\}_{n \geq 0} \), i.e.,

\[
a_\lambda^n(f) = \int_{-1}^{1} f(y) c_\lambda^n(y) d\mu_\lambda(y).
\]

The fractional powers of the operator \( \mathcal{L}_\lambda \) are defined by

\[
(-\mathcal{L}_\lambda)^{\sigma/2} f = \sum_{n=0}^{\infty} (n + \lambda)^\sigma a_\lambda^n(f) c_\lambda^n, \quad \sigma > 0.
\]

This operator should be the natural candidate to prove a Hardy type inequality for the ultraspherical expansion but, however, it is not the most appropriate in this setting. We have to consider other one with an analogous behaviour to \( (-\mathcal{L}_\lambda)^{\sigma/2} \), in order to deduce some results on the sphere. For each \( \sigma > 0 \) we define (spectrally) the operator

\[
A_\lambda^\sigma = \frac{\Gamma(\sqrt{\mathcal{L}_\lambda + \frac{1+\sigma}{2}})}{\Gamma(\sqrt{\mathcal{L}_\lambda + \frac{1-\sigma}{2}})}.
\]

Then for \( f \) defined on the interval \((-1,1)\)

\[
A_\lambda^\sigma f(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n + \lambda + \frac{1+\sigma}{2})}{\Gamma(n + \lambda + \frac{1-\sigma}{2})} a_\lambda^n(f) c_\lambda^n(x).
\]

Note that

\[
\frac{\Gamma(n + \lambda + \frac{1+\sigma}{2})}{\Gamma(n + \lambda + \frac{1-\sigma}{2})} \simeq (n + \lambda)^\sigma,
\]
then the behaviour of \( (-\mathcal{L}_\lambda)^{\sigma/2} \) and \( A_\lambda^\sigma \) is similar. The natural Sobolev space to analyse Hardy type inequalities is

\[
H_\lambda^\sigma = \left\{ f \in L_\lambda^2 : \|f\|_{H_\lambda^\sigma} := \left( \sum_{n=0}^{\infty} (n + \lambda)^\sigma (a_\lambda^n(f))^2 \right)^{1/2} < \infty \right\}.
\]

We have to note that \( H_\lambda^\sigma \) is equivalent to the space \( L^2_{\lambda,\sigma} \) introduced in [5].

With the previous notation our Hardy inequality for ultraspherical expansions is given in the following result.

**Theorem 1.** Let \( \lambda > 0 \) and \( 0 < \sigma < 1 \). Then for \( u \in H_\lambda^\sigma \)

\[
Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1 - x^2)^{\sigma/2}} d\mu_\lambda(x) \leq \int_{-1}^{1} u(x) A_\lambda^\sigma u(x) d\mu_\lambda(x),
\]
where

\[
Q_{\sigma,\lambda} = 2^\sigma \frac{\Gamma(\frac{\lambda}{2} + \frac{1+\sigma}{4})^2}{\Gamma(\frac{\lambda}{2} + \frac{1-\sigma}{4})^2}.
\]
Inequality (4) can be rewritten in terms of the Fourier coefficients

\[ Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1-x^2)^{\sigma/2}} d\mu_\lambda(x) \leq \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} (a_n^\lambda(u))^2, \]

which is a kind of Pitt inequality for the ultraspherical expansions (for other Pitt inequalities see [4, 11]). Note that for the right hand side of (4) we have, by (3),

\[ \int_{-1}^{1} u(x)A_n^\lambda u(x) d\mu_\lambda(x) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} (a_n^\lambda(u))^2 \simeq \|u\|_{H^\sigma_\lambda}^2, \]

so the space \( H^\sigma_\lambda \) is the adequate one.

The proof of Theorem 1 will be a consequence of a proper ground state representation in our setting, analogous to the given one in the Euclidean case in [9]. Following the ideas in that paper, we can see that the constant \( Q_{\sigma,\lambda} \) is sharp but not achieved. Similar ideas have been recently exploited in [7, 16].

