K-theory of 2-rank graphs associated to complete bipartite graphs

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Abstract

We associate to each complete connected bipartite graph \( \kappa \) a 2-dimensional square complex, which we call a tile complex, whose link at each vertex is \( \kappa \). We regard the tile complex in two different ways, each having a different structure as a 2-rank graph. To each 2-rank graph, we can associate a universal \( C^* \)-algebra, for which we compute the K-theory. We determine the homology of the tile complexes, and give generalisations of the procedures to complexes consisting of polygons with a higher number of sides.

1 Introduction

In [9], it was shown how to construct a two-dimensional square complex whose link at each vertex is a complete bipartite graph. In [4], generalising the work of [7], certain square complexes were associated a \( C^* \)-algebra, and their K-theory computed. We combine these two methods to build an infinite family of \( C^* \)-algebras corresponding to complete bipartite graphs.

We begin in Section 2 by detailing the method of construction of tile complexes, which we build exclusively for complete bipartite graphs. In Sections 3 and 4 we associate adjacency matrices to the tile complexes in two different ways: by considering the tiles as pointed, and as unpointed geometrical objects. By the fact that the adjacency matrices commute, they characterise the structure of a higher-rank graph [4], to which can be associated a generalisation of a graph algebra [6, Chapter 1]. We use a result of [2] to calculate the K-groups of these algebras (Theorems 3.11, 4.3).

In the brief Section 5 we show that the tile complexes have torsion-free homology groups given by \( H_1 \cong H_2 \cong \mathbb{Z}^{\alpha+\beta-2} \), and \( H_n = 0 \) otherwise.

Finally, we explore extensions of these methods to so-called \( 2t \)-polyhedra: two-dimensional complexes consisting entirely of \( 2t \)-gons. We discuss the naturality of these generalisations.

Throughout the paper, \( \alpha, \beta \) are finite positive integers, and \( \kappa(\alpha, \beta) \) denotes the complete connected bipartite graph on \( \alpha \) white and \( \beta \) black vertices.
2 The tile system associated to a bipartite graph

Definitions 2.1. Let $p \in \mathbb{Z}$ with $t \geq 2$, and let $A_1, \ldots, A_k$ be a sequence of solid $t$-gons, with directed edges labelled from some set $U$. By gluing together like-labelled edges (respecting their direction), we obtain a two-dimensional complex $P$. We call such a complex a $t$-polyhedron.

The link at a vertex $x$ of $P$ is the graph obtained as the intersection of $P$ with a small 2-sphere centred at $x$.

Theorem 2.2 (Vdovina, 2002). Let $G$ be a connected bipartite (undirected) graph on $\alpha$ white and $\beta$ black vertices, with edge set $E(G)$. Then we can construct a $2t$-polyhedron $P(G)$ which has $G$ as the link at each vertex, for each $t \geq 1$.

We reference [9], in which it was shown how to build such a $2t$-polyhedron. The general method is as follows:

Write $U' = \{u_1, \ldots, u_\alpha\}$ for the set of white vertices of $G$, and $V' = \{v_1, \ldots, v_\beta\}$ for the set of black vertices.

Let $U$ be a set with $2t\alpha$ elements, indexed $u_1^1, u_2^1, \ldots, u_1^2, \ldots, u_\alpha^2$ for each $u \in U'$, and let $V$ be the corresponding set with $2t\beta$ elements. Define fixed-point-free involutions $\varphi_U, \varphi_V$ on $U$ and $V$ respectively such that $\varphi_U(u_i^1) = \bar{u}_i^1$ and $\varphi_V(v_i^1) = \bar{v}_i^1$ for all $i, r$.

Each edge of the graph $G$ joins an element of $U'$ to an element of $V'$; choose such an edge joining, say, $u_i$ to $v_j$. Draw a $2t$-gon and distinguish a base vertex. Label its boundary anticlockwise, starting from the base, by the sequence $u_1^1, v_1^1, u_2^1, v_2^1, \ldots, u_\alpha^2, v_\beta^2$, giving each side of the boundary a forward-directed arrow. The involutions $\varphi_U$ and $\varphi_V$ reverse the direction of arrows. Draw one $2t$-gon $A_e$ for each edge $e$ in $E(G)$, and glue them together in the manner described above, in order to obtain a $2t$-polyhedron $P(G)$ (Figure 1). It is shown in [9] that each vertex of $P(G)$ has $G$ as its link.

![Figure 1: Construction of a 2t-polyhedron](image)

In this paper, we mainly concern ourselves with 4-polyhedra, that is, those constructed by gluing together squares. We will refer to 4-polyhedra as tile complexes.

For a connected bipartite graph $G$, Write $TC(G)$ for the tile complex which has the graph $G$ as the link at each vertex, and write $S'(G) := \{A_e \mid e \in E(G)\}$ for the set of geometric squares of which $TC(G)$ consists. We call elements of $S'(G)$ unpointed tiles, and denote them by 4-tuples of the form $A_e = (u_i^1, v_j^1, u_k^2, v_m^2) \in U \times V \times U \times V$. 

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We use square brackets, writing $A_e = [u_1, v_1, u_2, v_2]$ if we wish to emphasise that a square is labelled *anticlockwise and starting from the basepoint* by the sequence $u_1, v_1, u_2, v_2$ for some $u_1, u_2 \in U$, $v_1, v_2 \in V$. We define the set

$$S(G) := \{ [u_1^1, v_1^1, u_2^2, v_2^2], [u_1^2, v_1^2, u_2^1, v_2^1], \quad \quad \quad \quad \quad [u_1^3, v_1^3, u_2^4, v_2^4] \mid (u_1^i, v_1^i, u_2^i, v_2^i) \in S'(G) \},$$

and call elements of $S(G)$ pointed tiles. Notice that by placing the basepoint at the bottom-left vertex, we can arrange that the horizontal sides of each pointed tile be labelled by elements of $U$, and the vertical sides by elements of $V$, such that $S(G) \subseteq U \times V \times U \times V$. Indeed, the four tuples corresponding to each $(u_1^i, v_1^i, u_2^i, v_2^i) \in S'(G)$ in (1) are the four symmetries of a pointed tile which preserve this property (Figure 2).

Note also that by design, any two pointed tiles in $S(G)$ are distinct, and any two adjacent sides of a tile uniquely determine the remaining two sides.

![Figure 2: Visualisation of tiles: $A = [u_1, v_1, u_2, v_2]$, $B = [\bar{u}_1, \bar{v}_1, \bar{u}_2, \bar{v}_2]$](image)

**Definition 2.3.** Let $G$ be a connected bipartite graph on $\alpha$ white and $\beta$ black vertices. Let $U$, $V$ be sets with $|U| = 4\alpha$, $|V| = 4\beta$, and which are endowed with fixed-point-free involutions $u \mapsto \bar{u}$, $v \mapsto \bar{v}$ respectively. Let $S = S(G) \subseteq U \times V \times U \times V$ be the corresponding set of pointed tiles as constructed above. We call the data $(G, U, V, S)$ a tile system. This construction is closely related to that of a VH-datum, introduced in [10] and developed further in [1].

### 3 The $C^*$-algebra corresponding to a tile system

**Definition 3.1.** Let $(G, U, V, S)$ be a tile system, and let $A = [u_1, v_1, u_2, v_2]$, $B = [u_3, v_3, u_4, v_4]$ be pointed tiles in $S$. We define functions $f_1, f_2 : S \times S \rightarrow \{0, 1\}$ as follows:

$$f_1(A, B) = \begin{cases} 1 & \text{if } v_1 = \bar{v}_4 \text{ and } u_1 \neq \bar{u}_3, \\ 0 & \text{otherwise,} \end{cases}$$

$$f_2(A, B) = \begin{cases} 1 & \text{if } u_2 = \bar{u}_3 \text{ and } v_1 \neq \bar{v}_3, \\ 0 & \text{otherwise,} \end{cases}$$

as demonstrated in Figure 3. We define the horizontal adjacency matrix $M_1$ to be the $4\alpha \beta \times 4\alpha \beta$ matrix with $AB$-th entry $f_1(A, B)$, and the vertical adjacency matrix $M_2$ to be that with entries $f_2(A, B)$. 

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Definition 3.2. Let \((G, U, V, S)\) be a tile system, and let \(A, B, C\) be pointed tiles in \(S(G)\) such that \(M_1(A, B) = 1\) and \(M_2(A, C) = 1\). We say that the sets \(U\) and \(V\) satisfy the **Unique Factorisation Property** if there exists a unique \(D \in S\) such that \(M_2(B, D) = M_1(C, D) = 1\) (Figure 3).

