PICARD GROUP OF THE MODULI SPACES OF $G$–BUNDLES

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INTRODUCTION

Let $G$ be a simple simply-connected connected complex affine algebraic group and let $C$ be a smooth irreducible projective curve of genus $\geq 2$ over the field of complex numbers $\mathbb{C}$. Let $\mathcal{M}$ be the moduli space of semistable principal $G$-bundles on $C$ and let $\text{Pic} \, \mathcal{M}$ be its Picard group, i.e., the group of isomorphism classes of algebraic line bundles on $\mathcal{M}$. Following is our main result (which generalizes a result of Drezet-Narasimhan for $G = \text{SL}(N)$ [DN] to any $G$).

(A) Theorem. With the notation as above, $\text{Pic} \, (\mathcal{M}) \approx \mathbb{Z}$.

A more precise result is obtained in Theorem (2.4) together with Theorem (4.9) (see also Question 4.13).

We use the above result and a result of Grauert-Riemenschneider to prove the following second main result of this paper.

(B) Theorem. The dualizing sheaf $\omega$ of the moduli space $\mathcal{M}$ is locally free. In particular, $\mathcal{M}$ is a Gorenstein variety.

Further, for any finite dimensional representation $V$ of $G$, $H^i(\mathcal{M}, \Theta(V)) = 0$, for all $i > 0$, where $\Theta(V)$ is the theta bundle on the moduli space $\mathcal{M}$. In particular,
\[ \chi(\mathcal{M}, \Theta(V)) = \dim H^0(\mathcal{M}, \Theta(V)), \]
where $\chi$ is the Euler-Poincare characteristic.

In fact we have a sharper result than the above (cf. Theorem 2.8).

We make essential use of the flag variety $X$ associated to the affine Kac-Moody group corresponding to $G$, which parametrizes an algebraic family of $G$-bundles on $C$, and the fact that $\text{Pic} \, X \simeq \mathbb{Z}$. We also need to make use of the explicit construction of the moduli space $\mathcal{M}$ via GIT.

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1. Notation

Let $G$ be a simple simply-connected complex affine algebraic group and let $C$ be a smooth irreducible projective curve of genus $\geq 2$ over the field of complex numbers $\mathbb{C}$. As in [KNR, Theorem 3.4], let $\mathcal{M}$ be the moduli space of semistable principal $G$-bundles on $C$. Also, fix a point $p \in C$ and recall the definition of the generalized flag variety $X = G/P$ (associated to the affine Kac-Moody group $G$ corresponding to the group $G$) from [KNR, §2.1], its open subset $X^s$ and the morphism $\psi : X^s \to \mathcal{M}$ from [loc. cit., Definition 6.1]. Also, recall the notation $\tilde{W}, W, X_w$ from [loc. cit., §2.1].

For any ind-variety $Y$, by an algebraic vector bundle of rank $r$ over $Y$, we mean an ind-variety $E$ together with a morphism $\theta : E \to Y$ such that (for any $n$) $E_n \to Y_n$ is an algebraic vector bundle over the (finite dimensional) variety $Y_n$ of rank $r$, where $\{Y_n\}$ is the filtration of $Y$ giving the ind-variety structure and $E_n := \theta^{-1}(Y_n)$. If $r = 1$, we call $E$ an algebraic line bundle over $Y$.

Let $E$ and $F$ be two algebraic vector bundles over $Y$. Then a morphism (of ind-varieties) $\varphi : E \to F$ is called a bundle morphism if the following diagram is commutative:

$$
\begin{array}{ccc}
E & \xrightarrow{\varphi} & F \\
\downarrow & & \downarrow \\
X & \xrightarrow{\psi} & \mathcal{M}
\end{array}
$$

and moreover $\varphi|_{E_n} : E_n \to F_n$ is a bundle morphism for all $n$. In particular, we have the notion of isomorphism of vector bundles over $Y$.

We define $\text{Pic} Y$ as the set of isomorphism classes of algebraic line bundles on $Y$. It is clearly an abelian group under the tensor product of line bundles.

For any set $Y$, $I_Y$ denotes the identity map of $Y$.

2. Statement of the Main Theorems

We follow the notation from §1.

(2.1) Lemma. The morphism $\psi : X^s \to \mathcal{M}$ induces an injective map $\psi^* : \text{Pic}(\mathcal{M}) \longrightarrow \text{Pic}(X^s)$.

Proof. Let $\mathcal{L} \in \text{Pic}(\mathcal{M})$ be in the kernel of $\psi^*$, i.e., $\psi^*(\mathcal{L})$ admits a nowhere-vanishing regular section $\sigma$ on the whole of $X^s$. Fix $m \in \mathcal{M}$ and a trivialization for $\mathcal{L}|_m$. This canonically induces a trivialization for the bundle $\psi^*(\mathcal{L})|_{\psi^{-1}(m)}$. In particular, the section $\sigma|_{\psi^{-1}(m)}$ can be viewed as a (regular) map $\sigma_m : \psi^{-1}(m) \to \mathbb{C}^*$. But $\psi^{-1}(m)$ is a certain union of $\Gamma$-orbits say $\psi^{-1}(m) = \bigcup_{i \in I} \Gamma x_i$, for $x_i \in X$ and moreover $\Gamma x_i \cap \Gamma x_j \neq \emptyset$, for any $i, j \in I$ (cf. [KNR, Proof of Proposition 6.4]). Fixing $i \in I$, we get a regular map $\sigma_{m,i} : \Gamma \to \mathbb{C}^*$, defined as $\sigma_{m,i}(\gamma) = \sigma_m(\gamma x_i)$, for $\gamma \in \Gamma$. Now by [KNR, Corollary 2.6], $\sigma_{m,i}$ is a constant map for any $i \in I$, and hence $\sigma_m : \psi^{-1}(m) \to \mathbb{C}^*$ itself is a constant map. Thus the section $\sigma$ descends to a set theoretic section $\tilde{\sigma}$ of the line bundle $\mathcal{L}$, which is regular by [KNR, Proposition 4.1 and Lemma 6.2]. Of course, the section $\tilde{\sigma}$ does not vanish anywhere on $\mathcal{M}$ (since
σ was chosen to be nowhere-vanishing on $X^*$. This proves that $\mathcal{L}$ is a trivial line bundle on $\mathcal{M}$, thereby proving the lemma. □

It is clear that for any ind-variety $Y$, we have a natural map $\alpha : \text{Pic } Y \rightarrow \varprojlim_n \text{Pic } (Y_n)$.

(2.2) Lemma. $\text{Pic } X \approx \varprojlim_{\mathfrak{w} \in \tilde{W}/W} \text{Pic } (X_{\mathfrak{w}}) \approx \mathbb{Z}$ .

Proof. We will freely follow the notation from [KNR, §2.3]. Since the line bundles $\mathcal{L}(d\chi_0)$ (for $d \in \mathbb{Z}$) are, by construction, algebraic line bundles on $X$ and moreover for any $\mathfrak{w} \geq \mathfrak{s}_o$, $\mathcal{L}(\chi_0)|_{X_{\mathfrak{w}}}$ freely generates $\text{Pic } (X_{\mathfrak{w}})$, the surjectivity of the map $\alpha$ follows. Now we come to the injectivity of $\alpha$:

Let $\mathcal{L} \in \text{Ker } \alpha$. Fix a non-zero vector $v_o$ in the fiber of $\mathcal{L}$ over the base point $e \in X$. Then $\mathcal{L}|_{X_{\mathfrak{w}}}$ being a trivial line bundle on each $X_{\mathfrak{w}}$, we can choose a nowhere-vanishing section $s_{\mathfrak{w}}$ of $\mathcal{L}|_{X_{\mathfrak{w}}}$ such that $s_{\mathfrak{w}}(e) = v_o$. We next show that for any $\mathfrak{v} \geq \mathfrak{w}$, $s_{\mathfrak{v}|_{X_{\mathfrak{w}}}} = s_{\mathfrak{w}}$ : Clearly $s_{\mathfrak{v}|_{X_{\mathfrak{w}}}} = f s_{\mathfrak{w}}$, for some algebraic function $f : X_{\mathfrak{w}} \rightarrow \mathbb{C}^*$. But $X_{\mathfrak{w}}$ being projective and irreducible, $f$ is constant and in fact $f \equiv 1$ since $s_{\mathfrak{v}}(e) = s_{\mathfrak{w}}(e)$. This gives rise to a nowhere-vanishing section $s$ of $\mathcal{L}$ on the whole of $X$ such that $s|_{X_{\mathfrak{w}}} = s_{\mathfrak{w}}$. From this it is easy to see that $\mathcal{L}$ is isomorphic with the trivial line bundle on $X$. This proves that $\alpha$ is an isomorphism. Now the second isomorphism is proved in [KNR, Proposition 2.3]. □

We state the following very crucial ‘lifting’ result, the proof of which will be given in the next section.

