CONVEX NORMALITY OF RATIONAL POLYTOPES WITH LONG EDGES

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Abstract. We introduce the property of convex normality of rational polytopes and give a dimensionally uniform lower bound for the edge lattice lengths, guaranteeing the property. As an application, we show that if every edge of a lattice $d$-polytope $P$ has lattice length $\geq 2d^2(d+1)$ then $P$ is normal. This answers in the positive a question raised in 2007. If $P$ is a lattice simplex whose edges have lattice lengths $\geq d(d+1)$ then $P$ is even covered by lattice parallelepipeds. For the approach developed here, it is necessary to involve rational polytopes even for applications to lattice polytopes.

1. Integrally closed polytopes

All our polytopes are assumed to be convex. For a polytope $P$ the set of its vertices will be denoted by $\text{vert}(P)$.

A polytope $P \subset \mathbb{R}^d$ is lattice if $\text{vert}(P) \subset \mathbb{Z}^d$, and $P$ is rational if $\text{vert}(P) \subset \mathbb{Q}^d$.

Let $P \subset \mathbb{R}^d$ be a lattice polytope and denote by $L$ the subgroup of $\mathbb{Z}^d$, affinely generated by the lattice points in $P$; i. e., $L = \sum_{x,y \in P \cap \mathbb{Z}^d} \mathbb{Z}(x-y) \subset \mathbb{Z}^d$. Observe, $P \cap L = P \cap \mathbb{Z}^d$.

Definition 1.1. ([7, Def. 2.59]) Let $P \subset \mathbb{R}^d$ be a lattice polytope.

(a) $P$ is integrally closed if the following condition is satisfied:

$$c \in \mathbb{N}, \ z \in cP \cap \mathbb{Z}^d \implies \exists x_1, \ldots, x_c \in P \cap \mathbb{Z}^d \ x_1 + \cdots + x_c = z.$$

(b) $P$ is normal if the following condition is satisfied:

$$c \in \mathbb{N}, \ z \in cP \cap L \implies \exists x_1, \ldots, x_c \in P \cap L \ x_1 + \cdots + x_c = z.$$

The normality property is invariant under affine-lattice isomorphisms of lattice polytopes, and the property of being integrally closed is invariant under an affine change of coordinates, leaving the lattice structure $\mathbb{Z}^d \subset \mathbb{R}^d$ invariant.

A lattice polytope $P \subset \mathbb{R}^d$ is integrally closed if and only if it is normal and $L$ is a direct summand of $\mathbb{Z}^d$. Obvious examples of normal but not integrally closed polytopes are the s. c. empty lattice simplices of large volume. No classification of such simplices is known in dimensions $\geq 4$, the main difficulty being the lack of satisfactory characterization of their lattice widths; see [12, 18].

A normal polytope $P \subset \mathbb{R}^d$ can be made into a full-dimensional integrally closed polytope by changing the lattice of reference $\mathbb{Z}^d$ to $L$ and the ambient Euclidean
space $\mathbb{R}^d$ to the subspace $\mathbb{R}L$. In particular, normal and integrally closed polytopes refer to same isomorphism classes of lattice polytopes.

Normal/integrally closed polytopes enjoy popularity in algebraic combinatorics and they have been showcased on recent workshops ([11 2]). These polytopes represent the homogeneous case of the Hilbert bases of finite positive rational cones and the connection to algebraic geometry is that they define projectively normal embeddings of toric varieties. There are many challenges of number theoretic, ring theoretic, homological, and $K$-theoretic nature, concerning the associated objects: Ehrhart series’, rational cones, toric rings, and toric varieties; see [7].

If a lattice polytope is covered by (in particular, subdivided into) integrally closed polytopes then it is integrally closed as well. The simplest integrally closed polytopes one can think of are unimodular simplices, i.e., the lattice simplices $\Delta = \text{conv}(x_1, \ldots, x_k) \subset \mathbb{R}^d, \dim \Delta = k-1$, with $x_1-x_j, \ldots, x_{j-1}-x_j, x_{j+1}-x_j, \ldots, x_k-x_j$ a part of a basis of $\mathbb{Z}^d$ for some (equivalently, every) $j$.

Unimodular simplices are the smallest ‘atoms’ in the world of normal polytopes. But the latter is not built out exclusively of these atoms: not all 4-dimensional integrally closed polytopes are triangulated into unimodular simplices [9, Prop. 1.2.4], and not all 5-dimensional integrally closed polytopes are covered by unimodular simplices [9] – contrary to what had been conjectured before [17]. Further ‘negative’ results, such as [4] and [8] (disproving a conjecture from [10]), contributed to the current thinking in the area that there is no succinct geometric characterization of the normality property. One could even conjecture that in higher dimensions the situation gets as bad as it can; see the discussion at the end of [2, p. 2313].

‘Positive’ results in the field mostly concern special classes of lattice polytopes that are normal, or have unimodular triangulations or unimodular covers. Knudsen-Mumford’s classical theorem ([4 Sect. 3B], [13 Chap. III], ) says that every lattice polytope $P$ has a multiple $cP$ for some $c \in \mathbb{N}$ that is triangulated into unimodular simplices. The existence of a dimensionally uniform lower bound for such $c$ seems to be a very hard problem. More recently, it was shown in [6] that there exists a dimensionally uniform exponential lower bound for unimodularly covered multiple polytopes. By improving one crucial step in [6], von Thaden was able to cut down the bound to a degree 6 polynomial function in the dimension [7 Sect. 3C], [19].

For polytopes, arising in a different context and admitting unimodular triangulations as certificate of normality, see [3, 14, 15, 16].

The results above on multiple polytopes yield no new examples of normal polytopes, though. In fact, an easy argument ensures that for any lattice $d$-polytope $P$ the multiples $cP, c = d - 1, d, d + 1, \ldots$, are integrally closed [9 Prop. 1.3.3], [11]. However, that argument does not allow a modification that would apply to lattice polytopes with long edges of independent lengths. The following conjecture is proposed in [2 p. 2310]:

**Conjecture.** Simple lattice polytopes with ‘long’ edges are normal, where ‘long’ means some invariant, uniform in the dimension.

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1In the literature the difference between ‘normal’ and ‘integrally closed’ is sometimes blurred.
More precisely, let $P$ be a simple lattice polytope. Let $k$ be the maximum over the heights of Hilbert basis elements of tangent cones to vertices of $P$. Then, if any edge of $P$ has length $\geq k$, the polytope $P$ should be normal.

Here (i) the length is measured in the lattice sense, (ii) ‘tangent cones’ is the same as corner cones, and (iii) the heights of Hilbert basis elements of corner cones are normalized w.r.t. the extremal generators of the cones (leading, in particular, to non-integral heights).