Figure 3: Adjacency functions: (a) \(f_1(A, B) = 1\), (b) \(f_2(A, B) = 1\).

Proposition 3.3. Consider the complete bipartite graph \(\kappa = \kappa(\alpha, \beta)\) on \(\alpha \geq 2\) white and \(\beta \geq 2\) black vertices, and let \((\kappa, U, V, S(\kappa))\) be a tile system. Then the corresponding adjacency matrices \(M_1\) and \(M_2\) from Definition 3.1 commute, and the sets \(U, V\) satisfy the **Unique Factorisation Property**.

Proof. Without loss of generality, consider the pointed tile \(A = [u_i^1, v_j^1, u_i^2, v_j^2] \in S(\kappa)\), and define the sets

\[X_A := \{ T \in S(\kappa) \mid f_1(A, T) = 1 \}, \quad Y_A := \{ T \in S(\kappa) \mid f_2(A, T) = 1 \}.\]

So \(X_A\) comprises precisely those tiles of the form \([\bar{u}_k^1, \bar{v}_j^2, \bar{u}_k^2, \bar{v}_j^1]\), where \(k \neq i\), and \(Y_A\) only those of the form \([\bar{u}_l^2, \bar{v}_l^1, \bar{u}_l^1, \bar{v}_l^2]\), where \(l \neq j\). Since \(\alpha, \beta \geq 2\), these sets are non-empty.
Now we can define non-empty sets \((YX)_A := \bigcup_{T \in X_A} Y_T\) and \((XY)_A := \bigcup_{T \in Y_A} X_T\). Notice that \((YX)_A = (XY)_A = \{[u^1_l, v^2_l, u^1_k, v^1_l] \mid k \neq i \text{ and } l \neq j\}\) for all \(A \in \mathcal{S}(\kappa)\), and therefore that \(M_2M_1 = M_1M_2\).

Now, choose elements \(B = \bar{u}^1_{k_0}, v^2_{j_0}, \bar{u}^2_{k_0}, \bar{v}^1_{j_0} \in X_A\), and \(C = \bar{u}^2_i, \bar{v}^1_{l_0}, \bar{u}^1_i, \bar{v}^2{l_0} \in Y_A\). Then \(D = \bar{u}^2_{k_0}, v^1_{l_0}, u^1_{k_0}, v^2_{l_0}\) is the unique pointed tile adjacent to both \(B\) and \(C\).

Similarly, given \(B, C, D \in \mathcal{S}(\kappa)\) with \(f_2(B, D) = f_1(C, D) = 1\), there is a unique tile \(A\) adjacent to both \(B\) and \(C\), and so \(U, V\) satisfy the Unique Factorisation Property. \(\square\)

We will see shortly that a tile system is actually an example of a so-called \(k\)-\text{rank graph}, (specifically a 2-rank graph) which were introduced in [4] to build on work by [7].

**Higher-rank graphs**

**Definition 3.4.** Let \(\Lambda\) be a category such that \(\text{Ob}(\Lambda)\) and \(\text{Hom}(\Lambda)\) are countable sets (that is, a countable small category), and identify \(\text{Ob}(\Lambda)\) with the identity morphisms in \(\text{Hom}(\Lambda)\). For a morphism \(\lambda \in \text{Hom}_A(u, v)\), we define range and source maps \(r(\lambda) = v\) and \(s(\lambda) = u\) respectively.

Let \(d : \Lambda \to \mathbb{N}^k\) be a functor, called the **degree map**, and let \(\lambda \in \text{Hom}(\Lambda)\). We call the pair \((\Lambda, d)\) a \(k\)-**rank graph** (or simply a \(k\)-**graph**) if, whenever \(d(\lambda) = m + n\) for some \(m, n \in \mathbb{N}^k\), we can find unique elements \(\mu, \nu \in \text{Hom}(\Lambda)\) such that \(\lambda = \nu \lambda\), and \(d(\mu) = m, d(\nu) = n\). Note that for \(\mu, \nu\) to be composable, we must have \(r(\mu) = s(\nu)\).

For \(n \in \mathbb{N}^k\), we write \(\Lambda^n := d^{-1}(n)\); by the above property, we have that \(\Lambda^0 = \text{Ob}(\Lambda)\), and we call the elements of \(\Lambda^0\) the **vertices** of \((\Lambda, d)\). [4]

We include standard examples of higher-rank graphs in the event that the reader has not come across them. We direct the reader to e.g. [8] for further reference.

**Examples 3.5.**

1. Let \(E\) be a directed graph with vertex set \(E^0\), edge set \(E^1\), and maps \(r_E, s_E : E^1 \to E^0\) describing the range and source of edges, respectively. For \(n \geq 2\), define \(E^n\) to be the set of paths of length \(n\) on \(E\), and write \(E^* := \bigcup_{n \geq 0} E^n\). The maps \(r_E, s_E\) extend to \(E^*\) in a natural way. Then \(E^*\) is a small category, and we have a functor \(\ell : E^* \to \mathbb{N}\) which maps each path \(\lambda \in E^*\) to its length.

   If \(\ell(\lambda) = m + n\) for some \(m, n \in \mathbb{N}\), then there are unique paths \(\mu, \nu \in E^*\) such that \(\lambda = \nu \cdot \mu\), and \(\ell(\mu) = m, \ell(\nu) = n\).

   We must be wary that two finite paths \(\mu, \nu\) in such a directed graph \(E\) can be concatenated to give a path \(\nu \cdot \mu\) if and only if \(s_E(\mu) = r_E(\nu)\), so for the category \(E^*\) we choose range and source maps \(r(\lambda) = s_E(\lambda)\) and \(s(\lambda) = r_E(\lambda)\) respectively. In this way, \((E^*, \ell)\) is a 1-rank graph.

2. Let \((\Lambda, d)\) be a \(k\)-rank graph. If \(k = 0\), then \(d\) is the trivial functor, and \(\Lambda\) is a set.

   If \(k = 1\), then we can draw a directed graph in the manner above, with vertex set \(\Lambda^0\), edge set \(\Lambda^1\), and range and source maps \(r_E(\lambda) = s(\lambda)\) and \(s_E(\lambda) = r(\lambda)\) respectively.
Following the method of Theorem 2.2, label the elements of the sets $U$ where $u \in U = \{u_1, u_2, \ldots, u_\alpha, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_\alpha, \bar{u}_\bar{2} \}$ and $V = \{v_1, v_2, \ldots, v_\beta, \bar{v}_1, \bar{v}_2, \bar{v}_\beta, \bar{v}_\bar{2}\}$, where $u_1, \ldots, u_\alpha$ and $v_1, \ldots, v_\beta$ are the white and black vertices of $\kappa$, respectively. Construct the tile complex $TC(\kappa)$, and consider the set $S(\kappa) \subseteq U \times V \times U \times V$ of pointed tiles of $TC(\kappa)$. Since $\kappa$ is complete, there is for each $u_i$ and $v_j$ an edge joining them, hence:

$$S(\kappa) = \{[u_1^i, v_1^j, u_2^i, v_2^j], \bar{u}_1^i, v_1^j, v_2^j, \bar{v}_1^i], [u_2^i, v_2^j, u_1^i, v_1^j], [\bar{u}_1^i, \bar{v}_1^j, \bar{u}_2^i, \bar{v}_2^j] \mid 1 \leq i \leq \alpha, 1 \leq j \leq \beta \}.$$
Consider the corresponding adjacency matrices $M_1$ and $M_2$ as described in Definition 3.1 and note that they commute, by Proposition 3.3. We can draw directed graphs $E, F$ with the same vertex set $E^0 = F^0 = S(\kappa)$, and a directed edge joining vertex $A$ to $B$ if and only if $M_1(A, B) = 1$, $M_2(A, B) = 1$, respectively (Figure 5). Write $r_E, s_E$ (resp. $r_F, s_F$) for the maps describing the respective range and source of edges in $E^1$ (resp. $F^1$).

Define the following sets: $E^1 * F^1 := \{(\lambda, \mu) \in E^1 \times F^1 \mid r_E(\lambda) = s_F(\mu)\}$ and $F^1 * E^1 := \{(\mu, \lambda) \in F^1 \times E^1 \mid r_F(\mu) = s_E(\lambda)\}$. By the fact that $M_1, M_2$ commute, there is a bijection $\theta : E^1 * F^1 \to F^1 * E^1$ mapping $(\lambda, \mu) \mapsto (\mu', \lambda')$ such that $s_E(\lambda) = s_F(\mu')$ and $r_F(\mu) = r_E(\lambda')$.

