On the critical line zeros of \( L \) — functions attached to automorphic cusp forms.

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§1. Introduction. Statement of the main result

One of the most interesting questions in the theory of the Riemann zeta-function is the Riemann hypothesis which asserts that all non-trivial zeros of the Riemann zeta-function lie on the critical line. The Riemann hypothesis is not yet proved nor disproved. In 1942, Atle Selberg \[1\] showed that a positive proportion of non-trivial zeros of the Riemann zeta-function lie on the critical line (for the numerical estimates see works \[2\]–\[7\]). The same type result holds also for Dirichlet \( L \) - functions (\[8\]). In 1989, A. Selberg in his report at the conference in Amalfi conjectured that all functions from Selberg class \( S \) (which have decomposition in the Euler product and functional equation of the Riemann type as the necessary conditions) satisfy an analogue of the Riemann hypothesis (see \[9\]).

The Riemann zeta-function and Dirichlet \( L \) - functions are functions of degree one (for the definition of the degree, which is a characteristic of a functional equation, see \[9\]). In 1983, J.L. Hafner proved an analogue of Selberg’s theorem for a function of degree two. In his papers \[10\], \[11\] for \( L \) - series, whose coefficients are attached to those of holomorphic cusp forms (modular forms) or non-holomorphic cusp forms (Maass wave forms) for the full modular group with trivial character, the result on positivity of proportion of non-trivial zeros lying on the critical line is established (see the necessary definitions further in the text and in \[12\]). This work is a continuation of \[10\], \[11\]. Here we obtain an analogue of Selberg’s theorem for \( L \) - functions attached to automorphic cusp forms with respect to the Hecke congruence subgroup \( \Gamma_0(D) \) with arbitrary integral weight \( k \geq 1 \) (theorem \[1\]).

First, we recall some definitions and notation.

Suppose that \( f(z) \) is an automorphic cusp form of integral weight \( k \geq 1 \) for the group \( \Gamma_0(D) \) with a character \( \chi \) modulo \( D \) (briefly we write this as \( f \in S_k(\Gamma_0(D), \chi) \)), where \( \Gamma_0(D) \) is a subgroup of \( SL_2(\mathbb{Z}) \) that consists of all the matrices \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) satisfying the condition \( c \equiv 0 \pmod{D} \). In other words, let \( f \) be a holomorphic function on the upper half-plane, such that for every element \( \gamma \in \Gamma_0(D) \) the following relation is fulfilled:

\[
f(\gamma z) = \chi(\gamma)(cz + d)^k f(z), \quad \text{where} \quad \chi(\gamma) = \chi(d), \quad c \gamma z = \begin{pmatrix} az + b \\ cz + d \end{pmatrix}, \quad (1)
\]

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and also that \( f \) vanishes at every cusp of the group \( \Gamma_0(D) \). This entails that \( f(z) \) has an expansion

\[
f(z) = \sum_{n=1}^{+\infty} a(n)e^{2\pi inz} \quad \text{for} \quad \Re z > 0.
\]

Next, assume that \( f \) is not identically zero and is an eigenfunction of all the Hecke operators \( T_n \) for \( n = 1, 2, \ldots \), where

\[
T_n f(z) = \frac{1}{n} \sum_{a \equiv 1 \pmod{n}} \chi(a) a^k \sum_{0 \leq b < d} f\left(\frac{az + b}{d}\right).
\]

Without loss of generality we may assume that \( a(1) = 1 \). From the properties of the Hecke operators, the equality \( T_n f = a(n)f \) follows for every positive integer \( n \), and also if \( \Re s > 1 \), then for Dirichlet series

\[
L(s) = L_f(s) = \sum_{n=1}^{+\infty} \frac{r(n)}{n^s}
\]

with

\[
r(n) = a(n)n^{\frac{1-k}{2}},
\]

the identity

\[
L(s) = \prod_p \left( 1 - \frac{r(p)}{p^s} + \frac{\chi(p)}{p^{2s}} \right)^{-1}
\]

holds (here the product is carried over all consecutive prime numbers). For the normalized coefficients \( r(n) \) of the automorphic (for a congruence subgroup) cusp form of an integral weight, which is also an eigenfunction of all the Hecke operators, the estimate

\[
|r(n)| \leq \tau(n)
\]

is valid, where \( \tau(n) \) is the number of divisors of \( n \). This inequality was previously known as the Ramanujan - Petersson conjecture until its truth was proved in [13], [14]. The function \( L(s) \) satisfies the following functional equation (see [12], \$6.7, \$7.2):

\[
\Lambda(s) = \theta \cdot \overline{\Lambda(1 - \overline{s})},
\]

where \( \Lambda(s) \) is an entire function,

\[
\Lambda(s) = \left( \frac{2\pi}{\sqrt{D}} \right)^{s-k} \Gamma \left( s + \frac{k-1}{2} \right) L(s),
\]

and \(|\theta| = 1\), or, more precisely, \( \theta = i^{k/2} \eta \), where \( \eta \) is the eigenvalue of the operator \( \overline{W} \),

\[
\overline{W} f = (-1)^k D^{-k/2} z^{-k} f(1/Dz), \quad \overline{W} f = \eta f.
\]
The function $L(s)$ has zeros at $s = -\frac{k-1}{2}, -\frac{k-1}{2}-1, -\frac{k-1}{2}-2, \ldots$, which correspond to the poles of $\Gamma(s + \frac{k-1}{2})$ and are called “trivial zeros”. The remaining zeros lie in the strip $0 \leq \Re s \leq 1$ and are called “non-trivial”. $L(s)$ is a function of degree two and belongs to Selberg class. Therefore, an analogue of the Riemann hypothesis exists for this function which claims that all its non-trivial zeros lie on the critical line $\Re s = \frac{1}{2}$. In this work we prove the following theorem.

**Theorem 1.** Suppose that $f(z) = \sum_{n=1}^{+\infty} a(n) e^{2\pi inz}$ is an automorphic cusp form of integral weight $k \geq 1$ for the group $\Gamma_0(D)$ with character $\chi$ modulo $D$, which is also an eigenfunction of all the Hecke operators $T_n$ for $n = 1, 2, \ldots$. Define $L(s) = L_f(s)$ by the equalities (3) and (4), and set $N_0(T)$ to be the number of odd order zeros of $L(s)$ on the interval $\{s = \frac{1}{2} + it, 0 < t \leq T\}$. Then $N_0(T) \geq cT \ln T$ with some constant $c > 0$.

Notice that if $N(T)$ is the number of zeros of $L(s)$ in the rectangle $\{s \mid 0 \leq \Re s \leq 1, 0 < \Im s \leq T\}$, then the asymptotic formula holds

$$N(T) = \frac{T}{\pi} \ln \frac{T}{\pi c_1} + O(\ln T) \quad \text{when} \quad T \to +\infty;$$

thus we have

**Corollary 1.** A positive proportion of non-trivial zeros of $L_f(s)$ lie on the critical line.

An example of $L(s)$ considered in this work is a Hecke $L$-function with complex class group character on ideals of imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$ (since there is a corresponding form $f$ from $S_1(\Gamma_0(D), \chi_D)$).

