Knotted Solitons

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Abstract

The dynamical model on 3+1 dimensional space-time admitting soliton solutions is discussed. The proposal soliton is localized in the vicinity of a closed contour, which could be linked and/or knotted. The topological charge is Hopf invariant. Some applications in realistic physical systems are indicated.

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1. Introduction

The term “soliton” entered applied mathematics in 1965. It was coined by M. Kruskal and N. Zabusky for a special solution of nonlinear Korteweg-de Vries (KdV) equation, depicting solitary wave [1]. Use of convention of particle physics language shows that the author envisioned the particle-like interpretation for the object which they called soliton.

The attention of mathematical physicists to solitons was attracted after the inverse scattering method was devised by G. Gardner, J. Green, M. Kruskal and R. Miura for solving the KdV equation [2] and its extension to Nonlinear Schroedinger Equation was found by V. Zakharov and A. Shabat [3]. In the 1970’s this method and its generalizations got a lot of attention and involved quite a few active participants. Rather complete review can be found in [4]. In the end of that decade the quantum variant of the method was constructed and particle-like interpretations of solitons got natural confirmation in terms of quantum field theory, see review in [5].

The mathematical structure of the quantum method was deciphered in pure algebraic way leading in the 1980’s to notion of quantum groups with new applications in pure mathematics and mathematical physics.

The value of solitons for the particle physics consists in the possibility of going beyond the paradigm of the perturbation theory. Indeed, soliton solutions correspond to full nonlinear equations and disappear in their linearized form. Characteristic for solitons is that they interact strongly if the excitations of the linearized
fields interact weakly. Another attractive feature is the appearance of elementary topological characteristics for solitons topological charges.

This was understood already in the middle of 1970’s by several groups as I underlined in my lectures, when I was touring USA in 1975 (see e. g. [6]). However, all these tantalizing features of solitons had one very important drawback: the developed methods applied only in $1 + 1$ dimensional space-time.

Naturally the search for $3 + 1$ dimensional generalizations became eminent. General considerations showed that many features of $1 + 1$ dimensional systems, such as complete integrability and existence of exact many-particle solutions could not be generalized to $3 + 1$ dimensions. However, the mere existence of “one-particle” soliton solutions was not excluded. One particular example was introduced by Skyrme in a pioneer paper [7] long before the soliton rush. Another example was proposed by G. ’t Hooft and A. Polyakov in 1975 [8]. In the following years their solutions got real applications in nuclear and high energy physics.

In both examples the solitons are “point-like”, namely their deviation from the vacuum is concentrated around central point in space. Moreover they have spherical symmetry, allowing the separation of variables in the corresponding equations, reducing them to ODE, which one can treat on a usual PC.

In my lectures [9], already mentioned, I proposed one more possibility for $3 + 1$ dimensional system, allowing solitons. The model, which superficially looks as a slight modification of Skyrme model, has quite distinct features. The center of the would-be soliton is not a point, but a closed contour, possibly linked or knotted. However my proposal remained unnoticed. The reason was evident: the maximal symmetry for such a soliton is axial, reducing 3-dimensional nonlinear PDE to 2-dimensional one. Existing computers were not able to treat such a problem. Thus my proposal was in slumber for 20 years until my colleague Antti Niemi became interested and agreed to sacrifice a year to learn computing and devising the programm. The preliminary results published in [9] attracted the attention of professionals in computational physics and now we have an ample evidence, confirming my proposal [10], [11].

The development which followed showed unexpected universality of my model. The variables, used in it, were shown to enter the list of degrees of freedom for several systems, having realistic physical applications [12], [13].

In this talk I shall describe all these developments in detail. First I shall introduce the model, then briefly discuss its numerical treatment and finish with the description of the applications.

