INTRINSIC LINKING AND KNOTTING IN TOURNAMENTS

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ABSTRACT. A directed graph $G$ is intrinsically linked if every embedding of that graph contains a non-split link $L$, where each component of $L$ is a consistently oriented cycle in $G$. A tournament is a directed graph where each pair of vertices is connected by exactly one directed edge.

We consider intrinsic linking and knotting in tournaments, and study the minimum number of vertices required for a tournament to have various intrinsic linking or knotting properties. We produce the following bounds: intrinsically linked ($n = 8$), intrinsically knotted ($9 \leq n \leq 12$), intrinsically 3-linked ($10 \leq n \leq 23$), intrinsically 4-linked ($12 \leq n \leq 66$), intrinsically 5-linked ($15 \leq n \leq 154$), intrinsically $m$-linked ($3m \leq n \leq 8(2m - 3)^2$), intrinsically linked with knotted components ($9 \leq n \leq 107$), and the disjoint linking property ($12 \leq n \leq 14$).

We also introduce the consistency gap, which measures the difference in the order of a graph required for intrinsic $n$-linking in tournaments versus undirected graphs. We conjecture the consistency gap to be non-decreasing in $n$, and provide an upper bound at each $n$.

1. INTRODUCTION

A graph is called intrinsically linked if every embedding of that graph in $S^3$ contains cycles that form a non-split link, and intrinsically knotted if every embedding of that graph contains a cycle that is a non-trivial knot. These properties were first studied by Sachs [14] and by Conway and Gordon [3].

Researchers have studied variations of these properties, such as requiring every embedding of the graph to contain cycles that form a non-split $n$-component link [4], a non-split link where one of more of the components are non-trivial knots [5] [8], or even more complex structures [12] [2].

A directed graph is intrinsically $n$-linked as a directed graph (or intrinsically knotted as a directed graph) if every embedding of that graph in $S^3$ contains cycles that form a non-split $n$ component link (or non-trivial knot), and the edges that comprise each cycle of the link (or knot) have a consistent orientation. Examples of intrinsically 2-linked directed graphs are known [11], and intrinsically knotted, intrinsically 3-linked and 4-linked directed graphs have also been constructed [9].

In this work, we focus on a subset of directed graphs known as tournaments. A tournament on $n$ vertices is a directed graph with exactly one directed edge between each pair of vertices. Equivalently, a tournament on $n$ vertices is $K_n$ with a choice of orientation for each edge. We then ask, given an intrinsic linking property, what is the smallest $n$ such that there exists a tournament on $n$ vertices having that property? In Section 2, we study intrinsic linking and have the precise answer $n = 8$. That is, while $K_6$ is intrinsically linked, no tournament on 7 or fewer vertices is intrinsically linked as a digraph, and there exists a tournament on 8 vertices that is intrinsically linked as a digraph.
For intrinsic knotting we have the bounds $9 \leq n \leq 12$ in Section 3, and for the disjoint linking property we have $12 \leq n \leq 14$ in Section 9. In Sections 4, 5 and 6 we find $10 \leq n \leq 23$ for intrinsic 3-linking, $12 \leq n \leq 66$ for intrinsic 4-linking and $15 \leq n \leq 154$ for intrinsic 5-linking. We address $m$-linking for $m > 5$ in Section 7 and have $3m \leq n \leq 8(2m - 3)^2$. In Section 8, we construct a tournament that contains a non-split link where at least one of the components is a non-trivial knot, obtaining the bounds $9 \leq n \leq 107$ for this property.

The construction of the 4-linked tournament in Section 5 is similar in spirit to that of the 4-linked directed graph in [9] but is substantially less complicated and requires far fewer vertices and edges. Section 6 extends this construction to produce a 5-linked tournament. Adapting techniques of Flapan, Mellor and Naimi [5], we are able to demonstrate $n$-linked tournaments for all $n$. In very recent work, Mattman, Naimi and Pagano independently used similar techniques to construct examples of intrinsically $n$-linked complete symmetric directed graphs [13].

Given a tournament, we may ignore the edge orientations and consider it as an undirected complete graph. This graph may be intrinsically $n$-linked even if the tournament is not intrinsically $n$-linked as a directed graph. Thus the requirement that the components of the non-split link have a consistent orientation for an intrinsically $n$-linked digraph is restrictive and appears to require a larger and more complex graph to satisfy.

We introduce the consistency gap to measure this difference and denote it as $cg(n)$. We define $cg(n) = m' - m$ where $K_m$ is the smallest intrinsically $n$-linked complete graph, and $m'$ is the number of vertices in the smallest tournament that is intrinsically $n$-linked as a directed graph. In Section 10, we show that $cg(2) = 2$ and provide a bound on $cg(n)$ for all $n$.

2. INTRINSIC LINKING IN TOURNAMENTS

**Proposition 2.1.** No tournament on 6 vertices is intrinsically linked as a directed graph.

**Proof.** An orientation of $K_6$ gives a directed graph on 6 vertices with 15 edges. By Corollary 3.10 of [11], any directed graph on 6 vertices with 23 or fewer edges is not intrinsically linked as a directed graph. 

**Theorem 2.2.** No tournament on 7 vertices is intrinsically linked as a directed graph.

**Proof.** Let $T$ be a tournament on 7 vertices. We will break the proof into cases based on the maximum in degree of a vertex in $T$. We will check only the cases of max in degree equal to 6, 5, 4, and 3; as for the other cases we may consider the vertex of maximum out degree, and apply the same arguments.

We will demonstrate a linkless embedding in each case, relying primarily upon the embedding from [10] shown in Figure 1 that contains exactly 21 non-split links. We will call this the FMellor embedding. The links in the FMellor embedding are the following:

- 457-236 457-136 457-1362 457-1236
- 147-236 147-235 147-2356 147-12365
- 167-235 167-245 167-2435 167-2345
- 136-245 136-2547 136-2457
- 235-1467 235-1647

For intrinsic 3-linking we have the bounds $9 \leq n \leq 12$ in Section 3, and for the disjoint linking property we have $12 \leq n \leq 14$ in Section 9. In Sections 4, 5 and 6 we find $10 \leq n \leq 23$ for intrinsic 3-linking, $12 \leq n \leq 66$ for intrinsic 4-linking and $15 \leq n \leq 154$ for intrinsic 5-linking. We address $m$-linking for $m > 5$ in Section 7 and have $3m \leq n \leq 8(2m - 3)^2$. In Section 8, we construct a tournament that contains a non-split link where at least one of the components is a non-trivial knot, obtaining the bounds $9 \leq n \leq 107$ for this property.
Suppose $T$ has a vertex $v$ with in degree = 6. This vertex $v$ cannot be contained in a consistently oriented cycle, so any consistently oriented link must be contained in $T \setminus v$, which is a tournament on 6 vertices. By Proposition 2.1, a tournament on 6 vertices has a linkless embedding. Thus $T$ does as well.

