Rate-Distortion Bounds on Bayes Risk in Supervised Learning

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Abstract

An information-theoretic framework is presented for estimating the number of labeled samples needed to train a classifier in a parametric Bayesian setting. Ideas from rate-distortion theory are used to derive bounds on the average $L_1$ or $L_{\infty}$ distance between the learned classifier and the true maximum a posteriori classifier—which are well-established surrogates for the excess classification error due to imperfect learning—in terms of the differential entropy of the posterior distribution, the Fisher information of the parametric family, and the number of training samples available. The maximum a posteriori classifier is viewed as a random source, labeled training data are viewed as a finite-rate encoding of the source, and the $L_1$ or $L_{\infty}$ Bayes risk is viewed as the average distortion. The result is a complementary framework to the well-known probably approximately correct (PAC) framework. PAC bounds characterize worst-case learning performance of a family of classifiers whose complexity is captured by the Vapnik-Chervonenkis (VC) dimension. The rate-distortion framework, on the other hand, characterizes the average-case performance of a family of data distributions in terms of a quantity called the interpolation dimension, which represents the complexity of the family of data distributions. The resulting bounds do not suffer from the pessimism typical of the PAC framework, particularly when the training set is small. The framework also naturally accommodates multi-class settings. Furthermore, Monte Carlo methods provide accurate estimates of the bounds even for complicated distributions. The effectiveness of this framework is demonstrated in both a binary and multi-class Gaussian setting.

Index Terms

Supervised learning; Rate-distortion theory; Bayesian methods; Parametric statistics.

I. INTRODUCTION

A central problem in statistics and machine learning is supervised learning, in which a learning machine must choose a classifier using a sequence of labeled training samples drawn from an unknown distribution. The effectiveness of the learned classifier is measured by its accuracy in classifying future test samples drawn from the same distribution. Standard approaches to this problem include support vector machines, [1]–[3], random forests [4], and deep neural networks [5], [6].

In supervised learning, a fundamental question is the sample complexity: how many training samples are necessary to learn an effective classifier? The prevailing approach to characterizing the sample complexity is the the probably approximately correct (PAC) framework, which provides almost sure bounds on the sample complexity of families of classifiers irrespective of the data distribution. These bounds are expressed in terms of the Vapnik-Chervonenkis (VC) dimension, which captures combinatorially the complexity of
families of classifiers [7], [8]. A typical result goes as follows: for a classifier family with VC dimension $h$ and given $n$ training samples, the error probability of the learned classifier is $O(\sqrt{h/n})$ greater than the best classifier in the family. The PAC framework leads to the empirical risk minimization (ERM) and structural risk minimization (SRM) frameworks for model selection: The system designer considers a sequence of classifier families with increasing VC dimension, and chooses the family that minimizes the PAC bound over the available training set. PAC bounds are available for many popular classifiers, including SVMs and neural networks [1], [9], [10].

A drawback of PAC bounds is that they characterize the performance under the worst-case distribution. This leads to pessimistic predictions in many practical problems [11], [12], and practitioners often resort to cross-validation or regularization techniques to increase confidence that classifier performance generalizes to test samples. Furthermore, PAC bounds are only predictive when the number of training samples is relatively high. When the number of samples $n$ is smaller than the VC dimension, the PAC framework trivially predicts a generalization error bound of unity, and the predicted gap decays slowly in $n$. Finally, VC dimension bounds apply primarily to binary classification problems.

Previous research has addressed these challenges in part. Margin-dependent bounds based on the fat-shattering dimension and Rademacher complexity were developed in [13]–[15]. These bounds account for the margin by which the learned classifier separates the training samples, and they are tighter than VC dimension bounds. However, they are still worst-case over the data distribution. PAC-Bayes bounds, in which one imposes a prior over the set of classifiers, were developed in [16], [17]. The prior tailors the bound to classifiers with anticipated structure—such as sparse linear classifiers or neural networks with dropout. While the bounds account for averaging over the family of classifiers, these PAC bounds still hold for the worst-case data distribution.

Multi-class PAC bounds have been developed in terms of the Natarajan dimension, a multi-class analogue to the VC dimension [18], [19], but these bounds too are pessimistic. Furthermore, computing the Natarajan dimension is a challenge. For linear multi-class models of $M$ classes in $\mathbb{R}^d$, for example, [19] provides only an orderwise bound on the Natarajan dimension, $\Omega(Md) \leq d_N \leq O(Md \log Md)$. Finally, margin-dependent, multi-class bounds are derived [20]–[22]. Most of these bounds, however, have sample complexity that grows quadratically in the number of classes.

In this paper, we present a framework for computing sample complexity bounds which are not pessimistic, which are predictive when the training set is small, and which accommodate multi-class settings. We develop this framework in a Bayesian parametric setting. The joint distribution on data points $X$ and labels $Y$ belongs to a known parametric family $p(x, y; \theta)$, and the parameters that index the distribution are drawn from a known prior $q(\theta)$. An example is Gaussian classification, where for each class $y$, $p(x|y; \theta)$ is a multivariate Gaussian with fixed covariance and mean taken as a subvector of $\theta$. A suitable prior $q(\theta)$ is the conjugate prior, which in this case is itself a Gaussian.

The proposed framework provides lower bounds on the average $L_1$ and $L_\infty$ distance between the true posterior $p(y|x; \theta)$ and the posterior estimated from $n$ i.i.d. samples drawn from $p(x, y; \theta)$. Because the bounds are averaged over the prior $q(\theta)$, they do not exhibit the pessimism of the worst-case PAC bounds. Finally, because they bound the $L_1$ and $L_\infty$ error in estimating the posterior, they readily apply to multi-class parametric families. The $L_1$ and $L_\infty$ errors are well-known surrogates for the excess classification
error [23], [24], so bounds on these errors give insight into the performance of the learned classifier. This connects the problem of learning a good classifier to the problem of learning a distribution from samples. This problem has a rich history, dating back to the Good-Turing estimator [25] and continuing to recent results [26]–[29].

The proposed framework is inspired by a close relationship between supervised learning and lossy source coding. In the parametric Bayesian setting, the posterior distribution is a function of the random parameters $\theta$ and therefore is a random object. If we take the $L_1$ or $L_\infty$ distance as the distortion function, we can bound the number of bits needed to describe the posterior to within a specified tolerance. What follows is the main result of this paper: In order to drive the average $L_1$ or $L_\infty$ error below a threshold $\epsilon$, the mutual information between the training samples and the parameters $\theta$ must be at least as great as the differential entropy of the posterior plus a penalty term that depends on $\epsilon$ and a sample-theoretic quantity, called the interpolation dimension, which measures the number of data points from the posterior distribution needed to uniquely interpolate the entire function.

The resulting framework is a dual to the PAC framework. Whereas the PAC framework considers families of classifiers and provides generalization bounds that hold for any data distribution, the rate-distortion framework considers families of data distributions and provides generalization bounds that hold for any classifier. Whereas the VC dimension characterizes the combinatorial complexity of a family of classifiers, the interpolation dimension characterizes the sample complexity of a parametric family of data distributions. The larger the interpolation dimension, the more training samples are needed to guarantee classifier performance.

Previous works have exploited the relationship between learning and universal data compression and prediction. Rissanen proposed the minimum description length (MDL) criterion for model selection [30], which can be used to control the complexity of the model class and avoid overfitting. The MDL framework has since seen wide use in machine learning (see [31] and [32] and the references therein for a recent survey).

After formally laying out the problem statement in Section II, we present the main results in Section III. First, we establish $L_1$ and $L_\infty$ bounds that hold pointwise over high-probability sets of training and test samples. That is, these bounds provide necessary conditions over “almost” every possible training set and test point; if the bounds are violated, there exists a set of points having non-negligible probability for which the error exceeds the specified tolerance. Next, we relax the pointwise constraints and derive bounds that hold on the average over the training set and test sample.

Then, in Section IV we translate the error bounds into sample complexity bounds. Applying a Bayesian version of the capacity-redundancy theorem, [33], we find that the mutual information between the training set and the parameters $\theta$, which are random in our Bayesian setup, scales as $\log(n)$ and depends on the determinant of the Fisher information matrix averaged over the distribution family. Using this fact, we derive bounds on the number of samples needed to ensure small $L_1$ or $L_\infty$ error.

Finally, in Section V we evaluate the bounds for two examples. First, we consider a binary Gaussian setting, in which we can evaluate the bounds in closed form. Comparison with PAC bounds shows that the rate-distortion framework provides tighter, less pessimistic predictions of learning performance. Then, we consider a multi-class Gaussian setting. Here, we cannot evaluate the bounds in closed form. Instead, we
derive a general formula for computing multi-class bounds, of which individual terms can be evaluated numerically as necessary. Doing so for Gaussian classes, we again see that the rate-distortion framework provides accurate predictions of learning performance. We also see that the gap between the $L_\infty$ and $L_1$ bounds grows in the number of classes, due to “peakiness” in the estimation error of the posterior. We conclude that for $M > 2$, the derived bounds on the $L_1$ classification error provide a better estimate for the classification error probability compared with the derived bounds on the $L_\infty$ classification error, even though the $L_\infty$ error itself is a tighter bound on the classification error probability. We give our conclusions in Section VI.

Notation: Let $\mathbb{R}$ denote the field of real numbers. Let capital letters $X$ denote random variables and vectors, and lowercase letters $x$ denote their realizations. We let $E[\cdot]$ denote the expectation, with the subscript indicating the random variable over which the expectation is taken when necessary. Let $|\cdot|$ denote the cardinality of a set. For a function $f(x)$ and a finite set $S$, let $\{f(x)\}_S$ denote the $|S|$-length vector of function evaluations of $f$ at the points in $S$, suppressing the arguments when clarity permits. Let $[M] = \{1, \ldots, M\}$ for integer $M$. Let $I(\cdot;\cdot)$ denote the mutual information, $h(\cdot)$ denote the differential entropy, and $H(\cdot)$ denote the binary entropy function. We use the natural log throughout, so these quantities are measured in nats.