### §2. Main and auxiliary statements

Hereafter we suppose that $k$ (the weight of the form $f$) and $D$ (the level of the form $f$) are fixed numbers.

The main idea of the proof belongs to A. Selberg which is served by introducing a “mollifier”. Define numbers $\alpha(\nu)$ by the equality

$$\sum_{\nu=1}^{+\infty} \alpha(\nu) \nu^{-s} = \prod_{p>256} \left(1 - \frac{r(p)}{2p^s}\right). \quad (5)$$

Suppose that $X \geq 3$ and set

$$\beta(\nu) = \alpha(\nu) \left(1 - \frac{\ln \nu}{\ln X}\right)^+, \quad \text{where} \quad x^+ = \max(0, x). \quad (6)$$
Let us define a mollifier $\varphi(s)$ by the formula

$$\varphi(s) = \sum_{\nu=1}^{+\infty} \beta(\nu) \nu^{-s}. $$

Suppose $\delta > 0$,

$$\mathfrak{F}(t) = \frac{1}{\sqrt{2\pi}} \Lambda \left( \frac{1}{2} + it \right) \left| \varphi \left( \frac{1}{2} + it \right) \right|^2 \exp \left( \left( \frac{\pi}{2} - \delta \right) t \right). \quad (7)$$

The functional equation for $L(s)$ yields that $\theta^{-1/2} \mathfrak{F}(t)$ is real-valued for real $t$. Observe also, that odd order zeros of $\theta^{-1/2} \mathfrak{F}(t)$ are those of $L(1/2 + it)$.

For $0 < h_1 < 1$, put

$$I_1(t) = \int_{-h_1}^{h_1} |\theta^{-1/2} \mathfrak{F}(t + u)| du = \int_{-h_1}^{h_1} |\mathfrak{F}(t + u)| du,$$

$$I_2(t) = \left| \int_{-h_1}^{h_1} \theta^{-1/2} \mathfrak{F}(t + u) du \right| = \left| \int_{-h_1}^{h_1} \mathfrak{F}(t + u) du \right|.$$

Let $T > 1$ be a sufficiently large number. Define $E_1$ as a set of points $t \in (1, T)$, such that for $t \in E_1$ the inequality

$$I_1(t) > I_2(t)$$

holds. By $E_2$ denote the complementary set to $E_1$, i.e., the set of all points $t \in (1, T)$ with

$$I_1(t) = I_2(t).$$

If $\mu(E_1)$ denotes the measure of the set $E_1$, then the number of odd order zeros of the function $\theta^{-1/2} \mathfrak{F}(t)$ on the interval $(0, T)$ is not less than $\frac{\mu(E_1)}{2h_1} - 1$ (see [1] or [15]).

Now the main statement of this work is a consequence of the following theorem.

**Theorem 2.** Suppose that $0 < \delta < \delta_0$, where $\delta_0 < 1/10$ is some small positive number, $0 < h_1 < 1$, $X \geq 3$ and $\delta X^{86} e^{1/4} \leq 1$. Then the following estimates are valid:

a) $\int_{1}^{+\infty} I_1(t) dt \gg \delta^{-1} h_1$,

b) $\int_{-\infty}^{+\infty} I_2^2(t) dt \ll \delta^{-1} h_1^2 \ln \frac{\delta^{-1}}{\ln X}$,

c) $\int_{-\infty}^{+\infty} I_2^2(t) dt \ll \delta^{-1} \frac{h_1}{\ln X}$, where the constants implied in Vinogradov’s signs $\ll, \gg$ are absolute.
To derive the main result, set in theorem \[2\]
\[
\delta = T^{-1}, \quad X = T^{1/100}, \quad h_1 = \frac{A}{\ln X},
\]
where \(A\) is a sufficiently large positive constant, and write the following chain of relations
\[
I_3 = \int_1^T I_1(t)dt = \int_{E_1} I_1(t)dt + \int_{E_2} I_1(t)dt = \int_{E_1} I_1(t)dt + \int_{E_2} I_2(t)dt \leq I_1 + I_2,
\]
where
\[
I_1 = \int_{E_1} I_1(t)dt \leq (\mu(E_1))^{1/2} \left(\int_{-\infty}^{+\infty} I_1^2(t)dt\right)^{1/2},
\]
\[
I_2 = \int_1^T I_2(t)dt \leq T^{1/2} \left(\int_{-\infty}^{+\infty} I_2^2(t)dt\right)^{1/2}.
\]
The result c) of theorem \[2\] entails
\[
I_2 \ll T \left(\frac{h_1}{\ln X}\right)^{1/2}.
\]
Hence, by virtue of the estimate a), for sufficiently large \(A\) we have
\[
I_3 \geq 2I_2.
\]
Therefore, \(I_1 \geq I_3/2\), and thus the relations a) and b) of theorem \[2\] imply the estimates
\[
Th_1 \ll I_3 \ll (\mu E_1)^{1/2} \left(Th_1^2 \frac{\ln T}{\ln X}\right)^{1/2},
\]
or
\[
\mu(E_1) \gg T.
\]
Thereby, the number of odd order zeros of \(\theta^{-1/2} \zeta(t)\) (that is of \(L(1/2 + it)\)) on the interval \(0 < t \leq T\) is estimated from below by the quantity of order
\[
\mu(E_1)h_1^{-1} \gg T \ln X \gg T \ln T.
\]
Now it only remains to establish the statement of theorem \[2\]

Proof of the assertion a) of theorem \[2\] repeats the proof of a similar relation while considering the Riemann zeta-function instead of \(L(s)\) (see \[15\], §6.3).

To derive the assertions b) and c) of theorem \[2\] we employ the following auxiliary lemma.
Lemma 1. Suppose that $\mathcal{F}(y)$ is given by the formula (7), and

$$G(y) = \left| \sum_{n, \nu_1, \nu_2 \in \mathbb{N}} r(n)\beta(\nu_1)\beta(\nu_2) \frac{y}{\nu_2} \exp \left( -\frac{2\pi n\nu_1}{\sqrt{D}\nu_2} y \left( \sin \delta + i \cos \delta \right) \right) \right|^2. \quad (8)$$

Then

$$\int_{-\infty}^{+\infty} \left( \int_{-h_1}^{h_1} |\mathcal{F}(t+u)| du \right)^2 dt \leq 8h_1^2 \int_1^{+\infty} G(y) dy,$$

and

$$\int_{-\infty}^{+\infty} \left( \int_{-h_1}^{h_1} |\mathcal{F}(t+u)| du \right)^2 dt \leq 8h_1^2 \int_1^{+\infty} G(y) dy + 8 \int_{1}^{H} \frac{G(y)}{\ln^2 y} dy,$$

where $H = e^{1/h_1}$.

The proof of this result is contained in [16] (lemma 3 and 4).

