Deviation of the waves in an inhomogeneous medium

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Abstract

The wave deviation in an inhomogeneous medium with continuous variation of propagation velocity is deduced using the formula found by Noorbala and Sepehrinia. For electromagnetic waves (light) that propagate in the gravitational field, the deduced deviation is identical to that calculated from General Relativity. The method and its consequences are a good example that verifies the Noorbala-Sepehrinia’s formula, as well as the mecano-optics analogy (Hamilton’s principle/principle of stationary action and Fermat’s principle) for bodies’ movement in the gravitational field.

Keywords: Noorbala-Sepehrinia’s formula, Fermat’s principle, gravitational deviation of light

1. Introduction

In a paper [1], Noorbala and Sepehrinia (N-S) found a formula which correlates the refractive index $n$ to the angle $\theta$ of incidence (the angle between incident light ray and normal at the constant index surface) when the speed of the light varies continuously within a medium. This new relation is different to the one described by Snell’s law.

We will use it in our paper to compute the deviation of the wave (particularly, the light) when it travels through an inhomogeneous medium [2, 3, 4]. For this kind of medium we will assume that the refractive index depends only on the position vector magnitude, $n(r)$. Our result is similar to the one obtained in General Relativity for an isotropic metric [6, 7, 8].

All the derivations we will perform in the following sections confirm the correctness of the N-S relationship and demonstrate the analogy between mechanics and optics when a body moves through a gravitational field [4, 5, 11].

2. N-S relation for a medium with refractive index

Given an inhomogeneous medium with refractive index depending on the length of the position vector as

$$n(r) = \exp\left(\frac{N}{r^p}\right), N > 0,\ p = \frac{1}{2}, 1, \frac{3}{2}, ..., .$$

(1)

The above dependency is suggested by Rastall-Yilmaz-Rosen metrics [12, 13, 14, 15]. We will use it since it allows us to obtain an approach for the bending of the waves when the refractive index presents

$$n(r) = \exp\left(\frac{N}{r^p}\right) \approx 1 + \frac{N}{r^p},\ N > 0,\ \frac{N}{r^p} \ll 1.$$

(2)

For $p = 1$ and $N = r = 2Gm/c^2$ (where $m$ is the mass of a body placed at the origin of the reference frame), equation (2) becomes

$$n(r) = 1 + \frac{r}{r}.$$

(3)

According to the equation (10) of the paper of Noorbala and Sepehrinia [1], for continuous and inhomogeneous medium the law of refraction is

$$\sin \theta \frac{d (n \sin \theta)}{ds} = -n \cos \theta \left(\frac{dn}{ds} \dot{\theta}\right).$$

(4)
The surface of a medium where the refractive index is constant is spherical.

As shown in figure 1, \( \vec{r}(r, \varphi) \) is the position vector in polar coordinates, \( \hat{n} \) is perpendicular to the surface, \( \vec{v} \) is the direction of the vector \( \vec{v} = d\vec{r}/ds \), \( \theta \) is the angle between the vectors \( \hat{n} \) and \( \vec{v} \), while \( \delta \) is the deviation angle.

Since the refractive index depends only on the length of the position vector, then according to equation (2), the normal on the surface \( \hat{n} \), the position vector and the gradient of the index \( \nabla n = (\partial n/\partial r)(\vec{r}/r) = (\partial n/\partial r)\hat{n} \) are parallel. It follows

\[
\frac{d\hat{n}}{ds} = \frac{d}{ds} \left( \frac{\vec{r}}{r} \right) = \frac{d\varphi}{ds},
\]

where \( \varphi \) is, according to figure 1, the angular variable in polar coordinates \( (r, \varphi) \) in the plane of the wave path.

The vector \( d\hat{n}/ds \) is perpendicular on \( \hat{n} = \vec{r}/r \) and

\[
\frac{d\hat{n}}{d\varphi} = \frac{d\varphi}{ds} \cos \left( \frac{\pi}{2} - \theta \right) = \frac{d\varphi}{ds} \sin \theta.
\]

According to figure 1, to figure 2 and to the notations adopted in [2] and [1] one can establish the following correspondences

\[
dl \leftrightarrow ds = \sqrt{dr^2 + r^2 d\varphi^2}, \quad \alpha \leftrightarrow \theta,
\]

When substitute equation (6) in equation (4), it follows

\[
\sin \theta \frac{d(n \sin \theta)}{ds} = -n \frac{d\varphi}{ds} \cos \theta \sin \theta,
\]

or

\[
\frac{d(n \sin \theta)}{ds} = -n \cos \theta d\varphi,
\]

which is the N-S relation mentioned in the rows above.

3. N-S relation and the law of conservation of angular momentum

In the following we will derive the law of conservation of angular momentum and we will prove its compatibility with N-S relation when the refractive index depends on the length of the position vector.

The path of the light has the constant [2, 3, 4]. According to the equation (1) of the paper [2], this constant is

\[
L = rn(r) \sin \theta.
\]

At \( \theta = \pi/2 \) the radius becomes \( r(\pi/2) = R \). Using this radius and equations (1) and (9), the constant becomes

\[
L = Rn(R) = R \exp \left( \frac{N}{R^p} \right).
\]

One can differentiate (9), and so

\[
\frac{d}{r} (n \sin \theta) = -\frac{dr}{r} n \sin \theta.
\]

We will use the figure 1 of this paper and the figure 2 from [2] to establish the following relations

\[
\sin \theta = \frac{rd\varphi}{ds}, \quad \cos \theta = \frac{dr}{ds}.
\]

When substituting relations (12) within the right side of (11), one can obtain the N-S relation (8b). It is not surprising that these two relations are compatible.
The N-S relation was derived taking account of Fermat’s principle, which is in fact Hamilton’s principle for light [3, 4]. Using Hamilton’s principle, one derives the angular momentum conservation [5].

