Numerical calculation of dynamic models with non-power nonlinearities

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Abstract. The solution to the actual problem in the numerical calculation of dynamic models with nonlinearities described by non-power elementary functions is considered. The computational difficulties of using power series to solve the problem are analyzed. A procedure based on the analytical-numerical method developed by the authors is proposed to overcome these difficulties. The numerical procedure is applicable for models described by the system of nonlinear differential equations in the Cauchy normal form. The proposed procedure possesses all the advantages of power series, as well as eliminates the necessity of using power series compositions and developing special estimates. The achieved result increases the formalization level of nonlinear dynamics based on dynamic models with non-power nonlinearities. In the numerical procedure, the obtained solutions estimates are used and the boundaries of the one-dimensional solutions areas are adjusted in order to come to unknown precise solutions. As an example, the calculation of nonlinear autonomous dynamic model type “damped pendulum” is represented.

1. Introduction

The development of problem-oriented methods of calculating deterministic nonlinear dynamic models (NDM) resulted from improving in computing technologies and extending the application field of mathematical modeling method [1-4]. The synthesis of dynamic models for complex systems of various physical nature is often associated with the inclusion of nonlinear elementary functions. These functions are generally non-power. The arguments of elementary functions are desired solutions. Josephson junction and damped pendulum are the examples of the mentioned models in different research fields. [1, 5-8]. The estimation of dynamic models, the nonlinear parts of which contain non-power elementary functions with singularities, requires the modification of standard numerical methods [9].

To analyze nonlinear dynamics, functional series is widely used, the most known one is a power series [9-12]. The use of power series to estimate the dynamics of the nonlinear model has several advantages. First, the known coefficients of power series for the regular component of the sought-for solution allow to calculate the radius of the series convergence in each interval of the stepwise estimation. As a result, we obtain the answer to the question of whether the mentioned component of the sought-for solution exists. Establishing by the independent time variable \( t \) the length of the interval, where under the given pre-initial conditions, there is a regular component of the sought-for solution, the value of the convergence radius allows distinguishing the existing discontinuities of the...
second kind [1, 10-12]. Secondly, power series is highly applicable when the nonlinear part of a model is represented in the unified form (as generalized power series). Such generalization enables to apply the generalized integral Laplace transform. In each estimation interval, it determines a well-defined transition from the known approximate values of pre-initial conditions to the approximate values of initial conditions, highlighting the existing discontinuities of the first kind in the sought-for solutions [10-13]. Thirdly, the power series method allows one to formalize the estimation of the errors emerging and being accumulated during the estimation of approximate solution values, necessary for the analysis of nonlinear dynamics [10-13].

The application of functional power series to the description of the nonlinear non-power parts of dynamic models leads to the following computational difficulties. First, when the power series of the regular component to the sought-for solution becomes an independent variable of another power series, the coefficients formation of the power series compositions and studying convergence become much more complicated. Secondly, the application of power series composition requires respective mathematical transformations that decrease the level of the generalization and the formalization of calculation technique [10-13]. Thirdly, forced limitation of power series included in a composition by their partial sums, qualitatively changes the complexity of errors estimation at each step of the estimation, initiating new mathematical transformations [10-13].

In this paper we propose a calculation procedure to overcome the above-mentioned estimation difficulties of nonlinear dynamic models with non-power nonlinear parts. The proposed procedure is applicable to NDM, which are represented in the normal Cauchy form or are transformed to this form [1, 11-13]. The analytical-numerical method using power series has been chosen as the basis of the procedure proposed in [10].

