Polynomial approximation in weighted Dirichlet spaces

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Abstract We give an elementary proof of an analogue of Fejér’s theorem in weighted Dirichlet spaces with superharmonic weights. This provides a simple way of seeing that polynomials are dense in such spaces.

Keywords Dirichlet space · superharmonic weight · Fejér theorem

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1 Introduction and statement of main result

Let $D$ be the open unit disk and $T$ be the unit circle. We denote $\text{Hol}(D)$ the set of all holomorphic functions on $D$, and by $H^2$ the Hardy space on $D$.

Given $\zeta \in D$, we define $\mathcal{D}_\zeta$ to be the set of all $f \in \text{Hol}(D)$ of the form

$$f(z) = a + (z - \zeta)g(z),$$

where $g \in H^2$ and $a \in \mathbb{C}$. In this case, we set $\mathcal{D}_\zeta(f) := \|g\|_{H^2}^2$. We adopt the convention that, if $f \in \text{Hol}(D)$ but $f \notin \mathcal{D}_\zeta$, then $\mathcal{D}_\zeta(f) := \infty$.

Given a positive finite Borel measure $\mu$ on $\overline{D}$, we define $\mathcal{D}_\mu$ to be the set of all $f \in \text{Hol}(D)$ such that

$$\mathcal{D}_\mu(f) := \int_D \mathcal{D}_\zeta(f) d\mu(\zeta) < \infty.$$

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We endow $\mathcal{D}_\mu$ with the norm $\| \cdot \|_{\mathcal{D}_\mu}$ defined by

$$\|f\|_{\mathcal{D}_\mu}^2 := |f(0)|^2 + \mathcal{D}_\mu(f).$$

The spaces $\mathcal{D}_\mu$ are in fact precisely the weighted Dirichlet spaces on $\mathbb{D}$ with superharmonic weights. Though this identification is not required for a technical understanding of the results in this note, it serves as background and motivation for these results. The final section §4 contains a brief account of these spaces and their history.

When $\mu = \delta_\zeta$, the Dirac measure at $\zeta \in \mathbb{D}$, the space $\mathcal{D}_\mu$ reduces to $\mathcal{D}_\zeta$, which is sometimes called the local Dirichlet space at $\zeta$.

A fundamental property of the spaces $\mathcal{D}_\mu$ is that polynomials are dense. This fact was originally established by Richter [5] andAleman [1] using a type of wandering-subspace theorem. Other proofs followed, most recently a direct proof via a Fejér-type approximation theorem [3]. Our goal in this note is to give a more elementary proof of this last theorem. In fact we shall establish the following generalization.

**Theorem 1.1** Let $(w_{n,k})_{n,k \geq 0}$ be an array of complex numbers such that:

\begin{align}
w_{n,k} &= 0 \quad (k > n), \\
\lim_{n \to \infty} w_{n,k} &= 1 \quad (k \geq 0), \\
|w_{n,k}| &\leq M \quad (n, k \geq 0), \\
|w_{n,k} - w_{n,k+1}| &\leq L/n \quad (n, k \geq 0),
\end{align}

where $L, M$ are constants. Given $f \in \text{Hol}(\mathbb{D})$, say $f(z) = \sum_{k=0}^\infty a_k z^k$, set

$$p_n(z) := \sum_{k=0}^n w_{n,k} a_k z^k.\quad (1.5)$$

Then, for each positive finite measure $\mu$ on $\overline{\mathbb{D}}$, such that $f \in \mathcal{D}_\mu$, we have

$$\|f - p_n\|_{\mathcal{D}_\mu} \to 0 \quad (n \to \infty).$$

As a consequence, we deduce the Fejér-type theorem mentioned above. Given $f(z) = \sum_{k=0}^n a_k z^k$, we write

$$s_n(f)(z) := \sum_{k=0}^n a_k z^k \quad \text{and} \quad \sigma_n(f)(z) := \sum_{k=0}^n \left( 1 - \frac{k}{n+1} \right) a_k z^k.$$

**Corollary 1** If $\mu$ is a finite measure on $\overline{\mathbb{D}}$ and if $f \in \mathcal{D}_\mu$, then

$$\|\sigma_n(f) - f\|_{\mathcal{D}_\mu} \to 0 \quad (n \to \infty).$$

**Proof** Apply Theorem 1.1 with $w_{n,k} := 1 - k/(n+1)$ if $k \leq n$ and zero otherwise. \(\square\)

By contrast, it is known that, if $\mu = \delta_1$, the Dirac measure at $\zeta = 1$, then there exists $f \in \mathcal{D}_\mu$ such that $\|s_n(f) - f\|_{\mathcal{D}_\mu} \neq 0$ as $n \to \infty$ (see for example [2, p.117, Exercise 7.3.2]). This shows that Theorem 1.1 is no longer true if we omit the condition (1.4).
2 Approximation in local Dirichlet spaces

We begin with a simple lemma about approximation in $H^2$.

