SPECTRUM, TRACE AND OSCILLATION OF A STURM-LIOUVILLE TYPE RETARDED DIFFERENTIAL OPERATOR WITH INTERFACE CONDITIONS

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Abstract. In this study, a formula for regularized sums of eigenvalues of a Sturm-Liouville problem with retarded argument at the point of discontinuity is obtained. Moreover, oscillation properties of the related problem is investigated.

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1. Introduction

In this paper, we consider the boundary value problem for the differential equation with retarded argument

\[(1)\quad p(x)y''(x) + q(x)y(x - \Delta(x)) + \lambda^2 y(x) = 0,\]

on \([0, \pi/2) \cup (\pi/2, \pi]\), with boundary conditions

\[(2)\quad a_1 y(0) + a_2 y'(0) = 0,\]
\[(3)\quad y'(\pi) + dy(\pi) = 0\]

and interface conditions

\[(4)\quad \gamma_1 y\left(\frac{\pi}{2} - 0\right) - \delta_1 y\left(\frac{\pi}{2} + 0\right) = 0,\]
\[(5)\quad \gamma_2 y'(\frac{\pi}{2} - 0) - \delta_2 y'(\frac{\pi}{2} + 0) = 0,\]

where \(p(x) = p^2_1\) for \(x \in [0, \pi/2]\) and \(p(x) = p^2_2\) for \(x \in (\pi/2, \pi]\); the real-valued function \(q(x)\) is continuous in \([0, \pi/2) \cup (\pi/2, \pi]\) and has finite limits \(q(\pi/2 \pm 0) = \lim_{x \to \pi/2 \pm 0} q(x)\), the real-valued function \(\Delta(x) \geq 0\) is continuous in \([0, \pi/2) \cup (\pi/2, \pi]\) has finite limits \(\Delta(\pi/2 \pm 0) = \lim_{x \to \pi/2 \pm 0} \Delta(x)\), if \(x \in [0, \pi/2)\) then \(x - \Delta(x) \geq 0\); if \(x \in (\pi/2, \pi]\) then \(x - \Delta(x) \geq \pi/2\); \(\lambda\) is a spectral parameter; \(p_1, a_i, d, \gamma_i, \delta_i \ (i = 1, 2)\) are arbitrary real numbers such that \(p_i a_i d \neq 0 \ (i = 1, 2)\).

Differential equations with retarded argument appeared as far back as the eighteenth century in connection with the solution of the problem of Euler on the investigation of the general form of curves similar to their own evolutes. Differential equations with retarded argument arise in many application of mathematical modelling: for example, combustion in the chamber of a liquid propellant rocket engine \([1,2]\) and vibrations of the hammer in an electromagnetic circuit breaker \([3,4]\).
Boundary value problems with interface conditions arise in varied assortment of physical transfer problems (see [5]). Also, some problems with interface conditions arise in thermal conduction problems for a thin laminated plate (i.e., a plate composed by materials with different characteristics piled in the thickness). In this class of problems, interface conditions across the interfaces should be added since the plate is laminated. The study of the structure of the solution in the matching region of the layer with the basis solution in the plate leads to consideration of an eigenvalue problem for a second-order differential operator with piecewise continuous coefficients and interface conditions [6].

The asymptotic formulas for eigenvalues of boundary value problems with retarded argument obtained in [7-17]. In [17], the principal term of asymptotic distribution of eigenvalues of the problem (1)-(5) was obtained up to $O\left(\frac{1}{n}\right)$. But we need sharper asymptotic formulas. Therefore, we improve this formula up to $O\left(\frac{1}{n^3}\right)$.

The theory of regularized traces of Sturm-Liouville operators stems from the paper [18] of Gelfand and Levitan. Trace formulas for the Sturm-Liouville equation with a complex valued potential and with two point boundary conditions obtained in [19]. Regularized trace of the Sturm-Liouville problem with irregular boundary conditions investigated in [20]. A regularized trace formula for the matrix Sturm-Liouville operator found in [21]. The first regularized traces of boundary value problems with unbounded operator coefficient in a finite interval obtained in [22-24]. The second regularized trace of a differential operator with bounded operator coefficient and with the mixed boundary conditions can be found in [25]. For a comprehensive review of traces of Schrödinger operators, the interested reader is referred to [26].

