Wronskian solutions of the $T$, $Q$ and $Y$-systems related to infinite dimensional unitarizable modules of the general linear superalgebra $gl(M\mid N)$

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Abstract
In [1] (Z. Tsuboi, Nucl. Phys. B 826 (2010) 399 [arXiv:0906.2039]), we proposed Wronskian-like solutions of the T-system for $[M, N]$-hook of $gl(M\mid N)$. We have generalized these Wronskian-like solutions to the ones for the general T-hook, which is a union of $[M_1, N_1]$-hook and $[M_2, N_2]$-hook ($M = M_1 + M_1$, $N = N_1 + N_2$). These solutions are related to Weyl-type supercharacter formulas of infinite dimensional unitarizable modules of the general linear superalgebra $gl(M\mid N)$. Our solutions also include a Wronskian-like solution discussed in [2] (N. Gromov, V. Kazakov, S. Leurent, Z. Tsuboi, JHEP 1101 (2011) 155 [arXiv:1010.2720]) in relation to the $AdS_5/CFT_4$ spectral problem.

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1 Introduction

The T-system is a class of functional relations among transfer matrices of quantum integrable systems related to Yangians $Y(g)$ or quantum affine algebras $U_q(\hat{g})$ associated with simple Lie algebras $g$ [3]. In relation to finite dimensional representations of $g = gl(M|N)$ or $sl(M|N)$, it was proposed for $(M,N) = (2,0)$ case in [4], for $(M,N) = (M,0)$ in [5] and for any non-negative integers $(M,N)$ in [6, 7]. It is a certain Hirota bilinear difference equation [8] of the form:

$$T_{a,s}(u-1)T_{a,s}(u+1) = T_{a,s-1}(u)T_{a,s+1}(u) + T_{a-1,s}(u)T_{a+1,s}(u), \quad a, s \in \mathbb{Z}, \quad u \in \mathbb{C}$$

(1.1)

with specific boundary conditions on the mutually commuting dependent variables $T_{a,s}(u)$ which correspond to transfer matrices (T-operators) or their eigenvalues. In particular for finite dimensional representations of $g = gl(M|N)$, it is defined
on \([M, N]\)-hook (cf. Figure 1). Recently, T-system defined on a T-hook, which is a union of 2-copies of \([2, 2]\)-hooks (cf. Figure 2) appeared in the study of the \(AdS_5/CFT_4\) duality \cite{10} (see \cite{16} for an integral form of it). This T-system was further generalized \cite{11} to the T-system on generalized T-hook, which is a union of \([M_1, N_1]\)-hook and \([M_2, N_2]\)-hook \((M_1, M_2, N_1, N_2 \in \mathbb{Z}_{\geq 0};\) cf. Figure 2). It was pointed out \cite{17} that these T-systems on T-hooks are related to infinite dimensional unitarizable modules of \(gl(M_1 + M_2|N_1 + N_2)\) \cite{18}.

In our previous paper \cite{11}, we proposed Wronskian-like\(^2\) determinant solutions for the T-system defined on \([M, N]\)-hook of \(gl(M|N)\). These non-trivially generalize Wronskian-like solutions for \(gl(M)\) case \cite{19}. In contrast with determinant solutions \cite{6, 7} based on the quantum supersymmetric Jacobi-Trudi formula \(^3\), there is an upper bound for the size of the matrices for these Wronskian-like determinant solutions. And thus, they will be suited, for example, for the analysis of transfer matrices with large dimensional representations in the auxiliary space. In this paper, we will generalize our previous result to the generalized T-hook. The size of the matrices of our determinant formulas is \textbf{finitely bounded} although they are formulas for transfer matrices for \textbf{infinite} dimensional representations in the auxiliary space. Some of the determinant expressions for \(M_1 = N_1 = M_2 = N_2 = 2\) case have already been proposed in \cite{2} in the study of the \(AdS_5/CFT_4\) spectral problem.

Bäcklund transformations for the T-system were proposed for \(gl(M)\) case in \cite{19}, for \([M, N]\)-hook of \(gl(M|N)\) in \cite{9} and for the generalized T-hook in \cite{11}. Our Wronskian-like solutions also solve all the functional relations in the intermediate steps of the Bäcklund flows.

\(^1\)In this paper, we do not discuss integrable systems related to this duality explicitly. For an overview of the AdS/CFT integrability, see \cite{15}. Some explanations on the T-system (or Y-system) in AdS/CFT can be seen for example in review papers \cite{13, 12, 14, 3}.

\(^2\)To be precise, they are a kind of discrete analogue of Wronskian, which is called Casoratian.

\(^3\)This determinant formula for \(gl(M|0)\) case appeared in \cite{20}. It is often called 'Bazhanov-Reshetikhin formula' in literatures of physics. However, this formula essentially follows from resolutions of representations of the Yangian by Cherednik. In this sense, it may be called 'Cherednik-Bazhanov-Reshetikhin formula'.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{\([M, N]\)-hook for the T-system related to finite dimensional representations of \(gl(M|N)\) \cite{6, 7, 9}: \(T_{a,s}(u) = 0\) if \(\{a < 0\}\) or \(\{a > M, s > N\}\) or \(\{a > 0, s < 0\}\).

\label{fig:figure1}
\end{figure}
Figure 2: A generalized T-hook (a union of \([M_1, N_1]\)-hook and \([M_2, N_2]\)-hook) for the T-system for \(gl(M_1 + M_2|N_1 + N_2)\) \cite{10, 11}: \(T_{a,s}(u) = 0\) if \(\{a < 0\}\) or \(\{a > M_1, s > N_1\}\) or \(\{a > M_2, s < -N_2\}\).

The Q-system \footnote{One should not confuse the Q-system with functional relations among Baxter Q-operators.} is a degenerated version of the T-system. Namely, if one drops the spectral parameter dependence of the T-system, one obtains the Q-system. It was proposed as functional relations among characters of Yangians or quantum affine algebras (cf. \cite{21}). The Q-system for the T-hook was considered in \cite{22, 17} in relation to quasiclassical AdS/CFT, and an explicit Wronskian-like determinant (first Weyl-type) supercharacter solution was given \cite{17} for case \(M_1 = M_2 = 2, N_1 = 0, N_2 = 4\).

In section 2, we first consider a solution of the Q-system for the generalized T-hook in terms of supercharacters of infinite dimensional unitarizable modules of \(gl(M|N)\) and propose a Wronskian-like determinant expression for it.

The next step is to ‘Baxterize’ this supercharacter solution by Baxter Q-functions so that it satisfies the T-system. In this paper, ‘Baxterize’ means to put a spectral parameter into the supercharacter. This has been done in section 3, based on a map from the supercharacters to T-functions. The Baxter Q-operators were introduced by Baxter when he solved the 8-vertex model \cite{23}. And the Baxter Q-functions are their eigenvalues. However, ‘Baxter Q-functions’ here are any mutually commuting functions (or operators) which obey functional relations among Baxter Q-functions called QQ-relations (Eqs. \(3.1, 3.2\)). Thus one needs to impose additional conditions on analyticity for these functions for applications. This leads to the Bethe ansatz equations for the corresponding system. We also rewrite this determinant solution and obtained three equivalent expressions.

The Y-system is a system of functional relations related to the thermodynamic Bethe ansatz \cite{24}. It is related to the T-system by a dependent variable transformation. The Y-system is invariant under the gauge transformations for the T-functions (the dependent variables for the T-system). In this sense, the Y-system can be viewed as a gauge invariant form of the T-system (cf. \cite{19}). In section 4, we briefly comment on the Y-system for the generalized T-hook. Section 5 is devoted to concluding remarks. The formulas in the main text contain parameters to define the supercharacters. These parameters correspond to the boundary twist or the...
horizontal field whose crucial role in the construction of the Baxter Q-operators was recognized in [25] first. In appendix A (and section 3.8), we present formulas without these parameters. We will give a proof of the functional relations in appendix B. This supplements some omitted calculations of Ref. [2].

Although we did not discuss an explicit operator realization of the Baxter Q-operators, the formulas in this paper should also be valid for operators. In particular, the fact that our new determinant formulas satisfy the T-system is independent of whether \{Q_I\} are mutually commuting Baxter Q-operators or their eigenvalues.

2 Q-system

In this section, we introduce supercharacter formulas of some infinite dimensional representations of \( g = gl(M_1 + M_2 | N_1 + N_2) \), and consider functional relations (Q-system) among them. The supercharacters here as solutions of the Q-system should be interpreted as the ones for the quantum affine superalgebra \( U_q(\hat{\mathfrak{g}}) \) (or the super-Yangian \( Y(\mathfrak{g}) \)). This is possible since there is an evaluation map \( U_q(\hat{\mathfrak{g}}) \to U_q(\mathfrak{g}) \) (or \( Y(\mathfrak{g}) \to \mathfrak{g} \)). References relevant to our discussions on the representation theory are for example, [18, 26, 27, 28] (and also [17] for \( gl(4 | 4) \) case related to the AdS_5/CFT_4 duality, and [29] in relation to the string hypothesis on Bethe roots).

2.1 Supercharacters of unitarizable \( gl(M_1 + M_2 | N_1 + N_2) \) modules

Let us introduce sets

\[
\begin{align*}
\mathcal{B}_1 &= \{1, 2, \ldots, M_1\}, & \mathcal{B}_2 &= \{M_1 + 1, M_1 + 2, \ldots, M_1 + M_2\}, \\
\mathfrak{S}_1 &= \{M_1 + M_2 + 1, M_1 + M_2 + 2, \ldots, M_1 + M_2 + N_1\}, \\
\mathfrak{S}_2 &= \{M_1 + M_2 + N_1 + 1, M_1 + M_2 + N_1 + 2, \ldots, M_1 + M_2 + N_1 + N_2\}, \\
\mathcal{B} &= \mathcal{B}_1 \sqcup \mathcal{B}_2, & \mathfrak{S} &= \mathfrak{S}_1 \sqcup \mathfrak{S}_2, & I &= \mathcal{B} \sqcup \mathfrak{S}, & I_1 &= \mathcal{B}_1 \sqcup \mathfrak{S}_1, & I_2 &= \mathcal{B}_2 \sqcup \mathfrak{S}_2.
\end{align*}
\]

(2.1)

We will use a grading parameter on the set \( \mathcal{B} \sqcup \mathfrak{S} \):

\[
p_a = 1 \quad \text{for} \quad a \in \mathcal{B}, \quad p_a = -1 \quad \text{for} \quad a \in \mathfrak{S}.
\]

(2.2)

\textsuperscript{5}In general, there is no such evaluation map to the classical algebra for \( g \neq gl(M | N) \). Thus one have to consider the branching of the representations, and to take linear combinations of characters of the classical Lie algebra to obtain the solution of the Q-system [21].
Let us take any subsets of the above sets: \( B_1 \subset \mathfrak{B}_1, B_2 \subset \mathfrak{B}_2, F_1 \subset \mathfrak{F}_1, F_2 \subset \mathfrak{F}_2 \) and arrange them as tuples:

\[
B_1 = (b_1^{(1)}, b_2^{(1)}, \ldots, b_m^{(1)}), \quad B_2 = (b_1^{(2)}, b_2^{(2)}, \ldots, b_{m_2}^{(2)}), \\
F_1 = (f_1^{(1)}, f_2^{(1)}, \ldots, f_{n_1}^{(1)}), \quad F_2 = (f_1^{(2)}, f_2^{(2)}, \ldots, f_{n_2}^{(2)}).
\] (2.3)

For these tuples, we will also use notations:

\[
B = B_1 \cup B_2 = (b_1, b_2, \ldots, b_m) = (b_1^{(1)}, b_2^{(1)}, \ldots, b_{m_1}^{(1)}, b_1^{(2)}, b_2^{(2)}, \ldots, b_{m_2}^{(2)}), \\
F = F_1 \cup F_2 = (f_1, f_2, \ldots, f_n) = (f_1^{(1)}, f_2^{(1)}, \ldots, f_{n_1}^{(1)}, f_1^{(2)}, f_2^{(2)}, \ldots, f_{n_2}^{(2)}),
\] (2.4)

where we write the number of the elements as \( m = m_1 + m_2, n = n_1 + n_2 \). In the same way as above, we will use a symbol \( I \) for a tuple given by any subset of the full set \( \mathcal{I} \). We will distinguish tuples whose elements have different order. For example, \((1, 3, 4)\) and \((1, 4, 3)\) represent different tuples, while as sets, these are the same set \( \{1, 3, 4\} = \{1, 4, 3\} \). In case we need not mind order of the elements of these tuples, we will regard them just sets. For example, \( b \in B_1 \) means \( b \in \{b_1^{(1)}, b_2^{(1)}, \ldots, b_{m_1}^{(1)}\} \).

A sequence of \( a \) integers \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_a) \) is called a generalized partition of length \( a \) if they satisfy \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_a \). Here we do not assume that these are non-negative integers. If \( \lambda_a \geq 0 \), then \( \lambda \) is called a partition. For this generalized partition, we will use the following symbols:

\[
\langle x \rangle = \max(x, 0), \quad \lambda^+ = (\langle \lambda_1 \rangle, \langle \lambda_2 \rangle, \ldots, \langle \lambda_a \rangle), \quad \lambda^- = (\langle -\lambda_1 \rangle, \langle -\lambda_2 \rangle, \ldots, \langle -\lambda_a \rangle).
\] (2.5)

Here \( \lambda^+ \) is a partition and \( \lambda^- \) is a reverse ordered partition. For the generalized partition \( \lambda \), we assign a digram, called a generalized Young diagram (see figure 3) and will use the same symbol \( \lambda \) to denote this diagram. We will also use a notation for a partition \( \mu' = (\mu'_1, \mu'_2, \ldots) \) for any sequence of non-negative integers \( \mu = (\mu_1, \mu_2, \ldots) \), where \( \mu'_i = |\{k|\mu_k \geq i\}| \). If \( \mu \) is a partition, then \( \mu' \) is a transposition of it.

Let \( \{E_{ij}|i, j \in \mathfrak{B} \cup \mathfrak{F}\} \) be the generators of \( gl(M_1 + M_2|N_1 + N_2) \), which obey the super-commutation relation

\[
[E_{ij}, E_{kl}] = E_{ij}E_{kl} - (-1)^{(1-p_{l,p_j})(1-p_{k,p_l})} E_{kl}E_{ij} = \delta_{jk}E_{il} - (-1)^{(1-p_{l,p_j})(1-p_{k,p_l})} \delta_{il}E_{kj}.
\] (2.6)

For the fundamental representation, \( E_{ij} \) is \((M_1+M_2+N_1+N_2)(M_1+M_2+N_1+N_2)\) matrix whose \((k, l)\) matrix element is \( \delta_{ik}\delta_{jl} \). The generators \( \{E_{ij}|i, j \in B \cup F\} \) generate a subalgebra \( gl(m_1 + m_2|n_1 + n_2) \) of \( gl(M_1 + M_2|N_1 + N_2) \) by a natural embedding. A Cartan subalgebra \( \mathfrak{h}(B_1, B_2|F_1, F_2) \) of \( gl(m_1+m_2|n_1+n_2) \) is generated
Figure 3: A generalized Young diagram in a generalized T-hook: this T-hook is a union of $[3,3]$-hook and $[2,4]$-hook for $gl(3+2|3+4)$. The generalized Young diagram (of length 6) is given by a generalized partition $\lambda = (5,3,2,0,-2,-4)$, $\lambda^+ = (5,3,2,0,0,0)$, $\lambda^- = (0,0,0,2,4)$. To denote one ‘0’ in $\lambda$, a length one line is put on the bottom of the diagram.

by $\{E_{ii}|i \in B_2 \sqcup F_2 \sqcup B_1 \sqcup F_1\}$ and a Borel subalgebra $\mathfrak{b}(B_1, B_2|F_1, F_2)$ is generated by $\{E_{ii}|1 \leq a < b \leq m+n, (i_1, i_2, \ldots, i_{m+n}) = B_2 \sqcup F_2 \sqcup B_1 \sqcup F_1\}$. Let $\mathfrak{h}^*(B_1, B_2|F_1, F_2)$ be a dual space of $\mathfrak{h}(B_1, B_2|F_1, F_2)$ with a basis $\{\varepsilon_i\}$ such that $\varepsilon_i(E_{jj}) = \delta_{ij}$. A bilinear form in $\mathfrak{h}^*(B_1, B_2|F_1, F_2)$ is defined as $\langle \varepsilon_i|\varepsilon_j \rangle = p_i\delta_{ij}$. Let $\lambda$ be a generalized partition of length $a$ such that $\lambda_{m_1+i} \leq n_1$ and $-n_2 \leq \lambda_{a-m_2}$. Let us consider an irreducible highest weight representation of $gl(m_1+m_2|n_1+n_2)$ relative to the Borel subalgebra $\mathfrak{b}(B_1, B_2|F_1, F_2)$ with the highest weight:

$$\lambda = -\sum_{i=1}^{m_2}(\langle \lambda_{a-m_2+i}^- - n_2 \rangle + a)\varepsilon_{b(2)} - \sum_{i=1}^{n_2}(\langle \lambda^-\rangle_{n_2+1-i} - a)\varepsilon_{f(2)}$$

$$+ \sum_{i=1}^{m_1}\lambda^+_i\varepsilon_{a(1)} + \sum_{i=1}^{n_1}(\langle \lambda^+\rangle_i - m_1)\varepsilon_{f(1)}.$$ (2.7)

Here we used the same symbol for the partition and the highest weight corresponding to it. This representation is infinite dimensional for generic $\lambda$ and is called unitarizable module of $gl(m_1+m_2|n_1+n_2)$ (cf. [18]). This corresponds to a unitary representation of a real form $u(m_1, m_2|n_1, n_2)$ of $gl(m_1+m_2|n_1+n_2)$. We denote this representation as $W(B_1, B_2|F_1, F_2; \lambda)$. Let us introduce formal exponentials

$$z_j = \exp(\varepsilon_j).$$ (2.8)

$z_j$ can be regarded as complex numbers under the evaluation $z_j = z_j(h) = \exp(\varepsilon_j)(h) = \exp(\varepsilon_j(h))$ for any fixed $h \in \mathfrak{h}(B_1, B_2|F_1, F_2)$. The generating function of the supercharacters of the symmetric tensor representations of $gl(m_1+m_2|n_1+n_2)$ is defined as

$$w(t) = \frac{\prod_{f \in F_1}(1 - z_FT) \prod_{f \in F_2}(1 - z_FT)}{\prod_{b \in B_1}(1 - z_BT) \prod_{b \in B_2}(1 - z_BT)},$$ (2.9)
where \( t \in \mathbb{C} \). If we expand this with respect to non-negative power of \( t \), we obtain the supercharacters \( \chi_{(k)}^{B_1 \sqcup B_2, \emptyset, F_1 \sqcup F_2, \emptyset} = \text{sch}W(B_1 \sqcup B_2, \emptyset|F_1 \sqcup F_2, \emptyset; (k)) \) of the finite dimensional symmetric tensor representations of \( gl(m_1 + m_2|n_1 + n_2) \):

\[
w(t) = \sum_{k=0}^{\infty} \chi_{(k)}^{B_1 \sqcup B_2, \emptyset, F_1 \sqcup F_2, \emptyset} t^k,
\]

(2.10)

where \( \chi_{(k)}^{B_1 \sqcup B_2, \emptyset, F_1 \sqcup F_2, \emptyset} = 0 \) for \( k < 0 \). On the other hand \( ^{\text{6}} \), if we expand \( w(t) \) with respect to non-negative power of \( t \) (for the product indexed by \( B_1 \) and \( F_1 \)) and \( t^{-1} \) (for the product indexed by \( B_2 \) and \( F_2 \)), we obtain the supercharacters \( \chi_{(k)}^{B_1, B_2, F_1, F_2} = \text{sch}W(B_1, B_2|F_1, F_2; (k)) \) of infinite dimensional unitarizable representations of \( gl(m_1 + m_2|n_1 + n_2) \):

\[
w(t) = (-1)^{m_2} t^{n_2 - m_2} \left( \prod_{f \in F_2} \left( -z_f t \right) \right) \prod_{b \in B_2} z_b \prod_{f \in F_1} \left( 1 - z_f t \right) \prod_{b \in B_1} \left( 1 - z_b t \right) = (-1)^{m_2} \sum_{k \in \mathbb{Z}} \chi_{(k)}^{B_1, B_2, F_1, F_2} t^{k + n_2 - m_2}.
\]

(2.11)

Taking the coefficient of \( t^{s + n_2 - m_2} \) for \( s \in \mathbb{Z} \), one obtains

\[
\chi_{s}^{B_1, B_2, F_1, F_2} = \sum_{k=\max(0, -s)}^{\infty} \chi_{(s+k)}^{B_1, \emptyset, F_1, \emptyset} \chi_{(-k)}^{\emptyset, B_2, \emptyset, F_2},
\]

(2.12)

where

\[
\prod_{f \in F_1} \left( 1 - z_f t \right) \prod_{b \in B_1} \left( 1 - z_b t \right) = \sum_{k=0}^{\infty} \chi_{(k)}^{B_1, \emptyset, F_1, \emptyset} t^k,
\]

(2.13)

\[
\left( \prod_{f \in F_2} (-z_f) \right) \prod_{b \in B_2} z_b \prod_{f \in F_2} \left( 1 - z_f^{-1} t^{-1} \right) \prod_{b \in B_2} \left( 1 - z_b^{-1} t^{-1} \right) = \sum_{k=0}^{\infty} \chi_{(-k)}^{\emptyset, B_2, \emptyset, F_2} t^{-k},
\]

(2.14)

and \( \chi_{(-k)}^{\emptyset, B_2, \emptyset, F_2} = 0 \) for \( k > 0 \). Substituting (2.12) into the Jacobi-Trudi type formula (27), one can obtain the supercharacter \( \chi_{\lambda}^{B_1, B_2, F_1, F_2} = \text{sch}W(B_1, B_2|F_1, F_2; \lambda) \) labeled by the generalized partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_a) \) of length \( a \):

\[
\chi_{\lambda}^{B_1, B_2, F_1, F_2} = \det_{1 \leq i, j \leq a} \left( \chi_{\lambda_{-i+j}}^{B_1, B_2, F_1, F_2} \right).
\]

(2.15)

\(^{6}\)There is a discussion for quantum case (T-functions) in (30).
2.2 Wronskian solution of the Q-system

In this paper, we restrict our discussions mainly on the supercharacter for the rectangular diagram $\lambda = (s, s, \ldots, s) = (s^a)$: $\chi^B_{a,s} := \chi^{B_1,B_2,F_1,F_2}_{a,s}$ for $a \geq 0$ and $\chi^B_{a,s} := 0$ for $a < 0$. The above supercharacter satisfies the so-called Q-system

$$\left(\chi^B_{a,s} \right)^2 = \chi^B_{a,s-1} \chi^B_{a,s+1} + \chi^B_{a-1,s} \chi^B_{a+1,s}, \quad a, s \in \mathbb{Z} \quad (2.16)$$

for the generalized T-hook boundary condition:

$$\chi^B_{a,s} = 0 \text{ if } \{a < 0\} \text{ or } \{a > m_1, s > n_1\} \text{ or } \{a > m_2, s < -n_2\}, \quad (2.17)$$

and

$$\chi^B_{a,s} = \left( \frac{\prod_{b \in B} \chi^a_{b}}{\prod_{f \in F} (-\chi^a_{f})} \right)^a \chi^B_{a,s}, \quad (2.18)$$

$$\chi^B_{m_2,-n_2-a} = \chi^B_{m_2+a,-n_2} \text{ for } a \in \mathbb{Z}_{\geq 0}.$$

Here $\prod_{b \in B} \chi^a_{b}$ in the above equation is the superdeterminant of a group element of $GL(m|n)$ (cf. [32]). The above equation (2.16) is a special case of the Hirota equation. It is known that the shape of this equation is invariant under the following gauge transformation.

$$\tilde{\chi}^B_{a,s} = g_1 g_2 g_3 g_4 \chi^a_{a,s}, \quad (2.19)$$

where $g_1, g_2, g_3, g_4$ are arbitrary non-zero complex parameters.

