

Research Article

Polynomial Decay Rate for a Coupled Lamé System with Viscoelastic Damping and Distributed Delay Terms

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1. Introduction

In this work, we shall be concerned with studying the general decay rate of the following system in \( \Omega \times \mathbb{R}^+ \):

\[
\begin{aligned}
&u_t - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds - k_1\Delta u_t = \int_0^t \mu_1(\rho)\Delta u(x, t - \rho)d\rho = f_1(u, v), \\
v_t - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds - k_2\Delta v_t = \int_0^t \mu_2(\rho)\Delta v(x, t - \rho)d\rho = f_2(u, v).
\end{aligned}
\]

(1)

Equations (1) are associated with the following boundary and initial conditions

\[
\begin{aligned}
u(0, x, t) &= v(x, t) = 0, \text{ on } \partial\Omega \times \mathbb{R}^+, \\
u(0, x, 0) &= u_0(x), v(0, x, 0) = 0, u_t(0, x, 0) = u_1(x), v_t(0, x, 0) = v_1(x), x \in \Omega, \\
(u_t(x, t), v_t(x, t)) &= (f_1(u, t), f_2(u, t)), \text{ in } \Omega \times (0, 2).
\end{aligned}
\]

(2)

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n(n = 1, 2, 3) \), with smooth boundary \( \partial\Omega \). The elasticity differential operator \( \Delta_\varepsilon \) is given by

\[\Delta_\varepsilon u = \mu_\varepsilon \Delta u + (\mu + \lambda)\nabla (\nabla u),\]

(3)

and the Lamé constants \( \mu \) and \( \lambda \) are satisfying the following conditions

\[\mu > 0, \mu + \lambda > 0.\]

(4)

The parameters \( k_1, k_2, r_1, \) and \( r_2 \) are positive constants, with \( r_1 < r_2 \). The functions \( \mu_1, \mu_2 : [r_1, r_2] \rightarrow \mathbb{R} \) are bounded. The functions \( f_1(u, v) \) and \( f_2(u, v) \) which represent the source terms will be specified later.

After several authors have studied the problems of coupled systems and hyperbolic systems, their stability is associated with velocities and is proven under conditions imposed on the subgroup [1]. The researchers also studied behavior of the energy in a limited field with nonlinear damping and external force and a varying delay of time to find solutions to the Lame system [1–9].

Recently, problems that contain viscoelasticity have been addressed, and many results have been found regarding the global existence and stability of solutions (see [2, 9]), under conditions on the relaxation function, whether exponential or polynomial decay. In addition, in [10], Boulaaras obtained the stability result of the global solution to the Lamé system...
with the flexible viscous term by adding logarithmic nonlinearity, even though the kernel is not necessarily decreasing in contrast to what he studied [2].

Introducing a distributed delay term makes our problem different from those considered so far in the literature.

The importance of this term appears in many works, and this is due to the fact that many phenomena depends on their past. Also, it is influence on the asymptotic behavior of the solution for the different types of problems such that Timoshenko system [3, 11–13], transmission problem [14], wave equation [15], and thermoelastic system [16, 17].

In the present work, we extend the general decay result obtained by Feng in [18] to the case of distributed term delay, namely, we will make sure that the result is achieved if the distributed delay term exists.

This paper is organized as follows. In the second section, we give some preliminaries related to problem (1). In Section 3, we prove our main result.

2. Preliminaries

In this section, we provide some materials and necessary assumptions which we need in the prove of our results. We use the standard Lebesgue and Sobolev spaces with their scaler products and norms. For simplicity, we would write ||·|| instead of ||·||2. Throughout this work, we used a generic positive constant C.

For the relaxation functions g1 and g2, we assume, for i = 1, 2:

(A1) \( g_i(t) : \mathbb{R}^+ \to \mathbb{R}^+ \) are nonincreasing \( C^1 \) functions satisfying

\[
g_i(0) > 0 \quad \text{and} \quad \mu - \int_0^\infty g_i(s) \, ds = l_i > 0.
\]

We assume further that for i = 1, 2:

(A2) There exist two \( C^1 \) functions \( G_i : \mathbb{R}^+ \to \mathbb{R}^+ \), with \( G_i(0) = G_i(0) = 0 \), which are linear or are strictly increasing and strictly convex functions of class \( C^2(\mathbb{R}^+) \) on \( (0, r] \), \( r \leq g_i(0) \), such that

\[
g_i'(t) \leq -\xi_i(t) G_i(g_i(t)), \quad \forall t \geq 0,
\]

where \( \xi_i(t) \) are \( C^1 \) functions satisfying

\[
\xi_i(t) > 0, \quad \xi_i'(t) \leq 0, \quad \forall t \geq 0.
\]

(A3) For the source terms \( f_1 \) and \( f_2 \), we take

\[
f_1(u, v) = a(|u|^{p-1}u + v + \beta |u|^{p-1}v|v|^{p+1})/2, \quad \forall (u, v) \in \mathbb{R}^n \]

\[
f_2(u, v) = a(|u|^{p-1}u + v + \beta |v|^{p-1}v|u|^{p+1})/2, \quad \forall (u, v) \in \mathbb{R}^n \]

with \( a, \beta > 0 \). Clearly,

\[
u f_1(u, v) + v f_2(u, v) = (p + 1) F(u, v), \quad \forall (u, v) \in \mathbb{R}^n \]

where

\[
F(u, v) = \frac{1}{(p + 1)} \left[ a |u|^{p+1} + 2\beta |u|^{p+1}v|v|^{p+1}/2 \right], \quad \forall (u, v) \in \mathbb{R}^n \]

\[
f_1(u, v) = \frac{\partial F}{\partial u}, \quad f_2(u, v) = \frac{\partial F}{\partial v}.
\]

Further, we assume that there is \( C > 0 \), such that

\[
\left| \frac{df_i(u, v)}{du} \right| + \left| \frac{df_i(u, v)}{dv} \right| \leq C(|u|^{p+1} + |v|^{p+1}), \quad i = 1, 2 \quad \text{where} \quad 1 \leq p \leq 6.
\]

(A4)

\[
\text{if } n = 1, 2; p \geq 3, \text{ if } n = 3; p = 3.
\]

So, we have the embedding

\[
H_0^1(\Omega) \subseteq L^q(\Omega) \text{ for } 2 \leq q \leq \frac{2n}{n - 2} \text{ if } n \geq 3 \text{ or } q \geq 2 \text{ if } n = 1, 2,
\]

\[
L^r \text{ for } q < r.
\]

Let \( c_i \) the same embedding constant, so we have

\[
\|v\|_q \leq c_i\|\nabla v\|_2, \quad \|v\|_q \leq c_i\|v\|, \quad \forall v \in H_0^1(\Omega).
\]

Remark 1. There exist two constants \( A_1 > 0 \) and \( A_2 > 0 \) such that

\[
\int_{\Omega} |f_i(u, v)|^2 \, dx \leq A_i(l_1 \|\nabla u\|^2 + l_2 \|\nabla v\|^2)^p, \quad i = 1, 2.
\]

As in many papers, we introduce the following new variables

\[
\begin{align*}
\begin{cases}
z(x, \rho, \rho, t) = u_i(x, t - \rho \rho), \\
y(x, \rho, \rho, t) = v_i(x, t - \rho \rho),
\end{cases}
\end{align*}
\]

then, we obtain

\[
\begin{align*}
\begin{cases}
\rho z(x, \rho, \rho, t) + z_\rho(x, \rho, \rho, t) = 0, \\
z(x, 0, \rho, t) = u_i(x, t),
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
\rho y(x, \rho, \rho, t) + y_\rho(x, \rho, \rho, t) = 0, \\
y(x, 0, \rho, t) = v_i(x, t),
\end{cases}
\end{align*}
\]
Consequently, the problem (1) is equivalent to
\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta u + \int_0^t g_1(t-s)\Delta u(s)ds - k_1\Delta u_t - \int_0^t \mu_1(\rho)\Delta z(x, 1, \rho, t)d\rho &= f_1(u, v), \\
\frac{\partial v}{\partial t} - \Delta v + \int_0^t g_2(t-s)\Delta v(s)ds - k_2\Delta v_t - \int_0^t \mu_2(\rho)\Delta y(x, 1, \rho, t)d\rho &= f_2(u, v), \\
\rho z(x, \rho, \rho, t) + z_t(x, \rho, \rho, t) &= 0, \\
\rho y(x, \rho, \rho, t) + y_t(x, \rho, \rho, t) &= 0,
\end{align*}
\]
with the initial data and boundary conditions
\[
\begin{align*}
(u_0(x, 0), v(x, 0)) &= (u_0(x), v_0(x)), \text{ in } \Omega, \\
(u_0(x, 0), v(x, 0)) &= (u_1(x), v_1(x)), \text{ in } \Omega, \\
(u_0(x, t), v(x, t)) &= (f_1(x, t), g_1(x, t)), \text{ in } \Omega \times (0, \tau_2), \\
u(x, t) = v(x, t) = 0, \text{ in } \partial \Omega \times (0, \infty), \\
(z(x, \rho, \rho, 0), y(x, \rho, \rho, 0)) &= (f_2(x, \rho, \rho), g_2(x, \rho, \rho)), \text{ in } \Omega \times (0, 1) \times (0, \tau_2),
\end{align*}
\]
where
\[
(x, \rho, \rho, t) \in \Omega \times (0, 1) \times (\tau_1, \tau_2) \times (0, \infty).
\]