II. Problem Statement

Let each data point $X \in \mathbb{R}^d$ and its label $Y \in \{1, \ldots, M\}$ be distributed according to $p(x, y; \theta)$, where $\theta \in \Lambda \subset \mathbb{R}^k$ indexes a smooth parametric family of distributions $\mathcal{D} = \{p(x, y; \theta) : \theta \in \Lambda\}$. The learning machine obtains a sequence of $n$ samples, denoted $Z^n = (X^n, Y^n)$, where each pair $Z_i = (X_i, Y_i)$, $1 \leq i \leq n$ is drawn i.i.d. according to $p(x, y; \theta)$. The learning task is to select a classifier $\hat{y} = w(x)$ from the training samples $Z^n$. The classifier may be any function that maps $\mathbb{R}^d$ to $\{1, \ldots, M\}$; we do not restrict our attention to a particular family of classifiers (e.g., the family of linear classifiers) as in the PAC framework. If $\theta$ were known, one could choose the the maximum a posteriori classifier, which minimizes the classification error:

$$w_{\text{MAP}}(x) = \arg \max_y p(y|x; \theta),$$

where $p(y|x; \theta)$ is calculated according to Bayes’ rule. We suppose that the learning machine only knows the family of distributions $\mathcal{D}$, but not the specific distribution $p(y|x; \theta)$. The MAP classifier $w_{\text{MAP}}$ is unavailable, and the learning machine has to learn a classifier $w$ whose misclassification probability will necessarily be no smaller than that of $w_{\text{MAP}}$.

Our objective is to characterize the performance of the learned classifier $w(x)$ as a function of the number of training samples $n$. A natural performance metric is the gap between the misclassification error of the learned classifier and that of $w_{\text{MAP}}$:

$$L_c(x, \theta; w, w_{\text{MAP}}) = \Pr(Y \neq w(x)) - \Pr(Y \neq w_{\text{MAP}}(x)), \quad (1)$$

where the probabilities are computed according to $p(y|x, \theta)$. As discussed in the introduction, the minimax loss with respect to $L_c$ is characterized by the PAC framework. For a family of classifiers containing the MAP classifier and having with VC dimension $h$, $L_c(x, \theta; w, w_{\text{MAP}}) = O(\sqrt{h/n})$ for any distribution over $X$ and $Y$. The minimax loss, however, is pessimistic compared to typical loss.
In this work, we analyze the Bayes risk. Let \( q(\theta) \) be a prior distribution over the parametric family. The proposed framework presents performance bounds averaged over the family of distributions according to \( q(\theta) \). We can view the Bayes error in a few different ways. First, if \( q(\theta) \) represents the true distribution over the parameters space, then the Bayes risk is simply the average loss. Second, for any \( q \), the Bayes risk is smaller than the minimax risk, and it may better capture typical performance. Finally, if the pointwise risk is continuous in \( \theta \in \Theta \), then there is a \( \theta^* \in \Theta \) that “achieves” the Bayes risk, and any \( \theta \) more difficult than \( \theta^* \) will have risk no smaller than the Bayes risk.

The misclassification error gap turns out to be difficult to study under the proposed rate-distortion framework, and we instead study the \( L_1 \) and \( L_\infty \) losses. These losses are defined in terms of estimates of the posterior, which are then used to classify according to the MAP rule. For notational convenience, let \( W(y|x; \theta) \triangleq p(y|x; \theta) \) denote the posterior. Also let \( \delta(Z^n) = \hat{W}(y|x) \) be a learning rule that maps the training samples to an estimate of the posterior. Then, for every \( x \), the losses are defined as the \( L_1 \) and \( L_\infty \) distances between the \( M \)-dimensional vector formed by \( W(\cdot|x; \theta) \) and \( \hat{W}(\cdot|x) \):

\[
L_1(x, \theta; W, \hat{W}) = \sum_{y=1}^{M} |W(y|x; \theta) - \hat{W}(y|x)| \tag{2}
\]

\[
L_\infty(x, \theta; W, \hat{W}) = \max_{1 \leq y \leq M} 2|W(y|x; \theta) - \hat{W}(y|x)|. \tag{3}
\]

Note the factor of 2 in the definition of \( L_\infty \), which ensures that that \( L_1 \) and \( L_\infty \) are exactly the same when \( M = 2 \). It also ensures a relationship between \( L_\infty \) and \( L_c \).

A well-known fact (see, e.g. [23]) is that \( L_1 \), averaged over \( X \), bounds the classification loss \( L_c \) from above. A somewhat less well-known fact is that this relationship holds pointwise. For any point \( x \), a MAP classifier that uses \( \hat{W}(y|x) \) as the posterior will have a classification loss no larger than \( L_1 \). Also somewhat less well-known is that \( L_\infty \) also bounds the classification loss pointwise. (see [24] for a discussion). Indeed, the following relationship holds both on the average and pointwise:

\[
L_c \leq L_\infty \leq L_1. \tag{4}
\]

The relationship between \( L_1 \) and \( L_\infty \) does not hold for the usual definitions of these norms; it holds only because of the factor of 2 and the fact that the vectors are non-negative and sum to one. Also observe that each of these quantities is bounded as follows: \( L_c \leq 1, L_\infty \leq 2, \) and \( L_1 \leq M \).

The main contribution of this paper is a framework for computing lower bounds on the Bayes risk with respect to the loss functions \( L_1 \) and \( L_\infty \). We consider two risk functions that average over different random variables. In the first risk function, we make high-probability guarantees over the test point \( X \) and the training set \( Z^n \). These guarantees are akin to high-probability guarantees of the PAC framework, where the loss bounds hold everywhere except for a set of sufficiently low probability. In our case, however, these guarantees still hold on the average with respect to \( \theta \) rather than the worst-case. To make these high-probability guarantees, we consider subsets \( Z^n \) and \( \mathcal{X} \) of possible training sets and test points, respectively, having probability greater than \( (1 - \gamma) \) for \( \gamma > 0 \). Then, we consider the minimax Bayes risk, where we take the minimum risk over all high-probability subsets, the maximum risk over test points
Fig. 1: The analytical framework in this paper is by analogy to rate-distortion theory. In rate distortion, a source distribution $p(x)$ gives rise to an $n$-length sequence $X^n$, which is encoded to one of $2^{nR}$ indices. The decoder infers from this index a noisy reconstruction $\hat{X}^n$. In other words, we require that the Bayes risk be bounded for all test points and training sequences in a data distribution $p(x, y; \theta)$ and its associated posterior $W(y|x; \theta)$, and we treat the training samples $Z^n$ drawn from $p(x, y; \theta)$ as an imperfect encoding of the posterior. The learning machine infers from $Z^n$ a noisy estimate $\hat{W}(y|x)$, and the $L_1$ and $L_\infty$ estimation error depends on the number of samples via the bounds presented in Theorems 1-3.

and training sets in the high-probability subsets, and average over $\theta$. Concretely, define the $\gamma$-Bayes risk:

$$\hat{R}_1(\delta, \gamma) = \inf_{X \subseteq \mathbb{R}^d} \sup_{(x, z^n) \in X \times Z^n} \mathbb{E}_\theta[L_1(x, \theta; W, \hat{W})]$$

$$\hat{R}_\infty(\delta, \gamma) = \inf_{X \subseteq \mathbb{R}^d} \sup_{(x, z^n) \in X \times Z^n} \mathbb{E}_\theta[L_\infty(x, \theta; W, \hat{W})]$$

$$\hat{R}_c(\delta, \gamma) = \inf_{X \subseteq \mathbb{R}^d} \sup_{(x, z^n) \in X \times Z^n} \mathbb{E}_\theta[L_c(x, \theta; w, w_{\text{MAP}})]$$

In other words, we require that the Bayes risk be bounded for all test points and training sequences in a set of sufficiently high probability.

Just as the worst-case bounds from the PAC framework may be pessimistic, the high-probability guarantees associated with $\hat{R}$ may be too restrictive to describe typical behavior. Therefore, we define a second risk function, which we simply call the Bayes risk, in which we average over all of the random variables:

$$R_1(\delta) = E_{X, Z^n, \theta}[L_1(X, \theta; W, \hat{W})]$$

$$R_\infty(\delta) = E_{X, Z^n, \theta}[L_\infty(X, \theta; W, W)]$$

$$R_c(\delta) = E_{X, Z^n, \theta}[L_c(X, \theta; w, w_{\text{MAP}})]$$

In other words, the Bayes risk $R$ is simply the average performance.

For $\gamma$ sufficiently small, $\hat{R}(\delta, \gamma) \geq R(\delta)$, as a high-probability guarantee over $X$ and $Z^n$ is more stringent than an average-case guarantee. Regardless of $\gamma$, $\hat{R}$ reveals the robustness of learning performance to the test point and training set. One could consider additional permutations of high-probability and average-case risks, taking the minimax loss over $Z^n$ while averaging over $X$ and $\theta$. We omit these cases for brevity, while noting that bounds on these risks are not difficult to infer from the stated results. Finally, we emphasize that all of the Bayes risk functions are functions of the prior distribution $q$, however, this dependence is left implicit since we assume $q$ is known throughout this paper.
III. MAIN RESULTS

We present bounds on the Bayes risk functions defined in the previous section.

