Off-policy estimation of linear functionals: Non-asymptotic theory for semi-parametric efficiency

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Abstract

The problem of estimating a linear functional based on observational data is canonical in both the causal inference and bandit literatures. We analyze a broad class of two-stage procedures that first estimate the treatment effect function, and then use this quantity to estimate the linear functional. We prove non-asymptotic upper bounds on the mean-squared error of such procedures: these bounds reveal that in order to obtain non-asymptotically optimal procedures, the error in estimating the treatment effect should be minimized in a certain weighted $L^2$-norm. We analyze a two-stage procedure based on constrained regression in this weighted norm, and establish its instance-dependent optimality in finite samples via matching non-asymptotic local minimax lower bounds. These results show that the optimal non-asymptotic risk, in addition to depending on the asymptotically efficient variance, depends on the weighted norm distance between the true outcome function and its approximation by the richest function class supported by the sample size.

1 Introduction

A central challenge in both the casual inference and bandit literatures is how to estimate a linear functional associated with the treatment (or reward) function, along with inferential issues associated with such estimators. Of particular interest in causal inference are average treatment effects (ATE) and weighted variants thereof, whereas with bandits and reinforcement learning, one is interested in various linear functionals of the reward function (including elements of the value function for a given policy). In many applications, the statistician has access to only observational data, and lacks the ability to sample the treatment or the actions according to the desired probability distribution. By now, there is a rich body of work on this problem (e.g., [RRZ95, RR95, CCD+18, AK21, WAD17, MZJW22]), including various types of estimators that are equipped with both asymptotic and non-asymptotic guarantees. We overview this and other past work in the related work section to follow.

In this paper, we study how to estimate an arbitrary linear functional based on observational data. When formulated in the language of contextual bandits, each such problem involves a state space $X$, an action space $A$, and an output space $Y \subseteq \mathbb{R}$. Given a base measure $\lambda$ on the action space $A$—typically, the counting measure for discrete action spaces, or Lebesgue measure for continuous action spaces—we equip each $x \in X$ with a probability density function $\pi(x, \cdot)$ with respect to $\lambda$. This combination defines a probability distribution over $A$, known
either as the *propensity score* (in causal inference) or the *behavioral policy* (in the bandit literature). The conditional mean of any outcome \( Y \in \mathcal{Y} \) is specified as \( \mathbb{E}[Y \mid x, a] = \mu^*(x, a) \), where the function \( \mu^* \) is known as the *treatment effect* or the *reward function*, again in the causal inference and bandit literatures, respectively.

Given some probability distribution \( \xi^* \) over the state space \( \mathcal{X} \), suppose that we observe \( n \) i.i.d. triples \((X_i, A_i, Y_i)\) in which \( X_i \sim \xi^* \), and
\[
A_i \mid X_i \sim \pi(X_i, \cdot), \quad \text{and} \quad \mathbb{E}[Y_i \mid X_i, A_i] = \mu^*(X_i, A_i), \quad \text{for } i = 1, 2, \ldots, n. \tag{1}
\]
We also make use of the conditional variance function
\[
\sigma^2(x, a) := \mathbb{E} \left[ (Y - \mu^*(X, A))^2 \mid X = x, A = a \right], \tag{2}
\]
which is assumed to exist for any \( x \in \mathcal{X} \) and \( a \in \mathcal{A} \).

For a pre-specified weight function \( g : \mathcal{X} \times \mathcal{A} \rightarrow \mathbb{R} \), our goal is to estimate the linear functional
\[
\tau^* \equiv \tau(\mathcal{I}^*) := \int_{\mathcal{A}} \mathbb{E}_{\xi^*} \left[ g(X, a) \cdot \mu^*(X, a) \right] d\lambda(a), \tag{3}
\]
With this set-up, the pair \( \mathcal{I}^* := (\xi^*, \mu^*) \) defines a particular *problem instance*. Throughout the paper, we focus on the case where both the propensity score \( \pi \) and the weight function \( g \) are known to the statistician.

Among the interesting instantiations of this general framework are the following:

- **Average treatment effect**: The ATE problem corresponds to estimating the linear functional
  \[
  \tau^* = \mathbb{E}_{\xi^*} \left[ \mu^*(X, 1) - \mu^*(X, 0) \right].
  \]
  It is a special case of equation (3), obtained by taking the binary action space \( \mathcal{A} = \{0, 1\} \) with \( \lambda \) being the counting measure, along with the weight function \( g(x, a) := 2a - 1 \).

- **Weighted average treatment effect**: Again with binary actions, suppose that we adopt the weight function \( g(x, a) := (2a - 1) \cdot w(x) \), for some given function \( w : \mathcal{X} \rightarrow \mathbb{R}_+ \). With the choice \( w(x) := \pi(x, 1) \), this corresponds to average treatment effect on the treated (ATET).

- **Off-policy evaluation for contextual bandits**: For a general finite action space \( \mathcal{A} \), a *target policy* is a mapping \( x \mapsto \pi_{\text{tar}}(x, \cdot) \), corresponding to a probability distribution over the action space. If we take the weight function \( g(x, a) := \pi_{\text{tar}}(x, a) \) and interpret \( \mu^* \) as a reward function, then the linear functional (3) corresponds to the value of the target policy \( \pi_{\text{tar}} \). Since the observed actions are sampled according to \( \pi \)—which can be different than the target policy \( \pi_{\text{tar}} \)—this problem is known as *off-policy* evaluation in the bandit and reinforcement learning literature.

When the propensity score is known, it is a standard fact that one can estimate \( \tau(\mathcal{I}) \) at a \( \sqrt{n} \)-rate via an importance-reweighted plug-in estimator. In particular, under mild conditions,
the inverse propensity weighting (IPW) estimator, given by
\[ \hat{\tau}_{IPW}^n = \frac{1}{n} \sum_{i=1}^{n} \frac{g(X_i, A_i)}{\pi(X_i, A_i)} Y_i, \] (4)

is $\sqrt{n}$-consistent, in the sense that $\hat{\tau}_{IPW}^n - \tau^* = O_p(1/\sqrt{n})$.

However, the problem is more subtle than might appear at might first: the IPW estimator $\hat{\tau}_{IPW}^n$ fails to be asymptotically efficient, meaning that its asymptotic variance is larger than the optimal one. This deficiency arises even when the state space $X$ and action space $A$ are both binary; for instance, see §3 in Hirano et al. [Hir03]. Estimators that are asymptotically efficient can be obtained by first estimating the treatment effect $\mu^*$, and then using this quantity to form an estimate of $\tau(I)$. Such a combination leads to a semi-parametric method, in which $\mu^*$ plays the role of a nuisance function. For example, in application to the ATE problem, Chernozhukov et al. [CCD+18] showed that any consistent estimator of $\mu^*$ yields an asymptotically efficient estimate of $\tau_g(I)$; see §5.1 in their paper. In the sequel, so as to motivate the procedures analyzed in this paper, we discuss a broad range of semi-parametric methods that are asymptotically efficient for estimating the linear functional $\tau(I)$.

While such semi-parametric procedures have attractive asymptotic guarantees, they are necessarily applied in finite samples, in which context a number of questions remain open:

- As noted above, we now have a wide array of estimators that are known to be asymptotically efficient, and are thus “equivalent” from the asymptotic perspective. It is not clear, however, which estimator(s) should be used when working with a finite collection of samples, as one always does in practice. Can we develop theory that provides more refined guidance on the choice of estimators in this regime?

- As opposed to a purely parametric estimator (such as the IPW estimate), semi-parametric procedures involve estimating the treatment effect function $\mu^*$. Such non-parametric estimation requires sample sizes that scale non-trivially with the problem dimension, and induce trade-offs between the estimation and approximation error. In what norm should we measure the approximation/estimation trade-offs associated with estimating the treatment effect? Can we relate this trade-off to non-asymptotic and instance-dependent lower bounds on the difficulty of estimating the linear functional $\tau$?

The main goal of this paper is to give some precise answers to these questions. On the lower bound side, we establish instance-dependent minimax lower bounds on the difficulty of estimating $\tau$. These lower bounds show an interesting elbow effect, in that if the sample size is overly small relative to the complexity of a function class associated with the treatment effect, then there is a penalty in addition to the classical efficient variance. On the upper bound side, we propose a class of weighted constrained least-square estimators that achieve optimal non-asymptotic risk, even in the high-order terms. Both the upper and lower bounds are general, with more concrete consequences for the specific instantiations introduced previously.

**Related work:** Let us provide a more detailed overview of related work in the areas of semi-parametric estimation and more specifically, the literatures on the treatment effect problem as well as related bandit problems.

In this paper, we make use of the notion of local minimax lower bounds which, in its asymptotic instantiation, dates back to seminal work of Le Cam [LC60] and Hájek [Háj72].
These information-based methods were extended to semiparametric settings by Stein [Ste56] and Levit [Lev75, Lev78], among other authors. Under appropriate regularity assumptions, the optimal efficiency is determined by the worst-case Fisher information of regular parametric sub-models in the tangent space; see the monograph [BKRW93] for a comprehensive review.

Early studies of treatment effect estimation were primarily empirical [Ash78]. The unconfoundedness assumption was first formalized by Rosenbaum and Rubin [RR83], thereby leading to the problem setup described in Section 1. A series of seminal papers by Robins and Rotnitzky [RRZ95, RR95] made connections with the semi-parametric literature; the first semi-parametric efficiency bound, using the tangent-based techniques described in the monograph [BKRW93], was formally derived by Hahn [Hah98].

There is now a rich body of work focused on constructing valid inference procedures under various settings, achieving such semiparametric lower bounds. A range of methods have been studied, among them matching procedures [RT92, AI16], inverse propensity weighting [Hah98, HIR03, HNO08, WS20], outcome regression [CHT04, HW21], and doubly robust methods [RR95, CCD+18, MSZ18, FS19]. The two-stage procedure analyzed in the current paper belongs to the broad category of doubly robust methods.

In their classic paper, Robins and Ritov [RR97] showed that if no smoothness assumptions are imposed on the outcome model, then the asymptotic variance of the IPW estimator cannot be beaten. This finding can be understood as a worst-case asymptotic statement; in contrast, this paper takes an instance-dependent perspective, so that any additional structure can be leveraged to obtain superior procedures. Robins et al. [RTLvdV09] derived optimal rates for treatment effect estimation under various smoothness conditions for the outcome function and propensity score function. More recent work has extended this general approach to analyze estimators for other variants of treatment effect (e.g., [KBW22, AK21]). There are some connections between our proof techniques and the analysis in this line of work, but our focus is on finite-sample and instance-dependent results, as opposed to global minimax results.

Portions of our work apply to high-dimensional settings, of which sparse linear models are one instantiation. For this class of problems, the recent papers [BCNZ19, BWZ19, WS20] study the relation between sample size, dimension and sparsity level for which \( \sqrt{n} \)-consistency can be obtained. This body of work applies to the case of unknown propensity scores, which is complementary to our studies with known behavioral policies. To be clear, obtaining \( \sqrt{n} \)-consistency is always possible under our set-up via the IPW estimator; thus, our focus is on the more refined question of non-asymptotic sample size needed to obtain optimal instance-dependent bounds.

Our work is also related to the notion of second-order efficiency in classical asymptotics. Some past work [DGT06, DGT06, Cas07] has studied some canonical semi-parametric problems, including estimating the shift or period of one-dimensional regression functions, and established second-order efficiency asymptotic upper and lower bounds in the exact asymptotics framework. Our instance-dependent lower bounds do not lead to sharp constant factors, but do hold in finite samples. We view it as an important direction for future work to combine exact asymptotic theory with our finite-sample approach so as to obtain second-order efficiency lower bounds with exact first-order asymptotics.

There is also an independent and parallel line of research on the equivalent problem of off-policy evaluation (OPE) in bandits and reinforcement learning. For multi-arm bandits, the paper [LMS15] established the global minimax optimality of certain OPE estimators given a sufficiently large sample size. Wang et al. [WAD17] proposed the “switch” estimator, which switches between importance sampling and regression estimators; this type of procedure, with a particular switching rule, was later shown to be globally minimax optimal for any sample
size [MZJW22]. Despite desirable properties in a worst-case sense, these estimators are known to be asymptotically inefficient, and the sub-optimality is present even ignoring constant factors (see Section 3 of the paper [HIR03] for some relevant discussion). In the more general setting of reinforcement learning, various efficient off-policy evaluation procedures have been proposed and studied [JL16, YW20, ZWB21, KU22]. Other researchers [ZAW22, ZRAZ21, AW21, ZWB21] have studied procedures that are applicable to adaptively collected data. It is an interesting open question to see how the perspective of this paper can be extended to dynamic settings of this type.

Notation: Here we collect some notation used throughout the paper. Given a pair of functions $h_1, h_2 : A \rightarrow \mathbb{R}$ such that $|h_1 h_2| \in L^1(\lambda)$, we define the inner product

$$\langle h_1, h_2 \rangle_{\lambda} := \int h_1(a) h_2(a) \, d\lambda(a)$$

Given a set $A$ in a normed vector space with norm $\| \cdot \|_c$, we denote the diameter $\text{diam}_c(A) := \sup_{x,y \in A} \| x - y \|_c$. For any $\alpha > 0$, the **Orlicz norm** of a scalar random variable $X$ is given by

$$\|X\|_{\psi_\alpha} := \sup\left\{ u > 0 \mid \mathbb{E}[e^{(|X|/u)^\alpha}] \leq 1 \right\}.$$  

The choices $\alpha = 2$ and $\alpha = 1$ correspond, respectively, to the cases of sub-Gaussian and sub-exponential tails, respectively.

Given a metric space $(T, \rho)$ and a set $\Omega \subseteq T$, we use $N(\Omega, \rho; s)$ to denote the cardinality of a minimal $s$-covering of set $\Omega$ under the metric $\rho$. For any scalar $q \geq 1$ and closed interval $[\delta, D]$ we define the Dudley entropy integral

$$J_q(\Omega, \rho; [\delta, D]) := \int_\delta^D \left[ \log N(\Omega, \rho; s) \right]^{1/q} ds.$$  

Given a domain $X$, a bracket $[\ell, u]$ is a pair of real-valued functions on $X$ such that $\ell(x) \leq u(x)$ for any $x \in X$, and a function $f$ is said to lie in the bracket $[\ell, u]$ if $f(x) \in [\ell(x), u(x)]$ for any $x \in X$. Given a probability measure $Q$ over $X$, the size of the bracket $[\ell, u]$ is defined as $\|u - \ell\|_{L^2(Q)}$. For a function class $F$ over $X$, the bracketing number $N_{\text{br}}(F; L^2(Q); s)$ denotes the cardinality of a minimal bracket covering of the set $F$, with each bracket of size smaller than $s$. Given a closed interval $[\delta, D]$, the bracketed chaining integral is given by

$$J_{\text{br}}(F, L^2(Q); [\delta, D]) := \int_\delta^D \sqrt{\log N_{\text{br}}(F; L^2(Q); s)} \, ds.$$  

## 2 Non-asymptotic and instance-dependent upper bounds

We begin with a non-asymptotic analysis of a general class of two-stage estimators of the functional $\tau(I)$. Our upper bounds involve a certain weighted $L^2$-norm—see equation (8a)—which, as shown by our lower bounds in the sequel, plays a fundamental role.

### 2.1 Non-asymptotic risk bounds on two-stage procedures

We first provide some intuition for the class of two-stage estimators that we analyze, before turning to a precise description.
2.1.1 Some elementary intuition

We consider two-stage estimators obtained from simple perturbations of the IPW estimator (4). Given an auxiliary function \( f : \mathcal{X} \times \mathcal{A} \to \mathbb{R} \) and the data set \( \{ (X_i, A_i, Y_i) \}_{i=1}^n \), consider the estimate

\[
\tilde{\tau}_{n}^f = \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{g(X_i, A_i)}{\pi(X_i, A_i)} Y_i - f(X_i, A_i) + \langle f(X_i, \cdot), \pi(X_i, \cdot) \rangle_{\lambda} \right\}.
\]  

(5)

By construction, for any choice of \( f \in L^2(\xi^* \times \pi) \), the quantity \( \tilde{\tau}_{n}^f \) is an unbiased estimate of \( \tau \), so that it is natural to choose \( f \) so as to minimize the variance \( \text{var}(\tilde{\tau}_{n}^f) \) of the induced estimator. As shown in Appendix A.1, the minimum of this variational problem is achieved by the function

\[
f^*(x, a) := \frac{g(x, a)\mu^*(x, a)}{\pi(x, a)} - \langle g(x, \cdot), \mu^*(x, \cdot) \rangle_{\lambda},
\]  

(6a)

where in performing the minimization, we enforced the constraints \( \langle f(x, \cdot), \pi(x, \cdot) \rangle_{\lambda} = 0 \) for any \( x \in \mathcal{X} \). We note that this same function \( f^* \) also arises naturally via consideration of Neyman orthogonality.

The key property of the optimizing function \( f^* \) is that it induces an estimator \( \tilde{\tau}_{n}^{f^*} \) with asymptotically optimal variance—viz.

\[
v^2_* := \text{var} \left( \langle g(X, \cdot), \mu^*(X, \cdot) \rangle_{\lambda} \right) + \int_{\mathcal{A}} \mathbb{E}_{\xi^*} \left[ \frac{g^2(X, a)}{\pi(X, a)} \sigma^2(X, a) \right] d\lambda(a),
\]  

(6b)

where \( \sigma^2(x, a) := \text{var}(Y \mid x, a) \) is the conditional variance (2) of the outcome. See Appendix A.1 for details of this derivation.

2.1.2 A class of two-stage procedures

The preceding set-up naturally leads to a broad class of two-stage procedures, which we define and analyze here. Since the treatment effect \( \mu^* \) is unknown, the optimal function \( f^* \) from equation (6a) is also unknown to us. A natural approach, then, is the two-stage one:

(a) compute an estimate \( \tilde{\mu} \) using part of the data; and then (b) substitute this estimate in equation (6a) so as to construct an approximation to the ideal estimator \( \tilde{\tau}_{n}^{f^*} \). A standard cross-fitting approach (e.g., [CCD+18]) allows one to make full use of data while avoiding the self-correlation bias.

In more detail, we first split the data into two disjoint subsets \( \mathcal{B}_1 := \{ (X_i, A_i, Y_i) \}_{i=1}^{n/2} \) and \( \mathcal{B}_2 := \{ (X_i, A_i, Y_i) \}_{i=n/2+1}^{n} \). We then perform the following two steps:

Step I: For \( j \in \{1, 2\} \), compute an estimate \( \hat{\mu}_{n/2}^{(j)} \) of \( \mu^* \) using the data subset \( \mathcal{B}_j \), and compute

\[
\tilde{\tau}_{n/2}^{(j)}(x, a) := \frac{g(x, a)\hat{\mu}_{n/2}^{(j)}(x, a)}{\pi(x, a)} - \langle g(x, \cdot), \hat{\mu}_{n/2}^{(j)}(x, \cdot) \rangle_{\lambda}.
\]  

(7a)
**Step II:** Use the auxiliary functions \( \tilde{f}_{n/2}^{(1)} \) and \( \tilde{f}_{n/2}^{(2)} \) to construct the estimate

\[
\tilde{\tau}_n := \frac{1}{n} \sum_{i=1}^{n/2} \left\{ g(X_i, A_i) Y_i - \tilde{f}_{n/2}^{(2)}(X_i, A_i) \right\} + \frac{1}{n} \sum_{i=n/2+1}^{n} \left\{ g(X_i, A_i) Y_i - \tilde{f}_{n/2}^{(1)}(X_i, A_i) \right\}. \tag{7b}
\]

As described, these two steps should be understood as defining a meta-procedure, since the choice of auxiliary estimator \( \tilde{\mu}_{n/2}^{(j)} \) can be arbitrary.

The main result of this section is a non-asymptotic upper bound on the MSE of any such two-stage estimator. It involves the *weighted \( L^2 \)-norm* \( \| \cdot \|_\omega \) given by

\[
\| h \|_\omega^2 := \int \mathbb{E}_{\xi} \left[ \frac{g^2(X, a)}{\pi(X, a)} h^2(X, a) \right] d\lambda(a), \tag{8a}
\]

which plays a fundamental role in both upper and lower bounds for the problem. With this notation, we have:

**Theorem 1.** For any estimator \( \hat{\mu}_{n/2} \) of the treatment effect, the two-stage estimator (7) has MSE bounded as

\[
\mathbb{E} \left[ (\tilde{\tau}_n - \tau^*)^2 \right] \leq \frac{1}{n} \left\{ v_\omega^2 + 2\mathbb{E} \left[ \| \hat{\mu}_{n/2} - \mu^* \|_\omega^2 \right] \right\}. \tag{8b}
\]

See Section 4.1 for the proof of this claim.

Note that the upper bound (8b) consists of two terms, both of which have natural interpretations. The first term \( v_\omega^2 \) corresponds to the asymptotically efficient variance (6b); in terms of the weighted norm (8a), it has the equivalent expression

\[
v_\omega^2 = \text{var} \left( \langle g(X, \cdot), \mu^*(X, \cdot) \rangle_\lambda \right) + \| \sigma \|_\omega^2. \tag{8c}
\]

The second term corresponds to twice the average estimation error \( \mathbb{E} \left[ \| \hat{\mu}_{n/2} - \mu^* \|_\omega^2 \right] \), again measured in the weighted squared norm (8a). Whenever the treatment effect can be estimated consistently—so that this second term is vanishing in \( n \)—we see that the estimator \( \hat{\tau}_n \) is asymptotically efficient, as is known from past work [CCD+18]. Of primary interest to us is the guidance provided by the bound (8b) in the finite sample regime: in particular, in order to minimize this upper bound, one should construct estimators \( \hat{\mu} \) of the treatment effect that are optimal in the weighted norm (8a).

### 2.2 Some non-asymptotic analysis

With this general result in hand, we now propose some explicit two-stage procedures that can be shown to be finite-sample optimal. We begin by introducing the classical idea of an oracle inequality, and making note of its consequences when combined with Theorem 1. We then analyze a class of non-parametric weighted least-squares estimators, and prove that they satisfy an oracle inequality of the desired type.
2.2.1 Oracle inequalities and finite-sample bounds

At a high level, Theorem 1 reduces our problem to an instance of non-parametric regression, albeit one involving the weighted norm $\| \cdot \|_\omega$ from equation (8a). In non-parametric regression, there are many methods known to satisfy an attractive “oracle” property (e.g., see the books [Tsy08, Wai19]). In particular, suppose that we construct an estimate $\hat{\mu}$ that takes values in some function class $\mathcal{F}$. It is said to satisfy an oracle inequality for estimating $\mu^*$ in the norm $\| \cdot \|_\omega$ if

$$
E[\|\hat{\mu} - \mu^*\|_\omega^2] \leq c \inf_{\mu \in \mathcal{F}} \left\{ \|\mu - \mu^*\|_\omega^2 + \delta_n^2(\mu; \mathcal{F}) \right\}
$$

(9)

for some universal constant $c \geq 1$. Here the functional $\mu \mapsto \delta_n^2(\mu; \mathcal{F})$ quantifies the $\| \cdot \|_\omega^2$-error associated with estimating some function $\mu \in \mathcal{F}$, whereas the quantity $\|\mu - \mu^*\|_\omega^2$ is the squared approximation error, since the true function $\mu^*$ need not belong to the class. We note that the oracle inequality stated here is somewhat more refined than the standard one, since we have allowed the estimation error to be instance-dependent (via its dependence on the choice of $\mu$).

Given an estimator $\hat{\mu}$ that satisfies such an oracle inequality, an immediate consequence of Theorem 1 is that the associated two-stage estimator of $\tau^* \equiv \tau(I)$ has MSE upper bounded as

$$
E[|\hat{\tau}_n - \tau^*|^2] \leq \frac{1}{n} \left( v_n^2 + 2c \inf_{\mu \in \mathcal{F}} \left\{ \|\mu - \mu^*\|_\omega^2 + \delta_n^2(\mu; \mathcal{F}) \right\} \right).
$$

(10)

This upper bound is explicit, and given some assumptions on the approximability of the unknown $\mu^*$, we can use it to choose the “complexity” of the function class $\mathcal{F}$ in a data-dependent manner. See Section 2.4 for discussion and illustration of such choices for different function classes.

2.2.2 Oracle inequalities for non-parametric weighted least-squares

Based on the preceding discussion, we now turn to the task of proposing a suitable estimator of $\mu^*$, and proving that it satisfies the requisite oracle inequality (9). Let $\mathcal{F}$ be a given function class used to approximate the treatment effect $\mu^*$. Given our goal of establishing bounds in the weighted norm (8a), it is natural to analyze the non-parametric weighted least-squares estimate

$$
\hat{\mu}_m := \arg\min_{\mu \in \mathcal{F}} \left\{ \frac{1}{m} \sum_{i=1}^{m} \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} \left\{ \mu(X_i, A_i) - Y_i \right\}^2 \right\},
$$

(11)

where $\{(X_i, A_i, Y_i)\}_{i=1}^{m}$ constitute an observed collection of state-action-outcome triples.

Since the pairs $(X, A)$ are drawn from the distribution $\xi^*(x)\pi(x, a)$, our choice of weights ensures that

$$
E \left[ \frac{g^2(X, A)}{\pi^2(X, A)} \left\{ \mu(X, A) - Y \right\}^2 \right] = \|\mu - \mu^*\|_\omega^2 + E \left[ \frac{g^2(X, A)}{\pi^2(X, A)} \sigma^2(X, A) \right].
$$

so that (up to a constant offset), we are minimizing an unbiased estimate of $\|\mu - \mu^*\|_\omega^2$.

