Cayley (Di)Hypergraphs *

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Abstract

A Cayley (di)hypergraph is a hypergraph that its automorphism group contains a subgroup acting regularly on (hyper)vertices. In this paper, we study Cayley (di)hypergraph and its automorphism group.

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1 Introduction

The basic idea of the hypergraph concept is to consider such a generalization of a graph in which any subset of a given set may be an edge rather than two-element subsets.

In what follows we provide basic hypergraph definitions which generalize the respective graph concepts. Let \( V = \{v_1, v_2, \cdots, v_n\} \) be a finite set, and let \( E = \{e_1, e_2, \cdots, e_m\} \) be a family of subsets of \( V \). The pair \( \mathcal{H} = (V, E) \) is called a hypergraph with vertex set \( V \) also denoted by \( V(\mathcal{H}) \), and with edge set \( E \) also denoted by \( E(\mathcal{H}) \). Sometimes, the hypergraph \( \mathcal{H} = (V, E) \) is called a set-system.

\(|V| = n\) is called the order of the hypergraph, written also as \( n(\mathcal{H}) \). The elements \( v_1, v_2, \cdots, v_n \) are called the vertices and the sets \( e_1, e_2, \cdots, e_m \) are called the edges (hyperedges). The number of edges is usually denoted by \( m \) or \( m(\mathcal{H}) \).

Sometimes we will omit the indices when denoting the vertices and edges if this evidently does not lead to misunderstanding. To include the most general case (it may happen in some algorithms), we assume that the set of vertices \( V \) and/or the family \( E \) may be empty. A hypergraph which contains no vertices and no edges is called the empty set. Some edges may also be empty sets. Some edges may be the subsets of some other edges; in this case they are called included. In some cases some edges may coincide; they are then called multiple. A hypergraph is called simple if it contains no included edges. Hence simple hypergraphs do not have empty and multiple edges. Simple hypergraphs are also known as Sperner families.

In a hypergraph, two vertices are said to be adjacent if there is an edge \( e \in E \) that contains both vertices. The adjacent vertices are sometimes called neighbor to

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each other, and all the neighbors for a given vertex \( v \) are called the **neighborhood** of \( v \) in a graph or hypergraph. The neighborhood of \( v \) is denoted by \( N(v) \). Two edges are said to be **adjacent** if their intersection is not empty. If a vertex \( v_i \in V \) belongs to an edge \( e_j \in E \), then we say that they are **incident** to each other and denoted by \((v_i, e_j)\). Moreover, \((v_i, e_j)\) is called an **arc**. \( D(H) \) denote all the arcs of \( H \). As one can see, as in graph theory, the adjacency is referred to the elements of the same kind (vertices vs vertices, or edges vs edges), while the incidence is referred to the elements of different kind (vertices vs edges).

\[ E(v), v \in V, \] will denote all the edges containing the vertex \( v \). The number \(|E(v)|\) is called the **degree of the vertex** \( v \), the number \(|e_i|\) is called the **degree (size, cardinality) of the edge** \( e_i \). A hypergraph in which all vertices have the same degree \( k \geq 0 \) is called \( k \)-**regular**. A hypergraph in which all edges have the same degree \( r \geq 0 \) is called \( r \)-**uniform**. For a positive integer \( r \leq |V| \), we use \( V^{(r)} \) to denote the set of all \( r \)-**uniform hypergraph** (or \( r \)-**hypergraph**) with vertex set \( V \) and edge set \( E \) is a pair \((V, E)\), where \( E \) is a subset of \( V^{(r)} \).

An edge of a hypergraph which contains no vertices is called an **empty edge**. The degree of an empty edge is trivially 0. A vertex of a hypergraph which is incident to no edges is called an **isolated vertex**. The degree of an isolated vertex is trivially 0. An edge of cardinality 1 is called a **singleton (loop)**, a vertex of degree 1 is called a **pendant vertex**.

A simple hypergraph \( H \) with \(|e_1| = 2\) for each \( e_i \in E \) is thus a **simple graph**, maybe with isolated vertices.