As mentioned above partly, the literature is about the regularized traces of classic type differential operators is so rich and diverse. For a more comprehensive list, one can refer to the survey paper [27] and the paper [28]. However, there are only a few works on regularized traces and oscillation properties for differential operators with retarded argument. M. Pikula in [8] obtained trace formula of first order:

if $\tau \geq \pi$

$$
\sum_{n=1}^{\infty} (\lambda_n (\tau) - n^2) = q(0) \phi(-\tau) - \frac{h^2 + H^2}{2}
$$

and if $\tau \leq \pi$

$$
\sum_{n=1}^{\infty} \left( \lambda_n (\tau) - n^2 - \frac{2}{\pi} \left( h + H + \frac{\cos n\tau}{2} \int_{\tau}^{\pi} q(t) \, dt \right) \right) = q(0) \phi(-\tau) - \frac{h^2 + H^2}{2} + \frac{h + H}{2\pi} \int_{\tau}^{\pi} q(t) \, dt + \left[ \frac{1}{4} q(\tau) + \frac{1}{4} q(\pi) - q(\tau) \right] \frac{\pi - \tau}{\pi}
$$

$$
+ \left[ \frac{b}{4} - \left( \frac{1}{\sqrt{8}} \int_{\tau}^{\pi} q(t) \, dt \right)^2 \right] \frac{\pi - 2\tau}{\pi}
$$
for the boundary-value problem of second order with retarded argument:

\[-y'' + q(x)y(x - \tau) = \lambda y,\]
\[y'(0) - hy(0) = y'(\pi) + Hy(\pi) = 0,\]
\[y(x - \tau) = y(0)\varphi(x - \tau), \quad x \leq \tau, \quad \varphi(0) = 1.\]

C.F. Yang in [11] obtained formula of the first regularized trace, oscillations of the eigenfunctions and the solutions of inverse nodal problem for discontinuous boundary value problems with retarded argument and with interface conditions at the one point of discontinuity. F. Hira in [14] obtained a formula for regularized sums of eigenvalues for a Sturm-Liouville problem with retarded argument at the one point of discontinuity which contains a spectral parameter in the boundary conditions and as a most recent study in this topic, Şen studied regularized trace formula and oscillation of eigenfunctions of a Sturm-Liouville operator with retarded argument at two points of discontinuity [16].

The goals of this article are to calculate the regularized trace and to find the nodal points of eigenfunctions for the problem (1)-(5). We point out that our results are extension and/or generalization to those in [7-11, 17-18, 28, 30, 31]. For example, if the retardation function \(\Delta \equiv 0\) in (1) and \(p(x) \equiv 1, \delta = 1, \gamma = 1\) we have the formula of the first regularized trace for the classical Sturm-Liouville operator which is called Gelfand-Levitan formula (see [29]).

2. The spectrum

Let \(\omega_1(x, \lambda)\) be a solution of Eq. (1) on \([0, \frac{\pi}{2}]\), satisfying the initial conditions

(6) \[\omega_1(0, \lambda) = a_2 \quad \text{and} \quad \omega'_1(0, \lambda) = -a_1.\]

The conditions (6) define a unique solution of Eq. (1) on \([0, \frac{\pi}{2}]\) (see [7,17]).

After defining the above solution, then we will define the solution \(\omega_2(x, \lambda)\) of Eq. (1) on \([\frac{\pi}{2}, \pi]\) by means of the solution \(\omega_1(x, \lambda)\) using the initial conditions

(7) \[\omega_2\left(\frac{\pi}{2}, \lambda\right) = \gamma_1 \delta_1^{-1} \omega_1\left(\frac{\pi}{2}, \lambda\right) \quad \text{and} \quad \omega'_2\left(\frac{\pi}{2}, \lambda\right) = \gamma_2 \delta_2^{-1} \omega'_1\left(\frac{\pi}{2}, \lambda\right).\]

The conditions (7) define a unique solution of Eq. (1) on \([\frac{\pi}{2}, \pi]\) (see [17]).

