TRISECANT FLOPS, THEIR ASSOCIATED K3 SURFACES AND THE RATIONALITY OF SOME FANO FOURFOLDS

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Abstract. We prove the rationality of some Fano fourfolds via Mori Theory and the Minimal Model Program. This shows a connection between some admissible cubic fourfolds and some birational models of their associated K3 surfaces, pointing out that the rationality of admissible cubic fourfolds may be closely related to the construction of special projections of the associated K3 surfaces. We give several explicit examples of our construction, one of which relates the Fano divisor $C_{14}$ of cubic fourfolds with a divisor of rational Gushel-Mukai fourfolds inside the moduli space.

Introduction

The study of the rationality of higher dimensional Fano manifolds is a very active area of research. Many new and interesting contributions and conjectures appeared in the last decades, mostly concerning the irrationality of very general Fano hypersurfaces (see for example the recent survey [Kol19] and the references therein). Deep recent contributions in [KT19] imply that the locus of geometrically rational fibers in a smooth family of projective manifolds is closed under specialisation, improving substantially our understanding of the loci of rational objects in the corresponding moduli spaces, see [HPT18] for very significative examples in dimension four. Notwithstanding, the irrationality of the (very) general cubic fourfold and the complete description of the rational ones remain two of the most challenging open problems.

A great amount of recent theoretical work on cubic fourfolds (see for example the surveys [Has16, Kuz16]) lead to the expectation that the very general ones might be irrational and to the specification of infinitely many irreducible divisors $C_d$ of special admissible cubic fourfolds of discriminant $d$ in the moduli space $C$, whose union should be the locus of rational cubic fourfolds (Kuznetsov Conjecture). According to this conjecture, the rationality of cubics in $C_d$ depends on the existence of an associated K3 surface in the sense of Hassett/Kuznetsov, see [Has16, Kuz16].

The first admissible values are $d = 14, 26, 38, 42, 62, 74, 78, 86$. Fano showed the rationality of a general cubic fourfold in $C_{14}$, see [Fan43, BRS19]; while every cubic fourfold in the irreducible divisors $C_{26}$ and $C_{38}$ is rational by the main results of [RS19] (see also [RS18]). The proof was achieved by constructing surfaces $S_d \subset \mathbb{P}^5$, contained in a general cubic fourfold of $C_d$ and admitting a four dimensional family of $(3e - 1)$-secant curves of degree $e \geq 2$ parametrized by a rational variety, with the property that through a general point of $\mathbb{P}^5$ there passes a unique curve of the family. Then the cubics through $S_d$ become rational.
sections of the universal family and hence are birational to the rational parameter space (see also Theorem 1.1 here).

This method did not clarify the relation with the associated K3 surfaces and even the construction of explicit birational maps from general cubics \( X \subset \mathbb{P}^5 \) in \( C_{26} \) and in \( C_{38} \) to \( \mathbb{P}^4 \) (or to other notable rational smooth fourfolds \( W \)) in [RS18] apparently did not provide a birational incarnation in \( \mathbb{P}^4 \) (or in \( W \)) of a minimal K3 surface of genus 14, respectively of genus 20, determining the linear system of the inverse map. Indeed, the base loci of the linear systems of hypersurfaces of degree \( 3e - 1 \) having points of multiplicity \( e \) along the corresponding \( S_d \)'s giving the birational maps \( \mu : X \rightarrow \mathbb{P}^4 \) are intractable schemes while the base loci of the inverse maps \( \mu^{-1} : \mathbb{P}^4 \rightarrow X \subset \mathbb{P}^5 \) are even worse (see [RS18, Table 2]).

Here we study all these phenomena via Mori Theory and via the Minimal Model Program to explain the birational nature of the maps \( \mu : X \rightarrow W \) and of their inverses, to provide a description of their base loci and, finally, to illustrate the relations of these explicit birational maps with the associated K3 surfaces. We start with the observation that many of the known examples of surfaces \( S_d \subset \mathbb{P}^5 \) used to describe the special divisors \( C_d \) (not only the admissible ones) have ideal generated by cubic forms defining a map \( \varphi : \mathbb{P}^5 \rightarrow Z \subset \mathbb{P}^N \) which is birational onto its image, so that the restriction to a general cubic \( X \) through \( S_d \) defines a birational map \( \varphi : X \rightarrow Y \subset \mathbb{P}^{N-1} \) with \( Y \) a general hyperplane section of \( Z \). In many cases, the birational morphism \( \tilde{\varphi} : X' = \text{Bl}_{S_d} X \rightarrow Y \) is a small contraction, whose exceptional locus consists of (the union of) smooth surface(s) \( T' \subset X' \) ruled by the strict transforms of trisecant lines to \( S_d \). Since \( K_{X'} \) is zero on the strict transforms of trisecant lines, the map \( \tilde{\varphi} \) is a flop small contraction. In Theorem 2.6 we show that under the previous hypothesis there exists a flop \( \tilde{\psi} : W' \rightarrow Y \) of the surface \( T' \subset X' \) with \( W' \) a smooth projective fourfold. This Trisecant Flop \( \tau : X' \rightarrow W' \) is constructed by analyzing the splitting of \( N_{T'/X'} \) restricted to a general strict transform of a trisecant line to \( S_d \subset \mathbb{P}^5 \), see Remark 2.4. Then we remark that the existence of a congruence of \( (3e - 1) \)-secant rational curves to \( S_d \) of degree \( e \geq 2 \) produces an extremal ray on \( W' \) with divisorial locus, producing a birational morphism \( \nu : W' \rightarrow W \) with \( W \) a \( \mathbb{Q} \)-factorial Fano variety with \( \text{Pic}(W) \simeq \mathbb{Z} \) and index \( i(W) \), see Theorem 2.9. The birational morphism \( \nu \) is (generically) the inverse of the blow-up of an irreducible surface \( U \subset W \), which in the known admissible cases is a birational incarnation of the associated K3 surface to \( X \). Moreover, the map \( \mu = \nu \circ \tau \circ \lambda^{-1} : X \rightarrow W \) is given by a linear system of forms of degree \( 3e - 1 \) with points of multiplicity \( e \) along \( S_d \), while \( \mu^{-1} : W \rightarrow X \) is given by a linear system of divisors in \( |\mathcal{O}_W(e \cdot i(W) - 1)| \) having points of multiplicity \( e \) along \( U \).

Everything is captured by the following diagram:

\[
\begin{align*}
\text{Bl}_{T'} X' &= \text{Bl}_{R'} W' \\
\sigma &\quad \omega \\
\tau &\quad \nu \\
\lambda &\quad \phi \\
X &\quad Y &\quad W' \\
\phi &\quad \psi &\quad \psi \\
X &\quad Y &\quad W \\
\mu &
\end{align*}
\]
From these results (and from the examples considered in Sections 3 and 4) it emerges that the birational association between admissible cubic fourfolds and K3 surfaces passes through the construction of very particular (and in many cases also singular) non minimal birational models of these surfaces in the fourfolds $W$ by mean of peculiar linear systems of hyperplane sections (often with base points of high multiplicities) on the minimal K3 surfaces, as for the classical case considered by Fano. The discussion in Section 5 provides a list of possible candidates of K3 surfaces $U$ and of rational fourfolds $W$ with $U \subset W \subset \mathbb{P}^N$ (and with $N$ of relative small dimension), which might lead to the proof of other cases of the Kuznetsov Conjecture for more relatively small admissible values. Some of these examples of non minimal K3 surfaces have been studied by Voisin in [Voi19, §4] to prove some vanishing results related to Lehn’s Conjecture and have been later considered by Fontanari and Sernesi in [FS19, Theorem 10].

Similarly to Hassett’s analysis of cubic fourfolds, in [DIM15] and [DK18a, DK16, DK18b] the authors studied Gushel-Mukai fourfolds, that is quadratic sections of linear sections of cones over the Grassmannian $\mathbb{G}(1,4) \subset \mathbb{P}^9$, defined their coarse moduli space $\mathcal{G}M$ of dimension 24 and introduced, via Hodge Theory and via the Period map, the analogous definitions of countably many special divisors inside $\mathcal{G}M$. Our contribution here is an alternative description of a divisor $\mathcal{K}$ of rational GM-fourfolds inside the moduli space, studied firstly in [DIM15, Subsection 7.3]. We show the rationality of these fourfolds in Theorem 4.3 via a Trisecant Flop and via the contraction of the extremal ray determined by a congruence of secant rational curves. In particular we shall see that a general GM-fourfold $W \in \mathcal{K}$ is birational to a general cubic fourfold $X$ in the Fano divisor $\mathcal{C}_{14}$ via explicit birational maps given by linear systems with multiplicities along the associated K3 surfaces of $X$ and of the GM fourfold $W$. The maps $\mu: X \dashrightarrow W$ in this case provide a birational perspective to the association between Pfaffian fourfolds and smooth K3 surfaces of degree 14 and genus 8, developed originally by Beauville and Donagi in [BD85] and which has been the starting point of the modern study of special cubic fourfolds (and of hyperkähler manifolds) from different points of view.

Acknowledgements. This project started from an intuition of János Kollár that the maps $\mu: X \dashrightarrow W$ should be useful to describe the congruences (see [RS18, Subsection 1.1], Subsection 1.1 here, and [Kol19, Section 5, §29]). We are very indebted to him for his suggestion. We wish to thank Michele Bolognesi for his continuous support along the years and for many useful conversations on these (and on many other) topics. We are also grateful to Sandro Verra for several discussions on the subject.

1. Preliminaries

1.1. Congruences of $(3e-1)$-secant curves of degree $e$ to surfaces in $\mathbb{P}^5$. The following definitions have been introduced in [RS19, Section 1]. Let $\mathcal{H}$ be an irreducible proper family of (rational or of fixed arithmetic genus) curves of degree $e$ in $\mathbb{P}^5$ whose general element is irreducible. We have a diagram

$$
\begin{array}{c}
\mathcal{D} \\
\pi \\
\downarrow \\
\mathcal{H} \\
\mathbb{P}^5 \\
\end{array}
$$

...
where \( \pi : \mathcal{D} \to \mathcal{H} \) is the universal family over \( \mathcal{H} \) and where \( p : \mathcal{D} \to \mathbb{P}^5 \) is the tautological morphism. Suppose moreover that \( p \) is birational and that a general member \( [C] \in \mathcal{H} \) is \((re-1)\)-secant to an irreducible surface \( S \subset \mathbb{P}^5 \), that is \( C \cap S \) is a length \( re-1 \) scheme, \( r \in \mathbb{N} \). We shall call such a family \( \mathcal{H} \) (or \( \mathcal{D} \) or \( \pi : \mathcal{D} \to \mathcal{H} \)) a congruence of \((re-1)\)-secant curves of degree \( e \) to \( S \). Let us remark that necessarily \( \dim(\mathcal{H}) = 4 \).

An irreducible hypersurface \( X \in |H^0(\mathcal{I}_S(r))| \) is said to be transversal to the congruence \( \mathcal{H} \) if the unique curve of the congruence passing through a general point \( p \in X \) is not contained in \( X \). A crucial result is the following.

**Theorem 1.1.** [RS19, Theorem 1] Let \( S \subset \mathbb{P}^5 \) be a surface admitting a congruence of \((re-1)\)-secant curves of degree \( e \) parametrized by \( \mathcal{H} \). If \( X \in |H^0(\mathcal{I}_S(r))| \) is an irreducible hypersurface transversal to \( \mathcal{H} \), then \( X \) is birational to \( \mathcal{H} \).

If the map \( \Phi = \Phi|_{H^0(\mathcal{I}_S(r))} : \mathbb{P}^5 \to \mathbb{P}(H^0(\mathcal{I}_S(r))) \) is birational onto its image, then a general hypersurface \( X \in |H^0(\mathcal{I}_S(r))| \) is birational to \( \mathcal{H} \).

Moreover, under the previous hypothesis on \( \Phi \), if a general element in \( |H^0(\mathcal{I}_S(r))| \) is smooth, then every \( X \in |H^0(\mathcal{I}_S(r))| \) with at worst rational singularities is birational to \( \mathcal{H} \).

Since \( p : \mathcal{D} \to \mathbb{P}^5 \) is birational, we also have a rational map

\[
\varphi = \pi \circ p^{-1} : \mathbb{P}^5 \to \mathcal{H},
\]

whose fiber through a general \( p \in \mathbb{P}^5 \), \( F = \varphi^{-1}(\varphi(p)) \), is the unique curve of the congruence passing through \( p \).

It is natural to ask what linear systems on \( \mathbb{P}^5 \) are associated to the abstract birational maps \( \varphi : \mathbb{P}^5 \to \mathcal{H} \) as above or to their restrictions to a general \( X \in |H^0(\mathcal{I}_S(r))| \). The linear system \( |H^0(\mathcal{I}_S(re-1))| \), when not empty, contracts the fibers of \( \varphi \) and in [RS18] we showed that, quite surprisingly, in many cases they can provide a birational geometric realization of \( \varphi \) for \( r=3 \), yielding birational maps from cubic hypersurfaces through \( S \) to \( \mathcal{H} \) with \( \mathcal{H} = \mathbb{P}^4 \) or with \( \mathcal{H} \) a notable Fano fourfold. In the sequel we shall develop a theoretical framework for these phenomena in order to be able to understand also the birational maps defined by the previous linear systems.

### 1.2. Divisorial contractions, small contractions and flops.

We introduce some general definitions of the MMP, adapting them to our setting.

Let \( X \) be a smooth projective irreducible fourfold defined over the complex field with \( \rho(X) = 1 \) (here \( \rho(X) \) denotes the Picard number of \( X \)) and let \( \varphi : X \to W \) be a birational map onto a smooth (or at least \( \mathbb{Q} \)-factorial) irreducible projective fourfold, whose base locus scheme contains a surface \( S \) with at most a finite number of nodes.

Let \( \lambda : X' = \text{Bl}_S X \to X \) be the blow-up of \( S \) and consider the diagram:

\[
\begin{array}{ccc}
\text{Bl}_S X & \xrightarrow{\tilde{\varphi}} & W \\
\lambda \downarrow & & \downarrow \varphi \\
X & \xrightarrow{\varphi} & W.
\end{array}
\]

When \( W \) is smooth, the complexity of the birational map \( \varphi : X \to W \) depends on the base locus scheme of \( \tilde{\varphi} : \text{Bl}_S X \to W \). Surely the easiest case to be considered is when


$\tilde{\varphi} : \text{Bl}_S X \to W$ is a morphism, that is $\varphi$ is a special birational map in the sense of Semple and Tyrrell (solved by a single blow-up along a smooth irreducible variety).

If $X \subset \mathbb{P}^5$ is a cubic fourfold and if $S \subset X$ is smooth, few examples of special birational maps of the above type exist. Two examples of maps of this kind were firstly considered by Fano in [Fan43], have been revisited in modern terms in [AR04, BRS19] and played a fundamental role in the formulation of Kuznetsov Conjecture.

**Example 1.2.** Letting $\varphi : X \dasharrow W$ be a special birational map with $X \subset \mathbb{P}^5$ a cubic fourfold, letting $B \subset W$ be the base locus scheme of $\varphi^{-1}$ and letting $U = B_{\text{red}}$, Fano’s examples are the following:

(i) $S \subset \mathbb{P}^5$ is a smooth quintic del Pezzo surface, $W = \mathbb{P}^4$, $\varphi$ is given by $|H^0(I_S(2))|$ and $U \subset \mathbb{P}^4$ is a surface of degree 9 and sectional genus 8 having at most a finite number of singular points corresponding to planes in $X$ spanned by conics in $S$. If non singular, the surface $U$ is the projection from a 5-secant $\mathbb{P}^3 \subset \mathbb{P}^5$ of a smooth K3 surface of degree 14 and genus 8 and $\varphi^{-1}$ is given by $|H^0(I_U(4))|$. 

