Alternative dynamics in loop quantum Brans-Dicke cosmology

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Abstract

To inherit more features of full loop quantum Brans-Dicke theory, the Euclidean and Lorentzian terms of the Hamiltonian constraint are quantized independently in loop quantum Brans-Dicke cosmology. An alternative Hamiltonian constraint operator and its effective expression are obtained in the cosmological model. A residual quantum correction term is found in the effective Hamiltonian constraint, which has no analog in the effective Hamiltonian of the loop quantum cosmology from general relativity. The dynamics driven by this effective Hamiltonian constraint is analyzed in details. For the physically interesting case of \( \omega \gg 1 \), this effective Hamiltonian drives a bouncing evolution which associates an asymptotical de Sitter universe to a classical Brans-Dicke universe.

1 Introduction

How to unify general relativity (GR) with quantum mechanics by a theory of quantum gravity is a great challenge to theoretical physics. As a non-perturbative approach to quantum gravity, loop quantum gravity (LQG) has made remarkable progress in the past thirty years [1–4]. According to LQG, spacetime consists of fundamental units of spacetime quanta since the spectra of the operators corresponding to the classical length, area and volume turned out to be discrete [5–10]. Despite these achievements, the dynamics of LQG is still an open issue, as the problem of how to suitably quantize and solve the Hamiltonian constraint is still unsolved. There are some attempts to quantize the Hamiltonian constraints [11–16], and some properties of the resulted operators are studied [17–20]. The problems in the full LQG theory motivate us to consider the symmetry-reduced models, such as the homogeneous and isotropic cosmology, on which the loop quantization method is applied [21–23]. The consequent quantum cosmology is call loop quantum cosmology (LQC).

By cosmological observation, it is reasonable to consider the possibility that GR is not a valid theory of gravity on a cosmological scale because of the ‘dark energy’ and ‘dark matter’ problems. Thus a large variety of modified gravity theories have been studying. Among these theories, a well-known one is the Brans-Dicke theory [24], which is apparently compatible with Mach’s principle. Loop quantization of this theory was studied in [25], where not only the kinematical Hilbert space but also the Hamiltonian constraint operator were constructed. However, similar to the situation in LQG, it is still difficult to solve the Hamiltonian constraint in the full loop quantum Brans-Dicke theory (LQBDT). Then, the symmetry-reduced model of loop quantum Brans-Dicke cosmology (LQBDC) was developed afterward [26, 27]. By solving the effective Hamiltonian constraint, one obtained a symmetric bouncing evolution of the universe such that the classical big bang singularity was avoided in the quantum theory.

It should be noted that the Hamiltonian constraint in full LQG consists of two terms: the so-called Euclidean term and Lorentzian term. These two terms were first regularized and quantized as operators in [11]. Classically the Lorentzian term is proportional to the Euclidean term in the spatially flat cosmological models. Thus one could combine the two terms into one term and then quantize it to obtain the Hamiltonian constraint operator in the cosmological models. In both LQC and LQBDC, this treatment leads to the symmetric bounce of the universe [21, 22, 26, 27]. Alternatively, the Lorentzian term could also be quantized independently in the cosmological models by using the Thiemann’s trick as in full LQG and full LQBDT. This idea was first realized in [28], where an alternative

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Hamiltonian constraint operator was obtained in LQC. Notably, the effective Hamiltonian of this alternative operator was lately confirmed by the semiclassical analysis of Thiemann’s Hamiltonian constraint operator in full LQG, which leads to an asymmetric bounce scenario in LQC [29, 30]. This result relates the flat FLRW cosmological spacetime with an asymptotic de Sitter spacetime. Thus an effective cosmological constant and an effective Newton constant were obtained in LQC [29, 30]. This ambiguity also exists in LQBDC. To inherit more features of LQBDT, in this paper we will deal with the Euclidean and the Lorentzian terms independently in LQBDC. It will be shown that the main features of the effective dynamics of the alternative Hamiltonian in LQC are tenable by that of LQBDC.

The paper is arranged as follows. In Sec. 2 the classical Brans-Dicke cosmology with the coupling parameter \( \Omega \neq 3/2 \) will be briefly reviewed, and then the kinematics of LQBDC will be introduced. In Sec. 3, the Hamiltonian constraint of the Brans-Dicke cosmological model will be quantized by using the strategy to treat the Euclidean and Lorentzian terms independently as in full LQBDT. In Sec. 4, the effective Hamiltonian constraint of the alternative Hamiltonian operator will be derived by the path-integral method in LQBDC. Then in Sec. 5, the effective dynamics driven by the effective Hamiltonian will be studied. Finally, the results will be summarized and discussed in Sec. 6.

2 Brans-Dicke cosmology and its loop quantization

In the connection formulation of Brans-Dicke theory, the phase space consists of canonical pairs of geometrical conjugate variables \( (A^i_a, E^j_b) \) and scalar conjugate variables \( (\phi, \Pi) \), where \( A^i_a \) is a SU(2) connection and \( E^j_b \) is the densitized triad on the spatial manifold \( M \). The non-vanished Poisson brackets between the canonical variables read

\[
\{ A^i_a(x), E^j_b(y) \} = \kappa \gamma \delta^i_a \delta^j_b \delta(x,y), \\
\{ \phi(x), \Pi(y) \} = \delta(x,y),
\]

where \( \kappa = 8\pi G \) with \( G \) the Newtonian gravitational constant and \( \gamma \) is the Barbero-Immirzi parameter. In the case of the coupling parameter \( \Omega \neq -3/2 \), the Hamiltonian constraint in Brans-Dicke theory reads [26]

\[
H = \frac{\phi}{2} \left( F_{ab} - (\gamma^2 + \frac{1}{\phi^2}) \epsilon_{jmn} K^m_a \tilde{K}^n_b \right) \frac{\epsilon_{ikl} E^i_b E^k_l}{\sqrt{q}} + \frac{1}{3 + 2\omega} \left( \frac{(K_a^i E^a_i)^2}{\phi \sqrt{q}} + 2\kappa \left( K_a^i E^a_i \Pi \right) \frac{\Pi}{\sqrt{q}} + \kappa^2 \Pi^2 \phi \frac{\omega}{2\phi} \sqrt{q(D^a \phi)} D_a \phi + \sqrt{q} D_a D^a \phi = 0, \right.
\]

(2)

where \( F_{ab} = 2\theta_{[a} A^i_b + \epsilon^i_{kl} A^k_a A^l_b \) is the curvature of the connection \( A^i_a \), \( K^i_a \) is defined in [26], and \( q \) is the determinant of physical 3-metric on \( M \).