From (4), by using Cauchy-Schwarz inequality, we can obtain a Heisenberg type uncertainty principle as it was done for the sublaplacian of the Heisenberg group in [10], and for the fractional powers of the same sublaplacian in [16].

**Corollary 2.** Let \( \lambda > 0 \) and \( 0 < \sigma < 1 \). Then for \( u \in H^\sigma_\lambda \)

\[ Q_{\sigma,\lambda} \left( \int_{-1}^{1} u^2(x) d\mu_\lambda(x) \right)^2 \leq \int_{-1}^{1} u^2(x)(1-x^2)^{\sigma/2} d\mu_\lambda(x) \int_{-1}^{1} u(x)A_n^\lambda u(x) d\mu_\lambda(x), \]

where \( Q_{\sigma,\lambda} \) is the constant given in (5).

Pitt inequality (6) allows us to prove a logarithmic uncertainty principle for the ultraspherical expansions. The main idea comes from [3]. By an elementary argument, for a derivable function such that \( \phi(0) = 0 \) and \( \phi(\sigma) > 0 \) for \( \sigma \in (0,\varepsilon) \), with \( \varepsilon > 0 \), it is verified that \( \phi'(0_+) \geq 0 \). Then, taking the function

\[ \phi(\sigma) = \sum_{n=0}^{\infty} \frac{\Gamma(n+\lambda+\frac{1+\sigma}{2})}{\Gamma(n+\lambda+\frac{1-\sigma}{2})} (a_n^\lambda(u))^2 - Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1-x^2)^{\sigma/2}} d\mu_\lambda(x), \]

we have \( \phi(0) = 0 \) (this is Parseval identity) and, by (6), \( \phi(\sigma) > 0 \) for \( \sigma \in (0,1) \), then \( \phi'(0_+) \geq 0 \) and this inequality gives the logarithmic uncertainty principle, which is written as

\[ \left( \log 2 + \psi \left( \frac{\lambda}{2} + \frac{1}{4} \right) \right) \int_{-1}^{1} u^2(x) d\mu_\lambda(x) \]

\[ \leq \sum_{n=0}^{\infty} \psi \left( n + \lambda + \frac{1}{2} \right) (a_n(u))^2 + \int_{-1}^{1} \log(\sqrt{1-x^2})u^2(x) d\mu_\lambda(x), \]

where \( \psi(a) = \frac{\Gamma'(a)}{\Gamma(a)} \).

In next section we will show an application of Theorem 1 to obtain a Hardy inequality on the sphere. The results in Section 3 are the main ingredients in the proof of Theorem 1 which is given in last section of the paper.
2. An application to the sphere

It is well known that $L^2(S^d) = \oplus_{n=0}^\infty \mathcal{H}_n(S^d)$, where $\mathcal{H}_n(S^d)$ is the set of spherical harmonics of degree $n$ in $d + 1$ variables. If we consider the shifted Laplacian on the sphere

$$-\Delta_{S^d} = -\Delta_{S^d}^{\sigma} + \left(\frac{d-1}{2}\right)^2,$$

where $-\Delta_{S^d}$ is the Laplace-Beltrami operator on $S^d$, it is verified that

$$-\Delta_{S^d} \mathcal{H}_n(S^d) = \left(n + \frac{d-1}{2}\right)^2 \mathcal{H}_n(S^d).$$

In this way, the analogous of the operator $A^\lambda_n$ on $S^d$ is defined by

$$A^\lambda \sigma f = \frac{\Gamma(\sqrt{-\Delta_{S^d}^\sigma + \frac{1+\sigma}{2}})}{\Gamma(\sqrt{-\Delta_{S^d}^\sigma + \frac{1}{2}})} f$$

$$= \sum_{n=0}^\infty \frac{\Gamma(n + \frac{d-1}{2} + \frac{1+\sigma}{2})}{\Gamma(n + \frac{d-1}{2} + \frac{1}{2})} \text{proj}_{\mathcal{H}_n(S^d)} f,$$

where $\text{proj}_{\mathcal{H}_n(S^d)} f$ denotes the projection of $f$ onto the eigenspace $\mathcal{H}_n(S^d)$.

