Microscopic study of low energy collective states in even-even nuclei: a prospective analysis of dynamical corrections to vibrational mass parameters.

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Abstract. A systematic study of low energy nuclear structure at normal deformation has been carried out using the Generator Coordinate Method mapped onto a 5-Dimensional Collective quadrupole Hamiltonian (5DCH) by using the Gaussian Overlap Approximation (GCM-GOA). The collective space is spanned by Hartree Fock Bogoliubov (HFB) states under axial and triaxial quadrupole constraints deduced with the D1S Gogny force. In addition, our 5DCH includes the Thouless-Valatin dynamical corrections to its rotational kinetic terms. The work is described in detail elsewhere together with the corresponding comparisons with experimental data when it is available. Many properties show a satisfactory agreement with experiment, but there is an almost systematic overestimation of vibrational band head energies, which is the subject of the present paper. We show here the performance of the theory on related observables, and propose improvements of the theory to address the problem of the vibrational band heads. An important reason for the deficient is the treatment of vibrational inertial parameters. In particular, the theory needs to include the dynamical Thouless-Valatin corrections to the vibrational terms in the 5DCH. In present work, on the aim of a simple formula grounded by known symmetry rules within the 5DCH, these dynamical TV corrections are roughly estimated, allowing us to handle their possible effects in term of spectroscopic properties, to present a guess of the next model improvement, and to isolate some areas in the chart where states $I_\pi^\pi = 0^+_2$, $2^+_3$ or $3^+_3$ seem to have important components out of the scope of a pure collective quadrupole approach.

1. Introduction.

Until recently, the extensions beyond mean field needed to describe dynamical properties such as excitation energies have not been carried out in a systematic way on the whole chart of nuclides. However, a large scale study on the ground of a microscopical quadrupole 5- Dimensional Collective Hamiltonian (5DCH) has been recently proposed [1]. It relies upon quadrupole Constrained Hartree-Fock-Bogoliubov (CHFB) mean field deduced from the D1S Gogny force. Under the Gaussian Overlap Approximation, the Generator Coordinate Method is employed to reduce the so-called Griffin-Hill-Wheeler equation to a Shrödinger form. Local approximations as Inglis Belyaev cranking formulas and the self-consistent cranking model are then involved to
evaluate the ingredients of the deduced collective 5DCH. As a result, a large set of benchmarks have been obtained in this context, concerning ground state (gs) properties, and low energy quadrupole collective state spectroscopy up to I$^+$ = 6$^+$ at Normal Deformation (ND), in even-even nuclei with proton numbers Z=10 to 110, from dripline to dripline and neutron number N ≤ 200. This connected work is described in great details in [1] together with available comparisons with experimental data.

The aim of the present paper, is to report about a preliminary and prospective study about vibrational mass parameters within this 5DCH approach. As a matter of fact, the 5DCH in the state of art defined by [1] takes into account microscopically Thouless-Valatin (TV) dynamical corrections in its rotational terms. This is not the case in its vibrational term. The implementation of an Adiabatic Time Dependant HFB evaluation of the vibrational kinetic tensor remains a quite heavy task, in particular numerically for large scale calculations. This is the case in its vibrational term. The implementation of an Adiabatic Time Dependant HFB evaluation of the vibrational kinetic tensor remains a quite heavy task, in particular numerically for large scale calculations. This is not the case in its vibrational term.

In previous versions of the 5DCH belonging to the same context, in particular approaches of Super Deformation (SD) in the Hg - Pb [3] and the Actinide [4] regions, rotational TV corrections where approximately taken into account by a constant 4/3 scaling factor applied to Inglis Belyaev moments of inertia. That has been shown to be in general case a quite realistic assumption. This somewhat crude input has been overcrossed in [1], but we adopt now a similar strategy for the vibrational term. However, as it will be shown, the precise quantitative information obtained about TV effects on rotational term, through known symmetry relations in the 5DCH, may drive a somewhat more sophisticated scaling of the vibrational mass parameters.

2. Brief review of formalism.
We limit ourselves to a fast overview of the 5DCH method to fix the notation. Any detail can be found in [1] and references therein. In the second subsection, the method employed to renormalize vibrational mass parameters is examined in full detail.

