STOCHASTIC INTEGRATION AND BOUNDARIES
OF INFINITE NETWORKS

PALLE E. T. JORGENSEN AND ERIN P. J. PEARSE

ABSTRACT. We study the boundary theory of a connected weighted graph $G$ from the viewpoint of stochastic integration. For the Hilbert space $\mathcal{H}_E$ of Dirichlet-finite functions on $G$, we construct a Gel’fand triple $S \subseteq \mathcal{H}_E \subseteq S'$, which yields a probability measure $\mathbb{P}$ and an isometric embedding of $\mathcal{H}_E$ into $L^2(S', \mathbb{P})$, and hence gives a concrete representation of the boundary as a certain class of “distributions” in $S'$. In a previous paper, we proved a discrete Gauss-Green identity for infinite networks which produces a boundary representation for a harmonic function $u(x) = \sum_{bdG} v(x)\partial_n$, where the sum is understood in a limiting sense. In this paper, we use techniques from stochastic integration to make the boundary $bdG$ precise as a measure space, and replace the sum with an integral over $S'$, thus obtaining a boundary integral representation for the harmonic function $u$.

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1. INTRODUCTION

In this paper, we develop a boundary theory for an infinite network (connected weighted graph) $G$, using some techniques from the theory of stochastic integration. For the Hilbert space $\mathcal{H}_E$ of finite-energy functions on $G$, we construct a Gel’fand triple $S_G \subseteq \mathcal{H}_E \subseteq S'_G$, where both containments are strict, and the inclusion mappings are continuous. Here, $S_G$ is a space of “test functions” on the network and $S'_G$ is a class of “distributions” on the network, analogous to Schwartz’s classical functions of rapid decay and tempered distributions, respectively. A key result of this paper is Theorem 5.2, which establishes an...
isometric embedding of $\mathcal{H}_E$ into the Hilbert space $L^2(S'_G, \mathbb{P})$, where $\mathbb{P}$ is a Gaussian probability measure on $S'_G$. In Martin boundary theory, elements of the boundary are understood as certain minimal harmonic functions. We are studying finite-energy harmonic functions instead of positive harmonic functions, but the construction outlined above allows us to study elements of the boundary in an analogous fashion.

In a previous paper, we proved a discrete Gauss-Green identity for infinite networks:

$$E(u, v) = \sum_{G'} \pi \Delta v + \sum_{bdG} \pi \partial_n v.$$  \hfill (1.1)

Here, $E$ is a Dirichlet form and $\Delta$ is the graph Laplacian; see Theorem 2.21 and the discussion preceding it for precise definitions of the other terms. Formula (1.1) yields a boundary representation for a harmonic function $u$:

$$u(x) = \sum_{bdG} u \frac{\partial v}{\partial n} + C,$$  \hfill (1.2)

where $C$ is a constant and the sum is actually defined as a limit of “Riemann sums” over an increasing sequence of finite subnetworks of $G$; see Definition 2.20. In this paper, we use functional integration techniques from stochastic integration to make the boundary $bdG$ precise as a measure space, and replace the sum with an integral over $S'_G$, thus obtaining a boundary integral representation (in a sense analogous to that of Poisson or Martin boundary theory) for the harmonic function $u$.

Another key result of this paper is Corollary 6.1, which follows readily from Theorem 5.2 and gives a boundary integral representation for harmonic functions of finite energy:

$$u(x) = \int_{S'_G} u(x) h_x(\xi) d\nu(\xi) + u(o).$$  \hfill (1.3)

Here, $(h_x)_{x \in G}$ is a family of harmonic functions parametrized by the vertices and discussed in detail just below (and see also Lemma 2.17). Given a transient network, this allows one to identify a space of functions in $S'_G$ corresponding to the boundary of the network (in a manner reminiscent of the Martin boundary). Additionally, Example 7.1 presents the construction of a harmonic function of finite energy on a network with one “graph end” (in fact, a two-parameter family of such networks). The existence of such functions was first proved in [CW92], but we have never seen an explicit formula given before. We now proceed to describe these results in a bit more detail. The reader is also referred to the survey paper [JP09e] which gives an overview of how the results of the present paper fit into a larger investigation of functions of finite energy on resistance networks, and the effective resistance metric.

1.1. **Overview.** We define what it means for a function on a network to have finite energy in Definition 2.5. Then the (discrete) Dirichlet form $E$ (also given in Definition 2.5) is an inner product on the space of functions of finite energy, and in fact produces a Hilbert space which we denote by $\mathcal{H}_E$. The space $\mathcal{H}_E$ consists of equivalence classes of functions on the vertices of $G$, where $u \approx v$ if $u - v$ is a constant function. In a previous paper, we constructed a reproducing kernel $(v_x)_{x \in G}$ for this Hilbert space and used it to prove a discrete Gauss-Green identity which is recalled in Theorem 2.21.

The space $\mathcal{H}_E$ also enjoys an orthogonal decomposition into the subspace $\text{Fin}$ of ($E$-limits of) finitely supported functions and the subspace $\text{Harm}$ of harmonic functions; see Definitions 2.9–2.10 and Theorem 2.12. Since the reproducing kernel behaves well with
respect to projections, we also have reproducing kernels \( \{ f_x \}_{x \in G^0} \) for \( \mathcal{F}_{\text{fin}} \) and \( \{ h_x \}_{x \in G^0} \) for \( \mathcal{H}_{\text{Harm}} \), where

\[
f_x := P_{\mathcal{F}_{\text{fin}}} v_x, \quad \text{and} \quad h_x := P_{\mathcal{H}_{\text{Harm}}} v_x.
\]

It should be noted that these kernels are reproducing up to an additive constant; in other words, for some fixed reference vertex \( o \),

\[
(v_x, u)_E = u(x) - u(o),
\]

and similarly for \( f_x \) and \( h_x \).

Recall the classical result of Poisson that gives a kernel \( k : \Omega \times \partial\Omega \to \mathbb{R} \) from which a bounded harmonic function can be given via

\[
u(x) = \int_{\partial\Omega} u(y) k(x, d\beta), \quad y \in \partial\Omega. \tag{1.4}\]

We are motivated by the discrete analogue of this result appearing in (1.2). Comparison of (1.2) and (1.4) makes one optimistic that \( \text{bd} G \) can be realized as a measure space which supports a measure corresponding to \( \frac{\partial h}{\partial x} \), thus replacing the sum in (1.2) with an integral. In Corollary 6.1, we do precisely this.

Boundary theory of harmonic functions can roughly be divided three ways: the bounded harmonic functions (Poisson theory), the nonnegative harmonic functions (Martin theory), and the finite-energy harmonic functions studied in the present paper. While Poisson theory is a subset of Martin theory, the relationship between Martin theory and the study of \( \mathcal{H}_E \) is more subtle. For example, there exist unbounded functions of finite energy; cf. [JP09d, Ex. 13.10]. Some results detailing the interrelations are given in [Soa94, §3.7]. Whether the focus is on the harmonic functions which are bounded, nonnegative, or finite-energy, the goals of the associated boundary theory are essentially the same:

1. Construct a space \( \overline{D} \) which extends the original domain \( D \); this can be done by taking closure, compactifying, or similar operations.
2. One can then identify the boundary \( \text{bd} D \) as \( \overline{D} \setminus D \), or (if the boundary thus obtained would be larger than necessary/practical for the application in mind), as some subset of \( \overline{D} \setminus D \).
3. Define a procedure for extending harmonic functions \( u \) from \( D \) to \( \text{bd} D \). This extension \( \tilde{u} \) may be a measure (or other linear functional) on \( \text{bd} D \); it may not be a function.
4. Obtain a kernel \( k(x, \beta) \) defined on \( D \times \text{bd} D \) against which one can integrate the extension \( \tilde{u} \) so as to recover the value of \( u \) at a point in \( D \):

\[
u(x) = \int_{\text{bd} D} k(x, \beta) \tilde{u}(d\beta), \quad \forall x \in D,
\]

whenever \( u \) is a harmonic function of the given class.

Our approach to (1) is to use Gel’fand triples to extend the original domain, a method which is novel as far as we know. In a forthcoming work [JP10], we will introduce an interpolation formula that uses the analytic framework developed in this paper, and which turns \( G \) into a stochastic process. In particular, the interpolation formula allows one to find continua which naturally extend \( G \).
The difference between our boundary theory and that of Poisson and Martin is rooted in our focus on $\mathcal{H}_E$ rather than $\ell^2$: both classical theories concern harmonic functions with growth/decay restrictions. By contrast, provided they neither grow too wildly nor oscillate too wildly, elements of $\mathcal{H}_E$ may have values tending to both $+\infty$ and $-\infty$. See [JP09d, Ex. 13.10] for a function $h \in \text{Harm}$ which is unbounded in this way. Positive harmonic functions are naturally given to analysis based on probabilistic and potential-theoretic techniques, and the companion study of superharmonic (or subharmonic) functions is indispensable. Without positivity, however, one can get more mileage by considering the Dirichlet form $\mathcal{E}$ as an inner product and studying the resulting Hilbert space geometry.

Our boundary essentially consists of (equivalence classes of) infinite paths which can be distinguished by monopoles, i.e., two paths are not equivalent iff there is a monopole $w$ with different limiting values along each path. It is an immediate consequence that recurrent networks have no boundary, and transient networks with no nontrivial harmonic functions have exactly one boundary point (corresponding to the fact that the monopole at $x$ is unique). In particular, the integer lattices $\mathbb{Z}^d, \mathbb{I}$ each have 1 boundary point for $d \geq 3$ and 0 boundary points for $d = 1, 2$. In contrast, the Martin boundary of $(\mathbb{Z}^d, \mathbb{I})$ is homeomorphic to the unit sphere $S^{n-1}$ (where $S^0 = \{-1, 1\}$), and each $(\mathbb{Z}^d, \mathbb{I})$ has only one graph ends (except for $(\mathbb{Z}, \mathbb{I})$, which has two); cf. [PW90, §3.B], for example.

1.2. Outline. In our version of the program outlined above, we follow the steps in the order (2)-(3)-(1). A brief summary is given here; further introductory material and technical details appear at the beginning of each subsection.

§2 recalls basic definitions and some previously obtained results.

§3 describes two methods for constructing a Gel’fand triple. The technique presented in §3.1 works for any network $(G, c)$ and makes use of an orthonormal basis of $\mathcal{H}_E$ derived from the energy kernel $\{v_x\}_{x \in G}$ via the Gram-Schmidt algorithm, or equivalently, from the domain of a certain operator $\mathcal{N}$. The approach given in §3.2 works only for networks where $\Delta$ is an unbounded operator on $\mathcal{H}_E$. This version of $S_G$ is constructed in terms of the domain of $\Delta$.

§4 studies the structure of $S_G$ (the space of test functions) and $S'_G$ (the space of distributions) and establishes some key lemmas for later use.

§5 proves a key result: Theorem 5.2, which establishes the isometric embedding of $\mathcal{H}_E$ into $L^2(S'_G, \mathcal{P})$ given by the Wiener transform. Applying this isometry to the energy kernel $\{v_x\}$, we get a reproducing kernel $k(x, d\mathcal{P})$ given in terms of a version of Wiener measure. In fact, $\mathcal{P}$ is a Gaussian probability measure on $S'_G$ whose support is disjoint from $\text{Fin}$. The results in this section hold for any Gel’fand triple; in particular, for either of the ones constructed in §3 and §3.2.

§6 We consider certain measures $\mu_x$, defined in terms of the kernel and the Wiener measure just introduced, which are supported on $S'_G/\text{Fin}$ and indexed by the vertices $x \in G^0$. Then points of $\text{bd} G$ correspond to limits of sequences $(\mu_x)$ where $x_n \to \infty$, modulo a suitable equivalence relation.

Remark 1.1. While Doob’s martingale theory works well for harmonic functions in $L^\infty$ or $L^2$, the situation for $\mathcal{H}_E$ is different. The primary reason is that $\mathcal{H}_E$ is not immediately realizable as an $L^2$ space. A considerable advantage of our Gel’fand-Wiener-Ito construction is that $\mathcal{H}_E$ is isometrically embedded into $L^2(S'_G, \mathcal{P})$ in a particularly nice way: it corresponds to the polynomials of degree 1. See Remark 5.5.