In our analysis, we impose some natural conditions on the function class:

**(CC)** The function class $\mathcal{F}$ is a **convex and compact** subset of the Hilbert space $L^2_\omega$. 


We also require some tail conditions on functions $h$ that belong to the difference set
\[ \partial \mathcal{F} := \{ f_1 - f_2 \mid f_1, f_2 \in \mathcal{F} \}. \]

There are various results in the non-parametric literature that rely on functions being uniformly bounded, or satisfying other sub-Gaussian or sub-exponential tail conditions (e.g., [Wai19]). Here we instead leverage the less restrictive learning-without-concentration framework of Mendelson [Men15], and require that the following small probability condition holds:

(SB) There exists a pair $(\alpha_1, \alpha_2)$ of positive scalars such that
\[ P\left[ \left| \frac{g(X,A)}{\pi(X,A)} h(X,A) \right| \geq \alpha_1 \|h\|_{\omega} \right] \geq \alpha_2 \quad \text{for all } h \in \partial \mathcal{F}. \quad (12) \]

If we introduce the shorthand $\tilde{h} = \frac{2}{n} h$, then condition (12) can be written equivalently as
\[ P[\|\tilde{h}(X,A)\| \geq \alpha_1 \|\tilde{h}\|_{2} \geq \alpha_2], \]
so that it is a standard small-ball condition on the function $\tilde{h}$; see the papers [KM15, Men15] for more background.

As with existing theory on non-parametric estimation, our risk bounds are determined by the suprema of empirical processes, with “localization” so as to obtain optimal rates. Given a function class $\mathcal{H}$ and a positive integer $m$, we define the *Rademacher complexities*
\[
\mathcal{S}_m^2(\mathcal{H}) := E\left[ \sup_{f \in \mathcal{H}} \left\{ \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i g(X_i, A_i) (Y_i - \mu^*(X_i, A_i)) f(X_i, A_i) \right\}^2 \right], \quad \text{and} \quad (13a)
\]
\[
\mathcal{R}_m(\mathcal{H}) := E\left[ \sup_{f \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i g(X_i, A_i) f(X_i, A_i) \right], \quad (13b)
\]
where $(\varepsilon_i)_{i=1}^n$ are i.i.d. Rademacher random variables independent of the data.

With this set-up, we are now ready to state some oracle inequalities satisfied by the weighted least-squares estimator (11). As in our earlier statement (9), these bounds are indexed by some $\mu \in \mathcal{F}$, and our risk bound involves the solutions
\[
\frac{1}{s} \mathcal{S}_m((\mathcal{F} - \mu) \cap B_{\omega}(s)) \leq s, \quad \text{and} \quad (14a)
\]
\[
\frac{1}{r} \mathcal{R}_m((\mathcal{F} - \mu) \cap B_{\omega}(r)) \leq \frac{\alpha_1 \alpha_2}{32}. \quad (14b)
\]

Let $s_m(\mu)$ and $r_m(\mu)$, respectively, be the smallest non-negative solutions to these inequalities; see Proposition 4 in Appendix A.2 for their guaranteed existence.

**Theorem 2.** Under the convexity/compactness condition (CC) and small-ball condition (SB), the two-stage estimate (7) based on the non-parametric least-squares estimate (11) satisfies the oracle inequality
\[ E[(\hat{r}_n - r^*)^2] \leq \frac{1}{n} \left\{ v^2 + c \inf_{\mu \in \mathcal{F}} \left( \|\mu - \mu^*\|_{\omega}^2 + \delta^2_0(\mu; F) \right) \right\}, \quad (15a) \]

where the instance-dependent estimation error is given by
\[ \delta^2_0(\mu; F) := s_{n/2}^2(\mu) + r_{n/2}^2(\mu) + e^{-c'n} \operatorname{diam}_{\omega}^2(\mathcal{F} \cup \{\mu^*\}), \quad (15b) \]
for a pair $(c, c')$ of constants depending only on the small-ball parameters $(\alpha_1, \alpha_2)$. 

9
See Section 4.2 for the proof of this theorem.

A few remarks are in order. The bound (15a) arises by combining the general bound from Theorem 1 with an oracle inequality that we establish for the weighted least-squares estimator (11). Compared to the efficient variance $v_2^2$, this bound includes three additional terms: (i) the critical radii $s_{n/2}(\mu)$ and $r_{n/2}(\mu)$ that solve the fixed point equations; (ii) the approximation error under $\| \cdot \|_{\omega}$ norm; and (iii) an exponentially decaying term. For any fixed function class $\mathcal{F}$, if we take limits as the sample size $n$ tends to infinity, we see that the asymptotic variance of $\hat{\tau}_n$ takes the form

$$v_2^2 + c \inf_{\mu \in \mathcal{F}} \| \mu - \mu^* \|^2_{\omega}.$$  

Consequently, the estimator may suffer from an efficiency loss depending on how well the unknown treatment effect $\mu^*$ can be approximated (in the weighted norm) by a member of $\mathcal{F}$. When the outcome noise $Y_i = \mu^*(X_i, A_i)$ is of constant order, inspection of equations (14a) and (14b) reveals that—as $n$ tends to infinity—the critical radius $s_{n/2}(\mu)$ decays at a faster rate than $r_{n/2}(\mu)$. Therefore, the non-asymptotic excess risk—that is, any contribution to the MSE in addition to the efficient variance $v_2^2$—primarily depends on two quantities: (a) the approximation error associated with approximating $\mu^*$ using a given function class $\mathcal{F}$, and (b) the (localized) metric entropy of this function class. Interestingly, both of these quantities turn out to be information-theoretically optimal in an instance-dependent sense. More precisely, in Section 3, we show that an efficiency loss depending on precisely the same approximation error is unavoidable; we further show that a sample size depending on a local notion of metric entropy is also needed for such a bound to be valid.

### 2.3 A simulation study

We now describe a simulation study that helps to illustrate the elbow effect predicted by our theory, along with the utility of using reweighted estimators of the treatment effect. We can model a missing data problem by using $A \in \mathbb{A} := \{0, 1\}$ as a binary indicator variable for “missingness”—that is, the outcome $Y$ is observed if and only if $A = 1$. Taking $\xi$ as the uniform distribution on the state space $\mathbb{X} := [0, 1]$, we take the weight function $g(x, a) = a$, so that our goal is to estimate the quantity $E_\xi[\mu^*(X, 1)]$. Within this subsection, we abuse notation slightly by using $\mu$ to denote the function $\mu(\cdot, 1)$, and similarly $\mu^*$ for $\mu^*(\cdot, 1)$.

We allow the treatment effect to range over the first-order Sobolev smoothness class

$$\mathcal{F} := \left\{ f : [0, 1] \to \mathbb{R} \mid f(0) = 0, \| f \|^2_{H_1} := \int_0^1 (f'(x))^2 dx \leq 1 \right\},$$

corresponding (roughly) to functions that have a first-order derivative $f'$ with bounded $L^2$-norm. The function class $\mathcal{F}$ is a particular type of reproducing kernel Hilbert space (cf. Example 12.19 in the book [Wai19]), so it is natural to consider various forms of kernel ridge regression.

**Three possible estimators:** So as to streamline notation, we let $S_{\text{obs}} \subseteq \{1, \ldots, m\}$ denote the subset of indices associated with observed outcomes—that is, $a_i = 1$ if and only if $i \in S_{\text{obs}}$. Our first estimator follows the protocol suggested by our theory: more precisely, we estimate the function $\mu^*$ using a reweighted form of kernel ridge regression (KRR)

$$\hat{\mu}_{m, \omega} := \arg \min_{\mu \in L^2([0, 1])} \left\{ \sum_{i \in S_{\text{obs}}} \frac{g^2(X_i, 1)}{\pi^2(X_i, 1)} \{ Y_i - \mu(X_i) \}^2 + \lambda_m \| \mu \|^2_{H_1} \right\},$$

(16a)
where $\lambda_m \geq 0$ is a regularization parameter (to be chosen by cross-validation). Let $\hat{\tau}_{n,\omega}$ be the output of the two-stage procedure (7) when the reweighted KRR estimate is used in the first stage.

So as to isolate the effect of reweighting, we also implement the standard (unweighted) KRR estimate, given by

$$\hat{\mu}_{m, L^2} := \arg \min_{\mu \in L^2([0,1])} \left\{ \sum_{i \in S_{obs}} \{ Y_i - \mu(X_i) \}^2 + \lambda_m \| \mu \|_2 \right\}.$$  \hspace{1cm} (16b)

Similarly, we let $\hat{\tau}_{n, L^2}$ denote the estimate obtained by using the unweighted KRR estimate as a first-stage quantity.

Finally, so as to provide an (unbeatable) baseline for comparison, we compute the oracle estimate

$$\hat{\tau}_{n, \text{oracle}} := \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{(Y_i - \mu^*(X_i)) A_i}{\pi(X_i, 1)} + \mu^*(X_i) \right\}.$$ \hspace{1cm} (16c)

Here the term “oracle” refers to the fact that it provides an answer given the unrealistic assumption that the true treatment effect $\mu^*$ is known. Thus, this estimate cannot be computed based purely on observed quantities, but instead serves as a lower bound for calibrating. For each of these three estimators, we compute its $n$-rescaled mean-squared error

$$n \cdot \mathbb{E}[|\hat{\tau}_{n, \diamond} - \tau^*|^2] \quad \text{with } \diamond \in \{ \omega, L^2, \text{oracle} \}.$$ \hspace{1cm} (17)

**Variance functions:** Let us now describe an interesting family of variance functions $\sigma^2$ and propensity scores $\pi$. We begin by observing that if the standard deviation function $\sigma$ takes values of the same order as the treatment effect $\mu^*$, then the simple IPW estimator has a variance of the same order as the asymptotic efficient limit $v^2_*$. Thus, in order to make the problem non-trivial and illustrate the advantage of semiparametric methods, we consider variance functions of the following type: for a given propensity score $\pi$ and exponent $\gamma \in [0,1]$, define

$$\sigma^2(x, 1) := \sigma_0^2 \left[ \pi(x, 1) \right]^\gamma,$$ \hspace{1cm} (18)

where $\sigma_0 > 0$ is a constant pre-factor. Since the optimal asymptotic variance $v_*$ contains the term $\mathbb{E}\left[ \frac{\sigma^2(X, 1)}{\pi(X, 1)} \right]$, this family leads to a term of the form

$$\sigma_0^2 \mathbb{E}\left[ \frac{1}{\{\pi(X, 1)\}^{1-\gamma}} \right],$$

showing that (in rough terms) the exponent $\gamma$ controls the influence of small values of the propensity score $\pi(X, 1)$. At one extreme, for $\gamma = 1$, there is no dependence on these small values, whereas the other extreme $\gamma = 0$, it will be maximally sensitive to small values of the propensity score.

**Propensity and treatment effect:** We consider the following two choices of propensity scores

$$\pi_1(x, 1) := \frac{1}{2} - \left( \frac{1}{2} - \pi_{\min} \right) \sin(\pi x), \quad \text{and}$$

$$\pi_2(x, 1) := \frac{1}{2} - \left( \frac{1}{2} - \pi_{\min} \right) \sin(\pi x/2).$$ \hspace{1cm} (19a) \hspace{1cm} (19b)
where $\pi_{\text{min}} := 0.005$. At the same time, we take the treatment effect to be the “tent” function

$$
\mu^*(x) = \frac{1}{2} - \left| x - \frac{1}{2} \right| \quad \text{for } x \in [0, 1].
$$

(20)

Let us provide the rationale for these choices. Both propensity score functions take values in the interval $[\pi_{\text{min}}, 0.5]$, but achieve the minimal value $\pi_{\text{min}}$ at different points within this interval: $x = 1/2$ for $\pi_1$ and at $x = 1$ for $\pi_2$. Now observe that for the missing data problem, the risk of the naïve IPW estimator (4) contains a term of the form $E[\mu^*(X)^2]$. Since our chosen treatment effect function (20) is maximized at $x = 1/2$, this term is much larger when we set $\pi = \pi_1$, which is minimized at $x = 1/2$. Thus, the propensity score $\pi_1$ serves as a “hard” example. On the other hand, the treatment effect is minimized at $x = 1$, where $\pi_2$ achieves its minimum, so that this represents an “easy” example.

**Simulation set-up and results:** For each choice of exponent $\gamma \in \{0, 0.5, 1\}$ and each choice of propensity score $\pi \in \{\pi_1, \pi_2\}$, we implemented the reweighted estimator $\hat{\tau}_{n,\omega}$, the standard estimator $\hat{\tau}_{n,L^2}$ and the oracle estimator $\hat{\tau}_{n,\text{oracle}}$. For each simulation, we varied the sample size over the range $n \in \{2000, 4000, \ldots, 18000, 20000\}$. For each run, we use 5-fold cross validation to choose the value of regularization parameter $\lambda_n \in [10^{-1}, 10^2]$. For each estimator and choice of simulation parameters, we performed a total of 1000 independent runs, and used them to form a Monte Carlo estimate of the true MSE.

Figure 1 provides plots of the $n$-rescaled mean-squared error (17) versus the sample size $n$ for each of the three estimators in each of the six set-ups (three choices of $\gamma$, crossed with two choices of propensity score). In order to interpret the results, first note that consistent with the classical theory, the $n$-rescaled MSE of the oracle estimator stays at a constant level for different sample sizes. (There are small fluctuations, to be expected, since the quantity $\hat{\tau}_{n,\text{oracle}}$ itself is an empirical average over $n$ samples.) Due to the design of our problem instances, the naïve IPW estimator (4) has much larger mean-squared error; in fact, it is so large that we do not include it in the plot, since doing so would change the scaling of the vertical axis. On the other hand, both the reweighted KRR two-stage estimate $\hat{\tau}_{n,\omega}$ and the standard KRR two-stage estimate $\hat{\tau}_{n,L^2}$ exhibit the elbow effect suggested by our theory: when the sample size is relatively small, the high-order terms in the risk dominate, yielding a large normalized MSE. However, as the sample size increases, these high-order terms decay at a faster rate, so that the renormalized MSE eventually converges to the asymptotically optimal limit (i.e., the risk of the oracle estimator $\hat{\tau}_{n,\text{oracle}}$). In all our simulation instances, the weighted estimator $\hat{\tau}_{n,\omega}$, which uses a reweighted non-parametric least-squares estimate in the first stage, outperforms the standard two-stage estimator $\hat{\tau}_{n,L^2}$ that does not reweight the objective. Again, this behavior is to be expected from our theory: in our bounds, the excess MSE due to errors in estimating the treatment effect is measured using the weighted norm.

### 2.4 Implications for particular models

We now return to our theoretical thread, and illustrate the consequences of our general theory for some concrete classes of outcome models.

#### 2.4.1 Standard linear functions

We begin with the simplest case, namely that of linear outcome functions. For each $j = 1, \ldots, d$, let $\phi_j : \mathbb{X} \times \mathbb{A} \to \mathbb{R}$ be a basis function, and consider functions that are linear in this
Figure 1. Plots of the normalized MSE $n \cdot E[\hat{\tau}_{n, o} - \tau^*]$ for $o \in \{\omega, L^2, \text{oracle}\}$ versus the sample size. Each marker corresponds to a Monte Carlo estimate based on the empirical average of 1000 independent runs. As indicated in the figure titles, panels (a–f) show the normalized MSE of estimators for combinations of parameters: exponent $\gamma \in \{0, 0.5, 1\}$ in the top, middle and bottom rows respectively, and propensity scores $\pi_1$ and $\pi_2$ in the left and right columns, respectively. For each run, we used 5-fold cross validation to choose the value of regularization parameter $\lambda_n \in [10^{-1}, 10^2]$.

representation—viz. $f_\theta(x, a) = \sum_{j=1}^d \theta_j \phi_j(x, a)$ for some parameter vector $\theta \in \mathbb{R}^d$. For a
radius\(^1\) \(R_2 \succ 0\), we define the function class
\[
\mathcal{F} := \left\{ f_\theta \mid \|\theta\|_2 \leq R_2 \right\}.
\]

Our result assumes the existence of the following moment matrices:
\[
\Sigma := \mathbb{E} \left[ \frac{g^2(X,A)}{\pi^2(X,A)} \phi(X,A)\phi(X,A)^\top \right], \quad \text{and} \quad \Gamma_\sigma := \mathbb{E} \left[ \frac{g^4(X,A)}{\pi^4(X,A)} \sigma^2(X,A)\phi(X,A)\phi(X,A)^\top \right].
\]

With this set-up, we have:

**Corollary 1.** Under the small-ball condition (SB), given a sample size satisfying the lower bound \(n \geq c_0 \{ d + \log(R_2\lambda_{\max}(\Sigma)) \}\), the estimate \(\hat{\tau}_n\) satisfies the bound
\[
\mathbb{E}\left[ |\hat{\tau}_n - \tau^*|^2 \right] \leq \frac{1}{n} \left\{ v_n^2 + c \inf_{\mu \in \mathcal{F}} \|\mu - \mu^*\|_{2,\omega}^2 \right\} + \frac{c}{n^2} \text{trace} \left( \Sigma^{-1}\Gamma_\sigma \right),
\]
where the constants \((c_0, c)\) depend only on the small-ball parameters \((\alpha_1, \alpha_2)\).

See Appendix B.1 for the proof of this corollary.

A few remarks are in order. First, Corollary 1 is valid in the regime \(n \gtrsim d\), and the higher order term scales as \(\mathcal{O}\left( d/n^2 \right) \) in the worst case. Consequently, the optimal efficiency \(v_n^2 + \inf_{\mu \in \mathcal{F}} \|\mu - \mu^*\|_{2,\omega}^2\) is achieved when the sample size \(n\) exceeds the dimension \(d\) for linear models.\(^2\)

It is worth noting, however, that in the well-specified case with \(\mu^* \in \mathcal{F}\), the high-order term \(\frac{c}{n^2} \text{trace} \left( \Sigma^{-1}\Gamma_\sigma \right)\) in equation (21) does not necessarily correspond to the optimal risk for estimating the function \(\mu^*\) under the weighted norm \(\|\cdot\|_{\omega}\). Indeed, in order to estimate the function \(\mu^*\) with the optimal semi-parametric efficiency under a linear model, an estimator that reweights samples with the function \(\frac{1}{\sigma(X,A)}\) is the optimal choice, leading to a higher order term of the form \(\frac{c}{n^2} \text{trace} \left( \Sigma^{-1}\Sigma \right)\), where \(\Sigma := \mathbb{E} \left[ \frac{1}{\sigma^4(X,A)} \phi(X,A)\phi(X,A)^\top \right].\(^3\) In general, the question of achieving optimality with respect to both the approximation error and high-order terms (under the \(\|\cdot\|_{\omega}-\text{norm}\)) is currently open.

### 2.4.2 Sparse linear models

Now we turn to sparse linear models for the outcome function. Recall the basis function set-up from Section 2.4.1, and the linear functions \(f_\theta = \sum_{j=1}^d \theta_j \phi_j(x,a)\). Given a radius \(R_1 > 0\), consider the class of linear functions induced by parameters with bounded \(\ell_1\)-norm—viz.
\[
\mathcal{F} := \left\{ f_\theta \mid \|\theta\|_1 \leq R_1 \right\}.
\]

Sparse linear models of this type arise in many applications, and have been the subject of intensive study (e.g., see the books [HTW15, Wai19] and references therein).

\(^1\)We introduce this radius only to ensure compactness; in our final bound, the dependence on \(R_2\) is exponentially decaying, so that it is of little consequence.

\(^2\)We note in passing that the constant pre-factor \(c\) in front of the term \(n^{-1}\inf_{\mu \in \mathcal{F}} \|\mu - \mu^*\|_{2,\omega}^2\) can be reduced to 1 using the arguments in Corollary 5.

\(^3\)Note that \(\Sigma \Sigma^{\top} \Gamma_\sigma \geq \text{cov} \left( \frac{\sigma(X,A)^{-1}\phi(X,A)}{\sigma^4(X,A)} \phi(X,A)\phi(X,A)^\top \right) \gtrsim 0\). Taking the Schur complement we obtain that \(\Gamma_\sigma \geq \Sigma^{-1}\Sigma\), which implies that \(\text{trace}(\Sigma^{-1}\Gamma_\sigma) \geq \text{trace}(\Sigma^{-1}\Sigma)\).
We assume that the basis functions and outcome noise \( Y - \mu^*(X, A) \) satisfy the moment bounds
\[
\mathbb{E}
\left[
\frac{g(X, A)}{\pi(X, A)} (Y - \mu^*(X, A))\right] \leq (\bar{\sigma} \sqrt{\ell})^\ell, \quad \text{for any } \ell = 1, 2, \ldots, \quad (22a)
\]
\[
\max_{j=1, \ldots, d} \mathbb{E}
\left[
\frac{g(X, A)}{\pi(X, A)} \phi_j(X, A)\right] \leq (\nu \sqrt{\ell})^\ell, \quad \text{for any } \ell = 1, 2, \ldots. \quad (22b)
\]
Under these conditions, we have the following guarantee:

**Corollary 2.** Under the small-ball condition (SB) and the moment bounds (22), for any sparsity level \( k = 1, \ldots, d \) and sample size \( n \) such that \( n \geq c_0 \left\{ \frac{\nu^2 k \log(d)}{\lambda_{\min}(\Sigma)} + \nu^2 (\log d + \log(R_1 \cdot \lambda_{\max}(\Sigma))) \right\} \), we have
\[
\mathbb{E}
\left[
|\bar{\tau}_n - \tau^*|^2 \right] \leq \frac{\nu^2}{n} + \frac{c}{n} \inf_{\|\theta\|_{0} \leq k} \left\{ \|\mu^* - \langle \theta, \phi(\cdot, \cdot) \rangle \|_\infty^2 + \|\theta\|_0 \log(d) \right\},
\]
where the constants \( (c_0, c) \) depend only on the small ball parameters \( (\alpha_1, \alpha_2) \).

See Appendix B.2 for the proof of this corollary.

A few remarks are in order. First, the additional risk term compared to the semiparametric efficient limit \( \nu^2 / n \) is similar to existing oracle inequalities for sparse linear regression (e.g., §7.3 in the book [Wai19]). Notably, it adapts to the sparsity level of the approximating vector \( \bar{\theta} \). The complexity of the auxiliary estimation task is characterized by the sparsity level \( \|\theta\|_0 \) of the target function, which appears in both the high-order term of the risk bound and the sample size requirement. On the other hand, note that the \( \| \cdot \|_\infty \)-norm projection of the function \( \mu^* \) to the set \( \mathcal{F} \) may not be sparse. Instead of depending on the (potentially large) local complexity of such projection, the bound in Corollary 2 is adaptive to the trade-off between the sparsity level \( \|\bar{\theta}\|_0 \) and the approximation error \( \|\mu^* - \langle \bar{\theta}, \phi(\cdot, \cdot) \rangle\|_\infty \).

### 2.4.3 Hölder smoothness classes

Let us now consider a non-parametric class of outcome functions. With state space \( X = [0, 1]^d_x \) and action space \( A = [0, 1]^d_a \), define the total dimension \( p := d_x + d_a \). Given an integer order of smoothness \( k > 0 \), consider the class
\[
\mathcal{F}_k := \left\{ \mu : [0, 1]^p \rightarrow \mathbb{R} \mid \sup_{(x, a) \in [0, 1]^p} |\partial^\alpha \mu(x, a)| \leq 1 \quad \text{for any multi-index } \alpha \text{ satisfying } \|\alpha\|_1 \leq k \right\}.
\]
Here for a multi-index \( \alpha \in \mathbb{N}^p \), the quantity \( \partial^\alpha f \) denotes the mixed partial derivative
\[
\partial^\alpha f(x, a) := \left( \prod_{j=1}^p \frac{\partial^{\alpha_j} f}{\partial x_j^{\alpha_j}} \right) f(x, a).
\]
We impose the following assumptions on the likelihood ratio and random noise
\[
\mathbb{E}
\left[
\left| \frac{g(X, A)}{\pi(X, A)} (Y - \mu^*(X, A)) \right|^\ell \right] \leq (\bar{\sigma} \sqrt{\ell})^\ell \quad \text{and} \quad (23a)
\]
\[
\mathbb{E}
\left[
\left| \frac{g(X, A)}{\pi(X, A)} \right|^\ell \right] \leq (\nu \sqrt{\ell})^\ell, \quad \text{for any } \ell \in \mathbb{N}_+. \quad (23b)
\]
Additionally, we impose the $L_2 - L_4$ hypercontractivity condition

$$
\sqrt{\mathbb{E}\left[ \left( \frac{g(X,A)}{\pi(X,A)} f(X,A) \right)^4 \right]} \leq M_{2\rightarrow 4} \mathbb{E}\left[ \left( \frac{g(X,A)}{\pi(X,A)} f(X,A) \right)^2 \right] \text{ for any } f \in \mathcal{F}_k,
$$

(23c)

which is slightly stronger than the small-ball condition (SB).

Our result involves the sequences $\tilde{r}_n := c_{\nu,p/k} n^{-k/p} \log n$, and

$$
\tilde{s}_n := c_{\nu,p/k} \tilde{\sigma} \cdot \begin{cases} 
n^{-k/p} & \text{if } p < 2k \\
n^{-1/4} \sqrt{\log n} & \text{if } p = 2k, \\
n^{-k/p} & \text{if } p > 2k,
\end{cases}
$$

where the constant $c_{\nu,p/k}$ depends on the tuple $(\nu,p/k,M_{2\rightarrow 4})$.

