Broad Infinity and Generation Principles

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This work is aimed at each of the following communities.

- People such as ZFC users who accept AC.
- People such as ZF users who don’t accept this, but accept that there are just two subsets of 1.
- People such as IZF users who don’t accept this, but accept that there are set-many subsets of 1.
- People such as CZF users who don’t accept even this.

I belong to the first group, but find all of them interesting.
ZFC proves **Blass's axiom**: The class of regular cardinals is unbounded.

In particular, there is an uncountable regular cardinal.

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Should a choice sceptic believe these statements?

Maybe by a Reflection principle?

For the purposes of this talk: No.
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**Slogan**

*Any theory that proves the above statements is a bit choicy.*
Ord-is-Mahlo is a widely studied axiom scheme in set theory:

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But when I tried to understand Ord-is-Mahlo, these were my impressions:

- It seems unintuitive, unlike the ZFC axioms.
- It’s entangled with AC, as it implies Blass’s axiom.
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- It’s equivalent to Ord-is-Mahlo, assuming AC.
- It’s intuitively plausible, in my opinion.
- As far as I know, Broad ZF doesn’t imply the existence of an uncountable regular cardinal.
- Gitik-style result? Open question.
No familiarity with Ord-is-Mahlo is required 😊
Structure of talk

No familiarity with Ord-is-Mahlo is required 😊

1. What is Broad Infinity?
2. How to use Broad Infinity, if you accept AC.
3. How to use Broad Infinity, if you don’t accept AC.

Side remarks

If we allow the possibility of urelements and/or non-well-founded sets, everything still works.

Whenever AC is used in this story, a weak form called WISC (or a version of it) suffices.
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**Signature Infinity** is provable in IZF, whereas CZF users often adopt it as an axiom. It expresses an intuition: the set of $S$-terms.

**Broad Infinity** goes beyond ZFC. It expresses an intuition: the set of $G$-broad numbers.
Infinity—my favourite version

$\mathcal{T}$ is the class of all things (universal class).

First step: the constructors

We want Zero $\in \mathcal{T}$ and Succ : $\mathcal{T} \to \mathcal{T}$ such that Succ is injective and never yields Zero.
\( \mathcal{T} \) is the class of all things (universal class).

**First step: the constructors**

We want \( \text{Zero} \in \mathcal{T} \) and \( \text{Succ} : \mathcal{T} \to \mathcal{T} \) such that \( \text{Succ} \) is injective and never yields \( \text{Zero} \).

Zermelo’s definition achieves this:

\[
\begin{align*}
\text{Zero} & \overset{\text{def}}{=} \emptyset \\
\text{Succ}(x) & \overset{\text{def}}{=} \{x\}
\end{align*}
\]
A set $X$ is **nat-inductive** when

- $\text{Zero} \in X$

- for any $x \in X$, we have $\text{Succ}(x) \in X$.

A set of all natural numbers is a minimal (and therefore least) nat-inductive set.
A set $X$ is \textbf{nat-inductive} when

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Axiom of Infinity: There is a set of all natural numbers.

Note that this uniquely specifies a set.
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**Example of a natural number**

$\text{Succ(Succ(Succ(\text{Zero}))}$
A signature $S = (K_i)_{i \in I}$ is a family of sets. $I$ is a set of symbols, and $K_i$ is the arity of $i$. 
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\( I \) is a set of symbols, and \( K_i \) is the arity of \( i \).

A set \( X \) is \( S \)-inductive when

- for any symbol \( i \in I \) and \( K_i \)-tuple \([a_k]_{k \in K_i}\) within \( X \), we have \( \langle i, [a_k]_{k \in K_i} \rangle \in X \).

A set of all \( S \)-terms is a minimal (and therefore least) \( S \)-inductive set.

**Theorem (IZF): Signature Infinity**

For every signature \( S \), there’s a set of all \( S \)-terms.

Essentially proved by Słomiński in 1958.
### Example of an $S$-term

| Symbol | Arity          |
|--------|----------------|
| 5      | $\{0, 1, 2, 3\}$ |
| 6, 7   | $\emptyset$   |
| 8      | $\{0, 1, 2\}$  |

$\langle 8, \langle 0 \mapsto \langle 5, \langle 1 \mapsto \langle 6, [[]] \rangle \rangle, 2 \mapsto \langle 7, [[]] \rangle \rangle, 3 \mapsto \langle 7, [[]] \rangle \rangle \rangle$
$S$-term displayed as a well-founded 2D tree

Display conventions:

- Vertical dimension for tupling.
- Horizontal dimension for internal structure.
- Root at the left.
First step: the constructors

We want $\text{Start} \in \mathcal{T}$ and $\text{Build} : \mathcal{T}^3 \rightarrow \mathcal{T}$ so that $\text{Build}$ is injective and never yields $\text{Start}$.

$$
\text{Start} \quad \overset{\text{def}}{=} \quad \emptyset \\
\text{Build}(x, y, z) \quad \overset{\text{def}}{=} \quad \{\{x\}, \{x, \{\{y\}, \{y, z\}\}\}\}
$$

A broad signature $G$ is a function sending each $x \in \mathcal{T}$ to a signature $Gx$. 
Let $G$ be a broad signature.

