1. Introduction

Global singularity theory originates from problems in obstruction theory. Consider the following question: is there an immersion in a given homotopy class of maps between two smooth manifolds? We can reformulate this problem as follows. Suppose $M$ and $N$ are smooth real manifolds with $\dim N \geq \dim M$, and $f: M \to N$ is a sufficiently generic smooth map in a fixed homotopy class. The map $f$ is an immersion, if

$$\Sigma^1(f) \overset{def}{=} \{ p \in M \mid \dim \text{Ker}(d_pf) \geq 1 \} = \emptyset.$$ 

The set $\Sigma^1(f)$ is called the $\Sigma^1$-singularity locus, or simply the $\Sigma^1$-locus of $f$, i.e. the points in $M$ where $f$ has a $\Sigma^1$-singularity: the kernel of the differential of $f$ is non-zero. In the case of $\mathbb{Z}_2$-cohomology and a sufficiently generic map $f$, the set $\Sigma^1(f)$ represents a cohomology class via Poincaré duality. Clearly, if the Poincaré dual $\text{PD}[\Sigma^1(f)]$ is non-zero in $H^*(M, \mathbb{Z}_2)$, then $f$ is not an immersion.

In the 50s, René Thom proved the following statement, now known as Thom’s principle.

**Theorem 1.1 (Thom’s principle, [3])**. Let $\Theta$ be a singularity and let $m \leq n$ be non-negative integers. Suppose $\{a_1, \ldots, a_m\}$ and $\{a'_1, \ldots, a'_n\}$ are two sets of graded variables with $\deg a_i = \deg a'_i = i$. Then there exists a universal polynomial in $a_1, \ldots, a_m$ and $a'_1, \ldots, a'_n$

$$\text{Tp}[\Theta](a_1, \ldots, a_m, a'_1, \ldots, a'_n)$$

depending only on $\Theta$, $m$ and $n$, such that for all smooth compact real manifolds $M$ and $N$, $\dim(M) = m$, $\dim(N) = n$, and a sufficiently generic smooth map $f: M \to N$,

$$\Theta(f) = \{p \in M \mid f \text{ has a singularity of type } \Theta \text{ at } p\}$$

is a cycle in $M$, and

$$\text{PD}[\Theta(f)] = \text{Tp}[\Theta](w_1(TM), \ldots, w_m(TM), f^*w_1(TN), \ldots, f^*w_n(TN)) \in H^*(M, \mathbb{Z}_2),$$

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where \( w_i(TM) \) and \( w_j(TN) \) are the Stiefel-Whitney classes of the corresponding tangent bundles.

This universal polynomial is called the \emph{Thom polynomial} of \( \Theta \). In this paper we will give a rigorous construction of this polynomial in the case of complex manifolds.

Thom’s principle may also be translated from the real to the complex case.

**Theorem 1.2** (Thom’s principle in the complex case). Let \( \Theta \) be a singularity and let \( m \leq n \) be non-negative integers. Suppose \( \{a_1, \ldots, a_m\} \) and \( \{a_1', \ldots, a_n'\} \) are two sets of graded variables with \( \deg a_i = \deg a_i' = i \). Then there exists a universal polynomial in \( a_1, \ldots, a_m \) and \( a_1', \ldots, a_n' \)

\[
\text{Tp}[\Theta](a_1, \ldots, a_m, a_1', \ldots, a_n')
\]

depending only on \( \Theta \), \( m \) and \( n \), such that for all compact complex manifolds \( M \) and \( N \), \( \dim(M) = m \), \( \dim(N) = n \), and a sufficiently generic holomorphic map \( f: M \to N \),

\[
\Theta(f) = \{ p \in M \mid f \text{ has a singularity of type } \Theta \text{ at } p \}
\]

is a cycle in \( M \), and

\[
\text{PD}[\Theta(f)] = \text{Tp}[\Theta](c_1(TM), \ldots, c_m(TM), f^*c_1(TN), \ldots, f^*c_n(TN)) \in H^*(M, \mathbb{R}),
\]

where \( c_i(TM) \) and \( c_j(TN) \) are the Chern classes of the corresponding tangent bundles.

In fact, a result of Borel and Haefliger \cite{BH} implies that the real Thom polynomial for \( \Theta \) may be obtained by substituting the corresponding Stiefel-Whitney classes for the Chern classes in the corresponding Thom polynomial in the complex case.

Calculating Thom polynomials is difficult: some progress has been made in the works of Bérczi and Szentes \cite{BerSc}, Rimányi \cite{Rim}, Fehér and Rimányi \cite{FeRi}, Bérczi, Fehér and Rimányi \cite{BerFeRi}, Gaffney \cite{Gaf} and Ronga \cite{Ron}.

In this paper, we will focus on the properties of Thom polynomials of a particular class of singularities: \emph{contact singularities}. Let \( M \) and \( N \) be compact complex manifolds such that \( \dim(M) = m \), \( \dim(N) = n \) and \( m \leq n \). Let \( f: M \to N \) be a sufficiently generic (see Section 3 for the genericity condition) holomorphic map.

Let \( z_1, \ldots, z_m \) be the local coordinates on a chart \( U_p \) centered at a point \( p \in M \) and let \( y_1, \ldots, y_n \) be the local coordinates on a chart \( V_{f(p)} \) centered at \( f(p) \in N \). Let us denote \((f \circ y_j)(z_1, \ldots, z_m) = \sum \alpha_{i_1, \ldots, i_m} z_1^{i_1} \cdots z_m^{i_m} \) by \( f_j(z_1, \ldots, z_m) \). Denote the algebra of power series in \( z_1, \ldots, z_m \) without a constant term by \( \mathbb{C}_0[[z_1, \ldots, z_m]] \).

To each point \( p \in M \), we can associate the algebra

\[
A_f(p) = \mathbb{C}_0[[z_1, \ldots, z_m]]/I(f_1, \ldots, f_n),
\]
where \( I(f_1, \ldots, f_n) \) is the ideal generated by \( f_1, \ldots, f_n \).

