Connes-amenability and normal, virtual diagonals for measure algebras, I

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Dedicated to the memory of Barry E. Johnson, 1937–2002, on whose shoulders many of us stand.

Abstract

We prove that the measure algebra $M(G)$ of a locally compact group $G$ is Connes-amenable if and only if $G$ is amenable.

Keywords: locally compact group; group algebra; measure algebra; amenability; Connes-amenability; normal, virtual diagonal.

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Introduction

In [16], B. E. Johnson introduced the notion of an amenable Banach algebra, and proved that a locally compact group $G$ is amenable if and only if its group algebra $L^1(G)$ is amenable. The theory of amenable Banach algebras has been a very active field of research ever since. Once of the deepest results in this theory is due to A. Connes ([8]; see also [2]) and U. Haagerup ([12]): A $C^*$-algebra is amenable if and only if it is nuclear. In [24], S. Wassermann showed that a von Neumann algebra is nuclear/amenable if and only if it is subhomogeneous (see [22] for a proof that avoids the nuclearity-amenability nexus). This suggests that the definition of amenability from [16] has to be modified to yield a sufficiently rich theory for von Neumann algebras.

A variant of that definition — one that takes the dual space structure of a von Neumann algebra into account — was introduced in [18], but is most commonly associated

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with A. Connes’ paper \[4\]. For this reason, we refer to this notion of amenability as to Connes-amenability (the origin of this name seems to be A. Ya. Helemski˘ı’s paper \[13\]). As it turns out, Connes-amenability is the “right” notion of amenability for von Neumann algebras: It is equivalent to several other important properties such as injectivity and semidiscreteness (\[2\], \[1\], \[3\], \[9\], \[25\]; see \[23\], Chapter 6) for a self-contained exposition).

The definition of Connes-amenability makes sense for a larger class of Banach algebras (called dual Banach algebras in \[22\]). Examples of dual Banach algebras (other than \(W^*\)-algebras) are: \(B(E)\), where \(E\) is a reflexive Banach space; \(M(G)\), where \(G\) is a locally compact group; \(PM_p(G)\), where \(p \in (1, \infty)\) and \(G\) is a locally compact group (these algebras are called algebras of \(p\)-pseudomeasures). The investigation of Connes-amenability for dual Banach algebras which are not \(W^*\)-algebras is still in its initial stages. Some results on Connes-amenable \(W^*\)-algebras, carry over: For instance, in \[20\], A. T.-M. Lau and A. L. T. Paterson showed that, for an inner amenable group \(G\), the group von Neumann algebra \(VN(G) = PM_2(G)\) is Connes-amenable if and only if \(G\) is amenable; this is generalized to \(PM_p(G)\) for arbitrary \(p \in (1, \infty)\) in \[22\]. On the other hand, one cannot expect matters for general dual Banach algebras to turn out as nicely as for von Neumann algebras: In \[22\], it was shown that \(B(E)\) is not Connes-amenable if \(E = \ell^p \oplus \ell^q\) with \(p, q \in (1, \infty) \setminus \{2\}\) and \(p \neq q\).

The dual Banach algebra we are concerned with in this paper is the measure algebra \(M(G)\) of a locally compact group \(G\). As for von Neumann algebras, amenability in the sense of \[16\] is too strong a notion to deal with measure algebras in a satisfactory manner: In \[7\], H. G. Dales, F. Ghahramani, and A. Ya. Helemskiı prove that \(M(G)\) is amenable for a locally compact group \(G\) if and only if \(G\) is discrete and amenable. In contrast, Connes-amenability is a much less restrictive demand: Since the amenability of a locally compact group \(G\) implies the amenability of \(L^1(G)\), and since \(L^1(G)\) is \(w^*\)-dense in \(M(G)\), it follows easily that \(M(G)\) is Connes-amenable provided that \(G\) is amenable. In this paper, we prove the converse.

I am grateful to S. Tabaldyev for discovering a near fatal error in an earlier, stronger version of Lemma \[4.2\].

### 1 Connes-amenability and normal, virtual diagonals

This section is preliminary in character. We collect the necessary definitions we require in the sequel. All of it can be found in \[22\], but sometimes our choice of terminology here is different.

Let \(\mathfrak{A}\) and \(\mathfrak{B}\) be Banach algebras, and let \(E\) be a Banach \(\mathfrak{A}\)-\(\mathfrak{B}\)-bimodule. Then \(E^*\) becomes a Banach \(\mathfrak{B}\)-\(\mathfrak{A}\)-bimodule via

\[
\langle x, b \cdot \phi \rangle := \langle x \cdot b, \phi \rangle \quad \text{and} \quad \langle x, \phi \cdot a \rangle := \langle a \cdot x, \phi \rangle \quad (a \in \mathfrak{A}, b \in \mathfrak{B}, \phi \in E^*, x \in E).
\]
Definition 1.1 Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be Banach algebras. A Banach \( \mathfrak{A}\mathfrak{B} \)-bimodule \( E \) is called dual if there is a closed submodule \( E_\ast \) of \( E^\ast \) such that \( E = (E_\ast)^\ast \).

Remark There is no reason, in general, for \( E_\ast \) to be unique. If we refer to the \( w^\ast \)-topology on a dual Banach module \( E \), we always mean \( \sigma(E,E_\ast) \) with respect to a particular, fixed (often obvious) predual \( E_\ast \).

The following definition is due to B. E. Johnson ([16]):

Definition 1.2 A Banach algebra \( \mathfrak{A} \) is called amenable if, for every dual Banach \( \mathfrak{A}\mathfrak{B} \)-bimodule \( E \), every bounded derivation \( D: \mathfrak{A} \to E \) is inner.

We are interested in a particular class of Banach algebras:

Definition 1.3 A Banach algebra \( \mathfrak{A} \) is called dual if it is dual as a Banach \( \mathfrak{A} \)-bimodule.

