Helicity Basis for Spin 1/2 and 1

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We study the theory of the $(1/2, 0) \oplus (0, 1/2)$ and $(1, 0) \oplus (0, 1)$ representations in the helicity basis. The helicity eigenstates are not the parity eigenstates. This is in accordance with the idea of Berestetskii, Lifshitz and Pitaevskii. The behaviour of the helicity eigenstates with respect to the charge conjugation, $CP$- conjugation is also different comparing with the parity eigenstates.

I. INTRODUCTION.

Recently we generalized the Dirac formalism [1–4] and the Bargmann-Wigner formalism [5–7]. On this basis we proposed a set of equations for antisymmetric tensor (AST) field. Some of them imply parity-violating transitions. In this paper we are going to study transformations from the standard basis to the helicity basis in the Dirac theory and in the $(1, 0) \oplus (0, 1)$ Sankaranarayanan-Good theory [8,9]. The spin basis rotation changes properties of the corresponding states with respect to parity. The parity is a physical quantum number; so, we try to extract corresponding physical contents from considerations of the various spin bases.

II. THE $(1/2, 0) \oplus (0, 1/2)$ CASE.

Beginning the consideration of the helicity basis, we observe that it is well known that the operator $\hat{S}_3 = \sigma_3/2 \otimes I_2$ does not commute with the Dirac Hamiltonian unless the 3-momentum is aligned along with the third axis and the plane-wave expansion is used:

$$\left[\hat{H}, \hat{S}_3\right] = (\gamma^0\gamma^k \times \nabla)_3$$

Moreover, Berestetskii, Lifshitz and Pitaevskii wrote [10]: “... the orbital angular momentum $I$ and the spin $s$ of a moving particle are not separately conserved. Only the total angular momentum $j = l + s$ is conserved. The component of the spin in any fixed direction (taken as z-axis is therefore also not conserved, and cannot be used to enumerate the polarization (spin) states of moving particle.” The similar conclusion has been given by Novozhilov in his book [11]. On the other hand, the helicity operator $\sigma \cdot \tilde{p}/2 \otimes I$, $\tilde{p} = p/|p|$, commutes with the Hamiltonian (more precisely, the commutator is equal to zero when acting on the one-particle plane-wave solutions).

So, it is a bit surprising, why the 4-spinors have been studied so well when the basis have been chosen in such a way that they were eigenstates of the $\hat{S}_3$ operator:

$$u_{\pm \frac{1}{2}} = N_{\frac{1}{2}}^+ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, u_{\pm \frac{3}{2}} = N_{\frac{3}{2}}^+ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_{\pm \frac{1}{2}} = N_{\frac{1}{2}}^- \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, v_{\pm \frac{1}{2}} = N_{\frac{1}{2}}^- \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},$$

and, oppositely, the helicity basis case has not been studied almost at all (see, however, refs. [11–13]. Let me remind that the boosted 4-spinors in the ‘common-used’ basis are

$$u_{\pm \frac{1}{2}} = \frac{N_{\frac{1}{2}}^+}{\sqrt{2m(E + m)}} \begin{pmatrix} p^+ + m \\ p^- \end{pmatrix}, u_{\pm \frac{1}{2}} = \frac{N_{\frac{3}{2}}^+}{\sqrt{2m(E + m)}} \begin{pmatrix} p^- + m \\ p^+ \end{pmatrix},$$

$$v_{\pm \frac{1}{2}} = \frac{N_{\frac{1}{2}}^-}{\sqrt{2m(E + m)}} \begin{pmatrix} p^+ + m \\ p^- \end{pmatrix}, v_{\pm \frac{1}{2}} = \frac{N_{\frac{3}{2}}^-}{\sqrt{2m(E + m)}} \begin{pmatrix} p^- + m \\ p^+ \end{pmatrix},$$
\[ p^\pm = E \pm p_z, \quad p_{r,t} = p_x \pm ip_y. \] They are the parity eigenstates with the eigenvalues of \( \pm 1. \) The matrix \( \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is used in the parity operator.

