Abstract. This work focuses on the iterative solution of sequences of KKT linear systems arising in interior point methods applied to large convex quadratic programming problems. This task is the computational core of the interior point procedure and an efficient preconditioning strategy is crucial for the efficiency of the overall method. Constraint preconditioners are very effective in this context; nevertheless, their computation may be very expensive for large-scale problems, and resorting to approximations of them may be convenient. Here we propose a procedure for building inexact constraint preconditioners by updating a seed constraint preconditioner computed for a KKT matrix at a previous interior point iteration. These updates are obtained through low-rank corrections of the Schur complement of the (1,1) block of the seed preconditioner. The updated preconditioners are analyzed both theoretically and computationally. The results obtained show that our updating procedure, coupled with an adaptive strategy for determining whether to reinitialize or update the preconditioner, can enhance the performance of interior point methods on large problems.

Key words. convex quadratic programming, interior point methods, KKT systems, constraint preconditioners, matrix updates.

AMS subject classifications. 65F08, 65F10, 90C20, 90C51.

1. Introduction. Second-order methods for constrained and unconstrained optimization require, at each iteration, the solution of a system of linear equations. For large-scale problems it is common to solve each system by an iterative method coupled with a suitable preconditioner. Since the computation of a preconditioner for each linear system can be very expensive, in recent years there has been a growing interest in reducing the cost for preconditioning by sharing some computational effort through subsequent linear systems.

Updating preconditioner frameworks for sequences of linear systems have a common feature: based on information generated during the solution of a linear system in the sequence, a preconditioner for a subsequent system is generated. An efficient updating procedure is expected to build a preconditioner which is less effective in terms of linear iterations than the one computed from scratch, but more convenient in terms of cost for the overall linear algebra phase. The approaches existing in literature can be broadly classified as: limited-memory quasi-Newton preconditioners for symmetric positive definite and nonsymmetric matrices (see, e.g., [10, [34, [39]), recycled Krylov information preconditioners for symmetric and nonsymmetric matrices (see, e.g., [19, [28, [30, [36]), updates of factorized preconditioners for symmetric positive definite and nonsymmetric matrices (see, e.g., [23, [24, [38, [38]).
In this paper we study the problem of preconditioning sequences of KKT systems arising in the solution of the convex quadratic programing (QP) problem

\[
\text{minimize } \frac{1}{2} x^T Q x + c^T x, \\
\text{subject to } A_1 x - s = b_1, \quad A_2 x = b_2, \quad x + v = u, \quad (x, s, v) \geq 0
\]

by Interior Point (IP) methods \([32, 43]\). Here \(Q \in \mathbb{R}^{n \times n}\) is symmetric positive semidefinite, and \(A_1 \in \mathbb{R}^{m_1 \times n}, A_2 \in \mathbb{R}^{m_2 \times n}\), with \(m = m_1 + m_2 \leq n\). Note that \(s\) and \(v\) are slack variables, used to transform the inequality constraints \(A_1 x \geq b_1\) and \(x \leq u\) into equality constraints.

The updating strategy proposed here concerns constraint preconditioners (CPs) \([11, 14, 20, 22, 29, 35, 40]\). Using the factorization of a seed CP computed for some KKT matrix of the sequence, the factorization of an approximate CP is built for subsequent systems. The resulting preconditioner is a special case of inexact CP \([12, 25, 37, 40, 41]\) and is intended to reduce the computational cost while preserving effectiveness. To the best of our knowledge, this is the first attempt to study, both theoretically and numerically, a preconditioner updating technique for sequences of KKT systems arising from \((1.1)\). A computational analysis of the reuse of CPs in consecutive IP steps, which can be regarded as a limit case of preconditioner updating, has been presented in \([16]\).

Concerning preconditioner updates for sequences of systems arising from IP methods, we are aware of the work in \([1, 42]\), where linear programming problems are considered. In this case, the KKT linear systems are reduced to the so-called normal equation form; some systems of the sequence are solved by the Cholesky factorization, while the remaining ones are solved by the Conjugate Gradient method preconditioned with a low-rank correction of the last computed Cholesky factor.

Motivated by \([1, 42]\), in this work we adapt the low-rank corrections given therein to our problem. In our approach the updated preconditioner is an inexact CP where the Schur complement of the (1,1) block is replaced by a low-rank modification of the corresponding Schur complement in the seed preconditioner. The validity of the proposed procedure is supported by a spectral analysis of the preconditioned matrix and by numerical results illustrating its performance.

The paper is organized as follows. In section 2 we provide preliminaries on CPs and new spectral analysis results for general inexact CPs. In section 3 we present our updating procedure and specialize the spectral analysis conducted in the previous section to the updated preconditioners. In section 4 we discuss implementation issues of the updating procedure and present numerical results obtained by solving sequences of linear systems arising in the solution of convex QP problems. These results show that our updating technique is able to reduce the computational cost for solving the overall sequence whenever the updating strategy is performed in conjunction with adaptive strategies for determining whether to recompute the seed preconditioner or update the current one.

In the following, \(\| \cdot \|\) denotes the vector or matrix 2-norm and, for any symmetric matrix \(A\), \(\lambda_{\text{min}}(A)\) and \(\lambda_{\text{max}}(A)\) denote its minimum and maximum eigenvalues.

2. Inexact Constraints Preconditioners. In this section we discuss the features of the KKT matrices arising in the solution of problem \((1.1)\) by IP methods and present spectral properties of both CPs and inexact CPs. This analysis will be used to develop our updating strategy. Throughout the paper we assume that the matrix
\[ A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \in \mathbb{R}^{m \times n} \] is full rank.

The application of an IP method to problem (1.1) gives rise to a sequence of symmetric indefinite matrices differing by a diagonal, possibly indefinite, matrix. In fact, at the \( k \)th IP iteration, the KKT matrix takes the form

\[
A_k = \begin{bmatrix} Q + \Theta_k^{(1)} & A^T \\ A & -\Theta_k^{(2)} \end{bmatrix},
\]

where \( \Theta_k^{(1)} \in \mathbb{R}^{n \times n} \) is diagonal positive definite and \( \Theta_k^{(2)} \in \mathbb{R}^{m \times m} \) is diagonal positive semidefinite. In particular,

\[
\Theta_k^{(1)} = X_k^{-1} W_k + V_k^{-1} T_k,
\]

\[
\Theta_k^{(2)} = \begin{bmatrix} Y_k^{-1} S_k & 0 \\ 0 & 0 \end{bmatrix},
\]

where \((x_k, w_k)\), \((s_k, y_k)\), and \((v_k, t_k)\) are the pairs of complementary variables of problem (1.1) evaluated at the current iteration, and \( X_k, W_k, S_k, Y_k, V_k \) and \( T_k \) are the corresponding diagonal matrices according to the standard IP notation. If the QP problem has no linear inequality constraints then \( \Theta_k^{(2)} \) is the zero matrix; otherwise \( \Theta_k^{(2)} \) admits positive diagonal entries corresponding to slack variables for linear inequality constraints.

To simplify the notation, in the rest of the paper we drop the iteration index \( k \) from \( A_k, \Theta_k^{(1)} \) and \( \Theta_k^{(2)} \). Hence,

\[(2.1) \quad A = A_k,\]

and the CP for \( A \) is given by

\[(2.2) \quad \mathcal{P}_{ex} = \begin{bmatrix} G & A^T \\ A & -\Theta^{(2)} \end{bmatrix},\]

where \( G \) is an approximation to \( Q + \Theta^{(1)} \). We use the common choice where \( G \) is the diagonal matrix with the same diagonal entries as \( Q + \Theta^{(1)} \), i.e.,

\[ G = \text{diag}(Q + \Theta^{(1)}). \]

The matrix \( \mathcal{P}_{ex}^{-1} A \) has an eigenvalue at 1 with multiplicity \( 2m - p \), with \( p = \text{rank}(\Theta^{(2)}) \), and \( n - m + p \) real positive eigenvalues such that the better \( G \) approximates \( Q + \Theta^{(1)} \) the more clustered around 1 they are \([22, 35]\).