(2.3) Proposition. There exists a map $\overline{\psi}^* : \text{Pic } (\mathfrak{M}) \rightarrow \text{Pic } (X)$, making the following diagram commutative:

\[
\begin{array}{ccc}
\text{Pic } (\mathfrak{M}) & \xrightarrow{\overline{\psi}^*} & \text{Pic } (X) \\
\downarrow & & \downarrow \psi^* \\
\text{Pic } (X) & \xrightarrow{i^*} & \text{Pic } (X^*)
\end{array}
\]

where $i^*$ is the canonical restriction map.

As an easy consequence of the above proposition, Lemmas (2.1) and (2.2), we get the following main result of this paper.

(2.4) Theorem. For any smooth projective irreducible curve $C$ of genus $\geq 2$ and simple simply-connected connected affine algebraic group $G$, the map $\overline{\psi}^*$ (as in the above proposition) is an injective group homomorphism.

In particular, $\text{Pic } (\mathfrak{M}) \approx \mathbb{Z}$. 

Proof. Injectivity of $\overline{\psi}^*$ follows from the injectivity of $\psi^*$ (cf. Lemma 2.1) and the commutativity of the diagram in Proposition (2.3). By Proposition (2.3), Image $\psi^* \subset \text{Image } i^*$. But since $\text{Pic } X \approx \mathbb{Z}$ (by Lemma 2.2), Image $i^*$ is either finite or else Image $i^* \approx \mathbb{Z}$. Now since $\mathfrak{M}$ is a projective variety of dim $> 0$ (cf. [R1, Theorem 4.9]) and $\psi^*$ is injective (Lemma 2.1), Image $i^*$ can not be finite, in particular, $i^*$ is injective. Since $\psi^*$ and $i^*$ are group homomorphisms and $i^*$ is injective, we get that $\overline{\psi}^*$ is a group homomorphism. This proves the theorem. □
Let $n_{C,G} > 0$ be the least (positive) integer such that $\mathfrak{L}(n_{C,G} \chi_0) \in \text{Image } \overline{\psi}^*$. Then of course

$$\text{Image } \overline{\psi}^* = \{ \mathfrak{L}(dn_{C,G} \chi_0) \}_{d \in \mathbb{Z}}$$

We will be concerned with determining the number $n_{C,G}$ in §4.

**Remark.** In the case when $G = SL(n, \mathbb{C})$, it is a result of Drezet–Narasimhan [DN] that $\text{Pic } (\mathfrak{M}) \approx \mathbb{Z}$.

We recall the following well-known result. (We include a proof since we did not find it in the literature in this form.)

**Lemma.** Let $Y$ be a Cohen–Macaulay projective variety and let $U \subset Y$ be an open subset such that $\text{codim}_Y (Y \setminus U) \geq 2$. Now let $S_1$ and $S_2$ be two reflexive sheaves on $Y$ such that $S_{1|U} \approx S_{2|U}$. Then the sheaf $S_1$ is isomorphic with $S_2$ on the whole of $Y$.

**Proof.** We recall the following two facts from Commutative Algebra.

Fact 1: If $M, N$ are modules with depth $M, N > 1$, and $0 \to M \to N \to K \to 0$ is an exact sequence, then depth $K > 0$.

Fact 2: If $M$ is reflexive, then for any localisation $M_p$ of $M$ at a prime ideal $p$, depth $M_p > 1$, unless the dimension of the local ring itself is less than 2 (i.e. $M$ satisfies the ‘Serre condition’ $S_2$).

Let $i : U \hookrightarrow Y$ be the inclusion. Then from the above facts (and the assumptions of the lemma), one can check that $i_! i^* S_j = S_j$ (for $j = 1, 2$). Thus any homomorphism $i^* S_1 \to i^* S_2$ on $U$ gives rise to a homomorphism $S_1 \to S_2$, i.e., $\text{Hom } (S_1, S_2) \to \text{Hom } (i^* S_1, i^* S_2)$ is surjective. Injectivity is clear using reflexivity. This proves the lemma. □

We come to the following second main result of this paper.

**Theorem.** The dualizing sheaf $\omega$ of the moduli space $\mathfrak{M}$ is locally free. Moreover, $\overline{\psi}^*(\omega) = \mathfrak{L}(-2g\chi_0)$, where $g$ is the dual Coxeter number of the Lie algebra $\mathfrak{g}$ (cf. [KNR, Remark 5.3]).

In particular, $\mathfrak{M}$ is a Gorenstein variety. Further, for any line bundle $\mathfrak{L}$ on $\mathfrak{M}$ such that $\overline{\psi}^*(\mathfrak{L}) = \mathfrak{L}(d\chi_0)$ for some $d > -2g$, $H^i(\mathfrak{M}, \mathfrak{L}) = 0$, for all $i > 0$. So, for any finite dimensional representation $V$ of $G$, $H^i(\mathfrak{M}, \Theta(V)) = 0$, for all $i > 0$.

**Proof.** Let $\mathfrak{M}^o := \{ E \in \mathfrak{M}; E$ is a stable $G$–bundle and $\text{Aut} E = \text{centre of } G \}$. Then $\mathfrak{M}^o$ is an open subset of the smooth locus of $\mathfrak{M}$ and for any $E \in \mathfrak{M}^o$, the tangent space $T_E(\mathfrak{M}^o)$ can be identified with $H^1(C, \text{ad } E)$, where $\text{ad } E$ is the vector bundle on $C$ associated to the principal $G$-bundle $E$ via the adjoint representation of $G$ in its Lie algebra $\mathfrak{g}$. Also, on the set of stable bundles in the moduli space there are no identifications, i.e., if $E_1$ and $E_2$ are two stable $G$-bundles on $C$ such that $E_1$ is $S$-equivalent to $E_2$, then $E_1$ is isomorphic with $E_2$ (as follows from the definition of $S$-equivalence, cf. [KNR, §3.3]). Moreover, for any $E \in \mathfrak{M}^o$, $H^0(C, \text{ad } E) = 0$. In particular, the fiber of the canonical bundle of $\mathfrak{M}^o$ at $E$ can be identified with $\wedge^{\text{top}}(H^1(C, \text{ad } E)^*)$, where $\wedge^{\text{top}}$ is the top exterior power. This gives, from the definition of the determinant bundle and the $\Theta$-bundle (cf. [KNR, §3.8]), that

$$\text{Det } (\text{ad})^*|_{\mathfrak{M}^o} = \Theta(\text{ad})^*|_{\mathfrak{M}^o} = \omega|_{\mathfrak{M}^o}.$$
But \( \Theta(\text{ad})^* \) is a line bundle on the whole of \( \mathcal{M} \), in particular, it is a reflexive sheaf on \( \mathcal{M} \) by [H, Exercise 5.1, p. 123]. Since the dualizing sheaf \( \omega \) of a normal variety is always reflexive; the moduli space \( \mathcal{M} \) is Cohen–Macaulay and normal (cf. [R1, Theorem 4.9]); and \( \operatorname{codim}_{\mathcal{M}}(\mathcal{M}\setminus\mathcal{M}^0) \geq 2 \) (unless the curve \( C \) is of genus 2 and \( G = SL(2) \)) (cf. [F, Theorem II.6]); we obtain from Lemma (2.7):

\[
(1) \quad \omega \approx \Theta(\text{ad})^*, \text{ on the whole of } \mathcal{M}.
\]

(In the case of \( G = SL(2) \) the validity of (1) is well known.) This of course gives that \( \mathcal{M} \) is a Gorenstein variety (by definition). Now the assertion that \( \psi^*(\omega) = \mathcal{L}(-2g\chi_0) \) follows from [KNR, Theorem 5.4 and Lemma 5.2].

