Clifford operators in $SU(N)_1$; $N$ not odd prime

Howard J. Schnitzer

Martin Fisher School of Physics, Brandeis University, Waltham, Massachusetts 02453, USA
E-mail: schnitzr@brandeis.edu

Abstract: Farinholt gives a characterization of Clifford operators for qudits; $d$ both odd and even. In this comment it is shown that the necessary gates for the construction of Clifford operators; $N$ both odd and even, are obtained directly from operations that appear in $SU(N)_1$. A witness for $W_3$ states in $SU(2)_1$ is discussed. See e.g. [1–4].
1 Introduction

In applications there is a strong preference for qudits with \(d\) prime, in the construction of the Pauli group and Clifford operators. This is exemplified by applications of \(SU(N)_1\); \(N\) prime and it’s level-rank dual \(U(1)_N\). We show, following Farinholt [1], that the restriction to \(N\) prime is not necessary for \(SU(N)_1\) in the construction of the Pauli group and Clifford operators. The necessary operators are obtained from \(SU(N)_1\).

2 \(SU(d)_1\) Pauli group

Representations of \(SU(d)_1\)\(^1\) can be described by a single column Young tableau, with zero, one, ..., \((d-1)\) boxes. The fusion tensor of the theory is

\[
N^c_{ab}; \quad a + b = c \mod d
\]

so that

\[
N|a\rangle|b\rangle = |a\rangle|a + b \mod d\rangle.
\]

The modular transformation matrix \(S_{ab}\) satisfies

\[
|a\rangle = \sum_{b=0}^{d-1} S_{ab}|b\rangle, \quad a = 0 \text{ to } d - 1.
\]

Let \(\omega\) be a primitive \(d\)-th root of unity

\[
\omega = \exp\left(\frac{2\pi i}{d}\right)
\]

\(^1\)In what follows we denote the group as \(SU(d)_1\) rather than \(SU(N)_1\) to describe qudits.
then it can be shown \[2, 5\]

\[
S^* = \frac{1}{\sqrt{d}} \sum_{a=0}^{d-1} \sum_{b=0}^{d-1} \omega^{ab} |a\rangle \langle b|
\]  

(2.5)

which is the \(d\)-dimensional generalization of the Hadamard gate. Equation (2.5) can be rewritten as

\[
S^* |a\rangle = \frac{1}{\sqrt{d}} \sum_{b=0}^{d-1} \omega^{ab} |b\rangle
\]  

(2.6)

which is the \(d\)-dimensional discrete Fourier transform (QFT). With these ingredients, one can construct the qudit Pauli group.

\(n = 1\) qudits

Let

\[
Z_{ae} = \sum_{a,b=0}^{d-1} S_{bc} N_{b,1}^c (S_c^\dagger)
\]  

(2.7)

so that with (2.1)-(2.6),

\[
Z_{ac} = \sum_{b=0}^{d-1} S_{ab} (S_{b+1,a}^\dagger) \delta_{ac}
\]  

(2.8)

or

\[
Z = \sum_{a,b=0}^{d-1} S_{ab} (S_{b+1,a}^\dagger) |a\rangle \langle a|,
\]  

(2.9)

i.e.

\[
Z = \sum_{a=0}^{d-1} \omega^a |a\rangle \langle a|,
\]  

(2.10)

and

\[
Z |a\rangle = \omega^a |a\rangle,
\]  

(2.11)

which is the Pauli \(Z\). The modular transformation matrix is identical with the Pauli \(X\), since

\[
N_{a,1}^b |a\rangle = |a + 1, \; \text{mod} \; d\rangle,
\]  

(2.12)

which is identical to

\[
X |a\rangle = |a + 1, \; \text{mod} \; d\rangle,
\]  

(2.13)

or

\[
X = |a + 1\rangle \langle a| \; \text{mod} \; d
\]  

(2.14)
Therefore (2.11) and (2.14) are the basic ingredients for the single qudit Pauli group. From (2.10) and (2.14)

\[(XZ)^r = \omega^r(r-1)XZ\] (2.15)

when \(d\) is odd \(XZ\) has order \(d\), and when \(d\) is even \(XZ\) has order \(2d\). Define [1] \(\hat{\omega}\) the primitive \(D\)-th root of unity where

\[D = d; \ d \text{ odd}\]
\[D = 2d; \ d \text{ even}\] (2.16)

The single qudit Pauli group is the collection of operators

\[\hat{\omega}^r X^a Z^b; \ r \in \mathbb{Z}_D, \ a, b \in \mathbb{Z}_d.\] (2.17)

\[(X^a Z^b)(X'^a Z'^b) = \omega^{ab'-ba'}(X'^a Z'^b)(X^a Z^b),\] (2.18)

where the exponent of \(\omega\) is identified with a symplectic product.