Let me turn now your attention to the helicity spin basis. The 2-eigenspinors of the helicity operator can be defined as follows [14,15]:

\[
\phi_+^\pm = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}, \quad \phi_-^\pm = \begin{pmatrix} \sin \frac{\theta}{2} e^{-i\phi/2} \\ -\cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix},
\]

for \( \pm 1/2 \) eigenvalues, respectively.

We start from the Klein-Gordon equation, generalized for describing the spin-1/2 particles (i.e., two degrees of freedom); \( c = \hbar = 1: \)

\[
(E + \sigma \cdot p)(E - \sigma \cdot p) \phi = m^2 \phi.
\] (7)

It can be re-written in the form of the set of two first-order equations for 2-spinors. Simultaneously, we observe that they may be chosen as eigenstates of the helicity operator which present in (7):\(^2\)

\[
(E - (\sigma \cdot p))\phi_1 = (E - p)\phi_1 = m\chi_1,
\] (8)

\[
(E + (\sigma \cdot p))\chi_1 = (E + p)\chi_1 = m\phi_1,
\] (9)

and

\[
(E - (\sigma \cdot p))\phi_1 = (E + p)\phi_1 = m\chi_1,
\] (10)

\[
(E + (\sigma \cdot p))\chi_1 = (E - p)\chi_1 = m\phi_1.
\] (11)

If the \( \phi \) spinors are defined by the equation (6), then we can construct the corresponding \( u \) and \( v \) 4-spinors:

\[
u_+ = N_1^\pm \left( \begin{array}{c} \phi_1 \\ \psi_1 \end{array} \right), \quad u_+ = N_1^\pm \left( \begin{array}{c} \phi_1 \\ \psi_1 \end{array} \right),
\]

\[
u_- = N_1^\pm \left( \begin{array}{c} \phi_1 \\ -\psi_1 \end{array} \right), \quad u_- = N_1^\pm \left( \begin{array}{c} \phi_1 \\ -\psi_1 \end{array} \right),
\]

where the normalization to the unit (1) was used: \(^3\)

\[
u_+ u_+ = \delta_{\lambda\lambda'}, \quad \bar{v}_+ v_+ = -\delta_{\lambda\lambda'},
\]

\[
u_- u_- = 0 = \bar{v}_- v_-.
\]

One can prove that the matrix

\[
P = \gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\] (16)

can also be used in the parity operator as well as in the original Dirac basis. Indeed, the 4-spinors (12,13) satisfy the Dirac equation in the spinorial representation of the \( \gamma \)-matrices (see straightforwardly from (7)). Hence, the parity-transformed function \( \Psi'(t, -x) = P\Psi(t, x) \) must satisfy

\(^1\)The following method is due to the van der Waerden, Sakurai and Gersten works, see ref. [16].

\(^2\)This opposes to the choice of the basis (2), where 4-spinors are the eigenstates of the parity operator.

\(^3\)Of course, there are no any mathematical difficulties to change it to the normalization to \( \pm m \), which may be more convenient for the study of the massless limit.
\[ [i\gamma^\mu \partial^\mu_\mu - m] \Psi(t, -x) = 0, \]  

with \( \partial^\mu_\mu = (\partial/\partial t, -\nabla) \). This is possible when \( P^{-1}\gamma^0 P = \gamma^0 \) and \( P^{-1}\gamma^i P = -\gamma^i \). The matrix (16) satisfies these requirements, as in the textbook case.

