CONIC PROGRAMMING: INFEASIBILITY CERTIFICATES AND
PROJECTIVE GEOMETRY

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Abstract. We revisit facial reduction from the point of view of projective geometry. This leads us to a homogenization strategy in conic programming that eliminates the phenomenon of weak infeasibility. For semidefinite programs (and others), this yields infeasibility certificates that can be checked in polynomial time. Furthermore, we propose a refined type of infeasibility, which we call stably infeasible and we give several natural interpretations from the perspective of projective geometry.

Introduction

A fundamental algorithmic question in optimization theory is to detect whether a given problem is admissible, that is, whether the constraints yield a non-empty set. This is generally known as the feasibility problem. It usually amounts to the simultaneous verification of equalities and inequalities involving real functions. For the special class of conic programming, the admissible set is the intersection of a convex cone with an affine space in a finite-dimensional real vector space.

Our interest is focused on the feasibility problem in semidefinite programming (SDP), a subfamily of conic programming that is a central topic of modern mathematics. Semidefinite programming is a powerful extension of linear programming that enables to convexify hard non-convex optimization problems and to efficiently compute approximate solutions (e.g. the Goemans-Williamson semidefinite approximation of the MAX-CUT problem [9]). Semidefinite programming is used in several domains, ranging from control theory [5,11] to real algebra [2]. For instance, in the analysis of linear differential systems, finding a feasible point yields a Lyapunov function certifying asymptotic stability, while in algebraic settings, semidefinite programs are used to compute sum-of-squares certificates for positivity of polynomials over semi-algebraic sets.

The feasibility problem for semidefinite programs is the decision problem whether or not an affine space intersects the cone of positive semidefinite real symmetric matrices of some fixed size. It suffers, contrary to the special case of linear programming, from several pathological behaviors that appear quite frequently and can lead to numerical instabilities. A semidefinite program can be infeasible without admitting a strong separation between the cone and the affine space: this case is called weak infeasibility (cf. [7, Part II, 2.3] and Figure 2 below). The affine space in a weakly (in-)feasible program has (Euclidean) distance zero from the cone, which implies that numerical instabilities might occur.

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We propose to use the point of view of projective geometry to tackle these issues, i.e. we aim to homogenize the constraints defining the feasible region and to decide feasibility in the linear setup. The main advantage is that the linear setup is quite similar to the compact setup with respect to convex separation. This leads us to introduce the notion of stably infeasible conic programs, which is natural from two numerical points of view: They form the class of programs for which infeasibility is robust with respect to perturbations of the affine equations defining the conic program. Dually, they are the class of infeasible problems for which infeasibility certificates are also robust with respect to numerical errors.

Moreover, the homogeneous setup allows us to use separation arguments (similar to facial reduction) to provide infeasibility certificates (more precisely an interactive version) for any infeasible semidefinite program. This gives a new and elementary proof of Ramana’s theorem stating that the feasibility problem for semidefinite programming is in NP as well as co-NP in the Blum-Shub-Smale model of arithmetic with real numbers.

**Main results.** We outline the main results in our paper. In Section 1, we discuss homogenization in the context of conic programs and the behavior of the common feasibility types with respect to homogenization. The main contributions in this section are the Definition 1.9 of stably infeasible conic programs and their characterization using homogenization in Theorem 1.10, which establishes the two types of robustness with respect to numerical errors that stably infeasible conic programs exhibit (see also Corollary 1.11).

In the following Section 2, we study infeasibility certificates for conic programs in the general context of Pataki’s nice cones [18]. The main result is Theorem 2.4, which uses facial reduction on the homogenized problem to determine infeasibility. In the second part of this section, we discuss the existence of rational infeasibility certificates (given rational input data). We showcase an example essentially due to Scheiderer of a strongly infeasible semidefinite program that does not admit a rational infeasibility certificate.

We then discuss the general approach of homogenization in the special case of semidefinite programs in Section 3. The main result from Section 2 gives a new and elementary proof of Theorem 3.6, originally proved by Ramana using “extended Lagrange-Slater duals” of semidefinite programs.

**The feasibility problem in semidefinite programming.** We want to briefly discuss the major achievements related to the feasibility problem in semidefinite programming. All of them to date, as far as we are aware, are based on refinements of the dual conic program in one way or another. Several dual programs (different from the classical Lagrange dual) and corresponding theorems of the alternative have been proposed for semidefinite programming; see [21] for Ramana’s “extended Lagrange-Slater dual”, [22], [13] for Klep and Schweighofer’s SOS Dual, as well as [14,19]. They have in common that they are defined over the ground field, show no duality gap, and can be written down in polynomial-time with respect to the input size. The facial reduction method proposed by Borwein and Wolkowicz [4] can also be used to regularize weakly feasible semidefinite programs so that dual program has no duality gap. Approximate Farkas Lemmas that can deal with weakly feasible programs have been proposed in [20].
1. Homogenization: The general case

We first discuss basics of conic programming and convex separation before presenting homogenization in the context of general conic programming.

A set $K \subset \mathbb{R}^n$ is a cone if it is closed under multiplication by nonnegative scalars, and it is called pointed if it does not contain lines or equivalently if $K \cap (-K) = \{0\}$. A closed pointed cone with non-empty interior is called regular. In this section, we are interested in the feasibility of affine sections of regular cones in finite-dimensional real vector spaces. The dual vector space of a vector space $V$ is denoted by $V^*$, and the dual cone of a cone $K$ is denoted by $K^\vee = \{ \ell \in V^* : \forall x \in K \; \ell(x) \geq 0 \}$.

Let $K \subset \mathbb{R}^n$ be a regular convex cone, and let $L \subset \mathbb{R}^n$ be an affine subspace of dimension $d$. A (linear) conic programming problem is given by

$$\inf \ell(x) \; \text{s.t.} \; x \in K \cap L.$$  

The intersection $K \cap L$ is called the feasible set, and the objective function $\ell(x)$ is linear. We denote by $\text{int}(K)$ the Euclidean interior of $K$, and by $d(A, B) = \inf\{\|x-v\| : x \in A, v \in B\}$ the distance between two sets $A, B$. Generally speaking, there can exist different shades of feasibility for the feasible set of Problem (1.1).

**Definition 1.2.** We say that $K \cap L$ (or, equivalently, Problem (1.1)) is

1. feasible if $K \cap L$ is non-empty. In particular it is
   a. strongly feasible if $\text{int}(K) \cap L \neq \emptyset$.
   b. weakly feasible if it is feasible and $\text{int}(K) \cap L = \emptyset$.
2. infeasible if $K \cap L = \emptyset$.
   a. strongly infeasible if $d(K, L) > 0$.
   b. weakly infeasible if it is infeasible but not strongly infeasible.

We call any of the previous four subcases the feasibility type of $K \cap L$.

For linear programming, that is when $L = \{x \in \mathbb{R}^n : Ax = b\}$ is an affine space and $K = (\mathbb{R}_+)^n := \{x \in \mathbb{R}^n : x_i \geq 0, \forall i = 1, \ldots, n\}$ is the positive orthant, Farkas Lemma [8] implies that $K \cap L$ is infeasible if and only if it is strongly infeasible. By Farkas Lemma, there are only three feasibility types in linear programming, pictured in Figure 1.

![Figure 1. Feasibility types in Linear Programming: strong feasibility, weak feasibility and infeasibility](image-url)

In other words, the vector $y$ in the Farkas alternative $\{A^Ty \geq 0, y^Tb < 0\}$ is an infeasibility certificate, or improving ray, and corresponds geometrically to a linear functional strongly separating $b$ from the cone generated by the columns of $A$, according to the following definition.
**Definition 1.3.** Let $A, B \subset V$ be two sets and let $H = \{x \in V : \ell(x) = \lambda\} \subset V$ be an affine hyperplane defined by a linear functional $\ell \in V^*$ and $\lambda \in \mathbb{R}$. We say that the affine hyperplane $H$ strongly separates $A$ and $B$ if $\sup \{\ell(x) : x \in A\} < \lambda$ and $\lambda < \inf \{\ell(y) : y \in B\}$.

