The auxiliary field method in quantum mechanical four-fermi models: a study towards chiral condensation in QED

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Abstract

A study for checking the validity of the auxiliary field method (AFM) is made in quantum mechanical four-fermi models which act as prototypes of models for chiral symmetry breaking in quantum electrodynamics. It has been shown that AFM, defined by an insertion of Gaussian identity to path integral formulae and by the loop expansion, becomes more accurate when taking higher order terms into account under the bosonic model with a quartic coupling in 0- and 1-dimensions as well as the model with a four-fermi interaction in 0-dimension. The case is also confirmed in terms of two models with the four-fermi interaction among $N$ species in 1-dimension (the quantum mechanical four-fermi models): higher order corrections lead us towards the exact energy of the ground state for a whole region including the weak as well as the strong coupling. It is found that the second model belongs to a 'WKB exact class' that has no higher order corrections other than the lowest correction. Discussions are also made for unreliability on the continuous time representation of path integration and for a new model of QED as a suitable probe for chiral symmetry breaking.

1. Introduction

In a strong coupling region of QED, $e \to \infty$, an expectation for generating electron mass is awakened, that is, chiral condensation characterized by the expectation value of electron field operators $\langle \bar{\psi} \psi \rangle$ becoming nonzero is anticipated [1–3]. It is a nonperturbative phenomenon so that one must utilize some technique other than the naive strong coupling expansion [2]. Traditionally, efforts rely on the Schwinger–Dyson equation with for example a (quenched) ladder approximation [1, 3] which, however, conflicts with gauge invariance even when improved by a renormalization group [4]. Miransky in [3] points out the phase boundary around the critical coupling $e^2 \sim 4\pi$, which is further examined by the lattice QED [5].
Numerical simulations, however, in terms of lattice QED suffer from well-known species doublers [6] and are still far from the satisfactory level.

Meanwhile there is a way which can preserve gauge invariance and serve nonperturbatively. It is dispensed by Gross–Neveu and Kugo and Kikkawa [7] under Nambu–Jona-Lasinio [8] (NJL) type model and is evolved by inserting an identity in terms of the Gaussian integration with respect to a fictitious field called an auxiliary field or the Hubbard–Stratonovich field [9] into a path integral expression (for example, an auxiliary field \(\sigma(x) \sim \bar{\psi}(x)\psi(x)\) is apparently gauge invariant). The prescription combined with the loop expansion [10] is simple and transparent and is called the auxiliary field method (AFM) [11].

If the number of fermion \(N\) in NJL-type models is large enough quantum corrections of the auxiliary field are negligible so that the gap equation, the saddle point of the auxiliary field, exhibits chiral symmetry breaking. So far almost all the results in field theories are obtained under this assumption, in other words, all studies are restricted within the gap equation [12]. However, if \(N\) becomes smaller, quantum corrections, that is, higher loop contributions may change the phase structure (into the unbroken one [13]). It is therefore indispensable to investigate the whole structure of AFM.

To this end, we have studied the role of auxiliary fields in bosonic cases with a quartic interaction in 0- and 1-dimensions [14] and in the 0-dimensional four-fermi case [15]; since those models can be solved exactly or numerically we can check the accuracy of the AFM result. We find that AFM does work excellently for almost all regions of coupling: \(10^{-3} < g^2 < 10^3\) [14] or \(10^{-3} < \lambda < 10\) [15] when we take higher loops into account. (It should be noted that the loop expansion results in data with the inverse coupling \(1/g^2\), \(1/\lambda\) indicating its nonperturbative structure and is the asymptotic expansion so that we should stop considering higher loops somewhere.) The remaining task is therefore to study a quantum mechanical four-fermi model in terms of AFM. After confirming validity in this case, we can proceed to examine QED by use of AFM.

In order to illustrate AFM, an outline of the 0-dimensional fermionic model is given [15], the target quantity is the partition function \(Z\),

\[
Z = \int d^N\xi d^N\xi^* \exp \left[ -\omega(\xi^* \cdot \xi) + \frac{\lambda^2}{2N}(\xi^* \cdot \xi)^2 \right],
\]

(1)

where

\[
d^N\xi \equiv d\xi_1 \cdots d\xi_N, \quad d^N\xi^* \equiv d\xi_N^* \cdots d\xi_1^*, \quad (\xi^* \cdot \xi) \equiv \sum_{i=1}^{N} \xi_i^* \xi_i,
\]

(2)

and the coupling constant \(\lambda^2\) is supposed real. We have introduced \(2N\)-Grassmann variables and the notation is followed from the textbook of [16]. Introduce an auxiliary field, \(y\), to kill the \((\xi^* \cdot \xi)^2\) term in equation (1), which can be realized by inserting the identity

\[
1 = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left\{ y + \frac{\lambda}{\sqrt{N}}(\xi^* \cdot \xi) \right\}^2 \right]
\]

(3)

to give

\[
Z = \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \int d^N\xi d^N\xi^* \exp \left[ -\frac{y^2}{2} - \left( \omega + \frac{\lambda}{\sqrt{N}}y \right)(\xi^* \cdot \xi) \right]
\]

\[
\cdot \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \left( \omega + \frac{\lambda}{\sqrt{N}}y \right)^N \exp \left[ -\frac{y^2}{2} \right]
\]

As is mentioned shortly in the discussion QED Lagrangian can be mapped into a four-fermi-type model.
\[ y = \sqrt{N} \int_{-\infty}^{\infty} \sqrt{\frac{N}{2\pi}} \, dy \exp[-NI(y)]. \] (4)

with

\[ I(y) \equiv \frac{y^2}{2} - \ln(\omega + \lambda y). \] (5)

Find a solution \( y_0 \) of \( \frac{dI}{dy} = 0 \), called the classical solution or the saddle point, then expand \( I(y) \) around \( y_0 \):

\[ I(y) = I_0 + \frac{I_0^{(2)}}{2}(y - y_0)^2 + \frac{I_0^{(3)}}{3!}(y - y_0)^3 + \cdots, \quad I_0^{(n)} \equiv I^{(n)}(y_0), \] (6)

and finally make a change of variable such that \( (y - y_0) \mapsto \frac{y}{\sqrt{N}} \) to obtain

\[ Z = \exp(-NI_0) \int_{-\infty}^{\infty} \frac{dy}{\sqrt{2\pi}} \exp\left[-\frac{1}{2}I_0^{(2)}y^2 + \frac{1}{N}I_0^{(3)}y^3 + O\left(\frac{1}{N}\right)\right]. \] (7)

The rest of the work is to perform a perturbation with respect to \( 1/N \) called the loop expansion parameter. The loop expansion is well known as the semiclassical or WKB approximation. In almost all cases, \( I_0^{(2)} \neq 0 \) so that \( 1/N \) expansion can be performed with the aid of the Gaussian integration\(^4\).

In this paper, we consider the quantum mechanical models of four-fermi type with \( N \) species, which can be solved analytically. The first one is given by the anti-normal-ordered form; since its Hamiltonian in path integral formulae is expressed as a function of the Grassmann numbers, \( \xi_j \xi_j^* \) not of \( \xi_j \xi_j^* - 1 \) (the normal-ordered form) or of \( \xi_j^* (\xi_j + \xi_{j-1})/2 \) (the Weyl-ordered form) \(^{19}\). We establish the result that AFM works well for an almost whole coupling region including \( \lambda^2 = 0 \), \( 0 \leq |\lambda|^2 \leq 10 \), by taking higher orders in the loop expansion as were the cases in the bosonic \(^{14}\) and the 0-dimensional fermi cases \(^{15}\), which is the content of the section 2. The second model is simpler consisting only of the number operator. Classically, there is no difference between these two models but here arises an interesting situation: all the higher order corrections seem to vanish (although we have checked it up to the two-loop order), which reminds us of the WKB exact models (see the papers \(^{20}\) and the references therein). The result that the lowest order approximation fits well to the exact value is also found for \( 0 \leq |\lambda|^2 \leq 10 \), which is the content of the section 3. The final section is devoted to the discussion where the failure of the path integral representation in continuous time and a new trial towards chiral symmetry breaking in QED are presented. In the appendix, the exact energy eigenvalue of the ground state is discussed.

