Quantum Brownian motion

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We study the behavior of a subsystem (harmonic oscillator) in contact with a thermal reservoir (finite set of uncoupled harmonic oscillators). We exactly solve the eigenvalue problem and obtain the temporal evolution of the dynamical variables of interest. We show how the subsystem goes to equilibrium and give quantitative estimates of the Poincaré recurrence times. We study the behavior of the subsystem mean occupation number in the limit of a dense bath and compare it with the expected exponential decay law.
I. INTRODUCTION

One of the long standing paradoxical problems in theoretical physics lies in understanding how a macroscopic system reaches equilibrium departing from the reversible microscopical laws of nature. The simplest systems for which the origin of irreversibility can be studied on a microscopic basis are the linear ones [1,2]. In this work we investigate this behavior for a Brownian particle in a quantum-mechanical heat bath composed of a finite number of small oscillators. The key point of this analysis is the consideration of the linear coupling between the Brownian particle and the bath, which allows us to reduce the Hamiltonian to a set of uncoupled oscillators. This model is recurrently studied in the literature from many different approaches, such as the Langevin equation, the master equation, and the exact solution for the evolution operator, using different techniques. The importance of this model lies in its broad applicability in many fields of physics: condensed matter, statistical mechanics, quantum optics, quantum electrodynamics, quantum measurement theory, scattering and decay theory, etc. In the majority of previous works statistical fluctuations, dissipation, and equilibrium tendency in linear quantum-mechanical systems are shown to result from a projection of the total quantum system onto a restricted subspace. A macroscopic equation is obtained corresponding to a reduced description of the system. The restriction to the Markovian approximation and weak-coupling limit are usually undertaken as well as the analysis is carried out in the limit of a dense bath. Only few works are devoted to study the behavior of the system for a finite (discrete spectrum) bath [3,4]. In this case it is not possible to prove convergence to an equilibrium state in the limit $t \to \infty$ because of the existence of Poincaré recurrences, which become extremely infrequent for large systems. This is the reason why previous works prefer to eliminate them by passing to the limit of an infinite heat bath. However, quantitative estimates of the Poincaré recurrence times for finite systems can be made in a way compatible with a dissipative behavior. In this work we reinforce this fact. We study the time evolution of a finite system departing from the exact solution of the eigenvalue problem without appealing to approximations, assuming a
factorizable initial condition where the bath has reached a unique thermal equilibrium state (passive reservoir). No use is made of coarse graining, finite memory assumptions, randomly varying Hamiltonians or nonlinear modifications to the Schrödinger evolution. We perform our calculations for an arbitrary spectral density and temperature.

The work is organized as follows. In Sec. II we introduce the model and its exact solution. We also show a criterion to disregard a broad class of coupling functions. The exact solution is used in Sec. III in order to study the time evolution of the relevant variables of our problem, i.e. the mean occupation number (or energy) and mean position of the subsystem, the behavior of bath variables, etc. Sec. IV is devoted to derive an exact generalized form of the quantum Langevin equation, with time-dependent coefficients. Numerical results and their analysis are presented in Sec. V. We show that the mean position operator performs damping oscillations correlated with the mean energy of the subsystem, which decays in time towards a state of equilibrium with the bath. After reaching this state revivals occurs periodically in the subsystem. The bath remains almost unaltered in thermal equilibrium as a consequence of its passivity and robustness. In Sec. VI we take the limit of a dense bath and obtain, by means of an analytic continuation method, a complex frequency $z_0 = \Omega + \delta \Omega + i \Gamma / 2$ into which the unperturbed real frequency $\Omega$ of the oscillator is shifted by the heat bath. For a long period of time the exponential decay law dominates the evolution. In this period the standard form of the Langevin equation is derived, where the mean displacement of the Brownian particle undergoes a slowly damped harmonic oscillation corresponding to the complex frequency $z_0$. Deviations from the exponential decay law are also discussed.

II. BROWNIAN MOTION: THE MODEL AND ITS EXACT SOLUTION

Let us consider a harmonic oscillator interacting with a bath modeled by a set of harmonic oscillators. The Hamiltonian of the system is

$$H = \frac{p^2}{2M} + \frac{1}{2} M \Omega^2 X^2 + \sum_{n=1}^{N} \left( \frac{p_n^2}{2m_n} + \frac{1}{2} m_n \omega_n^2 x_n^2 \right) + H_I,$$
where $H_I$ represents the interaction. Capital and lower-case letters stand for subsystem and bath variables respectively. In our case $H_I$ only involves a linear coupling between the Brownian particle and the bath, i.e.

$$H_I = \sum_{n=1}^{N} c_n \left( X x_n + \frac{P p_n}{M \Omega m_n \omega_n} \right),$$

(2)

where all $c_n$ are real and small coupling constants. We define, as usual, creation and annihilation operators ($\hbar = 1$)

$$B = \sqrt{\frac{M \Omega}{2}} X + i \sqrt{\frac{1}{2M \Omega}} P,$$

$$b_n = \sqrt{\frac{m_n \omega_n}{2}} x_n + i \sqrt{\frac{1}{2m_n \omega_n}} p_n,$$

that satisfy the canonical commutation relations

$$[B, B^\dagger] = I,$$

$$[b_n, b_m^\dagger] = \delta_{nm},$$

(4)

the other commutators vanish. In terms of these operators the Hamiltonian reads

$$H = \Omega \left( B^\dagger B + \frac{1}{2} \right) + \sum_{n=1}^{N} \omega_n \left( b_n^\dagger b_n + \frac{1}{2} \right) + \sum_{n=1}^{N} g_n \left( Bb_n^\dagger + B^\dagger b_n \right),$$

(5)

where $g_n = c_n / \sqrt{M \Omega m_n \omega_n}$. The linear interaction allows us to find normal modes of $H$ in an exact way. This kind of coupling is known in the literature as the rotating wave approximation [5] (in general only the coupling between coordinates is taken into account).

In this model the interaction term preserves the total number of quanta. In fact, defining the number of quanta operators

$$N_\Omega = B^\dagger B,$$

$$N_n = b_n^\dagger b_n,$$

the total number of quanta given by
\[ N_T = N_\Omega + \sum_{n=1}^{N} N_n \]

is a constant of motion, due to \( dN_T/dt = i [H_I, N_T] = 0 \). Then we can resolve the Hamiltonian into sectors of definite number of quanta. For \( N_T = 1 \) (one-particle sector) and calling

\[ |\Omega\rangle \equiv B^\dagger |0\rangle = |1\rangle \otimes |0...0\rangle , \]

\[ |\omega_n\rangle \equiv b_n^\dagger |0\rangle = |0\rangle \otimes |0...\frac{1}{n_{\text{site}}...0\rangle , \]

we obtain

\[ H_1 = \Omega |\Omega\rangle \langle \Omega| + \sum_{n=1}^{N} \omega_n |\omega_n\rangle \langle \omega_n| + \sum_{n=1}^{N} g_n (|\Omega\rangle \langle \omega_n| + |\omega_n\rangle \langle \Omega|) + C, \]

(7)

where \( C = \frac{\Omega}{2} + \sum_{n=1}^{N} \frac{\omega_n}{2} \). This is the discrete version of the Friedrichs model [6].

Let us find now the normal modes of \( H \), i.e. the new set of uncoupled harmonic oscillators with normal frequencies \( \alpha_\nu \) (Greek subscripts run from 0 to \( N \), while Arabic ones run from 1 to \( N \)). Hence we write \( H \) as

\[ H = \sum_{\nu=0}^{N} \alpha_\nu c_\nu^\dagger c_\nu + C, \]

(8)

where the new creation operators, \( c_\nu \), are related to the old ones by means of a unitary (canonical) transformation

\[ c_\nu = \Phi_\nu B + \sum_{n=1}^{N} \phi_{\nu n} b_n, \]

(9)

which preserves the canonical commutation relations

\[ [c_\mu, c_\nu^\dagger] = \delta_{\mu\nu}. \]

(10)

Coefficients

\[ \Phi_\nu = \langle \alpha_\nu |\Omega\rangle , \]

\[ \phi_{\nu n} = \langle \alpha_\nu |\omega_n\rangle , \]
are the matrix elements of the unitary change of basis, from \{\ket{\alpha_\nu}\} to \{\ket{\Omega}, \ket{\omega_n}\}, where

\[
\ket{\alpha_\nu} = c^\dagger_\nu \ket{0}
\]
is an eigenvector of \(H_1\)

\[
H_1 = \sum_{\nu=0}^N \alpha_\nu \bra{\alpha_\nu} \alpha_\nu \ket{\alpha_\nu} + C.
\]

The canonical commutators (10) impose the following condition for \(\Phi_\nu\) and \(\phi_{\nu n}\)

\[
\Phi_\mu \Phi^*_\nu + \sum_{n=1}^N \phi_{\mu n} \phi^*_{\nu n} = \delta_{\mu\nu},
\]

which in the one-particle sector is a consequence of the orthogonality among eigenvectors of \(H_1\),

\[
\bra{\alpha_\mu} \alpha_\nu \ket{\alpha_\nu} = \delta_{\mu\nu}.
\]

By taking into account transformation (9) in the Heisenberg equation of motion for \(c_\nu\),

\[
i \frac{dc_\nu}{dt} = [c_\nu, H] = \alpha_\nu c_\nu,
\]

and using

\[
i \frac{dB}{dt} = [B, H] = \Omega B + \sum_{n=1}^N g_n b_n,
\]

\[
i \frac{db_n}{dt} = [b_n, H] = \omega_n b_n + g_n B,
\]

we obtain the following system of linear equations

\[
\Omega \Phi_\nu + \sum_{n=1}^N g_n \phi_{\nu n} = \alpha_\nu \Phi_\nu,
\]

\[
g_n \Phi_\nu + \omega_n \phi_{\nu n} = \alpha_\nu \phi_{\nu n}.
\]

