Lipschitzian Regularity of the Minimizing Trajectories for Nonlinear Optimal Control Problems

Delfim F. M. Torres

Department of Mathematics
University of Aveiro
3810-193 Aveiro, Portugal
fax: +351 234382014
e-mail: delfim@mat.ua.pt

Abstract

We consider the Lagrange problem of optimal control with unrestricted controls and address the question: under what conditions we can assure optimal controls are bounded? This question is related to the one of Lipschitzian regularity of optimal trajectories, and the answer to it is crucial for closing the gap between the conditions arising in the existence theory and necessary optimality conditions. Rewriting the Lagrange problem in a parametric form, we obtain a relation between the applicability conditions of the Pontryagin maximum principle to the later problem and the Lipschitzian regularity conditions for the original problem. Under the standard hypotheses of coercivity of the existence theory, the conditions imply that the optimal controls are essentially bounded, assuring the applicability of the classical necessary optimality conditions like the Pontryagin maximum principle. The result extends previous Lipschitzian regularity results to cover optimal control problems with general nonlinear dynamics.

Keywords: optimal control – Pontryagin maximum principle – boundedness of minimizers – nonlinear dynamics – Lipschitzian regularity.

1 Introduction

Given a Lagrangian $L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \rightarrow \mathbb{R}$, a dynamical equation $\dot{x}(t) = \varphi(t, x(t), u(t))$, and boundary conditions $x(a) = A, x(b) = B \in \mathbb{R}^n$, we

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\end{itemize}
consider the problem of finding a control \( u(\cdot) \in L_1([a, b]; \mathbb{R}^r) \) such that the corresponding state trajectory \( x(\cdot) \in W_{1,1}([a, b]; \mathbb{R}^n) \) of the dynamical equation satisfies the boundary conditions, and the pair \((x(\cdot), u(\cdot))\) minimizes the functional \( J[x(\cdot), u(\cdot)] := \int_a^b L(t, x(t), u(t)) \, dt \). We establish Lipschitzian regularity conditions for the minimizing trajectories of such optimal control problems. Lipschitzian regularity has a number of important implications. For example in control engineering applications, where optimal strategies are implemented by computer, the choice of discretization and numerical procedures depends on minimizer regularity \([3, 34]\). Lipschitzian regularity of optimal trajectories also precludes occurrence of the undesirable Lavrentiev phenomenon \([25, 6, 23, 22]\) and provides the validity of known necessary optimality conditions under hypotheses of existence theory \([11]\).

The techniques of the existence theory use compactness arguments which require to work with measurable control functions from \( L_p, 1 \leq p < \infty \) \([5]\).

On the other hand, standard necessary conditions for optimality, such as the classical Pontryagin maximum principle \([27]\), put certain restrictions on the optimal controls – namely, \textit{a priori} assumption that they are essentially bounded. Examples are known, even for polynomial Lagrangians and linear dynamics \([3]\), for which optimal controls predicted by the existence theory are unbounded and fail to satisfy the Pontryagin maximum principle \([12]\).

If we are able to assure that a minimizer \((\tilde{x}(t), \tilde{u}(t)), a \leq t \leq b\), of our problem is such that \( \tilde{u}(\cdot) \) is essentially bounded, then the solutions can be identified via the Pontryagin maximum principle. As far as \( \varphi(t, \tilde{x}(t), \tilde{u}(t)) \) is bounded, it also follows that the optimal trajectory \( \tilde{x}(\cdot) \) is Lipschitzian. Similarly, the Hamiltonian adjoint multipliers \( \tilde{\psi}(\cdot) \) of the Pontryagin maximum principle turn out to be Lipschitzian either. Thus, regularity theory justifies searching for minimizers among extremals and establishes a weaker form of the maximum principle in which the Hamiltonian adjoint multipliers are not required to be absolutely continuous but merely Lipschitzian.

The study of Lipschitzian regularity conditions has received few attention when compared with existence theory or necessary conditions, which have been well studied since the fifties and sixties. The question of Lipschitzian regularity, for the general Lagrange problem of optimal control, seems difficult, and attention have been on particular dynamics. Most part of results in this direction refers to problems of the calculus of variations. First results on Lipschitzian regularity for the basic problem of the calculus of variations \( -\varphi(u) = u \) – belong to L. Tonelli and S. Bernstein. Some further results have been obtained by C. B. Morrey and more recently by F. H. Clarke and R. B. Vinter among others. For a survey see \([3, \text{Ch. 2}]\) or \([35, \text{Ch. 11}]\). Less is known for the Lagrange problem of optimal control. Problems whose dynamics is linear and time invariant \(-\varphi(x, u) = Ax + Bu\) – were addressed in \([14]\). The result is obtained imposing conditions under
which the problem is reduced into a problem of the calculus of variations and then using the results available in the literature. Recently, a new approach to the Lipschitzian regularity has been developed by A. Sarychev and the author in [28], which allows to deal with a wide class of optimal control problems with control-affine dynamics – $\phi(t, x, u) = f(t, x) + g(t, x) u$.

As particular cases they include the problems of the calculus of variations treated before. The approach is based in the reduction of the problem to a time-optimal control problem, on the subsequent compactification of the space of admissible controls, and utilization of the Pontryagin’s maximum principle [27]. The conditions of Lipschitzian regularity arise from the conditions of applicability of the Pontryagin’s maximum principle [27] to the latter problem and from its equations Lipschitzian regularity of the corresponding minimizer of the initial problem is established. For a survey see [29]. The main result of [28] can be summarized in the following theorem:

**Theorem 1** ($\phi(t, x, u) = f(t, x) + g(t, x) u$). If $g(t, x)$ has complete rank $r$ for all $t$ and $x$; $L(\cdot, \cdot, \cdot), f(\cdot, \cdot), g(\cdot, \cdot)$ are $C^1$ smooth; and the following conditions are satisfied:

**coercivity** there exist a function $\theta : \mathbb{R} \to \mathbb{R}$ and $\zeta \in \mathbb{R}$ such that

$$L(t, x, u) \geq \theta(\|u\|) > \zeta, \quad \forall (t, x, u),$$

$$\lim_{r \to +\infty} \frac{\theta(r)}{r} = +\infty;$$

**growth condition** there exist constants $\gamma$, $\beta$, $\eta$, and $\mu$, with $\gamma > 0$, $\beta < 2$ and $\mu \geq \max \{\beta - 2, -2\}$, such that the inequality

$$(|L_t| + |L_{x_t}| + \|L \varphi_t - L_t \varphi\| + \|L \varphi_{x_t} - L_{x_t} \varphi\|) \|u\|^\mu \leq \gamma L^\beta + \eta,$$

holds for all $t \in [a, b]$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^r$, $i \in \{1, \ldots, n\}$;

then all the minimizers $\hat{u}(\cdot)$ of the problem, which are not abnormal extremal controls, are essentially bounded on $[a, b]$.