We will restrict ourselves to spatially flat, homogeneous and isotropic cosmology with the symmetry of \( S = \mathbb{R}^3 \times \text{SO}(3) \). Then the spatial 3-manifold \( M \) is diffeomorphic to \( \mathbb{R}^3 \). As in the standard treatment of LQC, we first introduce an “elementary cubic cell” \( V \) on \( M \) and restrict all integrals to this cell. Fix a fiducial 3-metric \( \tilde{q}_{ab} \) and denote the volume of \( V \) measured with \( \tilde{q}_{ab} \) by \( V_0 \). Let \( \tilde{e}_i^a \) and \( \tilde{\omega}_a^i \) be the triad and co-triad adapted to \( V \) and satisfying \( \tilde{\omega}_a^i \tilde{e}_i^b = \delta_a^b \) and \( \tilde{q}_{ab} = \delta_{ij} \tilde{\omega}_a^i \tilde{\omega}_b^j \). By fixing the local diffeomorphism and internal gauge freedom, the basic variables are reduced to

\[
A_i^a = \sqrt{V_0}^{-1/3} \tilde{\omega}_a^i, \quad E_j^b = p V_0^{-2/3} \sqrt{q} \tilde{e}_j^b, \quad \Pi = V_0^{-1} \sqrt{q} \Pi \phi.
\]

(3)

The nontrivial Poisson brackets among reduced variables \( c, b, \phi \) and \( \pi_\phi \) read

\[
\{ c, p \} = \frac{\kappa \gamma}{3}, \quad \{ \phi, \pi_\phi \} = 1.
\]

(4)

The remaining Hamiltonian constraint [2] is reduced to

\[
H = -3c^2 \sqrt{|p|} \frac{\kappa \gamma}{\gamma^2 \phi} + \frac{1}{(3 + 2\omega) \phi} \left( \frac{3cp}{\gamma} + \kappa \pi_\phi \phi \right)^2 = 0.
\]

(5)

The kinematical Hilbert space \( \mathcal{H} \) of the LQBDC can be given by the direct product of the geometric sector \( \mathcal{H}_{\text{geo}} = L^2(\mathbb{R}_{\text{bohr}}, d\mu_H) \) [31], where \( \mathbb{R}_{\text{bohr}} \) is the Bohr compactification of \( \mathbb{R} \) and \( d\mu_{\text{Bohr}} \) is the Haar measure, and the scalar field sector \( \mathcal{H}_{\text{sca}} = L^2(\mathbb{R}, d\mu) \), which is the usual Schrodinger representation, i.e.,

\[
\mathcal{H} = L^2(\mathbb{R}_{\text{bohr}}, d\mu_H) \otimes L^2(\mathbb{R}, d\mu).
\]

(6)

In \( \mathcal{H}_{\text{sca}} \), one has the configuration operator \( \hat{\phi} \) defined as multiplication and the momentum operator \( \hat{\pi}_\phi := i\hbar d/d\phi \). The generalized eigenstates \( |\phi\rangle \) of \( \hat{\phi} \) contribute a generalized basis of \( \mathcal{H}_{\text{sca}} \). In \( \mathcal{H}_{\text{geo}} \), there are two fundamental
operators, namely the momentum operator \( \hat{p} \) which represents the area of each side of \( \mathcal{V} \) and the configuration operator \( \exp(i\lambda c) \) which represents the holonomy of the reduced connection \( c \) along an edge parallel to an edge of \( \mathcal{V} \). Since we will follow the improved scheme as in [22], it is convenient to introduce a new operator

\[
\hat{v} = \frac{\text{sgn}(\hat{p})|\hat{p}|^{3/2}}{2\pi \gamma e^2_p \sqrt{\Delta}},
\]

where \( \ell_p = \sqrt{\frac{\hbar}{G}} \) is the Planck length and \( \Delta = 4\sqrt{3}\pi \gamma e^2_p \) denotes the area gap in full LQBDT. Note that \( \hat{v} \) is actually a dimensionless variable representing the physical volume of \( \mathcal{V} \). The eigenstates \( |v\rangle \) of the operator \( \hat{v} \) are labelled by real numbers \( v \) and contribute an orthonormal basis in \( \mathcal{H}_{\text{geo}} \) such that

\[
\langle v'|v\rangle = \delta_{v,v'},
\]

where \( \delta_{v,v'} \) is the Kronecker delta. A general state in \( \mathcal{H}_{\text{geo}} \) can be expressed as a countable sum: \( |\psi\rangle = \sum_n \psi_n |v_n\rangle \) and thus the inner product reads

\[
\langle \psi^{(1)}|\psi^{(2)}\rangle = \sum_n \bar{\psi}_n^{(1)} \psi_n^{(2)}.
\]

It should be noted that the operator which measures the physical volume \( V \) of \( \mathcal{V} \) is given by

\[
\hat{V} = 2\pi \gamma e^2_p \sqrt{\Delta} |\hat{v}|.
\]

where \( |\hat{v}| \) is the absolute value of the operator \( \hat{v} \). One prefers to use the holonomy operator \( e^{ib/2} \), where \( b := \mu c \) with \( \mu = \sqrt{\Delta/|p|} \). Note that \( e^{ib/2} \) represents the holonomy \( h_i^{(\mu)} \) of \( c \) along an edge parallel to the triad \( \hat{e}_i \) whose length with respect to the physical metric is \( \sqrt{\Delta} \). Thus the edge underlying \( h_i^{(\mu)} \) takes the minimal length of the quantum geometry. The variables \( b \) and \( v \) are conjugate to each other, since

\[
\{b, v\} = \frac{2}{\hbar}.
\]

Hence one has

\[
e^{ib/2} |v\rangle = |v + 1\rangle.
\]

Actually, the holonomy operator \( h_i^{(\mu)} \) can be expressed as

\[
\hat{h}_i^{(\mu)} = \frac{1}{2} \left( e^{ib/2} + e^{-ib/2} \right) - i \left( e^{ib/2} - e^{-ib/2} \right) \tau_i,
\]

where \( \tau_i \) are the generators of Lie algebra \( su(2) \) [22].