Finally we come to the proof of cohomology vanishing: By Serre duality [H, Corollary 7.7, Chap. III],

\[
H^i(\mathcal{M}, \mathcal{L})^* \approx H^{n-i}(\mathcal{M}, \mathcal{L}^\ast \otimes \omega)
\]

\[
= H^{n-i}(\mathcal{M}, \mathcal{L}^\ast \otimes \Theta(\text{ad})^*), \text{ by (1)}.
\]

But \( \overline{\psi}^*(\mathcal{L}^* \otimes \Theta(\text{ad})^*) = \mathcal{L}((-d-2g)\chi_0) \). Now since \( \operatorname{Pic}(\mathcal{M}) \approx \mathbb{Z} \) (by Theorem 2.4), we get that the line bundle \( \mathcal{L} \otimes \Theta(\text{ad}) \) is ample on \( \mathcal{M} \) (by assumption \( d > -2g \)).

The moduli space \( \mathcal{M} \) has rational singularities, as follows from [R, Proof of Theorem 4.9] and a result of Boutot [Bo]. Now the vanishing of \( H^i(\mathcal{M}, \mathcal{L}) \) (for \( i > 0 \)) follows from (2) and a result of Grauert-Riemenschneider [GR]. So the proof of the theorem is complete in view of [KNR, Theorem 5.4]. □

(2.9) Corollary. For any finite dimensional representation \( V \) of \( G \),

\[
\mathcal{X}(\mathcal{M}, \Theta(V)) = \dim H^0(\mathcal{M}, \Theta(V)),
\]

where \( \mathcal{X} \) is the Euler-Poincare characteristic:

\[
\mathcal{X}(\mathcal{M}, \Theta(V)) = \sum_i (-1)^i \dim H^i(\mathcal{M}, \Theta(V)).
\]

3. Extension of Line Bundles - Proof of Proposition (2.3)

(3.1). Recall the definition of the map \( \varphi : G \to X_0 \) from [KNR, §1] (where \( X_0 \) denotes the set of isomorphism classes of principal \( G \)-bundles on \( C \) which are algebraically trivial restricted to \( C^* \)). Fix an embedding \( G \hookrightarrow SL(n) \), for some \( n \). In particular, any principal \( G \)-bundle \( E \) on \( C \) gives rise to a vector bundle \( \overline{E} \) of rank \( n \) on \( C \) (associated to the standard representation of \( SL(n) \)). For any integer \( d \geq 1 \), define

\[
X_d = \{ gP \in X : H^1(C, \varphi(g) \otimes \mathcal{O}(-x + dp)) = 0 \text{ for all } x \in C \},
\]

where \( p \in C \) is the fixed base point. Then

\[
X_1 \subset X_2 \subset \cdots.
\]
(3.2) Lemma. Each $X_d$ is open in $X$. Moreover $X^s \subset X_{2g}$, where $X^s = \{g\mathcal{P} \in X : \varphi(g) \text{ is a semistable } G\text{-bundle}\}$, and $g$ is the genus of the curve $C$.

Proof. It suffices to prove that $X_d \cap X_m$ is open in $X_m$, for each $m \in \tilde{W}/W$.

Recall the definition of the family of $G$-bundles $\mathcal{U} \to C \times X$ from [KNR, Proposition 2.8]. Consider the restriction $\mathcal{U}_m$ of the $G$-bundle $\mathcal{U} \to C \times X$ to $C \times X_m$ and let $\mathcal{U}_m^\sim$ be the associated rank-$n$ vector bundle (corresponding to the embedding $G \hookrightarrow \text{SL}(n)$). Define a vector bundle $\mathcal{U}_m^\sim$ on $C \times C \times X_m$ such that $\mathcal{U}_m^\sim|_{x \times C \times X_m} = \mathcal{O}(-x + dp) \otimes \mathcal{U}_m^\sim$ for each $x \in C$; and let $\pi : C \times C \times X_m \to C \times X_m$ be the projection on the two extreme factors. Applying the upper semi-continuity theorem [H, Chapter III, §12] to the morphism $\pi$ and the locally free sheaf $\mathcal{U}_m^\sim$ on $C \times C \times X_m$, we get that the set

$$ S := \{(x, g\mathcal{P}) : H^1(C, \varphi(g) \otimes \mathcal{O}(-x + dp)) \neq 0\} $$

is a closed subset of $C \times X_m$. In particular, $\pi_2(S)$ is a closed subset of $X_m$, where $\pi_2 : C \times X_m \to X_m$ is the projection on the second factor. It is easy to see that $X_d \cap X_m = X_m \setminus \pi_2(S)$. This proves that $X_d$ is open in $X$.

For $g\mathcal{P} \in X^s$, $\varphi(g)$ is a semistable vector bundle, and hence the dual vector bundle $\varphi(g)^*$ is also semistable. Now, by the Serre duality,

$$ H^1(C, \varphi(g) \otimes \mathcal{O}(-x + dp)) \approx H^0(C, \varphi(g)^* \otimes \mathcal{O}(x - dp) \otimes K)^* . $$

Since $\varphi(g)^*$ is semistable, $H^0(C, \varphi(g)^* \otimes \mathcal{O}(x - dp) \otimes K) \neq 0$ implies that $d - 1 - \text{deg } K \leq 0$. In particular, if $d \geq 2 + \text{deg } K$, then $g\mathcal{P} \in X_d$. This proves the lemma since $\text{deg } K = 2g - 2$. □

We have

$$ \bigcup_{d \geq 1} X_d = X ; $$

in fact each Schubert variety $X_m$ is contained in some large enough $X_d$ ($d$ of course depending upon $m$). This follows by the upper semi-continuity theorem (using an argument similar to the one used in the proof of the above lemma).

(3.3) Fix any $d \geq 2g$. For all $m \geq d$ and $g\mathcal{P} \in X_d$, we have

1. $H^1(C, \varphi(g) \otimes \mathcal{O}(mp)) = 0$, and
2. $H^0(C, \varphi(g) \otimes \mathcal{O}(mp))$ generates the vector bundle $\varphi(g) \otimes \mathcal{O}(mp)$ at every point of $C$.

Let $q_d := \text{dim } H^0(C, \varphi(g) \otimes \mathcal{O}(dp))$. Then by Riemann-Roch theorem, $q_d = n(d + 1 - g)$. Denote by $\pi_d : \mathcal{F}_d \to X_d$ the $\text{GL}(q_d)$-bundle such that for $g\mathcal{P} \in X_d$, $\pi_d^{-1}(g\mathcal{P})$ is the set of all the frames of the vector space $H^0(C, \varphi(g) \otimes \mathcal{O}(dp))$. We call $\mathcal{F}_d$ the frame bundle associated to the family $\mathcal{U}|_{X_d}$ parametrized by $X_d$. Similarly, define the frame bundle $\pi_{d+1} : \mathcal{F}_{d+1} \to X_{d+1}$. Consider the parabolic subgroup $P = \{\theta \in \text{GL}(q_{d+1}) : \theta \mathbb{C}^{q_d} = \mathbb{C}^{q_d}\}$ of $\text{GL}(q_{d+1})$, where (for definiteness) $\mathbb{C}^{q_d} \hookrightarrow \mathbb{C}^{q_{d+1}}$ is sitting in the first $q_d$ coordinates. We define the principal $P$-subbundle $Q_d$ of $\mathcal{F}_{d+1}|_{X_d}$ by

$$ Q_d = \bigcup_{g\mathcal{P} \in X_d} \{s = (s_1, \ldots, s_{q_d+1}) \text{ a frame of } H^0(C, \varphi(g) \otimes \mathcal{O}((d+1)p)) \text{ such that} \} $$

$(s_1, \ldots, s_{q_d+1})$ is a frame of $H^0(C, \varphi(g) \otimes \mathcal{O}(dp)))$. 