Suppose \( A \) is an algebra. The \( \Theta_A \)-locus of a map \( f \) is defined as
\[
\Theta_A(f) = \{ p \in M \mid A f(p) \cong A \}.
\]

By definition, the Thom polynomial is a polynomial in two sets of variables. Damon proved that for contact singularities, the Thom polynomial may be expressed in a single set of variables. This is an important result, which plays a central role in recent advances in global singularity theory, but the proof of Damon’s theorem is hard to find in the literature. In this paper, we will give a modern proof of Damon’s result in the complex case.

**Theorem 1.3** (Damon, [4]). Let \( A \) be a finite-dimensional commutative algebra and let \( l \) be a non-negative integer. Suppose \( \{ b_i \}_{i \geq 0} \) is the set of graded variables with \( \deg(b_i) = i \). There exists a universal polynomial in \( \{ b_i \}_{i \geq 0} \) depending only on \( l \) and on \( A \)
\[
\tilde{T}_l^l A(b_1, b_2, \ldots),
\]
such that for all compact complex manifolds \( M \) and \( N \) with \( \dim(N) - \dim(M) = l \) and for every sufficiently generic holomorphic map \( f : M \to N \)
\[
T_p[\Theta_A](c_1(TM), c_2(TM), \ldots, f^* c_1(TN), f^* c_2(TN), \ldots) = \tilde{T}_l^l A(c_1(f), c_2(f), \ldots),
\]
where the variables \( c_i(f) \) are given by
\[
1 + c_1(f)t + c_2(f)t^2 + \ldots = \sum c_j(TM)^t \sum c_j(TN)^{t'}. \]

The universal polynomial \( \tilde{T}_l^l A(c_1(f), c_2(f), \ldots) \in \mathbb{R}[c_1(f), c_2(f), \ldots] \) has an important structural property: it is positive when expressed in the geometrically natural Schur basis of \( \mathbb{R}[c_1(f), c_2(f), \ldots] \) (we will discuss the geometric significance of this polynomial basis in Section 4).

The Schur polynomials form an important basis of the ring of symmetric polynomials parametrized by partitions. Given an integer partition \( \lambda = (\lambda_1, \ldots, \lambda_n) \) such that \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n > 0 \) define the conjugate partition \( \lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \) by taking \( \lambda_j^* \) to be the largest \( j \) such that \( \lambda_j \geq i \). Let us denote by \( s_\lambda (b_1, \ldots, b_n) \) the expression of Schur polynomials in elementary symmetric polynomials:
\[
s_\lambda (b_1, \ldots, b_n) = \det \{b_{\lambda_j^* + j - 1}\}_{i,j=1}. \]

The second result for which we give a new proof is the following theorem:

**Theorem 1.4** (Pragacz, Weber, [5]). Let \( A \) be a finite-dimensional commutative algebra and let \( m, l \) be non-negative integers. Let \( M \) and \( N \) be compact complex manifolds such that \( \dim(N) - \dim(M) = l \) and let \( f : M \to N \) be a sufficiently
generic holomorphic map. The Thom polynomial of $\Theta_A(f)$ expressed in $c_i(f)$ has positive coefficients in the Schur basis:

$$\tilde{T}_{\mathcal{P}}(c_1(f), c_2(f), \ldots) = \sum \alpha_{\lambda}s_{\lambda}(c_1(f), c_2(f), \ldots)$$

where $\alpha_{\lambda} \geq 0$.

2. Preliminaries

2.1. Singularity theory. Let $z_1, \ldots, z_n$ be the standard coordinates on $\mathbb{C}^n$. Denote by $J^n$ the algebra of formal power series in $z_1, \ldots, z_n$ without a constant term, i.e.

$$J^n = \{ h \in \mathbb{C}[[z_1, \ldots, z_n]] \mid h(0) = 0 \}.$$

The space of $d$-jets of holomorphic functions on $\mathbb{C}^n$ near the origin is the quotient of $J^n$ by the ideal of series with the lowest order term of degree at least $d+1$, i.e. the ideal generated by monomials $z_1^{i_1} \ldots z_n^{i_n}$ such that $\sum i_j = d + 1$. We will denote this ideal by $I(\mathcal{P}^{d+1})$:

$$J^n_d = J^n / I(\mathcal{P}^{d+1}).$$

As a linear space, the algebra $J^n_d$ may be identified with the space of polynomials in $z_1, \ldots, z_n$ of degree at most $d$ without a constant term. The space of $d$-jets of holomorphic maps from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^k, 0)$, or the space of map-jets, is denoted by $J_{d}^{n,k}$ and is naturally isomorphic to $J^n_d \otimes \mathbb{C}^k$. In this paper we will assume $n \leq k$.

Composition of map-jets together with cancellation of terms of degree greater than $d$ gives a well-defined map

$$J_d^{n,k} \times J_d^{k,m} \to J_d^{n,m}$$

$$(\Psi, \Phi) \mapsto \Psi \circ \Phi.$$

Consider a sequence of natural maps

$$J_d^{n,k} \to J_d^{n,k-1} \to \ldots \to J_1^{n,k} \cong \text{Hom}(\mathbb{C}^n, \mathbb{C}^k).$$

For $\Psi \in J_d^{n,k}$, the linear part of $\Psi$ is defined as the image of $\Psi$ in $J_1^{n,k}$: $\text{Lin} \Psi = \text{Im} \Psi \in J_1^{n,k}$.

Consider the set

$$\text{Diff}_d^{n} = \{ \Delta \in J_d^{n,n} \mid \text{Lin} \Delta \text{ invertible} \}.$$

Previously defined operation $\circ$ gives this set an algebraic group structure.