(ii) $S \subset \mathbb{P}^5$ is a smooth quartic rational normal scroll, $W = Q^4 \subset \mathbb{P}^5$ is a smooth quadric hypersurface, $\varphi$ is given by $|H^0(I_S(2))|$ and $U \subset \mathbb{P}^4$ is a surface of degree 10 and sectional genus 8 having at most a finite number of singular points corresponding to planes in $X$ spanned by conics in $S$. If non singular, the surface $U$ is the projection from the tangent plane of a smooth K3 surface of degree 14 and genus 8 and $\varphi^{-1}$ is given by $|H^0(I_U(3))|_{\mathbb{Q}}$.

**Remark 1.3.** The two surfaces $S \subset \mathbb{P}^5$ appearing in Example 1.2 are the only smooth surfaces in $\mathbb{P}^5$ admitting a congruence of secant lines ($r = 3$ and $e = 1$ in the definition), see for example [Rus00]. The lines of the congruence contained in $X$ describe the exceptional locus $\mathcal{E}$ of $\tilde{\varphi}$ (or equivalently the exceptional locus $\lambda(\tilde{\mathcal{E}})$ of $\varphi : X \dasharrow W$) and are birationally parametrized by the surfaces $U \subset W$.

The general MMP philosophy suggests that meaningful birational properties of (rational) cubic fourfolds might be related to small contractions from $X'$. So one can start to investigate birational properties of cubic fourfolds from the point of view of the MMP and, if they exist, to consider the most elementary links in the Sarkisov Program associated to small contractions, i.e. flops and flips (one may consult [HK13] for results about this program in arbitrary dimension).

**Definition 1.4.** Let $X$ be a smooth irreducible projective variety (from now on a projective manifold) and let $\tilde{\varphi} : X \to Y$ be a small contraction, i.e. $\tilde{\varphi}$ is a birational morphism onto a normal variety $Y$ inducing an isomorphism in codimension one and such that $\rho(X/Y) = 1$.

If $K_X \cdot C = 0$ for every irreducible curve contracted by $\tilde{\varphi}$, then $\tilde{\varphi} : X \to Y$ is called a small flop contraction. A small flop contraction $\tilde{\psi} : W \to Y$ with $W$ a projective manifold is called a flop of $\tilde{\varphi}$.

The resulting birational map $\tau = \tilde{\psi}^{-1} @ \tilde{\varphi} : X \dasharrow W$ is usually called a flop if it is not an isomorphism. Since we assume $\rho(X/Y) = 1 = \rho(W/Y)$, given $\tilde{\varphi}$ one can prove that the morphism $\tilde{\psi}$, if it exists, is unique as soon as $\tau$ is not an isomorphism.

One can flop the small contraction $\tilde{\varphi} : X \to Y$ by constructing a projective manifold $V$ and two birational morphisms $\sigma : V \to X$ and $\omega : V \to W$ such that $\sigma^*(K_X) = \omega^*(K_W)$. 
This means that the exceptional locus of $\sigma$, which is divisorial by the smoothness of $X$, is contracted by $\omega$ and that we have a commutative diagram:

\[
\begin{array}{ccc}
V & \xrightarrow{\omega} & W \\
\sigma \downarrow & & \downarrow \tau \\
X & \xrightarrow{\phi} & Y \\
\end{array}
\]

First of all one may ask if there exist flops of this kind on the fourfolds $X' = \text{Bl}_S X$ obtained from cubic fourfolds $X \subset \mathbb{P}^5$ by blowing-up a mildly singular surface $S \subset X$. As we shall see this is the case under some hypothesis and this occurrence is deeply related to the rationality of some special cubic fourfolds (or of other special fourfolds).

1.3. **Condition $K_3$ and examples of small contractions on cubic fourfolds.** Let us recall that, given homogeneous forms $f_i$ of degree $d_i \geq 1$, $i = 0, \ldots, M$, a vector of homogeneous forms $(g_0, \ldots, g_M)$ is a syzygy if $\sum_{i=0}^{M} f_i g_i = 0$. If $d_1 = \cdots = d_M = d$ and if $\deg(g_i) = h$ for every $i = 0, \ldots, M$, then we say that $(g_0, \ldots, g_M)$ is a syzygy of degree $h$ and for $h = 1$ we shall say that the syzygy is linear. For $i < j$ the syzygies $(0, \ldots, 0, f_j, 0, \ldots, 0, -f_i, 0, \ldots)$, corresponding to the trivial identity $f_j f_j + f_j(-f_i) = 0$ are called Koszul syzygies. We say that the Koszul syzygies are generated by the linear ones if they belong to the submodule generated by the linear syzygies. This is the condition $K_d$ introduced by Vermeire in [Ver01].

The next result provides a wide class of examples of rational maps with linear fibers (hence birational under mildly natural geometrical assumptions on their base locus scheme).

**Proposition 1.5.** ([Ver01, Proposition 2.8]) Let $f_0, \ldots, f_M$ be homogeneous forms in $N + 1$ variables of degree $d \geq 2$ satisfying condition $K_d$. Then the closure of each fiber of the rational map

$$\varphi = (f_0 : \cdots : f_M) : \mathbb{P}^N \to \mathbb{P}^M$$

is a linear space $\mathbb{P}^s$. For $s > 0$ the closure of the fiber intersects scheme theoretically the base locus scheme of $\varphi$ along a hypersurface of degree $d$.

**Remark 1.6.** Suppose that an irreducible surface $S \subset \mathbb{P}^5$ is scheme-theoretically defined by cubic equations satisfying condition $K_3$. Then, by Proposition 1.5, every positive dimensional fiber of $\varphi : \mathbb{P}^5 \to Z$ is a linear space $\mathbb{P}^s$ cutting $S$ in a cubic hypersurface $S \cap \mathbb{P}^s$ if $s > 0$. In particular $0 \leq s \leq 2$ (except some trivial cases) and $s = 2$ occurs only for planes spanned by cubic curves contained in $S$, which are mapped to a point by $\varphi$. Hence if condition $K_3$ for $S \subset \mathbb{P}^5$ holds and if a general cubic $X \subset \mathbb{P}^5$ through $S$ does not contain any plane spanned by cubic curves on $S$, the exceptional locus $T \subset X$ of the restriction of $\varphi$ to $X$ is ruled by proper trisecant lines. As we shall see in Section 2.1 the expected dimension of $T$ is two so that surfaces in $\mathbb{P}^5$ defined by cubic equations satisfying condition $K_3$ may naturally produce examples of small contractions on $X' = \text{Bl}_S X$.

2. **The Trisecant Flop and the Extremal Congruence Contraction**

We first introduce and study the behaviour of trisecant lines to a general non degenerate irreducible projective surface $S \subset \mathbb{P}^5$. 
2.1. The Hilbert scheme of trisecant lines to $S \subset \mathbb{P}^5$. For the generalities we shall follow the treatment in [Bau98]. Let $\text{Hilb}_r^c \mathbb{P}^5$ (respectively $\text{Hilb}_r^c S$) be the Hilbert scheme of $0$–dimensional length $r \geq 2$ subschemes of $\mathbb{P}^5$ (respectively of $S \subset \mathbb{P}^5$) and let $\text{Hilb}_r^c \mathbb{P}^5 \subset \text{Hilb}_r^c \mathbb{P}^5$ be the open non-singular subscheme consisting of curvilinear length $r$ subschemes, that is length $r$ subschemes which, locally around every point of their support, are contained in a smooth curve of $\mathbb{P}^5$. We can define $\text{Hilb}_r^c S$ as the scheme–theoretic intersection between $\text{Hilb}_r^c S$ and $\text{Hilb}_r^c \mathbb{P}^5$ inside $\text{Hilb}_r^c \mathbb{P}^5$.

Let $\text{Al}_r^c \mathbb{P}^5 \subset \text{Hilb}_r^c \mathbb{P}^5$ denote the subscheme consisting of aligned subschemes of length $r$, that is subschemes of length $r$ contained in a line. Finally

$$\text{Al}_r^c S = \text{Al}_r^c \mathbb{P}^5 \times_{\text{Hilb}_r^c \mathbb{P}^5} \text{Hilb}_r^c S$$

is the Hilbert scheme of length $r$ aligned subscheme of $S$. The schemes $\text{Hilb}_r^c \mathbb{P}^5$ and $\text{Al}_r^c \mathbb{P}^5$ are smooth of dimension $5r$ and $8 + r$, respectively. Moreover, if $S \subset \mathbb{P}^5$ is smooth, then $\text{Hilb}_r^c S$ is smooth of dimension $2r$. In particular, either $\text{Al}_3^c S$ is empty or every irreducible components of $\text{Al}_3^c S$ has dimension at least $11 + 6 - 15 = 2$, which is therefore the expected dimension of $\text{Al}_3^c S$. So, for an irreducible projective surface $S \subset \mathbb{P}^3$, one might expect that, with few exceptions, the Hilbert scheme of trisecant lines $\text{Al}_3^c S$ is generically reduced (and hence generically smooth) of pure dimension two.

There exists a natural morphism of schemes

$$\text{axis} : \text{Al}_r^c S \to \mathbb{G}(1,5),$$

sending each length $r \geq 2$ aligned subscheme of $S$ to the unique line containing its support, that is to the multiseicant line to $S$ determined by the subscheme of points (counted with multiplicity). Let $q : \mathcal{L} \to \mathbb{G}(1,5)$ be the universal family and let $p : \mathcal{L} \to \mathbb{P}^5$ be the tautological morphism. Then

$$\text{Trisec}(S) := p(q^{-1}(\text{axis}(\text{Al}_3^c S))) \subset \mathbb{P}^5$$

is called the Trisecant locus of $S \subset \mathbb{P}^5$. The previous count of parameters and analysis show that: the expected dimension of $\text{Trisec}(S)$ is three; that every irreducible component of $\text{Trisec}(S)$ has dimension at least two; that the irreducible components of dimension two of $\text{Trisec}(S)$ are either $S$ (in this case $S$ is ruled by lines) or planes cutting $S$ along a plane curve of degree at least three, see [Bau98].

By the Trisecant Lemma we deduce that $\dim(\text{Trisec}(S)) \leq 4$ for any irreducible surface $S \subset \mathbb{P}^5$. The known examples having $\dim(\text{Trisec}(S)) = 4$ are very few (and most of them very singular) and are described in [Rog]. The smooth surfaces $S \subset \mathbb{P}^5$ with $\dim(\text{Trisec}(S)) \leq 2$ are classified in [Bau98].

In our analysis we shall always consider the most general case $\dim(\text{Trisec}(S)) = 3$ and we shall also suppose that $\text{Al}_3^c S$ is generically smooth of pure dimension two. While the condition on the dimension is expected by the above parameter count, the smoothness of $\text{Al}_3^c S$ is related to the tangential behaviour of $S \subset \mathbb{P}^5$ at the points of intersection of a general trisecant line by [GP13, Proposition 4.3] (see also [CC93, Section 1] and [Ran15] for spectacular generalisations). We shall specialise this general result to our setting.

**Proposition 2.1.** Let $S \subset \mathbb{P}^5$ be an irreducible projective surface and let $\tilde{L} \subset \mathbb{P}^5$ be a proper trisecant line to $S$ such that $\tilde{L} \cap S = \{p_1, p_2, p_3\}$, with $p_1, p_2, p_3$ distinct smooth points of $S$, and with $[\tilde{L}]$ belonging to an irreducible component $\tilde{A}$ of $\text{Al}_3^c S$ of dimension two. Then $\text{Al}_3^c S$
is smooth at \([\tilde{L}]\) if and only if the tangent planes to \(S\) at the points \(p_i\)'s are in general linear position, that is \(T_{p_j}S \cap T_{p_k}S = \emptyset\) for any distinct \(p_j, p_k \in \tilde{L} \cap S\). In particular, if this condition holds for \([\tilde{L}] \in A\), then \(A\) is generically smooth and for a general \([L] \in \tilde{A}\) the tangent planes at the points \(p_i\)'s are in general linear position.

Moreover, in this case the irreducible component of \(\text{Trisec}(S)\) corresponding to \(\tilde{A}\) has dimension 3 and through a general point \(q\) of this irreducible component there passes a finite number of trisecant lines to \(S\), which are smooth points of the zero dimensional Hilbert scheme of trisecant lines to \(S\) passing through \(q\).

This result and the previous analysis motivates the next definition.

**Definition 2.2. (Expected Trisecant Behaviour)** Let \(S \subset \mathbb{P}^5\) be an irreducible non-degenerate projective surface. If \(A^3\) \(S\) is of pure dimension two and every irreducible component is generically reduced, then \(S \subset \mathbb{P}^5\) is said to have the expected trisecant behaviour.

We now start to study the consequences of this natural condition, which we shall assume until the end of this subsection. Let \(S_{\text{reg}} = S \setminus \text{Sing}(S)\) be the locus of smooth points of an irreducible non-degenerate surface \(S \subset \mathbb{P}^5\) and let \(\pi : \text{Bl}_S \mathbb{P}^5 \to \mathbb{P}^5\) be the blow-up of \(\mathbb{P}^5\) along \(S\). Then \(\pi^{-1}(\mathbb{P}^5 \setminus \text{Sing}(S))\) is a smooth variety. If \(L \subset \mathbb{P}^5\) is a proper trisecant line to \(S\), then \(L' \subset \text{Bl}_S \mathbb{P}^5\) denotes its strict transform. If \(\text{Sing}(S)\) is zero dimensional, then a general \([L] \in A^3\) \(S\) will cut \(S\) in three smooth distinct points and \(L'\) will be contained in the smooth locus of \(\text{Bl}_S \mathbb{P}^5\). In particular, \(N_{L'/\text{Bl}_S \mathbb{P}^5}\) will be locally free of rank four and \(\deg(N_{L'/\text{Bl}_S \mathbb{P}^5}) = -2\) by Adjunction Formula. In the same way for a closed irreducible variety \(V \subset \mathbb{P}^5\), we shall indicate by \(V' \subset \text{Bl}_S \mathbb{P}^5\) its strict transform.

