Equilibrium measures for the Hénon map at the first bifurcation

Samuel Senti\textsuperscript{1} and Hiroki Takahasi\textsuperscript{2}

\textsuperscript{1} Instituto de Matemática, Universidade Federal do Rio de Janeiro, C.P. 68 530, CEP 21941-909, R.J., Brasil\n\textsuperscript{2} Department of Mathematics, Keio University, Yokohama 223-8522, Japan

E-mail: senti@im.ufrj.br and hiroki@math.keio.ac.jp

Received 11 July 2012, in final form 23 April 2013
Published 14 May 2013
Online at stacks.iop.org/Non/26/1719

Recommended by R de la Llave

Abstract
We study the dynamics of strongly dissipative Hénon maps at the first bifurcation parameter where the uniform hyperbolicity is destroyed by the formation of tangencies inside the limit set. We prove the existence of an equilibrium measure which minimizes the free energy associated with the non-continuous potential $-t \log J_u$, where $t \in \mathbb{R}$ is in a certain interval of the form $(-\infty, t_0)$, $t_0 > 0$ and $J_u$ denotes the Jacobian in the unstable direction.

Mathematics Subject Classification: 37D25, 37D35, 37G25

1. Introduction

An important problem in dynamics is to describe how horseshoes are destroyed. A process of destruction through homoclinic bifurcations is modelled by the Hénon family\textsuperscript{3}

$$f_a: (x, y) \mapsto (1 - ax^2 + \sqrt{b}y, \pm \sqrt{b}x), \quad 0 < b \ll 1.$$  \hspace{1cm} (1)

For all large $a$, the non-wandering set is a uniformly hyperbolic horseshoe [7]. As one decreases $a$, the stable and unstable directions get increasingly confused, and at last it reaches a bifurcation parameter $a^*$ near 2. The non-wandering set of $f_a$ is a uniformly hyperbolic horseshoe for $a > a^*$, and $\{f_a\}$ generically unfolds a quadratic tangency at $a = a^*$ [2, 3, 6]. According to a general theory of global bifurcations (for instance, see [20] and references therein), a surprisingly rich array of complicated behaviours appear in the unfolding of the tangency. In this paper, instead of unfolding the tangency we study the dynamics of $f_{a^*}$ from a viewpoint of ergodic theory and thermodynamic formalism. The dynamics of $f_{a^*}$ is close to that of the

\textsuperscript{3} Our arguments and results also hold for Hénon-like families [6, 19], perturbations of the Hénon family.
uniformly hyperbolic horseshoe \([2, 6, 9, 26]\), yet already exhibits some complexities shared by those \(f_a, a < a^*\), and thus will provide an important insight into the bifurcation at \(a^*\).

Another motivation for the study of \(f_{a^*}\) is to develop an ergodic theory for non-attracting sets which are not uniformly hyperbolic. In the rigorous study of dynamical systems, a great deal of effort has been devoted to the study of chaotic attractors. A statistical approach has often been taken, i.e., to look for nice invariant probability measures which statistically predict the asymptotic ‘fate’ of positive Lebesgue measure sets of initial conditions. The non-wandering set of \(f_{a^*}\) behaves like a saddle, in that many orbits wander around it for a while due to its invariance, and eventually leave a neighbourhood of it \([26]\). Such non-attracting sets may be considered somewhat irrelevant, as they only concern transient behaviours. Although this point of view is justified for a wide variety of reasons, the study of non-attracting sets deserves our attention, because of their non-trivial influence on global dynamics. Moreover, important thermodynamic parameters relevant in this context, such as the Hausdorff dimension and escape rates, are not well-understood unless the uniform hyperbolicity is assumed.

We state our setting and goal in more precise terms. Write \(f\) for \(f_{a^*}\) and let \(\Omega\) denote the non-wandering set of \(f\). This set is closed, bounded, and so is a compact set. Let \(\mathcal{M}(f)\) denote the space of all \(f\)-invariant Borel probability measures endowed with the topology of weak convergence. For a given potential \(\varphi\) : \(\Omega \to \mathbb{R}\) (the minus of) the free energy function \(F_{\varphi}: \mathcal{M}(f) \to \mathbb{R}\) is given by

\[
F_{\varphi}(\mu) = h(\mu) + \int \varphi \, d\mu.
\]

where \(h(\mu)\) denotes the entropy of \(\mu\) and \(\mu(\varphi) = \int \varphi \, d\mu\). An equilibrium measure for the potential \(\varphi\) is a measure \(\mu_\varphi \in \mathcal{M}(f)\) which maximizes \(F_{\varphi}\), i.e.

\[
F_{\varphi}(\mu_\varphi) = \sup \{ F_{\varphi}(\mu) : \mu \in \mathcal{M}(f) \}.
\]

The existence and uniqueness of equilibrium measures depend upon the characteristics of the system and the potential. In our setting, the entropy map is upper semi-continuous (corollary 3.2) and so equilibrium measures exist for any continuous potential, and they are unique for a dense subset of continuous potentials \([27,\text{corollary 9.15.1}]\). However the most significant potentials often lack continuity and the above results do not apply, as is the case of the potential we are now going to introduce.

At a point \(z \in \mathbb{R}^2\), let \(E^u(z)\) denote the one-dimensional subspace such that

\[
\lim_{n \to \infty} \frac{1}{n} \log \|Df^{-n}|E^u(z)\| < 0.
\]

Since \(f^{-1}\) expands area, \(E^u(z)\) is unique when it makes sense. We call \(E^u\) an unstable direction. Denote the Jacobian in the unstable direction by

\[
J^u(z) := \|Df|E^u(z)\|.
\]

The geometric potential is then given by

\[
\varphi_t := -t \log J^u, \quad t \in \mathbb{R}.
\]

Due to the presence of the tangency, \(\varphi_t\) is merely bounded measurable and not continuous. Our goal is to prove the existence of equilibrium measures for \(\varphi_t\) with \(t\) in a certain interval containing all negative \(t\) and some (many) positive \(t\).

The (non-uniform) expansion along the unstable direction is responsible for the chaotic behaviour. Therefore, information on the dynamics of \(f\) as well as the geometry of \(\Omega\) is obtained by studying equilibrium measures for the geometric potentials \(\varphi_t\) and the associated pressure function \(t \in \mathbb{R} \mapsto P(t)\), where

\[
P(t) := \sup \{ F_{\varphi_t}(\mu) : \mu \in \mathcal{M}(f) \}.
\]
For instance, SRB measures when they exist should be equilibrium measures for $\varphi_1$. Those for $\varphi_0$ are the measures of maximal entropy. In addition, analogously to the case of basic sets of $C^2$ surface diffeomorphisms [18], one can show that the Hausdorff dimension of the non-wandering set along the unstable manifold is given by the first zero of the pressure function [23, theorem B]. As there is no SRB measure for the Hénon map $f$ at first bifurcation [26], the dimension is strictly less than 1.

Our study of $f$ heavily relies on the fact that $f$ may be viewed as a singular perturbation of the Chebyshev quadratic $x \in \mathbb{R} \rightarrow 1 - 2x^2$, because $0 < b \ll 1$ and $a^* \rightarrow 2$ as $b \rightarrow 0$. Hence, we introduce a small constant $\varepsilon > 0$ to quantify a proximity of $f$ to the Chebyshev quadratic. Define $t_0 = t_0(\varepsilon, b)$ by

$$t_0 = \inf\{t \in \mathbb{R}: P(t) \leq -(t/2) \log(4 - \varepsilon)\}.$$  \hspace{1cm} (3)

Observe that $0 < t_0 \leq +\infty$.

**Theorem.** For any small $\varepsilon > 0$ there exists $b_0 > 0$ such that if $0 < b < b_0$ and $t < t_0$, then there exists an equilibrium measure for $\varphi_t$.

The reason for restricting the range of $t$ to values for which the pressure of the system is sufficiently large is to exclude measures which charge too much weight to the fixed saddle $\bar{Q}$ where $\varphi_t$ is not continuous. The assumption $t < t_0$ guarantees that such measures are not equilibrium measures for $\varphi_t$.

Let us mention here some previous results closely related to ours which develop thermodynamics of systems at the boundary of uniform hyperbolicity. Makarov and Smirnov [16] studied rational maps on the Riemannian sphere for which every critical point in the Julia set is non-recurrent. Leplaideur, Oliveira and Rios [15] and Arbieto and Prudente [1] studied partially hyperbolic horseshoes treated in [8]. Leplaideur and Rios [13, 14] proved the existence and uniqueness of equilibrium measures for geometric potentials ($t$-conformal measures in their terms), for certain type 3 linear horseshoes in the plane (horseshoes with three symbols) with a single orbit of tangency studied in [21]. For this model, Leplaideur [12] proved the analyticity of the pressure function. Our map $f$ is similar in spirit to the model of [13, 14] introduced in [11, 21]. However, different arguments are necessary as $f$ does not satisfy the specific assumptions in [13, 14], such as the linearity and the balance between expansion/contraction rates.

The main difficulty is in handling the limit behaviour of a sequence of Lyapunov exponents. For $\mu \in \mathcal{M}(f)$, let $\lambda^u(\mu) = \mu(\log J^u)$, which we call the unstable Lyapunov exponent of $\mu$. Since $\log J^u$ is not continuous, the weak convergence $\mu_n \rightarrow \mu$ does not imply the convergence $\lambda^u(\mu_n) \rightarrow \lambda^u(\mu)$. We show that entropy and the unstable Lyapunov exponent are upper semi-continuous as functions of measures (corollary 3.2 and proposition 4.3). Hence, the existence of equilibrium measures for $t \leq 0$ follows from the upper semi-continuity of $F_{\varphi_t}$. For $t > 0$ we need a lower bound on the drop $\lim \lambda(\mu_n) - \lambda(\mu)$, as the unstable Lyapunov exponent may not be lower semi-continuous.

The structure of the paper is as follows. In section 2 we study the dynamics of $f$. Our approach follows the well-known line for Hénon-like systems [5, 19, 28], but now for the first bifurcation parameter. A key ingredient is the notion of critical points (see section 2.2). In brief terms, these are points where the fold of the map has the most dramatic effect. To compensate for contractions of derivatives suffered at returns to a critical neighbourhood, we develop a binding argument (proposition 2.5). In this argument we use a specific feature of the map $f$, namely that all critical points are non-recurrent, which does not hold for the maps treated in [5, 19, 28].
In section 3 we show that the dynamics on the non-wandering set is semi-conjugate to the full shift on two symbols. This implies the upper semi-continuity of entropy. Although this statement is not surprising, standard arguments do not work due to the presence of the tangency. At the first bifurcation parameter the non-wandering set has a product structure, in the sense that the stable and unstable curves always intersect each other at a unique point. This defines the semi-conjugacy.

In section 4 we use the results of section 2 to bound the amount of drop of the unstable Lyapunov exponents of sequences of measures (proposition 4.3). Using this bound and the assumption $t < t_0$, i.e., the pressure $P(t)$ is sufficiently large, we complete the proof of the theorem. In the appendix we show that $t_0$ can be made arbitrarily large by choosing small $\varepsilon$ and $b$.

