A Functional Integral Equation for the Complete Effective Action in Quantum Field Theory

K. Scharnhorst*

Universität Leipzig†
Naturwissenschaftlich-Theoretisches Zentrum
Augustusplatz 10/11
D-04109 Leipzig
Federal Republic of Germany

*E-mail: kjsch @ qft.physik.uni-leipzig.d400.de
†temporary address
Abstract

Based on a methodological analysis of the effective action approach certain conceptual foundations of quantum field theory are reconsidered to establish a quest for an equation for the effective action. Relying on the functional integral formulation of Lagrangian quantum field theory a functional integral equation for the complete effective action is proposed which can be understood as a certain fixed point condition. This is motivated by a critical attitude towards the distinction artificial from an experimental point of view between classical and effective action. While for free field theories nothing new is accomplished, for interacting theories the concept differs from the established paradigm. The analysis of this new concept is concentrated on gauge field theories treating QED as the prototype model. An approximative approach to the functional integral equation for the complete effective action of QED is exploited to obtain certain nonperturbative information about the quadratic kernels of the action. As particular application the approximative calculation of the QED coupling constant $\alpha$ is explicitly studied. It is understood as one of the characteristics of a fixed point given as a solution of the functional integral equation proposed. Finally, within the present approach the vacuum energy problem is considered and possible implications on the induced gravity concept are contemplated.
1 Introduction

Physical reality can be approached by means of quantum field theory from different perspectives. This in particular depends on the kind of information one is interested to extract in order to solve a problem under consideration but it is also influenced by the individual view toward the fundamental difficulties met in present day standard quantum field theory (and its generalized concepts like string theory). To a large extent, these different approaches reflect technical difficulties to fully (in particular, nonperturbatively) understand quantum field theoretical models rather than really differences in concept on a fundamental level. However, few pioneers of quantum field theory like Dirac [1],[2] and Feynman [3],[4] in particular pointing to the UV divergency problem always maintained the view that the right theory has not yet been found. This attitude has apparently not received majority support in time but in this respect it does not seem to exist any majority opinion at all. From this state of affairs we feel free to draw justification for a reconsideration of certain conceptual foundations of quantum field theory constituting the purpose of the present paper.

Notwithstanding above mentioned problems, it seems to exist wide agreement that the scattering matrix can be considered as the fundamental object for describing a particular quantum field theoretic model. This amounts to saying that full knowledge of the complete scattering matrix is considered equivalent to the solution of a quantum field theory and all interesting information, at least in principle, can be extracted from it. Construction of the scattering matrix can be attempted by different methods. For instance, the so-called S-matrix theory as studied in the 1950s in reaction to the emergence of the divergence problem in Lagrangian quantum field theory was designed to find the (finite) scattering matrix from rather general fundamental principles like causality, unitarity, Lorentz invariance using dispersion techniques without making reference to any Lagrangian underlying the theory (see [6],[7], e.g.). However, although quite general and interesting results have been obtained principles applied turned out not restrictive enough to completely fix the scattering matrix for realistic theories. Nowadays, after the successful re-emergence of (renormalizable) Lagrangian quantum field theory at the end of the 1960s description of the scattering matrix is supplied in a standard way in terms of the effective action of the theory considered [8]. In this sense, we may view the effective action as the genuine fundamental object of interest and will concentrate on its study in this article.

Historically, beyond the S-matrix theory already mentioned attempts to cure UV divergencies by nonlocal field theories played a particular role since the emergence of the divergence problem in the 1930s (for a review including references see

\footnote{For a description of the attitude in one large part of the community see ref. [5], e.g..}
Although it has been recognized early that nonlocal field theories may be accompanied by new, perhaps even more unpleasant difficulties, so with unitarity and (macro-)causality, theoretical thinking in this direction never ceased to exist. Most prominent, present day string theory although much more ambitious can be viewed as a particular way of giving preference to a special kind of nonlocality \[12\]. In recent years, few papers were again dealing with nonlocal quantum gauge field theories \[13\]–\[20\] (to mention only this subject) where in part the nonlocalities introduced are understood as regulators. Although having a different aim than fighting UV divergencies, also the average action concept proposed recently should be mentioned here \[21\],\[22\]. In principle, the drawback of all these nonlocal approaches however consists in the arbitrariness in the choice of the nonlocality introduced. So far, no unique recipe starting from first principles has been proposed.

However, the dominant paradigm in the field remains local renormalizable Lagrangian quantum field theory (Throughout the paper we will denote it by the term standard quantum field theory.). But, also there nonlocality is a well-known phenomenon because it is a feature of the effective action that can be derived for any quantum field theory (either local or nonlocal) and which also serves (in most cases) as generating functional of the one-particle-irreducible (1PI) Green functions. In general, the effective action is attributed different meanings by different authors. Few regard the effective action as some low energy representation of a quantum field theory obtained by integrating out certain (massive) degrees of freedom, while others consider the effective action as a full fledged description of the model under investigation from which arbitrary S-matrix elements (related to any observation one might be able to perform) can be derived. We will stick here to the latter view. To us, very pragmatically the effective action is that object which contains all the information ever to be measured under certain defined circumstances and there is no other (independent) object linking theory to physical reality. The shape of the effective action of course may depend on some of these circumstances (external conditions, e.g.). A similar point of view has recently been described with respect to the gravitational effective action by Vilkovisky \[23\]. The effective action concept we have in mind aims at quantum field theoretic models, especially those which are realistic like QED, and assumes that certain sectors of physical reality can be described in a consistent way independently of each other. It is therefore quite different from the TOE ('theory of everything') concept often related to superstring theory.

In short, the program of the present article can be described by saying that we intend to find a concept which allows to determine the structure of the (highly complex) observable 'effective action' without making reference to any other quantity not accessible to observation. In particular, the approach to quantum field theory
will be based on a critical attitude towards the distinction artificial from an experi-
mental point of view between the so-called classical action and the effective action. 
This way we will be lead to propose an equation for determining the (finite) effective 
action, which can be understood as a certain fixed point condition. It will be an 
equation for functionals of fields (actions) and is therefore designed to remove (to a 
certain extent — the field content has to be prescribed as usual) the arbitrariness 
in the choice of the Lagrangian standing at the beginning of any field theory. Such 
however can only be expected to happen for interacting theories, where our approach 
differs from the established paradigm. For free field theories, where this is not the 
case, nothing new is accomplished in this respect. As technical tool we rely on the 
functional integral formulation of Lagrangian quantum field theory which seems to 
be the appropriate and most convenient language for the description of our concept. 
While nonlocality will be an inherent feature of our approach in most cases, it is by 
no means the conceptual starting point of the present investigation. Of course, the 
program as just sketched is an abstract one. However, once we have proposed the 
general concept it will simply serve us as a guiding line for finding an appropriate 
approximative approach to perform explicit calculations (in this article: in QED as 
the prototype gauge field theory).

In the past decade the effective action concept has received interest from the 
point of view of its invariant geometrical formulation. This is an important step 
in ensuring the physical relevance of the effective action because its physical con-
sequences should not depend on the particular choice of coordinates for the field 
variables. Initial work in this direction traces back to Vilkovisky [24],[25] and 
DeWitt [26], for a recent discussion of the geometrical effective action see [27], for 
a review including further references [28]. For the purpose of the present article (to 
reduce complexity of the considerations) we simply bypass the subject and maintain 
that always those field coordinates are applied in terms of which the formalism takes 
it naive (non-geometrical) shape. Furthermore, for gauge field theories, a main con-
cern of the unique (geometrical) effective action concept, we find that generalized 
Landau gauge is the only sensible gauge. Inasmuch as for gauge field theories the 
geometrical effective action has been found to agree with the naive one (calculated 
by means of the standard background field method) exactly for generalized Landau 
gauge we feel free to ignore the subject also there [29]–[31].

The outline of the article is as follows. In chapter 2 we explain the general concept 
in some length. This is done in three steps. In section 2.1 based on a methodolog-
ical analysis we establish a quest for an equation for the complete effective action. 
While section 2.2 serves to suggest a particular answer to this question by imposing 
a certain fixed point condition in terms of a functional integral equation section 2.3 
discusses some features of this equation, among others the relation between stan-
standard quantum field theory and the present approach. Chapter 3 then applies the concept to gauge field theories. Specifically, there we formulate the functional integral equation for the complete effective action of QED and then in sections 3.1 and 3.2 Ward-Takahashi identities and Schwinger-Dyson equations are discussed respectively. Chapter 4 contains the major body of the explicit calculation performed. The model under investigation is QED in 4D Minkowski (Euclidean) space. While section 4.1 spells out what kind of approximative approach to the functional integral equation for the complete effective action of QED is applied in general section 4.2 and its subsections serve to establish a more concrete approximation strategy suited for explicit calculation and are concentrating on the quadratic kernels of the action. Subsections 4.2.1 and 4.2.2 discuss certain general requirements on the quadratic kernels of the fermion and gauge field actions respectively while subsection 4.2.3 explains the approximation strategy finally chosen. Section 4.3 also split into several subsections then presents the explicit calculation in some detail. Subsection 4.3.1 contains technical details of the functional integration performed. While for the quadratic kernel of the gauge field action we rely on a certain Ansatz subsection 4.3.2 establishes an integral equation for the quadratic kernel of the fermion action. This integral equation is then approximatively solved in subsections 4.3.2.1 and 4.3.2.2 in the asymptotic UV and IR regions respectively. This analysis yields certain nonperturbative information about the quadratic kernel of the fermion action. In the final subsection 4.3.3 of chapter 4 as particular application of the present method the approximative calculation of the QED coupling constant $\alpha$ is explicitly studied. It is understood as one of the characteristics of a fixed point given as a solution of the functional integral equation proposed. Certain technical details of the calculation described in chapter 4 are deferred to two Appendices at the end of the article. Chapter 5 shortly discusses the vacuum energy problem for QED on the 1-loop level. Final consideration then is devoted to the relevance of the proposed approach to the induced gravity concept. The article closes in chapter 6 with a discussion of some aspects of the results obtained.
2 An Equation for the Complete Effective Action

2.1 Do We Need an Equation? — Methodological Considerations

As introductory step let us begin with displaying key elements of the standard formulation of the effective action. We consider Lagrangian quantum field theory in flat (Minkowski) space-time and in this chapter we use scalar field theory to pursue the discussion. Hereby, it is understood that generalization to more complicated theories (in particular, gauge field theories) can be performed merely by standard means.

Construction starts with the generating functional of Green functions

\[ Z[J] = C \int D\phi \ e^{i\Gamma_0[\phi]} + i \int dx J(x)\phi(x) \]  \hspace{1cm} (2.1)

where \( \Gamma_0[\phi] \) is the so-called classical action of the theory and \( C \) some fixed normalization constant. Then, the generating functional of the connected Green functions is

\[ W[J] = -i \ln Z[J] \]  \hspace{1cm} (2.2)

The effective action \( \Gamma[\bar{\phi}] \) which also is the generating functional of the one-particle-irreducible (1PI) Green functions is obtained as the first Legendre transform of \( W[J] \).

\[ \Gamma[\bar{\phi}] = W[J] - \int dx J(x)\bar{\phi}(x) \]  \hspace{1cm} (2.3)

Here,

\[ \bar{\phi}(x) = \frac{\delta W[J]}{\delta J(x)} \]  \hspace{1cm} (2.4)

is understood which in turn leads to

\[ \frac{\delta \Gamma[\bar{\phi}]}{\delta \phi(x)} = - J(x) \]  \hspace{1cm} (2.5)

in analogy to the classical field equation for \( \Gamma_0[\phi] \). Equivalently, using above relations following formula for the effective action can be considered as the defining one

\[ e^{i\Gamma[\bar{\phi}]} = C \int D\phi \ e^{i\Gamma_0[\phi + \bar{\phi}]} + i \int dx J(x)\phi(x) \]  \hspace{1cm} (2.6)
where the r.h.s. of above equation has to be calculated at a current $J(x)$ which is a functional of $\tilde{\phi}$ and given by eq. (2.5). Therefore, as the r.h.s. is a functional of both $J$ and $\tilde{\phi}$ eqs. (2.5), (2.6) have to be understood as functional integro-differential equations for determining the (off-shell) effective action and give an implicit definition only. But, as we will argue below eq. (2.6) is not an equation in the narrow sense of the meaning of the word, instead it rather should be called a formula.

The latter point is barely discussed in the literature and shall now be considered from a methodological point of view. Observe that eq. (2.1) defines a map $g_1 : \Gamma_0[\phi] \rightarrow Z[J]$ from the class of functionals called classical actions to the class of functionals $Z$. Furthermore, we have mappings $g_2 : Z[J] \rightarrow W[J]$ (eq. (2.2), single-valued up to the uninteresting for the present purpose fixing of the sheet of the Riemann surface) and $g_3 : W[J] \rightarrow \Gamma[\tilde{\phi}]$ (eq. (2.3)). These three maps together define a map $g_3 \circ g_2 \circ g_1 = f : \Gamma_0[\phi] \rightarrow \Gamma[\tilde{\phi}]$ (eq. (2.6)) from the set of so-called classical actions to the set of effective actions. In total, this map is unique up to the renormalization problem which can always be treated in the present context by applying an appropriate regularization procedure for properly handling the divergencies. Inasmuch as this map $f$ is constructed explicitly eq. (2.6) is not a genuine equation with possibly a variety of solutions but rather expresses the image $\Gamma$ of $\Gamma_0$ with respect to the map $f$ — it is a formula.

Above consideration justifies following view. Once the functional integral measure is constructed (and typically this is done for a whole class of classical actions and then fixed forever) the classical action $\Gamma_0$ uniquely determines the corresponding effective action $\Gamma$. In other words, the effective action does not contain more information than (implicitly) contained in the classical action (supplemented by the functional integral measure). This point is usually not stressed in studying concrete models due to the calculational complexity involved. Although it is of next to no practical (i.e., calculational) relevance therefore, it involves important methodological implications. The most important one consist in the fact that the effective action does not appear as object in its own right but as a derived quantity only. Mere reformulations of the calculational tools used to determine the effective action, like Schwinger-Dyson equations, e.g., do not change this character.

Before proceeding further let us mention that formulas (2.1)–(2.6) reflect two features of modern quantum field theory. On one hand side, they stand for the convincing success of quantum field theory as witnessed in the last few decades, of a theory providing us with operational instruments producing numbers which agree with measurement to a degree not seen elsewhere in physics (or in any other science). On the other hand, they also stand for the fundamental conceptual difficulties inherent to local quantum field theory. The most important of them is manifesting
itself in terms of the well-known ultraviolet divergencies. Although there exist two or three (different) mainstream opinions with respect to this issue (and some other, related ones) numerous dissenting ones can also be found. From this observation one may conclude that research apparently has not yet lead to any generally accepted concept explaining and removing the problems in a finally convincing manner as judged from physics as an inherently consistent building combining theory and experiment. This amounts to saying that search in different directions seems justified and even certain doubt in the foundations of quantum field theory should not be rejected at once. With this in mind, in what follows we will apply the point of view that perhaps even certain foundations of quantum field theory are not understood up to their end and we will see whether we can throw different light on them. In this context, as outlined in the Introduction, we will focus on the effective action which we consider as the appropriate object to be studied.

Let us ask for principles effective actions should be governed by in general. While we have no problem in giving principles they should obey, like Lorentz invariance, CPT invariance, e.g., the answer to the question what they are in detail determined by in view of considerations given further above reduces to saying that they are uniquely given as image of the corresponding classical action by means of the map \( f \) containing information about the functional integral measure. This way, the question is traced back to the uncertainty in classical field theory what Lagrangian to choose. Although, one does not necessarily need to worry about this point here we will. Basically, we prescribe an effective action in terms of some low energy information rather than to find it from independent (quantum) principles not exhausted by fixing the classical action. And, if we are honest, at best we may say that our prescription is approximately right.

One may now confront the methodological insight obtained so far with the deductive idea often applied in theoretical physics that the special case (here the classical action) should be derived from the more general one (here the effective action) and not the other way around. In this sense, the complete effective action is the genuine fundamental object to be studied. If so, up to further investigation, one is willing to allow that the complete effective action might be an object in its own right. Then one has to find a method of determining the complete effective action differing from the established method. There are not so many methods available and to use

\[ \text{2Of course, any effective action has a certain classical limit but coincidence of its classical limit with that of another effective action does not necessarily entail identity of both effective actions then.} \]

\[ \text{3Certainly, also such a different method which does not start with the classical action may, at the end, lead to the conclusion that classical actions and effective actions are related to each other one-to-one, but then this is a result of the method and not the starting point.} \]
an equation for determining the complete effective action seems to be an approach natural within theoretical physics. Therefore, above view leads to the task to find such an equation for the complete effective action.

To the best of the authors knowledge such a question has not been raised so far in the existing literature. Independent of the kind of further answer to it, it should be emphasized that in view of the fundamental role of the effective action in quantum field theory it deserves one. Even rejection of the question (e.g., by closely sticking to the established formalism) has important methodological consequences as we have demonstrated above.

Search for an equation for the complete effective action needs to be ruled by a couple of principles. First, solutions of such an equation should be able to reproduce standard quantum field theoretic result with the required accuracy in order to stay in line with experiment. Obviously, this leaves not much room for an answer differing from the known one. Second, the formalism connected with such an equation should sufficiently differ from standard quantum field theory in order to be able to remove known problems, at least in part. And third, any sensible search for an equation for the complete effective action should take into account that the eventual result needs to be sufficiently general in order to be applicable to various situations and has to be restrictive enough at the same time in order to allow to derive from it concrete information.

While the call for an equation for the complete effective action still might be shared by a number of researchers and probably represents the least disputable part of the present investigation, to reach agreement with respect to an eventual answer to it very likely will be much more difficult. In the following section we are going to propose an answer which then shall be investigated in some further detail.

2.2 Proposing an Equation for the Complete Effective Action

Basically, there are two different routes to find the particular answer on the question put forward in the preceding section we prefer by proposing a specific equation for the complete effective action. One way is to discuss certain principles to be built in and then to write down an equation which embodies these. The other way which we will choose now is heuristically to motivate an equation which then will be analyzed with respect to its conceptual content.

Let us consider the map $f : \Gamma_0[\phi] \rightarrow \Gamma[\bar{\phi}]$ defined in section 2.1 mapping so-
called classical actions to effective actions. Although it is not necessarily well defined for the domain of classical actions (which are local functionals in general) we will not change the map \( f \) itself but instead we will now extend the domain of this map. For this purpose it suffices to mention that the set of so-called classical actions can be considered as a sub-set of the class of effective actions. From now on we understand the map \( f \) as a mapping of the set of effective actions into itself.

On the basis of formulas given in the preceding section we will now explicitly define the map \( f \) for the extended domain. Again, we define the generating functional of Green functions by

\[
Z_n[J_n] = C e^{-i\Gamma_{n-1}[0]} \int D\phi \ e^{i\Gamma_{n-1}[\phi]} + i \int dx J_n(x) \phi(x) ,
\]  

(2.7)

where as in eq. (2.1) \( C = C(\mu) \) is some fixed dimensional normalization constant depending on an arbitrary mass parameter \( \mu \) and compensating the dimension of the functional integral measure \( D\phi \). Changes in \( \mu \) correspond to changes in the normalization of the vacuum energy connected with \( \Gamma[0] \). In extending the domain of the map \( f \) we have introduced an additional normalization factor \( \exp(-i\Gamma_{n-1}[0]) \) (This is not a major point but worth to be appreciated from a conceptual point of view.). Classical actions typically are normalized to obey \( \Gamma_0[0] = 0 \). Then, eq. (2.6) tells us that \( \Gamma[0] \) is completely originated by vacuum fluctuations governed by the classical action \( \Gamma_0 \) (up to some normalization of the vacuum energy fixed for a whole class of actions). By including the additional normalization factor this principle is generalized to the map \( f \) acting in the extended domain and admits calculation of the vacuum energy as usual \(^4\).

The generating functional of the connected Green functions is

\[
W_n[J_n] = -i \ln Z_n[J_n] .
\]  

(2.8)

The generating functional of the 1PI Green functions (the image of \( \Gamma_{n-1} \)) is given by

\[
\Gamma_n[\bar{\phi}_n] = W_n[J_n] - \int dx J_n(x) \bar{\phi}_n(x) ,
\]  

(2.9)

where

\[
\bar{\phi}_n(x) = \frac{\delta W_n[J_n]}{\delta J_n(x)}
\]  

(2.10)

\(^4\)Having in mind standard quantum field theory, of course, here we refer to vacuum energy modifications under external conditions.
and consequently
\[ \frac{\delta \Gamma_n[\bar{\phi}_n]}{\delta \phi_n(x)} = - J_n(x), \] (2.11)
The generalization of eq. (2.6) reads
\[ e^{i \Gamma_n[\bar{\phi}_n]} = C e^{-i \Gamma_n[0]} \int D\phi \ e^{i \Gamma_{n-1}[\phi + \bar{\phi}_n]} + i \int dx J_n(x) \phi(x), \] (2.12)
where the r.h.s. of above equation is again to be calculated at a current \( J_n(x) \) which is a functional of \( \bar{\phi}_n \) given by eq. (2.11), and eqs. (2.12), (2.11) are acting as functional integro-differential equations for determining \( \Gamma_n \) (and the same accompanying comment as in section 2.1). The map \( g_3 \circ g_2 \circ g_1 = f : \Gamma_{n-1} \longrightarrow \Gamma_n \) is explicitly given by eqs. (2.7) \( (g_1 : \Gamma_{n-1} \longrightarrow Z_n) \), (2.8) \( (g_2 : Z_n \longrightarrow W_n) \), and (2.9) \( (g_3 : W_n \longrightarrow \Gamma_n) \).

Consider now iterations of the map \( f \) leading to some discrete series of effective actions \( \ldots \longrightarrow f \longrightarrow \Gamma_{n-1} \longrightarrow f \longrightarrow \Gamma_n \longrightarrow f \longrightarrow \Gamma_{n+1} \longrightarrow f \ldots \). Eventually this still can be combined with a certain truncation procedure, e.g., acting on the obtained effective action after each application of the map \( f \). It is worth noting that the successive calculation of higher loop contributions to the effective action in standard quantum field theory is such an iteration and truncation procedure. However, for the present purpose we do not consider any truncation procedure. Obviously, the most interesting question one may ask with respect to the iterations of the map \( f \) is whether it has any fixed point. It should be expected that the fixed point condition for the map \( f \) is not trivially fulfilled for any arbitrary action and should distinguish certain (complete) effective actions. Now, we propose that the fixed point condition for the map \( f \) defined above yields the equation for the complete effective action we are looking for.