**Lemma 2.3.** Let \(S \subset \mathbb{P}^5\) be an irreducible non-degenerate projective surface with the expected trisecant behaviour and with at most a finite number of singular points, let \(L \subset \mathbb{P}^5\) be a general trisecant line and let notation be as above. Then

\[
N_{L'/\text{Bl}_S \mathbb{P}^5} \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1).
\]

If \(\tilde{T} \subset \mathbb{P}^5\) denotes the unique irreducible component of \(\text{Trisec}(S)\) containing \(L\), then

\[
N_{T'/\text{Bl}_S \mathbb{P}^5} \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}.
\]

Furthermore, if \(T'\) is smooth along \(L'\), then

\[
N_{T'/\text{Bl}_S \mathbb{P}^5|_{L'}} \simeq O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1).
\]

**Proof.** The strict transforms of the trisecant lines to \(S\) determine a proper family of trisecant lines on \(\text{Bl}_S \mathbb{P}^5\) and the trisecant line \(L'\) represents a smooth point of this family, yielding \(N_{L'/\text{Bl}_S \mathbb{P}^5} \simeq O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1} \oplus O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(-1)\). Indeed, \(\deg(N_{L'/\text{Bl}_S \mathbb{P}^5}) = -2\), \(h^0(N_{L'/\text{Bl}_S \mathbb{P}^5}) = 2\) by the smoothness condition, and \(h^0(N_{L'/\text{Bl}_S \mathbb{P}^5}(-1)) = 0\) since through a general point of \(T'\) there passes a finite number of lines of the family. Then the exact sequence

\[
0 \to N_{L'/\tilde{T}'} \to N_{L'/\text{Bl}_S \mathbb{P}^5} \to N_{T'/\text{Bl}_S \mathbb{P}^5_{|_{L'}}}
\]

assures that \(N_{L'/\tilde{T}'}\) is locally free of rank 2 and that, letting \(N_{L'/\tilde{T}'} \simeq O_{\mathbb{P}^1}(a_1) \oplus O_{\mathbb{P}^1}(a_2)\), we have \(a_i \leq 0\) for \(i = 1, 2\). Since the line \(L'\) moves in a family of dimension two we have \(h^0(N_{L'/\tilde{T}'}) \geq 2\) and hence \(a_1 = a_2 = 0\).
If $T'$ is smooth along $L'$, then the morphism on the right of the exact sequence (2.1) is surjective and $N_{T'/\text{Bl}_{S} P^5_{|L'}} \cong O_{P^1}(-1) \oplus O_{P^1}(-1)$. □

**Remark 2.4.** We are interested in studying the birational properties of smooth cubic hypersurfaces $X \subset P^5$ passing through an irreducible projective surface $S \subset P^5$ having the expected trisecant behaviour and at most a finite number of singular points. Suppose $L \subset X \subset P^5$ is a general proper trisecant line to $S$ contained in $X$. Since $N_{\text{Bl}_{S} X/\text{Bl}_{S} P^5_{|L'}} \cong O_{P^1}$, since $N_{L'/\text{Bl}_{S} P^5} \cong O_{P^1} \oplus O_{P^1} \oplus O_{P^1}(-1) \oplus O_{P^1}(-1)$ and since we have the exact sequence:

$$0 \to N_{L'/\text{Bl}_{S} X} \to N_{L'/\text{Bl}_{S} P^5} \to N_{S} \to 0,$$

we deduce that either $N_{L'/\text{Bl}_{S} X} \cong O_{P^1} \oplus O_{P^1}(-1)$ or $N_{L'/\text{Bl}_{S} X} \cong O_{P^1} \oplus O_{P^1} \oplus O_{P^1}(-2)$. The last condition means that either the family of trisecant lines contained in $X$ is two dimensional and generically reduced or it is one dimensional but not generically reduced as a subscheme of the corresponding Hilbert scheme. If $X \subset P^5$ does not contain Trisc($S$), then the locus of trisecant lines to $S$ contained in $X$ is two dimensional and the family of trisecant lines contained in $X$ is one dimensional.

Thus when $X$ is **sufficiently general** and when $S$ has the expected trisecant behaviour and at most a finite number of singular points, one expects that the locus of trisecant lines to $S$ contained in $X$ is of pure dimension two and that the one dimensional families of trisecant lines to $S$ contained in $X$ are generically smooth as subschemes of the corresponding parameter space.

This expectation translates into the following conditions, letting notation be as above:

$$N_{L'/\text{Bl}_{S} X} \cong O_{P^1} \oplus O_{P^1}(-1) \oplus O_{P^1}(-1),$$

where $L' \subset \text{Bl}_{S} X$ denotes the strict transform of $L \subset X$; if $T \subset X$ denotes the unique two dimensional irreducible component of the locus of trisecant lines to $S$ contained in $X$ to which $L$ belongs, then

$$N_{L'/T} \cong O_{P^1}.$$ 

Furthermore, if $T'$ is smooth along $L'$, then

(2.2) $$N_{T'/\text{Bl}_{S} X_{|L'}} \cong O_{P^1}(-1) \oplus O_{P^1}(-1).$$

Condition (2.2) is crucial. Indeed, as we shall see in the next section, it essentially says that $T''$ can be flopped producing another four dimensional variety birational to $\text{Bl}_{S} X$ and hence to $X$ in a very natural way.

If an irreducible projective surface $S \subset P^5$ has the expected trisecant behaviour and if $S$ satisfies condition $K_3$, then, for a general cubic through $S$, the expected splittings listed above hold for a general proper trisecant line to $S$ contained in the cubic, see the proof of Theorem 2.6. There are also many other examples of different flavour for which the above conditions naturally hold and which naturally lead to flops of the trisecant locus contained in the cubic fourfold.

### 2.2. Assumptions and main definitions.

**Assumption 1.** Suppose we have a smooth irreducible projective surface (the treatment can be easily extended to surfaces with at most a finite number of singular points) $S \subset P^5$.
scheme-theoretically defined by cubic hypersurfaces and such that the associated rational map
\[ \varphi : \mathbb{P}^5 \dashrightarrow \mathbb{P}(H^0(I_S(3))) = \mathbb{P}^N \]
is birational onto the closure of its image \( Z = \overline{\varphi(\mathbb{P}^5)} \subset \mathbb{P}^N \).

Then the restriction of \( \varphi \) to a general \( X \in |H^0(I_S(3))| \) induces a birational map
\[ \varphi : X \dashrightarrow Y \subset \mathbb{P}^N - 1 \]
with \( Y \) a general hyperplane section of \( Z \subset \mathbb{P}^N \). On \( X' = \text{Bl}_S X \) we have
\[ -K_{X'} = \lambda^*(-K_X) - E = 3H' - E \]
and our hypothesis on the defining equations of \( S \) and on the birational map \( \varphi : \mathbb{P}^5 \dashrightarrow Z \)
can be reformulated by saying that \(-K_{X'}\) is a big divisor generated by its global sections. In particular, \(-K_{X'}\) is nef and big so that \( X' \) is a log Fano manifold.

The induced morphism
\[ \tilde{\varphi} : \text{Bl}_S X \rightarrow Y \]
is a small contraction (with very few exceptions). Indeed, the base locus scheme of \( \varphi \) is the surface \( S \) and \( \varphi \) contracts any irreducible (rational) curve \( C \subset X \) of degree \( e \geq 1 \) which is 3e–secant to \( S \), i.e. such that \( \text{length}(C \cap S) = 3e \) (proper 3e–secant curve to \( S \)). Let us indicate by \( T \subset X \) the closure of the locus of proper 3e-secant curves to \( S \) contained in \( X \). If \( L' \subset X' \) is the strict transform of a proper trisecant line to \( S \) contained in \( X \), let \([L']\) denote its numerical class in \( N_1(X') \).

The strict transform \( C' \subset X' \) of a proper 3e–secant curve \( C \subset X \) to \( S \) of degree \( e \geq 1 \) satisfies \([C'] = [eL']\). Therefore on \( X' = \text{Bl}_S X \) we have
\[ K_{X'} \cdot C' = (E - 3H') \cdot C' = 3e - 3e = 0 \]
for curves \( C' \subset X' \) as above.

**Definition 2.5. (Trisecant Flop)** Let notation and assumptions be as above. If \( \tilde{\varphi} : X' \rightarrow Y \)
is a small contraction of curves in \( \mathbb{R}[L'] \), then it is called a Trisecant flop contraction. If \( \tilde{\varphi} : X' \rightarrow Y \) is a Trisecant flop contraction and if there exists a flop \( \tilde{\psi} : W' \rightarrow Y \) of \( \tilde{\varphi} \) with \( W' \) a projective fourfold, then the resulting birational map \( \tau : X' \dashrightarrow W' \) will be called the Trisecant Flop (of \( \tilde{\varphi} : X' \rightarrow Y \)).

Let us remark that, by definition, if \( \tilde{\varphi} : X' \rightarrow Y \) is a Trisecant small contraction, then the exceptional locus of \( \tilde{\varphi} : X' \rightarrow Y \) has dimension at most two and the irreducible components of dimension two are covered by proper 3e–secant (rational) curves (in most cases they are ruled by these curves). By Zariski’s Main Theorem, a positive dimensional fiber is connected so that a general positive dimensional fiber is smooth and irreducible.

During our study of birational maps \( \varphi : \mathbb{P}^5 \dashrightarrow Z \) of the type described above, we constructed many examples of surfaces \( S \subset \mathbb{P}^5 \) inducing Trisecant flop contractions on a general cubic fourfold \( X \) through \( S \). In particular, surfaces satisfying condition \( K_3 \), which does not exhaust all the examples we know (see Table 1 for some examples).
2.3. Existence of the Trisecant Flop. For simplicity we shall now assume as above that $S$ is smooth. As always, let $\lambda : X' = \text{Bl}_S X \to X$ be the blow-up of $X$ along $S$, let $E \subset X'$ be the exceptional divisor and let $H' = \lambda^*(H)$, where $H \subset X$ is a hyperplane section.

The results in Subsection 2.1, in Remarks 1.6 and 2.4 suggest that, under some mild assumptions, Trisecant flop contractions might exist.

We shall now construct explicitly a flop of all the two dimensional irreducible components of $T$ determined by trisecant lines to $S$. When these loci exhaust the exceptional locus of a Trisecant Flop contraction we shall obtain a Trisecant Flop of $\tilde{\phi} : X' \to Y$. Flops of this kind have been also considered in [LLW10] in arbitrary dimension under the stronger assumption that the splitting (2.2) holds for every line of the ruling of $T'$.

Theorem 2.6. (Trisecant Flop) Let notation be as above, suppose that $S \subset \mathbb{P}^5$ satisfies Assumption 1, that

$$\tilde{\phi} : X' = \text{Bl}_S X \to Y$$

is a small contraction and let $T' \subset X'$ be its exceptional locus. Any irreducible smooth surface $\mathcal{T} \subset T'$, which is ruled via $\tilde{\phi}$ by trisecant lines to $S$ can be flopped, yielding a small contraction $\tilde{\psi} : W' \to Y$ with $W'$ a smooth projective fourfold.

In particular, if $\tilde{\phi} : X' \to Y$ is a Trisecant flop contraction and if every irreducible component of $T' \subset X'$ consists of smooth irreducible surfaces ruled via $\tilde{\phi}$ by trisecant lines, then the Trisecant flop $\tau : X' \dashrightarrow W'$ exists and $\tilde{\psi} : W' \to Y$ is given by $| - K_W |$.

Proof. Without loss of generality we can suppose that $S \subset X$ is smooth and that $\mathcal{T} = T'$ is a smooth irreducible surface ruled via $\tilde{\phi}$ and such that $\tilde{\phi}(T') = \mathcal{C} \subset Y$ is a curve. By definition of small contraction the general fiber of $\tilde{\phi} : T' \to \mathcal{C}$ is smooth and irreducible so that $\mathcal{C}$ generically coincides, as a scheme, with the parameter space of trisecant lines to $S$ contained in $X$. Let $L' \subset T'$ be a general fiber of the restriction of $\tilde{\phi}$ to $T'$. By Remark 2.4

$$\mathcal{N}_{T'/X'_W} \simeq \mathcal{O}_{\mathbb{P}^1} (1) \oplus \mathcal{O}_{\mathbb{P}^1} (1).$$

(2.3)

Let $\sigma : X'' = \text{Bl}_{T'} X' \to X'$ be the blow-up of $X'$ along $T'$ and let $E' \subset X''$ be the exceptional divisor. In particular $\sigma^{-1}(L') \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and there is another contraction of $\sigma^{-1}(L')$ to a smooth rational curve $L''$. The fibers of these contractions are smooth rational curves on the smooth threefold $E'$, which generate an extremal ray on $E'$. The locus of these curves covers $E'$ so that the extremal ray determines a contraction $\omega : E' \to R'$ with $R'$ a smooth surface, see [Mor82]. By the above analysis the surface $R'$ is ruled by the curves $L''$. The restriction of $N_{E'/X''}$ to every irreducible component of a positive dimensional fiber of $\omega$ is $\mathcal{O}_{\mathbb{P}^1} (1)$ so that every fiber of $\omega$ is smooth. There exists a morphism $\omega : X'' \to W'$ with $W'$ a smooth irreducible projective fourfold, which is the blow-up of $W'$ along the smooth surface $R'$, see for example [Art70, Nak71, FN71]. The smooth rational curves $L''$ are contracted to $\mathcal{C}$ by the nef and big linear system $| - K_W |$, yielding a morphism $\tilde{\psi} : W' \to Y$ such that $\mathcal{C} = \tilde{\psi}(R')$ and such that the surface $R'$ is ruled by $\tilde{\psi} : R' \to \mathcal{C}$. \hfill \Box

We now state a useful corollary, helpful for our applications and showing that the phenomenon described above really occurs.

Corollary 2.7. Let $S \subset \mathbb{P}^5$ be a surface satisfying Assumption 1, condition $K_3$ and such that $\dim(T') \leq 2$. If $X \subset \mathbb{P}^5$ is a cubic hypersurface through $S$ not containing any plane spanned...
by cubic curves on $S$ and if $\tilde{\varphi} : X' = Bl_S X \to Y$ is the associated Trisecant flop contraction, then there exists the Trisecant flop $\tilde{\psi} : W' \to Y$.

2.4. Trisecant Flop and congruences of $(3e - 1)$-secant rational curves of degree $e \geq 2$. The aim of this section is to relate the Trisecant Flop to (congruences of) $(3e - 1)$-secant curves to $S$. We start by an easy but very fruitful result.

**Proposition 2.8.** (Extremal ray generated by $(3e - 1)$-secant curves) Let notation be as above, suppose that $S \subset \mathbb{P}^5$ satisfies Assumption 1 and that there exists the Trisecant flop $\tilde{\psi} : W' \to Y$ of the small Trisecant flop contraction $\tilde{\varphi} : X' = Bl_S X \to Y$ with $X$ a general cubic fourfold through $S$.

If $C' \subset X'$ is the strict transform on $X'$ of a $(3e - 1)$-secant curve to $S$ contained in $X$, then the strict transform $\overline{C'}$ of $C'$ on $W'$ generates an extremal ray on $W'$.

**Proof.** By hypothesis there exists a Trisecant Flop of the Trisecant flop contraction $\tilde{\varphi} : X' = Bl_S X \to Y$ and hence a commutative diagram:

$$(2.4) \quad \begin{array}{ccc}
  V & \xrightarrow{\sigma} & W' \\
  \downarrow{\omega} & & \downarrow{\tau} \\
  X' & \xrightarrow{\varphi} & W \\
  \downarrow{\lambda} & & \downarrow{\psi} \\
  X & \xrightarrow{\tilde{\varphi}} & Y \end{array} \quad \text{.}$$

By definition $C'$ is the strict transform of a smooth rational curve of degree $e \geq 1$ which is $(3e - 1)$-secant to $S$. Thus

$$K_{X'} \cdot C' = (E - 3H') \cdot C' = 3e - 1 - 3e = -1.$$ Consider the possible degenerations of $C' \subset X'$ as sum of effective 1–cycles inside $X'$:

$$C' = C_1 + C_2.$$ The cycles $C_1$ and $C_2$ are supported on rational curves, have degree $e_1 = H' \cdot C_i$ and are $\beta_i = E \cdot C_i$ secant to $S$. In particular $e_1 + e_2 = e$ and $\beta_1 + \beta_2 = 3e - 1$. Since $-K_{X'}$ is nef and since

$$1 = -K_{X'} \cdot C' = -K_{X'} \cdot (C_1 + C_2),$$
either $K_{X'} \cdot C_1 = 0$ or $K_{X'} \cdot C_2 = 0$. In conclusion either $[C_2] = [e_2 L']$ or $[C_1] = [e_1 L']$ with $L'$ the strict transform of a trisecant line to $S$ contained in $X$. Let $\overline{C'} \subset W'$ be the strict transform of $C' \subset X'$. Then $[\overline{C'}]$ generates an extremal ray because $\tilde{\varphi}$ has contracted all the rational curves in $\mathbb{R}[L']$. \hfill \Box

The locus of the extremal ray $[\overline{C'}]$ will determine the type of the associated elementary Mori contraction from $W'$ onto a $\mathbb{Q}$-factorial Fano variety. Let us begin by considering the most relevant case for our applications. An example of fiber type contraction will be exhibited in Subsection 3.6 and others can be constructed as soon as $|{(3e - 1)H' - eE}| \neq \emptyset$ and $S \subset \mathbb{P}^5$ has a finite number $\rho \geq 2$ of $(3e - 1)$-secant curves of degree $e \geq 2$ passing through a general point of $\mathbb{P}^5$. The dimension of the general fiber of the contraction will be $\rho - 1 = 4 - \dim(\mu(X))$. We have constructed examples for every $\rho \leq 5$. 

