Determination of the modular Jacobian varieties $J_1(M, MN)$ with the Mordell–Weil rank zero

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Abstract

In this paper, we determine all modular Jacobian varieties $J_1(M, MN)$ over the cyclotomic fields $\mathbb{Q}(\zeta_M)$ with the Mordell–Weil rank zero assuming the Birch–Swinnerton-Dyer conjecture, following the method of Derickx, Etropolski, van Hoeij, Morrow, and Zureick-Brown.

Keywords: Torsion groups, Elliptic curves, Modular curves

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1 Introduction

The possible torsion groups of Mordell–Weil groups of elliptic curves over $\mathbb{Q}$ are completely classified by Mazur [14]. By generalizing Mazur’s method, the possible torsion groups over quadratic fields are also classified by Kenku and Momose [12] and by Kamienny [9]. Very recently a corresponding theorem for cubic fields is proven [6], but the higher degree cases are still unknown.

These are proven by considering the corresponding modular curves. More precisely, the existence of an elliptic curve with certain torsion points is essentially equivalent to the existence of certain rational points of the modular curve. Hence this kind of problem leads us to studying the rational points of modular curves.

On the other hand, in general the set of rational points of a curve is related to its Jacobian variety. For example, if the Mordell–Weil rank of the Jacobian variety is zero, then considering the Riemann–Roch spaces of divisors, we can determine all rational points of the curve by finite steps, at least in theory.

This observation leads us to considering the problem of determining the curves $X_1(M, MN)$ (for the definition see below) whose Jacobian varieties have the Mordell–Weil ranks zero. Following the method of [6, Theorem 3.1], in this paper we show the following (the cases of $M = 1$ and 2 are taken from [6, Theorem 3.1]):

**Theorem 1.1** (Main theorem) The rank of $J_1(M, MN)(\mathbb{Q}(\zeta_M))$ is zero if $(M, N)$ is in the following list:
If the Birch–Swinnerton-Dyer conjecture is true, then the converse holds.

Note that in [6] the authors claim that the converse of this theorem for \( M = 1, 2 \) also holds unconditionally. In its proof they use the converse of Kato’s theorem [10, Corollary 14.3], which is, according to a communication with the authors of the article, still an open problem, and hence the proof is incomplete.

Using this theorem we classify elliptic curves with certain kind of rational torsion points over cyclotomic fields.

In the final section, by generalizing the proof of Theorem 1.1, we prove a statement about the rank of \( J_1(M, MN)(\mathbb{Q}(\zeta_M)) \), and compute it concretely for some \((M, N)\) which are not on the list in Theorem 1.1. Also, assuming the Birch–Swinnerton-Dyer conjecture, we give a lower bound of the rank of \( J_0(N)(\mathbb{Q}) \).

For explicit computation of, for example the L-functions of the newforms, we use the computer algebra programs Magma [4] and Sagemath [19]. The code verifying our proofs is available at the Github repository https://github.com/kmatsuda111/rank0.

2 Preliminaries

Throughout this paper, \( N \) and \( M \) stand for positive integers unless otherwise stated.

For a subgroup \( \Gamma \) of \( \text{GL}_2(\mathbb{Z}/N) \), let \( X_\Gamma \) denote the modular curve over \( \mathbb{Z}[1/N][\zeta_N]^{\text{det} \Gamma} \) corresponding to \( \Gamma \) [5, IV. Section 3]. The space \( X_\Gamma \) is a smooth proper scheme of relative dimension 1 with geometrically connected fibers. Let \( J_\Gamma \) denote its Jacobian variety.

For particular subgroups

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/N) \right\},
\]

and

\[
\Gamma_1(M, MN) = \left\{ \begin{pmatrix} 1 & b \\ 0 & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/MN) \middle| b \equiv 0, d \equiv 1 \mod M \right\}
\]

respectively, we denote the curve \( X_\Gamma \) by \( X_0(N) \) and \( X_1(M, MN) \) respectively, and denote the curves \( X_1(1, N) \) and \( X_1(N, N) \) by \( X_1(N) \) and by \( X(N) \) respectively. The curve \( X_1(M, MN) \), which is defined over \( \mathbb{Z}[1/MN][\zeta_M] \), is the modular curve parametrizing elliptic curves and their independent two rational points of orders \( M \) and \( MN \). Moreover we denote similarly for their Jacobian varieties.
There is a canonical morphism $X_1(N) \rightarrow X_0(N)$, and it has automorphism group isomorphic to $(\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}$.

Moreover for a subgroup $\Delta$ of $(\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}$, we denote the curve $X_1(N)/\Delta$ by $X_\Delta$, and $J_\Delta$ its Jacobian variety. The curve $X_\Delta$ is isomorphic to $X_{\Gamma_\Delta}$ for the group

$$\Gamma_\Delta = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}/N\mathbb{Z}) \bigg| a \mod \pm 1 \in \Delta \right\}.$$ 

For a normalized eigenform $f$ of weight $2$ of level $\Gamma_1(N)$, let $K_f$ be the number field generated by the Fourier coefficients of $f$, and let $A_f$ denote the abelian variety associated to $f$. It is of dimension $[K_f : \mathbb{Q}]$ and there exists an order of $K_f$ acting on it [17, Theorem 1]. Then for a prime number $\ell$, since the homology group $H_1(X_1(N)(\mathbb{C}), \mathbb{Q})$ is free of rank 2 over $\mathbb{Q}$, we have that the Tate module $V_\ell(A_f) = T_\ell(A_f) \otimes \mathbb{Z}_\ell$ is free of rank 2 over $K_f \otimes \mathbb{Q}$, and by the construction, for a prime $p \nmid N$, the trace of a $p$-th arithmetic Frobenius on $V_\ell(A_f)$ is $a_p(f)$. Moreover, if $f$ is a newform, then the modular abelian variety $A_f$ is simple and $\text{End}_g A_f \otimes \mathbb{Q} = K_f$ [16, Corollary 4.2]. Moreover, for a subgroup $\Delta$ of $(\mathbb{Z}/N\mathbb{Z})^*/\{\pm 1\}$, the modular Jacobian variety $J_\Delta$ is isogenous to the product

$$\bigoplus_M \bigoplus_f A_f^{m_f},$$

where $M$ runs over positive divisors of $N$, $f$ runs over the set of Galois conjugacy classes of newforms of level $M$ whose characters $\epsilon$ satisfy that $\epsilon(a) = 1$ for all $a \in \mathbb{Z}/M$ with $a \mod \pm 1 \in \Delta \mod M$, and $m_f = s_0(N/M)$ is the number of positive divisors of $N/M$ [16, Proposition 2.3].