With this notation, when the outcome function is approximated by the class $\mathcal{F}_k$, we have the following guarantee for treatment effect estimation:

**Corollary 3.** Under the small-ball condition (SB) and the moment bounds (23)(a)–(c), we have

$$
\mathbb{E}\left[ |\hat{\tau}_n - \tau^*|^2 \right] \leq \frac{1}{n} \left\{ v_*^2 + c \inf_{\mu \in \mathcal{F}_k} \| \mu^* - \mu \|^2_{\omega} \right\} + \frac{c}{n} \left\{ s_n^2 + r_n^2 \right\},
$$

(24)

where the constant $c$ depends only on the small-ball parameters $(\alpha_1, \alpha_2)$.

See Appendix B.3 for the proof of this corollary.

It is worth making a few comments about this result. First, in the high-noise regime where $\tilde{\sigma} \gtrsim 1$, the term $\tilde{s}_n$ is dominant. This particular rate is optimal in the Donsker regime ($p < 2k$), but is sub-optimal when $p > 2k$. However, this sub-optimality only appears in high-order terms, and is of lower order for a sample size $n$ such that $\log n \gg (p/k)$. Indeed, even if the least-square estimators is sub-optimal for nonparametric estimation in non-Donsker classes, the reweighted least-square estimator (11) may still be desirable, as it is able to approximate the projection of the function $\mu^*$ onto the class $\mathcal{F}_k$, under the weighted norm $\| \cdot \|_{\omega}$.

### 2.4.4 Monotone functions

We now consider a nonparametric problem that involves shape constraints—namely, that of monotonic functions. Let $\phi: \mathcal{X} \times \mathcal{A} \rightarrow [0, 1]$ be a one-dimensional feature mapping. We consider the class of outcome functions that are monotonic with respect to this feature—namely, the function class

$$
\mathcal{F} := \left\{ (x,a) \rightarrow f(\phi(x,a)) \mid f: [0, 1] \rightarrow [0, 1] \text{ is non-decreasing} \right\}.
$$

We assume the outcome and likelihood ratio are uniformly bounded—specifically, that

$$
|Y_i| \leq 1 \quad \text{and} \quad \frac{g(X_i, A_i)}{\pi(X_i, A_i)} \leq b \quad \text{almost surely for } i = 1, 2, \ldots, n.
$$

(25)

Under these conditions, we have the following result:

---

4In fact, this lower bound cannot be avoided, as shown by our analysis in Section 3.2.1.
Corollary 4. Under the small-ball condition (SB) and boundedness condition (25), we have
\[ 
\mathbb{E}\left[ \left| \hat{\tau}_n - \tau^* \right|^2 \right] \leq \frac{1}{n} \left\{ v^2 + c \inf_{\mu \in \mathcal{F}} \| \mu - \mu^* \|_\omega \right\} + \frac{c}{n} \left( \frac{b^2}{n} \right)^{2/3}, 
\] where the constants \((c_0, c)\) depend only on the small-ball parameters \((\alpha_1, \alpha_2)\).

See Appendix B.4 for the proof of this corollary.

Note that compared to Corollaries 1–3, Corollary 4 requires a stronger uniform bound on the likelihood ratio \(g/\pi\): it is referred to as the strict overlap condition in the causal inference literature. In our analysis, this condition is required to make use of existing bracketing-based localized entropy control. Corollary 4 holds for any sample size \(n \geq 1\), and we establish a matching lower bound as a consequence of Proposition 1 to be stated in the sequel. It should be noted that the likelihood ratio bound \(b\) might be large, in which case the high-order term in Corollary 4 could be dominant (at least for small sample sizes). As with previous examples, optimal estimation of the scalar \(\tau^*\) requires optimal estimation of the function \(\mu^*\) under \(\| \cdot \|_\omega\)-norm. How to do so optimally for isotonic classes appears to be an open question.

2.5 Non-asymptotic normal approximation

Note that the oracle inequality in Theorem 2 involves an approximation factor depending on the small-ball condition in Assumption (SB), as well as other universal constants. Even with sample size \(n\) tending to infinity, the result of Theorem 2 does not ensure that the auxiliary estimator \(\hat{\mu}_{n/2}\) converges to a limiting point. This issue, while less relevant for the mean-squared error bound in Theorem 2, assumes importance in the inferential setting. In this case, we do need the auxiliary estimator to converge so as to be able to characterize the approximation error.

In order to address this issue, we first define the orthogonal projection within the class
\[ \bar{\mu} := \arg \min_{\mu \in \mathcal{F}} \| \mu - \mu^* \|_\omega. \] (27)

Our analysis also involves an additional squared Rademacher complexity, one which involves the difference \(\mu^* - \bar{\mu}\). It is given by
\[ D^2_m(\mathcal{H}) := \mathbb{E}\left[ \sup_{f \in \mathcal{H}} \left\{ \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} \left[ \mu^*(X_i, A_i) - \bar{\mu}(X_i, A_i) \right] f(X_i, A_i) \right\}^2 \right]. \] (28a)

We let \(d_m > 0\) be the unique solution to the fixed point equation
\[ \frac{1}{d} D_m((\mathcal{F} - \bar{\mu}) \cap B_\omega(d)) = d. \] (28b)

The existence and uniqueness is guaranteed by an argument analogous to that used in the proof of Proposition 4.

In order to derive a non-asymptotic CLT, we need a finite fourth moment
\[ M_4 := \mathbb{E}\left[ \left\{ \frac{g(X, A)}{\pi(X, A)} (Y - \bar{\mu}(X, A)) + \langle g(X, \cdot), \bar{\mu}(X, \cdot) \rangle_\lambda \right\}^4 \right]. \]

The statement also involves the excess variance
\[ v^2(\bar{\mu}) := \mathbb{E}\left[ \var\left\{ \frac{g(X, A)}{\pi(X, A)} \cdot \left\{ \mu^*(X, A) - \bar{\mu}(X, A) \right\} \big| X \right\} \right]. \]

With these definitions, we have the following guarantee:
Corollary 5. Under Assumptions (CC) and (SB), the two-stage estimator (7) satisfies the Wasserstein distance bound

$$W_1\left(\sqrt{n}\tilde{\tau}_n, Z\right) \leq \frac{\sqrt{M_p}}{\sqrt{n}} + c\left(r_{n/2} + s_{n/2} + d_n\right) + \text{diam}_{\omega}(\mathcal{F} \cup \{\mu^\star\}) \cdot e^{-c'n},$$

(29)

where $Z \sim \mathcal{N}(0, v^2_\mu + v^2(\bar{\mu}))$, and the pair $(c, c')$ of constants depend only on the small-ball parameters $(\alpha_1, \alpha_2)$.

See Section 4.3 for the proof of this corollary.

A few remarks are in order. First, in the limit $n \to +\infty$, Corollary 5 guarantees asymptotic normality of the estimate $\tilde{\tau}_n$, with asymptotic variance $v^2_\mu + v^2(\bar{\mu})$. In contrast, the non-asymptotic result given here makes valid inference possible at a finite-sample level, by taking into account the estimation error for auxiliary functions. Compared to the risk bound in Theorem 2, the right-hand-side of equation (29) contains two terms: the first term $\frac{\sqrt{4M_p}}{\sqrt{n}}$ is the Berry–Esseen error, and an additional critical radius $d_{n/2}$ depending on the localized multiplier Rademacher complexity. When the approximation error $\mu^* - \bar{\mu}$ is of order $o(1)$, the multiplier Rademacher complexity $\mathcal{D}_{n/2}$ becomes (asymptotically) smaller than the Rademacher complexity $\mathcal{S}_{n/2}$, resulting in a critical radius $d_{n/2}(\bar{\mu})$ smaller than $s_{n/2}(\bar{\mu})$. On the other hand, the efficiency loss in Corollary 5 is the exact variance $v^2(\bar{\mu})$ with unity pre-factor, which exhibits a smaller efficiency loss compared to Theorem 2.

Excess variance compared to approximation error: It should be noted that the excess variance term $v^2(\bar{\mu})$ in Corollary 5 is smaller than the best approximation error $\inf_{\mu^* \in \mathcal{F}} \|\mu - \mu^*\|^2_\omega$. Indeed, the difference $\Delta := \|\bar{\mu} - \mu^*\|^2_\omega - v^2(\bar{\mu})$ can be written as

$$\Delta = E\left[\left(\frac{g(X, A)}{\pi(X, A)} \cdot (\mu^* - \bar{\mu})(X, A)\right)^2\right] - E\left[\var\left(\frac{g(X, A)}{\pi(X, A)} \cdot (\mu^* - \bar{\mu})(X, A) \mid X\right)\right]$$

$$= E_{\pi^*}\left[\langle g(X, \cdot)\rangle, (\mu^* - \bar{\mu})(X, \cdot)\rangle\right].$$

(30)

When considering minimax risk over a local neighborhood around the function $\mu^*$, the difference term computed above is dominated by the supremum of the asymptotic efficient variance $v^2_\mu$ evaluated within this neighborhood. Consequently, the upper bound induced by Corollary 5 does not contradict the local minimax lower bound in Theorem 3; and since the difference $\|\bar{\mu} - \mu^*\|^2_\omega - v^2(\bar{\mu})$ does not involve the importance weight ratio $g/\pi$, this term is usually much smaller than the weighted norm term $\|\bar{\mu} - \mu^*\|^2_\omega$.

On the other hand, Corollary 5 and equation (30) provide guidance on the way of achieving the optimal pointwise exact asymptotic variance. In particular, when we choose a function class $\mathcal{F}$ such that $\langle h(x, \cdot), g(x, \cdot)\rangle_\lambda = 0$ for any $h \in \mathcal{F}$ and $x \in X$, the expression (30) becomes a constant independent of the choice of $\bar{\mu}$. For such a function class, a function $\bar{\mu}$ that minimizes the approximation error $\|\mu - \mu^*\|^2_\omega$ will also minimize the variance $v^2(\mu)$. Such a class can be easily constructed from any function class $\mathcal{H}$ by taking a function $h \in \mathcal{H}$ and replacing it with $f(x, a) := h(x, a) - \langle h(x, \cdot), g(x, \cdot)\rangle_\lambda$. And the optimal variance can still be written in the form of approximation error:

$$v(\bar{\mu}) = \|\bar{\mu} - \mu^*\|_\omega^*,$$

where $\mu^*(x, a) := \mu^*(x, a) - \langle \mu^*(x, \cdot), g(x, \cdot)\rangle_\lambda$. Indeed, the functional $v$ can be seen as the induced norm of $\|\cdot\|_\omega$ in the quotient space generated by $L^2_\omega$ modulo the subspace $L^2(\xi^*)$ that contains functions depending only on the state but not action.
3 Minimax lower bounds

Thus far, we have derived upper bounds for particular estimators of the linear functional \( \tau(I) \), ones that involve the weighted norm (8a). In this section, we turn to the complementary question of deriving local minimax lower bounds for the problem. Recall that any given problem instance is characterized by a quadruple of the form \((\xi^*, \pi, \mu^*, g)\). In this section, we state some lower bounds that hold uniformly over all estimators that are permitted to know both the policy \( \pi \) and the weight function \( g \). With \((\pi, g)\) known, the instance is parameterized by the pair \((\xi^*, \mu^*)\), and we derive two types of lower bounds:

- In Theorem 3, we study local minimax bounds in which the unknown probability distribution \( \xi^* \) and potential outcome function are allowed to range over suitably defined neighborhoods of a given target pair \((\xi^*, \mu^*)\), respectively, but without structural conditions on the function classes.
- In Proposition 1, we impose structural conditions on the function class \( F \) used to model \( \mu^* \), and prove a lower bound that involves the complexity of \( F \)—in particular, via its fat shattering dimension. This lower bound shows that if the sample size is smaller than the function complexity, then any estimator has a mean-squared error larger than the efficient variance.

3.1 Instance-dependent bounds under mis-specification

Given a problem instance \( I^* = (\xi^*, \mu^*) \) and an error function \( \delta : X \times A \to \mathbb{R} \), we consider the local neighborhoods

\[
N^\mu_\delta(\mu^*) := \left\{ \mu \mid |\mu(x,a) - \mu^*(x,a)| \leq \delta(x,a) \text{ for } (x,a) \in X \times A \right\},
\]

\[
N^\nu_\delta(\xi^*) := \left\{ \xi \mid D_{KL}(\xi \parallel \xi^*) \leq \frac{1}{n} \right\}.
\]

Our goal is to lower bound the local minimax risk

\[
M_n(C_\delta(\xi^*)) := \inf_{\hat{\tau}_n} \sup_{I \in C_\delta(\xi^*)} \mathbb{E}|\tau - \hat{\tau}_n|^2 \text{ where } C_\delta(\xi^*) := \left\{ (\xi, \mu) \in N^\nu_\delta(\xi^*) \times N^\mu_\delta(\mu^*) \right\}.
\]

Let us now specify the assumptions that underlie our lower bounds.

**Assumptions for lower bound:** First, we require some tail control on certain random variables, stated in terms of the \((2,4)\)-moment-ratio \( \|Y\|_{2\to4} := \frac{\mathbb{E}[Y^4]}{\mathbb{E}[Y^2]} \).

**MR** The random variables

\[
Z(X,A) := \frac{\delta(X,A)g(X,A)}{\pi(X,A)}, \quad \text{and} \quad Z'(X,A) := \langle \mu^*(X,\cdot), g(X,\cdot) \rangle_\lambda - \tau(I^*)
\]

have finite \((2,4)\)-moment ratios \( M_{2\to4} := \|Z\|_{2\to4} \) and \( M'_{2\to4} := \|Z'\|_{2\to4} \).

Second, we require the existence of a constant \( c_{\text{max}} > 0 \) such that the distribution \( \xi^* \) satisfies the following compatibility condition.

**COM** For a finite state space \( X \), we require \( \xi^*(x) \leq c_{\text{max}}/|X| \) for all \( x \in X \). If \( X \) is infinite, we require that \( \xi^* \) is non-atomic (i.e., \( \xi^*(\{x\}) = 0 \) for all \( x \in X \)), and set \( c_{\text{max}} = 1 \) for concreteness.
Finally, we impose a lower bound on the local neighborhood size:

(LN) The neighborhood function $\delta(x,a)$ satisfies the lower bound

$$\sqrt{n} \delta(x,a) \geq \frac{g(x,a)\sigma^2(x,a)}{\pi(x,a)\|\sigma\|_\omega} \quad \text{for any } (x,a) \in X \times A.$$  \hfill (34)

In the following statement, we use $c$ and $c'$ to denote universal constants.

**Theorem 3.** Under Assumptions (MR), (COM) and (LN), given a sample size lower bounded as $n \geq c' \max\{ (M_{2 \rightarrow 4})^2, M_{2 \rightarrow 4}^2 \}$, the local minimax risk over the class $C_\delta(\mathcal{I}^*)$ is lower bounded as

$$\mathcal{M}_n(C_\delta(\mathcal{I}^*)) \geq c n \left\{ \frac{v_\ast^2}{n} \right\} if n \geq \frac{|X|}{c_{\max}}$$

(35)

We prove this claim in Section 5.1.

It is worth understanding the reasons for each of the assumptions required for this lower bound to hold. The compatibility condition (COM) is needed to ensure that no single state can take a significant proportion of probability mass under $\xi^*$. If this condition is violated, then it could be possible to construct a low MSE estimate of the outcome function via an empirical average, which would then break our lower bound. The neighborhood condition (LN) ensures that the set of problems considered by the adversary is large enough to be able to capture the term $\|\sigma\|^2_\omega$ in the optimal variance $v_\ast^2$. Without this assumption, the “local-neighborhood” restriction on certain states-action pairs could be more informative than the data itself.

Now let us understand some consequences of Theorem 3. First, it establishes the information-theoretic optimality of Theorem 2 and Corollary 5 in an instance-dependent sense. Consider a function class $\mathcal{F}$ that approximately contains the true outcome function $\mu^*$; more formally, consider the $\delta$-approximate version of $\mathcal{F}$ given by

$$\mathcal{F}_\delta := \left\{ \overline{\mu} \in \mathbb{L}^2_\omega \mid \exists \mu \in \mathcal{F} \text{ such that } |\mu(x,a) - \overline{\mu}(x,a)| \leq \delta(x,a) \quad \text{for all } (x,a) \in X \times A \right\},$$

and let us suppose that $\mu^* \in \mathcal{F}_\delta$. With this notation, Theorem 3 implies a lower bound of the form

$$\inf_{\tau_n} \sup_{\mu \in \mathcal{F}_\delta} \mathbb{E}\left[ |\tau - \tau_n|^2 \right] \geq c n \left\{ \sup_{\mu \in \mathcal{F}} \text{var} \left( \langle g(X, \cdot), \mu(X, \cdot) \rangle \lambda + \|\sigma\|^2_\omega + \|\delta\|^2_\omega \right) \right\}. \hfill (36)$$

Thus, we see that the efficiency loss due to errors in estimating the outcome function is unavoidable; moreover, this loss is measured in the weighted norm $\|\cdot\|_\omega$ that also appeared centrally in our upper bounds.

It is also worth noting that for a finite cardinality state space $X$, Theorem 3 exhibits a “phase transition” in the following sense: for a sample size $n \gg |X|$, the lower bound is simply a non-asymptotic version of the semi-parametric efficiency lower bound (up to the pre-factor\(^5\) $c > 1$). On the other hand, when $n < |X|$, then the term $\|\delta\|^2_\omega/n$ starts to play a significant

\(^5\)Using slightly more involved argument, this pre-factor can actually be made arbitrarily close to unity.
role. For an infinite state space $\mathcal{X}$ without atoms, the lower bound (35) holds for any sample size $n$.

By taking $\mu^* = 0$ and $\delta(x, a) = 1$ for all $(x, a)$, equation (35) implies the global minimax lower bound

$$\inf_{\tilde{\tau}_n} \sup_{\|\mu\|_{\infty} \leq 1} \mathbb{E} [ |\tau - \tilde{\tau}_n|^2 ] \geq \frac{c}{n} \int_{\mathcal{X}} \mathbb{E}_{\xi^*} [ g^2(X, a) / \pi(X, a) ] d\lambda(a),$$

(37)
valid whenever $n \leq |\mathcal{X}|$.

The $\chi^2$-type term on the right-hand side of this bound is related to—but distinct from—results from past work on off-policy evaluation in bandits [WAD17, MZJW22]. In this past work, a term of this type arose due to noisiness of the observations. In contrast, our lower bound (37) is valid even if the observed outcome is noiseless, and the additional risk depending on the weighted norm arises instead from the impossibility of estimating $\mu^*$ itself.

3.2 Lower bounds for structured function classes

As we have remarked, in the special case of a finite state space ($|\mathcal{X}| < \infty$), Theorem 3 exhibits an interesting transition at the boundary $n \asymp |\mathcal{X}|$. On the other hand, for an infinite state space, the stronger lower bound in Theorem 3—namely, that involving $\|\delta\|_2^2$—is always in force. It should be noted, however, that this strong lower bound depends critically on the fact that Theorem 3 imposes no conditions on the function class $\mathcal{F}$ of possible treatment effects, so that the error necessarily involves the local perturbation $\delta$.

In this section, we undertake a more refined investigation of this issue. In particular, when some complexity control is imposed upon $\mathcal{F}$, then the lower bounds again exhibit a transition: any procedure pays a price only when the sample size is sufficiently small relative to the complexity of $\mathcal{F}$. In doing so, we assess the complexity of $\mathcal{F}$ using the fat-shattering dimension [KS94, ABDCBH97].

(FS) A collection of data points $(x_i)_{i=1}^N$ is shattered at scale $\delta$ by a function class $\mathcal{H} : \mathcal{X} \rightarrow \mathbb{R}$ means that for any subset $S \subseteq \{1, \ldots, N\}$, there exists a function $f \in \mathcal{H}$ and a vector $t \in \mathbb{R}^N$ such that

$$f(x_i) \geq t_i + \delta \quad \text{for all } i \in S, \quad \text{and} \quad f(x_i) \leq t_i - \delta \quad \text{for all } i \not\in S. \quad (38)$$

The fat-shattering dimension $\text{fat}_\delta(\mathcal{H})$ is the largest integer $N$ for which there exists some sequence $(x_i)_{i=1}^N$ shattered by $\mathcal{H}$ at scale $\delta$.

In order to illustrate a transition depending on the fat shattering dimension, we consider the minimax risk

$$M_n(\mathcal{F}) := \inf_{\tilde{\tau}_n} \sup_{\mu \in \mathcal{F}} \mathbb{E} [ |\tilde{\tau}_n - \tau(\xi, \mu)|^2 ],$$

specializing to the case of a finite action space $\mathcal{A}$ equipped with the counting measure $\lambda$. We further assume that the class $\mathcal{F}$ is a product of classes associated to each action, i.e., $\mathcal{F} = \bigotimes_{a \in \mathcal{A}} \mathcal{F}_a$, with $\mathcal{F}_a$ being a convex subset of real-valued functions on the state space $\mathcal{X}$. We also assume the existence$^6$ of a sequence $(x_j)_{j=1}^D$ that, for each action $a \in \mathcal{A}$, is shattered

$^6$Thus, per force, we have $D \leq \text{fat}_\delta(\mathcal{F}_a)$ for each $a \in \mathcal{A}$. 

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by $F_a$ at scale $\delta_a$. Analogous to the moment ratio assumption (MR), we need an additional assumption that

$$M_{2\rightarrow 4} := \| g(X, A) \delta_A \|_{2\rightarrow 4} < +\infty, \quad \text{for } X \sim U(\{x_j\}_{j=1}^D) \text{ and } A \sim \pi(X, \cdot).$$

(39)

**Proposition 1.** With the set-up given above, there are universal constants ($c, c'$) such that for any sample size satisfying $n \geq M_{2\rightarrow 4}$ and $n \leq c'D$, we have the lower bound

$$M_n(F) \geq c \frac{1}{n} \left\{ \frac{1}{D} \sum_{j=1}^D \sum_{a \in A} g_j^2(x_j, a) \delta_a^2 \right\}.$$  

(40)

See Section 5.2 for the proof of this claim.

A few remarks are in order. First, if we take $\xi$ to be the uniform distribution over the sequence $\{x_j\}_{j=1}^D$, the right-hand-side of the bound (40) is equal to $\frac{1}{n} \| \delta \|^2_2$. Thus, Proposition 1 is the analogue of our earlier lower bound (32) under the additional restriction that the treatment effect function $\mu^*$ belong the given function class $F$. This lower bound holds as long as $n \leq c'D$, so that the fat-shattering dimension $D$ as opposed to the state space cardinality $|\mathcal{X}|$ (for a discrete state space) demarcates the transition between different regimes.

An important take-away of Proposition 1 is that the sample size must exceed the “complexity” of the function class $F$ in order for the asymptotically efficient variance $v^2_*$ to be dominant. More precisely, suppose that—for some scale $\delta > 0$—the sample size is smaller than the fat-shattering dimension $\text{fat}_\delta(F)$. In this regime, the naïve IPW estimator (4) is actually instance-optimal, even when there is no noise. Observe that its risk contains a term of the form $\sum_{a \in A} \mathbb{E} \left[ \frac{g^2(X, a)}{\pi(X, a)} \right]$, which is not present in the asymptotically efficient variance $v^2_*$.

By contrast, suppose instead that the sample size exceeds the fat-shattering dimension. In this regime, it is possible to obtain non-trivial estimates of the treatment effect, so that superior estimates of $\tau^*$ are possible. From the point of view of our theory, one can use the fat shattering dimension $\text{fat}_\delta(F)$ to control the $\delta$-covering number [MV02], and hence the Rademacher complexities that arise in our theory. Doing so leads to non-trivial radii $(s_{n/2}, r_{n/2})$ in Theorem 2, and consequently, the asymptotically efficient variance will become the dominant term. We illustrate this line of reasoning via various examples in Section 2.4.

It should be noted that a sample size scaling with the fat-shattering dimension is also known to be necessary and sufficient to learn the function $\mu^*$ with $o(1)$ error [KS94, BLW94, ABDCBH97]. These classical results, in combination with our Proposition 1 and Theorem 2, exhibit that necessary conditions on the sample size for consistent estimation of the function $\mu^*$ are equivalent to those requiring for achieving the asymptotically efficient variance in estimating the scalar $\tau^*$.

**Worst-case interpretation:** It is worthwhile interpreting the bound (40) in a worst-case setting. Consider a problem with binary action space $\mathcal{A} = \{0, 1\}$ and $g(x, a) = 2a - 1$. Suppose that we use a given function class $\mathcal{H}$ (consisting of functions from the state space $\mathcal{X}$ to the interval $[0, 1]$) as a model$^7$ of both of the functions $\mu^*(\cdot, 0)$ and $\mu^*(\cdot, 1)$. Given a scalar $\pi_{\min} \in (0, 1/2)$, let $\Pi(\pi_{\min})$ be the set of propensity score functions such that $\pi(x, 1) \in [\pi_{\min}, 1 - \pi_{\min}]$ for any $x \in \mathcal{X}$. By taking the worst-case over this class, we find that

$^7$We write $\mu^* \in \mathcal{H}$ as a shorthand for this set-up.
there are universal constants $c, c' > 0$ such that
\[
\sup_{\pi \in \Pi(\pi_{min})} \inf_{\tau_n} \sup_{\mu^* \in \mathcal{H}} \mathbb{E} \left[ |\tau_n - \tau|^2 \right] \geq c \begin{cases} \frac{1}{n} + \frac{\delta^2}{n \pi_{min}} & \text{for } n \leq c' \text{fat}_{\delta}(\mathcal{F}), \\ \frac{1}{n} & \text{otherwise}, \end{cases}
\] (41)
for any $\delta \in (0, 1)$. The validity of this lower bound does not depend on noise in the outcome observations (and therefore applies to noiseless settings). Since $\pi_{min} \in (0, 1)$, any scalar $\delta \gg \sqrt{\pi_{min}}$ yields a non-trivial risk lower bound for sample sizes $n$ below the threshold fat$_{\delta}(\mathcal{F})$.

