On the Distribution of Products of Spherical Classes in Classical Symmetric Spaces of Rank One

Jafar Shaffaf\textsuperscript{1}

Abstract

The distribution of products of random matrices chosen from fixed spherical classes is determined for classical rank 1 symmetric spaces. It is observed that \( n \to \infty \) limit behaves approximately as in the abelian case. A theorem on the rate of convergence to the Haar measure in the case of \( SU(n) \) is also established.

AMS Subject Classification 2000: 22E46, 53C35, 43A90.

1 Introduction

A basic problem in random matrix theory is the determination of the distribution of products of matrices chosen randomly from specified classes. For example the problem of the support of products of random unitary matrices chosen from fixed conjugacy classes, or sums of Hermitian matrices of given eigenvalues is treated in [AW] and [F] (which is based on the work of Klyachko [Kl]). In this paper we investigate the distribution of products of random matrices chosen from spherical classes for classical rank 1 symmetric spaces. When matrices are chosen from conjugacy or spherical classes from a simple group of \( 2 \times 2 \) matrices a complete solution is given in [Sh1] and [Sh2].

The distribution of products of spherical classes may be interpreted as the determination of the algebra structure for the Hecke algebra generated by the singular measures concentrated on orbits of \( K \times K \) in \( G \) for a symmetric pair \((G, K)\). It can also be given an interpretation in geometric probability. Consider for example \( S^{2n-1} \) with the natural action of \( SU(n) \) on it. Fix a longitude \( \Lambda \subset S^{2n-1} \), let \( z_i \in \Lambda, i = 1, 2 \), and choose matrices \( g_i \) randomly from \( SU(n) \) (according to Haar measure) conditioned on the requirement that \( g_i.z_i \) lies in the same meridian as \( z_i \). Then one may inquire about the density function for \( g_2g_1 \).

For the case of classical symmetric spaces of rank 1 a complete solution to this problem is given in this paper (Theorems 3.1, 4.1, 5.1 and 5.2, 6.1). It is observed that as the

\textsuperscript{1}Institute for Studies in Theoretical Physics and Mathematics (IPM) and Sharif University of Technology, Tehran, Iran. Email: shaffaf@ipm.ir
dimension of the symmetric space of a given class tends to infinity, the distribution of the
product measures converges weakly to a singular continuous measure concentrated on a
single orbit of $K \times K$. This may be rephrased as the $n \to \infty$ limit behaves approximately as
in the abelian case. A similar limiting behavior is also observed for higher rank symmetric
spaces but this more complex limit theorem will be treated later in another publication.

In view of this limiting behavior, in section 7 the rate of convergence to Haar measure
on $SU(n)$ of products of the form

$$g_1h_1g_2h_2\ldots g_Nh_N,$$

where $g_i$'s (resp. $h_i$'s) are chosen from a fixed spherical class $O_a$ (resp. $O_b$), is determined.
In particular, it is shown that for $N \sim C \log n$ the product measure $\lambda_a \ast \lambda_b \ast \ldots \ast \lambda_a \ast \lambda_b$,
(product of $N$ copies of $\lambda_a \ast \lambda_b$) where $\lambda_a$ is the (probability) invariant measure on $O_a$,
tends to the Haar measure of $SU(n)$ as $n \to \infty$ in $L^p$ for $1 \leq p \leq 2$.

The methods used in this work are based on harmonic analysis on symmetric spaces. In
section 2 relevant integration formulae for symmetric spaces are stated in the appropriate
form. Sections 3, 4, 5 and 6 give the explicit density functions for products of spherical
classes in classical symmetric spaces of rank 1 of both compact and non-compact types.
The final section essential use is made of representation theory and harmonic analysis
for $SU(n) / S(U(1) \times U(n - 1))$ to obtain the rate of convergence. Similar results are also
valid for rank 1 symmetric spaces of the orthogonal groups but are not treated here.

The author wishes to thank Professor S. Shahshahani who gave him the opportunity
to pursue his interests, and especially Professor Mehrdad Shahshahani for suggesting the
problem and many stimulating discussions.

## 2 Integration Formulae

In this section we recall some basic integration formulae related to symmetric spaces. A
detailed treatment is given in [H1] and [H2].

Let $G$ be a connected compact semi-simple Lie group, $\mathfrak{g}$ its Lie algebra and $K$ be a
subgroup of $G$ with Lie algebra $\mathfrak{k}$ such that $(G, K)$ is a symmetric pair of compact type.
Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ the corresponding Cartan decomposition, $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace
and $\Sigma$ the corresponding set of restricted roots. Fix a Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$, let $\Sigma^+$ and
$\Phi = \{\alpha_1, \ldots, \alpha_l\} \subset \Sigma^+$ denote the corresponding sets of simple and positive roots. The
multiplicity of a root $\alpha$ will be denoted by $m_\alpha$. Let $M$ be the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, the
group $A = \exp \mathfrak{a}$ is a closed, and therefore a compact subgroup of $G$.

Let $dg$, $dk$, $dm$ and $da$ denote the Haar measures on the compact groups $G$, $K$, $M$
and $A$, and $du$ and $db$ be invariant measures on $U = G/K$, $B = K/M$, respectively. We
consider the surjective map

$$\Psi : (K/M) \times A \longrightarrow G/K, \quad \Psi(kM, a) = kaK.$$
The Jacobian of $\Psi$ is

$$\det(d\Psi_{(kM,a)}) = \prod_{\alpha \in \Sigma^+} |\sin \alpha(H)|^{m_\alpha}$$

(2.1)

where $a = \exp(H), H \in a$ and $m_\alpha$ is the multiplicity of the restricted root $\alpha$. Let $\delta(a)$ denote the right hand sight of (2.1). $\Psi$ is one-to-one and regular on an open dense set and we have:

$$\Psi^*(du) = c\delta(a)dbda$$

where $c$ is a constant depending on the normalization of measures.

**Theorem 2.1** Let $U = G/K$ be a Riemannian symmetric space of compact type. Then with the above notation we have

$$\int_G f(gK)dg = c \int_{K/M} \left( \int_A f(kaK)\delta(a)da \right) db$$

for all $f \in C(G/K)$. Moreover if $G$ is simply connected and $K$ is connected, then

$$\int_{G/K} f(gK)du = c \int_{K/M} db \left( \int_Q f(k(\exp H)K)\delta(\exp H)dH \right)$$

where

$$\delta(\exp(H)) = \prod_{\alpha \in \Sigma^+} |\sin \alpha(H)|^{m_\alpha}, \quad c^{-1} = \int_Q \prod_{\alpha \in \Sigma^+} (\sin(\alpha iH))^{m_\alpha} dH,$$

the measures on $U = G/K$, $B = K/M$ and $A$ are normalized to be 1, and $Q$ is the polyhedron

$$Q = \{ H \in a : \frac{1}{l} \mu_j(H) > 0, (1 \leq j \leq l), \frac{1}{l} \mu(H) < \pi \}.$$

According to the classification theory the classical symmetric spaces of rank 1 are

| Noncompact | Compact | Dimension of $G/K$ |
|------------|---------|--------------------|
| $(SU(n,1), S(U(1) \times U(n)))$ | $(SU(n + 1), S(U(1) \times U(n)))$ | $2n$ |
| $(SO(n,1), SO(n))$ | $(SO(n + 1), SO(n))$ | $n$ |
| $(Sp(n,1), Sp(1) \times Sp(n))$ | $(Sp(n + 1), Sp(1) \times Sp(n))$ | $4n$ |

There is also an exceptional symmetric pair of rank 1 associated to the exceptional Lie group $F_4$ which is not treated in this paper.
3 The Symmetric Pair \((SU(n+1), S(U(1) \times U(n)))\)

In this section \(G = SU(n+1)\) and \(K = S(U(1) \times U(n))\). In this case \(G\) is simply connected, \(K\) connected and \(G/K\) is irreducible. The Lie algebra of \(G\) is \(\mathfrak{g} = su(n+1)\), the space of traceless skew hermitian matrices and the Lie algebra of the subgroup \(K\) is

\[
\mathfrak{k} = \left\{ \begin{bmatrix} -\text{tr}(A) & 0 \\ 0 & A \end{bmatrix} ; \text{ } A \text{ is an } n \times n \text{ skew hermitian matrix} \right\}.
\]

Let \(\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\) be the Cartan decomposition then \(\mathfrak{p}\) is the subspace of matrices of the form

\[
\begin{bmatrix}
0 & \xi_1 & \ldots & \xi_n \\
-\xi_1 & 0_{n \times n} \\
\vdots & \phantom{0} & \phantom{0} & \phantom{0} \\
-\xi_n & \phantom{0} & \phantom{0} & \phantom{0}
\end{bmatrix}
\]

A maximal abelian subspace \(\mathfrak{a} \subset \mathfrak{p}\) is \(\mathfrak{a} = \{tH ; \ t \in \mathbb{R}\}\) where \(H\) is the matrix

\[
H = \begin{bmatrix}
0 & 1 & \ldots & 0 \\
-1 & 0 & \ldots & 0 \\
\vdots & \phantom{0} & \phantom{0} & \phantom{0} \\
0 & \phantom{0} & \phantom{0} & \phantom{0}
\end{bmatrix}
\]

The centralizer \(\mathfrak{m}\) of \(\mathfrak{a}\) in \(\mathfrak{k}\) is the subalgebra of \(\mathfrak{k}\) consisting of matrices of the form

\[
\begin{bmatrix}
-\text{tr}(A) & 0 \\
0 & A
\end{bmatrix}
\]

where \(A = [a_{ij}]\) is an \(n \times n\) skew hermitian matrix satisfying

\[
Av = \text{tr}(A) \ v
\]
where \( v = [1, 1, \ldots, 1]^T \). There is a unitary matrix \( P \) such that

\[
PAP^{-1} = \begin{bmatrix}
\text{tr}(A) & 0 \\
0 & B
\end{bmatrix}, \quad \text{tr}(B) = 0.
\]

Therefore we can identify the Lie algebra \( \mathfrak{m} \) with \( s(u(1) \times u(1)) \times su(n-1) \).