The equation for the complete effective action which is equivalent to the fixed point condition for the map \( f \) reads
\[ e^{i \Gamma[\bar{\phi}]} = C e^{-i \Gamma[0]} \int D\phi \ e^{i \Gamma[\phi + \bar{\phi}]} + i \int dx J(x) \phi(x), \] (2.13)
where
\[ J(x) = - \frac{\delta \Gamma[\bar{\phi}]}{\delta \phi(x)}. \] (2.14)
Eqs. (2.13) and (2.14) together define a genuine functional integro-differential equation for determining the complete (off-shell) effective action \( \Gamma \) of a quantum field
theory. Of course, this equation needs to be supplemented by additional information to specify the particular conditions under which it should be solved. Accumulated experience in quantum field theory tells us that in general solutions of eq. (2.13) – if there exists any at all – should be expected to be nonlocal and nonpolynomial functionals $\Gamma$ of the field $\phi$. Optimistically, one might think that above equation for the complete effective action is sufficiently restrictive in the case of interacting theories to enable us not only to find the structure of the effective action but hopefully also to determine dimensionless parameters it contains (e.g., coupling constants and mass ratios). What concerns its applicability, so the eventual range of theories remains to be explored. But it seems, that at least any theory which cannot be understood as being induced by some more fundamental one should be subject to the concept.

### 2.3 Exploring the Equation

Before we will analyze eq. (2.13) from the conceptual side let us ask whether it has any solution at all. The answer is that any free field theory solves eq. (2.13) (In saying so, of course, we neglect the vacuum energy problem.). For free field theories the formulation proposed in section 2.2 completely agrees with the standard formulation of quantum field theory displayed in section 2.1. However, the former obviously differs from the latter for interacting theories. In the future it remains to be seen whether there exists any interacting field theory which solves eq. (2.13).

Now, we will study eq. (2.13) with respect to its methodological content. Observe, that the proposed equation for the complete effective action is exclusively expressed in terms of an observable (at least, in principle) quantity namely the complete effective action which should be finite, of course. This specifies the concept of renormalizable quantum field theory by relying on observable objects only (Bare and dressed quantities agree here.). In this context, one may wonder whether the conceptual distinction between classical action and effective action is really a productive one. Although any theoretician may extract the classical limit from any solution of eq. (2.13) one may justified ask what does this tell an experimental physicist. In reality, vacuum fluctuations cannot be switched off (at best, they can be modified) and the experimentally relevant quantity is the effective action. Rather, the experimental physicist is interested in the leading (low energy, long distance, low intensity) terms of the derivative expansion of the effective action but these do not necessarily coincide with what is called the classical action although they will contain it in most cases. In view of our equation for the complete effective action also of limited sense is to ask which effective action term is induced and which is not because eq. (2.13) is a self-consistency condition.
Continuing above consideration, it should be mentioned that already in standard quantum field theory there is no difference in principle between a certain mode of vacuum fluctuations and macroscopic (external) fields. This is reflected by the insight that the effective action has a dual nature, namely on one hand it is considered as action governing the behaviour of macroscopic (external) fields and at the same time it is the generating functional of 1PI Green functions playing here-with a central role in describing vacuum fluctuations. In addition, any particular mode of vacuum fluctuations is acting in the background of all of them and merely experiences their total effective impact as described by the complete effective action. Therefore, the path integral construction should not rely on the classical action governing the weight of each path (mode) as is done in standard quantum field theory but the weight of each path (mode) should be determined by the complete effective action expressing the vacuum properties in total. Of course, this involves a certain self-referentiality which finds its adequate formulation in terms of a genuine equation. Concluding this we may say that eq. (2.13) is the theoretical expression of the dual nature of the complete effective action being effective action and generating functional at the same time. In other words, vacuum fluctuations are governed by one and the same action like macroscopic phenomena.

Having obtained certain insight into principles embodied in the proposed equation for the complete effective action in the following let us turn to eventual methods of its solution. To expect any final answer on this right now clearly would not be realistic, instead few aspects which come to mind immediately should be discussed only. Although there is no quick answer at hand to the question, one may ask whether the map $f$ has something like a contraction property in a certain neighborhood of a solution of eq. (2.13). If this is the case one could attempt its solution by iteration. With this concept in mind we will see how the relation of standard quantum field theory to the present formulation can be described. The standard formulation of quantum field theory can be viewed as first iteration of the map $f$ starting from a certain low energy (local) approximation (the so-called classical action) to the complete effective action. This can be considered as natural starting point which is expected to be close to a fixed point of the map $f$ for 'experimental' reasons. However, it is clear that in view of eq. (2.13) even the 'complete' (assuming we had summed up usual perturbation theory) effective action of standard quantum field theory given by eq. (2.6) is not the complete one in the sense of eq. (2.13) but remains just an approximation. The approximation method represented by standard local quantum field theory works reasonably good in lower spacetime dimensions, with considerable effort in 4 dimensions, but it becomes badly defined for most theories in higher dimensions. So, one may consider the properties of a theory with respect to renormalization as information about the possible quality of an approximate solution of eq. (2.13) obtained from some local Ansatz by iteration.
of the map $f$. Quantization of a classical theory can be understood as method for approximately solving eq. (2.13). However, simple extrapolation of the classical Lagrangian to arbitrary high energies leads to the well-known UV divergencies.

For practical (i.e., calculational) purposes the map $f$ is not a very convenient one. Instead, one may use a somewhat simpler map $\tilde{f}$ which differs from $f$ but, as one may see easily from eq. (2.13), it has one and the same set of fixed points like $f$. This simpler map $\tilde{f} : \Gamma_{n-1} \rightarrow \Gamma_n$ can be given by the following formula.

$$e^{i\Gamma_n[\bar{\phi}]} = C e^{-i\Gamma_{n-1}[0]} \int D\phi \ e^{i\Gamma_{n-1}[\phi + \bar{\phi}]} + i \int dx J_{n-1}(x) \phi(x) \quad (2.15)$$

The advantage of this formula is that it provides us with a compact and explicit representation of the $\tilde{f}$-image of $\Gamma_{n-1}$. However, in general an image of this map $\tilde{f}$ will not have the property to be generating functional of 1PI Green functions.

Concluding this section, let us express our view that the proposed equation for the complete effective action embodies a couple of features which seem reasonable and interesting from a physical point of view and also offers a guiding line for a re-evaluation of the established technical approach to quantum field theory and eventually its appropriate modification. From now on we simply will take eq. (2.13) as granted and consider it as starting point for further analysis.
3 Gauge Field Theories

In the present (and in the following) chapter we are going to study the equation for the complete effective action derived in chapter 2 in the case of gauge field theories. Although we will have in mind gauge field theories in general here we restrict ourselves to QED and comment only the case of non-Abelian gauge theories. In doing so it is understood that the Faddeev-Popov procedure used in standard quantum field theory for defining the functional integral measure can be applied in a slightly generalized way also in the present context, in particular, taking into account that in general solutions of eq. (2.13) are nonlocal and the gauge condition to be chosen will be, for convenience, nonlocal likewise.

We start by defining the generalized map $f$ for QED. The generating functional $Z$ of the Green functions is

$$Z_n[J_n, \bar{\eta}_n, \eta_n] = C e^{-i \Gamma_{n-1}[0,0]} \int D[a_\mu] D\psi D\bar{\psi} e^{i \Gamma_n[a, \psi, \bar{\psi}]} .$$

where

$$\Gamma_{gf}[a] = - \frac{1}{2\lambda} \int d^4y \ (F[a; y])^2 ,$$

$$F[a; y] = \int d^4x \ n_\mu(y-x) \ a^\mu(x) .$$

As usual, $\Gamma_{gf}$ is a gauge breaking term containing a linear, homogeneous functional $F$ of $a_\mu$ (for the moment $n_\mu$ is any arbitrary but appropriately chosen vector-valued distribution) and the brackets in $D[a_\mu]$ (eq. (3.1)) are thought to indicate that the Faddeev-Popov determinant has to be taken into account. $\Gamma_{n-1}$ is out of the class of gauge invariant effective actions.

Then, the $W$-functional is given by

$$W_n[J_n, \bar{\eta}_n, \eta_n] = -i \ln Z_n[J_n, \bar{\eta}_n, \eta_n] ,$$

and the image of $\Gamma_{n-1}$ is

\begin{footnote}{It is an almost trivial factor for Minkowski space QED, but already at finite temperature it becomes important. In addition, always having in mind possible generalization to non-Abelian gauge theories it serves as reminder for this complication then to be considered.}

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\[
\Gamma_n[A_n, \Psi_n, \bar{\Psi}_n] =
\]
\[
= W_n[J_n, \bar{\eta}_n, \eta_n] - \int d^4x \left[ J_{n\mu}(x) A_{n\mu}(x) + \bar{\eta}_n(x) \Psi_n(x) + \bar{\Psi}_n(x) \eta_n(x) \right].
\]

Again, we have the relations
\[
A_{n\mu}(x) = \frac{\delta W_n[J_n, \bar{\eta}_n, \eta_n]}{\delta J_n^\mu(x)}, \quad \frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta A_{n\mu}(x)} = - J_{n\mu}(x),
\]

\[
\Psi_n(x) = \frac{\delta W_n[J_n, \bar{\eta}_n, \eta_n]}{\delta \bar{\eta}_n(x)}, \quad \frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta \Psi_n(x)} = \bar{\eta}_n(x),
\]

\[
\bar{\Psi}_n(x) = - \frac{\delta W_n[J_n, \bar{\eta}_n, \eta_n]}{\delta \eta_n(x)}, \quad \frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta \bar{\Psi}_n(x)} = - \eta_n(x).
\]

Performing now shifts in the integration variables we find
\[
e^{i\Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]} =
\]
\[
= C e^{-i\Gamma_{n-1}[0, 0, 0]} \int D[a_\mu] D\psi D\bar{\psi} e^{i\Gamma_{n-1}[a + A_n, \psi + \bar{\psi}, \bar{\psi} + \bar{\Psi}_n]} e^{i\Gamma_{gf}[a + A_n]} + i \int d^4x \left[ J_{n\mu}(x) a^\mu(x) + \bar{\eta}_n(x) \psi(x) + \bar{\psi}(x) \eta_n(x) \right].
\]

Describing the map \( f \) from the gauge invariant effective action \( \Gamma_{n-1} \) to its image \( \Gamma_n \).

From the discussion leading to the background field method in gauge field theories we know that \( \Gamma_n \) is in general not gauge invariant because as one easily recognizes from eq. (3.9) the shift in the gauge field integration interferes with the gauge fixing term for the quantum fluctuations \(^6\). This is remedied in standard quantum field theory by starting in eq. (3.1) with a modified gauge fixing term \( \Gamma_{gf}[a - A] \) and the field \( A_\mu \) is fixed to obey \( A_\mu = A_{n\mu} \) (cf. [32] and references therein). But, in our approach the application of this procedure would entail that the map \( f \) (in particular, the gauge condition for the quantum fluctuations) had to be modified in each iteration step in dependence on the actual shape (gauge) of \( A_{n\mu} \), i.e., of \( F[A_n - A; y] \). While in standard quantum field theory \( A_{n\mu} \) can be understood as some fixed background field (essentially, this makes the background field method acceptable) our situation is worse because \( A_{n\mu} \) also contains pieces of arbitrary vacuum fluctuations to be

\(^6\)Further features, met in non-Abelian gauge field theories, we may disregard here.
integrated over later on. There is only one safe way to ensure that the gauge for \( A_{n\mu} \) and that for the vacuum fluctuations \( a_{\mu} \) do not interfere in a gauge dependent way (i.e., that the shift in the argument of the gauge field integration does not interfere with the gauge fixing term), namely one has to choose for \( A_{n\mu} \) the gauge

\[
F[A_n; y] = 0 \quad (3.10)
\]

If \( A_{n\mu} \) is a sum of independent pieces condition (3.10) applies to each component because \( F \) is linear and homogenous. Now, as already mentioned in general \( A_{n\mu} \) contains pieces of vacuum fluctuations to be integrated over in further iterations, consequently we have to impose condition (3.10) also onto these vacuum fluctuations. This argument of course applies to each iteration step of the map \( f \) and therefore the only consistent gauge is the generalized Landau gauge \( \lambda = 0 \). So, a ‘sharp’ gauge has to be imposed on all gauge fields, on external fields as well as on vacuum fluctuations, i.e., the whole system of functional relations is bound to one definite gauge. Of course, the gauge functional \( F \) can be chosen as convenience may require and the full gauge invariant effective action \( \Gamma_n \) consequently is obtained by letting \( F \) vary. The conclusion that only the generalized Landau gauge leads to sensible and invariant results well agrees with investigations dealing with the concept of the unique (geometrical) effective action [29]–[31].

From eq. (3.9) we read off now the equation for the complete (gauge invariant) effective action of QED.

\[
e^{i\Gamma[A, \Psi, \bar{\Psi}]} = C e^{-i\Gamma[0, 0, 0]} \int D[a_\mu] D\psi D\bar{\psi} e^{i\Gamma[a + A, \psi + \bar{\Psi}, \bar{\psi} + \bar{\Psi}]} \cdot e^{i\Gamma_{g\psi}[a] + i \int d^4x \left[ J_\mu(x)a^{\mu}(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x) \right]} \quad (3.11)
\]

\[
F[A; y] = 0 \quad ; \quad \lambda \rightarrow 0
\]

In any explicit calculation we will always leave the gauge parameter \( \lambda \) unfixed because this allows to better keep track of terms involved, and in the final results one may simply set \( \lambda = 0 \) then to find the correct answer.

Having defined above the notation we are prepared now to study in the following Ward-Takahashi identities and Schwinger-Dyson equations within the present
3.1 Ward-Takahashi Identities

In standard QED derivation of Ward-Takahashi identities merely relies on the fact that the classical action is gauge invariant. Therefore, generalization of this consideration to the present formulation is straightforward and reasoning proceeds without any major formal difference to the standard approach. Here, for convenience we will closely follow ref. [33], sect. 7.4., as an appropriate textbook treatment.

Consider in eq. (3.1) an infinitesimal gauge transformation

\[ a_\mu \rightarrow a_\mu + \partial_\mu \Lambda, \quad \psi \rightarrow \psi - ie\Lambda \psi, \quad \bar{\psi} \rightarrow \bar{\psi} + ie\Lambda \bar{\psi}. \]  

(3.12)

Then, we obtain in first order of \( \Lambda(x) \) (remember that \( F \) was chosen as a linear functional)

\[ \left\{ \frac{1}{\lambda} x \partial_\mu \int d^4y \frac{\delta F[A_n; y]}{\delta A_{n\mu}(x)} F \left[ -i \frac{\delta}{\delta J_n} ; y \right] - \partial_\mu \Gamma_n^{n\mu}(x) - e \left( \frac{\delta}{\delta \eta_n(x)} - \eta_n(x) \frac{\delta}{\delta \eta_n(x)} \right) \right\} Z_n[J_n, \bar{\eta}_n, \eta_n] = 0. \]  

(3.13)

By means of eqs. (3.4)-(3.8) above equation yields

\[ \frac{1}{\lambda} x \partial_\mu \int d^4y \frac{\delta F[A_n; y]}{\delta A_{n\mu}(x)} F[A_n; y] + \partial_\mu \frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta A_{n\mu}(x)} + + ie \frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta \Psi_n(x)} - ie \frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta \bar{\Psi}_n(x)} = 0. \]  

(3.14)

From this equation different Ward-Takahashi identities can be derived. As standard example let us consider the following. Taking functional derivatives of eq. (3.14) with respect to \( \bar{\Psi}_n(z') \), \( \Psi_n(z) \) and then setting \( \bar{\Psi}_n = \Psi_n = A_{n\mu} = 0 \) one finds

\[ x \partial_\mu \frac{\delta^3 \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta \Psi_n(z') \delta \Psi_n(z) \delta A_{n\mu}(x)} \bigg|_{\Psi_n = \bar{\Psi}_n = A_{n\mu} = 0} = \]

\[ = ie \left\{ \delta^{(4)}(x - z') \frac{\delta^2 \Gamma_n[0, \Psi_n, \bar{\Psi}_n]}{\delta \Psi_n(z') \delta \Psi_n(z)} - \delta^{(4)}(x - z) \frac{\delta^2 \Gamma_n[0, \Psi_n, \bar{\Psi}_n]}{\delta \bar{\Psi}_n(z') \delta \bar{\Psi}_n(z)} \right\} \bigg|_{\Psi_n = \bar{\Psi}_n = 0}. \]  

(3.15)
With

\[
\int d^4x \ d^4z \ e^{i(p'z' - pz - qx)} \frac{\delta^3 \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta \Psi_n(z') \delta \Psi_n(z) \delta A^\mu_n(x)} \bigg|_{\Psi_n = \bar{\Psi}_n = A_{n\mu} = 0} = e (2\pi)^4 \delta^{(4)}(p' - p - q) \bar{\Gamma}_{nm}(p,q,p') \quad (3.16)
\]

and

\[
\int d^4z \ d^4z' \ e^{i(p'z' - pz)} \frac{\delta^2 \Gamma_n[0, \Psi_n, \bar{\Psi}_n]}{\delta \Psi_n(z) \delta \Psi_n(z')} \bigg|_{\Psi_n = \bar{\Psi}_n = 0} = (2\pi)^4 \delta^{(4)}(p' - p) \bar{S}_n^{-1}(p) \quad (3.17)
\]

eq (3.15) yields the well-known Ward-Takahashi identity

\[q^\mu \bar{\Gamma}_{nm}(p, q, p + q) = \bar{S}_n^{-1}(p + q) - \bar{S}_n^{-1}(p) \quad . \quad (3.18)\]

We have seen that each image \( \Gamma_n \) of the map \( f \) respects the Ward-Takahashi identity (3.18) which is a consequence of the gauge invariance of its counter image \( \Gamma_n^{-1} \) (Beyond this property the counter image \( \Gamma_n^{-1} \) does not show up explicitly.). This in particular is also true for any solution of eq. (3.11).

Now, one may convince oneself that also in non-Abelian gauge field theories the derivation of generalized Ward identities (i.e., Slavnov-Taylor identities; cf. [8], sect. IV.7) remains unchanged and they also hold at each step of any iteration of the map \( f \). Violation of these (generalized) Ward identities (so, if anomalies occur) means that the equation for the complete effective action of such a theory will not have any solution. To see this note that the existence of an anomaly would entail that the image \( \Gamma_n = f(\Gamma_n^{-1}) \) of an action has a different behaviour than its counter image \( \Gamma_n^{-1} \), so blocking any attempt to solve the equation. In this sense, the well-known model builders requirement of anomaly cancellation (cf. [33], sect. 9.10., e.g.) can be understood as solvability condition for the functional integral equation for the complete effective action of a theory under consideration.

### 3.2 Schwinger-Dyson Equations

Let us start the study of Schwinger-Dyson equations with a comment concerning the nature of these equations in standard quantum field theory. They represent a chain of hierarchical equations connecting (1PI) Green functions of a theory and can

\[^7\text{Of course, as in standard quantum field theory this concerns only dynamical fields.}\]
be seen as a formulation standing in a certain equivalence to the functional integral representation given (for a scalar theory) by eqs. (2.1)-(2.3). However, as we have argued in section 2.1 the effective action $\Gamma$ is merely the image of the classical action $\Gamma_0$ with respect to the map $f$ and therefore Schwinger-Dyson equations are formulas to be viewed as a device for tackling the calculational complexity met in explicitly determining the effective action $\Gamma$, rather than genuine equations (Whether beyond this they also admit other solutions should not be further considered here.). From this it is clear that Schwinger-Dyson equations can be understood as a kind of representation of the map $f$ and they can also be formulated for the map $f$ acting in the extended domain of effective actions in general. Only, if we impose the fixed point condition for the map $f$ Schwinger-Dyson equations turn out to be genuine equations corresponding to the equation for the complete effective action (2.13).

In the present section we study QED Schwinger-Dyson equations for the map $f$ acting in the extended domain of (gauge invariant) effective actions. Only at the end we will specialize the result to the fixed point condition for the map $f$. For convenience, in deriving Schwinger-Dyson equations here we follow the textbook treatment given in ref. [34], sect. 10.1, as far as possible.