**Lemma 1** Let $(w_{n,k})_{n,k \geq 0}$ be an array of complex numbers satisfying the conditions (1.1), (1.2) and (1.3). Let $g \in H^2$, say $g(z) = \sum_{k=0}^\infty b_k z^k$, and for $n \geq 0$ let

$$g_n(z) := \sum_{k=0}^{n-1} w_{n,k+1} b_k z^k.$$  \hspace{1cm} (2.1)

Then

$$\|g - g_n\|_{H^2} \to 0 \quad (n \to \infty).$$  \hspace{1cm} (2.2)

Moreover, there exists a constant $C$, depending only on the array $(w_{n,k})$, such that

$$\|g_n\|_{H^2} \leq C \|g\|_{H^2} \quad (n \geq 0).$$  \hspace{1cm} (2.3)

**Proof** From condition (1.3) we have

$$\|g_n\|_{H^2}^2 = \sum_{k=0}^\infty |w_{n,k+1}|^2 |b_k|^2 \leq M^2 \sum_{k=0}^\infty |b_k|^2 = M^2 \|g\|_{H^2}^2.$$  

This gives (2.3) with $C = M$.

Also, given $\varepsilon > 0$, we can choose $N$ large enough so that $\sum_{k=N}^\infty |b_k|^2 < \varepsilon^2$, and then, for $n \geq N$, we have

$$\|g - g_n\|_{H^2}^2 \leq \sum_{k=0}^{N-1} |1 - w_{n,k+1}|^2 |b_k|^2 + (1 + M)^2 \varepsilon^2.$$  

From (1.2), the first term on the right-hand side tends to zero as $n \to \infty$. Thus we have $\limsup_{n \to \infty} \|g - g_n\|_{H^2} \leq (1 + M) \varepsilon$. As $\varepsilon$ is arbitrary, this gives (2.2). \hfill \Box

In light of the definition of $D_\zeta$, this lemma translates into the following approximation result for local Dirichlet spaces.

**Theorem 2.1** Let $(w_{n,k})_{n,k \geq 0}$ be an array of complex numbers satisfying conditions (1.1), (1.2) and (1.3). Let $\zeta \in \overline{D}$ and let $f \in D_\zeta$, say $f(z) = a + (z - \zeta)g(z)$, where $g \in H^2$ and $a \in \mathbb{C}$. Define $g_n$ as in (2.1) and set

$$f_n(z) := a + (z - \zeta)g_n(z).$$  \hspace{1cm} (2.4)

Then

$$D_\zeta(f - f_n) \to 0 \quad (n \to \infty),$$

and there exists a constant $C$, depending only on the array $(w_{n,k})$, such that

$$D_\zeta(f_n) \leq C^2 D_\zeta(f) \quad (n \geq 0).$$

**Proof** This is an immediate consequence of the identities $D_\zeta(f_n) = \|g_n\|_{H^2}^2$ and $D_\zeta(f - f_n) = \|g - g_n\|_{H^2}^2$. \hfill \Box
Let us compute the polynomials \( f_n \) explicitly. If we write \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( g(z) = \sum_{k=0}^{\infty} b_k z^k \) and equate coefficients of \( z^k \) in the relation \( f(z) = a + (z - \zeta)g(z) \), then we obtain
\[
\begin{align*}
  a_0 &= a - \zeta b_0, \\
  a_k &= b_{k-1} - \zeta b_k \quad (k \geq 1).
\end{align*}
\] (2.5)
Hence
\[
 f_n(z) = a + (z - \zeta)g_n(z) 
 = a + (z - \zeta) \sum_{k=0}^{n-1} w_{n,k+1} b_k z^k 
 = a + \sum_{k=1}^{n} w_{n,k} b_{k-1} z^k - \zeta \sum_{k=0}^{n} w_{n,k+1} b_k z^k 
 = a + \sum_{k=1}^{n} w_{n,k} b_{k-1} z^k + \zeta \sum_{k=0}^{n} (w_{n,k} - w_{n,k+1}) b_k z^k - \zeta w_{n,0} b_0 
 = a + \sum_{k=1}^{n} w_{n,k} a_k z^k + \zeta \sum_{k=0}^{n} (w_{n,k} - w_{n,k+1}) b_k z^k - \zeta w_{n,0} b_0,
\]
the last line using (2.5). Rearranging this slightly, we get
\[
 f_n(z) = \sum_{k=0}^{n} w_{n,k} a_k z^k + \zeta \sum_{k=0}^{n} (w_{n,k} - w_{n,k+1}) b_k z^k + (1 - w_{n,0}) a. 
\] (2.6)