**Directed Hypergraphs.** A directed hypergraph (or dihypergraph) \( H = (V, E) \) is a pair, where \( V \) is a (finite) set of vertices and \( E \) is a set of arcs. An arc \( a \in E \) is an ordered pair \((v, e)\), where \( v \in e \) and \( e \) is a subset of \( V \).

**Paths and Cycles.** In a hypergraph \( H = (V, E) \), an alternating sequence

\[ \mu = v_0 e_0 v_1 e_1 v_2 \cdots v_{t-1} e_{t-1} v_t \]

of distinct vertices \( v_0, v_1, v_2, \ldots, v_{t-1} \) and distinct edges \( e_0, e_1, e_2, \ldots, e_{t-1} \) satisfying \( v_i, v_{i+1} \in e_i, i = 0, 1, \cdots, t-1 \), is called a **path** connecting the vertices \( v_0 \) and \( v_t \), or, equivalently, \((v_0, v_t)\)-**path**; it is called a **cycle** if \( v_t = v_0 \). The value of \( t \) is called the **length of the path/cycle** respectively.

**Connected hypergraphs.** The hypergraph \( H = (V, E) \) is called **connected** if for any pair of its vertices there is a path connecting them. If \( H \) is not connected, then it consists of two or more **connected components** each of which is a connected hypergraph. An isolated vertex, a vertex incident to loops only and an empty edge are also considered connected components.

**Isomorphic hypergraphs.** Two (not necessarily simple) hypergraphs \( H = (V, E) \) and \( H' = (V', E') \) are called **isomorphic**, written \( H \cong H' \), if there is a one-to-one correspondence between the sets \( V \) and \( V' \) and a one-to-one correspondence between the sets \( E \) and \( E' \) such that for every vertex \( v \in V \) and for every edge \( e \in E \) we have that \( v \in e \) if and only if for the corresponding vertex \( v' \in V' \) and the corresponding
edge $e' \in E'$ the inclusion $v' \in e'$ holds. Using graph terminology, one could say that two hypergraphs are isomorphic if and only if their bipartite representations are isomorphic as graphs (preserving respective bipartition). An isomorphism between two hypergraphs without repeated hyperedge is a bijection

$$\sigma : V \rightarrow V'$$

such that

$$\text{for any } \{u_1, u_2, \cdots, u_r\} = e \in E \iff \{u_1^\sigma, u_2^\sigma, \cdots, u_r^\sigma\} = e^\sigma \in E'.$$

An isomorphism between two (di)hypergraphs without repeated arc is a bijection

$$\sigma : V \rightarrow V'$$

such that $|D(H)| = |D(H')|$ and for any

$$\{u_1, \{u_1, u_2, \cdots, u_r\}\} = (u_1, e) \in D(H)$$

$$\iff \{u_1^\sigma, \{u_1^\sigma, u_2^\sigma, \cdots, u_r^\sigma\}\} = (u_1^\sigma, e^\sigma) \in D(H').$$

The set of automorphisms of a hypergraph $H = (V, E)$ is a group under composition. We call it the automorphism group of $H$ and we denote it by Aut ($H$). A simple hypergraph is vertex transitive if for any pair of vertices $x, y \in V$ there is an automorphism $\sigma$ such that $x^\sigma = y$. A hypergraph is edge transitive if for any pair of edges $e_i, e_j \in E$ there is an automorphism $\sigma$ such that $e_i^\sigma = e_j$. In the same way a hypergraph is arc transitive if for any pair of arcs $(v, e_i), (u, e_j) \in D(H)$ there is an automorphism $\sigma$ such that $(v, e_i)^\sigma = (v^\sigma, e_i^\sigma) = (u, e_j)$. It is symmetric if it is arc transitive. Clearly, arc transitive 2-uniform hypergraph is an arc transitive graph.

Now, we assume that the hypergraphs are simple in this paper.

## 2 Cayley (di)hypergraphs

In this section, we define Cayley (di)hypergraphs and provide some basic properties.

We denote the set $\{1, 2, \cdots, n\}$ by $[n]$ for any positive integer $n$. By $|g|$, $|H|$, we denote the order of element $g$ and subgroup $H$ of a group $G$, respectively, and $|G : H|$ the index of $H$ in $G$.