We first define a few important concepts. The posterior $W(y|x; \theta)$ is an uncountable collection of random variables, one for each point $(x, y) \in \mathbb{R}^d \times \{1, \ldots, M\}$. The mutual information between, or the joint entropy of, uncountably many random variables is difficult to analyze directly, which makes it difficult to carry out the rate-distortion analysis on the Bayes risk. Therefore, we will analyze the information-theoretic quantities of a sampled version of $W(y|x; \theta)$, which acts as a sufficient statistic for the entire function. We capture this notion by defining the interpolation set and the interpolation dimension.

**Definition 1** (interpolation set/dimension). A finite set $S \subset \mathbb{R}^m \times \{1, \ldots, M\}$ is called an interpolation set for the posterior $W(y|x; \theta)$ if there is a deterministic mapping from the set $\{W(y|x; \theta)\}_S$ to the function $W(y|x; \theta)$ for any $\theta \in \mathbb{R}^d$. Furthermore, let the interpolation dimension $d_I(W)$ be the cardinality of the smallest interpolation set. If there is no such set, $d_I(W) = \infty$.

In the sequel, we will need a quantity that we call the multiplicity of the interpolation set.

**Definition 2** (multiplicity). Define the multiplicity of an interpolation set $S$ as the minimum over the number of times a point $x \in \mathbb{R}^d$ appears in $S$.

$$\mu(S) = \min_{x' \in \mathbb{R}^m} |\{(x, y) \in S : x = x'\}|.$$ 

If $S$ is an interpolation set with $|S| = d_I(W)$, the maximum multiplicity is $M - 1$. This is because the posterior is normalized. For a point $x \in \mathbb{R}$ appearing in $S$, the evaluation of $W(y|x; \theta)$ at $x$ for $M - 1$ values of $y$ suffices to determine $W$ for the remaining value of $y$.

The interpolation dimension characterizes the number of distinct evaluations of the posterior function needed to reconstruct the entire posterior function, and it is thus the analogue to the Nyquist sampling rate for families of parametric distributions. The interpolation dimension also serves as a measure of complexity similar to the VC dimension. The VC dimension characterizes the complexity of a family of classifiers by how many points it can shatter, whereas the interpolation dimension characterizes the complexity of a family of distributions by how many sample points of the posterior are needed to reconstruct it.

We emphasize that the number of function evaluations of the posterior needed for an interpolation set is entirely distinct from the number of independent samples drawn from the distribution needed to estimate the posterior. The interpolation set is akin to the Nyquist rate in sampling theory, and it describes the complexity of the posterior function. Although they are separate concepts, we will show that the number of training samples needed to learn the posterior is related to the interpolation dimension.

**A. Bounds involving $\check{R}(\delta)$**

The first result is a bound on the $L_1$ training set Bayes risk in terms of the interpolation dimension, the entropy of the posterior, and the mutual information between the training samples and the parameter $\theta$.

**Theorem 1.** Let $d_I(W) < \infty$ be the interpolation dimension of $W(y|x; \theta)$, and let $S$ be any interpolation
set with \(|S| = d_I(W)\). Then, there exists a learning rule \(\delta(Z^n)\) such that \(\hat{R}_1(\delta, \gamma) \leq \epsilon\) only if
\[
I(Z^n; \theta) \geq \inf_{\Pr(X) \geq 1-\gamma} \sup_{S \subset X \times \{1, \ldots, M\}} h(\{W\}_S) - d_I(W) \left(1 + \gamma \log \left(\frac{M}{\mu(S)}\right) + H(\gamma) + (1 - \gamma) \log \left(\frac{\epsilon}{\mu(S)}\right)\right),
\]
whenever \(h(\{W\}_S)\) exists and is finite. Furthermore, if \(\mu(S) = (M - 1)\), we get a stronger bound, and there exists a learning rule such that \(\hat{R}_1(\delta, \gamma) \leq \epsilon\) only if
\[
I(Z^n; \theta) \geq \inf_{\Pr(X) \geq 1-\gamma} \sup_{S \subset X \times \{1, \ldots, M\}} h(\{W\}_S) - d_I(W) \left(1 + H(\gamma) + (1 - \gamma) \log \left(\frac{\epsilon}{M}\right)\right).
\]

**Proof:** See Appendix A.

The content of Theorem 1 is that the information that the training samples \(Z^n\) provide about the parameter \(\theta\) must be at least as great as the differential entropy of \(W(y|x; \theta)\) less a penalty term involving the \(L_1\) tolerance, the interpolation dimension \(d_I(W)\), and the probability \(\gamma\). The higher the interpolation dimension, or the lower the tolerance, the more training samples are necessary. We note that the above bound holds for a subset \(S'\) of the interpolation set, with \(S'\) taking the place of \(S\) and \(|S'|\) taking the place of \(d_I\). If one underestimates the interpolation dimension, one can still obtain bounds on the Bayes risk; as one adds additional points to \(S'\), the bounds become tighter. However, if one overestimates the interpolation dimension, the result breaks down. For a set \(S'\) that overdetermines the posterior \(W(y|x; \theta)\), at least one point in \(\{W\}_{S'}\) is a deterministic function of the others, and the resulting probability density is singular with respect to the Lebesgue measure. This results in a differential entropy \(h(\{W\}_S)\) that is, depending on one’s definitions, either undefined or equal to \(-\infty\), and the bound conveys no information.

The statement and proof of Theorem 1 bear similarity to those of the classic rate-distortion theorem [34]. In rate-distortion theory, one wants to provide a high-fidelity representation of the source using a small number of bits. The rate-distortion theorem specifies the best possible tradeoff between the two desiderata, relating the number of bits required to represent the source, the amount of distortion introduced by compression, and the entropy of the source. The bounds in Theorem 1 take a similar form, where the mutual information \(I(Z^n; \theta)\) plays the role of the number of bits used to represent the source, the entropy of the posterior at the interpolation set \(h(\{W\}_{x \in S})\) takes the role of the source entropy, and the \(L_1\) or \(L_\infty\) Bayes risk takes the role of the distortion.

The next result is a bound on the \(L_\infty\) training set Bayes risk.

**Theorem 2.** Let \(d_I(W) < \infty\) be the interpolation dimension of \(W(y|x; \theta)\), and let \(S\) be an interpolation set with \(|S| = d_I(W)\). Then, there exists a learning rule \(\delta(Z^n)\) such that \(\hat{R}_\infty(\delta, \gamma) \leq \epsilon\) only if
\[
I(Z^n; \theta) \geq \inf_{\Pr(X) \geq 1-\gamma} \sup_{S \subset X \times \{1, \ldots, M\}} h(\{W\}_S) - d_I(W) \left(1 + H(\gamma) + (1 - \gamma) \log \left(\frac{\epsilon}{2}\right)\right),
\]
whenever \(h(\{W\}_S)\) exists and is finite.

**Proof:** See Appendix A.

The \(L_1\) and \(L_\infty\) bounds differ by the presence of the multiplicity \(\mu(S)\) of the interpolation set, which results from the definitions of the norms. Because the \(L_\infty\) norm is the maximum absolute value of the elements, whereas the \(L_1\) norm is their sum, the \(L_\infty\) bound involves only one element of \(W(\cdot|x)\), whereas
the $L_1$ bound involves each element of $W(\cdot|x)$ in $S$, which is at most $\mu(S)$. Recalling that the $L_\infty$ loss provides a tighter bound on the misclassification probability gap than the $L_1$ loss, and observing that the $L_1$ bounds are more restrictive whenever $\mu(S) > 2$, we expect the $L_\infty$ bounds to give tighter predictions on the classification error. Indeed, when $\mu(S) = (M - 1)$, the bound on the $L_1$ Bayes risk is roughly $M$ times that of the $L_\infty$ bound. Thus the bounds on $L_\infty$ distance predict loss that decays faster in the number of classes than those on the $L_1$ distance.

The accuracy of these predictions depends crucially on the relative tightness of the bounds. Consider a few scenarios, depicted in Figure 2. By (4), $R_\infty$ bounds the classification error $R_c$ more tightly than does $R_1$, as depicted in Figure 2(a). Let $R^-_\infty$ and $R^-_1$ denote the lower bounds, respectively. If the bounds are reasonably tight, and their gaps are roughly equal in magnitude, then $R^-_\infty$ provides a better prediction of $R_c$ than does $R^-_1$, as depicted in Figure 2(b). On the other hand, if one bound is significantly tighter than the other, as depicted in Figure 2(c), $R^-_\infty$ may not predict $R_c$ as well as $R^-_1$. Furthermore, $R^-_\infty$ and $R^-_1$ are lower bounds on quantities that are upper bounds on $R_c$, so in general $R^-_\infty$ and $R^-_1$ are neither lower nor upper bounds on $R_c$, as depicted in Figure 2(d).

In Section V we investigate these questions empirically. Somewhat surprisingly, we find that the $L_1$ and $L_\infty$ errors are similar, even when the number of classes $M$ is large. The loss vector turns out to be “peaky,” concentrated on one element and thus in the region where the $L_1$ and $L_\infty$ balls coincide. As a result, $R^-_\infty$ tends to be optimistic, and $R^-_1$ makes better predictions on all three loss functions $R_1, R_\infty$, and $R_c$. These findings come with caveats, detailed in Section V, so we cannot make general statements. However, the upshot appears to be that our bounds on the $L_1$ error are the best predictors of performance.

**B. Bounds involving $R(\delta)$**

The bounds presented in Theorems 1 and 2 involve strict conditions. A poor-performing set of test points and training sets with probability greater than $(1 - \gamma)$ forces a high value of the $\gamma$-Bayes risk. Therefore, we also bound the performance averaged over the test point and training sets. To do so, we carry out an analysis of the mutual information between $W$ and $\hat{W}$ averaged over an ensemble of interpolation sets. This requires additional machinery.

