Stable and Unstable Operations in Algebraic Cobordism

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Abstract

We describe additive (unstable) operations from a theory $A^*$ obtained from Algebraic Cobordism $\Omega^*$ of M.Levine-F.Morel by change of coefficients to any oriented cohomology theory $B^*$. We prove that there is 1-to-1 correspondence between the set of operations, and the set of transformations: $A^n((\mathbb{P}^\infty)\times r) \rightarrow B^m((\mathbb{P}^\infty)\times r)$ satisfying certain simple properties. This provides an effective tool of constructing such operations. As an application, we prove that (unstable) additive operations in Algebraic Cobordism are in 1-to-1 correspondence with the $L\otimes\mathbb{Q}$-linear combinations of Landweber-Novikov operations which take integral values on the products of projective spaces. On our way we obtain that stable operations there are exactly $L$-linear combinations of Landweber-Novikov operations. We also show that multiplicative operations $A^* \rightarrow B^*$ are in 1-to-1 correspondence with the morphisms of the respective formal group laws. We construct Integral (!) Adams Operations in Algebraic Cobordism, and all theories obtained from it by change of coefficients, giving classical Adams operations in the case of $K_0$. Finally, we construct Symmetric Operations for all primes $p$ (these operations in $\Omega^*$, previously known only for $p = 2$, are more subtle than the Landweber-Novikov operations, and have applications to rationality questions - [23],[22],[24]), as well as the T.tom Dieck - style Steenrod operations in Algebraic Cobordism.

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1 Introduction

In the current article we study operations between Generalized Oriented Cohomology Theories. An interest in this subject arose from the prominent role which cohomological operations play in Topology, as well as from the already known successful applications of them in Algebro-Geometric context. Not mentioning the classical applications of Adams operations in Algebraic K-theory, it starts with the crucial use of Milnor’s operations in motivic cohomology by V.Voevodsky in his proof of Milnor’s Conjecture ([27] and [26]). In the process he had to construct the whole Steenrod algebra in motivic cohomology with \( \mathbb{Z}/l \)-coefficients. The restriction of this algebra to Chow groups modulo \( l \) was later produced by P.Brosnan ([4]) by an elementary construction using equivariant Chow groups. Meanwhile, I.Panin and A.Smirnov ([15],[16]) were studying multiplicative operations between arbitrary theories, and their relation to the orientation, resulting in such statements as the general Riemann-Roch Theorem. Finally, M.Levine and F.Morel produced the universal generalized oriented cohomology theory - the Algebraic Cobordism \( \Omega^* \) ([12],[13]). The universality of \( \Omega^* \) combined with the reorientation procedure of I.Panin-A.Smirnov (following D.Quillen [18]) permitted to produce the multiplicative operations \( \Omega^* \rightarrow B^* \) easily and to classify them (in the ”invertible” case). In particular, one gets that all such operations are specializations of the Total Landweber-Novikov operation \( \Omega^* \rightarrow \Omega^*[b_1, b_2, \ldots] \). And the resulting Landweber-Novikov algebra plays an important role in the study of Algebraic Cobordism, and other free theories of M.Levine-F.Morel. In all the mentioned cases, with the exception of Adams operations, the operations were (essentially) stable. As a rare example of unstable operations (in the algebro-geometric context), the, so-called, Symmetric operations (mod 2) were introduced in [21] and [23]. Originally constructed with the aim of producing maps between Chow groups of different quadratic Grassmannians (of the same quadratic form), these operations in Algebraic Cobordism of M.Levine-F.Morel were successfully applied to the question of rationality of algebraic cycles ([22],[24]), where they provide the only known method to deal with 2-torsion. These operations can be combined into a Total one which is a ”formal half” of the ”negative part” of the Total Steenrod operation (mod 2) in Algebraic Cobordism. The topological counterpart of it was used by D.Quillen in [18]. Being more subtle than the Landweber-Novikov operations, the Symmetric ones (mod 2), in some sense, ”plug the gap” between \( L \) and \( H_*(MU) \) left by the Hurewicz map, plug 2-adically. To have an integral variant of such statements one would need Symmetric operations for all primes. Unfortunately, the case \( p = 2 \) was produced by an explicit geometric construction (using \( \text{Hilb}_2 \)), and it is unclear how to extend it for other primes. The desire to construct these operations was the main motivation behind the current article. In the end, it appeared that to produce Symmetric operation for \( p > 2 \) is about as ”simple” as to produce all (unstable) additive operations in Algebraic Cobordism. But to do it, one has to develop some new tools. To start with, one has to understand better the theory itself - what is it which distinguishes it from other theories of ”low quality”? What, I think, was underestimated previously, is that, in contrast to Topology, the general object satisfying the
Definition of the Generalized Oriented Cohomology Theory is not particularly good. And so, if one wants to obtain a result reminiscent of a topological one, one has to restrict attention to a very special class of theories. The picture which emerges suggests that there are "topological quality" theories parameterized by a formal group law $L \rightarrow A$, and a non-negative integer $n$, while "inbetween" and "across" there is an ocean of "low-grade" theories. The property of Algebraic Cobordism which permits us to deal with the unstable operations successfully, is that this theory can be defined inductively on dimension. Namely, that $\Omega^*(X)$ can be described in terms of $\Omega^*$ of smaller-dimensional varieties (and some explicit relations). This leads to the notion of a theory of rational type ($= \text{type } 0$). Such theories appear to be the same as free theories of M.Levine-F.Morel, and are the best theories available there. In particular, all the "standard" theories, like, $\text{CH}$, $\text{K}_0$, $\text{BP}$, higher Morava’s K-theories $K(n)$ are of this sort. At this stage I should recall that there are two types of cohomology theories in Algebraic Geometry: "large" ones $A_{j,i}^{i,j}$ - represented by some spectrum in $A^1$-homotopy theory, numbered by two indices, and "small" ones $A^i$, typically, represented by the "pure parts" $A^{2i,i}$ of large theories. The Algebraic Cobordism $\Omega$ of M.Levine-F.Morel belongs to the second type and, by the result of M.Levine ([11], see also [8]), is represented by the pure part of $MGL$ of V.Voevodsky. In this article, we work with "small" theories. The fact that $\Omega^*$ is a theory of rational type is non-trivial. Our proof uses the mentioned comparison result of M.Levine ([11]). Any theory $A^*$ of rational type can be inductively described in terms of generators (of smaller dimension) and relations. We provide three alternative descriptions here: two in terms of push-forwards, and one in terms of pull-backs - see Subsections 4.1, 4.2, 4.3. After that it becomes possible to construct operations inductively on dimension.

This enables us to show that an operation can be reconstructed from it’s action on $(\mathbb{P}^\infty)^r$, for all $r$. This is our Main result Theorem 5.1.

**Theorem 1.1** Let $A^*$ be a theory, obtained from $\Omega^*$ by change of coefficients, and $B^*$ be any theory in the sense of Definition 2.1. Fix $n, m \in \mathbb{Z}$. Then there is one-to-one correspondence between additive operations $A^n \xrightarrow{G} B^m$ and transformations $A^n((\mathbb{P}^\infty)^l) \xrightarrow{G} B^m((\mathbb{P}^\infty)^l)$, for $l \in \mathbb{Z}_{\geq 0}$

commuting with the pull-backs for:

(i) the action of $\mathfrak{S}_l$;

(ii) the partial diagonals;

(iii) the partial Segre embeddings;

(iv) $(\text{Spec}(k) \hookrightarrow \mathbb{P}^\infty) \times (\mathbb{P}^\infty)^r$, $\forall r$;

(v) the partial projections.

In Topology an analogous result was obtained by T.Kashiwara in [9, Theorem 4.2]. The "multiplicative" variant of our result (Proposition 5.18) says that multiplicative operations correspond to transformations as above commuting also with the external products of projective spaces. These results permit to describe and construct operations effectively, as one only needs to define them on $(\mathbb{P}^\infty)^r$, which is a cellular space. As a first application, we describe all additive (unstable) operations in Algebraic Cobordism of M.Levine-F.Morel. These appears to be exactly those $L \otimes_{\mathbb{Q}} \mathbb{Q}$-linear combinations (infinite, in general) of the Landweber-Novikov operations which take "integral" values on $\Omega^*((\mathbb{P}^\infty)^r)$, for all $r$. This is done in Theorem 6.1.
The subject. I want to thank O.Hauton, with whom we tried to produce the geometric construction discussions since our 2004-2005 common stay at IAS. These really influenced my way of thinking about geometric problems.

is cellular), we hint in Subsection 4.4 at possible direction which could help with the "real" algebro-

from Topology, thus creating a theory represented by a cellular spectrum (since everything in Topology

noting that topological application in Algebraic Geometry so far were restricted to simply pulling theories

extends to a unique stable operation), and consist of Steenrod operations only (Theorem 6.6). Finally,

if and only if the shifted FGL $F^\gamma = \psi$ in Theorem 6.9:

$(\varphi_G, \gamma_G)$ defines a 1-to-1 correspondence between the multiplicative operations $A^* \xrightarrow{G} B^*$ and the homomorphisms $(A, F_A) \to (B, F_B)$ of the respective formal group laws.

This reduces the classification of such multiplicative operations to algebra. In particular, we extend the result of I.Panin-A.Smirnov-M.Levine-F.Morel on multiplicative operations $\Omega^* \to B^*$ to the case where $b_0$ is not a zero divisor (compare with: "$b_0$-invertible", as in the original Theorem). This is done in Theorem 6.9.

Let $B^*$ be any theory in the sense of Definition 2.1, and $b_0 \in B$ be not a zero-divisor. Let $\gamma = b_0 x + b_1 x^2 + b_2 x^3 + \ldots \in B[[x]]$. Then there exists a multiplicative operation $\Omega^* \xrightarrow{G} B^*$ with $\gamma_G = \gamma$ if and only if the shifted FGL $F^\gamma_B \in B[b_0^{-1}][[x, y]]$ has coefficients in $B$ (that is, has no denominators). In this case, such an operation is unique.

As an immediate application of this we construct Integral Adams operations $\Psi_k$ in Algebraic Cobordism and all other theories of rational type. This is Theorem 6.15.

For any theory of rational type $A^*$, there are multiplicative (unstable) $A$-linear operations $\Psi_k : A^* \to A^*$, $k \in \mathbb{Z}$, such that $\gamma_{\Psi_k} = [k] \cdot A x$.

In the case of $K_0$ these are usual Adams operations.

As these unstable multiplicative operations are $A$-linear, they are all obtained from the ones in Algebraic Cobordism by change of coefficients. Previously, in the case of Algebraic Cobordism, such operations were known only $\otimes \mathbb{Q}$ (in which case they can be expressed through Landweber-Novikov operations), but this way, the torsion was lost.

Similar considerations permit to construct the T.tom Dieck - style Steenrod operations in Algebraic Cobordism - Theorem 6.17 (an object more subtle than the Quillen's style Steenrod operations - see Subsection 6.4). Finally, using the Main Theorem 6.1 itself we construct Symmetric operations for all primes $p$ - see Theorem 6.18. The last two results form a separate paper [25], not to overburden the given text. Aside from the mentioned major results we present various smaller ones - see Sections 6 and 7. In particular, we show that all operations in Chow groups mod $p$ are essentially stable (each extends to a unique stable operation), and consist of Steenrod operations only (Theorem 6.6). Finally, noting that topological application in Algebraic Geometry so far were restricted to simply pulling theories from Topology, thus creating a theory represented by a cellular spectrum (since everything in Topology is cellular), we hint in Subsection 6.4 at possible direction which could help with the "real" algebro-geometric problems.

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2 Algebraic Cobordism and other generalized oriented cohomology theories

2.1 Main definitions

Throughout the article $k$ will denote the base field of characteristic 0. $\text{Sm}_k$ will denote the category of smooth quasi-projective varieties over $k$, and $\text{Sch}_k$ - the category of separated schemes of finite type over $k$. Let $R^*$ be the category of graded commutative rings.

Following M.Levine-F.Morel ([12, Definition 1.1.2]) and D.Quillen ([18]) we introduce the notion of the generalized oriented cohomology theory on $\text{Sm}_k$. The only difference in comparison with [12, Definition 1.1.2] is that we impose the localization axiom $\text{(EXCI)}$. All the "standard" theories, like $\Omega^*$, $\text{CH}^*$ and $K_0$ do satisfy this axiom, but not their $A^1$-analogues $MGL^*, H^*, M$ and $K_*$. So, this axiom restricts our choice to pure parts of $A^1$-theories: $A^2*, M$. And I should point out that this axiom (together with the axiom $\text{(CONST)}$ below) is crucial for everything what follows - the main idea of the article is based on it.

**Definition 2.1** (cf. [12, Definition 1.1.2]) A generalized oriented cohomology theory on $\text{Sm}_k$ is given by:

1. **(D1)** An additive (pull-back) functor $A^*: \text{Sm}_k^{\text{op}} \to R^*$.

2. **(D2)** The structure of push forwards: for each projective morphism $f: Y \to X$ of relative codimension $d$, a homomorphism of graded $A^*(X)$-modules:

$$f_*: A^*(Y) \to A^{*+d}(X)$$

satisfying:

3. **(A1)** functoriality of push-forwards: $(\text{Id}_X)_* = \text{Id}_{A^*(X)}$, and

For projective morphisms $f: Y \to X$, $g: Z \to Y$ of relative codimensions $d$ and $e$,

$$(f \circ g)_* = f_* \circ g_*: A^*(Z) \to A^{*+d+e}(X).$$

4. **(A2)** For pair of transversal morphisms $f: X \to Z$, $g: Y \to Z$ fitting into the cartesian square

$$\begin{array}{ccc}
W & \xrightarrow{g'} & X \\
\downarrow f & & \downarrow f \\
Y & \xrightarrow{g} & Z
\end{array}$$

with $f$ projective of relative dimension $d$,

$$g^* f_* = f'_* g'^*.$$
(PB) For a rank $n$ vector bundle $E \rightarrow X$ with the canonical quotient line bundle $\mathcal{O}(1) \rightarrow \mathbb{P}(E)$, the zero section $s : \mathbb{P}(E) \rightarrow \mathcal{O}(1)$, and $\xi \in A^1(\mathbb{P}(E))$ defined by

$$\xi := s^*s_*(1),$$

one has: $A^*(\mathbb{P}(E))$ is a free $A^*(X)$-module with the basis

$$(1, \xi, \xi^2, \ldots, \xi^{n-1}).$$

(EH) For a vector bundle $E \rightarrow X$ and an $E$-torsor $p : V \rightarrow X$ one has: $p^* : A^*(X) \rightarrow A^*(V)$ is an isomorphism.

(EXCI) For a smooth quasi-projective variety $X$ with closed subscheme $Z \rightarrow X$ and open complement $U \rightarrow X$, one has an exact sequence:

$$A_*(Z) \xrightarrow{i_*} A_*(X) \xrightarrow{j^*} A_*(U) \rightarrow 0,$$

where $A_*(Z) := \lim_{\rightarrow V \rightarrow Z} A_*(V)$ - the limit taken over all projective maps from smooth varieties to $Z$, and for a $d$-dimensional variety $T$, $A_*(T) := A^{d-*}(T)$.

**Remark 2.2** 1) Notice, that (D2) contains the projection formula.

2) The extended homotopy property (EH) follows from the "usual" homotopy and (PB).

Whenever we refer to the generalized oriented cohomology theory in the sense of Definition 2.1, we will mean the theory satisfying the above set of axioms.

Quite often (especially, in our main results) we will need to impose an additional condition demanding our theory to be constant along field extensions. To formulate this condition, we need to remind that originally our theory was defined only for smooth quasi-projective varieties over $k$, in particular, for varieties of finite type. But one can extend it for localizations of such varieties by approximating them by finite-type ones. In particular, following M.Levine and F.Morel ([12, Subsection 4.4.1]), for any finitely generated field extension $L/k$ we can define $A^*(L)$ as $\text{colim}_{U \subset X} A^*(U)$ where $U$ runs over all open subsets of some smooth model $X$ with $k(X) = L$ (recall, that we are in characteristic zero, so all field extensions are separable). Then we have the notion of a generically constant theory of M.Levine-F.Morel - see [12 Definition 4.4.1]. I will call such theories just constant.

(CONST) The theory is called "constant" if the natural map $A^*(k) \rightarrow A^*(L)$ is an isomorphism, for each finitely generated field extension $L/k$.

All the standard theories are constant, but it is very easy to construct a non-constant one.

**Example 2.3** Let $A^*$ be any theory (say, a constant one), and $Y$ be a smooth quasi-projective variety over $k$. Then we can define a new theory: $A^*_{Y/k}(X) := A^*(Y \times_{\text{Spec}(k)} X)$. For example, we can take $Y = \text{Spec}(L)$, where $L/k$ is a finite field extension. This theory will not be constant. For example, if $L/k$ is Galois of degree $n$, then $A^*_{L/k}(\text{Spec}(L)) = \oplus_{i=1}^n A^*(\text{Spec}(L))$, while $A^*_{L/k}(\text{Spec}(k)) = A^*(\text{Spec}(L))$.

M.Levine and F.Morel coconstructed the universal generalized oriented cohomology theory $\Omega^*$ called Algebraic Cobordism (see [12 Theorem 1.2.6]). It has unique map to any other theory $A^*$. This theory satisfies (CONST). Being an analogue of complex-oriented cobordism theory $MU^{2*}$ in topology, (for fields possessing an embedding to $\mathbb{C}$) it has a topological realization functor $\Omega^*(X) \rightarrow MU^{2*}(X(\mathbb{C}))$, which is an isomorphism for $X = \text{Spec}(k)$.
2.2 An associated Borel-Moore theory

Each generalised oriented cohomology theory on $\text{Sm}_k$ can be extended to the Borel-Moore functor on $\text{Sch}_k$ in the sense of [12, Definition 2.1.2] - see [12, Remark 2.1.4]. We will not need most of the features of such a functor, only the push-forward maps which are completely straightforward, so will not list it’s axioms here. Later, in Subsection 4.3 in the case of theories of rational type we will need the refined pull-backs, but these will be supplied by the Algebraic Cobordism case done by M. Levine and F. Morel.

**Definition 2.4** For a quasi-projective scheme $Z$ define $A_*(Z) := \lim_{V \to Z} A_*(V)$, where the limit is taken over all $v : V \to Z$ with $v$ projective and $V$ smooth.

Clearly, $A_*(Z) = A_*(Z_{\text{red}})$, and if $Z = \bigcup_{i=1}^m Z_i$ is the decomposition into irreducible components, then we have an exact sequence:

$$0 \leftarrow A_*(Z) \leftarrow \bigoplus_{i=1}^m A_*(Z_i) \leftarrow \bigoplus_{i,j=1}^m A_*(Z_i \cap Z_j).$$

More generally, for a closed embedding $S \subset Z$ with the open compliment $U$, we have an excision sequence:

$$0 \leftarrow A_*(U) \leftarrow A_*(Z) \leftarrow A_*(S).$$

A’priori, $A_*(Z)$ for a singular scheme $Z$ is expressed in terms of $A_*$ of infinitely many smooth schemes. But Proposition 7.7 shows that one can construct a “finite” presentation related to the resolution of singularities.

2.3 Formal group law

Any theory in the sense of Definition 2.1 (even without (EXCI)) has Chern classes. Namely, if $V$ is a vector bundle of dimension $d$ on $X$, then $\xi \in A^1(\mathbb{P}_X(V^\vee))$ (as in the axiom (PB)) satisfies the unique equation:

$$\sum_{i=0}^d (-1)^i c_i^A(V) \cdot \xi^{d-i} = 0,$$

where $c_i^A(V) = 1$, and $c_i^A(V) \in A^i(X)$ are some elements. These satisfy the usual Cartan formula, and in the case of a linear bundle $L$, $c_1^A(L) = s^*s_*(1)$, where $s : X \to L$ is a zero section. By [12, Theorem 2.3.13], any theory $A^*$ as above satisfies the axiom:

**(DIM)** For any line bundles $L_1, \ldots, L_n$ on a smooth $X$ of dimension $< n$, one has: $c_1^A(L_1) \cdot \ldots \cdot c_1^A(L_n) = 0 \in A_*(X)$.

Thus, any power series in Chern classes can be evaluated on any element of $A_*(X)$.

To any theory $A^*$ as above one can associate the Formal Group Law (FGL, for short) $(A, F_A)$, where $A$ is the coefficient ring of $A^*$, and

$$F_A(x, y) = \text{Segre}^*(t) \in A[[x, y]] = A^*(\mathbb{P}^\infty \times \mathbb{P}^\infty),$$

where $\mathbb{P}^\infty \times \mathbb{P}^\infty \xrightarrow{\text{Segre}} \mathbb{P}^\infty$ is the Segre embedding, and $x, y, t$ are the 1-st Chern classes of $O(1)$ of the respective copies of $\mathbb{P}^\infty$. We will denote the coefficients of $F_A$ as $a_{i,j}^A$. Thus,

$$F_A(x, y) = \sum_{i,j} a_{i,j}^A \cdot x^i \cdot y^j.$$
The formal group law describes how to compute the 1-st Chern class of a tensor product of two line bundles in terms of the 1-st Chern classes of factors:

\[ c_1^A(L \otimes M) = F_A(c_1^A(L), c_1^A(M)). \]

The universal formal group law \((L, F_U)\) has canonical morphism to any other formal group law, in particular, to \((A, F_A)\). M.Levine and F.Morel have shown that, in the case of algebraic cobordism, the respective map is an isomorphism - see [12, Theorem 1.2.7]. In particular, \(\Omega^*(k) = L^*\), for any field \(k\).

We call the theory \(A^*\) additive, if its FGL is additive. By the result of M.Levine-F.Morel - see [12, Theorem 1.2.2], the Chow groups \(\text{CH}^*\) is the universal additive (= ordinary) theory.

### 3 Operations

#### 3.1 Category SmOp

As in topology, operation is just a morphism of theories considered as contravariant functors on \(\text{Sm}_k\) (thus, commuting with pull-backs). Such operations appear to be of "various quality", and the best behaving of them are, so-called, stable operations. Most of the operations used are stable (as exceptions I can only recall Adams operations in K-theory, and Symmetric operations in Algebraic Cobordism (see [23])). Although, stable operations are simpler, and it is easy to construct them, they are less subtle than unstable ones. The aim of the current article is to develop an effective method of producing unstable operations. And, although, in the end, stable operations is not what we are after (there are more or less no questions left about them), they provide an important "coordinate system" in which one can study unstable ones. To be able to talk about "stability" we need to introduce some notion of suspension. Let me remind that the typical example of our theory \(A^*\) is the pure part \(A_{2*}\) of some \(A_1\)-theory. And so, we need a \(\mathbb{P}^1\)-suspension. Following V.Voevodsky and I.Panin-A.Smirnov, let me introduce:

**Definition 3.1** Category \(\text{SmOp}\) has objects \((X, U)\), where \(X\) is a smooth quasi-projective variety over \(k\), and \(U \hookrightarrow X\) is an open subvariety. Morphisms from \((X, U)\) to \((Y, V)\) are maps \(X \xrightarrow{f} Y\) which map \(U\) to \(V\). We have a natural functor:

\[ N : \text{Sm}_k \longrightarrow \text{SmOp}, \]

sending \(X\) to \((X, 0)\).

In \(\text{SmOp}\) we can define Smash-product by the formula:

\[ (X, U) \wedge (Y, V) := (X \times Y, X \times V \cup U \times Y), \]

which permits to introduce the suspension:

**Definition 3.2**

\[ \Sigma_T(X, U) := (X, U) \wedge (\mathbb{P}^1, \mathbb{P}^1 \setminus 0). \]

Any theory \(A^*\) in the sense of Definition 2.1 can be extended to a contravariant functor \(A^* : \text{SmOp} \rightarrow \text{Ab}\) as follows:

\[ A^*((X, U)) := \ker(A^*(X) \xrightarrow{i^*} A^*(U)), \]

with the pull-backs naturally induced by those in \(\text{Sm}_k\). We have an external product:

\[ A^*((X, U)) \otimes A^*((Y, V)) \xrightarrow{\wedge} A^*((X, U) \wedge (Y, V)), \]
and a canonical element $\varepsilon^A = c^1_T(\mathcal{O}(1)) \in A^1(\mathbb{P}^1, \mathbb{P}^1(0))$ - the class of a rational point. We get the natural isomorphism:

$$
\Sigma_T : A^n((X, U)) \xrightarrow{\sim} A^{n+1}(\Sigma_T(X, U)) \quad x \mapsto x \wedge \varepsilon^A.
$$

**Definition 3.3** Let $A^*$ and $B^*$ be theories in the sense of Definition 2.1. An operation $G : A^n \rightarrow B^m$ is a morphism of contravariant functors of sets pointed by 0 (in other words, a transformation commuting with pull-backs, and sending zero to zero). An operation is called additive, if it is a homomorphism of abelian groups.

Note, that such an operation extends uniquely to a morphism of contravariant functors on $\text{SmOp}$. Moreover, the condition $0 \rightarrow 0$ is equivalent to the existence of such an extension (since $A^*((X, X)) = 0$, and there exists morphism $(X, U) \rightarrow (X, X)$).

**Definition 3.4** A stable operation $G : A^* \rightarrow B^{*+l}$ is a set of operations $\{G^n : A^n \rightarrow B^{n+l}, n \in \mathbb{Z}\}$, which commute with $\Sigma_T$.

As one would expect,

**Proposition 3.5** Any stable operation is additive.

**Proof:** Let $\alpha, \beta, \gamma : (\mathbb{P}^1, \mathbb{P}^1(0)) \rightarrow (\mathbb{P}^1, (\mathbb{P}^1(0))^2)$ be defined as follows: $\alpha = id \times \infty$, $\beta = \infty \times id$, $\gamma = \Delta$. Let $R \in Ob(\text{SmOp})$, and $\delta_R = \delta \wedge id_R$. Then one easily shows that $\gamma_R^* = \alpha_R^* + \beta_R^*$.

Denote: $\varepsilon_1 \cdot x + \varepsilon_2 \cdot y + \varepsilon_1 \cdot \varepsilon_2 \cdot 0 \in A^*((\mathbb{P}^1, (\mathbb{P}^1(0))^2 \wedge R)$ as $(x, y, 0)$. Then

$$G(\Sigma_T(x + y)) = G(\gamma^*(x, y, 0)) = \gamma^*G(x, y, 0) = (\alpha^* + \beta^*)G(x, y, 0) = G(\Sigma_TX) + G(\Sigma_TY),$$

and since $G$ is stable, $G(x + y) = G(x) + G(y)$. \qed

**Definition 3.6** An operation $G : A^* \rightarrow B^*$ is multiplicative if, for each $X$, the respective transformation is a homomorphism of rings.

To each multiplicative operation one can assign certain power series - the inverse Todd genus $\gamma_G = b_0 \cdot x + b_1 \cdot x^2 + b_2 \cdot x^3 + \ldots \in B[[x]]$, where, for $x^A = c^1_T(\mathcal{O}(1))$, $x^B = c^1_B(\mathcal{O}(1))$, one has: $G(x^A) = \gamma_G(x^B) \in B[[x^B]] = B(\mathbb{P}^\infty)$. Also, we have $\varphi_G : A \rightarrow B$ - the homomorphism of coefficient rings. The pair $(\varphi_G, \gamma_G)$ is a morphism of formal group laws: $(A, F_A) \rightarrow (B, F_B)$. In other words,

$$\varphi_G(F_A)(\gamma_G(u), \gamma_G(v)) = \gamma_G(F_B(u, v)).$$

Of course, the composition of multiplicative operations corresponds to the composition of morphisms of formal group laws:

$$(\varphi_{H\circ G}, \gamma_{H\circ G}(x)) = (\varphi_H \circ \varphi_G, \varphi_H(\gamma_G)(\gamma_H(x))).$$

In the case of $A^* = \Omega^*$, and $b_0$ invertible in $B$, the homomorphism $\varphi_G$ is completely determined by $\gamma_G$. Namely, $L$ is generated as a ring by universal coefficients $a_{ij}^\Omega$, and $\varphi_G(a_{ij}^\Omega)$ is the respective coefficient of the formal group law $F_B^\gamma_G(x, y) = \gamma_G(F_B(\gamma_G^{-1}(x), \gamma_G^{-1}(y)))$. Moreover, we have a method to construct multiplicative operations, in this case:

**Theorem 3.7** (Panin-Smirnov-Levine-Morel) If $b_0$ is invertible in $B$, then for each $\gamma = b_0x + b_1x^2 + b_2x^3 + \ldots \in B[[x]]$, there exists unique multiplicative operation $G : \Omega^* \rightarrow B^*$ with such $\gamma_G$. 

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Proof: I.Panin and A.Smirnov have shown in \[16\] (see also \[12\] and \[10\]) that any reparametrization as above gives rise to the change of orientation (push-forward structure) on $B^*$, while M.Levine and F.Morel have proven universality of $\Omega^*$ (\[12\] Theorem 1.2.6)), which gives the morphism of theories $\Omega^* \to B^*$, which provides the needed operation, since the pull-back structure on $B^*$ and $\tilde{B}^*$ is the same. \qed

Below we will be able to generalize this result substantially - see Theorems \[6.9\] and \[6.13\].

The following statement describes the relation between stable and multiplicative operations.

**Proposition 3.8** Let $G : A^* \to B^*$ be a multiplicative operation with $\gamma_G = b_0 x + b_1 x^2 + \ldots$. Then $G$ is stable if and only if $b_0 = 1$.

**Proof**: Since $G$ is multiplicative on $\text{Sm}_k$, it will be multiplicative on $\text{SmOp}$ (w.r.to $\wedge$). Then

$$G(\Sigma_T x) = G(x \wedge \varepsilon^A) = G(x) \wedge G(\varepsilon^A) = G(x) \wedge (b_0 \varepsilon^B) = b_0 \Sigma_T(G(x)).$$

Thus we get the needed identity (for all $x$) if and only if $b_0 = 1$. \qed

**Example 3.9** Let $S_{L-N}^{\text{Tot}} : \Omega^* \to \Omega^*[b_1, b_2, \ldots] = \Omega^*[\bar{b}]$ be the Total Landweber-Novikov operation. It is the multiplicative operation corresponding to the power series $x + b_1 x^2 + b_2 x^3 + \ldots$, where $b_i$ are independent variables (see \[12\] Example 4.1.25 and \[18\]). By Proposition 3.8 this operation is stable.

Any stable multiplicative operation $G : \Omega^* \to B^*$ is a specialization of $S_{L-N}^{\text{Tot}}$. Namely, for each such $G$ there exists unique morphism of theories $\theta_G : \Omega^*[b_1, b_2, \ldots] \to B^*$ such that $G = \theta_G \circ S_{L-N}^{\text{Tot}}$. This $\theta_G$ is the canonical morphism of theories on $\Omega^*$, and sends $b_i$’s to the coefficients of $\gamma_G$.

### 3.2 Stable operations in Algebraic Cobordism

We already have some examples of stable operations in $\Omega^*$ - the components of the Total Landweber-Novikov operation $S_{L-N}^{\text{Tot}}$. It appears that, basically, there is nothing else out there (as in topological case).

**Theorem 3.10** There exists natural 1-to-1 correspondence between the set $\text{Hom}_L(\mathbb{L}[\bar{b}], \mathbb{L})$, and the set of stable operations $\Omega^* \to \Omega^*$ given by: $\psi \leftrightarrow G_\psi$, where $G_\psi$ is the composition: $\Omega^* \xrightarrow{S_{L-N}^{\text{Tot}}} \Omega^*[\bar{b}] \xrightarrow{\otimes \psi} \Omega^*$.