Remark A Banach algebra which is also a dual space is a dual Banach algebra if and only if multiplication is separately \( w^\ast \)-continuous.

Examples 1. Every \( W^\ast \)-algebra is dual.

2. If \( E \) is a reflexive Banach space, then \( \mathcal{B}(E) = (E\hat{\otimes}E^\ast)^\ast \) is dual.

3. If \( G \) is a locally compact group, then \( \mathcal{M}(G) = \mathcal{C}_0(G)^\ast \) is dual.

4. If \( \mathfrak{A} \) is an Arens regular Banach algebra, then \( \mathfrak{A}^{**} \) is dual.

The following choice of terminology is motivated by the von Neumann algebra case:

Definition 1.4 Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be dual Banach algebras. A dual Banach \( \mathfrak{A}\mathfrak{B} \)-bimodule is called normal if, for each \( x \in E \), the maps

\[ \mathfrak{A} \to E, \quad a \mapsto a \cdot x \]

and

\[ \mathfrak{B} \to E, \quad b \mapsto x \cdot b \]

are \( w^\ast \)-continuous.

We can now define Connes-amenable, dual Banach algebras:

Definition 1.5 A dual Banach algebra \( \mathfrak{A} \) is called Connes-amenable if, for every normal, dual Banach \( \mathfrak{A} \)-bimodule \( E \), every bounded, \( w^\ast \)-continuous derivation \( D: \mathfrak{A} \to E \) is inner.
Amenability in the sense of [16], can be intrinsically characterized in terms of so-called approximate and virtual diagonals ([17]). There is a related notion for Connes-amenable, dual Banach algebras.

If $E_1, \ldots , E_n$ and $F$ are dual Banach spaces, we write $L^w_*(E_1, \ldots , E_n; F)$ for the bounded, separately $w^*$-continuous, $n$-linear maps from $E_1 \times \cdots \times E_n$ into $F$. In case $E_1 = \cdots = E_n =: E$, we simply let $L^w_*(E_1, \ldots , E_n; F)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be Banach algebras. Then $\mathfrak{A} \hat{\otimes} \mathfrak{B}$ becomes a Banach $\mathfrak{A}-\mathfrak{B}$-bimodule via

$$a \cdot (x \otimes y) := ax \otimes y \quad \text{and} \quad (x \otimes y) \cdot b := x \otimes yb \quad (a, x \in \mathfrak{A}, b, y \in \mathfrak{B}). \quad (1)$$

Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are dual. It is then routinely checked that $L^w_*(\mathfrak{A}, \mathfrak{B}; C)$ is a closed $\mathfrak{B}-\mathfrak{A}$-submodule of $(\mathfrak{A} \hat{\otimes} \mathfrak{B})^*$.

Let $\mathfrak{A}$ be a dual Banach algebra, and let $\Delta_\mathfrak{A}: \mathfrak{A} \hat{\otimes} \mathfrak{A} \to \mathfrak{A}$ denote the diagonal operator induced by $\mathfrak{A} \times \mathfrak{A} \ni (a, b) \mapsto ab$. Since multiplication in $\mathfrak{A}$ is separately $w^*$-continuous, we have $\Delta_\mathfrak{A}^* \mathfrak{A} \subset L^w_*(\mathfrak{A}, C)$. Taking the adjoint of $\Delta_\mathfrak{A}^* \mathfrak{A}$, we may thus extend $\Delta_\mathfrak{A}$ to $L^w_*(\mathfrak{A}, C)^*$ as an $\mathfrak{A}$-bimodule homomorphism (we denote this extension by $\Delta_{w^*}$).

**Definition 1.6** Let $\mathfrak{A}$ be a dual Banach algebra. An element $M \in L^w_*(\mathfrak{A}, C)^*$ is called a normal, virtual diagonal for $\mathfrak{A}$ if

$$a \cdot M = M \cdot a \quad \text{and} \quad a \Delta_{w^*} M = a \quad (a \in \mathfrak{A}).$$

One connection between Connes-amenability and the existence of normal, virtual diagonals is fairly straightforward ([8], [1]): If $\mathfrak{A}$ has a normal, virtual diagonal, then $\mathfrak{A}$ is Connes-amenable; in fact, it implies a somewhat stronger property ([22]). The main problem with proving the converse is that, in general, the dual module $L^w_*(\mathfrak{A}, C)^*$ need not be normal. For von Neumann algebras, however, Connes-amenability and the existence of normal, virtual diagonals are even equivalent ([8], [10]). We suspect, but have been unable to prove — except in the discrete case — that the same is true for the measure algebras of locally compact groups.

2 Separately $C_0$-functions on locally compact Hausdorff spaces

Our notation is standard: For a topological space $X$, we write $C_b(X)$ for the bounded, continuous functions on $X$; if $X$ is locally compact and Hausdorff, $C_0(X)$ (or rather $C(X)$ if $X$ is compact) denotes the continuous functions on $X$ vanishing at infinity, and $M(X) \cong C_0(X)^*$ is the space of regular Borel measures on $X$.

Let $X$ and $Y$ be locally compact Hausdorff spaces. In this section, we give a description of $L^w_*(M(X), M(Y); C)$ as a space of separately continuous functions on $X \times Y$. 
Definition 2.1 Let $X$ and $Y$ be locally compact Hausdorff spaces. A bounded function $f : X \times Y \to \mathbb{C}$ is called separately $C_0$ if:

(a) for each $x \in X$, the function

$$Y \to \mathbb{C}, \quad y \mapsto f(x, y)$$

belongs to $C_0(Y)$;

(b) for each $y \in Y$, the function

$$X \to \mathbb{C}, \quad x \mapsto f(x, y)$$

belongs to $C_0(X)$.

We define $SC_0(X \times Y)$ as the collection of all separately $C_0$-functions.