### 3 Alternative Hamiltonian constraint operator

In the homogeneous cosmological model, the Hamiltonian constraint [2] can be written as

\[
H = \frac{\phi}{2} \left( F_{ab} - (\gamma^2 + 1) \epsilon_{jmn} \hat{K}_a^m \hat{K}_b^n \right) + \frac{1}{2} \epsilon_{jmn} \frac{\hat{K}_a^m \hat{K}_b^n}{\sqrt{\bar{q}}} + \frac{1}{3} \frac{1}{2\omega} \left( \frac{K_{a}^{i} E_{i}^{a}}{\phi \sqrt{\bar{q}}} + 2\kappa \left( K_{a}^{i} E_{i}^{a} \right) \Pi_{\sqrt{\bar{q}}} + \kappa^2 \Pi_{\sqrt{\bar{q}}} \right) = 0.
\]

Similar to the case of full LQBDT, there is no operator corresponding to the connection \( A_{ij}^a(x) \) in LQBDC. Hence, one has to express the curvature \( F_{ab}^j \) in [11] by holonomies. This can be accomplished by using the Thiemann’s tricks [3]. Classically the curvature in our cosmological model can be regularized on the elementary cell as [22]

\[
F_{ab}^k = \lim_{\lambda \to 0} \text{Tr} \left( -\frac{h_i^{(\lambda)} K_{a}^{i}}{\lambda^2 V_{0}^{1/3}} \right) \tilde{\omega}_a^i \tilde{\omega}_b^i,
\]

where \( h_i^{(\lambda)} = h_i^{(\lambda)} h_j^{(\lambda)} (h_j^{(\lambda)})^{-1} (h_{j'}^{(\lambda)})^{-1} \) is the holonomy around the loop formed by the two edges of \( \mathcal{V} \) that are tangent to \( e_i^{(\lambda)} \) and \( e_j^{(\lambda)} \) whose length is \( \lambda V_{0}^{1/3} \) with respect to the fiducial metric \( q_{ab} \) respectively. To quantize the Hamiltonian constraint, we also need to use the regularizations

\[
\frac{\epsilon_{ijk} E_{k}^{c} E_{i}^{b}}{\sqrt{\det(q)}} = \lim_{\lambda \to 0} \frac{2\text{sgn}(p)\text{Tr}(h_m^{(\lambda)} (h_m^{(\lambda)})^{-1}, V \tau^i)}{\kappa^4 \lambda^2 V_{0}^{1/3}} \tilde{\omega}_a^m \epsilon_{abc},
\]

\[
\frac{2\text{sgn}(p)\text{Tr}(h_m^{(\lambda)} (h_m^{(\lambda)})^{-1}, V \tau^i)}{\kappa^4 \lambda^2 V_{0}^{1/3}} \tilde{\omega}_a^m \epsilon_{abc},
\]