A set $X$ is $G$-inductive when

- Start $\in X$
- for any $x \in X$ with $Gx = (K_i)_{i \in I}$, and any $i \in I$ and $K_i$-tuple $[a_k]_{k \in K_i}$ within $X$, we have $\text{Build}(x, i, [a_k]_{k \in K_i}) \in X$.

A set of all $G$-broad numbers is a minimal (and therefore least) $G$-inductive set.

Axiom scheme of Broad Infinity:
For any broad signature $G$, there’s a set of all $G$-broad numbers.
We can construct a class of all $G$-broad numbers, i.e. a least $G$-inductive class.
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This is an example of an introspectively generated class $C$, where we test $x \in C$ by looking at the $\in$-descendants of $x$. 

$G$-broad numbers form a class
We can construct a class of all $G$-broad numbers, i.e. a least $G$-inductive class.

This is an example of an introspectively generated class $C$, where we test $x \in C$ by looking at the $\in$-descendants of $x$.

This gives another way of stating Broad Infinity:
For any broad signature $G$, the class $\text{Broad}(G)$ is a set.
Example of an $G$-broad number

The broad signature $G$ sends
- $\text{Build}(\text{Start}, 6, [])$ to the signature with one symbol $8$ with arity $\{0, 1\}$
- everything else to the signature with nullary symbols $5, 6$.

Here is a $G$-broad number:

$$\text{Build} ( \text{Build} ( \text{Build} ( \text{Start}, 6, []), 8, \begin{bmatrix} 0 & \mapsto & \text{Start} \\ 1 & \mapsto & \text{Build} ( \text{Start}, 5, [])) \end{bmatrix}, 6, []))$$
$G$-broad number displayed as a well-founded 3D tree

\[
\text{Build}(\text{Build}(\text{Build}(\text{Start}, 6, []), 8, \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] \rightarrow \text{Start} \rightarrow \text{Build}(\text{Start}, 5, [])), 6, []))
\]

Display conventions:
- Vertical dimension for tupling.
- Horizontal dimension for Build.
- Depth dimension for internal structure.
- Root at the front.
- Start-marked leaves at the rear.
To build an $S$-term, we take a symbol $i \in I$ and a $K_i$-tuple of $S$-terms.

This is definite.

To build an ordinal, we take any (transitive) set of ordinals, however big.

Not definite.
To build an $S$-term, we take a symbol $i \in I$ and a $K_i$-tuple of $S$-terms. This is definite.

To build an ordinal, we take any (transitive) set of ordinals, however big. Not definite.

To build a $G$-broad number, we take either Start or a $G$-broad number $x$, with $Gx = (K_i)_{i \in I}$, and $i \in I$ and a $K_i$-tuple $[a_k]_{k \in K_i}$ of $G$-broad numbers. Definite? I think so!
Assuming $\Omega = \{0, 1\}$ the Broad Infinity scheme is equivalent to a reduced version, where the symbols are omitted.

Without assuming $\Omega = \{0, 1\}$ the reduced version might be weaker.
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Without assuming $\Omega = \{0, 1\}$ the reduced version might be weaker.

Although the full version is slightly more complicated, it’s just as intuitive, in my opinion.

So it would be strange to accept only the reduced version.
I look at the changing sea and sky
And try to picture Infinity.

(Noel Coward)

I look at all the trees outside
And picture Broad Infinity.

Paul Blain Levy (University of Birmingham)
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**Literary interlude**
I look at the changing sea and sky
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(Noel Coward)

2022 update

I look at all the trees outside
And picture Broad Infinity.
Broad Infinity is formulated to achieve maximum plausibility not ease of use.
Applying Broad Infinity

Broad Infinity is formulated to achieve maximum plausibility not ease of use.

So let’s look at equivalent schemes that are (less plausible but) easy to apply.
Given a class $C$, how do we generate a subset?

- Use a rubric. Assuming AC.
- Use a broad rubric. Assuming AC and Broad Infinity.
A *rubric* is a collection of rules telling us when to accept an element.
A **rubric** is a collection of rules telling us when to accept an element.

- **Rule 0.** If we accept $m_0$ and $m_1$, then for all $p \geq 2m_0$, we accept $m_0 + m_1 + p$.
- **Rule 1.** For all $p \geq 50$, we accept $2p$. 
A **rubric** is a collection of rules telling us when to accept an element.

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100, 102 and 402 are accepted. Every accepted number is $\geq 100$. 
A rule $\langle K, R \rangle$ on $C$ consists of
- a set $K$—the arity
- a function $R$ sending each $K$-tuple of elements $[a_k]_{k \in K}$
to a family of elements $(y_p)_{p \in P}$.