2. Model 1

The starting Hamiltonian\(^5\) is

\[ H(\hat{a}^\dagger, \hat{a}) = -\omega \hat{a}^\dagger \cdot \hat{a}^\dagger + \frac{\lambda^2}{2N} \sum_{i,j=1}^{N} \hat{a}_i \hat{a}_j \hat{a}_j^\dagger \hat{a}_i^\dagger, \] (8)

\(^4\) There occurs, however, an interesting situation: \( I_0^{(2)} = 0 \), when \( \lambda^2 < 0 \) and \( |\lambda| = \omega/2 \), called caustic \(^{17,18}\). Utilizing a standard prescription with the Airy function in this region and \( 1/N \) expansion under the Gaussian integration in the other region, we can conclude that in a fairly wide range of the coupling, \( 10^{-3} < |\lambda| \leq 10 \), AFM does work satisfactorily even within the next leading approximation but more excellently if higher order effects would be taken into account.

\(^5\) We employ the anti-normal ordering in the Hamiltonian (8) because in the path integral representation we can have matched subscripts between \( \xi_j \) and \( \xi_j^* \): see \(^{19}\) and equation (18).
with \( \hat{a} \cdot \hat{a}^\dagger \equiv \sum_{i=1}^{N} \hat{a}_i \hat{a}_i^\dagger \). Here, \( \hat{a}_i \) and \( \hat{a}_i^\dagger \) (\( i = 1, 2, \ldots, N \)) are the creation and the annihilation operators of \( i \)th fermion, satisfying
\[
\{ \hat{a}_i, \hat{a}_j^\dagger \} = \delta_{ij}, \quad \{ \hat{a}_i, \hat{a}_j \} = \{ \hat{a}_i^\dagger, \hat{a}_j^\dagger \} = 0 \quad (i, j = 1, 2, \ldots, N).
\]
Introduce the number operator
\[
\hat{n} \equiv \hat{a}^\dagger \cdot \hat{a} = \sum_{i=1}^{N} \hat{a}_i^\dagger \hat{a}_i,
\]
whose eigenstates are
\[
|n, r\rangle = n|n, r\rangle \quad (n = 0, 1, 2, \ldots, N; r = 1, 2, \ldots, \binom{N}{n})
\]
(11)
where \( r \) specifies the number of combinations of \( N \) elements taken \( n \) at a time without repetition.

The Hamiltonian (8) reads in terms of \( \hat{n} \) as
\[
H(\hat{a}^\dagger, \hat{a}) = -\left( \omega - \frac{\lambda^2}{2} + \frac{\lambda^2}{2N} \right) N + \left( \omega - \frac{\lambda^2}{2} + \frac{\lambda^2}{2N} \right) n + \frac{\lambda^2}{2N} n^2,
\]
whose eigenvalue, \( H(\hat{a}^\dagger, \hat{a})|n, r\rangle = E_n|n, r\rangle \), is
\[
E_n = -\left( \omega - \frac{\lambda^2}{2} + \frac{\lambda^2}{2N} \right) N + \left( \omega - \frac{\lambda^2}{2} + \frac{\lambda^2}{2N} \right) n + \frac{\lambda^2}{2N} n^2.
\]
The ground-state energy \( E_G \) is given as the lowest energy state out of \( N + 1 \) eigenenergies:
\[
E_G = \min_n E_n \quad (n = 0, 1, 2, \ldots, N),
\]
(15)
whose explicit calculations are relegated to the appendix.

Meanwhile \( E_G \) can be picked up from the partition function \( Z(T) \),
\[
Z(T) \equiv \text{Tr}(e^{-TH}) = \sum_{n=0}^{N} \sum_{r=1}^{\binom{N}{n}} |n, r\rangle e^{-TH} |n, r\rangle,
\]
(16)
such that
\[
E_G = -\lim_{T \to \infty} \frac{1}{T} \ln Z(T) = -\lim_{T \to \infty} \frac{1}{T} \ln \left( \sum_{n=0}^{N} \binom{N}{n} e^{-TE_n} \right),
\]
(17)
whose final expression apparently coincides with definition (15). \( Z(T) \) has a path integral representation,
\[
Z(T) = \lim_{N_t \to \infty} \prod_{j=1}^{N_t} \int d^N \xi_j \, d^N \xi_j^* \times \exp \left[ -\sum_{j=1}^{N} \left( (\xi_j^* \cdot \Delta \xi_j) + \Delta t \left( \omega (\xi_j^* \cdot \xi_j) + \frac{\lambda^2}{2N} (\xi_j^* \cdot \xi_j)^2 \right) \right) \right]_{\text{AP}},
\]
(18)
where \( \Delta t \equiv T/N_t \) and AP stands for the anti-periodic boundary condition, \( \xi_0 = -\xi_N \) [16].

Estimating \( Z(T) \) under AFM and comparing the results with the exact value, we can check the validity of AFM.
Introducing the auxiliary field $\sigma_j$ in terms of the Gaussian identity,

$$1 = \lim_{N_t \to \infty} \prod_{j=1}^{N_t} \int_{-\infty}^{\infty} \sqrt{\frac{\Delta t}{2\pi}} \, d\sigma_j \exp \left[ -\frac{\Delta t}{2} \sum_{j=1}^{N_t} \left\{ \sigma_j + \frac{i\lambda}{\sqrt{N}} (\xi_j^* \cdot \xi_j) \right\}^2 \right], \quad (19)$$

to erase the $(\xi^* \cdot \xi)^2$ term in equation (18) and performing the Grassmann integration $d^N \xi_j \, d^N \xi_j^*$, we obtain

$$Z(T) = \lim_{N_t \to \infty} \prod_{j=1}^{N_t} \int \sqrt{\frac{\Delta t}{2\pi}} \, d\sigma_j \exp \left[ -N I[\sigma] \right], \quad (20)$$

where

$$I[\sigma] = \frac{\Delta t}{2} \sum_{j=1}^{N_t} \sigma_j^2 - \ln \det \left\{ \delta_{ij} - \frac{i\lambda}{\sqrt{\Delta t}} (\xi_j^* \cdot \xi_j) \right\}. \quad (21)$$

As was explained in the introduction $1/N$ is the loop expansion parameter. When $N$ goes larger (although we will assume that $N$ is not large in the following), it is expected that the integral is dominated by the saddle point $\sigma_{0j}$ obeying the equation of motion,

$$0 = I^{(1)}_{j}[\sigma_0] \equiv \frac{\delta I[\sigma]}{\delta \sigma_j} \bigg|_{\sigma = \sigma_0} = \Delta t (\sigma_{0j} - i\lambda S_{jj}), \quad (22)$$

where the fermion propagator $S_{ij}$ obeys

$$\sum_{k=1}^{N_t} \{ \delta_{ik} - \delta_{i-1,k} + \Delta t (\omega + i\lambda \sigma_0) \delta_{ik} \} S_{kj} = \delta_{ij}. \quad (23)$$

Expand $I[\sigma]$ around $\sigma_{0j}$ such that

$$I[\sigma] = I[\sigma_0] + \frac{1}{2!} I^{(2)}_{ij}(\sigma_j - \sigma_{0j})(\sigma_j - \sigma_{0j}) \quad + \sum_{k=3}^{\infty} \frac{1}{k!} \sum_{j_1, j_2, \ldots, j_k=1}^{N_t} I^{(k)}_{j_1j_2\ldots j_k}(\sigma_{j_1} - \sigma_{0j})(\sigma_{j_2} - \sigma_{0j}) \cdots (\sigma_{j_k} - \sigma_{0j_k}), \quad (24)$$

where we have used the abbreviations

$$I^{(k)}_{j_1\ldots j_k} \equiv \frac{\delta^k I[\sigma]}{\delta \sigma_{j_1} \cdots \delta \sigma_{j_k}} \bigg|_{\sigma = \sigma_0}. \quad (25)$$

Shifting and scaling the integration variables, we obtain

$$Z(T) = e^{-N I[\sigma_0]} \lim_{N_t \to \infty} \prod_{j=1}^{N_t} \int \sqrt{\frac{\Delta t}{2\pi}} \, d\sigma_j \exp \left[ -\frac{1}{2!} \sum_{i,j=1}^{N_t} I^{(2)}_{ij} \sigma_i \sigma_j \right] \quad - \sum_{k=3}^{\infty} \frac{1}{k! N^{k/2-1}} \sum_{j_1, j_2, \ldots, j_k=1}^{N_t} I^{(k)}_{j_1j_2\ldots j_k} \sigma_{j_1} \sigma_{j_2} \cdots \sigma_{j_k}, \quad (26)$$

where

$$I^{(2)}_{ij} \equiv \Delta t \Delta_{ij}^{-1} = \Delta t (\delta_{ij} - \Delta t(\omega + i\lambda \sigma_0) \delta_{ij}), \quad (27)$$

with $\Delta_{ij}$ being a propagator of the auxiliary field and

$$I^{(3)}_{ijk} = i\lambda^3 (\Delta t)^3 (S_{ij} S_{kj} S_{lk} + S_{ik} S_{lj} S_{mj}), \quad (28)$$
Figure 1. Two-loop graphs: the vertices $I^{(4)}$ (28) and $I^{(4)}$ (29) are contracting to the points on the left-hand side to recognize the two-loop explicitly. ($\Delta$), the dotted line, denotes the propagator of the auxiliary field. On the right-hand side, the line designates the fermion propagator (23) and the numbers on the top in (a)–(e) denote multiplicity.