Results for general nonlinear dynamics, which is nonlinear both in state and control variables, are lacking. To deal with the problem we make use of a different auxiliary optimal control problem than the one in [28] (Section 3), which is obtained using an idea of time reparameterization that proved to be useful in many different contexts – see e.g. [19, Sec. 10], [14, 17], [20, Lec. 13], [5, p. 46], [3], [1], [10], [24, Ch. 5], [21, p. 29], and [32]. At the core of our proof techniques is the study of the relation between the minimizers and admissible trajectory-control pairs of the original and auxiliary problems (Section 3) and how the Pontryagin extremals are related (Section 4).
Applying weak necessary conditions than the Pontryagin’s maximum principle [27], for example the ones found in [8], to minimizers of the auxiliary problem, then one is able to obtain, from the established relations between minimizers and extremals, the desired regularity properties for the minimizers of the original problem (Section 5). Examples which possess minimizers according to the existence theory and to which our results are applicable while previously known Lipschitzian regularity conditions fail are provided (Section 6).

2 Formulation of Problems \((P), (P_\tau)\) and \((P_{\tau}[w(\cdot)])\)

We are interested in the study of Lipschitzian regularity conditions for the Lagrange problem of optimal control with arbitrary boundary conditions. For that is enough to consider the case when the boundary conditions are fixed: \(x(a) = A\) and \(x(b) = B\). Indeed, if \(\tilde{x}(\cdot)\) is a minimizing trajectory for a Lagrange problem with any other kind of boundary conditions, then \(\tilde{x}(\cdot)\) is also a minimizing trajectory for the corresponding fixed boundary problem with \(A = \tilde{x}(a)\) and \(B = \tilde{x}(b)\). The data for our problem is then

\[
\begin{array}{ll}
(a < b) & \quad a, b \in \mathbb{R} \ \ (a < b) \\
A, B \in \mathbb{R}^n & \quad A, B \in \mathbb{R}^n \\
L : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r & \rightarrow \mathbb{R} \\
\varphi : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r & \rightarrow \mathbb{R}^n.
\end{array}
\]

We assume \(L(\cdot, \cdot, \cdot), \varphi(\cdot, \cdot, \cdot) \in C, \) and \(\varphi(\cdot, \cdot, u), \varphi(\cdot, \cdot, u) \in C^1.\) (Smoothness hypotheses on \(L\) and \(\varphi\) can be weakened, as is discussed later in connection with the Pontryagin maximum principle.) The Lagrange problem of optimal control is defined as follows.

**Problem \((P)\).**

\[
I [x(\cdot), u(\cdot)] = \int_a^b L (t, x(t), u(t)) \ dt \rightarrow \text{min}
\]

\[
\begin{array}{l}
x(\cdot) \in W_{1, 1} ([a, b]; \mathbb{R}^n), \ u(\cdot) \in L_1 ([a, b]; \mathbb{R}^r) \\
\dot{x}(t) = \varphi (t, x(t), u(t)) , \ a.e. \ t \in [a, b] \\
x(a) = A, \ x(b) = B.
\end{array}
\]

The overdot denotes differentiation with respect to \(t\), while the prime will be used in the sequel to denote differentiation with respect to \(\tau\). To derive
conditions assuring that the optimal controls \( \tilde{u}(\cdot) \) of problem \((P)\) are essentially bounded, \( \tilde{u}(\cdot) \in L_{\infty} \), two auxiliary problems, defined with the same data \((\mathbb{1})\), will be used.

**Problem \((P_\tau)\).**

\[
J[t(\cdot), z(\cdot), v(\cdot)] = \int_a^b L(t(\tau), z(\tau), w(\tau)) \, v(\tau) \, d\tau \longrightarrow \min
\]

\[
\begin{align*}
\begin{aligned}
t(\cdot) &\in W_{1,\infty}([a, b]; \mathbb{R}) ,
z(\cdot) &\in W_{1,1}([a, b]; \mathbb{R}^n) \\
v(\cdot) &\in L_{\infty}([a, b]; [0.5, 1.5]) ,
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
t'(\tau) = v(\tau) \\
z'(\tau) = \varphi (t(\tau), z(\tau), w(\tau)) \, v(\tau)
\end{cases}
\end{align*}
\]

\(t(a) = a, \, t(b) = b\) \hspace{1cm} z(a) = A, \, z(b) = B.

**Remark 2.** The fact that the control variable \(v(\cdot)\) takes on its values in the set \([0.5, 1.5]\), guarantees that \(t(\tau)\) has an inverse function \(\tau(t)\).

Problem \((P)\) is actually equivalent to problem \((P_\tau)\), in the sense that problem \((P)\) can be formally transformed into problem \((P_\tau)\) by considering \(t\) as a dependent variable and introducing a one to one Lipschitzian transformation \([a, b] \ni t \mapsto \tau \in [a, b]\). Both problems have the same minimum value and a direct relation between admissible state-control pairs (cf. Section \([\mathbb{3}]\)).

The following problem is the same as problem \((P_\tau)\) except that \(w(\cdot) \in L_1([a, b]; \mathbb{R}^n)\) is fixed and the functional is to be minimized only over \(t(\cdot), z(\cdot)\) (the state variables) and \(v(\cdot)\) (the control variable).