Next, it is easy to prove that one can form the projection operators
\[
P_+ = + \sum_\lambda u_\lambda(p)\bar{u}_\lambda(p) = \frac{p_\mu\gamma^\mu + m}{2m},
\]
\[
P_- = - \sum_\lambda v_\lambda(p)\bar{v}_\lambda(p) = \frac{m - p_\mu\gamma^\mu}{2m},
\]
with the properties \( P_+ + P_- = 1 \) and \( P^2_\pm = P_\pm \). This permits us to expand the 4-spinors defined in the basis (2) in linear superpositions of the helicity basis 4-spinors and to find corresponding coefficients of the expansion:
\[
u_\sigma(p) = A_{\sigma\lambda}u_\lambda(p) + B_{\sigma\lambda}v_\lambda(p), \]
\[
u_\sigma(p) = C_{\sigma\lambda}u_\lambda(p) + D_{\sigma\lambda}v_\lambda(p).
\]
Multiplying the above equations by \( \bar{u}_\lambda, \bar{v}_\lambda \) and using the normalization conditions, we obtain \( A_{\sigma\lambda} = D_{\sigma\lambda} = \bar{u}_\lambda u_\sigma, \)
\( B_{\sigma\lambda} = C_{\sigma\lambda} = -\bar{v}_\lambda u_\sigma \). Thus, the transformation matrix from the common-used basis to the helicity basis is
\[
\left( \begin{array}{c}
u_\sigma \\ \nu_\sigma \end{array} \right) = \mathcal{U} \left( \begin{array}{c} u_\lambda \\ v_\lambda \end{array} \right), \quad \mathcal{U} = \left( \begin{array}{cc} A & B \\ B & A \end{array} \right)
\]

Neither \( A \) nor \( B \) are unitary:
\[
A = (a_{++} + a_{+-})(\sigma_\mu a^\mu) + (-a_{+-} + a_{--})(\sigma_\mu a^\mu)\sigma_3,
\]
\[
B = (-a_{++} + a_{+-})(\sigma_\mu a^\mu) + (a_{+-} + a_{--})(\sigma_\mu a^\mu)\sigma_3,
\]
where
\[
a^0 = -i\cos(\theta/2)\sin(\phi/2) \in \mathbb{R}, \quad a^1 = \sin(\theta/2)\cos(\phi/2) \in \mathbb{R},
\]
\[
a^2 = \sin(\theta/2)\sin(\phi/2) \in \mathbb{R}, \quad a^3 = \cos(\theta/2)\cos(\phi/2) \in \mathbb{R},
\]
and
\[
a_{++} = \frac{\sqrt{(E + m)(E + p)}}{2\sqrt{2m}}, \quad a_{+-} = \frac{\sqrt{(E + m)(E - p)}}{2\sqrt{2m}},
\]
\[
a_{+-} = \frac{-\sqrt{(E - m)(E + p)}}{2\sqrt{2m}}, \quad a_{--} = \frac{-\sqrt{(E - m)(E - p)}}{2\sqrt{2m}}.
\]

However, \( A^\dagger A + B^\dagger B = 1 \), so the matrix \( \mathcal{U} \) is unitary. Please note that the \( 4 \times 4 \) matrix acts on the spin indices \((\sigma, \lambda)\), and does not on the spinorial indices. Alternatively, the transformation can be written:
\[
u_\sigma^\alpha = [A_{\sigma\lambda} \otimes I_{\alpha\beta} + B_{\sigma\lambda} \otimes \gamma_{\alpha\beta}^5]u_\lambda^\beta,
\]
\[
u_\sigma^\alpha = [A_{\sigma\lambda} \otimes I_{\alpha\beta} + B_{\sigma\lambda} \otimes \gamma_{\alpha\beta}^5]v_\lambda^\beta.
\]

We now investigate the properties of the helicity-basis 4-spinors with respect to the discrete symmetry operations \( P \) and \( C \). It is expected that \( \lambda \to -\lambda \) under parity, as Berestetskii, Lifshitz and Pitaevskii claimed \([10]\).\(^4\) With respect to \( p \to -p \) (i.e., the spherical system angles \( \theta \to \pi - \theta, \phi \to \pi + \phi \)) the helicity 2-eigenspinors transform as follows: \( \phi_{\uparrow\downarrow} \to -i\phi_{\uparrow\downarrow}, \) ref. [15]. Hence,
\[
Pu_{\uparrow1}(\mathbf{p}) = -iu_{\uparrow1}(\mathbf{p}), \quad Pv_{\uparrow1}(\mathbf{p}) = +iv_{\uparrow1}(\mathbf{p}),
\]
\[
Pu_{\downarrow1}(\mathbf{p}) = -iu_{\downarrow1}(\mathbf{p}), \quad Pv_{\downarrow1}(\mathbf{p}) = +iv_{\downarrow1}(\mathbf{p}).
\]

\(^4\)Indeed, if \( x \to -x \), then the vector \( p \to -p \), but the axial vector \( S \to S \), that implies the above statement.
Thus, on the level of classical fields, we observe that the helicity 4-spinors transform to the 4-spinors of the opposite helicity.