The operator $A^\sigma$ becomes the fractional powers of the Laplacian in the Euclidean space through conformal transforms as was observed by T. P. Branson in [6]. So $A^\sigma$ will be defined in terms of $A^\lambda_n$ and is the main reason to consider $A^\lambda_n$ in the case of the ultraspherical expansions. An analogous of the Hardy-Littlewood-Sobolev inequality for $A^\sigma$ and some other inequalities for it were given by W. Beckner in [2]. The operators $A^\sigma$ also appear in [18, p. 151] and [17, p. 525].

Each point $x \in S^d$ can be written as

$$x = (t, \sqrt{1-t^2}x'_1, \ldots, \sqrt{1-t^2}x'_d),$$

for $t \in (-1, 1)$ and $x' = (x'_1, \ldots, x'_d) \in S^{d-1}$, and so

$$\int_{S^d} f(x) \, dx = \int_{-1}^1 \int_{S^{d-1}} f(t, \sqrt{1-t^2}x')(1-t^2)^{(d-2)/2} \, dx' \, dt.$$

With these coordinates, see [19, Section 3], we have that an orthonormal basis for each $\mathcal{H}_n(S^d)$ is given by

$$\phi_{n,j,k}(x) = \psi_{n,j}(t)Y_{j,k}^d(x'), \quad j = 0, \ldots, n,$$

with

$$\psi_{n,j}(t) = (1-t^2)^{j/2}c^{j+(d-1)/2}_{n-j}(t)$$

and $\{Y_{j,k}^d\}_{k=1}^{d(j)}$ an orthonormal basis of spherical harmonics on $S^{d-1}$ of degree $j$. The value $d(j)$ indicates the dimension of $\mathcal{H}_j(S^{d-1})$; i.e.,

$$d(j) = (2j + d - 2)\frac{(j + d - 3)!}{j!(d-2)!}.$$

Then, the orthogonal projection of $f$ onto the eigenspace $\mathcal{H}_n(S^d)$ can be written as

$$\text{proj}_{\mathcal{H}_n(S^d)} f = \sum_{j=0}^n \sum_{k=1}^{d(j)} f_{n,j,k} \phi_{n,j,k},$$
Then for 

It is known (see [1]) that for \(0 < x < y\) and \(j \geq 0\) we have that \(\frac{\Gamma(j+\sigma)}{\Gamma(j+x)} \geq \frac{\Gamma(x)}{\Gamma(y)}\). So, \(Q_{\sigma,j+(d-1)/2} \geq Q_{\sigma,(d-1)/2}\) and

The proof of (7) is finished by using the identity

\[
\sum_{j=0}^{\infty} \sum_{k=1}^{d(j)} \int_{-1}^{1} \frac{F_{j,k}^2(t)}{(1-t^2)^{\sigma/2}} \, dt \leq 2^{\sigma} \int_{S^d} \frac{f^2(x)}{|x - e_d||x + e_d|^\sigma} \, dx.
\]
The analogous role on the sphere of radially symmetric functions is played by functions which are invariant under the action of $SO(d-1)$. By $SO(d-1)$-invariance we mean that $f$ is invariant under the action of the group $SO(d - 1)$ on $S^{d-1}$ whenever $SO(d - 1)$ is embedded into $SO(d)$ in a suitable way. Each function $f$ of this kind can be written as $f(x) = g((x,e_d))$, for a certain function $g$ defined in $(-1,1)$. Then for this kind of functions Theorem 3 reduces to Theorem 1 with $\lambda = (d - 1)/2$, in this way we can deduce that the constant $2^\sigma Q_{\sigma,(d-1)/2}$ in (7) is sharp.