**Proof**: Since $S_{L-N}^{\text{Tot}}$ and $\otimes \psi$ are stable operations, so is their composition. Let now $G$ be some stable operation $\Omega^* \to \Omega^*$. Then $G$ is additive. In particular, $G|_{\text{Spec}(k)}$ is additive homomorphism $\mathbb{L} \to \mathbb{L}$. Consider the commutative diagram:

$$\begin{array}{ccc}
\mathbb{L} & \xrightarrow{S_{L-N}^{\text{Tot}}} & \mathbb{L}[ar{b}] \\
\downarrow & & \downarrow \\
\mathbb{Z}[ar{d}] & \xrightarrow{S} & \mathbb{Z}[ar{d}][\bar{b}],
\end{array}$$

where the vertical maps are induced by the natural embedding of rings $\mathbb{L} \hookrightarrow \mathbb{Z}[d_1, d_2, \ldots]$ corresponding to the twist of the additive formal group law by the change of parameter: $\delta(y) = y + d_1 y^2 + d_2 y^3 + \ldots$, and $S$ maps $d_i$ to $e_i$ - the coefficient of the power series $\rho(y) = \beta(\delta(y))$, where $\beta(x) = x + b_1 x^2 + b_2 x^3 + \ldots$. In particular, the vertical maps are isomorphisms $\otimes \mathbb{Q}$,

$$\mathbb{Z}[ar{d}][\bar{b}] = \mathbb{Z}[ar{d}][\bar{e}] \quad \text{and} \quad \mathbb{L}[ar{b}] \otimes \mathbb{Q} = \mathbb{L}[ar{e}] \otimes \mathbb{Q}.$$
Thus, for our $\mathbb{Z}$-linear map $L \xrightarrow{G|_L} L$ there exists unique $L$-linear map $L[i] \xrightarrow{\psi_G} L \otimes_{\mathbb{Z}} \mathbb{Q}$ such that the composition

$$L \xrightarrow{S_{L-N}^{\text{Tot}}} L[i] \xrightarrow{\psi_G} L \otimes_{\mathbb{Z}} \mathbb{Q}$$

coincides with $G|_L \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider the operation:

$$H = G - \psi_G \circ S_{L-N}^{\text{Tot}} : \Omega^* \to \Omega^* \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Let us show that $H = 0$.

**Lemma 3.11** Let $A^* \xrightarrow{H} B^*$ be stable operation such that $H|_X = 0$. Then $H|_{X \times \mathbb{P}^1} = 0$.

**Proof:** The maps $\mathbb{P}^1 \xrightarrow{\pi} \text{Spec}(k)$ define the decomposition: $C^*(X \times \mathbb{P}^1) = C^*(X) \oplus C^*(\Sigma X)$ respected by additive operations. Moreover, since $H$ is stable, and $H|_X$ is zero, so is $H|_{\Sigma X}$. Hence, $H|_{X \times \mathbb{P}^1} = 0$. □

**Lemma 3.12** Suppose, $B$ has no torsion, and $A^* \xrightarrow{H} B^*$ is stable operation such that $H|_{\text{Spec}(k)} = 0$. Then $H|_{(\mathbb{P}^\infty)^x \times r} = 0$, for all $r$.

**Proof:** We need to show that $H|_{(\mathbb{P}^N)^x \times r} = 0$, for all $N$ and $r$. Consider the natural projection $p : ((\mathbb{P}^1)^{xN})^{xN} \to (\mathbb{P}^N)^{xN}$. Since $p^* : B^*((\mathbb{P}^N)^{xN}) \to B^*((\mathbb{P}^1)^{xN})$ is injective, by Lemma 3.11 $H|_{(\mathbb{P}^\infty)^x \times r} = 0$.

**Remark 3.13** The condition that $B$ has no torsion is essential. Take, for example $A^* = B^* = \text{CH}^* / 2$, and $H = G_1 - G_2$, where $G_1 = \text{id}$ with $\gamma_{G_1} = x$ and $G_2 = \text{St}^{\text{Tot}}$ with $\gamma_{G_2} = x + x^2$ - the Total Steenrod operation. Then $\varphi_{G_1} = \varphi_{G_2}$ since there exists only one homomorphism of rings $\mathbb{Z}/2 \to \mathbb{Z}/2$, and so $H|_{\text{Spec}(k)} = 0$. At the same time, clearly, $H|_{\mathbb{P}^\infty} \neq 0$.

**Proposition 3.14** Let $A^*$ satisfies (CONST), and $B^*$ be any theory in the sense of Definition 2.7. Let $A^* \xrightarrow{H} B^*$ be an additive operation (not necessarily stable!) such that $H|_{(\mathbb{P}^\infty)^x \times r} = 0$, for all $r$. Then $H = 0$.

**Proof:** Let us prove by induction on the dimension of $X$ that $H|_{X \times (\mathbb{P}^\infty)^x \times r} = 0$, for all $r$. The base $(\dim(X) = 0)$ follows from our conditions. Suppose $\dim(X) = d$, and the statement is known for varieties of smaller dimension. Since $A^*(X \times (\mathbb{P}^N)^{xN})$ is a free module over $A^*(X)$ with basis consisting of monomials $\overline{\xi^m} = \prod_{i=1}^r \xi_i^{m_i}$, where $\xi_i = c_i^1(O(1)_i)$, it is sufficient to prove that $H(x \cdot \overline{\xi^m}) = 0$, for any $x \in A^*(X)$, for any $\overline{m}$. Because $A^*$ satisfies (CONST), we have: $H(x|_{\text{Spec}(k(X))} \cdot \overline{\xi^m}) = 0$, and by additivity of $H$ we can assume that $x|_{\text{Spec}(k(X))} = 0$, that is, $x$ is supported on some closed subvariety $Y \subset X$ (here we use (EXCI)). By the result of Hironaka (see Theorem 3.22), there exists a permitted blow up $\pi : \tilde{X} \to X$ with centers over $Y$ and of dimension $< \dim(Y)$, such that the proper preimage $\tilde{Y}$ of $Y$ is smooth. Since $\pi^* : B^*(X) \to B^*(\tilde{X})$ is injective, it is sufficient to show that $H(\pi^*(x) \cdot \overline{\xi^m}) = 0$. We have:

$$\overline{\tilde{Y}} = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = X$$

$$\overline{\tilde{Y}} = Y_n \xrightarrow{\pi'_n} Y_{n-1} \xrightarrow{\pi'_{n-1}} \ldots \xrightarrow{\pi'_1} Y_1 \xrightarrow{\pi'_0} Y_0 = Y,$$

where $X_{i+1} = Bl_{Z_i \subset X_i}$, $Z_i \subset X_i$ is smooth of dimension $< \dim(Y)$. Let $y_i \in A^*(X_i)$ be some element with support on $Y_i$. Then $\pi^*_i(y_i) = y_{i+1} + u_{i+1}$, where $y_{i+1}$ has support in $Y_{i+1}$ and $u_{i+1}$ has support in the special divisor $\mathbb{P}_{Z_i}(\mathcal{N}_{Z_i \to X_i})$. 


Lemma 3.15 Let $A^* \xrightarrow{H} B^*$ be an additive operation, $u \in A^*(Z)$, and 
$X \leftarrow \xymatrix{ Z \ar[r]^g & \mathbb{P}_Z(N_f \oplus O) }$ be regular embeddings. Then 
\[
\{ H(g_*(u)) = 0 \} \Rightarrow \{ H(f_*(u)) = 0 \}.
\]

Proof: Let us use the deformation to the normal cone construction. We have varieties $\widetilde{W} = B!_{Z \times \{0\} \subset X \times \mathbb{A}^1}$, 
$\widetilde{Z} = Z \times \mathbb{A}^1$, $W_0 = \mathbb{P}_Z(N_f \oplus O)$, $W_1 = X \times \{1\}$, fitting into the diagram: 
\[
\xymatrix{ W_0 \ar[r]^{i_0} \ar[d]_{j_0} & \widetilde{W} \ar[r]^{i_1} \ar[d]_{j_1} & W_1 \ar[d]^f \\
Z \ar[r]^g & \widetilde{Z} & Z }
\]
with both squares transversal cartesian. Let $\widetilde{Z} \xrightarrow{p} Z$ be the natural projection. Since $B^*$ satisfies 
$(EXCI)$, $H(h_*(p^*(u)))$ has support in $\widetilde{Z}$. That is, $H(h_*(p^*(u))) = h_*(v)$, for some $v \in B^*(\widetilde{Z})$. Then 
i_0^*H(h_*(p^*(u))) = H(i_0^*h_*(p^*(u))) = H(g_*j_0^*p^*(u)) = H(g_*(u)) = 0 should be equal to $i_0^*h_*(v) = g_*j_0^*(v)$. But 
j_0^* is an isomorphism, and $g_*$ is an injection. Hence, $v = 0$, and so $H(h_*(p^*(u))) = 0$. This implies that: 
$0 = i_0^*H(h_*(p^*(u))) = H(i_0^*h_*(p^*(u))) = H(f_*j_1^*p^*(u)) = H(f_*(u))$. \hfill \square

Lemma 3.16 Let $V$ be vector bundle on $Z$, and $V = \mathbb{P}_Z(V)$. Let $A^* \xrightarrow{H} B^*$ be additive operation s.t. 
$H|_{Z \times (\mathbb{P}^\infty)^r} = 0$, $\forall r$. Then $H|_{V \times (\mathbb{P}^\infty)^r} = 0$, $\forall r$.

Proof: $A^*(V)$ as an $A^*(Z)$-module is generated by powers of $c_1^{\mathbb{A}}(O(1))$. There are very ample line bundles 
$L_1, L_2$ on $V$ such that $O(1) = L_1 \otimes L_2^{-1}$. Hence, any element in $A^*(V)$ can be written as an $A^*(Z)$-linear 
combination of $c_1^{\mathbb{A}}(L_1)^{m_1} \cdot c_1^{\mathbb{A}}(L_2)^{m_2}$. And each such element is a pull-back of a certain element 
from $A^*(Z \times (\mathbb{P}^\infty)^r)$. Thus, any element from $A^*(V \times (\mathbb{P}^N)^{\times r})$ is a sum of elements pulled back from 
$A^*(Z \times (\mathbb{P}^N)^{\times r})$, and so $H$ must be trivial on it. \hfill \square

Lemma 3.17 Let $Z \xleftarrow{f} T$ be a regular embedding (of smooth varieties). Then 
$\{ H|_{Z \times (\mathbb{P}^\infty)^r} = 0$, $\forall r \} \Rightarrow H$ is zero on the image $(f \times id)_* \subset A^*(T \times (\mathbb{P}^\infty)^{\times r})$, $\forall r$.

Proof: Follows immediately from Lemmas 3.15 and 3.16. \hfill \square

Since $u_{i+1}$ has support on a smooth subvariety $\mathbb{P}_{Z_i}(N_{Z_i \subset X_i})$, it follows from the inductive assumption 
and Lemma 3.17 that $H(u_{i+1} \cdot \xi^m) = 0$, and the same will be true when we will lift $u_{i+1}$ to $\bar{X}$. Take now 
y_0 = x, and construct the elements $y_i, u_i$ as above. Then $\bar{y}_n$ has support in $\bar{Y}$, and by the above, 
$H(\bar{y}_n \cdot \xi^m) = H(\pi^*(x) \cdot \xi^m)$. Thus, we can reduce to the case where $x$ has support on a smooth subvariety 
$Y \subset X$, where it follows from the inductive assumption and Lemma 3.17. Induction step is done, and 
Proposition 3.14 is proven. \hfill \square

We have proven that the composition $\Omega^* \xrightarrow{G} \Omega^* \hookrightarrow \Omega^* \otimes_{\mathbb{Z}} \mathbb{Q}$ coincides with the composition $\Omega^* \xrightarrow{\mathcal{S}\mathcal{T}_{\text{ext}}^N} \Omega^* \xrightarrow{\mathcal{W}_G} \Omega^* \otimes_{\mathbb{Z}} \mathbb{Q}$, that is $G$ is a linear combination (infinite, in general) of the Landweber-Novikov operations. It remains to show that, in reality, this linear combination has coefficients in $\mathbb{L}$. Induction on 
the degree of the monomial $b^r = \prod_i b_i^{r_i}$, where $r = (r_1, r_2, \ldots)$, and $\deg(b_i) = i$. The base: the coefficient 
at $b^1 = 1$ is $G(1) \in \mathbb{L}$. Suppose, the coefficients at all monomial of smaller degrees are in $\mathbb{L}$. Consider
\[ X = x_i((\mathbb{P}^{i+1})^r), \text{ and } x = x_i(h)^r \in \Omega^{\sum r_i}(X), \text{ where } h = c_i^0(O(1)). \] The action of the operation \( S_{L-N}^T \) on the element \([v : V \to X]\) can be computed as follows: let \( \lambda_1, \lambda_2, \ldots \) be roots of \((-T_V + v^*T_X)\). Then
\[
S_{L-N}^T([v]) = \text{ coefficient at } b^\tau \text{ in } v_s(\prod_j (1 + \lambda_jb_1 + \lambda_j^2b_2 + \ldots)(1v))
\]

In our case, \((-T_V + v^*T_X) = \bigoplus O(1)\) - one summand per each copy of \(\mathbb{P}^{i+1}\), for all \(i\). Hence, \(S_{L-N}^T(x) = 0\), for all \(x\) of degree \(= \deg(\tau)\) different from \(\tau\), while \(S_{L-N}^T(x) = [\text{Spec}(k) \to X] = pt\) - the class of a rational point. By dimensional reasons, the operations corresponding to monomials of larger degrees also vanish on \(x\). Hence, \(G(x) = \mu_{\tau} \cdot pt + \sum_{\deg(\tau) < \deg(\tau)} \mu_{\tau} \cdot S_{L-N}^T(x)\). But \(G(x) \in \Omega^*(X)\), and all the summands aside from the first one belong to \(\Omega^*(X)\) by the inductive assumption. Hence, \(\mu_{\tau} \cdot pt \in \Omega^*(X)\) which implies that \(\mu_{\tau} \in L\). Induction step is proven, and so the coefficients belong to \(L\). It remains to run the Lemmas \[3.11\] \[3.12\] and Proposition \[3.14\] again (for \(L\)-linear combination, this time), and Theorem \[3.10\] is proven.

### 3.3 Unstable operations in Algebraic Cobordism (uniqueness)

Unstable operations can be described in terms of stable ones. In analogy with topology we have:

**Theorem 3.18** The correspondence: \(G_\psi \leftrightarrow \psi\), where \((G_\psi)_Q\) is the map: \(\Omega^* \xrightarrow{S_{L-N}^T} \Omega^* \bigoplus_{b} b_{\psi} \rightarrow \Omega^* \otimes \mathbb{Q}\), identifies the set of (unstable) additive operations \(\Omega^m \to \Omega^n\) with the subset of \(\text{Hom}_{\mathbb{L}}(\mathbb{L} \otimes \mathbb{Q}_{(m-n)})\). In other words, any such operations can be interpreted as a unique \(\mathbb{L} \otimes \mathbb{Q}\)-linear combination (infinite, in general) of the Landweber-Novikov operations.

**Proof:** By Proposition \[3.14\] we know that any additive operation \(\Omega^n \xrightarrow{G} \Omega^m\) is completely defined by its action on \((\mathbb{P}^{\infty})^r\), for all \(r\). Thus, it is sufficient to show that there exists such \(\mathbb{L} \otimes \mathbb{Q}\)-linear combination of the Landweber-Novikov operations which coincides with \(G_Q\) when restricted to \((\mathbb{P}^{\infty})^r\), \(\forall r\). We have mutually inverse operations:

\[
\Omega^* \otimes \mathbb{Q} \xrightarrow{G} \text{CH}^* \otimes \mathbb{Q}(d),
\]

where \(\gamma^{-1} = x + d_1x^2 + d_2x^3 + \ldots = \varphi(\log n), \) and \(\gamma_{\beta} = \log \beta\). Thus, we obtain a commutative diagram:

\[
\begin{array}{ccc}
\Omega^n \otimes \mathbb{Q} & \xrightarrow{G} & \Omega^m \otimes \mathbb{Q} \\
\beta \downarrow & & \alpha \downarrow \\
(\text{CH}^* \otimes \mathbb{Q}[d])_{(m)} & \xrightarrow{H} & (\text{CH}^* \otimes \mathbb{Q}[d])_{(n)},
\end{array}
\]

where \(H\) is an additive operation between additive theories.

Let \(A^*\) and \(B^*\) be two theories in the sense of Definition \[2.1\]. Let \(x_i = c_i^1(O(1), i)\), and \(y_i = c_i^B(O(1), i)\). Then \(A^*((\mathbb{P}^{\infty})^r)\) is a free \(A^*(\text{Spec}(k))\)-module with the basis \(\pi_I\), and \(B^*((\mathbb{P}^{\infty})^r)\) is a free \(B^*(\text{Spec}(k))\)-module with the basis \(\gamma_I\).

**Lemma 3.19** Let \(A^n \xrightarrow{H} B^m\) be an additive operation of additive theories. Suppose \(B\) has no torsion. Then there exists a homomorphism of abelian groups \(A^*(\text{Spec}(k)) \xrightarrow{H} B^{*+m-n}(\text{Spec}(k))\) such that \(H(u \cdot \pi_I) = H(u) \cdot \gamma_I\), for all \(I\) and all \(u \in A^{n-deg(I)}(\text{Spec}(k))\).
Proof: Because of the partial diagonals, it is sufficient to treat the case \( \overline{f} = x_1 \cdot x_2 \cdot \ldots \cdot x_r \). Thus, in any degree we have just one such monomial, and we only need to show that \( H(u \cdot \overline{f}) \) is \( \overline{f} \) times the linear function on \( u \). Changing our \( A^* \) and \( B^* \) by \( (A')^* = A^*[[x_1, \ldots, \hat{x}_i, \ldots, x_r]] \), and \( (B')^* = B^*[[x_1, \ldots, \hat{x}_i, \ldots, x_r]] \), we can assume that \( k = 1 \). Consider the Segre embedding \( \mathbb{P}^\infty \times \mathbb{P}^\infty \xrightarrow{f} \mathbb{P}^\infty \). Then \( f^*(u \cdot x) = u \cdot x_1 + u \cdot x_2 \). Let \( H(u \cdot x) = \gamma(y) = \gamma_0 + \gamma_1 y + \gamma_2 y^2 + \ldots \in B^*[[y]] \). Restricting to \( \text{Spec}(k) \hookrightarrow \mathbb{P}^\infty \), we see that \( \gamma_0 = 0 \). Let \( \gamma(y) = \gamma_1 y + \gamma_2 y^2 + \ldots \), that is, the next after the linear non-zero term has degree \( s \). Then from the equality: \( f^*(H(u \cdot x)) = H(f^*(u \cdot x)) \), we get:

\[
\gamma(y_1 + y_2) = \gamma(y_1) + \gamma(y_2).
\]

Comparing coefficients at \( y_1 \cdot y_2^{-1} \), we get: \( s \cdot \gamma_s = 0 \). Since \( B \) has no torsion, we get that \( \gamma(y) = \gamma_1 y \) is linear. Thus, we have shown that \( H(u \cdot (x_1 \cdot \ldots \cdot x_r)) = v \cdot (y_1 \cdot \ldots \cdot y_r) \), and the correspondence \( u \mapsto v \) defines an additive map \( A^{n-r}(\text{Spec}(k)) \xrightarrow{R} B^{m-r}(\text{Spec}(k)) \). □

The map \( L \rightarrow \mathbb{L} \otimes \mathbb{Q} \overset{\beta_0 \circ \text{Roa}|_{\text{Spec}(k)}}{\longrightarrow} \mathbb{L} \otimes \mathbb{Q} \) is additive. As we saw in the proof of Theorem 3.10 this map can be presented as the composition: \( \mathbb{L} \overset{\text{St}_n}{\longrightarrow} \mathbb{L}[I] \otimes_{\mathbb{Q}} \mathbb{L} \otimes \mathbb{Z} \otimes \mathbb{Z} \mathbb{Q} \), for some \( \psi \in \text{Hom}_\mathbb{L}(\mathbb{L}[I], \mathbb{L} \otimes \mathbb{Z} \mathbb{Q})_{(m-n)} \). Then Lemma 3.19 shows that on \( (\mathbb{P}^\infty)^r \), for all \( r \), \( H \) coincides with the above \( \mathbb{L} \otimes \mathbb{Q} \)-linear combination of the Landweber-Novikov operations. □

The natural question arises: which rational linear combinations of Landweber-Novikov operations are realized as (unstable) operations \( \Omega^a \rightarrow \Omega^m \)? It appears that exactly those which take integral values on \( \Omega^a((\mathbb{P}^\infty)^r) \), for all \( r \) - see Theorem 6.1. But this result is much more difficult than everything we discussed so far, and we will need the inductive description of the Algebraic Cobordism theory and various new tools in order to prove it.

4 Theories of rational type

In order to work with unstable operations we will need to produce the description of our theories which is inductive on the dimension. Not all theories admit a satisfactory description of the type we want. And the ones which do will be called theories of rational type. Later we will see that these are exactly the free theories of M. Levine-F. Morel. The needed description of the theory will be obtained in stages. The one which is actually used is provided by the bi-complex \( c \); but to get there we will need to introduce bi-complexes* \( a \) and \( b \), and to show that the Algebraic Cobordism of M. Levine-F. Morel is a theory of rational type.

4.1 The bi-complex* \( a \)

Everywhere in this and the next Subsection we will assume that \( A^* \) is a theory in the sense of Definition 2.1 satisfying (\text{CONST}) . Some statements are valid without the latter assumption, which will be indicated. Let \( X \) be smooth irreducible variety over \( k \). Consider the category \( S(X) \) whose objects are (isomorphism classes of) maps \( V \xrightarrow{v} X \), where \( V \) is smooth, \( v \) is projective, and \( \text{dim}(V) < \text{dim}(X) \), and morphisms are projective maps \( V_2 \xrightarrow{f} V_1 \) such that \( v_2 = v_1 \circ f \).

Similarly, we have a category \( S^1(X) \) whose objects are (isomorphism classes of) maps \( W \xrightarrow{w} X \times \mathbb{P}^1 \), where \( W \) is smooth, \( w \) is projective, \( \text{dim}(W) \leq \text{dim}(X) \), and \( W_0 = w^{-1}(X \times \{0\}) \xrightarrow{i_0} W \), \( W_1 = w^{-1}(X \times \{1\}) \xrightarrow{i_1} W \) are divisors with strict normal crossing. The morphisms can be defined as above, but we will not need them.
We have natural maps $\partial_0, \partial_1 : S^1(X) \to S(X)$ defined by:

$$\partial_i(W \xrightarrow{\text{pr}} X \times \mathbb{P}^1) = W_i \xrightarrow{\text{pr}} X.$$ 

In this formula we understand the divisor with strict normal crossing $W_i$ as the disjoint union $\bigsqcup_{J \subseteq \{1, \ldots, r\}} S_J$ of all of its faces (see Definition 7.13).

Consider the following bi-complex $a = a(A^*)$:

$$a_{1,0} \xrightarrow{d_{1,0}} a_{0,0} \xleftarrow{d_{0,1}} a_{0,1}$$

(such a "small" bi-complex will be called a bi-complex* below), where

- $a_{0,0} := \bigoplus_{\text{Ob}(S(X))} A_*(V)$;
- $a_{1,0} := \bigoplus_{\text{Mor}(S(X))} A_*(V_2)$;
- $a_{0,1} := \bigoplus_{\text{Ob}(S^1(X))} A_{*+1}(W)$.

and the differentials are defined as follows:

- $d_{1,0}(V_2 \xrightarrow{f} V_1, y) = (V_2, y) - (V_1, f_*(y))$;
- $d_{0,1}(U, z) = (\partial U, i^*_0(z)) - (\partial U, i^*_1(z))$ - see Definition 7.15 where we use the standard choice for the coefficients $F^j_{1, \ldots, r}$ (as soon as we pass to $\text{Coker}(d_{1,0}^a)$ the latter becomes irrelevant).

Let us denote as $H(a)$ the 0-th homology of the total complex $Tot(a)$ of $a$. In other words,

$$H(a) = \text{Coker}(a_{1,0} \oplus a_{0,1} \xrightarrow{d_{1,0}\oplus d_{0,1}} a_{0,0}).$$

If $A^*$ satisfies (CONST), then the restriction to the generic point defines the surjection $A^*(X) \to A \to 0$ which has a canonical splitting given by the pull-back $p^*_X : A = A^*(\text{Spec}(k)) \to A^*(X)$. Let us denote as $\overline{A}(X)$ the kernel $\text{Ker}(A^*(X) \to A)$. Thus, $A^*(X) = \overline{A}(X) \oplus A$ canonically.

The push-forwards define natural map $a_{0,0} \to \overline{A}_*(X)$, and it follows from Proposition 7.17 that it descends to a map $\theta_a : H(a) \to \overline{A}_*(X)$. By the (EXCI) and the resolution of singularities (Theorem 8.2), this map is surjective.

**Definition 4.1** Let $A^*$ be generalized oriented cohomology theory in the sense of Definition 2.1 satisfying (CONST). We say that $A^*$ is "of rational type" if the map $\theta_a : H(a) \to \overline{A}_*(X)$ is an isomorphism.

**Remark 4.2** Not all constant theories are of rational type. For example, CH$_{alg}$ - the Chow groups modulo algebraic equivalence is not such. Indeed, in this case, for a curve $C$ of genus $g > 0$, the map $\theta_a$ becomes $\overline{CH}(C) \to \overline{CH}_{alg}(C)$, and has large kernel.

But below we will see that all the "standard theories" are rational.

**Proposition 4.3** Algebraic cobordism $\Omega^*$ of M.Levine-F.Morel is a theory of rational type.

**Proof:** The main tool here is the comparison result of M.Levine stating that $\Omega^n$ is a "pure part" $MGL^{2n,n}$ of $MGL$. By the proof of Theorem 3.1 from [11] (take into account [8](8.14)), we have an exact sequence:

$$\mathbb{Z}[k(X)^\times] \otimes L_{\ast'} \xrightarrow{\text{div}^\Omega} \Omega^1_{\ast}(X) \to \Omega_{\ast}(X) \to \mathbb{L} \to 0.$$
Here $\Omega^{(1)}(X) = \lim_W \Omega_*(W)$, where the limit is taken over all closed subvarieties different from $X$. The map $\text{div}^\Omega$ is $L$-linear and is defined on $\mathbb{Z}[k(X)^	imes]$ as follows: for a rational function $f \in k(X)^	imes$ one resolves the indeterminacy of $f$ using Theorem 8.3 by the permitted blow up $\pi : X \to X$ making $\bar{f} : X \to \mathbb{P}^1$ a morphism, and $\bar{X}_0 = \bar{f}^{-1}(0) \to \bar{X}$, $X_\infty = \bar{f}^{-1}(\infty) \to \bar{X}$ - the divisors with normal crossing. Then $\text{div}^\Omega(f) = \pi_*([X_0] - [X_\infty])$, where $[X_0], [X_\infty] \in (\Omega^{(1)})^1(X)$ are classes of divisors with normal crossing - see Definition 7.14.

Consider the categories $\mathcal{S}'(X)$ and $\mathcal{S}^1(X)$ defined similar to $\mathcal{S}(X)$ and $\mathcal{S}^1(X)$, but with different dimension conditions: $\dim(\text{image}(v)) < \dim(X)$, $\dim(\text{image}(w)) \leq \dim(X)$.

For any theory $A^*$ we can define the following bi-complex $\alpha' = \alpha'(A^*)$:

\[
\begin{align*}
\alpha'_{0,0} &:= \bigoplus_{\text{Ob}(\mathcal{S}'(X))} A_*(V); &\alpha'_{1,0} &:= \bigoplus_{\text{Mor}(\mathcal{S}'(X))} A_*(V_2); &\alpha'_{0,1} &:= \bigoplus_{\text{Ob}(\mathcal{S}^1(X))} A_{*+1}(W),
\end{align*}
\]

where the differentials and $H(\alpha')$ are defined as for $\alpha$.

Now, let $A^* = \Omega^*$, and $\alpha' = \alpha'(\Omega^*)$. The push-forwards provide the natural map $\alpha'_{0,0} \to \Omega^{(1)}_*$, which clearly descends to the map $\alpha' : \text{Coker}(d_{1,0}^\Omega) \to \Omega^{(1)}_*(X)$.

**Lemma 4.4** The map $\alpha' : \text{Coker}(d_{1,0}^\Omega) \to \Omega^{(1)}_*(X)$ is an isomorphism.

**Proof:** Since $\text{Coker}(d_{1,0}^\Omega) = \lim_{V \to Z} \Omega_*(V)$, and $\Omega_*(Z) = \lim_{\bar{V} \to \bar{Z}} \Omega_*(\bar{V})$, we get:

\[
\Omega^{(1)}_*(X) = \lim_{\bar{V} \to \bar{Z}} \Omega_*(\bar{V}) = \lim_{V \to Z} \Omega_*(V) = \text{Coker}(d_{1,0}^\Omega).
\]

\[\square\]

In the same way, for any theory in the sense of Definition 7.1 we have:

\[
\lim_{Z \subset X} \text{Coker}(d_{1,0}^\Omega) = \text{Coker}(d_{1,0}^\Omega).
\]

From here it is easy to see that $H(\alpha') \to \Omega_*(X)$ is an isomorphism, but we will compare $\alpha'$ and $\alpha$ first.

We have a natural map of bi-complexes $\alpha \to \alpha'$ which gives us the map $\alpha : \text{Coker}(d_{1,0}^\Omega) \to \text{Coker}(d_{1,0}^\Omega')$, and $\hat{\alpha} : \text{Coker}(\hat{d}_{0,1}^\Omega) \to \text{Coker}(\hat{d}_{0,1}'^\Omega)$, where $\hat{d}_{0,1}^\Omega$ is the map $a_{0,1} \to \text{Coker}(d_{1,0}^\Omega)$ (and similar for $\alpha'$).

**Lemma 4.5** For any theory $A^*$ in the sense of Definition 7.1, the map $\alpha : \text{Coker}(d_{1,0}^\Omega) \to \text{Coker}(d_{1,0}^\Omega')$ is an isomorphism.

**Proof:** (surjectivity) Let $Z = \text{image}(V) \to X$, and $\pi_Z : \bar{Z} \to Z$ - Hironaka’s resolution of singularities (Theorem 8.2). Then by another result of Hironaka (Theorem 8.3) we can resolve the indeterminacy of the rational map $V \dashrightarrow \bar{Z}$ by blowing $V$ at smooth centers producing the following commutative diagram:

\[
\begin{array}{ccc}
\bar{V} & \xrightarrow{\pi_V} & V \\
\downarrow & & \downarrow v \\
\bar{Z} & \xrightarrow{\pi_z} & Z.
\end{array}
\]

(1)
Since \((\pi_v)_* : A_*(\tilde{V}) \to A_*(V)\) is surjective, we have: \((i \circ v, x) = (i \circ v \circ \pi_v, y) = (i \circ \pi_z, \tilde{v}_s(y))\), for some \(y \in A_*(\tilde{V})\), and \(\dim(\tilde{Z}) < \dim(X)\).

**injectivity** To prove injectivity, consider the filtration \(a^r\) on \(a'\) defined by additional condition: \(\dim(V) < r, \dim(W) < r + 1\). Then \(a'^\infty = a'\), and \(a'^{\dim(X)} = a\). Let us show that for \(r > \dim(X)\), the inclusion \(a'^{r-1} \subseteq a'\) induces an isomorphism \(\text{Coker}(d_{1,0}^{r-1}) = \text{Coker}(d_{1,0}^r)\). We need to prove injectivity only. Let \(x = \sum_v(v, x_v) \in a'^{r-1}\) is such that there exists \(y = \sum_v v(f, y_f) \in a_{1,0}^r\) such that \(d_{1,0}(y) = x\). Let us list all \(v : V \to X\) and \(v' : V' \to X\) participating in \(y\), and for each such \(V\) resolve the singularities of \(Z = \text{image}(V)\) by Hironaka, and resolving the indeterminacy of the rational map \(V \dashrightarrow Z\) using Theorem S.3 complete the commutative diagram \(\mathbb{I}\), where \(\pi_v\) is the permitted blow up with smooth centers.

**Lemma 4.6** Any map \(f : V' \to V\) and the choice of diagrams of type \(\mathbb{I}\) for \(V\) and \(V'\) can be completed to diagrams:

\[
\begin{array}{c}
\tilde{V}' & \xrightarrow{\pi_{v'}} V' & \xrightarrow{\pi_v} V \\
\tilde{V} & \xrightarrow{\pi_v} V & \xrightarrow{v} V' \\
Z' & \xrightarrow{\pi_{v'}} Z' & \xrightarrow{\pi_z} Z \\
\end{array}
\]

where \(\bar{f}_2, \epsilon_1, \epsilon_2\) are projective birational, \(\bar{f}_1\) is the composition of a projective birational and a closed embedding, \(j, j', e\) are regular embeddings, and \(\bar{X}, \bar{X}', Y_f\) are projective birational over \(X\).

