INFLUENCE OF VARIABLE COEFFICIENTS ON GLOBAL
EXISTENCE OF SOLUTIONS OF SEMILINEAR HEAT
EQUATIONS WITH NONLINEAR BOUNDARY CONDITIONS

ALEXANDER GLADKOV AND MOHAMMED GUEDDA

ABSTRACT. We consider semilinear parabolic equations with nonlinear boundary conditions. We give conditions which guarantee global existence of solutions as well as blow-up in finite time of all solutions with nontrivial initial data. The results depend on the behavior of variable coefficients as $t \to \infty$.

1. Introduction

We investigate the global solvability and blow-up in finite time for semilinear heat equation
\[ u_t = \Delta u + \alpha(t)f(u) \text{ for } x \in \Omega, \ t > 0, \] \hfill (1.1)
with nonlinear boundary condition
\[ \frac{\partial u(x, t)}{\partial \nu} = \beta(t)g(u) \text{ for } x \in \partial \Omega, \ t > 0, \] \hfill (1.2)
and initial datum
\[ u(x, 0) = u_0(x) \text{ for } x \in \Omega, \] \hfill (1.3)
where $\Omega$ is a bounded domain in $\mathbb{R}^n$ for $n \geq 1$ with smooth boundary $\partial \Omega$, $\nu$ is the unit exterior normal vector on the boundary $\partial \Omega$. Here $f(u)$ and $g(u)$ are nonnegative continuous functions for $u \geq 0$, $\alpha(t)$ and $\beta(t)$ are nonnegative continuous functions for $t \geq 0$, $u_0(x) \in C^1(\Omega)$, $u_0(x) \geq 0$ in $\Omega$ and satisfies boundary condition (1.2) as $t = 0$. We will consider nonnegative classical solutions of (1.1)–(1.3).

Blow-up problem for parabolic equations with reaction term in general form were considered in many papers (see, for example, [1] – [7] and the references therein). For the global existence and blow-up of solutions for linear parabolic equations with $\beta(t) \equiv 1$ in (1.2), we refer to previous studies [8] – [13]. In particular, Walter [9] proved that if $g(s)$ and $g'(s)$ are continuous, positive and increasing for large $s$, a necessary and sufficient condition for global existence is
\[ \int_{r_0}^{+\infty} \frac{ds}{g(s)g'(s)} = +\infty. \]

Some papers are devoted to blow-up phenomena in parabolic problems with time-dependent coefficients (see, for example, [14] – [20]). So, it follows from results of Payne and Philippin [15] blow-up of all nontrivial solutions for (1.1)–(1.3) with $\beta(t) \equiv 0$ under the conditions (2.15) and
\[ f(s) \geq z(s) > 0, \ s > 0, \]

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where \( z \) satisfies
\[
\int_a^{+\infty} \frac{ds}{z(s)} < +\infty \quad \text{for any} \quad a > 0
\]
and Jensen’s inequality
\[
\frac{1}{|\Omega|} \int_{\Omega} z(u) \, dx \geq z \left( \frac{1}{|\Omega|} \int_{\Omega} u \, dx \right) .
\] (1.4)

In (1.4), \(|\Omega|\) is the volume of \( \Omega \).

The aim of our paper is study the influence of variable coefficients \( \alpha(t) \) and \( \beta(t) \) on the global existence and blow-up of classical solutions of (1.1)–(1.3).

This paper is organized as follows. Finite time blow-up of all nontrivial solutions is proved in Section 2. In Section 3, we present the global existence of solutions for small initial data.

2. Finite time blow-up

In this section, we give conditions for blow-up in finite time of all nontrivial solutions of (1.1)–(1.3).

Before giving our main results, we state a comparison principle which has been proved in [21], [22] for more general problems. Let \( Q_T = \Omega \times (0, T), S_T = \partial \Omega \times (0, T), \Gamma_T = S_T \cup \Omega \times \{0\}, T > 0 \).

