Multistage Estimation of Bounded-Variable Means *

Xinjia Chen

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Abstract

In this paper, we develop a multistage approach for estimating the mean of a bounded variable. We first focus on the multistage estimation of a binomial parameter and then generalize the estimation methods to the case of general bounded random variables. A fundamental connection between a binomial parameter and the mean of a bounded variable is established. Our multistage estimation methods rigorously guarantee prescribed levels of precision and confidence.

1 Introduction

The estimation of the means of bounded random variables finds numerous applications in various fields of sciences and engineering. In particular, Bernoulli random variables constitute an extremely important class of bounded variables, since the ubiquitous problem of estimating the probability of an event can be formulated as the estimation of the mean of a Bernoulli variable. In many applications, one needs to estimate a quantity \( \mu \) which can be bounded in \([0, 1]\) after proper operations of scaling and translation. A typical approach is to design an experiment that produces a random variable \( Z \) distributed in \([0, 1]\) with expectation \( \mu \), run the experiment independently a number of times, and use the average of the outcomes as the estimate \([7]\). This technique, referred to as Monte Carlo method, has been applied to tackle a wide range of difficult problems.

Since the estimator of the mean of \( Z \) is obtained from finite samples of \( Z \) and is thus of random nature, for the estimator to be useful, it is necessary to ensure with a sufficiently high confidence that the estimation error is within certain margin. The well known Chernoff-Hoeffding bound \([3, 6]\) asserts that if the sample size is fixed and is greater than \( \frac{\ln \frac{2}{\delta}}{\epsilon^2} \), then, with probability at least \( 1 - \delta \), the sample mean approximates \( \mu \) with absolute error \( \epsilon \). The problem with Chernoff-Hoeffding bound is that the resultant sample size can be extremely conservative if the value of \( \mu \) is close to zero or one. In the case that \( \mu \) is small, it is more reasonable to seek an \((\epsilon, \delta)\) approximation for \( \mu \) in the sense that the relative error of the estimator is within a margin of

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*The author had been previously working with Louisiana State University at Baton Rouge, LA 70803, USA, and is now with Department of Electrical Engineering, Southern University and A&M College, Baton Rouge, LA 70813, USA; Email: chenxinjia@gmail.com
relative error $\varepsilon$ with probability at least $1 - \delta$. Since the mean value $\mu$ is exactly what we want to estimate, it is usually not easy to obtain reasonably tight lower bound for $\mu$. For a sampling scheme with fixed sample size, a loose lower bound of $\mu$ can lead to a very conservative sample size. For the most difficult and important case that no positive lower bound of $\mu$ is available, it is not possible to guarantee prescribed relative precision and confidence level by a sampling scheme with a fixed sample size. This forces us to look at sampling methods with random sample sizes.

The estimation techniques based on sampling schemes without fixed sample sizes have formed a rich branch of modern statistics under the heading of sequential estimation. Wald provided a brief introduction to this area in his seminal book [10]. Ghosh et al. offered a comprehensive exposition in [5]. In particular, Nadas proposed in [9] a sequential sampling scheme for estimating mean values with relative precision. Nadas’s sequential method requires no specific information on the mean value to be estimated. However, his sampling scheme is of asymptotic nature. The confidence requirement is guaranteed only as the margin of relative error $\varepsilon$ tends to 0, which implies that the actual sample size has to be infinity. This drawback severely circumvents the application of his sampling scheme.

In this paper, we revisit the sequential estimation of means of random variables bounded in $[0, 1]$. To overcome the limitations of existing methods, we have developed a new class of multistage sampling schemes. Our sampling schemes require no information of the unknown parameters and guarantees prescribed levels of precision and confidence. The remainder of the paper is organized as follows. Section 2 is devoted to the multistage estimation of a binomial parameter. In Section 3, we generalize the estimation methods of a binomial parameter to the mean of a bounded variable. In Section 4, we establish a link between a binomial parameter and the mean of a bounded variable. We demonstrate that the estimation methods for estimating a binomial parameter can be easily applied to the estimation of the mean of a bounded variable by virtue of this link. Section 5 is the conclusion. All proofs are given in the Appendices.

Throughout this paper, we shall use the following notations. The expectation of a random variable is denoted by $E[\cdot]$. The set of integers is denoted by $\mathbb{Z}$. The set of positive integers is denoted by $\mathbb{N}$. The ceiling function and floor function are denoted respectively by $\lceil \cdot \rceil$ and $\lfloor \cdot \rfloor$ (i.e., $\lceil x \rceil$ represents the smallest integer no less than $x$; $\lfloor x \rfloor$ represents the largest integer no greater than $x$). The notation $\text{sgn}(x)$ denotes the sign function which assumes value 1 for $x > 0$, value 0 for $x = 0$, and value $-1$ for $x < 0$. We use the notation $\Pr\{ \cdot \mid \theta \}$ to indicate that the associated random samples $X_1, X_2, \ldots$ are parameterized by $\theta$. The parameter $\theta$ in $\Pr\{ \cdot \mid \theta \}$ may be dropped whenever this can be done without introducing confusion. The other notations will be made clear as we proceed.

## 2 Estimation of Binomial Parameters

Let $X$ be a Bernoulli random variable defined in a probability space $(\Omega, \mathcal{F}, \Pr)$ such that $\Pr\{X = 1\} = 1 - \Pr\{X = 0\} = p \in (0, 1)$. It is a frequent problem to estimate the binomial parameter $p$
based on a sequence of i.i.d. random samples \( X_1, X_2, \cdots \) of \( X \). To solve this problem, we shall develop multistage sampling schemes of the following general structure. The sampling process is divided into \( s \) stages. The continuation or termination of sampling is determined by decision variables. For each stage with index \( \ell \), a decision variable \( D_\ell = D_\ell(X_1, \cdots, X_{n_\ell}) \) is defined based on samples \( X_1, \cdots, X_{n_\ell} \), where \( n_\ell \) is the number of samples available at the \( \ell \)-th stage. It should be noted that \( n_\ell \) can be a random number, depending on specific sampling schemes. The decision variable \( D_\ell \) assumes only two possible values 0, 1 with the notion that the sampling is continued until \( D_\ell = 1 \) for some \( \ell \in \{1, \cdots, s\} \). Since the sampling must be terminated at or before the \( s \)-th stage, it is required that \( D_s = 1 \). For simplicity of notations, we also define \( D_\ell = 0 \) for \( \ell = 0 \) throughout the remainder of the paper.

### 2.1 Control of Absolute Error

In many situations, it is desirable to construct an estimator for \( p \) with guaranteed absolute precision and confidence level. For this purpose, we have

**Theorem 1** Let \( 0 < \varepsilon < \frac{1}{2}, \ 0 < \delta < 1, \ \zeta > 0 \) and \( \rho > 0 \). Let \( n_1 < n_2 < \cdots < n_s \) be the ascending arrangement of all distinct elements of \( \left\{ \left(\left|\frac{24\varepsilon - 16\varepsilon^2}{9}\right| \ln\frac{1}{2\varepsilon} \right)^{1-\frac{1}{\rho}} : i = 0, 1, \cdots, \tau \right\} \) with \( \tau = \left\lfloor \ln\frac{6}{2(1+\varepsilon p)} \right\rfloor \). For \( \ell = 1, \cdots, s \), define \( K_\ell = \sum_{i=1}^{n_\ell} X_i \), \( \hat{\rho}_\ell = \frac{K_\ell}{n_\ell} \) and \( D_\ell \) such that \( D_\ell = 1 \) if \( (|\hat{\rho}_\ell - \frac{1}{2}| - \frac{\varepsilon}{2})^2 \geq \frac{1}{\delta} + \frac{\varepsilon}{2\ln(\delta)} \); and \( D_\ell = 0 \) otherwise. Suppose the stopping rule is that sampling is continued until \( D_\ell = 1 \) for some \( \ell \in \{1, \cdots, s\} \). Define \( \hat{\rho} = \frac{\sum_{i=1}^{n} X_i}{n} \) where \( n \) is the sample size when the sampling is terminated. Define

\[
\mathcal{Q}^+ = \bigcup_{\ell=1}^{s} \left\{ \frac{k}{n_\ell} + \varepsilon \in \left(0, \frac{1}{2}\right) : k \in \mathbb{Z} \right\} \bigcup \left\{ \frac{1}{2} \right\}, \quad \mathcal{Q}^- = \bigcup_{\ell=1}^{s} \left\{ \frac{k}{n_\ell} - \varepsilon \in \left(0, \frac{1}{2}\right) : k \in \mathbb{Z} \right\} \bigcup \left\{ \frac{1}{2} \right\}.
\]

Then, a sufficient condition to guarantee \( \Pr\{|\hat{\rho} - p| < \varepsilon \mid p\} > 1 - \delta \) for any \( p \in (0, 1) \) is that

\[
\sum_{\ell=1}^{s} \Pr\{|\hat{\rho}_\ell - p + \varepsilon, D_{\ell-1} = 0, D_\ell = 1 | p\} < \frac{\delta}{2} \quad \forall p \in \mathcal{Q}^- \quad \tag{1}
\]

\[
\sum_{\ell=1}^{s} \Pr\{|\hat{\rho}_\ell - p - \varepsilon, D_{\ell-1} = 0, D_\ell = 1 | p\} < \frac{\delta}{2} \quad \forall p \in \mathcal{Q}^+ \quad \tag{2}
\]

where both (1) and (2) are satisfied if \( 0 < \zeta < \frac{1}{2(\tau+1)} \).

### 2.2 Control of Absolute and Relative Errors

To construct an estimator satisfying a mixed criterion in terms of absolute and relative errors with a prescribed confidence level, we have

**Theorem 2** Let \( 0 < \delta < 1, \ \zeta > 0 \) and \( \rho > 0 \). Let \( \varepsilon_a \) and \( \varepsilon_r \) be positive numbers such that \( 0 < \varepsilon_a < \frac{3}{8} \) and \( \frac{6\varepsilon_a}{3 - 2\varepsilon_a} < \varepsilon_r < 1 \). Let \( n_1 < n_2 < \cdots < n_s \) be the ascending arrangement of all
distinct elements of \( \left\{ \left[ \frac{1}{2} \left( \frac{1}{e} - \frac{1}{e_r} - \frac{1}{3} \right) \right]^i : i = 0, 1, \ldots, \tau \right\} \) with \( \tau = \left\lfloor \frac{\ln(\log(1+p))}{\ln(1+p)} \right\rfloor \).

For \( \ell = 1, \ldots, s \), define \( K_\ell = \sum_{i=1}^{n_\ell} X_i \), \( \hat{p}_\ell = \frac{K_\ell}{n_\ell} \),

\[
D_\ell = \begin{cases} 
0 & \text{for } \frac{1}{2} - \frac{2}{3} \varepsilon_a - \sqrt{\frac{1}{4} + \frac{n_\ell \varepsilon_a^2}{2 \ln(\delta)}} < \hat{p}_\ell < \frac{6(1-\varepsilon_r)(3-\varepsilon_r) \ln(\delta)}{2(3-\varepsilon_r)^2 \ln(\delta) - 9n_\ell \varepsilon_a^2} \\
\frac{1}{2} + \frac{2}{3} \varepsilon_a - \sqrt{\frac{1}{4} + \frac{n_\ell \varepsilon_a^2}{2 \ln(\delta)}} < \hat{p}_\ell < \frac{6(1+\varepsilon_r)(3+\varepsilon_r) \ln(\delta)}{2(3+\varepsilon_r)^2 \ln(\delta) - 9n_\ell \varepsilon_a^2} \\
1 & \text{else}
\end{cases}
\]

for \( \ell = 1, \ldots, s-1 \) and \( D_s = 1 \). Suppose the stopping rule is that sampling is continued until \( D_\ell = 1 \) for some \( \ell \in \{1, \ldots, s\} \). Let \( \hat{p} = \frac{\sum_{i=1}^{n} X_i}{n} \) where \( n \) is the sample size when the sampling is terminated. Define \( p^* = \frac{\varepsilon_a}{\varepsilon_r} \) and

\[
\mathcal{D}_a^+ = \bigcup_{i=1}^{s} \left\{ \frac{k}{n_\ell} + \varepsilon_a : k \in \mathbb{Z} \right\} \cup \{p^*\}, \quad \mathcal{D}_a^- = \bigcup_{i=1}^{s} \left\{ \frac{k}{n_\ell} - \varepsilon_a : k \in \mathbb{Z} \right\} \cup \{p^*\},
\]

\[
\mathcal{D}_r^+ = \bigcup_{i=1}^{s} \left\{ \frac{k}{n_\ell(1+\varepsilon_r)} \in (p^*, 1) : k \in \mathbb{Z} \right\}, \quad \mathcal{D}_r^- = \bigcup_{i=1}^{s} \left\{ \frac{k}{n_\ell(1-\varepsilon_r)} \in (p^*, 1) : k \in \mathbb{Z} \right\}.
\]

Then, \( \Pr \left\{ \hat{p} - p < \varepsilon_a \text{ or } \left| \frac{\hat{p} - \rho}{p} \right| < \varepsilon_r \right\} > 1 - \delta \) for any \( p \in (0, 1) \) provided that

\[
\sum_{\ell=1}^{s} \Pr \left\{ \hat{p}_\ell \geq p + \varepsilon_a, D_{\ell-1} = 0, D_\ell = 1 \mid p \right\} < \frac{\delta}{2} \quad \forall p \in \mathcal{D}_a^-,
\]

\[
\sum_{\ell=1}^{s} \Pr \left\{ \hat{p}_\ell \leq p - \varepsilon_a, D_{\ell-1} = 0, D_\ell = 1 \mid p \right\} < \frac{\delta}{2} \quad \forall p \in \mathcal{D}_a^+,
\]

\[
\sum_{\ell=1}^{s} \Pr \left\{ \hat{p}_\ell \geq p(1+\varepsilon_r), D_{\ell-1} = 0, D_\ell = 1 \mid p \right\} < \frac{\delta}{2} \quad \forall p \in \mathcal{D}_r^+,
\]

\[
\sum_{\ell=1}^{s} \Pr \left\{ \hat{p}_\ell \leq p(1-\varepsilon_r), D_{\ell-1} = 0, D_\ell = 1 \mid p \right\} < \frac{\delta}{2} \quad \forall p \in \mathcal{D}_r^-,
\]

where these conditions are satisfied for \( 0 < \zeta < \frac{1}{2(\tau+1)} \).