The application of \( \mathcal{P}_{ex} \) requires its factorization, which can be computed either by using any \( LDL^T \) algorithm with a suitable pivoting strategy, e.g., the Bunch-Parlett one \([13]\), or by exploiting the block factorization

\[(2.3) \quad \mathcal{P}_{ex} = \begin{bmatrix} I_n \\ AG^{-1} \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & -S \end{bmatrix} \begin{bmatrix} I_n & G^{-1} A^T \\ 0 & I_m \end{bmatrix},\]

where \( I_r \) is the identity matrix of dimension \( r \) and \( S \) is the negative Schur complement of \( G \) in \( A \),

\[(2.4) \quad S = AG^{-1} A^T + \Theta^{(2)},\]
and forming a Cholesky-like factorization of $S$. In this work we consider the latter factorization.

In problems where a large part of the computational cost for solving the linear system depends on the computation of a Cholesky-like factorization of $S$, this matrix may be replaced by a computationally cheaper approximation of it \[25, 37, 40\]. Letting $S_{\text{inex}}$ be such approximation, the inexact CP takes the form

\[
P_{\text{inex}} = \begin{bmatrix} I_n & 0 \\ AG^{-1} & I_m \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & -S_{\text{inex}} \end{bmatrix} \begin{bmatrix} I_n & G^{-1}A^T \\ 0 & I_m \end{bmatrix}.
\]

(2.5)

Approximations of CPs may be also obtained by replacing the constraint matrix $A$ in (2.2) with a sparse approximations of it \[12\].

2.1. Spectral analysis. The spectral analysis of $P_{\text{inex}}^{-1}A$ has been addressed in the general context of saddle point problems \[8, 9, 41\]. Starting from these results, we provide further bounds which will be exploited to design preconditioner updates.

We are aware that the behaviour of many Krylov solvers, such as the SQMR one used in our numerical experiments, is not characterized by the distribution of the eigenvalues of the system matrix; however, in many practical cases the convergence of these solvers is determined by the spectrum of the coefficient matrix and therefore our updating procedure will be guided by the spectral analysis presented next.

Let us consider the eigenvalue problem

\[
\begin{bmatrix} Q + \Theta^{(1)} & A^T \\ A & -\Theta^{(2)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} P_{\text{inex}} x \\ y \end{bmatrix},
\]

and the Cholesky factorization

\[
S_{\text{inex}} = RR^T,
\]

with $R$ lower triangular. By using (2.5), the eigenvalue problem can be written as

\[
\begin{bmatrix} I_n & 0 \\ -AG^{-1} & I_m \end{bmatrix} \begin{bmatrix} Q + \Theta^{(1)} & A^T \\ A & -\Theta^{(2)} \end{bmatrix} \begin{bmatrix} I_n & -G^{-1}A^T \\ 0 & I_m \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \lambda \begin{bmatrix} G & 0 \\ 0 & -S_{\text{inex}} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix},
\]

where

\[
\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} I_n & G^{-1}A^T \\ 0 & I_m \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.
\]

The matrix at the left-hand side is equal to

\[
\begin{bmatrix} Q + \Theta^{(1)} & A(I_n - G^{-1}(Q + \Theta^{(1)})) \\ A(I_n - G^{-1}(Q + \Theta^{(1)})) & -A(2G^{-1} - G^{-1}(Q + \Theta^{(1)}G^{-1})A^T - \Theta^{(2)} \end{bmatrix},
\]

while, by (2.6), the matrix at the right-hand side can be written as

\[
\begin{bmatrix} G & 0 \\ 0 & -S_{\text{inex}} \end{bmatrix} = \begin{bmatrix} G^2 & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} G^2 & 0 \\ 0 & R^T \end{bmatrix}.
\]

Therefore, the eigenvalue problem becomes

\[
\begin{bmatrix} X & Y^T \\ Y & -Z \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix} = \lambda \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{v} \end{bmatrix},
\]

(2.7)
where

\begin{align}
    (2.8) & \quad X = G^{-\frac{1}{2}}(Q + \Theta^{(1)})G^{-\frac{1}{2}}, \\
    (2.9) & \quad Y = R^{-1}AG^{-\frac{1}{2}}(I_n - X), \\
    (2.10) & \quad Z = R^{-1}AG^{-\frac{1}{2}}(2I_n - X)G^{-\frac{1}{2}}A^TR^{-T} + R^{-1}\Theta^{(2)}R^{-T},
\end{align}

and

\[
\begin{bmatrix}
    \tilde{u} \\
    \tilde{v}
\end{bmatrix} = \begin{bmatrix}
    G_\frac{1}{2} & 0 \\
    0 & R^T
\end{bmatrix} \begin{bmatrix}
    u \\
    v
\end{bmatrix}.
\]

The eigenvalues of $P_{\text{inex}}^{-1}A$ can be characterized and bounded using Propositions 2.2, 2.3 and 2.12. Some of these results are summarized in the next theorem, where $\Re(\lambda)$ and $\Im(\lambda)$ denote the real and imaginary parts of $\lambda$, respectively. We note that the assumption of positive definiteness or semidefiniteness on $Z$ can be fulfilled by a proper scaling of $X$ enforcing its eigenvalues to be smaller than 2.

**Theorem 2.1.** Let $A$ and $P_{\text{inex}}$ be the matrices in (2.1) and (2.5), and let $X$, $Y$, $Z$ be the matrices in (2.8)–(2.10). Let $\lambda$ and $[\tilde{u}^T, \tilde{v}^T]^T$ be an eigenpair of the generalized eigenvalue problem (2.7). Then, $P_{\text{inex}}^{-1}A$ has at most $2m$ eigenvalues with non-zero imaginary part, counting conjugates. Furthermore, if $Y$ has full rank and $Z$ is positive semidefinite, it holds that

- if $\Im(\lambda) \neq 0$, then
  
  \[|\Im(\lambda)| \leq \|Y\|,\]
  \[\frac{1}{2}(\lambda_{\min}(X) + \lambda_{\min}(Z)) \leq \Re(\lambda) \leq \frac{1}{2}(\lambda_{\max}(X) + \lambda_{\max}(Z)),\]

- if $\Im(\lambda) = 0$, then either
  \[2\min\{\lambda_{\min}(X), \lambda_{\min}(Z)\} \leq \lambda \leq \max\{\lambda_{\max}(X), \lambda_{\max}(Z)\},\]
  for $\tilde{v} \neq 0$, or
  \[\lambda_{\min}(X) \leq \lambda \leq \lambda_{\max}(X),\]
  for $\tilde{v} = 0$.

The same result holds if $Y$ is rank deficient with $Z$ positive definite on $\ker(Y^T)$.

The forms (2.8)–(2.10) of $X$, $Y$ and $Z$ show that $X$ depends only on the approximation $G$ of $Q + \Theta^{(1)}$ while $Y$ and $Z$ depend on $S_{\text{inex}}$. In the next theorem we derive results which highlight the dependence of $\|Y\|$ and $\|Z\|$, and consequently of the spectrum of $P_{\text{inex}}^{-1}A_k$, on the quality of $S_{\text{inex}}$ as an approximation to $S$.