2. The dynamics

In this section we study the dynamics of $f$. In section 2.1 we state and prove basic geometric properties surrounding the invariant manifolds of fixed saddles. Although the dynamics outside of a fixed neighbourhood of the point of tangency is uniformly hyperbolic, returns to this neighbourhood are unavoidable. To control these returns, in section 2.2 we introduce critical points following the idea of Benedicks and Carleson [5]. In section 2.3 we analyse the dynamics near the orbits of the critical points. In section 2.4 and section 2.5 we discuss how to associate critical points to generic orbits which fall inside the neighbourhood of the tangency.

We use several positive constants whose purposes are as follows:

- $\varepsilon, \delta, b$ are small constants, chosen in this order; $\varepsilon$ is the constant specified in the theorem; $\delta$ is used to define a critical region (see section 2.2); $b$ is the constant from (1). We may shrink $\delta$ and $b$ if necessary, but only a finite number of times;
- the three constants below are used for estimates of derivatives:
  \[
  \sigma = 2 - \frac{\varepsilon}{2}, \quad \lambda_1 = 4 - \frac{\varepsilon}{2}, \quad \lambda_2 = 4 + \frac{\varepsilon}{2}.
  \]
  The $\sigma$ is used as a lower bound for derivatives far away from a critical region; $\lambda_1, \lambda_2$ are used as a lower and upper bound for derivatives near the fixed saddle near $(-1,0)$.
- any generic constant independent of $\varepsilon, \delta, b$ is simply denoted by $C$.

2.1. Basic geometric properties of the invariant manifolds

Let $P, Q$ denote the fixed saddles near $(1/2, 0)$ and $(-1, 0)$ respectively. If $f$ preserves orientation, let $W^u = W^u(Q)$. If $f$ reverses orientation, let $W^u = W^u(P)$. By a rectangle we mean any closed region bordered by two compact curves in $W^u$ and two in the stable manifolds of $P, Q$. By an unstable side of a rectangle we mean any of the two boundary curves in $W^u$. A stable side is defined similarly.

Let $R$ denote the largest possible rectangle determined by $W^u$ and $W^s(P)$, as indicated in figure 1. One of its unstable sides of $R$ contains the point of tangency near $(0, 0)$, which we denote by $\zeta_0$. Let $a^u_0$ denote the stable side of $R$ containing $f\zeta_0$, and let $a^u_0$ denote the other stable side of $R$. Since any point outside of $R$ diverges to infinity under positive or negative iteration [6], the non-wandering set $\Omega$ is contained in $R$.

Let $S$ denote the closed lenticular region bounded by the unstable side of $R$ and the parabola in $W^s(Q)$ containing $\zeta_0$ (figure 2). Points in the interior of $S$ are mapped to the outside of $R$, and they never return to $R$ under any positive iteration.

We need a couple of lemmas on the geometry of $W^u$. Let $a^u_1$ denote the component of $W^u(P) \cap R$ containing $P$. Let $a^u_1$ denote the one of the two components of $R \cap f^{-1}a^u_1$ which
Equilibrium measures for the Hénon map at the first bifurcation

Figure 1. Manifold organization for \( a = a^* \). There exist two hyperbolic fixed saddles \( P, Q \) near \((1/2, 0), (-1, 0)\) correspondingly. In the orientation preserving case (left), \( W^u(Q) \) meets \( W^s(Q) \) tangentially. In the orientation reversing case (right), \( W^u(P) \) meets \( W^s(Q) \) tangentially. The shaded regions represent the region \( R \). The point of tangency near the origin is denoted by \( \zeta_0 \) (see section 2.1).

Figure 2. The shaded closed lenticular region is denoted by \( S \) (left: orientation preserving case; right: orientation reversing case). The interior of \( S \) is mapped to the outside of \( R \), and its forward iterates do not intersect \( R \).

lies at the left of \( \zeta_0 \). Let \( \Theta \) denote the rectangle bordered by \( \alpha^+_1, \alpha^-_1 \) and the unstable sides of \( R \). The next lemma roughly states that ‘folds’ in \( W^u \) do not enter \( \Theta \). By a \( C^2(b) \)-curve we mean a compact, nearly horizontal \( C^2 \) curve for which the slopes of its tangent directions are \( \leq b^\frac{1}{4} \) and the curvature is everywhere \( \leq b^\frac{1}{4} \).

Lemma 2.1. [26, section 4] Any component of \( \Theta \cap W^u \) is a \( C^2(b) \)-curve with endpoints in \( \alpha^+_1, \alpha^-_1 \).

The next lemma will not be used for some time. For \( k \geq 0 \), let \( \Delta_k = \Theta \cap f^k R \). Observe that \( \Delta_k \) has \( 2^k \) components each of which is a rectangle, and by lemma 2.1, the unstable sides of \( \Delta_k \) are \( C^2(b) \)-curves. Also observe that \( \Delta_k \) is related to \( \Delta_{k-1} \) as follows: let \( Q_{k-1} \) denote any component of \( \Delta_{k-1} \). Then \( Q_{k-1} \cap f^k R \) has two components, each of which is a component of \( \Delta_k \).

Lemma 2.2. For \( k = 0, 1, \ldots \) and for each component \( Q_k \) of \( \Delta_k \), the Hausdorff distance between its unstable sides is \( O(b^k) \).

Proof. We argue by induction on \( k \). Assume the statement for \( 0 \leq k < j \). We regard the unstable sides of \( Q_j \) as graphs of functions \( \gamma_1, \gamma_2 \) defined on an interval \( I \). Let \( L(x) = |\gamma_1(x) - \gamma_2(x)| \). Since \( Q_j \) is contained in a component of \( \Delta_{j-1} \), the assumption of induction gives \( L^j(x) \leq (Cb)^\frac{j-1}{4} \) \(< \text{length}(I) \). Moreover \( |\gamma_1'(x) - \gamma_2'(x)| \leq L^j(x) \) holds, since \( \gamma \) is \( C^2 \) and so otherwise \( \gamma_1 \) would intersect \( \gamma_2 \). By this and the definition of \( C^2(b) \)-curves, \( L(y) \geq L(x) - (L^j(x) -Cb^\frac{j}{4} |x-y|)|x-y| \) holds for \( x, y \in I \), which is \( \geq L(x)/2 \) provided
Figure 3. Thick segments are part of \( W^s \) and \( W^i(P), W^s(Q) \). The shaded region is \( S \). The dots represent the critical points on \( \Theta \cap W^s \). The parabolas represent the pull-backs of the leaves of \( F^s \).

\[ |x - y| \leq L \hat{1}(x) \]. Hence, area(\( Q_j \)) \( \geq L \hat{1}(x) / 2 \) holds. If \( L(x) \geq b_j^2 \), then area(\( Q_j \)) \( \geq b_j^2 / 2 \), which yields a contradiction to area(\( Q_j \)) < area(\( f^j R \)) \( \leq (Cb)^j \). □

2.2. Critical points

We introduce a small neighbourhood of the tangency \( \zeta_0 \) as follows. Define

\[ I(\delta) = (-\delta, \delta) \times (-b, b) \].

Observe that, for any given \( \delta > 0 \), \( \zeta_0 \in I(\delta) \) provided \( b \) is sufficiently small.

The next lemma, which controls the growth of horizontal vectors outside of a fixed neighbourhood of the tangency, readily follows from viewing \( f \) as a perturbation of the Chebyshev quadratic which is smoothly conjugate to the tent map. We say a nonzero tangent vector \( v \) is \( b \)-horizontal if slope(\( v \)) \( \leq b \).

Lemma 2.3. For any \( \varepsilon > 0, \delta > 0 \) there exists \( b_0 = b_0(\varepsilon, \delta) > 0 \) such that the following holds for all \( 0 < b < b_0 \):

(a) if \( n \geq 1 \) and \( z \in R \) is such that \( z, f z, \ldots, f^{n-1} z \notin I(\delta) \), then for any \( b \)-horizontal vector \( v \) at \( z, Df^n(z)v \) is \( b \)-horizontal and \( \|Df^n(z)v\| \geq \delta \sigma^n \|v\| \). If moreover \( f^n z \in I(\delta) \), then \( \|Df^n(z)v\| \geq \sigma^n \|v\| \);

(b) if \( z \in [-2, 2] \setminus \Theta \), then for any \( b \)-horizontal vector \( v \) at \( z, Df(z)v \) is \( b \)-horizontal and \( \|Df(z)v\| \geq \sigma \|v\| \).

By virtue of lemma 2.3, the dynamics outside of the fixed neighbourhood \( I(\delta) \) is uniformly hyperbolic. To recover the loss of hyperbolicity due to returns to the inside of \( I(\delta) \), we mimic the strategy of Benedicks and Carleson [5] and develop a binding argument relative to critical points. For the rest of this subsection we introduce critical points, and perform preliminary estimates needed for the binding argument in the next subsection.

From the hyperbolicity of the saddle \( Q \) it follows that (using the centre manifold theorem [24] for the tangent bundle map) there exist two mutually disjoint connected open sets \( U^-, U^+ \) independent of \( b \) such that \( \alpha_0^- \subset U^-, \alpha_0^+ \subset U^+, U^+ \cap f U^- = \emptyset = U^- \cap f U^+ \) and a foliation \( F^s \) of \( U := U^- \cup U^+ \) by one-dimensional leaves such that:

(F1) \( F^s(Q) \), the leaf of \( F^s \) containing \( Q \), contains \( \alpha_0^- \);

(F2) if \( z, f z \in U \), then \( f(F^s(z)) \subset F^s(f z) \);
(F3) Let \( e^s(z) \) denote the unit vector in \( T_z F^s(z) \) with the positive second component. Then: 
\[ z \rightarrow e^s(z) \] is \( C^1 \) and \( \| Df e^s(z) \| \leq C b, \| \partial_z e^s(z) \| \leq C; \]

(F4) If \( z, f z \in U \), then slope\( (e^s(z)) \geq C \sqrt{b} \).

We call \( F^s \) a stable foliation on \( U \). From (F1), (F2) and \( f \alpha^+ \subset \alpha^- \) it follows that there is a leaf of \( F^s \) which contains \( \alpha^+ \). (F4) can be checked by contradiction: if it were false, then slope\( (e^s(f z)) \ll 1 \).

We say \( \zeta \in W^u \) is a critical point if \( f \zeta \in U^+ \) and \( T_{f \zeta} W^u = T_{f \zeta} F^s(f \zeta) \). From the results in [26] it follows that any component of \( \Theta \cap W^u \) admits a unique critical point, and it is contained in \( S \) (figure 3). Hence:

- \( \Omega \) does not contain any critical point other than \( \zeta_0 \);
- any critical point other than \( \zeta_0 \) is mapped by \( f \) to the outside of \( R \), and then escapes to infinity under positive iteration.

The second property implies that the critical orbits are contained in a region where the uniform hyperbolicity is apparent. Hence, by binding generic orbits which fall inside \( I(\delta) \) to suitable critical orbits, and then copying the exponential growth along the critical orbits, one shows that the horizontal slopes and the expansion are restored after suffering from the loss due to the folding behaviour near \( I(\delta) \). The times necessary for this recovery are called bound periods, introduced in section 2.3. This type of binding argument traces back to Jakobson [10] and Benedicks and Carleson [4, 5]. Our binding argument is an extension of Benedicks and Carleson’s to the first bifurcation parameter \( a^* \) which is not treated in [5].