The charge conjugation operation is defined as

\[ C = \begin{pmatrix} 0 & \Theta \\ -\Theta & 0 \end{pmatrix} \kappa. \] (33)

Hence, we observe

\[ Cu_\uparrow(p) = -v_\downarrow(p), \quad Cv_\uparrow(p) = +u_\downarrow(p), \] (34)
\[ Cu_\downarrow(p) = +v_\uparrow(p), \quad Cv_\downarrow(p) = -u_\uparrow(p). \] (35)

due to the properties of the Wigner operator \( \Theta \phi^* = -\phi \) and \( \Theta \phi = +\phi \). For the \( CP \) (and \( PC \)) operation we get:

\[ CPu_\uparrow(-p) = -PCu_\uparrow(-p) = +iv_\uparrow(p), \] (36)
\[ CPu_\downarrow(-p) = -PCu_\downarrow(-p) = -iv_\downarrow(p), \] (37)
\[ CPv_\uparrow(-p) = -PCv_\uparrow(-p) = +iu_\uparrow(p), \] (38)
\[ CPv_\downarrow(-p) = -PCv_\downarrow(-p) = -iu_\downarrow(p). \] (39)

which are different from the Dirac ‘common-used’ case.

Similar conclusions can be drawn in the Fock space.

**III. THE \((1,0) \oplus (0,1)\) CASE.**

In this Section we are going to investigate the behaviour of the field functions of the \((1,0) \oplus (0,1)\) representation in the helicity basis with respect to \( P, C \) and \( CP \) operations.

Let us start from the Klein-Gordon equation written for the 3-component function (\( \bar{h}c = 1 \)):

\[ (E^2 - p^2)\psi^{(3)} = m^2 \psi^{(3)}. \] (40)

The function \( \psi \) describe the particles, which is usually referred as spin 1; we refer to it as a “3-spinor”. On choosing the basis where \( S_{ij} = -i\epsilon^{ijk} \) one can derive the following property for any 3-vector \( \mathbf{a} \):

\[ (S \cdot \mathbf{a})_{ij} = a^2 \delta_{ij} - a^i a^j \] (41)

Then the equation (40) can be re-written in the form:

\[ (E - S \cdot \mathbf{p})(E + S \cdot \mathbf{p})\psi^j - p_i p_j \psi^j = m^2 \psi^j. \] (42)

In the coordinate space it is of the second order in the time derivative, but as in the spin-1/2 case [17] we can reduce it to the set of the 3-“spinor” equations of the first orders. The procedure permits us to consider the hamiltonian-like form

\[ i\hbar \frac{\partial \phi}{\partial t} = \hat{H} \psi \]

and make it easier to find the energy eigenstates of the problem.

We can denote:

\[ (E + S \cdot \mathbf{p})\psi = m\xi \] (43)
\[ p^i p^j \psi^j = \mathbf{p} \cdot \psi = m\mathbf{p} \varphi. \] (44)

Hence the equation (42) is written

\[ m(E - S \cdot \mathbf{p})\xi - m\mathbf{p} \varphi = m^2 \psi. \] (45)

Now, we insert the properties

\[ (S \cdot \mathbf{p})^i_j \psi^j = (\nabla \times \psi)^i, \quad p^i p^j \psi^j = -[\nabla (\nabla \cdot \psi)]^i, \] (46)

and define \( \psi = \mathbf{E} - i\mathbf{B} \). We can obtain (cf. with ref. [4])

\[ \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = -m \cdot \text{Im}(\xi), \quad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = m \cdot \text{Re}(\xi). \] (47)
respectively by means of separation of the equations (43,44) into the real and imaginary parts. Next, we fix \( \varphi = im\phi \) and \( \xi = imA \), with \( \phi \) and \( A \) being the electromagnetic-like potentials. The well-known Proca equation follows:

\[
\begin{align*}
\partial_\mu F^{\mu\nu} + m^2 A^\nu &= 0.
\end{align*}
\]