2. The field configurations and Hopf invariant

The space time is 4-dimensional Minkowski space $M$ with linear coordinates $x^\mu$, $\mu = 0, 1, 2, 3$, $x^0$ being time and $x^k$, $k = 1, 2, 3$ space variables. The field $\vec{n}(x)$ is defined on $M$ and has values on 2-dimensional sphere $S^2$:

$$\vec{n} : M \to S^2.$$
The boundary condition on spatial infinity is introduced

\[ \vec{n}|_{r=\infty} = \vec{n}_0, \quad (2.1) \]

where \( r = (x^1)^2 + (x^2)^2 + (x^3)^2 \) and \( \vec{n}_0 \) is a fixed vector, e.g. corresponding to the north pole

\[ \vec{n}_0 = (0, 0, 1). \]

We shall consider mostly the time independent configurations, corresponding to a soliton at rest. The boundary condition (2.1) effectively compactifies the space \( \mathbb{R}^3 \), turning it into sphere \( S^3 \), thus the stationary configurations realize the map

\[ \vec{n} : S^3 \to S^2, \quad (2.2) \]

which are known to be classified by Hopf invariant, sort of topological charge.

In general, the density of topological charge is the zero component \( J_0 \) of the current \( J_\mu \), which is conserved

\[ \partial_\mu J_\mu = 0 \]

independently of the equations of motion. Mathematically it is more natural to use the 3-form \( J \) dual to 1-form \( J^* = J_\mu \, dx^\mu \) and define the topological charge as an integral of \( J \) over space section

\[ Q = \int_{\mathbb{R}^3} J. \]

In our case the 3-form \( J \) is constructed as follows. The pull-back of the volume 2-form on \( S^2 \) via map (2.2) defines the closed 2-form on the space time

\[ H = H_{\mu\nu} \, dx^\mu \wedge dx^\nu, \]

where antisymmetric tensor \( H_{\mu\nu} \) is expressed via field configuration \( \vec{n}(x) \) as follows

\[ H_{\mu\nu} = (\partial_\mu \vec{n} \times \partial_\nu \vec{n}, \vec{n}). \quad (2.3) \]

Here I use usual notations of vector analysis in 3-space. In fact \( H \) is exact

\[ H = d\, C \]

and current 3-form is given by

\[ J = \frac{1}{4\pi} H \wedge C. \]

In more detail, we have the relations

\[ H_{ik} = \partial_i C_k - \partial_k C_i \]

and

\[ Q = \frac{1}{4\pi} \int \varepsilon_{ikj} H_{ik} C_j \, d^3x. \]

For regular configurations \( Q \) gets integer values. This integer has a nice interpretation in the description of which I shall use the terminology of magnetostatic.
Tensor $H_{ik}$ can be interpreted as a field strength of the stationary magnetic field in Maxwell theory. The corresponding lines of force are defined via equations
\[
\frac{d}{ds}x^i = \frac{1}{2} \varepsilon_{ikj} H_{kj},
\]
where $s$ is a local parameter along the line. It is easy to see that components of $\vec{n}(x)$ along these lines are constant
\[
\frac{d}{ds} \vec{n}(x) = 0,
\]
giving two “integrals of the motion”. In other words, the Maxwell lines of force are the preimages of points on $S^2$ under the map (2.2). Hopf invariant is the intersection number of any pair of such lines.

All these facts are well known and can be found in textbooks (see e.g. [14]). However I decided to include them into my text to make it more self-contained.

3. The dynamical model

I introduce the dynamical model by giving the relativistic action functional
\[
\mathcal{A} = a \int (\partial_\mu \vec{n})^2 d^4x + b \int (H_{\mu\nu})^2 d^4x.
\]
In the usual convention of high-energy physics $\mathcal{A}$ is dimensionless, so the parameter $a$ has dimension [length]$^{-2}$ and parameter $b$ is dimensionless. Corresponding static energy $E$ has the same form as $\mathcal{A}$ with space-time coordinates substituted by space variables only
\[
E = a \int (\partial_k \vec{n})^2 d^3x + b \int (H_{ik})^2 d^3x.
\]
(3.4)
and has proper dimension [length]$^{-1}$. The structure of $E$ is similar to that of Skyrme model, where the field variable having values in $S^3$ is used and corresponding topological charge is just a degree of map.

