Complete Classification of Local Conservation Laws for Generalized Kuramoto–Sivashinsky Equation

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August 20, 2021

Abstract

For an arbitrary number of spatial independent variables we present a complete list of cases when the generalized Kuramoto–Sivashinsky equation admits nontrivial local conservation laws of any order, and for each of those cases we give an explicit form of all the local conservation laws of all orders modulo trivial ones admitted by the equation under study.

Introduction

In the present paper we study the generalized Kuramoto–Sivashinsky equation, a PDE in $n + 1$ independent variables $t, x_1, \ldots, x_n$ and one dependent variable $u$ of the form

$$u_t = a \Delta^2 u + b(u) \Delta u + f(u) |\nabla u|^2 + g(u),$$

(1)

where $a$ is a nonzero constant, $b, f, g$ are smooth functions of $u$, $\Delta = \sum_{i=1}^{n} \partial^2 / \partial x_i^2$ is the Laplace operator, $|\nabla u|^2 = \sum_{i=1}^{n} (\partial u / \partial x_i)^2$, and $n$ is an arbitrary natural number.

Equation (1) is a natural generalization of the well-known Kuramoto–Sivashinsky equation

$$u_t + \Delta^2 u + \Delta u + |\nabla u|^2 / 2 = 0,$$

(2)

obtained from (1) by setting $a = -1, b = -1, f = -1/2$ and $g = 0$ and arising in a variety of physical and chemical contexts, describing inter alia flame propagation, reaction-diffusion systems and unstable drift waves in plasmas, see e.g. [6] and references therein. Equation (2) exhibits chaotic behavior and is an important model in the study of chaotic phenomena, cf. e.g. [6, 15].

Note that while many authors, see e.g. [2, 10, 16], have considered various generalizations of the Kuramoto-Sivashinsky equation (2), there is no generally accepted definition of the term ‘generalized Kuramoto–Sivashinsky equation’.

Our goal is to give below a complete classification of local conservation laws for (1). To the best of our knowledge, this was not yet done, especially for the arbitrary number $n$ of the space variables, although some partial results on conservation laws of several different generalizations of the Kuramoto–Sivashinsky equation for $n = 1, 2$ are known, see e.g. [2, 4, 16].

Conservation laws have numerous important applications, including for instance improving numerical solving the PDE under study using discretizations respecting known conservation laws, see e.g. [1, 3] and references therein, and construction of symmetries using Noether theorem and other methods, see for example [9, 11, 12, 17] and references therein.

On the other hand, obtaining complete description of local conservation laws for a given PDE, rather than finding a few low-order ones, is a difficult task that was achieved only for a rather small number of examples, cf. e.g. [5, 8, 13, 18] and references therein.

The rest of the paper is organized as follows. In Section 1 we recall some necessary definitions and set up notation, in Section 2 the main results are presented, while their proofs are given in Section 3.
1 Preliminaries

In this section we just recall standard definitions and introduce some notation mostly following [12] but adapted to the equation under study, i.e., (1).

We say that a differential function is a smooth function depending on $t, x_1, \ldots, x_n, u$ and finitely many $x$-derivatives of $u$. It should be stressed that the $t$-derivatives or mixed derivatives of $u$ involving both $x_i$ and $t$ are not included.

The symbols $D_t$ and $D_{x_i}$ will stand below for the so-called total derivatives, see e.g. [8, 9, 12] for more details on those.

A local conservation law for (1) is a differential expression

$$D_t(T) + \text{Div} X$$

which vanishes on all smooth solutions of (1). Here $X = (X_1, \ldots, X_n)$, $\text{Div} X = \sum_{i=1}^n D_{x_i}(X_i)$ stands for total divergence and $T, X_1, \ldots, X_n$ are differential functions.

The function $T$ in (3) is called a density and the vector function $X$ is called a flux for the conservation law under study.