12

F. RUSSO AND G. STAGLIANÒ
Theorem 2.9. (Extremal Contraction of the Congruence) Let notation be as above, suppose that $S \subseteq \mathbb{P}^5$ satisfies Assumption 1 and that there exists the Trisecant Flop $\tilde{\psi} : W' \rightarrow Y$ of the small Trisecant flop contraction $\tilde{\phi} : X' = Bl_S X \rightarrow Y$ with $X$ a general cubic fourfold through $S$.

If $S \subseteq \mathbb{P}^5$ admits a congruence $\pi : D \rightarrow \mathcal{H}$ of $(3e - 1)$-secant rational curves of degree $e \geq 2$, then the locus of curves of the congruence contained in $X \subseteq \mathbb{P}^5$ is a divisor $D \subset X$ and the following hold:

1. There exists a divisorial contraction $\nu : W' \rightarrow W$, with $W$ a locally $\mathbb{Q}$-factorial projective variety, whose exceptional locus $\overline{E}$ is the strict transform of $D$ on $W'$ and such that $\nu(D) = U$ is an irreducible surface supporting the base locus scheme $B$ of $\nu^{-1}$. In particular, $B$ is generically smooth and $\nu$ is generically the blow–up of the surface $U$.

2. The induced birational map $\mu' : X' \dashrightarrow W$ (or $\mu : X \dashrightarrow W$) is given by a linear system in $|(3e - 1)H' - eE|$.

3. Let $\overline{H} = \nu^*(\overline{H})$ with $\overline{H} \subset W$ a generator of $\text{Pic}(W)$ and let $-K_W = i(W)\overline{H}$. The induced birational map $\psi = \tilde{\psi} \circ \nu^{-1} : W \dashrightarrow Y$ is given by a linear system in $|i(W)\overline{H} - eE|$ while $\mu^{-1} : W \dashrightarrow X$ is given by a linear system in $|(i(W)\cdot e - 1)\overline{H} - eE|$ contracting the strict transform of $E$ in $W$, which is a locus of $(i(W) \cdot e - 1)$-secant curves of degree $e$ to $U \subset W$.

4. The irreducible components of $T$ are contained in the base locus scheme of $\mu$ and their flopped images on $W$ are contained in the base locus scheme of $\mu^{-1}$. The flopped images of the scrolls in $T$ are scrolls in $W$ ruled by curves of degree $e$ which are $(i(W) \cdot e)$-secant to $U$.

Proof. By hypothesis there exists a Trisecant Flop of the Trisecant flop contraction $\tilde{\phi} : X' = Bl_S X \rightarrow Y$ and hence the commutative diagram (2.4). Let $D' \subset X'$ be the strict transform of $D$ on $X'$ and let $C' \subset D'$ be the strict transform of a general curve of the congruence $\pi : D \rightarrow \mathcal{H}$, contained in $X$. By definition $C'$ is the strict transform of a smooth rational curve of degree $e \geq 2$ which is $(3e - 1)$-secant to $S$.

Let $\overline{C'} \subset W'$ be the strict transform of $C' \subset D'$ and let $\overline{E} \subset W'$ be the strict transform of $D$. Then $K_{W'} \cdot \overline{C'} = -1$ and $\overline{C'}$ generates an extremal ray by Proposition 2.8. The locus of the extremal ray $\mathbb{R}[\overline{C'}]$ is the irreducible divisor $E$. The extremal ray is contracted by the strict transform $\overline{M}$ of the divisor $M = (3e - 1)H' - eE = -e \cdot K_{X'} - H'$. There exists a contraction $\nu : W' \rightarrow W$ with $W$ a locally $\mathbb{Q}$-factorial projective variety of dimension four with $\text{Pic}(W) = \mathbb{Z}\langle \overline{H} \rangle$ and with $\overline{H}$ ample. If $i(W)$ is defined by $-K_W = i(W)\overline{H}$. Then $i(W) > 0$ since $W$ is birational to $X$ and hence covered by rational curves.

Let $U = \nu(\overline{E}) \subset W$. Since the $(3e - 1)$-secant curves to $S$ belong to a congruence, through a general point of $D$ there passes a unique curve of the congruence and the same holds for $\overline{E}$. This implies $\dim(U) = 2$ and that the general positive dimensional fiber of $\nu$ is a smooth rational curve of the congruence. In particular the base locus scheme of $\nu^{-1}$, which is supported on $U$, coincides generically with $U$ and hence it is generically smooth. All the assertions in 1) are now proved.

The birational map $\mu' = \nu \circ \tau : X' \dashrightarrow W$ is given by a linear system in $|a[(3e - 1)H' - eE]|$, $a \geq 1$. The irreducible components of $T$ are contained in the base locus scheme of this linear
system because
\[ a[(3e - 1)H - eE] \cdot F = -a < 0 \]
for every strict transform of a 3e-secant curve to \( S \) contained in \( X \). Since the map \( \mu' \) is compatible with the Trisecant Flop, necessarily \( a = 1 \) and we have a commutative diagram:

\[
\begin{array}{c}
\sigma \\
\downarrow \\
X' \quad \vdash \quad \tau \quad \dashrightarrow \quad W' \\
\downarrow \\
X \quad \mu' \quad \mu \\
\end{array}
\]

Let \( \overline{\Pi} \subset W \) be as above and let \( \overline{\Pi}' \) be its strict transform on \( W' \). Then \( F' \cdot \overline{\Pi}' = e \) and \(-K_{W'} = i(W)\overline{\Pi}' - \overline{E} \) so that \( i(W) \cdot e = \overline{E} \cdot F' \). Moreover, \( \overline{C}' \cdot \overline{E} = 1 \) and \( \overline{\Pi}' \cdot \overline{C}' = 0 \). Hence the birational morphism \( \tilde{\psi} : W' \to Y \) (or equivalently \( \psi : W \to Y \)) is given by a linear system in \( |i(W)\overline{\Pi}' - \overline{E}| \) (or equivalent by a linear system of divisors in \( |O_W(i(W))| \) vanishing on \( U \)).

The map \( \eta' = \mu^{-1} \circ \nu : W' \to X \) is given by a linear system in \( |a\overline{\Pi}' - e\overline{E}| \) because a general fiber \( \overline{C}' \) of \( \nu : \overline{E} \to U \) is sent into a curve of the congruence \( \mathcal{D} \), which by definition has degree \( e \geq 2 \). The compatibility with the Trisecant Flop gives
\[ -1 = (a\overline{\Pi}' - e\overline{E}) \cdot F' = a' - i(W) \cdot e. \]

In conclusion the birational map \( \eta' \) (and \( \mu^{-1} \)) is given by a linear system in \( |(i(W) \cdot e - 1)\overline{\Pi}' - e\overline{E}| \) of dimension 5. \( \square \)

**Remark 2.10.** Obviously, one can also revert the construction in Theorem 2.9 starting from suitable \( W \) and then producing the congruences of \((3e - 1)\)-secant curves of degree \( e \) to a surface \( S \subset X \subset \mathbb{P}^5 \) taking the image of \( \overline{E} \) in \( X \) and by taking \( S \subset X \subset \mathbb{P}^5 \) as the surface describing the linear system defining the inverse map \( \mu : X \to W \). In practice, as soon as the Trisecant Flop exists, the existence of a congruence of \((3e - 1)\)-secant lines to \( S \) is equivalent to the existence \( W \) and of the surface \( U \subset W \), which should be an incarnation of the associated K3 surface. From this point of view one associates to \( X \) the surface \( U \) as being the parameter space of the curves of the congruence \( \mathcal{D} \) contained in \( X \).

### 3. Applications to rationality of some cubic fourfolds and to the existence of some embeddings of non–minimal K3 surfaces

Now we apply the previous theoretical results to deduce that some cubic fourfolds are rational and, above all, also to find a birational incarnation of the associated K3 surfaces from the point of view of Hodge Theory or of Derived Category Theory. We summarize in Table 1 (see also Tables 2 and 3) the relevant information of some examples.
### Table 1

| i | 14 2 | Smooth minimal K3 surface of degree 8 and sectional genus 3, obtained as the image of $P^2$ via the linear system of quartic curves with 8 general base points | $P^4$ | Singular K3 surface of degree 10 and sectional genus 7, cut out by 12 quintics and having 8 singular points | 4-fold of degree 28 in $P^{11}$ cut out by 16 quadrics |
| ii | 26 2 | Rational scroll of degree 7 with 3 nodes | $P^4$ | Singular K3 surface of degree 10 and sectional genus 8, cut out by 12 quintics and one sextic, and having 3 singular points | 4-fold of degree 29 in $P^{11}$ cut out by 15 quadrics |
| iii | 38 2 | Smooth surface of degree 10 and sectional genus 6, obtained as the image of $P^2$ via the linear system of curves of degree 10 with 10 general triple points | $P^4$ | Smooth non-minimal K3 surface of degree 12 and sectional genus 14 cut out by 9 quintics | 4-fold of degree 20 in $P^8$ cut out by 16 cubics |
| iv | 26 2 | Projection of a smooth del Pezzo surface of degree 7 in $P^7$ from a line intersecting the secant variety in one general point | $P^7$ | Non-minimal K3 surface of degree 17 and sectional genus 11, cut out in $P^{11}$ by 5 quadrics and 13 cubics | 4-fold of degree 34 in $P^{12}$ cut out by 20 quadrics |
| v | 38 3 | Rational scroll of degree 8 with 6 nodes (≠) | $P^{10}$ | Smooth non-minimal K3 surface of degree 22 and sectional genus 14, cut out in $P^{15}$ by 24 quadrics | 4-fold of degree 17 in $P^9$ cut out by 3 quadrics and 4 cubics |
| vi | 14 3 | Projection from 3 general internal points of a minimal K3 surface of degree 14 and sectional genus 8 | Cubic fourfold | Projection from 3 general internal points of a minimal K3 surface of degree 14 and sectional genus 8 | Complete intersection in $P^7$ of 2 quadrics and one cubic |
| vii | 14 3 | Projection of a K3 surface of degree 10 and sectional genus 6 in $P^6$ from a general point on its secant variety | $P^6$ | Smooth minimal K3 surface of degree 14 and sectional genus 8 | Hypercubic section of a hyperplane section of $G(1,4)$ |
| viia | 14 5 | General hyperplane section of a conic bundle in $P^6$ of degree 13 and sectional genus 12 (≠) | Complete intersection of three quadrics in $P^7$ | Smooth non-minimal K3 surface of degree 13 and sectional genus 8, cut out by 9 quadrics | Hypersurface of degree 5 in $P^9$ |
| viix | 14 5 | General hyperplane section of a pfaffian threefold in $P^6$ of degree 14 and sectional genus 15 | $P^{14}$ | Smooth minimal K3 surface of degree 14 and sectional genus 8 embedded in $P^{15}$ | Hypersurface of degree 5 in $P^9$ |
| vii | 38 5 | Smooth surface of degree 11 and sectional genus 7, obtained as the image of $P^2$ via the linear system of curves of degree 12 with one general simple point, 4 general triple points, and 6 general quadrapole points (≠) | $P^{10}$ | Smooth non-minimal K3 surface of degree 25 and sectional genus 17, cut out in $P^{19}$ by 21 quadrics | Hypersurface of degree 7 in $P^9$ |

### Table 2

| i | 14 3 | Projection of an octic del Pezzo surface isomorphic to $F_4$ from a plane intersecting the secant variety in 3 general points | $G(1,3) \subset P^5$ | Non-minimal K3 surface of degree 13 and sectional genus 10, cut out in $P^8$ by one quadric, 9 quartics, and 3 quintics | 4-fold of degree 17 in $P^8$ cut out by 3 quadrics and 4 cubics |
| ii | 38 3 | Projection of an octic del Pezzo surface isomorphic to $F_8$ from a plane intersecting the secant variety in 3 general points, and 6 general quadrapole points (≠) | $L_{G4}(C^6) \cap P^{11} \subset P^{11}$ | Non-minimal K3 surface of degree 26 and sectional genus 17, cut out in $P^{11}$ by 30 quadrics | 4-fold of degree 18 in $P^8$ cut out by 2 quadrics and 8 cubics |
| iii | 14 3 | Isomorphic projection of a smooth surface in $P^9$ of degree 8 and sectional genus 2, obtained as the image of $P^2$ via the linear system of quartic curves with 4 simple base points and one double point | $P^9$ | Singular K3 surface of degree 14 and sectional genus 8, cut out in $P^9$ by 2 quadrics and 9 cubics, and having one singular point | Complete intersection of 4 quadrics in $P^9$ |
| iv | 26 5 | Rational scroll of degree 8 with 4 nodes (cut out by 8 cubics and 3 quartics) | $G(1,3) \subset P^5$ | Non-minimal K3 surface of degree 14 and sectional genus 11, cut out in $P^9$ by one quadric, 7 quartics, and 2 quintics | Complete intersection in $P^9$ of a quadric and a quartic |
**3.1. Determination of the surface $U$ in explicit examples.** The detection of the irreducible surface $U \subset W$ can be made very precise due to the knowledge of the invariants appearing in Theorem 2.9 and via the usual linkage arguments for the maps $\mu$ and $\varphi$. Nowadays a modern computer can find the invariants of $U$, detect its smoothness (or its singularities), and so on. In this way a birational incarnation of the associated K3 surface to $X$ appears inside $W$. Usually the birational map from the minimal K3 surface to $U$ is very particular and associated to a linear system of hyperplane sections with points of high multiplicity, as we shall see in many examples. A lot of cases (for relatively small $g$) naturally appear, see Table 1 and also Section 5 for some speculations. Let us recall that $d = 2g - 2$ is also the degree of the associated minimal K3 surface of genus $g$ in the embedding into $\mathbb{P}^{d+2}$.

**3.2. Trisecant flop associated to a degree 10 smooth surface $S_{38} \subset \mathbb{P}^5$ of sectional genus 6 (lines (iii) and (v) of Table 1).** Let us consider the surfaces $S_{38} \subset \mathbb{P}^5$ obtained as the image of $\mathbb{P}^2$ by the linear system of plane curves of degree 10 having 10 fixed triple points. These surfaces are contained in a general cubic fourfold in the admissible divisor $C_{38}$, as shown in [Nue15]. They were also studied in [RS19, RS18] to prove that every cubic in $C_{38}$ is rational. A general surface of this kind admits a congruence of 5–secant conics, we have $|5H' - 2E| = \mathbb{P}^4$ and in concrete general examples the associated map $\tilde{\mu} : \text{Bl}_{S_{38}} \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ is birational, see [RS19, RS18]. In [RS18] it was calculated that the base locus of $\mu^{-1}$ is a two dimensional scheme $B \subset \mathbb{P}^4$ of degree 42 and of sectional arithmetic genus 165.