We have a determinant expression of the supercharacter. Let us introduce a determinant labeled by tuples $B_1, B_2, F, R = (r_1, r_2, \ldots, r_a), S = (s_1, s_2, \ldots, s_b)$.

We are working on the generalization of Wronskian-like formulas on supercharacters and T-functions to any generalized Young diagram case and we already have some conjectures on it. Details will be published in a separate paper.

We use a notation for a matrix whose matrix elements are labeled by tuples $J = (j_1, j_2, \ldots, j_a), K = (k_1, k_2, \ldots, k_b)$:

$$(A_{j,k})_{j \in J} = \begin{pmatrix} A_{j_1,k_1} & A_{j_1,k_2} & \cdots & A_{j_1,k_b} \\ A_{j_2,k_1} & A_{j_2,k_2} & \cdots & A_{j_2,k_b} \\ \vdots & \vdots & \ddots & \vdots \\ A_{j_a,k_1} & A_{j_a,k_2} & \cdots & A_{j_a,k_b} \end{pmatrix}. \quad (2.20)$$

We also use a notation $(0)_{a \times b}$ for a $a$ by $b$ zero matrix.
\[ T_1 = (t_1^{(1)}, t_2^{(1)}, \ldots, t_c^{(1)}), T_2 = (t_1^{(2)}, t_2^{(2)}, \ldots, t_c^{(2)}) \), \( r_i, s_i, t_j^{(1)}, t_j^{(2)}, \eta \in \mathbb{C} : \]

\[
\Delta_{F,S,T_1,T_2}^{B_1,B_2,R,(s)} = \begin{vmatrix}
\frac{1}{z_b - z_f} & \left( \frac{1}{z_b - z_f} \right) & \left( \frac{1}{z_b - z_f} \right) & \cdots & \left( \frac{1}{z_b - z_f} \right) \\
\frac{1}{z_b - z_f} & \left( \frac{1}{z_b - z_f} \right) & \left( \frac{1}{z_b - z_f} \right) & \cdots & \left( \frac{1}{z_b - z_f} \right) \\
\frac{1}{z_b - z_f} & \left( \frac{1}{z_b - z_f} \right) & \left( \frac{1}{z_b - z_f} \right) & \cdots & \left( \frac{1}{z_b - z_f} \right) \\
\frac{1}{z_b - z_f} & \left( \frac{1}{z_b - z_f} \right) & \left( \frac{1}{z_b - z_f} \right) & \cdots & \left( \frac{1}{z_b - z_f} \right) \\
\end{vmatrix}, \tag{2.21}
\]

where the number of the elements of the sets \( |S| \) must satisfy \( |B_1| + |B_2| + |R| = |F| + |S| + |T_1| + |T_2| \). This is a minor determinant of an infinite size matrix. Let us introduce a notation:

\[
(a, b) = \begin{cases} 
(a, a + 1, a + 2, \ldots, b) & \text{for } b - a \in \mathbb{Z}_{\geq 0}, \\
\emptyset & \text{for } b - a \notin \mathbb{Z}_{\geq 0}.
\end{cases} \tag{2.22}
\]

The denominator formula of the supercharacter of \( gl(m|n) \) can be written as \( \text{[31]} \):

\[
D(B_1, B_2 | F_1, F_2) = D(B | F) = \Delta_{F,(1,m-n),\emptyset,\emptyset}^{B_1,B_2,(1,n-m),(0)} = \frac{\prod_{b,b' \in B_1 \cup B_2} (z_b - z_{b'}) \prod_{f,f' \in F} (z_{f'} - z_f)}{\prod_{(b,f) \in (B_1 \cup B_2) \times F} (z_b - z_f)}, \tag{2.23}
\]

where we used a notation for the product: \( \prod_{b,b' \in (b_1,b_2,\ldots,b_m), (z_b - z_{b'}) = \prod_{1 \leq i < j \leq m} (z_{b_i} - z_{b_j}) \). Then we find the following new determinant expression of the supercharacter (for \( (a, s) \) in the generalized T-hook. cf. Figure \[4] \[5] \[10] \[10] \[10].

\[
\chi_{a,s,F_1,F_2}^{B_1,B_2,F_1,F_2} = (-1)^{(s+\eta_1)(s+n_2)+\theta} \Delta_{F,(1,m-n),\emptyset,\emptyset}^{B_1,B_2,(a-s-\eta_1)} \frac{D(B_1, B_2 | F_1, F_2)}{D(B_1, B_2 | F_1, F_2)} \text{ for } a \geq \max\{s + \eta_1, -s + \eta_2, 0\}, \qquad -\eta_1 \leq s \leq \eta_2, \tag{2.24}
\]

\[
\chi_{a,s,F_1,F_2}^{B_1,B_2,F_1,F_2} = (-1)^{(s+\eta_1)(s+n_2)+\theta} \Delta_{F,(1,m-n),\emptyset,\emptyset}^{B_1,B_2,(a-s-\eta_1+1,a-\eta_1-\eta_2)} \frac{D(B_1, B_2 | F_1, F_2)}{D(B_1, B_2 | F_1, F_2)} \text{ for } a \geq \max\{s + \eta_1, -s + \eta_2, 0\}, \quad s \geq \max\{-\eta_1, \eta_2\}, \tag{2.25}
\]

\[
\chi_{a,s,F_1,F_2}^{B_1,B_2,F_1,F_2} = (-1)^{(s+\eta_1)n_2+\theta} \Delta_{F,(1,m-n),\emptyset,\emptyset}^{B_1,B_2,(a-s-\eta_1+1,a-\eta_1-\eta_2)} \frac{D(B_1, B_2 | F_1, F_2)}{D(B_1, B_2 | F_1, F_2)} \text{ for } a \geq \max\{s + \eta_1, -s + \eta_2, 0\}, \quad s \leq \min\{-\eta_1, \eta_2\}. \tag{2.26}
\]

\[|S| = \text{Card}(S) \text{ is the number of elements of the set } S.\]

\[\text{As far as discussions on the solution of the Q-system is concerned, the numbers } (n_1, n_2) \text{ are not very important if } n = n_1 + n_2 \text{ is fixed. To change } (n_1, n_2) \text{ for a fixed } n \text{ corresponds to change } s = 0 \text{ axis on the } (a, s) \text{ plane. The same remark can be applied to the solutions of the T-system in the next section.}\]
\[
\chi_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{(s+\eta_1)m_2+\theta} \frac{\Delta_{F,0,0}^{B_1,B_2,(1,2,...,s-\eta_1,a-s-\eta_1+1,a-s-\eta_1+2,...,a-\eta_2-1)}(a-s-\eta_1)}{D(B_1,B_2|F_1,F_2)}
\] for \( a \geq \max\{s+\eta_1,-s+\eta_2,0\} , \quad \eta_2 \leq s \leq -\eta_1 , \quad (2.27) \)

\[
\chi_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{am_2+\theta} \frac{\Delta_{F,0,0}^{B_1,B_2,(1,-a+m+\eta_2),(s-a+m+1,s+\eta_1),\emptyset}}{D(B_1,B_2|F_1,F_2)}
\] for \( a \leq \min\{s+\eta_1,\eta_1+\eta_2\} , \quad (2.28) \)

\[
\chi_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{a(m_1+n_2+1)+\theta} \frac{\Delta_{F,0,0}^{B_1,B_2,(1,a-\eta_1-\eta_2),(0)}}{D(B_1,B_2|F_1,F_2)}
\] for \( \eta_1+\eta_2 \leq a \leq s+\eta_1 , \quad (2.29) \)

\[
\chi_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{a(1+m_2)+\theta} \frac{\Delta_{F,0,0}^{B_1,B_2,(1,-a+m+\eta_2),(s-a+m+1,s+m),\emptyset}}{D(B_1,B_2|F_1,F_2)}
\] for \( a \leq \min\{-s+\eta_2,\eta_1+\eta_2\} , \quad (2.30) \)

\[
\chi_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{a(n_1+n_2)+\theta} \frac{\Delta_{F,0,0}^{B_1,B_2,(1,a-\eta_1-\eta_2),(0)}}{D(B_1,B_2|F_1,F_2)}
\] for \( \eta_1+\eta_2 \leq a \leq -s+\eta_2 , \quad (2.31) \)

where we introduced symbols \( \eta_1 = |B_1|-|F_1| = m_1-n_1 , \eta_2 = |B_2|-|F_2| = m_2-n_2 \) and

\[
\theta := \frac{(|B_1|+|B_2|)(|B_1|+|B_2|-1)}{2} + \frac{|F|(|F|-1)}{2} . \quad (2.32)
\]

If one changes the order of elements of the tuples \( B_1, B_2, F_1, F_2 \), signs of the denominator and the numerator of these formulas change. But they cancel one another, and thus the sign of the formulas do not change in total. In this sense, we can treat these tuples in \( \chi_{a,s}^{B_1,B_2,F_1,F_2} \) just sets. For \( F_2 = B_2 = \emptyset \), these determinants reduce to determinant expressions of character formulas of finite dimensional representations of \( gl(m_1|n_1) \) \[31\]. The above formulas \((2.24)-(2.31)\) take the following values at the boundary of the T-hook:

\[
\chi_{a,n_1}^{B_1,B_2,F_1,F_2} = \left( \frac{\prod_{b \in B_2} z_b^{m_1-a} \prod_{(b,f) \in B_1 \times F}(z_b-z_f)}{\prod_{(b',b) \in B_2 \times B_1}(z_{b'}-z_b)} \right)^{m_1-a} \quad \text{for} \quad a \geq m_1 , \quad (2.33)
\]

\[
\chi_{a,-n_2}^{B_1,B_2,F_1,F_2} = \frac{\prod_{b \in B_2} z_b^{-a-n_2} \prod_{(b,f) \in B_3 \times F}(z_b-z_f)}{\prod_{(b',b) \in B_2 \times B_1}(z_{b'}-z_b)} \quad \text{for} \quad a \geq m_2 , \quad (2.34)
\]

\[
\chi_{m_1,s}^{B_1,B_2,F_1,F_2} = \frac{\prod_{b \in B_1} z_b^{s-n_1} \prod_{(b,f) \in B_1 \times F}(z_b-z_f)}{\prod_{(b',b) \in B_2 \times B_1}(z_{b'}-z_b)} \quad \text{for} \quad s \geq n_1 , \quad (2.35)
\]

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Figure 4: The domain for the sparse determinant solutions of T-and Q-systems for $\eta_1 + \eta_2 > 0$ case. The lines corresponding to the boundaries of the domain of the definition on $(a, s)$ are drawn by thine lines. The lines $a = s + \eta_1$ and $a = -s + \eta_2$ intersect at $(a, s) = \left(\frac{m + n_2}{2}, \frac{m - n_1}{2}\right)$. The numbers (i)-(iv) correspond to the ones in Appendix B.
Figure 5: The domain for the sparse determinant solutions of T-and Q-systems for $\eta_1 + \eta_2 < 0$ case. The lines corresponding to the boundaries of the domain of the definition on $(a, s)$ are drawn by thine lines. The numbers (i) and (v) correspond to the ones in Appendix B.
\[
\begin{align*}
\chi_{B_1, B_2, F_1, F_2}^{B_1, B_2, F_1, F_2} &= \frac{\prod_{b \in B_2} z_b^{s-m_2+n_1} \prod_{(b, f) \in B_2 \times F}(z_b - z_f)}{\prod_{(b', b) \in B_2 \times B_1} (z_{b'} - z_b)} \quad \text{for } s \leq -n_2, \tag{2.36} \\
\chi_{B_0, s, F_1, F_2} &= 1 \quad \text{for } s \in \mathbb{Z}, \tag{2.37}
\end{align*}
\]

Then we arrived at

**Theorem 2.1.** Let \( \chi_{B_1, B_2, F_1, F_2}^{B_1, B_2, F_1, F_2} \) be defined by (2.24)-(2.31) and (2.17). Then, \( \chi_{B_1, B_2, F_1, F_2}^{B_1, B_2, F_1, F_2} \) solves the Q-system (2.16) with the boundary conditions (2.18) and (2.33)-(2.37).

A proof of this theorem will be given in the case of the T-functions in Appendix A.

In contrast to \( w(t) \) (2.11), the function \( w(-t)^{-1} \) generates the supercharacters of the anti-symmetric tensor representations if we expand it with respect to non-negative power of \( t \). Then we can repeat a similar calculation as (2.11)-(2.15) for \( w(-t)^{-1} \).

\[
\begin{align*}
w(-t)^{-1} &= ( -1 )^{n_2} t^{m_2-n_2} \left( \frac{\prod_{b \in B_2} z_b}{\prod_{f \in F_2} (1 + z_f)} \right) \frac{\prod_{b \in B_1} (1 + z_b t)}{\prod_{f \in F_1} (1 + z_f t)} \prod_{b \in B_2} (1 + z_b^{-1} t^{-1}) \\
&= ( -1 )^{n_2} \sum_{k \in \mathbb{Z}} \chi_{(1^k)}^{B_1, B_2, F_1, F_2} t^{k+m_2-n_2}, \tag{2.38}
\end{align*}
\]

where we have expanded each factor as follows

\[
\begin{align*}
\frac{\prod_{b \in B_1} (1 + z_b t)}{\prod_{f \in F_1} (1 + z_f t)} &= \sum_{k=0}^{\infty} \chi_{(1^k)}^{B_1, \emptyset, F_1, \emptyset} t^k, \tag{2.39} \\
\left( \frac{\prod_{b \in B_2} z_b}{\prod_{f \in F_2} (1 + z_f)} \right) \frac{\prod_{b \in B_2} (1 + z_b^{-1} t^{-1})}{\prod_{f \in F_2} (1 + z_f^{-1} t^{-1})} &= \sum_{k=0}^{\infty} \chi_{(1^{-k})}^{\emptyset, B_2, \emptyset, F_2} t^{-k}, \tag{2.40}
\end{align*}
\]

and \( \chi_{(1^k)}^{B_1, \emptyset, F_1, \emptyset} = \chi_{(1^{-k})}^{\emptyset, B_2, \emptyset, F_2} = 0 \) for \( k > 0 \). Here we formally extend the Young diagram \( (1^k) \) for \( k \geq 0 \) to negative direction \( (k < 0) \) with respect to \( s = 0 \) axis in \( (a, s) \)-plane. Then the coefficient of \( t^{a+m_2-n_2} \) for \( a \in \mathbb{Z} \) in (2.38) is given as follows:

\[
\hat{X}_{(1^a)}^{B_1, B_2, F_1, F_2} = \sum_{k=\max(0, -a)}^{\infty} \chi_{(1^{a+k})}^{B_1, \emptyset, F_1, \emptyset} \chi_{(1^{-k})}^{\emptyset, B_2, \emptyset, F_2}. \tag{2.41}
\]

We define more general function \( \hat{X}_{a, s}^{B_1, B_2, F_1, F_2} \) by substituting (2.41) into the Jacobi-Trudi type formula (which is ‘dual’ to (2.15)):

\[
\hat{X}_{a, s}^{B_1, B_2, F_1, F_2} = \det_{1 \leq i, j \leq s} (\hat{X}_{(1^{a+i-j})}^{B_1, B_2, F_1, F_2}), \tag{2.42}
\]

where \( s \in \mathbb{Z}_{\geq 0} \), and \( \hat{X}_{a, s}^{B_1, B_2, F_1, F_2} = 0 \) for \( s < 0 \). Note that the function \( \hat{X}_{a, s}^{B_1, B_2, F_1, F_2} \) satisfies Hirota equation defined on the 90 degree rotated T-hook (see Figure 6).
There are functional relations among supercharacters for subalgebras. These are a kind of Bäcklund transformations in the soliton theory, and connect supercharacters of $gl(M|N)$ to the trivial ones for $gl(0|0)$. We find the determinant formulas \((2.24)-(2.31)\) satisfy the following Bäcklund transformations. For $a,s \in \mathbb{Z}$, $b \in B_1$, $f \in F_1$, $B'_1 := B_1 \{b\}$ and $F'_1 := F_1 \{f\}$, the Bäcklund transformations for the ‘right wing’ are

\[
\begin{align*}
\chi_{a+1,s} & \cdot B_{1,B_2,F_1,F_2} - \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} &= z_b \chi_{a+1,s-1} \cdot B_{1,B_2,F_1,F_2} \cdot B_{1,B_2,F_1,F_2} = 0 \text{ if } s < 0 \text{ or } a > m_1, s > n_1 \lor a < -m_2, s > n_2. \\
\chi_{a+1,s} & \cdot B_{1,B_2,F_1,F_2} - \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} &= \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} = \chi_{a+1,s} \cdot B_{1,B_2,F_1,F_2} \cdot B_{1,B_2,F_1,F_2} = 0 \text{ if } s < 0 \text{ or } a > m_1, s > n_1 \lor a < -m_2, s > n_2. \\
\chi_{a+1,s} & \cdot B_{1,B_2,F_1,F_2} - \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} &= \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} = \chi_{a+1,s} \cdot B_{1,B_2,F_1,F_2} \cdot B_{1,B_2,F_1,F_2} = 0 \text{ if } s < 0 \text{ or } a > m_1, s > n_1 \lor a < -m_2, s > n_2.
\end{align*}
\]

For $b \in B_2$, $f \in F_2$, $B'_2 := B_2 \{b\}$ and $F'_2 := F_2 \{f\}$, the Bäcklund transformations for the ‘left wing’ are

\[
\begin{align*}
\chi_{a+1,s} & \cdot B_{1,B_2,F_1,F_2} - \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} &= \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} = \chi_{a+1,s} \cdot B_{1,B_2,F_1,F_2} \cdot B_{1,B_2,F_1,F_2} = 0 \text{ if } s < 0 \text{ or } a > m_1, s > n_1 \lor a < -m_2, s > n_2. \\
\chi_{a+1,s} & \cdot B_{1,B_2,F_1,F_2} - \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} &= \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} = \chi_{a+1,s} \cdot B_{1,B_2,F_1,F_2} \cdot B_{1,B_2,F_1,F_2} = 0 \text{ if } s < 0 \text{ or } a > m_1, s > n_1 \lor a < -m_2, s > n_2. \\
\chi_{a+1,s} & \cdot B_{1,B_2,F_1,F_2} - \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} &= \chi_{a,s} \cdot B_{1,B_2,F_1,F_2} = \chi_{a+1,s} \cdot B_{1,B_2,F_1,F_2} \cdot B_{1,B_2,F_1,F_2} = 0 \text{ if } s < 0 \text{ or } a > m_1, s > n_1 \lor a < -m_2, s > n_2.
\end{align*}
\]
The Bäcklund transformations for the supercharacters \([2.43]-[2.46]\) for \(B_2 = F_2 = 0\) were introduced in [32]. One can check that the boundary conditions \((2.17)\) and \((2.33)-(2.37)\) are compatible with the above Bäcklund transformations \((2.43)-(2.50)\). These equations can be written in a more symmetric form by the gauge transformation \([11]\) \((2.19)\) for \(g_2 = \prod_{i \in F_2}^{-z_b} i\) and \(g_1 = g_3 = g_4 = 1\). The shape of the equations for the right wing \((2.43)-(2.46)\) is invariant under this gauge transformation, while equations for the left wing \((2.47)-(2.50)\) become

\[
\begin{align*}
\tilde{\chi}_{a+1,s}^{B_1,B_2,F_1,F_2} & = z_b^{-1} \tilde{\chi}_{a,s}^{B_1,B_2,F_1,F_2} \chi_{a+1,s}^{B_1,B_2,F_1,F_2}, \\
\tilde{\chi}_{a,s}^{B_1,B_2,F_1,F_2} & = z_b^{-1} \tilde{\chi}_{a,s}^{B_1,B_2,F_1,F_2} \chi_{a+1,s}^{B_1,B_2,F_1,F_2}, \\
\tilde{\chi}_{a,s}^{B_1,B_2,F_1,F_2} & = z_f^{-1} \tilde{\chi}_{a,s}^{B_1,B_2,F_1,F_2} \chi_{a+1,s}^{B_1,B_2,F_1,F_2}, \\
\tilde{\chi}_{a,s-1}^{B_1,B_2,F_1,F_2} & = z_f^{-1} \tilde{\chi}_{a,s-1}^{B_1,B_2,F_1,F_2} \chi_{a+1,s-1}^{B_1,B_2,F_1,F_2}.
\end{align*}
\]  

One can see a ‘left and right symmetry’ \((z_b, z_f, s \pm 1) \leftrightarrow (z_b^{-1}, z_f^{-1}, s \mp 1)\) between \((2.43)-(2.46)\) and \((2.51)-(2.54)\).