For the sake of completeness let us substitute Petiau (DKP) equation [22]. It contains the part corresponding to the 4-vector potential also write transformation with which is in the symbolic form:

\[
\begin{align*}
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & -E & ip_z & -ip_y & p_x \\
0 & 0 & 0 & 0 & 0 & -ip_z & -E & ip_x & p_y \\
0 & 0 & 0 & 0 & 0 & ip_y & -ip_z & -E & p_z \\
0 & 0 & 0 & 0 & 0 & E & ip_z & -ip_y & -p_x \\
0 & 0 & 0 & 0 & 0 & -ip_z & E & ip_x & -p_y \\
-E & -ip_z & ip_y & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
ip_z & -E & -ip_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-ip_y & ip_x & -E & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-p_x & -p_y & -p_z & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\chi_1 \\
\chi_2 \\
\chi_3 \\
\psi_1 \\
\psi_2 \\
\psi_3 \\
\xi_1 \\
\xi_2 \\
\xi_3 \\
\varphi
\end{pmatrix}
= m
\begin{pmatrix}
\chi_1 \\
\chi_2 \\
\chi_3 \\
\psi_1 \\
\psi_2 \\
\psi_3 \\
\xi_1 \\
\xi_2 \\
\xi_3 \\
\varphi
\end{pmatrix},
\end{align*}
\]

which is in the symbolic form:

\[
\begin{pmatrix}
0_{3\times 3} & 0_{3\times 3} & -(E + S \cdot p)_{3\times 3} & p_{3\times 1} \\
0_{3\times 3} & 0_{3\times 3} & (E - S \cdot p)_{3\times 3} & -p_{3\times 1} \\
-(E - S \cdot p)_{3\times 3} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} \\
-p_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3}
\end{pmatrix}
\Xi = m\Xi
\]

for the 10-component field function \( \Xi = \text{column}(\tilde{\chi}, \tilde{\psi}, \tilde{\xi}, \varphi) \). This first-order equation is known as the Duffin-Kemmer-Petiau (DKP) equation [22]. It contains the part corresponding to the 4-vector potential. At first sight, for the construction of (54) we have used the equations (45) and (51-53), and omitted the equations (43,44). However, one can show that our DKP equation contains that information. If we write (43-45) and (51) in the matrix form, we can also write

\[
\begin{pmatrix}
0_{3\times 3} & 0_{3\times 3} & (E + S \cdot p)_{3\times 3} & p_{3\times 1} \\
0_{3\times 3} & 0_{3\times 3} & (E - S \cdot p)_{3\times 3} & -p_{3\times 1} \\
-p_{1\times 3} & 0_{1\times 3} & 0_{1\times 3} & 0_{1\times 3}
\end{pmatrix}
\begin{pmatrix}
\tilde{\psi} \\
\tilde{\chi} \\
\tilde{\xi} \\
\varphi
\end{pmatrix}
= 0,
\]

which is related to (55). It is more convenient to write this equation in terms of \( E, B, \phi \) and \( A \). We use the unitary transformation with

\[
\begin{align*}
\begin{pmatrix}
0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & -m_{3\times 3} \\
0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1} & m_{3\times 3} \\
-p_{3\times 1} & 0_{3\times 3} & 0_{3\times 3} & 0_{3\times 1}
\end{pmatrix}
\begin{pmatrix}
\tilde{\psi} \\
\tilde{\chi} \\
\tilde{\xi} \\
\varphi
\end{pmatrix}
= 0,
\end{align*}
\]

\[5\] It would be of interest to research the helicity basis for the DKP equation, as we did for the Dirac equation. However, we leave this task for the future works. Instead, we are going to consider the helicity basis of the solutions of the Weinberg-Tucker-Hammer second-order equations below.
\[ \mathbf{Y} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1_{3 \times 3} & 1_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 3} \\ i_{3 \times 3} & -i_{3 \times 3} & 0_{3 \times 1} & 0_{3 \times 3} \\ 0_{1 \times 3} & 0_{1 \times 3} & 2_{1 \times 1} & 0_{1 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 1} & 2_{3 \times 3} \end{pmatrix} . \]