For the ranks of Mordell–Weil groups of abelian varieties, Birch and Swinnerton-Dyer conjectured the following:

**Conjecture 2.1** (Birch–Swinnerton-Dyer conjecture) Let $A$ be an abelian variety over $\mathbb{Q}$, and assume that the L-function $L(s)$ of $A$ has an analytic continuation to $\mathbb{C}$. Then the order of zero of $L(s)$ at $s = 1$ and the rank of the Mordell–Weil group $A(\mathbb{Q})$ are the same.

Kato ([10, Corollary 14.3]) proves one direction of a special case: Namely, for a normalized eigenform $f$ of weight $2$ of level $\Gamma_1(N)$, if the order of the zero of the L-function $L(f,s)$ at $s = 1$ is zero, then the rank of $A_f(\mathbb{Q})$ is zero. In the proof of [6, Theorem 3.1] the authors use the converse, which is, according to a communication with the authors of the article, still an open problem, and hence the proof is incomplete. For the conditional results in this paper, we assume its converse.

For an abelian extension $K/\mathbb{Q}$ with the Galois group $G$, we identify the characters $\chi : G \rightarrow \mathbb{Q}^+$ with the Dirichlet characters corresponding to it.

For a Dirichlet character $\chi$ of conductor $M$ and for a modular form $f = \sum_n a_n q^n$ of weight $k$ of level $\Gamma_1(N)$ with the character $\varphi$ of conductor $N'$, we denote by $f_\chi$ the twist $\sum_n \chi(n) a_n q^n$ of the modular form $f$. This is a modular form of weight $k$ of level $\Gamma_1(N'')$ with the character $\chi^2 \varphi$ of conductor dividing $\text{lcm}(M, N')$, where $N'' = \text{lcm}(N, N'M, M^2)$ [3, Proposition 3.1]. Note that, by checking their Fourier coefficients, we have that if a modular form $f$ is a normalized eigenform, then so is the twist $f_\chi$.

Let $f$ be a normalized eigenform of weight $2$ of level $\Gamma_1(N)$. Then there exists a newform $g$ of level $M$ dividing $N$ such that $a_p(f) = a_p(g)$ for every prime $p$ not dividing $N/M$. Such $g$ is unique by the strong multiplicity one theorem (for example, see [15, Theorem
4.6.12]). We call such a newform the newform associated with \( f \), and denote it by \( f^{\text{new}} \). In this case, since the number field \( K_g \) is generated by \( a_p(g) \) for every \( p \) not dividing \( N \), the number field \( K_f \) contains \( K_g \).

Let \( \Delta \) be the group
\[
\ker((\mathbb{Z}/M^2N)^*/\pm 1 \to (\mathbb{Z}/MN)^*/\pm 1).
\]

Let \( \text{New}(M, MN) \) denote the set of newforms of weight 2, level dividing \( M^2N \), and conductor dividing \( MN \), and let \( \overline{\text{New}}(M, MN) \) be the set of Galois conjugacy classes of \( \text{New}(M, MN) \). Let \( D \) be the group of Dirichlet characters modulo \( M \). These play a crucial role in this paper.

**Lemma 2.2** The modular Jacobian variety \( J_\Delta \) is isogenous to the product
\[
\bigoplus_{G \in \text{New}(M, MN)} A_f^{\sigma_0(M^2N/N_f)},
\]
where \( N_f \) is the level of \( f \) and \( \sigma_0(M^2N/N_f) \) is the number of positive divisors of \( M^2N/N_f \) as in preliminaries.

**Proof** We show the statement by showing that the space \( S_2(\Gamma_\Delta) \) of the cusp forms of level \( \Gamma_\Delta \) is equal to the direct product \( \otimes_\epsilon S_2(\Gamma_1(M^2N), \epsilon) \), where \( \epsilon \) runs over the Dirichlet characters of conductor dividing \( MN \). The space \( S_2(\Gamma_\Delta) \) is a subspace of \( S_2(\Gamma_1(M^2N)) \), and the later space is the direct product \( \otimes \epsilon S_2(\Gamma_1(M^2N), \epsilon) \), where the sum is taken over all characters \( \epsilon \) mod \( M^2N \). Now by the definition of \( \Delta \), for \( f \in S_2(\Gamma_1(M^2N), \epsilon) \), we have that \( f \) is of level \( \Gamma_\Delta \) if and only if the character \( \epsilon \) is of conductor dividing \( MN \).

**Lemma 2.3** Sending a pair \((\chi, f)\), where \( \chi \in D \) and \( f \in \text{New}(M, MN) \), to \((f_\chi)^{\text{new}}\) defines a well-defined action of the group \( D \) on the set \( \text{New}(M, MN) \) from right.

**Proof** What we need to show is the well-definedness of the action. Let \( \chi \in D \) and let \( f \in \text{New}(M, MN) \). Then as we remarked in preliminaries, the twist \( f_\chi \) is a normalized eigenform of level \( \text{lcm}(M^2N, MN, M^2) = M^2N \) with the character of conductor dividing \( \text{lcm}(MN, M) = MN \). Thus its associated newform \((f_\chi)^{\text{new}}\) lies in \( \text{New}(M, MN) \). This shows the well-definedness.

### 3 Proof of the main theorem

In order to determine whether the Mordell–Weil rank of \( J_1(M, MN) \) is zero or not, we use the following corollary of the theorem of Kato: This seems to be well-known, but for the luck of reference we give a proof.

**Lemma 3.1** Let \( L/K \) be a finite abelian extension of number fields with Galois group \( G \) and \( F \) a number field containing all \( \exp(G) \)-th roots of unity. Assume, for every character \( \chi : G \to \mathbb{C}^* \), that there exists an abelian variety \( A_\chi \) over \( K \) on which an order of \( F \) acts. Also assume that there exists a prime \( \ell \) such that the Galois modules \( V_\ell(A_\chi) \) and \( \chi \otimes \mathbb{Q}_\ell \) \( V_\ell(A_1) \) over \( F \otimes \mathbb{Q}_\ell \) are isomorphic for all \( \chi \). Let \( B \) be the Weil restriction of \( (A_1)_L \) along with the extension \( L/K \). Then \( B \) is isogenous to \( \bigoplus_\chi A_\chi \).