The second part of the conjecture is a far reaching extension of the following well known problem, a.k.a. Oda’s question, that has attracted much interest recently: are all smooth polytopes normal? A lattice polytope $P \subset \mathbb{R}^d$ is called smooth if the primitive (i.e., with coprime components) edge vectors at every vertex of $P$ define a part of a basis of $\mathbb{Z}^d$. Such polytope are simple with $k = 1$. Oda’s question still remains wide open. The fact that so far no smooth polytope just without a unimodular triangulation has been found illustrates how limited our understanding in the area is. The second part of the conjecture yields also a dimensionally uniform lower bound, mentioned in the first part. In fact, it is known that the normalized heights of Hilbert basis elements of a simplicial rational $d$-cone, $d \geq 2$, are at most $d - 1$; this follows, for instance form [7, Prop. 2.43(d)].

In this paper we introduce the notion of $k$-convex-normality, $k \in \mathbb{Q}_{\geq 2}$, which is a ‘convex-rational’ version of Definition 1.1. Next is the main result of the paper:

**Theorem 1.2.** Let $P$ be a rational (not necessarily simple) polytope of dimension $d$. If the lattice length of every edge of $P$ is at least $0.5d^2(d+1)k$ then $P$ is $k$-convex-normal.

Although $k$-convex-normality concerns the multiple polytopes $cP$ with $c \in [2, k]_{\mathbb{Q}}$, when applied to lattice polytopes this is enough to cover the factors $c \in \mathbb{N}$ in Definition 1.1, even with $k = 4$. As an application to lattice polytopes, we prove the first part of the conjecture above in the following strong form:

**Theorem 1.3.** Let $P$ be a (not necessarily simple) lattice polytope of dimension $d$.

(a) If the lattice length of every edge of $P$ is at least $2d^2(d+1)$ then $P$ is integrally closed.

(b) If $P$ is a simplex and the lattice length of every edge of $P$ is at least $d(d+1)$ then $P$ is covered by lattice parallelepipeds. In particular, $P$ is integrally closed.

**Outline of the proof of Theorem 1.2.** Let $P$ be a rational polytope with long edges. Assuming Theorem 1.2 is true in dimension $d - 1$, we first show that the neighborhood of a certain width of the boundary surface of any multiple $cP$, $c \in [2, k]_{\mathbb{Q}}$, behaves as if $P$ were convex-normal. Then it is shown that the complement of this neighborhood is covered by certain parallel translates of lattice parallelepipeds inside $cP$. This does not require inductive assumptions and is achieved by propagating ‘corner parallelepipedral covers’ deep inside $cP$.

Actually, the situation is a bit subtler – one needs that (i) the involved parallelepipeds break up into strongly correlated families, and that (ii) the mentioned

\[\text{2}^{\text{Smooth polytopes correspond to projective embeddings of smooth projective toric varieties.}}\]
boundary neighborhood and the region, covered by the parallelepipeds, overlap in
certain nontrivial way; see Remark 5.3.

1.1. Notation and terminology. The affine and convex hulls of a subset \( X \subset \mathbb{R}^d \)
will be denoted, respectively, by \( \text{aff}(X) \) and \( \text{conv}(X) \).

The relative interior \( \text{int}(P) \) of a polytope \( P \subset \mathbb{R}^d \) is by definition the absolute
interior of \( P \) in \( \text{aff}(P) \).

If \( H \subset \mathbb{R}^d \) is an affine hyperplane then the closed half-spaces bounded by
\( H \) will be denoted by \( H^+ \) and \( H^- \), indicating if necessary which of the two is positive.

For a polytope \( P \subset \mathbb{R}^d \) the set of its facets will be denoted be \( F(P) \). If \( \dim P = d \)
and \( F \in F(P) \) the half-space \( H_F^+ \) and hyperplane \( H_F \) are defined from the unique
irredundant representation of the form (\cite[Thm. 1.6]{7}, \cite[Thm. 2.15(7)]{20})

\[
P = \bigcap_{F \in F(P)} H_F^+, \quad H_F = \text{aff}(F).
\]

For a vertex \( v \in \text{vert}(P) \) we let \( \mathcal{F}(P)^v \) denote the facets \( F \) of \( P \) that are visible
from \( v \), i.e., \( v \notin F \).

Let \( P \subset \mathbb{R}^d \) be a polytope, \( \dim P = d \), and \( F \in \mathcal{F}(P) \). The width of \( P \) w.r.t.
\( H_F \) in the Euclidean metric will be denoted \( \text{width}_F(P) \). We have \( \text{width}_F(P) = \max_{v \in \text{vert}(P)} \|v, H_F\| \)
where the Euclidean distance between a point \( x \in \mathbb{R}^d \) and an
affine hyperplane \( H \subset \mathbb{R}^d \) is denoted by \( \|x, H\| \). Further, for a real number \( \varepsilon > 0 \)
the \( \varepsilon \)-layer along \( F \) is the polytope

\[
U_P(F, \varepsilon) = \{ x \in P : \|x, H_F\| \leq \varepsilon \}.
\]

If \( \varepsilon < \text{width}_F(P) \) then \( U_P(F, \varepsilon) \) has a facet, different from \( F \) and parallel to \( F \). It
will be denoted by \( U_P(F, \varepsilon)^+ \).

The lattice length of a rational segment \( [x, y] \subset \mathbb{R}^d \) is the ratio of its Euclidean
length and that of the primitive integer vector in the direction of \( y - x \).

For a rational polytope \( P \) by \( E(P) \) we denote the minimum of the lattice lengths
of the edges of \( P \).

A polytope is simple if its edge directions at every vertex are linearly independent.

A parallelepiped is by definition the Minkowski sum of segments of linearly independent
directions.

Cones \( C \subset \mathbb{R}^d \) are always assumed to be finite and positive, i.e., they are
intersections of finite sets of homogeneous half-spaces and containing no nontrivial
subspace. A cone is simplicial if its one-dimensional faces determine linearly independent
directions.

Let \( C \subset \mathbb{R}^d \) be a rational cone, i.e., \( C \) is the intersection of half-spaces with rational
boundary hyperplanes. Then the primitive lattice points on the one-dimensional
faces are called the extremal generators of \( C \).

A \( d \)-polytope or \( d \)-cone is the same as a \( d \)-dimensional polytope or, respectively,
\( d \)-dimensional cone.