We recall the following notations
\[
\begin{align*}
(h \ast \varphi) &= \int_0^t h(t-s)\varphi(s)dsdx, \\
(h \circ \varphi)(t) &= \int_0^t h(t-s)\varphi(t) - \varphi(s)^2ds.
\end{align*}
\]
Thus, we have the following important property
\[
\int_{\Omega} (h \ast \varphi)\varphi dx = -\frac{1}{2} h(t)\|\varphi(t)\|^2 + \frac{1}{2} (h' \ast \varphi)(t) - \frac{1}{2} \int_0^t \left( h(t) \right) \|\varphi(t)\|^2 dsdx.
\]

The energy modified associated to the problem (19) is defined by
\[
E(t) = \frac{1}{2} \left[ \|u_t\|^2 + \left( \mu - \int_0^t g_1(s)ds \right) \|\nabla u\|^2 + (\lambda + \mu)\|\text{div}u\|^2 \right]
\]
\[
+ \frac{1}{2} \left[ (g_1'\nabla u)(t) + \eta \int_0^t \int_{\Omega} \rho_1(\rho)\|\nabla z(x, \rho, \rho, t)\|^2 d\rho dx \right]
\]
\[
+ \frac{1}{2} \left[ \|v_t\|^2 + \left( \mu - \int_0^t g_2(s)ds \right) \|\nabla v\|^2 + (\lambda + \mu)\|\text{div}v\|^2 \right]
\]
\[
+ \frac{1}{2} \left[ (g_2'\nabla v)(t) + \eta \int_0^t \int_{\Omega} \rho_2(\rho)\|\nabla y(x, \rho, \rho, t)\|^2 d\rho dx \right]
\]
\[
- \int_{\Omega} F(u, v)dx.
\]

First, we prove in the following theorem the result of energy identity.

**Lemma 2.** Assume that
\[
\int_{\mathcal{E}} |\mu_1(\rho)|d\rho < k_1, \quad i = 1, 2.
\]

Then, the energy modified defined by (24) satisfies, along the solution \((u, v, z, y)\) of (19), the estimate
\[
\frac{d}{dt} E(t) \leq -\frac{1}{2} g_1(t)\|\nabla u\|^2 + \frac{1}{2} \left( g_1'\nabla u \right)(t) - \frac{1}{2} \|g_2(\tau)\|\nabla v\|^2
\]
\[
+ \frac{1}{2} \left[ \|g_2'(\nabla v)(t) - \left( \frac{\eta + 1}{2} \right) \int_{\mathcal{E}} |\mu_1(\rho)|d\rho \right]
\]
\[
\cdot \|\nabla u\|^2 - \frac{\eta - 1}{2} \int_{\mathcal{E}} |\mu_2(\rho)|\|\nabla z(x, 1, \rho, t)\|^2 d\rho dx
\]
\[
- \frac{\eta - 1}{2} \int_{\mathcal{E}} \|\mu_2(\rho)\|\|\nabla y(x, 1, \rho, t)\|^2 d\rho dx \leq 0,
\]
for
\[
1 < \eta < \min \left( \frac{2k_1}{\int_{t_1}^{t_2} |\mu_1(\rho)|d\rho}, \frac{2k_2}{\int_{t_1}^{t_2} |\mu_2(\rho)|d\rho} \right) - 1.
\]

**Proof.** First multiplying the equation \((0,1)_u\) by \(u_t\) and integrating by parts over \(\Omega\), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left[ \|u_t\|^2 + \mu \|\nabla u\|^2 + (\lambda + \mu)\|\text{div}u\|^2 \right]
\]
\[
- \int_{\Omega} \nabla u \int_0^t g_1(t-s)\nabla u(s)ds + k_1\|\nabla u\|^2
\]
\[
+ \int_{\Omega} \nabla u \int_0^t \mu_1(\rho)\nabla z(x, 1, \rho, t)d\rho dx
\]
\[
= \int_{\Omega} f_1(u, v)udx,
\]
by using (23), we obtain
\[
1 - \frac{1}{2} \frac{d}{dt} \left[ \|u_t\|^2 + \left( \mu - \int_0^t g_1(s)ds \right)\|\nabla u\|^2 + (\lambda + \mu)\|\text{div}u\|^2 + (g_1'\nabla u)(t) \right]
\]
\[
= -\frac{1}{2} g_1\|\nabla u\|^2 + \frac{1}{2} \left( g_1'\nabla u \right)(t) + \int_{\Omega} u_0 f_1(u, v)dx - k_1\|\nabla u\|^2
\]
\[
+ \int_{\Omega} \nabla u \int_0^t \mu_1(\rho)\nabla z(x, 1, \rho, t)d\rho dx.
\]
Similarly, multiplying the equation (19) by \( v_1 \) and integrating over \( \Omega \), we obtain
\[
\frac{1}{2} \frac{d}{dt} \left[ \| v_1 \|^2 + \left( \mu - \int_0^t g_2(s) ds \right) \| \nabla v_1 \|^2 + (\lambda + \mu) \| \text{div} v_1 \|^2 + (g_3 \nabla v_1)(t) \right] \\
= \frac{1}{2} g_2 \| \nabla v_1 \|^2 + \frac{1}{2} \left( g_3 \nabla v_1 \right)(t) + \int_\Omega v_1 f_2(v_1) dx - k_2 \| \nabla v_1 \|^2 \\
+ \int_\Omega \nabla \Omega \int_{\tau_1}^{\tau_2} \mu_2(\rho) \nabla y(x, 1, \rho, t) d P dx.
\]

Multiplying the equation (19) by \(-\eta|\mu_1(\rho)|\Delta z(x, \rho, \rho, t)\) and integrating by parts over \( \Omega \times (0, 1) \times (\tau_1, \tau_2) \), we obtain
\[
\eta \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho |\mu_1(\rho)| \nabla x(x, \rho, \rho, t) \nabla z_1(x, \rho, \rho, t) d P dx d \rho d \rho
\\
= -\eta \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\rho)| \nabla x(x, \rho, \rho, t) \nabla z_1(x, \rho, \rho, t) d P dx d \rho d \rho,
\]
therefore
\[
\frac{d}{dt} \left[ \frac{1}{2} \eta \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \rho |\mu_1(\rho)| \nabla x(x, \rho, \rho, t) \nabla z_1(x, \rho, \rho, t) d P dx d \rho d \rho \right]
\\
= -\frac{\eta}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\rho)| d \rho \nabla x(x, \rho, \rho, t) \nabla z_1(x, \rho, \rho, t) d P dx d \rho d \rho
\\
= -\frac{\eta}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\rho)| \nabla u_1(x, t) d P dx d \rho d \rho
\\
+ \frac{\eta}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\rho)| \nabla u_1(x, t) d P dx d \rho d \rho.
\]

Using Young’s inequality, we have
\[
\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \mu_1(\rho) \nabla z_1(x, \rho, \rho, t) d P dx d \rho d \rho
\\
\leq \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \left( \nabla u_1 \sqrt{|\mu_1(\rho)|} \right) \left( \sqrt{|\mu_1(\rho)|} \nabla z_1(x, 1, \rho, t) \right) d P dx d \rho d \rho
\\
\leq \frac{1}{2} \left( \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} |\mu_1(\rho)| |P| \sqrt{\| \nabla u_1 \|^2} \right)
\\
+ \frac{1}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} \mu_1(\rho) \nabla z_1(x, 1, \rho, t) d P dx d \rho d \rho.
\]
\[ \int_{\Omega} \nabla v \cdot \mu_\gamma(\Omega) \nabla v(x, 1, \rho, t) \, dx \leq \frac{1}{2} \left( \int_{\Omega} |\mu_\gamma(\Omega)||\nabla v|| + \frac{1}{2} \int_{\Omega} |\mu_\gamma(\Omega)||\nabla v(x, 1, \rho, t)||\, dp \right)^2. \] (37)

This completes the proof.

