Weight systems for toric Calabi–Yau varieties and reflexivity of Newton polyhedra

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ABSTRACT

According to a recently proposed scheme for the classification of reflexive polyhedra, weight systems of a certain type play a prominent role. These weight systems are classified for the cases $n = 3$ and $n = 4$, corresponding to toric varieties with K3 and Calabi–Yau hypersurfaces, respectively. For $n = 3$ we find the well known 95 weight systems corresponding to weighted $\mathbb{P}^3$’s that allow transverse polynomials, whereas for $n = 4$ there are 184026 weight systems, including the 7555 weight systems for weighted $\mathbb{P}^4$’s. It is proven (without computer) that the Newton polyhedra corresponding to all of these weight systems are reflexive.
1 Introduction

A large class of Calabi–Yau manifolds can be described as hypersurfaces defined by transverse polynomials in weighted projective 4 spaces $\mathbb{P}^4$. These spaces could be classified \cite{1,2}, and the resulting list showed a remarkable property, known as mirror symmetry: To nearly each such variety there is a partner in the list such that the Hodge numbers $h^{1,1}$ and $h^{2,1}$ of the two varieties are interchanged. There are strong hints that mirror symmetry is not only a property concerning spectra, but a full symmetry of the corresponding conformal field theories. Batyrev \cite{3} has suggested a far more powerful technique for the construction of Calabi–Yau varieties: In his framework of reflexive polyhedra, mirror symmetry (at the level of Hodge numbers) is manifest, and it was checked by computer \cite{4,5} that the Newton polyhedra of all 7555 weight systems for $\mathbb{P}^4$’s are reflexive, i.e. that the older class of models is contained in this newer approach. This makes the following goals look desirable:

(i) The classification of reflexive polyhedra, and
(ii) an explanation of the reflexivity of Newton polyhedra corresponding to $\mathbb{P}^4$’s.

A big step towards the solution of problem (i) was taken in ref. \cite{6}: There it was shown that all reflexive polyhedra are bounded by polyhedra that can be described with the help of certain weight systems or combinations of weight systems, and an algorithm for the classification of these weight systems was proposed. In the present work, I present a far more efficient algorithm and use it to find all of the weight systems involved in the construction of reflexive polyhedra in $n \leq 4$ dimensions. Then I show that the Newton polyhedra corresponding to any of these weight systems (which contain the 7555 old ones as a small subset) are reflexive, thus solving problem (ii).

There is another recent development that should be mentioned here: It was found that, through black hole condensation, there seem to be physical transitions between string theories compactified on different Calabi–Yau manifolds \cite{7,8}, implying that there are connections between the various moduli spaces. In terms of toric geometry, such a transition may take place if a reflexive polyhedron describing some Calabi–Yau hypersurface is contained in the polyhedron describing some other Calabi–Yau hypersurface. In this way it was shown that the moduli spaces of all Calabi–Yau hypersurfaces of weighted $\mathbb{P}^4$’s are connected \cite{9,10}. The present work also provides a big step towards showing the connectedness of the moduli spaces of all toric Calabi–Yau varieties: Using the fact that the maximal Newton polyhedra corresponding to any of the weight systems found here are reflexive and that any reflexive polyhedron is contained in one of them, all that is left to do is to show the connectedness of the maximal Newton polyhedra, which should be a straightforward application of the tools developed in \cite{9,10}.

In section 2 I give a definition of reflexivity and a description of some of the main ideas of ref. \cite{6} used here. Then I describe the new algorithm for the construction of weight systems required for the classification of reflexive polyhedra and report the results of the implementation of this algorithm on a computer. In section 3 I give a proof that the Newton polyhedra corresponding to any such weight system (or combination of weight systems) is reflexive. I also give an explicit proof that the 7555 weight systems for weighted $\mathbb{P}^4$’s fall into the new set of weight systems.
2 The classification of weight systems

Consider a dual pair of lattices $\Gamma \simeq \mathbb{Z}^n$ and $\Gamma^*$ and their rational extensions $\Gamma_Q$ and $\Gamma_Q^*$; we denote the duality pairing $\Gamma_Q^* \times \Gamma_Q \to \mathbb{Q}$ by $\langle , \rangle$. A reflexive polytope $\Delta$ is an integer polytope in $\Gamma_Q$ (i.e., a polytope in $\Gamma_Q$ with vertices in $\Gamma$) with exactly one integer interior point (which we may choose to be the origin) such that

$$\Delta^* := \{ y \in \Gamma_Q^* : \langle y, x \rangle \geq -1 \forall x \in \Delta \}$$

is an integer polytope in $\Gamma_Q^*$. This is equivalent to the statement that all facets of $\Delta$ lie on hyperplanes of integer distance 1 to the interior point (a lattice hyperplane $H$ has integer distance $k$ to a lattice point $P$ if there are $k-1$ lattice hyperplanes parallel to $H$ between $H$ and $P$).

Let me briefly outline the basic ideas of the algorithm proposed in [6] for the classification of reflexive polyhedra: Given a reflexive polytope $\Delta$, we look for a set of hyperplanes $H_i$, $i = 1, \ldots, k$ carrying facets of $\Delta$ such that the $H_i$ define a (generically non–integer) bounded polyhedron $Q \subset \Gamma_Q$. We also assume that $Q$ is minimal in the sense that there is no polytope with the same properties but with a smaller number of facets. Each $H_i$ corresponds to some vertex $\mathbf{v}_i$ of $\Delta^*$, and $Q^*$ (the convex hull of $\mathbf{v}_i$) is an integer polytope with the interior point of $\Delta^*$ in its interior. In ref. [1] we have defined a redundant coordinate system where the $i$’th coordinate $P_i$ of some point $P \in \Gamma$ is given by its integer distance to the hyperplane $H_i$ (positive on the side of $\Delta$). In this way we get a natural embedding of $\Gamma$ in $\Gamma_Q \simeq \mathbb{Z}^k$. Whenever we use this type of coordinate system, we label the interior point, which has coordinates $(1, \ldots, 1)^T$, by $\mathbf{1}$. Making use of duality and the fact that the $H_i$ have distance 1 to $\mathbf{1}$, we see that $P_i = \langle \mathbf{v}_i, P \rangle + 1$. This coordinate system has several disadvantages: We require more coordinates than with a $\mathbb{Z}^n$ description, there is an ambiguity about the choice of $Q$, and even the lattice is not always completely determined. The advantages, however, seem to be greater: Our description naturally leads to pairing matrices between vertices of $\Delta^*$ and $\Delta$ which characterise dual pairs uniquely up to the choice of some sublattice, and even this finite ambiguity vanishes when we consider pairing matrices for all integer points of $\Delta^*$ and $\Delta$. In this way we avoid all the cumbersome considerations about equivalences of polyhedra that are mapped to each other by $GL(n, \mathbb{Z})$ transformations.