2.1. Overview on the 5DCH approach.
2.1.1. CHFB equations. The present 5DCH approach relies first of all upon the resolution of static and dynamic CHFB equations within the triaxial set of symmetries, namely:

$$\delta \left\langle \Phi^{eq} (q_0, q_2) \right| \hat{H} - \lambda_0 \hat{Q}_0 - \lambda_2 \hat{Q}_2 - \lambda_Z \hat{Z} - \lambda_N \hat{N} - \left( \omega \hat{I}_k \right) - \left( \dot{\hat{q}} \hat{P} \right) \left| \Phi^{eq} (q_0, q_2) \right\rangle = 0,$$

(1)

-where $\Phi^{eq} (q_0, q_2)$ is the Bogoliubov vacuum of quasiparticle, the corresponding total energy being named $E^{eq}_{CHFB} (q_0, q_2)$.
-where $\hat{H}$ is the two-body hamiltonian deduced from the Gogny effective interaction, here under its most usual parametrisation D1S [5]-[6].
-where Lagrange multipliers $\lambda_m$ for static constraints are fixed by the number of particles of each type, ($\langle \hat{Z} \rangle = Z$, $\langle \hat{N} \rangle = N$) and the quadrupole deformations ($\langle \hat{Q}_0 \rangle = q_0$, $\langle \hat{Q}_2 \rangle = q_2$).
-where the optional dynamical constraints (see section 2.1.3) refer to $\omega$ the angular velocity and $\dot{\hat{q}}$ the velocity in deformation variables and $\hat{P}$ the Baranger Veneroni impulsion operator for the adiabatic
collective motion [7], i.e. \( \hat{P} = i\hbar \left[ \frac{\partial R_0}{\partial q}, R_0 \right] \), \( R_0 \) being the generalized density matrix associated with the static Bogoliubov state \( \Phi_0(q) \).

As it is usual, quadrupole moments \( (q_0, q_2) \) are replaced by the dimensionless variables \( (\beta, \gamma) \) defined by \( \beta = \sqrt{5}\pi \sqrt{q_0^2 + 3q_2^2} / (3A^{5/2} r_0^4) \) and \( \gamma = \arctan(\sqrt{3} q_2 / q_0) \) with \( A = N + Z \) and \( r_0 = 1.2 \text{ fm} \). Associated cartesian coordinates \( (a_0 = \beta \cos \gamma, a_2 = \beta \sin \gamma) \) are also in use.

The CHFB equations are solved by expansion of single particle states onto a triaxial harmonic oscillator basis [8] [9]. Improvements for present large scale calculations are provided in [1].

2.1.2. The main 5DCH equation. Under the Gaussian Overlap Approximation, the Griffin-Hill-Wheeler equation [10] [11] for the five components of the quadrupole tensor in a space spanned by the CHFB states \( \Phi^{60} \) (and their rotational transforms) is reduced to a Schrödinger form (see a detailed discussion about the GCM-GOA approach in [12]). The deduced collective Hamiltonian \( \tilde{H}_{\text{col}} \) written in the intrinsic axes system is referred by three Euler angles (through operators \( \hat{I}_k \)) and the two quadrupole deformation variables \( (a_0, a_2) \). It takes the form of the so-called Bohr Hamiltonian in its most general version [13], i.e.

\[
\tilde{H}_{\text{col}} = \frac{1}{2} \sum_{k=1}^{3} \frac{\hat{I}_k^2}{J_k} + \sum_{m,n=0}^{2} D_{mn} \frac{\partial}{\partial a_m} D_{nm} (B_{mn})^{-1} \frac{\partial}{\partial a_n} + V, \tag{2}
\]

where \( D(a_0, a_2) \) is the metric and where the potential part \( V \) \( (a_0, a_2) \) writes [12] \( V = E^{60}_{\text{CHFB}} - \Delta V^{ZPE} \).

The six kinetic functions of the deformation \( (a_0, a_2) \), i.e. the three moments of inertia \( J_k \) along principal axes \( k \), the three mass parameters \( B_{00}, B_{20} = B_{20} \) and \( B_{22} \) together with the Zero Point Energy correction \( \Delta V^{ZPE} \) are evaluated under local approximations as explained in section 2.1.3.