First we exploit the gauge field integration. From eq. (3.1) we find

\[
\left\{ J_{n\mu}(x) + \frac{\delta \Gamma_{gf}}{\delta A_0^\mu(x)} \left[ -i \frac{\delta}{\delta J_n} \right] + \frac{\delta \Gamma_n^{-1}}{\delta A_0^\mu(x)} \left[ -i \frac{\delta}{\delta J_n}, -i \frac{\delta}{\delta \bar{\eta}_n}, i \frac{\delta}{\delta \eta_n} \right] \right\} Z_n[J_n, \bar{\eta}_n, \eta_n] = 0 .
\] (3.19)

Splitting $\Gamma_n^{-1}$ into a free (quadratic) and an interaction part (denoted by $\Gamma_n^{(int)}$) we obtain

\[
-\frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta A_0^\mu(x)} = \frac{1}{\lambda} \int d^4x' \frac{\delta F[A_n; x']}{\delta A_0^\mu(x')} F[A_n; x'] + \int d^4x' D_{n-1}^{-1\mu\nu}(x - x') A_0^\nu(x') + e^{-iW_n[J_n, \bar{\eta}_n, \eta_n]} \frac{\delta \Gamma_n^{(int)}[A_n, \Psi_n, \bar{\Psi}_n]}{\delta A_0^\mu(x)} \left[ -i \frac{\delta}{\delta J_n}, -i \frac{\delta}{\delta \bar{\eta}_n}, i \frac{\delta}{\delta \eta_n} \right] e^{iW_n[J_n, \bar{\eta}_n, \eta_n]} = 0
\] (3.20)

with

\[
\frac{\delta^2 \Gamma_m[A, 0, 0]}{\delta A^\mu(x) \delta A^\nu(x')} \bigg|_{A=0} = D_m^{-1\mu\nu}(x - x') .
\] (3.21)
Taking a functional derivative with respect to $A_{n\mu}$ and setting $\bar{\eta}_n = \eta_n = J_n = 0$ (and equivalently $\bar{\Psi}_n = \Psi_n = A_{n\mu} = 0$) eq. (3.20) yields

$$-D_{n-1}^{-1} \bar{\Psi}_{n \mu}(x-x') + D_{n-1}^{-1} \mu_{n\nu}(x-x') - e^{-iW_n[0,0,0]} \int d^4 z \ D_{n-1}^{-1} \nu_{n}(x'-z) \ .$$

Finally, the fixed point condition for the map $f$ leads to the following Schwinger-Dyson equation.

$$\frac{\delta \Gamma^{(int)}}{\delta A_{n\mu}(x)} \left[ -i \frac{\delta}{\delta J_n}, -i \frac{\delta}{\delta \bar{\eta}_n}, i \frac{\delta}{\delta \eta_n} \right] \frac{\delta}{\delta J_n(z)} e^{iW_n[J_n, \bar{\eta}_n, \eta_n]} \bigg|_{\bar{\eta}_n = \eta_n = J_n = 0} = 0 \ . (3.22)$$

Let us now exploit the fermionic integration. Likewise we obtain from eq. (3.1)

$$\left\{ \eta_n(x) + \frac{\delta \Gamma_n^{-1}}{\delta \Psi_n(x)} \left[ -i \frac{\delta}{\delta J_n}, -i \frac{\delta}{\delta \bar{\eta}_n}, i \frac{\delta}{\delta \eta_n} \right] \right\} Z_n[J_n, \bar{\eta}_n, \eta_n] = 0 \ . (3.24)$$

Again, splitting $\Gamma_n^{-1}$ into a free (quadratic) and an interaction part we find

$$-\frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta \Psi_n(x)} + \int d^4 x' \ S_{n-1}^{-1}(x-x') \ A_{n\mu}(x') +$$

$$+ e^{-iW_n[J_n, \bar{\eta}_n, \eta_n]} \frac{\delta \Gamma_n^{(int)}}{\delta \Psi_n(x)} \left[ -i \frac{\delta}{\delta J_n}, -i \frac{\delta}{\delta \bar{\eta}_n}, i \frac{\delta}{\delta \eta_n} \right] e^{iW_n[J_n, \bar{\eta}_n, \eta_n]} = 0 \ . (3.25)$$

with

$$\left. \frac{\delta^2 \Gamma_n[0, \Psi, \bar{\Psi}]}{\delta \Psi(x') \delta \bar{\Psi}(x)} \right|_{\bar{\Psi} = \Psi = 0} = S_{n-1}^{-1}(x-x') \ . (3.26)$$

Taking a functional derivative with respect to $\Psi_n$ and setting $\bar{\eta}_n = \eta_n = J_n = 0$ (and equivalently $\bar{\Psi}_n = \Psi_n = A_{n\mu} = 0$) eq. (3.25) yields

$$-S_{n-1}^{-1}(x-x') + S_{n-1}^{-1}(x-x') - e^{-iW_n[0,0,0]} \int d^4 z \ S_{n-1}^{-1}(z-x') \ .$$

$$\left. \frac{\delta \Gamma_n^{(int)}}{\delta \Psi_n(x)} \left[ -i \frac{\delta}{\delta J_n}, -i \frac{\delta}{\delta \bar{\eta}_n}, i \frac{\delta}{\delta \eta_n} \right] \frac{\delta}{\delta \eta_n(z)} e^{iW_n[J_n, \bar{\eta}_n, \eta_n]} \right|_{\bar{\eta}_n = \eta_n = J_n = 0} = 0 \ . (3.27)$$

And, the fixed point condition for the map $f$ leads to this Schwinger-Dyson equation.
\[ \frac{\delta \Gamma^{(int)}}{\delta \Psi(x)} \left[ -i \frac{\delta}{\delta J}, -i \frac{\delta}{\delta \bar{\eta}}, i \frac{\delta}{\delta \eta} \right] \frac{\delta}{\delta \eta(z)} e^{iW[J, \bar{\eta}, \eta]} \bigg|_{\bar{\eta}=\eta=J=0} = 0 \] (3.28)

The Schwinger-Dyson equations (3.22), (3.23), (3.27), (3.28) cannot be further studied unless the interaction part of the effective action has been specified, at least in a certain approximation. This in particular also concerns the final transition to relations between 1PI Green functions which hinges on this information. Therefore, presently it remains open how useful this kind of representation of the map \( f \) will be in future investigations.
4 QED — An Approximative Approach to the Equation for the Complete Effective Action

Besides structural investigation of the equation for the complete effective action of most interest appears to be whether the proposed approach is enabling us to extract concrete information for specific models not at all or not easily obtainable by established standard methods. We will focus here on QED as a realistic physical theory at the same time also being of major theoretical interest as simple prototype of a gauge field theory. The aim of this chapter is to demonstrate that the present approach indeed admits explicitly to find certain information about the complete effective action of QED that, in addition, can be seen to be of nonperturbative nature. Of course, the concrete study of the equation for the complete effective action of QED (eq. (3.11)) cannot be expected to be rigorous for the time being. It will be necessary to apply an approximation which however in certain respect should circumvent some of the problems appearing in standard quantum field theory. In particular, as far as possible we will take care that no inappropriate approximation giving rise to UV divergencies is introduced. Although most approximations we will exploit in this chapter can be expected to be reasonable for small values of the QED coupling constant $\alpha$, the explicit calculation we will undertake has to be understood in the first place as a model game to test in principle the calculational accessibility of the concept proposed. As a particular application of the new concept we will explicitly study how to determine the coupling constant $\alpha$ (i.e., the theoretical value of the fine structure constant) being understood as one of the characteristics of a fixed point of the map $f$. This is done using certain simple approximations (capable of future improvement) which at the end however turn out somewhat too simple yet to succeed numerically.

The approximative approach in general relied on in the present chapter will be as follows.

4.1 The Approximative Approach in General

We will study one iteration of the map $f$ starting from a certain Ansatz $\Gamma_I$ which is mapped by means of $f$ to its image $\Gamma_{II}$. The gauge invariant Ansatz for $\Gamma_I$ is chosen as a natural generalization of the so-called classical action $\Gamma_0$ (to obtain this replace $d_I, a_I, b_I$ by delta functions) which is the starting point for standard QED perturbation theory.

\[ \Gamma_I[A, \Psi, \bar{\Psi}] = \Gamma_I^G[A] + \Gamma_I^F[A, \Psi, \bar{\Psi}] \]  

(4.1)
\[ \Gamma^G_I[A] = \frac{1}{2} \int d^4x \, d^4x' \, A^\mu(x) \cdot [g_{\mu\nu} \, x^\square - \, x \partial_\mu \, x \partial_\nu] \, d_I(x - x') \, A^\nu(x') \] (4.2)

\[ \Gamma^F_I[A, \Psi, \bar{\Psi}] = \int d^4x \, d^4x' \, \Psi(x) \, e^{ie\int_x^{x'} dy_\mu \, A^\mu(y)} \cdot [a_I(x - x') \, (i \, \bar{\phi}_{x'} - e \, \mathcal{A}(x')) - m \, b_I(x - x')] \, \Psi(x') \] (4.3)

\( m \) is the electron mass, \( d_I, a_I, b_I \) are functions (distributions) arbitrary for the moment and the gauge functional \( F \) appearing in eq. (3.9) is to be chosen later in a way appropriate and convenient for explicit calculation. Furthermore, the line integration in the phase factor in eq. (4.3) is understood to be performed along a straight line connecting starting and end point. Eq. (4.3) is written in such shape as to keep contact with standard QED (\( \tilde{a} = \tilde{b} \equiv 1 \)) as close as possible.

Finally, the equation for the complete effective action (3.11) will be taken into account in such a way that we require at the end \( d_I = d_{II}, a_I = a_{II}, b_I = b_{II} \), at least in some approximation. All new structures of \( \Gamma_{II} \) not appearing in the Ansatz \( \Gamma_I \) will be viewed as induced ones within this approximation and remain beyond the scope of present interest.

It should find mention that an Ansatz similar to eq. (4.3) (with \( a_I = b_I \)) has unsuccessfully been explored earlier within the framework of nonlocal QED by

\[ \Gamma^F_I[A, \Psi, \bar{\Psi}] = \int d^4x \, d^4x' \, \Psi(x) \, S^{-1}(x, x') \, \Psi(x) \]

\( S^{-1}(0) = S^{-1}(x - x') = i \, a_I(x - x') \, \bar{\phi}_{x'} - m \, b_I(x - x') \) .

In general, we will alternatively write \( l(x) \) or \( l(r) \), \( r = -m^2 x^2 \) for one and the same function what however will not lead to any confusion in the context used respectively (All functions \( l(x) \) we are studying depend on \( x \) via \( x^2 \) only. \( l \) stands here for \( d, a, b \)). Fourier transforms are defined for \( l(x) \) by

\[ l(x) = \int \frac{d^4p}{(2\pi)^4} \, e^{ipx} \, \tilde{l}(p) \]

and equivalently we use the notation \( \tilde{l}(p) \) and \( \tilde{l}(s) \), \( s = -\frac{p^2}{m^2} \) for one and the same function respectively.
CHRÉTIEN and PEIERLS [35] (see also [36]). For a discussion and an explanation of the failure of the attempt turn to [37]. With reference to [35], the action (4.3) has also recently been studied in a different context (effective Lagrangians in nuclear theory) than ours [38]–[40].

4.2 Designing an Approximation Strategy

After having spelt out above what general kind of approximative approach we are going to rely on we need now to translate it into operational terms which are fundamental to the explicit calculation we are aiming at. So far, \( d_I, a_I, b_I \) are understood as completely arbitrary and clearly it is difficult to perform an explicit calculation based on such a general Ansatz. Therefore, below first we will discuss whether the mostly general Ansatz for \( d_I, a_I, b_I \) can sensibly be restricted to a certain subclass the final solution can be searched in. Of particular interest is whether these distributions can adequately be modelled by means of local operators. Let us start with the consideration of \( a_I, b_I \) characterizing the fermion action \( \Gamma^F \).

4.2.1 Consequences of Gauge Invariance for the Kernel of the Fermion Action \( \Gamma^F \)

One of the crucial solvability conditions of eq. (3.11) is that the map \( f \) should not violate gauge invariance. This in particular entails that the map \( f \) must not induce any mass term for the gauge field \( A_\mu \). Even a finite non-vanishing coefficient of such a mass term is not allowed not to speak about infinite ones which are pushed aside in standard QED by applying a gauge invariant regularization. Inasmuch as here we are aiming at finite solutions of the equation for the complete effective action (i.e., some approximation to it) even in a gauge non-invariant regularization scheme (like cut-off regularization) mass terms should not survive after lifting the regularization.

In the following we study restrictions arising from gauge invariance on the possible behaviour of the so far arbitrary kernel \( S_I^{-1} \) of the fermion action \( \Gamma^F_I \). In order to look for a mass term of the gauge field \( A_\mu \) we restrict ourselves to the class of constant gauge potentials \( A_\mu(x) = e^{-1}k_\mu \equiv \text{const.} \) the consideration of which is sufficient for this purpose. For this simple background \( \Gamma_{II} \) is given by the determinant of \( S_I^{-1} \) in the presence of the constant background \( k_\mu \) which can be viewed in momentum space representation as a constant external momentum. Because we cannot assume from the very beginning that the result in eq. (4.5) below will be finite (This is related to the vacuum energy problem which we will not consider in this chapter.) we are barred from simply using a shift \( p \rightarrow p - k \) (which would
make vanish the dependence on $k$ at once; this would only be applicable in a gauge invariant regularization). The effective action reads

$$\Gamma_{II}[e^{-1}k, 0, 0] = \text{const.} - i \ln \text{Det}_{\Lambda} \left( S^{-1}_{\Lambda}[e^{-1}k] \right) \quad (4.4)$$

$$= \text{const.} - 2i V_4 \int_{\Lambda} \frac{d^4p}{(2\pi)^4} \quad h\left(-\frac{(p+k)^2}{m^2}\right) \quad (4.5)$$

where

$$h(s) = \ln\left[ s \hat{a}_I^2(s) + \hat{b}_I^2(s) \right], \quad s = -\frac{p^2}{m^2} = \frac{p_E^2}{m^2}. \quad (4.6)$$

The subscript $\Lambda$ in eqs. (4.4), (4.5) indicates that we apply a cut-off regularization with a (radial) momentum space UV cut-off at $\Lambda$. The subscript $E$ in eq. (4.6) refers to the (Wick rotated) Euclidean momentum variable.

Now, let us further transform the integral appearing in eq. (4.5). First, we perform a Wick rotation and then we expand the integrand in powers of $k$ (up to $O(k^4)$; the notation is $h' = d/ds h$).

$$\int_{\Lambda} d^4p_E \ h\left(\frac{(p_E + k_E)^2}{m^2}\right) = \int_{\Lambda} d^4p \ \left\{ h(s) + 2 \frac{pk}{m^2} h'(s) + \frac{k^2}{m^2} h''(s) + \right.$$ \hspace{1cm} \left. + 2 \frac{(pk)^2}{m^4} h'''(s) + 2 \frac{k^2}{m^4} \frac{pk}{m^4} h''(s) + \right.$$ \hspace{1cm} \left. + \frac{4}{3} \frac{(pk)^3}{m^6} h''''(s) + \frac{1}{2} \frac{(k^2)^2}{m^4} h''(s) + \right.$$ \hspace{1cm} \left. + 2 \frac{k^2}{m^6} \frac{(pk)^2}{m^6} h''''(s) + \frac{2}{3} \frac{(pk)^4}{m^8} h''''(s) + \ldots \right\} \quad (4.7)$$

For convenience, we have omitted the index $E$ on the r.h.s.. Deleting in the integrand terms antisymmetric with respect to $p \rightarrow -p$ and applying following equivalences (valid under the 4D integral)

$$(pk)^2 \equiv \frac{1}{4} k^2 p^2 \quad (4.8)$$

$$(pk)^4 \equiv \frac{1}{8} (k^2)^2 (p^2)^2 \quad (4.9)$$

we find after some manipulations
\[ \Gamma_{II}[e^{-1}k, 0, 0] = \text{const.} + \frac{V_4}{8\pi^2} m^4 \left\{ \int_0^{\frac{\Lambda^2}{m^2}} ds \, s \, h(s) - \frac{1}{2} \frac{k^2}{m^2} \left[ s^2 \, h'(s) \right]_{0}^{\frac{\Lambda^2}{m^2}} + \frac{1}{12} \frac{(k^2)^2}{m^4} \left[ 3 \, s^2 \, h''(s) + s^3 \, h'''(s) \right]_{0}^{\frac{\Lambda^2}{m^2}} + \ldots \right\} \] (4.10)

where \( k_\mu \) denotes the constant (Minkowski space) gauge potential.

From the term proportional to \( k^2 \) in eq. (4.10) we see that the requirement of gauge invariance (i.e., vanishing of any mass term) yields that \( h(s) \) should behave for \( s \to \infty \) like

\[ h(s) \xrightarrow{s \to \infty} \text{const.} + O(s^\kappa) , \quad \kappa < -1 \quad (4.11) \]

Above condition obviously is also sufficient in order to make vanish all higher (in powers of \( k \)) gauge non-invariant structures. By translating information contained in (4.11) one finds following conditions to obey it \(^9\).

\[ \tilde{a}_I(s) \xrightarrow{s \to \infty} O(s^\kappa) \quad \kappa < -1 \quad (4.12) \]

\[ \tilde{b}_I(s) \xrightarrow{s \to \infty} \text{const.} + O(s^\kappa) \quad \text{const.} \neq 0 \quad \kappa < -1 \quad (4.13) \]

From these relations one recognizes that \( \tilde{a}_I \) and \( \tilde{b}_I \) should behave differently for \( s \to \infty \), i.e., they cannot be identical. This requirement is in line with results for the fermion self-energy calculated in lowest order of standard QED perturbation theory where \( \tilde{a} \) and \( \tilde{b} \) already differ (see, e.g., [34]).

Finally, let us come back to the purpose of this subsection. Although, with (4.12), (4.13) we have found certain expectations for the UV behaviour of \( a_I \) and \( b_I \) this result does not seem to improve our situation. Even worse, it indicates that \( a_I \) and \( b_I \) cannot adequately be approximated by any local operator Ansatz because it

\(^9\)We disregard here the somewhat weaker condition

\[ \tilde{a}(s) \xrightarrow{s \to \infty} s^{-1/2} + O(s^\kappa) \quad \kappa < -\frac{3}{2} \quad , \]

and all other variants requiring some fine tuning between \( \tilde{a} \) and \( \tilde{b} \).
would exhibit an unacceptable UV behaviour. So, from this analysis we conclude that for the moment \(a_I\) and \(b_I\) should indeed be kept arbitrary and the hope for simplifying our Ansatz is exclusively placed on the kernel of the gauge field action \(\Gamma^G_I\) which we will discuss now.

### 4.2.2 Requirements on the Kernel of the Gauge Field Action \(\Gamma^G_I\)

The requirements on the kernel of the gauge field action to be given below will not be made obvious immediately in this subsection but will be commented at the appropriate place in the course of the further calculation. Here we simply mention them in order to explain the approximation strategy and the reader is asked to find justification for them later on only.

The first requirement (cf. subsection 4.3.1, eq. (4.30)) is that we expect the (time integrated) self-energy (\(D^{\mu\nu}_I\) is the photon propagator derived from the action \(\Gamma^G_I + \Gamma_{gf}\))

\[
\frac{1}{2} \int d^4y \int d^4y' \quad \bar{J}_\mu(x,x';y) \quad D^{\mu\nu}_I(y-y') \quad J_\nu(x,x';y')
\]  

(4.14)

of a charged point particle represented by the current

\[
J_\mu(x,x';y) = e \int_0^1 d\tau \quad \dot{z}_\mu \delta^{(4)}(z(\tau) - y),
\]  

(4.15)

\[z_\mu(\tau) = (x' - x)_\mu \quad \tau + x_\mu,
\]

and propagating over a finite time interval to be finite. This is needed in order to properly define the map \(f\). Above requirement yields the condition

\[
\tilde{d}_I(s) \sim s^{-\kappa} \quad O(s^\kappa) \quad \kappa > \frac{1}{2}.
\]  

(4.16)

To take into account condition (4.16) is sufficient for most part of the explicit calculation we are attempting. However, it turns out that in finally imposing our approximation to the fixed point condition for the map \(f\) and then searching for a solution to it we need to request more in order to find some \(^{10}\). Specifically,

\(^{10}\)More precisely, this concerns the integral equation for the kernel of the fermion action to be studied further below (see subsection 4.3.2.1).
a solution correct in the asymptotic UV region can only be found if the photon propagator $D_i^{\mu\nu}(x)$ is finite in the coincidence limit $x \to 0$. This entails for the kernel of the gauge field action the stronger requirement

$$d_I(s) \sim s^{-\kappa} O(s^\kappa), \quad \kappa > 1. \quad (4.17)$$

We see that $d_I$ characterizing the kernel of the gauge field action should behave qualitatively quite different than $a_I$ and $b_I$ defining the kernel of the fermion action do. Conditions (4.16), (4.17) induce justified hope that $d_I$ can indeed be modelled by a local operator. Inasmuch as to respect condition (4.16) is sufficient for most of the further explicit calculation (i.e., in particular for the analysis of the asymptotic IR region) we choose the Ansatz

$$d_I(x) = \left[ 1 + \beta \frac{\Box}{m^2} \right] \delta^{(4)}(x), \quad (4.18)$$

where $\beta$ is an arbitrary real (positive) constant parameterizing the Ansatz. The analysis of the asymptotic IR region will be merely independent of further terms to be introduced in (4.18) to satisfy (4.17) and therefore they are ignored in the present Ansatz for calculational simplicity. Of course, the Ansatz introduces an additional (spurious) pole at $p^2 = \beta^{-1} m^2$ in the momentum space photon propagator representation. However, we will not be worried by this fact because we simply see eq. (4.18) as a model representation of an unknown and possibly complicated kernel of the gauge field action, and so it cannot be expected to be free of perhaps unpleasant properties in any respect. Also, eq. (4.18) can be understood as some low energy (i.e., IR) approximation that however can safely be extended to arbitrary high energies without severely misrepresenting the required true UV behaviour. For a discussion of some features and drawbacks of the particular model Ansatz (4.18) see [41],[42] and references therein. The analysis of the asymptotic UV region will not demand any further explicit knowledge of the photon propagator beyond condition (4.17) so that the Ansatz (4.18) can be used for most of the further calculation (that focuses on the IR analysis) and needs not to be supplemented by any specific UV Ansatz.

### 4.2.3 The Approximation Strategy in Ideal, and in Practice

After above considerations we are ready to define the approximation strategy to be followed in the explicit calculation. For reducing the calculational complexity
we will make use of the map $\tilde{f}$ (i.e., source terms are given by $\Gamma_I$ and not by $\Gamma_{II}$) instead of the map $f$. In practice, $\tilde{f}$ will be slightly modified still further as we will explain in section 4.3.1 below.

The local operator Ansatz (4.18) for the kernel of the gauge field action admits following procedure for applying the map $\tilde{f}$. First, starting from $\Gamma_I$ with eq. (4.18) inserted we will perform the functional integration over the gauge potentials. This can be done exactly, independently of the Ansatz (4.18). Then, we perform the integration over the fermion fields, and consequently we impose the fixed point approximation $a_I = a_{II}$, $b_I = b_{II}$. These integral equations are to be solved. In practice, solution of these coupled integral equations can be attempted in a certain approximation only. Specifically, we will explicitly solve them in the asymptotic UV region and in the asymptotic IR region respectively. Solutions $a$, $b$ of these integral equations are still parameterized by $\alpha$ ($\alpha = e^2/4\pi$)\textsuperscript{12} while we find that the parameter $\beta$ (of the kernel of the gauge field action) has to be considered as function of $\alpha$ in order to find any consistent solution at all. However, it remains to impose the third condition $d_I = d_{II}$ yet.