Relaxing the convexity requirement: Proposition 1 is based on the assumption each function class $\mathcal{F}_n$ is convex. This requirement can be relaxed if we require instead that the sequence $\{x_i\}_{i=1}^D$ be shattered with the inequalities (38) all holding with equality—that is, for any subset $S$, there exists a function $f \in \mathcal{H}$ and a vector $t \in \mathbb{R}^D$ such that
\[
f(x_i) = t_i + \delta \quad \text{for all } i \in S, \quad \text{and} \quad f(x_i) = t_i - \delta \quad \text{for all } i \notin S.
\] (42)
For example, any class of functions mapping $X$ to the binary set $\{0, 1\}$ satisfies this condition with $D = \text{VC}(\mathcal{F})$ and $\delta = 1/2$. In the following, we provide additional examples of non-convex function classes that satisfy equation (42).

### 3.2.1 Examples of fat-shattering lower bounds

We discuss examples of the fat-shattering lower bound (40) in this section. We first describe some implications for convex classes. We then treat some non-convex classes using the strengthened shattering condition (42).

**Example 1** (Smoothness class in high dimensions). We begin with a standard Hölder class on the domain $X = [-1, 1]^p$. For some index $k = 1, 2, \ldots$, we consider functions that are $k$-order smooth in the following sense
\[
\mathcal{F}_{k}^{(\text{Lip})} := \left\{ f : [-1, 1]^p \to \mathbb{R} \mid \sup_{x \in X} \max_{\alpha \in \mathbb{N}^p, \|\alpha\|_1 \leq k} |\partial^\alpha f(x)| \leq 1 \right\}.
\] (43)
By inspection, the class $\mathcal{F}$ is convex. We can lower bound its fat shattering dimension by combining classical results on $L^2$-covering number of smooth functions [KT59] with the relation between fat shattering dimension and covering number [MV02], we conclude that
\[
fat_t(\mathcal{F}_{k}^{(\text{Lip})}) \geq 2^{p/k}, \quad \text{for a sufficiently small scale } t > 0.
\] (44)
Consequently, for a function class with a constant order of smoothness (i.e., not scaling with the dimension $p$), the sample size required to approach the asymptotically optimal efficiency scales exponentially in $p$.

**Example 2** (Single index models). Next we consider a class of single index models with domain $X = [-1, 1]^p$. Since our main goal is to understand scaling issues, we may assume that $p$ is an integer power of 2 without loss of generality. Given a differentiable function $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$ and $\varphi'(x) \geq \ell > 0$ for all $x \in \mathbb{R}$, we consider ridge functions of the form $g_\beta(x) := \varphi(\langle \beta, x \rangle)$. For a radius $R > 0$, we define the class
\[
\mathcal{F}_R^{\text{GLM}} := \left\{ g_\beta \mid \|\beta\|_2 \leq R \right\}.
\] (45)
Let us verify the strengthened shattering condition (42). Suppose that the vectors \( \{x_j\}_{j=1}^p \) define the Hadamard basis in \( p \) dimensions, and so are orthonormal. Taking \( t_j = 0 \) for \( j = 1, \ldots, p \), given any binary vector \( \zeta \in \{-1,1\}^p \), we define the \( p \)-dimensional vector

\[
\beta(\zeta) = \frac{1}{p} \sum_{j=1}^p \varphi^{-1}(\zeta_j aR) x_j,
\]

Given the orthonormality of the vectors \( \{x_j\}_{j=1}^p \), we have

\[
\langle \beta(\zeta), x_\ell \rangle = \varphi^{-1}(\zeta_\ell aR)
\]

for each \( \ell = 1, \ldots, p \), and thus \( g_{\beta(\zeta)}(x_\ell) = \zeta_\ell aR \) for each \( \ell = 1, 2, \ldots, p \). Consequently, the function class \( F_{GLM}^{\text{GLM}} \) satisfies the strengthened shattering condition (42) with fat shattering dimension \( D = p \) and scale \( \delta = aR \). So when the outcome follows a generalized linear model, a sample size must be at least of the order \( p \) in order to match the optimal asymptotic efficiency.

\[\Box\]

**Example 3** (Sparse linear models). Once again take the domain \([-1,1]^p\), and consider linear functions of the form \( f_\beta(x) = \langle \beta, x \rangle \) for some parameter vector \( \beta \in \mathbb{R}^p \). Given a positive integer \( s \in \{1, \ldots, p\} \), known as the sparsity index, we consider the set of \( s \)-sparse linear functions

\[
F_{s,\text{sparse}} := \left\{ f_\beta \mid |\text{supp}(\beta)| \leq s, \text{ and } \|\beta\|_\infty \leq 1 \right\}.
\]

(46)

As noted previously, sparse linear models of this type have a wide range of applications (e.g., see the book [HTW15]).

In Appendix B.5, we prove that the strong shattering condition (42) holds with fat shattering dimension \( D \propto s \log \left( \frac{2p}{s} \right) \). Consequently, if the outcome functions \( \mu^* \) follow a sparse linear model, at least \( \Omega\left(s \log \left( \frac{2p}{s} \right) \right) \) samples are needed to make use of this fact.

\[\Box\]

4 Proofs of upper bounds

In this section, we prove the upper bounds on the estimation error (Theorem 1 and Theorem 2), along with corollaries for specific models.

4.1 Proof of Theorem 1

The error can be decomposed into three terms as \( \hat{\tau}_n - \tau^* = T_* - T_1 - T_2 \), where

\[
T_* := \frac{1}{n} \sum_{i=1}^n \left\{ \frac{g(X_i, A_i)}{\pi(X_i, A_i)} Y_i - \tau^* - f^*(X_i, A_i) \right\}.
\]

\[
T_1 := \frac{1}{n} \sum_{i=1}^{n/2} \left( \hat{f}^{(2)}_{n/2}(X_i, A_i) - f^*(X_i, A_i) \right), \quad \text{and} \quad T_2 := \frac{1}{n} \sum_{i=n/2+1}^n \left( \hat{f}^{(1)}_{n/2}(X_i, A_i) - f^*(X_i, A_i) \right).
\]

Since the terms in the summand defining \( T_* \) are i.i.d., a straightforward computation yields

\[
E[T_*^2] = \frac{1}{n} E \left[ \left( \frac{g(X_i, A_i)}{\pi(X_i, A_i)} Y_i - \tau^* - f^*(X_i, A_i) \right)^2 \right] = \frac{v^2}{n},
\]

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corresponding to the optimal asymptotic variance. For the cross term \( E[T_1 T_2] \), applying the Cauchy-Schwarz inequality yields

\[
|E[T_1 T_2]| \leq \sqrt{E[T_1^2]} \cdot \sqrt{E[T_2^2]} \leq \frac{1}{2n} E[\|\hat{\mu}_{n/2} - \mu^*\|_{\omega}^2].
\]

Consequently, in order to complete the proof, it suffices to show that

\[
E[T_1^2] = E[T_2^2] = \frac{1}{2n} E[\|\hat{\mu}_{n/2} - \mu^*\|_{\omega}^2], \quad \text{and} \quad (47a)
\]

\[
E[T_1 T_2] = E[T_2 T_2] = 0. \quad (47b)
\]

**Proof of equation (47a):** We begin by observing that \( E[T_1^2 \mid B_2] = \frac{1}{2n} \|\hat{f}_{n/2}^{(2)} - f^*\|^2_{\xi \times \pi}. \) Now recall equations (6a) and (7a) that define \( f^* \) and \( \hat{f}_{n/2}^{(2)} \) respectively. From these definitions, we have

\[
\|\hat{f}_{n/2}^{(2)} - f^*\|^2_{\xi \times \pi} = E_{X \sim \pi} \left[ \text{var}_{A \sim \pi(X, \cdot)} \left( \frac{g(X, A)}{\pi(X, A)} (\hat{\mu}_{n/2}^{(2)}(X, A) - \mu^*(X, A)) \mid X \right) \right] \mid B_2 \\
\leq E_{(X, A) \sim \pi \times \pi} \left[ \frac{g^2(X, A)}{\pi^2(X, A)} (\hat{\mu}_{n/2}^{(2)}(X, A) - \mu^*(X, A))^2 \right] \mid B_2 \right] = \|\hat{\mu}_{n/2}^{(2)} - \mu^*\|^2_{\omega}.
\]

Putting together the pieces yields \( E[T_1^2] \leq \frac{1}{2n} E[\|\hat{\mu}_{n/2}^{(2)} - \mu^*\|^2_{\omega}] \) as claimed. A similar argument yields the same bound for \( E[T_2^2] \).

**Proof of equation (47b):** We first decompose the term \( T_* \) into two parts:

\[
T_{*, j} := \frac{1}{n} \sum_{i=\lfloor (n-j-1)/2 \rfloor + 1}^{n j/2} \left\{ \frac{g(X_i, A_i)}{\pi(X_i, A_i)} Y_i - \tau^* - f^*(X_i, A_i) \right\}, \quad \text{for} \ j \in \{1, 2\}.
\]

Since for any \( x \in \mathbb{X} \), the functions \( f^*(x, \cdot) \) and \( \hat{f}_{n/2}^{(2)}(x, \cdot) \) are both zero-mean under \( \pi(x, \cdot) \), we have the following identity.

\[
E[T_{*, 2} T_1 \mid B_2] = \frac{1}{n} \sum_{i=1}^{n/2} E[T_{*, 2} \cdot E[\hat{f}_{n/2}^{(2)}(X_i, A_i) - f^*(X_i, A_i) \mid X_i] \mid B_2] = 0.
\]

Similarly, we have \( E[T_{*, 1} T_2] = 0. \) It remains to study the terms \( E[T_{*, j} T_j] \) for \( j \in \{1, 2\} \). We start with the following expansion:

\[
T_{*, 1} \cdot T_1 = \frac{1}{n^2} \sum_{i=1}^{n/2} \left\{ \frac{g(X_i, A_i)}{\pi(X_i, A_i)} Y_i - \tau^* - f^*(X_i, A_i) \right\} \cdot \left( \hat{f}_{n/2}^{(2)}(X_i, A_i) - f^*(X_i, A_i) \right)
\]

\[
+ \frac{1}{n^2} \sum_{1 \leq i \neq \ell \leq n/2} \left\{ \frac{g(X_i, A_i)}{\pi(X_i, A_i)} Y_i - \tau^* - f^*(X_i, A_i) \right\} \cdot \left( \hat{f}_{n/2}^{(2)}(X_\ell, A_\ell) - f^*(X_\ell, A_\ell) \right).
\]

For \( i \neq \ell \), by the unbiasedness of \( T_* \), we note that:

\[
E\left[ \left\{ \frac{g(X_i, A_i)}{\pi(X_i, A_i)} Y_i - \tau^* - f^*(X_i, A_i) \right\} \cdot \left( \hat{f}_{n/2}^{(2)}(X_\ell, A_\ell) - f^*(X_\ell, A_\ell) \right) \mid B_2, X_\ell \right] = 0.
\]
So we have that:

\[
\mathbb{E}[T_{s,t}T_1] = \frac{1}{2n} \mathbb{E}\left[ \left\{ \frac{g(X, A)}{\pi(X, A)} \mu^*(X, A) - \tau^* - f^*(X, A) \right\} \cdot (\hat{f}_{n/2}^{(2)}(X, A) - f^*(X, A)) \right]
\]

\[
= \frac{1}{2n} \mathbb{E}\left[ \left\{ \int \mu^*(X, \cdot) - \tau^* \right\} \cdot (\hat{f}_{n/2}^{(2)}(X, A) - f^*(X, A)) \right]
\]

\[
= \frac{1}{2n} \mathbb{E}\left[ \left\{ \mu^*(X, \cdot) - \tau^* \right\} \cdot \mathbb{E}[\hat{f}_{n/2}^{(2)}(X, A) - f^*(X, A) | X, \mathcal{B}_2] \right] = 0.
\]

### 4.2 Proof of Theorem 2

Based on Theorem 1 and the discussion thereafter, it suffices to prove an oracle inequality on the squared error \( \mathbb{E}[\|\hat{\mu}_m - \mu^*\|^2_\omega] \). So as to ease the notation, for any pair of functions \( f, g: \mathcal{X} \times \mathcal{A} \to \mathbb{R} \), we define the empirical inner product

\[
\langle f, g \rangle_m := \frac{1}{m} \sum_{i=1}^m \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} f(X_i, A_i) g(X_i, A_i), \quad \text{and the induced norm } \|f\|_m := \sqrt{\langle f, f \rangle_m}.
\]

With this notation, observe that our weighted least-squares estimator is based on minimizing the objective \( \|Y - \mu\|^2_m = \frac{1}{m} \sum_{i=1}^m \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} (Y_i - \mu(X_i, A_i))^2 \), where we have slightly overloaded our notation on \( Y \) — viewing it as a function such that \( Y(X, A_i) = Y_i \) for each \( i \).

By the convexity of \( \Omega \) and the optimality condition that defines \( \hat{\mu}_m, \) for any function \( \mu \in \mathcal{F} \) and scalar \( \beta \in (0, 1) \), we have \( \|Y - \mu\|^2_m \leq \|Y_i - (t \mu + (1-t)\hat{\mu}_m)\|^2_m \). Taking the limit \( t \to 0^+ \) yields the basic inequality

\[
\|\hat{\Delta}_m\|^2_m \leq \langle \mu^* - Y, \hat{\Delta}_m \rangle_m + \langle \hat{\Delta}_m, \tilde{\Delta} \rangle_m,
\]

where define the estimation error \( \hat{\Delta}_m := \hat{\mu}_m - \mu \), and the approximation error \( \tilde{\Delta} := \mu^* - \mu \). By applying the Cauchy–Schwarz inequality to the last term in equation (48), we find that

\[
\langle \hat{\Delta}_m, \tilde{\Delta} \rangle_m \leq \|\hat{\Delta}_m\|_m \cdot \|\tilde{\Delta}\|_m \leq \frac{1}{2} \|\hat{\Delta}_m\|^2_m + \frac{1}{2} \|\tilde{\Delta}\|^2_m.
\]

Combining with inequality (48) yields the bound

\[
\|\hat{\Delta}_m\|^2_m \leq \frac{2}{m} \sum_{i=1}^m W_i \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} \hat{\Delta}_m(X_i, A_i) + \|\tilde{\Delta}\|^2_m,
\]

(49)

where \( W_i := \mu^*(X_i, A_i) - Y_i \) is the outcome noise associated with observation \( i \).

The remainder of our analysis involves controlling different terms in the bound (49). There are two key ingredients in the argument:

- First, we need to relate the empirical \( L^2 \)-norm \( \| \cdot \|_m \) with its population counterpart \( \| \cdot \|_\omega \). Lemma 1 stated below provides this control.

- Second, using the Rademacher complexity \( \mathcal{S}_m \) from equation (13a), we upper bound the weighted empirical average term associated with the outcome noise \( W_i = \mu^*(X_i, A_i) - Y_i \) on the right-hand-side of equation (49). This bound is given in Lemma 2.

Define the event

\[
\mathcal{E}_\omega := \left\{ \|f\|^2_m \geq \frac{\alpha_2 \alpha_1^2}{16} \|f\|^2_\omega \right\} \text{ for all } f \in \mathcal{F}^* \setminus \mathbb{B}_\omega(r_m).
\]

The following result provides tail control on the complement of this event.
Lemma 1. There exists a universal constant \( c' > 0 \) such that
\[
\mathbb{P}(\mathcal{E}_c^c) \leq \exp\left(-\frac{\alpha^2}{2\hat{r}}m\right).
\] (51)

See Section 4.2.1 for the proof.

For any (non-random) scalar \( r > 0 \), we also define the event
\[
\mathcal{E}(r) := \{\|\hat{\Delta}_m\|_{\omega} \geq r\}.
\]

On the event \( \mathcal{E}_c \cap \mathcal{E}(r_m) \), our original bound (49) implies that
\[
\|\hat{\Delta}_m\|_\omega^2 \leq \frac{32}{\alpha_2\alpha_1^2m} \sum_{i=1}^{m} W_i \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} \hat{\Delta}_m(X_i, A_i) + \frac{16}{\alpha_2^2\alpha_1^2} \|\bar{\Delta}\|_m^2.
\] (52)

In order to bound the right-hand-side of equation (52), we need a second lemma that controls the empirical process in terms of the critical radius \( s_m \) defined by the fixed point relation (14a).

Lemma 2. We have
\[
\mathbb{E}\left[1_{\mathcal{E}(s_m)} \cdot \frac{2}{m} \sum_{i=1}^{m} W_i \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} \hat{\Delta}_m(X_i, A_i)\right] \leq s_m \sqrt{\mathbb{E}[\|\hat{\Delta}_m\|_\omega^2]}.
\] (53)

See Section 4.2.2 for the proof.

With these two auxiliary lemmas in hand, we can now complete the proof of the theorem itself. In order to exploit the basic inequality (52), we begin by decomposing the MSE as
\[
\mathbb{E}[\|\hat{\Delta}_m\|_\omega^2] \leq \sum_{j=1}^{3} T_j,
\]
where
\[
T_1 := \mathbb{E}[\|\hat{\Delta}_m\|_\omega^2 1_{\mathcal{E}_c \cap \mathcal{E}(r_m) \cap \mathcal{E}(s_m)}], \quad T_2 := \mathbb{E}[\|\hat{\Delta}_m\|_\omega^2 1_{\mathcal{E}(r_m) \cap \mathcal{E}(s_m)^c}], \quad \text{and} \quad T_3 := \mathbb{E}[\|\hat{\Delta}_m\|_\omega^2 1_{\mathcal{E}^c}].
\]

We analyze each of these terms in turn.

Analysis of \( T_1 \): Combining the bound (52) with Lemma 2 yields
\[
T_1 \leq \frac{32}{\alpha_2\alpha_1^2m} \mathbb{E}\left[1_{\mathcal{E}(r_m)} \cdot \sum_{i=1}^{m} W_i \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} \hat{\Delta}_m(X_i, A_i)\right] + \frac{16}{\alpha_2^2\alpha_1} \mathbb{E}[\|\bar{\Delta}\|_m^2]
\]
\[
\leq \frac{32}{\alpha_2\alpha_1^2} s_m \sqrt{\mathbb{E}[\|\hat{\Delta}_m\|_\omega^2]} + \frac{16}{\alpha_2^2\alpha_1} \mathbb{E}[\|\bar{\Delta}\|_m^2]
\]
\[
= \frac{32}{\alpha_2\alpha_1^2} s_m \sqrt{\mathbb{E}[\|\hat{\Delta}_m\|_\omega^2]} + \frac{16}{\alpha_2^2\alpha_1} \|\bar{\Delta}\|_\omega^2.
\] (54a)

where the final equality follows since \( \mathbb{E}[\|\bar{\Delta}\|_m^2] = \|\bar{\Delta}\|_\omega^2 \), using the definition of the empirical \( \ell^2 \)-norm, and the fact that the approximation error \( \bar{\Delta} \) is a deterministic function.

Bounding \( T_2 \): On the event \( [\mathcal{E}(r_m) \cap \mathcal{E}(s_m)]^c = \mathcal{E}^c(r_m) \cup \mathcal{E}^c(s_m) \), we are guaranteed to have
\[
\|\hat{\Delta}_m\|_\omega^2 \leq s_m^2 + r_m^2,
\]
and hence
\[
T_2 \leq s_m^2 + r_m^2.
\] (54b)
Analysis of $T_3$: Since the function class $\mathcal{F}$ is bounded, we have

$$T_3 \leq \text{diam}^2(\mathcal{F} \cup \{\mu^*\}) \cdot \mathbb{P}(\epsilon^c) \leq \text{diam}^2(\mathcal{F} \cup \{\mu^*\}) \cdot e^{-c \alpha^2_m}$$

for a universal constant $c > 0$.

Finally, substituting the bounds (54a), (54b) and (54c) into our previous inequality

$$\mathbb{E}[\|\tilde{\Delta}_m\|^2_\omega] \leq \sum_{j=1}^3 T_j$$

yields

$$\mathbb{E}[\|\tilde{\Delta}_m\|^2_\omega] \leq \frac{32}{\alpha_2 \alpha_1^2} s m \sqrt{\mathbb{E}[\|\tilde{\Delta}_m\|^2_\omega]} + \frac{16}{\alpha_2 \alpha_1^2} \|\tilde{\Delta}^2_\omega\|_\omega + (s^2_m + r^2_m) + \text{diam}^2(\mathcal{F} \cup \{\mu^*\}) \cdot e^{-c \alpha^2_m}.$$ 

Note that this is a self-bounding relation for the quantity $\mathbb{E}[\|\tilde{\Delta}_m\|^2_\omega]$. With the choice $m = n/2$, it implies the MSE bound

$$\mathbb{E}[\|\tilde{\Delta}_m\|^2_\omega] \leq 2 \mathbb{E}[\|\tilde{\Delta}_m\|^2_\omega] \leq 2 \mathbb{E}[\|\tilde{\Delta}_m\|^2_\omega]$$

$$\leq (2 + \frac{c}{\alpha_1 \alpha_2}) \|\tilde{\Delta}^2_\omega\|_\omega + \frac{c'}{\alpha_1 \alpha_2} s^2_n + \frac{c'}{\alpha_1 \alpha_2} r^2_n + \text{diam}^2(\mathcal{F} \cup \{\mu^*\}) \cdot e^{-c \alpha^2_n/2},$$

for a pair $(c, c')$ of positive universal constants. Combining with Theorem 1 and taking the infimum over $\mu \in \mathcal{F}$ completes the proof.

4.2.1 Proof of Lemma 1

The following lemma provides a lower bound on the empirical norm, valid uniformly over a given function class $\mathcal{H} \subseteq \{h/\|h\|_\omega \mid h \in \mathcal{F}^* \setminus \{0\}\}$.

**Lemma 3.** For a failure probability $\varepsilon \in (0, 1)$, we have

$$\inf_{h \in \mathcal{H}} \|h\|^2_m \geq \frac{\alpha_2 \alpha_1^2}{4} - 4 \alpha_1 \mathcal{R}_m(\mathcal{H}) - c \alpha^2 \left\{ \sqrt{\frac{\log(1/\varepsilon)}{m}} + \frac{\log(1/\varepsilon)}{m} \right\}$$

with probability at least $1 - \varepsilon$.

See Appendix A.3 for the proof of this lemma.

Taking it as given for now, we proceed with the proof of Lemma 1. For any deterministic radius $r > 0$, we define the set

$$\mathcal{H}_r := \{h/\|h\|_\omega \mid h \in \mathcal{F}^*, \text{ and } \|h\|_\omega \geq r\}.$$ 

By construction, the sequence $\{\mathcal{H}_r\}_{r > 0}$ consists of nested sets—that is, $\mathcal{H}_r \subseteq \mathcal{H}_s$ for $r > s$—and all are contained within the set $\{h/\|h\|_\omega \mid h \in \mathcal{F}^* \setminus \{0\}\}$. By convexity of the class $\mathcal{F}$, for any $h \in \mathcal{F}$ such that $\|h\|_\omega \geq r$, we have $r \cdot h/\|h\|_\omega \in \mathcal{F} \cap \mathbb{B}_\omega(r)$. Consequently, we can bound the Rademacher complexity as

$$\mathcal{R}_m(\mathcal{H}_r) = \mathbb{E} \left[ \sup_{h \in \mathcal{H}_r} \sum_{i=1}^m \epsilon_i g(X_i, A_i) h(X_i, A_i) \pi(X_i, A_i) \right] \leq \frac{1}{r} \mathbb{E} \left[ \sup_{h \in \mathcal{F}^* \cap \mathbb{B}_\omega(r)} \sum_{i=1}^m \epsilon_i g(X_i, A_i) h(X_i, A_i) \pi(X_i, A_i) \right]
= \frac{1}{r} \mathcal{R}_m(\mathcal{F}^* \cap \mathbb{B}_\omega(r)).$$
By combining this inequality with Lemma 3, we find that
\[
\inf_{f \in F^s \cap \mathbb{B}_\omega(r)} \left\{ \frac{\|f\|_m^2}{\|f\|_2^2} - \frac{4\alpha_1}{r} \mathcal{R}_m(F^s \cap \mathbb{B}_\omega(r)) - \alpha_1 \cdot \left\{ \sqrt{\frac{\log(1/\varepsilon)}{m}} + \frac{\log(1/\varepsilon)}{m} \right\} \right\} \geq \frac{16 \alpha_2^2}{4} - \frac{4\alpha_1}{r} \mathcal{R}_m(F^s \cap \mathbb{B}_\omega(r)) - \alpha_1 \cdot \left\{ \sqrt{\frac{\log(1/\varepsilon)}{m}} + \frac{\log(1/\varepsilon)}{m} \right\}
\] (56)
with probability at least \(1 - \varepsilon\). This inequality is valid for any deterministic radius \(r > 0\).