Thus all elements of the one-qudit Pauli group are obtained from basic operators of \(SU(d)_1\)

**n-qudits**

Up to a global phase [1]

\[X^a Z^b = X^{a_1} Z^{b_1} \otimes X^{a_2} Z^{b_2} \otimes ... \otimes X^{a_n} Z^{b_n}\] (2.19)

where

\[a = (a_1, a_2, ..., a_n)\] (2.20)

and

\[a = (b_1, b_2, ..., a_n)\] (2.21)

so that

\[(X^a Z^b)(X'^a Z'^b) = \omega^{\sum_{i=1}^{n} a_i b'_i - a'_i b_i}(X'^a Z'^b)(X^a Z^b).\] (2.22)

Consider the operator \(X^a Z^b\) along with all scalar multiples there of, where

\[\{\hat{\omega}^c X^a Z^b | c \in \mathbb{Z}_D\}\] (2.23)

defines the n-qudit Pauli group. From (2.22) this is isomorphic to the \(2n\) commutative ring

\[M_R = \mathbb{Z}_D \times \mathbb{Z}_D \times ... \times \mathbb{Z}_D.\] (2.24)

Multiplication in the Pauli group then corresponds to ring multiplication in (2.24).

Again all elements of the \(n\) qudit Pauli group are obtained from direct products of basic operators of \(SU(d)_1\). There ingredients allow one to construct \(n\) qudit Clifford operators following Farinholt [1].
SU($d$), Clifford operators

Single-qudit Clifford operators [1, 6]

The necessary gates are

i) The QFT gate (2.6)

$$\mathcal{P}|j\rangle = \omega^{\frac{(j-1)}{2}}|j\rangle, \quad j \text{ odd} \quad (2.25)$$

$$\mathcal{P}|j\rangle = \omega^{\frac{j^{2}}{2}}|j\rangle, \quad j \text{ even} \quad (2.26)$$

which alternatively can be written as

$$\mathcal{P}|j\rangle = Z^{\frac{(j-1)}{2}}|j\rangle, \quad j \text{ odd} \quad (2.27)$$

$$\mathcal{P}|j\rangle = \omega^{\frac{j^{2}}{2}}Z^{\frac{(j-1)}{2}}|j\rangle, \quad j \text{ even} \quad (2.28)$$

Multi-qudit Clifford operators [1, 6]

The QFT and phase-gate are obtained from the natural product generalization of (2.6) and (2.25) - (2.27). One also needs the sum gate for a $n$-qudit system, with $i$ as the control and $j$ as the target qudit. From (2.2) [1, 6]

$$C_{\text{sum}}|i\rangle|j\rangle = N|\bar{i}\rangle|\bar{j}\rangle, \quad d \text{ odd} \quad (2.29)$$

$$C_{\text{sum}}|i\rangle|j\rangle = |i\rangle|i+j, \mod d\rangle, \quad d \text{ odd} \quad (2.29)$$

$$C_{\text{sum}}|i\rangle|j\rangle = \omega^{\frac{i}{2}(i+j)}N|i\rangle|j\rangle, \quad d \text{ even} \quad (2.30)$$

$$C_{\text{sum}}|i\rangle|j\rangle = \omega^{\frac{i}{2}(i+j)}|i\rangle|i+j, \mod d\rangle, \quad d \text{ even} \quad (2.30)$$

Toffeli gate [3, 4, 6–10]

$$T^{(3)}|i, j, k\rangle = N_{(ij+k)}^{(ij+k)} = |i, j; ij + k\rangle \mod d, \quad d \text{ odd} \quad (2.31)$$

from equation (2.2), while

$$T^{(3)}|i, j, k\rangle = \omega^{\frac{i}{2}(ij+k)}N_{(ij+k)}^{(ij+k)} \quad (2.32)$$
Multi-Toffeli gate

\[
T^{(n)}|a_1, a_2, ..., a_{n-1}, b\rangle = N^{(a_1,a_2,\ldots,a_{n-1}+b)}_{a_1,a_2,\ldots,a_{n-1},b} |a_1, a_2, ..., a_{n-1}; a_1, a_2, ..., a_{n-1} + b\rangle \mod d, \quad d \text{ odd}
\]

\[
T^{(n)}|a_1, a_2, ..., a_{n-1}, b\rangle = \omega^{\frac{1}{2}(a_1, a_2, \ldots, a_{n-1}+b)} N^{(a_1,a_2,\ldots,a_{n-1}+b)}_{a_1,a_2,\ldots,a_{n-1},b}, \quad d \text{ even}
\]

Equations (2.25)-(2.34) provide the resources for fault-tolerant computation for both \(d\) odd and even.

