BOUND STATES OF STRINGS AND $p$-BRANES

Edward Witten

School of Natural Sciences, Institute for Advanced Study
Olden Lane, Princeton, NJ 08540, USA

The recent discovery of an explicit conformal field theory description of Type II $p$-branes makes it possible to investigate the existence of bound states of such objects. In particular, it is possible with reasonable precision to verify the prediction that the Type IIB superstring in ten dimensions has a family of soliton and bound state strings permuted by $SL(2,\mathbb{Z})$. The space-time coordinates enter tantalizingly in the formalism as non-commuting matrices.

October, 1995

1 Research supported in part by NSF Grant PHY92-45317.
1. Introduction

In many recent developments involving Type II superstrings, particles and $p$-branes carrying Ramond-Ramond charges have played an important role. In many instances it is important to know about possible bound states of such objects. For instance, to make sense of the strong coupling behavior of the Type IIA superstring in ten dimensions it appears to be necessary to assume that there are Ramond-Ramond or RR zero-branes of any (quantized) charge \[1,2\]. On the other hand, to make sense of the behavior of compactified Type II superstrings near certain conifold singularities it seems necessary to assume that two-branes and three-branes behave under certain circumstances as if there are no bound states \[3\].

One of the most interesting problems of this kind – and the main focus in this paper though we will also discuss other cases – concerns one-branes or strings in ten dimensions. The ten-dimensional Type IIB theory is believed to have an $SL(2,\mathbb{Z})$ S-duality symmetry \[1,2\]. The two two-forms of the theory, one from the NS-NS sector and one from the RR sector, transform as a doublet under $SL(2,\mathbb{Z})$.\[2\] The elementary Type IIB superstring is a source for the usual $B$-field from the NS-NS sector, and not for the RR field. Let us describe this by saying that it has charges $(1,0)$. $SL(2,\mathbb{Z})$ will map the elementary string to a string with charges $(m,n)$ for any relatively prime pair of integers $m,n$. The $SL(2,\mathbb{Z})$ prediction\[3\] for the mass of these solitonic and composite strings has been worked out in detail \[6\]. The mass formula is determined by the fact that the extended strings are BPS-saturated, invariant under half of the supersymmetries (which is also the reason that one can make sense of how the spontaneously broken $SL(2,\mathbb{Z})$ acts on these strings), so it will be enough in this paper to find a BPS-saturated string for every relatively prime pair $m,n$ without directly computing the masses.

What makes these questions accessible for study is that an explicit and amazingly simple description of strings (and more general $p$-branes) carrying RR charge has recently

\[2\] In general, the massless $p$-forms of low energy supergravity theories can be assigned to representations of the non-compact symmetry groups even though those groups are spontaneously broken. Otherwise, parallel transport in the moduli space of vacua would clash with Dirac quantization of the charges of $p-1$-branes.

\[3\] This prediction is perhaps somewhat analogous to the $S$-duality prediction of bound states of electrons and monopoles in $N = 4$ supersymmetric Yang-Mills theory, as checked in the two monopole sector by Sen \[5\].
been given \[7\] in the form of \(D\)-strings and \(D\)-branes. The basic \(D\)-string has charges \((0, 1)\), since it was seen in \[7\] to carry RR charge, but on symmetry grounds does not carry the NS-NS charge. In section 2, we show that the \((p, 1)\) bound states exist as an almost immediate consequence of the \(D\)-string structure. To see the \((m, n)\) strings with \(n > 1\) requires a more elaborate construction to which we then turn in section 3. In the process, two-dimensional gauge theory makes a perhaps unexpected appearance; the existence of the BPS-saturated \((m, n)\) strings is equivalent to the existence of certain previously unknown vacua with mass gap in two-dimensional \(N = 8\) supersymmetric gauge theory! A reasonably compelling argument for the existence of these vacua will be presented. We then go on in section 4 to discuss bound states of some of the other \(p\)-branes.

One of the most interesting aspects of the formalism is that the space-time coordinates normal to the \(p\)-brane world-sheet – or all of them for instantons, that is for \(p = −1\) – enter as non-commuting matrices (in the adjoint representation of \(SU(n)\) in the case of a bound state of \(n\) \(D\)-branes). Though perhaps easily motivated given the relation of \(D\)-branes to Chan-Paton factors via \(T\)-duality \[8-12\], this seems possibly significant for the general understanding of string theory.

2. The \((n, 1)\) Strings

First we recall the structure of \(D\)-branes, and verify that the \(D\)-string has the same low-lying world-sheet excitations as the fundamental Type IIB superstring and so can be interpreted as the desired \((0, 1)\) string.

We work in ten-dimensional Minkowski space, with a time coordinate \(x^0\) and space coordinates \(x^1, \ldots, x^9\). Ordinarily the Type II theory has closed strings only. The Dirichlet \(p\)-brane – with \(p\) even or odd for Type IIA or Type IIB – is simply an object whose presence modifies the allowed boundary conditions of strings so that in addition to the usual Neumann boundary conditions, one may also have Dirichlet boundary conditions with (for example) \(x^{p+1} = \ldots = x^9 = 0\). How to compute the mass per unit volume of a \(p\)-brane was explained in \[4\], while the fact that the \(p\)-brane couples to the appropriate RR \(p + 1\)-form is explained in \[7\].

Once such an object is introduced, in addition to the usual closed strings, one can have open strings whose ends are at any values of \(x^0, \ldots, x^p\) with \(x^j = 0\) for \(j > p\). These strings describe the excitations of the \(p\)-brane. The quantization of these open strings is isomorphic to the conventional quantization of an oriented open superstring. The massless states are
a vector and a spinor making up a ten-dimensional supersymmetric Yang-Mills multiplet with gauge group $U(1)$. As the zero modes of $x^j, j > p$ are eliminated by the boundary conditions, the massless particles are functions only of $x^0, \ldots, x^p$. The massless bosons $A_i(x^s), i, s = 0, \ldots, p$ propagate as a $U(1)$ gauge boson on the $p$-brane world-surface, while the other components $\phi_j(x^s)$ with $j > p, s = 0, \ldots, p$, are scalars in the $p+1$-dimensional sense. Note that the vectors have conventional open-string gauge boson vertex operators $V_A = \sum_{i=0}^p A_i(x^s)\partial_\tau X^i$, with $\partial_\tau$ the derivative tangent to the world-sheet boundary, while (as in [9]) the scalars have vertex operators of the form $V_\phi = \sum_{j>p} \phi_j(x^s)\partial_\sigma X^j$ with $\partial_\sigma$ the normal derivative to the boundary. For $\phi_j = \text{constant}$, the boundary integral of $V_\phi$ is the change in the world-sheet action upon adding a constant to $X^j, j > p$, so the scalars can be interpreted as oscillations in the position of the $p+1$-brane. This interpretation is developed in some detail in [13]. The theory on the $p+1$-dimensional world-volume is naturally thought of as the ten-dimensional $U(1)$ supersymmetric gauge theory dimensionally reduced to $p+1$ dimensions.