Then we have shown that $Q^*$ is composed of simplices (perhaps of lower dimension) that have $\mathbf{1}$ in their interiors. To each of these simplices there corresponds a weight system $\mathbf{q}$ in the following way: We define the weights $q_i$ to be the barycentric coordinates of $\mathbf{1}$ (the interior point of $\Delta^*$) w.r.t. the vertices $\mathbf{v}_i$ of the simplex, i.e. $\mathbf{1} = \sum q_i \mathbf{v}_i$ with $\sum q_i = 1$. The $q_i$ are positive because $\mathbf{1}$ is in the interior of the simplex defining them. Then $\langle \mathbf{1}, P \rangle = 0$ for $P \in \Gamma_Q$ implies $\sum q_i \langle \mathbf{v}_i, P \rangle = 0$, i.e. $\sum q_i (P_i - 1) = 0$ and $\sum q_i P_i = 1$. The number of independent equations of this type (i.e., the number of weight systems involved in the construction), is $k-n$. They define an $n$ dimensional lattice $\Gamma^n$ with $\Gamma \subseteq \Gamma^n \subset \Gamma^k$. Now it is easy to see that $\mathbf{1}$ is the only integer point in the interior of $Q$; for any point in the interior of $Q$ all coordinates have to be positive, i.e. for any integer point they have to be $\geq 1$. Comparing this with $\sum q_i (P_i - 1) = 0$ it is clear that this can be fulfilled only by $P_i = 1 \forall i$. For the construction of reflexive polyhedra we certainly need weight systems where $\mathbf{1}$ is in the interior not only of $Q$, but also in the interior of the maximal Newton polyhedron $\Delta_{\text{max}} = Q \cap \Gamma^n$ defined by a weight system. The classification algorithm proposed in [6] involved the consideration of minimal polytopes both in $\Gamma$ and in

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\[\Gamma^*\text{ and the construction of pairing matrices between these polytopes. There is, however, a far simpler way of constructing all allowed weight systems.}\]

The new algorithm is based on the following observation: Assume that a weight system \(q_1, \ldots, q_l\) allows a collection of points with coordinates \(x^i\), including the interior point with \(x^i = 1\) \(\forall i\). If these points fulfill an equation of the type \(\sum_{i=1}^{l} a_i x^i = 1\) with \(a \neq q\), then the weight system must also allow at least one point with \(\sum_{i=1}^{l} a_i x^i > 1\) and at least one point with \(\sum_{i=1}^{l} a_i x^i < 1\) to ensure that \(1\) is really in the interior of the maximal Newton polyhedron defined by \(q\). The latter inequality is the one that we actually use for the algorithm: Starting with the point \(1 = (1, \ldots, 1)^T\), we see that unless our weight system is \(q = (1/l, \ldots, 1/l)\), there must be at least one point with \(\sum_{i=1}^{l} x^i < l\). For \(l \leq 5\) there are only a few possibilities, and after choosing some point \(x_1\), we can look for some simple equation fulfilled by \(1\) and \(x_1\) and proceed in the same way.

For \(l = 3\) the classification is easily carried out by hand: Unless \(q = (1/3, 1/3, 1/3)\), we need at least one point with \(x^1 + x^2 + x^3 < 3\). As points where no coordinate is greater than 1 would be in conflict with the positivity of the weight system, we need the point \((2, 0, 0)^T\) (up to a permutation of indices). Now we note that \(1\) and \((2, 0, 0)^T\) both fulfill \(2x^1 + x^2 + x^3 = 4\), so \(q = (1/2, 1/4, 1/4)\) or we need a point with \(2x^1 + x^2 + x^3 < 4\). The only point allowed by this inequality which leads to a sensible weight system is \((0, 3, 0)^T\), leading to \(q = (1/2, 1/3, 1/6)\). One should note how easily we have reproduced all weight systems in comparison with the rather lengthy analysis in ref. [6].

For \(l = 4\) we can either get \(q = (1/4, 1/4, 1/4, 1/4)\) or we need a point with \(x^1 + x^2 + x^3 + x^4 < 4\). Up to permutations, all possibilities are exhausted by \((3, 0, 0, 0)^T, (2, 1, 0, 0)^T\) and \((2, 0, 0, 0)^T\). For the rest of the task the computer program requires only a few seconds. The result is a list of 99 weight systems which still have to be checked with respect to the property that the maximal Newton polyhedra defined by them must have 1 in their interiors.

For \(l = 5\) similar considerations show that, unless \(q_i = 1/5\) for \(i = 1, \ldots, 5\), the weight system must allow at least one of the points \((4, 0, 0, 0, 0)^T, (3, 1, 0, 0, 0)^T, (2, 2, 0, 0, 0)^T, (2, 1, 1, 0, 0)^T, (3, 0, 0, 0, 0)^T, (2, 1, 0, 0, 0)^T\) and \((2, 0, 0, 0, 0)^T\). Given these starting points, a C program running on an HP 735/125 required two days of system time to produce 200653 candidates for weight systems.