**Lemma 3.1** The Jacobian \( \delta \) for polar coordinates of the symmetric pair \( (G, K) \) is given by the formula

\[
\delta(\exp(tH)) = (\sin t)^{2(n-1)} \sin 2t.
\]

**Proof** - By a straightforward calculation the positive restricted roots are

\[
\alpha_1(tH) = it, \quad \alpha_2(tH) = 2\alpha_1(tH) = 2it
\]

with multiplicities \( m_{\alpha_1} = 2(n - 1), m_{\alpha_2} = 1 \). Therefore by theorem (2.1) for the Jacobian \( \delta \) we have

\[
\delta(\exp(tH)) = \prod_{\alpha \in \Sigma^+} |\sin \alpha(itH)|^{m_{\alpha}} = (\sin(t))^{2(n-1)} \sin(2t),
\]

and the lemma followed. \( \blacksquare \)

**Lemma 3.2** The stabilizer subgroup \( K_{z_0} \) of \( z_0 = [0 : 1 : 1 : \ldots : 1] \), under left action of \( K \) on \( \mathbb{CP}(n) \cong G/K \) is the centralizer subgroup \( M \).

**Proof** - Let \( p = [z, \xi] \in \mathbb{CP}(n) \) and note that \( \mathbb{CP}(n) \) is the quotient space \( \mathbb{CP}(n) = S^{2n+1}/S^1 \) under diagonal action of \( S^1 \). The orbit of a generic point \( z = [\zeta, \xi] \in \mathbb{CP}(n) \) under \( K = S(U(1) \times U(n)) \) is

\[
\begin{bmatrix}
e^{-i\theta} & 0 & \cdots & 0 \\
0 & B \\
\vdots \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\zeta \\
\xi
\end{bmatrix} = 
\begin{bmatrix}
e^{-i\theta} \zeta \\
B \xi
\end{bmatrix}.
\]

It is now clear that the first component of the above vector is a copy of \( S^1 \) and the second component is a \( (2n - 1) \)-sphere with equivalency under \( S^1 \) which is, by definition, a copy
of $\mathbb{CP}(n-1)$. Hence we showed that the orbit of a generic point $z \in \mathbb{CP}(n)$ under the action of the subgroup $K$ is equivalent to $S^1 \times \mathbb{CP}(n-1)$.

Fix a generic point $[\zeta, \xi] \in \mathbb{CP}(n)$ and consider the corresponding embedding of $S^1 \times \mathbb{CP}(n-1) \hookrightarrow \mathbb{CP}(n)$ given by $[354]$. Let $z \in \mathbb{CP}(n)$ we know $O_z = K . z = S^1 \times \mathbb{CP}(n-1)$ and let $K_z$ be the stabilizer subgroup of the point $z$ i.e. $K_z = \{ k \in K \mid k z = z \}$.

We want to show that this stabilizer subgroup $K_z$ for $z = z_0$ as in the theorem is isomorphic to the centralizer subgroup $M$. Recall that the Lie algebra $\mathfrak{m}$ of the Lie group $M$ is characterized by

$$\mathfrak{m} = \left\{ B = \begin{bmatrix} -\text{tr}(A) & 0 \\ 0 & A \end{bmatrix} \in \mathfrak{k} \mid A^* = -A , \ B \ v = \ \text{tr}(A) \ v \right\}$$

where $v = [0, 1, 1, \ldots , 1]$. So we obtain

$$e^B \ v = \det(e^A) \ v.$$ 

Therefore $v$ is an eigenvector for $M$ and conversely every element in $K$ which has $v$ as an eigenvector is necessarily an element of the centralizer subgroup $M$. Hence we have $K_{z_0} = M$. ■

**Remark 3.1** It is clear from the above lemma that the centralizer subgroup $M$ is a connected subgroup and therefore we have $M = S(U(1) \times U(1) \times U(n-1))$.

**Lemma 3.3** We have $K/M \simeq S^1 \times \mathbb{CP}(n-1)$.

**Proof**- According to the previous Lemma the orbit of the point $z_0 = [0 : 1 : 1 : \ldots : 1] \in \mathbb{CP}(n)$ under the action of the group $K$ can be identified with $S^1 \times \mathbb{CP}(n-1)$ and the stabilizer of this point is the centralizer subgroup $M$, so by the isomorphism theorem we have

$$O_{z_0} = K/K_{z_0} = K/M.$$ 

Hence $K/M = S^1 \times \mathbb{CP}(n-1)$. ■

Thus we are allowed to apply the polar coordinate on the pair $(K, M)$ and we bring it as the following lemma

**Lemma 3.4** The Jacobian of polar coordinates for the pair $(K, M)$ is

$$\delta_0(\exp(tH)) = (\sin(t))^{2(n-2)} \sin(2t)$$

$$6$$
Proof: Since the pair \((K, M)\) is symmetric in this case we have the Cartan decomposition \(\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{p}_0\) with \([\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{p}_0] \subseteq \mathfrak{p}_0, \quad [\mathfrak{p}_0, \mathfrak{p}_0] \subseteq \mathfrak{m}\).

Now \(\mathfrak{m} = \mathfrak{s}(u(1) \times u(1)) \times \mathfrak{su}(n - 1)\) which are the matrices of the form

\[
\begin{pmatrix}
-t\text{tr}(A) & 0 & 0 & \cdots & 0 \\
0 & t\text{tr}(A) & 0 & \cdots & 0 \\
0 & 0 & B \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & & & 0
\end{pmatrix}, \quad B \in \mathfrak{su}(n - 1)
\]

and the vector space \(\mathfrak{p}_0\) is matrices in the Lie algebra \(\mathfrak{k}\) of the form

\[
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \xi_1 & \cdots & \xi_{n-1} \\
0 & -\xi_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & -\xi_{n-1} & & & 0_{n-2}
\end{pmatrix}
\]

It is clear that the maximal subspace is spanned by

\[
H_0 = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
0 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & & & 0_{n-2}
\end{pmatrix}, \quad \in \mathfrak{p}_0.
\]

The positive restricted roots are

\[
\beta_1(tH_0) = it, \quad \beta_2(tH_0) = 2\beta_1 = 2it
\]

with multiplicities \(m_{\beta_1} = 2(n - 2), m_{\beta_2} = 1\). Therefore by theorem (2.1) we have

\[
\delta_0(\exp(tH_0)) = \prod_{\beta \in \Sigma^+} |\sin \alpha(itH)|^{m_{\alpha}} = (\sin(t))^{2(n-2)} \sin(2t).
\]

The lemma is proved. \(\blacksquare\)

Now we state the main theorem of this section.
Theorem 3.1 Let $\lambda_a$ and $\lambda_b$ be two (singular) spherical measures concentrated on the $K$-spherical classes $O_a$ and $O_b$ in the group $G = SU(n + 1)$ respectively. Then $\lambda_a \ast \lambda_b$ is absolutely continuous relative to the Haar measure on $SU(n+1)$. It is a spherical measure and for a continuous spherical function $f$ on $SU(n+1)$ we have

$$\lambda_a \ast \lambda_b(f) = (c\text{vol}(M)\text{vol}(K))^2 \delta(t_1)\delta(t_2) \int_{I_{a,b}} f(u)((a_2^2 - (u - a_1)^2)^{n-2} (a_1 - u) \, du,$$

where $a = \exp(t_1 H)$, $b = \exp(t_2 H)$, $\delta(t) = (\sin t)^{2(n-1)} \sin 2t$, $a_1 = \cos t_1 \cos t_2$, $a_2 = \sin t_1 \sin t_2$ and

$$I_{a,b} = [\cos(t_1 + t_2), \cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos(\frac{\pi}{2(n-2)})].$$

Proof: Since both $f$ and $\lambda_a$ are $K$-bi-invariant, $\lambda_a \ast f(x)$ is $K$-bi-invariant and therefore to compute $\mu_a \ast f(x)$ we can assume that $x$ is of the form $x = \exp(t'H)$. Let $\{\theta_n\}$ be a sequence of spherical functions converging weakly to the singular measure $\lambda_a$ on the orbit $O_a$. Applying the polar (Cartan) coordinate decomposition for the convolution $\lambda_a \ast f(x)$ we have

$$\lambda_a \ast f(x) = \int_{O_a} f(yx^{-1})dy$$

$$= \lim_{n \to \infty} \int_{G} \theta_n(g)f(gx^{-1})dg$$

$$= c \lim_{n \to \infty} \int_{K} \int_{A} \theta_n(k_1 a' k_2)f(k_1 a' k_2 x^{-1})\delta(a')da'dk_1dk_2$$

$$= c \lim_{n \to \infty} \int_{K} \int_{A} \theta_n(a')f(a'kx^{-1})\delta(a')da'dk$$

$$= c\delta(t_1) \int_{K} f(akx^{-1})dk$$

where

$$\delta(t) = \delta(\exp(tH)) = (\sin t)^{2(n-1)} \sin 2t.$$

Recall that $M$ is the centralizer group of $A$ in $K$ and it is easy to verify that the function $g$ defined by

$$g(k) = f(\exp(t_1 H)k \exp(-t'H))$$

is an $M$-spherical function. Applying the polar coordinates decomposition to the pair $(K,M)$ the above integral over $K$ becomes
\[
\lambda_a \ast \hat{f}(x) = c\delta(t_1) \int_K f(\exp(t_1 H) k \exp(-t'H)) dk
\]
\[
= c\delta(t_1) \int_M \int_M \int_{A_0} g(m_1 \ a \ m_2) \delta_0(a) dm_1 dm_2
\]
\[
= c (\text{vol}(M))^2 \delta(t_1) \int_{A_0} g(a) \delta_0(a) da ,
\]
where \( a = \exp(tH_0), \ H_0 = E_{23} - E_{32} \) is as above, \( A_0 \) is the corresponding real Cartan subgroup and \( \delta_0 \) is the Jacobian of the polar coordinates corresponding to the pair \((K, M)\).