**Theorem 2.2.** Let $A$ and $P_{\text{inex}}$ be the matrices in (2.1) and (2.5), and let $\lambda$ be an eigenvalue of $P_{\text{inex}}^{-1}A_k$. Let $X$, $Y$, and $Z$ be the matrices in (2.8)–(2.10) and suppose that $Z$ is positive definite. Then

\[\|Y\| \leq \sqrt{\lambda_{\max}(S_{\text{inex}}^{-1}AG^{-1}A^T)}\|I_n - X\|,\]

Furthermore, if $\Theta^{(2)} \neq 0$, then

\begin{align}
    (2.12) & \quad \lambda_{\max}(Z) \leq \lambda_{\max}(S_{\text{inex}}^{-1}S) \max\{2 - \lambda_{\min}(X), 1\}, \\
    (2.13) & \quad \lambda_{\min}(Z) \geq \lambda_{\min}(S_{\text{inex}}^{-1}S) \min\{2 - \lambda_{\max}(X), 1\},
\end{align}
\[
\text{otherwise}
\]
\[
\lambda_{\text{max}}(Z) \leq \lambda_{\text{max}}(S_{\text{inex}}^{-1}S)(2 - \lambda_{\text{min}}(X)),
\]
\[
\lambda_{\text{min}}(Z) \geq \lambda_{\text{min}}(S_{\text{inex}}^{-1}S)(2 - \lambda_{\text{max}}(X)).
\]

\textbf{Proof.} From (2.9) it follows that
\[
\|Y\| \leq \|R^{-1}AG^{-\frac{1}{2}}\| I_n - X\| = \sqrt{\lambda_{\text{max}}(R^{-1}AG^{-1}ATR^{-T})\|I_n - X\|}.
\]
The bound (2.11) follows noting that \(R^{-1}AG^{-1}ATR^{-T}\) and \(S_{\text{inex}}^{-1}AG^{-1}AT\) are similar.

To prove the bounds on the eigenvalues of \(Z\) when \(\Theta^{(2)} \neq 0\), we note that \(Z = R^{-1}AG^{-\frac{1}{2}}(\Theta^{(2)})^{\frac{1}{2}}\left[\begin{array}{cc} 2I_n - X & 0 \\ 0 & I_m \end{array}\right]\left[\begin{array}{c} G^{-\frac{1}{2}}A^T(\Theta^{(2)})^{\frac{1}{2}} \\ \Theta^{(2)} \end{array}\right]R^{-T}.
\]
Let \(VU\) be the rank-retaining factorization of \(G^{-\frac{1}{2}}A^T(\Theta^{(2)})^{\frac{1}{2}}\left[\begin{array}{cc} 2I_n - X & 0 \\ 0 & I_m \end{array}\right]\), where \(V \in \mathbb{R}^{(n+m)\times m}\) has orthogonal columns and \(U \in \mathbb{R}^{m\times m}\) is upper triangular and nonsingular. Then \(S = U^T U\) and
\[
Z = R^{-1}U^TV^T\left[\begin{array}{cc} 2I_n - X & 0 \\ 0 & I_m \end{array}\right]VUR^{-T};
\]
letting \(N = R^{-1}U^T\), we have
\[
\lambda_{\text{max}}(Z) = \|Z\| \leq \|N\|^2 \left\|\left[\begin{array}{cc} 2I_n - X & 0 \\ 0 & I_m \end{array}\right]\right\|.
\]
Inequality (2.12) follows observing that \(N^TN\) is similar to \(S_{\text{inex}}^{-1}S\) and that
\[
\left\|\left[\begin{array}{cc} 2I_n - X & 0 \\ 0 & I_m \end{array}\right]\right\| = \max\{\lambda_{\text{max}}(2I_n - X), 1\} = \max\{2 - \lambda_{\text{min}}(X), 1\}.
\]
In order to bound \(\lambda_{\text{min}}(Z)\), we observe that
\[
\frac{1}{\lambda_{\text{min}}(Z)} = \|Z^{-1}\| \leq \|N^{-1}\|^2 \left\|\left(V^T\left[\begin{array}{cc} 2I_n - X & 0 \\ 0 & I_m \end{array}\right]V\right)^{-1}\right\|.
\]
Since \(N^{-T}N^{-1}\) is similar to \(S^{-1}S_{\text{inex}}^{-1}\), we have
\[
\|N^{-1}\|^2 = \frac{1}{\lambda_{\text{min}}(S_{\text{inex}}^{-1}S)}.
\]
Furthermore, using the Courant-Fischer minimax characterization (see, e.g., [31, Theorem 8.1-2]), we get
\[
\left\|\left(V^T\left[\begin{array}{cc} 2I_n - X & 0 \\ 0 & I_m \end{array}\right]V\right)^{-1}\right\| = \frac{1}{\lambda_{\text{min}}\left(V^T\left[\begin{array}{cc} 2I_n - X & 0 \\ 0 & I_m \end{array}\right]V\right)}
\]
\[
\leq \frac{1}{\lambda_{\text{min}}\left[\begin{array}{cc} 2I_n - X & 0 \\ 0 & I_m \end{array}\right]}
\]
\[
= \min\{2 - \lambda_{\text{max}}(X), 1\}.
\]
which yields (2.13).

When \( \Theta^{(2)} = 0 \), inequalities (2.14) and (2.15) can be derived by assuming that 
\( VU \) is the rank-retaining factorization of \( G^{-\frac{1}{2}}A^T \), with \( V \in \mathbb{R}^{n \times m} \) having orthogonal columns and \( U \in \mathbb{R}^{m \times m} \) upper triangular and nonsingular. In this case equation (2.16) becomes
\[
Z = R^{-1}U^TV^T(2I_n - X)VUR^{-T}.
\]
and the thesis follows by reasoning as above.

**Remark 2.1.** If \( Q \) is diagonal then in Theorems 2.1 and 2.2 we have \( X = I_n, Y = 0 \) and \( Z \) positive definite. Hence, all the eigenvalues of \( P_{\text{inex}}^{-1}A \) are real. Furthermore, it results that \( P_{\text{inex}}^{-1}A \) has at least \( n \) unit eigenvalues with \( n \) associated independent eigenvectors of the form \( [x^T, 0^T]^T \) (corresponding to the case \( \bar{v} = 0 \) in Theorem 2.1), and that the remaining eigenvalues lie in the interval \( [\lambda_{\min}(S_{\text{inex}}^{-1}S), \lambda_{\max}(S_{\text{inex}}^{-1}S)] \) (see [25]).

3. Building an inexact constraint preconditioner by updates. In this section we design a strategy for updating the CP built for some seed matrix of the KKT sequence. The update is based on low-rank corrections and generates inexact constraint preconditioners for subsequent systems.

Let us consider a KKT matrix \( A_{\text{seed}} \) generated at some iteration \( r \) of the IP procedure,
\[
A_{\text{seed}} = \begin{bmatrix} Q + \Theta^{(1)}_{\text{seed}} & A^T \\ A & -\Theta^{(2)}_{\text{seed}} \end{bmatrix},
\]
where \( \Theta^{(1)}_{\text{seed}} \in \mathbb{R}^{n \times n} \) is diagonal positive definite and \( \Theta^{(2)}_{\text{seed}} \in \mathbb{R}^{m \times m} \) is diagonal positive semidefinite. The corresponding seed CP has the form
\[
P_{\text{seed}} = \begin{bmatrix} H & A^T \\ A & -\Theta^{(2)}_{\text{seed}} \end{bmatrix},
\]
where \( H = \text{diag}(Q + \Theta^{(1)}_{\text{seed}}) \). Assume that a block factorization of \( P_{\text{seed}} \) has been obtained by computing the Cholesky-like factorization of the negative Schur complement \( S_{\text{seed}} \) of \( H \) in \( P_{\text{seed}} \):
\[
S_{\text{seed}} = AH^{-1}A^T + \Theta^{(2)}_{\text{seed}} = LDL^T,
\]
where \( L \) is unit lower triangular and \( D \) is diagonal positive definite.

Let \( A \) be a subsequent matrix of the KKT sequence. We approximate the CP (2.3) corresponding to \( A \) by replacing \( S \) with a suitable update of \( S_{\text{seed}}, \) named \( S_{\text{upd}} \). Thus, we obtain the inexact preconditioner
\[
P_{\text{upd}} = \begin{bmatrix} I_n & 0 \\ AG^{-1} & I_m \end{bmatrix} \begin{bmatrix} G & 0 \\ 0 & -S_{\text{upd}} \end{bmatrix} \begin{bmatrix} I_n & G^{-1}A^T \\ 0 & I_m \end{bmatrix}.
\]
A guideline for building \( S_{\text{upd}} \) is provided by Theorem 2.2 and by the following result (see [1, Lemma 3.1]).