The next task is to find out whether the maximal Newton polyhedra defined by the weight systems really have 1 in their interiors. It is straightforward to construct all points allowed by some \(q\). Then one could in principle construct all facets and check that 1 does not lie on one of them or on the wrong side of one of them. I have used a different approach: Starting with \(l\) points of the maximal Newton polyhedron which are independent in \(Q^l\), it is easy to calculate the barycentric coordinates of 1 w.r.t these points. If all of the barycentric coordinates are positive (in this case we can identify them with the \(q\) system introduced in [6]), 1 is in the interior of the simplex defined by the \(l\) points. If some of the barycentric coordinates are negative, one can substitute the point corresponding to the smallest coordinate by a point on the other side of the hyperplane defined by the remaining points and try the same procedure again. The same strategy can be used in cases where only \(l - 1\) of the starting points are independent. If one of the barycentric coordinates is 0 while all others are positive, the points

\[\begin{align*}
&\text{In this paper I always denote the dimension of } \Gamma \text{ by } n \text{ and the number of weights in a weight system by } l; \\
&\text{if } \Gamma^n \text{ is defined by a single weight system, then } l = n + 1
\end{align*}\]
corresponding to the positive coordinates define a codimension one hyperplane with 1 in its interior, so one has to check whether there is at least one point on either side of this hyperplane. Depending on the starting points, this strategy produced results more or less quickly. The best good strategy is to use points with maximal exponents as starting points.

It turns out that exactly 95 of the 99 weight systems for \( l = 4 \) have the property that 1 is in the interior of the corresponding maximal Newton polyhedron. These are precisely the well known 95 weight systems for weighted \( \mathbb{P}^4 \)'s that have K3 hypersurfaces [11, 12].

For \( l = 5 \) the situation is completely different: The 7555 weight systems corresponding to weighted \( \mathbb{P}^4 \)'s that allow transverse polynomials are just a small subset of the 184026 different weight systems whose maximal Newton polyhedra contain 1. Later I will give a proof that in arbitrary dimensions weight systems corresponding to weighted \( \mathbb{P}^n \)'s have the property that their maximal Newton polyhedra contain 1. The simplest weight systems that do not correspond to weighted \( \mathbb{P}^4 \)'s are \((1, 1, 3, 4)/10 \) and \((1, 1, 1, 4, 5)/12 \). Note that in the latter system the first four weights are of Fermat type, whereas the last weight is such that no monomial of the type \( X^i(X^5)^\lambda \), which would be necessary for transversality, is allowed. The corresponding maximal Newton polyhedron has vertices \( V_1 = (12, 0, 0, 0, 0)^T \), \( V_2 = (0, 12, 0, 0, 0)^T \), \( V_3 = (0, 0, 12, 0, 0)^T \), \( V_4 = (0, 0, 0, 3, 0)^T \), \( V_5 = (0, 0, 0, 2, 0)^T \), \( V_6 = (0, 2, 0, 0, 0)^T \) and \( V_7 = (0, 0, 2, 0, 0)^T \). The facets correspond to the hyperplanes \( H_i : x^i = 0 \) for \( i = 1, \ldots, 5 \) and \( H_6 : 2x^4 + 3x^5 = 6 \) (spanned by the \( V_j \) with \( j \geq 4 \)). As 1 fulfils \( 2x^4 + 3x^5 = 5 \), it has integer distance 1 to \( H_6 \) and the maximal Newton polyhedron is reflexive. Its vertex pairing matrix, i.e. the matrix \( A_{ij} = \langle \nabla, V_j \rangle + 1 \) (with \( \nabla \) corresponding to \( H_i \) for \( i = 1, \ldots, 6 \)) is

\[
\begin{pmatrix}
12 & 0 & 0 & 2 & 0 & 0 \\
0 & 12 & 0 & 0 & 2 & 0 \\
0 & 0 & 12 & 0 & 0 & 2 \\
0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 2 & 2 & 2 \\
6 & 6 & 6 & 0 & 0 & 0 \\
\end{pmatrix}
\]

(2)

The zeroes in the matrix indicate incidence relations: \( A_{ij} = 0 \) means that \( V_j \) lies on \( H_i \).

With the help of a program that can test any weight system with respect to the property of having 1 in the interior of the maximal Newton polyhedron, there is an easy way to check the program for the construction of all such weight systems: Given a positive integer \( d \), one can consider all weight systems of the type \( q_i = n_i/d \) with \( n_i \in \mathbb{N} \) and \( \sum_{i=1}^d n_i = d \) and apply the interior point check. I have done this up to \( d = 230 \) for \( l = 5 \), resulting in approximately 50000 allowed weight systems which are identical with the ones the original program produced.

All of the weight systems found in this way might be interesting for theoretical reasons, in particular for a better understanding of the connection between older approaches to weighted \( \mathbb{P}^n \)'s and the toric framework and for dealing with the question of whether the moduli space of all Calabi–Yau varieties allowing a description in terms of reflexive polyhedra is connected. For our classification program, however, we need only those weight systems where every coordinate hyperplane is spanned by points of the maximal Newton polyhedron. It is easy to write a program that checks for this property. 58 of the 95 weight systems for \( l = 4 \) pass the test (see table II in the appendix). For \( l = 5 \) only approximately one fifth of the weight systems is
such that each coordinate hyperplane is spanned by points of the maximal Newton polyhedron. Among the 7555 weights corresponding to weighted $\mathbb{P}^4$’s slightly more than half have this property.