A numerical resolution of the collective Schrödinger equation is implemented, following methods explained in [3]. The collective wavefunction \( \Psi_{n}^{IM} \) associated with the \( n^{th} \)eigenvalue with angular momentum \( I \) (with projections and \( M \) and \( K \) on the third axis of the laboratory and intrinsic frame respectively is finaly such that:

\[
\tilde{H}_{\text{col}} \left| \Psi_{n}^{IM} \right\rangle = E_{n}^{I} \left| \Psi_{n}^{IM} \right\rangle \quad \text{and} \quad \left| \Psi_{n}^{IM} \right\rangle = \sum_{0 \leq K \leq I} g_{n}^{IK} (a_0, a_2) \left| IMK \right\rangle, \tag{3}
\]

where \( \left| IMK \right\rangle \) is a standard normalized combination of rotational Wigner matrices.

2.1.3. Evaluation of \( J_k, B_{mn} \) and \( \Delta V^{ZPE} \) functions. A selfconsistent cranking calculation is implemented for each deformation point. In this context, moments of inertia are calculated as:

\[
J_k = J_k^{TV} = \lim_{\omega \to 0} \frac{\left\langle \Phi^{\omega} \right| \hat{I}_k \left| \Phi^{\omega} \right\rangle}{\omega}, \tag{4}
\]

which has been shown to be equivalent to the Thouless-Valatin inertia [14]. A fixed value of \( \omega \), namely \( \omega = 0.002 \text{ MeV} \), is employed to approximate the limit numerically.

The static CHFB calculation of \( \Phi^{0} (a_0, a_2) \) leads us with its related quasiparticle (qp) states \( |\mu \rangle, |\nu \rangle \), ... and qp energies \( E_{\mu}, E_{\nu}, .... \) That allows to refer also in present work to standard Inglis-Belyaev inertia [15] [16] labelled as \( J_k^{\text{Inglos}} \), calculated as for instance in [3].
In an Adiabatic Time Dependant HFB frame, Thouless Valatin vibrational mass parameters would be similarly defined [2] [3] [7] as a limit at low velocity of the type:

\[ B_{TV}^{q} = \lim_{q \to 0} \frac{\langle \Phi^{0} q | \mathcal{P} | \Phi^{0} q \rangle}{q}. \quad (5) \]

Such calculation being not yet achieved, standard cranking formulas are employed. Cranking mass parameters are deduced within a linear response theory very similar to those involved for Inglis-Belyaev moment of inertia. Here, they are therefore also labelled as "Inglis" mass parameters. Defining in standard way the moment 

\[ M_{ij}^{\beta/\gamma} (q) \]

for the Bogoliubov state \( \Phi^{0} (q_{0}, q_{2}) \), mass parameters and ZPE correction are evaluated as [12] [1] :

\[ B_{Inglis}^{ij} (q) = \frac{1}{2} \frac{M_{ij}^{q} (q)}{M_{ij}^{q} (q)}, \quad \Delta V_{ZPE} (q) = \frac{1}{4} \sum_{i,j} \frac{M_{ij}^{q} (q)}{M_{ij}^{q} (q)}. \quad (6) \]

2.2. Renormalisation of vibrational mass parameters.

The symmetry rules within the quadrupole 5DCH (attached with both rotational invariance and intrinsic axes relabelling) have been studied in great detail long time ago in the pioneering work of ref. [13] and have been put by these authors under an elegant and compact form. With slight differences in notations, their conclusions for the kinetic part can be summarized as follows:

The kinetic term of the 5DCH is fully described in the most general case by two functions of deformation variables \( (a_{0}, a_{2}) \) named here \( B_{Rot} \) and \( B_{Vib} \). These two functions, which must be analytic in \( (a_{0}, a_{2}) \), are defined in the whole collective space and connected to usual moments of inertia \( J_{k} \) and vibrational mass parameters \( B_{mn} \) by:

\[ J_{k} (\beta, \gamma) = 4 (\beta \sin \gamma_{k})^{2} B_{Rot} (\beta, \gamma_{k}) \]
\[ B_{22} (\beta, \gamma) = B_{Vib} (\beta, \gamma_{3}) \]

with \( \gamma_{k} = \gamma - \frac{2\pi}{3} \) \( \quad (7) \)