Suppose $T$ has a vertex $v$ with in degree = 5. Label this vertex 7. There is a unique vertex $w$ such that the edge 7$w$ is oriented from 7 to $w$. Label $w$ as vertex 3. Then all of the links in the FMellor embedding have an inconsistently oriented cycle except possibly 136-245 245-1376 245-1736

Label one of the remaining vertices 2. Vertex 2 is adjacent to each of the four remaining unlabeled vertices, so at least two of these edges must have the same orientation, i.e. either both from 2 to the unlabeled vertices, or from the unlabeled vertices to 2. Label the end point of two edges with matching orientation 4 and 5. Then the cycle 245 is not consistently oriented, so the FMellor embedding of $T$ contains no consistently oriented non-split links.

Suppose $T$ has a vertex $v$ with in degree = 4. Label this vertex 7. There are four edges oriented from another vertex to 7. Label the end point of these edges 1, 4, 5, 6. Then all of the links in the FMellor embedding have a cycle with inconsistent orientation except possibly: 136-245 136-2547 136-2457 245-1376 245-1736

If two or more of the edges between \{1, 4, 5, 6\} and 2 are oriented from $w$ to 2, then we may rearrange the labels of \{1, 4, 5, 6\} so that two of these edges are 42 and 52. In this case, 245 has an inconsistent orientation, and the cycles 2547
and 2457 are inconsistently oriented as well. Thus, the FMellor embedding has no consistently oriented non-split links. As we may switch the labels of vertex 2 and vertex 3, at most one edge can be oriented from \{1, 4, 5, 6\} to vertex 3 as well.

Vertex 2 has at most one edge oriented from \{1, 4, 5, 6\} to 2, call it \(w_2\). Similarly 3 has at most one edge oriented from \{1, 4, 5, 6\} to 3, call it \(w'_3\). If no such \(w\) or \(w'\) exists, or if \(w \neq w'\) then we may choose to rearrange the labels \{1, 4, 5, 6\} such that edges 24 and 25 are oriented from 2 to 4 and from 2 to 5, and such that edges 31 and 36 are oriented from 3 to 1 and from 3 to 6. Then the cycles 136 and 245 are not consistently oriented and the FMellor embedding has no consistently oriented non-split links.

So we may assume that \(w = w' = 4\). Then cycle 136 is not consistently oriented, and the only possible links in the FMellor embedding are:

245-1376 245-1736

If edge 45 is oriented from 4 to 5, then the cycle 245 is not consistently oriented. So we may assume that edge 45 is oriented from 5 to 4. As we can exchange labels among \{1, 5, 6\}, we may assume that edges 41 and 46 are oriented to 4 as well. Thus, we have restricted the orientation of all edges in \(T\) except for the triangle 156, and edge 23. Notice that edge 2\(v\) is oriented from 2 to \(v\) if and only if 3\(v\) is oriented from 3 to \(v\) for all \(v\), so we may assume edge 23 is oriented from 2 to 3 (exchanging the labels of 2 and 3 if necessary). Consider the embedding \(f\) of \(T\) shown in Figure 2. As any link must be between two 3-cycles or a 3-cycle and a 4-cycle, to show \(f(T)\) contains no consistently oriented non-split links, it suffices to check that each 3-cycle in \(f(T)\) is either not consistently oriented, bounds a disk, or forms a link only with inconsistently oriented cycles.

Any 3-cycle that uses edge 23 is inconsistently oriented.
Any 3-cycle that contains vertex 7 is inconsistently oriented or bounds a disk.
Any consistent 3-cycle not using the above bounds a disk, or is 561.
If 561 is consistent, all of the cycles with which it forms a non-trivial link are inconsistently oriented.

Suppose the maximum in degree of a vertex in \( T \) is 3. Note that this implies that all vertices in \( T \) have in degree = out degree = 3. Choose a vertex, label it 7. Label the other vertices so that edges 17, 47 and 57 are oriented to 7.

As each of 6, 2, 3 have in degree 3, there must be nine edges that terminate on vertices \( \{2, 3, 6\} \). Three edges from vertex 7 terminate there, and the three edges that form a \( K_3 \) on \( \{2, 3, 6\} \) have end points there as well. Thus, there must be three edges from \( \{1, 4, 5\} \) to \( \{2, 3, 6\} \). Choose one such edge and rearrange the labels as necessary so that this edge is 16. Then all of the non-trivial links in the FMellor embedding have an inconsistently oriented cycle except possibly:

\[ 136-245 \quad 136-2547 \quad 136-2457 \quad 235-1467 \]

We break the remainder of the argument into three cases.

Case 1: one or both of 26 and 36 are oriented from 2 to 6 or 3 to 6. As 6 has in degree 3, and 76 and 16 are oriented to 6, at most one of 26 and 36 are oriented to 6. Switching labels if necessary, we may assume edge 36 is oriented from 3 to 6.

Then cycle 136 is not consistently oriented, so the only remaining potential link is 235-1467. Since 6 has in degree three, edges 64, 65 and 62 are oriented from 6 to the other vertex. We have edge 76 oriented from 7 to 6 and edge 64 oriented from 6 to 4. If 14 is oriented from 1 to 4, then 1467 has an inconsistent orientation, so there are no consistently oriented links in the FMellor embedding, and we are done. So we may assume edge 14 is oriented from 4 to 1. As we may switch the labels of vertex 4 and vertex 5, edge 15 must be oriented from 5 to 1 as well.