For a general conic programming problem, the natural generalization of Farkas Lemma fails dramatically. Indeed, a second shade of infeasibility as highlighted in Definition 1.2 might occur, namely weak infeasibility (Figure 2, third picture), for which the existence of improving rays is not guaranteed.

![Figure 2](image)

**Figure 2.** Feasibility types in conic programming: strong feasibility, weak feasibility, weak infeasibility and strong infeasibility

In order to study the feasibility types of general regular cones $K \subset \mathbb{R}^n$, one can use the following separation statement, for which we include a proof for completeness. It shows that strong infeasibility (as in Definition 1.2) is equivalent to the existence of strongly separating hyperplanes (as in Definition 1.3).

**Lemma 1.4** (Strong separation, see [23, Theorem 11.4]). Let $V$ be a real normed vector space, and let $A, B \subset V$ be closed convex sets. There is an affine hyperplane $H \subset V$ strongly separating $A$ from $B$ if and only if $d(A, B) > 0$.

**Proof.** Let $\ell \in V^*$ be a linear form on $V$ and suppose that $H = \{x \in V : \ell(x) = \lambda\}$ strongly separates $A$ from $B$. Write $a = \sup_{x \in A} \ell(x)$ and $b = \inf_{y \in B} \ell(y)$, so that $a < b$. Then for every $x \in A$ and $y \in B$ we get from Cauchy-Schwartz inequality that

$$
\|\ell\| \cdot d(x, y) = \|\ell\| \cdot \|x - y\| \geq \|\ell(x) - \ell(y)\| \geq |a - b|, 
$$

that is $d(A, B) \geq (b - a)/\|\ell\|$. Conversely, assume that $d(A, B) > 0$. Then there is $\varepsilon > 0$ such that $d(A + \mathbb{B}_\varepsilon, B + \mathbb{B}_\varepsilon) > 0$, where $\mathbb{B}_\varepsilon$ denotes the ball of radius $\varepsilon$ around the origin (for example, take $\varepsilon = d(A, B)/3$). The sets $A + \mathbb{B}_\varepsilon$ and $B + \mathbb{B}_\varepsilon$ are again convex as Minkowski sums of convex sets. By the separation theorem [1, Ch.III, Th.1.2], there exists an affine hyperplane $H = \{x \in V : \ell(x) = \lambda\} \subset V$ separating $A + \mathbb{B}_\varepsilon$ and $B + \mathbb{B}_\varepsilon$, that is $A + \mathbb{B}_\varepsilon \subset H^\leq = \{x \in V : \ell(x) \leq \lambda\}$ and $B + \mathbb{B}_\varepsilon \subset H^\geq = \{x \in V : \ell(x) \geq \lambda\}$. Since $\varepsilon > 0$, and by convexity of $A, A + \mathbb{B}_\varepsilon, B, B + \mathbb{B}_\varepsilon$ we get that

$$
\sup \{\ell(x) : x \in A\} < \lambda < \inf \{\ell(x) : x \in B\}
$$

which guarantees the separation.

We introduce now a point of view from projective geometry on the feasibility problem in conic programming. Let us consider the following setup for the rest of the section. Let $V$ be a finite-dimensional Euclidean space and let $\hat{K} \subset V$ be a regular convex cone. Let $U \subset V$ be an affine hyperplane with $0 \notin U$ and set $K = \hat{K} \cap U$. Let $L \subset U$ be a subspace. We assume that $K$ is also a cone in $U$ (after appropriate choice of coordinates), and consider the feasibility of $K \cap L$. 

From the point of view of projective geometry, $U$ determines an affine chart in the projective space $\mathbb{P}(V)$ and $K \subset U$ is the part of the cone $\hat{K}$ that we see on this affine chart. The set $\hat{K} \cap \text{lin}(U)$ is said to be at infinity with respect to $U$, where \text{lin}(U)$ is the linear space $U - v_0$ for any $v_0 \in U$.

Let us first describe this setup in the setting of linear programming. In this case, the above standard form can always be achieved for a linear program in equational form by simply homogenizing the linear constraints in the usual way. Concretely, let $L = \{ x \in \mathbb{R}^n : Ax = b \}$ for an $m \times n$ matrix $A$ of rank $m$ so that the feasible set of the linear program is the intersection of $L$ with the positive orthant $K = (\mathbb{R}_+)^n$. Then we can add a new variable $x_0$ and consider $L$ as the set of solutions of the homogeneous system $Ax = x_0b$ in $n + 1$ variables with the property that $x_0 = 1$. So the hyperplane $U$ in our general setup becomes the affine hyperplane defined by $x_0 = 1$ in $\mathbb{R}^{n+1}$ and $\hat{K} = (\mathbb{R}_+)^{n+1}$, the positive orthant in $\mathbb{R}^{n+1}$.

For semidefinite programming the setup is more special. Indeed, adding a single variable to a cone of positive semidefinite matrices is not sufficient to again obtain a cone of positive semidefinite matrices. Here, more care is necessary: we give more details about the semidefinite programming case in Section 3.

Using homogenization in this sense, we can conveniently characterize infeasibility of a conic program. From now on, the linear span of $L$ is denoted by $\hat{L}$.

**Proposition 1.5.** $L \cap K$ is infeasible if and only if $\hat{L} \cap \hat{K}$ is contained in $\text{lin}(U)$.

**Proof.** This follows from two simple facts: $L \cap K \subset \hat{L} \cap \hat{K}$ and $U \cap \text{lin}(U) = \emptyset$. □

The only implications for the feasibility types of $K \cap L$ and $\hat{K} \cap \hat{L}$ that hold in the general setup of conic programming are summarized in the following statement.

**Theorem 1.6.** Let $V$ be a finite-dimensional Euclidean space and let $\hat{K} \subset V$ be a regular cone. Let $U \subset V$ be an affine hyperplane with $0 \notin U$ and set $K = \hat{K} \cap U$. Let $L \subset U$ be a subspace and $\hat{L}$ the span of $L$ in $V$.

1. $\hat{K} \cap \hat{L}$ is strongly feasible if and only if $K \cap L$ is strongly feasible.
2. If $\hat{K} \cap \hat{L} = \{0\}$, then $K \cap L$ is strongly infeasible.

The proof reduces to the following two lemmas.

**Lemma 1.7.** In the setup of the above Theorem 1.6, if $\hat{K} \cap \hat{L}$ is strongly feasible then $K \cap L$ is strongly feasible.

**Proof.** If $\hat{L}$ intersects the interior of $\hat{K}$, then there exists an $x \in \text{int}(\hat{K}) \cap \hat{L}$ with $x \notin \text{lin}(U)$. This element can be rescaled such that it lies in $U$. So we can assume that $x \in \text{int}(\hat{K}) \cap \hat{L}$ and $x \in U$. It follows that $x \in \text{int}(K) \cap L$, proving the claim. □

**Lemma 1.8.** In the setup of the above Theorem 1.6, the assumption that $K \cap L$ be weakly infeasible implies that $\hat{K} \cap \hat{L}$ contains a non-zero vector.

**Proof.** Fix a norm $\|\| \cdot \|$ on $V$. Since $K \cap L$ is weakly infeasible, there exist sequences of points $(v_j)_{j \in \mathbb{N}} \subset K$ and $(w_j)_{j \in \mathbb{N}} \subset L$ such that $\|v_n - w_n\|$ goes to zero as $n$ goes to infinity because $d(K, L) = 0$.