As in the classic case, from Theorem 3 we deduce that in a distributional sense

$$A_\sigma = \frac{2^\sigma Q_{\sigma,(d-1)/2}}{(|x-e_d||x+e_d|)^\sigma} \geq 0.$$  

Note that in this case we are perturbing the operator $A_\sigma$ adding a potential with singularities in both poles of the sphere.

3. Auxiliary results

The following lemmas give the tools to prove Theorem 1. To be more precise, Lemma 1 provides a nonlocal representation of the operator $A_\sigma^λ$ with a kernel having nice properties for our target. Lemma 2 shows the action of the operator $A_\sigma^λ$ on the family of weights $(1 - x^2)^{-(\lambda/2 + (1-\sigma)/4)}$.

For $f, g \in L^2_\lambda$ we are going to set up the notation

$$\langle f, g \rangle_\lambda = \int_{-1}^1 f(x)g(x) d\mu_\lambda(x)$$

to simplify the writing.

**Lemma 1.** Let $\lambda > 0$ and $0 < \sigma < 1$. If $f$ is a finite linear combination of ultraspherical polynomials, then

$$A_\sigma^\lambda f(x) = \int_{-1}^1 (f(x) - f(y)) K_\sigma^\lambda(x,y) d\mu_\lambda(y) + E_{\sigma,\lambda} f(x), \quad x \in (-1,1),$$

where the kernel is given by

$$K_\sigma^\lambda(x,y) = D_{\sigma,\lambda} \int_{-1}^1 d\mu_{\lambda-1/2}(t) (1 - xy - \sqrt{1 - x^2} \sqrt{1 - y^2} t)^{\lambda+(1+\sigma)/2},$$

with

$$D_{\sigma,\lambda} = \frac{c_\lambda^2 (2^\lambda + (1+\sigma)/2)}{2^\lambda \Gamma(1+\lambda)}$$

and

$$E_{\sigma,\lambda} = \frac{\Gamma(\lambda+\frac{1+\sigma}{2})}{\Gamma(\lambda+\frac{1-\sigma}{2})}.$$

Moreover, for $f \in H_\sigma^\lambda$ we have

$$\langle A_\sigma^\lambda f, f \rangle_\lambda = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 (f(x) - f(y))^2 K_\sigma^\lambda(x,y) d\mu_\lambda(y) d\mu_\lambda(x) + E_{\sigma,\lambda} \langle f, f \rangle_\lambda.$$  

**Proof.** We start with the identity

$$\int_0^\infty \left( e^{-(n+\lambda)t} - e^{-(\sigma-1)t/2} \right) \left( \sinh t/2 \right)^{-\sigma-1} dt = 2^{1+\sigma} \Gamma(\sigma) \frac{\Gamma(n + \lambda + \frac{1+\sigma}{2})}{\Gamma(n + \lambda + \frac{1-\sigma}{2})}.$$
for $\lambda > 0$ (actually it is also true for values $\lambda > -1/2$) and $0 < \sigma < 1$. To deduce the previous identity it is enough to apply integration by parts with $u = e^{-(n+\lambda+1-\sigma)t}/t-1$ and $v = -2e^{-\sigma t/2}/t-1$, and use [14, eq. 8, p. 367]

$$\int_0^\infty e^{-\rho t} (\cosh(ct) - 1)^\nu dt = \frac{\Gamma(\rho - \nu)\Gamma(2\nu + 1)}{2^\nu c\Gamma(\frac{\rho}{c} + \nu + 1)}$$

for $c > 0$, $2\nu > -1$, and $\rho = cv$.