Similarly, both 234 and 235 must be consistently oriented or else we may switch the labels of 4 and 5 so that 235 has an inconsistent orientation, giving a linkless version of the FMellor embedding of \( T \). As 72 is oriented from 7 to 2, and 62 is oriented from 6 to 2, we cannot have both 42 and 52 oriented from 4 and 5 to 2. Thus, we must have 25 oriented from 2 to 5 and 24 oriented from 2 to 4. For 234 and 235 to be consistently oriented, edge 23 must be oriented from 3 to 2 and edges 35 and 34 must be oriented from 5 to 3 and from 4 to 3.

We have edges 72, 62, and 32 oriented to vertex 2. As vertex 2 has in degree 3, this implies edge 12 must be oriented from 2 to 1. We have edges 73, 53 and 43 oriented to 3. As vertex 3 has in degree 3, this implies that edge 13 must be oriented from 3 to 1. This is a contradiction, as vertex 1 has in degree 3, but edges 12, 13, 14 and 15 all terminate at 1. Thus, the edge orientations must be in a configuration that allows a linkless FMellor embedding.

Case 2: We may assume that 26 is oriented from 6 to 2 and 36 is oriented from 6 to 3, as otherwise we would be in Case 1. Assume one or both of 12 and 13 are oriented from 1 to 2 or 1 to 3. Exchanging labels if necessary, we may assume 13 is oriented from 1 to 3.

Then, the 3-cycle 136 is not consistently oriented, as edge 36 is oriented from 6 to 3 and edge 13 is oriented from 1 to 3 by assumption. Thus, the only potential link in the FMellor embedding is 235-1467. As the edge 76 is oriented from 7 to 6, the 4-cycle 1467 is only consistently oriented if edge 46 is oriented from 6 to 4. However, as edges 26 and 36 are oriented from 6 to 2 and from 6 to 3 by assumption, there is exactly one more edge oriented from a vertex 6 to \( v \). Thus, at least one of the edges 46 and 56 is oriented from \( v \) to 6. Thus, switching labels of vertices...
4 and 5 if necessary, we may assume that edge 46 is oriented from 4 to 6, so there
are no consistently oriented links in the FMellor embedding.

Case 3: We may assume that edge 26 is oriented from 6 to 2 and edge 36 is
oriented from 6 to 3, as otherwise we would be in Case 1. Further, we may assume
edge 12 is oriented from 2 to 1 and edge 13 is oriented from 3 to 1, as otherwise we
would be in Case 2.

There are three edges oriented from \{1, 4, 5\} to \{6, 2, 3\}, one of which is edge
16. Suppose the other two are of the form \(v6\) and \(v'6\), where \(v'\) may or may not be
equal to \(v\). Then edges 21, 24 and 25 are oriented from 2 to the other vertex, and
edges 31, 34 and 35 are oriented from 3 to the other vertex. Edge 23 is oriented
from 2 to 3 or from 3 to 2. As both vertex 2 and vertex 3 have out degree = 3,
either choice of orientation for edge 23 gives a contradiction. Thus there exists an
edge \(vw\) from \{1, 4, 5\} to \{6, 2, 3\} with \(w \neq 6\).

As 26 and 36 are oriented from 6 to 2 and 6 to 3 by assumption, edge \(w6\) is
oriented from 6 to \(w\). We may exchange the labels of \(w\) and 6, which gives either
36 oriented from 3 to 6 or 26 oriented from 2 to 6. We may exchange the labels \(v\)
and 1 if necessary so that edge 16 is oriented from 1 to 6, and so reduce to Case 1.

\(\Box\)

Theorem 2.3. There exists a tournament on 8 vertices that is intrinsically linked
as a digraph.

Proof. Label the vertices of \(K_8\) as \(\{a_1, a_2, a_3, b_1, b_2, b_3, x, y\}\). Consider the sub-
graph \(H\) of \(K_8\) isomorphic to \(K_{3,3,2}\) formed by choosing the vertex partitions
\(\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}\) and \(\{x, y\}\). Orient the edges of \(H\) as follows: from \(x\) and \(y\)
to \(a_i\), from \(b_j\) to \(x\) and \(y\), and from \(a_i\) to \(b_j\).

Every embedding of \(K_{3,3,2}\) contains a pair of disjoint 3-cycles that have non-zero
linking number [2]. As these 3-cycles are disjoint, they must be of the form
\(xa_ib_j\) and \(ya_kb_l\). By the construction of \(H\), these cycles are consistently oriented.
Choosing an arbitrary orientation for all edges of \(K_8 \setminus H\) gives a tournament that
is intrinsically linked as a directed graph. \(\Box\)

3. INTRINSIC KNOTTING IN TOURNAMENTS

Proposition 3.1. No tournament on 7 vertices is intrinsically knotted as a digraph.

Proof. By [3] there is an embedding \(f\) of \(K_7\) that contains a single knotted cycle
c, and further, \(c\) is a Hamiltonian cycle. Given an orientation on \(K_7\), put it in the
Conway-Gordon embedding \(f\).

Suppose \(c\) is consistently oriented. Then we may label the vertices of \(K_7\) so that
\(c\) is the cycle 1234567. Due to the symmetry of \(K_7\) we may also place \(K_7\) into
the Conway-Gordon embedding so that \(c\) is 1234576. In this embedding \(c\) has an
inconsistent orientation, and hence this embedding contains no consistently oriented
cycle that is a non-trivial knot. Thus any orientation of \(K_7\) yields a tournament that
is not intrinsically knotted as a directed graph. \(\Box\)

Proposition 3.2. No tournament on 8 vertices is intrinsically knotted as a digraph.
Proof. In [1], Abrams, Mellor and Trott demonstrate an embedding of $K_8$ with exactly 29 knotted cycles, which is shown in Figure 3. We will refer to this as the AMT embedding. The knotted cycles in the AMT embedding are:

$$1462375146238514723851586237158624724685371586234715862437158632471476238514723685146723851573264814623785146237851246853715862537148652371548623712735684164273581358624713724685145682371456823571468237514723856$$

Given a tournament $T$ on 8 vertices, we will show that it can be placed in the AMT embedding such that all of the knotted cycles are inconsistently oriented. We will break the proof into cases based on the maximum in degree of a vertex in the tournament. We need only check the cases of max in degree 7, 6, 5, and 4, as in the other cases we may repeat the argument using maximum out degree.