The escaping property motivates the following definition. For a critical point \( \zeta \) define
\[ n(\zeta) = \sup\{i \geq 1: f^i \zeta \in U\} \]
We have \( n(\zeta) \in [1, +\infty] \), and \( n(\zeta) = +\infty \) if and only if \( \zeta = \zeta_0 \). For \( i \geq 1 \) let
\[ w_i(\zeta) = D f^{i-1} (f \zeta) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \].
Since all forward iterates of \( \zeta \) up to time \( n(\zeta) \) are near the stable sides of \( R \), for every \( 1 \leq i \leq n(\zeta) \) we have
\[ \text{slope}(w_i(\zeta)) \leq \sqrt{b}, \]
and
\[ \lambda_1 \| w_i(\zeta) \| \leq \| w_{i+1}(\zeta) \| \leq 5 \| w_i(\zeta) \|. \] (5)

For \( p > 0 \) let
\[ B(r) = \{ x \in \mathbb{R}^2: \min\{|x - y|: y \in \alpha^+ \cup \alpha^- \} \leq r \}. \]
Choose a constant \( \tau > 0 \) independent of \( b \) such that
\[ B(22 \tau) \subset U \quad \text{and} \quad \tau \leq \frac{\sigma}{100} \log 2. \] (6)
For \( p \in [1, n(\zeta)] \) define
\[ D_p(\zeta) = \tau \left[ \sum_{i=1}^{p} d_i^{-1}(\zeta) \right]^{-1}, \quad \text{where} \quad d_i(\zeta) = \frac{\| w_{i+1}(\zeta) \|}{\| w_i(\zeta) \|^2}. \] (7)
The number \( D_p(\zeta) \) serves to define a strip around the leaf \( F^s(f \zeta) \) on which the distortion of \( f^{p-1} \) is controlled (see lemma 2.6 and (18)). The next lemma gives estimates on the size of this strip and its \( f^{p-1} \)-iterates.

**Lemma 2.4.** There exists \( p_0 = p_0(\varepsilon) \) such that if \( p \geq p_0 \), then

(a) \( (\lambda_2 + \varepsilon/2)^{-p} \leq D_p(\zeta) \leq \lambda_1^{-p} \);

(b) \( \tau/5 \leq \| w_p(\zeta) \| \leq 5 \tau \).
Proof. (5) yields
\[
\left(\lambda_2 + \frac{\varepsilon}{2}\right) \leq \frac{\tau}{\lambda_1 \lambda_2} \leq \frac{\tau \cdot \min_{1 \leq i \leq p} d_i(\xi)}{\lambda_1} \leq D_p(\xi) \leq \tau d_p(\xi) \leq 5\tau \lambda_1^{-p+1} \leq \lambda_1^{-p}.
\]

The first inequality holds for sufficiently large \( p \) depending only on \( \varepsilon \). As for (b) we have
\[
\|w_p(\xi)\|D_p(\xi) = \frac{\tau}{\lambda_1} \|w_{p+1}(\xi)\| \leq 5\tau.
\]

For the lower estimate, (5) yields
\[
\frac{1}{\|w_p(\xi)\|D_p(\xi)} = \frac{1}{\tau} \sum_{i=1}^{p} \frac{\|w_i(\xi)\|}{\|w_{p+1}(\xi)\|} \leq \frac{1}{\tau} \sum_{i=1}^{p} \lambda_1^{-i+p+1} \leq \frac{5}{\tau}.
\]

\[□\]

2.3. Recovering hyperbolicity

We now develop a binding argument for the map \( f \) at the first bifurcation in order to recover hyperbolicity. Throughout this subsection we assume \( \xi \) is a critical point, and \( \gamma \) is a \( C^2(\delta) \)-curve in \( I(\delta) \) which contains \( \xi \) and is tangent to \( E^u(\xi) \). Consider the leaf \( F^s(f \xi) \) of the stable foliation \( F^s \) through \( f \xi \). This leaf may be expressed as a graph of a smooth function: there exists an open interval \( J \) independent of \( b \) and a smooth function \( y \mapsto x(y) \) on \( J \) such that
\[
F^s(f \xi) = \{(x(y), y) \mid y \in J\}.
\]

For a point \( \zeta \in \gamma \\setminus \{\xi\} \) we associate two integers \( p(\zeta) \in [1, n(\zeta)] \), \( q(\zeta) \in [1, n(\zeta)] \) called bound and fold periods as follows. First, let \( p = p(\zeta) \) be such that
\[
fz = \{ (x, y) : D_p(\zeta) \leq |x - x(y)| \leq D_{p-1}(\zeta), y \in J \},
\]
when it makes sense. Next, define \( q = q(\zeta) \) by
\[
q = \min \left\{ 1 \leq i < p : |\zeta - z|^p \|w_{j+1}(\xi)\| \geq 1 \text{ for every } i \leq j < p \right\},
\]
where
\[
b = 2/\log (1/b) .
\]

Note that (8) (10) yield \(|\zeta - z|^p \|w_p(\xi)\| \geq 1\). So, \( q \) makes sense when \( p \) does, and \( q \leq p - 1 \). Also, note that if \( \zeta = \zeta_0 \), then \( p \) makes sense for all \( \zeta \in \gamma \\setminus \{\xi\} \) because \( n(\zeta_0) = +\infty \). Otherwise, \( p \) does not make sense when \( \zeta \) is too close to \( \xi \).

The purposes of these two periods are as follows: the fold period is used to restore large slopes of iterated tangent vectors to small slopes; the bound period is used to recover an expansion of derivatives.

We are in position to state a result we are leading up to. Let us agree that, for two positive numbers \( A, B, A \approx B \) indicates \( 1/C \leq a/b \leq C \) for some \( C \geq 1 \) independent of \( \varepsilon, \delta, b \).

Proposition 2.5. Let \( \xi \) be a critical point, and \( \gamma \) a \( C^2(\delta) \)-curve in \( I(\delta) \) which contains \( \xi \) and is tangent to \( E^u(\xi) \). If \( \zeta \in \gamma \\setminus \{\xi\} \) and \( p, q \) are the corresponding bound and fold periods, then:

(a) \( \log |\zeta - z|^{-b^q} \leq p \leq \log |\zeta - z|^{-b^{q+1}} \);
(b) \( \log |\zeta - z|^{-b^{q+1}} \leq q \leq \log |\zeta - z|^{-b^{q+1}} + 1 \).

Let \( v(z) \) denote any unit vector tangent to \( \gamma \) at \( z \). Then:

(c) \( \|Df^i v(z)\| \approx |\zeta - z| \cdot \|w_i(\xi)\| \) for every \( q < i \leq p \);
(d) \( \|Df^i v(z)\| \leq 1 \) for every \( 1 \leq i < q \).
(e) \( |DF^p v(z)| \geq (4 - \varepsilon)^j \);  
(f) slope\( (DF^p v(z)) \leq b^j \).

A proof of this proposition follows the line [5, 19, 28] that is now well-understood. We split \( DF v(z) \) into \( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \)-component and \( e^\ell(f \zeta) \)-component, and iterate them separately. The latter is contracted exponentially, and the former copies the growth of \( w_1(\xi), \ldots, w_p(\xi) \), and so is expanded exponentially. The contracted component is eventually dominated by the expanded one, and as a result the desired estimates holds.

The proof of proposition 2.5 will be given after the following

**Lemma 2.6.** Let \((x(y_0), y_0) \in \mathcal{F}(f \xi), \) and let \( \gamma_0 \) be the horizontal segment of the form \( \gamma_0 = \{(x, y_0) : |x - x(y_0)| \leq D_{p-1}(\xi)\}. \) Then:

(a) for all \( \xi, \eta \in \gamma_0 \) and every \( 1 \leq i < p \), \( |DF^i(\xi) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)| \leq 2 \cdot |DF^i(\eta) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)| ; 
(b) for every \( 1 \leq i < p \), \( f^i(\gamma_0) \) is a \( C^2(\eta) \)-curve and length\( (f^i \gamma_0) \leq 20 \tau \).

**Proof.** These estimates would hold if for all \( 0 \leq j < p - 1 \) we have

\[
(f^j \gamma_0) \subset [-2, 2]^2 \setminus \Theta, \quad \text{length } (f^j \gamma_0) \leq 20d_{j+1}(\xi)D_{p-1}(\xi) \leq 20 \tau.
\]

Indeed, let \( 1 \leq i < p \). Summing the inequality in (11) over all \( j = 0, 1, \ldots, i - 1 \) yields

\[
\log \frac{|DF^i(\xi) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)|}{|DF^i(\eta) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)|} = \sum_{j=0}^{i-1} \log \frac{|DF^j(\xi) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)|}{|DF^j(\eta) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)|} \leq \frac{1}{\sigma} \sum_{j=0}^{i-1} |DF(f^j \xi) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)| - |DF(f^j \eta) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)|
\]

\[
\leq \frac{5}{\sigma} \sum_{j=0}^{i-1} \text{length}(f^j \gamma_0) \leq \frac{100 \tau}{\sigma} \leq \log 2,
\]

where the last inequality follows from the second condition in (6).

We prove (11) by induction on \( j \). It is immediate to check it for \( j = 0 \). Let \( k > 0 \) and assume (11) for every \( 0 \leq j < k \). Then, from the form of our map (1), \( f^k \gamma_0 \) is a \( C^2(\eta) \)-curve. Summing the inequality in (11) over all \( 0 \leq j < k \) and then using (6) yields \( |DF^k(\xi) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)| \leq 2 \cdot |DF^k(\eta) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)| \) for all \( \xi, \eta \in \gamma_0 \). By a result of [19, section 6], \( |DF^k(z_0) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)| \leq 2 \cdot |DF^k(f \xi) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right)| \), where \( z_0 = (x(y_0), y_0) \). Hence

\[
\text{length}(f^k \gamma_0) \leq 4 |w_{k+1}(\xi)| D_{p-1}(\xi) = 4d_{k+1}(\xi)D_{p-1}(\xi) \frac{|w_{k+1}(\xi)|}{|w_{k+1}(\xi)|} \leq 20d_{k+1}(\xi)D_{p-1}(\xi) \leq 20 \tau.
\]

Since \( k < p \leq n(\xi) \), \( f^k \xi \in B(\tau) \) and thus \( f^k \gamma_0 \subset [-2, 2]^2 \setminus \Theta \) holds. Hence (11) holds for \( j = k \).

