NOTE ON THE COUNTEREXAMPLES FOR THE INTEGRAL TATE CONJECTURE OVER FINITE FIELDS

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Abstract. In this note we discuss some examples of non torsion and non algebraic cohomology classes for varieties over finite fields. The approach follows the construction of Atiyah-Hirzebruch and Totaro.

1. Introduction

Let $k$ be a finite field and let $X$ be a smooth and projective variety over $k$. Denote $k$ an algebraic closure of $k$ and $\mathfrak{g} = Gal(k/k)$. Let $\ell$ be a prime, $\ell \neq \text{char}(k)$. The Tate conjecture [19] predicts that the cycle class map

$$CH^i(X_{\bar{k}}) \otimes \mathbb{Q}_\ell \to \bigcup_U H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Q}_\ell(i))^U,$$

where the union is over all open subgroups $U$ of $\mathfrak{g}$, is surjective.

In the integral version one is interested in the cokernel of the cycle class map

$$(1.1) \quad CH^i(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_U H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^U.$$

This map is not surjective in general: the counterexamples of Atiyah-Hirzebruch [11], revisited by Totaro [20], to the integral version of the Hodge conjecture, provide also counterexamples to the integral Tate conjecture [3]. More precisely, one constructs an $\ell$-torsion class in $H^4_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell(2))$, which is not algebraic, for some smooth and projective variety $X$. However, one then wonders if there exists an example of a variety $X$ over a finite field, such that the map

$$(1.2) \quad CH^i(X_{\bar{k}}) \otimes \mathbb{Z}_\ell \to \bigcup_U H^2_{\text{ét}}(X_{\bar{k}}, \mathbb{Z}_\ell(i))^U/torsion$$

is not surjective ([12] [3]). In the context of an integral version of the Hodge conjecture, Kollár [11] constructed such examples of curve

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classes. Over a finite field, Schoen [17] has proved that the map (1.2) is always surjective for curve classes, if the Tate conjecture holds for divisors on surfaces.

In this note we follow the approach of Atiyah-Hirzebruch and Totaro and we produce examples where the map (1.2) is not surjective for \( \ell = 2, 3 \) or 5.

**Theorem 1.1.** Let \( \ell \) be a prime from the following list: \( \ell = 2, 3 \) or 5. There exists a smooth and projective variety \( X \) over a finite field \( k \), \( \text{char}k \neq \ell \), such that the cycle class map

\[
CH^2(X_k) \otimes \mathbb{Z}_\ell \to \bigcup_U H^4_{\text{et}}(X_k, \mathbb{Z}_\ell(2))^U/\text{torsion}
\]

is not surjective.

As in the examples of Atiyah-Hirzebruch and Totaro, our counterexamples are obtained as a projective approximation of the cohomology of classifying spaces of some simple simply connected groups, having \( \ell \)-torsion in its cohomology. The non algebraicity of a cohomology class is obtained by means of motivic cohomology operations: one establishes that the operation \( Q_1 \) does not vanish on some class of degree 4, but it always vanishes on the algebraic classes. This is done in section 2. Next, in section 3 we discuss some properties of classifying spaces in our context and finally we construct a projective variety approximating the cohomology of these spaces in small degrees in section 4.

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2. Motivic version of Atiyah-Hirzebruch arguments, revisited

2.1. Operations. Let \( k \) be a perfect field with \( \text{char}(k) \neq \ell \) and let \( \mathcal{H}(k) \) be the motivic homotopy theory of pointed \( k \)-spaces (see [14]).
For $X \in \mathcal{H}(k)$, denote by $H^{*,*}(X, \mathbb{Z}/\ell)$ the motivic cohomology groups with $\mathbb{Z}/\ell$-coefficients (loc.cit.). If $X$ is a smooth variety over $k$, note that one has an isomorphism $CH^*(X)/\ell \cong H^{2*,*}(X, \mathbb{Z}/\ell)$.