Another contrast is that $\Delta$ may, in general, be unbounded in our context. Recall that when studying an operator, an important subtlety is that “the” adjoint $\Delta^*$ depends on the choice of domain, i.e., the linear subspace $\text{dom}(\Delta) \subseteq \mathcal{H}$.
Boundary theory is a well-established subject; the deep connections between harmonic analysis, probability, and potential theory have led to several notions of boundary and we will not attempt to give complete references. However, we recommend [Saw97] for introductory material on Martin boundary and [DS84, LP09] for introductory material on resistance networks. Additionally, [Lyo83, Car73, Woe00], and the foundational paper [NW59] provide more specific background. With regard to infinite graphs and finite-energy functions, see [Soa94, SW91, CW92, Dod06, PW90, PW88, Woe86, Tho90]. But we ask different questions here, and the operator theory we use is different; it does not easily compare to earlier literature. For some recent related areas, see e.g., [AL08, AAL08, AD06] reproducing kernels, [Arv86] Markov operators, [Cho08] graph analysis, and [AP09] operator theory.

There has been a recent interest in analysis and potential theory on infinite-dimensional spaces, and the use of stochastic integration in conjunction with reproducing kernels [HNS09, Xia10, CdtU10, XZZ09], and Gel’fand triples [HLW10, BK07]. Although our setting here is different, we are able to adapt these tools for the task at hand. This is nontrivial because, in the classical case, there is a natural differentiable structure around, and therefore the choice of a Schwartz space going into a useful Gel’fand triple is often rather conventional. But by contrast, we deal with discrete structures, and so we must give up differential operators. Nonetheless, we exhibit Schwartz spaces that yield Gel’fand triples which accomplish what we need.

2. Basic terms and previous results

We now proceed to introduce the key notions used throughout this paper: resistance networks, the energy form $E$, the Laplace operator $\Delta$, and their elementary properties. Our approach is somewhat different from existing studies of networks in the literature, and so we take this opportunity to introduce the tools we will need: an unbounded Laplace operator with dense domain in a Hilbert space, a two-point reproducing kernel for this Hilbert space, the quadratic form associated to the Laplacian, and Gelfand triples. Since these are tools not commonly used in geometric analysis, we include their definitions and some theorems from earlier papers which will be needed later. Additionally, we will use the theorems of Bochner (Theorem 2.23), and Minlos (Theorem 2.24).

**Definition 2.1.** A resistance network is a connected graph $(G, c)$, where $G$ is a graph with vertex set $G^0$, and $c$ is the conductance function which defines adjacency by $x \sim y$ iff $c_{xy} > 0$, for $x, y \in G^0$. We assume $c_{xy} = c_{yx} \in [0, \infty)$, and write $c(x) := \sum_{y \sim x} c_{xy}$. We require $c(x) < \infty$ (note that we allow vertices of infinite degree), but $c(x)$ need not be a bounded function on $G^0$. The notation $c$ may be used to indicate the multiplication operator $(cv)(x) := c(x)v(x)$, i.e., the diagonal matrix with entries $c(x)$ with respect to the (vector space) basis $\{\delta_x\}$.

In this definition, connected means simply that for any $x, y \in G^0$, there is a finite sequence $\{x_i\}_{i=0}^n$ with $x = x_0$, $y = x_n$, and $c_{x_{i-1}x_i} > 0$, $i = 1, \ldots, n$. Conductance is the reciprocal of resistance, so one can think of $(G, c)$ as a network of nodes $G^0$ connected by resistors of resistance $c_{xy}^{-1}$. We may assume there is at most one edge from $x$ to $y$, as two conductors $c_{xy}^{-1}$ and $c_{xy}^{-2}$ connected in parallel can be replaced by a single conductor with
conductance \( c_{xy} = c_{yx}^1 + c_{yx}^2 \). Also, we assume \( c_{xx} = 0 \) so that no vertex has a loop, as electric current will never flow along a conductor connecting a node to itself.\(^1\)

**Definition 2.2.** The Laplacian on \( G \) is the linear difference operator which acts on a function \( v : G^0 \to \mathbb{R} \) by

\[
(\Delta v)(x) := \sum_{y \sim x} c_{xy}(v(x) - v(y)).
\]  

A function \( v : G^0 \to \mathbb{R} \) is harmonic iff \( \Delta v(x) = 0 \) for each \( x \in G^0 \).

We have adopted the physics convention (so that the spectrum is nonnegative) and thus our Laplacian is the negative of the one commonly found in the PDE literature. The network Laplacian \((2.1)\) should not be confused with the stochastically renormalized Laplace operator \( c^{-1} \Delta \) which appears in the probability literature, or with the spectrally renormalized Laplace operator \( c^{-1/2} \Delta c^{-1/2} \) which appears in the literature on spectral graph theory (e.g., \([\text{Chu01}]\)).

**Definition 2.3.** An exhaustion of \( G \) is an increasing sequence of finite and connected subgraphs \( \{G_k\}_{k=1}^{\infty} \), so that \( G_k \subseteq G_{k+1} \) and \( G = \bigcup G_k \). Since any vertex or edge is eventually contained in some \( G_k \), there is no loss of generality in assuming they are contained in \( G_1 \), for the purposes of a specific computation.

**Definition 2.4.** The notation

\[
\sum_{x \in G^0} := \lim_{k \to \infty} \sum_{x \in G_k}
\]  

is used whenever the limit is independent of the choice of exhaustion \( \{G_k\} \) of \( G \). This is clearly justified, for example, whenever the sum has only finitely many nonzero terms, or is absolutely convergent as in the definition of \( E \) in Definition 2.5.

**Definition 2.5.** The energy of functions \( u, v : G^0 \to \mathbb{C} \) is given by the (closed, bilinear) Dirichlet form

\[
E(u, v) := \frac{1}{2} \sum_{x \in G^0} \sum_{y \in G^0} c_{xy}(\overline{m}(x) - \overline{m}(y))(v(x) - v(y)),
\]  

with the energy of \( u \) given by \( E(u) := E(u, u) \). The domain of the energy is

\[
\text{dom} E := \{u : G^0 \to \mathbb{C} : E(u) < \infty\}.
\]  

Since \( c_{xy} = c_{yx} \) and \( c_{xy} = 0 \) for nonadjacent vertices, the initial factor of \( \frac{1}{2} \) in \((2.3)\) implies there is exactly one term in the sum for each edge in the network.

2.1. **The energy space** \( \mathcal{H}_E \). Let \( 1 \) denote the constant function with value 1 and recall that \( \ker E = \mathbb{C} 1 \).

\(^1\)Nonetheless, self-loops may be useful for technical considerations: one can remove the periodicity of a random walk by allowing self-loops. This can allow one to obtain a “lazy walk” which is ergodic, and hence more tractable. See, for example, \([\text{LPW08, LP09}]\).
Definition 2.6. The energy form $E$ is symmetric and positive definite on $\text{dom} E$. Then $\text{dom} E/\mathbb{C}1$ is a vector space with inner product and corresponding norm given by
\[
\langle u, v \rangle_E := E(u, v) \quad \text{and} \quad \|u\|_E := E(u, u)^{1/2}.
\]
The energy Hilbert space $\mathcal{H}_E$ is $\text{dom} E/\mathbb{C}1$ with inner product (2.5).

Definition 2.7. Let $v_x$ be defined to be the unique element of $\mathcal{H}_E$ for which
\[
\langle v_x, u \rangle_E = u(x) - u(o), \quad \text{for every } u \in \mathcal{H}_E.
\]
The collection $\{v_x\}_{x \in G}$ forms a reproducing kernel for $\mathcal{H}_E$ ([JP09b, Cor. 2.7]); we call it the energy kernel and (2.6) shows its span is dense in $\mathcal{H}_E$. Note that $v_x$ corresponds to a constant function, since $\langle v_x, u \rangle_E = 0$ for every $u \in \mathcal{H}_E$. Therefore, $v_x$ is often ignored or omitted.

Definition 2.8. A dipole is any $v \in \mathcal{H}_E$ satisfying the pointwise identity $\Delta v = \delta_x - \delta_y$ for some vertices $x, y \in G^0$. One can check that $\Delta v_x = \delta_x - \delta_y$; cf. [JP09b, Lemma 2.13].

Definition 2.9. For $v \in \mathcal{H}_E$, one says that $v$ has finite support if there is a finite set $F \subseteq G^0$ for which $v(x) = k \in \mathbb{C}$ for all $x \notin F$, i.e., the set of functions of finite support in $\mathcal{H}_E$ is
\[
\text{span}\{\delta_x\} = \{u \in \text{dom} E : u(x) = k \text{ for some } k, \text{ for all but finitely many } x \in G^0\},
\]
where $\delta_x$ is the Dirac mass at $x$, i.e., the element of $\mathcal{H}_E$ containing the characteristic function of the singleton $\{x\}$. It is immediate from (2.3) that $E(\delta_x) = c(x)$, whence $\delta_x \in \mathcal{H}_E$.

Define $\text{Fin}$ to be the closure of $\text{span}\{\delta_x\}$ with respect to $E$.

Definition 2.10. The set of harmonic functions of finite energy is denoted
\[
\mathcal{H}_\text{Harm} := \{v \in \mathcal{H}_E : \Delta v(x) = 0, \text{ for all } x \in G^0\}.
\]

Note that this is independent of choice of representative for $v$ in virtue of (2.1).

Lemma 2.11 ([JP09b, 2.11]). For any $x \in G^0$, one has $\langle \delta_x, u \rangle_E = \Delta u(x)$.

The following result follows easily from Lemma 2.11; cf. [JP09b, Thm. 2.15].

Theorem 2.12 (Royden decomposition). $\mathcal{H}_E = \mathcal{F}_\text{in} \oplus \mathcal{H}_\text{Harm}$.

Definition 2.13. A monopole at $x \in G^0$ is an element $w_x \in \mathcal{H}_E$ which satisfies $\Delta w_x(y) = \delta_{xy}$, where $\delta_{xy}$ is Kronecker’s delta. In case the network supports monopoles, let $w_x$ always denote the unique energy-minimizing monopole at the origin.

When $\mathcal{H}_E$ contains monopoles, let $\mathcal{M}_x$ denote the vector space spanned by the monopoles at $x$. This implies that $\mathcal{M}_x$ may contain harmonic functions; see [JP09b, Lemma 4.1]. With $v_x$ and $f_x = P_{\text{fin}}v_x$ as above, we indicate the distinguished monopoles
\[
w_x := v_x + w_o \quad \text{and} \quad w'_x := f_x + w_o.
\]

Remark 2.14. Note that $w_o \in \mathcal{F}_\text{in}$, whenever it is present in $\mathcal{H}_E$, and similarly that $w'_x$ is the energy-minimizing element of $\mathcal{M}_x$. To see this, suppose $w_x$ is any monopole at $x$. Since $w_x \in \mathcal{H}_E$, write $w_x = f + h$ by Theorem 2.12, and get $E(w_x) = E(f) + E(h)$. Projecting away the harmonic component will not affect the monopole property, so $w'_x = P_{\text{fin}}w_x$ is the unique monopole of minimal energy. The Green function is $g(x, y) = w'_x(y)$, where $w''_y$ is the representative of $w'_y$ satisfying $w''_y(o) = 0$. 


Definition 2.15. The dense subspace of $\mathcal{H}_E$ spanned by monopoles (or dipoles) is

$$M := \text{span}[v_x]_{x \in G} + \text{span}[w^x_x, w^y_x]_{x \in G}. \quad (2.10)$$

Let $\Delta_M$ be the closure of the Laplacian when taken to have the dense domain $M$. Since $\Delta$ agrees with $\Delta_M$ pointwise, we may suppress reference to the domain for ease of notation.

Lemma 2.16 (JP09b, Lemma 3.5). $\Delta_M$ is Hermitian with $\langle u, \Delta_M u \rangle_E \geq 0$ for all $u \in M$.

Lemma 2.17 (JP09b, Lemma 3.6). When the network is transient, $M$ contains the spaces $\text{span}[v_x], \text{span}[f_x]$, and $\text{span}[h_x]$, where $f_x = P_{f_{in}}v_x$ and $h_x = P_{f_{out}}v_x$. When the network is not transient, $M = \text{span}[v_x] = \text{span}[f_x]$.

Remark 2.18 (Monopoles and transience). The presence of monopoles in $\mathcal{H}_E$ is equivalent to the transience of the simple random walk on the network with transition probabilities $p(x,y) = c_{xy}/c(x)$: note that if $w_x$ is a monopole, then the current induced by $w_x$ is a unit flow to infinity with finite energy. It was proved in [Lyo83] that the network is transient if and only if there exists a unit current flow to infinity; see also [LP09, Thm. 2.10].

2.2. The discrete Gauss-Green identity. The space $M$ is introduced as a dense domain for $\Delta$ and as the scope of validity for the discrete Gauss-Green identity of Theorem 2.21.