Now, we consider the Poisson operator for ultraspherical expansions. It is given by

$$e^{-t\sqrt{-\lambda}}f(x) = \sum_{n=0}^\infty e^{-(n+\lambda)t}a_n^\lambda (f)c_n^\lambda(x) = \int_{-1}^1 f(y)P_\lambda^\lambda(x, y) d\mu_\lambda(y),$$

with

$$P_\lambda^\lambda(x, y) = \sum_{n=0}^\infty e^{-(n+\lambda)t}c_n^\lambda(x)c_n^\lambda(y).$$

By the product formula for ultraspherical polynomials [8, eq. B.2.9, p. 419]

$$\frac{C_n^\lambda(x)C_n^\lambda(y)}{C_n^\lambda(1)} = c_\lambda \int_{-1}^1 C_n^\lambda(xy + \sqrt{1-x^2}\sqrt{1-y^2}) d\mu_{\lambda-1/2}(t), \quad \lambda > 0,$$

the identity [8, eq. B.2.8, p. 419]

$$\sum_{n=0}^\infty \frac{n+\lambda}{\lambda}C_n^\lambda(x)r^n = \frac{1-r^2}{(1-2xr+r^2)^{\lambda+1}}, \quad 0 \leq r < 1,$$

and the relation $d_n^\lambda = \frac{\lambda}{c_\lambda(n+\lambda)}C_n^\lambda(1)$, we deduce the expression

$$P_\lambda^\lambda(x, y) = \frac{c_\lambda^2}{\lambda} \int_{-1}^1 \sinh t \frac{\sinh t}{(\cosh t - w(s))^{\lambda+1}} d\mu_{\lambda-1/2}(s),$$

with $w(s) = xy + \sqrt{1-x^2}\sqrt{1-y^2}s$. The previous identity for $P_\lambda^\lambda$ is not new, it appears as formula (2.12) in [12].

Combining (10) and the definition of the Poisson operator, it is clear that

$$A_\lambda^\lambda f(x) = \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\lambda}}f(x) - f(x)e^{-(\sigma-1)t/2}\right)(\sinh t/2)^{-\sigma-1} dt,$$

which can be splitted in

$$A_\lambda^\lambda f(x) = \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\lambda}}f(x) - f(x)e^{-(\sigma-1)t/2}\right)(\sinh t/2)^{-\sigma-1} dt$$

$$+ \frac{f(x)}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^\infty \left(e^{-t\sqrt{-\lambda}}1(x) - e^{-(\sigma-1)t/2}\right)(\sinh t/2)^{-\sigma-1} dt.$$

From the obvious identity

$$e^{-t\sqrt{-\lambda}}1(x) = \int_{-1}^1 P_\lambda^\lambda(x, y) d\mu_\lambda(y) = e^{-x^2},$$

the article continues.
for the second term in (11) we have
\[
\frac{f(x)}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^{\infty} \left( e^{-t\sqrt{-C x}} 1(x) - e^{-(\sigma-1)t/2} \right) (\sinh t/2)^{-\sigma-1} \ dt
\]
\[
= \frac{f(x)}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^{\infty} \left( e^{-\lambda t} - e^{-(\sigma-1)t/2} \right) (\sinh t/2)^{-\sigma-1} \ dt
\]
\[
= E_{\sigma,\lambda} f(x),
\]
where we have used (10) with \( n = 0 \).

The first integral in (11) verifies
\[
\frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^{\infty} \left( e^{-t\sqrt{-C x}} f(x) - f(x)e^{-t\sqrt{-C x}} 1(x) \right) (\sinh t/2)^{-\sigma-1} \ dt
\]
\[
= \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^{\infty} \int_{-1}^{1} P_t^\lambda(x, y) (f(x) - f(y)) \ d\mu_\lambda(y) (\sinh t/2)^{-\sigma-1} \ dt
\]
\[
= \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_{-1}^{1} (f(x) - f(y)) \int_0^{\infty} P_t^\lambda(x, y) (\sinh t/2)^{-\sigma-1} \ dt \ d\mu_\lambda(y)
\]
\[
= \int_{-1}^{1} (f(x) - f(y)) K_\sigma^\lambda(x, y) \ d\mu_\lambda(y),
\]
with
\[
K_\sigma^\lambda(x, y) = \frac{1}{2^{1+\sigma}\Gamma(-\sigma)} \int_0^{\infty} P_t^\lambda(x, y) (\sinh t/2)^{-\sigma-1} \ dt.
\]

