Differential dissipativity analysis of reaction-diffusion systems

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Abstract
This note shows how classical tools from linear control theory can be leveraged to provide a global analysis of nonlinear reaction-diffusion models. The approach is differential in nature. It proceeds from classical tools of contraction analysis and recent extensions to differential dissipativity.

Keywords: Differential analysis, reaction-diffusion systems, dominance theory, spatial homogeneity.

1. Introduction
Reaction-diffusion equations are broadly used for modeling the spatio-temporal evolution of processes appearing in many fields of science such as propagation of electrical activity on cells in cellular biology; reactions between substances on active media in chemistry; transport phenomena in semiconductor devices in electronics; and combustion processes and heat propagation in physics, to name a few. They have attracted recent interest in control, most notably in [2] and [1], because the close link between reaction-diffusion systems and synchronization models under diffusive coupling: the linear diffusion term in reaction-diffusion partial differential equations is the continuum limit of the diffusive (or incrementally passive) interconnection a network of agents sharing the same reaction dynamics. In that sense, the results in [2] and [1] are infinite dimensional generalizations of classical finite dimensional results pertaining to synchronization [19, 22, 23, 26]. Our contribution in the present note is twofold. First, we model reaction diffusion systems as the interconnection of a linear spatially and time-invariant (LTSI) model with a static nonlinearity. This natural decomposition calls for a linear control theory in the analysis and design of reaction-diffusion systems. Our observation is that linear control theory can be leveraged to provide a global analysis of nonlinear reaction-diffusion systems as the interconnection of a linear system and a static nonlinearity: the linear differential dissipativity theory [9] to characterize the attractor of two classical reaction-diffusion models: Nagumo model of bistability [17], and Fitzugh-Nagumo model of oscillation [8].

Some notation
Let \( L^2_n(\Omega) \) denote the Hilbert space of square integrable functions mapping \( \Omega \subset \mathbb{R} \) to \( \mathbb{R}^n \) with the conventional inner product \( \langle x,y \rangle_{L^2_n(\Omega)} = \int_{\Omega} x(\theta)^\top y(\theta) d\theta \) and norm denoted by \( \| \cdot \|_{L^2_n(\Omega)} \). When clear from the context, we will drop the subindex. For vector \( \xi, \psi \) in \( \mathbb{R}^n \), the inner product is denoted as \( \xi^\top \psi \) and the associated norm as \( \| \cdot \| \). The set \( \mathbb{C}_+ := \{ a + jb \in \mathbb{C} | a \geq 0 \} \) denotes the set of complex numbers with non-negative real part, whereas \( \mathbb{R}_+ \) denotes the set of non-negative real numbers. A symmetric, positive (semi-) definite matrix \( \Pi \) is denoted as \( (\Pi \geq 0) \Pi > 0 \), whereas, \( I_n \) represents the identity matrix of dimension \( n \).

2. Reaction-diffusion systems
We consider reaction-diffusion systems regarded as the feedback interconnection of a linear system and a static nonlinearity:

\[
\begin{align}
\Sigma : \quad & \frac{\partial}{\partial t} \phi(\theta,t) = D \nabla^2 x(\theta,t) + Ax(\theta,t) + Bu(\theta,t) \quad (1a) \\
& y(\theta,t) = C x(\theta,t) \quad (1b) \\
& u(\theta,t) = -\varphi(y(\theta,t))
\end{align}
\]

We denote by \( x(\theta,t) \in \mathbb{R}^n \) the state of the system at position \( \theta \in \Omega \subset \mathbb{R} \) and time \( t \geq 0 \), whereas \( u(\theta,t) \in \mathbb{R}^m \),
$y(\theta,t) \in \mathbb{R}^m$ are the distributed input and output, respectively. For simplicity we consider the spatial domain $\Omega \subset \mathbb{R}$ as the boundary of the unit circle $\partial \Omega$. Thus, \( \theta \in [0,2\pi] \) and we have the following periodic boundary conditions

$$x(0,t) = x(2\pi,t) \quad (2a)$$

$$\frac{\partial x}{\partial \theta}(0,t) = \frac{\partial x}{\partial \theta}(2\pi,t) \quad (2b)$$

Spatial diffusion is modelled via the matrix \( D \in \mathbb{R}^{n \times n} \) which is symmetric and positive definite and the Laplace operator \( \nabla^2 : \text{Dom}(\nabla^2) \subset L^2_0(\Omega) \rightarrow L^2_0(\Omega) \) with domain

$$\text{Dom}(\nabla^2) = \{ x(\cdot,t) \in H^2(\Omega;\mathbb{R}^n) \mid (2) \text{ holds} \}. \quad (3)$$

Here \( H^2(\Omega;\mathbb{R}^n) = H^2(\Omega) \times \cdots \times H^2(\Omega) \) denotes the Sobolev space of functions in \( L^2_0(\Omega) \) such that the \( i \)-th component \( x_i(\cdot,t) \) is differentiable (in the generalized sense) with derivatives in \( L^2(\Omega) \). The static nonlinearity \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m \) is assumed to be continuously differentiable (i.e., \( \varphi \in C^1(\mathbb{R}^n) \)) and satisfies the following standing assumption.

