On the generation of random ensembles of qubits and qutrits

Computing separability probabilities for fixed rank states

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Abstract. The question of the generation of random mixed states is discussed, aiming for the computation of probabilistic characteristics of composite finite dimensional quantum systems. Particularly, we consider the generation of the random Hilbert-Schmidt and Bures ensembles of qubit and qutrit pairs and compute the corresponding probabilities to find a separable state among the states of a fixed rank.

1 Introduction

The presentation addresses a special topic of quantum theory of finite dimensional systems that is of prime importance in the theory of quantum information and quantum communications. Namely, we study the problem of calculation of the probability for a random state of a binary composite quantum system, such as qubit-qubit or qubit-qutrit pairs, to be a separable or an entangled state. According to the “Geometric Probability Theory”, this «separability/entanglement probability», is determined by the relative volume of separable states with respect to the volume of the whole state space. In order to avoid subtle numerical calculation of the multidimensional integrals, (for a generic 2-qubit case the integrals are over the semi-algebraic domain of 15-dimensional Euclidean space), the Monte-Carlo ideology with a specific method of generation of random variables has been used (see e.g.. [4]–[7], and references therein). Our report aims to present several results on the numeric studies of separability probabilities for random qubit-qubit and qubit-qutrit Hilbert-Schmidt and Bures ensembles. Particularly, the distribution properties of the “probability of entanglement” will be described with respect to the subsystem’s Bloch vectors for the qubit-qubit and qubit-qutrit random Hilbert-Schmidt and Bures states of all possible ranks.

2 The separability probability

There are two ways to analyze the probabilistic aspects of entanglement characteristics, such as the so-called separability probability, $P_{\text{sep}}$. The latter is the probability to find a separable state among...
all possible states. The first approach is to consider the state space of a quantum system as the Riemannian space and, following the strategy of the theory of geometric probability, identify the separability probability with with a two volumes ratio:

\[
P_{\text{sep}} = \frac{\text{Vol}(\text{Separable states})}{\text{Vol}(\text{All states})}.
\]  

(1)

The second approach consists of dealing with the ensemble of random states, whose distribution is in correspondence with the volume measure in \([1]\). In this case the separability probability can be equivalently written as:

\[
P_{\text{sep}} = \frac{\text{Vol}(\text{Number of separable states in ensemble})}{\text{Vol}(\text{Total number of states in ensemble})}.
\]  

(2)

The first method is suitable for analytic calculations, however it presents a big computational issue (see e.g., \([2]\) and references therein). The second method allows us to use highly effective modern numerical computational techniques. Below we will give results on the separability probability of the qubit-qubit and qubit-qutrit systems obtained within the latter approach.

### 3 Generating the Hilbert-Schmidt and Bures random states

Having in mind calculations of the separability probability of states with different ranks we shortly present the algorithm for the generation of density matrices from the Hilbert-Schmidt and the Bures ensembles of a \(n\)–level quantum system.

- The first step in the construction of both ensembles is the generation of complex \(n \times n\) matrices \(Z\) from the Ginibre ensemble, i.e., the matrices whose entries have real and imaginary parts distributed as normal random variables;
- Density matrices \(\varrho_{\text{HS}}\) describing states from the Hilbert-Schmidt ensemble are:

\[
\varrho_{\text{HS}} = \frac{ZZ^+}{\text{tr}(ZZ^+)};
\]  

(3)

- Density matrices \(\varrho_{\text{B}}\) from the Bures ensemble are generated with the aid of the Ginibre matrices \(Z\) and matrices \(U \in SU(n)\), distributed over the \(SU(n)\) group according to the Haar measure,

\[
\varrho_{\text{B}} = \frac{(I + U)ZZ^+ (I + U^+)}{\text{tr}((I + U)ZZ^+ (I + U^+))}.
\]  

(4)

Since we are interested in studying the entanglement of states with a fixed rank of density matrices, the above algorithm requires a certain specification. For states of a lower rank than the maximal one we proceed as follows.

- **Rank-3 states** We start with the generation of complex, rank-3 Ginibre matrices. It is known that any such matrix admits the following representation:

\[
Z = P_Z \begin{pmatrix}
A & x_1 \\
y_1 & x_2 \\
y_2 & x_3 \\
y_3 & D
\end{pmatrix} Q_Z,
\]  

(5)
Table 1: The separability probability for the maximal rank-4 states in qubit-qubit and qubit-qutrit systems

| States       | System       | Separable |
|--------------|--------------|-----------|
| HS ensemble  | $2 \otimes 2$ | 0.2424    |
|              | $2 \otimes 3$ | 0.0270    |
| Bure ensemble| $2 \otimes 2$ | 0.0733    |
|              | $2 \otimes 3$ | 0.0014    |