The fermionic integration finally has induced a contribution $\Delta \Gamma^G_I$ to the gauge field action as follows\textsuperscript{13}.

$$\Delta \Gamma^G_I [A] =$$

$$= \frac{\alpha}{4\pi} \int d^4 x \ A^\mu(x) \left[ g_{\mu\nu} \Box - \partial_\mu \partial_\nu \right] \left[ C_{1a} + C_{2a} \frac{\Box}{m^2} + \ldots \right] A^\nu(x) \quad (4.19)$$

$C_{1a}$, $C_{2a}$ are functionals of the distributions $a$ and $b$. Therefore, they can also be viewed as certain functions of $\alpha$ and of the parameter $\beta(\alpha)$. For the moment let us vary the parameter $\beta$ independently of $\alpha$ although we believe that the necessity to consider the parameter $\beta$ as a function of $\alpha$ in course of solving the integral equation for the quadratic kernel of the fermion action is not bound to the particular method we will apply. The condition $d_I = d_{II}$ then reads

$$C_{1a}(\alpha, \beta) = 0 \quad (4.20)$$

$$C_{2a}(\alpha, \beta) = 0 \quad (4.21)$$

\textsuperscript{12}Let us assume that there is an unique solution $a$, $b$ only what is supported (in practice) by explicit calculation to be discussed further below.

\textsuperscript{13}Gauge non-invariant structures do not occur because the solutions $a$ and $b$ exhibit an UV behaviour as will be shown preventing those from occurring even in a gauge non-invariant regularization (at removing the cut-off).
and both these equations define an implicit function $\alpha(\beta)$ (or $\beta(\alpha)$), i.e., certain curves in the $\alpha$-$\beta$-plane. The crossing points of these curves correspond to the set of allowed values $(\alpha, \beta)$. So far, the functional $C_{1a}$ has been explicitly calculated (see Appendix A) with considerable effort in 1-loop approximation only (i.e., taking into account the quadratic kernel of the fermion action in the presence of an arbitrary gauge potential). To determine $C_{2a}$ in 1-loop approximation along the same lines is a trivial but extremely laborious task reserved to be undertaken in the future. But, if as mentioned the parameter $\beta$ has to be viewed as a function of $\alpha$ in advance of imposing $d_I = d_{II}$ eqs. (4.20), (4.21) cannot be satisfied simultaneously anyway (To expect that they are degenerate seems not to be very realistic.). Requiring that at least in the asymptotic IR (long distance, long wavelength) region the fixed point condition should be fulfilled we choose eq. (4.20) as condition to be respected. So, in principle the equation

$$C_{1a}(\alpha, \beta(\alpha)) = 0 \quad (4.22)$$

admits us to determine the QED coupling constant $\alpha$ within the present approximative approach. It is clear that the above method can easily be accommodated to the inclusion of additional terms in the Ansatz (4.18).

So, at the end of this section we are equipped with a plan for the explicit calculations, and we will now proceed along the lines just discussed.

### 4.3 Bringing the Approximation Strategy to Work: Explicit Calculation

#### 4.3.1 Performing the Functional Integration

According to our approximation strategy first we have to calculate the functional integral (cf. eqs. (2.15), (3.9))

$$e^{i\Gamma_{II}[A, \Psi, \bar{\Psi}]} =$$

$$= C \int D[a_{\mu}] D\psi D\bar{\psi} e^{i\Gamma_I[a + A, \psi + \bar{\psi}, \bar{\psi} + \bar{\bar{\Psi}}]} .$$

$$\cdot e^{i\Gamma_{gf}[a] + i \int d^4x \left[ J_{I\mu}(x) a^\mu(x) + \bar{\eta}_I(x) \psi(x) + \bar{\psi}(x) \eta_I(x) \right]} \quad (4.23)$$

with
\[
\frac{\delta \Gamma_I[A, \Psi, \bar{\Psi}]}{\delta A^\mu(x)} = - J_{I\mu}(x) , \quad (4.24)
\]

\[
\frac{\delta \Gamma_I[A, \Psi, \bar{\Psi}]}{\delta \Psi(x)} = \bar{\eta}_I(x) , \quad (4.25)
\]

\[
\frac{\delta \Gamma_I[A, \Psi, \bar{\Psi}]}{\delta \bar{\Psi}(x)} = - \eta_I(x) , \quad (4.26)
\]

inserted. In calculating \( J_{I\mu} \) we may neglect the term stemming from \( \Gamma^F_I \) because in \( \Gamma_{II} \) it gives rise to fermion interactions only \(^{14}\). Furthermore, by using a partial integration we rewrite eq. (4.3) in the following manner (for the definition of \( \bar{J} \) see eq. (4.15))

\[
\Gamma^F_I[A, \Psi, \bar{\Psi}] = \int d^4x \, d^4x' \, \bar{\Psi}(x) \, e^{i \int d^4y \, \bar{\Psi}(x) \, A^\mu(y)} \cdot [i \partial_x \, a_I (x - x') - m \, b_I (x - x')] \, \Psi(x') \quad (4.27)
\]

This will admit us to represent the result of the gauge field integration to be treated first in a very convenient way. To perform the gauge field integration we temporarily expand in eq. (4.23) the term \( e^{i \Gamma^F_I} \) in a power series,

\[
e^{i \Gamma^F_I} = 1 + i \Gamma^F_I - \frac{1}{2} \left( \Gamma^F_I \right)^2 + \ldots , \quad (4.28)
\]

what is a very natural procedure in view of the Grassmann integration. This way it turns out that the result of the gauge field integration can be given as an infinite sum of Gaussian integrals. Each term of this sum corresponds to a certain power \( n \) of \( \Gamma^F_I \) and contains the expression

\[
\int D[a_\mu] \quad e^{-i \Gamma^G_I[a] + i \Gamma_{gf}[a] + i \sum_{k=1}^n \int d^4y \, \bar{\Psi}(x_k, x'_k; y) \, a^\mu(y) , \quad (4.29)
\]

where the arguments \( \{x_k, x'_k\} \) refer to the integration variables in the \( k \)-th copy of \( \Gamma^F_I \). Having performed the Gaussian integration eq. (4.29) reads

\(^{14}\)Incidentally, it should be noted that reasoning leading to this fact also makes use of Furry’s theorem (i.e., an appropriate generalization of it) which applies to our situation. It excludes a closed fermion loop tadpole contribution.
\[ C \cdot e^{- \frac{i}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \int d^4y \int d^4y' \; \bar{J}_\mu(x_k, x'_k; y) D^{\mu\nu}_I(y - y') J_\nu(x_l, x'_l; y')} \]  

(4.30)

Terms with \( k = l \) are self-energy contributions while off-diagonal terms of the double sum in the exponent generate fermion interactions. We see that the requirement (4.16) arises naturally in the course of the functional integration. Let us define the following function from the self-energy term.

\[ g(x - x') = e^{- \frac{i}{2} \int d^4y \int d^4y' \; \bar{J}_\mu(x, x'; y) D^{\mu\nu}_I(y - y') J_\nu(x, x'; y')} \]  

(4.31)

\( g \) can be calculated explicitly, and for the Ansatz (4.18) this is done in Appendix B. Using \( g \) we introduce the new functions \( a_{Ig}, b_{Ig} \) by defining a map \( g : a \rightarrow a_g, \ b \rightarrow b_g \) specified by the prescriptions

\[ a'_{Ig}(x) = g(x) \ a'_I(x) \]  

(4.32)

\[ b_{Ig}(x) = g(x) \ b_I(x) \]  

(4.33)

Here, the notation \( a'(x) = d/dr \ a(r) , \ r = -m^2 x^2 \) is used. The uncertainty in \( a_{Ig} \) due to the free integration constant is removed by noting that \( g(0) = 1 \) (this follows from condition (4.16)) and consequently requiring the same behaviour for \( a_{Ig}(x) \) and \( a_I(x) \) at \( x \to 0 \).

Now, we may reverse the procedure indicated in eq. (4.28) and re-exponentiate the terms of the infinite sum under the remaining fermionic integration which however cannot be done in a closed form. Proceeding this way we obtain

\[ e^{i \Gamma_{II}[A, \Psi, \bar{\Psi}]} = C \cdot e^{i \Gamma_I^{G}[A]} \int D\bar{\psi} D\psi e^{i \int d^4x \left[ \bar{\eta}_I(x) \bar{\psi}(x) + \bar{\psi}(x) \eta_I(x) \right]} . \]

\[ \cdot \exp \left\{ i \int d^4x \int d^4x' \left[ \bar{\psi}(x) + \bar{\psi}(x) \right] e^{i e \int_x^{x'} dy_\mu \; A^\mu(y)} \right\} . \]
\[
\cdot \left[ a_{1g}(x - x') \left( i \not\partial_x - e A(x') \right) - m b_{1g}(x - x') \right] (\psi(x') + \Psi(x')) - \\
- \frac{1}{2} \int d^4x \ d^4x' \ d^4z \ d^4z' \cdot \\
\cdot \left[ \left( \bar{\psi}(x) + \Psi(x) \right) \left[ i \not\partial_x a_{1g}(x - x') - m b_{1g}(x - x') \right] (\psi(x') + \Psi(x')) \cdot \\
\cdot \left( \bar{\psi}(z) + \bar{\Psi}(z) \right) \left[ i \not\partial_z a_{1g}(z - z') - m b_{1g}(z - z') \right] (\psi(z') + \Psi(z')) \cdot \\
\cdot \left( e^{-i \int d^4y \ d^4y' \ J_\mu(x, x'; y) D_\mu^{\nu}(y - y') J_\nu(z, z'; y') - 1} \right) \right] + \ldots \right\} (4.34)
\]

In the last term of eq. (4.34) we have already put \( A_\mu = 0 \) because we will consider 1-loop contributions (i.e., those stemming in eq. (4.34) from the quadratic kernel of the fermion action in the presence of the arbitrary gauge potential \( A_\mu \)) to the quadratic kernel of the gauge field action \( \Gamma_{II}^G \) only \(^{15}\). In eq. (4.34) the remaining fermionic integration is now done (in the sense of perturbation theory, and which after integration is formally summed up again). In performing the Gaussian integration (i.e., treating the last term (and all further terms) in eq. (4.34) as perturbation) for calculational simplicity we neglect the source terms (linear in \( \Psi, \bar{\Psi} \); others are not of our present interest) that contain \( [g(x) - 1] \) factors. For our envisaged study due to \( g(0) = 1 \) these source terms are irrelevant in the asymptotic UV region and in the asymptotic IR region they will lead to certain changes which are apparently small however as long as \( \alpha \) is sufficiently small. Now, without appealing to the eventual range of \( \alpha \) we simply understand this neglect as a certain further modification of the map \( \tilde{f} \) but which preserves all important features (In particular, it does not lead to any change in the asymptotic UV region.). So we obtain for eq. (4.34) the following result \(^{16}\).

\[
e^{i \Gamma_{II}[A, \Psi, \bar{\Psi}]} = \\
= C \ e^{i \Gamma^G[A]} + i \Delta \Gamma^G[A] \cdot \\
\cdot \exp \left\{ i \int d^4x \ d^4x' \bar{\Psi}(x) \left[ i \not\partial_x a_{1g}(x - x') - m b_{1g}(x - x') \right] \Psi(x') - \\
\right\}
\]

\(^{15}\)As long as \( \alpha \) is sufficiently small higher loop contributions will only lead to small quantitative changes.

\(^{16}\)We display only non-interaction terms of \( \Gamma_{II} \) which we are exclusively interested in. Furthermore, on the r.h.s. only the term containing one photon propagator is shown.
- \int d^4x \, d^4x' \, d^4z \, d^4z' \cdot \\
\cdot \bar{\Psi}(x) \left[ i \, \partial_x \, a_{Ig} \left( x - x' \right) - m \, b_{Ig} \left( x - x' \right) \right] \cdot \\
\cdot S_{I(g)}(x' - z) \left[ i \, \partial_z \, a_{Ig} \left( z - z' \right) - m \, b_{Ig} \left( z - z' \right) \right] \Psi(z') \cdot \\
\cdot \int d^4y \, d^4y' \, \bar{J}_\mu(x, x'; y) \, D^\mu\nu(y - y') \, J_\nu(z, z'; y') + \ldots \} \quad (4.35)

Here, \( \Delta \Gamma^G_I \) is defined by eq. (A.1). However, as is clear from eq. (4.34) for the present purpose in eq. (A.2) \( a_I, b_I \) have to be replaced by \( a_{Ig}, b_{Ig} \) respectively. Accordingly, the fermion propagator \( S_{I(g)}(x) \) used here reads

\[
S_{I(g)}(x) = - \int \frac{d^4p}{(2\pi)^4} e^{ipx} \frac{\not{p} \, \tilde{a}_{Ig}(p) - m \, \tilde{b}_{Ig}(p)}{p^2 \tilde{a}_{Ig}^2(p) - m^2 \tilde{b}_{Ig}^2(p) + i\epsilon} \quad (4.36)
\]

Eq. (4.35) provides us with those terms of the image \( \Gamma_{II} \) of \( \Gamma_I \) we need to know for our further investigation. So we may now proceed to apply the fixed point condition to the kernel of the fermion action.

### 4.3.2 The Integral Equation for the Kernel of the Fermion Action

Considering \( \Gamma_{II}[0, \Psi, \bar{\Psi}] = \Gamma^F_{II}[0, \Psi, \bar{\Psi}] \) and writing the quadratic terms as

\[
\Gamma^F_{II}[0, \Psi, \bar{\Psi}] = \\
= \int d^4x \, d^4x' \, \bar{\Psi}(x) \left[ i \, a_{II} \left( x - x' \right) \, \partial_x' - m \, b_{II} \left( x - x' \right) \right] \Psi(x') \quad (4.37)
\]

eq (4.35) provides us with expressions for \( a_{II}, b_{II} \). Consequently, we may explicitly write down the fixed point condition \( a_I = a_{II}, b_I = b_{II} \). For convenience, we do it in terms of \( a_g, b_g \), but any information obtained for these quantities can be translated into terms of \( a, b \) by means of relations (4.32), (4.33). The integral equation reads\(^{17}\)

\[
[g(x - z') - 1] \left[ i \, \partial_z \, a_g \left( x - z' \right) - m \, b_g \left( x - z' \right) \right] =
\]

\(^{17}\)Note, that the fixed point condition has been multiplied by \( g(x - z') \) yet.
\[ = -i \ g(x - z') \left\{ \int d^{1}x' \ d^{1}z \left[ i \ \partial_{x} a_{g}(x - x') - m \ b_{g}(x - x') \right] \cdot S_{(g)}(x' - z) \left[ i \ \partial_{z} a_{g}(z - z') - m \ b_{g}(z - z') \right] \cdot \right. \]

\[ \cdot \int d^{4}y \ d^{4}y' \ \tilde{J}_{\mu}(x, x'; y) \ D_{I}^{\mu\nu}(y - y') \ \tilde{J}_{\nu}(z, z'; y') + \ldots \right\} \] . \quad (4.38)

Eq. (4.38) represents two coupled integral equations for \( a_{g}, b_{g} \) and needs now to be solved. In general, this is a complicated task and we will restrict ourselves to the solution of eq. (4.38) in the asymptotic UV (i.e., \(-m^{2}(x - z')^{2} \rightarrow 0\)) and IR (i.e., \(-m^{2}(x - z')^{2} \rightarrow \infty\)) regions respectively. Before studying these cases let us mention that eq. (4.38) has an exact but trivial solution, namely

\[ a_{g}(x) = a(x) \equiv 0 \quad , \quad (4.39) \]

\[ b_{g}(x) = b(x) = \tilde{b}(\infty) \ \delta^{(4)}(x) \quad , \quad (4.40) \]

where \( \tilde{b}(\infty) \) is some arbitrary real constant. Of course, this solution corresponds to the non-interacting case where the gauge and fermion sectors are decoupled and it is not very interesting therefore. However, in the following we will search the interacting solution of eq. (4.38) as sum of the trivial solution (4.39), (4.40) and some additional nontrivial contribution. As already mentioned it seems to be rather complicated to find a nontrivial and exact solution of eq. (4.38), but it appears possible to analyze it merely exactly at least in the asymptotic UV region and for small \( \alpha \) to leading order in the IR region solely based on those terms explicitly displayed in eq. (4.38). First we will turn to the asymptotic UV region.

### 4.3.2.1 Solving the Integral Equation in the Asymptotic UV Region

Playing around with eq. (4.38) one soon recognizes, that to find a solution correct in the asymptotic UV region one needs to assume that the photon propagator \( D_{I}^{\mu\nu}(x) \) is finite in the coincidence limit \( x \rightarrow 0 \). Consequently, the photon propagator written as

\[ D_{I}^{\mu\nu}(x) = - \int \frac{d^{4}p}{(2\pi)^{4}} \frac{e^{ipx}}{p^{2} + i\epsilon} \frac{1}{d(p)} \left[ g^{\mu\nu} - (1 - \lambda) \frac{p^{\mu}p^{\nu}}{p^{2} + i\epsilon} \right] \quad (4.41) \]

We always have the Euclidean region in mind, of course.

An appropriate gauge fixing term \( \Gamma_{gf} \) has been added to the gauge field action \( \Gamma_{G}^{Q} \), i.e., it has been chosen \( \tilde{n}_{\mu}(p) = ip_{\mu} \ d(p)^{1/2} \).
reads in the coincidence limit

\begin{equation}
D_{\mu\nu}^{I}(0) = i g^{\mu\nu} \frac{3 + \lambda}{4} K_{A} m^{2},
\end{equation}

\begin{equation}
K_{A} = \frac{1}{4\pi^{2}} \int_{0}^{\infty} ds \frac{1}{d(s)},
\end{equation}

where \( K_{A} \) is some finite, real constant.

The analysis of the integral equation (4.38) in the asymptotic UV region now starts by replacing the photon propagator (4.41) by its leading short distance term (4.42). Consequently, the current-current interaction then reads in the short distance limit

\begin{equation}
\int d^{4}y d^{4}y' \tilde{J}_{\mu}(x, x'; y) D_{\nu}^{I}(y - y') \tilde{J}_{\nu}(z, z'; y') = i \alpha \pi \frac{3 + \lambda}{2} K_{A} m^{2} (x - x')(z - z') + \ldots,
\end{equation}

and the function \( g \) has the short distance behaviour

\begin{equation}
g(x) = 1 + \frac{\alpha \pi}{2} (3 + \lambda) K_{A} m^{2} x^{2} + \ldots.
\end{equation}

The leading short distance terms (4.43), (4.44) have to be inserted into the integral equation (4.38) yielding

\begin{equation}
\frac{1}{2} \left( x-z' \right)^{2} \left[ i \partial_{z'} a_{g}(x - z') - m b_{g}(x - z') \right] = \int d^{4}x' d^{4}z \left[ i \partial_{x} a_{g}(x - x') - m b_{g}(x - x') \right] S_{(g)}(x' - x) \cdot \left[ i \partial_{z} a_{g}(z - z') - m b_{g}(z - z') \right] (x - x')(z - z') + \ldots.
\end{equation}

Here, certain constants have been divided out. Conveniently, we will now further consider above integral equation in momentum space. For this purpose we translate coordinate difference factors occurring (i.e., \((x - z')^{2}, (x - x')(z - z')\)) into momentum space derivatives. Having this in mind one may convince oneself that to leading order terms (indicated by dots \ldots) containing more than just one photon propagator and which are not all coupled to a closed fermion loop do not contribute
because they are related to a higher number of derivatives in momentum space (and those terms then are falling faster off in the UV (i.e., high momentum) region). Effectively, to those terms shown in eq. (4.45) only diagrams additionally contribute where all photon propagators are coupled to closed fermion loops. However, these closed fermion loops can always be summed up to give an effective (modified) photon propagator. As long as its coincidence limit remains finite eq. (4.45) stays in effect. So, the UV analysis can be done merely exactly. In addition, already from eq. (4.45) we recognize that the leading UV term of the solution we are in search of is independent of the coupling constant $\alpha$ as well as of the structure of the gauge field action (beyond condition (4.17)) determining the constant $K_A$ what will give the UV behaviour a kind of universal character.

Eq. (4.45) now reads in momentum space (the subscript $g$ is omitted for the moment)

\[
\tilde{p} \left[ s \tilde{a}'' + 3 \tilde{a}' \right] + m \left[ s \tilde{b}'' + 2 \tilde{b}' \right] =
\]

\[
= \frac{2}{\tilde{s}^2 \tilde{a}^2 + b^2} \cdot \left\{ \tilde{p} \left[ s^2 \tilde{a} \left( \tilde{a}' \right)^2 + s \tilde{a}^2 \tilde{a}' + 2 s \tilde{a}' \tilde{b}' - s \tilde{a} \left( \tilde{b}' \right)^2 - \frac{1}{2} \tilde{a}^2 + b \tilde{b}' \right] + 
+ m \left[ 2 s^2 \tilde{a}^2 \tilde{b}' - s^2 \left( \tilde{a}' \right)^2 \tilde{b} + s \tilde{a} \tilde{b}' - s \tilde{a}' \tilde{b} + s \tilde{b} \left( \tilde{b}' \right)^2 - a^2 \tilde{b} \right] \right\} + 
+ \ldots \quad (4.46)
\]

Here, the notation is $\tilde{a} = \tilde{a}(s)$, $\tilde{a}' = d/ds \tilde{a}$, $s = -p^2/m^2$. We will now solve the two coupled differential equations represented by eq. (4.46) in the asymptotic UV region $s \to \infty$. Our Ansatz in accordance with conditions (4.12), (4.13) will be $\tilde{a} = \tilde{a}_s$, $\tilde{b} = \tilde{b}(\infty) + \tilde{b}_s$, where $\tilde{a}_s$, $\tilde{b}_s$ are assumed to vanish power-like in leading order for $s \to \infty$. Neglecting all clearly nonleading terms the two coupled differential equations yielded by eq. (4.46) then read \(^{20}\)

\[
\frac{1}{2} \tilde{a}''_s + \frac{3}{2} \tilde{a}'_s \quad \quad s \to \infty
\]

\[
= s^2 \tilde{a}_s \left( \tilde{a}'_s \right)^2 + s \tilde{a}_s^2 \tilde{a}'_s - \frac{1}{2} \tilde{a}^2_s + \left[ 2 s \tilde{a}'_s + \tilde{a}_s \right] \tilde{b}' + \ldots \quad , (4.47)
\]

\(^{20}\)Note, that also a temporary transition $\tilde{a}_s \to \tilde{b}(\infty) \tilde{a}_s$, $\tilde{b}_s \to \tilde{b}(\infty) \tilde{b}_s$ has been applied and then the factor 2 $\tilde{b}(\infty)$ has been divided out of the equations below.
\[
\frac{1}{2} s \tilde{b}_s'' + \tilde{b}_s' \xrightarrow{s \to \infty} - s^2 (\tilde{a}_s')^2 - s \tilde{a}_s \tilde{a}_s' - \tilde{a}_s^2 + \ldots .
\] (4.48)

Let us first discuss eq. (4.47) and its consequences on the asymptotic UV behaviour of $\tilde{b}_s$. As long as the term on the l.h.s. of eq. (4.47) does not vanish to leading order we are forced to conclude that $\tilde{b}_s \xrightarrow{s \to \infty} 1/s$, i.e., $\tilde{b}_s \xrightarrow{s \to \infty} \ln s$. But, such a behaviour is in conflict with gauge invariance because it is not in line with condition (4.13). So, we are forced to conclude that the l.h.s. of eq. (4.47) should vanish to leading order, consequently it must hold ($C_\tilde{a}$ is some constant)

\[
\tilde{a}_s \xrightarrow{s \to \infty} \frac{C_\tilde{a}}{s^2} + \ldots .
\] (4.49)

This information is sufficient to determine the leading behaviour of $\tilde{b}_s$ from eq. (4.48), and we find

\[
\tilde{b}_s \xrightarrow{s \to \infty} - \frac{C_\tilde{a}^2}{s^3} + \ldots .
\] (4.50)

We may now come back to eq. (4.47) and determine the next-to-leading term of $\tilde{a}_s$. Writing $\tilde{a}_s$ without any loss of generality as

\[
\tilde{a}_s = \frac{C_\tilde{a}}{s^{2}} \tilde{v}(s)
\] (4.51)

\[
\tilde{v}(\infty) = 1
\]

and taking into account (4.49), (4.50) eq. (4.47) then reads \textsuperscript{22}

\[
\frac{C_\tilde{a}}{2} \frac{1}{s^2} \left[ s \tilde{v}'' - \tilde{v}' \right] \xrightarrow{s \to \infty} - \frac{15}{2} \frac{C_\tilde{a}^3}{s^6} + \ldots .
\] (4.52)

\textsuperscript{21}Of course, one could also try the assumption that the term in front of $\tilde{b}_s'$ vanishes (i.e., $\tilde{a}_s \xrightarrow{s \to \infty} s^{-1/2}$), however eq. (4.48) immediately leads to the same result then.