**Assumption 2.1.**

1. There exists \( 0 < M < \infty \) such that \( \eta^\top \varphi(\eta) < 0 \) for all \( \eta \in \mathbb{R}^m \) for all \( \|\eta\| \geq M \).
2. The nonlinear function \( \varphi : \mathbb{R}^m \rightarrow \mathbb{R}^m \) satisfies the differential dissipation inequality

$$\left[ \begin{array}{cc} I_m & -J_{\varphi}(\eta) \\ -J_{\varphi}(\eta) & L \end{array} \right] \left[ \begin{array}{c} Q \\ R \end{array} \right] \left[ \begin{array}{c} I_m \\ -J_{\varphi}(\eta) \end{array} \right] \leq 0 \quad (4)$$

for all \( \eta \in \mathbb{R}^m \), where \( J_{\varphi}(\eta) \in \mathbb{R}^{m \times m} \) denotes the Jacobian matrix of \( \varphi \) and the matrices \( Q, R \in \mathbb{R}^{m \times m} \) are constant.

The dissipation inequality (4) is a classical differential sector condition. In the scalar case \( (m = 1) \), it reduces to

$$\left( J_{\varphi}(\eta) - K_1 \right) \left( J_{\varphi}(\eta) - K_2 \right) \preceq 0 \quad (5)$$

with \( Q = \frac{1}{2} \left( K_1^\top K_1 + K_2^\top K_2 \right), \quad L = \frac{1}{2} \left( K_1^\top + K_2^\top \right) \) and \( R = I_n \). Condition (5) then expresses that the slope of \( \varphi \) at any point lies in the interval \( [K_1, K_2] \), whenever \( K_1 < K_2 \). See Figure 1 for an illustration. The reader will note that model (1) reduces to

$$\frac{\partial x}{\partial t}(\theta,t) = D\nabla^2 x(\theta,t) - \varphi(x(\theta,t)) \quad (6)$$

in the special case defined by \( m = n, \quad A = 0, \quad B = C = I_n \). This latter form is the classical form of a reaction-diffusion system in the literature [20].

**Remark 2.2.** Assumption 2.1 ensures that the system (1)-(2) admits a unique (classical) solution for any initial condition \( x(\theta,0) = x_0(\theta) \) which is defined in the whole time interval \( t \in [0, +\infty) \), given as

$$x(\theta,t) = T(t)x_0(\theta) + \int_0^t T(t-\tau)F(x(\theta,\tau))d\tau$$

where \( T(t) : L^2_0(\Omega) \rightarrow L^2_0(\Omega) \) is the \( C_0 \)-semigroup generated by the operator \( D\nabla^2 \) and \( F(x(\theta,t)) = Ax(\theta,t) + B\varphi(Cx(\theta,t)) \). See e.g., [18, Theorems 1.4 and 1.5, Chapter 6].

3. Differential analysis of reaction diffusion systems

Differential analysis consists in analyzing the properties of infinitesimal variations \( \delta x(\theta,t) \) around an arbitrary solution \( x(\theta,t) \) of (1). Such variations satisfy the variational equation [6]

$$\delta \Sigma : \begin{cases} \delta x(\theta,t) = D\nabla^2 \delta x(\theta,t) + A\delta x(\theta,t) + B\delta u(\theta,t) \\ \delta y(\theta,t) = C\delta x(\theta,t) \\ \delta u(\theta,t) = -J_f(y(\theta,t))\delta y(\theta,t) \end{cases} \quad (7)$$

with boundary conditions

$$\delta x(0,t) = \delta x(2\pi,t) \quad (8a)$$

$$\frac{\partial \delta x}{\partial \theta}(0,t) = \frac{\partial \delta x}{\partial \theta}(2\pi,t) \quad (8b)$$

The variational system is linear. It is the interconnection of the same LTSI model (1) with a time-varying output feedback gain evaluated along an arbitrary solution \( x(\theta,t) \). In the following subsections we focus on the analysis of the differential model (7)-(8). We analyze spatial and temporal variations separately.

3.1. Differential spatial dynamics

The spatial infinitesimal variation of the solution \( x(\theta,t) \) at time \( t \) is

$$\lim_{\Delta \theta \rightarrow 0} \frac{x(\theta + \Delta \theta,t) - x(\theta,t)}{\Delta \theta} = \nabla x(\theta,t)$$

Hence, the gradient vector is solution of the variational dynamics. In addition, it satisfies the integral constraint

$$\frac{1}{2\pi} \int_0^{2\pi} \delta x(\theta,t) d\theta = 0. \quad (9)$$

which simply rewrites (2a) in terms of \( \delta x = \nabla x \). We refer to the subsystem (7)-(8) as the differential spatial dynamics. Notice that condition (4) imposes an orthogonality condition in the space \( L^2_0(\Omega) \). Namely, let \( T : L^2_0(\Omega) \rightarrow L^2_0(\Omega) \) be the bounded linear operator mapping

$$\delta x \mapsto \int_\Omega \delta x(\theta,t)d\theta := \delta \bar{x} \quad (10)$$

Thus, \( T \) is a projection operator onto the space of constant (in space) functions. Moreover, the space \( L^2_0(\Omega) \) is decomposed as

$$L^2_0(\Omega) = W \oplus W^\perp$$
where $W := R(T)$ and $W^\perp := N(T)$, the range and the null space of the operator $T$, respectively. Hence, any solution of the differential dynamics (7)–(9) has the decomposition $\delta x = (I - T)\delta x + T\delta x$. It is straightforward to verify that indeed $(I - T)\delta x = \nabla x \in W^\perp$ and $T\delta x = \delta x \in W$. We recall the definition of spatial homogeneity introduced in [1].