Lemma 2.2 Let $X$ and $Y$ be locally compact Hausdorff spaces, and let $f \in SC_0(X \times Y)$. Then the following hold:

(i) for each $\mu \in M(X)$, the function

$$Y \to \mathbb{C}, \quad y \mapsto \int_X f(x, y) \, d\mu(x)$$

belongs to $C_0(Y)$;

(ii) for each $\nu \in M(Y)$, the function

$$X \to \mathbb{C}, \quad x \mapsto \int_Y f(x, y) \, d\nu(y)$$

belongs to $C_0(X)$.

Proof We only prove (i).

Let $\mu \in M(X)$. Since the measures with compact support are norm dense in $M(X)$, there is no loss of generality if we suppose that $X$ is compact. Suppose that $Y$ is not compact (the compact case is easier), and let $Y_\infty$ be its one-point-compactification. Extend $f$ to $X \times Y_\infty$ by letting

$$f(x, \infty) = 0 \quad (x \in X),$$

so that $f$ is separately continuous on $X \times Y_\infty$. Let $\tau$ be the topology of pointwise convergence on $C(X)$. Since the map

$$Y_\infty \to C(X), \quad y \mapsto f(\cdot, y)$$

is continuous with respect to the given topology on $Y_\infty$ and to $\tau$ on $C(X)$, the set

$$K := \{f(\cdot,y) : y \in Y_\infty \}$$

is $\tau$-compact. By [11, Théorème 5], this means that $K$ is weakly compact, so that the weak topology and $\tau$ coincide on $K$. Let $(y_\alpha)_{\alpha}$ be a convergent net in $Y_\infty$ with limit $y$. Since $f(\cdot,y_\alpha) \overset{\tau}{\to} f(\cdot,y)$, it follows that

$$\lim_{\alpha} \int_X f(x,y_\alpha) \, d\mu(x) = \int_X f(x,y) \, d\mu(x).$$

This means that the function

$$Y_\infty \to \mathbb{C}, \quad y \mapsto \int_X f(x,y) \, d\mu(x)$$

is continuous on $Y_\infty$; since it vanishes at $\infty$ by definition, this establishes (i). \hfill \Box

Remark For compact spaces, Lemma 2.2 is well known ([3, Theorem A.20]).

**Lemma 2.3** Let $X$ and $Y$ be locally compact Hausdorff spaces, and let $f \in SC_0(X \times Y)$. Then the bilinear map

$$\Phi_f : M(X) \times M(Y) \to \mathbb{C}, \quad (\mu,\nu) \mapsto \int_Y \int_X f(x,y) \, d\mu(x) \, d\nu(y) \quad (2)$$

belongs to $L^2_{w^*}(M(X),M(Y);\mathbb{C})$.

**Proof** Clearly, $\Phi_f$ is bounded, and it is immediate from Lemma 2.2(i) that it is $w^*$-continuous in the second variable. Since $f$ is separately continuous, it is $\mu \otimes \nu$-measurable for all $\mu \in M(X)$ and $\nu \in M(Y)$ by [15]. It follows that the integral in (2) not only exists, but — by Fubini’s theorem — is independent of the order of integration, i.e.

$$\Phi_f(\mu,\nu) = \int_X \int_Y f(x,y) \, d\nu(y) \, d\mu(x) \quad (\mu \in M(X), \nu \in M(Y)).$$

It then follows from Lemma 2.2(ii) that $\Phi_f$ is also $w^*$-continuous in the first variable. \hfill \Box

**Proposition 2.4** Let $X$ and $Y$ be locally compact Hausdorff spaces. Then

$$SC_0(X \times Y) \to L^2_{w^*}(M(X),M(Y);\mathbb{C}), \quad f \mapsto \Phi_f \quad (3)$$

is an isometric isomorphism.
Clearly, \( \| \Phi f \| \leq \| f \|_{\infty} \) for all \( f \in SC_0(X \times Y) \). On the other hand,

\[
\| \Phi f \| \geq \sup \{|\Phi f(\delta_x, \delta_y)| : x \in X, y \in Y\} = \sup \{|f(x, y)| : x \in X, y \in Y\} = \| f \|_{\infty} \quad (f \in SC_0(X \times Y)),
\]

so that (2) is an isometry.

Let \( \Phi \in L_{w^*}(M(X), M(Y); \mathbb{C}) \) be arbitrary, and define

\[ f : X \times Y \to \mathbb{C}, \quad (x, y) \mapsto \Phi(\delta_x, \delta_y). \]

It is immediate that \( f \in SC_0(X \times Y) \) such that \( \Phi f(\delta_x, \delta_y) = \Phi(\delta_x, \delta_y) \) for all \( x \in X \) and \( y \in Y \). Separate \( w^* \)-continuity yields that \( \Phi = \Phi f \). \( \square \)

We shall, from now on, identify \( SC_0(X \times Y) \) and \( L_{w^*}(M(X), M(Y); \mathbb{C}) \) as Banach spaces.

**Proposition 2.5** Let \( X \) and \( Y \) be locally compact Hausdorff spaces. Then the map

\[ M(X \times Y) \to L_{w^*}^2(M(X), M(Y); \mathbb{C})^*, \quad \mu \mapsto \Psi_\mu, \]

where

\[ \Psi_\mu(f) := \int_{X \times Y} f(x, y) \, d\mu(x, y) \quad (\mu \in M(X \times Y), f \in SC_0(X \times Y)), \quad (4) \]

is an isometry.

**Proof** By [13] again, the integral in (4) is well-defined. Since \( C_0(X \times Y) \subset SC_0(X \times Y) \), it follows at once that \( \| \Psi_\mu \| = \| \mu \| \) holds for all \( \mu \in M(X \times Y) \). \( \square \)

### 3 Separately \( C_0 \)-functions on locally compact groups

Let \( G \) and \( H \) be locally compact groups. Then \( SC_0(G \times H) \) becomes a Banach \( M(H) \)-\( M(G) \)-bimodule through the following convolution formulae for \( f \in SC_0(G \times H), \mu \in M(H) \), and \( \nu \in M(G) \):

\[
(\mu \cdot f)(g, h) := \int_H f(g, hk) \, d\mu(k) \quad (g \in G, h \in H)
\]

and

\[
(f \cdot \nu)(g, h) := \int_G f(kg, h) \, d\nu(k) \quad (g \in G, h \in H).
\]

The following extension of Proposition 2.4 is then routinely checked:
Proposition 3.1 Let $G$ and $H$ be locally compact groups. Then

$$SC_0(G \times H) \rightarrow L^2_w(M(G), M(H); \mathbb{C}), \quad f \mapsto \Phi f$$

as defined in Proposition 2.4 is an isometric isomorphism of Banach $M(H)$-$M(G)$-bimodules.

