Geometry and Spectral Variation: the Operator Norm

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http://dx.doi.org/10.22147/jusps-A/301003

Acceptance Date 08th July, 2018, Online Publication Date 2nd October, 2018

Abstract

In this paper, we will obtain if $A$ is a $q$-$k$-normal matrix and $B$ is any matrix close to $A$, then the optimal matching distance $d(\sigma(A), \sigma(B))$ is bounded by $\|A-B\|$.

Key words: $q$-$k$-Hermitian, $q$-$k$-Skew-Hermitian, $q$-$k$-normal path, $q$-$k$-unitary

AMS Classifications: 15A09, 15A57, 15A24, 15A33, 15A15

Introduction

We will use the notation $\sigma(A)$ for both the subset of the quaternion plane that consists of all the $q$-$k$-eigenvalues on $n \times n$ matrix $A$, and for the unordered $n$-tuple whose entries are the $q$-$k$-eigenvalues of $A$ counted with multiplicity. Since we will be taking of the distances $s(\sigma(A), \sigma(B)), h(\sigma(A), \sigma(B))$ and $d(\sigma(A), \sigma(B))$, it will be clear which of the two objects is being represented by $\sigma(A)$.

We explore, how fare, these results can we carried over the $q$-$k$-normal matrices. The first difficulty we face is that, if the matrices re not $q$-$k$-Hermitian, there is no natural way to order their $q$-$k$-eigenvalues. So, the problem has to be formulated in terms of optimal matchings even after this has been done, analogues of the inequalities above turn out to be a little more complicated. Though several good results are known, many await discovery.

Definitions and Some Theorems

Theorem 2.1:

Let $A$ be a $q$-$k$-normal and let $B$ be any matrix such that $\|A-B\|$ is smaller half of the distance between

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any two-distinct q-k eigenvalues of A. Then $d(\sigma(A), \sigma(B)) \leq \|A - B\|$. 

Proof:

Let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be all the distinct q-k eigenvalues of A.

Let $\delta = \|A - B\|$, all the q-k eigenvalues of B lie in union of the disks $\overline{D}(\alpha_j, \delta)$. By the hypothesis, these disks are mutually disjoint.

We will show that if q-k eigenvalue $'\alpha_j'$ has multiplicity $m_j$, then the disk $\overline{D}(\alpha_j, \delta)$ contains exactly $m_j$ q-k eigenvalues of B, counted with their respective multiplicities. Once this is established, the statement of the theorem is seen to follow easily.

Let $A(t) = (1-t)A + B; \ 0 \leq t \leq 1$.

$\Rightarrow$ This is a continuous map from $[0,1]$ into the space of quaternion matrices.

$\Rightarrow A(0) = A \text{ and } A(1) = B$

$\Rightarrow \|A - B\| = \|A(0) - A(1)\|$

$\Rightarrow \|A - A(t)\| = t\delta$

So, all the q-k eigenvalues of $A(t)$ also lie in the disks $\overline{D}(\alpha_j, \delta)$ for each $0 \leq t \leq 1$, as $t$ moves from 0 to 1 the q-k eigenvalues of $A(t)$ trace continuous curves can jump from one of the disks $\overline{D}(\alpha_j, \delta)$ to another. So, if we start off with $m_j$ such curves in the disk $\overline{D}(\alpha_j, \delta)$. We must end up with exactly as many.

Hence proved.

Remark 2.2:

Let $H_{n \times n}$ denote the set of q-k normal of a fixed size $n$. If A is an element of $H_{n \times n}$, then so is $tA$ for all real $'t'$. Thus the set $H_{n \times n}$ is path connected. However, $N$ is not an affine set.

Definition 2.3:

A continuous map $'\gamma'$ from any interval $[a,b]$ into $H_{n \times n}$ will be called a q-k normal path or a q-k normal curve. If $\gamma(a) = A$ and $\gamma(b) = B$, We say that $\gamma$ is a path joining A and B, then A and B are end prints of $\gamma$. The length of $\gamma$ is defined with respect to the norm $\| \cdot \|$ by $1_{\| \cdot \|} (\gamma) = \sup_{k=0}^{n-1} \| \gamma(t_{k+1}) - \gamma(t_k) \| (1)$

Where the supremum is taken over all partitions of $[a,b]$ as $a = t_0 < t_1 < \ldots < t_m = b$.