Definition 3.1 (Spatial homogeneity [1]). The system (1)-(2) is spatially homogeneous with rate $\mu > 0$ if for any given initial condition $x(0) = x_0$,

$$\|\nabla x(\cdot, t)\|_{L^2(\Omega)} \leq M e^{-\mu t} \|\nabla x(\cdot, 0)\|_{L^2(\Omega)},$$  

where $M > 0$.

Spatial homogeneity is thus equivalent to contraction of the spatial differential dynamics.

Proposition 3.2. System (1)-(2) is spatially homogeneous with rate $\mu > 0$, if and only if, the origin of the system (7)-(8)-(9) is uniformly exponentially stable with the same rate $\mu > 0$.

Proof. The proof is a direct consequence of Definition 3.1 and the fact that $\delta x = \nabla x$ is the solution of (7)-(9).

Conditions guaranteeing the exponential homogeneity of (1)-(2) have been studied extensively [1, 2, 4, 11]. The dissipativity formulation of those conditions is as follows. Let $\sigma : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}$ be the quadratic differential supply rate

$$\sigma(\delta y, \delta u) := \begin{bmatrix} \delta y(\theta, t) \\ \delta u(\theta, t) \end{bmatrix}^\top \begin{bmatrix} Q & L \\ L^\top & R \end{bmatrix} \begin{bmatrix} \delta y(\theta, t) \\ \delta u(\theta, t) \end{bmatrix},$$

where the matrices $Q, L, R$ are constant.

Definition 3.3. The LTSI system (11)-(12) is uniformly differential dissipative with rate $\mu > 0$ and with respect to the supply rate (12), if there exists a matrix $\Pi = \Pi^\top > 0$ such that the following inequality holds for all admissible $\delta u$ with $(\delta x, \delta y)$ satisfying (11)-(12):

$$\int_{\Omega} \begin{bmatrix} \frac{\partial}{\partial \theta} \delta x \\ \Pi \delta x \end{bmatrix}^\top \begin{bmatrix} 0 & 0 \\ \Pi & 2\mu \Pi + \varepsilon I_n \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \theta} \delta x \\ \Pi \delta x \end{bmatrix} d\theta \leq -\int_{\Omega} \sigma(\delta y, \delta u) d\theta.$$  

Additionally, if (13) holds in an invariant subspace $\mathcal{V} \subset L^2(\Omega)$ of $\delta x$ then we say that the system is uniformly differential dissipative in $\mathcal{V}$.

Henceforth, dissipativity is always assumed with respect to the supply rate (12). With those definitions in place, the dissipativity analysis of spatial homogeneity of (1)-(2) is an infinite-dimensional version of the classical circle criterion.

Theorem 3.4. Let $\varphi$ satisfy the dissipation inequality (4).

If the LTSI system (11)-(12) is uniformly differential dissipative with rate $\mu > 0$ in $W^\perp$, then the closed-loop system (1)-(2) is spatially homogeneous with the same rate $\mu$.

Proof. Let $S(\delta x) = (\delta x(\cdot, t), \Pi \delta x(\cdot, t))_{L^2(\Omega)}$, where $\Pi = \Pi^\top > 0$ satisfies (13). Then (13) is equivalent to

$$\frac{d}{dt} S(\delta x) \leq \int_{\Omega} \sigma(\delta y, \delta u) d\theta - 2\mu S(\delta x) - \varepsilon \|\delta x(\cdot, t)\|_{L^2(\Omega)}.$$  

Using $\delta u(\theta, t) = -J_s(y(\theta, t)) \delta y(\theta, t)$ and the sector bound (4) leads to

$$\frac{d}{dt} S(\delta x) \leq -2\mu S(\delta x) - \varepsilon \|\delta x(\cdot, t)\|_{L^2(\Omega)}$$  

Now, multiplying both sides of (14) by $\varepsilon^2 \mu^2$ and integrating from $\tau = 0$ up to $\tau = t$, yields

$$\|\delta x(\cdot, t)\|_{L^2(\Omega)} \leq \sqrt{\frac{\lambda_{\text{max}}(\Pi)}{\lambda_{\text{min}}(\Pi)}} \varepsilon \mu^2 \|\delta x(\cdot, 0)\|_{L^2(\Omega)}$$  

where $\lambda_{\text{max}}(\Pi)$ and $\lambda_{\text{min}}(\Pi)$ denote the maximum and minimum eigenvalue of $\Pi$. The result follows from the identity $\delta x = \nabla x$ in $W^\perp$.

The following theorem provides a numerical test for uniform differential dissipativity of the LTSI system (11)-(12).

Theorem 3.5. Let $\Pi \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix such that

$$\Pi D + D^\top \Pi \geq 0$$  

and

$$\begin{bmatrix} \Theta_{1,1} & \Pi B - C^\top L \\ B^\top \Pi - L^\top C & -R \end{bmatrix} \succeq 0,$$  

where $\Theta_{1,1} = (A - \lambda_2 D)^\top \Pi (A - \lambda_2 D) + 2\Pi B - C^\top Q C + \varepsilon I_n$. Then the LTSI system (11)-(12) is uniformly differential dissipative with rate $\mu \geq 0$ in $\mathcal{V}^\perp$, the orthogonal complement of $\mathcal{V} \doteq \text{span}\{\nu_1\}$, where $\lambda_2, \nu_2$ are the $q$-th eigenvalue and eigenvector of the operator $\mathcal{V}^2$ with domain (3), respectively.

