Birefringent break up of Dirac fermions in a square optical lattice

Malcolm P. Kennett, Nazanin Komeilizadeh, Kamran Kaveh, and Peter M. Smith
Physics Department, Simon Fraser University, 8888 University Drive, Burnaby, British Columbia, V5A 1S6, Canada
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We generalize a proposal by Sørensen et al. [Phys. Rev. Lett. 94, 086803 (2005)] for creating an artificial magnetic field in a cold atom system on a square optical lattice. This leads us to an effective lattice model with tunable spatially periodic modulation of the artificial magnetic field and the hopping amplitude. When there is an average flux of half a flux quantum per plaquette the spectrum of low-energy excitations can be described by massless Dirac fermions in which the usually doubly degenerate Dirac cones split into cones with different “speeds of light” which can be tuned to give a single Dirac cone and a flat band. These gapless birefringent Dirac fermions arise because of broken chiral symmetry in the kinetic energy term of the effective low energy Hamiltonian. We characterize the effects of various perturbations to the low-energy spectrum, including staggered potentials, interactions, and domain wall topological defects.

With the discovery of graphene [1] and topological insulators [2], there has been much recent interest in systems in which low energy excitations can be described using Dirac fermions. A parallel area of interest has been the exploration of the possibility of generating artificial magnetic fields for cold atoms confined in an optical lattice. Neutral bosonic cold atoms cannot couple to a magnetic field directly, so there have been numerous proposals [3–6] of approaches to couple atoms to an artificial magnetic field, several of which have been implemented experimentally [7,8].

The problem of the spectrum of quantum particles in a uniform magnetic field on a lattice has the well-known Hofstadter spectrum [9]. Our modification of the proposal by Sørensen et al. [6] leads to an effective Hamiltonian with a tunable Hofstadter-like spectrum that arises from the combination of hopping and an artificial magnetic field with a non-zero average that are both periodically modulated in the \(x\) and \(y\) directions. The presence of spatial periodicity in the amplitude as well as the phase of the hopping is the key difference between the model we consider here and previous work on the spectrum of particles in the presence of magnetic fields that are periodic in both the \(x\) and \(y\) directions [10]. This difference facilitates the unusual Dirac-like spectrum that we discuss in this Letter.

In our effective model, when there is an average of half a flux quantum per plaquette, and at half-filling, the low energy degrees of freedom can be described by a Dirac Hamiltonian with the unusual property that chiral symmetry is broken in the kinetic energy rather than via mass terms. This has the consequence that the doubly degenerate Dirac cone for massless fermions splits into two cones with tunable distinct slopes, analagous to a situation in which there are two speeds of light for fermionic excitations, similar to birefringence of light in crystals such as calcite. We discuss the meaning of broken chiral symmetry in our effective model and explore the effects of various perturbations, such as staggered potentials, domain walls, and interactions between fermions.

The approach to obtain an artificial magnetic field for cold atoms in an optical lattice suggested by Sørensen et al. [6] was presented in the context of the Bose-Hubbard model, but ignores interactions and is not specific to bosons. We consider a model of spinless fermions (corresponding to only one available hyperfine state for cold atoms) with Hamiltonian

\[
H = -J \sum_{\langle i,j \rangle} (\hat{c}_i^\dagger \hat{c}_j + \hat{c}_j^\dagger \hat{c}_i),
\]

where \(\hat{c}_i^\dagger\) and \(\hat{c}_i\) are fermionic creation and annihilation operators respectively at site \(i\), \(\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i\) is the number operator, and the notation \(\langle i,j \rangle\) indicates that we restrict the sum in the hopping term to nearest neighbours only. There can be no Hubbard-like interaction for spinless fermions, and since nearest neighbour interactions in an optical lattice system are weak, we postpone our discussion of interactions.

In Ref. [6], two steps are required to generate an artificial magnetic field. First, a time-varying quadrupolar potential \(V(t) = V_{qp} \sin(\omega t) \hat{x}\hat{y}\) is applied to the system, and second, the hopping is modulated as a function of time. During the course of one oscillation of the quadrupolar potential, hopping in the \(x\) direction is turned for a very short period of time \(\tau \ll t_0 = \frac{2\pi}{\omega}\) at times \(t = n t_0\), where \(n\) is an integer, and hopping in the \(y\) direction is turned on for time \(\tau\) around \(t = (n + \frac{1}{2}) t_0\). Due to the periodic oscillation in the Hamiltonian, the time evolution operator after \(m\) periods may be written as \(U(t = m t_0) = U(t = t_0)^m\).