2. Computational procedure of the analytical-numerical method for estimating a nonlinear dynamic model

Let us consider NDM with non-power nonlinearities. The model dynamics is described as:

\[
\dot{x}_i(t) = \varphi_i(x_1(t), x_2(t), \ldots, x_{L_i}(t), t); \quad i = 1, \ldots, L
\]

where “\(\dot{\;}\)” is the time derivative sign; \(\varphi_i(x_1(t), x_2(t), \ldots, x_{L_i}(t), t)\) is the sum of multiplied results \(x_i(t)\), \(r = 1, 2, \ldots, L_i\) non-stationary parameters, inputs and non-power elementary functions, the argument of which is the argument of which is the sought-for solution \(x_i(t)\), \(l \in [1, L_i]\) in arbitrary fractional rational degrees. The pre-initial conditions \(x_i^0(0^-)\), \(r = 1, 2, \ldots, L_i\), \(0^- = t_0^-\) are assigned to estimate the NDM (1). The determination of the NDM (1) is as follows. Basing on the initial system (1) we form a conjugate system of nonlinear differential equations with respect to a new independent simulation variable \(x = x_i(t), l \in [1, L_i]\).

This new variable is the sought-for solution \(x_i(t), l \in [1, L_i]\) of the NDM (1) that is included in the nonlinear NDM part as the argument of elementary functions. We assume the absence of a singular component in the above solution. In accordance with the known procedure [14], the conjugate system of equations has the following form:

\[
\frac{dx_i}{dx} = \frac{\varphi_i(x_1(t), x_2(t), \ldots, x_{L_i-1}(t), x_{L_i+1}(t), \ldots, x_{L_i}(t), t(x), x)}{\varphi_i(x_1(t), x_2(t), \ldots, x_{L_i+1}(t), x_{L_i+2}(t), \ldots, x_{L_i}(t), t(x), x)};
\]

\[
\frac{dt}{dx} = \frac{1}{\varphi_i(x_1(t), x_2(t), \ldots, x_{L_i-1}(t), x_{L_i+1}(t), \ldots, x_{L_i}(t), t(x), x)};
\]

\[
x' = \mp 1,
\]

where “\(\mp\)” is the sign of derivative with respect to the variable \(x\).
Pre-initial conditions for conjugate system (2) with respect to the new independent variable \( x = x_i(t) \) are the following: 
\[
  x_i^i (x_i^{i-1}) = x_i^i (0^-), \quad x_i^{i+1} = x_i^i (t_{k-1}); \quad t(x_i^{i-1}) = t_{k-1}; \quad 0^- = t_{k-1}; \quad k \in [1, K]; \quad r = 1, 2, ..., l-1, \ l + 1, ..., L_i [10].
\]

The double sign in the right part of the last differential equation in system (2) indicates that the positive estimation of step \( h \) with respect to the initial variable \( t \) in model (1) can be corresponded to both positive and negative increment of the new independent variable \( x = x_i(t) \) of the conjugate equation system (2). The rule for choosing this sign is as follows. Exact values given for the first estimation interval or approximate values obtained for the subsequent estimation intervals \([0^-, 0^+ + \tau], \ 0^- = t_{k-1}\) of pre-initial conditions determine the formula when inequality holds:
\[
  \varphi_i \left( x_i^i (t), x_i^2 (t), ..., x_i^{L_i} (t), t \right) > 0. \tag{3}
\]
In this case one can choose the plus sign “+” in the last differential equation of system (2). If pre-initial conditions comply with the conditions:
\[
  \varphi_i \left( x_i^i (t), x_i^2 (t), ..., x_i^{L_i} (t), t \right) < 0. \tag{4}
\]
One should choose the minus sign “−” in the last differential equation of the system (2).

As a result of the transition to the new independent variable \( x = x_i(t) \), the directed transformation of the nonlinear part description, external inputs and non-stationary parameters of the studied NDM (1) are gained. Consequently, in system (2) the nonlinear part, parameters and external inputs are described by power functions with respect to the variable \( x = x_i(t) \).

