OUTER ACTIONS OF MEASURED QUANTUM GROUPOIDS

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Abstract. Mimicking a recent article of Stefaan Vaes, in which was proved that every locally compact quantum group can act outerly, we prove that we get the same result for measured quantum groupoids, with an appropriate definition of outer actions of measured quantum groupoids. This result is used to show that every measured quantum groupoid can be found from some depth 2 inclusion of von Neumann algebras.
1. Introduction

1.1. In two articles ([Val1], [Val2]), J.-M. Vallin has introduced two notions (pseudo-multiplicative unitary, Hopf-bimodule), in order to generalize, up to the groupoid case, the classical notions of multiplicative unitary [BS] and of Hopf-von Neumann algebras [ES] which were introduced to describe and explain duality of groups, and led to appropriate notions of quantum groups ([ES], [W1], [W2], [BS], [MN], [W3], [KV1], [KV2], [MNW]).

In another article [EV], J.-M. Vallin and the author have constructed, from a depth 2 inclusion of von Neumann algebras $M_0 \subset M_1$, with an operator-valued weight $T_1$ verifying a regularity condition, a pseudo-multiplicative unitary, which led to two structures of Hopf bimodules, dual to each other. Moreover, we have then constructed an action of one of these structures on the algebra $M_1$ such that $M_0$ is the fixed point subalgebra, the algebra $M_2$ given by the basic construction being then isomorphic to the crossed-product. We construct on $M_2$ an action of the other structure, which can be considered as the dual action.

If the inclusion $M_0 \subset M_1$ is irreducible, we recovered quantum groups, as proved and studied in former papers ([EN], [E2]).

Therefore, this construction leads to a notion of ”quantum groupoid”, and a construction of a duality within ”quantum groupoids”.

1.2. In a finite-dimensional setting, this construction can be mostly simplified, and is studied in [NV1], [BSz1], [BSz2], [Sz], [Val3], [Val4], [Val5], and examples are described. In [NV2], the link between these ”finite quantum groupoids” and depth 2 inclusions of $II_1$ factors is given, and in [D] had been proved that any finite-dimensional connected $C^*$-quantum groupoid can act outerly on the hyperfinite $II_1$ factor.

1.3. In [E3], the author studied, in whole generality, the notion of pseudo-multiplicative unitary introduced par J.-M. Vallin in [Val2]; following the strategy given by [BS], with the help of suitable fixed vectors, he introduced a notion of ”measured quantum groupoid of compact type”. Then F. Lesieur in [L], using the notion of Hopf-bimodule introduced in [Val1], then there exist a left-invariant operator-valued weight and a right-invariant operator-valued weight, mimicking in this wider setting the axioms and the technics of Kustermans and Vaes ([KV1], [KV2]), obtained a pseudo-multiplicative unitary, which, as in the quantum group case, ”contains” all the informations about the object (the von Neumann algebra, the coproduct, the antipod, the co-inverse). Lesieur gave the name of ”measured quantum groupoids” to these objects. A new set of axioms for these had been given in an appendix of [E5]. In [E4] had been shown that, with suitable conditions, the objects constructed in [EV] from depth 2 inclusions, are ”measured quantum groupoids” in the sense of Lesieur.

1.4. In [E5] have been developed the notions of action (already introduced in [EV]), crossed-product, etc, following what had been done for locally compact quantum groups in ([E1], [ES1], [V1]); a biduality theorem for actions had been obtained in ([E5], 11.6). Moreover, we proved in ([E5] 13.9) that, for any action of a measured quantum groupoid, the inclusion of the initial algebra (on which the measured quantum groupoid is acting) into the crossed-product is depth 2, which leads, thanks to [E4], to the construction of another measured quantum groupoid ([E5] 14.2). In [E6] was proved a generalization of Vaes’ theorem ([V1], 4.4) on the standard implementation of an action of a locally compact quantum group; namely, we had obtained such a result when there exists a normal semi-finite faithful operator-valued weight from the von Neumann algebra on which
the measured quantum groupoid is acting, onto the copy of the basis of this measured
quantum groupoid which is put inside this algebra by the action.

1.5. One question remained open: can any measured quantum groupoid be constructed
from a depth 2 inclusion? For locally compact quantum groups, the answer is positive
([E5] 14.9), but the most important step in that proof was Vaes’ [V2], who proved that
any locally compact quantum group has an outer action.

1.6. In that article, we answer positively to that question, and we follow the same
strategy than for locally compact groups: thanks to the construction of the measured
quantum groupoid associated to an action ([E5], 14.2), we show that this question is
equivalent to the existence of an outer action; to prove that last result, we mimick again
what was done in [V2], by proving that any measured quantum groupoid has a faithful
action. In [V2], it was constructed on some free product of factors; here we clearly
need to construct this action on an amalgated free product of von Neumann algebras, as
described, for instance, by Ueda [U].

1.7. This article is organized as follows:
In chapter 2, we recall very quickly all the notations and results needed in that article;
we have tried to make these preliminaries as short as possible, and we emphazise that
this article should be understood as the continuation of [E5] and [E6].
In chapter 3, we define outer actions of a measured quantum groupoid, and, in chapter 4,
faithful actions and minimal actions, and obtain links between faithful and outer actions.
In chapter 5, we construct an outer action of any measured quantum groupoid on some
amalgated free product of von Neumann algebras. Finally, in chapter 6, we study if and when it is possible for a measured quantum groupoid
to act outerly on a semi-finite (or finite) von Neumann algebra, or a finite factor.

2. Preliminaries

This article is the continuation of [E5]; preliminaries are to be found in [E5], and we
just recall herafter the following definitions and notations:

2.1. Spatial theory; relative tensor products of Hilbert spaces and fiber products
of von Neumann algebras ([C1], [S], [T], [EV]). Let \( N \) be a von Neumann
algebra, \( \psi \) a normal semi-finite faithful weight on \( N \); we shall denote by \( H_\psi, \mathfrak{H}_\psi, \ldots \) the
canonical objects of the Tomita-Takesaki theory associated to the weight \( \psi \); let \( \alpha \) be a
non degenerate faithful representation of \( N \) on a Hilbert space \( \mathcal{H} \); the set of \( \psi \)-bounded
elements of the left-module \( _\alpha\mathcal{H} \) is:

\[
D(\alpha, \mathcal{H}, \psi) = \{ \xi \in \mathcal{H}; \exists C < \infty, \|\alpha(y)\xi\| \leq C\|\Lambda_\psi(y)\|, \forall y \in \mathfrak{H}_\psi \}
\]

Then, for any \( \xi \) in \( D(\alpha, \mathcal{H}, \psi) \), there exists a bounded operator \( R^{\alpha, \psi}(\xi) \) from \( H_\psi \) to \( \mathcal{H} \),
defined, for all \( y \in \mathfrak{H}_\psi \) by:

\[
R^{\alpha, \psi}(\xi)\Lambda_\psi(y) = \alpha(y)\xi
\]

which intertwines the actions of \( N \).
If \( \xi, \eta \) are bounded vectors, we define the operator product

\[
<\xi, \eta>_{\alpha, \psi} = R^{\alpha, \psi}(\eta)^* R^{\alpha, \psi}(\xi)
\]

belongs to \( \pi_\psi(N)' \), which, thanks to Tomita-Takesaki theory, will be identified to the
opposite von Neumann algebra \( N^\circ \).
If \( y \in N \) is analytical with respect to \( \psi \), and if \( \xi \in D(\alpha H, \psi) \), then we get that \( \alpha(y)\xi \) belongs to \( D(\alpha H, \psi) \) and that:

\[
R^{\alpha,\psi}(\alpha(y)\xi) = R^{\alpha,\psi}(\xi)J_{\psi}\sigma_{\gamma}(y^*)J_{\psi}
\]

If now \( \beta \) is a non degenerate faithful antirepresentation of \( N \) on a Hilbert space \( K \), the relative tensor product \( K \otimes D(\alpha H, \psi) \) by the scalar product defined, if \( \xi_1, \xi_2 \) are in \( K \), \( \eta_1, \eta_2 \) are in \( D(\alpha H, \psi) \), by the following formula:

\[(\xi_1 \otimes \eta_1 | \xi_2 \otimes \eta_2) = (\beta(\eta_1, \eta_2 >_{\alpha,\psi})\xi_1 | \xi_2)\]

If \( \xi \in K \), \( \eta \in D(\alpha H, \psi) \), we shall denote \( \xi_1 \otimes \eta \) the image of \( \xi \otimes \eta \) into \( K \otimes N \), and, writing \( \rho^{\beta,\alpha}_\psi(\xi) = \xi \otimes \eta \), we get a bounded linear operator from \( \psi \) into \( \beta \otimes \alpha H \), which is equal to \( 1_K \otimes \rho^{\alpha,\psi}(\eta) \).

Changing the weight \( \psi \) will give an isomorphic Hilbert space, but the isomorphism will not exchange elementary tensors!

We shall denote \( \sigma_{\psi} \) the relative flip, which is a unitary sending \( K_{\beta \otimes \alpha} \) onto \( K_{\alpha \otimes \beta} \), defined, for any \( \xi \) in \( D(K_{\beta}, \psi) \), \( \eta \) in \( D(\alpha H, \psi) \), by:

\[\sigma_{\psi}(\xi \otimes \eta) = \eta \otimes \beta \xi \]

In \( x \in \beta(N)' \), \( y \in \alpha(N)' \), it is possible to define an operator \( x \beta \otimes \alpha y \) on \( K \beta \otimes \alpha H \), with natural values on the elementary tensors. As this operator does not depend upon the weight \( \psi \), it will be denoted \( x \beta \otimes \alpha y \). If \( P \) is a von Neumann algebra on \( H \), with \( \alpha(N) \subset P \), and \( Q \) a von Neumann algebra on \( K \), with \( \beta(N) \subset Q \), then we define the fiber product \( Q \beta_{\alpha} P \) of \( \{ x \beta \otimes \alpha y, x \in Q', y \in P' \}^\prime \).

It is straightforward to verify that, if \( Q_1 \) and \( P_1 \) are two other von Neumann algebras satisfying the same relations with \( N \), we have:

\[Q \beta_{\alpha} P \cap Q_1 \beta_{\alpha} P_1 = (Q \cap Q_1) \beta_{\alpha} (P \cap P_1)\]

In particular, we have:

\[Q \beta_{\alpha} \alpha(N) = (Q \cap \beta(N)') \beta \otimes \gamma \]

Moreover, this von Neumann algebra can be defined independently of the Hilbert spaces on which \( P \) and \( Q \) are represented; if \( i = 1, 2 \), \( \alpha_i \) is a faithful non degenerate homomorphism from \( N \) into \( P_i \), \( \beta_i \) is a faithful non degenerate antihomomorphism from \( N \) into \( Q_i \), and \( \Phi \) (resp. \( \Psi \)) an homomorphism from \( P_i \) to \( P_2 \) (resp. from \( Q_1 \) to \( Q_2 \)) such that \( \Phi \circ \alpha_1 = \alpha_2 \) (resp. \( \Psi \circ \beta_1 = \beta_2 \)), then, it is possible to define an homomorphism \( \Psi \beta_{\alpha} \Phi \) from \( Q_1 \beta_{\alpha} \alpha_1 P_1 \) into \( Q_2 \beta_{\alpha} \alpha_2 P_2 \).

The operators \( \theta^{\alpha,\psi}(\xi, \eta) = R^{\alpha,\psi}(\xi)R^{\alpha,\psi}(\eta)^* \), for all \( \xi, \eta \) in \( D(\alpha H, \psi) \), generates a weakly dense ideal in \( \alpha(N)' \). Moreover, there exists a family \( \{ e_i \} \) of vectors in \( D(\alpha H, \psi) \) such that the operators \( \theta^{\alpha,\psi}(e_i, e_i) \) are 2 by 2 orthogonal projections \( \theta^{\alpha,\psi}(e_i, e_i) \) being then the
projection on the closure of $\alpha(N)e_i)$. Such a family is called an orthogonal $(\alpha, \psi)$-basis of $\mathcal{H}$.

2.2. **Measured quantum groupoids ([L], [E5])**. A quintuplet $(N, M, \alpha, \beta, \Gamma)$ will be called a Hopf-bimodule, following ([Val2], [EV] 6.5), if $N, M$ are von Neumann algebras, $\alpha$ a faithful non-degenerate representation of $N$ into $M$, $\beta$ a faithful non-degenerate anti-representation of $N$ into $M$, with commuting ranges, and $\Gamma$ an injective involutive homomorphism from $M$ into $M_{\beta*\alpha}M$ such that, for all $X$ in $N$:

(i) $\Gamma(\beta(X)) = 1_{\beta \otimes_{\alpha} \beta(X)}$

(ii) $\Gamma(\alpha(X)) = \alpha(X)_{\beta \otimes_{\alpha} 1}$

(iii) $\Gamma$ satisfies the co-associativity relation:

$$\Gamma_{\beta*\alpha} \text{id} \Gamma = (\text{id}_{\beta*\alpha} \Gamma)\Gamma$$

This last formula makes sense, thanks to the two preceding ones and 2.1. The von Neumann algebra $N$ will be called the basis of $(N, M, \alpha, \beta, \Gamma)$.

If $(N, M, \alpha, \beta, \Gamma)$ is a Hopf-bimodule, it is clear that $(N^o, M, \beta, \alpha, \varsigma_N \circ \Gamma)$ is another Hopf-bimodule, we shall call the symmetrized of the first one. (Recall that $\varsigma_N \circ \Gamma$ is a homomorphism from $M$ to $M_{\beta*\alpha}M$).

If $N$ is abelian, $\alpha = \beta$, $\Gamma = \varsigma_N \circ \Gamma$, then the quadruplet $(N, M, \alpha, \alpha, \Gamma)$ is equal to its symmetrized Hopf-bimodule, and we shall say that it is a symmetric Hopf-bimodule.

A measured quantum groupoid is an octuplet $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ such that ([E5], 3.8):

(i) $(N, M, \alpha, \beta, \Gamma)$ is a Hopf-bimodule,

(ii) $T$ is a left-invariant normal, semi-finite, faithful operator valued weight $T$ from $M$ to $\alpha(N)$, which means that, for any $x \in \mathcal{M}_T^+$, we have $(\text{id}_{\beta*\alpha} T)\Gamma(x) = T(x)_{\beta \otimes_{\alpha} 1}$.

(iii) $T'$ is a right-invariant normal, semi-finite, faithful operator-valued weight $T'$ from $M$ to $\beta(N)$, which means that, for any $x \in \mathcal{M}_{T'}^+$, we have $(T'_{\beta*\alpha} \text{id})\Gamma(x) = 1_{\beta \otimes_{\alpha} T'}(x)$.

(iv) $\nu$ is normal semi-finite faithfull weight on $N$, which is relatively invariant with respect to $T$ and $T'$, which means that the modular automorphisms groups of the weights $\Phi = \nu \circ \alpha^{-1} \circ T$ and $\Psi = \nu \circ \beta^{-1} \circ T'$ commute.

We shall write $H = H_\nu$, $J = J_\Phi$, and, for all $n \in N$, $\tilde{\beta}(n) = J_\alpha(n)J$, $\tilde{\alpha}(n) = J_\beta(n^*)J$. The weight $\Phi$ will be called the left-invariant weight on $M$.

Examples are explained in 2.5 and 2.6.