The cubic generators of the homogeneous ideal of $S_{38}$ satisfy condition $K_3$ so that the linear system $|H^3(I_{S_{38}}(3))|$ defines a birational map onto the image $\varphi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^9$ by Proposition 1.5. The positive dimensional fibers of 

$$\tilde{\varphi} : \text{Bl}_{S_{38}} \mathbb{P}^5 \rightarrow Z \subset \mathbb{P}^9$$

are 10 planes (spanned by the 10 elliptic cubic curves on $S_{38}$ determined by 9 of the 10 base points) and trisecant lines describing a three dimensional irreducible variety $\tilde{T} \subset \text{Trisec}(S)$ mapped by $\tilde{\varphi}$ to a Veronese surface $V \subset Z$. In particular, the locus $\text{Trisec}(S_{38}) \subset \mathbb{P}^5$ consists of the irreducible three dimensional variety $\tilde{T}$ and of ten planes.
A general cubic hypersurface \( X \subset \mathbb{P}^5 \) through \( S_{38} \) is smooth and does not contain any plane in \( \text{Trisec}(S_{38}) \). The restriction of \( \tilde{\varphi} \) to \( X' = \text{Bl}_{S_{38}} X \) induces a small contraction:

\[
\tilde{\varphi} : X' = \text{Bl}_{S_{38}} X \rightarrow Y \subset \mathbb{P}^8,
\]

with \( Y \) a general hyperplane section of \( Z \). From the previous description we deduce that the locus \( T \subset X \) of trisecant lines contained in \( X \) is an irreducible scroll surface such that \( \varphi(T) = \overline{C} \subset Y \) is a smooth rational normal quartic curve (a general hyperplane section of the Veronese surface \( V \subset Z \)). It turns out that \( T \) has degree 8 and six nodes. The scroll \( T' \subset X' \) is smooth and \( \tilde{\varphi}|_{T'} : T' \rightarrow \overline{C} \) is a \( \mathbb{P}^1 \)-bundle. The birational morphism \( \tilde{\varphi} \) is an isomorphism between \( X' \setminus T' \) and \( Y \setminus \overline{C} \) and hence it is a Trisecant flop contraction.

Theorem 2.6 and Theorem 2.9 assure the existence of the the Trisecant flop \( \tilde{\psi} : W' \rightarrow Y \) and of the commutative diagram (0.1). The scroll \( \overline{R} = \nu(R') \subset \mathbb{P}^4 \) has degree 6 (recall the we tensor with \( \mathcal{O}_{\mathbb{P}^4}(-1) \)). Let \( U \subset \mathbb{P}^4 \) be the support of the base locus of \( \nu^{-1} \). The lines of the scroll \( \overline{R} \) are 5-secant to \( U \) and \( \psi(\overline{R}) = \tilde{\psi}(R') = \overline{C} \).

Now we have all the information to look for \( U \subset \mathbb{P}^4 \). Let \( \Pi \subset \mathbb{P}^4 \) be a general plane. Then \( \Pi \cap U \) consists of \( \deg(U) \) general points. Since \( \psi(\Pi) = U \subset Y \) is a non-degenerate surface of degree \( 25 - \deg(U) \), the linear system of quintics defining \( U \) restricted to \( \Pi \) determines a linear system of plane quintics of dimension 8. Then \( \deg(U) = 12 \) and \( \overline{U} \) is a smooth surface of degree 13.

By taking two general cubic hypersurfaces \( X_1, X_2 \) through \( S_{38} \) we get a surface \( S' \subset X \) of degree \( 17 = 27 - 10 \) defined by \( X \cap X_1 \cap X_2 = S_{38} \cup S' \). Then \( \mu(S') = U'' \subset \mathbb{P}^4 \) is a surface of degree \( 13 = 25 - \deg(U) \). Let \( V_1, V_2 \) be two general quintic hypersurfaces through \( U \). By taking two general quintic hypersurfaces through \( U'' \) we deduce that \( U \subset \mathbb{P}^4 \) is also smooth and that \( q = 0 \) and \( p_g = 1 \). Hence \( U \subset \mathbb{P}^4 \) is a degree 12 and genus 14 non-minimal K3 surface, whose ideal has at least 9 forms of degree 5 defining the map \( \psi : \mathbb{P}^4 \dashrightarrow Y \subset \mathbb{P}^9 \). In particular \( W' \simeq \text{Bl}_U \mathbb{P}^4 \) and

\[
\mu^{-1} : \mathbb{P}^4 \dashrightarrow X \subset \mathbb{P}^9
\]

is given by the linear system \( |9\overline{R'} - 2\overline{E}| \).

This smooth non minimal K3 surface \( U \subset \mathbb{P}^4 \) of degree 12 and genus 14 had been also constructed in [DES93, Section 2.7], where the authors showed that it is the blow-up in eleven points of a smooth K3 surface \( U' \subset \mathbb{P}^{20} \) of degree 38 and genus 20 with ideal generated by 9 forms of degree 5 satisfying condition \( K_5 \). The linear system on \( U' \) is given by \( |D' - 4E_1 - \sum_{i=2}^{11} E_i| \) with \( D' \) the strict transform of a hyperplane section of \( U' \subset \mathbb{P}^{20} \).

In conclusion via the Trisecant Flop and via the contraction of the congruence we found the associated surface \( U \) to a general pair \((X, S_{38})\) which is a birational incarnation of the Hodge Theoretically or Derived Categorically smooth minimal K3 surface of degree 38 and genus 20 associated to \( X \).

Starting from a general non minimal K3 surface \( U \subset \mathbb{P}^4 \) of degree 12 and genus 14 as above and by considering the linear system \( |9\overline{R'} - 2\overline{E}| \), we get a birational map

\[
\eta : \mathbb{P}^4 \dashrightarrow X \subset \mathbb{P}^5
\]

with \( X \subset \mathbb{P}^5 \) a general smooth cubic fourfold in \( C_{38} \).

A general cubic \( X \) through \( S_{38} \) contains also the octic scroll with 6 nodes \( T \subset X \subset \mathbb{P}^5 \) as part of the base locus scheme of the linear system \( |5H' - 2E| \). The study of the
associate birational map $\varphi_T : \mathbb{P}^5 \dashrightarrow Z_T \subset \mathbb{P}^9$ given by $|H^0(I_T(3))|$ reveals the existence of a congruence of 8-secant twisted cubics to $T$. The linear system of octic hypersurfaces with tripe points along $T$ gives a map $\mu_T : X \dashrightarrow W_T \subset \mathbb{P}^{10}$, with $W_T$ a linear section of $\mathbb{G}(1,5) \subset \mathbb{P}^{14}$. The surface $U_T \subset W_T$ is a smooth non minimal K3 surface of degree 22 and genus 14 isomorphic to the blow-up in one point $p_1$ of a minimal K3 surface $U' \subset \mathbb{P}^{20}$ of degree 38 and genus 20. It has ideal generated by 24 quadratic forms, which restricted to $W_T$ yields a birational map $\psi : W_T \dashrightarrow Y \subset \mathbb{P}^8$ ($W_T$ has ideal generated by 15 forms of degree 2 which provide 15 of the 24 generators of the ideal of $U_T$). The linear system on the blow-up of $U'$ at the point $p_1$ is given by $|D' - 4E_1|$, with $D'$ the strict transform of a hyperplane section of $U' \subset \mathbb{P}^{20}$. This linear system is very ample by [FS19, Theorem 10] and it gives an embedding into $\mathbb{P}^{10}$ by [Vo19, Proposition 4.1]. From this description we deduce that the surface $U \subset \mathbb{P}^4$ considered above is the linear projection of $U_T \subset \mathbb{P}^{10}$ from a $\mathbb{P}^5$ cutting $U_T$ in 10 points. As far as we know the construction of $U_T \subset W_T \subset \mathbb{P}^{10}$ was not studied explicitly before. The birational map $\mu^{-1}_T : W_T \dashrightarrow X \subset \mathbb{P}^5$ is given by a linear system of divisors in $|O_{W_T}(5)|$ having points of multiplicity 3 along $U_T$, as prescribed by Theorem 2.9.

We shall now describe other examples, leaving some verifications to the reader.

### 3.3. Trisecant flop associated surface to a projected degree 8 smooth surface $S_{14} \subset \mathbb{P}^5$ of sectional genus 3 (line (i) of Table 1).

Let $S_{14} \subset \mathbb{P}^5$ be a smooth isomorphic projection of an octic smooth surface $S \subset \mathbb{P}^6$ of sectional genus 3 in $\mathbb{P}^6$, obtained as the image of $\mathbb{P}^2$ via the linear system of quartic curves with 8 general base points. These surfaces are contained in a general cubic fourfold in the admissible divisor $\mathcal{C}_{14}$ and they were studied in [RS19, RS18], where it is also proved that: a general surface of this kind admits a congruence of 5–secant conics; that in concrete general examples the associate map $\mu : X \dashrightarrow \mathbb{P}^4$ is birational. In [RS18] it was computed the base locus of $\mu^{-1}$, which is a two dimensional scheme $B \subset \mathbb{P}^4$ of degree 52 and of sectional arithmetic genus 256.

Since the homogeneous ideal of $S_{14}$ satisfies condition $K_3$, the linear system $|H^0(I_{S_{14}}(3))|$ defines a birational map onto the image $\varphi : \mathbb{P}^5 \dashrightarrow Z \subset \mathbb{P}^{12}$ by Proposition 1.5. The locus $T \subset X$ is two dimensional for $X$ general cubic through $S_{14}$. By Corollary 2.7, there exists the Trisecant Flop of the restriction of $\varphi$ to a general $X$ through $S_{14}$ and Theorem 2.9 assures the existence of the birational contraction $\nu : W' \rightarrow W$.

Let $U \subset \mathbb{P}^4$ be the support of the base locus of $\nu^{-1}$. We have all the information to look for a possible $U \subset \mathbb{P}^4$: it may be a birational (smooth) K3 surface whose ideal contains at least 12 forms of degree 5 defining a map $\psi : \mathbb{P}^4 \dashrightarrow Y \subset \mathbb{P}^{11}$. The singular (non minimal K3) surface $U \subset \mathbb{P}^4$ of degree 10 and sectional genus 7 with 8 nodes, obtained by projecting the minimal K3 surface $U' \subset \mathbb{P}^8$ of degree 14 and genus 8 from a general tangent plane into $\mathbb{P}^5$ and then from a general external point, has ideal generated by 12 forms of degree 5 satisfying condition $K_3$. The linear system of forms of degree 9 with points of multiplicity 2 along $U \subset \mathbb{P}^4$ defines a birational map $\eta : \mathbb{P}^4 \dashrightarrow X \subset \mathbb{P}^5$, which is $\mu^{-1}$.

### 3.4. Trisecant flop associated surface to a Farkas-Verra septimic scroll $S_{26} \subset \mathbb{P}^5$ with three nodes (line (ii) of Table 1).

Let $S_{26} \subset \mathbb{P}^5$ be a projection with three nodes of a general septimic scroll $S(3,4) \subset \mathbb{P}^8$ (the projection is made by a plane generated by three general points on the secant variety $S(3,4)$). These surfaces are contained in a general cubic fourfold in the admissible divisor $\mathcal{C}_{26}$, as shown in [FV18] and they were also studied in
where it is also proved that: a general surface of this kind admits a congruence of 5–secant conics; that \( |5H' - 2E| = \mathbb{P}^4 \); that in concrete general examples the associate map \( \mu : X \rightarrow \mathbb{P}^4 \) is birational. In [RS18] it was computed the base locus of \( \mu^{-1} \), which is a two dimensional scheme \( B \subset \mathbb{P}^4 \) of degree 51 and of sectional arithmetic genus 246.

Since the homogeneous ideal of \( S_{26} \) satisfies condition \( K_3 \), the linear system \( |H^0(I_{S_{26}}(3))| \) defines a birational map onto the image \( \varphi : \mathbb{P}^5 \rightarrow Z \subset \mathbb{P}^{12} \) by Proposition 1.5. The locus \( T \subset X \) is two dimensional for \( X \) general cubic through \( S_{14} \). By Corollary 2.7, there exists the Trisecant Flop of the restriction of \( \tilde{\varphi} \) to a general \( X \) through \( S_{14} \) and Theorem 2.9 assures the existence of the birational contraction \( \nu : W' \rightarrow \mathbb{P}^4 \).

Let \( U \subset \mathbb{P}^4 \) be the support of the base locus of \( \nu^{-1} \). We have all the information to find \( U \subset \mathbb{P}^4 \): it may be a birational K3 surface whose ideal contains at least 12 forms of degree 5 defining a map \( \psi : \mathbb{P}^4 \rightarrow Y \subset \mathbb{P}^{11} \). By studying the image of the divisor of 5–secant conics contained in \( X \) and via liaison, we found that \( U \subset \mathbb{P}^4 \) has degree 10, sectional genus 8 and 3 nodes. The surface is obtained by the associated minimal K3 surface \( U' \subset \mathbb{P}^{14} \) of degree 26 in this way. On the blow-up of \( U' \) at a point \( p_i \), one considers the linear system \( |D' - 4E_1| \), with \( D' \) the pull-back of a hyperplane section of \( U' \subset \mathbb{P}^{14} \). It gives a birational morphism onto \( U \subset \mathbb{P}^4 \). The ideal of \( U \subset \mathbb{P}^4 \) is generated by 12 forms of degree 5 satisfying condition \( K_5 \). The birational map \( \mu^{-1} : \mathbb{P}^4 \rightarrow X \subset \mathbb{P}^5 \) is given by a linear system of hypersurfaces of degree 9 having points of multiplicity 2 along \( U \) as prescribed by Theorem 2.9.

3.5. Two Trisectant flops coming from special Cremona transformations (lines (viii) and (ix) of Table 1). We give two examples of surfaces in \( \mathbb{P}^5 \) which admit a congruence of 14-secant rational normal quintic curves and then we study the associated Trisecant Flop and the resulting Extremal Contraction of the congruences.

Let \( \Phi : \mathbb{P}^6 \rightarrow \mathbb{P}^6 \) be a special cubic Cremona transformation, that is a Cremona transformation defined by cubic forms and whose base locus scheme is smooth and connected. From the main result in [Sta18] (see also [Sta19]), it follows that the base locus of \( \Phi \) can be of the two following types:

1. a threefold \( B_1 \) of degree 14, sectional genus 15 with trivial canonical bundle which is Pfaffian, i.e. given by the Pfaffians of a skew–symmetric matrix;
2. a conic bundle \( B_2 \) over \( \mathbb{P}^2 \), embedded in \( \mathbb{P}^6 \) as a threefold of degree 13 and sectional genus 12.

Let \( S_i \subset \mathbb{P}^5 \) be a general hyperplane section of one of the two types of threefolds \( B_i \subset \mathbb{P}^6 \) as above, \( i = 1, 2 \). Then \( S_i \subset \mathbb{P}^5 \) is a smooth surface with ideal generated by 7 cubic forms and it is contained in a smooth cubic fourfold. The surface \( S_1 \) satisfies condition \( K_3 \) while the surface \( S_2 \) does not satisfy this condition and there exist irreducible components of the associated exceptional locus ruled by 6-secant conics. Moreover, from a standard parameter count, one sees that the closure of the locus of smooth cubics containing one of the two types of surfaces \( S_i \) is \( \mathcal{C}_{14} \) (see [Sta19, Section 4] and Table 3).

Let \( \varphi : \mathbb{P}^5 \rightarrow Z \subset \mathbb{P}^6 \) be the map defined by the linear system of cubics through \( S_i \), that is the restriction of the special Cremona transformation \( \Phi : \mathbb{P}^6 \rightarrow \mathbb{P}^6 \) to a general hyperplane \( \mathbb{P}^5 \subset \mathbb{P}^6 \). We have that \( Z \subset \mathbb{P}^6 \) is a quintic hypersurface since the inverse of \( \Phi \) is defined by forms of degree 5. Through a general point \( z = \varphi(p) \) there passes 120 = 5! lines \( L_i \subset Z \). Let \( B' \subset Z \) be the irreducible fourfold, base locus of \( \Phi^{-1} \). Every line \( L_i \) is mapped by \( \varphi^{-1} \) onto a smooth curve of degree \( e = 5 - \text{length}(L_i \cap B') \) which is \((3e - 1)\)-secant to \( S \), being
mapped back to the line \( L \) by \( \tilde{\Phi} \). In both examples there is a unique line \( T \) through \( z \) with \( T \cap B' = \emptyset \), yielding that these surfaces admit a congruence of 14-secant rational normal curves of degree 5.

We have the birational maps \( \varphi : X \dasharrow Y \subset \mathbb{P}^5 \) with \( X \) a general cubic fourfold in \( C_{14} \) and with \( Y \subset \mathbb{P}^5 \) a quintic hypersurface which is a general hyperplane section of \( Z \).

