Prisoners’ dilemma in the presence of collective dephasing

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Abstract
We quantize prisoners’ dilemma in the presence of collective dephasing with a dephasing rate $\gamma$. It is shown that for a two-parameter set of strategies, $Q \otimes Q$ is Nash equilibrium below a cut-off value of time. Beyond this cut-off it bifurcates into two new Nash equilibria $Q \otimes D$ and $D \otimes Q$. Furthermore, for the maximum value of decoherence $C \otimes D$ and $D \otimes C$ also become Nash equilibria. At this stage the game has four Nash equilibria. On the other hand, for a three-parameter set of strategies, there is no pure strategy Nash equilibrium; however, there is a mixed strategy (non-unique) Nash equilibrium that is not affected by collective dephasing.

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1. Introduction
In game theory, the Nash equilibrium (NE) is a central solution concept. It is a set of strategies from which unilateral deviation of any player reduces his/her payoff. However, some shortcomings are also associated with this solution concept. First, it is not necessarily true for each game to have unique NE. Battle of Sexes (BoS) and Chicken game (CG) are well-known examples in this regard. Second, in some cases NE could result in an outcome that is far from the benefit of the players. Prisoners’ dilemma (PD) is an example where the rational reasoning forces the players to fall into a dilemma with the worst outcomes. Quantum game theory helps to resolve such dilemmas [1, 2] and shows that quantum strategies can be advantageous over classical strategies [1, 3, 4]. One of the foremost and elegant steps in this direction is by Eisert et al [1] to remove the dilemma in PD. In this quantization scheme, the strategy space of the players is a two-parameter set of $2 \times 2$ unitary matrices. Starting with a maximally entangled initial quantum state, the authors showed that for a suitable quantum strategy the dilemma disappears from the game. They also pointed out a quantum strategy which always wins over all the classical strategies. Later on, Marinatto and Weber [2] introduced another interesting and simple scheme for the analysis of non-zero sum games in the quantum domain. They gave
Hilbert structure to the strategic spaces of the players. They also used a maximally entangled initial state and allowed the players to play their tactics by applying the probabilistic choice of unitary operators. Applying their scheme to an interesting game of BoS, they found the strategy for which both the players can achieve equal payoffs. For both these schemes, the role of the initial quantum state remained important \[4, 6–10\]. In our earlier work on the subject, we introduced a generalized quantization scheme for two-person non-zero sum games that gives a relationship between these two apparently different quantization schemes \[11\]. A separate set of parameters was identified for which this scheme reduces to that of Marinatto and Weber \[2\] and Eisert et al \[1\] quantization schemes. Furthermore, some other interesting situations were identified which were not apparent within the existing two quantization schemes.