Proof. The reader is addressed to [2] for a detailed proof of this fact.

The differential spatial dynamics (12) implies that $W^\perp = \text{span}\{\nu_1\}$ and therefore uniform differential dissipativity of (11)-(12) in $W^\perp$ is tested by solving (15)-(16) with $\lambda_2 = 1$.

3.2. Differential temporal dynamics

The differential dynamics (7)-(8) constrained to $W^\perp$ describes the temporal evolution of variations in space. The complementary dynamics of the differential system constrained to $W$, i.e., $T \delta x$, describes the average behavior of the differential dynamics.
**Theorem 3.6.** The dynamics of (1)-(3) in the invariant set $W$ reduce to
\[
\begin{align*}
\frac{d}{dt}\delta x(t) &= A\delta x(t) + B\delta u(t) \\
\delta y(t) &= C\delta x(t) \\
\delta u(t) &= -J_\varphi(\delta y(t))
\end{align*}
\] (17a)

where $\delta y(t)$ satisfies
\[
\begin{align*}
\frac{d}{dt}\delta y(t) &= A_\delta x(t) + B\delta u(t) \\
y(t) &= C_\delta x(t) \\
\dot{u}(t) &= -\varphi(\delta y(t))
\end{align*}
\] (17b)

Proof. We first compute the dynamics of $T\delta x = \delta \dot{x}$. Thus applying the projection $T$ in (10) on both sides of (13) yields
\[
\begin{align*}
\frac{d}{dt}\delta x(t) &= A_\delta x(t) + B\delta u(t) \\
\delta y(t) &= C_\delta x(t)
\end{align*}
\] (19)

where $\delta x$ denotes the average in space of the differential variable $\delta x$, that is
\[
\delta x(t) = \int_\Omega \delta x(\theta, t)d\theta,
\]
similar for $\delta y$, and $\delta u$. We recall that $\delta x \in W$ implies $\nabla x = 0$, since $\nabla x$ is indeed in $W_{\perp}$. This last fact means that $x$ is independent of the spatial coordinate. Hence, for $\delta x \in W$, we also have that $\nabla^2 x = 0$. It thus follows that (11)-(2) reduces to the ODE (18). Finally, the feedback control (16) projects into
\[
\delta u(t) = -\int_\Omega J_\varphi(\delta y(t))\delta y(\theta, t)d\theta = -J_\varphi(\delta y(t))
\]

The above result agrees with the traditional approach of [3,11] in which spatial homogeneity reduces a PDE into an ODE. Thus, (17) describes the differential dynamics of the homogeneous behavior which we identify as the differential temporal dynamics. We now illustrate the use of differential dissipativity analysis to study non-equilibrium asymptotic behaviors of the temporal dynamics. We make use of a recent development of the theory in [9,17,16,15]. We recall that the inertia of a symmetric matrix $P \in \mathbb{R}^{n \times n}$ is the triple $(\nu, \zeta, \pi)$ where each entry denotes the number of negative, zero and positive eigenvalues, respectively.

**Definition 3.7.** The linear system (18a) is $p$-dissipative with rate $\lambda \geq 0$ if there exists a symmetric matrix $P = P^T$ with inertia $(p, 0, n-p)$ such that for all admissible $\delta u$ and all $(\delta x, \delta y)$ satisfying (17) the following holds
\[
\begin{bmatrix}
\frac{\delta x}{\delta x}^T \\
\frac{\delta y}{\delta u}
\end{bmatrix} \begin{bmatrix}
0 & P \\
P & 2\lambda P + \varepsilon I_n
\end{bmatrix} \begin{bmatrix}
\frac{\delta x}{\delta x} \\
\frac{\delta y}{\delta u}
\end{bmatrix} \leq
\begin{bmatrix}
\delta y^T \\
\delta u
\end{bmatrix} \begin{bmatrix}
Q & L \\
L^T & R
\end{bmatrix} \begin{bmatrix}
\delta y \\
\delta u
\end{bmatrix}
\]

The following theorem, taken from [17] and repeated here for completeness, provides useful information for characterizing the homogenous part of the asymptotic behavior.

**Theorem 3.8.** Let $\varphi$ satisfy the dissipation inequality (1). If the LTSE system (1a) is strictly $p$-dissipative with rate $\lambda \geq 0$ in $W$. Then the homogeneous dynamics of the closed-loop (1)-(2) is $p$-dominant. In particular, each bounded solution asymptotically converges to an equilibrium for $p = 1$ and to a simple limit set (equilibrium, closed orbit, or connected arc of equilibria) for $p = 2$.

Proof. The homogeneous dynamics of the closed-loop (1)-(2) is given by (18), which is a lumped Lur’e system, Theorem 3.6. The result thus follows from [15, Theorem 4.2].

It follows from Definition 3.7 that in $W$ (1a) is strictly $p$-dissipative with rate $\lambda \geq 0$ if there exist $\varepsilon > 0$ and a matrix $P = P^T$ with inertia $(p, 0, n-p)$ satisfying
\[
\begin{bmatrix}
\hat{\Theta}_{1,1} & PB - C^TL \\
B^TP - L^TC & -R
\end{bmatrix} \preceq 0
\]

where $\hat{\Theta}_{1,1} = A^TP + PA + 2\lambda P - C^TQC + \varepsilon I_n$. In this way, the differential model (17)-(18) contains all of the information needed for the study of the global behavior of (11)-(2).