**Proof** Let \( A = A_1 \) and \( \overline{K} \) an algebraic closure of \( K \). First we claim that the Galois module \( V_\ell(B) \) is isomorphic to \( V_\ell(A) \otimes \mathbb{Q}_\ell \mathbb{Q}_L[G] \). Since \( L/K \) is Galois, we have that \( \overline{K} \otimes_K L \cong \overline{K}[G] \).
Under this isomorphism, for $\sigma \in G_K$, the automorphism $\sigma \otimes \text{id}$ on $\overline{\mathcal{O}} \otimes_K L$ corresponds to the canonical action on $\overline{\mathcal{O}}[G]$. Hence we have that
\[ B(\overline{\mathcal{O}}) = A(\overline{\mathcal{O}} \otimes_K L) = A \left( \prod_{\sigma \in G} \overline{\mathcal{O}} \right) = \bigoplus_{\sigma \in G} A(\overline{\mathcal{O}}) = A(\overline{\mathcal{O}}) \otimes \mathbb{Z}[G]. \]

Thus the claim follows. Therefore since $G$ is abelian, we have that
\[ V_{\ell}(B) \cong V_{\ell}(A) \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell[G]. \]
\[ \cong \bigoplus_{\chi} V_{\ell}(A) \otimes \mathbb{Q}_\ell[G], \]
\[ \cong \bigoplus_{\chi} V_{\ell}(A). \]

Therefore by Faltings [7, Korollar1] we have the result. \( \square \)

**Corollary 3.2** Let $f$ be a normalized eigenform of weight 2 of level $\Gamma_1(N)$, and $K/\mathbb{Q}$ be an abelian extension with the Galois group $G$. Then we have
\[ \dim_{K_f} A_f(K) \otimes \mathbb{Q} = \sum_{\chi} \dim_{K_f} A_f(\mathbb{Q}) \otimes \mathbb{Q}, \]
where the sum is taken over all Dirichlet’s characters of $K/\mathbb{Q}$.

**Proof** Let $F$ be a finite extension of $K_f$ which contains all $\exp(G)$-th roots of unity. The endomorphism ring End$^0 \oplus (F[K_f]) = \text{End}_{\text{iso}}[F[K_f]]$ up to isogeny is isomorphic to $M_{[F:K_f]}(\text{End}^0 A_f)$, which contains $M_{[F:K_f]}(K_f)$. Hence fixing a $K_f$-algebra homomorphism $F \rightarrow M_{[F:K_f]}(K_f)$, there exists and we fix an order of $F$ acting on $A_f$. With this action, for every prime $\ell$, the Galois module $V_{\ell}(A_f)$ is isomorphic to $V_{\ell}(A_f) \otimes_{K_f} F$ over $F \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$, and for a prime $p$ not dividing $\ell N$, a $p$-th arithmetic Frobenius $\psi_p$ has trace $\chi(p)a_p(f)$. Therefore by the semi-simplicity of the Tate modules of abelian varieties ([7, Satz3]) and by Chebotarev’s density theorem, we have $V_{\ell}(A_f) \cong V_{\ell}(A_f \otimes_{[F:K_f]} F)$. Thus by the lemma above we have $[F : K_f] \text{rank} A_f(K) = \sum_{\chi} [F : K_f] \text{rank} A_f(\mathbb{Q}).$ \( \square \)

**Lemma 3.3** Let $f$ be a normalized eigenform of weight 2 of level $\Gamma_1(N)$, and let $g$ be the newform associated with $f$. Then $A_f$ is isogenous to $A_g \oplus [K_f : K_g]$, and we have $\dim_{K_f} A_f(\mathbb{Q}) \otimes \mathbb{Q} = \dim_{K_g} A_g(\mathbb{Q}) \otimes \mathbb{Q}$.

**Proof** Let $L = K_f$. Then for a prime $\ell$ and for a prime $p$ not dividing $\ell N$, the traces of a $p$-th arithmetic Frobenius on the Galois modules $V_{\ell}(A_f)$ and $V_{\ell}(A_g) \otimes_{K_g} L$ over $L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ are $a_p(f) = a_p(g)$. Therefore considering the decomposition of $L \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ into a product of fields and corresponding decomposition of these modules, we have, by semi-simplicity and by Chebotarev’s density theorem, that these two modules are isomorphic. In particular $V_{\ell}(A_f)$ and $V_{\ell}(A_g) \otimes [K_f : K_g]$ are isomorphic over $\mathbb{Q}_\ell$. Thus by Faltings’ theorem we have that $A_f$ and $A_g \otimes [K_f : K_g]$ are isogenous. \( \square \)

Let $\Delta$ be the group $\ker((\mathbb{Z}/M^2 N)^*/ \pm 1 \rightarrow (\mathbb{Z}/MN)^*/ \pm 1)$.

Then by the third paragraph of preliminaries of [8], we have that $(X_\Delta \otimes_{\mathbb{Q}} M) \cong X_1(M,MN)$. Recall that New$(M,MN)$ is the set of newforms of weight 2, level dividing $M^2 N$, and
conductor dividing $MN$, that $\overline{\text{New}(M, MN)}$ is the set of Galois conjugacy classes of $\text{New}(M, MN)$, and that $D$ is the group of Dirichlet characters modulo $M$.

Using these statements we deduce the following crucial proposition.

**Proposition 3.4** Consider the following statements:

1. The rank of $J_1(M, MN)(\mathbb{Q}(\zeta_M))$ is zero.
2. The rank of $J_\Delta(\mathbb{Q})$ is zero.
3. For every newform $f \in \text{New}(M, MN)$, the special value of the $L$-function $L(f, s)$ at $s = 1$ is nonzero.

Then we have $(1) \iff (2) \iff (3)$. Moreover if the Birch–Swinnerton-Dyer conjecture is true, then these statements are equivalent.

**Proof** Since $(X_\Delta)_{\mathbb{Q}(\zeta_M)} \simeq X_1(M, MN)$, the condition (1) implies (2). By Lemma 2.2, the modular Jacobian variety $J_\Delta$ is isogenous to

$$\bigoplus_{G \in \text{Gal}(F:M, MN)} A_f^\sigma_{G(f)}(M^2N/\mathcal{N}_f).$$

Thus by the theorem of Kato [10, Corollary 14.3], we have that (3) implies (2), and that if the Birch–Swinnerton-Dyer conjecture is true, then these two statements are equivalent.