Suppose there is a vertex $v$ in $T$ with in degree = 7. Then no consistently oriented cycle can contain vertex $v$, and so any consistently oriented cycle must be contained in $T \setminus v$, which is a tournament on seven vertices. By Proposition 3.1, a tournament on seven vertices has an embedding with no consistently oriented cycle that forms a non-trivial knot. Thus $T$ does as well, and hence $T$ is not intrinsically knotted as a directed graph.

Suppose that $v$ is a vertex of maximal in degree, with in degree 4, 5, or 6. Label $v$ as vertex 1. There is a vertex 6 such that the edge 16 is oriented from 1 to 6. Let $W$ be the set of vertices where edge $w_i 1$ is oriented from $w_i$ to 1. As 1 has maximal in degree, there exists some $w_i$ such that the edge 6$w_i$ is oriented from 6 to $w_i$. Label this $w_i$ as vertex 4. There are at least 3 more elements of $W$. There is an edge between $w_1$ and $w_2$, label them 5 and 8 so that the edge 58 is oriented from 8 to 5. Label one of the remaining elements of $W$ as vertex 7.
Embed $T$ in the AMT embedding shown in Figure 3. We have edge orientations 41 51 71 81 85 and 64. With these orientations, all of the knotted cycles in the AMT embedding have inconsistent orientations, and hence $T$ has an embedding that does not contain a consistently oriented nontrivial knot, making $T$ not intrinsically knotted as a directed graph.

One method to prove a graph is intrinsically knotted is to find a $D_4$ graph minor in every embedding that satisfies certain linking conditions for its cycles [15] [7]. The authors extended this technique to directed graphs in [9]. However, in the directed graph case, the $D_4$ must satisfy certain edge orientation conditions and rather than being a minor, it must be found by deleting edges and identifying vertices through an operation called consistent edge contraction. Vertices $v$ and $w$ may be identified by consistent edge contraction if edge $vw$ is oriented from $v$ to $w$ and either $v$ is a sink in $G \setminus vw$ or $w$ is a source in $G \setminus vw$.

**Proposition 3.3.** There exists a tournament on 12 vertices that is intrinsically knotted as a digraph.

**Proof.** We will choose an orientation of the edges of $K_{12}$ so that every embedding of the resulting tournament $T$ allows the construction of the $D_4$ graph from Corollary 3.1 of [9], and hence must be intrinsically knotted as a digraph.

Label the 12 vertices $x_1, x_2, x_3, y_1, y_2, y_3, a_1, a_2, a_3, b_1, b_2, b_3$. Orient the edges from $x_i$ to $y_j$, from $x_i$ to $x_j$ for $j > i$, and from $y_i$ to $y_j$ for $j > i$. Orient the edges from $y_i$ to $a_j$, from $x_3$ to $a_i$, from $y_3$ to $b_i$, from $a_i$ to $b_j$, from $b_i$ to $x_j$, and $b_i$ to $y_1$. Choose arbitrary orientations for the remaining edges.

Fix an embedding of $T$. Consider the subgraph formed by the $x_i$ and $y_j$. The underlying graph is $K_6$, so it contains a pair of 3-cycles $T_1, T_3$ with odd linking number. Up to switching the labels of $T_1$ and $T_3$, there are three cases:

- $T_1$ is $x_1x_2x_3$. In this case, $x_1$ is a source in $T_1$, $x_3$ is a sink in $T_1$, $y_1$ is a source in $T_3$, and $y_3$ is a sink in $T_3$.
- $T_1$ is $x_i, y_1, y_2$. In this case, $x_i$ is a source in $T_1$, $y_2$ is a sink in $T_1$, $x_j$ is a source in $T_3$, and $y_3$ is a sink in $T_3$.
- $T_1$ is $x_i, x_j, y_k$ with $k \neq 3$ and $j > i$. Then $x_i$ is a source in $T_1$, $y_k$ is a sink in $T_1$, $x_m$ is a source in $T_3$ and $y_3$ is a sink in $T_3$.

So in all cases, $y_3$ is a sink in $T_3$, the sink of $T_1$ is one of $x_3, y_1, y_2$ and the sources of $T_1$ and $T_3$ are elements of $\{x_1, x_2, x_3, y_1\}$.

Consider the subgraph formed by the vertices $y_3, a_i, b_j$. This graph contains $K_{3,3,1}$, with $y_3$ as the preferred vertex, and $a_i, b_j$ as the other partitions. Thus the embedding of $T$ contains a 3-cycle $T_4$ and a 4-cycle $S$ with odd linking number, where the 3-cycle is $y_3a_mb_n$ and the 4-cycle is of the form $a_i,b_j,a_k,b_l$. In the 3-cycle, $y_3$ is a source and $b_m$ is a sink. In the 4-cycle, the $a$ vertices have out degree 2, and the $b$ vertices have in degree 2. There is an edge between $b_j$ and $b_k$, we may assume it is oriented from $b_j$ to $b_k$. This edge divides $S$ into two 3-cycles, $T_2$ and $T_2'$, at least one of which has odd linking number with $T_4$. We may assume it is $T_2$. Note that $a_i$ is a source in $T_2$ and $b_k$ is a sink.

We may now construct the $D_4$, from $T_1, T_2, T_3$, and $T_4$. Note that $y_3$ is the sink of $T_3$ and the source in $T_4$. The sink of $T_4$ is $b_l$ for some $i$, and the source of $T_1$ is one of $x_1, x_2, x_3$. By construction, there is an edge oriented from $b_l$ to the source of $T_1$. Similarly, the sink of $T_1$ is one of $x_3, y_1, y_2$ and the source of $T_2$ is one of the
and so by construction there is an edge from the sink of \(T_1\) to the source of \(T_2\). The sink of \(T_2\) has an edge to any possible source of \(T_3\).

We now delete all edges except those in \(T_i\) and the edges we have chosen connecting the \(T_i\). Using consistent edge contraction, we may contract edges to complete the construction of the \(\overline{T_i}\). Thus \(T\) contains a consistently oriented nontrivial knot in any embedding, and hence is intrinsically knotted as a digraph.