We know that $0 \notin L$, because $0 \notin U \supset L$. Therefore, there exists a $\delta$ such that $\|w_n\| > \delta$ for all $n \in \mathbb{N}$. So we can estimate

$$\left| \frac{1}{\|v_n\|} v_n - \frac{1}{\|w_n\|} w_n \right| \leq \frac{1}{\|w_n\|} \|v_n - w_n\| \leq \frac{1}{\|w_n\|} \frac{\|w_n\|}{\|v_n\|} \|v_n - w_n\|,$$
which goes to 0, because $\|w_n\|/\|v_n\|$ goes to 1. Indeed, 
\[ 1 - \frac{\|v_n\|}{\|w_n\|} = \frac{\|w_n\| - \|v_n\|}{\|w_n\|} \leq \frac{1}{\delta} \|w_n - v_n\|. \]

So the claim follows from the fact that $K \cap S$ and $\hat{L} \cap S$ are compact, where $S = \{x \in V : \|x\| = 1\}$. Indeed, the sequences $(v_i/\|v_i\|)_{i \in \mathbb{N}} \subset \hat{L} \cap S$ and $(w_j/\|w_j\|)_{j \in \mathbb{N}} \subset \hat{K} \cap S$ have convergent subsequences and their limits must be equal by the above computation. □

This concludes the proof of Theorem 1.6.

**Proof of Theorem 1.6.** If $K \cap L$ is strongly feasible, then clearly $\emptyset \neq \text{int}_U(K) \cap L \subset \text{int}(\hat{K}) \cap \hat{L}$. So $\hat{K} \cap \hat{L}$ is also strongly feasible. The other implication is Lemma 1.7. Claim (2) of the theorem follows from Lemma 1.8 because $K \cap L \subset \hat{K} \cap \hat{L}$. □

We distinguish the following subclass of (strongly) infeasible conic programs.

**Definition 1.9.** Let $K$ be a cone and let $L$ be a $d$–dimensional affine space. We say that $K \cap L$ (or, equivalently, Problem (1.1)) is *stably infeasible* if there exists an open neighborhood $N$ of $L$ in the Grassmannian of $d$–dimensional affine spaces in $\mathbb{R}^n$ such that $K \cap L'$ is infeasible for all $L' \in N$.

From a numerical point of view, this definition means that the conic program $K \cap L$ remains infeasible under small perturbations of the affine space $L$. For another justification of the word stable in this context, see Corollary 1.11 below.

It is easy to check that every stably infeasible conic program must be strongly infeasible, but the converse is false even for linear programs. In the right picture of Figure 3 the affine space $L$ is parallel to one of the “asymptotes” of the feasible set, hence arbitrary perturbations of $L$ may result both in feasible and infeasible programs. In other words, unstable conic programs are infeasible programs that are arbitrarily close to feasible ones (sharing this property with weakly infeasible ones).

![Figure 3. Stable and unstable infeasibility](image)

Homogenization as described in this section distinguishes stably infeasible conic programs from not stably infeasible ones.

**Theorem 1.10.** Let $V$ be a finite-dimensional Euclidean space and let $\hat{K} \subset V$ be a regular cone. Let $U \subset V$ be an affine hyperplane with $0 \notin U$ and set $K = \hat{K} \cap U$. Let $L \subset U$ be a subspace and $\hat{L}$ the span of $L$ in $V$. The conic program $K \cap L$ is stably infeasible if and only if $\hat{K} \cap \hat{L} = \{0\}$.

**Proof.** First suppose that $\hat{K} \cap \hat{L} = \{0\}$. This means that there is a linear form $\ell \in \text{int}(\hat{K}^\vee)$ such that $\ell(x) = 0$ for all $x \in \hat{L}$. So Lemma A.2 implies that $K \cap L$ is stably infeasible.
Conversely, if $K \cap L$ were stably infeasible and $\hat{K} \cap \hat{L}$ contained a nonzero vector, then, for every neighborhood of $L$ in the Grassmannian of affine subspaces of $H$, there would exist a subspace $L'$ such that $\hat{K} \cap \hat{L}'$ is strongly feasible, that is a contradiction. □

The first paragraph of the above proof implies the following alternative definition of stable infeasibility.

**Corollary 1.11.** Let $K \subset V$ be a regular cone and let $L$ be an affine subspace. There exists a separating hyperplane $\ell \in \operatorname{int}(K^\vee)$ with $\ell(x) < 0$ for all $x \in L$ if and only if $K \cap L$ is stably infeasible. □

This statement gives another motivation for calling this infeasibility type stable, because the normal vector $\ell$ of a separating hyperplane can be chosen in the interior of the dual cone $K^\vee$. For applications, when $K^\vee$ is a positive orthant or semidefinite cone, for instance, this means that the problem of testing the membership of $\ell$ in $K^\vee$ is stable in regards to small perturbations.

We conclude this discussion with an example of an infeasible, but not stably infeasible, conic program that can be transformed into both weakly feasible and infeasible programs by simply translating the affine space.

**Example 1.12.** Let $C$ be the convex set $\{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \leq 1, xy - 1 \geq 0, z = 1\}$ and let $K = \{t v \in \mathbb{R}^3 : t \in \mathbb{R}^+, v \in C\}$ be the conical hull of $C$ (the smallest cone containing $C$). Let $L$ be the line $\{(x, y, z) \in \mathbb{R}^3 : y = -1, z = 1\}$ included (together with $C$) in the hyperplane $\{(x, y, z) \in \mathbb{R}^3 : z = 1\}$. Then it is easy to check that

- $K \cap L$ is not stably infeasible
- $K \cap (L + (0, 1, 0)^T)$ is weakly infeasible
- $K \cap (L + (0, 2, 0)^T)$ is weakly feasible.

2. INFEASIBILITY CERTIFICATES AND FACIAL REDUCTION

In this section, our goal is to provide infeasibility certificates for general conic programs using homogenization.

**Definition 2.1.** Let $K \subset V$ be a regular cone, and let $L \subset V$ be an affine space. An affine function $f$ on $V$ is called an infeasibility certificate of $K \cap L$ whenever $f(x) \geq 0$ on $K$ and $f(x) < 0$ on $L$.

An infeasibility certificate exists if and only if $L \cap K$ is strongly infeasible, see Lemma 1.4. Our goal is to establish an iterative version of infeasibility certificates, relying on homogenization, that can also be used for weakly infeasible programs. We begin with a technical consequence of the separation theorem.

**Lemma 2.2.** Let $L \subset V$ be a linear subspace and let $K \subset V$ be a regular convex cone. Let $H$ be a supporting hyperplane of $K$ containing $L$. If $L \cap K$ is contained in the relative boundary of the face $H \cap K$ of $K$, then the dimension of $L \cap \operatorname{span}(H \cap K)$ is strictly smaller than the dimension of $L$.

**Proof.** By contraposition, if $L \cap \operatorname{span}(H \cap K) = L$, then $L$ must intersect the relative interior of the face $H \cap K$ by the separation theorem [1, Ch.III, Th.1.2]. □
The following definition goes back to work of Pataki in the context of facial reduction, see [18] and [19].

**Definition 2.3.** We call a convex cone \( K \subset V \) nice, if \( K^\vee + F^\perp \) is closed for every face \( F \subset K \).

**Theorem 2.4.** Let \( K \subset V \) be a regular, nice convex cone. Let \( L \subset V \) be an affine space properly contained in an affine hyperplane \( U \subset V \) with \( 0 \notin U \). If \( L \cap K = \emptyset \), there exists a sequence of elements \( \ell_1, \ell_2, \ldots, \ell_k \in K^\vee \) with the following properties:

Set \( F_i = \{ x \in K : \ell_i(x) = 0 \} \) and \( L_i = L_{i-1} \cap \operatorname{span}(F_{i-1}) \) for \( i > 1 \) with \( L_1 = \hat{L} \).

We have

1. \( k \leq 1 + \dim(L) \),
2. \( F_i \supset F_{i+1} \),
3. \( F_i \supset L_i \cap K \supset \hat{L} \cap K \), and
4. \( F_k \subset \operatorname{lin}(U) \).

**Proof.** If \( L \cap K \) is stably infeasible, then \( \hat{L} \cap K = \{ 0 \} \) and there exists an element \( \ell \in \text{int}(K^\vee) \) with \( L \subset \{ x \in V : \ell(x) = 0 \} \). This is a certificate of the claimed form with \( k = 1 \).

By Theorem 1.10, we are left with the case \( \hat{L} \cap K \neq \{ 0 \} \).