In particular, consider the one-brane or $D$-string of the Type IIB theory, in the conventional Lorentz background with the RR scalar vanishing. In this case the $U(1)$ gauge field $A_i, i = 0, 1$ describes no propagating degrees of freedom, though it will nonetheless play an important role latter. The massless bosons are the transverse oscillations of the string. As for the massless fermions, after the GSO projection they are a spinor $\psi_{\alpha}, \alpha = 1 \ldots 16$, with definite chirality in the ten-dimensional sense. The left and right-moving fermions on the $D$-brane world-sheet are thus components of ten-dimensional spinors of the same chirality (they are even components of the same spinor), as is usual for Type IIB. Thus, the $D$-string has, as expected, the same world-sheet structure as the elementary Type IIB superstring, making possible its interpretation as the $(0,1) SL(2,\mathbb{Z})$ partner of the $(1,0)$ elementary string.

**Bound States Of Fundamental Strings With A D-String**

Now we begin our study of bound states by looking for the $(m,1)$ states, that is, the bound states of $m$ fundamental strings with a $D$-string. To see that something special must be at work, it is enough to consult the expected mass formula [8]. For the case in which the RR scalar vanishes, the tension in string units of an $(m,n)$ string is

$$T_{m,n} = T \sqrt{m^2 + \frac{n^2}{\lambda^2}}$$

(2.1)
with $T$ a constant and $\lambda$ the string coupling constant. In particular, $T_{0,1} \sim 1/\lambda$, an example of the fact that RR charges have mass of order $1/\lambda$ in string units.

The point about (2.1) which is perhaps surprising at first sight is that the binding energy of an elementary string with a $D$-string to form a $(1,1)$ string is almost one hundred per cent. In fact, an elementary string has a tension of order one, but the difference in tension between a $(0,1)$ $D$-string and a $(1,1)$ string is of order $\lambda$, so to first order the tension energy of an elementary string completely disappears when it combines with a $D$-string.

While this may sound surprising, it is easy to see in the $D$-string picture why it is so. To make the discussion clean, compactify the $x^1$ direction, so that we are working on $\mathbb{R} \times S^1 \times \mathbb{R}^8$, where $\mathbb{R}$ is the time direction, $x^1$ is the spatial direction of the strings we will consider, and $\mathbb{R}^8$ encompasses the normal directions. Consider first, in the absence of $D$-strings, a collection of ordinary strings wrapping around $S^1$. The total string winding number is conserved because the closed strings are not permitted to break, or alternatively because after integrating over $S^1$ there is a gauge field on the non-compact low energy world $\mathbb{R} \times \mathbb{R}^8$ – namely the component $B_{11}$ of the Neveu-Schwarz $B$ field – whose conserved charge is the total winding number of strings.

Now suppose that there is also a $D$-string wrapping around the $S^1$, described as above by allowing strings to terminate at $x^2 = \ldots = x^9 = 0$. In the presence of such a $D$-string, the winding number of elementary strings around the $S^1$ is not conserved! In fact, let $V_W$ be the vertex operator of an elementary string wrapping a certain number of times around the $S^1$. In the presence of the $D$-string, $V_W$ has a non-zero one point function on the disc, because the boundary of the disc, while fixed at $x^j = 0$, $j > 1$, is permitted to wrap around $S^1$ any required number of times.

Since there is a conserved gauge charge coupled to the elementary string winding states, how can it be that the winding states can disappear in the $D$-brane sector? Obviously, the conserved charge must be carried also by some degrees of freedom on the $D$-brane world-sheet. This in fact happens as a result of a mechanism that has long been known [14] in the context of oriented open and closed bosonic strings. The Neveu-Schwarz $B$ field couples in bulk to the string world-sheet $\Sigma$ by an interaction

$$I_B = \frac{1}{2} \int_{\Sigma} d^2 \sigma \epsilon^{\alpha \beta} \partial_{\alpha} X^i \partial_{\beta} X^j B_{ij},$$

(2.2)
Under a gauge transformation $B_{ij} \rightarrow B_{ij} + \partial_i \Lambda_j - \partial_j \Lambda_i$, (2.2) is invariant if the boundary of $\Sigma$ is empty. If not, then $I_B$ transforms as

$$I_B \rightarrow I_B + \int_{\partial \Sigma} d\tau A_i \frac{dX^i}{d\tau}. \quad (2.3)$$

Before concluding that the theory with the $B$ field is inconsistent in the presence of boundaries, one must remember that the oriented open string has a $U(1)$ gauge field $A$, encountered above, which couples to the boundary by

$$I_A = \int_{\partial \Sigma} d\tau A_i \frac{dX^i}{d\tau}. \quad (2.4)$$

Gauge invariance is therefore restored if the gauge transformation of $B$ is accompanied by $A \rightarrow A - \Lambda$. The gauge-invariant field strength is therefore not $F_{ij} = \partial_i A_j - \partial_j A_i$ but $\mathcal{F}_{ij} = F_{ij} - B_{ij}$.

In the case at hand, the $B$ kinetic energy is an integral over all space $\mathcal{M}$, while the gauge field “lives” only on the $D$-brane world-sheet $\Sigma$. The relevant part of the effective action therefore scales like

$$I = \int_{\mathcal{M}} d^{10}x \sqrt{g} \frac{1}{2\lambda^2} |dB|^2 + \int_{\Sigma} d^2x \sqrt{g_{\Sigma}} \frac{1}{2\lambda} \mathcal{F}^2 \quad (2.5)$$

where now $\mathcal{F} = \epsilon^{ij} \mathcal{F}_{ij}/2$ and $g, g_{\Sigma}$ are the metrics on $\mathcal{M}$ and on $\Sigma$. The scaling with $\lambda$ comes from the fact that the first term comes from the two-sphere, and the second from the disc.