Players have to share qubits to play quantum games and the qubits are prone to decoherence. Many authors obtained interesting results by quantizing games in the presence of noise \[4, 5, 12–15\]. Chen et al \[13\] analyzed PD in the presence of three prototype quantum channels and showed that the payoffs gradually decrease with increasing noise without affecting the NE of the game. Later on, Flitney and Abbott \[4, 5\] studied quantum games in the presence of decoherence to find the advantage that a quantum player could have over a classical one. In their scheme, this advantage of a quantum player is termed the measure of ‘quantumness’ of a quantum game subjected to decoherence. They showed that the advantage of the quantum player reduces with increasing decoherence and at the maximum value of decoherence it disappears completely. In a \(2 \times 2\) symmetric game for the maximum value of decoherence, the payoffs of both players become the same and at this stage the classical game cannot be reproduced. Then, Nawaz and Toor \[14\] analyzed quantum games in the presence of correlated noise. They also restricted one of the players to play classical strategies. They showed that the effects of memory and decoherence become effective only when the game starts from a maximally entangled state and the measurement is performed in an entangled basis. In this case the quantum player outperforms the classical one. They also highlighted the fact that memory controls payoff reduction due to decoherence, and for the maximum value of memory decoherence becomes ineffective. Following the same lines, Ramzan et al \[15\] quantized PD, BoS and CG in the presence of three prototype quantum-correlated channels using a generalized quantization scheme. They also observed that the effects of the memory and decoherence become effective for the maximally entangled initial state and entangled measurement for which the quantum player outperforms the classical player. They also noted that memory has no effect on the NE. An important type of noise that has been studied extensively in quantum information theory is collective dephasing \[16–21\]. It plays a crucial role in physical systems like trapped ions, quantum dots and atoms inside a cavity. It also allows the existence of the decoherence-free subspace \[22\]. In this paper, we quantize PD in the presence of collective dephasing using our generalized quantization scheme for games, which allows us to perform measurements in an entangled as well as a product basis \[11\]. The game starts with a maximally entangled state that has decohered by the collective dephasing channel of the dephasing rate \(\gamma\). We show that for measurements in an entangled basis when the players are allowed to play the two-parameter set of strategies, \(Q \otimes Q\), which is the NE at \(t = 0\) (no noise) \[1\], remains a NE for \(e^{-2\gamma t} > \frac{1}{2}\). With increasing time \(t\) when \(e^{-2\gamma t} \leq \frac{1}{2}\), \(Q \otimes Q\) does not remain a NE but there appear two new NE \(Q \otimes D\) and \(D \otimes Q\) simultaneously. Furthermore, when \(e^{-2\gamma t} \rightarrow 0\) then besides \(Q \otimes D\) and \(D \otimes Q\) the strategy pairs \(C \otimes D\) and \(D \otimes C\) also become NE. At this stage there are four NE in the game. For the measurement in a product basis, \(C \otimes D\) and \(D \otimes C\) are decoherence-free NE. For the three-parameter set of strategies, there exists no pure strategy NE because for every strategy of one player, the other also has a counter strategy. However, there can be a mixed
strategy (non-unique) NE [23, 24] which also remains unaffected by collective dephasing for both entangled and product basis measurements.

The paper is organized as follows. In section 2, after a brief introduction to PD we quantize it in the presence of collective dephasing and section 3 concludes the results.

2. Quantization of PD with collective dephasing

PD is based on a story of two suspects, say Alice and Bob, who have allegedly committed a crime together. They have been arrested and are being interrogated in two separate cells. Each suspect will have to decide whether to confess the crime or to deny the crime without any communication between them. In game theory, the players’ decision to confess the crime is termed ‘to defect’ (strategy D) and to deny the crime is ‘to cooperate’ (strategy C). This situation can be depicted in the form of a bimatrix shown as follows:

\[
\begin{array}{cc}
\text{Alice} & \text{Bob} \\
\hline \\
C & (3, 3) \\
D & (5, 0)
\end{array}
\]

Depending upon their decisions, the players get payoffs according to the above payoff matrix. It is clear from this payoff matrix that D is the dominant strategy for both players. Therefore, rational reasoning forces each of them to play D resulting in (D, D) as the NE with payoffs (1, 1), which is not Pareto optimal. However, it was possible for the players to get better payoffs (3, 3) if they played C instead of D. This is generally known as the dilemma of this game.

For the quantum version of PD, the classical strategies C (cooperate) and D (defect) are assigned two basis vectors |C⟩ and |D⟩, respectively, in a Hilbert space of a two-level system. The state of the game at any instant is a vector in four-dimensional Hilbert space spanned by the basis vectors |CC⟩, |CD⟩, |DC⟩ and |DD⟩. Here, the entries in ket refer to the qubits possessed by Alice and Bob, respectively. But the qubits are prone to decoherence due to their interaction with environment. For the environment of the form of collective dephasing the state \( \hat{\rho} \) of the system is transformed by the following master equation [19]:

\[
\frac{d\hat{\rho}}{dt} = \gamma \left( 2\hat{J}_z \hat{\rho} \hat{J}_z - \hat{J}^2 \hat{\rho} - \hat{\rho} \hat{J}^2 \right),
\]

where \( \gamma \) is the dephasing rate and \( \hat{J}_z \) are the collective spin operators defined as

\[
\hat{J}_z = \sum_{i=1}^{2} \hat{\sigma}_z^{(i)} / 2
\]

with \( \hat{\sigma}_z \) as the Pauli matrices. Under the action of collective dephasing (2), the maximally entangled state

\[
|\phi^+\rangle = \frac{|00\rangle + i|11\rangle}{\sqrt{2}}
\]

shared by the players becomes

\[
\rho(t) = \frac{1}{2} (1 - e^{-2\gamma t})(|00\rangle\langle 00| + |11\rangle\langle 11|) + |\phi^+\rangle\langle \phi^+| e^{-2\gamma t}.
\]