Let \( F \subset K \) be the smallest face containing \( \hat{L} \cap K \). Since \( L \cap K = \emptyset \), we have \( F \subset \operatorname{lin}(U) \) and \( F \neq K \). So there exists a supporting hyperplane \( H = \{ x \in V : \ell(x) = 0 \} \) with \( \ell \in K^\vee \) and \( \hat{L} \subset H \). Set \( L_1 = \hat{L}, \ell_1 = \ell \), and \( F_1 = H \cap K \). We have \( F \subset F_1 \).

If \( F_1 = F \), we are done for \( k = 1 \). If \( F_1 \neq F \), put \( L_2 = L_1 \cap \operatorname{span}(F_1) \), which is a proper subspace of \( L_1 \) by the previous Lemma 2.2. Since \( F_1 \neq F \), we know that \( L_2 \cap F_1 = F \) is a proper face of \( F_1 \). By [1, Ch.III, Th.1.2], there is a supporting hyperplane of the cone \( F_1 \) containing \( L_2 \). Its normal vector is in the dual convex cone \( F_1^\vee \), which is \( K + F_1^\perp \) because the cone \( K \) is nice. So there exists \( \ell_2 \in K^\vee \) such that \( L_2 \subset \{ x \in V : \ell_2(x) = 0 \} \). Set \( F_2 = \{ x \in K : \ell_2(x) = 0 \} \). If \( F_2 = F \), we are done for \( k = 2 \). Otherwise, we proceed iteratively to obtain the sequence \( \ell_1, \ell_2, \ldots, \ell_k \) in the claim. The bound \( k \leq 1 + \dim(L) \) follows from the inequalities \( 1 \leq \dim(F) = \dim(F_k) < \dim(L_{k-1}) < \ldots < \dim(L_1) = 1 + \dim(L) \). \( \square \)

**Remark 2.5.** The essential step in the proof of the previous Theorem 2.4 is closely related to facial reduction [4,22] on the weakly feasible conic program \( \hat{L} \cap K \). In fact, facial reduction algorithms compute \( \operatorname{span}(F) \) by computing the supporting hyperplanes \( \ell_i \) in the above theorem. For semidefinite programming, this is often done by rank maximization.

If the conic program is stably infeasible, the facial reduction is unnecessary, in the sense that \( k = 1 \). If it is not stably infeasible, regardless of whether it is strongly or weakly infeasible, it might require \( k \geq 2 \). We give explicit examples of semidefinite programs in Section 3.

We can apply this theorem certainly to the positive orthants (linear programming). More interestingly, it applies to semidefinite programs, see below. Other families of nice cones include second order cones. Moreover, given a family of nice regular convex cones \( K_n \subset V_n \) such that we can check membership in \( K_n \) and in \( K_n^\vee \) in polynomial time, the feasibility problem for this family is in \( \text{NP}_R \) and co-\( \text{NP}_R \).

We give details below for semidefinite programs, see Theorem 3.6.

We first turn to the existence of rational infeasibility certificates.
2.1. **Rational Infeasibility Certificates.** In this section, we suppose that the cone $K$ (resp. the affine space $L$) in Problem 1.1 is a $\mathbb{Q}$-definable semialgebraic set (resp. affine space), that is defined by polynomial inequalities (resp. equalities) with coefficients in $\mathbb{Q}$. The semialgebraic model includes the case of linear and semidefinite programming, together with a large range of other optimization problems, while the rationality of the defining (in)equalities reflects the usual assumption that the model can be represented by rational data. Under these assumptions, we address the question whether one can compute infeasibility certificates that are again definable over $\mathbb{Q}$.

If the infeasibility certificate $f$ in Definition 2.1 can be defined with rational coefficients, we say that the certificate is *rational*.

**Remark 2.6.** Let $K$ be a cone and let $L$ be an affine space. If $K \cap L$ is stably infeasible, then there exists a rational infeasibility certificate. Indeed, this is a direct consequence of Corollary 1.11 and the fact that $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

In general, even for strongly infeasible programs, rational certificates need not exist, as we demonstrate below. In the case of linear programming, it is well-known that rational infeasibility certificates always exist, by Farkas Lemma. Here, $K = (\mathbb{R}_+)^n$ and the affine space $L$ is usually defined by $Ax = b$. Recall the definition of strong separation in Definition 1.3.

**Proposition 2.7.** Suppose that the entries of $A, b$ are in $\mathbb{Q}$. If \{ $x \in \mathbb{R}^n$: $Ax = b$ $\} \cap (\mathbb{R}_+)^n$ is infeasible, there exists $y \in \mathbb{Q}^n$ and $\lambda \in \mathbb{Q}$ such that $H = \{ x \in \mathbb{R}^n$: $y^T(Ax - b) = \lambda \}$ strongly separates $L$ and $(\mathbb{R}_+)^n$.

**Proof.** The vector $y$ can be chosen to be rational because it is the solution of linear inequalities with coefficients in $\mathbb{Q}$: indeed, the condition $y^Tb < 0$ can be weakened as a closed condition $y^Tb \leq \varepsilon$ for some negative $\varepsilon \in \mathbb{Q}$ (under the assumption that the open inequality has a solution). The weakened system of inequalities is a feasible (by Farkas Lemma) linear program defined over $\mathbb{Q}$, hence it has at least one rational solution. So, let $y \in Q^n$ satisfy $A^Ty \geq 0$, $y^Tb < 0$ and set $\ell(x) = y^T(Ax - b)$. Then $\ell$ vanishes on $L$ and we have $\ell(x) = y^TAx - y^Tb > 0$ for all $x \in (\mathbb{R}_+)^n$. Since $A^Ty \geq 0$, we know that $r := \inf_{x \in (\mathbb{R}_+)^n} \ell(x) \geq -y^Tb > 0$. Let $\lambda \in (0, r) \cap \mathbb{Q}$. Then

$$\sup_{x \in L}\ell(x) = 0 < \lambda < r = \inf_{x \in (\mathbb{R}_+)^n}\ell(x)$$

hence $H$ strongly separates $L$ and $(\mathbb{R}_+)^n$. \hfill $\square$

The infeasibility certificate $f(x) = \ell(x) - \lambda$ in Proposition 2.7 is rational and exists independently of the stability of the infeasibility, that is even if $L$ is contained in a hyperplane intersecting the cone $(\mathbb{R}_+)^n$ at infinity (in which case $(\mathbb{R}_+)^n \cap L$ is not stably infeasible).

We now turn to semidefinite programming. We illustrate with the following example that there are strongly infeasible semidefinite programs that do not admit rational infeasibility certificate.

**Example 2.8.** The underlying reason for this example is the existence of linear spaces $U$ defined over $\mathbb{Q}$ with the property that $U \cap S^n_{\mathbb{Q}}$ is non-empty but does not contain any rational points. Examples for such linear spaces are given by Scheiderer in [24] in the context of sum-of-squares certificates of positive polynomials.
Let \( v \) be the column vector containing the monomial basis \( \{x^2, y^2, z^2, xy, xz, yz\} \). The explicit example [24, Example 2.8] consists of the linear space \( L' \subset S^6 \), which is the span of the affine space of symmetric matrices \( M \) defined by the affine equations

\[
v^T M v = (x^4 + xy^3 + y^4 - 3x^2yz - 4xy^2z + 2x^2z^2 + xz^3 + yz^3 + z^4).
\]

The linear space \( L' \) is a 7-dimensional subspace of the 21-dimensional space \( S^6 \) such that \( L' \cap S^6_+ \) is a 2-dimensional cone with no rational points. Indeed, the right hand side in the previous equality is a positive polynomial with rational coefficients, that cannot be written as a sum of squares of polynomials with rational coefficients.