The following special case is worthy of note.

**Corollary 2** Let \( \zeta \in \mathbb{W} \) and let \( f \in D_\zeta \), say \( f(z) = \sum_{k=0}^{\infty} a_k z^k \). Then \( \sum_{k=0}^{\infty} a_k z^k \) converges and, setting
\[
 f_n(z) := \sum_{k=0}^{n-1} a_k z^k + \left( \sum_{k=n}^{\infty} a_k z^k \right) \zeta^n,
\]
we have
\[
 D_\zeta (f - f_n) \rightarrow 0.
\]

**Proof** Let \( w_{n,k} := 1 \) for \( k \leq n \) and zero otherwise. This satisfies (1.1), (1.2) and (1.3), so by Corollary 2 we have \( D_\zeta (f - f_n) \rightarrow 0 \) as \( n \rightarrow \infty \). Formula (2.6) shows that
\[
 f_n(z) = \sum_{k=0}^{n} a_k z^k + \zeta b_n z^n = \sum_{k=0}^{n-1} a_k z^k + b_{n-1} z^n,
\]
where the last equality is from (2.5).

All that remains is to identify \( b_{n-1} \). For this, we note that, by (2.5), for all \( N \geq n \) we have
\[
\sum_{k=n}^{N} a_k z^k = z^1 - z \sum_{k=n}^{N} (b_{k-1} z^{k-1} - b_k z^k) = b_{n-1} - b_N z^{N+1-n}.
\]
Since the sequence \( (b_k) \) is square summable, we also have \( b_N \rightarrow 0 \) as \( N \rightarrow \infty \). Hence \( b_{n-1} = \sum_{k=n}^{\infty} a_k z^{k-n} \), as desired. \( \square \)
### 3 Proof of Theorem 1.1

Unfortunately, the polynomials \( f_n \) that approximate \( f \) in Theorem 2.1 and Corollary 2 depend upon \( \zeta \) (note that the \( (b_k) \) also depend upon \( \zeta \)). This makes them unsuitable for approximation in \( D_\mu \). To circumvent this difficulty, we return to the formula (2.6). Notice that, although \( f_n \) itself depends on \( \zeta \), the first term on the right-hand side of (2.6) does not. If we can somehow show that the other terms tend to zero in an appropriate way, then the first term approximates \( f_n \) and hence also \( f \). This is the strategy for the proof of Theorem 1.1. To implement it, we need the following general estimate.

**Lemma 2** Let \( q \) be a polynomial of degree \( n \). Then

\[
\mathcal{D}_\zeta(q) \leq n^2 \|q\|^2_{H^2} \quad (\zeta \in \mathbb{D}).
\]

**Proof** We have

\[
\mathcal{D}_\zeta(q) = \sum_{k=0}^{n-1} |d_k|^2,
\]

where the \( (d_k) \) are determined by the relation

\[
q(z) = q(\zeta) + (z - \zeta) \sum_{k=0}^{n-1} d_k z^k.
\]

Writing \( q(z) = \sum_{j=0}^{n} c_j z^j \), and equating coefficients of powers of \( z \) gives

\[
c_n = d_{n-1}
\]

and

\[
c_k = d_{k-1} - \zeta d_k \quad (1 \leq k \leq n-1).
\]

Solving for \( d_k \) in terms of \( c_k \), we get

\[
d_k = \sum_{j=k+1}^{n} c_j \xi^{n-j} \quad (0 \leq k \leq n-1).
\]