![FIG. 1: Time dependence of the hopping and the quadrupolar potential during the course of one period of the quadrupolar potential.](image-url)
Our modification to the proposal in Ref. [4] is that when hopping is turned on in the x-direction at time \( t = nt_0 \), hopping is also turned on in the y-direction with an amplitude \( 0 \leq \beta \leq 1 \) relative to the hopping in the x-direction. At time \( t = (n + \frac{1}{2}) t_0 \), hopping is turned on in the y direction, and hopping in the x-direction is turned on with amplitude \( \beta \) relative to the hopping in the x-direction as illustrated in Fig. 1.

The operators for hopping in the x and y directions, are \( \hat{T}_x = -J \sum_{x,y} (\hat{c}_{x+1,y} \hat{c}_{x,y} + \text{h.c.}) \) and \( \hat{T}_y = -J \sum_{x,y} (\hat{c}_{x,y+1} \hat{c}_{x,y} + \text{h.c.}) \) respectively, and we may write the time evolution operator as

\[
U (t = nt_0) = \left[ e^{\frac{-i\pi}{\hbar}(\hat{T}_x + \beta \hat{T}_y)} e^{2\pi i \alpha x \hat{\gamma}} e^{-\frac{i\pi}{\hbar}(\beta \hat{T}_x + \hat{T}_y)} \times e^{-2\pi i \alpha x \hat{\gamma}} e^{\frac{i\pi}{\hbar}(\hat{T}_x + \beta \hat{T}_y)} \right]^m, (2)
\]

where \( \alpha = \nu x / \pi \hbar \) and we have set the lattice constant to unity. To lowest order in \( Jt/\hbar \) we can write this in the form

\[
U = e^{-i\frac{\beta e \pi}{\hbar} T x},
\]

where the effective Hamiltonian is

\[
H_{\text{eff}} = -J_0 \sum_{x,y} \left\{ \left[ (1 + \beta e^{2 \pi i \alpha x}) \hat{c}_{x,y+1}^\dagger \hat{c}_{x,y} + \text{h.c.} \right] + \left[ (\beta e^{2 \pi i \alpha y}) \hat{c}_{x,y+1}^\dagger \hat{c}_{x,y} + \text{h.c.} \right] \right\}, (3)
\]

with \( J_0 = \tau J / t_0 \). A more conventional way to write this Hamiltonian is in the form

\[
H_{\text{eff}} = -\sum_{ij} t_{ij} e^{\frac{i\pi}{\hbar} J_0} A_{ij} \hat{c}_i^\dagger \hat{c}_j + \text{h.c.}, (4)
\]

from which we may identify the amplitude of the hopping:

\[
t_{x+1,y} = J_0 \sqrt{1 + \beta^2 + 2 \beta \cos(2 \pi \alpha y)}, \quad \text{(5)}
\]

\[
t_{x,y+1} = J_0 \sqrt{1 + \beta^2 + 2 \beta \cos(2 \pi \alpha x)}, \quad \text{(6)}
\]

and the artificial magnetic field

\[
B_z = \frac{2 \pi \alpha \hbar}{e} \left\{ \frac{\beta^2 + \beta \cos(2 \pi \alpha x)}{1 + \beta^2 + 2 \beta \cos(2 \pi \alpha x)} - \frac{1 + \beta \cos(2 \pi \alpha y)}{1 + \beta^2 + 2 \beta \cos(2 \pi \alpha y)} \right\}. (7)
\]

