Lifting Normal Elements in Nonseparable Calkin Algebras

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Abstract. We use the remarkable distance estimate of Ilya Kachkovskiy and Yuri Safarov\cite{6} to show that if $H$ is a nonseparable Hilbert space and $K$ is any closed ideal in $B(H)$ that is not the ideal of compact operators, then any normal element of $B(H)/K$ can be lifted to a normal element of $B(H)$.

Suppose $H$ is a Hilbert space with $\dim H = d \geq \aleph_0$. We let $B(H)$ denote the set of all (bounded linear) operators on $H$. For each cardinal $m$ with $\aleph_0 \leq m \leq d$, we let $\mathcal{F}_m(H) = \{T \in B(H) : \text{rank}T = \dim T(H) < m\}$, and let $\mathcal{K}_m(H)$ be the norm closure of $\mathcal{F}_m(H)$. The set $\{\mathcal{K}_m(H) : \aleph_0 \leq m \leq d\}$ is the collection of all nonzero proper norm-closed ideals of $B(H)$. The quotient C*-algebra $\mathcal{C}_m(H) = B(H)/\mathcal{K}_m(H)$ is called the $m$-Calkin algebra, and the quotient map $\pi_m : B(H) \to \mathcal{C}_m(H)$ is called the $m$-Calkin map.

The properties of these Calkin algebras depend significantly on the properties of the cardinal $m$. We say that $m$ is countably cofinal if $m$ is the supremum of a countable collection of smaller cardinals. It was shown in \cite{4}, Thm. 4.11 that if $m$ is not countably cofinal (which holds exactly when $\mathcal{K}_m(H) = \mathcal{F}_m(H)$), then, for any separable unital C*-subalgebra $A$ of $\mathcal{C}_m(H)$, there is a unital $*$-homomorphism $\rho : A \to B(H)$ such that $\pi_m \circ \rho$ is the identity on $A$. Hence, in this case, for any element $t$ of $\mathcal{C}_m(H)$ there is a $T \in B(H)$ so that $\pi_m : C^*(T) \to C^*(t)$ is a $*$-isomorphism sending $T$ to $t$. Hence $t$ lifts to an element $T$ in $B(H)$ that shares all the properties preserved under $*$-isomorphisms, i.e., being normal, subnormal, hyponormal, isometric, or unitary.

When the cardinal $m$ is countably cofinal, the situation is not so clear cut. When $m = \aleph_0$, the classical result of L. G. Brown, R. G. Douglas, and P. A. Fillmore $\cite{3}$ shows that the Fredholm index is the only obstruction to lifting norm elements. When $m > \aleph_0$, it is still possible to lift invertibles and unitaries, but the case of normality has remained open since 1981. In this paper we prove that when $m > \aleph_0$ is countably cofinal, then, for any normal element $t \in \mathcal{C}_m(H)$ there is a normal operator $T \in B(H)$ such that $\pi_m(T) = t$.

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A key ingredient in our work is the wonderful new theorem (the Kachkovskiy-Safarov inequality) of [6], which estimates the distance \( d(a, N_f (A)) \) of an element of \( a \) in a C*-algebra \( A \) with real rank zero to the set \( N_f (A) \) of normal operators in \( A \) with finite spectrum:

\[
(\#) \quad d(a, N_f (A)) \leq C \left( \|a^*a - aa^*\|^{1/2} + d_1(a) \right),
\]

for some universal constant \( C \), and where

\[
d_1(a) = \sup_{\lambda \in \mathbb{C}} \text{dist} (a - \lambda, GL(A)),
\]

where \( GL(A) \) is the connected component of 1 in the group \( GL(A) \) of invertible elements of \( A \).

When \( A \) is a von Neumann algebra, \( GL_0(A) = GL(A) \), so

\[
d_1(a) = \sup_{\lambda \in \mathbb{C}} \text{dist} (a - \lambda, GL(A))
\]

A nice formula for the \( \sup \text{dist} (b, GL(A)) \) is given by C. L. Olsen in [7] and more general results appear in the works of R. Bouldin [1, 2]. We will not need these characterizations here. If \( H \) is an infinite-dimensional Hilbert space, then \( H \) is isomorphic to \( H \oplus H \oplus \cdots \), so if \( T \in B(H) \), we can identify \( T(\infty) = T \oplus T \oplus \cdots \) with an operator in \( B(H) \). Similarly, if \( A \) is an infinite von Neumann algebra, then there is an orthogonal sequence \( \{P_n\} \) of projections summing to 1 so that each \( P_n \) is Murray-von Neumann equivalent to 1, so if \( T \in A \), we can still view \( T(\infty) \) as an element of \( A \).