By the definition (14b) of the critical radius \(r_m\), inequality (14b) holds for any \(r > r_m\). We now set \(r = r_m\) in equation (56). Doing so allows us to conclude that given a sample size satisfying \(m \geq \frac{1024\alpha_2^2}{\alpha_2^2} \log(1/\varepsilon)\), we have
\[
\frac{4\alpha_1}{r_m} \mathcal{R}_m(F^s \cap \mathbb{B}_\omega(r_m)) \leq \frac{\alpha_2^2}{16}, \quad \text{and} \quad \alpha_1 \cdot \left\{ \sqrt{\frac{\log(1/\varepsilon)}{m}} + \frac{\log(1/\varepsilon)}{m} \right\} \leq \frac{\alpha_2^2}{16}.
\]
Combining with equation (56) completes the proof of Lemma 1.

### 4.2.2 Proof of Lemma 2

Recall our notation \(W_i := \mu^*(X_i, A_i) - Y_i\) for the outcome noise. Since the set \(\Omega\) is convex, on the event \(E(s_m)\), we have
\[
\frac{1}{\|\Delta_m\|_2} \sum_{i=1}^m W_i g^2(X_i, A_i) \Delta_m(X_i, A_i) \leq \frac{1}{s_m} \sup_{h \in F^s \cap \mathbb{B}_\omega(s_m)} \sum_{i=1}^m W_i g^2(X_i, A_i) h(X_i, A_i).
\] (57)
Define the empirical process supremum
\[
Z_m(s_m) := \sup_{h \in F^s \cap \mathbb{B}_\omega(s_m)} \frac{1}{m} \sum_{i=1}^m W_i g^2(X_i, A_i) h(X_i, A_i).
\]
Since the all-zeros function \(0\) is an element of \(F^s \cap \mathbb{B}_\omega(s_m)\), we have \(Z_m(s_m) \geq 0\). Equation (57) implies that
\[
\mathbb{E}\left[1_{E(s_m)} \cdot \frac{2}{m} \sum_{i=1}^m g^2(X_i, A_i) \mu^*(X_i, A_i) - Y_i \right] \Delta_m(X_i, A_i) \leq \mathbb{E}\left[ \frac{\|\Delta_m\|_2}{s_m} Z_m(s_m) \right]\]
\[
\leq \sqrt{\mathbb{E}\left[ \|\Delta_m\|_2^2 \right]} \cdot \sqrt{s_m^2 \mathbb{E}\left[ Z_m^2(s_m) \right]},
\] (58)
where the last step follows by applying the Cauchy–Schwarz inequality.

Define the symmetrized random variable
\[
Z'_m(s_m) := \sup_{h \in F^s \cap \mathbb{B}_\omega(s_m)} \frac{1}{m} \sum_{i=1}^m \varepsilon_i g^2(X_i, A_i) \mu^*(X_i, A_i) - Y_i h(X_i, A_i),
\]
where \(\{\varepsilon_i\}_{i=1}^m\) is an i.i.d. sequence of Rademacher variables, independent of the data. By a standard symmetrization argument (e.g., §2.4.1 in the book [Wai19]), there are universal constants \((c, c')\) such that
\[
\mathbb{P}[Z_m(s_m) > t] \leq c' \mathbb{P}[Z'_m(s_m) > ct], \quad \text{for any} \ t > 0.
\]
Integrating over \(t\) yields the bound
\[
\mathbb{E}[Z_m^2(s_m)] \leq c'^2 c \mathbb{E}[Z'_m^2(s_m)] = c'^2 c' \mathbb{E}[Z^2_m(s_m)] \leq c'^2 c'^2 s_m^2,
\]
where equality (i) follows from the definition of \(s_m\). Substituting this bound back into equation (58) completes the proof of Lemma 2.
4.3 Proof of Corollary 5

Define the function $\tilde{f}(x,a) := \frac{g(x,a)}{\pi(x,a)}\bar{f}(x,a) - \langle g(x,\cdot), \bar{f}(x,\cdot) \rangle \lambda$, which would be optimal if $\bar{f}$ were the true treatment function. It induces the estimate

$$\tilde{\tau}_{n,f} = \frac{1}{n} \sum_{i=1}^{n} \left\{ \frac{g(X_i, A_i)}{\pi(X_i, A_i)} (Y_i - \bar{f}(X_i, A_i)) + \langle g(X_i, \cdot), \bar{f}(X_i, \cdot) \rangle \lambda \right\},$$

which has (n-rescaled) variance $n \cdot \mathbb{E} \left[ \left| \tilde{\tau}_{n,f} - \tau^* \right|^2 \right] = v^2 + v^2(\bar{f})$, where $v^2$ is the efficient variance, and $v^2(\bar{f}) := \mathbb{E} \left( \frac{g(X,A)}{\pi(X,A)} \right) \left( \mu^* - \bar{f} \right)(X,A)$. Let us now compare two-stage estimator $\tilde{\tau}_n$ with this idealized estimator. We have

$$\mathbb{E}[|\tilde{\tau}_{n,f} - \tilde{\tau}_n|^2] \leq \frac{2}{n^2} \mathbb{E}\left[ \sum_{i=1}^{n/2} (\tilde{f} - \tilde{f}_{n/2}^{(2)})(X_i, A_i)^2 \right] + \frac{2}{n} \mathbb{E}\left[ \sum_{i=n/2+1}^{n} (\tilde{f} - \tilde{f}_{n/2}^{(1)})(X_i, A_i)^2 \right]$$

$$\leq \frac{4}{n} \mathbb{E}[\|\tilde{\mu}_{n/2} - \bar{f}\|^2].$$

Thus, we are guaranteed the Wasserstein bound

$$\mathcal{W}_1(\sqrt{n}\tilde{\tau}_n, \sqrt{n}\tilde{\tau}_{n,f}) \leq \mathcal{W}_2(\sqrt{n}\tilde{\tau}_n, \sqrt{n}\tilde{\tau}_{n,f}) \leq 2\sqrt{\mathbb{E}[(\|\tilde{\mu}_{n/2} - \bar{f}\|^2)]}.$$

Consequently, by the triangle inequality for the Wasserstein distance, it suffices to establish a normal approximation guarantee for the idealized estimator $\tilde{\tau}_{n,f}$, along with control on the error induced by approximating the function $\bar{f}$ using an empirical estimator.

**Normal approximation for $\tilde{\tau}_{n,f}$**: We make use of the following non-asymptotic central limit theorem:

**Proposition 2** ([Ros11], Theorem 3.2 (restated)). Given i.i.d. zero-mean random variables $\{X_i\}_{i=1}^{n}$ with finite fourth moment, the rescaled sum $W_n := \sum_{i=1}^{n} X_i / \sqrt{n}$ satisfies the Wasserstein bound

$$\mathcal{W}_1(W_n, Z) \leq \frac{1}{\sqrt{n}} \left( \frac{\mathbb{E}|X_1|^3}{\mathbb{E}[X_1^2]} + \sqrt{\frac{2\mathbb{E}[X_1^4]}{\pi^2 \mathbb{E}[X_1^2]}} \right) \quad \text{where } Z \sim \mathcal{N}(0, \mathbb{E}[X_1^2]).$$

Since we have $\mathbb{E}[|X_1|^3] \leq \sqrt{\mathbb{E}[X_1^2] \cdot \mathbb{E}[X_1^2]}$, this bound implies that $\mathcal{W}_1(W_n, Z) \leq \frac{2}{\sqrt{n}} \cdot \sqrt{\mathbb{E}[X_1^4] / \mathbb{E}[X_1^2]}$. Applying this bound to the empirical average $\tilde{\tau}_{n,f}$ yields

$$\mathcal{W}_1(\sqrt{n}\tilde{\tau}_{n,f}, Z) \leq \frac{2}{\sqrt{n}} \cdot \sqrt{\frac{M_4}{v^2 + v^2(\bar{f})}},$$

as claimed.

**Bounds on the estimation error $\|\tilde{\mu}_{n/2} - \bar{f}\|_2$**: From the proof of Theorem 2, recall the basic inequality (48)—viz.

$$\|\tilde{\Delta}\|_m^2 \leq \frac{1}{m} \sum_{i=1}^{m} W_i \frac{g^2(X_i, A_i)}{\pi^2(x, A_i)} \tilde{\Delta}_m(X_i, A_i) + \langle \tilde{\Delta}_m, \vec{\Delta} \rangle_m,$$

where $W_i = \mu^*(X_i, A_i) - Y_i$ is the outcome noise.
As before, we define the approximation error $\tilde{\Delta} := \mu^* - \tilde{\mu}$. Since $\tilde{\mu} = \arg\min_{h \in \mathcal{F}} \| h - \mu^* \|_\omega$ is the projection of $\mu^*$ onto $\mathcal{F}$, and $\tilde{\mu}_m \in \mathcal{F}$ is feasible for this optimization problem, the first-order optimality condition implies that $\langle \tilde{\Delta}_m, \tilde{\Delta} \rangle_\omega \leq 0$. By adding this inequality to our earlier bound (59) and re-arranging terms, we find that

$$\| \Delta_m \|^2 \leq \frac{1}{m} \sum_{i=1}^m g^2(X_i, A_i) (\mu^*(X_i, A_i) - Y_i) \hat{\Delta}_m(X_i, A_i) + (\langle \Delta_m, \Delta \rangle_m - \langle \hat{\Delta}_m, \Delta \rangle_\omega).$$ (60)

Now define the empirical process suprema

$$Z_m(r) := \sup_{h \in \mathcal{F} \cap \mathbb{B}_\omega(r)} \frac{1}{m} \sum_{i=1}^m g^2(X_i, A_i) (\mu^*(X_i, A_i) - Y_i) h(X_i, A_i),$$

and

$$Z_m'(s) := \sup_{h \in \mathcal{F} \cap \mathbb{B}_\omega(s)} \frac{1}{m} \sum_{i=1}^m g^2(X_i, A_i) h(X_i, A_i) \Delta(X_i, A_i) - \langle \Delta, h \rangle_\omega).$$

From the proof of Theorem 2, recall the events

$$\mathcal{E}_\omega := \left\{ \| f \|^2_m \geq \alpha_2 \alpha_1^2 \| f \|^2_m, \text{ for any } f \in \mathcal{F} \setminus \mathbb{B}_\omega(r_m) \right\}, \quad \text{and} \quad \mathcal{E}(r) := \left\{ \| \Delta_m \|_\omega \geq r \right\}.$$

Introduce the shorthand $u_m = \max\{r_m, s_m, d_m\}$. On the event $\mathcal{E}_\omega \cap \mathcal{E}(u_m)$, the basic inequality (60) implies that

$$\frac{1}{m} \| \tilde{\Delta}_m \|^2 \leq \frac{1}{m} \| \hat{\Delta}_m \|^2 \leq Z_m(\| \hat{\Delta}_m \|_\omega) + Z_m'(\| \hat{\Delta}_m \|_\omega) \leq Z_m'(s_m) Z_m(s_m),$$

where step (ii) follows from the non-increasing property of the functions $r \mapsto r^{-1} Z_m(r)$ and $s \mapsto s^{-1} Z_m'(s)$.

So there exists a universal constant $c > 0$ such that

$$\mathbb{E} \left[ \| \tilde{\Delta}_m \|^2 \mathbb{1}_{\mathcal{E}_\omega \cap \mathcal{E}(u_m)} \right] \leq \frac{c}{\alpha_2 \alpha_1^2} \left\{ \frac{1}{s_m^2} \mathbb{E} \left[ Z_m'(s_m) \right] + \frac{1}{d_m^2} \mathbb{E} \left[ \{ Z_m'(d_m) \}^2 \right] \right\}.$$

Via the same symmetrization argument as used in the proof of Theorem 2, there exists a universal constant $c > 0$ such that

$$\mathbb{E} \left[ Z_m'(s_m) \right] \leq c S_m^2(\mathcal{F} \cap \mathbb{B}_\omega(s_m)), \quad \text{and} \quad \mathbb{E} \left[ \{ Z_m'(d_m) \}^2 \right] \leq c D_m^2(\mathcal{F} \cap \mathbb{B}_\omega(d_m)).$$

By the definition of the critical radius $s_m$, we have

$$\frac{1}{s_m} S_m(\mathcal{F} \cap \mathbb{B}_\omega(s_m)) = s_m, \quad \text{and} \quad \frac{1}{d_m} D_m(\mathcal{F} \cap \mathbb{B}_\omega(d_m)) = d_m.$$

Combining with the moment bound above, we arrive at the conclusion:

$$\mathbb{E}[\| \tilde{\Delta}_m \|_\omega^2] \leq \mathbb{E}[\| \tilde{\Delta}_m \|_\omega^2 \mathbb{1}_{\mathcal{E}_\omega \cap \mathcal{E}(u_m)}] + \mathbb{E}[\| \tilde{\Delta}_m \|_\omega^2 \mathbb{1}_{\mathcal{E}(u_m)^c}] + \mathbb{E}[\| \tilde{\Delta}_m \|_\omega^2 \mathbb{1}_{\mathcal{E}(u_m)^c}]$$

$$\leq \left(1 + \frac{c}{\alpha_2 \alpha_1^2} \right) \cdot \left( r_m^2 + s_m^2 + d_m^2 \right) + \text{diam}_\omega^2(\mathcal{F} \cup \{ \mu^* \}) \cdot \mathbb{P}(\mathcal{E}_\omega^c)$$

$$\leq \left(1 + \frac{c}{\alpha_2 \alpha_1^2} \right) \cdot \left( r_m^2 + s_m^2 + d_m^2 \right) + \text{diam}_\omega^2(\mathcal{F}) \cdot e^{-c\alpha_2 m}.$$
5 Proofs of minimax lower bounds

In this section, we prove the two minimax lower bounds—namely, Theorem 3 and Proposition 1.

5.1 Proof of Theorem 3

It suffices to show that the minimax risk
\[ M_n \equiv M_n(C_\delta(I^*), \nu^*(X, \cdot)) \]
for the following three lower bounds:

\[ M_n \geq c n \var_{\xi^*} \left( \langle g(X, \cdot), \mu^*(X, \cdot) \rangle \lambda \right) \text{ for } n \geq 4(M'_2 \to 4)^2, \quad (61a) \]

\[ M_n \geq c n \| \sigma \|_\omega^2 \text{ for } n \geq 16, \quad (61b) \]

\[ M_n \geq c n \| \delta \|_\omega^2 \text{ for } n \in \left[ M'_2 \to 4, c' |X|/c_{\max} \right]. \quad (61c) \]

Given these three inequalities, the minimax risk \( M_n \) can be lower bounded by the average of the right-hand side quantities, assuming that \( n \) is sufficiently large. Since \( c \) is a universal constant, these bounds lead to the conclusion of Theorem 3.

Throughout the proof, we use \( P_{\mu^*, \xi} \) to denote the law of a sample \((X, A, Y)\) under the problem instance defined by outcome function \( \mu^* \) and data distribution \( \xi \). We further use \( P_{\otimes n \mu^*, \xi} \) to denote its \( n \)-fold product, as is appropriate given our i.i.d. data \((X_i, A_i, Y_i)_{i=1}^n\).

5.1.1 Proof of the lower bound (61a)

The proof is based on Le Cam’s two-point method: we construct a family of probability distributions \( \{ \xi_s \mid s > 0 \} \), each contained in the local neighborhood \( N_{\nu^*}(\xi^*) \). We choose the parameter \( s \) small enough to ensure that the probability distributions \( P_{\otimes n \mu^*, \xi_s} \) and \( P_{\otimes n \mu^*, \xi^*} \) are “indistinguishable”, but large enough to ensure that the functional values \( \tau(\xi_s, \mu^*) \) and \( \tau(\xi^*, \mu^*) \) are well-separated. See §15.2.1–15.2.2 in the book [Wai19] for more background.

More precisely, Le Cam’s two-point lemma guarantees that for any distribution \( \xi_s \in N_{\nu^*}(\xi^*) \), the minimax risk is lower bounded as
\[ M_n \geq \frac{1}{4} \left( 1 - d_{TV}(P_{\otimes n \mu^*, \xi_s}, P_{\otimes n \mu^*, \xi^*}) \right) \cdot \left( \tau(\xi_s, \mu^*) - \tau(\xi^*, \mu^*) \right)^2, \quad (62) \]

Recall that throughout this section, we work with the sample size lower bound
\[ n \geq 4(M'_2 \to 4)^2. \quad (63) \]

Now suppose that under the condition (63), we can exhibit a choice of \( s \) within the family \( \{ \xi_s \mid s > 0 \} \) such that the functional gap satisfies the lower bound
\[ \tau(\xi_s, \mu^*) - \tau(\xi^*, \mu^*) \geq \frac{1}{16 \sqrt{n}} \left( \frac{1}{\nu^*}(g(X, \cdot), \mu^*(X, \cdot)) \lambda \right), \quad (64a) \]

whereas the TV distance satisfies the upper bound
\[ d_{TV}(P_{\otimes n \mu^*, \xi_s}, P_{\otimes n \mu^*, \xi^*}) \leq \frac{1}{3}. \quad (64b) \]

These two inequalities, in conjunction with Le Cam’s two-point bound (62), imply the claimed lower bound (61a).

With this overview in place, it remains to define the family \( \{ \xi_s \mid s > 0 \} \), and prove the bounds (64a) and (64b).
Family of perturbations: Define the real-valued function

\[ h(x) := \langle \mu^*(x, \cdot), g(x, \cdot) \rangle_\lambda - \mathbb{E}_{\xi^*} \left[ \langle \mu^*(X, \cdot), g(X, \cdot) \rangle_\lambda \right], \]

along with its truncated version

\[ h_{tr}(x) := \begin{cases} h(x) & \text{if } |h(x)| \leq 2M_{2 \to 4} \cdot \sqrt{\mathbb{E}_{\xi^*}[h^2(X)]}, \\ \text{sgn}(h(x)) \cdot \sqrt{\mathbb{E}_{\xi^*}[h^2(X)]} & \text{otherwise.} \end{cases} \]

For each \( s > 0 \), we define the tilted probability measure

\[ \xi_s(x) := Z_s^{-1} \xi^*(x) \exp \left( sh_{tr}(x) \right), \quad \text{where } Z_s = \sum_{x \in \mathbb{X}} \xi^*(x) \exp \left( sh_{tr}(x) \right). \]

It can be seen that the tilted measure satisfies the bounds

\[ \exp \left( -s \| h_{tr} \|_\infty \right) \leq \frac{\xi_s(x)}{\xi^*(x)} \leq \exp \left( s \| h_{tr} \|_\infty \right) \quad \text{for any } x \in \mathbb{X}, \]

whereas the normalization constant is sandwiched as

\[ \exp \left( -s \| h_{tr} \|_\infty \right) \leq Z_s \leq \exp \left( s \| h_{tr} \|_\infty \right). \]

Throughout this section, we choose

\[ s := (4\| h_{tr} \|_{L^2(\xi^*)})^{-1}, \quad (65a) \]

which ensures that

\[ s \| h_{tr} \|_\infty = \frac{1}{4\sqrt{n}} \cdot \| h_{tr} \|_{L^2(\xi^*)} \overset{(i)}{\leq} \frac{1}{\sqrt{8n}} \cdot \| h \|_{L^2(\xi^*)} \overset{(ii)}{\leq} \frac{1}{8}, \quad (65b) \]

where step (i) follows from the definition of the truncated function \( h_{tr} \), and step (ii) follows from the sample size condition (63).

Proof of the lower bound (64a): First we lower bound the gap in the functional. We have

\[ \tau(\xi_s, \mu^*) - \tau(\xi^*, \mu^*) = \mathbb{E}_{\xi^*} \left[ \langle \mu^*(X, \cdot), g(X, \cdot) \rangle_\lambda \right] - \mathbb{E}_{\xi^*} \left[ \langle \mu^*(X, \cdot), g(X, \cdot) \rangle_\lambda \right] \]

\[ = \mathbb{E}_{\xi^*} \left[ h(X)e^{sh_{tr}(X)} \right] / \mathbb{E}_{\xi^*} \left[ e^{sh_{tr}(X)} \right]. \quad (66) \]

Note that \( |sh_{tr}(X)| \leq 1/8 \) almost surely by construction. Using the elementary inequality \( |e^z - 1 - z| \leq z^2 \), valid for all \( z \in [-1/4, 1/4] \), we obtain the lower bound

\[ \mathbb{E}_{\xi^*} \left[ h(X)e^{sh_{tr}(X)} \right] \geq \mathbb{E}_{\xi^*} \left[ h(X) \right] + s\mathbb{E}_{\xi^*} \left[ h(X)h_{tr}(X) \right] - s^2 \mathbb{E}_{\xi^*} \left[ |h(X)| \cdot |h_{tr}(X)|^2 \right] \quad (67) \]

Now we study the three terms on the right-hand-side of equation (67). By definition, we have \( \mathbb{E}_{\xi^*}[h(X)] = 0 \). Since the quantities \( h(X) \) and \( h_{tr}(X) \) have the same sign almost surely, the second term admits a lower bound

\[ \mathbb{E}_{\xi^*} \left[ h(X)h_{tr}(X) \right] \geq \mathbb{E}[h_{tr}^2(X)] \geq \frac{1}{2} \mathbb{E}[h^2(X)] \]

where the last step follows from Lemma 7.
Focusing on the third term in the decomposition (67), we note that Cauchy-Schwarz inequality yields
\[ \mathbb{E}_{\xi^*}[|h(X)\cdot h_{tr}(X)|^2] \leq \sqrt{\mathbb{E}[h^2(X)]} \cdot \sqrt{\mathbb{E}[h^{4}(X)]} \leq \sqrt{M'_{2 \rightarrow 4} \cdot \left\{ \mathbb{E}[h^2(X)] \right\}^{3/2}}, \]
where the last step follows from the definition of the constant $M'_{2 \rightarrow 4}$.

Combining these bounds with equation (67) and substituting the choice (65a) of the parameter $s$, we obtain the following lower bound on the functional gap
\[ \mathbb{E}_{\xi^*}[h(X)e^{sh_{tr}(X)}] \geq s\|h\|^2_{L^2(\xi^*)} - s^2 \sqrt{M'_{2 \rightarrow 4}} \|h\|^3_{L^2(\xi^*)} \]
\[ \geq \frac{1}{8\sqrt{n}} \|h\|^2_{L^2(\xi^*)} - \frac{\sqrt{M'_{2 \rightarrow 4}}}{16n} \|h\|^3_{L^2(\xi^*)} \]
\[ \geq \frac{3}{32\sqrt{n}} \|h\|^2_{L^2(\xi^*)}, \]
where the last step follows because $n \geq 4(M'_{2 \rightarrow 4})^2$.

On the other hand, since $|sh_{tr}(X)| \leq 1/8$ almost surely, we have $\mathbb{E}_{\xi^*}[e^{sh_{tr}(X)}] \leq 3/2$. Combining with the bound above and substituting into the expression (66), we find that we find that
\[ \tau(\xi_s, \mu^*) - \tau(\xi^*, \mu^*) \geq \frac{3}{32\sqrt{n}} \|h\|^2_{L^2(\xi^*)} / \mathbb{E}_{\xi^*}[e^{sh_{tr}(X)}] \geq \frac{1}{16\sqrt{n}} \|h\|^2_{L^2(\xi^*)}, \]
which is equivalent to the claim (64a).

**Proof of the upper bound (64b):** Pinsker’s inequality ensures that
\[ d_{TV}(\mathbb{P}^{\otimes n}_{\mu^*,\xi_s}, \mathbb{P}^{\otimes n}_{\mu^*,\xi^*}) \leq \sqrt{\frac{1}{2} \chi^2(\mathbb{P}^{\otimes n}_{\mu^*,\xi_s} \| \mathbb{P}^{\otimes n}_{\mu^*,\xi^*})}, \tag{68} \]
so that it suffices to bound the $\chi^2$-divergence. Beginning with the divergence between $\xi_s$ and $\xi^*$ (i.e., without the tensorization over $n$), we have
\[ \chi^2(\xi_s \| \xi^*) = \var_{\xi^*} \left( \xi_s(X)/\xi^*(X) \right) = \frac{1}{Z_s^2} \var_{\xi^*} \left( e^{sh_{tr}(X)} - 1 \right) \]
\[ \leq \exp \left( 2s\|h_{tr}\|_{\infty} \right) \cdot \mathbb{E}_{\xi^*} \left[ |e^{sh_{tr}(X)} - 1|^2 \right] \]
\[ \leq \exp \left( 4s\|h_{tr}\|_{\infty} \right) \cdot s^2 \mathbb{E}_{\xi^*} \left[ h_{tr}^2(X) \right]. \tag{69} \]
where the last step follows from the elementary inequality $|e^x - 1| \leq e^{|x|} \cdot |x|$, valid for any $x \in \mathbb{R}$. Given the choice of tweaking parameter $s$, we have $\exp \left( 4s\|h_{tr}\|_{\infty} \right) \leq 2$.

The definition of the truncated function $h_{tr}$ implies that $\mathbb{E}_{\xi^*} \left[ h_{tr}^2(X) \right] \leq \mathbb{E}_{\xi^*} \left[ h^2(X) \right]$. Combining this bound with our earlier inequality (69) yields
\[ \chi^2(\xi_s \| \xi^*) \leq 2s^2 \mathbb{E}_{\xi^*} \left[ h_{tr}^2(X) \right] \leq \frac{1}{8n}, \]
which certifies that $\xi_s \in \mathcal{N}^{\mu^*}(\xi^*)$, as required for the validity of our construction.