**Problem \((P_\tau[w(\cdot)])\).**

\[
K[t(\cdot), z(\cdot), v(\cdot)] = \int_a^b F(\tau, t(\tau), z(\tau), v(\tau)) \, d\tau \longrightarrow \min
\]

\[
\begin{align*}
\begin{aligned}
t(\cdot) &\in W_{1,\infty}([a, b]; \mathbb{R}) ,
z(\cdot) &\in W_{1,1}([a, b]; \mathbb{R}^n) \\
v(\cdot) &\in L_{\infty}([a, b]; [0.5, 1.5])
\end{aligned}
\end{align*}
\]

\[
\begin{align*}
\begin{cases}
t'(\tau) = v(\tau) \\
z'(\tau) = f (t(\tau), z(\tau), v(\tau))
\end{cases}
\end{align*}
\]

\(t(a) = a, \, t(b) = b\) \hspace{1cm} z(a) = A, \, z(b) = B,

where \(F(\tau, t, z, v) = L(t, z, w(\tau)) \, v, \, f (t, z, v) = \varphi (t, z, w(\tau)) \, v\).

**Remark 3.** Problem \((P_\tau)\) is autonomous while \((P)\) and \((P_\tau[w(\cdot)])\) are not.

The relation between problem \((P)\) and problem \((P_\tau[w(\cdot)])\) is discussed in the following two sections.
3 Relation Between the Solutions of the Problems

Let us begin to determine the relation between admissible pairs for problem \((P)\) and admissible quadruples for problem \((P_\tau)\).

**Definition 4.** The pair \((x(\cdot), u(\cdot))\) is said to be *admissible for \((P)\)* if all conditions in (2) are satisfied. Similarly, \((t(\cdot), z(\cdot), v(\cdot), w(\cdot))\) is said to be *admissible for \((P_\tau)\)* if all conditions in (3) are satisfied.

Next lemma specifies a passage from an admissible state-control pair of the problem \((P)\) to an admissible state-control pair of the problem \((P_\tau)\). Importantly, the values of the two problems are the same.

**Lemma 5.** Let \((x(\cdot), u(\cdot))\) be admissible for \((P)\). Then, for any function \(v(\cdot)\) satisfying

\[
v(\cdot) \in L_\infty ([a, b]; [0.5, 1.5]), \quad \int_a^b v(s) \, ds = b - a,
\]

\[
t(\tau) = a + \int_a^{\tau} v(s) \, ds, \quad z(\tau) = x(t(\tau)) \text{ and } w(\tau) = u(t(\tau)), \quad \text{are such that}
\]

\((t(\cdot), z(\cdot), v(\cdot), w(\cdot))\) is admissible for \((P_\tau)\). Moreover,

\[
J [t(\cdot), z(\cdot), v(\cdot), w(\cdot)] = I [x(\cdot), u(\cdot)].
\]

**Proof.** All conditions in (3) become satisfied:

- Function \(t(\cdot)\) is Lipschitzian: \(\frac{dt(\tau)}{d\tau} = v(\cdot) \in L_\infty ([a, b]; [0.5, 1.5])\);
- Function \(z(\cdot)\) is absolutely continuous since it is a composition of the absolutely continuous function \(x(\cdot)\) with the strictly monotonous Lipschitzian continuous function \(t(\cdot)\):
  \[
  \frac{dt(\tau)}{d\tau} = v(\tau) > 0;
  \]
- Function \(w(\cdot)\) is Lebesgue measurable, \(w(\cdot) \in L_1\), because \(u(\cdot)\) is measurable and \(t(\cdot)\) is a strictly monotonous absolutely continuous function;
- Differentiating \(z(\cdot)\) we obtain:
  \[
  z'(\tau) = \frac{dz(\tau)}{d\tau} = \frac{dx(t(\tau))}{dt} \frac{dt(\tau)}{d\tau}.
  \]

In view of (2) and (7), one concludes from this last equality that

\[
z'(\tau) = \varphi (t(\tau), x(t(\tau)), u(t(\tau))) \, v(\tau) = \varphi (t(\tau), z(\tau), w(\tau)) \, v(\tau);
\]
From (5) and from the definition of \( t(\tau) \) we have \( t(a) = a \) and \( t(b) = b \). It follows that

\[
z(a) = x(t(a)) = x(a) = A; \\
z(b) = x(t(b)) = x(b) = B.
\]

It remains to prove equality (6). Since

\[
J[t(\cdot), z(\cdot), v(\cdot), w(\cdot)] = \int_a^b L(t(\tau), z(\tau), w(\tau)) \, v(\tau) \, d\tau
\]

from the change of variable \( t(\tau) = t, \)
\[
\begin{bmatrix}
dt = v(\tau) \, d\tau \\
\tau = a \iff t = a \\
\tau = b \iff t = b,
\end{bmatrix}
\]

it follows from (8) the pretended conclusion:

\[
J[t(\cdot), z(\cdot), v(\cdot), w(\cdot)] = \int_a^b L(t, x(t), u(t)) \, dt = I[x(\cdot), u(\cdot)].
\]

The passage established by Lemma 5 is not unique because, compared to \((P)\), problem \((P_{\tau})\) has one more state variable and one more control variable. However, as far as the right-hand side of the control system for the problem \((P_{\tau})\) is autonomous, does not depend on \( \tau \), the set of admissible state-control pairs of \((P_{\tau})\) is invariant under translations of \( \tau \). Lemma 6 asserts that to each admissible state-control pair for \((P_{\tau})\) there corresponds a unique state-control pair for the problem \((P)\) with the same value for the cost functionals.