**Definition 3 (interpolation map).** Let the map $S : \mathbb{R}^d \times \{1, \ldots, M \} \to (\mathbb{R}^d \times [M])^{d_I}$ send sample point $(x, y)$ to an interpolation set $S(x, y)$, such that $(x, y) \in S(x, y)$. We call $S$ an interpolation map if $S(x, y)$...
if the following conditions hold, except for a set of Lebesgue measure zero: (1) \( S(x, y) \) is defined, (2) each point \((x', y') \in \mathbb{R}^d \times \{1, \ldots, M\} \) appears in a constant, nonzero, and finite number of sets \( S(x, y) \), and (3) for each \((x', y') \in S(x, y)\), \( p(x') = p(x) \). Furthermore, let \( S(x, y) = \{s_1(x, y), \ldots, s_{d_I}(s, y)\} \) denote the mapping pointwise, where \( s_1(x, y) = (x, y) \) without loss of generality.

An interpolation map may not always exist, but the conditions for existence are not too restrictive. For example, in Section V we consider a binary Gaussian scenario in which any basis in \( \mathbb{R}^d \) is an interpolation set. In this case, \( S(x, y) \) maps to an orthogonal basis containing \((x, y)\). The map fails only to map the point \( x = 0 \) to an interpolation set.

However, the exclusion of a measure-zero set from the conditions of the interpolation map may impact the bounds on the Bayes risk. For example, in Section V we consider multi-class Gaussian classification, in which case every interpolation set contains \( \{0\} \times [M-1] \). This measure-zero set violates the conditions of Definition 3, and the resulting Bayes risk bounds exclude it from the analysis. To this end, we define an effective interpolation dimension of an interpolation map.

**Definition 4 (interpolation subdimension).** For an interpolation map \( S(x, y) \), let \( \mathcal{W} \subset \mathbb{R}^d \times [M] \) be the measure-zero set on which the conditions in Definition 3 do not hold. Then, define the interpolation subdimension of \( S \):

\[
d_S(S) = \min_{(x,y)} |S(x, y) \setminus \mathcal{W}|,
\]

where the minimization is over the \((x,y)\) pairs for which \( S(x, y) \) is defined.

When an interpolation map exists, we can bound the Bayes risk with expressions similar to those of Theorems 1 and 2.

**Theorem 3.** Suppose the density \( p(x) \) is Riemann integrable, and suppose an interpolation map \( S(x, y) \) exists with interpolation dimension \( D_I \) and interpolation subdimension \( d_S \). Then, there exists a learning rule \( \delta(Z^n) \) with Bayes risk \( \bar{R}_1 \leq \epsilon \) only if

\[
I(Z^n; \theta) \geq \max_y E_X[h(\{W\}_{S(X,Y)})] + d_S \left( \log \left( \frac{M}{2\epsilon} \right) - 1 \right) - (d_I - d_S)(\log(2) + 1).
\]

Similarly, there exists a learning rule \( \delta(Z^n) \) with Bayes risk \( \bar{R}_\infty \leq \epsilon \) only if

\[
I(Z^n; \theta) \geq \max_y E_X[h(\{W\}_{S(X,Y)})] + d_S \left( \log \left( \frac{1}{\epsilon} \right) - 1 \right) - (d_I - d_S)(\log(2) + 1).
\]

**Proof:** See Appendix B

The difference between the bounds on \( \bar{R}(\delta) \) and \( R(\delta) \) corresponds to the difference in the risk functions. Instead of taking the supremum over interpolation sets to get a high-probability bound on the performance, we take the expectation over interpolation sets to get a bound on the average performance. The penalty term associated with the \( L_1/L_\infty \) tolerance is similar to that of Theorems 1 and 2 except that \( \gamma = 1 \), because there is no low-probability set excluded from the analysis.
IV. Sample Complexity Bounds

To evaluate the bounds in Theorems 1-3, one must evaluate two quantities: the mutual information $I(Z^n; \theta)$ between the training set and the parameterization $\theta$, and the differential entropy $h(\{W\}_S)$ of the posterior evaluated at points in the interpolation set, perhaps averaged over an interpolation map $S(x, y)$. Under appropriate conditions, these quantities can be expressed in simpler terms, permitting the explicit computation of sample complexity bounds. We present the expressions for $I(Z^n; \theta)$ and $h(\{W\}_S)$, respectively, after which we derive sample complexity bounds.

A. An Expression for $I(Z^n; \theta)$

Using results from universal source coding, we express $I(Z^n; \theta)$ in terms of the differential entropy of $\theta$ and the Fisher information matrix of the distribution family. Let $\alpha = s(\theta)$ be a minimal sufficient statistic of $\theta$, meaning both that we can recover $\theta$ from $\alpha$ and that $\alpha$ is a function of any other sufficient statistic.

Then, let $I(\alpha)$ denote the Fisher information matrix with respect to $\alpha$:

$$I_{ij} = -E_{X,Y} \left[ \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} \log(p(X, Y; \theta)) \right].$$

The Fisher information roughly quantifies the amount of information, on the average, that each training sample conveys about $\theta$. Under appropriate regulatory conditions, we can make this notion precise and bound the mutual information in terms of the number of training samples $n$.

**Theorem 4.** Let the parametric family $p(x, y; \theta)$ have minimal sufficient statistic $\alpha \in \mathbb{R}^t$. Suppose that the Fisher information matrix $I(\alpha)$ exists, is non-singular, and has finite determinant. Further suppose that the central limit theorem holds for the maximum-likelihood estimator of $\alpha$. That is, $(\hat{\alpha}(Z^n) - \alpha) \sqrt{n}$ converges to a normal distribution with zero mean and covariance matrix $I^{-1}(\alpha)$. Then, the following expression holds

$$I(Z^n; \theta) = \frac{t}{2} \log \frac{n}{2\pi e} + E_{\alpha} \log |I(\alpha)|^{\frac{1}{2}} + h(\alpha) + o_n(1). \quad (17)$$

**Proof:** This follows from the celebrated redundancy-capacity theorem (see [35]–[37]). Averaging the bounds derived by Clarke and Barron [33] over $q(\alpha)$ yields the result. 

The upshot is that the information conveyed by the training set grows as $\log(n)$, with a prelog that depends on the effective dimension of the parameter space. Further constants are determined by the sensitivity of the distribution to the parameters, as quantified by the Fisher information matrix, and the prior uncertainty of the parameters, as expressed by $h(\alpha)$. The expression is intuitive in light of the assumption that the central limit theorem holds. The maximum-likelihood estimator of $\alpha$ approaches a Gaussian distribution in the limit of increasing $n$, and the resulting mutual information includes a term with the associated differential entropy.

B. Expressions for $h(\{W\}_S)$

The differential entropy $h(\{W\}_S)$ is an unusual quantity. To compute it, we must evaluate the density of the posterior distribution $W(y|x; \theta)$, evaluated at finitely many points, and take the expected logarithm.
This “density of a distribution” will often be difficult to evaluate in closed form, and evaluating the expected logarithm will be more difficult still. Therefore, we cannot expect that a closed-form expression for \( h(\{W\}_S) \) will be available for problems of interest.

Nevertheless, we can develop intuition for \( h(\{W\}_S) \). Suppose that a minimal interpolation set has the form \( S = \{x_1, \cdots, x_l\} \times [M - 1] \). That is, the interpolation set is \( l \) points in \( \mathbb{R}^d \) evaluated at the first \( M - 1 \) values of \( y \); because of the normalization of the posterior, the \( M \)th value is determined by the other samples. For example, for the binary Gaussian classifier, the interpolation set was a basis of \( \mathbb{R}^m \) evaluated at \( y = 1 \). Indeed, a common case is where \( l = d \), and where the \( x_i \)s form a basis of \( \mathbb{R}^m \). Either way, the interpolation dimension is \( d_I = l(M - 1) \).

In this case, the differential entropy has the following expression.

**Theorem 5.** Define the random variables \( N_{iy} = p(x_i, y; \theta) \), for every \( 1 \leq i \leq k \) and every \( 1 \leq y \leq M \). Then, under the preceding assumptions, the differential entropy of the posterior is

\[
h(\{W\}_S) = -\sum_{i=1}^k (M - 1) E \left[ \log \left( \frac{1 + 2S_i}{1 + S_i} \right) \right] - \sum_{i=1}^k E \left[ \log \left( 1 + \frac{S_i}{(1 + S_i)(1 + 2S_i)} \right) \right] + h(N),
\]

where \( S_i = \sum_{y=1}^M N_{iy} \), and where \( N \) is the matrix of the random variables \( N_{iy} \), supposing that \( h(N) \) exists.

**Proof:** See Appendix C

We make a few remarks about the preceding expression. First, it is a function of both the parametric family \( p(x, y; \theta) \) and the prior \( q(\theta) \). If the resulting distribution \( p(N) \) is simple, then one can compute \( h(\{W\}_S) \) relatively easily; otherwise, one may need to resort to numerics. Whether or not it is simpler to estimate numerically the terms in the expression or to directly estimate \( h(\{W\}_S) \) depends on the specific distributions in question. Second, as \( M \to \infty \), for continuous distributions the first two terms converge with probability one on \( d_I \log(2) \). Finally, even if the expression is difficult to compute, it establishes scaling laws on \( h(W) \), showing that it increases linearly in \( d_I \) for parametric families satisfying the preceding conditions and corresponding to continuous distributions.

We also can bound the entropy from above.

**Theorem 6.** For an interpolation set satisfying the conditions stated above, the posterior entropy satisfies

\[
h(\{W\}_S) \leq -d_I \log(M - 1).
\]

**Proof:** See Appendix C

A fortiori, this expression bounds the expected entropy over an interpolation map \( S(x, y) \) or the infimum over interpolation sets inside high-probability sets. As before, the entropy grows linearly with the interpolation dimension. Further, here we see that the entropy decays at least as fast as \( -M \log(M) \). While this bound is inexact, it is useful for rule-of-thumb estimation of the bounds when the true entropy is difficult to obtain. For example, in the multi-class Gaussian case, considered in Section V, we find that substituting the preceding bound in place of the exact differential entropy only negligibly impacts the resulting bounds.