\[ I^{(4)}_{ijkl} = \lambda^4 (\Delta t)^4 (S_{ij}S_{jk}S_{kl}S_{li} + S_{ij}S_{ji}S_{lk}S_{il} + S_{ik}S_{kj}S_{ji}S_{li} + S_{il}S_{lk}S_{kl}S_{ji} + S_{il}S_{lk}S_{kj}S_{ji}). \] (29)

Armed with these we have

\[ Z_{\text{tree}} = e^{-NI[\sigma_0]}, \] (30)

\[ Z_{1\text{-loop}} = Z_{\text{tree}} \frac{1}{\sqrt{\det \Delta^{-1}}}, \] (31)

\[ Z_{2\text{-loop}} = Z_{1\text{-loop}} \left[ 1 + \frac{1}{N} \right. \text{(two-loop graphs)} \left. \right], \] (32)

where the two-loop graphs are shown in figure 1.

Now let us start a detailed estimation: take a time-independent solution, \( \sigma_{0j} \mapsto \bar{\sigma}_0 \),

\[ \sigma_{0j} \mapsto \bar{\sigma}_0, \] (33)

with the overlined symbol, since we are interested in the ground state (= vacuum). The fermion propagator (23) can be calculated with the aid of the anti-periodic eigenfunctions \( f_p(0) = -f_p(N_t) \):

\[ f_p(j) = \frac{1}{\sqrt{N_t}} e^{i(x_pj)/N_t} = \frac{1}{\sqrt{N_t}} (x_p)^j \quad (1 \leq j \leq N_t, 1 \leq p \leq N_t), \] (34)

with \( x_p \equiv e^{i(2p+1)/N_t} \) being the \( N_t \)th root of \((-1)\). These obey

\[ \sum_{p=1}^{N_t} f_p(j)f_p^*(k) = \delta_{jk}, \quad \sum_{j=1}^{N_t} f_p^*(j)f_p(j) = \delta_{pq}, \] (35)
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\[ \bar{S}_{jk} = \sum_{p=1}^{N_t} f_p^* (j) \bar{S}(p) f_p(k) \]

\[ = -\frac{1}{N_t} \sum_{p=1}^{N_t} \frac{(x_p)^{j-k}}{x_p - (1 + \Omega \Delta t)} \]

\[ = \frac{1}{(1 + \Omega \Delta t)^{N_t} + 1} [\theta_{jk} (1 + \Omega \Delta t)^{N_t+N_t-j} - \theta_{k,j+1} (1 + \Omega \Delta t)^{N_t+j-1}], \quad (36) \]

where

\[ \tilde{\Omega} \equiv \omega + i \lambda \delta_0, \quad (37) \]

\[ \theta_{ij} = \begin{cases} 1: & i \geq j \\ 0: & i < j. \end{cases} \quad (38) \]

The equation of motion (22), now called the gap equation, with the help of the propagator (36) becomes

\[ \bar{\sigma} = i \lambda \tilde{S} = i \lambda \frac{(1 + \tilde{\Omega} \Delta t)^{N_t-1}}{(1 + \tilde{\Omega} \Delta t)^{N_t} + 1} \overset{T \to \infty}{\longrightarrow} i \lambda \frac{e^{\tilde{\Omega} T}}{e^{\tilde{\Omega} T} + 1}. \quad (39) \]

In view of equation (37), this reads

\[ \tilde{\Omega} - \omega = -\lambda^2 \frac{e^{\Delta T}}{e^{\Delta T} + 1}, \quad (40) \]

whose right-hand side becomes a step function as \( T \to \infty \) (see figure 2).

The overlined propagator \( \tilde{\Delta}_{ij} \) of the auxiliary field (27) can be expressed as

\[ \tilde{\Delta}_{ij} = \sum_{p=1}^{N_t} F_p^* (i) \tilde{\Delta}(p) F_p(j), \quad \tilde{\Delta}(p) \equiv \frac{1}{a} - \frac{T b}{a(a + T b)} \delta_{pN_t}, \quad (41) \]

where

\[ a \equiv 1 - \Delta \lambda^2 \frac{(1 + \Omega \Delta t)^{N_t-2}}{(1 + \Omega \Delta t)^{N_t} + 1}, \quad b \equiv \lambda^2 \frac{(1 + \Omega \Delta t)^{N_t-2}}{[(1 + \Omega \Delta t)^{N_t} + 1]^2}, \quad (42) \]

and use has been made of the periodic eigenfunctions \( F_p(0) = F_p(N_t) \):

\[ F_p(j) = \frac{1}{\sqrt{N_t}} e^{2 \pi i pj/N_t} \quad (1 \leq j \leq N_t, 1 \leq p \leq N_t), \quad (43) \]
\[
\sum_{p=1}^{N_t} F_p(j) F_p^*(k) = \delta_{jk}, \quad \sum_{j=1}^{N_c} F_p^*(j) F_q(j) = \delta_{pq}.
\]

Therefore, the tree part (30) is found as
\[
\tilde{Z}(T)_{\text{tree}} = \exp(-N \tilde{I}[\tilde{\sigma}_0]);
\]
\[
\tilde{I}[\tilde{\sigma}_0] = \frac{T}{2} \tilde{\sigma}_0^2 - \ln \det[\delta_{ij} - \delta_{i-1,j} + \Delta t \tilde{\Omega}] = -\frac{T}{2\lambda^2} (\tilde{\Omega} - \omega)^2 - \ln(1 + e^{\tilde{\Omega}T}).
\]

We need to know \(\det \tilde{\Delta}^{-1}_{ij}\) in order to obtain the one-loop part (31):
\[
\det \tilde{\Delta}^{-1}_{ij} \equiv \left[ 1 - \Delta \lambda_2^2 \left( \frac{1 + \tilde{\Omega} \Delta t}{\tilde{\Omega} + 1} \right)^{N_i - 2} \right]^{N_i - 1} \\
\times \left[ 1 - \Delta \lambda_2^2 \left( \frac{1 + \tilde{\Omega} \Delta t}{\tilde{\Omega} + 1} \right)^{N_j - 2} + T \lambda_2^2 \left( \frac{1 + \tilde{\Omega} \Delta t}{\tilde{\Omega} + 1} \right)^{N_j - 2} \right]^{N_j - 1}
\]
\[
\times \exp \left[ -\lambda_2^2 \frac{e^{\tilde{\Omega}T}}{e^{\tilde{\Omega}T} + 1} T \right] \left[ 1 + \lambda_2^2 \frac{e^{\tilde{\Omega}T}}{(e^{\tilde{\Omega}T} + 1)^2} T \right].
\]

Note the term proportional to \(\Delta t\) in the second line vanishes when \(N_t \rightarrow \infty\).