A rubric on $C$ is a family of rules $(\langle K_i, R_i \rangle)_{i \in I}$, indexed by a set.
Assuming AC, every rubric $\mathcal{R}$ on a class $C$ generates a subset of $C$. This is a minimal (and therefore least) $\mathcal{R}$-inductive subset. This scheme is called Set Generation.
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It implies the existence of an uncountable regular ordinal so AC is needed (assuming ...).
Any accepted element triggers a rubric.

The basic rubric is as follows.

- **Rule 0.** Arity $= \{0, 1\}$, sends $[m_0, m_1] \mapsto (m_0 + m_1 + p)p \geq 2m_0$.
- **Rule 1.** Arity $= \emptyset$, sends $[] \mapsto (2p)p \geq 50$.

The rubric triggered by 7 is as follows.

- **Rule 0.** Arity $= \{0, 1\}$, sends $[m_0, m_1] \mapsto (m_0 + m_1 + 500p)p \geq 9$.

The rubric triggered by 100 is as follows.

- **Rule 0.** Arity $= \{0, 1, 2\}$, sends $[m_0, m_1, m_2]$ $\mapsto (m_0 + m_1m_2 + p)p \geq 17$.
- **Rule 1.** Arity $= \emptyset$, sends $[] \mapsto (p)p \geq 1000$.
- **Rule 2.** Arity $= \{0\}$, sends $[m_0] \mapsto (m_0 + p)p \geq 4$.

All other natural numbers trigger the empty rubric.
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- **Rule 1.** Arity = \emptyset, sends \([] \mapsto (p)_{p \geq 1000}\).
- **Rule 2.** Arity = \{0\}, sends \([m_0] \mapsto (m_0 + p)_{p \geq 4}\).

All other natural numbers trigger the empty rubric.

100, 102, 402 and 107 are accepted. Every accepted number is \(\geq 100\).
A broad rubric $\mathcal{B} = (\mathcal{B}_0, \mathcal{B}_1)$ consists of

- a basic rubric $\mathcal{B}_0$
- for each $x \in C$, a triggered rubric $\mathcal{B}_1(x)$. 
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Assuming AC and Broad Infinity, every broad rubric on $C$ generates a subset of $C$.

This scheme is called **Broad Set Generation**, and is easy to apply.
A Grothendieck universe extending a set $X$ is a transitive set $\mathcal{U}$ such that

- $X \subseteq \mathcal{U}$.
- $\mathbb{N} \in \mathcal{U}$.
- For every set of sets $A \in \mathcal{U}$, we have $\bigcup A \in \mathcal{U}$.
- For every set $A \in \mathcal{U}$, we have $\mathcal{P}A \in \mathcal{U}$.
- For every set $K \in \mathcal{U}$ and $K$-tuple $[a_k]_{k \in K}$ within $\mathcal{U}$, we have $\{a_k \mid k \in K\} \in \mathcal{U}$.

Broad Set Generation implies that every set generates a Grothendieck universe. 
(The “Universe Axiom” of Grothendieck and Verdier.)
If AC is not assumed

Without AC, we can prove that every rubric on a class $C$ generates a family, where accepted elements are indexed by derivations.

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This scheme is called Family Generation.

Assuming Broad Infinity, every broad rubric on $C$ generates a family.

This scheme is called Broad Family Generation, and is easy to apply.

Adapted from the literature on induction-recursion in type theory.
A Tarski-style universe extending a family of sets $(B_a)_{a \in A}$ is a family of sets $(D_m)_{m \in M}$ such that

- For all $a \in A$, we have $\text{embed}(a) \in M$ with $D_{\text{embed}(a)} = B_a$.
- We have $\text{zero} \in M$ with $D_{\text{zero}} = \emptyset$.
- ... 
- For any $m \in M$ and function $g : D_m \to M$, we have $\pi(m, g) \in M$ with $D_{\pi(m, g)} = \prod_{k \in D_m} D_g(m)$.

Broad Family Generation implies that every family of sets generates a Tarski-style universe.
I think that Broad Infinity is intuitive. Hopefully you do too!
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Assuming AC, it’s equivalent to Broad Set Generation.

Without assuming AC, it’s equivalent to Broad Family Generation.

Both are easy to apply.
Conclusions and further steps

- I think that Broad Infinity is intuitive. Hopefully you do too!
- Assuming AC, it’s equivalent to Broad Set Generation.
- Without assuming AC, it’s equivalent to Broad Family Generation.
- Both are easy to apply.
- They extend Set Generation and Family Generation respectively, which don’t require Broad Infinity.
Conclusions and further steps

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- Next part of the story: ordinals.
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They extend **Set Generation** and **Family Generation** respectively, which don’t require Broad Infinity.

Next part of the story: ordinals.

This leads to the equivalence of Broad Infinity and Ord-is-Mahlo, assuming AC.