A COMPREHENSIVE REVIEW ON GRAPH PEBBLING AND RUBBLING

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Abstract. A graph pebbling is a combinatorial game played over a fixed graph and a pebbling move is the method of removing two pebbles from one endpoint and sets one pebble at the other endpoint, while the remaining pebble is dropped. A graph’s rubbling number is the smallest number necessary to ensure that any vertex can be reached from any pebble distribution of pebbles. A vertex is reachable if a pebble can be placed using pebbling or rubbling moves. In this paper we have attempted to give a summary of how graph pebbling have undergone a lot of challenges to derive many strong results in proof complexity. A comprehensive review of the existing works in pebbling and rubbling is carried out in a sequential order.

Keywords: Graph pebbling, The No Cycle Lemma, Graham’s Conjecture, Optimal Pebbling, Cover Pebbling, Graph Rubbling, $t$-Pebbling Number

1. Introduction

Graph pebbling was first proposed by Lagarias and Saks as a method to address a specific problem in number theory. Chung introduced the concept as an alternative test to a scientific problem in additive number theory. Pebbling theory is used as a design of network optimization for the transport of consumable goods in which some of it is used during the travel. Some of the oil it transports during its transit is used by oil tankers. The pebbles can be considered as oil barrels. Pebbling, peg solitaire, chip frying and checker jumping are remarkable connective games on graphs. Everyone has its roots in fields including number theory, statistical mechanics, and economics and of course recreational mathematicians. Graph pebbling was started in 1989 by Chung [1] and is practiced on a connected graph. Let us assume $q$ pebbles are shared to the vertices in the graph $G$. The pebbling stage includes eliminating 2 pebbles from a vertex and then placing one of the pebbles at a neighboring vertex. Pebbling game is a prototype for moving of consumable items, moving army troops and so on. Distribution $q$ is considered ‘u solvable’ in the event that $u$ consists of a pebble after certain sequence of pebbling beginning from $q$. For the vertex $u$ of graph $G$ considering $f(G, u)$to be the minimum $m$ such that each distribution of $m$ pebbles on $G$can be solved by $u$. The distribution $q$ can be solved if each vertex
can be reached under $q$. For the graph $G$ pebbling number $f(G)$ is defined as the minimum integer $q$ in a manner that for a distribution of the pebbles $(q)$ to vertices of graph, a single pebble can be transferred to a particular target (root) vertex.

**Figure 1.** Graph pebbling number in a cube

Figure 1 presents the pebbling numbers to be assigned to each vertex in a cube.

$$f(G) = \max f(G, u); u \in V(G)$$

The following Figure 2 shows pebbles located at the vertices of a graph after two pebbling moves. Number present at every vertex shows the number of pebbles placed at the vertex.

**Figure 2.** A maximum path partition of a tree
In 1989, Lemke and Kleitman showed a hypothesis of Erdos and Lemke from additive number theory. [2]

It is discussed that for any set \( N = \{n_1, n_2, \ldots, n_r\} \) for \( r \) natural numbers, there is a non-empty index set \( I \subset \{1, 2, 3, \ldots, r\} \) such that \( \sum_{i \in I} k_i \). The conjecture asserted that the additional condition \( \sum_{i \in I} k_i \leq \text{lcm}(r, n_1, n_2, \ldots, n_r) \) could also be needed [7]. Although a suitable proof was developed, Lemke and Kleitman gave a piece of specific evidence and included a substantial amount of case analysis. Saks and Lagarias planned to start graph pebbling as an instinctive medium to prove the theorem. Specifically, Erdos and Lemke’s addition would follow if the assumed pebbling number for the Cartesian path product was right. The use of pebbling as a medium to build the conjecture would be clear only if we append the correction made by Clarke et al. In this complete paper, G is a connected graph and \( f(G) \) denotes pebbling number for \( G \). There exist various known results associated with \( f(G) \).

1. \( f(P_{g+1}) = 2^g \)
2. \( f(K_n) = n \) for a complete graph \( K_n \).
3. If \( G' \) is a spanning connected sub-graph of \( G \) then \( f(G') \geq f(G) \).
4. \( f(D_{2k+1}) = 2^k, k \geq 2 \)
5. \( f(D_{2k+1}) = 2^{(2k+1)/2} + 1, k \geq 1 \).