Finally, once an interpolation map \( S(x, y) \) is identified, the expected entropy \( E_X[h(\{W\}_{S(X,y)})] \) can
be computed numerically via Monte Carlo methods, such as those presented in [38]. As long as one can sample easily from \( q(\theta) \), one can produce arbitrarily many samples of \( W(y|x) \) with which to estimate the entropy. One can further take the average over an interpolation map or the infimum over high-probability sets numerically. We emphasize that this computation need only be carried out once for a given distribution model. Then, the risk bounds can be evaluated for any \( n \).

### C. Sample Complexity Bounds

From Theorem 4, we can derive explicit sample complexity bounds for the \( L_1 \) and \( L_\infty \) Bayes risk. Substituting (17) into (15) and (16), we obtain the necessary conditions

\[
\log(n) \geq \frac{1}{t}E_\alpha[\log |I(\alpha)^{-1}|] + \frac{2}{t}(EX[h(\{W\}_{S(x,y)})] - h(\alpha)) + \frac{2d_I}{t} \left( \log \left( \frac{M}{2R_1} \right) - 1 \right) + \log(2\pi e) + o(1)
\]

for any \( y \) and for an interpolation map \( S(x,y) \). Similar bounds hold for the \( \gamma \)-Bayes risk. We can describe intuitively the terms in this more complicated expression. The expression involving the Fisher information matrix describes how many samples are needed on average to learn the minimal sufficient statistic \( \alpha \) and thus \( \theta \). However, the objective is to learn \( W \), and it is sufficient but perhaps not necessary to learn \( \theta \) in order to learn \( W \). The second term captures this notion with the difference between the entropies \( h(W_S) \) and \( h(\alpha) \); when \( h(\alpha) \) is bigger, the term corrects the number of samples needed. The following term is the slack term associated with the tolerance desired in the risks \( \tilde{R} \), and the final \( o(1) \) term arises from the approximation of \( I(Z^n; \theta) \).

Further, if we impose regularity conditions on the eigenvalues of \( I(\alpha) \) and the differential entropies \( h(\{W\})_S \) and \( h(\alpha) \), we derive the following bounds on the sample complexity.

**Proposition 1.** In addition to the conditions of Theorem 4, suppose that \( \frac{1}{t}E_\alpha[\log |I(\alpha)^{-1}|] \geq c_1 \), \( \mu(S) = M - 1 \), \( EX[h(W_{S(x,y)})]/d_I \geq -\log(M - 1) \), and \( h(\alpha)/t \leq c_2 \), for positive constants \( c_1 \) and \( c_2 \). Then,

\[
\tilde{R}_1 \geq \frac{1}{2} \left( \sqrt{\frac{1}{n}} \right)^{t/d_I} \cdot c_3(1 + o(1))
\]

\[
\tilde{R}_\infty \geq \frac{1}{M} \left( \sqrt{\frac{1}{n}} \right)^{t/d_I} \cdot c_3(1 + o(1))
\]

where \( c_3 = (2\pi e)^{t/2d_I} \exp(t/(2d_I)c_1 - t/d_Ic_2) \).

**Proof:** Algebraic manipulation on Theorem 4.

For the special case \( t = d_I \), in which the dimensionality of the minimal sufficient statistic \( \alpha \) is equal to the interpolation dimension, we obtain the order-wise rule \( \epsilon = \Omega(\sqrt{1/n}) \), which agrees with PAC bounds on sample complexity. In this case, it is both necessary and sufficient for the number of training samples to grow quadratically in the required precision. Even when the scaling laws are correct, however, precise estimates of the constants and higher-order terms are necessary to make useful predictions for small \( n \). Furthermore, we see that the \( L_1 \) and \( L_\infty \) errors differ by a factor of \( M/2 \).
D. Achievability

A natural question is whether matching upper bounds on the Bayes risk hold. The random coding and joint typicality arguments that prove achievability bounds for rate distortion do not apply to the supervised learning setting. Joint typicality arguments depend on repeated i.i.d. draws from the source distribution. The analogous situation in supervised learning would be to encode multiple i.i.d. posteriors, each associated with a different draw from $p(\theta)$. Of course, we consider only a single draw from $p(\theta)$. Furthermore, random coding arguments presuppose design control over the source code, which would correspond to design control over the distribution of the labeled data. We do not have such control, so these arguments do not apply.

The worst-case bounds of the PAC framework serve as upper bounds on the Bayes risk. When the interpolation dimension, the dimension of the parameter space, and the VC dimension are identical, both frameworks are order optimal in that they predict a $\sqrt{1/n}$ decay in the Bayes risk. However, the constants and higher order terms differ significantly, primarily because VC bounds are distribution agnostic while our bounds take the prior distribution $p(\theta)$ into account.

Asymptotic upper bounds on the Bayes risk can be derived via analysis of the plug-in estimate of the posterior from $\theta$. Suppose the posterior is Lipschitz continuous in the $L_1$ or $L_\infty$ norm with respect to $\theta$ and the conditions of Theorem 4 hold. Then central limit theorem implies that the estimation error of $\hat{\theta}$ has variance scaling as $1/n$ and depending on the Fisher information matrix. By the Lipschitz assumption, the Bayes risk scales as $1/\sqrt{n}$, with constants and higher-order terms depending on the Fisher information matrix and the Lipschitz constant.

V. Numerical Examples

A. Binary Gaussian Classifier

First, we consider a binary Gaussian setting. Let $M = 2$ and $p(y) = 1/2$, $y \in \{1, 2\}$, and let $\Lambda = \mathbb{R}^d$ parameterize the data distributions. The class-conditional densities are Gaussian with antipodal means:

$$
p(x|y = 1; \theta) = \mathcal{N}(\theta, \sigma^2 I)
$$

$$
p(x|y = 2; \theta) = \mathcal{N}(-\theta, \sigma^2 I),
$$

where $\sigma^2 > 0$ is the known variance. We choose the prior $q(\theta) = \mathcal{N}(0, (1/d)I)$.

The MAP classifier for this problem is a hyperplane passing through the origin and normal to $\theta$. The VC dimension for this family of classifiers is $d$, suggesting that $d$ samples are necessary for good learning performance. This may be pessimistic. For $\sigma \ll d$, a single labeled sample is sufficient to obtain $\theta$ and recover the optimum classifier.

In this setting, we can derive the rate-distortion bounds on the $L_1$ and $L_\infty$ Bayes risk in closed form. First, one can verify that $\theta$ is itself a minimal sufficient statistic, and $I(\theta) = 1/\sigma^2 I$. It is also immediate that $h(\theta) = d/2 \log(2\pi e/d)$. Combining these facts with Theorem 4, we obtain

$$I(Z^n; \theta) = \frac{d}{2} \log \left( \frac{n}{d\sigma^2} \right) + o(1).$$
However, in this case we can evaluate the mutual information in closed form. Because the prior \( q(\theta) \) is Gaussian, the posterior is not only asymptotically Gaussian but also Gaussian for any \( n \). Let \( T_i = \theta + N_i \), where \( N_i \sim \mathcal{N}(0, \sigma^2 I) \). Simple calculation reveals that

\[
I(Z^n; \theta) = I(T^n; \theta) = \frac{d}{2} \log \left( 1 + \frac{n}{d\sigma^2} \right).
\]

The discrepancy between the estimated and exact mutual information is negligible unless \( n \ll d\sigma^2 \).

Next, we bound the entropy of the posterior, which by Bayes rule is the sigmoid of the inner product of \( \theta \) with \( x \):

\[
W(y = 1|x; \theta) = \frac{1}{1 + \exp(-2/\sigma^2 x^T \theta)}.
\]

To interpolate \( W \), it is sufficient to sample \( W \) at any basis of \( \mathbb{R}^d \) for any value of \( y \). Therefore, we choose the interpolation map

\[
S(x, y) = \{x, x^2, \ldots, x^d\} \times \{y\},
\]

where \( \{x, x^2, \ldots, x^d\} \) is an orthogonal basis and where \( \|x_i\| = \|x\| \) for every \( i \). The interpolation dimension and the interpolation subdimension are both equal to \( d \). Choosing orthogonal points makes the samples statistically independent, and in the following theorem we present a bound on the expected differential entropy.

**Theorem 7.** The expected differential entropy of the posterior evaluated at the interpolation map is

\[
E_X[h(\{W\}_S(x, y))] \geq \frac{d}{2} \psi(d/2) + \frac{d}{2} \log \left( \frac{16\pi(1/(d\sigma^2) + 1)}{d\sigma^2} \right) - \frac{d\Gamma((d+1)/2)}{\Gamma(d/2)} \sqrt{\frac{4(1/(d\sigma^2) + 1)}{\pi d\sigma^2}} - \frac{3d}{2} - 2d \log(2),
\]

where \( \Gamma(\cdot) \) is the Gamma function, and \( \psi(\cdot) \) is the digamma function, and where \( \nu(d, \sigma^2) \) is defined as the lower bound divided by \( d \). Let \( \nu(d, \sigma^2) \) denote the preceding bound divided by \( d \).

Combining these results, and recalling that the \( L_1 \) and \( L_\infty \) errors are identical for \( M = 2 \), we obtain the following bound on the \( L_1 \) Bayes risk:

\[
R_1(\delta) = R_\infty(\delta) \geq \sqrt{\frac{\sigma^2 d}{\sigma^2 d + n}} \exp(\nu(d, \sigma^2) - 1).
\]

Therefore, we expect expect the \( L_1 \) and \( L_\infty \) Bayes risk, and therefore the expected loss \( L(w, w_{MAP}) \), to be small when \( n \gg d\sigma^2 \). This result is intuitive. When the signal dimension is high, it will in general require more samples to train a classifier, but this requirement can be mitigated by small model noise variance. We emphasize that the preceding bound holds for all \( n \); one does not need to ensure that \( n > d \), as is necessary in the PAC framework.