As an illustration for the fact that weight systems without the above mentioned property are redundant in the classification scheme, consider the system (40, 41, 486, 1134, 1701)/3402. Its maximal Newton polyhedron $\Delta_{\text{max}}$ is the simplex whose vertices are the columns of the matrix

$$
\begin{pmatrix}
83 & 1 & 0 & 0 & 0 \\
2 & 82 & 0 & 0 & 0 \\
0 & 0 & 7 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

(3)

The lines of this matrix correspond to the points in $\Gamma$ dual to the coordinate hyperplanes. Obviously the first two lines cannot correspond to vertices of $\Delta^*_{\text{max}}$. The lacking vertices of $\Delta^*_{\text{max}}$ are easily found to be (84, 0, 0, 0, 0) and (0, 84, 0, 0, 0). Thus the vertex pairing matrix for $\Delta^*_{\text{max}}$ and $\Delta_{\text{max}}$ is given by

$$
\begin{pmatrix}
84 & 0 & 0 & 0 & 0 \\
0 & 84 & 0 & 0 & 0 \\
0 & 0 & 7 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 2
\end{pmatrix}
$$

(4)

which obviously corresponds to the Fermat type weight system $(1/84, 1/84, 1/7, 1/3, 1/2)$.

We finish this section with a table of the numbers of various types of weight systems for $l = 5$. In this table, “span” means the weight systems where each coordinate hyperplane is spanned by points of the maximal Newton polyhedron, $\mathbb{P}^4$ means that the weights correspond to weighted $\mathbb{P}^4$’s that allow transverse polynomials, and in addition I have given the numbers of weight systems containing and not containing a weight of 1/2.

|       | $\mathbb{P}^4$, $q_5 = 1/2$ | $\mathbb{P}^4$, $q_5 < 1/2$ | $\mathbb{P}^4$ | $q_5 = 1/2$ | $q_5 < 1/2$ | total |
|-------|---------------------------|-----------------------------|---------------|-------------|-------------|-------|
| span  | 1309                      | 2860                        | 4169          | 14872       | 23858       | 38730 |
| total | 2390                      | 5165                        | 7555          | 97036       | 86990       | 184026|

Table I: Numbers of various types of weight systems with 5 weights

Tables III and IV in the appendix contain small sublists of the complete list of weight systems (with small and large $d$).

3 Some results derived without a computer

The main aim of this section is to show that the maximal Newton polyhedra corresponding to weights or combinations of weights constructed in the way reported in the previous section are all reflexive. As a prerequisite, we first need a technical lemma.

Lemma 1: Consider an integer pyramid $\text{Pyr}$ of height $h \geq 2$ in a lattice $\Gamma \simeq \mathbb{Z}^4$ and the pyramid $\text{Pyr}_{\text{double}}$, which has the same peak and the same shape as $\text{Pyr}$, but double height $2h$. Then $\text{Pyr}_{\text{double}}$ contains integer lattice points which are neither in $\text{Pyr}$ nor in the base of
**Remarks:** The same proof works for lattices \(\mathbb{Z}^3\) (otherwise triangulate the base and pick any simplex). Then the base of \(Pyr_{\text{double}}\) is a simplex in \((2\mathbb{Z})^3\) and we may choose its vertices to be \(\vec{e}_1, 2\vec{e}_2, 2\vec{e}_3\). The peak has coordinates \((2x, 2y, 2z, 2h)^T\). The points in \(Pyr_{\text{double}}\) can be parameterized as

\[
\lambda \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \nu \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \rho \begin{pmatrix} 2x \\ 2y \\ 2z \\ 2h \end{pmatrix}
\]

with \(\lambda \geq 0, \cdots, \rho \geq 0\) and \(\lambda + \mu + \nu + \rho \leq 1\). With \(x' = x \mod h, y' = y \mod h, z' = z \mod h\), (with \(0 \leq x' < h\) etc.) it is easily checked that the points with \((\lambda_1, \mu_1, \nu_1, \rho_1) = (h - x', h - y', h - z', 1)/2h\) and \((\lambda_{h-1}, \mu_{h-1}, \nu_{h-1}, \rho_{h-1}) = (x', y', z', h - 1)/2h\) are integer points at heights \(1\) and \(h - 1\), respectively. Clearly all parameters \(\lambda_1, \cdots\) are positive. With \(\lambda_1 + \mu_1 + \nu_1 + \rho_1 + \lambda_{h-1} + \mu_{h-1} + \nu_{h-1} + \rho_{h-1} = 2\), at least one of the inequalities \(\lambda_1 + \mu_1 + \nu_1 + \rho_1 \leq 1\) and \(\lambda_{h-1} + \mu_{h-1} + \nu_{h-1} + \rho_{h-1} \leq 1\) must be fulfilled. This means that at least one of the two points is a point of \(Pyr_{\text{double}}\), and because of the height it is neither in \(Pyr\) nor in the base. \(\square\)

**Remarks:** The same proof works for lattices \(\mathbb{Z}^n\) with \(n < 4\). For \(n = 5\) there is the following counterexample: Let the base again be given by \(\vec{0}\) and \(2\vec{e}_i\) and the peak by \((2, 2, 2, 2, 4)^T\). With \(h = 2\), a point fulfilling the criterion of the lemma would have to be at height \(1\). With the same ansatz as in the proof, we would have \(\rho = 1/4\) and \(\lambda \geq 0, \mu \geq 0, \cdots\) would have to be at least \(1/4\), resulting in \(\lambda + \cdots + \rho \geq 5/4\) and a point outside \(Pyr_{\text{double}}\). Thus there is no integer point between the bases of \(Pyr\) and \(Pyr_{\text{double}}\).

**Theorem:** Four or lower dimensional maximal Newton polyhedra with \(1\) in their interior are reflexive.

**Proof:** Consider a collection of points in a maximal Newton polyhedron \(\Delta_{\text{max}}\) spanning a hyperplane at distance \(h \geq 2\) from \(1\). We take these points to define the base of the pyramid \(Pyr\) of lemma 1 and \(1\) as the peak. Then \(Pyr_{\text{double}}\) lies in \((\Delta_{\text{max}})_{\text{double}} \subseteq \{ x \in \Gamma^n : x^i \geq -1\} \). Only the base of \(Pyr_{\text{double}}\) can intersect with the boundary of \((\Delta_{\text{max}})_{\text{double}}\), so the integer points of \(Pyr_{\text{double}}\) that are not in the base must have nonnegative coordinates, i.e. they must be in \(\Delta_{\text{max}}\). Thus the lemma ensures that there are points in the cone defined by \(Pyr\) which are outside of \(Pyr_{\text{double}}\), but within \(\Delta_{\text{max}}\). This means that the hyperplane defined by the base of \(Pyr\) is not a bounding hyperplane of \(\Delta_{\text{max}}\). Therefore every bounding hyperplane of \(\Delta_{\text{max}}\) must be at distance \(1\), i.e. \(\Delta_{\text{max}}\) is reflexive. \(\square\)

**Remark:** The theorem holds not only for maximal Newton polyhedra defined by a single weight system with \(l = n + 1\), as mainly considered in this paper, but also for maximal Newton polyhedra defined by several weight systems with \(l < n + 1\) involved in the classification scheme of [3].