In last computation we have used Fubini theorem. This is justified for finite combinations of ultraspherical polynomials by using the estimate
\[
P_t^\lambda(x, y) \leq C \frac{\sinh t}{(1-x^2)^{\lambda/2}(1-y^2)^{\lambda/2}(\cosh t - xy - \sqrt{1-x^2 \sqrt{1-y^2}})},
\]
which follows from the elementary inequality
\[
\int_{-1}^{1} \frac{(1 - s^2)^{\lambda-1}}{(A - Bs)^{\lambda+1}} \ ds \leq \frac{C}{B^\lambda(A - B)^{\lambda+1}}, \quad A > B > 0, \quad \lambda > 0,
\]
and the mean value theorem. Indeed, taking \( C_f = \max \{|f'(x)| : x \in [-1,1] \} \) and using the inequality \( 1 - xy - \sqrt{1-x^2 \sqrt{1-y^2}} \geq C|x-y|^2 \), we have
\[
\int_0^{\infty} \int_{-1}^{1} P_t^\lambda(x, y) |f(x) - f(y)| \ d\mu_\lambda(y) (\sinh t/2)^{-\sigma-1} \ dt
\]
\[
\leq \frac{C_f}{(1-x^2)^{\lambda/2}} \left( C_1 \int_{-1}^{1} \int_{-1}^{1} t^{-\sigma} |x-y| \ dt^2 + |x-y|^2 \right) (1-y^2)^{\lambda/2-1/2} dy \ dt
\]
\[
+ C_2 \int_{-1}^{1} \int_{-1}^{1} e^{-(\sigma+1)t/2} |x-y|(1-y^2)^{\lambda/2-1/2} dy \ dt
\]
\[
=: \frac{C_f}{(1-x^2)^{\lambda/2}} (I_1 + I_2).
\]

Obviously, \( I_2 \) is a finite integral. For \( I_1 \) the change of variable \( t = |x-y|s \) gives
\[
I_1 \leq C_1 \int_{-1}^{\infty} \frac{s^{-\sigma}}{s^2 + 1} \ ds \int_{-1}^{1} |x-y|^{-\sigma} (1-y^2)^{\lambda/2-1/2} dy \ < \infty.
\]
To obtain the expression of $K^\lambda_\omega(x,y)$ we observe that

$$
K^\lambda_\omega(x,y) = \frac{c^2_\lambda}{2^{\lambda+1+\sigma}} \int_0^\infty \int_{-1}^1 \sinh t \frac{d\mu_{\lambda-1/2}(s)}{(\sinh t - w(s))^{\lambda+1}} \sinh t d\mu_{\lambda-1/2}(s) \langle \sinh t/2 \rangle^{-\sigma-1} dt
$$

where we have applied Fubini theorem and the change of variable $2(\sinh t/2)^2 = z(1 - w(s))$ in last equality. With the last identity we have concluded the proof of (8).

To prove (9) we follow the argument in [16, Lemma 5.1]. First, we observe that the kernel $K^\lambda_\omega(x,y)$ is positive and symmetric in the sense that $K^\lambda_\omega(x,y) = K^\lambda_\omega(y,x)$. Then, (9) is clear when $f$ is a finite linear combination of ultraspherical polynomials. For $f \in H^\omega_\lambda$ we consider a sequence of finite linear combinations of ultraspherical polynomials $\{p_k\}_{k \geq 0}$ such that $p_k$ converges to $f$ in $H^\omega_\lambda$. Then, by using the definition of $A^\lambda_\omega$, it is clear that $(A^\lambda_\omega p_k, p_k)_\lambda$ converges to $(A^\lambda_\omega f, f)_\lambda$.