Voevodsky [22] defined the reduced power operations $P^i$ and the Milnor’s operations $Q_i$ on $H^{*,*}(X, \mathbb{Z}/\ell)$:

$$P^i : H^{*,*}(X, \mathbb{Z}/\ell) \to H^{*+2i(\ell-1),*+i(\ell-1)}(X, \mathbb{Z}/\ell), i \geq 0$$

$$Q_i : H^{*,*}(X, \mathbb{Z}/\ell) \to H^{*+2\ell i-1,*+(\ell i-1)}(X, \mathbb{Z}/\ell), i \geq 0,$$

where $Q_0 = \beta$ is the Bockstein operation of degree $(1,0)$ induced from the short exact sequence $0 \to \mathbb{Z}/\ell \xrightarrow{\times \tau} \mathbb{Z}/\ell^2 \to \mathbb{Z}/\ell \to 0$ (see also [16]).

One of the key ingredients for this construction is the following computation of the motivic cohomology of the classifying space $B\mu_\ell$ ([22]):

**Lemma 2.1.** ([22 §6]) For each object $X \in \mathcal{H}(k)$, the graded algebra $H^{*,*}(X \times B\mu_\ell, \mathbb{Z}/\ell)$ is generated over $H^{*,*}(X, \mathbb{Z}/\ell)$ by $x, \deg(x) = (1,1)$ and $y, \deg(y) = (2,1)$ with $\beta(x) = y$ and $x^2 = \begin{cases} 0 & \ell \text{ is odd} \\ \tau y + \rho x & \ell = 2 \end{cases}$

where $0 \neq \tau \in H^{0,1}(\text{Spec}(k), \mathbb{Z}/\ell) \cong \mu_\ell$ and $\rho = (-1) \in k^*/(k^*)^\ell \cong K^M_1(k)/2 \cong H^{1,1}(\text{Spec}(k), \mathbb{Z}/2)$.

For what follows, we assume that $k$ contains a primitive $\ell^2$-th root of unity $\xi$, so that $B\mathbb{Z}/\ell \xrightarrow{\sim} B\mu_\ell$ and $\beta(\tau) = \xi^\ell$ ($= \rho$ for $p = 2$) is zero in $k^*/(k^*)^\ell = H^{1,1}_{et}(\text{Spec}(k); \mathbb{Z}/\ell)$.

We will need the following properties:

**Proposition 2.2.** (i) $P^i(x) = 0$ for $i > m/2$ and $x \in H^{m,n}(X, \mathbb{Z}/\ell)$;

(ii) $P^i(x) = x^\ell$ for $x \in H^{2i,j}(X, \mathbb{Z}/\ell)$;

(iii) for $X$ smooth the operation

$$Q_i : CH^m(X)/\ell = H^{2m,m}(X, \mathbb{Z}/\ell) \to H^{2m+2\ell i-1,m+(\ell i-1)}(X, \mathbb{Z}/\ell)$$

is zero;

(iv) $Op.(\tau x) = \tau Op.(x)$ for $Op. = \beta, Q_i$ or $P^i$;

(v) $Q_i = [P^{\ell i-1}, Q_{i-1}]$.

**Proof.** See [22 §9]; for (iii) one uses that $H^{m,n}(X, \mathbb{Z}/\ell) = 0$ if $m-2n > 0$ and $X$ is a smooth variety over $k$, (iv) follows from the Cartan formula for the motivic cohomology.
2.2. Computations for $BZ/\ell$. The computations in this section are similar to [1, 20, 21].

Lemma 2.3. In $H^{*,*}(BZ/\ell, Z/\ell)$, we have $Q_i(x) = y^\ell$ and $Q_i(y) = 0$.

Proof. By definition $Q_0(x) = \beta(x) = y$. Using induction and Proposition 2.2, we compute

$$Q_i(x) = \ell^{i-1}Q_{i-1}(x) - Q_{i-1}\ell^{i-1}(x) = \ell^{i-1}Q_{i-1}(x) = \ell^{i-1}(y^{\ell^{i-1}}) = y^\ell.$$ 

Then $Q_1(y) = -Q_0P_1(y) = -\beta(y^\ell) = 0$. For $i > 1$, using induction and Proposition 2.2 again, we conclude that $Q_i(y) = -Q_{i-1}\ell^{i-1}(y) = 0$.