From the second of these equations we can obtain \(\phi_{\nu n}\) as

\[
\phi_{\nu n} = \frac{g_n \Phi_\nu}{\alpha_\nu - \omega_n},
\]
which is valid only if $\alpha_\nu \neq \omega_n$, $\forall \nu, n$. Replacing it into the first equation of (13) we have

$$\Phi_\nu \left( \alpha_\nu - \Omega - \sum_{n=1}^{N} \frac{g_n^2}{\alpha_\nu - \omega_n} \right) = 0. \quad (15)$$

Since we are looking for non-trivial solutions the expression between brackets must be identically zero. Then we have an equation for the normal frequencies of the new set of harmonic oscillators

$$\alpha_\nu - \Omega = \sum_{n=1}^{N} \frac{g_n^2}{\alpha_\nu - \omega_n}. \quad (16)$$

The procedure developed above is equivalent to solve the eigenvalue problem $H_1 |\alpha_\nu\rangle = \alpha_\nu |\alpha_\nu\rangle$ for the matrix that represents the Friedrichs Hamiltonian in the basis $\{|\Omega\rangle, |\omega_n\rangle\}$

$$\begin{pmatrix} \Omega & g_1 & \cdots & g_N \\ g_1 & \omega_1 \\ \vdots & \vdots & \ddots & \vdots \\ g_N & \omega_N \end{pmatrix} \begin{pmatrix} \Phi_\nu^* \\ \Phi_{\nu 1}^* \\ \Phi_{\nu N}^* \end{pmatrix} = \begin{pmatrix} \Phi_\nu \\ \Phi_{\nu 1} \\ \Phi_{\nu N} \end{pmatrix}. \quad (17)$$

In fact (17) is the complex conjugated matrix of the matrix form of Eqs. (13) because $g_n$ is real. From Eq. (11) for $\mu = \nu$ [normalization of the eigenvectors of (17)] we obtain

$$|\Phi_\nu|^2 = \frac{1}{1 + \sum_{n=1}^{N} \left( \frac{g_n}{\alpha_\nu - \omega_n} \right)^2}. \quad (18)$$

which can be completely determined if we know the set of eigenvalues $\alpha_\nu$. The normal frequencies $\alpha_\nu$ can be obtained numerically or by analytic perturbative methods (in some special cases can even be obtained exactly), so we assume that they are well known. In Fig. 1 we show where these values are located.
The normal frequencies correspond to the intersection of the straight line $\alpha_\nu - \Omega$ and the summation of the hyperboles $g_n^2 / (\alpha_\nu - \omega_n)$. From this picture we see that the normal frequencies always lie between consecutive frequencies of the unperturbed Hamiltonian, except for the two extremum values which lie outside the interval delimited by $\omega_1$ and $\omega_N$. \{\alpha_\nu\}_{\nu=0,...,N} never coincide with \{\omega_n\}_{n=1,...,N}, and are very close to each $\omega_n$ near the extrema and move away from $\omega_n$ in the centrum. In order to guarantee the positivity of the Hamiltonian as the lowest frequency approaches zero and the convergence of the series $\sum_{n=1}^{N} \frac{g_n^2}{\alpha_\nu - \omega_n}$, it is required that

$$\sum_{n=1}^{N} \frac{g_n^2}{\omega_n - \omega_1 + \delta} < \Omega - \omega_1 + \delta \quad \text{and} \quad \sum_{n=1}^{N} \frac{g_n^2}{\omega_N + \delta - \omega_n} < \omega_N + \delta - \Omega,$$

where $\delta$ is an infinitesimal parameter (e.g. the distance between contiguous unperturbed frequencies), and also that $g(\omega)$ behaves smoothly around $\omega_1$ and $\omega_N$, having a small value at these points. Then the coupling privileges the interaction with the subsystem oscillator of frequency $\Omega$. These conditions express the fact that the interaction is small. However, for
small $N$, the interaction must not decrease very fast from the centrum since if such were the case the subsystem oscillator would be coupled to few bath oscillators and then the bath would not be effective. These problems disappear approaching to the continuum (Sec. VI).

### III. TIME EVOLUTION

We are looking for the way in which the relevant variables of the system evolve in time. That is to know the time evolution of $B$ and $b_n$ and, from them, all other related dynamical variables. Coming back to Eq. (12) we can integrate it and obtain the temporal evolution of $c_\nu$

$$c_\nu(t) = e^{-i\alpha_\nu t}c_\nu(0). \quad (20)$$

Using the closure relation

$$\sum_{\nu=0}^{N} |\alpha_\nu\rangle \langle\alpha_\nu| = I$$

we can obtain the following identities:

$$\sum_{\nu=0}^{N} \Phi_\nu^* \Phi_\nu = 1,$$

$$\sum_{\nu=0}^{N} \Phi_\nu^* \phi_{\nu m} = 0,$$

$$\sum_{\nu=0}^{N} \phi_{\nu n}^* \phi_{\nu m} = \delta_{nm}. \quad (21)$$

Eqs. (21) allow us to perform the inverse of the transformation (9),

$$B = \sum_{\nu=0}^{N} \Phi_\nu^* c_\nu. \quad (22)$$

$$b_n = \sum_{\nu=0}^{N} \phi_{\nu n}^* c_\nu. \quad (23)$$

We are interested in knowing the explicit form of $B(t)$ and $b_n(t)$. Then, from Eqs. (20), (22), (23), and (9) we have
\[ B(t) = \sum_{\nu=0}^{N} \Phi^{*}_{\nu} e^{-i\alpha_{\nu}t} C_{\nu}(0) = \sum_{\nu=0}^{N} \Phi^{*}_{\nu} e^{-i\alpha_{\nu}t} \left[ \Phi^{*}_{\nu} B(0) + \sum_{n=1}^{N} \phi_{\nu n} b_{n}(0) \right], \tag{24} \]

\[ b_{n}(t) = \sum_{\nu=0}^{N} \phi_{\nu n}^{*} e^{-i\alpha_{\nu}t} C_{\nu}(0) = \sum_{\nu=0}^{N} \phi_{\nu n}^{*} e^{-i\alpha_{\nu}t} \left[ \Phi^{*}_{\nu} B(0) + \sum_{m=1}^{N} \phi_{\nu m} b_{m}(0) \right]. \]

We have expressed the time evolution of the unperturbed annihilation operators only in terms of their initial values. By considering Eq. (14) we write Eq. (24) as

\[ B(t) = \sum_{\nu=0}^{N} |\Phi_{\nu}|^2 e^{-i\alpha_{\nu}t} \left[ B(0) + \sum_{n=1}^{N} \frac{g_{n}}{\alpha_{\nu} - \omega_{n}} b_{n}(0) \right], \tag{25} \]

\[ b_{n}(t) = \sum_{\nu=0}^{N} |\Phi_{\nu}|^2 \frac{g_{n}}{\alpha_{\nu} - \omega_{n}} e^{-i\alpha_{\nu}t} \left[ B(0) + \sum_{m=1}^{N} \frac{g_{m}}{\alpha_{\nu} - \omega_{m}} b_{m}(0) \right], \]

and a similar set of equations for the Hermitian conjugate operators. Eqs. (25) are determined uniquely by knowing the eigenvalues \( \alpha_{\nu} \).

These are the exact solutions of our problem. From them we can obtain all the relevant information under consideration. For example, one of the relevant variables is the position of the subsystem oscillator, \( X(t) = \frac{1}{\sqrt{2M\Omega}} \left[ B^{\dagger}(t) + B(t) \right] \), given by

\[ X(t) = \sum_{\nu=0}^{N} |\Phi_{\nu}|^2 \left\{ \left[ \cos (\alpha_{\nu} t) X(0) + \sin (\alpha_{\nu} t) \tilde{P}(0) \right] \right. \]

\[ + \frac{1}{\sqrt{M\Omega}} \sum_{m=1}^{N} \frac{g_{m}}{\alpha_{\nu} - \omega_{m}} \left[ \sqrt{m_{n}\omega_{n}} \cos (\alpha_{\nu} t) x_{n}(0) + \frac{\sin (\alpha_{\nu} t)}{\sqrt{m_{n}\omega_{n}}} p_{n}(0) \right] \} \tag{26} \]

where \( \tilde{P} \equiv \frac{P}{\sqrt{M\Omega}} \). Similarly, from \( \tilde{P} = i\sqrt{\frac{1}{2M\Omega}} \left[ B^{\dagger}(t) - B(t) \right] \) we have

\[ \tilde{P}(t) = \sum_{\nu=0}^{N} |\Phi_{\nu}|^2 \left\{ \left[ -\sin (\alpha_{\nu} t) X(0) + \cos (\alpha_{\nu} t) \tilde{P}(0) \right] \right. \]

\[ + \frac{1}{\sqrt{M\Omega}} \sum_{n=1}^{N} \frac{g_{n}}{\alpha_{\nu} - \omega_{n}} \left[ -\sqrt{m_{n}\omega_{n}} \sin (\alpha_{\nu} t) q_{n}(0) + \frac{\cos (\alpha_{\nu} t)}{\sqrt{m_{n}\omega_{n}}} p_{n}(0) \right] \} \tag{27} \]