**Lemma 3.1.** Let \( B \in \mathbb{R}^{m \times n} \) be full rank and let \( E, F \in \mathbb{R}^{n \times n} \) be symmetric and positive definite. Then, any eigenvalue \( \lambda \) of \((BEB^T)^{-1}BFB^T\) satisfies
\[
\lambda_{\min}(E^{-1}F) \leq \lambda((BEB^T)^{-1}BFB^T) \leq \lambda_{\max}(E^{-1}F).
\]
In the remaining of this section we show that specific choices of $S_{\text{upd}}$ provide easily computable bounds on the eigenvalues of $S_{\text{upd}}^{-1}S$ and make the spectral analysis in section 2 useful for constructing practical inexact preconditioners by updating techniques. For ease of presentation, we consider the cases $\Theta^{(2)} = 0$ and $\Theta^{(2)} \neq 0$ separately.

3.1. Updated preconditioners for $\Theta^{(2)} = 0$. We consider a low-rank update/downdate of $S_{\text{seed}} = AH^{-1}A^T$ that generates a matrix $S_{\text{upd}}$ of the form

$$S_{\text{upd}} = AJ^{-1}A^T,$$

(3.6)

for some given diagonal positive definite matrix $J \in \mathbb{R}^{n \times n}$.

Since the matrix $G$ in (2.4) and the matrix $J$ in (3.6) are diagonal, by Lemma 3.1 the eigenvalues of $S_{\text{upd}}^{-1}S$ can be bounded by easily computable scalars, as shown next. Let $G_{ii}$ and $J_{ii}$ be the diagonal entries of $G$ and $J$, and $\gamma(J) = (\gamma_1(J), \ldots, \gamma_n(J))$ the vector with entries given by the diagonal entries of $JG^{-1}$ sorted in nondecreasing order, i.e.,

$$\min_{1 \leq i \leq n} \frac{J_{ii}}{G_{ii}} = \gamma_1(J) \leq \gamma_2(J) \leq \cdots \leq \gamma_n(J) = \max_{1 \leq i \leq n} \frac{J_{ii}}{G_{ii}}.$$

The eigenvalues of $S_{\text{upd}}^{-1}S$ satisfy

$$\gamma_1(J) \leq \lambda(S_{\text{upd}}^{-1}S) \leq \gamma_n(J).$$

(3.7)

These bounds can be combined with the results of Theorem 2.2 as follows.

**Corollary 3.2.** Let $A$, $P_{\text{upd}}$ and $S_{\text{upd}}$ be the matrices in (2.1), (3.4), (3.6), and $RR^T$ be the Cholesky factorization of $S_{\text{upd}}$, with $R$ lower triangular. Let (2.7) be the generalized eigenvalue problem equivalent to the eigenvalue problem for $P_{\text{upd}}^{-1}A$, where $X$, $Y$ and $Z$ are the matrices in (2.8)–(2.10). Let $\gamma_1(J)$ and $\gamma_n(J)$ be the first and last component of $\gamma(H)$. Then, if $Z$ is positive definite, we have

$$\|Y\| \leq \sqrt{\gamma_n(J)} \|I_n - X\|,$$

(3.8)

and

$$\lambda_{\max}(Z) \leq \gamma_n(J) (2 - \lambda_{\min}(X)),$$

$$\lambda_{\min}(Z) \geq \gamma_1(J) (2 - \lambda_{\max}(X)).$$

(3.9) \quad (3.10)

**Proof.** The proof follows straightforwardly from (2.11), (2.14), (2.15), with $S_{\text{upd}}$ replacing $S_{\text{inex}}$, and from (3.7). \qed

Motivated by the previous results, we build $S_{\text{upd}}$ as a low-rank correction of $S_{\text{seed}}$, based on the vector $\gamma(H)$ whose entries are the diagonal entries of $HG^{-1}$ sorted in nondecreasing order, i.e.,

$$\min_{1 \leq i \leq n} \frac{H_{ii}}{G_{ii}} = \gamma_1(H) \leq \gamma_2(H) \leq \cdots \leq \gamma_n(H) = \max_{1 \leq i \leq n} \frac{H_{ii}}{G_{ii}}.$$

(3.11)

Following the procedure proposed by Baryamureeba et al. in [1], the matrix $J$ is chosen as a diagonal matrix which accounts for changes from $H$ to $G$. More precisely,
let \( l = (l_1, \ldots, l_n) \) be the vector of indices obtained by a permutation of the vector \((1, 2, \ldots, n)\) so that \( l_i \) is the position of the scalar \( H_{ii}/G_{ii} \) in the vector \( \gamma(H) \) according to (3.11), i.e.,

\[
\gamma_{l_i}(H) = \frac{H_{ii}}{G_{ii}}.
\]

Let \( q_1 \) and \( q_2 \) be nonnegative integers, with \( q = q_1 + q_2 \leq n \), and let \( \Gamma \) be the set of the indices of the diagonal entries of \( HG^{-1} \) corresponding to the \( q_1 \) largest entries of \( \gamma(H) \) that are greater than one and the \( q_2 \) smallest entries of \( \gamma(H) \) that are smaller than one, i.e.,

\[
\Gamma = \{ i : 1 \leq l_i \leq q_2 \text{ and } \gamma_{l_i}(H) < 1 \} \cup \{ i : n - q_1 + 1 \leq l_i \leq n \text{ and } \gamma_{l_i}(H) > 1 \}.
\]

Then, \( J \) is set as

\[
(3.12) \quad J_{ii} = \begin{cases} 
G_{ii}, & \text{if } i \in \Gamma, \\
H_{ii}, & \text{otherwise.}
\end{cases}
\]

By the previous definition of \( J \), \( S_{\text{upd}} \) is positive definite and \( S_{\text{upd}} - S_{\text{seed}} \) is a low-rank matrix if the cardinality of \( \Gamma \) is small. Moreover,

\[
(3.13) \quad \gamma_1(J) = \min\{1, \min_{i \notin \Gamma} \gamma_{l_i}(H)\} = \min\{1, \gamma_{q_2+1}(H)\},
\]

\[
(3.14) \quad \gamma_{n}(J) = \max\{1, \max_{i \notin \Gamma} \gamma_{l_i}(H)\} = \max\{1, \gamma_{n-q_1}(H)\},
\]

and an improvement on the bounds (3.8)–(3.10) may be expected as long as \( \gamma_{q_1}(H) \) and \( \gamma_{n-q_1+1}(H) \) are well separated from \( \gamma_{q_2+1}(H) \) and \( \gamma_{n-q_1}(H) \), respectively. Another consequence of this low-rank correction is that \( S - S_{\text{upd}} \) has \( q \) zero eigenvalues; thus \( P_{\text{inex}}^{-1} A_k \) has \( 2q \) unit eigenvalues with geometric multiplicity \( q \) [41, Theorem 3.3].

By (3.12) and (3.14),

\[
(3.15) \quad AJ^{-1}A^T = AH^{-1}A^T + AKAT = LDL^T + \bar{A}K\bar{A}^T,
\]

where \( K \) is the diagonal matrix

\[
(3.16) \quad K_{ii} = \begin{cases} 
G_{ii}^{-1} - H_{ii}^{-1}, & \text{if } i \in \Gamma, \\
0, & \text{otherwise,}
\end{cases}
\]

\( \bar{A} \in \mathbb{R}^{m \times q} \) consists of the columns of \( A \) with indices in \( \Gamma \), and \( \bar{K} \in \mathbb{R}^{q \times q} \) is the diagonal matrix having on the diagonal the nonzero entries of \( K \) corresponding to those indices. Thus, the Cholesky-like factorization of \( S_{\text{upd}} \) can be computed by updating or downdating the factorization \( LDL^T \) of \( S_{\text{seed}} \). Specifically, an update is performed if \( H_{ii} > G_{ii} \), while a downdate is performed if \( H_{ii} < G_{ii} \). This task can be accomplished by either using efficient procedures for updating and downdating the (sparse) Cholesky factorization [23], or by the Sherman-Morrison-Woodbury formula (see, e.g., [31, section 2.1.3]). Clearly, once \( S_{\text{upd}} \) has been factorized, the factorization of \( P_{\text{upd}} \) is readily available.