Let us formulate now the main lemmas, from which the statement of theorem 2 will follow easily.

Lemma 2. Assume that $G(y)$ is defined by the formula (8). Then under the conditions of theorem 2, for $1 \leq x \leq e^{1/h_1}$ the estimate

$$J(x, \theta) = \int_{x}^{+\infty} G(u) u^{-\theta} du \ll \frac{\delta^{-1}}{\theta x^\theta \ln X}$$

is valid uniform in $\theta$ from the interval $0 < \theta \leq 1/4$.

The core of the proof of lemma 2 relies on the following two lemmas. Lemma 3 is related to an estimation of a “diagonal” term (estimation of Selberg sums), and lemma 4 accordingly to a “non-diagonal” term. Let us adopt further the following notation:

- $\overline{K}(s) = \overline{K(\overline{s})}$,

- $p^\alpha | m$ means that $p^\alpha$ divides $m$, but $p^{\alpha+1} \nmid m$.

Lemma 3. Let $0 \leq \theta \leq 1/4$, and define the sum $S(\theta)$ by the equality

$$S(\theta) = \sum_{\nu_1, \ldots, \nu_4 \leq X} \frac{\beta(\nu_1)\beta(\nu_2)\beta(\nu_3)\beta(\nu_4)}{\nu_2 \nu_4} \left( \frac{q}{\nu_1 \nu_3} \right)^{1-\theta} K \left( \frac{\nu_1 \nu_4}{q}, 1 - \theta \right) K \left( \frac{\nu_2 \nu_3}{q}, 1 - \theta \right),$$

where

$$q = (\nu_1 \nu_4, \nu_2 \nu_3).$$
\[ K(m, s) = \prod_{p|m} \left( 1 + \frac{|r^2(p)|}{p^s} + \frac{|r^2(p^2)|}{p^{2s}} + \ldots \right)^{-1} \times \prod_{p^\alpha|\nu} \left( \frac{r(p^{\alpha}) + \frac{r(p^{\alpha+1})}{p^s} + \frac{r(p^{\alpha+2})}{p^{2s}} + \ldots}{p^s} \right). \]

Then the estimate
\[ S(\theta) \ll \frac{X^{2\theta}}{\ln X} \]
holds uniformly in \( \theta \).

**Lemma 4.** Let \( N \gg 1, (m_1, m_2) = 1, m_1^8 m_2^9 \leq N, \ l \leq N^{10/11}, \)
\[ S = \sum_{n=1}^{N-1} r(n) \left( \frac{m_1 n + l}{m_2} \right) \]
(assuming that the function \( r(\cdot) \) vanishes for non-integral argument). Then for arbitrary \( \varepsilon > 0 \) the following estimate is valid:
\[ S \ll \varepsilon N^{10/11 + \varepsilon} m_1^{8/11} m_2^{-2/11}. \]

Lemma 4 implies the following

**Corollary 2.** The function
\[ D_{m_1, m_2}(s, l) = \sum_{n=1}^{+\infty} \frac{r(n) r\left( \frac{m_1 n + l}{m_2} \right)}{(m_1 n + l/2)^s} \]
has an analytic continuation in the half-plane \( \text{Re } s > 10/11, \) and, moreover, for \( 0 < \varepsilon_0 \leq 1/11, \) in the region \( \text{Re } s \geq 10/11 + \varepsilon_0 \) the estimate
\[ D_{m_1, m_2}(s, l) \ll \varepsilon_0 \frac{|s|}{m_1^{4/11} + \varepsilon_0} \left( (m_1^8 m_2^9)^{1/11 - 4/9} + l^{1/11 - 14/9} \right) \]
holds true.

The deduction of corollary 2 from lemma 4 is contained in [16]. The main tool for obtaining the statement of lemma 4 is Jutila’s variant of circle method ([17], [18]), which we also used to prove similar result [16] (lemma 7) for coefficients \( r(n) \) of automorphic cusp forms of weight \( k = 1. \) A modification in the proof of the statement for an arbitrary weight \( k \) is that one has to use the following analogue of lemma 10 in [16]: let \( q \equiv 0(\mod D), \ (a, q) = 1, aa^* \equiv 1(\mod q), \) and let \( k(t) \) be a smooth (for example, twice continuously differentiable) function with compact support on \((0, +\infty)\). Then
\[ \sum_{n=1}^{+\infty} r(n) k(n) e^{2\pi i \frac{s}{q} n} = \chi(a) \sum_{n=1}^{+\infty} r(n) e^{-2\pi i \frac{s}{q} n} k(n), \]
where \( \tilde{k}(n) = \frac{2\pi i}{q} \int_0^{+\infty} k(t) J_{k-1} \left( \frac{4\pi \sqrt{nt}}{q} \right) dt \), and \( J_{k-1}(t) \) is Bessel function.

To establish the above relation one has to notice for an automorphic cusp form \( f \in S_k(\Gamma_0(D), \chi) \),
\[
f(z) = \sum_{n=1}^{+\infty} a(n) e^{2\pi i n z},
\]
that the equality
\[
f \left( \frac{a}{q} + i w \right) = \frac{\chi(a)}{(-iqw)^k} f \left( -\frac{a}{q} - \frac{1}{iq^2 w} \right), \quad \text{Re} \ w > 0,
\]
holds, which implies the relation
\[
\sum_{n=1}^{+\infty} a(n) k(n) e^{2\pi i \frac{a}{q} n} = i^{k-1} \frac{\chi(a)}{q^k} \sum_{n=1}^{+\infty} a(n) e^{-2\pi i \frac{a}{q} n} \int k(t) \left( \int_{c-i\infty}^{c+i\infty} \frac{1}{w^{k-1}} e^{-\frac{2\pi}{q} \sqrt{nt}} e^{2\pi tw} dw \right) dt.
\]

Thus, by virtue of the identity
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{as-bs^{-1}} \frac{ds}{s^k} = \left( \frac{a}{b} \right)^{\frac{k-1}{2}} J_{k-1}(2\sqrt{ab}),
\]
which is valid for \( a, b, c > 0, \ k \geq 1 \) (see [19], §7.3.3), we have that the integral enclosed in parenthesis in the previous relation is equal to
\[
2\pi i q^{k-1} \left( \frac{t}{n} \right)^{\frac{k-1}{2}} J_{k-1} \left( \frac{4\pi \sqrt{nt}}{q} \right).
\]

To complete the proof of that auxiliary statement one has to set \( k(t) = k_1(t) t^{\frac{1-k}{2}} \) in the formula (10) and recall that \( r(n) = a(n) n^{\frac{1-k}{2}} \).

§3. Proof of statements b) and c) of theorem 2

Let us prove assertions b) and c) of theorem 2 using the lemmas formulated above.