Thus we have
\[
\lambda_a \ast \hat{f}(x) = c\delta(t_1) (\text{vol}(M))^2 \delta(t_1) \int_{Q_0} g(a) \delta_0(\exp(tH_0)) dt
\]
where the polyhedron \( Q_0 \) is the interval \([0, \frac{\pi}{2(n-2)}]\) in this case. So the above convolution integral becomes
\[
\lambda_a \ast \hat{f}(x) = c\delta(t_1) (\text{vol}(M))^2 \int_0^{\frac{\pi}{2(n-2)}} f(\exp(t_1 H) \ \exp(tH_0) \ \exp(-t'H) ) \delta_0(\exp(tH_0)) dt
\]
where \( \delta(t_1) = (\sin t_1)^{2(n-1)} \sin 2t_1 \) and \( \delta_0(t) = (\sin t)^{2(n-2)} \sin 2t \). We know that the function \( f \) is a spherical function and so it depends only to the norm of the first entry of the product matrix
\[
\exp(t_1 H) \ \exp(tH_0) \ \exp(-t'H).
\]
After a simple calculation we obtain
\[
a_{11} = \cos t_1 \cos t' - \cos t \sin t_1 \sin t'.
\]
Set \( a_1 = \cos t_1 \cos t' \) and \( a_2 = \sin t_1 \sin t' \) to obtain
\[
\lambda_a \ast \hat{f}(x) = c\delta(t_1) (\text{vol}(M))^2 \int_0^{\frac{\pi}{2(n-2)}} f(a_1 - a_2 \cos t) \delta_0(\exp(tH_0)) dt
\]
Next we compute the convolution \( \lambda_a \ast \lambda_b(f) \) with \( a = \exp(t_1 H) \) and \( b = \exp(t_2 H) \). Assume that \( h(x) = \lambda_b \ast \hat{f}(x) \) then
\[
(\lambda_a \ast \lambda_b)(f) = (\lambda_a \ast (\lambda_b \ast \hat{f}))(e) = (\lambda_a \ast h)(e) = \int_{O_a} h(x) d\lambda_a(x) = h(a) \text{vol}(O_a),
\]
Applying polar coordinates on $G$ for the volume of the spherical class $O_a$, $a = \exp(t_1 H)$, we obtain

$$\text{vol}(O_a) = c \int_K \int_K \delta(\exp(t_1 H)) \, dk \, dk' = c(\text{vol}(K))^2 (\sin(t_1))^{2(n-1)} \sin 2t_1.$$ 

For $h(a)$ we have

$$h(a) = \lambda_b \ast f(a) = c \delta(t_2)(\text{vol}(M))^2 \int_0^{\pi/(2(n-2))} f(a_1 - a_2 \cos t)(\sin t)^{2(n-2)} \sin 2t \, dt.$$ 

We make change of variable $u = a_1 - a_2 \cos t$. Then $u$ is an increasing function on the interval $[0, \frac{\pi}{2n-2}]$ and it maps this interval onto the interval

$$I_{a,b} = [\cos(t_1 + t_2), \cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos(\frac{\pi}{2(n-2)})].$$

Substituting the new variable $u$ and simplifying the above integral we obtain

$$h(a) = \lambda_b \ast f(a) = \frac{c\delta(t_2)(\text{vol}(M))^2}{a_2^{2(n-1)}} \int_{I_{a,b}} f(u)(a_1^2 - (u - a_1)^2)^{n-2}(a_1 - u) \, du.$$ 

Finally for $\lambda_a \ast \lambda_b(f)$ we obtain

$$\lambda_a \ast \lambda_b(f) = h(a)\text{vol}(O_a) = \frac{(\text{vol}(M)\text{vol}(K))^2}{a_2^{2(n-1)}} \delta(t_1)\delta(t_2) \int_{I_{a,b}} f(u)(a_1^2 - (u - a_1)^2)^{n-2}(a_1 - u) \, du$$

which completes the proof of the theorem.

\textbf{Corollary 3.1} Choosing matrices $A$ and $B$ according to the (singular) invariant measures on the spherical classes $O_a$ and $O_b$ respectively and normalized to be probability measures, then the support of the distribution of the product $AB$ is the interval $[\cos(t_2 + t_1), \cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos(\frac{\pi}{2(n-2)})]$ and its density function is

$$\frac{2n-2}{(\sin t_1 \sin t_2 \sin(\frac{\pi}{2(n-2)}))^{2n-2}(a_2^2 - (u - a_1)^2)^{n-2}(a_1 - u)},$$

where $a_1 = \cos t_1 \cos t_2$ and $a_2 = \sin t_1 \sin t_2$. 

10
Corollary 3.2 With the notation and hypotheses of Corollary 3.1 the density function for the convolution of probability measures \( \lambda_a \) and \( \lambda_b \) converges weakly to the singular invariant measure on the spherical class through \( \exp(t_1 + t_2)H \) as \( n \to \infty \).

Proof - Since \( \lambda_a \) and \( \lambda_b \) are probability measures, so is \( \lambda_a \ast \lambda_b \). The support of this measure is the interval \( I_{a,b} \), in the appropriate coordinate system, which tends to the single point \( \cos(t_1 + t_2) \) from which the required result follows.

4 The Symmetric Pair \((SU(1, n), S(U(1) \times U(n)))\).

Let \( G = Su(1, n) \) and \( K = S(U(1) \times U(n)) \), and \( g = k \oplus p \) be the corresponding Cartan decomposition where \( g \) and \( k \) are the Lie algebras of \( G \) and \( K \). Let \( a \subset p \) be

\[
a = \{ tH : t \in \mathbb{R} \}
\]

where \( H = E_{12} + E_{21} \) and the restricted roots are given by \( \alpha(tH) = t \) and \( 2\alpha \) with multiplicities \( 2(n-1) \) and \( 1 \) respectively. The centralizer of \( a \) in \( k \) is \( m = s(u(1) \times u(1) \times u(n-1)) \) and the corresponding Lie subgroup is \( M = S(U(1) \times U(1) \times U(n-1)) \). Therefore the pair \((K, M)\) in the case of the non-compact symmetric pair \((SU(1, n), S(U(1) \times U(n)))\) is same as in the case of the compact symmetric pair \((SU(n+1), S(u(1) \times U(n)))\) treated in the preceding section.

Theorem 4.1 Let \( \lambda_a \) and \( \lambda_b \) be two (singular) spherical measures concentrated on the \( K \)-spherical classes \( O_a \) and \( O_b \) in the group \( G = SU(1, n) \) respectively. Then \( \lambda_a \ast \lambda_b \) is absolutely continuous relative to the Haar measure on \( SU(1, n) \). It is a spherical measure and for a continuous spherical function \( f \) on \( SU(1, n) \) we have

\[
\lambda_a \ast \lambda_b(f) = (cvol(M)vol(K))^2 \delta(t_1)\delta(t_2) \int_{I_{a,b}} f(u)((a_2^2 - (u - a_1)^2)^{n-2}(a_1 - u)du,
\]

where \( a = \exp(t_1 H), b = \exp(t_2 H) \), \( \delta(t) = (\sinh t)^{2(n-1)} \sinh 2t \), and

\[
I_{a,b} = [\cosh(t_1 - t_2), \cosh t_1 \cosh t_2 - \sinh t_1 \sinh t_2 \cos\left(\frac{\pi}{2(n-2)}\right)].
\]

Proof - Since both \( f \) and \( \lambda_a \) are \( K \)-bi-invariant, \( \lambda_a \ast f(x) \) is \( K \)-bi-invariant and therefore to compute \( \lambda_a \ast f(x) \) we can assume that \( x \) is of the form \( x = \exp(t'H) \). Let \( \{\theta_n\} \) be a sequence of spherical functions converging weakly to the singular measure \( \lambda_a \) on the orbit.
Applying the polar coordinates (Cartan) decomposition for the convolution $\lambda_a \star \hat{f}(x)$ we obtain

$$
\lambda_a \star \hat{f}(x) = \int_{O_a} f(yx^{-1})dy
$$

$$
= \lim_{n \to \infty} \int_{G} \theta_n(g)f(gx^{-1})dg
$$

$$
= e \lim_{n \to \infty} \int_{K} \int_{K} \int_{A} \theta_n(k_1a'k_2)f(k_1a'k_2x^{-1})\delta(a')da'dk_1dk_2
$$

$$
= e \lim_{n \to \infty} \int_{K} \int_{A} \theta_n(a')f(a'kx^{-1})\delta(a')da'dk
$$

$$
= c\delta(t_1) \int_{K} f(akx^{-1})dk
$$

where

$$
\delta(t) = \delta(\exp(tH)) = (\sinh t)^{2(n-1)} \sinh 2t.
$$

Recall that $M$ is the centralizer group of $A$ in $K$ and it is easy to verify that the function $g$ defined by

$$
g(k) = f(\exp(t_1H)k\exp(-t'H))
$$

is an $M$-spherical function. Using polar coordinates, as in the previous section, the above integral over $K$ reduces to

$$
\lambda_a \star \hat{f}(x) = c\delta(t_1) \int_{K} f(\exp(t_1H)k\exp(-t'H))dk
$$

$$
= c\delta(t_1) \int_{M} \int_{M} \int_{A_0} g(m_1 a m_2)\delta_0(a)dm_1dadm_2
$$

$$
= c(\text{vol.}(M))^2\delta(t_1) \int_{A_0} g(a)\delta_0(a)da,
$$

where $H_0 = E_{12} - E_{21}$, $A_0 = \{\exp(tH_0)\}$ is the real Cartan subgroup and $\delta_0$ is as in the theorem (2.1). Thus we have:

$$
\lambda_a \star \hat{f}(x) = c(\text{vol.}(M))^2\delta(t_1) \int_{Q_0} g(a)\delta_0(\exp(tH_0))dt,
$$

where the polyhedra $Q_0$ is the interval $[0, \frac{\pi}{2^{n-2}}]$. Simplifying we obtain