3. General Decay

In this section we will prove that the solution of problems (19)–(20) decay generally to trivial solution. Using the energy method and suitable Lyapunov functional.

In the following, we will present our main stability result:

**Theorem 3** (Decay rates of energy). Assume that (A1)-(A3) hold. Then, for every \( t_0 > 0 \), there exist two positive constants \( a_1 \) and \( a_2 \) such that the energy defined by (24) satisfies the following decay

\[ E(t) \leq a_2 G_A^2 \left( \alpha_1 \int_{g_i^{-1}(r)} \xi(s) \, ds \right), \forall t \geq g_i^{-1}(r), \] (38)

where

\[ G_A(t) = \int_t^\infty \frac{1}{sG_\gamma(s)} \, ds, G_\gamma(t) = \min \left\{ G'_i(t), G''_i(t) \right\}, \] (39)

and \( \xi(t) = \min \{ \xi_1(t), \xi_2(t) \}, g(t) = \max \{ g_1(t), g_2(t) \}. \)

This theorem will be proved later after providing some remarks.

**Remark 4.**

1. In case \( \int_0^\infty \xi_i(t) \, dt = \infty \), Theorem 3 ensures \( \lim_{t \to \infty} E(t) = 0 \).
2. From (A2), we infer that \( \lim_{t \to \infty} g_i(t) = 0 \). Then, there exists some \( t_1 \geq 0 \) large enough such that
   \[ g_i(t_1) = r \Rightarrow g_i(t) \leq r, \forall t \geq t_1. \] (40)

As \( G_i \) is positive continuous functions, and \( g_i \) and \( \xi_i \) are positive nonincreasing continuous functions, then, for all \( 0 \leq t \leq t_1 \),

\[ 0 \leq g_i(t_1) \leq g_i(t) \leq g_i(0) \text{ and } 0 < \xi_i(t_1) \leq \xi_i(t) \leq \xi_i(0), \] (41)

which implies for some positive constants \( a_i \) and \( b_i \),

\[ a_i \leq \xi_i(t)G_i(g_i(t)) \leq b_i. \] (42)

Consequently,

\[ g_i(t) \leq -\xi_i(t)G_i(g_i(t)) \leq -\frac{a_i}{g_i(0)}g_i(0) - \frac{a_i}{g_i(0)}g_i(t), \text{ for } t \in [0, t_1]. \] (43)

(3) We also mention Johnson’s inequality, which is very important for proving our result. If \( G \) is a convex function on \([a, b], g : \Omega \to a, b]\), we have

\[ G \left[ \frac{1}{k} \int \sum g(x)h(x) \, dx \right] \leq \frac{1}{k} \int \sum G[g(x)]h(x) \, dx, \] (44)

where \( h \) is a function that satisfies

\[ h(x) \geq 0 \text{ and } \int \sum h(x) \, dx = k > 0. \] (45)

To prove the desired result, we create a Lyapunov functional equivalent to \( E \). For this, we define some functions that allow us to construct this Lyapunov function.

As in Baowei [18] and Mustafa ([19, 20]), we define

\[ C_{ij} = \int_0^\infty \frac{g_i''(s)}{\sqrt{\xi g_i(s) - g_i''(s)}}, \text{ and } h_i(t) = \xi g_i(t) - g_i''(t), i = 1, 2. \] (46)

for any \( 0 < \xi < 1 \).

**Lemma 5.** Let \((u, v, z, y)\) be a solution of the problem (19). Then, the functional

\[ \varphi(t) = \int_\Omega u(t)u_i(t) \, dx + \int_\Omega v(t)v_i(t) \, dx, \] (47)

satisfies the estimate

\[ \varphi'(t) \leq -\frac{1}{2} \|\nabla u(t)\|^2 + \frac{1}{2} \|\nabla v(t)\|^2 + \|u_i(t)\|^2 + \|v_i(t)\|^2 \]

\[ - (\lambda + \mu) \|\nabla u(t)\|^2 - (\lambda + \mu) \|\nabla v(t)\|^2 \]

\[ + \frac{3C_{ij}}{2I_i} (h_i + \nabla u)(t) + \frac{3k^2}{2I^2_\gamma} \|\nabla v_i\|^2 \]

\[ \varphi'(t) + \frac{3k^2}{2I^2_\gamma} \int_\Omega |\mu_\gamma(\Omega)||\nabla v(x, 1, \rho, t)||\, dp \]

\[ + \frac{3k^2}{2I^2_\gamma} \int_\Omega |\mu_\gamma(\Omega)||\nabla y(x, 1, \rho, t)||\, dp \] (48)

**Proof.** Taking the derivative of (47), we obtain

\[ \varphi'(t) = \int_\Omega |u(t)|^2 \, dx + \int_\Omega u(t)u_i(t) \, dx + \int_\Omega |v(t)|^2 \, dx \]

\[ + \int_\Omega v(t)v_i(t) \, dx. \] (49)
From problem (19) and using integration by parts, we get

\[
\psi'(t) = \|u(t)\|^2 + \|v(t)\|^2 + \int_{\Omega} u(t) \left( \Delta u - \int_0^t g_1(t-s) \Delta u(s) ds \right) dx \\
+ k_1 \Delta u + \int_{\Omega} \mu_1(t) \Delta z(x, 1, \varphi, t) dx + f_1(u, v) dx \\
+ \int_{\Omega} \mu_2(t) \Delta y(x, 1, \varphi, t) dx + f_2(u, v) dx \\
= \|u(t)\|^2 + \|v(t)\|^2 - k_1 \int_{\Omega} \nabla u \nabla u dx - k_1 \int_{\Omega} \nabla v \nabla v dx \\
- \mu \int_{\Omega} g_1(t) \|\Delta u\|^2 - (\lambda + \mu) \|\nabla u\|^2 \\
+ \int_{\Omega} \nabla u (t) \int_{\Omega} g_1(t-s) \nabla u(s) ds dx - \int_{\Omega} \nabla u (t) \int_{\Omega} \mu_1(t) \nabla z(x, 1, \varphi, t) dx \\
+ \int_{\Omega} \nabla v (t) \int_{\Omega} g_1(t-s) \nabla v(s) ds dx - \int_{\Omega} \nabla v (t) \int_{\Omega} \mu_2(t) \nabla y(x, 1, \varphi, t) dx \\
- k_1 \int_{\Omega} \nabla v \nabla u dx - \mu \int_{\Omega} g_1(t) \|\nabla v\|^2 - (\lambda + \mu) \|\nabla v\|^2 \\
+ \int_{\Omega} \nabla u (t) \int_{\Omega} g_1(t-s) \nabla (u-v) ds dx \\
- \int_{\Omega} \nabla u (t) \int_{\Omega} \mu_1(t) \nabla z(x, 1, \varphi, t) dx \\
- \int_{\Omega} \nabla v (t) \int_{\Omega} \mu_2(t) \nabla y(x, 1, \varphi, t) dx \\
+ \int_{\Omega} \nabla u (t) \int_{\Omega} f_1(u, v) dx + \int_{\Omega} \nabla v (t) \int_{\Omega} f_2(u, v) dx.
\]

(50)

By using Hölder and Young’s inequalities, we have

\[
\int_{\Omega} \nabla u (t) \int_{\Omega} g_1(t-s) \nabla (u-v) ds dx \\
\leq \frac{l_1}{6} \|\nabla u(t)\|^2 + \frac{3}{2l_1} \int_{\Omega} \left( \int_{\Omega} \left( g_1(t-s) \nabla u(s) - \nabla u(t) \right) ds \right)^2 dx \\
\leq \frac{l_1}{6} \|\nabla u(t)\|^2 + \frac{3}{2l_1} \int_{\Omega} \left( \int_{\Omega} \left( g_1(s) \right) ds \right)^2 dx \\
\times \left( \int_{\Omega} \left( \sqrt{g_1(t-s)} - \sqrt{g_1(s)} \right) ds dx \right)^2 \\
\leq \frac{l_1}{6} \|\nabla u(t)\|^2 + \frac{3}{2l_1} \int_{\Omega} \left( \int_{\Omega} \left( g_1(s) \right) ds \right)^2 dx \\
\times \left( \int_{\Omega} \left( \sqrt{g_1(t-s)} - \sqrt{g_1(s)} \right) ds dx \right)^2 \\
\leq \frac{l_1}{6} \|\nabla u(t)\|^2 + \frac{3C_{g1}^2}{2l_1} \|u(t)\|
\]