Another interesting magnitude is the occupation number of the oscillator representing the Brownian particle (subsystem dynamics). We have
\[
\langle (B^\dagger B) (t) \rangle = \sum_{\mu,\nu=0}^{N} |\Phi_\mu|^2 |\Phi_\nu|^2 e^{i(\alpha_\mu - \alpha_\nu)t} \left[ \langle (B^\dagger B) (0) \rangle + \sum_{n=1}^{N} \frac{g_n}{\alpha_\mu - \omega_n} \langle (B^\dagger b_n) (0) \rangle \right] 
\]

\[
+ \sum_{n=1}^{N} \frac{g_n}{\alpha_\mu - \omega_n} \langle (B^\dagger b_n) (0) \rangle + \sum_{n,m=1}^{N} \frac{g_n g_m}{(\alpha_\mu - \omega_n)(\alpha_\nu - \omega_m)} \langle (b_n^\dagger b_m) (0) \rangle \right].
\]

Let us first consider the case in which the set of harmonic oscillators modeling the bath is in thermal equilibrium inside a big reservoir and the Brownian oscillator is isolated from the rest. At \( t = 0 \) we extract the bath from the reservoir at temperature \( T \), and put it in contact with the subsystem oscillator, in such a way that the bath becomes a thermal reservoir for the Brownian particle. In this situation the initial state of the total system is represented by a time-independent density matrix which is a direct (tensorial) product of the matrices representing the isolated harmonic oscillator \( \rho_B(0) \) and the environment degrees of freedom \( \rho_b(0) \) in thermal equilibrium at temperature \( T \)

\[
\rho(0) = \rho_B(0) \otimes \frac{e^{-\beta H_b}}{\text{tr}_b \{e^{-\beta H_b}\}}, \tag{29}
\]

where \( H_b = \sum_{n=1}^{N} \omega_n \left( b_n^\dagger b_n + \frac{1}{2} \right) \) is the bath Hamiltonian and \( \text{tr}_b \) is the partial trace over the reservoir. If it is the case we have no correlations among the initial states of subsystem and reservoir. To obtain the time evolution of \( \langle (B^\dagger B) (t) \rangle \equiv \text{tr} \{\rho(0) (B^\dagger B) (t)\} \), we need to specify the initial values of Eq. \( (28) \) in the state \( \rho \) of Eq. \( (29) \). They are given by

\[
\langle (B^\dagger B) (0) \rangle = \kappa,
\]

\[
\langle (B b_n^\dagger) (0) \rangle = 0 = \langle (B^\dagger b_n) (0) \rangle,
\]

\[
\langle (b_n^\dagger b_m) (0) \rangle = (e^{\beta \omega_n} - 1)^{-1} \delta_{nm}, \tag{30}
\]

where \( \kappa \) is the initial number of quanta in the subsystem oscillator. Therefore the subsystem dynamics is given by

\[
\langle (B^\dagger B) (t) \rangle = \sum_{\mu,\nu=0}^{N} \frac{2 |\Phi_\mu|^2 |\Phi_\nu|^2 \cos [(\alpha_\mu - \alpha_\nu) t]}{(\alpha_\mu - \omega_n)(\alpha_\nu - \omega_m)} \left[ \langle (B^\dagger B) (0) \rangle + \sum_{n=1}^{N} \frac{g_n^2}{\alpha_\mu - \omega_n} \frac{1}{e^{\beta \omega_n} - 1} \right] 
\]

\[
+ \sum_{\nu=0}^{N} |\Phi_\nu|^4 \left[ \langle (B^\dagger B) (0) \rangle + \sum_{n=1}^{N} \frac{g_n^2}{\alpha_\nu - \omega_n} \frac{1}{e^{\beta \omega_n} - 1} \right]. \tag{31}
\]
In a similar way we obtain the mean value of the number of quanta operator for the $n$–oscillator of the bath, i.e.

$$
\langle \left( b_n^\dagger b_n \right) (t) \rangle = \sum_{\mu, \nu = 0}^{N} \frac{2 |\Phi_\mu| |\Phi_\nu|^2 g_n^2 \cos [(\alpha_\mu - \alpha_\nu) t]}{N \sum_{m = 1}^{N} (\alpha_\mu - \omega_n)(\alpha_\nu - \omega_n)} \left[ \kappa + \sum_{m = 1}^{N} \frac{g_m^2}{(\alpha_\mu - \omega_m)(\alpha_\nu - \omega_m)} e^{\beta \omega_m} - 1 \right]
$$

$$+ \sum_{\nu = 0}^{N} |\Phi_\nu|^4 \frac{g_n^2}{(\alpha_\nu - \omega_n)^2} \left[ \kappa + \sum_{m = 1}^{N} \left( \frac{g_m}{\alpha_\nu - \omega_m} \right)^2 \frac{1}{e^{\beta \omega_m} - 1} \right].
$$

(32)

The expressions obtained can be formally rewritten as

$$
\langle N_\Omega(t) \rangle = P_{\Omega \Omega}(t) \langle N_\Omega(0) \rangle + \sum_{n = 1}^{N} P_{\Omega n}(t) \langle N_n(0) \rangle,
$$

$$
\langle N_n(t) \rangle = P_{n \Omega}(t) \langle N_\Omega(0) \rangle + \sum_{m = 1}^{N} P_{nm}(t) \langle N_m(0) \rangle,
$$

(33)

where $\langle N_\Omega \rangle = \langle B^\dagger B \rangle$ and $\langle N_n \rangle = \langle b_n^\dagger b_n \rangle$. In a forthcoming paper we will show that this is a general result of this kind of models, which allows us to derive the Pauli master equation. It can be proved that $P_{\Omega \Omega}$ and $P_{\Omega n}$ are respectively, the transition probability of the one-particle state $|\Omega\rangle$ remaining unchanged (survival probability) and the transition probability from the state $|\omega_n\rangle$ to the state $|\Omega\rangle$. They represent the probability that at time $t$ the contribution to the oscillator occupation number comes from itself and from the bath, respectively. $P_{n \Omega}$ and $P_{nm}$ are the probability that the $n$–bath occupation number has contribution from the oscillator and from the bath, respectively. These probabilities satisfy the normalization condition

$$
P_{\Omega \Omega} + \sum_{n = 1}^{N} P_{\Omega n} = 1, \quad P_{n \Omega} + \sum_{m = 1}^{N} P_{nm} = 1,
$$

and are explicitly given by
\[ P_{\Omega\Omega}(t) = \left| \langle \Omega | e^{-iHt} | \Omega \rangle \right|^2 = 2 \sum_{\mu, \nu=0}^{N} |\Phi_\mu|^2 |\Phi_\nu|^2 \cos [(\alpha_\mu - \alpha_\nu) t] + \sum_{\nu=0}^{N} |\Phi_\nu|^4, \]

\[ P_{\Omega n}(t) = P_{n\Omega}(t) \equiv \left| \langle \Omega | e^{-iHt} | \omega_n \rangle \right|^2 = 2 \sum_{\mu, \nu=0}^{N} |\Phi_\mu|^2 |\Phi_\nu|^2 \frac{g_n^2 \cos [(\alpha_\mu - \alpha_\nu) t]}{(\alpha_\mu - \omega_n)(\alpha_\nu - \omega_n)} + \sum_{\nu=0}^{N} |\Phi_\nu|^4 \left( \frac{g_n}{\alpha_\nu - \omega_n} \right)^2, \]

\[ P_{nm}(t) \equiv \left| \langle \omega_n | e^{-iHt} | \omega_m \rangle \right|^2 = 2 \sum_{\mu, \nu=0}^{N} |\Phi_\mu|^2 |\Phi_\nu|^2 \frac{g_n^2 g_m^2 \cos [(\alpha_\mu - \alpha_\nu) t]}{(\alpha_\mu - \omega_n)(\alpha_\nu - \omega_n)(\alpha_\mu - \omega_m)(\alpha_\nu - \omega_m)} + \sum_{\nu=0}^{N} |\Phi_\nu|^4 \left( \frac{g_n g_m}{(\alpha_\nu - \omega_n)(\alpha_\nu - \omega_m)} \right)^2. \]

We can see from the second equation of (33) that, although there is no interaction term in the Hamiltonian among the bath oscillators themselves, the time evolution for a bath oscillator has contributions coming from the whole bath. This fact was noticed in Ref. [4] [cf. Eqs. (2.2e) and (2.2f)].

The decomposition made in Eq. (33) is useful for studying the different contributions to the time evolution of the mean number operators (Sec. V).

In the limit of low temperatures \( T \to 0 \), \( \langle N_n(0) \rangle = \left( e^{\beta \omega_n} - 1 \right)^{-1} \to 0 \), and then \( \langle N_\Omega(t) \rangle = P_{\Omega\Omega}(t) \langle N_\Omega(0) \rangle = \kappa P_{\Omega\Omega}(t) \). In the case \( \kappa = 1 \) it is the survival probability (the probability of no decay of the state \( |\Omega\rangle \)),

\[ \langle N_\Omega(t) \rangle |_{T=0} = \left| \langle \Omega | e^{-iHt} | \Omega \rangle \right|^2, \quad \text{for} \quad \langle N_\Omega(0) \rangle = 1. \]  

In Sec. VI for a dense bath we show that the asymptotic behavior of this probability obeys a power-law decay. This is a well known fact in decay theory of unstable quantum systems and was reported as an anomaly in statistical treatments of quantum open systems (see, e.g., Ref. [4]).