### 2.4 Discrete transformations on the solutions

Let us consider the following maps:

\[
\sigma(\varepsilon_i) = \begin{cases} 
-\varepsilon_{M+1-i} & \text{for } i \in \mathfrak{B}, \\
-\varepsilon_{2M+N+1-i} & \text{for } i \in \mathfrak{F},
\end{cases}
\]  

\[
\tau(\varepsilon_i) = -\varepsilon_{M+N+1-i} & \text{for } i \in \mathfrak{J}.
\]  

Here \(\tau\) changes the parity in the sense \((\varepsilon_{\tau(i)}|\varepsilon_{\tau(j)}) = -p_{ij} \delta_{ij}\). We assume that these maps can be lifted to the ones for character variables \(z_i = e^{\varepsilon_i}\) \((i \in \mathfrak{J})\):

\[
\sigma(z_i) = \begin{cases} 
z_{M+1-i}^{-1} & \text{for } i \in \mathfrak{B}, \\
z_{2M+N+1-i}^{-1} & \text{for } i \in \mathfrak{F},
\end{cases}
\]  

\[
\tau(z_i) = z_{M+N+1-i}^{-1} & \text{for } i \in \mathfrak{J}.
\]

We also define \(\sigma(t) = t^{-1}\) and \(\tau(t) = -t^{-1}\) for \(t \in \mathbb{C}\). \(\sigma\) is an automorphism of \(gl(M|N)\) and in addition to \(\sigma\), \(\tau\) also becomes an automorphism if \(M = N\). They correspond to \(\mathbb{Z}_2\) automorphism of \(gl(M|N)\) and \(\mathbb{Z}_2 \times \mathbb{Z}_2\) automorphism of \(gl(M|M)\) (see for example, [33]).

\(^{11}\) This corresponds to consider supercharacters of a central extension of \(gl(m_1 + m_2|n_1 + n_2)\). See Remark 3.2 in [20].
We also introduce a permutation group over any subset $J$ of $\mathfrak{H}$ and denote it as $S(J)$. We suppose $\rho \in S(J)$ acts on $\varepsilon_i$ and $z_i$ as $\rho(\varepsilon_i) = \varepsilon_{\rho(i)}$ and $\rho(z_i) = z_{\rho(i)}$ for any $i \in J$. In particular, $S(\mathfrak{B}_2) \times S(\mathfrak{F})$ corresponds to the Weyl group symmetry of $g\ell(M|N)$. Our supercharacter solution $\chi^{\mathfrak{B}_2,\mathfrak{B}_1,\mathfrak{H}_1,\mathfrak{H}_2}$ is invariant under $S(\mathfrak{B}_1) \times S(\mathfrak{B}_2) \times S(\mathfrak{H}_1) \times S(\mathfrak{H}_2)$. It does not have full Weyl group symmetry.

The map $\sigma$ induces a reflection of the T-hook with respect to $s = 0$ axis, and the map $\tau$ induces 90 degree rotation and a reflection with respect to $a = 0$ axis of the T-hook. Let us define $\hat{B}_1 = \{M + 1 - b_i^{(2)}\}_{i=1}^{m_2}$, $\hat{B}_2 = \{M + 1 - b_i^{(1)}\}_{i=1}^{m_1}$, $\hat{F}_1 = \{2M + N + 1 - f_i^{(2)}\}_{i=1}^{n_2}$, $\hat{F}_2 = \{2M + N + 1 - f_i^{(1)}\}_{i=1}^{n_1}$, $\hat{B} = \hat{B}_1 \sqcup \hat{B}_2$, $\hat{F} = \hat{F}_1 \sqcup \hat{F}_2$. Then from (2.57), (2.13) and (2.14), we obtain

$$\sigma(\chi_{(s)}^{B_1,0,F_1,0}) = \left(\prod_{b \in B_2} z_b \prod_{f \in F_2} (-z_f)\right)^{a \hat{B}_1,0,F_1,0} \chi_{(-s)}^{\hat{B}_1,0,F_1,0} \chi_{(-s)}^{a \hat{B}_2,0,F_2,0},$$

where $s \in \mathbb{Z}_{\geq 0}$. Therefore by (2.12) and (2.15), we have

$$\sigma(\chi_{a,s}^{B_1,B_2,F_1,F_2}) = \left(\prod_{b \in \hat{B}} z_b \prod_{f \in \hat{F}} (-z_f)\right)^{a \hat{B}_1,0,F_1,0} \chi_{-s,a}^{\hat{B}_1,B_2,F_1,F_2},$$

where $a \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}$. Let us define $\hat{B}_1 = \{M + N + 1 - f_i^{(2)}\}_{i=1}^{m_2}$, $\hat{B}_2 = \{M + N + 1 - f_i^{(1)}\}_{i=1}^{m_1}$, $\hat{F}_1 = \{M + N + 1 - b_i^{(2)}\}_{i=1}^{n_2}$, $\hat{F}_2 = \{M + N + 1 - b_i^{(1)}\}_{i=1}^{n_1}$, $\hat{B} = \hat{B}_1 \sqcup \hat{B}_2$, $\hat{F} = \hat{F}_1 \sqcup \hat{F}_2$. Then from (2.58), (2.13), (2.14), (2.39) and (2.40), we obtain

$$\tau(\chi_{(s)}^{B_1,0,F_1,0}) = (-1)^s \left(\prod_{f \in \hat{F}_2} (-z_f) \prod_{b \in \hat{B}_2} z_b\right)^{\hat{B}_1,0,F_1,0} \chi_{(-s)}^{\hat{F}_1,0},$$

$$\tau(\chi_{(-s)}^{0,B_2,0,F_2}) = (-1)^{s+n_2} \left(\prod_{f \in \hat{F}_1} (-z_f) \prod_{b \in \hat{B}_1} z_b\right)^{\hat{B}_1,0,F_1,0} \chi_{(s)}^{\hat{F}_2,0},$$

where $s \in \mathbb{Z}_{\geq 0}$ and $n_2 = m_2 - n_2$. Therefore by (2.12), (2.15), (2.41) and (2.42), we have

$$\tau(\chi_{a,s}^{B_1,B_2,F_1,F_2}) = (-1)^{(s+n_2)\alpha} \left(\prod_{f \in \hat{F}} (-z_f) \prod_{b \in \hat{B}} z_b\right)^{a \hat{B}_1,B_2,F_1,F_2} \chi_{-s,a}^{\hat{B}_1,B_2,F_1,F_2} \chi_{-s,a},$$

where $a \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}$. In the same way as above, we obtain

$$\sigma(\chi_{a,s}^{B_1,B_2,F_1,F_2}) = \left(\prod_{f \in \hat{F}} (-z_f) \prod_{b \in \hat{B}} z_b\right)^{a \hat{B}_1,B_2,F_1,F_2} \chi_{-s,a}^{\hat{B}_1,B_2,F_1,F_2} \chi_{-s,a},$$

(2.65)
and
\[ \tau(\chi_{a,s}^{B_1,B_2,F_1,F_2}) = (-1)^{(a+n_2)s} \left( \prod_{b \in B} \mathfrak{z}_b \right)^s \chi_{s,a}^{B_1,B_2,F_1,F_2}, \]  
(2.66)

where \( s \in \mathbb{Z}_{\geq 0} \) and \( a \in \mathbb{Z} \). One can check that \( \sigma \) and \( \tau \) are involutions (\( \sigma^2 = \tau^2 = 1 \)). One can also check the commutativity \( \sigma \tau = \tau \sigma \). Explicitly, we have
\[ \sigma \tau(\chi_{a,s}^{B_1,B_2,F_1,F_2}) = \tau \sigma(\chi_{a,s}^{B_1,B_2,F_1,F_2}) = (-1)^{(a+n_2)s} \chi_{s,a}^{B_1,B_2,F_1,F_2}, \]  
(2.67)

where \( a \in \mathbb{Z}_{\geq 0} \) and \( s \in \mathbb{Z} \) and \( \hat{B}_1 = \{ f_i^{(1)} - M \}_{i=1}^{n_1}, \hat{B}_2 = \{ f_i^{(2)} - M \}_{i=1}^{n_2}, \hat{F}_1 = \{ b_i^{(1)} + N \}_{i=1}^{m_1}, \hat{F}_2 = \{ b_i^{(2)} + N \}_{i=1}^{m_2} \). We also have
\[ \sigma \tau(\chi_{a,s}^{B_1,B_2,F_1,F_2}) = \tau \sigma(\chi_{a,s}^{B_1,B_2,F_1,F_2}) = (-1)^{(a+n_2)s} \chi_{s,a}^{B_1,B_2,F_1,F_2}, \]  
(2.68)

where \( s \in \mathbb{Z}_{\geq 0} \) and \( a \in \mathbb{Z} \). In this way, we obtain 4 type of solutions of the Q-system from the ‘seed’ solution \( \chi_{a,s} = \chi_{a,s}^{\mathfrak{B}_1,\mathfrak{B}_2,\mathfrak{F}_1,\mathfrak{F}_2} \) (see Figure 7). There are one to one correspondence among these solutions. Two of them \( \{ \chi_{a,s}, \sigma(\chi_{a,s}) \} \) are solutions of the Q-system for \( gl(M|N) \), which are ‘conjugate’ one another, while the other two \( \{ \tau(\chi_{a,s}), \sigma(\chi_{a,s}) \} \) are rather the ones for \( gl(N|M) \). Only \( M = N \) case, these are solutions for the same algebra \( gl(M|M) \).

### 3 T-system

In this section, we propose Wronskian-like determinant solutions of the T-system for the generalized T-hook. Our formulas generalize formulas in our previous papers on \([M,N]-\)hook for \( gl(M|N) \) \([11,38]\) and the T-hook for \( gl(4|4) \) \([2]\), and also a Wronskian-like determinant for the non-super case \( gl(M) \) \([19]\) (see also \([25,34,35,36,37,39]\)). To generalize the Wronskian-like formulas for the non-super case \( gl(M) \) in \([19]\) to the super case \( gl(M|N) \) is a non-trivial task since matrix elements (Q-functions) of the determinants are non-trivially related one another by functional relations (QQ-relations) \((3.1)\) and \((3.2)\). In addition, our results here and the ones in our previous papers \([11,38]\) are conceptually closer to the one in \([25]\) than the one in \([19]\).

In the context of the representation theory, our formulas will be examples of q-(super)characters \([40]\) (resp. Gelfand-Tsetlin (super)characters \([41]\)) for infinite

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\(^{12}\) For the full sets \((m_1 = M_1, m_2 = M_2, n_1 = N_1, n_2 = N_2)\), we have \( \mathfrak{B}_1 = \{ i \}_{i=1}^{M_2}, \mathfrak{B}_2 = \{ 2 + i \}_{i=1}^{M_2}, \mathfrak{F}_1 = \{ M + 2i \}_{i=1}^{N_2}, \mathfrak{F}_2 = \{ M + 2i \}_{i=1}^{N_2} \).

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Figure 7: Generalized T-hooks from discrete transformations for supercharacters (section 2.4) and T-functions (section 3.6): (I): $\chi_{B_1,B_2,F_1,F_2} = \mathbb{T}_{a,s}^{B_1,B_2,F_1,F_2} = 0$ if \( \{a < 0\} \), or \( \{a > m_1, s > n_1\} \) or \( \{a > m_2, s < -n_2\} \), (II): $\hat{\chi}_{B_1,B_2,F_1,F_2} = \mathbb{T}_{a,s}^{B_1,B_2,F_1,F_2} = 0$ if \( \{a < 0\} \), or \( \{a > m_2, s > n_2\} \) or \( \{a > m_1, s < -n_1\} \), (III): $\hat{\chi}_{B_1,B_2,F_1,F_2} = \mathbb{T}_{a,s}^{B_1,B_2,F_1,F_2} = 0$ if \( \{s < 0\} \), or \( \{s > m_2, a > n_2\} \) or \( \{s > m_1, a < -n_1\} \), (IV): $\hat{\chi}_{B_1,B_2,F_1,F_2} = \mathbb{T}_{a,s}^{B_1,B_2,F_1,F_2} = 0$ if \( \{s < 0\} \), or \( \{s > m_1, a > n_1\} \) or \( \{s > m_2, a < -n_2\} \).
dimensional representations of $U_q(\widehat{gl}(M|N))$ (resp. $Y(gl(M|N))$). The q-character for finite dimensional representations of the quantum affine algebra is a relatively well understood object. However, much is not known about the q-character for infinite dimensional representations of the quantum affine algebra or for even finite dimensional representations of the quantum affine superalgebra.

Let us consider an arbitrary function $f(u)$ of $u \in \mathbb{C}$ (the spectral parameter). Throughout this paper we use a notation on a shift of the spectral parameter such as $f^{[a]} = f(u + a\hbar)$ for an additive shift, and $f^{[a]} = f(uq^{ah})$ for a multiplicative shift ($q$-difference), where $a \in \mathbb{Z}$. Here the unit of the shift $\hbar$ is any non-zero fixed complex number. If there is no shift ($a = 0$), we often omit $[0]$: $f^{[0]} = f = f(u)$.

### 3.1 QQ-relations

Let us consider complex functions $Q_I(u)$ of $u \in \mathbb{C}$ (the spectral parameter), which are labeled by any subset $I$ of the full set $\mathcal{I}$. We assume that these functions $\{Q_I\}$ satisfy the following functional relations:

\[
(z_i - z_j)Q_I Q_{I,ij} = z_iQ_{I,i}Q^{-p_i}_{I,j} - z_jQ^{-p_i}_{I,i}Q_{I,j} \quad \text{for} \quad p_i = p_j, \tag{3.1}
\]

\[
(z_i - z_j)Q_{I,i} Q_{I,j} = z_iQ_I^{-p_i}Q_{I,ij} - z_jQ_I^{p_i}Q^{-p_i}_{I,j} \quad \text{for} \quad p_i = -p_j, \tag{3.2}
\]

where $i, j \in \mathcal{I} \setminus I$ ($i \neq j$); $\{z_i\}_{i \in \mathcal{I}}$ are complex parameters. Here we used an abbreviation $Q_{I,i} = Q_{(i_1,i_2,\ldots,i_a,i,j)}$ for $I = (i_1,i_2,\ldots,i_a)$. In addition, we will not mind the order of the elements of the index set $I$ for $Q_I$. For example, $Q_{(1,3,4)} = Q_{(4,3,1)} = Q_{(3,4,1)} = Q_{(1,3,4)}$. These functional relations (QQ-relations) are known as relations among the Baxter Q-operators or Q-functions for quantum integrable systems. Functional relations related to these can be seen, for example, in $[34, 35, 42, 30, 43, 36, 9, 32, 38]$. Here we used expressions based on index sets on Hasse diagram $\mathcal{I}$ (cf. Figure 8). The second equation (3.2) is sometimes called `fermionic duality’ in recent papers, which came from earlier papers on the ‘particle-hole transformation’ in statistical physics $[4, 15, 7]$. And the parameters $\{z_i\}_{i \in \mathcal{I}}$ correspond to the boundary twist of the transfer matrix in this context. These also correspond to the parameters for the supercharacters $[2,8]$. And in this paper, we will call the functions $\{Q_I\}$, the Q-functions. As remarked in our previous paper $[1]$, there are $2^{M+N}$ Q-functions in total corresponding to the number of the choices $\{z_i\}_{i \in \mathcal{I}}$.

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13$\hbar = \frac{i}{2}$ is often used for an additive shift in literatures.

14It also appeared in the context of ‘Gauge/Bethe Correspondence’ in $[14]$.

15The Baxter Q-functions have additional analytical structures on the spectral parameter such as polynomiality for some models. However in this paper, we will not assume such structures on $\{Q_I\}$ and discuss functional relations among them, which are independent of detailed function form of $\{Q_I\}$. Then the formulas in this paper are also valid for the Baxter Q-operators. What is important for us is that $Q_I(u)$, $Q_J(v)$ and the parameters $z_i$ commute each other for any $I, J \subset \mathcal{I}$, $i \in \mathcal{I}$ and $u, v \in \mathbb{C}$.
of the subsets of the full set \( \mathcal{I} \). It is known that the shape of these equations is invariant under the following gauge transformation.

\[
\tilde{Q}_I = g_1^{[\sum_{j \in I} p_j]} g_2^{[-\sum_{j \in I} p_j]} Q_I, \tag{3.3}
\]

where \( g_1, g_2 \) are arbitrary gauge functions of the spectral parameter. In this paper, we assume, without loss of generality,

\[
Q_\emptyset = 1. \tag{3.4}
\]

Note that the \( \{\tilde{Q}_I\} \) satisfies the QQ-relations with a generic \( \tilde{Q}_\emptyset \) if the gauge functions are chosen as \( g_1 = \tilde{Q}_\emptyset, g_2 = 1 \) (or \( g_1 = 1, g_2 = \tilde{Q}_\emptyset \)) in (3.3). Using the gauge freedom (3.3), one may normalize instead of (3.4). If both (3.4) and (3.5) are imposed, the Q-functions are considered to be in the normalization of the universal \( R \)-matrix. We have several determinant expressions of the solution of the QQ-relations. One of them is the following \( \text{[4]} \).

\[
Q_{I,B,F} = \frac{\prod_{(b,f) \in B \times F} (z_b - z_f)}{\prod_{b \in B, b' \in B'} (z_{b'} - z_b) \prod_{f \in F, f' \in F'} (z_f - z_{f'}) (Q_I^{[n-m]})^n \prod_{k=1}^{m-1} Q_I^{[n-m+2k]}}
\]

\[\times \left| \frac{Q_{I,b,f}^{[n-m]}}{z_b - z_f} \right|_{b \in B, f \in F} \left( z_b^{j-1} Q_{I,b}^{[2j-1+n-m]} \right)_{1 \leq j \leq m-n} \quad \text{for } m \geq n, \tag{3.6}\]

\[
Q_{I,B,F} = \frac{\prod_{(b,f) \in B \times F} (z_b - z_f)}{\prod_{b \in B, b' \in B'} (z_{b'} - z_b) \prod_{f \in F, f' \in F'} (z_f - z_{f'}) (Q_I^{[n-m]})^m \prod_{k=1}^{m-1} Q_I^{[n-m+2k]}}
\]

\[\times \left| \frac{Q_{I,b,f}^{[n-m]}}{z_b - z_f} \right|_{b \in B, f \in F} \left( (-z_f)^{j-1} Q_{I,f}^{[-2j+1+n-m]} \right)_{1 \leq j \leq n-m, f \in F} \quad \text{for } m \leq n, \tag{3.7}\]

where \( I \subset \mathcal{I}, B \subset \mathcal{B} \) and \( F \subset \mathcal{F} \) (\( m := |B|, n := |F| \)) are mutually disjoint sets.