**Proof of proposition 2.5.** Split

\[
DF v(z) = A_0 \cdot \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) + B_0 \cdot e^\ell(f \xi).
\]

By [25, lemma 2.2],

\[
|A_0| \approx |\xi - z| \quad \text{and} \quad |B_0| \leq Cb^j.
\]

For a point \( r \) near \( f \xi \), write \( r = f \xi + \xi(r)w_1(\xi)^\top + \eta(r)e^\ell(f \xi)^\top \), where \( \top \) denotes the transpose. The integrations of the inequalities in (12) along \( \gamma \) from \( \xi \) to \( z \) give

\[
|\xi(f \xi)| \approx |\xi - z|^2 \quad \text{and} \quad |\eta(f \xi)| \leq Cb^j |\xi - z|.
\]
Write \( f_\zeta = (x_0, y_0) \) and \( f_\zeta = (x_1, y_1) \). Since \( f_\gamma \) is tangent to \( \mathcal{F}(f_\zeta) \) at \( f_\zeta \) we have \( \frac{df_\zeta}{dy}(y_1) = 0 \). (F1) gives \( \left| \frac{df_\zeta}{dy}(y_1) \right| \leq C \). Then
\[
|\xi(x(y), y)| \leq C|y_0 - y_1|^2.
\]
(13) gives
\[
|y_0 - y_1|^2 \leq C|\eta(f_\zeta)|^2 \leq C\sqrt{b}|\zeta - z|^2.
\]
Since \( |x_0 - x(y_0)| = |\xi(f_\zeta) - \xi(x(y_0), y)| \), the above two inequalities and (13) yield
\[
|x_0 - x(y_0)| \approx |\zeta - z|^2.
\]
(14)
Using (8) (14) and lemma 2.4(a) we have
\[
|\zeta - z|^2 \leq C \cdot D_{p-1}(\zeta) \leq C \cdot \lambda_1^{-p}.
\]
(15)
Taking logs, rearranging the results and then shrinking \( \delta \) if necessary we get
\[
p \log \lambda_1 \leq \log C - 2 \log |\zeta - z| \leq -3 \log |\zeta - z|,
\]
which yields the upper estimate in (a). For the lower one, using (8) (14) and lemma 2.4(a) again we have
\[
|\zeta - z|^2 \geq C \cdot D_p(\zeta) \geq C \cdot (\lambda_2 + \varepsilon/2)^{-p}.
\]
(16)
Taking logs of both sides, rearranging the results and then shrinking \( \delta \) if necessary we get
\[
-2 \log |\zeta - z| \leq p \log(\lambda_2 + \varepsilon/2) + \log C \leq p \log 5.
\]
The last inequality is due to the fact that the lower bound of \( p \) becomes larger as \( \delta \) gets smaller. This completes the proof of (a).

As for (b), (5) and the definition of \( q \) give
\[
\lambda_1^{q-1} \leq \|w_q(\xi)\| \leq |\zeta - z|^{-\beta}.
\]
Taking logs of both sides and then rearranging the result yields the upper estimate in (b). For the lower one, using (5) and the definition of \( q \) again we have
\[
\lambda_q^{q-1} \geq \|w_{q+1}(\xi)\| \geq |\zeta - z|^{-\beta}.
\]
Taking logs of both sides yields the lower estimate in (b). This completes the proof of (b).

Before proceeding further, we establish a bounded distortion in the strip
\[
\left\{ (x, y) : |x - x(y)| \leq D_{p-1}(\xi), y \in J \right\}.
\]
(17)
Take arbitrary two points \( \xi_1, \xi_2 \) in the strip (17), and denote by \( \eta_\sigma \) the point of \( \mathcal{F}(f_\xi) \) with the same \( y \)-coordinate as that of \( \eta_\sigma \) (\( \sigma = 1, 2 \)). By the result of [19, Section 6],
\[
\|Df^{i}(\eta_1) (1) \| \leq 2 \cdot \|Df^{i}(\eta_2) (1) \| \text{ holds for every } 1 \leq i < p.
\]
This and lemma 2.6(a) yield
\[
\|Df^{i}(\xi_1) (1) \| \leq 8 \cdot \|Df^{i}(\xi_2) (1) \| \quad \text{for every } 1 \leq i < p.
\]
(18)
We now move on to proving the rest of the items of proposition 2.5. Consider another splitting
\[
Df v(z) = A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + B \cdot e^i(fz),
\]
and write
\[
e^i(fz) = \begin{pmatrix} \cos \theta(z) \\ \sin \theta(z) \end{pmatrix} \quad \text{and} \quad \rho \cdot Df v(z) = \begin{pmatrix} \cos \psi \\ \sin \psi \end{pmatrix},
\]
where \( \theta, \psi \in [0, \pi) \) and \( \rho > 0 \) is the normalizing constant. (12) implies \(|\theta(\xi) - \psi| \approx \rho^{-1}|\xi - z| \gg |\xi - z|\). (F1) gives \(|\theta(\xi) - \theta(z)| \leq C|\xi - z| \ll |\theta(\xi) - \psi|\), which implies \(|\theta(z) - \psi| \approx |\theta(\xi) - \psi|\). Hence

\[
|A| \approx |\psi - \theta(z)| \approx |\theta(\xi) - \psi| \approx |\xi - z|.
\]

Using (18) (19) we have

\[
|A| \cdot \|Df^{i-1}(f z)(\xi)\| \approx |\xi - z| \cdot \|w_i(\xi)\|.
\]

If \( i > q \), then we have

\[
|A| \cdot \|Df^{i-1}(f z)(\xi)\| \approx |\xi - z| \cdot \|w_i(\xi)\| > |\xi - z|^{1-\beta}.
\]

and

\[
|B| \cdot \|Df^{i-1}e'(f z)\| \leq (C b)^{i-1} \leq (C b)^q \leq (\xi - z)^{2q}.
\]

The inequality in (21) follows from the definition of \( q \). The last inequality in (22) follows from the lower estimate of \( q \), the definition of \( \beta \) and proposition 2.5(b). For the first inequality in (22) we have used the invariance (F2) of the stable foliation \( \mathcal{F}^s \) and the contraction in (F3) for the iterates of \( z \). This argument is justified by the next claim. Recall that \( U \) is the domain where \( e' \) makes sense (see section 2.2).

**Claim 2.7.** For every \( 1 \leq i \leq p - 1 \), \( f^iz \in U \).

**Proof.** The inclusion for \( i = 1 \) holds provided \( \delta \) is sufficiently small. Let \( i \geq 2 \). Since \( f^iz \in R \), \( f^iz \) is at the right of \( W^s_\text{loc}(Q) \). On the other hand, since \( z \in S \), \( f^iz \in W^s_\text{loc}(Q) \). Let \( \ell \) denote the straight segment connecting \( f^iz \) and \( f^iz \). Let \( y \) denote the point of intersection between \( \ell \) and \( W^s_\text{loc}(Q) \). Since \( b \ll 1 \), \( f^iz \) and \( f^iz \) are near the \( x \)-axis, and so \( y \in B(\tau) \) holds. Hence \( f^iz \in B(2\tau) \subset U \).

(20) (22) yield

\[
\|Df^i v(z)\| \approx |A| \cdot \|Df^{i-1}(f z)(\xi)\| \approx |\xi - z| \cdot \|w_i(\xi)\|,
\]

and hence (c).

Let \( i \leq q \). The definition of \( q \) and \( \|w_i(\xi)\| \leq \|w_q(\xi)\| \) give

\[
|A| \cdot \|Df^{i-1}(f z)\| \leq |\xi - z| \cdot \|w_i(\xi)\| \leq |\xi - z| \cdot \|w_q(\xi)\| \leq |\xi - z|^{1-\beta} \ll 1.
\]

This and \( |B| \cdot \|Df^{i-1}e'(f z)\| \leq (C b)^{i-1} \) yield (d).

As for (e) (15) gives \( |\xi - z|^{-1} \geq C \lambda_1^{-\varepsilon} \). (14) and the first inequality of lemma 2.4(b) give \( \|w_p(\xi)\| \cdot |\xi - z|^2 \geq C \|w_p(\xi)\| \cdot D_p(\xi) \geq C r \). Hence

\[
\|Df^p v(z)\| \geq C \|w_p(\xi)\| \cdot |\xi - z| \geq C r |\xi - z|^{-1} \geq C r \lambda_1^{-\varepsilon} \geq (4 - \varepsilon)^{-\varepsilon},
\]

where the last inequality holds provided \( \delta \) is sufficiently small. (f) follows from (c).
2.4. Unstable leaves

In order to use proposition 2.5 for a global analysis of the dynamics on $\Omega$, we have to find critical points in a suitable position for each return to $I(\delta)$. To this end we show that part of $\Omega$ is contained in the union of one-dimensional leaves, which are accumulated by sufficiently long $C^2(b)$-curves in $W^u$.

Let $\bar{\Gamma}^u$ denote the collection of $C^2(b)$-curves in $W^u$ with endpoints in the stable sides of $\Theta$. Let

$$\Gamma^u = \{\gamma^u; \gamma^u \text{ is the pointwise limit of a sequence in } \bar{\Gamma}^u\}.$$ 

Any curve in $\Gamma^u$ is called an unstable leaf. By the $C^2(b)$-property, the pointwise convergence is equivalent to the uniform convergence. Since two distinct curves in $\Gamma^u$ do not intersect each other, the uniform convergence is equivalent to the $C^1$ convergence. Hence, any unstable leaf is a $C^1$ curve with endpoints in the stable sides of $\Theta$ and the slopes of its tangent directions are $\leq \sqrt{B}$. Let $W^u$ denote the union of all unstable leaves.

**Lemma 2.8.** $\Theta \cap \Omega \subset W^u$.

**Proof.** Let $z \in \Theta \cap \Omega$. Then there exists an arbitrarily large integer $k$ such that $f^{-k}z \notin I(\delta)$. Since $z \in \Omega$, $f^{-k}z \in R$. Hence, $z \in \Delta_k$ holds. Since $k$ can be made arbitrarily large, from lemma 2.2 $z$ is accumulated by curves in $\bar{\Gamma}^u$. Hence $z$ is contained in an unstable leaf.

2.5. Bound/free structure

Let $z \in \Omega \cap I(\delta)$. To the forward orbit of $z$ we associate inductively a sequence of integers $0 =: n_0 < n_0 + p_0 < n_1 < n_1 + p_1 < n_2 < n_2 + p_2 < \cdots$, and then introduce useful terminologies along the way.

**Lemma 2.9.** If $z \in \Omega \cap I(\delta)$, then there exists a critical point $\zeta$ and a $C^2(b)$-curve $\gamma$ which contains $z$, $\zeta$ and is tangent to $E^u(z)$, $E^u(\zeta)$.

**Proof.** Since $z \in \Omega \cap I(\delta)$, by lemma 2.8 it is accumulated by $C^2(b)$-curves in $W^u$ with endpoints in the stable sides of $\Theta$, each of which admits a critical points. Hence the claim follows.

Given $n_i$ with $f^{n_i}z \in I(\delta)$, in view of lemma 2.9 take a critical point $\zeta$ and a $C^2(b)$-curve $\gamma$ in $I(\delta)$ which contains $f^{n_i}z$, $\zeta$ and is tangent to $E^u(f^{n_i}z)$, $E^u(\zeta)$. Let $p_i = p(f^{n_i}z)$ denote the bound period of $f^{n_i}z$ given by the definition in section 2.3 applied to $(\zeta, \gamma)$.

We claim that $p_i$ makes sense. This is clear if $\zeta = \zeta_0$. Consider the case $\zeta \neq \zeta_0$. Then $n(\zeta) < +\infty$. If $p_i$ does not make sense, then $f^{n_i}z$ comes too close to $\zeta$, so that $|\zeta - f^{n_i}z| \leq C \cdot D_{n_i}(\zeta)$ for some $C > 0$. The estimate in lemma 2.6(b) implies $|f^{n_i+1}\zeta - f^{n_i+1}\zeta| \leq 21\tau$. Since $f^{n_i+1}\zeta \notin U$ and $B(22\tau) \subset U$, $f^{n_i+1}\zeta \notin R$ holds. This yields a contradiction to the assumption that $z \in \Omega$. Hence the claim follows.