Table 2: The separability probability for non-maximal rank qubit-qubit states

| States       | Rank | Separable |
|--------------|------|-----------|
| HS ensemble  | 3    | 0.1652    |
|              | 2    | 0         |
|              | 1    | 0         |
| Bure ensemble| 3    | 0.0494    |
|              | 2    | 0         |
|              | 1    | 0         |

In (5) permutations of entries by $P_Z$ and $Q_Z$ are such that $A$ is a regular element of the Ginibre ensemble of $3 \times 3$ matrices. Apart from that, if 3-tuples $Y = (y_1, y_2, y_3)$ and $X = (x_1, x_2, x_3)$ are composed of the normal random complex variables, while the entry $D$ is

$$D = Y A^{-1} X,$$

we arrive at a rank-3 random matrix $Z$ from the Ginibre ensemble of complex $4 \times 4$ matrices.

- **Rank-2 states**: Similarly, the $4 \times 4$ Ginibre matrix of rank-2 can be written as,

$$Z = P_Z \begin{pmatrix} A & B \\ C & D \end{pmatrix} Q_Z,$$

where $A, B$ and $C$ are $2 \times 2$ complex Ginibre matrices, while $2 \times 2$ matrix $D$ reads, $D = C A^{-1} B$.

- **Rank-1 states**: Finally, a $4 \times 4$ complex Ginibre matrix $Z$ of rank-1 admits the representation:

$$Z = P_Z \begin{pmatrix} a & y_1 & y_2 & y_3 \\ x_1 & y_1 & y_2 & y_3 \\ x_2 & y_2 & y_3 \\ x_3 & y_3 \end{pmatrix} Q_Z,$$

where $a, x_1, x_2, x_3$ and $y_1, y_2, y_3$ are normal random complex variables, while $3 \times 3$ matrix $D$ is:

$$D = \frac{1}{a} \begin{pmatrix} x_1 y_1 & x_1 y_2 & x_1 y_3 \\ x_2 y_1 & x_2 y_2 & x_2 y_3 \\ x_3 y_1 & x_3 y_2 & x_3 y_3 \end{pmatrix}.$$

Now, using the generic Ginibre matrix and representations (5)–(7) for non-maximal rank matrices we can build either the Hilbert-Schmidt or Bures states of the required rank according to either (3) or (4).

4 Computing the separability probability

Computation of the sought-for separability probability (2) requires the test of generated states on their separability. With this goal the Peres-Horodecki criterion [1] has been used in our calculations.

4.1 Results of the computations

First of all, in table 1 we collect results of the calculations of the separability probability of the generic (rank-4) states from the Hilbert-Schmidt and Bures ensembles of qubit-qubit and qubit-qutrit pairs.
Furthermore, results of the computations of the separability probabilities of non-maximal rank states of a qubit-qubit system, for both ensembles, are given in table 2. Here, it is worth making a comment on the zero separability probability for rank-2 and rank-1 states. That can be understood noting the following: If a density matrix of 2-qubits $\rho$ is such that $\text{rank}(\rho) < d_A = \text{rank}(\rho_A)$, then $\rho$ is not separable [8]. Here, $\rho_A$ denotes the reduced density matrix obtained by taking the partial trace with respect to the second qubit. Indeed, since during the generation of rank-2 and rank-1, almost all reduced matrices are not singular, the above mentioned inequality is true.

Figure 1: The separability probability for the Hilbert-Schmidt qubit-qubit states as a function of the first qubit’s Bloch vector.

5 Concluding remarks

Having the above described algorithms as a tool for the generation of ensembles of random states with different ranks, the diverse characteristics of entanglement in composite quantum systems can be studied. As an example, on figures 1a and 1b we illustrate the separability probability in 2-qubit systems as a function of the Bloch radius of the constituent qubit for the states of maximal and sub-maximal ranks. Our analysis shows that the distribution of the separability probability with respect to the Bloch radius of the qubit is uniform for maximal rank states (see figure 1a), while for the rank-3 states the deviation from the total separability probability is depicted on figure 1b.

References

[1] I.Bengtsson and K.Zyczkowski, *Geometry of Quantum States: An Introduction to Quantum Entanglement*, Cambridge University Press, 2006.
[2] P.B.Slater, arXiv:1701.01973v7, (2017).
[3] D.A.Klain and G.C.Rota, *Introduction to Geometric Probability*, Cambridge University Press, 1997.
[4] S.L.Braunstein, A 219 169 (1996).
[5] K.Zyczkowski and H-J.Sommers, J. Phys. A: Math. Theor. 34, 7111 (2001).
[6] V.A.Osipov, H.J.Sommers and K.Zyczkowski, J. Phys. A: Math. Theor. 43, 055302, (2010)
[7] A.Khvedelidze and I.Rogojin, J. of Math. Sciences, 209, 6, 988–1004, (2015)
[8] M.Ruskai and E.M.Werner, J.Phys.A.Math.Theor., 42, 095303, (2009)