3 \(W_3\) states are magical

\(W_3\) is magical by definition, since it is not a stabilizer state. The discussion of magic states for qubits is limited by the absence of the discrete Wigner function for qubits. However, there exist entanglement witnesses \([11]\) with non-local stabilizing operators which can detect three qubits states which are close to a \(|W_3\rangle\) state,

\[
|W_3\rangle = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |001\rangle),
\]

which is not a stabilizer state. A witness for this state is \([11]\]

\[
\tilde{W}^{(W_3)} = \frac{2}{3}\mathbb{I} - |W_3\rangle\langle W_3|
\]

Any witness for a \(|W_3\rangle\) state has the property that

\[
\text{Tr}(\rho \mathcal{W}) < 0
\]

for a state which is close to \(|W_3\rangle\). Therefore from (3.2) one considers

\[
\text{Tr}(\rho \tilde{W}^{W_3}) = \frac{2}{3} - \text{Tr}(\rho \rho_{W_3}) < 0
\]

for states normalized to \(\text{Tr} \rho = 1\). In particular

\[
\text{Tr}(\rho_{W_3} \tilde{W}^{W_3}) = \frac{2}{3} - \text{Tr}(\rho_{W_3}^2) < 0
\]

or

\[
\text{Tr}(\rho_{W_3}^2) > \frac{2}{3}.
\]

Following Tóth and Gühne \([11]\), one can create \(|W_3\rangle\) from \(|000\rangle\) using unitary operator. The generators of the stabilizer for \(|000\rangle\) are

\[
S_k^{(000)} = Z^{(k)}; \quad k = 1, 2, 3.
\]
One can stabilize $|W_3\rangle$ by

$$S_k^{(W_3)} = U S_k^{(000)} U^\dagger \quad (3.8)$$

The $U$ is not unique but one choice is [11]

$$U = \frac{1}{\sqrt{3}} [X^{(1)} Z^{(2)} + X^{(2)} Z^{(3)} + Z^{(1)} X^{(3)}] \quad (3.9)$$

In (3.7) and (3.9), the $X^{(i)}$ and $Z^{(i)}$ are the Pauli operators for 3-qubits, obtained as direct products of the Pauli operators (2.11) and (2.13), and are constructed as operations in $SU(2)_1$. The generators of stabilizing operators, based on (3.8) are [11]

$$S_1^{(W_3)} = \frac{1}{3} [Z^{(1)} + 2Y^{(1)} Y^{(2)} Z^{(3)} + 2X^{(1)} Z^{(2)} X^{(3)}]$$

$$S_2^{(W_3)} = \frac{1}{3} [Z^{(2)} + 2Z^{(1)} Y^{(2)} Y^{(3)} + 2X^{(1)} X^{(2)} Z^{(3)}] \quad (3.10)$$

$$S_3^{(W_3)} = \frac{1}{3} [Z^{(3)} + 2Y^{(1)} Z^{(2)} Y^{(3)} + 2Z^{(1)} X^{(2)} X^{(3)}]$$

which are non-local. Tóth and Gühne [11] present other witnesses for $|W_3\rangle$.

Magic states can be distilled by Toffeli gates, such as those presented above, as operations in $SU(2)_1$. Akers and Rath [12] have argued that holographic CFT states require a large amount of tripartite entanglement. Witnesses will be helpful in pursuing that issue.

4 Comments

For $d$ prime, only a linear number of gates are needed to implement a Clifford operation in $d$-dimensional Hilbert space, while in general $O(D \log D)$ are needed to implement a Clifford operator for $d$ even [1]. A strong preference for $d$ prime emerges in terms of the number of resources required to construct gates, using Clifford operations and stabilizer states, and for magic state models [13–20].

The comments of this note apply to Chern-Simons $SU(d)_1$ as well as its level-rank dual $U(1)_d$ [21], which then extends Theorem 1 of [22] to $d$ even.

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