\textsuperscript{22}To be more precise, the vanishing of the leading term on the l.h.s. of eq. (4.52) (eq. (4.51) inserted) rests on the relation ($\hat{p} = (-p_0, \mathbf{p})$)

\[
p \Box \frac{\hat{p}}{[p^2]^{3/2}} = i 2 \pi^2 \partial_\hat{p} \delta^{(4)}(p)
\]

accompanied by certain reasonable assumptions about $\tilde{a}_s(s \to 0)$ (i.e., $\tilde{v} \sim s^2, s \to 0$; or even some weaker condition).
And we find

\[ \tilde{v}(s) \xrightarrow{s \to \infty} 1 - \frac{C_0^2}{s^2} + \ldots \]  

(4.53)

Summarizing above results, one can say that eq. (4.38) admits a (unique) solution respecting conditions (4.12), (4.13). It behaves in the asymptotic UV region as follows:

\[ \tilde{a}_g(s) \xrightarrow{s \to \infty} \frac{C_0}{s^2} \tilde{b}(\infty) \left[ 1 - \frac{C_0^2}{s^3} + \ldots \right] \]  

(4.54)

\[ \tilde{b}_g(s) \xrightarrow{s \to \infty} \tilde{b}(\infty) \left[ 1 - \frac{C_0^2}{s^3} + \ldots \right] \]  

(4.55)

Most important, in qualitative respect this asymptotic UV behaviour is independent of the coupling constant \( \alpha \) and of any specific details of the photon propagator structure beyond condition (4.17). Furthermore, due to \( g(0) = 1 \) (cf. eq. (4.31)) \( \tilde{a}, \tilde{b} \) exhibit the same leading UV behaviour like \( \tilde{a}_g, \tilde{b}_g \). We will discuss consequences of above results further below (see subsection 4.3.3 and chapter 5). In the next subsection we will study eq. (4.38) in the asymptotic IR region.

### 4.3.2.2 Solving the Integral Equation in the Asymptotic IR Region

For the IR analysis of the integral equation (4.38) we need to apply our Ansatz (4.18) to the photon propagator \(^{23}\). Consequently, the current-current interaction reads in the long distance limit to leading order \(^{24}\)

\[
\int d^4y \, d^4y' \, \bar{J}_\mu(x, x'; y) \, D_{I}^{\mu\nu}(y - y') \, \bar{J}_\nu(z, z'; y') =
\]

\[
= i \frac{\alpha}{\pi} \left\{ \frac{(1 + \lambda)}{2} \frac{(x - x')(z - z')}{(x - z')^2} + \right. 
\]

\[
+ \left. \frac{(1 - \lambda)}{2} \frac{(x - x')(x - z')(z - z')}{[(x - z')^2]^2} \right\} + \ldots, \quad (4.56)
\]

\(^{23}\)To obtain this propagator a gauge fixing term \( \Gamma_{gf} \) with \( \tilde{n}_\mu = ip_\mu \tilde{d}(p)^{1/2} \) has been added to the gauge field action \( \Gamma_G^I \).

\(^{24}\)Of course, it is not specifically related to the Ansatz (4.18), only next-to-leading terms will be influenced.
Here, \((x - x')^2\), \((z - z')^2\) are understood to be small compared with \((x - z')^2\). The function \(g\) has the long distance behaviour (We give it here right for Euclidean space. For the full expression and its derivation see Appendix B.)

\[
g(x_E) = C_g \left( m^2 x_E^2 \right)^{\alpha(3 - \lambda)/4\pi} \exp \left\{ -\frac{\alpha}{2\sqrt{\beta}} m|x_E| \left[ 1 + \ldots \right] \right\}, \quad (4.57)
\]

\[
C_g = (4\beta)^{-\alpha(3 - \lambda)/4\pi} \exp \left\{ \frac{\alpha}{4\pi} \left[ (3 + \lambda) + 2 (3 - \lambda) \gamma \right] \right\}.
\]

Please note, that eq. (4.57) contains the Bloch-Nordsieck contribution (cf. [43],[44] and references therein) exhibiting a power-like behaviour with the well-known exponent \(\alpha(3 - \lambda)/4\pi\). It appears justified to assume that the leading IR behaviour displayed in eqs. (4.56), (4.57) will depend on additional terms to be introduced in the Ansatz (4.18) in order also to satisfy condition (4.17) very weakly only. For the purpose of calculational simplicity those terms can be safely disregarded therefore.

We may now insert eqs. (4.56), (4.57) into the integral equation (4.38). Having in mind IR analysis in Euclidean space on the l.h.s. of eq. (4.38) we replace the factor \([1 - g(x - z')]\) simply by 1 because this is the leading contribution due to the exponential decay (i.e., oscillation in Minkowski space) of \(g(x_E)\) for \(m^2 x_E^2 \to \infty\). Furthermore, coordinate difference factors (i.e., \((x - x')_\mu\), \((z - z')_\nu\)) occurring on the r.h.s. of eq. (4.56) are translated into momentum space derivatives acting on the Fourier transform of the kernel \(S^{-1}\) of the fermion action. So, eq. (4.38) reads now

\[
[i \partial_\mu a_\mu(x - z') - m b_\mu(x - z')] =
\]

\[
= \frac{\alpha}{\pi} g(x - z') \cdot \left( \frac{(1 + \lambda)}{2} \frac{g^{\mu\nu}}{(x - z')^2} + (1 - \lambda) \frac{(x - z')^\mu (x - z')^\nu}{[(x - z')^2]^2} \right) \cdot \int \frac{dp}{(2\pi)^4} \exp[ip(x - z')] \left[ p_\mu \partial_\mu \left( \not{p} \not{a}_\mu(p) + m \not{b}_\mu(p) \right) \right].
\]

\^[25] We always have in mind the region \(-m^2(x - z')^2 \to \infty\). More precisely, for any large but fixed value of \((x - z')^2\) contributions from integration regions in the integral equation (4.38) where \((x - x')^2\), \((z - z')^2\) are not small compared to \((x - x')^2\) can be expected to be small due to the expected decay of \(a_\mu\), \(b_\mu\) there. Furthermore, terms containing higher powers of \(1/(x - z')^2\) are suppressed in the asymptotic IR region whatever their coefficient numerically might be.\]
What concerns the contribution of terms containing more than just one photon propagator (indicated by dots \(...\)) following comments are due. Most of those terms will finally yield higher powers of $1/(x-z')^2$ at least and these can therefore be neglected in the asymptotic IR region. However, one should expect that also terms are occurring which are of the same order as the 1-loop term given above. However, such terms should be expected to only weakly contribute numerically as long as $\alpha$ is sufficiently small because each additional photon propagator is accompanied by an additional factor of $\alpha$. This argument is what is left within the present approximative approach of the line of reasoning applied in standard QED perturbation theory. Of course, the belief based on this reasoning may turn out wrong by nonperturbative mechanisms which are not easily seen at the present stage of the investigation. Anyway, in the region where $\alpha$ is of order 1 terms containing more than just one photon propagator cannot be neglected anymore in principle. But for the purpose of the present model calculation (without appealing to the eventual range of $\alpha$) we simply ignore all terms containing more than just one photon propagator also in the region where $\alpha$ is not small.

To determine the IR tail of $a_g, b_g$ (i.e., the l.h.s. of eq. (4.58)) it remains to find the leading long distance contribution of the Fourier integral on the r.h.s. of eq. (4.58). To further proceed we would preferably need to know the analytic structure of the integrand, in particular that of the denominator. We do not have any reliable information on this, but as appears reasonable we will assume that the integrand has a simple pole at some $p_0 = \pm \sqrt{D^2 - s_0 m^2}$ with

$$ s_0 = - \frac{\bar{b}_g^2(s_0)}{\bar{a}_g^2(s_0)} , \quad (s_0 < 0) , \quad (4.59) $$

and that just this pole determines the leading long distance behaviour of the Fourier integral. Consequently, we may exploit the residue of this pole and the leading long distance contribution of the Fourier integral is simply given by the product of the nominator of its integrand (appropriately treated by considering $p_\kappa$ factors occurring as configuration space derivatives acting on eq. (4.60)) taken at $p^2/m^2 = -s_0$ and the leading long distance term of

$$ \frac{1}{\bar{a}_g^2(s_0)} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-z')}}{p^2 + s_0 m^2 + i\epsilon} . \quad (4.60) $$
The explicit calculation is now straightforward but somewhat tedious and we will comment few points only. So, as one intermediate step one calculates the following useful relation $\left( \tilde{a}' = \frac{d}{ds} \tilde{a}(s), s = -p^2/m^2 \right)$.

\[
\left[ \mu \partial_\mu \left( \not p \tilde{a}(p) + m \tilde{b}(p) \right) \right] \cdot \left[ \mu \partial_\nu \left( \not p \tilde{a}(p) + m \tilde{b}(p) \right) \right] = \\
= \left[ \not p \tilde{a} - m \tilde{b} \right] \left[ \gamma_\mu \gamma_\nu \tilde{a}^2 + 4 \frac{p_\mu p_\nu}{m^2} \left( \tilde{a}' \right)^2 + 4 \frac{p_\mu p_\nu}{m^2} \left( \tilde{b}' \right)^2 - \frac{2}{m^2} (\gamma_\mu p_\nu + \gamma_\nu p_\mu) \not p \tilde{a} \tilde{a}' - \frac{2}{m^2} (\gamma_\mu p_\nu + \gamma_\nu p_\mu) \tilde{a} \tilde{b}' + \\
+ 8 \frac{p_\mu p_\nu}{m^2} \tilde{a} \tilde{b}' + (\gamma_\mu \not p \gamma_\nu - \not p \gamma_\mu \gamma_\nu) \tilde{a}^3 - 2 \gamma_\mu p_\nu \frac{p_\mu p_\nu}{m^2} \tilde{a} \tilde{a}' + \\
+ 2 \not p \gamma_\mu \not p \gamma_\nu \frac{p_\mu p_\nu}{m^2} \tilde{a}^2 \tilde{a}' \right] + (\gamma_\mu \not p \gamma_\nu - \not p \gamma_\mu \gamma_\nu) \tilde{a}^3 - 2 \gamma_\mu p_\nu \frac{p_\mu p_\nu}{m^2} \tilde{a} \tilde{a}' + \\
+ 2 \not p \gamma_\mu \not p \gamma_\nu \frac{p_\mu p_\nu}{m^2} \tilde{a}^2 \tilde{a}' - 2 (\gamma_\mu \not p - \not p \gamma_\mu) \frac{p_\mu p_\nu}{m^2} \tilde{a} \tilde{a}' \tilde{b}'
\] (4.61)

In performing the calculation we always keep track of those terms contributing in the long distance region to leading order only. In particular, the leading long distance term of eq. (4.60) is read off from the relation (written for Euclidean space here)

\[
\int \frac{d^4 p_E}{(2\pi)^4} \frac{e^{ip_E x_E}}{p_E^2 + m^2} = \frac{m}{4\pi^2 |x_E|} K_1 \left( m |x_E| \right) = \frac{\sqrt{m}}{2 (2\pi)^{3/2} |x_E|^{3/2}} e^{-m |x_E|} \left[ 1 + \ldots \right] \] (4.62)

The result obtained this way for the IR tail of $a_g$, $b_g$ then is (We give this and all further results for Euclidean space.)

\[
a_g(x_E) \xrightarrow{m^2 x_E^2 \rightarrow \infty} m^4 \frac{\alpha C_g G}{(2\pi)^{5/2}} \frac{(-s_0)^{3/4} \tilde{a}_g(s_0)}{\sqrt{-s_0} + \alpha/2\sqrt{\beta}} (m |x_E|)^{-7/2 + \alpha(3-\lambda)/2\pi} .
\]

\[
. \cdot e^{-(\sqrt{-s_0} + \alpha/2\sqrt{\beta} \cdot m |x_E|} \left[ 1 + \ldots \right] \] (4.63)

\[
b_g(x_E) \xrightarrow{m^2 x_E^2 \rightarrow \infty} m^4 \frac{\alpha C_g H}{(2\pi)^{5/2}} \frac{(-s_0)^{3/4} \tilde{a}_g(s_0)}{\sqrt{-s_0} + \alpha/2\sqrt{\beta}} (m |x_E|)^{-7/2 + \alpha(3-\lambda)/2\pi} .
\]
\[
G = -\frac{3}{2} (1 + \lambda) + 2 (3 - \lambda) \left[ s_0 \frac{\tilde{a}_g(s_0)}{\tilde{\alpha}_g(s_0)} + \sqrt{-s_0} \frac{\tilde{b}_g(s_0)}{\tilde{\alpha}_g(s_0)} + \frac{1}{2} \right]^2
\]

\[
H = -3 (1 + \lambda) - G
\]

It should find mention that eq. (4.65) provides us with an implicit expression for \(G\) only because in view of eqs. (4.64), (4.66) its r.h.s. also depends on \(G\) via the term \(\tilde{b}_g(s_0)/\tilde{\alpha}_g(s_0)\). Therefore, eq. (4.65) represents a cubic equation for the value of \(G\) which has always at least one (real) solution. From eq. (4.65) one recognizes that \(G\) is a RG invariant quantity, i.e., it is invariant against (finite) mass and (fermion) wave function renormalizations (We will discuss the normalization issue further below.).

Taking into account the definitions (4.32), (4.33) we find from eqs. (4.63), (4.64) the IR tail of \(a, b\).

\[
a(x_E) \xrightarrow{m^2 x_E^2 \to \infty} m^4 \frac{\alpha G}{(2\pi)^{5/2}} (-s_0)^{1/4} \tilde{a}_g(s_0) \ (m|x_E|)^{-7/2} \ .
\]

\[
\cdot \ e^{-\sqrt{-s_0} \ m|x_E| \ [1 + \ldots ]}
\]

\[
b(x_E) \xrightarrow{m^2 x_E^2 \to \infty} m^4 \frac{\alpha H}{(2\pi)^{5/2}} (-s_0)^{3/4} \tilde{a}_g(s_0) \ (m|x_E|)^{-7/2} \ .
\]

\[
\cdot \ e^{-\sqrt{-s_0} \ m|x_E| \ [1 + \ldots ]}
\]

From above equations we see that the IR tails of \(a, b\) agree qualitatively (The same is true for \(a_g, b_g\).)

After having obtained the functional dependence of the kernel of the fermion action in the asymptotic IR region we still need to fix the arbitrary constants involved (In particular, this will require to discuss the normalization issue not touched so far.). For this purpose we have to calculate the Fourier transforms of \(a_g, b_g\), and that of \(a, b\) the latter of which are determined by the solution of the integral equation (4.38) via eqs. (4.32), (4.33). It appears reasonable to represent these Fourier
transforms in the low $s$ region\footnote{In the following we will deliberately leave open the precise meaning of this term and we will return to the issue in section 4.3.3 only.} appropriate for the normalization purposes we are aiming at by the sum of the Fourier transforms of the trivial solution (4.39), (4.40) and the Fourier transforms $\tilde{a}_g$, $\tilde{b}_g$, $\tilde{a}_s$, $\tilde{b}_s$ of the IR tails of $a_g$, $b_g$ and $a$, $b$ given in eqs. (4.63), (4.64) and (4.67), (4.68) respectively. So, we simply extend the long distance representations (4.63), (4.64), (4.67), (4.68) to the whole configuration space and expect that this procedure will give reasonable results in the low $s$ region at least.

To the calculation of the Fourier transforms following formula applies \[45].

\[
\int d^4x_E \ e^{-ip_E x_E} \left( x_E^2 \right) \rho e^{-\rho|x_E|} =
\]
\[
= -\frac{4\pi^2 \Gamma(4+2\kappa)}{|p_E| (\rho^2 + p_E^2)^{3/2} + \kappa} \left( P_{2(1+\kappa)} - \rho P_{2(1+\kappa)} \left( \frac{\rho}{\sqrt{\rho^2 + p_E^2}} \right) \right)
\]
\[
= \frac{4\pi^2 \Gamma(3+2\kappa)}{p_E^2 (\rho^2 + p_E^2)^{3/2} + \kappa} \cdot \left[ \sqrt{\rho^2 + p_E^2} P_{1+2\kappa} \left( \frac{\rho}{\sqrt{\rho^2 + p_E^2}} \right) - \rho P_{2(1+\kappa)} \left( \frac{\rho}{\sqrt{\rho^2 + p_E^2}} \right) \right]
\]
(4.69)

Having in mind continuation to Minkowski space, please note that (more precisely) the condition $|\text{Im} \ |p_E|| < Re \rho$ is to be respected. Although it is less compact in the following we will always exploit the lower representation of eq. (4.69) because we find it more convenient for an eventual transition back to Minkowski space.

For $\tilde{a}_g$ given in the low $s$ region as Fourier transform of eq. (4.63) we obtain the following result.

\[
\tilde{a}_g(s) = \frac{\alpha C_g G}{\sqrt{2\pi}} \Gamma \left( -\frac{1}{2} + \frac{(3-\lambda)}{2\pi} \right) (-s_0)^{3/4} \hat{a}_g(s_0) \cdot
\]
\[
\cdot \frac{1}{s} \left[ (\sqrt{-s_0} + \alpha/2\sqrt{\beta})^2 + s \right]^{1/4} - \alpha(3-\lambda)/4\pi
\]
\[
\cdot \left[ \frac{s}{s_0} + \frac{s}{\sqrt{-s_0} + \alpha/2\sqrt{\beta}} \right].
\]
\[ P - 5/2 + \alpha(3-\lambda)/2\pi \left( \left( 1 + \frac{s}{\sqrt{-s_0 + \alpha/2\sqrt{\beta}}} \right)^{-1/2} \right) - \]

\[ - P - 3/2 + \alpha(3-\lambda)/2\pi \left( \left( 1 + \frac{s}{\sqrt{-s_0 + \alpha/2\sqrt{\beta}}} \right)^{-1/2} \right) \]

(4.70)

By specifying \( s = s_0 \) (this corresponds to an analytic continuation to Minkowski space) above equation leads to a consistency equation (the value of \( \tilde{a}_g(s_0) \) drops out) yielding a first relation among the parameters of the IR solution. It reads

\[
1 = - \alpha^1 - \alpha(3-\lambda)/2\pi \frac{G}{\sqrt{2\pi}} \Gamma \left( -\frac{1}{2} + \alpha \frac{(3-\lambda)}{2\pi} \right) \cdot \exp \left\{ \frac{\alpha}{4\pi} \left[ (3 + \lambda) + 2 (3 - \lambda) \gamma \right] \right\} w^{-1/2} (1 + 2w)^{1/4} - \frac{\alpha(3-\lambda)}{4\pi} .
\]

\[
\cdot \left[ \sqrt{1+2w} \frac{P - 5/2 + \alpha(3-\lambda)/2\pi \left( \frac{1+w}{\sqrt{1+2w}} \right) - \left( \frac{1+w}{\sqrt{1+2w}} \right) \right] \right] ,
\]

(4.71)

Here, \( G \) is understood as a function of \( w \) and \( \alpha \) (and \( \lambda \)). It is given as solution of the following cubic equation derived from eq. (4.65).

\[
G^3 + \left\{ \frac{3}{2} \frac{(1+\lambda)}{2} - 2 (3 - \lambda) \left[ \left( 2 + \frac{1}{w} \right) L(w,\alpha) + \frac{1}{2} \right] \right\} G^2 - \frac{12 (3 - \lambda)}{2} \left( 1 + \frac{1}{w} \right) \left[ \left( 2 + \frac{1}{w} \right) L(w,\alpha) + \frac{1}{2} \right] L(w) G - \frac{18 (3 - \lambda)}{2} \left( 1 + \frac{1}{w} \right)^2 L(w,\alpha)^2 = 0
\]

(4.73)

\[
L(w,\alpha) = s_0 \frac{\tilde{a}'_g(s_0)}{\tilde{a}_g(s_0)}
\]

To obtain this cubic equation we have made use of the relation
\[
\tilde{b}_g(s) = -\sqrt{-s_0} \left( 1 + \frac{1}{w} \right) \left[ 1 + \frac{3 (1 + \lambda)}{G} \right] \tilde{a}_g(s) + \tilde{b}(\infty) \quad (4.74)
\]

Based on eqs. (4.63), (4.64) and therefore valid in the low \( s \) region only. We see that solutions \( G \) of equation (4.73) are functions of \( w, \alpha \) while solutions \( w \) of eq. (4.71) exclusively depend on \( \alpha \) (and on \( \lambda \), in principle, if for conceptual reasons we were not to set it to zero as outlined in chapter 3). Clearly, they do not depend on \( \tilde{b}(\infty) \).