Then, $\mathfrak{G}$ can be equipped with a pseudo-multiplicative unitary $W$ which is a unitary from $H_{\beta \otimes_{\alpha} H}$ onto $H_{\alpha \otimes_{\tilde{\beta}} H}$ ([E5], 3.6), which intertwines $\alpha$, $\tilde{\beta}$, $\beta$ in the following way: for all $X \in N$, we have:

$$W(\alpha(X)_{\beta \otimes_{\alpha} 1}) = (1_{\alpha \otimes_{\tilde{\beta}} \alpha(X)})W$$

$$W(1_{\beta \otimes_{\alpha} \beta(X)}) = (1_{\alpha \otimes_{\tilde{\beta}} \beta(X)})W$$

$$W(\tilde{\beta}(X)_{\beta \otimes_{\alpha} 1}) = (\tilde{\beta}(X)_{\alpha \otimes_{\tilde{\beta}} 1})W$$
and the operator \( W \) satisfies : 
\[
(1_{\beta} \otimes \beta W)_{N^o}(W \beta \otimes N^o(1_{\beta})) = (W_{\alpha} \otimes \beta)(1_{\beta} \otimes \alpha)_{N^o}(1_{\beta} \otimes \beta W)_{N^o}
\]

Here, \( \sigma_{\alpha, \beta} \) goes from \((H_{\alpha} \otimes \beta H)_{\alpha, \beta} H\) to \((H_{\beta} \otimes \alpha H)_{\alpha, \beta} H\), and \(1_{\beta} \otimes \alpha\) goes from \(H_{\beta} \otimes \alpha H\) to \(H_{\beta} \otimes \alpha H\).

All the intertwining properties allow us to write such a formula, which will be called the "pentagonal relation". Moreover, \( W, M \) and \( \Gamma \) are related by the following results :

(i) \( M \) is the weakly closed linear space generated by all operators of the form \((id \ast \omega_{\xi, \eta})(W)\), where \( \xi \in D(aH, \nu) \), and \( \eta \in D(H_{\beta}, \nu^o) \).

(ii) for any \( x \in M \), we have \( \Gamma(x) = W(1_{\alpha} \otimes \beta x)W \).

Moreover, it is also possible to construct many other data, namely a co-inverse \( R \), a scaling group \( \tau_t \), an antipod \( S \), a modulus \( \delta \), a scaling operator \( \lambda \), a managing operator \( P \), and a canonical one-parameter group \( \gamma_t \) of automorphisms on the basis \( N \). Instead of \( \mathfrak{G} \), we shall mostly use \((N, M, \alpha, \beta, \Gamma, T, R_{T}, \nu)\) which is another measured quantum groupoid, denoted \( \hat{\mathfrak{G}} \), which is equipped with the same data \((W, R, \ldots)\) as \( \mathfrak{G} \).

A dual measured quantum groupoid \( \hat{\mathfrak{G}} \), which is denoted \((N, \hat{M}, \alpha, \beta, \hat{\Gamma}, \hat{T}, \hat{R}_{RT}, \nu)\), can be constructed, and we have \( \hat{\mathfrak{G}} = \mathfrak{G} \).

Canonically associated to \( \mathfrak{G} \), can be defined also the opposite measured quantum groupoid \( \mathfrak{G}^o = (N^o, M, \beta, \alpha, \varsigma_N \Gamma, R_{T^r}, T, \nu^o) \) and the commutant measured quantum groupoid \( \mathfrak{G}^c = (N^c, M^c, \beta, \alpha, \Gamma^c, R^c, R^c T^c R^c, \nu^o) \); we have \((\mathfrak{G}^o)^c = (\mathfrak{G}^c)^o = \mathfrak{G} \), \( \hat{\mathfrak{G}}^o = (\hat{\mathfrak{G}})^c \), \( \hat{\mathfrak{G}}^c = (\hat{\mathfrak{G}})^o \), and \( \mathfrak{G}^{oo} = \mathfrak{G}^{cc} \) is canonically isomorphic to \( \mathfrak{G} \) ([E5], 3.12).

The pseudo-multiplicative unitary of \( \hat{\mathfrak{G}} \) (resp. \( \mathfrak{G}^o \), \( \mathfrak{G}^c \)) will be denoted \( \hat{W} \) (resp. \( W^o \), \( W^c \)). The left-invariant weight on \( \hat{\mathfrak{G}} \) (resp. \( \mathfrak{G}^o \), \( \hat{\mathfrak{G}}^c \)) will be denoted \( \hat{\nu} \) (resp. \( \nu^o \), \( \hat{\nu}^c \)).

Let \( _a\mathfrak{F}_b \) be a \( N - N \)-bimodule, i.e. a Hilbert space \( \mathfrak{F} \) equipped with a normal faithful non degenerate representation \( a \) of \( N \) on \( \mathfrak{F} \) and a normal faithful non degenerate antirepresentation \( b \) on \( \mathfrak{F} \), such that \( b(N) \subset a(N) \). A corepresentation of \( \mathfrak{G} \) on \( _a\mathfrak{F}_b \) is a unitary \( V \) from \( _a\mathfrak{F}_b \otimes \beta H \) onto \( _a\mathfrak{F}_b \otimes \alpha H \), satisfying, for all \( n \in N \) :

\[
V(b(n)_{\alpha} \otimes \beta \nu)_{N^o} = (1_{\beta} \otimes \alpha (\beta(n)))V
\]

\[
V(1_{\alpha} \otimes \beta \alpha (x))_{N^o} = (a(n)_{b} \otimes \alpha 1)V
\]

such that, for any \( \xi \in D(a\mathfrak{F}, \nu) \) and \( \eta \in D(b\mathfrak{F}, \nu^o) \), the operator \((\omega_{\xi, \eta} \ast \nu)(V)\) belongs to \( M \) (then, it is possible to define \((id \ast \theta)(V)\), for any \( \theta \in M_{\nu}^{\alpha, \beta} \) which is the linear set generated by the \( \omega_{\xi, \eta} \), with \( \xi \in D(a\mathfrak{F}, \nu) \cap D(b\mathfrak{F}, \nu^o) \)), and such that the application \( \theta \rightarrow (id \ast \theta)(V) \) from \( M_{\nu}^{\alpha, \beta} \) into \( L(\mathfrak{F}) \) is multiplicative ([E5] 5.1, 5.5).

2.3. Action of a measured quantum groupoid ([E5]). An action ([E5], 6.1) of \( \mathfrak{G} \) on a von Neumann algebra \( A \) is a couple \((b, a)\), where :

(i) \( b \) is an injective \( * \)-antihomomorphism from \( N \) into \( A \);

(ii) \( a \) is an injective \( * \)-homomorphism from \( A \) into \( A \).
(iii) \(b\) and \(a\) are such that, for all \(n\) in \(N\):

\[ a(b(n)) = 1_b \otimes a \beta(n) \]

(which allow us to define \(a \beta^* a\) \(id\) from \(A^{\beta^* a} M^{\beta^* a} M\) into \(A^{\beta^* a} M^{\beta^* a} M\)) and such that:

\[ (a \beta^* a \alpha) a = (id_b \otimes a \Gamma) a \]

The invariant subalgebra \(A^\omega\) is defined by:

\[ A^\omega = \{x \in A \cap b(N)^\prime; a(x) = x b \otimes a 1\} \]

Let us write, for any \(x \in A^+\), \(T_a(x) = (id_b \otimes a \Phi) a(x)\); this formula defines a normal faithful operator-valued weight from \(A\) onto \(A^\omega\); the action \(a\) will be said integrable if \(T_a\) is semi-finite ([E5], 6.11, 12, 13 and 14).

If the von Neumann algebra acts on a Hilbert space \(\mathfrak{H}\), and if there exists a representation \(\alpha\) of \(N\) on \(\mathfrak{H}\) such that \(b(N) \subset A \subset a(N)\)', a corepresentation \(V\) of \(\mathfrak{G}\) on the bimodule \(_a\mathfrak{H}_b\) will be called an implementation of \(a\) if we have \(a(x) = V(x \otimes b 1)\), for all \(x \in A\) ([E5], 6.6); moreover, if \(\psi\) is a normal semi-finite faithful weight on \(A\), and \(V\) an implementation of \(a\) on \(a(H\psi)_b\) (with \(a(n) = J\psi b(n^*) J\psi\)), \(V\) will be called a standard implementation of \(a\) if ([E5], 6.9):

\[ V^\ast = (J\psi \alpha^\otimes \beta J\tilde{\Phi}) V(J\psi b \otimes a J\tilde{\Phi}) \]

2.4. Crossed-product ([E5]). The crossed-product of \(A\) by \(\mathfrak{G}\) via the action \(a\) is the von Neumann algebra generated by \(a(A) \otimes b \otimes a M\)' ([E5], 9.1) and is denoted \(A \rtimes a \mathfrak{G}\); then there exists ([E5], 9.3) an integrable action \((1_b \otimes a \hat{\alpha}, \hat{a})\) of \((\hat{\mathfrak{G}})^c\) on \(A \rtimes a \mathfrak{G}\), called the dula action.

The biduality theorem ([E5], 11.6) says that the bicrossed-product \((A \rtimes a \mathfrak{G}) \rtimes a \mathfrak{G}^\circ\) is canonically isomorphic to \(A b^\ast a \mathcal{L}(H)\); more precisely, this isomorphism is given by:

\[ \Theta(a b^\ast a id)(A b^\ast a \mathcal{L}(H)) = (A \rtimes a \mathfrak{G}) \rtimes a \mathfrak{G}^\circ \]

where \(\Theta\) is the spatial isomorphism between \(\mathcal{L}(\hat{\mathfrak{H}}) b^\beta a \mathcal{L}(H)\) and \(\mathcal{L}(\hat{\mathfrak{H}}) b^\beta a \mathcal{L}(H)\) implemented by \(1_b \otimes a \sigma_{\nu} W^\alpha \sigma_{\nu}\); the biduality theorem says also that this isomorphism sends the action \((1_b \otimes a \hat{\alpha}, \hat{a})\) of \(\mathfrak{G}\) on \(A b^\ast a \mathcal{L}(H)\), defined, for any \(X \in A b^\ast a \mathcal{L}(H)\), by:

\[ \mathfrak{a}(X) = (1_b \otimes a \sigma_{\nu} W^\alpha \sigma_{\nu})(id \otimes a \leq a)(a b^\ast a \otimes id)(X)(1 \otimes a \sigma_{\nu} W^\alpha \sigma_{\nu})^\ast \]

on the bidual action (of \(\mathfrak{G}^{\circ}\)) on \((A \rtimes a \mathfrak{G}) \rtimes a \mathfrak{G}^\circ\);

We have \((A \rtimes a \mathfrak{G})^{\hat{a}} = a(A)\) ([E5] 11.5), and, therefore, the normal faithful semi-finite operator-valued weight \(T_{\hat{a}}\) sends \(A \rtimes a \mathfrak{G}\) onto \(a(A)\); therefore, starting with a normal semi-finite weight \(\psi\) on \(A\), we can construct a dual weight \(\tilde{\psi}\) on \(A \rtimes a \mathfrak{G}\) by the formula \(\tilde{\psi} = \psi \circ a^{-1} \circ T_{\hat{a}}\) ([E5] 13.2).

Moreover ([E5] 13.3), the linear set generated by all the elements \((1_b \otimes a a)(x)\), for all \(x \in \mathfrak{N}_\psi\), \(a \in \mathfrak{N}_{\hat{a}_c} \cap \mathfrak{N}_{Fe}\), is a core for \(\Lambda_{\tilde{\psi}}\), and it is possible to identify the GNS
representation of \( A \rtimes \mathfrak{G} \) associated to the weight \( \tilde{\psi} \) with the natural representation on 
\( H_{\psi} \mathbf{b} \otimes_{\nu} H_{\Phi} \) by writing 
\[
\Lambda_{\psi}(x) \mathbf{b} \otimes_{\nu} \Lambda_{\tilde{\psi}}(a) = \Lambda_{\tilde{\psi}}[\mathbf{1}_b \otimes \mathbf{a}(x)]
\]
which leads to the identification of \( H_{\tilde{\psi}} \) with \( H_{\psi} \mathbf{b} \otimes_{\nu} H \).

Then, the unitary \( U^a_{\psi} = J_{\psi}(J_{\psi} a \otimes_{\delta} J_{\Phi}) \) from \( H_{\psi} \mathbf{a} \otimes_{\nu} \mathbf{H} \) onto \( H_{\psi} \mathbf{b} \otimes_{\nu} H_{\Phi} \) satisfies 
\[
U^a_{\psi}(J_{\psi} \mathbf{b} \otimes_{\alpha} J_{\Phi}) = (J_{\psi} \mathbf{b} \otimes_{\alpha} J_{\Phi})(U^a_{\psi})^*
\]
and we have ([E5] 13.4):

(i) for all \( y \in A \):
\[
a(y) = U^a_{\psi}(y \mathbf{a} \otimes_{\ delta} \mathbf{1})(U^a_{\psi})^*
\]
(ii) for all \( b \in M \):
\[
(1_b \otimes_{\alpha} J_{\Phi} b J_{\Phi}) U^a_{\psi} = U^a_{\psi}(1 \mathbf{a} \otimes_{\ delta} J_{\Phi} b J_{\Phi})
\]
(iii) for all \( n \in N \):
\[
U^a_{\psi}(b(n) \mathbf{a} \otimes_{\ delta} \mathbf{1}) = (1_b \otimes_{\ alpha} \beta(n)) U^a_{\psi}
\]
\[
U^a_{\psi}(1 \mathbf{a} \otimes_{\ alpha} \alpha(n)) = (a(n) \mathbf{b} \otimes_{\ delta} \mathbf{1}) U^a_{\psi}
\]

Finally, if there exists a normal faithful semi-finite operator-valued weight \( \mathfrak{T} \) from \( A \) on \( b(N) \) such that \( \psi = \nu^a \circ b^{-1} \circ \mathfrak{T} \), then, we can prove ([E6] 5.7 and 5.8) that \( U^a_{\psi} \) is a corepresentation, and, therefore a standard implementation of \( a \).

2.5. Depth 2 Inclusions. Let \( M_0 \subset M_1 \) be an inclusion of \( \sigma \)-finite von Neumann algebras, equipped with a normal faithful semi-finite operator-valued weight \( T_1 \) from \( M_1 \) to \( M_0 \) (to be more precise, from \( M_1^+ \) to the extended positive elements of \( M_0 \) (cf. [T] IX.4.12)). Let \( \psi_0 \) be a normal faithful semi-finite weight on \( M_0 \), and \( \psi_1 = \psi_0 \circ T_1 \); for \( i = 0, 1 \), let \( H_i = H_{\psi_i} \), \( J_i = J_{\psi_i} \), \( \Delta_i = \Delta_{\psi_i} \) be the usual objects constructed by the Tomita-Takesaki theory associated to these weights. We shall write \( j_i \) for the mirroring on \( \mathcal{L}(H_i) \) defined by \( j_i(x) = J_i x^* J_i \). We shall write also \( j_1 \) for the restriction of the mirroring to \( M'_0 \cap M_2 \) (which is an anti-automorphism of \( M'_0 \cap M_2 \)), or for the restriction of the mirroring to \( M'_0 \cap M_1 \) (which is an injective anti-homomorphism from \( M'_0 \cap M_1 \) into \( M'_0 \cap M_2 \)).

Following ([J], 3.1.5(i)), the von Neumann algebra \( M_2 = J_1 M'_0 J_1 \) defined on the Hilbert space \( H_1 \) will be called the basic construction made from the inclusion \( M_0 \subset M_1 \). We have \( M_1 \subset M_2 \), and we shall say that the inclusion \( M_0 \subset M_1 \subset M_2 \) is standard.

Using Haagerup’s theorem ([T], 4.24), we can construct from \( T_1 \) another normal faithful semi-finite operator-valued weight \( T'_1 \) from \( M'_1 \) onto \( M'_1 \), and, by definition of \( M_2 \), a normal faithful semi-finite operator-valued weight \( T_2 \) from \( M_2 \) onto \( M_1 \); \( T_2 \) will be called the basic construction made from \( T_1 \); we can go on and construct \( M_2 \subset M_3 \) and \( T_3 \) by the basic construction made from \( M_1 \subset M_2 \) and \( T_2 \).