□

Let $G = (G/\ell)^3$. As above, we assume that $k$ contains a primitive $\ell^2$-th root of unity. From Lemma 2.1, we have an isomorphism

$$H^{*,*}(BG, Z/\ell) \cong H^{*,*}(Spec(k), Z/\ell)[y_1, y_2, y_3] \otimes \Lambda(x_1, x_2, x_3)$$

where $\Lambda(x_1, x_2, x_3)$ is isomorphic to the $Z/\ell$-module generated by 1 and $x_i...x_s$ for $i_1 < ... < i_s$ and $x_i.x_j = -x_jx_i (i \leq j)$, with $\beta(x_i) = y_i$ and $x_i^2 = T y_i$ for $\ell = 2$.

Lemma 2.4. Let $x = x_1x_2x_3$ in $H^{3,3}(BG, Z/\ell)$. Then

$$Q_iQ_jQ_k(x) \neq 0 \in H^{2*,*}(BG, Z/\ell) \quad \text{for } i < j < k.$$ 

Proof. Using Proposition 2.2(v) and Cartan formula (2.2(iv)), we get

$$Q_k(x) = y_1^kx_2x_3 - y_2^kx_1x_3 + y_3^kx_1x_2.$$ 

Then we deduce

$$Q_iQ_jQ_k(x) = \sum_{\sigma \in S_3} \pm y_\sigma^k y_\sigma(2) y_\sigma(3) \neq 0 \in Z/\ell[y_1, y_2, y_3].$$ 

□

3. Exceptional Lie Groups

Let $(G, \ell)$ be a simple simply connected Lie group and a prime number from the following list:

$$\begin{cases} 
G_2, \ell = 2, \\
F_4, \ell = 3, \\
E_8, \ell = 5. 
\end{cases}$$ 

(3.1)
Then $G$ is 2-connected and $H^3(G, \mathbb{Z}) \cong \mathbb{Z}$. Hence $BG$, viewed as a topological space, is 3-connected and $H^4(BG, \mathbb{Z}) \cong \mathbb{Z}$ (see [13] for example). We write $x_4(G)$ for a generator of $H^4(BG, \mathbb{Z})$.

Given a field $k$ with $\text{char}(k) \neq \ell$, let us denote by $G_k$ the (split) reductive algebraic group over $k$ corresponding to the Lie group $G$.

The Chow ring $CH^*(BG_k)$ has been defined by Totaro [21]. More precisely, one has

\begin{equation}
BG_k = \lim_{\to}(U/G_k),
\end{equation}

where $U \subset W$ is an open set in a linear representation $W$ of $G_k$, such that $G_k$ acts freely on $U$. One can then identify $CH^i(U/G_k)$ if $\text{codim}_W(W \setminus U) > i$, the group $CH^i(BG_k)$ is then independent of a choice of such $U$. Similarly, one can define the étale cohomology groups $H^i_{\text{ét}}(BG_k, \mathbb{Z}_{\ell}(j))$ and the motivic cohomology groups $H^{*,*}(BG_k, \mathbb{Z}/\ell)$ (see [7], the latter coincide with the motivic cohomology groups of [14] (cf. [7, Proposition 2.29 and Proposition 3.10]). We also have the cycle class map

\begin{equation}
\text{cl} : CH^*(BG_k) \otimes \mathbb{Z}_{\ell} \to \bigcup_U H^2_{\text{ét}}(BG_k, \mathbb{Z}_{\ell}(*)^U),
\end{equation}

where the union is over all open subgroups $U$ of $\text{Gal}(\bar{k}/k)$.

The following proposition is known.

**Proposition 3.1.** Let $(G, \ell)$ be a group and a prime number from the list [3.4]. Then

(i) the group $G$ has a maximal elementary non toral subgroup of rank 3:

\[ i : A \cong (\mathbb{Z}/\ell)^3 \subset G; \]

(ii) $H^4(BG, \mathbb{Z}/\ell) \cong \mathbb{Z}/\ell$, generated by the image $x_4$ of the generator $x_4(G)$ of $H^4(BG, \mathbb{Z}) \cong \mathbb{Z};$

(iii) $Q_1(i^*x_4) = Q_1Q_0(x_1x_2x_3)$, in the notations of Lemma 2.4. In particular, $Q_1(i^*x_4)$ is non zero.