**Proof:** Since all the fibers of \(\pi_v\) are rational varieties, there exists a rational map \(g : \tilde{V}' \dashrightarrow \tilde{V}\) lifting \(f\). Let \(R_f\) be the closure of the image of

\[
\tilde{V}' = \text{image}(\tilde{V}') \to \tilde{Z} \times Z Z'.
\]

Then \(R_f \xrightarrow{g_1} Z \xrightarrow{j} \bar{X}\), where \(j\) is a regular embedding, and \(g_1\) is a closed one. Also, the rational map \(\bar{X} \dashrightarrow \bar{X}'\) is defined in the generic point of \(R_f\), and we have commutative diagram:

\[
\begin{array}{c}
\tilde{Z}' & \xrightarrow{j'} \bar{X}' \\
\tilde{X}' & \xrightarrow{j\circ g_1} \bar{X} \\
\end{array}
\]

Then, by Theorems S.2, S.3 there exists a permitted blow up \(Y_f \to \bar{X}\), which resolves indeterminacy of
this map, and makes the proper preimage $T_f$ of $R_f$ smooth. We get the commutative diagram

$$
\begin{array}{c}
\tilde{Z} \\
\pi_v
\end{array}
\xymatrix{
\tilde{Z} \\
\pi_v \ar[r]^-{T_f} & \tilde{Z}'
}
$$

Then, by Theorem 8.3, there exists a blow up with smooth centers $\tilde{f}_2 : W_f \rightarrow \tilde{V}'$ which fits into the diagram (2).

Let $f : V' \rightarrow V$ be a map appearing in $y$. Apply Lemma 4.6 to $f$ and diagrams of type (1) chosen for $V \rightarrow X$ and $V' \rightarrow X$. We get diagram (2).

Notice, that the front and the rear faces of our diagram do not depend on $f$, but only on $v$ and $v'$. Let $y = \sum f(y_f)$. The maps $\pi_v$ and $\pi_{v'}$ are birational, so $(\pi_v)_*(1)$ and $(\pi_{v'})_*(1)$ are invertible. Consider $b_f := (\pi_{v'})^*(\pi_v y_f) \in A_*\tilde{V}'$, and $c_f := (\pi_{v'})^* ((\pi_v y_f)) \in A_*\tilde{V}$). Since $\tilde{f}_2$ is birational, we can find $a_f \in A_*W_f$ such that $(\tilde{f}_2)_*(a_f) = b_f$. Define $d_f = (h_f)_*(a_f)$.

The map $\pi_v$ is the permitted blow up, so by Proposition 7.6(1), we have an exact sequence:

$$
0 \leftarrow A_*(V) \xrightarrow{(\pi_v)_*} A_*(\tilde{V}) \leftarrow \text{Ker}(A_*(E) \xrightarrow{p} A_*(R)),
$$

where $E = \coprod E_i$ - the disjoint union of the components of the exceptional divisor of the blow up, and $R = \coprod R_i$ is the disjoint union of the respective smooth centers. Since $(\pi_v)_*(c_f - (\tilde{f}_1)_*(a_f)) = 0$, there exists $e_f \in A_*(E)$ such that $p_*(e_f) = 0$ and $\tilde{q}_*(e_f) = c_f - (\tilde{f}_1)_*(a_f)$, where $p$ and $\tilde{q}$ fit into the commutative diagram:

$$
\begin{array}{c}
E \xrightarrow{\tilde{q}} \tilde{V} \\
p \ar[r] & \pi_v \ar[r] & \tilde{V}
\end{array}
$$

Consider $t = \sum f(t_f)$, where

$$
t_f := (\pi_v, c_f) - (\tilde{v}, c_f) + (\tilde{f}_1, d_f) - (\tilde{f}_2, d_f) + (\tilde{v}', b_f) - (\pi_{v'}, b_f) + (\tilde{v} \circ \tilde{q}, e_f) - (p, e_f).
$$

Since $d_1(y) \in a_{r,0}^{r-1}$, we have that for every $v : V \rightarrow X$ of dimension $r$, we have: $\sum \int_V f(y_f) - \sum y_f = 0$.

Then the sum $\sum ((\pi_v, c_f) - (\tilde{v}, c_f) + (\tilde{v}', b_f) - (\pi_{v'}, b_f))$ is equal to

$$
\sum_v \left( \left( \int_V f - \frac{\sum f(y_f)}{\pi_v}_* \right) - \left( \int_V f - \sum \frac{\sum y_f}{(\pi_v)_*(1)} \right) \right) = 0.
$$

And the remaining terms of $t_f$ are of dimension $\leq r - 1$ (notice, that $\tilde{f}$ is a closed embedding, and $\text{dim}(T_f) \leq \text{dim}(Z) < \text{dim}(X)$). Hence, $t \in a_{1,0}^{r-1}$. On the other hand, $d_1(t) = d_{1,0}(y)$ (instead of the path $V' - V$ we are moving: $V' - \tilde{V}' - \tilde{Z}' - T_f - \tilde{Z} - \tilde{V} - V$, and correct the discrepancy on $\tilde{Z}$ with the $e_f$-terms). Hence, the map Coker($d_{1,0}^{r-1}$) $\rightarrow$ Coker($d_{1,0}^{r}$) is injective, which implies that Coker($d_{1,0}^{r}$) = Coker($d_{1,0}^{r-1}$).
Return to the case $A^* = \Omega^*$. We obtain the commutative diagram with exact columns:

$$
\begin{array}{cccc}
\hat{a}_{0,1} & \rightarrow & a'_{0,1} & \rightarrow \rightarrow \\
\downarrow & & \downarrow & \rightarrow \rightarrow \\
Coker(d^a_{1,0}) & \alpha \rightarrow & Coker(d'^a_{1,0}) & \alpha' \rightarrow \rightarrow \\
\downarrow & & \downarrow & \rightarrow \rightarrow \\
H(a) & \hat{\alpha} \rightarrow & H(a') & \hat{\alpha}' \rightarrow \rightarrow \\
\downarrow & & \downarrow & \rightarrow \rightarrow \\
0 & & 0 & 0 \\
\end{array}
$$

where $\alpha$ and $\alpha'$ are isomorphisms. It remains to observe that the map $\text{div}$ can be factored through $a_{0,1}$ by the very definition. This shows that the maps $H(a) \xrightarrow{\hat{\alpha}} H(a') \xrightarrow{\hat{\alpha}'} \Omega^*_s(X)$ are isomorphisms. □

Note, that more generally, we have:

**Lemma 4.7** For any theory $A^*$ (Definition 2.1 + (CONST)) the map $H(a) \xrightarrow{\hat{\alpha}} H(a')$ is an isomorphism.

**Proof:** We already know by Lemma 4.5 that $\text{Coker}(d^a_{1,0}) \xrightarrow{\alpha} \text{Coker}(d'^a_{1,0})$ is an isomorphism. So, we need to prove only the injectivity of $\hat{\alpha}$.

Let $(i \circ u, x) \in a'_{0,1}$, where $U \xrightarrow{i} T = \text{image}(U) \xrightarrow{\hat{\iota}} X \times \mathbb{P}^1$ s.t. the preimages $U_0$ and $U_1$ of $X \times \{0\}$ and $X \times \{1\}$ are divisors with normal crossing on $U$, and $T$ has dimension $\leq \dim(X)$. Let $\hat{T} \xrightarrow{\pi_T} T$ - Hironaka’s resolution of singularities (see Theorem 8.2) such that the preimages $\hat{T}_0$ and $\hat{T}_1$ of 0 and 1 on $\hat{T}$ are divisors with strict normal crossing, which using the resolution of the indeterminacy of the rational map $(\pi_T)^{-1} \circ u$ (see Theorem 8.3) can be embedded into the commutative diagram:

$$
\begin{array}{ccc}
\hat{U} & \xrightarrow{\hat{\iota}} & \hat{T} \\
\downarrow{\pi_U} & & \downarrow{\pi_T} \\
U & \xrightarrow{\iota} & T,
\end{array}
$$

where the preimages $\hat{U}_0$ and $\hat{U}_1$ of 0 and 1 on $\hat{U}$ are also divisors with strict normal crossing. For $l = 0, 1$ we have the diagram with Cartesian squares:

$$
\begin{array}{ccc}
\hat{T} & \xrightarrow{\hat{\iota}} & \hat{U} \\
\downarrow{i_{\hat{T}}} & \downarrow{i_{\hat{U}}} & \downarrow{i_{\hat{U}}} \\
T_l & \xrightarrow{i_l} & U_l
\end{array}
$$

Since $\pi_U$ is birational, the map $(\pi_U)_*$ is surjective. Hence, there exists $y \in A_*(\hat{U})$ such that $(\pi_U)_*(y) = x$. By the "Multiple points excess intersection formula" (Proposition 7.20), $d_{0,1}(i \circ u, x) = d_{0,1}(i \circ u \circ \pi_U, y) = d_{0,1}(i \circ \pi_T, \hat{u}_*(y))$, and $(i \circ \pi_T, \hat{u}_*(y)) \in a_{0,1}$, since $\dim(T) \leq \dim(X)$. Thus, $\hat{\alpha}$ is an isomorphism. □
Remark 4.8 The above Lemma is valid without the (CONST) assumption - see Remark 7.22

Proposition 4.3 permits to describe all theories of rational type. It appears that these are exactly the free theories in the sense of M.Levine and F.Morel - see [12, Remark 2.4.14].

**Proposition 4.9** Let $A^*$ be a theory (Definition 2.1 + (CONST)). Then

$$A^* \text{ is of rational type } \iff A^* = \Omega^* \otimes_L A.$$  

In particular, there is a 1-to-1 correspondence between theories of rational type $A^*$ and formal group laws $(A, F_{A^*})$.

**Proof:** Since the tensor product functor is exact from the right, any theory of the type $\Omega^* \otimes_L A$ will be of rational type by the very definition, since $\Omega^*$ is (Proposition 4.3).

Conversely, suppose $A^*$ is of rational type. By the result of M.Levine-F.Morel, $\Omega^*$ is the universal theory, so we get a canonical morphism $\Omega^* \to A^*$ which extends to $\Omega^* \otimes_L A \xrightarrow{g} A^*$ (since $A$ acts on $A^*$). This morphism is an isomorphism for $X$ of dimension zero. Consider the following commutative diagram with exact rows:

$$
\begin{array}{c}
(a_{1,0} \oplus a_{0,1})(\Omega^* \otimes_L A) & \longrightarrow & a_{0,0}(\Omega^* \otimes_L A) & \longrightarrow & (\Omega^* \otimes_L A)(X) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \gamma & & \\
(a_{1,0} \oplus a_{0,1})(A^*) & \longrightarrow & a_{0,0}(A^*) & \longrightarrow & A^*(X) & \longrightarrow & 0
\end{array}
$$

Since $a_{0,0}(B^*) = \bigoplus_{\dim(V) < \dim(X)} B_*(V)$, and $g$ is an isomorphism on points, by induction on the dimension of $X$ we see that $g$ is surjective for all $X$ (recalling that $B^*(Y) = \overline{B^*}(Y) \oplus B$). Again using the above diagram, we prove by induction on the dimension that $g$ is an isomorphism for all $X$. □

**Remark 4.10** Although, in the end, our theories of rational type appear to be the same as free theories of M.Levine-F.Morel, there are several justifications for this alternative approach. The main ones being that it gives the definition which is internal, and permits to argue inductively on dimension, which will be crucial in our dealing with unstable operations below. Also, this definition is rather flexible and permits generalizations - see Subsection 4.4.

The following result shows that the set of theories of rational type is closed w.r.to reparametrization and w.r.to multiplicative projectors.

**Proposition 4.11** Let $A^*$ be a theory of rational type. Then:

1) For any $\gamma(x) = a_0x + a_1x^2 + \ldots \in A[[x]]$ with invertible $a_0$, the reparametrization $(A^*)^\gamma$ is a theory of rational type.

2) For any multiplicative projector $\rho : A^* \to A^*$, the theory $\rho A^*$ (with the quotient structure) is a theory of rational type.

**Proof:** 1) Evidently, $(A^*)^\gamma$ satisfies (CONST). One can construct an isomorphism of bi-complexes $a((A^*)^\gamma) \xrightarrow{\sim} a(A^*)$ defined by multiplication by $T_a(N_g)$, where $N_g$ is the virtual normal bundle of $g$, $T_a(N_g) = \prod \left(\frac{\gamma(x)}{x}\right)(\lambda_i)$ (cf. [16]), where $\lambda_i$ are $A$-roots of $V$, and $g = v : V \to X$ on $a_{0,0}, g = v_1 \circ f : V_2 \to X$ on $a_{1,0}, f = u : W \to X \times \mathbb{P}^1$ on $a_{0,1}$. This implies that $(A^*)^\gamma$ is of rational type as well. Alternatively, using Theorem 3.7 we get a multiplicative operation $\Omega^* \to \Omega^* \otimes_L A = A^*$ corresponding to $\gamma$. Restricted
to Spec($k$) it gives the formal group law $\mathbb{L} \to A^\gamma$, where $A^\gamma = A$ and $F_{A^\gamma}(x, y) = \gamma(F_A(\gamma^{-1}(x), \gamma^{-1}(y)))$.

Hence, our operation extends to a multiplicative operation

$$G_\gamma : \Omega^* \otimes_\mathbb{L} A^\gamma \to \Omega^* \otimes_\mathbb{L} A.$$ 

It is an isomorphism (of pull-back structures) since there is a multiplicative inverse $G_{\gamma^{-1}}$. Thus, $\Omega^* \otimes_\mathbb{L} A^\gamma$ is just a reparametrization of $\Omega^* \otimes_\mathbb{L} A$. By construction, the morphism of FGL’s corresponding to $G_\gamma$ is $(id, \gamma)$. Hence, $\Omega^* \otimes_\mathbb{L} A^\gamma = (A^*)^\gamma$ - see [16], and the latter theory is of rational type.

2) Let $(\varphi_{\rho}, \gamma_{\rho})$ be the respective morphism of FGL’s. From the condition $\rho \circ \rho = \rho$ (and the fact that $\gamma$ is invertible) we immediately get: $\varphi_{\rho}(\gamma_{\rho})(x) = x$. We have an invertible multiplicative operation $G_{\gamma_{\rho}} : (A^*)^\gamma \to A^*$ which permits to decompose $\rho$ as $G_{\gamma_{\rho}} \circ \pi$, where $FGL(\pi) = (\varphi_{\rho}, x)$ and $FGL(G_{\gamma_{\rho}}) = (id, \gamma_{\rho})$. Then $FGL(\pi \circ G_{\gamma_{\rho}}) = (\varphi_{\rho}, x)$. Hence, $\mu := \pi \circ G_{\gamma_{\rho}} : (A^*)^\gamma \to (A^*)^\gamma$ and $\pi : A^* \to (A^*)^\gamma$ are morphisms of theories (commute with pull-backs and push-forwards). And since $G_{\gamma_{\rho}}$ is invertible, $\mu$ is a projector. Hence, $\rho A^*$ is realized as a quotient of $A^*$ and as a direct summand of $(A^*)^\gamma$ under a projector endomorphism $\mu$. By 1), $(A^*)^\gamma$ is a theory of rational type, hence so is $\rho A^*$ by the very Definition 4.11 Clearly, $FGL(\rho A^*) = (\varphi_{\rho}(A), \varphi_{\rho}(F_A(x, y)))$. □

By results of M. Levine and F. Morel ([12, Theorems 1.2.18, 1.2.19]), the (usual) Chow groups $CH^*$ and $K_0$ are free theories, and hence, theories of rational type. It follows from Proposition 4.11 that other ”standard (pure) theories” such as $BP^*$ and higher Morava K-theories $K(n)$ are of rational type as well. It looks like everything which was ”pulled” from topology gives pure theories of rational type. This seems to be related to the fact that in topology all spectra are made of ”cellular” spaces, so their algebraic counterparts have ”cellular” spectra as well (and this is what seems to be responsible for rational type).

Let $A^*$ be any theory satisfying (CONST). Let us denote as $(A^{(0)})^*$ the theory $\Omega^* \otimes_\mathbb{L} A$. Then we have the canonical map $(A^{(0)})^* \overset{g}{\longrightarrow} A^*$.

**Proposition 4.12** (1) $g : (A^{(0)})^* \to A^*$ is surjective;

(2) (cf. [12, Remark 2.4.14]) Any morphism $A^* \xrightarrow{f} B^*$ of theories extends to a canonical commutative diagram:

$$
\begin{array}{ccc}
(A^{(0)})^* & \xrightarrow{f^{(0)}} & (B^{(0)})^* \\
g_A \downarrow & & \downarrow g_B \\
A^* & \xrightarrow{f} & B^*.
\end{array}
$$

In particular, the assignment $A^* \mapsto (A^{(0)})^*$ is a functor on the category of theories. The assignment: $f \mapsto f^{(0)}$ is injective.

(3) ([12, Remark 2.4.14]) Morphisms $A^* \to B^*$ between theories of rational type are in 1-to-1 correspondence with $L$-algebra morphisms $A \to B$.

**Proof:** (1) Since $a_{0,0} : (A^*)^* \to \overline{A}^*(X)$ is surjective for any theory satisfying (EXCI), the surjectivity of $g$ follows by induction on the dimension of $X$.

(2) Restriction to Spec($k$) gives the morphism of $L$-algebras $A \to B$, and hence a morphism of theories $(A^{(0)})^* \xrightarrow{f^{(0)}} (B^{(0)})^*$. The fact that the respective diagram is commutative follows from universality of $\Omega^*$, as the composition $\Omega^* \to (A^{(0)})^* \to A^* \xrightarrow{f} B^*$ can also be decomposed as $\Omega^* \to (B^{(0)})^* \to B^*$, and $(B^{(0)})^* = (A^{(0)})^* \otimes_A B$. The injectivity $f \mapsto f^{(0)}$ follows from (1).

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(3) Follows immediately from universality of $\Omega^*$. □

Let us now give some alternative descriptions of rational theories, which will permit us to work with unstable operations effectively.

4.2 The bi-complex* $b$.

The bi-complex* $b(A^*)$ describes the theory $A^*$ of $X$ in terms of $A^*$ of smaller-dimensional varieties and push-forward maps. But, to be able to work with cohomological operations we will need to find the presentation in terms of pull-backs. This will be done in two steps. First, we will enhance the classes $(U \xrightarrow{\pi} X, \delta)$ from $a$ by requiring $U$ to sit in some $\tilde{X}$ projective bi-rational over $X$, but still keeping push-forwards - this is done in the current Section. And then, in the next one, we will switch from push-forwards to pull-backs by roughly "going down and up" via the projection $\tilde{X} \to X$.

Let us move to the first step. Consider the following resolution category $\mathcal{RC}(X)$ of $X$.

Objects of $\mathcal{RC}(X)$ are diagrams $Z \xrightarrow{z} X \xleftarrow{\rho} \tilde{X}$, where $z$ is an embedding of a closed proper subscheme, and $\rho$ is a projective birational morphism, which is an isomorphism outside $Z$ and such that $V = \rho^{-1}(Z)$ is a divisor with strict normal crossing.

Morphisms are commutative diagrams:

$$
\begin{array}{ccc}
Z_2 & \xrightarrow{z_2} & X \xleftarrow{\rho_2} \tilde{X}_2 \\
\downarrow & & \downarrow \pi \\
Z_1 & \xrightarrow{z_1} & X \xleftarrow{\rho_1} \tilde{X}_1,
\end{array}
$$

(4)

where $\pi$ is projective. Among these we will distinguish ones of especially simple kind:

- type I: $i = id$, $\pi$ is a single blow-up over $V_1$ permitted w.r.t $V_1$;

- type II: $\pi = id$.

We will denote respective morphisms as $\mathcal{Mor}_I$ and $\mathcal{Mor}_{II}$, respectively. Note, that for morphisms of type I, $\pi^{-1}(V_1) = V_2$.

Consider also the category $\mathcal{RC}^1(X)$ of diagrams $Z \xrightarrow{z} X \times \mathbb{P}^1 \xleftarrow{\rho} X \times \mathbb{P}^1$, where $z$ is an embedding of a closed subscheme, and $\rho$ is projective birational map, isomorphic outside $Z$, where $W = \rho^{-1}(Z)$ is a divisor with strict normal crossing having no components over 0 and 1, such that the preimages $\tilde{X}_0 = \rho^{-1}(X \times 0)$ and $\tilde{X}_1 = \rho^{-1}(X \times 1)$ are smooth divisors on $X \times \mathbb{P}^1$, and such that $W \cap \tilde{X}_0 \to \tilde{X}_0$ and $W \cap \tilde{X}_1 \to \tilde{X}_1$ are divisors with strict normal crossing. Morphisms can be defined in the same way as for $\mathcal{RC}(X)$, but we will not need them.

We have maps $\partial_0, \partial_1: Ob(\mathcal{RC}^1(X)) \to Ob(\mathcal{RC}(X))$ defined by:

$$
\partial_l(Z \xrightarrow{z} X \times \mathbb{P}^1 \xleftarrow{\rho} X \times \mathbb{P}^1) = (Z \xrightarrow{z_l} X \xleftarrow{\rho} \tilde{X}_l),
$$

where $Z_l = (X \times \{l\}) \cap Z$.

Consider the bi-complex* $b = b(A^*)$:

- $b_{0,0} := \bigoplus_{Ob(\mathcal{RC}(X))} A_*(V)$;  
  $b_{1,0} := \bigoplus_{\mathcal{Mor}_I \cup \mathcal{Mor}_{II}} A_*(V_2)$;  
  $b_{0,1} := \bigoplus_{Ob(\mathcal{RC}^1(X))} A_{*+1}(W)$.

And the differentials are defined as follows:

- $d_{1,0}(V_2 \xrightarrow{f} V_1, y) = (V_1, (\pi_V(f))_* (y)) - (V_2, y)$

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Lemma 4.13

1. The subgroup $im_Z$ does not depend on the choice of the resolution, but on $Z$ only.

2. $im_Z = im_{Z_{red}}$.

Proof: Let $\rho_1 : \bar{X}_1 \to X$ and $\rho_2 : \bar{X}_2 \to X$ be two resolutions as above (with the same $X \setminus Z$), and $x_1 \in A_*(V_1)$ be some element. We have a birational map $\rho_2^{-1} \circ \rho_1 : \bar{X}_1 \to \bar{X}_2$ which is an isomorphism outside $V_1$ and $V_2$. By the Weak Factorization Theorem (see Theorem 8.6(6)), there exists a diagram

\[
\begin{array}{ccccccc}
\bar{X}_1 & \xleftarrow{Y_1} & \bar{X}_2 \\
\xrightarrow{Y_2} & \bar{Y}_3 & \xrightarrow{Y_4} & \cdots & \xrightarrow{Y_{n-2}} & \bar{X}_{n-1} & \xrightarrow{Y_n} \bar{X}_2 \\
\pi & & & & & & \\
\end{array}
\]

of smooth projective varieties over $X$ where each blow up is permitted w.r. to all components of the preimage of $Z$. Since the respective push-forwards are surjective, and each projection $Y_i \to X$ is an isomorphism outside $Z$, we can "transfer" $x_1$ from $\bar{X}_1$ to $\bar{X}_2$ using elements from part $I$ of the image($d_{1,0}^b$).

$\square$

Lemma 4.14 Let $T \subset X$ be a closed subscheme, and $Y \subset X$ be a divisor smooth outside $T$. Let $(V, x) \in b_{0,0}$ be an element defined over $T$. Then there exists an element $(V', x') \in b_{0,0}$ defined over $T \cup Y$ such that

$$(V, x) \equiv (V', x') \mod image(d_{1,0}^b), \quad \text{and} \quad pr_T(x)|_{T \cup Y} = pr_{T \cup Y}(x').$$

Proof: By permitted blow up $\rho' : \bar{X} \to X$ with centers over $T$ we can make the preimages $V_1 \subset V_2$ of $T \subset T \cup Y$ simultaneously divisors with strict normal crossing. By Lemma 4.13 we can assume that $x$ is defined for this resolution $(T \to X \xleftarrow{\rho'} \bar{X})$. And using the second part of the image($d_{1,0}^b$), we get that $((T \cup Y \to X \xleftarrow{\rho'} \bar{X}), x')$, where $x' := j_*(x)$ represent the same element in the cokernel, and $pr_{T \cup Y}(x') = pr_T(x)|_{T \cup Y}$. $\square$

Lemma 4.15 Let $Z \subset X$ be a proper closed subscheme. Then there exist divisors $Y_i, i = 1, \ldots, m$ such that $Z_{red} \subset \cup_i Y_i$, and $Y_j$ is smooth outside $\cup_{i=1}^{j-1} Y_i$.

Proof: Use Noetherian induction. The base $(Z = \emptyset)$ is trivial. Suppose, we know the statement for all proper closed subschemes of $Z_{red}$. By Proposition 7.12 there exist a divisor $Y$ of $X$ which contains $Z_{red}$, and is smooth outside $Z$, and in the generic points of the components of $Z$. Thus, the locus of singular points $S$ of $Y$ is a proper subscheme of $Z_{red}$. By induction, there exist divisors $Y_i, i = 1, \ldots, m$ such that $S_{red} \subset \cup_i Y_i$, and $Y_j$ is smooth outside $\cup_{i=1}^{j-1} Y_i$. Then the divisors $Y_i, i = 1, \ldots, m$ together with $Y$ satisfy the conditions for $Z_{red}$. $\square$
Lemma 4.16 Let $V = (Z \to X \xleftarrow{\rho} \tilde{X}) \in \text{Ob}(\mathcal{RC}(X))$ with $V = \rho^{-1}(Z)$, and $x \in A_*(V)$ be such element that $\text{pr}_Z(x) = 0 \in \lim_{Z' \subset X} A_*(Z')$. Then there exists $Z' \supset Z$ and $V' = (Z' \to X \xleftarrow{\rho'} \tilde{X}') \in \text{Ob}(\mathcal{RC}(X))$ with $V' = \rho^{-1}(Z')$, and $x' \in A_*(V')$ such that $(V', x') \equiv (V, x) \mod \text{image}(d^b_{1,0})$, and $\text{pr}_Z(x') = 0 \in A_*(Z')$.

Proof: Let $Z'' \supset Z$ be such closed subvariety that $\text{pr}_Z(x)|_{Z''} = 0$. By Lemma 4.15, starting from any closed subvariety $Z$ and adding divisors as in Lemma 4.14 one can get a closed subvariety $Z''$ containing any given $Z''$ and an element $x'$ defined over it and representing the same class $\mod \text{image}(d^b_{1,0})$. Since $\text{pr}_Z(x)|_{Z''} = 0$, so is $\text{pr}_Z(x)|_{Z'}$, which is equal to $\text{pr}_Z(x')$, by Lemma 4.14.

Lemma 4.17 Let $u \in b_{0,0}$ be an element whose image in $\lim_{Z' \subset X} A_*(Z')$ is zero. Then $u \in \text{image}(d^b_{1,0})$.

Proof: Let $Z_1, Z_2$ be two closed subvarieties. Then it follows from Lemma 4.15 that by adding divisors as in Lemma 4.14 one can obtain from $Z_1$ and $Z_2$ the same closed subvariety $Z$ (equal to the union of the respective $Y_i$’s for both sets). Then it follows from Lemma 4.14 that we can assume that $u = (V, x)$ is defined over one set $Z$. From Lemma 4.16 we can assume that $\text{pr}_Z(x) = 0 \in A_*(Z)$. Let us prove by Noetherian induction (or induction on the dimension of $Z$, if you want) that then $u \in \text{image}(d^b_{1,0})$. If $Z$ is empty, there is nothing to prove. Suppose, we know the statement for proper closed subvarieties of $Z$. By Proposition 7.12, there exists a divisor $Y$ containing $Z$ and smooth outside some proper closed subvariety $S$ of $Z$. By Lemma 4.14, we can assume that $u$ is defined over $Y$. Now, we can choose a different resolution $V' = (Y \to X \xleftarrow{\rho'} \tilde{X}')$ of $Y$ which involves only blow ups with centers over $S$ and resolves $S$ simultaneously. By Proposition 4.13, we can assume that $u$ is defined for this resolution. Let $\bar{Y}$ be the proper preimage of $Y$, $E_i$ be all the components of the exceptional divisor of $\rho$. We have: $u = (V', x)$, where $\text{pr}_Y(x) = 0$. By Lemma 7.9 we have an exact sequence:

$$0 \leftarrow A_*(Y) \leftarrow (A_*(\bar{Y}) \oplus A_*(S)) \leftarrow \oplus_i A_*(E_i \cap \bar{Y}).$$

Since $x$ is supported on $\bar{Y} \cup (\cup_i E_i)$, from the diagram:

$$\begin{array}{ccc}
\bar{Y} & \leftarrow & (A_*(\bar{Y}) \oplus A_*(S)) \\
\rho' \downarrow & & \oplus_i A_*(E_i \cap \bar{Y}) \\
Y \leftarrow & & S \\
\rho \downarrow & & \downarrow j \\
Y \leftarrow & & \cup_i E_i
\end{array}$$

we see that $x$ can be represented by an element from $A_*(\cup_i E_i)$ which projects to zero in $A_*(S)$. Thus, modulo the image($d^b_{1,0}$), $u$ is equal to an element defined over $S$ which vanishes in $A_*(S)$. By induction, $u \in \text{image}(d^b_{1,0})$.

Corollary 4.18 Let $A^*$ be any theory (Definition 2.7 + (CONST)). Then the natural map

$$\text{Coker}(d^b_{1,0}) \to \text{Coker}(d^b_{1,0})$$

is an isomorphism.

Proof: The surjectivity follows from the existence of resolution making given closed subvariety $Z$ a divisor with strict normal crossing by permitted blow up over $Z$ (Theorem 8.4), and the fact that all fibers of such a resolution are rational varieties (so, admit sections). The injectivity follows from Lemma 4.17.

Remark 4.19 The above Corollary is valid without the (CONST) assumption - see Remark 7.22.
Proposition 4.20 For any theory $A^∗$ (Definition 2.1 + (CONST)), the natural map

$$\hat{\beta} : H(b) \to H(a)$$

is an isomorphism.

Proof: By Corollary 4.18 and Lemma 4.5, the map $\beta : \text{Coker}(d_{1,0}^b) \to \text{Coker}(d_{1,0}^a)$ is an isomorphism. So we need to prove the injectivity of $\hat{\beta}$ only.

Let us introduce category $\mathcal{RC}^{\mathbb{P}_1}(X)$ whose objects are diagrams $(Z \xrightarrow{\rho} X \times \mathbb{P}_1 \xleftarrow{\pi} X \times \mathbb{P}_1)$, where $\rho$ is projective birational map isomorphic outside $Z$, such that the preimage $W = \rho^{-1}(Z)$ is a divisor with strict normal crossing, which is in good position w.r.t. the preimages of $X \times \{0\}$ and $X \times \{1\}$ (that is, the union of these divisors is a divisor with strict normal crossing) which are also divisors with strict normal crossing. In particular, for each component $S$ of $W$, either $S$ is over 0 or 1, or: $S_0 = s^{-1}(X \times \{0\})$ $\xrightarrow{i_0} S$, $S_1 = s^{-1}(X \times \{1\})$ $\xrightarrow{i_1} S$ are divisors with strict normal crossing. We will not need morphisms of $\mathcal{RC}^{\mathbb{P}_1}(X)$. We have maps $\partial_0, \partial_1 : \text{Ob}(\mathcal{RC}^{\mathbb{P}_1}(X)) \to \text{Ob}(S(X))$ defined by:

$$\partial_i(Z \xrightarrow{\rho} X \times \mathbb{P}_1 \xleftarrow{\pi} X \times \mathbb{P}_1) = (W_l \to X),$$

where $X_l$ is the proper preimage of $X \times \{l\}$ and $W_l = W \cap X_l$.

Define:

- $b_{0,1}^l := \bigoplus_{\text{Ob}(\mathcal{RC}^{\mathbb{P}_1}(X))} A_{*+1}(W)$

and differential $d_{0,1}^b : b_{0,1}^l \to a_{0,0}$ by the formula:

$$d_{0,1}(U, \sum_S z_s) = (\partial_0(U), \sum_{S \to \mathbb{P}_1 -, \text{dom.}} i_0^*(z_s)) - (\partial_1(U), \sum_{S \to \mathbb{P}_1 -, \text{dom.}} i_1^*(z_s)).$$

The basic difference between $b_{0,1}$ and $b_{0,1}$ is that in the latter we do not permit blow ups with centers located over 0 and 1, but we do not require components of $W$ to be in normal crossing with the preimages of 0 and 1.