Let \( L = (L')^\perp - I_6 := \{P - I_6 : P \in (L')^\perp\} \subset (S^6_+)^* \). We claim that \( L \cap S^6_+ \) (after the identification \( S^6_+ = (S^6_+)^* \)) is strongly infeasible but that there is no rational certificate for this fact. Indeed, let \( A \in L' \cap S^6_+ \). Then \( \langle A, Q \rangle = \langle A, P \rangle - \langle A, I_6 \rangle < 0 \) for every \( Q = P - I_6 \in L, P \in (L')^\perp \). This shows that \( L \cap S^6_+ \) is strongly infeasible.

To see that there is no rational infeasibility certificate, let \( A \) be such that \( \langle A, M \rangle \geq 0 \) for all \( M \in S^6_+ \) and \( \langle A, Q \rangle < 0 \) for all \( Q \in L \). Since \( S^6_+ \) is self-dual, it follows that \( A \in S^6_+ \). Since \( A \) is bounded from above (as a linear form) on \( L \) and \( L \) is an affine space, it follows that \( A \) has to be constant on \( L \), i.e. \( A \) must vanish on \( (L')^\perp \). We conclude that \( A \) lies in \( L' \cap S^6_+ \), which does not contain any rational points.

3. Homogenization of semidefinite programs

In this section, we apply homogenization as discussed in Section 1 for general conic programs in the special case of semidefinite programs. A semidefinite program (SDP) in standard implicit form (see e.g. [7, Chapter 2]) is given by

\[
\inf \langle C, X \rangle \text{ s.t. } X \in K, \text{ and } \langle M_i, X \rangle = b_i, i = 1, \ldots, c.
\]

Above, \( C, M_1, \ldots, M_c \) are elements of \( S^d \), the vector space of real symmetric \( d \times d \) matrices. We fix the inner product \( \langle \cdot, \cdot \rangle : S^d \times S^d \to \mathbb{R}, \langle A, B \rangle := \text{trace}(AB) = \sum_{i,j} a_{ij} b_{ij}, \) on \( S^d \). We are concerned with the regular cone \( K = S^d_+ := \{X \in S^d : X \succeq 0\} \) of positive semidefinite real symmetric matrices.

A linear matrix inequality (usually abbreviated as LMI) gives a parametric representation for the feasible set of a semidefinite program (instead of the implicit representation used above). So let \( A_1, \ldots, A_n \in S^d \) be linearly independent symmetric matrices and let \( A_0 \in S^d \) be a fixed matrix. A linear matrix inequality is an expression of the form

\[
A_0 + x_1 A_1 + x_2 A_2 + \ldots + x_n A_n \succeq 0.
\]

The solution set of this inequality is the set of points \( (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \) such that the eigenvalues of the matrix on the left hand side of the inequality are nonnegative. Such a set is called a spectrahedron. We say that a linear matrix inequality is (weakly or strongly) (in-)feasible if \( S^d_+ \cap L \) is (weakly or strongly) (in-)feasible in the sense of Definition 1.2, where \( L \) is the affine space \( A_0 + \langle A_1, A_2, \ldots, A_n \rangle \). An implicit description of the feasible set as given in (3.1) can be made explicit by linear algebra operations over the ground field (the smallest field containing the entries of the \( M_i \)).

We first comment on a standard example in the literature of a weakly infeasible linear matrix inequality and on the corresponding typical behavior of numerical solvers (see for instance [7, Example 2.2]).
Example 3.2 (Standard weakly infeasible LMI). We consider the 1-dimensional linear matrix inequality $A(x_1) \succeq 0$ with

$$A(x_1) = \begin{bmatrix} 0 & 1 \\ 1 & x_1 \end{bmatrix}$$

The linear matrix inequality has no solution since, for instance, $\det A = -1$. Solving generic (randomly generated) semidefinite programs over this linear matrix inequality using SeDuMi [25] or SDPT3 [26] as solvers through Matlab/Yalmip [15], we typically get either “Numerical problems” (error 4 in SeDuMi) or “Unbounded function” (error 2 in SeDuMi). The same holds when trying to solve the SDP through Sage via CVXOPT [6], a software targeted to conic optimization: similarly to the above case, the solver stops after a few iterations since the objective function is considered unbounded over the admissible set.

For every non-zero $n \in \mathbb{N}$, the matrix

$$X_n = \begin{bmatrix} 1/n & 1 \\ 1 & n \end{bmatrix}$$

is positive semidefinite, and the previous linear matrix inequality $A(x_1) \succeq 0$ is given as the intersection of $S^2$ with $L = \{ X \in S^2 : \langle M_1, X \rangle = 0, \langle M_2, X \rangle = 2 \}$, for

$$M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$  

We have $\langle M_1, X_n \rangle = 1/n$ and $\langle M_2, X_n \rangle = 2$ for all $n$, from which we deduce that $d(X_n, L) \to 0$, for $n \to +\infty$ that is the linear matrix inequality is weakly infeasible. The same holds for every linear matrix inequality whose matrix contains $A(x_1)$ as a principal submatrix. □

A last example shows another weakly infeasible linear matrix inequality. It appears as a pathological case of Lasserre relaxations in the context of multivariate polynomial optimization.

Example 3.3 (Motzkin polynomial). We consider the Motzkin sextic polynomial

$$f = x_1^4x_2^2 + x_1^2x_2^4 + 1 - 3x_1^2x_2^2,$$

which is globally non-negative but does not admit a certificate as sum of squares of polynomials. Moreover, $f - \lambda$ is not a sum of squares for any $\lambda \in \mathbb{R}$ [2, Sec.3.1.2]. Applying [27, Cor.3.3], one gets that the Lasserre relaxations that one associates to the global optimization problem

$$\inf f(x) \quad \text{s.t. } x \in \mathbb{R}^3$$

are weakly infeasible. Since weak infeasibility can be turned into strong feasibility or strong infeasibility by small perturbations, it is not surprising that the numerical solvers have difficulty handling this problem: When trying to solve (3.4) using the software Gloptipoly [12] under Matlab, the Gloptipoly command $msdp(min(f))$ stops at the third relaxation without computing solutions, but forcing it to go through the seventh relaxation, one gets feasible solutions that yield the four minima of the Motzkin polynomial. That is, the LMI solver which is called by Gloptipoly computes the correct solution even though the corresponding relaxation is infeasible: indeed, because of rounding errors, there is no difference between feasible and infeasible problems that are not stably infeasible. □
3.1. Membership in \textit{co-NP}\(_{\mathbb{R}}\). The goal of this section is to apply our homogenization scheme in order to prove that the SDP feasibility problem belongs to the class \textit{NP}\(_{\mathbb{R}}\) \cap \textit{co-NP}\(_{\mathbb{R}}\) (the \(\mathbb{R}\) index stands for the Blum-Shub-Smale model of computation, see [3]) This was first proved by Ramana [21] using the so-called Extended Lagrange-Slater Dual of a semidefinite program.

The basic idea to show that the SDP feasibility problem is in \textit{co-NP} is to find an infeasibility certificate (of polynomial size) as in Proposition 1.5. This is in general not possible (see Example 3.8 below) but rather, we need an iterative version of such certificates, as developed in Theorem 2.4.

**Corollary 3.5.** Let \(L \subset S^d\) be an affine space properly contained in an affine hyperplane \(U \subset S^d\) with \(0 \notin U\). If \(L \cap S^d_+\) is infeasible, there exists a sequence of matrices \(C_1, C_2, \ldots, C_k \in S^d_+\) with the following properties: For every \(i = 1, 2, \ldots, k\), set \(F_i = \{M \in S^d_+: \langle C_i, M \rangle = 0\}\), the face of \(S^d_+\) supported by \(C_i\). Set \(L_1 = \hat{L}\) and \(L_i = L_{i-1} \cap \text{span}(F_{i-1})\) for \(i > 1\). We have

\begin{enumerate}[(1)]  
  \item \(k \leq \min\{d - 1, 1 + \dim(L)\}\),
  \item \(F_i \supset F_{i+1}\),
  \item \(F_i \supset L_i \cap S^d_+ \supset \hat{L} \cap S^d_+\), and
  \item \(F_k \subset \text{lin}(U)\).
\end{enumerate}

**Proof.** The cone of positive semidefinite matrices is nice, see [17]. The bound of \(d - 1\) in (1) follows from the fact that the rank of \(C_i\) is strictly bigger than the rank of \(C_{i-1}\). \(\square\)

We show later (Example 3.12) that the bound \(d - 1\) in (1) is sharp in general and we give a geometric explanation in terms of the tangent cone.