\( \mathbb{R}_+, \mathbb{Q}_+, \) and \( \mathbb{Z}_+ \) refer to the corresponding sets of nonnegative numbers.
For an interval $I \subset \mathbb{R}$ and number $\lambda \in \mathbb{R}$ we let
\[
I_\mathbb{Q} = I \cap \mathbb{Q}, \quad I_\mathbb{N} = I \cap \mathbb{N}, \quad \mathbb{Q}_{\geq \lambda} = [\lambda, \infty)_{\mathbb{Q}}, \quad \mathbb{Q}_{> \lambda} = (\lambda, \infty)_{\mathbb{Q}}, \\
\mathbb{N}_{\geq \lambda} = [\lambda, \infty)_{\mathbb{N}}, \quad \mathbb{N}_{> \lambda} = (\lambda, \infty)_{\mathbb{N}}.
\]

For a subset $X \subset \mathbb{R}^d$ we put $\mathbb{R}_+X = \{\lambda x : \lambda \in \mathbb{R}_+, \ x \in X\}$.

2. Convex normality

Lemma 2.1. Let $P$ be a rational $d$-polytope, $d \in \mathbb{N}$. Then
\[
cP = \bigcup_{v \in \text{vert}(P)} v + (c-1)P, \quad c \in \mathbb{Q}_{\geq d+1}.
\]

Proof. For each $c \in \mathbb{Q}_{d+1}$ the equality says that $cP$ is covered by the homothetic images $P_v$ of $cP$ with factor $\frac{c-1}{c}$ and centered at $v, v \in \text{vert}(cP)$.

When $P$ is a simplex, this follows from the fact that at least one of the barycentric coordinates of every point $x \in P$ w.r.t. the vertices of $P$ is $\leq \frac{1}{d+1}$.

Consider the general case. By Carathéodory Theorem ([7, Thm. 1.53], [20, Prop. 1.15]) there exists a cover (and even a triangulation [7, Thm. 1.51]) $cP = \bigcup_j \Delta_j$ where each $\Delta_j$ is a $d$-simplex, spanned by vertices of $cP$. Then Lemma 2.1 is valid for the $\Delta_j, j \in J$, and, simultaneously,
\[
P = \bigcup_j \bigcup_{\text{vert}(\Delta_j)} (\Delta_j)_w \subset \bigcup_{\text{vert}(cP)} P_v,
\]
where $(\Delta_j)_w$ refers to the homothetic transformation of $\Delta_j$, with factor $\frac{c-1}{c}$ and centered at $w \in \text{vert}(\Delta_j)$.

The crucial definition below uses covers of $cP$ which are, in a sense, dual to those in Lemma 2.1.

Definition 2.2. Assume $d \in \mathbb{N}, k \in \mathbb{Q}_{\geq 2}$, and $P$ is a rational $d$-polytope.

$P$ is said to be $k$-convex-normal if the following condition is satisfied:
\[
(cP) \ni cP = \bigcup_{x \in (c-1)P \cap ((c-1)v + \mathbb{Z}^d)} x + P, \quad c \in [2, k]_{\mathbb{Q}}.
\]

(See Remarks 3.3 and 5.3.)

Here is a convenient equivalent reformulation. For $c \in \mathbb{Q}_{\geq 2}$ and $v \in \text{vert}(P)$ denote by $Q(v)$ the parallel translate of $(c-1)P$ that moves $(c-1)v$ to $cv$. Put $R(v, c) = \bigcup_{x \in Q(v) \cap (cv + \mathbb{Z}^d)} (x - v + P)$. Then $P$ is convex-normal iff
\[
(1) \quad cP = \bigcup_{\text{vert}(P)} R(v, c), \quad c \in [2, k]_{\mathbb{Q}}.
\]

Remark 2.3. (a) The ‘$\supset$’ part of the equality in Definition 2.2 is valid for any polytope $P$ and any $c \in \mathbb{Q}_{\geq 1}$.

(b) Unimodular simplices are not convex-normal, not even in dimension 2.
(c) The property $\text{CN}(d,k)$ is invariant under a linear transformation, corresponding to a matrix in $\text{SL}_d(\mathbb{Z})$, followed by a parallel translation by a rational vector.

(d) The reason why we do not consider $\text{CN}(d,\infty)$ in Definition 2.2 will be explained in Remark 5.3(b).

**Lemma 2.4.** (a) An $(d+1)$-convex-normal lattice $d$-polytope is integrally closed.

(b) The equality in Definition 2.2 holds true for $c \in \mathbb{Q}_{\geq 1}$ if $P$ is a rational parallelepiped $\boxtimes$ with $E(\boxtimes) \geq 1$.

(c) The equality in Definition 2.2 holds true for any rational $d$-polytope $P$ and any $c \in [1, \frac{d+1}{d}]$.

**Proof.** (a) Let $P$ be a lattice $d$-polytope. Then, trivially,
\[ v + \mathbb{Z}^d = cv + \mathbb{Z}^d = \mathbb{Z}^d, \quad v \in \text{vert}(P), \quad c \in \mathbb{N}. \]

Assume $P$ is a lattice $d$-polytope, satisfying $\text{CN}(d,d+1)$, and let $c \in [2,d]_{\mathbb{N}}$. Then, in view of (2), for every $z \in cP \cap \mathbb{Z}^d$ there exist $x \in (c-1)P \cap \mathbb{Z}^d$ and $x_c \in x + P$ with $z = x + x_c$. Then necessarily $x_c \in P \cap \mathbb{Z}^d$, and the descending induction from $c$ to $1$ implies $z = x_1 + \cdots + x_c$ with $x_1, \ldots, x_c \in P \cap \mathbb{Z}^d$.

Now assume $c \in \mathbb{N}_{\geq d+1}$. Then, by Lemma 2.1 and (2), for every $z \in cP \cap \mathbb{Z}^d$ there exist $x_c \in \text{vert}(P)$ and $x \in (c-1)P$ with $z = x + x_c$. Then necessarily $x_c \in (c-1)P \cap \mathbb{Z}^d$. So the descending induction from $c$ to $d+1$ brings us to the situation $c = d$.

(b) Let $c,l \in \mathbb{Q}_{\geq 1}$. Then we have
\[ \frac{cl}{2} - l \leq [(c-1)l]. \]

In fact, for $c \leq 2$ the l.h.s. of the inequality is non-positive and for $c \geq 2$ we have
\[ \frac{(c-2)l}{2} \leq \frac{(c-2)l}{2} + \frac{cl - 2}{2} = (c-1)l - 1 \leq [(c-1)l]. \]

Let $\boxtimes$ be a rational $d$-parallelepiped with edge lengths $\geq 1$. We can assume $\boxtimes \subset \mathbb{R}^d$. By applying a parallel translation that moves a vertex of $\boxtimes$ to 0, followed by an appropriate rational change of coordinates, and then taking the image of $\mathbb{Z}^d$ under the composite map as the lattice of reference, we can without loss of generality assume $\boxtimes = \{(y_1, \ldots, y_d) : 0 \leq y_i \leq l_i, \ i = 1, \ldots, d\}$ for some $l_1, \ldots, l_d \in \mathbb{Q}_{\geq 1}$.