(Observe that $H^0(C, \varphi(g) \otimes O(dp))$ sits canonically inside $H^0(C, \varphi(g) \otimes O((d+1)p))$ induced from the embedding $\varphi(g) \otimes O(dp) \hookrightarrow \varphi(g) \otimes O((d+1)p)$.) Then we have the following commutative diagram:

$$
\begin{array}{ccc}
F_d & \xrightarrow{\beta_d} & Q_d \\
\pi_d \downarrow & & \downarrow \pi_{d+1} \\
X_d & \hookrightarrow & X_{d+1},
\end{array}
$$

where $\beta_d$ takes any $s = (s_1, \ldots, s_{qd+1}) \in Q_d$ to the frame $(s_1, \ldots, s_q)$ of $H^0(C, \varphi(g) \otimes O(dp))$. It is clear that $\beta_d$ is a principal $U$-bundle, where $U := \{ \theta \in \text{GL}(qd+1) : \theta|_{C^q} = I \} \subset P$. Then clearly $U$ is a normal subgroup of $P$.

As in [KNR, §7.8], we have an irreducible smooth quasi-projective variety $R_d$ with an action of $\text{GL}(qd)$, a family $W_d$ of $G$-bundles on $C$ parametrized by $R_d$ and a lift of the $\text{GL}(qd)$-action to $W_d$ (as bundle automorphisms) such that there exists a $\text{GL}(qd)$-equivariant morphism $\varphi_d : F_d \rightarrow R_d$ with the property that the families $\pi_d^*(U_{x_d})$ and $\varphi_d^*(W_d)$ are isomorphic. Moreover, let $R_d^s = \{ x \in R_d : W_d|_{C \times x} \text{ is a semistable } G\text{-bundle} \}$ be the $\text{GL}(qd)$-invariant open subset of $R_d$. Then the canonical map $\theta_d : R^s_d \rightarrow \mathfrak{M}$ is surjective. Moreover, $\theta_d$ is $\text{GL}(qd)$-equivariant with respect to the trivial action of $\text{GL}(qd)$ on the moduli space $\mathfrak{M}$ (of semistable $G$-bundles on $C$). We recall the construction of $R_d$ for its use in the sequel [R1, §§3.8,3.13.3]:

Let $R_d^o$ be the set of locally free quotients $E$ of $\mathbb{C}^{qd} \otimes_C \mathcal{O}_C$ of rank $n$ and degree $nd$ such that the canonical map $\mathbb{C}^{qd} \approx H^0(\mathbb{C}^{q_d} \otimes \mathcal{O}_C) \rightarrow H^0(E)$ is an isomorphism. Then $R_d^o$ supports a tautological family $\tilde{W}_d^o$ of rank-$n$ vector bundles on $C$. Set $W_d^o := \tilde{W}_d^o \otimes_{\mathcal{O}_C \times R_d^o} \mathcal{O}_C(-dp)$. Now let

$$R_d = \{(x, \sigma) : x \in R_d^o \text{ and } \sigma \text{ is a reduction of the structure group of } W_d^o|_{C \times x} \text{ to } G \}.$$ 

Then clearly $R_d$ supports a family $W_d$ of $G$-bundles on $C$ and moreover $\text{GL}(qd)$ acts canonically on $W_d$ via its action on $\mathbb{C}^{qd}$.

Using $H^1(C, E) = 0$, one proves that $R_d$ is smooth and that the infinitesimal deformation map $T_t(R_d) \rightarrow H^1(C, \text{Ad } W_d|_{C \times t})$ is surjective, where $T_t(R_d)$ is the tangent space at $t$ to $R_d$.

### (3.4) Proposition. For any $d \geq 2g$, the codimension of $R_d \setminus R_d^s$ in $R_d$ is at least 2, where $R_d$ is explicitly constructed as above.

To prove the above proposition, we need the notion of the canonical reduction (or filtration) of a principal $G$-bundle on $C$. We choose a Borel subgroup $B$ of $G$ and a maximal torus $T \subset B$. By a standard parabolic subgroup we mean a parabolic subgroup $P$ containing $B$. The following result is due to Ramanathan [R2, Proposition 1] (see also [Be]).

### (3.5) Theorem. Let $E$ be a principal $G$-bundle on $C$. Then there exists a unique standard parabolic subgroup $P$ of $G$ and a unique reduction $E_P$ of $E$ to the subgroup $P$ such that the following conditions hold:

1. If $U$ is the unipotent radical of $P$, then the $P/U$-bundle $E_{P/U}$ obtained from $E_P$ by extension of the structure group via $P \rightarrow P/U$ is semi-stable. (Observe that $P/U$ is reductive.)
(2) For any non-trivial character $\chi$ of $P$ which is a non-negative linear combination of simple roots of $B$, the line bundle on $C$ associated to $E_P$ by $\chi$ has strictly positive degree.

The unique reduction $E_P$ of $E$ as above is called the canonical reduction.

(3.6) Lemma. Let $E_P$ be the canonical reduction of a principal $G$-bundle $E$ on $C$. Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of $G$ and $P$ respectively. Denote by $E_\mathfrak{s}$ the vector bundle associated to $E_P$ by the natural representation of $P$ on the vector space $\mathfrak{s} := \mathfrak{g}/\mathfrak{p}$. Then we have

$$H^0(C, E_\mathfrak{s}) = 0.$$  

Proof. We may assume that $P \neq G$. Let $0 = V_0 \subset V_1 \cdots \subset V_k = \mathfrak{s}$ be a filtration of $\mathfrak{s}$ by $P$-submodules $V_i$ such that for any $1 \leq i \leq k$, the $P$-module $W_i := V_i/V_{i-1}$ is irreducible. In particular, $U$ acts trivially on $W_i$ (cf. [Ku, Lemma 1]). If $V_i$ is the vector bundle on $C$ associated to $E_P$ by the representation of $P$ on $V_i$, then $E_\mathfrak{s}$ is filtered by the subbundles $V_i$. We now show that $H^0(C, W_i) = 0$ for all $1 \leq i \leq k$, where $W_i := V_i/V_{i-1}$. This will of course prove the lemma.

Since the action of $U$ on $W_i$ is trivial, we obtain an (irreducible) representation of the reductive group $P/U$ on $W_i$. Since $E_{P/U}$ is semi-stable, the vector bundles $W_i$ are semi-stable, and hence it is sufficient to show that $\deg(W_i) < 0$. Now the weights of $T$ on $\mathfrak{s}$ are of the form $\sum c_\alpha \alpha$ with $c_\alpha \leq 0$ and $c_\alpha < 0$ for at least one $\alpha \notin I$, where $I$ is the subset of simple roots of $P = \{\alpha\}$ defining the parabolic subgroup $P$ (i.e. $I$ is the set of simple roots for $P/U$). It follows from this that the character of $P$ defined by the determinant of the representation of $P$ on $W_i$ is non-trivial and is a non-positive linear combination of $\{\alpha\}_{\alpha \notin I}$. By Condition 2) of Theorem (3.5), we see that $\deg(W_i) < 0$. This completes the proof of the lemma. □

Let $P$ be a standard parabolic subgroup of $G$ and $E_P$ be a reduction of the $G$-bundle $E$ to $P$. For any character $\chi$ of $P$, denote by $E_{P,\chi}$ the line bundle on $C$ associated to $E_P$ by $\chi$. Let $X(P)$ (resp. $X(T)$) denote the character group of $P$ (resp. $T$). Then $X(T) = \oplus_{\alpha \in \Pi} \mathbb{Z} \omega_\alpha$, where $\omega_\alpha$ is the fundamental weight defined by $\omega_\alpha(\beta^\vee) = \delta_{\alpha,\beta}$, for any simple coroot $\beta^\vee$. Moreover (since $G$ is simply-connected) $X(P) = \oplus_{\alpha \notin I} \mathbb{Z} \omega_\alpha$. The map $\chi \mapsto \deg(E_{P,\chi})$ defines an element of $\text{Hom}_{\mathbb{Z}}(X(P), \mathbb{Z})$, which in turn can be lifted to the element $\mu$ of $\text{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$ defined by $\mu(\omega_\alpha) = \deg(E_{P,\alpha})$ if $\alpha \notin I$ and $\mu(\omega_\alpha) = 0$ if $\alpha \in I$. We call $\mu$ the type of the reduction $E_P$.

Using the above lemma, one can prove the following proposition; the proof being similar to that of [PV, Theorem 4, p. 90].