Moreover, the result for polynomial functions implies

$$
\langle A^\lambda_\omega p_k, p_k \rangle_\lambda = \frac{1}{2} \int_{-1}^1 \int_{-1}^1 (p_k(x) - p_k(y))^2 K^\lambda_\omega(x,y) d\mu_\lambda(y) d\mu_\lambda(x) + E_{\sigma,\lambda}(p_k, p_k)_\lambda < \infty.
$$

Consequently, the functions $P_k(x,y) = p_k(x) - p_k(y)$ form a Cauchy sequence in $L^2((-1,1) \times (-1,1), d\omega)$ where $d\omega(x,y) = K^\lambda_\omega(x,y) d\mu_\lambda(x) d\mu_\lambda(y)$ which converges to $f(x) - f(y)$ in this norm. Hence, passing to the limit in (12), we complete the proof of the lemma.

**Lemma 2.** Let $\lambda > 0$ and $2\lambda + 1 > \sigma > 0$. Then

$$
A^\lambda_\omega \left( \frac{1}{(1 - x^2)^{\lambda/2 + (1 - \sigma)/4}} \right) = \frac{Q_{\sigma,\lambda}}{(1 - x^2)^{\lambda/2 + (1 + \sigma)/4}},
$$

where $Q_{\sigma,\lambda}$ is the constant given in (5).

**Proof.** First of all, we have to realize that the ultraspherical polynomial $C^\lambda_{2m}(x)$ is odd for $n = 2m + 1$, $m \in \mathbb{Z}^+$; therefore, for $\beta > 0$, the function $(1 - x^2)^{\beta - 1} C^\lambda_{2m+1}(x)$ is an odd function and its integral over the interval $(-1,1)$ is zero. For $n = 2m$ we use [15, eq. 15, p. 519] to obtain

$$
\int_{-1}^1 (1 - x^2)^{\beta - 1} C^\lambda_{2m}(x) dx = \frac{\sqrt{\pi} (2\lambda)_{2m} \Gamma(\beta) \Gamma(\beta + 1/2)^{-1}}{(2m)! \Gamma(\beta + 1/2)} F_2(-2m, 2\lambda + 2m, \beta; 2\beta, \lambda + 1/2; 1)
$$

where in last identity we have evaluated the hypergeometric function with the so-called Watson formula [13, eq. 16.4.6, p. 406]. Therefore, if we denote $\alpha = \lambda/2 + (1 - \sigma)/4$, we obtain that

$$
\int_{-1}^1 (1 - x^2)^{\alpha - 1} C^\lambda_{2m}(x) dx = R_{\sigma,\lambda} \int_{-1}^1 (1 - x^2)^{\alpha + \sigma/2 - 1} C^\lambda_{2m}(x) dx,
$$
with
\[ R_{\sigma,\lambda} = \frac{\Gamma(\alpha)\Gamma(\alpha - \lambda + 1/2)}{\Gamma(\alpha + \sigma/2)\Gamma(\alpha - \lambda + 1/2 + \sigma/2)} \times \frac{\Gamma(\alpha + m + 1/2 + \sigma/2)\Gamma(\alpha - \lambda - m + 1/2 + \sigma/2)}{\Gamma(\alpha + m + 1/2)\Gamma(\alpha - \lambda - m + 1/2)}. \]

In this way, if we prove the identity
\[ R_{\sigma,\lambda} = Q_{\sigma,\lambda}^{-1} \frac{\Gamma(2m + 2\alpha + \sigma)}{\Gamma(2m + 2\alpha)} \]
we will conclude the proof, because (14) implies
\[ a_n^{\lambda} \left( \frac{1}{(1 - x^2)^{\alpha/2}} \right) = Q_{\sigma,\lambda}^{-1} \frac{\Gamma(n + 2\alpha + \sigma)}{\Gamma(n + 2\alpha)} a_n^{\lambda} \left( \frac{1}{(1 - x^2)^{\alpha}} \right), \]
where we have had in mind that the \( n \)-th Fourier coefficient is null when \( n = 2m+1 \).