Finally by Corollary 3.2 and Lemma 2.2, the rank of $J_1(M, MN)(\mathbb{Q}(\zeta_M)) = J_\Delta(\mathbb{Q}(\zeta_M))$ is zero if and only if the rank of $A_f(\mathbb{Q})$ is zero for every Dirichlet character $\chi$ modulo $M$ and for every newform $f \in \text{New}(M, MN)$. Now such a twist is a normalized eigenform of level $\text{lcm}(M^2N, MNM, M^2) = M^2N$ with the character of conductor dividing $\text{lcm}(MN, M) = MN$. Therefore by Lemma 3.3 we have that (2) implies (1). $\Box$

From this proposition, we easily prove the main theorem following the method of [6].

**Lemma 3.5** For $M \geq 3$, both of the ranks of $J_1(MN)(\mathbb{Q})$ and of $J_0(M^2N)(\mathbb{Q})$ are zero if $(M, N)$ is in the following list:

- $M = 3$, $N \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 16, 20\}$,
- $M = 4$, $N \in \{1, 2, 3, 4, 5, 6, 9\}$,
- $M = 5$, $N \in \{1, 2, 3, 4, 6\}$,
- $M = 6$, $N \in \{1, 2, 3, 4, 5\}$,
- $M = 7$, $N \in \{1, 2\}$,
- $M = 8$, $N = 1$,
- $M = 9$, $N = 1$,
- $M = 10$, $N = 1$,
- $M = 12$, $N = 1$.

Moreover if the Birch–Swinnerton-Dyer conjecture is true, then the converse holds.
For $M \geq 3$, the rank of $J_1(M^2N)(\mathbb{Q})$ is zero if $(M, N)$ is in the following list:

- $M = 3$, $N \in \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12, 20\}$,
- $M = 4$, $N \in \{1, 2, 3, 4, 6\}$,
- $M = 5$, $N \in \{1, 2, 3, 4, 6\}$,
- $M = 6$, $N \in \{1, 2, 3, 5\}$,
- $M = 7$, $N \in \{1, 2\}$,
Moreover if the Birch–Swinnerton-Dyer conjecture is true, then the converse holds.

Proof By [6, Theorem 3.1]. As we remark in the introduction, the proof of [6, Theorem 3.1] seems to be invalid since it requires the unproved part of the Birch–Swinnerton-Dyer conjecture, however the ‘if’ part is still valid. □

Proof of the main theorem First of all, for the group $\Delta$ as in the Proposition 3.4, there are morphisms $X_1(M^2N) \to X_\Delta \to X_0(M^2N)$ over $\mathbb{Q}$ and $X_1(M,MN) \to X_1(MN)$ over $\mathbb{Q}(\zeta_M)$. Hence using Proposition 3.4, we have that, if the rank of $J_1(M^2N)(\mathbb{Q})$ is zero then the rank of $J_1(M,MN)(\mathbb{Q}(\zeta_M))$ is zero, and conversely if the rank of $J_1(M,MN)(\mathbb{Q}(\zeta_M))$ is zero then the ranks of $J_1(MN)(\mathbb{Q})$ and of $J_0(M^2N)(\mathbb{Q})$ are zero.

Thus by Lemma 3.5 it remains to show that the special values of the L-functions of $J_1(M,MN)(\mathbb{Q}(\zeta_M))$ at $s = 1$ are nonzero for $(M,N) = (3,7),(3,14),(3,16),(4,5),(6,4)$ and for $(12,1)$, and that the special value of the L-function of $J_1(M,MN)(\mathbb{Q}(\zeta_M))$ for $(M,N) = (4,9)$ is zero.

Using the algorithm of [2, Theorem 4.5.], we can compute for a newform $f$ whether $L(f,1)$ is zero or not. (For example use Magma [4].) According to it we have that $L(f,1) \neq 0$ for every newform of level dividing $M^2N$ with the character of conductor dividing $MN$, except the newforms in the Galois conjugacy class of newforms labeled “G1N1441” by Magma [4], whose character has the conductor 36, and conversely for those newforms $f$, we have $L(f,1) = 0$. Therefore by Proposition 3.4 we have the desired result. □

Lastly from this theorem we deduce a corollary about the existence of certain kinds of elliptic curves.

Corollary 3.6 For $N = 1, \ldots, 5$, there exist infinitely many elliptic curves over $\mathbb{Q}(\zeta_N)$ whose Mordell–Weil groups contain subgroups isomorphic to $(\mathbb{Z}/N)^2$. For $N = 6, \ldots, 10, 12$, there are no such elliptic curves over $\mathbb{Q}(\zeta_N)$. For the other $N$, there are at most finitely many such elliptic curves over $\mathbb{Q}(\zeta_N)$.

Proof Consider the modular curves $X(N)$ over $\mathbb{Q}(\zeta_N)$ which classifies generalized elliptic curves with their full level $N$ structures. The third statement is just Faltings’ theorem. So assume $N = 1, \ldots, 10$ or 12. For each $N$ we consider, by the theorem, the Mordell–Weil rank of its Jacobian variety is zero. For $N = 1, \ldots, 4$ or 5, since the curve $X(N)$ has genus 0 and of course has a rational point (the $\infty$-cusp), it is isomorphic to $\mathbb{P}^1$. Therefore the result in this case is trivial. Next assume $N = 6, \ldots, 10$ or 12. For such an $N$, the curve $X(N)$ has genus greater than 0 and is the fine moduli scheme of the corresponding stack. Hence $X(N)(\mathbb{Q}(\zeta_N)) \to J(N)(\mathbb{Q}(\zeta_N))$ is injective. By [11, Appendix], for a good prime $p$ of the curve $X(N)$ over $\mathbb{Q}(\zeta_N)$ above $p$ satisfying $e(p/p) < p - 1$, since the Mordell–Weil rank is zero, the reduction map $J(N)(\mathbb{Q}(\zeta_N)) \to J(N)(\mathbb{F}_q)$ at $p$ is injective. Note that the condition $e(p/p) < p - 1$ is almost automatic: the good primes of $X(N)/\mathbb{Q}(\zeta_N)$ are exactly the primes which do not divide the level $N$, and the unramified primes of $\mathbb{Q}(\zeta_N)$ are exactly the primes which do not divide $N$, except the characteristic 2. Thus in this case, by the
diagram

\[
\begin{align*}
X(N)(\mathbb{Q}(\zeta_N)) & \xrightarrow{\subseteq} J(N)(\mathbb{Q}(\zeta_N)) \\
X(N)(\mathbb{F}_q) & \xrightarrow{} J(N)(\mathbb{F}_q)
\end{align*}
\]

