ALMOST SQUARE DUAL BANACH SPACES

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Abstract. We show that finite dimensional Banach spaces fail to be uniformly non locally almost square. Moreover, we construct an equivalent almost square bidual norm on \( \ell_\infty \). As a consequence we get that every dual Banach space containing \( c_0 \) has an equivalent almost square dual norm. Finally we characterize separable real almost square spaces in terms of their position in their fourth duals.

1. Introduction

A Banach space \( X \) with unit sphere \( S_X \) is said to be
(a) almost square (ASQ) if for every \( \varepsilon > 0 \) and every finite set \( (x_i)_{i=1}^n \subset S_X \) there exists \( h \in S_X \) such that
\[
\|x_i \pm h\| \leq 1 + \varepsilon \quad \text{for every } i = 1, \ldots, n.
\]
(b) locally almost square (LASQ) if for every \( \varepsilon > 0 \) and every \( x \in S_X \) there exists \( h \in S_X \) such that
\[
\|x \pm h\| \leq 1 + \varepsilon.
\]

These notions were introduced in [ALL16] and have later on been considered in several papers, among others [BGLPRZ16], [GLRZ17], and [LLRZ17]. The ASQ property is connected to the intersection property, introduced by Harmand and Behrends in [BH85] when studying proper M-ideals. A Banach space has the intersection property (IP) if for every \( \varepsilon > 0 \) there are \( (x_i)_{i=1}^n \subset X \) with \( \|x_i\| < 1 \) for \( i = 1, \ldots, n \) such that \( \|x_i - h\| \leq 1 \) implies \( \|h\| \leq \varepsilon \). The connection is that a Banach space is ASQ if and only if it fails the IP for all \( 0 < \varepsilon < 1 \) [ALL16, Proposition 6.1]. The main result in the present paper is that there exists a dual (in fact bidual) ASQ space. Using this we give a positive answer to a question from [ALL16, Question 6.6] (later posed in [BGLPRZ16, p. 1039] and [GLRZ17, p. 130]) whether ASQ spaces can be dual. Also we give a negative answer to a question from [BH85, p. 167] whether a space that can never be a proper M-ideal must have

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the IP. The latter follows since, as noted in [BH85, p. 167], dual spaces can never be proper M-ideals.

Let us now provide some background on ASQ and LASQ spaces. A typical example of an ASQ space is $c_0$. Just take the $h$ in the definition to be a standard unit vector $e_n$ with big enough $n$. The reason why this works is that functions in $c_0$ have a common zero at infinity. That is, "having a common zero" is in some sense a general feature of the vectors in ASQ spaces while for LASQ spaces it is that of "having a zero". The latter can be exemplified by the space $X = \{ f \in C[0,1] : f(0) = -f(1) \}$ (As usual $C[0,1]$ denotes the space of continuous functions on the unit interval with the max-norm). In $X$ all functions have a zero, but in this space two different functions can certainly have different zeros, so $X$ is LASQ, but not ASQ [ALL16, Example 3.7].

Both ASQ and LASQ spaces possess diameter two properties, so they belong to a class of spaces which, in a sense, is diametrically opposed to the class of spaces with the Radon-Nikodym property. If a Banach space $X$ is ASQ, then all finite convex combinations of slices of the unit ball $B_X$ of $X$ have diameter two [ALL16, Proposition 2.5 and Theorem 1.3], and if $X$ is LASQ then all slices of $B_X$ have diameter two [Kub13, Proposition 2.5]. Recall that a closed subspace $X$ of Banach space $Z$ is an M-ideal in $Z$ provided there exists some Banach space $Y$ such that $Z^* = Y \oplus_1 X^\perp$ where $X^\perp$ is the annihilator of $X$ (see the book [HWW93] for the theory of M-ideals). A non-reflexive space which is an M-ideal in its bidual (i.e. an M-embedded space) is ASQ [ALL16, Corollary 4.3], so the class of ASQ spaces is quite big and in fact quite a lot bigger than the class of non-reflexive M-embedded spaces. Take e.g. $c_0(\ell_1)$, the $c_0$-sum of $\ell_1$. This space is ASQ by using the same idea as for $c_0$, but the space is not M-embedded since the property of being M-embedded is inherited by subspaces [HWW93, III.1.Theorem 1.6]. However, because of its $c_0$-like nature, it is perhaps not so surprising that ASQ spaces happen to share several properties by non-reflexive M-embedded spaces. E.g. besides having quite strong diameter two properties, they contain copies of $c_0$ and are stable by taking $c_0$-sums [ALL16, Lemma 2.6 and Example 3.1]. Non-reflexive M-embedded spaces are never dual spaces, and the same is true for a large class of ASQ spaces. For example, weakly compactly generated dual spaces are never ASQ [ALL16, p. 1564]. Moreover, in [BGLPRZ16, Theorem 2.5] it was shown that if you put constraints on the vectors $h$ in (a) in the definition of an ASQ space, then the space can never be dual. More precisely it was shown that if a Banach space $X$ satisfies that for any $\delta > 0$ there exists a set $(h_\gamma)_{\gamma \in \Gamma} \subset S_X$ such that