Usual check based on the scale transformation is favorable for (3.4) in the same way as in Skyrme model. Indeed
\[
E = E_2 + E_4,
\]
where $E_2$ and $E_4$ are quadratic and quartic in derivatives of $\vec{n}$ correspondingly. Thus under scaling $x \to \lambda x$ we have
\[
E_2 \to \lambda E_2, \quad E_4 \to \frac{1}{\lambda^2} E_4
\]
and the virial theorem states that on the minimal configuration (if any)
\[
E_2 = E_4.
\]
In terms of quantum theory $E_2$ has a standard interpretation of the energy of nonlinear sigma-model whereas $E_4$ is rather exotic. On the contrary in the magnetic interpretation, mentioned above, $E_4$ is a natural term — it is just the Maxwell magnetic energy, whereas the nature of $E_2$ is not that clear. However in what follows the presence of both $E_2$ and $E_4$ is crucial for the existence of solitons as the scaling argument already showed.

This is confirmed also by a beautiful estimate, obtained in [15]

$$E \geq c|Q|^{3/4},$$

which shows that in the sectors with nonzero $Q$ the minimum of energy is strongly positive. Thus the soliton solutions should be obtained by the minimizing of $E$ with $Q \neq 0$ fixed.

Unfortunately until now there exists no proof of the compactness of the minimizing sequence in general case. For the case of axial symmetry encouraging result are obtained in [16]. So the main argument for the evidence of solitons in my model is based on the numerical work.

4. Numerical work

To find the numerical evidence of the existence of localized solitons it is not necessary to solve the nonlinear elliptic equation, obtained by the variational principle

$$\frac{\delta E}{\delta \vec{n}} = 0. \quad (4.5)$$

Instead one can introduce an auxiliary time $s$ and consider the parabolic equation

$$\frac{d\vec{n}}{ds} = \frac{\delta E}{\delta \vec{n}} \quad (4.6)$$

with initial value $\vec{n}_{\text{init}}$

$$\vec{n}|_{s=0} = \vec{n}_{\text{init}}$$

being a configuration with the prescribed Hopf invariant. Of course to simulate (4.6) on the computer one is to use some difference scheme. If for large $s$ solution of (4.6) stabilizes it gives the solution of (4.5). In other words the soliton appears as an attractor for the evolution equation.

There are of course many important practical details how to discretize equation, how to take into account the normalization condition $\vec{n}^2 = 1$ and how to choose the initial configuration $\vec{n}_{\text{init}}$. The main papers [10] and [11] use different prescription for all this, however quite satisfactorily the final results coincide. I refer to these papers for the details of calculations and proceed to describe the results.

The iterative process was performed for the configuration with $Q = 1, 2, \ldots, 7$. The results are as follows: for $Q = 1$ and $Q = 2$ the solutions are axial symmetric. The center line — the preimage of the point $n = (0, 0, -1)$ — is a circle. The surfaces $n_3 = \alpha, -1 < \alpha < 1$ are toroidal and they are spanned by the lines of force
wrapping the torus once for \( Q = 1 \) and twice for \( Q = 2 \). In other words the soliton can be viewed as a filament of lines of force, closed and twisted once or twice.

The solution for \( Q = 3 \) is similar but not axial symmetric any more, the corresponding “cable” is warped. For \( Q = 4 \) the soliton is a link of two twisted filaments. Especially beautiful case is \( Q = 7 \), the central line of the corresponding soliton is a trefoil knot.

The file [17] contains impressive moving pictures illustrating the convergence of the iterations. I plan to show these movies in my talk, but unfortunately can not do it in a written text.

Thus the numerical work gives the compelling evidence of the existence of string-like solitons in my model. There remains an important mathematical challenge to provide the rigorous existence theorem. Another interesting direction is to find some realistic applications of the model. Some progress in this direction is already obtained and I proceed to the description of it.