**Definition 2.1** Let $G$ be a group and let $X$ be a set of subsets $x_1, x_2, \cdots, x_d$ of $G$ such that $1_G \in x_i$ for each $i \in [d]$, where $1_G$ is the identity element in $G$. A Cayley dihypergraph $CD(G, X)$ has vertex set $G$ and arc set $\{(g, xg) \mid g \in G, x \in X\}$. Moreover, $X$ is called a Cayley hyperset.

Obviously, the 2-uniform Cayley dihypergraph $CD(G, X)$ is the Cayley digraph $\text{Cay} (G, S)$, where $S = \bigcup_{i=1}^d x_i \setminus \{1_G\}$.

The following facts are basic for Cayley dihypergraphs.
Proposition 2.2 Let \( \mathcal{H} = \text{CD}(G, X) \) be a Cayley dihypergraph of \( G \) with respect to \( X \). Then

1. \( \text{Aut}(\mathcal{H}) \) contains the right regular representation \( G_R \) of \( G \), so \( \mathcal{H} \) is vertex transitive.
2. \( \mathcal{H} \) is connected if and only if \( G = \langle \bigcup_{x \in X} x \rangle \).
3. \( \mathcal{H} \) is undirected if and only if \( X = \{ xg^{-1} \mid x \in X, g \in x \} \).

Proof (1) Recall that \( G_R = \langle \hat{h} : x \mapsto xh \mid x, h \in G \rangle \leq \text{Sym}(G) \) and that \( G_R \) is a regular subgroup of \( \text{Sym}(G) \). So it suffices to prove that every element \( \hat{h} \in G_R \) is an automorphism of \( \mathcal{H} \). Let \((g, xg)\) be an arc of \( \mathcal{H} \), where \( g \in G \) and \( x \in X \). Then

\[
(g, xg)^h = (g^h, (xg)^h) = (gh, xgh).
\]

And so \((g, xg)^h \in D(\mathcal{H})\), that is \( \hat{h} \) is an automorphism of \( \mathcal{H} \).

Since \( G_R \) is transitive on \( G = V(\mathcal{H}) \), we have \( \mathcal{H} \) is vertex transitive.

(2) Since \( \mathcal{H} \) is vertex transitive, \( \mathcal{H} \) is connected if and only if for any vertex \( g \) there is a path between \( 1_G \) and \( g \), if and only if there are some elements \( s_i \in x_i \) and \( x_i \in X \) for \( i \in [n] \) such that

\[
1_G, x_1, s_1, x_2s_1, s_2s_1, \ldots, x_ns_{n-1} \cdots s_2s_1, s_n s_{n-1} \cdots s_2s_1 = g.
\]

This is true if and only if \( G = \langle \bigcup_{x \in X} x \rangle \).

(3) For any \( x \in X \), since \( \mathcal{H} \) is undirected and \((1_G, x) \in D(\mathcal{H})\), we have \((g, x) \in D(\mathcal{H})\) for each \( g \in X \). So there exists \( y \in X \) such that \((g, yg) = (g, x)\). Hence \( yg = x \), and so \( xg^{-1} \in X \). It follows that \( X = \{ xg^{-1} \mid x \in X, g \in x \} \).

Note that \( \mathcal{H} \) is vertex transitive. Thus to prove \( \mathcal{H} \) is undirected, it is sufficient to prove that for every element \( x \in X \) and each \( g \in x \), \((g, x)\) is an arc. Since \( y = xg^{-1} \in X \), \((g, x) = (y, yg)\) is an arc. Therefore, \( \mathcal{H} \) is undirected. \( \square \)

Definition 2.3 Let \( X \) be a Cayley hyperset of group \( G \). \([X] = \{ xg^{-1} \mid x \in X, g \in x \}\) is called a Cayley closure of \( X \). If \( X = [X] \), then \( X \) is called Cayley closed.

Definition 2.4 Let \( 1_G \in x \) be a subset of group \( G \), and let \( X = \{ xg^{-1} \mid g \in x \} \). Then \( X \) is called single Cayley closed.