Let $c \in \mathbb{Q}_{\geq 1}$. Then, by (3),
\[ c\boxtimes = \bigcup_{X_i \in \{A_i, B_i\}} X_1 \times \cdots \times X_d, \quad A_i = [0, [(c-1)l_i] + l_i], \]
\[ B_i = [cl_i - [(c-1)l_i] + l_i], \]
\[ i = 1, \ldots, d. \]

Because $l_i \geq 1, i = 1, \ldots, d$, for each of the boxes $B' = X_1 \times \cdots \times X_d$ above we have
\[ B' \subset \bigcup_{v \in \text{vert}(\boxtimes)} x + \boxtimes \quad x \in (c-1)\boxtimes \cap ((c-1)v + \mathbb{Z}^d) \]
(c) It is enough to show
\[ cP = \bigcup_{v \in \text{vert}((c-1)P)} v + P, \quad c \in \left[ 1, \frac{d+1}{d} \right]_Q. \]

But for the mentioned range of \( c \) the polytope \( P \) is a homothetic image of \( cP \) with factor \( \geq \frac{d}{d+1} \), and the argument with Carathéodory Theorem in the proof of Lemma 2.1 applies.

\[ \square \]

3. CN in dimension \( d-1 \Rightarrow \) Boundary CN in dimension \( d \)

**Definition 3.1.** Assume \( k \in \mathbb{Q}_{\geq 2} \) and \( P \subset \mathbb{R}^d \) is a rational \( d \)-polytope. \( P \) is said to be \( k \)-boundary-convex-normal if the following condition is satisfied:

\[ U_{cP}(cF, \varepsilon_F) \subset \bigcup_{v \in \text{vert}(P)} x + P, \quad c \in [2, k]_Q, \]

\[ (\text{BCN}(d, k)) \]

\[ F \in \mathbb{F}(P), \quad \varepsilon_F = \frac{\text{width}_F(P)}{d+1}. \]

**Lemma 3.2.** Let \( d \in \mathbb{N}_{\geq 2}, \, k \in \mathbb{Q}_{\geq 2}, \) and \( \lambda \in \mathbb{Q}_{>0} \). Assume every rational \((d-1)\)-polytope \( Q \) with \( E(Q) \geq \frac{d}{d+1} \lambda \) satisfies \( \text{CN} \left( d-1, k + \frac{k-1}{d} \right) \). Let \( P \) be a rational \( d \)-polytope with \( E(P) \geq \lambda \), \( w \in \text{vert}(P) \), and \( F \in \mathbb{F}(P)^w \). Then for the rational \( d \)-pyramid \( \Delta = \text{conv}(w, F) \) we have

\[ U_{c\Delta}(cF, \varepsilon) \subset \bigcup_{v \in \text{vert}(P)} x + P, \quad c \in [2, k]_Q, \]

\[ \varepsilon = \frac{\|w, H_F\|}{d+1}. \]

**Proof.** We can assume \( P \subset \mathbb{R}^d \). Denote:

\[ \Sigma = \bigcup_{v \in \text{vert}(P)} x + P, \quad x \in (c-1)P \cap ((c-1)v + \mathbb{Z}^d) \]

\[ \Pi = \bigcup_{v \in \text{vert}(F)} x + U_{\Delta}(F, \varepsilon). \]

Since \( \Pi \subset \Sigma \), it is enough to show

\[ (4) \quad U_{c\Delta}(cF, \varepsilon) \subset \Pi. \]

Let \( G = U_{\Delta}(F, \varepsilon)^+ \in \mathbb{F}(U_{\Delta}(F, \varepsilon)) \). Then \( G \) is a homothetic image of \( F \) with factor \( d/(d+1) \). In particular, \( G \) is a rational \((d-1)\)-polytope whose every edge has lattice length \( \geq \frac{d}{d+1} \lambda \). By the assumption in the theorem, \( G \) satisfies \( \text{CN} \left( d-1, k + \frac{k-1}{d} \right) \).
The rational polytope $K = U_{c\Delta}(cF, \varepsilon)^+$ is a homothetic image of $F$ with factor $\frac{cd + c - 1}{d+1}$. So $K$ is a homothetic image of $G$ with factor

$$c_1 = \frac{cd + c - 1}{d+1}, \frac{d+1}{d} = c + \frac{c-1}{d} \in \left[2 + \frac{k-1}{d}, k + \frac{k-1}{d}\right]_Q.$$ 

The polytope $(c-1)F$ is a rational homothetic image of $G$ with factor $c_2 = d^{-1}(d+1)(c-1)$. Since $c_1 - 1 = c_2$, in view of (I), we have

$$\bigcup_{v \in \text{vert}(F)} y + G = K,$$

or, equivalently, $K \subset \Pi$. To put in other words, the lid of the truncated pyramid $U_{c\Delta}(cF, \varepsilon)$ is covered by the relevant parallel translates of the lid of the smaller truncated pyramid $U_{\Delta}(F, \varepsilon)$.

Pick a point $z \in U_{c\Delta}(cF, \varepsilon)$. The ray $cw + \mathbb{R}_+ (z - cw)$ intersects $K$ at some point $z_K$. Let $z_K \in x + G$ for some $x$ as in the index set in the definition of $\Pi$.

The pyramid $(z_K + \mathbb{R}_+ (-w + \Delta)) \cap U_{c\Delta}(cF, \varepsilon)$ contains $z$ and, simultaneously, is a homothetic image of $\Delta$. In particular, it has ‘side’ facets parallel to the corresponding side facets of the truncated pyramid $x + U_{\Delta}(F, \varepsilon)$. Consequently,

$$z \in (z_K + \mathbb{R}_+ (-w + \Delta)) \cap U_{c\Delta}(cF, \varepsilon) \subset x + U_{\Delta}(F, \varepsilon) \subset \Pi.$$

\[\square\]

**Remark 3.3.** (a) In the proof of Lemma 3.2 there are two places that make it necessary to involve rational polytopes in Definition 2.2: the polytope $G$, to which the assumption on $(d-1)$-polytopes is applied, is usually not lattice, and the number $c_1$ is usually not an integer.

(b) If one defined the convex normality by the ‘dual’ equalities

$$cP = \bigcup_{v \in \text{vert}(P)} x + (c-1)P, \quad c \in [2, k]_Q,$$

then the lower bound for the analogue of $c_1$ in the proof of Lemma 3.2 would have been $2 - \frac{1}{d+1}$, blocking the possibility for induction on $d$.