**Lemma 6.** Let \((t(\cdot), z(\cdot), v(\cdot), w(\cdot))\) be admissible for \((P_{\tau})\). Then the pair \((x(\cdot), u(\cdot)) = (z(\tau(\cdot)), w(\tau(\cdot)))\), where \(\tau(\cdot)\) is the inverse function of \(t(\cdot)\), is admissible for \((P)\). Moreover

\[
I[x(\cdot), u(\cdot)] = J[t(\cdot), z(\cdot), v(\cdot), w(\cdot)].
\]

**Proof.** Similar arguments to the ones in the proof of Lemma 5 show that function \(x(\cdot)\) is absolutely continuous and that \(u(\cdot) \in L_1([a, b]; \mathbb{R}^r)\). Differentiating \(x(\cdot)\) we obtain:

\[
\dot{x}(t) = \frac{dx(t)}{dt} = \frac{dz(\tau(t))}{d\tau} \frac{d\tau(t)}{dt},
\]
and from (3), and from the fact that \( \frac{dr(t)}{dt} = \frac{1}{v(r(t))} > 0 \), one concludes that

\[
\dot{x}(t) = \frac{\varphi(t, x(t), z(t), w(t)) v(t)}{v(t)} = \varphi(t, x(t), u(t)).
\]

As far as \( \tau(a) = a \) and \( \tau(b) = b \), it comes \( x(a) = z(\tau(a)) = x(a) = A, x(b) = z(\tau(b)) = z(b) = B \), and all conditions in (2) are satisfied. Equality (10) follows by direct calculations from the change of variable \( \tau(t) = \tau \) and relations (9):

\[
I [x(\cdot), u(\cdot)] = \int_a^b L(t, x(t), u(t)) \, dt = \int_a^b L(t, z(t), w(t)) \, dt = \int_a^b L(t, z(t), w(t)) \, d\tau = J [t(\cdot), z(\cdot), v(\cdot), w(\cdot)].
\]

\[\square\]

From Lemmas 3 and 4, the following two corollaries are obvious. They establish the relation between the minimizers of problems \( P \) and \( P_\tau \). As a consequence, solving the problem \( P \) turns out to be the same as solving the problem \( P_\tau \).

**Corollary 7.** If \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) is a minimizer of problem \( P \), then, for any function \( \tilde{v}(\cdot) \) satisfying (1) and (3) (e.g. \( \tilde{v}(\tau) \equiv 1 \)), the 4-tuple

\[
(\tilde{t}(\cdot), \tilde{z}(\cdot), \tilde{v}(\cdot), \tilde{w}(\cdot)),
\]

defined by \( \tilde{t}(\tau) = a + \int_a^\tau \tilde{v}(s) \, ds \), \( \tilde{z}(\tau) = \tilde{x}(\tilde{t}(\tau)) \), \( \tilde{w}(\tau) = \tilde{u}(\tilde{t}(\tau)) \), is a minimizer to problem \( P_\tau \).

**Corollary 8.** If \((\check{t}(\cdot), \check{z}(\cdot), \check{v}(\cdot), \check{w}(\cdot))\) is a minimizer of problem \( P_\tau \), then the pair \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) defined from \((\check{t}(\cdot), \check{z}(\cdot), \check{v}(\cdot), \check{w}(\cdot))\) as in Lemma 4 is a minimizer to problem \( P \).

Thus, let \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) be a minimizer of problem \( P \). From Corollary 6 we know how to construct a minimizer \((\check{t}(\cdot), \check{z}(\cdot), \check{v}(\cdot), \check{w}(\cdot))\) to problem \( P_\tau \). Obviously, as far as problem \( P_\tau[\check{w}(\cdot)] \) is the same as problem \( P_\tau \) except that \( \check{w}(\cdot) \) is fixed, \((\check{t}(\cdot), \check{z}(\cdot), \check{v}(\cdot), \check{w}(\cdot))\) furnishes a minimizer to problem \( P_\tau[\check{w}(\cdot)] \). Choosing \( \check{v}(\tau) \equiv 1 \) we obtain.

**Proposition 9.** If \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) is a minimizer of problem \( P \), then the triple \((\check{t}(\tau), \check{z}(\tau), \check{v}(\tau)) = (\tau, \tilde{x}(\tau), 1)\) furnishes a minimizer to problem \( P_\tau[\check{u}(\cdot)] \).

Proposition 8 gives a minimizer to problem \( P_\tau[\check{u}(\cdot)] \), where \( \check{u}(\cdot) \) is an optimal control to the corresponding problem \( P \). As far as for the problem \( P_\tau[\check{u}(\cdot)] \) the admissible controls are already bounded, one can check that
the pertinent hypotheses on functions $F$ and $f$ (on functions $L$ and $\phi$) required by the Pontryagin maximum principle are valid, that is, one can obtain conditions under which $(\tilde{t}(\tau), \tilde{z}(\tau), \tilde{v}(\tau)) = (\tau, \tilde{x}(\tau), 1)$ is a Pontryagin extremal. To conclude that the original minimizer $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ of $(P)$ is also an extremal, one needs to know how the extremals of the problems are related. This will be addressed in the next section.

4 Relation Between the Extremals

At the core of optimal control theory is the celebrated Pontryagin maximum principle. The maximum principle is a first order necessary optimality condition for the optimal control problems. It first appear in the book [27]. Since then, several versions have been obtained by weakening the hypotheses. For example, in [27] it is assumed that functions $L(\cdot, \cdot, \cdot)$ and $\phi(\cdot, \cdot, \cdot)$ are continuous, and have continuous derivatives with respect to the state variables $x$: $L(t, \cdot, u) \in C^1$. Instead of the continuity assumption of $L(\cdot, \cdot, \cdot)$ and $\phi(\cdot, \cdot, \cdot)$, a version only requiring that functions $L(\cdot, x, \cdot)$ and $\phi(\cdot, x, \cdot)$ are Borel measurable can be found in book [4, Ch. 5]. There, in order to assure the applicability of the maximum principle, the following assumption is imposed: there exists an integrable function $\alpha(\cdot)$ defined on $[a, b]$ such that the bound

$$\left\| \frac{\partial L}{\partial x}(t, x, u(t)) \right\| \leq \alpha(t) \quad (11)$$

$$\left\| \frac{\partial \phi_i}{\partial x}(t, x, u(t)) \right\| \leq \alpha(t) \quad (12)$$