(51)

Similarly, we obtain

\[
\int_{\Omega} \nabla v (t) \int_{\Omega} g_1(t-s) \nabla (v-v) ds dx \\
\leq \frac{l_1}{6} \|\nabla v(t)\|^2 + \frac{3}{2l_1} \int_{\Omega} \left( \int_{\Omega} \left( g_1(t-s) \nabla v(s) - \nabla v(t) \right) ds \right)^2 dx \\
\leq \frac{l_1}{6} \|\nabla v(t)\|^2 + \frac{3}{2l_1} \int_{\Omega} \left( \int_{\Omega} \left( g_2(s) \right) ds \right)^2 dx \\
\times \left( \int_{\Omega} \left( \sqrt{g_1(t-s)} - \sqrt{g_2(s)} \right) ds dx \right)^2 \\
\leq \frac{l_1}{6} \|\nabla v(t)\|^2 + \frac{3C_{g2}^2}{2l_1} \|v(t)\|
\]

(52)

The Young’s inequality gives

\[
k_1 \int_{\Omega} \nabla u(t) \nabla u_1(t) dx \leq \frac{l_1}{6} \|\nabla u(t)\|^2 + \frac{3k_1^2}{2l_1} \|\nabla u_1(t)\|^2, \\
k_2 \int_{\Omega} \nabla v(t) \nabla v_1(t) dx \leq \frac{l_1}{6} \|\nabla v(t)\|^2 + \frac{3k_2^2}{2l_1} \|\nabla v_1(t)\|^2.
\]

(53)

For the source term, we have

\[
\int_{\Omega} u(t) f_1(u, v) dx + \int_{\Omega} v(t) f_2(u, v) dx = (p+1) \int_{\Omega} F(u, v) dx.
\]

(54)

Combining the equations (51)–(54), thus, our proof is completed.

Lemma 6. Let \((u, v, z, y)\) be a solution of the problem (19). Then, the functional

\[
\psi(t) = \int_{\Omega} u_1(t) \int_{\Omega} g_1(s) (u(s) - u(t)) ds dx \\
+ \int_{\Omega} v_1(t) \int_{\Omega} g_2(s) (v(s) - v(t)) ds dx = \psi_1(t) + \psi_2(t),
\]

(55)
satisfies for any \( \delta > 0 \) the estimate

\[
\psi'(t) \leq (\delta + \delta \Lambda_3) \| \nabla u(t) \|^2 + \delta \| \text{div} u \|^2 + \delta \Lambda_4 \| \nabla v(t) \|^2 \\
+ \left( \delta - \int_0^t g_1(s) ds \right) \| u(t) \|^2 + \frac{c \left[ C_{c_2} + 1 \right]}{\delta} (h_2 \nabla u)(t) \\
+ \delta k_1 \int_0^t \mu_1(\rho) |\nabla z(x, t, \rho, s)|^2 dx ds + \delta k_2 \| \nabla v(t) \|^2 \\
+ (\delta + \delta \Lambda_4) \| \nabla v(t) \|^2 + \delta \| \text{div} v \|^2 + \delta \Lambda_4 \| \nabla u(t) \|^2 \\
+ \left( \delta - \int_0^t g_1(s) ds \right) \| v(t) \|^2 + \frac{c \left[ C_{c_2} + 1 \right]}{\delta} (h_2 \nabla v)(t) \\
+ \delta k_2 \int_0^t \mu_2(\rho) |\nabla y(x, t, \rho, s)|^2 dx ds + \delta k_1 \| \nabla v(t) \|^2,
\]

(56)

where \( \Lambda_3 \) and \( \Lambda_4 \) are two positive constants.

Proof. First, we begin to estimate \( \psi'(t) \)

\[
\psi'(t) = \int_\Omega u(t) \left( \int_0^t g_1(s) ds \right) \| u(t) \|^2 ds \\
+ \int_\Omega u(t) \left( \int_0^t g_1(s) ds \right) \| u(t) \|^2 ds - \left( \int_0^t g_1(s) ds \right) \| u(t) \|^2 \\
= \int_\Omega \Delta u - \int_0^t g_1(t-s) \Delta u(s) ds \\
+ k_1 \Delta u + \int_0^t \mu_1(\rho) \Delta z(x, t, \rho, s) dp + f_1(u, v) \\
\times \left( \int_0^t g_1(t-s) (u(s) - u(t)) ds \right) \\
+ \int_\Omega \left( \int_0^t g_1(t-s) (u(s) - u(t)) ds \right) \\
- \left( \int_0^t g_1(s) ds \right) \| u(t) \|^2 \\
- \left( \int_0^t g_1(s) ds \right) \| u(t) \|^2 - \left( \int_0^t g_1(t-s) (\nabla u(s) - \nabla u(t)) ds \right) \\
= \int_\Omega \left( \int_0^t g_1(t-s) (u(s) - u(t)) ds \right) \\
- \left( \int_0^t g_1(s) ds \right) \| u(t) \|^2 \\
- \left( \int_0^t g_1(t-s) (\nabla u(s) - \nabla u(t)) ds \right) \\
- \mu_1(\rho) \text{div}(u(t))ds \\
- \left( \int_0^t g_1(t-s) (\nabla u(s) - \nabla u(t)) ds \right) \\
+ \int_\Omega \left( \int_0^t \mu_1(\rho) \nabla z(x, t, \rho, s) \right) \left( \int_0^t g_1(t-s) (\nabla u(s) - \nabla u(t)) ds \right) dx \\
+ \int_\Omega f_1(u, v) \left( \int_0^t g_1(t-s) (u(s) - u(t)) ds \right) dx \\
- k_1 \int_\Omega \nabla u(t) \left( \int_0^t g_1(t-s) (\nabla u(s) - \nabla u(t)) ds \right) dx.
\]

(57)

As in previous proof and by using Young’s inequality, we conclude that for any \( \delta > 0 \),

\[
\psi'(t) = \int_\Omega u(t) \left( \int_0^t g_1(t-s) (u(s) - u(t)) ds \right) dx \\
- \left( \int_0^t g_1(s) ds \right) \| u(t) \|^2 \\
- \left( \int_0^t g_1(s) ds \right) \| u(t) \|^2 \\
- \left( \int_0^t g_1(t-s) (\nabla u(s) - \nabla u(t)) ds \right) ^2 \\
- \left( \int_0^t g_1(t-s) (\nabla u(s) - \nabla u(t)) ds \right) ^2 \\
- \left( \int_0^t g_1(t-s) (\nabla u(s) - \nabla u(t)) ds \right) ^2
\]

(58)

Similarly and by using the fact \( \| \text{div} u \|^2 \leq c \| \nabla u \|^2 \), we have

\[
(\lambda + \mu) \int_\Omega \text{div}(u(t)) \left( \int_0^t g_1(t-s) (\nabla u(s) - \nabla u(t)) ds \right) dx \\
\leq \delta \| \text{div} \|^2 + \frac{c C_{c_1} l_2 \delta}{\delta} (h_1 \nabla u)(t).
\]

(60)

The same argument for

\[
\int_\Omega f_1(u, v) \left( \int_0^t g_1(t-s) (u(s) - u(t)) ds \right) dx \\
\leq \delta \int_\Omega \left( f_1(u, v) \right)^2 dx + \frac{1}{4 \delta} \left( \int_0^t g_1(t-s) (u(s) - u(t)) ds \right) ^2 dx \\
\leq \delta \Lambda_1 (I_{1\| \nabla u \|^2} + l_2 \| \nabla u \|^2)^2 + \frac{c C_{c_1} l_2 \delta}{\delta} (h_1 \nabla u)(t) \\
\leq \delta \Lambda_3 (I_{l_1 \| \nabla u \|^2} + l_2 \| \nabla u \|^2)^2 + \frac{c C_{c_1} l_2 \delta}{\delta} (h_1 \nabla u)(t),
\]

(61)