Let $\Delta_n \in \text{Diff}_d^{n}$, $\Delta_k \in \text{Diff}_d^{k}$, and $\Psi \in J_d^{n,k}$. The left-right action of $\text{Diff}_d^{n} \times \text{Diff}_d^{k}$ on $J_d^{n,k}$ is given by

$$(\Delta_n, \Delta_k)\Psi = \Delta_n \circ \Psi \circ \Delta_k^{-1}.$$
Definition 2.1. Left-right invariant algebraic subsets of $J_d^{n,k}$ are called singularities.

To a given element $\Psi \in J_d^{n,k} \cong J_d^n \otimes \mathbb{C}^k$, presented as $(\Psi_1, \ldots, \Psi_k), \Psi_i \in J_d^n$, we can associate an algebra $A_{\Psi} = J_d^n / I(P_1, \ldots, P_k)$. This algebra is nilpotent: there exists a natural number $m$ such that $A_{\Psi}^m = 0$, in other words, a product of any $m$ elements of $A_{\Psi}$ is equal to 0. $A_{\Psi}$ is nilpotent because $J_d^n$ itself is nilpotent: $(J_d^n)^{d+1} = 0$.

Definition 2.2. Suppose $A$ is a nilpotent algebra. The subset $\Theta_A^{n,k} = \{ \Psi \in J_d^{n,k} \mid A_{\Psi} \cong A \}$ is called a contact singularity.

Proposition 2.3 (\cite{6}). Let $A$ be a nilpotent algebra: $A^{d+1} = 0$. For $n \geq \text{dim}(A/A^2)$ and $k$ sufficiently large, $\Theta_A^{n,k}$ is a non-empty, left-right invariant, irreducible quasi-projective algebraic subvariety of $J_d^{n,k}$.

2.2. Equivariant Poincaré dual. Suppose a topological group $G$ acts continuously on a topological space $M$, and $Y$ is a closed $G$-invariant subvariety in $M$. In this section we will define an analog of a Poincaré dual of $Y$, which reflects the $G$-action: the equivariant Poincaré dual of $Y$.

Let $G$ be a topological group and let $\pi: EG \to BG$ be the universal $G$-bundle, i.e. a principle $G$-bundle such that if $p: E \to B$ is any principle $G$-bundle, then there is a map $\zeta: B \to BG$ unique up to homotopy and $E \cong \zeta^*EG$. The universal $G$-bundle exists, is unique up to homotopy and can be constructed as a principle $G$-bundle with contractible total space.

Now we can construct the space with a free $G$-action and the same homotopy type as a fixed before topological space $M$, the Borel construction:

Definition 2.4. The Borel construction (also homotopy quotient or homotopy orbit space) for a topological group $G$ acting on a topological space $M$ is the space $EG \times_G M$, i.e. the factor of $EG \times M$ by the diagonal action: $(gx, gy) \sim (x, y)$, where $g \in G, x \in EG, y \in M$.

Definition 2.5. The equivariant cohomology of $M$ is the ordinary cohomology for the Borel construction:

$$H_G^c(M) = H^*(EG \times_G M).$$

Note that since $(EG \times pt)/G = EG/G = BG$, the equivariant cohomology of a point is $H_G^c(pt) = H^*(BG)$. 
We would like to define an analog of a Poincaré dual in the equivariant case, i.e. when a group $G$ acts on a space $M$ and $Y \subset M$ is a closed $G$-invariant subvariety. We constructed a ‘substitute’ for the orbit space of $G$-action on $M$: the Borel construction $EG \times_G M$. Now, $EG \times_G Y$ is again a $G$-invariant subvariety of $EG \times_G M$, and we want to define a dual of $EG \times_G Y$ in $H^*(EG \times_G M) \cong H_G^*(M)$. However, first we have to deal with the fact that $EG$ is usually infinite-dimensional by introducing an approximation.

**Lemma 2.6 ([7]).** Suppose $E_m$ - any connected space with a free $G$-action, such that $H^i E_m = 0$ for $0 < i < k(m)$, where $k(m)$ is some integer. Then for any $M$, there are natural isomorphisms

$$H^i(E_m \times_G M) \cong H^i(EG \times_G M)$$

for $i < k(m)$.

Let us fix $EG$, $BG$ and the approximations $EG_1 \subset EG_2 \subset \ldots \subset EG$ together with $BG_i = EG_i/G$. We can now consider $EG_m \times_G Y \subset EG_m \times_G M$ – two finite dimensional spaces. Let $D$ be the codimension of $EG_m \times_G Y$ in $EG_m \times_G M$.

Every irreducible closed subvariety of a non-singular variety has a well-defined Borel-Moore homology class [8], so we can define the equivariant Poincaré dual of $Y$ as follows:

$$eP(Y) = [EG_m \times_G Y]_{BM} \in H^{2D}(EG_m \times_G M) = H^2_G(M)$$

for $m$ large enough.

### 2.3. The Thom polynomial.

We want to study the equivariant Poincaré dual of a closure of a singularity $\overline{\Theta} \subset J_{d}^{n,k}$. It is convenient to restrict the action of $\text{Diff}_d^n \times \text{Diff}_d^k$ on $J_{d}^{n,k}$ to the action of $G = \text{Gl}_n \times \text{Gl}_k$.

First, we need to fix $EG$, $BG$ and the corresponding approximations. Recall that $C^\infty$ is defined as $\{(z_1, z_2, \ldots) \mid z_i \in C, \text{ only finite number of } z_i \text{ is non-zero}\}$. Fix $E \text{Gl}_n = \text{Fr}(n, \infty)$, the manifold of $n$-frames of orthonormal vectors in $C^\infty$, and $B \text{Gl}_n = \text{Gr}(n, \infty)$, the Grassmannian of $n$-planes in $C^\infty$. So, in our case $EG = \text{Fr}(n, \infty) \times \text{Fr}(k, \infty)$ and $BG = \text{Gr}(n, \infty) \times \text{Gr}(k, \infty)$. The approximations are given by $EG_i = \text{Fr}(n, i) \times \text{Fr}(k, i)$ and $BG_i = \text{Gr}(n, i) \times \text{Gr}(k, i)$.