We say that a conservation law is trivial if either $T$ and $X$ themselves vanish on all (smooth) solutions of (1) or $D_t(T) + \text{Div} X \equiv 0$ holds identically no matter whether (1) holds.

Two conservation laws are said to be equivalent if their difference is a trivial conservation law.

A conservation law (3) for (1) is in characteristic form if

$$D_t(T) + \text{Div} X = Q(u_t - (a\Delta^2 u + b(u)\Delta u + f(u)|\nabla u|^2 + g(u)))$$

where the differential function $Q$ is called a characteristic of the conservation law in question.

Note that a local conservation law for (1) is nontrivial if and only if its characteristic is nonzero.

In closing note that it is readily verified, cf. e.g. [12], that we lose no generality in not allowing $X_i$ and $T$ to depend on the $t$-derivatives or mixed derivatives of $u$ involving both $x_i$ and $t$, as disallowing such dependence only removes certain trivial parts from the conservation laws under study.

2 Main results

Theorem 1 Let equation (1) with $a \neq 0$ and smooth functions $b(u), f(u),$ and $g(u)$ satisfy one of the following conditions:

1. $f \neq \frac{\partial b}{\partial u}$,

2. $f = 0, \frac{\partial^2 g}{\partial u^2} \neq 0$ and $b$ is constant,

3. $f = \frac{\partial b}{\partial u} \neq 0, \frac{\partial^2 g}{\partial u^2} \neq 0$ and $\frac{\partial^3 g}{\partial u^3} \frac{\partial b}{\partial u} - \frac{\partial^2 g}{\partial u^2} \frac{\partial^2 b}{\partial u^2} \neq 0$.

Then equation (1) admits no nontrivial local conservation laws.

In particular, we have the following

Corollary 1 The Kuramoto–Sivashinsky equation (2) has no nontrivial local conservation laws.

Thus, under the conditions of Theorem 1 equation (1) and, in particular, the original Kuramoto–Sivashinsky equation (2), admit no nontrivial local conservation laws. This result has a number of important consequences. For one, as there are no local conservation laws in the cases under study, while discretizing (2) and (1) under the assumptions of Theorem 1 in order to solve the equations in question numerically, one does not have to worry about consistency of the chosen discretization with the conservation laws, cf. the discussion in the introduction and in [5].
Theorem 2 If $f = \frac{\partial b}{\partial u}$ then (1) with $a \neq 0$ and smooth functions $b(u), f(u),$ and $g(u)$ admits nontrivial local conservation laws, all of which are listed below modulo trivial ones, only in the following three cases:

I. Let $f = 0$, $b = c_1$, and $g = c_2u + c_3$, where $c_1, c_2, c_3$ are arbitrary constants. Then the densities of all nontrivial local conservation laws for (1) have the form

$$T^I = uQ^I + \frac{1}{c_2}Q^I$$

where $Q^I = Q^I(t, x_1, \ldots, x_n)$ is any (smooth) solution of the linear PDE with constant coefficients

$$\frac{\partial Q^I}{\partial t} + a\Delta^2(Q^I) + c_2Q^I + c_1\Delta(Q^I) = 0.$$  \hspace{1cm} (5)

II. Let $f = \frac{\partial b}{\partial u} \neq 0$ and $g = c_2u + c_3$, where $c_2$ and $c_3$ are arbitrary constants. Then the densities of all nontrivial local conservation laws for (1) have the form

$$T^{II} = \tilde{Q}^{II} e^{-c_2t}u$$

where $\tilde{Q}^{II} = \tilde{Q}^{II}(x_1, \ldots, x_n)$ is any (smooth) solution of the linear Laplace equation

$$\Delta \tilde{Q}^{II} = 0.$$  \hspace{1cm} (7)