As a result we have
\[ \begin{pmatrix} E_{3 \times 3} \\ i(S \cdot p)_{3 \times 3} \\ p_{1 \times 3} \\ -i p_{1 \times 3} \\ im_{3 \times 3} \\ -2m_{1 \times 1} \end{pmatrix} = \begin{pmatrix} \mathbf{E} \\ \mathbf{B} \end{pmatrix} = \begin{pmatrix} \mathbf{A} \end{pmatrix} , \]
where \( p_{1 \times 3} = (p_x, p_y, p_z) \) is the row and \( p_{3 \times 1} \) is the column. It is equivalent to the Proca set.

Taking into account the Proca equations (49,50), the definitions of \( \mathbf{E}^i = F^{i0} \), \( \mathbf{B}^i = -\frac{1}{2}\epsilon^{ijk} F^{jk} \) and the definition of the Levi-Civita tensor, we can obtain the Tucker-Hammer equation [18]:
\[ \begin{pmatrix} E^2 - p^2 - 2m^2 \\ E^2 - p^2 + 2E(S \cdot p) + 2(S \cdot \mathbf{p})^2 \\ E^2 - p^2 - 2m^2 \end{pmatrix} = 0. \]

In the covariant form the equation (59) is written:
\[ (\gamma^\mu p_\mu + p^\mu - 2m^2) \Psi(p^\mu) = 0. \]
with the \( 6 \times 6 \) matrices [19])
\[ \gamma^{00} = \begin{pmatrix} 0 & 1_{3 \times 3} \\ 1_{3 \times 3} & 0 \end{pmatrix}, \quad \gamma^{0i} = \gamma^{i0} = \begin{pmatrix} 0 & -S^i \\ S^i & 0 \end{pmatrix}, \]
\[ \gamma^{ij} = \begin{pmatrix} 0 & -\delta_{ij} + S_i S_j + S_j S_i \\ -\delta_{ij} + S_i S_j + S_j S_i & 0 \end{pmatrix} . \]

In the coordinate space we have:
\[ (\gamma^\mu \partial_\mu + \partial^\mu \partial_\mu + 2m^2) \Psi(x^\mu) = 0. \]

If we set the condition \( \partial_\mu \partial_\mu \rightarrow -m^2 \) we can recover the Weinberg equation, ref. [20]6:
\[ \Gamma \begin{pmatrix} \chi \\ \psi \end{pmatrix} = \begin{pmatrix} m^2 + 2E(S \cdot p) + 2(S \cdot \mathbf{p})^2 \\ m^2 - 2E(S \cdot p) + 2(S \cdot \mathbf{p})^2 \end{pmatrix} \begin{pmatrix} \chi \\ \psi \end{pmatrix} = 0, \]
which is in the covariant form
\[ (\gamma^\mu \partial_\mu + m^2) \Psi(x^\mu) = 0. \]

Thus, from what we have seen above, we can conclude that the Duffin-Kemmer-Petiau, Proca, Weinberg and Tucker-Hammer equations are all related one another. They can be obtained by various transformations from the relativistic dispersion relation, \( E^2 - p^2 = m^2 \).

Let us consider the equation (59) as a set of equations for the bivector components in the helicity basis. Then, we have \( (p = |\mathbf{p}|) \):
\[ (E^2 - p^2 + 2Ep + 2p^2)\psi_1 = (2m^2 - (E^2 - p^2))\chi_1, \]
\[ (E^2 - p^2 - 2Ep + 2p^2)\chi_1 = (2m^2 - (E^2 - p^2))\psi_1, \quad (h = 1) \]
\[ (E^2 - p^2 + 2Ep + 2p^2)\psi_1 = (2m^2 - (E^2 - p^2))\chi_1, \]
\[ (E^2 - p^2 - 2Ep + 2p^2)\chi_1 = (2m^2 - (E^2 - p^2))\psi_1, \quad (h = -1) \]
\[ (E^2 - p^2)\psi_- \rightarrow (2m^2 - (E^2 - p^2))\chi_- , \]
\[ (E^2 - p^2)\chi_- \rightarrow (2m^2 - (E^2 - p^2))\psi_- , \quad (h = 0). \]