\[
\frac{2\text{sgn}(p)\text{Tr}(h_m^{(\lambda)} (h_m^{(\lambda)})^{-1}, V \tau^i)}{\kappa^4 \lambda^2 V_{0}^{1/3}} \tilde{\omega}_a^m \epsilon_{abc},
\]
and
\[ \tilde{K}^i_a(x) = \frac{1}{2\gamma(\kappa\gamma)^2} \{ A^i_a(x), \{ C, V \} \}, \]  
where \( \text{sgn}(p) \) denotes the sign of \( p \) and \( C = \int d^3x \epsilon^{ijk} F^a_{\mu
u}(x) E_j^a(x) E_k^a(x) / \sqrt{g(x)} \). The integration of the Hamiltonian \( H^\lambda \) reads
\[ \mathcal{C} = \int V d^3x H(x) = \lim_{\lambda \to 0} H^{(\lambda)} \]
where
\[ H^{(\lambda)} = -\frac{\text{sgn}(p)}{2\pi G \gamma \lambda^3} \text{Tr}(h^{(\lambda)} \tau^i) \text{Tr}(h^{(\lambda)} \{ (h^{(\lambda)}_m)^{-1}, \{ C, V \} \} \tau_i) e^{kjm} \]
\[ + \frac{\text{sgn}(p)}{\gamma^2(8\pi G \gamma)^3 \lambda^3} (\gamma^2 + \frac{1}{\phi^2}) e^{ijk} \text{Tr}(h_i^{(\lambda)} \{ h_j^{(\lambda)}, \{ C, V \} \} h_j^{(\lambda)} \{ h_k^{(\lambda)}, \{ C, V \} \} \{ h_k^{(\lambda)}, V \} ) \]
\[ + \frac{1}{2\omega + 3} \left( \frac{(\{ C, V \})^2}{4\gamma^2(\kappa\gamma)^2 \phi V} + \frac{1}{4\gamma^2} \pi \phi \right). \]
However, the family of operators \( \hat{H}^{(\lambda)} \) do not converge as \( \lambda \to 0 \). Thus, in the so-called \( \hat{\mu} \)-scheme, one fixed the length \( \lambda \) of the edge underlying the holonomies in the Hamiltonian to \( \hat{\mu} = \sqrt{\Delta / p} \), which implies that the curvature is smeared over the elementary faces with the physical area \( A_{\Box} = \Delta \). By this treatment, we obtain the Hamiltonian constraint operator as
\[ \mathcal{C} = \lim_{\lambda \to \hat{\mu}} \hat{H}^{(\lambda)}. \]
It should be noted that classically one has
\[ \lim_{\lambda \to \hat{\mu}} \{ h^{(\lambda)}, \tilde{K} \} = \frac{2}{3} \{ h^\mu, \tilde{K} \}, \]
where \( \tilde{K} = \int d^3x \tilde{K}^a_a E_a \). Hence, in the expression of \( \lim_{\lambda \to \hat{\mu}} \), the commutator \( \{ \tilde{h}^{(\lambda)}, \tilde{K} \} \) would be replaced by \( \frac{2}{3} [\tilde{h}^{\mu}, \tilde{K}] \). It is convenient to split the expression of \( \mathcal{C} \) into three parts as \( \mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3 \). Their actions on the basis \( |v, \phi\rangle = |v\rangle \otimes |\phi\rangle \) of \( \mathcal{H} \) are given by:
\[ \mathcal{C}_1 |v, \phi\rangle = \phi \sin(b) \hat{A} \sin(b) |v, \phi\rangle \]
\[ - \frac{1}{8} \alpha \phi (f_+(v)|v + 4, \phi\rangle + f_0(v)|v, \phi\rangle + f_-(v)|v - 4, \phi\rangle) \]
with \( \hat{A} = -i\hat{v} \left( \sin \frac{\beta}{2} |\phi\rangle \cos \frac{\beta}{2} |\phi\rangle \cos \frac{\beta}{2} |\phi\rangle \sin \frac{\beta}{2} \right) \), \( \alpha = 6\pi \gamma E_\mu / \sqrt{\Delta} \), \( f_+(v) = (v + 2)(|v + 1| - |v + 3|) \), \( f_-(v) = f_+(v - 4) \), and \( f_0(v) = -f_+(v) - f_-(v) \),
\[ \mathcal{C}_2 |v, \phi\rangle = \frac{\alpha}{256\gamma^2} \phi (\gamma^2 + \frac{1}{\phi^2}) \hat{\beta} \hat{\beta} \hat{\beta} |v, \phi\rangle \]
\[ = - \frac{\alpha}{16^3 \times 2\gamma^2} \phi (\gamma^2 + \frac{1}{\phi^2}) \left( g_\Delta^\Delta(v) A(v + 4) g_\Delta^\Delta(v + 4) |v + 8, \phi\rangle \right) \]
\[ - (g_\Delta^\Delta(v) A(v + 4) g_\Delta^\Delta(v + 4) + g_\Delta^\Delta(v - 4) A(v - 4) g_\Delta^\Delta(v)) |v, \phi\rangle \]
\[ + g_\Delta^\Delta(v - 4) A(v - 4) g_\Delta^\Delta(v) |v - 8, \phi\rangle \]
with \( \hat{A} = 2 \left( \sin \frac{\beta}{2} |\phi\rangle \cos \frac{\beta}{2} |\phi\rangle \cos \frac{\beta}{2} |\phi\rangle \sin \frac{\beta}{2} \right) \), \( \hat{\beta} = 2 \sin(b) \hat{A} \sin(b) \), \( g_+(v) := (f_+(v) |v - |v + 4|) \), \( g_-(v) := f_-(v) (|v - 4| - |v|) \) and \( g_\Delta^\Delta(v) := g_+(v + 1) - g_\Delta^\Delta(v - 1) \), and
\[ \mathcal{C}_3 |v, \phi\rangle = \frac{\alpha}{3 \times 2\omega + 3} \phi \left( \frac{-3|\hat{c}, |\phi\rangle|^2}{64\gamma^2\phi} + 3 |\hat{c}, |\phi\rangle |\hat{\pi}_\phi + \kappa^2 \frac{3}{4\alpha^2 \Delta} (\hat{\pi}_{\phi}^2 + \phi \hat{\pi}_{\phi}^2) \right) \sqrt{|v - 1|} |v, \phi\rangle \]
\[ = (\sqrt{3} \frac{3\alpha}{32\gamma}) \phi \left( \frac{g_+(v) g_+(v + 4) + g_+(v) g_-(v + 4) + g_-(v) g_-(v - 4)}{\sqrt{|v + 8|}} |v + 8, \phi\rangle \right) \]
\[ + \frac{g_-(v) g_-(v - 4)}{\sqrt{|v - 8|}} |v - 8, \phi\rangle \right) + \kappa \frac{3}{16\gamma \sqrt{\Delta}} \phi \left( \frac{g_+(v) g_+(v + 4)}{\sqrt{|v + 4|}} |v + 4, \phi\rangle \right) \]
\[ - \frac{g_-(v) g_-(v - 4)}{\sqrt{|v - 4|}} |v - 4, \phi\rangle \right) + \kappa^2 \frac{3}{2\alpha^2 \Delta} \phi \left( \frac{\hat{\pi}_{\phi}^2 + \phi \hat{\pi}_{\phi}^2}{|v - 1|} \right) |v, \phi\rangle. \]
where $v^{-1}$ is defined by $v^{-1}|v\rangle = v^{-1}|v\rangle$ if $v \neq 0$, and $v^{-1}|v\rangle = 0$ if $v = 0$. 

### 4 The effective Hamiltonian constraint

To get an effective Hamiltonian constraint, we calculate the transition amplitude of the Hamiltonian constraint operator $\hat{A}$ as

$$A(v_f, \phi_f; v_i, \phi_i) = \langle v_f, \phi_f | v_i, \phi_i \rangle_{\text{phy}} = \lim_{\alpha \to \infty} \int_{-\alpha}^\alpha d\alpha \langle v_f \phi_f | e^{i\alpha \hat{A}} | v_i, \phi_i \rangle. \tag{22}$$

Dividing the path into $N$ parts by setting $\alpha = \sum_{n=1}^{N} \epsilon_n$ and inserting the basis, we have

$$\langle v_f, \phi_f | e^{i\alpha \hat{A}} | v_i, \phi_i \rangle = \sum_{v_{N-1}, \ldots, v_1} \int d\phi_{N-1} \cdots d\phi_1 \prod_{n=1}^{N} \langle \phi_n, v_n | e^{i\epsilon_n \hat{A}} | v_{n+1}, \phi_{n+1} \rangle, \tag{23}$$

where $\langle \phi_n, v_n | e^{i\epsilon_n \hat{A}} | v_{n+1}, \phi_{n+1} \rangle$ can be calculated by using the formula

$$\int d\phi_n \langle \phi_n, v_n | e^{i\epsilon_n \hat{A}} | v_{n+1}, \phi_{n+1} \rangle = \delta_{v_n, v_{n+1}} - i\epsilon_n \int d\phi_n \langle \phi_n, v_n | (\hat{A}_1 + \hat{A}_2 + \hat{A}_3) | v_{n+1}, \phi_{n+1} \rangle. \tag{24}$$