Microscopic calculations providing moments of inertia and mass parameters in the first sextant \( 0 \leq \gamma \leq \pi/3 \), functions \( B_{Rot} \) and \( B_{Vib} \) are finally defined on this ground in the whole collective space,

i) using the subsidiary relations:

\[ B_{00} (\beta, \gamma) = \frac{2}{3} B_{Vib} (\beta, \gamma_{1}) + \frac{2}{3} B_{Vib} (\beta, \gamma_{2}) - \frac{1}{3} B_{Vib} (\beta, \gamma_{3}) \]
\[ B_{02} (\beta, \gamma) = (B_{Vib} (\beta, \gamma_{1}) - B_{Vib} (\beta, \gamma_{2})) / \sqrt{3} \quad (8) \]

ii) knowing they must be even functions of \( a_{2} \) and therefore invariant in the change \( \gamma \to -\gamma \).

iii) and knowing they must be equal to each other on the axis \( a_{2} = 0 \) and therefore have to verify:

\[ B_{Rot} (\beta, \gamma = 0) = B_{Vib} (\beta, \gamma = 0) ; B_{Rot} (\beta, \gamma = \pi) = B_{Vib} (\beta, \gamma = \pi). \quad (9) \]

Conditions eq.9 constitute a quite weak constraint, because the \( B_{Rot} \) function is related to moment of inertia (eq.7) through a factor \( \sin^{2} \gamma \) which vanishes identically on the axis. However, we
have tried to exploit them to renormalize our Inglis-Belyaev mass parameters using a simple
precription, which is defined by the following steps:

i) For each nuclear system, and for each deformation $\beta$, microscopical functions $B$ are
decomposed in two parts, an even ($A^\oplus$)and an odd ($A^\ominus$) part in the change $\gamma \rightarrow (\pi - \gamma)$, (i.e.
also $a_0 \rightarrow -a_0$). We define therefore:

$$A^\oplus (\beta, \gamma) = \frac{1}{2} (B(\beta, \gamma) + B(\beta, \pi - \gamma)) \quad A^\ominus (\beta, \gamma) = \frac{1}{2} (B(\beta, \gamma) - B(\beta, \pi - \gamma))$$

Due to their $a_0$-parity property, functions $A$ can be easily scaled to have a given value along
the $a_2 = 0$ axis.

ii) At the issue of our microscopical calculations functions $B^\text{Inglis}_{\text{Rot}}$ and $B^\text{TV}_{\text{Rot}}$ are known. The
effect of TV dynamical correction on rotational part are now approximated at each deformation
$\beta$ on applying scaling factors $\alpha^\oplus (\beta)$ and $\alpha^\ominus (\beta)$ to the corresponding Inglis Belyaev functions
$A^\text{Inglis}_\text{Rot} (\beta, \gamma)$ and $A^\text{Inglis}_\text{Rot} (\beta, \gamma)$. In practice, $\alpha^i$ factors are determined through a least square
process on the numerical mesh $\{\beta_m, \gamma_n\}$, on minimizing

$$\forall \beta_m, \{i = \oplus, \ominus\}, \delta \sum_n |\alpha^i (\beta_m) \, A^\text{Inglis}_\text{Rot} (\beta_m, \gamma_n) - A^\text{TV}_\text{Rot} (\beta_m, \gamma_n)|^2 = 0$$

That of course involves points of the mesh with $\gamma \neq 0$ and $\gamma \neq \pi$ only, where the $B^\text{TV}_{\text{Rot}}$ function
and its deduced parts $A^\text{TV}_{\text{Rot}} k$ are completely defined. This first step of the prescription can be
shown to be very satisfactory, on comparing true TV moments of inertia (which are kept
in the 5DCH calculations) and those deduced (using eq.7 after inverting eq.10) from the Inglis
scaled functions $A^\text{Rot} TV i = \alpha^i (\beta) \, A^\text{Inglis}_\text{Rot} i (\beta, \gamma)$.