Following now ([GHJ] 4.6.4), we shall say that the inclusion \( M_0 \subset M_1 \) is depth 2 if the inclusion (called the derived tower):
\[
M'_0 \cap M_1 \subset M'_0 \cap M_2 \subset M'_0 \cap M_3
\]
is also standard, and, following ([EN], 11.12), we shall say that the operator-valued weight \( T_1 \) is regular if both restrictions \( T'_2 = T_{2|M'_0 \cap M_2} \) and \( T'_3 = T_{2|M'_0 \cap M_3} \) are semi-finite.

In [EV] was proved that, with such an hypothesis, there exists a coproduct \( \Gamma \) from \( M'_0 \cap M_2 \)
into $(M'_0 \cap M_2)_{j_1 \ast id} (M'_0 \cap M_2)$ (where here $id$ means the injection of $M'_0 \cap M_1$ into $M'_0 \cap M_2$) such that $(M'_0 \cap M_1, M'_0 \cap M_2, id, j_1, \Gamma)$ is a Hopf-bimodule; moreover, $\tilde{T}_2$ is then a left-invariant weight, and $j_1 \circ \tilde{T}_2 \circ j_1$ a right-invariant weight; if there exists a normal faithful semi-finite weight $\chi$ on $M'_0 \cap M_1$ invariant under the modular automorphism group $\sigma_1^{T_1}$ ([E4] 8.2 and 8.3), we get that:

$$\mathfrak{G}_1 = (M'_0 \cap M_1, M'_0 \cap M_2, id, j_1, \Gamma, \tilde{T}_2, j_1 \circ \tilde{T}_2 \circ j_1, \chi)$$

is a measured quantum groupoid. We shall write $\mathfrak{G}_1 = \mathfrak{G}(M_0 \subset M_1)$.

Moreover, the inclusion $M_1 \subset M_2$ satisfies the same hypothesis, and leads to another measured quantum groupoid $\mathfrak{G}_2$, which can be identified with $\mathfrak{G}_1^{\sigma_1}$, and there exists a canonical action $a$ of $\mathfrak{G}_2$ on $M_1$ ([EV], 7.3), which can be described as follows: the anti-representation of the basis $\mathcal{M}'_1 \cap M_2$ (which, using $j_1$, is anti-isomorphic to $M'_0 \cap M_1$), is given by the natural inclusion of $M'_0 \cap M_1$ into $M_1$, and the homomorphism from $M_1$ is given by the natural inclusion of $M_1$ into $M_3$ (which is, thanks to ([EV], 4.6), isomorphic to $M_1 \times_{\alpha} \mathcal{L}(H_{\chi_2})$, where $\chi_2 = \chi \circ \tilde{T}_2$). We then get that $M_0 = M_1^a$ and that $M_2$ is isomorphic to $M_1 \times_{\alpha} \mathfrak{G}_2$ ([EV], 7.5 and 7.6).

So, from a depth 2 inclusion $M_0 \subset M_1$ equipped with a regular operator-valued weight, and an invariant weight on the first relative commutant, one can construct a measured quantum groupoid $\mathfrak{G}_3$, given, in fact, by a specific action $a$ of $\mathfrak{G}_2$ on $M_1$, with $M_0$ being the invariant elements under this action.

If $\mathfrak{G}$ is any measured quantum groupoid, and $(b, a)$ an action of $\mathfrak{G}$ on a von Neumann algebra $A$; then the inclusion $a(A) \subset A \times_a \mathfrak{G}$ is depth 2 ([E5], 13.9), and the operator-valued weight $T_\alpha$ is regular ([E5], 13.10); so, we can construct a Hopf-bimodule from this depth 2 inclusion, equipped with a left-invariant operator-valued weight and a right-invariant operator-valued weight; moreover, if there exists a weight $\chi$ on $A \times_a \mathfrak{G} \cap a(A)'$, invariant by $\sigma_1^{T_\alpha}$, we get another measured quantum groupoid $\mathfrak{G}(a) = \mathfrak{G}(a(A) \subset A \times_a \mathfrak{G})$ ([E5], 14.2), which contains, in a sense, $\mathfrak{G}^{oc}$ ([E5], 14.7).

More precisely, the inclusion $a(A) \subset A \times_a \mathfrak{G} \subset A_{b \ast a} \mathcal{L}(H)$ is standard, and, if we write $B = A \times_a \mathfrak{G} \cap a(A)'$ and $b = a_B$, the derived inclusion $B \subset A_{b \ast a} \mathcal{L}(H) \cap a(A)'$ is isomorphic to $b(B) \subset B \times_b \mathcal{G}^c$ ([E5], 13.9), and there exist a $\ast$-anti-automorphism $j_1$ of $B \times_b \mathcal{G}^c$ and a coproduct $\Gamma_1$ such that ([E5] 14.2):

$$\mathfrak{G}(a) = (B, B \times_b \mathcal{G}^c, b, j_1 \circ b, \Gamma_1, T_b, j_1 \circ T_b \circ j_1, \chi)$$

2.6. **Examples of measured quantum groupoids.** Examples of measured quantum groupoids are the following:

(i) locally compact quantum groups, as defined and studied by J. Kustermans and S. Vaes ([KV1], [KV2], [V1]); these are, trivially, the measured quantum groupoids with the basis $N = \mathbb{C}$.

(ii) measured groupoids, equipped with a left Haar system and a quasi-invariant measure on the set of units, as studied mostly by T. Yamanouchi ([Y1], [Y2], [Y3], [Y4]); it was proved in [E6] that these measured quantum groupoids are exactly those whose underlying von Neumann algebra is abelian.

(iii) the finite dimensional case had been studied by D. Nikshych and L. Vainermann ([NV1], [NV2]) and J.-M. Vallin ([Val3], [Val4]); in that case, non trivial examples are given.

(iv) continuous fields of $(\mathbb{C}^\ast$-version of) locally compact quantum groups, as studied by E.
Blanchard in ([B11], [B12]); it was proved in [E6] that these measured quantum groupoids are exactly those whose basis is central in the underlying von Neumann algebras of both the measured quantum groupoid and its dual.

(v) in ([L], 17.1), be given a family $\mathfrak{G}_i = (N_i, M_i, \alpha_i, \beta_i, \Gamma_i, T_i, T'_i, \nu_i)$ a measured quantum groupoids, Lesieur showed that it is possible to construct another measured quantum groupoid $\mathfrak{G} = \bigoplus_{i \in \mathcal{I}} \mathfrak{G}_i = (\bigoplus_{i \in \mathcal{I}} N_i, \bigoplus_{i \in \mathcal{I}} M_i, \bigoplus_{i \in \mathcal{I}} \alpha_i, \bigoplus_{i \in \mathcal{I}} \beta_i, \bigoplus_{i \in \mathcal{I}} \Gamma_i, \bigoplus_{i \in \mathcal{I}} T_i, \bigoplus_{i \in \mathcal{I}} T'_i, \bigoplus_{i \in \mathcal{I}} \nu_i)$.

(vi) in [DC], K. De Commer proved that, in the case of a monoidal equivalence between two locally compact quantum groups (which means that each of these locally compact quantum group as an ergodic and integrable action on the other one), it is possible to construct a measurable quantum groupoid of basis $\mathbb{C}^2$ which contains all the data. Moreover, this construction was useful to prove new results on locally compact quantum groups, namely on the deformation of a locally compact quantum group by a unitary 2-cocycle; he proved that these measured quantum groupoids are exactly those whose basis $\mathbb{C}^2$ is central in the underlying von Neumann algebra of the measured quantum groupoid, but not in the underlying von Neumann algebra of the dual measured quantum groupoid.

(vi) in 2.5 was described how, from an action $(b, a)$ of a measured quantum groupoid $\mathfrak{G}$, it is possible to construct another measured quantum groupoid $\mathfrak{G}(a)$; as a particular case, this allows to canonically associate to any action $a$ of a locally compact quantum group $\mathfrak{G}$ on a von Neumann algebra $A$, a measured quantum groupoid $\mathfrak{G}(a)$.

3. Outer actions of a measured quantum groupoid

In this chapter, we define (3.2) outer actions of a measured quantum groupoid, and prove (3.9) that a measured quantum groupoid can be constructed by a geometric construction from a depth 2 inclusion if and only if it has an outer action on some von Neumann algebra.

3.1. Theorem. Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum group, and $(b, a)$ an action of $\mathfrak{G}$ on a von Neumann algebra $A$; then, are equivalent:

(i) $A \rtimes_a \mathfrak{G} \cap a(A)' = 1_b \otimes_a \hat{\alpha}(N)$;

(ii) $A_b \rtimes_a \mathcal{L}(H) \cap a(A)' = 1_b \otimes_a M'$;

(iii) it is possible to define the measured quantum groupoid $\mathfrak{G}(a)$, and $\mathfrak{G}(a) = \mathfrak{G}^{oc}$.

Proof. Let us suppose (i); using then ([E5], 14.1 (iii)), we get (ii).

Let us suppose (ii); using ([E5], 14.7), we see the application $x \mapsto 1_b \otimes_a x$ from $M'$ onto $A_b \rtimes_a \mathcal{L}(H) \cap a(A)'$ is an isomorphism of Hopf-bimodules, from $\mathfrak{G}^{oc}$ onto the Hopf-bimodule constructed from the depth 2 inclusion $a(A) \subset A \rtimes_a \mathfrak{G}$, and sends the left- (resp. right-) invariant operator-valued weights of $\mathfrak{G}^{oc}$ on the left (resp. right-) invariant operator-valued weights of Hopf-bimodule constructed from the depth 2 inclusion $a(A) \subset A \rtimes_a \mathfrak{G}$; therefore, we get that the weight $\nu$ is invariant under $\sigma_{T,a}^T$, which means that we can define the measured quantum groupoid $\mathfrak{G}(a)$, and that $\mathfrak{G}(a) = \mathfrak{G}^{oc}$, which is (iii). Let us suppose (iii); the application $x \mapsto 1_b \otimes_a x$ from $M'$ onto $A_b \rtimes_a \mathcal{L}(H) \cap a(A)'$ is an isomorphism between $\mathfrak{G}^{oc}$ and $\mathfrak{G}(a)$; in particular, we get (i). \qed

3.2. Definition. Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum group, and $(b, a)$ an action of $\mathfrak{G}$ on a von Neumann algebra $A$; we shall say that the action $(b, a)$ is outer if it satisfies one of the equivalent conditions of 3.1.
3.3. Theorem. Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum group, and $(b, a)$ an outer action of $\mathfrak{G}$ on a von Neumann algebra $A$; then, the dual action of the measured quantum groupoid $\hat{\mathfrak{G}}^c$ on the crossed product $A \rtimes_a \mathfrak{G}$ is outer.

Proof. Let us put the von Neumann algebra $A$ on its standard Hilbert space $L^2(A)$; we have, using 3.1 and ([E5], 3.11):

$$A_{b *^M_N} L(H) \cap (A \rtimes_a \mathfrak{G})' = A_{b *^M_N} L(A)^\prime \cap L(L^2(A))_{b *^M_N} \hat{M}$$

$$= 1_b \otimes a_{N} M' \cap L(L^2(A))_{b *^M_N} \hat{M}$$

$$= 1_b \otimes a_{N} M' \cap \hat{M}$$

$$= 1_b \otimes \hat{\beta}(N)$$

from which we get the result, using 3.1 again. \hfill \Box

3.4. Proposition. Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum group, and $(b, a)$ an outer action of $\mathfrak{G}$ on a von Neumann algebra $A$; then, we have:

$$Z(A) = \{b(n), n \in Z(N), \beta(n) \in Z(M)\}$$

Moreover, we have:

$$Z(A \rtimes_a \mathfrak{G}) = \{1_b \otimes a_{N} \hat{\alpha}(n), \alpha(n) \in Z(\hat{M})\}$$

Proof. As we have $a(Z(A)) \subset A \rtimes_a \mathfrak{G} \cap a(A)'$, we get that, for any $z \in Z(A)$, there exists $n \in N$ such that $a(z) = 1_b \otimes a_{N} \hat{\alpha}(n)$. But, we then infer that $\hat{\alpha}(n)$ belongs to $M \cap M' \cap \hat{M}'$; therefore, we have $\hat{\alpha}(n) = \beta(n) \in Z(M)$, $n \in Z(N)$, and $a(z) = 1_b \otimes a_{N} \beta(n) = a(b(n))$, from which we get that

$$Z(A) \subset \{b(n), n \in Z(N), \beta(n) \in Z(M)\}$$

Conversely, if $n \in Z(N)$, such that $\beta(n) \in Z(M)$, we get that $a(b(n)) = 1_b \otimes a_{N} \beta(n)$ commutes with all elements $a(x) \in A_{b *^M_N} M$, for any $x \in A$; therefore, we get that $b(n) \in Z(A)$. Applying this result to the outer action $\hat{a}$, we get that

$$Z(A \rtimes_a \mathfrak{G}) = \{1_b \otimes a_{N} \hat{\alpha}(n), \hat{\alpha}(n) \in Z(\hat{M})\}$$

and, as $\hat{\alpha}(n) = J \hat{\alpha}(n) \hat{J}$ ([E5], 3.11), we get the result. \hfill \Box

3.5. Corollary and Definition. Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum group, and $(b, a)$ an outer action of $\mathfrak{G}$ on a von Neumann algebra $A$;

(i) the algebra $A$ is a factor if and only if we have:

$$\{n \in N, \alpha(n) \in Z(M)\} = \{n \in N, \beta(n) \in Z(M)\} = \alpha(N) \cap \hat{\beta}(N) = \mathbb{C}$$

Such a measured quantum groupoid is called connected. Then, the scaling operator of $\mathfrak{G}$ is a scalar.

(ii) $A \rtimes_a \mathfrak{G}$ is a factor if and only if $\hat{\mathfrak{G}}$ is connected.

(iii) both $A$ and $A \rtimes_a \mathfrak{G}$ are factors if and only if both $\mathfrak{G}$ and $\hat{\mathfrak{G}}$ are connected; then, we shall say that $\mathfrak{G}$ is biconnected. We have then:

$$\alpha(N) \cap \beta(N) = \alpha(N) \cap \hat{\beta}(N) = \mathbb{C}$$
Proof. We clearly have, by definition of \( \beta \), that :

\[
\alpha(N) \cap \beta(N) \subset \{ n \in N, \alpha(n) \in Z(M) \} = \{ n \in N, \alpha(n) = \beta(n) \} \subset \alpha(N) \cap \beta(N)
\]

moreover, using the co-inverse \( R \), it is clear that \( \{ n \in N, \alpha(n) \in Z(M) \} = \mathbb{C} \) is equivalent to \( \{ n \in N, \beta(n) \in Z(M) \} = \mathbb{C} \); then, the first part of (i) is given by 3.4. In that situation, we get immediately that the scaling operator of \( G \), which belongs to \( \alpha(N) \cap Z(M) \), must be a scalar, which finishes the proof of (i).

By applying (i) to the action \( \alpha \) of \( \bar{G}^\circ \) (which is outer by 3.3), we get (ii), and (i) and (ii) give (iii). \( \square \)

3.6. Corollary. Let \( \mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu) \) be a measured quantum group, such that \( \alpha(N) \subset Z(M) \), and \( b, a \) an outer action of \( \mathcal{G} \) on a von Neumann algebra \( A \); then, we have \( Z(A) = b(N) \); let us write \( N = L^\infty(X, \nu) \); the von Neumann algebra \( A \) is decomposable and can be written \( A = \int_X A^x d\nu(x) \), and, for \( \nu \) almost all \( x \in X \), the algebras \( A^x \) are factors.