Although numerically the discriminant of eq. (4.73) turns always out to be negative in the relevant domain, only one of the three real solutions of eq. (4.73) then proves appropriate to find a solution of eq. (4.71) furthermore. In general, solutions \( G, w \) of above equations can be found numerically only (For a plot of numerical results see figs. 1, 2.). But, for sufficiently small \( \alpha (\alpha \ll 1) \) \( w(\alpha) \) turns out to be large \( (w(\alpha) \gg 1) \) and eq. (4.71) admits an analytical solution in this region. This asymptotic solution will be studied now.

We investigate the case \( \alpha \ll 1 \) (We assume that the solution \( w(\alpha) \) in this region will be much larger than one.). Let us start with the following asymptotic representation [46].

\[
z^{-1/2} + \kappa \left[ z^{-1} P_{-5/2} + \kappa(z) - P_{-3/2} + \kappa(z) \right] =
\]

\[
= \left( \frac{1}{2} - \kappa \right) \frac{\Gamma (1 - \kappa)}{\Gamma (\frac{3}{2} - \kappa)} \frac{2^{1/2 - \kappa}}{2^{1/2}} \cdot
\]

\[
\cdot \left[ 1 - \frac{1}{4\kappa} z^2 \frac{(\frac{1}{2} - \kappa)}{(\frac{3}{2} - \kappa)} \right] \cdot
\]

\[
\cdot \left( \frac{(2z)^{2\kappa}}{1 - \kappa} \frac{\Gamma (\frac{1}{2} - \kappa)}{\Gamma (\frac{1}{2} + \kappa)} \frac{\Gamma (1 + \kappa)}{\Gamma (1 - \kappa)} - 1 \right) + O\left( z^{-2(2 - \kappa)} \right) \quad (4.75)
\]

\[
\kappa > 0 \ , \ |z| \gg 1
\]

\[
= \frac{2 \sqrt{2}}{3\pi} \left[ 1 - \frac{3}{16} z^{-2} \left[ 2 \ln 8z + 1 \right] + O(z^{-4 \ln z}) \right] , \quad (4.76)
\]

\[
\kappa = 0 \ , \ |z| \gg 1
\]

Then, from eq. (4.71) one finds (Here, \( \ln w(\alpha) \) is thought to grow for small \( \alpha \) like \( \alpha^{-1/2} \) at most.)
\[ G = \frac{3\pi}{4\alpha} \left\{ 1 - \frac{\alpha}{4\pi} \left[ (3 + \lambda) + 2(3 - \lambda) \left( \frac{8}{3} - \ln \left( \frac{2^5 w(\alpha)}{\alpha} \right) \right) \right] + \right. \\
\left. + \frac{1}{2} \left( \frac{\alpha(3 - \lambda)}{2\pi} \right)^2 \ln w(\alpha) \ln \left[ \alpha^4 w(\alpha) \right] + O(\alpha^{3/2}) \right\}. \tag{4.77} \]

Taking into account (4.77) eq. (4.74) can then be inserted on the r.h.s. of eq. (4.65) and eq. (4.77) on its l.h.s. The solution of the resulting equation for \( w(\alpha) \) is now straightforward. One finds for small \( \alpha \)

\[ w(\alpha) = \frac{1}{32} \exp \left\{ \frac{2}{3} \sqrt{\frac{2\pi}{\alpha(1 - \lambda/3)}} + 4 + \sqrt{\frac{\alpha(1 - \lambda/3)}{2\pi}} \ln \alpha - \right. \\
\left. - \frac{1}{6} \sqrt{\frac{\alpha(1 - \lambda/3)}{2\pi}} \left[ \frac{59}{3} + \frac{38\lambda}{(3 - \lambda)} \right] + O(\alpha) \right\}, \tag{4.78} \]

\( \alpha \ll 1 \).

Note, that higher loop contributions possibly to be taken into account in the integral equation (4.38) will influence above result via the last term in the exponent only. To see this simply replace in the first term in the exponent \( \alpha \) by \( \alpha[1 + O(\alpha)] \). Finally, using (4.78) one finds from eq. (4.77) following expression for \( G(\alpha) \).

\[ G(\alpha) = \frac{3\pi}{4\alpha} \left\{ 1 + 2 \sqrt{\frac{\alpha(1 - \lambda/3)}{2\pi}} + \frac{\alpha(3 - \lambda)}{2\pi} \ln \alpha + \right. \\
\left. + \frac{\alpha(9 - 5\lambda)}{4\pi} + 15 \left( \frac{\alpha(1 - \lambda/3)}{2\pi} \right)^{3/2} \ln \alpha + O(\alpha^{3/2}) \right\}. \tag{4.79} \]

\( \alpha \ll 1 \).

The next task is to find the solution \( s_0 \) of eq. (4.59). But, any solution \( s_0 \) can sensibly be related to physics only if the mass normalization to be used is specified. So, before attempting the task to find \( s_0 \) we discuss the normalization issue in somewhat greater detail now.

Let us assume we had determined \( s_0 \). Then, whatever normalization of \( \tilde{a}_g(s_0) \) is applied eq. (4.59) yields the value of \( \tilde{b}_g(s_0) \), and in our specific case the value of \( \tilde{b}(\infty) \) because eq. (4.64) is not independent of eq. (4.63). Now, let a certain function...
\( \hat{g} = \hat{g}(-m^2x^2) \) with \( \hat{g}(0) = 1 \) define a map \( \hat{g} : a_g \rightarrow a_{\hat{g}g}, \ b_g \rightarrow b_{\hat{g}g} \) by applying the prescriptions (4.32), (4.33) to \( \hat{g} \). Considering the equation

\[
s_1 = -\frac{b_{\hat{g}g}(s_1)}{a_{\hat{g}g}(s_1)} \quad \text{, } (s_1 < 0)
\]  

(4.80)

the map \( \hat{g} \) obviously induces a map \( \hat{g}_s : s_0 \rightarrow s_1 \). If \( \hat{g} \equiv 1 \), \( \hat{g} \) and \( \hat{g}_s \) are the identity maps. If we specifically choose \( \hat{g} = g^{-1} \), then \( \hat{g} \) is the inverse of \( g \) and it holds \( a_{\hat{g}g} = a, \ b_{\hat{g}g} = b \) (cf. eqs. (4.32), (4.33)). However, \( a, \ b \) are related to physics and we would like to formulate normalization conditions in their terms, i.e., we naturally prefer to impose standard normalization conditions on \( \tilde{a}, \ \tilde{b} \) (i.e., mass shell normalization at the physical electron mass \( m \)):

\[
\tilde{a}(s_1 = -1) = \pm \tilde{b}(s_1 = -1) = N_2^{-1} = 1 \quad . \tag{4.81}
\]

By other words, we of course require that the fermion propagator derived from the effective action we are in search of has a pole related to the physical electron mass \( m \). In eq. (4.81) \( N_2 \) is the (fermion) wave function normalization constant \(^{27}\). Note, that it is always possible to choose \( s_1 = -1 \) because in our set-up there exists a scaling symmetry \( m \rightarrow \tau m \) \( (s \rightarrow s/\tau^2) \), \( \beta \rightarrow \tau^2 \beta, \ b \rightarrow b/\tau \) for any non-zero real parameter \( \tau \) (RG invariance against (finite) mass renormalizations). Consequently, we now apply the inverse map \( \hat{g}_s^{-1} : s_1 \rightarrow s_0 \) to determine \( s_0 \).

Taking into account (cf. eqs. (4.67), (4.68))

\[
\tilde{b}(s) = -\sqrt{-s_0} \left[ 1 + \frac{3}{2} \frac{(1 + \lambda)}{G} \right] \tilde{a}(s) + \tilde{b}(\infty) \quad \tag{4.82}
\]

(valid in the low \( s \) region) and the low \( s \) result for the Fourier transform of \( a \)

\[
\tilde{a}(s) = \sqrt{2} \alpha \ G(\alpha) \ \tilde{a}(s_0) \ \frac{s_0}{s} \left( 1 - \frac{s}{s_0} \right)^{1/4} \cdot \[ \sqrt{1 - \frac{s}{s_0}} P_{-5/2} \left( \left( 1 - \frac{s}{s_0} \right)^{-1/2} \right) - P_{-3/2} \left( \left( 1 - \frac{s}{s_0} \right)^{-1/2} \right) \] \quad \tag{4.83}
\]

we conveniently calculate for the \( s_1 \)-pole via \( \sqrt{-s_1} \ \tilde{a}(s_1) = \pm \tilde{b}(s_1) \) the value of the RG invariant quantity \( \tilde{b}(\infty)/(\sqrt{-s_0} \ \tilde{a}_g(s_0)) \) (i.e., the value of the RG variant quantity \( \tilde{b}(\infty) \) expressed in terms of \( s_0 \) and \( \tilde{a}_g(s_0) \)). We find

\(^{27}\)A (finite) wave function renormalization corresponds to a change in \( N_2 \).
\[
\frac{\tilde{b}(\infty)}{\sqrt{-s_0} \tilde{a}_g(s_0)} =
\]
\[
= \sqrt{2} u \alpha G(\alpha) \left\{ \pm 1 + \sqrt{u} \left[ 1 + \frac{3(1 + \lambda)}{G(\alpha)} \right] \right\} \left( 1 - u^{-1} \right)^{1/4} \cdot
\]
\[
\cdot \left[ \sqrt{1 - u^{-1}} P_{-5/2} \left( \left( 1 - u^{-1} \right)^{-1/2} \right) - P_{-3/2} \left( \left( 1 - u^{-1} \right)^{-1/2} \right) \right], \quad (4.84)
\]
\[
u = \frac{s_0}{s_1}.
\]

The same quantity can now be found from the \(s_0\)-pole via \(\sqrt{-s_0} \tilde{a}_g(s_0) = \tilde{b}_g(s_0)\) and both values have to agree, of course, what provides us with an equation for \(s_0\) measured in units of \(s_1\), which is in our case \((s_1 = -1)\) related to the physical electron mass \(m\). The equation reads

\[
1 + \left( 1 + \frac{1}{w(\alpha)} \right) \left[ 1 + \frac{3(1 + \lambda)}{G(\alpha)} \right] =
\]
\[
= \sqrt{-2} s_0 \alpha G(\alpha) \left\{ \pm 1 + \sqrt{-s_0} \left[ 1 + \frac{3(1 + \lambda)}{G(\alpha)} \right] \right\} \left( 1 + s_0^{-1} \right)^{1/4} \cdot
\]
\[
\cdot \left[ \sqrt{1 + s_0^{-1}} P_{-5/2} \left( \left( 1 + s_0^{-1} \right)^{-1/2} \right) - P_{-3/2} \left( \left( 1 + s_0^{-1} \right)^{-1/2} \right) \right], \quad (4.85)
\]

Again, in general solutions \(s_0(\alpha)\) of this equation can be studied numerically only (see fig. 3). However, for very small \(\alpha\) \((\alpha \ll 1)\) where \(s_0\) is very close to -1 it can also be investigated analytically and one finds (choose the upper sign in eq. (4.85))

\[
\sqrt{\alpha} \left[ 1 + O\left( \sqrt{\alpha \ln \alpha} \right) \right] = \sqrt{\frac{2\pi}{(1 - \lambda/3)}} \left[ \frac{3(1 + s_0)}{32} \ln \left( \frac{1 + s_0}{64} \right) + 3 \right], \quad (4.86)
\]

\[
\alpha \ll 1.
\]

It should be noted that for eq. (4.85) a critical value \(\alpha = \alpha_c\) exists which separates the \(\alpha\) regions in which the upper and lower signs in eq. (4.85) apply. For \(\alpha < \alpha_c\) only in case of the upper sign a solution \(s_0\) exists \(^{29}\) while for \(\alpha > \alpha_c\) only the

\(^{28}\)We omit the other root \(\sqrt{-s_0} \tilde{a}_g(s_0) = -\tilde{b}_g(s_0)\) because one does not find any solution \(s_0\) in this case.

\(^{29}\)It is clear that for small \(\alpha\) (i.e., \(\alpha \to 0\)) a smooth transition from \(\sqrt{-s_0} \tilde{a}_g(s_0) = \tilde{b}_g(s_0)\) to \(\sqrt{-s_1} \tilde{a}(s_1) = \pm \tilde{b}(s_1)\) must exist, consequently the upper sign holds.
lower sign admits to find a solution $s_0$. This critical value $\alpha_c$ corresponds to the singularity $s_0(\alpha \to \alpha_c) \to -\infty$. Consequently, we find from (4.85) the equation for determining $\alpha_c$ by considering $s_0 \to -\infty$. It reads

$$1 + \left[ 1 + \frac{1}{w(\alpha_c)} - \frac{\alpha_c}{2\sqrt{2}} G(\alpha_c) \right] \left[ 1 + \frac{3 (1 + \lambda)}{G(\alpha_c)} \right] = 0 \quad . \quad (4.87)$$

Numerically, one finds $\alpha_c \simeq 0.70$ (see fig. 3). Furthermore, there exists a maximal value $\alpha = \alpha_{\text{max}} > \alpha_c$ beyond which no solution $s_0$ can be found. The value of $\alpha_{\text{max}}$ corresponds to the limit $s_0(\alpha \to \alpha_{\text{max}}) \to -1$. The corresponding equation for $\alpha_{\text{max}}$ reads

$$\left[ 1 + \frac{4}{3\pi} \alpha_{\text{max}} G(\alpha_{\text{max}}) \right] + \left[ 1 + \frac{1}{w(\alpha_{\text{max}})} - \frac{4}{3\pi} \alpha_{\text{max}} G(\alpha_{\text{max}}) \right] \left[ 1 + \frac{3 (1 + \lambda)}{G(\alpha_{\text{max}})} \right] = 0 \quad . \quad (4.88)$$

The numerical calculation yields $\alpha_{\text{max}} \simeq 2.64$ (see fig. 3).

From above considerations it is clear that to find a consistent IR solution of the integral equation (4.38) requires to understand the parameter $\beta$ of our Ansatz (4.18) as some function of $\alpha$ and therefore it cannot be left arbitrary up to the point where we are going to impose the fixed point condition for the kernel of the gauge field action. It will be true in general that one parameter of any Ansatz (containing, say, $n$ parameters) for the kernel of the gauge field action needs to be reserved to allow to find a consistent IR solution of the integral equation (4.38). We have only one parameter at hand and from eq. (4.72) we immediately find its dependence on $\alpha$ (for a plot see fig. 4).

$$\beta = \beta(\alpha) = -\frac{\alpha^2 w(\alpha)^2}{4 s_0(\alpha)} \quad . \quad (4.89)$$

Here, $w(\alpha)$, $s_0(\alpha)$ are solutions of eqs. (4.71), (4.85) respectively. One easily recognizes (cf. fig. 4) that for small $\alpha$ the parameter $\beta$ assumes unrealistic large values what underscores the point that the present approximative calculation has to be understood as a model calculation only.

After having applied the normalization condition (4.81) and having fixed the parameters $G$, $s_0$, $\beta$ the functions $\tilde{a} = \tilde{a}_s$, $\tilde{b} = \tilde{b}(\infty) + \tilde{b}_s$ can be written in the low $s$ region as follows ($s_0 \leq -1$).
\[ \tilde{a}(s) = -\frac{1}{s} \left(1 - \frac{s}{s_0}\right)^{1/4} \left(1 + \frac{1}{s_0}\right)^{-1/4} \cdot \left[ \sqrt{1 - \frac{s}{s_0}} P_{-5/2} \left(\left(1 - \frac{s}{s_0}\right)^{-1/2}\right) - P_{-3/2} \left(\left(1 - \frac{s}{s_0}\right)^{-1/2}\right) \right] - \left[ \sqrt{1 + \frac{1}{s_0}} P_{-5/2} \left(\left(1 + \frac{1}{s_0}\right)^{-1/2}\right) - P_{-3/2} \left(\left(1 + \frac{1}{s_0}\right)^{-1/2}\right) \right]^{-1} \cdot \]

\[ \tilde{b}(s) = (\pm 1 - \tilde{b}(\infty)) \tilde{a}(s) + \tilde{b}(\infty) \cdot \tag{4.91} \]

The parameter \( \tilde{b}(\infty) \) in the normalization applied reads (for a plot see fig. 5)

\[ \tilde{b}(\infty) = \pm 1 - \sqrt{-s_0} \frac{H}{G} = \pm 1 + \sqrt{-s_0} \left[ 1 + \frac{3(1 + \lambda)}{G(\alpha)} \right] . \tag{4.92} \]

For small \( \alpha \) we immediately find from eq. (4.79)

\[ \tilde{b}(\infty) = 1 + \sqrt{-s_0} \left[ 1 + 4 (1 + \lambda) \frac{\alpha}{\pi} + O\left(\alpha^{3/2}\right) \right] , \tag{4.93} \]

\[ \alpha \ll 1 \quad \text{and} \quad \alpha < \alpha_c . \]

Taking into account eq. (4.86) \((s_0 \simeq -1, \alpha \ll 1)\) we recognize that for small \( \alpha \) \((\alpha \ll 1, \alpha < \alpha_c)\) it holds \( \tilde{b}(\infty) \simeq 2 \). From a physical point of view this might be interpreted such a way that at low energies the fermion action merely describes individual real fermions \((\tilde{b} \simeq 1)\), i.e., a single particle interpretation is possible, while at high energies it reflects collective properties of the vacuum which are related to fermion (electron-positron) pairs, consequently \( \tilde{b} \sim \tilde{b}(\infty) \simeq 2 \). Apparently, such an interpretation breaks down at stronger coupling.

Now, the appropriately normalized \( \tilde{a}_g(s) \) (eq. (4.70)) reads in the low \( s \) region

\[ \tilde{a}_g(s) = \]

\[ = \frac{C_g}{2\sqrt{\pi}} \Gamma\left(-\frac{1}{2} + \alpha \frac{3 - \lambda}{2\pi} \right) (-s_0 - 1)^{-1/4} \cdot \left[ \sqrt{1 + \frac{1}{s_0}} P_{-5/2} \left(\left(1 + \frac{1}{s_0}\right)^{-1/2}\right) - P_{-3/2} \left(\left(1 + \frac{1}{s_0}\right)^{-1/2}\right) \right]^{-1} . \]
\[
\cdot \frac{1}{s} \left[ (\sqrt{-s_0} + \alpha/2\sqrt{\beta})^2 + s \right]^{1/4} - \frac{\alpha(3-\lambda)}{4\pi}.
\]
\[
\cdot \frac{s}{\sqrt{1 + \frac{s}{(\sqrt{-s_0} + \alpha/2\sqrt{\beta})^2}}}
\]
\[
\cdot P_{-5/2} + \frac{\alpha(3-\lambda)}{2\pi} \left( \left( 1 + \frac{s}{(\sqrt{-s_0} + \alpha/2\sqrt{\beta})^2} \right)^{-1/2} \right) -
\]
\[
- P_{-3/2} + \frac{\alpha(3-\lambda)}{2\pi} \left( \left( 1 + \frac{s}{(\sqrt{-s_0} + \alpha/2\sqrt{\beta})^2} \right)^{-1/2} \right) \right] .
\] (4.94)

And eq. (4.74) can be written as

\[
\tilde{b}_g(s) = \left( \pm 1 - \tilde{b}(\infty) \right) \left( 1 + \frac{1}{w(\alpha)} \right) \tilde{a}_g(s) + \tilde{b}(\infty) .
\] (4.95)

Clearly, \( s_0, \beta, \tilde{b}(\infty) \) are functions of \( \alpha \) (\( \lambda = 0 \) as explained in chapter 3).

Finally, the correctly normalized IR tails of \( a, b \) characterizing the kernel of the fermion action are

\[
a(x_E) \xrightarrow{m^2x_E^2 \to \infty} \frac{m^4}{\sqrt{2} (2\pi)^{5/2}} (-s_0 - 1)^{-1/4} .
\]
\[
= \frac{m^4}{\sqrt{2} (2\pi)^{5/2}} (-s_0 - 1)^{-1/4} .
\]
\[
\cdot \left[ \sqrt{1 + \frac{1}{s_0} P_{-5/2} \left( \left( 1 + \frac{1}{s_0} \right)^{-1/2} \right) - P_{-3/2} \left( \left( 1 + \frac{1}{s_0} \right)^{-1/2} \right) \right]^{-1} .
\]
\[
\cdot (m|x_E|)^{-7/2} e^{-\sqrt{-s_0} m|x_E|} \left[ 1 + \ldots \right] ,
\] (4.96)
\[
b(x_E) \xrightarrow{m^2x_E^2 \to \infty} \left( \pm 1 - \tilde{b}(\infty) \right) a(m^2x_E^2 \to \infty) ,
\] (4.97)

where \( s_0 \) and \( \tilde{b}(\infty) \) are to be considered as functions of \( \alpha \). Clearly, in qualitative respect eqs. (4.96), (4.97) agree with the long distance representation of the 1-loop fermion self-energy calculated in standard QED perturbation theory.

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To conclude this subsection it should be emphasized that in analyzing the integral equation (4.38) for the kernel of the fermion action in the asymptotic UV and IR regions respectively based on certain reasonable assumptions we have obtained a qualitative and nonperturbative understanding of the behaviour of its solution. Furthermore, the IR analysis even yields approximative quantitative, nonperturbative results which combined with the information about the UV behaviour of the kernel of the fermion action obtained admits to attempt the approximative calculation of the QED coupling constant $\alpha$. This we will study now.