### 2.3 Control of Relative Error

In many situations, it is desirable to design a sampling scheme to estimate \( p \) such that the estimator satisfies a relative error criterion with a prescribed confidence level. By virtue of the function

\[
g(\varepsilon, \gamma) = 1 - \sum_{i=0}^{\gamma-1} \frac{1}{i!} \left( \frac{\gamma}{1+\varepsilon} \right)^i \exp \left( - \frac{\gamma}{1+\varepsilon} \right) + \sum_{i=0}^{\gamma-1} \frac{1}{i!} \left( \frac{\gamma}{1-\varepsilon} \right)^i \exp \left( - \frac{\gamma}{1-\varepsilon} \right),
\]

we have developed a simple sampling scheme as described by the following theorem.

**Theorem 3** Let \( 0 < \varepsilon < 1, \quad 0 < \delta < 1, \quad \zeta > 0 \) and \( \rho > 0 \). Let \( \gamma_1 < \gamma_2 < \cdots < \gamma_s \) be the ascending arrangement of all distinct elements of \( \left\{ \left[ \frac{1}{2} \left( \frac{1}{e} + 1 \right) \right]^i : i = 0, 1, \ldots, \tau \right\} \) with \( \tau = \left\lfloor \frac{\ln(\log(1+p))}{\ln(1+p)} \right\rfloor \). Let \( \hat{p}_\ell = \frac{\sum_{i=1}^{n_\ell} X_i}{n_\ell} \) where \( n_\ell \) is the minimum number of samples such that
of stage at which the sampling is terminated. Then, for any \(p\) small to guarantee this purpose, we have high since samples are obtained one by one when inverse sampling is involved. In view of this continued until \(D_{\ell} = 1\) for some \(\ell \in \{1, \cdots, s\}\). Define estimator \(\hat{p} = \frac{\sum_{i=1}^{n} X_i}{n}\) where \(n\) is the sample size when the sampling is terminated. Then, \(\Pr \left\{ \left| \frac{\hat{p} - p}{p} \right| \leq \varepsilon \mid p \right\} \geq 1 - \delta\) for any \(p \in (0,1)\) provided that \(\zeta > 0\) is sufficiently small to guarantee \(g(\varepsilon; \gamma_s) < \delta\) and

\[
\ln(\delta) < \left[ \frac{(1 + \varepsilon + \sqrt{1 + 4\varepsilon + \varepsilon^2})^2}{4\varepsilon^2} + \frac{1}{2} \right] \left[ \frac{\varepsilon}{1 + \varepsilon} - \ln(1 + \varepsilon) \right],
\]

(7)

\[\sum_{\ell=1}^{s} \Pr \{ \hat{p}_{\ell} \leq (1 - \varepsilon)p, D_{\ell-1} = 0, D_{\ell} = 1 \mid p \} \leq \frac{\delta}{2} \quad \forall p \in \mathcal{D}_{r}^-, \]

(8)

\[\sum_{\ell=1}^{s} \Pr \{ \hat{p}_{\ell} \geq (1 + \varepsilon)p, D_{\ell-1} = 0, D_{\ell} = 1 \mid p \} \leq \frac{\delta}{2} \quad \forall p \in \mathcal{D}_{r}^+ \]

(9)

where \(\mathcal{D}_{r}^+ = \bigcup_{\ell=1}^{s} \left\{ \frac{\gamma_{\ell}}{m_{(1+\varepsilon)}} \in (p^*, 1) : m \in \mathbb{N} \right\}\) and \(\mathcal{D}_{r}^- = \bigcup_{\ell=1}^{s} \left\{ \frac{\gamma_{\ell}}{m_{(1-\varepsilon)}} \in (p^*, 1) : m \in \mathbb{N} \right\}\) with \(p^* \in (0, z_{s-1})\) denoting the unique number satisfying

\[
g(\varepsilon; \gamma_s) + \sum_{\ell=1}^{s-1} \exp \left( \frac{\gamma_{\ell}}{z_{\ell}} 2 \left( \frac{2p^* - z_{\ell}}{2p^* - z_{\ell}} \right) \left( \frac{2p^* - z_{\ell}}{2p^* - z_{\ell}} - 1 \right) \right) = \delta
\]

where \(z_{\ell} = 1 + \frac{2\varepsilon}{3 + \varepsilon} + \frac{9\varepsilon^2 \gamma_{\ell} \ln(\delta)}{2(3 + \varepsilon)^2 \ln(\delta)}\) for \(\ell = 1, \cdots, s - 1\).

In this section, we have proposed a multistage inverse sampling plan for estimating a binomial parameter, \(p\), with relative precision. In some situations, the cost of sampling operation may be high since samples are obtained one by one when inverse sampling is involved. In view of this fact, it is desirable to develop multistage estimation methods without using inverse sampling. For this purpose, we have

**Theorem 4** Let \(0 < \varepsilon < 1, 0 < \delta < 1\) and \(\zeta > 0\). Let \(\tau\) be a positive integer. For \(\ell = 1, 2, \cdots\), let \(\hat{p}_{\ell} = \frac{\sum_{i=1}^{n_{\ell}} X_i}{n_{\ell}}\), where \(n_{\ell}\) is deterministic and stands for the sample size at the \(\ell\)-th stage. For \(\ell = 1, 2, \cdots\), define \(D_{\ell}\) such that \(D_{\ell} = 1\) if \(\hat{p}_{\ell} \geq \frac{6(1 + \varepsilon)(3 + \varepsilon) \ln(\delta_{\ell})}{(3 + \varepsilon)^2 \ln(\delta_{\ell}) - 9n_{\ell} \varepsilon^2}\); and \(D_{\ell} = 0\) otherwise, where \(\delta_{\ell} = \delta\) for \(1 \leq \ell \leq \tau\) and \(\delta_{\ell} = \delta 2^{\tau - \ell}\) for \(\ell > \tau\). Suppose the stopping rule is that sampling is continued until \(D_{\ell} = 1\) for some stage with index \(\ell\). Define estimator \(\hat{p} = \hat{p}_{l}\), where \(l\) is the index of stage at which the sampling is terminated. Then, \(\Pr \{ l < \infty \} = 1\) and \(\Pr \{ \left| \frac{\hat{p} - p}{p} \right| \leq \varepsilon \mid p \} \geq 1 - \delta\) for any \(p \in (0,1)\) provided that \(2(\tau + 1) \zeta \leq 1\) and \(\inf_{\ell > 0} \frac{n_{\ell+1}}{n_{\ell}} > 0\).

### 2.4 Fixed-width Confidence Intervals

In some literature, the estimation of \(p\) has been formulated as the problem of constructing a fixed-width confidence interval \((L, U)\) such that \(U - L \leq 2\varepsilon\) and that \(\Pr \{ L < p < U \mid p \} > 1 - \delta\) for any \(p \in (0,1)\) with prescribed \(\varepsilon \in (0, \frac{1}{2})\) and \(\delta \in (0,1)\). For completeness, we shall develop multistage sampling schemes in this setting.

Making use of the Clopper-Pearson confidence interval [4], we have established the following sampling scheme.
Theorem 5 For $\alpha \in (0, 1)$ and integers $0 \leq k \leq n$, define

$$
\mathcal{L}(n, k, \alpha) = \begin{cases} 
0 & \text{if } k = 0 \\
\frac{p}{\alpha} & \text{if } k > 0
\end{cases}$$

and

$$
\mathcal{U}(n, k, \alpha) = \begin{cases} 
1 & \text{if } k = n \\
\frac{p}{\alpha} & \text{if } k < n
\end{cases}
$$

with $p \in (0, 1)$ satisfying $\sum_{j=k}^{n} \binom{n}{j} p^j (1 - p)^{n-j} = \frac{\alpha}{2}$ and $\bar{p} \in (0, 1)$ satisfying $\sum_{j=0}^{k} \binom{n}{j} \bar{p}^j (1 - \bar{p})^{n-j} = \frac{\alpha}{2}$. Let $\zeta > 0$ and $\rho > 0$. Let $n_1 < n_2 < \cdots < n_s$ be the ascending arrangement of all distinct elements of $\left\{ \left( \frac{2z}{\ln \frac{1}{1+p}} \right)^{1/\zeta} \ln \frac{1}{1+p} : i = 0, 1, \cdots, \tau \right\}$ with $\tau = \frac{\ln \frac{1}{1+p}}{\ln \frac{1}{1+p}}$. For $\ell = 1, \cdots, s$, define $K_\ell = \sum_{i=1}^{n_\ell} X_i$ and $D_\ell$ such that $D_\ell = 1$ if $\mathcal{U}(n_\ell, K_\ell, \zeta \delta) - \mathcal{L}(n_\ell, K_\ell, \zeta \delta) \leq 2\varepsilon$; and $D_\ell = 0$ otherwise. Suppose the stopping rule is that sampling is continued until $D_\ell = 1$ for some $\ell \in \{1, \cdots, s\}$. Define $L = \mathcal{L}(n, \sum_{i=1}^{n} X_i, \zeta \delta)$ and $U = \mathcal{U}(n, \sum_{i=1}^{n} X_i, \zeta \delta)$, where $n$ is the sample size when the sampling is terminated. Define

$$
\mathcal{D}_L = \bigcup_{\ell=1}^{s} \{ \mathcal{L}(n_\ell, k, \zeta \delta) \in (0, 1) : 0 \leq k \leq n_\ell \}, \quad \mathcal{D}_U = \bigcup_{\ell=1}^{s} \{ \mathcal{U}(n_\ell, k, \zeta \delta) \in (0, 1) : 0 \leq k \leq n_\ell \}.
$$

Then, a sufficient condition to guarantee $\Pr \{ L < p < U \mid p \} > 1 - \delta$ for any $p \in (0, 1)$ is that

$$
\sum_{\ell=1}^{s} \Pr \{ \mathcal{L}(n_\ell, K_\ell, \zeta \delta) \geq p, D_{\ell-1} = 0, D_\ell = 1 \mid p \} < \frac{\delta}{2} \quad \forall p \in \mathcal{D}_L,
$$

$$
\sum_{\ell=1}^{s} \Pr \{ \mathcal{U}(n_\ell, K_\ell, \zeta \delta) \leq p, D_{\ell-1} = 0, D_\ell = 1 \mid p \} < \frac{\delta}{2} \quad \forall p \in \mathcal{D}_U
$$

where both (10) and (11) are satisfied if $0 < \zeta < \frac{1}{2\ln(1+p)}$.

Making use of Chernoff-Hoeffding inequalities [3, 3], we have established the following sampling scheme.

Theorem 6 For $\alpha \in (0, 1)$ and integers $0 \leq k \leq n$, define

$$
\mathcal{L}(n, k, \alpha) = \begin{cases} 
\bar{p} & \text{for } 0 < k < n, \\
\left( \frac{\alpha}{2} \right)^{1/n} & \text{for } k = n
\end{cases}$$

and

$$
\mathcal{U}(n, k, \alpha) = \begin{cases} 
1 - \left( \frac{\alpha}{2} \right)^{1/n} & \text{for } k = 0, \\
\bar{p} & \text{for } k = n
\end{cases}
$$

with $\bar{p} \in (0, \frac{k}{n})$ satisfying $\mathcal{M}_B(\frac{k}{n}, \bar{p}) = \frac{\ln(\zeta \delta)}{n}$ and $\bar{p} \in (\frac{k}{n}, 1)$ satisfying $\mathcal{M}_B(\frac{k}{n}, \bar{p}) = \frac{\ln(\zeta \delta)}{n}$, where $\mathcal{M}_B(\ldots)$ is a function such that $\mathcal{M}_B(z, \theta) = z \ln \frac{\alpha}{2} + (1-z) \ln \frac{1-\alpha}{2}$ for $z \in (0, 1)$ and $\theta \in (0, 1)$. Let $\zeta > 0$ and $\rho > 0$. Let $n_1 < n_2 < \cdots < n_s$ be the ascending arrangement of all distinct elements of $\left\{ \left( \frac{2z}{\ln \frac{1}{1+p}} \right)^{1/\zeta} \ln \frac{1}{1+p} : i = 0, 1, \cdots, \tau \right\}$ with $\tau = \frac{\ln \frac{1}{1+p}}{\ln \frac{1}{1+p}}$. For $\ell = 1, \cdots, s$, define $K_\ell = \sum_{i=1}^{n_\ell} X_i$ and $D_\ell$ such that $D_\ell = 1$ if $\mathcal{U}(n_\ell, K_\ell, \zeta \delta) - \mathcal{L}(n_\ell, K_\ell, \zeta \delta) \leq 2\varepsilon$; and $D_\ell = 0$ otherwise. Suppose the stopping rule is that sampling is continued until $D_\ell = 1$ for some $\ell \in \{1, \cdots, s\}$. Define $L = \mathcal{L}(n, \sum_{i=1}^{n} X_i, \zeta \delta)$ and $U = \mathcal{U}(n, \sum_{i=1}^{n} X_i, \zeta \delta)$, where $n$ is the sample size when the sampling is terminated. Define

$$
\mathcal{D}_L = \bigcup_{\ell=1}^{s} \{ \mathcal{L}(n_\ell, k, \zeta \delta) \in (0, 1) : 0 \leq k \leq n_\ell \}, \quad \mathcal{D}_U = \bigcup_{\ell=1}^{s} \{ \mathcal{U}(n_\ell, k, \zeta \delta) \in (0, 1) : 0 \leq k \leq n_\ell \}.
$$
Then, a sufficient condition to guarantee $\Pr \{ L < p < U \mid p \} > 1 - \delta$ for any $p \in (0, 1)$ is that
\[
\sum_{\ell=1}^{s} \Pr \{ \mathcal{L}(n_{\ell}, K_{\ell}, \zeta \delta) \geq p, D_{\ell-1} = 0, D_{\ell} = 1 \mid p \} < \frac{\delta}{2} \quad \forall p \in \mathcal{Q}_{L},
\]
\[
\sum_{\ell=1}^{s} \Pr \{ \mathcal{U}(n_{\ell}, K_{\ell}, \zeta \delta) \leq p, D_{\ell-1} = 0, D_{\ell} = 1 \mid p \} < \frac{\delta}{2} \quad \forall p \in \mathcal{Q}_{U}
\]
where both (12) and (13) are satisfied if $0 < \zeta < \frac{1}{2(\tau+1)}$.

Making use of Massart’s inequality, we have established the following sampling scheme.