Here we have to interpret the product as \( \prod_{k=1}^{1/f(0)} f(k) = 1/f(0), \prod_{k=1}^{0} f(k) = 1 \) for any complex function \( f \) (the same remark should be applied for (3.55)-(3.60) and (A.18)-(A.20)). These determinant formulas (3.6)-(3.7) for \( m, n \geq 1 \) were introduced in [1] for \( I = \emptyset \) case and were generalized to \( I \neq \emptyset \) case in [2]. For any complement set

\[\text{[16]} \quad \text{The other determinant expressions based on ‘bosonization’ or ‘fermionization’ trick (which correspond to changes of the basis) was proposed in Appendix A of [2].}\]
\[ T := \mathcal{J} \setminus I, \] we define \( \mathcal{Q}_T = \mathcal{Q}_T \). Note that \( \mathcal{Q}_T \) satisfy QQ-relations whose shift of the spectral parameter looks opposite to the one for the original relations (3.1) and (3.2):

\begin{align*}
(z_i - z_j) \mathcal{Q}_{I, ij} = z_i \mathcal{Q}_{I, i}^{[p_i]} \mathcal{Q}_{I, j}^{[p_j]} - z_j \mathcal{Q}_{I, j}^{[p_j]} \mathcal{Q}_{I, i}^{[p_i]} & \quad \text{for } p_i = p_j, \quad (3.8) \\
(z_i - z_j) \mathcal{Q}_{I, i} \mathcal{Q}_{I, j} = z_i \mathcal{Q}_{I}^{[p_i]} \mathcal{Q}_{I, ij}^{[-p_i]} - z_j \mathcal{Q}_{I}^{[-p_j]} \mathcal{Q}_{I, ij}^{[p_j]} & \quad \text{for } p_i = -p_j. \quad (3.9)
\end{align*}

Now we want to lift the maps \( \sigma (2.55) \) and \( \tau (2.56) \) to the ones for the \( \mathcal{Q} \)-functions so that the QQ-relations (3.4) and (3.5) are invariant under \( \sigma \) and \( \tau \). For any \( \{b_i\}_{i=1}^m \subset \mathcal{B}, \{f_i\}_{i=1}^n \subset \mathcal{F}, \) we define

\begin{align*}
\sigma(\{b_i\}_{i=1}^m \sqcup \{f_i\}_{i=1}^n) &= \mathcal{J} \setminus (\{M + 1 - b_i\}_{i=1}^m \sqcup \{2M + N + 1 - f_i\}_{i=1}^n), \quad (3.10) \\
\tau(\{b_i\}_{i=1}^m \sqcup \{f_i\}_{i=1}^n) &= \{M + N + 1 - f_i\}_{i=1}^n \sqcup \{M + N + 1 - b_i\}_{i=1}^m. \quad (3.11)
\end{align*}

Then for \( I = \{b_i\}_{i=1}^m \sqcup \{f_i\}_{i=1}^n \), if we define

\[ \sigma(\mathcal{Q}_I) = \mathcal{Q}_{\sigma(I)}, \]

we find that the QQ-relations (3.1) and (3.2) are invariant under (2.57) and (3.12). For simplicity, here we use the normalization (3.4) and (3.5) at the same time (or without using both of them). If not, one has to multiply with the \( \mathcal{Q} \)-functions by the gauge functions (cf. (3.3)). As for the map \( \tau \), if we define

\[ \tau(\mathcal{Q}_I) = \mathcal{Q}_{\tau(I)}, \]

we find that the QQ-relations (3.1) and (3.2) for \( \text{gl}(M|N) \) are mapped to QQ-relations for \( \text{gl}(N|M) \) under (2.58) and (3.13), and for \( \text{gl}(M|M) \) (\( M = N \)) case, they are invariant (see Figure 8).

The other option is to change (3.11) as

\[ \tau(\{b_i\}_{i=1}^m \sqcup \{f_i\}_{i=1}^n) = \mathcal{J} \setminus (\{M + N + 1 - f_i\}_{i=1}^n \sqcup \{M + N + 1 - b_i\}_{i=1}^m) \] (3.14)

and in stead of (3.13), to define for example

\[ \tau(\mathcal{Q}_I(u)) = \mathcal{Q}_{\tau(I)}(-u) \]

for an additive spectral parameter \( u \), and

\[ \tau(\mathcal{Q}_I(x)) = \mathcal{Q}_{\tau(I)}(x^{-1}) \]

for a multiplicative spectral parameter \( x \). As we remarked for (3.8) and (3.9), if we take a complement of the index set, the shift of the spectral parameter of the QQ-relations looks opposite to the original one. Then a manipulation to revert the
Figure 8: Hasse diagrams for the Q-functions and discrete transformations on \( gl(2|1) \sim gl(1|2) \): \( \tau \) changes the parity of the index sets. In (I) and (II), \( \{1, 2\} \) are bosonic and \( \{3\} \) is fermionic \( (gl(2|1)) \), while in (III) and (IV), \( \{1\} \) is bosonic and \( \{2, 3\} \) are fermionic \( (gl(1|2)) \).
shift of the spectral parameter is necessary to make the QQ-relation invariant under \( \tau \). A similar remark can be applied for \( \sigma \). Namely, we may change (3.10) and (3.12) as

\[
\sigma(\{b_i\}_{i=1}^m \sqcup \{f_i\}_{i=1}^n) = \{M + 1 - b_i\}_{i=1}^m \sqcup \{2M + N + 1 - f_i\}_{i=1}^n, \quad (3.17)
\]

\[
\sigma(Q_I(u)) = Q_{\sigma(I)}(-u) \quad (3.18)
\]

for an additive spectral parameter \( u \), and

\[
\sigma(Q_I(x)) = Q_{\sigma(I)}(x^{-1}) \quad (3.19)
\]

for a multiplicative spectral parameter \( x \).

3.2 Baxterizing the supercharacter by Baxter Q-functions

Let us introduce differential operators \( d_a = 2z_a \frac{\partial}{\partial z_a} + 1 \) which evaluate the degrees of monomials on \( \{z_i\} \). We will use these operators to produce shifts of the spectral parameter of the Baxter Q-functions:

\[
Q_a[\{p_a d_a\} \cdot z_{k_1}^{1} z_{k_2}^{2} \cdots z_{k_{M+N}}^{M+N}, \quad \text{where} \quad k_1, k_2, \ldots, k_{M+N} \in \mathbb{C}. \]

Consider an operator \( B^I \prod \) which acts on any function \( f(\{z_i\}) \) of \( \{z_i\}_{i \in I} \):

\[
B^I \cdot f(\{z_i\}) = \frac{1}{D(I)} \prod_{a \in I} Q_a[\{p_a d_a\} \cdot [D(I) f(\{z_i\})]], \quad (3.21)
\]

where \( D(I) = D(I \cap \mathcal{B}|I \cap \mathcal{F}) \). Our observation is that this operator produces T-functions (transfer matrices) or \( q \)-(super)characters when it acts on (super)characters. More specifically, it produces solutions of the T-system when it acts on the solutions of the Q-system. For example, the T-function for the \((s + 1)\)-dimensional representation \( V_s \) of \( gl(2) \) can be obtained by acting \( B^I \) on the character \( \chi(s) = (z_1^{s+1} - z_2^{s+1})/(z_1 - z_2) \) as:

\[
B^{(1,2)} \cdot \chi(s) = \frac{1}{z_1 - z_2} Q_1^{[d_1]} Q_2^{[d_2]} \cdot [(z_1 - z_2) \chi(s)]
\]

\[
= \frac{1}{z_1 - z_2} Q_1^{[d_1]} Q_2^{[d_2]} \cdot (z_1^{s+1} - z_2^{s+1})
\]

\[
= \frac{z_1^{s+1} Q_1^{[2s+3]} Q_2^{[1]} - z_2^{s+1} Q_1^{[1]} Q_2^{[2s+3]}}{z_1 - z_2} = T_s. \quad (3.22)
\]

\footnote{for this, one may use complex conjugation if the unit of shift is a complex number}
\( \mathcal{T} \) corresponds to the well known Wronskian-type formula on the T-operator in \[25\] if \( Q_1 \) and \( Q_2 \) are the Baxter Q-operators for \( U_q(\hat{sl}_2) \) and \( z_1, z_2 \) are boundary twist parameters with \( z_1 z_2 = 1 \). One can apply the operator \( B^I \) to the supercharacter for the Verma modules, which leads

\[
\mathcal{T}^+ B, F = \frac{1}{\mathcal{D}(B|F)} \prod_{i=1}^{m} z_{b_i}^{\Lambda_{b_i} - m - n - i} Q_{b_i}^{[2(\Lambda_{b_i} + m - n - i) + 1]} \prod_{i=1}^{n} (-z_{f_i})^{\Lambda_{f_i} + n - i} Q_{f_i}^{[-2(\Lambda_{f_i} + n - i) - 1]},
\]

where \( \Lambda \) is the highest weight of the form \( \Lambda = \sum_{i=1}^{m} \Lambda_{b_i} \epsilon_{b_i} + \sum_{i=1}^{n} \Lambda_{f_i} \epsilon_{f_i} \) (\( \Lambda_{b_i}, \Lambda_{f_i} \in \mathbb{C} \)). Most of the T-operators/functions can be given as summation over (3.23).

In this way, (super)characters can be ‘Baxterized’ by the Baxter Q-functions. And obtained T-functions are ‘Q-deformation’ of supercharacters. The operator \( B^I \) works for both T-operators and T-functions in the sense that the definition of this operator does not depend on if \( Q_a \) are operators or their eigenvalues. The way to Baxterize the supercharacter is not unique. Instead of (3.20) and (3.21), we can use

\[
\mathcal{Q}_a^{[-p_a d_a]} \cdot z_1^{k_1} z_2^{k_2} \cdots z_{M+N}^{k_{M+N}} = \mathcal{Q}_a^{[-(2k_a + 1)p_a]} \cdot z_1^{k_1} z_2^{k_2} \cdots z_{M+N}^{k_{M+N}},
\]

\[
\mathcal{B}^I \cdot f(\{z_i\}) = \frac{1}{\mathcal{D}(I)} \prod_{a \in I} \mathcal{Q}_a^{[-p_a d_a]} \cdot |D(I) f(\{z_i\})|.
\]

In this case, the formula looks nice if one uses the normalization (3.5). In the next section, we will Baxterize the solution of the Q-system in section 2 and obtain determinant solutions of the T-system.

### 3.3 Sparse expression of the determinants

Let us introduce a determinant labeled by the same tuples as (2.21):

\[\text{\textsuperscript{18}}\text{As for the atypical representations of superalgebras, this summation can be infinite sum. In this case, one can use supercharacters of infinite dimensional representations which are smaller than Verma modules and obtain finite sum formulas. Examples can be seen in (3.55)-(3.57).} \]

\[\text{\textsuperscript{19}}\text{There are infinitely many ways to put spectral parameter into the supercharacters. However, we expect that the ways to produce solutions of the T-system from those of the Q-system will be classified up to the gauge transformations and trivial rescaling of the spectral parameter. In this paper, we are only interested in the Baxterizations which produce solutions of the T-system. We speculate that the number of the right Baxterizations is related to the number of the automorphisms of the underlying algebras. The notations bar and without bar are borrowed from notations for two different kind of evaluation representations of a quantum affine algebra (they came from automorphisms of the algebra) in [35].} \]
\[
\Delta_{F,ST_1,T_2}^{B_1,B_2,R,[\eta ; \xi]} = \begin{vmatrix}
(\frac{Q^{[\xi]}_{\xi-f}}{z_{b} - z_f})_{b \in B_1, f \in F} & Q^{[\xi]}_{B_1,S} & Q^{[\xi]}_{B_1,T_1} & (0)_{|B_1| \times |T_2|} \\
(\frac{-s f^{2}}{z_{b} - z_f})_{b \in B_2, f \in F} & Q^{[\xi]}_{B_2,S} & (0)_{|B_2| \times |T_1|} & Q^{[\xi]}_{B_2,T_2} \\
(-z_{f})^{-(s - 2r + 1)} & (0)_{|R| \times |S|} & (0)_{|R| \times |T_1|} & (0)_{|R| \times |T_2|}
\end{vmatrix}
\]  

(3.25)

where we used a notation for a matrix

\[
Q^{[\xi]}_{X,Y} = \left( z_{b}^{-1} Q^{[\xi+s-1]}_{b,f} \right)_{b \in X, f \in Y},
\]

and the number of the elements of the sets must satisfy \(|B_1| + |B_2| + |R| = |F| + |S| + |T_1| + |T_2|\). Let us Baxterize the supercharacter formulas \((2.21),(2.31)\) by the operator \(B^I\) (for \(I = B_1 \cup B_2 \cup F_1 \cup F_2\), \(B^I := B^{B_1,B_2,F_1,F_2}\)).

\[
\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2} := (B^{B_1,B_2,F_1,F_2} \cdot \chi_{a,s}^{B_1,B_2,F_1,F_2})^{[a-s-\eta_1-\eta_2]}, \quad a, s \in \mathbb{Z},
\]

(3.26)

where \(\eta_1 = m_1 - n_1, \eta_2 = m_2 - n_2\) and an overall shift \((a - s - \eta_1 - \eta_2)\) on the spectral parameter is introduced just for a normalization \(^{20}\). Explicitly, we obtain the following determinant formula (for \((a, s)\) in the generalized T-hook. cf. Figure 4[3], 21).

\[
\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{(s+\eta_1)(s+\eta_2)+\theta} \frac{\Delta_{F,\emptyset,(1,s+\eta_1),(s+a+\eta_1+1-a+\eta_2)\, B_1, B_2,F_1,F_2}}{D(B_1, B_2, F_1, F_2)}
\]

for \(a \geq \max\{s + \eta_1, -s + \eta_2, 0\}, \quad -\eta_1 \leq s \leq \eta_2\),  

(3.27)

\[
\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{(s+\eta_1)(s+\eta_2)+\theta} \frac{\Delta_{F,\emptyset,(1,s+\eta_1),(a-s-\eta_1-a-s-\eta_2)\, B_1, B_2,F_1,F_2}}{D(B_1, B_2, F_1, F_2)}
\]

for \(a \geq \max\{s + \eta_1, -s + \eta_2, 0\}, \quad s \geq \max\{-\eta_1, \eta_2\}\),  

(3.28)

\[
\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{(s+\eta_1)m_2+\theta} \frac{\Delta_{F,\emptyset,(1,s+\eta_1),(a-s-\eta_1-a-s-\eta_2)\, B_1, B_2,F_1,F_2}}{D(B_1, B_2, F_1, F_2)}
\]

for \(a \geq \max\{s + \eta_1, -s + \eta_2, 0\}, \quad s \leq \min\{-\eta_1, \eta_2\}\),  

(3.29)

\[
\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{(s+\eta_1)m_2+\theta} \frac{\Delta_{F,\emptyset,(1,2,\ldots,-s-\eta_1,a-s-\eta_1+1-a-s-\eta_1+2,\ldots,a-m_2-\eta_2),(a-s-\eta_1,a-s-m_2-\eta_2)\, B_1, B_2,F_1,F_2}}{D(B_1, B_2, F_1, F_2)}
\]

for \(a \geq \max\{s + \eta_1, -s + \eta_2, 0\}, \quad \eta_2 \leq s \leq -\eta_1\),  

(3.30)

\(^{20}\) For more general Young diagram \(\lambda\), the T-function will be given by \(\mathcal{T}_{\lambda}^{B_1,B_2,F_1,F_2} = B^{B_1,B_2,F_1,F_2}, \chi_{\lambda}^{B_1,B_2,F_1,F_2}\) up to the overall shift of the spectral parameter.  

\(^{21}\) The T-function \(\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2}\) in the normalization of the spectral parameter of our previous paper \([1]\) corresponds to \(\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2} = \frac{\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2[\eta_1-\eta_2]}}{\xi} (\text{for } B_2 = F_2 = \emptyset, \eta_2 = 0)\).
\[ T_{a,s,F_2} = (-1)^{m_2+a+\theta} \frac{\Delta F_{(1, -a+s+1, s+\eta_1, \theta)}(B_1, B_2, F_1, F_2)}{D(B_1, B_2, F_1, F_2)} \]

for \( a \leq \min\{s + \eta_1, \eta_1 + \eta_2\} \), \( (3.31) \)

\[ T_{a,s,F_2} = (-1)^{a(m_2+1)+\theta} \frac{\Delta F_{(1, -a+s+1, s+\eta_1, \theta)}(B_1, B_2, F_1, F_2)}{D(B_1, B_2, F_1, F_2)} \]

for \( a \leq \min\{-s + \eta_2, \eta_1 + \eta_2\} \), \( (3.33) \)

\[ T_{a,s,F_2} = (-1)^{a(m_2+1)+\theta} \frac{\Delta F_{(1, -a+s+1, s+\eta_1, \theta)}(B_1, B_2, F_1, F_2)}{D(B_1, B_2, F_1, F_2)} \]

for \( \eta_1 + \eta_2 \leq a \leq -s + \eta_2 \), \( (3.34) \)

where \( \theta \) is defined in \((2.32)\). Due to \((2.17), (3.26)\) naturally satisfies the generalized T-hook boundary condition:

\[ T_{a,s,F_2} = 0 \text{ if } \{a < 0\} \text{ or } \{a > m_1, s > n_1\} \text{ or } \{a > m_2, s < -n_2\}. \]

\( (3.35) \)

At the boundaries of the T-hook, \((3.27)-(3.34)\) take the following ‘boundary values’:

\[ T_{a,n_1,F_2} = \left( \prod_{b \in B_2} z_b^{m_1-a} \prod_{(b,f) \in B_1 \times F}(z_b - z_f) \prod_{(b,f) \in B_1 \times B_1}(z_b - z_f) \right) Q_B^{[-a+m_1]} Q_{B_2,F_1,F_2} \]

for \( a \geq m_1 \), \( (3.36) \)

\[ T_{a,-n_2,F_2} = \prod_{b \in B_2} z_b^{n_2-a-n_2} \prod_{(b,f) \in B_2 \times F}(z_b - z_f) \prod_{(b,f) \in B_2 \times B_1}(z_b - z_f) \]

for \( a \geq m_2 \), \( (3.37) \)

\[ T_{s,m_1,F_2} = \frac{\prod_{b \in B_1} z_b^{s-m_1}}{\prod_{(b',b) \in B_2 \times B_1}(z_{b'} - z_b)} Q_{B_1,F_1,F_2} \]

for \( s \geq n_1 \), \( (3.38) \)

\[ T_{s,m_2,F_2} = \frac{\prod_{b \in B_2} z_b^{s-m_2+n_2}}{\prod_{(b',b) \in B_2 \times B_1}(z_{b'} - z_b)} Q_{B_2,F_1,F_2} \]

for \( s \leq -n_2 \), \( (3.39) \)

\[ T_{0,s,F_2} = Q_{B_1,B_2,F_1,F_2} \text{ for } s \in \mathbb{Z}, \]

\( (3.40) \)
and satisfy
\[
T_{m_1, n_1 + a}^{B_1, B_2, F_1, F_2} = \left( \prod_{b \in B} \frac{z_b}{1 - z_b} \right)^a T_{m_1 + a, n_1}^{B_1, B_2, F_1, F_2}, \tag{3.41}
\]
\[
T_{m_2, -n_2 - a}^{B_1, B_2, F_1, F_2} = T_{m_2 + a, -n_2}^{B_1, B_2, F_1, F_2} \quad \text{for} \quad a \in \mathbb{Z}_{\geq 0}.
\]

The Baxterization of the Q-system (2.16) is the following T-system (Hirota equation):
\[
T_{a, s}^{B_1, B_2, F_1, F_2} = T_{a, s-1}^{B_1, B_2, F_1, F_2} + T_{a, s+1}^{B_1, B_2, F_1, F_2} + T_{a+1, s}^{B_1, B_2, F_1, F_2},
\]
\[
a, s \in \mathbb{Z}. \tag{3.42}
\]

In fact, we find

**Theorem 3.1.** Let \( T_{a, s}^{B_1, B_2, F_1, F_2} \) be defined by (3.27)-(3.35). Then \( T_{a, s}^{B_1, B_2, F_1, F_2} \) solves the T-system with the boundary conditions (3.36)-(3.41).

A proof of this theorem will be given in Appendix B, where the coefficient free form of the T- and Q-functions defined in section 3.8 and Appendix A will be used. The above ‘sparse’ determinant solution is a natural generalization of the solution for the \([M, N]\)-hook [1]. At the boundaries of the T-hook, the Hirota equation (3.42) becomes discrete dAlembert equations:
\[
T_{m_1, s}^{B_1, B_2, F_1, F_2} = T_{m_1, s-1}^{B_1, B_2, F_1, F_2} + T_{m_1, s+1}^{B_1, B_2, F_1, F_2} \quad \text{for} \quad s > n_1,
\]
\[
T_{a, n_1}^{B_1, B_2, F_1, F_2} = T_{a-1, n_1}^{B_1, B_2, F_1, F_2} \quad \text{for} \quad a > m_1,
\]
\[
T_{m_2, s}^{B_1, B_2, F_1, F_2} = T_{m_2, s-1}^{B_1, B_2, F_1, F_2} + T_{m_2, s+1}^{B_1, B_2, F_1, F_2} \quad \text{for} \quad s < -n_2, \tag{3.43}
\]
\[
T_{a, -n_2}^{B_1, B_2, F_1, F_2} = T_{a-1, -n_2}^{B_1, B_2, F_1, F_2} \quad \text{for} \quad a > m_2,
\]
\[
T_{0, s}^{B_1, B_2, F_1, F_2} = T_{0, s-1}^{B_1, B_2, F_1, F_2} + T_{0, s+1}^{B_1, B_2, F_1, F_2} \quad \text{for} \quad s \in \mathbb{Z}.
\]

One can check that (3.36)-(3.40) satisfy the above equations (3.43). It is known that the shape of the Hirota equation (3.42) is invariant under the following gauge transformation (cf. [19]).
\[
\widetilde{T}_{a, s}^{B_1, B_2, F_1, F_2} = g_3^{a+s} g_4^{[a-s]} g_5^{[-a+s]} g_6^{[-a-s]} T_{a, s}^{B_1, B_2, F_1, F_2}, \tag{3.44}
\]
where \( g_3, g_4, g_5, g_6 \) are arbitrary functions of the spectral parameter. The boundary conditions (3.36)-(3.40) become more symmetric form (with a generic \( \widetilde{Q}_0 \)) if the gauge functions are chosen as \( g_3 = \widetilde{Q}_0^{[n_1-n_2]}, g_4 = g_5 = 1, g_6 = \widetilde{Q}_0^{[m+n_2]} \) in (3.44) and
There is one to one correspondence between
and generalized for \( [M, N] = 1 \) in (3.44) and \( g_1 = 1, g_2 = \tilde{Q}_0 \) in (3.3)). In fact, (3.40) becomes
\[
\prod_{s}^{B_{1},B_{2},F_{1},F_{2}} = \tilde{Q}_0^{[s+n_1-n_2]} \tilde{Q}_0^{[-s]}_{B_{1},B_{2},F_{1},F_{2}} \text{ for } s \in \mathbb{Z},
\]
while the shape of (3.36)-(3.39) is unchanged. \( \prod_{a,s}^{B_{1},B_{2},F_{1},F_{2}} \) for the full set \( B_1 = \mathcal{B}_1 \), \( B_2 = \mathcal{B}_2 \), \( f_1 = \mathfrak{F}_1 \) and \( f_2 = \mathfrak{F}_2 \) gives the solution of the T-system for the generalized T-hook.