III. Let $f = \frac{\partial b}{\partial u} \neq 0$, $\frac{\partial^2 g}{\partial u^2} \neq 0$ and $\frac{\partial^2 g}{\partial u^2} \frac{\partial b}{\partial u} = \frac{\partial^2 g}{\partial u^2} \frac{\partial^2 b}{\partial u^2}$. Then there exist constants $c_4 \neq 0$ and $c_5$ such that $b = \frac{1}{c_4} \frac{\partial g}{\partial u} + c_5$, and the densities of all nontrivial local conservation laws for (1) have the form

$$T^{III} = e^{(c_4c_5-a_2^2)t}u\tilde{Q}^{III}$$

where $\tilde{Q}^{III} = \tilde{Q}^{III}(x_1, \ldots, x_n)$ is any (smooth) solution of a linear PDE with constant coefficients

$$\Delta \tilde{Q}^{III} + c_4\tilde{Q}^{III} = 0.$$  \hspace{1cm} (9)

The following result is readily verified by straightforward computation.

Corollary 2 The fluxes for the conservation laws listed in the above theorem have the following form, up to the obvious trivial contributions:

I. If $f = 0$, $b = c_1$, and $g = c_2u + c_3$, where $c_1, c_2, c_3$ are constants, then the fluxes associated with the conservation laws with densities of the form $T^I$ are

$$X^I_i = c_1 \left( \frac{1}{c_2} + u \right) \frac{\partial Q^I}{\partial x_i} - u_{x_i}Q^I + a \sum_{j=1}^{n} \left( \frac{1}{c_2} + u \right) \frac{\partial^2 Q^I}{\partial x_i \partial x_j} - u_{x_i} \frac{\partial^2 Q^I}{\partial x_j^2} + u_{x_j x_i} \frac{\partial Q^I}{\partial x_i} - u_{x_j x_i x_i}Q^I.$$  \hspace{1cm} (10)

II. If $f = \frac{\partial b}{\partial u} \neq 0$ and $g = c_2u + c_3$, where $c_2, c_3$ are constants, then the fluxes associated with the conservation laws with densities of the form $T^{II}$ are

$$X^{II}_i = -c_3x_i\tilde{Q}^{II} + \frac{c_3}{2} \frac{\partial \tilde{Q}^{II}}{\partial x_i} - \frac{c_3x_i}{2} \frac{\partial \tilde{Q}^{II}}{\partial x_i}.$$  \hspace{1cm} (11)

where $\tilde{b} = \tilde{b}(u)$ is such that $\frac{\partial \tilde{b}}{\partial u} = b$. 

3
III. If $f = \frac{\partial b}{\partial u} \neq 0$, $\frac{\partial^2 g}{\partial u^2} \neq 0$ and $\frac{\partial^3 g \ \partial b}{\partial u^3} = \frac{\partial^2 g \ \partial b}{\partial u^2} \partial u^2$, so $b = \frac{1}{c_4} \frac{\partial g}{\partial u} + c_5$ for suitable constants, then the fluxes associated with the conservation laws with densities of the form $T^I$ are

$$X^I_i = e^{(c_4c_5 - ac_4^2)w} \left( a\tilde{Q}^I u_{x_i} - c_5\tilde{Q}^I u_{x_i} + c_5u \frac{\partial \tilde{Q}^I}{\partial x_i} - \frac{1}{c_4} \tilde{Q}^I \frac{\partial g}{\partial u} u_i + \frac{1}{c_4} g \frac{\partial \tilde{Q}^I}{\partial x_i} + a \sum_{j=1}^{n} \left( \frac{\partial \tilde{Q}^I}{\partial x_j} u_{x_i x_j} - \tilde{Q}^I u_{x_i x_j} \right) \right).$$

(12)

3 Proof of the main results

Since the proofs of both Theorems 1 and 2 is based on the analysis of the same determining equation for the conservation law characteristics of (1) we will prove the theorems in question simultaneously.

Proof of Theorems 1 and 2. Let a differential function $Q$ be a characteristic of a conservation law for (1).