\[ \text{6We should mention that this procedure is not quite clear, because the dispersion relations of the Weinberg equation and the Tucker-Hammer equation may be different (see \[21\]). The Weinberg equation permits, in general, the tachyonic solutions, } E^2 - p^2 = -m^2. \]
where the 3-“spinors” are in the helicity basis (see [14, p.192]):

\[
\chi^\dagger = \begin{pmatrix}
\frac{1+\cos \theta}{\sqrt{2}} e^{-i\phi} \\
\frac{\sin \theta}{\sqrt{2}} e^{i\phi}
\end{pmatrix}, \quad \chi = \begin{pmatrix}
\frac{-\sin \theta}{\sqrt{2}} e^{-i\phi} \\
\frac{\cos \theta}{\sqrt{2}} e^{i\phi}
\end{pmatrix}, \quad \chi_1 = \begin{pmatrix}
\frac{1-\cos \theta}{\sqrt{2}} e^{-i\phi} \\
\frac{-\sin \theta}{\sqrt{2}} e^{i\phi}
\end{pmatrix}.
\] (68)

The normalization condition is chosen \( \chi^\dagger \chi = 1 \).

Taking into account (65-67) we can write the bivectors \( u_{\uparrow,\downarrow,-} = \begin{pmatrix} \chi_{\uparrow,\downarrow,-} \\ \bar{u}_{\uparrow,\downarrow,-} \end{pmatrix} \) in the following way:

\[
u_{\uparrow,\downarrow} = N_{\uparrow} \begin{pmatrix} \chi_{\uparrow} \\ \frac{2m^2-(E^2-p^2)}{E^2-p^2+2Ep+2p^2} \chi_{\downarrow} \end{pmatrix}, \quad u_{\uparrow,\downarrow,-} = N_{\downarrow} \begin{pmatrix} \chi_{\downarrow} \\ \frac{2m^2-(E^2-p^2)}{E^2-p^2+2Ep+2p^2} \chi_{\uparrow} \end{pmatrix}, \quad u_{\uparrow,\downarrow} = N_{\uparrow} \begin{pmatrix} \chi_{\uparrow} \\ \frac{2m^2-(E^2-p^2)}{E^2-p^2+2Ep+2p^2} \chi_{\downarrow} \end{pmatrix}.
\] (69)

Let us now introduce \( \pi_\lambda = u_{\uparrow}^\dagger \gamma^{00}, \nu_\lambda = \gamma^5 u_\lambda \) (where \( \gamma^5 = \begin{pmatrix} 1_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & -1_{3\times3} \end{pmatrix} \)). After the normalization to the unit and imposing \( m^2 = E^2-p^2 \), our bivectors are then

\[
u_{\uparrow,\downarrow} = \frac{1}{\sqrt{2}} \begin{pmatrix} E+p \\ m \end{pmatrix} \chi_{\uparrow}, \quad u_{\uparrow,\downarrow,-} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_{\downarrow} \\ \frac{m}{E+p} \chi_{\uparrow} \end{pmatrix}, \quad u_{\uparrow,\downarrow} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{m}{E+p} \chi_{\downarrow} \\ \frac{m}{E+p} \chi_{\uparrow} \end{pmatrix}.
\] (70)

Now we study the discrete symmetry operations for spin-1 case (as we did for spin-1/2 case in the previous Section). The bivectors have the following properties:

1. The Parity \( (p \rightarrow -p, \theta \rightarrow \pi-\theta, \phi \rightarrow \pi+\phi) \). We note that the 3-“spinors” are transformed as \( \chi_h \rightarrow -\chi_{-h}; \) the parity operator is \( P = \gamma^{00} \) (it is analogous to that which was used for spin-1/2 (see (16)). Therefore,

\[ P_{u_{\uparrow,\downarrow}}(-p) = -u_{\uparrow,\downarrow}(p), \quad P_{u_{\uparrow,-}}(-p) = -u_{\uparrow,-}(p), \quad P_{u_{\uparrow}}(-p) = -u_{\uparrow}(p). \] (72)