In order to limit the computational cost for building the updated factorization of \( S_{\text{upd}} \), \( q \) must be kept fairly small. We note that in the limit case \( q = 0 \) the set \( \Gamma \) is empty; hence \( S_{\text{upd}} = S_{\text{seed}} \) and

\[
\gamma_{l_i}(H) = \frac{Q_{ii} + (\Theta_{\text{seed}}^{(1)})_{ii}}{Q_{ii} + \Theta_{ii}^{(1)}}.
\]
The element $\gamma_i(H)$ is expected to be close to 1 if $(\Theta^{(1)})_{ii} - (\Theta^{(1)}_{\text{seed}})_{ii}$ is small. More generally, if $\Theta^{(1)}_{ii}$ tends to zero or infinity, as it happens when the IP iterate approaches an optimal solution where strict complementarity holds, the quality of the preconditioner may significantly deteriorate.

We conclude this section showing the spectrum of $A$, $P^{-1}_{ex}A$, and $P^{-1}_{\text{upd}} A$ arising in the solution of problem CVXQP1 in the CUTEst collection [33], with dimensions $n = 1000$ and $m = 500$ (see Figure 3.1). The matrix $A$ has been obtained at the 10th iteration of the IP method used for the numerical experiments (see section 4), whereas $A_{\text{seed}}$ has been obtained at the 6th iteration. The preconditioner $P_{\text{upd}}$ has been built for $q_1 = q_2 = 25$; in this case $\gamma_{q_2+1} = 1.85e-3$, while $\gamma_{n-q_1} = 2.73e+0$. We see that, unlike $P_{ex}$, $P_{\text{upd}}$ moves some eigenvalues from the real to the complex field. Nevertheless, $P_{\text{upd}}$ tends to cluster the eigenvalues of $A$ around 1, and $\gamma_{q_2+1}$ and $\gamma_{n-q_1}$ provide approximate bounds on the real and imaginary parts of the eigenvalues of the preconditioned matrix, according to Theorem 2.1, Corollary 3.2, and equalities (3.13)-(3.14). Of course, this clustering is less effective than the one performed by $P_{ex}$, but it is useful in several cases, as shown in section 4.

**3.2. Updated preconditioners for $\Theta^{(2)} \neq 0$.** The updating strategy described in the previous section can be generalized to the case $\Theta^{(2)} \neq 0$. To this end, we note that the sparsity pattern of $\Theta^{(2)}$ does not change throughout the IP iterations.
and the set $\mathcal{L} = \{ i : \Theta^{(2)}_{ii} \neq 0 \}$ has cardinality equal to the number $m_1$ of linear inequality constraints in the QP problem. Let $\tilde{\Theta}^{(2)}_{seed}$ and $\tilde{\Theta}^{(2)}$ be the $m_1 \times m_1$ diagonal submatrices containing the nonzero diagonal entries of $\Theta^{(2)}_{seed}$ and $\Theta^{(2)}$, respectively, and let $I_m$ be the rectangular matrix consisting of the columns of $I_m$ with indices in $\mathcal{L}$. Then, we have

$$S_{seed} = A H^{-1} A^T + \Theta^{(2)}_{seed} = \tilde{A} H^{-1} \tilde{A}^T,$$
$$S = AG^{-1} A^T + \Theta^{(2)} = \tilde{A} G^{-1} \tilde{A}^T,$$

where

$$\tilde{A} = \begin{bmatrix} A & I_m \end{bmatrix}, \quad \tilde{H}^{-1} = \begin{bmatrix} H^{-1} & 0 \\ 0 & \tilde{\Theta}^{(2)}_{seed} \end{bmatrix}, \quad \tilde{G}^{-1} = \begin{bmatrix} G^{-1} & 0 \\ 0 & \tilde{\Theta}^{(2)} \end{bmatrix}. $$

Analogously, letting

$$S_{upd} = AJ^{-1} A^T + \Theta^{(2)}_{upd},$$

we have

$$S_{upd} = \tilde{A} J \tilde{A}^T,$$

where

$$\tilde{J} = \begin{bmatrix} J^{-1} & 0 \\ 0 & \tilde{\Theta}^{(2)}_{upd} \end{bmatrix},$$

and $\tilde{\Theta}^{(2)}_{upd}$ is the $m_1 \times m_1$ diagonal submatrix of $\Theta^{(2)}_{upd}$ containing its nonzero diagonal entries. Thus, we can choose $\tilde{J}$ using the same arguments as in the previous section. In particular, if $\tilde{\gamma}(\tilde{J}) = (\tilde{\gamma}_1(\tilde{J}), \ldots, \tilde{\gamma}_{n+m_1}(\tilde{J}))$ is the vector with elements given by the diagonal entries of $\tilde{J} \tilde{G}^{-1}$ sorted in nondecreasing order, then, by Lemma 3.1, the eigenvalues of $S_{upd}^{-1} S$ satisfy

$$\tilde{\gamma}_1(\tilde{J}) \leq \lambda(S_{upd}^{-1} S) \leq \tilde{\gamma}_{n+m_1}(\tilde{J}),$$

and the following result holds.

**Corollary 3.3.** Let $A$, $P_{upd}$ and $S_{upd}$ be the matrices in (2.1), (3.4), (3.17), and $RR^T$ be the Cholesky factorization of $S_{upd}$, with $R$ lower triangular. Let (2.7) be the generalized eigenvalue problem equivalent to the eigenvalue problem for $P_{upd}^{-1} A$, where $X$, $Y$ and $Z$ are the matrices in (2.8)–(2.10). Let $\tilde{\gamma}_1(\tilde{J})$ and $\tilde{\gamma}_{n+m_1}(\tilde{J})$ be the first and the last component of $\tilde{\gamma}(\tilde{J})$. Then, if $Z$ is positive definite, we have

$$\|Y\| \leq \sqrt{\tilde{\gamma}_{n+m_1}(\tilde{J})} \|I_n - X\|,$$

and

$$\lambda_{max}(Z) \leq \tilde{\gamma}_{n+m_1}(\tilde{J}) \max\{2 - \lambda_{min}(X), 1\},$$
$$\lambda_{min}(Z) \geq \tilde{\gamma}_1(\tilde{J}) \min\{2 - \lambda_{max}(X), 1\}.$$
Proof. Since $\Theta^{(2)}$ is positive semidefinite, for any vector $w \in \mathbb{R}^n$ we have
\[
w^T S_{\text{upd}}^{-\frac{1}{2}} A G^{-1} A^T S_{\text{upd}}^{-\frac{1}{2}} w \leq w^T S_{\text{upd}}^{-\frac{1}{2}} (A G^{-1} A^T + \Theta^{(2)}) S_{\text{upd}}^{-\frac{1}{2}} w,
\]

Then, by matrix similarity,
\[
\lambda_{\max}(S_{\text{upd}}^{-\frac{1}{2}} A G^{-1} A^T) \leq \lambda_{\max}(S_{\text{upd}}^{-\frac{1}{2}} S^{\frac{1}{2}}).
\]

(3.22)

Therefore, by using Theorem 2.2 and (3.18), we obtain (3.19). The bounds (3.20) and (3.21) directly follow from Theorem 2.2 and (3.18).