Proof of the statement c). For \( H = e^{1/h_1} \), applying lemma 2 with \( \theta = 1/4 \), we get the estimate
\[
\int_1^H G(x) dx = - \int_1^H x^\theta \frac{d}{dx} J(x, \theta) dx = - x^\theta J(x, \theta) \bigg|_{x=1}^{x=H} + \theta \int_1^H x^{\theta-1} J(x, \theta) dx \leq \frac{\delta^{-1}}{\theta \ln X} + \frac{\delta^{-1} \ln H}{\ln X} \leq \frac{\delta^{-1}}{h_1 \ln X}.
\]
Similarly, with $\theta = 1/4$ we have

$$\int_{\ln^2 x}^{+\infty} G(x) dx = -\int_{\ln^2 x}^{+\infty} x^\theta d\frac{d}{dx} J(x, \theta) dx$$

$$= -\frac{x^\theta}{\ln^2 x} J(x, \theta) \bigg|_{x=H}^{x=+\infty} + \int_{H}^{+\infty} J(x, \theta) x^\theta \left( \frac{\theta}{x \ln^2 x} - \frac{2}{x \ln^3 x} \right) dx$$

$$\ll \frac{\delta^{-1}}{\theta \ln X \ln^2 H} + \frac{\delta^{-1}}{\ln X \ln H} + \frac{\delta^{-1}}{\theta \ln X \ln^2 H} \ll \frac{\delta^{-1}h_1}{\ln X}.$$ 

Now the statement $)$ follows from lemma $\blacksquare$

**Proof of the statement b).** Application of lemma 2 with $x = 1$, $\theta = \frac{1}{\ln \delta^{-1}}$ gives

$$\int_{1}^{\delta^{-2}} G(u) du \ll \int_{1}^{+\infty} G(u) u^{-1/\ln \delta^{-1}} du \ll \frac{\delta^{-1} \ln \delta^{-1}}{\ln X}.$$ 

Using formula (8), we estimate the integral on the interval $(\delta^{-2}, +\infty)$ of $G(u)$ by the following expression

$$\int_{\delta^{-2}}^{+\infty} G(u) du \ll \sum_{n_1, n_2, \nu_1, \nu_2, \nu_3, \nu_4} \frac{|r(n_1) r(n_2) \beta(\nu_1) \beta(\nu_2) \beta(\nu_3) \beta(\nu_4)|}{\nu_2 \nu_4} \times$$

$$\times \int_{\delta^{-2}}^{+\infty} \text{exp} \left( -\frac{2\pi}{\sqrt{D}} \left( \frac{n_1 \nu_1}{\nu_2} + \frac{n_2 \nu_3}{\nu_4} \right) \delta x \right) dx.$$ 

Let $A = \frac{2\pi}{\sqrt{D}} \left( \frac{n_1 \nu_1}{\nu_2} + \frac{n_2 \nu_3}{\nu_4} \right)$. From the equality

$$\int_{\delta^{-2}}^{+\infty} \text{exp}(-Ax) dx = \frac{e^{-A\delta^{-2}}}{A}$$

we obtain the following estimate:

$$\int_{\delta^{-2}}^{+\infty} G(u) du \ll \delta^{-1} \sum_{n_1, n_2, \nu_1, \nu_2, \nu_3, \nu_4} \frac{|r(n_1) r(n_2) \beta(\nu_1) \beta(\nu_2) \beta(\nu_3) \beta(\nu_4)|}{(n_1 \nu_1 \nu_4 + n_2 \nu_2 \nu_3)} \text{exp} \left( -\frac{2\pi}{\sqrt{D}} \delta^{-1} \left( \frac{n_1 \nu_1}{\nu_2} + \frac{n_2 \nu_3}{\nu_4} \right) \right)$$

$$\ll \delta^{-1} \sum_{n_1, n_2, \nu_1, \nu_2, \nu_3, \nu_4} \frac{|r(n_1) r(n_2) \beta(\nu_1) \beta(\nu_2) \beta(\nu_3) \beta(\nu_4)|}{\sqrt{n_1 n_2 \nu_1 \nu_2 \nu_3 \nu_4}} \text{exp} \left( -\frac{2\pi}{\sqrt{D}} \delta^{-1} \left( \frac{n_1 \nu_1}{\nu_2} + \frac{n_2 \nu_3}{\nu_4} \right) \right)$$

$$\leq \delta^{-1} \left( \sum_{n_1 \nu_2 \leq \nu_1} \frac{1}{\nu_1 \nu_2} \sum_{n} |r(n)| \sqrt{n} \text{exp} \left( -\frac{2\pi}{\sqrt{D}} n \delta^{-1} \nu_1 \nu_2 \nu_3 \nu_4 \right) \right)^2.$$
Since $|r(n)| \leq \tau(n)$, and $\nu_2 \leq X \leq \delta^{-1/3}$, then

$$\sum_{n=1}^{+\infty} \frac{|r(n)|}{\sqrt{n}} \exp\left(-\frac{2\pi \sqrt{D}}{\sqrt{\nu_2}} \delta^{-1}\right) \ll \exp\left(-\sqrt{\delta^{-1}}\right).$$

Hence,

$$\int_{\delta^{-2}}^{+\infty} G(u) du \ll \delta^{-1} X \exp\left(-\sqrt{\delta^{-1}}\right) \ll 1.$$

Now the statement b) of theorem 2 follows from lemma 1. □

§4. Proof of main lemma 2

Similarly to the proof in [10], we introduce a non-negative smooth factor $\phi(u) \in C^2(0, +\infty)$, having the property

$$\phi(u) = \begin{cases} 0, & u \leq 1/2 \\ 1, & u \geq 1. \end{cases}$$

Then

$$J(x, \theta) \leq \int_{0}^{+\infty} \phi\left(\frac{u}{x}\right) G(u) u^{-\theta} du = x^{1-\theta} \int_{0}^{+\infty} \phi(u) G(u x) u^{-\theta} du. \quad (11)$$

We need several auxiliary relations for estimation of the last integral. For $\text{Re } s > 1$, define

$$\Phi(s, y) = \int_{0}^{+\infty} \phi(u) u^{-s} \exp(-2\pi i y u) du.$$

We then have

$$\Phi^*(s) := \int_{0}^{+\infty} \phi'(u) u^{-s+1} du = (s - 1) \Phi(s, 0).$$

Since $\phi'$ is a function with compacts support, then $\Phi^*(s)$ is an entire function. Therefore, $\Phi^*(s)$ is the analytical continuation of $(s - 1) \Phi(s, 0)$ to the whole complex plane. Moreover, for $\text{Re } s$ from any bounded interval, the estimate

$$\Phi^*(s) = O(1)$$

holds, where constant in $O$ - symbol is absolute. For $y > 0$, integration by parts twice gives

$$\Phi(s, y) = O(|s|^2 y^{-2}).$$
Square out the modulus of the sum in the expression (8) for $G(x)$, we get

$$G(x) = \sum_{\nu_1, \nu_2, \nu_3, \nu_4} \frac{\beta(\nu_1)\beta(\nu_2)\beta(\nu_3)\beta(\nu_4)}{\nu_2\nu_4} \left( \sum_{n=1}^{+\infty} r(n)r \left( \frac{nm_1}{m_2} \right) \exp \left( -\frac{4\pi}{\sqrt{D}} \frac{nm_1}{Q} x \sin \delta \right) \right) + 2 \text{Re} \sum_{l \geq 1} \exp \left( -\frac{2\pi i}{\sqrt{D}} x \cos \delta \right) \sum_{n=1}^{+\infty} r(n)r \left( \frac{nm_1 + l}{m_2} \right) \exp \left( -\frac{4\pi}{\sqrt{D}} \frac{(nm_1 + l/2)}{Q} x \sin \delta \right),$$

where the following notion were used

$$q = (\nu_1\nu_4, \nu_2\nu_3), \quad Q = \frac{\nu_2\nu_4}{q}, \quad m_1 = \frac{\nu_1\nu_4}{q}, \quad m_2 = \frac{\nu_2\nu_3}{q}.$$