Let \( \tilde{\varphi}_1 : \text{Bl}_{S_1} X \dasharrow W \subset \mathbb{P}^{14} \) be the map defined by the linear system \( [14H' - 5E] \). Then \( W \subset \mathbb{P}^{14} \) is a prime Fano fourfold with \( i(W) = 1 \), which is a linear section of \( G(1,6) \subset \mathbb{P}^{20} \).

The surface \( U \subset W \subset \mathbb{P}^{14} \) is a smooth minimal K3 surface of degree 14 and genus 8. The map \( \psi : W \dasharrow Y \subset \mathbb{P}^5 \) is given by the linear forms in \( |O_W(1)| \) defining the linear span \( \langle U \rangle = \mathbb{P}^8 \subset \mathbb{P}^{14} \) while \( \mu^{-1} : W \dasharrow X \subset \mathbb{P}^5 \) is given by a linear system of divisors in \( |O_W(4)| \) having points of multiplicity 5 along \( U \).

Let \( \tilde{\varphi}_2 : \text{Bl}_{S_2} X \dasharrow W \subset \mathbb{P}^7 \) be the map defined by the linear system \( [14H' - 5E] \). Then \( W \subset \mathbb{P}^7 \) is a prime Fano fourfold with \( i(W) = 2 \), which is the complete intersection of three quadric hypersurfaces. It is worth noticing that the \( W \)'s obtained in this way are rational (being birational to a cubic fourfold in \( C_{14} \)) and that they describe a locus of codimension three in the moduli space of complete intersections of three quadrics in \( \mathbb{P}^7 \), see [HPT17]. The surface \( U \subset W \subset \mathbb{P}^7 \) is a smooth surface of degree 13 and sectional genus 8, which is the projection from a point on it of a smooth minimal K3 surface \( U' \subset \mathbb{P}^8 \) of degree 14 and sectional genus 8. The ideal of \( U \subset \mathbb{P}^7 \) is generated by 9 quadratic forms, whose restriction to \( W \) yields the birational map \( \psi : W \dasharrow Y \subset \mathbb{P}^5 \). The birational map \( \mu^{-1} : W \dasharrow X \subset \mathbb{P}^5 \) is given by a linear system of divisors in \( |O_W(9)| \) having points of multiplicity 5 along \( U \), as prescribed by Theorem 2.9.

3.6. Trisecant flops and contractions of fiber type. Let \( S_{32} \subset \mathbb{P}^5 \) be a smooth non-degenerate surface of degree 10 and sectional genus 6 obtained as the image of \( \mathbb{P}^2 \) by the rational map defined by the linear system of plane curves of degree 9 having eleven general base points \( p_1, \ldots, p_{11} \), of which \( p_1, \ldots, p_6 \) are at least triple; \( p_7, \ldots, p_{10} \) are at least double; \( p_{11} \) is simple. The irreducible divisor \( C_{32} \) can be described as the closure of the locus of cubic fourfolds containing a surface \( S_{32} \subset \mathbb{P}^5 \), see [Nue15, §3]. In [RS19, Remark 7] we previously pointed out that a general \( S_{32} \) satisfies property \( K_3 \) and that it has two \( 5 \)-secant conics through a general points of \( \mathbb{P}^5 \). The Trisecant Flop \( \tau : X' \rightarrow W' \) of the associated Trisecant small flop contraction \( \tilde{\varphi} : X = \text{Bl}_{S_32} \rightarrow Y \). A general \( X \) through \( S_{32} \) contains some \( 5 \)-secant conics, which determine an extremal ray on \( W' \).

Consider the linear system \( |H^0(\mathcal{I}_{S_{32}}^2(5))| \). One can verify that for a general \( S_{32} \) we have \( h^0(\mathcal{I}_{S_{32}}^2(5)) = 4 \) and that this linear system induces a dominant rational map \( \varphi : \mathbb{P}^5 \rightarrow \mathbb{P}^3 \). The restriction of \( \varphi \) to a general \( X \) through \( S_{32} \) induces a dominant rational map \( \varphi : X \rightarrow \mathbb{P}^3 \), whose general fiber is a quartic rational normal curve which is 10-secant to \( S_{32} \). Thus the elementary Mori contraction of the extremal ray on \( W' \) induced by the class of the strict transform of a \( 5 \)-secant conic to \( S_{32} \) contained in \( X \) is a conic bundle \( \nu : W' \rightarrow \mathbb{P}^3 \), whose general fiber is transformed into a general fiber of \( \varphi \).

4. A divisor of rational Gushel–Mukai fourfolds via the Trisecant Flop

By definition a Gushel-Mukai fourfold \( Z \subset \mathbb{P}^8 \), GM fourfold for short, is a quadratic section of a linear section of a cone over the Grassmannian \( G(1,4) \subset \mathbb{P}^9 \). Equivalently, we can consider a smooth prime Fano fourfold \( Z \) of degree 10 and index 2, that is \( \text{Pic}(Z) \simeq \mathbb{Z}(H) \)
is generated by the class of an ample divisor $H$ such that $H^4 = 10$ and $-K_Z \equiv 2H$. Then $H$ is very ample and embeds $Z$ in $\mathbb{P}^8$ either as a quadratic section of a hyperplane section of $\mathbb{G}(1, 4) \subset \mathbb{P}^9$ (Mukai fourfold, see [Muk89, DIM15]) or as a double cover of $\mathbb{G}(1, 4) \cap \mathbb{P}^7$ branched along its intersection with a quadric (Gushel fourfold, see [Gus82, DIM15]). Gushel fourfolds are specializations of Mukai fourfolds of degree 10 and genus 6.

There exists a 24 dimensional coarse moduli space $\mathcal{G}M$ of GM fourfolds, where the locus of Gushel fourfolds is of codimension 2, see [DIM15, DK16, DK18b]. There exists a period map

$$p : \mathcal{G}M \to \mathcal{D},$$

with $\mathcal{D}$ a quasi-projective variety of dimension 20, called the period domain. The map $p$ is a dominant submersion by [DIM15, Theorem 4.4]. In particular, for a very general GM fourfold $Z \subset \mathbb{P}^8$, the natural inclusion

$$(4.1) \quad A(\mathbb{G}(1, 4)) = H^4(\mathbb{G}(1, 4), \mathbb{Z}) \cap H^{2,2}(\mathbb{G}(1, 4)) \subseteq A(Z) = H^4(Z, \mathbb{Z}) \cap H^{2,2}(Z)$$

is an equality, see [DIM15, Corollary 4.6]. If $A(\mathbb{G}(1, 4)) \subsetneq A(Z)$ holds, then $Z$ is said to be a special GM fourfold. Following [DIM15, Section 7], suppose that an ordinary GM fourfold $Z \subset \mathbb{P}^8$ contains a (smooth) surface $S$ such that $[S] \in A(Z) \setminus A(\mathbb{G}(1, 4))$. Write $[S] = a\sigma_{3,1} + b\sigma_{2,2}$ in $\mathbb{G}(1, 4)$. Then the Double Points Formula for $S \subset Z$ yields:

$$(4.2) \quad S^2 = 3a + 4b + 2K_S \cdot \sigma_{1,1} - 2K_S^2 - 12\chi(O_S).$$

The determinant (or discriminant) of the intersection matrix in the basis $(\sigma_{1,1}S, \sigma_{2,2}S - \sigma_{1,1}S, [S])$ is

$$(4.3) \quad d = 4S^2 - 2(b^2 + (a-b)^2).$$

Inside $\mathcal{D}$ there exist countably many arithmetic hypersurfaces $\mathcal{D}_d$, $d \in \mathbb{N}$, expressing the previous strict inclusion of lattices, whose union is the so called Noether-Lefschetz locus. A standard argument yields that $\mathcal{D}_d \neq \emptyset$ implies $d \equiv 0, 2, 4 \pmod{8}$, see [DIM15, Lemma 6.1]. If $d \equiv 2 \pmod{8}$, then $\mathcal{D}_d = \mathcal{D}_d^p \cup \mathcal{D}_d^q$ with $\mathcal{D}_d^p$ and $\mathcal{D}_d^q$ irreducible hypersurfaces interchanged by the natural involution $r_\mathcal{D} : \mathcal{D} \to \mathcal{D}$.

Some loci of rational Gushel-Mukai fourfolds of different codimension inside $\mathcal{G}M$ are known since the classical work of Roth, see [Rot49] and also [DIM15, Section 7] for recent contributions. As for cubic fourfolds the rationality/irrationality of a (very) general Gushel-Mukai fourfold is unknown and there are striking relations between the two problems.

**4.1. Smooth quintic del Pezzo fivefolds in $\mathbb{P}^8$ through a K3 surfaces of degree 14 and genus 8.** Let $Y \subset \mathbb{P}^8$ be a quintic del Pezzo fivefold. As it is well known $Y \subset \mathbb{P}^8$ is a general hyperplane section of $\mathbb{G}(1, 4) \subset \mathbb{P}^9$, $\text{Aut}(Y)$ has dimension 15 and it coincides with the group of projective transformations of $\mathbb{P}^8$ leaving $Y$ fixed. Moreover, two quintic del Pezzo fivefolds are projectively equivalent so that the Hilbert scheme $\mathcal{D}P$ parametrizing these manifolds is irreducible of dimension 65 and generically smooth.

Let $\mathcal{S}$ be the Hilbert scheme parametrizing K3 surfaces of degree 14 and genus 8 in $\mathbb{P}^8$. Then $\dim(\mathcal{S}) = 99$. $\mathcal{S}$ is generically smooth and a general $[S] \in \mathcal{S}$ is a transversal linear section of the Plücker embedding $\mathbb{G}(1, 5) \subset \mathbb{P}^{14}$. Let $\Pi_p \subset \mathbb{G}(1, 5)$ be a $\mathbb{P}^4$ representing lines passing through a general $p \in \mathbb{P}^5$ and parametrized by a fixed hyperplane $H \subset \mathbb{P}^5$ not passing through $p$. Let $\pi_p : \mathbb{G}(1, 5) \dashrightarrow \mathbb{G}(1, H) \subset \mathbb{P}^9$ be the projection which to a line $[L] \subset \mathbb{G}(1, 5) \setminus \Pi_p$ associates its projection from $p$ into $H$. Then one verifies that a
general linear space $M = \mathbb{P}^8 \subset \mathbb{P}^{14}$ is such that $M \cap \Pi_p = \emptyset$ (see also [Uga02] for general results on smoothness of the projection of surfaces in $G(1, 5)$ via $\pi_p$). Thus the K3 surface $S = M \cap G(1, 5)$ is projected isomorphically from $\Pi_p$ onto a smooth surface $S \subset G(1, H) \subset \mathbb{P}^9$ contained in a hyperplane $H_p = \pi_{\Pi_p}(M)$. For $p \in \mathbb{P}^5$ general, $Y_p = H_p \cap G(1, H)$ is a smooth del Pezzo fivefold. Thus, fixing $S \subset G(1, 4) \subset \mathbb{P}^9$ one has an irreducible five dimensional family of del Pezzo fivefolds containing $S$ parametrised by an open subset of $\mathbb{P}^5$.

Let

$$I = \{([S], [Y]) : S \subset Y\} \subset S \times \mathcal{D}P$$

and let

$$I \xleftarrow{p_1} S \xrightarrow{p_2} \mathcal{D}P$$

We have seen that $[S] \in S$ general is contained in at least a smooth del Pezzo fivefold, i.e. $p_1$ is a surjective morphism, and that $p_1^{-1}([S])$ is irreducible and of dimension 5, yielding that $I$ is irreducible of dimension 104. Hence, on a fixed quintic del Pezzo fivefold $Y \subset \mathbb{P}^8$ we have a 39 dimensional family of K3 surfaces of degree 14 and genus 8 contained in $Y$, a fact that can be also verified by an explicit computation in an example. In particular, every quintic del Pezzo fivefold $Y \subset \mathbb{P}^8$ contains such a family and $\dim(p_2^{-1}([Y])) = 39$.

**Remark 4.1.** The five dimensional family of del Pezzo fivefolds containing a general smooth K3 surface $S \subset \mathbb{P}^8$ of degree 14 and genus 8 can be considered as the dual of the five dimensional family of smooth quintic del Pezzo surfaces contained in the associated cubic pfaffian fourfold. Indeed, following [BD85], let $V$ a vector space of dimension 6 and let

$$G(2, V) \subset \Delta \subset \mathbb{P}(\Lambda^2V), \ G(2, V^*) \subset \Delta^* \subset \mathbb{P}(\Lambda^2V^*)$$

be the Plücker embeddings of the Grassmann manifolds $G(2, V)$, respectively $G(2, V^*)$. These manifolds are the loci of tensors of rank at most two while the cubic hypersurfaces $\Delta$, respectively $\Delta^*$, which are the secant varieties of $G(2, V)$ and of $G(2, V^*)$ respectively, are the loci of tensors of rank at most four.

Let $L = \mathbb{P}^8 \subset \mathbb{P}(\Lambda^2V)$ be general. Then $S = L \cap G(2, V) \subset L$ is a general smooth K3 surface of degree 14 and genus 8, while $X = L^\perp \cap \Delta^*$ is a smooth cubic hypersurface in $L^\perp = \mathbb{P}^5$. Let $L^\perp = \mathbb{P}(U)$ with $U \subset \Lambda^2V^*$ of dimension 6.

To a general subspace $W \subset U \subset \Lambda^2V^*$ of dimension 5 there corresponds a surjection $\Lambda^2V^* \to \Lambda^2W^*$. Consider the set of $[\alpha] \in X$ such that $\ker(\alpha) \subset W$. Then $\alpha_{|W}$ is decomposable and since $X \cap G(2, V^*) = \emptyset$, the inclusion $U \subset \Lambda^2V^*$ yields an embedding $U \subset \Lambda^2W^*$ and hence an embedding $L^\perp \subset \mathbb{P}(\Lambda^2W^*)$ such that $L^\perp \cap G(2, W)$ is a smooth quintic del Pezzo surface contained in $X$ given exactly by the $[\alpha] \in X$ such that $\ker(\alpha) \subset W$. In this way one constructs the well known five dimensional family of smooth quintic del Pezzo surfaces contained in the Pfaffian cubic $X$, see [BD85].

**4.2. GM fourfolds containing a K3 surface of degree 14 and genus 8.** Let $Y \subset \mathbb{P}^8$ be a smooth quintic del Pezzo fivefold and let

$$T = H^0(\mathcal{I}_Y(2)) \subset H^0(\mathcal{O}_{\mathbb{P}^8}(2)) = R.$$
Then \( \dim(T) = 5, \dim(R) = 45, \dim(R/T) = 40 \) and for every class \([Q] \in \mathbb{P}(R/T)\) general we obtain a general GM fourfold \( Z = Z[Q] \subset \mathbb{P}^8 \). The Hilbert scheme \( GM \) parametrizing GM fourfolds in \( \mathbb{P}^8 \) has dimension 104 = 65 + 39. In particular, we deduce that GM fourfolds depend on 24 = 104 − 80 moduli since their automorphism group is finite by [DIM15, Proposition 4.1].