We can also Baxterize the solution of the Q-system by the operator \( \overline{B}^{B_{1},B_{2},F_{1},F_{2}} \) in the same way as (3.26):
\[
\prod_{a,s}^{B_{1},B_{2},F_{1},F_{2}} := \left( \overline{B}^{B_{1},B_{2},F_{1},F_{2}}, \chi_{a,s}^{B_{1},B_{2},F_{1},F_{2}} \right)^{[-a+s+n_1+n_2]}, \quad (3.46)
\]
There is one to one correspondence between \( \prod_{a,s}^{B_{1},B_{2},F_{1},F_{2}} \) and \( \prod_{a,s}^{B_{1},B_{2},F_{1},F_{2}} \), and they can be transformed to one another by a discrete transformation. Then, we can repeat the same discussion for \( \prod_{a,s}^{B_{1},B_{2},F_{1},F_{2}} \) as \( \prod_{a,s}^{B_{1},B_{2},F_{1},F_{2}} \).

### 3.4 Bäcklund transformations for the T-system

The Bäcklund transformations of the T-system for \( gl(M) \) were introduced in [19], and generalized for \([M, N]\)-hook of \( gl(M|N) \) in [9] and for the general T-hook in [11]. Hirota equation (3.42) is a consistency condition for the above equations. For \( a, s \in \mathbb{Z}, b \in B_1, f \in F_1, B'_1 := B_1 \setminus \{b\} \) and \( F'_1 := F_1 \setminus \{f\} \), Bäcklund transformations for the ‘right wing’ are
\[
\prod_{a+1,s}^{B_{1},B_{2},F_{1},F_{2}[-1]} \prod_{a,s}^{B'_{1},B_{2},F_{1},F_{2}[1]} - \prod_{a,s}^{B_{1},B_{2},F_{1},F_{2}} \prod_{a+1,s}^{B'_{1},B_{2},F_{1},F_{2}} = z_{b}^{-a+1,s-1} \prod_{a+1,s}^{B_{1},B_{2},F_{1},F_{2}} \prod_{a,s}^{B'_{1},B_{2},F_{1},F_{2}} \quad (3.47)
\]
\[
\prod_{a,s+1}^{B_{1},B_{2},F_{1},F_{2}} \prod_{a,s}^{B'_{1},B_{2},F_{1},F_{2}} - \prod_{a,s}^{B_{1},B_{2},F_{1},F_{2}[-1]} \prod_{a,s+1}^{B'_{1},B_{2},F_{1},F_{2}[1]} = z_{b}^{-a+1,s} \prod_{a+1,s+1}^{B_{1},B_{2},F_{1},F_{2}} \prod_{a,s+1}^{B'_{1},B_{2},F_{1},F_{2}} \quad (3.48)
\]
\[
\prod_{a+1,s}^{B_{1},B_{2},F'_{1},F_{2}[-1]} \prod_{a,s}^{B_{1},B_{2},F'_{1},F_{2}[-1]} - \prod_{a,s}^{B_{1},B_{2},F_{1},F_{2}} \prod_{a+1,s}^{B'_{1},B_{2},F_{1},F_{2}} = z_{f}^{-a+1,s-1} \prod_{a+1,s}^{B_{1},B_{2},F_{1},F_{2}} \prod_{a,s}^{B'_{1},B_{2},F_{1},F_{2}} \quad (3.49)
\]
\[
\prod_{a,s+1}^{B_{1},B_{2},F'_{1},F_{2}} \prod_{a,s}^{B_{1},B_{2},F'_{1},F_{2}} - \prod_{a,s}^{B_{1},B_{2},F'_{1},F_{2}[-1]} \prod_{a,s+1}^{B_{1},B_{2},F_{1},F_{2}} = z_{f}^{-a+1,s} \prod_{a+1,s+1}^{B_{1},B_{2},F_{1},F_{2}} \prod_{a,s+1}^{B_{1},B_{2},F_{1},F_{2}} \quad (3.50)
\]
and for $b \in B_2$ and $f \in F_2$, $B'_2 := B_2 \setminus \{b\}$ and $F'_2 := F_2 \setminus \{f\}$, Bäcklund transformations for the ‘left wing’ are

\[
\begin{align*}
2b z_t & B_1 B_2 F_1 F_2 | B_{a,s} B'_{a,s} F_{a,s} | = T_{a+1,s} B_1 B_2 F_1 F_2 | B_{a,s} B'_{a,s} F_{a,s}, \\
2a z_t & B_1 B_2 F_1 F_2 | B_{a,s} B'_{a,s} F_{a,s} | = T_{a+1,s} B_1 B_2 F_1 F_2 | B_{a,s} B'_{a,s} F_{a,s}. \\
2f z_t & B_1 B_2 F_1 F_2 | B_{a,s} B'_{a,s} F_{a,s} | = T_{a+1,s} B_1 B_2 F_1 F_2 | B_{a,s} B'_{a,s} F_{a,s},
\end{align*}
\]

These are Baxterization of the Bäcklund transformations for the supercharacters (2.43)-(2.54). We find that our solution of the T-system (3.27)-(3.34) satisfies the above Bäcklund transformations (3.47)-(3.54). We will not give a direct proof of this fact. Rather, we have checked consistency of the boundary conditions (3.35) and (3.36)-(3.40) with these equations (3.47)-(3.54). Note that the solution of the Bäcklund transformation of the T-system with a given boundary condition is unique as remarked in [9] for the case $m_2 = n_2 = 0$.

### 3.5 Other expressions of the solutions

There are several Wronskian-like expressions for the solution of the T-system. This is because the Q-functions are, in general, not independent as there are QQ-relations (3.1)-(3.2), and thus there are many choices of which Q-functions one uses to express the T-functions. One can obtain these by applying the Laplace expansion on the determinant to the sparse determinant expression (3.27)-(3.34) and determinant expressions for the solution of the QQ-relations such as (3.1)-(3.2). As examples, here we only present two kind of expressions, which are equivalent to the sparse determinant expression (3.27)-(3.34). These are the same quantity but different expressions (based on different basis on the Q-functions).

Let us apply the Laplace expansion on the determinant to the sparse determinant expression (3.27)-(3.34) and rewrite this based on the determinant expression (3.6)-(3.10)
Then we obtain the following simple expression of the solution.

\[
\mathbb{T}^B_{a,s} = \sum_{I \cup J = B_1 \cup B_2, |I| = n_1 - s} \left( \prod_{b \in B_2} z_b \right)^{s - a + \eta_1} \times \prod_{(b,j) \in B_1 \times J} \left( z_b - z_j \right) \prod_{(i,j) \in J \times I} \left( z_i - z_j \right) Q_{B_1,I}^{[a - \eta_2]} Q_{B_2,J}^{[-a + \eta_1]}
\]

for \( a \geq \max\{s + \eta_1, -s + \eta_2, 0\} \), \( (3.55) \)

\[
\mathbb{T}^B_{a,s} = \sum_{I \cup J = B_1 \cup B_2, |I| = a} \prod_{(i,b) \in B_2 \times I} \left( z_b - z_i \right) \prod_{(i,j) \in I \times J} \left( z_i - z_j \right) Q_I^{[s + \eta_1 - \eta_2]} Q_{J,B_2,F_1,F_2}^{[-s]} \quad \text{for} \ a \leq s + \eta_1 \), \( (3.56) \)

\[
\mathbb{T}^B_{a,s} = \sum_{I \cup J = B_1 \cup B_2, |I| = a} \prod_{(i,b) \in B_2 \times I} \left( z_b - z_i \right) \prod_{(i,j) \in I \times J} \left( z_i - z_j \right) Q_I^{[s + \eta_1 - \eta_2]} Q_{B_1,I,F_1,F_2}^{[-s]} \quad \text{for} \ a \leq -s + \eta_2 \), \( (3.57) \)

where the summation is taken over any possible decomposition of the original set into two disjoint sets \( I \) and \( J \) with fixed sizes. Note that the right hand side of \( (3.55) \) is well defined even for any \( a \in \mathbb{C} \), and the right hand sides of \( (3.56) \) and \( (3.57) \) are well defined even for any \( s \in \mathbb{C} \). Thus one can consider analytic continuation of these functions with respect to \( a \) or \( s \). We can further rewrite the above expression. Let us substitute the determinant expression \( (3.6)-(3.7) \) for non-trivial \( I \) into \( (3.55)-(3.57) \), and apply the Laplace expansion on the determinant. Then we obtain the

\footnote{In the representation theoretical context, these expressions should be interpreted in term of a kind of Bernstein-Gel’fand-Gel’fand (BGG) resolution of infinite dimensional modules. In this case, each term of these formulas of the form \( \mathbb{T}^B_{j} = \chi^B_{J} Q_{F,J}^{[sh_1]} Q_{B,F \setminus J}^{[sh_2]} \) (\( J \subset B \cup F \); \( sh_1, sh_2 \in \mathbb{C} \); \( \chi_{J}^{B} \) : the character part) corresponds to a supercharacter (or rather \( T \)-function or \( q \)-(super)character) of an infinite dimensional highest weight representation which is smaller than the Verma module. Examples for these expressions for \( T \)-operators can be seen in \( (35) \) for \( U_q(\mathfrak{sl}(3)) \), in \( (38) \) for \( U_q(\mathfrak{sl}(2|1)) \). In this way, the \( T \)-functions \( \mathbb{T}^B_{j} \) are building blocks of our solutions. One of the questions is whether it is possible to construct solutions of the Hirota equation as summations over \( \mathbb{T}^B_{j} \) when the boundary condition is more complicated than the \( T \)-hook, such as a ‘star hook’ (a union of a \( T \)-hook and an upside-down \( T \)-hook).}

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following ‘dense determinant expression’ of the solution.

\[
\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2} = \left( -1 \right)^{m_{2a}} \prod_{b,b' \in B_1} (z_{b'} - z_b) \prod_{k=1}^{m_{1-s-1}} Q_{B_1}^{a-s-m_1+2k} \times \left( \begin{array}{c} \left( z_{b'}^{-1} Q_{b,b',F_1,F_2} \right)_{\substack{b \in B_1, \ 1 \leq j \leq m_{1-s}}} \left( \frac{z_{b'}^{-1+\eta_{s,m_1+2j-1}} \prod_{\nu \in B_2} (z_{b'} - z_{\nu})}{\prod_{\nu' \in B_2} (z_b - z_{\nu'})} Q_{b}^{s-a+2j-1+\eta_2} \right)_{\substack{b \in B_1, \ 1 \leq j \leq m_{1-s}}} \end{array} \right)
\]

for \( a \leq s + \eta_1 \). \( (3.58) \)

\[
\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2} = \left( -1 \right)^{m_{2a}} \prod_{k=1}^{m_{1-s-1}} Q_{B_1}^{a-s+m_1-2k} \prod_{k=1}^{m_{2-s-1}} Q_{B_2}^{a-s+n_2+\eta_1-2k} \times \left( \begin{array}{c} \prod_{b \in B_1,f \in F} (z_b - z_f) \prod_{f < f'} (z_{f'} - z_f) \left( \frac{z_f^{1+\eta_{s,m_1+2j-1}} Q_{B_1,f}^{a-s-n_1-\eta_2-2i+1}}{\prod_{b \in B_1} (z_b - z_f) Q_{B_2,f}^{a-s+n_2+\eta_1-2i+1}} \right)_{\substack{1 \leq i \leq n_1 - s + \eta_2, \ f \in F}} \end{array} \right)
\]

for \( a \geq \max\{s + \eta_1, -s + \eta_2, 0\} \). \( (3.59) \)

\[
\mathcal{T}_{a,s}^{B_1,B_2,F_1,F_2} = \left( -1 \right)^{m_{2a}} \prod_{b,b' \in B_2} (z_{b'} - z_b) \prod_{k=1}^{m_{2-s-1}} Q_{B_1}^{a-s-m_2+2k} \times \left( \begin{array}{c} \left( \frac{z_b^{1+\eta_{s,m_1+2j-1}} Q_{B_1,b,F_1,F_2}^{a-s-m_2+2j-1}}{\prod_{\nu \in B_1} (z_{b'} - z_{\nu})} \right)_{\substack{b \in B_2, \ 1 \leq j \leq m_{2-s}}} \end{array} \right)
\]

for \( a \leq -s + \eta_2 \). \( (3.60) \)

These expressions \( (3.55)-(3.60) \) of the solution of the T-system for \( m_1 = m_2 = n_1 = n_2 = 2 \) case were previously reported in \cite{2} in the context of the T-system for AdS/CFT. Note that these expression satisfies the Q-system \( (2.16) \) if we formally put Q-functions \{Q\} to 1. Thus these are other new expressions of the supercharacters. In particular for \( m_1 = m_2 = n_1 = n_2 = 2 \), \( (3.58)-(3.60) \) reduce to the determinant solution \cite{17} of the Q-system for AdS/CFT.

### 3.6 Discrete transformations on the solutions

Let us consider how the solutions are transformed under the map \( \sigma \) and \( \tau \) defined in section 2.4 and \( (3.10)-(3.13) \). For this purpose, we introduce other solutions of

\( \sigma \) and \( \tau \) for the Q-functions \( (3.14)-(3.19) \)
the T-systems defined on T-hooks and 90 degree rotated T-hooks (see Figure 6). For \( a, s \in \mathbb{Z} \), we define:

\[
\mathbb{T}^{B_1,B_2,F_1,F_2}_{a,s} = \sum_{I \cup J = F_1 \cup F_2, |I| = n_1 - s} \left( \prod_{b \in B_2, j \in I} z_b \right)^{s-a+\eta_1} \times \prod_{(b,j) \in B_1 \times I} (z_b - z_j) \prod_{(i,j) \in I \times J} (z_i - z_j) Q^{-a+\eta_2}_{B_1,J} Q^{a-\eta_1}_{B_2,J} 
\]

for \( a \geq \max\{s + \eta_1, -s + \eta_2, 0\} \), \( 3.61 \)

\[
\mathbb{T}^{B_1,B_2,F_1,F_2}_{a,s} = \sum_{I \cup J = B_1 \cup B_2, |I| = a} \prod_{i \in I} z_i^{s-a+\eta_1} \prod_{(i,j) \in I \times F} (z_i - z_j) \times \prod_{(i,b) \in B_2 \times I} (z_i - z_b) \prod_{(j,i) \in J \times I} (z_j - z_i) Q^{-s-\eta_1+\eta_2}_{B_1,J} Q^{s}_{J,B_2,F_1,F_2} 
\]

for \( a \leq s + \eta_1 \), \( 3.62 \)

\[
\mathbb{T}^{B_1,B_2,F_1,F_2}_{a,s} = \sum_{I \cup J = B_1 \cup B_2, |I| = a} \prod_{i \in I} z_i^{s-a+\eta_1} \prod_{(i,j) \in I \times F} (z_i - z_j) \times \prod_{(i,b) \in B_2 \times I} (z_i - z_b) \prod_{(j,i) \in J \times I} (z_j - z_i) Q^{-s-\eta_1+\eta_2}_{B_1,J} Q^{s}_{J,B_2,F_1,F_2} 
\]

for \( a \leq -s + \eta_2 \), \( 3.63 \)

where \( \mathbb{T}^{B_1,B_2,F_1,F_2}_{a,s} = 0 \) if \( \{a < 0\} \) or \( \{a > m_1, s > n_1\} \) or \( \{a > m_2, s < -n_2\} \).

\[
\mathbb{T}^{B_1,B_2,F_1,F_2}_{a,s} = \sum_{I \cup J = B_1 \cup B_2, |I| = m_2 + a} \left( \prod_{f \in F_2} (-z_f) \right)^{a-s-\eta_1} \prod_{i \in I} z_i \times \prod_{(i,f) \in I \times F_2} (z_i - z_f) \prod_{(i,j) \in I \times F_1} (z_i - z_j) \times \prod_{(i,b) \in B_2 \times I} (z_i - z_b) \prod_{(j,i) \in J \times I} (z_j - z_i) Q^{s+\eta_1}_{F_2,J} Q^{s-\eta_2}_{F_1,J} 
\]

for \( s \geq \max\{a - \eta_1, -a - \eta_2, 0\} \), \( 3.64 \)

\[
\mathbb{T}^{B_1,B_2,F_1,F_2}_{a,s} = \sum_{I \cup J = F_1 \cup F_2, |I| = s} \prod_{i \in I} (z_i)^{a-s-\eta_1} \prod_{(i,f) \in I \times F_2} (z_i - z_f) \times \prod_{(i,b) \in B_2 \times I} (z_i - z_b) \prod_{(j,i) \in J \times I} (z_j - z_i) \times Q^{-a+\eta_1-\eta_2}_{I} Q^{a}_{B_1,B_2,J,F_2} 
\]

for \( s \leq a - \eta_1 \), \( 3.65 \)

\[
\mathbb{T}^{B_1,B_2,F_1,F_2}_{a,s} = \sum_{I \cup J = F_2 \cup F_1, |I| = s} \prod_{i \in I} (z_i)^{a-s-\eta_1} \prod_{(i,f) \in I \times F_2} (z_i - z_f) \times \prod_{(i,b) \in B_2 \times I} (z_i - z_b) \prod_{(j,i) \in J \times I} (z_j - z_i) \times Q^{-a+\eta_1-\eta_2}_{I} Q^{a}_{B_1,B_2,J,F_2} 
\]

for \( s \leq -a - \eta_2 \), \( 3.66 \)
\[
\begin{align*}
\tilde{T}_{a,s}^{B_1,B_2,F_1,F_2} &= \sum_{I \cup J = B_1 \cup B_2, |I| = m_2 + a} \left( \frac{\prod_{f \in F_2}(-z_f)}{\prod_{i \in I} z_i} \right)^{a-s-\eta_1} \\
&\quad \times \frac{\prod_{(j,f) \in J \times F_2}(z_j - z_f)}{\prod_{(i,f) \in I \times F_1}(z_i - z_f)} \prod_{(i,j) \in I \times J}(z_i - z_j)
\end{align*}
\]

for \( s \geq \max\{a - \eta_1, -a - \eta_2, 0\}, \quad (3.67)

\[
\begin{align*}
\tilde{T}_{a,s}^{B_1,B_2,F_1,F_2} &= \sum_{I \cup J = F_1, |I| = s} \frac{\prod_{i \in I}(-z_i)^{a-s-m_1}}{\prod_{(i,f) \in I \times F_2}(z_i - z_f)} \prod_{(i,j) \in I \times J}(z_i - z_j)
\end{align*}
\]

\[
\times \frac{\prod_{(i,f) \in I \times F_1}(z_i - z_f)}{\prod_{(i,j) \in I \times J}(z_i - z_j)}
\]

for \( s \leq a - \eta_1, \quad (3.68)\)

\[
\begin{align*}
\tilde{T}_{a,s}^{B_1,B_2,F_1,F_2} &= \sum_{I \cup J = F_2, |I| = s} \frac{\prod_{i \in I}(-z_i)^{a-s-m_2}}{\prod_{(i,f) \in I \times F_2}(z_i - z_f)} \prod_{(i,j) \in I \times J}(z_i - z_j)
\end{align*}
\]

\[
\times \frac{\prod_{(i,j) \in I \times J}(z_i - z_j)}{\prod_{(i,f) \in I \times F_1}(z_i - z_f)}
\]

for \( s \leq -a - \eta_2, \quad (3.69)\)