Proof. If \( \alpha(N) \subset Z(M) \), we have, using the co-inverse \( R \), that \( \beta(N) \subset Z(M) \), and, therefore, using 3.4, \( Z(A) \subset b(N) \), and, as \( \alpha(b(n)) = 1 \otimes a \beta(n) \) commutes with \( A \otimes a M \), and, therefore, with \( \alpha(A) \), we get that \( b(N) \subset Z(A) \), which finishes the proof. \( \square \)

3.7. Example. Let \( M_0 \subset M_1 \) a depth 2 inclusion, equipped with a regular operator-valued weight \( T_1 \) from \( M_1 \) onto \( M_0 \), and a normal semi-finite faithful weight \( \chi \) on \( M_0 \cap M_1 \), invariant under \( \sigma^T_1 \); let us use all notations of 2.5. There exists a measured quantum groupoid \( \mathcal{G}_2 \) and an action \( a \) of \( \mathcal{G}_2 \) on \( M_1 \), with \( M_0 = M_1^a \). Then, this action \( a \) is outer: in fact, the crossed-product \( M_1 \rtimes_a \mathcal{G}_2 \) is isomorphic with \( M_2 \), and this isomorphism, described in ([EV] 7.6), sends \( a(M_1) \) on \( M_1 \), and \( 1 \otimes a \hat{\alpha}(n) \) on \( n \), for any \( n \in M_1 \cap M_2 \).

Here \( b \) is the restriction of the mirroring \( j_1 \) to \( M_1 \cap M_2 \), which sends the basis \( M_1 \cap M_2 \) on \( M_0 \cap M_1 \), \( \alpha \) is the injection of \( M_1 \cap M_2 \) into \( M_1 \cap M_3 \), and \( \hat{\alpha} \) is the restriction to \( M_1 \cap M_2 \) of the standard representation of \( M_0 \cap M_2 \).

3.8. Example. (i) Let \( G \) be a locally compact quantum group; then an action of \( G \) is outer (in the sense of 3.2) if and only if it is strictly outer in the sense of Vaes ([V2] 2.5).

(ii) let \( \mathcal{G}_i \) be a family of measured quantum groupoids, and \( (b_i, a_i) \) an action of \( \mathcal{G}_i \) on a von Neumann algebra \( A_i \). Let us construct \( \mathcal{G} = \bigoplus_{i \in I} \mathcal{G}_i \) (2.6(\nu)); then, let us define \( b = \bigoplus_{i \in I} b_i \), which will be an injective \( * \)-antihomomorphism from \( \bigoplus_{i \in I} N_i \) into \( \bigoplus_{i \in I} A_i \), and \( a = \bigoplus_{i \in I} a_i \), which will be an injective \( * \)-homomorphism from \( \bigoplus_{i \in I} A_i \) into \( \bigoplus_{i \in I} (A_i b_i * a_i, M_i) = (\bigoplus_{i \in I} A_i) b_i * a_i M \), where \( \alpha = \bigoplus_{i \in I} \alpha_i \) and \( M = \bigoplus_{i \in I} M_i \); then \( (b, a) \) is an action of \( \mathcal{G} \) on \( \bigoplus_{i \in I} A_i \), and this action is outer if and only if all the actions \( a_i \) are outer.

3.9. Theorem. Let \( \mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu) \) be a measured quantum group; then, are equivalent :

(i) there exists a depth 2 inclusion \( M_0 \subset M_1 \), equipped with a regular operator-valued weight \( T_1 \) from \( M_1 \) onto \( M_0 \), and a normal semi-finite faithful weight \( \chi \) on \( M_0 \cap M_1 \), invariant under \( \sigma^T_1 \), such that \( \mathcal{G} = \mathcal{G}(M_0 \subset M_1) \);

(ii) there exists a von Neumann algebra \( A \), and \( (b, a) \) a outer action of \( \mathcal{G} \) on \( A \).

Proof. Let us suppose (i); let \( M_0 \subset M_1 \subset M_2 \subset \ldots \) be Jones’ tower associated to the inclusion \( M_0 \subset M_1 \); then (2.5), \( \hat{\mathcal{G}}^o = \mathcal{G}(M_1 \subset M_2) \), and, therefore, \( \mathcal{G}^{oc} = \mathcal{G}(M_2 \subset M_3) \), and \( \mathcal{G} = \mathcal{G}(M_4 \subset M_5) \). Applying 3.7 to the inclusion \( M_3 \subset M_4 \), we get (ii).
Let us suppose (ii); then, using 3.1, we have $\mathcal{G}^{oc} = \mathcal{G}(a) = \mathcal{G}(a(A) \subset A \rtimes_a \mathcal{G})$. Using ([E5] 13.9), we get that Jones’ tower associated to the incusion $a(A) \subset A \rtimes_a \mathcal{G}$ is:

$$a(A)_{\alpha} \otimes \beta L_{\mathcal{N}} 1 \subset \tilde{\alpha}(A \rtimes_a \mathcal{G}) \subset (A \rtimes_a \mathcal{G}) \rtimes_a \tilde{\mathcal{G}}_c \subset (A \rtimes_a \mathcal{G})_{\alpha} \otimes \beta L_{\mathcal{H}}$$

and, therefore, we get that $\mathcal{G} = \mathcal{G}[(A \rtimes_a \mathcal{G}) \rtimes_a \tilde{\mathcal{G}}_c \subset (A \rtimes_a \mathcal{G})_{\alpha} \otimes \beta L_{\mathcal{H}}]$, which gives (i).

$\square$

4. Faithful actions of a measured quantum groupoid

In this chapter, we define faithful actions of a measured quantum groupoid (4.1), and we prove some links between faithful and outer actions (4.4, 4.8).

4.1. Definition. Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum group, and $(b, a)$ an action of $\mathcal{G}$ on a von Neumann algebra $A$; we shall say that the action $(b, a)$ is faithful if

$$\{(\omega_{\eta, \beta, \alpha} \ast id) a(x), \eta \in D(L_2(A)_b, \nu^o), x \in A\}^\prime = M$$

We shall say that $(b, a)$ is minimal if it is faithful and if $A \cap (A^a)' = b(N)$.

4.2. Example. Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum group, $(\beta, \Gamma)$ the action of $\mathcal{G}$ on $M$ defined in ([E5], 6.10). Using [E5] 3.6 (ii) and 3.8 (vii), we get that the von Neumann algebra generated by the set $\{(\omega_{\eta, \beta, \alpha} \ast id) \Gamma(x), \eta \in D(H_\beta, \nu^o), x \in M\}$ is equal to $M$, which says that $(\beta, \Gamma)$ is faithful.

4.3. Proposition. Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum group, and $(b, a)$ an action of $\mathcal{G}$ on a von Neumann algebra $A$; let $A_1$ be a von Neumann subalgebra of $A$ such that $b(N) \subset A_1 \subset A$, and such that $a(A_1) \subset A_1 b^* a M$; therefore $(b, a|_{A_1})$ is an action of $\mathcal{G}$ on $A_1$; moreover, if $(b, a|_{A_1})$ is faithful, then $(b, a)$ is faithful.

Proof. Trivial. $\square$

4.4. Proposition. Let $\mathcal{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum group, and $(b, a)$ a minimal action of $\mathcal{G}$ on a von Neumann algebra $A$; then $(b, a)$ is an outer action.

Proof. Let $z \in A_{b^* a} \mathcal{L}(H) \cap (a(A))'$; then, using 4.1 and 2.1, $z$ belongs to:

$$A_{b^* a} \mathcal{L}(H) \cap (A^a b \otimes_{\alpha} 1_H)' = A_{b^* a} \mathcal{L}(H) \cap (A^a)' b^* a \mathcal{L}(H)$$

$$= A \cap (A^a)' b^* a \mathcal{L}(H) = b(N) b^* a \mathcal{L}(H) = 1 b \otimes_{\alpha} \alpha(N)'$$

So, there exists $y \in \alpha(N)'$ such that $z = 1 b \otimes_{\alpha} y$. But, as $z$ commutes with $a(A)$, we get that $y$ commutes with all elements of the form $((\omega_{\eta, b^* a} \ast id) a(x))$, for all $\eta \in D(L_2(A)_b, \nu^o)$ and $x \in A$. Therefore, by 4.1, we get that $y \in M'$, which finishes the proof, by 3.1. $\square$
4.5. **Definition.** Let $\mathfrak{G}$ be a measured quantum groupoid, $A$ be a von Neumann algebra, and $\theta$ be a normal faithful state on $A$; let us denote $id_N$ the canonical anti-homomorphism from $N$ into $N^\circ \otimes B$, and $id_A$ the identity of $A$; then, as the fiber product $(A \otimes N^\circ) \ast_{\alpha} M$ can be identified with $A \otimes (M \cap \alpha(N^\circ))$, we get that $(id_N, id_A \otimes \beta)$ is an action of $\mathfrak{G}$ on $A \otimes N^\circ$, we shall call the trivial action of $\mathfrak{G}$ on $A \otimes N^\circ$; this generalizes the example ([E5], 6.2). If $\theta$ denote a faithful state on $A$, we shall denote $E_\theta$ the normal faithful conditional expectation from $A \otimes N^\circ$ onto $N^\circ$ given by the slice map $\theta \otimes id_N$; this conditional expectation satisfies $(E_\theta \circ id_N, id_A) = (id_A \otimes \beta) \circ E_\theta$. Moreover, we have, trivially $(A \otimes N^\circ) id_A \otimes \beta = A \otimes C$.

4.6. **Proposition.** Let $\mathfrak{G}$ be a measured quantum groupoid; for $i = 1, 2$, let $(b_i, a_i)$ be an action of $\mathfrak{G}$ on a von Neumann algebra $A_i$: let us suppose that there exists a normal faithful conditional expectation $E_i$ from $A_i$ onto $b_i(N)$, invariant under $a_i$, i.e. ([E6], 7.6) such that $(E_i \circ b_i \ast_{\alpha} id)a_i = a_i \circ E_i$. Then, there exists an action $(b, a)$ of $\mathfrak{G}$ on the amalgamated free product $A_1 \star_{N^\circ} A_2$, as defined in ([U], 2), where $b$ is the anti-isomorphism from $N$ into the canonical subalgebra $b(N)$ of $A_1 \star_{N^\circ} A_2$, and $a_i$ is given by the composition of the isomorphism $a_1 \star a_2$ from $A_1 \star_{N^\circ} A_2$ onto $a_1(A_1) \star_{N^\circ} a_2(A_2)$ constructed, as ([U], p.366), using ([U], 2.5), and the inclusion:

$$a_1(A_1) \star_{N^\circ} a_2(A_2) \subset (A_1 \star_{N^\circ} A_2) b_i \ast_{\alpha} M$$

which is given by the formulae $(E_i \circ b_i \ast_{\alpha} id)a_i = a_i \circ E_i$. For any $x_i \in A_i$, considered as a subalgebra of $A_1 \star_{N^\circ} A_2$, we have $a(x_i) = a_i(x_i)$.

**Proof.** The construction of the application $a$ is an application of ([U] 2.5). Then, it is straightforward to get it is an action by verifying it on each $A_i$. \hfill $\Box$

4.7. **Definition.** Let $\mathfrak{G}$ be a measured quantum groupoid, and let $(b, a)$ be a faithful action of $\mathfrak{G}$ on a von Neumann algebra $A$; let us suppose that there exists a normal faithful conditional expectation $E$ from $A$ onto $b(N)$, invariant under $a$. Moreover, let $A$ be a von Neumann algebra, and $\theta$ a normal faithful state on $A$, and let us consider the trivial action on $\mathfrak{G}$ on $A \otimes N^\circ$, as defined in 4.5.

Let us construct now the action $a_1$ of $\mathfrak{G}$ on the amalgamated free product $A \star_{N^\circ} (A \otimes N^\circ)$ of $A$ over its subalgebra $b(N)$ with $(A \otimes N^\circ)$ over its subalgebra $N^\circ$, constructed as in 4.6 using the normal faithful conditional expectation $E$ from $A$ onto $b(N)$ and the normal faithful conditional expectation $E_\theta$ from $A \otimes N^\circ$ onto $N^\circ$ defined in 4.5, which are invariant, respectively, towards the action $a$ and the trivial action.

As the action $a$ is a restriction of $a_1$ and is faithful, we get that the action $a_1$ is faithful also. Moreover, we have trivially $A \otimes C \subset [A \star_{N^\circ} (A \otimes N^\circ)]^a$.

4.8. **Theorem.** Let $\mathfrak{G}$ be a measured quantum groupoid; let us suppose that $\mathfrak{G}$ has a faithful action on a von Neumann algebra $A$, such that there exists a normal faithful conditional expectation $E$ from $A$ onto $b(N)$, invariant under $a$; then $\mathfrak{G}$ has an outer action.

**Proof.** Let’s use Barnett’s result ([Ba], th. 2); using the notations of 4.6, let us take $A = (A_1, \theta_1) \star (A_2, \theta_2)$, each $\theta_i$ being a faithful state on the von Neumann algebra $A_i$, and let us suppose that there exists $a$ in the centralizer $A_1^{\theta_1}$ such that $\theta_1(a) = 0$, and $b, c$
in the centralizer $A_2^\omega$ such that $\theta_2(b) = \theta_2(c) = \theta_2(b^*c) = 0$; let’s use the normal faithful conditional expectations $(\theta_1 \otimes id)$ from $A_1 \otimes N^\circ$ onto $N^\circ$ and $(\theta_2 \otimes id)$ from $A_2 \otimes N^\circ$ onto $N^\circ$; it is straightforward to get that the amalgated free product $(A_1 \otimes N^\circ) \boxplus_N (A_2 \otimes N^\circ)$ is equal to $A \otimes N^\circ$, which, by the associativity of the amalgamated free product, leads to:

$$
\mathcal{A}(A \otimes N^\circ) = \mathcal{A}(A_1 \otimes N^\circ) \boxplus_N (A_2 \otimes N^\circ)
$$

Then, we get that the elements $a \otimes 1, b \otimes 1, c \otimes 1$ satisfy the conditions of ([U], condition I-A of Appendix I), which leads to ([U], Prop. I-C of Appendix I):

$$
\mathcal{A}(A \otimes N^\circ) \cap \{a \otimes 1, b \otimes 1, c \otimes 1\}' = N^\circ
$$

from which we get:

$$
\mathcal{A}(A \otimes N^\circ) \cap [\mathcal{A}(A \otimes N^\circ)] a_1 \subset \mathcal{A}(A \otimes N^\circ) \cap (A \otimes C)'
$$

$$
\subset \mathcal{A}(A \otimes N^\circ) \cap \{a \otimes 1, b \otimes 1, c \otimes 1\}' = N^\circ
$$

which proves that the action $a_1$ is minimal, in the sense of 4.1, and, therefore, outer, using 4.4. □

5. ANY MEASURED QUANTUM GROUPOID HAS AN OUTER ACTION

In this chapter, following the strategy of [V2], we construct a faithful action of a measured quantum groupoid $G$ on a von Neumann algebra acting on a "relative Fock space". We prove that this action leaves invariant some conditional expectation; then, using 4.8, inspired again by [V2], we construct a strictly outer action on some amalgamated free product.