By Eqs. (19), (20), and (21), we obtain

$$\int d\phi_n \langle \phi_n, v_n | \hat{A}_1 | v_{n+1}, \phi_{n+1} \rangle = -\frac{1}{2\pi \hbar} \int d\phi_n \int d\pi_n e^{i\frac{3\pi}{8} (\phi_n - \phi_{n+1})} \phi_n (v_n + v_{n+1}) (\delta_{v_n, v_{n+1}+4} - 2\delta_{v_n, v_{n+1}} + \delta_{v_n, v_{n+1}-4}),$$

and

$$\int d\phi_n \langle \phi_n, v_n | \hat{A}_2 | v_{n+1}, \phi_{n+1} \rangle = \int d\phi_n \int d\pi_n e^{i\frac{3\pi}{8} (\phi_n - \phi_{n+1})} \phi_n (\gamma^2 + \frac{1}{\phi_n}) (v_n + v_{n+1}) (\delta_{v_n, v_{n+1}+8} - 2\delta_{v_n, v_{n+1}} + \delta_{v_n, v_{n+1}-8}),$$

and

$$\int d\phi_n \langle \phi_n, v_n | \hat{A}_3 | v_{n+1}, \phi_{n+1} \rangle = \int d\phi_n \int d\pi_n e^{i\frac{3\pi}{8} (\phi_n - \phi_{n+1})} \phi_n \left( \frac{4}{\sqrt{v_n v_{n+1}}} \left( \delta_{v_n, v_{n+1}+8} - 2\delta_{v_n, v_{n+1}} + \delta_{v_n, v_{n+1}-8} \right) \right), \tag{25}$$

Combining these equations and the formulas

$$\delta_{v_n, v_{n+1}+4} - 2\delta_{v_n, v_{n+1}} + \delta_{v_n, v_{n+1}-4} = -\frac{1}{\pi} \int_0^\pi d\theta_n 4e^{-ib_n (v_n - v_{n+1})} \sin^2(b_n),$$

$$\delta_{v_n, v_{n+1}+4} - \delta_{v_n, v_{n+1}-4} = \frac{i}{\pi} \int_0^\pi d\theta_n 2e^{-ib_n (v_n - v_{n+1})} \sin(2b_n),$$

$$\delta_{v_n, v_{n+1}} = \frac{1}{\pi} \int_0^\pi d\theta_n e^{-i\frac{1}{2}b_n (v_n - v_{n+1})},$$

5
we get

\[
\langle \phi_n^e \alpha \phi_n^e | \phi_{n+1}, v_{n+1} \rangle = \frac{1}{2\pi \hbar} \int d\pi_n e^{i\frac{\pi}{\phi_n^e}(\phi_n^e - \phi_{n-1})} \left( \frac{\alpha}{8} \phi_n^e (v_n + v_{n+1})^4 \sin^2(b_n) - \frac{\alpha}{32\gamma^2} \phi_n^e (\gamma^2 + \frac{1}{\phi_n^e}) (v_n + v_{n+1})^4 \sin^2(2b_n) \right) + \frac{1}{3 + 2\omega} \left( \frac{\sqrt{3}}{4\gamma} \frac{\alpha}{\phi_n^e} (v_n v_{n+1} + \frac{4}{v_n v_{n+1}}) \right) 4 \sin^2(2b_n) - \left( \frac{\sqrt{3}}{4\gamma} \right) \frac{\alpha}{\phi_n^e} \frac{8}{v_n v_{n+1}} 2 \cos(4b_n) + \kappa \frac{3}{4\gamma \sqrt{\Delta}} \pi_n \frac{v_n + v_{n+1}}{v_n v_{n+1}} \sin(2b_n) + \kappa^2 \frac{3}{2\omega} \left( \phi_n^e + \phi_{n+1} \right) \frac{\pi_n^2}{v_n v_{n+1}} \right).
\]

Hence the transition amplitude \[22\] can be expressed as

\[
A(v_f, \phi_f; v_i, \phi_i) = \lim_{\alpha \to 0} \lim_{N \to \infty} \sum_{\phi_{N-1}, \ldots, \phi_1} \int d\phi_{N-1} \cdots d\phi_1 \prod_{n=1}^N \langle \phi_n, v_n | e^{-i\epsilon_n C} | \phi_{n-1}, v_{n-1} \rangle \int D\alpha \int D\phi \int D\pi \int Db \int Dv \exp \left\{ \frac{i}{\hbar} \int \left( \pi^2 - \frac{h}{2} b^2 + \frac{h}{2} \phi_n \sin^2(b) - \frac{\alpha}{4\gamma^2} \phi_n \sin^2(2b) \right) \right. - \frac{1}{3 + 2\omega} \left( \frac{\sqrt{3}}{2\gamma} \frac{1}{\phi_n^e} (v^2 + \frac{4}{v}) \sin^2(2b) - \frac{3\kappa}{\sqrt{\Delta}} \phi_n \sin(2b) + \kappa^2 \frac{3}{2\omega} \phi_n^e \frac{1}{v} \right) \right\}.
\]

Therefore, the effective Hamiltonian constraint can be read from Eq. \[28\] as

\[
H_{\text{eff}} = -\alpha \phi v \sin^2(b) + \frac{\alpha}{4\gamma^2} \phi (\gamma^2 + \frac{1}{\phi^e}) \sin^2(2b) + \frac{1}{3 + 2\omega} \frac{\alpha}{\phi v} \left( \frac{\sqrt{3}}{\gamma} \sin(2b) + \frac{\sqrt{3} \kappa}{\alpha \sqrt{\Delta}} \phi \pi_\phi \right)^2 - \frac{3\alpha}{3 + 2\omega} \frac{1}{\gamma^2 v^2} \sin^2(2b).
\]

In the limit \(b \to 0\), we have

\[
H_{\text{eff}} = -\frac{\alpha}{\phi \gamma^2} v b^2 + \frac{1}{3 + 2\omega} \left( \frac{\sqrt{3}}{\gamma} v b + \frac{\sqrt{3} \kappa}{\alpha \sqrt{\Delta}} \phi \pi_\phi \right)^2 - \frac{3\alpha}{3 + 2\omega} \frac{1}{\gamma^2 v^2} (1 - 12 b^2).
\]