**Examples:** The weight system \(q_i = 1/5, i = 1, \cdots, 5\) contains the points \((2, 0, 0, 0, 3)^T\), \((0, 2, 0, 3)^T\), \((0, 0, 2, 3)^T\), \((0, 0, 0, 2, 3)^T\) defining the hyperplane \(x^5 = 3\) (at distance 2 to \(1\)). The base of \(Pyr_{\text{double}}\) is the convex hull of the points \((3, -1, -1, -1, 5)^T\), \((-1, 3, -1, -1, 5)^T\), \((-1, -1, 3, -1, 5)^T\), \((-1, -1, -1, 3, 5)^T\), and the “height one” points \((1, 0, 0, 0, 4)^T\), \((0, 1, 0, 0, 4)^T\), \((0, 0, 1, 0, 4)^T\), \((0, 0, 0, 1, 4)^T\) ensure that \(x^5 = 3\) is not a bounding hyperplane. In the same way the hyperplane \(x^5 = 4\) (at distance 3 to \(1\)), with the points \((1, 0, 0, 0, 4)^T\), \((0, 1, 0, 0, 4)^T\), \((0, 0, 1, 0, 4)^T\), \((0, 0, 0, 1, 4)^T\), gives rise to a pyramid of height 6 with base points \((1, -1, -1, -1, 7)^T\), \((1, -1, -1, 1, 7)^T\), \((1, 1, -1, -1, 7)^T\), \((1, 1, 1, -1, 7)^T\), \((1, 1, 1, 1, 7)^T\).
defined by the columns of $A$, because the diagonal elements are involved and without the help of a computer, we still have to show that the list of 7 555 weights for $\mathbb{P}^4$'s is contained in our complete list of weights whose maximal Newton polyhedra have 1 in the interior. The analogous statement holds in any dimension and also for abelian orbifolds (with the transversality condition applied to polynomials that are invariant under the twist group); although it looks quite obvious to anyone who has worked with weighted $\mathbb{P}^n$'s for some time, the proof turns out to be rather technical.

**Lemma 2:** Maximal Newton polyhedra corresponding to weighted $\mathbb{P}^n$'s or abelian orbifolds of weighted $\mathbb{P}^n$'s that allow transverse polynomials have 1 in their interiors.

**Proof:** A transverse polynomial contains monomials of the type $M^i = (X^i)^{a_i}$ or $M^i = (X^i)^{a_i}X^j$ (with $a_i \geq 2$) for each $i$. These monomials define points which can be arranged in the matrix

$$A = \begin{pmatrix} a_1 & x & x & x & \cdots \\ x & a_2 & x & x & \cdots \\ x & x & a_3 & x & \cdots \\ \cdots \end{pmatrix},$$

where in each column at most one of the $x$'s can be 1, whereas all others are zero. Let us first see that $A$ is regular: Assuming it were singular, we could find a nontrivial vanishing linear combination of its lines, i.e. we would have $\lambda \neq 0$ with $\sum_i \lambda_i A^i = 0$. The specific form of our matrix implies that $\lambda_i = 0$ if $M^i = (X^i)^{a_i}$ and $a_i \lambda_i + \lambda_{p(i)} = 0$ if $M^i = (X^i)^{a_i}X^j$. Iterating this, we get $a_i a_{p(i)} \lambda_i - \lambda_{p(p(i))} = 0$ etc. At some point either $\lambda_{p(\cdots(i)\cdots)} = 0$ or $p(\cdots(i)\cdots) = i$, showing that indeed $\lambda_i = 0$. Thus $A$ is regular and we can solve $\sum_i A^i i^j = 1$ for $i^j$. Then $\sum_j i^j = \sum_i (A^{-1})^j_i = \sum_i q_i = 1$, showing that the $i^j$ are the barycentric coordinates of 1 with respect to the columns of $A$. If all of the $i^j$ are positive, 1 is in the interior of the simplex defined by the columns of $A$. Let us now assume that not all of the $i^j$ are positive: Let $i^j \leq 0$ for $j \in I_-$ and $i^j > 0$ for $j \in I_+$, with $I_- \cup I_+ = \{1, \cdots, n\}$. Now sum the equations defining the $i^j$ over $i \in I_-$ to get

$$\sum_{i \in I_-} \sum_j A^i j i^j = |I_-|$$

and split $\sum_j$ in $\sum_{j \in I_-} + \sum_{j \in I_+}$. Then

$$\sum_{i \in I_-} \sum_{j \in I_-} A^i j i^j \leq 2 \sum_{j \in I_-} i^j$$

because the diagonal elements are involved and

$$\sum_{i \in I_-} \sum_{j \in I_+} A^i j i^j \leq \sum_{j \in I_+} i^j$$

because each column contains at most a single 1. Thus the l.h.s. of eq. (7) fulfills

$$\sum_{i \in I_-} \sum_j A^i j i^j \leq 2 \sum_{j \in I_-} i^j + \sum_{j \in I_+} i^j = 1 + \sum_{j \in I_-} i^j \leq 1$$