Let us denote the images of the respective maps:

$$a_{0,1} \xrightarrow{\hat{d}_{0,1}^a} \text{Coker}(d_{1,0}^a); \quad b_{0,1}^l \xrightarrow{\hat{d}_{0,1}^b} \text{Coker}(d_{1,0}^b); \quad b_{0,1} \xrightarrow{\hat{d}_{0,1}^b} \text{Coker}(d_{1,0}^b)$$

as $\text{im}(\hat{d}_{0,1}^a), \text{im}(\hat{d}_{0,1}^b), \text{im}(\hat{d}_{0,1}^b)$, respectively.

Let $(Z \xrightarrow{\rho} X \times \mathbb{P}_1 \xleftarrow{\pi} X \times \mathbb{P}_1) \in \text{Ob}(\mathcal{RC}^{\mathbb{P}_1}(X))$ with $W = \rho^{-1}(Z)$. Let us denote as $\text{im}_Z$ the image $(A_{*+1}(W) \xrightarrow{\hat{d}_{0,1}^b} \text{Coker}(d_{1,0}^a))$.

Lemma 4.21

(1) The subgroup $\text{im}_Z$ does not depend on the choice of the resolution, but on $Z$ only.

(2) $\text{im}_Z = \text{im}_{Z, \text{red}}$. 
Proof: Let \((Z_1 \xrightarrow{\iota_1} X \times \mathbb{P}^1 \xrightarrow{\pi_1} Q_1)\) and \((Z_2 \xrightarrow{\iota_2} X \times \mathbb{P}^1 \xrightarrow{\pi_2} Q_2)\) be two such resolutions with \(\rho_1^{-1}(Z_1) = W_1, \rho_2^{-1}(Z_2) = W_2\) with \((Z_1)\text{red} = (Z_2)\text{red}\. In particular, \(Q_1\) and \(Q_2\) are isomorphic outside the preimages of \(Z_i \cup X \times \{0\} \cup X \times \{1\}\). By the Weak Factorization Theorem (Theorem 8.4), there exists a diagram:

\[
\begin{array}{ccccccc}
Y_1 & \xrightarrow{\pi} & Y_2 & \xrightarrow{\pi} & Y_3 & \cdots & Y_{n-2} & \xrightarrow{\pi} & Y_{n-1} & \xrightarrow{\pi} & Y_n & \xrightarrow{\pi} & Q_2, \\
\end{array}
\]

of smooth projective varieties over \(X \times \mathbb{P}^1\) where each blow up is permitted w.r. to all components of the preimage of \(Z_i \cup X \times \{0\} \cup X \times \{1\}\). The respective push-forwards are surjective and commute with \(d_{0,1}\) by Proposition 7.20 (although, \(Y_i\)'s are of slightly more general type than \(X \times \mathbb{P}^1\) from \(b_{0,1}\), we can consider the respective elements in \(a_{0,1}\) and the differential \(d_{0,1}^a\) on them). Notice also, that components dominant over \(\mathbb{P}^1\) are mapped to dominant ones. Hence, \(im_Z\) will be the same. □

**Lemma 4.22** Let \((U \xrightarrow{f} X \times \mathbb{P}^1, x) \in a_{0,1},\) and \(T = f(U) \subset X \times \mathbb{P}^1\). Then

\[d_{0,1}^a(f, x) \in im_T.\]

**Proof:** Let \((f, x) \in a_{0,1},\) where \(U \xrightarrow{f} X \times \mathbb{P}^1\), is such that the preimages \(U_0\) and \(U_1\) of \(X \times \{0\}\) and \(X \times \{1\}\) are divisors with strict normal crossing on \(U\), and \(T = f(U) \xrightarrow{\iota} X \times \mathbb{P}^1\) has dimension \(\leq \dim(X)\). We can assume \(U\) irreducible. Let \((T \xrightarrow{\iota} X \times \mathbb{P}^1 \xrightarrow{\pi} X \times \mathbb{P}^1) \in Ob(\mathcal{RC}(X))\) be the resolution of \(T\) with \(\tilde{T} = \pi^{-1}(T)\) (which always exists by Theorem 8.3, Proposition 8.5). Since the fibers of \(\pi\) are rational varieties (and so, admit sections), we get a rational map \(U \dasharrow S\) to some component \(S \xrightarrow{\pi_S} \tilde{T}\). Notice, that \(S\) is dominant over \(\mathbb{P}^1\). Resolving the indeterminacy of this map and making the preimages \(\tilde{U}_0\) and \(\tilde{U}_1\) of 0 and 1 on \(\tilde{U}\) divisors with strict normal crossing (Theorems 8.3 and 8.4), we get commutative diagram:

\[
\begin{array}{ccc}
\tilde{U} & \xrightarrow{\tilde{u}} & S \\
\pi_U \downarrow & & \pi_S \\
U & \xrightarrow{u} & T.
\end{array}
\]

For \(l = 0, 1\) we have the diagram with Cartesian squares:

\[
\begin{array}{ccc}
S & \xrightarrow{\tilde{u}} & \tilde{U} \\
\iota_{l,S} \downarrow & & \iota_{l,U} \\
S_l & \xrightarrow{\tilde{u}_l} & \tilde{U}_l.
\end{array}
\]

Since \(\pi_u\) is birational, the map \((\pi_u)_*\) is surjective. Hence, there exists \(y \in A_*(\tilde{U})\) such that \((\pi_u)_*(y) = x\). By the ”Multiple points excess intersection formula” (Proposition 7.20),

\[d_{0,1}^a(f, x) = d_{0,1}^a(f \circ \pi_U, y) = d_{0,1}^a(i \circ \pi_S, \tilde{u}_*(y)) = d_{0,1}^a((T \xrightarrow{i} \pi \circ \pi^{-1})((j_S)_* \circ \tilde{u}_*(y))).\]

Thus, \(d_{0,1}^a(U, x) \in im_T.\) □

**Lemma 4.23** For any theory \(A^*\) (Definition 2.7 + (CONST)),

\[im(d_{0,1}^a) = im(d_{0,1}^b).\]
Proof: Clearly, $\text{im}(\hat{b}'_{0,1}) \subset \text{im}(\tilde{d}_{0,1})$, since the diagram:

$$
\begin{array}{ccc}
b'_0 & \xrightarrow{\hat{b}'_{0,1}} & a_0,0 \\
\downarrow & \leftarrow & \downarrow \\
a_0 & \xrightarrow{\tilde{d}_{0,1}} & a_0,0,
\end{array}
$$

is commutative. The other inclusion follows from Lemma 4.22. \qed

Remark 4.24 The above Lemma is valid without the (CONST) assumption - see Remark 7.22.

Lemma 4.25 Let $Z \subset X \times \mathbb{P}^1$ be a closed subvariety, and $Y \subset X \times \mathbb{P}^1$ be a divisor such that $Y \cup X \times \{0\} \cup X \times \{1\}$ is a divisor with strict normal crossing outside $Z$. Then

$$
\text{im} Z \subset \text{im} Z_{\cup Y}.
$$

Proof: By Proposition 8.5 and Theorem 8.4, we can find a blow up $\widetilde{X \times \mathbb{P}^1} \xrightarrow{\rho} X \times \mathbb{P}^1$ permitted w.r.t. $X \times \{0\}$ and $X \times \{1\}$ with centers over $Z$, such that the preimages of $Z$, $Z \cup Y$, $X \times \{0\}$, and $X \times \{1\}$ are strict normal crossing divisors, and so is their union. By Lemma 4.21, we can use this resolution to define $\text{im} Z$ and $\text{im} Z_{\cup Y}$. The inclusion follows. \qed

Lemma 4.26 Let $Z \subset X \times \mathbb{P}^1$ be a closed subvariety whose projection to $X$ has dimension $< \dim(X)$ (for example, $\dim(Z) < \dim(X)$). Then

$$
\text{im} Z \subset \text{im} (\hat{b}'_{0,1}).
$$

Proof: We have: $\dim(p(Z)) < \dim(X)$, where $p : X \times \mathbb{P}^1 \to X$ is the projection. By Lemma 4.15, there exist divisors $T_i$, $i = 1, \ldots, m$ on $X$ such that $T_i$ is smooth outside $\cup_{j=1}^{i-1} T_j$, and $\cup_{i=1}^m T_i =: T = p(Z)_{\text{red}}$. Then $Y_i = T_i \times \mathbb{P}^1$, $i = 1, \ldots, m$ will have the property: $Y_i \cup X \times \{0\} \cup X \times \{1\}$ is a divisor with strict normal crossing outside $\cup_{j=1}^{i-1} Y_j$, and $\cup_{i=1}^m Y_i \supset Z_{\text{red}}$. By Lemma 4.25, $\text{im} Z \subset \text{im} Y$, where $Y = \cup_{i=1}^m Y_i = T \times \mathbb{P}^1$. Now we can make the preimage of $Y \cup X \times \{0\} \cup X \times \{1\}$ a divisor with strict normal crossing by blowing $X \times \mathbb{P}^1$ at smooth centers of the form $B \times \mathbb{P}^1$ located over $T \times \mathbb{P}^1$. Thus, the preimage $\widetilde{Y}$ of $Y$ will not have components over 0 or 1, and the preimages of $X \times \{0\}$ and $X \times \{1\}$ will be birational to $X$ (and consist each of one component only). Hence, the elements of $\text{im} Y$ will be coming from $b_{0,1}$. \qed

Lemma 4.27 Let $\mathcal{U} = (Z \xrightarrow{\pi} X \times \mathbb{P}^1 \xrightarrow{\rho} \mathbb{P}^1) \in \text{Ob}(RC'(X))$ with $W = \rho^{-1}(Z)$, and $u = (\mathcal{U}, x) \in b'_{0,1}$ be such element that $(\rho_W)_*(x) = 0 \in A_{n+1}(Z)$. Then

$$
\tilde{a}_{0,1}'(u) \in \text{im}(\hat{d}'_{0,1}).
$$

Proof: Using Lemma 4.25 and Lemma 4.15 (as in the proof of the Lemma 4.26), we can make $Z$ a divisor. Let $T \subset Z$ be a closed subvariety outside which $Z \cup X \times \{0\} \cup X \times \{1\}$ is a divisor with strict normal crossing. Then $\dim(T) < \dim(Z) = \dim(X)$. Let $\widetilde{X \times \mathbb{P}^1} \xrightarrow{\pi} X \times \mathbb{P}^1$ be the resolution making the preimages of $Z \cup X \times \{0\} \cup X \times \{1\}$ and of $T \cup X \times \{0\} \cup X \times \{1\}$ - divisors with strict normal crossing by blowing at smooth centers permitted w.r.t. $X \times \{0\}$ and $X \times \{1\}$ and located over $T$. By Lemma 4.21, we can assume that $u$ is defined for this resolution $\tilde{\mathcal{U}} = (Z \xrightarrow{\pi} X \times \mathbb{P}^1 \xrightarrow{\rho} \mathbb{P}^1)$. Let $W' = \pi^{-1}(Z)$ be
the preimage, \( \tilde{Z} \) be the proper preimage of \( Z \), and \( E_i \) be all the components of the exceptional divisor of \( \rho \). We have: \( u = (V, x) \), where \( (\pi_W)_*(x) = 0 \). By Lemma 7.9 we have an exact sequence:

\[
0 \leftarrow A_*(Z) \leftarrow \left( A_*(\tilde{Z}) \oplus A_*(T) \right) \leftarrow \oplus_i A_*(E_i).
\]

From the diagram:

\[
\begin{array}{c}
\tilde{Z} \\
\rho' \\
Z \\
\phantom{\rho'}
\end{array} \xleftarrow{\cap_i(E_i \cap \tilde{Z})} \xleftarrow{T} \xleftarrow{\cup_i E_i} T
\]

we see that \( x \) can be represented by an element from \( A_*(\cup_i E_i) \) which projects to zero in \( A_*(T) \). Thus, we can assume that \( u \) is defined over \( T \). Since \( \dim(T) < \dim(X) \), by Lemma 4.26 \( d_{0,1}'(u) \in \text{im}(d_{0,1}) \).

**Lemma 4.28** Let \( Z \subset X \times \mathbb{P}^1 \) be a proper closed subvariety. Then there exist divisors \( Y_i, i = 1, \ldots, m \) such that \( Z \subset \cup_i Y_i \), and \( Y_j \cup X \times \{0\} \cup X \times \{1\} \) is a divisor with strict normal crossing outside \( \cup^{i-1}_{j=1} Y_i \).

**Proof:** Let \( Z' \subset Z \) be the closure of \( Z \setminus (X \times \{0\} \cup X \times \{1\}) \), and \( S' = (Z' \cap (X \times \{0\} \cup X \times \{1\})) \cup \text{Sing}(Z) \cup Z_{<\dim(X)} \) (where the latter is the union of the smaller dimensional (non-divisorial) components). Let \( S := p(S')_{\text{red}} \), where \( p : X \times \mathbb{P}^1 \to X \) is the projection. Then \( \dim(S) < \dim(X) \). By Lemma 4.15 there are divisors \( T_i, i = 1, \ldots, m \) on \( X \) such that \( T_i \) is smooth outside \( \cup^{i-1}_{j=1} T_j \), and \( \cup^{m}_{i=1} T_i \supset S \). Then, clearly, the divisors \( Y_i := T_i \times \mathbb{P}^1 \) on \( X \times \mathbb{P}^1 \) will have the property that \( Y_i \cup X \times \{0\} \cup X \times \{1\} \) is a divisor with normal crossing outside \( \cup^{i-1}_{j=1} Y_j \), and on \( (X \times \mathbb{P}^1) \setminus \cup^{m}_{i=1} Y_i \) our closed subvariety \( Z \) is a smooth divisor not meeting \( (X \times \{0\} \cup X \times \{1\}) \). It remains to add the divisorial part of \( Z \) to \( Y_i, i = 1, \ldots, m \).

**Lemma 4.29**

1) Let \( Z = Z_1 \cup Z_2 \), where \( Z_1, Z_2 \) be closed subvarieties of \( X \times \mathbb{P}^1 \). Then \( \text{im}_{Z_1} + \text{im}_{Z_2} \mod \text{im}(d_{0,1}') \).

2) Let \( u = (U, x) \in b_{0,1}' \) be an element defined over a closed subvariety \( Z \) such that the restriction of \( (\rho_W)_*(x) \) to the generic points of all the components of \( Z \) of dimension \( = \dim(X) \) is zero. Then \( d_{0,1}'(u) \in \text{im}(d_{0,1}') \).

**Proof:** By Lemma 4.28 we can find divisors \( Y_i, i = 1, \ldots, m \) such that \( Y_i \cup X \times \{0\} \cup X \times \{1\} \) is a divisor with strict normal crossing outside \( \cup^{i-1}_{j=1} Y_j \), and \( Y = \cup^{m}_{i=1} Y_i \supset Z \).

(1) We have: the map \( A_{i+1}(Z_1) \oplus A_{i+1}(Z_2) \to A_{i+1}(\tilde{Z}) \) is surjective, and so are the maps \( (\rho_W)_* : A_{i+1}(W) \to A_{i+1}(Z), (\rho_{W_1})_* : A_{i+1}(W_1) \to A_{i+1}(Z_1), \) and \( (\rho_{W_2})_* : A_{i+1}(W_2) \to A_{i+1}(Z_2) \) (the fibers are rational varieties). But if we have elements \( u, u_1, u_2 \) defined over \( Z, Z_1, \) and \( Z_2, \) respectively, such that \( (\rho_W)_* u = (\rho_{W_1})_* u_1 |Z + (\rho_{W_2})_* u_2 |Z \), then by the proof of Lemma 4.25 there are elements \( u', u'_1, u'_2 \) defined over \( Y \) which represent the same elements in \( \text{Coker}(d_{i,0}') \) as \( u, u_1, u_2 \) and such that their images in \( A_{i+1}(Y) \) are \( (\rho_W)_* u |Y, (\rho_{W_1})_* u_1 |Y, (\rho_{W_2})_* u_2 |Y \), respectively. It remains to apply Lemma 4.27

(2) By condition, \( (\rho_W)_*(x) \) is defined over some subvariety \( S \subset Z \) of dimension \( < \dim(X) \). Using the above \( Y \) again together with Lemmas 4.25 4.26 and 4.27 we get what we need.

**Lemma 4.30** Let \( Z \subset X \times \mathbb{P}^1 \) be a divisor of degree 1 over \( X \). Then:

\[
\text{im}_Z \in \text{im}(d_{0,1}').
\]
Proof: It will be convenient to change a parametrization of $\mathbb{P}^1$, so that the points 0 and $\infty$ be marked (instead of 0 and 1). By our condition, $Z$ is a closure of the graph of some meromorphic function $f \in k(X)^\times$. Let $D_0$ and $D_\infty$ be the divisors of zeroes and poles of $f$ (we can clearly assume that $f$ is not constant). By blowing $\tilde{X} \xrightarrow{\pi} X$ at smooth centers located over $D_0 \cup D_\infty$ we can make the proper preimages $\tilde{D}_0$ and $\tilde{D}_\infty$ (we take the proper preimages of components keeping the multiplicities), the special divisor $E$ of $\pi$, and $\tilde{D}_0 \cup E$, $\tilde{D}_\infty \cup E$ to be divisors with strict normal crossing, and the function $f : \tilde{X} \to \mathbb{P}^1$ be everywhere defined (by Theorem 8.3 and Proposition 8.5). Let $X \times \mathbb{P}^1 = \tilde{X} \times \mathbb{P}^1$. Thus, it is obtained from $X \times \mathbb{P}^1$ by blow ups constant along $\mathbb{P}^1$. We have the regular embedding: $\tilde{X} \cong \Gamma_f \to X \times \mathbb{P}^1$, and $\Gamma_f \cap (X \times 0) = \tilde{D}_0 + E_0$, and $\Gamma_f \cap (X \times \infty) = \tilde{D}_\infty + E_\infty$, where $E_0$ and $E_\infty$ are some $\mathbb{Z}_{\geq 0}$-linear combinations of the components of $E$. Thus, these are divisors with strict normal crossing on $\Gamma_f$. Let $F = E \times \mathbb{P}^1$ be the special divisor on $\mathbb{X} \times \mathbb{P}^1$. By our condition, the divisor $\Gamma_f \cup F$ has strict normal crossing. The map $\varepsilon$ is the blow up of some closed subscheme $S \to X$ of codimension $\geq 2$. Then $\varepsilon^{-1}(S) = \mathcal{E}$ is some positive linear combination of (all of) the exceptional divisors, and for $T := S \times \mathbb{P}^1$, and $\rho = (\varepsilon \times \text{id})$, $\rho^{-1}(T) = F' = E' \times \mathbb{P}^1$. On the other hand, $\rho^{-1}(Z) = \Gamma_f + F''$, where $F''$ is another linear combination of the components of $F$. Hence, $W := \rho^{-1}(Z \times T)$ is $\Gamma_f$ plus some linear combination of (all) the components of $F$, and $\rho$ is an isomorphism on $X \times \mathbb{P}^1$. By blowing $X \times \mathbb{P}^1 \xrightarrow{\pi} X \times \mathbb{P}^1$ further at smooth centers located over $W \cap (\tilde{X} \times \{0\})$ and $W \cap (\tilde{X} \times \{\infty\})$ we can make the preimage $\tilde{W}$ of $W$, as well as it’s union with the preimages of $(\tilde{X} \times \{0\})$ and $(\tilde{X} \times \{\infty\})$ to be divisors with strict normal crossing. Let $\tilde{u} \in b_{0,1}$ be an arbitrary element defined over $Z \cup T$. By Lemma 4.21), we can assume that it has the form: $\tilde{u} = (\tilde{Z} \cup T \to X \times \mathbb{P}^1 \xrightarrow{\rho \pi} X \times \mathbb{P}^1, \tilde{x})$, for some $\tilde{x} \in A_{s+1}(\tilde{W})$. Let $u := ((Z \cup T \to X \times \mathbb{P}^1 \xrightarrow{\rho \pi} X \times \mathbb{P}^1), (\pi_{\tilde{W}})_s(\tilde{x})) \in b_{0,1}$. By Proposition 7.20 the push-forward $\pi_*$ commutes with the pull-backs $i_0^*$ and $i_\infty^*$. Hence, $\bar{d}_{0,1}(\tilde{u}) = \bar{d}_{0,1}(u) \in \text{Coker}(d_{0,1}^2)$. By Lemma 4.29 $\text{im}_Z \subset \text{im}_{Z \cup T} \subset \text{im}(\bar{d}_{0,1})$. 

Let $Y \subset X \times \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible divisor. We will say that $Y$ is "in general position", if it’s intersections with all the faces of $\mathbb{P}^1 \times \mathbb{P}^1$ have the right dimension (that is, $Y_{s=1}$ and $Y_{l=1}$ are divisors in $X \times \mathbb{P}^1$, for $l = 0, 1$, and $Y_{s=l,t=m}$ are divisors in $X$, for $l, m = 0, 1$, and $Y$ is smooth in the generic points of all the components of the intersection with the large faces. We call a divisor $Z \subset X \times \mathbb{P}^1$ "constant" if it has the form $Q \times \mathbb{P}^1$.

Lemma 4.31 Let $A^*$ be any theory as in Definition 2.7 satisfying (CONST). Let $Y \subset X \times \mathbb{P}^1 \times \mathbb{P}^1$ be an irreducible divisor in general position, and such that $Y_{s=1} = Z \cup \{ \text{constant div.} \}$, where $Z$ is irreducible, and the divisors $Y$ and $(s = 1)$ are transversal in the generic point of $Z$. Then:

$$\text{im}_Z \subset \text{im}_{Y_{s=0}} + \text{im}_{Y_{t=0}} + \text{im}_{Y_{l=1}} + \text{im}(\bar{d}_{0,1})$$

Proof: Let $X \times \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\pi} X \times \mathbb{P}^1 \times \mathbb{P}^1$ be the permitted blow up with centers over $Y$ making the proper preimage $\tilde{Y}$ of $Y$ a smooth divisor, the proper preimage of $(\cup_{l=0} \{X \times \mathbb{P}^1 \times \{l\}) \cup (\cup_{m=0} \{X \times \{m\} \times \mathbb{P}^1\}$ - a divisor with strict normal crossing, and it’s intersection with $\tilde{Y}$ - a divisor with strict normal crossing on $\tilde{Y}$. Then $\pi^{-1}(X \times \mathbb{P}^1 \times \{l\})$ consist of the proper preimage $\tilde{T}_{s=1}$ of $X \times \mathbb{P}^1 \times \{l\}$ and, possibly, of some components $\tilde{E}_{s=1}$ located over some closed subvarieties $R_{s=1}$ of $Y \cap X \times \mathbb{P}^1 \times \{l\}$. Moreover, since this intersection is smooth in the generic point, the mentioned subvarieties will be of codimension $\geq 2$.
on $X \times \mathbb{P}^1 \times \{1\}$ (that is, of dimension $< \dim(X)$). And the same happens to $\pi^{-1}(X \times \{m\} \times \mathbb{P}^1)$. Let

$$X \times \{m\} \times \{1\} \xrightarrow{j_{s=t}} X \times \mathbb{P}^1 \times \{1\} \quad \text{and} \quad \tilde{Y}_{t=m,s=t} \xrightarrow{i_{s=t}} \tilde{Y}_{s=t}. $$

be the face maps and their restriction on $\tilde{Y}$. By the functoriality of $i^*$ we have:

$$(i_{t=m}^*)^* \circ (i_{s=t})^* = (i_{t=m}^*)^* \circ (i_{t=m})^* : A_{s+2}(\tilde{Y}) \rightarrow A_s(\tilde{Y}_{t=m,s=t}). \quad (5)$$

If $x \in A_{s+2}(\tilde{Y})$, then $(i_{s=t})^*(x)$ will have components on $\tilde{T}_{s=t}$ representing some element of $b'_{0,1}$, and, possibly, some components on $\tilde{E}_{s=t}$. Considering the latter components as elements of $a_{0,1}$, we see from Lemma 4.22 that $d_{0,1}^b$ of these in Coker($d_{1,0}^a$) will belong to $im R_{s=t}$, which is a subset of $im(d_{0,1}^b)$, by Lemma 4.26 since $\dim(R_{s=t}) < \dim(X)$. Thus, modulo $im(d_{0,1}^b)$, we can ignore these components. In the same way, by the same Lemma we can ignore the components located over constant divisors on $X \times \mathbb{P}^1$.

Let $(i_{s=t})^*(x) \in b'_{0,1}$ denotes the $\tilde{T}_{s=t}$-part of $(i_{s=t})^*(x)$ with constant components ignored (well-defined modulo elements coming from $\tilde{E}_{s=t}$ and modulo constant components). Similar considerations apply to $(i_{t=m})^*(x)$. From (5) we get:

$$d_{0,1}^b((i_{s=t})^*(x) - d_{0,1}^b((i_{s=0})^*(x) \equiv d_{0,1}^b((i_{t=1})^*(x) - d_{0,1}^b((i_{t=0})^*(x) \mod im(d_{0,1}^b)$$

Let $\tilde{i} : \tilde{Z} \rightarrow \tilde{Y}$ be the proper preimage of $Z$. Then $\tilde{Z}$ is a component of $\tilde{Y} \cap \tilde{T}_{s=t}$, and by our condition (on the restriction of faces to $\tilde{Y}$), $\tilde{Z}$ is a smooth divisor on $\tilde{Y}$. Moreover, since the intersection of $Y$ and $X \times \mathbb{P}^1 \times \{1\}$ was transversal in the generic point of $Z$, we have:

$$\pi^*(O(s = 1))|_{\tilde{Y}} = O(\tilde{Z}) \otimes O(C) \otimes O(\tilde{E}_{s=1}'),$$

where $C$ is some constant divisor, and $\tilde{E}_{s=1}'$ is some $\mathbb{Z}_{>0}$-linear combination of the components of $\tilde{E}_{s=1}$, which implies that we can set: $(i_{s=1})^*(x) = (\tilde{i})^*(x)$. The map $\tilde{Z} \rightarrow Z$ is birational, and the composition

$$A_{s+2}(\tilde{Y}) \xrightarrow{(\tilde{i})^*} A_{s+1}(\tilde{Z}) \xrightarrow{A^*} A_s(k(\tilde{Z}))$$

is surjective, because $A^*$ is constant (this is the only (!) place where we are using the $(CONST)$ axiom).

Thus, for any $a \in A_{s+1}(k(Z))$ we can find an element $x \in A_{s+2}(\tilde{Y})$ such that $\pi^*((i_{s=1})^*(x))|_{k(Z)} = a$, while (by the above identity) $d_{0,1}^b((i_{s=1})^*(x)) \in im Y_{s=0} + im Y_{s=1} + im Y_{s=0} + im(d_{0,1}^b)$. It remains to apply Lemma 4.29(2). \hfill $\square$

**Lemma 4.32** For any theory $A^*$ (Definition 2.7 + $(CONST)$),

$$im(d_{0,1}^b) = im(d_{0,1}^b).$$

**Proof:** By Lemma 4.26 it remains to show that $im Z \subset im(d_{0,1}^b)$, for each divisor $Z \subset X \times \mathbb{P}^1$. By Lemmas 4.29 and 4.26 we can assume that $Z$ is irreducible, and has positive degree over $X$, and by definition of $d_{0,1}^b$ we can assume that $Z$ is different from $X \times \{0\}$ and $X \times \{1\}$, and so, meets these divisors in proper codimension. Let $Z'' \subset \mathbb{P}^1_{k(X)}$ be the restriction of $Z$ w.r.t. the open embedding $\text{Spec}(k(X)) \rightarrow X$. Then $Z''$ is some closed point of some degree $n > 0$ on the projective line over $k(X)$. Let $\sum_{i=0}^n a_i t^i$ be the homogeneous equation of $Z''$. By our condition, $a_0 \neq 0$ and $\sum_{i=0}^n a_i \neq 0$. Consider
Y^n \subset (\mathbb{P}^1 \times \mathbb{P}^1)_{k(X)}$ defined by the equation: $\sum_{i=0}^{n} a_i t_i^{n-i} s_1 + a_0 s_0$. Then $Y^n = Z^n, Y^0 = 0$ is given by the equation $a_0^0, Y^0 = 0$ - by the equation $a_0 s_0$, and $Y^0 = 0$ - by the equation $(\sum_{i=1}^{n} a_i) s_1 + a_0 s_0$. Thus, $Y^n$ is a curve in general position on $(\mathbb{P}^1 \times \mathbb{P}^1)_{k(X)}$, whose restriction to ($s = 1$) is $Z^n$, whose other three restrictions are points of degree 1 (some with multiplicities), and which does not meet "corners" ($s = l, t = m$), $l, m = 0, 1$. Moreover, $Y^n$ is smooth at all the points $Y^n, Y^n, Y^n, Y^n$; and transversal to ($s = 1$) at $Z^n$ (as our point is separable). Denote as $Y$ the closure of $Y^n$ in $X \times \mathbb{P}^1 \times \mathbb{P}^1$. Then, modulo constant divisors, $Y_s = Y$, while $Y_s = Y_{t=0}$, and $Y_{t=1}$ are equivalent to some divisors of degree 1 over $X$ (some of these with multiplicities), and $Y$ is transversal to ($s = 1$) at the generic point of $Z$. It follows from Lemmas $1.3.1$, $1.3.2$ and $1.2.1$ that $\text{im} Z \subset \text{im} (d_{0,1})$.

Combining Lemmas $1.2.7$ and $1.3.2$ we see that $\text{im}(d_{0,1}) = \text{im}(d_{0,1}) \subset \text{Coker}(d_{1,0})$, which implies that the natural map $\hat{\beta}: H(b) \to H(a)$ is an isomorphism. Proposition $1.2.1$ is proven.

4.3 The bi-complex $\mathfrak{c}$.

Now we are ready to construct the description of $A^*$ in terms of pull-backs. In this section we will assume that $A_*$ is a theory of rational type. By Proposition $4.1.3$ this means that $A_*$ is free in the sense of M.Levine-F.Morel, that is, it can be obtained from Algebraic Cobordism theory by change of coefficients: $A_* = \Omega_* \otimes A$. In particular, we can use the tools constructed by M.Levine and F.Morel for Algebraic Cobordism in [12]. Among them we will need the refined pull-backs for locally complete intersection morphisms.

Consider the bi-complex $\mathfrak{c} = c(A^*)$:

- $c_{0,0} := \bigoplus_{\text{Ob}(\text{RC}(X))} A_*(V) \cap \text{im}(\rho)$, where $\rho : A_*(Z) \to A_*(V)$ is the refined pull-back;

- $c_{1,0} := \left( \bigoplus_{\text{Mor}_{\text{I}}} A_*(V_1) \cap \text{im}(\rho) \right) \bigoplus \left( \bigoplus_{\text{Mor}_{\text{II}}} A_*(V_2) \cap \text{im}(\rho) \right) \quad \text{see [14]}$;

- $c_{0,1} := \bigoplus_{\text{Ob}(\text{RC}^1(X))} A_{s+1}(W) \cap \text{im}(\rho)$.

and the differentials are defined as follows:

- $d_{1,0}(V_2 \xrightarrow{f} V_1, x) = (V_2, (\pi_V(f))^1(x)) - (V_1, x)$ and
- $d_{1,0}(V_2 \xrightarrow{f} V_1, y) = (V_2, y) - (V_1, (i_V(f))_s(y))$.

- $d_{0,1}(U, \sum_S z_s) = (\partial_0 U, \sum_S i^1_S(z_s)) - (\partial_1 U, \sum_S i^0_S(z_s))$.