Suppose \( S \subset Z \) is a smooth K3 surface of degree 14 and genus 8. By the analysis in subsection 4.1 we deduce \([S] = 9\sigma_{3,1} + 5\sigma_{2,2}\). Moreover, we have \( S^2 = 23 \) by (4.2) and \( d = 10 \) by (4.3). Consider the locus \( K \) in the Hilbert scheme of GM fourfolds in \( \mathbb{P}^8 \) corresponding to the closure of those containing a smooth K3 surface of degree 14 and genus 8. Since such a general K3 surface is contained in an irreducible five dimensional family of del Pezzo fivefolds, it is contained in an irreducible 14 dimensional family of GM fourfolds. Consider the incidence correspondence:

\[
J = \{( [S], [Z] ) : S \subset Z \} \subset S \times GM
\]

and let

\[
\begin{array}{c}
J \\
p_1 \downarrow \quad \downarrow p_2 \\
S \quad GM
\end{array}
\]

where by abusing notation we indicate by \( GM \) the Hilbert scheme of GM fourfolds in \( \mathbb{P}^8 \) (and not only the coarse moduli space as before). By the previous analysis \( J \) is irreducible of dimension 113 = 99 + 14. In a specific example of GM fourfold \( Z \subset \mathbb{P}^8 \) in \( K \) we verified that \( h^0(N_{S/Z}) = 10 \) for a general K3 surface \( S \subset Z \) of degree 14 and genus 8, yielding \( \dim(p_2(J)) \geq 113 - 10 = 103 \). Since \( p_2(J) \subset GM \) (a very general GM fourfold cannot contain a K3 surface of degree 14 and genus 8 by [DIM15, Corollary 4.6]) and since \( \dim(GM) = 104 \), we deduce that \( K = p_2(J) \) is an irreducible divisor in \( GM \). Moreover, a general \([Z] \in K \) contains a 10 dimensional family of K3 surfaces of degree 14 and genus 8, whose general element is smooth and of Picard number one. We also have \( p(K) = D_{10}' \).

**Remark 4.2.** Since \( h^0(O_S(2)) = 30 \) and since a del Pezzo fivefold containing a GM fourfold \( X \) has a 39 dimensional family of K3 surfaces of degree 14 and genus 8, a pure enumeration of the parameters would suggest that the dimension of the family of K3 surfaces of degree 14 and sectional genus 8 contained in \( X \) is 9. Since this would imply \( K = p_2(J) = GM \), we see that the 30 conditions imposed to quadrics to contain a K3 surface of degree 14 and genus 8 are not independent. We shall come back on this parameter count in the next section.

### 4.3. Surfaces of degree 10 and genus 6 with a node in \( \mathbb{P}^5 \) obtained as projections of general K3 surfaces of degree 10 and genus 6.

Let \( S' \subset \mathbb{P}^6 \) be a smooth K3 surface of degree 10 and genus 6. Let \( S_{14} \subset \mathbb{P}^5 \) be the projection of \( S' \) from a general point on the secant variety of \( S' \). Then \( S_{14} \subset \mathbb{P}^5 \) is a degree 10 and sectional genus 6 surface with a node, contained in smooth cubic hypersurfaces and whose ideal is generated by 10 cubic forms.

On a cubic fourfold \( X \in H^0(\mathcal{I}_{S_{14}}(3)) \) we have \((S_{14})^2 = 38 \) and \( d = 3 \cdot 38 - 100 = 14 \). The surfaces \( S_{14} \) depend on at least 19 + 5 moduli so that the corresponding Hilbert scheme \( S_{14} \) has dimension at least 59. By an explicit computation we find that for a (general) \( S_{14} \) of the previous kind we have \( h^0(N_{S_{14}/\mathbb{P}^5}) = 59 \), proving that \( S_{14} \) is generically smooth of dimension 59 and that the surfaces \( S_{14} \)’s depend on 24=59-35 moduli.
Since \( h^0(\mathcal{I}_{S_{14}}(3)) = 10 \), since there exists a cubic fourfold containing a 14-dimensional family of such surfaces and since \( \dim(C_{14}) = 19 \), we conclude that
\[
\mathcal{C}_{14} = \{ [X] \in \mathcal{C} \text{ for which } \exists [S_{14}] \in S_{14} : S_{14} \subset X \}.
\]
In particular, a general cubic \( X \in \vert H^0(\mathcal{I}_{S_{14}}(3)) \vert \) is rational for a general \( S_{14} \) and the dimension of the family of surfaces \( S_{14} \)'s contained in a general \( X \in \mathcal{C}_{14} \) is \( 14 = 59 + 9 - 54 \).

The projection from the node \( q \) of \( S_{14} \) yields a birational map \( j : S_{14} \rightarrow S_0 \subset \mathbb{P}^4 \) onto the projection \( S_0 \subset \mathbb{P}^4 \) of \( S' \subset \mathbb{P}^6 \) from two general points on it, which is a singular surface of degree 8 and sectional genus 6, cut out by one cubic and four quartics. The birational map \( j \) can also be described by (a six-dimensional family of) linear systems of quadrics on \( S_{14} \) whose base loci are the union of the point \( q \) with a smooth quadratic section of a smooth quintic del Pezzo surface. From this one can find smooth quintic del Pezzo surfaces intersecting the given \( S_{14} \) along a smooth curve of degree 10 and sectional genus 6 (we give some computational details in the Macaulay2 package mentioned in Section 6).

4.4. **Rationality of GM-fourfolds in the divisor \( \mathcal{K} \).** We now apply Theorems 2.6 and 2.9 to prove the rationality of GM-fourfolds in \( \mathcal{K} \) and to illustrate an unknown birationality between paffian cubic fourfolds determined by the associated K3 surfaces.

**Theorem 4.3.** Let notation be as above. Then:

(i) A general \( S_{14} \subset \mathbb{P}^5 \) as above admits a congruence of 8-secant twisted cubics and the linear system \( \vert H^0(\mathcal{I}_{S_{14}}^3(8)) \vert \) of octic hypersurfaces with triple points along \( S_{14} \) restricted to a general \( X \in \vert H^0(\mathcal{I}_{S_{14}}(3)) \vert \) defines a birational map \( \mu : X \rightarrow W \subset \mathbb{P}^8 \) with \( W \) a smooth GM-fourfold.

(ii) The map \( \mu^{-1} : W \rightarrow X \) is given by a linear system of divisors in \( \vert O_W(5) \vert \) having triple points along a smooth K3 surface \( S \subset W \subset \mathbb{P}^8 \) of degree 14 and genus 8. In particular \( W \) is a general GM fourfold in the irreducible divisor \( \mathcal{K} \subset \mathcal{G}M \).

(iii) Every \( [W] \in \mathcal{K} \) is rational and \( \mathcal{K} \) is an irreducible component of the inverse image in \( \mathcal{G}M \) of the period domain \( \mathcal{D}_{10} \) under the period map.

**Proof.** Let notation be as above and let \( S_{14} \subset S_{14} \) be general. The linear system \( \vert H^0(\mathcal{I}_{S_{14}}(3)) \vert \) satisfies condition \( K_3 \) and defines a map \( \varphi = \varphi_{S_{14}} : \mathbb{P}^5 \rightarrow \mathbb{P}^9 \) which is birational onto a cubic section \( Z = \varphi(\mathbb{P}^5) \subset \mathbb{P}^8 \) of a del Pezzo fivefold in \( \mathbb{P}^8 \). Through a general point \( z = \varphi(p) \in Z \) there passes 18 lines contained in \( Z \). Of these 11 are images of the eleven secant lines to \( S_{14} \) passing through \( p \). The remaining 7 come from six 5-secant conics to \( S_{14} \) passing through \( p \) and a single 8-secant twisted cubic to \( S_{14} \) passing through \( p \), which is thus transversal to a general \( X \in \vert H^0(\mathcal{I}_{S_{14}}(3)) \vert \) by Theorem 1.1. We deduce that a general \( S_{14} \) admits a congruence of 8-secant twisted cubics. Moreover, there exists the Trisecant Flop of the small flop contraction \( \tilde{\varphi} : X' = \text{Bl}_{S_{14}} X \rightarrow Y \subset \mathbb{P}^8 \) with \( X \) general cubic through \( S_{14} \) and with \( Y \) the corresponding general hyperplane section of \( Z \) (this is the example corresponding to the line (vii) of Table 1).

By an explicit computation we verified that \( h^0(\mathcal{I}_{S_{14}}^3(8)) = 9 \) for a general \( S_{14} \subset S_{14} \). Furthermore, considering the map \( \mu : X \rightarrow W \subset \mathbb{P}^8 \), defined by the linear system \( \vert H^0(\mathcal{I}_{S_{14}}^3(8)) \vert \), we verified that \( W \subset \mathbb{P}^8 \) is a smooth GM-fourfold and that the image of the twisted cubics of the congruence contained in \( X \) is a smooth K3 surface \( S \subset W \subset \mathbb{P}^8 \) of degree 14 and genus 8. The surface \( S \subset \mathbb{P}^5 \) is defined by 15 quadratic forms while \( W \) is defined by 6 quadratic forms.
forms. The restriction of $H^0(I_S(2))$ to $W$ induces a birational map $\psi: W \dasharrow Y \subset \mathbb{P}^8$ and the Trisecant Flop $\tilde{\psi}: W' = \text{Bl}_S W \to Y \subset \mathbb{P}^8$. The linear system of divisors in $|O_W(5)|$ having triple points along $S$ define $\mu^{-1}: W \dasharrow X \subset \mathbb{P}^5$.

The conclusion in (ii) follows from the parameter count in subsection 4.3 and can be also explained in the following way. The restriction of $\varphi$ to a general cubic fourfold $X \in H^0(I_{S_{14}}(3))$ induces a birational map onto $W$. Since a general cubic in $C_{14}$ contains a 14 dimensional family of surfaces $S_{14}$, we deduce that the family of $W$’s produced in this way has dimension $19 + 14 - 10 = 23$. The ten dimensional family of K3 surfaces of degree 14 and genus 8 in $W$ arises as the base loci of the inverses of $\varphi: X \dasharrow Z_{S_{14}}$, as shown above. The claim in (iii) about the rationality of every GM fourfold in $\mathcal{K}$ follows from the rationality of a general $W \in \mathcal{K}$ and from the main result of [KT19].

Remark 4.4. In the notation of Theorem 4.3, let $\mu: \mathbb{P}^5 \dasharrow W \subset \mathbb{P}^8$ denote the rational map defined by the linear system $|H^0(I_{S_{14}}^3(8))|$ of octic hypersurfaces with triple points along a general $S_{14}$. If $D \subset \mathbb{P}^5$ is a general quintic del Pezzo surface intersecting $S_{14}$ along a smooth curve of degree 10 and genus 6, as constructed at the end of Subsection 4.3, we have that $\mu(D) \subset W$ is a smooth quadric surface $Q$ which is 5-secant a general K3 surface $S \subset W$ of degree 14 and genus 8. Then $Q$ is a $\tau$-quadric according to the terminology in [DIM15] and it has the following property: the projection from the linear span of $Q$ gives a birational map $W \dasharrow \mathbb{P}^4$ whose inverse is defined by the linear system of quartics passing through a singular surface $S_0 \subset \mathbb{P}^4$ obtained by projecting a K3 surface in $\mathbb{P}^6$ of degree 10 and genus 6 from two points on it (see [DIM15, Proposition 7.4]). It turns out that via this birational map the image of a general K3 surface $S \subset W$ of degree 14 and genus 8 is a smooth surface in $\mathbb{P}^4$ of degree 9 and sectional genus 8 cut out by 6 quartics, as in Example 1.2(i).

In particular, the divisor $\mathcal{K}$ of GM fourfolds coincides with the divisor of GM fourfolds containing a $\tau$-quadric studied in [DIM15, Subsection 7.3].

5. SOME QUESTIONS ON THE RATIONALITY OF ADMISSIBLE CUBIC FOURFOLDS

The previous results and the analysis of a lot of examples suggest to formulate some questions of birational flavour about the rationality of cubic fourfolds.

5.1. Flops and rationality. First of all one can ask whether it is true that a cubic fourfold $X \subset \mathbb{P}^5$ is rational if and only if it contains an irreducible surface $S \subset \mathbb{P}^5$ such that there exist a flop $\tau: X' = \text{Bl}_S X \dasharrow W'$ and a congruence of $(3e - 1)$-secant curves to $S$ of degree $e \geq 2$ transversal to $X$, determining an extremal divisorial contraction $\nu: W' \to W$ to a prime $\mathbb{Q}$-factorial rational Fano variety. Here $S \subset \mathbb{P}^5$ is not necessarily scheme-theoretically defined by cubic equations but one can expect, as it occurs in many examples, that the scheme defined by cubics through $S$ is able to determine the Trisecant flop and also the congruence (see Table 2).

One might also ask whether a cubic fourfold $X \subset \mathbb{P}^5$ is rational if and only if there exist an irreducible surface $S \subset X$ and an integer $e \geq 2$ such that the linear system $|H^0(I_S^e(3e - 1))|$ determines a map $\mu: X \dasharrow W$ birational onto the image.

5.2. Looking for the associated K3 surfaces $U \subset W$. Let $d$ be an admissible value, let $g = (d + 2)/2$ and let $U' \subset \mathbb{P}^g$ be a general K3 surface of degree $d$ and genus $g$. One can ask whether there exists a birational map $\psi: U' \dasharrow U \subset \mathbb{P}^r$ such that $U$ is contained in a
rational prime \( \mathbb{Q} \)-factorial Fano variety \( W \subset \mathbb{P}^r \) of index \( i(W) \). Then one might look for an integer \( e \geq 2 \) such that the linear system of divisors in \( |O_W(e \cdot i(W) − 1)| \) having points of multiplicity \( e \) along \( U \) determines a birational map onto the image. Finally, one hopes that the image of this map is a general cubic fourfold in \( C_d \).

Let us analyze better this last question with a look to Table 1 and to the examples considered by Fano. In some cases the requirement \( W = \mathbb{P}^4 \) forces the appearance of mild singularities on \( U \), typically nodes, see lines (i) and (ii). Also the case of the smooth non minimal K3 surface of degree 9 and sectional genus 8 in \( \mathbb{P}^4 \) considered by Fano for \( d = 14 \), see Subsection 1.2, depends on the existence of 5 points on \( U' \) imposing only four conditions to hyperplane sections. Very few smooth non minimal K3 surfaces \( U \subset \mathbb{P}^4 \) are known and for most of them the degree \( d = 2g − 2 \) of the surfaces \( U' \subset \mathbb{P}^g \) is not admissible, see [DES93, §B4]). Let us also recall that, according to a long standing conjecture, the degree of such a smooth \( U \subset \mathbb{P}^4 \) should be bounded by 15, see [DES93].

The other lines of Table 1 suggest to look for surfaces \( U' \subset \mathbb{P}^r \) contained in rational prime Fano fourfolds. We are aware of a unique technique to produce interesting smooth non minimal K3 surfaces as above with \( r \) small (if \( r \) is big, then the four dimensional variety \( W \) containing \( U \) might be singular or a Fano fourfold of index 1). It is the following result due to Voisin and to Fontanari-Sernesi.

\textbf{Theorem 5.1.} ([Voi19, Proposition 4.1], [FS19, Theorem 10]) \textit{Let} \( U' \subset \mathbb{P}^g \) \textit{be a minimal primitive K3 surface of degree} \( d = 2g − 2 \), \textit{let} \( m \geq 1 \) \textit{be an integer, let} \( p \in U' \) \textit{be a point, let} \( H' \subset U = \text{Bl}_p U' \) \textit{be the pull-back of a general hyperplane section of} \( U' \) \textit{and let} \( E \subset U \) \textit{be the exceptional divisor}. \textit{If} \( d = 2g − 2 \geq (m + 1)^2 + 3 \), \textit{then the linear system} \( |H' − mE| \) \textit{is very ample on} \( U \) \textit{and gives an embedding} \( U \subset \mathbb{P}^g − \frac{m(m+1)}{2} \). \textit{Moreover}, \( \deg(U) = 2g − 2 − m^2 \) \textit{and the sectional genus of} \( U \) \textit{is} \( g − \frac{m(m−1)}{2} \).