where \( \tilde{T}_{a,s}^{B_1,B_2,F_1,F_2} = \tilde{T}_{a,s}^{B_1,B_2,F_1,F_2} = 0 \) if \( \{s < 0\} \), or \( \{a > m_1, s > n_1\} \) or \( \{a < -m_2, s > n_2\} \). For the full sets of the index sets, these coincide with the T-functions without over-line up to the overall shift of the spectral parameter: \( \tilde{T}_{a,s}^{B_1,B_2,\tilde{\alpha}_1,\tilde{\alpha}_2} = \tilde{T}_{a,s}^{B_1,B_2,\tilde{\alpha}_1,\tilde{\alpha}_2[n_2-n_1]} \), \( \tilde{T}_{a,s}^{B_1,B_2,\tilde{\alpha}_1,\tilde{\alpha}_2} = \tilde{T}_{a,s}^{B_1,B_2,\tilde{\alpha}_1,\tilde{\alpha}_2[n_2-n_1]} \). We find the following transformation property of the T-functions under \( \sigma \) and \( \tau \):

\[
\begin{align*}
\sigma(\tilde{T}_{a,s}^{B_1,B_2,F_1,F_2}) &= \left( \frac{\prod_{b \in B} z_b}{\prod_{f \in F}(-z_f)} \right)^{a} \tilde{T}_{a-s}^{B_1,B_2,F_1,F_2}, \quad (3.70)
\end{align*}
\]

\[
\begin{align*}
\tau(\tilde{T}_{a,s}^{B_1,B_2,F_1,F_2}) &= (-1)^{(s+\eta_2)a} \left( \frac{\prod_{b \in B} z_b}{\prod_{f \in F}(-z_f)} \right)^{a} \tilde{T}_{a-s,a}^{B_1,B_2,F_1,F_2}, \quad (3.71)
\end{align*}
\]

\[
\begin{align*}
\sigma(\tilde{T}_{a,s}^{B_1,B_2,F_1,F_2}) &= \left( \frac{\prod_{b \in B} z_b}{\prod_{f \in F}(-z_f)} \right)^{a} \tilde{T}_{a-s}^{B_1,B_2,F_1,F_2}, \quad (3.72)
\end{align*}
\]

\[
\begin{align*}
\tau(\tilde{T}_{a,s}^{B_1,B_2,F_1,F_2}) &= (-1)^{(s+\eta_2)a} \left( \frac{\prod_{b \in B} z_b}{\prod_{f \in F}(-z_f)} \right)^{a} \tilde{T}_{a-s,a}^{B_1,B_2,F_1,F_2}, \quad (3.73)
\end{align*}
\]

where \( a \in \mathbb{Z}_{\geq 0} \) and \( s \in \mathbb{Z} \), and

\[
\begin{align*}
\sigma(\tilde{T}_{a,s}^{B_1,B_2,F_1,F_2}) &= \left( \frac{\prod_{f \in F}(-z_f)}{\prod_{b \in B} z_b} \right)^{s} \tilde{T}_{a-s,a}^{B_1,B_2,F_1,F_2} \quad (3.74)
\end{align*}
\]
where $s \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}$. We also have

\begin{align}
\sigma \tau(\Pi_{a,s}^{B_1,B_2,F_1,F_2}) &= \tau \sigma(\Pi_{a,s}^{B_1,B_2,F_1,F_2}) = (-1)^{(a+\eta_2)s} \hat{\Pi}_{s,a}^{B_1,B_2,F_1,F_2}, \tag{3.78} \\
\sigma \tau(\Pi_{a,s}^{B_1,B_2,F_1,F_2}) &= \tau \sigma(\Pi_{a,s}^{B_1,B_2,F_1,F_2}) = (-1)^{(a+\eta_2)s} \hat{\Pi}_{s,a}^{B_1,B_2,F_1,F_2}, \tag{3.79}
\end{align}

where $a \in \mathbb{Z}_{\geq 0}$ and $s \in \mathbb{Z}$, and

\begin{align}
\sigma \tau(\Pi_{a,s}^{B_1,B_2,F_1,F_2}) &= \tau \sigma(\Pi_{a,s}^{B_1,B_2,F_1,F_2}) = (-1)^{(a+\eta_2)s} \hat{\Pi}_{s,a}^{B_1,B_2,F_1,F_2}, \tag{3.80} \\
\sigma \tau(\Pi_{a,s}^{B_1,B_2,F_1,F_2}) &= \tau \sigma(\Pi_{a,s}^{B_1,B_2,F_1,F_2}) = (-1)^{(a+\eta_2)s} \hat{\Pi}_{s,a}^{B_1,B_2,F_1,F_2}, \tag{3.81}
\end{align}

where $s \in \mathbb{Z}_{\geq 0}$ and $a \in \mathbb{Z}$. In this way, we can obtain 4 type of the solutions for the T-hook (see Figure 7). There is one to one correspondence among them.

### 3.7 Reductions of solutions by automorphisms

In this section, we briefly announce our idea on how to obtain solutions of T-systems for other algebras. We will discuss details elsewhere.

We find that reductions on the QQ-relations by $\sigma$ or $\tau$ (and some dualities among different superalgebras (cf. [46])) produce solutions of the T-systems for different algebras. The reductions here are basically accomplished by identifying the image of the Q-functions and the parameters $\{z_a\}$ by the maps $\sigma$ or $\tau$ with the original ones (up to the gauge and manipulations on the spectral parameter in some cases). Let us consider ‘$sl(M|N)^{(2)}$ type reduction’ by $\sigma$:

$$\sigma(Q_I) = Q_{I} \quad \text{for} \quad I \subset \mathcal{I}, \quad \sigma(z_a) = z_a \quad \text{for} \quad a \in \mathcal{I}. \tag{3.82}$$

If $M$ or $N$ are odd, fixed points by $\sigma$ appear. For example for $N = 2r + 1$ case, we have $\sigma(z_{M+r+1}) = z_{M+r+1}^{-1} = z_{M+r+1}$. Then $z_{M+r+1} = \pm 1$. The minus sign $z_{M+r+1} = -1$ effectively changes the sign of $p_{M+r+1}$ from the grading of the superalgebra, which induces a duality among a superalgebra ($z_{M+1} = 1$) and an ordinary algebra ($z_{M+1} = -1$) for the case $N = 1$. In particular, $sl(2r|1)^{(2)}$ type reduction for $z_{2r+1} =$
−1 produces QQ-relations (and then Bethe equations) and Wronskian solutions \(^{26}\) of the T-system for \(U_q(\mathfrak{g}^{(1)})\) or \(Y(\mathfrak{g})\), where \(\mathfrak{g} = \text{so}(2r + 1)\). \(sl(0|2r + 1)^{(2)}\) type reduction for \(z_{r+1} = -1\) corresponds to \(\mathfrak{g} = \text{osp}(1|2r)\) (cf. \(^{27}\)), \(sl(2r + 1|0)^{(2)}\) type reduction \(^{27}\) for \(z_{r+1} = 1\) corresponds to \(U_q(A_{2r}^{(2)})\), and \(sl(2r|0)^{(2)}\) type reduction corresponds to \(U_q(A_{2r-1}^{(2)})\). These reduction produce additional functional relations among T-functions, which do not exist before the reductions. There will also be interesting reductions by \(\tau\) \(^{28}\).

### 3.8 Removing the twist

Bazhanov, Lukyanov, Zamolodchikov \(^{25}\) defined the Baxter Q-operators as trace of the universal \(R\)-matrix over \(q\)-oscillator representations of the quantum affine algebra \(U_q(\widehat{sl}(2))\). And importance of boundary twists or horizontal fields to regularize the trace over the infinite dimensional space was recognized in \(^{25}\) for the first time. In this context, the parameters \(\{z_i\}\) correspond to these boundary twists or horizontal fields. In this paper, these parameters were used to define supercharacters. We can eliminate these parameters \(\{z_i\}\) by the following transformation:

\[
Q_i^{[0]} = a_{I} T_i^{[0]} Q_I^{[0]}, \tag{3.83}
\]

\[
T_i^{[0]} = a_{I} f_i^{[a-s]} T_i^{[a,s]}, \tag{3.84}
\]

where \(I\) is a tuple (as a set, it is a subset of the full set \(\mathcal{I}\)); \(a_I = \prod_{j,k \in I : j < k} (\frac{z_i - z_k}{(z_i - z_j)^2})\) for \(|I| \geq 2\), \(a_I = 1\) for \(|I| = 0,1\); \(f_i^{[s]} = \prod_{i \in I} f_i^{[0]}\), \(T_i^{[a,s]} := T_i^{[\mathcal{I} \cap \mathcal{B}_1, \mathcal{I} \cap \mathcal{B}_2, \mathcal{I} \cap \mathcal{B}_3, \mathcal{I} \cap \mathcal{S}_2}\). Now the order of the elements of the index set (a tuple \(I\) of \(T_i^{[a,s]}\) and \(Q_i^{[0]}\) affects overall sign of the functions (as oppose

\(^{26}\) For \(s \in \mathbb{Z}_{\geq 1}\), \(B = \{1, 2, \ldots, 2r\}\), \(F = \{2r + 1\}\), we define \(T_i^{[s]} := T_i^{[B, 0, F, \theta]}\) for \(a \in \{1, 2, \ldots, r - 1\}\), \(T_i^{[2s]} := T_{i, r, s}^{[B, 0, F, \theta]}\) and \(T_i^{[r]} := \prod_{k=1}^{r} (\sqrt{z_i} + \frac{1}{\sqrt{z_i}}) T_{i, r, s}^{[B, 0, F, \theta]}\). Then we find that \(\{T_i^{[a]}\}\) solve the T-system for \(\mathfrak{g} = \text{so}(2r + 1)\) up to some rescaling of the functions. It is interesting to see that the T-functions \(T_i^{[r]}\) related to the spin representations are given by T-functions one level lower than the original one in the sense of the Bäcklund transformations. Thus the T-function \(T_i^{[r]}\) is proportional to the Q-functions \(Q_B = Q_F\). The T-system for \(\text{so}(2r + 1)\) was proposed in \(^{25}\). However, the Wronskian like solution for it was not known in the literatures.

\(^{27}\) In this case, we have to add a shift of the spectral parameter to \(^{3.82}\) as \(\sigma(Q_I) = Q_i^{[2z_i]}\) (\(h \in \mathbb{C} \setminus \{0\}\) cf. \(^{27}\)).

\(^{28}\) A question is whether it is possible to treat the Q-functions and the T-system for \(AdS_5/CFT_4\) efficiently in relation to reductions similar to the ones based on \(\sigma\) and \(\tau\) for \(gl(4|4)\). For this, one will be need careful analysis on the analyticity on the spectral parameter. The reduction may accompany manipulation on the Riemann sheets on the spectral parameter. For the usual twisted quantum affine algebras case, the corresponding manipulation was just a shift of the spectral parameter. But in this case, it could be more involved (cf. \(^2\)).
Then the QQ-relations (3.1) and (3.2) become a coefficient free form:

\[ Q_{I,i} Q_{I,j} = Q_{I,i}^{[p]} Q_{I,j}^{[-p]} - Q_{I,i}^{[-p]} Q_{I,j}^{[p]} \]

for \( p_i = p_j \),

\[ Q_{I,i} Q_{I,j} = Q_{I,i}^{[p]} Q_{I,j}^{[p]} - Q_{I,i}^{[-p]} Q_{I,j}^{[-p]} \]

for \( p_i = -p_j \).

We remark that the coefficient free form of (3.86) for \((M + 1)(N + 1)\) Q-functions was discussed in detail in [9]. In this form, one can see a \( GL(M) \times GL(N) \) symmetry of the QQ-relations. In fact, the following transformation (3.87)-(3.90) preserve the shape of the QQ-relations.

\[ Q_{b'} = \sum_{b \in \mathcal{B}} A_{b'b} Q_b, \quad Q'_{f'} = \sum_{f \in \mathcal{F}} B_{f'f} Q_f \]

for \( b' \in \mathcal{B}, \ f' \in \mathcal{F} \),

\[ Q_{b'_1, \ldots, b'_m, f'_1, \ldots, f'_n} = \sum_{b_1, \ldots, b_m \in \mathcal{B}} \sum_{f_1, \ldots, f_n \in \mathcal{F}} A_{b'_1 b_1} \ldots A_{b'_m b_m} B_{f'_1 f_1} \ldots B_{f'_n f_n} Q_{b_1 \ldots b_m, f_1 \ldots f_n} \]

for \( b'_1, \ldots, b'_m \in \mathcal{B}, \ f'_1, \ldots, f'_n \in \mathcal{F} \).

In particular,

\[ Q'_{\emptyset} = Q_{\emptyset}, \]

\[ Q'_{1 \ldots M, M+1, \ldots, M+N} = \frac{\det_{1 \leq i,j \leq M} (A_{i,j}) \det_{1 \leq i,j \leq N} (B_{i,j+M+j})}{Q_{1 \ldots M, M+1, \ldots, M+N}}. \]

This \( GL(M) \times GL(N) \) symmetry for \( M = 2, N = 0 \) case was used to take without twist limit \((z_i \to 1)\) in [50] (see also, [39]). As in Figure 8, we put \( 2^{M+N} \) Q-functions on the Hasse diagram. Now the diagram is on a hyper-sphere or a hyper-oval sphere due to the above symmetry.

In appendix A, we list the coefficient free form of the solutions of the T-system by the transformations (3.83)-(3.84). Once the they are obtained, one can forget about the relations (3.83) and (3.84), and regard \( \{Q_I\} \) as any complex functions of the spectral parameter (with the normalization \( Q_{\emptyset} = 1 \)) which satisfy (3.85) and (3.86).

4 \ Y-system

The Y-system is a system of functional relations related the thermodynamic Bethe ansatz. The T-system (3.42) is related to the Y-system by the following standard
T-hook is a union of $[3,3]$-hook and $[2,4]$-hook for $gl(3 + 2)$. The Y-system is defined on the dots.

Then the 'Y-functions' $Y_{a,s}$ satisfy the following Y-system (cf. Figure 9):

$$Y_{a,s}^{B_1,B_2,F_1,F_2} = \frac{(1 + Y_{a,s-1}^{B_1,B_2,F_1,F_2})(1 + Y_{a,s+1}^{B_1,B_2,F_1,F_2})}{(1 + (Y_{a-1,s}^{B_1,B_2,F_1,F_2})^{-1})(1 + (Y_{a+1,s}^{B_1,B_2,F_1,F_2})^{-1})}. \quad (4.2)$$

Thus we can obtain the solution of the Y-system through (4.1) based on the solution of the T-system presented in section 3. It is known that the Y-system (4.2) is invariant under the gauge transformation (3.44). This Y-system for the T-hook for $M_1 = M_2 = N_1 = N_2 = 2$ was proposed in [10] in the study of $AdS_5/CFT_4$ duality, and was generalized for the general T-hook in [11]. It was also studied [29] in relation to the string hypothesis on Bethe roots. These generalize Y-systems for super spin chains [51] (cf. [52]). We have also found Y-system like equations for the Bäcklund transformations (3.47)-(3.54). Explicitly, they are written as [29]:

$$\mathcal{A}_{a+1,s}^{(1)B_1,B_2,F_1,F_2} = \frac{(1 + \mathcal{A}_{a+1,s-1}^{(1)B_1,B_2,F_1,F_2})(1 + \mathcal{A}_{a,s+1}^{(1)B_1,B_2,F_1,F_2})}{(1 + (\mathcal{A}_{a,s}^{(1)B_1,B_2,F_1,F_2})^{-1})(1 + (\mathcal{A}_{a+1,s}^{(1)B_1,B_2,F_1,F_2})^{-1})}, \quad (4.3)$$

---

**Note:** There are several other definitions of the Y-functions. For example, we can define $\mathcal{A}_{a,s}^{(2)B_1,B_2,F_1,F_2} := \frac{z_{a+1,s+1}^{B_1,B_2,F_1,F_2}z_{a+1,s+1}^{B_1,B_2,F_1,F_2}}{z_{a+1,s+1}^{B_1,B_2,F_1,F_2}z_{a+1,s+1}^{B_1,B_2,F_1,F_2}}$ instead of (4.4). This satisfies $\mathcal{A}_{a,s+1}^{(2)B_1,B_2,F_1,F_2} = \frac{(1 + \mathcal{A}_{a,s+1}^{(2)B_1,B_2,F_1,F_2})(1 + \mathcal{A}_{a+1,s}^{(2)B_1,B_2,F_1,F_2})}{(1 + (\mathcal{A}_{a,s}^{(2)B_1,B_2,F_1,F_2})^{-1})(1 + (\mathcal{A}_{a+1,s}^{(2)B_1,B_2,F_1,F_2})^{-1})}$. In the main text, we defined the Y-functions so that the shift of the spectral parameter appears in the left hand side of the equations in the same way as the Y-system (4.2).
\begin{align*}
\mathcal{A}_{a,s}^{(2)}(B_1, B_2, F_1, F_2[1])&= \frac{(1 + \mathcal{A}_{a+1,s}^{(2)}(B_1, B_2, F_1, F_2))(1 + \mathcal{A}_{a-1,s+1}^{(2)}(B_1, B_2, F_1, F_2))}{(1 + \mathcal{A}_{a,s}^{(2)}(B_1, B_2, F_1, F_2))}. \\
\mathcal{A}_{a,s}^{(2)}(B_1, B_2, F_2) &= \frac{z^2 B_{a+1,s} m_{B_1, B_2, F_2} m_{B_1, B_2, F_2}}{T_{a,s}}. \quad (4.4) \\
\mathcal{A}_{a,s}^{(3)}(B_1, B_2, F_2[1]) &= \frac{(1 + \mathcal{A}_{a+1,s}^{(3)}(B_1, B_2, F_2))(1 + \mathcal{A}_{a-1,s+1}^{(3)}(B_1, B_2, F_2))}{(1 + \mathcal{A}_{a,s}^{(3)}(B_1, B_2, F_2))}. \quad (4.5) \\
\mathcal{A}_{a,s}^{(4)}(B_1, B_2, F_2[1]) &= \frac{(1 + \mathcal{A}_{a+1,s}^{(4)}(B_1, B_2, F_2))(1 + \mathcal{A}_{a-1,s+1}^{(4)}(B_1, B_2, F_2))}{(1 + \mathcal{A}_{a,s}^{(4)}(B_1, B_2, F_2))}. \quad (4.6) \\
\mathcal{A}_{a,s}^{(5)}(B_1, B_2, F_2[1]) &= \frac{(1 + \mathcal{A}_{a+1,s}^{(5)}(B_1, B_2, F_2))(1 + \mathcal{A}_{a-1,s+1}^{(5)}(B_1, B_2, F_2))}{(1 + \mathcal{A}_{a,s}^{(5)}(B_1, B_2, F_2))}. \quad (4.7) \\
\mathcal{A}_{a,s}^{(6)}(B_1, B_2, F_2[1]) &= \frac{(1 + \mathcal{A}_{a+1,s}^{(6)}(B_1, B_2, F_2))(1 + \mathcal{A}_{a-1,s+1}^{(6)}(B_1, B_2, F_2))}{(1 + \mathcal{A}_{a,s}^{(6)}(B_1, B_2, F_2))}. \quad (4.8) \\
\mathcal{A}_{a,s}^{(7)}(B_1, B_2, F_2[1]) &= \frac{(1 + \mathcal{A}_{a+1,s}^{(7)}(B_1, B_2, F_2))(1 + \mathcal{A}_{a-1,s+1}^{(7)}(B_1, B_2, F_2))}{(1 + \mathcal{A}_{a,s}^{(7)}(B_1, B_2, F_2))}. \quad (4.9) \\
\mathcal{A}_{a+1,s}^{(8)}(B_1, B_2, F_2[1]) &= \frac{(1 + \mathcal{A}_{a,s}^{(8)}(B_1, B_2, F_2))(1 + \mathcal{A}_{a-1,s+1}^{(8)}(B_1, B_2, F_2))}{(1 + \mathcal{A}_{a,s}^{(8)}(B_1, B_2, F_2))}. \
\end{align*}
Here notation on the index set is basically same as the one in (3.47)-(3.54). The double prime on the index set should be interpreted as follows. For example, $A_{a,s}^{(1)}B_1,B_2,F_1,F_2 = z_{b}B_{a+1,s}^{B_1,B_2,F_1,F_2}$ contains $B''_1$. Here we use notations $B_1' := B_1 \setminus \{b\}$, $B_1'' := B_1 \setminus \{b,b'\}$ for any fixed elements $b,b' \in B_1$ ($b \neq b'$). The double prime for the other sets should be interpreted in the same manner.

An integral form of the Y-system (4.2) is the thermodynamic Bethe ansatz equation. It is an infinite system of nonlinear integral equations with an infinite number of unknown functions for the free energy of quantum integrable systems at finite temperatures. Thus to find a finite system of nonlinear integral equations (NLIE) with a finite number of unknown functions which is equivalent to the thermodynamic Bethe ansatz equation will be an important problem. There are at least two type of NLIE in literatures. One of them is NLIE of Takahashi-type [53]. We derived NLIE of Takahashi-type for several integrable systems in statistical mechanics whose underlying algebras have arbitrarily rank [54]. NLIE of Takahashi-type is suited for calculations of high temperature thermodynamics. However, for the analysis of the low temperature thermodynamics, another type of NLIE proposed by Destri-de Vega or Klumper [55] seems to be better than Takahashi-type NLIE. Attempts to generalize this type of NLIE have been made by several authors, and in the case of integrable field theoretical models, NLIE of this type are known even for models whose underlying algebras have arbitrarily rank [56]. On the other hand for quantum integrable spin chains in statistical mechanics, this type of NLIE for arbitrary rank has not been established yet. This is because one needs a considerable trial and errors to find auxiliary functions with good analytical properties which play a key role (with a method to modify the Y-system proposed in [58]) in the derivation of the NLIE. In this context, we remark that the above ‘Y-functions’ $\{A_{a,s}^{(b)}B_1,B_2,F_1,F_2\}_{b=1}^8$ for the Bäcklund transformations are similar to such auxiliary functions for NLIE [51]. It will be important to clarify exact relation among them.