Consequently, the function \(\omega(x, \lambda)\) is defined on \([0, \frac{\pi}{2}] \cup \left(\frac{\pi}{2}, \pi\right]\) by the equality

\[\omega(x, \lambda) = \begin{cases} \omega_1(x, \lambda), & x \in [0, \frac{\pi}{2}], \\ \omega_2(x, \lambda), & x \in \left(\frac{\pi}{2}, \pi\right] \end{cases}\]

is a solution of (1) on \([0, \frac{\pi}{2}] \cup \left(\frac{\pi}{2}, \pi\right]\), which satisfies one of the boundary conditions and the interface conditions (4)-(5) Then the following integral equations hold:

\[\omega_1(x, \lambda) = a_2 \cos \frac{\lambda}{p_1} x - \frac{a_1 p_1}{\lambda} \sin \frac{\lambda}{p_1} x \]
\[-\frac{1}{\lambda p_1} \int_0^x q(\tau) \sin \frac{\lambda}{p_1} (x - \tau) \omega_1(\tau - \Delta(\tau), \lambda) d\tau,\]

(8)
\[ \omega_2(x, \lambda) = \frac{\gamma_1 a_2}{\delta_1} \cos \frac{\lambda}{p_2} \left( \frac{\pi}{2} - \frac{p_2}{2p_1} - x \right) - \frac{\gamma_1 a_1 p_2}{\lambda \delta_2} \sin \frac{\lambda}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} - x \right) \]

\[ \frac{1}{\lambda p_2} \int_{0}^{x} q(\tau) \sin \frac{\lambda}{p_2} (x - \tau) \omega_2(\tau - \Delta(\tau), \lambda) \, d\tau. \]

Solving the equations (8)-(9) by the method of successive approximation, we obtain the following asymptotic equalities for \(|\lambda| \to \infty\):

\[ \omega_1(x, \lambda) = a_2 \cos \frac{\lambda}{p_1} x - a_1 \sin \frac{\lambda}{p_1} x - \frac{a_2}{2p_1} \int_{0}^{x} q(\tau) \sin \frac{\lambda}{p_1} (x - \Delta(\tau)) \, d\tau \]

\[ \frac{1}{\lambda p_1} \int_{0}^{x} q(\tau) \cos \frac{\lambda}{p_1} (x - (2\tau - \Delta(\tau))) \, d\tau + O \left( \frac{1}{\lambda^2} \right). \]

Differentiating (10) with respect to \(x\), we get

\[ \omega'_1(x, \lambda) = \frac{a_2}{p_1} \sin \frac{\lambda}{p_1} x - a_1 \cos \frac{\lambda}{p_1} x - \frac{a_2}{2p_1^2} \int_{0}^{x} q(\tau) \cos \frac{\lambda}{p_1} (x - \Delta(\tau)) \, d\tau \]

\[ \frac{1}{\lambda p_1^2} \int_{0}^{x} q(\tau) \cos \frac{\lambda}{p_1} (x - (2\tau - \Delta(\tau))) \, d\tau + O \left( \frac{1}{\lambda} \right). \]

Using the fact that,

\[ O \left( \frac{1}{\lambda} \right) = \begin{cases} \int_{0}^{x} q(\tau) \sin \frac{\lambda}{p_1} (2\tau - \Delta(\tau)) \, d\tau, & x \in [0, \frac{\pi}{2}] ; \\ \int_{0}^{x} q(\tau) \cos \frac{\lambda}{p_1} (2\tau - \Delta(\tau)) \, d\tau, & x \in [0, \frac{\pi}{2}] ; \\ \int_{\frac{\pi}{2}}^{x} q(\tau) \sin \frac{\lambda}{p_1} (2\tau - \Delta(\tau)) \, d\tau, & x \in [\frac{\pi}{2}, \pi] ; \\ \int_{\frac{\pi}{2}}^{x} q(\tau) \cos \frac{\lambda}{p_1} (2\tau - \Delta(\tau)) \, d\tau, & x \in [\frac{\pi}{2}, \pi] \end{cases} \]

(see [9]), (10) and (11) we have

\[ \omega_2(x, \lambda) = \frac{\gamma_1 a_2}{\delta_1} \cos \frac{\lambda}{p_2} \left( \frac{\pi}{2} - \frac{p_2}{2p_1} - x \right) - \frac{\gamma_1 a_1 p_2}{\lambda \delta_2} \sin \frac{\lambda}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} - x \right) \]

\[ - \frac{\gamma_1 a_2 (p_1 + p_2)}{2\lambda \delta_1 p_1 p_2} \left\{ \left[ B \left( \frac{\pi}{2}, \lambda \right) + D(x, \lambda) \right] \sin \frac{\lambda}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} + x \right) \right\} + O \left( \frac{1}{\lambda^2} \right). \]
Here,