In the next section we derive an exact equation of motion of the mean value of the position operator \( X \) (a generalized form of the Langevin equation).
IV. LANGEVIN EQUATION

Let us consider Eqs. (26) and (27) for the bath in thermal equilibrium at the initial time. In this case let \(|N_n\rangle\) be a basis of eigenvectors of \(N_n\). Taking mean values in the state (29) we have

\[
\langle b_n(0) \rangle = \text{tr} \left\{ b_n(0) \frac{\rho_B(0) \exp \left[-\beta \omega_n (N_n + 1/2)\right] \prod_{i=1, i \neq n}^{N} \exp \left[-\beta \omega_i (N_i + 1/2)\right]}{\text{tr} \{ e^{-\beta H_B} \}} \right\}.
\]

There is a vanishing factor \(\sum_{N_n} \langle N_n| b_n(0) \exp \left[-\beta \omega_n (N_n + 1/2)\right]|N_n\rangle\), since \(\langle N_n| b_n(0) |N_n\rangle = 0\). Then \(\langle b_n(0) \rangle = 0\).

Similarly \(\langle b_n^\dagger(0) \rangle = 0\). So we have \(\langle q_n(0) \rangle = 0 = \langle p_n(0) \rangle\) . Thus

\[
\langle X(t) \rangle = \sum_{\nu=0}^{N} |\Phi_{\nu}|^2 \left[ \cos (\alpha_{\nu} t) \langle X(0) \rangle + \sin (\alpha_{\nu} t) \langle \bar{P}(0) \rangle \right],
\]

\[
\langle \bar{P}(t) \rangle = \sum_{\nu=0}^{N} |\Phi_{\nu}|^2 \left[ -\sin (\alpha_{\nu} t) \langle X(0) \rangle + \cos (\alpha_{\nu} t) \langle \bar{P}(0) \rangle \right].
\]

The initial and instantaneous variables are related by a generalized sum of rotations

\[
\begin{pmatrix}
\langle X(t) \rangle \\
\langle \bar{P}(t) \rangle
\end{pmatrix} = \sum_{\nu=0}^{N} |\Phi_{\nu}|^2
\begin{pmatrix}
\cos (\alpha_{\nu} t) & \sin (\alpha_{\nu} t) \\
-\sin (\alpha_{\nu} t) & \cos (\alpha_{\nu} t)
\end{pmatrix}
\begin{pmatrix}
\langle X(0) \rangle \\
\langle \bar{P}(0) \rangle
\end{pmatrix},
\]

a transformation which can be summarized as

\[
\begin{pmatrix}
\langle X(t) \rangle \\
\langle \bar{P}(t) \rangle
\end{pmatrix} = \begin{pmatrix} a(t) & b(t) \\ -b(t) & a(t) \end{pmatrix}
\begin{pmatrix}
\langle X(0) \rangle \\
\langle \bar{P}(0) \rangle
\end{pmatrix},
\]

(38)

where \(a(t) = \sum_{\nu=0}^{N} |\Phi_{\nu}|^2 \cos (\alpha_{\nu} t)\) and \(b(t) = \sum_{\nu=0}^{N} |\Phi_{\nu}|^2 \sin (\alpha_{\nu} t)\).

We can invert the matrix of Eq. (38) to obtain \(\langle X(0) \rangle\) and \(\langle \bar{P}(0) \rangle\) as functions of \(\langle X(t) \rangle\) and \(\langle \bar{P}(t) \rangle\):

\[
\begin{pmatrix}
\langle X(0) \rangle \\
\langle \bar{P}(0) \rangle
\end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} a(t) & -b(t) \\ b(t) & a(t) \end{pmatrix}
\begin{pmatrix}
\langle X(t) \rangle \\
\langle \bar{P}(t) \rangle
\end{pmatrix},
\]

(39)
where $\Delta(t) = a^2(t) + b^2(t)$. By taking time derivatives in Eq. (38) and replacing the initial mean values by those of Eq. (39) we have

$$\begin{pmatrix} \langle X(t) \rangle \\ \langle \tilde{P}(t) \rangle \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \dddot{a} a + \dddot{b} b & \dddot{b} a - \dddot{a} b \\ \dddot{a} b - \dddot{b} a & \dddot{a} a + \dddot{b} b \end{pmatrix} \begin{pmatrix} \langle X(t) \rangle \\ \langle \tilde{P}(t) \rangle \end{pmatrix}. \quad (40)$$

Similarly, from the second derivative of (38) we have

$$\begin{pmatrix} \langle \dddot{X}(t) \rangle \\ \langle \dddot{\tilde{P}}(t) \rangle \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} \dddot{a} a + \dddot{b} b & \dddot{b} a - \dddot{a} b \\ \dddot{a} b - \dddot{b} a & \dddot{a} a + \dddot{b} b \end{pmatrix} \begin{pmatrix} \langle X(t) \rangle \\ \langle \tilde{P}(t) \rangle \end{pmatrix}. \quad (41)$$

Finally eliminating $\langle \tilde{P}(t) \rangle$ from Eq. (40) we obtain a generalized form of the Langevin equation with time-dependent coefficients,

$$\langle \dddot{X}(t) \rangle + \Omega^2(t) \langle X(t) \rangle + \Gamma(t) \langle \dddot{X}(t) \rangle = 0, \quad (42)$$

where

$$\Omega^2(t) = \frac{\dddot{a} \dddot{b} - \dddot{b} \dddot{a}}{\ddot{a} \ddot{b} - \ddot{b} \ddot{a}}, \quad \Gamma(t) = \frac{b \dddot{a} - a \dddot{b}}{a \ddot{b} - b \ddot{a}}. \quad (43)$$

The standard stochastic force $f_{\text{stoch}}$ does not appear in Eq. (42) since it is included in the terms containing the operators $q_n(0)$ and $p_n(0)$, which were eliminated by taking the mean values in a thermal equilibrium initial state of the bath. That is $\langle f_{\text{stoch}}(t) \rangle = 0$. Eq. (42) contrasts with the equivalent, but non-local in time, standard integro-differential form of the equation of motion of $\langle X \rangle$. Eq. (42) is actually a rather complicated expression since the coefficients are not easy of evaluating. In Sec. VI we estimate them in the continuous limit.

V. EXAMPLES AND RESULTS

Let us describe the model we used for obtaining the numerical results of this section.
A. Choice of parameters

The model described in Secs. II and III consists of three main ingredients: the subsystem and the bath, the interaction, and the initial conditions. We have considered a subsystem represented by a harmonic oscillator with natural frequency $\Omega$ and mass $M$ (heavy Brownian particle), a bath of small oscillators with frequencies $\omega_n$ varying in a range between $\omega_{\min}$ and $\omega_{\max}$, a small linear coupling between system and bath, $g_n = \lambda c'_n$, where $\lambda = (M\Omega)^{-1/2}$ and $c'_n = c_n (m_n \omega_n)^{-1/2}$, and the whole composed system prepared in such a way that at $t = 0^-$ there is no correlation between subsystem and bath. The bath is in equilibrium with an external heat source at temperature $T = (k_B \beta)^{-1}$, where $k_B$ is the Boltzmann constant, and at $t = 0^+$ the bath is extracted from the thermal source, put in contact with the subsystem and the total system is left isolated. Each parameter mentioned above defines a typical time scale. These are: the scale associated with the natural frequency of the isolated subsystem, $\Omega^{-1}$; the scale defined by the lowest frequency of the bath, $\omega_{\min}^{-1}$, related to the reaction of the system when the interaction is switched on; the decay time $\Gamma^{-1}$ [see Eq. (69) in Sec. VI] in which the subsystem dissipates its energy into the reservoir, and which is related to the squared of the perturbation parameter $\lambda$; the memory time related with the highest frequency present in the bath, $\omega_{\max}^{-1}$; the time scale $\beta$ associated with thermal effects (relative to quantum ones); the Poincaré recurrence time given by the minimal difference between contiguous normal frequencies, specifically $t_P \simeq \frac{2\pi}{\min(\alpha_{\nu+1} - \alpha_\nu)}$ [see Eq. (48) below]; and two time scales related with quantum deviations form the exponential decay law, a very short time $t_Z$ (Zeno period, which is responsible for no decaying of the subsystem under a continuous succession of measurements and occurs because of the temporal derivative of the survival amplitude vanishes at $t = 0$) and a very long one $t_K$ (Khalfin period of power series tails, which is a consequence of the lower bound of the energy) (see Sec. VI). In order to have a manifestation of these time scales the parameters and variables must be chosen with certain criterion. An important condition we must take into account and which is frequently overlooked in the literature is condition (19). For example, in the case of a semi-infinite
frequency spectrum, $\omega \in (0, \infty)$, the often used ohmic spectral density, $g^2(\omega) \sim \omega$, does not satisfy condition (19), since it has a logarithmic type divergency. As we want to obtain a pictorial image of the temporal evolution of the main magnitudes of Sec. III, then we specify the parameters appearing in these magnitudes [e.g. Eqs. (26), (27), (31), and (32)] as follows:

$$\begin{align*}
\Omega &= 1, \\
\beta &= \frac{1}{\Omega}, \\
\kappa &= 1. 
\end{align*}$$

This choice of $\beta$ fixes the thermal time scale to the same value of the $\Omega^{-1}$ scale, and then purely quantum-mechanical and thermal effects are comparable, so we are far of the classical limit $\hbar \Omega \ll k_B T$. The choice of $\kappa$ facilitates the comprehension of the one-particle sector several times studied in decay theory.