Definition 2.19. If $H$ is a subgraph of $G$, then the boundary of $H$ is

$$\text{bd } H := \{x \in H : \exists y \in H^c, y \sim x\}. \quad (2.11)$$

The interior of a subgraph $H$ consists of the vertices in $H$ whose neighbours also lie in $H$:

$$\text{int } H := \{x \in H : y \sim x \implies y \in H\} = H \setminus \text{bd } H. \quad (2.12)$$

For vertices in the boundary of a subgraph, the normal derivative of $v$ is

$$\frac{\partial v}{\partial n}(x) := \sum_{y \in H} c_{xy}(v(x) - v(y)), \quad \text{for } x \in \text{bd } H. \quad (2.13)$$

Thus, the normal derivative of $v$ is computed like $\Delta v(x)$, except that the sum extends only over the neighbours of $x$ which lie in $H$.

Definition 2.19 will be used primarily for subgraphs that form an exhaustion of $G$, in the sense of Definition 2.3: an increasing sequence of finite and connected subgraphs $\{G_k\}$, so that $G_k \subseteq G_{k+1}$ and $G = \bigcup G_k$. Also, recall that $\sum_{\text{bd } G} := \lim_{k \to \infty} \sum_{\text{bd } G_k}$ from Definition 2.20.

Definition 2.20. A boundary sum is computed in terms of an exhaustion $\{G_k\}$ by

$$\sum_{\text{bd } G} := \lim_{k \to \infty} \sum_{\text{bd } G_k}. \quad (2.14)$$

whenever the limit is independent of the choice of exhaustion, as in Definition 2.4.

On a finite network, all harmonic functions of finite energy are constant, so that $\mathcal{H}_E = \mathcal{F}_{in}$ by Theorem 2.12, and one has $\mathcal{E}(u, v) = \sum_{x \in G^p} u(x)\Delta v(x)$, for all $u, v \in \mathcal{H}_E$. In fact, this remains true for recurrent infinite networks, as shown in [JP09b, Thm. 4.4]; see also [KY89]. However, the possibilities are much richer on an infinite network, as evinced by the following theorem.
Theorem 2.21 (Discrete Gauss-Green identity). If \( u \in \mathcal{H}_E \) and \( v \in \mathcal{M} \), then

\[
(u, v)_E = \sum_{G'} \pi \Delta v + \sum_{bd G} \pi \| v \|_{bd G}.
\] (2.15)

The discrete Gauss-Green formula (2.15) is the main result of [JP09b]; that paper contains several consequences of the formula, especially as pertains to transience.

2.3. Gel'fand triples and duality. One would like to obtain a (probability) measure space to serve as the boundary of \( G \). It is shown in [Gro67, Gro70, Min63] that no Hilbert space of functions \( \mathcal{H} \) is sufficient to support a Gaussian measure \( \mathbb{P} \) (i.e., it is not possible to have \( 0 < \mathbb{P}(\mathcal{H}) < \infty \) for a \( \sigma \)-finite measure). However, it is possible to construct a Gel'fand triple (also called a rigged Hilbert space): a dense subspace \( S \) of \( \mathcal{H} \) with

\[
S \subseteq \mathcal{H} \subseteq S',
\] (2.16)

where \( S \) is dense in \( \mathcal{H} \) and \( S' \) is the dual of \( S \). Additionally, \( S \) and \( S' \) must also satisfy some technical conditions: \( S \) is a dense subspace of \( \mathcal{H} \) with respect to the Hilbert norm, but also comes equipped with a strictly finer “test function” topology. When \( S \) is a Fréchet space equipped with a countable system of seminorms (stronger than the norm on \( \mathcal{H} \)), then the inclusion map of \( S \) into \( \mathcal{H} \) is continuous; in fact, it is possible to chose the seminorms in such a way that one gets a nuclear embedding (details below). Therefore, when the dual \( S' \) is taken with respect to this finer (Fréchet) topology, one obtains a strict containment \( \mathcal{H} \subseteq S' \). It turns out that \( S' \) is large enough to support a nice probability measure, even though \( \mathcal{H} \) is not.

It was Gel'fand's idea to formalize this construction abstractly using a system of nuclearity axioms [GMŠ58, Min58, Min59]. Our presentation here is adapted from quantum mechanics and the goal is to realize \( bdG \) as a subset of \( S' \). We will give a “test function topology” as a Fréchet topology defined via a specific sequence of seminorms, using either an onb for \( \mathcal{H}_E \) (in §3) or the domain of \( \Delta^p \) (in §3.2).

Remark 2.22 (Tempered distributions and the Laplacian). There is a concrete situation when the Gel'fand triple construction is especially natural: \( \mathcal{H} = L^2(\mathbb{R}, dx) \) and \( S \) is the Schwartz space of functions of rapid decay. That is, each \( f \in S \) is \( C^\infty \) smooth functions which decays (along with all its derivatives) faster than any polynomial. In this case, \( S \) is the space of tempered distributions and the seminorms defining the Fréchet topology on \( S \) are

\[
p_m(f) := \sup\{|x^k f^{(n)}(x)| : x \in \mathbb{R}, 0 \leq k, n \leq m\}, \quad m = 0, 1, 2, \ldots,
\]

where \( f^{(n)} \) is the \( n \)-th derivative of \( f \). Then \( S' \) is the dual of \( S \) with respect to this Fréchet topology. One can equivalently express \( S \) as

\[
S := \{ f \in L^2(\mathbb{R}) : (\tilde{P}^2 + \tilde{Q}^2)^n f \in L^2(\mathbb{R}), \forall n\},
\] (2.17)

where \( \tilde{P} : f(x) \mapsto \frac{1}{i} \frac{df}{dx} \) and \( \tilde{Q} : f(x) \mapsto x f(x) \) are Heisenberg's operators. The operator \( \tilde{P}^2 + \tilde{Q}^2 \) is often called the quantum mechanical Hamiltonian, but some others (e.g., Hida, Gross) would call it a Laplacian, and this perspective tightens the analogy with the present study. In this sense, (2.17) could be rewritten \( S := \text{dom} \Delta^\infty \); compare to (3.15). We show that a general network \((G, c)\) always has a harmonic oscillator; in fact, we discuss
an operator $\mathcal{N}$ in Definition 3.7 which is unitarily equivalent to $P^2 + Q^2$ and hence has the same spectrum.

The duality between $S$ and $S'$ allows for the extension of the inner product on $\mathcal{H}$ to a pairing of $S$ and $S'$:

$$\langle \cdot, \cdot \rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \quad \text{to} \quad \langle \cdot, \cdot \rangle_{\tilde{\mathcal{H}}} : S \times S' \to \mathbb{R}.$$  

In other words, one obtains a Fourier-type duality restricted to $S$. Moreover, it is possible to construct a Gel'fand triple in such a way that $\mathbb{P}(S') = 1$ for a Gaussian probability measure $\mathbb{P}$. When applied to $\mathcal{H} = \mathcal{H}_c$, the construction yields three main outcomes:

1. The next best thing to a Fourier transform for an arbitrary graph.
2. A concrete representation of $\mathcal{H}_c$ as an $L^2$ measure space $\mathcal{H}_c \cong L^2(S', \mathbb{P})$.
3. A boundary integral representation for harmonic functions of finite energy.

As a prelude, we begin with Bochner’s Theorem, which characterizes the Fourier transform of a positive finite Borel measure on the real line. The reader may find [RS75] helpful for further information.

**Theorem 2.23** (Bochner). Let $G$ be a locally compact abelian group. Then there is a bijective correspondence $\mathcal{F} : \mathcal{M}(G) \to \mathcal{PD}(\hat{G})$, where $\mathcal{M}(G)$ is the collection of measures on $G$, and $\mathcal{PD}(\hat{G})$ is the set of positive definite functions on the dual group of $G$. Moreover, this bijection is given by the Fourier transform

$$\mathcal{F} : \nu \mapsto \varphi_\nu \quad \text{by} \quad \varphi_\nu(\xi) = \int_G e^{i\langle \xi, x \rangle} d\nu(x).$$  

(2.18)

For our representation of the energy Hilbert space $\mathcal{H}_c$ in the case of general resistance network, we will need Minlos’ generalization of Bochner’s theorem from [Min63, Sch73]. This important result states that a cylindrical measure on the dual of a nuclear space is a Radon measure iff its Fourier transform is continuous. In this context, however, the notion of Fourier transform is infinite-dimensional, and is dealt with by the introduction of Gel’fand triples [Lee96].

**Theorem 2.24** (Minlos). Given a Gel’fand triple $S \subseteq \mathcal{H} \subseteq S'$, Bochner’s Theorem may be extended to yield a bijective correspondence between the positive definite functions on $S$ and the Radon probability measures on $S'$. Moreover, in a specific case, this correspondence is uniquely determined by the identity

$$\int_{S'} e^{\frac{1}{2} \langle u, x \rangle} d\mathbb{P}(\xi) = e^{-\frac{1}{2} \langle u, u \rangle}.$$  

(2.19)

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the original inner product on $\mathcal{H}$ and $\langle \cdot, \cdot \rangle_{\tilde{\mathcal{H}}}$ is its extension to the pairing on $S \times S'$.

Formula (2.19) may be interpreted as defining the Fourier transform of $\mathbb{P}$; the function on the right-hand side is positive definite and plays a special role in stochastic integration, and its use in quantization.

3. **Gel’fand triples for $\mathcal{H}_c$**

In this section, we describe two methods for construction a Gel’fand triple for $\mathcal{H}_c$. The first method is applicable to all networks, but relies on the choice of some enumeration of the vertices of $G$, and the Gram-Schmidt algorithm for producing an orthonormal basis. However,
we will see that the Gram-Schmidt algorithm yields a much more explicit formula than usual, in the present context. The second method is applicable only when the Laplacian is unbounded. However, in this case the construction does not require any enumeration (or onb) and may provide for more feasible computations.

**Remark 3.1.** Note that \( S_0 \) and \( S'_0 \) consist of \( \mathbb{R} \)-valued functions in this section. This technical detail is important because we do not expect the integral \( \int_{\mathbb{R}} e^{\langle u, \cdot \rangle} dF \) from (2.19) to converge unless it is certain that \( \langle u, \cdot \rangle \) is \( \mathbb{R} \)-valued. After the Wiener embedding is carried out in Theorem 5.2, all results can be complexified.

### 3.1. Gel’fand triples via Gram-Schmidt

In this section, we describe a Gel’fand triple for \( \mathcal{H}_G \) where the class of test functions \( S_G \) is described in terms of the decay properties of a certain orthonormal basis (onb) for \( \mathcal{H}_G \). We will see in Remark 5.9 that this onb corresponds to a system of i.i.d. random variables (which are, in fact, Gaussian with mean 0 and variance 1).

The onb \( \{e_n\}_{n \in \mathbb{N}} \) comes by applying the Gram-Schmidt process to the reproducing kernel \( \{v_{x_n}\}_{n \in \mathbb{N}} \), where we have fixed some enumeration \( \{x_{n}\}_{n \in \mathbb{N}} \) of the vertices \( G \setminus \{o\} \). That is, we put \( x_0 = o \) and henceforth exclude \( x_0 \) from the discussion, as it will not be relevant. Given \( \{e_1, \ldots, e_{n-1}\} \), one obtains \( e_n \) via

\[
\begin{pmatrix}
|v_{x_1}|^{-1} & 0 & 0 & \ldots & 0 \\
M_{2,1} & M_{2,2} & 0 & \ldots & 0 \\
M_{3,1} & M_{3,2} & M_{3,3} & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M_{n,1} & M_{n,2} & M_{n,3} & \ldots & M_{n,n}
\end{pmatrix}
\begin{pmatrix}
e_1 \\
e_2 \\
e_3 \\
\vdots \\
e_n
\end{pmatrix}
= 
\begin{pmatrix}
v_{x_1} \\
v_{x_2} \\
v_{x_3} \\
\vdots \\
v_{x_n}
\end{pmatrix}.
\]

Consequently, for each \( N \in \mathbb{N} \), the triangular nature of \( M \) gives

\[
\text{span}\{v_{x_1}, \ldots, v_{x_N}\} = \text{span}\{e_1, \ldots, e_N\}.
\]

**Remark 3.2.** Note that the reproducing kernel gives one an explicit formula for the entries of the inverse of this particular Gram-Schmidt matrix:

\[
(M^{-1})_{i,j} = \langle v_{x_i}, e_j \rangle_{\mathcal{E}} = e_j(x_i) - e_j(o).
\]

This is certainly in distinct contrast with the general case, and allows us to find a formula for the entries of \( M \) itself in Lemma 3.3.