Proof The $M(H)$-$M(G)$-bimodule action on $SC_0(G \times H)$ induces an $M(G)$-$M(H)$-bimodule action on $SC_0(G \times H)^*$. Embedding $M(G) \hat{\otimes} M(H)$ into $SC_0(G \times H)^*$, we need to show that $M(G) \hat{\otimes} M(H)$ is a $M(G) \hat{\otimes} M(H)$-submodule of $SC_0(G \times H)^*$ such that the module actions are the canonical ones (see (1)).

Let $\kappa, \mu \in M(G)$, and let $\nu \in M(H)$. Then we have for $f \in SC_0(G \times H)$:

$$\langle f, \kappa \cdot (\mu \otimes \nu) \rangle = \langle f \cdot \kappa, \mu \otimes \nu \rangle$$

$$= \int_H \int_G \int_G f(kg, h) \, d\kappa(k) \, d\mu(g) \, d\nu(h)$$

$$= \int_H \int_G f(g, h) \, d(\kappa \ast \mu)(g) \, d\nu(h)$$

$$= \langle f, \kappa \ast (\mu \otimes \nu) \rangle.$$

An analogous property holds for the right $M(H)$-module action on $SC_0(G \times H)^*$. □

Remark It is easy to see that $C_0(G \times H)$ is a closed $M(H)$-$M(G)$-submodule of $SC_0(G \times H)$, so that $M(G \times H) \cong C_0(G \times H)^*$ is a quotient of $SC_0(G \times H)^*$. It is easily checked that

$$\mu \cdot \nu = (\mu \otimes \delta_e) \ast \nu \quad (\mu \in M(G), \nu \in M(G \times H))$$

and

$$\nu \cdot \mu = \nu \ast (\delta_e \otimes \mu) \quad (\mu \in M(H), \nu \in M(G \times H)).$$

We have the following:

Proposition 3.2 Let $G$ and $H$ be locally compact groups. Then:

(i) $M(G \times H)$ is a normal, dual Banach $M(G)$-$M(H)$-bimodule.

(ii) The map

$$M(G \times H) \rightarrow L^2_w(M(G), M(H); \mathbb{C})^*, \quad \mu \mapsto \Psi_\mu,$$  \quad (6)

as defined in Proposition 2.5, is an isometric homomorphism of Banach $M(G)$-$M(H)$-bimodules.
The maps

\[ M(G) \to M(G \times H), \quad \mu \mapsto \mu \otimes \delta_e \]

and

\[ M(H) \to M(G \times H), \quad \nu \mapsto \delta_e \otimes \nu \]

are \( w^* \)-continuous. In view of the preceding remark and the fact that \( M(G \times H) \) is a dual Banach algebra, (i) is immediate.

For (ii), let \( \mu \in M(G) \) and \( \nu \in M(G \times H) \). Then we have for any \( f \in SC_0(G \times H) \):

\[
\langle f, \mu \cdot \Psi \nu \rangle = \langle f \cdot \mu, \Psi \nu \rangle = \int_{G \times H} \int_G f(kg, h) d\mu(k) d\nu(g, h) = \int_{G \times H} \int_{G \times H} f(kg, k'h) d(\mu \otimes \delta_e)(k, k') d\nu(g, h) = \int_{G \times H} f(g, h) d((\mu \otimes \delta_e) \ast \nu)(g, h) = \int_{G \times H} f(g, h) d(\mu \cdot \nu)(g, h) = \langle f, \Psi_{\mu \cdot \nu} \rangle. \]

Hence, (6) is a left \( M(G) \)-module homomorphism.

Analogously, one shows that (6) is a right \( M(H) \)-module homomorphism. \( \square \)

With these preparations made, we can already give an alternative proof of [22, Proposition 5.2].

For any locally compact group \( G \), the operator \( \Delta_\ast := \Delta_\ast^{*} |_{C_0(G)} \) is given by

\[
(\Delta_\ast f)(g, h) = f(gh) \quad (f \in C_0(G), \ g, h \in G). \]

If \( G \) is compact, \( \Delta_\ast \) maps \( C_0(G) = C(G) \) into \( C(G \times G) = C_0(G \times G) \). Hence, \( \Delta_{w^*} \) drops to an \( M(G) \)-bimodule homomorphism \( \Delta_{0, w^*} : M(G \times G) \to M(G) \).

**Proposition 3.3** Let \( G \) be a compact group. Then there is a normal, virtual diagonal for \( M(G) \).

**Proof** Since \( G \) is amenable, \( M(G) \) is Connes-amenable (this is the easy direction of Theorem [5.3]).

Define a \( w^* \)-continuous derivation

\[ D : M(G) \to M(G \times G), \quad \mu \mapsto \mu \otimes \delta_e - \delta_e \otimes \mu. \]
It is immediate that $D$ attains its values in $\ker \Delta_{0,w^*}$. Being the kernel of a $w^*$-continuous $M(G)$-bimodule homomorphism between normal, dual Banach $M(G)$-bimodules, $\ker \Delta_{0,w^*}$ is a normal, dual Banach $M(G)$-bimodule in its own right. Since $M(G)$ is Connes-amenable, there is $N \in \ker \Delta_{0,w^*}$ such that

$$D\mu = \mu \cdot N - N \cdot \mu \quad (\mu \in M(G)).$$

Letting $M := \delta_e \otimes \delta_e - N$, and embedding $M$ into $SC_0(G \times G)^*$ via Proposition 3.2, we obtain a normal, virtual diagonal for $M(G)$. \phantomsection \tag*{\hfill \Box}

**Remark** The proof of Proposition 3.3, does not carry over to non-compact, locally compact groups with Connes-amenable measure algebra because, for non-compact $G$, we no longer have $\Delta_4 C_0(G) \subset C_0(G \times G)$; in fact, it is easy to see that $\Delta_4 C_0(G) \cap C_0(G \times G) = \{0\}$ whenever $G$ is not compact.