This field is the sum of a spatially uniform piece with magnitude \( \frac{2 \pi \alpha \hbar}{e} \) and a piece that is spatially periodic in both the x and y directions. If \( \beta = 0 \), the hopping amplitude is \( J_0 \) and the field is uniform with strength \( \frac{2 \pi \alpha \hbar}{e} \), corresponding to a flux of \( \alpha \phi_0 \) per plaquette (where \( \phi_0 \) is the flux quantum) as found in Ref. [4] and there is a Hofstadter spectrum. If \( \beta = 1 \), then \( B_z = 0 \), but the hopping parameters are still spatially periodic. At \( \beta \) intermediate between 0 and 1, both the hopping and the magnetic field are spatially periodic in x and y. This illustrates the essential difference between the model we consider and previous work on quantum particles in a periodic magnetic field on a lattice – there is spatial periodicity of \( 1/\alpha \) in the amplitude of the hopping as well as in the magnetic field. For finite \( \beta \) the spectrum (illustrated for \( \beta = 1 \) in Fig. 3) as a function of \( \alpha \) is reminiscent of the Hofstadter spectrum.

When \( \alpha = 1/2 \) there is an average of half a flux quantum per plaquette, and the theory is time reversal symmetric as fermions cannot detect the sign of the flux \( \pm \). The effective Hamiltonian Eq. (3) simplifies to a tight binding model with four sites in the unit cell as shown in Fig. 3(a).

Labelling the four sites in the unit cell as A, B, C, and D, and Fourier transforming in space, we may rewrite the effective Hamiltonian in the following form:

\[
H = \sum_k \psi_k^\dagger [E_k - \mathcal{H}_k] \psi_k, \quad \text{(8)}
\]

with

\[
\mathcal{H}_k = 2 \begin{pmatrix}
0 & J_+ \cos k_y & J_+ \cos k_x & 0 \\
J_+ \cos k_y & 0 & 0 & -J_- \cos k_x \\
J_+ \cos k_x & 0 & 0 & J_- \cos k_y \\
0 & -J_- \cos k_x & J_- \cos k_y & 0
\end{pmatrix},
\]
where \( J_\pm = J_0(1 \pm \beta) \) and (in row vector form) \( \psi_k = (c_{Ak}, c_{Bk}, c_{Ck}, c_{Dk}) \). We find that the dispersion is
\[
E_k = \pm J_+ \sqrt{\cos^2 k_x + \cos^2 k_y},
\]
and we note that when \( \beta = 1, J_- = 0 \), so there will be a flat band at \( \epsilon = 0 \) and a dispersing band associated with \( J_+ \).

We can also see that in the vicinity of the points \( K_{\pm, \pm} = (\pm \frac{\pi}{2}, \pm \frac{\pi}{2}) \), the spectrum is linear
\[
E_q = \pm J_+ \sqrt{q_x^2 + q_y^2},
\]
where \( q = k - (\pm \frac{\pi}{2}, \pm \frac{\pi}{2}) \), and there are cones with two different slopes, corresponding to \( J_+ \) respectively as illustrated in Fig. 3 b). When \( \beta = 0 \), the two slopes are identical, whereas as \( \beta \to 1 \), the \( J_- \) band becomes flat, and the \( J_+ \) band remains as a cone. Several authors recently considered lattice models for cold atoms that are equivalent to the \( \beta = 1 \) limit of our model, in which there are three bands, one flat, and one Dirac like [12 13].

To start in this direction, we expand around the Dirac points and represent the low energy theory (with \( k \) measured with respect to \( K \))
\[
H_k = 2J_0 \left[ (\gamma^0 \gamma^1 + i \beta \gamma^3) k_x + (\gamma^0 \gamma^2 + i \beta \gamma^5) k_y \right],
\]
where we use a non-standard representation of the gamma matrices in which \( \gamma^0 = \sigma_3 \otimes I_2, \gamma^1 = i \sigma_2 \otimes I_2, \gamma^2 = i \sigma_3 \otimes \sigma_2, \gamma^3 = -i \sigma_1 \otimes I_2 \), and \( \gamma^5 = -\gamma^0 \gamma^1 \gamma^2 \gamma^3 = -i \sigma_3 \otimes \sigma_1 \).

The matrices \( \gamma^0, \gamma^1, \gamma^2 \) satisfy the Clifford algebra \( \gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu} \) with Minkowski metric \( g^{\mu\nu} \).