**Lemma 3.3.** The entries of the Gram-Schmidt matrix \( M \) are given by

\[
M_{i,j} = \begin{cases} 
(\Delta e_i)(x_j), & j \leq i \\
0, & \text{else, for } i, j = 1, 2, \ldots
\end{cases} \tag{3.4}
\]

**Proof.** For \( j \leq i \), an application of (2.11) gives

\[
\Delta e_i(x_j) = \langle \delta_{x_j}, e_i \rangle_{\mathcal{E}} = \left\langle \delta_{x_j}, \sum_{k \leq i} M_{i,k} v_{x_k} \right\rangle_{\mathcal{E}} = \sum_{k \leq i} M_{i,k} \langle \delta_{x_j}, v_{x_k} \rangle_{\mathcal{E}} = \sum_{k \leq i} M_{i,k} (\delta_{x_j}(x_k) - \delta_{x_j}(o)),
\]

where the last equality comes by (2.6). Note that \( \delta_{x}(y) = \delta_{x}(o) \) for every \( y \) except \( y = x \), so the last sum above has a nonzero term only for \( k = j \), and the result follows. \( \square \)
From (3.3) and Lemma 3.3, we have the handy conversion formulas:

\[ e_i = \sum_{j \leq i} \Delta e_i(x_j)v_{x_j} \quad \text{and} \quad v_{x_i} = \sum_{j \leq i} (e_k(x_i) - e_k(o)) e_j. \]  

(3.5)

**Lemma 3.4.** We have the identity

\[ \sum_{j \leq k} (e_k(x_i) - e_k(o)) \Delta e_k(x_j) = \delta_{i,j}, \quad \text{for } i, j = 1, 2, \ldots. \]  

(3.6)

**Proof.** By formula (3.3), the left side of (3.6) is equal to

\[ \sum_{j \leq k} (e_k(x_i) - e_k(o)) \Delta e_k(x_j) = \Delta \left( \sum_{k \leq j} \langle v_{x_i}, e_j \rangle e_k(x_j) \right) = \Delta v_{x_i}(x_j) = \delta_{x_i}(x_j) - \delta_{o}(x_j). \]

Note that \( \Delta e_k(x_j) = 0 \) for \( j > k \), so the second sum runs over all \( k \leq i \). Also, note that \( \delta_{x_i}(x_j) - \delta_{o}(x_j) = \delta_{i,j} \) for \( i, j > 0 \) (and the indexing of \( M \) begins at 1, not 0). \( \square \)

Lemma 3.4 can also be proven by combining the identities in (3.5).

**Lemma 3.5.** Let \( V_{x,y} := \langle v_x, v_y \rangle_{\mathcal{E}} \) and let \( E = M^{-1} \) be defined as in (3.3). Then \( EE^* = V \).

**Proof.** Computing entrywise,

\[ (EE^*)_{i,j} = \sum_k E_{x_i,x_k} E_{y_j,y_k} = \sum_k (e_k(x_i) - e_k(o))(e_k(x_j) - e_k(o)) = \sum_k \langle v_{x_i}, e_k \rangle \langle v_{x_j}, e_k \rangle, \]

which is equal to \( \langle v_{x_i}, v_{x_j} \rangle_{\mathcal{E}} \) by Parseval’s identity. \( \square \)

**Definition 3.6.** The space of test functions and the space of distributions corresponding to the onb \( \{e_n\}_{n \in \mathbb{N}} \) are defined by

\[ S_G = \{ s = \sum_{n \in \mathbb{N}} s_n e_n : \forall p \in \mathbb{N}, \exists C > 0 \text{ such that } |s_n| \leq C/n^p \}, \quad \text{and} \]

\[ S'_G = \{ \xi = \sum_{n \in \mathbb{N}} \xi_n e_n : \exists p \in \mathbb{N}, \exists C > 0 \text{ such that } |\xi_n| \leq Cn^p \}. \]

(3.7)

(3.8)

Thus, \( S_G = \bigcap_{p \in \mathbb{N}} \{ s : ||s||_p < \infty \} \) where the Fréchet \( p \)-seminorm of \( s = \sum_{n \in \mathbb{N}} s_n e_n \) is

\[ ||s||_p := \left( \sum_{n \in \mathbb{N}} n^p |s_n|^2 \right)^{1/2}, \quad s \in S_G, \ p \in \mathbb{N}. \]

(3.9)

Note that the system of seminorms (3.9) is equivalent to system of seminorms defined by

\[ ||s||_p := \sup_{n \in \mathbb{N}} n^p |s_n|, \quad s \in S_G, \ p \in \mathbb{N}, \]

(3.10)

in the sense that both define the same Fréchet topology on \( S_G \). (Each seminorm in one system is dominated by one from the other, but with a different \( p \).) We occasionally find it more convenient to calculate with (3.10) instead of (3.9).
Definition 3.7. Let $\mathcal{V} := \text{span}\{v_{\alpha}\}_{\alpha \in G}$ and define a mapping $\mathcal{N} : \mathcal{V} \to \mathcal{H}_E$ by

$$\mathcal{N}v_{x_{\alpha}} = \sum_{k=1}^{n} k\epsilon_k(x_{\alpha})e_k.$$(3.11)

Remark 3.8. From (3.11), one has

$$||\mathcal{N}v_{x_{\alpha}}||^2_E = \sum_{k=1}^{n} k^2|\epsilon_k(x_{\alpha})|^2 \quad \text{and} \quad \langle v_{x_{\alpha}}, \mathcal{N}v_{x_{\alpha}} \rangle_E = \sum_{k=1}^{n} k\epsilon_k(x_{\alpha})\epsilon_k(x_{\alpha}).$$ (3.12)

Note that $\epsilon_k \in \mathcal{V}$ by (3.2), and that $\mathcal{N}\epsilon_k = k\epsilon_k$ for each $k \in \mathbb{N}$. We use the symbol $\mathcal{N}$ for the operator discussed in this section by way of analogy with the number operator $\mathcal{N}$ from quantum mechanics. Indeed, $\mathcal{N}$ can also be defined as $\mathcal{N}^\dagger a$ for a suitable operator $\mathcal{N}$ and its adjoint.

In the following lemma, we use the symbol $\tilde{\mathcal{N}}$ to denote the closure of the operator $\mathcal{N}$ (i.e., the domain is the closure of span$\{\epsilon_n\}$ with respect to the graph norm).

Lemma 3.9. The mapping $\mathcal{N}$ is essentially self-adjoint, and is unbounded if and only if $G$ is infinite. Moreover, if we define the seminorms $\rho_{\alpha}(u) := ||(\tilde{\mathcal{N}})^\alpha u||_E$, then $\{\rho_{\alpha}\}$ and $\{|| \cdot ||_p\}$ induce equivalent topologies on $\mathcal{S}_G$, so that

$$\mathcal{S}_G = \bigcap_{n \in \mathbb{N}} \text{dom}(\tilde{\mathcal{N}})^n$$ (3.13)

and $u \in \mathcal{S}_G$ if and only if $\rho_{\alpha}(u) < \infty$ for each $n \in \mathbb{N}$.

Proof. Unitary equivalence of $\mathcal{H}_E$ with $\ell^2(\mathbb{Z}_+)$ is given by $U : \epsilon_n \mapsto \delta_n$, where $\delta_n(m) := \delta_{n,m}$ (Kronecker $\delta$) for $n, m \in \mathbb{Z}_+$. Define $N_{\alpha} : \ell^2(\mathbb{Z}_+) \to \ell^2(\mathbb{Z}_+)$ by $N_{\alpha}\delta_n = n\delta_n$ so that $N_{\alpha}U = U\mathcal{N}$ holds on the dense subspace span$\{\epsilon_n\}$. The rest follows by [Sim79, §2]. □

Corollary 3.10. The inclusion mapping $\mathcal{S}_G \hookrightarrow \mathcal{H}_E$ is nuclear, and so $\mathcal{S}_G \subseteq \mathcal{H}_E \subseteq \mathcal{S}_G'$ is a Gel'fand triple.

Proof. When the space of test functions is defined as dom$T^{\infty}$ for some operator $T$ with pure point spectrum (as in (3.13)), then nuclearity follows if there is a $p \in \mathbb{Z}_+$ such that the reciprocal eigenvalues of $T$ are $p$-summable; see [Sim79]. Since $\mathcal{N}$ has spectrum $\mathbb{Z}_+$ and $\sum_{n=1}^{\infty} n^{-p} \in (\mathbb{R}^+)^*$ for $p \geq 2$, the conclusion follows. □

Lemma 3.11. The energy kernel $\{v_{x\alpha}\}_{\alpha \in G}$ is a Fréchet-dense subset of $\mathcal{S}_G$.

Proof. In the expansion with respect to the onb as in (3.7), the basis element $\epsilon_k$ has coefficients $s_n = \delta_{nk}$. Since this sequence $\{s_n\}$ vanishes after $n = k$, it clearly satisfies the required decay condition $|s_n| \leq Cn^{-p}$. From (3.2), the same clearly holds for $v_{x_{\alpha}}$. This shows that the kernel is contained in $\mathcal{S}_G$.

To see that $\{v_{x_{\alpha}}\}$ is dense in $\mathcal{S}_G$, it suffices by (3.2) to show that the onb $\{\epsilon_n\}_{n \in \mathbb{N}}$ is dense. Given any $u = \sum u_{k}\epsilon_{k} \in \mathcal{S}_G$ and any $p \in \mathbb{N}$, there is a $C = C_{p}$ such that $|u_{k}| \leq C/k^{p+1}$. Now if $u_{N} = \sum_{k=1}^{N} u_{k}\epsilon_{k}$ is the $N^{\text{th}}$ truncation of $u$, then

$$\|u - f\|_{p} = \sup_{k} k^{p}|u_{k} - f_{k}| = \sup_{k > N} k^{p}|u_{k} - f_{k}| \leq k^{p} \frac{C}{k^{p+1}} \rightarrow_{k \rightarrow \infty} 0.$$

Thus, one can always approximate $u \in \mathcal{S}_G$ by $u_{N} \in \text{span}\{\epsilon_{1}, \ldots, \epsilon_{N}\}$. □
3.2. Gel’fand triples in the case when $\Delta$ is unbounded. In the case when $\Delta : \mathcal{H}_E \rightarrow \mathcal{H}_E$ is unbounded, there is an alternative construction of $S_G$ and $S'_G$, which begins by identifying a certain subspace of $M = \text{dom} \Delta_M$ (as given in Definition 2.15) to act as the space of test functions.

**Definition 3.12.** Let $\Delta_V$ be a self-adjoint extension of $\Delta_M$: since $\Delta_M$ is Hermitian and commutes with conjugation (since $c$ is $\mathbb{R}$-valued), a theorem of von Neumann’s states that such an extension exists.

Let $\Delta^p_V u := (\Delta_V \Delta_V \ldots \Delta_V) u$ be the $p$-fold product of $\Delta_V$ applied to $u \in \mathcal{H}_E$. Define $\text{dom}(\Delta^p_V)$ inductively by

$$\text{dom}(\Delta^p_V) := \{ u : \Delta^{p-1}_V u \in \text{dom}(\Delta_V) \}.$$  

**Definition 3.13.** The (Schwartz) space of potentials of rapid decay is

$$S_G := \text{dom}(\Delta^\infty_V),$$

where $\text{dom}(\Delta^p_V) := \bigcap_{p=1}^\infty \text{dom}(\Delta^p_V)$ consists of all $\mathbb{R}$-valued functions $u \in \mathcal{H}_E$ for which $\Delta^p_V u \in \mathcal{H}_E$ for any $p$. The space of Schwartz distributions or tempered distributions is the dual space $S'_G$ of $\mathbb{R}$-valued continuous linear functionals on $S_G$.

**Remark 3.14.** Note that $S_G$ is dense in $\text{dom}(\Delta_V)$ with respect to the graph norm, by standard spectral theory. For each $p \in \mathbb{N}$, there is a seminorm on $S_G$ defined by

$$\|u\|_p := \|\Delta^p_V u\|_{\mathcal{H}_E}.$$  

Since $(\text{dom} \Delta^p_V, \|\cdot\|_p)$ is a Hilbert space for each $p \in \mathbb{N}$, the subspace $S_G$ is a Fréchet space. Note also that since $\Delta$ is unbounded, $S_G$ is a proper subspace of $\mathcal{H}_E$.

**Lemma 3.15.** If $\deg(x)$ is finite for each $x \in G^0$, then one has $v_x \in S_G$.

**Proof.** If $\deg(x) < \infty$ then Lemma 3.16 shows that $\delta_x \in \text{span}(v_x).$ \hfill $\square$

Note that we take $\deg(x) < \infty$ as a blanket assumption, in fact, as part of the definition of a network. However, it is emphasized in Lemma 3.15 because this is the only place where it is really necessary. (But note that $\deg(x)$ may be unbounded.)