We construct a 2-rank graph $(\Lambda, d)$ as follows: let $\Lambda^0 = S(\kappa)$, and for $(m, n) \in \mathbb{N}^2$, write $W(m, n) := \{(p, q) \in \mathbb{N}^2 \mid p \leq m, q \leq n\}$. Let $A(p, q) \in S(\kappa)$, for $(p, q) \in W(m, n)$, let $\lambda(p, q) \in E^1$, for $(p, q) \in W(m - 1, n)$, and let $\mu(p, q) \in F^1$, for $(p, q) \in (m, n - 1)$. Then an element of $\Lambda^{(m, n)}$ is given by a triple $(A, \lambda, \mu) = (A(p, q), \lambda(p, q), \mu(p, q))$ such that:

(a) $s_E(\lambda(p, q)) = r_F(\mu(p, q)) = A(p, q)$,
(b) $r_E(\lambda(p, q)) = A(p + 1, q)$ and $r_F(\mu(p, q)) = A(p, q + 1)$,
(c) $\theta(\lambda(p, q), \mu(p + 1, q)) = (\mu(p, q), \lambda(p, q + 1))$,

whenever these conditions make sense. We write $\Lambda := \bigcup_{m,n \geq 0} \Lambda^{(m, n)}$, and define range and source maps $r(A, \lambda, \mu) := A(0, 0)$, $s(A, \lambda, \mu) := A(m, n)$ respectively (again, take care over the change in direction).

![Figure 6: An element $(A, \lambda, \mu)$ of $\Lambda^{(m, n)}$ can be represented as an $m \times n$ grid. The isomorphism $\theta$ defines commuting squares. Here is an element of $\Lambda^{(2,3)}$.](image)

If $\varphi, \psi$ are paths of nonzero length $m, n$ in $E, F$ respectively, with $r_E(\varphi) = s_E(\psi)$, then there is a unique element $\varphi \psi = (A, \lambda, \mu) \in \Lambda^{(m, n)}$ such that $\varphi = \lambda(0, 0) \cdots \lambda(m - 1, 0)$, and $\psi = \mu(m, 0) \cdots \mu(m, n - 1)$. If instead (or as well) $r_F(\psi) = s_E(\varphi)$, then there is a unique element $\psi \varphi$ such that $\varphi = \lambda(0, n) \cdots \lambda(m - 1, n)$ and $\psi = \mu(0, 0) \cdots \mu(0, n - 1)$ (Figure 6).

Then, given elements $(A_1, \lambda_1, \mu_1) \in \Lambda^{(m_1, n_1)}$ and $(A_2, \lambda_2, \mu_2) \in \Lambda^{(m_2, n_2)}$ such that $A_1(m_1, n_1) = A_2(0, 0)$, we can find a unique element $(A_1, \lambda_1, \mu_1)(A_2, \lambda_2, \mu_2) = (A_3, \lambda_3, \mu_3)$ in $\Lambda^{(m_1 + m_2, n_1 + n_2)}$ such that:
(a) $A_3(p, q) = A_1(p, q)$, and $A_3(m + p, n + q) = A_2(p, q)$,
(b) $\lambda_3(p, q) = \lambda_1(p, q)$, and $\lambda_3(m + p, n + q) = \lambda_2(p, q)$,
(c) $\mu_3(p, q) = \mu_1(p, q)$, and $\mu_3(m + p, n + q) = \mu_2(p, q)$,

whenever these conditions make sense. In this way, composition is defined in $\Lambda$, and by construction we have associativity and the factorisation property of Definition 3.4. Thus $\Lambda$, together with obvious degree functor $d : (A, \lambda, \mu) \mapsto (p, q)$, has the structure of a 2-rank graph, and we write $(\Lambda, d) = \Lambda(\kappa)$.

**Definitions 3.7.** Let $(\Lambda, d)$ be a $k$-rank graph, let $n \in \mathbb{N}^k$, and let $v \in \Lambda^0$. Write $\Lambda^n(v)$ for the set of morphisms in $\Lambda^n$ which map onto the vertex $v$, that is, $\Lambda^n(v) := \{ \lambda \in \Lambda^n \mid r(\lambda) = v \}$. We say that $(\Lambda, d)$ is **row-finite** if each set $\Lambda^n(v)$ is finite, and that $(\Lambda, d)$ has **no sources** if each $\Lambda^n(v)$ is non-empty.

As an extension of the concept of a **graph algebra** (c.f. [6]), we can associate a $C^*$-algebra to a $k$-rank graph as follows:

**Definition 3.8.** Let $\Lambda = (\Lambda, d)$ be a row-finite $k$-rank graph with no sources. We define $C^* (\Lambda)$ to be the universal $C^*$-algebra generated by a family $\{ s_\lambda \mid \lambda \in \Lambda \}$ of **partial isometries** (that is, operators $s_\lambda$ whose restriction to $(\ker s_\lambda)^{\perp}$ are isometries) which have the following properties:

(a) The set $\{ s_v \mid v \in \Lambda^0 \}$ satisfies $(s_v)^2 = s_v = s_v^*$ and $s_us_v = 0$ for all $u \neq v$.
(b) If $r(\lambda) = s(\mu)$ for some $\lambda, \mu \in \Lambda$, then $s_\lambda s_\mu = s_\mu s_\lambda$.
(c) For all $\lambda \in \Lambda$, we have $s_\lambda^* s_\lambda = s_{s(\lambda)}$.
(d) For all vertices $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have:

$$s_v = \sum_{\lambda \in \Lambda^n(v)} s_\lambda s_\lambda^*.$$  

Note that without the row-finiteness condition, property (d) is not well-defined.

**Theorem 3.9 (Evans, 2008).** Let $\Lambda$ be a row-finite 2-graph with no sources, finite vertex set $\Lambda^0$ with $|\Lambda^0| = n$, and edge matrices $M_E, M_F$. Then:

$$K_0(C^*(\Lambda)) \cong \text{rk} \left( \text{coker} \left( 1 - M_E^T, 1 - M_F^T \right) \right)$$

$$+ \text{rk} (\text{coker}(1 - M_E, 1 - M_F)) + \text{tor} \left( \text{coker} \left( 1 - M_E^T, 1 - M_F^T \right) \right), \quad (2)$$

$$K_1(C^*(\Lambda)) \cong \text{rk} \left( \text{coker} \left( 1 - M_E^T, 1 - M_F^T \right) \right)$$

$$+ \text{rk} (\text{coker}(1 - M_E, 1 - M_F)) + \text{tor}(\text{coker}(1 - M_E, 1 - M_F)), \quad (3)$$

where $1$ is the $n \times n$ identity matrix, $(\ast, \ast)$ denotes the corresponding block $n \times 2n$ matrix, $\text{rk}(\mathfrak{G})$ denotes the torsion-free rank of a finitely-generated Abelian group $\mathfrak{G}$, and $\text{tor}(\mathfrak{G})$ denotes the torsion part of $\mathfrak{G}$. [3, Proposition 4.4]
Corollary 3.10. Let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on $\alpha \geq 2$ white and $\beta \geq 2$ black vertices, and let $(\kappa, U, V, S(\kappa))$ be a tile system with adjacency matrices $M_1$, $M_2$ as in Definition [3.7]. As an abuse of notation, we write $C^*(\kappa) = C^*(\Lambda(\kappa))$. Then

$$K_0(C^*(\kappa)) = K_1(C^*(\kappa)) = \text{coker} \left( 1 - M_1^T, 1 - M_2^T \right) \oplus \text{rk} \left( \text{coker} \left( 1 - M_1^T, 1 - M_2^T \right) \right).$$

Proof. It is straightforward to verify that the matrices $M_1$ and $M_2$ are symmetric (that is, $f_1(A, B) = f_1(B, A)$ and $f_2(A, B) = f_2(B, A)$ for all $A, B \in S(\kappa)$). Furthermore, $\alpha, \beta < \infty$ by assumption, and by the Unique Factorisation Property of $U$ and $V$ (Proposition 3.3) we know that each row and column of $M$ is row-finite, has no sources, and is such that $|\Lambda(\kappa)| = 4\alpha\beta$, whence the result follows from Theorem 3.9 \QED

Theorem 3.11 (K-groups for pointed tile systems). Let $a, b \geq 0$, and let $\kappa(a + 2, b + 2)$ be the complete bipartite graph on $a + 2$ white and $b + 2$ black vertices. Without loss of generality, we assume that $a \leq b$. Write $l := \text{lcm}(a,b)$, and $g := \text{gcd}(a,b)$. Then, for $\epsilon = 1, 2$:

(i) If $a = b = 0$, then $K_\epsilon(C^*(\kappa(a + 2, b + 2))) = K_\epsilon(C^*(\kappa(2, 2))) \cong \mathbb{Z}^8$.