**Theorem 7** For $\alpha \in (0, 1)$ and integers $0 \leq k \leq n$, define
\[
\mathcal{L}(n, k, \alpha) = \max \left\{ 0, \frac{k}{n} + \frac{3}{4} \frac{1 - \frac{2k}{n}}{\sqrt{\frac{1}{2} n \ln \frac{4}{\alpha}}} k(1 - \frac{k}{n}) \right\},
\]
and
\[
\mathcal{U}(n, k, \alpha) = \min \left\{ 1, \frac{k}{n} + \frac{3}{4} \frac{1 - \frac{2k}{n}}{\sqrt{\frac{1}{2} n \ln \frac{4}{\alpha}}} k(1 - \frac{k}{n}) \right\}.
\]

Let $\zeta > 0$ and $\rho > 0$. Let $n_{1} < n_{2} < \cdots < n_{s}$ be the ascending arrangement of all distinct elements of $\left\{ \left( \frac{8}{9} \left( \frac{3}{2} \right)^{i} + 1 \right) \left( \frac{3}{2} - 1 \right) \ln \frac{1}{\zeta \delta} : i = 0, 1, \cdots, \tau \right\}$ with $\tau = \left\lfloor \frac{\ln (\frac{9}{4} \tau + 1)}{\ln (1 + \rho)} \right\rfloor$. For $\ell = 1, \cdots, s$, define $K_{\ell} = \sum_{i=1}^{n_{\ell}} X_{i}$ and $D_{\ell}$ such that $D_{\ell} = 1$ if
\[
1 - \frac{9}{2 \ln (\zeta \delta)} K_{\ell} \left( 1 - \frac{K_{\ell}}{n_{\ell}} \right) \leq \varepsilon^{2} \left[ \frac{4}{3} - \frac{3n_{\ell}}{2 \ln (\zeta \delta)} \right]^{2},
\]
and $D_{\ell} = 0$ otherwise. Suppose the stopping rule is that sampling is continued until $D_{\ell} = 1$ for some $\ell \in \{1, \cdots, s\}$. Define $L = \mathcal{L}(n, \sum_{i=1}^{n} X_{i}, \zeta \delta)$ and $U = \mathcal{U}(n, \sum_{i=1}^{n} X_{i}, \zeta \delta)$, where $n$ is the sample size when the sampling is terminated. Define
\[
\mathcal{Q}_{L} = \bigcup_{\ell=1}^{s} \{ \mathcal{L}(n_{\ell}, k, \zeta \delta) \in (0, 1) : 0 \leq k \leq n_{\ell} \}, \quad \mathcal{Q}_{U} = \bigcup_{\ell=1}^{s} \{ \mathcal{U}(n_{\ell}, k, \zeta \delta) \in (0, 1) : 0 \leq k \leq n_{\ell} \}.
\]
Then, a sufficient condition to guarantee $\Pr \{ L < p < U \mid p \} > 1 - \delta$ for any $p \in (0, 1)$ is that
\[
\sum_{\ell=1}^{s} \Pr \{ \mathcal{L}(n_{\ell}, K_{\ell}, \zeta \delta) \geq p, D_{\ell-1} = 0, D_{\ell} = 1 \mid p \} < \frac{\delta}{2} \quad \forall p \in \mathcal{Q}_{L},
\]
\[
\sum_{\ell=1}^{s} \Pr \{ \mathcal{U}(n_{\ell}, K_{\ell}, \zeta \delta) \leq p, D_{\ell-1} = 0, D_{\ell} = 1 \mid p \} < \frac{\delta}{2} \quad \forall p \in \mathcal{Q}_{U}
\]
where both (12) and (13) are satisfied if $0 < \zeta < \frac{1}{2(\tau+1)}$.

It should be noted that the interval estimation methods described in Theorems 5-7 can be made less conservative by using tight bounds of $C(p, \varepsilon) = 1 - \Pr \{ L < p < U \mid p \}$ for $p \in [a, b] \subseteq \Theta$ in Theorem 5. Based on such bounds, a branch-and-bound type strategy described in section 2.8 of [1] can be used to facilitate the search of an appropriate value of $\zeta$ such that the coverage probability associated with interval $(L, U)$ is no less than $1 - \delta$. 

7
Theorem 8 Let $L_\ell = L(\hat{p}^\ell, \zeta, \delta)$ and $U_\ell = U(\hat{p}^\ell, \zeta, \delta)$ for $\ell = 1, \cdots, s$. Then,

\[
C(p, \varepsilon) \leq \Pr\{L \geq a \mid b\} + \Pr\{U \leq b \mid a\} \\
\leq \sum_{\ell=1}^{s} \Pr\{L_\ell \geq a, D_{\ell-1} = 0, D_\ell = 1 \mid b\} + \sum_{\ell=1}^{s} \Pr\{U_\ell \leq b, D_{\ell-1} = 0, D_\ell = 1 \mid a\},
\]

\[
C(p, \varepsilon) \geq \Pr\{L \geq b \mid a\} + \Pr\{U \leq a \mid b\} \\
\geq \sum_{\ell=1}^{s} \Pr\{L_\ell \geq b, D_{\ell-1} = 0, D_\ell = 1 \mid a\} + \sum_{\ell=1}^{s} \Pr\{U_\ell \leq a, D_{\ell-1} = 0, D_\ell = 1 \mid b\}
\]

for any $p \in [a, b]$. Moreover, if the open interval $(a, b)$ contains no element of the supports of $L$ and $U$, then

\[
C(p, \varepsilon) \leq \Pr\{L \geq b \mid b\} + \Pr\{U \leq a \mid a\} \\
\leq \sum_{\ell=1}^{s} \Pr\{L_\ell \geq b, D_{\ell-1} = 0, D_\ell = 1 \mid b\} + \sum_{\ell=1}^{s} \Pr\{U_\ell \leq a, D_{\ell-1} = 0, D_\ell = 1 \mid a\},
\]

\[
C(p, \varepsilon) \geq \Pr\{L > a \mid a\} + \Pr\{U < b \mid b\} \\
\geq \sum_{\ell=1}^{s} \Pr\{L_\ell > a, D_{\ell-1} = 0, D_\ell = 1 \mid a\} + \sum_{\ell=1}^{s} \Pr\{U_\ell < b, D_{\ell-1} = 0, D_\ell = 1 \mid b\}
\]

for any $p \in (a, b)$.

We would like to note that Theorems 1 and 2 of [1] play important roles in the establishment of the theorems in this section. As can be seen from Theorems 1–6, the confidence requirements can be satisfied by choosing $\zeta$ to be sufficiently small. The application of the double-decision-variable method and the single-decision-variable method is obvious. To determine $\zeta$ as large as possible and thus make the sampling schemes most efficient, the computational techniques such as bisection confidence tuning, domain truncation, triangular partition developed in [1] can be applied.

With regard to the tightness of the double-decision-variable method, we can develop results similar to Theorems 13, 18 and 23 of [1].

With regard to the asymptotic performance of our sampling schemes, we can develop results similar to Theorems 14, 19 and 24 of [1].

3 Estimation of Bounded-variable Means

The method proposed for estimating binomial parameters can be generalized for estimating means of random variables bounded in interval $[0, 1]$. Formally, let $Z \in [0, 1]$ be a random variable with expectation $\mu = \mathbb{E}[Z]$. We can estimate $\mu$ based on i.i.d. random samples $Z_1, Z_2, \cdots$ of $Z$ by virtue of the following results.
Theorem 9 Let $0 < \varepsilon < \frac{1}{2}$ and $0 < \delta < 1$. Let $n_1 < n_2 < \cdots < n_s$ be a sequence of sample sizes such that $n_s \geq \frac{\ln \frac{2s}{\delta^2}}{2 \varepsilon^2}$. Define $\hat{\mu}_\ell = \frac{\sum_{i=1}^{n_\ell} z_i}{n_\ell}$ for $\ell = 1, \cdots, s$. Suppose the stopping rule is that sampling is continued until $(|\hat{\mu}_\ell - \frac{1}{2}| - \frac{\varepsilon}{2})^2 \geq \frac{1}{4} - \frac{\varepsilon^2 n_\ell}{2 \ln(2s/\delta)}$ for some $\ell \in \{1, \cdots, s\}$. Define $\hat{\mu} = \frac{\sum_{i=1}^{n_s} z_i}{n_s}$ where $n$ is the sample size when the sampling is terminated. Then, $\Pr \{|\hat{\mu} - \mu| < \varepsilon\} \geq 1 - \delta$.

This theorem can be shown by a variation of the argument for Theorem 1.

Theorem 10 Let $0 < \delta < 1$, $0 < \varepsilon_\ell < \frac{3}{s}$ and $\frac{6s \varepsilon_\ell}{s - 2s_\ell} < \varepsilon_\ell < 1$. Let $n_1 < n_2 < \cdots < n_s$ be a sequence of sample sizes such that $n_s \geq 2 \left(\frac{1}{\varepsilon_r} + \frac{1}{3}\right) \left(\frac{1}{\varepsilon_r} - \frac{1}{3}\right) \ln \left(\frac{2s}{\delta}\right)$. Define $\hat{\mu}_\ell = \frac{\sum_{i=1}^{n_\ell} z_i}{n_\ell}$ for $\ell = 1, \cdots, s$. Define

$$D_\ell = \begin{cases} 0 & \text{for } \frac{1}{2} - \frac{2}{s} \varepsilon_\ell - \sqrt{\frac{n_\ell}{2} + \frac{2}{s} \varepsilon_\ell} < \hat{\mu}_\ell < \frac{n_\ell}{2} + \frac{2}{s} \varepsilon_\ell - \sqrt{\frac{n_\ell}{2} + \frac{2}{s} \varepsilon_\ell} < \hat{\mu}_\ell < \frac{n_\ell}{2} + \frac{2}{s} \varepsilon_\ell \text{ or } n_\ell \in \{1, \cdots, s\}\}, \\
1 & \text{else} \end{cases}$$

for $\ell = 1, \cdots, s - 1$ and $D_s = 1$. Suppose the stopping rule is that sampling is continued until $D_\ell = 1$ for some $\ell \in \{1, \cdots, s\}$. Define $\hat{\mu} = \frac{\sum_{i=1}^{n_s} z_i}{n_s}$ where $n$ is the sample size when the sampling is terminated. Then, $\Pr \{|\hat{\mu} - \mu| < \varepsilon_\ell \text{ or } |\hat{\mu} - \mu| < \varepsilon_r | \mu\} \geq 1 - \delta$.

This theorem can be shown by a variation of the argument for Theorem 2. In the general case that $Z$ is a random variable bounded in $[a, b]$, it is useful to estimate the mean $\mu = \mathbb{E}[Z]$ based on i.i.d. samples of $Z$ with a mixed criterion. For this purpose, we shall introduce the function

$$\mathcal{M}(z, \mu) = \begin{cases} \frac{(\mu - z)^2}{2(\frac{\mu}{3} + \frac{z}{3})(\frac{\mu}{3} + \frac{z}{3} - 1)} & \text{for } 0 \leq z \leq 1 \text{ and } \mu \in (0, 1), \\
-\infty & \text{for } 0 \leq z \leq 1 \text{ and } \mu \notin (0, 1) \end{cases}$$

and propose the following multistage estimation method.

Theorem 11 Let $0 < \delta < 1$, $\varepsilon_\ell > 0$ and $0 < \varepsilon_r < 1$. Let $n_1 < n_2 < \cdots < n_s$ be a sequence of sample sizes such that $n_s \geq \frac{(b - a)^2}{2 \varepsilon^2} \ln \left(\frac{2s}{\delta^2}\right)$. Define $\hat{\mu}_\ell = \frac{\sum_{i=1}^{n_\ell} z_i}{n_\ell}$, $\bar{\mu}_\ell = a + \frac{1}{b - a} \hat{\mu}_\ell$, $\underline{\mu}_\ell = a + \frac{1}{b - a} \min \left\{ \hat{\mu}_\ell - \varepsilon_\ell, \frac{\hat{\mu}_\ell}{1 + \text{sgn}(\hat{\mu}_\ell) \varepsilon_\ell} \right\}$, $\bar{\mu}_\ell = a + \frac{1}{b - a} \max \left\{ \hat{\mu}_\ell + \varepsilon_\ell, \frac{\hat{\mu}_\ell}{1 - \text{sgn}(\hat{\mu}_\ell) \varepsilon_\ell} \right\}$

for $\ell = 1, \cdots, s$. Suppose the stopping rule is that sampling is continued until $\mathcal{M}(\bar{\mu}_\ell, \underline{\mu}_\ell) \leq \frac{1}{n_\ell} \ln \frac{\delta}{2\varepsilon}$ and $\mathcal{M}(\bar{\mu}_\ell, \underline{\mu}_\ell) \leq \frac{1}{n_\ell} \ln \frac{\delta}{2\varepsilon}$ for some $\ell \in \{1, \cdots, s\}$. Define $\hat{\mu} = \frac{\sum_{i=1}^{n_s} z_i}{n_s}$ where $n$ is the sample size when the sampling is terminated. Then, $\Pr \{|\hat{\mu} - \mu| < \varepsilon_\ell \text{ or } |\hat{\mu} - \mu| < \varepsilon_r | \mu\} \geq 1 - \delta$. 