we have that the reduction map \(X(N)(\mathbb{Q}(\zeta_N)) \to X(N)(\mathbb{F}_q)\) is injective, where \(q\) is the norm of \(p\). On the other hand, by the Hasse bound, if \(N^2 > (1 + \sqrt{\eta})^2\), i.e., if \(q < (N - 1)^2\), then the later set consists of cusps. Since for a field \(k\) with the characteristic prime to \(N\), the cusps of \(X(N)(\overline{k})\) correspond to the full level \(N\) structures of the Neron \(N\)-gon over \(\overline{k}\), both of \(X(N)(\mathbb{Q}(\zeta_N))\) and \(X(N)(\mathbb{F}_q)\) contain all cusps. Since of course the number of cusps of \(X(N)(\overline{k})\) is independent of \(k\), we have that if \(q < (N - 1)^2\) then the reduction map \(X(N)(\mathbb{Q}(\zeta_N)) \to X(N)(\mathbb{F}_q)\) is surjective. Hence now what we need to show is the claim that, for our \(N\), there is a good prime \(p\) of \(\mathbb{Q}(\zeta_N)\) whose norm over \(\mathbb{Q}\) is less than \((N - 1)^2\) and whose characteristic is odd. This is easy, and we have done.

\[\square\]

4 Higher rank

In the previous section we proved that the rank of \(J_1(M, MN)(\mathbb{Q}(\zeta_M))\) is zero if and only if the rank of \(J_\Delta(\mathbb{Q})\) is zero, where \(\Delta\) is the subgroup as in Proposition 3.4. With more careful computation it is possible to show a more general statement. Recall that \(\text{New}(M, MN)\) is the set of newforms of weight 2, level dividing \(M^2N\), and conductor dividing \(MN\), and \(\text{New}(M, MN) = \mathbb{G}_Q \setminus \text{New}(M, MN)\) is the set of Galois conjugacy classes of \(\text{New}(M, MN)\). Also recall that \(D\) is the group of the Dirichlet characters modulo \(M\).

**Proposition 4.1** We have the following formula:

\[
\text{rank } J_1(M, MN)(\mathbb{Q}(\zeta_M)) = \sum_{G_{qf} \in \text{New}(M, MN)} \sum_{\chi \in D} \sigma_0(M^2N / N_{(f_q)^\text{new}}) \text{ rank } A_f(\mathbb{Q}),
\]

where \(N_{(f_q)^\text{new}}\) is the level of \((f_q)^\text{new}\), and \(\sigma_0\) is the divisor function.

Note that, decomposing \(J_\Delta\) into simple factors as in Lemma 2.2, we have the following formula:

\[
\text{rank } J_1(M, MN)(\mathbb{Q}(\zeta_M)) = \sum_{G_{qf} \in \text{New}(M, MN)} \sigma_0(M^2N / N_f) \text{ rank } A_f(\mathbb{Q}(\zeta_M)).
\]

Thus using Corollary 3.2 we can compute the rank of \(J_1(M, MN)(\mathbb{Q}(\zeta_M))\). However, in order to compute rank \(J_1(M, MN)(\mathbb{Q}(\zeta_M))\) using this formula, we need to compute the newform \(g\) associated with the twist \(f_x\) for every \(\chi : (\mathbb{Z}/M)^* \to \mathbb{C}^*\) and for every \(f\). The importance of Proposition 4.1 is that we only need to compute the newform associated with the twist \(f_x\) only for \(f\) satisfying that \(\text{rank } A_f(\mathbb{Q})\) is nonzero.

Before proving the proposition, we compute rank \(J_1(M, MN)(\mathbb{Q}(\zeta_M))\) for some \((M, N)\) using Proposition 4.1.

**Example 4.2** [The rank of \(J(11)(\mathbb{Q}(\zeta_{11}))\)] Let \(\Delta\) be the subgroup as in Proposition 3.4 for \((M, N) = (11, 1)\). According to [18], \(J_\Delta\) has only one simple factor, say \(A\), whose rank over \(\mathbb{Q}\) is nonzero. (\(A = A_f\) for \(f\) a newform in the Galois conjugacy class of newforms labeled “121.2.a.b” in [18].) The newform \(f\) has analytic rank 1, and the abelian variety \(A\) is of dimension 1. Therefore by [13, Corollary C], we have that \(\text{rank } A(\mathbb{Q}) = 1\).
Again according to [18], for every \( \chi : (\mathbb{Z}/11)^* \to \mathbb{C}^* \), the newform associated with \( f_{\chi} \) has level 121. Therefore since there are 10 characters \( \chi : (\mathbb{Z}/11)^* \to \mathbb{C}^* \), we have that rank \( J(11)(\mathbb{Q}(\zeta_{11})) \) = 10.

**Example 4.3** [The rank of \( J(14)(\mathbb{Q}(\zeta_{14})) \)] Let \( \Delta \) be the subgroup as in Proposition 3.4 for \((M, N) = (14, 1)\). According to [18], \( J_\Delta \) has only one simple factor, say \( A \), whose rank over \( \mathbb{Q} \) is nonzero. \( A = A_f \) for \( f \) a newform in the Galois conjugacy class of newforms labeled “196.2.a.a” in [18].) The newform \( f \) has analytic rank 1, and the abelian variety \( A \) is of dimension 1. Therefore by [13, Corollary C], we have that rank \( A(\mathbb{Q}) = 1 \). Again according to [18], the newforms associated with \( f_{\chi} \) have level 28 for characters \( \chi : (\mathbb{Z}/7)^* \to \mathbb{C}^* \) of order 6 (there are 2 such characters), and the newforms associated with \( f_{\chi} \) have level 196 for other characters (there are 4 such characters). Therefore we have that rank \( J(14)(\mathbb{Q}(\zeta_{14})) = 2\sigma_0(196/28) + 4\sigma_0(196/196) = 8 \).