A gauge field in two dimensions has no propagating degrees of freedom, and indeed the equation of motion of $A$ asserts that $\mathcal{F}$ is constant. The equation of motion for $B$ contains $\mathcal{F}$ as a source, so $D$-brane states with non-zero $\mathcal{F}$ carry the $B$-charge; these are the sought-for $D$-brane states carrying the charge that in the absence of $D$-branes is carried only by string winding states.

The momentum conjugate to $A$ is $\pi = \mathcal{F}/\lambda$, and as we will see momentarily $\pi$ is quantized in integer units, so $\mathcal{F} = m\lambda$ for integer $m$. With this value of $\mathcal{F}$, the $\mathcal{F}$ dependent part of the energy is of order of order $m^2\lambda$ as expected from (2.1).

Quantization of $\pi$ as claimed above is equivalent to the statement that the gauge group is $U(1)$ rather than $\mathbb{R}$. The Hamiltonian constraints of two-dimensional gauge theory require that the states be invariant under infinitesimal gauge transformations and so are linear combinations of the states $\psi_m(A) = \exp(im\int_S A)$. The state $\psi_m$ — which
has $\pi = m$ – is only invariant under topologically non-trivial $U(1)$ gauge transformations if $m$ is an integer. It must be the case that the gauge group is $U(1)$ and $m$ is quantized, because if $D$-strings could carry a continuum of values of the elementary string winding number, then annihilation of $D$-strings and antistrings would yield an inexistent continuum of string winding number states in the vacuum sector.

If one wants the allowed values of $\pi$ to be not integers $m$, but rather to be of the form $m + \theta/2\pi$ with some fixed $\theta$, then the method to achieve that is well known \[15\]. One must add to the action a term of the form $\int_{\Sigma} \text{constant} \cdot F$. In our problem the “constant” must be the scalar field $\beta$ from the RR sector; the desired coupling should come from a $\beta - A$ two point function on the disc. This shift in the allowed values of the elementary string winding number for $D$-strings appears in the general $SL(2, \mathbb{Z})$-invariant formula \[8\] for the tension of $D$-strings. It is analogous to the $\theta$-dependent shift in the allowed electric charges of a magnetic monopole \[16\].

There actually is a good analogy between the problem we have just analyzed and certain questions about monopoles in, say, $N = 4$ super Yang-Mills theory in four dimensions. In that theory, the BPS formula for masses of electrons and dyons shows that for weak coupling the rest mass of an electron nearly disappears when it combines with a monopole. That occurs because the classical monopole solution is not invariant under charge rotations; there is therefore a collective coordinate which for weak coupling can carry charge at very little cost in energy. For strings, the gauge transformation that measures string winding number is a constant mode of the gauge parameter $\Lambda$ of the $B$ field. The equation $\delta A = -\Lambda$ shows that the $D$-string is not invariant under this gauge transformation, so that there is a collective coordinate – namely $\int_{S^1} A$ – that can carry string winding number at very little cost in energy.

3. Bound States Of $D$-Strings

We will here describe the general setting for analyzing bound states of $D$-branes and then apply it to $D$-strings.

First of all, to find a bound state of $n$ parallel Dirichlet $p$-branes, one needs a description of the relevant low energy physics that is valid when the branes are nearby. Because

\[4\] Though the principles for $p$-forms of $p > 1$ are less clear, an analogous mechanism involving compactness of the gauge group may well cause the vacuum parameter discussed at the end of \[7\] to be quantized.
of the relation of $D$-branes to Chan-Paton factors, there is an obvious guess for what happens when the $D$-branes are precisely on top of each other: this should correspond to an unbroken $U(n)$ gauge symmetry, with the effective action being ten-dimensional $U(n)$ supersymmetric gauge theory, dimensionally reduced to the $p+1$-dimensional world-volume of the $D$-brane.

This ansatz can be justified by considering two parallel Dirichlet $p$-branes, say one at $x^j = 0$, $j > p$, and one at $x^j = a^j$, $j > p$. To find the excitation spectrum in the presence of two $p$-branes, in addition to closed strings, we must consider strings that start and end on one or the other $p$-brane. Strings that start and end on the same brane give a $U(1) \times U(1)$ gauge theory (with one $U(1)$ living on each $p$-brane) as we have discussed at the beginning of section 2. We must also consider strings starting at the first brane and ending at the second, or vice versa. Such strings have $U(1) \times U(1)$ charges $(-1, 1)$ or $(1, -1)$, respectively. The ground state in this sector (as usual for supersymmetric open strings) is a vector multiplet; as the string must stretch from the origin to $a = (a^{p+1}, \ldots, a^9)$, the vector multiplet mass, which is the ground state energy in this sector, is $T|a|$ with $T$ the elementary string tension. Thus as $a \to 0$, world-volume vector bosons of charges $(\mp 1, \pm 1)$ becomes massless, giving the restoration of $U(2)$ gauge symmetry on the world-volume. Likewise, as $n$ parallel branes become coincident, one has restoration of a $U(n)$ gauge symmetry on the world-volume of the $p$-brane. Filling out the vector multiplets, the low energy theory on the world-volume is simply the dimensional reduction of ten-dimensional supersymmetric Yang-Mills theory with gauge group $U(n)$ from ten to $p + 1$ dimensions.

To make the physical interpretation somewhat clearer, let us consider in some detail the bosonic fields of this theory. In addition to the world-volume $U(n)$ gauge field $A$, there are $9 - p$ scalar fields in the adjoint representation of $U(n)$. Changing notation slightly from section 2, we will call them $X^j$, $j = p+1, \ldots, 9$, because of their interpretation, which will be clear momentarily, as position coordinates of the $D$-branes. The potential energy for the $X^j$ in the dimensional reduction of ten-dimensional supersymmetric Yang-Mills theory is

$$V = \frac{T^2}{2} \sum_{i,j=p+1}^{9} \text{Tr} [X^i, X^j]^2. \quad (3.1)$$

Classical states of unbroken supersymmetry or zero energy thus have $[X^i, X^j] = 0$, so the $X^i$ can be simultaneously diagonalized, giving $X^i = \text{diag}(a^i_{(1)}, a^i_{(2)}, \ldots, a^i_{(n)})$. We interpret such a configuration as having $n$ parallel $p$-branes with the position of the $\lambda^{th}$ $p$-brane,
$\lambda = 1, \ldots, n$ being the space-time point $a_\lambda$ with coordinates $a^i_\lambda$. As support for this interpretation, note that the masses obtained by expanding (3.1) around the given classical solution are just $T|a_\lambda - a_\mu|$, $1 \leq \lambda < \mu \leq n$, corresponding to strings with one end at $a_\lambda$ and one at $a_\mu$. Note that the diagonalization of the $X^j$ is not unique; the Weyl group of $U(n)$ acts by permuting the $a_\lambda$’s, corresponding to the fact that the $n$ $p$-branes are identical bosons (or fermions).