\[
A(x, \lambda) = \int_0^x q(\tau) \sin \frac{\lambda \Delta(\tau)}{p_1} d\tau \quad x \in [0, \pi/2];
\]

\[
B(x, \lambda) = \int_0^x q(\tau) \cos \frac{\lambda \Delta(\tau)}{p_1} d\tau \quad x \in [0, \pi/2];
\]

\[
C(x, \lambda) = \int_{\pi/2}^x q(\tau) \sin \frac{\lambda \Delta(\tau)}{p_2} d\tau, \quad x \in [\pi/2, \pi];
\]

\[
D(x, \lambda) = \int_{\pi/2}^x q(\tau) \cos \frac{\lambda \Delta(\tau)}{p_2} d\tau, \quad x \in [\pi/2, \pi].
\]

Differentiating (12) with respect to \(x\), we get

\[
\omega_\lambda^2(x, \lambda) = \frac{\gamma_1 a_2 \lambda}{\delta_1 p_2} \sin \frac{\lambda}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} + x \right) - \frac{\gamma_1 a_2 (p_1 + p_2)}{2 \delta_1 p_1 p_2^2} \left\{ \left[ B \left( \frac{\pi}{2}, \lambda \right) + D(x, \lambda) \right] \cos \frac{\lambda}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} + x \right) \right\} + O \left( \frac{1}{\lambda} \right).
\]

(13)

The solution \(\omega(x, \lambda)\) defined above is a nontrivial solution of (1) satisfying conditions (2) and (4)-(5). Putting \(\omega(x, \lambda)\) into (3), we get the characteristic equation

\[(14) \quad \Theta(\lambda) \equiv \omega' (\pi, \lambda) + d\omega(\pi, \lambda).
\]

The set of eigenvalues of boundary value problem (1)-(5) coincides with the set of the squares of roots of (14), and eigenvalues are simple (see [17]).

From (12), (13) and (14), we obtain

\[
\Theta(\lambda) \equiv -\frac{\gamma_1 a_2 \lambda}{\delta_1 p_2} \sin \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right) - \frac{\gamma_1 a_1 p_1}{\delta_1 p_2} \cos \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right)
\]

\[
- \frac{\gamma_1 a_2 (p_1 + p_2)}{2 \delta_1 p_1 p_2^2} \left\{ \left[ B \left( \frac{\pi}{2}, \lambda \right) + D(\pi, \lambda) \right] \cos \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right) \right\}
\]

\[
- \left[ A \left( \frac{\pi}{2}, \lambda \right) + C(\pi, \lambda) \right] \sin \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right)
\]

\[
+ \frac{d\gamma_1 a_2}{\delta_1} \cos \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right) + O \left( \frac{1}{\lambda} \right).
\]

Define

\[
\Theta_0(\lambda) \equiv -\frac{\gamma_1 a_2 \lambda}{\delta_1 p_2} \sin \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right).
\]

Denote by \(\lambda_n^0 = \frac{2p_1 p_2}{p_1 + p_2} n, n \in \mathbb{Z}\), zeros of the function \(\Theta_0(\lambda)\). It is simple algebraically except for \(\lambda^0_{\pm 0}\) and we have
Denote by $C_n$ the circle of radius, $0 < \varepsilon < \frac{1}{2}$, centered at the origin $\lambda_0$ and by $\Gamma_{N_0}$ the counterclockwise square contours with four vertices $K = N_0 + \varepsilon + N_0i$, $L = -N_0 - \varepsilon + N_0i$, $M = -N_0 - \varepsilon - N_0i$, $N = N_0 + \varepsilon - N_0i$, where $i = \sqrt{-1}$ and $N_0$ is a natural number. Obviously, if $\lambda \in C_n$ or $\lambda \in \Gamma_{N_0}$, then $|\Theta_0(\lambda)| \geq M |\lambda| e^{\text{Im} \lambda \pi}$ ($M > 0$) by using a similar method in [11, 32]. Thus, on $\lambda \in C_n$ or $\lambda \in \Gamma_{N_0}$, we have

\[
\frac{\Theta(\lambda)}{\Theta_0(\lambda)} = 1 + \frac{1}{\lambda} \left\{ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda \right) + D(\pi, \lambda) \right) \cot \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right) - \frac{1}{2\lambda^2} \left[ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda \right) + D(\pi, \lambda) \right) - dp_2 \right]^2 \right\}
\]