We now define the greatest path partition \( W \) of a tree \( S \). Let us assume the partition \( W = (W_1, W_2, \ldots, W_m) \) if the edges of \( S \) into paths \( W_1, W_2, \ldots, W_m \), such that \( w_i \geq w_{i+1} \) where \( w_i = (W_i) \) Any selection of root vertex \( v \) introduces an edge orientation \( S \) for every path \( W_i \). The direction on \( W_i \) defines the root \( r_i \) of \( W_i \) that may not or may be the endpoint of \( W_i \). If \( r_i \) is the endpoint of \( W_i \), then, it can be said that \( W_i \) is redirected. \( W_i \) is referred to as an \( r \) path partition of \( T \) provided every path \( W_i \) is \( r \) directed, and called as path partition if it’s a \( v \) path partition for some \( v \). The partition of path \( W \) majorises the other say \( W' \) if its path sequence lengths majorises, that is, if \( r_j > r'_j \) where \( j = \min \{i: r_i \neq r'_i\} \). A partition of path (resp. \( r \) path) \( S \) is maximum (resp. \( r \) maximum) if there is no other path (resp. \( r \) path) partition majorising it. Let \( \{r_1, r_2, \ldots, r_m\} \) be a non-increasing length of the path sequence for a maximum partition \( W = (W_1, W_2, \ldots, W_m) \) of a tree \( T \). For example, a maximum path partition of the tree in figure 2 is \( W = \{(c, e, f, g, c, r), \{a, b, c\}, \{d, b\}, \{h, f\}\} \). The pebbling number \( f(S, u) \) for vertex \( u \) in a tree is \( 2^{a_1} + 2^{a_2} + 2^{a_3} - r + 1 \) where \( a_1, a_2, a_3 \) is the sequence of the path size in a maximum path partition of \( S \). Chung defined the 2 pebbling and \( t \) pebbling properties for a graph \( G \). It can be said that the property of 2-pebbling is satisfied by the graph \( G \) if the two pebbles could be transferred to a particular vertex while total number of starting pebbles is a minimum of \( 2f(G) - r + 1 \) where \( r \) indicates vertex number with a minimum of one pebble. The property of 2-pebbling is satisfied by the \( n \)-cube and paths [3].

In rubbling, unlike pebbling the total of two pebbles from some neighbor \( v \) of a vertex \( v \) of graph \( G \) is removed from the move, and one pebble is placed on \( v \). In case of a series of moves in rubbling starting from the pebble distribution until reaching \( v \), the vertex can be reached from pebble distribution. Rubbling number of the graph \( G \) is the number of the pebbles for any distribution of total pebbles where any given vertex can be reached. The optimal pebbling number is represented as the least value such that all the vertices can be reached due to the existence of distribution of pebbles [4–6]. The cost involved in sequencing the pebbling steps depends on the number of steps traversed. From different theorems, including Weight Argument, it has been observed that every graph satisfies fractional pebbling number. The Janus Graph project has been developed from the rubble of Titan Graph, [7].

The availability of an additional move-in rubbling makes it different from pebbling. In strict rubbling transfer, two pebbles are removed along with collecting the removed pebbles from two separate vertices, followed by placing 1 pebble in one of the common neighboring vertices. The rubbling transfer can be either a normal rubbling or pebbling move. On removing pebbling
moves, we get optimal rubbling moves. A solvable distribution is found on using $2^{diam}(G)$, but in general, few pebbles are sufficient to develop a distribution that can be solved [8].