In sum, then, the naive configuration space of $n$ parallel, identical $p$-branes should be interpreted as the moduli space of classical vacua of the supersymmetric $U(n)$ gauge theory. When the $p$-branes are nearby, the massive modes cannot be ignored; one must then study the full-fledged quantum $U(n)$ gauge theory, and not just the slow motion on the classical moduli space of vacua. The appearance of non-commuting matrices $X^j$ which can be interpreted as space-time coordinates to the extent that they commute is highly intriguing for the future. In what follows, we use this formalism to study string and $p$-brane bound states.

The Mass Gap

In fact, a BPS-saturated state of $n$ $p$-branes would correspond simply to a supersymmetric ground state of the effective Hamiltonian, that is to a supersymmetric vacuum of the $p + 1$-dimensional supersymmetric gauge theory. Let us discuss just what kind of vacuum would correspond to a bound state.

A bound state is a state that is normalizable except for the center of mass motion. In the problem at hand, there is no difficulty in separating out the overall motion of the center of mass. A $U(n)$ gauge theory is practically the same thing as the product of a $U(1)$ gauge theory and an $SU(n)$ gauge theory. (There is a subtle global difference, which will be important later but not in the preliminary remarks that we are about to make.) In our problem, the $U(1)$ factor of the gauge group describes the motion of the string center of mass. In fact, the $U(1)$ theory has the right degrees of freedom to do that, since in section two we saw that the $p$-brane world-volume theory at low energies was simply $U(1)$ supersymmetric gauge theory. And it is clear that adding a constant to the position of all $p$-branes shifts the $U(1)$ variable $\text{Tr}X^i$ without changing the traceless $SU(n)$ part.

In our problem of finding string bound states, since the $U(1)$ theory already has massless excitations in correspondence with those of the elementary string, a supersymmetric $SU(n)$ vacuum that could give an $SL(2, \mathbb{Z})$ transform of the elementary string must have a mass gap, as any extra massless excitations would spoil the correspondence with the
elementary string. The possible existence of vacua with a mass gap is rather delicate and surprising from several points of view. For instance, note that we are here dealing with $N = 8$ supersymmetric Yang-Mills theory, obtained by dimensional reduction from ten dimensions. In the $N = 2$ or $N = 4$ theories, one could immediately exclude the possibility of a vacuum with a mass gap because those theories have chiral $R$-symmetries with anomalous two point functions which (according to a well-known argument by 't Hooft) imply that there can be no mass gap in any vacuum. For $N = 8$, there is an $SO(8)$ $R$-symmetry that acts chirally in the sense that the left-moving fermions transform in one spinor representation and the right-movers in the other; as these representations have the same quadratic Casimir, there is no anomaly in the two point function and no immediate way to disprove on such grounds the existence of a vacuum with a mass gap.

**Topological Sectors**

Since we want ultimately to study bound states of $m$ elementary strings and $n$ D-strings, we must determine precisely what question in the gauge theory corresponds to such a collection of strings. It is convenient to answer this by first asking what topological sectors the $U(n)$ theory has, and then interpreting these for our problem.

In general, in two-dimensional gauge theory with any gauge group $G$, topological sectors can be introduced by placing a charge at infinity in any desired representation $\mathcal{R}$. Because of screening, only the transformation of $\mathcal{R}$ under the center of the group really matters. For instance, for the free theory, $G = U(1)$ without matter, we can place at infinity a particle of any desired charge $m$. We then get a state, encountered in section 2, in which the canonical momentum of the gauge field is $\pi = m$. For $SU(n)$, with all fields being in the adjoint representation, we can place at infinity a charge of any desired “$n$-ality,” so there are $n$ sectors to consider, as discussed in [17].

Let us consider our problem with gauge group $U(n)$ (we take at face value the Chan-Paton structure that indicates that the global structure of the gauge group is precisely $U(n)$). Consider what happens if one places at infinity a “quark” in the fundamental representation of $U(n)$. To see the physical interpretation, consider a vacuum consisting of $n$ widely separated parallel D-strings with commuting matrices $X^j$. This breaks $U(n)$ to $U(1)^n$, one $U(1)$ factor for each D-strings. Any given state of the quark has charge one for one $U(1)$ and zero for the others, so one of the $n$ D-strings is bound with one elementary string and the others with none. This topological sector therefore has the quantum numbers of $n$ D-strings and one elementary one. By a similar reasoning, if we
put at infinity a tensor product of \( m \) quarks, we get \( m \) elementary strings together with \( n \) \( D \)-strings.

As a special case, an antisymmetric combination of \( n \) copies of the fundamental representation is trivial as an \( SU(n) \) representation -- and so does not affect the center of mass or \( SU(n) \) part of the theory at all -- but by shifting the \( U(1) \) field strength adds \( n \) elementary quarks. Thus the center of mass dynamics of \( m \) elementary strings and \( n \) \( D \)-strings depends only on the value of \( m \) modulo \( n \). This is one of the predictions of \( SL(2,\mathbb{Z}) \), as the matrix

\[
\begin{pmatrix}
1 & t \\
0 & 1
\end{pmatrix},
\]

with arbitrary integer \( t \), can in acting on

\[
\begin{pmatrix}
m \\
n
\end{pmatrix}
\]

shift \( m \) by any desired multiple of \( n \). This result is analogous to the fact that in Sen’s study of bound states in \( N = 4 \) supersymmetric Yang-Mills theory in four dimensions \cite{Sen}, one sees without any detailed calculation that the number of BPS-saturated bound states of \( m \) electrons and \( n \) monopoles depends only on the value of \( m \) modulo \( n \).