On the base of the previous results the generalization of the updating procedure to the case $\Theta^{(2)} \neq 0$ is straightforward. Let $l$ be the vector of indices such that $\tilde{\gamma}_l(\tilde{H}) = \tilde{H}_{ii}/\tilde{G}_{ii}$, $q_1$ and $q_2$ be nonnegative integers, with $q = q_1 + q_2 \leq n + m_1$, and $\tilde{\Gamma}$ be the set of the indices of the diagonal entries of $\tilde{H} \tilde{G}^{-1}$ corresponding to the $q_1$ largest entries of $\tilde{\gamma}(\tilde{H})$ that are greater than one and the $q_2$ smallest entries of $\tilde{\gamma}(\tilde{H})$ that are smaller than one, i.e.,
\[
\tilde{\Gamma} = \{i : 1 \leq i \leq q_2 \text{ and } \tilde{\gamma}_l(\tilde{H}) < 1\} \cup \{i : n + m_1 - q_1 + 1 \leq i \leq n + m_1 \text{ and } \tilde{\gamma}_l(\tilde{H}) > 1\}.
\]

We set $\tilde{J}$ as
\[
\tilde{J}_{ii} = \begin{cases} 
\tilde{G}_{ii}, & \text{if } i \in \tilde{\Gamma}, \\
\tilde{H}_{ii}, & \text{otherwise},
\end{cases}
\]

(3.23)

thus, $\tilde{J}$ accounts for changes from $H$ to $G$ and from $\Theta_{\text{seed}}^{(2)}$ to $\Theta^{(2)}$. For $q$ small enough, $S_{\text{upd}}$ is a low-rank correction of $S_{\text{seed}}$.

4. Numerical results. We tested the effectiveness of our updating procedure by solving sequences of KKT systems arising in the solution of convex QP problems with $m \leq n$ and $A$ with full rank. These problems were either taken from the CUTEst collection [33] or obtained by modifying CUTEst problems as explained next, and were chosen to have Schur complements with different factorization costs. Since most of the available large convex QP problems with $\Theta^{(2)} \neq 0$ have Schur complements with band structure and their factorization is low cost, we modified some CUTEst QP problems with non-band Schur complement and linear constraint $Ax = b$, by changing this constraint into $Ax \geq b$. The selected problems, along with their dimensions and the number of nonzero entries of the Schur complement, are listed in Table 4.1; the modified problems are identified by appending -M to their name in the problem collection. In the first five problems $\Theta^{(2)} = 0$, while in the remaining ones $\Theta^{(2)} \neq 0$. The Hessian matrices of UBH1 and LISWET5 are diagonal and consequently $\mathcal{P}_{\text{ex}}$ is equal to the KKT matrix $A$. For all the problems the Schur complements are very sparse; they are non-banded for the CVXQP and UBH1 problems only. CVXQP1, CVXQP2 and CVXQP3 differ by the number of linear constraints, thus having Schur complements of different dimensions; clearly the same holds for the corresponding modified problems.

The sequences of linear systems were obtained by applying the PRQP solver to the selected problems and extracting the KKT matrices arising at each IP iteration, along with the corresponding right-hand sides. PRQP is a Fortran 90 code based on an Inexact Potential Reduction IP method [14] [17] [20] (see also http://www.dimat.unina2.it/diserafino/prqp.htm). The starting point for this code was chosen as explained in [21] and the tolerances on the relative duality gap and infeasibilities were set to $10^{-7}$.
and $10^{-8}$, respectively. Within PRQP the KKT systems were solved by the Conjugate (CG) method coupled with the exact CP, using an adaptive tolerance in the stopping criterion, which relates the accuracy in the solution of the KKT system to the quality of the current IP iterate, in the spirit of inexact IP methods [2, 15]. The tolerance associated with each system was extracted too, to be used in our experiments.

We implemented our updated preconditioners, as well as the exact CP, in Matlab, using the CHOLMOD library [23] to compute the sparse $LDL^T$ factorization of the negative Schur complement $S_{seed}$ and the low-rank updating and downdating required to build $S_{upd}$. This library was called through its MEX interface. The systems were solved using a Matlab implementation of the SQMR method without look-ahead [26], since the CG method cannot generally be applied to KKT systems coupled with inexact CPs. For each system, the SQMR iterations were stopped when the residual was lower than the associated tolerance, as in the solution of the KKT system within the IP code. A maximum number of 500 iterations was considered too. We note that only one matrix-vector product per iteration is performed in our SQMR implementation, except in the last few iterations, where an additional matrix-vector product per iteration is computed, to use the residual instead of the so-called BCG-residual in the stopping criterion, as in the QMRPACK code [27].

In our updating strategy, we set $q_1 = q_2 = q/2$ and $q = 50, 100$, in order to keep low the overhead of the updating/downdating phase (the dimension of the Schur complement in our experiments is significantly larger than $q$). We also considered “the limit case” $q = 0$, corresponding to $S_{upd} = S_{seed}$ in the updated preconditioner $P_{upd}$. In the case $\Theta^{(2)} = 0$, we observed that including in $\Gamma$ the indices corresponding to values of $\gamma_i(H)$ that are too close to 1 does not lead to significant benefits. Therefore, we put in $\Gamma$ the indices corresponding to the $q_1$ largest $\gamma_i(H)$’s such that $\gamma_i(H) \geq 10$ and the $q_2$ smallest $\gamma_i(H)$’s such that $\gamma_i(H) \leq 0.1$. When the number of such $\gamma_i(H)$’s was lower than $q/2$, we chose $q_1$ and $q_2$ so that $q_1 + q_2$ was as large as possible. The previous observation also holds for $\tilde{\Gamma}$ and the ratios $\tilde{\gamma}_i(\tilde{H})$ in the case where $\Theta^{(2)} \neq 0$.

On the base of preliminary numerical experiments, we decided to refresh the preconditioner, i.e., to build $P_{ex}$ instead of $P_{upd}$, when the time for computing $P_{upd}$ and solving the linear system exceeded 90% of the time for building the last exact preconditioner and solving the corresponding system. When for a specific system of the sequence this situation occurred, the next system of the sequence was solved using the preconditioner $P_{ex}$. We also set a maximum number, $k_{max}$, of consecutive preconditioner updates, after which the refresh was performed anyway. This strategy aims at avoiding possible situations in which the time saved by updating the Schur complement instead of re-factorizing it, is offset by an excessive increase in the number of SQMR iterations, due to deterioration of the quality of the preconditioner. In the experiments discussed here $k_{max} = 4$ was used for all the test problems but the ones with diagonal Hessian matrix $Q$, where $k_{max}$ was set to 3. The latter choice is motivated by the fact that $A = P_{ex}$ if $Q$ is diagonal, and hence only one iteration of SQMR has to be performed, which generally has a lower cost than the setup of $P_{ex}$. Actually, in this case, we computed the solutions of the KKT systems by solving the block-triangular systems resulting from the block factorization of $P_{ex}$. Finally, if SQMR with $P_{upd}$ achieved the maximum number of iterations without satisfying the required accuracy, the preconditioner $P_{ex}$ corresponding to the current KKT system was computed and used to solve that system again. In our experiments, this only happened to the KKT system associated with the second-to-last PRQP iteration applied to CVXQP3-M, for all the values of $q$. 

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We performed our experiments using Matlab R2011b (v. 7.13) on an Intel Core i7 processor, with clock frequency of 2.67 GHz, 12 GB of RAM and 8 MB of cache memory. Matlab was installed under the Linux Ubuntu operating system (Linux version 3.2.0-35-generic) and CHOLMOD was compiled using gcc 4.3.4. The tic and toc Matlab commands were used to measure elapsed times.