$$
\lambda_a \star \hat{f}(x) = c\delta(t_1)(\text{vol.}(M))^2 \int_{0}^{\frac{\pi}{2^{n-2}}} f(\exp(t_1H)\exp(tH_0)\exp(-t'H))\delta_0(\exp(tH_0))dt
$$
where \( \delta(t_1) = (\sinh t_1)^{2(n-1)} \sinh 2t_1 \) and \( \delta_0(t) = (\sin t)^{2(n-2)} \sin 2t \). The function \( f \) is spherical and so it depends only on the norm of the first entry of the product matrix
\[
\exp(t_1 H) \exp(t H_0) \exp(-t'H).
\]
Now
\[
a_{11} = \cosh t_1 \cosh t' - \cos t \sinh t_1 \sinh t'.
\]
Set \( a_1 = \cosh t_1 \cosh t' \) and \( a_2 = \sinh t_1 \sinh t' \) to obtain
\[
\lambda_a \ast \tilde{f}(x) = c \delta(t_1)(\text{vol}(M))^2 \int_0^{2(n-2)} f(a_1 - a_2 \cos t) \delta_0(\exp(t H_0))dt.
\]
Let \( h(x) = \lambda_b \ast \tilde{f}(x) \), then
\[
(\lambda_a \ast \lambda_b)(f) = (\lambda_a \ast (\lambda_b \ast \tilde{f}))(e) = (\lambda_a \ast h)(e) = \int_{O_a} h(x) d\lambda_a(x) = h(a)\text{vol}(O_a).
\]
Using the decomposition \( G = KAK \) we obtain
\[
\text{vol}(O_a) = c \int_K \int_K \delta(\exp(t_1 H)) \, dk \, dk' = c(\text{vol}(K))^2(\sinh(t_1))^{2(n-1)} \sinh 2t_1.
\]
Therefore
\[
h(a) = \lambda_b \ast \tilde{f}(a) = c \delta(t_2)(\text{vol}(M))^2 \int_0^{2(n-2)} f(a_1 - a_2 \cos t)(\sin t)^{2(n-2)} \sin 2tdt
\]
The change of variable \( u = a_1 - a_2 \cos t \) maps the interval \([0, \frac{\pi}{2(n-2)}]\) onto the interval
\[
I_{a,b} = [\cosh(t_1 - t_2), \cosh t_1 \cosh t_2 - \sinh t_1 \sinh t_2 \cos(\frac{\pi}{2(n-2)})]
\]
Therefore
\[
h(a) = \lambda_b \ast \tilde{f}(a) = \frac{c\delta(t_2)(\text{vol}(M))^2}{a_2^{2(n-1)}} \int_{I_{a,b}} f(u)(a_2^2 - (u - a_1)^2)^{n-2} (a_1 - u)du
\]
Finally for \( \lambda_a \ast \lambda_b(f) \) we obtain
\[
\lambda_a \ast \lambda_b(f) = h(a)\text{vol}(O_a) = \frac{(c\text{vol}(M)\text{vol}(K))^2}{a_2^{2(n-1)}} \delta(t_1) \delta(t_2) \int_{I_{a,b}} f(u)(a_2^2 - (u - a_1)^2)^{n-2} (a_1 - u)du
\]
which completes the proof of the theorem. \( \blacksquare \)
Corollary 4.1 Choosing matrices $A$ and $B$ according to the (singular) invariant measures on the spherical classes $O_a$ and $O_b$ respectively and normalized to be probability measures, then the support of the distribution of the product $AB$ is the interval $[\cosh(t_2 - t_1), \cosh t_1 \cosh t_2 - \sinh t_1 \sinh t_2 \cos(\frac{\pi}{2(n-2)})]$ and its density function is

$$\frac{2n-2}{(\sinh t_1 \sinh t_2 \sin \frac{\pi}{2(n-2)})^{2n-2}} (a_2^2 - (u - a_1)^2)^{n-2} (a_1 - u),$$

where $a_1 = \cosh t_1 \cosh t_2$ and $a_2 = \sinh t_1 \sinh t_2$. Furthermore

$$\lim_{n \to \infty} \lambda_a \star \lambda_b = \lambda_c, \text{ weakly,}$$

where $\lambda_c$ is the singular invariant probability measure on the spherical class through $\exp((t_1 - t_2)H)$.

Remark 4.1 Note that $\exp(\pm tH)$ are in the same spherical class and therefore $\exp((t_1 - t_2)H)$ and $\exp((t_2 - t_1)H)$ are in the same spherical class.

5 The symmetric pairs $(SO(n+1), SO(n))$, and $(SO(1, n) \circlearrowright, SO(n))$, $n \geq 3$

The Lie algebra of the orthogonal group $G = SO(n + 1)$ is the algebra of skew symmetric matrices i.e.

$$\mathfrak{g} = \mathfrak{so}(n + 1) = \{ A \in M_{n+1}(\mathbb{R}) \mid A^t = -A \}$$

For the Cartan decomposition of $\mathfrak{g}$ we have

$$\mathfrak{so}(n + 1) = \mathfrak{so}(n) \oplus \mathfrak{p},$$

where $\mathfrak{p}$ is the subspace spanned by the matrices of the form

$$\begin{bmatrix}
0 & \xi_1 & \cdots & \xi_n \\
-\xi_1 & & & \\
\vdots & & 0_{n \times n} & \\
-\xi_n & & & 
\end{bmatrix}$$

A maximal abelian subspace of $\mathfrak{p}$ is:

$$a = \{ tH : t \in \mathbb{R} \}$$
where $H$ is the matrix

$$H = \begin{bmatrix}
0 & 1 & \ldots & 0 \\
-1 & 0 & & \\
0 & 0 & \ddots & 0 \\
\vdots & \vdots & \ddots & 0 \\
0 & 0 & \ldots & 0
\end{bmatrix}_{n \times n}$$

The centralizer $\mathfrak{m}$ of $\mathfrak{a}$ in $\mathfrak{k} = \mathfrak{so}(n-1)$ is exactly the Lie algebra $\mathfrak{so}(n-1)$. It is straightforward that the centralizer subgroup $M$ is connected and therefore we have $M = SO(n-1)$ and

$$K/M = SO(n)/SO(n-1) \cong S^{n-1}.$$ 

By straightforward calculation the eigenvalues of the operator

$$adH : \mathfrak{so}(n) \to \mathfrak{so}(n)$$

are $\pm i$ with multiplicity $n-1$. Thus for $(SO(n+1), SO(n))$ we have one positive restricted root $\alpha(tH) = it$ whose multiplicity is $m_\alpha = n-1$. Hence

$$\delta(\exp(tH)) = \prod_{\alpha \in \Sigma^+} |\sin \alpha(iH)|^{m_\alpha} = (\sin(t))^{n-1}.$$ 

For the pair $(K, M) = (SO(n), SO(n-1))$ the maximal abelian subspace $\mathfrak{a}_0$ is

$$\mathfrak{a}_0 = \{tH_0 \mid t \in \mathbb{R}\},$$

where $H_0$ is

$$H_0 = \begin{bmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & -1 & 1 & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & 0 \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix}_{n \times n-3}$$

Therefore the corresponding Jacobian for the pair $(K, M)$ is

$$\delta_0(\exp(tH_0)) = \prod_{\alpha \in \Sigma^+} |\sin \alpha(iH_0)|^{m_\alpha} = (\sin(t))^{n-2}.$$ 

Since $SO(n+1)$ is not simply connected we work with the double cover $\text{Spin}(n+1)$ which is simply connected, and let

$$\pi : \text{Spin}(n+1) \to SO(n+1).$$
Since $\text{spin}(n+1) = \mathfrak{so}(n+1)$ we have $\text{spin}(n+1) = \text{spin}(n) \oplus p$. Note that we identify $\text{Spin}(n-1)$ with the pre-image of the subgroup $K = SO(n-1)$ under the covering map $\pi$. We denote this subgroup by $\tilde{K} = \text{Spin}(n) = \pi^{-1}(SO(n))$.

The computation of the restricted positive root and its multiplicity is the same as in the case of $\mathfrak{so}(n+1)$. Now Theorem (2.1) is applicable to the symmetric pair $(\text{Spin}(n+1), \text{Spin}(n))$, but note that a function on the group $SO(n+1)$ can be considered as a function on $\text{Spin}(n)$ and its integral over the group $G = \text{Spin}(n+1)$ is equal to twice its integral over $G = SO(n+1)$.

**Theorem 5.1** Let $\lambda_a$ and $\lambda_b$ be two (singular) spherical measures concentrated on the $K$-spherical classes $O_a$ and $O_b$ in the group $G = SO(n)$ respectively. Then $\lambda_a \ast \lambda_b$ is absolutely continuous relative to the Haar measure on $SO(n)$. It is a spherical measure and for a continuous spherical function $f$ on $SO(n)$ we have

$$\lambda_a \ast \lambda_b(f) = (c \text{vol}(M) \text{vol}(K))^2 \delta(t_1) \delta(t_2) \int_{I_{a,b}} f(u)((a_2^2 - (u - a_1)^2)^{\frac{n-4}{2}} du),$$

where $a = \exp(t_1 H)$, $b = \exp(t_2 H)$, $\delta(t) = (\sin t)^{n-2}$, and

$$I_{a,b} = [\cos(t_1 + t_2), \cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos(\frac{\pi}{n-3})].$$

**Proof:** Since both $f$ and $\lambda_a$ are $K$-bi-invariant, $\lambda_a \ast f(x)$ is $K$-bi-invariant and therefore to compute $\lambda_a \ast f(x)$ we can assume that $x$ is of the form $x = \exp(t'H)$. Let $\{\theta_n\}$ be a sequence of spherical functions converging weakly to the singular measure $\lambda_a$ on the orbit $O_a$. Applying the polar coordinates (Cartan) decomposition for the convolution $\lambda_a \ast \tilde{f}(x)$ we have

$$\lambda_a \ast \tilde{f}(x) = \lim_{n \to \infty} \int_G \theta_n(g)f(gx^{-1})dg
= \frac{1}{2} \lim_{n \to \infty} \int_G \theta_n(g)f(gx^{-1})dg
= \frac{c}{2} \lim_{n \to \infty} \int_K \int_{A} \int_K \int_{A} \theta_n(k_1a'k_2)f(k_1a'k_2x^{-1})\delta(a')da'dk_1dk_2
= \frac{c}{2} \lim_{n \to \infty} \int_K \int_{A} \int_K \int_{A} \theta_n(a')f(a'x^{-1})\delta(a')da'dk
= \frac{c}{2} \delta(t_1) \int_K f(ax^{-1})dk
= c\delta(t_1) \int_K f(ax^{-1})dk
= \frac{c}{2} \delta(t_1) \int_K f(ax^{-1})dk$$

16
where
\[ \delta(t) = \delta(\exp(tH)) = (\sin t)^{n-2}. \]

Recall that \( M \) is the centralizer of \( A \) in \( K \) and that the function \( g \) defined by
\[ g(k) = f(\exp(t_1H)k\exp(-t'H)) \]
is an \( M \)-spherical function. Applying the polar coordinates decomposition to the pair \((K, M)\) the above integral becomes
\[
\lambda_a \ast \hat{f}(x) = c\delta(t_1) \int_K f(\exp(t_1H)k\exp(-t'H))dk = c\delta(t_1)\int_M \int_M \int_{A_0} g(m_1 a m_2)\delta_0(a)dm_1dadm_2
\]
\[
= c(\text{vol}(M))^2\delta(t_1)\int_{A_0} g(a)\delta_0(a)da,
\]
where \( a = \exp(tH_0) \), where \( H_0 = E_{23} - E_{32} \) is as above, \( A_0 \) is the real Cartan subgroup and \( \delta_0 \) is the Jacobian of the polar coordinates corresponding to the pair \((K, M)\). Thus we have:
\[
\lambda_a \ast \hat{f}(x) = c(\text{vol}(M))^2\delta(t_1)\int_{Q_0} g(a)\delta_0(\exp(tH_0))dt,
\]
where the polyhedra \( Q_0 \) is the interval \([0, \frac{\pi}{n-2}]\). Therefore
\[
\lambda_a \ast \hat{f}(x) = c\delta(t_1)(\text{vol}(M))^2\int_0^{\frac{\pi}{n-2}} f(\exp(t_1H) \exp(tH_0) \exp(-t'H))\delta_0(\exp(tH_0))dt
\]
where \( \delta(t_1) = (\sin t_1)^{n-1} \) and \( \delta_0(t) = (\sin t)^{n-2} \). The function \( f \) is spherical and so it depends only on the norm of the first entry of the product matrix \( \exp(t_1H) \exp(tH_0) \exp(-t'H) \), and is given by \( a_{11} = \cos t_1 \cos t' - \cos t \sin t_1 \sin t' \). Set \( a_1 = \cos t_1 \cos t' \) and \( a_2 = \sin t_1 \sin t' \) to obtain
\[
\lambda_a \ast \hat{f}(x) = c\delta(t_1)(\text{vol}(M))^2\int_0^{\frac{\pi}{n-2}} f(a_1 - a_2 \cos t)\delta_0(\exp(tH_0))dt
\]
Let \( h(x) = \lambda_b \ast \hat{f}(x) \) with \( a = \exp(t_1H) \) and \( b = \exp(t_2H) \), then