**Known Vacua**

Now let us discuss the known supersymmetric vacua of this theory. The only known vacua, roughly speaking, are the ones in which the matrices \( X^j \) commute, with large and distinct eigenvalues, breaking \( SU(n) \) to \( U(1)^{n-1} \). Supersymmetric non-renormalization theorems imply that the energy is exactly zero in that region, not just in the classical approximation.

Speaking of a family of vacua is actually a rough way of describing the situation; because of two-dimensional infrared divergences, the space of eigenvalues of the \( X^j \) up to permutation is more like the target space of a string theory than the parameter space of a family of vacua (as it would be if we were discussing \( p \)-branes of \( p > 1 \)). At any rate, it is true that -- at least if we do not introduce a charge at infinity -- the \( X^j \) can become large at no cost in energy, and therefore that the thermodynamic quantities diverge as if there were a family of vacua.

What happens if there is a charge at infinity? Since a few tricky issues will arise, let us begin by considering some special cases. Suppose that \( G = SU(2) \) and that the representation at infinity is a quark doublet. If the \( X^j \) go to infinity, \( SU(2) \) is broken to
\textit{U(1).} The \textit{U(1)} theory is free at energies low compared to the mass scale determined by $\langle X^j \rangle$, so subsequent calculations can be made semiclassically. Higgsing of $SU(2)$ to $U(1)$ does not lead to screening of the charge at infinity, since every component of the $SU(2)$ doublet is charged under $U(1)$. Thus the $SU(2)$ theory with a quark (or any representation odd under the center of $SU(2)$) at infinity has the property that there is an energetic barrier to going to large $X^j$ (or more exactly the energy at infinity is bounded strictly above the supersymmetric value). This does not answer the question of whether the $SU(2)$ theory with a quark at infinity has a supersymmetric ground state; it merely says that such a state, if it exists, decays exponentially for large $X^j$.

For an example that gives a different answer, take $n = 4$, that is $G = SU(4)$, and let $X^j$ go to infinity in a direction breaking $SU(4)$ to $SU(2) \times SU(2) \times U(1)$. For $m = 1$, we can take the representation at infinity to be the fundamental representation $S$ of $SU(4)$. Every component of $S$ is charged under the $U(1)$, so for $m = 1$, $n = 4$, there is an energetic barrier to going to infinity in this direction (or any other, as one can easily verify). Now try $m = 2$, $n = 4$. We can take the representation at infinity to be the antisymmetric product $\wedge^2 S$ of two copies of $S$. Screening now occurs as $\wedge^2 S$ contains a component that is neutral under $U(1)$. This component transforms as $(2, 2)$ under $SU(2) \times SU(2)$, so that each $SU(2)$ has $m = 1$ – a quark at infinity. Thus, we cannot determine without understanding the $SU(2)$ dynamics – which is given by a strongly coupled quantum field theory near the origin – whether in the $SU(4)$ theory with $m = 2$ there is an energetic barrier to going to large $X^j$. If – as we argue later – there is a supersymmetric ground state of the $m = 1$, $n = 2$ system, then there is no energetic barrier to taking $X^j$ to infinity in this particular direction.

In each case, what we have described in the gauge theory language is perfectly obvious in terms of the original $D$-strings. The discussion of $m = 1, n = 2$ amounted to saying that (as is obvious from the BPS formula) if the $(m, n) = (1, 2)$ system is separated into $(0, 1)$ and $(1, 1)$ subsystems, then the energy is greater than the supersymmetric or BPS value for the $(1, 2)$ system. The discussion of the $(2, 4)$ case amounted to showing that if there is an $m = 1, n = 2$ supersymmetric bound state, then there is no energetic barrier to separating the $(2, 4)$ system into two $(1, 2)$ subsystems.

Now we discuss the general case. First we show in gauge theory language that if $m$ is prime to $n$, there is an energetic barrier to taking $X^j$ to infinity in any direction. Taking $X^j$ to infinity in any direction will break $SU(n)$ to a subgroup that contains at least one $U(1)$ factor that can be treated semiclassically; it is enough to show that one can pick
a $U(1)$ for which every component of the representation at infinity is charged. Indeed, one can always pick an unbroken $U(1)$ that has only two eigenvalues in the fundamental representation $S$ of $SU(n)$, say of multiplicity $a$ and $n-a$, with $1 \leq a \leq n$. The eigenvalues are $(n-a)$ and $-a$. For given $m$, if we place at infinity the tensor product of $m$ copies of $S$, the possible $U(1)$ eigenvalues are $s(n-a) + (m-s)(-a)$ with $0 \leq s \leq m$; this is congruent to $-ma$ modulo $n$, and so can never vanish if $m$ is prime to $n$. Thus the expected energetic barrier exists when $m$ and $n$ are relatively prime.

For the converse, we must show that if $m$ and $n$ are not relatively prime, then without knowledge of strongly coupled non-abelian gauge dynamics, one cannot prove that there is an energetic barrier to going to large $X$. Place at infinity the $m^{th}$ antisymmetric tensor power of $S$. Let $d$ be the greatest common divisor of $m$ and $n$. Take the $X^j$ to infinity in a direction that breaks $SU(n)$ to $SU(n/d)^d$ times a product of $U(1)$’s. The $m^{th}$ antisymmetric power of $S$ contains a component neutral under all of the $U(1)$’s, so without knowledge of strongly coupled nonabelian gauge dynamics one cannot show that there is an energetic barrier to taking the $X^j$ to infinity in the chosen direction (which corresponds to separating the $(m,n)$ system into $d$ separate $(m/d,n/d)$ subsystems).

The Perturbation Argument

In what follows, we will use an argument that only works when there is an energetic barrier to taking the $X^j$ large, and so only works when $m$ and $n$ are relatively prime. When $m$ and $n$ are not relatively prime, one would be dealing – because the energetic barrier to separation is absent – with bound states at threshold, which are extremely delicate things (though they are possible in quantum mechanics and duly appear when required by duality, as in [8,19]). In the present case, as there are apparently no bound states at threshold of elementary strings, $SL(2,\mathbb{Z})$ requires the absence of bound states whenever $m,n$ are not relatively prime. But here I will focus only on the relatively prime case in which the energetic barrier makes things easier.