A comparison among the exact and the updated preconditioners is presented in Table 4.1. For each preconditioner we show the total number of SQMR iterations and the overall computation time, in seconds, needed to solve the whole sequence. The number of iterations is not reported for the problems with diagonal Hessian, because in this case the systems are solved by a direct method, as previously explained. For most of the problems the number of iterations obtained with $P_{\text{upd}}$ decreases as $q$ increases; thus, updating the Schur complement by low-rank information appears to be beneficial in terms of iterations. In the cases where the number of SQMR iterations is practically constant when going from $q = 0$ to $q = 100$, we verified that either the number of ratios $\gamma_l(H)$ (or $\tilde{\gamma}_l(\tilde{H})$) that do not belong to $(0, 10)$ is very small, or all the ratios are very close each other, so that increasing $q$ does not lead to any improvement. For similar reasons the reduction of the number of iterations from $q = 50$ to $q = 100$ is less significant than from $q = 0$ to $q = 50$. We note that the increase of iterations observed in CVXQP2-M when passing from $q = 0$ to $q = 50$ depends on the refresh strategy; when $q = 0$ more refreshes are required because of the lower quality of the preconditioner, i.e., more exact preconditioners are computed, thus yielding a lower number of iterations.

As expected, the exact preconditioner requires less iterations than the updated ones, but this does not imply smaller execution times. Actually, the updating tech-

| Problem   | $n, m$ | nnz(S) | $P_{\text{ex}}$ | $P_{\text{upd}}$ ($q = 0$) | $P_{\text{upd}}$ ($q = 50$) | $P_{\text{upd}}$ ($q = 100$) |
|-----------|--------|--------|-----------------|--------------------------|--------------------------|--------------------------|
| CVXQP1    | 20000  | 10000  | 67976           | 232 5.76e+1            | 740 3.79e+1            | 530 3.72e+1            |
| CVXQP2    | 20000  | 5000   | 15994           | 273 1.29e+0            | 352 1.65e+0            | 330 1.57e+0            |
| CVXQP3    | 20000  | 15000  | 155942          | 497 8.32e+2            | 2006 4.82e+2           | 1166 3.67e+2           |
| STCQP2    | 16385  | 8190   | 114660          | 259 1.44e+0            | 267 1.47e+0            | 267 1.49e+0            |
| UBH1      | 17997  | 12000  | 59988           | — 8.68e+2              | 834 3.59e+2            | 831 3.93e+2            |
| CVXQP1-M  | 20000  | 10000  | 67976           | 962 1.29e+2            | 2960 1.03e+2           | 2474 1.02e+2           |
| CVXQP2-M  | 20000  | 5000   | 15994           | 1042 5.28e+0           | 1531 7.01e+0           | 1653 7.64e+0           |
| CVXQP3-M  | 15000  | 11250  | 116910          | 970 1.28e+3            | 3054 4.14e+2           | 2563 3.88e+2           |
| LISWET5   | 20001  | 20000  | 59998           | — 7.31e−1              | 226 1.77e+0            | 226 1.84e+0            |
| MOSARQP1  | 22500  | 20000  | 257166          | 74 7.67e+0             | 202 8.87e+0            | 184 9.36e+0            |

Table 4.1
Comparison between $P_{\text{ex}}$ and $P_{\text{upd}}$. 
unique yields a significant reduction of the overall time for CVXQP1, CVXQP3, UBH1, CVXQP1-M, and CVXQP3-M. In these cases, the factorization of the Schur complement is more expensive than the solution via SQMR; thus, the increase in the number of SQMR iterations, due to the use of updated preconditioners instead of exact ones, is largely offset by the time saving obtained by performing low-rank corrections of an available factorization. In the solution of these problems the best tradeoff between the effectiveness of the updated preconditioner and the overall computational cost is obtained with $q = 50$. The results with UBH1 also show that for problems with diagonal Hessian, if the cost of the factorization of the Schur complement is high, the iterative solution of the KKT systems with the use of updated preconditioners may be a convenient alternative to a direct approach. For STCQP2 all the preconditioners are practically equivalent in terms of iterations and execution times. For the remaining problems the updating strategy is not beneficial; the reason is that the factorization of the Schur complement is not expensive, and hence the computation of $P_{ex}$ does not significantly affect the execution time.

To provide more insight into the behaviour of the updated preconditioners, in Tables 4.2-4.5 we show some details concerning the solution of the sequences of KKT systems arising from four problems, i.e., CVXQP1, STCQP2, UBH1, and MOSARQP2. For each IP iteration we report the number $nit$ of iterations of SQMR, as well as the time $T_{prec}$ for building the preconditioner, the time $T_{solve}$ for solving the linear system, and the sum $T_{sum}$ of these times. The last row contains the total iterations and times over all IP iterations, while the rows in bold correspond to the IP iterations at which the preconditioner is refreshed. These tables clearly support the previous observation that the updating strategy is efficient when the computation of $P_{ex}$ is expensive, as it is for CVXQP3 and UBH1. Conversely, when the time for building $P_{ex}$ is modest, recomputing $P_{ex}$ is a natural choice. It also appears that the refresh strategy plays a significant role in achieving efficiency, since it prevents the preconditioner from excessive deterioration. In particular, the refresh is crucial for UBH1, because the number of SQMR iterations obtained with $P_{upd}$ tends to rapidly increase from an IP iteration to the next one. It is also worth noting that when the time for computing $P_{ex}$ is not dominant, the refresh tends to occur more frequently, since a small increase in the number of iterations obtained with $P_{upd}$ may easily raise the execution time over the 90% of the time corresponding to the last application of the exact preconditioner.

5. Conclusion. We proposed a preconditioner updating procedure for the solution of sequences of KKT systems arising in IP methods for convex QP problems. The preconditioners built by this procedure belong to the class of inexact constraint preconditioners and are obtained by updating a given seed constraint preconditioner. The updating is performed by using low-rank corrections of the Schur complement of the $(1,1)$ block in the seed preconditioner and yields to a factorized updated preconditioner. Starting from the spectral analysis given in [8, 41] and exploiting results presented in [4], we provide bounds on the eigenvalues of the preconditioned matrix in terms of the “quality” of the approximation of the Schur complement. These results drive the design of effective low-rank correction strategies. The numerical experiments show that our updated preconditioners, combined with a suitable preconditioner refreshing, can be rather successful. In practice, it is shown that the higher the cost of the Schur complement factorization, the more advantageous the updating procedure becomes. Finally, we believe that the updating strategy proposed here paves the way to the definition of preconditioner updating procedures for sequences of KKT systems.
where the Hessian and constraint matrices change from one iteration to the next.

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systems in potential reduction software for large-scale quadratic problems

IP it $P_{ex}$ $P_{upd}$ ($q=50$)
\begin{tabular}{cccc|cccc}
| IP it | $T_{it}$ | $T_{solve}$ | $T_{sum}$ | $nit$ | $T_{prec}$ | $T_{solve}$ | $T_{sum}$ |
|-------|---------|-------------|-----------|------|------------|-------------|-----------|
| 1     | 5.11e+1 | 2.60e-1     | 5.14e+1   | 1    | 5.11e+1    | 2.60e-1     | 5.14e+1   |
| 2     | 5.11e+1 | 2.60e-1     | 5.14e+1   | 42   | 5.31e+0    | 5.31e+0     | 1.06e+1   |
| 3     | 5.10e+1 | 2.32e-1     | 5.12e+1   | 128  | 4.70e+0    | 1.57e+1     | 2.04e+1   |
| 4     | 5.11e+1 | 2.34e-1     | 5.13e+1   | 365  | 4.21e+0    | 4.61e+1     | 5.03e+1   |
| 5     | 5.10e+1 | 2.35e-1     | 5.12e+1   | 2    | 5.10e+1    | 2.35e-1     | 5.12e+1   |
| 6     | 5.08e+1 | 2.39e-1     | 5.10e+1   | 40   | 3.64e-1    | 5.11e+0     | 5.47e+0   |
| 7     | 5.10e+1 | 2.38e-1     | 5.12e+1   | 82   | 5.03e+0    | 9.97e+0     | 1.50e+1   |
| 8     | 5.08e+1 | 2.34e-1     | 5.10e+1   | 118  | 4.92e+0    | 1.41e+1     | 1.90e+1   |
| 9     | 5.08e+1 | 2.40e-1     | 5.10e+1   | 2    | 5.08e+1    | 2.40e-1     | 5.10e+1   |
| 10    | 5.06e+1 | 2.38e-1     | 5.08e+1   | 18   | 1.25e-3    | 2.32e+0     | 2.32e+0   |
| 11    | 5.07e+1 | 2.41e-1     | 5.09e+1   | 14   | 5.06e+0    | 1.81e+0     | 6.87e+0   |
| 12    | 5.06e+1 | 2.39e-1     | 5.08e+1   | 13   | 5.05e+0    | 1.72e+0     | 6.77e+0   |
| 13    | 5.06e+1 | 2.34e-1     | 5.08e+1   | 2    | 5.06e+1    | 2.34e-1     | 5.08e+1   |
| 14    | 5.06e+1 | 2.39e-1     | 5.08e+1   | 1    | 1.27e-3    | 2.35e-1     | 2.36e-1   |
| 15    | 5.06e+1 | 2.34e-1     | 5.08e+1   | 1    | 1.26e-3    | 2.33e-1     | 2.34e-1   |
| 16    | 5.06e+1 | 2.47e-1     | 5.08e+1   | 1    | 1.26e-3    | 2.34e-1     | 2.35e-1   |
| 17    | 5.06e+1 | 2.50e-1     | 5.09e+1   | 1    | 5.06e+1    | 2.50e-1     | 5.09e+1   |
\end{tabular}