(a) for every $\varepsilon > 0$ and $x_1, \ldots, x_n \in S_X$ there exists $\gamma \in \Gamma$ such that $\|x_i \pm h_\gamma\| \leq 1 + \varepsilon$ for every $i = 1, \ldots, n$, and
(b) for every finite subset $F$ of $\Gamma$ and any choice of signs $\xi_\gamma \in \{1,-1\}$ we have $\|\sum_{\gamma \in F} \xi_\gamma h_\gamma\| \leq 1 + \delta$,
then it cannot be a dual space. Such a space is said to be unconditioned almost square (UASQ). Among UASQ spaces we find separable ASQ spaces [BGLPRZ16, Corollary 2.4]. In [BGLPRZ16, Question 6.1] it was left open whether dual ASQ spaces are UASQ. Our construction of a dual ASQ space answers also this in the negative.

Let us now describe the content of the paper. In Section 2 we show that finite dimensional Banach spaces with a fixed dimension are uniformly non LASQ, but we construct examples showing that this is not so if you allow the dimension to grow.

In Section 3 the technique developed in Section 2 for finite dimensional spaces is refined to construct an equivalent bidual ASQ norm on $\ell_\infty$. This in turn is used to show that every dual Banach space containing $c_0$ admits an equivalent dual ASQ norm.

In Section 4 we characterize separable real ASQ spaces in terms of their position in their fourth dual.

2. Finite dimensional spaces

We start this section by showing that for a fixed dimension, finite dimensional spaces are uniformly non LASQ.

**Proposition 2.1.** Given $n \in \mathbb{N}$. Then there exists $\delta > 0$ such that for all Banach spaces $X$ with dimension $n$, there exists $x \in S_X$ such that for all $h \in S_X$ we have

$$\max \|x \pm h\| \geq 1 + \delta.$$  

**Proof.** We can assume that $X = \mathbb{R}^n$ with any given norm $\| \cdot \|$. We will show that there exists $x \in S_X$ such that (1) holds for any positive $\delta \leq (1 + n^{-1})^{1/2} - 1$. To prove this we use John’s Theorem (see e.g. [FHH11, Theorem 6.22]) that there exists an ellipsoid $J$ of minimal volume such that

$$n^{-1/2}J \subset B_X \subset J,$$

i.e.

$$\|x\|_J \leq \|x\| \leq n^{1/2}\|x\|_J,$$

where $\| \cdot \|_J$ denotes the Euclidian norm on $X$ we get when using $J$ as the unit ball. We have $S_X \cap \partial J \neq \emptyset$ where $\partial J = \{(x_i)_{i=1}^n \in X : \sum_{i=1}^n x_i^2 = 1\} \subset J$ since otherwise $J$ could not be of minimal volume.
Take $x \in S_X \cap \partial J$. Then for $h \in S_X$ we have

$$
\max \|x \pm h\|_2^2 \geq \max \|x \pm h\|_J^2
$$

$$
= \max \{\|x\|_J^2 + \|h\|_J^2 \pm 2 \langle x, h \rangle\}
$$

$$
\geq \|x\|_J^2 + (n^{-1/2}\|h\|_J)^2
$$

$$
\geq \|x\|_J^2 + n^{-1}\|h\|_J^2 = 1 + n^{-1},
$$

so $\max \|x \pm h\| \geq (1 + n^{-1})^{1/2} \geq 1 + \delta$. □

If you allow the dimensions of the finite dimensional spaces to grow, the conclusion of Proposition 2.1 is no longer true. Indeed, we will construct finite dimensional spaces for which, roughly speaking, the corners of the unit balls flattens out as the dimensions grow. Let us introduce some notation and definitions that we need. By $(e_j)_{j=1}^\infty$ denote, as usual, the standard unit vectors in $c_00$ (the vector space of finitely supported vectors). Let $k, n, m \in \mathbb{N}$ with $k \leq n \leq m$ and set