In this case, the game starts with the state (5) that has decohered by a collective dephasing channel of the dephasing rate \( \gamma \) for time \( t \). The players apply their strategies on this decohered state. The strategy of the players is represented by the unitary operator \( U_i \) given as [11]

\[
U_i = \cos \frac{\theta_i}{2} R_i + \sin \frac{\theta_i}{2} C_i,
\]

\[
|\phi^+\rangle = \frac{|00\rangle + i|11\rangle}{\sqrt{2}}
\]

where $i = A$ or $B$ and $R_i$, $C_i$ are the unitary operators defined as

$$R_i |0⟩ = e^{iθ} |0⟩, \quad R_i |1⟩ = e^{-iθ} |1⟩,$$

$$C_i |0⟩ = - |1⟩, \quad C_i |1⟩ = |0⟩.$$  \hspace{1cm} (7)

After the application of the strategies, the initial state given by equation (5) transforms into

$$ρ_f = (U_A ⊗ U_B)ρ(t)(U_A ⊗ U_B)^†.$$  \hspace{1cm} (8)

The payoffs for Alice and Bob are

$$P^A = 3P_{00} + P_{11} + 5P_{10},$$ \hspace{1cm} (9a)

$$P^B = 3P_{00} + P_{11} + 5P_{01},$$ \hspace{1cm} (9b)

where

$$P_{00} = |ψ_{00}⟩⟨ψ_{00}|, \quad |ψ_{00}⟩ = \cos \frac{δ}{2} |00⟩ + i \sin \frac{δ}{2} |11⟩,$$ \hspace{1cm} (10a)

$$P_{11} = |ψ_{11}⟩⟨ψ_{11}|, \quad |ψ_{11}⟩ = \cos \frac{δ}{2} |11⟩ + i \sin \frac{δ}{2} |00⟩,$$ \hspace{1cm} (10b)

$$P_{10} = |ψ_{10}⟩⟨ψ_{10}|, \quad |ψ_{10}⟩ = \cos \frac{δ}{2} |10⟩ - i \sin \frac{δ}{2} |01⟩,$$ \hspace{1cm} (10c)

$$P_{01} = |ψ_{01}⟩⟨ψ_{01}|, \quad |ψ_{01}⟩ = \cos \frac{δ}{2} |01⟩ - i \sin \frac{δ}{2} |10⟩,$$ \hspace{1cm} (10d)

and $δ \in [0, \frac{π}{2}]$ is the entanglement of measurement basis. These payoff operators reduce to that of Eisert’s scheme for $δ = \frac{π}{2}$ [1], and for $δ = 0$ these transform into that of Marinatto and Weber’s scheme [2]. The payoffs for the players are obtained as

$$S_A(θ_A, φ_A, θ_B, φ_B) = \text{Tr}(P^A ρ_f),$$

$$S_B(θ_A, φ_A, θ_B, φ_B) = \text{Tr}(P^B ρ_f),$$ \hspace{1cm} (11)

where Tr represents the trace of a matrix. Using equations (1), (5), (8), (9) and (11), the payoffs are obtained as

$$S_A(θ_A, φ_A, θ_B, φ_B) = [2 + e^{-2γt} \cos 2(φ_A + φ_B) \sin δ] \cos^2 \frac{θ_A}{2} \cos^2 \frac{θ_B}{2}$$

$$+ [2 - e^{-2γt} \sin δ] \sin^2 \frac{θ_A}{2} \sin^2 \frac{θ_B}{2}$$

$$+ \frac{5}{2} [1 - e^{-2γt} \cos 2φ_A \sin δ] \cos^2 \frac{θ_A}{2} \sin^2 \frac{θ_B}{2}$$

$$+ \frac{5}{2} [1 + e^{-2γt} \cos 2φ_B \sin δ] \sin^2 \frac{θ_A}{2} \cos^2 \frac{θ_B}{2}$$