By definition,

$$eP(\overline{\Theta}) \in H^*_G(J_{d}^{n,k}) = H^*(BG) = H^*(\text{Gr}(n, \infty) \times \text{Gr}(k, \infty)),$$

since $J_{d}^{n,k}$ is contractible.
Let $L_n$ denote the tautological vector bundle over $\text{Gr}(n, \infty)$, i.e.
$$\text{Gr}(n, \infty) \times \mathbb{C}^\infty \ni \{(V, q) \mid q \in V\}.$$ Then we can identify $H^\ast(\text{Gr}(n, \infty), \mathbb{C})$ with $\mathbb{C}[c_1, \ldots, c_n]$, where $c_i$ are the Chern classes of $L_n^\ast$ – the dual tautological bundle. This observation allows us to define the Thom polynomial as follows:

**Definition 2.7.** Let $d, n, k \in \mathbb{N}$ and let $n \leq k$. Let $\Theta \subset J_{d}^{n,k}$ be a singularity. The Thom polynomial of $\Theta$ is defined as
$$\text{Tp}[\Theta](c, c') = e\pi(\Theta) \in H^\ast(\text{Gr}(n, \infty) \times \text{Gr}(k, \infty)) \cong \mathbb{C}[c_1, \ldots, c_k] \otimes \mathbb{C}[c'_1, \ldots, c'_k],$$ where $c_i$ are the Chern classes of $L_n^\ast$ and $c'_i$ – the Chern classes of $L_k^\ast$.

The notation $\text{Tp}[\Theta](c, c')$ comes from the total Chern class: $c = \sum c_i$.

The Thom polynomial defined above coincides with the universal polynomial from the Thom’s principle due to the following theorem.

**Theorem 2.8 ([16]).** Let $M$ and $N$ be compact complex manifolds, $\dim(M) \leq \dim(N)$, and let $f : M \rightarrow N$ be a holomorphic map. If the $\Theta$-locus of $f$ has expected dimension and is reduced, then its fundamental class in $M$ represents the Thom polynomial $\text{Tp}[\Theta](c, c')$ (defined in 2.7) evaluated in $c_1(TM)$ and $f^\ast c_j(TN)$:
$$\text{Tp}[\Theta](c_1(TM), c_2(TM), \ldots, f^\ast c_1(TN), f^\ast c_2(TN), \ldots).$$

In this paper we will think of Thom polynomial as defined in 2.7. For a detailed discussion of the relation between this definition and the Thom’s principle, see [2] and [16].

### 3. Damon’s theorem

Before stating and proving Damon’s theorem, let us first discuss the relation between Thom polynomials for different singularities.

Suppose $A$ is a nilpotent algebra. Fix $k, n, k', n' \in \mathbb{N}$ such that $k \geq n$, $k' \geq n'$ and $k - n = k' - n'$. Consider $\Theta_{A}^{n,k} \subset J_{d}^{n,k}$ and $\Theta_{A}^{n',k'} \subset J_{d}^{n',k'}$ and the corresponding approximations for $N, N' \ni 0$ of the Borel constructions $EG_{N} \times_{G} \Theta_{A}^{n,k} \subset EG_{N} \times_{G} J_{d}^{n,k}$ and $EG'_{N'} \times_{G'} \Theta_{A}^{n',k'} \subset EG'_{N'} \times_{G'} J_{d}^{n',k'}$ for $G = \text{Gl}_n \times \text{Gl}_k$ and $G' = \text{Gl}_{n'} \times \text{Gl}_{k'}$.

Suppose $\varphi$ and $h$ in the following diagram are holomorphic.

$$\begin{array}{ccc}
\Sigma_1 = EG_{N} \times_{G} \Theta_{A}^{n,k} & \xrightarrow{\pi} & EG_{N} \times_{G} J_{d}^{n,k} & \xrightarrow{\varphi} & \text{Gr}(n, N) \times \text{Gr}(k, N) \\
& \downarrow{h} & & \downarrow{} & \\
\Sigma_2 = EG'_{N'} \times_{G'} \Theta_{A}^{n',k'} & \xrightarrow{\pi'} & EG'_{N'} \times_{G'} J_{d}^{n',k'} & \xrightarrow{\varphi'} & \text{Gr}(n', N') \times \text{Gr}(k', N')
\end{array}$$
If the following conditions \([9]\) are satisfied:

- the square on the right commutes,
- \(h^{-1}(\Sigma_2) = \Sigma_1\),
- \(h\) is transversal to the smooth points of \(\Sigma_2\),

then \(h^* \text{PD}[\Sigma_2] = \text{PD}[h^{-1}(\Sigma_2)] = \text{PD}[\Sigma_1]\). From the commutativity of the right square we obtain the equality

\[
\text{TP}[\Theta_A^{n,k}] = \varphi^* \text{TP}[\Theta_A^{n',k'}].
\]

Let now \(n' = n + 1\), \(k' = k + 1\). Define the map \(\varphi\) as follows:

\[
\varphi: \text{Gr}(n, N) \times \text{Gr}(k, N) \longrightarrow \text{Gr}(n + 1, N + 1) \times \text{Gr}(k + 1, N + 1)
\]

\[
(V_1, V_2) \mapsto (V_1 \oplus \mathbb{C}, V_2 \oplus \mathbb{C}).
\]

Define \(h\) in a similar way. Let \((e_1, e_2, \ldots, e_{N+1})\) be a fixed orthonormal basis of \(\mathbb{C}^{N+1}\), and let \((t_1, \ldots, t_n)\) be an orthonormal \(n\)-frame in \(\mathbb{C}^N\) such that \(e_{n+1} \notin \langle t_1, \ldots, t_n \rangle\). Let \((\Psi_1, \ldots, \Psi_k) \in J_d^{n,k}\), i.e. \(\Psi_j(z_1, \ldots, z_n) \in J_d^n\).