Then, we have

where \( \Lambda_3 = \left( (2p+1)/(p-1) \right) E(0)^{p-1} \).
From (46), we have
\[
\int_{\Omega} u_i(t) \int_0^t g_i'(t-s) (u(s) - u(t)) ds \, dx \\
= \int_{\Omega} u_i(t) \int_0^t h_i(t-s) (u(s) - u(t)) ds \, dx \\
- \int_{\Omega} u_i(t) \int_0^t \xi \, g_i'(t-s) (u(s) - u(t)) ds \, dx \leq \delta \|u_i(t)\|^2 \\
+ \frac{c}{\delta} \int_{\Omega} \left( \int_0^t \sqrt{h_1(t-s)} \sqrt{h_1(t-s)} (u(s) - u(t)) ds \right)^2 \, dx \\
+ \frac{c^2}{\delta} \int_{\Omega} \left( \int_0^t g_i(t-s) (u(s) - u(t)) ds \right)^2 \, dx \leq \delta \|u_i(t)\|^2 \\
+ \frac{c}{\delta} \left( \int_0^t h_1(s) ds \right) (h_1 + u(t)) + \frac{c^2 C_{L_1}}{\delta} (h_1 + u(t)) \leq \delta \|u_i(t)\|^2 \\
+ \frac{c}{\delta} \left( h_1 \circ \nabla u(t) \right) + \frac{c^2 C_{L_1}}{\delta} (h_1 \circ \nabla u(t)) \leq \delta \|u_i(t)\|^2 \\
+ \frac{c^2}{\delta} (h_1 \circ \nabla u(t))
\]

The same steps can be taken to get the next estimate for \(\psi_2'(t)\),
\[
\psi_2'(t) \leq \left( \delta - \left( \int_0^t g_i(s) ds \right) \right) \|u_i(t)\|^2 + \left( \delta + \delta \Lambda_1 L_2 \right) \|\nabla u\|^2 \\
+ \delta \|\text{div} u\|^2 + \delta \Lambda_1 L_2 \|\nabla u\|^2 + \frac{c^2}{\delta} (h_1 \circ \nabla u(t))
\]

where \(\Lambda_1 = \Lambda_2 \left[ \left( 2(p+1)/(p-1) \right) E(0) \right]^{p-1}\).

**Lemma 7.** Let \((u, v, z, y)\) be a solution of the problem (19). Then, the functional
\[
I(t) = \int_{\Omega} \int_{t_1}^t \int_{0}^{r_2} \rho e^{-rp} \left[ |\mu_1| (|\nabla z(x, \rho, \rho, t)|^2 \\
+ |\mu_2| (|\nabla y(x, \rho, \rho, t)|^2 \right] dxdpdr,
\]

satisfies the estimate
\[
I'(t) \leq -e^{-r_2} \int_{\Omega} \int_{t_1}^t \left| \mu_1 \right| (\|\nabla z(x, I, \rho, t)\|^2 \right| dxdpdr \\
- e^{-r_2} \int_{\Omega} \int_{t_1}^t \left| \mu_2 \right| (\|\nabla y(x, I, \rho, t)\|^2 \right| dxdpdr \\
+ k_1 \|\nabla u_i(t)\|^2 + k_2 \|\nabla v_i(t)\|^2 - I(t)
\]

**Proof.** Differentiating (66) with respect to \(t\), we get
\[
\frac{d}{dt} I(t) = 2 \int_{\Omega} \int_{t_1}^t \int_{0}^{r_2} \rho e^{-rp} \left[ |\mu_1| (|\nabla z(x, \rho, \rho, t)|^2 \\
+ |\mu_2| (|\nabla y(x, \rho, \rho, t)|^2 \right] dxdpdr.
\]

By using (17) and (18), we have
\[
\frac{d}{dt} I(t) = - \int_{\Omega} \int_{t_1}^t \int_{0}^{r_2} e^{-rp} \left[ |\mu_1| \left( \frac{\partial}{\partial \rho} \right) (|\nabla z(x, \rho, \rho, t)|^2 \\
+ |\mu_2| \left( \frac{\partial}{\partial \rho} \right) (|\nabla y(x, \rho, \rho, t)|^2 \right) dxdpdr \\
= - \int_{\Omega} \int_{t_1}^t \int_{0}^{r_2} \left[ |\mu_1| \left( \frac{\partial}{\partial \rho} \right) (e^{-rp} |\nabla z(x, \rho, \rho, t)|^2 \\
+ |\mu_2| \left( \frac{\partial}{\partial \rho} \right) (e^{-rp} |\nabla y(x, \rho, \rho, t)|^2 \right) dxdpdr \\
- \int_{\Omega} \int_{t_1}^t \int_{0}^{r_2} \rho e^{-rp} \left[ |\mu_1| (|\nabla z(x, \rho, \rho, t)|^2 \\
+ |\mu_2| (|\nabla y(x, \rho, \rho, t)|^2 \right] dxdpdr.
\]
Thus,
\[
\frac{d}{dt}I(t) = - \int_{\Omega} \int_{r_1}^{r_2} e^{p} d\mu_1(\rho)\|\nabla s(x, \rho, t)\|^2 dx dp
\]
\[
+ \left( \int_{r_1}^{r_2} e^{p} d\mu_1(\rho) \right) \|\nabla s(x, \rho, t)\|^2
\]
\[
- \int_{\Omega} \int_{r_1}^{r_2} e^{p} d\mu_1(\rho) \|\nabla s(x, \rho, t)\|^2 dx dp
\]
\[
+ \left( \int_{r_1}^{r_2} d\mu_1(\rho) \right) \|\nabla s(x, \rho, t)\|^2
\]
\[
+ \left( \int_{r_1}^{r_2} d\mu_1(\rho) \right) (\mu_1(\rho)) \|\nabla s(x, \rho, t)\|^2 dx dp
\]
\[
\cdot \left( \mu_1(\rho)\|\nabla s(x, \rho, t)\|^2 + \mu_1(\rho)\|\nabla s(x, \rho, t)\|^2 \right) dx dp.
\]
(70)

Since \( e^{p} \) is decreasing function over \((r_1, r_2)\), the desired estimate (67) follows immediately from (25).

The following lemmas are needed to prove the general decay when the functions \( G_i(t) \) \((i = 1, 2)\) are nonlinear. The proof can be found in Mustafa [19].

**Lemma 8. The functional**

\[
\theta_1(t) = \int_{\Omega} \int_{t_0}^{t} \sigma_1(s-t) |\nabla u(s)|^2 ds dx,
\]
(71)

where \( \sigma_1(t) = \int_{t_0}^{t} g_1(s) ds \) satisfies

\[
\theta_1'(t) \leq - \frac{1}{2} (g_1 e \nabla u)(t) + 3(\mu - I_1) \|\nabla u\|^2.
\]
(72)

**Lemma 9. The functional**

\[
\theta_2(t) = \int_{\Omega} \int_{t_0}^{t} \sigma_2(s-t) |\nabla v(s)|^2 ds dx,
\]
(73)

where \( \sigma_2(t) = \int_{t_0}^{t} g_2(s) ds \) satisfies

\[
\theta_2'(t) \leq - \frac{1}{2} (g_2 e \nabla v)(t) + 3(\mu - I_2) \|\nabla v\|^2.
\]
(74)

Now, we define the following functional

\[
\mathcal{F}(t) = NE(t) + N_1 \phi(t) + N_2 \psi(t) + I(t),
\]
(75)

where \( N, N_1, \) and \( N_2 \) are positive constants. It is easy to prove \( F(t) \) and \( E(t) \) are equivalent, namely, there exist two positive constants \( \kappa_1 \) and \( \kappa_2 \) such that

\[
\kappa_1 E(t) \leq \mathcal{F}(t) \leq \kappa_2 E(t).
\]
(76)

By Young's inequality, we get

\[
\mathcal{F}(t) \leq \left( \frac{N}{2} + \frac{N_1}{2} + \frac{N_2}{2} \right) \left[ \|u\|^2 + \|v\|^2 \right]
\]
\[
+ \left( \frac{N}{2} \left( \mu - \int_{t_0}^{t} g_1(s) ds \right) + \frac{N_1}{2} + \frac{N_2}{2} \right) \|\nabla u\|^2
\]
\[
+ \left( \frac{N}{2} \left( \mu - \int_{t_0}^{t} g_2(s) ds \right) + \frac{N_1}{2} + \frac{N_2}{2} \right) \|\nabla v\|^2
\]
\[
+ \left( \frac{N}{2} + \frac{N_2}{2} \int_{t_0}^{t} \left( g_1(s) ds + g_1 e \nabla u(t) \right)
\]
\[
+ \left( \frac{N}{2} + \frac{N_2}{2} \int_{t_0}^{t} \left( g_2(s) ds + g_2 e \nabla v(t) \right)
\]
\[
+ \left( \frac{N}{2} + \frac{N_2}{2} \int_{t_0}^{t} \left( \rho |\mu_1(\rho)| \|\nabla s(x, \rho, t)\|^2 d\rho dx \right)
\]
\[
+ \left( \frac{N}{2} + \frac{N_2}{2} \int_{t_0}^{t} \left( \rho |\mu_2(\rho)| \|\nabla s(x, \rho, t)\|^2 d\rho dx \right)
\]
\[
+ \left( \frac{N}{2} \left( \lambda + \mu \right) \|\nabla u\|^2 + \|\nabla v\|^2 \right) - N \int_{\Omega} F(u, v) dx.
\]
(77)