Finally, by the tensorization property of the $\chi^2$-divergence, we have
\[ \chi^2(\mathbb{P}^{\otimes n}_{\mu^*,\xi_s} \| \mathbb{P}^{\otimes n}_{\mu^*,\xi^*}) \leq \left( 1 + \frac{1}{8n} \right)^n - 1 \leq \frac{3}{20}. \]
Combining with our earlier statement (68) of Pinsker’s inequality completes the proof of the upper bound (64b).
5.1.2 Proof of equation (61b)

The proof is also based on Le Cam’s two-point method. Complementary to equation (61a), we take the source distribution \( \xi^* \) to be fixed, and perturb the outcome function \( \mu^* \). Given a pair \( \mu(s), \mu(-s) \) of outcome functions in the local neighborhood \( \mathcal{N}^\text{val}_\delta \), Le Cam’s two-point lemma implies

\[
M_n \geq \frac{1}{4} \left\{ 1 - d_{TV} \left( \mathbb{P}^{\otimes n}_{\mu(s), \xi^*}, \mathbb{P}^{\otimes n}_{\mu(-s), \xi^*} \right) \right\} \cdot \left\{ \tau(\xi^*, \mu(s)) - \tau(\xi^*, \mu(-s)) \right\}^2, \tag{70}
\]

With this set-up, our proof is based on constructing a pair \( (\mu(s), \mu(-s)) \) of outcome functions within the neighborhood \( \mathcal{N}^\text{val}_\delta(\mu^*) \) such that

\[
\tau(\xi^*, \mu(s)) - \tau(\xi^*, \mu(-s)) \geq \frac{1}{2\sqrt{n}} \| \sigma \|_\omega, \quad \text{and} \tag{71a}
\]

\[
d_{TV} \left( \mathbb{P}^{\otimes n}_{\mu(s), \xi^*}, \mathbb{P}^{\otimes n}_{\mu(-s), \xi^*} \right) \leq \frac{1}{3}. \tag{71b}
\]

**Construction of problem instances:** Consider the noisy Gaussian observation model

\[ Y_i \mid X_i, A_i \sim \mathcal{N} \left( \mu^*(X_i, A_i), \sigma^2(X_i, A_i) \right) \quad \text{for} \ i = 1, 2, \ldots, n. \tag{72} \]

We construct a pair of problem instances as follows: for any \( s > 0 \), define the functions

\[
\mu^*_s(x, a) = \mu^*(x, a) + s \frac{g(x, a)}{\pi(x, a)} \sigma^2(x, a), \quad \text{and} \quad \mu^*_{-s}(x, a) = \mu^*(x, a) - s \frac{g(x, a)}{\pi(x, a)} \sigma^2(x, a)
\]

for any \((x, a) \in \mathbb{X} \times \mathbb{A}\).

Throughout this section, we make the choice \( s := \frac{1}{4\| \sigma \|_\omega \sqrt{n}} \). Under such choice, the compatibility condition (34) ensures that

\[
|\mu_{zs}(x, a) - \mu^*(x, a)| = s \frac{g(x, a)}{\pi(x, a)} \sigma^2(x, a) \leq \delta(x, a) \quad \text{for any} \ (x, a) \in \mathbb{X} \times \mathbb{A} \ \text{and} \ z \in \{-1, 1\}.
\]

This ensures that both \( \mu(s) \) and \( \mu(-s) \) belong to the neighborhood \( \mathcal{N}^\text{val}_\delta(\mu^*) \). It remains to prove the two bounds required for Le Cam’s two-point arguments.

**Proof of equation (71a):** For the target linear functional under our construction, we note that

\[
\tau(\xi^*, \mu(s)) - \tau(\xi^*, \mu(-s)) = 2s \mathbb{E}_{\xi^*} \left[ \left( \frac{g(X, \cdot)}{\pi(X, \cdot)} \sigma^2(X, \cdot), g(X, \cdot) \right) \right] = 2s \| \sigma \|_\omega^2 = \frac{1}{2\sqrt{n}} \| \sigma \|_\omega,
\]

which establishes the bound (71a).

**Proof of the bound (71b):** It order to bound the total variation distance, we study the KL divergence between the product distributions \( \mathbb{P}^{\otimes n}_{\mu(s), \xi^*} \) for \( z \in \{-1, 1\} \). Indeed, we have

\[
D_{KL} \left( \mathbb{P}^{\otimes n}_{\mu(s), \xi^*} \mid \mathbb{P}^{\otimes n}_{\mu(-s), \xi^*} \right) \overset{(i)}{=} n D_{KL} \left( \mathbb{P}_{\mu(s), \xi^*} \mid \mathbb{P}_{\mu(-s), \xi^*} \right) \overset{(ii)}{\leq} n \mathbb{E} \left[ D_{KL} \left( \mathcal{L}(Y \mid X, A)_{\mu(s)} \mid \mathcal{L}(Y \mid X, A)_{\mu(-s)} \right) \right], \tag{73}
\]

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where in step (i), we use the tensorization property of KL divergence, and in step (ii), we use convexity of KL divergence. The expectation is taken with respect to $X \sim \xi^*$ and $A \sim \pi(X, \cdot)$.

Noting that the conditional law $\mathcal{L}(Y \mid X, A)\mid_{\mu_{(z,s)}}$ is Gaussian under both problem instances, we have

$$D_{\text{KL}} \left( \mathcal{L}(Y \mid x, a)\mid_{\mu_{(z,s)}} \mid \mathcal{L}(Y \mid x, a)\mid_{\mu_{(-s)}} \right) = \frac{4s^2g^2(x, a)}{\pi^2(x, a)} \sigma^2(x, a).$$

Substituting into equation (73), we find that $D_{\text{KL}} \left( \mathbb{P}_{\mu_{(z,s)}} \xi^* \mid \mathbb{P}_{\mu_{(-s)}} \xi^* \right) \leq 4n\sigma^2 \|\sigma\|^2_\omega$. For a sample size $n \geq 16$, with the choice of the perturbation parameter $s = \frac{1}{4\sqrt{n} \|\sigma\|_\omega}$, an application of Pinsker’s inequality leads to the bound

$$d_{\text{TV}} \left( \mathbb{P}_{\mu_{(z,s)}} \xi^*, \mathbb{P}_{\mu_{(-s)}} \xi^* \right) \leq \sqrt{\frac{1}{2} D_{\text{KL}} \left( \mathbb{P}_{\mu^*, \xi^*} \mid \mathbb{P}_{\mu_{(-s)}} \xi^* \right)} \leq \frac{1}{2\sqrt{2}},$$

which completes the proof of equation (71b).

### 5.1.3 Proof of equation (61c)

The proof is based on Le Cam’s mixture-vs-mixture method (cf. Lemma 15.9, [Wai19]). We construct a pair $(Q_1, Q_{-1})$ of probability distributions supported on the neighborhood $N_{\delta}^{\text{val}}(\mu^*)$; these are used to define two mixture distributions with the following properties:

- The mixture distributions have TV distance bounded as
  $$d_{\text{TV}} \left( \int \mathbb{P}_{\mu_{(z,s)}} \| \mathbb{P}_{\mu_{(-s)}} \right) \leq \frac{1}{4}.$$  

See Lemma 4 for details.

- There is a large gap in the target linear functional when evaluated at functions in the support of $Q_1^*$ and $Q_{-1}^*$. See Lemma 5 for details.

For any binary function $\zeta : X \times A \rightarrow \{-1, 1\}$, we define the perturbed outcome function

$$\mu_\zeta(x, a) := \mu^*(x, a) + \zeta(x, a) \cdot \delta(x, a) \text{ for all } (x, a) \in X \times A.$$

By construction, we have $\mu_\zeta \in N_{\delta}^{\text{val}}(\mu^*)$ for any binary function $\zeta$. Now consider the function

$$\rho(x, a) := \begin{cases} 
\frac{g(x, a)\delta(x, a)}{\|\delta\|_\omega \pi(x, a)} & \text{if } \frac{|g(x, a)|\delta(x, a)}{\pi(x, a)} \leq 2M_{2\rightarrow 4} \|\delta\|_\omega \\
\sgn(g(x, a)) & \text{otherwise}.
\end{cases}$$

It can be seen that $\mathbb{E}[\rho^2(X, A)] \leq 1$ where the expectation is taken over a pair $X \sim \xi^*$ and $A \sim \pi(X, \cdot)$.

For a scalar $s \in (0, \frac{1}{2M_{2\rightarrow 4}}]$ and a sign variable $z \in \{-1, 1\}$, we define the probability distribution

$$Q^*_s := \mathcal{L}(\mu_\zeta), \text{ where } \zeta \sim \prod_{x \in X, a \in A} \text{Ber} \left( \frac{1 + zs\rho(x, a)}{2} \right).$$

Having constructed the mixture distributions, we are ready to prove the lower bound (61c). The proof relies on the following two lemmas on the properties of the mixture distributions:
Lemma 4. The total variation distance between mixture-of-product distributions is upper bounded as
\[
\text{d}_{TV} \left( \int P_{\mu, \xi} \otimes dQ^s_1(\mu), \int P_{\mu, \xi} \otimes dQ^s_{-1}(\mu) \right) \leq 2s\sqrt{n} + 4 \cdot e^{-n/4}.
\] (77)

Lemma 5. Given a state space with cardinality lower bounded as \(|X| \geq 128c_{\max}/s^2\), we have
\[
\mathbb{P}_{\mu \sim Q^s_1} \left\{ \tau(\xi^s, \mu) \geq \tau(\xi^s, \mu^s) + \frac{s}{8}\|\delta\|_0 \right\} \geq 1 - 2 \cdot e^{-4}, \quad \text{and} \quad (78a)
\]
\[
\mathbb{P}_{\mu \sim Q^s_{-1}} \left\{ \tau(\xi^s, \mu) \leq \tau(\xi^s, \mu^s) - \frac{s}{8}\|\delta\|_0 \right\} \geq 1 - 2 \cdot e^{-4}. \quad (78b)
\]

We prove these lemmas at the end of this section.

Taking these two lemmas as given, we now proceed with the proof of equation (61c). Based on Lemma 5, we define two sets of functions as follows:
\[
E_1 := \left\{ \mu \xi \mid \xi \in \{-1, 1\}^{X \times A}, \tau(\xi^s, \mu^s, \xi^s, \mu^s) \geq \tau(\xi^s, \mu^s) + \frac{s}{8}\|\delta\|_0 \right\}, \quad \text{and} \\
E_{-1} := \left\{ \mu \xi \mid \xi \in \{-1, 1\}^{X \times A}, \tau(\xi^s, \mu^s) \leq \tau(\xi^s, \mu^s) - \frac{s}{8}\|\delta\|_0 \right\}.
\]

When the sample size requirement in equation (61c) is satisfied, Lemma 5 implies that \(Q^s_z(E_z) \geq 1 - e^{-4}\) for \(z \in \{-1, 1\}\). We set \(s = \frac{1}{16\sqrt{n}}\), and define
\[
Q^s_z := Q^s_1|E_z, \quad \text{for} \ z \in \{-1, 1\}. \quad (79)
\]

By construction, the probability distributions \(Q^s_z\) and \(Q^s_{-1}\) have disjoint support, and for any pair \(\mu \in \text{supp}(Q^s_1) \subseteq E_1\) and \(\mu' \in \text{supp}(Q^s_{-1}) \subseteq E_{-1}\), we have:
\[
\tau(\xi^s, \mu) \geq \tau(\xi^s, \mu^s) + \frac{\|\delta\|_0}{128\sqrt{n}}, \quad \text{and} \quad \tau(\xi^s, \mu') \leq \tau(\xi^s, \mu^s) - \frac{\|\delta\|_0}{128\sqrt{n}}. \quad (80)
\]

Furthermore, combining the conclusions in Lemma 4 and Lemma 5 using Lemma 6, we obtain the total variation distance upper bound:
\[
d_{TV} \left( \int P_{\mu, \xi} \otimes dQ^s_1(\mu), \int P_{\mu, \xi} \otimes dQ^s_{-1}(\mu) \right)
\leq \frac{1}{1 - 2 \cdot e^{-4}} \left[ \int P_{\mu, \xi} \otimes dQ^s_1(\mu), \int P_{\mu, \xi} \otimes dQ^s_{-1}(\mu) \right] + 4 \cdot e^{-4}
\leq \frac{1/8 + 4 \cdot e^{-n/4}}{1 - 2 \cdot e^{-4}} + 4 \cdot e^{-4} \leq \frac{1}{4},
\]
which completes the proof of equation (75).

Combining equation (80) and (75), we can invoke Le Cam’s mixture-vs-mixture lemma, and conclude that
\[
M_n \geq \frac{1}{4} \left\{ 1 - d_{TV} \left( \int P_{\mu, \xi} \otimes dQ^s_1(\mu), \int P_{\mu, \xi} \otimes dQ^s_{-1}(\mu) \right) \right\} \cdot \inf_{\mu \in \text{supp}(Q^s_1)} \left[ \inf_{\mu' \in \text{supp}(Q^s_{-1})} \left[ \tau(\xi^s, \mu) - \tau(\xi^s, \mu') \right] \right]^2
\geq \frac{c||\delta||^2_0}{n},
\]
for a universal constant \(c > 0\). This completes the proof of equation (61c).
Proof of Lemma 4: Our proof exploits a Poissonization device, which makes the number of observations random, and thereby simplifies calculations. For \( z \in \{-1, 1\} \), denote the mixture-of-product distribution:

\[
Q_z(s, \otimes n) := \int P_{\mu, \xi} \cdot dQ_z^e(\mu).
\]

We construct a pair \((Q_1(s, Poi), Q_{-1}(s, Poi))\) of mixture distributions as follows: randomly draw the sample size \( \nu \sim \text{Poi}(2n) \) independent of \( \zeta \) and random sampling of data. For each \( z \in \{-1, 1\} \), we let \( Q_z(s, Poi) \) be the mixture distribution:

\[
Q_z(s, Poi) := \sum_{k=0}^{\infty} Q_z(s, \otimes k) \cdot P(\nu = k).
\]

By a known lower tail bound for a Poisson random variable (c.f. [BLM13], §2.2), we have

\[
P(\nu \geq n) \geq 1 - e^{-n/4}, \quad \text{(81)}
\]

We note that on the event \( \tilde{\mathcal{E}}_n \), the probability law \( Q_z(s, \otimes n) \) is actually the projection of the law \( Q_z(s, \text{Poi}) \big| \tilde{\mathcal{E}}_n \) on the first \( n \) observations. Consequently, we can use Lemma 6 to bound the total variation distance between the original mixture distributions using that of the Poissonized models:

\[
d_{TV}(Q_1(s, \otimes n), Q_{-1}(s, \otimes n)) \leq d_{TV}\left(Q_1(s, \text{Poi}) \big| \tilde{\mathcal{E}}_n, Q_{-1}(s, \text{Poi}) \big| \tilde{\mathcal{E}}_n\right) + 4P(\tilde{\mathcal{E}}_n) \\
\leq \frac{1}{P(\tilde{\mathcal{E}}_n)} d_{TV}(Q_1(s, \text{Poi}), Q_{-1}(s, \text{Poi})) + 4 \cdot e^{-n/4} \\
\leq 2d_{TV}(Q_1(s, \text{Poi}), Q_{-1}(s, \text{Poi})) + 4 \cdot e^{-n/4}, \quad \text{(82)}
\]

valid for any \( n \geq 4 \).

It remains to bound the total variation distance between the Poissonized mixture distributions. We start by considering the empirical count function

\[
M(x, a) := \sum_{i=1}^{\nu} 1\{X_i = x, A_i = a\} \quad \text{for all } (x, a) \in \mathcal{X} \times \mathcal{A},
\]

Note that conditionally on the value of \( \nu \), the vector \((M(x, a))_{x \in \mathcal{X}, a \in \mathcal{A}}\) follows a multinomial distribution. Since \( \nu \sim \text{Poi}(2n) \), we have

\[
\forall x \in \mathcal{X}, a \in \mathcal{A} \quad M(x, a) \sim \text{Poi}(2n \xi^*(x) \pi(x, a)), \quad \text{independent of each other.}
\]

For each \((x, a) \in \mathcal{X} \times \mathcal{A}\) and \( z \in \{-1, 1\} \), we consider a probability distribution \( Q_z'(x, a) \) defined by the following sampling procedure:

(a) Sample \( M(x, a) \sim \text{Poi}(2n \xi^*(x) \pi(x, a)) \).

(b) Sample \( \zeta(x, a) \sim \text{Ber}\left(\frac{1+z \rho(x, a)}{2}\right) \).

(c) Generate a (possibly empty) set of \( M(x, a) \) independent observations from the conditional law of \( Y \) given \( X = x \) and \( A = a \).
By independence, for any $z \in \{-1, 1\}$, it is straightforward to see that:

$$Q_z^{(s, \text{Poi})} = \prod_{(x,a) \in X \times A} Q_z'(x,a),$$

Pinsker’s inequality, combined with the tensorization of the KL divergence, guarantees that

$$d_{TV}(Q_1^{(s, \text{Poi})}, Q_{-1}^{(s, \text{Poi})}) \leq \sqrt{\frac{1}{2} D_{KL}(Q_1^{(s, \text{Poi})} \| Q_{-1}^{(s, \text{Poi})})} = \sqrt{\frac{1}{2} \sum_{x \in X, a \in A} D_{KL}(Q_1'(x, a) \| Q_{-1}'(x, a))}. \quad (83)$$

Note that the difference between the probability distributions $Q'(x, a)$ and $Q'_{-1}(x, a)$ lies only in the parameter of the Bernoulli random variable $\zeta(x, a)$, which is observed if and only if $M(x, a) > 0$. By convexity of KL divergence, we have:

$$D_{KL}(Q_1'(x, a) \| Q_{-1}'(x, a)) \leq \mathbb{P}(M(x, a) > 0) \cdot D_{KL}(\text{Ber}(\frac{1 + s \rho(x, a)}{2}) \| \text{Ber}(\frac{1 - s \rho(x, a)}{2}))$$

$$\leq 4(1 - e^{-2n\xi^*(x)\pi(x,a)}) \cdot s^2 \rho^2(x, a)$$

$$\leq 8n \xi^*(x)\pi(x,a) s^2 \rho^2(x)$$

$$\leq 8ns^2 \xi^*(x) \frac{g(x,a)\delta^2(x,a)}{\pi(x,a)\|\delta\|_\omega^2}$$

Substituting back to the decomposition result (83), we conclude that

$$d_{TV}(Q_1^{(s, \text{Poi})}, Q_{-1}^{(s, \text{Poi})}) \leq \sqrt{\frac{1}{2} \sum_{x \in X, a \in A} 8ns^2 \xi^*(x) \frac{g^2(x,a)\delta^2(x,a)}{\pi(x,a)\|\delta\|_\omega^2} \leq 2s\sqrt{n}}.$$

Finally, combining with equation (82) completes the proof.

**Proof of Lemma 5:** Under our construction, we can compute the expectation of the target linear functional $\tau(I)$ under both distributions. In particular, for $z = 1$, we have

$$\mathbb{E}_{\mu \sim Q_1^{(s, \text{Poi})}}[\tau(\xi^*, \mu)] = \tau(\xi^*, \mu^*) + \frac{s}{2} \cdot \mathbb{E}_{\xi^*} \left[ \int_A \delta(x, a) g(x, a) \rho(X, A) d\lambda(a) \right]$$

$$\geq \tau(\xi^*, \mu^*) + \frac{s}{2}\|\delta\|_\omega \cdot \mathbb{E} \left[ \frac{\delta^2(X, A) g^2(X, A)}{\pi^2(X, A)} \mathbb{1} \left\{ \frac{|g(X, A)|\delta(X, A)}{\pi(X, A)} \leq 2M_{2\rightarrow 4}\|\delta\|_\omega \right\} \right],$$

where the last expectation is taken with respect to $X \sim \xi^*$ and $A \sim \pi(X, \cdot)$.

Applying Lemma 7 to the random variable $g(X, A)\delta(X, A)/\pi(X, A)$ yields

$$\mathbb{E} \left[ \frac{\delta^2(X, A) g^2(X, A)}{\pi^2(X, A)} \mathbb{1} \left\{ \frac{|g(X, A)|\delta(X, A)}{\pi(X, A)} \leq 2M_{2\rightarrow 4}\|\delta\|_\omega \right\} \right] \leq \frac{1}{2} \mathbb{E} \left[ \frac{\delta^2(X, A) g^2(X, A)}{\pi^2(X, A)} \right].$$

Consequently, we have the lower bound on the expected value under $Q_1^{(s, \text{Poi})}$

$$\mathbb{E}_{\mu \sim Q_1^{(s, \text{Poi})}}[\tau(\xi^*, \mu)] \geq \tau(\xi^*, \mu^*) + \frac{s}{4}\|\delta\|_\omega. \quad (84a)$$

Similarly, under the distribution $Q_{-1}^{(s, \text{Poi})}$, we note that:

$$\mathbb{E}_{\mu \sim Q_{-1}^{(s, \text{Poi})}}[\tau(\xi^*, \mu)] \leq \tau(\xi^*, \mu^*) - \frac{s}{4}\|\delta\|_\omega. \quad (84b)$$
We now consider the concentration behavior of random function \( \mu \sim Q_\xi \) for each choice of \( z \in \{-1, 1\} \). Since the random signs are independent at each state-action pair \((x, a) \in X \times A\), we can apply Hoeffding’s inequality: more precisely, with with probability \( 1 - 2e^{-2t} \), we have

\[
|\tau(\xi^*, \mu) - \mathbb{E}_{\mu \sim Q_\xi} [\tau(\xi^*, \mu)]| \leq \sqrt{\frac{\ln \frac{1}{\delta}}{2n}} \sum_{x \in X, a \in A} \xi^2(x) g^2(x, a) \delta^2(x, a)
\]

where in step (i), we use the compatibility condition \( \xi^*(x) \leq \frac{c_{\max}}{|X|} \) for any \( x \in X \).

Given a state space with cardinality lower bounded as \( |X| \geq 128c_{\max}/s^2 \), we can combine the concentration bound with the expectation bounds (84) so as to obtain

\[
\mathbb{P}_{\mu \sim Q_\xi} \{ \tau(\xi^*, \mu) \geq \tau(\xi^*, \mu^*) + \sqrt{\frac{\ln \frac{1}{\delta}}{2n}} \|\delta\|_\omega} \} \geq 1 - 2 \cdot e^{-2t}, \quad \text{and}
\]

\[
\mathbb{P}_{\mu \sim Q_\xi} \{ \tau(\xi^*, \mu) \leq \tau(\xi^*, \mu^*) - \sqrt{\frac{\ln \frac{1}{\delta}}{2n}} \|\delta\|_\omega} \} \geq 1 - 2 \cdot e^{-2t}.
\]

Setting \( t = 2 \) completes the proof of Lemma 5.

5.2 Proof of Proposition 1

Let the input distribution \( \xi^* \) be the uniform distribution over the sequence \( \{x_j\}_{j=1}^D \). It suffices to show that

\[
\inf \sup_{\tau_n, \mu \in \mathcal{F}} \mathbb{E} \left[ |\tau_n - \tau(\xi^*, \mu)|^2 \right] \geq \frac{c}{n} \left\{ \frac{1}{D} \sum_{j=1}^D \sum_{a \in A} g^2(x_j, a) \pi(x_j, a) \delta_a^2 \right\}.
\]

Recall that we are given a sequence \( \{x_j\}_{j=1}^D \) such that for each \( a \in A \), the function class \( \mathcal{F}_a \) shatters it at scale \( \delta_a \). Let \( \{t_{j,a}\}_{j=1}^D \) be the sequence of function values in the fat-shattering definition (38). Note that since the class \( \mathcal{F} \) is convex, we have

\[
\bigotimes_{j=1}^D \bigotimes_{a \in A} [t_{j,a} - \delta_a, t_{j,a} + \delta_a] \subseteq \bigotimes_{a \in A} \left\{ (f_a(x_j))_{j \in [D]} \mid f_a \in \mathcal{F}_a \right\}.
\]

Note that this distribution satisfies the compatibility condition with \( c_{\max} = 1 \) and the hyper-contractivity condition with a constant \( M_{2 \rightarrow 4} = \|g(X, A) \delta_A/\pi(X, A)\|_{2 \rightarrow 4} \). Invoking equation (61c) over the local neighborhood \( N_{\delta}(t) \) yields the claimed bound (85).

6 Discussion

We have studied the problem of evaluating linear functionals of the outcome function (or reward function) based on observational data. In the bandit literature, this problem corresponds to off-policy evaluation for contextual bandits. As we have discussed, the classical notion of semi-parametric efficiency characterizes the optimal asymptotic distribution, and the finite-sample analysis undertaken in this paper enriches this perspective. First, our analysis uncovered the importance of a particular weighted \( L^2 \)-norm for estimating the outcome function \( \mu^* \). More precisely, optimal estimation of the scalar \( \tau^* \) is equivalent to optimal estimation of the
outcome function $\mu^*$ under such norm, in the sense of minimax risk over a local neighborhood. Furthermore, when the outcome function is known to lie within some function class $\mathcal{F}$, we showed that a sample size scaling with the complexity of $\mathcal{F}$ is necessary and sufficient to achieve such bounds non-asymptotically.