$$
A_n = \left\{ \frac{1}{n} \sum_{j \in A} e_j \in \ell_1^m : A \subseteq \{1, \ldots, m\}, |A| = n \right\},
$$

$$
B_k = \left\{ \frac{1}{k} e_j^* \in \ell_1^m : j = 1, \ldots, m \right\},
$$

where $\ell_1^m$ denotes the vector space $\mathbb{R}^m$ with the $\ell_1$-norm. Put $C_{k,n} := B_k \cup A_n$. Define $\| \cdot \|_{k,n} : \ell_\infty^m \to \mathbb{R}$ by

$$
\|f\|_{k,n} = \sup_{x \in C_{k,n}} |f(x)|,
$$

where

$$
f(x) = \begin{cases} 
  n^{-1} \sum_{j \in \text{supp} x} f_j, & x \in A_n, \\
  k^{-1} f_j, & x = k^{-1} e_j^* \in B_k.
\end{cases}
$$

Clearly $\| \cdot \|_{k,n}$ is a norm on $\ell_\infty^m$ satisfying

$$
k^{-1}\|f\|_\infty \leq \|f\|_{k,n} \leq \|f\|_\infty.
$$

Put $F_{k,n} : (\ell_\infty^m, \| \cdot \|_{k,n})$.

**Lemma 2.2.** Let $k, n, m \in \mathbb{N}$ with $k^2 \leq n$ and $m = 2n$. Let $f \in F_{k,n}$ of norm 1. Then there is $h \in F_{k,n}$ of norm 1 such that

$$
\|f \pm h\|_{k,n} \leq 1 + \frac{1}{k}.
$$

**Proof.** Let $f \in S_{F_{k,n}}$. We want to find $h \in S_{F_{k,n}}$ such that

$$
\|f \pm h\|_{k,n} \leq 1 + \frac{1}{k}.
$$

To this end note that by the pigeonhole principle there is a subset $E$ of $\{1, 2, \ldots, m\}$ with cardinality $\geq n$ on which $f$ has the same sign.
Assume without loss of generality that $f$ is non-negative here. Since for any $x \in A_n$ with $\text{supp } x \subseteq E$ we have

$$1 = \|f\|_{k,n} \geq |f(x)| = \frac{1}{n} \sum_{j \in \text{supp } x} |f_j|,$$

there exists $l \in E$ such that $0 \leq f_l \leq 1$. Now, let $h = ke_l$. Then

$$\|f \pm h\|_{k,n} = \max \left\{ \sup_{x \in B_k} |(f \pm h)(x)|, \sup_{x \in C_n} |(f \pm h)(x)| \right\}$$

$$= \max \left\{ \sup_{j \in \mathbb{N}} \frac{|f_j \pm ke_l(e_j^*)|}{k}, \sup_{x \in C_n} \frac{\sum_{j \in \text{supp } x} f_j \pm ke_l(e_j^*)}{n} \right\}$$

$$\leq \max \left\{ \max \left\{ \sup_{j \neq l} \frac{|f_j|}{k}, \frac{|f_l| + k}{k} \right\}, \sup_{x \in C_n} \frac{\sum_{j \in \text{supp } x} f_j \pm ke_l(e_j^*)}{n} \right\}$$

$$\leq \max \left\{ 1 + \frac{k}{k}, \frac{1}{k} \frac{\sum_{j \in \text{supp } x} |f_j|}{n} + \frac{k}{k} \right\}$$

$$\leq \max \left\{ 1 + \frac{1}{k}, 1 + \frac{k}{n} \right\} \leq 1 + \frac{1}{k},$$

as wanted. \(\square\)

**Remark 2.3.** Note that we could get (9) in Lemma 2.2 to hold for any given number $N$ of norm 1 vectors simply by choosing the dimension $m = m(N)$ of the space $F_{k,n}$ to be sufficiently big.

3. A bidual ASQ norm on $\ell_\infty$

An alert reader will have noticed that the lower isomorphism constant in (5) changes as the dimension grows. In order to construct an equivalent bidual ASQ norm on $\ell_\infty$ we want to avoid this. Thus we need to refine a bit the technique developed in the previous section.