5.1. NOTATIONS. Let $G = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid; let us use all the notations of 2.2; moreover, let us take $\omega$ a faithful state on $N$; let us consider the normal semi-finite faithful weight $\varphi = \omega \circ \alpha^{-1} \circ T$ on $M$, and the Hilbert space $H_\varphi$, and the antilinear bijective isometry $J_\varphi$ on $H_\varphi$ given by the Tomita-Takesaki theory. Using ([St], 3.16), we get that there exists a unitary $u$ from $H_\varphi$ onto $H$, which sends $\pi_\varphi$ on $\pi_\varphi$, and $J_\varphi$ on $J$. Therefore, this unitary sends $\pi_\varphi \circ \alpha$ on $\pi_\varphi \circ \alpha$ (which we shall both write $\alpha$ for simplification), $\pi_\varphi \circ \beta$ on $\pi_\varphi \circ \beta$ (which we shall both write $\beta$ for simplification), and we have, for all $n \in N$:

$$
u_\varphi(n^*) = u^* J_\varphi(n^*) u = u^* J_\varphi(n^*) u J_\varphi = J_\varphi \pi_\varphi(\alpha(n)) J_\varphi
$$

Therefore, $u$ sends the antirepresentation $n \mapsto J_\varphi(\alpha(n)) J_\varphi$ on $\hat{\beta}$, and we shall write again $\hat{\beta}(n) = J_\varphi(\alpha(n)) J_\varphi$ for simplification.

Using now Lesieur’s theorem ([L], 3.51), we get that there exists a pseudo-multiplicatif unitary $W_\omega$ from $H_{\varphi \otimes \beta} H_\varphi$ onto $H_{\varphi \otimes \beta} H_\varphi$.

More generally, for any representation $\pi$ of $M$ on a Hilbert space $\mathcal{H}$, there exists a unitary $U_{\mathcal{H}}$ from $\mathcal{H}_{\varphi_{\pi \alpha \beta}} H_{\varphi}$ onto $\mathcal{H}_{\varphi_{\pi \alpha \beta \circ \alpha}} H_{\varphi}$ which satisfies, for any $x \in M$ ([L], 3.39):

$$
(\pi_{\varphi_{\pi \alpha \beta \circ \alpha}} \pi_\varphi) \Gamma(x) = U_{\mathcal{H}}(1_{\pi \alpha \beta \circ \alpha} x) U_{\mathcal{H}}^*
$$

Let us apply that construction to the representation $\pi_\varphi$ on $\mathcal{H}$; the definition of $W_\omega$ and $U_{\mathcal{H}}$ ([L], 3.14) gives then that $U_{\mathcal{H}} = (u^* \otimes \beta) W_\omega^* (u^* \otimes \alpha)$.
5.2. Definition and notations. For any \( n \in \mathbb{N} \), let us write \( H^{(n)} \) for \( L^2(\mathcal{N}) \) (that we shall identify with the Hilbert space \( H_\nu \) given by the G.N.S. construction made from the weight \( \nu \)) if \( n = 0 \), for \( H \) if \( n = 1 \), for \( H_\alpha \otimes \beta H \) for \( n = 2 \), and, if \( n \geq 3 \), for the relative tensor product \((n\text{-times})\) \( H_\alpha \otimes \beta H \).

Each of these Hilbert spaces is equipped with a surjective involutive antilinear isometry, \( J_\nu \) on \( H^{(0)} \), \( J \) on \( H^{(1)} \), \( \sigma_\nu (J_\alpha \otimes \beta J) \) on \( H^{(2)} \), and \( \Sigma_n (J_\alpha \otimes \beta J_\alpha \otimes \beta \ldots \otimes \beta J) \) on \( H^{(n)} \) where \( \Sigma_n \) means \( \Sigma_n (\xi_1 \otimes \beta \xi_2 \otimes \beta \ldots \otimes \beta \xi_n) = \xi_n \otimes \beta \xi_{n-1} \otimes \beta \ldots \otimes \beta \xi_1 \).

Let us write \( \mathcal{F}(H) = \bigoplus_n H^{(n)} \), and let \( \mathcal{B} \) be the surjective involutive antilinear isometry constructed by taking the direct sum of all these isometries on \( H^{(n)} \).

Let us consider the canonical representation of \( \mathcal{N} \) on \( H^{(0)} \), the representation \( 1_\alpha \otimes \beta \alpha \) on \( H^{(2)} \), and the representations \( \alpha_n = 1_\alpha \otimes \beta 1_\alpha \otimes \beta \ldots \otimes \beta 1 \) on \( H^{(n)} \), and let us write \( a \) for the direct sum of all these representations, which is a normal faithful representation of \( \mathcal{N} \) on \( \mathcal{F}(H) \).

We define on \( \mathcal{F}(H) \) bounded operator \( l(\xi) \) by:

- for any \( n \in \mathbb{N} \), \( l(\xi) \Lambda_\nu(n) = \alpha(n) \xi \);
- for any \( \eta \in H_\nu \), \( l(\xi) \eta = \xi \otimes \beta \eta \);
- for any \( \xi_1 \alpha \otimes \beta \xi_2 \otimes \beta \ldots \otimes \beta \xi_n \in H^{(n)} \), \( l(\xi)(\xi_1 \alpha \otimes \beta \xi_2 \otimes \beta \ldots \otimes \beta \xi_n) = \xi_1 \alpha \otimes \beta \xi_1 \alpha \otimes \beta \xi_2 \otimes \beta \ldots \otimes \beta \xi_n \).

Then, we get that \( l(\xi) \) belongs to \( a(\mathcal{N})' \), and, for \( \xi, \xi' \in D(\alpha H, \nu) \), we have \( l(\xi')^* l(\xi) = b(\xi, \xi', \alpha, \nu) \).

We can easily check that \( l(\xi) l(\xi)^* \) is equal to \( 0 \) on \( H^{(0)} \), is equal to \( \theta^{\alpha,\nu}(\xi, \xi) \) on \( H^{(1)} \), and to \( \theta^{\alpha,\nu}(\xi) \alpha \otimes \beta 1 \) on \( H^{(n)} \). Therefore, if \((\xi_i)_{i \in I}\) is an orthogonal \((\alpha, \nu)\) basis of \( H \), we get that \( \sum_i l(\xi_i)^* l(\xi_i) = 1 - P_{H^{(0)}} \).

Let us write \( \mathcal{A} \) for the von Neumann algebra generated by all the operators \( l(\xi) \), for \( \xi \in D(\alpha H, \nu) \). From these remarks, we infer that \( b(\mathcal{N}) \subset \mathcal{A} \subset a(\mathcal{N})' \), and that \( P_{H^{(0)}} \in \mathcal{A} \).

Taking the final support of \( l(\xi) P_{H^{(0)}} \), we get that \( \theta^{\alpha,\nu}(\xi, \xi) P_{H^{(1)}} \) belongs to \( \mathcal{A} \), and taking again an \((\alpha, \nu)\)-orthogonal basis of \( H \), we get that \( P_{H^{(1)}} \) belongs to \( \mathcal{A} \). By recurence, we get that, for all \( n \in \mathbb{N} \), \( P_{H^{(n)}} \) belongs to \( \mathcal{A} \).

For any \( \eta \in D(H_\beta, \nu) \), we define on \( \mathcal{F}(H) \) a bounded operator \( r(\eta) \) by:

- for any \( n \in \mathbb{N} \), \( r(\eta) \Lambda_\nu(n) = \beta(n^* \eta) \);
- for any \( \xi \in H_\nu \), \( r(\eta) \xi = \xi \otimes \beta \eta \);
- for any \( \xi_1 \omega \otimes \beta \xi_2 \otimes \beta \ldots \otimes \beta \xi_n \in H^{(n)} \), \( r(\eta)(\xi_1 \otimes \beta \xi_2 \otimes \beta \ldots \otimes \beta \xi_n) = \xi_1 \alpha \otimes \beta \xi_1 \omega \otimes \beta \xi_2 \otimes \beta \ldots \otimes \beta \xi_n \).

Then, we easily get that \( r(\eta) = \beta \mathcal{J}(J \eta) \beta \), and that \( r(\eta) \in \mathcal{A}' \), from which we get that \( \beta \mathcal{A} \mathcal{J} \subset \mathcal{A}' \).

Let us now consider a faithful normal state \( \omega \) on \( \mathcal{N} \) and the G.N.S. construction \((H_\omega, \pi_\omega, \Lambda_\omega(1))\) made from \( \omega \). There exists a unique unitary \( u \) from \( H_\omega \) onto \( H_\nu = L^2(\mathcal{N}) \), such that \( u^* nu = \pi_\omega(n) \), for all \( n \in \mathbb{N} \), and \( u J_\omega = J_\nu u \); then, for any \( p \in \mathbb{N} \), analytic with
respect to $\nu$, using 2.1 and these properties of $u$, we have:

$$l(\alpha(p)\xi)u\Lambda_\omega(1) = l(\xi)J_\nu\sigma^{r-i/2}(p^*)J_\nu u\Lambda_\omega(1)$$

$$= l(\xi)uJ_\omega\pi_\omega(\sigma^{r-i/2}(p^*)))J_\omega\Lambda_\omega(1)$$

Using the weak density of the analytic elements in $N$, we get that the closure of $l(A)u\Lambda_\omega(1)$ contains, for any $\xi \in D(\alpha H, \nu)$, the subspace $l(\xi)u\pi_\omega(N)\Lambda_\omega(1)$; therefore, it contains $l(\xi)L^2(N)$, and, by definition, it contains $\xi$.

On the other hand, we have $J_\nu u\Lambda_\omega(1) = uJ_\omega\Lambda_\omega(1) = u\Lambda_\omega(1)$; in the sequel, we shall skip the unitary $u$, and consider the vector $\Lambda_\omega(1)$ as an element of $L^2(N)$, invariant by $J_\nu$.

5.3. Proposition. Let’s take the notations of 5.2; we have:

(i) the state $\Omega(X) = (X\Lambda_\omega(1)|\Lambda_\omega(1))$ on $A$ is faithful;

(ii) let $E(X) = b(X\Lambda_\omega(1),\Lambda_\omega(1) >_{a,\omega})$; then, $E$ is a normal faithful conditional expectation from $A$ onto $b(N)$.

Proof. We had seen in 5.2 that any $\xi \in D(\alpha H, \nu)$ belongs to $A\Lambda_\omega(1)$; therefore, $A\Lambda_\omega(1)$ contains $H$; the same way, we get that, for all $n \in N$, $A\Lambda_\omega^{(n)}$ contains $H^{(n+1)}$, and, therefore, the vector $\Lambda_\omega(1)$ is cycling for $\Lambda$; as $\mathbb{g}\Lambda_\omega(1) = \Lambda_\omega(1)$, by 5.2 again, we see that this vector is cycling also for $\mathbb{g}\Lambda_\omega^{(n)}$, and, therefore, also for $\Lambda^{(n)}$; so, $\Lambda_\omega(1)$ is separating for $\Lambda$, from which we get (i).

Let us write, for $X \in A$, $E(X) = b(X\Lambda_\omega(1),\Lambda_\omega(1) >_{a,\omega})$; $E$ is a positive bounded application from $\Lambda$ on $b(N)$; moreover, for any $n \in N$, we have $b(n)\Lambda_\omega(1) = J_\omega n^*J_\omega\Lambda_\omega(1) = \sigma_r^{r-i/2}(n)\Lambda_\omega(1)$, and $R^{\omega}(b(n)\Lambda_\omega(1)) = R^{\omega}(\Lambda_\omega(1))J_\omega n^*J_\omega$; so, we get that $E(b(n)) = b(n)$, and, therefore $E^2 = E$, and $E$ is a conditional expectation. As $\Omega(X) = \Omega \circ E(X)$, we get, using (i), that $E$ is faithful, which finishes the proof.

5.4. Proposition. Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid; let us use the notations of 5.1 and 5.2. Then:

(i) $\sigma_{\nu^\alpha}W\sigma_{\nu^\beta}$ is a corepresentation of $\mathfrak{G}$ on the $N - N$ bimodule $\alpha H_{\beta}$;

(ii) there exists a unitary

$$(\sigma_{\nu^\alpha}W\sigma_{\nu^\beta})_{1,n}(\sigma_{\mu^\alpha}W\sigma_{\mu^\beta})_{2,n} ... (\sigma_{\nu^\alpha}W\sigma_{\nu^\beta})_{n-2,n}(\sigma_{\nu^\alpha}W\sigma_{\nu^\beta})_{n-1,n}$$

from $H^{(n-1)}_{\alpha \otimes \beta} H$ to $H^{(n-1)}_{\beta^\alpha \otimes \alpha} H$, which is a corepresentation of $\mathfrak{G}$ on the $N - N$ bimodule $\beta_{\alpha} H_{\alpha}^{(n)}$.

(iii) by taking the sum of all these, we can define a corepresentation $\mathfrak{F}(\sigma_{\nu^\alpha}W\sigma_{\nu^\beta})$ of $\mathfrak{G}$ on the $N - N$ bimodule $\nu \mathfrak{F}(H)_\alpha$.

Proof. Result (i) is nothing but ([E5]5.6). It is then easy to get, at least formally, that $(\sigma_{\nu^\alpha}W\sigma_{\nu^\beta})_{1,2}(\sigma_{\nu^\alpha}W\sigma_{\nu^\beta})_{2,3}$ is, using ([E5], 5.1), a corepresentation of $\mathfrak{G}$ on the $N - N$ bimodule $\beta_{\alpha} H_{\alpha}^{(2)}$, and we can get by recurrence a proof of (ii). The proof of (iii) is then straightforward.

5.5. Theorem. Let $\mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu)$ be a measured quantum groupoid; let us use the notations of 5.1, 5.3 and 5.4. Then the corepresentation $\mathfrak{F}(\sigma_{\nu^\alpha}W\sigma_{\nu^\beta})$ of $\mathfrak{G}$ on the $N - N$ bimodule $\nu \mathfrak{F}(H)_\alpha$ implements, in the sense of ([E5], 6.6) an action $(b, \alpha)$ of $\mathfrak{G}$ on $A$ defined, for all $X \in A$ by:

$$a(X) = \mathfrak{F}(\sigma_{\nu^\alpha}W\sigma_{\nu^\beta})(X) \otimes_{\alpha} 1)\mathfrak{F}(\sigma_{\nu^\alpha}W\sigma_{\nu^\beta})^*$$

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Moreover, this action is faithful, and we have \((E_{b*\alpha} id)a = a \circ E\).