An example is displayed in the left hand side (a) of figure 1 where the two sets of moments
of inertia as function of the triaxiality $\gamma$ are compared for the $^{110}\text{Ru}$ nucleus at $\beta = 0.3$. Scaling
factors $\alpha^i$ are found to be very smooth functions of $\beta$ varying typically between 1.0 and 3/2.
That means that the overall behaviour of the $B^\text{TV}_{\text{Rot}}$ function is mainly contained in the $B^\text{Inglis}_{\text{Rot}}$
one, and that these two functions can be related each other with a very good accuracy by a
weak and regular scaling transformation.

iii) The last step of the prescription consists in applying the same "scaling" on vibrational
function $B^\text{Inglis}_{Vib}$, securing thus the condition on axis $a_2 = 0$. Thus, on the ground of this
minimum of information trick, one defines the new vibrational functions:

$$A^\text{Rot} TV i (\beta, \gamma) = \alpha^i (\beta) \, A^\text{Inglis}_{Vib} i (\beta, \gamma), \text{ with } \{i = \oplus, \ominus\}$$

Deduced kinetic functions $B^\text{Rot} TV_{Vib}$ and other related functions $B$ are displayed for our example
(i.e. $^{110}\text{Ru}$ at $\beta = 0.3$) on the right hand side (b) of figure 1. As shown in there, the condition
on axis $a_2 = 0$ (eq. 9) is pretty well verified for the fully coherent description with $B^\text{Inglis}_{\text{Rot}}$ and
$B^\text{Inglis}_{Vib}$, but is broken if only rotational kinetic terms includes the dynamical TV contribution
as it the case in the systematic calculation of [1]. Finally, the ad hoc scaled function $B^\text{Rot} TV_{Vib}$
verifies the equalities on axis to $B^\text{TV}_{Vib}$.

3. Results.
3.1. The low energy spectrum of $^{110}\text{Ru}$.
We have taken $^{110}\text{Ru}$ (Z=44,N=66) nucleus as an example. It is therefore convenient to see
on this particular case the differences in spectrum induced by various hypotheses employed as
Figure 1. On side (a), for the $^{110}$Ru nucleus, at deformation $\beta = 0.3$, Thouless-Valatin moments of inertia $J_{TV}^{k=1,2,3}$ (red lines) are compared as function of the triaxiality $\gamma$ with corresponding "scaled" (see text) Inglis-Belyaev moments of inertia (blue lines). On side (b)$P_{Rot}^{Inglis}$ (full triangles, blue line ), $B_{Rot}^{TV}$ (open triangles, pink line), $B_{Vib}^{Inglis}$ (full dots, red line) and $B_{Vib}^{RnTV}$ (open dots, green line) -see text for definitions- are displayed as functions of the triaxiality $\gamma$.

input of the 5DCH. This nucleus is a triaxial-$\gamma$–soft nucleus -see its Potential Energy Surface in figure 2-. As shown in this figure, a significative progress in the description of in gs- quasi-$\beta$ and quasi-$\gamma$ bands is obtained, step by step, from the pure Inglis-Belyaev calculation of the kinetic tensor " $\left(J_{k}^{Inglis}, B_{mn}^{Inglis}\right)$ " , to the calculation including exact TV corrections for the rotational term only " $\left(J_{k}^{TV}, B_{mn}^{Inglis}\right)$ " -i.e. those obtained in [1]-, and finally to the present calculation including also the approximate estimation of the TV vibrational term labelled here " $\left(J_{k}^{TV}, B_{mn}^{RnTV}\right)$ " . It is noticeable that in such a soft system, a correction of vibrational term affects not only the band head energies but also the rotational sequence (in particular the gs band). In this nucleus, there is no available experimental data for the quasi-$\beta$ band head energy. It is predicted here at 0.950 MeV. In this nucleus, as it is the most general case in the chart, the level $I_{\pi n} = 2^{+}_{2}$, with more than 80% of the norm belonging to the $K=2$ part of the space (in any of these calculations) is clearly the quasi-$\gamma$ band head. That seems to be consistent with the experimental spectrum.