We have noted above that the supersymmetric vacua we want for the relatively prime case should have a mass gap, to avoid giving massless world-sheet modes that do not have counterparts for the elementary string. It is in fact highly plausible that any supersymmetric ground state of the $(m,n)$ system in the relatively prime case would have a mass gap. We at least know that any such state would have a true normalizable vacuum, since there is an energetic barrier against the $X^j$ becoming large. There are no known $N > 4$ supersymmetric systems with $SO(N)$ symmetry, massless particles and a normalizable vacuum;
the lack of an $N > 4$ superconformal algebra with unitary representation theory tends to be an obstruction since it means that there is no conformal field theory for such a system to flow to in the infrared. So we assume that supersymmetric vacua of the $(m, n)$ system all have a mass gap. There can be only finitely many of them as none can be at large $X^i$; the problem is to determine the number.

To argue that the number is one, we use the fact that, once it is given that there is a mass gap, the number of ground states cannot change under a small perturbation. (With massless particles, it would be possible for a vacuum to split into several under a weak perturbation, or to break supersymmetry and disappear from the list of supersymmetric vacua.) We will simply pick a judicious perturbation that makes things simple.

$N = 1$ supersymmetric Yang-Mills theory reduces to $N = 4$ in four dimensions. The $N = 4$ theory, viewed as an $N = 1$ theory, has in addition to the vector multiplet three chiral superfields $A, B,$ and $C$ in the adjoint representation. The superpotential is $\text{Tr} A[B, C]$. We make a perturbation to a system with superpotential

$$W = \text{Tr} \left( A[B, C] - \frac{\epsilon}{2} (A^2 + B^2 + C^2) \right).$$

(3.4)

(The special features of this perturbation were exploited previously in [20,21].)

Of course, we are really interested in the dimensional reduction of this system to two dimensions. Notice that, by rescaling of $A, B, C$, and the world-sheet coordinates, and an $R$-transformation, operations that only change the Lagrangian by an irrelevant operator $\int d^2 \sigma d^4 \theta \ldots$, one can adjust $\epsilon$ to any desired value, given that it is non-zero. Thus while the mass gap permits us to introduce a small $\epsilon$ without disturbing the vacuum structure, we may in fact take $\epsilon$ large and use semi-classical reasoning.

The condition for a vacuum state

$$[A, B] = \epsilon C$$
$$[B, C] = \epsilon A$$
$$[C, A] = \epsilon B$$

(3.5)

asserts that the quantities $A/\epsilon$, $B/\epsilon$, and $C/\epsilon$ obey the commutation relations of $SU(2)$. At the classical level, a vacuum is given by specifying an $n$-dimensional representation of $SU(2)$, together with the values of the scalar fields $\phi, \bar{\phi}$ that are in the vector multiplet.
in four dimensions, but become scalars in two dimensions. The two-dimensional scalar potential has extra terms

$$\Delta V = \sum_{X=A,B,C} \text{Tr} \left( [X, \phi][X, \bar{\phi}] + [X, \bar{\phi}][X, \phi] \right) + \text{Tr}[\phi, \bar{\phi}]^2,$$

which imply that the energy can vanish classically only if $\phi, \bar{\phi}$ commute with each other and with $A, B, C$.

We want to know whether with a representation at infinity of $n$-ality equal to $m$, this system has a supersymmetric ground state with mass gap at the quantum level. We can split the discussion into three parts, depending on whether the $SU(2)$ representation is (i) trivial, (ii) non-trivial but reducible, or (iii) irreducible. In the trivial case, as $A, B, C$ are massive in expanding around $A = B = C = 0$, the low energy theory is the two-dimensional $N = 2$ $SU(n)$ theory (associated with dimensional reduction from four-dimensional $N = 1$). By an argument given above, that theory does not have a vacuum with a mass gap\(^5\) (and presumably breaks supersymmetry because of the inability to screen the charge at infinity). In case (ii), the low energy theory has at least one unbroken $U(1)$ under which (by an argument given above) no component of the charge at infinity is neutral, so supersymmetry is spontaneously broken at the quantum level.

The interesting case therefore is case (iii). In this case, the gauge group is completely broken – screening the charge at infinity – and all fields, including $\phi$ and $\bar{\phi}$, get a mass. This therefore is the desired unique vacuum with mass gap corresponding to the ($m, n$) string.

\(^5\) There is a subtlety here: the argument for absence of mass gap in the $N = 2$ theory depends on a chiral symmetry which is here explicitly broken by coupling to the massive fields. The breaking is, however, irrelevant at low energies unless a twisted chiral superpotential is generated. This can be excluded by a variety of standard arguments. For instance, the twisted chiral superpotential would have to be independent of the mass of the massive particles, as $\epsilon$ is chiral rather than twisted chiral. In two-dimensional superrenormalizable theories like this one, the effects of massive particles on the massless ones vanish for large mass, except sometimes for ill-convergent one-loop tadpoles (absent here because as all fields are in the adjoint representation of $SU(n)$, the tadpoles vanish). So the fact that the twisted chiral superpotential must be independent of $\epsilon$ means that it cannot be generated at all.
4. Bound States Of Other $D$-Branes And $p$-Branes

In this section, we briefly apply some of the same methods to other questions involving bound states of $p$-branes. For any $p$, one has to look at the dimensional reduction of ten-dimensional supersymmetric Yang-Mills theory to $p + 1$ dimensions. We first consider the Type IIB theory, and then more superficially Type IIA.

4.1. Type IIB

Instantons

One might begin with the Type IIB $-1$-branes or instantons. They are given by the dimensional reduction of supersymmetric Yang-Mills theory all the way to zero dimensions. Thus, in the $n$-instanton sector, all ten space-time coordinates become matrices $X^i$, $i = 0, \ldots, 9$, while the fermion zero modes in the instanton sector will be represented by the supersymmetric partners $\psi_\alpha$, $\alpha = 1, \ldots, 16$, also in the adjoint representation. The instanton measure is an integral over $X$ and $\psi$ weighted by $e^{-I}$, with $I$ a multiple of the dimensionally-reduced supersymmetric Yang-Mills action:

$$I = \sum_{i<j} \text{Tr}[X^i, X^j]^2 + \sum_{i,\alpha,\beta} \Gamma^i_{\alpha\beta} \text{Tr}\psi^\alpha [X^i, \psi^\beta].$$

(4.1)

(The $\Gamma$’s are gamma-matrices.) This is the pure case of seeing the space-time coordinates as non-commuting matrices. For a supersymmetric minimum of the action, of course, we minimize the action with $[X^i, X^j] = 0$, and then the matrices commute. One can ask whether non-supersymmetric instantons can be found as higher critical points of $I$ with non-commuting $X$’s. A scaling argument shows that there are none: $I$ scales as $t^4$ under $X \rightarrow tX$, so any critical point has $I = 0$.