Remark 2.4:
If this length is finite, the path $\gamma$ is said to be rectifiable. If the function $\gamma$ is piecewise $H'$ function then
\[ l_{\| \cdot \|}(\gamma) = \int_a^b \| \gamma'(t) \| dt \]  

(2)

**Theorem 2.5:**
Let $A$ and $B$ be q-k normal matrices, and let $\gamma$ be rectifiable q-k normal path joining them $T$, then
\[ d(\sigma(A), \sigma(B)) \leq l_{\| \cdot \|}(\gamma) \]  

(3)

**Proof:**
For our convenience, let us choose the parameter $t$ to vary in $[0,1]$.

For $0 \leq r \leq 1$, let $\gamma_r$ be that part of the curve which is parameterised by $[0, r]$.

Let $G = \{ r \in [0,1] : d(\sigma(A), \sigma(\gamma(r))) \leq l_{\| \cdot \|}(\gamma_r) \}$. The theorem will be proved if we show that the point 1 is in $G$.

Since the function $\gamma$, the arc length, and the distance $d$ are all continuous in their arguments, the set $G$ is closed. So it contains the point $g = \sup G$.

We have to show that $g = 1$. Suppose $g < 1$, let $S = \gamma(g)$ lying theorem (2.1). We can find a point $t$ in $(g,1]$. Such that, if $T = \gamma(t)$, then $d(\sigma(B), \sigma(T)) \leq \| S - T \|$. But then
\[ d(\sigma(A), \sigma(\gamma(t))) \leq d(\sigma(A), \sigma(S)) + d(\sigma(S), \sigma(T)) \]
\[ \leq l_{\| \cdot \|}(\gamma_g) + \| S - T \| \]
\[ \leq l_{\| \cdot \|}(\gamma_r) \]

By the definition of $g$, this is not possible. So $g = 1$. Hence proved.

**Remark 2.6:**
An effective estimate of $d(\sigma(A), \sigma(B))$ can thus be obtained if one could find that the length of the shortest normal path joining $A$ and $B$. This is a difficult problem since the geometry of the set $H_{n \times n}$ is poorly understood. However, the theorems above have several interesting consequences.

**Definition 2.7:**
Let $S$ be any subset of $H_{n \times n}$. We will say that $S$ is metrically flat in the metric induced by the norm $\| \cdot \|$. If any two points $A$ and $B$ in $S$ can be joined by a path that lies entirely within $S$ and has length $\| A - B \|$.

**Remark 2.8:**
Every affine set in metrically flat. A non-trivial exchange of a $\| \cdot \|$ flat set is given by the theorem below. Let $U$ be the set of $n \times n$. q-k unitary matrices and $H.U$ the set of all constant multiple of q-k unitary matrices.
Theorem 2.9: The set $H.U$ is $\| \cdot \|$ flat.

Proof: First note that $H.U$ consists of just non-negative real multiplies of $q$-$k$ unitary matrices.

Let $A_0 = r_0 U_0$ and $A_1 = r_1 U_1$ be any two elements of this set, where $r_0, r_1 \geq 0$.

Choose an orthonormal basis in which the $q$-$k$ unitary matrix is $U_1 U_0^{-1}$ diagonal.

$U_1 U_0^{-1} = \text{dia}(e^{i\theta_1}, \ldots, e^{i\theta_n})$ with $|\theta_1| \leq |\theta_{n-1}| \leq \ldots \leq |\theta_1| \leq \pi$.

We, Reduce to such a form can be achieved by a $q$-$k$ unitary conjugation. Such a process changes neither $q$-$k$ eigenvalues nor norms. So, we may assume that all $q$-$k$ matrices are written with respect to the above orthonormal basis.

Let $K = \text{dia}(i\theta_1, i\theta_2, \ldots, i\theta_n)$, then $K$ is $q$-$k$ Skew-Hermitian matrix whose $q$-$k$ eigenvalues are in the interval $(-i\pi, i\pi]$.