17
\[(\lambda_a \star \lambda_b)(f) = (\lambda_a \star (\lambda_b \star \hat{f}))(e) = (\lambda_a \star h)(e) = \int_{O_a} h(x) d\lambda_a(x) = h(a) \text{vol}(O_a),\]

Using the Cartan decomposition we obtain

\[
\text{vol}(O_a) = \frac{c}{2} \int_K \int_K \delta(\exp(t_1 H)) \, dk \, dk' = 2c(\text{vol}(K))^2 (\sin(t_1))^{n-1}
\]

Now

\[
h(a) = \lambda_b \star \hat{f}(a) = c \delta(t_2)(\text{vol}(M))^2 \int_0^{\frac{\pi}{2}} f(a_1 - a_2 \cos t)(\sin t)^{n-2} \, dt
\]

The change of variable \(u = a_1 - a_2 \cos t\) maps the interval \([0, \frac{\pi}{n-2}]\) onto the interval

\[
I_{a,b} = [\cos(t_1 + t_2), \cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos(\frac{\pi}{n-2})]
\]

The expression for \(h(a)\) becomes

\[
h(a) = \lambda_b \star \hat{f}(a) = \frac{c\delta(t_2)(\text{vol}(M))^2}{a_2^{n-2}} \int_{I_{a,b}} f(u)(a_2^2 - (u - a_1)^2)^{\frac{n-3}{2}} \, du
\]

Finally for \(\lambda_a \star \lambda_b(f)\) we obtain

\[
\lambda_a \star \lambda_b(f) = h(a) \text{vol}(O_a) = \frac{c\text{vol}(M) \text{vol}(K))^2 \delta(t_1) \delta(t_2)}{a_2^{n-2}} \int_{I_{a,b}} f(u)(a_2^2 - (u - a_1)^2)^{\frac{n-3}{2}} \, du
\]

which completes the proof of the theorem. ■

**Corollary 5.1** Choosing matrices \(A\) and \(B\) according to the (singular) invariant measures on the spherical classes \(O_a\) and \(O_b\) respectively and normalized to be probability measure, then the support of the distribution of the product \(AB\) is the interval

\[\left[\cos(t_2 + t_1), \cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos(\frac{\pi}{n-2})\right]\]
and its density function is
\[
\frac{\left( a_2^2 - (u - a_1)^2 \right)^{n-3}}{a_2^{n-2} \int_0^{\pi/2} (\sin t)^{n-2} dt} ,
\]
where \( a_1 = \cos t_1 \cos t_2 \) and \( a_2 = \sin t_1 \sin t_2 \). Furthermore
\[
\lim_{n \to \infty} \lambda_a \ast \lambda_b = \lambda_c , \quad \text{weakly,}
\]
where \( \lambda_c \) is the singular invariant probability measure on the spherical class through \( \exp((t_1 + t_2)H) \).

For the symmetric pair \((SO(1, n)^\circ, SO(n))\) the calculations are similar and therefore are not repeated. We obtain

**Theorem 5.2** Let \( \lambda_a \) and \( \lambda_b \) be two (singular) spherical measures concentrated on the \( K \)-spherical classes \( \mathcal{O}_a \) and \( \mathcal{O}_b \) in the group \( G = SO(1, n) \) respectively. Then \( \lambda_a \ast \lambda_b \) is absolutely continuous relative to the Haar measure on \( SO(1, n) \). It is a spherical measure and for a continuous spherical function \( f \) on \( SO(1, n) \) we have
\[
\lambda_a \ast \lambda_b (f) = (c\textnormal{vol}(\mathcal{M}) \textnormal{vol}(K))^2 \delta(t_1) \delta(t_2) \int_{I_{a,b}} f(u) \left( (a_2^2 - (u - a_1)^2)^{n-3} \right) du ,
\]
where \( a = \exp(t_1 H) \), \( b = \exp(t_2 H) \), \( \delta(t) = (\sinh t)^{n-1} \), and
\[
I_{a,b} = \left[ \cosh(t_1 - t_2), \cosh t_1 \cosh t_2 - \sinh t_1 \sinh t_2 \cos(\pi n^{-2}) \right] .
\]

**Corollary 5.2** Choosing matrices \( A \) and \( B \) according to the (singular) invariant measures on the spherical classes \( \mathcal{O}_a \) and \( \mathcal{O}_b \) respectively and normalized to be probability measure, then the support of the distribution of the product \( AB \) is the interval
\[
[\cosh(t_1 - t_2), \cosh t_1 \cosh t_2 - \sinh t_1 \sinh t_2 \cos(\pi n^{-2})]
\]
and its density function is
\[
\frac{\left( a_2^2 - (u - a_1)^2 \right)^{n-3}}{a_2^{n-2} \int_0^{\pi/2} (\sin t)^{n-2} dt} ,
\]
where \( a_1 = \cosh t_1 \cosh t_2 \) and \( a_2 = \sinh t_1 \sinh t_2 \). Furthermore
\[
\lim_{n \to \infty} \lambda_a \ast \lambda_b = \lambda_c , \quad \text{weakly,}
\]
where \( \lambda_c \) is the singular invariant probability measure on the spherical class through \( \exp((t_1 + t_2)H) \).
6 The symmetric pairs \((Sp(n + 1), Sp(1) \times Sp(n))\), and \\
\((Sp(1, n), Sp(1) \times Sp(n))\)

Let \(G = Sp(n + 1)\) and \(K = Sp(1) \times Sp(n)\), then \(G\) is simply connected, \(K\) connected 
and \(G/K\) is the quaternionic projective space. The Lie algebra of \(G\) is \(g = sp(n + 1)\), the 
space of complex matrices \(X\) satisfying \(JX + X^tJ = 0\). If we write \(X\) in the form 
\[
\begin{bmatrix}
X_1 & X_2 \\
X_3 & X_4
\end{bmatrix}
\]
where \(X_1, X_2, X_3, X_4\) are matrices of degree \(n + 1\), the condition \(JX + X^tJ = 0\) gives 
\[
X_4 = -X_1^t \quad X_3 = X_3^t \quad X_2 = X_2^t
\]
and the Lie algebra of the subgroup \(K\) is

\[
\mathfrak{k} = \left\{ \begin{bmatrix}
x_{11} & x_{12} & 0 & 0 \\
-x_{12} & x_{11} & 0 & 0 \\
0 & 0 & Y_{11} & Y_{12} \\
0 & 0 & -Y_{12} & Y_{11}
\end{bmatrix} \right\} 
\]

where \(x_{ij} \in \mathbb{C}\) and \(Y_{ij} \in u(n)\), \(Y_{12}\) is \(n \times n\) symmetric.

Let \(g = \mathfrak{k} \oplus \mathfrak{p}\) be the Cartan decomposition then \(\mathfrak{p}\) is the subspace of matrices of the form

\[
\begin{bmatrix}
0 & 0 & Z_{13} & -Z_{14} \\
0 & 0 & -Z_{14} & Z_{13} \\
-Z_{13} & Z_{14} & 0 & 0 \\
Z_{14} & -Z_{13} & 0 & 0
\end{bmatrix},
\]

where \(Z_{ij}\) are \(1 \times n\) complex matrices. A maximal abelian subspace \(\mathfrak{a} \subset \mathfrak{p}\) is \(\mathfrak{a} = \{tH \mid t \in \mathbb{R}\}\) where \(H\) is the matrix

\[
H = E_{32} + E_{41} - E_{23} - E_{14}.
\]

The restricted roots are given by \(\alpha(tH) = it\) and \(2\alpha\) with multiplicities \(8(n - 1)\) and \(2\) 
respectively. The centralizer of \(\mathfrak{a}\) in \(\mathfrak{k}\) is \(\mathfrak{m} = \mathfrak{sp}(1) \times \mathfrak{sp}(1) \times \mathfrak{sp}(n - 1)\), the corresponding 
Lie subgroup \(M\) is connected and \(M = Sp(1) \times Sp(1) \times Sp(n - 1)\).