The fact that $d_{1,0}$ lands in $c_{0,0}$ follows from functoriality of the refined pull-backs, and from commutativity of them with push-forwards ([12] Theorem 6.6.6(3),(2)(a))

Our $W_l = \bigcup_S S_l$ fits into the cartesian diagram (with $Z_l = Z \cap X \times \{l\}$):

$$
\begin{array}{c}
\text{Z}_l & \xrightarrow{\rho_l} & \text{W}_l & \xrightarrow{j_l} & \text{W} & \xrightarrow{\rho} & \text{Z} & \xrightarrow{\rho_l} & \text{X} \\
\downarrow{\text{w}_l} & & \downarrow{w} & & \downarrow{\text{z}} & & \downarrow{\text{w}} & & \downarrow{\text{z}} \\
X \times \{l\} & \xrightarrow{\rho_l} & \text{X} & \xrightarrow{\rho} & \text{X} \times \mathbb{P}^1 & \xrightarrow{\rho} & \text{X} \\
\end{array}
$$

And since the blow up $\text{X} \times \mathbb{P}^1 \to X \times \mathbb{P}^1$ has no centers located over 0 and 1, it is transversal to the embeddings $X \times \{l\} \xrightarrow{k_l} X \times \mathbb{P}^1, l = 0, 1$. This implies that $(k_l)^1 = (j_l)^1 : A_{s+1}(W) \to A_*(W_l)$.
Now, the fact that $d_{0,1}$ lands in $c_{0,0}$ follows from the equality (for $l = 0, 1$):
$$\sum_{S} i^{*}_{l}(z_{s}) = (k_{l})^{\dagger}(\sum_{S} z_{s}),$$
and from functoriality of the refined pull-backs:
$$(j_{l})^{\dagger} \circ \rho^{j} = (\rho \circ j_{l})^{\dagger} = (\rho_{l})^{\dagger} \circ (k_{l})^{\dagger}.$$ 

Consider the map:
$$\rho^{j} \circ \rho_{*} : b \to c.$$ 

The fact that it commutes with the first half of $d_{1,0}$ follows from the identity $(\rho \circ \pi)^{1} \circ (\rho \circ \pi)_{*} = \pi^{1} \circ (\rho^{j} \circ \rho_{*}) \circ \pi_{*}$ (which uses the functoriality of push-forwards and refined pull-backs - see [12, Theorem 6.6.6(3)]). The commutativity with the second half of $d_{1,0}$ follows from the fact that the push-forwards are functorial, and that the refined pull-backs commute with push-forwards - see [12, Theorem 6.6.6(2)(a)]. The commutativity with $d_{0,1}$ follows from the equality: $(k_{l})^{\dagger} = (j_{l})^{\dagger} : A_{s+1}(W) \to A_{s}(W_{l})$, and the identity:
$$j_{l}^{\dagger} \circ \rho^{j} \circ \rho_{*}^{l} = \rho_{l}^{j} \circ k_{l}^{\dagger} \circ \rho_{*}^{l} = \rho_{l}^{j} \circ \rho_{l}^{j} \circ k_{l}^{\dagger} = \rho_{l}^{j} \circ \rho_{l}^{j} \circ j_{l}^{\dagger},$$
which again uses [12, Theorem 6.6.6(3),(2)(a)].

Let us denote as $H(c)$ the 0-th homology of the total complex $\text{Tot}(c)$ of $c$. We have natural maps:
$$\gamma : \text{Coker}(d_{1,0}^{0}) \to \text{Coker}(d_{1,0}^{k}), \quad \text{and} \quad \hat{\gamma} : H(b) \to H(c).$$

Since all the fibers of $\rho$ are rational varieties, and $V = \rho^{-1}(Z)$, the map $\rho_{*} : A_{s}(V) \to A_{s}(Z)$ is surjective, (and similar for $W$). Hence, the map $\rho^{j} \circ \rho_{*} : b \to c$ is surjective.

Since $A^{*}$ is a theory of rational type, by Definition 4.1 and Proposition 4.20 we have an identification:
$$A^{*} = H(b),$$
which gives the map
$$\varphi : A^{*} \to H(c).$$

On the other hand, we have the map
$$\psi = \frac{\rho_{*}}{\rho_{*}(1)} : H(c) \to A^{*}.$$ 

The fact that $\psi$ is well defined follows from commutativity of push-forwards with the refined pull-backs, functoriality of push-forwards, and projection formula. From the same projection formula we see that the composition $\psi \circ \varphi : A^{*} \to A^{*}$ is the identity. Thus, we have proven:

**Theorem 4.33** Let $A^{*}$ be a theory of rational type. Then we have the natural identification:
$$A^{*} = H(c).$$

### 4.4 Theories of higher types

The Definition 4.1 is sufficiently flexible. In the sense, that one can modify the term $a_{0,1}$ somewhat. In the current form, it imposes rational equivalence.

Consider the bi-complex $a^{(1)} = a^{(1)}(A^{*})$, where $a_{0,0}^{(1)} = a_{0,0}, a_{1,0}^{(1)} = a_{1,0}$, while $a_{0,1}^{(1)} := \bigoplus_{W \to X \times C} A_{s+1}(W)$, where the sum is taken over all projective maps $w : W \to X \times C$ with $W$ smooth, $C$ - smooth projective curve with two fixed points $p_{0} \xrightarrow{j_{0}} C \xleftarrow{j_{1}} p_{1}$ of the same degree on it, $\dim(W) \leq \dim(X)$, and $W_{0} = w^{-1}(X \times p_{0}) \xrightarrow{j_{0}} W$, $W_{1} = w^{-1}(X \times p_{1}) \xleftarrow{j_{1}} W$ are divisors with strict normal crossing, where the differential $d_{0,1}$ is defined as before.

Define $H(a^{(1)})$ as the zero-th homology of $\text{Tot}(a^{(1)})$. We have a natural surjection $H(a) \to H(a^{(1)})$. 32
Definition 4.34 Let $A^*$ be a theory (in the sense of Definition 2.1) satisfying (CONST). We call $A^*$ a "theory of algebraic type", if the natural homomorphism $\theta_a : H(a) \to A^*$ descends to an isomorphism $\theta_{a(1)} : H(a(1)) \to A^*$.

Example of such a theory is provided by $\text{CH}_{\text{alg}}$ - the Chow groups modulo algebraic equivalence.

Notice, that there is a lot of freedom in imposing relations similar to that of $a^0$, $1$ producing a theory in the sense of Definition 2.1 inductively on the dimension of $X$. This way one obtains a huge number of "theories", which suggests that the general object satisfying this Definition is not particularly good, and additional restrictions are needed if one wants to prove any result of interest.

It is natural to consider the rational type as type $0$, and algebraic type as type $1$. It is tempting to extend this to arbitrary $n$, producing a tower:

$$(A^{(0)})^* \to (A^{(1)})^* \to \ldots \to (A^{(n)})^* \to \ldots,$$

where $(A^{\infty})^* = \text{image}((A^{(0)})^* \to A^*)$, and $A^{(n)}$-equivalence on $m$-motives, $m \leq n$ would coincide with the topological equivalence. In particular, I would expect the "proper" theories of type $n$ to be again in 1-to-1 correspondence with the formal group laws $\mathbb{L} \to A$ (and so produced by change of coefficients from the Algebraic Cobordism of type $n$). We will not pursue this direction in the current paper.

5 From products of projective spaces to $\text{Sm}_k$

In this section, $A^*$ is a theory of rational type, and $B^*$ is any theory in the sense of Definition 2.1. Our aim here is to prove the main result of the article:

Theorem 5.1 Let $A^*$ be a theory of rational type, and $B^*$ be any theory in the sense of Definition 2.1. Fix $n,m \in \mathbb{Z}$. Then any additive transformation

$$A^n((\mathbb{P}^\infty)^{\times l}) \xrightarrow{G} B^m((\mathbb{P}^\infty)^{\times l}), \text{ for } l \in \mathbb{Z}_{\geq 0}$$

commuting with the pull-backs for:

(i) the action of $\mathfrak{S}_l$;

(ii) the partial diagonals;

(iii) the partial Segre embeddings;

(iv) $(\text{Spec}(k) \hookrightarrow \mathbb{P}^\infty)^{\times r}$, $\forall r$;

(v) the partial projections

extends to a unique additive operation $A^n \xrightarrow{G} B^m$ on $\text{Sm}_k$.

Remark 5.2 1) The condition on $A^*$ is necessary. For example, the identity transformation

$$\text{CH}^*_{\text{alg}} |_{(\mathbb{P}^\infty)^{\times l}} \xrightarrow{id} \text{CH}^*_{\text{rat}} |_{(\mathbb{P}^\infty)^{\times l}}$$

cannot be extended to a morphism of theories.

2) In Topology an analogous result was obtained by T.Kashiwabara - see [9, Theorem 4.2].
Proof:

The transformation $A^n \rightrightarrows B^m$ on $(\mathbb{P}^\infty)^{\times l}$ commuting with the partial diagonals and partial projections is completely determined by its action on $\alpha \cdot (\prod_{i=1}^l z_i^A)$, where $z_i^A = c_i^1(\mathcal{O}(1)_{B/j})$. Let

$$G(\alpha \cdot \prod_{i=1}^l z_i^A) =: G_l(\alpha)(z_i^B, \ldots, z_l^B) \in B[[z_i^B, \ldots, z_l^B]](m).$$

Conditions (i), (iii) and (iv) impose certain restrictions on these. More precisely, (i) implies (a_i), (iii) implies (a_ii) and (iv) implies (a_ii) below for $X = \text{Spec}(k)$. Starting with the power series $G_l(\alpha)(z_1, \ldots, z_l)$ we will extend $G$ to $X \times (\mathbb{P}^\infty)^{\times l}$ by induction on the dimension of $X$.

**Definition 5.3** Let $X$ be smooth quasi-projective variety. Denote as $G(X) = \{G_l, l \in \mathbb{Z}_{\geq 0}\}$ the following data:

$$G_l \in \text{Hom}_{\mathbb{Z}-\text{lin}}(\mathbb{A}^{n-l}(X), B^*(X)[[z_1, \ldots, z_l]](m))$$

satisfying:

1. $G_l$ is symmetric with respect to $\mathcal{G}_l$;

2. $G_l(\alpha) = \prod_{i=1}^l z_i \cdot F_l(\alpha)$, for some $F_l(\alpha) \in B^*(X)[[z_1, \ldots, z_l]](m-l)$.

3. $G_l(\alpha)(x + B y, z_2, \ldots, z_l) = \sum G_{l+i+j-1}(\alpha \cdot a^{A}_{i,j})(x^{i}, y^{j}, z_2, \ldots, z_l)$, where $a^{A}_{i,j}$ and $a^{B}_{i,j}$ are the coefficients of the formal group laws of $A^*$ and $B^*$.

Let $\chi_A(x) = (-A x) = \sum_{i \geq 0} c_i^A \cdot x^{i+1}, x - A y = \sum_{i,j} b_{i,j}^A x^i y^j$, and similar for $B$. Then it follows from 1) and 2) (by (simultaneous for all $l$) induction on the degree of terms of power series) that:

$$G_l(\alpha)(-B x, z_2, \ldots, z_l) = \sum G_{l+i}(\alpha \cdot c_i^A)(x^{i+1}, z_2, \ldots, z_l),$$

and

$$G_l(\alpha)(x - B y, z_2, \ldots, z_l) = \sum G_{l+i+j-1}(\alpha \cdot b_{i,j}^A)(x^{i}, y^{j}, z_2, \ldots, z_l).$$

If $V$ is some vector bundle with $B$-roots $\lambda_1^B, \ldots, \lambda_r^B$, then it follows from (a_i) that $F_{l+r}(\alpha)(\lambda_1^B, \ldots, \lambda_r^B, z_1, \ldots, z_l)$ is a function of $c_1^B(V), \ldots, c_r^B(V)$, and so, it does not depend on the choice of roots.

**Definition 5.4** Let $d$ be some natural number. We say that $G(d)$ is defined, if, for all $X$ smooth quasi-projective of dimension $\leq d$, $G(X)$ is defined, and these satisfy:

1. For any $f : X \to Y$ with $\dim(X), \dim(Y) \leq d$, and any $\alpha \in A^{n-l}(Y)$,

$$G_l(f^*_A(\alpha)) = f^*_B G_l(\alpha).$$

2. For any regular embedding $j : X \to Y$ of codimension $r$ with normal bundle $\mathcal{N}_j$ with $B$-roots $\mu_1^B, \ldots, \mu_r^B$, for any $\alpha \in A^{n-l-r}(X)$, one has:

$$F_l(j_*(\alpha))(z_1, \ldots, z_l) = j_*(F_{l+r}(\alpha)(\mu_1^B, \ldots, \mu_r^B, z_1, \ldots, z_l)).$$

The condition (b_ii) can be rewritten as:

$$G_l(j_*(\alpha))(z_1, \ldots, z_l) = j_* \text{Res}_{l=0} G_{l+r}(\alpha)(t + B \mu_1^B, \ldots, t + B \mu_r^B, z_1, \ldots, z_l) \cdot \omega_l^B \frac{(t + B \mu_1^B) \cdots (t + B \mu_r^B) \cdot t}{(t + B \mu_r^B) \cdots (t + B \mu_1^B) \cdot t}.$$

In such a situation we have the following specialization result. To shorten the notations, we will denote $z_1, \ldots, z_l$ as $\overline{z}$.
Lemma 5.5 Let $G(d)$ be defined, $X$ be smooth quasi-projective variety of dimension $\leq d$, and $L$ be a linear bundle on $X$ with $\lambda^L = c_1^L(L)$, $\lambda^B = c_1^B(L)$. Then, for any $\alpha \in A^{n-l-1}(X)$,

$$G_l(\alpha \cdot \lambda^L)(z) = G_{l+1}(\alpha)(\lambda^B, z).$$  \hfill (6)

Proof: 1) Let $L$ be very ample. Then $\lambda^L = j_*(1)$, where $Y \rightarrow X$ is a smooth divisor, and:

$$G_l(\alpha \cdot \lambda^L)(z) = F_l(j_*(j^*(\alpha))(z)) \cdot \prod_{i=1}^{l} z_i = j_* F_l(j^*(\alpha))(\lambda^B, z) \cdot \prod_{i=1}^{l} z_i = j_* j^* F_l(\alpha)(\lambda^B, z) \cdot \prod_{i=1}^{l} z_i = G_l(\alpha)(\lambda^B, z).$$

2) Let now $L$ be arbitrary. Since $X$ is quasi-projective, $L = \mathcal{L}_1 \otimes \mathcal{L}^{-1}_2$, for some very ample line bundles $\mathcal{L}_i$. Using 1) and the analogue of $(a_{iii})$ for the formal difference, we get:

$$G_l(\alpha \cdot \lambda^L)(z) = \sum_{i,j} G_l(\alpha \cdot (\lambda^L)^i \cdot (\lambda^A)^j \cdot b_{i,j})(z) = \sum_{i,j} G_l(\alpha \cdot b_{i,j}((\lambda^L)^i, (\lambda^A)^j, z) = G_{l+1}(\alpha)(\lambda^L - B \lambda^L, z) = G_{l+1}(\alpha)(\lambda^B, z).$$

If $G(X)$ is defined, we can define $G(X \times \mathbb{P}^\infty)$ as follows. We have:

$A^\ast(X \times \mathbb{P}^\infty) = A^\ast(X)[[t]]$, where $t = c_1^A(\mathcal{O}(1))$. For $\alpha(t) = \sum_{i=0}^{\infty} \alpha_i \cdot t^i$, set:

$$G_l(\alpha(t))(z) = \sum_i G_{l+i}(\alpha_i)(t^{x_i}, z) \in B^\ast[[t]][[z_1, \ldots, z_i]],$$

which converges by $(a_{ii})$. It follows immediately from the definition that $(a_{ii,iii})$ are satisfied.

Lemma 5.6 Suppose that $G(X)$ is defined, and satisfies (6). Then the above definition of $G(X \times \mathbb{P}^\infty)$ satisfies (6) as well.

Proof: Arbitrary linear bundle $L$ on $X \times \mathbb{P}^\infty$ has the form $\mathcal{M}(r)$, for some $r \in \mathbb{Z}$, and some linear bundle $\mathcal{M}$ on $X$. Let $\mu^A = c_1^A(\mathcal{M})$, $\mu^A + A [r] \cdot A t = \sum_{i,j} c_{ij}(\mu^A)^i \cdot t^j$, and $\gamma \in A^\ast(X)$. Then, by the definition of $G(X \times \mathbb{P}^\infty)$, the condition (6) for $X$, and $(a_{iii})$, we get:

$$G_l(\gamma \cdot t^p \cdot (\mu^A + A [r] \cdot A t))(z) = \sum_{i,j} G_{l+p+i+j}(\gamma \cdot c_{i,j}^A)((t^{x+p+j}, (\mu^B)^{x_i}, z) = G_{l+p+1}(\gamma \cdot t^p)((\mu^B + B [r] \cdot B t), z).$$

This extends to arbitrary element $\alpha(t)$ of $A^\ast(X \times \mathbb{P}^\infty)$ by linearity. 

Suppose $G(d-1)$ is defined, and $X \rightarrow \text{Spec}(k)$ is a smooth quasi-projective variety of dimension $\leq d$. Let $D \rightarrow X$ be divisor with strict normal crossing (see Subsection 7.2) with components $D_i \rightarrow D$ (and $d_i = d \circ d_i : D_i \rightarrow X$), and $\gamma = \sum_i (d_i)_* (\gamma_i) \in A^{n-l-1}(D)$. Let $\lambda^B = c_1^B(\mathcal{O}(D_i))$. Then, let us define:

$$F_l(\gamma | D)(z) := \sum_i (d_i)_* F_{l+1}(\gamma_i)(\lambda^B_i, z) \in B^\ast(X)[[z_1, \ldots, z_i]], \quad (*)$$
Notice, that \( \dim(D_i) \leq d - 1 \), so \( G(D_i) \) is defined. Applying \((b_{ni})\) to \( D_{\{i,j\}} \overset{d_{\{i,j\}_{/i}}}{\longrightarrow} D_i \), we get:

\[
F_{l+1}((d_{\{i,j\}_{/i}})_*)\delta_\lambda(B, \bar{z}) = (d_{\{i,j\}_{/i}})_*F_{l+2}(\delta_\lambda_{\gamma_j}^B, \lambda_i^B, \bar{z}),
\]

which implies that our definition does not depend on the presentation of \( \gamma \) as a sum of \( (\hat{d}_i)_*(\gamma_i) \). Also it follows from \((b_{ni})\) that, in the case \( \dim(X) \leq (d - 1) \) we have:

\[
F_l(\gamma|D)(\bar{z}) = F_l(d_*(\gamma))(\bar{z}).
\]

**Proposition 5.7** Suppose, we have a cartesian diagram \((\mathfrak{D})\) (Subsection 7.2) with \( X \) and \( Y \) of dimension \( \leq d \), and \( D \) and \( E \) - divisors with strict normal crossing. Then:

\[
f^*F_l(\gamma|D)(\bar{z}) = F_l(\bar{f}^*(\gamma)|E)(\bar{z}).
\]

**Proof:** From the definition \((\mathfrak{D})\) above and the definition of \( \bar{T}^* \) (Definition 7.18), it is clear that it is sufficient to treat the case of a smooth \( D \). Let \( E = \sum_{j=1}^s m_j \cdot E_j \), \( \lambda^A = c_1(O_X(D)) \), \( \mu_j^A = c_1(O_Y(E_j)) \) (and similar for \( \lambda^B, \mu_j^B \)). Let us denote \((F_{j|m_{1,..,m}_j})A \in A[\mu_1^A, \ldots, \mu_s^A]\) (Definition 7.14) as \( C_j^A \) (and similar for \( B \)).

**Lemma 5.8** Suppose, we have a cartesian square \((\mathfrak{D})\) with \( \dim(Y) \leq d \), and \( D \) - smooth divisor, \( E \) - divisor with strict normal crossing. Then, for any choice of coefficients \( C_j^{A,B} \),

\[
F_l(\bar{f}^*(\gamma)|E)(\bar{z}) = \sum_{J \subset \{1, \ldots, s\}} (e_J)_*(C_j^B \cdot F_{l+1}(f_j^*(\gamma))(f_j^*(\lambda^B), \bar{z})).
\]

**Proof:** We will denote the 1-st Chern class of the bundle \( O(1) \) on \( \mathbb{P}^\infty \) (in both \( A^* \) and \( B^* \)-theory) by \( t \). Let \( \tilde{\mu}_j^A = t \cdot A \mu_j^A \), and similar for \( B \). Let us denote: \( \mu_j^A = \sum_{\ell \in I}^\infty \lambda_j^A \cdot A \mu_j^A \), and \( \tilde{\mu}_j^A = \sum_{\ell \in I}^\infty \lambda_j^A \cdot A \tilde{\mu}_j^A \) (and similar for \( \mu_j^B \)). For each subset \( \emptyset \neq J \subset \{1, \ldots, s\} \), let us denote \( \times_{\ell \in I} \mu_j^B \) as \( (\mu_j^B)^J \). From the definition \((\ast)\), Definitions 7.14 and 7.18, and \((b_{ni})\), \((b_{ni})\) it is clear that both parts do not depend on the choice of coefficients \( C_j^{A,B} \). Let us use the standard choice for \( C_j^{A,B} \). Recall, that

\[
C_j^A = \frac{\sum_{I \subset J} (-1)^{|J|-|I|} \mu_j^A}{(\mu_j^A)^J}. \quad \text{Denote as } \widetilde{C}_j^A \text{ the analogous coefficients for } \tilde{\mu}_j. \quad \text{We have:}
\]

\[
F_l(\bar{f}^*(\gamma)|E)(\bar{z}) = \sum_{J \subset \{1, \ldots, s\}} (e_J)_*F_{l+1}(f_j^*(\gamma) \cdot C_j^A)((\mu_j^B)^J, \bar{z}) = \sum_{J \subset \{1, \ldots, s\}} (e_J)_* \text{Res } R_J \cdot \omega_t^B , \quad \text{where}
\]

\[
R_J = \frac{G_{l+1}(f_j^*(\gamma) \cdot \widetilde{C}_j^A)(\tilde{\mu}_j^B)^J, \bar{z}}{t \cdot (\tilde{\mu}_j^B)^J \cdot \prod_{i=1}^l \tilde{z}_i} = \frac{G_l(f_j^*(\gamma) \cdot (\sum_{I \subset J} (-1)^{|J|-|I|} \tilde{\mu}_j^A))(\bar{z})}{t \cdot (\tilde{\mu}_j^B)^J \cdot \prod_{i=1}^l \tilde{z}_i} = \frac{\sum_{I \subset J} (-1)^{|J|-|I|} G_{l+1}(f_j^*(\gamma)) (\tilde{\mu}_j^B, \bar{z})}{t \cdot (\tilde{\mu}_j^B)^J \cdot \prod_{i=1}^l \tilde{z}_i} = \frac{\sum_{I \subset J} (-1)^{|J|-|I|} \tilde{\mu}_j^B \cdot F_{l+1}(f_j^*(\gamma))(\tilde{\mu}_j^B, \bar{z})}{t},
\]

\[
\sum_{l \subset J} \frac{\widetilde{C}_j^B}{t \cdot (\tilde{\mu}_j^B)^J} \sum_{l \subset I \subset J} (-1)^{|I|-|I|} F_{l+1}(f_j^*(\gamma))(\tilde{\mu}_j^B, \bar{z}).
\]

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Since, for fixed \( L \subset J \), \( \sum_{L \subset J} F_{l+1}(f^*_l(\gamma))(\tilde{\mu}^B, \bar{\tau}) \cdot (-1)^{|J|-|I|} \) is divisible by \((\tilde{\mu}^B)^{J/L}\) (this is true for any power series \( F(x) \)), we have:

\[
(e_{J/L})_* \text{Res}_{t=0} \frac{\tilde{C}_L \cdot \sum_{L \subset J} \frac{1}{t} \cdot (\tilde{\mu}^B)^{J/L}}{\sum_{L \subset J} \frac{1}{t} \cdot (\tilde{\mu}^B)^{J/L}} \cdot (-1)^{|J|-|I|} F_{l+1}(f^*_l(\gamma))(\tilde{\mu}^B, \bar{\tau}) \cdot \omega^B_t = \]

\[
\text{Res}_{t=0} \frac{\tilde{C}_L \cdot \sum_{L \subset J} \frac{1}{t} \cdot (\tilde{\mu}^B)^{J/L}}{\sum_{L \subset J} \frac{1}{t} \cdot (\tilde{\mu}^B)^{J/L}} \cdot (-1)^{|J|-|I|} F_{l+1}(f^*_l(\gamma))(\tilde{\mu}^B, \bar{\tau}) \cdot \omega^B_t, \quad \text{and so}
\]

\[
\sum_{J \subset \{1, \ldots, s\}} (e_J)_* \text{Res}_{t=0} R_j \cdot \omega^B_t = \sum_{J \subset \{1, \ldots, s\}} (e_J)_* \text{Res}_{t=0} S_j \cdot \omega^B_t, \quad \text{where}
\]

\[
S_j = \sum_{J \subset K} \frac{\tilde{C}_L}{t} \cdot \sum_{J \subset N \subset K} (-1)^{|K|-|N|} F_{l+1}(f^*_l(\gamma))(\tilde{\mu}^B_N, \bar{\tau}) = \frac{\tilde{C}_L}{t} F_{l+1}(f^*_l(\gamma))(\tilde{\mu}^B_{\{1, \ldots, s\}}, \bar{\tau}). \quad \text{Thus,}
\]

\[
F_l(\mathcal{J}^*_l(\gamma)|E)(\bar{\tau}) = \sum_{J \subset \{1, \ldots, s\}} (e_J)_* \left( C^B_j \cdot F_{l+1}(f^*_l(\gamma))(\gamma^B, \bar{\tau}) \right).
\]

\[
\square
\]

It remains to observe that our expression is equal to \( f^*F_l(\gamma|D)(\bar{\tau}) \), by Proposition 7.20 and (b). \( \square \)

Suppose \( G(d-1) \) is defined, and \( X \xrightarrow{p^X} \text{Spec}(k) \) is a smooth quasi-projective variety of dimension \( \leq d \). Let us define \( G(X) \) as follows. Since \( A^* \) is a theory of rational type, by Theorem 4.3, \( A^*(X) = A \oplus H(\epsilon) \), where \( \epsilon \) is the bi-complex* of Subsection 4.3.

For the constant part, set: \( G_l(p^X_\lambda(\alpha))(z_1, \ldots, z_l) := p^X_\lambda G_l(\alpha)(z_1, \ldots, z_l) \).

For the \( A^*(X) \)-part, consider an element of \( c_{0,0} \). It is represented by the pair

\[
((Z \xrightarrow{\rho} X \xleftarrow{\rho} \tilde{X}), \gamma),
\]

where \( Z \xrightarrow{\rho} X \) is a closed subscheme, \( \rho \) is projective birational morphism, isomorphic outside \( V = \rho^{-1}(Z) \), which is a divisor with strict normal crossing on \( \tilde{X} \), and \( \gamma \in A^{n-l-1}(V) \cap \text{im}(\rho^*) \). Recall, that the respective element \( \alpha \in \mathcal{A}^{n-l}(X) \) is \( \frac{\rho_\lambda \gamma}{\rho_\lambda(1^B)} \).

Define:

\[
F_l(\alpha)(z_1, \ldots, z_l) := \frac{\rho_\lambda F_l(\gamma|V)(z_1, \ldots, z_l)}{\rho_\lambda(1^B)}.
\]

Notice, that \( \text{dim}(V_i) \leq d-1 \), so \( G(V_i) \) is defined. So, \( F_l \) is well-defined on \( c_{0,0} \). Also, for \( \text{dim}(X) \leq (d-1) \), it follows from (b), (b.i), and the fact that \( v_*(\gamma) \in \text{im}(\rho^*) \) that the "new" definition of \( F_l \) agrees with the "old" one.

Now we need to check that it is trivial on the images of \( d_{1,0}^l \) and \( d_{0,1}^l \).

**Proposition 5.9** In the above situation, \( F_l(\gamma|V) \in \text{im}(\rho^*) \).

**Proof:** Consider first the case where \( \rho \) is the permitted blow up with smooth centers \( R_j \). Let \( R_j \xrightarrow{\tilde{E}_j} E_j \xrightarrow{\epsilon} \tilde{X} \) be the components of the special divisor of \( \rho \). Then, by our condition on \( V \), \( E_j \) is among the components \( V_i \) of \( V \). Then, by Proposition 7.3, to prove that \( F_l(\gamma|V) \in \text{image}(\rho^*) \) we need to show that
$e_j^*(F_i(\gamma|V)) \in \text{image}(\varepsilon_j^*)$, for each $j$. Since $V$ is a divisor with strict normal crossing on $\bar{X}$, and $E_j$ is a component of it, for any other component $V_i$ of $V$, the left cartesian diagram below is transversal:

\[
\begin{array}{ccc}
H_{i,j} & \xrightarrow{u_{i,j}} & E_j \\
\downarrow h_{i,j} & & \downarrow e_j \\
V_i & \xrightarrow{v} & \bar{X}.
\end{array}
\]

\[
\begin{array}{ccc}
V & \xrightarrow{v} & \bar{X} \\
\downarrow \rho & & \downarrow e_j \\
Z & \xrightarrow{z} & X \\
\downarrow \tau_j & & \downarrow e_j \\
R_j & & R_j
\end{array}
\]

By (the very simple case of) Proposition 5.7, $e_j^*F_i(h_{i,j}^*(\gamma_i)|H_{i,j})$. But $\dim(H_{i,j})$, $\dim(V_i)$, $\dim(E_j) \leq (d - 1)$. Thus,

$$F_i(h_{i,j}^*(\gamma_i)|H_{i,j}) = F_i((u_{i,j})^*h_{i,j}^*(\gamma_i)) = F_i(e_j^*(v_i)^*(\gamma_i)).$$

And the same is true for the $E_j$-component: $e_j^*F_i(\gamma_j|E_j) = F_i(e_j^*(e_j)^*(\gamma_j))$. Hence, by [12] Theorem 6.6.6 (2)(a),

$$e_j^*F_i(\gamma|V) = \sum_i e_j^*F_i(\gamma_i|V_i) = \sum_i F_i(e_j^*(v_i)^*(\gamma_i)) = F_i(e_j^*v_*(\gamma)).$$

Thus,

$$F_i(e_j^*\rho^*z_*(\beta)) = F_i(e_j^*\rho^*z_*(\beta)) = F_i(e_j^*\rho^*z_*(\beta)) = e_j^*F_i(\gamma|V) \in \text{image}(\rho^*) \quad \square$$

Let now $\rho$ be just projective bi-rational map. Let $Z \subset X$ be closed subset $\rho(V)$ outside which $\rho$ is an isomorphism. Then there exists a permitted blow up $\rho' = \rho \circ \pi$ with centers over $Z$. Let $V' = \pi^*(V)$. By Proposition 5.7, $\pi^*F_i(\gamma|V) = F_i(\pi_V^*(\gamma)|V')$, which is in the image($\pi^*\rho^*$) by the case proven. Since $\pi^*$ is injective, $F_i(\gamma|V) \in \text{image}(\rho^*)$.

The 1-st part of $(d_{1,0}^r)$: Let

\[
\begin{array}{ccc}
V' & \xrightarrow{\nu'} & \bar{X} \\
\downarrow \pi_V & & \downarrow \pi \\
V & \xrightarrow{\nu} & \bar{X}
\end{array}
\]

be the cartesian square, with $V$ and $V'$ divisors with strict normal crossing, with $\pi$ the blow up over $V$ permitted w.r.t. $V$, and $V = \rho^{-1}(Z)$ for some closed subscheme $Z \xrightarrow{\sim} X$. We need to check that the pairs:

\[
((Z \xrightarrow{\sim} X \xleftarrow{\rho} \bar{X}), \gamma) \quad \text{and} \quad ((Z \xrightarrow{\sim} X \xleftarrow{\rho \circ \pi} \bar{X}), \pi_V^*(\gamma))
\]

produce the same result. From Propositions 5.7 and 5.9

$$\frac{\rho_*\pi_*(F_i(\pi_V^*(\gamma)|V'))(\pi)}{\rho_*\pi_*(1_B)} = \frac{\rho_*\pi_*(F_i(\gamma|V)(\pi))}{\rho_*\pi_*(1_B)} = \frac{\rho_*F_i(\gamma|V)(\pi)}{\rho_*\pi_*(1_B)}$$

Thus, $F_i$ is trivial on the image of the 1-st part of $(d_{1,0}^r)$.