(ii) If $a = 0, 1$ and $b \geq 1$, then

$$K_\epsilon(C^*(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/b)^2 \oplus \mathbb{Z}^{4(b+1)}.$$

(iii) If $a, b \geq 2$ and $a, b$ are coprime, then

$$K_\epsilon(C^*(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/a)^{b-a} \oplus (\mathbb{Z}/ab)^{\alpha+1} \oplus \mathbb{Z}^{2(a+1)(b+1)}.$$

(iv) If $a, b \geq 2$ and $a, b$ are not coprime, then

$$K_\epsilon(C^*(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/a)^{b-a} \oplus (\mathbb{Z}/l)^{\alpha+1} \oplus (\mathbb{Z}/g)^{\alpha+2} \oplus \mathbb{Z}^{2(a+1)(b+1)},$$

where $(\mathbb{Z}/a)^0$ is defined to be the trivial group in the case that $a = b$.

Proof. We begin by proving (iii) and (iv), since (i) and (ii) are special cases thereof.

So, assume that $a, b \geq 2$. Write $\alpha = a + 2$ and $\beta = b + 2$, and for $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$, let $A_{ij}$ denote the pointed tile $[u^1_i, v^2_j, u^2_i, v^2_j] \in S(\kappa)$. Similarly, write $B_{ij} := [\bar{u}^1_i, \bar{u}^2_j, \bar{u}^2_i, \bar{u}^2_j], C_{ij} := [\bar{u}^2_i, \bar{u}^1_j, \bar{u}^1_i, \bar{u}^2_j], D_{ij} := [u^2_i, v^2_j, u^1_i, v^2_j]$ for the horizontal reflection, vertical reflection, and rotation by $\pi$ of $A_{ij}$, respectively. Then $S(\kappa) = \{A_{ij}, B_{ij}, C_{ij}, D_{ij} \mid 1 \leq i \leq \alpha, 1 \leq j \leq \beta\}$, and

$$\text{coker} = \text{coker} \left( 1 - M_1^T, 1 - M_2^T \right) = \left\{ S \in S(\kappa) \left| S = \sum_{T \in S(\kappa)} M_1(S, T) \cdot T \right. \right\} = \sum_{T \in S(\kappa)} M_2(S, T) \cdot T.$$ (4)

Now fix $p \in \{1, \ldots, \alpha\}$, $q \in \{1, \ldots, \beta\}$, and notice that:
Hence the relations of (4) are given by equations of the form

\[ \text{ord}(\mathbb{B}_i) = \text{ord}(\mathbb{A}_j) = \text{ord}(\mathbb{C}_k) = 1, \text{iff } T = B_{ij}; M_1(B_{pq}, T) = 1, \text{iff } T = A_{iq}, \text{ for some } i \neq p, \]

\[ M_1(C_{pq}, T) = 1, \text{ iff } T = D_{ij}; M_1(D_{pq}, T) = 1, \text{ iff } T = C_{iq}, \text{ for some } i \neq p, \]

\[ M_2(A_{pq}, T) = 1, \text{ iff } T = C_{pj}; M_2(B_{pq}, T) = 1, \text{ iff } T = D_{pj}, \text{ for some } j \neq q, \]

\[ M_2(C_{pq}, T) = 1, \text{ iff } T = A_{pj}; M_2(D_{pq}, T) = 1, \text{ iff } T = B_{pj}, \text{ for some } j \neq q. \]

Hence the relations of (4) are given by equations of the form

\[ A_{pq} = (\alpha - 1)A_{pq} + (\alpha - 2) \sum_{i \neq p} A_{iq} \quad \text{and} \quad A_{pq} = (\beta - 1)A_{pq} + (\beta - 2) \sum_{j \neq q} A_{pj}. \]

In particular, we can write

\[ B_{pq} = \sum_{i \neq p} A_{iq} \quad \text{and} \quad C_{pq} = \sum_{j \neq q} A_{pj} \]

such that

\[ A_{pq} = (\alpha - 1)A_{pq} + (\alpha - 2) \sum_{i \neq p} A_{iq} \quad \text{and} \quad A_{pq} = (\beta - 1)A_{pq} + (\beta - 2) \sum_{j \neq q} A_{pj}. \]

Define \( J_q := \sum_{i=1}^{\alpha} A_{iq} \), and \( I_p := \sum_{j=1}^{\beta} A_{pj} \). Then \((\alpha - 2)J_q = (\beta - 2)I_p = 0, \) and by considering the way we have constructed the relations of (4), we deduce that \( \text{ord}(J_q) = \alpha - 2 \) and \( \text{ord}(I_p) = \beta - 2 \). Viewing the sum of all the tiles \( A_{ij} \) both as the sum of all the \( I_i \) and of the \( J_j \), we conclude that

\[
\text{ord} \left( \sum_{i,j} A_{ij} \right) = \text{gcd}(\alpha - 2, \beta - 2).
\]

For convenience, we write \( g := \text{gcd}(\alpha - 2, \beta - 2) \), and the sum of all \( A_{ij} \) as \( \Sigma \). Now, we can also write \( D_{pq} \) in terms of the \( A_{ij} \), namely \( D_{pq} = \sum_{i \neq p} \sum_{j \neq q} A_{ij} \). Hence we can remove all the \( B_{pq}, C_{pq} \), and \( D_{pq} \) from the list of generators of \( \text{coker} \), such that

\[
\text{coker} = \langle A_{pq} \mid (\alpha - 2)J_q = (\beta - 2)I_p = 0, \text{ for } 1 \leq p \leq \alpha, 1 \leq q \leq \beta \rangle.
\]

Notice that \( A_{p1} = I_p - \sum_{j=2}^{\beta} A_{pj} \), and \( B_{1q} = J_q - \sum_{i=2}^{\alpha} A_{iq} \), and that we can write \( I_1 = \Sigma - \sum_{i=2}^{\alpha} I_i \), and \( J_1 = \Sigma - \sum_{j=2}^{\beta} J_j \). So after a series of Tietze transformations on (4), we have

\[
\text{coker} = \langle \Sigma, I_p, J_q, A_{pq} \mid (\alpha - 2)J_q = (\beta - 2)I_p = g\Sigma = 0, \text{ for } 2 \leq p \leq \alpha, 2 \leq q \leq \beta \rangle.
\]

which, after substituting \( a = \alpha - 2, \ b = \beta - 2, \) gives a presentation for \((\mathbb{Z}/b)^{a+1} \oplus (\mathbb{Z}/g)^{b+1} \oplus (\mathbb{Z}/a)^{\alpha+1} \oplus (\mathbb{Z}/\alpha)^{\beta+1}\). In particular, we have \( a + 1 \) copies of \((\mathbb{Z}/b) \oplus (\mathbb{Z}/a)\). It is well-known that if \(a\) and \(b\) are not coprime, \((\mathbb{Z}/b) \oplus (\mathbb{Z}/a) \cong (\mathbb{Z}/l) \oplus (\mathbb{Z}/g)\); in case (iv), this together with Corollary 3.10 immediately gives the desired result. In case (iii), where \(a\) and \(b\) are coprime, we instead have that \((\mathbb{Z}/b) \oplus (\mathbb{Z}/a) \cong (\mathbb{Z}/ab)\), and we are done.

Now consider case (i), where \(a = 2\). Then, following the method above, coker is generated by \( \langle A_{pq} \mid p, q = 1, 2 \rangle \) with trivial relations, and so \( \text{coker} \cong \mathbb{Z}^4 \). Hence by Corollary 3.10 \( K_e(C^*(\kappa)) \cong \mathbb{Z}^8 \).