**Theorem 2.1.** Let \( v(x, t), w(x, t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T) \) satisfy the inequalities:
\[
v_t - \Delta v - \alpha(t)f(v) < w_t - \Delta w - \alpha(t)f(w) \quad \text{in} \quad Q_T,
\]
\[
\frac{\partial v(x, t)}{\partial \nu} - \beta(t)g(v) < \frac{\partial w(x, t)}{\partial \nu} - \beta(t)g(w) \quad \text{on} \quad S_T,
\]
\[
v(x, 0) < w(x, 0) \quad \text{in} \quad \Omega.
\]
Then
\[
v(x, t) < w(x, t) \quad \text{in} \quad Q_T.
\]

The first our blow-up result is the following.

**Theorem 2.2.** Let \( g(s) \) be a nondecreasing positive function for \( s > 0 \) such that
\[
\int_0^{+\infty} \frac{ds}{g(s)} < +\infty
\]
and
\[
\int_0^{+\infty} \beta(t) \, dt = +\infty.
\] (2.2)

Then any nontrivial nonnegative solution of (1.1)–(1.3) blows up in finite time.

**Proof.** We suppose that \( u(x, t) \) is a nontrivial nonnegative solution which exists in \( Q_T \) for any positive \( T \). Then for some \( T > 0 \) there exists \((x, T) \in Q_T\) such that \( u(x, T) > 0 \). Since \( u_t - \Delta u = \alpha(t)f(u) \geq 0 \), by strong maximum principle \( u(x, t) > 0 \) in \( Q_T \setminus \overline{Q_T} \). Let \( u(x, t_*) = 0 \) in some point \((x_*, t_*) \in S_T \setminus \overline{Q_T}\). According to Theorem 3.6 of [23] it yields \( \partial u(x_*, t_*) / \partial \nu < 0 \), which contradicts the boundary condition (1.2). Thus, \( u(x, t) > 0 \) in \( Q_T \cup \overline{S_T} \setminus \overline{Q_T} \). Then there exists \( t_0 > T \) such that \( \beta(t_0) > 0 \) and
\[
\min_{\overline{\Omega}} u(x, t_0) > 2\sigma,
\] (2.3)
where \( \sigma \) is a positive constant.
Let $G_N(x, y; t - \tau)$ denote the Green’s function for the heat equation given by
\[ u_t - \Delta u = 0 \quad \text{for} \quad x \in \Omega, \ t > 0 \]
with homogeneous Neumann boundary condition. We note that the Green’s function has the following properties (see, for example, [24], [25]):
\[ G_N(x, y; t - \tau) \geq 0, \ x, y \in \Omega, \ 0 \leq \tau < t, \]  
\[ \int_{\Omega} G_N(x, y; t - \tau) dy = 1, \ x \in \Omega, \ 0 \leq \tau < t, \]  
\[ G_N(x, y; t - \tau) \geq c_1, \ x, y \in \overline{\Omega}, \ t - \tau \geq \varepsilon, \]  
\[ |G_N(x, y; t - \tau) - 1/|\Omega|] \leq c_2 \exp[-c_3(t - \tau)], \ x, y \in \overline{\Omega}, \ t - \tau \geq \varepsilon, \]
for some small $\varepsilon > 0$. Here by $c_i$ ($i \in \mathbb{N}$) we denote positive constants.

Now we introduce several auxiliary functions. We suppose that $h(s) \in C^1((0, +\infty)) \cap C([0, +\infty))$, $h(s) > 0$ for $s > 0$, $h'(s) \geq 0$ for $s > 0$, $g(s) \geq h(s)$ and
\[ \int_{-\infty}^{+\infty} ds \frac{h(s)}{h(s)} < +\infty. \]

Let $\xi(t)$ be a positive continuous function for $t \geq t_0$ such that
\[ \int_{t_0}^{+\infty} \xi(t) dt < \frac{\sigma}{2} \]  
and $\gamma(t)$ be a positive continuous function for $t \geq t_0$ such that $\gamma(t_0) = \beta(t_0)h(2\sigma)$ and
\[ \int_{t_0}^{t} \gamma(\tau) \int_{\partial\Omega} G_N(x, y; t - \tau) dS_y d\tau < \frac{\sigma}{2} \]  
for $x \in \overline{\Omega}, \ t \geq t_0$.  