This description of the effective action for $-1$-branes has an intriguing similarity to the ADHM description of Yang-Mills instantons in four dimensions, which are determined by matrices $X^i$, $i = 1, \ldots, 4$, obeying not $[X^i, X^j] = 0$ but the self-dual counterpart $[X^i, X^j] = \frac{1}{2} \varepsilon^{ijkl} [X_k, X_l]$.

Three-branes

We next move on to consider bound states of Dirichlet three-branes. A bound state with the same world-volume structure as the basic three-brane would correspond to a vacuum with mass gap of $N = 4$ $SU(n)$ supersymmetric Yang-Mills theory in four dimensions.
There are none, since anomalous triangle diagrams of the $SU(4)$ $R$-symmetry currents imply – by ’t Hooft’s old argument – that this theory can have no vacuum with mass gap. In fact, it is strongly believed that in this theory the moduli space of vacua is exactly given by the classical answer. In three-brane language, this means that the vacua are labeled by the positions of the $n$ three-branes, up to permutation, so that there are no bound states even with exotic world-volume structure.

One might think that this result on absence of three-brane bound states is what is needed to justify the assumption in [3] of considering in conifold physics only simple, and not multiple, wrapping of three-branes around collapsing three-cycles. But this would be too hasty a conclusion; we have argued here only the absence of three-brane bound-states in flat space, and the conclusion does not immediately carry over to the Calabi-Yau context where the three-branes are wrapped around a curved cycle. The opposite result in flat space would, however, have been undesirable; a bound state in flat space would have led to a bound state in the conifold problem when the radius of curvature is large enough (larger than the largest length scale important in the structure of the flat space bound state), and therefore, by BPS-saturation, also when it is small. By studying the soft modes in $N = 4$ super Yang-Mills theory on $\mathbb{R} \times S^3$, it can be seen that the bound state problem for $n$ three-branes wrapped around an $S^3$ is equivalent to the question – which will not be addressed here – of whether there are bound states at threshold in the dimensional reduction to 0 + 1 dimensions of four-dimensional $N = 1$ super Yang-Mills theory with gauge group $SU(n)$.

Five-Branes

For Dirichlet five-branes the story is formally the same. For bound states with the same structure as the elementary $D$-brane, one must consider vacua with mass gap in six-dimensional supersymmetric Yang-Mills theory. There can be none because of anomalous four-point functions of the $SU(2) \times SU(2)$ $R$-symmetry. One might worry about the unrenormalizability of six-dimensional super Yang-Mills theory, but this issue seems inessential as the string theory provides some sort of cutoff, and the anomaly argument is an infrared argument. The unrenormalizability – and weak coupling in the infrared – strongly suggest that the moduli space of vacua is given by the classical answer, and hence that there are also no bound states with exotic world-volume structure.

But the absence of bound states of Dirichlet five-branes with each other is not the whole story for five-branes. One must also consider the solitonic five-brane [22-24]. In
fact the two sorts of five-brane form an $SL(2, \mathbb{Z})$ doublet just like the two kinds of string. Electric-magnetic duality in ten dimensions pairs the five-branes with strings. $SL(2, \mathbb{Z})$ predicts the existence of a bound state of $m$ Dirichlet five-branes with $n$ solitonic ones for every relatively prime pair $m, n$. The prediction cannot be fully tested at the moment because the solitonic five-branes are not sufficiently well understood, but a few simple remarks are possible.

Note that in string units, the elementary string and $D$-string have tensions of order 1 and $1/\lambda$, respectively; in Einstein units (where duality acts naturally) these become $\sqrt{\lambda}$ and $1/\sqrt{\lambda}$. The Dirichlet and solitonic five-brane have tensions of order $1/\lambda$ and $1/\lambda^2$, respectively, in string units; in Einstein units the tensions are of order $\sqrt{\lambda}$ and $1/\sqrt{\lambda}$. There is thus a parallel between strings and five-branes with the elementary string mapped to the Dirichlet five-brane and the $D$-string mapped to the solitonic five-brane. A BPS-saturated bound state of $m$ Dirichlet five-branes and $n$ solitonic ones would have tension in string units

$$\tilde{T}_{m,n} = \frac{1}{\lambda} \sqrt{m^2 + n^2 / \lambda^2}, \quad (4.2)$$

a formula quite analogous to (2.1) for strings.

The first bound state problem that we considered in section two was a bound state of one $D$-string and $m$ elementary strings. The main qualitative issue was that the tension of the elementary string completely disappears (in the weak coupling limit) in the presence of a $D$-string. There is a similar issue for a bound state of one solitonic five-brane with $m$ Dirichlet five-branes. The $m$ Dirichlet five-branes in vacuum would have a tension of order $m/\lambda$, but the above formula says that for weak coupling this energy practically disappears in the field of a solitonic five-brane. Adding the $m$ Dirichlet five-branes to the solitonic one should increase the ground state energy (in string units) by an amount only of order $m^2$, and not $m/\lambda$, for small $\lambda$.

How can this be? The key is that the soliton five-brane is given explicitly by a four-dimensional solution which contains a region (an infinite tube that represents a sort of hole in space-time) in which the dilaton blows up and the effective value of $\lambda$ goes to infinity. The energy of a Dirichlet five-brane therefore vanishes as it falls down the hole. To make this quantitative, one would have to understand better the strong-coupling region of the soliton, but one can at least assert that the main surprising feature of the soliton solution, which is existence of the strong coupling end, is just what is needed to make the $SL(2, \mathbb{Z})$ prediction possible.
Seven-branes

The Type IIB theory also has a Dirichlet seven-brane. Its bound states with standard world-volume structure would correspond to vacua with mass gap in the dimensional reduction of ten-dimensional super Yang-Mills theory to eight dimensions. Such vacua do not exist, because the $U(1)$ global symmetry (which arises in the dimensional reduction from ten to eight dimensions) has an anomalous five point function, which would be impossible in a vacuum with mass gap. The weak infrared coupling of the theory strongly suggests that bound states with exotic world-volume structure are also absent.