As a direct consequence of Corollary 3.5, we get that the feasibility problem for semidefinite programs is in \textit{co-NP}\(_{\mathbb{R}}\).

**Theorem 3.6.** The feasibility problem for semidefinite programming is in \textit{NP}\(_{\mathbb{R}}\) \cap \textit{co-NP}\(_{\mathbb{R}}\) (Blum-Shub-Smale model).

**Proof.** Let us first recall that the feasibility problem for semidefinite programming is in \textit{NP}\(_{\mathbb{R}}\). Let \(L = A_0 + \langle A_1, \ldots, A_n \rangle \subset S^d\) be the given affine space and let \(n = \dim L\). Given \(x \in \mathbb{R}^n\), evaluating \(A(x) = A_0 + \sum x_i A_i\) has a cost of \(O(nd^2)\) and deciding whether \(A(x) \succeq 0\) can be done in \(O(d^3)\) (see [21, Th. 25, (iii)]).

To show that the feasibility problem is in \textit{co-NP}\(_{\mathbb{R}}\), we distinguish two cases. If \(L\) is an affine hyperplane, then \(L \cap S^d_+ = \emptyset\) implies that every normal vector \(C\) of \(L\) is positive or negative semidefinite. Assume that \(C\) is positive semidefinite and pick \(M \in L\). With these assumptions, \(L \cap S^d_+ = \emptyset\) holds if and only if \(\langle C, M \rangle < 0\). The normal vector \(C\) is given in the implicit form for semidefinite programs or it can be computed in polynomial time from a spanning set of \(L\). Computing a matrix \(M\) in \(L\) and evaluating \(\langle C, M \rangle\) can also be done in polynomial time.

This leaves the case \(\text{codim}(L) \geq 2\). In this case, the certificate given by Theorem 3.5 is of size \((n + 1)\left(\begin{array}{c} d+1 \\ 2 \end{array}\right)\) \((k\) symmetric \(d \times d\) matrices, with \(k \leq n + 1\)). The conditions that the \(C_i\) are positive semidefinite can be verified in polynomial time (as recalled above). The same is true for the inclusions \(F_i \supset F_{i+1}\) because this can be checked in terms of the kernels of \(C_i\) and \(C_{i+1}\). Finally, \(\text{span}(F_k) \subset \text{lin}(U)\) can also be checked in polynomial time by a computation of a basis of \(\text{span}(F_k)\). \(\square\)
To give a geometric explanation of why we need such a hierarchy of certificates for the feasibility problem for semidefinite programming (as opposed to the feasibility problem in linear programming, for instance), we discuss some general convexity theory (in particular tangent cones).

**Definition 3.7.** Let $K \subset V$ be a regular convex cone and let $F \subset K$ be a face. The tangent cone to $K$ at $F$ is the convex cone

$$ TC_F(K) = \bigcap \left\{ H^+ : K \subset H^+, H \cap K \supset F \right\}, $$

the intersection of all closed half-spaces supporting $K$ in a face containing $F$.

Equivalently, $TC_F(K)$ is the closure of the cone generated by all differences $w - v$ for a vector $v$ in the relative interior of $F$. The tangent cone determines what kind of supporting hyperplane to $K$ exists that separates $K$ and a linear space. We illustrate this fact for $K = S^3_+$. We discuss a geometric way to understand this example in the remainder of this section.

**Example 3.8.** Consider the 2-dimensional linear space

$$ L = \text{span} \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\} \subset S^3. $$

The intersection $R = S^3_+ \cap L$ is the ray spanned by the first generator of $L$. Consider the tangent cone

$$ TC_R(S^3_+) = \left\{ \begin{pmatrix} * & * \\ * & B \end{pmatrix} : B \in S^2_+ \right\}. $$

The intersection $TC_R(S^3_+) \cap L$ also contains the second generator of $L$. This geometric fact shows that there is not supporting hyperplane $H$ of $S^3_+$ separating $L$ and $S^3_+$ with $H \cap S^3_+ = R$. In fact, there is a unique supporting hyperplane $H$ of $S^3_+$ containing $L$, and its normal vector is

$$ C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

The intersection of $L$ with the span of the face of $S^3_+$ exposed by $C$ is the line spanned by the first generator of $L$.

**Lemma 3.9** (separation lemma). Let $K \subset V$ be a regular convex cone and let $F \subset K$ be a face. Let $\pi : V \to V/\text{span}(F)$ be the canonical projection. The closure of $\pi(K)$ is exactly $\pi(TC_F(K)).$

**Proof.** This follows from biduality.

$$ \text{clos}(\pi(K)) = (\pi(K)^\vee)^\vee = (K^\vee \cap \text{span}(F)^\perp)^\vee = $$

$$ = \bigcap \left\{ \overline{\Pi}^+ : K \subset \overline{\Pi}^+, \text{span}(F) \subset H \right\} = \bigcap \left\{ \overline{\Pi}^+ : K \subset \overline{\Pi}^+, H \cap K \supset F \right\} $$

where the dual at the end of the first line is taken with respect to $V/\text{span}(F)$ using $(V/\text{span}(F))^* \cong \text{span}(F)^\perp \subset V^*$. □

For a discussion of the face lattice of the cone of positive semidefinite matrices used in the following statement, we refer to [1].
Lemma 3.10. Let $F$ be a face of $S^d_+$ corresponding to a subspace $U \subset \mathbb{R}^d$ via the anti-isomorphism of the face lattice of $S^d_+$ with the lattice of subspaces of $\mathbb{R}^d$, given by $U \mapsto F_U = \{ A \in S^d_+ : U \subset \ker(A) \}$. Let $M$ be in the relative interior of $F$, and let $r = \rank(M)$. Then

$$TC_F(S^d_+) = S^d_+ + T_M$$

where $T_MV_r$ is the tangent space at $M$ to the variety of symmetric matrices of rank at most $r$. In particular, $T_MV_r$ is the lineality space of $TC_F(S^d_+)$. Moreover, we have that the intersection of the lineality space of $TC_F(S^d_+)$ with $S^d_+$ recovers the face $F$ for every proper face $F$ of $S^d_+$.

Proof. Up to conjugation by the orthogonal group, we can assume that $U$ is the coordinate subspace defined by the linear equations $x_{d-r+1} = 0, x_{d-r+2} = 0, \ldots, x_d = 0$. That is $U = \text{span}(x_1, \ldots, x_{d-r})$ and $F_U$ is the set of matrices of the form

$$M = \begin{pmatrix} M' & 0 \\ 0 & 0 \end{pmatrix}$$

where $M' \succeq 0$ and has size $r \times r$. So the tangent cone to $S^d_+$ at $F_U$ is

$$TC_{F_U}(S^d_+) = \left\{ \begin{pmatrix} * & * \\ * & B \end{pmatrix} : B \in S^{d-r}_+ \right\}.$$

On the other hand, the tangent space $T_MV_r$ to the variety of matrices of rank at most $r$ at $M$ is the linear space of all matrices whose bottom right $(d-r) \times (d-r)$ block is 0.