5. The applications

Nonlinear fields such as \( \vec{n}(x) \) rarely enter the dynamical models directly. However they can appear as a part of degrees of freedom in a suitable parameterization of the original fields. For example in condensed matter theory one uses the complex valued functions \( \psi_\alpha(x) \), \( \alpha = 1, \ldots, N \) to describe the density amplitudes of Bose gas or the gap function of superconductor. The interaction supports the configurations, for which

\[
\rho^2 = \sum_{\alpha=1}^{N} |\psi_\alpha|^2 \tag{5.7}
\]

is nonvanishing. In this case it is natural to use \( \rho \) as one of the independent variables and introduce new variables

\[
\chi_\alpha = \psi_\alpha / \rho
\]

such that

\[
\sum_{\alpha=1}^{N} |\chi_\alpha|^2 = 1. \tag{5.8}
\]

In this way the compact target (I use the slang of the string theory) \( S^{2N-1} \) appears.

When magnetic interaction is introduced the invariance with respect to the phase transformation

\[
\psi_\alpha(x) \rightarrow e^{i\lambda(x)} \psi_\alpha(x)
\]

is invoked. This means that the target \( S^{2N-1} \) changes

\[
S^{2N-1} \rightarrow S^{2N-1} / U(1).
\]

In particular for \( N = 2 \) we have

\[
S^3 / U(1) \sim S^2
\]
and the field $\vec{n}(x)$ naturally appears. Quite satisfactorily the tensor $H_{ik}$ also emerges as a contribution to the magnetic field strength.

Let us illustrate it in more detail. From the beginning we shall treat the stationary system, so no electric field will be used.

The magnetic field is described in a usual way by means of the vector potential $A_k(x)$ and its interaction with $\psi$-fields is introduced via covariant derivatives

$$\nabla_k \psi = \partial_k \psi + iA_k \psi.$$  

The energy density (of Landau-Ginsburg-Gross-Pitaevsky type) looks as follows

$$E = \sum_{\alpha=1}^{2} |\nabla_k \psi_{\alpha}|^2 + \frac{1}{2} F_{ik}^2 + V(|\psi_{\alpha}|), \quad (5.9)$$

where

$$F_{ik} = \partial_i A_k - \partial_k A_i$$

is the field strength of the magnetic field. The energy is invariant with respect to the gauge transformations

$$\psi_{\alpha} \rightarrow e^{i\lambda} \psi_{\alpha}, \quad A_k \rightarrow A_k - \partial_k \lambda.$$  

We shall make the change of the field variables so that only gauge invariant ones will remain. For that observe that the first term in the RHS of (5.9) is a quadratic form in $A$

$$\sum_{\alpha=1}^{2} |\nabla_k \psi_{\alpha}|^2 = \sum_{\alpha=1}^{2} |\partial_k \psi_{\alpha}|^2 + A_k J_k + \rho^2 A_k^2,$$

where we use variable $\rho$ from (5.7) and introduce current

$$J_k = -i \sum_{\alpha=1}^{2} (\bar{\psi}_{\alpha} \partial_k \psi_{\alpha} - \partial_k \bar{\psi}_{\alpha} \psi_{\alpha}).$$

It is easy to check, that under the gauge transformations the current $J_k$ changes as follows

$$J_k \rightarrow J_k + 2\rho^2 \partial_k \lambda,$$

so that the sum

$$C_k = A_k + \frac{1}{2\rho^2} J_k$$

is gauge invariant. We shall use this variable instead of $A_k$. Another gauge invariant combination is given by the quadratic form

$$\vec{n} = (\bar{\chi}_1, \bar{\chi}_1) \vec{\tau} \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \quad (5.10)$$

where $\vec{\tau} = (\tau_1, \tau_2, \tau_3)$ are Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
Normalization (5.8) for $\chi_\alpha$ implies that $\vec{n}$ is a real unit vector. In fact the map