### 4 A left introverted subspace of separately $C_0$-functions

For general, possibly non-compact, locally compact groups, we need a Banach $M(G)$-bimodule that can play the rôle of $M(G \times G)$ in the proof of Proposition 3.3.

Let $G$ be a locally compact group. For a function $f : G \to \mathbb{C}$ and for $g \in G$, define functions $L_g f, R_g f : G \to \mathbb{C}$ through

$$(L_g f)(h) := f(gh) \quad \text{and} \quad (R_g f)(h) := f(hg) \quad (h \in G).$$

A closed subspace $E$ of $\ell^\infty(G)$ is called *left invariant* if $L_g f \in E$ for each $f \in E$ and $g \in G$. A left invariant subspace $E$ of $\ell^\infty(G)$ is called *left introverted* if, for each $\phi \in E^*$, the function

$$\phi \cdot f : G \to \mathbb{C}, \quad g \mapsto \langle L_g f, \phi \rangle$$

belongs again to $E$.

**Examples**

1. $\ell^\infty(G)$ is trivially left introverted.

2. $C_0(G)$ is left introverted ([14], (19.5) Lemma)).

3. The space

$$\mathcal{LUC}(G) := \{f \in C_b(G) : G \ni g \mapsto L_g f \text{ is norm continuous}\}$$

of *left uniformly continuous functions* on $G$ is left introverted ([21], (2.11) Proposition)).
If $E$ is a left introverted subspace of $\ell^\infty(G)$, then $E^*$ is a Banach algebra in a natural manner:

$$\langle \phi \ast \psi, f \rangle := \langle \psi \bullet f, \phi \rangle \quad (\phi, \psi \in E^*, f \in E).$$

In case $E = C_0(G)$, this is the usual convolution product on $M(G)$.

We now define a certain space of separately $C_0$-functions which is, as we shall see, left introverted. For any locally compact group $G$, let $G_{\text{LUC}}$ denote the character space of the commutative $C^*$-algebra $\mathcal{LUC}(G)$. The multiplication $\ast$ on $\mathcal{LUC}(G)^\ast$ turns $G_{\text{LUC}}$ into a compact semigroup with continuous right multiplication that contains $G$ as a dense subsemigroup ([1]). Also, we use $G^{\text{op}}$ to denote the same group, but with reversed multiplication.

**Definition 4.1** For locally compact groups $G$ and $H$, let

$$\mathcal{LUCS}_0(G \times H^{\text{op}}) := \{ f \in \mathcal{LUC}(G \times H^{\text{op}}) : \omega \bullet f \in SC_0(G \times H) \text{ for all } \omega \in (G \times H^{\text{op}})_{\text{LUC}} \}.$$

**Remark** If both $G$ and $H$ are compact, then $\mathcal{LUCS}_0(G \times H^{\text{op}}) = C(G \times H)$.

**Lemma 4.2** Let $G$ and $H$ be locally compact groups, let $f \in \mathcal{LUCS}_0(G \times H^{\text{op}})$, and let $h \in H$. Then $\{ L_{(g,h)}f : g \in G \}$ is relatively weakly compact.

**Proof** The claim is clear for compact $G$, so that we may suppose without loss of generality that $G$ is not compact.

By [11, Théorème 5], it is sufficient to show that $\{ L_{(g,h)}f : g \in G \}$ is relatively compact in $\mathcal{LUC}(G \times H^{\text{op}})$ with respect to the topology of pointwise convergence on $(G \times H^{\text{op}})_{\text{LUC}}$. Also, we may suppose without loss of generality that $h = e$.

Let $\hat{f} \in C((G \times H^{\text{op}})_{\text{LUC}})$ denote the Gelfand transform of $f$. The map

$$G \to \mathbb{C}, \quad g \mapsto \hat{f}((\delta_g \otimes \delta_e) \ast \omega)$$

is continuous for each $\omega \in (G \times H^{\text{op}})_{\text{LUC}}$. Let $G_\infty$ denote the one-point-compactification of $G$. Let $(g_\alpha)_{\alpha}$ be a net in $G$ with $g_\alpha \to \infty$. For any $\omega \in (G \times H^{\text{op}})_{\text{LUC}}$, we then have

$$\hat{f}((\delta_{g_\alpha} \otimes \delta_e) \ast \omega) = (\omega \bullet f)(g_\alpha, e) \to 0$$

because $\omega \bullet f \in SC_0(G \times H)$. Hence,

$$G \to \mathcal{LUC}(G \times H^{\text{op}}), \quad g \mapsto L_{(g,e)}f$$

(7)

extends as a continuous map to $G_\infty$, where $\mathcal{LUC}(G \times H^{\text{op}})$ is equipped with the topology of pointwise convergence on $(G \times H^{\text{op}})_{\text{LUC}}$. As the continuous image of the compact space $G_\infty$, the range of $\hat{f}$ is compact in the topology of pointwise convergence on $(G \times H^{\text{op}})_{\text{LUC}}$. 

\qed
Proposition 4.3 Let $G$ and $H$ be locally compact groups. Then $\mathcal{LUCSC}_0(G \times H^{\text{op}})$ is left introverted.