**Lemma 3.16.** For any $x \in G^0$, $\delta_x = c(x)v_x - \sum_{y \sim x} c_{xy}v_y$.

**Proof.** Lemma 2.11 implies $\langle \delta_x, u \rangle_E = \langle c(x)v_x - \sum_{y \sim x} c_{xy}v_y, u \rangle_{\mathcal{H}_E}$ for every $u \in \mathcal{H}_E$, so apply this to $u = v_z$, $z \in G^0$. Since $\delta_x, v_x \in \mathcal{H}_E$, it must also be that $\sum_{y \sim x} c_{xy}v_y \in \mathcal{H}_E$. \hfill $\square$

**Remark 3.17.** When the hypotheses of Lemma 3.15 are satisfied, it should be noted that $\text{span}(v_x)$ is dense in $S_G$ with respect to $\mathcal{E}_c$, but not with respect to the Fréchet topology induced by the seminorms (3.16), nor with respect to the graph norm. One has the inclusions

$$\left\{ \left[ \begin{array}{c} v_x \\ \Delta_M v_x \end{array} \right] \right\} \subseteq \left\{ \left[ \begin{array}{c} s \\ \Delta_V s \end{array} \right] \right\} \subseteq \left\{ \left[ \begin{array}{c} u \\ \Delta_V u \end{array} \right] \right\}$$

where $s \in S_G$ and $u \in \mathcal{H}_E$. The second inclusion is dense but the first is not.
4. The structure of $S_G$ and $S'_G$

From this point on, we assume that a Gel’fand triple has been chosen, using either of the methods described in the previous section. Henceforth, we use the symbol $\Lambda$ to denote the operator $\hat{N} = N^*$ or the operator $\Delta_V$, depending on how the Gel’fand triple was constructed:

$$\Lambda := \begin{cases} \hat{N}, & \text{Definition 3.7} \\ \Delta_V, & \text{Definition 3.12}. \end{cases}$$

4.1. The structure of $S_G$. We establish that $S_G$ is a dense analytic subset of $H_E$, and that the energy product can be extended not just to a pairing on $S_G \times S'_G$, but all the way to a pairing on $H_E \times S'_G$. Parts of this subsection closely parallel the general theory, and a good reference would be [Sim79] or [Hiid80].

**Definition 4.1.** Let $\chi_{[a,b]}$ denote the usual indicator function of the interval $[a, b] \subseteq \mathbb{R}$, and let $\mathcal{F}$ be the spectral transform in the spectral representation of $\Lambda$, and let $E$ be the associated projection-valued measure. Then define $E_n$ to be the *spectral truncation operator* acting on $H_E$ by

$$E_n u := \mathcal{F}^{-1} \chi_{[\frac{1}{n},n]} \mathcal{F} u = \int_{1/n}^n E(dt) u.$$

**Lemma 4.2.** With respect to $E$, $S_G$ is a dense analytic subspace of $H_E$.

**Proof.** This essentially follows immediately once it is clear that $E_n$ maps $H_E$ into $S_G$. For $u \in H_E$, and for any $p = 1, 2, \ldots,$

$$\|\Lambda^n E_n u\|^2_{E} = \int_{1/n}^{n} \lambda^{2p} \|E(d\lambda) u\|^2_{E} \leq n^{2p} \|u\|^2_{E}, \quad (4.2)$$

So $E_n u \in S_G$. It follows that $\|u - E_n u\|_{E} \to 0$ by standard spectral theory. \qed

**Theorem 4.3.** $S_G \subseteq H_E \subseteq S'_G$ is a Gel’fand triple, and the energy form $\langle \cdot, \cdot \rangle_E$ extends to a pairing on $S_G \times S'_G$ defined by

$$\langle u, v \rangle := \langle \Lambda^n u, \Lambda^{-n} v \rangle_E, \quad (4.3)$$

where $p$ is any integer such that $|v(u)| \leq K \|\Lambda^n u\|_{E}$ for all $u \in S_G$, for some $K > 0$.

**Proof.** In combination with (3.15)–(3.16), Lemma 4.2 establishes that $S_G \subseteq H_E \subseteq S'_G$ is a Gel’fand triple. If $v \in S'_G$, then there is a $C$ and $p$ such that $|\langle s, v \rangle_{E}| \leq C \|\Lambda^n s\|_{E}$ for all $s \in S_G$. Set $\varphi(\Lambda^n s) := \langle s, v \rangle_{E}$ to obtain a continuous linear functional on $H_E$ (after extending to the orthogonal complement of span$(\Lambda^n s)$ by 0 if necessary). Now Riesz’s lemma gives a $w \in H_E$ for which $\langle s, v \rangle_{E} = \langle \Lambda^n s, w \rangle_{E}$ for all $s \in S_G$, and we define $\Lambda^{-n} v := w \in H_E$ to make the meaning of the right-hand side of (4.3) clear. \qed

**Lemma 4.4.** The pairing on $S_G \times S'_G$ is equivalently given by

$$\langle u, \xi \rangle := \lim_{n \to \infty} \langle E_n u, \xi \rangle, \quad (4.4)$$

where the limit is taken in the topology of $S'_G$. Moreover, $\bar{u}(\xi) = \langle u, \xi \rangle_{E}$ is $\mathbb{R}$-valued on $S'_G$. 

Proof. $E_n$ commutes with $\Lambda$. This is a standard result in spectral theory, as $E_n$ and $\Lambda$ are unitarily equivalent to the two commuting operations of truncation and multiplication, respectively. Therefore,

$$\xi(E_n u) = \langle E_n u, \xi \rangle_w = \langle \Lambda^p E_n s, \Lambda^{-p} \xi \rangle_E = \langle E_n \Lambda^p s, \Lambda^{-p} \xi \rangle_E = \langle \Lambda^p s, \Lambda^{-p} \xi \rangle_E.$$ 

Standard spectral theory also gives $E_n v \to v$ in $\mathcal{H}_E$, so

$$\lim_{n \to \infty} \xi(E_n u) = \lim_{n \to \infty} \langle \Lambda^p s, E_n \Lambda^{-p} \xi \rangle_E = \langle \Lambda^p u, \Lambda^{-p} v \rangle_E.$$ 

Note that the pairing $\langle \cdot, \cdot \rangle_w$ is a limit of real numbers, and hence is real. □

Corollary 4.5. $E_n$ extends to a mapping $E_n : S'_G \to \mathcal{H}_E$ defined via $\langle u, E_n \xi \rangle_E := \xi(E_n u)$.

Thus, we have a pointwise extension of $\langle \cdot, \cdot \rangle_w$ to $\mathcal{H}_E \times S'_G$ given by

$$\langle u, \xi \rangle_w = \lim_{n \to \infty} \langle u, E_n \xi \rangle_E.$$ (4.5)

4.2. The structure of $S'_G$. The next results are structure theorems akin to those found in the classical theory of distributions; see [Str03, §6.3] or [AG92, §3.5]. If $\mathcal{H}_E \subseteq S'_G$, then Theorem 4.6 would say $S'_G = \bigcup_p \Lambda^p(\mathcal{H}_E)$ (of course, this is typically false when $\mathcal{H}_{\text{arm}} \neq 0$).

Theorem 4.6. The distribution space $S'_G$ is

$$S'_G = \{ \xi(u) = \langle \Lambda^p u, v \rangle_E : \exists v \in \mathcal{H}_E, p \in \mathbb{Z}^+, \forall u \in S_E \}. \quad (4.6)$$

Proof. It is clear from the Schwarz inequality that $\xi(u) = \langle \Lambda^p u, v \rangle_E$ defines a continuous linear functional on $S_E$, for any $v \in \mathcal{H}_E$ and nonnegative integer $p$. For the other direction, we use the same technique as in Lemma 4.3. Observe that if $\xi \in S'_G$, then there exists $K, p$ such that $|\xi(u)| \leq K\|\Lambda^p u\|_E$ for every $u \in S_E$. This implies that the map $\xi : \Lambda^p u \mapsto \xi(u)$ is continuous on the subspace $Y = \text{span}\{\Lambda^p u : u \in \mathcal{H}_E, p \in \mathbb{Z}^+\}$. This can be extended to all of $\mathcal{H}_E$ by precomposing with the orthogonal projection to $Y$. Now Riesz’s lemma gives a $v \in \mathcal{H}_E$ for which $\xi(u) = \langle \Lambda^p u, v \rangle_E$.

Note that $v \not\in \mathcal{H}_E$ may not lie in the domain of $\Lambda^p$. If it did, one would have $\langle \Lambda^p u, v \rangle_E = \langle u, \Lambda^p v \rangle_w = \langle u, \Lambda^p f \rangle_w$, where $f = P_{\mathcal{H}_E}v$. The theorem could then be written $S'_G = \bigcup_{p=0}^\infty \Lambda^p(\mathcal{H}_E)$. However, this turns out to have contradictory implications.

We now provide two results enabling one to recognize certain elements of $S'_G$.

Lemma 4.7. A linear functional $f : S_E \to \mathbb{C}$ is an element of $S'_G$ if and only if there exists $p \in \mathbb{N}$ and $F_0, F_1, \ldots, F_p \in \mathcal{H}_E$ such that

$$f(u) = \sum_{k=0}^p \langle F_k, \Lambda^k u \rangle_E, \quad \forall u \in \mathcal{H}_E.$$ (4.7)

Proof. By definition, $f \in S'_G$ iff $\exists p, C < \infty$ for which $|f(u)| \leq C\|u\|_p$ for every $u \in S_E$. Therefore, the linear functional

$$\Phi : \bigoplus_{k=0}^p \text{dom}(\Lambda^k) \to \mathbb{C} \quad \text{by} \quad \Phi(u, \Lambda u, \Lambda^2 u, \ldots, \Lambda^p u) = f(u)$$

is continuous and Riesz’s Lemma gives $F = (F_k)_{k=0}^p \in \bigoplus_{k=0}^p \mathcal{H}_E$, with
Corollary 4.8. If $\Lambda : \mathcal{H}_E \to \mathcal{H}_E$ is bounded, then $S_G' = \mathcal{H}_E$.

Proof. We always have the inclusion $\mathcal{H}_E \hookrightarrow S_G'$ by taking $p = 0$. If $\Lambda$ is bounded, then the adjoint $\Lambda^*$ is also bounded, and (4.7) gives

$$f(u) = \left( \sum_{k=0}^{p} (\Lambda^*)^k F_k, u \right)_{\mathcal{H}_E}, \quad \forall u \in S_G'. \quad (4.8)$$

Since $S_G$ is dense in $\mathcal{H}_E$ by Lemma 4.2, we have $f = \sum_{k=0}^{p} (\Lambda^*)^k F_k \in \mathcal{H}_E$. \hfill $\square$

Remark 4.9. In view of Lemma 3.9, Corollary 4.8 shows that $S_G'$ is a proper extension of $\mathcal{H}_E$ on any infinite network.

In the case when the Gel’fand triple is constructed from the domain of $\Delta_M$, as in Definition 3.12, then one can extend $\Delta$ to distributions.

Definition 4.10. Extend $\Delta$ to $S_G'$ by defining

$$\Delta \xi(v_x) := \langle \delta_x, \xi \rangle_{\mathcal{W}}, \quad (4.9)$$

so that $\Delta \xi(v_x) = \sum_{y \neq x} c_{xy}(\xi(v_x) - \xi(v_y))$ follows readily from Lemma 3.16.

Now extend $\Delta$ to $\tilde{\Delta}$ defined on $\tilde{v}_x \in L^2(\frac{S_G'}{\mathcal{T}_M}, \mathbb{P}^Q)$ by $\tilde{\Delta}(\tilde{v}_x)(\xi) := \tilde{\Delta}v_x(\xi)$, so that

$$\tilde{\Delta} : \tilde{v}_x \mapsto c(x)\tilde{v}_x - \sum_{y \neq x} c_{xy}\tilde{v}_y. \quad (4.10)$$

Since $v_x \mapsto \tilde{v}_x$ is an isometry, it is no great surprise that

$$\langle \tilde{v}_x, \tilde{\Delta}v_y \rangle_{L^2} = \int_{S_G'} \tilde{v}_x(\xi)\tilde{v}_y(\Delta \xi) d\mathbb{P}(\xi) = \langle v_x, \Delta v_y \rangle_E. \quad (4.11)$$

5. The Wiener embedding and the space $S_G'$

We have now obtained a Gel’fand triple $S_G \subseteq \mathcal{H}_E \subseteq S_G'$ (from either Lemma 3.10 or Theorem 4.3), and we are ready to apply the Minlos Theorem to a particularly lovely positive definite function on $S_G$, in order that we may obtain a particularly nice measure on $S_G'$. This allows us to realize $\text{bd} \ G$ as a subset of $S_G'$. Recall that $S_G$ contains the energy kernel; see Lemma 3.11 or Lemma 3.15.