Let us check that (15) actually holds. Using the reflection formula [1, eq. 6.1.17, p. 256] twice we have
\[ \frac{\Gamma(\alpha - \lambda - m + 1/2 + \sigma/2)}{\Gamma(\alpha - \lambda - m + 1/2)} = \frac{\Gamma(\alpha + m + \sigma/2)}{\Gamma(\alpha + m)} \frac{\sin(\pi(\alpha - \lambda - m + 1/2))}{\sin(\pi(\alpha - \lambda - m + 1/2 + \sigma/2))} = \frac{\Gamma(\alpha + m + \sigma/2)}{\Gamma(\alpha + m)} \frac{\Gamma(\alpha - \lambda + 1/2 + \sigma/2)}{\Gamma(\alpha + \sigma/2)\Gamma(\alpha - \lambda + 1/2)}, \]
and then
\[ R_{\sigma,\lambda} = \frac{\Gamma(\alpha)^2}{\Gamma(\alpha + \sigma/2)^2} \frac{\Gamma(\alpha + m + \sigma/2)\Gamma(\alpha + m + \sigma/2 + 1/2)}{\Gamma(\alpha + m)\Gamma(\alpha + m + 1/2)} = Q_{\sigma,\lambda}^{-1} \frac{\Gamma(2m + 2\alpha + \sigma)}{\Gamma(2m + 2\alpha)}, \]
by the duplication formula [1, eq. 6.1.18, p. 256].

4. PROOF OF THEOREM 1

Polarizing the identity (9) in Lemma 1 we obtain
\[ \langle g, A_{\sigma}^\lambda f \rangle_\lambda = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} F(x,y)K_{\sigma}^\lambda(x,y) d\mu_\lambda(y) d\mu_\lambda(x) + E_{\sigma,\lambda}(g,f)_\lambda, \]
with \( F(x,y) = (g(x) - g(y))(f(x) - f(y)). \)

Let us take \( g(x) = (1 - x^2)^{-\lambda/2 - (1 - \sigma)/4} \) and \( f(x) = u^2(x)/g(x) \) for \( u \in H_\lambda^\sigma. \)

Then
\[ F(x,y) = (u(x) - u(y))^2 - g(x)g(y) \left( \frac{u(x)}{g(x)} - \frac{u(y)}{g(y)} \right)^2 \]
and (16) becomes
\[ \langle g, A_{\sigma}^\lambda f \rangle_\lambda = \langle u, A_{\sigma}^\lambda u \rangle_\lambda - \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} g(x)g(y) \left( \frac{u(x)}{g(x)} - \frac{u(y)}{g(y)} \right)^2 K_{\sigma}^\lambda(x,y) d\mu_\lambda(y) d\mu_\lambda(x). \]

Now, by (13), we have
\[ \langle g, A_{\sigma}^\lambda f \rangle_\lambda = \langle A_{\sigma}^\lambda g, f \rangle_\lambda = Q_{\sigma,\lambda} \int_{-1}^{1} \frac{u^2(x)}{(1 - x^2)^{\sigma/2}} d\mu_\lambda(x) \]
and then we can deduce the ground state representation

\begin{equation}
\langle u, A_\sigma^\lambda u \rangle - Q_{\sigma,\lambda} - \int_{-1}^{1} \left( \frac{u^2(x)}{(1 - x^2)^{\sigma/2}} \right) d\mu_\lambda(x) = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} \frac{g(x)g(y)}{g(x)g(y)} \left( \frac{u(x)}{g(x)} - \frac{u(y)}{g(y)} \right)^2 K_\lambda^\sigma(x, y) d\mu_\lambda(y) d\mu_\lambda(x).
\end{equation}

So, due to the positivity of the kernel $K_\lambda^\sigma$, we conclude the proof.

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