For the sake of simplicity we consider for the variables the case in which the bath frequencies are equidistant around the frequency $\Omega$, i.e.

$$\omega_n = \Omega + A \left( n - \frac{N + 1}{2} \right), \quad n = 1, \ldots, N, \tag{45}$$

where $A$ is the spacing between contiguous frequencies of the bath, $A = \omega_{n+1} - \omega_n$, being the band width $\omega_N - \omega_1 = A(N - 2)$, and the number of small oscillators $N$ is an odd integer.

The coupling function is given by a Lorentzian-like function

$$g_n = \frac{Da^2}{a^2 + (\omega_n - \Omega)^2}. \tag{46}$$

This function is plotted in Fig. 2. $D$ is related to the coupling strength $\lambda$ and is taken equal to $A$, for reasons which will become clear below. We fix $a = \frac{A(N-2)}{2}$ in order to have half of the maximum value of $g_n$ at the extrema. Finally we take the band width equal to 0.018, for all $N$, which is the value that allows us to compare our numerical results with those obtained by Gruver et al. [4] (who solved a set of coupled differential equations coming from a maximum entropy principle approach) in the case $N + 1 = 32$. In their work the choice
of the coupling function is different and also different the criterion to increase the number of small oscillators. While we maintain fixed the band width, they maintain fixed the value of the spacing $A$. As one of the purposes of this work is to study the way of reaching the continuous limit (see Sec. VI), we must have $A \to 0$, $N \to \infty$, and $AN = \text{const.}$.

![FIG. 2. The coupling function.](image)

Let us see the criterion used for fixing $D$. It is a consequence of conditions (19). Taking $\delta = A$ we have

$$\sum_{n=1}^{N} \frac{g_n^2}{\omega_N + A - \omega_n} < \omega_N + A - \Omega.$$  

From Eq. (13) we have $\omega_N + A - \Omega = \frac{A(N+1)}{2}$, $\omega_N + A - \omega_n = A(N + 1 - n)$, and for a large bath we can approximate $g_n^2 \simeq \frac{D^2 N^4}{16[(N/2)^2 + (N/2 - n)^2]}$. Then it must be satisfied

$$\frac{D^2 N^3}{8A^2} \sum_{n=1}^{N} \frac{1}{(N + 1 - n) \left[ \left( \frac{N}{2} \right)^2 + \left( \frac{N}{2} - n \right)^2 \right]^2} < 1.$$  

The summation can be bounded by $N$ times its maximum value, which is reached for $n = N$. Thus
\[
\frac{D^2 N^3}{8A^2} N \left( \frac{N}{2} \right)^2 = \frac{D^2}{2A^2} < 1,
\]
which implies
\[
D < \sqrt{2} A.
\] (47)

So we find an upper boundary for \( D \). In general we choose \( D = A \).

The \( N + 1 \) normal frequencies \( \alpha_\nu \) are obtained from a matrix-diagonalization routine.

**B. Numerical results**

In Figs. 3 to 7 we plot \( \langle N_\Omega \rangle \) vs. \( \Omega t \). We show that \( \langle N_\Omega \rangle \) decays in time to an asymptotic value, when the subsystem reaches equilibrium with the bath, given by \((e - 1)^{-1} \approx 0.582\) [see Eq. (77) in Sec. VI, recall \( \beta \Omega = 1 \)]. After a long period a revival appears reaching again a similar value to the initial condition. Since for positive values of \( t \) the arguments of the cosines in Eq. (31) never are exactly in phase, the revival does not fully reconstruct the initial condition, so that the peak is smaller than the initial one and slightly broadens.

Then, in the continuous limit, the time of revival goes to infinity and the peak gets out of sight among thermal fluctuations. As shown in Fig. 9, this revival is periodic in time. It corresponds to the Poincaré recurrence time and is given by the inverse of the smallest difference of normal frequencies, since Eq. (31) is a quasi-Fourier series because of the quasi-equidistance of the normal frequencies. Specifically

\[
t_P \simeq \frac{2\pi}{\min (\alpha_{\nu+1} - \alpha_\nu)},
\] (48)

where obviously \( \min (\alpha_\mu - \alpha_\nu) = \min (\alpha_{\nu+1} - \alpha_\nu) \). Table I shows this time for different values of \( N + 1 \).

| \( N + 1 \) | 10 | 32 | 100 | 500 |
|-------------|----|----|-----|-----|
| \( t_P \)   | 3370 | 11190 | 37311 | 177994 |

**Tab. I.** Poincaré recurrence time.
Fig. 3 shows that for a small number of bath oscillators, it does not result effective and loses the necessary robustness to break the natural oscillations of the subsystem. For $N + 1 = 10$ (Fig. 4) the energy lost of the subsystem oscillator is close to a dissipative behavior and the bath begins to be effective. It is surprising that for so few bath oscillators the subsystem already dissipates. With increasing $N$, fluctuations get smaller and, since the spacing between frequencies decreases, $t_P$ grows. In Fig. 8 we draw $\langle N_\Omega \rangle$ for different values of $N + 1$ vs. a re-scaled time with respect to $t_P$ ($\Omega = 1$). It shows the tendency to an exponential decay when the model approaches to the continuum. Nevertheless in the continuous limit the exponential decay law is not exact as we show in Sec. VI.

**FIG. 3.** $N + 1 = 6$.

**FIG. 4.** $N + 1 = 10$. 
Figs. 9 and 10 show the behavior of $\langle N_\Omega \rangle$ for very long times in order to see the periodicity of this magnitude and the recurrence of $t_P$. In Fig. 9 we see that the height of the peaks monotonously decreases and afterwards it begins to oscillate. We do not have an explanation
of this fact. For even longer times we see that there does not exist a definite pattern repeating itself (Fig. 10, notice that in this picture only the envelopement of the peaks is plotted).

Fig. 11 shows the form of a peak \( (N + 1 = 32) \) which is non-symmetrical. The growing side of the peak is steeper than the subsequent quasi-exponential decay. In Fig. 12 we choose \( D = 20A \). In this case conditions (19) are not satisfied and a non-dissipative behavior occurs with very quick oscillations. For Figs. 13 and 14 we have selected the value \( D = 2A \) for \( N + 1 = 32 \) and 100 respectively. This value satisfies conditions (19) [remember that the bound (47) is excessive]. However this worsening of \( D \) is reflected in the fact that \( \langle N_\Omega \rangle \) presents more fluctuations.

---

**FIG. 9.** Behavior of the peaks of \( \langle N_\Omega \rangle \).

**FIG. 10.** Peaks for long times.
FIG. 11. Form of a peak.

FIG. 12. $N + 1 = 32, D = 20A.$

FIG. 13. $N + 1 = 32, D = 2A.$

FIG. 14. $N + 1 = 100, D = 2A.$

In Figs. 15 to 18 we plot $\langle X \rangle$ superposed to $\langle N_\Omega \rangle$ for times around the second peak centered in $t_P$. We see how the behavior of $\langle X \rangle$ and $\langle N_\Omega \rangle$ are correlated and how after the revival the subsystem oscillator is damped [see Eq. (74) in Sec. VI]. In Fig. 16 the choice
of $D = 2A$ shows again the growth of the fluctuations.

**FIG. 15.** $N + 1 = 32$, $D = A$.

**FIG. 16.** $N + 1 = 32$, $D = 2A$.

**FIG. 17.** $N + 1 = 100$.

**FIG. 18.** $N + 1 = 500$. 
Fig. 19 contains the behavior of $P_{\Omega\Omega}$ and $P_{\Omega n}$ for $N + 1 = 32$. We see that the bath contribution remains almost constant except at times around $t_P$ in which the survival probability has a maximum. In Fig. 20 we compare the behavior of $\langle N_n \rangle$ for a value near to the central frequency $\Omega$ and for a value far of it. We see that while $\langle N_2 \rangle$ smoothly fluctuates around a constant value $\left( e^{\beta \omega_2} - 1 \right)^{-1}$, $\langle N_{16} \rangle$ is sensible to what happens with $\langle N_{\Omega} \rangle$ and then it is displaced with respect to $\left( e^{\beta \omega_{16}} - 1 \right)^{-1}$. This is an indication that the transference of energy from the subsystem to the bath is more effective for frequencies near to $\Omega$. In Fig. 21 we confirm the hypothesis that the central oscillators are those which receive the energy of the Brownian particle, since going to the continuous limit the distribution of $P_{\Omega n}$ approaches to a delta function. The asymptotic value of $\langle N_{\Omega} \rangle$ is given by

$$\langle N_{\Omega}(\infty) \rangle = \sum_{n=1}^{N} P_{\Omega n}(\infty) \frac{1}{e^{\beta \omega_n} - 1},$$

where $P_{\Omega n}(\infty) = \sum_{\nu=0}^{N} \left( |\Phi_{\nu}|^2 \frac{g_{\nu_{0}}}{\alpha_{\nu_{0}} - \omega_n} \right)^2 \equiv \theta_N(\omega_n)$. In the continuum we show, in Sec. VI, that $P_{\Omega \omega}(\infty) = \delta(\omega - \Omega)$ up to the first order in a re-scaled parameter $\tau = \lambda^2 t$ (which is known as $\lambda^2 t$ approximation). Fig. 21 thus plots $\theta_N$ vs. $\omega_n$.

**FIG. 19.** Survival probability and bath contribution.  
**FIG. 20.** Bath population.
FIG. 21. Energy transfer to the central bath oscillator.