The 2-nd part of $(d_{1,0}^r)$: It follows immediately from the definition of $F_i(\gamma|V)$ that $F_i$ is trivial on the image of the 2-nd part of $(d_{1,0}^r)$.

$(d_{0,1}^r)$: Let $\Xx = X \xleftarrow{\rho} X \times \mathbb{P}^1$ be projective birational map, isomorphic outside the strict normal crossing divisor $W$, where $W = \rho^{-1}(Z)$ for some closed subscheme $Z \xrightarrow{\sim} X \times \mathbb{P}^1$, $W$ has no components over 0 and 1, such that the preimages $X_0 = \rho^{-1}(X \times \{0\})$, and $X_1 = \rho^{-1}(X \times \{1\})$ are smooth divisors on $\Xx$, and $W \cap X_0 \leadsto X_0$ and $W \cap X_1 \leadsto X_1$ are divisors with strict normal crossing. In particular,
for each component $S$ of $W$, $S_0 = s^{-1}(X \times \{0\}) \overset{i_0}{\rightarrow} S$ and $S_1 = s^{-1}(X \times \{1\}) \overset{i_1}{\rightarrow} S$ are divisors with strict normal crossing.

Let $\delta = \sum S_\delta \in A^{n-1}(W) \cap im(p')$. We need to show that $F_l$ takes the same values on the pairs

\[(Z_0 \rightarrow X \overset{\iota \circ p}{\rightarrow} X_0), \sum_S i_0^*(\delta_S)) \quad \text{and} \quad ((Z_1 \rightarrow X \overset{\iota \circ p}{\rightarrow} X_1), \sum_S i_1^*(\delta_S)).\]

Let $S$ be some component of $W$, and $S_0, k \overset{i_{0,k}}{\rightarrow} S_0, S_1, k \overset{i_{1,k}}{\rightarrow} S_1$ be the components of $S_0, S_1$. Let $\tilde{S} \overset{p_0, k}{\rightarrow} S$ be some projective bi-rational morphism, which is an isomorphism outside some divisor with normal crossing $H \overset{h}{\rightarrow} \tilde{S}$, and such that $p \circ h$ does not contain components of $S_0$ and $S_1$. Let $\lambda^{A, B} = c_1^{A, B} (\mathcal{O}_{X \times \mathbb{P}^1}(S))$, and $\mu^{A, B} = c_1^{A, B} (\mathcal{O}_{X_0}(S_{0, k})).$ Let $\gamma = \gamma(u) \in A^{n-1}(H)$, and $\beta \in A^{n-1}(S)$ be the image of $u$ (in particular, $h_\ast (\gamma) = p^\ast (\beta)$). Let us denote:

$$\tilde{F}_l(\beta | S)(\overline{z}) := s_\ast \left( \frac{p_\ast F_{l+1}(\gamma | H)(p^\ast (\lambda^B) \ast \overline{z}))}{p_\ast (1)} \right)$$

**Lemma 5.10** In the above situation,

$$\tilde{i}_0^*(\tilde{F}_l(\beta | S)) = F_l(i_0^*(\beta) | S_0) \quad \text{and} \quad \tilde{i}_1^*(\tilde{F}_l(\beta | S)) = F_l(i_1^*(\beta) | S_1)$$

**Proof:** It is sufficient to treat the $S_0$ case. By our condition, $p \circ h$ itersects each component of $S_0$ and $S_1$ in positive codimension.

Consider one of these components $S_{0,k}$. By Theorems 8.3 and 8.4 we can find a permitted blow up $\tilde{S}_{0,k} \overset{p_{0,k}}{\rightarrow} S_{0,k}$, which fits into the diagram:

\[
\begin{array}{ccc}
H_{0,k} & \overset{h_{0,k}}{\longrightarrow} & \tilde{S}_{0,k} \\
\downarrow i_{0,H} & & \downarrow \tilde{i}_{0,k} \\
H & \overset{h}{\longrightarrow} & S \\
\downarrow s & & \downarrow \iota_0 \\
X & \overset{s}{\longrightarrow} & X \times \mathbb{P}^1,
\end{array}
\]

where the left square is cartesian, and $H_{0,k}$ is a divisor with strict normal crossing on $\tilde{S}_{0,k}$.

**Lemma 5.11** Let

$$\begin{array}{ccc}
\tilde{T} & \overset{j}{\longrightarrow} & \tilde{S} \\
\downarrow q & & \downarrow p \\
T & \overset{j}{\longrightarrow} & S
\end{array}$$

be commutative diagram with $p$ and $q$ - projective bi-rational. Let $x \in im(p^\ast)$. Then:

$$\frac{q_\ast (j^\ast (x))}{q_\ast (1)} = j^\ast \left( \frac{p_\ast (x)}{p_\ast (1)} \right).$$

**Proof:** Let $x = p^\ast (y)$. Then $q_\ast (j^\ast (p^\ast (y))) = q_\ast (j^\ast (y)) = q_\ast (1) \cdot j^\ast (y) = q_\ast (1) \cdot j^\ast (p^\ast (y))$, which implies what we need.
By Proposition 5.9, \( F_{i+1}(\gamma|H)(p^*(\lambda^B), \overline{\tau}) \in \text{im}(p^*) \). Then, by Lemma 5.11 Proposition 5.7 and (b_i), we have:

\[
\frac{\iota_{0,k}^* (p_* F_{i+1}(\gamma|H)(p^*(\lambda^B), \overline{\tau}))}{p_* (1)} = \frac{(p_{0,k}*) (\iota_{0,k}^* h_* (\gamma)) (\iota_{0,k}^* p^*(\lambda^B), \overline{\tau}))}{(p_{0,k})_* (1)} = \frac{(\iota_{0,k}^* (F_{i+1} (p_{0,k}^* i_{0,k}^* (\beta)) (p_{0,k}^* i_{0,k}^* (\lambda^B), \overline{\tau})))}{(p_{0,k})_* (1)} = F_{i+1} (\iota_{0,k}^* (\beta)) (\iota_{0,k}^* (\lambda^B), \overline{\tau}).
\]

We can assume that coefficients \( C_{A,B}^J \) are chosen to be zero, for \(|J| > 1\). Then, by Proposition 7.20 and Lemma 5.8,

\[
\tilde{i}_{0}^* (\overline{\tau}) = \sum_k (s_{0,k})_* (C_k^B \cdot F_{i+1} (\iota_{0,k}^* (\beta)) (\iota_{0,k}^* (\lambda^B), \overline{\tau})) = F_{i} (\iota_{0}^* (\lambda^B), S_0 (\overline{\tau}).
\]

\[\square\]

Let \( S \xrightarrow{\pi} \text{Spec}(k) \) be the natural projection. Denote:

\[
\tilde{F}_{i} (1|S)(\overline{\tau}) := s_* (\pi_S^* F_{i+1}(1)(\lambda^B, \overline{\tau})).
\]

**Lemma 5.12** In the above situation,

\[
\tilde{i}_{0}^* (\overline{\tau}) = F_{i} (\iota_{0}^*(1)|S_0) \quad \text{and} \quad \tilde{i}_{1}^* (\overline{\tau}) = F_{i} (\iota_{1}^*(1)|S_1)
\]

**Proof:** We treat \( S_0 \) only. By Proposition 7.20 and Lemma 5.8 we have:

\[
\tilde{i}_{0}^* (\pi_S^* F_{i+1}(1)(\lambda^B, \overline{\tau})) = \sum_k (s_{0,k})_* (C_k^B \cdot i_{0,k}^* \pi_S^* F_{i+1}(1)(\lambda^B, \overline{\tau})) = \sum_k (s_{0,k})_* (C_k^B \cdot F_{i+1}(1)(\lambda^B, \overline{\tau})) = F_{i} (\iota_{0}^*(1)|S_0(\overline{\tau})).
\]

\[\square\]

**Proposition 5.13** In the above situation,

\[
\frac{(p_0)_* F_{i} (\sum_j i_j^*(\delta_j)|W_0)(\overline{\tau})}{(p_0)_* (1)} = \frac{(p_1)_* F_{i} (\sum_j i_j^*(\delta_j)|W_1)(\overline{\tau})}{(p_1)_* (1)}.
\]

**Proof:**

**Lemma 5.14** Let \( S \) be quasi-projective variety, and \( T \) be some divisor on it. Then any element \( \beta \in \overline{A}_s(S) \) can be represented by an element from \( A_s(Y) \), where \( Y \to S \) is a closed subscheme containing no components of \( T \).

**Proof:** Since \( A^* \) is obtained from Algebraic Cobordism theory by the change of coefficients, it is sufficient to treat the case of \( A^* = \Omega^* \), and \( \beta = \pi_*(\delta) \), where \( T_1 \xrightarrow{\pi} T_1 \) is the resolution of singularities of some component of \( T \). Since \( S \) is quasi-projective, \( 1_{T_1}^{CH} \) can be represented as \( \sum_k \pm 1_{R_k}^{CH} \), where \( R_k \) are irreducible divisors different from components of \( T \). Taking resolutions \( R_k \xrightarrow{\phi} R_k \), we have: \( \pi_*(\delta) = \sum_k (\phi_k)_* (\pm 1_{R_k}^\gamma) + \gamma \), where \( \gamma \) has support of codimension \( \geq 2 \). \(\square\)
Lemma 5.14 together with Theorem 8.3 implies that any element \( \beta \in \overline{A}^{n-l-1}(S) = H(\mathfrak{b}) \) can be represented by an element \( ((Y \to S, \delta', S), x) \), where \( p \) is projective bi-rational, isomorphism outside \( Y \subset S \), where \( Y \) is a closed sub-scheme containing no components of \( S_0 \) and \( S_1 \). Then the respective element in \( H(c) \) will be \( ((Y \to S, \delta', S), p'(\rho_H) \ast (x)) \). Taking \( \gamma = p'(\rho_H) \ast (x) \) and \( \beta = p \ast h \ast (x) \) as above, we obtain from Lemma 5.10 that
\[
\overline{i}_0^*(\overline{F}_l(\beta|S)) = F_l((i_0^*)\ast (\beta)|S_0) \quad \text{and} \quad \overline{i}_1^*(\overline{F}_l(\beta|S)) = F_l((i_1^*)\ast (\beta)|S_1).
\]
And the same is true for \( \beta = 1 \), by Lemma 5.12. Since \( \rho \) has no centers over 0 and 1, the following cartesian diagram is transversal:

\[
\begin{array}{ccc}
X_0 & \longrightarrow & X \times \mathbb{P}^1 \\
\rho_0 & \downarrow & \rho \\
X \times \{0\} & \longrightarrow & X \times \{1\} \\
\end{array}
\]

This implies that, for any \( \delta \in A^{n-l-1}(W) \),
\[
\frac{(\rho_0)_* F_l(\sum_S i_0^*\delta_S) | W_0)(\overline{\tau})}{(\rho_0)_*(1)} = i_0^* \left( \frac{\rho_* F_l(\delta | W)(\overline{\tau})}{\rho_*(1)} \right) = i_1^* \left( \frac{\rho_* F_l(\delta | W)(\overline{\tau})}{\rho_*(1)} \right) = \frac{(\rho_1)_* F_l(\sum_S i_1^* \delta_S) | W_1)(\overline{\tau})}{(\rho_1)_*(1)}.
\]

\( \square \)

It follows from Proposition 5.13 that \( F_l \) is trivial on the image of \( d^0_{0,1} \). Thus, we obtain:

**Proposition 5.15** Suppose, \( G(d-1) \) is defined, and \( X \) is smooth quasi-projective variety of dimension \( d \). Then the above definition of \( F_l \) is well-defined and satisfies the conditions of \( G(X) \).

**Proof:** We already have proven that \( F_l \) is well-defined on \( A^{n-l}(X) \). The conditions \( (a_i), (a_{ij}) \) follow immediately from the definition of \( F_l \). As for \( (a_{ii}), (a_{ij}) \), it is clearly sufficient to treat the case, where \( V \overset{\varphi}{\rightarrow} X \) is a smooth divisor, and \( \alpha = v_\ast(\gamma) \). Let \( \lambda^{A,B} = c_{1}^{A,B}(\mathcal{O}_X(V)) \). Then
\[
G_l(v_\ast(\gamma))(\gamma, x + B, y, z_2, \ldots, z_l) = v_\ast \left( \frac{\overline{G}_{l+1}(\gamma) | (\gamma, x + B, y, z_2, \ldots, z_l) \omega_{1}^{B}}{(t + B \lambda^{B}) \cdot t} \right) = v_\ast \left( \sum_{i,j} \frac{\overline{G}_{l+i+j-1}(\gamma, a_{ij}^{A})(\gamma, x^{i}, y^{j}, z_2, \ldots, z_l) \omega_{1}^{B}}{(t + B \lambda^{B}) \cdot t} \right).
\]

\( \square \)

**Proposition 5.16** Suppose, \( G(d-1) \) is defined. Then it extends to \( G(d) \).

**Proof:** Above we have defined \( G(X) \), for each \( X \) of dimension \( \leq d \), which, in the case of \( \dim(X) \leq (d-1) \), coincides with the "old" definition. It remains to check the conditions \( (b_{i}, \ldots) \). Let \( X, Y \) be smooth varieties of dimension \( \leq d \). For constant \( \alpha \in A^{n-l}(X) \), \( (b_{i}) \) is evident. Consider now the case \( \alpha \in \overline{A}^{n-l}(X) \). Let \( \overline{X} \overset{\varphi}{\rightarrow} X \) be a projective-birational map, which is an isomorphism outside the strict normal crossing divisor \( V_X \overset{\nu_X}{\rightarrow} \overline{X} \), and \( \gamma \in A^{n-l-1}(V_X) \cap \rho_X \). We can assume that \( \alpha = (\varphi)\ast (\nu_X) \ast (\gamma) \). By Theorems 8.3 and 8.4
we can find a projective bi-rational map \( \bar{Y} \xrightarrow{\rho_Y} Y \), which fits into the commutative diagram with the left square cartesian, and \( V_Y \xrightarrow{\nu_Y} \bar{Y} \) - a divisor with strict normal crossing:

\[
\begin{array}{ccc}
V_Y & \xrightarrow{\nu_Y} & \bar{Y} \\
\downarrow{f_Y} & & \downarrow{\rho_Y} \\
V_X & \xrightarrow{\nu_X} & X \\
\end{array}
\]

By Proposition 5.9, Lemma 5.11, Proposition 5.7, and Proposition 7.20:

\[
f^* \left( \left( \frac{\rho_X)_* (v_X)_* (\gamma) \right) \right) = f^* \left( \left( \frac{\rho_Y)_* (\gamma | V^X) \right) \right) = \left( \frac{\rho_Y)_* (f^* (\gamma) | V_Y) \right).
\]

This proves (b\(_i\)).

Let now \( X \xrightarrow{j} Y \) be regular embedding of codimension \( r \) with normal bundle \( N_j \), with \( \dim(Y) \leq d \). Consider the blow-up diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{j} & \bar{Y} \\
\downarrow{\varepsilon} & & \downarrow{\pi} \\
X & \xrightarrow{j} & Y,
\end{array}
\]

where \( E = \mathbb{P}(X(N_j)) \), and \( N_{\bar{Y}} = \mathcal{O}(\bar{Y}). \) Let \( M = \varepsilon^* N_j / \mathcal{O}(\bar{Y}) \), \( \nu_1^{A,B}, \ldots, \nu_r^{A,B} \) be roots of \( M \), \( \zeta^{A,B} \) - root of \( \mathcal{O}(\bar{Y}) \), and \( \alpha \in A^{n-l-r}(X) \). Then, by the already proven (b\(_i\)), the Excess Intersection Formula (Proposition 7.11), the definition of \( G(\bar{Y}) \), Lemma 5.5 again (b\(_i\)), and Proposition 7.14 again, we get:

\[
\pi^* \left( \left( \frac{\rho_X)_* (v_X)_* (\gamma) \right) \right) = \pi^* \left( \left( \frac{\rho_Y)_* (\gamma | V^X) \right) \right) = \left( \frac{\rho_Y)_* (f^* (\gamma) | V_Y) \right).
\]

\[
\begin{array}{l}
\pi^* \left( \left( \frac{\rho_X)_* (v_X)_* (\gamma) \right) \right) = \pi^* \left( \left( \frac{\rho_Y)_* (f^* (\gamma) | V_Y) \right) \right) = \left( \frac{\rho_Y)_* (f^* (\gamma) | V_Y) \right).
\end{array}
\]

And since \( \pi^* \) is injective, we obtain (b\(_i\)).

Now we can finish the proof of Theorem 5.1.

By the conditions of the Theorem, \( G(0) \) is defined. Then it follows from Proposition 5.16 that it can be extended to \( G(\infty) \). Consider now \( G_0 : A^n(X) \rightarrow B^m(X) \), for all \( X \in \text{Sm}_k \). By (b\(_i\)), this is an additive operation. It remains to see that it extends the original \( A^n \left( (\mathbb{P}^\infty)^\times l \right) \xrightarrow{G} B^m \left( (\mathbb{P}^\infty)^\times l \right) \). From commutativity with the pull-backs for partial diagonals and partial projections, it is sufficient to compare the results on \( \alpha \cdot \prod_{i=1}^l z_i^A \in A^n \left( (\mathbb{P}^\infty)^\times l \right) \), where \( \alpha \in A^{n-l} \), and \( z_i^A = c_i^A \left( \mathcal{O}(1)_i \right) \). Let \( j : (\mathbb{P}^\infty)^\times l \rightarrow (\mathbb{P}^\infty)^\times l \) be the product of hyperplane section embeddings. Then \( G_0(j_*(\alpha)) = G_1(\alpha)(z_1^B, \ldots, z_l^B) = G(\alpha \cdot \prod_{i=1}^l z_i^A) \), by (b\(_i\)) and the definition of \( G(\text{Spec}(k)) \). Thus, \( G_0 \) extends the original transformation on products of projective spaces. The uniqueness follows from Proposition 3.14.

**Remark 5.17** Notice that we simultaneously have proven that our operation satisfies (b\(_i\)). This can be considered as a Riemann-Roch type result for additive operations. It generalizes the multiplicative case (see [15]).
We will also need the following multiplicative version:

**Proposition 5.18** Suppose, in the situation of Theorem 5.1, the original transformation $A^*((\mathbb{P}^\infty)^\times l) \overset{G}{\rightarrow}\ B^*((\mathbb{P}^\infty)^\times l)$ commutes with the external products of projective spaces. Then the resulting operation $G$ is multiplicative.

**Proof:** We need to prove that $G$ respects external products of varieties:

$$G_{t+m}(\alpha \times \beta)(x_1, \ldots, x_l, y_1, \ldots, y_m) = G_t(\alpha)(x_1, \ldots, x_l) \times G_m(\beta)(y_1, \ldots, y_m),$$

for any $\alpha \in A^*(X)$, $\beta \in A^*(Y)$. We first prove it for the case $Y = \text{Spec}(k)$ by induction on the dimension of $X$. The base and the case where $\alpha$ is constant follow from our condition. In the case $\alpha \in \mathcal{A}(X)$, we can find a projective bi-rational morphism $\tilde{\psi} \overset{\psi}{\rightarrow} X$ such that $\rho^*(\alpha)$ is supported on some divisor with strict normal crossing. Since $\rho^*$ is injective, without loss of generality, we can assume that $\alpha = v_x(\alpha')$, where $V \overset{\psi}{\rightarrow} X$ is a smooth divisor. Let $\lambda^B = c^B_1(\mathcal{O}_X(V))$. Then

$$G_{t+m}(\alpha \times \beta)(\mathcal{F}, \mathcal{G}) = (v \times \text{id})_* G_{t+m+1}(\alpha' \times \beta)(\lambda^B, \mathcal{F}, \mathcal{G}) =$$

$$(v \times \text{id})_* (G_{t+1}(\alpha')(\lambda^B, \mathcal{F}) \times G_m(\beta)(\mathcal{G})) = G_t(\alpha)(\mathcal{F}) \times G_m(\beta)(\mathcal{G}),$$

which proves the induction step. Now, by the induction on the $\dim(Y)$, using similar arguments, we prove the general case. \qed

### 6 Applications

#### 6.1 Unstable operations in Algebraic Cobordism

As a first application of our main result (Theorem 5.1), let us finish the description of unstable operations in Algebraic Cobordism:

**Theorem 6.1** The correspondence: $G_\psi \leftrightarrow \psi$, where $(G_\psi)_Q$ is the map: $\Omega^k \overset{\text{Tot}}{\rightarrow} \Omega^k \otimes \mathbb{Q}$ identifies the set of (unstable) additive operations $\Omega^n \rightarrow \Omega^m$ with the subset of $\text{Hom}_L(L[L], L \otimes \mathbb{Q})_{m-n}$, corresponding to exactly those linear combinations $S$ of the Landweber-Novikov operations which satisfy the integrality conditions: $S(\Omega^n((\mathbb{P}^\infty)^\times r)) \subseteq \Omega^m((\mathbb{P}^\infty)^\times r)$, for all $r$.

**Proof:** It follows immediately from Theorems 3.18 and 5.1. \qed

If $A^*$ is a theory of rational type, and $B^*$ is any theory in the sense of Definition 2.1, then (unstable) additive operations $A^n \rightarrow B^m$ can be described as follows.

**Theorem 6.2** Let $A^*$ be theory of rational type, and $B^*$ be any theory. Then there is 1-to-1 correspondence between the set of (unstable) additive operations $A^n \overset{G}{\rightarrow} B^m$ and the set consisting of the following data $\{G_l, l \in \mathbb{Z}_{\geq 0}\}$:

$$G_l \in \text{Hom}_{\mathbb{Z} \text{-lin}}(A^{n-l}, B[[z_1, \ldots, z_l]](m))$$

satisfying:

(a) $G_l$ is symmetric with respect to $G_l$;

(a) $G_l(\alpha) = \prod_{i=1}^l z_i \cdot F_i(\alpha)$, for some $F_i(\alpha) \in B[[z_1, \ldots, z_l]](m-l)$.

(a) $G_l(\alpha)(x + B y, z_2, \ldots, z_l) = \sum_{i,j \geq 1} G_{i+j+l-1}(\alpha \cdot a_{i,j}^A)(x^{x_i}, y^{x_j}, z_2, \ldots, z_l)$, where $a_{i,j}^A$ are the coefficients of the FGL of $A^*$. 

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Proof: It follows immediately from Proposition\[3.14\] Theorem\[5.1\] and the discussion right after it. □

With any data \{G_l, l ∈ Z≥0\} one can associate the data \{\tilde{G}_l, l ∈ Z≥0\}, where \tilde{G}_l : A^{n-l} → B^{m-l} is the constant term of the \(F_l : A^{n-l} → B[[z_1, \ldots, z_i]]_{(m-l)}\). We have:

**Proposition 6.3** If \(B\) has no torsion, then \{\tilde{G}_l, l ∈ Z≥0\} carries the same information as \{G_l, l ∈ Z≥0\}.

Proof: Let \(F_l(α) = \sum_i h_{i, l}(α)\tilde{z}^i\). Let us prove by induction on the degree of \(\tilde{7}\) (simultaneously for all \(l\)) that \(h_{i, l}\) is determined by \(h_{i, r}\), for all \(r\). Base is evident. Consider the equation \((a_{ii})\). Let \(\tilde{z} = z_1^i \cdot \ldots \cdot z_l^i\), where \(i_1 \geq i_2 \geq \ldots \geq i_l\) (by \(a_i\), it is sufficient to treat this case). Using the fact that \(x + y = x + y + \) higher terms, comparing coefficients at \(x^{i_1}y^{i_2+1} \cdot \ldots \cdot z_l^{i_l+1}\), we get that \(i_1 \cdot h_{i, l}(α)\) is expressible in terms of \(h_{i, j}(something)\), for \(|i| < |l|\), and all \(r\). Since \(B\) has no torsion, and \(i_1 ≠ 0\), \(h_{i, l}(α)\) is determined by these smaller terms. □

**Corollary 6.4** Let \(A^*\) be a theory satisfying (CONST), and \(B^*\) be any theory in the sense of Definition\[2.7\] with \(B\) torsion-free. Then an additive (unstable) operation \(A^n → B^m\) is determined by its action on the image of \((j_l)\), for all \(l\), where \(j_l : \text{Spec}(k) → (\mathbb{P}^1)^l\) is an embedding of a rational point.

Proof: This follows from Propositions\[3.14\] and \[6.3\]. □

But if \(B\) has torsion, then \{\tilde{G}_l, l ∈ Z≥0\} does not determine \{G_l, l ∈ Z≥0\}.

**Example 6.5** Consider \(A^* = B^* = \text{CH}^*/p, p\)-prime. Then \(F_B(x, y) = x + y\) is additive, and \(A^{n-1} = 0\), for \(l ≠ n\). Thus, \(G_l = 0\), for \(l ≠ n\), and the only conditions on \(G_n\) are: symmetry and additivity. Thus, \(G_n(z_1, \ldots, z_n)\) is an arbitrary symmetric polynomial with \(Z/p\)-coefficients of degree \(m\) containing monomials where each \(z_i\) enters in degree \(p^{r_i}\), where \(r_i ≥ 0\). And \(G_n\) is the coefficient at \(z_1 \cdot \ldots \cdot z_n\) (so, it is zero if \(n ≠ m\), and an element of \(Z/p\), if \(n = m\)). Of course, it does not determine \(G_n\).

In the case of Chow groups modulo \(p\) we can describe all the operations explicitly. These appear to be essentially stable, and so expressible in term of Steenrod operations (defined by V.Voevodsky\[26\] and P.Brosnan\[4\]).

**Theorem 6.6** Any additive operation \(\text{CH}^n/p → \text{CH}^m/p\) extends to a stable operation. The \(\mathbb{F}_p\)-vector space of such operations has a basis consisting of Steenrod operations\[S^k\], where \(k = (k_1, \ldots, k_s)\) is a partition with \(k_i = p^{r_i} - 1, r_i ≥ 0, |k| = (m - n), s ≤ n\).

Proof: In the example\[6.5\] we saw that any additive operation \(\text{CH}^n/p → \text{CH}^m/p\) is determined by some symmetric polynomial \(F_n(z_1, \ldots, z_n)\) of degree \((m - n)\), where each variable \(z_i\) enters in degree \(p^{r_i} - 1\), for some \(r_i ≥ 0\). The value of \(G\) on the class \(x_n = \prod_{i=1}^n h_i \in \text{CH}^n((\mathbb{P}^\infty)^{\times n})\) is equal to \(x_n \cdot F_n(h_1, \ldots, h_n)\), which coincides with the value of the (stable!) Steenrod operation\[S^k\]. Since \(\text{CH}^{n-1}(\text{Spec}(k))/p = 0\), for \(l ≠ n\), these two operations \(\text{CH}^n/p → \text{CH}^m/p\) coincide on \(\text{Sm}_k\), by Proposition\[3.14\]. Clearly, the \(\mathbb{F}_p\)-vector space of mentioned polynomials \(F_n\) has a basis consisting of the symmetrizations of monomials corresponding to partitions as above. □

**Remark 6.7** In particular, Theorem\[6.6\] provides another construction of Steenrod operations in Chow groups.
6.2 Multiplicative operations between theories of rational type

The following result reduces the study of multiplicative operations on theories of rational type to the study of morphisms of FGLs (recall, that such theories are in 1-to-1 correspondence with FGLs).

**Theorem 6.8** Let $A^*$ be theory of rational type, and $B^*$ be any theory in the sense of Definition 2.7. The assignment $G \mapsto (\varphi_G, \gamma_G)$ defines a 1-to-1 correspondence between the multiplicative operations $A^* \xrightarrow{G} B^*$ and the homomorphisms $(A, F_A) \to (B, F_B)$ of the respective formal group laws.

*Proof:* Any multiplicative operation $G$ defines the homomorphism $(\varphi_G, \gamma_G) : (A, F_A) \to (B, F_B)$ of FGLs. On the other hand, any homomorphism $(\varphi, \gamma)$ defines the transformation $A^*((\mathbb{P}^\infty)^{\times r}) \xrightarrow{H} B^*((\mathbb{P}^\infty)^{\times r})$ by the rule:

$$H(f(z_1^A, \ldots, z_r^A)) := \varphi(f)(\gamma(z_1^B), \ldots, \gamma(z_r^B)),$$

where $f \in A[[z_1^A, \ldots, z_r^A]] = A^*((\mathbb{P}^\infty)^{\times r})$. Clearly, this transformation commutes with the pull-backs for the action of $\mathbb{G}_r$, and for partial diagonals. As for partial Segre embeddings, let $Seg = (Segre \times id^{(r-1)})$. Then we have:

$$Seg^* f(z_1^A, \ldots, z_r^A) = f(F_A(x^A, y^A), z_2^A, \ldots, z_r^A), \quad \text{while} \quad Seg^* \varphi(f)(\gamma(z_1^B), \ldots, \gamma(z_r^B)) = \varphi(f)(\gamma(F_B(x^B, y^B)), \gamma(z_2^B), \ldots, \gamma(z_r^B)).$$

Since $\varphi(F_A)(\gamma(x^B), \gamma(y^B)) = \gamma(F_B(x^B, y^B))$, we get that our transformation commutes with the pull-backs for Segre embeddings as well. Commutativity with the morphisms (Speck $\leftrightarrow \mathbb{P}^\infty \times (\mathbb{P}^\infty)^{\times r}$ follows from the fact that $\gamma$ has no constant term. Thus, it extends to a unique operation $A^* \xrightarrow{H} B^*$. Since our transformation on $(\mathbb{P}^\infty)^{\times r}$ commutes with the external products of projective spaces, it follows from Proposition 5.18 that the resulting operation will be multiplicative. It follows from Proposition 3.14 that the above two assignments are inverse to each other. \qed

The situation here is more simple than in Topology. It is the reflection of the same phenomenon as the fact that our theories of rational type are in 1-to-1 correspondence with the formal group laws. In Topology some special cases of the above result are known - see, for example [5], Theorem 3.7.

Consider now the case where $A^* = \Omega^*$. We can extend the Theorem 6.7.

**Theorem 6.9** Let $B^*$ be any theory in the sense of Definition 2.7, and $b_0 \in B$ be not a zero-divisor. Let $\gamma = b_0 x + b_1 x^2 + b_2 x^3 + \ldots \in B[[x]]$. Then there exists a multiplicative operation $\Omega^* \xrightarrow{G} B^*$ with $\gamma_G = \gamma$ if and only if the shifted FGL $F_G^\gamma \in B[b_0^{-1}][[x, y]]$ has coefficients in $B$ (that is, has no denominators). In this case, such an operation is unique.

*Proof:* Since $\varphi_G(F_G)(\gamma(x^B), \gamma(y^B)) = \gamma(F_B(x^B, y^B))$, and $\varphi_G(F_G)$ has coefficients in $B$, the above condition is necessary. On the other hand, if $F_B^\gamma$ has coefficients in $B$, by universality of the FGL $(\mathbb{L}, F_\Omega)$, we get a ring homomorphism $\mathbb{L} \xrightarrow{\varphi} B$ such that $\varphi(F_\Omega) = F_B^\gamma$, and we get a morphism of FGLs which provides the needed operation by Theorem 6.8. \qed

The above two results provide an effective tool in constructing multiplicative operations. We will use them below to construct Integral Adams Operations and T.tom Dieck - style Steenrod operations in Algebraic Cobordism.

Let us describe the morphisms of FGLs (and so, the multiplicative operations between the respective theories) in some situations.
For $r > 1$, denote: $d(r) := G.C.D.((i^r)_i, 0 < i < r)$. Then

$$d(r) = \begin{cases} p, & \text{if } r = p^k, \text{ for some } k; \\ 1, & \text{otherwise.} \end{cases}$$

**Lemma 6.10** Let $(\varphi_G, \gamma_G) : (A, F_A) \rightarrow (B, F_B)$ be a morphism of FGLs. Then either $b_0 \neq 0$, or the first non-zero coefficient $b_{r−1}$ of $\gamma_G$ satisfies: $d(r) \cdot b_{r−1} = 0$.

*Proof:* Suppose, $b_0 = 0$, and $b_{r−1}$ is the first non-zero coefficient of $\gamma_G$. From the equality:

$$\varphi_G(F_A)(\gamma_G(x), \gamma_G(y)) = \gamma_G(F_B(x, y)),$$

we get: $b_{r−1}x^r + b_{r−1}y^r + \text{higher terms} = b_{r−1}(x+y)^r + \text{higher terms}$, which implies that $d(r) \cdot b_{r−1} = 0$. \(\square\)

Suppose, now $B$ is an integral domain. Then the characteristic $\text{char}(B)$ is either a prime $p$, or 0.