Let $n_{i+1}$ denote the next return time of the orbit of $z$ to $I(\delta)$ after $n_i + p_i$. Then lemma 2.9 applies to $f^{n_{i+1}}z$. A recursive argument allows us to decompose the forward orbit of $z$ into segments corresponding to time intervals $(n_i, n_{i+1})$ and $[n_i + p_i, n_{i+1}]$, during which we describe the points in the orbit of $z$ as being ‘bound’ and ‘free’ states respectively. Each $n_i$ is called a free return time.

Let us record the following derivative estimates:

$$\|Df^{p_i}E^u(f^{n_i}z)\| \geq (4 - \epsilon)^{\frac{1}{2}} \quad \text{and} \quad \|Df^{n_{i+1}-n_{i+1}-p_i}E^u(f^{n_{i+1}}z)\| \geq \sigma^{n_{i+1}-n_i-p_i}. \quad (23)$$
Figure 4. The regions $R_0$, $R_1$ and the $s/u$-rectangles.

The first one is a consequence of proposition 2.5. The second one follows from lemma 2.3 and the fact that $E^s(f^n p z)$ is spanned by a $b$-horizontal vector, which in turn follows from proposition 2.5(f).

3. Symbolic coding

In this section we show that $f|\Omega$ is semi-conjugate to the full shift on two symbols. As a corollary we obtain an upper semi-continuity of entropy. In section 3.1 we give precise statements of the main results in this section. In section 3.2 we introduce some relevant definitions, and in section 3.3 we construct the semi-conjugacy.

3.1. Upper semi-continuity of entropy

The region $R \setminus \text{int} S$ consists of two rectangles, intersecting each other only at $\xi_0$. Let $R_0$ denote the one at the left of $\xi_0$ and let $R_1$ denote the one at the right (figure 4). Let $\Sigma_2 = \{0, 1\}^\mathbb{Z}$ denote the shift space endowed with the product topology of the discrete topology in $\{0, 1\}$. Let $K = \{z \in \mathbb{R}^2: \{f^n z\}_{n \in \mathbb{Z}} \text{ is bounded}\}$.

Since any point outside of $R$ goes to infinity under positive or negative iteration, $K = \bigcap_{n \in \mathbb{Z}} f^n R$. Let $\pi: \Sigma_2 \to K$ denote the coding map, namely, for $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Sigma_2$ let $\pi(\omega) = \{x \in K: f^n x \in R_{\omega_n} \ \forall n \in \mathbb{Z}\}$.

Let $\sigma: \Sigma_2 \circlearrowleft$ denote the left shift.

**Proposition 3.1.** For any $\omega \in \Sigma_2$, $\pi(\omega)$ is a singleton. In addition, $\pi$ is surjective, continuous, 1-1 except on $\bigcup_{m=-\infty}^{\infty} f^m \xi_0$ where it is 2-1. It gives a semi-conjugacy $\pi \circ \sigma = f \circ \pi$.

It follows that any point in $K$ is non-wandering, and thus $K \subset \Omega$. Since $\Omega$ is bounded, $\Omega \subset K$. Hence we obtain $K = \Omega$, and the following.

**Corollary 3.2.** The entropy map $\mu \in \mathcal{M}(f) \mapsto h(\mu)$ is upper semi-continuous. In particular, there exists an equilibrium measure for any continuous potential. Moreover, for a dense set of continuous potentials this equilibrium measure is unique.

**Proof.** Let $\mathcal{M}(\sigma)$ denote the space of $\sigma$-invariant Borel probability measures endowed with the topology of weak convergence. The push-forward $\pi_*: \mathcal{M}(\sigma) \to \mathcal{M}(f)$ is a continuous
map from a compact space to a Hausdorff space. To show that $\pi_+$ is bijective, we use the following, the proof of which is left as an exercise.

**Claim 3.3.** Let $X_i$ be a topological space and $B_i$ its Borel $\sigma$-algebra, $i = 1, 2$. Let $h: X_1 \rightarrow X_2$ be a bijective map which sends open sets to Borel sets. Then $h^{-1}$ is measurable.

Let $K_0 = \pi^{-1} \left( K \setminus \bigcup_{n=-\infty}^{\infty} f^n \zeta_0 \right)$ and $\pi_0 = \pi|_{K_0}$. Since $\pi_0$ is bijective and sends open sets to measurable sets, by claim 3.3 it is a measurable bijection, and thus the pull-back $\pi_0^*$ is well-defined. Since $\zeta_0$ is not a periodic point, any $v \in M(f)$ gives full weight to $K_0$, and so $\pi_0^*(v) \in M(\sigma)$. Hence $\pi_+$ is bijective. In particular $\pi_+$ is a homeomorphism, and the inverse is $\pi_0^*$. Then the existence of equilibrium measures for any continuous potential follows directly from the upper semi-continuity of the entropy map of $\sigma$. The uniqueness follows from [27, corollary 9.15.1].

### 3.2. s/u-rectangles

By an $s$-rectangle we mean a rectangle in $R$ whose unstable sides belong to the unstable sides of $R$. A $u$-rectangle is a rectangle in $R$ whose stable sides belong to the stable sides of $R$. Let $\omega = (\omega_n)_{n \in \mathbb{Z}} \in \Sigma_2$ and write $\omega = (\omega^-, \omega^+) \in \Sigma_2$, where $\omega^- = (\omega_n)_{n < 0}$ and $\omega^+ = (\omega_n)_{n \geq 0}$. For $k \leq l$, let

$$[\omega_k, \omega_{k+1}, \ldots, \omega_l] = \bigcap_{k \leq n \leq l} f^{-n}(R_{\omega_n}).$$

If $k \geq 0$, then this set is an $s$-rectangle in $R_{\omega_0}$, if $l \leq -1$, then it is a $u$-rectangle. Set $V^u(\omega^-) = \bigcap_{n < 0} [\omega_{n-1}, \ldots, \omega_{-1}]$ and $V^s(\omega^+) = \bigcap_{n \geq 0} [\omega_0, \ldots, \omega_n]$. We have $\pi(\omega) = V^u(\omega^-) \cap V^s(\omega^+)$.  

### 3.3. Proof of proposition 3.1

Let $\omega \in \Sigma_2$. We show that $\pi(\omega)$ is a singleton. In the coding of the uniformly hyperbolic horseshoe, one considers families of stable and unstable strips ($s/u$-rectangles in our terms) and show that their boundary curves converge to curves, intersecting each other exactly at one point. In our situation, due to the presence of tangency, the convergence of the stable sides of $s$-rectangles is not clear. To circumvent this point, we take advantage of the fact that $f = f_\alpha^*$ and $\alpha^*$ is the first bifurcation parameter.

**Lemma 3.4.** If $\Theta \cap \pi(\omega) \neq \emptyset$, then $\pi(\omega)$ is a singleton.

**Proof.** For each $n > 0$ let $\partial [\omega_0, \ldots, \omega_n]$ denote any stable side of the $s$-rectangle $[\omega_0, \ldots, \omega_n]$. Since $W^u(P)$ does not intersect itself, either $\partial [\omega_0, \ldots, \omega_n] \subset \Theta$ or $\subset R \setminus \Theta$. The next sublemma implies that if $\partial [\omega_0, \ldots, \omega_n] \subset \Theta$, then it does not wind around $\Theta \cap V^u(\omega^-)$ (see figure 5). For $\gamma \in W^p$, let $D(\gamma)$ denote the closed domain bordered by $\gamma$, the unstable side of $\Theta$ containing $\zeta_0$ and the stable sides of $\Theta$. If $\gamma$ is one of the unstable sides of $\Theta$, then let $D(\gamma) = \Theta$.

**Sublemma 3.5.** If $\partial [\omega_0, \ldots, \omega_n] \subset \Theta$, then $\partial [\omega_0, \ldots, \omega_n] \cap D(\Theta \cap V^u(\omega^-))$ is connected.

**Proof.** Suppose this intersection is not connected. By lemma 2.2, the unstable sides of $\Theta \cap [\omega_{-m}, \ldots, \omega_{-1}]$ are $C^2(b)$-curves, and converge in $C^1$ to the curve $\Theta \cap V^u(\omega^-) \in W^u$. Hence, it is possible to choose an integer $m > 0$ and an unstable side $\gamma \cap [\omega_{-m}, \ldots, \omega_{-1}]$ such that $\partial [\omega_0, \ldots, \omega_n] \cap D(\gamma) \cap \Theta$ is not connected.

Since the endpoints of $\gamma$ and $\partial [\omega_0, \ldots, \omega_n]$ are transverse homoclinic or heteroclinic points, and the transversality persists under small modifications of the parameter, for $\alpha$ bigger than
and close to \(a^*\) one can consider the continuations \(\gamma(a), \partial^*[\omega_0 \cdots \omega_n](a)\) of these two curves. For the same reason, the domain \(D(\cdot)\) makes sense for \(f_a\). Since \(a^*\) is the first bifurcation parameter, \(f_a\) for \(a > a^*\) is Smale’s horseshoe map. Hence, \(f^*[\omega_0 \cdots \omega_n](a)\) has to be connected. By the continuous parameter dependence of invariant manifolds, there must come a parameter \(\omega_0 > a^*\) such that \(\partial^*[\omega_0 \cdots \omega_n](a^*)\) meets \(\gamma(a^*)\) tangentially. This yields a contradiction to the fact that \(a^*\) is the first bifurcation parameter.

Since \(\Theta \cap \pi(\omega) \neq \emptyset\), at least one of the stable sides of \([\omega_0 \cdots \omega_n]\) is contained in \(\Theta\), and so intersects \(\Theta \cap V^u(\omega^-)\). By sublemma 3.5, \(\Theta \cap V^u(\omega^-) \cap [\omega_0 \cdots \omega_n]\) is a strictly decreasing sequence of closed curves in \(\Theta \cap V^u(\omega^-)\). Hence, \(\Theta \cap V^u(\omega^-) \cap V^u(\omega^+) = \Theta \cap \pi(\omega)\) is a singleton, or else a closed curve. We argue by contradiction to eliminate the latter alternative.

Suppose that \(\gamma := \Theta \cap \pi(\omega)\) is not a singleton. Then it is a closed curve. Since \(\gamma\) is \(C^1\) accumuluated by curves in \(\Gamma^u\), one can define a bound/free structure for any point in \(\gamma\). Suppose that \(x, y \in \gamma\), \(n > 0\) are such that \(f^n x\) is bound and \(f^n y \in I(\delta)\). Then \(f^n x\) is near \(Q\), and thus \(f^n + 1 \gamma\) intersects both \(R_0\) and \(R_1\). This yields a contradiction. Hence, it follows that if \(x \in \gamma\), \(n > 0\) and \(f^n x\) is bound, then \(f^n \gamma \cap I(\delta) = \emptyset\). Then one can take an arbitrarily large integer \(n\) such that all points on \(f^n \gamma\) are free. Proposition 2.5 yields length\((f^n \gamma) \geq \delta(4 - \varepsilon)^\delta \cdot \text{length}(\gamma)\), and that the tangent vectors of \(\gamma\) are \(b\)-horizontal. Hence, some forward iterates of \(\gamma\) intersect both \(R_0\) and \(R_1\), a contradiction. This completes the proof of lemma 3.4.