Similarly, when \(a = 2\) and \(b \geq 3\), it is straightforward to show that

\[ \text{coker} = \langle I_p, A_{pq} \mid (\beta - 2)I_p = 0, \text{ for } p = 1, 2 \text{ and } 2 \leq q \leq \beta \rangle, \]

and when \(a = 3\) and \(b \geq 3\), we have

\[ \text{coker} = \langle \Sigma, I_p, J_q, A_{pq} \mid J_q = (\beta - 2)I_p = \Sigma = 0, \text{ for } p = 2, 3 \text{ and } 2 \leq q \leq \beta \rangle, \]

both of which are presentations for \((\mathbb{Z}/(\beta - 2))^2 \oplus \mathbb{Z}^{(\beta - 1)}\); hence by Corollary 3.10 (ii) is proved. \( \square \)
4 Unpointed tiles

There is an alternative way we could have defined the adjacency matrices above, which will lead to a different 2-rank graph structure.

Define an unpointed tile system $(G, U, V, S')$ in the same way as Definition 2.8 but replacing $S = S(G)$ with the set of unpointed tiles $S' = S'(G)$. Analogues of the results in Section 3 also hold for unpointed tile systems.

**Definition 4.1.** Let $(G, U, V, S')$ be an unpointed tile system, and identify the unpointed tile $A = (u_1, v_1, u_2, v_2) \in S'$ with the set of pointed tiles

$$A := \{[u_1, v_1, u_2, v_2], [\bar{u}_1, \bar{v}_2, \bar{u}_1], [u_2, v_2, u_1, v_1], [\bar{u}_2, \bar{v}_1, \bar{u}_1, \bar{v}_2]\} \subseteq S.$$ 

Identify $B = (u_3, v_3, u_4, v_4) \in S'$ with its corresponding set $B \subseteq S$, and recall the functions $f_1, f_2 : S \times S \to \{0, 1\}$ from Definition 3.1. We define functions $f_1', f_2' : S' \times S' \to \{0, 1\}$ as follows:

$$f_1'(A, B) = \begin{cases} 1 & \text{if } f_1(A_*, B_*) = 1 \text{ for some } A_*, B_* \in B, \\ 0 & \text{otherwise}, \end{cases}$$

$$f_2'(A, B) = \begin{cases} 1 & \text{if } f_2(A_*, B_*) = 1 \text{ for some } A_*, B_* \in B, \\ 0 & \text{otherwise}. \end{cases}$$

We define adjacency matrices $M_1'$, $M_2'$ accordingly.

**Proposition 4.2.** Consider the complete bipartite graph $\kappa = \kappa(\alpha, \beta)$ on $\alpha \geq 2$ white and $\beta \geq 2$ black vertices, and let $(\kappa, U, V, S'(\kappa))$ be an unpointed tile system. Then the corresponding adjacency matrices $M_1'$ and $M_2'$ commute, and $U$, $V$ satisfy the Unique Factorisation Property.

Hence $(\kappa, U, V, S'(\kappa))$ has a 2-rank graph structure.

**Proof.** Given two unpointed tiles $A, B \in S'(\kappa)$, consider their respective sets of pointed tiles $A, B \in S(\kappa)$ as defined in Definition 1.1. Notice that $f_1'(A, B) = 1$ if and only if, for every $A_* \in A$, we can find some $B_* \in B$ such that $f_1(A_*, B_*) = 1$. The same is true for $f_2'$. Write $A = (u^1_i, v^1_j, u^2_k, v^2_l)$, and define sets

$$X_A := \{T \in S'(\kappa) \mid f_1'(A, T) = 1\}, \quad Y_A := \{T \in S'(\kappa) \mid f_2'(A, T) = 1\}.$$

Then $X_A$ contains precisely those tiles of the form $(u^1_i, v^1_j, u^2_k, v^2_l)$, where $k \neq i$, and $Y_A$ only those of the form $(u^1_i, v^1_j, u^2_k, v^2_l)$, where $l \neq j$. The proof then proceeds in a similar fashion to that of Proposition 3.3 and the 2-rank graph structure follows immediately from [4] §6 as in Theorem 3.9.

We write $\Lambda'(\kappa)$ for the 2-rank graph induced from the adjacency matrices $M_1'$ and $M_2'$. It is not difficult to verify that $\Lambda'(\kappa)$ is row-finite, with finite vertex set and no sources. Hence we can apply Evans’ Theorem 3.9 and we derive the following result:

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Theorem 4.3 (K-groups for unpointed tile systems). Let \(a, b \geq 0\), and let \(\kappa(a + 2, b + 2)\) be the complete bipartite graph on \(a + 2\) white and \(b + 2\) black vertices. Again, without loss of generality, we can assume that \(a \leq b\). Write \(C^*(\kappa) := C^*(\Lambda'(\kappa))\). Then, for \(\epsilon = 1, 2\):

(i) If \(a = b = 0\), then 
\[
K_\epsilon(C^*(\kappa(a + 2, b + 2))) = K_\epsilon(C^*(\kappa(2, 2))) \cong \mathbb{Z}^2.
\]

(ii) If \(a = 0\) and \(b \geq 1\), then 
\[
K_\epsilon(C^*(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/2)^b \oplus (\mathbb{Z}/(2b)).
\]

(iii) If \(a, b \geq 1\), then 
\[
K_\epsilon(C^*(\kappa(a + 2, b + 2))) \cong (\mathbb{Z}/2)^{(a+1)(b+1)} - 1 \oplus (\mathbb{Z}/g),
\]
where \(g := \gcd(2a, 2b)\).

Proof. Again, we start by proving (iii), as the first two cases follow. Write \(\alpha := a + 2\), \(\beta := b + 2\), and let \(\alpha, \beta \geq 3\). For \(1 \leq i \leq \alpha\), \(1 \leq j \leq \beta\), write \(A_{ij}\) for the unpointed tile \((u_{1i}, v_{1j}, u_{2i}, v_{2j}) \in S'(\kappa)\). Then

\[
coker = coker \left( 1 - (M_1')^T, 1 - (M_2')^T \right) = \left\{ A_{ij} \in S'(\kappa) \mid A_{ij} = \sum_{T \in S'(\kappa)} M_1'(A_{ij}, T) \cdot T = \sum_{T \in S'(\kappa)} M_2'(A_{ij}, T) \cdot T \right\}. \tag{7}
\]

Fix \(p \in \{1, \ldots, \alpha\}, q \in \{1, \ldots, \beta\}\), and notice that:

- \(M_1'(A_{pq}, T) = 1\) if and only if \(T = A_{1q}\), for some \(i \neq p\),
- \(M_2'(A_{pq}, T) = 1\) if and only if \(T = A_{pj}\), for some \(j \neq q\).

Hence the relations of (7) are given by \(A_{pq} = \sum_{i \neq p} A_{iq} = \sum_{j \neq q} A_{pj}\). Define

\[
J_{pq} := \left( \sum_{i=2}^{\alpha} A_{iq} \right) - A_{pq} \quad \text{and} \quad I_{pq} := \left( \sum_{j=2}^{\beta} A_{pj} \right) - A_{pq},
\]

for \(p, q \geq 2\). Then

\[
2J_{pq} = 2 \left( \sum_{i=2}^{\alpha} A_{iq} \right) - 2A_{pq}
= 2(A_{2q} + \cdots + A_{aq} - A_{pq}) + A_{1q} - A_{1q}
= (A_{1q} + A_{2q} + \cdots + A_{aq} - A_{pq}) + (-A_{1q} + A_{2q} + \cdots + A_{aq}) - A_{pq}
= A_{pq} + 0 - A_{pq}
= 0,
\]
and similarly $2I_{pq} = 0$. Note that $J_{pq} = 0$ or $I_{pq} = 0$ only if $A_{pq} = A_{1q}$ or $A_{pq} = A_{p1}$ respectively. But since $\alpha, \beta \geq 3$, these equivalences are not relations of (7), and so $\text{ord}(J_{pq}) = \text{ord}(I_{pq}) = 2$. Notice that we can write each $A_{1q}$ and $A_{p1}$ in terms of the other $A_{ij}$, for $p,q \geq 2$; hence we can remove these from the list of generators by a sequence of Tietze transformations.

Also notice that we can write $A_{2q} = J_{2q} - \sum_{i=3}^{\alpha} A_{iq}$. Proceeding inductively, we can write each $A_{pq}$ in terms of the $J_{iq}$ and the $A_{iq}$ for $i > p$. Similarly, we can write each $A_{pq}$ in terms of the $I_{pj}$ and the $A_{pj}$ for $j > q$. Hence we can rewrite the generators of coker as $A_{11}, I_{pq}, J_{pq}$, for $p,q \geq 2$.