The two-loop parts in figure 1 are obtained in terms of the overlapped quantities \(\tilde{\Delta}_{ij}\) and \(\tilde{S}_{ij}\) in \(I_{ij}^{(3)} (28)\) and \(I_{ijkl}^{(4)} (29)\):

**figure 1(a)**:
\[
\frac{\lambda^4}{4} (\Delta t)^2 \sum_{i,j,k,l} \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{kl} \tilde{S}_{li} \tilde{\Delta}_{ij} \tilde{\Delta}_{kl}
\]
\[
\sum_{i,j,k,l} \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{kl} \tilde{S}_{li} \tilde{\Delta}_{ij} \tilde{\Delta}_{kl}
\]
\[
\frac{\lambda^4}{4} (\Delta t)^2 \sum_{i,j,k,l} \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{kl} \tilde{S}_{li} \tilde{\Delta}_{ij} \tilde{\Delta}_{kl}
\]

**figure 1(b)**:
\[
\frac{\lambda^4}{4} (\Delta t)^2 \sum_{i,j,k,l} \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{kl} \tilde{S}_{li} \tilde{\Delta}_{ij} \tilde{\Delta}_{kl}
\]
\[
\frac{\lambda^4}{4} (\Delta t)^2 \sum_{i,j,k,l} \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{kl} \tilde{S}_{li} \tilde{\Delta}_{ij} \tilde{\Delta}_{kl}
\]

**figure 1(c)**:
\[
\frac{\lambda^4}{4} (\Delta t)^2 \sum_{i,j,k,l} \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{kl} \tilde{S}_{li} \tilde{\Delta}_{ij} \tilde{\Delta}_{kl}
\]
\[
\frac{\lambda^4}{4} (\Delta t)^2 \sum_{i,j,k,l} \tilde{S}_{ij} \tilde{S}_{jk} \tilde{S}_{kl} \tilde{S}_{li} \tilde{\Delta}_{ij} \tilde{\Delta}_{kl}
\]
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$$E_{\text{tree}} = N \left[ -\frac{1}{2\lambda^2} (\Omega - \omega)^2 - \lim_{T \to \infty} \frac{1}{T} \ln (1 + e^{\Omega T}) \right].$$  

$$E_{\text{1-loop}} = E_{\text{tree}} - \lim_{T \to \infty} \frac{-\lambda^2}{2 e^{\Omega T} + 1} + \lim_{T \to \infty} \frac{1}{2T} \ln \left( 1 + \frac{\lambda^2 T e^{\Omega T}}{(e^{\Omega T} + 1)^2} \right).$$  

$$E_{\text{2-loop}} = E_{\text{1-loop}} - \frac{1}{N} \lim_{T \to \infty} \left[ \frac{\lambda^4 T (2 e^{\Omega T} - 1) e^{\Omega T}}{4 (e^{\Omega T} + 1)^4} \left( 1 + \frac{\lambda^2 T e^{\Omega T}}{(e^{\Omega T} + 1)^2} \right)^{-1} - \frac{\lambda^4 T (7 e^{2\Omega T} - 11 e^{\Omega T} + 1) e^{2\Omega T}}{12 (e^{\Omega T} + 1)^6} \left( 1 + \frac{\lambda^2 T e^{\Omega T}}{(e^{\Omega T} + 1)^2} \right)^{-2} + \frac{5\lambda^6 T (e^{\Omega T} - 1)^2 e^{3\Omega T}}{24 (e^{\Omega T} + 1)^8} \left( 1 + \frac{\lambda^2 T e^{\Omega T}}{(e^{\Omega T} + 1)^2} \right)^{-3} \right].$$  

Estimation is made for the two cases that (i) \( \lambda^2 > 0 \) and (ii) \( \lambda^2 < 0 \).

(i) \( \lambda^2 > 0 \): the solution of the gap equation (40) can be categorized into three cases (ia), (ib) and (ic), according to the value of \( \omega \), see figure 3.
Figure 3. The gap equation, $\lambda^2 > 0$: the step function stands for the right-hand side of the gap equation (40) and the line, $\bar{\Omega} - \omega$, for the left-hand side. There is one crossing point in each case: (ia) $\omega > 0$ and $\lambda^2 \geq \omega$; (ib) $\omega > 0$ and $0 < \lambda^2 < \omega$; (ic) $\omega < 0$.

Figure 4. The case of $\lambda^2 > 0$, $\omega = 1$, corresponding to (ia) and (ib). The solid, dotted, dashed-dotted, dashed-double dotted lines designate the exact, tree, one-loop and two-loop results, respectively. We put $N = 2$ (left), 4 (right). The approximation becomes better when taking higher loops into account.

(ia) $\omega > 0$ and $\lambda^2 \geq \omega : \bar{\Omega} = 0$ (under $T \to \infty$) then from equation (40)

$$e^{\bar{\Omega}T} = \frac{\omega}{\lambda^2 - \omega},$$

so that from equation (55)

$$E_G^{2\text{-loop}} = -N \frac{\omega^2}{2\lambda^2} - \frac{\omega}{2} - \frac{1}{N} \frac{\lambda^2}{8}. \quad (56)$$

(ib) $\omega > 0$ and $0 < \lambda^2 < \omega : \bar{\Omega} = \omega - \lambda^2$, then $e^{\bar{\Omega}T} \to \infty$, so that there is no correction from the two-loop in equation (55) to give

$$E_G^{1\text{-loop}} = E_G^{2\text{-loop}} = -N \left( \frac{\omega - \lambda^2}{2} \right) - \frac{\lambda^2}{2}. \quad (57)$$

There are two domains in $\lambda^2$ divided by $\omega$, which exposes a striking difference to the exact case (A.3) where $N + 1$ domains exist. Due to these, the curve of exact energy has many discontinuities arising from the boundaries of domains. We plot the results of (ia) and (ib) with $N = 2$, $\omega = 1$, $0 \leq \lambda^2 \leq 10$ on the left-hand side of figure 4. (Note that in (ib), equation (57), we can put $|\lambda^2| = 0!$, which is a striking difference from the previous results [14, 15] where the zero in the coupling constant plane is an essential singularity.) Although the deviations in the tree and one-loop results from the exact energy are substantial, the two-loop contribution cures the situation; whose effect is very clearly seen with $N = 4$ on the right-hand side of figure 4.
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Figure 5. The gap equation, $\lambda^2 < 0$: (iia) $\omega > 0$; (iib) $\omega < 0$ and $0 < |\lambda^2| < |\omega|$; (iic) $\omega < 0$ and $|\lambda^2| \geq |\omega|$, where three solutions (iic1)–(iic3) exist.

(iia) $\omega > 0$: $\tilde{\Omega} = \omega + |\lambda^2|$, then $e^{\tilde{\Omega}T} \xrightarrow{T \to \infty} \infty$, so that there is also no correction from the two-loop, yielding

$$E^{\text{1-loop}}_G = E^{\text{2-loop}}_G = 0,$$

(59) whose result coincides with the exact energy (A.6) in the appendix. The tree result has a slight deviation but the one-loop correction fits in the curve. Again there is no singularity in $|\lambda^2|$ space to be able to put $|\lambda^2| = 0$ (see the left-hand side graph in figure 6).

(ii) $\lambda^2 < 0$: five different solutions of equation (40) are found according to the value of $\omega$ (see figure 5).

(iia) $\omega > 0$: $\tilde{\Omega} = \omega + |\lambda^2|$, then $e^{\tilde{\Omega}T} \xrightarrow{T \to \infty} \infty$, so that there is also no correction from the two-loop, yielding

$$E^{\text{1-loop}}_G = E^{\text{2-loop}}_G = 0,$$

(59) which is nothing but the result obtained in equation (A.6) in the appendix. The tree result has a slight deviation but the one-loop correction fits in the curve. Again there is no singularity in $|\lambda^2|$ space to be able to put $|\lambda^2| = 0$ (see the left-hand side graph in figure 6).

(iib) $\omega < 0$ and $0 < |\lambda^2| < |\omega|$: $\tilde{\Omega} = -|\omega|$, then $e^{\tilde{\Omega}T} \xrightarrow{T \to \infty} 0$, so that in equation (55) the one- and two-loop corrections vanish to give

$$E^{\text{tree}}_G = E^{\text{1-loop}}_G = E^{\text{2-loop}}_G = 0.$$

(60) The result coincides with the exact energy (A.5) in the appendix.

(iiic) $\omega < 0$ and $|\lambda^2| \geq |\omega|$: from figure 5, there are three different solutions:

(iic1) $\tilde{\Omega} = -|\omega|$, then $e^{\tilde{\Omega}T} \xrightarrow{T \to \infty} 0$, so that there is again no correction from the higher orders to obtain

$$E^{\text{tree}}_G = E^{\text{1-loop}}_G = E^{\text{2-loop}}_G = 0.$$

(61) (iiic2) $\tilde{\Omega} = 0$ (under $T \to \infty$), then from equation (40)

$$e^{\tilde{\Omega}T} = \frac{|\omega|}{|\lambda^2| - |\omega|},$$

so that

$$E^{\text{2-loop}}_G = N \frac{|\omega|^2}{2|\lambda^2|} + \frac{|\omega|}{2} + \frac{1}{N} \frac{|\lambda^2|}{8}.$$

(62) from equation (55), which is apparently positive to be greater than (iic1) and should be discarded.

(iic3) $\tilde{\Omega} = -|\omega| + |\lambda^2| (> 0)$, then $e^{\tilde{\Omega}T} \xrightarrow{T \to \infty} \infty$, so that in equation (55) the two-loop correction vanishes, yielding
Figure 6. Left: $\lambda^2 < 0$ and $\omega = 1$ case, corresponding to (iia). Here also $N = 2$. The solid, dotted, dashed-dotted, dashed-double dotted lines designate the exact, tree, one-loop and two-loop results, respectively. The tree approximation slightly deviates from the exact value but including the higher orders reproduces it: the one-loop approximation is accurate enough. Right: $\lambda^2 < 0$, $\omega = -1$ and $N = 2$ case, corresponding to (iib) and (iic): the two- as well as the one-loop approximation matches with the exact value.