The system (2) with respect to the new variable \( x = x_i(t) \) describes NDM with the unselected linear part uniquely corresponding to model (1). Having transformed system (2) in accordance to the analytical part of the analytical-numerical method for regular constituent solutions \( x_i(x) \), \( t(x) \) in the given estimation interval one can obtain the following power series [10]:
\[
  x_i^i (x) = \sum_{i=0}^{\infty} P_{r,i} x^i / i!; \quad t(x) = \sum_{i=0}^{\infty} P_{r,i} x^i / i!,
\]
where \( P_{r,i} \), \( r = 1, 2, ..., l-1, \ l + 1, ..., L_i \) and \( P_{r,i} \) are the expansion coefficients of regular components of the sought-for solutions into power series in the right semi-neighborhoods of points with the abscissa \( x = x_{k-1} = x_i^i (0^-), \ 0^- = t_{k-1} \), \( l \in [1, L_i] \). Regular components are calculated taking into account the sign in the right part of the last differential equation of system (2) using formulas from [10].

After compiling converging power series (5) for the regular components of the sought-for solutions in the conjugate system (2), at the discrete beginning instant of the current estimation interval \([0^-, 0^+ + \tau], \ 0^- = t_{k-1}\) one will obtain the following generalized canonical form of system (2) [10]:
\[
  \frac{dx_i^i (x)}{dx} = \sum_{i=1}^{\infty} P_{r,i} x^{i-1} / (i-1)!!; \quad \frac{dx}{dx} = \sum_{i=1}^{\infty} P_{r,i} x^{i-1} / (i-1)!!; \quad x^i = \mp 1, \\
  x_i^i (x_{k-1}) = P_{r,0}; \quad t(x_{k-1}) = P_{r,0} = t_{k-1}.
\]

Transforming the conjugate system of nonlinear differential equations (2) to the generalized canonical form (6) gives two important results: the transition from pre-initial to initial conditions with interconnected new and original variables \( x = x_{k-1} = x_i^i (0^-), \ 0^- = t_{k-1} \) as well as the expansion of the
regular solution components \( x_i(t), \ t(x), \ x = x_i'(t), \ i \in \{1, L_1\} \) into converging power series (5). In this transition, it is necessary to check if inequalities (3) and (4) hold for the obtained values of initial conditions (exact for the first estimate interval and approximately for all subsequent ones). If, after checking the result, it is necessary to change the sign in the right part of the last differential equation in (2), we do it and repeat the coefficients calculation of power series (5).

Using generalized canonical form (6), one can form the conjugate system of differential equations with respect to the initial time variable \( t \) [14]. The conjugate system of differential equations that is obtained in accordance to the power series operation rules has the form [11]:

\[
\begin{align*}
\dot{x}_i^e(t) &= \left[ \sum_{i=1}^{\infty} P_{ij} x_i^{e-1}/(i-1)! \right] \left[ \sum_{i=1}^{\infty} P_{ij} x_j^{e-1}/(i-1)! \right]^{-1} = \sum_{i=1}^{\infty} R_{ij} t^{i-1}/(i-1)!, \\
\dot{x}_i^e(t) &= \left[ \sum_{i=1}^{\infty} P_{ij} x_i^{e-1}/(i-1)! \right]^{-1} = \sum_{i=1}^{\infty} R_{ij} t^{i-1}/(i-1)!, \\
\end{align*}
\]

\( i = 1, \)

\( x_i^e(t_k') = R_{e,j} = P_{e,j}, \)

\( t_k = P_{e,j}, \)

\( x_i^e(t_k') = R_{e,0} = x_i^e(t_k'), \)

\( x_i^e(t_k') = x_i^e(t_k'), \)

\( r = 1, 2, ..., l - 1, l + 1, ..., L_n, \ l \in \{1, L_n\}. \)

In the current interval \([0', 0' + \tau], \) \( 0' = t_k' \) system (7) is the generalized canonical form of the investigated NDM (1), the nonlinear properties of which are described by non-power elementary functions. In the mentioned interval, the solutions of system (7) are the following power series:

\[
\begin{align*}
x_i^e(t) &= \sum_{i=0}^{\infty} R_{ij} t^{i}/!; \ r = 1, 2, ..., L_n.
\end{align*}
\]

Power series (8) in the current interval \([0', 0' + \tau], \) \( 0' = t_k' \) describe the regular components of the sought-for solutions of NDM (1). Such a result is identical to the one obtained when applying the standard procedure of an analytical part of the analytical-numerical method regarding NDM with power nonlinearities [10]. The subsequent numerical part of the analytical-numerical method including the errors estimation is similar to the common method procedure.