**Theorem 6.1** Let \(\lambda_a\) and \(\lambda_b\) be two (singular) spherical measures concentrated on the 
\(K\)-spherical classes \(O_a\) and \(O_b\) in the group \(G = Sp(n + 1)\) respectively. Then \(\lambda_a \ast \lambda_b\) is 
absolutely continuous relative to the Haar measure on \(Sp(n + 1)\). It is a spherical measure 
and for a continuous spherical function \(f\) on \(Sp(n + 1)\) we have
\[ \lambda_a \star \lambda_b(f) = \frac{(\text{vol}(M)\text{vol}(K))^2}{a_2^{8n-12}} \delta(t_1) \delta(t_2) \int_{I_{a,b}} f(u)(a_2^2 - (u - a_1)^2)^{4n-\frac{18}{2}} (a_1 - u)^2 du , \]

where \( c \) is a constant, \( a = \exp(t_1 H) \), \( b = \exp(t_2 H) \), \( \delta(t) = (\sin t)^{8(n-1)} (\sin 2t)^2 \), \( a_1 = \cos t_1 \cos t_2 \), \( a_2 = \sin t_1 \sin t_2 \) and

\[ I_{a,b} = [\cos(t_1 + t_2), \cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos \left(\frac{\pi}{8(n-2)}\right)]. \]

**Proof:** Since both \( f \) and \( \lambda_a \) are \( K \)-bi-invariant, \( \lambda_a \star f(x) \) is \( K \)-bi-invariant and therefore to compute \( \mu_a \star f(x) \) we can assume that \( x \) is of the form \( x = \exp(t'H) \). Let \( \{\theta_n\} \) be a sequence of spherical functions converging weakly to the singular measure \( \lambda_a \) on the orbit \( O_a \). Applying the polar (Cartan) coordinate decomposition for the convolution \( \lambda_a \star \hat{f}(x) \) we have

\[
\lambda_a \star \hat{f}(x) = \int_{O_a} f(yx^{-1}) dy \\
= \lim_{n \to \infty} \int_G \int_K \int_A \theta_n(g) f(gx^{-1}) dg \\
= c \lim_{n \to \infty} \int_K \int_K \int_A \theta_n(k_1 a' k_2) f(k_1 a' k_2 x^{-1}) \delta(a') da' dk_1 dk_2 \\
= c \lim_{n \to \infty} \int_K \int_A \theta_n(a') f(a' k x^{-1}) \delta(a') da' dk \\
= c \delta(t_1) \int_K f(akx^{-1}) dk
\]

where

\[ \delta(t) = \delta(\exp(tH)) = (\sin t)^{8(n-1)}(\sin 2t)^2. \]

The function \( g \) defined by

\[ g(k) = f(\exp(t_1 H) k \exp(-t'H)) \]

is an \( M \)-spherical function. Applying the polar coordinates decomposition to the pair \( (K, M) \), the above integral over \( K \) becomes

\[
\lambda_a \star \hat{f}(x) = c \delta(t_1) \int_K f(\exp(t_1 H) k \exp(-t'H)) dk \\
= c \delta(t_1) \int_M \int_M \int_{A_0} g(m_1 a m_2) \delta_0(a) dm_1 da m_2 \\
= c (\text{vol}(M))^2 \delta(t_1) \int_{A_0} g(a) \delta_0(a) da ,
\]

21
where \( a = \exp(tH_0), \) \( H_0 = E_{74} + E_{83} - E_{56} - E_{65} \) is as above, \( A_0 \) is the corresponding real Cartan subgroup and \( \delta_0 \) is the Jacobian of the polar coordinates corresponding to the pair \((K, M)\). Thus we have

\[
\lambda_a \star \tilde{f}(x) = c(\text{vol}(M))^2 \delta(t_1) \int_{Q_0} g(a)\delta_0(\exp(tH_0))dt
\]

where the polyhedron \( Q_0 \) is the interval \([0, \frac{\pi}{8(n-2)}]\) in this case. So the above convolution integral becomes

\[
\lambda_a \star \tilde{f}(x) = c\delta(t_1)(\text{vol}(M))^2 \int_0^{\frac{\pi}{8(n-2)}} f(\exp(t_1H) \exp(tH_0) \exp(-t'H))\delta_0(\exp(tH_0))dt
\]

where \( \delta(t_1) = (\sin t_1)^{8(n-1)}(\sin 2t_1)^2 \) and \( \delta_0(t) = (\sin t)^{8(n-2)}(\sin 2t)^2 \). We know that the function \( f \) is a spherical function and so it depends only on the norm of the first entry \( a_{11} \) of the product matrix \( \exp(t_1H) \exp(tH_0) \exp(-t'H) \), and after a simple calculation we obtain

\[
a_{11} = \cos t_1 \cos t' - \cos t \sin t_1 \sin t'.
\]

Set \( a_1 = \cos t_1 \cos t' \) and \( a_2 = \sin t_1 \sin t' \) to obtain

\[
\lambda_a \star \tilde{f}(x) = c\delta(t_1)(\text{vol}(M))^2 \int_0^{\frac{\pi}{8(n-2)}} f(a_1 - a_2 \cos t)\delta_0(\exp(tH_0))dt
\]

Next we compute the convolution \( \lambda_a \star \lambda_b(f) \) with \( a = \exp(t_1H) \) and \( b = \exp(t_2H) \). Assume that \( h(x) = \lambda_b \star \tilde{f}(x) \) then

\[
(\lambda_a \star \lambda_b)(f) = (\lambda_a \star (\lambda_b \star \tilde{f}))(e) = (\lambda_a \star h)(e) = \int_{O_a} h(x)d\lambda_a(x) = h(a)\text{vol}(O_a),
\]

Applying polar coordinates on \( G \) for the volume of the spherical class \( O_a, a = \exp(t_1H) \), we obtain

\[
\text{vol}(O_a) = c \int_{K} \int_{K} \delta(\exp(t_1H))\,dk\,dk' = c(\text{vol}(K))^2(\sin(t_1))^{8(n-1)}(\sin 2t_1)^2.
\]

For \( h(a) \) we have

\[
h(a) = \lambda_b \star \tilde{f}(a) = c\,\delta(t_2)(\text{vol}(M))^2 \int_0^{\frac{\pi}{8(n-2)}} f(a_1 - a_2 \cos t)(\sin t)^{8(n-2)}(\sin 2t)^2 dt.
\]
We make change of variable 
\[ u = a_1 - a_2 \cos t. \]
Then \( u \) is an increasing function on the interval \([0, \frac{\pi}{8(n-2)}]\) and it maps this interval onto the interval
\[ I_{a,b} = [\cos(t_1 + t_2), \cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos(\frac{\pi}{8(n-2)})] \]
Substituting the new variable \( u \) and simplifying the above integral we obtain
\[
h(a) = \lambda_b \ast \tilde{f}(a) = \frac{c \delta(t_2) (\text{vol}(M))^2}{a_2^{8n-12}} \int_{I_{a,b}} f(u)(a_2^2 - (u - a_1)^2)^{4n-15} \frac{1}{2^7} (a_1 - u)^2 du
\]
Finally for \( \lambda_a \ast \lambda_b(f) \) we obtain
\[
\lambda_a \ast \lambda_b(f) = h(a) \text{vol}(O_a)
= \frac{(c \text{vol}(M) \text{vol}(K))^2}{a_2^{8n-12}} \delta(t_1) \delta(t_2) \int_{I_{a,b}} f(u)(a_2^2 - (u - a_1)^2)^{4n-15} \frac{1}{2^7} (a_1 - u)^2 du
\]
which completes the proof of the theorem. ■

**Corollary 6.1** Choosing matrices \( A \) and \( B \) according to the (singular) invariant means on the spherical classes \( \mathcal{O}_a \) and \( \mathcal{O}_b \) respectively and normalized to be probability measures, then the support of the distribution of the product \( AB \) is the interval \([\cos(t_2 + t_1), \cos t_1 \cos t_2 - \sin t_1 \sin t_2 \cos(\frac{\pi}{8(n-2)})]\) and its density function is
\[
\frac{1}{a_2^{8n-12}} \int_{\mathbb{R}^n} (\sin t)^{8(n-2)} (\cos t)^2 dt
= \frac{1}{4n-15} (a_2^2 - (u - a_1)^2)^{4n-15} \frac{1}{2^7} (a_1 - u)^2,
\]
where \( a_1 = \cos t_1 \cos t_2 \) and \( a_2 = \sin t_1 \sin t_2 \). Furthermore
\[
\lim_{n \to \infty} \lambda_a \ast \lambda_b = \lambda_c, \quad \text{weakly},
\]
where \( \lambda_c \) is the singular invariant probability measure on the spherical class through \( \exp((t_1 + t_2)H) \).

For the symmetric pair \((Sp(1, n), Sp(1) \times Sp(n))\) the calculations are similar and therefore are not repeated. We obtain
**Theorem 6.2** Let $\lambda_a$ and $\lambda_b$ be two (singular) spherical measures concentrated on the $K$-spherical classes $O_a$ and $O_b$ in the group $G = Sp(1,n)$ respectively. Then $\lambda_a \ast \lambda_b$ is absolutely continuous relative to the Haar measure on $Sp(1,n)$. It is a spherical measure and for a continuous spherical function $f$ on $Sp(1,n)$ we have

$$
\lambda_a \ast \lambda_b(f) = \frac{(\text{vol}(M) \text{vol}(K))^2}{a_2^{8n-12}} \delta(t_1) \delta(t_2) \int_{I_{a,b}} f(u)(a_2^2 - (u - a_1)^2)^{4n-12} (a_1 - u)^2 \, du,
$$

where $a = \exp(t_1 H)$, $b = \exp(t_2 H)$, $\delta(t) = (\sinh t)^{8(n-1)}(\sinh 2t)^2$, and $a_1 = \cosh t_1 \cosh t_2$, $a_2 = \sinh t_1 \sinh t_2$

$$
I_{a,b} = [\cosh(t_1 - t_2), \cosh t_1 \cosh t_2 - \sinh t_1 \sinh t_2 \cos(\pi/8(n-2))].
$$

**Corollary 6.2** Choosing matrices $A$ and $B$ according to the (singular) invariant measures on the spherical classes $O_a$ and $O_b$ respectively and normalized to be probability measures, then the support of the distribution of the product $AB$ is the interval $[\cosh(t_2 - t_1), \cosh t_1 \cosh t_2 - \sinh t_1 \sinh t_2 \cos(\pi/8(n-2))]$ and its density function is

$$
\frac{1}{a_2^{8n-12}} \frac{\pi}{\pi(n-2)} (\sin t)^{8n-14}(\cos t)^2 dt (a_2^2 - (u - a_1)^2)^{4n-12} (a_1 - u)^2,
$$

where $a_1 = \cosh t_1 \cosh t_2$ and $a_2 = \sinh t_1 \sinh t_2$. Furthermore

$$
\lim_{n \to \infty} \lambda_a \ast \lambda_b = \lambda_c,
$$

where $\lambda_c$ is the singular invariant probability measure on the spherical class through $\exp((t_1 - t_2)H)$.

### 7 Convergence to Haar Measure

It was noted that $\lambda_a \ast \lambda_b$ behaves approximately as in the abelian case when $n \to \infty$. In this section we determine the rate of convergence of $(\lambda_a \ast \lambda_b)^l(n)$ to the Haar measure as $n \to \infty$. More precisely we prove

**Theorem 7.1** Let $\lambda_a$ denote the invariant measure on the spherical class $O_a$. Then $(\lambda_a \ast \lambda_b)^l(n)$ converges to the Haar measure on $SU(n)$ as $n \to \infty$ if $l(n) \geq c \log n$ where $c$ is a constant depending on the choice of the spherical classes $O_a$ and $O_b$.  