Therefore, we have,

$$\|A_0 - A_1\| = \|r_0 U_0 - r_1 U_1\|$$

$$= \|r_0 I - r U_1 U_0^{-1}\|$$

$$= \max_j |r_0 - r_k e^{i\theta_j}|$$

$$= |r_0 - r_k|.$$

This last quantity is the length of the straight line joining the points $r_0$ and $r_k e^{i\theta_j}$ in the quaternion space. Parameterise this line segment as $r(t)e^{i\theta_j}$, $0 \leq t \leq 1$. This can be done except when $|\theta_i| = \pi$, an exceptional case to which we will return later. The equation above can then be written as

$$\|A_0 - A_1\| = \frac{1}{0} \left| \int r(t)e^{i\theta_j} \right| dt$$

$$= \int_0^1 \left| r'(t) + r(t)i\theta_j \right| dt$$

Now, let $A(t) = r(t)e^{(k)U_0}$, $0 \leq t \leq 1$.

This is a smooth curve in $H.U$ with end points $A_0$ and $A_1$. The length of this curve is

$$\int_0^1 \|A'(t)\|dt = \int_0^1 \left| r'(t)e^{(k)U_0} + r(t)ke^{(k)U_0} \right| dt$$

$$= \int_0^1 \|r'(t)I + r(t)k\| dt.$$
Since, $e^{(tk)U_0}$ is a q-$k$ unitary matrix.

$$
\| r'(t)I + r(t)K \| = \max_{j} \left| r'(t) + ir(t)\theta_j \right|
$$

$$
= \left| r'(t) + ir(t)\theta_1 \right|
$$

We put the last equation together, we see that the path $A(t)$ joining $A_0$ and $A_1$ has length $\|A_0 - A_1\|$.

The exceptional case $\|\theta_1\| = \pi$ is much simpler. The piecewise linear path that joins $A_0$ to 0 and then to $A_1$ has length $r_0 + r_1$.

This is equal to $\left| r_0 + r_1 e^{i\theta_1} \right|$ and hence to $\|A_0 - A_1\|$.

Thus H.U is flat

Hence proved.

**Theorem 2.10:**

The set $H_{mn}$ q-$k$ normal matrices is $\| . \|$ flat if and only if $n \leq 2$.

**Proof:**

Let $A$ and $B$ be $2 \times 2$ q-$k$ normal matrices. If the q-$k$ eigenvalues of $A$ and these of $B$ lie on two parallel lines, We assume that these two lines are parallel to real axis.

Then the q-$k$ Skew-Hermitian part of $A - B$ is scalar and hence $A - B$ is q-$k$ normal.

The straight line joining $A$ and $B$ then they are lying in $H_{mn}$.

If the q-$k$ eigenvalues of $A$ and $B$ do not lie on parallel lines, then they lie on two concentric circles.

If $\alpha$ is common centre of these circles then $A$ and $B$ are in the set $\alpha + H.U$.

This set is $\| . \|$ flat. Thus, in either case, $A$ and $B$ can be joined by q-$k$ normal path of length $\|A - B\|$.

Hence proved.

**Remark 2.11:**

If $n \geq 3$ then $H_{mn}$ cannot be $\| . \|$ flat because of theorem (2.5).

**Example 2.12:**

Here is an example of a q-$k$ Hermitian $A$ and a q-$k$ Skew-Hermitian matrix $B$ that cannot be joined by a q-$k$ normal path of length $\|A - B\|$.

Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ ; $B = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

Then $\|A - B\| = 2$.

If there were a q-$k$ normal path of length 2 joining $A,B$ then the midpoint of this path would be a normal matrix $C$ such that $\|A - C\| = \|B - C\| = 1$. 
Since each entry of a matrix is dominated by its norm, this implies that \( |C_{21}| \leq 1 \) and \( |C_{21} + 1| \leq 1 \)

Hence \( C_{21} = 0 \).

By the same argument, \( C_{32} = 0 \).

So \[ A - C = \begin{pmatrix} * & * & * \\ 1 & * & * \\ * & 1 & * \end{pmatrix} \]

Where \( * \) represents an entry whose value is not yet known. But if \( \|A - C\| = 1 \).

We must have \[ A - C = \begin{pmatrix} 0 & 0 & * \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \]

Hence, \[ C = \begin{pmatrix} 0 & 1 & * \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

But then \( C \) could not have been normal.

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