The results of the described procedure at the current step \( h = h_k \) are one-dimensional domains enclosed in double inequalities and containing unknown exact values \( x_i^e(t_k'), \ r = 1, 2, ..., L_n, \)

\( t_k = t_k' + h_k \) of the regular solution components \( x_i^e(t) \) of system (1).

3. Calculation of a nonlinear autonomous dynamic model “damped pendulum”

Let us consider the nonlinear dynamic model of a “damped pendulum” [12, 13] that complies with the nonlinear electric circuit. This circuit comprises three parallel-connected elements: capacity, inductance and nonlinear resistor. There are the normalized parameters of the circuit:

\( L = 1; \ C = 1; \ G(t) = G_0 + (\sin i_c(t) - i_c(t))/u_c(t), \) \( G_0 = 0.4. \) Pre-initial conditions are the following:

\( i_c(0') = 3, \) \( u_c(0') = 2, \) \( 0' = t_0, \) \( t_0 = 0. \) The electric circuit is described by the system that contains three nonlinear differential equations:

\[
\begin{align*}
\frac{dt_{i_c}}{dt} &= C^{-1} \left[ -G_0 u_c(t) - \sin i_c(t) \right]; \\
\frac{dt_i}{dt} &= L^{-1} u_c(t); \quad i = 1.
\end{align*}
\]

The nonlinear part of system (9) contains the sine function, the argument of which is the sought-for solution \( i_c(t) \) . The solutions singularity of system (9) is that their phase plane due to an infinite number of equilibrium positions, contains “the basins of attraction and the basins of repelling”. This
fact causes the necessity to apply numerical methods for the NDM estimation, which have a fairly perfect error control procedure [12, 13].

To solve the posed task according to the proposed computational procedure, let us first form system (2) conjugate with the system (9) of differential equations. As a new independent modelling variable \( x \), we should choose the regular component \( i^*_k(t) \) of the sought-for solution in initial system (9), which is the argument of the sine function in the nonlinear part of this system. In the first estimation step the system conjugate with system (9) for the new independent simulation variable \( x = i^*_k(t) \) has the form:

\[
\frac{du^*_k}{dx} = \frac{L}{C} \left[ -G_0 - \sin x \cdot u^*_k(x) \right]; \quad \frac{dt}{dx} = \frac{L}{w^*_k(x)}; \quad x' = 1, \tag{10}
\]

where \( x_0 = x(0^-) = i^*_k(0^-) = 3 \), \( u^*_k(x_0^-) = u^*_k(0^-) = 2 \), \( t(x_0^-) = t_0 = 0 \).

The sign “+” in the third equation of system (10) is caused by the fact that inequality (3) holds under pre-initial conditions for the function in the right part of the second equation of system (9). Having transformed the system of differential equations (10) according to the analytical part of analytical-numerical method according to form (5) we will get the description of its sought-for solutions:

\[
u^*_k(x) = \sum_{i=0}^\infty P_i x^i / i!, \quad t(x) = \sum_{i=0}^\infty P_{i,j} x^i / i!, \tag{11}\]

where \( P_0 = u^*_k(x_0^-) \), \( P_1 = L/C \left[ -G_0 - \sin x / u^*_k(x_0^-) \right] + L/C \left[ -\cos x / u^*_k(x_0^-) \right] \sin x / \left( u^*_k(x_0^-)^2 \right) \), \( P_2 = L/C \left[ -\cos x / u^*_k(x_0^-) \right] \sin x / \left( u^*_k(x_0^-)^2 \right)^2 \), \( P_{i,0} = t(x_0^-) \), \( P_{i,j} = L / u^*_k(x_0^-) \), \( i \neq 0 \).