Expanding $\ln \frac{\Theta(\lambda)}{\Theta_0(\lambda)}$ by the Maclaurin formula, we find that

\[
\ln \frac{\Theta(\lambda)}{\Theta_0(\lambda)} = \frac{1}{\lambda} \left\{ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda \right) + D(\pi, \lambda) \right) - dp_2 \right\} \times \cot \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right) \frac{p_1 + p_2}{2p_1p_2} \left( A \left( \frac{\pi}{2}, \lambda \right) + C(\pi, \lambda) \right) \cot \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right) + O \left( \frac{1}{\lambda^2} \right).
\]

Using the well-known Rouche Theorem, we get that $\Theta(\lambda)$ has the same number of zeros inside $\Gamma_{N_0}$ as $\Theta_0(\lambda)$ (see [11]). Using the residue theorem, we have

\[
\lambda_n - \lambda_0 = -\frac{1}{2\pi i} \oint_{C_n} \ln \frac{\Theta(\lambda)}{\Theta_0(\lambda)} \, d\lambda
\]

\[
= -\frac{1}{2\pi i} \oint_{C_n} \left[ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda \right) + D(\pi, \lambda) \right) - dp_2 \right] \cot \frac{\lambda \pi}{p_2} \left( \frac{p_2 - p_1}{2p_1} + 1 \right) \, d\lambda
\]

\[
+ \frac{1}{2\pi i} \oint_{C_n} \frac{p_1 + p_2}{2p_1p_2} \left( A \left( \frac{\pi}{2}, \lambda \right) + C(\pi, \lambda) \right) \frac{d\lambda}{\lambda}.
\]
Theorem 2.1. The spectrum of the problem (1)-(5) has the
\[ \lambda_n = \lambda_n^0 - \frac{1}{\lambda_n^0} \left( \frac{a_1 p_1}{a_2} + \frac{p_1 + p_2}{2 p_1 p_2} \left( B \left( \frac{\pi}{2}, \lambda_n^0 \right) + D(\pi, \lambda_n^0) \right) - dp_2 \right) \]
\[ + \frac{1}{\lambda_n^0} \left\{ \frac{p_1 + p_2}{p_1 p_2} \left( A \left( \frac{\pi}{2}, \lambda_n^0 \right) + C(\pi, \lambda_n^0) \right) \right\} \]
\[ \times \left( A \left( \frac{\pi}{2}, \lambda_n \right) + C(\pi, \lambda_n) \right) \frac{\cot \frac{\lambda_n}{p_2} \left( \frac{p_2 - p_1}{2 p_1} + 1 \right)}{2 \lambda^2} d\lambda + O \left( \frac{1}{n^3} \right). \]

Thus, using (15) and residue calculation we have proven the following theorem.

\[ \lambda_n = \lambda_n^0 - \frac{1}{\lambda_n^0} \left( \frac{a_1 p_1}{a_2} + \frac{p_1 + p_2}{2 p_1 p_2} \left( B \left( \frac{\pi}{2}, \lambda_n^0 \right) + D(\pi, \lambda_n^0) \right) - dp_2 \right) \]
\[ + \frac{1}{\lambda_n^0} \left\{ \frac{p_1 + p_2}{p_1 p_2} \left( A \left( \frac{\pi}{2}, \lambda_n^0 \right) + C(\pi, \lambda_n^0) \right) \right\} \]
\[ \times \left( A \left( \frac{\pi}{2}, \lambda_n \right) + C(\pi, \lambda_n) \right) \frac{\cot \frac{\lambda_n}{p_2} \left( \frac{p_2 - p_1}{2 p_1} + 1 \right)}{2 \lambda^2} d\lambda + O \left( \frac{1}{n^3} \right). \]

asymptotic distribution for sufficiently large |n|.

3. The regularized trace formula

In this section, we will get regularized trace formula for the problem (1)-(5).