Fix $k \in \mathbb{N}$ such that $k \geq 2$. For $n \in \mathbb{N}$ let $E_n = \{2^n, \ldots, 2^{n+1} - 1\}$ and $C_n = \{2^{-1} e_l^* \pm 2^{-1} e_m^* \in \ell_1 : l, m \in E_n, l \neq m\}$. For $N \in \mathbb{N}$ let

$$A_N = \left\{ \frac{1}{N} \sum_{j \in A} e_j^* \in \ell_1 : A \subset \mathbb{N}, |A| = kN \right\}$$

$$B = \left\{ \frac{1}{k} e_j^* \in \ell_1 : j \in \mathbb{N} \right\}.$$

$$C = \bigcup_{n=1}^\infty C_n.$$

Put $D_N := C \cup B \cup A_N$. Define $\| \cdot \|_N : \ell_\infty \to \mathbb{R}$ by

$$\|f\|_N = \sup_{x \in D_N} |f(x)|,$$
Lemma 3.1. The space \( (\ell_\infty, \| \cdot \|_N) \) is the bidual of \( (c_0, \| \cdot \|_N) \).

Proof. Note that \( (e_j)_{j=1}^\infty \) is a basis for \( (c_0, \| \cdot \|_N) \) which is monotone and shrinking. Thus \( (c_0, \| \cdot \|_N)^{**} \) can be identified with space of sequences \( f = (f_j)_{j=1}^\infty \) such that \( \|f\| = \sup_K \| \sum_{j=1}^K f_j e_j \|_N < \infty \). Now observe that

\[
\|f\| = \sup_K \| \sum_{j=1}^K f_j e_j \|_N = \sup_N \| P_K f \|_N
\]

\[
= \sup_K \sup_{x \in D_N} |P_K f(x)| = \lim_{K} \sup_{x \in D_N} |P_K f(x)| = \|f\|_N,
\]

where \( P_K \) denotes the projection on \( \ell_\infty \) that projects vectors onto their first \( K \) coordinates. \( \square \)

Put \( X_N := (c_0, \| \cdot \|_N) \) and \( X_N^{**} := (c_0, \| \cdot \|_N)^{**} = (\ell_\infty, \| \cdot \|_N) \). We will need the following lemma.

Lemma 3.2. For \( N \in \mathbb{N}, \varepsilon > 0 \), and \( (f^i)_{i=1}^K \subset S_{X_N^{**}} \), there exists \( n \in \mathbb{N} \) and distinct \( l, m \in E_n \) such that

(a) \( |f^i_l| \leq k^{-1} \) for all \( j \in E_n \) and \( i \in \{1, \ldots, K\} \),

(b) \( |f^i_l - f^i_m| < \varepsilon \) for all \( i \in \{1, \ldots, K\} \).

Proof. Let \( \varepsilon > 0 \). Note that from the definition of the norm each \( f^i \) has only finitely many coordinates \((< 2kN)\) of absolute value \( > k^{-1} \). Thus there exists \( n_0 \geq 2 \) such that \( [a] \) holds for all \( n \geq n_0 \). Now choose an integer \( p > n_0 \) such that \( 2k^{-1}/p < \varepsilon \). By the pigeonhole principle we can choose \( n_1 > p \) such that \( |f^1_l - f^1_m| < \varepsilon \) for at least \( p^K \) different pairs of distinct integers \( l \) and \( m \) in \( E_{n_1} \). Now again by the pigeonhole principle \( [b] \) must hold for at least one of these pairs. \( \square \)

Let us prove that \( X_N^{**} \) is getting closer and closer to being ASQ as \( N \) grows. Still it is not LASQ for any even number \( N \geq k \).

Lemma 3.3. Let \( N \in \mathbb{N} \) and \( (f^i)_{i=1}^K \subset S_{X_N^{**}} \). Then there is \( h \in S_{X_N} \) such that

\[
\|f^i \pm h\|_N \leq 1 + \frac{1}{N}
\]
for every \( i = 1, \ldots, K \). Nevertheless, if \( N \) is an even number \( \geq k \), then \( X_N \) is not LASQ.

**Proof.** Find \( n \in \mathbb{N} \) such that the conclusion in Lemma 3.2 holds for \( \varepsilon = N^{-1} \). Put \( h = e_l - e_m \). Then \( 1 \geq \| h \|_N = h(2^{-1}e_l - 2^{-1}e_m) = 1 \), so \( \| h \|_N = 1 \). Note that

\[
m_C := \sup_{x \in C} |(f^i + h)(x)| = \max \left\{ \sup_{x \in C_n} |(f^i + h)(x)|, \sup_{x \in C} |f^i(x)| \right\}
\]

\[
\leq \max \left\{ \max_{j \in E_n \setminus \{l, m\}} |(f^i + h)(2^{-1}e_j^* + 2^{-1}e_m^*)|, \max_{p \in \{l, m\}} |(f^i + h)(2^{-1}e_p^*)|, |f^i|_N \right\}
\]