**Proof.** For (i) see [3], for the computation of the cohomology groups with $\mathbb{Z}/\ell$-coefficients in (ii) see [13] VII 5.12; (iii) follows from [10] for $\ell = 2$ and [8, Proposition 3.2] for $\ell = 3, 5$ (see [9] as well). \( \square \)

4. **Algebraic approximation of $BG$**

Write

\begin{equation}
BG_k = \lim_{\to}(U/G_k)
\end{equation}

as in the previous section. Using proposition 3.1 and a specialization argument, we will first construct a quasi-projective algebraic variety $X$
over \( k \) as a quotient \( X = U/G_k \) (where \( \text{codim}_U(W \setminus U) \) is big enough), such that the cycle class map (1.2) is not surjective for such \( X \). However, if one is interested only in quasi-projective counterexamples for the surjectivity of the map (1.2), one can produce more naive examples, for instance as a complement of some smooth hypersurfaces in a projective space. Hence we are interested to find an approximation of Chow groups and the étale cohomology of \( BG_k \) as a smooth and projective variety. In the case when the group \( G \) is finite, this is done in [3, Théorème 2.1]. In this section we give such an approximation for the groups we consider here, this construction is suggested by B. Totaro.

**Proposition 4.1.** Let \( G \) be a compact Lie group as in (3.1). For all but finitely many primes \( p \) there exists a smooth and projective variety \( X_k \) over a finite field \( k \) with \( \text{char} k = p \), an element \( x_{4,k} \in H^4_{\text{ét}}(B(G_m \times G_k), \mathbb{Z}_\ell(2)) \), invariant under the action of \( \text{Gal}(\bar{k}/k) \) and a map \( \tau : X_k \to B(G_m \times G_k) \) in the category \( H_\cdot(k) \) such that

(i) \( y_{4,k} = \tau^* \text{pr}_2^* x_{4,k} \) is a non zero class in \( H^4_{\text{ét}}(X_k, \mathbb{Z}_\ell(2))/\text{torsion} \), where \( \text{pr}_2 : G_m \times G_k \to G_k \) is the projection on the second factor;

(ii) the operation \( Q_1(\bar{y}_{4,k}) \) is non zero, where we write \( \bar{y}_{4,k} \) for the image of \( y_{4,k} \) in \( H^4_{\text{ét}}(X_k, \mathbb{Z}/\ell) \).

**Remark 4.2.** For the purpose of this note, the proposition above is enough. See also [6] for a general statement on the projective approximation of the cohomology of classifying spaces.

Theorem 1.1 now follows from the proposition above:

**Proof of theorem 1.1.**
For \( k \) a finite field and \( X_k \) as in the proposition above, we find a non-trivial class \( y_{4,k} \) in its cohomology in degree 4 modulo torsion, which is not annihilated by the operation \( Q_1 \). This class can not be algebraic by proposition 2.2(iii). \( \square \)

**Proof of proposition 4.1.**
We proceed in three steps. First, we construct a quasi-projective approximation in a family parametrized by \( \text{Spec} \mathbb{Z} \). Then, for the geometric generic fibre we produce a projective approximation, by a topological argument. We finish the proof by specialization.

**Step 1:** construction of a family.
Let \( \mathcal{G} \) be a split reductive group over \( B = \text{Spec} \mathbb{Z} \) corresponding to \( G \),
such a group exists by [SGA3] XXV 1.3. As $B$ is an affine scheme of dimension 1, we can embed $G$ as a closed subgroup of $GL_{d,B}$ for some $d$ (see [SGA3] VI 13.2 and 13.5). Moreover, one can assume that $G \hookrightarrow PGL_{d,B}$ such that this embedding lifts to $H = GL_{d,B}$, up to replacing $B$ by an open subset (e.g. using the map $A \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & A \end{pmatrix}$ and changing $d$ by $d + 2$).