8.68e+2 5.39e+0 8.68e+2 831 2.89e+2 1.04e+2 3.93e+2

Table 4.3

UBH1: details for $P_{ex}$ and $P_{upd}$ with $q = 50$.

IP it $P_{ex}$ $P_{upd}$ ($q=50$)
\begin{tabular}{cccc|cccc}
| IP it | $nit$ | $T_{it}$ | $T_{solve}$ | $T_{sum}$ | $nit$ | $T_{prec}$ | $T_{solve}$ | $T_{sum}$ |
|-------|------|---------|-------------|-----------|------|------------|-------------|-----------|
| 1     | 13   | 1.08e-2 | 8.78e-2     | 9.86e-2   | 13   | 1.08e-2    | 8.78e-2     | 9.86e-2   |
| 2     | 11   | 1.01e-2 | 5.88e-2     | 6.89e-2   | 13   | 3.83e-3    | 7.05e-2     | 7.43e-2   |
| 3     | 11   | 1.01e-2 | 5.89e-2     | 6.90e-2   | 12   | 3.78e-3    | 6.44e-2     | 6.82e-2   |
| 4     | 11   | 1.02e-2 | 6.00e-2     | 7.02e-2   | 10   | 3.86e-3    | 5.40e-2     | 5.79e-2   |
| 5     | 11   | 1.01e-2 | 6.00e-2     | 7.01e-1   | 13   | 3.84e-3    | 6.76e-2     | 7.14e-2   |
| 6     | 15   | 1.00e-2 | 7.68e-2     | 8.68e-2   | 15   | 1.00e-2    | 7.68e-2     | 8.68e-2   |
| 7     | 19   | 1.01e-2 | 9.73e-2     | 1.07e-1   | 22   | 4.07e-3    | 1.20e-1     | 1.24e-1   |
| 8     | 22   | 1.01e-2 | 9.48e-2     | 1.05e-1   | 22   | 1.01e-2    | 9.48e-2     | 1.05e-1   |
| 9     | 24   | 1.01e-2 | 1.19e-1     | 1.29e-1   | 24   | 4.14e-3    | 1.33e-1     | 1.37e-1   |
| 10    | 24   | 1.00e-2 | 1.17e-1     | 1.27e-1   | 24   | 1.00e-2    | 1.17e-1     | 1.27e-1   |
| 11    | 29   | 1.00e-2 | 1.42e-1     | 1.52e-1   | 29   | 4.35e-3    | 1.63e-1     | 1.67e-1   |
| 12    | 33   | 1.00e-2 | 1.58e-1     | 1.68e-1   | 33   | 1.00e-2    | 1.58e-1     | 1.68e-1   |
| 13    | 36   | 1.75e-1 | 1.85e-1     | 3.60e-1   | 37   | 4.39e-3    | 1.97e-1     | 2.01e-1   |
\end{tabular}

259 1.32e-1 1.31e+0 1.61e+0 267 8.33e-2 1.42e+0 1.49e+0

Table 4.4

STCQP2: details for $P_{ex}$ and $P_{upd}$ with $q = 50$.

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Table 4.5

MOSARQPI: details for $P_{\text{ex}}$ and $P_{\text{upd}}$ with $q = 50$.

| IP it | $P_{\text{ex}}$ | $P_{\text{upd}}$ (q=50) |
|-------|----------------|--------------------------|
|       | nit | $T_{\text{prec}}$ | $T_{\text{solve}}$ | $T_{\text{sum}}$ | nit | $T_{\text{prec}}$ | $T_{\text{solve}}$ | $T_{\text{sum}}$ |
| 1     | 2   | 2.79e−1 | 1.05e−1 | 3.84e−1 | 2 | 2.79e−1 | 1.05e−1 | 3.84e−1 |
| 2     | 2   | 2.78e−1 | 1.11e−1 | 3.89e−1 | 5 | 8.49e−2 | 1.76e−1 | 2.61e−1 |
| 3     | 3   | 2.78e−1 | 1.37e−1 | 4.15e−1 | 9 | 1.17e−1 | 2.94e−1 | 4.11e−1 |
| 4     | 3   | 2.75e−1 | 1.32e−1 | 4.07e−1 | 3 | 2.75e−1 | 1.32e−1 | 4.07e−1 |
| 5     | 3   | 2.78e−1 | 1.34e−1 | 4.12e−1 | 15 | 4.96e−1 | 4.77e−1 | 4.96e−1 |
| 6     | 4   | 2.76e−1 | 1.63e−1 | 4.39e−1 | 4 | 2.76e−1 | 1.63e−1 | 4.39e−1 |
| 7     | 4   | 2.81e−1 | 1.64e−1 | 4.45e−1 | 15 | 1.75e−1 | 4.67e−1 | 6.42e−1 |
| 8     | 4   | 2.79e−1 | 1.62e−1 | 4.41e−1 | 4 | 2.79e−1 | 1.62e−1 | 4.41e−1 |
| 9     | 4   | 2.79e−1 | 1.61e−1 | 4.40e−1 | 19 | 4.90e−1 | 5.82e−1 | 1.07e+0 |
| 10    | 5   | 2.81e−1 | 1.92e−1 | 4.73e−1 | 5 | 2.81e−1 | 1.92e−1 | 4.73e−1 |
| 11    | 5   | 2.80e−1 | 1.90e−1 | 4.70e−1 | 15 | 8.42e−2 | 4.69e−1 | 5.53e−1 |
| 12    | 6   | 2.80e−1 | 2.24e−1 | 5.04e−1 | 6 | 2.80e−1 | 2.24e−1 | 5.04e−1 |
| 13    | 5   | 2.77e−1 | 1.88e−1 | 4.65e−1 | 30 | 3.58e−1 | 8.97e−1 | 1.26e+0 |
| 14    | 6   | 2.79e−1 | 2.26e−1 | 5.05e−1 | 6 | 2.79e−1 | 2.26e−1 | 5.05e−1 |
| 15    | 6   | 2.76e−1 | 2.17e−1 | 4.93e−1 | 10 | 8.48e−2 | 3.25e−1 | 4.10e−1 |
| 16    | 6   | 2.79e−1 | 2.21e−1 | 5.00e−1 | 30 | 2.88e−1 | 3.25e−1 | 6.13e−1 |
| 17    | 6   | 2.74e−1 | 2.18e−1 | 4.92e−1 | 6 | 2.74e−1 | 2.18e−1 | 4.92e−1 |
| 74    | 4.73e+0 | 2.93e+0 | 7.67e+0 | 184 | 4.40e+0 | 4.96e+0 | 9.36e+0 |

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