(-1, 1, -1, 1, 7)^T, (-1, -1, 1, 1, 7)^T, (-1, -1, 1, 1, 1)^T. This time the integer point we are looking for is (0, 0, 0, 0, 1)^T.
with equality iff \( \sum_{j \in I_-} q^j = 0 \) and \( \sum_{i \in I_-} \sum_{j \in I_+} A^i_j q^j = 1 \). Therefore \( I_- \) can have at most 1 element corresponding to some \( q^j = 0 \), and the corresponding line may contain no 0, i.e. up to permutations our matrix is

\[
A = \begin{pmatrix}
a_1 & 1 & 1 & 1 & \cdots \\
0 & a_2 & 0 & 0 & \cdots \\
0 & 0 & a_3 & 0 & \cdots \\
\cdots
\end{pmatrix}.
\] (11)

1 is in the interior of the \( n - 1 \) dimensional simplex defined by all columns except the first. This simplex lies on the hyperplane \( x^1 = 1 \). The first point is above this hyperplane, and the transversality condition ensures that there is also a point with \( x^1 = 0 \), i.e. a point below this hyperplane. Therefore 1 is again in the interior of the maximal Newton polyhedron. \( \square \)

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Appendix: Various Tables

In table II I list the 95 weight systems for $l = 4$. The last column indicates whether a weight system has the property that coordinate hyperplanes are spanned by points of the maximal Newton polyhedron.

Table III contains all weight systems for $l = 5$ with $d \leq 20$, whereas table IV contains the weight systems with the largest values of $d$ for the cases $q_5 = 1/2$ and $q_5 < 1/2$. The last columns indicate whether a weight system corresponds to a weighted $\mathbb{P}^4$ that allows transverse polynomials. All weights in table III have the property that coordinate hyperplanes are spanned by points of the maximal Newton polyhedron, whereas none of the weight systems in table IV fulfil this criterion.

| $n_1$ | $n_2$ | $n_3$ | $n_4$ | d  | span | $n_1$ | $n_2$ | $n_3$ | $n_4$ | d  | span | $n_1$ | $n_2$ | $n_3$ | $n_4$ | d  | span |
|-------|-------|-------|-------|----|------|-------|-------|-------|-------|----|------|-------|-------|-------|-------|----|------|
| 1     | 1     | 1     | 1     | 4  | y    | 2     | 3     | 5     | 7     | 17 | y    | 5     | 6     | 7     | 9     | 27 | n    |
| 1     | 1     | 1     | 2     | 5  | y    | 3     | 4     | 5     | 6     | 18 | y    | 2     | 5     | 9     | 11    | 27 | n    |
| 1     | 1     | 2     | 2     | 6  | y    | 1     | 4     | 6     | 7     | 18 | y    | 4     | 6     | 7     | 11    | 28 | n    |
| 1     | 1     | 1     | 3     | 6  | y    | 2     | 3     | 5     | 8     | 18 | y    | 3     | 4     | 7     | 14    | 28 | y    |
| 1     | 1     | 2     | 3     | 7  | y    | 2     | 3     | 4     | 9     | 18 | y    | 1     | 4     | 9     | 14    | 28 | n    |
| 1     | 2     | 2     | 3     | 8  | y    | 1     | 3     | 5     | 9     | 18 | y    | 5     | 6     | 8     | 11    | 30 | n    |
| 1     | 1     | 2     | 4     | 8  | y    | 1     | 2     | 6     | 9     | 18 | y    | 3     | 4     | 10    | 13    | 30 | n    |
| 1     | 2     | 3     | 3     | 9  | y    | 3     | 4     | 5     | 7     | 19 | y    | 4     | 5     | 6     | 15    | 30 | y    |
| 1     | 1     | 3     | 4     | 9  | y    | 2     | 5     | 6     | 7     | 20 | y    | 2     | 6     | 7     | 15    | 30 | n    |
| 1     | 2     | 3     | 4     | 10 | y    | 3     | 4     | 5     | 8     | 20 | y    | 1     | 6     | 8     | 15    | 30 | n    |
| 1     | 2     | 2     | 5     | 10 | y    | 2     | 4     | 5     | 9     | 20 | y    | 2     | 3     | 10    | 15    | 30 | y    |
| 1     | 1     | 3     | 5     | 10 | y    | 2     | 3     | 5     | 10    | 20 | y    | 1     | 4     | 10    | 15    | 30 | y    |
| 1     | 2     | 3     | 5     | 11 | y    | 1     | 4     | 5     | 10    | 20 | y    | 4     | 5     | 7     | 16    | 32 | n    |
| 2     | 3     | 3     | 4     | 12 | y    | 3     | 5     | 6     | 7     | 21 | n    | 2     | 5     | 9     | 16    | 32 | n    |
| 1     | 3     | 4     | 4     | 12 | y    | 1     | 5     | 7     | 8     | 21 | n    | 3     | 5     | 11    | 14    | 33 | n    |
| 2     | 2     | 3     | 5     | 12 | y    | 2     | 3     | 7     | 9     | 21 | y    | 4     | 6     | 7     | 17    | 34 | n    |
| 1     | 2     | 4     | 5     | 12 | y    | 1     | 3     | 7     | 10    | 21 | n    | 3     | 4     | 10    | 17    | 34 | n    |
| 1     | 2     | 3     | 6     | 12 | y    | 2     | 4     | 5     | 11    | 22 | n    | 7     | 8     | 9     | 12    | 36 | n    |
| 1     | 1     | 4     | 6     | 12 | y    | 1     | 4     | 6     | 11    | 22 | y    | 3     | 4     | 11    | 18    | 36 | n    |
| 1     | 3     | 4     | 5     | 13 | y    | 1     | 3     | 7     | 11    | 22 | n    | 1     | 5     | 12    | 18    | 36 | n    |
| 2     | 3     | 4     | 5     | 14 | y    | 3     | 6     | 7     | 8     | 24 | n    | 5     | 6     | 8     | 19    | 38 | n    |
| 2     | 2     | 3     | 7     | 14 | y    | 4     | 5     | 6     | 9     | 24 | y    | 3     | 5     | 11    | 19    | 38 | n    |
| 1     | 2     | 4     | 7     | 14 | y    | 1     | 6     | 8     | 9     | 24 | y    | 5     | 7     | 8     | 20    | 40 | n    |
| 3     | 3     | 4     | 5     | 15 | y    | 3     | 4     | 7     | 10    | 24 | y    | 3     | 4     | 14    | 21    | 42 | y    |
| 2     | 3     | 5     | 5     | 15 | y    | 2     | 3     | 8     | 11    | 24 | n    | 2     | 5     | 14    | 21    | 42 | n    |
| 1     | 3     | 5     | 6     | 15 | y    | 3     | 4     | 5     | 12    | 24 | y    | 1     | 6     | 14    | 21    | 42 | y    |
| 1     | 3     | 4     | 7     | 15 | y    | 2     | 3     | 7     | 12    | 24 | y    | 4     | 5     | 13    | 22    | 44 | n    |
| 1     | 2     | 5     | 7     | 15 | y    | 1     | 3     | 8     | 12    | 24 | y    | 3     | 5     | 16    | 24    | 48 | n    |
| 1     | 4     | 5     | 6     | 16 | y    | 4     | 5     | 7     | 9     | 25 | n    | 7     | 8     | 10    | 25    | 50 | n    |
| 2     | 3     | 4     | 7     | 16 | y    | 2     | 5     | 6     | 13    | 26 | n    | 4     | 5     | 18    | 27    | 54 | n    |
| 1     | 3     | 4     | 8     | 16 | y    | 1     | 5     | 7     | 13    | 26 | n    | 5     | 6     | 22    | 33    | 66 | n    |
| 1     | 2     | 5     | 8     | 16 | y    | 2     | 3     | 8     | 13    | 26 | n    |       |       |       |       |     |      |