(3.7) Proposition. Let $\mathcal{W}$ be a family of $G$-bundles on $C$ parametrized by a smooth variety $S$. Assume that at each point $t \in S$ the infinitesimal deformation map

$$T_t(S) \to H^1(C, \text{Ad}(\mathcal{W}_t))$$

is surjective, where $\mathcal{W}_t = \mathcal{W}|_{C \times t}$ and $T_t(S)$ is the tangent space at $t$ to $S$. For $\mu \in \text{Hom}(X(T), \mathbb{Q})$, let $S_\mu$ be the subset of $S$ consisting of those points $t \in S$ such that the canonical reduction of $\mathcal{W}_t$ is of type $\mu$. Then $S_\mu$ is non-empty only for finitely many $\mu$. Moreover, $S_\mu$ is locally closed and smooth, and the normal space...
at \( t \in S_\mu \) is given by \( H^1(C, \mathcal{W}_{t,s}) \), where \( \mathcal{W}_{t,s} \) is the vector bundle associated to the canonical reduction \( \mathcal{W}_{t,P} \) by the representation of \( P \) on \( s := g/p \).

**Proof of Proposition (3.4).** The family \( \mathcal{W} = \mathcal{W}_d \) parametrized by \( R_d \) satisfies the hypothesis of the above proposition (3.7). So it suffices to prove that for \( t \in R_d \setminus R^*_d \), we have \( \dim H^1(C, \mathcal{W}_{t,s}) \geq 2 \). By Lemma (3.6), \( H^0(C, \mathcal{W}_{t,s}) = 0 \) and we have by Riemann-Roch theorem,

(1) \[ \dim H^1(C, \mathcal{W}_{t,s}) = -\deg \mathcal{W}_{t,s} + \dim(s)(g - 1), \]

where recall that \( g \) is the genus of \( C \). Further, since \( t \in R_d \setminus R^*_d \), we have \( g \neq p \). The weight of \( \bigwedge^{top}s = -2\rho_o \), where \( \rho_o := \sum_{\alpha \notin \Gamma} \omega_\alpha \). Write

(2) \[ 2\rho_o = \sum_{\alpha \in \Gamma} n_\alpha \alpha, \]

for some non-negative integers \( n_\alpha \). So

(3) \[ \deg \mathcal{W}_{t,s} = -2\deg(E_{P,\rho_o}). \]

(Observe that \( \rho_o \in X(P) \) and moreover it is a non-trivial character.) By Condition (2) of Theorem (3.5) and (2), \( \deg(E_{P,\rho_o}) \geq 1 \), and hence by (3)

\[ \deg \mathcal{W}_{t,s} \leq -2. \]

This gives (using 1) that \( \dim H^1(C, \mathcal{W}_{t,s}) \geq 2 \), proving Proposition (3.4). \( \square \)

**Lemma (3.9).** Let \( H \) be an affine algebraic group acting algebraically on a smooth variety \( Y \) and let \( U \) be a \( H \)-stable open subset such that \( \text{codim}_Y(Y \setminus U) \geq 2 \). Then the canonical restriction map \( \text{Pic}^H(Y) \to \text{Pic}^H(U) \) is an isomorphism, where \( \text{Pic}^H(Y) \) denotes the set of isomorphism classes of \( H \)-equivariant line bundles on \( Y \).

**Proof.** Let \( L \) be an \( H \)-equivariant line bundle on \( U \). Since \( Y \) is smooth and \( \text{codim}_Y(Y \setminus U) \geq 2 \), \( L \) extends uniquely to a line bundle \( \tilde{L} \) on \( Y \). We show that \( \tilde{L} \) is \( H \)-equivariant:

Fix \( h \in H \) and an open subset \( V \subset Y \) such that \( \tilde{L}|_V \) is a trivial line bundle. In particular, the line bundle \( \tilde{L}|_{hV} \) also is trivial (since by the \( H \)-equivariance of \( L \), \( \tilde{L}|_{h(U \cap V)} \) is trivial and moreover \( \text{codim}_V(V \setminus U) \geq 2 \)). Take a nowhere-vanishing section \( s_1 \) of \( \tilde{L}|_V \) and \( s_2 \) of \( \tilde{L}|_{hV} \). Now for any \( x \in U \cap V \), \( f_h(x)s_2(hx) = h(s_1(x)) \), for some (unique) \( f_h(x) \in \mathbb{C}^* \). Clearly the map \( U \cap V \to \mathbb{C}^* \), taking \( x \mapsto f_h(x) \) is a regular map, which extends to a regular map \( f_h : V \to \mathbb{C}^* \) (since \( \text{codim}_V(V \setminus U) \geq 2 \)). Define an action of \( h \) on \( \tilde{L}|_V \) by

\[ h(s_1(x)) = f_h(x)s_2(hx), \quad \text{for all } x \in V. \]

By the uniqueness of extension, this action of \( h \) on \( \tilde{L}|_V \) patches-up to give an action of \( h \) on the whole of \( \tilde{L} \). Further, as can be easily seen, this is a regular action of \( H \) on \( \tilde{L} \).

The injectivity of \( \text{Pic}^H(Y) \to \text{Pic}^H(U) \) is easy to see: An \( H \)-equivariant section, which does not vanish anywhere on \( U \), extends to a nowhere-vanishing section on \( Y \) (and by uniqueness of extension it is \( H \)-equivariant). \( \square \)
(3.10) Lifting of line bundles from $\mathcal{M}$ to $X_d$. Take any $d \geq 2g$. Let $\mathcal{L}$ be a line bundle on $\mathcal{M}$. Pull back the line bundle $\mathcal{L}$ via the $\text{GL}(q_d)$-equivariant morphism $\theta_d : R^*_d \to \mathcal{M}$ to get a $\text{GL}(q_d)$-equivariant line bundle $\theta^*_d(\mathcal{L})$ on $R^*_d$ (cf. §3.3). By the above Lemma (3.9) and Proposition (3.4), $\theta^*_d(\mathcal{L})$ extends to a $\text{GL}(q_d)$-equivariant line bundle $\widehat{\theta^*_d(\mathcal{L})}$ on $R_d$. Consider the diagram, where all the maps are $\text{GL}(q_d)$-equivariant (the map $i_d$ is the inclusion, $\varphi_d$ and $\pi_d$ are as in §3.3, and $\text{GL}(q_d)$ acts trivially on $X_d$):

$$
\begin{array}{ccc}
\mathcal{F}_d & \xrightarrow{\varphi_d} & R^*_d \\
\downarrow \pi_d & & \downarrow \theta_d \\
X_d & & \mathcal{M}
\end{array}
$$

Now $\varphi^*_d(\widehat{\theta^*_d(\mathcal{L})})$ being a $\text{GL}(q_d)$-equivariant line bundle and $\pi_d$ is a principal $\text{GL}(q_d)$-bundle, this descends to give a line bundle (denoted) $\mathcal{L}_d$ on $X_d$ (cf. [Kr, Proposition 6.4]).

(3.11) Lemma. For any line bundle $\mathcal{L}$ on $\mathcal{M}$ and $d \geq 2g$

$$
\mathcal{L}_{d+1|X_d} \approx \mathcal{L}_d ,
$$

and $\mathcal{L}_{d|X^s} \approx \psi^*(\mathcal{L})$, where $\psi : X^s \to \mathcal{M}$ is the morphism as in §1 (cf. Lemma 3.2).

Proof. We will freely use the notation from §3.3. Let $X_m$ be a fixed Schubert variety, and denote the (reduced) variety $X_m \cap X_d$ by $Y = Y_{d,m}$. Then $Y^s := Y \cap X^s$ is an open non-empty (irreducible) subvariety of $X_m$. We denote by $\mathcal{F}_{d,Y}$, $\mathcal{F}_{d+1,Y}$ and $Q_{d,Y}$ the restrictions of $\mathcal{F}_d$, $\mathcal{F}_{d+1}$ and $Q_d$ to $Y$, where $Q_d$ is the $P$-subbundle of $\mathcal{F}_{d+1|X_d}$ as in §3.3. We show that $\mathcal{L}_{d|Y} \approx \mathcal{L}_{d+1|Y}$ and $\mathcal{L}_{d|Y^s} \approx \psi^*(\mathcal{L})|Y^s$. This will of course prove the lemma.