In Figure 3, we evaluate the bounds numerically. We select \( d = 50 \) and \( \sigma^2 \in \{0.1, 0.5, 1\} \), and estimate the relevant quantities from \( 200 \times 200 \) draws of \( \theta \), the training set \( Z^n \), and test points \((x, y)\). We plot the
bound on $R$, the empirical average of the losses, the empirical average of the excess classification error $L_c(w; w_{\text{MAP}})$, and the PAC bound derived from the VC dimension. For the empirical quantities, we find the MAP estimate of $\theta$ from the $Z^n$ and plug the result into the posterior $W(y|x; \theta)$. The gap between the rate-distortion bound and empirical performance depends on $\sigma^2$, and the gap narrows with increasing $\sigma^2$. In either case, the PAC bound significantly overestimates the classification error, whereas the Bayes risk bound is on the same order as $R_c$.

**B. Multi-class Gaussian**

Next, we consider a multi-class Gaussian setting. Let $p(x|y; \theta) = \mathcal{N}(\theta_y, \sigma^2 I)$, where $\theta \in \mathbb{R}^{dM}$, and each $\theta_y \in \mathbb{R}^d$ is the $y$th sub-vector of length $d$ and denotes the mean of the $y$th class. Let $p(y) = 1/M$ and $q(\theta) = \mathcal{N}(0, 1/dI)$. The Fisher information matrix is again $I(\theta) = 1/\sigma^2 I$, and $h(\theta) = Md/2 \log(2\pi e/d)$. 

Fig. 3: Bayes risk bounds and empirical performance for binary Gaussian classification.
However, again we can compute the mutual information directly, yielding
\[
I(Z^n; \theta) = \frac{Md}{2} \log \left( 1 + \frac{n}{d\sigma^2 M} \right)
\] (29)

To find the interpolation dimension and an interpolation map, consider the posterior:
\[
W(y|x; \theta) = \frac{\exp \left( -\frac{1}{2\sigma^2} \|x\|^2 + \|\theta_y\|^2 - 2x^T\theta_y \right)}{\sum_{y'=1}^M \exp \left( -\frac{1}{2\sigma^2} \|x\|^2 + \|\theta_{y'}\|^2 - 2x^T\theta_{y'} \right)}.
\] (30)

Due to normalization, it is necessary and sufficient to be able to recover the numerator of \(W(y|x; \theta)\) for every \(x\) up to a multiplicative constant in \(y\). To do so, we first evaluate \(W(y|x; \theta)\) at \(\{0\} \times [M-1]\), which gives us access to \(\exp(-1/(2\sigma^2) \|\theta_y\|^2)\) up to a constant. Then, we evaluate \(W(y|x; \theta)\) at the Cartesian product of \([M-1]\) and any orthogonal basis of \(\mathbb{R}^d\). This gives us access to \(\exp(-1/(2\sigma^2)(1+\|\theta_y\|^2-2\theta_y)).\) Because we have isolated a basis for \(\mathbb{R}^d\), as well as the mean norms \(\|\theta_y\|^2\), we can solve for the numerator of the posterior for any \((x, y)\) pair up to a constant that disappears upon normalization. For \(M \leq d+1\), this is both sufficient and necessary to recover the posterior, and the interpolation dimension is \(d_I = (M-1)(d+1)\). Therefore, define the interpolation map as
\[
S(x, y) = \{0, x, x_2, x_d\} \times [M-1],
\] (31)

where \(x, x_2, \ldots, x_d\) are an orthogonal basis, and \(\|x_i\| = \|x\|\) for every \(i\). The measure-zero set \(\mathcal{W} = \{0\} \times [M-1]\) appears in infinitely many interpolation sets, in violation of the conditions of Definition 3. Therefore, the interpolation subdimension is \(d_S = (M-1)d\).

In this case, we cannot solve for the differential entropy of the posterior in closed form. We estimate \(EX[h(\{W\}_{S(X,y)})]\) numerically as described in [38]. The resulting bounds on the \(L_1\) and \(L_\infty\) Bayes risk are
\[
\tilde{R}_1 \geq \frac{M}{2} \left( \sqrt{\frac{dM\sigma^2}{n + dM\sigma^2}} \right)^{M-1} \exp \left( \frac{EX[h(\{W\}_{S(X,y)})]}{d(M-1)} - 1 - 1/d \right) \left( \frac{1}{2} \right)^{1/d}
\]
\[
\tilde{R}_\infty \geq \left( \sqrt{\frac{dM\sigma^2}{n + dM\sigma^2}} \right)^{M-1} \exp \left( \frac{EX[h(\{W\}_{S(X,y)})]}{d(M-1)} - 1 - 1/d \right) \left( \frac{1}{2} \right)^{1/d}.
\]

Observe that the bounds on the Bayes risk scale slightly faster than \(\sqrt{1/n}\) as suggested in Theorem 4 owing to the discrepancy between the dimension of the parameter space and the interpolation dimension. Careful numerical experiments, omitted for brevity, lead us to believe that both the \(L_1\) and \(L_\infty\) risk scale as \(\sqrt{1/n}\). Thus the scaling law predicted by the rate-distortion bounds is slightly optimistic, with the discrepancy decreasing as \(M\) increases.

In Figures 4-6, we evaluate the bounds numerically. We again choose \(d = 100\), \(\sigma^2 \in \{0.01, 0.1, 0.5\}\), and here we take \(M \in \{5, 10, 15\}\). We plot the bounds and empirical Bayes risk and the empirical average of \(L(w, w_{\text{MAP}})\), again over \(200 \times 200\) samples of \(\theta\), the training set, and the test point, and again using the MAP estimate of \(\theta\) from \(Z^n\) to compute the empirical quantities. We also consider the PAC bound given by the Natarajan dimension [18]. Following [19], we estimate the Natarajan dimension as \(Md\), and we plot the resulting bound.
We observe a few phenomena. First, the gap between the predicted $L_1$ and $L_\infty$ risk grows with increasing $M$, as predicted. As $M$ increases, the bound on the $L_\infty$ Bayes risk becomes optimistic by a few orders of magnitude, while the bound on the $L_1$ risk remains a reasonable prediction of the classification risk for all values of $M$. The PAC bound, on the other hand, is pessimistic in all regimes. Finally, the empirical $L_1$ and $L_\infty$ errors are closer together than predicted by theory, although the gap does grow in $M$. This is because the estimation error of the posterior $W(y|x; \theta)$ tends, empirically, to be “peaky” in $y$, concentrating around a single value, leading to similar errors regardless of norm. The bounds on $\tilde{R}_1$ and $\tilde{R}_\infty$, on the other hand, optimistically suppose the error is evenly spread among the classes, in which case the norms differ substantially. The extent to which this result generalizes beyond this scenario is a topic for further investigation. One possibility is that an estimate of the posterior chosen explicitly to minimize the Bayes risk, instead of simply taking the MAP estimate of $\theta$, may bring the empirical performance more in line with theory.

![Graphs](image-url)

Fig. 4: Bayes risk bounds and empirical performance for Gaussian classification with $M = 5$. 
VI. CONCLUSION

We have presented a framework, inspired by rate-distortion theory, for computing Bayesian estimates of the sample complexity in supervised learning. By contrast the existing PAC framework, the bounds are relevant when the training set is small, depend on the distribution of the data, and apply readily to multi-class settings. We evaluated the bounds in the binary and multi-class Gaussian settings. In the multi-class setting, we found empirically that the $L_1$ and $L_\infty$ Bayes risk are comparable. The bounds on the $L_1$ Bayes risk, which are strictly larger than the $L_\infty$ bounds for $M > 2$, are therefore tighter and appear to be more useful in practice.

Future work includes extending the rate-distortion framework to unsupervised learning and non-parametric Bayes settings, as well as developing criteria for model selection, similar to structural risk minimization, based on quantities such as the interpolation dimension and the posterior entropy.
Fig. 6: Bayes risk bounds and empirical performance for Gaussian classification with $M = 15$.

APPENDIX A

PROOF OF THEOREMS 1 AND 2

The proof of Theorem 1 involves the maximization of the entropy of the posterior estimation error

$U(y|x) \triangleq W(y|x; \theta) - W(y|x)$ subject to the $L_1$ constraint, for which we prove a lemma.

Lemma 8. Let $S$ be an interpolation set with $|S| = d_I(W)$. Then, for any learning rule $\delta$ such that $\bar{R}_1(\delta, \gamma) \leq \epsilon$, the differential entropy of $\{U\}_S = \{W - \hat{W}\}_S$ satisfies

$$h(\{U\}_S) \leq d_I(W) \left( 1 + H(\gamma) + (1 - \gamma) \log \left( \frac{\epsilon}{M} \right) \right).$$

When $\mu(S) = M - 1$, the differential entropy further satisfies

$$h(\{U\}_S) \leq d_I(W) \left( 1 + H(\gamma) + (1 - \gamma) \log \left( \frac{\epsilon}{M} \right) \right).$$
Proof: We start by bounding
\[ h(\{U\}_S) \leq \sum_{(x,y) \in S} h(U(y|x)), \]  
which holds with equality if and only if each \( U(y|x) \) is independent for all \((x,y) \in S\). By hypothesis, \( U(y|x) \) has \( L_1 \) norm no greater than \( \epsilon \) with probability \((1 - \gamma)\), or
\[ \sum_{y=1}^{M} |U(y|x)| \leq \epsilon, \]  
for every \( x \in \mathbb{R}^d \), and its \( L_1 \) norm is no greater than \( M \) for the remaining set, which has probability \( \gamma \).