Proof Let $f \in \mathcal{LUCSC}_0(G \times H^{\text{op}})$, and let $\phi \in \mathcal{LUCSC}_0(G \times H^{\text{op}})^\ast$. Since $\mathcal{LUC}(G \times H^{\text{op}})$ is left introverted, it is immediate that $\phi \star f \in \mathcal{LUC}(G \times H^{\text{op}})$.

We first claim that $\phi \star f \in SC_0(G \times H)$. Fix $h \in H$; we will show that $(\phi \star f)(\cdot,h)$, i.e. the function

$$G \to \mathbb{C}, \quad g \mapsto \langle L_{(g,h)} f, \phi \rangle$$

belongs to $C_0(G)$. Since $(\phi \star f)(\cdot,h)$ is clearly continuous, all we have to show is that it vanishes at $\infty$. Suppose without loss of generality that $G$ is not compact, and let $(g_\alpha)_{\alpha}$ be a net in $G$ such that $g_\alpha \to \infty$. Let $\tau$ denote the topology of pointwise convergence on $G \times H$. It is clear that $L_{(g_\alpha,h)} f \xrightarrow{\tau} 0$. Since $\{ L_{(g,h)} f : g \in G \}$ is relatively weakly compact by Lemma 4.2, the weak topology and $\tau$ coincide on the weak closure of $\{ L_{(g,h)} f : g \in G \}$, so that, in particular, $\langle L_{(g_\alpha,h)} f, \phi \rangle \to 0$.

Analogously, one sees that $(\phi \star f)(g,\cdot) \in C_0(H)$ for each $g \in G$.

Let $\omega \in (G \times H^{\text{op}})^{\mathcal{LUC}}$. Since, by the foregoing,

$$\omega \star (\phi \star f) = (\omega \star \phi) \star f \in SC_0(G \times H),$$

it follows that $\phi \star f \in \mathcal{LUCSC}_0(G \times H^{\text{op}})$. \qed

Theorem 4.4 Let $G$ and $H$ be locally compact groups. Then we have:

(i) $\mathcal{LUCSC}_0(G \times H^{\text{op}})$ is a closed $\mathcal{M}(H)\mathcal{M}(G)$-submodule of $SC_0(G \times H^{\text{op}})$.

(ii) $\mathcal{LUCSC}_0(G \times H^{\text{op}})^\ast$ is a normal, dual Banach $\mathcal{M}(G)\mathcal{M}(H)$-bimodule.

(iii) If $H = G$, then $\Delta_*$ maps $C_0(G)$ into $\mathcal{LUCSC}_0(G \times G^{\text{op}})$.

Proof For (i), first note that it is routinely checked that $\mu \cdot f, f \cdot \nu \in \mathcal{LUC}(G \times H^{\text{op}})$ for all $f \in \mathcal{LUCSC}_0(G \times H^{\text{op}})$ and all $\mu \in \mathcal{M}(G)$ and $\nu \in \mathcal{M}(H)$. Fix $f \in \mathcal{LUCSC}_0(G \times H^{\text{op}})$, $\mu \in \mathcal{M}(G)$, $\nu \in \mathcal{M}(H)$, and let $\omega \in (G \times H^{\text{op}})^{\mathcal{LUC}}$. Since

$$\omega \star (\mu \cdot f)(g,h) = \langle \mu \cdot L_{(g,h)} f, \omega \rangle \quad ((g,h) \in G \times H^{\text{op}}),$$

an application of Lemma 4.2 as in the proof of Proposition 4.3 yields that $\omega \star (\mu \cdot f) \in SC_0(G \times H^{\text{op}})$. A similar, but easier argument yields that $\omega \star (f \cdot \nu) \in SC_0(G \times H^{\text{op}})$.

For (ii), first observe that the canonical embedding of $\mathcal{M}(G \times H^{\text{op}})$ into $\mathcal{LUC}(G \times H^{\text{op}})^\ast$ via integration is an algebra homomorphism. If we view $\mathcal{M}(G \times H^{\text{op}})$ as a $\mathcal{M}(G)\mathcal{M}(H)$-submodule of $\mathcal{LUCSC}_0(G \times H^{\text{op}})^\ast$ (through Proposition 4.2(ii)), we see routinely that

$$\mu \cdot \nu = (\mu \otimes \delta_e) \ast \nu \mid_{\mathcal{LUCSC}_0(G \times H)} \quad (\mu \in \mathcal{M}(G), \nu \in \mathcal{M}(G \times H^{\text{op}})). \quad (8)$$
Fix $\mu \in M(G)$. By (the simple direction of) Equation (19) — actually already proven in [26] —, the map
\[
\mathcal{LUC}(G \times H^{\text{op}})^* \to \mathcal{LUC}(G \times H^{\text{op}})^*, \quad \phi \mapsto (\mu \otimes \delta_e) \ast \phi \tag{9}
\]
is $w^*$-continuous. Let $\phi \in \mathcal{LUC}(G \times H^{\text{op}})^*$ be arbitrary, and choose a net $(\nu_{\alpha})_{\alpha}$ in $M(G \times H^{\text{op}})$ that converges to $\phi$ in the $w^*$-topology (the existence of such a net follows with a simple Hahn–Banach argument). Then (9) and the $w^*$-continuity of (9), yield that
\[
\mu \cdot \phi = w^*\lim_{\alpha} \mu \cdot \nu_{\alpha} = w^*\lim_{\alpha} (\mu \otimes \delta_e) \ast \nu_{\alpha} = (\mu \otimes \delta_e) \ast \phi.
\]
Let $f \in \mathcal{LUCSC}_0(G \times H^{\text{op}})$. Then we have
\[
\langle f, \mu \cdot \phi \rangle = \langle f, (\mu \otimes \delta_e) \ast \phi \rangle = \langle \phi \cdot f, \mu \otimes \delta_e \rangle \tag{10}
\]
By Proposition 4.3, $\mathcal{LUCSC}_0(G \times H^{\text{op}})$ is left introverted, so that, in particular, $\phi \cdot f \in \mathcal{SC}_0(G \times H^{\text{op}})$. Let $(\mu_{\alpha})_{\alpha}$ be a net in $M(G)$ that converges to $\mu$ in the $w^*$-topology. Then (10) yields:
\[
\lim_{\alpha} \langle f, \mu_{\alpha} \cdot \phi \rangle = \lim_{\alpha} \langle \phi \cdot f, \mu_{\alpha} \otimes \delta_e \rangle = \langle \phi \cdot f, \mu \cdot \phi \rangle.
\]
It follows that, for any $\phi \in \mathcal{LUCSC}_0(G \times H^{\text{op}})^*$, the map
\[
M(G) \to \mathcal{LUCSC}_0(G \times H^{\text{op}})^*, \quad \mu \mapsto \mu \cdot \phi
\]
is $w^*$-continuous. Noting that
\[
\phi \cdot \nu = \phi \ast (\delta_e \otimes \nu) \quad (\nu \in M(H)),
\]
we see analogously that
\[
M(H) \to \mathcal{LUCSC}_0(G \times H^{\text{op}})^*, \quad \nu \mapsto \phi \cdot \nu
\]
is $w^*$-continuous for all $\phi \in \mathcal{LUCSC}_0(G \times H^{\text{op}})^*$. This proves (ii).