\[
h: EG_N \times_G J_d^{n,k} \longrightarrow EG'_{N+1} \times_{G'} J_d^{n+1,k+1}
\]

\[
((t_1, \ldots, t_n), (\Psi_1, \ldots, \Psi_k)) \mapsto ((t_1, \ldots, t_n, e_{n+1}), (\Psi_1, \ldots, \Psi_k, z_{n+1}))
\]

Let us denote the set of Chern classes of the dual tautological bundle \(L^*_n\) on \(\text{Gr}(n, N)\) by \(c = c_1, \ldots, c_n\), the Chern classes of \(L^*_k\) by \(c' = c'_1, \ldots, c'_k\), the Chern classes of \(L^*_{n+1}\) on \(\text{Gr}(n + 1, N + 1)\) by \(\tau = \tau_1, \ldots, \tau_{n+1}\) and the Chern classes of \(L^*_{k+1}\) by \(\tau' = \tau'_1, \ldots, \tau'_{k+1}\). The transversality and the commutativity of the square on the right are straightforward, so the following is true:

\[
\text{TP}[\Theta_A^{n,k}](c, c') = \varphi^* \text{TP}[\Theta_A^{n+1,k+1}](\tau, \tau').
\]

We can also show how the pullback of \(\varphi\) acts on the Chern classes \(\tau_i\) and \(\tau'_i\):

\[
\varphi^*(\tau_i) = c_i \text{ for } i \leq n \text{ and } \varphi^*(\tau_{n+1}) = 0,
\]

\[
\varphi^*(\tau'_i) = c'_i \text{ for } i \leq k \text{ and } \varphi^*(\tau'_{k+1}) = 0.
\]

Using the properties of a pullback map we conclude the following.

**Lemma 3.1.** In the above notations,

\[
\text{TP}[\Theta_A^{n,k}](c_1, \ldots, c_n, c'_1, \ldots, c'_k) = \text{TP}[\Theta_A^{n+1,k+1}](\varphi^*(\tau_1), \ldots, \varphi^*(\tau_{n+1}), \varphi^*(\tau'_1), \ldots, \varphi^*(\tau'_{k})) =
\]

\[
= \text{TP}[\Theta_A^{n+1,k+1}](c_1, \ldots, c_n, 0, c'_1, \ldots, c'_k, 0)
\]
We can iterate the same procedure for $\text{Tp}[\Theta_A^{n+k+2}]$, $\text{Tp}[\Theta_A^{n+k+3}]$, etc, but since the Thom polynomial has a fixed degree, there will be a stabilization. This conclusion proves that the Thom polynomial depends only on the difference $k - n$ but not on $n$ and $k$, it also allows us to define the notion that generalizes the Thom polynomial.

**Definition 3.2.** Fix a nilpotent algebra $A$ and the difference between the dimensions of the source and the target of the map-jets, i.e. $n - k$ in our previous notations, denote this number by $j$. Let $m > \text{codim}(\Theta_A^{n,k})$ in $J_d^{n,k}$. Define the universal Thom polynomial as

$$\text{UTp}[\Theta_A^j](c_1, \ldots, c_m, e_1, \ldots, e_{m+j}) = \text{Tp}[\Theta_A^{n+m+j}](c_1, \ldots, c_m, e_1, \ldots, e_{m+j})$$

for $n > m$.

For all $n, k$ such that $k - n = j$ we obtain

$$\text{Tp}[\Theta_A^j](c_1, \ldots, c_n, e_1, \ldots, e_k) = \text{UTp}[\Theta_A^j](c_1, \ldots, c_n, 0, \ldots, 0, e_1, \ldots, e_k, 0, \ldots, 0).$$

Let us show an important property of the universal Thom polynomial. Let

$$f: \text{Gr}(n, N) \to \text{Gr}(l, M)$$

be any holomorphic map. Consider the diagram:

$$\begin{array}{ccc}
\text{EG}_N \times G \text{Gr}_d^{n,k} & \xrightarrow{\pi} & \text{Gr}(n, N) \times \text{Gr}(k, N) \\
\downarrow h & & \downarrow \varphi \\
\text{EG}_N \times G \text{Gr}_d^{n+l,k+l} & \xrightarrow{\pi'} & \text{Gr}(n + l, N + M) \times \text{Gr}(k + l, N + M)
\end{array}$$

Define $\varphi$ as

$$\varphi(V_1, V_2) = (V_1 \oplus f(V_1), V_2 \oplus f(V_1)), \quad V_1 \in \text{Gr}(n, N), \quad V_2 \in \text{Gr}(k, N).$$

Let $(e_1, \ldots, e_n)$ be the orthonormal basis for $V_1$, $(e'_1, \ldots, e'_k)$ – the orthonormal basis for $V_2$, and $(\overline{e}_1, \ldots, \overline{e}_l)$ – the orthonormal basis for $f(V_1)$. Let $\Psi = (\Psi_1, \ldots, \Psi_k) \in J_d^{n,k}$. Define $h$ as follows:

$$h[(e_1, \ldots, e_n, e'_1, \ldots, e'_k), \Psi] =$$

$$= [(e_1, \ldots, e_n, \overline{e}_1, \ldots, \overline{e}_l, e'_1, \ldots, e'_k, \overline{e}_1, \ldots, \overline{e}_l), (\Psi_1, \ldots, \Psi_k, z_{k+1}, \ldots, z_{k+l})]$$

Let $c$ be the total Chern class of $L_n^*$, $c'$ – the total Chern class of $L_k^*$, and $d_f$ – the total Chern class of $f^*(L_1^*)$. We have the following formulae for the pullbacks:

$$\varphi^*c(L_n^*) = c(L_n^* \oplus f^*L_1^*) = cd_f$$

$$\varphi^*c(L_k^*) = c(L_k^* \oplus f^*L_1^*) = c'd_f$$
On the level of the universal Thom polynomials we obtain the following.