We consider the following problem
\[
\begin{cases}
\xi = \Delta v - \xi(t) & \text{for} \ x \in \Omega, \ t > t_0, \\
\frac{\partial v(x, t)}{\partial \nu} = \beta(t)h(v) - \gamma(t) & \text{for} \ x \in \partial\Omega, \ t > t_0, \\
v(x, t_0) = 2\sigma & \text{for} \ x \in \Omega.
\end{cases}
\]

To find lower bound for $v(x, t)$ we represent in equivalent form
\[ v(x, t) = 2\sigma \int_{\Omega} G_N(x, y; t) dy - \int_{t_0}^{t} \int_{\Omega} G_N(x, y; t - \tau) \xi(\tau) dy d\tau \]
\[ + \int_{t_0}^{t} \int_{\partial\Omega} G_N(x, y; t - \tau) (\beta(t)h(v) - \gamma(t)) dS_y d\tau. \]  

Using (2.7), (2.8) and the properties of the Green’s function (2.3), (2.5), we obtain from (2.10)
\[ v(x, t) \geq 2\sigma - \int_{t_0}^{t} \xi(\tau) d\tau - \int_{t_0}^{t} \gamma(\tau) \int_{\partial\Omega} G_N(x, y; t - \tau) dS_y d\tau > \sigma. \]

As in [13] we put
\[ m(t) = \int_{\Omega} \int_{v(x, t)}^{+\infty} \frac{ds}{h(s)} dx. \]
We observe that \( m(t) \) is well defined and positive for \( t \geq t_0 \). Since \( v(x,t) \) is the solution of (2.9), we get

\[
m'(t) = -\int_\Omega \frac{v_t}{h(v)} \, dx = -\int_\Omega \frac{\Delta v}{h(v)} \, dx + \xi(t) \int_\Omega \frac{dx}{h(v)}.\]

Applying the inequality \( h'(v) \geq 0 \), Gauss theorem, the boundary condition in (2.9) and (2.11), we obtain for \( t \geq t_0 \)

\[
m'(t) \leq -\int_{\partial\Omega} \frac{1}{h(v)} \frac{\partial v}{\partial\nu} \, dS + \xi(t) \frac{|\Omega|}{h(\sigma)} \leq -|\partial\Omega| \beta(t) + \frac{|\Omega| \xi(t) + |\partial\Omega| \gamma(t)}{h(\sigma)}. \tag{2.12}
\]

Due to (2.2), (2.6) – (2.8) \( m(t) \) is negative for large values of \( t \). Hence \( v(x,t) \) blows up in finite time \( T_0 \). Applying Theorem 2.1 to \( v(x,t) \) and \( u(x,t) \) in \( Q_T \setminus Q_{t_0} \) for any \( T \in (t_0,T_0) \), we prove the theorem. \( \Box \)

**Remark 2.3.** If \( u_0(x) \) is positive in \( \overline{\Omega} \) we can obtain an upper bound for blow-up time of the solution. We put \( t_0 = 0 \) and \( v(x,0) = u_0(x) - \varepsilon \) in (2.9) for \( \varepsilon \in (0, \min_{\Omega} u_0(x)) \). Integrating (2.12) over \([0,T]\), we have

\[
m(t) \leq m(0) - |\partial\Omega| \int_0^T \beta(t) \, dt + \int_0^T \frac{|\Omega| \xi(t) + |\partial\Omega| \gamma(t)}{h(\sigma)} \, dt.
\]

Since \( m(t) > 0 \) and \( \varepsilon \), \( \xi(t) \), \( \gamma(t) \) are arbitrary we conclude that the solution of (1.1)–(1.3) blows up in finite time \( T_0 \), where \( T_0 \leq T \) and

\[
\int_\Omega \int_{\bar{u}_0(x)}^{+\infty} \frac{ds}{h(s)} \, dx = |\partial\Omega| \int_0^T \beta(t) \, dt.
\]

**Remark 2.4.** We note that (1.1)–(1.3) with \( u_0(x) \equiv 0 \) may have trivial and blow-up solutions under the assumptions of Theorem 2.2. Indeed, let the conditions of Theorem 2.2 hold, \( \alpha(t) \equiv 0 \), \( \beta(t) \equiv 1 \) and \( g(u) = u^p \), \( u \in [0, \gamma] \) for some \( \gamma > 0 \) and \( 0 < p < 1 \). As it was proved in [26], problem (1.1)–(1.3) has trivial and positive for \( t > 0 \) solutions and last one blows up in finite time by Theorem 2.2.