$(i = 1, \ldots, n)$ holds for all $(t, x) \in [a, b] \times \mathbb{R}^n$. The existence and integrability of $\alpha(\cdot)$, and the bound $(11)$–$(12)$, are guaranteed under the hypotheses that $L$ and $\phi$ possess derivatives $\frac{\partial L}{\partial x}$ and $\frac{\partial \phi}{\partial x}$ which are continuous in $(t, x, u)$, and $u(\cdot)$ is essentially bounded (these are the hypotheses found in [27]). Alternative hypotheses are the following growth conditions (see [8, Sec. 4.4 and p. 212]):

$$\left\| \frac{\partial L}{\partial x} \right\| \leq c |L| + k, \quad \left\| \frac{\partial \phi_i}{\partial x} \right\| \leq c |\phi_i| + k, \quad (13)$$

with constants $c$ and $k$, $c > 0$. In the context of the Lipschitzian regularity, conditions $(13)$ are particularly important: they are easy to check in practice. Those who are familiar with the Lipschitzian regularity conditions for the basic problem of the calculus of variations ($\phi(u) = u$), will recognize $(13)$ as a generalization and a weak version of the classical Tonelli–Morrey Lipschitzian regularity condition (cf. e.g. [29, §7.1]) where no restriction is
imposed to $\frac{\partial L}{\partial u}$:

**Tonelli-Morrey Lipschitzian regularity condition.** If the Lagrangian $L$ satisfies the growth condition $\|\frac{\partial L}{\partial x}\| + \|\frac{\partial L}{\partial u}\| \leq c |L| + k$, $c > 0$, then any solution $\tilde{x}(\cdot)$ to the basic problem of the calculus of variations, in the class of absolutely continuous functions, is indeed Lipschitzian ($\tilde{u}(\cdot) = \dot{\tilde{x}}(\cdot) \in L_{\infty}$) and satisfy the Pontryagin maximum principle.

From the fact that (13) is a generalization of the Tonelli–Morrey Lipschitzian regularity condition, one can guess a link between the applicability conditions of the maximum principle and the Lipschitzian regularity conditions. The link between the applicability conditions of the classical Pontryagin maximum principle [27] and the Lipschitzian regularity conditions for optimal control problems with control-affine dynamics, was established in [28] using a reduction of the problem to a time-minimal control problem. Here, to deal with general nonlinear dynamics, we use completely different auxiliary problems and we will need to apply the maximum principle under weaker hypotheses than those in [27]. This is due to the fact that when we fix $w(\cdot) \in L_1([a, b]; \mathbb{R}^r)$, functions $F(\tau, t, z, v)$ and $f(\tau, t, z, v)$ of problem $(P_\tau[w(\cdot)])$ are not continuous in $\tau$ but only measurable. Hypotheses (13) are suitable, as far as they can be directly verifiable for a given problem. Weaker hypotheses than (11) and (12) can also be considered. In this respect, important improvements are obtained from the use of nonsmooth analysis. For example, one can substitute (11) and (12) by the weaker conditions

$$
|L(t, x_1, u(t)) - L(t, x_2, u(t))| \leq \alpha(t) \|x_1 - x_2\| \quad (14)
$$

$$
|\varphi_i(t, x_1, u(t)) - \varphi_i(t, x_2, u(t))| \leq \alpha(t) \|x_1 - x_2\|
$$

and formulate the maximum principle in a nonsmooth setting, in terms of generalized gradients (see [7, 8]). Proving general versions of the maximum principle under weak hypotheses is still in progress and the interested reader is referred to the recent paper [30].

**Definition 10.** Let $(x(\cdot), u(\cdot))$ be admissible for $(P)$. We say that the quadruple $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$, $\psi_0 \in \mathbb{R}_0^+$ and $\psi(\cdot) \in W_{1,1}([a, b]; \mathbb{R}^n)$, is an extremal of $(P)$, if the following two conditions are satisfied for almost all $t \in [a, b]$:

the adjoint system

$$
\dot{\psi}(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), \psi_0, \psi(t)); \quad (15)
$$

the maximality condition

$$
H(t, x(t), u(t), \psi_0, \psi(t)) = \sup_{u \in \mathbb{R}^r} H(t, x(t), u, \psi_0, \psi(t)); \quad (16)
$$
where the Hamiltonian equals

\[ H(t, x, u, \psi_0, \psi) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u). \]

**Definition 11.** Let \((t(\cdot), z(\cdot), v(\cdot), w(\cdot))\) be admissible for \((P_\tau)\). The 7-tuple \((t(\cdot), z(\cdot), v(\cdot), w(\cdot), p_0, p_t(\cdot), p_z(\cdot))\), \(p_0 \in \mathbb{R}_0^r, p_t(\cdot) \in W_{1,\infty}([a,b]; \mathbb{R})\) and \(p_z(\cdot) \in W_{1,1}([a,b]; \mathbb{R}^n)\), is said to be an extremal of \((P_\tau)\), if the following two conditions are satisfied for almost all \(\tau \in [a,b]\):

the adjoint system

\[
\begin{cases}
    p_t'(\tau) = -\frac{\partial H}{\partial t}(t(\tau), z(\tau), v(\tau), w(\tau), p_0, p_t(\tau), p_z(\tau)) , \\
    p_z'(\tau) = -\frac{\partial H}{\partial z}(t(\tau), z(\tau), v(\tau), w(\tau), p_0, p_t(\tau), p_z(\tau)) ;
\end{cases}
\]  

(17)

the maximality condition

\[
\mathcal{H}(t(\tau), z(\tau), v(\tau), w(\tau), p_0, p_t(\tau), p_z(\tau)) = \sup_{\substack{v \in [0.5,1.5] \, w \in \mathbb{R}^r}} \mathcal{H}(t(\tau), z(\tau), v, w, p_0, p_t(\tau), p_z(\tau)) ;
\]  

(18)

where the Hamiltonian equals

\[
\mathcal{H}(t, z, v, w, p_0, p_t, p_z) = (p_0 L(t, z, w) + p_t + p_z \cdot \varphi(t, z, w)) \, v.
\]

**Remark 12.** Functions \(H\) and \(\mathcal{H}\), respectively the Hamiltonians in Definitions 10 and 11, are related by the following equality:

\[
\mathcal{H}(t, z, v, w, p_0, p_t, p_z) = (H(t, z, w, p_0, p_z) + p_t) \, v .
\]  

(19)

From it one concludes that

\[
\begin{align*}
\frac{\partial \mathcal{H}}{\partial t} &= \frac{\partial H}{\partial t} \, v, \\
\frac{\partial \mathcal{H}}{\partial z} &= \frac{\partial H}{\partial x} \, v.
\end{align*}
\]  

(20)

(21)

**Definition 13.** An extremal is called normal if the cost multiplier \((\psi_0\) in the Definition 10 and \(p_0\) in the Definition 11) is different from zero and abnormal if it vanishes.