\(\square\)

4. Intrinsic 3-Linking in Tournaments

**Proposition 4.1.** No tournament on 9 vertices is intrinsically 3-linked as a digraph.

**Proof.** \(K_9\) has an embedding with no 3-link [6], so any 9 vertex tournament has an embedding with no non-split consistently oriented 3 component link. \(\square\)

**Proposition 4.2.** There exists a tournament on 23 vertices that is intrinsically 3-linked as a digraph.

**Proof.** Start with \(K_{3,3,2}\), with vertex partitions \(A = \{a_1, a_2, a_3\}\), \(B = \{b_1, b_2, b_3\}\) and \(C = \{c_1, c_2\}\), where edges are directed from \(A\) to \(B\), from \(B\) to \(C\), and from \(C\) to \(A\). Expand vertex \(c_1\) into edge \(d_1d_2\), so that vertices in \(B\) are connected only to \(d_1\), and directed to \(d_1\), and vertices in \(A\) are only connected to \(d_2\), and such edges are directed from \(d_2\) to \(A\). Denote this directed graph \(D\). It follows readily from work of Chan et al [2] that the graph \(D\) is intrinsically linked, and in every spatial embedding, there is a pair of linked cycles \(C_1, C_2\) with \(C_1\) containing the edge \(d_1d_2\). We similarly expand vertex \(c_1\) in a second copy of \(K_{3,3,2}\) with the same edge orientations and denote the resulting graph \(D'\).

Glue \(D\) and \(D'\) along \(d_1d_2\) to get \(DDD\). Now, let \(\hat{D}\) be a directed graph that is defined similarly to \(D\), except the edge \(e = d_1d_2\) is oriented the other way, from \(d_2\) to \(d_1\). Glue \(\hat{D}\) to \(DDD\) along \(d_1d_2\), so that the glued edge goes from \(d_1\) to \(d_2\) in the copies of \(D\), but from \(d_2\) to \(d_1\) in \(\hat{D}\). Call the resulting graph \(\hat{DDD}\). Embed \(\hat{DDD}\).

In each copy of \(D\) and \(D'\), there is a pair of consistently oriented links, call them \(C_1\) and \(C_2\) (and \(C'_1\) and \(C'_2\) in \(D'\)), where \(C_1\) and \(C'_1\) share the edge \(d_1d_2\). Similarly, in \(\hat{D}\), there is a pair of linked cycles \(C''_1\) and \(C''_2\), where \(C''_2\) is consistent, \(C''_1\) is not, and \(C'_1\) shares edge \(d_1d_2\). Consider the set of cycles \(S = \{C_1, C'_1, C''_1, C_2, C_1 + C'_1\}\), then \(\left|C_1\right| + \left|C'_1\right| + \left|C''_1\right| - \left|C_1 + C'_1\right| = 0\) in \(H_1(\mathbb{R}^3 - C_2, \mathbb{Z})\) and in \(H_1(\mathbb{R}^3 - C''_2, \mathbb{Z})\) and in \(H_1(\mathbb{R}^3 - C'_2, \mathbb{Z})\) (see Figure 4). It follows that \(C_2, C'_2\) and \(C''_2\) each have non-zero linking number with at least two cycles in \(S\). By the pigeonhole principle, there is a non-split 3-link of consistently oriented cycles.

The digraph \(DDD\) has at most one edge between any pair of vertices, so we may add directed edges (of arbitrary orientation) to form a tournament \(T\) on 23 vertices. As \(T\) has \(DDD\) as a subgraph, it is intrinsically 3-linked as a directed graph.

\(\square\)

5. Intrinsic 4-Linking in Tournaments

**Proposition 5.1.** No tournament on 11 vertices is intrinsically 4-linked as a digraph.
Figure 4. The cycle $C_2$ is linked with at least two consistently oriented cycles in this figure.

Proof. A 4 component link must contain 4 disjoint cycles. Any cycle in a tournament must contain at least 3 vertices. Thus any 4-linked tournament must have 12 or more vertices.

Proposition 5.2. There exists a tournament on 66 vertices that is intrinsically 4-linked as a digraph.

Proof. Let $H$ be the intrinsically 3-linked graph constructed in Proposition 4.2. In any embedding of $H$, there is a 3 component link $(L_1, L_2, L_3)$ where $L_1$ is an element of $S = \{C_1, C_1', C_1'' + C_1', C_1' + C_1''\}$, and $L_2$ and $L_3$ have non-zero linking number with $L_1$. Further, $L_1$ must contain vertices $d_1$ and $d_2$. As $L_1$ is consistently oriented, we may consider $L_1$ to be composed of two paths, $P_1$ from $d_1$ to $d_2$, and $P_2$ from $d_2$ to $d_1$.

Let $G$ be a directed graph formed from 3 copies of $H$. Label them $H_1, H_2,$ and $H_3$. To form $G$, identify $d_1$ from $H_1$ to $d_2$ in $H_2$, $d_1$ in $H_2$ to $d_2$ in $H_3$ and $d_1$ in $H_3$ to $d_2$ in $H_1$.

In any embedding of $G$, we may find a 3-component link $(L_{i1}, L_{i2}, L_{i3})$ in each $H_i$. Using the decomposition of $L_{i1}$ into paths, we may form the consistently oriented cycles $Z = P_{11} \cup P_{21} \cup P_{31}$ and $W = P_{12} \cup P_{22} \cup P_{32}$. Let $S' = \{L_{11}, L_{21}, L_{31}, Z, W\}$ and note that $[L_{11}] + [L_{21}] + [L_{31}] - [Z] - [W] = 0$ in $H_1(\mathbb{R}^3 - L_{i2}, Z)$ and in $H_1(\mathbb{R}^3 - L_{i3}, Z)$. It follows that $L_{12}, L_{13}, L_{22}, L_{23}, L_{32}$ and $L_{33}$ each have non-zero linking number with at least 2 cycles in $S'$. By the pigeonhole principle, there is a non-split 4-link of consistently oriented cycles.

The digraph $G$ has at most one edge between any pair of vertices, so we may add directed edges (of arbitrary orientation) to form a tournament $T$ on 66 vertices. As $T$ has $G$ as a subgraph, it is intrinsically 4-linked as a directed graph.