The following two propositions are trivial.

Proposition 2.5 Let \( X \) be a Cayley hyperset of group \( G \). Suppose every elements of \( X \) is a subgroup of \( G \). Then \( X = [X] \) and the Cayley dihypergraph \( \mathcal{H} = \text{CD}(G, X) \) is undirected. Moreover, \(|E(\mathcal{H})| = \sum_{x \in X} |G : x|\).
Proposition 2.6 Let $G$ be a group, and let $X$ be Cayley closed. Then the Cayley dihypergraph $\mathcal{H} = CD(G, X)$ is undirected.

Definition 2.7 Let $1_G \in x$ and $1_G \in y$ be two subsets of group $G$. $x$ and $y$ are called Cayley equivalent if there exists $g \in x$ such that $xg^{-1} = y$. Let $Y$ be a Cayley hyperset of $G$. $Y$ is called non-Cayley equivalent if any two different elements of $Y$ are not Cayley equivalent.

If $|Y| = 1$, then we also say $Y$ is non-Cayley equivalent.

Definition 2.8 Let $X$ be a Cayley hyperset. $Y$ is called non-Cayley equivalent hyperset of $X$ if
1) $Y \subseteq X$,
2) $Y$ is non-Cayley equivalent,
3) and for any $x \in X$ there exists $y \in Y$ such that $x$ and $y$ are Cayley equivalent.

Definition 2.9 Let $G$ be a group and let $Y$ be a non-Cayley equivalent hyperset. $\Gamma = CH(G, Y)$ is an undirected hypergraph with vertex set $V(\Gamma) = G$ and edge set $E(\Gamma) = \{xg \mid x \in Y, g \in G\}$.

Remark 2.10 In the paper [1], Lee and Kwon introduced Cayley hypergraphs which are similar to definition 2.9.

Proposition 2.11 Let $G$ be a group and let $Y$ be a non-Cayley equivalent hyperset. Let $\Gamma = CH(G, Y)$. Then $\Gamma$ is the undirected Cayley dihypergraph $\mathcal{H} = CD(G, \{Y\})$.

Definition 2.12 Let $\mathcal{H} = (V, D)$ be a directed hypergraph. The underlying hypergraph of $\mathcal{H}$ is the hypergraph $\mathcal{H}^u = (V, D^u)$ where for every arc $(x, e)$ the edge $e \in D^u$, and every edge of $\mathcal{H}^u$ has a corresponding arc in $\mathcal{H}$.

Proposition 2.13 Let $\mathcal{H} = CD(G, X)$ be a Cayley dihypergraph and let $Y_1$ and $Y_2$ be non-Cayley equivalent hypersets of $X$. Let $\mathcal{H}_1 = CH(G, Y_1)$ and $\mathcal{H}_2 = CH(G, Y_2)$. Then
1) $\mathcal{H}_1 = \mathcal{H}_2$,
2) the underlying hypergraph of $\mathcal{H}$ is $\mathcal{H}^u = \mathcal{H}_1$.

Proof 1) Since $Y_1$ and $Y_2$ are non-Cayley equivalent hypersets of $X$, for any $y_1 \in Y_1$, there exists $y_2 \in Y_2$ such that $y_1 = y_2a_2^{-1}$ and $y_2 = y_1a_1^{-1}$ for some $a_2 \in Y_2$ and $a_1 \in Y_1$. And so $y_1g \in E(\mathcal{H}_1)$ if and only if $y_1g = y_2a_2^{-1}g \in E(\mathcal{H}_2)$ for any $g \in G$. It follows that $E(\mathcal{H}_1) = E(\mathcal{H}_2)$. Hence $\mathcal{H}_1 = \mathcal{H}_2$.