For \(n \in \mathbb{Z}\), let \(A_n(\omega) = \{x \in \pi(\omega); f^n x \in \Theta\}.

\textbf{Lemma 3.6.} The following holds for all \(m, n \in \mathbb{Z}\):

(a) \(A_n(\omega)\) is a singleton unless it is empty;
(b) either (i) \(A_n(\omega) = A_m(\omega)\), or (ii) \(A_n(\omega) = \emptyset\) or \(A_n(\omega) = \emptyset\).

\textbf{Proof.} We have \(f^n A_n(\omega) = \Theta \cap \pi(a^n \omega)\). Hence lemma 3.4 gives (a). To prove (b) we need

\textbf{Sublemma 3.7.} If \(x \in R \setminus \Theta\) and \(y \in \Theta\), then \(\pi^{-1}(x) \cap \pi^{-1}(y) = \emptyset\).

We finish the proof of lemma 3.6(b) assuming sublemma 3.7. If (i) (ii) do not hold, then \(f^m A_m(\omega) \subset \Theta\) and \(f^m A_m(\omega) \subset R \setminus \Theta\). We have \(\pi^{-1}(f^m A_m(\omega)) = \pi^{-1}(f^m A_m(\omega))\), while sublemma 3.7 gives \(\pi^{-1}(f^m A_m(\omega)) \cap \pi^{-1}(f^m A_m(\omega)) = \emptyset\). This yields a contradiction.

It is left to prove sublemma 3.7. For \(x \in K\) and \(n \in \mathbb{Z}\), define \(o_n(x) \in \{0, 1\}\) by \(f^n x \in R_{o_n(x)}\). In the case \(f^n x = \zeta_0\) we let \(o_n(x) = 0\) or 1. It suffices to claim that if \(x \in R \setminus \Theta\) and \(y \in \Theta\), then there exists \(n \geq 0\) such that \(o_n(x) \neq o_n(y)\). To see this, define

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure5}
\caption{The situation eliminated by sublemma 3.5 in which \(\partial^*[\omega_0 \cdots \omega_n] \cap D(\Theta \cap V^u(\omega^-))\) is not connected.}
\end{figure}
rectangles $S_1, S_2, S_3, S_4$ as follows: $S_1$ (respectively $S_2$) is the component of $R \setminus \text{Int}\Theta$ at the left (respectively right) of $S_0$; $S_2 = R_0 \setminus \text{int}S_1$ and $S_3 = R_1 \setminus \text{int}S_4$ (see figure 6). Observe that: $fS_1 \subset S_1 \cup S_2 \cup S_3$; $fS_2 \subset S_2$; $fS_3 \subset S_3$; $fS_4 \subset S_1 \cup S_2 \cup S_3$. Either: (i) $x \in S_1$, $y \in S_2$; (ii) $x \in S_1$, $y \in S_3$; (iii) $x \in S_2$, $y \in S_3$; (iv) $x \in S_3$, $y \in S_4$. In cases (ii) and (iii) we have $\omega_0(x) \neq \omega_0(y)$, and so the claim holds with $n = 0$. In case (i) either $\omega_0(x)\omega_1(x) = 00$, $\omega_0(y)\omega_1(y) = 01$ and the claim holds with $n = 1$, or else $fx \in S_3$, $fy \in S_4$ which is reduced case (iv).

We now consider case (iv). Then either $\omega_0(x)\omega_1(x) = 10$, $\omega_0(y)\omega_1(y) = 11$ and the claim holds with $n = 2$, or else $fx \in S_1$, $fy \in S_4$ and $\omega_0(x)\omega_1(x) = 11 = \omega_0(y)\omega_1(y)$. If $fx \in I(\delta)$ then $\omega_0(x)\omega_1(x)\omega_2(x)\omega_3(x) = 1110$, $\omega_0(y)\omega_1(y)\omega_2(y)\omega_3(y) \in \{1100, 1101, 1111\}$. Hence the claim holds with $n = 2$ or 3.

Let us now assume that $fx \notin I(\delta)$. Let $z$ denote the point of intersection between $V^+(\omega^x)$ and the unstable leaf containing $x$. Let $L$ denote the segment connecting $z$ and $x$. Lemma 2.3 implies that the lengths of the forward images of $L$ grow exponentially as long as the images does not meet $I(\delta)$. Let $k > 1$ be the smallest positive integer such that $I(\delta) \cap f^kL \neq \emptyset$. Since $f\alpha_1^x \subset \alpha_1^x$ and $L$ intersects $\alpha_1^x$, $\omega_0(x) = 1 = \omega_0(y)$ for $0 \leq i \leq k - 1$. If $f^kx \in I(\delta)$ then $f^kz \in S_4$, and so $\omega_k(x)\omega_{k+1}(y)\omega_{k+2}(y) \in \{100, 101, 111\}$. For $x$, $\omega_k(x) = 0$, or else $\omega_k(x)\omega_{k+1}(x)\omega_{k+2}(x) = 110$. Hence the claim holds with $n = k, k + 1$ or $k + 2$. The same reasoning holds for the case $f^3y \in I(\delta)$.

We are in position to complete the proof of proposition 3.1. Let $E = \{x \in K: f^n z \notin \Theta \forall n \in \mathbb{Z}\}$. We have

$$\pi(\omega) = \{x \in E: \pi^{-1}(x) = \omega\} \cup \bigcup_{n \in \mathbb{Z}} A_n(\omega). \quad (24)$$

It is easy to see that $E$ is contained in the stable sides of $R$. In addition, lemma 2.2 implies that if $x, y \in K$ belong to the same stable side of $R$, then $\pi^{-1}(x) \neq \pi^{-1}(y)$. Hence the first set in (24) is a singleton unless it is empty. By lemma 3.6, the second set in (24) is a singleton unless it is empty. Either the first or the second set is empty, for otherwise sublemma 3.7 yields a contradiction. Consequently, $\pi(\omega)$ is a singleton.

Since $R_0 \cap R_1 = \{\xi_0\}$, $\pi$ is 1-1 except on $\bigcup_{n = -\infty}^{\infty} f^n \xi_0$ where it is 2-1. Observe that, since $\sigma$ sends cylinder sets to cylinder sets, the continuity of $\pi$ at a point $\omega$ implies the continuity of $\pi$ at $\sigma^n\omega$, $n \in \mathbb{Z}$. The continuity of $\pi$ on $\pi^{-1}\Theta$ follows from the proof of lemma 3.4. By the above observation, $\pi$ is continuous on $\Sigma_2 \setminus \pi^{-1}E$. The continuity on $\pi^{-1}E$ is obvious. Since $K \subset R_0 \cup R_1$, $\pi$ is surjective. □
4. Proof of the theorem

In this section we finish the proof of the theorem. In section 4.1 we study the regularity of the unstable direction $E^u$ defined in (2). In section 4.2 we estimate the amount of drop of unstable Lyapunov exponents in the weak convergence of measures. In section 4.3 we prove the theorem.

4.1. Regularity of the unstable direction

We first show that $E^u$ is Borel measurable. For two positive integers $i, j > 1$, let $\Omega_{i,j}$ denote the set of all $z \in \Omega$ for which there exists $v \in T_z \mathbb{R}^2 \setminus \{0\}$ such that $\|Df^{-n}(z)v\| \leq i j^{-n} \|v\|$ holds for every $n \geq 0$. Clearly, $\Omega_{i,j}$ is a closed set. Observe that $E^u(z)$ makes sense if and only if there exist $i, j$ such that $z \in \Omega_{i,j}$. Since $E^u$ is continuous on $\Omega_{i,j}$, it is Borel measurable on $\bigcup_{i,j} \Omega_{i,j}$.

Due to the presence of the tangency, $E^u$ is not continuous at $Q$. We show that $E^u$ makes sense, and is continuous on a large subset of $\Omega$. Let $\partial^s R$ denote the union of the stable sides of $R$ and let $\Omega' = \Omega \setminus \partial^s R$.

**Proposition 4.1.** $E^u$ is well-defined on $\Omega$, and is continuous on $\Omega'$.

**Proof.** We first prove that $E^u$ makes sense on $\mathcal{W}^u$, and is spanned by the tangent directions of the unstable leaves in $\Gamma^u$. Since any unstable leaf is a $C^1$ limit of a sequence of curves in $\Gamma^u$, these statements follow from the next uniform backward contraction on curves in $\Gamma^u$.

**Lemma 4.2.** There exists $C > 0$ such that for any $\gamma \in \Gamma^u$, $z \in \gamma$ and $n > 0$, $\|Df^n|E^u(f^{-n}z)\| \geq C(4 - \epsilon)\frac{1}{2}$.

**Proof.** Take a large integer $M \geq n$ so that $f^{-M}z$ is contained in the local unstable manifold of the saddle. We introduce a bound/free structure for the forward orbit of $f^{-M}z$. Observe that $z \in \Theta$ must be free, as the forward orbit of a critical point never returns close to $\Theta$.

We first consider the case where $f^{-n}z$ is free. Splitting the orbit $f^{-n}z, f^{-n+1}z, \ldots, z$ into bound and free segments, and then applying the derivative estimates in (23) we get the desired inequality.

We now consider the case where $f^{-n}z$ is bound. Let $i$ denote the smallest $j > n$ such that $f^{-j}z \in I(\delta)$. Let $p, q$ denote the corresponding bound and fold periods. We have $-n < -i + p$. There are two cases, $-n$ being either inside or outside of the fold period. If $-n < -i + q$, then

$$\|Df^n|E^u(f^{-n}z)\| = \frac{\|Df^i|E^u(f^{-i}z)\|}{\|Df^{i-n}|E^u(f^{-i}z)\|} \geq \|Df^i|E^u(f^{-i}z)\| \geq (4 - \epsilon)\frac{1}{2} > (4 - \epsilon)\frac{1}{2}.$$  

For the first inequality we have used proposition 2.5 (d). If $-n \geq -i + q$, then by proposition 2.5 (c) and (5) for some $C \in (0, 1)$ we have

$$\frac{\|Df^p|E^u(f^{-i}z)\|}{\|Df^{i-n}|E^u(f^{-i}z)\|} \geq C \frac{\|w_\theta(\xi)\|}{\|w_{i-n}(\xi)\|} \geq C \lambda_{ij}^{i-q}.$$  

Since both $f^{i-n}z$ and $f^{-i}z$ are free, proposition 2.5 (e) and lemma 2.3 yield

$$\|Df^i|E^u(f^{-i}z)\| \geq (4 - \epsilon)\frac{1}{2} \|Df^p|E^u(f^{-i}z)\|.$$  

We do not make any claim on the continuity of $E^u$ on $\partial^s R \setminus \{Q\}$. This does not matter because $f$-invariant probability measures do not charge this set.
Multiplying these two inequalities and then using $p - i + n > 0$, $\lambda_1 > (4 - \varepsilon)^{\frac{1}{2}} > 1$ we obtain

$$\|Df^n|E^u(f^{-n}z)\| \geq C\lambda_1^{p-i+n}(4-\varepsilon)^{\frac{i}{2}} \geq C(4 - \varepsilon)^{\frac{i}{2}}.$$

Lemma 4.2 shows that $E^u$ is well-defined on $W^u$. By lemma 2.8, $E^u$ is defined everywhere on $\Theta \cap \Omega$. Due to the $Df$-invariance, $E^u$ is well-defined everywhere on $\bigcup_{n=\infty}^{+\infty} f^n(\Theta \cap \Omega)$. If $z \in \Omega$ is not contained in $\bigcup_{n=\infty}^{+\infty} f^n(\Theta \cap \Omega)$, then $z \in \partial^u R$. Since the dynamics outside of $\Theta$ is uniformly hyperbolic, the standard cone field argument shows that $E^u(z)$ is well-defined. Therefore, $E^u$ is well-defined on $\Omega$. Lemma 2.2 implies that $E^u$ is uniformly continuous on $\Theta \cap W^u$, and thus it is continuous on $W^u$. Let $z \in \Omega'$. Then there exists $n \geq 0$ such that $f^n z \in \Theta$. We first consider the case where $f^n z$ is not in the stable sides of $\Theta$. Then $E^u$ is continuous at $f^n z$, and so the $Df$-invariance of $E^u$ and the continuity of $Df$ together imply that $E^u$ is continuous at $z$ as well.