**Example 3.9.** We illustrate the above analysis with an application to the Nagumo model describing the spatio-temporal dynamics of a bistable transmission line [17],
\[
\frac{\partial x}{\partial t}(\theta, t) = D\nabla^2 x(\theta, t) + Ax(\theta, t) - \varphi(x(\theta, t))
\]

where $x(\theta, t) \in \mathbb{R}$, $D > 0$, and $\varphi : \mathbb{R} \to \mathbb{R}$ is an “N-shape” function as the one shown in Figure 1. Thus, $\varphi$ satisfies (3) for some $K_1 < 0 < K_2$. The boundary conditions are the same as in (2). In this example, condition (15) reduces to $\Pi > 0$ and by using Schur’s complement formula it follows that (16) is equivalent to the condition,
\[
\Pi^2 + 2\left(A + \mu - D - \frac{K_1 + K_2}{2}\right)\Pi + \left(K_1 - K_2\right)^2 < 0
\]
Straightforward computations reveal that uniform differential dissipativity of the LTSI dynamics \((1a)\) in \(W^\perp\) and with rate at least \(\mu\) is guaranteed whenever

\[
D > A + \mu - K_1
\]

which implies spatial homogeneity of the closed-loop \((1)-(2)\) according to Theorem \[5\]. Now, the complementary dynamics in \(W\) is given by \((18)\), whose dissipativity property is verified by \((21)\), which in this case reduces into

\[
P^2 + 2 \left( A + \lambda - \frac{K_1 + K_2}{2} \right) P + \frac{(K_1 - K_2)^2}{4} < 0 \quad (24)
\]

It is easy to verify that if \(K_1 > A\) then \((21)\) admits a positive solution \(P > 0\), that is, the LTSI system \((1a)\) is \(0\)-dissipative in \(W\) with rate \(0 < \lambda < K_1 - A\). In such case, there is a unique equilibrium for \((1)-(2)\) that is globally asymptotically stable, that is, the complete spatio-temporal behavior goes towards the unique equilibrium. On the other hand, if \(A > K_1\), then \((21)\) admits a negative solution \(P < 0\), that is, the LTSI system \((1a)\) is \(1\)-dissipative in \(W\) with positive rates \(\lambda\) satisfying \(\lambda > K_2 - A\). Further, from a conventional local stability analysis one gets that the origin of the dynamics in \(W\) is unstable whenever \(A > K_1\). Thus, when condition \((23)\) and \(A > K_1\) hold, then the PDE \((22)\) will have a homogeneous bistable behavior. Figure 3 shows the spatio-temporal evolution of the system to two different initial conditions with the following parameters \(A = 0\), \(D = 1.1\), \(K_1 = -1\), and \(K_2 = 1\).

![Figure 2: Spatio-temporal evolution of trajectories of Nagumo’s equation (22) to two different initial conditions showing both the spatial homogeneity and the bistable nature of the transmission line.](image)

4. Analysis in the frequency domain

The linear system \((1)-(2)\) is both spatially and time invariant (LTSI): solutions shifted in time and in space satisfy the same equation \([3]\). Spatial and time invariance properties of linear systems allow for insightful frequency domain analysis. In this section, we briefly illustrate the frequency-domain interpretation of the results of the previous sections.

4.1. Differential spatial dynamics

Spatial invariance allows to analyze a linear PDE as a family of ODEs parametrized by the spatial frequency \(\zeta\). By taking the Fourier transform of \((1a)\) with respect to the spatial variable \(\theta\), we transform the PDE \((1a)\) into the family of linear systems

\[
\begin{align*}
\delta_x \dot{x}_\zeta(t) &= (A - \zeta^2 D) \delta x_\zeta(t) + B \delta u_\zeta(t) \\
\delta y_\zeta(t) &= C \delta x_\zeta(t)
\end{align*}
\]

where, for the case of \(\Omega = \partial D\), \(\zeta \in \mathbb{Z}\) (the dual group to \(\partial D\)). Notice that each \(\delta x_\zeta(t)\), \(\zeta \in \mathbb{Z}\), is a coefficient on the Fourier series expansion of \(\delta x(t)\). The splitting between spatial and temporal differential dynamics in the previous section has an obvious interpretation in the frequency domain: \((26)\) reduces to the differential temporal dynamics for the uniform spatial mode, that is \(\zeta = 0\), whereas the differential spatial dynamics correspond to all other modes \(\zeta \in \mathbb{Z} \setminus \{0\}\). The following theorem provides sufficient conditions that guarantee the spatial homogeneity of the closed-loop \((1)-(2)\) via the family of ODEs \((22)\).

**Theorem 4.1.** Suppose that for each \(\zeta \in \mathbb{Z} \setminus \{0\}\), the linear system \((25)\) is \(0\)-dissipative with rate \(\mu \geq 0\) and with the same storage function \(S(\delta x_\zeta) = \delta x_\zeta \Pi \delta x_\zeta\). Then the closed-loop \((1)-(2)\) is spatially homogeneous with the same rate \(\mu\).