**Remark 14.** As far as the Hamiltonian is homogeneous with respect to the Hamiltonian multipliers, for normal extremals one can always consider, by scaling, that the cost multiplier takes value \(-1\).
Lemma 16. Let \( (x(\cdot), u(\cdot), \psi_0, \psi(\cdot)) \) be an extremal of \((P)\). Then, for any function \( v(\cdot) \in L_\infty([a, b]; [0.5, 1.5]) \) satisfying \( \int_a^b v(s) \, ds = b - a \), the 7-tuple \((t(\cdot), z(\cdot), v(\cdot), p_0, p_t(\cdot), p_z(\cdot))\) defined by

\[
\begin{align*}
t(\tau) &= a + \int_a^\tau v(s) \, ds, \\
z(\tau) &= x(t(\tau)), \quad w(\tau) = u(t(\tau)), \\
p_0 &= \psi_0, \quad p_z(\tau) = \psi(t(\tau)), \\
p_t(\tau) &= -H(t(\tau), x(t(\tau)), u(t(\tau)), \psi_0, \psi(t(\tau)))
\end{align*}
\]

is an extremal of \((P_t)\) with \( H(t(\tau), z(\tau), v(\tau), w(\tau), p_0, p_t(\tau), p_z(\tau)) \equiv 0 \).

Proof. From Lemma 15 we know that such 7-tuple is admissible for \((P_t)\). The maximality condition \((18)\) is trivially satisfied since we are in the singular case: from \((15)\) the Hamiltonian \( H \) vanishes for \( p_t = -H(t, z, w, p_0, p_z) \). It remains to prove the adjoint system \((17)\). Since \( \frac{dH}{dt} = \frac{\partial H}{\partial t} \) along the extremals (see e.g. [27] or [4]) the derivative of \( p_t(\tau) \) with respect to \( \tau \) is given by

\[
\frac{dp_t}{d\tau} = \frac{dH}{d\tau} = -\frac{\partial H}{\partial t} \frac{dt}{d\tau} + \frac{\partial H}{\partial t} \frac{dt}{d\tau} = -\frac{\partial H}{\partial t} v.
\]

From relation \((20)\) the first of the equalities \((17)\) is proved: \( p_t' = -\frac{\partial H}{\partial t} \).

Similarly, as far as \( p_z(\tau) = \psi(t(\tau)) \) and from \((15)\) \( \frac{d}{dt} \psi(t) = -\frac{\partial H}{\partial x} \), it follows from \((21)\) that \( p_z' = \frac{d\psi(t)}{dt} \frac{dt}{d\tau} = -\frac{\partial H}{\partial x} v = -\frac{\partial H}{\partial x} \).

It is also possible to construct an extremal of problem \((P)\) given an extremal of \((P_t)\).

Lemma 17. Let \((t(\cdot), z(\cdot), v(\cdot), w(\cdot), p_0, p_t(\cdot), p_z(\cdot))\) be an extremal of \((P_t)\). Then \((x(\cdot), u(\cdot), \psi_0, \psi(\cdot)) = (z(t(\cdot)), w(t(\cdot)), p_0, p_z(t(\cdot)))\) is an extremal of \((P)\) with \( \tau(\cdot) \) the inverse function of \( t(\cdot) \).
Proof. From Lemma 6 we know that the pair \((x(\cdot), u(\cdot))\) is admissible for \((P)\). Direct calculations show that

\[
\dot{\psi} = \frac{d}{dt}p_z(\tau) = \frac{dp_z(\tau)}{d\tau} \frac{d\tau}{dt} = -\frac{\partial H}{\partial z} v.
\]

From (21) the required adjoint system is obtained: \(\dot{\psi} = -\frac{\partial H}{\partial x}\). Maximal condition \((18)\) implies that

\[
H(t(\tau), z(\tau), w(\tau), p_0, p_t(\tau), p_z(\tau)) = \sup_{w \in \mathbb{R}^r} H(t(\tau), z(\tau), w, p_0, p_t(\tau), p_z(\tau))
\]

for almost all \(\tau \in [a, b]\). Given the relation \((19)\) one can write that

\[
H(t(\tau), z(\tau), w(\tau), p_0, p_z(\tau)) = \sup_{w \in \mathbb{R}^r} H(t(\tau), z(\tau), w, p_0, p_z(\tau))
\]

Putting \(\tau = \tau(t)\) we obtain the maximality condition \((16)\). □

Lemmas 16 and 17 establish a correspondence between abnormal extremals of problems \((P)\) and \((P_\tau)\).

**Corollary 18.** If there are no abnormal extremals of problem \((P)\) then there are no abnormal extremals of problem \((P_\tau)\). If there are no abnormal extremals of \((P_\tau)\) then there are also no abnormal extremals of \((P)\).

**Definition 19.** We call a control an abnormal extremal control if it corresponds to an abnormal extremal.

**Proposition 20.** If \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) is a minimizer of problem \((P)\) and \(\tilde{u}(\cdot)\) is not an abnormal extremal control, then the minimizing control \(\tilde{v} \equiv 1\) of Proposition 9 is not an abnormal extremal control too.