4.3.3 The Fixed Point Condition for the Kernel of the Gauge Field Action and the Approximative Calculation of the QED Coupling Constant $\alpha$

From eq. (4.35) we recognize that the functional integration induces a change $\Delta\Gamma^G_I[A]$ to be added to the gauge field action $\Gamma^G_I[A]$ to obtain $\Gamma^G_{II}[A]$. In accordance with our approximation strategy we display only those terms that match our Ansatz (4.18).

$$\Delta\Gamma^G_I[A] = \frac{\alpha}{4\pi} \int d^4x \ A^\mu(x) \left[ g_{\mu\nu} \Box - \partial_\mu \partial_\nu \right] \left[ C_{1a} + C_{2a} \frac{\Box}{m^2} + \ldots \right] A^\nu(x)$$

Because $a_g$, $b_g$ respect conditions (4.12), (4.13) (cf. eqs. (4.54), (4.55)) no terms violating gauge invariance occur and eq. (A.10) applies. $C_{1a}$ reads (see Appendix A; as explained in section 4.3.1 we confine ourselves to 1-loop contributions)

$$C_{1a} = \frac{2}{3} \ln \left[ \frac{\tilde{b}(\infty)}{\tilde{b}_g(0)} \right]^2 - \int_0^\infty ds \ M(s),$$

$$M(s) = \frac{1}{s\tilde{a}_g^2 + \tilde{b}_g^2} \left[ s \ \tilde{a}_g^2 \tilde{a}_g^{''} + s \ \tilde{a}_g^{'''} + \tilde{b}_g\tilde{b}_g'' \right] +$$

$$+ \frac{2}{3} s^3 \tilde{a}_g^{'''} + 3 s^2 \tilde{a}_g^{''} + 2 \frac{2}{3} s^2 \tilde{b}_g^{'''} +$$

$$+ 2 s \tilde{a}_g \tilde{a}_g' + 3 s \tilde{b}_g \tilde{b}_g' - s (\tilde{b}_g')^2 + 3 \tilde{b}_g \tilde{b}_g' \right].$$

From above expression one recognizes that $C_{1a}$ is a RG invariant quantity, i.e., it is invariant against (finite) mass and (fermion) wave function renormalizations. $C_{2a}$ has not yet been calculated in terms of $\tilde{a}_g$, $\tilde{b}_g$ but it will have an analogous
representation. Because $\tilde{a}_g, \tilde{b}_g$ exclusively depend on $\alpha$ the coefficients $C_{1a}, C_{2a}$ can both be understood as functions of this parameter. Then, the fixed point condition $d_I = d_{II}$ according to our approximation strategy reads (cf. subsection 4.2.3)

$$C_{1a}(\alpha) = 0,$$

(4.101)

$$C_{2a}(\alpha) = 0.$$

(4.102)

It is clear that within our approximative approach we do not have enough parameters left to satisfy both of these equations (if they are not degenerate, perhaps by accident). We decide to choose eq. (4.101) as fixed point equation because we require that at least in the asymptotic IR (long distance, long wavelength) region the fixed point condition for the map $f$ should be fulfilled. Consequently, to determine the QED coupling constant $\alpha$ we have to find the zero(s) of $C_{1a}(\alpha)$.

The explicit calculation of $C_{1a}$ has of course to be based on information obtained in the preceding sections. The first point to be made is that we will take eq. (4.99) as it stands. In principle, one could identically reformulate it by exploiting partial integrations for functions that obey conditions (4.12), (4.13) (or the even somewhat weaker conditions $a_g(s) = O(s^\kappa), \kappa < -1/2, b_g(s) = O(1), s \to \infty$). We choose the present representation for its ‘minimal’ shape (Of course, this is merely a matter of taste.). Let us also emphasize that it turns out advantageous because a certain piece is already integrated out and it therefore depends on the boundary values of $\tilde{b}_g$ only. This term contains certain nonperturbative information from the solution of the integral equation (4.38) for the kernel of the fermion action not easily incorporated otherwise. Finally, one should keep in mind that although different representations of eq. (4.99) are equivalent in a rigorous mathematical sense, they may lead to different answers if approximative information is taken into account only (and this is what we will do).

Now, the first guess might be simply to insert into eq. (4.99) the IR representation found for $\tilde{a}_g, \tilde{b}_g$ (eqs. (4.94), (4.95)). But, as comes as no surprise the integral in eq. (4.99) is not convergent for $\alpha \leq \pi/3$ (it is logarithmically UV divergent then). In other words, this approximation would be so crude as to even not deliver finite results. So, in the parameter region $\alpha \leq \pi/3$ at least one has to proceed differently. Without any problem we may always insert the value of $\tilde{b}(\infty)$ determined by the normalization conditions applied within the IR analysis. For $\tilde{b}_g(0)$ and in the low $s$ integration region of the integral we will insert $\tilde{a}_g, \tilde{b}_g$ as given by eqs. (4.94), (4.95). In the large $s$ region $\tilde{a}_g, \tilde{b}_g$ will be taken from eqs. (4.54), (4.55). One immediately recognizes that this is a better approximation because the integral in
eq. (4.99) then gives finite results. Now, of course, the practical question arises which intermediate value of the integration variable $s$ in eq. (4.99) should split the application regions of the IR and UV representation of $\tilde{a}_g, \tilde{b}_g$. Perhaps, one could choose to fit together the IR and UV representations at some value of $s$ to be determined by a certain condition. For the purpose of the present numerical calculation we select another way. The UV tail of the integrand $M(s)$ in eq. (4.99) will not contribute significantly and we therefore ignore it by simply cutting the integration over the IR representation of the integrand at some upper value $s = s_x$. This value is determined as follows. Observe that the exact integrand $M(s)$ in eq. (4.99) is positive for $s \to \infty$. To see this one may insert eqs. (4.54), (4.55) into (4.100) and one finds to leading order

$$M(s) = 11 \frac{C_a^2}{s^4} + \ldots > 0 , \quad s \to \infty .$$

On the other hand, one may easily convince oneself that for $\alpha \leq \pi/3$ the integrand $M(s)$ of eq. (4.99) turns negative for $s \to \infty$ if the low $s$ representations (4.94), (4.95) are inserted. One now detects that the integrand with the low $s$ representation inserted is positive for $s = 0$. Consequently, there exists a zero of the integrand taken in the IR representation (cf. fig. 6). Obviously, this zero determines the point beyond which the IR (low $s$) representation starts to strongly misrepresent the true integrand and we therefore choose this zero as upper cut-off $s_x$ of the numerical integration (See fig. 7 for the dependence of $s_x$ on $\alpha$).\footnote{Another choice might be to fit the IR and UV representations of the integrand together at some $s_y < s_x$. Here, one way is to require continuity of the integrand at $s = s_y$ and to determine $s_y$ by extremizing the value of the integral. However, in doing so one detects that the contribution of the UV tail is negligible numerically.} It is clear that this recipe leads to a certain slightly lower value of the integral than if the UV region was not neglected.

Now, the result of the numerical calculation of $C_{1a}(\alpha)$ is shown in fig. 8, while fig. 9 displays the behaviour of the two contributions $C_{1a}(\alpha)$ derives from (cf. eq. (4.99)). Unfortunately, within the approximation applied we do not find any zero of $C_{1a}(\alpha)$, but from fig. 9 one recognizes that both contributions to be taken into account are indeed comparable numerically. We believe that the contribution of the integral in eq. (4.99) is underestimated within the approximation applied compared with the exact one which relates to the exact solution of the integral equation (4.38). The contribution of the first term in eq. (4.99) is probably determined to a more reliable degree because only the boundary values of $\tilde{b}_g(s)$ contribute to it. Furthermore, the smaller $\alpha$ the more the approximation applied for the second term in eq. (4.99) miscalculates it. This can easily be seen from fig. 6 (and fig. 7). The true
integrand (the exact solution of eq. (4.38), which we do not know presently, inserted) would likely contribute more because we expect the integrand $M(s)$ to be positive for large $s$. This would shift curve 2 in fig. 9 to larger values and consequently a zero of $C_{1a}(\alpha)$ might occur.

To conclude, the mechanism proposed has explicitly been shown capable to attempt the calculation of the QED coupling constant $\alpha$. However, the approximation applied turns out too simple yet to obtain any specific value of $\alpha$. In particular, for small values of $\alpha$ where most of the approximations applied within the calculation given in the present chapter appear to be most justified no zero of $C_{1a}(\alpha)$ is found. But it is clear that more advanced approximations may lead to a different picture. This needs to be studied in the future. We postpone further discussion of this issue to chapter 6.
5 The Vacuum Energy, and Related Problems

In this chapter we discuss the vacuum energy issue and some related problems we did not mention so far. The consideration will not be aimed at the most general theoretical set-up eventually possible which very likely would turn out fruitless, but we restrict consideration to QED and in particular to that approximative approach to it studied in chapter 4. It might be hoped that this special case yields certain new insight into the problem useful at least for gauge field theories in general.

In standard QED in 4D Minkowski space the vacuum energy density originating from fermion as well as from photon fluctuations and their interactions is a divergent quantity but it is considered as unimportant because it can either be removed by applying normal ordering (in operator quantization) or by appropriately normalizing the functional integral defining the theory. No physical quantity depends on it. But, it is also known that modifications of the vacuum energy density as occurring when external conditions are applied (boundary conditions, temperature, external fields) do matter and in certain cases consequences are even observable in experiment (so, the Casimir effect) [47]-[50]. Few changes of the vacuum energy density turn out to be finite immediately (e.g., the Casimir energy density, or the free energy density for QED at finite temperature). Others require renormalization, like the QED effective potential for (say) a constant magnetic field. Even more care is needed in the study of QED in a gravitational background field we will return to later. However, large part of the motivation for studying the vacuum energy density derives from this situation because it gives rise to the concept of induced (classical) gravity [51] understood as some kind of gravitational (metric) Casimir effect (for a review of recent work and further references see [52],[53], also note [54],[55]).

First, let us compare the calculation of the vacuum energy density in standard QED and within the present approach. We restrict ourselves to the 1-loop level which contains all important features. We apply the simplest regularization possible, namely cut-off regularization (with a (radial) momentum space UV cut-off at \(\Lambda\)), which is most suited for our purposes. The vacuum energy density \(\rho_{vac}\) is given by

\[
\Gamma_{II}[0,0,0] = -V_4 \rho_{vac} \\
= \text{const.} - i \ln \text{Det}_\Lambda \left(S^{-1}_t\right) - \\
- i \ln \text{Det}_\Lambda \left(D^{-1}_{gh} \right) + \frac{i}{2} \ln \text{Det}_\Lambda \left(D^{-1}_{\mu\nu} \right). \quad (5.1)
\]

Here,
\[ S_I^{-1}(x - x') = i \partial_x a_I(x - x') - m b_I(x - x') \] (5.2)

\[ D_{gh}^{-1}(x - x') = \frac{1}{\sqrt{\lambda}} x \partial_\mu n^\mu(x - x') \] (5.3)

\[ D_{I,\mu\nu}(x - x') = [g_{\mu\nu} x \Box - x \partial_\mu x \partial_\nu] d_I(x - x') - \frac{1}{\lambda} \int d^4y \: n_\mu(y - x) n_\nu(y - x') \] (5.4)

are the quadratic kernels of the fermion, ghost (contributing in QED to the vacuum energy only), and gauge field actions respectively. From eq. (5.1) we find accordingly

\[ \Gamma_{II}[0,0,0] = \]

\[ = const. - 2i V_4 \int \frac{d^4p}{(2\pi)^4} \ln \left[ -p^2 \tilde{a}_I(p)^2 + m^2 \tilde{b}_I(p)^2 \right] - 
\]

\[ - i V_4 \int \frac{d^4p}{(2\pi)^4} \ln \left[ i \lambda^{-1/2} \tilde{p} \tilde{n}(p) \right] + 
\]

\[ + \frac{i}{2} V_4 \int \frac{d^4p}{(2\pi)^4} \ln \left[ \det \left[ (g_{\mu\nu} p^2 - p_\mu p_\nu) \tilde{d}_I(p) - \lambda^{-1} \tilde{n}_\mu(p) \tilde{n}_\nu(p) \right] \right] \] (5.5)

Taking into account the relation

\[ \det \left[ (g_{\mu\nu} p^2 - p_\mu p_\nu) \tilde{d} - \lambda^{-1} \tilde{n}_\mu \tilde{n}_\nu \right] = - \frac{d^3}{\lambda} \left[ p \tilde{n} \right]^2 \left[ p^2 \right]^2 \] (5.6)

and applying a Wick rotation one finds after some manipulations

\[ \Gamma_{II}[0,0,0] = const. + \]

\[ ^{31}n_\mu \text{ can be here any vector-valued distribution, e.g., perhaps a derivative } \partial_\mu \text{ acting on some scalar function leading to a Lorentz type gauge, or any constant vector times a scalar function yielding an axial type gauge.} \]

\[ ^{32}\text{We have absorbed certain } \ln m \text{ terms into the first (normalization) constant on the r.h.s. of eq. (5.5).} \]
+ \frac{V_4}{8\pi^2} m^4 \int_0^{\Lambda^2/m^2} ds \ s \ \left\{ \ln \left[ s \ \tilde{a}_I(s)^2 + \tilde{b}_I(s)^2 \right] - \frac{1}{2} \ln \left[ s \ \tilde{d}_I(s)^{3/2} \right] \right\} \ . \ (5.7)

There is no trace left of the gauge condition because we have correctly included in the kernel of the ghost action (5.3) the gauge parameter \( \lambda \) (For a related discussion see [56], [57]). One immediately recognizes the well-known fact that in standard QED \( \tilde{a}_I = \tilde{b}_I = \tilde{d}_I \equiv 1 \) the vacuum energy density \( \rho_{\text{vac}} \) diverges \(^{33}\). Now, QED in a background field (electromagnetic or gravitational; we restrict consideration to these external conditions most interesting in view of standard QED difficulties) will change the quantity \( s \) (stemming from differential operators in configuration space) appearing in the argument of the logarithms above to some \( s + \Delta s \) where for large \( s \) the change \( \Delta s \) behaves like \( \Delta s \sim \infty \) const. \(^{34}\). Of course, as already mentioned one can always absorb the divergent terms characteristic for 4D Minkowski space and displayed in eq. (5.7) on the r.h.s. into the normalization constant of the functional integral. But, for QED in a background field the logarithm in the integrand of eq. (5.7) then reads for large \( s \)

\[
\ln \left[ 1 + \frac{\Delta s}{s} + \ldots \right] \sim \ln \left[ 1 + O(s^{-1}) \right] = O(s^{-1}) \quad (5.8)
\]

and the vacuum energy density depending on the background field is still divergent (This even holds up to \( \Delta s \sim \infty 1/s \)).

Now, compare this with our approximative approach to the equation for the complete effective action of QED. From eqs. (4.54), (4.55) we know that it holds

\[
s \ \tilde{a}_I(s)^2 + \tilde{b}_I(s)^2 \ \sim \ \tilde{b}(\infty)^2 \left[ 1 - \frac{C_a^2}{s^3} + \ldots \right] \ . \ (5.9)
\]

Absorbing a \( \ln \tilde{b}(\infty) \) term into the normalization constant of the functional integral we see that the part of the vacuum energy density originating from fermion fluctuations (the first term in the integrand of eq. (5.7)) is even finite without any further appeal to this constant. As we have explained in subsection 4.3.2 this is true.

\(^{33}\)Incidentally, one may always formally (by ignoring finiteness/convergence requirements of properly applied mathematics) transform eq. (5.5) in the 'sum over the spectrum' formula for the vacuum energy density by exploiting the cut in the appropriate variable (i.e., \( p_0 \)) connected with the logarithms, starting at the lowest energy eigenvalue of the spectrum, and extending to infinity.

\(^{34}\)Considering a connection in the covariant derivatives this naively yields \( \Delta s \sim \infty \sqrt{s} \), but symmetry reasons finally lead to the somewhat weaker behaviour \( \Delta s \sim \infty \) const.
irrespectively of the particular approximation applied (i.e., whether we first perform the gauge field integration or the fermionic integration). Consequently, any change of the fermionic part of the vacuum energy density under the influence of external (electromagnetic as well as gravitational) fields will also be finite. But, in view of condition (4.17) the part of the vacuum energy density originating from photon fluctuations (the second term in the integrand of eq. (5.7)) is still divergent and equally as in standard QED we need to absorb this divergency for 4D Minkowski space into the normalization constant of the functional integral in order to properly define the equation for the complete effective action of QED. This can be done without any problem. The only concern remaining is the behaviour of the gauge field determinant in the presence of a gravitational background field. We do not have any quick answer on this, but let us speculate for a moment. Assume we had for 4D Minkowski space absorbed the UV divergence stemming from the gauge field determinant into the normalization constant of the functional integral by using a certain power of the determinant of the d’Alembertian\[35\]. If one now generalizes the 4D Minkowski space functional integral to an arbitrary gravitational background this has to be done for the whole functional integral measure, i.e., also the (normalization) determinant of the d’Alembertian has to be generalized covariantly. Then of course, using this recipe the vacuum energy density of QED would be finite in electromagnetic as well as in gravitational background fields. If one is to reject above recipe one has to further discuss the determinant of the d’Alembertian in the presence of a gravitational background field what is a problem of long standing concern, in particular the gauge field conformal anomaly and its regularization dependence [58]. Finally, it appears not unreasonable to expect that above discussion persists to apply also if further contributions (higher loops) are taken into account.

Above consideration now admits to compare standard QED in a gravitational background field and the present approach. In standard QED the structure of the first few terms of the effective gravitational action (i.e., up to a minus sign the (time integrated) vacuum energy) is known [47],[58],[59].

\[\Gamma_{II}[0, 0, 0] = \]
\[= \int d^4x \sqrt{-g} \left\{ m^4 c_1 + m^2 c_2 R + c_3 \square R + c_4 R^2 + \right.\]
\[+ c_5 R_{\mu\nu} R^{\mu\nu} + c_6 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \ldots \} \quad (5.10)\]

\[35\]One may well imagine that \(d_I\) behaves for high energies such a way that this recipe removes all divergencies. If not, perhaps the determinant of \(d_I\) in whole has to be included in the normalization constant.
$c_1$ to $c_6$ are certain divergent dimensionless constants. We have already discussed above $c_1$ (i.e., $-\rho_{\text{vac}}$ for 4D Minkowski space), $c_2$ is a quadratically (in the cut-off $\Lambda$) divergent quantity while $c_3$ to $c_6$ diverge logarithmically. All further terms are finite. Consistency requires to start in the standard QED functional integral with a certain bare gravitational action (included in $\Gamma_I$) containing all terms displayed in eq. (5.10) in order to be able to absorb the divergencies into the bare constants in front of them. Consequently, induced gravity is not a consistent concept within standard QED. In contradistinction to standard QED, by taking into account the UV behaviour of the quadratic kernel of the fermion action (a consequence of the equation for the complete effective action of QED) we have demonstrated above that whatever the technical approach to calculate $c_2$ to $c_6$ will be in detail \textsuperscript{36} these coefficients will come out finite (at least at the 1-loop level). The contribution from the determinant of the gauge field kernel will depend on the choice one is willing to make for the normalization of the functional integral. Therefore, within the present framework induced gravity might under certain circumstance turn out to be a valid concept. Of course, as has been pointed out by Sakharov in his pioneering paper [51] the (induced) gravitational action will very likely not be dominated by contributions stemming from QED but from the heaviest excitations (particles) existent in nature. If one would like to attempt the calculation of the induced gravitational action within the concept proposed in the present paper one would first have to study the equation for the complete effective action of the standard model at least. If one is willing to do so this will require much effort and certainly results cannot be obtained quickly. But, in view of the possible outcome perhaps it might be worth to be done.

\textsuperscript{36}In cut-off regularization they will have representations analogous to eqs. (A.6)–(A.8).
6 Discussion and Conclusions

Before turning to some matters of principle let us further discuss the approximative approach to the functional integral equation for the complete effective action of QED. We have seen that the general approximative approach chosen (cf. section 4.1) admits to find certain nonperturbative information about the quadratic kernels of the QED action. Particular emphasis deserves the fact that the information found indicates that there exists an unique solution to the functional integral equation only (at least within the approximative approach studied). Of course, this point has to be studied further using more advanced approximations in order to see whether for the QED coupling constant $\alpha$ only one admissible value exists (if any at all – but nature appears to allow for some). Furthermore, within the approximative approach divergencies as they are characteristic for standard QED do not show up (at least, as far as the present study runs). It should perhaps also be said that the nonlocal character of the fermion action admits to employ nonperturbative techniques which are not quickly applicable in standard QED. For example, as we have seen this way the well-known Bloch-Nordsieck contribution can be obtained easily and it contains important IR (long distance) information crucial to the further calculation.

However, so far the concept proposed in the present article has not yet successfully passed the crucial test attempted in subsection 4.3.3, namely the approximative calculation of the QED coupling constant $\alpha$. As we have seen the approach used is indeed suited for explicit calculation but inasmuch as within the simple approximation applied we did not find any zero of $C_{1a}(\alpha)$ the question remains presently open. How might a better approximation look like? First, it should be noted that by imposing eq. (4.71) independently of the value of $\alpha$ a strong coupling condition has been enforced which annihilates the hope that higher loop contributions can really be neglected in the integral equation for the quadratic kernel of the fermion action (4.38). But, to take into account higher loop contributions would add complications to the formalism not easily to be resolved in analytical calculations. One way out of this dilemma might be to relax for approximative purposes the fixed point condition for the quadratic kernel of the fermion action to $a_{II} = C \, a_I, \, b_{II} = C \, b_I$ where $C$ is some arbitrary real constant, instead of immediately enforcing $C = 1$. This requirement of structural similarity perhaps could be sufficient to keep the conceptual content alive and at the same time admits to count indeed (not only seemingly) in any arguments on the eventual smallness of $\alpha$. The parameter $\beta$ then would also be unconstrained as long as the fixed point condition $d_{II} = d_I$ is not enforced. To finally fix both $\alpha$ and the parameter $\beta$ the conditions (4.20), (4.21) can be applied simultaneously. Whether this recipe yields a more effective approximation remains to be seen in future investigations. It might perhaps also be necessary to include some higher loop contributions to $C_{1a}$ and $C_{2a}$. Certainly, the solution of the inte-
The integral equation for the kernel of the fermion action (4.38) has to be studied further. May be, it will also be advisable to improve the Ansatz (4.18). These are few of the changes in the approximation strategy which can be implemented most easily along the lines of chapter 4. Perhaps, still more severe changes are required. Finally, it should be said that the calculation discussed in chapter 4 should merely be understood as a first (naive) attempt to extract information out of the functional integral equation for the complete effective action by means of a simple approximation which however admits mostly analytical investigation. It is clear, of course, that the present understanding is poor and much remains to be learned.