By Mellin’s transform formula for Euler Gamma-function, $e^{-x} = \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} \Gamma(s)x^{-s}ds$, we find

$$J(x, \theta) \leq J_1(x, \theta) + J_2(x, \theta),$$

where

$$J_1(x, \theta) = x^{1-\theta} \sum_{\nu_1, \nu_2, \nu_3, \nu_4} \frac{\beta(\nu_1)\beta(\nu_2)\beta(\nu_3)\beta(\nu_4)}{\nu_2\nu_4} \times \left( \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D_{m_1, m_2}(s, 0) \Gamma(s) B^{-s}\Phi(s + \theta, 0) ds \right),$$

$$J_2(x, \theta) = 2x^{1-\theta} \sum_{\nu_1, \nu_2, \nu_3, \nu_4} \frac{\beta(\nu_1)\beta(\nu_2)\beta(\nu_3)\beta(\nu_4)}{\nu_2\nu_4} \times \left( \text{Re} \sum_{l \geq 1} \frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D_{m_1, m_2}(s, l) \Gamma(s) B^{-s}\Phi(s + \theta, B_1l) ds \right),$$

$$D_{m_1, m_2}(s, l) = \sum_{n=1}^{+\infty} r(n)r \left( \frac{m_1n+l}{m_2} \right) \left( \frac{m_1n+l}{m_2} \right)^s,$$

$$B = \frac{4\pi x \sin \delta}{\sqrt{DQ}}, \quad B_1 = \frac{x \cos \delta}{\sqrt{DQ}}.$$

For a given positive integer $m$, denote by $M_1(m)$ the subset of positive integers constituted by 1 and by all numbers with the same set of prime divisors as $m$ has. Since $(m_1, m_2) = 1$, we have

$$D_{m_1, m_2}(s) = D_{m_1, m_2}(s, 0) = \frac{1}{(m_1 m_2)^s} \sum_{n=1}^{+\infty} \frac{r(nm_2)r(nm_1)}{n^s}$$

$$= \frac{1}{(m_1 m_2)^s} K(m_1, s) K(m_2, s) D(s),$$

$$K(m, s) = \frac{1}{(m)^s} \sum_{n=1}^{+\infty} \frac{r(n)}{n^s}.$$
where

\[ D(s) = \sum_{n=1}^{+\infty} \frac{|r(n)|^2}{n^s}, \]

\[ K(m, s) = \prod_{p|m} \left( 1 + \frac{|r(p)|^2}{p^s} + \frac{|r(p^2)|^2}{p^{2s}} + \ldots \right)^{-1} \times \prod_{p^\alpha|m} \left( \frac{r(p^\alpha)}{p^s} + \frac{r(p^\alpha+1)r(p)}{p^{2s}} + \frac{r(p^\alpha+2)r(p^2)}{p^{3s}} + \ldots \right) \]

\[ = \sum_{k \in M_1(m)} \frac{r(mk)r(k)}{k^s} \left( \sum_{k \in M_1(m)} \frac{|r(k)|^2}{k^s} \right)^{-1}. \]

If a prime \( p \) divides \( m_j \), then \( p > 256 \); this entails that, for \( \text{Re } s \geq 1/2 \),

\[ \left| 1 + \frac{|r(p)|^2}{p^s} + \frac{|r(p^2)|^2}{p^{2s}} + \ldots \right| > 0 \]

and, therefore, for a fixed \( m_j \), that \( K(m_j, s) \) is an analytic function in the region \( \text{Re } s \geq 1/2 \). Moreover, for either \( m = m_1 \) or \( m = m_2 \), in this region the following estimate is valid

\[ |K(m, s)| \leq \prod_{p|m} \left( 1 - \frac{2}{256^{1/2}} - \frac{3}{256} - \frac{4}{256^{3/2}} - \ldots \right)^{-1} \times \tau(m) \prod_{p^\alpha|m} \left( 1 + \frac{\alpha+2}{\alpha+1} \cdot \frac{2}{256^{1/2}} + \frac{\alpha+3}{\alpha+1} \cdot \frac{3}{256} + \frac{\alpha+4}{\alpha+1} \cdot \frac{4}{256^{3/2}} + \ldots \right) \]

\[ \leq \tau(m) \prod_{p|m} \left( 1 - \frac{1}{4} - \frac{1}{4^2} - \frac{1}{4^3} - \ldots \right)^{-1} \left( 1 + \frac{3}{16} \cdot \frac{2}{16} + \frac{4}{16^2} + \frac{3}{16^3} + \frac{5}{16^4} + \ldots \right) \]

\[ = \tau(m) \prod_{p|m} \frac{3}{2} \left( \sum_{n \geq 2} \frac{n(n-1)}{2} \cdot \frac{1}{16^{n-2}} \right) \leq \tau(m) \prod_{p|m} 3 \leq \tau(m) \tau_3(m) \leq \tau_6(m), \]

(12)

where \( \tau_i(n) = \sum_{n_1 \ldots n_i = n} 1 \). Also, \( D(s) \) can be meromorphically continued to the whole complex plane with the pole of first order at \( s = 1 \) (see [20]). This gives a meromorphic continuation of \( D_{m_1, m_2}(s) \) into half-plane \( \text{Re } s > 1/2 \).