To prove next blow-up result for (1.1)–(1.3) we need a comparison principle with unstrict inequality in the boundary condition.

**Theorem 2.5.** Let \( \delta > 0 \) and \( v(x,t), w(x,t) \in C^{2,1}(Q_T) \cap C^{1,0}(Q_T \cup \Gamma_T) \) satisfy the inequalities:

\[
v_t - \Delta v - \alpha(t)f(v) + \delta < w_t - \Delta w - \alpha(t)f(w) \quad \text{in} \; Q_T,
\]

\[
\frac{\partial v(x,t)}{\partial\nu} \leq \frac{\partial w(x,t)}{\partial\nu} \quad \text{on} \; S_T,
\]

\[
v(x,0) < w(x,0) \quad \text{in} \; \overline{\Omega}.
\]

Then

\[
v(x,t) \leq w(x,t) \quad \text{in} \; Q_T.
\]

**Proof.** Let \( \tau \) be any positive constant such that \( \tau < T \) and a positive function \( \gamma(x) \in C^2(\overline{\Omega}) \) satisfy the following inequality

\[
\frac{\partial \gamma(x)}{\partial\nu} > 0 \quad \text{on} \; \partial\Omega.
\]
For positive $\varepsilon$ we introduce
\[ w_\varepsilon(x, t) = w(x, t) + \varepsilon \gamma(x). \] (2.13)

Obviously,
\[ v(x, 0) < w_\varepsilon(x, 0) \text{ in } \Omega, \quad \frac{\partial v(x, t)}{\partial \nu} < \frac{\partial w_\varepsilon(x, t)}{\partial \nu} \text{ on } S_\tau. \]

Moreover,
\[ v_t - \Delta v - \alpha(t)f(v) < w_\varepsilon t - \Delta w_\varepsilon - \alpha(t)f(w_\varepsilon) \text{ in } Q_\tau, \]
if we take $\varepsilon$ so small that
\[ \delta > \varepsilon \Delta \gamma + \alpha(t)[f(w + \varepsilon \gamma) - f(w)] \text{ in } Q_\tau. \]

Applying Theorem 2.1 with $\beta(t) \equiv 0$, we obtain
\[ v(x, t) < w_\varepsilon(x, t) \text{ in } Q_\tau. \]

Passing to the limits as $\varepsilon \to 0$ and $\tau \to T$, we prove the theorem. \[ \square \]

**Theorem 2.6.** Let $f(s) > 0$ for $s > 0$,
\[ \int^{+\infty} ds f(s) < +\infty \] (2.14)
and
\[ \int^{+\infty}_0 \alpha(t) dt = +\infty. \] (2.15)

Then any nontrivial nonnegative solution of (1.1) – (1.3) blows up in finite time.

*Proof.* We suppose that $u(x, t)$ is a nontrivial nonnegative solution which exists in $Q_T$ for any positive $T$. In Theorem 2.2 we proved (2.3). Let $\xi(t)$ be a positive continuous function for $t \geq t_0$ such that
\[ \max_{[\sigma, 2\sigma]} f(s) \int^{+\infty}_t \xi(t) dt < \sigma. \] (2.16)

We consider the following auxiliary problem
\[ \begin{cases} v'(t) = \alpha(t)f(v) - \xi(t)f(v), & t > t_0, \\ v(t_0) = 2\sigma. \end{cases} \] (2.17)

We prove at first that
\[ v(t) > \sigma \text{ for } t \geq t_0. \] (2.18)

Suppose there exist $t_1$ and $t_2$ such that
\[ t_2 > t_1 \geq t_0, \quad v(t_1) = 2\sigma, \quad v(t_2) = \sigma, \]
and
\[ \quad v(t) > \sigma \text{ for } t \in [t_0, t_2) \text{ and } v(t) \leq 2\sigma \text{ for } t \in [t_1, t_2]. \]

Integrating the equation in (2.17) over $[t_1, t_2]$, we have due to (2.16)
\[ v(t_2) \geq -\max_{[\sigma, 2\sigma]} f(s) \int^{t_2}_{t_1} \xi(t) dt + v(t_1) > \sigma. \]

A contradiction proves (2.18).