**Lemma 1.** Suppose \( A \) is an infinite von Neumann algebra acting on a separable Hilbert space and \( T \in A \). Then

1. If \( A \) is of type III, then
   \[
   d_1(T) \leq \|T^*T - TT^*\|^{1/2},
   \]
   and
   \[
   d(T, N_f (A)) \leq 2C \|T^*T - TT^*\|^{1/2}.
   \]

2. If \( A \) is of type I\(_\infty\) or type II\(_\infty\), then
   \[
   d_1(T(\infty)) \leq \|T^*T - TT^*\|^{1/2},
   \]
   and
   \[
   d(T(\infty), N_f (A)) \leq 2C \|T^*T - TT^*\|^{1/2}.
   \]

**Proof.** Note that \( \|T^*T - TT^*\|^{1/2} \) is unchanged if \( T \) is replaced with \( T - \lambda \), \( T(\infty) \) or with \( T^* \). Hence it will suffice to show \( \text{dist} (T, GL(A)) \leq \|T^*T - TT^*\|^{1/2} \) in part (1) and \( \text{dist} (T(\infty), GL(A)) \leq \|T^*T - TT^*\|^{1/2} \) in part (2). The parts involving the distance to \( N_f (A) \) follow immediately from the Kachkovskiy-Safarov inequality [6] (see (\#) above). If both \( T^*T \) and \( TT^* \) are invertible, then \( T \) is invertible and the desired inequalities are trivially true. Since nothing changes when \( T \) is replaced by \( T^* \), there is no harm in assuming that \( T^*T \) is not invertible. Then, for every \( \varepsilon > 0 \), if we let \( P_\varepsilon \) be the spectral projection for \( T^*T \) corresponding to the interval \( [0, \varepsilon] \), we have \( P_\varepsilon \neq 0 \),

\[
\|TP_\varepsilon\|^2 = \|P_\varepsilon T^*TP_\varepsilon\| \leq \varepsilon
\]
and

\[ \|P_T\|^2 = \|P_T T^* P_T\| \leq \|P_T T^* P_T\| + \|T^* T - TT^*\|. \]

Hence \( \|T - P_T T^* P_T\| \leq \sqrt{\varepsilon} + \sqrt{\varepsilon} + \|T^* T - TT^*\|. \)

We first consider the case when \( \mathcal{A} \) is an infinite factor. If \( \mathcal{A} \) is a type III factor, then the projections onto \( \ker \left( P_T T^* P_T \right) \) and \( \ker \left( P_T T^* P_T \right)^* \) are nonzero and equivalent, which implies that \( P_T T^* P_T \) is a norm limit of invertible elements \(^2\). Thus, for every \( \varepsilon > 0 \)

\[ \text{dist} \left( T, GL(\mathcal{A}) \right) \leq \sqrt{\varepsilon} + \sqrt{\varepsilon} + \|T^* T - TT^*\|, \]

which implies

\[ \text{dist} \left( T, GL(\mathcal{A}) \right) \leq \|T^* T - TT^*\|^{1/2}. \]

If \( \mathcal{A} \) is a type I\(_{\infty} \) or type II\(_{\infty} \) factor, then the projections onto \( \ker \left( P_T T^* P_T \right) \) \(^{(\infty)}\) and \( \ker \left( \left( P_T T^* P_T \right)^* \right) \) \(^{(\infty)}\) are nonzero and have the form \( Q^{(\infty)} \) and are equivalent, and we get

\[ \text{dist} \left( T^{(\infty)}, GL(\mathcal{A}) \right) \leq \|T^* T - TT^*\|^{1/2}. \]

Hence we have proved statements (1) and (2) when \( \mathcal{A} \) is an infinite factor von Neumann algebra on a separable Hilbert space.

For the general case, using the central decomposition \(^5\), there is a direct integral decomposition \( \mathcal{A} = \int_{\Omega} \mathcal{A}_\omega d\mu(\omega) \), where each \( \mathcal{A}_\omega \) is a factor von Neumann algebra. If \( \mathcal{A} \) has type III, then each \( \mathcal{A}_\omega \) is a type III factor. If \( T = \int_{\Omega} T_\omega d\mu(\omega) \in \mathcal{A} \) and \( \varepsilon > 0 \), then, using standard measurable cross-section arguments, it is easy to measurable choose, for each \( \omega \in \Omega \), an invertible operator \( S_\omega \in \mathcal{A}_\omega \) such that

\[ \|T_\omega - S_\omega\| \leq \|T^{*}_\omega T_\omega - T_\omega T^{*}_\omega\|^{1/2} + \varepsilon. \]