In the next section we will use Proposition 20 to show that, for non-abnormal minimizers, the Lipschitzian regularity conditions we are looking for, assuring that the minimizing controls, predicted by Tonelli’s existence theorem, are indeed bounded, appear from the applicability conditions of the maximum principle to problem \((P_\tau[\tilde{u}(\cdot)])\).

### 5 The General Regularity Result

Filippov [18] gave the first general existence theorem for optimal control (the original paper, in Russian, appears in 1959). There exist now an extensive literature on the existence of solutions to problems of optimal control. We refer the interested reader to the book [4] for significant results, various
formulations, and detailed discussions. Follows a set of conditions, of the type of Tonelli [31], that guarantee existence of minimizer for problem \((P)\).

**“Tonelli” existence theorem for \((P)\).** Problem \((P)\) has a minimizer \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) with \(\tilde{u}(\cdot) \in L_1([a, b]; \mathbb{R}^r)\), provided there exists at least one admissible pair, functions \(L(\cdot, \cdot, \cdot)\) and \(\varphi(\cdot, \cdot, \cdot)\) are continuous, and the following convexity and coercivity conditions hold:

**(convexity)** Functions \(L(t, x, \cdot)\) and \(\varphi(t, x, \cdot)\) are convex for all \((t, x)\);

**(coercivity)** There exists a function \(\theta : \mathbb{R}_+^\infty \to \mathbb{R}\), bounded below, such that

\[
L(t, x, u) \geq \theta (\|\varphi(t, x, u)\|) \quad \text{for all } (t, x, u); \quad (22)
\]

\[
\lim_{r \to +\infty} \frac{\theta(r)}{r} = +\infty; \quad (23)
\]

\[
\lim_{\|u\| \to +\infty} \|\varphi(t, x, u)\| = +\infty \quad \text{for all } (t, x). \quad (24)
\]

**Remark 21.** For the basic problem of the calculus of variations one has \(\varphi = u\) and the theorem above coincides with the classical Tonelli existence theorem.

Analyzing the hypotheses of both necessary optimality conditions and existence theorem, one comes to the conclusion that the requirements of existence theory do not imply those of the maximum principle. Given a problem, it may happen that the necessary optimality conditions are valid while existence is not guaranteed; or it may happen that the minimizers predicted by the existence theory fail to be extremals. Follows the main results of the paper.

**Theorem 22.** Under the above hypothesis of coercivity, all control minimizers \(\tilde{u}(\cdot)\) of \((P)\), which are not abnormal extremal controls, are essentially bounded on \([a, b]\) if the applicability conditions of the maximum principle (for example \((13), (11) - (12)\) or \((14)\)) to functions \(F\) and \(f\) of problem \((P, [\tilde{u}(\cdot)])\) are assured.

**Remark 23.** Convexity is not required in the regularity result of Theorem 22 in order to establish the Lipschitzian regularity of the (non-abnormal) minimizing trajectories \(\tilde{x}(\cdot)\). Convexity is only required to establish the existence of minimizers, not the regularity. This fact is important since existence theorems without the convexity assumptions are a question of great interest (see e.g. [26] and the references therein).

\[\text{1 A precise statement of possible regularity conditions are found in Theorem 24.}\]
Applying the hypotheses (13) of the maximum principle to functions \( F \) and \( f \) of problem \((P_\tau [\bar{u}(\cdot)])\), the following result is trivially obtained.

**Theorem 24.** Under the hypothesis of coercivity, the growth conditions: there exist constants \( c > 0 \) and \( k \) such that

\[
\left| \frac{\partial L}{\partial t} \right| \leq c |L| + k, \quad \left| \frac{\partial L}{\partial x} \right| \leq c |L| + k, \quad \left| \frac{\partial \varphi}{\partial t} \right| \leq c \|\varphi\| + k, \quad \left| \frac{\partial \varphi_i}{\partial x} \right| \leq c |\varphi_i| + k \quad (i = 1, \ldots, n);
\]

imply that all minimizers \( \bar{u}(\cdot) \) of \((P)\), which are not abnormal extremal controls, are essentially bounded on \([a, b]\).

In the special cases of a time-invariant linear control system, or for the problems of the calculus of variations, Theorem 24 gives the typical growth conditions, of the type of Tonelli–Morrey, obtained in previous results [13, 15, 14]. For the case of control-affine dynamics, the growth condition of Theorem 25 is less restrictive.

A minimizer \( \bar{u}(\cdot) \) which is not essentially bounded may fail to satisfy the Pontryagin Maximum Principle. As far as essentially bounded minimizers are concerned, the Pontryagin Maximum Principle is valid.

**Theorem 25.** Under the hypotheses of Theorem 24, all minimizers of \((P)\) are Pontryagin extremals. Furthermore, it is valid a weaker form of the Pontryagin maximum principle in which the adjoint multipliers \( \psi(\cdot) \) are not required to be absolutely continuous, but are required instead to be merely Lipschitzian.

**Proof.** (Theorem 23) Let \((\bar{x}(\cdot), \bar{u}(\cdot))\) be a minimizer of \((P)\). From Propositions 4 and 20 and by the assumptions of the theorem, we know that there exist absolutely continuous functions \( \bar{p}_t(\cdot) \) and \( \bar{p}_z(\cdot) \) such that for almost all points \( \tau \in [a, b] \)

\[
v \mapsto [-L(\tau, \bar{x}(\tau), \bar{u}(\tau)) + \bar{p}_t(\tau) + \bar{p}_z(\tau) \cdot \varphi(\tau, \bar{x}(\tau), \bar{u}(\tau))] v
\]

is maximized at \( v = 1 \) on the interval \([0.5, 1.5]\). This implies that

\[
L(\tau, \bar{x}(\tau), \bar{u}(\tau)) = \bar{p}_t(\tau) + \bar{p}_z(\tau) \cdot \varphi(\tau, \bar{x}(\tau), \bar{u}(\tau)). \quad (25)
\]