But $A_{11} = -(A_{p1} + J_{p1}) = -(A_{1q} + I_{1q})$ for all $p,q \geq 2$, so

$$(\alpha - 2)A_{11} = - \sum_{i=3}^{\alpha} (A_{i1} + J_{i1}) = - \left( J_{21} + \sum_{i=3}^{\alpha} J_{i1} \right),$$

and so $2(\alpha - 2)A_{11} = 0$. Similarly, we find that $2(\beta - 2)A_{11} = 0$, and hence that $\text{gcd}(2(\alpha - 2), 2(\beta - 2))A_{11} = 0$. Write $g := \text{gcd}(2(\alpha - 2), 2(\beta - 2))$. By considering the procedural construction of relations in (7), it becomes clear that in fact $\text{ord}(A_{11}) = g$.

Observe that, since $I_{pq}$ is defined in terms of the $A_{pj}$, and each $A_{pj}$ can be written in terms of the $J_{ij}$, we can remove the $I_{pq}$ from the list of generators of coker. Finally, the sum of all the tiles $A_{ij}$ is 0, and in particular we can write $A_{22}$ in terms of the remaining tiles. Hence we can rewrite (7) as

\[
\text{coker} = \langle J_{2q}, J_{p2}, J_{pq}, A_{11} \mid 2J_{2q} = 2J_{p2} = 2J_{pq} = gA_{11} = 0, \text{ for } 3 \leq p \leq \alpha, 3 \leq q \leq \beta \rangle,
\]

which, after substituting $a = \alpha - 2$, $b = \beta - 2$, gives a presentation for $(\mathbb{Z}/2)^{(a+1)(b+1)-1} \oplus (\mathbb{Z}/g)$. Since there is no torsion-free part, this proves (iii).

If $\alpha = 2$, then $A_{1q} = A_{2q}$ for all $1 \leq q \leq \beta$, so we can write

\[
\text{coker} = \langle A_{1q} \mid A_{1q} = \sum_{j \neq q} A_{1j}, \text{ for } 1 \leq q \leq \beta \rangle.
\]

We adjust the proof above accordingly to obtain the result of (ii). Finally, in case (i) where $\alpha = \beta = 2$, we have $A_{11} = A_{12} = A_{21} = A_{22}$ with no further relations, such that $\text{coker} = \langle A_{11} \rangle \cong \mathbb{Z}$, and the result follows from Theorem 5.2

\[\square\]

5 The homology of a tile complex

Theorem 5.1. Let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on $\alpha \geq 2$ white and $\beta \geq 2$ black vertices, let $(\kappa, U, V, S'(\kappa))$ be an unpointed tile system, and let $TC(\kappa)$ be its associated tile complex. Then the homology groups of $TC(\kappa)$ are given by:

\[
H_n(TC(\kappa)) \cong \begin{cases} 
0 & \text{for } n = 0, \\
\mathbb{Z}^{\alpha + \beta - 2} & \text{for } n = 1, 2, \\
0 & \text{for } n \geq 3.
\end{cases}
\]

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Proof. Since $TC(\kappa)$ is a path-connected, 2-dimensional CW-complex by construction, clearly $H_n(TC(\kappa)) \cong 0$ for $n = 0$ and $n \geq 3$.

The proof uses as its basis that of [3 Proposition 3]. The boundary of each square in $TC(\kappa)$ is given by an element of $S^1(\kappa)$; write these elements as $(u_1^1, v_j^1, u_2^1, v_j^2)$. By construction, $TC(\kappa)$ has four vertices: each one the origin of all directed edges labelled $u_1^1$, $v_j^1$, $u_2^1$, and $v_j^2$ respectively. Each tile is homotopy equivalent to a point; pick tile $(u_1^1, v_1^1, u_2^1, v_2^2)$ and contract it, thereby identifying the four vertices. Call the resulting tile complex $TC_1(\kappa)$.

This is a 2-dimensional CW-complex whose edges are loops, and whose 2-cells comprise:

- $(\alpha - 1)(\beta - 1)$-many unpointed tiles $A_{ij} = (u_1^1, v_j^1, u_2^1, v_j^2)$,
- $(\alpha - 1)$-many 2-gons $B_i$ with boundaries described analogously by $(u_1^1, u_2^1)$,
- $(\beta - 1)$-many 2-gons $C_j$ with boundaries described by $(v_j^1, v_j^2)$,

for $2 \leq i \leq \alpha$, $2 \leq j \leq \beta$. Consider the chain complex associated to $TC_1(\kappa)$:

$$\cdots \to C_3 \overset{\partial_3}{\to} C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0 \overset{\partial_0}{\to} 0.$$ 

Since $TC_1(\kappa)$ is 2-dimensional and has one vertex, this boils down to

$$0 \to C_2 \overset{\partial_2}{\to} C_1 \overset{\partial_1}{\to} C_0 \overset{\partial_0}{\to} 0,$$

and so $H_1(TC_1(\kappa)) \cong C_1 / \text{im}(\partial_2)$, and $H_2(TC_1(\kappa)) \cong \text{ker}(\partial_2)$. We have $\partial_2(A_{ij}) = u_1^1 + v_j^1 + u_2^1 + v_j^2$, $\partial_2(X_i) = u_1^1 + u_2^1$, and $\partial_2(Y_j) = v_j^1 + v_j^2$. Clearly $\ker(\partial_2)$ is generated by $\{A_{ij} - B_i - C_j \mid 2 \leq i \leq \alpha, 2 \leq j \leq \beta\}$, but each $A_{ij}$ can be written as $A_{ij} = B_i + C_j$, such that $\ker(\partial_2) \cong \mathbb{Z}^{\alpha + \beta - 2}$.

Similarly, we have an Abelian group presentation for $H_1(TC_1(\kappa))$ as follows:

$$H_1(TC_1(\kappa)) \cong \langle u_i^1, v_j^1, u_i^2, v_j^2 \mid u_1^1 + v_1^1 + u_2^1 + v_2^2 = u_1^1 + u_2^1 = v_1^1 + v_2^2 = 0, \text{for } 2 \leq i \leq \alpha, 2 \leq j \leq \beta \rangle,$$

which, after substituting $u_2^2 = -u_1^1$ and $v_2^2 = -v_1^1$, gives

$$H_1(TC_1(\kappa)) \cong \langle u_1^1, v_1^1, \text{for } 2 \leq i \leq \alpha, 2 \leq j \leq \beta \rangle.$$

This is a presentation for $\mathbb{Z}^{\alpha + \beta - 2}$, and since $TC_1(\kappa)$ is homotopy equivalent to $TC(\kappa)$, we are done.

6 Pointed and unpointed $2t$-polyhedra

In this section we suggest generalisations of the methods above for constructing $C^*$-algebras associated to $2t$-polyhedra, both for even and arbitrary $t \geq 1$.

When $t = 2$, we have an innate idea of what it means for two $2t$-gons to be stackable: functions we called horizontal and vertical adjacency in Definition 3.1. We extend this notion to all even $t \geq 2$ in as natural a way possible.
Consider the adjacency functions \( f \) on a polygon system. Let \( A \) be a pointed polygon, and only if, after reflecting \( A \) through an axis connecting the midpoints of sides labelled \( u \) and \( v \), we can obtain \( B \) starting from a distinguished basepoint by the sequence \( u \rightarrow v \rightarrow u \rightarrow v \rightarrow \ldots \). Reflect \( A \) through an axis connecting the midpoints of sides labelled \( u \) and \( v \), and denote them by \( A_u \) and \( A_v \). Write \( \bar{A} = \{u, v\} \) for the set of 2-gons which comprise \( A \), and \( \bar{B} = \{\bar{u}, \bar{v}\} \) for a pointed polygon, that is, one labelled anticlockwise and starting from a distinguished basepoint. We can view two pointed tiles \( (G, U, V, \mathcal{S}) \) as being horizontally adjacent (that is, \( f(A, B) = 1 \)) if and only if, after reflecting \( A \) through an axis connecting the midpoints of sides labelled \( u \) and \( v \), and then changing the labels of those edges, we say that \( A \) and \( B \) are vertically adjacent.