From (2.17) we obtain
\[ \int^{v(t)}_{2\sigma} \frac{ds}{f(s)} = \int^{t}_{t_0} [\alpha(\tau) - \xi(\tau)] d\tau. \] (2.19)
By (2.14) – (2.16) the left side of (2.19) is finite and the right side of (2.19) tends to infinity as \( t \to \infty \). Hence the solution of (2.17) blows up in finite time \( T_0 \). Applying Theorem 2.5 to \( v(t) \) and \( u(x,t) \) in \( Q_T \setminus Q_{t_0} \) for any \( T \in (t_0, T_0) \), we prove the theorem.

Remark 2.7. If \( u_0(x) \) is positive in \( \Omega \) we can obtain an upper bound for blow-up time of the solution. Taking \( t_0 = 0 \), we conclude from (2.19) that the solution of (1.1)–(1.3) blows up in finite time \( T_b \), where

\[
\int_{\min \Omega}^{+\infty} \frac{ds}{f(s)} = \int_0^T \alpha(t) \, dt.
\]

Remark 2.8. Theorem 2.6 does not hold if \( f(s) \) is not positive for \( s > 0 \). To show this we suppose that \( f(u_1) = 0 \) for some \( u_1 > 0 \), \( \beta(t) \equiv 0 \), \( u_0(x) = u_1 \). Then problem (1.1) – (1.3) has the solution \( u(x, t) = u_1 \).

Remark 2.9. We note that (2.14) is necessary condition for blow-up of solutions of (1.1)–(1.3) with \( \beta(t) \equiv 0 \). Let \( f(s) > 0 \) for \( s > 0 \) and

\[
\int_{s_0}^{+\infty} \frac{ds}{f(s)} = +\infty.
\]

Then any solution of (1.1)–(1.3) is global. Indeed, let \( u(x, t) \) be a nontrivial solution of (1.1)–(1.3). Then there exist \( t_0 \geq 0 \) and \( x \in \Omega \) such that \( u(x, t_0) > 0 \).

We consider the following problem

\[
\begin{align*}
\text{v}'(t) &= (\alpha(t) + \xi(t))f(v), \quad t > t_0, \\
v(t_0) &= \max_{\Omega} u(x, t_0) > 0,
\end{align*}
\]  

(2.20)

where \( \xi(t) \) is some positive continuous function for \( t \geq t_0 \). Obviously, \( v(t) \) is global solution of (2.20). Applying Theorem 2.5 to \( u(x, t) \) and \( v(t) \) in \( Q_T \setminus Q_{t_0} \) for any \( T > t_0 \), we prove the theorem.

Remark 2.10. Problem (1.1)–(1.3) with \( u_0(x) \equiv 0 \) may have trivial and blow-up solutions under the assumptions of Theorem 2.6. Indeed, let the conditions of Theorem 2.6 hold, \( \beta(t) \equiv 0 \), \( f(s) \) be a nondecreasing Hölder continuous function on \( [0, \epsilon] \) for some \( \epsilon > 0 \) and

\[
\int_0^\epsilon \frac{ds}{f(s)} < +\infty.
\]

As it was proved in [27], problem (1.1)–(1.3) has trivial and positive for \( t > 0 \) solutions and last one blows up in finite time by Theorem 2.6.

3. Global existence

To formulate global existence result for problem (1.1)–(1.3) we suppose:

\( f(s) \) is a nonnegative locally Hölder continuous function for \( s \geq 0 \),

(3.1)

there exists \( p > 0 \) such that \( f(s) \) is a positive nondecreasing function for \( s \in (0, p) \),

(3.2)

\[
\int_0^s \frac{ds}{f(s)} = +\infty, \quad \lim_{s \to 0} \frac{g(s)}{s} = 0,
\]

(3.3)

\[
\int_0^{+\infty} (\alpha(t) + \beta(t)) \, dt < +\infty
\]

(3.4)
and there exist positive constants \( \gamma, t_0 \) and \( K \) such that \( \gamma > t_0 \) and
\[
\int_{t-t_0}^{t} \frac{\beta(\tau)}{\sqrt{t-\tau}} d\tau \leq K \text{ for } t \geq \gamma.
\tag{3.5}
\]

**Theorem 3.1.** Let \((3.1)–(3.5)\) hold. Then problem \((1.1)–(1.3)\) has bounded global solutions for small initial data.