The dimension of the minimal representation of the Clifford algebra in 2+1 dimensions is 2, allowing for the \( 2 \times 2 \) Pauli matrices as a choice for the \( \gamma \)'s. A non-minimal 4 \times 4 representation as we have used above leads to a freedom in the choice of the \( \gamma^0 \) matrix, i.e. a matrix with \( (\gamma^0)^2 = I_4 \) that anticommutes with \( \gamma^1 \) and \( \gamma^2 \). Candidates for \( \gamma^0 \) are then \( \{ \gamma^0, \gamma^0 \gamma^1, \gamma^0 \gamma^2, \gamma^0 \gamma^3 \} \). The matrices \( \{ \gamma^0, \gamma^0 \gamma^1, \gamma^0 \gamma^2, \gamma^0 \gamma^3 \} \) form a triplet and \( \gamma^1 \gamma^2 \) forms a singlet with respect to the SU(2) “chiral”-symmetry group with generators \( \{ \frac{1}{2} \gamma^3, \frac{1}{2} \gamma^5, \frac{1}{2} \gamma^{35} \} \) (where \( \gamma^{35} \equiv \gamma^3 \gamma^5 \)). Each different choice of \( \gamma^0 \) corresponds to a different labelling of the four sites in the unit cell. The elements of the chiral group generate transitions between each labelling. For example, the generator \( \gamma^3 \) translates the plauette indices to the labelling of the neighboring lattice cell along the \( y \)-direction, whilst \( \gamma^5 \) translates the plauette indices to the neighbouring cell in the \( x \)-direction.

\[
e^{i \gamma^3} \begin{pmatrix} c_A \\ c_B \\ c_C \\ c_D \end{pmatrix} = i \begin{pmatrix} c_B \\ c_A \\ -c_D \\ -c_C \end{pmatrix},
\]
\[
e^{i \gamma^5} \begin{pmatrix} c_A \\ c_B \\ c_C \\ c_D \end{pmatrix} = i \begin{pmatrix} c_C \\ c_D \\ c_A \\ c_B \end{pmatrix}.
\]

Similarly, \( \gamma^{35} \) translates the plauette one lattice cell along the \( x \)- and one lattice cell along the \( y \)-direction. When \( \beta = 0 \), the elements of the chiral group are symmetries of \( H_k \).

When \( \beta \neq 0 \), the \( \gamma^3 \) and \( \gamma^5 \) terms in \( H_k \) break the chiral symmetry and shifts along either the \( x \)- or \( y \)-directions do not leave \( H_k \) invariant. This manifest chiral symmetry breaking is inherently different from the conventional notion of spontaneous chiral symmetry breaking in field theoretical models which is the signature of mass generation [15].

An additional discrete symmetry of the \( H_k \) (that arises from the hopping structure in \( H_k \)) that holds even when \( \beta \neq 0 \) is
\[
\Gamma = \frac{i}{2}(\gamma^1 \gamma^3 + \gamma^2 \gamma^5) - \frac{i}{2}(\gamma^2 \gamma^3 - \gamma^1 \gamma^5),
\]
which corresponds to a reflection about the diagonal \( AD \) in the unit cell, with \( c_A \to c_A, c_B \to c_C, c_C \to c_B \) and \( c_D \to -c_D \). The action of \( \Gamma \) on \( H_k \) is to exchange \( k_x \) and \( k_y \).

Fermion birefringence: As illustrated in Fig. 3 b) the dispersion Eq. (9) admits massless fermions with two different “speeds of light” controlled by \( \beta \). The eigenvectors (written as row vectors) for the positive and negative energy \( J_+ \) bands are \( \Psi_1 = \frac{1}{\sqrt{2}}(1, -\sin \theta, -\cos \theta, 0) \) and \( \Psi_2 = \frac{1}{\sqrt{2}}(1, \sin \theta, \cos \theta, 0) \); whilst the eigenvectors for the \( J_- \) bands are \( \Psi_3 = \frac{1}{\sqrt{2}}(0, \cos \theta, -\sin \theta, 1) \) and \( \Psi_4 = \frac{1}{\sqrt{2}}(0, -\cos \theta, \sin \theta, 1) \), where we write \( k_x = k \cos \theta \) and \( k_y = k \sin \theta \).

Any other state will break up into fast (\( J_+ \)) and slow (\( J_- \)) fermionic excitations, analogous to fast and slow modes in a birefringent medium.