Therefore, the ground-state energy as well as the number of domains coincides with the exact energy (A.7) in the appendix. After including the one-loop effect, the ground-state energy as well as the number of domains coincides with the exact energy (A.7) in the appendix. 

So far the tree or one-loop result matches with the exact curve except the case $\lambda^2 > 0$, $\omega = 1$ (cases (ia) and (ib)), where the number of domains (two) is different from the exact one ($N + 1$), equation (A.3). The two-loop effect, however, cures the situation and the deviation becomes smaller in $N = 4$ (figure 4). The reason is seen from figure 7 where we can recognize that the tree approximation approaches closer to the exact value when $N$ moves from 2 to 4 and 10, and moreover that when $N$ goes larger, discontinuities in the exact curve fade away. (As the number of domains in $\lambda^2$ (A.3) increases sharpness at the boundary wears off gradually.) Needless to say that AFM is analytic so that unless a boundary of domains in the loop expansion happens to coincide as of $\lambda^2 < 0$; $\omega < 0$ (cases (iib) and (iic): figure 6 (right)), the deviation is inevitable as of $\lambda^2 > 0$; $\omega > 0$ (cases (ia) and (ib)). Therefore, we conclude that AFM works still well even in $\lambda^2 > 0$; $\omega > 0$: the deviation emerges not from the failure of AFM but from the model with discontinuities in the energy curve.

3. Model 2

In this section, we adopt a slightly different model:

$$H(\hat{a}^\dagger, \hat{a}) = \omega(\hat{a}^\dagger \cdot \hat{a}) + \frac{\lambda^2}{2N}(\hat{a}^\dagger \cdot \hat{a})(\hat{a}^\dagger \cdot \hat{a}) = \omega \hat{n} + \frac{\lambda^2}{2N} \hat{n}^2.$$  (65)
Classically, there is no difference between models 1 and 2, but the reason for considering this model is that all the higher order corrections seem to vanish (although we have checked up to the two-loop) to give us another example of WKB exact model \[20\]. The energy eigenvalue is obtained as before:
\[ E_n = \omega n + \frac{\lambda^2}{2N} n^2. \]  
(66)

The partition function in this case becomes
\[
Z(T) = \lim_{N_t \to \infty} N_t^{\frac{N}{2}} \int \cdots \int \exp \left[ - N \sum_{j=1}^{N_t} \left\{ (\xi_j^* \cdot \Delta \xi_j) + (\omega + \lambda^2 - \frac{\lambda^2}{2N}) (\xi_j^* \cdot \xi_j) + \frac{\lambda^2}{2N} (\xi_j^* \cdot \xi_j)^2 \right\} \right]_{\text{AP}}
\]
\[ = \lim_{N_t \to \infty} N_t^{\frac{N}{2}} \int \cdots \int \sqrt{\frac{N \Delta t}{2\pi}} \exp[-N \tilde{I}[\sigma]], \]  
(67)

\[ \tilde{I}[\sigma] \equiv T \left( \omega + \frac{\lambda^2}{2} \right) \]
\[ + \frac{\Delta t}{2} \sum_{j=1}^{N_t} \sigma_j^2 - \ln \det \left\{ \delta_{ij} - \delta_{i-1,j} + \Delta t \left( \omega + \lambda^2 + i\lambda \sigma_i - \frac{\lambda^2}{2N} \right) \delta_{ij} \right\}. \]  
(68)

The saddle point \( \sigma_{0j} \) is given as
\[
0 = \frac{\delta \tilde{I}[\sigma]}{\delta \sigma_j} \bigg|_{\sigma_{0j}} = \Delta t (\delta_{0j} - i\lambda \delta_{jj}). \]  
(69)
with \( \tilde{S}_{jk} \) obeying
\[
\sum_{k=1}^N \left\{ \delta_{jk} - \delta_{j-1,k} + \Delta t \left( \omega + \lambda^2 + i\lambda \delta_{00} - \frac{\lambda^2}{2N} \right) \delta_{ik} \right\} \tilde{S}_{kj} = \delta_{ij}. 
\]  
(70)

Note that contrary to the previous situation, \( O(1/N) \) term has already appeared in the expression. Therefore, the loop expansion should be distinguished from the \( 1/N \) expansion in this model: we should perform the loop expansion and then arrange the results in the order of \( 1/N \). All the procedures to the two-loop ground-state energy are, therefore, the same except that all quantities are tilded; for example, the time-independent solution of the gap equation (40) should read \( \tilde{\sigma}_0 \rightarrow \tilde{\sigma}_0 \). The fermion propagator, equation (70), is
\[
\tilde{S}_{jk} = \frac{1}{(1 + \tilde{\Omega} \Delta t)^N + 1} \{ \delta_{jk} (1 + \tilde{\Omega} \Delta t)^N \delta_{j-1,k} - \delta_{k,j-1} (1 + \tilde{\Omega} \Delta t)^{-j-1} \},
\]  
(71)

with
\[
\tilde{\Omega} \equiv \omega + \lambda^2 + i\lambda \tilde{\sigma}_0 - \frac{\lambda^2}{2N}.
\]  
(72)

The gap equation (69) becomes
\[
\tilde{\Omega} - \left( \omega + \lambda^2 - \frac{\lambda^2}{2N} \right) = -\lambda^2 \frac{e^{\tilde{\Omega} T}}{e^{\tilde{\Omega} T} + 1}.
\]  
(73)

If we expand \( \tilde{\Omega} \) as
\[
\tilde{\Omega} = \tilde{\Omega}_0 + \frac{\tilde{\Omega}_1}{N} + \frac{\tilde{\Omega}_2}{N^2} + \cdots,
\]  
(74)

we obtain from equation (73)
\[
\Omega_0 - (\omega + \lambda^2) = -\lambda^2 \frac{e^{\tilde{\Omega}_0 T}}{e^{\tilde{\Omega}_0 T} + 1},
\]  
(75)

\[
\tilde{\Omega}_1 = -\frac{\lambda^2}{2} \left( 1 + T \lambda^2 \frac{e^{\tilde{\Omega}_0 T}}{(e^{\tilde{\Omega}_0 T} + 1)^2} \right)^{-1}.
\]  
(76)

\[
\tilde{\Omega}_2 = -T^2 \left( \frac{\lambda^2}{2} \right) \frac{e^{\tilde{\Omega}_0 T} (1 - e^{\tilde{\Omega}_0 T})}{(e^{\tilde{\Omega}_0 T} + 1)^3} \left( 1 + T \lambda^2 \frac{e^{\tilde{\Omega}_0 T}}{(e^{\tilde{\Omega}_0 T} + 1)^2} \right)^{-3}.
\]  
(77)

The tree part of the partition function (30) now reads
\[
\tilde{Z}(T)_{\text{tree}} = \exp(-N \tilde{I}[\tilde{\sigma}_0]);
\]  
(78)

\[
\tilde{I}[\tilde{\sigma}_0] = T \left( \omega + \frac{\lambda^2}{2} \right) - \frac{T}{2N} \left( \tilde{\Omega} - \omega + \frac{\lambda^2}{2N} \right)^2 - \ln(1 + e^{\tilde{\Omega} T}).
\]  
(79)

The ground-state energy up to the two-loop is obtained as
\[
E^{2\text{-loop}}_G = N \left[ \left( \omega + \frac{\lambda^2}{2} \right) - \frac{1}{2N} \left( \tilde{\Omega} - \omega + \frac{\lambda^2}{2N} \right)^2 - \lim_{T \to \infty} \frac{1}{T} \ln(1 + e^{\tilde{\Omega} T}) \right]
\]  
\[
- \lim_{T \to \infty} \frac{\lambda^2}{2} e^{\tilde{\Omega} T} + \frac{1}{2T} \ln \left( 1 + \frac{\lambda^2 T e^{\tilde{\Omega} T}}{(e^{\tilde{\Omega} T} + 1)^2} \right).
\]
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The expansion equation (74) with equations (75)–(77) give

\[ E^{(i)G} = \frac{\lambda^2 T (2e^{\Delta T} - 1) e^{\Delta T}}{4(e^{\Delta T} + 1)^4} \left( 1 + \frac{\lambda^2 T e^{\Delta T}}{(e^{\Delta T} + 1)^2} \right)^{-1} \]

\[ - \frac{\lambda^6 T^2(7e^{2\Delta T} - 11e^{\Delta T} + 1)e^{2\Delta T}}{12(e^{\Delta T} + 1)^6} \left( 1 + \frac{\lambda^2 T e^{\Delta T}}{(e^{\Delta T} + 1)^2} \right)^{-2} \]

\[ + \frac{5\lambda^8 T^3(e^{\Delta T} - 1)^2 e^{3\Delta T}}{24(e^{\Delta T} + 1)^8} \left( 1 + \frac{\lambda^2 T e^{\Delta T}}{(e^{\Delta T} + 1)^2} \right)^{-3} \]  