24
To interpret this theorem let $O_a$ and $O_b$ denote spherical classes where $a = \exp(t_1 H)$ and $b = \exp(t_2 H)$. We had observed that as $n \to \infty$ the Product $O_a \cdot O_b$ converges to the spherical measure concentrated on the spherical class passing through $\exp(t_1 + t_2)H$. The measure $(\lambda_a \star \lambda_b)^{(n)}$ represents the empirical measure of products

$$A_1 B_1 A_2 B_2 \cdots A_{l(n)} B_{l(n)}$$

where $A_i$’s are chosen randomly on $O_a$ and similarly for $B_j$’s. Theorem 7.1 asserts that for $l(n)$ of the stated form this empirical measure converges weakly to the Haar measure on $G = SU(n)$ as $n \to \infty$.

The proof of this theorem requires Schur-Weyl theory on the representations of $SU(n)$ (see [B] or [W] for an account of Schur-Weyl theory). To fix notation we recall the relevant facts. Let $T$ denote a Young diagram, i.e., a graphical representation of a partition $m = m_1 + \cdots + m_k$, $k \leq n$, with $m_i \geq m_{i+1}$. A Young diagram $T$ filled with integers $1, 2, \ldots, m$ is denoted by $\{T\}$ and called a Young tableau. A standard Young tableau is one such that the integers are strictly increasing along rows and columns. We fix the enumeration of the boxes in a Young diagram by starting at the upper left corner and moving along columns consecutively. With this enumeration of the squares in a Young diagram $T$, two subgroups of the symmetric group $S_m$ specified, namely, the subgroup $H = H_T$ consisting of all permutations preserving the rows, and $H' = H'_T$ consisting of all permutations preserving the columns. Let $\mathbb{Z}[S_m]$ be the integral group algebra of the symmetric group, i.e., formal linear combinations with integers coefficients and multiplication inherited from the group law in $S_m$. Define the Young symmetrizer $C = C_T \in \mathbb{Z}[S_m]$ as

$$C = C_T = \left( \sum_{\tau \in H'} \epsilon_{\tau} \tau \right) \left( \sum_{\sigma \in H} \sigma \right) = \sum_{\sigma \in H, \tau \in H'} \epsilon_{\tau} \tau \sigma.$$

The symmetric group $S_m$ and therefore its group algebra $\mathbb{Z}[S_m]$ act on the tensor space $T^m(V)$. In fact, given a tensor $v_{i_1} \otimes \cdots \otimes v_{i_m}$, $v_{ij} \in V$, and $\sigma \in S_m$, the action of $\sigma$ is given by

$$v_{i_1} \otimes \cdots \otimes v_{i_m} \xrightarrow{\sigma} v_{i_{\sigma(1)}} \otimes \cdots \otimes v_{i_{\sigma(m)}}.$$

Notice that this action of the permutation group commutes with the induced action of $G = SU(n)$ on $T^m(V)$, and therefore we have a representation $\tau_m$ of $G \times S_m$ on $T^m(V)$. It also follows that image of $T^m(V)$ under a Young symmetrizer is invariant under $G$. It is well-known that

**Theorem 7.2** Every partition $T : m = m_1 + \cdots + m_k$ with $m_1 \geq m_2 \geq \cdots \geq m_k$ determines a unique irreducible representation $\lambda_T$ of $S_n$, and every irreducible representation of $S_n$ is of the form $\lambda_T$. The degree of $\lambda_T$ is the number of standard Young tableaux whose underlying Young diagram is $T$. 

25
The basic result of Schur-Weyl theory can be summarized as follows:

**Theorem 7.3** The representation \( \rho_T \) of \( G \) is irreducible. For every Young diagram \( T \) corresponding to a partition of \( m \), let \( Z_T \subseteq T^m(V) \) be the minimal linear subspace containing \( \text{Im}C_T \) and invariant under action of \( SU(n) \times S_m \). \( Z_T \) has dimension \( \text{deg}(\rho_T) \cdot \text{deg}(\lambda_T) \) and is irreducible under the representation \( \tau_T = \rho_T \otimes \lambda_T \) of \( SU(n) \times S_m \). \( \text{deg}(\rho_T) \) is equal to the number of semi-standard Young tableaux whose underlying diagram is \( T \). Furthermore \( T^m(V) \) admits of the decomposition, as a \( G \times S_m \)-module (under \( \tau_m \)),

\[
\sum_T Z_T,
\]

where the summation is over all partitions of \( T \) of \( m \) with \( k \leq n \) parts. Let \( A_T(G) \) and \( A_T(S_m) \) denote the algebras of linear transformations of \( Z_T \) generated by the matrices \( \rho_T(g) \otimes I \), \( (g \in G) \), and \( I \otimes \lambda_T(\sigma) \), \( (\sigma \in S_m) \). Then the full matrix algebra on \( Z_T \) has the decomposition \( A_T(G) \otimes A_T(S_m) \).

An irreducible representation \( \rho \) of \( SU(n) \) occurs in \( L^2(G/K) \) if and only if it has a \( K \)-fixed vector (Frobenius reciprocity), and since \((G, K)\) is a symmetric pair the space of \( K \)-fixed vectors is one dimensional. We have

**Proposition 7.1** Let \( G = SU(n) \) and \( K = S(U(k) \times U(n-k)) \) where \( k \geq n-k \). An irreducible representation \( \rho \) of \( G \) has a \( K \)-fixed vector if and only if the corresponding Young diagram is of the form

\[
\begin{array}{cccccc}
\vdots & \vdots & \vdots & & \\
\vdots & \vdots & \vdots & & \\
\vdots & \vdots & \vdots & & \\
\end{array}
\]

where there are \( k \) squares in the first column and \( n-k \) squares in the last, and the number of columns of lengths \( k \) and \( n-k \) are equal.

**Proof** - Let \( T \) be a Young diagram of the form specified in the lemma and let \( r \) denote the number of columns of length \( k \) (or \( n-k \)), and \( \{T\} \) denote the semi-standard Young tableau where the first \( r \) columns are filled with integers \( 1, \ldots, k \) and the last \( r \) columns are filled with integers \( k+1, \ldots, n \). The action \( g \in U(k) \times U(n-k) \) on the vector \( v_T \) corresponding to \( \{T\} \) is given by

\[
v_T \rightarrow (\det g_1 \det g_2)^r, \quad \text{where} \quad g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}
\]

Therefore \( v_T \) is fixed by \( K \). The converse statement that the existence of \( K \)-fixed vector implies the corresponding Young diagram is of the required form will be proven only for \( k = n-1 \) which is the only case needed here. We make use of the following simple
Lemma 7.1 Let $\mathcal{T} \subset SU(n) = G$ be the maximal torus of diagonal matrices. An irreducible representation $\rho_T$ of $G$ contains a $T$-fixed vector if and only if the corresponding Young diagram contains $rn$ squares for some positive integer $r$, and the fixed vector is represented by a Young tableau with the same number of 1’s, 2’s, . . . , $n$’s.

Proof of Lemma - Let $e^{i\tau_j}$ denote the diagonal entries of a matrix in $\mathcal{T}$. Then the only relation among $\tau_j$’s is $\sum \tau_j = 0$. Therefore there is a vector in the representation space of $\rho_T$ fixed by $\mathcal{T}$ if and only if there is a semi-standard Young tableau $\{T\}$ containing the same number of 1’s, 2’s, . . . , $n$’s which proves the lemma.

Let $\rho_T$ denote the irreducible representation $U(n)$ and the Young diagram $T$ correspond to the partition $m = m_1 + \ldots + m_n$. According to the Branching Law [Kn, page 569] the restriction of $\rho_T$ to $U(n-1)$ decomposes into a direct sum of irreducible representations with multiplicity one according to partitions $l = l_1 + \ldots + l_{n-1}$ such that

$$m_1 \geq l_1 \geq m_2 \geq l_2 \geq \ldots \geq l_{n-1} \geq m_n.$$  \hspace{1cm} (7.1)

In order for the restriction of $\rho_T$ to $U(n-1)$ to contain the representation $\text{det}^r$ it is necessary and sufficient that one of the partitions of $l$ be of the form

$$l = l' + l' + \ldots + l', \quad \text{that is} \quad l = rl'.$$

Therefore by (7.1), the representation $\text{det}^r$ occurs in the restriction of $\rho_T$ to $U(n-1)$ if and only if

$$l' = m_2 = m_3 = \ldots = m_{n-1}.$$ \hspace{1cm} (7.2)

Since for irreducible representation of $SU(n)$ it is only necessary to consider Young diagrams with $n - 1$ rows, it follows from (7.2) that the restriction of an irreducible representation $\rho_T$ of $SU(n)$ to $U(n-1)$ contains $\text{det}^r$ if and only if

$$m_1 \geq m_2 = m_3 = \ldots = m_{n-1}.$$ 

By Lemma 7.1 such a representation contains a $T$-fixed vector if and only if there is a Young tableau $\{T\}$ with the same number of 1’s, 2’s, . . . , $n$’s. Furthermore the one dimensional invariant subspace transforming according to $\text{det}^r$ under $U(n-1)$ is spanned by the Young tableau $\{T\}$ where the number of 1’s and $n$’s in the first row is $l' = m_2$. Therefore $m_1 = 2l'$ and the proof of the Proposition is complete. ■

Proof of Theorem 7.1 - In order to prove the theorem we recall the relevant aspect of the Plancherel theorem for a compact connected semi-simple Lie group. The Fourier transform of the spherical measures $\lambda_a$ is

$$F(\lambda_a)(\rho) = \check{\lambda}_a(\rho) = \int_{O_a} \rho(x)d\lambda_a = \phi_\rho(a)\text{vol}(O_a),$$ \hspace{1cm} (7.3)
with similar expression for $\hat{\lambda}_b$, where $\phi_\rho$ is the elementary $K$-spherical function corresponding to the irreducible representation $\rho$ containing a $K$-fixed vector. It is well-known that every elementary $K$-spherical function on $G$ is of the form $\rho_{11}(g)$ in which $\rho_{11}$ is the $(11)$-entry of the matrix of $\rho$ relative to an orthonormal basis $v_1, \ldots, v_N$ where $\rho(K)v_1 = v_1$ (see [H2], page 414). According to Proposition 7.1 irreducible representations of $SU(n)$ containing a $K$ fixed vector are parameterized by integers $m$ corresponding to partitions

$$N = 2m + m + m + \ldots + m \quad (n-1) \text{ summands.}$$

We need

**Lemma 7.2** Let $\rho$ be a spherical representation of the group $G$ (i.e., containing a $K$-fixed vector $v$), then the corresponding elementary spherical function

$$|\phi_m(\exp tH)| = C \frac{(n-1)^{n-\frac{1}{2}}m^{m+\frac{1}{2}}}{(n+m-1)^{n+m-\frac{1}{2}}} t^{-n+\frac{1}{2}} \frac{t}{\sqrt{m}}.$$

for some constant $C$ independent of $m$ and $n$.