Staggered potentials: Staggered on-site potentials are a natural perturbation to \( H_k \) in the context of cold atoms on an optical lattice. We can write the most general form of such a potential as
\[
\Delta = \sum_k \frac{1}{2} \left[ \Delta_0 I_4 + \Delta_1 \gamma^0 + \Delta_2 (i \gamma^1 \gamma^3 + i \gamma^2 \gamma^5) \right] \psi_k,
\]
where we may set \( \Delta_0 = 0 \) since this just corresponds to a uniform shift of the chemical potential. The \( \Delta_1 \) term violates chiral symmetry in the usual way but is Lorentz invariant and hence introduces a gap in the dispersion of the fermions
\[
E_k = \pm \sqrt{\Delta_1^2 + 4J_0^2 k^2}.
\]
When \( \beta = 1 \) there are flat bands at \( E = \pm \Delta_1 \) that intersect the \( J_+ \) bands only at \( (k_x, k_y) = (0, 0) \). The birefringence property discussed above is unaffected by the \( \Delta_1 \) term. We combine \( i \gamma^1 \gamma^3 \) and \( i \gamma^2 \gamma^5 \) into a Lorentz invariant term \( \Delta_2 \) and a Lorentz violating term \( \Delta_3 \). There are two cases in which we have obtained simple analytic solutions for the spectrum: case I) \( \Delta_1 \neq 0, \Delta_2 \neq 0, \Delta_3 = 0 \), for which
\[
E_k = \left\{ \begin{array}{ll}
\Delta_2 \pm \sqrt{\Delta_1^2 + \Delta_2^2 + 4J_0^2 k^2} \\
-\Delta_2 \pm \sqrt{\Delta_1^2 - \Delta_2^2 + 4J_0^2 k^2}
\end{array} \right.,
\]
and case II): \( \Delta_1 \neq 0, \Delta_2 = 0, \Delta_3 \neq 0 \), for which
\[
E_k = \begin{cases} \\
\Delta_3 \pm \sqrt{(\Delta_1 - \Delta_3)^2 + 4J^2_k k^2_x + 4J^2_k k^2_y} \\
-\Delta_3 \pm \sqrt{(\Delta_1 + \Delta_3)^2 + 4J^2_k k^2_x + 4J^2_k k^2_y}
\end{cases}.
\]

In case I) the dispersion remains isotropic in momentum space and there are flat bands when \( \beta = 1 \), whereas in case II), the dispersion becomes anisotropic, with the anisotropy governed by \( \beta \) through \( J \). In both cases, there is a shift in the spectrum and there will be at least one set of massive modes (however in both cases there can be a set of massless modes whose dispersion is given by the upper half of a cone if \( \Delta_1 = \pm \Delta_2,3 \) and \( \Delta_0 = \mp \Delta_2,3 \).

**Interactions:** as we consider spinless fermions, there will be no on-site Hubbard interaction, so we consider nearest neighbour interactions of the extended Hubbard type (for cold atoms in an optical lattice these will generally be weak):
\[
H_{\text{int}} = \sum_{(ij)} V_{ij} n_in_j.
\]