By the Cauchy–Schwarz inequality, it follows that

\[
|d_k|^2 \leq (n-k) \sum_{j=k+1}^{n} |c_j|^2 \leq n^2 \|q\|^2_{H^2} \quad (0 \leq k \leq n-1).
\]

Hence, finally

\[
\mathcal{D}_\zeta(q) = \sum_{k=0}^{n-1} |d_k|^2 \leq n^2 \|q\|^2_{H^2}.
\]

This completes the proof of the lemma. \( \square \)

**Completion of proof of Theorem 1.1** Let \( \mu \) be a finite positive Borel measure on \( \mathbb{D} \), and let \( f \in D_\mu \). Then

\[
\int_{\mathbb{D}} \mathcal{D}_\zeta(f) \, d\mu(\zeta) < \infty,
\]

and in particular \( \mathcal{D}_\zeta(f) < \infty \) for \( \mu \)-almost every \( \zeta \) in \( \mathbb{D} \). We claim that, for each such \( \zeta \), and with \( p_n \) as defined as in (1.5), we have

\[
\lim_{n \to \infty} \mathcal{D}_\zeta(f - p_n) = 0 \quad \text{and} \quad \sup_{n \geq 1} \mathcal{D}_\zeta(f - p_n) \leq C^2 \mathcal{D}_\zeta(f), \quad (3.1)
\]
where $C$ is a constant depending only on the array $(w_{n,k})$. If so, then, by the dominated convergence theorem, we have
\[
\mathcal{D}_\mu(f - p_n) = \int_\mathcal{D} \mathcal{D}_\xi(f - p_n) d\mu(\xi) \to 0 \quad (n \to \infty).
\]
Since also $f(0) - p_n(0) = a_0(1 - w_{n,0}) \to 0$ as $n \to \infty$, it follows that
\[
\|f - p_n\|_{\mathcal{D}_\mu} \to 0 \quad (n \to \infty),
\]
thereby establishing the theorem.

It remains to verify the claim (3.1). Fix $\xi$ with $\mathcal{D}_\xi(f) < \infty$, and define $g_n$ and $f_n$ as in (2.1) and (2.4) respectively. As $\mathcal{D}_\xi(\cdot)_{1/2}$ is a seminorm, we have
\[
\mathcal{D}_\xi(f - p_n)^{1/2} \leq \mathcal{D}_\xi(f - f_n)^{1/2} + \mathcal{D}_\xi(f_n - p_n)^{1/2}.
\]
By Corollary 2 we have
\[
\lim_{n \to \infty} \mathcal{D}_\xi(f - f_n) = 0 \quad \text{and} \quad \sup_{n \geq 1} \mathcal{D}_\xi(f - f_n) \leq C_1^2 \mathcal{D}_\xi(f),
\]
where $C_1$ depends only on $(w_{n,k})$. Therefore it remains to show that
\[
\lim_{n \to \infty} \mathcal{D}_\xi(f_n - p_n) = 0 \quad \text{and} \quad \sup_{n \geq 1} \mathcal{D}_\xi(f_n - p_n) \leq C_2^2 \mathcal{D}_\xi(f), \quad (3.2)
\]
where $C_2$ depends only on $(w_{n,k})$. For this we use the formula (2.6), according to which
\[
f_n(z) - p_n(z) = \xi \sum_{k=0}^n (w_{n,k} - w_{n,k+1}) b_k z^k + \text{constant}.
\]
Using Lemma 3 and the fact that $\mathcal{D}_\xi(\cdot)$ is zero on constants, we get
\[
\mathcal{D}_\xi(f_n - p_n) \leq n^2 \sum_{k=0}^n |w_{n,k} - w_{n,k+1}| |b_k|^2.
\]
Using condition (1.4), we have
\[
\mathcal{D}_\xi(f_n - p_n) \leq L^2 \sum_{k=0}^N |b_k|^2 \leq L^2 \|g\|^2_{H^1} = L^2 \mathcal{D}_\xi(f),
\]
which yields the second part of (3.2) with $C_2 = L$. As for the first part, given $\varepsilon > 0$, we can choose $N$ so large that $\sum_{k=N}^\infty |b_k|^2 < \varepsilon^2$. Then, for all $n \geq N$, we have
\[
\mathcal{D}_\xi(f_n - p_n)^{1/2} \leq \mathcal{D}_\xi\left(\sum_{k=0}^{N-1} (w_{n,k} - w_{n,k+1}) b_k z^k\right)^{1/2} + \mathcal{D}_\xi\left(\sum_{k=N}^n (w_{n,k} - w_{n,k+1}) b_k z^k\right)^{1/2} \leq \left(N^2 \sum_{k=0}^{N-1} |w_{n,k} - w_{n,k+1}|^2 |b_k|^2\right)^{1/2} + \left(n^2 \sum_{k=N}^n |w_{n,k} - w_{n,k+1}|^2 |b_k|^2\right)^{1/2} \leq \left(N^2 \sum_{k=0}^{N-1} |w_{n,k} - w_{n,k+1}|^2 |b_k|^2\right)^{1/2} + L\varepsilon.
\]
From condition (1.2), the first term on the right-hand side tends to zero as $n \to \infty$. Thus $\limsup_{n \to \infty} \mathcal{D}_\xi(f_n - p_n) \leq L\varepsilon$. As $\varepsilon$ is arbitrary, this establishes the first part of (3.2), and completes the proof of the theorem. \qed
4 Background on weighted Dirichlet spaces