And,

\[ P_{v_{\uparrow,\downarrow}}(-p) = +v_{\uparrow,\downarrow}(p), \quad P_{v_{\uparrow,-}}(-p) = +v_{\uparrow,-}(p), \quad P_{v_{\uparrow}}(-p) = +v_{\uparrow}(p). \] (73)

2. The Charge Conjugation is defined

\[ C = e^{i\alpha} \begin{pmatrix} 0 & \Theta \\ -\Theta & 0 \end{pmatrix} \mathcal{K} \] (74)

(similarly to (33)) with \( \Theta_{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \). Hence, \( \Theta \chi_\uparrow = \chi_\downarrow, \Theta \chi_\downarrow = \chi_\uparrow, \Theta \chi_\pm = -\chi_\mp \). Finally, we have

\[ C_{u_{\uparrow,\downarrow}}(p) = +e^{i\alpha} v_{\uparrow,\downarrow}(p), \quad C_{u_{\uparrow,-}}(p) = -e^{i\alpha} v_{\uparrow,-}(p), \quad C_{u_{\uparrow}}(p) = +e^{i\alpha} v_{\uparrow}(p). \] (75)

And

\[ C_{v_{\uparrow,\downarrow}}(p) = -e^{i\alpha} u_{\uparrow,\downarrow}(p), \quad C_{v_{\uparrow,-}}(p) = +e^{i\alpha} u_{\uparrow,-}(p), \quad C_{v_{\uparrow}}(p) = -e^{i\alpha} u_{\uparrow}(p). \] (76)

3. The \( CP \) and \( PC \) Operations:

\[ CP_{u_{\uparrow,\downarrow}}(-p) = -PC_{u_{\uparrow,\downarrow}}(-p) = -e^{i\alpha} v_{\uparrow,\downarrow}(p), \] (77)

\[ CP_{v_{\uparrow,\downarrow}}(-p) = -PC_{v_{\uparrow,\downarrow}}(-p) = -e^{i\alpha} u_{\uparrow,\downarrow}(p), \] (78)

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\[ CP_{u,1}(-p) = -PC_{u,1}(-p) = -e^{i\alpha}v_{1,1}(p), \]  
\[ CP_{v,1}(-p) = -PC_{v,1}(-p) = -e^{i\alpha}u_{1,1}(p), \]  
\[ CP_{u,-1}(-p) = -PC_{u,-1}(-p) = +e^{i\alpha}v_{1,-1}(p), \]  
\[ CP_{v,-1}(-p) = -PC_{v,-1}(-p) = +e^{i\alpha}u_{1,-1}(p). \]  

We found within the classical field theory that the properties of particle and anti-particle of spin-1 are different comparing with the known cases (when the basis is chosen in such a way that the solutions are the eigenstates of the parity).

IV. THE CONCLUSIONS.

• Similarly to the \((\frac{1}{2}, \frac{1}{2})\) representation [12], the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) and \((1, 0) \oplus (0, 1)\) field functions in the helicity basis are not eigenstates of the common-used parity operator; \(|p, \lambda> \Rightarrow |-p, -\lambda>\) on the classical level. This is in accordance with the earlier consideration of Berestetskii, Lifshitz and Pitaevskii.

• Helicity field functions may satisfy the ordinary Dirac equation with \(\gamma\)'s to be in the spinorial representation.

• Helicity field functions can be expanded in the set of the Dirac 4-spinors by means of the matrix \(U^{-1}\) given in this paper.

• \(P\) and \(C\) operations anticommute in this framework on the classical level.

• The different formulations of the spin-1 particles are all connected by algebraic transformations.

• The properties of spin-1 solutions in the helicity basis with respect to \(P\), \(C\), \(CP\) are similar to those in the spin-1/2 case.

In order to make the above conclusions to be more firm one should repeat the calculations in the Fock space within the “secondary quantization” framework (see [17] for the spin-1/2 case).

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