**Lemma 3.3.** In the above notations,

$$\text{UTp}[\Theta_A^j](c, c') = \text{UTp}[\Theta_A^j](cd_f, c'd_f).$$

Let us formulate the Damon’s theorem.

**Theorem 3.4 (Damon, [4]).** Let $d, n, k \in \mathbb{N}$ and let $n \leq k$. Suppose $A$ is a nilpotent algebra and $\Theta_A^{n,k} \subset J_d^{n,k}$ a contact singularity. The Thom polynomial of $\Theta_A^{n,k}$ depends only on the difference $k - n$ and can be expressed in a single set of variables $\tilde{c}$ given by the generating series

$$1 + \tilde{c}t + \tilde{c}^2t^2 + \ldots = \sum_{i=0}^{k} c_i t^i \sum_{j=0}^{n} c_j t^j.$$

These new variables are called the relative Chern classes. We will denote the Thom polynomial expressed in the relative Chern classes by $\text{Tp}[\Theta_A^{n,k}](c'/c)$.

**Proof.** The previous discussion implies that if there existed a map $f$ such that $df = 1/c$, the Damon’s theorem would be proved since

$$\text{UTp}[\Theta_A^j](c, c') = \text{UTp}[\Theta_A^j](1, c'/c) = \text{UTp}[\Theta_A^j](c'/c).$$

In fact, such a map does not exist. The equality $c(L^*) = 1/c(Q^*)$ holds for a finite Grassmannian, so $df$ should be $c(Q^*)$, but the Chern classes of the dual tautological bundle can not be pulled back to $Q^*$ via a holomorphic map because $c(L^*)$ is positive (i.e. the Chern classes of $L^*$ are linear combinations with non-negative coefficients of the Poincaré duals to analytic subvarieties) and $c(Q^*)$ is not.

Let $S$ be an ample line bundle over $\text{Gr}(n, N)$. Then for $\alpha$ big enough, $Q_n^* \otimes S^{\otimes \alpha}$ is generated by its global holomorphic section and thus has positive Chern classes. There exists a holomorphic map

$$f_\alpha : \text{Gr}(n, N) \rightarrow \text{Gr}(n + l_\alpha, N + M_\alpha)$$

such that $f_\alpha^*(L_n^*) = Q_n^* \otimes S^{\otimes \alpha}$.

Let us compute the total Chern class of this twisted bundle. Denote the bundles from the splitting principle for $Q_n^*$ by $E_1, \ldots, E_n$ and their first Chern classes by $y_1, \ldots, y_n$, denote the first Chern class of $S$ by $z$. Then the following identity holds:

$$c(Q_n^* \otimes S^{\otimes \alpha}) = c(E_1 \otimes S^{\otimes \alpha} \oplus \ldots \oplus E_n \otimes S^{\otimes \alpha}) =$$

$$= \prod_{i=1}^{n} (y_i + \alpha z + 1) = \prod_{i=1}^{n} (y_i + 1) + \alpha P(\alpha) = c(Q_n^*) + \alpha \cdot P(\alpha),$$
where \( \alpha \cdot P(\alpha) \) is a polynomial in \( \alpha \) that contains all the summands of \( \prod_{i=1}^{n}(x_{i} + \alpha y + 1) \) that depend on \( \alpha \). Define

\[
\varphi_{\alpha}: \text{Gr}(n, N) \times \text{Gr}(k, N) \rightarrow \text{Gr}(n + l_{\alpha}, N + M_{\alpha}) \times \text{Gr}(k + l_{\alpha}, N + M_{\alpha})
\]

\[
(V_{1}, V_{2}) \mapsto (V_{1} \oplus f_{\alpha}(V_{1}), V_{2} \oplus f_{\alpha}(V_{1}))
\]

Denote the total Chern class of the dual tautological bundle \( L_{n+l_{\alpha}}^{*} \) on \( \text{Gr}(n + l_{\alpha}, N + M_{\alpha}) \) by \( \mathbf{\Sigma} \) and the total Chern class of the dual tautological bundle \( L_{k+l_{\alpha}}^{*} \) on \( \text{Gr}(k + l_{\alpha}, N + M_{\alpha}) \) by \( \mathbf{\Sigma}' \). Then by the previous discussion we have the following relations between the Chern classes:

\[
\varphi^{*}(\mathbf{\Sigma}) = c \cdot (c(Q_{n}^{*}) + \alpha P(\alpha)) = 1 + c \cdot \alpha P(\alpha)
\]

\[
\varphi^{*}(\mathbf{\Sigma}') = c' \cdot (c(Q_{n}^{*}) + \alpha P(\alpha)) = c'/c + c' \cdot \alpha P(\alpha).
\]

Or, on the level of the universal Thom polynomials:

\[
\text{UTp}[\Theta_{A}^{j}](1 + c \cdot \alpha P(\alpha), c'/c + c' \cdot \alpha P(\alpha)) = \text{UTp}[\Theta_{A}^{j}](1, c'/c) + \alpha P_{2}(\alpha) = \text{UTp}[\Theta_{A}^{j}](c, c'),
\]

where \( \alpha P_{2}(\alpha) \) denotes all the summands that depend on \( \alpha \).