The asymptotic formula (15) for the eigenvalues implies that for all sufficiently large \( N_0 \), the numbers \( \lambda_n \) with \( |n| \leq N_0 \) are inside \( \Gamma_{N_0} \), and the numbers \( \lambda_n \) with \( |n| > N_0 \) are outside \( \Gamma_{N_0} \). It follows that

\[ \lambda_n^2 + \lambda_n^0 + \sum_{0 \neq n = -N_0}^{N_0} \left( \lambda_n^2 - \left( \lambda_n^0 \right)^2 \right) = -\frac{1}{2\pi i} \oint_{\Gamma_n} 2\lambda \ln \frac{\Delta(\lambda)}{\Delta_0(\lambda)} d\lambda \]
\[ = \frac{1}{2\pi i} \oint_{\Gamma_n} 2 \left[ \frac{a_1 p_1}{a_2} + \frac{p_1 + p_2}{2 p_1 p_2} \left( B \left( \frac{\pi}{2}, \lambda_n \right) + D(\pi, \lambda_n) \right) - dp_2 \right] \]
\[ \times \cosh \frac{\lambda p_2}{p_2} \left( \frac{p_2 - p_1}{2 p_1} + 1 \right) d\lambda + \frac{1}{2\pi i} \oint_{\Gamma_n} 2 \left[ \frac{p_1 + p_2}{2 p_1 p_2} \left( A \left( \frac{\pi}{2}, \lambda \right) + C(\pi, \lambda) \right) \right] d\lambda \]
\[ + \frac{1}{2\pi i} \oint_{\Gamma_n} \left[ \frac{a_1 p_1}{a_2} + \frac{p_1 + p_2}{2 p_1 p_2} \left( B \left( \frac{\pi}{2}, \lambda_n \right) + D(\pi, \lambda_n) \right) - dp_2 \right]^2 \frac{\cot^2 \frac{\lambda_n}{p_2} \left( \frac{p_2 - p_1}{2 p_1} + 1 \right)}{\lambda} d\lambda \]
\[ + \frac{1}{2\pi i} \oint_{\Gamma_n} \left( \frac{p_1 + p_2}{4p_1^2p_2^2} \right) \left( A \left( \frac{\pi}{2}, \lambda \right) + C(\pi, \lambda) \right)^2 \frac{d\lambda}{\lambda} \]

\[ + \frac{1}{2\pi i} \oint_{\Gamma_n} \frac{p_1 + p_2}{p_1p_2} \left[ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda \right) + D(\pi, \lambda) \right) - dp_2 \right] \]

\[ \times \left( A \left( \frac{\pi}{2}, \lambda \right) + C(\pi, \lambda) \right) \frac{\cot \frac{p_2-p_1}{2p_1} + 1}{\lambda} d\lambda + O \left( \frac{1}{N_0} \right), \]

by calculations, which implies that

\[ \lambda^2 - \lambda^2_0 + \sum_{0\neq n=-N_0}^{N_0} \lambda^2_n - (\lambda^0_n)^2 \]

\[ = -\frac{2}{\pi} \sum_{0\neq n=-N_0}^{N_0} \left[ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda^0_n \right) + D(\pi, \lambda^0_n) \right) - dp_2 \right] \]

\[ - \frac{2}{\pi} \left[ a_1p_1 + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, 0 \right) + D(\pi, 0) \right) - dp_2 \right] \]

\[ + \sum_{0\neq n=-N_0}^{N_0} \frac{p_1 + p_2}{p_1p_2} \left[ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda^0_n \right) + D(\pi, \lambda^0_n) \right) - dp_2 \right] \]

\[ \times \left( A \left( \frac{\pi}{2}, \lambda^0_n \right) + C(\pi, \lambda^0_n) \right) \frac{1}{\lambda^0_n} + R \]

\[ - \left[ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, 0 \right) + D(\pi, 0) \right) - dp_2 \right]^2 \]

\[ + \frac{(p_1 + p_2)^2}{4p_1^2p_2^2} \left( A \left( \frac{\pi}{2}, 0 \right) + C(\pi, 0) \right)^2 + O \left( \frac{1}{N_0} \right), \]

(16)

where,

\[ R = \text{Res}_{\lambda=0} \left\{ \frac{p_1 + p_2}{p_1p_2} \left[ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda \right) + D(\pi, \lambda) \right) - dp_2 \right] \]

\[ \times \left( A \left( \frac{\pi}{2}, \lambda \right) + C(\pi, \lambda) \right) \frac{\cot \frac{p_2-p_1}{2p_1} + 1}{\lambda} d\lambda \right\} \]