Eq. \[30\] is different from the classical Brans-Dicke Hamiltonian constraint \[5\] by the residual term \(\frac{3\alpha}{3 + 2\omega} \frac{1}{\gamma^2 v^2} (1 - 12 b^2)\). In order to compare this term with the others, it is convenient to introduce a new variable

\[
B = \frac{b}{4\pi G \gamma \sqrt{\Delta}}.
\]

which is canonically conjugate to the physical volume \(V\) of the elementary cell \(V\) as

\[
\{ B, V \} = 1.
\]

Then Eq. \[30\] can be re-expressed in terms of \(B\) and \(V\) as

\[
H_{\text{eff}} = -\frac{3\kappa^2}{4\phi} V B^2 + \frac{\kappa}{3 + 2\omega} \frac{1}{\phi V} \left( \frac{3}{2} B V + \pi_\phi \phi \right)^2 - \frac{h^2}{3 + 2\omega} \frac{9\kappa^2}{16V} \left( 1 - 3\kappa^2 \phi^2 \Delta B^2 \right).
\]

It is obvious from Eq. \[33\] that the residual term in \[30\] is of order \(h^2\), which is certainly a quantum correction. By checking the derivation procedure of the effective Hamiltonian, one can find that the residual term comes from the effect of \(\hat{c}, \hat{\epsilon}\) in \(\hat{c}_3\). Thus this is a particular term existing in the effective theory of LQBDC, since there is no square term of a commutator in the expression of the Hamiltonian constraint operator in the usual LQC. For semiclassical consideration, one may get rid of this term and obtain the following effective Hamiltonian constraint

\[
H_{\text{eff}} = -\alpha \phi v \sin^4(b) + \frac{\alpha}{4\gamma^2} \phi (\gamma^2 + \frac{1}{\phi^e}) \sin^2(2b) + \frac{1}{3 + 2\omega} \left( \frac{\sqrt{3}}{2\gamma} v \sin(2b) + \frac{\sqrt{3} \kappa}{\alpha \sqrt{\Delta}} \phi \pi_\phi \right)^2.
\]

As we will show in next section, the dynamics driven by this effective Hamiltonian can be obtained analytically.
5 The effective dynamics

To simplify the calculation of the dynamics determined by the effective Hamiltonian (34), we choose a lapse function $N = \nu \phi/\alpha$, such that the effective Hamiltonian constraint can be re-expressed as

$$C = NH_{\text{eff}} = \phi^2 v^2 \sin^4(\beta) - \frac{1}{4\gamma^2} v^2 \sin^2(2\beta) + \frac{1}{3 + 2\omega} \left( \frac{\sqrt{3}}{2\gamma} v \sin(2\beta) + \frac{\sqrt{3\kappa}}{\alpha \sqrt{\Delta}} \phi \pi \right)^2 = 0. \quad (35)$$

Let $X = \nu \sin(2\beta)$, $Y = \phi \pi \phi$ and $Z = \phi \nu \sin^2(\beta)$. Then two constants of motion with respect to $C$ can be obtained as

$$\xi_1 = hX/4 - Y,$$
$$\xi_2 = Z^2 + AY^2 + BXY,$$

where

$$A = \frac{8(3\omega + 2)}{3\gamma^2(2\omega + 3)\hbar^2},$$
$$B = -\frac{4(\omega - 1)}{\gamma^2(2\omega + 3)\hbar}. \quad (37)$$

However, the constraint (35) implies that $\xi_1$ and $\xi_2$ are related by

$$\xi_2 = \frac{8\omega}{\gamma^2(2\omega + 3)\hbar^2} \xi_1^2. \quad (38)$$

The equations of motion for $X$, $Y$ and $Z$ can be easily derived by using the Hamilton’s equations with the Hamiltonian $C$, which, together with the Hamiltonian constraint (35), leads to

$$\dot{Y} = -2Z^2,$$
$$Z^2 = -(A + \frac{4B}{\hbar})Y^2 - \frac{4B}{\hbar} \xi_1 Y + \xi_2 =: aY^2 + bY + c,$$

where we defined

$$a = \frac{8(3\omega - 8)}{3\gamma^2(2\omega + 3)\hbar^2},$$
$$b^2 - 4ac = \frac{256\xi_1^2}{3\gamma^4(2\omega + 3)\hbar^4}. \quad (40)$$

Thus the types of the solutions $Y(t)$ depend on the sign of $b^2 - 4ac$. For $b^2 - 4ac < 0$, $Y(t)$ takes the form of a tangent function, while for $b^2 - 4ac > 0$, it takes the form of a hyperbolic tangent function. We are interested in the case with the coupling parameter $\omega \gg 1$, which coincides with the solar system experiments [33, 34]. In this case Eq. (40) ensures that $b^2 - 4ac > 0$. Then Eq. (39) gives

$$\dot{Y} = -2a(Y - y_1)(Y - y_2), \quad (41)$$

where $y_1$ and $y_2$ ($y_1 > y_2$) are the roots of the equation $aY^2 + bY + c = 0$. Thus the solutions to Eq. (41) can be obtained as

$$Y_{\pm}(t) = y_1 + \frac{y_2 - y_1}{1 + e^{2a(y_1 - y_2)t}}. \quad (42)$$

Taking account of the fact that $Z^2 = a(Y - y_1)(Y - y_2) \geq 0$, we conclude the following two cases.

(i) For $a > 0$, i.e., $\omega > 8/3$, the solution is

$$Y_{-}(t) = \frac{3(\omega - 1) - \sqrt{3(2\omega + 3)}}{8 - 3\omega} \xi_1 + \frac{2\sqrt{3(2\omega + 3)}}{8 - 3\omega} \xi_1 \left( 1 - e^{-\frac{32\xi_1 t}{\sqrt{3}(2\omega + 3)}} \right)^{-1}. \quad (43)$$