5.1. The Wiener embedding. In [JP09c, §5], we constructed $\mathcal{H}_E$ from the resistance metric by making use of negative definite functions. We now apply this to a famous result of Schoenberg which may be found in [BCR84, SW49].

Theorem 5.1 (Schoenberg). Let $X$ be a set and let $Q : X \times X \to \mathbb{R}$ be a function. Then the following are equivalent.

1. $Q$ is negative definite.
2. $\forall t \in \mathbb{R}^+$, the function $p_t(x,y) := e^{-Q(x,y)}$ is positive definite on $X \times X$.
3. There exists a Hilbert space $\mathcal{H}$ and a function $f : X \to \mathcal{H}$ such that $Q(x,y) = \|f(x) - f(y)\|^2_{\mathcal{H}}$. 

\[f(u) = \langle F, (u, \Lambda u, \ldots, \Lambda^p u) \rangle_{\mathcal{H}_E} = \sum_{k=0}^{p} (\Lambda^*)^k F_k, u \rangle_{\mathcal{H}_E}. \]
In the proof of the following theorem, we apply Schoenberg’s Theorem with $t = \frac{1}{2}$ to the resistance metric in the form

$$R^E(x, y) = \|v_x - v_y\|_E^2,$$  \hfill (5.1)

which appears in [JP09c, Thm. 2.13]. The proof of Theorem 5.2 also uses the notation $\mathbb{E}_\xi(f) := \int_{S'_G} f(\xi) d\mathbb{P}(\xi)$.

**Theorem 5.2** (Wiener embedding). The Wiener transform $\mathcal{W} : \mathcal{H}_E \to L^2(S'_G, \mathbb{P})$ by

$$\mathcal{W} : v \mapsto \tilde{v}, \quad \tilde{v}(\xi) := \langle v, \xi \rangle_E,$$  \hfill (5.2)

is an isometry. The extended reproducing kernel $\{\tilde{v}_\lambda\}_{\lambda \in G}$ is a system of Gaussian random variables which gives the resistance distance by

$$R^E(x, y) = \mathbb{E}_\xi((\tilde{v}_x - \tilde{v}_y)^2).$$  \hfill (5.3)

Moreover, for any $u, v \in \mathcal{H}_E$, the energy inner product extends directly as

$$\langle u, v \rangle_E = \mathbb{E}_\xi(\overline{v} u) = \int_{S'_G} \overline{v} u \, d\mathbb{P}.$$  \hfill (5.4)

**Proof.** Since $R^E(x, y)$ is negative semidefinite (see [JP09c, Thm. 5.4]), we may apply Schoenberg’s theorem and deduce that $\exp(-\frac{1}{2} |u - v|_E^2)$ is a positive definite function on $\mathcal{H}_E \times \mathcal{H}_E$. Consequently, an application of the Minlos correspondence to the Gel’fand triple established in Lemma 4.2 yields a Gaussian probability measure $\mathbb{P}$ on $S'_G$.

Moreover, (2.19) gives

$$\mathbb{E}_\xi(e^{i \langle u, \xi \rangle_E}) = e^{-\frac{1}{2} |u|_E^2},$$  \hfill (5.5)

whence one computes

$$\int_{S'_G} \left(1 + \frac{i}{2} \langle u, \xi \rangle_E - \frac{1}{2} \langle u, \xi \rangle_E^2 + \cdots \right) d\mathbb{P}(\xi) = 1 - \frac{1}{2} \langle u, u \rangle_E + \cdots.$$  \hfill (5.6)

Now it follows that $\mathbb{E}(\tilde{v}_u^2) = \mathbb{E}_\xi((u, \xi)_E^2) = |u|_E^2$ for every $u \in S'_G$, by comparing the terms of (5.6) which are quadratic in $u$. Therefore, $\mathcal{W} : \mathcal{H}_E \to S'_G$ is an isometry, and (5.6) gives

$$\mathbb{E}_\xi(|\tilde{v}_x - \tilde{v}_y|^2) = \mathbb{E}_\xi(|v_x - v_y|^2) = \|v_x - v_y\|_E^2,$$  \hfill (5.7)

whence (5.3) follows from (5.1). Note that by comparing the linear terms, (5.6) implies $\mathbb{E}_\xi(1) = 1$, so that $\mathbb{P}$ is a probability measure, and $\mathbb{E}_\xi((u, \xi)) = 0$ and $\mathbb{E}_\xi((u, \xi)^2) = |u|_E^2$, so that $\mathbb{P}$ is actually Gaussian.

Finally, use polarization to compute

$$\langle u, v \rangle_E = \frac{1}{4} (|u + v|_E^2 - |u - v|_E^2)$$

$$= \frac{1}{4} \left(\mathbb{E}_\xi((u + v)^2) - \mathbb{E}_\xi((u - v)^2)\right)$$

by (5.7)

$$= \frac{1}{4} \int_{S'_G} |\tilde{u} + \tilde{v}|^2(\xi) - |\tilde{u} - \tilde{v}|^2(\xi) \, d\mathbb{P}(\xi).$$
\[ = \int_{S_G} \overline{u}(\xi) \overline{v}(\xi) \, d\mathbb{P}(\xi). \]

This establishes (5.4) and completes the proof. \[\Box\]

It is important to note that since the Wiener transform \( W : S_G \to S_G' \) is an isometry, the conclusion of Minlos’ theorem is stronger than usual: the isometry allows the energy inner product to be extended isometrically to a pairing on \( \mathcal{H}_E \times S_G' \) instead of just \( S_G \times S_G' \).

**Remark 5.3.** With the embedding \( \mathcal{H}_E \to L^2(S_G', \mathbb{P}) \), we obtain a maximal abelian algebra of Hermitian multiplication operators \( L^\infty(S_G') \) acting on \( L^2(S_G', \mathbb{P}) \). For a sharp contrast, note that the multiplication operators on \( \mathcal{H}_E \) are trivial, by \([JP09b, \text{Lem. 2.3}]\). This result states that if \( \varphi : G^0 \to \mathbb{R} \) and \( M_\varphi \) denotes the multiplication operator defined by \((M_\varphi u)(x) = \varphi(x)u(x)\), then \( M_\varphi \) is Hermitian if and only if \( M_\varphi = kI_k \) for some \( k \in \mathbb{R} \).

**Remark 5.4.** The reader will note that we have taken pains to keep everything \( \mathbb{R} \)-valued in this section (especially the elements of \( S_G \) and \( S_G' \)), primarily to ensure the convergence of

\[ \int_{S_G} e^{\langle u, \xi \rangle_W} \, d\mathbb{P}(\xi) \text{ in } (5.5). \]

However, now that we have established the fundamental identity

\[ \langle u, v \rangle_E = \int_{S_G} \overline{u} \overline{v} \, d\mathbb{P} \text{ in } (5.4) \]

and extended the pairing \( \langle \cdot, \cdot \rangle_W \) to \( \mathcal{H}_E \times S_G' \), we are at liberty to complexify our results via the standard decomposition into real and complex parts: \( u = u_1 + iu_2 \) with \( u_i \) \( \mathbb{R} \)-valued elements of \( \mathcal{H}_E \), etc.

**Remark 5.5.** The polynomials are dense in \( L^2(S_G', \mathbb{P}) \). More precisely, if \( \varphi(t_1, t_2, \ldots, t_k) \) is an ordinary polynomial in \( k \) variables, then

\[ \varphi(\xi) := \varphi(\langle u_1, \xi \rangle_W, \langle u_2, \xi \rangle_W, \ldots, \langle u_k, \xi \rangle_W) \quad (5.8) \]

is a polynomial on \( S_G' \) and

\[ \mathcal{P}_{\text{Poly}} := \{ \varphi(\overline{u_1}(\xi), \overline{u_2}(\xi), \ldots, \overline{u_k}(\xi)) : \deg(\varphi) \leq n, \, u_j \in \mathcal{H}_E, \, \xi \in S_G' \} \quad (5.9) \]

is the collection of polynomials of degree at most \( n \), and \( \{ \mathcal{P}_{\text{Poly}} \}_{n=0}^{\infty} \) is an increasing family whose union is all of \( S_G' \). One can see that the monomials \( \langle u, \xi \rangle_W \) are in \( L^2(S_G', \mathbb{P}) \) as follows: compare like powers of \( u \) from either side of (5.6) to see that

\[ \mathbb{E}_\xi \left( |\langle u, \xi \rangle_W^2 | \right) = 0 \]

and

\[ \mathbb{E}_\xi \left( |\langle u, \xi \rangle_W^2 | \right) = \int_{S_G} |\langle u, \xi \rangle_W^2 | \, d\mathbb{P}(\xi) = \frac{(2n)!}{2^n n!} ||u||_{E}^{2n}, \quad (5.10) \]

and then apply the Schwarz inequality.

To see why the polynomials \( \{ \mathcal{P}_{\text{Poly}} \}_{n=0}^{\infty} \) should be dense in \( L^2(S_G', \mathbb{P}) \) observe that the sequence \( \{ P_{\mathcal{P}_{\text{Poly}}} \}_{n=0}^{\infty} \) of orthogonal projections increases to the identity, and therefore, \( \{ P_{\mathcal{P}_{\text{Poly}}} \} \) forms a martingale, for any \( u \in \mathcal{H}_E \) (i.e., for any \( \tilde{u} \in L^2(S_G', \mathbb{P}) \)).

Denote the “multiple Wiener integral of degree \( n \)” by

\[ H_n := \left\{ \text{cl span}\{ \langle u, \gamma \rangle_W : u \in \mathcal{H}_G \} \right\} \bigoplus \{ \langle u, \gamma \rangle_W : k < n, u \in \mathcal{H}_G \}, \]

for each \( n \geq 1 \), and \( H_0 := \mathbb{C}1 \) for a vector \( 1 \) with \( ||1||_2 = 1 \). Then we have an orthogonal decomposition of the Hilbert space

\[ L^2(S_G', \mathbb{P}) = \bigoplus_{n=0}^{\infty} H_n. \quad (5.11) \]
See [Hid80, Thm. 4.1] for a more extensive discussion.

A physicist would call (5.11) the Fock space representation of \( L^2(S'_G, \mathbb{P}) \) with “vacuum vector” \( 1 \); note that \( H_1 \) has a natural (symmetric) tensor product structure. Familiarity with these ideas is not necessary for the sequel, but the decomposition (5.11) is helpful for understanding two key things:

(i) The Wiener isometry \( W : \mathcal{H}_E \to L^2(S'_G, \mathbb{P}) \) identifies \( \mathcal{H}_E \) with the subspace \( H_1 \) of \( L^2(S'_G, \mathbb{P}) \), in particular, \( L^2(S'_G, \mathbb{P}) \) is not isomorphic to \( \mathcal{H}_E \). In fact, it is the second quantization of \( \mathcal{H}_E \).

(ii) The constant function \( 1 \) is an element of \( L^2(S'_G, \mathbb{P}) \) but does not correspond to any element of \( \mathcal{H}_E \). In particular, constant functions in \( \mathcal{H}_E \) are equivalent to 0, but this is not true in \( L^2(S'_G, \mathbb{P}) \).

It is somewhat ironic that we began this story by removing the constants (via the introduction of \( \mathcal{E} \)), only to reintroduce them with a certain amount of effort, much later. Item (ii) explains why it is not nonsense to write things like \( E(S'_G) = \int_{S'_G} 1 \ d\mathbb{P} = 1 \), and will be helpful when discussing boundary elements in §6.1.

**Corollary 5.6.** For \( e_s(\xi) := \delta^{(v_s,\xi)}W \), one has \( E(e_s) = e^{-\frac{1}{2}R'(v,s)} \) and hence

\[
E(e_s e_t) = \int_{S'_G} e_s(\xi)e_t(\xi) \ d\mathbb{P} = e^{-\frac{1}{2}R'(v,t)}.
\]  

**(5.12)**

**Proof.** Substitute \( u = v_s \) or \( u = v_s - v_y \) in (5.5) and apply (5.1). \( \square \)

**Remark 5.7.** Free resistance is interpreted as the reciprocal of an integral over a path space in [JP09c, Rem. 3.15]; Corollary 5.6 provides a variation on this theme:

\[
R'(x, y) = -2 \log E(e_x e_y) = -2 \log \int_{S'_G} e_x(\xi)e_y(\xi) \ d\mathbb{P}.
\]  

**(5.13)**

Observe that Theorem 5.2 was carried out for the free resistance, but all the arguments go through equally well for the wired resistance; note that \( \mathcal{W}' \) is similarly negative semi-definite by Theorem 5.1 and [JP09c, Cor. 5.5]. Thus, there is a corresponding Wiener transform \( \mathcal{W} : \mathcal{F} \to L^2(S'_G, \mathbb{P}) \) defined by

\[
\mathcal{W} : v \mapsto \tilde{f}, \quad f = P_{\mathcal{F}v} \quad \text{and} \quad \tilde{f}(\xi) = \langle f, \xi \rangle_w.
\]  

**(5.14)**

Again, \( \{f_x\}_{x \in \mathbb{G}} \) is a system of Gaussian random variables which gives the wired resistance distance by \( R^w(x, y) = E(\tilde{f}_{x} - \tilde{f}_{y}^2) \).