Equalities (11) determine the transition from the given values of pre-initial conditions \( u^*_k(x_0^-) = u^*_k(i^*_k(0^-)) \), \( t(x_0^-) = t(i^*_k(0^-)) \) to the values of initial conditions \( u^*_k(x_0^+) = P_0 \), \( t(x_0^+) = P_{i,0} \).

Being obtained using power series (11), the generalized canonical form of conjugate system (10) according to the form (6) can be written as:

\[
\frac{du^*_k(x)}{dx} = \sum_{i=0}^\infty P_i x^i / (i-1)!; \quad \frac{dt(x)}{dx} = \sum_{i=1}^\infty P_{i,j} x^i / (i-1)!, \quad x' = 1, \tag{12}\]

where \( x_0 = x(0^-) = i^*_k(0^-) \); \( u^*_k(x^+) = P_0 \); \( t(x^+) = P_{i,0} \).

The differential equation system conjugate with system (12), according to the form (7) for the original variable \( t \) is written as:

\[
\frac{du^*_k(t)}{dt} = \left[ \sum_{i=0}^\infty P_i x^i / (i-1)! \right] \left[ \sum_{i=0}^\infty P_{i,j} x^i / (i-1)! \right]^{i-1} = \sum_{i=1}^\infty R_{1,i} t^{i-1} / (i-1)!, \tag{13}\]

\[
\frac{dt^*_k(t)}{dt} = \left[ \sum_{i=0}^\infty P_i x^i / (i-1)! \right]^{i-1} = \sum_{i=1}^\infty R_{2,i} t^{i-1} / (i-1)!, \quad i = 1.
\]

The system (13) with power series is the generalized canonical form of the initial NDM (9) for the current interval \( [0^-; 0^+ + \tau] \), \( 0^+ = i^*_k(t_{i-1}) \), \( k = 1 \). In this interval if the calculation step does not exceed the convergence radii of the power series in (13), the series converge to the expanded functions. The sought-for solutions of the NDM (9) have the form:
unknown exact solutions of the original equations system. Thus, with the help of double system calculation steps, the approximate solutions, the calculation procedure for which is described in [10], are obtained for the several inequalities, and they contain unknown exact values of the sought solutions. In the considered example, one-dimensional regions have the following description:

\[
\begin{align*}
    u_c^r(t;I_1) &- |\Delta u_c^r(t;I_1)| \leq u_c^r(t;I_1) + |\Delta u_c^r(t;I_1)|; \\
    i_L^r(t;I_2) &- |\Delta i_L^r(t;I_2)| \leq i_L^r(t;I_2) + |\Delta i_L^r(t;I_2)|.
\end{align*}
\]

Using these inequalities, the approximate circuit responses \( u_c(t) \) and \( i_L(t) \) are obtained for the several calculation steps represented in Table 1.

4. Conclusion

The numerical calculation method of dynamic models with nonlinearities described by non-power elementary functions is described. The goal is to form the system of nonlinear differential equations, conjugate to the original system of equations (the original model). The conjugate system is reduced to the generalized classical form. Further, the system of nonlinear differential equations is formed, conjugate to the system in the generalized classical form. The solutions to the last conjugate system of equations are power series describing the boundaries of one-dimensional domains that contain unknown exact solutions of the original equations system. Thus, with the help of double system
conjugation, the main result is achieved. First, we obtain that the ready estimates of the sought-for solutions are used and, secondly, we find that the boundaries of one-dimensional domains of exact solutions can be controlled, bringing them infinitely close to unknown exact solutions.

The numerical procedure, having all advantages of power series apparatus, eliminates the necessity of using power series compositions and increases the estimate formalization level of nonlinear dynamics based on dynamic models with non-power nonlinearities. The proposed method is illustrated with the calculation of the nonlinear autonomous dynamic model of the “damped pendulum”.

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