6. INTRINSIC 5-LINKING IN TOURNAMENTS

Proposition 6.1. No tournament on 14 vertices is intrinsically 5-linked as a directed graph.
Proof. A 5 component link must contain 5 disjoint cycles. Any cycle in a tournament must contain at least 3 vertices. Thus any 5-linked tournament must have 15 or more vertices. □

**Proposition 6.2.** There exists a tournament on 154 vertices that is intrinsically 5-linked as a digraph.

Proof. We proceed as in the 4-linking case.

Let $H$ be the intrinsically 3-linked graph constructed in Proposition 4.2. In any embedding of $H$, there is a 3 component link $(L_1, L_2, L_3)$ where $L_1$ is an element of $S = \{ C_1, C_1', C_1'' + C_1', C_1 + C_1'' \}$, and $L_2$ and $L_3$ have non-zero linking number with $L_1$. Further, $L_1$ must contain vertices $d_1$ and $d_2$. As $L_1$ is consistently oriented, we may consider $L_1$ to be composed of two paths, $P_1$ from $d_1$ to $d_2$, and $P_2$ from $d_2$ to $d_1$.

Let $G$ be a directed graph formed from 7 copies of $H$. Label them $H_1, H_2, \ldots, H_7$. To form $G$, identify $d_1$ from $H_1$ to $d_2$ in $H_2$, $d_1$ in $H_2$ to $d_2$ in $H_3$ and so on to $d_1$ in $H_7$ to $d_2$ in $H_1$.

In any embedding of $G$, we may find a 3-component link $(L_{11}, L_{12}, L_{13})$ in each $H_i$. Using the decomposition of $L_{1i}$ into paths, we may form the consistently oriented cycles $Z = P_{11} \cup P_{21} \cup \ldots \cup P_{71}$ and $W = P_{12} \cup P_{22} \cup \ldots \cup P_{72}$. Let $S' = \{ L_{11}, L_{21}, \ldots, L_{71}, Z, W \}$ and note that $[L_{11}] + [L_{21}] + \ldots + [L_{71}] - [Z] - [W] = 0$ in $H_1(\mathbb{R}^3 - L_{12}, \mathbb{Z})$ and in $H_1(\mathbb{R}^3 - L_{13}, \mathbb{Z})$. It follows that $L_{12}, L_{13}, L_{22}, L_{23}, \ldots, L_{72}$ and $L_{73}$ each have non-zero linking number with at least 2 cycles in $S'$. By the pigeonhole principle, there is a non-split 5-link of consistently oriented cycles.

The digraph $G$ has at least one edge between any pair of vertices, so we may add directed edges (of arbitrary orientation) to form a tournament $T$ on 154 vertices. As $T$ has $G$ as a subgraph, it is intrinsically 5-linked as a directed graph. □

7. INTRINSIC N-LINKING IN TOURNAMENTS

For the general case, we will rely on the construction from Lemma 1 of Flapan, Mellor and Naimi [5]. We first note the obvious lower bound.

**Proposition 7.1.** No tournament on $3n - 1$ vertices is intrinsically n-linked as a directed graph.

Proof. An $n$-component link must contain $n$ disjoint cycles. Any cycle in a tournament must contain at least 3 vertices. Thus any $n$-linked tournament must have $3n$ or more vertices. □

The following lemma establishes the building block we need for the full construction.

**Lemma 7.2.** There exists a tournament $T'$ on 8 vertices such that every embedding of $T'$ contains a link $(L_1, L_2)$ such that the linking number $lk(L_1, L_2)$ is odd, $L_2$ is consistently oriented, and $L_1$ contains two vertices $a$ and $b$ such that it may be decomposed into two paths $P_1$ and $P_2$ such that each $P_i$ is consistently oriented from vertex $a$ to vertex $b$.

Proof. Start with $K_{3,3,2}$ with vertex partitions $A = \{ a_1, a_2, a_3 \}$, $B = \{ b_1, b_2, b_3 \}$ and $C = \{ c_1, c_2 \}$, where all edges are directed from $A$ to $B$, from $B$ to $c_2$, from $c_2$
to A, from A to c1 and c1 to B. Give all remaining edges an arbitrary orientation, and label this tournament T′.

As every embedding of $K_{3,3,2}$ contains a pair of 3-cycles with odd linking number, so does every embedding of T′. Each 3-cycle must contain exactly one of the $c_i$. Label the 3-cycle that contains $c_i$ as $C_i$. Then $lk(L_1, L_2)$ is odd, and $L_2$ is consistently oriented. The cycle $L_1$ contains the vertices $c_1, a_j, b_k$, so it may be decomposed into the path $P_1 = a_jb_k$ and the path $P_2 = a_jc_kb$, each of which are consistently oriented from $a_j$ to $b_k$.

We now construct the n-linked tournament.

**Proposition 7.3.** There exists a tournament $T$ on $8(2n - 3)^2$ vertices such that $T$ is intrinsically n-linked as a directed graph.

**Proof.** We begin with $(2n - 3)^2$ copies of the tournament $T'$ from Lemma 7.2 and label them $T'_1$. We add edges oriented from $b_k$ to $a_{(i+1)j}$ and edges oriented from $b_{(2n-3)k}$ to $a_{1j}$. We then add edges of arbitrary orientation to complete the construction of the tournament $T$.

Embed $T$. We may now follow the proof of Lemma 1 from [5]. In each copy of $T'_1$, we may find a 2-link $L_{i1}, L_{i2}$ and edges $e_i$ from $L_{i1}$ to $L_{(i+1)1}$. Let $C = \bigcup P_{e} \cup e_i$. Notice that $C$ is consistently oriented. If $lk(C, L_{i2}) \neq 0$ for $n - 1$ cycles $L_{i2}$, we are done. If not, we will construct an index set $I$ and form a new cycle $Z = \bigcup P_{\epsilon} \cup e_i$ where $\epsilon = 2$ if $i \in I$ and $\epsilon = 1$ otherwise. Note that for any choice of $I$, the cycle $Z$ is consistently oriented.

Let $M$ be a $(2n - 3)^2$ by $(2n - 3)^2$ matrix where the entry $m_{ij} = lk(L_{i1}, L_{j})$ modulo 2. By the construction of $T'$, $m_{ii} = 1$. Let $M'$ be the reduced row echelon form of $M$ modulo 2. Let $r$ be the rank of $M$. If $r \geq (2n - 3)$ then let $V$ be the modulo 2 sum of the rows of $M'$. If $r < (2n - 3)$, then each column of $M$ contains a 1, some row of $M'$ contains at least 2n - 3 non-zero entries. Let $V$ be this row.