5 Concluding remarks

In this paper, we have continued our trials [1, 38] (and also [6, 7, 60, 48, 17, 2, 62]) to construct T- and Q-functions/operators for integrable models related to quantum affine (super)algebras and establish Wronskian-type formulas for any representations.

\[^{30}\] There are also NLIE related to algebras with arbitrarily rank [57] in rather different context.

\[^{31}\] Tableau sum expressions of the T-functions $T_{a,s}^{B_1,B_2,F_1,F_2}$ for $B_2 = F_2 = \emptyset$ are available in [1]. In term of these, $\{A_{a,s}^{(b)}B_1,B_2,F_1,F_2\}_{b=1}^8$ have similar structure as auxiliary functions for the NLIE (cf. [59]).
and functional relations among them. This paper is a small step toward our goal, and we focused our discussion on solutions of the T-system in terms of Q-functions in relation to infinite dimensional unitarizable modules of $gl(M|N)$. Most of our discussions here do not depend on whether the Q-functions are Baxter Q-operators or their eigenvalues. One of the next steps is to realize our formulas as operators. This will give us information on more precise algebraic and analytical structures of our formulas. There are several methods to perform this. One of them is to use the co-derivate $[61]$ on the generating function $w(t)$ of the supercharacters of the symmetric tensor representations of $gl(M|N)$, which produces a generating operator for T-operators. We used $[62]$ it to define T-and Q-operators. Generating operators of the T-operators satisfy a kind of Hirota equation, which we call a master identity. Many functional relations among T-and Q-operators follow from the master identity by easy manipulations. Our T-operators in $[62]$ were the ones for finite dimensional representations on both quantum and auxiliary spaces of the models (they are formulas on $[M, N]$-hook of $gl(M|N)$). It seems plausible that we can realize our formulas for infinite dimensional representations in the auxiliary space as operators just by changing the expansion point of the generating functions (as in $(2.11)$) and apply the co-derivative in the same way as $[62]$. Another approach relevant to our formulas is to use oscillator representations of the quantum affine superalgebras (or superYangians), which was proposed by $[25]$ for $U_q(\hat{sl}(2))$ case $[32]$ and developed in various directions $[35, 37, 38, 64, 65]$. In this construction, Q-operators are defined as supertrace of monodromy matrices over some oscillator representations. To construct a generating operator $[33]$ for T-operators in this approach seems to be one of the key steps toward our goal. Although the Master identity in $[62]$ was proposed for T-operators whose quantum space is fundamental representation on each site, notion of it is independent of the quantum space (and also independent of the coderivative), and will be generalizable to any quantum space $[34]$. We hope to address these issues step by step near future.

$[32]$As for T-operators related to the T-hook, there is an approach $[63]$ which is different but conceptually closer to $[25]$. $[33]$Note that the generating function of the characters is a kind of partition function of harmonic oscillators. The ‘co-derivative approach’ and the ‘oscillator approach’ will be unified in this context. In addition, a duality among two different groups seems to play a role to establish the master identity. Here we mean a duality that parameters $\{t_k\}$ in the product $\prod_k w(t_k)$ also generate group characters which are different from the original group characters generated by $w(t)$ itself. $[34]$In this context, the generating operators of the T-operators will be a kind of (super)determinant over a function of a L-operator which generates the super-Yangian or the quantum affine superalgebra. Some related quantities are known in literatures (for example, $[66]$).
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A Coefficient free form of the solutions

In this section, we list the coefficient free form of the solutions of the T-system given in section 3.3 and 3.5 by the transformations (3.83)-(3.84). For any sets $I$ and $J$, we will use a notation $\epsilon_{I,J} := (-1)^{\text{Card} \{(i,j) \in I \times J | i > j\}}$. Then (3.55)-(3.57) reduce to

$$T_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{(a+m_1)(s+n_2)+m_1m_2} \sum_{|I\cup J|=F_1\cup F_2, |I|=n_1-s} \epsilon_{I,J} Q_{B_1,I}^{[a-\eta_2]} Q_{B_2,J}^{[-a+\eta_1]}$$

for $a \geq \max\{s+\eta_1, -s+\eta_2, 0\}$, \hspace{1cm} (A.1)

$$T_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{m_2a} \sum_{|I\cup J|=B_1, |I|=a} \epsilon_{I,J} Q_{I,B_2,F_1,F_2}^{[s+\eta_1-\eta_2]} Q_{I,B_1}^{[-s]}$$

for $a \leq s+\eta_1$, \hspace{1cm} (A.2)

35 first, as a poster at ‘Integrability in Gauge and String Theory 2010’, Nordita, Sweden, 28 June 2010 - 2 July, and seminars at AEI, Potsdam, 9 August 2010; at IPMU, Japan, 7 September 2010
36 V. Kazakov, S. Leurent, D. Volin rewrote some of our formulas, which we gave to them, in a compact form in terms of the differential form [67]. This is based on the fact that any determinant can be expressed in terms of the differential form. And the differential form is particularly useful to express Wronskian type determinant formulas. We used Laplace expansion on determinants and Plücker identities in our proof of the formulas (in appendix B). One can prove Laplace expansion formula and Plücker identities elegantly based on the differential form. Thus, the differential form expression will be useful to make our proof of our formulas transparent. On the other hand, when one evaluates the formulas (for example, numerically), one has to extract coefficients of Grassmann numbers, and reproduces Wronskian like determinant formulas.
where the number of the elements of the sets must satisfy
\[ |I| \leq |J| = a \]
and

\[ \Delta_{T_1, T_2, \emptyset, (a-s-n_1, a-s-n_1-n_2)} \]

for \( a \leq -s + \eta_2 \), \hfill (A.3)

The sparse determinant (3.25) becomes in the following form.

\[
\Delta_{B_1, B_2, R, |\eta|, \xi} =
\begin{vmatrix}
\left( Q_{b, f}^{\xi} \right)_{b \in B_1, f \in F} & \left( Q_{b}^{\xi+2s-1} \right)_{b \in B_1, s \in S} & \left( Q_{b}^{\xi+2t-1} \right)_{b \in B_1, t \in T_1} & (0)_{|B_1| \times |T_2|} \\
\left( -1 \right)^{n_2} Q_{b, f}^{\xi-2n} & \left( Q_{b}^{\xi+2s-1} \right)_{b \in B_2, s \in S} & (0)_{|B_2| \times |T_1|} & \left( Q_{b}^{\xi+2t-1} \right)_{b \in B_2, t \in T_2} \\
\left( -1 \right)^{n_1} Q_{f}^{\xi-2r+1} & (0)_{|R| \times |S|} & (0)_{|R| \times |T_1|} & (0)_{|R| \times |T_2|}
\end{vmatrix}
\hfill (A.9)

The sparse determinant (3.25) becomes in the following form.

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\[ T_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{(s+\eta_1)m_2+\Theta} \Delta_{F_1,F_2,\emptyset,\emptyset} \]

for \( a \geq \max\{s + \eta_1, -s + \eta_2, 0\}, \quad \eta_2 \leq s \leq -\eta_1, \quad (A.13) \]

\[ T_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{m_2 a + \Theta} \Delta_{F_1,F_2,\emptyset,\emptyset,\emptyset} \]

for \( a \leq \min\{s + \eta_1, \eta_1 + \eta_2\}, \quad (A.14) \]

\[ T_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{a(m_2+1)+\Theta} \Delta_{F_1,F_2,\emptyset,\emptyset,\emptyset} \]

for \( \eta_1 + \eta_2 \leq a \leq s + \eta_1, \quad (A.15) \]

\[ T_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{a(m_2+1)+\Theta} \Delta_{F_1,F_2,\emptyset,\emptyset,\emptyset} \]

for \( a \leq \min\{-s + \eta_2, \eta_1 + \eta_2\}, \quad (A.16) \]

\[ T_{a,s}^{B_1,B_2,F_1,F_2} = (-1)^{a(m_2+1)+\Theta} \Delta_{F_1,F_2,\emptyset,\emptyset,\emptyset} \]

for \( \eta_1 + \eta_2 \leq a \leq -s + \eta_2, \quad (A.17) \]

where we introduced a symbol \( \Theta := \frac{(|B_1|+|B_2|)(|B_1|+|B_2|-1)}{2} \).

The dense determinant expression of the solution \([3.58]-[3.60]\) becomes in the following form.

\[ T_{a,s}^{B_1,B_2,F_1,F_2} = \frac{(-1)^{m_1(m_1-1)+m_2 a}}{\prod_{k=1}^{m_1-a-1} Q_{a-s-m_1+2k}^{B_2,F_1,F_2}} \times \left| \left( Q_{b,B_2,F_1,F_2}^{[-s+a-m_1+2j-1]} \right)_{b \in B_1, 1 \leq j \leq m_1-a} \left( Q_{b}^{[s-a+2j-1+\eta_1-\eta_2]} \right)_{b \in B_2, 1 \leq j \leq a} \right| \]

for \( a \leq s + \eta_1, \quad (A.18) \]

\[ T_{a,s}^{B_1,B_2,F_1,F_2} = \frac{(-1)^{m_1(m_1-1)+m_2 a}}{\prod_{k=1}^{m_1-a-1} Q_{a-s-n_1-n_2-2k}^{B_1} \prod_{k=1}^{m_2-s-1} Q_{a-s-n_1+n_2+\eta_1-2k}^{B_2}} \times \left| \left( Q_{B_1,f}^{[a-s+n_1-n_2-2i+1]} \right)_{1 \leq i \leq n_1-s, f \in F} \right| \left| \left( Q_{B_2,f}^{[-a+s+n_1+n_2+\eta_1-2i+1]} \right)_{1 \leq i \leq n_2+s, f \in F} \right| \]

for \( a \geq \max\{s + \eta_1, -s + \eta_2, 0\}, \quad (A.19) \]

\[ T_{a,s}^{B_1,B_2,F_1,F_2} = \frac{(-1)^{m_2(m_2-1)+m_1 a}}{\prod_{k=1}^{m_2-a-1} Q_{a-s-m_2+2k}^{B_1,F_1,F_2}} \times \left| \left( Q_{b}^{[s-a+2j-1+\eta_1-\eta_2]} \right)_{b \in B_2, 1 \leq j \leq a} \left( Q_{B_1,b,F_1,F_2}^{[-s-a-m_2+2j-1]} \right)_{b \in B_2, 1 \leq j \leq m_2-a} \right| \]

for \( a \leq -s + \eta_2. \quad (A.20) \]
B Proof of the functional relations

In this appendix, we will prove that the T-function $T_{a,s}^{B_1,B_2,F_1,F_2}$ in Appendix A satisfies the Hirota equation:

$$T_{a,s}^{B_1,B_2,F_1,F_2[-1]}T_{a,s}^{B_1,B_2,F_1,F_2[1]} = T_{a,s-1}^{B_1,B_2,F_1,F_2}T_{a,s+1}^{B_1,B_2,F_1,F_2} + T_{a-1,s}^{B_1,B_2,F_1,F_2}T_{a+1,s}^{B_1,B_2,F_1,F_2}$$  \(B.1\)

on the generalized T-hook. We will use QQ-relation for $I = \emptyset$ with a normalization $Q_\emptyset = 1$.

$$Q_b Q_f = Q_b^{[1]} - Q_b^{[-1]}, \quad b \in \mathfrak{B}, \quad f \in \mathfrak{F}. \quad \text{(B.2)}$$

We will use the following lemma in the proof.

**Lemma B.1.** The following relations are valid for the determinant \(A.9\) under the relation \(B.2\), where $|B_1| + |B_2| + a = |F| + b + c_1 + c_2$.

(i) If there are components such that $t_i^{(1)} = 0$ for some $i \in \{1, 2, \ldots, c_1\}$ or $r_j = 1$ for some $j \in \{1, 2, \ldots, a\}$, the following relation holds

$$\Delta_{B_1,B_2,(r_1,r_2,\ldots,r_a),[\eta|\xi]} F,(t_i^{(1)},t_j^{(2)},\ldots,t_k^{(2)}) = (-1)^{a+m_2+c_2} \Delta_{B_1,B_2,(r_1-1,r_2-1,\ldots,r_a-1),[\eta-1|\xi-2]} F,(t_i^{(1)},t_j^{(2)},1+1,\ldots,t_k^{(2)},1+1) \quad \text{(B.3)}$$

(ii) If there is a component such that $s_i = 0$ for some $i \in \{1, 2, \ldots, b\}$, the following relation holds

$$\Delta_{B_1,B_2,(r_1,r_2,\ldots,r_a),[0|\xi]} F,(s_i,t_i^{(2)},\ldots,t_k^{(2)}) = (-1)^a \Delta_{B_1,B_2,(r_1-1,r_2-1,\ldots,r_a-1),[0|\xi-2]} F,(s_i+1,t_i^{(2)},1+1,\ldots,t_k^{(2)},1+1) \quad \text{(B.4)}$$

(iii) If there are components such that $t_i^{(2)} = -\eta + 1$ for some $i \in \{1, 2, \ldots, c_2\}$ or $r_j = \eta$ for some $j \in \{1, 2, \ldots, a\}$, the following relation holds

$$\Delta_{B_1,B_2,(r_1,r_2,\ldots,r_a),[\eta|\xi]} F,(t_i^{(1)},t_j^{(1)},t_k^{(2)},\ldots,t_k^{(2)}) = (-1)^{m_2+c_2} \Delta_{B_1,B_2,(r_1,r_2,\ldots,r_a),[\eta-1|\xi]} F,(t_i^{(1)},t_j^{(1)},t_k^{(2)},\ldots,t_k^{(2)}) \quad \text{(B.5)}$$

For any $N \times (N+2)$ matrix, we will write a minor determinant whose $a$-th and $b$-th columns are removed from it as $D[a, b]$. We will use the following Plücker identity among determinants.

$$D[k_1, k_2] D[k_3, k_4] - D[k_1, k_3] D[k_2, k_4] + D[k_1, k_4] D[k_2, k_3] = 0 \quad \text{(B.6)}$$

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where \( k_1 < k_2 < k_3 < k_4 \).

At first, let us check the determinant formulas in Appendix A satisfy the boundary conditions \((A.4)-(A.8), (A.4)-(A.7)\) trivially follow from \((A.1)-(A.3), (A.8)\) for \( s \geq -\eta_1 \) or \( s \leq \eta_2 \) also trivially follows from \((A.2)-(A.3)\). For \( \eta_1 + \eta_2 \geq 0 \), this covers the boundary condition at \( a = 0 \). Let us consider the case \( \eta_1 + \eta_2 < 0 \). All we have to do is to check \((A.8)\) for \( \eta_2 < s < -\eta_1 \). In this case,

\[
T_{B_1, B_2, F_1, F_2} = (-1)^{(s+\eta_1)m_2+1} \Delta_{B_1, B_2, \{1,-\eta_1-\eta_2\},\{-s-\eta_1:s-m-\eta_2\}} \tag{B.7}
\]

follows from \((A.13)\). One can rewrite \((3.6)\) and \((3.7)\) for \( I = \emptyset, B = B_1 \sqcup B_2 \) and \( F = F_1 \sqcup F_2 \) in the following form\(^{37}\)

\[
Q_{B_1, B_2, F_1, F_2} = (-1)^{\frac{m(m-1)}{2}} \Delta_{B_1, B_2, \emptyset, \{0: \eta_1-\eta_2\}, \{1, \eta_1+\eta_2\}, \emptyset, \emptyset} \quad \text{for } \eta_1 + \eta_2 \geq 0, \tag{B.8}
\]

\[
Q_{B_1, B_2, F_1, F_2} = (-1)^{\frac{m(m-1)}{2}} \Delta_{B_1, B_2, \{1,-\eta_1-\eta_2\}, \{0: \eta_1-\eta_2\}, \emptyset, \emptyset, \emptyset} \quad \text{for } \eta_1 + \eta_2 \leq 0. \tag{B.9}
\]

Applying \((B.5)\) to \((B.7)\) and comparing this with \((B.9)\), we find that \((A.8)\) is valid.

Except on the lines defined by \( a = s + \eta_1 \) or \( a = -s + \eta_2 \) and the boundaries in the T-hook, that \((A.13)-(A.20)\) satisfy the Hirota equation \((B.1)\) reduces to the Plücker identity \((B.3)\).

Then the rest of our task is to prove \((B.1)\) on the lines defined by \( a = s + \eta_1 \) or \( a = -s + \eta_2 \) in the T-hook. For this, we consider the following five cases depending on the values of \( m_1, m_2, n_1, n_2 \) (cf. Figure 4, 5). We remark that some of them become void for some specific values of \( m_1, m_2, n_1, n_2 \). From now on, we will abbreviate the index sets \( B_1, B_2, F_1, F_2 \) except when the size of them changes. For example, we use abbreviation:

\[
B_1 \setminus b_{a}^{(1)} = (b_1^{(1)}, b_2^{(1)}, \ldots, b_{\alpha-1}^{(1)}, b_{\alpha+1}^{(1)}, \ldots, b_{m_2}^{(1)}).
\]

**(i) the case** \( a = s + \eta_1 \), \( \max\{-\eta_1, \eta_2\} + 1 \leq s \leq n_1 \), \( m_1, n \geq 1 \)

That \((A.11)\) and \((A.15)\) satisfy \((B.1)\) is equivalent to

\[
\Delta_{\emptyset, \{1, s+\eta_1\}, \emptyset} \Delta_{\emptyset, \{1, s-\eta_2\}, \emptyset}
\]

\[
= (-1)^{\eta_1+n_2+1} \left( \Delta_{\emptyset, \{2, s-\eta_2\}, \emptyset} \Delta_{\emptyset, \{1, s+\eta_1-1\}, \emptyset} \Delta_{\emptyset, \{1, s-\eta_2\}, \emptyset} \Delta_{\emptyset, \{1, s+\eta_1\}, \emptyset} \Delta_{\emptyset, \{2, s+\eta_1+1\}, \emptyset} \right). \tag{B.10}
\]

Let us apply \((B.5)\) for the left hand side of \((B.10)\) and expand the determinants,

\(^{37}\)One may unify these as \( Q_{B_1, B_2, F_1, F_2} = (-1)^{\frac{m(m-1)}{2}} \Delta_{F_1, F_2, \{1, \eta_1+\eta_2\}, \emptyset, \emptyset} \).
and apply (B.2): 

\[
\Delta_{0,1}^1 \Delta_{1,0}^1 + \Delta_{0,1}^1 \Delta_{2,0}^1 = (-1)^{m_2} \Delta_{0,1}^1 \Delta_{1,0}^1 (-1)^{n_2+1} + (-1)^{m_2} \Delta_{0,1}^1 \Delta_{2,0}^1 (-1)^{n_2+1} \times \sum_{\alpha=1}^{m_1} \sum_{\beta=1}^{n_2} (-1)^{n_2+1} \Delta_{0,1}^1 \Delta_{1,0}^1 (-1)^{m_2+1} + (-1)^{m_2} \sum_{\alpha=1}^{m_1} \sum_{\beta=1}^{n_2} (-1)^{n_2+1} \Delta_{0,1}^1 \Delta_{1,0}^1 (-1)^{m_2+1} 
\]

Subtracting the \(\beta\)-th column from the \((n+1)\)-th column in the determinant in the first summand in the right hand side of (B.11), we obtain

\[
\Delta_{0,1}^1 \Delta_{1,0}^1 (-1)^{n_2+1} \times \sum_{\alpha=1}^{m_1} \sum_{\beta=1}^{n_2} (-1)^{n_2+1} \Delta_{0,1}^1 \Delta_{1,0}^1 (-1)^{m_2+1} + (-1)^{m_2} \sum_{\alpha=1}^{m_1} \sum_{\beta=1}^{n_2} (-1)^{n_2+1} \Delta_{0,1}^1 \Delta_{1,0}^1 (-1)^{m_2+1} 
\]
\[
= - \sum_{\gamma=1}^{m_2} (-1)^{\gamma+m_1+n+1} Q_{\beta_i (2)}^{(0)} \Delta_{\emptyset, (1, s-\eta_2), [0;0]}^{R_2 \backslash \gamma_2, (1, s-\eta_2), [0;0]} \n - \sum_{\gamma=1}^{s-\eta_2} (-1)^{\gamma+m_1+m_2+n+1} (-1)^{\gamma-1} Q_{\beta_i (2)}^{[-2\gamma+1]} \Delta_{\emptyset, (2, s+\eta_1), \emptyset}^{\emptyset, (1, s-\eta_2) \backslash \gamma, [0;0]}. \quad (B.12)
\]

Subtracting the \(\alpha\)-th row from the \((m_1 + m_2 + 1)\)-th row in the determinant in the second summand in the right hand side of \((B.11)\), we find:

\[
\begin{vmatrix}
(Q_{\beta_i (2)}^{[2]} f_j)_{1 \leq i \leq m_1, 1 \leq j \leq n} & (Q_{\beta_i (1)}^{[2j+1]} f_j)_{1 \leq i \leq m_1, 1 \leq j \leq s+\eta_1} \\
(-Q_{\beta_i (2)}^{[0]} f_j)_{1 \leq i \leq m_2, 1 \leq j \leq n} & (0)_{m_2 \times (s+\eta_1)} \\
(Q_{\beta_i (2)}^{[2]} f_j)_{1 \leq j \leq n} & (0)_{1 \times (s+\eta_1)} \\
((-1)^{i-1} Q_{\beta_i (2)}^{[-2i+3]} f_j)_{2 \leq i \leq s-\eta_2, 1 \leq j \leq n} & (0)_{(s-\eta_2-1) \times (s+\eta_1)}
\end{vmatrix}
\]

\[
= - \sum_{\gamma=1}^{s+\eta_1} (-1)^{\gamma+n+m_1+m_2+1} Q_{\beta_i (1)}^{[2\gamma+1]} \Delta_{\emptyset, (2, s-\eta_2), [1;2]}^{\emptyset, (1, s+\eta_1) \backslash \gamma, \emptyset}. \quad (B.13)
\]

Substituting \((B.12)\) and \((B.13)\) into the right hand side of \((B.11)\), and taking summation over \(\alpha\) and \(\beta\), we find that only a few terms in the summation over \(\gamma\) are non-zero (since there are the same rows or columns in determinants except for a few values of \(\gamma\)), and finally we obtain the right hand side of \((B.10)\).