Table II: Weights for $l = 4$
| $n_1$ | $n_2$ | $n_3$ | $n_4$ | $n_5$ | $d$ | $P^+$ |
|-------|-------|-------|-------|-------|-----|-------|
| 1     | 1     | 1     | 1     | 1     | 5   | y     |
| 1     | 1     | 1     | 1     | 2     | 6   | y     |
| 1     | 1     | 1     | 2     | 2     | 7   | y     |
| 1     | 1     | 1     | 1     | 3     | 7   | y     |
| 1     | 1     | 2     | 2     | 2     | 8   | y     |
| 1     | 1     | 1     | 2     | 3     | 8   | y     |
| 1     | 1     | 1     | 1     | 4     | 8   | y     |
| 1     | 1     | 2     | 2     | 3     | 9   | y     |
| 1     | 1     | 1     | 3     | 3     | 9   | y     |
| 1     | 1     | 1     | 2     | 4     | 9   | y     |
| 1     | 2     | 2     | 2     | 3     | 10  | y     |
| 1     | 2     | 3     | 3     | 10   | y   |       |
| 1     | 1     | 2     | 2     | 4     | 10  | y     |
| 1     | 1     | 1     | 3     | 4     | 10  | n     |
| 1     | 1     | 1     | 2     | 5     | 10  | y     |
| 1     | 2     | 2     | 3     | 3     | 11  | y     |
| 1     | 1     | 2     | 3     | 4     | 11  | y     |
| 1     | 1     | 2     | 2     | 5     | 11  | n     |
| 1     | 1     | 1     | 3     | 5     | 11  | y     |
| 2     | 2     | 2     | 3     | 3     | 12  | y     |
| 1     | 2     | 3     | 3     | 3     | 12  | y     |
| 1     | 2     | 2     | 3     | 3     | 12  | y     |
| 1     | 1     | 3     | 3     | 4     | 12  | y     |
| 1     | 1     | 2     | 4     | 4     | 12  | y     |
| 1     | 2     | 2     | 2     | 5     | 12  | y     |
| 1     | 1     | 2     | 3     | 5     | 12  | y     |
| 1     | 1     | 1     | 4     | 5     | 12  | n     |
| 1     | 1     | 2     | 2     | 6     | 12  | y     |
| 1     | 1     | 1     | 3     | 6     | 12  | y     |
| 1     | 2     | 3     | 3     | 4     | 13  | y     |
| 1     | 1     | 3     | 4     | 4     | 13  | y     |
| 1     | 2     | 2     | 3     | 5     | 13  | y     |
| 1     | 1     | 3     | 3     | 5     | 13  | y     |
| 1     | 1     | 2     | 4     | 5     | 13  | n     |
| 1     | 1     | 2     | 3     | 6     | 13  | y     |
| 1     | 1     | 1     | 4     | 6     | 13  | y     |
| 2     | 2     | 3     | 3     | 4     | 14  | y     |
| 1     | 2     | 3     | 4     | 4     | 14  | n     |
| 2     | 2     | 2     | 3     | 5     | 14  | n     |
| 1     | 2     | 3     | 3     | 5     | 14  | n     |
| 1     | 2     | 2     | 4     | 5     | 14  | y     |
| 1     | 1     | 3     | 4     | 5     | 14  | n     |
| 1     | 2     | 2     | 3     | 6     | 14  | y     |
| 1     | 1     | 2     | 4     | 6     | 14  | y     |
| 1     | 2     | 2     | 2     | 7     | 14  | y     |
| $n_1$ | $n_2$ | $n_3$ | $n_4$ | $n_5$ | d | $\mathbb{F}_l$ |
|-------|-------|-------|-------|-------|---|----------------|
| 1     | 1     | 3     | 5     | 7     | 17 | y              |
| 1     | 2     | 3     | 3     | 8     | 17 | y              |
| 1     | 1     | 3     | 4     | 8     | 17 | y              |
| 1     | 1     | 2     | 5     | 8     | 17 | y              |
| 3     | 3     | 3     | 4     | 5     | 18 | n              |
| 2     | 3     | 4     | 4     | 5     | 18 | n              |
| 2     | 3     | 3     | 5     | 5     | 18 | y              |
| 1     | 3     | 4     | 5     | 5     | 18 | n              |
| 2     | 3     | 3     | 4     | 6     | 18 | y              |
| 1     | 3     | 4     | 4     | 6     | 18 | n              |
| 2     | 2     | 3     | 5     | 6     | 18 | y              |
| 1     | 3     | 3     | 5     | 6     | 18 | y              |
| 1     | 2     | 4     | 5     | 6     | 18 | n              |
| 1     | 2     | 3     | 6     | 6     | 18 | y              |
| 1     | 1     | 4     | 6     | 6     | 18 | y              |
| 2     | 2     | 3     | 4     | 7     | 18 | y              |
| 1     | 3     | 3     | 4     | 7     | 18 | n              |
| 1     | 2     | 4     | 4     | 7     | 18 | n              |
| 1     | 2     | 3     | 5     | 7     | 18 | n              |
| 1     | 1     | 4     | 5     | 7     | 18 | n              |
| 1     | 2     | 2     | 6     | 7     | 18 | n              |
| 1     | 1     | 3     | 6     | 7     | 18 | n              |
| 2     | 2     | 3     | 3     | 8     | 18 | y              |
| 1     | 2     | 3     | 4     | 8     | 18 | n              |
| 1     | 2     | 2     | 5     | 8     | 18 | y              |
| 1     | 1     | 3     | 5     | 8     | 18 | n              |
| 1     | 1     | 2     | 6     | 8     | 18 | n              |
| 2     | 2     | 2     | 3     | 9     | 18 | y              |
| 1     | 2     | 3     | 3     | 9     | 18 | y              |
| 1     | 2     | 2     | 4     | 9     | 18 | y              |
| 1     | 1     | 3     | 4     | 9     | 18 | n              |
| 1     | 1     | 2     | 5     | 9     | 18 | n              |
| 1     | 1     | 1     | 6     | 9     | 18 | y              |
| 3     | 3     | 4     | 4     | 5     | 19 | y              |
| 2     | 3     | 4     | 5     | 5     | 19 | n              |
| 1     | 3     | 4     | 5     | 6     | 19 | y              |
| 3     | 3     | 4     | 7     | 19    | n  | y              |
| 1     | 3     | 4     | 4     | 7     | 19 | n              |
| 2     | 2     | 3     | 5     | 7     | 19 | n              |
| 1     | 3     | 3     | 5     | 7     | 19 | n              |
| 1     | 2     | 4     | 5     | 7     | 19 | y              |
| 1     | 2     | 3     | 6     | 7     | 19 | n              |
| 1     | 1     | 4     | 6     | 7     | 19 | n              |
| 1     | 3     | 3     | 4     | 8     | 19 | y              |