Let \( |\bar{p}_t(\tau)| \leq M \) and \( |\bar{p}_z(\tau)| \leq M \) on \([a, b]\). Dividing both sides of inequality (23) by \( \|\varphi(\tau, \bar{x}(\tau), \bar{u}(\tau))\| \) and using the coercivity hypothesis (22), one obtains

\[
\frac{\Theta(\|\varphi(\tau, \bar{x}(\tau), \bar{u}(\tau))\|)}{\|\varphi(\tau, \bar{x}(\tau), \bar{u}(\tau))\|} \leq M \frac{1 + \|\varphi(\tau, \bar{x}(\tau), \bar{u}(\tau))\|}{\|\varphi(\tau, \bar{x}(\tau), \bar{u}(\tau))\|}.
\]

The coercivity condition (23)–(24) yields the essential boundedness of \( \bar{u}(\cdot) \) on \([a, b]\).
6 An Example

As far as Theorem 24 is able to cover optimal control problems with dynamics which is nonlinear both in the state and in the control variables, plenty of examples possessing minimizers according to the existence theory can be easily constructed for which our result is applicable while previously known Lipschitzian regularity conditions, such as those in [14] and [28], fail. Follows one such example with \( n = r = 2 \).

Example 26.

\[
\int_0^1 (u_1^2(t) + u_2^2(t)) \left( e^{2(x_1(t)+x_2(t))} + 1 \right) \, dt \rightarrow \min
\]

\[
\begin{cases}
  \dot{x}_1(t) = \sqrt{u_1^2(t) + u_2^2(t)} \\
  \dot{x}_2(t) = u_2(t) e^{x_1(t)+x_2(t)}
\end{cases}
\]

\( x_1(0) = 0, x_1(1) = 1, x_2(0) = 1, x_2(1) = 1. \)

Here we have:

\[
L(x_1, x_2, u_1, u_2) = (u_1^2 + u_2^2) \left( e^{2(x_1+x_2)} + 1 \right);
\]

\[
\varphi(x_1, x_2, u_1, u_2) = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix} = \begin{bmatrix} \sqrt{u_1^2 + u_2^2} \\ u_2 e^{x_1+x_2} \end{bmatrix}.
\]

Clearly, all conditions of Tonelli’s existence theorem are satisfied: an admissible quadruple is \((x_1(t), x_2(t), u_1(t), u_2(t)) = (t, 1, 1, 0)\); functions \( L(\cdot, \cdot, \cdot, \cdot) \) and \( \varphi(\cdot, \cdot, \cdot, \cdot) \) are continuous in \( \mathbb{R}^4 \); function \( L(x_1, x_2, \cdot, \cdot) \) is strictly convex; \( \varphi(x_1, x_2, \cdot, \cdot) \) is convex; from the inequality \( L = (u_1^2 + u_2^2) \left( e^{2(x_1+x_2)} + 1 \right) \geq u_1^2 + u_2^2 + u_2 e^{2(x_1+x_2)} \) we have quadratic coercivity \( (\theta(r) = r^2) \). Therefore the problem has a solution for \( x_1(\cdot), x_2(\cdot) \in W_{1,1}([0, 1]; \mathbb{R}) \) and \( u_1(\cdot), u_2(\cdot) \in L_1([0, 1]; \mathbb{R}) \). Smooth assumptions on data \([l] \) are satisfied, since \( L(\cdot, \cdot, \cdot, \cdot) \) and \( \varphi(\cdot, \cdot, u_1, u_2) \) are of class \( C^\infty \). Theorem 24 allow us to conclude that all minimizing controls, which are not abnormal extremal controls, are bounded:

- The conditions on \( \frac{\partial L}{\partial t} \) and \( \frac{\partial \varphi}{\partial t} \) are trivially satisfied as far as the problem is autonomous: \( L \) and \( \varphi \) do not depend explicitly on the time variable.

- The growth conditions on \( \frac{\partial L}{\partial x} \) and \( \frac{\partial \varphi}{\partial x} \) are also satisfied: \( \frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial x_2} = 2e^{2(x_1+x_2)} (u_1^2 + u_2^2) \leq 2L; \frac{\partial \varphi_1}{\partial x_1} = \frac{\partial \varphi_1}{\partial x_2} = 0; \frac{\partial \varphi_2}{\partial x_1} = \frac{\partial \varphi_2}{\partial x_2} = \varphi_2. \)

Unbounded minimizers, if there are any, are abnormal extremal controls. From Theorem 25 all minimizing controls of the problem (normal or abnormal) can be identified via the Pontryagin maximum principle.
7 Final Remarks

In this paper we study properties of minimizing trajectories for general problems of optimal control in the cases where controls are unconstrained (like in the calculus of variations). We provide conditions which guarantee Lipschitzian regularity of the minimizing trajectories for the Lagrange problem of optimal control in the general nonlinear case. These conditions solve the discrepancy between the optimality and existence results, assuring that minimizers predicted by the existence theory satisfy the optimality conditions. At the same time, undesirable phenomena, like the Lavrentiev one, are naturally precluded. We show that the conditions of Lipschitzian regularity are related with the applicability conditions of Pontryagin’s maximum principle. To deal with dynamics which are control-affine, the classical Pontryagin maximum principle \[27\] is enough (see \[28\]). To treat the general case, a maximum principle under weak assumptions, like the one in \[1\], is necessary. Our approach is based on the relationship of the extremals of the Lagrange problem with the extremals of an auxiliary problem, and on the subsequent utilization of Pontryagin’s maximum principle to the later problem. The maximality condition of Pontryagin’s maximum principle together with the coercivity assumption of the existence theorem imply the Lipschitzian regularity of the corresponding minimizer of the original problem. This approach allows us to deal with more general class of problems of optimal control with nonlinear dynamics.

It remains to clarify the interconnection between Lipschitzian regularity and abnormal extremality. For the problems of the calculus of variations studied in \[13\] and \[15\] no abnormal extremals exist. For the optimal control problems considered in \[14\] and \[28\], abnormal extremals are, like here, put aside. The question of how to establish Lipschitzian regularity for the abnormal minimizing trajectories seems to be a completely open question.

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