9
4 A Link between Binomial and Bounded Variables

There exists an inherent connection between a binomial parameter and the mean of a bounded variable. In this regard, we have

**Theorem 12** Let $Z$ be a random variable bounded in $[0, 1]$. Let $U$ a random variable uniformly distributed over $[0, 1]$. Suppose $Z$ and $U$ are independent. Then,

$$
\mathbb{E}[Z] = \Pr\{Z \geq U\}.
$$

**Proof.** Let $F_{Z,U}$ be the joint distribution of $Z$ and $U$. Let $F_Z$ be the cumulative distribution function of $Z$. Since $Z$ and $U$ are independent, using Riemann-Stieltjes integration, we have

$$
\Pr\{Z \geq U\} = \int_{z=0}^{1} \int_{u=0}^{z} dF_{Z,U} = \int_{z=0}^{1} \int_{u=0}^{z} du \ dF_{Z} = \int_{z=0}^{1} z \ dF_{Z} = \mathbb{E}[Z].
$$

To see why Theorem 12 reveals a relationship between the mean of a bounded variable and a binomial parameter, we define

$$
X = \begin{cases} 
1 & \text{for } Z \geq U, \\
0 & \text{otherwise.} 
\end{cases}
$$

Then, by Theorem 12 we have $\Pr\{X = 1\} = 1 - \Pr\{X = 0\} = \mathbb{E}[Z]$. This implies that $X$ is a Bernoulli random variable and $\mathbb{E}[Z]$ is actually a binomial parameter. As a consequence, the techniques of estimating a binomial parameter can be useful for estimating the mean of a bounded variable. Specially, for a sequence of i.i.d. random samples $Z_1, Z_2, \cdots$ of bounded variable $Z$ and a sequence of i.i.d. random samples $U_1, U_2, \cdots$ of uniform variable $U$ such that that $Z_i$ is independent with $U_i$ for all $i$, we can define a sequence of i.i.d. random samples $X_1, X_2, \cdots$ of Bernoulli random variable $X$ by

$$
X_i = \begin{cases} 
1 & \text{for } Z_i \geq U_i, \\
0 & \text{otherwise.} 
\end{cases}
$$

5 Conclusion

We have established a new multistage approach for estimating the mean of a bounded variable. Our approach can provide an estimator for the unknown mean which rigorously guarantees prescribed levels of precision and confidence. Our approach is also very flexible in the sense that the precision can be expressed in terms of different types of margins of errors.
A Preliminary Results for Proofs of Theorems

We need some preliminary results, especially some properties of function $\mathcal{M}(z, \mu)$ defined in Section 3.

**Lemma 1** $\mathcal{M}(z, z + \varepsilon)$ is monotonically increasing with respect to $z \in (0, \frac{1}{2} - \frac{2\varepsilon}{3})$, and is monotonically decreasing with respect to $z \in (\frac{1}{2} - \frac{2\varepsilon}{3}, 1 - \varepsilon)$. Similarly, $\mathcal{M}(z, z - \varepsilon)$ is monotonically increasing with respect to $z \in (\varepsilon, \frac{1}{2} + \frac{2\varepsilon}{3})$, and is monotonically decreasing with respect to $z \in (\frac{1}{2} + \frac{2\varepsilon}{3}, 1)$.

**Proof.** The lemma can be established by checking the partial derivatives

$$
\frac{\partial \mathcal{M}(z, z + \varepsilon)}{\partial z} = \frac{\varepsilon^2}{[(z + \frac{2\varepsilon}{3})(1 - z - \frac{2\varepsilon}{3})]^2} \left( \frac{1}{2} - \frac{2\varepsilon}{3} - z \right),
$$

$$
\frac{\partial \mathcal{M}(z, z - \varepsilon)}{\partial z} = \frac{\varepsilon^2}{[(z - \frac{2\varepsilon}{3})(1 - z + \frac{2\varepsilon}{3})]^2} \left( \frac{1}{2} + \frac{2\varepsilon}{3} - z \right).
$$

**Lemma 2** Let $0 < \varepsilon < \frac{1}{2}$. Then, $\mathcal{M}(z, z + \varepsilon) \geq \mathcal{M}(z, z - \varepsilon)$ for $z \in [0, \frac{1}{2}]$, and $\mathcal{M}(z, z + \varepsilon) < \mathcal{M}(z, z - \varepsilon)$ for $z \in (\frac{1}{2}, 1]$.

**Proof.** By the definition of the function $\mathcal{M}(., .)$, we have that $\mathcal{M}(z, \mu) = -\infty$ for $z \in [0, 1]$ and $\mu \notin (0, 1)$. Hence, the lemma is trivially true for $0 \leq z \leq \varepsilon$ or $1 - \varepsilon \leq z \leq 1$. It remains to show the lemma for $z \in (\varepsilon, 1 - \varepsilon)$. This can be accomplished by noting that

$$
\mathcal{M}(z, z + \varepsilon) - \mathcal{M}(z, z - \varepsilon) = \frac{2\varepsilon^3(1 - 2z)}{3(z + \frac{2\varepsilon}{3})(1 - z - \frac{2\varepsilon}{3})(z - \frac{2\varepsilon}{3})(1 - z + \frac{2\varepsilon}{3})},
$$

where the right-hand side is seen to be positive for $z \in (\varepsilon, \frac{1}{2})$ and negative for $z \in (\frac{1}{2}, 1 - \varepsilon)$.

**Lemma 3** $\mathcal{M}\left(z, \frac{z}{1+\varepsilon}\right) > \mathcal{M}\left(z, \frac{z}{1-\varepsilon}\right)$ for $0 < \varepsilon < 1 - \varepsilon < 1$.

**Proof.** It can be verified that

$$
\mathcal{M}\left(z, \frac{z}{1+\varepsilon}\right) - \mathcal{M}\left(z, \frac{z}{1-\varepsilon}\right) = \frac{2\varepsilon^3z(2 - z)}{3(1 + \frac{\varepsilon}{3})(1 - z + \varepsilon)(1 - \frac{\varepsilon}{3})(1 - z - \varepsilon)(1 - \frac{-\varepsilon}{3})},
$$

from which it can be seen that $\mathcal{M}\left(z, \frac{z}{1+\varepsilon}\right) > \mathcal{M}\left(z, \frac{z}{1-\varepsilon}\right)$ for $z \in (0, 1 - \varepsilon)$.

**Lemma 4** $\mathcal{M}(\mu - \varepsilon, \mu) < \mathcal{M}(\mu + \varepsilon, \mu)$ for $0 < \varepsilon < \mu < \frac{1}{2} < 1 - \varepsilon$. 
Proof. The lemma follows from the fact that
\[ M(\mu - \varepsilon, \mu) - M(\mu + \varepsilon, \mu) = \frac{\varepsilon^3(2\mu - 1)}{3(\mu - \frac{\varepsilon}{3})(1 - \mu + \frac{\varepsilon}{3})(\mu + \frac{\varepsilon}{3})(1 - \mu - \frac{\varepsilon}{3})}, \]
where the right-hand side is negative for \( 0 < \varepsilon < \mu < \frac{1}{2} < 1 - \varepsilon \).

\[ \square \]

Lemma 5 \( M \left( z, \frac{z}{1+\varepsilon} \right) \) is monotonically decreasing with respect to \( z \in (0, 1) \). Similarly, \( M \left( z, \frac{z}{1-\varepsilon} \right) \) is monotonically decreasing with respect to \( z \in (0, 1 - \varepsilon) \).

Proof. The lemma can be shown by verifying that
\[ \frac{\partial}{\partial z} M \left( z, \frac{z}{1+\varepsilon} \right) = -\frac{\varepsilon^2}{2(1+\frac{\varepsilon}{3})} \frac{1+\varepsilon}{(1+\varepsilon)(1-z) + \frac{2\varepsilon z}{3}} < 0 \]
for \( z \in (0, 1) \) and that
\[ \frac{\partial}{\partial z} M \left( z, \frac{z}{1-\varepsilon} \right) = -\frac{\varepsilon^2}{2(1-\frac{\varepsilon}{3})} \frac{1-\varepsilon}{(1-\varepsilon)(1-z) - \frac{2\varepsilon z}{3}} < 0 \]
for \( z \in (0, 1 - \varepsilon) \).

\[ \square \]

Lemma 6 For any fixed \( z \in (0, 1) \), \( M(z, \mu) \) is monotonically increasing with respect to \( \mu \in (0, z) \), and is monotonically decreasing with respect to \( \mu \in (z, 1) \). Similarly, for any fixed \( \mu \in (0, 1) \), \( M(z, \mu) \) is monotonically increasing with respect to \( z \in (0, \mu) \), and is monotonically decreasing with respect to \( z \in (\mu, 1) \).

Proof. The lemma can be shown by checking the following partial derivatives:
\[ \frac{\partial M(z, \mu)}{\partial \mu} = \frac{(z - \mu)[\mu(1-z) + z(1-\mu) + z(1-z)]}{3 \left[ \left( \frac{2\mu}{3} + \frac{z}{3} \right)(1 - \frac{2\mu}{3} - \frac{z}{3}) \right]^2}, \]
\[ \frac{\partial M(z, \mu)}{\partial z} = \frac{(\mu - z) \left[ \mu(1 - \frac{2\mu}{3} - \frac{z}{3}) + \frac{z - \mu}{6} \right]}{\left[ \left( \frac{2\mu}{3} + \frac{z}{3} \right)(1 - \frac{2\mu}{3} - \frac{z}{3}) \right]^2} = \frac{(\mu - z) \left[ (1 - \mu) \left( \frac{2\mu}{3} + \frac{z}{3} \right) + \frac{\mu - z}{6} \right]}{\left[ \left( \frac{2\mu}{3} + \frac{z}{3} \right)(1 - \frac{2\mu}{3} - \frac{z}{3}) \right]^2}. \]

\[ \square \]

The following result, stated as Lemma 7, is due to Massart [8].

Lemma 7 Let \( \overline{X}_n = \sum_{i=1}^{n} X_i \) where \( X_1, \ldots, X_n \) are i.i.d. random variables such that \( 0 \leq X_i \leq 1 \) and \( \mathbb{E}[X_i] = \mu \in (0, 1) \) for \( i = 1, \ldots, n \). Then, \( \Pr \{ \overline{X}_n \geq z \} < \exp(nM(z, \mu)) \) for any \( z \in (\mu, 1) \).

Similarly, \( \Pr \{ \overline{X}_n \leq z \} < \exp(nM(z, \mu)) \) for any \( z \in (0, \mu) \).

Lemma 8 Let \( \overline{X}_n = \sum_{i=1}^{n} X_i \) where \( X_1, \ldots, X_n \) are i.i.d. random variables such that \( 0 \leq X_i \leq 1 \) and \( \mathbb{E}[X_i] = \mu \in (0, 1) \) for \( i = 1, \ldots, n \). Then, \( \Pr \{ \overline{X}_n \geq \mu, M(\overline{X}_n, \mu) \leq \frac{\ln n}{n} \} \leq \alpha \) for any \( \alpha > 0 \).
**Proof.** Since the lemma is trivially true for \( \alpha \geq 1 \), it remains to show it for \( \alpha \in (0, 1) \). It can be checked that \( \mathcal{M}(\mu, \mu) = 1 \) and \( \mathcal{M}(1, \mu) = \frac{9(\mu-1)}{4(2\mu+1)} \). Since \( \frac{\partial \mathcal{M}(z, \mu)}{\partial z} = (\mu - z)[\mu(1 - \frac{2\mu}{3} - \frac{z}{3}) + \frac{z - \mu}{6}] / \left((\frac{2\mu}{3} + \frac{z}{3})(1 - \frac{2\mu}{3} - \frac{z}{3})\right)^2 < 0 \) for \( z \in (0, \mu) \), we have that \( \mathcal{M}(z, \mu) \) is monotonically decreasing from 0 to \( \frac{9(\mu-1)}{4(2\mu+1)} \) as \( z \) increases from \( \mu \) to 1. To show the lemma, we need to consider three cases as follows.

Case (i): \( \frac{9(\mu-1)}{4(2\mu+1)} > \frac{\ln \alpha}{n} \). In this case, we have that \( \{X_n \geq \mu, \mathcal{M}(X_n, \mu) \leq \frac{\ln \alpha}{n}\} \) is an impossible event and the corresponding probability is 0. This is because the minimum of \( \mathcal{M}(z, \mu) \) with respect to \( z \in (\mu, 1] \) is equal to \( \frac{9(\mu-1)}{4(2\mu+1)} \), which is greater than \( \frac{\ln \alpha}{n} \).

Case (ii): \( \frac{9(\mu-1)}{4(2\mu+1)} = \frac{\ln \alpha}{n} \). In this case, we have that \( \Pr \{X_n \geq \mu, \mathcal{M}(X_n, \mu) \leq \frac{\ln \alpha}{n}\} = \prod_{i=1}^n \Pr(X_i = 1) \leq \prod_{i=1}^n \Pr(X_i = 1) = \alpha \), where the last inequality is due to the fact that \( \ln \mu < \frac{9(\mu-1)}{4(2\mu+1)} \). To prove this fact, we define \( g(\mu) = \ln \mu - \frac{9(\mu-1)}{4(2\mu+1)} \).

Then, the first derivative of \( g(\mu) \) is \( g'(\mu) = \frac{5\mu^2+4-11\mu(1-\mu)}{4\mu(2\mu+1)^2} \geq 0 \) for any \( \mu \in (0, 1) \). This implies that \( g(\mu) \) is monotonically increasing with respect to \( \mu \in (0, 1) \). By virtue of such monotonicity and the fact that \( g(1) = 0 \), we can conclude that \( g(\mu) < 0 \) for any \( \mu \in (0, 1) \). This establishes \( \ln \mu < \frac{9(\mu-1)}{4(2\mu+1)} \).

Case (iii): \( \frac{9(\mu-1)}{4(2\mu+1)} < \frac{\ln \alpha}{n} \). In this case, there exists a unique number \( z^* \in (\mu, 1) \) such that \( \mathcal{M}(z^*, \mu) = \frac{\ln \alpha}{n} \). Since \( \mathcal{M}(z, \mu) \) is monotonically decreasing with respect to \( z \in (\mu, 1) \), it must be true that any \( x \in (\mu, 1) \) satisfying \( \mathcal{M}(x, \mu) \leq \frac{\ln \alpha}{n} \) is no less than \( z^* \). This implies that \( \{X_n \geq \mu, \mathcal{M}(X_n, \mu) = \frac{\ln \alpha}{n}\} \leq \{X_n \geq z^*\} \) and \( \Pr \{X_n \geq \mu, \mathcal{M}(X_n, \mu) \leq \frac{\ln \alpha}{n}\} \leq \exp(n \mathcal{M}(z^*, \mu)) = \alpha \), where the last inequality follows from Lemma 7. This completes the proof of the lemma.

\[ \square \]

**Lemma 9** Let \( X_n = \sum_{i=1}^n X_i \) where \( X_1, \ldots, X_n \) are i.i.d. random variables such that \( 0 \leq X_i \leq 1 \) and \( \mathbb{E}[X_i] = \mu \in (0, 1) \) for \( i = 1, \cdots, n \). Then, \( \Pr(X_n \leq \mu, \mathcal{M}(X_n, \mu) \leq \frac{\ln \alpha}{n}) \leq \alpha \) for any \( \alpha > 0 \).