**Example 4.4** [The rank of \( J(15)(\mathbb{Q}(\zeta_{15})) \)] Let \( \Delta \) be the subgroup as in Proposition 3.4 for \((M, N) = (15, 1)\). According to [18], \( J_\Delta \) has two simple factors, say \( A_1 \) and \( A_2 \), whose ranks over \( \mathbb{Q} \) are nonzero. \( A_1 = A_{f_1} \) for \( f_1 \) a newform in the Galois conjugacy class of newforms labeled “225.2.a.a”, and \( A_2 = A_{f_2} \) for \( f_2 \) in the class labeled “225.2.a.c” in [18].) The newforms \( f_i \) have analytic ranks 1, and the abelian varieties \( A_i \) are of dimension 1. Therefore by [13, Corollary C], we have that rank \( A_i(\mathbb{Q}) = 1 \). Again according to [18], for \( f = f_1 \), the newform associated with \( f_{\chi} \) has level 75 for every character \( \chi : (\mathbb{Z}/15)^* \to \mathbb{C}^* \) of conductor 3 or 15 (there are 4 such characters), and the newforms associated with \( f_{\chi} \) have level 225 for other characters (there are 4 such characters). Next for \( f = f_2 \), a newform associated with \( f_{\chi} \) has level 225 for every \( \chi : (\mathbb{Z}/15)^* \to \mathbb{C}^* \). Therefore we have that rank \( J(15)(\mathbb{Q}(\zeta_{15})) = 4\sigma_0(225/75) + 12\sigma_0(225/225) = 20 \).

For the subgroup \( \Delta \) as in Proposition 3.4 for \((M, N) = (13, 1)\), the modular Jacobian variety \( J_\Delta \) has only one simple factor \( A_f \) whose Mordell–Weil rank over \( \mathbb{Q} \) is nonzero. The abelian variety \( A_f \) has the dimension three and \( f \) has analytic rank one. If Birch–Swinnerton-Dyer conjecture is true, then it follows that rank \( A_f(\mathbb{Q}) = 3 \), and thence we also can compute the rank of \( J(13)(\mathbb{Q}(\zeta_{13})) \). For a modular abelian variety \( A_f \) if its L-function has a simple pole at \( s = 1 \), then it seems that we obtain \( \dim A_f A_f(\mathbb{Q}) \otimes \mathbb{Q} = 1 \) unconditionally. We, however, could not find a reference. Newforms of low levels have low analytic ranks. For example, the newforms of level less than 389 have analytic ranks at most 1. (See [18].) Hence with this proposition we can easily compute the Mordell–Weil ranks of many modular Jacobian varieties.

**Proof of Proposition 4.1** First by Lemma 2.2 one obtains

\[
\text{rank } J_1(M, MN)(\mathbb{Q}(\zeta_M)) = \sum_{G \neq f \in \text{New}(M MN)} \sigma_0(M^2 N / N_f) \text{rank } A_f(\mathbb{Q}(\zeta_M)).
\]

For newforms \( f \) and \( g \), if these are Galois conjugate to each other, then the fields \( K_f \) and \( K_g \) generated by their Fourier coefficients are isomorphic to each other, and also the modular abelian varieties \( A_f \) and \( A_g \) are isogenous to each other. Moreover for a newform \( f \), the dimension of \( A_f \) equals to \([K_f : \mathbb{Q}]\). Therefore we have

\[
\text{rank } J_1(M, MN)(\mathbb{Q}(\zeta_M)) = \sum_{f \in \text{New}(M, MN)} \sigma_0(M^2 N / N_f) \dim_{K_f} A_f(\mathbb{Q}(\zeta_M)) \otimes \mathbb{Q}. \quad (1)
\]
For each \( f \in \text{New}(M, MN) \), by Corollary 3.2 AND Lemma 3.3 we have

\[
\dim_{K_f} A_f(Q(\zeta_M)) \otimes \mathbb{Q} = \sum_{v \in D} \dim_{K(v)_{\text{new}}} A(v)_{\text{new}}(Q) \otimes \mathbb{Q}.
\]

Thus combining them together we obtain

\[
\text{rank } J_1(M, MN)(Q(\zeta_M)) = \sum_{f \in \text{New}(M, MN)} \sum_{v \in D} \sigma_0(M^2N/N_f) \dim_{K(v)_{\text{new}}} A(v)_{\text{new}}(Q) \otimes \mathbb{Q}.
\]

Since for every \( v \in D \), the map

\[
\text{New}(M, MN) \to \text{New}(M, MN) \quad f \mapsto (f)_{\text{new}}
\]

is well-defined and bijective by Lemma 2.3, we obtain

\[
\text{rank } J_1(M, MN)(Q(\zeta_M)) = \sum_{f \in \text{New}(M, MN)} \sum_{v \in D} \sigma_0(M^2N/N_f) \dim_{K_f} A_f(Q) \otimes \mathbb{Q}.
\]

Thus again by the same reason as in equation (1), the result follows. \( \square \)

Using the same method as in [6, Theorem 3.1, Remark 3.4], for fixed integer \( r \), we can get a necessary condition for that rank \( J_0(N)(\mathbb{Q}) = r \), assuming the Birch–Swinnerton-Dyer conjecture. For a prime \( p \), let \( g_0^+(p) \) be the genus of the modular curve \( X_0^+(p) \), which is the quotient of the modular curve \( X_0(p) \) by the Atkin–Lehner involution.

**Lemma 4.5** If the Birch–Swinnerton-Dyer conjecture is true, then the following inequality holds:

\[
\text{rank } J_0(p)(\mathbb{Q}) \geq g_0^+(p).
\]

**Proof** The modular Jacobian variety \( J_0(p) \) is isogenous to \( J_0^-(p) \times J_0^+(p) \). Hence we have rank \( J_0(p)(\mathbb{Q}) \geq \text{rank } J_0^+(p)(\mathbb{Q}) \). The abelian variety \( J_0^+(p) \) is the Jacobian variety of the modular curve \( X_0^+(p) \), and is isogenous to a product \( \bigoplus_f A_f \), where the direct sum is taken over all Galois conjugacy classes of the newforms of level \( \Gamma_0(p) \) fixed by the Atkin–Lehner involution. Thus for every simple factor of \( J_0^+(p) \), its analytic rank is odd, and in particular is nonzero. Hence if the Birch–Swinnerton-Dyer conjecture is true, then every simple factor of \( J_0^+(p) \) has nonzero Mordell–Weil rank. Moreover, for every such newform \( f \), since an order of \( K_f \) acts on \( A_f \), we have rank \( A_f(\mathbb{Q}) \geq [K_f : \mathbb{Q}] \), which is equal to the dimension of \( A_f \). Thus we obtain

\[
\text{rank } J_0^+(p)(\mathbb{Q}) = \sum_f \text{rank } A_f(\mathbb{Q})
\]

\[
\geq \sum_f \dim A_f
\]

\[
= \dim J_0^+(p),
\]

where the sum is taken over all Galois conjugacy classes of the newforms of level \( \Gamma_0(p) \) fixed by the Atkin–Lehner involution. Thus the result. \( \square \)

Note that Lemma 4.5 does not hold for composite numbers, for example rank \( J_0(28)(\mathbb{Q}) = 0 \) but \( g_0^+(28) = 1 \). We also note that we know the complete list of prime numbers \( p \) so that the genera \( g_0^+(p) \) are less than 7, see [1, Proposition 4.5].
Lemma 4.6 Let $N$ be a positive integer.