1) $\text{char}(B) = 0$:

**Corollary 6.11** Let $A^*$ and $B^*$ be any theories in the sense of Definition 2.1 with torsion-free $B$, and $A^* \overset{G}{\rightarrow} B^*$ be a multiplicative operation. Then either $\gamma_G = 0$, or $b_0 \neq 0$. \(\square\)

We will call operations with $b_0 \neq 0$ - operations of the main type. The respective $\gamma$ will also be called of the main type.

2) $\text{char}(B) = p$:

Let $B^*$ be a theory, where $B$ is a ring of characteristic $p$. We can obtain a new theory $\text{Fr}(B)^*$ from $B^*$ by the change of coefficients: $B \overset{Fr}{\rightarrow} B$. In particular, $\text{Fr}(B^*) = \text{Fr}(F_B)$. We have natural multiplicative operation: $\text{Fr} : \text{Fr}(B)^* \rightarrow B^*$ defined by: $\text{Fr}(u \otimes b) = u^p \cdot b$. The respective morphism of formal group laws will be: $(id, x^p)$.

Let $A^* \overset{G}{\rightarrow} B^*$ be a multiplicative operation, and $(\varphi, \gamma) : (A, F_A) \rightarrow (B, F_B)$ be a morphism of FGLs, such that $\gamma = b_{r−1}x^r + \ldots$, and $b_{r−1} \neq 0$. Then it follows from Lemma 6.10 that $r = p^k$, for some $k \geq 0$.

**Lemma 6.12** In the above situation, $\gamma(x) = \delta(x^{p^k})$, for some $\delta \in B[[y]]$ with $\delta_0 \neq 0$.

*Proof:* We need to show that degrees of all non-zero terms of $\gamma$ are divisible by $p^k$. From the contrary, let $b_{s−1}$ be the smallest coefficient with $p^k \nmid s$. Then looking at the degree $s$ component of the equality:

$$\varphi(F_A)(\gamma(x), \gamma(y)) = \gamma(F_B(x, y)),$$

we get: $b_{s−1}x^s + b_{s−1}y^s = b_{s−1}(x+y)^s$, which implies that $d(s) \cdot b_{s−1} = 0$. But since $p \nmid s$, and $B$ has characteristic $p$, this implies that $b_{s−1} = 0$. \(\square\)

Thus, any such morphism $(\varphi, \gamma)$ of FGLs can be presented as the composition

$$(A, F_A) \overset{(\varphi, \delta)}{\rightarrow} (B, F_{\text{Fr}^k(B)}) \overset{(id, x^{p^k})}{\rightarrow} (B, F_B).$$

Return now to the situation where $A^*$ is a theory of rational type. Then the morphism $(\varphi, \delta)$ of formal group laws defines a multiplicative operation $A^* \overset{H}{\rightarrow} \text{Fr}^k(B)^*$, and we get that $G = \text{Fr}^k \circ H$, where $H$ is an operation of the main type.

Combining Theorem 6.9 with the above considerations, we get:

**Theorem 6.13** Let $B^*$ be any theory in the sense of Definition 2.1 with $B$ - an integral domain. Then:
get a multiplicative operation $\Psi^k$.

2) If $\text{char}(B) = p$, then the assignment $G \mapsto (k, \gamma_H)$, where $G = \widetilde{Fr}^k \circ H$, with $H$ of the main type, provides a 1-to-1 correspondence between multiplicative operations $\Omega^* \xrightarrow{G} B^*$ and pairs $(k, \gamma)$, where either $(k, \gamma) = (\infty, 0)$, or $k \in \mathbb{Z}_{\geq 0}$, and $\gamma = b_0 x + \ldots \in B[[x]]$ has $b_0 \neq 0$, and $(\widetilde{Fr}^k(F_B))^\gamma \in B[b_0^{-1}][[x, y]]$ has coefficients in $B$.

One can compose the morphisms of FGLs. Moreover, if $(\varphi, \gamma)$ and $(\varphi, \beta)$ have common homomorphism of coefficient rings, we can also "add" such morphisms of FGLs (just as one can add morphisms into an abelian group). Namely, we can set: $(\varphi, \beta) + (\varphi, \gamma) = (\varphi, \delta)$, where $\delta(x) = \varphi(F_A)(\beta(x), \gamma(x))$.

In particular, if $A^*$ is a theory of rational type, and there exists only one endomorphism $A \xrightarrow{\gamma} A$, then the set of multiplicative operations $A^* \xrightarrow{\gamma} A^*$ has a natural ring structure with multiplication = the composition, and addition as above. This happens for Chow groups, and for $K_0$. In the case of $\text{CH}^*/p$, we get:

**Theorem 6.14** The ring of multiplicative operations $\text{CH}^*/p \to \text{CH}^*/p$ is $\mathbb{Z}/p[[\widetilde{Fr}]]$. In particular, the composition is commutative.

**Proof:** Since there is only one ring homomorphism $\mathbb{Z}/p \to \mathbb{Z}/p$, the multiplicative operations $\text{CH}^*/p \to \text{CH}^*/p$ are in 1-to-1 correspondence with the additive power series $\gamma(x) = \sum_r b_r x^r$. Moreover, $\text{Fr}((\text{CH}^*/p)^*) = \text{CH}^*/p$, and $\widetilde{Fr} : \text{CH}^*/p \to \text{CH}^*/p$ is given by the power series $x^p$. The composition of operations corresponds to the composition of $\gamma$'s, and addition is the usual addition of $\gamma$'s. Thus, our ring can be naturally identified with $\mathbb{Z}/p[[\widetilde{Fr}]]$.

Under the identification above, the total Steenrod operation $\text{St}^{Tot} = \text{id} + S^1 + S^2 + \ldots$ corresponds to $1 + \widetilde{Fr}$, and the Integral Adams Operation $\Psi_k$ (see below) corresponds to $k$. In particular, $\Psi_0$ which is identity on $\text{CH}^0$ and zero on $\text{CH}^i$, $i > 0$ corresponds to 0.

### 6.3 Integral Adams Operations

Adams operations $\Psi_k$ provide an important tool in studying $K$-groups. In topology, analogous operations were constructed by S.P.Novikov for complex-oriented cobordisms $MU$ in [14]. This construction required inverting $k$, since $\Psi_k$ were basically expressed in terms of Landweber-Novikov operations, and the respective formulas do have $k$-denominators. Only much later it was shown by W.S.Wilson that these operations can be defined integrally and are naturally multiplicative unstable operations - see [28, Theorem 11.53]. Using our main results we can construct similar operations in Algebraic Cobordism and all other theories of rational type (it is worth noting, that although we produce a similar object, our methods are completely different as we are working with the theories themselves, not with spectra).

**Theorem 6.15** For any theory of rational type $A^*$, there are multiplicative (unstable) $A$-linear operations $\Psi_k : A^* \to A^*$, $k \in \mathbb{Z}$, such that $\gamma_{\Psi_k} = [k] \cdot_A x$.

In the case of $K_0$ these are usual Adams operations.

**Proof:** Consider $\gamma_k = [k] \cdot_A x$. Since $(id, \gamma_k)$, is an endomorphism of FGL $(A, F_A)$, by Theorem 6.8 we get a multiplicative operation $\Psi_k : A^* \to A^*$ with such $\gamma$.

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As the above operations are $A$-linear they can be obtained from the ones in Algebraic Cobordism by change of coefficients.

The set of Adams operations has common homomorphism of coefficient rings equal to the identity, and so form a ring $R_{\Psi,A}$. Clearly, $\Psi_k$ is just the image of $k$ under the canonical surjective ring homomorphism $\mathbb{Z} \rightarrow R_{\Psi,A}$. The operation $\Psi_0$ can be described as follows: it acts as $id$ on constant elements, and as zero on $A$. Thus, it is responsible for the decomposition which we used throughout the paper.

Adams operations can be used in the study of the graded Algebraic Cobordism (see [12 Subsection 4.5.2]). Being operations, they respect the codimension of support of an element: $\Psi_k(F^n\Omega^*(X)) \subset F^n\Omega^*(X)$, and so act on the graded ring $Gr^*\Omega^*(X)$. We have the natural surjection:

$$\mathrm{CH}^* \otimes_{\mathbb{Z}} \mathbb{L}^* \rightarrow Gr^*\Omega^*,$$

which commutes with the action of $\Psi_k$ (recall, that these operations are $\mathbb{L}$-linear). Thus, $\Psi_k|_{Gr^*\Omega^*}$ is the multiplication by $k^n$. Suppose now, $X \xrightarrow{f} Y$ is a morphism of smooth varieties. Then we get the morphism of the respective filtrations: $f^*: F^n\Omega^*(Y) \rightarrow F^n\Omega^*(X)$. This provides the spectral sequence computing Ker and Coker of $f^*$:

$$E_r^{p,q,n} = H^p(Gr(f)^* : Gr^q\Omega^q(Y) \rightarrow Gr^q\Omega^q(X), \quad p = 0, 1, \text{ and } d_r : E_r^{0,q,n} \rightarrow E_r^{1,q,n+r-1}. \quad \text{Adams operations permit to estimate the exponent of } d_r. \quad \text{Denote: } e(n, r) = G. C. D. (k^n(k^{r-1} - 1), k \in \mathbb{Z}).$$

**Proposition 6.16** $e(n, r) \cdot d_r|_{E_r^{0,q,n}} = 0.$

**Proof:** Since the Adams operation respects the filtration, it acts on the spectral sequence. Then $\Psi_k$ must act as multiplication by $k^n$ on $E_r^{p,q,n}$. Since $d_r : E_r^{0,q,n} \rightarrow E_r^{1,q,n+r-1}$, we get that, for any $k$, $k^n(k^{r-1} - 1)$ multiplied by such $d_r$ is zero. \[\square\]

It is easy to see that $e(0, r) = 1$, and $e(n, 2s) = 2$, for all $n, s \geq 1$. And prime factors of $e(n, r)$ are exactly those $p$ for which $(p - 1)|r - 1$. In particular, these do not depend on $n$. But the powers of these primes do. Thus, the "unstable information" is concentrated in these powers.

In particular, the above considerations apply to the extension of fields morphism.

### 6.4 Symmetric Operations for all primes, and T.tom Dieck - style Steenrod operations

These topics represent the main content of the paper [25]. Here we just present briefly the main results and ideas. The construction of Symmetric Operations for all primes was the main motivation behind the current paper. For about 5 years the author tried to construct them, until he realized that it is about as simple as constructing all unstable operations in Algebraic Cobordism. But let me start with the Steenrod operations.

Steenrod operations provide an important structure on $\mathrm{CH}^*/p$ which permits to do more elaborate tricks with algebraic cycles than the usual addition and multiplication. Individual Steenrod operations can be organized into "larger" multiplicative operations. One of the possible approaches is to consider the multiplicative operation: $St : \mathrm{CH}^*/p \rightarrow \mathrm{CH}^*/[t]$ given by the morphism of FGLs (see Theorem 6.8): $(\varphi, \gamma)$, where $\varphi : \mathbb{Z}/p \rightarrow \mathbb{Z}/p[[t]]$ is the natural embedding (the unique morphism of rings), and $\gamma = -t^{p-1}x + x^p$ (notice, that our $\gamma$ is additive in $x$). Then the individual Steenrod operation $S_r|_{\mathrm{CH}^*/p}$ will be the coefficient of $S|_{\mathrm{CH}^*/p} \frac{1}{p}$ at $t^{(m-r)(p-1)}$. At the first glance it looks like we complicate things by making our operation unstable (the coefficient at $x$ is not 1), but it appears to be convenient in various respects.
The original approach to Steenrod operations in Chow groups due to P.Brosnan (see [4]) is through \( \mathbb{Z}/p \)-equivariant Chow groups. In this construction, one produces the multiplicative operation \( Sq : \text{CH}^\ast(X)/p \to \text{CH}^\ast(X)/p \otimes_{\mathbb{Z}/p} \text{CH}^\ast(B\mathbb{Z}/p)/p \). We have \( \text{CH}^\ast(B\mathbb{Z}/p)/p = \mathbb{Z}/p[[t]] \), and one can show (see [4]) that the only non-trivial coefficients of \( Sq \) will be at \( t^r(p-1) \), \( r \geq 0 \). The fact that the two constructions agree follows from Theorem 6.8 (the morphism of FGLs for \( Sq \) is easy to compute).

All of the above was known in topology for quite a while. And both mentioned constructions were extended to complex-oriented cobordism \( MU \). The equivariant version is due to T. tom Dieck ([20]), and it goes completely parallel to the \( H^*/p \) (and \( \text{CH}^*/p \)) case. Here \( MU^*(X \times B\mathbb{Z}/p) = MU^*(X)[[t]]/(p \cdot MU(t)) \), and one gets a multiplicative operation

\[
Sq : MU^*(X) \to MU^*(X \times B\mathbb{Z}/p) \to MU^*(X)[[t]]/(p \cdot MU(t)).
\]

Notice, that this time, we have to invert \( t \) and \( (p-1)! \), since the shifted formal group law \( F_{MU[[t]]}^\gamma \) has denominators. Also, \( St \) has non-trivial coefficients at \( t^j \), for \( j \) not divisible by \( (p-1) \). It was shown by D.Quillen that his approach agrees with the one of T. Tom Dieck. More precisely, one has the following commutative diagram:

\[
\begin{array}{ccc}
MU^* & \xrightarrow{St} & MU^*((p-1)!^{-1})[[t]][t^{-1}] \\
\downarrow_{Sq} & & \downarrow \\
MU^*[[t]]/(p \cdot MU(t)) & \rightarrow & MU^*[[t]][t^{-1}] / (p \cdot MU(t)).
\end{array}
\]

Let us try to extend these constructions to the case of Algebraic Cobordism \( \Omega^* \). The Quillen’s version is completely straightforward. Here one needs only the universality of \( \Omega^* \) supplied by M.Levine-F.Morel ([12] Theorem 1.2.6) and the change of orientation of I.Panin-A.Smirnov ([16]). Let us do a more general case (suggested by D.Quillen). Namely, choose representatives \( \{i_j, 0 < j < p\} \) of all non-zero cosets modulo \( p \), and denote \( i := \prod_{j=1}^{p-1} i_j \). Then we can consider the power series \( \gamma = \prod_{j=0}^{p-1} (x + \Omega [i_j] \cdot t) \in \mathbb{L}[[t]][[x]] \), which, by Theorem 3.7, defines the multiplicative operation

\[
St(i) : \Omega^* \to \Omega^*[i^{-1}][[t]][t^{-1}].
\]

The situation with the version of T.tom Dieck is rather different. Although one can easily define the \( \mathbb{Z}/p \)-equivariant Algebraic Cobordism \( \Omega^*_\mathbb{Z}/p(X) \), one encounters problems trying to prove that the natural map \( \Omega^n(X) \to \Omega^p_{\mathbb{Z}/p}(X \times p) \) is well-defined. It is easy to show that the standard cobordism relations are respected, but the author was unable to handle the double-points relations. The only case where the author succeeded was \( p = 2 \), where he had to employ the Symmetric Operations (modulo 2) constructed in [21], [23]. These operations, which are more subtle than the Steenrod ones, until now were unavailable for \( p > 2 \).

Hopefully, our Theorem 6.8 permits to construct what we need.

**Theorem 6.17** ([25] Theorem 6.4) There is the multiplicative operation \( Sq \) which fits into the commu-
**Theorem 6.18** ([25 Theorem 7.1]) There is unique additive operation $\Phi(\overline{t}) : \Omega^* \to \Omega^*[i^{-1}][t^{-1}]$ such that

$$(St(\overline{t}) - \frac{[p]}{t} \cdot \Phi(\overline{t})) : \Omega^* \to \Omega^*[i^{-1}][t].$$

Some traces of the $MU$-analogue of this operation were used by D. Quillen in [18], and they provide the main tool of the mentioned article.

In Algebraic Cobordism the described operation appeared originally in the works [21] and [23] of the author in the case $p = 2$ in a different form. Namely, in the form of "slices", which were constructed geometrically. Only substantially later the author had realized that these slices can be combined into the "formal half" of the "negative part" of some multiplicative operation, which had a power series $\gamma = x \cdot (x - \Omega t)$ reminiscent of a Steenrod operation in Chow groups mod 2. How to view the operation $\Phi(\overline{t})$? The natural approach would be to consider the coefficients of it at particular monomials $t^{-n}$, or, equivalently, $\text{Res}_{t=0} \frac{t^n \cdot \Phi(\overline{t})}{t} \omega_t$, for all $n$. And, if one thinks about it, there is no point restricting yourself to monomials, so one can consider

$$\Phi(\overline{t})^{q(t)} := \text{Res}_{t=0} \frac{q(t) \cdot \Phi(\overline{t}) \omega_t}{t},$$

where $q(t) = q_1 t + q_2 t^2 + \ldots \in \mathbb{L}[t][t]$ is any power series without the constant term. Of course, there are various relations among these slices which bind them together into something "larger" - the operation $\Phi(\overline{t})$. For $p = 2$, these are exactly the Symmetric operations $\Phi^{q(t)}$ of [23].

**Proposition 6.19** ([25 Proposition 7.2]) In the case $p = 2$, with $\overline{t} = \{-1\}$, for any power series as above, we have:

$$\Phi(\overline{t})^{q(t)} = -\Phi^{q(t)}.$$

Notice, that for $p = 2$, there is, in addition, a non-additive operation $\Phi^1$ (see [23]). At the moment, we can not produce it’s analogues for $p > 2$ as our methods so far are restricted to additive operations only. Hopefully, additive Symmetric Operations are sufficient for most applications.

The cases $p = 2$ and 3 are special, since we can choose our representatives $\overline{t}$ to be invertible in $\mathbb{Z}$. For $p = 2$, we have two such choices: $\{1\}$, or $\{-1\}$ (in [23], $\{-1\}$ was "chosen"). For $p = 3$, the choice is canonical: $\{1, -1\}$. Thus, we get integral operations $\Phi(\overline{t}) : \Omega^* \to \Omega^*[t^{-1}]$. And, for arbitrary $p$, we can choose our remainders to be the powers of some fixed prime $l$ (generating $(\mathbb{Z}/p)^*$), so that only one prime would be inverted. Moreover, this prime can be chosen in infinitely many ways, so, in a sense, the picture is as good as integral.

For $p = 2$ the Symmetric operations were applied to the study of 2-torsion effects in Chow groups - they provide the only known method to get "clean results" on rationality - see [22] and [24].
similar applications are expected for other primes. Other applications involve the study of the structure of the \( L \)-module \( \text{Gr}\Omega^*(X) \). Here the construction of Symmetric Operations for all primes changes the statements \( \otimes \mathbb{Z}_(2) \) into integral ones.

7 Basic tools

Here we present various results which permit to work effectively with cohomology theories.

7.1 Projective bundle and blow-up results

We start with the excess intersection formula - see \[23, \text{Theorem 5.19}\] and \[12, \text{Theorem 6.6.9}\]. Consider cartesian square

\[
\begin{array}{ccc}
W & \xrightarrow{f'} & Z \\
\downarrow{g'} & & \downarrow{g} \\
Y & \xrightarrow{f} & X
\end{array}
\]

with \( f, f' \) - regular embeddings, and \((g')^*(N_{Y \subset X})/N_{W \subset Z} =: \mathcal{M}\) the vector bundle of dimension \( d \).

**Proposition 7.1** Let \( A^* \) be any theory in the sense of Definition \[2.7\]. In the above situation,

\[
g^*f_*(v) = f'_*(c^A_d(M) \cdot (g')^*(v));
\]

If \( g \) is projective, then also:

\[
f^*g_*(u) = g'_*(c^A_d(M) \cdot (f')^*(u)).
\]

**Proof:** Both of the above references are dealing with the \( \Omega^* \)-case. Although, the statement of \[12, \text{Theorem 6.6.9}\] is more general, it requires the development of the whole theory of refined pull-backs. For Algebraic Cobordism such a theory is constructed in \[12\], but it requires some work to extend it to a more general context. In contrast, the proof of \[23, \text{Theorem 5.19}\] does not use any specifics of \( \Omega^* \) and works in general.

Another important tool is the formula of Quillen - see \[17, \text{Theorem 1}\], \[16, \text{Formula (24)}\], and \[23, \text{Theorem 5.35}\]. It describes push-forwards for projective bundles.

Recall that, for an \( n \)-dimensional vector bundle \( V \), the roots are elements \( \lambda_i \in A^1(\text{Flag}_X(V)) \), \( i = 1, \ldots n \) such that \( \prod_{i=1}^n(t + \lambda_i) = \sum_{i=0}^n c^i_V t^{n-i} \), where \( \text{Flag}_X(V) \) is a variety of complete flags of \( V \), and \( c^i_V \) are Chern classes in the theory \( A^* \). The important point here is that the pull-back map \( A^*(X) \to A^*(\text{Flag}_X(V)) \) is split injective.

Recall also that \( \omega_A \in A[[x]]dx \) is the canonical invariant 1-form satisfying: \( w_A(0) = dx \). Such a form can be obtained from the formal group law \( F_A(x, y) \) of \( A^* \) by the formula: \( \omega_A = \left( \frac{\partial F_A}{\partial y} \right)_{y=0}^{-1} dx \). By the formula of Mistchenko it can be expressed as:

\[
([\mathbb{P}^0]_A + [\mathbb{P}^1]_A \cdot x + [\mathbb{P}^2]_A \cdot x^2 + \ldots) \, dx,
\]

where \([\mathbb{P}^r]_A\) is the class of \( \mathbb{P}^r \) in \( A^*(\text{Spec}(k)) = A \).
Proposition 7.2 Let $A^*$ be any theory in the sense of Definition 2.1. Let $X$ be some smooth quasi-projective variety, $V$ be some $n$-dimensional vector bundle on it, and $\pi : \mathbb{P}_X(V) \to X$ be the corresponding projective bundle. Let $f(t) \in A^*(X)[[t]]$, and $\xi = c_1^A(O(1))$. Then

$$\pi_!(f(\xi)) = \text{Res}_{t=0} \frac{f(t) \cdot \omega_A}{\prod_i (t + A \lambda_i)}.$$

where $\lambda_i$ are roots of $V$, and $+_A$ is the formal addition in the sense of $F_A$.

Proof: Clearly, both parts of the formula are $A^*$-linear, so it is sufficient to prove the result in the case: $f(t) = t^r$ - a monomial. Then it formally follows from the $\Omega^*$-case proven in [23, Theorem 5.35] (using the universality of $\Omega^*$ - [12, Theorem 1.2.6]). \hfill \Box

We will need various results concerning the blow up morphism.

Let $X$ be smooth variety, $R$ it’s smooth closed subvariety, $\tilde{X} = \text{Bl}_X R$ - the blow up of $X$ at $R$, and $E$ - the exceptional divisor on $\tilde{X}$. These fit into the blow-up diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{j} & \tilde{X} \\
\varepsilon \downarrow & & \downarrow \pi \\
R & \xrightarrow{i} & X.
\end{array}
$$

Let $N$ be the normal bundle of $R$ in $X$, then $E \cong \mathbb{P}_R(N)$. Let $d = \text{dim}(N) = \text{codim}(R \subset X)$. Denote $\tilde{N} = N \oplus O(1)$, and $\tilde{E} = \mathbb{P}_R(\tilde{N}) \xrightarrow{\tilde{\varepsilon}} R$.

The following result of M.Levine and F.Morel describes the class of the blow up in the $A^*$ of the base.

Proposition 7.3 ([12, Proposition 2.5.2]) Let $A^*$ be any theory in the sense of Definition 2.1. Then

$$
\pi_!(1) = 1 + i_* \tilde{\varepsilon}^* \left( \frac{c_1^A(O(1))}{c_1^A(O(-1))} \right).
$$

In particular, the above class is invertible. More generally, we have:

Proposition 7.4 Let $A^*$ be any theory in the sense of Definition 2.1, and $\pi : \tilde{X} \to X$ be projective bi-rational morphism of smooth varieties. Then

(1) $\pi_*(1)$ is invertible in $A^*(X)$.

(2) $\pi_* : A_*(\tilde{X}) \to A_*(X)$ is surjective.

Proof: By universality of $\Omega^*$ ([12, Theorem 1.2.6]), we have the canonical map of theories $\Omega^* \to A^*$, and $\pi_*(1)$ is in the image of this map. So, it is sufficient to treat the case $A^* = \Omega^*$. Since $\pi$ is bi-rational, we have a closed subscheme $Z \subset X$ of positive codimension, such that $\pi$ is an isomorphism outside $Z$. Then $\pi_*(1) = 1 + u$, where $u$ is supported on $Z$. That means that $u$ has positive codimension of support, and so is nilpotent by [24, Statement 5.2]. Hence, $\pi_*(1)$ is invertible. It remains to apply the projection formula. \hfill \Box

The following result describes what happens to the whole $A^*$ when you blow up some smooth variety at a smooth center.
Proposition 7.5 (cf. [23 Proposition 5.24]) Let $A^*$ be any theory in the sense of Definition 2.7. Then we have split exact sequences:

(1) \[ 0 \leftarrow A_*(X) \xrightarrow{\pi_{e^{-1}e}} A_*(\X) \oplus A_*(R) \xrightarrow{j_{\epsilon_{e^*}}} A_*(E) \leftarrow 0; \]

(2) \[ 0 \rightarrow A^*(X) \xrightarrow{\pi_{e^{-1}e}} A^*(\X) \oplus A^*(R) \xrightarrow{j^*_{\epsilon_{e^*}}} A^*(E) \rightarrow 0. \]

Proof: In the case $A^* = \Omega^*$, (1) was proven in [23 Proposition 5.24], and the same proof works for arbitrary $A^*$. Let us recall some details. Let $K = \epsilon^*N/O(-1)$ be the excess bundle on $E$. It is easy to see (see [23 Proposition 5.22]) that the class of the diagonal on $E \times_R E$ is given by $c_{d-1}^A(K_1 \otimes O(1)_2)$, where $V_l$ denotes the bundle $V$ lifted from the $l$-th component. This class can be written as $c_{d-1}^A(K) \times 1 + \sum_{j \geq 1} \gamma_{d-1-i} \times \zeta^j$, where $\gamma_j \in A^j(E)$ are some elements, and $\zeta = c_{d-1}^A(O(-1))$. Let us introduce the elements $\alpha := c_{d-1}^A(K) \in A^*(E)$, and $\beta := \epsilon^*\left( \frac{c_{d-1}^A(O(1))}{c_{d-1}^A(O(-1))} \right) \in A^*(R)$. Then for any $u \in A^*(E)$, we have:

\[
\begin{align*}
  u &= \alpha \cdot e^*\epsilon_*(u) + \sum_{j \geq 1} \gamma_{d-1-j} \cdot e^*\epsilon_*(u \cdot \zeta^j); \\
  u &= e^*\epsilon_*(u \cdot \alpha) + \sum_{j \geq 1} \zeta^j \cdot e^*\epsilon_*(u \cdot \gamma_{d-1-j}),
\end{align*}
\]

Consider the maps $F : A_*(E) \rightarrow A_*(E)$ and $G : A^*(E) \rightarrow A^*(E)$ given by:

\[
F(u) = \sum_{j \geq 0} \gamma_{d-2-j} \cdot e^*\epsilon_*(u \cdot \zeta^j); \quad G(u) = \sum_{j \geq 0} \zeta^j \cdot e^*\epsilon_*(u \cdot \gamma_{d-2-j}).
\]

Consider the diagram:

\[
\begin{array}{ccc}
E \times_R E & \xrightarrow{id \times e} & E \times R \E \\
\downarrow {p_1} & & \downarrow \varrho \\
E & \rightarrow & R \\
\end{array}
\]

Let $E \xrightarrow{\varrho} \E$ be the natural embedding. Then $c_*(1) = c_{d-1}^A(O(1))$. We get:

\[
F(1) = (p_1)_* \left( \frac{c_{d-1}^A(K_1 \otimes O(1)_2) - c_{d-1}^A(K_1)}{c_{d-1}^A(O(-1)_2)} \right)
\]

\[
= \varrho_* \left( \frac{c_{d-1}^A(K_1 \otimes O(1)_2) - c_{d-1}^A(K_1)}{c_{d-1}^A(O(-1)_2)} \right) \cdot \frac{c_{d-1}^A(O(1)_2)}{c_{d-1}^A(O(-1)_2)}
\]

\[
= \varrho_* \left( \frac{c_{d-1}^A(K_1 \otimes O(1)_2)}{c_{d-1}^A(O(-1)_2)} - c_{d-1}^A(K) \cdot \varrho_* \rho^* \left( \frac{c_{d-1}^A(O(1))}{c_{d-1}^A(O(-1))} \right) \right)
\]

\[
= \Res_{t=0} \frac{c_{d-1}^A(K)(t) \cdot t \cdot \omega_A}{c_{d-1}^A(N)(t) \cdot (-A)} - c_{d-1}^A(K) \cdot \epsilon^*\epsilon_*(\frac{c_{d-1}^A(O(1))}{c_{d-1}^A(O(-1))})
\]

\[
= \Res_{t=0} \frac{c_{d-1}^A(N)(t) \cdot (-A)}{c_{d-1}^A(K)(t) \cdot \omega_A} - \alpha \cdot \epsilon^*\epsilon_*(\beta) = -\alpha \cdot \epsilon^*\epsilon_*(\beta).
\]

Now we can construct contracting homotopies $\lambda$ and $\mu$ for (1) and (2):

\[
\begin{array}{ccc}
A_*(E) & \xrightarrow{d_2} & A_*(\X) \\
\lambda_1 \left\{ \begin{array}{c} d_1 \\ d_2 \end{array} \right\} \lambda_2 \left\{ \begin{array}{c} d_3 \\ d_4 \end{array} \right\} & \rightarrow & A_*(X),
\end{array}
\]

\[
\begin{array}{ccc}
A^*(E) & \xrightarrow{d_2} & A^*(\X) \\
\mu_1 \left\{ \begin{array}{c} d_1 \\ d_2 \end{array} \right\} \lambda_3 & \rightarrow & A^*(X),
\end{array}
\]

\[
\begin{array}{ccc}
A_*(R) & \xrightarrow{d_3} & A_*(X) \\
\mu_3 \left\{ \begin{array}{c} d_1 \\ d_3 \end{array} \right\} & \rightarrow & A^*(R),
\end{array}
\]

\[
\begin{array}{ccc}
A^*(R) & \xrightarrow{d_3} & A^*(X),
\end{array}
\]

\[
\begin{array}{ccc}
A^*(R) & \xrightarrow{d_3} & A^*(X),
\end{array}
\]

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in the following way: $\lambda_4 = \pi^*; \lambda_3 = \beta \cdot i^*, \lambda_1 = \alpha \cdot \varepsilon^*; \lambda_2 = F \circ j^*$, while $\mu_4 = \pi_*; \mu_3 = i_*(\beta \cdot )$, $\mu_1 = \varepsilon_*(\alpha \cdot );$ and $\mu_2 = j_\circ G$.

From the equality $F(1) = -\alpha \cdot \varepsilon^*(\beta)$ (using several times the projection formula) one easily obtains the left ones of the following identities:

$$\lambda_2 \circ \lambda_4 = -\lambda_1 \circ \lambda_3; \quad d_2 \circ \lambda_1 = -\lambda_4 \circ d_3; \quad (7)$$

$$\mu_4 \circ \mu_2 = -\mu_3 \circ \mu_1; \quad \mu_1 \circ d_2 = -d_3 \circ \mu_4, \quad (8)$$

while the right ones are the Excess Intersection Formula (Proposition 7.1). The identity: $d_3 \circ \lambda_3 + d_4 \circ \lambda_4 = id_{A_*(X)}$ is just the Proposition 7.3 (plus the projection formula). The identity: $d_1 \circ \lambda_1 + \lambda_3 \circ d_3 = id_{A_*(R)}$ follows from the Excess Intersection Formula and Proposition 7.3. The identity: $\lambda_1 \circ d_1 + \lambda_2 \circ d_2 = id_{A_*(E)}$ follows from the definition of $F$. Finally, the identity: $d_2 \circ \lambda_2 + \lambda_4 \circ d_4 = id_{A_*(\tilde{X})}$ follows from the ones already proven, plus (7), plus the fact that the map $(j, \pi^*) : A_*(E) \oplus A_*(X) \rightarrow A_*(\tilde{X})$ is surjective, which follows from the (EXCI) axiom (see the proof of 23, Proposition 5.24).