Equation (1) satisfies the conditions of Theorem 6 from [8] and hence $Q$ can depend at most on $t, x_1, \ldots, x_n$.

Then without loss of generality we can [12] assume that the density $T$ of the associated conservation law depends at most on $t, x_1, \ldots, x_n$ and $u$ and we have

$$\partial T/\partial u = Q,$$

(13)

whence we can readily find $T$ if given $Q$.

With this in mind it is readily verified using the results from [12] that $Q$ a characteristic of conservation law for (1) if and only if it satisfies the following equation:

$$\frac{\partial Q}{\partial t} + a \sum_{i,j=1}^{n} \frac{\partial^4 Q}{\partial x_i^2 \partial x_j^2} + b \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} + \left( 2 \frac{\partial b}{\partial u} - 2f \right) \sum_{i=1}^{n} \left( u_{x_i} Q + u_x \frac{\partial Q}{\partial x_i} \right) +$$

$$+ \left( \frac{\partial^2 b}{\partial u^2} - \frac{\partial f}{\partial u} \right) \sum_{i=1}^{n} u_{x_i}^2 Q + \frac{\partial g}{\partial u} Q = 0.$$  

(14)

As $Q$ is independent of $u$ and its derivatives, applying $\partial/\partial u_{x_i}$ to (14) for any $i$ we get

$$\left( 2 \frac{\partial b}{\partial u} - 2f \right) Q = 0,$$

(15)

which means that we must have

$$f = \frac{\partial b}{\partial u},$$

(16)

otherwise there exist no nontrivial conservation laws. This establishes part 1 of Theorem 1.

For the rest of the proof we assume that (16) holds.

Using (16) one can simplify equation (14) to

$$\frac{\partial Q}{\partial t} + a \sum_{i,j=1}^{n} \frac{\partial^4 Q}{\partial x_i^2 \partial x_j^2} + \frac{\partial g}{\partial u} Q + b \sum_{i=1}^{n} \partial^2 Q = 0.$$  

(17)

Upon applying $\partial/\partial u$ to (17) we get

$$\frac{\partial^2 g}{\partial u^2} Q + \frac{\partial b}{\partial u} \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0.$$  

(18)

We can split the analysis of (18) into three cases labelled as A, B and C.

Case A: $\frac{\partial b}{\partial u} = 0$, meaning that $b$ is constant and $f = 0$.

Then from (18) we get

$$\frac{\partial^2 g}{\partial u^2} Q = 0,$$

(19)
which means that we must have

$$\frac{\partial^2 g}{\partial u^2} = 0$$  \hspace{1cm} (20)$$

otherwise there exist no nontrivial conservation laws. This establishes part 2 from Theorem 1.

Setting \(b = c_1\) and \(g = c_2 u + c_3\), where \(c_1, c_2\) and \(c_3\) are arbitrary constants, we can simplify (17) to

$$\frac{\partial Q}{\partial t} + a \sum_{i,j=1}^{n} \frac{\partial^4 Q}{\partial x_i^2 \partial x_j^2} + c_2 Q + c_1 \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0.$$  \hspace{1cm} (21)$$

Now using (13) we can readily find the associated \(T\), and thus establish part I from Theorem 2. For convenience we denote in Theorem 2 the relevant \(Q\) and \(T\) as \(Q^I\) and \(T^I\) to indicate their relation to part I of the theorem in question, and adopt similar notation for parts II and III. Note that the way we used the residual freedom in the choice of the form of \(T^I\), making it inhomogeneous in \(u\), is motivated by the desire to keep the form of \(X^I\) reasonably simple.