By a construction of [21, Remark 1.4] and [2, Lemme 9.2], there exists $n > 0$, a linear $H$-representation $O_B^{\oplus n}$ and an $H$-invariant open subset $U \subset O_B^{\oplus n}$, which one can assume flat over $B$, such that the action of $H$ is free on $U$. Let $V_N = O_B^{\oplus n}$. Then the group $PGL_{n,B}$ acts on $\mathbb{P}(V_N)$ and, taking $N$ sufficiently large, one can assume that the action is free outside a subset $S$ of high codimension $s \geq 4$.

By restriction, the group $G$ acts on $\mathbb{P}(V_N)$ as well, let $Y = \mathbb{P}(V_N)/G$ be the GIT quotient for this action [15, 18]. The scheme $Y$ is projective over $B$ and we fix an embedding $Y \subset \mathbb{P}^M_B$. Let

$$f: W \rightarrow B$$

be the open set of $Y$ corresponding to the quotient of the open set $U$ as above where $G$ acts freely. From the construction, $Y \setminus W$ has high codimension in $Y$.

For any point $b \in B$ with residue field $\kappa(b)$, the fibre $W_b$ is a smooth quasi-projective variety and if $N$ is big enough, we have isomorphisms by lifting $G$ to $GL_{n,B}$ (cf. p. 263 in [21])

$$W_b \cong (\mathbb{P}(V_N) - S)_b/G_b \cong ((V_N - \{0\})/G_m - S)_b/G_b \cong (V_N - S')_b/(G_m \times G_b)$$

where $S' = pr^{-1}S \cup \{0\}$ for the projection $pr: (V_N - \{0\}) \rightarrow \mathbb{P}(V_N)$. Hence we have isomorphisms

$$H^i(W_b, \mathbb{Z}_\ell) \cong H^i(B(G_m \times G_b, \mathbb{Z}_\ell))$$

for $i \leq s, \ell \neq char \kappa(b)$, induced by a natural map $W_b \rightarrow B(G_m \times G_b)$ from the presentation (4.1).

**Step 2: the generic fibre.**

Let $Y = \mathcal{Y}_C$ and $W = \mathcal{W}_C$ be the geometric generic fibres of $\mathcal{Y}$ and $\mathcal{W}$ over $B$. Consider a general linear space $L$ in $\mathbb{P}^M$ of codimension equal to $1 + dim(Y - W)$. Then $L \cap Y = L \cap W$ so $X := L \cap W$ is a smooth projective variety. Note that one can assume that $L$ is defined over $\mathbb{Q}$.

By a version of the Lefschetz hyperplane theorem for quasi-projective varieties, established by Hamm (as a special case of Theorem II.1.2 in [4]), for $V \subset \mathbb{P}^M$ a closed complex subvariety of dimension $d$, not
necessarily smooth, $Z \subset V$ a closed subset, and $H$ a hyperplane in $\mathbb{P}^M$, if $V - (Z \cup H)$ is local complete intersection (e.g. $V - Z$ is smooth) then
\[
\pi_i((V - Z) \cap H) \to \pi_i(V - Z)
\]
is an isomorphism for $i < d - 1$ and surjective for $i = d - 1$. In particular, $H^i((V - Z) \cap H, \mathbb{Z}) \to H^i(V - Z, \mathbb{Z})$ is an isomorphism for $i < d - 1$ and surjective for $i = d - 1$ by the Whitehead theorem.

We then deduce that
\[
\text{(4.4)} \quad H^i(X, R) \simeq H^i(B(\mathbb{G}_m \times G), R) \text{ for } i \leq s \text{ and } R = \mathbb{Z} \text{ or } \mathbb{Z}/n.
\]
Hence $H^i_\text{et}(X, \mathbb{Z}/n) \simeq H^i_\text{et}(B(\mathbb{G}_m \times G), \mathbb{Z}/n), i \leq s$. Note that as the cohomology of $BG$ is a direct factor in the cohomology of $B(\mathbb{G}_m \times G)$, we get that $x_4(G)$ (with the notations of the previous section) generates a direct factor isomorphic to $\mathbb{Z}_\ell$ in the cohomology group $H^4_\text{et}(X, \mathbb{Z}_\ell)$.