For the convenience of the reader, we list in Table 4 the first admissible values \( d \) with the maximal value of \( m \) allowed by Theorem 5.1. It should be noted that \( r = g − \frac{m(m+1)}{2} \) tends to remain rather small.

| \( d = 2g − 2 \) | \( m \) | \( \deg(U) = 2g − 2 − m^2 \) | \( g(U) = g − \frac{m(m−1)}{2} \) | \( r = g − \frac{m(m+1)}{2} \) | \( h^0(\mathcal{O}_{\mathbb{P}^r}(2)) − \chi(\mathcal{O}_U(2)) \) |
|---|---|---|---|---|---|
| 14 | 2 | 10 | 7 | 5 | 1 |
| 26 | 3 | 17 | 11 | 8 | 12 |
| 38 | 4 | 22 | 14 | 10 | 24 |
| 42 | 5 | 17 | 12 | 7 | 5 |
| 62 | 6 | 26 | 17 | 11 | 30 |
| 74 | 7 | 25 | 17 | 10 | 21 |
| 78 | 7 | 29 | 19 | 12 | 38 |
| 86 | 8 | 22 | 16 | 8 | 7 |
| 98 | 8 | 34 | 22 | 14 | 58 |
| 114 | 9 | 33 | 22 | 13 | 46 |
| 122 | 9 | 41 | 26 | 17 | 96 |
| 134 | 10 | 34 | 23 | 13 | 45 |

\textbf{Table 4.} Embeddings in \( \mathbb{P}^r \) of blow-ups at one point of K3 surfaces of degree \( d \) as described by Theorem 5.1. Here \( d \) runs over all admissible values < 140.
Let us remark that applying Theorem 5.1 for \( g = 8, d = 14 \) and \( m = 2 \) one obtains Fano's example of a smooth surface \( U \subset \mathbb{P}^5 \) of degree 10 and genus 7. In this case \( h^0(\mathcal{I}_U(2)) = 1 \) and the unique quadric \( W \) containing \( U \) is smooth. Moreover, \( h^0(\mathcal{I}_U(3)) = 12 \) and the linear system \( |H^0(\mathcal{I}_U(3))|_W \) yields a birational map to a smooth cubic fourfold \( X \subset \mathbb{P}^5 \) in \( C_{14} \). A general external projection of \( U \subset \mathbb{P}^5 \) into \( \mathbb{P}^4 \) has eight nodes and it appears in line (i) of Table 1. So either the complete linear system \( |H' - 2E| \) or a general sublinear system of dimension four are able to determine birational representations of a general cubic fourfold in \( C_{14} \).

Let us pass to the next admissible value \( d = 26 \) and hence \( g = 14 \). We take \( m = 3 \) and we get a smooth non minimal K3 surface \( \tilde{U} \subset \mathbb{P}^8 \). There exists a surface \( U \subset \mathbb{P}^7 \), which is a suitable linear projection of \( \tilde{U} \) and whose ideal is generated by 5 quadratic forms, defining a smooth del Pezzo fourfold \( W \subset \mathbb{P}^7 \) of degree 5, and by 13 cubics defining by restriction a small contraction \( \text{Bl}_U W \to Y \subset \mathbb{P}^{12} \). Then \( W \subset \mathbb{P}^7 \) is birational to a general cubic fourfold in \( C_{26} \) via the linear system of quintic forms having points of multiplicity 2 along \( U \subset W \), see line (iv) of Table 1.

If we take \( d = 38 \) (and hence \( g = 20 \)) and \( m = 4 \), then \( U \subset \mathbb{P}^{10} \) is a smooth non minimal K3 surface of degree 22 and sectional genus 14 contained in the Mukai fourfold \( W = G(1,5) \cap \mathbb{P}^{10} \). Via the Trisectant Flop one deduces that a general cubic in \( C_{38} \) is birational to this \( W \), see line (v) of Table 1.

So one tries to find possible candidates \( U \subset W \subset \mathbb{P}^r \) for the next admissible values following the same path. The first interesting case to be investigated is \( d = 42 \) for which the Kuznetsov Conjecture is open. If \( U' \subset \mathbb{P}^{22} \) is a general primitive K3 surface of degree 42 and genus 22, then taking \( m = 5 \) we produce a smooth non minimal K3 surface \( U \subset \mathbb{P}^7 \) of degree 17 and genus 12. We suspect that the five quadrics vanishing on \( U \) define a del Pezzo fourfold \( W \subset \mathbb{P}^7 \) containing \( U \). If this were the case, one might expect that the linear systems of forms of degree \( 3e - 1 \) having point of multiplicity \( e \geq 2 \) along \( U \) restricted to \( W \) give birational maps onto general cubic fourfolds in \( C_{12} \). We are not aware of a surface contained in a general cubic fourfold in \( C_{12} \) able to define maps \( \mu \) onto a rational fourfold (see [Lai17] for the construction of rational normal scrolls of degree 9 with 8 nodes in these cubic fourfolds).

Continuing with the next admissible value \( d = 62 \) (and \( g = 32 \)), we can take \( m = 6 \) and obtain a surface \( U \subset \mathbb{P}^{11} \) of degree 26 and sectional genus 17. We suspect that in this case \( W \subset \mathbb{P}^{11} \) is a linear section of the Lagrangian Grassmannian \( LG_3(\mathbb{C}^6) \subset \mathbb{P}^{13} \). Indeed, we constructed some examples of non minimal K3 surfaces in \( \mathbb{P}^{11} \) contained in such \( W \), obtained by a minimal K3 surface of degree 38 and genus 20 by imposing three double points to the linear system of hyperplane sections and defined by 30 quadratic equations (see the line (xii) of Table 2). So also in this case one hopes to verify Kuznetsov Conjecture.

We also point out that a general projection of such a K3 surface \( U \subset LG_3(\mathbb{C}^6) \subset \mathbb{P}^{11} \) from one of its points (respectively, from one of its tangent planes) yields an example of surface with the same numerical invariants as in the line \( d = 74 \) (respectively, \( d = 86 \)) of Table 4.
6. Explicit examples of trisecant flops in Macaulay2

We provide a Macaulay2 package [GS19] named TrisecantFlops,\(^1\) which produces explicit examples of Trisecant Flops in accordance to Tables 1 and 2. Once the package is loaded, typing example\(i\) (where \(i\) is an integer between 1 and 14), will build up a birational map \(X \to W\) as in the \(i\)-th line of Tables 1 and 2. For instance, we now consider the third example.

Macaulay2, version 1.14
with packages: ConwayPolynomials, Elimination, IntegralClosure, InverseSystems,
LLL, PrimaryDecomposition, ReesAlgebra, TangentCone, Truncations

\begin{verbatim}
i1 : needsPackage "TrisecantFlops";
i2 : mu = example 3;
o2 : SpecialRationalMap (birational map from hypersurface in PP^5 to PP^4)
i3 : describe inverse mu
o3 = rational map defined by forms of degree 9
   source variety: PP^4
   target variety: smooth cubic hypersurface in PP^5
   birationality: true
   projective degrees: {1, 9, 27, 15, 3}
\end{verbatim}

We can obtain the smooth surface \(S \subset \mathbb{P}^5\) of degree 10 and sectional genus 6 by giving the following command.

\begin{verbatim}
i4 : S = specialBaseLocus mu;
i5 : (codim S, degree S, (genera S)_1)
o5 = (3, 10, 6)
\end{verbatim}

Analogous, the K3 surface \(U \subset \mathbb{P}^4\) is obtained as follows.

\begin{verbatim}
i6 : U = specialBaseLocus inverse mu;
i7 : (codim U, degree U, (genera U)_1)
o7 = (2, 12, 14)
\end{verbatim}

Finally, the following command yields an extension to \(\mathbb{P}^5\) of the map \(\mu : X \to W = \mathbb{P}^4\) whose general fibre is a 5-secant conic to the surface \(S\).

\begin{verbatim}
i8 : extend mu;
o8 : SpecialRationalMap (dominant rational map from PP^5 to PP^4)
\end{verbatim}

References

[AR04] A. Alzati and F. Russo, Some elementary extremal contractions between smooth varieties arising from projective geometry, Proc. Lond. Math. Soc. 89 (2004), 25–53.

[Art70] M. Artin, Algebraization of formal moduli: II. Existence of modifications, Ann. of Math. 91 (1970), no. 1, 88–135.

[Bau98] I. Bauer, The classification of surfaces in \(\mathbb{P}^5\) having few trisecant lines, Rend. Sem. Mat. Univ. Pol. Torino 56 (1998), 1–20.

[BD85] A. Beauville and R. Donagi, La variété des droites d’une hypersurface cubique de dimension 4 (in French), C. R. Math. Acad. Sci. Paris 301 (1985), no. 14, 703–706.

[BRS19] M. Bolognesi, F. Russo, and G. Staglianò, Some loci of rational cubic fourfolds, Math. Ann. 373 (2019), no. 1, 165–190.

[CC93] L. Chiantini and C. Ciliberto, A few remarks on the lifting problem, Astérisque 218 (1993), 95–109.

[DES93] W. Decker, L. Ein, and F.-O. Schreyer, Construction of surfaces in \(\mathbb{P}^4\), J. Algebraic Geom. 2 (1993), no. 2, 185–237.

[DIM15] O. Debarre, A. Iliev, and L. Manivel, Special prime Fano fourfolds of degree 10 and index 2, Recent Advances in Algebraic Geometry: A Volume in Honor of Rob Lazarsfeld’s 60th Birthday (C. Hacon,

\(^1\)It is available at https://www.dropbox.com/sh/0xzja09okz3ak3/AAABpedeC_cL68whLQIPGkva?dl=0.
M. Mustata, and M. Popa, eds.), London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, 2015, pp. 123–155.

[DK16] O. Debarre and A. Kuznetsov, Gushel–Mukai varieties: linear spaces and periods, preprint: https://arxiv.org/abs/1605.05648, 2016.

[DK18a] O. Debarre and A. Kuznetsov, Gushel–Mukai varieties: Classification and birationalities, Algebr. Geom. 5 (2018), 15–76.

[DK18b] O. Debarre and A. Kuznetsov, Gushel–Mukai varieties: moduli, preprint: https://arxiv.org/abs/1812.09186, 2018.

[Fan43] G. Fano, Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4° ordine, Comment. Math. Helv. 15 (1943), no. 1, 71–80.

[FN71] A. Fujiki and S. Nakano, Supplement to “on the inverse of monoidal transformation”, Publ. RIMS Kyoto Univ. 7 (1971), 637–644.

[FS19] C. Fontanari and E. Sernesi, Non-surjective Gaussian maps for singular curves on K3 surfaces, Collect. Math. 70 (2019), no. 1, 107–115.

[Fan43] G. Fano, Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4° ordine, Comment. Math. Helv. 15 (1943), no. 1, 71–80.

[FN71] A. Fujiki and S. Nakano, Supplement to “on the inverse of monoidal transformation”, Publ. RIMS Kyoto Univ. 7 (1971), 637–644.

[FS19] C. Fontanari and E. Sernesi, Non-surjective Gaussian maps for singular curves on K3 surfaces, Collect. Math. 70 (2019), no. 1, 107–115.

[FV18] G. Farkas and A. Verra, The universal K3 surface of genus 14 via cubic fourfolds, J. Math. Pures Appl. 111 (2018), 1–20.

[GP13] L. Gruson and C. Peskine, On the smooth locus of aligned Hilbert schemes, the k-secant lemma and the general projection theorem, Duke Math. J. 162 (2013), no. 3, 553–578.

[GS19] D. R. Grayson and M. E. Stillman, MACAULAY2 — A software system for research in algebraic geometry (version 1.14), Home page: http://www.math.uiuc.edu/Macaulay2/, 2019.

[Gus82] N. P. Gushel, Fano varieties of genus 6 (in russian), Izv. Akad. Nauk USSR Ser. Mat. 46 (1982), no. 6, 1159–1174, English transl.: Math. USSR-Izv. 21 3 (1983), 445–459.

[Has16] B. Hassett, Cubic fourfolds, K3 surfaces, and rationality questions, Rationality Problems in Algebraic Geometry: Levico Terme, Italy 2015 (R. Pardini and G. P. Pirola, eds.), Springer International Publishing, Cham, 2016, pp. 29–66.

[HK13] C. D. Hacon and J. Mc Kernan, The Sarkisov program, J. Algebraic Geom. 22 (2013), 389–405.

[HPT17] B. Hassett, A. Pirutka, and Y. Tschinkel, Intersections of three quadrics in $\mathbb{P}^7$, Surv. Differ. Geom. 22 (2017), 259–274.

[HPT18] B. Hassett, A. Pirutka, and Y. Tschinkel, Stable rationality of quadric surface bundles over surfaces, Acta Math. 220 (2018), no. 2, 341–365.

[Kol19] J. Kollár, Algebraic hypersurfaces, Bull. Amer. Math. Soc. (2019), https://doi.org/10.1090/bull/1663.

[KT19] M. Kontsevich and Y. Tschinkel, Specialization of birational types, to appear in Invent. Math., preprint: https://arxiv.org/abs/1708.05699, 2019.

[Kuz16] A. Kuznetsov, Derived categories view on rationality problems, Rationality Problems in Algebraic Geometry: Levico Terme, Italy 2015 (R. Pardini and G. P. Pirola, eds.), Springer International Publishing, Cham, 2016, pp. 67–104.

[Lai17] K. Lai, New cubic fourfolds with odd-degree unirational parametrizations, Algebra & Number Theory 11 (2017), 1597–1626.

[LLW10] Y.-P. Lee, H.-W. Lin, and C.-L. Wang, Flops, motives, and invariance of quantum rings, Ann. of Math. 172 (2010), no. 1, 243–290.

[Mor82] S. Mori, Threefolds whose canonical bundles are not numerically effective, Ann. of Math. 116 (1982), no. 1, 133–176.

[Muk89] S. Mukai, Biregular classification of Fano 3-folds and Fano manifolds of coindex 3, Proc. Natl. Acad. Sci. USA 86 (1989), no. 9, 3000–3002.

[Nak71] S. Nakano, On the inverse of monoidal transformation, Publ. RIMS Kyoto Univ. 6 (1971), no. 3, 483–502.

[Nue15] H. Nuer, Unirationality of moduli spaces of special cubic fourfolds and K3 surfaces, Algebr. Geom. 4 (2015), 281–289.

[Ran15] Z. Ran, Unobstructedness of filling secants and the Gruson–Peskine general projection theorem, Duke Math. J. 164 (2015), no. 4, 697–722.

[Rog] E. Rogora, On projective varieties for which a family of multisecant lines has dimension larger than expected, preprint n. 28/96, Dip. di Matematica, Università degli Studi di Roma La Sapienza, available at http://www1.mat.uniroma1.it/people/rogora/pdf/28.pdf.
[Rot49] L. Roth, *Algebraic varieties with canonical curve sections*, Ann. Mat. Pura Appl. 29 (1949), no. 1, 91–97.

[RS18] F. Russo and G. Staglianò, *Explicit rationality of some cubic fourfolds*, available at https://arxiv.org/abs/1811.03502, 2018.

[RS19] ______, *Congruences of 5-secant conics and the rationality of some admissible cubic fourfolds*, Duke Math. J. 168 (2019), no. 5, 849–865.

[Rus00] F. Russo, *On a theorem of Severi*, Math. Ann. 316 (2000), no. 1, 1–17.

[Sta18] G. Staglianò, *Special cubic Cremona transformations of $\mathbb{P}^6$ and $\mathbb{P}^7$*, Adv. Geom. 19 (2018), no. 2, 191–204.

[Sta19] G. Staglianò, *Special cubic birational transformations of projective spaces*, to appear in Collect. Math., doi:10.1007/s13348-019-00251-8, preprint: https://arxiv.org/abs/1901.01203, 2019.

[Uga02] L. Ugaglia, *Subvarieties of the Grassmannian $G(1,n)$ with small secant variety*, Comm. Algebra 30 (2002), 4059–4083.

[Ver01] P. Vermeire, *Some results on secant varieties leading to a geometric flip construction*, Compos. Math. 125 (2001), no. 3, 263–282.

[Voi19] C. Voisin, *Segre classes of tautological bundles on Hilbert schemes of surfaces*, Algebr. Geom. 6 (2019), no. 2, 186–195.

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