**Proof.** Since the lemma is trivially true for \( \alpha \geq 1 \), it suffices to show it for \( \alpha \in (0, 1) \). It can be checked that \( \mathcal{M}(\mu, \mu) = 1 \) and \( \mathcal{M}(0, \mu) = \frac{9\mu}{4(2\mu-3)} \). Since \( \frac{\partial \mathcal{M}(z, \mu)}{\partial z} = (\mu - z)[(\mu - z)(1 - \frac{2\mu}{3} - \frac{z}{3}) + \frac{\mu - z}{6}] / \left((\frac{2\mu}{3} + \frac{z}{3})(1 - \frac{2\mu}{3} - \frac{z}{3})\right)^2 > 0 \) for \( z \in (0, \mu) \), we have that \( \mathcal{M}(z, \mu) \) is monotonically increasing from \( \frac{9\mu}{4(2\mu-3)} \) to 0 as \( z \) increases from 0 to \( \mu \). Now there are three cases:

Case (i): \( \frac{9\mu}{4(2\mu-3)} > \frac{\ln \alpha}{n} \). In this case, we have that \( \{X_n \leq \mu, \mathcal{M}(X_n, \mu) \leq \frac{\ln \alpha}{n}\} \) is an impossible event and the corresponding probability is 0. This is because the minimum of \( \mathcal{M}(z, \mu) \) with respect to \( z \in (0, \mu) \) is equal to \( \frac{9\mu}{4(2\mu-3)} \), which is greater than \( \frac{\ln \alpha}{n} \).

Case (ii): \( \frac{9\mu}{4(2\mu-3)} = \frac{\ln \alpha}{n} \). In this case, we have that \( \Pr \{X_n \leq \mu, \mathcal{M}(X_n, \mu) \leq \frac{\ln \alpha}{n}\} = \Pr(X_i = 0, i = 1, \cdots, n) = \prod_{i=1}^n \Pr(X_i = 0) = \prod_{i=1}^n (1 - \Pr(X_i \neq 0)) = (1 - \mu)^n < \exp \left(n \cdot \frac{9\mu}{4(2\mu-3)}\right) = \alpha \), where the last inequality is due to the fact that \( \ln(1 - \mu) < \frac{9\mu}{4(2\mu-3)} \). To prove this fact, we define \( h(\mu) = \ln(1 - \mu) - \frac{9\mu}{4(2\mu-3)} \). Then, the first derivative of \( h(\mu) \) is \( h'(\mu) = -\frac{16\mu^2 + 21\mu - 9}{4(1-\mu)(2\mu-3)^2} < 0 \) for any \( \mu \in (0, 1) \). This implies that \( h(\mu) \) is monotonically
deciding with respect to \( \mu \in (0, 1) \). By virtue of such monotonicity and the fact that \( h(0) = 0 \), we can conclude that \( h(\mu) < 0 \) for any \( \mu \in (0, 1) \). This establishes \( \ln(1 - \mu) < \frac{9\mu}{4(2\mu - 3)} \).

Case (iii): \( \frac{3\mu}{4(2\mu - 3)} < \frac{\ln \alpha}{n} \). In this case, there exists a unique number \( Z^* \in (0, \mu) \) such that \( M(Z^*, \mu) = \frac{\ln \alpha}{n} \). Since \( M(z, \mu) \) is monotonically increasing with respect to \( z \in (0, \mu) \), it must be true that any \( z \) satisfies \( M(z, \mu) \leq \frac{\ln \alpha}{n} \) is no greater than \( Z^* \). This implies that \( \{X_n \leq \mu, M(X_n, \mu) \leq \frac{\ln \alpha}{n}\} \subseteq \{X_n \leq Z^*\} \) and thus \( \Pr\{X_n \leq \mu, M(X_n, \mu) \leq \frac{\ln \alpha}{n}\} \leq \Pr\{X_n \leq Z^*\} \leq \exp(nM(Z^*, \mu)) = \alpha \), where the last inequality follows from Lemma 7. This completes the proof of the lemma.

\[ \square \]

\section{Proof of Theorem 1}

Throughout the proof of Theorem 1, we define random variables \( D_\ell, \ell = 1, \ldots, s \) such that \( D_\ell = 1 \) if \( (|\hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3}| \geq \frac{1}{4} + \frac{n_{\ell} \varepsilon^2}{2 \ln(\zeta \delta)} \) and \( D_\ell = 0 \) otherwise. Then, the stopping rule can be restated as “sampling is continued until \( D_\ell = 1 \) for some \( \ell \in \{1, \ldots, s\} \).”

**Lemma 10** \( D_s = 1 \).

**Proof.** By the definition of \( D_s \), we have that \( \{D_s = 1\} = \left\{ (|\hat{p}_s - \frac{1}{2} - \frac{2\varepsilon}{3}| \geq \frac{1}{4} + \frac{n_s \varepsilon^2}{2 \ln(\zeta \delta)} \right\} \). By the definition of sample sizes, we have \( n_s = \left\lceil \frac{\ln \frac{1}{2\varepsilon}}{2\zeta \delta} \right\rceil \geq \frac{\ln \frac{1}{2\varepsilon}}{2\zeta \delta} \), which implies that \( \frac{1}{4} + \frac{n_s \varepsilon^2}{2 \ln(\zeta \delta)} \leq 0 \). Since \( \left\{ (|\hat{p}_s - \frac{1}{2} - \frac{2\varepsilon}{3}| \geq 0 \right\} \) is a sure event, it follows that \( \left\{ (|\hat{p}_s - \frac{1}{2} - \frac{2\varepsilon}{3}| \geq \frac{1}{4} + \frac{n_s \varepsilon^2}{2 \ln(\zeta \delta)} \right\} \) is a sure event and consequently \( D_s = 1 \). This completes the proof of the lemma.

\[ \square \]

**Lemma 11** \( \{\hat{p}_\ell \leq p - \varepsilon, D_\ell = 1\} \subseteq \{\hat{p}_\ell < p, M(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_\ell}\} \) for \( \ell = 1, \ldots, s \).

**Proof.** Since \( \{D_\ell = 1\} = \left\{ (|\hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3}| \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \right\} \), it suffices to show

\[
\left\{ \hat{p}_\ell \leq p - \varepsilon, \left| \hat{p}_\ell - \frac{1}{2} \right| - \frac{2\varepsilon}{3} \right\} \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \right\} \subseteq \left\{ \hat{p}_\ell < p, M(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_\ell} \right\}.
\]

For this purpose, we let \( \omega \in \{\hat{p}_\ell \leq p - \varepsilon, (|\hat{p}_\ell - \frac{1}{2} - \frac{2\varepsilon}{3}| \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \right\} \), \( \hat{p}_\ell = \hat{p}_\ell(\omega) \) and proceed to show \( \hat{p}_\ell < p, M(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_\ell} \). Clearly, \( \hat{p}_\ell < p \) follows immediately from \( \hat{p}_\ell \leq p - \varepsilon \). To show \( M(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_\ell} \), we need to establish

\[
\left( \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \right)^2 \geq \frac{1}{4} + \frac{n_\ell \varepsilon^2}{2 \ln(\zeta \delta)} \tag{16}
\]
based on
\[
\left( \hat{p}_\ell - \frac{1}{2} \right)^2 - \frac{2\varepsilon}{3} \geq \frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)}.
\] (17)

It is obvious that (16) holds if \( \frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)} \leq 0 \). It remains to show (16) under the condition that \( \frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)} > 0 \). Note that (17) implies either
\[
\left| \hat{p}_\ell - \frac{1}{2} \right| - \frac{2\varepsilon}{3} \geq \sqrt{\frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)}}.
\] (18)
or
\[
\left| \hat{p}_\ell - \frac{1}{2} \right| - \frac{2\varepsilon}{3} \leq -\sqrt{\frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)}}.
\] (19)

Since (18) implies either \( \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \geq \frac{4\varepsilon}{3} + \sqrt{\frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)}} > \sqrt{\frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)}} \) or \( \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \leq -\sqrt{\frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)}} \),

it must be true that (18) implies (16). On the other hand, (19) also implies (16) because (19)
implies \( \sqrt{\frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)}} \leq \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \). Hence, we have established (16) based on (17).

Since \( -\frac{1}{2} < \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \leq p - \frac{1}{2} + \frac{2\varepsilon}{3} < \frac{1}{2} \), we have \( \frac{1}{4} - \left( \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3} \right)^2 > 0 \) and, by virtue of (16),
\[
\mathcal{M}(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) = -\frac{\varepsilon^2}{2\left(\frac{1}{4} - (\hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon}{3})^2\right)} \leq \frac{\ln(\zeta)}{n_\ell}.
\]

Since \( \hat{p}_\ell \leq p - \varepsilon \), we have \( 0 \leq \hat{p}_\ell < \hat{p}_\ell + \varepsilon \leq p < 1 \). Hence, using the fact that \( \mathcal{M}(z, \mu) \) is monotonically decreasing with respect to \( \mu \in (z, 1) \) as asserted by Lemma 8, we have \( \mathcal{M}(\hat{p}_\ell, p) \leq \mathcal{M}(\hat{p}_\ell, \hat{p}_\ell + \varepsilon) \leq \frac{\ln(\zeta)}{n_\ell} \). The proof of the lemma is thus completed. \(\square\)

**Lemma 12** \( \{\hat{p}_\ell \geq p + \varepsilon, \ D_\ell = 1\} \subseteq \{\hat{p}_\ell > p, \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta)}{n_\ell}\} \) for \( \ell = 1, \ldots, s \).

**Proof.** Since \( \{D_\ell = 1\} = \left\{ \left( \frac{1}{4} - \left( \hat{p}_\ell - \frac{1}{2} \right)^2 - \frac{2\varepsilon}{3} \right)^2 \geq \frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)} \right\} \), it suffices to show
\[
\left\{ \hat{p}_\ell \geq p + \varepsilon, \left( \frac{1}{4} - \left( \hat{p}_\ell - \frac{1}{2} \right)^2 - \frac{2\varepsilon}{3} \right)^2 \geq \frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)} \right\} \subseteq \left\{ \hat{p}_\ell > p, \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta)}{n_\ell} \right\}.
\]

For this purpose, we let \( \omega \in \left\{ \hat{p}_\ell \geq p + \varepsilon, \left( \hat{p}_\ell - \frac{1}{2} \right)^2 - \frac{2\varepsilon}{3} \right\} \); \( \hat{p}_\ell = \hat{p}_\ell(\omega) \) and proceed to show \( \hat{p}_\ell > p, \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta)}{n_\ell} \). Clearly, \( \hat{p}_\ell > p \) follows immediately from \( \hat{p}_\ell \geq p + \varepsilon \). To show \( \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta)}{n_\ell} \), we need to establish
\[
\left( \hat{p}_\ell - \frac{1}{2} \right)^2 - \frac{2\varepsilon}{3} \geq \frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)}.
\] (20)

based on
\[
\left( \hat{p}_\ell - \frac{1}{2} \right)^2 - \frac{2\varepsilon}{3} \geq \frac{1}{4} + \frac{n\varepsilon^2}{2\ln(\zeta)}.
\] (21)
It is obvious that (20) holds if \( \frac{1}{4} + \frac{n \varepsilon^2}{2 \ln(\zeta \delta)} \leq 0 \). It remains to show (20) under the condition that \( \frac{1}{4} + \frac{n \varepsilon^2}{2 \ln(\zeta \delta)} > 0 \). Note that (21) implies either

\[
\left| \hat{p}_\ell - \frac{1}{2} \right| - \frac{2 \varepsilon}{3} \geq \sqrt{\frac{1}{4} + \frac{n \ell \varepsilon^2}{2 \ln(\zeta \delta)}},
\]

or

\[
\left| \hat{p}_\ell - \frac{1}{2} \right| - \frac{2 \varepsilon}{3} \leq -\sqrt{\frac{1}{4} + \frac{n \ell \varepsilon^2}{2 \ln(\zeta \delta)}}.
\]

Since (22) implies either \( \hat{p}_\ell - \frac{1}{2} - \frac{2 \varepsilon}{3} \leq -\frac{2 \varepsilon}{3} - \sqrt{\frac{1}{4} + \frac{n \ell \varepsilon^2}{2 \ln(\zeta \delta)}} < -\frac{1}{4} + \frac{n \ell \varepsilon^2}{2 \ln(\zeta \delta)} \) or \( \hat{p}_\ell - \frac{1}{2} - \frac{2 \varepsilon}{3} \geq 0 \), it must be true that (22) implies (20). On the other hand, (23) also implies (20) because (23) implies \( \hat{p}_\ell - \frac{1}{2} - \frac{2 \varepsilon}{3} \leq -\sqrt{\frac{1}{4} + \frac{n \ell \varepsilon^2}{2 \ln(\zeta \delta)}} \). Hence, we have established (20) based on (21).

Since \( -\frac{1}{2} < p + \varepsilon - \frac{1}{2} - \frac{2 \varepsilon}{3} \leq \hat{p}_\ell - \frac{1}{2} - \frac{2 \varepsilon}{3} \leq 1 - \frac{1}{2} - \frac{2 \varepsilon}{3} < \frac{1}{2} \), we have \( \frac{1}{4} - (\hat{p}_\ell - \frac{1}{2} - \frac{2 \varepsilon}{3})^2 > 0 \) and, by virtue of (20),

\[
\mathcal{M}(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) = -\frac{\varepsilon^2}{2 \left( \frac{1}{4} - (\hat{p}_\ell - \frac{1}{2} - \frac{2 \varepsilon}{3})^2 \right)} \leq \ln(\zeta \delta) \frac{n_\ell}{n}. 
\]

Since \( \hat{p}_\ell \geq \mu + \varepsilon \), we have \( 0 < p \leq \hat{p}_\ell - \varepsilon < \hat{p}_\ell \leq 1 \). Hence, using the fact that \( \mathcal{M}(\hat{p}_\ell, \mu) \) is monotonically increasing with respect to \( \mu \in (0, 1] \) as asserted by Lemma [6], we have \( \mathcal{M}(\hat{p}_\ell, p) \leq \mathcal{M}(\hat{p}_\ell, \hat{p}_\ell - \varepsilon) \leq \ln(\zeta \delta) \frac{n_\ell}{n} \). The proof of the lemma is thus completed.