**Lemma 3.4.** Let $d \in \mathbb{N}_{\geq 2}$, $k \in \mathbb{Q}_{\geq 2}$, and $\lambda \in \mathbb{Q}_{>0}$. If every rational $(d-1)$-polytope $Q$ with $E(Q) \geq \frac{d}{d+1} \lambda$ satisfies $\text{CN}(d-1, k + \frac{k-1}{d})$ then every rational $d$-polytope $P$ with $E(P) \geq \lambda$ satisfies $\text{BCN}(d, k)$.

**Proof.** Let $P$ be a rational $d$-polytope with edge lengths $\geq \lambda$, $F \in \mathbb{F}(P)$, and

$$\varepsilon_F = \frac{\text{width}_F(P)}{d+1}.$$

Fix a vertex $w \in \text{vert}(P) \setminus F$ with $\|w, H_F\| = \text{width}_F(P)$.

For every facet $G \in \mathbb{F}(P)^w$ denote

$$\Delta(G) = \text{conv}(w, G), \quad \varepsilon_{w, G} = \frac{\|w, H_G\|}{d+1}.$$
By Lemma 3.2 we have the inclusions
\[ \bigcup_{G \in \mathcal{F}} U_{c\Delta(G)}(cG, \varepsilon_{w,G}) \subseteq \bigcup_{v \in \text{vert}(P)} x + P, \quad c \in [2, k]_\mathbb{Q}. \]

But for every \( c \in [2, k]_\mathbb{Q} \) we also have
\[ U_{cP}(cF, \varepsilon_{w,F}) = U_{cP}(cF, \varepsilon_{F}) \subset P \setminus H(cP) = \bigcup_{G \in \mathcal{F}} U_{c\Delta(G)}(cG, \varepsilon_{w,G}), \]

where \( H(cP) \) denotes the homothetic image of \( cP \), centered at \( cw \) and with factor \( \frac{cd + c - 1}{cd + c} \). (The inclusion in the middle, essentially, amounts to convexity of \( cP \).)

4. Deep parallelepipedal covers from vertices

4.1. Gauging \( \Delta_d \). The standard unimodular rectangular \( d \)-simplex \( \Delta_d \) is the convex hull of \( \{0, e_1, \ldots, e_d\} \), where \( e_i \) is the \( i \)th standard basic vector in \( \mathbb{R}^d \).

Lemma 4.1. Let \( H \subset \mathbb{R}^d \) be an affine \( d \)-hyperplane, not intersecting the interior of \( \Delta_d \). Then
\[ \max \left( \|v, H\| : v \in \text{vert}(\Delta_d) \right) \geq \frac{1}{\sqrt{d}}. \]

Proof. Let \( H \) be given by an equation
\[ a_1X_1 + \cdots + a_dX_d + b = 0, \quad a_1^2 + \cdots + a_d^2 = 1. \]

We write \( H = H(a_1X_1 + \cdots + a_dX_d + b) \).

The standard point-hyperplane distance formula in our case becomes
\[ \|v, H\| = |a_1v_1 + \cdots + a_dv_d + b|, \quad v = (v_1, \ldots, v_d). \]

Without loss of generality we can assume \( |a_1| \leq \ldots \leq |a_d| \).

The condition \( \text{int}(\Delta_d) \cap H = \emptyset \) is equivalent to the condition that the numbers \( a_1 + b, \ldots, a_d + b \) are all either nonnegative or nonpositive. In particular,
\[ b \neq 0 \quad \Rightarrow \quad |a_1| \leq \ldots \leq |a_d| \leq |b|. \]

We have
\[ \|e_1, H\| = |a_1 + b|, \ldots, \|e_d, H\| = |a_d + b|, \|0, H\| = |b|. \]

For every \( d \)-tuple
\[ \bar{\alpha} = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d, \quad \alpha_1^2 + \cdots + \alpha_d^2 = 1, \]
the function
\[ \phi(\bar{\alpha}) : \mathbb{R} \to \mathbb{R}_+, \quad \beta \mapsto \max \left( \|v, H(\bar{\alpha}, \beta)\| : v \in \text{vert}(\Delta_d) \right), \]
\[ H(\bar{\alpha}, \beta) = H(\alpha_1X_1 + \cdots + \alpha_dX_d + \beta), \]
attains its minimum at a \( \beta \) for which \( H(\bar{\alpha}, \beta) \) contains a vertex of \( \Delta_d \): slide the hyperplane \( H \) towards \( \Delta_d \) until it hits the simplex! Therefore, and in view of (5), there is no loss of generality in assuming that either \( b = 0 \), corresponding to \( 0 \in H \),
or $0 \neq b = -a_i$, corresponding to $e_i \in H$ for some $i$ and $0 \notin H$. In the latter case
the inequalities (5) imply $|a_i| = \cdots = |a_d| = |b|$

By (6) we only need to show the inequalities:

(i) $|a_d| \geq \frac{1}{\sqrt{d}}$

(ii) $\max(|a_1 - a_i|, \ldots, |a_{i-1} - a_i|, |a_{i+1} - a_i|, \ldots, |a_d - a_i|, |a_d|) \geq \frac{1}{\sqrt{d}}$

or, equivalently, just (i). We have the optimization problem: minimize $a_d$ subject to

$0 \leq a_1 \leq \ldots \leq a_{d-1} \leq a_d, \quad a_1^2 + \cdots + a_{d-1}^2 + a_d^2 = 1.$

This is equivalent to the linear optimization problem: minimize $a'_d = a_d^2$, subject to

$0 \leq a'_1 \leq \ldots \leq a'_{d-1} \leq a'_d, \quad a'_1 + \cdots + a'_{d-1} + a'_d = 1.$

The answer is $a'_d = 1/d$. \hfill \qed

4.2. Gauging corner parallelepipedal covers. For a rational simplicial cone $C$
with extremal generators $x_1, \ldots, x_d$, $d = \dim C$, we let $\Box(C)$ denote the parallelepiped
$
\left\{ \sum_{i=1}^d \lambda_i x_i : 0 \leq \lambda_i \leq 1 \right\}.$

Let $P \subset \mathbb{R}^d$ be a rational $d$-polytope, $v \in \text{vert}(P)$, and $C$ a simplicial cone that
is spanned by a system of vectors of the form

$v_1 - v, \ldots, v_d - v, \quad v_1, \ldots, v_d \in \text{vert}(P),
\quad \text{rank}_\mathbb{Q}(v_1 - v, \ldots, v_d - v) = d.$

Let $x_i$ be the primitive integer vector in the direction of $v_i - v$, $i = 1, \ldots, d$. Denote by $P(v, C)$ the union of all parallelepipeds of the form

$v + \sum_{i=1}^d a_i x_i + \Box(C) \subset P, \quad a_1, \ldots, a_d \in \mathbb{Z}_+.$

Fix $l \in \mathbb{Q}_{\geq 1}$ and denote

$\bar{\varepsilon} = (\varepsilon_{\mathbb{F}(P)}) \bar{\varepsilon}(P), \quad \varepsilon_{\mathbb{F}} = \frac{\text{width}_{\mathbb{F}}(P)}{l(d+1)}, \quad P(v, \bar{\varepsilon}) = P \setminus \bigcup_{F \in \mathbb{F}(P)^{\lor}} \mathbb{U}(F, \varepsilon_{\mathbb{F}}),$ 

where the ‘bar’ refers to the closure in the Euclidean topology.