Then, for any \( N \), there exists \( \kappa_1 > 0 \) such that

\[
\mathcal{F} \leq \kappa_1 E(t).
\]
(78)

On the other hand, we can find

\[
\mathcal{F}(t) \geq \left( \frac{N}{2} - \frac{N_1}{2} - \frac{N_2}{2} \right) \left[ \|u\|^2 + \|v\|^2 \right]
\]
\[
+ \left( \frac{N}{2} \left( \mu - \int_{t_0}^{t} g_1(s) ds \right) - \frac{N_1}{2} - \frac{N_2}{2} \right) \|\nabla u\|^2
\]
\[
+ \left( \frac{N}{2} \left( \mu - \int_{t_0}^{t} g_2(s) ds \right) - \frac{N_1}{2} - \frac{N_2}{2} \right) \|\nabla v\|^2
\]
\[
+ \left( \frac{N}{2} - \frac{N_2}{2} \int_{t_0}^{t} \left( g_1(s) ds \right) \right)
\]
\[
+ \left( \frac{N}{2} - \frac{N_2}{2} \int_{t_0}^{t} \left( g_2(s) ds \right) \right)
\]
\[
+ \left( \frac{N}{2} \left( \lambda + \mu \right) \|\nabla u\|^2 + \|\nabla v\|^2 \right) - N \int_{\Omega} F(u, v) dx.
\]
(79)

We choose \( N \) large enough so that

\[
\frac{N}{2} - \frac{N_1}{2} - \frac{N_2}{2} > 0, \quad \frac{N}{2} \left( \mu - \int_{t_0}^{t} g_i(s) ds \right) - \frac{N_1}{2} - \frac{N_2}{2} > 0,
\]
\[
\frac{N}{2} - \frac{N_2}{2} \int_{t_0}^{t} g_i(s) ds > 0, \quad i = 1, 2.
\]
(80)
Then, there exist $\kappa_2 > 0$ such that

$$\mathcal{F}(t) \geq \kappa_2 E(t). \quad (81)$$

**Lemma 10.** The functional $\mathcal{F}(t)$ satisfies for any $t \geq t_1$,

$$\mathcal{F}'(t) \leq -4(\mu - L_1)\|\nabla u(t)\|^2 - 4(\mu - L_2)\|\nabla v(t)\|^2 - \|u(t)\|^2$$

$$- \|v(t)\|^2 - \|\operatorname{div} u(t)\|^2 - \|\operatorname{curl} v(t)\|^2 + \frac{1}{4}(\sigma_1 \nabla u(t)) + \frac{c}{\Omega} F(u(t), v(t)) \, dx$$

$$- \int_\Omega \int_{t_1}^t \rho \mu(t) \|\nabla z(x, \rho, t)\|^2 \, dp \, dx$$

$$- \int_\Omega \int_{t_1}^t \rho \mu(t) \|\nabla y(x, \rho, t)\|^2 \, dp \, dx.$$  \quad (82)

**Proof.**

$$g_0 = \min \left\{ \int_0^l g_1(s) \, ds \right\} \int_0^l g_2(s) \, ds \right\}. \quad (83)$$

From Lemmas 5, 6, and 7, noting that $g_1 = \zeta g_1 - h_i$ we have for any $t \geq t_1$,

$$\mathcal{F}'(t) \leq -\left(\frac{l_2}{2} N_1 - N_2 \delta(1 + \Lambda_3 l_1) - N_2 \delta \Lambda_4 l_1 \right) \|\nabla u(t)\|^2$$

$$- \left(\frac{l_2}{2} N_1 - N_2 \delta(1 + \Lambda_3 l_2) - N_2 \delta \Lambda_4 l_2 \right) \|\nabla v(t)\|^2$$

$$- (g_0 N_2 - \delta N_2 - N_1) \|u(t)\|^2 + \|v(t)\|^2$$

$$+ \frac{\zeta N}{2} \left( (g_1 \nabla u(t) + (g_2 \nabla v(t) \right)$$

$$+ \left( N_1 \mu(t) \right) \|\nabla z(x, \rho, t)\|^2 \, dp \, dx$$

$$- \left[ \frac{N}{2} - N_2 \delta \frac{C_2 + 1}{C_2} - \frac{3N}{2} \frac{C_1}{C_2} \right] (h_1 \nabla u(t))$$

$$- \left[ \frac{N}{2} - N_2 \delta \frac{C_2 + 1}{C_2} - \frac{3N}{2} \frac{C_1}{C_2} \right] (h_2 \nabla v(t))$$

$$- [(\lambda + \mu) N_1 - \delta N_2] \left[ \|\operatorname{div} u(t)\|^2 + \|\operatorname{curl} v(t)\|^2 \right]$$

$$- \left( N_{\sigma} + e^{-t_1} - N_1 \frac{3k_2}{2l_1} k_1 - \frac{k_1}{2} \right) \int_{t_1}^t \mu_1 \left( \frac{\|\nabla z(x, 1, \rho, t)\|^2}{2} \right) \, dp \, dx$$

$$- \left( N_{\sigma} + e^{-t_1} - N_1 \frac{3k_2}{2l_1} k_1 - \frac{k_2}{2} \right) \int_{t_1}^t \mu_2 \left( \frac{\|\nabla y(x, 1, \rho, t)\|^2}{2} \right) \, dp \, dx$$

$$- \left[ N_{\sigma_1} - N_1 \frac{3k_2^2}{2l_1} - \delta N_2 k_1 - k_1 \right] \|\nabla u(t)\|^2$$

$$- \left[ N_{\sigma_2} - N_1 \frac{3k_2^2}{2l_1} - \delta N_2 k_2 - k_2 \right] \|\nabla v(t)\|^2 - I(t). \quad (84)$$

where

$$\sigma_1 = \left[ k_1 - \frac{\eta + 1}{2} \left( \int_{t_1}^t \mu_1 \, dp \right) \right],$$

$$\sigma_2 = \left[ k_2 - \frac{\eta + 1}{2} \left( \int_{t_1}^t \mu_2 \, dp \right) \right] \quad (85)$$

Taking $\delta = 1/2 N_2$, we can get

$$\mathcal{F}'(t) \leq -\left( \frac{l_2}{2} N_1 - \frac{1}{2} (1 + \Lambda_3 l_1) - \frac{1}{2} \Lambda_4 l_1 \right) \|\nabla u(t)\|^2$$

$$- \left( \frac{l_2}{2} N_1 - \frac{1}{2} (1 + \Lambda_3 l_2) - \frac{1}{2} \Lambda_4 l_2 \right) \|\nabla v(t)\|^2$$

$$- (g_0 N_2 - \frac{1}{2} - N_1) \|u(t)\|^2 + \|v(t)\|^2$$

$$+ \frac{\zeta N}{2} \left( (g_1 \nabla u(t) + (g_2 \nabla v(t) \right)$$

$$+ \left( N_{\sigma} + e^{-t_1} - N_1 \frac{3k_2}{2l_1} k_1 - \frac{k_1}{2} \right) \int_{t_1}^t \mu_1 \left( \frac{\|\nabla z(x, 1, \rho, t)\|^2}{2} \right) \, dp \, dx$$

$$- \left[ \frac{N}{2} - 2cN_2 - C_{c_1} \left( 2cN_2^2 + \frac{3N}{2} \frac{C_1}{C_2} \right) \right] (h_1 \nabla u(t))$$

$$- \left[ \frac{N}{2} - 2cN_2 - C_{c_2} \left( 2cN_2^2 + \frac{3N}{2} \frac{C_2}{C_1} \right) \right] (h_2 \nabla v(t))$$

$$- [(\lambda + \mu) N_1 - \frac{1}{2}] \|\operatorname{div} u(t)\|^2 + \|\operatorname{curl} v(t)\|^2$$

$$- \left[ N_{\sigma} + e^{-t_1} - N_1 \frac{3k_2}{2l_1} k_2 - \frac{k_2}{2} \right] \int_{t_1}^t \mu_2 \left( \frac{\|\nabla y(x, 1, \rho, t)\|^2}{2} \right) \, dp \, dx$$

$$- \left[ N_{\sigma_1} - N_1 \frac{3k_2^2}{2l_1} - \frac{3k_2}{2} \right] \|\nabla u(t)\|^2$$

$$- \left[ N_{\sigma_2} - N_1 \frac{3k_2^2}{2l_1} - \frac{3k_2}{2} \right] \|\nabla v(t)\|^2 - I(t). \quad (86)$$