In order to estimate \( J_1(x, \theta) \), move the contour of integration in

\[ \int_{2-i\infty}^{2+i\infty} D_{m_1, m_2}(s)\Gamma(s)B^{-\theta}\Phi(s+\theta, 0)ds = \int_{2-i\infty}^{2+i\infty} \frac{K(m_1, s)K(m_2, s)D(s)\Gamma(s)\Phi^*(s+\theta)(Bm_1m_2)^{-s}}{s+\theta-1}ds \]
to the line \( \Re s = 2/3 \). We, therefore, pass simple poles at \( s = 1 \), \( s = 1 - \theta \). Denoting as \( \mathcal{D} \) the residue of \( D(s) \) at \( s = 1 \), we get

\[
\frac{1}{2\pi i} \int_{2-i\infty}^{2+i\infty} D_{m_1,m_2}(s)\Gamma(s)B^{-s}\Phi(s,\theta,0)ds = \frac{\mathcal{D}K(m_1,1)\overline{K}(m_2,1)\Phi^*(1+\theta)}{Bm_1m_2}\theta
\]

\[
+ \frac{K(m_1,1-\theta)\overline{K}(m_2,1-\theta)D(1-\theta)\Gamma(1-\theta)\Phi^*(1)}{(Bm_1m_2)^{1-\theta}} + O\left(\frac{(m_1m_2)^{1/3}}{(Bm_1m_2)^{2/3}}\right)
\]

\[
= \frac{c(\theta)K(m_1,1)\overline{K}(m_2,1)}{\theta x_1^\delta} \sum_{\nu_1} \frac{1}{\nu_1^\nu_3} + \frac{c'(\theta)K(m_1,1-\theta)\overline{K}(m_2,1-\theta)D(1-\theta)}{(x_1^\delta)^{1-\theta}} \left(\frac{q}{\nu_1^\nu_3}\right)^{1-\theta}
\]

\[
+ O\left(\frac{1}{(x_1^\delta)^{2/3} (\nu_1^\nu_3)^{1/3}}\right),
\]

where \( c(\theta) \), \( c'(\theta) \) are constants that depend on \( \theta \) and, for \( 0 < \theta \leq 1/4 \), are bounded. Applying estimate \( D(1-\theta) \ll \theta^{-1} \) and lemma \( \text[3] \) we arrive at the following relation:

\[
J_1(x, \theta) \ll \frac{|S(0)|}{\theta x_1^\delta} + \frac{|S(\theta)|}{\theta^{-1}} + \frac{x_1^{1/3}}{x_1^\delta} \sum_{\nu_1 \leq X} \frac{1}{(\nu_1^\nu_3)^{1/3} (\nu_1^\nu_4)^{2/3}} \ll \frac{\delta^{-1}}{\theta x_1^\delta \ln X} \left(1 + (X^2 e^{\frac{1+1/12}{3}} \delta^2 + X^2 (\ln X) e^{\frac{1+1/12}{3}} \delta^{1/3})\right) \ll \frac{\delta^{-1}}{\theta x_1^\delta \ln X},
\]

since we used \( X^3 e^{\frac{1+1/12}{3}} \delta^{-1} \ll 1 \).

Now let us estimate the “non-diagonal” term \( J_2(x, \theta) \). It follows from corollary \( \text[2] \) to lemma \( \text[4] \) that we can move the path of integration in the integral \( J_2(x, \theta) \) to the line \( \Re s = 11/12 \). Using the statement of lemma \( \text[4] \) we obtain the estimate

\[
J_2(x, \theta) \ll x_1^{-\theta} \sum_{\nu_1 \leq X} \frac{1}{\nu_1^\nu_4} \sum_{l \geq 1} \frac{1}{B_1^{11/12} B_1^{2/12}} (m_1 m_2)^{-2/11 + l/10} (m_1^{-10/11} m_2^{-1})
\]

\[
\ll \frac{\delta^{-11/12}}{x_1^\theta} \sum_{\nu_1 \leq X} \frac{Q^{2+11/12}}{\nu_1^\nu_4} (m_1 m_2)^{-2/11} \ll \frac{\delta^{1+11/12}}{x_1^\theta} \sum_{\nu_1 \leq X} \frac{Q^{1+11/12}}{m_1 m_2} \ll \frac{\delta^{-11/12}}{x_1^\theta} (X^2)^{1+1/12} (X^{3+11/12})^4
\]

\[
\ll \frac{\delta^{-11/12}}{x_1^\theta} X^{4/3} \ll \frac{\delta^{-1}}{x_1^\theta \ln X},
\]

since \( X \leq \delta^{-1/80} \). The lemma is proved. \( \Box \)

§5. Proof of lemma \( \text[3] \) estimation of Selberg sums

Application of Möbius inversion formula to \( f \), that is

\[
f(q) = \sum_{d|q} \sum_{m|d} \mu(m) f \left( \frac{d}{m} \right),
\]
gives the identity
\[ q^{1-\theta} K \left( \frac{\nu_1 \nu_4}{q}, 1 - \theta \right) K \left( \frac{\nu_2 \nu_3}{q}, 1 - \theta \right) = \sum_{d|q} \sum_{m|d} \mu(m) \left( \frac{d}{m} \right)^{1-\theta} \times \]
\[ \times K \left( \frac{\nu_1 \nu_4 m}{d}, 1 - \theta \right) K \left( \frac{\nu_2 \nu_3 m}{d}, 1 - \theta \right). \]

Inserting this formula to the expression for \( S(\theta) \), we find that
\[ S(\theta) = \sum_{\nu_1, \ldots, \nu_4 \leq X} \frac{\beta(\nu_1) \beta(\nu_2) \beta(\nu_3) \beta(\nu_4)}{\nu_2 \nu_4 (\nu_1 \nu_3)^{1-\theta}} \sum_{d|q} \sum_{m|d} \mu(m) \left( \frac{d}{m} \right)^{1-\theta} \times \]
\[ \times K \left( \frac{\nu_1 \nu_4 m}{d}, 1 - \theta \right) K \left( \frac{\nu_2 \nu_3 m}{d}, 1 - \theta \right) = \sum_{d \leq X^2} \sum_{m|d} \mu(m) \left( \frac{d}{m} \right)^{1-\theta} |g(d, m)|^2, \]
where
\[ g(d, m) = \sum_{\nu_1 \nu_4 \equiv 0 (\mod d)} \frac{\beta(\nu_1) \beta(\nu_4)}{\nu_1^{1-\theta} \nu_4} K \left( \frac{\nu_1 \nu_4 m}{d}, 1 - \theta \right). \]

Further, represent numbers \( \nu_j \) in the form \( \nu_j = \delta_j \nu_j' \), where \( (\nu_j', d) = 1 \), and all prime divisors of \( \delta_j \) are those (coincident) of \( d \), i.e., \( \delta_j \in M_1(d) \). Then
\[ g(d, m) = \sum_{\delta_1 \delta_4 \equiv 0 (\mod d)} \frac{1}{(\delta_1^{1-\theta} \delta_4)} \sum_{\delta_j \leq X, (\nu_j, d) = 1} \frac{\beta(\delta_1 \nu_1) \beta(\delta_4 \nu_4)}{\nu_1^{1-\theta} \nu_4} K \left( \frac{\delta_1 \delta_4 m}{d}, \nu_1 \nu_4, 1 - \theta \right). \]

From the definition (3) of \( \beta(\nu) \) it follows, that
\[ g(d, m) = \frac{1}{\ln^2 X} \sum_{\delta_1 \delta_4 \equiv 0 (\mod d)} \frac{\alpha(\delta_1) \alpha(\delta_4)}{\delta_1^{1-\theta} \delta_4} K \left( \frac{\delta_1 \delta_4 m}{d}, 1 - \theta \right) \times \]
\[ \times \sum_{\nu_j \leq X/\delta_1, (\nu_j, d) = 1} \frac{\alpha(\nu_j) \alpha(\nu_4)}{\nu_1^{1-\theta} \nu_4} K \left( \nu_1 \nu_4, 1 - \theta \right) \ln \frac{X}{\delta_1 \nu_1} \ln \frac{X}{\delta_4 \nu_4}. \]