We first show that

$$
\mathcal{L}_{d|Y^s} \approx \psi^*(\mathcal{L})|Y^s : 
$$

From the commutativity of the diagram (where $\mathcal{F}_{d,Y}^s := \pi^*_d(Y^s)$, and $\pi_d$, $\varphi_d$, and $\psi$ are the corresponding restriction maps, which we denote by the same symbols)

$$
\begin{array}{ccc}
\mathcal{F}_{d,Y}^s & \xrightarrow{\varphi_d} & R^*_d \\
\downarrow \pi_d & & \downarrow \theta_d \\
Y^s & & \mathcal{M}
\end{array}
$$

we see that the $\text{GL}(q_d)$-linearizations on $\pi^*_d(\psi^*\mathcal{L})$ and $\varphi^*_d(\theta^*_d\mathcal{L})$ are the same. This shows that $\mathcal{L}_{d|Y^s} \approx \psi^*(\mathcal{L})|Y^s$ (since $\pi_d$ is a principal $\text{GL} (q_d)$-bundle).

If $H$ is an affine algebraic group and $\mathcal{H}$ an $H$-linearized line bundle on a principal $H$-bundle, we denote by $\mathcal{H}^H$ the line bundle on the base space (of the $H$-bundle) obtained by descending $\mathcal{H}$.

Let $\tilde{\mathcal{W}}^d_0$ be the vector bundle on $C \times R^*_d$ which is the pull-back of $\mathcal{W}^d_0$ by the map $L \times \theta : C \times R^*_d \rightarrow C \times \mathbb{P}^d$, where $\theta : R^*_d \to \mathbb{P}^d$ is the canonical map.
Let \( \pi''_d : \mathcal{F}''_d \to R_d \) (resp. \( \pi'_d : \mathcal{F}'_d \to R_d \)) be the frame bundle of the vector bundle \( (p_{R_d})_*(W_d^0 \otimes \mathcal{O}(p)) \) (resp. \( (p_{R_d})_*(W_d^0) \)), where \( p_{R_d} : C \times R_d \to R_d \) is the projection on the second factor. Just as in §3.3, the inclusion

\[
(p_{R_d})_*(\tilde{W}_d^0) \hookrightarrow (p_{R_d})_*(\tilde{W}_d^0 \otimes \mathcal{O}(p))
\]
defines a \( P \)-subbundle \( Q'_d \subseteq \mathcal{F}'_d \) on \( R_d \) and a morphism \( \beta'_d : Q'_d \to \mathcal{F}'_d \). Further, analogous to the map \( \varphi_d : \mathcal{F}_d \to R_d \) (cf. §3.10), there is a \( \text{GL}(q_{d+1}) \)-equivariant morphism \( \varphi'_d : \mathcal{F}'_d \to R_{d+1} \). Thus we have the diagram:

\[
\begin{array}{ccc}
Q'_d & \xleftarrow{\beta'_d} & \mathcal{F}'_d \\
\downarrow & & \downarrow \varphi'_d \\
\mathcal{F}''_d & \xrightarrow{\pi''_d} & R_{d+1} \\
\end{array}
\]

(Observe that \( \beta'_d \) is a principal \( U \)-bundle, \( \pi'_d \) is a principal \( \text{GL}(q_d) \)-bundle and \( \pi''_d \) is a principal \( \text{GL}(q_{d+1}) \)-bundle.) Considering the commutative diagram (where \( \mathcal{F}''_d := \pi''_d^{-1}(R^s_d) \))

\[
\begin{array}{ccc}
\mathcal{F}''_d & \xleftarrow{\varphi''_d} & \mathcal{F}'_d \\
\downarrow & & \downarrow \varphi'_d \\
R^s_d & \xrightarrow{\pi'_d} & R^s_{d+1} \\
\theta_d & \xleftarrow{\beta'_d} & \theta_{d+1} \\
\end{array}
\]

we see, as above, that

\[
(\varphi'_d \ast \theta_{d+1}^s)^{\text{GL}(q_{d+1})} \approx \theta_d^s(\mathcal{L}).
\]

Since \( \text{codim}_{R_d}(R^s_d \setminus R^s_d) \geq 2 \) and \( R_d \) is smooth, we have

\[
\bar{\theta}_d^s \mathcal{L} \approx (\varphi'_d \ast (\bar{\theta}_{d+1}^s \mathcal{L}))^{\text{GL}(q_{d+1})}.
\]

Now

\[
(\varphi'_d \ast (\bar{\theta}_{d+1}^s \mathcal{L}))^{\text{GL}(q_{d+1})} \\
\approx (\gamma_d^s(\bar{\theta}_{d+1}^s \mathcal{L}))^P \\
\approx ((\gamma_d^s(\bar{\theta}_{d+1}^s \mathcal{L}))^U)^{\text{GL}(q_d)} \\
\approx \sigma^s((\gamma_d^s(\bar{\theta}_{d+1}^s \mathcal{L}))^U),
\]

where \( \gamma_d : Q'_d \to R_{d+1} \) is the restriction of \( \varphi'_d \) to \( Q'_d \) and \( \sigma : R_d \to \mathcal{F}'_d \) is the canonical section, given by the isomorphism

\[
\mathbb{C}^{q_d} = H^0(C, \mathbb{C}^{q_d} \otimes \mathcal{O}_C) \to H^0(C, \tilde{W}_d^0|_{C \times t})
\]

for \( t \in R_d \). Thus

\[
(2) \quad \bar{\theta}_d^s \mathcal{L} \approx \sigma^s((\bar{\theta}_{d+1}^s \mathcal{L})^U).
\]
Consider the following commutative diagram
\[
\begin{array}{ccc}
Q_{d,Y} & \xrightarrow{\alpha} & Q'_{d,Y} \\
\beta_d \downarrow & & \downarrow \phi_d \\
F_{d,Y} & \xrightarrow{\delta} & F'_{d,Y} \\
\pi_d \downarrow & & \downarrow R_{d+1} \\
Y & \xrightarrow{\phi} & R_d \\
\end{array}
\]

\[(D_4)\]

where \(\delta := \sigma \circ \varphi_d\), and the map \(\alpha\) is defined as follows: Let \(gP \in Y\) and \(s = (s_1, \ldots, s_{q_d}, \ldots, s_{q_{d+1}})\) be a frame of \(H^0(C, \varphi(g) \otimes O((d+1)p))\) such that \(s = (s_1, \ldots, s_{q_d})\) is a frame of \(H^0(C, \varphi(g) \otimes O(dp))\). We have a commutative diagram:

\[
\begin{array}{ccc}
0 & \rightarrow & H^0(C, \varphi(g) \otimes O(dp)) \\
\downarrow & & \downarrow \\
0 & \rightarrow & H^0(C, \tilde{\varphi}(g) \otimes O((d+1)p)) \\
\end{array}
\]

where the vertical maps are isomorphisms. Observe that, under the first vertical isomorphism, the frame \(s\) is mapped to the frame \(\delta(s)\). Now define \(\alpha(s)\) to be the frame in \(H^0(C, \tilde{\varphi}(g) \otimes O(p))\) which is the image of the frame \(s\) under the second vertical isomorphism. Then \(\alpha\) is a \(P\)-equivariant morphism.

We claim that (as line bundles on \(F_{d,Y}\))

\[(3)\]

\[\varphi_d^*(L) \cong (\alpha^* \gamma_d^*(\hat{\theta}_{d+1}^* L))^U.\]

This follows since

\[
(\alpha^* \gamma_d^*(\hat{\theta}_{d+1}^* L))^U \\
\cong \delta^* (\gamma_d^*(\hat{\theta}_{d+1}^* L))^U \\
\cong \varphi_d^* \sigma^* (\gamma_d^*(\hat{\theta}_{d+1}^* L))^U \\
\cong \varphi_d^* (\hat{\theta}_{d}^* L), \text{ using (2)}.\]