(80)

Insert the expansion equation (74) with equations (75)–(77) to give

\[ E^N_G = N \left( \omega + \frac{\lambda^2 T}{2} \right) + \frac{1}{2\lambda^2 T} \mathcal{G}_0 - (\omega + \lambda^2)^2 \frac{\lambda^2 T}{2}\left( 1 + \frac{\lambda^2 T e^{\Delta T}}{(e^{\Delta T} + 1)^2} \right) \]  

(81)

\[ E^1_G = E^N_G + \frac{1}{T} \ln \left( 1 + \frac{\lambda^2 T e^{\Delta T}}{(e^{\Delta T} + 1)^2} \right). \]  

(82)

\[ E^{1/N}_G = E^1_G = \frac{1}{N} \lim_{T \to \infty} \left[ \frac{\lambda^4 T (e^{2\Delta T} - 4e^{\Delta T} + 1)e^{\Delta T}}{8(e^{\Delta T} + 1)^4} \left( 1 + \frac{\lambda^2 T e^{\Delta T}}{(e^{\Delta T} + 1)^2} \right)^{-1} \right] \]

\[ - \frac{\lambda^6 T^2(4e^{2\Delta T} - 11e^{\Delta T} + 4)e^{2\Delta T}}{12(e^{\Delta T} + 1)^6} \left( 1 + \frac{\lambda^2 T e^{\Delta T}}{(e^{\Delta T} + 1)^2} \right)^{-2} \]

\[ + \frac{5\lambda^8 T^3(e^{\Delta T} - 1)^2 e^{3\Delta T}}{24(e^{\Delta T} + 1)^8} \left( 1 + \frac{\lambda^2 T e^{\Delta T}}{(e^{\Delta T} + 1)^2} \right)^{-3} \]  

(83)

Again estimation is made for two cases that (i) \( \lambda^2 > 0 \) and (ii) \( \lambda^2 < 0 \).

(i) \( \lambda^2 > 0 \): the solution of the gap equation (73) can be categorized into three cases (ia), (ib) and (ic) according to the value of \( \omega \), see the left-hand side graph of figure 8.

(ia) \( \omega > 0 \): \( \tilde{\omega}_0 = \omega \), then from equation (75) \( e^{\tilde{\omega}_0 T} \to \infty \), so that all higher orders of \( 1/N \) in equation (83) vanish, leaving the leading \( O(N) \) term,

\[ E^N_G = E^1_G = E^{1/N}_G = 0, \]  

(84)

which matches with the exact energy (A.10) in the appendix.

(ib) \( \omega < 0 \) and \( \lambda^2 \geq |\omega| \): \( \tilde{\omega}_0 = 0 \) (under \( T \to \infty \), then from equation (75)

\[ e^{\tilde{\omega}_0 T} = -1 + \frac{\lambda^2}{|\omega|}, \]

so that there remains only the leading term in equation (83) to give

\[ E^N_G = E^1_G = E^{1/N}_G = -N \frac{\omega^2}{2\lambda^2}. \]  

(85)

(ic) \( \omega < 0 \) and \( 0 < \lambda^2 < |\omega| \) case: \( \tilde{\omega}_0 = -|\omega| + \lambda^2 < 0 \), then \( e^{\tilde{\omega}_0 T} \to 0 \), so that the leading term gives us only

\[ E^N_G = E^1_G = E^{1/N}_G = N \left( -|\omega| + \frac{\lambda^2}{2} \right). \]  

(86)

Here again we can put \( \lambda^2 = 0 \) in view of (ic), equation (86), then plot the result with \( N = 2, \omega = -1, 0 \leq \lambda^2 \leq 10 \) on the right-hand side of figure 8, from which we
see that the leading approximation almost matches the exact curve. The deviation comes from the boundary of $\lambda^2$ domains, where discontinuity in the exact energy curve emerges. Note that the number of domains is three in the energy eigenvalue (generally $N + 1$: see equation (A.15) in the appendix) but two in AFM.

(ii) $\lambda^2 < 0$: five different solutions of equation (73) are found according to the value of $\omega$, see figure 9.

(iia) $\omega > 0$ and $0 < |\lambda^2| < \omega$: $\Omega_0 = \omega$, then $e^{\Omega_0 T \rightarrow \infty} \rightarrow \infty$, so that the leading term in equation (83) gives us only

$$E_N^G = E^1_G = E_G^{1/N} = 0.$$  \hfill (87)

(iiib) $\omega > 0$ and $|\lambda^2| \geq \omega$:

(iib1) $\Omega_0 = \omega$, then $e^{\Omega_0 T \rightarrow \infty} \rightarrow \infty$, so that again the leading term gives us

$$E_N^G = E^1_G = E_G^{1/N} = 0.$$  \hfill (88)

(iib2) $\Omega_0 = 0$ (under $T \rightarrow \infty$), then from equation (75)

$$e^{\Omega_0 T} = -1 + \frac{|\lambda^2|}{\omega},$$

so that there is no correction other than the leading term yielding

$$E_N^G = E^1_G = E_G^{1/N} = N \frac{\omega^2}{2|\lambda^2|}.$$  \hfill (89)

which is positive to be greater than (iib1) and should be discarded.
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\[ \bar{\Omega}_0 = \omega - |\lambda^2| (< 0), \text{ then } e^{\bar{\Omega}_0 T} \xrightarrow{T \to \infty} 0, \text{ so that there remains only the leading term in equation (83) to give} \]
\[ E_N^G = E_G^1 = E_G^{1/N} = N \left( \omega - \frac{|\lambda^2|}{2} \right) = \begin{cases} \text{positive: } 0 \leq |\lambda^2| < 2\omega \\ \text{negative: } |\lambda^2| \geq 2\omega. \end{cases} \]
\[ (90) \]

Therefore, we should adopt the solution (iia) up to \(|\lambda^2| = 2\omega\) then switch it to (iib3):
\[ E_G = \begin{cases} N \left( \omega - \frac{|\lambda^2|}{2} \right) & |\lambda^2| \geq 2\omega \\ 0 & 0 \leq |\lambda^2| < 2\omega, \end{cases} \]
\[ (91) \]

which is nothing but the exact energy \((A.17)\) in the appendix.

\(\omega < 0\): \(\bar{\Omega}_0 = -|\omega| - |\lambda^2|, \text{ then } e^{\bar{\Omega}_0 T} \xrightarrow{T \to \infty} 0, \text{ so that there is no correction from the higher orders to obtain} \]
\[ E_N^G = E_G^1 = E_G^{1/N} = -N \left( |\omega| + \frac{|\lambda^2|}{2} \right), \]
\[ (92) \]

matching with the exact energy \((A.18)\) in the appendix.

We have checked the model up to the \(O(1/N^2)\) to find that all higher order corrections vanish. Therefore, we can conclude that the model belongs to the ‘WKB exact class’ \([20]\), although the expansion is performed with respect to \(1/N\) which differs from the loop expansion \((=WKB)\) in this case. We have also found that the leading term reproduces the exact value except the case \(\lambda^2 > 0, \omega < 0\), whose slight discrepancy emerges from the boundaries where discontinuity is eminent (see the right-hand side of figure 8). As was stated before, AFM is analytic so that if the number of the domains in \(\lambda^2\) differs a slight deviation could inevitably occur.

4. Discussion

In this paper, we discuss the validity of AFM in terms of quantum mechanical four-fermi models. The model with the anti-normal-ordered form shows us that when \(\lambda^2 > 0, \omega > 0\), the two-loop result almost fits the exact value (figure 4), when \(\lambda^2 > 0, \omega < 0\), the tree result becomes exact and when \(\lambda^2 < 0, \omega > 0\) (figure 6 (left)) as well as when \(\lambda^2 < 0, \omega < 0\) (figure 6 (right)), the one-loop correction reproduces the exact energy. Even in \(\lambda^2 > 0, \omega > 0\), we should expect the exact fit of the one-loop or tree result, but the non-analytic structure of the exact energy curve causes a deviation.