Lemma 4.2. If $E(P) \geq l \dim P(d + P + 1)$ then

$P(v, \bar{\varepsilon}) \cap C \subset P(v, C).$

Proof. By shifting $P$ by $-v$ we can assume $v = 0$.

Let $x \in P(0, \bar{\varepsilon}) \cap C$. There is a parallelepiped of the form

$\Box = \sum_{j=1}^d a_j x_j + \Box(C), \quad a_j \in \mathbb{Z}_+,$

containing $x$. We want to show $\Box \subset P$. 


By applying a linear transformation to $\mathbb{R}^d$, we can make $\square$ into the standard unit $d$-cube $\Box_d = \left\{ \sum_{i=1}^d \lambda_i e_i : 0 \leq \lambda_i \leq 1, i = 1, \ldots, d \right\}$. (Usually, such a transformation does not respect the lattice $\mathbb{Z}^d$.) For simplicity we will use the same notation $P$ and $C$ for the corresponding images under the linear transformation. Then the inclusion $\square \subset P$ follows if we show
\[ \|x, H_{F'}\| \geq \text{diam}(\square_d) = \sqrt{d}, \quad F' \in \mathbb{F}(P \cap C)^0. \]

Every facet $F' \in \mathbb{F}(P \cap C)^0$ is contained in some facet $F \in \mathbb{F}(P)^0$. So it is enough to show
\[ \varepsilon_F \geq \sqrt{d}, \quad F \in \mathbb{F}(P)^0. \]

Pick $F \in \mathbb{F}(P)^0$. The function \[ \|x, H_F\| : \text{conv}(0, v_1, \ldots, v_d) \to \mathbb{R}_+ \]
is maximized at a vertex of the rectangular simplex $\text{conv}(0, v_1, \ldots, v_d)$. The latter contains $l(d^2 + d)\Delta_d$ because $E(P) \geq l(d^2 + d)$ and the segments $[0, v_i], i = 1, \ldots, d,$ are all edges of $P$. So (7) follows, because by Lemma 4.1 we have
\[ l(d+1)\varepsilon_F = \text{width}_F(P) \geq \max_{w \in \{0, v_1, \ldots, v_d\}} \|w, H_F\| \geq \frac{l(d^2 + d)}{\sqrt{d}}. \]

**Corollary 4.3.** In the notation of Lemma 4.2, if $0 \in \text{vert}(P)$ then the union of all lattice parallelepipeds inside $P$ contains $P(0, \varepsilon)$.

**Proof.** This follow from Lemma 4.2 and the existence of a cover of the form $\mathbb{R}_+P = \bigcup_j C_j$, where the $C_j, j \in J$, are simplicial $d$-cones, spanned by extremal generators of $\mathbb{R}_+P$ – Carathéodory Theorem for cones; see [7, Thm. 1.55]. One can even choose the cover to be a triangulation of $\mathbb{R}_+P$; see [7, Thm. 1.54], [20, Prop. 1.15(i)]. \qed

**5. Recursion rules for CN**

Let $d \in \mathbb{N}, k \in \mathbb{Q}_{\geq 2}$, and $P$ denote a general rational $d$-polytope. Define:
\[ \text{cn}(d, k) = \inf \{ l \in \mathbb{Q} \mid E(P) \geq l \implies P \text{ satisfies CN}(d, k) \}; \]
\[ \text{bcn}(d, k) = \inf \{ l \in \mathbb{Q} \mid E(P) \geq l \implies P \text{ satisfies BCN}(d, k) \}. \]

It is not a priori clear that these are finite numbers. What makes them finite and, in fact, the whole strategy work is the following recursion rules:

**Lemma 5.1.** For $d \in \mathbb{N}_{\geq 2}$ and $k \in \mathbb{Q}_{\geq 2}$ we have:

(a) \[ \text{cn}(1, k) \leq 1. \]

(b) \[ \text{bcn}(d, k) \leq \frac{d+1}{d} \text{cn} \left( d-1, k + \frac{k-1}{d} \right), \]

(c) \[ \text{cn}(d, k) \leq \max \left( kd(d+1), \text{bcn}(d, k) \right). \]
Proof. (a) This follows from the one-dimensional case of Lemma 2.4(b).

(b) This follows from Lemma 3.4.

Notice. It is easily shown that $\frac{k - 2}{2(k - 1)} \leq cn(1, k)$ for $k \geq 2$ and $cn(1, 2) = 0$. Finding the exact value of $cn(1, k)$ seems an interesting question.

(c) Let $P \subset \mathbb{R}^d$ be a rational $d$-polytope with $E(P) > \max \left(kd(d + 1), bcn(d, k)\right)$. We want to show that $P$ satisfies $CN(d, k)$.

Pick $v \in \text{vert}(P)$. Applying the parallel translation by $-v$, there is no loss of generality in assuming $v = 0$; see Remark 2.3(c).

Fix a cover of the form $\mathbb{R}_+ P = \bigcup_{J} C_j$, where the $C_j$, $j \in J$, are simplicial $d$-cones, spanned by extremal rays of $\mathbb{R}_+ P$; see the proof of Corollary 4.3.