Setting all of the \( V_{ij} = V_0 \), we can write the interaction Hamiltonian in terms of spinors as
\[
H_{\text{int}} = \frac{V_0}{16} \sum_k \left[ (\bar{\psi}_k \gamma^0 \psi_k)^2 - (\bar{\psi}_k \psi_k)^2 \right],
\]
with \( \bar{\psi}_k = \psi_k^\dagger \gamma^0 \). The identity and \( \gamma^0 \) that appear in the kernels of the quartic interaction terms are the only elements of the Clifford algebra that either commute or anticommute with all of the elements of the Lorentz group and the chiral group, ensuring that the interactions remain invariant under any rotation of the lattice by the Lorentz group or relabelling of the plaquette indices by the chiral group. At the mean field level, the \( (\bar{\psi}\psi)^2 \) term breaks the chiral symmetry by introducing an effective mass term \( m_0 \gamma^0 \), and the \( (\bar{\psi}^{(i)\dagger}\gamma^0)^2 \) term renormalizes the chemical potential as \( \delta I_4 \) and is otherwise uninteresting. In the limit of weak interactions, the mean field interaction Hamiltonian is:
\[
H_{\text{int}}^{\text{MF}} = \sum_k \psi_k^\dagger \left[ (\delta I_4 + m_0 \gamma^0) + (m_1 \gamma^0 \gamma^1 + m_2 \gamma^0 \gamma^2 + m_3 i \gamma^3 + m_5 \gamma^5) \right] \psi_k,
\]
where \( \delta = \langle n_A \rangle + \langle n_B \rangle + \langle n_C \rangle + \langle n_D \rangle \), and the order parameter for staggered charge density wave order \( m_0 = \langle n_A \rangle - \langle n_B \rangle - \langle n_C \rangle + \langle n_D \rangle \) arise from the Hartree term. The remaining masses, \( m_1, m_2, m_3 \) and \( m_5 \) arise from the Fock term – if these are dropped and \( \beta = 0 \), we recover the mean-field approximation of the Gross-Neveu model \[16\]. Similarly to a \( \Delta_1 \gamma^0 \) staggered potential, the Hartree term leads to massive excitations, but does not destroy fermion birefringence. The detailed study of interactions when \( \beta \neq 0 \) is a topic for future investigation.

For small values of \( \beta \), when \( H_{\text{int}} \) is added to Eq. \[13\] there is a mapping between the weak interaction strength regime considered above to the strong interaction strength limit that preserves the property of birefringence:
\[
E_k(\beta, V_0) = \beta E_k(\beta^{-1}, \beta^{-1} V_0).
\]

This arises from the appearance of the chiral symmetry generators, \( \gamma^3 \) and \( \gamma^5 \) in the kinetic energy and their duality with Lorenz group generators \( \gamma^0 \gamma^3, \gamma^0 \gamma^5 \). Upon choosing a different representation of Clifford algebra elements, one can transform \( \gamma^0 \gamma^1 \leftrightarrow \gamma^3, \gamma^0 \gamma^2 \leftrightarrow \gamma^5 \).

**Topological defects:** broken chiral symmetry at \( \beta \neq 0 \) implies that there cannot be vortices, but domain walls of the form \( \Delta_1(x) \gamma^0 \) where \( \lim_{x \to \infty} \Delta_1(x) = \Delta \) and \( \lim_{x \to -\infty} \Delta_1(x) = -\Delta \) can occur. If \( \beta = 0 \) then the form of the solutions with energy \( |\epsilon| < \Delta \) is well known. When \( \beta \neq 0 \) we can find zero energy bound states with different spatial extents for the + and - solutions:
\[
\psi_+(x) = e^{-\kappa_+ \int_0^x ds \Delta_i(s)} u_+; \quad \psi_-(x) = e^{-\kappa_- \int_0^x ds \Delta_i(s)} u_-,
\]
where \( u_+ = (1, 0, i, 0), u_- = (0, -i, 0, 1) \), and \( \kappa_+ = 1/2J \).

In this Letter we have demonstrated a model whose low energy excitations are birefringent fermions that arise from broken chiral symmetry. We discuss the low energy properties of the model and illustrate the meaning of broken chiral symmetry in our model. We argue that such a model could be realised by cold atoms in an optical lattice. This might not be the only route – as noted in a similar context in Ref. \[11\] another approach might be through an appropriately engineered semiconductor heterostructure.

An important feature of the birefringent fermion dispersion that we find here is that the slopes of the \( J_+ \) and \( J_- \) bands can be controlled by the parameter \( \beta \). In particular when \( \beta = 1 \), there can be flat bands in the spectrum and these flat bands are robust to the addition of a staggered potential \( \Delta_1 \gamma^0 \) and weak nearest neighbour Hubbard interactions at the Hartree level. Flat bands such as Landau levels can lead to interesting correlated phases when interactions beyond mean field are taken into account \[12\]. This suggests that future avenues for research on this model could include the study of such correlated phases when \( \beta \neq 0 \), and the generalization of the model to fermions with spin. Including spin would allow for on-site Hubbard interactions, which would be considerably complicate matters and require techniques similar to those that have been used to study high temperature superconductors \[18\].

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