2) For any $yg \in E(\mathcal{H}_1)$ where $y \in Y_1$ and $g \in G$, since $Y_1$ is a subset of $X$, $(g, yg)$ is an arc of $\mathcal{H}$. For any arc $(g, xg)$ where $x \in X$, there exists $y \in Y$ such that $x = ya^{-1}$. And so $xg = ya^{-1}g$ is an edge of $\mathcal{H}_1$. Therefore, the underlying hypergraph of $\mathcal{H}$ is $\mathcal{H}_1$. □
By Proposition 2.2, for each $h \in G$, the right translation $h_R$ that sends vertex $g$ into the vertex $gh$ is an automorphism of the Cayley (di)hypergraph $H = CD(G, X)$. The group of right translations $G_R = \{h = h_R \mid h \in G\}$ (isomorphic to $G$) is a subgroup of $\text{Aut}(H)$ acting regularly on the vertex set. The following theorem shows that the converse is also true.

**Theorem 2.14** Let $H$ be a (di)hypergraph. Then the following statements are equivalent:

1. The hypergraph $H$ is isomorphic to a Cayley (di)hypergraph.
2. The group $\text{Aut}(H)$ contains a subgroup which acts regularly on the vertex set of $H$.

**Proof** (1) $\Rightarrow$ (2): If $H = CD(G, X)$, then $G_R = \{h_R \mid h \in G\}$ (isomorphic to $G$) is a subgroup of $\text{Aut}(H)$ acting regularly on the vertex set of $H$.

(2) $\Rightarrow$ (1): Let $G$ be a subgroup of $\text{Aut}(H)$ that acts regularly on the vertex set. Take one vertex $v$ of $H$ and label this vertex as $1_G$ where $1_G$ is the identity element of $G$. For any $g \in G$, label the vertex $v^g$ as $g$. Let us consider $G$ as the vertex set of $H$. Let $[(1_G, e_1)], [(1_G, e_2)], \cdots, [(1_G, e_d)]$ be the orbits of arcs under the action of $G$ on the arc set. Since $G$ acts regularly on the vertex set, each orbit contains at least one arc containing $1_G$. Choose exactly one arc containing $1_G$ from each orbit $[(1_G, e_i)], i \in [d]$. Let $(1_G, e_1), (1_G, e_2), \cdots, (1_G, e_d)$ be those arcs. Let $x_i = e_i$, $i \in [d]$. Then for any $(g, e) \in D(H)$, there exists $j \in [d]$ such that $(g, e)^{\bar{g}} = (1_G, e_j)$, which means that $(g, e) = (1_G, e_j)^g = (g, e_jg)$. Conversely, for any $j \in [d]$ and $g \in G$, $(g, e_jg) = (1_G, e_j)^g$ is an arc of $H$. Therefore the (di)hypergraph $H$ is isomorphic to the Cayley (di)hypergraph $CD(G, X)$, where $X = \{x_1, x_2, \cdots, x_d\}$.

**Example 2.15** Let $H = (V, D)$ be a Cayley dihypergraph $CD(G, X)$, where $G = Z_7, X = \{\{0, 1, 3\}, \{0, 4, 5\}, \{0, 2, 6\}\}$. Clearly, $X$ is single Cayley closed. Now $V = Z_7$,

$$E(H) = \{\{0, 1, 3\}, \{1, 2, 4\}, \{2, 3, 5\}, \{3, 4, 6\}, \{4, 5, 0\}, \{5, 6, 1\}, \{6, 0, 2\}\}$$

and $D = \{(v, e) \mid e \in E(H), v \in e\}$.

This hypergraph is the Fano plane and the undirected hypergraph $\text{CH}(G, Y)$ where $Y = \{\{0, 1, 3\}\}$.

### 3 Automorphisms of Cayley (di)hypergraph

We will show the normalizer of $G_R$ in $\text{Aut}(H)$.

Some automorphisms of $G$ may induce automorphisms of the Cayley (di)hypergraph $H$. Such automorphisms lie in the following subgroup.

$$\text{Aut}(G, X) = \{\sigma \in \text{Aut}(G) \mid x^\sigma \in X, \text{ for any } x \in X\}$$
Lemma 3.1 For a Cayley dihypergraph $\mathcal{H} = \text{CD}(G, X)$, we have

$$\text{Aut}(G) \cap \text{Aut}(\mathcal{H}) = \text{Aut}(G, X),$$

$$\text{Inn}(G) \cap \text{Aut}(\mathcal{H}) = \text{Inn}(G, X).$$

Proof Let $\sigma \in \text{Aut}(G)$. Then $\sigma$ is a permutation on the vertex set $G$ and fixes the vertex $1_G$, the identity of $G$. If $\sigma \in \text{Aut}(\mathcal{H})$, then $\sigma$ fixes $X$ (all the arcs containing the vertex $1_G$), and so $\sigma \in \text{Aut}(G, X)$, and $\text{Aut}(G) \cap \text{Aut}(\mathcal{H}) \leq \text{Aut}(G, X)$.