**Proof** - This lemma is probably well-known to experts in spherical functions but since the author does not know of a specific reference for it in the form suitable for this work, a proof is sketched. However, in [SC] estimates for Jacobi polynomials are used to establish precise rates of convergence for certain diffusion processes. By applying the radial part of the Laplacian to the elementary spherical function $\phi_m$, one obtains a second order linear ordinary differential equation with regular singular points for it. Consequently one obtains [H2]

$$\phi_m(\exp tH) = F(m + n, -m, n; \sin^2 t)$$

where $F$ is the hypergeometric function which reduces to the Jacobi polynomial $P_m^{n-1,n}(\cos^2 t)$ (except for normalization by a constant) (see [AAR] for explanation of notation and extensive treatment of Jacobi polynomials). Now

$$P_m^{n-1,n}(1) = \frac{(n + m - 1)!}{(n-1)!m!}.$$

Since $\phi_m(e) = 1$, $P_m^{n-1,n}$ should be normalized accordingly. Estimates for Jacobi polynomials are obtained by examining the behavior of their generating function on the unit circle and applying standard methods for obtaining estimates from generating functions. In fact one obtains (see [AAR] especially page 350)

$$P_m^{n-1,n}(\cos \theta) = \begin{cases} \theta^{-n+\frac{1}{2}}O(\frac{1}{\sqrt{m}}), & \text{for } \frac{\pi}{n} \leq \theta \leq \frac{\pi}{2}; \\ O(m^{n-1}), & \text{for } 0 \leq \theta \leq \frac{\pi}{n}; \end{cases} \quad (7.4)$$
for a suitable constant $c$ as $m \to \infty$. Substituting in the expression for $\phi_m$ in terms of Jacobi polynomials we obtain the desired estimate. ■

Because $\phi_m$ is a spherical function, the value of this function on the spherical class $O_a$ is constant and so we can take $a = \exp(t_1 H)$. Now by (7.3) for the Fourier transform of the spherical measure $\lambda_a$ and $\lambda_b$ we obtain:

$$F(\lambda_a)(\rho_m) = \phi_m(a)\text{vol}(O_a) = \phi_m(\exp t_1 H)\text{vol}(O_a)$$

$$F(\lambda_b)(\rho_m) = \phi_m(b)\text{vol}(O_b) = \phi_m(\exp t_2 H)\text{vol}(O_b),$$

where $\rho_m$ is the spherical representation corresponding to $m$.

Now applying the Plancherel theorem for the function $(\lambda_a \star \lambda_b)^{(n)}$ we obtain

$$\| (\lambda_a \star \lambda_b)^{(n)} - 1 \|^2_{L^2} = \sum_{m>0} d_{\rho_m}(\phi_m(t_1)\phi_m(t_2))^{2(n)},$$

We want to find $l(n)$ such that the sequence

$$c_n = \sum_m d_{\rho_m}(\phi_m(t_1)\phi_m(t_2))^{2(n)}$$

(7.5)

converges to zero when $n$ goes to infinity. This will ensure that $(\lambda_a \star \lambda_b)^{(n)}$ converges to the Haar measure in $L^p$-norm for $1 \leq p \leq 2$ as $n \to \infty$ since on compact groups of fixed finite volume $L^2$ norm dominates $L^p$ for $p \leq 2$.

For analyzing the sequence $\{c_n\}$ we need to compute the dimension of the representation $\rho_m$. By Weyl’s dimension formula the dimension of the irreducible representation $\rho_T$ of $U(n)$ determined by the Young diagram $T : m = m_1 + m_2 + \ldots + m_n$ is

$$D(a_1, a_2, \ldots, a_n) = D(m-1, m-2, \ldots, 0),$$

where $a_k = m_k + n - k$ and $D(a_1, a_2, \ldots, a_n) = \prod_{j<k}(a_j - a_k)$. In our situation the spherical representation $\rho_m$ has the Young diagram characterized in Proposition (7.4) and the corresponding partition is $m n = 2m + m + \ldots + m$ where $2m$ is the number of columns in the corresponding Young diagram. Therefore

$$a_1 = 2m + n - 1, \quad a_2 = m + n - 2, \quad a_3 = m + n - 3, \ldots, a_{n-1} = m + 1, \quad a_n = 0$$

By the Weyl’s dimension formula the dimension $d(m, n)$ of the representation $\rho_m$ is

$$d(m, n) = (2m + n - 1)\prod_{k=2}^{n-1} \frac{(n-k+1)!((m+n-k)^2)}{(n-2)!(n-1)! \ldots 2!}$$

$$= (2m + n - 1)\prod_{k=2}^{n-1} \frac{(m+n-k)^2}{(n-2)!(n-1)!}$$

$$= \frac{((m+n-2)!)^2}{(m!)^2(n-2)!(n-1)!} (2m + n - 1).$$
Applying the Sterling estimate $n! \sim \frac{1}{\sqrt{2\pi}} n^{n+\frac{1}{2}} e^{-n}$ we obtain

$$d(m, n) \sim \frac{e}{2\pi} \frac{(m + n - 2)^{2(m+n)-3}}{m^{2m+1}n^{2n-2}}(2m + n - 1)$$

Substituting for $\phi_m(t)$ from the Lemma 7.2 we obtain

$$c_n = b^{(2n-1)l(n)} \sum_{m=1}^{(m + n - 2)^{2(m+n)-3}(2m + n - 1)} \left( \frac{(n - 1)^{2n-1}m^{2m}}{(n + m - 1)^{2(m+n)-1}} \right)^{2l(n)}$$

where $b = \frac{1}{\epsilon_1 t_2}$. Now we decompose the summation $c_n$ into two parts $s_1$ and $s_2$ as follows:

$$c_n = b^{(2n-1)l(n)} (s_1 + s_2) = b^{(2n-1)l(n)} \left( \sum_{m \leq n-1} + \sum_{m > n-1} \right)$$

Since in $s_1$ the summation is over $m \leq n - 1$ we have

$$s_1 = \sum_{m \leq n-1} (m + n - 1)^{2(m+n)-3-2l(n)(2m+n-1)}(2m + n - 1)m^{4ml(n)-2m-1}n^{2(2n-1)l(n)-2n+2}$$

$$\leq n^{2(2n-1)l(n)-2n+2} \sum_{m \leq n-1} (2(n - 1))^{2(m+n)-3-2l(n)(2m+n-1)}(n - 1)^{4ml(n)-2m-1}$$

$$= 2^{2n-4ml(n)+2l(n)-3} (n - 1)^{2n-4ml(n)+2l(n)-3} n^{2(2n-1)l(n)-2n+2} \sum_{m \leq n-1} 2^{2m-4ml(n)}(n - 1)^{2m-4ml(n)}(n - 1)^{4ml(n)-2m}$$

$$= 2^{2n-4ml(n)+2l(n)-3} n^{n-1} \sum_{m \leq n-1} (4^{1-2l(n)})^m$$

Therefore

$$b^{(2n-1)l(n)} s_1 \leq 2^{2n-4ml(n)+2l(n)-3} n^{n-1} b^{(2n-1)l(n)} a^n - 1 \frac{1}{a - 1},$$

where $a = 4^{1-2l(n)}$. Now for $l(n) \geq C_1 \log n$ and $C_1$ sufficiently large and depending on $t_1$ and $t_2$, we obtain

$$s_1 \leq \frac{C_2}{n^\epsilon}, \quad (7.6)$$

for some $\epsilon > 0$ and some constant $C_2$ depending only on $t_1$ and $t_2$.

Now we estimate $s_2$ where the summation is over $m > n - 1$. It is clear that
\[ s_2 \leq \sum_{m>n-1} 3m(2m)^{2(m+n)-2l(n)(2m+n-1)} m^{4nl(n)-2m+n-2l(n)-2(n+2)} \]

\[ = b(2n-1)l(n)2^{2n-4nl(n)+2l(n)-3} \sum_{m>n-1} m^{2n+2l(n)-4nl(n)-3} 2^{m-4nl(n)} \]

\[ \leq b(2n-1)l(n)2^{2n-4nl(n)+2l(n)-3} \sum_{m=n}^{\infty} a^m \]

\[ = b(2n-1)l(n)2^{2n-4nl(n)+2l(n)-3} \frac{a^n}{1-a} \]

where \( a = 4^{1-2l(n)} \) as before. Consequently for \( l(n) \geq C_3 \log n \) and \( C_3 \) sufficiently large and depending on \( t_1 \) and \( t_2 \) we have

\[ s_2 \to 0 \quad \text{as} \quad n \to \infty. \quad (7.7) \]

Therefore \( c_n = s_1 + s_2 \) tends to zero and the proof of the theorem is complete. \( \blacksquare \)
References

[AAR] Andrew, G., R. Askey and R. Roy - *Special Functions*. Cambridge University Press, (1999).

[AW] Agnihotri, S. and C. T. Woodward - Eigenvalues of products of unitary matrices and quantum Schubert calculus, *Math. Res. Lett.*, 5, (2002), pp. 817-836.

[B] Boerner, H. - *Representations of groups, with special consideration for the needs of modern physics*. American Elsevier Publishing Co., Inc., New York (1970).

[F] Fulton, W. - Eigenvalues, invariant factors, highest weights and Schubert Calculus, *Bull. Amer. Math. Soc.*, 37, no. 3, (2000), pp. 209-249.

[H1] Helgason, S. - *Differential Geometry, Lie Groups, and Symmetric Spaces*, (2002).

[H2] Helgason, S. - *Groups and Geometric Analysis*, (1984).

[Kl] Klyachko. A. A. - Stable bundles, representation theory and Hermitian operators, *Selecta Math.*, 4, no.3 (1998), pp. 419-445.

[Kn] Knapp, A. - *Lie Groups Beyond an Introduction*, Birkhauser, (2004).

[SC] Saloff-Coste, L. - Precise estimates on the rate at which certain diffusions tend to equilibrium, *Math. Zeit.*, 217, (1994), pp.641-677.

[Sh1] Shaffaf, J. - *Dissertation*, Sharif University of Technology.

[Sh2] Shaffaf, J. - On Products of Random Matrices and certain Hecke Algebras associated with Groups of $2 \times 2$ Matrices, *submitted*.

[W] Weyl, H. - *The Classical Groups*, Princeton University Press 1966.

Institute for Studies in Theoretical Physics and Mathematics, Tehran, Iran, and Sharif University of Technology, Tehran, Iran.