We apply the following identity, which is valid for two multiplicative functions \( f_1 \) and \( f_2 \), that are non-zero only for square free numbers:
\[ \sum_{l_1 \leq y_1, (l_1, a) = 1, (l_1, t_2) = 1} f_1(l_1) f_2(l_2) f(l_1, l_2) = \sum_{r, a = 1} \mu(r) f_1(r) f_2(r) \sum_{\lambda_1 r \leq y_1, (\lambda_1, ra) = 1} \sum_{\lambda_2 r \leq y_2, (\lambda_2, ra) = 1} f_1(\lambda_1) f_2(\lambda_2) f(r \lambda_1, r \lambda_2). \]
Therefore, we get
\[
g(d, m) = \frac{1}{\ln^2 X} \sum_{\substack{\delta_1 \delta_4 \equiv 0 \pmod{d} \\
\delta_j \in \mathcal{M}_1(d)}} \frac{\alpha(\delta_1) \alpha(\delta_4)}{\delta_1 \delta_4} K \left( \frac{\delta_1 \delta_4 m}{d}, 1 - \theta \right) \times
\]
\[
\times \sum_{(n, d) = 1} \frac{\alpha^2(n)}{n^{2 - \vartheta}} K(n^2, 1 - \theta) \sum_{(r, dn) = 1} \mu(r) \alpha^2(r) K^2(r, 1 - \theta) \times
\]
\[
\left( \sum_{\lambda_1 \delta_1 nr \leq X, (\lambda_1, dnr) = 1} \frac{\alpha(\lambda_1) K(\lambda_1, 1 - \theta)}{\lambda_1^{1 - \vartheta}} \ln \frac{X}{\delta_1 nr \lambda_1} \right) \times
\]
\[
\left( \sum_{\lambda_4 \delta_4 nr \leq X, (\lambda_4, dnr) = 1} \frac{\alpha(\lambda_4) K(\lambda_4, 1 - \theta)}{\lambda_4^{1 - \vartheta}} \ln \frac{X}{\delta_4 nr \lambda_4} \right).
\]

Further we show that, for \(0 \leq \theta, \gamma \leq \frac{1}{4}, N \geq 1, X_1 \geq 1\), the following estimate holds
\[
S_\theta(X_1, \gamma, N) = \sum_{\lambda \leq X_1, (\lambda, N) = 1} \frac{\alpha(\lambda) K(\lambda, 1 - \theta)}{\lambda^{1 - \gamma}} \ln \frac{X_1}{\lambda} \ll X_1^\gamma \sqrt{\ln(X_1 + 2)} \prod_{p \mid N} \left( 1 + \frac{1}{p} \right)^2.
\]
(13)

The equality
\[
\prod_{p > 256} \left( 1 + \frac{|r(p)|}{p^s} + \frac{|r(p)|^2}{4p^{2s}} \right) = \prod_{p > 256} \left( 1 + \frac{|r(p)|^2}{2p^s} \right)^2
\]
\[
= \left( \sum_{n=1}^{\infty} \frac{|\alpha(n)|}{n^s} \right)^2 = \sum_{n=1}^{\infty} \frac{1}{n^s} \left( \sum_{n_1 n_2 = n} |\alpha(n_1) \alpha(n_2)| \right)
\]
entails that the function \(b(n) = \sum_{n_1 n_2 = n} |\alpha(n_1) \alpha(n_2)|\) is multiplicative, and that also due to the inequality \(|r(p)| < 4\) (where \(p\) is any prime number), for \(n \mid d\),
\[
b(nd) \leq b(d).
\]

Using this inequality and the estimate \(|K(n, 1 - \theta)| \leq \tau_6(n)\) (see (12)), we find that
\[
\sum_{\delta_1 \delta_4 \equiv 0 \pmod{d}, \delta_j \in \mathcal{M}_1(d)} \frac{|\alpha(\delta_1) \alpha(\delta_4)|}{\delta_1 \delta_4} K \left( \frac{\delta_1 \delta_4 m}{d}, 1 - \theta \right) \leq \frac{1}{d} \sum_{n \mid d} \frac{\tau_6(n)}{n} |K(nm, 1 - \theta)| \sum_{\delta_1 \delta_4 = nd} |\alpha(\delta_1) \alpha(\delta_4)|
\]
\[
\leq \tau_6(m) \frac{b(d)}{d} \prod_{p \mid d} \left( 1 + \frac{\tau_6(p)}{p} \right).
\]
From this and (13) we get

$$S(\theta) \ll X^{2\theta} (\ln X)^{-2} \sum_{d \leq X^2} \frac{b^2(d)}{d^{1+\theta}} \prod_{p | d} \left(1 + \frac{1}{p}\right)^8 \left(1 + \frac{\tau_6(p)}{p}\right)^2 \left(1 + \frac{\tau_6^2(p)}{p^{1-\theta}}\right) \sum_{m | d} \frac{\mu^2(m)\tau_6^2(m)}{m^{1-\theta}}$$

$$\leq X^{2\theta} (\ln X)^{-2} \sum_{d \leq X^2} \frac{b^2(d)}{d^{1+\theta}} \prod_{p | d} \left(1 + \frac{1}{p}\right)^8 \left(1 + \frac{\tau_6(p)}{p}\right)^2 \left(1 + \frac{\tau_6^2(p)}{p^{1-\theta}}\right).$$

Now if $p$ is sufficiently large, and $0 \leq \theta \leq \frac{1}{4}$, then

$$\left(1 + \frac{1}{p}\right)^8 \left(1 + \frac{\tau_6(p)}{p}\right)^2 \left(1 + \frac{\tau_6^2(p)}{p^{1-\theta}}\right) \leq 1 + \frac{1}{\sqrt{p}};$$

whence,

$$\prod_{p | d} \left(1 + \frac{1}{p}\right)^8 \left(1 + \frac{\tau_6(p)}{p}\right)^2 \left(1 + \frac{\tau_6^2(p)}{p^{1-\theta}}\right) \ll \prod_{p | d} \left(1 + \frac{1}{\sqrt{p}}\right) \leq \sum_{m | d} \frac{1}{\sqrt{m}}.$$

From this inequality we find:

$$S(\theta) \ll X^{2\theta} (\ln X)^{-2} \sum_{d \leq X^2} \frac{b^2(d)}{d^{1+\theta}} \sum_{m | d} \frac{1}{\sqrt{m}}$$

$$\leq X^{2\theta} (\ln X)^{-2} \sum_{m \leq X^2} \frac{1}{\sqrt{m}} \sum_{d \leq X^2} \frac{b^2(d)}{d^{1+\theta}}$$

$$\leq X^{2\theta} (\ln X)^{-2} \sum_{m \leq X^2} \frac{b^2(m)}{m^{3/2+\theta}} \sum_{d \leq X^2} \frac{b^2(d)}{d^{1+\theta}}$$

$$\ll X^{2\theta} (\ln X)^{-2} \sum_{d \leq X^2} \frac{b^2(d)}{d} \ll X^{2\theta} (\ln X)^{-1}.$$