In the second model, all the higher order corrections vanish other than the lowest one, showing us another example of the WKB exact class: when \(\lambda^2 > 0, \omega > 0\), the leading term reproduces the exact value; when \(\lambda^2 > 0, \omega < 0\), it fits well except the region where non-analytic structure becomes eminent (figure 8 (right)) and when \(\lambda^2 < 0, \omega > 0\) (equation (90)) as well as when \(\lambda^2 < 0, \omega < 0\) (equation (92)), it fits exactly. The only deviation in (figure 8 (right)) comes from the non-analytic structure of the exact energy curve.

Our study covers a wide range of coupling, \(0 \leq |\lambda^2| \leq 10\), including the free case \(|\lambda^2| = 0\). This is a peculiar outcome in the quantum fermi model; since under the loop expansion results always contain terms of inverse coupling \(1/|\lambda|^2\) (see, for example, equations \((56)\) or \((85), (89)\)). The reason for being free from the singularity is that the right-hand side of the gap equation \((40)\) takes the form of a step function to bring us several solutions and then divide the coupling domain into several pieces.
So far discussions are made under the discrete time path integral representation which keep \( N_t \) finite until the end of calculations, but in perturbation or WKB approximation people rely on the continuous time path integral which takes \( N_t \to \infty \) first. Of course, it is simpler and easier, but in some cases \([21]\) we are forced to use the discrete time representation. Here, we give another example: model 1 with \( N_t \to \infty \). The partition function (18) becomes

\[
Z(T) = \int \mathcal{D}^N \vec{\xi} \mathcal{D}^N \vec{\xi}^* \exp \left[ - \int_0^T \left\{ \xi^*(t) \cdot \dot{\xi}(t) \right\} \right] A_P,
\]

where \( A_P \) implies \( \vec{\xi}(0) = -\vec{\xi}(T) \). After introducing the auxiliary field \( \sigma(t) \) it gives

\[
Z(T) = \int \mathcal{D} \sigma \exp \left[ -NI[\sigma] \right],
\]

where

\[
I[\sigma] = \int_0^T \frac{d\sigma^2(t)}{2} - \text{Tr} \ln \left( \frac{d}{dt} + \omega + i\lambda \sigma(t) \right),
\]

with \( \text{Tr} \) being the functional trace. Find the constant solution \( \sigma(t) = \bar{\sigma}_0 \), yielding the gap equation

\[
\left. \frac{\delta I[\sigma]}{\delta \sigma(t)} \right|_{\bar{\sigma}_0} = \bar{\sigma}_0 - i\lambda \bar{S}(t,t) = 0,
\]

and the fermion propagator,

\[
\bar{S}(t,t') = \frac{1}{2 \cosh \frac{\Omega T}{2}} \left[ \theta(t - t') \exp \left( \bar{\Omega} \left\{ \frac{T}{2} - (t - t') \right\} \right) - \theta(t' - t) \exp \left( -\bar{\Omega} \left\{ \frac{T}{2} + (t - t') \right\} \right) \right],
\]

with

\[
\bar{\Omega} \equiv \omega + i\lambda \bar{\sigma}_0.
\]

By noting \( \theta(0) = 1/2 \), equation (96) reads

\[
\bar{\Omega} - \omega = -\frac{\lambda^2}{2} \tanh \frac{\Omega T}{2}.
\]

The propagator of the auxiliary field (41) turns out to be

\[
\Delta(p) = 1 - \frac{Tb}{1 + Tb} \delta_{p0}, \quad p \equiv \frac{2m\pi}{T} \quad (m = 0, \pm 1, \pm 2, \ldots),
\]

with

\[
b = \frac{\lambda^2}{4} \left( 1 - \tanh^2 \frac{\Omega T}{2} \right).
\]

The ground-state energy up to the two-loop (equation (55) under the discrete time) is

\[
E_{G}^{2\text{-loop}} = N \left[ -\frac{1}{2\lambda^2} (\Omega - \omega)^2 - \lim_{T \to \infty} \frac{1}{T} \ln \cosh \frac{\Omega T}{2} \right] + \frac{1}{2} \lim_{T \to \infty} \frac{1}{T} \ln \left[ 1 + \frac{T\lambda^2}{4} \left( 1 - \tanh^2 \frac{\Omega T}{2} \right) \right].
\]
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\[ \bar{\Omega}^2 = \omega - \frac{\lambda^2}{2}, \]

so that

\[ E^\text{tree}_G = E^\text{1-loop}_G = E^\text{2-loop}_G = N \left( \frac{\lambda^2}{8} - \frac{\omega}{2} \right). \]  

(ii) \( \omega > 0 \) and \( \lambda^2 \geq 2\omega : \bar{\Omega} = 0 \), then

\[ \tanh \frac{\bar{\Omega}T}{2} \xrightarrow{T \to \infty} -\frac{2\omega}{\lambda^2}, \]

so that

\[ E^\text{tree}_G = E^\text{1-loop}_G = E^\text{2-loop}_G = -\frac{N\omega^2}{2\lambda^2}. \]  

In continuum all the higher order corrections vanish. The result for \( N = 2 \) and \( \omega = 1 \) is plotted in figure 10: disagreement with the exact value is apparent. In a discrete time (in figure 4), the tree result is improved by higher loop effects but there are no higher loops.

Here, we reach the final goal of checking the validity of AFM: we can convince ourselves that AFM is very reliable in both Bose- as well as Fermi-quantum mechanical cases for a huge range of the coupling constants. Even if the number of species \( N \) under consideration is not so
large, results become more and more accurate in a wide range of the coupling constant when taking higher and higher orders into account. We will then proceed to examine QED by means of AFM. As long as the local auxiliary fields are concerned, we do not worry about gauge dependence at all; since it is well known that the loop expansion does not break any symmetry. An example would be described as follows: a starting Lagrangian of massless QED reads

$$\mathcal{L} = -\bar{\psi} \gamma^\mu (\partial^\mu - ieA^\mu) \psi - \frac{1}{4} F_{\mu\nu}^2, \quad F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu,$$

(105)

so that the partition function,

$$Z = \int DA_\mu D\psi D\bar{\psi} \exp \left( \int d^4x \mathcal{L} \right),\quad (106)$$

can be reduced, after introducing the auxiliary fields $B_\mu, C_\mu$ and integrating with respect to $A_\mu$, to

$$Z = \int DB_\mu DC_\mu D\psi D\bar{\psi} \exp \left( \int d^4x \mathcal{L}' \right),\quad (107)$$

$$\mathcal{L}' = -\bar{\psi} \gamma^\mu (\partial^\mu - ieB^\mu + e\gamma_5 C^\mu) \psi - \frac{e^2}{2m^2} \{ (\bar{\psi} \psi)^2 - (\bar{\psi} \gamma_5 \psi)^2 \} - \frac{m^2}{2} B_\mu \left( -\delta_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{-\partial^2 + m^2} \right) B_\nu - \frac{m^2}{2} C_\mu C_\nu \left( \partial^2 \right),$$

(108)

where use has been made of the Fierz identity. The mass parameter $m$ emerges from adjusting the dimensionality of the auxiliary fields $B_\mu, C_\mu$. Note that there is no gauge fixing: the remnant of which can be seen from the invariance, $B_\mu \rightarrow B_\mu + \partial_\mu \Lambda$. The four-fermi form looks similar to the Nambu–Jona-Lasinio model \[8\] so that the chiral structure, whether the quantity, $\langle \bar{\psi} \psi \rangle$, is zero or nonzero, would be explored in a parallel manner.

In non-Abelian cases without fermions AFM could also play a drastic role to investigate the puzzle of colour confinement dynamically. These are our future tasks.

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Appendix. The exact ground-state energy

In this appendix, calculations are made for the ground-state energy (15).

A.1. Model 1

First complete the square in the energy eigenvalue (14) to give

$$E_n = \frac{\lambda^2}{2N} \left( n - \left( N - \frac{1}{2} - \frac{N\omega}{\lambda^2} \right) \right)^2 - \frac{N}{2\lambda^2} \left( \omega + \frac{\lambda^2}{2N} \right).$$

(A.1)

The vertex is thus

$$n_{\text{vert}} = N - \frac{1}{2} - \frac{N\omega}{\lambda^2},$$

(A.2)

where the minimum or maximum occurs if there is no restriction for $n$. However, $0 \leq n \leq N$ so we need a detailed inspection.
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Figure 11. The patterns of the ground-state energy for $\lambda^2 > 0$. (a) The case of $n_{\text{vert}} < \frac{1}{2}$. The minimum occurs at $n = 0$. (b) The case of $n_{\text{vert}} \geq \frac{2N-1}{2}$. The minimum occurs at $n = N$. (c) The case of $\frac{1}{2} \leq n_{\text{vert}} < \frac{2N-1}{2}$, when $\frac{2m-1}{2} \leq n_{\text{vert}} < \frac{2m+1}{2}$ ($m = 1, 2, \ldots, N-1$), the minimum occurs at $n = m$.