These two facts combined give the claim. \hfill \qed

Corollary 3.11. Let $L \subset S^d_+$ be a linear space and let $F \subset S^d_+$ be the smallest face of $S^d_+$ containing $S^d_+ \cap L$. There exists a supporting hyperplane $H$ of $S^d_+$ with $L \subset H$ and $H \cap S^d_+ = F$ if and only if $L \cap TC_F(S^d_+)$ is contained in the lineality space of $TC_F(S^d_+)$.\addtocounter{corollary}{1}

Proof. We consider the canonical projection $\pi : V \to V/\text{span}(F)$. The existence of a supporting hyperplane $H$ of $S^d_+$ with $L \subset H$ and $H \cap S^d_+ = F$ is equivalent to the existence of a supporting hyperplane $\overline{H} \subset V/\text{span}(F)$ of $\pi(S^d_+)$ with $\pi(L) \subset \overline{H}$ and $\overline{H} \cap \pi(S^d_+) = \{ \}$. By Lemma 3.10, the closure of $\pi(S^d_+)$ is $\pi( TC_F(S^d_+))$. So $\overline{H} \cap \pi(S^d_+) = \{ \}$ and $\pi(L) \subset \overline{H}$ imply that $L$ is contained in the lineality space of $TC_F(S^d_+)$. Conversely, there exists a supporting hyperplane $H$ of $\pi(S^d_+)$ such that $\overline{H} \cap \pi(S^d_+) = \{ \}$ is the lineality space of $\pi(S^d_+)$. So, if $L$ is contained in the lineality space of $TC_F(S^d_+)$, there is a supporting hyperplane $H = \pi^{-1}(\overline{H})$ of $S^d_+$ that contains $L$ and $H \cap S^d_+$ is contained in the intersection of the lineality space of $TC_F(S^d_+)$ with $S^d_+$. By Lemma 3.10, we have

$$TC_F(S^d_+) \cap S^d_+ = F.$$ \hfill \qed

We can extend Example 3.8 to show that the bound $d-1$ for the length of the iterative infeasibility certificate in Corollary 3.5 is tight, using the tangent cone. The original example is the special case of the following for $d = 3$.

Example 3.12. Let $E_{11}$ be the matrix whose $(1,1)$ entry is 1 and all other entries are equal to 0. For $i \in \{2, \ldots, d-1\}$, set $A_i$ to be the matrix whose $(i,i)$ entry is 1,
whose \((1, i + 1)\) and \((i + 1, 1)\) entries are \(-1\) and all others equal to 0. Let \(L\) be the linear space spanned by \(E_{11}\) and \(A_2, A_3, \ldots, A_{d-1}\). Similar to Example 3.8, there is a unique supporting hyperplane to \(S^d_+\) that contains \(L\). Namely, its normal vector \(C_1\) is the matrix whose \((d, d)\) entry is 1 (and all others are 0). So we now intersect with the span of the face supported by \(C_1\), which is to say that we set the last row and column equal to 0. The intersection of \(L\) with this linear space it spanned by \(E_{11}, A_2, A_3, \ldots, A_{d-2}\). By induction, we see that the infeasibility certificate as in Corollary 3.5 needs \(k = d - 1\).

The main difference between the cone \(S^d_+\) of positive semidefinite matrices and the positive orthant \((\mathbb{R}_+)^n\) from the point of view of this chapter, is in the tangent cones to proper faces. The tangent cone of \((\mathbb{R}_+)^n\) at a proper face \(F\) is simply \((\mathbb{R}_+)^n + \text{span}(F)\), i.e. the lineality space is the span of the face itself. For the cone \(S^d_+\), the lineality space of \(TC_F(S^d_+)\) is bigger than just the span of the face. These tangent directions prevent an immediate separation that is possible in the polyhedral case. This can be seen as the geometric reason for the differences between the two tangent directions prevent an immediate separation that is possible in the polyhedral case.

### 3.2. An alternative homogenization of SDPs

In this final section, we give a characterization of infeasible semidefinite programs, based on a lift of the cone \(S^d_+\) to the larger semidefinite cone \(S^{d+2n}_+\). It relies on an alternative way to homogenize linear matrix inequalities, which was used in [16]. As before, let \(L = A_0 + \langle A_1, \ldots, A_n \rangle\) and let \(\hat{L} = \langle A_0, A_1, \ldots, A_n \rangle\) be the linear span of \(L\). We also assume that \(A_1, A_2, \ldots, A_n\) are linearly independent so that \(\dim(L) = n\). Then we have

\[
S^d_+ \cap \hat{L} = \{ X \in S^d : X \succeq 0, \exists x_i \in \mathbb{R}, X = x_0 A_0 + x_1 A_1 + \cdots + x_n A_n \}.
\]

**Theorem 3.13.** The program \(S^d_+ \cap L\) is infeasible if and only if

\[
\begin{cases}
(X, r) \in S^d \times \mathbb{R} : X \in S^d_+ \cap \hat{L}, \\
\begin{bmatrix} x_0 & x_1 \\ z_i \\ r \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} x_0 & x_n \\ z_i \\ r \end{bmatrix} \succeq 0
\end{cases}
= \{0\} \times \mathbb{R}_+.
\]

Above, \(\oplus\) denotes the block sum of the 2 \times 2 matrices into a 2n \times 2n matrix.

**Proof:** Let \((X, r)\) be in the set in (3.14). Since \(X \in \hat{L}\), there is \((x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}\) with \(X = x_0 A_0 + \sum_{i=1}^n x_i A_i\). From the semidefinite constraint on the 2 \times 2 blocks we deduce that \(x_0 \geq 0\). If \(x_0 = 0\), then the 2 \times 2 blocks being positive semidefinite imply that \((x_0, x_1, \ldots, x_n) = 0\). If \(x_0 > 0\), then we can rescale to get a point \(A_0 + \sum_{i=1}^n (x_i/x_0) A_i\) in \(S^d_+ \cap L\). We deduce that \(S^d_+ \cap L\) is infeasible if and only if the projection of the set (3.14) in \(S^d\) is \(\{0\}\). Over this point, again by the additional semidefinite constraints, \(r\) can take any nonnegative value.

**Remark 3.15.** The size of the additional 2n \times 2n semidefinite constraint in the set (3.14) of Theorem 3.13 grows linearly in the dimension of \(L\) and one needs to add a constant number of variables (namely, 2) with respect to the original linear matrix inequality. This implies that the extra cost for checking the condition of Theorem 3.13, that can be used, combined with the homogenization and Theorem 1.6, to compute the feasibility type of \(K \cap L\), is controlled. Moreover the lifted LMI in (3.14) is defined over the same field as that of original one.
Example 3.16 (Example 3.2 continued). Homogenizing the linear matrix inequality in Example 3.2 following Theorem 3.13, we get the homogeneous linear matrix inequality

\[
A^{(h)}(x_0, x_1, r) = \begin{bmatrix}
0 & x_0 & 0 \\
 x_0 & x_1 & \end{bmatrix} \bigoplus \begin{bmatrix}
x_0 & x_1 \\
x_1 & r \\
\end{bmatrix} \succeq 0.
\]

Recall that Theorem 3.13 predicts that the unique solution to the homogenized linear matrix inequality above is the ray \((x_0, x_1, r) = (0, 0, r)\), so one gets only 4 significant digits for \(x_1\). Minimizing generic linear forms on this linear matrix inequality, we get either error “Unbounded function”, or an approximation of the feasible solutions \((0, 0, r)\) (with good precision). This reflects the fact that a linear function \(\ell\) is either unbounded from below on a cone or its minimum is zero, and it is attained at the origin. □

Example 3.17 (Example 4.6.2 in [13]). Consider the linear matrix inequality

\[
A(x_1, x_2) = \begin{bmatrix}
0 & x_1 & 0 \\
x_1 & x_2 & 1 \\
0 & 1 & x_1 \\
\end{bmatrix} \succeq 0.
\]

This is weakly infeasible, but without a linear certificate in the sense of [13, Definition 4.3.2 and Remark 4.3.6]. Indeed, it follows by [13] that one can associate to the linear matrix inequality \(A \succeq 0\) a quadratic module \(M_A\), containing polynomials that are positive over the associated spectrahedron. The infeasibility certificate is given by the membership \(-1 \in M_A\), which contradicts the feasibility of the linear matrix inequality. Klep and Schweighofer show in [13, Example 4.6.2] that the SOS-multipliers in the membership certificate \(-1 \in M_A\) have degree at least 4 for this example (so squares of linear forms are not enough).