| $n_1$ | $n_2$ | $n_3$ | $n_4$ | $n_5$ | d | $\mathbb{F}_l$ |
|-------|-------|-------|-------|-------|---|----------------|
| 1     | 2     | 3     | 5     | 8     | 19 | n              |
| 1     | 1     | 3     | 6     | 8     | 19 | y              |
| 1     | 2     | 3     | 4     | 9     | 19 | y              |
| 1     | 2     | 2     | 5     | 9     | 19 | y              |
| 1     | 1     | 3     | 5     | 9     | 19 | y              |
| 1     | 1     | 2     | 6     | 9     | 19 | y              |
| 3     | 4     | 4     | 4     | 5     | 20 | y              |
| 3     | 3     | 4     | 5     | 5     | 20 | y              |
| 2     | 4     | 4     | 5     | 5     | 20 | y              |
| 2     | 3     | 5     | 5     | 5     | 20 | y              |
| 1     | 4     | 5     | 5     | 5     | 20 | y              |
| 2     | 2     | 5     | 5     | 5     | 20 | y              |
| 1     | 3     | 5     | 5     | 6     | 20 | y              |
| 1     | 2     | 5     | 5     | 6     | 20 | n              |
| 1     | 3     | 4     | 4     | 7     | 20 | n              |
| 2     | 3     | 3     | 5     | 7     | 20 | n              |
| 2     | 2     | 4     | 5     | 7     | 20 | n              |
| 1     | 3     | 4     | 5     | 7     | 20 | n              |
| 1     | 2     | 5     | 5     | 7     | 20 | n              |
| 2     | 2     | 3     | 6     | 7     | 20 | y              |
| 1     | 2     | 4     | 6     | 7     | 20 | y              |
| 1     | 1     | 5     | 6     | 7     | 20 | n              |
| 2     | 3     | 3     | 4     | 8     | 20 | y              |
| 1     | 3     | 4     | 4     | 8     | 20 | y              |
| 1     | 3     | 5     | 5     | 8     | 20 | n              |
| 2     | 2     | 3     | 5     | 8     | 20 | n              |
| 1     | 3     | 4     | 4     | 8     | 20 | y              |
| 2     | 3     | 3     | 4     | 9     | 20 | y              |
| 1     | 3     | 5     | 5     | 9     | 20 | y              |
| 1     | 2     | 4     | 4     | 9     | 20 | y              |
| 1     | 2     | 5     | 5     | 9     | 20 | y              |
| 1     | 1     | 4     | 5     | 9     | 20 | y              |
| 1     | 2     | 2     | 6     | 9     | 20 | y              |
| 1     | 3     | 4     | 4     | 7     | 19 | n              |
| 2     | 2     | 3     | 3     | 10    | 20 | y              |
| 1     | 2     | 3     | 4     | 10    | 20 | y              |
| 1     | 1     | 4     | 4     | 10    | 20 | y              |
| 1     | 1     | 3     | 5     | 10    | 20 | y              |
| 1     | 1     | 2     | 6     | 10    | 20 | y              |

Table III: Weights for $l = 5$ and $d \leq 20$
Table IV: Weights for $l = 5$ and large $d$ (for $q_5 = 1/2$ and $q_5 < 1/2$)
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