The purpose of this section is to explain the origin of the spaces $P_\mu$ and their connection to weighted Dirichlet spaces. As mentioned in the introduction, though this is not required for a technical understanding of the results in this note, it serves as background and motivation for these results.

Given a positive integrable function $\omega$ on $\mathbb{D}$, the $\omega$-weighted Dirichlet space consists of those $f \in \text{Hol}(\mathbb{D})$ such that

$$
\int_\mathbb{D} |f'(z)|^2 \omega(z) \, dA(z) < \infty,
$$

where $dA$ denotes normalized area measure on $\mathbb{D}$. The classical Dirichlet space corresponds to taking $\omega \equiv 1$.

One class of weights that has been much studied over the years are the power weights $\omega(z) := (1 - |z|^2)^{1-\alpha}$, where $0 \leq \alpha \leq 1$. One can show that, if $f(z) = \sum_{k=0}^{\infty} a_k z^k$, then

$$
\int_\mathbb{D} |f'(z)|^2 (1 - |z|^2)^{1-\alpha} \, dA(z) \asymp \sum_{k=1}^{\infty} k^\alpha |a_k|^2.
$$

Thus, as $\alpha$ runs from 0 to 1, we obtain a scale of spaces between the Hardy space ($\alpha = 0$) and the classical Dirichlet space ($\alpha = 1$).

Another important class of weights $\omega$ are the harmonic weights. These were introduced by Richter [5] in connection with his analysis of shift-invariant subspaces of the classical Dirichlet space, and further studied by Richter and Sundberg in [6].

Subsequently Aleman [1] introduced the class of superharmonic weights, which subsumes the power weights and the harmonic weights.

Let $\omega$ be a positive superharmonic function on $\mathbb{D}$. Then there exists a unique positive finite Borel measure $\mu$ on $\mathbb{D}$ such that, for all $z \in \mathbb{D},$

$$
\omega(z) = \int_\mathbb{D} \log \left| \frac{1 - \zeta z}{\zeta - z} \right| \frac{2}{1 - |\zeta|^2} \, d\mu(\zeta) + \int_T \frac{1 - |\zeta|^2}{|\zeta - z|^2} \, d\mu(\zeta),
$$

(see e.g. [4, Theorem 4.5.1]). Defining $P_\mu(f)$ as in §1 we have the following result, which was established by Richter and Sundberg [6, Proposition 2.2] in the case when $\omega$ is harmonic on $\mathbb{D}$ (this corresponds to $\mu$ being supported on $T$) and by Aleman [1, §IV, Theorem 1.9] for general superharmonic weights.

**Theorem 4.1** Let $\omega$ be a superharmonic weight on $\mathbb{D}$, and let $\mu$ be the associated measure on $\mathbb{D}$. Then, for all $f \in \text{Hol}(\mathbb{D})$, we have

$$
P_\mu(f) = \int_\mathbb{D} |f'(z)|^2 \omega(z) \, dA(z).
$$

Thus $P_\mu$ is exactly the $\omega$-weighted Dirichlet space, justifying the assertion made in the introduction.

Further information on weighted Dirichlet spaces can be found in [6] and [1], as well as in Chapter 7 of the monograph [2].
Conflict of interest

The authors declare that they have no conflict of interest.

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