Now we are in a position to prove Theorem 1. Since \( \frac{\varepsilon}{24 \varepsilon - 16 \varepsilon^2} > 1 \) for any \( \varepsilon \in (0, \frac{1}{2}] \), we have \( \tau > 0 \). Hence, the sequence of sample sizes \( n_1, \ldots, n_s \) is well-defined. By Lemma [10] the sampling must stopped at some stage with index \( \ell \in \{1, \ldots, s\} \). This shows that the sampling scheme is well-defined. Noting that \( \{n = n_\ell\} \subseteq \{D_\ell = 1\} \) for \( \ell = 1, \ldots, s \), we have

\[
\Pr\{|\hat{p} - p| \geq \varepsilon\} = \sum_{\ell=1}^s \{\hat{p}_\ell \leq p - \varepsilon, n = n_\ell\} + \sum_{\ell=1}^s \{\hat{p}_\ell \geq p + \varepsilon, n = n_\ell\} \\
\leq \sum_{\ell=1}^s \{\hat{p}_\ell \leq p - \varepsilon, D_\ell = 1\} + \sum_{\ell=1}^s \{\hat{p}_\ell \geq p + \varepsilon, D_\ell = 1\}. 
\]

By Lemmas [11] and [9]

\[
\sum_{\ell=1}^s \{\hat{p}_\ell \leq p - \varepsilon, D_\ell = 1\} \leq \sum_{\ell=1}^s \left\{\hat{p}_\ell < p, \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_\ell}\right\} \leq s \zeta \delta \leq (\tau + 1) \zeta \delta. \quad (25)
\]

By Lemmas [12] and [8]

\[
\sum_{\ell=1}^s \{\hat{p}_\ell \geq p + \varepsilon, D_\ell = 1\} \leq \sum_{\ell=1}^s \left\{\hat{p}_\ell > p, \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_\ell}\right\} \leq s \zeta \delta \leq (\tau + 1) \zeta \delta. \quad (26)
\]

Combining (24), (25) and (26) yields \( \Pr\{|\hat{p} - p| \geq \varepsilon\} \leq 2(\tau + 1) \zeta \delta \). Hence, if we choose \( \zeta \) to be a positive number less than \( \frac{1}{2(\tau + 1)} \), we have \( \Pr\{|\hat{p} - p| < \varepsilon\} > 1 - \delta \). This completes the proof of Theorem 1.
C Proof of Theorem 2

Throughout the proof of Theorem 2, we define

\[ p_\ell = \min \left\{ \hat{p}_\ell - \varepsilon_a, \frac{\hat{p}_\ell}{1 + \varepsilon_r} \right\}, \quad \overline{p}_\ell = \max \left\{ \hat{p}_\ell + \varepsilon_a, \frac{\hat{p}_\ell}{1 - \varepsilon_r} \right\}. \]

By tedious computation, we can show the following lemma.

Lemma 13 For \( \ell = 1, \ldots, s \),

\begin{equation}
\left\{ \hat{p}_\ell \geq \frac{6(1 + \varepsilon_r)(3 + \varepsilon_r) \ln(\zeta \delta)}{2(3 + \varepsilon_r)^2 \ln(\zeta \delta) - 9n_r \varepsilon_r^2} \right\} = \left\{ M \left( \hat{p}_\ell, \frac{\hat{p}_\ell}{1 + \varepsilon_r} \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \right\}, \tag{27}
\end{equation}

\begin{equation}
\left\{ \hat{p}_\ell \geq \frac{6(1 - \varepsilon_r)(3 - \varepsilon_r) \ln(\zeta \delta)}{2(3 - \varepsilon_r)^2 \ln(\zeta \delta) - 9n_r \varepsilon_r^2} \right\} = \left\{ M \left( \hat{p}_\ell, \frac{\hat{p}_\ell}{1 - \varepsilon_r} \right) \leq \frac{\ln(\zeta \delta)}{n_\ell} \right\}. \tag{28}
\end{equation}

Lemma 14 \( \{ \overline{p}_s \geq p \} \subseteq \{ \hat{p}_s > p, M(\hat{p}_s, p) \leq \frac{\ln(\zeta \delta)}{n_s} \} \).

Proof. To prove the lemma, we let \( \omega \in \{ \overline{p}_s \geq p \} \), \( \hat{p}_s = \hat{p}_s(\omega) \), \( p_s = p_s(\omega) \) and proceed to show \( \hat{p}_s > p \), \( M(\hat{p}_s, p) \leq \frac{\ln(\zeta \delta)}{n_s} \). Clearly, \( \hat{p}_s > p \) follows immediately from \( \overline{p}_s \geq p > 0 \). To show \( M(\hat{p}_s, p) \leq \frac{\ln(\zeta \delta)}{n_s} \), we shall first show \( M(\hat{p}_s, p) \leq \frac{\ln(\zeta \delta)}{n_s} \). For simplicity of notations, we denote \( p^* = \frac{\ln(\zeta \delta)}{n_s} \). We need to consider three cases as follows.

Case (i): \( \hat{p}_s \leq p^* - \varepsilon_a \). In this case,

\[ M(\hat{p}_s, p_s) = M(\hat{p}_s, \hat{p}_s - \varepsilon_a) < M(\hat{p}_s, \hat{p}_s + \varepsilon_a) \leq M(p^* - \varepsilon_a, p^*) < M(p^* + \varepsilon_a, p^*) \leq \frac{\ln(\zeta \delta)}{n_s}. \]

Here the first inequality is due to \( \varepsilon_a < p + \varepsilon_a \leq p_s + \varepsilon_a = \hat{p}_s \leq p^* - \varepsilon_a < \frac{1}{2} \) and the fact that \( M(z, z + \varepsilon) \leq M(z, z - \varepsilon) \) for \( \varepsilon < z < \frac{1}{2} \), which is asserted by Lemma 2. The second inequality is due to \( \varepsilon_a < p + \varepsilon_a \leq p_s + \varepsilon_a = \hat{p}_s < p^* - \varepsilon_a < \frac{1}{2} - \varepsilon_a \) and the fact that \( M(z, z + \varepsilon) \) is monotonically increasing with respect to \( z \in (0, \frac{1}{2} - \varepsilon) \), which can be seen from Lemma 1. The third inequality is due to \( \varepsilon_a < p^* < \frac{1}{2} \) and the fact that \( M(p + \varepsilon, p) > M(p - \varepsilon, p) \) for \( \varepsilon < p < \frac{1}{2} \) as a result of Lemma 4. The last inequality is due to the fact that \( n_s = \left\lceil \frac{\ln(\zeta \delta)}{M(p^* + \varepsilon_a, p^*)} \right\rceil \geq \frac{\ln(\zeta \delta)}{M(p^* + \varepsilon_a, p^*)} \), which follows from the definition of \( n_s \).

Case (ii): \( p^* - \varepsilon_a < \hat{p}_s < p^* + \varepsilon_a \). In this case,

\[ M(\hat{p}_s, p_s) = M(\hat{p}_s, \hat{p}_s - \varepsilon_a) \leq M(p^* + \varepsilon_a, p^* + \varepsilon_a - \varepsilon_a) = M(p^* + \varepsilon_a, p^*) \leq \frac{\ln(\zeta \delta)}{n_s} \]

where the first inequality is due to \( \varepsilon_a < p + \varepsilon_a \leq p_s + \varepsilon_a = \hat{p}_s < p^* + \varepsilon_a < \frac{1}{2} - \varepsilon_a \) and the fact that \( M(z, z - \varepsilon) \) is monotonically increasing with respect to \( z \in (\varepsilon, \frac{1}{2} + \frac{2\varepsilon}{3}) \), which can be seen from Lemma 1.

Case (iii): \( \hat{p}_s \geq p^* + \varepsilon_a \). In this case,

\[ M(\hat{p}_s, p_s) = M(\hat{p}_s, \frac{\hat{p}_s}{1 + \varepsilon_r}) \leq M\left(p^* + \varepsilon_a, \frac{p^* + \varepsilon_a}{1 + \varepsilon_r}\right) = M(p^* + \varepsilon_a, p^*) \leq \frac{\ln(\zeta \delta)}{n_s}. \]
where the first inequality is due to the fact that $M(z, z/(1+\varepsilon))$ is monotonically decreasing with respect to $z \in (0, 1)$, which can be seen from Lemma [3].

Therefore, we have shown $M(\hat{p}_s, p_s) \leq \frac{\ln(\zeta \delta)}{n_s}$ for all cases. As a result of Lemma [6], $M(z, \mu)$ is monotonically increasing with respect to $\mu \in (0, z)$. By virtue of such monotonicity and the fact that $\hat{p}_s \geq p_s \geq p > 0$, we have $M(\hat{p}_s, p) \leq M(\hat{p}_s, p_s) \leq \frac{\ln(\zeta \delta)}{n_s}$. This completes the proof of the lemma.

\[ \square \]

Lemma 15 \[ \{p_s \leq p\} \subseteq \{\hat{p}_s < p, M(\hat{p}_s, p) \leq \frac{\ln(\zeta \delta)}{n_s}\} \].

\textbf{Proof.} To prove the lemma, we let $\omega \in \{\hat{p}_s \leq p\}$, $\hat{p}_s = \hat{p}_s(\omega)$, $p_s = p_s(\omega)$ and proceed to show $\hat{p}_s < p$, $M(\hat{p}_s, p) \leq \frac{\ln(\zeta \delta)}{n_s}$. Clearly, $\hat{p}_s < p$ follows immediately from $p_s \leq p < 1$. To show $M(\hat{p}_s, p) \leq \frac{\ln(\zeta \delta)}{n_s}$, we shall first show $M(\hat{p}_s, p_s) \leq \frac{\ln(\zeta \delta)}{n_s}$ by considering three cases as follows.

Case (i): $\hat{p}_s \leq p^* - \varepsilon_a$. In this case,

\[ M(\hat{p}_s, p_s) = M(\hat{p}_s, \hat{p}_s + \varepsilon_a) \leq M(p^* - \varepsilon_a, p^* - \varepsilon_a + \varepsilon_a) = M(p^* - \varepsilon_a, p^*) < M(p^* + \varepsilon_a, p^*) \leq \frac{\ln(\zeta \delta)}{n_s}. \]

Here the first inequality is due to $0 \leq \hat{p}_s \leq p^* - \varepsilon_a < \frac{1}{2} - \varepsilon_a$ and the fact that $M(z, z + \varepsilon)$ is monotonically increasing with respect to $z \in (0, \frac{1}{2} - \varepsilon)$, which is asserted by Lemma [1]. The second inequality is due to $\varepsilon_a < p^* < \frac{1}{2}$ and the fact that $M(p + \varepsilon, p) > M(p - \varepsilon, p)$ for $\varepsilon < p < \frac{1}{2}$, which can be seen from Lemma [3].

Case (ii): $p^* - \varepsilon_a < \hat{p}_s < p^* + \varepsilon_a$. In this case,

\[ M(\hat{p}_s, p_s) = M\left(\hat{p}_s, \frac{\hat{p}_s + \varepsilon_a}{1 - \varepsilon_r}\right) < M(p^* - \varepsilon_a, \frac{p^* - \varepsilon_a + \varepsilon_a}{1 - \varepsilon_r}) = M(p^* - \varepsilon_a, p^*) < M(p^* + \varepsilon_a, p^*) \leq \frac{\ln(\zeta \delta)}{n_s} \]

where the first inequality is due to $0 < p^* - \varepsilon_a < \hat{p}_s = (1 - \varepsilon_r)p_s \leq (1 - \varepsilon_r)p < 1 - \varepsilon_r$ and the fact that $M(z, z/(1 - \varepsilon))$ is monotonically decreasing with respect to $z \in (0, 1 - \varepsilon)$, which is asserted by Lemma [5].

Case (iii): $\hat{p}_s \geq p^* + \varepsilon_a$. In this case,

\[ M(\hat{p}_s, p_s) = M\left(\hat{p}_s, \frac{\hat{p}_s + \varepsilon_a}{1 + \varepsilon_r}\right) < M\left(\hat{p}_s, \frac{\hat{p}_s + \varepsilon_a}{1 + \varepsilon_r}\right) \leq M\left(p^* + \varepsilon_a, \frac{p^* + \varepsilon_a + \varepsilon_a}{1 + \varepsilon_r}\right) = M(p^* + \varepsilon_a, p^*) \leq \frac{\ln(\zeta \delta)}{n_s}. \]

Here the first inequality is due to $0 < \hat{p}_s = (1 - \varepsilon_r)p_s \leq (1 - \varepsilon_r)p < 1 - \varepsilon_r$ and the fact that $M(z, z/(1 + \varepsilon)) > M(z, z/(1 - \varepsilon))$ for $0 < z < 1 - \varepsilon$, which can be seen from Lemma [8]. The second inequality is due to $p^* + \varepsilon_a \leq \hat{p}_s$ and the fact that $M(z, z/(1 + \varepsilon))$ is monotonically decreasing with respect to $z \in (0, 1)$, which is asserted by Lemma [5].

Therefore, we have shown $M(\hat{p}_s, p_s) \leq \frac{\ln(\zeta \delta)}{n_s}$ for all cases. As a result of Lemma [6], $M(z, \mu)$ is monotonically decreasing with respect to $\mu \in (z, 1)$. By virtue of such monotonicity and the
fact that \( \hat{p}_s \leq \bar{p}_s \leq p < 1 \), we have \( \mathcal{M}(\hat{p}_s, p) \leq \mathcal{M}(\hat{p}_s, \bar{p}_s) \leq \frac{\ln(\zeta \delta)}{n_r} \). This completes the proof of the lemma.

\[ \]

**Lemma 16** \( \{ \bar{p}_\ell \leq p, D_\ell = 1 \} \subseteq \{ \hat{p}_\ell < p, \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_r} \} \) for \( 1 \leq \ell < s \).

**Proof.** To show the lemma, we let \( \omega \in \{ \bar{p}_\ell \leq p, D_\ell = 1 \} \), \( \hat{p}_\ell = \hat{p}_\ell(\omega), \bar{p}_\ell = \bar{p}_\ell(\omega) \) and proceed to show \( \hat{p}_\ell < p, \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_r} \). Clearly, \( \hat{p}_\ell < p \) follows immediately from \( \bar{p}_\ell \leq p < 1 \). To show \( \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_r} \), we shall first show \( \mathcal{M}(\hat{p}_\ell, \bar{p}_\ell) \leq \frac{\ln(\zeta \delta)}{n_r} \) by considering three cases as follows.