4.2. Type IIA Superstrings

Now we move on to the Type IIA superstring in ten dimensions. There are Dirichlet $p$-branes for $p$ even, while elementary and solitonic $p$-branes only exist for odd $p$ (in fact, $p$ equal to one or five). So the only bound states to inquire about are the bound states of $D$-branes with themselves, which correspond to vacua with mass gap in ten-dimensional $SU(n)$ super Yang-Mills theory dimensionally reduced to $p + 1$ dimensions.

Predictions for small $p$ seem to follow from results about string dynamics. For $p = 0$, where one is dealing with ordinary particles (carrying Ramond-Ramond electric charge), precisely one bound state for each $n$ is apparently needed to agree with the Kaluza-Klein spectrum of eleven-dimensional supergravity. For $p = 2$, to make sense of the physics of conifolds [3], one wants no bound state when two-branes are wrapped around a two-cycle, and therefore (taking the limit as the Calabi-Yau manifold is scaled up) no bound state in flat space. For $p > 2$ there seem to be no known predictions.

For $p = 0$, one has ordinary quantum mechanics, albeit supersymmetric quantum mechanics of a rather special sort. One wants to know whether there are bound states at threshold, and these might be accessible to analysis, though the question is beyond the reach of the present paper.

For $p > 0$, one is dealing with odd-dimensional quantum field theory, and anomaly-based arguments to exclude ground states with mass gap are not nearly as powerful as they are in even dimensions. In some cases, however, some results can be obtained using discrete anomalies. For instance, for $p = 2$, the relevant three-dimensional super Yang-Mills theory has an $SO(7)$ global symmetry (obtained by dimensional reduction from ten dimensions). If one weakly gauges the $SO(7)$, then one can consider whether the effective action of the theory is even or odd under a topologically non-trivial $SO(7)$ gauge transformation. For $n$ even (so that the dimension of $SU(n)$ is odd), reasoning given on p. 309 of [2] shows
that the effective action is odd, behavior that cannot be reproduced in a parity-conserving theory with mass gap. (The global anomaly can be reproduced in a theory with mass gap by adding a parity-violating Chern-Simons interaction.) So any vacua with mass gap have spontaneously broken parity and are paired by the action of parity; the total number of bound states with standard world-volume structure is therefore even.
References

[1] P. Townsend, “The Eleven-Dimensional Super-Membrane Revisited,” Phys. Lett. B350 (1995) 184, hep-th/9501068.
[2] E. Witten, “String Theory Dynamics In Various Dimensions,” Nucl.Phys. B443 (1995) 85, hep-th/9503124.
[3] A. Strominger, “Massless Black Holes And Conifolds In String Theory,” Nucl. Phys. B451 (1995) 96, hep-th/9504090.
[4] C. Hull and P. Townsend, “Unity Of Superstring Dualities,” Nucl. Phys. B438 (1995) 109, hep-th/9410167.
[5] A. Sen, “Dyon-Monopole Bound States, Self-dual Harmonic Forms On The Multi-Monopole Moduli Space, and $SL(2,\mathbb{Z})$ Invariance In String Theory,” Phys. Lett. B329 (1994) 217, hep-th/9402032.
[6] J. H. Schwarz, “An $SL(2,\mathbb{Z})$ Multiplet Of Type II Superstrings,” hep-th/9508143, “Superstring Dualities,” hep-th/9509148.
[7] J. Polchinski, “Dirichlet Branes And Ramond-Ramond Charges,” hep-th/9510017.
[8] J. Dai, R. G. Leigh, and J. Polchinski, “New Connections Between String Theories,” Mod. Phys. Lett. A4 (1989) 2073.
[9] J. Polchinski, “Combinatorics of Boundaries In String Theory,” Phys. Rev. D50 (1994) 6041, hep-th/9407031.
[10] P. Horava, “Strings On World-Sheet Orbifolds,” Nucl. Phys. B327 (1989) 461, “Background Duality Of Open String Models,” Phys. Lett. B321 (1989) 251.
[11] M. B. Green, “Space-Time Duality And Dirichlet String Theory,” Phys. Lett. B266 (1991) 325.
[12] A. Sagnotti, “Open Strings And Their Symmetry Groups,” in Cargese ‘87, “Nonperturbative Quantum Field Theory,” ed. G. Mack et. al. (Pergamon Press, 1988) p. 527.
[13] R. Leigh, “Dirac-Born-Infeld Action From Dirichlet Sigma Model,” Mod. Phys. Lett. A4 (1989) 2767.
[14] E. Cremmer and J. Scherk, “Spontaneous Dynamical Breaking Of Gauge Symmetry In Dual Models,” Nucl. Phys. B72 (1974) 117.
[15] S. Coleman, “More About The Massive Schwinger Model,” Ann. Phys. 101 (1976) 239.
[16] E. Witten, “Dyons Of Charge $e\theta/2\pi$,” Phys. Lett. 86B (1979) 283.
[17] E. Witten, “Theta Vacua In Two-Dimensional Quantum Chromodynamics,” Nuovo Cim. 51A (1979) 325.
[18] J. A. Harvey and J. P. Gauntlett, “$S$-Duality And The Dyon Spectrum In $N = 2$ Super-Yang-Mills Theory,” hep-th/9508156.
[19] S. Sethi, M. Stern, and E. Zaslow, “Monopole and Dyon Bound States In $N = 2$ Supersymmetric Yang-Mills Theories,” hep-th/9508117.

[20] C. Vafa and E. Witten, “A Strong Coupling Test Of S-Duality,” Nucl. Phys. B431 (1994) 3, hep-th/9408074.

[21] R. Donagi and E. Witten, “Supersymmetric Yang-Mills Theory and Integrable Systems,” hep-th/9510101, section 4.

[22] A. Strominger, “Heterotic Solitons,” Nucl. Phys. B343 (1990) 167.

[23] C. G. Callan, Jr., J. A. Harvey, and A. Strominger, “World-Sheet Approach To Heterotic Instantons And Solitons,” Nucl. Phys. B359 (1991) 611.

[24] M. J. Duff, R. R. Khuri, and J. X. Lu, “String Solitons,” Phys. Rept. 259 (1995) 213-326,1995, hep-th/9412184.

[25] L. Alvarez-Gaumé and E. Witten, “Gravitational Anomalies,” Nucl. Phys. B234 (1983) 269.