**Remark 5.8.** For \( u \in \mathcal{H}_r \) and \( \xi \in S'_G \), let us abuse notation and write \( u \) for \( \tilde{u} \). That is, \( u(\xi) = \tilde{u}(\xi)_w \). Unnecessary tildes obscure the presentation and the similarities to the Poisson kernel in §6.

**Remark 5.9.** Theorem 5.2 showed that \( \{e_s\} \) forms a system of Gaussian random variables. Since the Wiener transform is an isometry,

\[
E(e_s) = 0 \quad \text{and} \quad E(e_s e_t) = \delta_{s,t}.
\]  

**(5.15)**

Since independence of Gaussian random variables is determined by the first two moments, it follows that \( \{e_s\} \) forms a system of i.i.d. Gaussian random variables with mean 0 and variance 1. This is noteworthy because while independence implies orthogonality, the converse does not hold without the additional hypothesis that the distributions be Gaussian.
6. The resistance boundary of a transient network

With the tools developed in §3 and §5, we now construct the resistance boundary \( \text{bd} G \) as a set of equivalence classes of infinite paths. Recall that we began with a comparison of the Poisson boundary representation for bounded harmonic functions with the boundary sum representation recalled in (1.2):

\[
u(x) = \int_{\partial G} u(y) k(x, dy) \quad \leftrightarrow \quad u(x) = \sum_{\text{bd} G} u \frac{\partial u}{\partial n} + u(o).
\]

In this section, we replace the sum with an integral and complete the parallel.

**Corollary 6.1** (Boundary integral representation for harmonic functions).

For any \( u \in \mathcal{H} \text{arm} \) and with \( h_x = P_0 \text{arm}v_x \),

\[
u(x) = \int_{\partial G} u(\xi) h_x(\xi) d\mathbb{P}^O(\xi) + u(o).
\]

**Proof.** Starting with (2.6), compute

\[
u(x) - u(o) = \langle h_x, u \rangle_E = \overline{\langle u, h_x \rangle_E} = \int_{S'_{G}} \overline{u h_x} d\mathbb{P}^O,
\]

where the last equality comes by substituting \( \nu = h_x \) in (5.4). It is shown in [JP09b, Lem. 2.24] that \( \overline{h_x} = h_x \).

**Remark 6.2** (A Hilbert space interpretation of \( \text{bd} G \)). In view of Corollary 6.1, we are now able to “catch” the boundary between \( S_G \) and \( S'_G \) by using \( \Lambda \) and its adjoint. The boundary of \( G \) may be thought of as (a possibly proper subset of) \( S'_G \). Corollary 6.1 suggests that \( k(x, d\xi) := h_x(\xi) d\mathbb{P}^O \) is the discrete analogue in \( \mathcal{H}_E \) of the Poisson kernel \( k(x, dy) \), and comparison of (1.2) with (6.1) gives a way of understanding a boundary integral as a limit of Riemann sums:

\[
u(x) = \lim_{k \to \infty} \sum_{\text{bd} G} u(x) \frac{\partial u}{\partial n}(x).
\]

(We continue to omit the tildes as in Remark 5.8.) By a theorem of Nelson, \( \mathbb{P}^O \) is fully supported on those functions which are Hölder-continuous with exponent \( \alpha = \frac{1}{2} \), which we denote by \( \text{Lip}(\frac{1}{2}) \subseteq S'_G \); see [Nel64]. Recall from [JP09c, Cor. 2.16] that \( \mathcal{H}_E \subseteq \text{Lip}(\frac{1}{2}) \).

See [Arv76a, Arv76b, Min63, Nel69].

6.1. The boundary as equivalence classes of paths. We are finally able to give a concrete representation of elements of the boundary. We continue to use the measure \( \mathbb{P}^O \) from Theorem 5.2. Recall the Fock space representation of \( L^2(S'_G, \mathbb{P}) \) discussed in Remark 5.5:

\[
u \left( \frac{S'_G}{\text{Fin}}, \mathbb{P}^O \right) \cong \bigoplus_{n=0}^{\infty} \mathcal{H}_E^{(n)}.
\]

where \( \mathcal{H}_E^{(0)} := \mathbb{C}1 \) for a unit “vacuum” vector \( 1 \) corresponding to the constant function, and \( \mathcal{H}_E^{(n)} \) denotes the \( n \)-fold symmetric tensor product of \( \mathcal{H}_E \) with itself. Observe that \( 1 \) is orthogonal to \( \text{Fin} \) and \( \mathcal{H} \text{arm} \), but is not the zero element of \( L^2(S'_G, \mathbb{P}^O) \).
Lemma 6.3. For all \( v \in \mathcal{H} \text{arm} \), \( \int_{S_G} v d\mathbb{P}^Q = 0 \).

Proof. The integral \( \int_{S_G} v d\mathbb{P}^Q = \int_{S_G} 1v d\mathbb{P}^Q \) is the inner product of two elements in \( L^2(S_G, \mathbb{P}^Q) \) which lie in different (orthogonal) subspaces; see (5.11).

Alternatively, Lemma 6.3 holds because the expectation of every odd-power monomial vanishes by (5.6); see also (5.10) and the surrounding discussion of Remark 5.5.

Recall that we abuse notation and write \( h_x = \langle h_x, \cdot \rangle_W = \tilde{h}_x \) for elements of \( S_G \).

Definition 6.4. Denote the measure appearing in Corollary 6.1 by

\[
d\mu_x := (1 + h_x) d\mathbb{P}^Q. \tag{6.5}\]

The function \( 1 \) does not show up in (6.1) because it is orthogonal to \( \mathcal{H} \text{arm} \):

\[
\int_{S_G} u(1 + h_x) d\mathbb{P}^Q = \int_{S_G} u d\mathbb{P}^Q + \langle u, h_x \rangle_E = \langle u, h_x \rangle_E, \quad \text{for } u \in \mathcal{H} \text{arm},
\]

where we used Lemma 6.3. Nonetheless, its presence is necessary,

\[
\int_{S_G} 1d\mu_x = \int_{S_G} 1(1 + h_x) d\mathbb{P}^Q = \int_{S_G} 1 d\mathbb{P}^Q + \int_{S_G} h_x d\mathbb{P}^Q = 1,
\]

again by Lemma 6.3.

Remark 6.5. We have shown that as a linear functional, \( \mu_x[1] = 1 \). It follows by standard functional analysis that \( \mu_x \geq 0 \) \( \mathbb{P}^Q \)-a.e. on \( S_G \). Thus, \( \mu_x \) is absolutely continuous with respect to \( \mathbb{P}^Q \) (\( \mu_x \ll \mathbb{P}^Q \)) with Radon-Nikodym derivative \( d\mu_x = 1 + h_x \).

Definition 6.6. Recall that a path in \( G \) is an infinite sequence of successively adjacent vertices. We say that a path \( \omega = (x_0, x_1, x_2, \ldots) \) is a path to infinity, and write \( \omega \to \infty \), iff \( \omega \) eventually leaves any finite set \( F \subseteq G^0 \), i.e.,

\[
\exists N \text{ such that } n \geq N \implies x_n \notin F. \tag{6.6}\]

If \( \omega_1 = (x_0, x_1, x_2, \ldots) \) and \( \omega_2 = (y_0, y_1, y_2, \ldots) \) are two paths to infinity, define an equivalence relation by

\[
\omega_1 \equiv \omega_2 \iff \lim_{n \to \infty} (h(x_n) - h(y_n)) = 0, \quad \text{for every } h \in M. \tag{6.7}\]

In particular, all paths to infinity are equivalent when \( \mathcal{H} \text{arm} = 0 \).

If \( \beta = [\omega] \) is any such equivalence class, pick any representative \( \omega = (x_0, x_1, x_2, \ldots) \) and consider the associated sequence of measures \( \{\mu_{x_n}\} \). These probability measures lie in the unit ball in the weak-\( \star \) topology, so Alaoglu’s theorem gives a weak-\( \star \) limit

\[
\nu_\beta := \lim_{n \to \infty} \mu_{x_n}. \tag{6.8}\]

For any \( h \in \mathcal{H} \text{arm} \), this measure satisfies

\[
h(x_n) = \int_{S_G} \tilde{h} d\mu_{x_n} \xrightarrow{n \to \infty} \int_{bd G} \tilde{h} d\nu_\beta. \tag{6.9}\]
Thus, we define $\text{bd} G$ to be the collection of all such $\beta$, and extend harmonic functions to $\text{bd} G$ via

$$\tilde{h}(\beta) := \int_{\text{bd} G} \tilde{h} \, d\nu_G. \tag{6.10}$$

**Definition 6.7.** For $u \in \mathcal{H}_C$, denote $\|u\|_\infty := \sup_{x \in C^0} |u(x) - u(o)|$, and say $u$ is bounded iff $\|u\|_\infty < \infty$.

**Lemma 6.8.** If $v \in \mathcal{H}_C$ is bounded, then $P_{\text{fin}} v$ is also bounded.

**Proof.** Choose a representative for $v$ with $0 \leq v \leq K$. Then by Corollary 6.1 and (6.5),

$$P_{\text{Harm}} v(x) = \int_S v(\xi) h_x(\xi) d\nu_G(\xi) + o(u(o)) = \int_S v(\xi) d\mu_x(\xi) + o(u(o)).$$

Since $\mu_x$ is a probability measure (cf. Remark 6.6), we have $P_{\text{Harm}} v \geq 0$, and hence the finitely supported component $P_{\text{fin}} v = v - P_{\text{Harm}} v$ is also bounded. □

**Lemma 6.9.** Every $v \in M$ is bounded. In particular, $\|v\|_\infty \leq R^F(o, x)$.

**Proof.** According to Definition 2.15, it suffices to check that $v_x$, $w_x^v$ and $w_x^f$ are bounded for each $x$. Furthermore, $\|w_x^v\|_\infty = \|P_{\text{fin}} w_x^v\|_\infty \leq \|w_x^f\|_\infty$ by Lemma 6.8, and $w_x^o = v_x + w_x^o$ by definition, so it suffices to check $v_x$ and $w_x^o$. By [Soo94, Lem. 3.70], $w_x^o$ has a representative which is bounded, taking only values between 0 and $w_x^o(o) > 0$. It remains only to check $v_x$. The following approach is taken from the “proof” of [JP09c, Conj. 3.18].

Fix $x, y \in G^0$ and an exhaustion $\{G_k\}$, and suppose without loss of generality that $o, x, y \in G_1$. Also, let us consider the representative of $v_x$ specified by $v_x(o) = 0$. On a finite network, it is well-known that

$$v_x = R(o, x) u_x, \tag{6.11}$$

where $u_x(y)$ is the probability that a random walker (RW) started at $y$ reaches $x$ before $o$, that is, $u_x(y) := P_x[\tau_x < \tau_o]$, where $\tau_x$ denotes the hitting time of $x$. This idea is discussed in [DS84, LPW08, LP09].

Therefore, one can write (6.11) on $G_k$ as $v_x^{(k)} = R^{G_k}(o, x) u_x^{(k)}$. In other words, $v_x^{(k)}$ is the unique solution to $\Delta v = \delta_x - \delta_o$ on the finite subnetwork $G_k^k$. Consequently, for every $k$ we have $v_x^{(k)}(y) \leq R^{G_k}(o, x)$ for all $y \in G_k$. Since $R^F(x, y) = \lim_{k \to \infty} R^{G_k}(x, y)$ by [JP09c, Def. 2.9], we have $\|v_x\|_\infty \leq R^F(o, x)$ for every $x \in G^0$. □

**Theorem 6.10.** Let $\beta \in \text{bd} G$ and let $\omega = (x_0, x_1, x_2, \ldots)$ be any representative of $\beta$. Then $\beta \in \text{bd} G$ defines a continuous linear functional on $S_G$ via

$$\beta(v) := \lim_{n \to \infty} \int_{S_G} \tilde{v} \, d\mu_{x_n}, \quad v \in S_G. \tag{6.12}$$

In fact, the action of $\beta$ is equivalently given by

$$\beta(v) = \lim_{n \to \infty} P_{\text{Harm}} v(x_n) - P_{\text{Harm}} v(o), \quad v \in S_G. \tag{6.13}$$
Proof. To see that (6.12) and (6.13) are equivalent, compute

\[
\int_{S_\mathcal{G}} \tilde{\eta} (1 + h_\nu) \, d\mathcal{P} = \int_{S_\mathcal{G}} \tilde{\eta} \, d\mathcal{P} + \int_{S_\mathcal{G}} \tilde{\eta} h_\nu \, d\mathcal{P} = \langle v, h_\nu \rangle \mathcal{E} = P_{\mathcal{H}_\nu} v(x_\nu) - P_{\mathcal{H}_\nu} v(\alpha),
\]

because 1 is orthogonal to \(\mathcal{H}_\nu\) in \(L^2(S_\mathcal{G}, \mathcal{P})\); see (5.11).