**Proof.** Using ([E5], 6.6), we get that \(\mathcal{F}(\sigma_{\nu}W\sigma_{\nu'})\) implements an action on \(a(N)'\). Moreover, we get, for \(\xi\) and \(\eta\) in \(D(\alpha H, \nu)\), and \(\eta' \in D(\alpha H, \nu) \cap D(\beta H, \nu')\), that:

\[
(id_{b*\alpha} \omega_{\eta,\eta'})[\mathcal{F}(\sigma_{\nu}W\sigma_{\nu})(l(\xi)_{a\otimes b})] \mathcal{F}(\sigma_{\nu}W\sigma_{\nu'})] = l[(i \ast \omega_{\eta,\eta'})(\sigma_{\nu}W\sigma_{\nu'})\xi]
\]

From which we get that

\[
\mathcal{F}(\sigma_{\nu}W\sigma_{\nu})(a_{\alpha}(\omega_{\eta,\eta'})_{\nu}) \subset A_{b*\alpha} M
\]

which gives that \((b, a)\) is an action of \(\mathcal{G}\) on \(A\). Moreover, using the formula:

\[
(id_{b*\alpha} \omega_{\eta,\eta'})a(l(\xi)) = l((id \ast \omega_{\eta,\eta'})(\sigma_{\nu}W\sigma_{\nu'})\xi)
\]

we get that, for any \(\zeta \in D(H_{\beta}, \nu')\) and \(n \in \mathfrak{A}_{\nu}\), we get:

\[
((\omega_{\lambda_{\nu}(n), \zeta} b*\alpha id)a(l(\xi))\eta|\eta') = (l((id \ast \omega_{\eta,\eta'})(\sigma_{\nu}W\sigma_{\nu'})\xi)\Lambda_{\nu}(n)|\zeta)
\]

\[
= (\alpha(n)(id \ast \omega_{\eta,\eta'})(\sigma_{\nu}W\sigma_{\nu'})\xi|\zeta) = ((i \ast \omega_{\alpha(n), \eta'})\Lambda_{\nu}(n)|\zeta)
\]

\[
= (\omega_{\xi, \zeta} \ast id)(\sigma_{\nu}W\sigma_{\nu'})\alpha(n)|\eta|\eta')
\]

from which we get that:

\[
(\omega_{\Lambda_{\nu}(n), \zeta} b*\alpha id)a(l(\xi)) = (\omega_{\xi, \zeta} \ast id)(\sigma_{\nu}W\sigma_{\nu'})\alpha(n)
\]

So, using 2.2, we get that the weak closure of all the elements of the form \((\omega_{\eta} b*\alpha id)a(x)\), for \(\eta \in D(L^{2}(A)_{b}, \nu')\) and \(x \in A\), contains all elements in \(M\), and, therefore, this action \(a\) is faithful. Finally, we have, for any \(X \in A\):

\[
(\Omega_{b*\alpha} id)a(X) = \beta(<X\Lambda_{\omega}(1), \Lambda_{\omega}(1)>)_{\alpha, \omega})
\]

and, therefore:

\[
(E_{b*\alpha} id)a(X) = 1_{b\otimes a}(\Omega_{b*\alpha} id)a(X)
\]

\[
= 1_{b\otimes a} \beta(<X\Lambda_{\omega}(1), \Lambda_{\omega}(1)>)_{\alpha, \omega})
\]

\[
= a(b(<X\Lambda_{\omega}(1), \Lambda_{\omega}(1)>)_{\alpha, \omega}))
\]

\[
= a(E(X))
\]

which finishes the proof. \(\square\)

5.6. **Theorem.** Let \(\mathcal{G}\) be a measured quantum groupoid; there exists a depth 2 inclusion \(M_{0} \subset M_{1}\), equipped with a regular operator-valued weight \(T_{1}\) from \(M_{1}\) onto \(M_{0}\), and a normal semi-finite faithful weight \(\chi\) on \(M_{0} \cap M_{1}\), invariant under \(\sigma_{T_{1}}\), such that \(\mathcal{G} = \mathcal{G}(M_{0} \subset M_{1})\)

**Proof.** Clear by 4.8 and 3.9. \(\square\)
6. Outer actions on semi-finite and finite von Neumann algebras

In this chapter, we study the case when a measured quantum groupoid is acting outerly on a semi-finite von Neumann algebra (6.3, 6.5, 6.8), or a finite von Neumann algebra (6.9, 6.10). S. Vaes had proved ([V2], 3.5) that, if a locally compact quantum group acts outerly on a II$_1$ factor, then its scaling group $\tau_1$ is trivial. Here the situation is much more complicated, as it is known, since M.-C. David’s result ([D]), that any connected finite dimensional measured quantum groupoid (with an antipod which is involutive on the two basis) acts outerly on the hyperfinite II$_1$ factor (and, instead of the locally compact quantum case, there are finite dimensional quantum groupoids with a non trivial scaling group).

6.1. Definition. Let $\mathfrak{G}$ be a measured quantum groupoid, and let $(b,a)$ be an action on a von Neumann algebra $A$. We shall say ([E6], 4.1) that this action is weighted if there exists a normal, semi-finite faithful operator-valued weight $\mathcal{T}$ from $A$ onto $b(N)$. Then, the weight $\psi = \nu^a \circ b^{-1} \circ \mathcal{T}$ will be called lifted from $\nu^a$ (or lifted). Then, for any lifted weight $\psi$ on $A$, it is possible to define a 2-cocycle $(D\psi \circ a : D\psi)_t = \Delta^\psi_\phi (\Delta^{-\psi}_\phi b \otimes \alpha \Delta^\psi_\phi)$ in $A_{b^\alpha} (M \cap \beta(N'))$ which satisfies, for all $s,t \in \mathbb{R}$ ([E6], 7.2 and 7.3):

$$(D\psi \circ a : D\psi)_{s+t} = (D\psi \circ a : D\psi)_s (\sigma^\psi_{s b^\alpha} \tau_s) ((D\psi \circ a : D\psi)_t)$$

$$(\text{id} b^\alpha \Gamma)((D\psi \circ a : D\psi)_t) = (a b^\alpha \text{id}) ((D\psi \circ a : D\psi)_t) ((D\psi \circ a : D\psi)_t)$$

This 2-cocycle is, by definition, Connes’ cocycle derivative $(\overline{D\psi} : D\psi)_t$, where $\overline{\psi}$ is the bidual weight of $\psi$, defined on $A_{b^\alpha} \mathcal{L}(H)$ which is canonically isomorphic to the bicrossed product ([E5], 11.6), and the weight $\overline{\psi}$ is equal to $\overline{\nu^a} \circ (\psi b^\alpha \text{id})$, where $\overline{\nu^a}$ is a normal semi-finite faithful weight on $\alpha(N)'$, such that $\frac{d\overline{\nu^a}}{d\nu^a} = \Delta^{-1/2}_\phi$ ([E6], 4.6), and $\psi b^\alpha \text{id}$ is a normal semi-finite faithful operator-valued weight from $A_{b^\alpha} \mathcal{L}(H)$ onto $\alpha(N)'$ ([E6], 4.4); moreover, we have then $\frac{d\overline{\psi}}{d\psi} = \Delta^{1/2}_\phi$, where $\overline{\psi}$ is the dual weight of $\psi$, defined on the cross-product $A \rtimes_a \mathfrak{G}$, and $\frac{d\psi}{d\overline{\psi}} = \Delta^{1/2}_\psi b \otimes \alpha \Delta^{-1/2}_\phi$, which leads to the result.

6.2. Proposition. Let $\mathfrak{G}$ be a measured quantum groupoid, and let $(b,a)$ be a weighted strictly outer action of $\mathfrak{G}$ on a von Neumann algebra $A$. Let $t \in \mathbb{R}$ be in Connes’invariant $T(A)$ ([St], 27.1); then, there exists $v \in M \cap \alpha(N)' \cap \beta(N)'$ such that $\Gamma(v) = v b^\alpha v$, and $\tau_1(v) = v x v^*$, for all $x \in M$; moreover, we have $\sigma_t^\phi = \text{id}.$

Proof. Let $\psi$ be a lifted weight; as $\sigma_t^\psi$ is interior, there exists a unitary $w \in A$ such that $\sigma_t^\psi(x) = wxw^*$ for all $x \in A$; therefore, we get that $\Delta^\psi_\phi = w J_\psi w J_\psi$, and, using ([E6], 7.1 and 7.2), we get that:

$$(D\psi \circ a : D\psi)_t = \Delta^\psi_\phi (w J_\psi w J_\psi b \otimes \alpha \Delta^\psi_\phi)$$

One should note that it is possible to define the unitary $w^* \otimes \alpha \Delta^\psi_\phi$ on elementary tensors (and then extends it to the Hilbert space $H_\phi \otimes H$) because we have, for all $n \in N,$
\[ w^* b(n) w = \sigma_{-t}(b(n)) = b(\sigma_t'(n)) \] and \[ \Delta_{-it}^\Phi \alpha(n) \Delta_{-it}^\Phi = \tau_t(\alpha(n)) = \alpha(\sigma_t'(n)). \] Moreover, we have:

\[
(D \psi \circ a : D \psi)_t(w J_{\psi} w J_{\psi} b_{N} \Delta_{-it}^\Phi) a(x) = \Delta_{-it}^\Psi a(x)
\]

which is equal to \( a(w) a(x) a(w^*)(D \psi \circ a : D \psi)_t(w J_{\psi} w J_{\psi} b_{N} \Delta_{-it}^\Phi) \).

From which we get that \( a(w^*)(D \psi \circ a : D \psi)_t(w b_{N} \Delta_{-it}^\Phi) \) (which belongs to \( A b_{N} L(H) \)) commutes with \( a(x) \), for all \( x \in M \). Using then \( 3.2 \) and \( 3.1 \), we get that there exists \( u \in M' \) such that:

\[
a(w^*)(D \psi \circ a : D \psi)_t(w b_{N} \Delta_{-it}^\Phi) = 1 b_{N} u
\]

or:

\[
(D \psi \circ a : D \psi)_t = a(w)(w^* b_{N} u \Delta_{-it}^\Phi)
\]

from which we deduce that \( u \Delta_{-it}^\Phi = v \) belongs to \( M \cap \beta(N)' \); so \( v \Delta_{-it}^\Phi \) belongs to \( M' \), and \( vw^* = \Delta_{-it}^\Phi x \Delta_{-it}^\Phi = \tau_t(x) \). So, the automorphism \( \tau_t \) is interior; as \( v \in \beta(N)' \), we get that \( \beta(\sigma_{-t}^\psi(n)) = \tau_t(\beta(n)) = \beta(n) \), which implies that \( \sigma_{-t} \) is interior. So, we get that \( wb(n)w^* = \sigma_{-t}^\psi(b(n)) = b(\sigma_{-t}^\psi(n)) = b(n) \), and, therefore, \( w \in A \cap b(N)' \). Moreover, as:

\[
(D \psi \circ a : D \psi)_t = a(w)(w^* b_{N} v)
\]

and \( w \) commutes with \( \beta(N) \), we get that \( v \in \alpha(N)' \); moreover, the cocycle property with respect to \( a \) gives that:

\[
(id_{b_{N} \alpha} \Gamma)(a(w)(w^* b_{N} \Gamma(v))) = (a_{b_{N} \alpha} id)(a(w)(a(w^*)(b_{N} v))(a(w)(w^* b_{N} v)) \beta_{N} 1)
\]

\[
= ((a_{b_{N} \alpha} id)(a(w)(w^* b_{N} v)) v \beta_{N} v)
\]

from which we deduce that \( \Gamma(v) = v \beta_{N} v \).

\[ \square \]

6.3. Proposition. Let \( \mathcal{G} \) be a measured quantum groupoid, and let \((b, a)\) be a weighted outer action of \( \mathcal{G} \) on a semi-finite von Neumann algebra \( A \); then:

(i) there exists a positive non singular operator \( \rho \) affiliated to \( M \cap \alpha(N)' \cap \beta(N)' \), such that, for any \( t \in \mathbb{R}, x \in M \), we have:

\[
\Gamma(\rho) = \rho_{\beta_{N} \alpha} \rho
\]

\[
\tau_t(x) = \rho^{it} x \rho^{-it}
\]

from which we deduce that \( \rho \) commutes with \( \Delta_{\Phi} \).

(ii) the weight \( \nu \) is a trace, and, for any normal semi-finite faithful trace \( \theta \) on \( A \), \( \theta \) is lifted, and \( (D\theta : D\theta)_t = 1_{b_{N} \alpha} \rho^{it} \), with the notations of 6.1.

(iii) we have \( \hat{\nu}(A) \subset A^a \), \( \hat{\nu} Z(A) b_{N} \subset Z(A \rtimes_{a} \mathcal{G}) \), and \( \alpha(N) \cap Z(M) \subset \alpha(N) \cap Z(\hat{M}) \).
Proof. Let us apply 6.2 to the hypothesis; we get that \( \nu \) is a trace, and, therefore, that any normal semi-finite faithful trace \( \theta \) on \( A \) is lifted from \( \nu^\rho \); we obtain then, for any \( t \in \mathbb{R} \), the existence of a unitary \( v_t \in M \cap \alpha(N)'' \cap \beta(N)' \) such that \( \Gamma(v_t) = v_t \beta \otimes \alpha v_t \), \( \tau_t(x) = v_t x v_t^* \), for all \( x \in M \), and \( (D\theta \circ \alpha : D\theta)_t = 1 \beta \otimes \alpha v_t \); it is therefore clear that the application \( t \mapsto v_t \) is continuous; moreover, the cocycle relation (with respect to \( (id \beta \otimes \alpha \tau_t) \)) of \( (D\theta \circ \alpha : D\theta)_t \) leads to \( v_{s+t} = v_s \tau_s(v_t) \).

But we get also that, for all \( t \in \mathbb{R} \), we have \( \Delta^{it}_\theta = 1_b \otimes \alpha v_t \Delta^{-it}_\theta \). From which we deduce that \( t \mapsto v_t \Delta^{-it}_\theta \) is a one-parameter group of unitaries. So, for any \( s, t \in \mathbb{R} \), we have \( v_s \Delta^{-is}_\theta v_t \Delta^{-it}_\theta = v_{s+t} \Delta^{-i(s+t)}_\theta \). From the cocycle relation of \( v_t \), we then get that \( \Delta^{-is}_\theta v_t = \tau_s(v_t) \Delta^{-is}_\theta \), and, therefore, that \( \tau_s(v_t) = \tau_s(v_t) \); from which we deduce that the unitaries \( v_t \) are invariant under \( \tau_s \), and, therefore, that \( t \mapsto v_t \) is a one-parameter group of unitaries in \( M \cap \alpha(N)'' \cap \beta(N)' \). From which, with the help of 6.2, we finish the proof of (i) and (ii).

Moreover, let now \( k \in Z(N)^+ \), such that \( \alpha(k) \) belongs to \( Z(M) \); then \( \beta(k) = R(\alpha(k)) \) belongs also to \( Z(M) \), using 3.4, we get that \( b(k) \in Z(A) \); Let us write \( k = \int_0^{\|k\|} \lambda de\lambda \), and \( k_n = \int_1^{\|k\|} \lambda de\lambda \); then \( k_n \) is invertible, and, for any normal semi-finite faithful trace \( \theta \) on \( A \), there exists a normal semi-finite faithful trace \( \theta_n \) on \( A \) such that \( (D\theta_n : D\theta)_t = b(k_n)^it \).

We then obtain that:

\[
(D\overline{\theta_n} : D\overline{\theta})_t = a(b(k_n)^it) = 1_b \otimes \alpha \beta(k)^it
\]

and, on the other hand:

\[
(D\theta_n : D\theta)_t = b(k_n)^it \otimes \alpha 1 = 1_b \otimes \alpha \alpha(k)^it
\]

Applying then (ii) to the traces \( \theta \) and \( \theta_n \), as \( \rho \) commutes with \( \alpha(N) \) and \( \beta(N) \), we get that \( \alpha(k) = \beta(k_n) \), and, when \( n \) goes to \( \infty \), \( \alpha(k) = \beta(k) \), from which we get that \( \alpha(k) \) belongs to \( Z(M) \).

Moreover, let now \( x \in Z(A) \); using again 3.4, we get that there exists \( k \in Z(N) \) such that \( \alpha(k) \) belongs to \( Z(M) \), and \( x = b(k) \); but, now, we have, as we proved that \( \alpha(k) = \beta(k) \):

\[
a(x) = 1_b \otimes \alpha \beta(k) = 1_b \otimes \alpha \alpha(k) = b(k) \otimes \alpha \alpha 1
\]

which proves that \( Z(A) \subset A^\alpha \). On the other hand, as \( \beta(k) \) belongs to \( Z(M) \), we have also:

\[
x \otimes \alpha 1 = a(x) = 1_b \otimes \alpha \beta(k) = 1_b \otimes \alpha \hat{\alpha}(n)
\]

and, using again 3.4, we get that \( x \otimes \alpha 1 \) belongs to \( Z(A \times_a \mathfrak{G}) \), which finishes the proof. \( \square \)

6.4. Corollary. Let \( \mathfrak{G} \) be a measured quantum groupoid, and let \( (b, a) \) be a weighted outer action of \( \mathfrak{G} \) on a semi-finite von Neumann algebra \( A \); then, if \( A \times_a \mathfrak{G} \) is a factor, then \( A \) is a factor; equivalently, if \( \mathfrak{G} \) is connected, then \( \mathfrak{G} \) is connected also.