Our result lies at the intersection of decision-making problems and the classical semi-parametric theories, which motivates several promising directions of future research on both threads:

- Our analysis reduces the problem of obtaining finite-sample optimal estimates for linear functionals to the nonparametric problem of estimating the outcome function under a weighted norm. Although the re-weighted least-square estimator (11) converges to the best approximation of the treatment effect function in the class, it is not clear whether it always achieves the optimal trade-off between the approximation and estimation errors. How to optimally estimate the nonparametric component under weighted norm (so as to optimally estimate the scalar $\tau^*$ in finite sample) for a variety of function classes is an important direction of future research, especially with weight functions.

- The analysis of the current paper was limited to i.i.d. data, but similar issues arise with richer models of data collection. There are recent lines of research on how to estimate linear functionals with adaptively collected data (e.g. when the data are generated from an exploratory bandit algorithm [ZHHA21, SRR20, KDMW21]), or with an underlying Markov chain structure (e.g. in off-policy evaluation problems for reinforcement learning [JL16, YW20, KU22, ZWB21]). Many results in this literature build upon the asymptotic theory of semi-parametric efficiency, so that it is natural to understand whether extensions of our techniques could be used to obtain finite-sample optimal procedures in these settings.

- The finite-sample lens used in this paper reveals phenomena in semi-parametric estimation that are washed away in the asymptotic limit. This paper has focused on a specific class of semi-parametric problems, but more broadly, we view it as interesting to see whether such phenomena exist for other models in semi-parametric estimation. In particular, if a high-complexity object—such as a regression or density function—needs to be estimated in order to optimally estimate a low-complexity object—such as a scalar—it is important to characterize the minimal sample size requirements, and the choice of nonparametric procedures for the nuisance component that are finite-sample optimal.

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A Proofs of auxiliary results in the upper bounds

In this section, we state and prove some auxiliary results used in the proofs of our non-asymptotic upper bounds.

A.1 Some properties of the estimator $\hat{\tau}_n^f$

In this appendix, we collect some properties of the estimator $\hat{\tau}_n^f$ defined in equation (5).

**Proposition 3.** Given any deterministic function $f \in L^2(\xi \times \pi)$ for any $a \in A$, we have $\mathbb{E}[\hat{\tau}_n^f] = \tau_f(I)$. Furthermore, if $\langle f(x, \cdot), \pi(x, \cdot) \rangle_\lambda = 0$ for any $x \in X$, we have

$$n \cdot \mathbb{E}\left[|\hat{\tau}_n^f - \tau_f(I)|^2\right] = \text{var}_\xi \left(\langle g(X, \cdot), \mu^*(X, \cdot) \rangle_\lambda \right) + \int_A \mathbb{E}\left[\sigma^2(X, a) g^2(X, a) \frac{\pi(X, a)}{\pi(X, a)} \right] d\lambda(a)$$

$$+ \int_A \mathbb{E}\left[\pi(X, a) f(X, a) - \frac{g(X, a) \mu^*(X, a)}{\pi(X, a)} + \langle g(X, \cdot), \mu^*(X, \cdot) \rangle_\lambda \right]^2 d\lambda(a). \quad (86)$$

This decomposition immediately implies the claims given in the text. The only portion of the MSE decomposition (86) that depends on $f$ is the third term, and by inspection, this third term is equal to zero if and only if

$$f(x, a) = \frac{g(x, a) \mu^*(x, a)}{\pi(x, a)} - \langle g(x, \cdot), \mu^*(x, \cdot) \rangle_\lambda \quad \text{for all} \ (x, a) \in X \times A.$$

**Proof.** Since the action $A_i$ follows the probability distribution $\pi(X_i, \cdot)$ conditionally on $X_i$, we have $\mathbb{E}[f(X_i, A_i) | X_i] = \langle \pi(X_i, \cdot), f(X_i, \cdot) \rangle_\lambda$, and the estimator $\hat{\tau}_n^f$ is always unbiased. Since the function $f$ is square-integrable with respect to the measure $\xi \times \pi$, the second moment can be decomposed as follows:

$$\mathbb{E}\left[\left|\frac{g(X_i, A_i)}{\pi(X_i, A_i)} Y_i - f(X_i, A_i) + \int_A \pi(X_i, a) f(X_i, a) d\lambda(a)\right|^2\right]$$

$$= \int_A \mathbb{E}\left[\pi(X_i, a) \left| \frac{g(X_i, a)}{\pi(X_i, a)} Y_i - f(X_i, a) \right|^2\right] d\lambda(a)$$

$$= \int_A \mathbb{E}\left[\sigma^2(X, a) g^2(X, a) \frac{\pi(X, a)}{\pi(X, a)} \right] d\lambda(a) + \int_A \mathbb{E}\left[\pi(X, a) \left| \frac{g(X, a) \mu^*(X, a)}{\pi(X, a)} - f(X, a) \right|^2\right] d\lambda(a)$$

Conditionally on the value of $X$, we have the bias-variance decomposition

$$\int_A \pi(X, a) \left| \frac{g(X, a) \mu^*(X, a)}{\pi(X, a)} - f(X, a) \right|^2 d\lambda(a)$$

$$= \langle g(X, \cdot), \mu^*(X, \cdot) \rangle_\lambda^2 + \int_A \pi(X, a) \left| f(X, a) - \frac{g(X, a) \mu^*(X, a)}{\pi(X, a)} + \langle g(X, \cdot), \mu^*(X, \cdot) \rangle_\lambda \right|^2 d\lambda(a).$$

Finally, we note that

$$\mathbb{E}\left[\left(\langle g(X, \cdot), \mu^*(X, \cdot) \rangle_\lambda \right)^2\right] - \tau_f^2(I) = \left(\mathbb{E}\left[\langle g(X, \cdot), \mu^*(X, \cdot) \rangle_\lambda \right]\right)^2 = \text{var}_\xi \left(\langle g(X, \cdot), \mu^*(X, \cdot) \rangle_\lambda \right).$$

Putting together the pieces completes the proof. \(\square\)
A.2 Existence of critical radii

In this section, we establish the existence of critical radii $s_m(\mu)$ and $r_m(\mu)$ defined in equations (14a) and (14b), respectively.

**Proposition 4.** Suppose that the compatibility condition (CC) holds, and that the Rademacher complexities $S_m((F - \mu) \cap B_\omega(r_0))$ and $R_m((F - \mu) \cap B_\omega(r_0))$ are finite for some $r_0 > 0$. Then:

(a) There exists a unique scalar $s_m = s_m(\mu) > 0$ such that inequality (14a) holds for any $s \geq s_m$, with equality when $s = s_m$, and is false when $s \in [0, s_m)$.

(b) There exists a scalar $r_m = r_m(\mu) > 0$ such that inequality (14b) holds for any $r \geq r_m$.

**Proof.** Denote the shifted function class $F^* := F - \mu$. Since the class $F$ is convex by assumption, for positive scalars $r_1 < r_2$ and any function $f \in F^* \cap B_\omega(r_2)$, we have $\frac{r_1}{r_2} f \in F^* \cap B_\omega(r_1)$.

$$\frac{1}{r_2} R_m(F^* \cap B_\omega(r_2)) \leq \frac{1}{r_2} R_m\left(\frac{r_2}{r_1} \cdot (F^* \cap B_\omega(r_1))\right) = \frac{1}{r_1} R_m(F^* \cap B_\omega(r_1)).$$

So the function $r \mapsto r^{-1} R_m(F^* \cap B_\omega(r))$ is non-increasing in $r$. A similar argument ensures that the function $s \mapsto s^{-1} S(F^* \cap B_\omega(s))$ is also non-increasing in $s$.

Since the function class $F$ is compact in $L^2_\omega$, we have $D := \text{diam}_\omega(F \cup \{\mu^*\}) < +\infty$, and hence

$$R_m(F^*) = R_m(F^* \cap B_\omega(D)) \leq \frac{D}{r_0} R_m(F^* \cap B_\omega(r_0)) < +\infty,$$

which implies that $R_m(F^* \cap B_\omega(r)) < +\infty$ for any $r > 0$. Similarly, the Rademacher complexity $S(F^* \cap B_\omega(s))$ is also finite.

For the inequality (14a), the left-hand side is a non-increasing function of $s$, while the right-hand side is strictly increasing and diverging to infinity as $s \to +\infty$. Furthermore, the right-hand-side is equal to zero at $s = 0$, while the left-hand side is always finite and non-negative for $s > 0$. Consequently, a unique fixed point $s_m \geq 0$ exists, and we have

$$s^{-1} S(F^* \cap B_\omega(s)) \begin{cases} < s, & \text{for } s > s_m, \\ > s, & \text{for } s \in (0, s_m). \end{cases}$$

As for inequality (14b), the left-hand-side is non-increasing, and we have

$$\lim_{r \to +\infty} r^{-1} R(F^* \cap B_\omega(r)) \leq \lim_{r \to +\infty} r^{-1} R(F^*) = 0.$$

So there exists $r_m \geq 0$ such that inequality (14b) holds for any $r \geq r_m$.

\[\square\]

A.3 Proof of Lemma 3

We define the auxiliary function

$$\phi(t) := \begin{cases} 0 & t \leq 1, \\ t - 1 & 1 \leq t \leq 2, \\ 1 & t > 2. \end{cases}$$
First, observe that for any scalar \( u > 0 \), we have
\[
\frac{1}{m} \sum_{i=1}^{m} \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} h^2(X_i, A_i) \geq \frac{1}{m} \sum_{i=1}^{m} u^2 \cdot \mathbb{I}[\left| \frac{g(X_i, A_i)h(X_i, A_i)}{\pi(X_i, A_i)} \right| \geq u] \\
\geq \frac{1}{m} \sum_{i=1}^{m} u^2 \cdot \phi(\left| \frac{g(X_i, A_i)h(X_i, A_i)}{\pi(X_i, A_i)} \right|) =: Z_m^{(\phi)}(h).
\]

Second, for any function \( h \in \mathcal{H} \), we have
\[
\mathbb{E}\left[Z_m^{(\phi)}(h)\right] = u^2 \cdot \sum_{a \in \mathcal{A}} \mathbb{E}_\xi\left[\pi(X, a) \phi\left(\left| \frac{g(X, a)h(X, a)}{\pi(X, a)} \right|\right)\right] \\
\geq u^2 \sum_{a \in \mathcal{A}} \mathbb{E}_\xi\left[\pi(X, a) \cdot \mathbb{I}\left[\left| \frac{g(X, a)h(X, a)}{\pi(X, a)} \right| \geq 2\right]\right] \\
= u^2 \cdot \mathbb{P}_{X \sim \xi, A \sim \pi(X, \cdot)}\left(\left| \frac{g(X, A)h(X, A)}{\pi(X, A)} \right| \geq 2u \right).
\]

Recall that the constant \( \alpha_1 \) is the constant factor in the small-ball probability condition (SB). Choosing the threshold \( u := \frac{\epsilon}{2} \) and using the equality \( \|h\|_\omega = 1 \), we see that the small-ball condition implies that
\[
\mathbb{P}_{X \sim \xi, A \sim \pi(X, \cdot)}\left(\left| \frac{g(X, A)h(X, A)}{\pi(X, A)} \right| \geq 2u \right) \geq \alpha_2.
\]

Now we turn to study the deviation bound for \( Z_m^{(\phi)}(h) \). Using known concentration inequalities for empirical processes [Ada08]—see Proposition 6 in Appendix D for more detail—we are guaranteed to have
\[
\sup_{h \in \mathcal{H}} \left( Z_m^{(\phi)}(h) - \mathbb{E}\left[Z_m^{(\phi)}(h)\right] \right) \leq 2\mathbb{E} \sup_{h \in \mathcal{H}} \left( Z_m^{(\phi)}(h) - \mathbb{E}\left[Z_m^{(\phi)}(h)\right] \right) + \frac{\alpha_1^2}{m} \left\{ \frac{\log(1/\epsilon)}{m} + \frac{\log(1/\epsilon)}{m} \right\}.
\]

with probability at least \( 1 - \epsilon \).

For the expected supremum term, standard symmetrization arguments lead to the bound
\[
\mathbb{E} \sup_{h \in \mathcal{H}} \left( Z_m^{(\phi)}(h) - \mathbb{E}\left[Z_m^{(\phi)}(h)\right] \right) \leq \frac{\alpha_1^2}{m} \cdot \mathbb{E} \left[ \sup_{h \in \mathcal{H}} \sum_{i=1}^{m} \varepsilon_i \phi\left(\frac{2|h(X_i, A_i)g(X_i, A_i)|}{\alpha_1 \pi(X_i, A_i)}\right)\right].
\]

Note that since \( \phi \) is a 1-Lipschitz function, we may apply the Ledoux-Talagrand contraction (e.g., equation (5.6.1) in the book [Wai19]) so as to obtain
\[
\mathbb{E}\left[\sum_{h \in \mathcal{H}} \sum_{i=1}^{m} \varepsilon_i \phi\left(\frac{2|h(X_i, A_i)g(X_i, A_i)|}{\alpha_1 \pi(X_i, A_i)}\right)\right] \leq \frac{4}{\alpha_1} \mathbb{E}\left[\sup_{h \in \mathcal{H}} \sum_{i=1}^{m} \varepsilon_i g(X_i, A_i) h(X_i, A_i)\right].
\]

Combining the pieces yields the lower bound
\[
\frac{1}{m} \sum_{i=1}^{m} \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} h^2(X_i, A_i) \\
\geq \frac{\alpha_2 \alpha_1^2}{4} - \frac{4\alpha_1}{m} \mathbb{E}\left[\sup_{h \in \mathcal{H}} \sum_{i=1}^{m} \varepsilon_i g(X_i, A_i) h(X_i, A_i)\right] - \frac{\alpha_1^2}{2} \cdot \left\{ \frac{\log(1/\epsilon)}{m} + \frac{\log(1/\epsilon)}{m} \right\}, \quad (87)
\]

uniformly holding true over \( h \in \mathcal{H} \), with probability \( 1 - \epsilon \), which completes the proof of the lemma.
B Proofs of the corollaries

This section is devoted to the proofs of Corollaries 1—4, as stated in Section 2.4.

B.1 Proof of Corollary 1

Let us introduce the shorthand \( f_\theta(x, a) := \langle \theta, \phi(x, a) \rangle \) for functions that are linear in the feature map. Moreover, for a vector \( \theta \in \mathbb{R}^d \) and radius \( r > 0 \), we define the recentering function \( \tilde{\mu}(x, a) := \langle \tilde{\theta}, \phi(x, a) \rangle \).

Our proof strategy is to bound the pair of critical radii \((s_m, r_m)\), and we do so by controlling the associated Rademacher complexities. By a direct calculation, we find that

\[
(F - \tilde{\mu}) \cap \mathbb{B}_\omega(r) \subseteq \left\{ f_\theta \mid \theta^\top \Sigma \theta \leq r^2 \right\}, \quad \text{where } \Sigma := \mathbb{E}\left[ \frac{g^2(X, A)}{\pi^2(X, A)} \phi(X, A) \phi(X, A)^\top \right].
\]

We can therefore bound the Rademacher complexities as

\[
\mathcal{S}_m^2 \left( (F - \tilde{\mu}) \cap \mathbb{B}_\omega(r) \right) \leq \mathbb{E} \left[ \sup_{\|\theta\| \leq r} \left\{ \frac{1}{m} \left( \sum_{i=1}^m \frac{\varepsilon_i g(X_i, A_i)}{\pi(X_i, A_i)} (Y_i - \mu(X_i, A_i)) \phi(X_i, A_i) \right) \right\}^2 \right] = \frac{r^2}{m} \text{trace} \left( \Sigma^{-1} \Gamma_\sigma \right),
\]

and

\[
\mathcal{R}_m \left( (F - \tilde{\mu}) \cap \mathbb{B}_\omega(r) \right) \leq \mathbb{E} \left[ \sup_{\|\theta\| \leq r} \frac{1}{m} \left( \sum_{i=1}^m \frac{\varepsilon_i g(X_i, A_i)}{\pi(X_i, A_i)} \phi(X_i, A_i) \right) \right] \leq r \sqrt{\frac{d}{m}}.
\]

By definition of the fixed point equations, the critical radii can be upper bounded as

\[
s_m \leq \sqrt{\frac{1}{m} \text{trace} \left( \Sigma^{-1} \Gamma_\sigma \right)}, \quad \text{and } r_m \leq \begin{cases} +\infty, & m \leq \frac{1024}{\alpha^2} \frac{d}{\sigma^2}, \\ 0, & m > \frac{1024}{\alpha^2} \frac{d}{\sigma^2}. \end{cases}
\]

Combining with Theorem 2 completes the proof of this corollary.

B.2 Proof of Corollary 2

We introduce the shorthand \( f_\theta(x, a) = \langle \theta, \phi(x, a) \rangle \) for functions that are linear in the feature map. Given any vector \( \tilde{\theta} \in \mathbb{R}^d \) such that \( \|\tilde{\theta}\|_1 = R_1 \), define the set \( S = \text{supp}(\tilde{\theta}) \subseteq [d] \) along with the function \( \mu(x, a) = \langle \tilde{\theta}, \phi(x, a) \rangle \). For any radius \( r > 0 \) and vector \( \theta \in (F - \tilde{\mu}) \cap \mathbb{B}_\omega(r) \), we note that

\[
\|\theta_{S^c}\|_1 = \|\theta_{S^c} + \tilde{\theta}_{S^c}\|_1 = \|\theta + \tilde{\theta}\|_1 - \|\theta_{S} + \tilde{\theta}_{S}\|_1 \leq R_1 - \|\theta_{S}\|_1 + \|\theta_{S}\|_1 \leq \|\theta_{S}\|_1.
\]

Recalling that \( \Sigma = \mathbb{E}\left[ \frac{g^2(X, A)}{\pi^2(X, A)} \phi(X, A) \phi(X, A)^\top \right] \), we have the inclusions

\[
(F - \tilde{\mu}) \cap \mathbb{B}_\omega(r) \subseteq r \cdot \left\{ f_\theta \mid \|\theta_{S^c}\|_1 \leq \|\theta_{S}\|_1, \|\theta\|_\Sigma \leq 1 \right\} \subseteq r \cdot \left\{ f_\theta \mid \|\theta\|_1 \leq 2\sqrt{|S|/\lambda_{\min}(\Sigma)} \right\} \subseteq 2r \sqrt{|S|/\lambda_{\min}(\Sigma)} \cdot \text{conv} \left\{ \{ \pm \phi_j\}_{j=1}^d \right\}.
\]
where the second step follows from the bound $\|\theta_S\|_1 \leq \|\theta_S\|_2 \sqrt{|S|} \leq \|\theta\|_\Sigma \sqrt{|S|/\lambda_{\min}(\Sigma)}$, valid for any $\theta \in \mathbb{R}^d$.

For each coordinate $j = 1, \ldots, d$, we can apply the Hoeffding inequality along with the sub-Gaussian tail assumption (22) so as to obtain

$$\Pr\left[ \left| \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i g(X_i, A_i) \phi(X_i, A_i) \right| \geq t \right] \leq 2e^{-\frac{mt^2}{2\nu^2}} \quad \text{for any } t > 0.$$ 

Taking the union bound over $j = 1, 2, \ldots, d$ and then integrating the resulting tail bound, we find that

$$\mathbb{E}\left[ \max_{j=1,\ldots,d} \left| \frac{1}{m} \sum_{i=1}^{m} g(X_i, A_i) \phi_j(X_i, A_i) \right| \right] \leq \sqrt{\frac{\log d}{m}}.$$ 

Combining with equation (88), we conclude that

$$\mathcal{R}_m((\mathcal{F} - \bar{\mu}) \cap \mathbb{B}_\omega(r)) \leq 2\nu \sqrt{\frac{|S| \cdot \log(d)}{m \lambda_{\min}(\Sigma)}} \quad \text{for any } \bar{\mu}(x, a) = \langle \bar{\theta}, \phi(x, a) \rangle \text{ with } \bar{\theta} \text{ supported on } S.$$ 

Consequently, defining the constant $c_0 = \frac{4996}{\alpha_1^2}$, when the sample size satisfies $m \geq c_0 |S| \frac{\nu^2 \log(d)}{\lambda_{\min}(\Sigma)}$, the critical radius $r_m$ is 0.

Now we turn to bound the critical radius $s_m$. By the sub-Gaussian condition (22), we have the Orlicz norm bound

$$\left\| \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} \varepsilon_i \phi_j(X_i, A_i)(Y_i - \mu^*(X_i, A_i)) \right\|_{\psi_1} \leq \left\| \frac{g(X_i, A_i)}{\pi(X_i, A_i)} \phi_j(X_i, A_i) \right\|_{\psi_1} \cdot \|g(X_i, A_i)\|_{\psi_1} \leq \nu \bar{\sigma}.$$ 

Invoking a known concentration inequality (see Proposition 6 in Appendix D), we conclude that there exists a universal constant $c_1 > 0$ such that

$$\Pr\left( \left| \frac{1}{m} \sum_{i=1}^{m} \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} \varepsilon_i \phi_j(X_i, A_i)(Y_i - \mu^*(X_i, A_i)) \right| \geq t \right) \leq 2 \exp\left( \frac{-c_1 mt^2}{\nu^2 \bar{\sigma}^2 + t \nu \bar{\sigma} \log(m)} \right),$$

for any scalar $t > 0$.

Taking the union bound over $j = 1, 2, \ldots, d$ and integrating out the tail yields

$$\mathbb{E}\left[ \max_{j \in [d]} \left| \frac{1}{m} \sum_{i=1}^{m} \frac{g(X_i, A_i)}{\pi(X_i, A_i)} \varepsilon_i \phi_j(X_i, A_i) \right|^2 \right] \leq c_2 \nu^2 \bar{\sigma}^2 \left\{ \sqrt{\frac{\log d}{m}} + \frac{\log d \cdot \log m}{m} \right\}^2,$$

Given a sample size lower bounded as $m \geq \log^2 d$, the derivation above guarantees that the Rademacher complexity is upper bounded as

$$\mathcal{S}((\mathcal{F} - \bar{\mu}) \cap \mathbb{B}_\omega(r)) \leq c \nu \bar{\sigma} \sqrt{\frac{|S| \cdot \log(d)}{m \lambda_{\min}(\Sigma)}},$$

and consequently, the associated critical radius satisfies an upper bound of the form $s_m \leq c \nu \bar{\sigma} \sqrt{\frac{|S| \log(d)}{m \lambda_{\min}(\Sigma)}}$.

Combining with Theorem 2 completes the proof of Corollary 2.
B.3 Proof of Corollary 3

Clearly, the function class $\mathcal{F}_k$ is symmetric and convex. Consequently, for any $\mu \in \mathcal{F}_k$, we have

$$(\mathcal{F}_k - \mu) \cap B_\omega(r) \subseteq (2\mathcal{F}_k) \cap B_\omega(r).$$

For any pair $\mu_1, \mu_2 \in (2\mathcal{F}) \cap B_\omega(r)$, by the sub-Gaussian assumption in equations (23), we have that

$$E\left[\frac{g(X_i, A_i)}{\pi(X_i, A_i)} \varepsilon_i(\mu_1 - \mu_2)(X_i, A_i)^2\right] = \|\mu_1 - \mu_2\|_2^2,$$

and

$$\|\frac{g(X_i, A_i)}{\pi(X_i, A_i)} \varepsilon_i(\mu_1 - \mu_2)(X_i, A_i)\|_1 \leq \nu \|\mu_1 - \mu_2\|_\infty.$$ 

By a known concentration inequality (see Proposition 6 in Appendix D), for any $t > 0$, we have

$$P\left(\left|\frac{1}{m} \sum_{i=1}^m g(X_i, A_i) \varepsilon_i(\mu_1 - \mu_2)(X_i, A_i)\right| \geq t\right) \leq 2 \exp\left(-\frac{-c_1 mt^2}{\|\mu_1 - \mu_2\|_2^2 + t\nu \|\mu_1 - \mu_2\|_\infty \log(m)}\right),$$

We also note that the Cauchy–Schwarz inequality implies that

$$E\left[\sup_{\|\mu_1 - \mu_2\|_2 \leq \delta} \frac{1}{m} \sum_{i=1}^m g(X_i, A_i) \varepsilon_i(\mu_1 - \mu_2)(X_i, A_i)\right] \leq \delta.$$ 

By a known mixed-tail chaining bound (see Proposition 7 and equation (94) in Appendix D), we find that

$$R_m((\mathcal{F}_k - \mu) \cap B_\omega(r)) \leq \frac{c}{\sqrt{m}} J_2((2\mathcal{F}_k) \cap B_\omega(r), \|\cdot\|_\omega; [\delta, r])$$

$$+ \frac{c\nu \log m}{m} J_1((2\mathcal{F}_k) \cap B_\omega(r), \|\cdot\|_\infty; [\delta, 2]) + 2\delta,$$  

for any scalar $\delta \in [0, 2]$. Observing the norm domination relation $\|f\|_\omega \leq \nu \|f\|_\infty$ for any function $f$, we have $J_2((2\mathcal{F}_k) \cap B_\omega(r), \|\cdot\|_\omega; [\delta, r]) \leq J_2(2\mathcal{F}_k, \|\cdot\|_\infty; [\delta, r])$. As a result, in order to control the right-hand-side of equation (89), it suffices to bound the covering number of the class $\mathcal{F}_k$ under the $\|\cdot\|_\infty$-norm.