**Proof.** The hypothesis on the family of systems \((25)\) is equivalent to the existence of a matrix \(\Pi = \Pi^\top \geq 0\) satisfying the following family of parametrized LMIs

\[
\Theta_\zeta(\Pi) := \begin{bmatrix}
\Theta_{1,1}(\zeta) \\
B^\top \Pi - L^\top C
\end{bmatrix}
\]

\[\Pi B - C^\top L \leq 0 \quad (26)\]

were \(\Theta_{1,1}(\zeta) = (A - \zeta^2 D)^\top \Pi + \Pi (A - \zeta^2 D) + 2 \mu \Pi + \epsilon I_n\). The rest of the proof consists in showing that \((27)\) is equivalent to conditions \((15)-(16)\). To that end, let \(\tau = \frac{\pi}{\zeta} \in (0,1]\). It then follows that condition \((26)\) holds for all \(\zeta \in \mathbb{Z} \setminus \{0\}\) if and only if

\[
\tau \left( B^\top \Pi - L^\top C \right) \leq 0 \quad (27)
\]

holds for all \(\tau \in (0,1]\), where \(\hat{\Theta}_{1,1}(\tau) = (\tau A - D)^\top \Pi + \Pi (\tau A - D) - \tau (C^\top QC + 2 \mu \Pi + \epsilon I_n)\). Now, let us assume first that \((27)\) holds. Thus, setting \(\tau = 1\) in \((27)\) implies \((16)\). Next, a necessary condition for \((27)\) to hold is

\[-D^\top \Pi - \Pi D + (A^\top \Pi + \Pi A - C^\top QC + 2 \mu \Pi + \epsilon I_n) \leq 0\]

for all \(\tau \in (0,1]\). Such condition is possible only if \((15)\) holds. The converse statement follows directly by noting

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there exists $\mu \geq \Pi = \Pi$ varying gains $\delta x = -\dot{J}_c(t)\delta x$ satisfying

$$\begin{bmatrix} I_m & -\dot{J}_c(t) \end{bmatrix}^T Q L R \begin{bmatrix} I_m \\ -\dot{J}_c(t) \end{bmatrix} \leq 0. \tag{28}$$

For each value of $\zeta$, the storage $S(\delta x) = \delta x^T \Pi \delta x$, where $\Pi = \Pi \geq 0$, satisfies

$$\frac{d}{dt} S(\delta x) = \delta x^T \Pi \delta x,$$

and the application of the S-procedure yields $\Pi$ as a sufficient condition for the uniform exponential stability of the family of closed-loops. It is worth stressing that in general $\dot{J}_c(t)$ is not the spatial Fourier transform of the term $\delta x^T \Pi \delta x$ in (27).

In the previous subsection we analyzed spatial homogeneity via the LMIs (15)-(19). The analysis in the spatial frequency domain in this section provides an alternative: because the Fourier transform is an isometry between $L^2_\alpha(\Omega)$ and $L^2_\alpha(\mathbb{Z})$, it is sufficient to show that the dynamics of each Fourier coefficient, given by (25), converges exponentially to zero with rate at least $\mu$ for each $\zeta \in \mathbb{Z} \setminus \{0\}$. Additionally, Parseval’s theorem \cite{12} Lemma 1.5] implies that for any quadratic function $\sigma(\delta y, \delta u)$, as defined in (12), and satisfying $\sigma(\delta y, \delta u) \leq 0$ then $\sigma(\delta y, \delta u) \leq 0$ for all $\zeta \in \mathbb{Z}$. Therefore, it is enough to verify the stability of (25) subject to the quadratic constraint $\sigma(\delta y, \delta u) \leq 0$. That is, to verify only the individual dissipativity properties of each Fourier coefficient. To this end, let us introduce the family of transfer functions associated to (25) as

$$G_\zeta(s) = C(sI - (A - \zeta^2 D))^{-1} B \tag{29}$$

where $s \in \mathbb{C}$ and $\zeta \in \mathbb{Z} \setminus \{0\}$. In the SISO case, graphical tests (circle criterion) can be derived. Let $\mathcal{D}(K_1, K_2)$ be the disk in the complex plane given by the set

$$\mathcal{D}(K_1, K_2) := \left\{ x + jy \in \mathbb{C} \mid \left(x + \frac{K_1 + K_2^2}{2K_1K_2} \right)^2 + y^2 \leq \frac{K_2 - K_1}{2K_1K_2} \right\} \tag{30}$$

**Theorem 4.3.** Let $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ be such that it satisfies the differential sector condition (5). If for each $\zeta \in \mathbb{Z} \setminus \{0\}$ there exists $\mu \geq 0$ such that

1. $G_\zeta(s - \mu)$ has no poles on the closure of $\mathbb{C}_+$;
2. one of the following conditions is satisfied
   a) $0 < K_1 < K_2$ and the Nyquist plot of $G(s - \mu)$ lies outside the disk $\mathcal{D}(K_1, K_2)$.
   b) $K_1 < 0 < K_2$ and the Nyquist plot of $G(s - \mu)$ lies inside the disk $\mathcal{D}(K_1, K_2)$.
   c) $K_1 < K_2 < 0$ and the Nyquist plot of $G(s - \mu)$ lies outside the disk $\mathcal{D}(K_1, K_2)$.

Then the closed-loop (11)-(2) is spatially homogeneous with rate $\mu$.

**Proof.** The proof is the same as in the standard circle criterion, see e.g., [10, 13].

**Remark 4.4.** It is worth to stress that in Theorem 4.3 we have disregarded the cases in which the Nyquist plot make encirclements of any given disk.

### 4.2. Differential temporal dynamics

The second part of the analysis concerns the asymptotic behavior of the model (15), which is finite dimensional. In such case the frequency domain approach is explored in [15], where sufficient conditions are guaranteed. The analysis is now centered around the feedback interconnection of (25) with $\zeta = 0$ and a nonlinear term $\varphi : \mathbb{R}^m \to \mathbb{R}^m$ satisfying (5). For the sake of completeness we state the main result for the case of SISO systems, whose proof can be found in [15].