Applying the homogenization scheme of Theorem 3.13, we get

\[
A^{(h)}(x_0, x_1, x_2, r) = \begin{bmatrix}
0 & x_1 & 0 \\
x_1 & x_2 & x_0 \\
0 & x_0 & x_1 \\
\end{bmatrix} \bigoplus \begin{bmatrix}
x_0 & x_1 \\
x_1 & r \\
\end{bmatrix} \bigoplus \begin{bmatrix}
x_0 & x_2 \\
x_2 & r \\
\end{bmatrix} \succeq 0.
\]

One can check by hand that this linear matrix inequality has as solution the half-line \((0, 0, 0, r)\), with \(r \geq 0\). Hence we deduce that the original linear matrix inequality is infeasible. □

Appendix A. Grassmannian

In this section, we want to summarize useful facts about the real and affine Grassmannians and give detailed pointers to the literature. The section includes proofs of facts that we have used in preceding sections, most importantly Section 1.

We begin with a technically precise explanation of what we mean by the Grassmannian of \(d\)-dimensional affine subspaces of \(\mathbb{R}^n\) based on the construction in geometry for the projective case.

Remark A.1. Denote by \(\mathbb{P}^n\) the \(n\)-dimensional real projective space (often denoted \(\mathbb{RP}^n\) or \(\mathbb{P}^n(\mathbb{R})\)), i.e. \(\mathbb{R}^{n+1} \setminus \{0\}/\mathbb{R}^*\), where \(\mathbb{R}^* = \mathbb{R} \setminus \{0\}\) acts diagonally on \(\mathbb{R}^{n+1}\). We can specify a point of \(\mathbb{P}^n\) by homogeneous coordinates \((x_0 : x_1 : \ldots : x_n)\), not all \(x_i\) equal to 0. These coordinates represent the equivalence class \(t(x_0, x_1, \ldots, x_n), t \in \mathbb{R}^*\), i.e. the line spanned by the vector \((x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}\). A \(d\)-dimensional linear subspace \(L\) of \(\mathbb{P}^n\) is a subset of points that come from a \((d + 1)\)-dimensional
linear space $\hat{L} \subset \mathbb{R}^{n+1}$, i.e. $L = \hat{L}/\mathbb{R}^\ast$. Such a linear space can be generated by $d + 1$ vectors $v_0, v_1, \ldots, v_d$, namely a basis of $\hat{L}$.

In this way, the coordinates on $\mathbb{R}^{n+1}$ give local coordinates on the Grassmannian $\mathbb{G}(d, n)$ of $d$-dimensional subspaces of $\mathbb{P}^n$. Indeed, we represent a $d$-dimensional linear subspace $L$ of $\mathbb{P}^n$ by the matrix

$$(v_0 \ v_1 \ \ldots \ v_d),$$

where $v_0, v_1, \ldots, v_d$ is any basis of $\hat{L} \subset \mathbb{R}^{n+1}$. Of course, a different basis should represent the same point in $\mathbb{G}(d, n)$. Therefore, we mod out the equivalence relation of column operations, which take us from one basis of $\hat{L}$ to any other. So if we write $V_{d,n}$ for the set of $(n + 1) \times (d + 1)$ matrices of rank $d + 1$, the Grassmannian is

$$\mathbb{G}(d, n) = V_{d,n} / \text{GL}_{d+1}(\mathbb{R}),$$

where $\text{GL}_{d+1}(\mathbb{R})$ is the general linear group of invertible $(d+1) \times (d+1)$ real matrices.

Based on this projective discussion, we want to explain the Grassmannian $\text{Gr}(d, n)$ of $d$-dimensional affine subspaces of $\mathbb{R}^n$. For this, we fix the embedding

$$\iota: \begin{cases} \mathbb{R}^n \to \mathbb{P}^n \\ (x_1, x_2, \ldots, x_n) \mapsto (1 : x_1 : x_2 : \ldots : x_n) \end{cases}$$

so that the hyperplane $H_0 = \{(x_0 : x_1 : \ldots : x_n) \in \mathbb{P}^n : x_0 = 0\}$ plays the special role of the “hyperplane at infinity”. A $d$-dimensional affine subspace $L = \text{aff}(v_0, v_1, \ldots, v_d) \subset \mathbb{R}^n$ defines via $\iota$ a $d$-dimensional linear subspace $L_\ast$ of $\mathbb{P}^n$, namely $\hat{L}_\ast = \text{span}(L)$. A basis of this projective linear space is $\{\iota(v_i) : i = 0, 1, \ldots, d\}$. Conversely, every $d$-dimensional projective linear space that is not contained in $H_0$ comes from a unique $d$-dimensional affine subspace of $\mathbb{R}^n$ by the above construction. By the Grassmannian $\text{Gr}(d, n)$ of $d$-dimensional affine subspaces of $\mathbb{R}^n$, we mean the complement of the $d$-dimensional projective subspaces contained in $H_0$ in $\mathbb{G}(d, n)$. Technically, this is a quasi-projective variety. More importantly, it is an open subset of $\mathbb{G}(d, n)$ and as such a smooth manifold.

Above, we need a basic topological fact that we prepare here. We use the dual projective space $(\mathbb{P}^n)^\ast$ of hyperplanes in $\mathbb{P}^n$, where we identify a hyperplane with its normal vector. Since a normal vector of a hyperplane is uniquely determined up to non-zero scaling, this is indeed a point in an $n$-dimensional projective space.

**Lemma A.2.** Let $U$ be an open set in $(\mathbb{P}^n)^\ast$ of hyperplanes in $\mathbb{P}^n$. Then the set of $d$-dimensional projective subspaces of $\mathbb{P}^n$ that are contained in a hyperplane $H$ that lies in $U$ is an open subset of $\mathbb{G}(d, n)$.

**Proof.** Consider the incidence correspondence

$$\Sigma = \{(L, [H]) : L \subset H\} \subset \mathbb{G}(d, n) \times (\mathbb{P}^n)^\ast$$

of $d$-dimensional projective spaces $L$ and hyperplanes $H \subset \mathbb{P}^n$ such that $L$ is contained in $H$; together with the two projections $\pi_1: \mathbb{G}(d, n) \times (\mathbb{P}^n)^\ast \to \mathbb{G}(d, n)$ and $\pi_2: \mathbb{G}(d, n) \times (\mathbb{P}^n)^\ast \to (\mathbb{P}^n)^\ast$. This incidence correspondence is in fact a projective bundle over $\mathbb{G}(d, n)$ of rank $n - d - 1$. Indeed, this is a simple linear algebra computation: By changing the basis of the ambient projective space, we can assume that $L$ is represented by the matrix

$$\begin{pmatrix} I_{d+1} \\ 0_{n-d} \end{pmatrix}$$
by choosing a basis of $L$ and extending it to any basis of the ambient space. Then a neighborhood of $L$ in $G(d,n)$ consists of all linear subspaces with basis of the form

$$M = \begin{pmatrix} I_{d+1} \\ A \end{pmatrix}$$

for any $(n-d) \times (d+1)$-matrix $A$. In fact, this is a standard affine chart of the Grassmannian $W$, see e.g. [10]. A point $v = (v_0 : v_1 : \ldots : v_d : w) \in (\mathbb{P}^n)^*$ is the normal vector of a hyperplane containing the linear space represented by the above matrix $M$ if and only if $v$ is in the left kernel of $M$. So the local trivialization of $\pi_1: \Sigma \to G(d,n)$ around $L$ is the map

$$\begin{cases} W \times \mathbb{P}^{n-d-1} \to \pi_1^{-1}(W) \\ (M, w) \mapsto (M, [-wA, w]) \end{cases}.$$ 

With this structure in mind, the proof of the claim is elementary topology. By continuity of $\pi_2$, the set $\pi_2^{-1}(U) \subset \Sigma$ is an open subset of $\Sigma$. We claim that $\pi_1(\pi_2^{-1}(U)) \subset G(d,n)$ is also open. Being open is a local property, so we can locally trivialize the projection $\pi_1$ around a point $L \in \pi_1(\pi_2^{-1}(U))$ and conclude the claim from the fact that coordinate projections are open maps. □

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