$$- \frac{1}{4} [e^{-2γt} + 2 \sin δ] \sin θ_A \sin θ_B \sin (φ_A + φ_B)$$

$$- \frac{5}{4} \sin θ_A \sin θ_B \sin (φ_A - φ_B) \sin δ,$$ \hspace{1cm} (12)

$$S_B(θ_A, φ_A, θ_B, φ_B) = [2 + e^{-2γt} \cos 2(φ_A + φ_B) \sin δ] \cos^2 \frac{θ_A}{2} \cos^2 \frac{θ_B}{2}$$

$$+ [2 - e^{-2γt} \sin δ] \sin^2 \frac{θ_A}{2} \sin^2 \frac{θ_B}{2}$$

$$+ \frac{5}{2} [1 - e^{-2γt} \cos 2φ_A \sin δ] \cos^2 \frac{θ_A}{2} \sin^2 \frac{θ_B}{2}$$

$$+ \frac{5}{2} [1 + e^{-2γt} \cos 2φ_B \sin δ] \sin^2 \frac{θ_A}{2} \cos^2 \frac{θ_B}{2}$$

$$- \frac{1}{4} [e^{-2γt} + 2 \sin δ] \sin θ_A \sin θ_B \sin (φ_A + φ_B)$$

$$- \frac{5}{4} \sin θ_A \sin θ_B \sin (φ_A - φ_B) \sin δ.$$ \hspace{1cm} (13)
These payoffs transform to that of Eisert et al [1] for \( t = 0 \) and \( \delta = \frac{\pi}{2} \). As in our generalized quantization scheme, measurements can be performed in entangled \((\delta = \frac{\pi}{2})\) as well as in product basis \((\delta = 0)\). Therefore, we take both cases one by one.

2.1. Entangled measurement

For entangled measurement, i.e. \( \delta = \frac{\pi}{2} \) the payoffs given in equations (12) and (13) become

\[
\begin{align*}
S_A(\theta_A, \phi_A, \theta_B, \phi_B) &= \left[2 + e^{-2\gamma t} \cos 2(\phi_A + \phi_B) - 2 \sin^2 \frac{\theta_A}{2} - \sin^2 \frac{\theta_B}{2}\right] \\
&\quad + \frac{5}{2} \left[1 - e^{-2\gamma t} \cos 2\phi_A\right] \cos^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} \\
&\quad + \frac{5}{2} \left[1 + e^{-2\gamma t} \cos 2\phi_B\right] \sin^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} \\
&\quad - \frac{1}{4} \left[2 + e^{-2\gamma t}\right] \sin \theta_A \sin \theta_B \sin(\phi_A + \phi_B) \\
&\quad - \frac{5}{4} \sin \theta_A \sin \theta_B \sin(\phi_A - \phi_B)
\end{align*}
\]

\[
S_B(\theta_A, \phi_A, \theta_B, \phi_B) = \left[2 + e^{-2\gamma t} \cos 2(\phi_A + \phi_B) - 2 \sin^2 \frac{\theta_A}{2} - \sin^2 \frac{\theta_B}{2}\right] \\
+ \frac{5}{2} \left[1 + e^{-2\gamma t} \cos 2\phi_A\right] \cos^2 \frac{\theta_A}{2} \sin^2 \frac{\theta_B}{2} \\
+ \frac{5}{2} \left[1 - e^{-2\gamma t} \cos 2\phi_B\right] \sin^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2} \\
- \frac{1}{4} \left[2 + e^{-2\gamma t}\right] \sin \theta_A \sin \theta_B \sin(\phi_A + \phi_B) \\
- \frac{5}{4} \sin \theta_A \sin \theta_B \sin(\phi_A - \phi_B).
\]

For this case, \( Q \otimes \bar{Q} \) with \( (\theta_A, \phi_A, \theta_B, \phi_B) = (0, \frac{\pi}{2}, 0, \frac{\pi}{2}) \) is the NE of the game with payoffs \( S_A(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) = S_B(0, \frac{\pi}{2}, 0, \frac{\pi}{2}) = 2 + e^{-2\gamma t} [1] \). For the analysis of this NE, we apply the following NE conditions:

\[
S_A(0, \pi, 0, \pi) - S_A(\theta_A, \phi_A, 0, \pi) \geq 0 \\
S_B(0, \pi, 0, \pi) - S_B(0, \pi, \theta_B, \phi_B) \geq 0.
\]