**Proof.** It is well known that problem \((1.1)–(1.3)\) has a local nonnegative classical solution \( u(x,t) \). Let \( y(x,t) \) be a solution of the following problem
\[
\begin{align*}
y_t &= \Delta y, \quad x \in \Omega, \quad t > 0, \\
\frac{\partial y}{\partial \nu}(x,t) &= \xi(t) + \beta(t), \quad x \in \partial \Omega, \quad t > 0, \\
y(x,0) &= 1, \quad x \in \Omega,
\end{align*}
\tag{3.6}
\]
where \( \xi(t) \) is a positive continuous function that satisfies \((3.4), (3.5)\) with \( \beta(t) = \xi(t) \). According to Lemma 3.3 of [28] there exists a positive constant \( Y \) such that
\[
1 \leq y(x,t) \leq Y, \quad x \in \Omega, \quad t > 0.
\]
Due to \((3.2), (3.3)\) for any \( a \in (0,p) \), there exist \( \varepsilon(a) \) and a positive continuous function \( \eta(t) \) such that
\[
0 < \varepsilon(a) < \frac{a}{Y}, \quad \int_{0}^{a} \eta(t) dt < \infty \text{ and } \int_{\varepsilon Y}^{a} ds \int_{0}^{\infty} (\alpha(t) + \eta(t)) dt.
\]
for any \( \varepsilon \in (0, \varepsilon(a)) \). Now for any \( T > 0 \) we construct a positive supersolution of \((1.1)–(1.3)\) in \( Q_T \) in such a form that
\[
\bar{u}(x,t) = \varepsilon z(t)y(x,t),
\]
where function \( z(t) \) is defined in the following way
\[
\int_{\varepsilon Y}^{eYz(t)} \frac{ds}{f(s)} = Y \int_{0}^{t} (\alpha(\tau) + \eta(\tau)) d\tau.
\]
It is easy to see that \( \varepsilon Yz(t) < a \) and \( z(t) \) is the solution of the following Cauchy problem
\[
z'(t) - \frac{1}{\varepsilon} (\alpha(t) + \eta(t)) f(\varepsilon Yz(t)) = 0, \quad z(0) = 1.
\]
After simple computations it follows that
\[
\begin{align*}
\bar{u}_t - \Delta \bar{u} - \alpha(t)f(\bar{u}) &= \varepsilon z'(t) + \varepsilon z \Delta y - \alpha(t)f(\varepsilon z) \\
&\geq \alpha(t)(f(\varepsilon Yz(t)) - f(\varepsilon z)) + \eta(t)f(\varepsilon Yz(t)) > 0, \quad x \in \Omega, \quad t > 0,
\end{align*}
\]
and
\[
\frac{\partial \bar{u}}{\partial \nu} - \beta(t)g(\bar{u}) = \varepsilon z(\xi(t) + \beta(t)) - \beta(t)g(\varepsilon z(y(x,t))) \geq 0
\]
for small values of \( a \). Thus, by Theorem 2.1 there exist bounded global solutions of \((1.1)–(1.3)\) for any initial data satisfying the inequality
\[
u_0(x) < \varepsilon.
\]
\( \square \)
Remark 3.2. We suppose that \( g(s) \) is a nondecreasing positive function for \( s > 0 \), \( f(s) > 0 \) for \( s > 0 \) and (2.1), (2.14) hold. Then by Theorem 2.2 and Theorem 2.6 (3.4) is necessary for global existence of solutions of (1.1)–(1.3).

Let for any \( a > 0 \) \( g(s) > \delta(a) > 0 \) if \( s > a \). Then arguing in the same way as in the proof of Lemma 3.3 of [28] it is easy to show that (3.5) is necessary for the existence of nontrivial bounded global solutions of (1.1)–(1.3).

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Alexander Gladkov, Department of Mechanics and Mathematics, Belarusian State University, 4 Nezavisimosti Avenue, 220030 Minsk, Belarus and Peoples’ Friendship University of Russia (RUDN University), 6 Miklukho-Maklaya street, 117198 Moscow, Russian Federation
E-mail address: gladkoval@mail.ru

Mohammed Guedda, Université de Picardie, LAMFA, CNRS, UMR 7352, 33 rue Saint-Leu, F-80039, Amiens, France
E-mail address: mohamed.guedda@u-picardie.fr