In order to estimate the Dudley chaining integral for the localized class, we begin with the classical bound [KT59]

$$\log N(\mathcal{F}_k, \|\cdot\|_\infty; \varepsilon) \leq \left(\frac{c}{\varepsilon}\right)^{p/k},$$

where $c > 0$ is a universal constant. Using this bound, we can control the Dudley entropy integrals for any $\alpha \in \{1, 2\}$, $q > 0$, and interval $[\delta, u]$ with $u \in \{r, 2\}$. In particular, for any interval $[\delta, u]$ of the non-negative real line, we have

$$J_\alpha\left(q, \mathcal{F}_k, \|\cdot\|_\infty; [\delta, u]\right) \leq \int_{\delta}^{u} \left(\frac{cq}{\varepsilon}\right)^{\frac{p}{\alpha k}} d\varepsilon \leq cq^{\frac{p}{\alpha k}} \begin{cases} \frac{\alpha k}{\alpha k - p} u^{1 - \frac{p}{\alpha k}} & \text{if } p < \alpha k, \\ \log \left(\frac{u}{\delta}\right) \left(\frac{\alpha k}{p - \alpha k}\right)^{\frac{p}{\alpha k} - 1} & \text{if } p = \alpha k, \\ \frac{\alpha k}{p - \alpha k} \left(\frac{\varepsilon}{\delta}\right)^{\frac{p}{\alpha k} - 1} & \text{if } p > \alpha k. \end{cases}$$

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We set $\delta = \left( \frac{2}{m} \right)^{k/p}$, and use the resulting upper bound on the Dudley integral to control the Rademacher complexity; doing so yields

\[
R_m((\mathcal{F}_k - \bar{\mu}) \cap B_\omega(r)) \leq c_{\nu, p/k} \cdot \begin{cases} 
\frac{r^{1 - \frac{p}{k}}}{\pi} / \sqrt{m} + \log m \cdot m^{-k/p} & \text{if } p < 2k, \\
\log(m) / \sqrt{m} & \text{if } p = 2k, \\
m^{-k/p} & \text{if } p > 2k.
\end{cases}
\]

Solving the fixed point equation (14b) yields

\[
s_m \leq c'_{\nu, p/k} m^{-k/p} \cdot \log m,
\]

where the constant $c_{\nu, p/k}$ and $c'_{\nu, p/k}$ depend on the parameters $(\nu, p/k)$, along with the small ball constants $(\alpha_1, \alpha_2)$.

Turning to the critical radius $s_m$, we note that each term in the empirical process associated with the observation noise satisfies

\[
\mathbb{E} \left[ \left( \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} (Y_i - \mu^*(X_i, A_i)) \varepsilon_i(\mu_1 - \mu_2)(X_i, A_i) \right)^2 \right] 
\leq \sqrt{\mathbb{E} \left[ \left( \frac{g(X_i, A_i)}{\pi(X_i, A_i)} (Y_i - \mu^*(X_i, A_i)) \right)^4 \right]} \cdot \sqrt{\mathbb{E} \left[ \left( \frac{g(X_i, A_i)}{\pi(X_i, A_i)} (\mu_1 - \mu_2)(X_i, A_i) \right)^4 \right]} 
\leq \sigma^2 M_{2 \rightarrow 4} \| \mu_1 - \mu_2 \|_\omega^2,
\]

and

\[
\left\| \frac{g^2(X_i, A_i)}{\pi^2(X_i, A_i)} (Y_i - \mu^*(X_i, A_i)) \varepsilon_i(\mu_1 - \mu_2)(X_i, A_i) \right\|_{\psi_1} 
\leq \left\| \frac{g(X_i, A_i)}{\pi(X_i, A_i)} (Y_i - \mu^*(X_i, A_i)) \right\|_{\psi_2} \cdot \left\| \frac{g(X_i, A_i)}{\pi(X_i, A_i)} (\mu_1 - \mu_2)(X_i, A_i) \right\|_{\psi_2} 
\leq \sigma \nu \| \mu_1 - \mu_2 \|_{\infty}.
\]

Following the same line of derivation in the bound for the Rademacher complexity $R_m$, we use the mixed-tail chaining bound to find that

\[
S_m((\mathcal{F}_k - \bar{\mu}) \cap B_\omega(r)) \leq \frac{c \sigma \sqrt{M_{2 \rightarrow 4}}}{\sqrt{m}} J_2((2\mathcal{F}_k) \cap B_\omega(r), \| \cdot \|_\omega; [\delta, r]) 
+ \frac{c \sigma \nu \log m}{m} J_1((2\mathcal{F}_k) \cap B_\omega(r), \| \cdot \|_\infty; [\delta, 2]) + 2\delta,
\]

valid for all $\delta \in [0, 2]$. The Dudley integral bound (90) then implies

\[
S_m((\mathcal{F}_k - \bar{\mu}) \cap B_\omega(r)) \leq c_{\nu, p/k} \sigma \cdot \begin{cases} 
\frac{r^{1 - \frac{p}{k}}}{\pi} / \sqrt{m} + \log m \cdot m^{-k/p} & \text{if } p < 2k, \\
\log(m) / \sqrt{m} & \text{if } p = 2k, \\
m^{-k/p} & \text{if } p > 2k.
\end{cases}
\]

where the constant $c_{\nu, p/k}$ depends on the parameters $(\nu, p/k)$ and the constant $M_{2 \rightarrow 4}$. Solving the fixed point equation yields

\[
s_m \leq c_{\nu, p/k} \sigma \cdot \begin{cases} 
m^{-\frac{k}{p}} & \text{if } p < 2k, \\
m^{-1/4} \sqrt{\log m} & \text{if } p = 2k, \\
m^{-\frac{k}{p}} & \text{if } p > 2k.
\end{cases}
\]

Combining with Theorem 2 completes the proof of Corollary 3.
B.4 Proof of Corollary 4

For any $\bar{\mu} \in \mathcal{F}$, define the function class

$$\mathcal{H} := \left\{ (x, a) \rightarrow \frac{g(x, a)}{\pi(x, a)} f(x, a) \mid f \in \mathbb{B}_\omega(r) \cap (\mathcal{F} - \bar{\mu}) \right\}.$$ 

Clearly, the class $\mathcal{H}$ is uniformly bounded by $b$, and for any $f \in \mathcal{F}$, we have the upper bound

$$\mathbb{E}\left[ \left| \frac{g(x, a)}{\pi(x, a)} f(X, A) \right|^2 \right] = \|f\|^2_{\omega} \leq r^2.$$

Invoking a known bracketing bound on empirical processes (cf. Prop. 8 in Appendix D), we have

$$\mathbb{E}\left[ \sup_{h \in \mathcal{H}} \frac{1}{m} \sum_{i=1}^{m} \epsilon_i h(X_i, A_i) \right] \leq \frac{c}{\sqrt{m}} J_{\text{bra}}(\mathcal{H}, \|\cdot\|_{L^2}; [0, r]) \left\{ 1 + \frac{b J_{\text{bra}}(\mathcal{H}, \|\cdot\|_{L^2}; [0, r])}{r^2 \sqrt{m}} \right\}$$ \hspace{1cm} (91)

For functions $\ell, f, u : [0, 1] \rightarrow \mathbb{R}$, such that $f$ is contained in the bracket $[\ell, u]$, we let:

$$\tilde{\ell}(x, a) := \frac{g(x, a)}{\pi(x, a)} \left\{ \ell(\phi(x, a)) 1_{g(x, a) > 0} + u(\phi(x, a)) 1_{g(x, a) < 0} - \bar{\mu}(x, a) \right\},$$

$$\tilde{u}(x, a) := \frac{g(x, a)}{\pi(x, a)} \left\{ u(\phi(x, a)) 1_{g(x, a) > 0} + \ell(\phi(x, a)) 1_{g(x, a) < 0} - \bar{\mu}(x, a) \right\}.$$ 

It is easily observed that the function $(x, a) \rightarrow \frac{g(x, a)}{\pi(x, a)}(f - \bar{\mu})(x, a)$ lies in the bracket $[\tilde{\ell}, \tilde{u}]$, and for any probability law $Q$ on $X \times A$, we have $\|\tilde{u} - \tilde{\ell}\|_{L^2(Q)} \leq b \cdot \|u - \ell\|_{L^2(Q_\phi)}$, where $Q_\phi$ is the probability law of $\phi(X, A)$ for $(X, A) \sim Q$.

It is known (cf. Thm 2.7.5 in the book [vdVW96]) that the space of monotonic functions from $[0, 1]$ to $[0, 1]$ has $\varepsilon$-bracketing number under any $L^2$-norm bounded by $\exp(c/\varepsilon)$ for any $\varepsilon > 0$. Substituting back into the bracketing entropy bound (91) yields

$$\mathcal{R}_m(\mathbb{B}_\omega(r) \cap (\mathcal{F} - \bar{\mu})) \leq c \left\{ \sqrt{\frac{br}{m}} + \frac{b^2}{rm} \right\}.$$ 

From the definition of the fixed point equation, we can bound the critical radius $r$ as

$$r_m \leq \frac{cb}{m} + \frac{cb}{\sqrt{m}},$$

where $c > 0$ is a universal constant.

Turning to the squared Rademacher process associated with the outcome noise, we construct the function class

$$\mathcal{H}' := \left\{ (x, a, y) \rightarrow y \cdot \frac{g^2(x, a)}{\pi^2(x, a)} f(x, a) \mid f \in \mathbb{B}_\omega(r) \cap (\mathcal{F} - \bar{\mu}) \right\}.$$ 

For functions $\ell, f, u : [0, 1] \rightarrow \mathbb{R}$, such that $f$ is contained in the bracket $[\ell, u]$, we can similarly construct

$$\tilde{\ell}(x, a, y) := y \cdot \frac{g^2(x, a)}{\pi^2(x, a)} \left\{ \ell(\phi(x, a)) 1_{y > 0} + u(\phi(x, a)) 1_{y < 0} - \bar{\mu}(x, a) \right\},$$

$$\tilde{u}(x, a, y) := y \cdot \frac{g^2(x, a)}{\pi^2(x, a)} \left\{ u(\phi(x, a)) 1_{y > 0} + \ell(\phi(x, a)) 1_{y < 0} - \bar{\mu}(x, a) \right\}.$$
It is easily observed that the function \((x, ay) \mapsto y \cdot \frac{\phi(x,a)}{p(x,a)}(f - \bar{f})(x, a)\) lies in the bracket \([\bar{t}, \bar{u}]\), and for any probability law \(Q\) on \(X \times A \times \mathbb{R}\), we have \(\|\bar{u} - \bar{t}\|_{L^2(Q)} \leq b^2 \cdot \|u - t\|_{L^2(Q)}\), where \(Q_\phi\) is the probability law of \(\phi(X, A)\) for \((X, A, Y) \sim Q\). Applying the bracketing bound yields

\[
\mathbb{E}\left[\sup_{h \in H'} \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i h(X_i, A_i, Y_i - \mu^*(X_i, A_i))\right] \leq \frac{c}{\sqrt{m}} \mathcal{J}_{\text{br}}(H', \| \cdot \|_{L^2}; [0, br]) \left\{ 1 + \frac{b \mathcal{J}_{\text{br}}(H', \| \cdot \|_{L^2}; [0, br])}{(br)^2 \sqrt{m}} \right\}
\leq cb\left(\sqrt{r/m} + \frac{1}{rm}\right).
\]

Denote \(Z_m := \sup_{h \in H'} \frac{1}{m} \sum_{i=1}^{m} \varepsilon_i h(X_i, A_i, Y_i - \mu^*(X_i, A_i))\). By a standard functional Bernstein bound (e.g., Thm. 3.8 in the book [Wai19]), we have the tail bound

\[
\mathbb{P}\left[Z_m \geq 2\mathbb{E}[Z_m] + t\right] \leq 2 \exp\left(\frac{-mt^2}{56(br)^2 + 4b^2t}\right) \quad \text{for any } t > 0.
\]

Combining with the expectation bound, we conclude that \(S_m = \sqrt{\mathbb{E}[Z_m^2]} \leq 2cb\left(\sqrt{r/m} + \frac{1}{rm}\right)\).

By definition of fixed point equation, the critical radius can be upper bounded \(s_m \leq c(b_2/m)^{1/3}\), and substituting this bound into Theorem 2 completes the proof of this corollary.

### B.5 Strong shattering for sparse linear models

In this section, we state and prove the claim from Example 3 about the size of the fat shattering dimension for the class of sparse linear models.

**Proposition 5.** There is a universal constant \(c > 0\) such that the function class \(\mathcal{F}_s\) of \(s\)-sparse linear models over \(\mathbb{R}^p\) satisfies the strong shattering condition (42) with fat shattering dimension \(D = cs \log(ep/s)\) at scale \(\delta = 1\).

**Proof.** We assume without loss of generality (adjusting constants as needed) that \(p/s = 2^k\) is an integer power of two. Our argument involves constructing a set of vectors by dividing the \(p\) coordinates into \(s\) blocks. Let the matrix \(A \in \{0, 1\}^{k \times 2^k}\) be such that by sequentially writing down the elements in \(j\)-th column, we get the binary representation of the integer \((j - 1)\), for \(j = 1, 2, \ldots, 2^k\). Let \((a_i^j)_{1 \leq i \leq k}\) be the row vectors of the matrix \(A\). For \(i \in [k]\) and \(j \in s\), we construct the \(p\)-dimensional data vector as \(x_{ij} = a_i \otimes e_j\), where the \(e_j \in \mathbb{R}^s\) is the indicator vector of \(j\)-th coordinate. The cardinality of this set is given by

\[
|\{x_{ij} : i \in [k], j \in s\}| = ks = \frac{1}{\log 2} \cdot s \log(p/s).
\]

It suffices to construct a hypercube packing for this set. Given a binary vector \(v \in \{0, 1\}^k\), we let \(J(v) \in \{1, 2, \ldots, 2^k\}\) such that the \(J(v)\)-th column of the matrix \(A\) is equal to \(v\). (Note that our construction ensures that such a column always exists and is unique.)

Given any binary vector \(\zeta \in \{0, 1\}^{k \times s}\), we construct the following vector:

\[
\beta_\zeta := \sum_{i=1}^{s} e(J(\zeta_1, \zeta_2, \ldots, \zeta_k)) \otimes e_i
\]

where the function \(e : [2^k] \to \mathbb{R}^{2^k}\) maps the integer \(j\) to the indicator vector of \(j\)-th coordinate.
We note that the vector $\beta$ is supported on $s$-coordinates, with absolute value of each coordinate bounded by 1. Moreover, our construction ensures that $i \in [k]$ and $j \in [s],$

$$\beta_i \top x_{i,j} = a_i^\top e(J(\zeta_{i,1}, \zeta_{i,2}, \ldots, \zeta_{i,k})) = \zeta_{i,j}.$$  

Therefore, we have planted a hypercube $\prod_{i \in [k], j \in [s]} (x_{i,j}, \{0,1\})$ in the graph of the function class $\mathcal{F}_{\text{sparse}}$, which completes the proof of the claim. \hfill \Box

## C Some elementary inequalities and their proofs

In this section, we collect some elementary results used throughout the paper, as well as their proofs.

### C.1 Bounds on conditional total variation distance

The following lemma is required for the truncation arguments used in the proofs of our minimax lower bounds. In particular, it allows us to make small modifications on a pair of probability laws by conditioning on good events, without inducing an overly large change in the total variation distance.

**Lemma 6.** Let $(\mu, \nu)$ be a pair of probability distributions over the same Polish space $\mathcal{S}$, and consider a subset $\mathcal{E} \subseteq \mathcal{S}$ such that $\min \{\mu(\mathcal{E}), \nu(\mathcal{E})\} \geq 1-\varepsilon$ for some $\varepsilon \in [0,1/4]$. Then the conditional distributions $(\mu \mid \mathcal{E})$ and $(\nu \mid \mathcal{E})$ satisfy the bound

$$d_{TV}(\mu, \nu) - 4\varepsilon \leq d_{TV}[(\mu \mid \mathcal{E}), (\nu \mid \mathcal{E})] \leq \frac{1}{1-\varepsilon} d_{TV}(\mu, \nu) + 2\varepsilon.$$  

**(92)**

**Proof.** Recall the variational definition of the TV distance as the supremum over functions $f : \mathcal{X} \to \mathbb{R}$ such that $\|f\|_\infty \leq 1$. For any such function $f$, we have

$$|\mathbb{E}_\mu[f(X)] - \mathbb{E}_\nu[f(X)]| \leq |\mathbb{E}_\mu[f(X)1_{X \in \mathcal{E}}] - \mathbb{E}_\nu[f(X)1_{X \in \mathcal{E}}]| + \mathbb{E}_\mu[|f(X)| 1_{\mathcal{E}^c}] + \mathbb{E}_\nu[|f(X)| 1_{\mathcal{E}^c}]$$

$$\leq \left| \frac{\mathbb{E}_\mu[f(X)1_{X \in \mathcal{E}}]}{\mu(\mathcal{E})} - \frac{\mathbb{E}_\nu[f(X)1_{X \in \mathcal{E}}]}{\nu(\mathcal{E})} \right| + \frac{1}{\mu(\mathcal{E})} - \frac{1}{\nu(\mathcal{E})} \mathbb{E}_\nu[|f(X)|] + 2\varepsilon$$

$$\leq d_{TV}((\mu \mid \mathcal{E}), (\nu \mid \mathcal{E})) + 4\varepsilon,$$

and re-arranging yields the lower bound (i).

On the other hand, in order to prove the upper bound (ii), we note that

$$|\mathbb{E}_{\mu \mid \mathcal{E}}[f(X)] - \mathbb{E}_{\nu \mid \mathcal{E}}[f(X)]| = \frac{1}{\mu(\mathcal{E})} \left| \mathbb{E}_\mu[f(X)1_{X \in \mathcal{E}}] - \mathbb{E}_\nu[f(X)1_{X \in \mathcal{E}}] \right| \mu(\mathcal{E})$$

$$\leq \frac{1}{\mu(\mathcal{E})} \left| \mathbb{E}_\mu[f(X)1_{X \in \mathcal{E}}] - \mathbb{E}_\nu[f(X)1_{X \in \mathcal{E}}] \right| + \mathbb{E}_\nu[|f(X)|] \cdot \left| \frac{\mu(\mathcal{E})}{\nu(\mathcal{E})} - 1 \right|$$

$$\leq \frac{1}{1-\varepsilon} d_{TV}(\mu, \nu) + 2\varepsilon,$$

which completes the proof. \hfill \Box
C.2 A second moment lower bound for truncated random variable

The following lemma is frequently used in our lower bound constructions.

**Lemma 7.** Let $X$ be a real-valued random variable with finite fourth moment, and define the $(2\rightarrow 4)$-moment constant $M_{2\rightarrow 4} := \sqrt{\mathbb{E}[X^4]/\mathbb{E}[X^2]}$. Then we have the lower bound

$$
\mathbb{E}\left[X^2 \cdot 1\{|X| \leq 2M_{2\rightarrow 4}\sqrt{\mathbb{E}[X^2]}\}\right] \geq \frac{1}{2}\mathbb{E}[X^2].
$$

**Proof.** Without loss of generality, we can assume that $\mathbb{E}[X^2] = 1$. Applying Cauchy–Schwarz inequality implies that

$$
\mathbb{E}\left[X^2 1\{|X| \geq 2M_{2\rightarrow 4}\}\right] \leq \sqrt{\mathbb{E}[X^4]} \cdot \sqrt{\mathbb{P}(\{|X| \geq 2M_{2\rightarrow 4}\})} \leq M_{2\rightarrow 4} \cdot \sqrt{\mathbb{P}(\{|X| \geq 2M_{2\rightarrow 4}\})}.
$$

By Markov’s inequality, we have

$$
\mathbb{P}(\{|X| \geq 2M_{2\rightarrow 4}\}) \leq \frac{\mathbb{E}[X^2]}{4M_{2\rightarrow 4}^2} = \frac{1}{4M_{2\rightarrow 4}^2}.
$$

Substituting back to above bounds, we conclude that $\mathbb{E}\left[X^2 1\{|X| \geq 2M_{2\rightarrow 4}\}\right] \leq \frac{1}{2}$, and consequently,

$$
\mathbb{E}\left[X^2 1\{|X| \leq 2M_{2\rightarrow 4}\}\right] = \mathbb{E}[X^2] - \mathbb{E}\left[X^2 1\{|X| \geq 2M_{2\rightarrow 4}\}\right] \geq \frac{1}{2},
$$

which completes the proof. □

D Empirical process results from existing literature

In this appendix, we collect some known bounds on the suprema of empirical processes.

D.1 Concentration for unbounded empirical processes

We use a concentration inequality for unbounded empirical processes. It applies to a countable class $\mathcal{F}$ of measurable functions, and a supremum of the form

$$
Z := \sup_{f \in \mathcal{F}} \left|\sum_{i=1}^{n} f(X_i)\right|
$$

where $\{X_i\}_{i=1}^{n}$ is a sequence of independent random variables such that $\mathbb{E}[f(X_i)] = 0$ for any $f \in \mathcal{F}$.

**Proposition 6** (Theorem 4 of [Ada08], simplified). There exists a universal constant $c > 0$ such that for any $t > 0$ and $\alpha \geq 1$, we have

$$
\mathbb{P}(Z > 2\mathbb{E}(Z) + t) \leq \exp\left(-\frac{t^2}{4v^2}\right) + 3 \exp\left(-c\left(\frac{t}{\max_{i=1,\ldots,n} \sup_{f \in \mathcal{F}} |f(X_i)| \psi_{1/\alpha}}\right)^{1/\alpha}\right),
$$

where $v^2 := \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \mathbb{E}[f^2(X_i)]$ is the maximal variance.
The countability assumption can be easily relaxed for separable spaces. A useful special case of Proposition 6 is by taking the class \( \mathcal{F} \) to be a singleton and letting \( \alpha = 1 \), in which case the bound becomes
\[
\frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbb{E}[f(X)] \leq c \sqrt{\text{var}(f(X))} \log(1/\delta) \frac{\log n}{n} + c \log n \cdot \|f(X)\|_{\psi_1} \cdot \log(1/\delta),
\]
with probability \( 1 - \delta \).

### D.2 Some generic chaining bounds

We also use a known generic chaining tail bound. It involves a separable stochastic process \((Y_t)_{t \in T}\) and a pair \((d_1, d_2)\) of metrics over the index set \( T \). We assume that there exists some \( t_0 \in T \) such that \( Y_{t_0} \equiv 0 \).

**Proposition 7** (Theorem 3.5 of Dirksen [Dir15]). Suppose that for any pair \( s, t \in T \), the difference \( Y_s - Y_t \) satisfies the mixed tail bound
\[
\mathbb{P} \left( |Y_s - Y_t| \geq \sqrt{ud_1(s, t)} + ud_2(s, t) \right) \leq 2e^{-u} \quad \text{for any } u > 0.
\]
Then for any \( \ell \geq 1 \), we have the moment bound
\[
\left\{ \mathbb{E} \left[ \sup_{t \in T} |Y_t|^\ell \right] \right\}^{1/\ell} \leq c \left( \gamma_2(T, d_1) + \gamma_1(T, d_2) \right) + 2 \sup_{t \in T} \left( \mathbb{E} |Y_t|^\ell \right)^{1/\ell},
\]
where \( \gamma_\alpha(T, d) \) is the generic chaining functional of order \( \alpha \) for the metric space \((T, d)\).

For a set \( T \) with diameter bounded by \( r \) under the metric \( d \), the generic chaining functional can be upper bounded in terms of the Dudley entropy integral as
\[
\gamma_\alpha(T, d) \leq c \mathcal{J}_\alpha(T, d; [0, r]) \quad \text{for each } \alpha \in \{1, 2\}
\]
(e.g., cf. Talagrand [Tal06]). Furthermore, suppose that the norm domination relation
\[
d_1(s, t) \leq a_0 d_2(s, t)
\]
holds true for any pair \( s, t \in T \). Let \( r_1, r_2 \) be the diameter of the set \( T \) under the metrics \( d_1, d_2 \), respectively. If we apply Proposition 7 to a maximal \( \delta \)-packing for the set \( T \) under metric \( d_1 \), we immediately have
\[
\left\{ \mathbb{E} \left[ \sup_{t \in T} |Y_t|^p \right] \right\}^{1/p} \leq c \left\{ \mathcal{J}_2(T, d_1; [\delta, r_1]) + \mathcal{J}_1(T, d_2; [\delta/a_0, r_2]) \right\} + \left\{ \mathbb{E} \sup_{s, t \in T \atop d_1(s, t) \leq \delta} |Y_s - Y_t|^p \right\}^{1/p} + 2 \sup_{t \in T} \left( \mathbb{E} |Y_t|^p \right)^{1/p}.
\]

### D.3 Bracketing entropy bounds

Finally, we use the following bracketing integral bound for empirical processes:

**Proposition 8** (Lemma 3.4.2 of [vdVW96]). Let \( \mathcal{F} \) be a class of measurable functions, such that \( \mathbb{E}[f^2(X)] \leq r^2 \) and \( |f(X)| \leq M \) almost surely for any \( f \in \mathcal{F} \). Given \( n \) i.i.d. samples \( \{X_i\}_{i=1}^{n} \), we have
\[
\mathbb{E} \left[ \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} f(X_i) - \mathbb{E}[f(X)] \right] \leq c \sqrt{n} \mathcal{J}_n(\mathcal{F}, \| \cdot \|_{L^2}; [0, r]) \left\{ 1 + \frac{M \mathcal{J}_n(\mathcal{F}, \| \cdot \|_{L^2}; [0, r])}{r^2 \sqrt{n}} \right\}.
\]