We end this section by obtaining the results of Ref. [4] for $N + 1 = 32$. In Fig. 22 we show $\langle N_\Omega \rangle$ vs. $\Omega t$ (cf. Fig. 1 (b) of Ref. [4]). In Fig. 23 we see the damping oscillations, in Fig. 24 we compare the survival probability with the bath contribution, and in Fig. 25 we show the behavior of different values of $\langle N_n \rangle$. 
FIG. 22. $\langle N_\Omega \rangle$ vs. $\Omega t$.  

FIG. 23. $\langle X \rangle$ vs. $\Omega t$.

FIG. 24. Survival probability and bath contribution.  

FIG. 25. $\langle N_n \rangle$ vs. $\Omega t$.  

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VI. ASYMPTOTIC LIMIT AND CONTINUOUS BATH

In Sec. V we have studied a particular model for a finite but increasing number of bath oscillators with the aim of approaching to the continuous limit. In this section we analytically take this limit for frequencies spanning a segment into the positive real axis. At first glance the continuous limit of Eq. (31) can be made by changing summations for integrals. However it cannot be straightforwardly taken because of the appearance of a continuous set of divergencies along the integration domain. We must give a criterion to avoid these singularities, which is related to the choice of the boundary conditions. Following Ullersma’s pioneering work [1] we propose a way, based on an analytic continuation method, to do that.

We can define the function $R_d(z)$ (the reduced resolvent operator in the energy representation, where $d$ stands for discrete case) of the complex variable $z$ departing from Eq. (16) as

$$R_d^{-1}(z) = z - \Omega - \sum_{n=1}^{N} \frac{g_n^2}{z - \omega_n},$$

(50)

where the normal frequencies $\alpha_\nu$ are given by the simple poles of $R_d : R_d^{-1}(\alpha_\nu) = 0$. Eq. (50) can be rewritten in terms of $R_d$ as

$$|\Phi_\nu|^2 = \frac{1}{(R_d^{-1})'(\alpha_\nu)}.$$  

(51)

Eq. (51) allows us to write the first equation of Eq. (25) as

$$B(t) = \sum_{\nu=0}^{N} \frac{e^{-i\alpha_\nu t}}{(R_d^{-1})'(\alpha_\nu)} \left[ B(0) + \sum_{n=1}^{N} \frac{g_n}{\alpha_\nu - \omega_n} b_n(0) \right].$$

(52)

In order to perform the continuous limit let us consider the following identities:

$$\sum_{\nu=0}^{N} \frac{e^{-i\alpha_\nu t}}{(R_d^{-1})'(\alpha_\nu)} = \frac{1}{2\pi i} \oint_C dz \frac{e^{-izt}}{R_d^{-1}(z)},$$

(53)

$$\sum_{\nu=0}^{N} \frac{e^{-i\alpha_\nu t}}{(R_d^{-1})'(\alpha_\nu)} \sum_{n=1}^{N} \frac{g_n}{\alpha_\nu - \omega_n} b_n(0) = \frac{1}{2\pi i} \oint_C dz \frac{e^{-izt}}{R_d^{-1}(z)} \sum_{n=1}^{N} \frac{g_n}{z - \omega_n} b_n(0),$$

(54)
where \(C\) is a counterclockwise contour in the \(z\)-plane that encircles the \(N+1\) singularities of \(R_d\) in the positive real axis (see Fig. 26).

**FIG. 26.** Counterclockwise contour \(C\).

We make use of the residues theorem: 
\[
\oint_C \frac{P(z)}{Q(z)} \, dz = 2\pi i \sum_{k=1}^r \text{Res}\left[\frac{P}{Q}, z_k\right] = 2\pi i \sum_{k=1}^r \frac{P(z_k)}{Q'(z_k)},
\]
where \(z_k\) are the simple zeros of \(Q(z)\). Eq. (53) is a direct consequence of the residues theorem and Eq. (54) follows from the same theorem and from the fact that 
\[
\sum_n e^{-i\omega nt} R_d^{-1}(\omega_n) g_n b_n(0) = 0,
\]
since \(R_d^{-1}(\omega_n)\) diverges for all \(n\).

When the bath frequencies form a dense set, the normal frequencies are also dense. This limit of a continuous bath is valid for times that satisfy
\[
t \ll \min (\alpha_{\nu+1} - \alpha_{\nu})^{-1}.
\]
In this approximation \(R_d^{-1}(z)\) goes to
\[
R^{-1}(z) = z - \Omega - \int_{\omega_{\min}}^{\omega_{\max}} d\omega \frac{g^2(\omega)}{z - \omega},
\]
where \(\omega_{\min}\) and \(\omega_{\max}\) are the extrema of the continuous set (lower and upper cutoff respectively) and avoid infrared and ultraviolet divergencies respectively.
$g^2(\omega)$ is defined by

$$g^2(\omega) \Delta \omega = \sum_{\omega_n < \omega < \omega + \Delta \omega} g^2_n.$$  

The function $R^{-1}(z)$ has a cut along $(\omega_{\text{min}}, \omega_{\text{max}})$, corresponding to the continuous spectrum of normal frequencies. In order to ensure that the equation $R^{-1}(z) = 0$ has no real roots it is necessary that $\Omega \in (\omega_{\text{min}}, \omega_{\text{max}})$, which together with conditions (19) adequately generalized to this case, provide a necessary and sufficient criterion of feasibility for dissipation in linear models. Generalized conditions (19) are given by

$$\int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{g^2(\omega)}{\omega - \omega_{\text{min}}} < \Omega - \omega_{\text{min}}, \quad \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{g^2(\omega)}{\omega_{\text{max}} - \omega} < \omega_{\text{max}} - \Omega,$$

assuming that the integrals are well defined. The cut of the function $R^{-1}(z)$ can be reached from above and below the positive real axis, giving the limiting values

$$R^{-1}(\alpha \pm i \epsilon) = \alpha - \Omega - \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{g^2(\omega)}{\alpha - \omega \pm i \epsilon},$$

where $\alpha \in (\omega_{\text{min}}, \omega_{\text{max}})$. Then, contraction of the contour $C$ in Eq. (53) yields

$$\frac{1}{2\pi i} \oint_C \frac{e^{-izt}}{R^{-1}(z)} dz = \frac{1}{2\pi i} \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha e^{-i\alpha t} \left[ \frac{1}{R^{-1}(\alpha - i \epsilon)} - \frac{1}{R^{-1}(\alpha + i \epsilon)} \right].$$

On the other hand, by taking into account the well known identity between distributions

$$\frac{1}{x \pm i \epsilon} = \text{PV} \frac{1}{x} \mp i \pi \delta(x),$$

where PV stands for the principal value and $\delta$ is the Dirac delta distribution, we have

$$R^{-1}(\alpha + i \epsilon) - R^{-1}(\alpha - i \epsilon) = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{g^2(\omega)}{\alpha - \omega - i \epsilon} - \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{g^2(\omega)}{\alpha - \omega + i \epsilon} = 2i\pi g^2(\alpha).$$

Then using (59) Eq. (57) is reduced to

$$\frac{1}{2\pi i} \oint_C \frac{e^{-izt}}{R^{-1}(z)} dz = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha \frac{g^2(\alpha)}{|R^{-1}(\alpha + i \epsilon)|^2} e^{-i\alpha t},$$

where we consider that $R^{-1}(\alpha - i \epsilon) = R^{-1*}(\alpha + i \epsilon)$. Contracting the contour $C$ in Eq. (54) and taking the continuous limit we have
\[
\frac{1}{2\pi i} \oint_C dz e^{-izt} \int_{\omega_{\min}}^{\omega_{\max}} d\omega \frac{g(\omega)}{z - \omega} b_\omega(0) = \frac{1}{2\pi i} \int_{\omega_{\min}}^{\omega_{\max}} d\alpha e^{-i\alpha t} \int_{\omega_{\min}}^{\omega_{\max}} d\omega g(\omega) \\
\times \left[ \frac{1}{R^{-1}(\alpha - \omega - i\epsilon)} - \frac{1}{R^{-1}(\alpha - \omega + i\epsilon)} \right] b_\omega(0),
\]

and after performing a straightforward calculation we obtain

\[
\frac{1}{2\pi i} \oint_C dz e^{-izt} \int_{\omega_{\min}}^{\omega_{\max}} d\omega \frac{g(\omega)}{z - \omega} b_\omega(0) = \int_{\omega_{\min}}^{\omega_{\max}} d\alpha g(\alpha) \frac{1}{R^{-1}(\alpha - i\epsilon)} e^{-i\alpha t} b_\alpha(0) \\
+ \int_{\omega_{\min}}^{\omega_{\max}} d\alpha \int_{\omega_{\min}}^{\omega_{\max}} d\omega \frac{g^2(\alpha)g(\omega)}{|R^{-1}(\alpha + i\epsilon)|^2 (\alpha - \omega + i\epsilon)} e^{-i\alpha t} b_\omega(0),
\]

(61)

where we have used identity (58) for \( x = \alpha - \omega \) to have the \( \delta \)-function expressed as

\[
\frac{1}{(\alpha - \omega - i\epsilon)} - \frac{1}{(\alpha - \omega + i\epsilon)} = 2\pi i\delta(\alpha - \omega).
\]

Then \( B(t) \) can be written in the continuous limit as

\[
B(t) = \int_{\omega_{\min}}^{\omega_{\max}} d\omega \phi_\alpha(\omega) b_\alpha(0).
\]

(62)

where

\[
\Phi_\alpha = \frac{g(\alpha)}{R^{-1}(\alpha + i\epsilon)},
\]

(63)

\[
\phi_\alpha(\omega) = \delta(\alpha - \omega) + \frac{g(\alpha)g(\omega)}{R^{-1}(\alpha + i\epsilon)(\alpha - \omega + i\epsilon)}.
\]

(64)

Compare Eqs. (63) and (64) with those obtained in Ref. [9] [Eqs. (4.8a) and (4.8b); in this work \( \omega_{\min} = 0 \) and \( \omega_{\max} = \infty \)], which correspond to the Lippmann-Schwinger coefficients of the eigenvectors of the continuous generalization of the one-particle Hamiltonian (7) [2,10].