Suppose that $H = G$. It is well known $C_0(G) \subset \mathcal{LUC}(G) \cap \mathcal{RUC}(G)$, where
\[
\mathcal{RUC}(G) := \{ f \in C_b(G) : G \ni g \mapsto R_g f \text{ is norm continuous} \}.
\]
Let $f \in C_0(G)$, and note that
\[
L_{(g,h)} \Delta_s f = \Delta_s (L_g R_h f) \quad ((g, h) \in G \times G^{\text{op}}).
\]
The norm continuity of \( \Delta_* \) shows that \( \Delta_*f \in \mathcal{LUC}(G \times G^{\text{op}}) \). To show that \( \Delta_*f \in \mathcal{LUCSC}_0(G \times G^{\text{op}}) \), let \( \omega \in (G \times G^{\text{op}})^{\mathcal{LUC}} \). Let \( ((g_\alpha, h_\alpha))_\alpha \) be a net in \( G \times G^{\text{op}} \) such that \( (g_\alpha, h_\alpha) \to \omega \). Passing to a subnet, we may suppose that \( (g_\alpha h_\alpha)_\alpha \) converges to some \( k \in G \) or tends to infinity. In the first case, we have

\[
(\omega \cdot \Delta_* f)(g, h) = \lim_\alpha \Delta_* f (gg_\alpha, h_\alpha h) = f(kh) \quad ((g, h) \in G \times H^{\text{op}})
\]

and in the second one

\[
(\omega \cdot \Delta_* f)(g, h) = \lim_\alpha \Delta_* f (gg_\alpha, h_\alpha h) = \lim_\alpha f (gg_\alpha h_\alpha h) = 0 \quad ((g, h) \in G \times H^{\text{op}}).
\]

In either case, \( \omega \cdot \Delta_* f \in \mathcal{SC}_0(G \times H^{\text{op}}) \) holds. This proves (iii). \( \square \)

5 Connes-amenability of \( M(G) \)

Let \( G \) be a locally compact group. As a consequence of Theorem 4.4(iii), \( \Delta_M(G) \) extends to an \( M(G) \)-bimodule homomorphism \( \Delta_{0,w^*} : \mathcal{LUCSC}_0(G \times G^{\text{op}})^* \to M(G) \):

**Proposition 5.1** Let \( G \) be a locally compact group such that \( M(G) \) is Connes-amenable. Then there is \( M \in \mathcal{LUCSC}_0(G \times G^{\text{op}})^* \) such that

\[
\mu \cdot M = M \cdot \mu \quad (\mu \in M(G)) \quad \text{and} \quad \Delta_{0,w^*}M = \delta_e.
\]

**Proof** Define a derivation

\[
D : M(G) \to \mathcal{LUCSC}_0(G \times G^{\text{op}})^*, \quad \mu \mapsto \mu \otimes \delta_e - \delta_e \otimes \mu.
\]

It is easy to see that \( D \) is \( w^* \)-continuous and attains its values in \( \ker \Delta_{0,w^*} \). Being the kernel of a \( w^* \)-continuous bimodule homomorphism, \( \ker \Delta_{0,w^*} \) is a \( w^* \)-closed submodule of the normal, dual Banach \( M(G) \)-module \( \mathcal{LUCSC}_0(G \times G^{\text{op}})^* \) and thus a normal, dual Banach \( M(G) \)-module in its own right. Since \( M(G) \) is Connes-amenable, there is thus \( N \in \ker \Delta_{0,w^*} \) such that

\[
D\mu = \mu \cdot N - N \cdot \mu \quad (\mu \in M(G)).
\]

The element

\[
M := \delta_e \otimes \delta_e - N
\]

then has the desired properties. \( \square \)

**Remark** Since \( \mathcal{LUCSC}_0(G \times G)^* \) is only a quotient of \( \mathcal{SC}_0(G \times G)^* \), Proposition 5.1 does not allow us to conclude that \( M(G) \) has a normal, virtual diagonal.
Lemma 5.2 Let $G$ and $H$ be locally compact groups. Then $\mathcal{LUCSC}_0(G \times H^{\text{op}})$ is an essential ideal of $\mathcal{LUC}(G \times H^{\text{op}})$.