**Theorem 4.5 (Extended circle criterion).** Consider the closed-loop system (18). Let $G_0(s)$ be the transfer function associated to (18) and let $\varphi : \mathbb{R} \to \mathbb{R}$ satisfy the differential sector condition (5). Then the closed-loop system (18) is p-dominant with rate $\lambda > 0$ if

1. $G_0(s - \lambda)$ has $q$ poles on the interior of $\mathbb{C}_+$ and no poles on the $\omega$-axis;
2. The Nyquist plot of $G_0(s - \lambda)$ makes $E = p - q$ clockwise encirclements of the point $-1/K_1$.
3. one of the following conditions is satisfied
   a) $0 < K_1 < K_2$ and the Nyquist plot of $G(s - \lambda)$ lies outside the disk $\mathcal{D}(K_1, K_2)$.
   b) $K_1 < 0 < K_2$ and the Nyquist plot of $G(s - \lambda)$ lies inside the disk $\mathcal{D}(K_1, K_2)$.
   c) $K_1 < K_2 < 0$ and the Nyquist plot of $G(s - \lambda)$ lies outside the disk $\mathcal{D}(K_1, K_2)$.

Theorem 4.3 gives us a sufficient condition for spatial homogeneity of reaction-diffusion systems, whereas Theorem 4.5 gives us a sufficient condition for the type of homogeneous motion.
Example 4.6. We apply our approach to the FitzHugh-Nagumo equation

\[
\begin{align*}
\frac{\partial x_1}{\partial t}(\theta,t) &= D_{1,1} \nabla^2 x_1(\theta,t) - x_2(\theta,t) + x_1(\theta,t), \\
\frac{\partial x_2}{\partial t}(\theta,t) &= D_{2,2} \nabla^2 x_2(\theta,t) + ax_1(\theta,t) - bx_2(\theta,t), \\
y(\theta,t) &= x_1(\theta,t), \\
u(\theta,t) &= -\varphi(y(\theta,t))
\end{align*}
\]

where \( \varphi : \mathbb{R} \to \mathbb{R} \) is a nonlinear “N-shape” function in the differential sector \([K_1, K_2]\), as the one shown in Figure 7. We first focus on the analysis of spatial homogeneity. The family of transfer functions \( G_\zeta(s) \) has the form

\[
G_\zeta(s) = \frac{s + \frac{1}{\varepsilon}(b + \zeta^2 D_{2,2})}{(s + \varepsilon(2D_{1,1}))(s + \frac{1}{\varepsilon}(b + \zeta^2 D_{2,2})) + \frac{1}{\varepsilon}}
\]

(32)

Now, we check the conditions stated in Theorem 4.3. Thus, if

\[
\mu < \min \left\{ D_{1,1}, \frac{1}{\varepsilon}(b + D_{2,2}) \right\}
\]

then condition 1 holds for all \( \zeta \in \mathbb{Z} \setminus \{0\} \). Setting the pa-

rameters as, \( D_{1,1} = 0.5, D_{2,2} = 0.02, \varepsilon = 0.1, a = 0.1, b = 0.05, K_1 = -1.0 \) and \( K_2 = 1.0 \), we now look for the values of \( \mu \) for which condition 2-(b) also holds. Thus, setting \( \mu = 0.01 \), we get the family of Nyquist plots depicted in Figure 9. From Figure 2, it follows that, with our choice of parameters, we can expect a rate of convergence of the synchronization error of at least \( \mu = 0.01 \). With that information on the rate \( \mu \), we verify a solution to the LMI conditions (15)-(16) and we get a positive definite solution \( \Pi \) as

\[
\Pi = \begin{bmatrix}
1.16451 & -0.61023 \\
-0.61023 & 1.1594
\end{bmatrix}
\]

which confirms the spatial homogeneity of the FitzHugh-Nagumo equation with the selected parameters. The fol-

Figure 3: Family of Nyquist plots of (32) for \( \zeta \in \mathbb{Z} \setminus \{0\} \) and with parameters \( D_{1,1} = 0.5, D_{2,2} = 0.02, \varepsilon = 0.1, a = 0.1, b = 0.05, K_1 = -1.0 \) and \( K_2 = 1.0 \), we now look for the values of \( \mu \) for which condition 2-(b) also holds. Thus, setting \( \mu = 0.01 \), we get the family of Nyquist plots depicted in Figure 9. From Figure 2, it follows that, with our choice of parameters, we can expect a rate of convergence of the synchronization error of at least \( \mu = 0.01 \). With that information on the rate \( \mu \), we verify a solution to the LMI conditions (15)-(16) and we get a positive definite solution \( \Pi \) as

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Figure 4: Nyquist plot of the transfer function \( G_0(s - \lambda) \) associated to system (15) with temporal rate \( \lambda = 0.8 \).