To maximize the sum of individual entropies, we apply the \( L_1 \) constraint only to those points \( x \in \mathbb{R}^d \) that appear in \( S \). To that end, let
\[ S(x) = \{ y : (x,y) \in S \}, \]  
be the values of \( y \) corresponding to each \( x \) represented in \( S \). This leads to the optimization problem
\[
\begin{align*}
\text{maximize} & \quad \sum_{y \in S(x)} h(U(y|x)) \\
\text{subject to} & \quad \sum_{y \in S(x)} |U(y|x)| \leq \epsilon \text{ with probability } (1 - \gamma), \\
& \quad \epsilon < \sum_{y \in S(x)} |U(y|x)| \leq M \text{ with probability } \gamma.
\end{align*}
\]

Each \( U(y|x) \) is subject to an absolute value constraint. It is straightforward to show that the maximum-entropy distribution is a mixture of uniform distributions, one corresponding to the constraint that holds with \( 1 - \gamma \), and one corresponding to the constraint that holds with probability \( \gamma \). The resulting density is
\[
p_U(U(y|x)) = \begin{cases} 
\frac{(1-\gamma)\mu(S)}{2\epsilon} & \text{if } |U| \leq \frac{\epsilon}{\mu(S)} \\
\frac{\gamma\mu(S)}{2(M-\epsilon)} & \text{if } \frac{\epsilon}{\mu(S)} < |U| \leq \frac{M}{\mu(S)}.
\end{cases}
\]

Therefore, each entropy is bounded by
\[
h(U(y|x)) \leq \gamma \log \left( \frac{2(M-\epsilon)}{\gamma\mu(S)} \right) + (1 - \gamma) \log \left( \frac{2\epsilon}{(1-\gamma)\mu(S)} \right) \\
\leq 1 + \gamma \log \left( \frac{M}{\mu(S)} \right) + H(\gamma) + (1 - \gamma) \log \left( \frac{\epsilon}{\mu(S)} \right).
\]

Substituting this bound into (32), we obtain the result
\[
h(\{U\}_S) \leq d_I \left( 1 + \gamma \log \left( \frac{M}{\mu(S)} \right) + H(\gamma) + (1 - \gamma) \log \left( \frac{\epsilon}{\mu(S)} \right) \right).
\]

Next, consider the case \( \mu(S) = M - 1 \). Let \( S_i(x) = [M] \setminus \{i\} \), where \([M] := \{1, \ldots, M\}\). Since \( \mu(S) = (M - 1) \), for all \( i \in [M] \), the set of pairs \((x,y)\) such that \( y \in S_i(x) \) would also lead to an interpolation set. We need to simultaneously solve the maximization of (32) subject to (33) for all interpolation sets \( S_i \) for \( i \in [M] \) by solving several parallel problems, one for each \( x \) represented in \( S \).
Due to symmetry we can express the resulting problem as:

$$\begin{align*}
\text{maximize}_{p(U)} & \quad \frac{M-1}{M} \sum_{y \in [M]} h(U(y|x)) \\
\text{subject to} & \quad \sum_{y \in [M]} |U(y|x)| \leq \epsilon \text{ with probability } (1 - \gamma), \\
& \quad \epsilon < \sum_{y \in [M]} |U(y|x)| \leq M \text{ with probability } \gamma,
\end{align*}$$

where the sum for the $L_1$ constraint only includes those $y \in S(x)$ because those points do not appear in the entropy expression; for points $(x, y)$ such that $y \neq S(x)$, we can set $U(y|x) = 0$ without loss in the objective function. Again each $U(y|x)$ is subject to an absolute value constraint, so the maximum-entropy distribution is again a mixture of uniforms. Here, however, there are $M$ random variables in the sum constraint, and the maximizing density is

$$f_U(U(y|x)) = \begin{cases} 
\frac{1-\gamma}{2M} & \text{if } |U| \leq \frac{\epsilon}{M} \\
\frac{\gamma}{2(M-\epsilon/M)} & \text{if } \frac{\epsilon}{M} < |U| \leq 1.
\end{cases}$$

Computing the entropy, we obtain the result.

Now we are ready to prove Theorem 1.

**Proof of Theorem 1.** First, observe the following Markov chain:

$$\{W(y|x; \theta)\}_S \rightarrow \theta \rightarrow Z^n \rightarrow \{\hat{W}(y|x)\}_S.$$  \hfill (37)

Therefore, the data processing inequality yields

$$I(Z^n; \theta) \geq I(\{W(y|x; \theta)\}_S; \{\hat{W}(y|x)\}_S) \geq \inf_{p(W|\theta)} I(\{W(y|x; \theta)\}_S; \{\hat{W}(y|x)\}_S),$$

where the latter infimum is over all conditional distributions of $\hat{W}$ satisfying the constraint on $\hat{R}_1(\delta)$.

$$I(Z^n; \theta) \geq \inf_{p(W|\theta)} h(\{W(y|x; \theta)\}_S) - h(\{W(x; \theta)\}_S|\{W(y|x; \theta) + U(y|x)\}_S)$$

$$\geq \inf_{p(U)} h(\{W(y|x; \theta)\}_S) - h(\{U(y|x)\}_S).$$

Using Lemma 8 to bound $h(\{U(y|x)\}_S)$ (or $h(\{U(y|x)\}_{S'})$ for some other interpolation set $S'$), and taking the infimum over all sets of probability at least $1 - \gamma$, we obtain the result.

We remark briefly on the tightness of the data processing step. Because $S$ is an interpolation set, $\{W\}_S$ completely characterizes $W$; similarly, if we suppose that $\hat{W}$ is chosen from the parametric family, then $\{\hat{W}\}_S$ also completely characterizes $\hat{W}$, and $I(W; \hat{W}) = I(\{W\}_S; \{\hat{W}\}_S)$. However, the bound in Theorem 1 holds even when $\hat{W}$ is not chosen from the parametric family. This suggests that the bound is even looser for $\hat{W}$ chosen outside of the family, meaning that the error can only be worse if we relax the family of distributions over which we estimate the posterior.
To prove Theorem 2, we need to bound the entropy of $U$ under the $L_{\infty}$ loss, which we accomplish in the following lemma.

**Lemma 9.** Let $S$ be an interpolation set with $|S| = d_I(W)$. For any learning rule such that $\hat{R}_{\infty}(\delta) \leq \epsilon$, the differential entropy of $\{U\}_S = \{W - \hat{W}\}_S$ satisfies

$$h(\{U\}_S) \leq d_I(W) \left(1 + H(\gamma) + (1 - \gamma) \log(\epsilon)\right).$$

**Proof:** As in the $L_1$ case, we bound the differential entropy by supposing each $U(y|x)$ to be independent:

$$h(\{U\}_S) \leq \sum_{(x,y) \in S} h(U(y|x)),$$

where here $U(y|x)$ is subject to an $L_{\infty}$ constraint, thus $U(y|x)$ must satisfy a magnitude constraint for each point $(x,y) \in S$ with probability $(1 - \gamma)$:

$$2|U(x,y)| \leq \epsilon, \forall (x,y) \in S,$$

and with probability $\gamma$, $|U(x,y)| \leq 1$. Again each differential entropy is maximized via a uniform distribution conditioned on each of the intervals, but in this case each point independently satisfies the same constraint. Substituting this result into the sum of entropies, the result follows.

Next, we give the proof of Theorem 2.

**Proof of Theorem 2.** As in the proof of Theorem 1, we bound the mutual information $I(Z^n; \theta)$ via the data-processing inequality and the bounds on conditional entropy. Following that argument, we arrive at the condition

$$I(Z^n; \theta) \geq \inf_{p(U)} h(\{W(y|x; \theta)\}_S) - h(\{U(y|x)\}_S), \quad (43)$$

where the infimum is subject to the $L_{\infty}$ constraint. Using Lemma 9 to bound $h(\{U(y|x)\}_S)$, and taking the infimum over all sets of probability at least $1 - \gamma$, we obtain the result.

**APPENDIX B**

**PROOF OF THEOREM 3**

**Proof of Theorem 3.** Following the data-processing inequality arguments of the previous theorems, we obtain

$$I(Z^n; \theta) \geq \sup_S \inf_{p(U)} h(\{W\}_S) - h(\{U\}_S) \geq \inf_{x,y} h(\{W\}_{S(x,y)}) - h(\{U\}_{S(x,y)}) \geq E_x \left[ \inf_{p(U)} h(\{W\}_{S(x,y)}) - h(\{U\}_{S(x,y)}) \right],$$

for any $y$. Similar to the proofs of Theorems 1 and 2, each $U(y|x)$ is subject to an absolute value constraint. However, in this case the constraints hold in expectation rather than pointwise. Therefore, for any interpolation set $S(x,y)$, the differential entropy $h(\{U\}_{S(x,y)})$ is maximized when each $U(y|x)$ has a
Laplace distribution. Let the scale parameter of each Laplace distribution be \( \epsilon(y|x) \). Then, we have
\[
I(Z^n; \theta) \geq E_X[h(\{W\}_S(x,y))] - \sup_{\epsilon(y|x)} E_X \left[ \sum_{i=1}^{d_l} (\log(2\epsilon(s_i(X,y))) + 1) \right]. \tag{44}
\]

It remains to take the supreme of the sum of logarithm terms while respecting the Bayes risk constraint. For simplicity, we perform this maximization over a countable partition of \( \mathbb{R}^n \); thus \( x \) takes countably many values and \( p(x) \) is a mass function rather than a density. By the Riemann integrability of \( p(x) \), the result follows from taking the limit of such partitions.