In either case, $V$ can be written as the modulo 2 sum of rows of $M$, so $V = \sum_{i \in I} r_i$, where $r_i$ are the rows of $M$. Let $Z = \bigcup P_{\epsilon} \cup e_i$ where $\epsilon = 2$ if $i \in I$ and $\epsilon = 1$ otherwise. Let $V_j$ be the $j$th entry of $V$. Notice that that $V_j = \sum_{i \in I} lk(L_{i1}, L_{j})$ modulo 2 and that $lk(Z, L_{j2}) = lk(C, L_{j2}) + \sum_{i \in I} lk(L_{i1}, L_{j2})$ modulo 2.

Thus, $lk(Z, L_{j2}) = lk(C, L_{j2}) + V_j$ modulo 2. At least $2n - 3$ of the $V_j$ are odd, and there exist at most $n - 2$ components $L_{j2}$ that have non-zero linking number with $C$. Thus, $lk(Z, L_{j2}) = 1$ modulo 2 for at least $n - 1$ components $L_{j2}$, giving a non-split link with $n$ consistently oriented components.

Flapan, Mellor and Niami introduced the idea of linking patterns in the study of intrinsically linked graphs in [5]. The linking pattern of a link $L_1, \ldots, L_n$ is the graph $\Gamma$ with vertices $v_1, \ldots, v_n$ and an edge between $v_i$ and $v_j$ if $lk(L_i, L_j) \neq 0$. They then show that for any linking pattern $\Gamma$ there exists a graph $G$ such that every embedding of $G$ contains a link whose linking pattern contains $\Gamma$. The first step to obtain this general result is to show that for any $n$, there exists a graph $G$ such that every embedding of $G$ contains a link whose linking pattern contains $K_{n,n}$. This result requires only the iterative application of Lemma 1 of [5]. As Proposition 7.3 is the direct analogue of Lemma 1 of [5] for tournaments, we have the analogous result as a corollary.
Corollary 7.4. For all \( n \), there exists a tournament \( T \) such that every embedding of \( T \) contains a consistently oriented link whose mod 2 linking pattern contains \( K_{n,n} \).

It is likely that the techniques of Mattman, Naimi and Pagano [13] can be extended to find examples of tournaments with arbitrary linking patterns.

8. Intrinsic Linking with Knotted Components

If every embedding of a graph \( G \) contains a non-split \( n \)-component link where at least \( m \) components of the link are non-trivial knots, we will say that \( G \) is intrinsically \( n \)-linked with \( m \)-knotted components.

If every embedding of a directed graph contains a consistently oriented non-split \( n \)-component link where at least \( m \) components of the link are non-trivial knots, we will say that \( G \) is intrinsically \( n \)-linked with \( m \)-knotted components as a directed graph.

In [8], the first author demonstrated graphs that are intrinsically \( n \)-linked with \( m \)-knotted components for \( m < n \). Flapan, Naimi and Mellor used different techniques to construct examples for all \( m \), including \( m = n \) in [5].

We will use the techniques of [8] to construct a tournament that is intrinsically 2-linked with 1-knotted component as a directed graph.

Proposition 8.1. No tournament on 8 vertices is intrinsically 2-linked with 1-knotted component as a directed graph.

Proof. If every embedding of a tournament contains a consistently oriented non-split link where at least one component of the link is a non-trivial knot, then that tournament is intrinsically knotted as a digraph.

By Proposition 3.2, no tournament on 8 vertices is intrinsically knotted as a digraph. \( \square \)

Lemma 8.2. There exists a tournament \( T' \) on 14 vertices that is intrinsically knotted as a digraph, and further the pair of adjacent edges \( y_3 \alpha, \alpha y_3' \) are contained in a consistently oriented non-trivial knot in every embedding of \( T' \).

Proof. We construct \( T' \) based on the knotted tournament from Proposition 3.3.

The tournament \( T' \) contains 14 vertices. Label them \( x_1, x_2, x_3, y_1, y_2, y_3, \alpha, y_3', a_1, a_2, a_3, b_1, b_2, b_3 \). Orient the edge from \( y_3 \) to \( \alpha \) and the edge from \( \alpha \) to \( y_3' \). Orient the edges from \( y_3' \) to \( a_j \), and from \( y_3' \) to \( b_i \).

Continue to orient the edges as in Proposition 3.3 that is from \( x_i \) to \( y_j \), from \( x_i \) to \( x_j \) for \( j > i \), and from \( y_i \) to \( y_j \) for \( j > i \). Orient the edges from \( y_i \) to \( a_j \), from \( x_3 \) to \( a_i \), from \( a_i \) to \( b_j \), from \( b_i \) to \( x_j \), and \( b_i \) to \( y_1 \). Give the remaining edges an arbitrary orientation.

Note that \( x_1, x_2, x_3, y_1, y_2, y_3 \) form a \( K_6 \) subgraph, and hence must contain a pair of 3-cycles with non-zero linking number in every embedding. Label the 3-cycle containing \( y_3 \) as \( T_3 \). The vertices \( y_3', a_1, a_2, a_3, b_1, b_2, b_3 \) form a subgraph isomorphic to \( K_{3,3,1} \), and hence contain a 3-cycle and 4-cycle with non-zero linking number in every embedding, with \( y_3' \) in the 3-cycle. Label this 3-cycle \( T_4 \). Note that \( y_3 \) is a sink in \( T_3 \) and \( y_3' \) is a source in \( T_4 \).

We may now continue as in Proposition 3.3 using consistent edge contraction to construct a \( \overline{D}_4 \) graph. If we choose to contract all edges except \( y_3 \alpha \) and \( \alpha y_3' \), we obtain a \( \overline{D}_4 \) graph, with one vertex expanded into a consistently oriented 2-path.
Thus, the edges \( y_3 \alpha \) and \( \alpha y'_3 \) are contained in a consistently oriented cycle that is a non-trivial knot in every embedding of \( T' \).

\[ \square \]

**Proposition 8.3.** There exists a tournament \( T \) on 107 vertices that is intrinsically 2-linked with 1-knotted component as a directed graph.