**ii) the case** \(a = s+\eta_1, \frac{\eta_2-\eta_1}{2} < s \leq \eta_2 \quad (\eta_1 + \eta_2 \geq 1)\)

At first, we assume \(\eta_1 + \eta_2 \geq 2\). Due to \((B.4)\) and \((B.4)\), that \((A.10)\) and \((A.14)\) satisfy \((B.1)\) is equivalent to

\[
\Delta_{\emptyset, [0;0]}^{\emptyset, [0;0]} \Delta_{(0, \eta_2-s-1), (0, s+\eta_1-1), \emptyset}^{\emptyset, [0;0]} = (-1)^{(s+\eta_1)(s+\eta_2+1)} \Delta_{\emptyset, [0;0]}^{\emptyset, [0;0]} \Delta_{(0, \eta_2-s-1), (1, s+\eta_1), \emptyset}^{\emptyset, [0;0]} \Delta_{(0, \eta_2-s), (1, s+\eta_1-1), \emptyset}^{\emptyset, [0;0]} \Delta_{\emptyset, [0;0]}^{\emptyset, [0;0]} \Delta_{(0, \eta_2-s), (1, s+\eta_1), \emptyset}^{\emptyset, [0;0]} \Delta_{\emptyset, [0;0]}^{\emptyset, [0;0]} \Delta_{(0, \eta_2-s), (1, s+\eta_1-1), \emptyset}^{\emptyset, [0;0]}, \quad (B.14)
\]

Due to the following identities follow from manipulations on columns of the deter-
minants:
\[
\Delta_{(0,\eta_2-s-1),(0,s+\eta_1-1),0}^{\emptyset,[0;0]} = (-1)^{\eta_1+s} \Delta_{(0,\eta_2-s-1),(1,s+\eta_1-1),0}^{\emptyset,[0;0]},
\]
\[
\Delta_{(0,\eta_1+s-1),(0,\eta_2-s)}^{\emptyset,[0;0]} = (-1)^{\eta_1+s} \eta_2+s) \Delta_{(1,\eta_2-s),(1,s+\eta_1-1),0}^{\emptyset,[0;0]},
\]
\[
\Delta_{(0,\eta_1+s),(0,\eta_2-s-1)}^{\emptyset,[0;0]} = (-1)^{\eta_1+s} \eta_2+s+\eta_2+1 \Delta_{(1,\eta_2-s-1),(1,s+\eta_1),0}^{\emptyset,[0;0]}.
\]

(B.14) is equivalent to
\[
\Delta_{(0,\eta_2-s-1),(1,s+\eta_1-1),0}^{\emptyset,[0;0]} \Delta_{(0,\eta_2-s-1),(1,s+\eta_1-1),0}^{\emptyset,[0;0]} = \Delta_{(0,\eta_2-s-1),(1,s+\eta_1-1),0}^{\emptyset,[0;0]} \Delta_{(0,\eta_2-s-1),(1,s+\eta_1-1),0}^{\emptyset,[0;0]}
\]
\[
- \Delta_{(0,\eta_2-s-1),(1,s+\eta_1-1),0}^{\emptyset,[0;0]} \Delta_{(1,\eta_2-s-1),(1,s+\eta_1),0}^{\emptyset,[0;0]}.
\]

This is nothing but the Plücker identity (B.6).

Next we consider the case \(\eta_1 + \eta_2 = 1\). In this case, only \(a = 1\), \(s = \eta_2\) is allowed. After manipulation on rows and columns of the determinants based on (B.2), that (A.10) and (A.14) satisfy (B.1) reduces to the Plücker identity (B.6) for the following \((m_1 + m_2 + 1) \times (n + 4)\) matrix

\[
\begin{pmatrix}
1 & 1 & 1 & (Q_{f_j}^{[-1]})_{1 \leq j \leq n} & 0 \\
(0)_{m_1 \times 1} & (0)_{m_1 \times 1} & (Q_{\eta_2}^{[-1]})_{1 \leq i \leq m_1} & (Q_{\eta_1}^{[-1]} f_j)_{1 \leq i \leq m_1} & (Q_{\eta_2}^{[-1]})_{1 \leq i \leq m_1} \\
(0)_{m_2 \times 1} & (Q_{\eta_2}^{[-1]})_{1 \leq i \leq m_2} & (Q_{\eta_1}^{[-1]} f_j)_{1 \leq i \leq m_2} & (Q_{\eta_2}^{[-1]})_{1 \leq i \leq m_2} & (0)_{m_2 \times 1} \\
\end{pmatrix}
\]

(B.17)

at \(k_1 = 1\), \(k_2 = 2\), \(k_3 = 3\), \(k_4 = n + 4\).

(iii) the case \(a = \frac{m+1}{2}\), \(s = \frac{m-1}{2}\) \(\eta_1 + \eta_2 \in 2\mathbb{Z}_{\geq 1}\)

Due to (B.4) and (B.5), that (A.10), (A.14) and (A.16) satisfy (B.1) is equivalent to
\[
\Delta_{(0,\eta_2^{(-1)}),(0,\eta_2^{(-1)}),0}^{\emptyset,[0;0]} \Delta_{(1,\eta_2^{(-1)}),(1,\eta_2^{(-1)}),0}^{\emptyset,[0;0]}
\]
\[
= (-1) \Delta_{(0,\eta_2^{(-1)}),(0,\eta_2^{(-1)}),0}^{\emptyset,[0;0]} \Delta_{(0,\eta_2^{(-1)}),(0,\eta_2^{(-1)}),0}^{\emptyset,[0;0]}
\]
\[
+ \Delta_{(0,\eta_2^{(-1)}),(0,\eta_2^{(-1)}),0}^{\emptyset,[0;0]} \Delta_{(1,\eta_2^{(-1)}),(0,\eta_2^{(-1)}),0}^{\emptyset,[0;0]}.
\]

(B.18)

Then we find (B.18) is equivalent\(^{35}\) to (B.16) at \(s = \frac{m-\eta_1}{2}\) since there are identities

\(^{35}\)Note that \(m_1 + m_2 = n + 1\). This matrix also suggest us yet another determinant expression of the solution.

\(^{36}\)Note that (B.14) is not equivalent to (B.10) at \(s = \frac{m-\eta_1}{2}\).
follow from manipulations on columns of the determinants:

\[
\Delta_{(0,91+2r-1),(0,91+2r-1),0}^{\emptyset,[0:0]} = (-1)^{\frac{91+2r}{2}} \Delta_{(0,91+2r-1),(1,91+2r-1),(0)}^{\emptyset,[0:0]},
\]

\[
\Delta_{(1,91+2r),(0,91+2r-1),0}^{\emptyset,[0:0]} = \Delta_{(1,91+2r),(1,91+2r-1),(0)}^{\emptyset,[0:0]}, \tag{B.19}
\]

\[
\Delta_{(1,91+2r),(0,91+2r-1),0}^{\emptyset,[0:0]} = (-1)^{\frac{91+2r}{2}-1} \Delta_{(1,91+2r),(1,91+2r-1),(0)}^{\emptyset,[0:0)}. \tag{B.20}
\]

(iv) the case \( a = -s + \eta_2, \ -\eta_1 \leq s < \frac{n_2-n_1}{2} \) \( (\eta_1 + \eta_2 \geq 1) \)

At first, we assume \( \eta_1 + \eta_2 \geq 2 \). Using \([B.3]\) and \([B.5]\) repeatedly, we obtain

\[
\Delta_{(1,a+\eta_1+1),(2s+\eta_1-\eta_2+2,s+\eta_1)}^{\emptyset,[0:0]} = (-1)^{(m_2-s+n_2-1)(s+n_1)} \Delta_{(1,-s+\eta_2-1,0)}^{\emptyset,[0:0]} \Delta_{(1,-s+\eta_2-1,1),(s-\eta_2+2,0)}^{\emptyset,[0:0]}, \tag{B.21}
\]

\[
\Delta_{(1,a+\eta_1+1),(2s+\eta_1-\eta_2,s+\eta_1-1)}^{\emptyset,[0:0]} = (-1)^{(n_2+s)(n_1+n_2+1)} \Delta_{(1,-s+\eta_1+1,1),(s-\eta_2+2,0)}^{\emptyset,[0:0]} \tag{B.22}
\]

We can also obtain the following relations based on \([B.4]\):

\[
\Delta_{(1,a+\eta_1),(2s+\eta_1-\eta_2+2,s+\eta_1)}^{\emptyset,[0:0]} = \Delta_{(1,-s+\eta_1+1,0),(s-\eta_2+2,1)}^{\emptyset,[0:0]}, \tag{B.23}
\]

Then, based on \([B.20]-[B.21]\), we can show that \([A.10]\) and \([A.16]\) satisfy \([B.1]\) is equivalent to

\[
\Delta_{(-s-\eta_1+1,0),(s-\eta_2+2,1)}^{\emptyset,[0:0]} \Delta_{(1,s+\eta_1+1,0),(s-\eta_2+2,1)}^{\emptyset,[0:0]} = (-1)^{n_2-s} \Delta_{(-s+\eta_1+1,1),(s-\eta_2+2,1)}^{\emptyset,[0:0]} \Delta_{(-s-\eta_1+1,0),(s-\eta_2+2,0)}^{\emptyset,[0:0]} \tag{B.24}
\]

After manipulations on columns of determinants similar to \([B.19]\), we find \([B.22]\) reduces to

\[
\Delta_{(-s-\eta_1+1,0),(s-\eta_2+2,1)}^{\emptyset,[0:0]} \Delta_{(1,s+\eta_1+1,0),(s-\eta_2+2,1)}^{\emptyset,[0:0]} = \Delta_{(-s-\eta_1+2,1),(s-\eta_2+2,1)}^{\emptyset,[0:0]} \Delta_{(-s-\eta_1+1,0),(s-\eta_2+2,0)}^{\emptyset,[0:0]} \tag{B.25}
\]

This is nothing but the Plücker identity \([B.6]\).

Next we consider the case \( \eta_1 + \eta_2 = 1 \). In this case, only \( a = 1, \ s = -\eta_1 \) is allowed. After manipulation on rows and columns of the determinants based on
Using (B.3) and (B.5) repeatedly, we obtain:

Let us expand the determinants in the left hand side of (B.26), and apply (B.2):

\[
\begin{pmatrix}
-1 & -1 & (Q^{[1]}_{f_f})_{1 \leq j \leq n} & 0 \\
(0)_{m_1 \times 1} & (Q^{[1]}_{h_1})_{1 \leq i \leq m_1} & (Q^{[1]}_{h_1(j)})_{1 \leq i \leq m_1} & (0)_{m_1 \times 1} \\
(0)_{m_2 \times 1} & (0)_{m_2 \times 1} & (Q^{[1]}_{h_2})_{1 \leq i \leq m_2} & (Q^{[2]}_{h_2(j)})_{1 \leq i \leq m_2}
\end{pmatrix}
\]

(B.24)

at \(k_1 = 1, k_2 = 2, k_3 = 3, k_4 = n + 4\).

(v) the case \(a = -s + \eta_2, -n_2 \leq s \leq \min\{-\eta_1, \eta_2\} - 1, (n, m_2 \geq 1)\)

Using (B.3) and (B.5) repeatedly, we obtain:

\[
\Delta^{(1,-s-\eta_1),[0;-2(s+\eta_1)-2(s+\eta_1)]}_{\emptyset,\emptyset,(2s+\eta_1+A,\eta_1)} = (-1)^{(n_2+s)(s+\eta_2)} \Delta^{(1,-s-\eta_1),[0;-2(s+\eta_1)-2(s+\eta_1)]}_{\emptyset,\emptyset,(2s+\eta_1+A,\eta_1)}
\]

\[
= (-1)^{(n_2+s)(s+\eta_2)} \Delta^{(1,-s-\eta_1),[0;-2(s+\eta_1)-2(s+\eta_1)]}_{\emptyset,\emptyset,(2s+\eta_1+A,\eta_1)}
\]

Due the above relations (B.25), that (A.12) and (A.17) satisfy (B.1) is equivalent to

\[
\begin{align*}
\Delta^{(s+\eta_1+1,0),[0;2]}_{\emptyset,\emptyset,(s-\eta_2+1,-1)} & - \Delta^{(s+\eta_1+1,0),[0;2]}_{\emptyset,\emptyset,(s-\eta_2+1,-1)} \\
\end{align*}
\]

(B.26)

Let us expand the determinants in the left hand side of (B.26), and apply (B.2):

\[
\begin{align*}
\Delta^{(s+\eta_1+1,0),[0;2]}_{\emptyset,\emptyset,(s-\eta_2+1,-1)} &= \sum_{\alpha=1}^{n} (-1)^{s+\eta_1+2m_2-s-\eta_1} Q^{[1]}_{f_{\alpha}} \Delta^{(s+\eta_1+2,0),[1;2]}_{\emptyset,\emptyset,(s-\eta_2+1,-1)} \\
& \times \sum_{\beta=1}^{m_2} (-1)^{\beta+m_1+n-s-\eta_2} Q^{[1]}_{h_{\beta}} \Delta^{(s+\eta_1+1,0),[0;2]}_{\emptyset,\emptyset,(s-\eta_2+2,-1)} \\
\end{align*}
\]

Note that \(m_1 + m_2 = n + 1\).
$$= \sum_{\alpha=1}^{n} \sum_{\beta=1}^{m_2} (-1)^{\alpha+m_1+m_2-s-\eta_1} (-1)^{\beta+m_1+n-s+\eta_2} (Q^{[2]}_{b_{\beta}} f_\alpha - Q^{[0]}_{b_{\beta}} f_\alpha) \times \Delta^{(s+\eta_1+2,0),[1:2]} F_{\eta_2-\eta_1} \Delta^{B_2 \setminus \{b_{\beta}\}, (s+\eta_1+1,0),[0:2]} \Delta^{0,0,(s-\eta_2-1,-1)}$$

$$= - \sum_{\beta=1}^{m_2} (-1)^{\beta+m_1+n-s+\eta_2} \Delta^{B_2 \setminus \{b_{\beta}\}, (s+\eta_1+1,0),[0:2]} \Delta^{0,0,(s-\eta_2+1,-1)}$$

$$\times \begin{vmatrix}
(Q^{[2]}_{b_{(1)}^i} f_j)_{1 \leq i \leq m_1, 1 \leq j \leq n} & (0)_{m_1 \times (\eta_2-s)} \\
(-Q^{[0]}_{b_{(1)}^i} f_j)_{1 \leq i \leq m_2, 1 \leq j \leq n} & (Q^{[2]}_{b_{(2)}^i} f_\alpha)_{1 \leq i \leq m_2, s-\eta_2 \leq j \leq -1} \\
((-1)^{i-1}Q^{[-2i+3]} f_j)_{s+\eta_1+2 \leq i \leq 0, 1 \leq j \leq n} & (0)_{(-s-\eta_1-1) \times (\eta_2-s)} \\
(Q^{[0]}_{b_{(2)}^i} f_j)_{1 \leq j \leq n} & (0)_{1 \times (\eta_2-s)}
\end{vmatrix}.$$

Adding the $(\beta + m_1)$-th row to the $(m_1 + m_2 - s - \eta_1)$-th row in the determinant in the first summand in the right hand side of (B.27), we obtain

$$\times \begin{vmatrix}
(Q^{[2]}_{b_{(1)}^i} f_j)_{1 \leq i \leq m_1, 1 \leq j \leq n} & (0)_{m_1 \times (\eta_2-s)} \\
(-Q^{[0]}_{b_{(1)}^i} f_j)_{1 \leq i \leq m_2, 1 \leq j \leq n} & (Q^{[2]}_{b_{(2)}^i} f_\alpha)_{1 \leq i \leq m_2, s-\eta_2 \leq j \leq -1} \\
((-1)^{i-1}Q^{[-2i+3]} f_j)_{s+\eta_1+2 \leq i \leq 0, 1 \leq j \leq n} & (0)_{(-s-\eta_1-1) \times (\eta_2-s)} \\
(Q^{[0]}_{b_{(2)}^i} f_j)_{1 \leq j \leq n} & (0)_{1 \times (\eta_2-s)}
\end{vmatrix}.$$

$$= \begin{vmatrix}
(Q^{[2]}_{b_{(1)}^i} f_j)_{1 \leq i \leq m_1, 1 \leq j \leq n} & (0)_{m_1 \times (\eta_2-s)} \\
(-Q^{[0]}_{b_{(1)}^i} f_j)_{1 \leq i \leq m_2, 1 \leq j \leq n} & (Q^{[2]}_{b_{(2)}^i} f_\alpha)_{1 \leq i \leq m_2, s-\eta_2 \leq j \leq -1} \\
((-1)^{i-1}Q^{[-2i+3]} f_j)_{s+\eta_1+2 \leq i \leq 0, 1 \leq j \leq n} & (0)_{(-s-\eta_1-1) \times (\eta_2-s)} \\
(0)_{1 \times n} & (Q^{[2]}_{b_{(2)}^i} f_\alpha)_{s-\eta_2 \leq j \leq -1}
\end{vmatrix}$$

$$= \sum_{\gamma=s-\eta_2}^{n} (-1)^{m_1+m_2-s-\eta_1+n+\gamma-s+\eta_2+1} Q^{[2\gamma+1]}_{b_{(2)}^i} \Delta^{(s+\eta_1+2,0),[1:2]} F_{\eta_2-\eta_1} \Delta^{0,0,(s-\eta_2,-1)\setminus \gamma}.$$ (B.28)

Subtracting the $\alpha$-th column from the $(n - s + \eta_2)$-th column in the determinant in
the second summand in the right hand side of (B.27), we obtain

\[
\begin{pmatrix}
(Q[^2_{b_i^{(1)}}]_{f_j})_{1 \leq i \leq m_1, 1 \leq j \leq n} & (Q[^{2j+1}_{b_i^{(2)}}]_{f_j})_{1 \leq i \leq m_2, s-\eta_2+1 \leq j \leq -1} & (Q[^{2}_{b_i^{(1)}}]_{f_\alpha})_{1 \leq i \leq m_1} \\
(((-1)^{-1}Q[^{-2i+3}_{f_j}])_{s+\eta_1+1 \leq i \leq 0, 1 \leq j \leq n} & (0)^{m_1 \times (\eta_2-s-1)} & (0)^{m_1 \times 1} \\
& (0)^{(-s-\eta_1) \times (\eta_2-s-1)} & (0)^{(-s-\eta_1) \times 1}
\end{pmatrix}
\]

\[
= - \sum_{\gamma=1}^{m_1} (-1)^{\gamma+n+\eta_2-s} Q[^{2}_{b_i^{(1)}}]_{f_\alpha} \Delta_{B_1 \{b_i^{(1)},(s+\eta_1+1,0),[0;2]\},B_1,\emptyset,(s-\eta_2+1,-1)} \\
- \sum_{\gamma=s+\eta_1+1}^{0} (-1)^{m_1+m_2+\gamma-s-\eta_1+n+\eta_2-s} (-1)^{\gamma-1} Q[^{-2\gamma+3}_{f_\alpha} \Delta_{B_1 \{b_i^{(1)},(s+\eta_1+1,0),[0;2]\},\emptyset,(s-\eta_2+1,-1))}. \tag{B.29}
\]

Substituting (B.28) and (B.29) into the right hand side of (B.27), and taking summation over \(\alpha\) and \(\beta\), we find that only a few terms in the summation over \(\gamma\) are non-zero (since there are the same rows or columns in determinants except for a few values of \(\gamma\)), and finally we obtain the right hand side of (B.26).

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In section 2.4 of this paper, solutions of the Yang-Baxter relation (L-operators) for the Q-operators (for trigonometric models) were presented. The Q-operators were given as the (super)trace of these L-operators over some oscillator representations. A preliminary idea on the derivation of the L-operators was presented first in many conferences (these include the following two: “Workshop and Summer School: From Statistical Mechanics to Conformal and Quantum Field Theory”, the university of Melbourne, January, 2007; La 79eme Rencontre entre physiciens theoriciens et mathematiciens “Supersymmetry and Integrability”, IRMA Strasbourg, June, 2007) in 2007. Recently, a method closely related to this (for rational models) has been developed rapidly.
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