Since \( \alpha P_{2}(\alpha) = \text{UTp}[\Theta_{A}^{j}](c, c') - \text{UTp}[\Theta_{A}^{j}](1, c'/c) \) their expressions in the Schur polynomial basis are also equal:

\[
\alpha P_{2}(\alpha) = \alpha \sum W_{\lambda \mu}(\alpha)s_{\lambda}(c)s_{\mu}(c')
\]

\[
\text{UTp}[\Theta_{A}^{j}](c, c') - \text{UTp}[\Theta_{A}^{j}](1, c'/c) = \sum B_{\lambda \mu} s_{\lambda}(c)s_{\mu}(c')
\]

\[
\alpha \sum W_{\lambda \mu}(\alpha)s_{\lambda}(c)s_{\mu}(c') = \sum B_{\lambda \mu} s_{\lambda}(c)s_{\mu}(c')
\]

This equation holds if and only if

\[
B_{\lambda \mu} = \alpha W_{\lambda \mu}(\alpha)
\]

for all \( \lambda \) and \( \mu \). However, since this is true for all sufficiently big \( \alpha \), the polynomial \( B_{\lambda \mu} - \alpha W_{\lambda \mu}(\alpha) \) has infinite number of roots. Thus, it is zero for all \( \alpha \). This implies that \( B_{\lambda \mu} = 0 \) for all \( \lambda \) and \( \mu \), i.e. \( \text{UTp}[\Theta_{A}^{j}](c, c') = \text{UTp}[\Theta_{A}^{j}](1, c'/c) \). \( \square \)

4. Positivity

The Schur polynomials serve as a natural basis for the cohomology ring of Grassmannians. Given an integer partition \( \lambda = (\lambda_{1}, \ldots, \lambda_{n}) \), such that \( N \geq \lambda_{1} \geq \lambda_{2} \geq \ldots \geq \lambda_{n} > 0 \) define the conjugate partition \( \lambda^{*} = (\lambda_{1}^{*}, \ldots, \lambda_{n}^{*}) \) by taking \( \lambda_{i}^{*} \) to be the largest \( j \) such that \( \lambda_{j} \geq i \). Denote by \( s_{\lambda}(b_{1}, \ldots, b_{n}) \) the expression of the Schur polynomials in elementary symmetric polynomials:

\[
s_{\lambda}(b_{1}, \ldots, b_{n}) = \det\{b_{\lambda_{i+j-1}}\}_{i,j=1}^{n}.
\]
The Schur polynomials of degree $d$ in $n$ variables form a linear basis for the space of homogeneous degree $d$ symmetric polynomials in $n$ variables.

Consider the finite Grassmannian $\text{Gr}(n, N)$. The Schur polynomials indexed by $\lambda$ such that $N \geq \lambda_1 \geq \ldots \geq \lambda_n > 0$, evaluated in the Chern classes $c_1, \ldots, c_n$ of the dual tautological vector bundle $L_n^*$ are the Poincaré duals of the Schubert cycles – homological classes of Schubert varieties $\sigma_{\lambda}$, special varieties forming a basis for the homology of the Grassmannian $[8]$:

$$s_{\lambda}(c_1, \ldots, c_n) = \text{PD}[\sigma_{\lambda}].$$

The following result was first proved by Pragacz and Weber. Here we give a new proof of this result.

**Theorem 4.1** (Pragacz, Weber, [5]). Let $d, n, k \in \mathbb{N}$ and let $n \leq k$. Suppose $A$ is a nilpotent algebra and $\Theta_{A}^{n,k} \subset J_{d}^{n,k}$ a contact singularity. The Thom polynomial of $\Theta_{A}^{n,k}$ expressed in the relative Chern classes is Schur-positive:

$$\text{Tp}[\Theta_{A}^{n,k}](c', c) = \sum_{\lambda} \alpha_{\lambda} s_{\lambda}(c'/c)$$

where $\alpha_{\lambda} \geq 0$.

**Proof.** By Damon’s theorem, Thom polynomials for contact singularities can be written as follows:

$$\text{Tp}[\Theta_{A}^{n,k}](c, c') = \text{Tp}[\Theta_{A}^{n+M,k+M}](1, c'/c) = \sum_{\lambda} \alpha_{0\lambda}s_{0}(c'/c) = \sum_{\lambda} \alpha_{\lambda}s_{\lambda}(c'/c)$$

for $M$ big enough. To prove the positivity we show that $\alpha_{0\lambda} \geq 0$ for all $\lambda$.

Fix a plane $V_0 \in \text{Gr}(n, N)$ and define the map

$$h: \text{Gr}(k, N) \longrightarrow \text{Gr}(n, N) \times \text{Gr}(k, N)$$

$$h(V) = (V_0, V).$$

Let $\varphi$ be the unique map making the following diagram commutative:

$$\begin{array}{ccc}
\varphi^{-1}(E_{G}N \times_{G} \Theta_{A}^{n,k}) & \longrightarrow & h^*(E_{G}N \times_{G} J_{d}^{n,k}) \\
\downarrow \varphi & & \downarrow p_2 \\
\Sigma = E_{G}N \times_{G} \Theta_{A}^{n,k} & \longrightarrow & E_{G}N \times_{G} J_{d}^{n,k} \\
\downarrow 1 & & \downarrow p_1 \\
& & \text{Gr}(n, N) \times \text{Gr}(k, N)
\end{array}$$

The idea of the proof is to show that

$$\sum_{\lambda} \alpha_{\lambda}s_{\lambda}(c') = h^*(\text{Tp}[\Theta_{A}^{n,k}](c, c')) = \text{PD}[X],$$

where $X$ is an analytic cycle in $\text{Gr}(k, N)$. 

Let $\sigma_{\lambda'}$ be a homology class of a Schubert variety of dimension complementary to $\dim X$. $\text{Gl}(k)$ acts transitively on $\text{Gr}(k, N)$, so by Kleiman's theorem \cite{10} there exists $A \in \text{Gl}_k$ such that $(AX) \cap \sigma_{\lambda'}$ is of expected dimension (so, discrete) and $AX$ is homologous to $X$.

$$\#(X \cap \sigma_{\lambda'}) = \text{PD}[X] \cdot \text{PD}[\sigma_{\lambda'}] = \sum_{\mu} \alpha_{0\mu} \cdot s_{\mu}(e') \cdot s_{\lambda'}(c') = \alpha_0 = \sum_{x \in \text{CX} \cap \sigma_{\lambda'}} \text{mult}_x \geq 0.$$  

Here $\text{mult}_x$ is an intersection multiplicity, which is non-negative for two analytic cycles.