Throughout the paper we have preliminary applied the standard point of view that the space-time structure is prescribed to the functional integral equation for the complete effective action. In a certain sense, it is considered as ‘classical’ and as prior to quantum effects (at least for flat space-time). However, the criticism spelled out in section 2.3 with respect to the artificial distinction between classical action and effective action also applies to this view on the space-time structure. Therefore, more adequate the structure of space-time should be understood as some characteristics of the quantum field theoretic vacuum. Basically, this is the point of view applied within the concept of induced gravity although this aspect is hardly discussed in the literature. But, also in flat space-time the idea applies. Recent investigations of propagation of light in a Casimir vacuum indicate that this concept is already implicitly entailed in standard QED [60]-[62]. As discussed in ref. [62], although lack of appropriate nonperturbative calculational tools leaves the question so far unsettled in the strict sense the only conceptually viable (as far as present knowledge is concerned) of the alternatives allowed by the Kramers-Kronig relation for the refractive index \( n(\omega) \) of the Casimir vacuum (\( \omega \) is the frequency of the test wave) is that \( n(\infty) < 1 \) holds for propagation of light perpendicular to two parallel mirrors in the slab between them (This entails a signal velocity of light larger than in the free space vacuum.). While the result is often viewed as something like a paradox in standard QED it is easily understandable by means of the concept put forward in the present article (where it may count as a special application). If the map \( f \) is modified such a way that it is no longer fully Lorentz invariant then also the solution of the functional integral equation for the complete effective action is no longer fully Lorentz invariant and the dispersion analysis in accordance with the effective Maxwell action may well reveal a change in the signal velocity of light. The point is that only one situation can be considered as the one where normalization is performed (and we typically choose free Minkowski space as reference situation and the signal velocity of light there as reference standard, although of course also any less symmetrical set-up could be used). But, in view of the discus-

\[ ^{37} \text{For an appropriate functional integral formulation of standard QED in the presence of two parallel mirrors see [63].} \]
sion performed in section 2.3 it makes no sense to consider any normalized value of
a certain quantity (mass, charge, velocity of light, e.g.) as classical because this is
a concept not accessible to experiment. We can only denote certain values defined
by a certain measurement scenario under defined circumstances as reference values.
Any changes of these values measured under different circumstances are certainly of
quantum nature but equally well these values could have served as initial reference
values. Consequently, it appears most sensible to consider these quantities from the
very beginning as characteristics of the quantum field theoretic vacuum and their
changes as parameterizing changes of it with respect to some reference situation.

Summarizing the concept proposed in the present article let us point out that it
proposes a view on quantum field theory which differs from the established one, but
the established standard paradigm finds it natural explanation and place within this
new approach. In particular, it incorporates and continues in modified shape certain
ideas used in local renormalizable quantum field theory such as the unobservability
of bare quantities and the hypothesis that the vanishing of the beta function(s)
(corresponding to a fixed point of the renormalization group) defines the physical
coupling constant(s) of a model. The functional integral equation for the complete
effective action proposed ensures (merely by definition) that any of its solutions is
finite (It is not a solution, otherwise.). This removes to a certain extent the concern
of divergencies standard quantum field theory is beset by, but the price to pay for
this is the present uncertainty whether the functional integral equation proposed has
beyond free field theories any other nontrivial solution (i.e., any nonlinear (interact-
ing) field theory). The most natural place to find out whether the proposed concept
is physically correct should be QED because unlike some other model theories it is
a theory for phenomena definitely present in nature. QED is certainly structurally
more complex than scalar model field theories, e.g., but if for QED something new
can be learned we may feel sure that our physical understanding has advanced. The
approximative approach to the functional integral equation for the QED effective
action presented has proved its calculational accessibility. Although the particular
approximation studied is still quite simple it has yielded certain nonperturbative
information what indicates that the present approach also has certain calculational
advantages. However, only further investigation will show whether any obviously
appropriate approximation can be found which yields with reasonable calculational
effort the correct value of the fine structure constant. In a certain sense this should
be viewed as a crucial test because in principle the present approach if really phys-
ically correct and adequate should be able to pass it.
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Appendix A

Consider the following formula

\[ e^{i\Delta \Gamma^G_I[A]} = C \int D\psi D\bar{\psi} \ e^{i\Gamma^F_I[A, \psi, \bar{\psi}]} , \]  

(A.1)

where \( \Gamma^F_I[A, \psi, \bar{\psi}] \) is given by eq. (4.3). In the present Appendix we are going to calculate the coefficients of the first two quadratic terms of the derivative expansion of \( \Delta \Gamma^G_I[A] \), i.e., the coefficient of the mass term \( A_\mu A^\mu \) and the coefficients of \( (\partial_\mu A^\mu)^2 \) and \( \partial_\mu A_\nu \partial^\mu A^\nu \). For this purpose we rewrite \( \Gamma^F_I[A, \psi, \bar{\psi}] \) in the following symmetrized form.

\[ \Gamma^F_I[A, \psi, \bar{\psi}] = \]

\[ = \frac{1}{2} \int d^4x \ d^4x' \ \bar{\psi}(x) \ e^{ie \int_x^{x'} dy_\mu \ A^\mu(y)} \cdot \]

\[ \cdot \left[ a_I(x-x') \left( i \overset{\rightarrow}{\partial}_\mu - e \ A^\mu(x') \right) - m \ b_I(x-x') \right] \psi(x') + \]

\[ + \frac{1}{2} \int d^4x \ d^4x' \ \bar{\psi}(x) \left[ - (i \overset{\leftarrow}{\partial}_\mu + e \ A^\mu(x)) a_I(x-x') - m \ b_I(x-x') \right] \cdot \]

\[ e^{ie \int_x^{x'} dy_\mu \ A^\mu(y)} \psi(x') \]  

(A.2)

We now expand the r.h.s. of eq. (A.2) in powers of \( A_\mu \) up to \( O(A^2) \) (i.e., \( O(e^2) \)) and insert following expansions (the upper obtained by using \( y_\mu(\tau) = (x'-x)_\mu \, \tau + x_\mu, \ \tau \in [0,1] \)).

\[ \int_x^{x'} dy_\mu \ A^\mu(y) = (x'-x)^\mu \left\{ A_\mu(y) + \right. \]

\[ + \frac{1}{24} (x'-x)^\nu (x'-x)^\lambda \ \partial_\nu \partial_\lambda \ A_\mu(y) + \ldots \} \bigg|_{y=\frac{x+x'}{2}} \right\} , \]  

(A.3)

\[ A_\mu(x) + A_\mu(x') = \]

\[ = 2 \left\{ A_\mu(y) + \frac{1}{8} (x'-x)^\nu (x'-x)^\lambda \ \partial_\nu \partial_\lambda \ A_\mu(y) + \ldots \} \bigg|_{y=\frac{x+x'}{2}} \right\} . \]  

(A.4)

For calculating the coefficients of \( A_\mu A^\mu, (\partial_\mu A^\mu)^2 \), and \( \partial_\mu A_\nu \partial^\mu A^\nu \) in \( \Delta \Gamma^G_I[A] \) it is sufficient to keep at most two derivatives acting on the gauge potentials in \( \Gamma^F_I[A, \psi, \bar{\psi}] \).
The expression obtained this way for $\Gamma^F_I$ (we will not give this rather long expression) now serves as the starting point for deriving Feynman rules and calculating the effective action terms desired. One should take notice that $\Gamma^F_I$ also contains terms quadratic in $A_\mu$ what leads to the situation that besides the standard photon polarization diagram also a tadpole contribution to the photon self-energy is to be taken into account.

The explicit calculation of the terms we are aiming at is quite tedious and shall not be displayed here. We only comment few points of the calculation. Coordinate differences as occurring in eqs. (A.3), (A.4) are translated into momentum space as derivatives with respect to a corresponding momentum variable acting on certain functions in momentum space. This of course involves partial integrations in momentum space for which as usual boundary contributions are assumed not to occur. The photon polarization function is a nonlocal distribution. Therefore, from the formal expression derived by the Feynman rules the local structures we are interested in have to be extracted. In order to properly define this procedure we apply a (radial) momentum space UV cut-off at $\Lambda$ for the loop integration. The final result will be given within this gauge non-invariant cut-off regularization. Furthermore, a Wick rotation for the loop integration is performed and such equivalences like (4.8), (4.9) are used. Then, the final result reads

$$\Delta \Gamma^G_I[A] = \text{const.} + \frac{e^2}{16\pi^2} \int d^4x \left\{ C_0 \, m^2 A_\mu(x) A^\mu(x) + \\
+ \left[ C_{1s} \left[ g_{\mu\nu} g_{\alpha\beta} + g_{\mu\alpha} g_{\nu\beta} \right] + C_{1a} \left[ g_{\mu\nu} g_{\alpha\beta} - g_{\mu\alpha} g_{\nu\beta} \right] \right] \partial^\alpha \partial^\beta A^\nu(x) + \\
+ \ldots \right\} \quad (A.5)$$

where $(h' = d/ds \, h)$

$$C_0 = -s^2 \, h' \left\lceil \frac{\Lambda^2}{m^2} \right\rceil_0$$

$$C_{1s} = -\frac{1}{6} \, s^3 \, h''' - \frac{1}{2} \, s^2 \, h'' + \\
+ \frac{1}{2} \left( e^{-h} \left[ s^4 \, \dddot{a}a'' + 2 \, s^3 \, \dddot{a}a' + s^3 \, \dddot{b}b'' + s^2 \, \dddot{b}b' \right] \right)' - \\
e^{-h} \left[ \frac{1}{3} \, s^4 \, \dddot{a}a''' + 2 \, s^3 \, \dddot{a}a'' + \frac{1}{3} \, s^3 \, \dddot{b}b''' + 2 \, s^2 \, \dddot{a}a' + \\
\right]$$

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\[ C_{1a} = \frac{1}{18} s^3 h'' - \frac{1}{6} s^2 h'' - \frac{2}{3} s h' + \frac{2}{3} h + \]
\[ + \frac{1}{2} \left( e^{-h} \left[ -\frac{1}{3} s^4 \tilde{a} \tilde{a}'' + \frac{2}{3} s^3 \tilde{a} \tilde{a}' - \frac{1}{3} s^3 \tilde{b} \tilde{b}'' + s^2 \tilde{b} \tilde{b}' \right] \right)' + \]
\[ + e^{-h} \left[ \frac{1}{9} s^4 \tilde{a} \tilde{a}'' + \frac{4}{3} s^3 \tilde{a} \tilde{a}'' + \frac{1}{9} s^3 \tilde{b} \tilde{b}'' + 2 s^2 \tilde{a} \tilde{a}' + \right. \]
\[ \left. + \frac{7}{6} s^2 \tilde{b} \tilde{b}' + 2 s \tilde{b} \tilde{b}' \right] \left| \frac{\Lambda^2}{m^2} \right|_0 - \]
\[ - \int_0^\Lambda ds \frac{1}{s^2 + \tilde{b}^2} \left[ \frac{s \tilde{a}^2}{s^2 + \tilde{b}^2} \left[ s \tilde{a} \tilde{a}' + \tilde{b} \tilde{b}' \right] + \right. \]
\[ + \frac{2}{3} s^3 \tilde{a} \tilde{a}'' + 3 s^2 \tilde{a} \tilde{a}' + \frac{2}{3} s^3 \tilde{b} \tilde{b}'' + \]
\[ \left. + 2 s \tilde{a} \tilde{a}' + 3 s \tilde{b} \tilde{b}'' - s (\tilde{b}')^2 + 3 \tilde{b} \tilde{b}' \right] \], \quad (A.8)

For convenience, in the equations we have omitted the index \( I \) for \( \tilde{a} \) and \( \tilde{b} \). The result given above is exact for any value of the cut-off \( \Lambda \), so far no term vanishing at removing the cut-off has been neglected. A comparison of the mass term with eq. (4.10) shows that both results although obtained by different methods agree as expected. Also the first line of eq. (A.7) can be re-identified in eq. (4.10). For \( \tilde{a} = \tilde{b} \equiv 1 \) the standard QED result is reproduced (cf. [64]; [65], eq. (9-64), for \( \Lambda \rightarrow \infty \) the coefficient \( C(0) \) there is related to our expressions by the equation
\[ C(0) = -e^2 \left( \frac{5C_{1s} + 3C_{1a}}{24\pi^2} \right). \]

Now, if conditions (4.12), (4.13) are fulfilled above result significantly simplifies. Then, the UV cut-off can be lifted without any problem (\( \Lambda \rightarrow \infty \)), the coefficients \( C_0 \) and \( C_{1s} \) connected with terms spoiling gauge invariance are vanishing and the final completely gauge invariant result reads
\[ \Delta \Gamma'^G [A] = \text{const. +} \]
\[ + C_{1a} \frac{e^2}{16\pi^2} \int d^4 x \ A^\mu (x) \left[ g_{\mu \nu} \Box - \partial_\mu \partial_\nu \right] A^\nu (x) + \ldots, \quad (A.9) \]
with

\[ C_{1a} = \frac{2}{3} \ln \left[ \frac{\bar{b}(\infty)}{b(0)} \right]^2 - \]

\[ - \int_0^\infty ds \frac{1}{s\tilde{a}^2 + \tilde{b}^2} \left[ \frac{s \tilde{a}^2}{s\tilde{a}^2 + \tilde{b}^2} \left[ s \tilde{a}\tilde{a}' + \tilde{b}\tilde{b}' \right] + \right. \]

\[ + \frac{2}{3} s^3 \tilde{a}\tilde{a}''' + 3 s^2 \tilde{a}\tilde{a}'' + \frac{2}{3} s^2 \tilde{b}\tilde{b}''' + \]

\[ + 2 s \tilde{a}\tilde{a}' + 3 s \tilde{b}\tilde{b}'' - s (\tilde{b}')^2 + 3 \tilde{b}\tilde{b}' \]

. (A.10)

It is worth noting that the coefficient \( C_{1a} \) is finite due to conditions (4.12), (4.13). Gauge invariance and UV finiteness are closely related here\(^{38}\) (For a further discussion see [37]).

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\(^{38}\)Were it not for the first term (\( \sim (s\tilde{a}^2 + \tilde{b}^2)^{-2} \)) in the integral in eqs. (A.8), (A.10), also the weaker condition given in footnote 9 on p. 29 then replacing (4.12) would lead to gauge invariance and UV finiteness at the same time.
Appendix B

In this Appendix we explicitly calculate the function

\[ g(x - x') = e^{-\frac{i}{2} \int d^4y d^4y' \, \bar{J}_\mu(x', y') D_\mu^\nu(x', y') \, J_\nu(x, y) } \]  

for the Ansatz (4.18)

\[ d_I(x) = \left[ 1 + \beta \frac{\Box}{m^2} \right] \delta^{(4)}(x) . \]  

Eq. (B.1) can easily be rewritten as

\[ g(x - x') = e^{-i e^2 (x - x')^2 \int_0^1 d\tau (1 - \tau) \, D_I((x - x') \, \tau) } . \]

\[ \cdot e^{i e^2 (1 - \lambda) \left[ D_I^*(x - x') - D_I^*(0) \right]} , \]  

where

\[ D_I^*(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{(p^2 + i\epsilon)^2} \frac{1}{d_I(p)} , \]  

and

\[ D_I(x) = \Box D_I^*(x) . \]  

For simplicity, let us perform the calculation for \( g \) in Euclidean space. Results then can be read off for Minkowski space whenever needed by rotating back the fourth coordinate. In Euclidean space \( D_I^* \) and \( D_I \) read

\[ D_I^*(x_E) = -\frac{i}{16\pi^2} \ln \left( \mu^2 x_E^2 \right) - \frac{\beta}{m^2} D_I(x_E) \]  

39 Of course, this transformation is not specific to the Ansatz (B.2). To obtain eq. (B.3) a gauge fixing term \( \Gamma_{GF} \) with \( \tilde{a}_\mu = i p_\mu \, \tilde{d}_I(p)^{1/2} \) has been added to the gauge field action \( \Gamma^Q_I \).

40 The IR divergence can be regularized and drops then out for \( g(x) \). The spurious pole generated by the model Ansatz \( \tilde{d}_I(p) \) is understood as also supplied with the \( i\epsilon \)-prescription.
\(\mu^2\) is the temporary IR cut-off applied, and

\[
D_I(x_E) = \frac{i}{4\pi^2 x_E^2} - \frac{i m}{4\pi^2 \sqrt{\beta} |x_E|} K_1 \left( \frac{m|x_E|}{\sqrt{\beta}} \right) . \tag{B.7}
\]

For the further calculation following integral turns out to be useful (\(L_\nu\) are Struve functions) [45], vol. 2.

\[
\int \frac{d\tau}{\tau} K_1(\tau) = - K_1(\tau) - \tau K_0(\tau) - \frac{\pi}{2} \tau \left[ K_1(\tau) L_0(\tau) + K_0(\tau) L_1(\tau) \right] \tag{B.8}
\]

Consequently, we find (\(\gamma\) is the Euler constant)

\[
-x_E^2 \int_0^1 d\tau (1 - \tau) D_I(x_E \tau) = i m \frac{\tau}{4\pi^2} \left\{ 1 + \gamma + \frac{1}{2} \ln \left( \frac{m^2 x_E^2}{4\beta} \right) + \right. \\
+ \left( 1 - \frac{m^2 x_E^2}{\beta} \right) K_0 \left( \frac{m|x_E|}{\sqrt{\beta}} \right) - \frac{m|x_E|}{\sqrt{\beta}} K_1 \left( \frac{m|x_E|}{\sqrt{\beta}} \right) - \frac{\pi}{2} \frac{m^2 x_E^2}{\beta} - \right. \\
\left. \cdot \left[ K_1 \left( \frac{m|x_E|}{\sqrt{\beta}} \right) L_0 \left( \frac{m|x_E|}{\sqrt{\beta}} \right) + K_0 \left( \frac{m|x_E|}{\sqrt{\beta}} \right) L_1 \left( \frac{m|x_E|}{\sqrt{\beta}} \right) \right] \right\} . \tag{B.9}
\]

The final result for \(g(x_E)\) is then (\(t = m|x_E|/\sqrt{\beta}\))

\[
g(x_E) = \exp \left\{ \frac{\alpha}{\pi} \left[ 1 + \gamma + \frac{1}{2} \ln \left( \frac{t^2}{4} \right) + \left( 1 - t^2 \right) K_0(t) - t K_1(t) - \right. \\
- \frac{\pi}{2} t^2 \left[ K_1(t) L_0(t) + K_0(t) L_1(t) \right] \right. \\
+ \left. \alpha \frac{1 - \lambda}{\pi} \left[ \frac{1}{t^2} - \frac{1}{t} K_1(t) + \frac{1}{4} (2\gamma - 1) + \frac{1}{4} \ln \left( \frac{t^2}{4} \right) \right] \right\} . \tag{B.10}
\]

In the long distance limit (\(t \gg 1\)) eq. (B.10) reads
\[ g(x_E) = \exp \left\{ \frac{\alpha}{2\pi} \left[ -\pi t + \frac{(3 - \lambda)}{2} \ln \frac{i^2}{4} + \right. \right. \\
\left. \left. + \frac{(3 + \lambda)}{2} + (3 - \lambda) \gamma + \ldots \right] \right\} \] . \quad (B.11)

So, in the long distance region we are mainly interested in the function \( g(x_E) \) can be written as follows.

\[ g(x_E) = C_g \left( m^2 x_E^2 \right)^{\alpha(3 - \lambda)/4\pi} e^{-\frac{\alpha}{2\sqrt{\beta}} m|x_E|} \left[ 1 + \ldots \right] \quad (B.12) \]

\[ C_g = (4\beta)^{-\alpha(3 - \lambda)/4\pi} \exp \left\{ \frac{\alpha}{4\pi} \left[ (3 + \lambda) + 2 (3 - \lambda) \gamma \right] \right\} \quad (B.13) \]

One easily recognizes in eq. (B.12) the well-known exponent of the (power-like) Bloch-Nordsieck contribution (cf. [43],[44] and references therein).
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Fig. 1: Solution $w$ of eq. (4.71) as function of $\alpha$
$G(\alpha)$

**Fig. 2:** Solution $G$ of eq. (4.79) (with $w$ as solution of eq. (4.71) inserted) as function of $\alpha$
$-s_0(\alpha)$

**Fig. 3**: Solution $s_0$ of eq. (4.85) as function of $\alpha$. The dashed line is located at $\alpha_c \simeq 0.70$ corresponding to the singularity $-s_0 \to \infty$. It separates the regions where the upper ($\alpha < \alpha_c$) and the lower ($\alpha > \alpha_c$) sign in the normalization condition (4.81) is applied respectively. Beyond $\alpha_{max} \simeq 2.64$ eq. (4.85) does not have any solution as one recognizes from the dotted line drawn at $-s_0 = 1$. 
Fig. 4: The parameter $\beta$ as function of $\alpha$ (cf. eq. (4.89)). For further comments see fig. 3.
Fig. 5: The parameter $\tilde{b}(\infty)$ as function of $\alpha$ (cf. eq. (4.92)). Note, that $\tilde{b}(\infty)$ is close to 2 for small $\alpha$. For further comments see fig. 3.
Fig. 6: Typical behaviour of the integrand $M(s)$ in eq. (4.99) for small arguments where eqs. (4.94), (4.95) are inserted ($M(s)$ is drawn here for $\alpha = 0.05$.). The zero (we denote it by $s_x$) of the function $M(s)$ is understood as defining the applicability region of the low $s$ representation (4.94), (4.95). $s_x$ as function of $\alpha$ is shown in fig. 7.
Fig. 7: Zero $s_x$ of $M(s)$ as function of $\alpha$ (see fig. 6). The dashed line is drawn at $\alpha_c$ while the dotted line is located at $\alpha \simeq 1.55$. In both cases one finds numerically $s_x \to \infty$. 
Fig. 8: The coefficient $C_{1a}$ (see eq. (4.99)) as function of $\alpha$. Above curve is the difference of the contributions represented by the curves 1 and 2 shown in fig. 9 (Curve 1 stands for the first term in eq. (4.99) while curve 2 is the contribution of the integral.). Please note that $C_{1a}$ is a completely smooth function at $\alpha = \alpha_c$ (dashed line) although certain parameters involved (see figs. 3, 5) are singular there.
Fig. 9: Contributions to $C_{1a}$ as functions of $\alpha$. For a further explanation see fig. 8.
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