**Definitions 6.1.** Let \( G \) be a connected bipartite graph on \( \alpha \) white and \( \beta \) black vertices. Let \( U, V \) be sets with \( |U| = 2t\alpha \), \( |V| = 2t\beta \), and which are gifted with fixed-point-free involutions \( u \mapsto \bar{u} \), \( v \mapsto \bar{v} \) respectively. Construct the 2t-polyhedron \( P(G) \) from Theorem 2.2 which has \( G \) as its link at each vertex, using \( U \) and \( V \), and write \( S'(G) := \{ A_e \mid e \in E(G) \} \) for the set of 2t-gons which comprise \( P(G) \). We call elements of \( S'(G) \) unpointed polygons, and denote them by \( A_e = (u_1, v_1, \ldots, u_t, v_t) \). Analogously to Section 2, we write \( [u_1, v_1, \ldots, u_t, v_t] \) for a pointed polygon, that is, one labelled anticlockwise and starting from a distinguished basepoint by the sequence \( u_1, v_1, \ldots, u_t, v_t \), for some \( u_i \in U \), \( v_i \in V \). Write \( S_t = S_t(G) \) for the set of 2t\alpha\beta pointed polygons. We call the tuple \((G, U, V, S_t)\) a polygon system. Similarly we call a tuple \((G, U, V, S'_t)\) an unpointed polygon system.

Consider the adjacency functions \( f_1, f_2 \) from Definition 3.1. We can view two pointed tiles \( (4\text{-gons}) \) \( A = [u_1, v_1, u_2, v_2] \) and \( B \) as being horizontally adjacent (that is, \( f_1(A, B) = 1 \)) if and only if, after reflecting \( A \) through an axis connecting the midpoints of sides labelled \( u_1 \) and \( u_2 \), and then replacing \( u_1, u_2 \) by some \( u'_1 \neq u_1, u'_2 \neq u_2 \) respectively, we can obtain \( B \). Likewise, if and only if we can obtain \( B \) by reflecting \( A \) through an axis joining the midpoints of the \( v \) edges, and then changing the labels of those edges, do we say that \( A \) and \( B \) are vertically adjacent.

**Definition 6.2.** Let \( t \) be an even integer, let \((G, U, V, \mathcal{S}_t)\) be a polygon system, and let \( A = [u_1, v_1, \ldots, u_t, v_t] \in \mathcal{S}_t \) be a pointed polygon. Reflect \( A \) through an axis joining the midpoints of sides labelled \( u_1 \) and \( u_{(t/2)+1} \) to obtain a new pointed polygon \([\bar{u}_1, \bar{u}_t, \bar{v}_1, v_{t-1}, \ldots, \bar{u}_2, \bar{v}_1] \). We say that a pointed polygon \( B \in \mathcal{S}_t \)
is $V$-adjacent to $A$ if $B = [u'_1, v_1, u'_2, v_{t-1}, \ldots, u'_2, v_1]$, for some $u'_i \neq u_i$.

Similarly, reflect $A$ such that $u_1 \mapsto \bar{u}_{(t/2)+1}$; we obtain a new pointed polygon
\[ [\bar{u}_{(t/2)+1}, \bar{v}_{(t/2)+1}, \bar{u}_{t/2}, \bar{v}_{t/2}, \ldots, \bar{u}_1, \bar{v}_1, \bar{u}_t, \bar{v}_t, \ldots, \bar{u}_{(t/2)+2}, \bar{v}_{(t/2)+1}] . \tag{8} \]

We say that a pointed polygon $B \in S_t$ is $U$-adjacent to $A$ if $B$ is of the form [9], but with all elements $v_i$ replaced with some $v'_i \neq v_i$ (Figure [7]).

We define the $U$- and $V$-adjacency matrices, $M_U$ and $M_V$ respectively, to be the $2t \alpha \beta \times 2t \alpha \beta$ matrices with $AB$-th entry 1 if $A$ and $B$ are $U$-adjacent (resp. $V$-adjacent), and 0 otherwise.

**Proposition 6.3.** Let $t$ be even, and $(\kappa, U, V, S_t(\kappa))$ be a polygon system with adjacency matrices $M_U$, $M_V$. Then these matrices commute, and $U$, $V$ satisfy the Unique Factorisation Property.

Hence $(\kappa, U, V, S_t(\kappa))$ has a 2-rank graph structure.

**Proof.** Without loss of generality, consider the pointed polygon $A = [u^1_1, v_1, \ldots, u^1_i, v^1_i] \in S_t(\kappa)$. Then a pointed polygon $B$ is $V$-adjacent to $A$ if and only if $B = [\bar{u}^1_k, \bar{v}^1_l, \ldots, \bar{u}^2_k, \bar{v}^2_l]$, for some $k \neq i$. Suppose $B$ is such a polygon $V$-adjacent to $A$; then a pointed polygon $D$ is $U$-adjacent to $B$ if and only if
\[
D = \left[ u^{(t/2)+1}_k, v^{(t/2)+1}_l, \ldots, u^1_k, v^1_l, u^2_k, v^2_l \right] , \tag{9} \]

for some $l \neq j$. Likewise, $C$ is $U$-adjacent to $A$ if and only if
\[
C = \left[ \bar{u}^{(t/2)+1}_i, \bar{v}^{(t/2)+1}_i, \ldots, \bar{u}^1_i, \bar{v}^1_i, \bar{u}^2_i, \bar{v}^2_i \right] , \]

for some $l \neq j$. Clearly if $C$ is such a polygon, then $D$ is $V$-adjacent to $C$ if and only if it is of the form [9]. Exactly one such $D$ exists in $S_t(\kappa)$, hence $U$, $V$ have the Unique Factorisation Property, and the 2-rank graph structure follows from [4, §6].

Recall the 2-rank graph $\Lambda(\kappa)$ induced from a tile system and its adjacency matrices $M_1$, $M_2$ in Section 3 and recall its associated universal $C^*$-algebra $C^*(\Lambda)$ from Definition 3.8. Similarly, we write $\Lambda_t(\kappa)$ for the 2-rank graph induced from the $U$- and $V$-adjacency matrices $M_U$ and $M_V$, and observe that $\Lambda_t(\kappa)$ is row-finite, with finite vertex set and no sources. Hence from Evans’ Theorem 3.9, we can deduce:

**Theorem 6.4** (K-groups for pointed polygon systems, $t$ even). Let $\alpha, \beta \geq 2$, let $t \geq 2$ be even, and let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on $\alpha$ white and $\beta$ black vertices. Then
\[
K_\epsilon(C^*(\Lambda_t(\kappa))) \cong (K_\epsilon(C^*(\Lambda(\kappa))))^{t/2} ,
\]

for $\epsilon = 1, 2$.

**Proof.** Fix $t$ and assume without loss of generality that $\alpha \leq \beta$. Analogously to in the proof of Theorem 6.11 we denote the pointed polygons in $S_t(\kappa)$ as follows:

- $(A_r)_{ij} := [u^1_i, v^r_j, \ldots, u^1_i, v^r_j, u^{-1}_i, v^{-1}_j, \ldots, u^{-1}_i, v^{-1}_j]$,
for $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$, $1 \leq r \leq t/2$, and with addition in superscript indices defined modulo $t$. Note that each $S \in S_t(\kappa)$ takes one of the above forms. Then

$$\operatorname{coker}(1 - M_U^T, 1 - M_V^T) = \left\langle (A_r)_{pq}, (B_r)_{pq}, (C_r)_{pq}, (D_r)_{pq} \right\rangle$$

$$= \sum_{i \neq p} (B_r)_{iq} = \sum_{j \neq q} (C_r)_{pj},$$

$$= \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (D_r)_{pj},$$

$$= \sum_{i \neq p} (D_r)_{iq} = \sum_{j \neq q} (A_r)_{pj},$$

$$= \sum_{i \neq p} (C_r)_{iq} = \sum_{j \neq q} (B_r)_{pj},$$

for $1 \leq p \leq \alpha, 1 \leq q \leq \beta$, and $1 \leq r \leq t/2$.

But, comparing this to the presentation (4), we see this is precisely a presentation for the direct sum of $t/2$ copies of $\operatorname{coker}(I - M_U^T, I - M_V^T)$ as in Theorem 5.11 and the result follows.

If we extend the concept of $U$- and $V$-adjacency from Definition 6.2 in the obvious way, we can obtain a similar result for unpointed polygon systems of complete bipartite graphs. Write $\Lambda^\prime_t(\kappa)$ for the induced 2-rank graph. We realise that the proof of Theorem 4.3 does not depend on the number of sides $2t$ of the polygons; hence nor do the K-groups associated to $\Lambda^\prime_t(\kappa)$.

**Corollary 6.5** (to Theorem 4.3) K-groups for unpointed polygon systems. Let $\alpha, \beta \geq 2$, and let $\kappa = \kappa(\alpha, \beta)$ be the complete bipartite graph on $\alpha$ white and $\beta$ black vertices. Then

$$K_\epsilon(C^\ast(\Lambda^\prime_t(\kappa))) \cong K_\epsilon(C^\ast(\Lambda^\prime(\kappa))),$$

for $\epsilon = 1, 2$, and all $t \geq 1$.