3.2. Vibrational states in the Gd ($Z=64$) isotopic chain.

Now we consider another example taken in the so-called $N = 90$ $\gamma$–soft-rigid prolate transition in the rare earths, namely in the Gd ($Z = 64$) isotopic chain : In figure 3, the evolution in the gs band energies and of $I_{\pi n} = 0^{+}_{2}, 2^{+}_{2}$ and $2^{+}_{3}$ state energies are displayed as a function of the neutron number $N$ for the $\left(J_{k}^{TV}, B_{mn}^{Inglis}\right)$ and the $\left(J_{k}^{TV}, B_{mn}^{RnTV}\right)$ calculations and for the experiment. For these nuclei, in the gs band, the renormalization of vibrational part of the kinetic tensor is without any significative consequences. As discussed in [1], and specialy in the gs band, the vicinity of the shell closure for $N=82$ induces some discrepancies with experiment. For all these nuclei, in both calculations (with the noticeable exception of the $N = 90$ $^{154}$Gd), the $I_{n}^{\pi} = 2^{+}_{2}$ has the properties a $\gamma$-band head with more than 75% of the norm in the $K = 2$ part of the space. An inversion with the $I_{n}^{\pi} = 2^{+}_{3}$ occurs at $N = 90$ where the two states are
Figure 2. For the $^{110}$Ru nucleus, the experimental low energy spectrum (in red color) for the gs band (on the left), the quasi-$\beta$ band -NB the 0+ band head is unknown- (on center) and the quasi-$\gamma$ band (on the right) is compared with three theoretical spectra deduced respectively with kinetic functions ($J_{k}^{Inglis}$, $B_{mn}^{Inglis}$) -levels in black color-, ($J_{k}^{TV}$, $B_{mn}^{Inglis}$) -levels in blue color-, and ($J_{k}^{TV}$, $B_{mn}^{RnTV}$) -levels in green color-. Experimental data are extracted from the compilation [17]. Potential energy surface is also displayed in a sextant. Isolines are separated by 1 MeV.

very close in energy. The $N = 90$ $\gamma$–soft-rigid prolate transition is well localized: The gs deformation ($\langle \beta \rangle$, $\langle \gamma \rangle$) is (0.17, 25$^\circ$) at $N = 86$, reaches (0.34, 12$^\circ$) at $N = 90$, and then follows a regular behaviour to more prolate rigid system, reaching (0.37, 4$^\circ$) at $N = 100$. As shown, the improvment of the description on renormalizing vibrational kinetic term is noticeable but is not sufficient to lower enough the band head energies in the most deformed Gd nuclei. Thus, if around $N=90$ in Gd nuclei, a quadrupole collective space seems to be rich enough to describe the low energy collective modes, -this is confirmed by a very good agreement with experiment obtained for E0 and E2 transition probabilities in $^{154}$Gd-, it is highly probable that other degrees of freedom take some importance with increasing deformation and have therefore to be also taken into account in the future.

3.3. An overview of results in the whole chart.

A statistic approach has been performed in a large area of the chart ($18 \leq Z \leq 98$). Nuclei included in the panel are those i) for wich the experimental energy of the considered state ($I_{n}^\pi = 0^+_2$, $2^+_2$ or $2^+_3$) is known in the Brookheaven nuclear data table [17] and ii) and those for which the calculated spectroscopy has been validated. (some nuclei at or close the doubly shell closure are not considered -see the discussion of this point in [1]-. Each of these excluded nuclei corresponds to an empty point in the map displayed in figure 5. Results of this statistical analysis are displayed in figure 4. One can see in this figure the almost systematic overestimation of excitation energies for states $I_{n}^\pi = 0^+_2$, $2^+_2$ and $2^+_3$ respectively, obtained in the standard calculation referred as ($J_{k}^{TV}$, $B_{mn}^{Inglis}$). As a result, clouds of points obtained in the
Figure 3. Evolution in gs band energies (a) and of \( I^\pi_n = 0^+_2 \) (b), \( 2^+_2 \) (c) and \( 2^+_3 \) (d) state energies as function of the neutron number for the Gd isotopic chain. Experimental values, extracted from [17], are joined by red lines. Blue lines with full triangle down correspond to the standard calculation \( (\mathcal{J}^TV_k, B^{Inglis}_{mn}) \), whereas green lines with open triangle up correspond to the present calculation with renormalized vibrational mass parameters i.e. \( (\mathcal{J}^TV_k, B^{RenTV}_{mn}) \).