Case (i): \( \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a \). In this case, by the definition of the stopping rule, we have \( \hat{p}_\ell \leq \frac{1}{2} - \frac{2\varepsilon_a}{3} - \sqrt{\frac{1}{3} + \frac{m \bar{a}^2 r^2}{2\ln(\zeta \delta)}} \), which implies \( \frac{1}{2} - (\hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon_a}{3})^2 \leq -\frac{m \varepsilon_a^2}{2\ln(\zeta \delta)} \). Observing that \( \bar{p}_\ell = \hat{p}_\ell + \varepsilon_a \leq p < 1 \), we have

\[
-\frac{1}{2} < \hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon_a}{3} = \bar{p}_\ell - \varepsilon_a - \frac{1}{2} + \frac{2\varepsilon_a}{3} \leq p - \varepsilon_a - \frac{1}{2} + \frac{2\varepsilon_a}{3} < \frac{1}{2}.
\]

Hence, \( \frac{1}{2} - (\hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon_a}{3})^2 > 0 \) and \( \mathcal{M}(\hat{p}_\ell, \bar{p}_\ell) = -\frac{\varepsilon_a^2}{2\left[\frac{1}{2} - (\hat{p}_\ell - \frac{1}{2} + \frac{2\varepsilon_a}{3})\right]^2} \leq \frac{\ln(\zeta \delta)}{n_r} \).

Case (ii): \( \hat{p}_\ell - \frac{\varepsilon_a}{\varepsilon_r} < \varepsilon_a \). In this case, by the definition of the stopping rule and (28) of Lemma 13 we have \( \mathcal{M}(\hat{p}_\ell, \bar{p}_\ell) \leq \frac{\ln(\zeta \delta)}{n_r} \) with \( \bar{p}_\ell = \frac{\hat{p}_\ell}{1 - \varepsilon_r} \).

Case (iii): \( \hat{p}_\ell \geq \frac{\varepsilon_a}{\varepsilon_r} + \varepsilon_a \). In this case, we have \( \bar{p}_\ell = \frac{\hat{p}_\ell}{1 - \varepsilon_r} \) and \( \mathcal{M}(\hat{p}_\ell, \bar{p}_\ell) = \mathcal{M}\left(\hat{p}_\ell, \frac{\hat{p}_\ell}{1 - \varepsilon_r}\right) \leq \frac{\ln(\zeta \delta)}{n_r} \). Here, the first inequality follows from Lemma 3 and the fact that \( \hat{p}_\ell = (1 - \varepsilon_r)p \leq (1 - \varepsilon_r)p < 1 - \varepsilon_r \). The second inequality follows from the definitions of the stopping rule and (27) of Lemma 13.

Therefore, we have shown \( \mathcal{M}(\hat{p}_\ell, \bar{p}_\ell) \leq \frac{\ln(\zeta \delta)}{n_r} \) for all three cases. As a result of Lemma 6 \( \mathcal{M}(z, \mu) \) is monotonically decreasing with respect to \( \mu \in (z, 1) \). By virtue of such monotonocity and the fact that \( 0 < \hat{p}_\ell < \bar{p}_\ell \leq p < 1 \), we have \( \mathcal{M}(\hat{p}_\ell, p) \leq \mathcal{M}(\hat{p}_\ell, \bar{p}_\ell) \leq \frac{\ln(\zeta \delta)}{n_r} \). This completes the proof of the lemma.

\[ \]

**Lemma 17** \( \{ \bar{p}_\ell \geq p, D_\ell = 1 \} \subseteq \{ \hat{p}_\ell > p, \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_r} \} \) for \( 1 \leq \ell < s \).

**Proof.** To show the lemma, we let \( \omega \in \{ \bar{p}_\ell \geq p, D_\ell = 1 \} \), \( \hat{p}_\ell = \hat{p}_\ell(\omega), \bar{p}_\ell = \bar{p}_\ell(\omega) \) and proceed to show \( \hat{p}_\ell > p, \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_r} \). Clearly, \( \hat{p}_\ell > p \) follows immediately from \( \bar{p}_\ell \geq p > 0 \). To show \( \mathcal{M}(\hat{p}_\ell, p) \leq \frac{\ln(\zeta \delta)}{n_r} \), we shall first show \( \mathcal{M}(\hat{p}_\ell, \bar{p}_\ell) \leq \frac{\ln(\zeta \delta)}{n_r} \) by considering three cases as follows.

Case (i): \( \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a \). In this case, we have \( \bar{p}_\ell = \hat{p}_\ell - \varepsilon_a \) and

\[
\varepsilon_a < p + \varepsilon_a \leq \bar{p}_\ell + \varepsilon_a = \hat{p}_\ell \leq \frac{\varepsilon_a}{\varepsilon_r} - \varepsilon_a \leq \frac{1}{2} - \frac{4\varepsilon_a}{3},
\]

where the last inequality follows from the assumption about \( \varepsilon_a \) and \( \varepsilon_r \). By virtue of the fact that \( \varepsilon_a < \hat{p}_\ell < \frac{1}{2} < 1 - \varepsilon_a \) and Lemma 2 we have \( \mathcal{M}(\hat{p}_\ell, \bar{p}_\ell) = \mathcal{M}(\hat{p}_\ell, \hat{p}_\ell + \varepsilon_a) \). Since
\[- \frac{1}{2} < \varepsilon_a - \frac{1}{2} + \frac{2 \varepsilon_a}{3} < \frac{\hat{p}_\ell - \frac{1}{2} + \frac{2 \varepsilon_a}{3}}{\frac{1}{4} + \frac{n \varepsilon_a^2}{2 \ln(\zeta)}}, \text{ which implies } \frac{1}{4} - (\hat{p}_\ell - \frac{1}{2} + \frac{2 \varepsilon_a}{3})^2 \leq - \frac{n \varepsilon_a^2}{2 \ln(\zeta)}. \]

By the definition of the stopping rule, we have \( \hat{p}_\ell \leq \frac{1}{2} - \frac{2 \varepsilon_a}{3} - \frac{n \varepsilon_a^2}{2 \ln(\zeta)}, \) which implies \( \frac{1}{4} - (\hat{p}_\ell - \frac{1}{2} + \frac{2 \varepsilon_a}{3})^2 \leq - \frac{n \varepsilon_a^2}{2 \ln(\zeta)} \) and thus \( M(\hat{p}_\ell, p_\ell) < M(\hat{p}_\ell, \hat{p}_\ell + \varepsilon_a) = \frac{\ln(\zeta)}{n_\ell}. \)

Case (ii): \( \frac{\hat{p}_\ell - \frac{1}{2}}{\varepsilon_a} > \varepsilon_\alpha. \) In this case, since \( p_\ell = \hat{p}_\ell - \varepsilon_a \geq p > 0, \) we have

\[- \frac{1}{2} < p + \varepsilon_a - \frac{1}{2} + \frac{2 \varepsilon_a}{3} \leq \hat{p}_\ell + \varepsilon_a - \frac{1}{2} - \frac{2 \varepsilon_a}{3} = \hat{p}_\ell - \frac{1}{2} \leq \frac{1}{2}, \]

which implies \( \frac{1}{4} - (\hat{p}_\ell - \frac{1}{2} + \frac{2 \varepsilon_a}{3})^2 > 0. \) By the definition of the stopping rule, we have \( \hat{p}_\ell \leq \frac{1}{2} + \frac{2 \varepsilon_a}{3} - \frac{n \varepsilon_a^2}{2 \ln(\zeta)}, \) which implies \( \frac{1}{4} - (\hat{p}_\ell - \frac{1}{2} + \frac{2 \varepsilon_a}{3})^2 \leq - \frac{n \varepsilon_a^2}{2 \ln(\zeta)}. \) It follows that \( M(\hat{p}_\ell, p_\ell) = M(\hat{p}_\ell, \hat{p}_\ell - \varepsilon_a) \leq \frac{\ln(\zeta)}{n_\ell} \) because \( \frac{1}{4} - (\hat{p}_\ell - \frac{1}{2} + \frac{2 \varepsilon_a}{3})^2 > 0. \)

Case (iii): \( \hat{p}_\ell \leq \frac{\varepsilon_\alpha}{\varepsilon_a} + \varepsilon_\alpha. \) In this case, by the definition of the stopping rule and \( (27) \) of Lemma \( 13, \) we have \( M(\hat{p}_\ell, p_\ell) \leq \frac{\ln(\zeta)}{n_\ell} \) with \( p_\ell = \frac{\hat{p}_\ell}{1 + \varepsilon_\alpha}. \)

Therefore, we have shown \( M(\hat{p}_\ell, p_\ell) \leq \frac{\ln(\zeta)}{n_\ell} \) for all three cases. As a result of Lemma \( 6, M(z, \mu) \) is monotonically increasing with respect to \( \mu \in (0, z). \) By virtue of such monotonicity and the fact that \( 0 < p \leq p_\ell < \hat{p}_\ell < 1, \) we have \( M(\hat{p}_\ell, p_\ell) \leq M(\hat{p}_\ell, p_\ell) \leq \frac{\ln(\zeta)}{n_\ell}. \) This completes the proof of the lemma.

Now we are in a position to prove Theorem 2. As a direct consequence of the assumption that \( 0 < \varepsilon_a < \frac{3}{2} \) and \( \frac{6 \varepsilon_a}{3 - 2 \varepsilon_a} < \varepsilon_r < 1, \) we have \( \frac{3}{2} \left( \frac{1}{\varepsilon_r} - \frac{1}{\varepsilon_a} - \frac{1}{3} \right) > 1, \) which implies that \( \tau > 0. \) This shows that the sequence of sample sizes \( n_1, \ldots, n_s \) is well-defined and it follows that the sampling scheme is well-defined. Invoking the definitions of \( p_\ell, \hat{p}_\ell \) and noting that \( \{ n = n_\ell \} \subseteq \{ D_\ell = 1 \} \) for \( \ell = 1, \ldots, s, \) we have

\[
\Pr\{|\hat{p} - p| \geq \varepsilon_a, |\hat{p} - p| \geq \varepsilon_r p\} = \sum_{\ell=1}^{s} \Pr\{|\hat{p}_\ell - p| \geq \varepsilon_a, |\hat{p}_\ell - p| \geq \varepsilon_r p, n = n_\ell\} = \sum_{\ell=1}^{s} \Pr\{|\hat{p}_\ell \leq p, n = n_\ell\} + \sum_{\ell=1}^{s} \Pr\{|\hat{p}_\ell \geq p, n = n_\ell\} \leq \sum_{\ell=1}^{s} \Pr\{|\hat{p}_\ell \leq p, D_\ell = 1\} + \sum_{\ell=1}^{s} \Pr\{|\hat{p}_\ell \geq p, D_\ell = 1\}. \tag{29}
\]

By Lemmas \( 16, 15 \) and \( 9, \)

\[
\sum_{\ell=1}^{s} \Pr\{|\hat{p}_\ell \leq p, D_\ell = 1\} \leq \sum_{\ell=1}^{s} \left\{ \hat{p}_\ell < p, M(\hat{p}_\ell, p) \leq \frac{\ln(\zeta)}{n_\ell} \right\} \leq s \zeta \delta \leq (\tau + 1) \zeta \delta. \tag{30}
\]

By Lemmas \( 17, 14 \) and \( 8, \)

\[
\sum_{\ell=1}^{s} \Pr\{|\hat{p}_\ell \geq p, D_\ell = 1\} \leq \sum_{\ell=1}^{s} \left\{ \hat{p}_\ell > p, M(\hat{p}_\ell, p) \leq \frac{\ln(\zeta)}{n_\ell} \right\} \leq s \zeta \delta \leq (\tau + 1) \zeta \delta. \tag{31}
\]

Combining \( (29), (30) \) and \( (31) \) yields \( \Pr\{|\hat{p} - p| \geq \varepsilon_a, |\hat{p} - p| \geq \varepsilon_r p\} \leq 2(\tau + 1) \zeta \delta. \) Hence, if we choose \( \zeta \) to be a positive number less than \( \frac{1}{2(\tau + 1)}, \) we have \( \Pr\{|\hat{p} - p| < \varepsilon_a \text{ or } |\hat{p} - p| < \varepsilon_r p\} > 1 - \delta. \) This completes the proof of Theorem 2.
D Proof of Theorem 3

In the course of proving Theorem 3, we need to use the following lemma regarding inverse binomial sampling, which has been established by Chen in [1].

Lemma 20 Let $X_1, X_2, \ldots$ be a sequence of i.i.d. Bernoulli random variables such that $\Pr\{X_i = 1\} = 1 - \Pr\{X_i = 0\} = p \in (0, 1)$ for $i = 1, 2, \ldots$. Let $n$ be the minimum integer such that $\sum_{i=1}^n X_i = \gamma$ where $\gamma$ is a positive integer. Then, for any $\alpha > 0$,

$$\Pr\left\{ \frac{\gamma}{n} \leq p, \mathcal{M}_1\left(\frac{\gamma}{n}, p\right) \leq \frac{\ln \alpha}{\gamma} \right\} \leq \alpha,$$

$$\Pr\left\{ \frac{\gamma}{n} \geq p, \mathcal{M}_1\left(\frac{\gamma}{n}, p\right) \leq \frac{\ln \alpha}{\gamma} \right\} \leq \alpha$$

where

$$\mathcal{M}_1(z, \mu) = \begin{cases} \ln \frac{\mu}{z} + (\frac{1}{z} - 1) \ln \frac{1-\mu}{1-z} & \text{for } z \in (0, 1) \text{ and } \mu \in (0, 1), \\ \ln \mu & \text{for } z = 1 \text{ and } \mu \in (0, 1), \\ -\infty & \text{for } z = 0 \text{ and } \mu \in (0, 1). \end{cases}$$

Lemma 21 Let $X_1, X_2, \ldots$ be a sequence of i.i.d. Bernoulli random variables such that $\Pr\{X_i = 1\} = 1 - \Pr\{X_i = 0\} = p \in (0, 1)$ for $i = 1, 2, \ldots$. Let $n$ be the minimum integer such that $\sum_{i=1}^n X_i = \gamma$ where $\gamma$ is a positive integer. Then, for any $\alpha > 0$,

$$\Pr\left\{ \frac{\gamma}{n} \leq p, \mathcal{M}_1\left(\frac{\gamma}{n}, p\right) \leq \frac{\ln \alpha}{\gamma} \right\} \leq \alpha,$$

$$\Pr\left\{ \frac{\gamma}{n} \geq p, \mathcal{M}_1\left(\frac{\gamma}{n}, p\right) \leq \frac{\ln \alpha}{\gamma} \right\} \leq \alpha$$

where $\mathcal{M}_1(z, \mu) = \frac{\mathcal{M}(z, \mu)}{z}$ for $0 < z \leq 1$ and $0 < \mu < 1$.