2. Literature Review

(i) Gilbert & Tarjan (1978) examines two variations of a one-person pebble game played on directed graphs which is considered as register allocation model. [9]

(ii) Pippenger (1982) showed the survey of some unknown results on pebbling to show the new results along with representing some open problems. [10]

(iii) Chung et al. (1987) evaluated the class formation problems while adjusting the data structures. [11]

(iv) Kirousis & Papadimitriou (1986) introduced the node-searching method where an edge is cleared by putting searchers at endpoints. [12]

(v) Megiddo et al. (1988) provided the structural characterization of the graphs proposed by T. Parsons. [13]

(vi) Moews (1998) explained accurately the optimal pebbling number of cartesian power of a random graph in the same manner. [14]

(vii) Clarke et al. (1997) presented a technical improvement of Chung’s alternate proof of a number theoretic result of Lemke and Kleitman through pebbling. [15]

(viii) Hurlbert (2000) proved the result incorporating a more generic product along with providing a generalization of Chung. [16, 17]

(ix) S.S.Wang (2001) explained the concept of pebbling and Graham’s Conjecture. [18, 19]

(x) H L Fu, C L Shiue (2002) proved the optimal pebbling number of a caterpillar graph. [20]

(xi) Herscovici (2003) proved Graham’s conjecture where the cycles for a variety of graphs are shown. [21–23]

(xii) A Bekmetjev, G Brightwell, A Czygrinow, G Hurlbert explained Thresholds for families of multi sets, with an application to graph pebbling. [22]

(xiii) Czygrinow & Hurlbert (2004) proved that the graphs on n greater than 9 vertices with a minimum degree of n/2 and are Class 0 with bipartite nature in greater than 36 vertices in every part having a minimum degree of m/2 +1. [24]

(xiv) Elledge & Hurlbert (2004) proved the conjecture in the case of Abelian groups. Graph pebbling is used to prove that for every element in the sequence of a finite Abelian group there exists a nonempty sub sequence. [25]

(xv) Vuong & Wyckoff (2004) gave some essential conditions for the distribution of pebbles covering a weight function of a connected graph. [26]

(xvi) Watson (2005) discussed the complications of the graph pebbling along with providing a comparison with the traditional pebbling and newly introduced cover pebbling. The challenge of finding the cover pebbling number for a random demand is found to be NP-hard. [27]

(xvii) Wyels & Friedman (2005) provided a simple discussion on Pachter, Snevily and Voxman’s determination of the optimal pebbling number of paths for establishing the optimal pebbling number of cycles. [28, 29]

(xviii) Gartner et al. (2005) introduced the notion of domination cover pebbling achieved by associating graph cover pebbling with domination theory in graphs. [30]

(xix) Milans & Clarke (2006) showed the computational complexity of computing along with providing characterization for unordered set of pebbling moves to form a valid sequence of pebbling moves. [31]
Bunde et al. (2007) computed the pebbling number in special families including prisms and Mobius ladders. [32]

Shrenu T & Theran (2008) discussed a new algorithm where the pebble game consists of colors and this helps to understand the features of the family of the sparse graphs and algorithmic solutions related to tree decomposition of graphs. [20]

Blasiak & Schmitt (2008) discussed elaborately the degree sum condition for a graph to check it as C0S 0. [33]

Lee & Streinu (2008) discussed the characterization of the sparse graphs through a family of simple, elegant and efficient algorithms known as (k-l)n pebble games. [34]

Bellford & Seiben (2009) discussed the rubbling and optimal rubbling number of some graphs along with showing that the Graham’s conjecture never hold for rubbling numbers. [5]

Bekmetev & Cusack (2009) considered the algorithms to determine the solvability of a pebbling configuration of graphs having diameter two. [35]

Shiue and Fu (2009) found the optimal pebbling number of the caterpillars. [36]

Surynek (2009) discussed a case when the graph modeling the bi connected environment and reviews the popular problems of moving pebbles on graphs with multi-robot path planning. [37]

Sieben (2010) presented an algorithm to find the pebbling number of weighted graphs. This algorithm is used along with graph simplifications to recognize the regular pebbling number of all connected graphs with nine vertices. [38]

Postle et al. (2013) improved the (Bukh) bound by discussing the pebbling number of a graph of diameter three on n vertices at [3n/2]+2 and it is the best. Further an asymptotic bound was obtained for the pebbling number of graphs of diameter four. [39]

Gyori et al. (2017) discussed the lower bounds on optimal pebbling as well as rubbling numbers by the distance domination number. [40, 41]