1. Write $N = Mp^e$ for a positive integer $M$, for a prime $p$, and for $e \geq 1$. For a newform $f$ of level $N_f$ dividing $M$, let $m_f$ be the prime-to-$p$-part of $M/N_f$. Then $J_0(N)$ contains $J_0(M) \oplus \left( \oplus_f A_f^{\sigma_0(m_f)} \right)$ up to isogeny, where $f$ runs over the Galois conjugacy classes of the newforms of level dividing $M$.

2. Assume that we can write $N = M_1M_2$ for relatively prime positive integers. Then $J_0(N)$ contains $J_0(M_1)^{\sigma_0(M_2)} \oplus J_0(M_2)^{\sigma_0(M_1)}$ up to isogeny.

3. Let $M$ be a positive proper divisor of $N$. If the rank of $J_0(N)(\mathbb{Q})$ is nonzero, then $\text{rank } J_0(M)(\mathbb{Q}) < \text{rank } J_0(N)(\mathbb{Q})$.

Proof 1. First we show the first statement. Let $M$ be a positive integer, $p$ a prime, $e \geq 1$ an integer, and $N = Mp^e$. The modular Jacobian variety $J_0(M)$ decomposes as $\oplus_f A_f^{\sigma_0(M/N_f)}$, where $f$ runs over the Galois conjugacy classes of the newforms of level dividing $M$. Since $M$ divides $N_f$ for each such $f$, the modular Jacobian variety $J_0(N)$ also contains $A_f$ up to isogeny, with the multiplicity $\sigma_0(N/N_f)$. Thus it suffices to show, for each $f$ as above and for $m_f$ as in the statement, that $\sigma_0(N/N_f) = \sigma_0(M/N_f) + e\sigma_0(m_f)$, which is elementary: Write $M/N_f = p^b m_f$ for a nonnegative integer $b$. Then since $m_f$ is prime-to-$p$, we obtain $\sigma_0(N/N_f) = \sigma_0(p^{e+b})\sigma_0(m_f) = (e + b + 1)\sigma_0(m_f) = e\sigma_0(m_f) + (b + 1)\sigma_0(m_f) = e\sigma_0(m_f) + \sigma_0(M/N_f)$.

2. For the second statement let $M_1$ be positive integers as in the statement. Since $M_1$ and $M_2$ are relatively prime to each other, considering the simple decomposition of $J_0(N)$, we obtain that $J_0(N)$ contains $\left( \oplus_f A_f^{\sigma_0(N/N_f)} \right) \oplus \left( \oplus_g A_g^{\sigma_0(N/N_g)} \right)$ up to isogeny, where $f$ runs over the Galois conjugacy classes of the newforms of level dividing $M_1$, and where $g$ runs over those of level dividing $M_2$. Thus again since $M_1$ and $M_2$ are relatively prime to each other, we obtain $\sigma_0(N/N_f) = \sigma_0(M_1/N_f)\sigma_0(M_2)$ and $\sigma_0(N/N_g) = \sigma_0(M_2/N_g)\sigma_0(M_1)$. Since $\sigma_0(M_1/N_f)$ (and $\sigma_0(M_2/N_g)$) is the multiplicity of the modular abelian variety $A_f$ (and $A_g$) as a simple factor of $J_0(M_1)$ (and $J_0(M_2)$ respectively), the result follows.

3. For the third statement, let $M$ be a positive proper divisor of $N$, and assume that the rank of $J_0(N)(\mathbb{Q})$ is nonzero. If $\text{rank } J_0(M)(\mathbb{Q}) = 0$, then there is nothing to show. Assume that $\text{rank } J_0(M)(\mathbb{Q}) > 0$. In this case there exists a simple factor $A$ of $J_0(M)$ whose Mordell–Weil rank is nonzero. By the first statement, for each simple factor $A$ of $J_0(M)$, we have that $J_0(M) \times A$ is contained in $J_0(N)$ up to isogeny. Thus the result.

\[ \square \]

Proposition 4.7 Assume that the Birch–Swinnerton-Dyer conjecture is true. Let $r$ be an integer. Then there exist only finitely many integers $N$ such that $\text{rank } J_0(N)(\mathbb{Q}) = r$, i.e., the rank of $J_0(N)(\mathbb{Q})$ tends to infinity as the level $N$ tends to infinity.

Proof We show it by induction on $r$. First by [6, Theorem 3.1], the statement for $r = 0$ is true. Next assume that $r > 0$ and suppose that there exist only finitely many integers $N$ such that $\text{rank } J_0(N)(\mathbb{Q}) < r$. Define

\[ h(x) = \frac{x - 5\sqrt{x} + 4}{24} - \frac{\sqrt{x}}{\pi} (\log(16x) + 2). \]
We can compute that the function $h(x)$ is monotonically increasing for sufficiently large $x$, for example for $x > 10^6$. Since $h(x)$ tends to infinity as $x$ tends to infinity, we have that there exist only finitely many integers $N$ such that $h(N) < r$. Therefore by Lemma 4.5 and by [1, Proposition 4.4], there exist only finitely many primes $p$ such that $\text{rank } J_0(p)(\mathbb{Q}) = r$. By Lemma 4.6 (3), if $N$ is a composite number then $N$ is of the form of $M_1 M_2$ for positive integers $M_i > 1$ satisfying that $\text{rank } J_0(M_i)(\mathbb{Q}) < r$. Therefore the induction hypothesis yields the result. 

Moreover the proof above inductively constructs, for each integer $r$, a finite set $S_r$ such that if the rank of $J_0(N)(\mathbb{Q})$ is equal to $r$ then $N \in S_r$. Namely:

**Definition 4.8** Let $S_0$ be the set of positive integers $N$ such that the rank of $J_0(N)(\mathbb{Q})$ is zero ([6, Theorem 3.1]). For $r \geq 1$, assuming that we have defined the set $S_r$ for every $r' < r$, we define $S_r$ to be the set of positive integers $N$ satisfying that, $N$ is either a prime such that $g_0^+(N) \leq r$, or a composite number such that every positive proper divisor $M$ of $N$ lies in the set $\bigcup_{i=0}^{r-1} S_i$.