Proof. This is clear, using 6.3(iii). \( \square \)
6.5. **Theorem.** Let \( \mathcal{G} \) be a measured quantum groupoid, and let \( (b, a) \) be a weighted outer action of \( \mathcal{G} \) on a semi-finite von Neumann algebra \( A \); let \( \theta \) be a normal semi-finite faithful trace on \( A \); then, \( \nu \) is a trace, there exists a normal semi-finite faithful operator-valued weight \( T \) from \( A \) onto \( b(N) \) such that \( \theta = \nu \circ b^{-1} \circ T \), and we have, for all \( x \in \mathcal{N}_\theta \cap \mathcal{N}_T \):

\[
(\theta b^*_N \text{id})a(x^*x) = \beta \circ b^{-1} \circ T(x^*x)\rho^{-1}
\]

where \( \rho \) is a non singular positive operator affiliated to \( M \cap \alpha(N)^r \cap \beta(N)^r \) had been defined in 6.3 and satisfies, for any \( t \in \mathbb{R} \), \( x \in M \):

\[
\Gamma(\rho) = \rho \beta \otimes_\alpha \rho_N
\]

\[
\tau_t(x) = \rho^{it}x\rho^{-it}
\]

**Proof.** We had got in 6.3 that \( \nu \) is a trace, and the existence of the operator \( \rho \); as \( \rho \) is affiliated to \( M \cap \alpha(N)^r \cap \beta(N)^r \), if we write \( \rho = \int_0^\infty \lambda d\lambda \) and, for all \( n \in \mathbb{N} \), \( f_n = \int_{1/n}^\infty d\lambda \), we get, for any \( \xi \in D(aH, \nu) \cap D(H_\beta, \nu^o) \), that \( f_n \xi \) belongs to \( D(aH, \nu) \cap D(H_\beta, \nu^o) \cap D(\rho^{-1/2}) \), and that \( \rho^{-1/2} f_n \xi \) belongs to \( D(aH, \nu) \cap D(H_\beta, \nu^o) \).

As \( (D\theta : D\theta)_t = 1 b \otimes_\alpha \rho^{it} \), by 6.3, we have, for all \( \zeta, \zeta' \) in \( D(aH, \nu) \) and \( t \in \mathbb{R} \):

\[
\sigma_t^\theta(1 b \otimes_\alpha \theta^{a,\nu}(\zeta, \zeta')) = (1 b \otimes_\alpha \rho^{it})\sigma_t^\theta(1 b \otimes_\alpha \theta^{a,\nu}(\zeta, \zeta'))(1 b \otimes_\alpha \rho^{-it})
\]

\[
= 1 b \otimes_\alpha \rho^{it} \sigma_t^\theta(\theta^{a,\nu}(\zeta, \zeta')) \rho^{-it}
\]

\[
= 1 b \otimes_\alpha \rho^{it} \Delta_\phi^{-it} \theta^{a,\nu}(\zeta, \zeta') \Delta_\phi^{it} \rho^{-it}
\]

\[
= 1 b \otimes_\alpha \theta^{a,\nu}(\rho^{it} \Delta_\phi^{-it} \zeta, \rho^{it} \Delta_\phi^{-it} \zeta')
\]

Let \( x \in \mathcal{N}_\theta, \xi \in D(aH, \nu) \cap D(H_\beta, \nu^o) \cap D(\rho^{-1/2}), \eta \in D(\Delta_\phi^{-1/2}) \cap D(\rho^{-1/2}), \) \( \eta \) in \( D(\Delta_\phi^{-1/2}) \cap D(\rho^{-1/2}), \) such that \( \Delta_\phi^{-1/2} \eta \) and \( \rho^{-1/2} \eta \) belong to \( D(\alpha H, \nu) \); we have then :

\[
\|\Lambda_\theta(x) \otimes_\beta \rho^{-1/2} \xi \otimes_\beta J_\phi \Delta_\phi^{-1/2} \eta \| = \|\Lambda_\theta(x^*) \otimes_\nu \rho^{-1/2} \xi \otimes_\nu J_\phi \Delta_\phi^{-1/2} \eta \|
\]

which, using ([E6], 4.11), is equal to :

\[
\|\Lambda_\theta(1 b \otimes_\alpha \theta^{a,\nu}(\Delta_\phi^{-1/2} \eta, \Delta_\phi^{1/2} \rho^{-1/2} \xi)) \| = \|\Lambda_\theta(1 b \otimes_\alpha \theta^{a,\nu}(\Delta_\phi^{-1/2} \eta, \Delta_\phi^{1/2} \rho^{-1/2} \xi)) \|
\]

The hypothesis about \( \xi \) and \( \eta \) give that \( (1 b \otimes_\alpha \theta^{a,\nu}(\Delta_\phi^{-1/2} \eta, \Delta_\phi^{1/2} \rho^{-1/2} \xi))^* \) belongs to \( D(\sigma_{-1/2}) \), and, therefore, we get that :

\[
\|\Lambda_\theta(x) \otimes_\beta \rho^{-1/2} \xi \otimes_\beta J_\phi \Delta_\phi^{-1/2} \eta \| = \|\Lambda_\theta(1 b \otimes_\alpha \theta^{a,\nu}(\xi, \rho^{-1/2} \eta)) \|
\]

which, using again the Radon-Nykodym derivative between \( \theta \) and \( \theta_\zeta \), is equal to :

\[
\|\Lambda_\theta(a(x) \otimes_\nu \theta^{a,\nu}(\xi, \eta)) \| = \|\theta^{a,\nu}(\xi, \eta)^* (\theta b^*_N \text{id})a(x^*x)\theta^{a,\nu}(\xi, \eta) \|
\]

We had got in ([E6], 4.11) that \( \Lambda_\theta(a(x) \otimes_\nu \theta^{a,\nu}(\xi, \eta)) = \xi \otimes_\beta J_\phi \Delta_\phi^{-1/2} \eta \), and, therefore, we get :

\[
\|\Lambda_\theta(x) \otimes_\beta \rho^{-1/2} \xi \otimes_\beta J_\phi \Delta_\phi^{-1/2} \eta \| = ((\theta b^*_N \text{id})a(x^*x)\xi \otimes_\beta J_\phi \Delta_\phi^{-1/2} \eta) \xi \otimes_\beta J_\phi \Delta_\phi^{-1/2} \eta
\]
which, by density gives, for all \( n \in N \):
\[
(\Lambda_\theta(x) a_{\omega^\mu} \alpha(n) \rho^{-1/2} \Lambda_\theta(x) a_{\omega^\mu} \rho^{-1/2}) = ((\theta b_{\omega}^* \alpha_N \mathrm{id}) a(x^* x) \alpha(n) \rho^{-1/2} \xi | \xi)
\]
and, therefore \( \| \Lambda_\theta(x) a_{\omega^\mu} \rho^{-1/2} \xi \|^2 = ((\theta b_{\omega}^* \alpha_N \mathrm{id}) a(x^* x) \rho^{-1/2} \xi | \xi) \). If now \( x \) belongs to \( \mathcal{N}_\theta \cap \mathcal{N}_\tau \), \( \Lambda_\theta(x) \) belongs to \( D(\omega H_\theta, \nu) \), and \( \Lambda_\theta(x), \Lambda_\theta(x) > a, \nu = b^{-1}(\mathcal{T}(x^* x)) \). We get then that :
\[
(\beta \circ b^{-1}(\mathcal{T}(x^* x)) \rho^{-1/2} \xi | \xi) = ((\theta b_{\omega}^* \alpha_N \mathrm{id}) a(x^* x) \rho^{-1/2} \xi | \xi)
\]
and, as we deal on both sides with positive closed operators, by density, we get the result.  

6.6. Notations. On the constructive point of view, we shall now prove that any locally compact groupoid acts outerly on the semi-finite von Neumann algebra (6.8); in the case of finite groupoids, this result had been obtained by J.-M. Vallin in ([Val5], 3.3.11); let’s fix the notations: let \( \mathcal{G} \) be a locally compact groupoid, in the sense of [R], equipped with a left Haar system \((\Lambda^\mu)_{\mu \in \mathcal{G}(0)}\) and a quasi-invariant measure \( \nu \) on the set of units \( \mathcal{G}(0) \). Let us denote \( \mu = \int_{\mathcal{G}(0)} \lambda d\nu(x) \). Let us consider the left regular representation \( \lambda(g) \) of \( \mathcal{G} \); for any \( g \in \mathcal{G} \), \( \lambda(g) \) is a unitary from \( L^2(\mathcal{G}(g), \lambda^g) \) onto \( L^2(\mathcal{G}(g), \lambda^g) \), where, as usual, for any \( x \in \mathcal{G}(0), \mathcal{G} = r^{-1}(x) \). We can consider as well \( \lambda^g \) as an orthogonal operator from the real Hilbert space \( L^2(\mathcal{G}(g), \lambda^g) \) onto the real Hilbert space \( L^2(\mathcal{G}(g), \lambda^g) \); this operator extends to an isomorphism from the Clifford algebra \( \mathrm{Cl}(L^2(\mathcal{G}(g), \lambda^g)) \) (see [Bla] for details) onto the Clifford algebra \( \mathrm{Cl}(L^2(\mathcal{G}(g), \lambda^g)) \); On each of these algebras, there exists a finite trace, and each GNS representation of these algebras generate a factor, which is the hyperfinite \( II_1 \) factor if \( \mathcal{G}(g) \) is infinite (and a finite dimensional factor if it is finite). This construction is just a generalization of [Bla] up to groupoids. 

So, for any \( x \in \mathcal{G}(0) \), we get a copy \( A^x \) of the hyperfinite \( II_1 \) factor \( \mathcal{R} \) (or a finite dimensional factor), and, for any \( g \in \mathcal{G} \), an isomorphism \( a_g \) from \( A^g \) onto \( A^g \); we obtain then an action \( a \) of the groupoid \( \mathcal{G} \) on the von Neumann algebra \( A = \int_{\mathcal{G}(0)} A^x d\nu(x) \) (which is hyperfinite \( II_1 \)); let us write \( \tau^x \) for the canonical finite trace on \( A^x \). By the unicity of the normalized trace \( \tau^x \) on \( A^x \), we have, for any \( g \in \mathcal{G} \), and \( y \) positive in \( A^g \), \( \tau^x(a_g(y)) = \tau^g(x) \). If we consider now \( \mathfrak{G}(\mathcal{G}) \) the canonical abelian measured groupoid associated to \( \mathcal{G} \), we get that the application \( a \) given by the formula \( a(f_{\mathcal{G}(0)} a^x d\nu(x) = \int_{\mathcal{G}} a_g(a^g) d\mu(g) \) is an action of \( \mathfrak{G}(\mathcal{G}) \) on \( A \), together with the isomorphism \( b \) of \( L^\infty(\mathcal{G}(0), \nu) \) with \( Z(A) \). ([E6] 6.3); it is clear that the formula \( E(f_{\mathcal{G}(0)} a^x d\nu(x)) = b(x \mapsto \tau^x(a^x)) \) is a normal faithful conditional expectation from \( A \) onto \( Z(A) \); moreover, we have then \( (E \otimes_{\mathcal{G}(0)} \nu) a = a \). 

6.7. Proposition. Let \( \mathcal{G} \) be a locally compact separable groupoid, and let \( a \) be a faithful action of \( \mathcal{G} \) on a von Neumann algebra \( A = \int_{\mathcal{G}(0)} A^x d\nu(x) \), where the algebras \( A^x \) are factors, equipped with a faithful state \( \omega^x \), invariant by \( a \), i.e. such that \( \omega(a^g \circ a_g = \omega^g \), for all \( g \in \mathcal{G} \). Let us consider the infinite tensor product \( (B^x, \omega^x_{\infty}) = \otimes_n (A^x, \omega^x) \) of copies
of \((A^x, \omega^x)\). It is clear that \((B^x)_{x \in \mathcal{G}(0)}\) is a continuous field of factors. Then the action 
\[\tilde{a}_g = \otimes_N(a_g, \omega^{r(g)})\] defines an outer action of \(\mathcal{G}\) on \(B = \int_{\mathcal{G}(0)} B^x d\nu(x)\).

**Proof.** The proof is completely taken from ([V2], 5.1), and we shall give the arguments only when it differs.

Let’s take \(a \in B \rtimes \tilde{a} \mathcal{G} \cap \tilde{a}(B)\); as in ([V2], 5.1), we can prove that \(a\) commutes with all elements of the form 
\[1 \otimes_{L^\infty(\mathcal{G}(0))} (\omega_{\xi} \otimes_{L^\infty(\mathcal{G}(0))} \text{id} a(x))\], for all \(x \in B\), and \(\xi \in D(\otimes_N L^2(A^x, \omega^x)_b, \nu)\), where \(b\) means the isomorphism from \(L^\infty(\mathcal{G}(0), \nu)\) onto \(Z(B)\), and \(r\) the injection of \(L^\infty(\mathcal{G}(0), \nu)\) into \(L^\infty(\mathcal{G}, \mu)\) given the the range function. As \(a\) is faithful, these elements are functions on \(\mathcal{G}\) which separate the points of \(\mathcal{G}\), and we get that the commutant of these elements is equal to \(\mathcal{L}(L^2(B))\) \(\tilde{b}^*\) \(L^\infty(\mathcal{G}, \mu)\). Using then ([E5], 9.4 and 11.5), we get that \(a\) belongs to \(\tilde{a}(B)\), and, therefore, to \(\tilde{a}(Z(B)) = 1 \otimes_{L^\infty(\mathcal{G}(0), \nu)} s(L^\infty(\mathcal{G}(0), \nu))\), where \(s\) is the injection of \(L^\infty(\mathcal{G}(0), \nu)\) into \(L^\infty(\mathcal{G}, \mu)\) given the the source function, which is here equal to \(\tilde{r}\) (because \(s(L^\infty(\mathcal{G}(0), \nu))\) is central in \(L^\infty(\mathcal{G}, \mu)\)). So, we get that \((b, \tilde{a})\) is outer.

6.8. **Theorem.** Let \(\mathcal{G}\) be locally compact separable infinite groupoid; then \(\mathcal{G}\) has an outer action on a hyperfinite semi-finite von Neumann algebra.

**Proof.** Let us apply 6.7 to the action constructed in 6.6. 

6.9. **Theorem.** Let \(\mathcal{G}\) be a measured quantum groupoid, and let \((b, a)\) be an outer action of \(\mathcal{G}\) on a finite von Neumann algebra \(A\); let \(\theta\) be a faithful tracial state on \(A\); then :

(i) there exists a positive non singular operator \(h\) affiliated to the center of \(N\), such that \((D\theta b ; D\nu)_1 = h^it\); moreover, for all \(x \in M\), we have \(\tau_i(x) = \alpha(h^{-it})\beta(h^it)x\alpha(h^it)\beta(h^{-it})\).

(ii) there exists a normal faithful conditional expectation from \(A\) onto \(b(N)\), which is invariant under \(a\).

**Proof.** As \(\theta \circ b\) is a trace on \(N\), we get that there exists a normal faithful conditional expectation \(E\) from \(A\) onto \(b(N)\); therefore, the action \((b, a)\) is weighted, and we can apply 6.3, from which we get that \(\nu\) is a trace, and, threfore, the existence of the operator \(h\) defining the Radon-Nikodym derivative between \(\theta \circ b\) and \(\nu\). Moreover, as, for any positive \(n\) in \(N\), we have \(\theta \circ b(n) = \nu(hn)\), we get that there exists also a normal semi-finite faithful operator-valued weight \(\mathcal{T}\) from \(A\) onto \(b(N)\) such that \(\theta = \nu \circ b^{-1} \circ \mathcal{T}\), which verify, for all positive \(x\) in \(A\), \(\mathcal{T}(x) = b(h)E(x)\); moreover, using then 6.5, we get, for any \(x \in \mathcal{M}^+_\mathcal{T}\):

\[\left(\mathcal{E} b^* a(x) 1 \mathcal{B}_a \alpha(h)\right) = \left(\mathcal{E} b^* a(x) b(h) 1 \mathcal{B}_a \right)\]

\[\mathcal{T} \mathcal{B}_a \alpha(h)\]

\[\mathcal{B}_a \beta (h^{-1})\]

\[\mathcal{B}_a \beta (h^{-1}) \mathcal{E}(x)\]

And, making now \(x\) increase towards 1, we get that \(\rho = \beta(h)\alpha(h^{-1})\), which is (i).