First, we take $N_1 > 0$ large such that

$$\left( \lambda + \mu \right) N_1 - \frac{1}{2} > 0, \left( \frac{l_2}{2} N_1 - \frac{1}{2} (1 + \Lambda_3 l_1) - \frac{1}{2} \Lambda_4 l_1 \right) > 4(\mu - l_1),$$

$$\left( \frac{l_2}{2} N_1 - \frac{1}{2} (1 + \Lambda_3 l_2) - \frac{1}{2} \Lambda_4 l_2 \right) > 4(\mu - l_2). \quad (87)$$
We choose \( N_2 > 0 \) large enough so that
\[
g_0 N_2 > \frac{1}{2} - N_1 > 1. \tag{88}
\]

Note that
\[
0 < \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i(s)} < \frac{\zeta g_i^2(s)}{-g_i(s)}, \quad i = 1, 2. \tag{89}
\]

Then, for any \( s \in [0, \infty) \), we get
\[
\lim_{\zeta \to 0} \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i(s)} = 0, \quad i = 1, 2. \tag{90}
\]

By using the fact \( \zeta g_i^2(s)/(\zeta g_i(s) - g_i(s)) < g_i(s), \quad i = 1, 2 \) and using Lebesgue-dominated convergence theorem, we can get
\[
\lim_{\zeta \to 0} \zeta C_{\zeta, i} = \lim_{\zeta \to 0} \int_0^\infty \frac{\zeta g_i^2(s)}{\zeta g_i(s) - g_i(s)} = 0, \quad i = 1, 2. \tag{91}
\]

Thus, there exist some \( \zeta_0 (0 < \zeta_0 < 1) \) such that if \( \zeta < \zeta_0 \), then
\[
\zeta C_{\zeta, i} < \frac{1}{8[(N_1/2l_1) + 2cN_2]} \quad \text{and} \quad \zeta C_{\zeta, 2} < \frac{1}{8[(N_1/2l_1) + 2cN_2]} . \tag{92}
\]

At last, we choose \( N \) large enough and choose \( \zeta \) satisfying
\[
\frac{1}{4} N - 2cN_2^2 > 0 \quad \text{and} \quad \zeta = \frac{1}{2N} > \zeta_0 . \tag{93}
\]

so, we arrive at
\[
\left[ \frac{N}{2} - 2cN_2^2 - C_{\zeta, i} \left( 2cN_2^2 + \frac{3N_1}{2l_1} \right) \right] > 0 \quad \text{and} \quad \left[ \frac{N}{2} - 2cN_2^2 - C_{\zeta, i} \left( 2cN_2^2 + \frac{3N_1}{2l_1} \right) \right] > 0. \tag{94}
\]

Therefore, we choose \( N \) even larger (if needed) so that
\[
\left[ N_1 + e^{-\zeta} - N_1 \frac{3}{2l_1} k_1 - \frac{k_1}{2} \right] > 0, \quad \left[ N_1 + e^{-\zeta} - N_1 \frac{3}{2l_1} k_2 - \frac{k_2}{2} \right] > 0, \quad \left[ N_1 + e^{-\zeta} - N_1 \frac{3}{2l_1} k_3 - \frac{k_3}{2} \right] > 0,
\]
\[
\left[ N_1 + e^{-\zeta} - N_1 \frac{3}{2l_1} k_1 - \frac{k_1}{2} \right] > 0, \quad \left[ N_1 + e^{-\zeta} - N_1 \frac{3}{2l_1} k_2 - \frac{k_2}{2} \right] > 0, \quad \left[ N_1 + e^{-\zeta} - N_1 \frac{3}{2l_1} k_3 - \frac{k_3}{2} \right] > 0. \tag{95}
\]

Thus, (82) is established.

**Proof of Theorem 11.** Taking into account (43) and (26), we obtain that for any \( t \geq t_1 \),
\[
\int_0^{t_1} g_i(s) \int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dx ds 
\leq - \frac{g_i(0)}{a_1} \int_0^{t_1} \int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dx ds \leq -cE'(t),
\]
\[
\int_0^{t_1} g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dx ds 
\leq - \frac{g_2(0)}{a_2} \int_0^{t_1} \int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dx ds \leq -cE'(t). \tag{96}
\]

Noting (82), we shall see that there exists a constant \( m > 0 \) such that for all \( t \geq t_1 \),
\[
\mathcal{F}'(t) \leq -mE(t) - cE' + c \int_{t_1}^t g_i(s) \int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dx ds 
+ c \int_{t_1}^t g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dx ds. \tag{97}
\]

Denote \( \mathcal{L}(t) = \mathcal{F}(t) + cE(t) \). It is obvious that \( \mathcal{L}(t) \) is equivalent to \( E(t) \). It follows from (97) that
\[
\mathcal{L}'(t) \leq -mE(t) + c \int_{t_1}^t g_i(s) \int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dx ds 
+ c \int_{t_1}^t g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dx ds. \tag{98}
\]

We consider two cases.

**Case 11.** \( G(t) \) is linear: By multiplying (98) by \( \xi(t) \) and using (A2) and (26), we obtain
\[
\xi(t) \mathcal{L}'(t) \leq -m \xi(t) E(t) + c \xi(t) \int_{t_1}^t g_i(s) \int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dx ds 
+ c \xi(t) \int_{t_1}^t g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dx ds 
\leq -m \xi(t) E(t) + c \int_{t_1}^t \xi_1(t) g_i(s) \int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dx ds 
+ c \int_{t_1}^t \xi_2(t) g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dx ds 
\leq -m \xi(t) E(t) \xi'(t) E(t) + c \int_{t_1}^t \xi_1(t) g_i(s) \int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dx ds 
+ c \int_{t_1}^t \xi_2(t) g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dx ds 
\leq -m \xi(t) E(t) - c \int_{t_1}^t g_i(s) \int_\Omega |\nabla u(t) - \nabla u(t - s)|^2 dx ds 
- c \int_{t_1}^t g_2(s) \int_\Omega |\nabla v(t) - \nabla v(t - s)|^2 dx ds 
\leq -m \xi(t) E(t) - cE'(t). \tag{99}
\]
which gives as \( \xi(t) \) is nonincreasing,

\[
[\xi(t) \mathcal{L}(t) + cE(t)]' \leq \xi(t) \mathcal{L}'(t) + cE'(t) \leq -m\xi(t)E(t) \quad \forall t \geq t_1. 
\]

(100)

Denote \( K(t) = \xi(t) \mathcal{L}(t) + cE(t) \). We get

\[
K(t)' \leq -m\xi(t)E(t). 
\]

(101)

Hence, using the fact that \( K(t) \sim E(t) \), we easily obtain

\[
E(t) \leq c_1 \exp \left( -c_3 \int_{t_1}^{t} \xi(s)ds \right). 
\]

(102)

**Case 12.** \( G(t) \) is nonlinear: First, we use Lemmas 8 and 9 to deduce that

\[
J(t) = \mathcal{F}(t) + \theta_1(t) + \theta_2(t),
\]

(103)

is nonnegative, and it satisfies for some positive constant \( k \) and for any \( t \geq t_1 \),

\[
J'(t) \leq -(\mu - l_1)\|\nabla u\|^2 - (\mu - l_2)\|\nabla v\|^2 - \|u_t\|^2 - \|v_t\|^2 - \frac{1}{4}(g_1 \nabla \xi - g_2 \nabla v)
+ c\int_{\Omega} F(u, v)dx - \int_{\Omega} \int_{t_1}^{t} \rho|\mu_1(\rho)|\|\nabla z(x, \rho, \xi)\|^2 dpdxds
- \int_{\Omega} \int_{t_1}^{t} \rho|\mu_2(\rho)|\|\nabla y(x, \rho, \xi)\|^2 dpdxds \leq -kE(t) \leq 0.
\]

(104)

Therefore,

\[
k \int_{t_1}^{t} E(s)ds \leq J(t_1) - J(t) \leq J(t_1),
\]

(105)

this implies that

\[
\int_{0}^{\infty} E(s)ds < \infty.
\]

(106)