For example we obtain that

\[
S_0 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 38, 39, 40, 41, 42, 44, 45, 46, 47, 49, 50, 51, 52, 54, 55, 56, 59, 60, 62, 63, 64, 66, 68, 70, 71, 72, 75, 76, 78, 80, 81, 84, 87, 90, 94, 95, 96, 98, 100, 104, 105, 108, 110, 119, 120, 126, 132, 140, 144, 150, 168, 180\}
\]

by [6, Theorem 3.1], and hence by definition we obtain

\[
S_1 = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 68, 69, 70, 71, 72, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 98, 99, 100, 101, 102, 104, 105, 108, 110, 112, 115, 117, 118, 119, 120, 121, 123, 124, 125, 126, 128, 131, 132, 133, 135, 136, 138, 140, 141, 142, 143, 144, 145, 147, 150, 152, 153, 155, 156, 160, 161, 162, 165, 168, 169, 175, 177, 180, 187, 188, 189, 190, 192, 196, 200, 203, 205, 207, 208, 209, 210, 213, 216, 217, 220, 221, 225, 235, 238, 240, 243, 245, 247, 252, 253, 261, 275, 280, 287, 288, 289, 295, 299, 300, 315, 319, 323, 329, 341, 343, 355, 357, 360, 361, 377, 391, 403, 413, 437, 451, 475, 493, 497, 517, 527, 529, 533, 551, 589, 611, 649, 667, 697, 713, 767, 779, 781, 799, 833, 841, 893, 899, 923, 943, 961, 1003, 1081, 1121, 1189, 1207, 1271, 1349, 1357, 1363, 1457, 1633, 1681, 1711, 1829, 1927, 2059, 2201, 2209, 2419, 2773, 2911, 3337, 3481, 4189, 5041\}.
\]

From statements above we can obtain a rough but easy-to-understand necessary condition for that rank $J_0(N)(\mathbb{Q}) \leq r$, in the form of a lower bound of rank $J_0(N)(\mathbb{Q})$. 


Proposition 4.9 Assume that the Birch–Swinnerton-Dyer conjecture is true. Then rank\(J_0(N)(\mathbb{Q}) > -1 + \log_{180} N\).

Proof We show the statement by showing that, for a nonnegative integer \(r\), if rank\(J_0(N)(\mathbb{Q}) \leq r\), then we obtain \(N \leq 180^{r+1}\). First we show a sharper result for prime numbers: Namely, for a prime number \(p\) and for a positive integer \(r\), if rank\(J_0(p)(\mathbb{Q}) \leq r\) then \(p \leq 180^r\), and if rank\(J_0(p)(\mathbb{Q}) = 0\) then \(p \leq 180\). Let \(r\) be a nonnegative integer and \(p\) a prime. Define
\[
h(x) = \frac{x - 5\sqrt{x} + 4}{24} - \frac{\sqrt{x}}{\pi}(\log(16x) + 2).
\]
Let \(f(x) = h(180^x) - x\). Then we can compute that \(f(x)\) is monotonically increasing for \(x \geq 2\), and that \(f(7) > 0\). Hence for \(r \geq 7\), if a prime \(p\) satisfies \(p > 180^r\) then \(r < h(180^r) < h(p)\). Therefore by Lemma 4.5 and by [1, Proposition 4.4], it shows our claim for \(r \geq 7\). For \(1 \leq r \leq 6\), by [1, Proposition 4.5] we have that if \(g_0^+(p) \leq r\) then \(p \leq 180^r\), hence our claim for \(1 \leq r \leq 6\). Finally if rank\(J_0(p)(\mathbb{Q}) = 0\) then by [6, Theorem 3.1] we have \(p \leq 180\).

We show the statement for general case by induction on \(r\). Let \(N\) be an integer so that rank\(J_0(N)(\mathbb{Q}) \leq r\). We may assume that \(N\) is a composite number. In the case \(r = 0\), the statement follows from [6, Theorem 3.1]. For \(r = 1\), we computed \(S_1\) explicitly above, hence the result. Let \(r \geq 2\) and suppose that the statement holds for every \(r' < r\). By the induction hypothesis, we may assume that rank\(J_0(N)(\mathbb{Q}) = r\).

First assume that we can write \(N = M_1M_2\) for relatively prime integers \(M_i > 1\). Then by Lemma 4.6 (2) we have that
\[
2 \text{rank } J_0(M_1)(\mathbb{Q}) + 2 \text{rank } J_0(M_2)(\mathbb{Q}) \leq \text{rank } J_0(N)(\mathbb{Q}).
\]
Thus for each \(i\) the rank of \(J_0(M_i)(\mathbb{Q})\) is at most \(r/2\), and so the induction hypothesis implies that \(M_i \leq 180^{(r/2)+1}\), where \([\cdot]\) is the floor function. If rank\(J_0(M_2)(\mathbb{Q}) = 0\), then \(M_2 \leq 180 \leq 180^{r/2}\), and thus \(N \leq 180^{r+1}\). If rank\(J_0(M_2)(\mathbb{Q}) \neq 0\), then rank\(J_0(M_1)(\mathbb{Q}) \leq -1 + r/2\) and hence \(M_1 \leq 180^{r/2}\), which shows that \(N \leq 180^{r+1}\).

Next assume that \(N = p^e\) for a prime \(p\) and for an integer \(e \geq 2\). In this case by Lemma 4.6 (1) the rank of \(J_0(p)(\mathbb{Q})\) does not exceed \(r/e\). If \(r/e < 1\), then rank\(J_0(p)(\mathbb{Q}) = 0\), and hence \(p \leq 71\). On the other hand by Lemma 4.6 (3) we have rank\(J_0(p^{e-1})(\mathbb{Q}) \leq r - 1\), which implies \(p^{e-1} \leq 180^r\) by the induction hypothesis. Thus in this case \(N \leq 180^{r+1}\).

If \(r/e \geq 1\) then we have shown that in this case we obtain \(p \leq 180^{\lfloor r/e \rfloor}\) in the argument above. Hence \(N \leq 180^r\). This completes the proof. \(\Box\)

Note that, although the statements in this section treat the higher rank, these, as we have remarked in preliminaries, assume only the converse of Kato’s theorem [10, Corollary 14.3], but do not require the full strength of Birch–Swinnerton-Dyer conjecture.

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