Assume $c \in [2, k]_{\mathbb{Q}}$. Because $c - 1 \geq 1$ we have $E\left((c - 1)P\right) \geq E(P) > kd(d + 1)$ and by (twofold application of) Lemma 4.2, for every $j \in J$ we have the inclusions:

\begin{align}
P(0, \bar{\varepsilon}) \cap C_j & \subset P(0, C_j), \quad (c - 1)P(0, \bar{\varepsilon}) \cap C_j \subset (c - 1)P(0, C_j), \\
(8) \quad P(0, \bar{\varepsilon}) \cap C_j & \subset (c - 1)P(0, \bar{\varepsilon}) \cap C_j \subset (c - 1)P(0, C_j).
\end{align}

notation as in Lemma 4.2 with

$$\bar{\varepsilon} = (\varepsilon_F)_{F \in \mathbb{F}(P)}, \quad \varepsilon_F = \frac{\text{width}_F(P)}{k(d + 1)}, \quad F \in \mathbb{F}(P).$$

Let $(c - 1)\bar{\varepsilon} = ((c - 1)\varepsilon_F), \quad F \in \mathbb{F}(P)$. Because $c - 1 \geq 1$, we have

$$(c - 1)P(0, (c - 1)\bar{\varepsilon}) \subset (c - 1)P(0, \bar{\varepsilon}),$$

which, together with the second inclusion in (8), gives

$$\quad (c - 1)P(0, (c - 1)\bar{\varepsilon}) \cap C_j \subset (c - 1)P(0, C_j).$$

For every $j \in J$ we also have

$$(c - 1)P(0, C_j) + P(0, C_j) \subset$$

$$\bigcup_{(\square_1, \square_2) \in A \times B} \square_1 + \square_2 = \bigcup_{\square_1 \in A, \square_2 \in B} \bigcup_{x \in \text{vert}(\square_1)} x + \square_2 =$$

$$\bigcup_{\square \in A, x \in \text{vert}(\square)} x + P(0, C_j) \subset \bigcup_{x \in ((c - 1)P(0, C_j) \cap \mathbb{Z}^d)} x + P(0, C_j) \subset$$

$$\bigcup_{x \in ((c - 1)P(0, C_j) \cap \mathbb{Z}^d)} x + P \cap C_j,$$

(10)
where $A$ and $B$ are, respectively, the sets of congruent parallelepipeds of the form
\[
\sum_{i=1}^{d} a_i x_{ji} + \Box(C_j) \subset (c - 1)P(0, C_j), \quad \sum_{i=1}^{d} a_i x_{ji} + \Box(C_j) \subset P(0, C_j),
\]
where $x_{j1}, \ldots, x_{jd} -$ the extremal generators of $C_j$,
\[a_1, \ldots, a_d \in \mathbb{Z}_+.
\]

On the other hand for every $j \in J$ the following equality (in self-explanatory notation) holds true for reasons of homothety w.r.t. 0:
\begin{equation}
P(0, \bar{\varepsilon}) \cap C_j + (c - 1)P(0, (c - 1)\bar{\varepsilon}) \cap C_j = cP(0, c\bar{\varepsilon}) \cap C_j.
\end{equation}
Then, integrating over $j \in J$, (8) – the first inclusion, (9), (10), and (11) imply
\begin{equation}
cP(0, c\bar{\varepsilon}) \subset \bigcup_{x \in (c-1)P \cap \mathbb{Z}_d^d} x + P.
\end{equation}

For every $F \in \mathcal{F}(P)^0$ we have $c_F \leq \frac{\text{width}_F(P)}{d+1}$. Therefore,
\begin{equation}
cP(0, \bar{\sigma}) \subset cP(0, c\bar{\varepsilon}), \quad \bar{\sigma} = (\sigma_F), \quad \sigma_F = \frac{\text{width}_F(P)}{d+1}, \quad F \in \mathcal{F}(P)^0.
\end{equation}

Because $\mathcal{E}(P) > \text{bcn}(d, k)$, (12) and (13) together imply $\text{CN}(d, k)$ for $P$.

**Corollary 5.2.** (a) For all $d \in \mathbb{N}_{\geq 2}$ and $k \in \mathbb{Q}_{\geq 2}$ we have
\[
\text{cn}(d, k) \leq \max \left( kd(d+1), \frac{d+1}{d} \left\lfloor \left( d - 1, k + \frac{k-1}{d} \right) \right\rfloor \right).
\]

(b) For all $d \in \mathbb{N}$ and $k \in \mathbb{Q}_{\geq 2}$ we have $\text{cn}(d, k) < \infty$.

In fact, (a) follows from Lemma 5.1(b,c) and (b) follows from (a) and Lemma 5.1(a).

**Remark 5.3.** (a) In the proof above we used twice that $c - 1 \geq 1$. This explains why in Definition 1.1 we choose $k \geq 2$ and $c \in [2, k]_\mathbb{Q}$, and not $k \geq 1$ and $c \in [1, k]_\mathbb{Q}$.

(b) The general ‘non-homothetic’ version of (11) in the form of the equation
\[
P(0, \bar{\rho}) \cap C + Q(0, \bar{\sigma}) \cap C = (P + Q)(0, \bar{\rho} + \bar{\sigma}) \cap C,
\]
where $P$ and $Q$ are polytopes, sharing the corner cone $C$ at 0, and
\[
\bar{\rho} = (\rho, \ldots, \rho), \quad \bar{\sigma} = (\sigma, \ldots, \sigma), \quad \bar{\rho} + \bar{\sigma} = (\rho + \sigma, \ldots, \rho + \sigma), \quad \rho, \sigma > 0,
\]
is not possible, not even in dimension 2. Even more is true: one easily finds polygonal examples of the form
\[
(P + Q)(0, \bar{\tau}) \cap C \nsubseteq P(0, \bar{\rho}) \cap C + Q(0, \bar{\sigma}) \cap C,
\]
\[
\bar{\tau} = (\tau, \ldots, \tau), \quad \tau \gg \rho + \sigma.
\]

This observation explains the need of some sort of control on the $\bar{\varepsilon}$-cut-off facet layers in the Minkowski sums. Such is provided by the presence of homothety in (11).
The drawback of the homothetic control is that we need the original cut-off layers to be ‘$k$-thin’, compared to the facet layers mentioned in BCN$(d,k)$ (Definition 3.1). The net effect of this is the appearance of the factor $k$ in the statement of Theorem 1.2. But the effect is essentially cancelled by an arithmetic shortcut in the proof of Theorem 1.3 in Section 6.2 below. As noticed there, without such a shortcut we would still get a dimensionally uniform bound for the property of integrally closed, but $O(d^4)$ instead of $O(d^3)$.

6. Proofs

6.1. Proof of Theorem 1.2. The limit case will be taken care off by

**Lemma 6.1.** Let $d \in \mathbb{N}$ and $k \in \mathbb{Q}_{\geq 2}$. If $P$ is a rational $d$-polytope with $E(P) \geq \mathfrak{c}(k,d)$ then $P$ satisfies $\text{CN}(d,k)$.