Using this result in the calculation above, we get, for any \(x \in \mathcal{M}^+_\mathcal{T}\):

\[\left(\mathcal{E} b^* a(x) \mathcal{B}_a \right) = \mathcal{B}_a \beta (h^{-1}) \mathcal{E}(x) = \mathcal{A}(\mathcal{E}(x))\]

which, by increasing limits, remains true for any positive \(x\) in \(A\); which is (ii).
6.10. Theorem. Let $\mathfrak{G}$ be a measured quantum groupoid, and let $(b, a)$ be an outer action on a von Neumann algebra $A$; let us suppose that the crossed-product $A \rtimes_a \mathfrak{G}$ is a finite von Neumann algebra. Then:

(i) $\hat{\mathfrak{G}}$ is a measured quantum groupoid of compact type, in the sense of ([L], 13.2) and ([E3], 5.11), i.e. there exists a left-invariant normal conditional expectation on the Hopf-bimodule $(\hat{M}, N, \alpha, \beta, \Gamma)$, which implies that $\hat{\delta} = \lambda = 1$. Moreover, we get also that $M$ is semi-finite.

(ii) we have :

\[ Z(A)_{b\otimes_\alpha N} = Z(A \rtimes_a \mathfrak{G}). \]

\[ \alpha(N) \cap Z(M) = \alpha(N) \cap Z(\hat{M}) \]

Therefore, $A$ is a factor if and only if $A \rtimes_a \mathfrak{G}$ is a factor, and $\mathfrak{G}$ is connected if and only if $\hat{\mathfrak{G}}$ is connected.

(iii) let us suppose $A$ is a factor; then, $\mathfrak{G}$ is finite dimensional if and only if the depth 2 inclusion $a(A) \subset A \rtimes_a \mathfrak{G}$ is of finite index.

Proof. Let $\theta$ be a faithful tracial normal state on $A \rtimes_a \mathfrak{G}$, then, the restriction of $\theta$ to $a(A)$ is also a normal faithful state, and, there exists a normal faithful conditional expectation $E$ from $A \rtimes_a \mathfrak{G}$ onto $a(A)$. On the other hand, using ([E5], 9.8), we get that there exists a normal semi-finite faithful operator-valued weight $T_\alpha$ from $a$ onto $A \rtimes_a \mathfrak{G}$, and we get that $(DT_\alpha : DE)_t$ belongs to $A \rtimes_a \mathfrak{G}$, which is, as $a$ is outer, $1_{b\otimes_\alpha N}$.

Moreover, let us consider $\theta \circ a$ which is is a faithful tracial normal state on $A$ (and allows us to apply 6.9); we have :

\[ (DT_\alpha : DE)_t = (D\theta \circ a : D\theta)_t \]

and, as $\theta$ is a trace, we get that there exists $k$ positive invertible affiliated to $N$ such that :

\[ (DT_\alpha : DE)_t = 1_{b\otimes_\alpha N} \hat{\alpha}(k^u) \]

which gives that, for any positive $X$ in $A \rtimes_a \mathfrak{G}$, we have :

\[ T_\alpha(X) = E((1_{b\otimes_\alpha N} \hat{\alpha}(k^{1/2}))X(1_{b\otimes_\alpha N} \hat{\alpha}(k^{1/2}))) \]

and, taking now a positive $y$ in $\hat{M}$, we have, using ([E5], 9.8) :

\[ 1_{b\otimes_\alpha N} \hat{\alpha}(y) = E(1_{b\otimes_\alpha N} \hat{\alpha}(k^{1/2})y\hat{\alpha}(k^{1/2})) \]

From which, by taking the restriction of $E$ to $1_{b\otimes_\alpha N}$, we get a normal faithful conditional expectation $F$ from $\hat{M}$ onto $\alpha(N)$ which satisfies, for all positive $y$ in $\hat{M}$ :

\[ \hat{T}(y) = F(\hat{\alpha}(k^{1/2}))y\hat{\alpha}(k^{1/2})) \]

Or, equivalently, we get the existence of a normal faithful conditional expectation $G$ from $\hat{M}$ onto $\alpha(N)$ such that, for all positive $z$ in $\hat{M}$, we have :

\[ \hat{T}(z) = G(\hat{\alpha}(k^{1/2})z\hat{\alpha}(k^{1/2})) \]

It is then straightforward to verify that $G$ is left-invariant, which gives the beginning of (i). As $\hat{\mathfrak{G}}$ is of compact type, we have $\hat{\Phi} = F \circ \hat{R}$, which implies $\hat{\delta} = \hat{\lambda} = 1$; as $\hat{\lambda} = \lambda^{-1}$, ([E5], 3.10 (vii)), we get that $\lambda = 1$. Moreover, we get also that $\Delta_\Phi = P$, as $P^u$ is the
standard implementation of \( \tau_t \), which is, thanks to 6.9, equal to the interior automorphism implemented by \( \alpha(h^{-it})\beta(h^{it}) \), where \( h \) is defined as \( \theta(1_b \otimes_\alpha \beta(n)) = \nu(hn) \), for all positive \( n \) in \( N \). So, \( \sigma_t^\Phi = \tau_t \) is interior, which finishes the proof of (i).

By applying again 6.9, we get that the action \( a \) is weighted, so that we may apply 6.3 to the action \( a \); but, we may also apply 6.9 and 6.3 to the action \( \tilde{a} \); so, we get that:

\[
Z(A)_{b \otimes_\alpha} N, C \subset Z(A \rtimes_a \mathfrak{G}) \subset Z(A_{b \otimes_\alpha} \mathcal{L}(H)) = Z(A)_{b \otimes_\alpha} N
\]

and that:

\[
\alpha(N) \cap Z(M) \subset \alpha(N) \cap Z(\hat{M}) \subset \alpha(N) \cap Z(M)
\]

which finishes the proof of (ii).

When \( A \) is a factor, it is well known ([GHJ], 4.6.2) that, if the inclusion \( a(A) \subset A \rtimes_a \mathfrak{G} \) is of finite index, then all relative commutants in the tower are finite-dimensional; in particular \( A_{b \otimes_\alpha} \mathcal{L}(H) \cap a(A)' \) is finite-dimensional, and we get that \( \mathfrak{G} \) is finite-dimensional by 3.1(ii).

Conversely, if \( \mathfrak{G} \) is finite-dimensional, so is \( H \), and the factor \( A_{b \otimes_\alpha} \mathcal{L}(H) \) is finite, so is \( a(A)' \), and the index of \( a(A) \subset A \rtimes_a \mathfrak{G} \) is finite ([I], 2.1.7). \( \square \)

6.11. **Remarks.** (i) in ([NSzW] 4.2.5) was proved that any connected finite dimensional quantum groupoid outerly acting on a factor is biconnected.

(ii) in [NV2] was proved that any depth 2 finite index subfactor of the hyperfinite II\(_1\) factor \( \mathcal{R} \) leads to a biconnected finite dimensional quantum groupoid outerly acting on \( \mathcal{R} \) (such that the subfactor is the algebra of invariants elements by this action).

(iii) in [D] was proved that any finite dimensional biconnected quantum groupoid, whose antipod is involutive on the two copies of the basis (called target and source Cartan subalgebras, or target and source co-unital subalgebras) is outerly acting on \( \mathcal{R} \); this hypothesis on the antipod is equivalent to the fact of having a finite normal quasi-invariant trace on the basis).

6.12. **Examples.** (i) as recalled in 6.11(iii), any connected measured quantum groupoid \( \mathfrak{G} = (N, M, \alpha, \beta, \Gamma, T, T', \nu) \) such that \( \dim M < \infty \), and \( \nu \) is a trace has an outer action \( a \) on \( \mathcal{R} \) such that the crossed-product \( \mathcal{R} \rtimes_\alpha \mathfrak{G} \) is isomorphic to \( \mathcal{R} \), and the index of the inclusion \( a(\mathcal{R}) \subset \mathcal{R} \rtimes_\alpha \mathfrak{G} \) is finite. Moreover, by 6.11(ii), these are the only outer actions of finite index of measured quantum groupoids on \( \mathcal{R} \).

(ii) let \( \tau \) be the tracial normal faithful state on \( \mathcal{R} \), and let us write \( H \) for \( H_\tau \), \( J \) for \( J_\tau \), \( Tr \) the canonical semi-finite faithful trace on \( \mathcal{L}(H) \). Let us denote by \( T_\tau \) the normal faithful semi-finite operator-valued weight from \( \mathcal{L}(H) \) onto \( \mathcal{R} \), such that \( \tau \circ T_\tau = Tr \). Let us recall the construction of the "\( \mathcal{R} \)-quantum groupoid" as made in ([L],14). Let us consider the von Neumann algebra \( \mathcal{R}^o \otimes \mathcal{R} \), equipped with its canonical structure of \( \mathcal{R} \)-bimodule, i.e. we define, for any \( x \in \mathcal{R} \), \( \alpha(x) = 1 \otimes x \), and \( \beta(x) = x^o \otimes 1 \); then, the fiber product \( \mathcal{R}^o \otimes \mathcal{R} \otimes_\alpha \mathcal{R}^o \otimes \mathfrak{G} \) is canonically isomorphic to \( \mathcal{R}^o \otimes \mathcal{R} \); moreover \( \mathfrak{G}(\mathcal{R}) = (\mathcal{R}, \mathcal{R}^o \otimes \mathcal{R}, \alpha, \beta, id, id \otimes \tau, \tau^o \otimes id, \tau) \) is a measured quantum groupoid, which is of compact type, because \( id \otimes \tau \) is a conditional expectation. Constructing its dual, we obtain the von Neumann algebra \( \mathcal{L}(H) \), with its canonical structure of \( \mathcal{R} \)-bimodule; let us write \( id_{|\mathcal{R}} \) for the inclusion of \( \mathcal{R} \) into \( \mathcal{L}(H) \), and \( id_{\mathcal{R}}^o \) for the canonical anti-homomorphism \( x \mapsto Jx^*J \) from \( \mathcal{R} \) into \( \mathcal{L}(H) \); then, we get that the fiber product \( \mathcal{L}(H)_{id_{\mathcal{R}}^o \otimes id_{|\mathcal{R}}} \mathcal{R} \mathcal{L}(H) \) is canonically isomorphic to \( \mathcal{L}(H) \), and we get that \( \widehat{\mathfrak{G}(\mathcal{R})} \) is equal to \( (\mathcal{R}, \mathcal{L}(H), id_{|\mathcal{R}}, id_{\mathcal{R}}^o, id, T_\tau, (T_\tau)^o, \tau) \), where \( id \) means the identity of \( \mathcal{L}(H) \), and \( (T_\tau)^o \) is
defined, for any \( y \in \mathcal{L}(H) \), by \( (T_r)^o(y) = JT_r(JyJ) \).

Let us consider now the trivial action \( (id_{\mathcal{R}}, id_{\mathcal{R}}^o) \) of \( \hat{\mathcal{G}}(\mathcal{R}) \) on \( \mathcal{R} \) \([E5], 6.2\): it’s crossed-product \([E5], 9.5\) is the von Neumann algebra of \( \hat{\mathcal{G}}(\mathcal{R})^o \), i.e. \( \mathcal{R} \otimes \mathcal{R}^o \), which is isomorphic to \( \mathcal{R} \); moreover, the relative commutant of \( \mathcal{R} \) into the crossed-product is \( \mathcal{R}^o \), and, so, this action is outer. The inclusion of \( \mathcal{R} \) into the crossed-product is isomorphic to \( \mathcal{R} \otimes \mathcal{C} \subset \mathcal{R} \otimes \mathcal{R}^o \), which is of infinite index \([J], 2.1.19\).

(iii) Let \( \mathcal{G} \) be a finite dimensional measured quantum groupoid, with a relatively invariant trace \( \nu \) on the basis \( N \), and \( (b, a) \) an outer action of \( \mathcal{G} \) on \( \mathcal{R} \); let \( \hat{\mathcal{G}}(\mathcal{R}) \) be the dual \( \mathcal{R} \)-quantum groupoid, and \( (id, id) \) its trivial action on \( \mathcal{R} \), which is outer by (ii). We can construct now the measured quantum groupoid \( \mathcal{G} \oplus \hat{\mathcal{G}}(\mathcal{R}) \) \((2.6(v))\) and construct the action \( (b \oplus id, a \oplus id) \) of \( \mathcal{G} \oplus \hat{\mathcal{G}}(\mathcal{R}) \) on \( \mathcal{R} \oplus \mathcal{R} \). We obtain this way \((3.8(ii))\) an outer action of \( \mathcal{G} \oplus \hat{\mathcal{G}}(\mathcal{R}) \) on \( \mathcal{R} \oplus \mathcal{R} \) (or, equivalently, on \( \mathcal{R} \)), whose crossed-product is also isomorphic to \( \mathcal{R} \). This action is clearly of infinite index.

(iv) Let \( \mathcal{G} \) be a locally compact quantum group having a strictly outer action (in the sense of \([V2]\)) on \( \mathcal{R} \); for instance, any locally compact group \( G \) \(([V2], 5.2)\), or any amenable Kac algebra of discrete type \(([V2], 8.1)\) (or, equivalently, using \([To], 3.17\)\), any Kac algebra of discrete type, such that the underlying von Neumann algebra of the dual Kac algebra of compact type is injective), or, by duality, any Kac algebra of compact type whose underlying von Neumann algebra is injective. Using again \(3.8(ii)\) and the example given in (iii), we obtain the existence of an outer action of \( \mathcal{G} \oplus \hat{\mathcal{G}}(\mathcal{R}) \) on \( \mathcal{R} \), for any such \( \mathcal{G} \), any finite dimensional measured quantum groupoid \( \mathcal{G} \), with a relative invariant trace on the basis. Using \(6.10(i)\), we get that its crossed-product is finite if and only if \( \mathcal{G} \) is of discrete type. This crossed-product is then a finite factor, by \(6.10(ii)\), and, in the case when this Kac algebra of discrete type has an invariant mean, (which, by \([To], 3.17\), implies that the underlying von Neumann algebra of the dual Kac algebra of compact type is injective), we get that the crossed-product is injective and is therefore isomorphic to \( \mathcal{R} \), and this inclusion will be of infinite index.

(v) Let’s now give examples of outer actions of measured quantum groupoids on semi-finite von Neumann algebras. We had got in \(6.8\) that any locally compact separable infinite groupoid has an outer action on a hyperfinite semi-finite von Neumann algebra; therefore, using (iv), we easily get that \( \mathcal{G}(\mathcal{G}) \oplus \mathcal{G} \oplus \hat{\mathcal{G}}(\mathcal{R}) \) has on outer action on a hyperfinite semi-finite von Neumann algebra, where \( \mathcal{G}(\mathcal{G}) \) is the measured quantum groupoid constructed from \( \mathcal{G} \), \( \mathcal{G} \) is any locally compact quantum group having a strictly outer action on \( \mathcal{R} \), \( \mathcal{G} \) is any finite dimensional measured quantum groupoid, with a relatively invariant trace on the basis, and \( \hat{\mathcal{G}}(\mathcal{R}) \) has been defined in (ii).

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