**Proof.** We will construct \( T \) from 9 copies of the tournament \( T' \) from Lemma 8.2 and an additional vertex \( \beta \), in a manner similar to the construction of Proposition 2.2 of [8].

Label the 9 copies of \( T' \) as \( T'_i \). Identify all \( \alpha_i \) to form a single vertex \( \alpha \). We will identify edges \( y_3 \alpha \) and \( \alpha y'_3 \) as follows. Identify \( y_{31} \alpha, y_{32} \alpha, y_{33} \alpha \) and label the new vertex \( v_1 \). Identify \( y_{34} \alpha, y_{35} \alpha, y_{36} \alpha \) and label the new vertex \( v_2 \). Identify \( y_{37} \alpha, y_{38} \alpha, y_{39} \alpha \) and label the new vertex \( v_3 \).

Identify \( \alpha y'_{31}, \alpha y'_{34}, \alpha y'_{37} \) and label the new vertex \( w_1 \). Identify \( \alpha y'_{32}, \alpha y'_{35}, \alpha y'_{38} \) and label the new vertex \( w_2 \). Identify \( \alpha y'_{33}, \alpha y'_{36}, \alpha y'_{39} \) and label the new vertex \( w_3 \).

Orient the edges from \( \beta \) to the \( w_i \) and from \( v_i \) to \( \beta \). Add edges of arbitrary orientation to complete the construction of \( T \).

Embed \( T \). In each copy of \( T'_i \) there exists a consistently oriented knotted cycle \( c_i \) that contains edges \( y_{3i} \alpha \) and \( \alpha y'_{3i} \). The cycle \( c_i \) is composed of \( y_{3i} \alpha, \alpha y'_{3i} \) and a path \( P_i \) from \( y_{3i} \) to \( y_{3i} \).

Consider the vertices \( \alpha, \beta, v_i, w_i \) together with the edges \( \alpha w_i, v_i \alpha, \beta w_i, v_i \beta \) and the paths \( P_i \). This graph is isomorphic to \( K_{3,3,2} \), and thus must contain a pair 3-cycles with non-zero linking number [2]. Further, the 3-cycles must be of the form \( \alpha w_i v_j \) and \( \beta w_j v_i \). These cycles are consistently oriented, and by construction \( \alpha w_i v_j \) is a non-trivial knot.

\[ \square \]

9. The Disjoint Linking Property in Tournaments

Chan et al (2) demonstrated graphs that have a disjoint pair of links in every spatial embedding (have the disjoint linking property) but do not contain disjoint copies of intrinsically linked graphs. We modify their construction here for tournaments.

**Proposition 9.1.** No tournament on 11 vertices has the disjoint linking property.

**Proof.** A disjoint pair of links must have at least 4 disjoint cycles. Any cycle in a tournament must contain at least 3 vertices. Thus any tournament with the disjoint linking property must have 12 or more vertices.

\[ \square \]

**Proposition 9.2.** There exists a tournament on 14 vertices that has the disjoint linking property, but does not contain two disjoint tournaments that are intrinsically linked.

**Proof.** Consider \( K_{5,5,4} \) with vertex set \( A = \{a_1, ..., a_5\} \), \( B = \{b_1, ..., b_5\} \) and \( C = \{c_1, c_2, c_3, c_4\} \). Form a directed graph by orienting edges from \( C \) to \( A \), from \( A \) to \( B \), and from \( B \) to \( C \). Call the resulting graph \( G \). Embed \( G \). Let \( D \) be the subgraph of \( G \) formed by \( \{a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2\} \). Note that \( D \) is an orientation of \( K_{3,3,2} \), and hence contains a pair of 3-cycles with non-zero linking number. Further, these 3-cycles are consistently oriented. Since the graph \( D \) is a subgraph of \( G \), \( G \) has a pair of consistently-oriented linked 3-cycles. Removing the linked 3-cycles and all
edges incident to their vertices results in an embedded copy of $D$, which will again have a disjoint pair of linked 3-cycles. Thus $G$ has the disjoint linking property.

As $G$ has at most one edge between any pair of vertices, we may add edges of arbitrary orientation to $G$ to form a tournament $T$ on 14 vertices. As $G$ is a subgraph of $T$, $T$ has the disjoint linking property. Since no tournament on 7 (or fewer) vertices is intrinsically linked, $T$ cannot contain two disjoint copies of intrinsically linked tournaments.

\[ \square \]

10. The Consistency Gap

Given a tournament, we may ignore the edge orientations and consider it as an undirected complete graph. The complete graph $K_6$ is intrinsically linked \[ [14] \[ 3 \], but by Theorem 2.3, the smallest tournament that is intrinsically linked as a digraph has 8 vertices. This implies that every embedding of a tournament on 6 or 7 vertices contains one or more non-split links, but in some embeddings one or more the components of each non-split link has inconsistent edge orientations.

Thus the requirement that the components of the non-split link have a consistent orientation for an intrinsically linked digraph is restrictive and appears to require a larger and more complex graph to satisfy. We propose to measure how restrictive this condition is using the consistency gap.

We define the consistency gap, denoted $cg(n)$, as $m' - m$ where $K_m$ is the smallest intrinsically $n$-linked complete graph, and $m'$ is the number of vertices in the smallest tournament that is intrinsically $n$-linked as a directed graph. Note that $cg(n) \geq 0$ for all $n$. By the preceding discussion, we have the following corollary.

Corollary 10.1. The consistency gap at 2, $cg(2) = 2$.

By work of Flapan, Naimi, and Pommersheim, the smallest intrinsically 3-linked complete graph is $K_{10}$ \[ [6] \]. Combining that with Proposition 1.2, we have the following.

Corollary 10.2. The consistency gap at 3, $cg(3) \leq 13$.

An $n$-linked complete graph must contain at least $3n$ vertices. Combining this with Proposition 5.2 and Proposition 6.2 we have the following.

Corollary 10.3. The consistency gap at 4, $cg(4) \leq 54$, and the consistency gap at 5, $cg(5) \leq 139$.

Finally, the results of Section 7 give a bound for the general case.

Corollary 10.4. For $n > 5$, the consistency gap at $n$, $cg(n) \leq 8(2n - 3)^2 - 3n$.

We conjecture the consistency gap to be non-decreasing in $n$. Does it increase without bound?

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