(i) $\lambda^2 > 0$ and $\omega > 0$: $E_n$ is concave and $n_{\text{vert}} < \frac{2N-1}{2}$. The pattern of $E_n$ is given in figure 11. From figure 11(a), if

$$n_{\text{vert}} < \frac{1}{2} \Leftrightarrow 0 < \lambda^2 < \omega \left( \frac{N}{N-1} \right),$$

$n = 0$ is the minimum to give

$$E_G = E_0 = \frac{N-1}{2}\lambda^2 - N\omega.$$

Generally, from figure 11(c), if

$$\frac{2m-1}{2} \leq n_{\text{vert}} < \frac{2m+1}{2} \Leftrightarrow \omega \left( \frac{N}{N-m} \right) \leq \lambda^2 < \omega \left( \frac{N}{N-m-1} \right),$$

$n = m(1, 2, \ldots, N-1)$ is the minimum to give

$$E_G = E_m = \frac{(N-m)(N-m-1)}{2N}\lambda^2 - (N-m)\omega.$$

Cases can be summarized to give

$$\omega \left( \frac{N}{N-m} \right) \leq \lambda^2 < \omega \left( \frac{N}{N-m-1} \right) \quad (m = 0, 1, 2, \ldots, N-1), \quad (A.3)$$

then

$$E_G = E_m = \frac{(N-m)(N-m-1)}{2N}\lambda^2 - (N-m)\omega. \quad (A.4)$$
Figure 12. The patterns of the ground-state energy for $\lambda^2 < 0$. (a) The case of $n_{\text{vert}} \leq \frac{N}{2}$. The minimum occurs at $n = N$. (b) The case of $\frac{N}{2} < n_{\text{vert}}$. The minimum occurs at $n = 0$.

(Note that when $m = 0$ the lower bound $\omega$ of $\lambda^2$ should be discarded.) As $N$ goes large the number of domains given by equation (A.3) increases, which smoothes the discontinuity of the energy curve around the boundaries as seen from figure 7 with $N = 2, 5$ and 10.

(ii) $\lambda^2 > 0$ and $\omega < 0$:

$$n_{\text{vert}} = N - 1 + \frac{N|\omega|}{\lambda^2} > N - \frac{1}{2}.\ldots(A.5)$$

From figure 11(b), $n = N$ is the lowest. $E_N = 0$ from equation (A.1), so that

$$E_G = E_N = 0.$$  

(iii) $\lambda^2 < 0$ and $\omega > 0$: $E_n$ is convex and the pattern of $E_n$ is shown in figure 12:

$$n_{\text{vert}} = N - \frac{1}{2} + \frac{N\omega}{|\lambda^2|} > N - \frac{1}{2},$$

since $N \geq 2$. Therefore from figure 12(b), $n = 0$ is the minimum and

$$E_G = E_0 = -\frac{N}{2}|\lambda^2| - N\omega.\ldots(A.6)$$

(iv) $\lambda^2 < 0$ and $\omega < 0$:

$$n_{\text{vert}} = N - \frac{1}{2} - \frac{N|\omega|}{|\lambda^2|}.$$  

From figure 12(a), if

$$n_{\text{vert}} \leq \frac{N}{2} \Leftrightarrow 0 < |\lambda^2| \leq |\omega| \frac{2N}{N - 1},$$

$n = N$ is the minimum to give $E_N = 0$. Further from figure 12(b), if

$$\frac{N}{2} < n_{\text{vert}} \Leftrightarrow |\omega| - \frac{2N}{N - 1} < |\lambda^2|,$$  

$n = 0$ is the minimum to give $E_0 = -\frac{N-1}{2}|\lambda^2| + N|\omega|$. Therefore,

$$E_G = \begin{cases} 0: & 0 < |\lambda^2| \leq \frac{2N}{N - 1}|\omega| \\ -\frac{N-1}{2}|\lambda^2| + N|\omega|: & |\lambda^2| > \frac{2N}{N - 1}|\omega|. \ldots(A.7) \end{cases}$$
A.2. Model 2

Again complete the square in the energy eigenvalue (66) to yield

\[ E_n = \frac{\lambda^2}{2N} \left( n + \frac{N\omega}{\lambda^2} \right)^2 - \frac{N\omega^2}{2\lambda^2}. \]  

(A.8)

Then the vertex is

\[ n_{\text{vert}} = -\frac{N\omega}{\lambda^2}. \]  

(A.9)

(i) \( \lambda^2 > 0 \) and \( \omega > 0 \): \( E_n \) is concave. If

\[ n_{\text{vert}} = -\frac{N\omega}{\lambda^2} < 0; \]

from figure 11(a), \( n = 0 \) is the minimum to give

\[ E_G = E_0 = 0. \]  

(A.10)

(ii) \( \lambda^2 > 0 \) and \( \omega < 0 \):

\[ n_{\text{vert}} = \frac{N|\omega|}{\lambda^2}, \]

which is positive. From figure 11(a), if

\[ n_{\text{vert}} < \frac{1}{2} \Leftrightarrow \lambda^2 > 2N|\omega|, \]

the minimum occurs at \( n = 0 \) to give

\[ E_G = E_0 = 0. \]

Generally, if

\[ \frac{2m - 1}{2} \leq n_{\text{vert}} < \frac{2m + 1}{2} \Leftrightarrow \frac{2N}{2m + 1}|\omega| < \lambda^2 \leq \frac{2N}{2m - 1}|\omega|; \]  

(A.12)

from figure 11(c), \( n = m \) \((m = 1, 2, \ldots, N - 1)\) is the minimum to give

\[ E_G = E_m = -|\omega|m + \frac{\lambda^2}{2N}m^2. \]  

(A.13)

When \( n_{\text{vert}} \geq \frac{2N - 1}{2} \)

\[ 0 < \lambda^2 \leq \frac{2N}{2N - 1}|\omega|; \]  

(A.14)

from figure 11(b), the minimum occurs at \( n = N \) to give

\[ E_G = E_N = -N|\omega| + \frac{N\lambda^2}{2}. \]

In view of equations (A.11), (A.12) and (A.14), we can take the ground-state energy as equation (A.13),

\[ E_G = E_m = -|\omega|m + \frac{\lambda^2}{2N}m^2, \quad m = 0, 1, \ldots, N, \]  

(A.15)

in the region

\[ \frac{2N}{2m + 1}|\omega| < \lambda^2 \leq \frac{2N}{2m - 1}|\omega|, \quad m = 0, 1, \ldots, N, \]  

(A.16)

under the conditions \( \frac{2N}{2m + 1}|\omega| \rightarrow 0 \) and \( \frac{2N}{2m - 1}|\omega| \rightarrow \infty. \)
(iii) $\lambda^2 < 0$ and $\omega > 0$: $E_n$ is convex:

$$n_{\text{vert}} = N\omega / |\lambda^2|.$$  

The situation is similar to (iv) in model 1. If

$$n_{\text{vert}} \leq N \iff |\lambda^2| \geq 2\omega;$$

then from figure 12(a), $n = N$ is the minimum to give

$$E_G = E_N = N \left( \omega - \frac{|\lambda^2|}{2} \right).$$

Further if

$$N / 2 < n_{\text{vert}} \iff 0 < |\lambda^2| < 2\omega;$$

from figure 12(b), $n = 0$ is the minimum to give

$$E_G = E_0 = 0.$$  

Therefore,

$$E_G = \begin{cases} N \left( \omega - \frac{|\lambda^2|}{2} \right) : & |\lambda^2| \geq 2\omega \\ 0 : & 0 < |\lambda^2| < 2\omega. \end{cases} \quad (A.17)$$

(iv) $\lambda^2 < 0$ and $\omega < 0$:

$$n_{\text{vert}} = -N|\omega| / |\lambda^2|,$$

which is always negative so that from figure 12(a) $n = N$ is the minimum:

$$E_G = E_N = -N \left( |\omega| + \frac{|\lambda^2|}{2} \right). \quad (A.18)$$

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