Now the bundle \(\varphi_d^*(\hat{\theta}_{d}^* L)\) has a \(GL(q_d)\)-linearization coming from the action of \(GL(q_d)\) on \(\hat{\theta}_{d}^* L\) and (by definition of \(L_d\)) the bundle on \(Y\) obtained by descent is \(L_{d|Y}\). On the other hand, the bundle \((\alpha^* \gamma_d^*(\hat{\theta}_{d+1}^* L))^U\) has a \(GL(q_d)\)-action given by the action of \(P/U \cong GL(q_d)\) arising from the action of \(P\) on \(\alpha^* \gamma_d^*(\hat{\theta}_{d+1}^* L)\) (which in turn comes from the action of \(GL(q_{d+1})\), in particular, \(P\) on \(\hat{\theta}_{d+1}^* L\)) and the bundle on \(Y\) obtained by descent via \(\pi_d\) is \(L_{d+1|Y}\). Now on \(F_{d,Y}^* := \pi_d^{-1}(Y^*)\), these two actions of \(GL(q_d)\) coincide (i.e. the isomorphism \(\eta\) of line bundles on \(F_{d,Y}\) as guaranteed by (3) is \(GL(q_d)\)-equivariant on \(F_{d,Y}^*\)), as is seen from the commutative diagram (got from the diagrams \(D_1\) and \(D_4\)) (where \(Q_{d,Y}^* := \beta_d^{-1}(F_{d,Y}^*)\)):
Since $Y^s$ is dense in $Y$, we have that $F_{d,Y}^s$ is dense in $F_{d,Y}$; in particular, the isomorphism $\eta$ is $GL(q_d)$-equivariant on the whole of $Y$. Hence $\mathfrak{L}_{d|Y} \approx \mathfrak{L}_{d+1|Y}$. Denote this isomorphism by $\mu$. Then the restriction of $\mu$ to $Y^s$ is the identity identification (1). From this it is easy to see that $\mathfrak{L}_d \approx \mathfrak{L}_{d+1|X_d}$. This completes the proof of the lemma. □

Finally we come to the

(3.12) Proof of Proposition (2.3). For any Schubert variety $X_m$, there exists a large enough $d(w)$ such that $X_m \subset X_{d(w)}$. Let $\hat{\mathfrak{L}_m}$ be the line bundle on $X_m$ defined by $\hat{\mathfrak{L}_m} = L_{d(w)}|_{X_m}$. By Lemma (3.11), $\hat{\mathfrak{L}_m}$ is well defined and $\hat{\mathfrak{L}_m}|_{X_s} \approx \psi^*(\mathfrak{L})|_{X_s}$, where $\psi : X^s \to \mathfrak{M}$ is the morphism as in §1. Moreover, for $v \leq w$, $\hat{\mathfrak{L}_w}|_{X_v} \approx \hat{\mathfrak{L}_v}$. In particular, by Lemma (2.2), we get a line bundle $\hat{\mathfrak{L}}$ on $X$ with $\hat{\mathfrak{L}}|_{X_s} \approx \psi^*(\mathfrak{L})$. This proves the proposition. □

4. Determination of Pic ($\mathfrak{M}$)

(4.1) Definition [D, §2]. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two (finite dimensional) complex simple Lie algebras and $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra homomorphism. There exists a unique number $m_\varphi \in \mathbb{C}$, called the Dynkin index of the homomorphism $\varphi$, satisfying

$$\langle \varphi(x), \varphi(y) \rangle = m_\varphi \langle x, y \rangle, \text{ for all } x, y \in \mathfrak{g}_1,$$

where $\langle \cdot, \cdot \rangle$ is the Killing form on $\mathfrak{g}_1$ (and $\mathfrak{g}_2$) normalized so that $\langle \theta, \theta \rangle = 2$ for the highest root $\theta$.

It is easy to see from [KNR, Lemma 5.2] that for a finite dimensional representation $V$ of $\mathfrak{g}_1$ given by a Lie algebra homomorphism $\varphi : \mathfrak{g}_1 \to sl(V)$, we have $m_\varphi = m_V$, where $m_V$ is as in [KNR, §5.1] and $sl(V)$ is the Lie algebra of trace 0 endomorphisms of $V$.

By taking a representation $V$ of $G_2$ such that $m_V \neq 0$, and using [KNR, Corollary 5.6], the following proposition follows easily.

(4.2) Proposition. Let $G_1, G_2$ be two connected complex simple algebraic groups. Then for any algebraic group homomorphism $\varphi : G_1 \to G_2$, the induced map at the third homotopy group level

$$\varphi_* : \pi_3(G_1) \approx \mathbb{Z} \longrightarrow \pi_3(G_2) \approx \mathbb{Z}$$

is given by the multiplication via the Dynkin index $m_{d\varphi}$ of the induced Lie algebra homomorphism $d\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$, where $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) is the Lie algebra of $G_1$ (resp. $G_2$).

In particular, $m_\varphi$ is an integer.

(4.3) Remark. The integrality of $m_\varphi$ is proved by Dynkin [D, Theorem 2.2], and so is the following lemma [D, Theorem 2.5], by a quite different (and long) argument.
Lemma. Let $g$ be a complex simple Lie algebra and let $V(\lambda)$ be an irreducible representation of $g$ with highest weight $\lambda$. Then the Dynkin index $m_{V(\lambda)}$ of the representation $V(\lambda)$ is given by

$$m_{V(\lambda)} = (\|\lambda + \rho\|^2 - \|\rho\|^2) \frac{\dim C V(\lambda)}{\dim C g},$$

where $\rho$ is the half sum of positive roots and the Killing form on $g$ is normalized (as earlier) so that $\|\theta\|^2 = 2$ for the highest root $\theta$.

Proof. The representation $V = V(\lambda)$ of course gives rise to a Lie algebra homomorphism $\varphi = \varphi_V : g \to sl(V)$. Since $m_V = m_\varphi$ (cf. §4.1), for any $x, y \in g$

$$m_V(x, y) = \text{trace} (\varphi(x) \circ \varphi(Y)).$$

Choose a basis $\{e_i\}$ of $g$ and let $\{e^i\}$ be the dual basis of $g$ with respect to the Killing form $\langle , \rangle$. Consider the Casimir element $\Omega := \sum_i e_i e^i \in U(g)$. Then $\Omega$ acts on $V$ via

$$\Omega_V := \sum_i \varphi(e_i) \circ \varphi(e^i).$$

But $V$ being irreducible of highest weight $\lambda$,

$$\Omega_V = (\|\lambda + \rho\|^2 - \|\rho\|^2) I_V,$$

where $I_V$ is the identity operator of $V$. In particular,

$$m_V = \frac{1}{\dim g} \sum_i \text{trace} (\varphi(e_i) \circ \varphi(e^i)), \quad \text{by (1)}$$

$$= \frac{1}{\dim g} \text{trace} \Omega_V, \quad \text{by (2)}$$

$$= \frac{1}{\dim g} (\|\lambda + \rho\|^2 - \|\rho\|^2) \dim V, \quad \text{by (3)}.$$

This proves the lemma. □

We also need the following

Lemma. Let $g$ be a complex simple Lie algebra and let $V$ and $W$ be two finite dimensional representations of $g$. Then

$$m_{V \otimes W} = m_V \dim W + m_W \dim V.$$

Proof. Write the characters

$$\text{ch } V = \sum_\lambda n_\lambda e^\lambda, \quad \text{and}$$

$$\text{ch } W = \sum_\mu m_\mu e^\mu, \quad \text{for some } n_\lambda, m_\mu \in \mathbb{Z}.$$
Then
\[
\text{ch } (V \otimes W) = \sum_{\lambda, \mu} n_{\lambda\mu} e^{\lambda + \mu}.
\]
Hence by [KNR, Lemma 5.2],
\[
2m_{V \otimes W} = \sum_{\lambda, \mu} n_{\lambda\mu} \langle \lambda + \mu, \theta^\vee \rangle^2
\]
\[
= \sum_{\lambda, \mu} n_{\lambda\mu} \langle \lambda, \theta^\vee \rangle^2 + \sum_{\lambda, \mu} n_{\lambda\mu} \langle \mu, \theta^\vee \rangle^2 + 2 \sum_{\lambda, \mu} n_{\lambda\mu} \langle \lambda, \theta^\vee \rangle \langle \mu, \theta^\vee \rangle
\]
\[
= 2 \left( \sum_{\mu} m_{\mu} \right) m_V + 2 \left( \sum_{\lambda} n_{\lambda} \right) m_W + 2 \left( \sum_{\lambda} n_{\lambda} \langle \lambda, \theta^\vee \rangle \right) \left( \sum_{\mu} m_{\mu} \langle \mu, \theta^\vee \rangle \right)
\]
\[
= 2(\dim W)m_V + 2(\dim V)m_W + 2 \left( \sum_{\lambda} n_{\lambda} \langle \lambda, \theta^\vee \rangle \right) \left( \sum_{\mu} m_{\mu} \langle \mu, \theta^\vee \rangle \right).
\]