Proof Let $f \in \mathcal{LUCSC}_0(G \times H^{\text{op}})$, let $F \in \mathcal{LUC}(G \times H^{\text{op}}) \subset \mathcal{LUC}(G \times H^{\text{op}})$, and let $\omega \in (G \times H^{\text{op}})\mathcal{LUC}$. Let $((g_\alpha, h_\alpha))_\alpha$ be a net in $G \times H^{\text{op}}$ converging to $\omega$. Since

$$\omega \cdot (fF) = \lim_\alpha R_{(g_\alpha, h_\alpha)}(fF) = \lim_\alpha (R_{(g_\alpha, h_\alpha)} f)(R_{(g_\alpha, h_\alpha)} F) = (\omega \cdot f)(\omega \cdot F)$$

with pointwise convergence on $G \times H$ and since $\omega \cdot F \in \mathcal{LUC}(G \times H^{\text{op}})$, it follows that $\omega \cdot (fF) \in SC_0(G \times H)$. Hence, $\mathcal{LUCSC}_0(G \times H^{\text{op}})$ is an ideal of $\mathcal{LUC}(G \times H^{\text{op}})$. Since $C_0(G \times H) \subset \mathcal{LUCSC}_0(G \times H^{\text{op}})$, it is even an essential ideal. \qed

Theorem 5.3 For a locally compact group $G$, the following are equivalent:

1. $G$ is amenable.
2. $M(G)$ is Connes-amenable.

Proof (i) $\implies$ (ii): By \cite[Theorem 2.5]{16}, $L^1(G)$ is amenable. Since $L^1(G)$ is $w^*$-dense in $M(G)$, \cite[Proposition 4.2]{22} yields the Connes-amenability of $M(G)$.

(ii) $\implies$ (i): Let $M \in \mathcal{LUCSC}_0(G \times G^{\text{op}})^*$ be as in Proposition 5.1. View $M$ as a measure on the character space of the commutative $C^*$-algebra $\mathcal{LUCSC}_0(G \times G^{\text{op}})$, so that $|M| \in \mathcal{LUCSC}_0(G \times G^{\text{op}})^*$ can be defined in terms of measure theory. It is routinely checked that $|M| \neq 0$, and

$$\delta_g \cdot |M| = |M| \cdot \delta_g \quad (g \in G). \quad (11)$$

By Lemma 5.2 $\mathcal{LUCSC}_0(G \times G^{\text{op}})$ is an essential, closed ideal in $\mathcal{LUC}(G \times G^{\text{op}})$. We may therefore view $\mathcal{LUC}(G \times G^{\text{op}})$ as a $C^*$-subalgebra of the multiplier algebra $\mathcal{M}(\mathcal{LUCSC}_0(G \times G^{\text{op}}))$. Since $\mathcal{M}(\mathcal{LUCSC}_0(G \times G^{\text{op}}))$, in turn, embeds canonically into $\mathcal{LUCSC}_0(G \times G^{\text{op}})^*$, we may view $\mathcal{M}(\mathcal{LUCSC}_0(G \times G^{\text{op}}))$ and thus $\mathcal{LUC}(G \times G^{\text{op}})$ as a $C^*$-subalgebra of $\mathcal{LUCSC}_0(G \times G^{\text{op}})^*$, so that, in particular, $\langle f, |M| \rangle$ is well-defined for each $f \in \mathcal{LUC}(G \times G^{\text{op}})$. Note that the embedding of $\mathcal{LUC}(G \times G^{\text{op}})$ into $\mathcal{LUCSC}_0(G \times G^{\text{op}})^*$ is an $M(G)$-bimodule homomorphism (where the $M(G)$-bimodule action on $\mathcal{LUC}(G \times G^{\text{op}})$ is defined as on $SC_0(G \times G)$). Define

$$m: \mathcal{LUC}(G) \to \mathbb{C}, \quad f \mapsto \langle f \otimes 1, |M| \rangle.$$ 

Since $f \otimes 1 \in \mathcal{LUC}(G \times G^{\text{op}})$ for each $f \in \mathcal{LUC}(G)$, $m$ is a well-defined, positive, linear
functional. For $f \in \mathcal{LUC}(G)$ and $g \in G$, we have:

$$\langle L_{g} f, m \rangle = \langle L_{(g,e)} (f \otimes 1), |M| \rangle = \langle f \otimes 1, \delta_{g} \cdot |M| \rangle = \langle f \otimes 1, |M| \rangle = \langle f \otimes 1, |M| \cdot \delta_{g} \rangle = \langle L_{(e,g)} (f \otimes 1), |M| \rangle = \langle f \otimes 1, |M| \rangle = \langle f, m \rangle.$$ 

Normalizing $m$, we thus obtain a left invariant mean on $\mathcal{LUC}(G)$. Hence, $G$ is amenable by [23, Theorem 1.1.9].

We believe that assertions (i) and (ii) in Theorem 5.3 are equivalent to:

(iii) $M(G)$ has a normal virtual diagonal.

Although we have been unable to prove this, Proposition 3.3 as well as the following corollary support this belief:

**Corollary 5.4** Let $G$ be a discrete group. Then the following are equivalent:

(i) $G$ is amenable.

(ii) $\ell^{1}(G)$ is Connes-amenable.

(iii) There is a normal, virtual diagonal for $\ell^{1}(G)$.

**Proof** (i) $\Longrightarrow$ (iii): If $G$ is amenable, $\ell^{1}(G)$ is amenable, so that there is a virtual diagonal $M \in (\ell^{1}(G) \hat{\otimes} \ell^{1}(G))^{**}$ for $\ell^{1}(G)$. Let $\rho : (\ell^{1}(G) \hat{\otimes} \ell^{1}(G))^{**} \to \mathcal{L}_{\text{w}^{*}}^{2}(\ell^{1}(G), \mathbb{C})^{*}$ denote the restriction map. Then $\rho(M)$ is a normal, virtual diagonal for $\ell^{1}(G)$.

Since (i) $\iff$ (ii) by Theorem 5.3, and since (iii) $\implies$ (ii) for any dual Banach algebra, this proves the corollary.

**Remark** Since discrete groups are trivially inner amenable, the equivalence of (i) and (ii) in Corollary 5.4 can alternatively be deduced from [20]: If $\ell^{1}(G)$ is Connes-amenable, then so is $\text{VN}(G)$ by [22, Proposition 4.2], which, by [20], establishes the amenability of $G$.

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