(ii) For $a < 0$, i.e., $-3/2 < \omega < 8/3$, the solution is

$$Y_{+}(t) = \frac{3(\omega - 1) - \sqrt{3(2\omega + 3)}}{8 - 3\omega} \xi_1 + \frac{2\sqrt{3(2\omega + 3)}}{8 - 3\omega} \xi_1 \left( 1 + e^{-\frac{32\xi_1 t}{\sqrt{3}(2\omega + 3)}} \right)^{-1}. \quad (44)$$
By Eq. (39) we can obtain the expression of \( Z_{\pm}(t) \) corresponding to \( Y_{\pm}(t) \) as

\[
Z_-(t) = \frac{2\sqrt{2}\xi_1}{\hbar \gamma \sqrt{3\omega - 8}} \left| \sinh \left( \frac{16\xi_1}{\gamma^2 h^2 \sqrt{3(2\omega + 3)}} t \right) \right|^{1/5}, \\
Z_+(t) = \frac{2\sqrt{2}\xi_1}{\hbar \gamma \sqrt{8 - 3\omega}} \left| \cosh \left( \frac{16\xi_1}{\gamma^2 h^2 \sqrt{3(2\omega + 3)}} t \right) \right|^{1/5}.
\]

(45)

The equation of motion for \( \phi \), which can be derived by the Hamilton’s equation as well as the Hamiltonian constraint \([45]\), reads

\[
\dot{\phi}_\pm = \frac{16\phi_\pm}{3\gamma^2(2\omega + 3)h^2}(5Y_\pm + 3\xi_1).
\]

(46)

The solutions of Eq. (46) can be obtained as

\[
\phi_-(t) = \phi_0 2^{5/8} e^{-\frac{16\xi_1 t}{5\gamma^2(2\omega - 8)}} \left| \sinh \left( \frac{16\xi_1 t}{\sqrt{3(2\omega + 3)\gamma^2 h^2}} \right) \right|^{\frac{5}{\sqrt{\omega - 8}}}, \\
\phi_+(t) = \phi_0 2^{5/8} e^{-\frac{16\xi_1 t}{5\gamma^2(2\omega - 8)}} \left| \cosh \left( \frac{16\xi_1 t}{\sqrt{3(2\omega + 3)\gamma^2 h^2}} \right) \right|^{\frac{5}{\sqrt{\omega - 8}}},
\]

(47)

where \( \phi_0 \) is an integration constant. The dynamical evolution of \( v \) and \( b \) can be obtained by using the functions \( X \), \( Y \), \( Z \) and \( \phi \) as

\[
v = \frac{\phi X^2}{4Z} + \frac{Z}{\phi}, \quad \sin(2b) = \frac{X}{v}, \quad \cos(2b) = 1 - \frac{2Z}{v\phi}.
\]

(48)

It should be noted that in the solutions obtained so far we adopted the coordinate time \( t \) corresponding to the lapse function in Eq. (35). However, the Hubble parameter is defined with respect to the cosmological proper time \( \tau \), which is related to the coordinate time by \( d\tau = 8\pi G N dt \). By denoting \( \dot{\iota} := dv/dt \), the Hubble parameter \( H \) can be expressed as

\[
H = \frac{\alpha \dot{\iota}}{24\pi G v^2 \phi} = \frac{4\alpha \dot{\phi}^2 X (\phi X Z + 2Z\phi X - \phi X\dot{Z}) + 16\alpha Z^2 (\dot{Z}\phi - Z\dot{\phi})}{24\pi G \phi (\dot{\phi}^2 X^2 + 4Z^2)^2}.
\]

(49)

Taking account of the solar system experiments, we consider the case \( \omega > 8/3 \). In this case, the dynamics is described by the functions \( Y_-(t) \), \( Z_-(t) \) and \( \phi_-(t) \). Since the functions \( Y_-(t) \) and \( Z_-(t) \) are ill-defined at \( t = 0 \), they are valid in the domain \( t \in (-\infty, 0) \cup (0, \infty) \), so is the lapse function \( N = \dot{\phi}v/\alpha \). Because \( N \) dose not vanish in this domain, as a time coordinate \( t \) is well defined in either the branch \( (-\infty, 0) \) or \( (0, \infty) \). Moreover, for a given \( t_0 > 0 \), the integrals \( \int_{\pm t_0}^{\pm \infty} N(t) dt \) and \( \int_{\pm t_0}^{\pm \infty} N(t) dt \) diverge. Hence the cosmological time \( \tau \) ranges over \( (-\infty, \infty) \) in either the branch of domain of \( t \). Thus we can choose one of the branches, say \( t \in (0, \infty) \), to cover the whole spacetime. Thanks to the divergence of the integrals, the hypersurfaces of \( t = 0 \) and \( t = \infty \) are actually the past and future timelike infinities respectively. Furthermore, the effective dynamics will return to the classical one for \( v \gg 1 \). This happens in the classical regions of \( \frac{1}{t} \ll 1 \) and \( t \gg 1 \) respectively.

Now we consider the dynamical behavior of the universe with \( t \in (0, \infty) \). As \( t \to 0 \), the leading terms of the functions \( Y_-, Z_-, X_-, \) and \( \phi_- \) read respectively

\[
Y_-(t) \cong \frac{3\gamma^2 h^2(2\omega + 3)}{16(3\omega - 8)} \frac{1}{t}, \\
Z_-(t) \cong \frac{\gamma h \sqrt{6(2\omega + 3)}}{8\sqrt{3\omega - 8}} \frac{1}{|t|}, \\
X_-(t) \cong -\frac{3\gamma^2 h(2\omega + 3)}{4(3\omega - 8)} \frac{1}{t}, \\
\phi_-(t) \cong \phi_0 \frac{32\xi_1}{\sqrt{3(2\omega + 3)\gamma^2 h^2}} t^{5/(3\omega - 8)}.
\]

(50)
Thus, their derivatives with respect to $t$ are respectively

$$\dot{Y}_-(t) \equiv -\frac{1}{t} Y_-(t), \quad \dot{Z}_-(t) \equiv -\frac{1}{t} Z_-(t),$$

$$\dot{X}_-(t) \equiv -\frac{1}{t} X_-(t), \quad \dot{\phi}_-(t) \equiv \frac{5}{3\omega - 8} \frac{1}{t} \phi_-(t).$$

Hence, as $t \to 0$, by Eq. (49), the Hubble parameter approaches

$$H \equiv -\frac{8\alpha(\omega - 1)}{8\pi\gamma\ell^2} \sqrt{6(3\omega - 8)(2\omega + 3)} < 0.$$  \hspace{1cm} (52)