In the case of the constraint \( \bar{R}_1 \leq \epsilon \), the associated Lagrangian is
\[
L(\epsilon(y|x), \lambda) = \sum_x p(x) \sum_{i=1}^{d_l} (\log(\epsilon(s_i(x,y))) + 1) + \lambda \left( \sum_x p(x) \sum_{y=1}^{M} \epsilon(y|x) - \epsilon \right). \tag{45}
\]
Let \( \mathcal{W} \) denote the measure-zero set for which the conditions in Definition 3 do not hold. For such points, we can set \( \epsilon(y|x) = 1 \) without impacting the Bayes risk constraint. For any point \( (x, y) \notin \mathcal{W} \), differentiating with respect to \( \epsilon(y|x) \) and setting the result to zero yields
\[
\frac{k p(x)}{\epsilon(y|x)} + \lambda p(x) = 0, \tag{46}
\]
where \( k \) is the number of interpolation sets in which \( (x, y) \) appears. By Definition 3, \( k \) is nonzero, finite, and constant. This implies that the optimum \( \epsilon(y|x) \) is a constant in \( x \). Furthermore, the preceding holds for any \( y \), so \( \epsilon(y|x) \) is constant in \( y \). In order to satisfy the constraint \( \bar{R}_1 \leq \epsilon \), this implies \( \epsilon(y|x) = \epsilon/M \). Substituting this into (44) yields the result.

In the case of the constraint \( \bar{R}_\infty \leq \epsilon \), it is clearly optimum to choose \( \epsilon(y|x) \) to be constant in \( x \), or \( \epsilon(y|x) = \epsilon(x) \). With a slight abuse of notation, the Lagrangian in this case is
\[
L(\epsilon(x), \lambda) = \sum_x p(x) \sum_{i=1}^{d_l} (\log(\epsilon(s_i(x)))) + 1) + \lambda \left( 2 \sum_x p(x) \epsilon(x) - \epsilon \right). \tag{47}
\]
Differentiating and setting equal to zero again shows that the optimum \( \epsilon(x) \) is a constant, which implies \( \epsilon(y|x) = \epsilon/2 \) and yields the result.

\section*{Appendix C}
\textbf{Proof of Theorem 5}

Proof of Theorem 5. By Bayes’ rule, the posterior at each pair in the interpolation set is
\[
W(x_i|y_i; \theta) = \frac{N_{iy}}{\sum_{y=1}^{M} N_{iy}}.
\]
In order to compute \( h(\{W\}_S) \), we need to compute the density \( p(\{W\}_S) \). A standard result \cite{39} is that the density is
\[
p_W(\{W\}_S) = |J| p_N(f(\{W\}_S)), \tag{48}
\]
where \( f \) is the one-to-one function mapping the samples \( \{W\}_S \) to the terms \( N_{iy} \), and were \( J \) is the Jacobian matrix of \( f^{-1} \). \( W_i = (W_{i1}, \ldots W_{iM-1}) \) and \( N_i = (N_{i1}, \ldots, N_{iM-1}) \). Because of the normalization
constraint, the random variables $W_{iy}$ are overdetermined by the random variables $N_{iy}$. Therefore, without loss of generality we can take $N_{iM} = 1$ for every $i$. Then, it is straightforward to show that the mapping between $W_i$ and $N_i$ is

$$N_i = W_i \left(1 + \frac{S_i}{1 + S_i} \right) \triangleq g(W_i),$$

(49)

where $S_i = \sum_{y=1}^{M-1} W_{iy}$. Taking derivatives, the determinant of the Jacobian, denoted $|J_i|$, is

$$|J_i| = \left(\frac{1 + 2S_i}{1 + S_i}\right)^{M-1} \left(1 + \frac{S_i}{(1 + S_i)(1 + 2S_i)}\right).$$

(50)

Now, let $W$ be the matrix of all vectors $W_i$ and $N$ be the matrix of all vectors $N_i$. Clearly the Jacobian is block diagonal, thus the Jacobian of the entire mapping, denoted $|J|$, is

$$|J| = \prod_{i=1}^{k} \left(\frac{1 + 2S_i}{1 + S_i}\right)^{M-1} \left(1 + \frac{S_i}{(1 + S_i)(1 + 2S_i)}\right).$$

(51)

The density of the matrix $W$ is thus $|J|p(N)$, where $p(N)$ is the density of the matrix $N$. The resulting entropy is therefore

$$h(\{W\}_S) = -E[\log p(W)] = -\sum_{i=1}^{k} (M - 1)E\left[\log \left(\frac{1 + 2S_i}{1 + S_i}\right)\right] - \sum_{i=1}^{k} E\left[\log \left(1 + \frac{S_i}{(1 + S_i)(1 + 2S_i)}\right)\right] + h(N).$$

(52)

Next, we prove Theorem 6.

**Proof of Theorem 6.** The proof follows the same structure as the proof of Lemma 8. The entropy $h(\{W\}_S)$ is bounded above by the sum of the individual entropies. Furthermore, as $W(y|x; \theta)$ is normalized and non-negative, the sum of the posterior over $y$ for a fixed $x$ must be equal to one. As the interpolation dimension is equal to $M - 1$ by hypothesis, the entropy is maximized by letting each random variable be uniformly distributed across $[0, 1/(M - 1)]$, and the result follows.

**Appendix D**

**Proof of Theorem 7.** For the interpolation set $S(x, y)$, let $c = \|x\|$, and let $W_i = W(s_i(x, y))$ denote the posterior evaluated at the $i$th element of the set determined by the interpolation map. Without loss of generality, suppose $y = 1$. Further, let $Z_i = (2x_i^T \theta)/\sigma^2$. Straightforward computation shows that

$$Z_i \sim \mathcal{N} \left(0, \frac{4c^2}{\sigma^4 d}\right),$$

(53)

and

$$W_i = \frac{1}{1 + \exp(-Z_i)}.$$

(54)
Our first objective is to find the density \( p(W_i) \). Using the Jacobian formula,

\[
p(W_i) = J(f) \cdot p_N(f(W_i)),
\]

where

\[
f(W_i) = \log \left( \frac{W_i}{1-W_i} \right)
\]
is the mapping from \( W_i \) to \( N_i \), and \( J(f) \) is the Jacobian of \( f \), which is equal to

\[
J(f) = \frac{\partial}{\partial W_i} f(W_i) = \frac{1}{W_i(1-W_i)}.
\]

Therefore,

\[
p(W_i) = J(f) \cdot N \left( f(W_i), \frac{4e^2}{\sigma^2 d} \right)
= \frac{1}{W_i(1-W_i)} \exp \left( -\frac{\sigma^2 d}{8c^2} \log^2 \left( \frac{W_i}{1-W_i} \right) \right).
\]

Next, the differential entropy is

\[
h(W_i) = -E[\log(p(W_i))]
= \frac{1}{2} \log \left( \frac{8\pi c^2}{d\sigma^4} \right) + E[\log(W_i(1-W_i))] + E \left[ \frac{\sigma^2 d}{8c^2} \log^2 \left( \frac{W_i}{1-W_i} \right) \right].
\]

Observe that

\[
W_i(1-W_i) = \frac{\exp(-Z_i)}{(1+\exp(-Z_i))^2},
\]

and

\[
\frac{W_i}{1-W_i} = \exp(Z_i).
\]

Therefore,

\[
h(W_i) = \frac{1}{2} \log \left( \frac{8\pi c^2}{d\sigma^4} \right) - E[Z_i] + \frac{\sigma^4 d}{8c^2} E[Z_i^2] - 2E[\log(1+\exp(-Z_i))]
\ge \frac{1}{2} \log \left( \frac{8\pi c^2}{d\sigma^4} \right) + \frac{1}{2} - 2E[\log(1+\exp(-Z_i))]
= \frac{1}{2} \log \left( \frac{8\pi c^2}{d\sigma^4} \right) - 2 \log(2) - \sqrt{\frac{2e^2}{\pi d\sigma^4}}
\]

where (66) follows from the fact that \( 1 + e^{-x} < 2e^{-x} \) for any \( x < 0 \) and \( 1 + e^{-x} \leq 2 \) for any \( x \geq 0 \). Furthermore, (67) follows from the fact that \( E[|Z_i|^+] \) is exactly half the expectation of the “folded” Gaussian, which is well known. This bound is tight to within a constant gap of \( \log(2) \).

Our final step is to take the expectation over \( X \), which has a white Gaussian distribution with per-element variance \( 1/d + \sigma^2 \). Define the random variables \( C = \|X\| \) and \( V = C^2/(1/d + \sigma^2) \), which
yields
\[ E_X[h(W_i)] \geq E \left[ \frac{1}{2} \log \left( \frac{8\pi (1/(d\sigma^2) + 1)V}{d\sigma^2} \right) - \sqrt{\frac{2(1/(d\sigma^2) + 1)V}{\pi d\sigma^2}} \right] - \frac{3}{2} - 2\log(2). \] (68)

By definition, \( V \sim \chi^2(d) \), so in order to evaluate the preceding expectation we need to compute the mean of a \( \chi \)-distributed random variable and the expected logarithm of a \( \chi^2 \) random variable. These quantities are well-known, and applying them yields
\[ E_X[h(W_i)] \geq \frac{1}{2} \psi(d/2) + \frac{1}{2} \log \left( \frac{16\pi (1/(d\sigma^2) + 1)}{d\sigma^2} \right) - \frac{\Gamma((d+1)/2)}{\Gamma(d/2)} \sqrt{\frac{4(1/(d\sigma^2) + 1)}{\pi d\sigma^2}} - \frac{3}{2} - 2\log(2), \] (69)

where \( \Gamma(\cdot) \) is the Gamma function, and \( \psi(\cdot) \) is the digamma function.

\[ \beth \]

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