Curtis et al. (2018) used the block structure of the graphs to evaluate pebbling numbers and the accurate number for pebbling complete graphs. Along with this, the upper bound is provided for k-pebbling of diameter -two graphs which can further be extended for blocks having a diameter of two. [42, 43]

Herscovici et al. (2018) proved the generalization of Graham’s conjecture for optimal pebbling random sets of target distributions. Bounds are provided on optimal pebbling numbers of products of full graphs and the optimal t-pebbling numbers for particular products are searched. [22]

Hurlbert (2017) developed a tool named the Weight Function Lemma to compute the upper bounds along with computing the accurate values for the pebbling numbers when linear optimization assists them. [44]

Beeler et al. (2019) discussed the initial positioning of a minimum total number of pebbles on the vertices such that no vertex receives multiple pebbles by performing a sequence of pebbling and rubbling. The 1-restricted optimal rubbling numbers are determined for Cartesian products. [45, 46]

Sreedevi. S (2019) determined the pebbling number of Watkin Snark graph by constructing a Watkin flower snark. [47]

3. Methods
It has been observed that the potential of graph pebbling to show similar impact as shown to positional games by graphs including Cops-and-Robbers and Chip-Firing along with the high possibility of applying to structural graph theory as well as theoretical computer science [48].
the pebbling step, weight is removed from one vertex and placed at an adjacent vertex. Pebbling in graphs aims to put the starting weight at any random root to ensure that there is sufficient fuel, money, energy and information at the location where the network exists. However, rubbling fails to provide the benefits offered by pebbling and hence is restricted in applications only to the Janus Graph project which has been developed from the rubble of Titan Graph. The Janus Graph only runs on multiple platforms, most suitably Cassandra, H Base and Berkeley. D.B.

| Reference Number | Method                                                                 | Results                                                                 | Observations                                                                 |
|------------------|------------------------------------------------------------------------|-------------------------------------------------------------------------|----------------------------------------------------------------------------|
| 7                | Graph pebbling                                                         | It involves a simple graph without loops and multiple edges.            | More research needs to be done. Very limited application is available.     |
| 8                | Pebbling move                                                          | It provides the effective graphs through the 2-pezbling property       | The pebbling move needs to be explored more for real-time applications.    |
| 1                | An algorithm for determining pebbling number for weighted graphs       | It finds the regular pebbling number of all connected graphs with at most nine vertices. | Simplification of the graph with the open ear is very difficult.          |
| 2                | A pebble distribution on a graph                                        | The graph is simple and connected with a minimum number of edges.       | The construction fails on changing the starting graph.                    |
| 16               | A method for transferring the pebble to a particular vertex from any initial distribution of 2 pebbles within a hypercube. | The n-cube satisfies the property of 2-pebbling and the paths possess the property of 2-pebbling | The number of original pebbling number and the calculated pebbling number is not the same. |
| 7                | Existing graph pebbling methods                                       | The inspiration derives from the transfer of information from initial points to final points, allowing them to track the entire graph. | The first weight needs to be placed at any predetermined root to ensure the flow of money, fuel, information or energy at the network location. |
| 31               | A Class 0 graph where pebbling numbers is the same as the number of vertices. | This paper enhanced the property of class 0 on the random graph threshold [49] | Since it is unknown if a threshold exists for any arbitrary value so Class 0 is taken as monotone property. |
| 21               | Two graphs are pebbled in diameter.                                    | Determined that the reachability of the vertex is NP-hard, for graphs including of diameter 2. [50] | More research needs to be done in graph pebbling through the diameter.    |
4. Conclusion
The review gives a brief of how graph pebbling have undergone a lot of challenges to derive many strong results in proof complexity. The results are presented in an ordered fashion and some of the topics are analyzed, which can pave the way for future research. The objectives are achieved by explaining how the encoding pebble games played on graphs can be shown to obtain trade-off properties for proof length and space from pebbling time-space trade-offs for these graphs. While these connections have turned out to be very useful, the reductions are far from tight. It would be interesting to clarify the true relationship between pebble games and pebbling formulas for resolution-based proof systems.

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