Passing to the limit as \( N_0 \to \infty \) in (16), we have

\[ \lambda^2 - \lambda^2_0 + \sum_{0\neq n=-\infty}^{\infty} \left\{ \lambda^2_n - (\lambda^0_n)^2 \right\} \]

\[ + \frac{2}{\pi} \left[ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda^0_n \right) + D(\pi, \lambda^0_n) \right) - dp_2 \right] \]

\[ - \frac{p_1 + p_2}{p_1p_2} \left[ \frac{a_1p_1}{a_2} + \frac{p_1 + p_2}{2p_1p_2} \left( B \left( \frac{\pi}{2}, \lambda^0_n \right) + D(\pi, \lambda^0_n) \right) - dp_2 \right] \]

\[ \times \left( A \left( \frac{\pi}{2}, \lambda^0_n \right) + C(\pi, \lambda^0_n) \right) \frac{1}{\lambda^0_n} \]
\[
\begin{aligned}
= -\frac{2}{\pi} \left[ \frac{a_1 p_1}{a_2} + \frac{p_1 + p_2}{2 p_1 p_2} \left( B \left( \frac{\pi}{2}, 0 \right) + D(\pi, 0) \right) - dp_2 \right] + R \\
\left[ \frac{a_1 p_1}{a_2} + \frac{p_1 + p_2}{2 p_1 p_2} \left( B \left( \frac{\pi}{2}, 0 \right) + D(\pi, 0) \right) - dp_2 \right]^2 \\
+ \frac{(p_1 + p_2)^2}{4 p_1^2 p_2^2} \left( A \left( \frac{\pi}{2}, 0 \right) + C(\pi, 0) \right)^2.
\end{aligned}
\]

The series on the left side of (17) is called the regularized trace of the problem (1)-(5).

4. The oscillation

In this chapter, we will find nodal points of eigenfunctions of the problem (1)-(5).

Let us rewrite the equation (10) and replace \( \lambda \) by \( \lambda_n \)

\[
\omega_1(x, \lambda_n) = a_2 \cos \frac{\lambda_n x}{p_1} - \frac{a_1 p_1}{\lambda_n} \sin \frac{\lambda_n x}{p_1} - \frac{a_2 \sin \frac{\lambda_n x}{p_1}}{2 \lambda_n p_1} \int_0^x q(\tau) \cos \left( \frac{\lambda_n \Delta(\tau)}{p_1} \right) d\tau + O \left( \frac{1}{\lambda_n^2} \right).
\]

Let us assume that \( x^j_n \) are the nodal points of the eigenfunction \( \omega_1(x, \lambda_n) \).

Taking \( \cot \left( \frac{\lambda_n x}{p_1} \right) \neq 0 \) into account for sufficiently large \( n \), we get

\[
\cot \left( \frac{\lambda_n x}{p_1} \right) \left[ 1 + \frac{A(x, \lambda_n)}{2 \lambda_n p_1} \right] = \frac{a_1 p_1}{a_2 \lambda_n} \left( \frac{B(x, \lambda_n)}{2 \lambda_n p_1} + O \left( \frac{1}{\lambda_n^2} \right) \right).
\]

It follows easily that

\[
\tan \left( \frac{\lambda_n x}{p_1} + \frac{\pi}{2} \right) = -\frac{a_1 p_1}{a_2 \lambda_n} \frac{B(x, \lambda_n)}{2 \lambda_n p_1} + O \left( \frac{1}{\lambda_n^2} \right).
\]

Thus, solving the equation (18), one obtains

\[
x^j_n = (j - \frac{1}{2}) \frac{\pi p_1}{\lambda_n} - \frac{a_1 p_1^2}{a_2 \lambda_n^2} \frac{B \left( x^j_n, \lambda_n \right)}{2 \lambda_n^2} + O \left( \frac{1}{\lambda_n^3} \right).
\]

Note that

\[
\lambda_n^{-1} = \frac{1}{\lambda_n^0} + \frac{\frac{a_1 p_1}{a_2} + \frac{p_1 + p_2}{2 p_1 p_2} \left( B \left( \frac{\pi}{2}, \lambda_n^0 \right) + D(\pi, 0) \right) - dp_2}{(\lambda_n^0)^3} \pi + O \left( \frac{1}{n^2} \right)
\]

and

\[
\lambda_n^{-2} = \frac{1}{(\lambda_n^0)^2} + O \left( \frac{1}{n^2} \right).
\]