In the case where $f^n z$ is in the stable sides of $\Theta$, the above argument is slightly incomplete, because the continuity of $E^u$ in a neighbourhood of $f^n z$ is not proved yet. However, we can prove this by slightly extending the region $\Theta$ and repeating the same arguments. $\Box$

4.2. Unstable Lyapunov exponents of limit points

A main result in this subsection is as follows. Let $\mathcal{M}'(f)$ denote the set of all ergodic $f$-invariant Borel probability measures and let $\delta_Q$ denote the Dirac measure at $Q$.

Proposition 4.3. If $\{\mu_n\} \subset \mathcal{M}'(f)$, $\mu_n \rightarrow \mu$, $\mu = u \delta_Q + (1-u)\nu$, $0 \leq u \leq 1$, $\nu \in \mathcal{M}(f)$ and $\nu\{Q\} = 0$, then:

$$\frac{\mu}{2} \log(4 - \varepsilon) + (1-u)\lambda^u(\nu) \leq \lim_{n \rightarrow \infty} \lambda^u(\mu_n);$$

$$\lim_{n \rightarrow \infty} \lambda^u(\mu_n) \leq u\lambda^u(\delta_Q) + (1-u)\lambda^u(\nu).$$

Proof. We first introduce a family of delimiting curves which allow us to relate the proximity of an orbit’s return close to the tangency with the time it will subsequently spend near $Q$. Let $\tilde{a}_0$ denote the component of $W^s(P) \cap R$ containing $P$. Define a sequence $\{\tilde{a}_k\}_{k \geq 1}$ of compact curves in $W^s(P) \cap R$ inductively as follows. Let $\tilde{a}_0 = \alpha_1^-$ and $\tilde{a}_0 = \alpha_1^+$. Given $\tilde{a}_{k-1}$, $k > 1$, define $\tilde{a}_k$ to be the one of the two components of $R \cap f^{-1}\tilde{a}_{k-1}$ which lies at the left of $\zeta_0$ (figure 7). The curves obey the following diagram

$$\tilde{a}_k \rightarrow f \tilde{a}_{k-1} \rightarrow f \tilde{a}_{k-2} \rightarrow \cdots \rightarrow f \tilde{a}_1 = \alpha_1^- \rightarrow f \tilde{a}_0 = \alpha_1^+.$$
For each \( k \geq 1 \), let \( \tilde{V}_k \) denote the rectangle containing \( Q \) which is bordered by \( \tilde{\alpha}_k \) and \( \partial R \) (see section 2.1 for the definitions of \( \tilde{\alpha}_k \)). Let \( M > 0 \) be a large integer, and define \( V_k = V_{k,M} \) by
\[
V_k = \bigcup_{i=0}^{Mk} f^i \tilde{V}_{2Mk}.
\]
Observe that \( \{V_k\} \) is a nested sequence, and \( \bigcap_{k=1}^{\infty} V_k = \alpha_0^- \).

Fix a partition of unity \( \{\rho_{0,k}, \rho_{1,k}\} \) on \( R \) such that
\[
\text{supp}(\rho_{0,k}) = \{x \in R : \rho_{0,k}(x) \neq 0\} \subset V_k \quad \text{and} \quad \text{supp}(\rho_{1,k}) \subset R \setminus \overline{V_{2k}}.
\]
We argue with subdivision into two cases.

**Case I:** \( u = 0 \). The desired inequalities are direct consequences of the next

**Lemma 4.4.** If \( \{\mu_n\} \subset \mathcal{M}(f) \), \( \mu_n \to \mu \) and \( \mu(Q) = 0 \), then \( \lambda^u(\mu_n) \to \lambda^u(\mu) \).

**Proof.** Set \( \overline{T} = \lim_{n \to \infty} \lambda^u(\mu_n) \) and \( L = \lim_{n \to \infty} \lambda^u(\mu_n) \). Taking subsequences if necessary we may assume \( \overline{T} = \lim_{n \to \infty} \lambda^u(\mu_n) \). Since \( \lambda^u(\mu_n) = \mu_n(\rho_{0,k} \log J^u) + \mu_n(\rho_{1,k} \log J^u) \) and \( \rho_{1,k} \log J^u \) is continuous by proposition 4.1, the limit \( \lim_{n \to \infty} \mu_n(\rho_{1,k} \log J^u) \) exists. Hence, for every \( k \),
\[
\overline{T} = \lim_{n \to \infty} \mu_n(\rho_{0,k} \log J^u) + \lim_{n \to \infty} \mu_n(\rho_{1,k} \log J^u).
\]
(25)
Since \( \mu(Q) = 0 \) and \( \mu \in \mathcal{M}(f) \) we have \( \mu(\partial V_k) = 0 \), and thus \( \lim_{n \to \infty} \mu_n(V_k) = \mu(V_k) \).

Then
\[
\lim_{n \to \infty} \mu_n(\rho_{0,k} \log J^u) \leq \log 5 \cdot \lim_{n \to \infty} \mu_n(V_k) = \log 5 \cdot \mu(V_k).
\]
We also have \( \lim_{n \to \infty} \mu_n(V_k) \to 0 \) as \( k \to \infty \). The weak convergence gives
\[
\lim_{n \to \infty} \mu_n(\rho_{1,k} \log J^u) = \mu(\rho_{1,k} \log J^u).
\]
From the dominated convergence theorem, the second term of the right-hand-side of (25) goes to \( \lambda^u(\mu) \) as \( k \to \infty \). Hence we obtain \( \overline{T} = \lambda^u(\mu) \). The same reasoning gives \( L = \lambda^u(\mu) \). □

**Case II:** \( u \neq 0 \). The next lemma allows us to estimate contributions of the iterates near the saddle \( Q \) to the unstable Lyapunov exponents.

**Lemma 4.5.** There exist large integers \( M_0, k_0 \) such that the following holds for all \( M \geq M_0 \) and \( k \geq k_0 \): if \( z \in \Omega, m > 0 \) are such that \( f^{-1}z \notin V_{k,M}, z, fz, \ldots, f^{m-1}z \in V_{k,M}, f^mz \notin V_{k,M} \), then
\[
\frac{1}{2} \log(4 - \varepsilon) \leq \frac{1}{m} \sum_{i=0}^{m-1} \log J^u(f^iz) \leq \lambda^u(\delta_Q).
\]

**Proof.** Let \( z \in \Omega, m > 0 \) be as in the statement. We have \( z \notin f^i \tilde{V}_{2Mk} \) for every \( 0 < i \leq Mk \), for otherwise \( f^{-i}z \in V_{k,M} \). Since \( z \in V_{k,M} \), we have \( z \in \tilde{V}_{2Mk} \). Hence
\[
m - 1 \geq Mk \quad \text{and} \quad f^{-1}z \in I(\delta).
\]
(26)
where the latter holds provided \( k_0 \) is chosen sufficiently large.
Set \( y = f^{-2} z \). By lemma 2.9 there exist a critical point \( \zeta \) and a \( C^2(b) \)-curve which contains \( \zeta, y \), and is tangent to both \( E^u(\zeta) \) and \( E^u(y) \). Let \( p = p(y) \) denote the corresponding bound period.

In the following we argue as in the proof of proposition 2.5. Fix a \( C^2(b) \)-curve \( \gamma \) which connects \( f(\zeta) \) and \( f^s(\zeta) \). Similarly to the proof of (14) we have \( \text{length}(\gamma) \approx |\zeta - y|^2 \). Since \( f^i(\gamma) (i = 0, 1, \ldots, m + 1) \) are \( C^2(b) \)-curves located near the stable sides of \( R \), and \( f^{m+1} y = f^{m-1} z \in V_k \), \( f^{m+2} y = f^m z \notin V_k \), there exists \( C \geq 1 \) such that
\[
C^{-1} \lambda_2^{-k} \leq \text{length}(f^{m+1}y) \leq C\lambda_1^{-k+1}.
\] (27)

The bounded distortion gives
\[
|\zeta - y|^2 \|w_{m+2}(\zeta)\| \approx \text{length}(f^{m+1}y).
\] (28)

From (15) (16), there exists \( C \geq 1 \) such that
\[
C^{-1}(\lambda_2 + \varepsilon/2)^{-\frac{p}{2}} \leq |\zeta - y| \leq C\lambda_1^{-\frac{p}{2}}.
\] (29)

Let \( v(y) \) denote any vector which spans \( E^u(y) \). Split \( Df v(y) = A \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} + B \cdot e^i(fy) \).