We have obtained the continuous generalization of \( B(t) \). By an straightforward but similar calculation we can obtain the continuous version of \( b_n(t) \).

From Eq. (62) we can give an approximate expression of the Langevin equation. By taking mean values in a thermal initial state for the bath we have
\[ \langle B(t) \rangle = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha |\Phi_\alpha|^2 e^{-i\alpha t} \langle B(0) \rangle . \] (65)

Performing an analytical continuation to the complex plane, one can extract the contribution of the poles of $|\Phi_\alpha|^2$. Then, let us analyze the analytic structure of $|\Phi_\alpha|^2$ as a complex function. The function $R^{-1}(z)$ has no zeros in the complex plane. It can be easily seen by considering $z = a + ib$,

\[ a + ib - \Omega - \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{g^2(\omega)}{a + ib - \omega} = a - \Omega - \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{(a - \omega)g^2(\omega)}{(a - \omega)^2 + b^2} + ib \left( 1 + \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{g^2(\omega)}{(a - \omega)^2 + b^2} \right), \]

where the imaginary part is equal to zero only if $b = 0$. Thus only real zeroes can exist, but in this case $R^{-1}(z)$ is only well defined by its limiting values $R^{-1}(\alpha \pm i\epsilon)$. The discontinuity of $R^{-1}(z)$ in $(\omega_{\text{min}}, \omega_{\text{max}})$ is given by Eq. (59). $R^{-1}(\alpha \pm i\epsilon)$ has no zeroes because the imaginary part is not null, since $g^2(\alpha) \neq 0$ for $\alpha \in (\omega_{\text{min}}, \omega_{\text{max}})$. Then $R^{-1}(z)$, which is analytic in the complex plane except for the cut discontinuity along $(\omega_{\text{min}}, \omega_{\text{max}})$, has no zeroes in its definition range. In order to define an analytic function in all the complex plane we must analytically extend $R^{-1}(z)$ into the second Riemann sheet. To see this let us consider the function $f(z) = (z - \omega)^{-1}$ with its limiting values $(\alpha - \omega \pm i\epsilon)^{-1}$. We can consider $f(z)$ as a multivalued function or take the limiting values as defining two different functions (complex distributions). Then we define

\[ f_\pm(z) = \begin{cases} \frac{1}{z - \omega}, & \text{for } \text{Im}z \geq 0, \\ \frac{1}{\alpha - \omega \pm i\epsilon}, & \text{for } \alpha \in (\omega_{\text{min}}, \omega_{\text{max}}), \\ \frac{1}{z - \omega} \mp 2\pi i\delta(z - \omega), & \text{for } \text{Im}z \leq 0. \end{cases} \] (66)

$f_\pm(z)$ is analytic in all the complex plane. Let us check this fact for $f_+(z)$. Approaching the real axis from above and below we have

\[ f_+(\alpha + i\epsilon) - f_+(\alpha - i\epsilon) = \frac{1}{\alpha - \omega + i\epsilon} - \frac{1}{\alpha - \omega - i\epsilon} + 2\pi i\delta(\alpha - \omega), \]

but that is exactly zero due to identity (58). Putting the functions $f_\pm(z)$ into $R^{-1}(z)$ we get two analytic continuations, $R_\pm^{-1}(z)$ (see, e.g., Ref. [11]). We can analytically continue the
function \( g^2(\alpha) \) across \((\omega_{\min}, \omega_{\max})\). We call \( G(z) \) this extension, i.e. there is an open region \( \Delta \) of the complex plane containing \((\omega_{\min}, \omega_{\max})\) and a meromorphic function \( G : \Delta \to \mathbb{C} \) such that \( g^2(\alpha) = G(\alpha) \) for \( \alpha \in (\omega_{\min}, \omega_{\max}) \). For notational convenience we write \( g^2(z) = G(z) \) also for non-real \( z \). So, we are now ready to use the Cauchy theorem and change the contour of integration in Eq. (65) by a contour \( \Sigma \) in the lower complex plane as shown in Fig. 27, which leaves the singularities of \( G(z) \) outside, namely

\[
\langle B(t) \rangle = \int_{\Sigma} dz \frac{G(z)e^{-itz}}{R_+^{-1}(z)R_-^{-1}(z)} \langle B(0) \rangle .
\]  

(67)

\[ \text{FIG. 27. Contour } \Sigma. \]

The last part of definition (66) can be considered as the values of \( f(z) \) into the second sheet. The relevant fact is that \( R_+^{-1}(z) \) has now a complex zero formally given by the zero of

\[
z - \Omega - \int_{\omega_{\min}}^{\omega_{\max}} d\omega \frac{g^2(\omega)}{z - \omega} + 2i\pi G(z) = 0.
\]

This zero can be estimated up to the second order as
\[ z_0 = \Omega + \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{g^2(\omega)}{\Omega - \omega - i\epsilon} + 2i\pi G(\Omega) = \Omega + \delta\Omega - i\frac{\Gamma}{2}. \]

That is, \( \Omega \) has two corrections, a real one, which provides the frequency shift, given by

\[ \delta\Omega = \text{PV} \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \frac{g^2(\omega)}{\Omega - \omega}, \quad (68) \]

and a negative imaginary part, which represents the frequency width, given by

\[ \Gamma = 2\pi g^2(\Omega). \quad (69) \]

For the model we have developed in Sec. V this second order correction should vanish due to the symmetrical distribution of both the bath frequencies and the interaction \( g(\omega) \).

Coming back to our original purpose let us evaluate the integral of Eq. (67) for the case in which only a simple zero \( z_0 \) is present. By taking the contour \( \Sigma \) lying below \( z_0 \) and using the residues theorem we have

\[ \langle B(t) \rangle = \left[ \frac{e^{-iz_0 t}}{(R_+)^{-1}(z_0)} + \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha \frac{g^2(\alpha)}{R_+^{-1}(\alpha)R_-^{-1}(\alpha)} e^{-i\alpha t} \right] \langle B(0) \rangle, \quad (70) \]

where the tilde over \( R_+^{-1}(z) \) stands for

\[ \frac{1}{R_+^{-1}(\alpha)} = \frac{1}{R_+^{-1}(\alpha)} + 2\pi i \frac{\delta(\alpha - z_0)}{(R_+^{-1})'(z_0)}, \]

being \( \delta(\alpha - z_0) \) the complex extension of the Dirac delta, defined by

\[ \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha f(\alpha)\delta(\alpha - z_0) = f(z_0), \quad \text{if} \quad z_0 \in \text{Int} C_0, \]

which actually means \( \frac{1}{2\pi i} \oint_{C_0} dz' f(z') \frac{1}{z' - z_0} = f(z_0) \), for \( C_0 \) a contour encircling \( z_0 \) into the second sheet as depicted by Fig. 28.
FIG. 28. Contour $C_0$; dotted line encircles $z_0$ into the second sheet.

The second term inside the brackets of Eq. (70) is called the background. It is responsible for deviations from the exponential law and its contribution is relevant only for either very short or very long times. In the regime where the exponential decay dominates we can neglect the background. Then we obtain the approximate expression

$$\langle B(t) \rangle = e^{-iz_0 t} \left( R^{-1}_+ \right)'(z_0) \langle B(0) \rangle,$$

(71)

which satisfies the following differential equation

$$\langle \dot{B}(t) \rangle + (\Omega + \delta \Omega)^2 \langle B(t) \rangle = 0.$$

(72)

Eq. (72) has the form of a harmonic oscillator equation, however $z_0$ is now complex. Keeping in mind that $z_0 = \Omega + \delta \Omega - i \frac{\Gamma}{2}$, and neglecting the terms in $\Gamma^2$, it is derived

$$\langle \ddot{B}(t) \rangle + (\Omega + \delta \Omega)^2 \langle B(t) \rangle + \Gamma \langle \dot{B}(t) \rangle = 0,$$

(73)
which corresponds to the equation of a damped harmonic oscillator. Making the same for
the creation operator and since \( X = \frac{1}{\sqrt{2M\Omega}} (B + B^\dagger) \) we finally reach the standard form
of the Langevin equation (in mean values):

\[
\langle \dot{X}(t) \rangle + (\Omega + \delta\Omega)^2 \langle X(t) \rangle + \Gamma \langle \dot{X}(t) \rangle = 0. \tag{74}
\]

The first two terms on the left hand side represents the Hamiltonian evolution of an oscil-
lator with the renormalized frequency \( \Omega + \delta\Omega \), while the third one represents the ‘friction’
(dissipative) part of the damped linear oscillator with the damping factor \( \Gamma \). This \( \Gamma \) is the
one we could have obtained if we had performed the same analytical continuation made in
this section and if we had neglected the background in Eq. (43).