$$(\chi_1, \chi_2) \to \vec{n}$$

defined in (5.10) is a standard Hopf map. Variable $\vec{n}$ is manifestly gauge invariant and the set of variables $(\rho, \vec{n}, C_k)$ is our gauge invariant choice, substituting for the initial set $(\psi_\alpha, A_k)$. The energy density can be explicitly expressed via $\rho, \vec{n}$ and $C_k$ as follows

$$E = (\partial_k \rho)^2 + \rho^2 ((\partial_k \vec{n})^2 + C_k^2) + \frac{1}{2} (\partial_k C_i - \partial_i C_k + H_{ik})^2 + v(\rho, n_3).$$

The most notable feature is the appearance of the tensor $H_{ik}$, defined in (2.3). The model, described in the main text, emerges if we put $\rho = \text{const}$ and $C = 0$. Hopefully nontrivial $\rho$ and $C$, at least confined to some range, do not spoil the soliton picture. This problem is under discussion now, see [13], [18].

Let us stress, that the use of two fields $\psi_\alpha, \alpha = 1, 2$ is most essential in this example. If $N = 1$ only variables $\rho$ and $C$ remain after the reduction, similar to just described. If $N > 2$ the $CP(N - 1)$ field generalizing $\vec{n}$ has no topological characteristics.

Another application, considered recently [24], deals with the parameterization for the $SU(2)$ Yang-Mills field $A^a_\mu(x), \mu = 0, 1, 2, 3, a = 1, 2, 3$. The Yang-Mills Lagrangian is invariant with respect to the nonabelian gauge transformations

$$\delta A^a_\mu = \partial_\mu \varepsilon^a + f^{abc} A^b_\mu \varepsilon^c.$$ 

However in some treatments one reduces this invariance by the partial gauge fixing to the abelian one

$$\delta B_\mu = i \varepsilon B_\mu, \quad \delta A^3_\mu = \partial_\mu \varepsilon,$$

where $B_\mu = A^1_\mu + i A^2_\mu$ is a complex vector field. I shall not discuss the reason for this reduction here and proceed assuming that it is done. Observe, that two vector fields $A^1_\mu, A^2_\mu$ in generic situation define a plane in Minkowski space and introduce an orthonormalized basis in this plane $e^\mu_\alpha, \alpha = 1, 2$.

$$e^\mu_\alpha e^\mu_\beta = \delta_{\alpha\beta}.$$ 

Let $e_\mu = e^1_\mu + i e^2_\mu$. The basis is defined up to rotation

$$e_\mu \to e^{\mu \omega} e_\mu.$$ 

The fields $B_\mu$ can be written in terms of this basis as

$$B_\mu = \psi_1 e_\mu + \psi_2 \bar{e}_\mu,$$

and thus two complex valued fields $\psi_1$ and $\psi_2$ appear. The situation becomes quite similar to the previous example and indeed in [12] the complete parameterization of
the Yang-Mills variables is introduced with appearance of \( \vec{n} \)-field and corresponding \( H \)-tensor. This is an indication that the Yang-Mills theory can have string-like excitations. However the situation is not that simple. The classical Yang-Mills theory is conformally invariant and has no dimensional parameters. Thus no hope for the localized regular classical solution exists. Nevertheless this complication could be lifted by quantum corrections. The famous “dimensional transmutation”, which leads to the appearance of dimensional parameter in quantum effective action, could favor the nonvanishing value of the corresponding \( \rho \)-variable. All these considerations at the moment are rather speculative and need much more work to become reasonable. Personally I am quite impressed by this possibility and continue to work on it.

6. Conclusions

I think that the topic of my talk is quite instructive. It connects different domains in mathematics and mathematical physics: nonlinear PDE, elementary topology, quantum field theory, numerical methods. It illustrates the essential unity of mathematics, theoretical and applied. Finally it could lead to the realistic physical applications. For all these reasons I decided to present it to the ICM2002.

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