Figure 5: Family of Nyquist plots of (32) for \( \zeta \in \mathbb{Z} \setminus \{0\} \) and with parameters \( D_{1,1} = 0.5, D_{2,2} = 0.02, \varepsilon = 0.1, a = 0.1, b = 0.05, K_1 = -1.0 \) and \( K_2 = 1.0 \), we now look for the values of \( \mu \) for which condition 2-(b) also holds. Thus, setting \( \mu = 0.01 \), we get the family of Nyquist plots depicted in Figure 9. From Figure 2, it follows that, with our choice of parameters, we can expect a rate of convergence of the synchronization error of at least \( \mu = 0.01 \). With that information on the rate \( \mu \), we verify a solution to the LMI conditions (15)-(16) and we get a positive definite solution \( \Pi \) as

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\[
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which confirms the spatial homogeneity of the FitzHugh-Nagumo equation with the selected parameters. The fol-

5. Conclusions

We illustrated the potential of differential dissipativity analysis for the analysis of nonlinear reaction-diffusion systems. The differential dynamics naturally decompose into two components, the differential spatial dynamics and the differential temporal dynamics. We illustrated sufficient conditions for spatial homogeneity, that is, contraction of the differential spatial dynamics, and for \( p \)-differential dissipativity of the differential temporal dynamics. Future work will explore the same framework to analyze asymptotic spatiotemporal behaviors that are homogeneous neither in space nor in time. Such behaviors include traveling waves and spatiotemporal patterns.
Figure 5: Spatio-temporal evolution of state trajectories of FitzHugh-Nagumo model showing an homogeneous oscillatory behavior.

References

[1] Z. Aminzare and E. D. Sontag. Some remarks on spatial uniformity of solutions of reaction-diffusion PDEs. *Nonlinear Analysis*, 147:125–144, 2016.

[2] M. Arcak. Certifying spatially uniform behavior in reaction-diffusion PDE and compartmental ODE systems. *Automatica*, 47:1219–1229, 2011.

[3] B. Bamieh, F. Paganini, and M. A. Dahleh. Distributed control of spatially invariant systems. *IEEE Transactions on Automatic Control*, 47(7):1091–1107, 2002.

[4] E. Conway, D. Hoff, and J. Smoller. Large time behavior of solutions of systems of nonlinear reaction-diffusion equations. *SIM Journal of Applied Mathematics*, 35(1):1–16, 1978.

[5] J. B. Conway. *A Course in Functional Analysis*. Graduate Texts in Mathematics. Springer, 1990.

[6] P. E. Crouch and A. J. van der Schaft. *Variational and Hamiltonian Control Systems*. Springer-Verlag, Berlin, Germany, 1987.

[7] R. Curtain, O. V. Iftime, and H. Zwart. System theoretic properties of a class of spatially invariant systems. *Automatica*, 45:1619–1627, 2009.

[8] R. FitzHugh. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal*, 1(6):445–466, 1961.

[9] F. Forni and R. Sepulchre. Differential dissipativity theory for dominance analysis. *IEEE Transactions on Automatic Control*, 64(6):2340–2351, 2019.

[10] W M Haddad and V Chellaboina. *Nonlinear dynamical systems and control: a Lyapunov-based approach*. Princeton University Press, USA, 2008.

[11] J. K. Hale. Large diffusivity and asymptotic behavior in parabolic systems. *Journal of Mathematical Analysis and Applications*, 118:455–466, 1986.

[12] J. Keener and J. Sneyd. *Mathematical Physiology I: Cellular Physiology*. Springer, New York, USA, 2009.

[13] H. K. Khalil. *Nonlinear Systems*. Prentice Hall, 3rd edition, 2002.

[14] Y. Kuramoto. *Chemical oscillations, waves and turbulence*. Springer-Velag, Berlin, 1984.

[15] F. A. Miranda-Villatoro, F. Forni, and R. Sepulchre. Analysis of Lur’e dominant systems in the frequency domain. *Automatica*, 98:76–85, 2018.

[16] F. A. Miranda-Villatoro, F. Forni, and R. Sepulchre. Dissipativity analysis of negative resistance circuits. Preprint: https://arxiv.org/abs/1908.11193, 2020.

[17] J. Nagumo, S. Yoshizawa, and S. Arimoto. Bistable transmission lines. *IEEE Transactions on Circuit Theory*, 12(3):400–412, 1965.

[18] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Applied Mathematical Sciences. Springer, 1983.

[19] A. V. Proskurnikov, F. Zhang, M. Cao, and J. M. A. Scherpen. A general criterion for synchronization of incrementally dissipative nonlinearly coupled agents. In 2015 European Control Conference (ECC), pages 581–586, Linz, Austria, July 2015.

[20] J. C. Robinson. *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*. Cambridge University Press, New York, USA, 2001.

[21] E. Schöll. *Nonlinear Spatio-Temporal Dynamics and Chaos in Semiconductors*. Cambridge University Press, New York, USA, 2001.

[22] J.-J. E. Slotine and W. Wang. A study of synchronization and group cooperation using partial contraction theory. In V. Kumar, N. Leonard, and A. S. Morse, editors, *Cooperative Control*, volume 309 of *Lecture Notes in Control and Information Science*, pages 207–228. Springer, 2005.

[23] G. B. Stan and R. Sepulchre. Analysis of interconnected oscillators by dissipativity theory. *IEEE Transactions on Automatic Control*, 52(2):256–270, 2007.

[24] E. M. Stein and R. Shakarchi. *Fourier Analysis: An introduction*. Princeton Lectures in Analysis I. Princeton University Press, 2002.

[25] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*. Springer-Verlag, New York, second edition, 1997.

[26] N. van de Wouw, A. Pavlov, and H. Nijmeijer. Controlled synchronisation of continuous PWA systems. In K. V. Pettersen, J. T. Gravdahl, and H. Nijmeijer, editors, *Group coordination and cooperative control*. Springer, 2006.