As we have pointed out in Eq. (35) the survival probability of the state \( |\Omega\rangle \) is given
by the square modulus of the integral of Eq. (65). Let us study its short and long time
behavior. From general grounds, if we have a Hamiltonian given by \( H = H_0 + V \), where
\( H_0 \) is an unperturbed Hamiltonian and \( V \) a small perturbation, and \( \{ |\psi_n\rangle \} \) is a set of
eigenvectors of \( H_0 \), \( H_0 |\psi_n\rangle = E_n |\psi_n\rangle \), the survival probability of the state \( |\psi_k\rangle \) is given
by \( P_k(t) = |\langle \psi_k | e^{-iHt} |\psi_k\rangle|^2 \). At very short times the exponential can be approximated by
its first two terms in the Taylor expansion, \( e^{-iHt} \approx 1 - iHt \). Then the survival probability
behaves as \( P_k(t) \approx 1 + O(t^2) \), which does not correspond to an exponential decay behavior
like \( 1 - \Gamma t \). Thus, at very short times, we have a non-exponential behavior, as it was shown
is Sec. V, which is known as Zeno’s period [4]. For very long times we also have a non-
exponential contribution to the survival probability. As \( t \) goes to infinity the integral of Eq.
(65) (survival amplitude) goes to zero as a consequence of the Riemann-Lebesgue theorem.
Then the behavior of the survival probability depends on the small-frequency behavior of
\( g^2(\alpha) \). For small frequencies \( R^{-1}(\omega_{\min} + i\epsilon) = \omega_{\min} - \Omega - \int_{\omega_{\min}}^{\omega_{\max}} d\omega \frac{g^2(\omega)}{\omega_{\min} - \omega + i\epsilon} \approx \omega_{\min} - \Omega \),
since the integral is bounded by condition (56). The behavior of \( g(\alpha) \) is model dependent.
We consider Ullersma’s spectral strength of the kind \( g(\alpha) = \frac{c_1\alpha}{\sqrt{c_2^2 + \alpha^2}} \), where \( c_1 \) and \( c_2 \) are
constants, so the small-frequency behavior is given by \( g(\alpha) \approx (c_1/c_2)\alpha \). Therefore we have,
for \( \frac{1}{\omega_{\text{max}}} \ll t \ll \frac{1}{\omega_{\text{min}}} \),

\[
\int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha |\Phi_\alpha|^2 e^{-i\alpha t} \approx \left( \frac{c_1}{c_2} \right)^2 \int_{\omega_{\text{min}}}^{1/t} \frac{d\alpha}{\omega_{\text{min}} - \Omega} e^{-i\alpha t} \sim \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha \alpha^2 e^{-i\alpha t}.
\]

Calling \( \alpha t = u \) we obtain for the survival amplitude

\[
\int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha \alpha^2 e^{-i\alpha t} = t^{-3} \int_{\omega_{\text{min}}^t}^{\omega_{\text{max}}t} du u^2 e^{-iu} = it^{-3} \left( u^2 e^{-iu} - 2iue^{-iu} - 2e^{-iu} \right)|_{\omega_{\text{min}}^t}^{\omega_{\text{max}}t}.
\]

Since the squared modulus behaves as \( t^{-6} \left( (\omega_{\text{max}}^4 - \omega_{\text{min}}^4) t^4 + 4 \right) \), the survival probability gives a power law decay

\[
P_{\Omega\Omega}(t) \sim \Lambda^4 t^{-2} \quad \text{as } t \to \infty,
\]

where \( \Lambda = \sqrt{\omega_{\text{max}}^4 - \omega_{\text{min}}^4} \) is a measure of an upper cutoff frequency. This deviation from the exponential decay law given by a power series tail is known from Khalfin’s original work [12].

Let us now study the asymptotic behavior of \( \langle N_\Omega(t) \rangle \) in the case of a dense bath, using the \( \lambda^2 t \) approximation. Going back to Eq. (52) and its complex conjugate we write

\[
\langle N_\Omega(t) \rangle = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha' e^{i(\alpha - \alpha')t} |\Phi_\alpha|^2 |\Phi_{\alpha'}|^2 \langle N_\Omega(0) \rangle
\]

\[
+ \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha' e^{i(\alpha - \alpha')t} \psi_{\alpha'}^* \psi_{\alpha'} \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\omega \phi^*_{\alpha'}(\omega) \phi_{\alpha'}(\omega) \langle N_\omega(0) \rangle,
\]

where we have considered an uncorrelated initial state between subsystem and bath, with

the bath in thermal equilibrium, i.e. \( \langle N_\omega(0) \rangle = \left( e^{\beta \omega} - 1 \right)^{-1} \). From the Riemann-Lebesgue theorem all the oscillating terms vanish as \( t \to \infty \), so the only survival term is that with the product of deltas contained in \( \phi' \)'s coefficients [Eq. (54)], namely

\[
\langle N_\Omega(\infty) \rangle = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha \frac{g^2(\alpha)}{R_+^{-1}(\alpha)^2} \frac{1}{e^{\beta \omega} - 1}.
\]

We now perform the limit \( \lambda \to 0 \). In such a limit \( R_+^{-1}(\alpha) \approx \alpha - \Omega \pm i\epsilon \) and taking into account Eq. (59) we obtain
\[ \langle N_\Omega(\infty) \rangle = \int_{\omega_{\text{min}}}^{\omega_{\text{max}}} d\alpha \frac{1}{2\pi i} \left[ \frac{1}{\alpha - \Omega - i\epsilon} - \frac{1}{\alpha - \Omega + i\epsilon} \right] \frac{1}{e^{\beta\omega} - 1} = \frac{1}{e^{\beta\Omega} - 1}, \]  

(77)

since the expression between square brackets is \(2\pi i \delta(\alpha - \Omega)\). We then see that the subsystem oscillator reaches thermal equilibrium with the bath. In Sec. V we have studied the behavior of the corresponding asymptotic form of \(\langle N_\Omega \rangle\) for a finite size bath [Eq. (49)]. In that case we have seen that, when approaching to the continuous limit, \(\sum_{\nu=0}^{N} \left( \frac{\varphi_{\nu} g_{\nu}}{\omega_{\nu} - \omega} \right)^2\) behaves as a delta function. It is important to remark that Eq. (77) shows that the transfer of energy mainly occurs between two oscillators in resonance. In Fig. 29 we have a diagrammatic representation of the time evolution of \(\langle N_\Omega(t) \rangle\). First, there is a deviation from the exponential decay law due to the Zeno period, after that the exponential decay dominates for long time the evolution until the Khalfin power series tail, finishing in the asymptotic value of thermal equilibrium given by Eq. (77). Note that the continuous limit carries Zeno time to zero and Khalfin time to infinity but nevertheless the exponential decay law is not valid at all. Numerical estimates will be given elsewhere.

**FIG. 29.** Time scales.
VII. CONCLUDING REMARKS

This work provides an exhaustive analysis of the most popular model of Brownian motion in a way which has not been deeply explored in the literature on the subject till now. The exact solution of the eigenvalue problem allows us to study the time behavior of the magnitudes of interest without resorting to approximations. No doubts about numerical errors can arise, since the diagonalization method used, which has a powerful speed of calculus, does not have recursive increasing deviations. Moreover the continuous limit is performed in an analytic manner, obtaining the standard results found in the literature.

Figures of Sec. V clearly show all the properties expected for a dissipative system (a damped oscillator in this case), with estimates of the Poincaré recurrence time, fluctuations and equilibrium. The Poincaré period arises as an exact time of revival and not as an statistical property of the ensemble. The validity of the exponential decay law is also enlightened. In a forthcoming paper we will study other statistical properties of the model, such as correlation functions, and we will consider other spectral densities and different ways to distribute the unperturbed frequencies of the bath oscillators.

On the other hand, the problem of irreversibility can be traced in the following way. Even for a finite system one can objectively ‘see’ an irreversible evolution, which only depends on the system and not on the ability of the observer. This irreversibility is not a consequence of a coarse-grained distribution or due to approximations. Nevertheless the time evolution is not strictly irreversible but it is practically irreversible for our scale of observation. Moreover, if we consider a real system with a large number of degrees of freedom (e.g. Avogadro’s number), it is easy to convince ourselves that we will see a time asymmetrical evolution for the Brownian particle, since the Poincaré time becomes larger than the age of the Universe.
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[1] P. Ullersma, Physica 32, 27-55 (1966).
[2] M.A. Castagnino, F.H. Gaioli, and E. Gunzig, Fund. Cosmic Phys. 16, 221-375 (1996) and references therein.
[3] R. Davidson and J.J. Kozak, J. Math. Phys. 12, 903-917 (1971).
[4] J.L. Gruver, J. Aliaga, H.A. Cerdeira, and A.N. Proto, Phys. Rev. E 51, 6263-6266 (1995).
[5] K. Lindenberg and B.J. West, Phys. Rev. A 30, 568-582 (1984) and references therein.
[6] K.O. Friedrichs, Commun. Pure Appl. Math. 1, 361-406 (1948).
[7] F. Haake and R. Reibold, Phys. Rev. A 32, 2462-2475 (1985).
[8] L. van Hove, Physica 21, 517-540 (1955). See also E.B. Davies, Commun. Math. Phys. 33, 171-186 (1973).
[9] E.C.G. Sudarshan, C.B. Chiu, and V. Gorini, Phys. Rev. D 18, 2914-2929 (1978).
[10] M.A. Castagnino and R. Laura, Phys. Rev. A 56, 108-119 (1997).
[11] P. Exner, Open Quantum Systems and Feynman Integrals (Reidel, Amsterdam, 1985).
[12] L.A. Khalfin, Zh. Eksp. Teor. Fiz. 33, 1371-1382 (1957) [Sov. Phys. JETP 6, 1053-1063 (1958)].