\textit{Step 3: specialization argument.}
We can now specialize the construction above to obtain the statement over a finite field.

More precisely, one can find a dense open set $B' \subset B$ and a linear space $L \subset \mathbb{P}^M_{B'}$ such that $L_\mathbb{C} \simeq L$ and such that for any $b \in B'$ the fibre $\mathcal{X}_b$ of $\mathcal{X} = L \cap \mathcal{Y}$ is smooth. Up to passing to an étale cover of $B'$, one can assume that the inclusion $(\mathbb{Z}/\ell)^3 \subset G_\mathbb{C}$ from proposition 3.1 extends an inclusion $i : \mathcal{A} = (\mathbb{Z}/\ell)^3_{B'} \hookrightarrow \mathcal{G}_{B'}$ (cf. [SGA3] XI.5.8).

Let $b \in B'$ and let $k = \kappa(b)$. As the schemes $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{U}/\mathcal{A}$ are smooth over $B'$, we have the following commutative diagram, where the vertical maps are induced by the specialisation maps:

\[
\begin{array}{cccccc}
H^4_\text{et}(X, \mathbb{Z}/\ell(2)) & \longrightarrow & H^4_\text{et}(Y, \mathbb{Z}/\ell(2)) & \longrightarrow & H^4_\text{et}(\mathcal{U}_C/((\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) & \longrightarrow & H^4_\text{et}(B((\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H^4_\text{et}(\mathcal{X}_k, \mathbb{Z}/\ell(2)) & \longrightarrow & H^4_\text{et}(\mathcal{Y}_k, \mathbb{Z}/\ell(2)) & \longrightarrow & H^4_\text{et}(\mathcal{U}_k/((\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell) & \longrightarrow & H^4_\text{et}(B((\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell)
\end{array}
\]

The left vertical map is an isomorphism since $\mathcal{X}$ is proper. Hence we get a class $y_{4,k} \in H^4_\text{et}(\mathcal{X}_k, \mathbb{Z}/\ell(2))$, corresponding to $x_4(G) \in H^4_\text{et}(X, \mathbb{Z}/\ell(2))$. The map $H^4_\text{et}(Y, \mathbb{Z}/\ell(2)) \to H^4_\text{et}(X, \mathbb{Z}/\ell(2))$ is an isomorphism by step 2, so that $y_{4,k}$ comes from an element $x_{4,k} \in H^4_\text{et}(\mathcal{Y}_k, \mathbb{Z}/\ell(2))$. Let $z_{4,k} \in H^4_\text{et}(B((\mathbb{Z}/\ell)^3, \mathbb{Z}/\ell)$ be the image of $x_{4,k}$. From the diagram and proposition 3.1 we deduce that $Q_1(z_{4,k}) = Q_1Q_0(x_1x_2x_3) \neq 0$, hence $Q_1(y_{4,k})$ is non zero as well. From the construction, the class $y_{4,k}$ generates a subgroup of $H^4_\text{et}(\mathcal{X}_k, \mathbb{Z}/\ell(2))$, which is a direct factor isomorphic to $\mathbb{Z}_\ell$, and is Galois-invariant. Letting $\mathcal{X}_k = \mathcal{X}_k$ this finishes the proof of the proposition.
Remark 4.3. We can also adapt the arguments of [3, Théorème 2.1] to produce projective examples with higher torsion non-algebraic classes, while in loc.cit. one constructs ℓ-torsion classes. Let $G(n)$ be the finite group $G(F_{ℓ^n})$, so that we have

$$\lim_{\leftarrow} H^*(BG(n), \mathbb{Z}_ℓ) = H^*(BG_{\overline{k}}, \mathbb{Z}_ℓ).$$

Then, following the construction in loc.cit. one gets

For any $n > 0$, there exit a positive integer $i_n$ and a Godeaux-Serre variety $X_{n,\overline{k}}$ for the finite group $G(n)$ such that

1. $x \in H^4_{\text{ét}}(X_{n,\overline{k}}, \mathbb{Z}_ℓ(2))$ generates $\mathbb{Z}/ℓ^n'$ for some $n' \geq n$;
2. $x$ is not in the image of the cycle class map (1.1).

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