The identity: $\mu_3 \circ d_3 + \mu_4 \circ d_4 = id_{A_*(X)}$ follows from Proposition 7.3 and the projection formula. The identity: $\mu_1 \circ d_1 + d_3 \circ \mu_2 = id_{A_*(R)}$ follows from the Excess Intersection Formula and Proposition 7.3. The identity: $d_1 \circ \mu_1 + \mu_2 \circ d_2 = id_{A_*(E)}$ follows from the definition of $G$. Finally, the identity: $\mu_2 \circ d_2 + d_4 \circ \mu_4 = id_{A_*(\tilde{X})}$ follows the ones already proven, plus (8), plus the fact that the map $(j^*, \pi_*) : A_*(\tilde{X}) \rightarrow A_*(E) \oplus A_*(X)$ is injective, which follows from the fact that $\lambda$ is a contracting homotopy for the complex (1).

In the case of multiple blow-ups we get:

**Proposition 7.6** Let $A^*$ be any generalized oriented cohomology theory in the sense of Definition 2.1, and $\pi : \tilde{V} \rightarrow V$ be the permitted blow up of a smooth variety with smooth centers $R_i$ and the respective components of the exceptional divisor $E_i \xrightarrow{\xi} R_i$. Then one has exact sequences:

1. $0 \leftarrow A_*(V) \xleftarrow{\pi^*} A_*(\tilde{V}) \leftarrow \oplus_i \text{Ker}(A_*(E_i) \xrightarrow{(\xi_i)_*} A_*(R_i))$.

2. $0 \rightarrow A^*(V) \xrightarrow{\pi^*} A^*(\tilde{V}) \rightarrow \oplus_i \text{Coker}(A^*(R_i) \xrightarrow{(\xi_i)^*} A^*(E_i))$.

**Proof:** The Proposition 7.5 settles the case where $\pi$ is a single blow up. Let us use induction on the number of blowings. Suppose, $\tilde{V}$ is the result of $n$ blowings, and $V$ is the result of $(n - 1)$ (first) of them. Then $\rho : \tilde{V} \rightarrow Y$ is a single blow up with the center $R$. Let $F_i$, $i = 1, \ldots, n - 1$ be the components of the exceptional divisor of $Y$, and $E_i$, $i = 1, \ldots, n - 1$ be there proper preimages under $\rho$, and $E$ be the exceptional divisor of $\rho$. By inductive assumption and Proposition 7.5 we have exact sequences:

$$0 \leftarrow A_*(V) \xleftarrow{\pi^*} A_*(Y) \leftarrow \oplus_{i=1}^{n-1} \text{Ker}(A_*(F_i) \xrightarrow{(\xi_i)_*} A_*(R_i));$$

$$0 \leftarrow A_*(Y) \xleftarrow{\rho^*} A_*(\tilde{V}) \leftarrow \text{Ker}(A_*(E) \xrightarrow{\xi} A_*(R)).$$

Taking into account that the map:

$$\text{Ker}(A_*(F_i) \rightarrow A_*(R_i)) \leftarrow \text{Ker}(A_*(E_i) \rightarrow A_*(R_i))$$

is surjective, we get the first exact sequence. The second one can be proven in a similar fashion.

The following "singular" variant of the above result is an important tool in our calculations, and it permits to present $A_*(Z)$ in terms of $A_*$ of finitely many smooth varieties.
Proposition 7.7 Let $Z$ be a variety, and $	ilde{Z} \xrightarrow{\pi} Z$ be the blow up with centers $R_i$ and exceptional divisors $E_i$. Then we have an exact sequence:

$$0 \leftarrow A_*(Z) \leftarrow \left( A_*(\tilde{Z}) \oplus (\oplus_i A_*(R_i)) \right) \leftarrow \oplus_i A_*(E_i).$$

Proof:

Lemma 7.8 Let $\pi : \tilde{V} \to V$ be projective birational map of smooth varieties, which is an isomorphism outside the closed subvariety $T \to V$, and such that $W = \pi^{-1}(T)$ is a divisor with strict normal crossing with components $E_i$. Then we have an exact sequence:

$$0 \leftarrow A_*(V) \leftarrow \left( A_*(\tilde{V}) \oplus (\oplus_i A_*(E_i)) \right) \leftarrow \oplus_i A_*(E_i) \leftarrow A_*(T).$$

Proof: Let $\pi' : \tilde{V}' \to V$ be the permitted blow up with centers over $T$ resolving $T$ to a divisor $W'$ with strict normal crossing (Theorem 8.4). Let $E'_j$ be the components of $W'$, and $R'_j$ be the respective smooth centers. Then, by Proposition 7.6 we have an exact sequence:

$$0 \leftarrow A_*(V) \leftarrow \left( A_*(\tilde{V}') \oplus (\oplus_i A_*(E'_i)) \right) \leftarrow \oplus_i A_*(E'_i) \leftarrow A_*(T).$$

Since the map $A_*(\tilde{V}')/(\oplus_i A_*(E'_i) \to A_*(T))$ factors through $A_*(\tilde{V}')/(\oplus_i A_*(E'_i) \to A_*(T))$, we have the statement for $\tilde{V}'$. Let us denote $B(\tilde{V}) := \text{Coker} \left( \oplus_i A_*(E'_i) \to A_*(T) \right) \to A_*(\tilde{V})$.

We have a natural surjective map $B(\tilde{V}) \to A_*(V)$. Since $\tilde{V}$ and $\tilde{V}'$ are isomorphic outside $W$ and $W'$, by the Weak Factorization Theorem (Theorem 8.6(6)), we have a diagram:

$$\begin{array}{cccccccc}
\tilde{V}' & \leftarrow & Y_1 & \leftarrow & Y_2 & \leftarrow & Y_3 & \leftarrow & \ldots & \leftarrow & Y_{n-3} & \leftarrow & Y_{n-2} & \leftarrow & Y_{n-1} & \leftarrow & \tilde{V},
\end{array}$$

where all $Y_i$’s are projective either over $\tilde{V}'$, or $\tilde{V}$, and all the maps are blowings up/down w.r.t smooth centers which belong to exceptional divisor, and meet all of it’s components properly. In particular, each $Y_i$ has a natural map to $V$, which is an isomorphism outside $T$, and the preimage of $T$ is the exceptional divisor (with normal crossing) on $Y_i$. Since the maps $Y_{2n-1} \to Y_{2n} \leftarrow Y_{2n+1}$ are blowings up/down with centers belonging to an exceptional divisor, we see (using Proposition 7.3) that the maps $B(Y_{2n-1}) \to B(Y_{2n}) \leftarrow B(Y_{2n+1})$ are isomorphisms. Clearly, these identifications are compatible with the maps $B(Y_i) \to A_*(V)$. Since the map $B(\tilde{V}') \to A_*(V)$ is an isomorphism, so is the map $B(\tilde{V}) \to A_*(V)$. \qed

Lemma 7.9 Let $\tilde{Z} \xrightarrow{\pi} Z$ be a projective map of varieties, which is an isomorphism outside the closed subvariety $R \to Z$ with the preimage $E = \pi^{-1}(R)$. Then one has an exact sequence:

$$0 \leftarrow A_*(Z) \leftarrow \left( A_*(\tilde{Z}) \oplus A_*(R) \right) \leftarrow A_*(E).$$

Proof: The fact that it is a complex is evident.

Let us construct the map

$$A_*(Z) \xrightarrow{\varphi} \text{Coker} \left( A_*(E) \to A_*(\tilde{Z}) \oplus A_*(R) \right)$$

inverse to our projection. Let $v : V \to Z$ be some projective map with $V$ smooth irreducible. If image of $v$ is contained in $R$, then we get a natural map $A_*(V) \to A_*(R) \to \text{Coker} (A_*(E) \to A_*(\tilde{Z}) \oplus A_*(R))$. 

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Otherwise, we have a birational map $V \to \tilde{Z}$, which can be resolved by blowing smooth centers over $v^{-1}(R)$. Then the exceptional set on $\tilde{V}$ is a divisor with strict normal crossing $W$, which is mapped to $E$ via $\tilde{v}$. Moreover, we can assume that $W$ coincides with the preimage of $R$. If $F_j$ are components of $W$, and $S_j$ are the respective smooth centers, then by Lemma 7.8:

$$0 \leftarrow A_*(V) \xleftarrow{\pi_*} A_*(\tilde{V}) \leftarrow \oplus_j \text{Ker}(A_*(F_j) \to A_*(v^{-1}(R))).$$

Since $F_j$ are mapped to $E$, and $v^{-1}(R)$ to $R$, the map $(\tilde{v})_* : A_*(\tilde{V}) \to A_*(\tilde{Z})$ provides a well-defined map $A_*(V) \xrightarrow{\sim} \text{Coker} \left( A_*(E) \to A_*(\tilde{Z}) \oplus A_*(R) \right)$.

Let $\tilde{V}_1, \tilde{V}_2$ be two resolutions as above, with the exceptional divisors $W_1$ and $W_2$. Then $\tilde{V}_1 \backslash W_1 \cong V \backslash v^{-1}(R) \cong \tilde{V}_2 \backslash W_2$. Hence, by the Weak Factorization Theorem (Theorem 8.6(6)), there exists a diagram:

$$\begin{array}{c c c c c c}
Y_1 & Y_2 & \ldots & Y_{n-2} & Y_{n-1} & Y_n \\
\tilde{V}_1 & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow \tilde{V}_2,
\end{array}$$

where all $Y_i$’s are projective either over $\tilde{V}_1$, or $\tilde{V}_2$, and all the maps are blowings up/down w.r.t smooth centers which belong to exceptional divisor, and meet all of it’s components properly. In particular, each $Y_i$ has a natural map to $\tilde{Z}$, so that the preimage of $E$ is the exceptional divisor.

Using notations from the proof of Lemma 7.8 let us define

$$B(Y_i) := \text{Coker}((\oplus_j \text{Ker}(A_*(G_j) \to A_*(v^{-1}(R)))) \to A_*(Y_i)),$$

where $G_j$ are components of the exceptional divisor of $Y_i$. Then we have a natural map $B(Y_i) \to \text{Coker}(A_*(E) \to A_*(\tilde{Z}) \oplus A_*(R))$, which is compatible with the identifications: $B(Y_{2n-1}) = B(Y_{2n}) = B(Y_{2n+1})$ (as in the proof of Lemma 7.8). This shows that the map

$$A_*(V) \xrightarrow{\sim} \text{Coker} \left( A_*(E) \to A_*(\tilde{Z}) \oplus A_*(R) \right)$$

does not depend on the choice of the resolution $\tilde{V} \to V$.

Let $V_1 \xrightarrow{f} V_2 \xrightarrow{g} Z$ be some projective maps with $V_1$ and $V_2$ smooth, and $v_1 = v_2 \circ f$. We can assume $V_1$ and $V_2$ irreducible. If the image $(v_2) \subset R$, then both maps $\varphi_{v_1}$ and $\varphi_{v_2}$ are passing through $A_*(R)$ and are clearly compatible with $f_*$. So, we can assume that $image(v_2) \not\subset R$. Let $\tilde{V}_2 \to V_2$ be the permitted blow up resolving indeterminacy of $\pi^{-1} \circ v_2$, and resolving $v_2^{-1}(R)$ to a divisor with normal crossing $W_2$.

If $image(v_1) \subset R$, then since the fibers of the projection $\tilde{V}_2 \to V_2$ are rational, we get a rational map $V_1 \dashrightarrow W_2$. Resolve the indeterminacies of this map: $V_1 \xleftarrow{\rho_*} \tilde{V}_1 \xrightarrow{f'} W_2$, which gives $f : \tilde{V}_1 \to \tilde{V}_2$. Since the map $\rho_* : A_*(\tilde{V}_1) \to A_*(V_1)$ is surjective, and the compatibility of this map with $\varphi_{v_1}$, $\varphi_{v_2}$ is already known (the image is in $R$), we can substitute $V_1$ by $\tilde{V}_1$. Since $W_2$ is mapped to $E$, we get that

$$\varphi_{v_2} \circ \tilde{f}_* = \varphi_{v_1} : A_*(\tilde{V}_1) \to \text{Coker} \left( A_*(E) \to A_*(\tilde{Z}) \oplus A_*(R) \right).$$

Finally, if $image(v_1) \not\subset R$, then we get a rational map $V_1 \dashrightarrow \tilde{V}_2$ with indeterminacies only over $v_1^{-1}(R)$ which can be resolved by $\tilde{V}_1 \to \tilde{V}_2$, making the preimage of $R$ a divisor with normal crossing $W_1$. We get a map $\tilde{f} : \tilde{V}_1 \to \tilde{V}_2$. Then we can take $v_1 = \tilde{v}_2 \circ \tilde{f}$, and so $\varphi_{v_1} = \varphi_{v_2} \circ f_*$. Thus, we get a well-defined map:

$$A_*(Z) \xrightarrow{\varphi} \text{Coker} \left( A_*(E) \to A_*(\tilde{Z}) \oplus A_*(R) \right)$$

It is easy to see that it is inverse to the natural projection:

$$A_*(Z) \xleftarrow{\psi} \text{Coker} \left( A_*(E) \to A_*(\tilde{Z}) \oplus A_*(R) \right)$$

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\[ \text{On the left: let } v : V \to Z \text{ be some projective map with } V \text{ smooth irreducible. There are two cases: 1) } \text{image}(v) \subset R; \text{ 2) } \text{image}(v) \not\subset R. \text{ In both cases, the fact that } \psi \circ \varphi|_{A_s(V)} \text{ is the identity is evident from the very definition.} \]

\[ \text{On the right: the fact that } \varphi \circ \psi \text{ is the identity on the } A_s(R)-\text{component is evident. As for the } A_s(\tilde{Z})-\text{component, if we have some projective map } v : V \to \tilde{Z} \text{ with the resolution } \rho : \tilde{V} \to V \text{ of the closed subvariety } v^{-1}(E), \text{ then the map } \varphi_v \circ \psi|_{A_s(V)} \text{ is just } v_* : A_s(V) \to \text{Coker } \left(A_s(E) \to A_s(\tilde{Z}) \oplus A_s(R)\right) \text{ (since } \rho \circ v \text{ factors through } v). \text{ Thus, we get the identity map on the } A_s(\tilde{Z})-\text{component as well. Hence, our complex is exact.} \]

\[ \text{Remark 7.10 Of course, if } A^* \text{ can be extended to a "large" theory, the above Lemma follows automatically from the respective localization (excision) axiom. The point is that it is true for any theory in the sense of Definition 2.1.} \]

\[ \text{Lemma 7.9 settles the case where } \pi \text{ is a single blow up. The rest is done by the induction on the number of blowings in the same way as the proof of Proposition 7.6.} \]

\[ \text{Remark 7.11 1) In particular, this applies when } \tilde{Z} \to Z \text{ is the resolution of } Z \text{ as in Theorems 8.2, 8.3, that is the permitted blow up with smooth centers which meet the components of the exceptional divisor properly, and resolves the singularities of } Z, \text{ and then makes the special divisor the one with the strict normal crossing. In this case, all the varieties aside from } Z \text{ participating in the formula are smooth, and we get the "finite" presentation of } A_s(Z) \text{ in terms of smooth varieties.} \]

\[ \text{2) The map } A_s(\tilde{Z}) \to A_s(Z) \text{ is not surjective, in general, if } Z \text{ is not smooth. Take, for example, } Z \text{- the cone over an anisotropic conic, and } R \text{- it’s vertex. Then } \tilde{Z} \text{ has no zero cycles of odd degree, while } Z \text{ has a rational point.} \]

We will also need the following Bertini-type result.

\[ \text{Proposition 7.12 Let } X \text{ be smooth quasi-projective variety, and } Z \subset X \text{ be a proper closed subvariety of it. Then there exists a divisor } Y \text{ of } X \text{ which contains } Z, \text{ and is smooth outside } Z, \text{ as well as in the generic points of the components of } Z. \]

\[ \text{7.2 Multiple points excess intersection formula} \]

In this subsection, \( A^* \) is any theory in the sense of Definition 2.1. Our main aim here is Proposition 7.20. This analogue of the usual Excess Intersection Formula, where regular embeddings (of smooth varieties) are substituted by strict normal crossing divisors, is a very useful computational tool. To state it, one needs to define the pull-back maps for such divisors. In the case of Algebraic Cobordism, or any theory obtained from it by change of coefficients, this is just a (small) piece of the theory of refined pull-backs developed by M.Levine-F.Morel (following W.Fulton [6]). But this piece is much more explicit than the general one and is sufficient for almost all applications we need. The exception is Subsection 4.3, where the refined pull-backs of more general type appear, and where we have to restrict to \( \text{theories of rational type} \) (= free theories of M.Levine-F.Morel) as a result. The formula is valid for arbitrary theory in the sense of Definition 2.1 (see the original version of the current text), but since our main statements are valid only for \( \text{theories of rational type} \), we formulate it only for constant theories and refer to the case of Algebraic Cobordism done by M.Levine and F.Morel (see [12, Theorem 6.6.6(2)(a)]).

We recall:
Definition 7.13 ([12, Definition 3.1.4]) Let $X$ be a smooth variety, and $D = \sum_{i=1}^{r} l_i D_i$ be an effective Weil divisor on $X$. We call $D$ a divisor with strict normal crossing, if for any $J \subset \{1, \ldots, r\}$, the intersection scheme $\cap_{i \in J} D_i$ is a smooth subvariety of $X$ of codimension $\#(J)$.

Denote as $|D| \xrightarrow{d} X$ the support $(\cup_{i=1}^{r} D_i)_{\text{red}}$. By $A_{*}(D)$ we will always mean $A_{*}(|D|)$. In particular, it does not depend on the multiplicity of the components as long as one is positive. Recall, that we have an exact sequence:

$$0 \leftarrow A_{*}(D) \leftarrow \oplus_{i} A_{*}(D_i) \leftarrow \oplus_{i \neq j} A_{*}(D_i \cap D_j).$$

Thus, an element of $A_{*}(D)$ can be thought of as a collection of elements of $A_{*}(D_i)$ modulo some identifications.

The strict normal crossing divisor has a divisor class $[D] \in A^0(D)$ such that $d_*([D]) = c_1^A(O(D)) \in A^1(X)$. Having $\lambda_i = c_1^A(O(D_i))$, the idea is to write $[l_1] : F_A \lambda_1 + F_A [l_2] : F_A \lambda_2 + \ldots + F_A [l_r] : F_A \lambda_r$ as $\sum_{I \subset \{1, \ldots, r\}} (\prod_{i \in I} \lambda_i) \cdot F_I^{l_1, \ldots, l_r}(\lambda_1, \ldots, \lambda_r)$, where $F_I^{l_1, \ldots, l_r}$ is some power series in $r$ variables with $A$-coefficients, and then define:

**Definition 7.14 ([12, Definition 3.1.5])**

$$[D] := \sum_{I \subset \{1, \ldots, r\}} (\hat{d}_I)_* (1) \cdot F_I^{l_1, \ldots, l_r}(\lambda_1, \ldots, \lambda_r),$$

where $\hat{d}_I : D_I = \cap_{i \in I} D_i \rightarrow |D|$ is the closed embedding.

The result does not depend on how you subdivide the above formal sum into pieces, but there is some standard way. The convention is (see [12, Subsection 3.1]) to define $F_I^{l_1, \ldots, l_r}$ as the sum of those monomials which are made exactly of $\lambda_i$, $i \in I$ divided by the $\prod_{i \in I} \lambda_i$. For our purposes, though, it will be convenient to be flexible in choosing $F_I^{l_1, \ldots, l_r}$, so below it will be any collection of power series satisfying the above equation.

**Definition 7.15** Having a divisor $D = \sum_{i=1}^{r} l_i D_i$ with strict normal crossing on $X$, we can define the pull-back:

$$d^* : A_{*}(X) \rightarrow A_{*-1}(D)$$

by the formula

$$d^*(x) = \sum_{I \subset \{1, \ldots, r\}} (\hat{d}_I)_* x \cdot F_I^{l_1, \ldots, l_r}(\lambda_1, \ldots, \lambda_r),$$

where $d_I : D_I \rightarrow X$ is the regular embedding of the $I$-th face of $D$.

Notice, that such a pull-back clearly depends on the multiplicity of the components (in our notations it is manifested only by the target). Also, since for $I \subset J$, for $d_{J/I} : D_J \rightarrow D_I$, we have: $(d_{J/I})_* (1) = \prod_{i \in J \setminus I} \lambda_i$, the projection formula shows that it does not matter, how one chooses the $F_I^{l_1, \ldots, l_r}$ (in particular, one can choose these to be zero for $\#(I) > 1$).

Immediately from the definition, we obtain:

**Lemma 7.16** The composition $d_* \circ d^* : A_{*}(X) \rightarrow A_{*-1}(X)$ is the multiplication by $c_1^A(O(D))$. 

Let $w : W \rightarrow X \times \mathbb{P}^1$ be a projective map, with $W$ smooth, such that $W_0 = w^{-1}(X \times 0) \xrightarrow{w_0} W$ and $W_1 = w^{-1}(X \times 1) \xrightarrow{w_1} W$ are divisors with strict normal crossing. Let $W_0 \xrightarrow{w_0} X$, $W_1 \xrightarrow{w_1} X$ be natural maps. As a corollary of Lemma 7.16 we get:
Proposition 7.17 In the above situation, \((i_0)_* \circ i_0^* = (i_1)_* \circ i_1^* \) in \(A_*(W)\). In particular, \((w_0)_* \circ i_0^* = (w_1)_* \circ i_1^* \)

Proof: Observe that \(\mathcal{O}_W(W_0) \cong \pi^*(\mathcal{O}_{\mathcal{F}_1}(1)) \cong \mathcal{O}_W(W_1)\).

Let

\[
\begin{array}{c}
E \xrightarrow{\phi} Y \\
\downarrow \quad \downarrow \\
D \xrightarrow{d} X.
\end{array}
\]

be a Cartesian square, where \(X\) and \(Y\) are smooth and \(D \xrightarrow{d} X\) and \(E \xrightarrow{\phi} Y\) are divisors with strict normal crossing (closed codimension 1 subschemes given by principal ideals whose \(\text{div}\) is a strict normal crossing divisor). Then we can define:

\[
\mathcal{T}^* : A^*(D) \to A^*(E)
\]

as follows. Suppose, \(D = \sum_{i=1}^r l_iD_i, E = \sum_{j=1}^s m_jE_j\), where \(D_i\) and \(E_j\) are irreducible components; \(\lambda_i = c^A(\mathcal{O}(D_i)), \mu_j = c^A(\mathcal{O}(E_j))\), and \(f^*(D_i) = \sum p_{i,j}E_j\). If \(P, L, M\) are matrices \((p_{i,j}), (l_i), (m_j)\), then we have: \(L \cdot P = M\). Notice, that if \(p_{i,j} \neq 0\), for some \(i\) and \(j\), then we have the natural map \(f_{j,i} : E_j \to D_i\), and so the map \(f_{j,i} : E_j \to D_i\), for any \(j \supset j\). Assume that \(F^{p_{i,1},\ldots,p_{i,s}}_j = 0\), if \(p_{i,j} = 0\), for some \(j \in J\) (notice, that there are no monomials divisible by \(\mu_j\) in the \(\sum_{j} F^A[p_{i,j}] \cdot F_A \mu_j\); so any "reasonable" choice will do).

Definition 7.18 Let \(x = \sum_{i}(\hat{e}_i)_*(x_i)\), for some \(x_i \in A^*(D_i)\). Define:

\[
\mathcal{T}^*(x) := \sum_{i=1}^r \sum_{J \subseteq \{1,\ldots,s\}} (\hat{e}_i)_* \sum_{J \subseteq \{1,\ldots,s\}} (\hat{e}_j)_* f_{j,i}^*(x_i) \cdot F^{p_{i,1},\ldots,p_{i,s}}(\mu_1, \ldots, \mu_s) \in A^*(E),
\]

where we ignore the terms with the zero \(F^{p_{i,1},\ldots,p_{i,s}}_j\).

Again, \(,\) since for \(I \subseteq J\), for \(e_{j/i} : E_j \to E_I\), we have: \((e_{j/i})_*(1) = \prod_{j \in J \setminus i} \mu_j\), the projection formula shows that it does not matter, how we choose the \(F^{p_{i,1},\ldots,p_{i,s}}_j\). Also, it is clear that we get a well-defined map in the case \(D \cdot \text{smooth irreducible (of multiplicity 1)}\). One can show that this map is well-defined in general (see the original version of this text), but we will spare the reader from that.

Finally, in the case of a free theory in the sense of Levine-Morel - see \cite{12} Remark 2.4.14 (by Proposition 4.4 these theories are exactly our theories of rational type), \(\Omega^*\) appears to be the same as a refined pull-back morphism.

Lemma 7.19 Let \(A^* = \Omega^* \otimes_L A\) be a theory obtained from Algebraic Cobordism by change of coefficients. Then for any square \(1\), we have:

\[
e^* = d^! \quad \text{and} \quad \mathcal{T}^* = f^!.
\]

Proof: The first identity follows from Lemma 6.6.2, Lemma 6.5.6, Definition 6.5.1, and definitions of Subsection 6.2.1 of \cite{12}. The second identity needs to be checked only for the case where \(D\) is a smooth divisor and \(f\) is a regular embedding, where, in the case of codimension 1, it follows from the first identity, and the general case follows from the deformation to the normal cone construction.

Proposition 7.20 (Multiple points excess intersection formula)

Let \(A^*\) be a theory satisfying \((\text{CONST})\). Then, in the above situation, we have:
(1) \( e_\ast \circ f^\ast = f^\ast \circ d_\ast. \)

(2) Suppose, \( f \) is projective. Then \( f^\ast \circ e = d^\ast \circ f_\ast. \)

**Proof:** If \( A^\ast = \Omega^\ast \otimes_{\mathbb{L}} A \) is obtained from Algebraic Cobordism by change of coefficients, this is a particular case of \([12, \text{Theorem 6.6.6(2)(a)}]\). The general case follows from Proposition \([12, \text{Proposition 4.12(1)}]\). \( \square \)

As another partial case of results of M.Levine-F.Morel we get the following functoriality statement. Suppose,

\[
\begin{array}{ccc}
D & \xrightarrow{d} & E \\
\downarrow e & & \downarrow f \\
X & \xrightarrow{u} & Y \\
\downarrow v & & \downarrow v \\
& & Z \\
\end{array}
\]

be the cartesian diagram, where \( X, Y, Z \) are smooth, and \( D, E \) and \( F \) are divisors with strict normal crossing.

**Proposition 7.21** Let \( A^\ast \) satisfies (CONST). Then, in the above situation,

\[
(u \circ v)^\ast = v^\ast \circ u^\ast.
\]

**Proof:** If \( A^\ast = \Omega^\ast \otimes_{\mathbb{L}} A \) is obtained from Algebraic Cobordism by change of coefficients, this is a particular case of \([12, \text{Theorem 6.6.6(3)}]\). The general case follows from Proposition \([12, \text{Proposition 4.12(1)}]\). \( \square \)

**Remark 7.22** Propositions \([7.20] \) and \([7.21] \) are valid for any theory in the sense of Definition \([2.1] \) - see the previous version of this text.

## 8 Resolution of singularities

In this section we list the results related to Resolution of Singularities and the Weak Factorization Theorem which are widely used throughout the text.

**Definition 8.1** Let \( X \) be a smooth variety and \( D \) - a divisor with strict normal crossing on it. By a permitted blow-up w.r.to \( D \) we will understand such a sequence of blow-ups with smooth centers \( R_i \subset X_i \):

\[
\widetilde{X} = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \ldots \xrightarrow{\pi_1} X_1 \xrightarrow{\pi_1} X
\]

such that, for the exceptional divisor \( E_i \) of \( \pi_i = \pi_1 \circ \ldots \circ \pi_i : X_i \rightarrow X \), and the total transform \( \pi_i^\ast(b) \),
the divisor \( E_i + (\pi_i)^\ast(b) \) has strict normal crossing, and \( R_i \) has normal crossing with it.

If \( D \) is empty, we will call it just a permitted blow-up.

**Theorem 8.2** (Hironaka, \([7]\)) Let \( Z \) be a subvariety of a smooth variety \( X \). Then there exists a permitted blow-up \( \pi : \widetilde{X} \rightarrow X \) such that:

1. All the centers \( R_i \) are lying over the singular locus of \( Z \).
2. The strict transform \( \widetilde{Z} \subset \widetilde{X} \) of \( Z \) is smooth and has normal crossing with \( E_n \).
Theorem 8.3 (Hironaka, [7]) Let \( f : X \to Y \) be a rational map of reduced varieties. Then there is a permitted blow-up \( \pi : \bar{X} \to X \) such that:

1. All the centers \( R_i \) are lying over the locus of \( X \) where it is not smooth, or \( f \) is not a morphism.
2. The rational map \( f \circ \pi : \bar{X} \to Y \) is a morphism.

Theorem 8.4 (Hironaka, [7], see also [1 1.2.3] and [2]) Let \( \mathcal{I} \) be a sheaf of ideals on a smooth variety \( X \), and \( U \subset X \) be an open subvariety such that \( \mathcal{I}|_U \) is an ideal sheaf of a divisor with strict normal crossing. Then there is a permitted blow-up \( \pi : \bar{X} \to X \) with centers outside \( U \) such that the total transform \( \pi^*(\mathcal{I}) \) is an ideal of a strict normal crossing divisor \( \bar{E} \).

There is also a relative to divisor \( D \) version (see [1 1.2.2] and [2]).

Proposition 8.5 Let \( X \) be smooth quasi-projective variety, \( Z \subset X \) - a closed subvariety, and \( D \) - a divisor with strict normal crossing on \( X \). Then there exists a permitted w.r.to \( D \) blow up \( X \) with centers over \( Z \) such that \( \pi^{-1}(Z) \cup \pi^{-1}(D) \) is a divisor with strict normal crossing.

The following result is the Weak Factorization Theorem - [1 Theorem 0.3.1], see also [29].

Theorem 8.6 (Abramovich-Karu-Matsuki-Wlodarczyk) Let \( \theta : X_1 \to X_2 \) be birational map of smooth proper varieties over \( k \), which is an isomorphism on the open set \( U \subset X_1 \). Then \( \theta \) can be factored into a sequence of blowings up and blowings down with nonsingular centers disjoint from \( U \). Namely, to any such \( \theta \) we can associate a diagram:

\[
X_1 = Y_0 \to \varphi_1 Y_1 \to \varphi_2 Y_2 \to \cdots \to \varphi_{i-1} Y_{i-1} \to \varphi_i Y_i \to \cdots \to \varphi_{i+1} Y_{i+1} \to \cdots \to \varphi_n Y_n = Y_2
\]

where

1. \( \theta = \varphi_l \circ \varphi_{l-1} \circ \cdots \circ \varphi_2 \circ \varphi_1 \),
2. \( \varphi_i \) are isomorphisms on \( U \), and
3. either \( \varphi_i \), or \( \varphi_i^{-1} \) is a blow up morphism with smooth center disjoint from \( U \).
4. Functoriality: if \( g : \theta \to \theta' \) is an absolute isomorphism carrying \( U \) to \( U' \), and \( \varphi'_i : Y'_l \to Y_i \) is the factorization of \( \varphi'_i \), then the resulting rational maps \( g_i : Y'_i \to Y'_i \) is an absolute isomorphism.
5. There is an index \( i_0 \) such that, for \( i \leq i_0 \), the map \( Y_i \to X_1 \) is projective map, while for \( i \geq i_0 \), \( Y_i \to X_2 \) is projective map.
6. Let \( E_i \subset Y_i \) be the exceptional divisor of \( Y_i \to X_1 \) (respectively, of \( Y_i \to X_2 \)) in case \( i \leq i_0 \) (respectively, \( i \geq i_0 \)). Then the above centers of blow up have normal crossing with \( E_i \). If, moreover, \( X_1 \setminus U \) (respectively, \( X_2 \setminus U \)) is a normal crossing divisor, then the centers of blow up have normal crossing with the inverse images of this divisor.

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