Conversely, if $\sigma \in \text{Aut}(G, X)$, then $X^\sigma = X$. For any $(g, e) \in D(\mathcal{H})$, there exists $x \in X$ such that $(g, e) = (1_G, x)^\sigma$. Then

$$(g, e) \in D(\mathcal{H}) \iff (1_G, x) \in D(\mathcal{H})$$

$$\iff x \in X$$

$$\iff x^\sigma \in X$$

$$\iff (1_G, x^\sigma) \in D(\mathcal{H})$$

$$\iff (1_G, x^\sigma)^\sigma \in D(\mathcal{H})$$

$$\iff (g^\sigma, (x^\sigma)^\sigma) \in E(\mathcal{H})$$

$$\iff (g, e)^\sigma \in D(\mathcal{H}).$$

So $\sigma$ is an automorphism of $\mathcal{H}$, and so $\text{Aut}(G, X) \leq \text{Aut}(G) \cap \text{Aut}(\mathcal{H})$.

Finally, $\text{Inn}(G) \cap \text{Aut}(\mathcal{H}) = \text{Inn}(G) \cap (\text{Aut}(G) \cap \text{Aut}(\mathcal{H})) = \text{Inn}(G) \cap \text{Aut}(G, X) = \text{Inn}(G, X)$. The following theorem shows the normalizer of $G_R$ in $\text{Aut}(\mathcal{H})$.

Theorem 3.2 For a Cayley (di)hypergraph $\mathcal{H} = \text{CD}(G, X)$, the normalizer of the regular subgroup $G_R$ in $\text{Aut}(\mathcal{H})$ is $G_R \rtimes \text{Aut}(G, X) \leq \text{Aut}(\mathcal{H})$.

Proof Since the normalizer of $G_R$ in $\text{Sym}(G)$ is $G_R \rtimes \text{Aut}(G)$, $N_{\text{Aut}(\mathcal{H})}(G_R) = \text{Aut}(\mathcal{H}) \cap (G_R \rtimes \text{Aut}(G)) = G_R(\text{Aut}(\mathcal{H}) \cap \text{Aut}(G))$. By last Lemma, we have $N_{\text{Aut}(\mathcal{H})}(G_R) = G_R \rtimes \text{Aut}(G, X) \leq \text{Aut}(\mathcal{H})$.

Corollary 3.3 Let $G$ be a group and let $Y$ be a non-Cayley equivalent hyperset. For a Cayley undirected hypergraph $\mathcal{H} = \text{CH}(G, Y)$, the normalizer of the regular subgroup $G_R$ in $\text{Aut}(\mathcal{H})$ is $G_R \rtimes \text{Aut}(G, [Y]) \leq \text{Aut}(\mathcal{H})$.

Example 3.4 Let $\mathcal{H}$ be the Cayley dihypergraph $\text{CD}(G, X)$, where $G = \mathbb{Z}_7, X = \{\{0, 1, 3\}, \{0, 4, 5\}, \{0, 2, 6\}\}$. Note that $A = \text{Aut}(G) \cong \mathbb{Z}_6$ and $|X| = 3$. If $X^\sigma = X$ for some $\sigma \in A$, then $|\sigma| = 1$ or $|\sigma| = 3$. Clearly, there exists an automorphism $\sigma$ of order 3 such that $X^\sigma = X$. Therefore, the normalizer of the regular subgroup $G_R$ in $\text{Aut}(\mathcal{H})$ is isomorphic to $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$. 

7
References

[1] J. Lee, Y.S. Kwon, Cayley hypergraphs and Cayley hypermaps, *Discrete Math.* 313(4) (2013), 540–549.