**Case B:** \(\frac{\partial b}{\partial u} \neq 0\) but \(\frac{\partial^2 g}{\partial u^2} = 0\).

Then from (18) we get

$$\sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0.$$  \hspace{1cm} (22)$$

Setting \(g = c_2 u + c_3\) and using equation (22) we can simplify (17) to

$$\frac{\partial Q}{\partial t} + c_2 Q = 0.$$  \hspace{1cm} (23)$$

Solving these we get \(Q = e^{-c_2 t} \tilde{Q}\), where \(\tilde{Q}\) is a (smooth) function of independent variables \(x_1, \ldots, x_n\) which needs to satisfy the Laplace equation \(\Delta(\tilde{Q}) = 0\), establishing part II from Theorem 2 upon another application of (13).

**Case C:** \(\frac{\partial b}{\partial u} \neq 0\) and \(\frac{\partial^2 g}{\partial u^2} \neq 0\).

Then we can divide (18) by \(\frac{\partial b}{\partial u}\) and apply \(\frac{\partial}{\partial u}\), then we get

$$\frac{\partial^3 g}{\partial u^3} \frac{\partial b}{\partial u} - \frac{\partial^2 g}{\partial u^2} \frac{\partial^2 b}{\partial u^2} Q = 0,$$

$$\left(\frac{\partial b}{\partial u}\right)^2 Q = 0,$$  \hspace{1cm} (24)$$

which means that we must have

$$\frac{\partial^3 g}{\partial u^3} \frac{\partial b}{\partial u} = \frac{\partial^2 g}{\partial u^2} \frac{\partial^2 b}{\partial u^2}$$  \hspace{1cm} (25)$$

otherwise there exist no nontrivial conservation laws. This establishes part 3 from Theorem 1.

Assuming (25) holds, we can solve (25) for \(b\):

$$b = \frac{1}{c_4} \frac{\partial g}{\partial u} + c_5,$$  \hspace{1cm} (26)$$

where \(c_4, c_5\) are arbitrary constants, and simplify (18) to

$$Q + \frac{1}{c_4} \sum_{i=1}^{n} \frac{\partial^2 Q}{\partial x_i^2} = 0.$$  \hspace{1cm} (27)$$

Using (26) and (27) we can simplify (17) to

$$\frac{\partial Q}{\partial t} + (ac_4^2 - c_4 c_5) Q = 0.$$  \hspace{1cm} (28)$$

Hence \(Q = e^{-(ac_4^2 - c_4 c_5)t} \tilde{Q}\), where \(\tilde{Q}\) is a (smooth) function of independent variables \(x_1, \ldots, x_n\) which needs to satisfy \(\tilde{Q} + \frac{1}{c_4} \Delta \tilde{Q} = 0\). Making use of (13) yields the associated density \(T\), thus establishing part III of Theorem 2 and completing the proof. \(\square\)
Note that while for $n > 1$ in all three cases listed in Theorem 2 equation (1) admits infinitely many local conservation laws, for $n = 1$ the situation is strikingly different: in the case I there is still infinitely many nontrivial local conservation laws, while in the cases II and III there are just two, because for $n = 1$ equations (7) and (9) become linear ordinary differential equations of second order while (5) remains a linear partial differential equation in two independent variables. Notice also that for $n = 1$ the results of case I of Theorem 2 readily follow, up to a shift of $u$ by a suitable constant to turn the special case of (1) under study into a linear homogeneous PDE, from Theorem 3 of [15].

In fact, the presence of infinitely many local conservation laws in case I of Theorem 2 is not unexpected, because this is a degenerate case of sorts, when (1) becomes a linear inhomogeneous partial differential equation. Even for $n > 1$ there still is a significant difference between the cases in Theorem 2 as it is readily verified that for case I the associated infinite set of local conservation laws is parametrized by an arbitrary (smooth) function of $n$ independent variables $x_1, \ldots, x_n$ while for cases II and III the associated infinite sets of associated local conservation laws are parameterized by pairs of arbitrary (smooth) functions of $n – 1$ variables.

Acknowledgments

This research was supported in part by the Specific Research Grant SGS/13/2020 of Silesian University in Opava.

I would like to thank my advisor Artur Sergyeyev, for the patient guidance, encouragement and advice.

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