With the help of equations (14) and (15) the above inequalities give

\[
7e^{-2\gamma t} \cos^2 \frac{\theta_i}{2} - 1 \geq 0,
\]

where \( i = A, B \). These inequalities are satisfied for all \( \phi \)'s and \( \theta \)'s if \( e^{-2\gamma t} \geq \frac{1}{7} \). With increasing \( \gamma t \) when \( e^{-2\gamma t} < \frac{1}{7} \), then inequalities (17) are not satisfied and \( Q \otimes \bar{Q} \) does not remain a NE, but \( Q \otimes D \) and \( D \otimes \bar{Q} \) appear as NE in the game. For \( Q \otimes D \) as NE we require

\[
S_A(0, \frac{\pi}{2}, \pi, 0) - S_A(\theta_A, \phi_A, \pi, 0) \geq 0 \\
S_B(0, \frac{\pi}{2}, \pi, 0) - S_B(0, \frac{\pi}{2}, \theta_B, \phi_B) \geq 0.
\]
Using equations (14) and (15), inequalities (18) become
\[
7 - (2 - 5 \cos 2\phi_A) \cos^2 \frac{\theta_A}{2} e^{-2y_t} + \sin^2 \frac{\theta_A}{2} \geq 0
\] (19)
\[
[1 - (5 - 2 \cos 2\phi_B) e^{-2y_t}] \cos^2 \frac{\theta_B}{2} \geq 0.
\] (20)

The above inequalities are satisfied for all \(\theta\)'s and \(\phi\)'s subject to the condition \(0 \leq e^{-2y_t} \leq \frac{1}{4}\).

It shows that \(Q \otimes D\) remains a NE for all values of \(y_t\) for which \(0 \leq e^{-2y_t} \leq \frac{1}{4}\). By similar reasoning \(D \otimes Q\) can be proved to be a NE for all values of \(y_t\) for which \(0 \leq e^{-2y_t} \leq \frac{1}{4}\). However, when \(e^{-2y_t} \to 0\) then besides \(Q \otimes D\) and \(D \otimes Q\) the strategy pairs \(C \otimes D\) and \(D \otimes C\) also become NE. At this stage we have four NE in the game. The NE conditions for \((C, D)\) are
\[
S_A(0, 0, \pi, 0) - S_A(\theta_A, \phi_A, \pi, 0) \geq 0
\]
\[
S_B(0, 0, \pi, 0) - S_B(0, 0, \theta_B, \phi_B) \geq 0.
\] (21)

With the help of equations (14) and (15), the above inequalities become
\[
\sin^2 \frac{\theta_A}{2} - \left[3 + (2 - 5 \cos 2\phi_A) \cos^2 \frac{\theta_A}{2}\right] e^{-2y_t} \geq 0
\]
\[
[1 + (5 - 2 \cos 2\phi_B) e^{-2y_t}] \cos^2 \frac{\theta_B}{2} \geq 0.
\] (22)

Inequalities (22) are satisfied for all \(\theta\)'s and \(\phi\)'s only if \(e^{-2y_t} \to 0\). Similarly, it can be proved that \(D \otimes C\) is NE when \(e^{-2y_t} \to 0\).

2.2. Product measurement

For the measurement in product basis, i.e. \(\delta = 0\), the payoffs given in equations (12) and (13) for player \(i = A\) or \(B\) become
\[
S_i(\theta_A, \phi_A, \theta_B, \phi_B) = 2 - \cos^2 \frac{1}{2} \theta_A \cos^2 \frac{1}{2} \theta_B + \frac{1}{2} \cos^2 \frac{1}{2} \theta_B
\]
\[
+ \frac{1}{2} \cos^2 \frac{1}{2} \theta_A - \frac{1}{2} e^{-2y_t} \sin \theta_A \sin \theta_B \sin (\phi_A + \phi_B).
\] (23)