Note that

\[ \|S_\omega\| \leq \|T_\omega - S_\omega\| + \|T_\omega\| \leq \|T^{*}_\omega T_\omega - T_\omega T^{*}_\omega\|^{1/2} + \varepsilon + \|T_\omega\| \leq \|T^* T - TT^*\|^{1/2} + \varepsilon + \|T\| \text{ a.e.}(\mu), \]

so \( \int_{\Omega} S_\omega d\mu(\omega) \) is defined. Although each \( S_\omega \) is invertible, the operator \( S = \int_{\Omega} S_\omega d\mu(\omega) \in \mathcal{A} \) might not be invertible. However, each \( S_\omega \) has a polar decomposition \( U_\omega |S_\omega| \) with \( U_\omega \) unitary. Thus \( S = U |S| \) with \( U = \int_{\Omega} U_\omega d\mu(\omega) \) unitary. Hence, \( S \) is the limit of the sequence \( \{U(|S| + 1/n)\} \) of invertible operators. Thus,

\[ \text{dist} \left( T, GL(\mathcal{A}) \right) \leq \|T - S\| \leq \|T^* T - TT^*\|^{1/2} + \varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, we have the desired result. The case when the \( \mathcal{A}_\omega \)'s are type I\(_{\infty} \) or type II\(_{\infty} \) factors is handled similarly.

\[ \square \]

**Corollary 1.** If \( H \) is a Hilbert space and \( T \in B(H) \) and \( C^* (T) \cap K_\aleph_0 (H) = \{0\} \), then

\[ d_1 (T) \leq \|T^* T - TT^*\|^{1/2}. \]

**Proof.** For every \( 0 \neq A \subseteq C^* (T) \) we have \( \text{rank} A \geq \aleph_0 \), so \( \text{rank} A = \text{rank} A^{(\infty)} \). Thus, by \(^4\) Thm. 3.14, There is a sequence \( \{U_n\} \) of unitary operators in \( B(H) \) such that \( \|U_n^* T^{(\infty)} U_n - T\| \to 0 \), which means \( d_1 (T) = d_1 (T^{(\infty)}) \). \[ \square \]
We are now ready to prove our main theorem.

**Theorem 1.** Suppose $H$ is an infinite-dimensional Hilbert space and $\aleph_0 < m \leq \dim H$ is countably cofinal. If $t \in C_m (H)$ is normal, then there is a normal operator $T \in B (H)$ such that $\pi_m (T) = t$.

**Proof.** We first choose an $S \in B (H)$ such that $\pi_m (S) = t$. We can write $H$ as a direct sum $H = \sum_{j \in J} H_j$ with each $\dim H_j = \aleph_0$ and with $H_j$ a reducing subspace for $S$. Hence, we can write $S = \sum_{j \in J} S_j$ with respect to this decomposition. Since $S^* S - S S^* = \sum_{j \in J} (S_j^* S_j - S_j S_j^*) \in K_m (H)$, we see that $E = \{ j \in J : S_j^* S_j - S_j S_j^* \neq 0 \}$ has cardinality at most $m$ and $S_j$ is normal when $j \notin E$. If $\text{Card} E < m$, then $T = \sum_{j \in J \notin E} S_j \chi_j (j)$ is normal and $\pi_m (T) = \pi_m (S) = t$. Hence we can assume $J = E$, so $\dim H = \text{Card} E = m$.

It follows from [4] Cor. 3.11, Thm. 4.6 that there is a unitary operator $U \in B (H)$ and irreducible operators $A_1, A_2, \ldots$ and cardinals $k_1, k_2, \ldots$ such that

$$U^* S U - \sum_{1 \leq n < \infty} A_n^{(k_n)} \in K_m (H).$$

Hence we can assume that $S = \sum_{1 \leq n < \infty} A_n^{(k_n)}$. Since each $A_n$ is irreducible, it must act on a separable Hilbert space. If $k_n = m$, then $A_n$ must be normal. Hence we can write $S = N \oplus \sum_{n \in F} A_n^{(k_n)}$, where $k_n < m$ whenever $n \in F$. If $F$ is finite, $\pi_m (N \oplus 0) = \pi_m (S) = t$. Hence, we can assume $F = \{ n_1, n_2, \ldots \}$ with $\aleph_0 \leq k_{n_1} \leq k_{n_2} \leq \cdots < m$ and $m = \sup \{ k_j : j \in \mathbb{N} \}$. It follows now that

$$\lim_{j \to \infty} \left\| A_{n_j} A_{n_j}^* - A_j A_j^* \right\| = 0.$$

However, $A_j^{(k_n)}$ is unitarily equivalent to $(A_j^{(\infty)})^{(k_n)}$ for each $j$, which implies by Lemma [4]

$$\lim_{j \to \infty} \text{dist} \left( A_j^{(k_n)}, N \right) = 0.$$

Hence there is a sequence $\{ B_j \}$ of normal operators such that

$$\lim_{j \to \infty} \left\| A_j^{(k_n)} - B_j \right\| = 0.$$

Hence $T = N \oplus \sum_{1 \leq j < \infty} B_j$ is normal and $\pi_m (T) = \pi_m (S) = t$. \hfill $\square$

**Remark 1.** If we suppose $m$ is not countably cofinal in the preceding proof, we see that $\text{Card} E$ must be less than $m$ and the proof is complete.

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