Let us consider the details. We should construct the algebraic variety $X$. First, denote $\text{EG}^N \times G J^{n,k}_d$ by $E$ and $\text{EG}^N \times G \Theta^{n,k}_d$ by $\Sigma$ for short. It is clear that $\varphi^{-1}(\Sigma) \subset h^*(E)$. If $\varphi$ is also transversal to $\Sigma$, then we have that

$$\varphi^* \text{PD}[\Sigma] = \text{PD}[\varphi^{-1}(\Sigma)].$$

By definition, we need to show that:

$$\text{Im}(d_x(\varphi)) + T_{\varphi(x)} \Sigma = T_{\varphi(x)} E$$

for $x \in \varphi^{-1}(\Sigma)$. Locally

$$T_{(z,y)} E = T_z(\text{Gr}(n, N) \times \text{Gr}(k, N)) \oplus T_y J^{n,k}_d$$

for $z \in \text{EG}^N = \text{Gr}(n, N) \times \text{Gr}(k, N)$ and $y \in J^{n,k}_d$. With this interpretation the transversality is obvious since $\text{Im}(d_x(\varphi))$ has $T_y J^{n,k}_d$ as a direct summand and $T_{\varphi(x)} \Sigma$ has $T_z(\text{Gr}(n, N) \times \text{Gr}(k, N))$ as a direct summand.

Let us show that the vector bundle $h^*(E)$ has enough holomorphic sections to find a holomorphic section $s$ transversal to $\varphi^{-1}(\Sigma)$.

**Lemma 4.2.** $\text{EG}^N \times_G J^{n,k}_d = \left( \bigoplus_{i=1}^d \text{Sym}^i L_n \right) \boxtimes L_k^*$

**Proof.** An element of a fiber of $\text{EG}^N \times_G J^{n,k}_d$ is a class $[(e_n, e_k), f]$, where $f \in J^{k,n}_d$, $e_n$ is a frame, i.e. a linear injective map form $\mathbb{C}^n$ to $\mathbb{C}^N$, and $e_k$ is a linear injective map form $\mathbb{C}^k$ to $\mathbb{C}^N$. We consider a class $[(e_n, e_k), f]$ with the equivalence relation

$$[(e_n, e_k), f] \sim [(e_n A_n^{-1}, e_k A_k^{-1}), A_k f A_n^{-1}],$$

where $A_n \in \text{Gl}_n$, $A_k \in \text{Gl}_k$.

An element of the fiber of $\left( \bigoplus_{i=1}^d \text{Sym}^i L_n \right) \boxtimes L_k^*$ is a polynomial function of degree at most $d$ without a constant term between $V_n \in \text{Gr}(n, N)$ and $V_k \in \text{Gr}(k, N)$.

The map $[(e_k, e_n), f] \mapsto e_n \circ f \circ e_k^{-1}$ is correctly defined and is a bijection. \hfill \Box
We use this lemma to 'decompose' $h^*(E)$:

$$h^*(E) = \bigoplus_{i=1}^{d} \text{Sym}^i(\text{Triv}_n) \otimes L^*_k = \text{Triv}^{(d+n)}_n \otimes L^*_k,$$

where $\text{Triv}_n$ is a trivial vector bundle whose fiber is a complex vector space of dimension $n$.

We use the following theorem to show that this bundle has enough global holomorphic sections to find one transversal to $\varphi^{-1}(\Sigma)$.

**Theorem 4.3** (Parametric transversality theorem, [11]). Let $M$, $K$, $Z$, $S$ be smooth manifolds. Consider $F: M \times S \to K \supset Z$, smooth map transversal to $Z$. Then for almost all $s \in S$ the map $F_s$ is transversal to $Z$.

Let $D = (d+n) - 1$. In the notations of Thom’s transversality theorem, let

$$M = \text{Gr}(k, N), \quad K = \text{Hom}(L_k, \mathbb{C}^D) \cong h^*(EG_N \times_G J_d^{n,k}),$$

$$Z = \varphi^{-1}(\Sigma), \quad S = \Gamma(\text{Hom}(L_k, \mathbb{C}^D)) = \text{Hom}(\mathbb{C}^N, \mathbb{C}^D).$$

Then, the map $F$ from the theorem is the following:

$$F: \text{Gr}(k, N) \times \text{Hom}(\mathbb{C}^N, \mathbb{C}^D) \longrightarrow \text{Hom}(L_k, \mathbb{C}^D).$$

$$(V, f) \mapsto f|_V.$$

The transversality of $F$ to $\varphi^{-1}(\Sigma)$ obviously follows from the fact that $d(V, f)F$ is surjective for all $V$ and $f$.

Now, by Parametric transversality theorem, the set of holomorphic sections of $h^*(E)$ transversal to smooth points of $\varphi^{-1}(\Sigma)$ is open and dense in all holomorphic sections of this bundle. The set of holomorphic sections of $h^*(E)$ transversal to smooth points of the set of singular points of $\varphi^{-1}(\Sigma)$ is open and dense in the set of holomorphic sections transversal to smooth point of $\varphi^{-1}(\Sigma)$, and so on. Since this procedure drops the dimension of the variety, it is the finite process and the intersection of a finite number of open and dense sets is again open and dense. So, we can choose a holomorphic section $s$ transversal to $\varphi^{-1}(\Sigma)$.

The analytic subvariety $X$ from the discussion at the beginning of the proof is $s^{-1}\varphi^{-1}(\Sigma)$:

$$\text{PD}[s^{-1}\varphi^{-1}(\Sigma)] = s^* \text{PD}[\varphi^{-1}(\Sigma)] = (pr_2^*)^{-1} \varphi^* \text{PD}[\Sigma] = h^*(pr_1^*)^{-1} \text{PD}[\Sigma] = h^* \text{TP}[\Theta^{n,k}_{\Delta}](c, c') = \sum_{\lambda} \alpha_{\lambda}s_{\lambda}(c')$$

and the proof of positivity is complete. \qed
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