**Corollary 6.6** (to Theorem 5.11). Let $(\kappa, U, V, S_t(\kappa))$ be an unpointed polygon system, and let $P(\kappa)$ be its associated $2t$-polyhedron. Then the homology groups of $P(\kappa)$ do not depend on $t$, that is:

$$H_n(P(\kappa)) \cong \begin{cases} 0 & \text{for } n = 0, \\ \mathbb{Z}^{\alpha+\beta-2} & \text{for } n = 1, 2, \\ 0 & \text{for } n \geq 3. \end{cases}$$

$$\square$$
Questions on canonicality

Corollary 6.5 gives us a collection of K-groups corresponding to systems of polygons with an arbitrary even number of sides $2t$, whereas in the pointed case, Theorem 6.4 insists on $2t$ being divisible by four. This is due to how we define adjacency in each instance: in the $2t$-polyhedron $P(\kappa)$, each face is adjacent to every other, and since the number of faces is not dependent on $t$, nor are the $U$- and $V$-adjacency matrices in an unpointed polygon system.

Adjacency in the pointed case is more difficult to define canonically. When $t = 2$, and we are dealing with tiles, there is an obvious pair of adjacency functions. We extended these in Definition 6.2 thinking of two polygons as adjacent if we can reflect one horizontally or vertically in order to obtain the form of the other. This works since horizontal and vertical reflections commute, and so the adjacency matrices will satisfy the Unique Factorisation Property. If $t$ is not even, then there are no two distinct reflections of $2t$-gons which commute, and preserve the structure of pointed polygons. We must pick the same two reflections for both adjacency functions, else some combination of rotations and identity transformations. None of these options is a direct extension of our horizontal and vertical adjacency functions from Section 3, and so there is no natural choice.

We suggest that the following definitions of $U$- and $V$-adjacency for pointed $2t$-gons are the most intuitive, based on the idea that adjacent polygons should have opposite orientations.

**Definition 6.7.** Let $t \geq 1$ be a fixed arbitrary integer, let $(G, U, V, S_t)$ be a polygon system, and let $A = [u_1, v_1, \ldots, u_t, v_t] \in S_t$ be a pointed polygon.

We say that a pointed polygon $B \in S_t$ is $V^*$-adjacent to $A$ if and only if $B = [\bar{u}_1', \bar{v}_1', \ldots, \bar{u}_t', \bar{v}_t']$, for some $u_i' \neq u_i$.

Similarly, we say that a pointed polygon $C \in S_t$ is $U^*$-adjacent to $A$ if and only if $C = [u_1', \bar{v}_1', \ldots, \bar{u}_t', \bar{v}_t']$, for some $v_i' \neq v_i$. We define the $U^*$- and $V^*$-adjacency matrices $M_U^*$ and $M_V^*$ respectively, as above.

The proof of the following is almost identical to that of Proposition 6.3 together with Proposition 3.6. From this, along with Theorem 3.9, we can deduce Theorem 6.9.

**Proposition 6.8.** Let $(\kappa, U, V, S_t(\kappa))$ be a polygon system with adjacency matrices $M_U^*$, $M_V^*$. Then $(\kappa, U, V, S_t(\kappa))$ induces a 2-rank graph $\Lambda_t^*(\kappa)$, which is row-finite, with finite vertex set and no sources.

**Theorem 6.9** (K-groups for pointed polygon systems, $t$ arbitrary). Let $a, b \geq 0$, let $t \geq 1$, and let $\kappa = \kappa(a+2, b+2)$ be the complete bipartite graph on $a+2$ white and $b+2$ black vertices. Without loss of generality, we assume that $a \leq b$. Then, for $\epsilon = 1, 2$:

(i) If $a = b = 0$, then $K_\epsilon(C^* (\Lambda_t^*(\kappa))) \cong \mathbb{Z}^{4t}$.

(ii) If $b \geq 1$ and $a, b$ are coprime, then $K_\epsilon(C^* (\Lambda_t^*(\kappa))) \cong \mathbb{Z}^{2t(a+1)(b+1)}$.

(iii) If $b \geq 1$ and $a, b$ are not coprime, then

$$K_\epsilon(C^* (\Lambda_t^*(\kappa))) \cong \mathbb{Z}^{2t(a+1)(b+1)} \oplus (\mathbb{Z}/g)^t,$$

where $g := \gcd(a, b)$.


**Proof.** The proof unsurprisingly follows the same lines as those of Theorems 3.11, 4.3 and 6.4. Write \( \alpha := a + 2, \beta := b + 2, \) and let \( \beta \geq 3 \). We denote the pointed polygons in \( S_t(\kappa) \) as:

\[
\begin{align*}
(A)_{ij} & := [u_i^r, v_j^r, \ldots, u_i^r, v_j^r, u_i^r, v_j^r, \ldots, u_i^{r-1}, v_j^{r-1}], \\
(B)_{ij} & := [\bar{u}_i^r, \bar{v}_j^r, \ldots, \bar{u}_i^r, \bar{v}_j^r, \ldots, \bar{u}_i^{r+1}, \bar{v}_j^{r+1}],
\end{align*}
\]

for \( 1 \leq i \leq \alpha, 1 \leq j \leq \beta, 1 \leq r \leq t \), and with addition in superscript indices defined modulo \( t \). Observe that each \( S \in S_t(\kappa) \) is either of the form \((A)_{ij}\) or \((B)_{ij}\). Then

\[
coker = \text{coker } (1 - (M_U^T)T, 1 - (M_V^T)T) = \left\{ (A_r)_{pq}, (B_r)_{pq} \mid \begin{align*}
(A_r)_{pq} &= \sum_{i \neq p} (B_r)_{iq} = \sum_{j \neq q} (B_r)_{pj}, \\
(B_r)_{pq} &= \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (A_r)_{pj},
\end{align*} \right. \text{ for } 1 \leq p \leq \alpha, 1 \leq q \leq \beta, \text{ and } 1 \leq r \leq t \right\}.
\]

As in the proof of Theorem 3.11 define \((J_r)_q := \sum_{i=1}^\alpha (A_r)_{iq}, \) and \((I_r)_p := \sum_{j=1}^\beta (A_r)_{pj} \). By means of a sequence of Tietze transformations, and using some observations from previous proofs, we see that the above presentation is equivalent to

\[
coker = \left\{ (A_r)_{pq} \right\} \left\{ (A_r)_{pq} = \sum_{i \neq p} \sum_{k \neq i} (A_r)_{ki} = \sum_{j \neq q} \sum_{l \neq j} (A_r)_{pl}, \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (A_r)_{pj} \right\} = \left\{ (A_r)_{pq} \right\} \left\{ (\alpha - 2)(J_r)_q = (\beta - 2)(I_r)_p = 0, \sum_{i \neq p} (A_r)_{iq} = \sum_{j \neq q} (A_r)_{pj} \right\} = \left\{ (A_r)_{pq} \right\} \left\{ (\alpha - 2)(J_r)_q = (\beta - 2)(I_r)_p = 0, (J_r)_q = (I_r)_p, \text{ for all } p, q \right\}.
\]

We can rewrite each \((A_r)_{1i}\) and \((A_r)_{ij}\) in terms of the other \((A_r)_{ij}\), the \((J_r)_q\), and the \((I_r)_p\), and hence remove them from the list of generators. Then, since \((J_r)_q = (I_r)_p\) for all \( 1 \leq p \leq \alpha, 1 \leq q \leq \beta \) we can remove all-but-one of these from the list of generators as well, leaving:

\[
coker = \langle (A_r)_{pq}, (J_r)_1 \mid (\alpha - 2)(J_r)_1 = (\beta - 2)(J_r)_1 = 0, \text{ for } 2 \leq p \leq \alpha, 2 \leq q \leq \beta, \text{ and } 1 \leq r \leq t \rangle. \quad (10)
\]

We substitute \( a = \alpha - 2, b = \beta - 2, \) and write \( q := \gcd(a, b) \). Then (10) is a presentation for \( \mathbb{Z}^{((a+1)(b+1))} \oplus (\mathbb{Z}/q)^t \) if \( q > 1, \) and \( \mathbb{Z}^{((a+1)(b+1))} \) otherwise. If \( a = \beta = 2, \) then (10) gives a presentation for \( \mathbb{Z}^2 \). Together with Theorem 3.9 this gives the desired result. \( \square \)
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