In this case, the strategy pairs \(D \otimes C\) and \(C \otimes D\) are two NE of the game with payoffs \(\left(\frac{2}{7}, \frac{2}{7}\right)\). For \(D \otimes C\) the NE conditions \(S_A(\pi, 0) - S_A(\theta_A, 0) \geq 0\) and \(S_B(\pi, 0) - S_B(\theta_B, 0) \geq 0\) give \(\cos^2 \frac{\theta_A}{2} \geq 0\) and \(\sin^2 \frac{\phi_B}{2} \geq 0\), respectively. Both these conditions are always satisfied and are independent of decoherence effects. Therefore \(D \otimes C\) is NE for all values of \(y_t\). Similarly, it can be proved that \(C \otimes D\) is a NE with same properties. It is to be noted that the classical game cannot be reproduced in this case. It is due to the fact that when a quantum game starts with an entangled state of the form \(\psi_{in} = \cos \frac{\xi}{2} |00\rangle + i \sin \frac{\xi}{2} |11\rangle\) and a measurement is performed in a product basis, then Marinatto and Weber’s quantization scheme results [2]. In this scheme, the classical results can be reproduced with an unentangled initial quantum state with \(\xi = 0\). But in this case the game starts with a maximally entangled state that has decohered by collective dephasing with the dephasing rate \(y\) (see (5)). At \(t = 0\) (i.e. \(e^{-2y_t} = 1\)) the initial state of a game is a maximally entangled state and for all \(t > 0\) the initial state becomes mixed. No value of \(y\) can be found that can transform equation (5) to a state that is required to reproduce the classical game. This highlights the fact that initial quantum state plays a crucial role in the solution of quantum games [4, 6–10].
2.3. Three-parameter set of strategies

For a three-parameter set of strategies, the players are equipped with unitary operators of the form

\[
\hat{U}(\theta, \phi, \psi) = \begin{bmatrix}
\cos \frac{\theta}{2} & i e^{i\phi} \sin \frac{\theta}{2} \\
\cos \frac{\phi}{2} & i e^{i\psi} \sin \frac{\phi}{2} \\
\cos \frac{\psi}{2} & i e^{i\theta} \sin \frac{\psi}{2} \\
\end{bmatrix},
\]

and the payoffs of players for the initial state given by equation (5) become

\[
S_A(\theta_A, \phi_A, \psi_A, \theta_B, \phi_B, \psi_B) = [2 + e^{-2\gamma t} \cos 2(\phi_A + \phi_B) \sin \delta] \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2}
\]
\[
+ \frac{1}{4} \sin \theta_A \sin \theta_B \left[ e^{-2\gamma t} \sin (\phi_A + \phi_B - \psi_A - \psi_B) \right]
\]

\[
S_B(\theta_A, \phi_A, \psi_A, \theta_B, \phi_B, \psi_B) = [2 + e^{-2\gamma t} \cos 2(\phi_A + \phi_B) \sin \delta] \cos^2 \frac{\theta_A}{2} \cos^2 \frac{\theta_B}{2}
\]
\[
+ \frac{1}{4} \sin \theta_A \sin \theta_B \left[ e^{-2\gamma t} \sin (\phi_A + \phi_B - \psi_A - \psi_B) \right].
\]

In this case, there is no pure strategy NE because for every strategy of Alice, Bob also has a counter strategy. However there can be a mixed strategy (non-unique) NE [23, 24]. The NE occurs when Alice chooses the operators \(A_1 = (\theta_A = 0, \phi_A = 0, \psi_A = \psi)\) and \(A_2 = (\theta_A = 0, \phi_A = \frac{\pi}{2}, \psi_A = \psi)\) with equal probability and Bob chooses \(B_1 = (\theta_B = \pi, \phi_B = \phi, \psi_B = 0)\) and \(B_2 = (\theta_B = \pi, \phi_B = \phi, \psi_B = \frac{\pi}{2})\) with equal probability. The payoffs corresponding to each pair of strategies are