Now, we define \( I_1(t) \) by

\[
I_1(t) = q \int_{t_1}^{t} \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds,
\]

(107)

\[
I_2(t) = q \int_{t_1}^{t} \int_{\Omega} |\nabla v(t) - \nabla v(t-s)|^2 dxds,
\]

where (106) allows for a constant \( 0 < q < 1 \) chosen so that for all \( t \geq t_1 \)

\[
I_i(t) < 1, \quad i = 1, 2.
\]

(108)

We also assume without loss of generality that \( I_1(t) > 0 \) for all \( t \geq t_1 \); otherwise, (98) yields an exponential decay. Also, we define \( \lambda_1(t) \) and \( \lambda_2(t) \) by

\[
\lambda_1(t) := -\int_{t_1}^{t} g_1(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds,
\]

(109)

\[
\lambda_2(t) := -\int_{t_1}^{t} g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t-s)|^2 dxds.
\]

(109)

We observe that \( \lambda_i(t) \leq -cE'(t) \) \( i = 1, 2 \). Noting that \( G_i(t) \) is strictly convex on \( (0, r) \) and \( G_i(0) = 0 \), then

\[
G_i(\nu x) \leq \nu G_i(x), \quad i = 1, 2,
\]

(110)

provided \( 0 \leq \nu \leq 1 \) and \( x \in (0, r] \). By using (A2), (108), and Jensen's inequality, we can obtain

\[
\lambda_1(t) = -\frac{1}{q G_1(t)} \int_{t_1}^{t} I_1(t) (\xi(t)) |\nabla u(t) - \nabla u(t-s)|^2 dxds
\]

\[
\geq -\frac{\xi_1(t)}{q G_1(t)} \int_{t_1}^{t} I_1(t) G_1(t_1) (g_1(t_1)) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds,
\]

(111)

where \( G_1 \) is an extension of \( G_1 \) such that \( G_1 \) is strictly increasing and strictly convex \( C^2 \) function on \( (0, +\infty) \), see [19]. We have from (111)

\[
\int_{t_1}^{t} g_1(s) \int_{\Omega} |\nabla u(t) - \nabla u(t-s)|^2 dxds \leq \frac{1}{q} \tilde{G}_1^{-1}(\frac{\lambda_1(t)}{\xi_1(t)}),
\]

(112)

Similarly, we have

\[
\int_{t_1}^{t} g_2(s) \int_{\Omega} |\nabla v(t) - \nabla v(t-s)|^2 dxds \leq \frac{1}{q} \tilde{G}_2^{-1}\left(\frac{\lambda_2(t)}{\xi_2(t)}\right).
\]

(113)

We infer from (98), (112), and (113) that for any \( t \geq t_1 \)

\[
\mathcal{L}'(t) \leq -mE(t) + cG_1^{-1}\left(\frac{\lambda_1(t)}{\xi_1(t)}\right) + cG_2^{-1}\left(\frac{\lambda_2(t)}{\xi_2(t)}\right).
\]

(114)
Let us denote

$$G_0(t) = \min \left\{ G_1^{-1}, G_2^{-1} \right\}. \quad (115)$$

For $\varepsilon_0 < r$, the functional $\mathcal{K}_4(t)$ defined by:

$$\mathcal{K}_4(t) = G_0 \left( \frac{E(t)}{E(0)} \right) \mathcal{L}(t) + E(t), \quad (116)$$

is equivalent to $E$, and by using the fact that $E' \leq 0$, $G_i' > 0$, and $G_i'' > 0$, we infer from (114) that

$$\mathcal{K}_4(t) = \varepsilon_0 G_0 \left( \frac{E(t)}{E(0)} \right) \mathcal{L}(t) + G_0 \left( \frac{E(t)}{E(0)} \right) \mathcal{L}'(t) \quad (117)$$

Let $G^*$ be the convex conjugate of $G$ in the sense of Young (see Arnold [21]), then

$$\tilde{G}_i^*(s) = s \left( \tilde{G}_i' \right)^{-1}(s) - \tilde{G}_i \left( \tilde{G}_i'^{-1}(s) \right), \quad i = 1, 2, \quad (118)$$

and $G^*$ satisfies the following Young’s inequality

$$MD_i \leq \tilde{G}_i^*(M) + \tilde{G}_i(D_i), \quad i = 1, 2. \quad (119)$$

With $M = G_0(\varepsilon_0(E(t)/E(0)))$ and $D_i = \tilde{G}_i'^{-1}(\varepsilon_0 q \lambda_i(t) / \xi_i(t))$ and noting $\tilde{G}_i^*(t) \leq t (\tilde{G}_i'^{-1})^{-1}(t)$ and (117), we conclude

$$\mathcal{K}_4(t) \leq -mE(t)G_0 \left( \frac{E(t)}{E(0)} \right) + cG_i^* \left( G_0 \left( \frac{E(t)}{E(0)} \right) \right) \quad \text{and}$$

$$\frac{d}{dt} \mathcal{K}_4(t) \leq -mE(t)G_0 \left( \frac{E(t)}{E(0)} \right) \quad \text{where} \quad G_0(t) = \varepsilon_0 G_0(t). \quad (120)$$

Multiplying by $\xi(t)$, we get

$$\xi(t) R(t) \leq -(mE(t) - c\varepsilon_0) \lambda_0(t) + cq \lambda_1(t) + \lambda_2(t) \quad \text{and}$$

$$\xi(t) R(t) \leq -(mE(t) - c\varepsilon_0) \lambda_0(t) - cE'. \quad (121)$$

Consequently, with $\mathcal{K}_2(t) = G_1(t) + cE(t)$, which satisfies for some $\beta_1, \beta_2 > 0$,

$$\beta_1 \mathcal{K}_2(t) \leq E(t) \leq \beta_2 \mathcal{K}_2(t). \quad (122)$$

Choosing a suitable $\varepsilon_0$, we can get from (121) that there exists a constant $k > 0$,

$$\mathcal{K}_2(t) \leq -k\xi(t) E(t) / E(0) \quad \text{and}$$

$$\mathcal{K}_2(t) \geq k\xi(t) E(t) / E(0). \quad (123)$$

With $0 \leq \varepsilon_0(E(t)/E(0))$, we conclude that for any $t > 0$,

$$G_0 \left( \frac{E(t)}{E(0)} \right) = \min \left\{ G_1^* \left( \frac{E(t)}{E(0)} \right), G_2^* \left( \frac{E(t)}{E(0)} \right) \right\} \quad \text{and}$$

$$G_0 \left( \frac{E(t)}{E(0)} \right) = \min \left\{ G_1^* \left( \frac{E(t)}{E(0)} \right), G_2^* \left( \frac{E(t)}{E(0)} \right) \right\}. \quad (124)$$

Denote $R(t) = (\beta_1 \mathcal{K}_2(t)/E(0))$. Using (122), then

$$R(t) \sim E(t). \quad (125)$$

Since $G_1'(t) = G_0(\varepsilon_0 t^2) + 2\varepsilon_0 t G_0'(\varepsilon_0 t^2)$, then, using the strict convexity of $G_0$ on $(0, r)$, we arrive at $G_0'(t)$, $G_1(t) > 0$ on $(0, 1]$. We infer from (123) that there exists a constant $b_1 > 0$ such that for all $t \leq t_1$,

$$R'(t) \leq -b_1\xi(t) G_1(R(t)). \quad (126)$$

Then, by integration over $(t_1, t)$, we have

$$\int_{t_1}^{t} \frac{R'(s)}{G_1(R(s))} ds \geq b_1 \int_{t_1}^{t} \frac{\xi(s) ds}{s G_0(s)} \quad \text{and}$$

$$\int_{t_1}^{t} \frac{\xi(s) ds}{s G_0(s)} \leq \frac{1}{e_0} G_0^{-1} \left( b_1 \int_{t_1}^{t} \xi(s) ds \right), \quad (127)$$

where $G_0(t) = \int_{t_1}^{t} (1/s G_0(s)) ds$ is strictly decreasing on $(0, r]$ and $\lim_{t \to 0} G_0(t) = +\infty$. A combination of (125) and (127), estimate (38), is established.
4. Conclusion
In this work, we have proved a general energy decay of a coupled Lamé system with distributed time delay. This result is an natural extension of Feng’s work in [18]. In order to complete this work, the study of the global existence and the blow-up of the solutions of (1) and (2) will be the subject of forthcoming works.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that there is no conflict of interests regarding the publication of this manuscript.

Authors’ Contributions
The authors contributed equally in this article. They have all read and approved the final manuscript.

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