**Proof.** We can assume $P \subset \mathbb{R}^d$. For every $\varepsilon \in \mathbb{Q}_{> 0}$ the polytope $(1 + \varepsilon)P$ satisfies $\text{CN}(d,k)$. On the one hand, for all $c \in [2,k]_{\mathbb{Q}}$ and all sufficiently small $\varepsilon \in \mathbb{Q}_{> 0}$, depending on $k$ and $P$, the sets

$$\bigcup_{v \in \text{vert}((1 + \varepsilon)P)} ((c - 1)(1 + \varepsilon)v + \mathbb{Z}^d) \subset \mathbb{R}^d$$

are, respectively, the parallel translates by $\varepsilon(c - 1)v$ of the fixed set

$$\bigcup_{v \in \text{vert}(P)} ((c - 1)v + \mathbb{Z}^d) \subset \mathbb{R}^d.$$

On the other hand, $(1 + \varepsilon)P \to P$ as $\varepsilon \to 0$. Consequently, for every number $c \in [2,k]_{\mathbb{Q}}$, the complement

$$X \setminus Y, \quad X = cP, \quad Y = \bigcup_{v \in \text{vert}(P)} x + P$$

$$x \in (c - 1)P \cap ((c - 1)v + \mathbb{Z}^d) \subset \mathbb{R}^d$$

has Lebesgue $d$-measure 0. In view of the facts that $X$ and $Y$ are both closed in $\mathbb{R}^d$ and $Y \subset X$, we have the desired equality $X = Y$. $\square$

Now we turn to Theorem 1.2 proper. By Corollary 5.2(b) the function $\mathfrak{c}(d,k) : \mathbb{N} \times \mathbb{Q}_{\geq 2} \to \mathbb{R}_+$ is well defined. For any fixed $d \in \mathbb{N}$ the function $\mathfrak{c}(d,k) : \mathbb{Q}_{\geq 2} \to \mathbb{R}_+$ is non-decreasing. So by Corollary 5.2(a) we have the (simpler) inequalities:

$$\mathfrak{c}(d,k) \leq \max\left( d(d + 1)k, \frac{d + 1}{d} \mathfrak{c}(d - 1, \frac{(d + 1)k}{d}) \right), \quad d \in \mathbb{N}_2, \quad k \in \mathbb{Q}_{\geq 2}.$$

By induction on $i$ from 1 to $d - 1$, the latter imply

$$\mathfrak{c}(d,k) \leq \max\left( \left\{ \frac{d^2(d + 1)k}{d + 1 - j} \right\}_{j=1,\ldots,i}, \frac{d + 1}{d + 1 - i} \mathfrak{c}(d - i, \frac{(d + 1)k}{d + 1 - i}) \right),$$

$$i = 1, \ldots, d - 1.$$
Therefore,
\[
\mathfrak{c}_n(d, k) \leq \max \left( \left\{ \frac{d^2(d + 1)k}{d + 1 - j} \right\} \right)_{j=1}^{d-1} \cdot 0.5(d + 1) \mathfrak{c}_n \left( 1, \frac{(d + 1)k}{d + 1 - i} \right) \leq \max \left( 0.5d^2(d + 1)k, \ 0.5(d + 1) \right) = 0.5d^2(d + 1)k.
\]

This already proves the version of Theorem 1.2 with the strict inequality $E(P) > 0.5d^2(d + 1)k$, and the non-strict inequality is covered by Lemma 6.1.  \[\square\]

6.2. **Proof of Theorem 1.3(a).** Notice, Theorem 1.2 and Lemma 2.4(a) already imply the weaker version of Theorem 1.3(a) with the inequality $E(P) \geq 0.5d^2(d+1)^2$. But Theorem 1.2 can be used in a more efficient way. All we need is Lemma 6.2.  \[\Box\]

**Lemma 6.2.** Every lattice $d$-polytope $P$ with $E(P) \geq \mathfrak{c}_n(d, 4)$ is integrally closed.

**Proof.** Let $P \subset \mathbb{R}^d$ be as in the lemma. We show the equality in Definition 1.1(a) by induction on $c \in \mathbb{N}$. Assume it has been shown for all natural factors $< c$.

Denote
\[
I_n = [2^n, 2^{n+1}]_\mathbb{N}, \quad P_n = 2^{n-1}P, \quad L_n = 2^{n-1}\mathbb{Z}^d \subset \mathbb{Z}^d, \quad n \in \mathbb{N}.
\]

Then $P_n$ is a rational polytope with $E_n(P) \geq \mathfrak{c}_n(d, 4)$, where the subindex $n$ indicates that the lattice of reference in $L_n$.

Let $c \in I_n$ for some $n \in \mathbb{N}$, and pick $z \in cP \cap \mathbb{Z}^d$. We have
\[
cP = \begin{cases} 
  c'P_n \text{ with } c' = c2^{-n+1} \in [2, 4]_\mathbb{Q} \text{ if } n > 1, \\
  c \in [2, 4]_\mathbb{Q} \text{ if } n = 1.
\end{cases}
\]

If $n > 1$ then $P_n$ satisfies $\text{CN}(d, 4)$ w.r.t. the lattice $L_n$ (Lemma 5.1). So $z = x + y$ for some $x \in (c' - 1)P_n \cap (c' - 1)v + L_n$, $v \in \text{vert} P_n$, and $y \in P_n$. Then, necessarily, $y \in P_n \cap \mathbb{Z}^d$. In particular, $((c - 2^{n-1})P) \cap \mathbb{Z}^d + (2^{n-1}P) \cap \mathbb{Z}^d = cP \cap \mathbb{Z}^d$.

If $n = 1$ then we have $z \in ((c - 1)P) \cap \mathbb{Z}^d + P$ (Lemma 5.1) and, hence, $((c - 1)P) \cap \mathbb{Z}^d + P \cap \mathbb{Z}^d = cP \cap \mathbb{Z}^d$.

In both cases the induction assumption applies.  \[\square\]

6.3. **Proof of Theorem 1.3(b).** Lattice parallelepipeds are integrally closed – a consequence of Lemma 2.4(b). Therefore, we only need to show that $P$ is covered by lattice parallelepipeds. In view of Corollary 4.3 it is enough to show
\[
\bigcup_{v \in \text{vert}(P)} P(v, \varepsilon) = P, \quad \varepsilon = \frac{\text{width}_F(P)}{d + 1}.
\]

(Notation as in that corollary.) Since $P$ is a simplex, for every vertex $v \in \text{vert}(P)$ the polytope $P(v, \varepsilon)$ is the homothetic image of $P$ with factor $\frac{d}{d+1}$ and centered at $v$. But such covers were considered in the proof of Lemma 2.1.  \[\square\]

**Notice.** The results in this paper extend to all polytopes whose edges are parallel to rational directions and all real factors $\geq 2$. 

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