\[
S_A(A_1, B_1) = \frac{5}{7} \left( 1 + e^{-2\gamma t} \sin \delta \right), \quad S_A(A_1, B_2) = \frac{5}{7} \left( 1 - e^{-2\gamma t} \sin \delta \right)
\]
\[
S_A(A_2, B_1) = \frac{5}{7} \left( 1 - e^{-2\gamma t} \sin \delta \right), \quad S_A(A_2, B_2) = \frac{5}{7} \left( 1 + e^{-2\gamma t} \sin \delta \right)
\]
\[
S_B(A_1, B_1) = \frac{5}{7} \left( 1 + e^{-2\gamma t} \sin \delta \right), \quad S_B(A_1, B_2) = \frac{5}{7} \left( 1 - e^{-2\gamma t} \sin \delta \right)
\]
\[
S_B(A_2, B_1) = \frac{5}{7} \left( 1 - e^{-2\gamma t} \sin \delta \right), \quad S_B(A_2, B_2) = \frac{5}{7} \left( 1 + e^{-2\gamma t} \sin \delta \right)
\]

and the payoff for both players at NE is \(\frac{5}{7}\). Although the payoffs against an individual pair of strategies depend upon decoherence, the average payoff at NE remains independent of decoherence for both entangled (\(\delta = \frac{\pi}{2}\)) and product measurements (\(\delta = 0\)).
3. Conclusion

We quantized PD using the generalized quantization scheme for the case when the initial state of a game is a maximally entangled state that has decohered by collective dephasing of the dephasing rate $\gamma$. In the generalized quantization scheme, we have the options of performing a measurement in an entangled as well as a product basis. In the case of entangled basis measurements when the players are allowed to play two-parameter set of strategies then $Q \otimes Q$ is a NE for $e^{-2\gamma t} > \frac{1}{2}$. With increasing time when $e^{-2\gamma t} < \frac{1}{2}$, then $Q \otimes Q$ disappears as a NE and two new NE $Q \otimes D$ and $D \otimes Q$ appear simultaneously. $Q \otimes D$ and $D \otimes Q$ remain NE for $0 \leq e^{-2\gamma t} \leq \frac{1}{2}$; however, when $e^{-2\gamma t} \to 0$, then $C \otimes D$ and $D \otimes C$ also become NE resulting in four NE in the game. On the other hand, for the measurement in a product basis there are two NE $C \otimes D$ and $D \otimes C$ in the game which are not affected by decoherence. For a three-parameter set of strategies there is mixed strategy NE that is also independent of decoherence for measurement in an entangled as well as a product basis.

Du et al [26] also generalized Eisert’s quantization scheme for PD to study the effects of entanglement on NE by taking an initial quantum state of the form

$$\psi_{\text{in}} = \cos \frac{\xi}{2} |00\rangle + i \sin \frac{\xi}{2} |11\rangle,$$

(27)

where $\xi \in [0, \frac{\pi}{2}]$ is the measure of entanglement. They showed that, for a two-parameter set of strategies, $Q \otimes Q$ is only a NE of PD when the entanglement of the initial quantum state is greater than a certain threshold value $\xi_{\text{th1}} = 0.685$. When the entanglement of the initial state becomes less than this threshold value, then two new NE $Q \otimes D$ and $D \otimes Q$ appear in the game. This feature of the game holds up to $\xi_{\text{th2}} = 0.464$. For any entanglement in the range $0 \leq \xi \leq 0.464$, the game shows classical behavior with $D \otimes D$ as the NE. Our results are consistent with Du et al’s results [26] with two exceptions. First, the threshold value of the entanglement parameter for a particular NE is different. Second, at some minimum value of entanglement Du et al’s game behaves like a classical PD with $D \otimes D$ being the NE, whereas in our case this feature is absent. We see that it is due to the fact that in Du et al’s [26] scheme the game starts from a maximally entangled pure state of the form (27) that remains pure for all values of $\xi$ $\geq$ 0. But in our case the game starts from an initial state of the form (5) that is a maximally entangled pure state only at $\gamma t = 0$. For all $\gamma t > 0$, it becomes mixed, and for the maximum decoherence (i.e. when $e^{-2\gamma t} \to 0$) it transforms to $\frac{1}{2} (|00\rangle \langle 00| + |11\rangle \langle 11|)$.

As the initial state plays an important role in the solution of quantum games [1, 2, 27], the features of PD in our case are somewhat different from that of Du et al [26].

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