(1)

For any \( h \in \mathfrak{h} \), define \( \beta_V(h) = \sum_{\lambda} n_{\lambda} \langle \lambda, h \rangle \). Then the map \( \beta_V : \mathfrak{h} \to \mathbb{C}, \ h \mapsto \beta_V(h) \) is \( W \)-equivariant (with the trivial action of \( W \) on \( \mathbb{C} \)). But \( \mathfrak{h} \) being an irreducible \( W \)-module,
\[
\beta_V \equiv 0.
\]

Combining (1) and (2), the lemma follows. □

**4.6 Definition.** Let \( \mathfrak{g} \) be a complex simple Lie algebra and let \( \theta \) be the highest root (with respect to some choice of the set of positive roots). Express the associated coroot \( \theta^\vee \) in terms of the simple coroots:
\[
\theta^\vee = \sum_{i=1}^{\ell} m_i \alpha_i^\vee.
\]

Now define \( d = d(\mathfrak{g}) \) to be the least common multiple of \( \{m_i\}_{i=1, \ldots, \ell} \). Then the number \( d \) is given as follows:

| Type of \( \mathfrak{g} \)       | \( d(\mathfrak{g}) \) |
|---------------------------------|------------------------|
| \( A_\ell (\ell \geq 1) \), \( B_\ell \) (\( \ell \geq 3 \)) | 1                      |
| \( C_\ell (\ell \geq 2) \), \( D_\ell \) (\( \ell \geq 4 \)) | 2                      |
| \( G_2 \)                       | 2                      |
| \( F_4 \)                       | 6                      |
| \( E_6 \)                       | 6                      |
| \( E_7 \)                       | 12                     |
| \( E_8 \)                       | 60                     |

**4.7 Proposition.** For any finite dimensional representation \( V \) of \( \mathfrak{g} \), the number \( d(\mathfrak{g}) \) divides \( m_V \). Moreover, there exists an irreducible representation \( V_0 \) of \( \mathfrak{g} \) such that \( d(\mathfrak{g}) = m_{V_0} \).

**Proof.** Unfortunately, our proof is case by case. We follow the indexing convention as in [P, Blanche LI-IX]. We denote the \( i \) th fundamental weight \( (1 \leq i \leq \ell) \) by \( \omega_i \).
Case 1 – $A_{\ell}(\ell \geq 1), C_{\ell}(\ell \geq 2)$: As in [KNR, Lemma 5.2], $m_{V_o} = 1$, for the standard $(\ell + 1)$-dimensional representation $V_o$ of $A_{\ell}$. Similarly for the standard $2\ell$-dimensional representation $V_o$ of $C_{\ell}$ (with highest weight $\omega_1$), $m_{V_o} = 1$ (as can be seen from Lemma 4.4).

For a simply-connected group $G$, since the fundamental representations $\{V(\omega_i)\}_{1 \leq i \leq \ell}$ generate the representation ring $R(G)$ as an algebra (cf [A]), to prove that $d(\mathfrak{g})$ divides $m_V$ for any $\mathfrak{g}$-module $V$, it suffices to show that $d(\mathfrak{g})$ divides $m_i := m_{V(\omega_i)}$ for all $1 \leq i \leq \ell$ (cf. Lemma 4.5). In the following calculations, we make use of Lemma (4.4) and [B, Planche I-IX] freely.

Case 2 – $B_{\ell}$ ($\ell \geq 3$): For $1 \leq i \leq \ell - 1$, $m_i = 2\left(\frac{2\ell - 1}{i - 1}\right)$, since $\dim V(\omega_i) = \left(\frac{2\ell + 1}{i}\right)$; and $m_{\ell - 1} = 2\ell - 2$.

In particular, $m_1 = 2$, so take $V_o = V(\omega_1)$.

Case 3 – $D_{\ell}$ ($\ell \geq 4$): For $1 \leq i \leq \ell - 2$, $m_i = 2\left(\frac{2\ell - 2}{i - 1}\right)$, since $\dim V(\omega_i) = \left(\frac{2\ell}{i}\right)$; and $m_{\ell - 1} = m_2 = 2\ell - 3$.

In particular, $m_1 = 2$.

In the following calculations, $\dim V(\omega_i)$ is taken from [BMP].

Case 4 – $G_2$: $m_1 = 2$, $m_2 = 8$.

(Observe that $V(\omega_2)$ is the adjoint representation of $G_2$ and hence $m_2$ can be calculated from [KNR, Lemma 5.2 and Remark 5.3].)

Case 5 – $F_4$: $m_1, \ldots, m_4$ are respectively 18, 9×98, 126, and 6.

Case 6 – $E_6$: $m_1, \ldots, m_6$ are respectively 6, 24, 150, 1800, 150, and 6.

Case 7 – $E_7$: $m_1, \ldots, m_7$ are respectively 36, 360, 65×72, 2750×108, 104×165, 8×81, and 12.

Case 8 – $E_8$: $m_1, \ldots, m_8$ are respectively 12×125, 4750×18, 49×108000, 75×111275472, 30×4720170, 45×39520, 15×980, and 60. ⊢

(4.8) Remark. The values of $m_i$ given above are also contained in [D], but many of his values are incorrect.

Combining Proposition (4.7) and Theorem (2.4) with the chart in Definition (4.6), we get the following strengthening of Theorem (2.4).

(4.9) Theorem. With the notation and assumptions as in Theorem (2.4), consider the injective map $\overline{\psi^*} : \text{Pic}(\mathfrak{M}) \hookrightarrow \text{Pic}(X) \approx \mathbb{Z}$. Then

1. $\overline{\psi^*}$ is surjective in the case where $G$ is of type $A_{\ell}$ ($\ell \geq 1$), and $C_{\ell}$ ($\ell \geq 2$).
2. The order $\gamma = \gamma_G$ of the cokernel of $\overline{\psi^*}$ is bounded as follows:
   a. $G = B_{\ell}$ ($\ell \geq 3$), $\gamma \leq 2$
   b. $G = D_{\ell}$ ($\ell \geq 4$), $\gamma \leq 2$
   c. $G = G_2$, $\gamma \leq 2$
   d. $G = F_4$, $\gamma \leq 6$
   e. $G = E_6$, $\gamma \leq 6$
   f. $G = E_7$, $\gamma \leq 12$
(g) $G = E_8$, $\gamma \leq 60$.

4.10 Definition. A (complex) representation $V$ of $G$ is said to be orthogonal if there exists a $G$-invariant non-degenerate symmetric $\mathbb{C}$-bilinear form on $V$.

Clearly, an orthogonal representation is isomorphic with its dual. Of course the adjoint representation $g$ of $G$ is orthogonal.

Even though we do not have a full proof as yet (nor do we know any place in the literature where it is proved), we believe that the following proposition is true. (G. Faltings has written to the first author that this should be true. He has suggested that one should show that the cohomology is locally representable by a complex $\mathcal{E} \rightarrow S \rightarrow \mathcal{E}^*$, where $S$ corresponds to an alternating form. Thus giving $\det (S) \approx \text{Pfaff} (S)^2$.

4.11 “Proposition”. Let $V$ be an orthogonal representation of $G$. Then the theta bundle $\Theta (V)$ on $M$ admits a square root as a line bundle, i.e., there exists an (algebraic) line bundle $L$ on $M$ such that $L^2 \approx \Theta (V)$.

4.12 Remark. The validity of the above proposition will show that the map $\psi^*$ (as in the above theorem 4.9) is surjective for $G = B_\ell$ ($\ell \geq 3$), $G = D_\ell$ ($\ell \geq 4$), and $G = G_2$. (Observe that for $G_2$, $V(\omega_1)$ is seven dimensional orthogonal representation.) Moreover the bounds for $\gamma$ in the cases $F_4, E_7$, and $E_8$ can be improved to 3, 6, and 30 respectively.

We feel that the following question has an affirmative answer.

4.13 Question. For any $C, G$ as in Theorem (4.9) (with genus $C \geq 2$), is it true that the bounds for the order of $\gamma$ as in Remark (4.12) are in fact equalities?

For $C = \mathbb{P}^1(\mathbb{C})$, since the moduli space $M$ reduces to a point, the map $\psi^*$ is clearly an isomorphism.

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