Using (27) (28) (29), for some \( C > 0 \) we have
\[
|A| \cdot \|Df^{m+1}(fy)\| \leq |\zeta - y| \cdot \|w_{m+2}(\zeta)\| \geq C\lambda_2^{-k} |\zeta - y|^{-1} \geq C\lambda_2^{-k}\lambda_1^{\frac{p}{2}}.
\]

On the other hand we have
\[
|B| \cdot \|Df^{m+1}e^i(fy)\| \leq (Ch)^{m+1}.
\]

Since \( \|Df^2v(y)\| \leq 1 \), if \( p \geq m + 2 \) then using \( p > Mk \) which follows from (26) and then choosing sufficiently large \( M_0 \) if necessary, we have
\[
\frac{\|Df^{m+2}v(y)\|}{\|Df^2v(y)\|} \geq C\lambda_2^{-k}\lambda_1^{\frac{p}{2}} - (Ch)^{m+1} \geq (4 - \varepsilon)^{\frac{p}{2}}.
\]

Now, observe that
\[
\frac{1}{m} \sum_{i=0}^{m-1} \log J^u(f^i z) = \frac{1}{m} \log \frac{\|Df^{m+2}v(y)\|}{\|Df^2v(y)\|}.
\]

Hence the first inequality in lemma 4.5 holds. In the case \( p < m + 2 \) the first inequality follows from proposition 2.5(e)(f) and lemma 2.3. The second inequality in the lemma is obvious. \( \square \)

Returning to the proof of proposition 4.3 in the case \( u \neq 0 \), choose \( M \geq M_0 \) and \( k \geq k_0 \) for which the estimates in lemma 4.5 hold. From the ergodic theorem one can choose a point \( \xi_n \in \Omega \) such that
\[
\lim_{m \to \infty} \frac{1}{m} \# \{ 0 \leq i < m : f^i \xi_n \in V_{k,M} \} = \mu_n(V_{k,M}).
\]

If \( \mu_n \neq \delta_Q \), then the positive orbit of \( \xi_n \) is a concatenation of segments in \( V_{k,M} \) and those out of \( V_{k,M} \). Lemma 4.5 yields
\[
\mu_n(\rho_{0,k} J^u) = \lim_{m \to \infty} \frac{1}{m} \sum_{i=0}^{m-1} \rho_{0,k} \log J^u(f^i(\xi_n)) \geq \mu_n(V_{k,M}) \cdot \frac{1}{2} \log(4 - \varepsilon).
\]
Observe that the same inequality still holds in the case $\mu_n = \delta_Q$. Since $\lim_{n \to \infty} \mu_n(V_k, M) \geq u > 0$ we get
\[
\lim_{n \to \infty} \mu_n(\rho_0, k \log J^u) \geq \frac{u}{2} \log(4 - \varepsilon).
\]
If $u \neq 1$, then the weak convergence for the sequence $\{\frac{\mu_n - u \delta_Q}{1 - u}\} \subset \mathcal{M}(f)$ implies
\[
\lim_{n \to \infty} \mu_n(\rho_{1,k} \log J^u) = (1 - u)v(\rho_{1,k} \log J^u).
\]
The same inequality remains true in the case $u = 1$. Consequently,
\[
\lim_{n \to \infty} \lambda^n(\mu_n) \geq \lim_{n \to \infty} \mu_n(\rho_{0,k} \log J^u) + \lim_{n \to \infty} \mu_n(\rho_{1,k} \log J^u)
\]
\[
\geq \frac{u}{2} \log(4 - \varepsilon) + (1 - u)v(\rho_{1,k} \log J^u).
\]
Since $v(Q) = 0$, $\rho_{1,k} \log J^u \to \log J^u$ as $k \to \infty$. Letting $k \to \infty$ and then using the dominated convergence theorem gives the first estimate in the proposition. A proof of the second one is completely analogous, with the second inequality in lemma 4.5. $\square$

4.3. Existence of equilibrium measures for $\varphi_t$

We now complete the proof of the theorem.

**Proof of the theorem.** By the ergodic decomposition theorem [17], the unstable Lyapunov exponent of $\mu$ is written as a linear combination of the unstable Lyapunov exponents of its ergodic components. Since the same property holds for entropies and $\mathcal{M}(f)$ is compact, one can choose a convergent sequence $\{\mu_n\} \subset \mathcal{M}(f)$ such that $F_{\varphi_t}(\mu_n) > P(t) - 1/n$. Let $\mu \in \mathcal{M}(f)$ denote the limit point. In the case $t \leq 0$, the upper semi-continuity of entropy and proposition 4.3 yield $P(t) = \lim_{n \to \infty} F_{\varphi_t}(\mu_n) \leq F_{\varphi_t}(\mu)$. Namely $\mu$ is an equilibrium measure for $\varphi_t$.

We now consider the case $t > 0$. Write $\mu = u \delta_Q + (1 - u)v$ where $0 \leq u \leq 1, v \in \mathcal{M}(f)$ and $v(Q) = 0$. The upper semi-continuity of entropy gives
\[
P(t) = \lim_{n \to \infty} F_{\varphi_t}(\mu_n) \leq h(\mu) - t \lim_{n \to \infty} \lambda^n(\mu_n).
\]
If $u = 1$ then $\mu = \delta_Q$ and thus $h(\mu) = 0$. Proposition 4.3 gives $P(t) \leq -(t/2) \log(4 - \varepsilon)$ and a contradiction arises because $P(t) > -(t/2) \log(4 - \varepsilon)$ from (3) and $t < t_0$. Hence $u \neq 1$ holds. If $u \neq 0$, then using proposition 4.3 and $h(\mu) = (1 - u)h(v)$ we have
\[
P(t) \leq h(\mu) - t \left( \frac{u}{2} \log(4 - \varepsilon) + (1 - u)\lambda^u(v) \right)
\]
\[
= (1 - u)F_{\varphi_t}(v) - \frac{tu}{2} \log(4 - \varepsilon) < (1 - u)F_{\varphi_t}(v) + uP(t).
\]
Rearranging this gives $(1 - u)P(t) < (1 - u)F_{\varphi_t}(v)$, and thus $P(t) < F_{\varphi_t}(v)$, a contradiction. Hence $u = 0$, and $P(t) \leq F_{\varphi_t}(v) = F_{\varphi_t}(\mu)$. Namely $v$ is an equilibrium measure for $\varphi_t$.

**Acknowledgments**

We thank anonymous referees for useful comments. SS is partially supported by the CNPq Brazil. HT is partially supported by the Grant-in-Aid for Young Scientists (B) of the JSPS, Grant No 23740121. This research is partially supported by the Kyoto University Global COE Programme. We thank Renaud Leplaideur, Isabel Rios, Paulo Varandas, and Michiko Yuri for fruitful discussions.
Appendix: on the size of $t_0$.

Since the topological entropy of $f$ is $\log 2$, the variational principle shows $P(0) = \log 2$. By Ruelle’s inequality [22], $P(1) \leq 0$. Since $f$ has no SRB measure [26], $P(1) < 0$. Hence, there equation $P(t) = 0$ has the unique solution in $(0, 1)$, which is denoted by $t^\dagger$. Observe that $t^\dagger < t_0$. From the next lemma and the fact that $t^\dagger \to 1$ as $b \to 0$ [23, theorem B] it follows that $t_0$ can be made arbitrarily large by choosing sufficiently small $\varepsilon$ and $b$.

Lemma 4.6. $t_0 \geq (1/t^\dagger) \log 2 - (1/2) \log (4 - \varepsilon)$.

Proof. Consider the pressure function $t \in \mathbb{R} \mapsto P(t)$ and its graph. The two points $(0, \log 2)$ and $(t^\dagger, 0)$ lie on the graph. Since the graph is concave up, $[(t, P(t)) : t > t^\dagger]$ lies above the straight line through the two points. In other words, $P(t) > -(1/t^\dagger)(t - t^\dagger) \log 2 + \log 2$. A direct computation shows that $-(1/t^\dagger)(t - t^\dagger) \log 2 + \log 2 > -(t/2) \log (4 - \varepsilon)$ provided $t < (1/t^\dagger) \log 2 - (1/2) \log (4 - \varepsilon)$. □

References

[1] Arbieto A and Prudente L 2012 Uniqueness of equilibrium states for some partially hyperbolic horseshoes Discrete Contin. Dyn. Syst. 32 27–40
[2] Bedford E and Smillie J 2004 Real polynomial diffeomorphisms with maximal entropy: tangencies Ann. Math. 160 1–25
[3] Bedford E and Smillie J 2006 Real polynomial diffeomorphisms with maximal entropy: II. Small Jacobian Ergod. Theory Dyn. Syst. 26 1259–83
[4] Benedicks M and Carleson L 1991 The dynamics of the Hénon map Ann. Math. 133 73–169
[5] Benedicks M and Carleson L 1985 On iterations of $1 - ax^2$ on $(-1, 1)$ Ann. Math. 122 1–25
[6] Cao Y, Luzzatto S and Rios I 2008 The boundary of hyperbolicity for Hénon-like families Ergod. Theory Dyn. Syst. 28 1049–80
[7] Devaney R and Nitecki Z 1979 Shift automorphisms in the Hénon mapping Commun. Math. Phys. 67 137–46
[8] Díaz L, Horita V, Sambarino M and Rios I 2009 Destroying horseshoes via heterodimensional cycles: generating bifurcations inside homoclinic classes Ergod. Theory Dyn. Syst. 29 433–74
[9] Hoensch U A 2008 Some hyperbolicity results for Hénon-like diffeomorphisms Nonlinearity 21 587–611
[10] Jakobson M 1981 Absolutely continuous invariant measures for one-parameter families of one-dimensional maps Commun. Math. Phys. 81 39–88
[11] Kirk S 1996 The Palis–Takens problem on the first homoclinic tangency inside the horseshoe Int. J. Bifurc. Chaos Appl. Sci. Eng. 6 737–44
[12] Leplaideur R 2011 Thermodynamic formalism for a family of nonuniformly hyperbolic horseshoes and the unstable Jacobian Ergod. Theory Dyn. Syst. 31 423–47
[13] Leplaideur R and Rios I 2005 Invariant manifolds and equilibrium states for non-uniformly hyperbolic horseshoes Nonlinearity 18 2847–80
[14] Leplaideur R and Rios I 2009 On $t$-conformal measures and Hausdorff dimension for a family of non-uniformly hyperbolic horseshoes Ergod. Theory Dyn. Syst. 29 1917–30
[15] Leplaideur R, Oliveira K and Rios I 2011 Equilibrium states for partially hyperbolic horseshoes Ergod. Theory Dyn. Syst. 31 179–95
[16] Makarov N and Smirnov S 2003 On thermodynamics of rational maps II.: Non-recurrent maps J. Lond. Math. Soc. 67 417–32
[17] Mañé R 1987 Ergodic Theory and Differentiable Dynamics (Ergebnisse der Mathematik und ihrer Grenzgebiete vol 3) (Berlin: Springer)
[18] Manning A and McCluskey H 1983 Hausdorff dimension for horseshoes Ergod. Theory Dyn. Syst. 3 251–260
[19] Mora L and Viana M 1993 Abundance of strange attractors Acta Math. 171 1–71
[20] Palis J and Takens F 1993 Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations. (Cambridge Studies in Advanced Mathematics vol 35) (Cambridge: Cambridge University Press)
[21] Rios I 2001 Unfolding homoclinic tangencies inside horseshoes: hyperbolicity, fractal dimensions and persistent tangencies Nonlinearity 14 431–62
[22] Ruelle D 1978 An inequality for the entropy of differentiable maps Bol. Soc. Brasil. Math. 9 83–7
[23] Senti S and Takahasi H 2012 Equilibrium measures for the Hénon map at the first bifurcation: uniqueness and geometric/statistical properties (arXiv:1209.2224)
[24] Shub M 1987 Global Stability of Dynamical Systems (New York: Springer)
[25] Takahasi H 2011 Abundance of nonuniform hyperbolicity in bifurcations of surface endomorphisms Tokyo J. Math. 34 53–113
[26] Takahasi H 2012 Prevalent dynamics at the first bifurcation of Hénon-like families Commun. Math. Phys. 312 37–85
[27] Walters P 1982 An Introduction to Ergodic Theory (Graduate Texts in Mathematics vol 79) (New York: Springer)
[28] Wang Q-D and Young L-S 2001 Strange attractors with one direction of instability Commun. Math. Phys. 218 1–97