Let us consider the other side. As $t \to \infty$, the leading terms of those functions become respectively

$$Y_-(t) \equiv \frac{3(\omega - 1) + \text{sgn}(t\xi_1)\sqrt{3(2\omega + 3)}}{8 - 3\omega} \xi_1,$$

$$X_-(t) \equiv \frac{4 \left(5 + \text{sgn}(t\xi_1)\sqrt{3(2\omega + 3)}\right)}{(3\omega - 8)\hbar} \xi_1,$$

$$Z_-(t) \equiv \frac{2\sqrt{2}|\xi_1|}{\hbar\gamma\sqrt{3\omega - 8}} \exp \left(\frac{-16|\xi_1 t|}{\gamma^2\hbar^2 \sqrt{3(2\omega + 3)}}\right),$$

$$\dot{\phi}_-(t) \equiv \exp \left(\frac{16\xi_1 t}{\gamma^2\hbar^2(3\omega - 8)} \left(\frac{5\text{sgn}(t\xi_1)}{\sqrt{3(2\omega + 3)}} - 1\right)\right).$$

Then their time derivatives are respectively

$$\dot{Y}_-(t) \equiv 0,$$

$$\dot{X}_-(t) \equiv 0,$$

$$\dot{Z}_-(t) \equiv -\text{sgn}(t\xi_1) \frac{16\xi_1}{\gamma^2\hbar^2 \sqrt{3(2\omega + 3)}} Z_-(t),$$

$$\dot{\phi}_-(t) \equiv \frac{16\xi_1}{\gamma^2\hbar^2(3\omega - 8)} \left(\frac{5\text{sgn}(t\xi_1)}{\sqrt{3(2\omega + 3)}} - 1\right) \phi_-(t).$$

Hence the asymptotic behavior of the Hubble parameter for $t \to \infty$ reads

$$H \equiv \lim_{t \to \infty} \frac{256\alpha\xi^2 e^{-\frac{16\xi_1 t}{\gamma^2\hbar^2 \sqrt{3(2\omega + 3)}}}}{24\pi G\gamma^3\hbar^3 \sqrt{3\omega - 8} \sqrt{3\omega + \frac{9}{2}}} = 0.$$ \hspace{1cm} (55)

Eqs. (52) and (55) imply that there exists at least one moment $t_0 \in (0, \infty)$ such that $H(t_0) = 0$. Hence a bounce of the universe may happen at $t = t_0$. On one side, the negative Hubble constant around $t = 0^+$ implies that the universe goes through an asymptotical de Sitter epoch there. On the other side, the fact that $H(t)$ approaches to $0^+$ as $t \to \infty$ implies that the effective theory returns to the classical Brans-Dicke cosmology at late time. It is easy to check that the asymptotic behaviours of the universe would not change if the residual term in the effective Hamiltonian (29) \hspace{1cm} was taken into account. However, the detailed evolution around the bounce would be influenced by that term. The numerical simulations for the evolution of $H(t)$ are plotted in Fig. 1 where the dynamics of $H(t)$ driven by the Hamiltonian constraints (29) and (34) are compared. According to the result, there is only a single bounce with $H(t) = 0$. Around the bounce, the residual term does affect the dynamics. It is straightforward to check that the dynamics of $H(t)$ for $t \in (-\infty, 0)$ behaves similar to that for $t \in (0, \infty)$.

### 6 Discussion

In the previous sections, to inherit more features of full LQBDT, we dealt with the Euclidean and Lorentzian terms of the Hamiltonian constraint independently in LQBDT. The Hamiltonian constraint operator (17) alternative to the one obtained in (26) was constructed in section 3. The effective Hamiltonian constraint (29) was also derived.
Figure 1: Comparison of the evolutions of $H(t)$ driven by (34) (the solid line) and by (29) (the red dashed line): The difference between the two evolutions of $H(t)$ is also given (the black dot-dashed line). The parameters in this plot is chosen as $\omega = 10^4$, $\gamma = 0.2357$, $\hbar = 1$, $\xi = 5$ and $\phi_0 = 1$.

from the alternative Hamiltonian operator by the semiclassical analysis in section 4. It turns out that there exists a residual quantum correction term in the effective Hamiltonian, which could not be obtained simply by replacing $b \to \sin(b)$ or $b \to \sin(2b)/2$ in the classical Hamiltonian constraint. This is a particular property of our LQBDC. The dynamics given by the effective Hamiltonian constraint was analyzed in section 5. The evolution equation of the universe was solved analytically by getting rid of the residual term which is of $\hbar^2$-order. The dynamical behaviors of the Hubble parameter for the physically interesting case of $\omega \gg 1$ was considered. It turns out that the classical singularity is resolved by a quantum bounce which relates a de Sitter epoch to a usual classical Brans-Dicke cosmology. Both the evolutions driven by the effective Hamiltonian (34) and by the original (29) with the residual term were numerically computed and plotted in Fig. 1. The comparison of the two evolutions shows that the two Hamiltonians determine the qualitatively same dynamics. However, the residual term affected the evolution around the bounce, while they give the same asymptotic behaviours.

Since an asymptotical de Sitter epoch appears in our cosmological model, it is interesting to see whether that epoch of the model can match the observation of current accelerating universe. By substituting (50) into (49), the Hubble parameter in the asymptotical de Sitter epoch can be expressed as

$$H(t) \cong -\frac{2\alpha}{\sqrt{6\pi \gamma}} \frac{\sqrt{3\omega - 8}}{2\omega + 3} \frac{2(\omega - 1)(3\omega - 8) + (2\omega + 3)(3\omega - 13)\gamma^2 \phi_-(t)^2}{(2(3\omega - 8) + 3(2\omega + 3)\gamma^2 \phi_-(t)^2)^2}. \quad (56)$$

Hence, if one asked $H(t)$ at some fixed $t$ to match the observation, the value of $\phi_-(t)$ would have to be sufficiently large. For instance, letting $\omega = 10^4$, one has $\phi_-(t) = 8.899 \times 10^{30}$. Moreover, $H(t)$ should change slowly at the moment $t$. Such a requirement could be achieved by choosing $\phi_0$ and $\xi_1$ in the expression of $\phi_-(t)$ properly. However, it is straightforward to check that in this case, the effective gravitational constant $G/\phi_-(t)$ in the Brans-Dicke theory is far away from the observational value because of the huge value of $\phi_-(t)$. Thus there is no evidence that the emerged asymptotical de Sitter epoch could match our current universe.

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