4. The bending of the light traveling in an inhomogeneous medium

We will apply N-S relation for computing the deviation of a ray of light which is parallel to $Ox$, starts from a point with coordinates $r \to +\infty$, $\varphi \to 0$ and is directed towards the point coordinates $r = R$, $\varphi = \pi/2$ (see figure 1!). Here $R$ is the minimum length of the position vector related to the reference frame. Here the bending angle is zero, $\delta = 0$ and $\theta = \pi/2$, since these two angles are related by

$$\varphi + \delta + \theta = \pi$$

and

$$d\theta = -d\varphi - d\delta$$

When expressing equation (8b) using $r$, $n$ and $\varphi$ (Appendix 1, equation A1.3), we get

$$d\delta = \frac{r \, dn}{n \, dr} d\varphi.$$  \hspace{1cm} (14)

Deriving equation (1), yields

$$1 + p \, \frac{dn}{r \, dr} = - \frac{pN}{r^{p+1} n}.$$  \hspace{1cm} (15)

Replacing equation (15) in equation (14), one obtains

$$d\delta = \frac{pN}{r^{p} \, d\varphi}.$$  \hspace{1cm} (16)

To integrate equation (16) it is necessary to find out how $\varphi$ depends on $r$. According to [2, 3, 4], the path of the light in a medium is described by (equation 2 from [2])

$$d\varphi = \pm \frac{L \, dr}{r \sqrt{r^2 - L^2}}$$

where $L$ is the constant from equations (9) and (10).

Then we will do two replacements: first equations (1) and (10) in equation (17)

$$d\varphi = \pm \frac{R \exp\left(\frac{N}{R^p}\right) \, dr}{r \sqrt{r^2 \exp\left(2N/r^p\right) - R^2 \exp\left(2N/R^p\right)}}$$

and second equation (18) in equation (16)

$$d\delta = \mp \frac{pN \exp\left(\frac{N}{R^p}\right) \, dr}{r^{p+1} \sqrt{r^2 \exp\left(2N/r^p\right) - R^2 \exp\left(2N/R^p\right)}}.$$  \hspace{1cm} (19)

Integrating equation (19) between $r \to +\infty$ and $r = R$ we obtain the maximum deviation $\delta_n = \delta(r \to +\infty)$.

Here, the integral (20)

$$\delta_n = \mp \frac{\sqrt{\pi} NT \left(\frac{p+1}{2}\right) e^{\frac{N}{2R^p}}}{R^p T \left(\frac{p}{2}\right)}$$

or the approximate form

$$\delta_n \equiv \mp \frac{\sqrt{\pi} NT \left(\frac{p+1}{2}\right) \left(1 + \frac{N}{2R^p} + \frac{N^2}{8R^{2p}}\right)}{R^p T \left(\frac{p}{2}\right)}$$

is an analytical formula using WolframAlpha or Wolfram Programming Lab [16].

5. The deviation of the light in a gravitational field

Assuming a gravitational field of a body with mass $m$, the refractive index can be expressed like (3), for the first-order isotropic Schwarzschild metric [6, 7, 8]. Therefore, the maximum deviation, with $p = 1$ and $N = r_g$ in equation (21), [16], is

$$\delta_n \approx \pm \frac{r_g}{R}.$$  \hspace{1cm} (21)

The entire deviation, which occurs when the wave travels from $r \to +\infty$ to $r \to -\infty$ and passing at minimum distance $R$, is

$$\delta_g = 2\delta_n = 2\frac{r_g}{R}.$$  \hspace{1cm} (22)

This result is just the relativistic deviation [6, 9]. That is, an inhomogeneous optic medium with a refractive index of the form (3) behaves like a spherical lens and this lens mimics a gravitational field.

6. Conclusions

According to the general relativity, half of the bending of the light in a gravitational field depends on the curvature of the space and the other half on the variation of the velocity of the light along its path [9].

An optical approach must assume that, for the inhomogeneous medium, $n(r)$, the deviation of the waves, no matter their type [10], depends on the optical properties of the medium, as we have shown in this paper.

Since in classic physics, path bending results from the action of a force, then a gravitational force (i.e. which is directly proportional to the mass and inversely proportional to the distance) is also the effect of a refractive index which depends on the position vector. Therefore, describing how a vacuum around a particle becomes inhomogeneous, leads to a phenomenological-causal approach of the gravitational interaction into the electromagnetic world, i.e. the world where the main interaction is the electromagnetic one.
Intuitively, we may assume that a standing wave packet, travelling through an inhomogeneous medium, has a null tangential velocity. This tangential velocity is the velocity of energy along the path. But for the same wave packet, the velocity and the acceleration along radial direction are nonzero. The radial acceleration is directly proportional to the gradient of the refractive index.

For a refractive index from equation (3), the acceleration is directly proportional to \( \frac{g_{rr}}{r^2} \), i.e. a gravitational type of acceleration.

This kind of approach helps explain the success of applying mechano-optics of the gravitational interaction to the research of light and of particles with rest mass [11].

**Appendix 1**

Equation (8b) can also be described as

\[
\frac{dn}{n} \sin \theta + n \cos \theta d\theta = -n d\phi \cos \theta.
\]  
(A1.1)

Replacing \( d\theta \) from equation (13b) in equation (A1.1), one gets

\[
d\delta = \frac{dn}{n} \sin \theta \cos \theta.
\]  
(A1.2)

Finally using equation (12) in equation (A2.2), it follows

\[
d\delta = \frac{\alpha}{n} d\phi.
\]  
(A1.3)

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