Proof. By Massart’s inequality (i.e., Theorem 2 at page 1271 of [8]), we have $\mathcal{M}_1(z, \mu) < \mathcal{M}_1(z, \mu)$ for any $z \in (0, p)$. By virtue of this fact and Lemma 18, we have

$$\Pr\left\{ \frac{\gamma}{n} \leq p, \mathcal{M}_1\left(\frac{\gamma}{n}, p\right) \leq \frac{\ln \alpha}{\gamma} \right\} \leq \Pr\left\{ \frac{\gamma}{n} \leq p, \mathcal{M}_1\left(\frac{\gamma}{n}, p\right) \leq \frac{\ln \alpha}{\gamma} \right\} \leq \alpha,$$

$$\Pr\left\{ \frac{\gamma}{n} \geq p, \mathcal{M}_1\left(\frac{\gamma}{n}, p\right) \leq \frac{\ln \alpha}{\gamma} \right\} \leq \Pr\left\{ \frac{\gamma}{n} \geq p, \mathcal{M}_1\left(\frac{\gamma}{n}, p\right) \leq \frac{\ln \alpha}{\gamma} \right\} \leq \alpha.$$ 

This completes the proof of the lemma.

In the sequel, we define random variables $D_\ell$, $\ell = 1, \ldots, s$ such that $D_\ell = 1$ if $\gamma_\ell \geq \frac{6n_1(1+\varepsilon)(3+s)}{2(1+\varepsilon)\ln(\zeta/\delta) + 9-\mu_0}$ and $D_\ell = 0$ otherwise. Then, the stopping rule can be restated as “sampling is continued until $D_\ell = 1$ for some $\ell \in \{1, \ldots, s\}$”. For simplicity of notations, we also define $D_0 = 0$.

By tedious computation, we can show the following lemma.

Lemma 20 \(\{D_\ell = 1\} = \left\{ \mathcal{M}_1\left(\bar{p}_\ell, \frac{\bar{p}}{1+p}\right) \leq \frac{\ln(\zeta/\delta)}{\gamma_\ell} \right\} \) for $\ell = 1, \ldots, s$.

Lemma 21 \(\{\bar{p}_\ell \leq p(1-\varepsilon), D_\ell = 1\} \subseteq \left\{ \bar{p}_\ell < p, \mathcal{M}_1(\bar{p}_\ell, p) \leq \frac{\ln(\zeta/\delta)}{\gamma_\ell} \right\} \) for $\ell = 1, \ldots, s$. 

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**Lemma 23**

Let \( \omega \in \{ \hat{\rho}_\ell \leq p(1 - \varepsilon), \ D_\ell = 1 \} \) and \( \hat{\rho}_\ell = \hat{\rho}_\ell(\omega) \). To show the lemma, it suffices to show \( \hat{\rho}_\ell \leq p \) and \( M_1(\hat{\rho}_\ell, p) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \). By Lemma 20,

\[
\{ \hat{\rho}_\ell \leq p(1 - \varepsilon), \ D_\ell = 1 \} = \left\{ \hat{\rho}_\ell \leq p(1 - \varepsilon), \ M_1 \left( \hat{\rho}_\ell, \frac{\hat{\rho}_\ell}{1 + \varepsilon} \right) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \right\}
\]

which implies \( \hat{\rho}_\ell \leq p(1 - \varepsilon) \) and \( M_1 \left( \hat{\rho}_\ell, \frac{\hat{\rho}_\ell}{1 + \varepsilon} \right) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \). Clearly, \( \hat{\rho}_\ell \leq p(1 - \varepsilon) \) implies \( \hat{\rho}_\ell < p \). To show \( M_1(\hat{\rho}_\ell, p) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \), we shall consider two cases as follows:

- In the case \( \hat{\rho}_\ell = 0 \), we have \( M_1(\hat{\rho}_\ell, p) = -\infty < \frac{\ln(\zeta \delta)}{\gamma \ell} \).
- In the case of \( \hat{\rho}_\ell > 0 \), we have \( 0 < \hat{\rho}_\ell \leq p(1 - \varepsilon) < 1 - \varepsilon \). Since

\[
M_1 \left( \frac{z}{1 + \varepsilon} \right) - M_1 \left( \frac{z}{1 - \varepsilon} \right) = \frac{2e^3(2 - z)}{3(1 + \frac{z}{3}) \left[ 1 - z + \varepsilon (1 - \frac{z}{3}) \right] \left[ 1 - z - \varepsilon (1 - \frac{z}{3}) \right]} > 0
\]

for \( 0 < z < 1 - \varepsilon \), we have \( M_1 \left( \frac{\hat{\rho}_\ell}{1 + \varepsilon} \right) < M_1 \left( \frac{\hat{\rho}_\ell}{1 - \varepsilon} \right) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \). Note that

\[
\frac{\partial M_1(z, \mu)}{\partial \mu} = \frac{(z - \mu) [\mu(1-z) + z(1-\mu) + z(1-z)]}{3z \left[ (\frac{2\mu}{3} + \frac{z}{3}) (1 - \frac{2\mu}{3} - \frac{z}{3}) \right]^2},
\]

from which it can be seen that \( M_1(z, \mu) \) is monotonically decreasing with respect to \( \mu \in (z, 1) \).

By virtue of such monotonicity and the fact that \( 0 < \hat{\rho}_\ell < \frac{\hat{\rho}_\ell}{1 + \varepsilon} \leq p < 1 \), we have \( M_1(\hat{\rho}_\ell, p) \leq M_1 \left( \frac{\hat{\rho}_\ell}{1 - \varepsilon} \right) < \frac{\ln(\zeta \delta)}{\gamma \ell} \). This completes the proof of the lemma.

\[\square\]

**Lemma 22**

\( \{ \hat{\rho}_\ell \geq p(1 + \varepsilon), \ D_\ell = 1 \} \subseteq \{ \hat{\rho}_\ell > p, \ M_1(\hat{\rho}_\ell, p) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \} \) for \( \ell = 1, \cdots, s \).

**Proof.** Let \( \omega \in \{ \hat{\rho}_\ell \geq p(1 + \varepsilon), \ D_\ell = 1 \} \) and \( \hat{\rho}_\ell = \hat{\rho}_\ell(\omega) \). To show the lemma, it suffices to show \( \hat{\rho}_\ell > p \) and \( M_1(\hat{\rho}_\ell, p) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \). By Lemma 20,

\[
\{ \hat{\rho}_\ell \geq p(1 + \varepsilon), \ D_\ell = 1 \} = \left\{ \hat{\rho}_\ell \geq p(1 + \varepsilon), \ M_1 \left( \hat{\rho}_\ell, \frac{\hat{\rho}_\ell}{1 + \varepsilon} \right) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \right\}
\]

which implies \( \hat{\rho}_\ell \geq p(1 + \varepsilon) \) and \( M_1 \left( \hat{\rho}_\ell, \frac{\hat{\rho}_\ell}{1 + \varepsilon} \right) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \). Clearly, \( \hat{\rho}_\ell \geq p(1 + \varepsilon) \) implies \( \hat{\rho}_\ell > p \). Since \( 1 \geq \hat{\rho}_\ell \geq p(1 + \varepsilon) \), we have \( 0 < p < \frac{\hat{\rho}_\ell}{1 + \varepsilon} < \hat{\rho}_\ell \leq 1 \). Noting that \( \frac{\partial M_1(z, \mu)}{\partial \mu} = \frac{(z - \mu) [\mu(1-z) + z(1-\mu) + z(1-z)]}{3z \left[ (\frac{2\mu}{3} + \frac{z}{3}) (1 - \frac{2\mu}{3} - \frac{z}{3}) \right]^2} > 0 \) for \( 0 < \mu < z < 1 \), we have \( M_1(\hat{\rho}_\ell, p) \leq M_1 \left( \frac{\hat{\rho}_\ell}{1 + \varepsilon} \right) \leq \frac{\ln(\zeta \delta)}{\gamma \ell} \). This completes the proof of the lemma.

\[\square\]

**Lemma 23**

\( D_s = 1 \).
Proof. To show $D_s = 1$, it suffices to show $M_1 \left(z, \frac{\epsilon}{1+\epsilon}\right) \leq \frac{\ln(\zeta\delta)}{\gamma_s}$ for any $z \in (0, 1]$. This is because $0 < \hat{p}_s(\omega) \leq 1$ for any $\omega \in \Omega$ and $\{D_s = 1\} = \{M_1 \left(\hat{p}_s, \frac{\epsilon}{1+\epsilon}\right) \leq \frac{\ln(\zeta\delta)}{\gamma_s}\}$ as asserted by Lemma 20.

By the definition of sample sizes, we have $\gamma_s = \left[\frac{\ln(\zeta\delta)}{-\epsilon^2 (2(1+\epsilon)(1+\epsilon))}\right] \geq \frac{\ln(\zeta\delta)}{-\epsilon^2 (2(1+\epsilon)(1+\epsilon))}$ since $\lim_{z \to 0} M_1 \left(z, \frac{\epsilon}{1+\epsilon}\right) = -\epsilon^2 \left[2 (1 + \frac{\epsilon}{2}) (1 + \epsilon)\right]^{-1} < 0$, we have $\lim_{z \to 0} M_1 \left(z, \frac{\epsilon}{1+\epsilon}\right) \leq \frac{\ln(\zeta\delta)}{\gamma_s}$. Note that $M_1 \left(z, \frac{\epsilon}{1+\epsilon}\right) = -\epsilon^2 \left(\frac{\epsilon}{2(1+\epsilon)(1+\epsilon)-(1-\frac{\epsilon}{2})}\right)$, from which it can be seen that $M_1 \left(z, \frac{\epsilon}{1+\epsilon}\right)$ is monotonically decreasing with respect to $z \in (0, 1)$. Hence, $M_1 \left(z, \frac{\epsilon}{1+\epsilon}\right) < \lim_{z \to 0} M_1 \left(z, \frac{\epsilon}{1+\epsilon}\right) \leq \frac{\ln(\zeta\delta)}{\gamma_s}$ for any $z \in (0, 1)$. Since $M_1 \left(z, \frac{\epsilon}{1+\epsilon}\right)$ is a continuous function with respect to $z \in (0, 1)$ and $M_1 \left(1, \frac{\epsilon}{1+\epsilon}\right) = \lim_{z \to 1} M_1 \left(z, \frac{\epsilon}{1+\epsilon}\right)$, it must be true that $M_1 \left(1, \frac{\epsilon}{1+\epsilon}\right) \leq \frac{\ln(\zeta\delta)}{\gamma_s}$. This completes the proof of the lemma.

Lemma 24 $\Pr\{\hat{p} \leq p(1-\epsilon)\} \leq \sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \leq p(1-\epsilon), \ D_{\ell-1} = 0, \ D_\ell = 1\} \leq (\tau+1)\zeta\delta$ for any $p \in (0, 1)$.

Proof. By Lemma 23 the sampling must stop at some stage with index $\ell \in \{1, \cdot \cdot \cdot , s\}$. This implies that the stopping rule is well-defined. Let $\gamma = \sum_{i=1}^{n} X_i$. Then, we can write $\Pr\{\hat{p} \leq p(1-\epsilon)\} = \sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \leq p(1-\epsilon), \ \gamma = \gamma_\ell\}$. By the definition of the stopping rule, we have $\{\gamma = \gamma_\ell\} \subseteq \{D_{\ell-1} = 0, \ D_\ell = 1\}$. Hence,

$$\Pr\{\hat{p} \leq p(1-\epsilon)\} \leq \sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \leq p(1-\epsilon), \ D_{\ell-1} = 0, \ D_\ell = 1\} \leq \sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \leq p(1-\epsilon), \ D_\ell = 1\}. \quad (34)$$

Applying Lemma 21 and 32 of Lemma 19 we have

$$\sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \leq p(1-\epsilon), \ D_\ell = 1\} \leq \sum_{\ell=1}^{s} \Pr\left\{\hat{p}_\ell < p, \ M_1 (\hat{p}_\ell, p) \leq \frac{\ln(\zeta\delta)}{\gamma_\ell}\right\} \leq s\zeta\delta \leq (\tau+1)\zeta\delta. \quad (35)$$

Finally, the lemma can be established by combining (34) and (35).

Lemma 25 $\Pr\{\hat{p} \geq p(1+\epsilon)\} \leq \sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \geq p(1+\epsilon), \ D_{\ell-1} = 0, \ D_\ell = 1\} \leq (\tau+1)\zeta\delta$ for any $p \in (0, 1)$.

Proof. Note that

$$\Pr\{\hat{p} \geq p(1+\epsilon)\} \leq \sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \geq p(1+\epsilon), \ D_{\ell-1} = 0, \ D_\ell = 1\} \leq \sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \geq p(1+\epsilon), \ D_\ell = 1\}. \quad (36)$$

Applying Lemma 22 and 33 of Lemma 19 we have

$$\sum_{\ell=1}^{s} \Pr\{\hat{p}_\ell \geq p(1+\epsilon), \ D_\ell = 1\} \leq \sum_{\ell=1}^{s} \Pr\left\{\hat{p}_\ell > p, \ M_1 (\hat{p}_\ell, p) \leq \frac{\ln(\zeta\delta)}{\gamma_\ell}\right\} \leq s\zeta\delta \leq (\tau+1)\zeta\delta. \quad (37)$$
Combining (36) and (37) proves the lemma.

Finally, we are in a position to prove Theorem 3. Noting that \( \Pr\{|\hat{p} - p| \geq \varepsilon p\} = \Pr\{\hat{p} \leq p(1 - \varepsilon)\} + \Pr\{\hat{p} \geq p(1 + \varepsilon)\} \) and making use of Lemmas 24 and 25, we have \( \Pr\{|\hat{p} - p| \geq \varepsilon p\} \leq (\tau + 1)\zeta \delta + (\tau + 1)\zeta \delta = 2(\tau + 1)\zeta \delta \) for any \( p \in (0,1) \). Hence, if we choose \( \zeta \) to be a positive number less than \( \frac{1}{2(\tau + 1)} \), we have \( \Pr\{|\hat{p} - p| \geq \varepsilon p\} < \delta \) and thus \( \Pr\{|\hat{p} - p| < \varepsilon p\} > 1 - \delta \) for any \( p \in (0,1) \). This completes the proof of Theorem 3.

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