\((\mathcal{J}^TV_k, B^{RenTV}_{mn})\) approach are closer to the line of "perfection" (defined by \( E^{Th} = E^{Exp} \)) and more symmetrically distributed around it. Of course, there is also a certain amount of points quite far from this line in any case. Experimental states have been considered here only for their spin and order in the spectrum. Clearly a certain amount of them does not belong (at all or for a part) to quadrupole collective modes.

To localize and have in hand discrepancies between experimental and theoretical values obtained with the present \((\mathcal{J}^TV_k, B^{RenTV}_{mn})\) calculation, a map in the plan (N,Z) of the ratio \( E^{Th}/E^{Exp} \) for the state \( I^\pi_n = 0^+_2 \) is produced in figure 5. A simple overview on this map shows a quite satisfactorying agreement. However, apart ponctual and isolated problems, apart defaults directly connected with double shell closures, a quite large and noticeable area of descrepancies experience-theory appears in the well deformed actides (Points of the map with \( 90 \leq Z \leq 98, 134 \leq N \leq 152 \) ) where, even with such renormalized mass parameter, the \( 0^+_2 \) state energy is overestimated by at least 30% in this region, these results seems to indicate, that the experimental \( 0^+_2 \) state is not the quasi−\( \beta \) band head (a quasi−\( \beta \) band head could exists at higher energy), or that this state is a more complicate mixing, involving other degrees of freedom. Due to the systematic character of the descripancy in the region, these degrees of freedom are probably of higher collective multipole type. Further studies will be devoted to understand more deeply these facts.

4. Conclusions.

Clearly, the function \( B^{Vib} \) including its TV contributions should be evaluated microscopically instead of being simply rescaled over the Inglis-Belyaev function. However, it is believable that the present renormalized kinetic function \( B^{Vib}_{RenTV} \), which exhibits the general behaviour of the \( B^{Vib}_{Inglis} \) one, drives a lowering of vibrational band heads which has at least the good order of magnitude. Under this assumption, it has been shown that a global quantitative improvment
Figure 4. States \( I_n^= 0^+ \) (a,b), \( 2^+ \) (c,d) and \( 2^+ \) (e,f) for even-even nuclei with \( 20 \leq Z \leq 100 \) are displayed as function of their theoretical energies (vertical axis) and experimental ones (horizontal axis) On the left (a,c,e) with blue points (resp right (b,d,f) with green points) theoretical values are issueing from the standard calculation with \( (J^{TV}_k, B_{Inglis}^{mn}) \) (resp the calculation with renormalized vibrational mass parameters with \( (J^{TV}_k, B_{RnTV}^{mn}) \). In each plot, as a guide for eyes, the dashed line corresponds to \( E^{Th} = E^{Exp} \).

of the theory can be expected at next step, precisely on evaluating within an Adiabatic Time Dependant HFB approach these vibrational TV contributions. As known, beyond the pure quadrupole collective space considered here, a microscopic theory considering single particle modes is clearly needed in the vicinity of shell closures. But moreover, it has been shown here, that for some well deformed nuclei, (for instance in the actinide region), a convincing description of \( I_n^= 0^+ \) states presumably implies to take also into account collective degrees of freedom of higher multipoles.

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Figure 5. Map \((N, Z)\) of the ratio \(\text{Test} = E_{\text{Th}}/E_{\text{exp}}\), where \(E_{\text{Th}}\) is the theoretical energy of the \(I^n_0 = 0^+_2\) state deduced with renormalized vibrational mass parameters \(B_{\text{renTV}}\) (see text for definition) and \(E_{\text{exp}}\) is the corresponding experimental energy extracted from the compilation [17].

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