Now, to see that (6.12) or (6.13) defines a bounded linear functional, we only need to check that \(\sup_{v \in S_\mathcal{G}} \| \beta(v) \|_{\mathcal{E}} = 1\) is bounded, but this is the content of Lemma 6.9. Note that the equivalence relation (6.7) ensures that the limit is independent of the choice of representative \(\omega\). \(\square\)

Remark 6.11. In light of (6.13), one can think of \(v_\beta\) in (6.8) as a Dirac mass. Thus, \(\beta \in \text{bd} \mathcal{G}\) is a boundary point, and integrating a function \(f\) against \(v_\beta\) corresponds to evaluation of \(f\) at that boundary point.

7. Examples

Our presentation of \(\text{bd} \mathcal{G}\) may appear somewhat abstract in the general case. However, we now illustrate the concept with a simple and entirely explicit example where the representation by equivalence classes given at the end of §6 takes on an especially concrete and visual form. Moreover, the computations can be completed without the direct construction of \(S_\mathcal{G}, S'_\mathcal{G}\), or any discussion of \(L^2(S_\mathcal{G}, \mathcal{P})\); we can obtain the boundary simply by constructing certain functions on the network. We feel this is an especially nice feature of our approach.

Example 7.1 (One-sided infinite ladder network). Consider two copies of the nearest-neighbour graph on the nonnegative integers \(\mathbb{Z}^+\), one with vertices labelled by \(\{x_n\}\), and the other with vertices labelled by \(\{y_n\}\). Fix two positive numbers \(\alpha > 1 > \beta > 0\). In addition to the edges \(c_{x_n, x_{n+1}} = \alpha^n\) and \(c_{y_n, y_{n+1}} = \beta^n\), we also add “rungs” to the ladder by defining \(c_{x_n, y_n} = \beta^n\):

\[
\begin{array}{cccccccc}
\alpha & x_0 & \alpha^2 & x_1 & \alpha^3 & x_2 & \ldots & x_n & \alpha^{n+1} & \ldots \\
\beta & y_0 & \beta^2 & y_1 & \beta^3 & y_2 & \ldots & y_n & \beta^{n+1} & \ldots \\
\end{array}
\]

This network was suggested to us by Agelos Georgakopoulos as an example of a one-ended network with nontrivial \(\mathcal{H}_\text{arm}\). The function \(u\) constructed below is the first example of an explicitly computed nonconstant harmonic function of finite energy on a graph with one end (existence of such a phenomenon was proved in [CW92]). Numerical experiments indicate that this function is also bounded (and even that the sequences \(\{u(x_n)\}_{n=0}^{\infty}\) and \(\{u(y_n)\}_{n=0}^{\infty}\) actually converge very quickly), but we have not yet been able to prove this. Numerical evidence also suggests that \(\Delta\) is not essentially self-adjoint on this network, but we have not yet proved this, either.

This graph clearly has one end. We will show that such a network has nontrivial resistance boundary if and only if \(\alpha > 1\) and in this case, the boundary consists of one point for \(\beta = 1\), and two points for \(\beta\) such that \((1 + \frac{1}{\alpha})^2 < \alpha/\beta^2\). It will be made clear that the paths \(\omega_x = (x_1, x_2, x_3, \ldots)\) and \(\omega_y = (y_1, y_2, y_3, \ldots)\) are equivalent in the sense of Definition 6.6 if and only if \(\beta = 1\).

For presenting the construction of \(u\), choose \(\beta < 1\) satisfying \(4\beta^2 < \alpha\) (at the end of the construction, we explain how to adapt the proof for the less restrictive condition \((1 + \frac{1}{\alpha})^2 < \alpha/\beta^2\)).
\( \alpha/\beta^2 \). We now construct a nonconstant \( u \in \mathcal{H}arn \) with \( u(x_0) = 0 \) and \( u(y_0) = -1 \). If we consider the flow induced by \( u \), the amount of current flowing through one edge determines \( u \) completely (up to a constant). Once it is clear that there are two boundary points in this case, it is clear that specifying the value of \( u \) at one (and grounding the other) determines \( u \) completely.

Due to the symmetry of the graph, we may abuse notation and write \( n \) for \( x_n \) or \( y_n \), and \( \tilde{n} \) for the vertex “across the rung” from \( n \). For a function \( u \) on the ladder, denote the horizontal increments and the vertical increments by

\[
\delta u(n) := u(n+1) - u(n) \quad \text{and} \quad \sigma u(n) := u(n) - u(\tilde{n}),
\]

respectively. Thus, for \( n \geq 1 \), we can express the equation \( \Delta u(n) = 0 \) by

\[
\Delta u(n) = \alpha^2 \delta u(n-1) - \alpha^{n+1} \delta u(n) + \beta^n \sigma u(n) = 0,
\]

which is equivalent to

\[
\delta u(n) = \frac{1}{\alpha} \delta u(n-1) + \frac{\beta^n}{\alpha^{n+1}} \sigma u(n).
\]

Since symmetry allows one to assume that \( u(\tilde{n}) = 1 - u(n) \), we may replace \( \sigma u(n) \) by \( 2u(n) + 1 \) and obtain that any \( u \) satisfying

\[
u(n+1) = u(n) + \frac{\delta u(n-1)}{\alpha} + \frac{\beta^n}{\alpha} u(n) + \frac{\beta^n}{\alpha} \quad (7.2)
\]

is harmonic. It remains to see that \( u \) has finite energy.

Our estimate for \( E(u) < \infty \) requires the assumption that \( \alpha > 4\beta^2 \), but numerical computations indicate that \( u \) defined by (7.2) will be both bounded and of finite energy, for any \( \beta < 1 < \alpha \). First, note that \( u(1) = \frac{1}{\alpha} \) and so an immediate induction using (7.2) shows that \( \delta u(n) = u(n+1) - u(n) > 0 \) for all \( n \geq 1 \), and so \( u \) is strictly increasing. Since \( \beta < 1 < \alpha \), we may choose \( N \) so that

\[
n \geq N \quad \implies \quad \left( \frac{\beta}{\alpha} \right)^n < \frac{\alpha - 1}{2}.
\]

Then \( n \geq N \) implies

\[
u(n+1) \leq 2u(n) + \frac{1}{\alpha}, \quad (7.3)
\]

by using (7.2) and the fact that \( u(n) \) is increasing and \( \frac{\beta}{\alpha} < 1 \). Now use (7.2) to write

\[
\delta u(n) = \frac{1}{\alpha} \delta u(n-1) + \left( \frac{2}{\alpha} u(n) + \frac{1}{\alpha} \right) \left( \frac{\beta}{\alpha} \right)^n
\]

\[
= \frac{1}{\alpha^n} \delta u(0) + \sum_{k=0}^{n-1} \frac{1}{\alpha^n} \left( \frac{2}{\alpha} u(n-k) + \frac{1}{\alpha} \right) \left( \frac{\beta}{\alpha} \right)^{n-k}
\]

\[
= \frac{1}{\alpha^n} + \frac{\beta^{1-n} \delta u(0)}{\alpha^{n+1} (1-\beta)} + \frac{2}{\alpha^n} \sum_{k=1}^{n} \beta^k u(k),
\]

where the second line comes by iterating the first, and the third by algebraic simplification.

Applying the estimate (7.3) gives
2 \sum_{k=1}^{n} \beta^k u(k) \leq 2^2 \sum_{k=1}^{n} \beta^k u(k-1) + \frac{2}{3} \sum_{k=1}^{n} \beta^k = 2^n \sum_{k=1}^{n} \beta^k u(k-1) + 2^{n+2} \cdot \frac{1-\beta^2}{1-\beta},

and iterating gives
\[ \delta u(n) \leq \frac{1}{\alpha^{n+1}} \left( 1 + \frac{\beta(1-\beta^2)}{1-\beta} + \frac{(2\beta)^n}{\alpha} + \frac{2\beta^n}{\alpha^{n+1}} \sum_{k=0}^{n-1} 2^k \beta^k - \beta^2 \right). \] (7.4)

Now the energy \( E(u) = \sum_{n=0}^{\infty} \alpha^{n+1} (\delta u(n))^2 \) can be estimated by using (7.4) as follows:
\[ E(u) \leq \sum_{n=0}^{\infty} \frac{1}{\alpha^{n+1}} \left( 1 + \frac{\beta(1-\beta^2)}{1-\beta} + \frac{(2\beta)^n}{\alpha} + \frac{2\beta^n}{\alpha^{n+1}} - 2^{n+2} \beta^{n+1} - 2^{n+2} \beta^{n+2} + (2\beta)^n \right)^2. \]

and the condition \( \alpha > 4\beta^2 \) ensures convergence.

Note that this computations above can be slightly refined: instead of \( \alpha > 4\beta^2 \), one need only assume that \( \alpha > (1 + \frac{1}{\alpha})^2 \beta^2 \). Then, fix \( \varepsilon > 0 \) for which \( \alpha/\beta^2 > (1 + \frac{1}{\alpha})^2 + \varepsilon \) and choose \( N \) so that \( n \geq N \) implies \( (\beta/\alpha)^n < 1 + \frac{1}{\alpha} + \varepsilon (1 + 2\alpha + \alpha\varepsilon) \). Then the calculations can be repeated, with most occurrences of 2 replaced by \( 1 + \frac{1}{\alpha} + \varepsilon \).

Remark 7.2. [Comparison of Example 7.1 to the 1-dimensional integer lattice] In [JP09b, Ex. 6.3], we showed that the “nonnegative geometric integers” network

\[ \begin{array}{cccccccccccc}
0 & a & a^2 & a^3 & a^4 & \cdots
\end{array} \]

supports a monopole but not a harmonic function of finite energy, for \( \alpha > 1 \). These conductances correspond to the biased random walk where, at each vertex, the walker has transition probabilities

\[ p(n, m) = \begin{cases} \frac{1}{1+\alpha}, & m = n - 1, \\
\frac{a}{1+\alpha}, & m = n + 1. \end{cases} \]

In particular, this is a spatially homogeneous distribution. In contrast, the random walk corresponding to Example 7.1 has transition probabilities

\[ p(n, m) = \begin{cases} \frac{1}{1+\alpha+\frac{\beta}{\alpha}}, & m = n - 1, \\
\frac{\beta/\alpha}{1+\alpha+\frac{\beta}{\alpha}}, & m = n + 1, \\
\frac{1}{1+\alpha+\frac{\beta}{\alpha}}, & m = n. \end{cases} \]

Thus, Example 7.1 is asymptotic to the nonnegative geometric integers.

One can even think of Example 7.1 as describing the \textit{scattering theory} of the geometric half-integer model, in the sense of [LP89]. In this theory, a wave (described by a function) travels towards an obstacle. After the wave collides with the obstacle, the original function is transformed (via the “scattering operator”) and the resulting wave travels away from the obstacle. The scattering is typically localized in some sense, corresponding to the location of the collision.
To see the analogy with the present scenario, consider the current flow defined by the harmonic function \( u \) constructed in Example 7.1, i.e., induced by Ohm’s law: \( I(x, y) = e_{xy}(u(x) - u(y)) \). With \( \text{div}_{y}(x) := \frac{1}{2} \sum_{\{z : I(x, z) > 0\}} I(x, z) \), this current defines a Markov process with transition probabilities

\[
P(x, y) = \frac{I(x, y)}{\text{div}_{y}(x)}, \quad \text{if} \quad I(x, y) > 0,
\]

and \( P(x, y) = 0 \) otherwise; see [JP09d, JP09a]. This describes a random walk where a walker started on the bottom edge of the ladder will tend to step leftwards, but with a geometrically increasing probability of stepping to the upper edge, and then walking rightwards off towards infinity. The walker corresponds to the wave, which is scattered as it approaches the geometrically localized obstacle at the origin.

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University of Iowa, Iowa City, IA 52246-1419 USA
E-mail address: jorgen@math.uiowa.edu

University of Iowa, Iowa City, IA 52246-1419 USA
E-mail address: epearse@math.uiowa.edu