The last estimate in the previous formula is provided by the following one

$$\sum_{n=1}^{+\infty} \frac{b^2(n)}{n^s} = \prod_{p>256} \left(1 + \frac{|r(p)|^2}{p^s} + \frac{|r(p)|^4}{16p^{2s}}\right) = \prod_{p>256} \left(1 + \frac{|r(p)|^4}{16p^{2s}(1 + \frac{|r(p)|^2}{p^s})}\right)$$

$$= \sum_{n=1}^{+\infty} \frac{1}{n^s} \left( \sum_{n_1n_2=n} b_1(n_1)b_2(n_2) \right),$$

where

$$\sum_{n=1}^{+\infty} \frac{b_1(n)}{n^s} = \prod_{p>256} \left(1 + \frac{|r(p)|^2}{p^s}\right), \quad \sum_{n=1}^{+\infty} \frac{b_2(n)}{n^s} = \prod_{p>256} \left(1 + \frac{|r(p)|^4}{16p^{2s}(1 + \frac{|r(p)|^2}{p^s})}\right)$$

and, thus,

$$\sum_{d \leq X^2} \frac{b^2(d)}{d} \leq \sum_{n_1n_2 \leq X^2} \frac{b_1(n_1)b_2(n_2)}{n_1n_2} \leq \prod_p \left(1 + \frac{|r(p)|^4}{16p^2(1 + \frac{|r(p)|^2}{p})}\right) \sum_{n_1 \leq X^2} \frac{|r(n_1)|^2}{n_1} \ll \ln X,$$
in view of the equality (20)

$$\sum_{n \leq x} |r(n)|^2 = cx + O(x^{3/5}), \quad c > 0$$

(see 21 for $O$-estimate of the present sum, which is also enough to obtain the desired result).

We then left to show the truth of estimate (13) for the sum $S_\theta(X_1, \gamma, N)$. Without loss of generality, we may assume that $X_1 \geq 10$, and that, for every $p|N$, the condition $p > 256$ is fulfilled. For $\text{Re } s > 1$, consider the generated function

$$h_{\theta,N}(s) = \sum_{(n,N)=1} \frac{\alpha(n)K(n,1-\theta)}{n^s} = \prod_{p|N} \left(1 + \frac{\alpha(p)K(p,1-\theta)}{p^s}\right)$$

$$= \prod_{p>256 \atop (p,N)=1} \left(1 - \frac{r(p)K(p,1-\theta)}{2p^s}\right) = \prod_{p} \left(1 + \frac{|r(p)|^2}{p^s} + \frac{|r(p^2)|^2}{p^{2s}} + \ldots\right)^{-1/2} \times$$

$$\times \prod_{p>256 \atop (p,N)=1} \left(1 + \frac{|r(p)|^2}{p^s} + \frac{|r(p^2)|^2}{p^{2s}} + \ldots\right)^{1/2} \left(1 - \frac{r(p)K(p,1-\theta)}{2p^s}\right) \times$$

$$\times \prod_{p|(256)!N} \left(1 + \frac{|r(p)|^2}{p^s} + \frac{|r(p^2)|^2}{p^{2s}} + \ldots\right)^{1/2}.$$

If $0 \leq \theta \leq 1/4$ then the product

$$N_1(s,\theta) = \prod_{p>256 \atop (p,N)=1} \left(1 + \frac{|r(p)|^2}{p^s} + \frac{|r(p^2)|^2}{p^{2s}} + \ldots\right)^{1/2} \left(1 - \frac{r(p)K(p,1-\theta)}{2p^s}\right)$$

$$= \prod_{p>256 \atop (p,N)=1} \left(1 + \frac{|r(p)|^2}{2p^s} + O\left(\frac{1}{p^{2\sigma}}\right)\right) \left(1 - \frac{|r(p)|^2}{2p^s} + O\left(\frac{1}{p^{\sigma+1-\theta}}\right)\right)$$

$$= \prod_{p>256 \atop (p,N)=1} \left(1 + O\left(\frac{1}{p^{2\sigma}}\right) + O\left(\frac{1}{p^{\sigma+1-\theta}}\right)\right), \quad s = \sigma + it,$$

defines an analytic function in the half-plane $\text{Re } s > 1/2$. For the generated function we have the identity

$$h_{\theta,N}(s) = (D(s))^{-1/2}N_1(s,\theta)G_N(s),$$

where

$$D(s) = \sum_{n=1}^{+\infty} \frac{|r(n)|^2}{n^s},$$
\[ G_N(s) = \prod_{n \neq 256N} \left(1 + \frac{|r(p)|^2}{p^s} + \frac{|r(p^2)|^2}{p^{2s}} + \ldots \right)^{1/2}. \]

By means of Perron summation formula we find that

\[ S_\theta(X_1, \gamma, N) = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} h_{\theta, N}(s + 1 - \gamma) \frac{X_1^s}{s^2} ds = \frac{X_1^\gamma}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{X_1^s}{(s + \gamma)^2} \frac{N_1(s + 1, \theta)G_N(s + 1)}{(D(s + 1))^{1/2}} ds. \]

For \( \text{Re } s \geq 1 \), the following estimates are valid:

\[ N_1(s, \theta) \ll 1, \quad 0 \leq \theta \leq 1/4, \]

\[ G_N(s) \ll \prod_{p \mid N} \left(1 + \frac{1}{p}\right)^2 = G_N. \]

Move the path of integration from the line \( \text{Re } s = 1 \) to the contour constructed by the semicircle \( \{|s| = (\ln X_1)^{-1}, \ \text{Re } s \geq 0\} \) and the two rays \( \{s = \pm \pm \sqrt{t}, |t| \geq (\ln X_1)^{-1}\} \). For \( D(s) \) in the region \( \text{Re } s \geq 1 \) we shall use an estimate \( |D(s)|^{-1} \ll |s - 1| \). The integral over the semicircle (which we denote as \( K_1 \)) can be estimated in the following way

\[ K_1 \ll X_1^\gamma G_N \frac{X_1^{(\ln X_1)^{-1}}}{(\ln X_1)^2} \ll X_1^\gamma (\ln X_1)^{1/2} G_N. \]

For the integral over the rays (which we denote as \( K_2 \)), the relation

\[ K_2 \ll X_1^\gamma G_N \int_{(\ln X_1)^{-1}}^{\infty} \frac{t^{1/2}}{t^2} dt \ll X_1^\gamma (\ln X_1)^{1/2} G_N. \]

holds. Thus, the estimate (13) is obtained and, therefore, the lemma is proved. \( \square \)

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