Substituting (20) and (21) into (19) we have

\[
x^j_n = (j - \frac{1}{2}) \frac{\pi p_1}{\lambda_n^0} - \frac{(j - \frac{1}{2}) p_1}{\lambda_n^0} \left[ \frac{a_1 p_1}{a_2} + \frac{p_1 + p_2}{2 p_1 p_2} \left( B \left( \frac{\pi}{2}, \lambda_n^0 \right) + D(\pi, 0) \right) - dp_2 \right].
\]
Thus, solving the equation (23), one obtains
\[
\frac{x_n}{\gamma_1} = \frac{a_2 \cos \lambda_n}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} + x \right) - \frac{a_1 p_1}{\lambda_n} \sin \frac{\lambda_n}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} + x \right)
\]
\[
- \frac{a_2 (p_1 + p_2)}{2 \lambda_n p_1 p_2} \left\{ \left[ B \left( \frac{\pi}{2}, \lambda_n \right) + D(x, \lambda_n) \right] \sin \frac{\lambda_n}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} + x \right) \right. 
\]
\[
\left. + \left[ A \left( \frac{\pi}{2}, \lambda_n \right) + C(x, \lambda_n) \right] \cos \frac{\lambda_n}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} + x \right) \right\} + O \left( \frac{1}{\lambda_n^2} \right).
\]

For nodal points of \( \omega_2(x, \lambda_n) \), again, taking \( \sin \lambda_n/p_2 \left( \frac{p_2 - p_1}{2p_1} + x \right) \neq 0 \) into account for sufficiently large \( n \), we get
\[
\cot \frac{\lambda_n}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} + x \right) \left[ 1 - \frac{\left( A \left( \frac{\pi}{2}, \lambda_n \right) + C(x, \lambda_n) \right) (p_1 + p_2)}{2 \lambda_n p_1 p_2} \right] = \frac{a_1 p_1}{a_2 \lambda_n} + \frac{(p_1 + p_2) \left( B \left( \frac{\pi}{2}, \lambda_n \right) + D(x, \lambda_n) \right)}{2 \lambda_n p_1 p_2} + O \left( \frac{1}{\lambda_n^2} \right),
\]
and thus
\[
\tan \left( \frac{\lambda_n}{p_2} \left( \frac{\pi (p_2 - p_1)}{2p_1} + x \right) + \frac{\pi}{2} \right) = - \frac{a_1 p_1}{a_2 \lambda_n} - \frac{(p_1 + p_2) \left( B \left( \frac{\pi}{2}, \lambda_n \right) + D(x, \lambda_n) \right)}{2 \lambda_n^2 p_1 p_2} + O \left( \frac{1}{\lambda_n^2} \right). \tag{23}
\]
Thus, solving the equation (23), one obtains
\[
\frac{x_n}{\gamma_1} = - \frac{\pi (p_2 - p_1)}{2p_1} + \left( j - \frac{1}{2} \right) \frac{\pi}{2} p_2 
\]
\[
- \frac{(p_1 + p_2) \left( B \left( \frac{\pi}{2}, \lambda_n \right) + D \left( \frac{\pi}{2}, \lambda_n \right) \right)}{2 \lambda_n^2 p_1 p_2} + O \left( \frac{1}{\lambda_n^2} \right).
\]
\[
\frac{x_n}{\gamma_1} = - \frac{\pi (p_2 - p_1)}{2p_1} + \left( j - \frac{1}{2} \right) \frac{\pi p_2}{\lambda_n} 
\]
\[
- \frac{(p_1 + p_2) \left[ a_1 p_1 + \frac{a_2 p_2 p_1}{4 p_1^2} \left( B \left( \frac{\pi}{2}, \lambda_n \right) + D \left( \frac{\pi}{2}, \lambda_n \right) \right) - dp_2 \right]}{\lambda_0^2} + O \left( \frac{1}{\lambda_n^2} \right), \quad j = \left\lfloor \frac{n}{2} \right\rfloor + 1, n.
\]
\[
\frac{x_n}{\gamma_1} = - \frac{\pi (p_2 - p_1)}{2p_1} + \left( j - \frac{1}{2} \right) \frac{p_2}{\lambda_n} 
\]
\[
- \frac{(p_1 + p_2) \left[ B \left( \frac{\pi}{2}, \lambda_n \right) + D \left( \frac{\pi}{2}, \lambda_n \right) \right]}{2 \lambda_n^2 p_1} + O \left( \frac{1}{n^2} \right), \quad j = \left\lfloor \frac{n}{2} \right\rfloor + 1, n.
\]
Thus we have proven the following theorem:

**Theorem 4.1.** For sufficiently large \( n \), we have the formulas (22) and (25) of the nodal points for the problem (1)-(5).
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