\[
= \max \left\{ \max_{j \in E_n \setminus \{l, m\}} |(2^{-1}(f_j^i + 1) + 2^{-1}(f_m^i + 1)|, \max_{p \in \{l, m\}} |(2^{-1}f^i_p + 2^{-1}(f_p + 1)|, 1 \right\}
\]

\[
\leq \max \left\{ \max\{k^{-1}, 1 + 2^{-1}\varepsilon\}, k^{-1} + 2^{-1}, 1 \right\} = 1 + 2^{-1}N^{-1},
\]

\[
m_B := \sup_{x \in B} |(f^i + h)(x)| = \max\{\sup_{j \neq l, m} k^{-1}|f_j^i|, k^{-1} \max_{j = l, m} |f_j^i + 1|\}
\]

\[
\leq k^{-1} \max\{|f^i|_N, 1 + k^{-1}\} \leq 1,
\]

and

\[
m_{A_N} := \sup_{x \in A_N} |(f^i + h)(x)|
\]

\[
= \max \left\{ \sup_{x \in A_N} |(f^i + h)(x)|, \sup_{x \in A_N} |f^i(x)| \right\}
\]

\[
\leq \max\{1 + N^{-1}, 1\} = 1 + N^{-1}.
\]

Thus

\[
|f^i \pm h|_N = \sup\{|(f^i \pm h)(x) : x \in D_N|\}
\]

\[
= \max \left\{ \sup_{x \in C} |((f^i \pm h)(x)|, \sup_{x \in B} |(f^i \pm h)(x)|, \sup_{x \in A_N} |(f^i \pm h)(x)| \right\}
\]

\[
\leq \max\{m_C, m_B, m_{A_N}\} \leq 1 + N^{-1}
\]

for every \( i = 1, \ldots, K \).

The LASQ part: Let \( E \subset \mathbb{N} \) with \( |E| = N/2 \) such that \( E \cap E_1 = \emptyset \) and \( |E \cap E_n| = 1 \) for all \( 2 \leq n \leq N/2 + 1 \). Put \( f = \sum_{j \in E} 2e_j \). Then \( f \) has norm 1. Let \( 0 < \varepsilon < 1/3N \) and assume there is \( h \in S_{X_N} \) such that \( \| f \pm h \| \leq 1 + \varepsilon \). We claim that for every \( 2 \leq n \leq N/2 + 1 \), every \( l, j \in E_n \) with \( j \neq l \) and \( l \in E_n \cap E \), we have \( |h_l| + |h_j| \leq 2\varepsilon \). In
particular $|h_j| \leq 2\varepsilon$. Indeed, let $x = 2^{-1}(e^*_i \pm e^*_j) \in C_n$. Then

$$1 + \varepsilon \geq \max \|f \pm h\|$$

$$\geq \max (f \pm h)(x)$$

$$\geq 2^{-1}(2 + |h| + |h_m|) = 1 + 2^{-1}(|h| + |h_j|),$$

so $|h| + |h_j| \leq 2\varepsilon$ and thus $|h_j| \leq 2\varepsilon$. Hence for every $2 \leq n \leq N/2 + 1$ and every $x = 2^{-1}(e^*_i \pm e^*_m) \in C_n$ we have

$$|h(x)| \leq 2^{-1}(|h| + |h_m|) \leq 2\varepsilon,$$

and for $x = k^{-1}e^*_j$ where $j \in E_n$ we have

$$|h(x)| \leq 2\varepsilon k^{-1}.$$

Thus we have three possibilities;

(a) there exists $y \in A_N$ such that $|h(y)| = |N^{-1} \sum_{j \in A} h_j| = 1$, or

(b) there exists $y \in B$ such that $|h(y)| = |h_j|/k = 1$, or

(c) there exists $n \geq N/2 + 2$ and $y \in C_n$ for which $h(y) = 2^{-1}(|h| + |h_m|) = 1$.

Case (a). We must have $|N^{-1} \sum_{j \in A \setminus E} h_j| \geq 1 - \varepsilon$. Since $|A \setminus E| \geq (2k - 1)N/2$, there must exist a subset $A_1 \subset A \setminus E$ with $|A_1| = kN - N/2 = (2k - 1)N/2$ such that $|N^{-1} \sum_{j \in A_1} h_j| \geq 1 - \varepsilon - 1/(2k - 1)$. Put $A_2 = E \cup A_1$ and $x = N^{-1} \sum_{j \in A_2} e^*_j$. Then

$$1 + \varepsilon \geq \|f \pm h\|$$

$$\geq (f \pm h)(x)$$

$$\geq N^{-1} \max \left\{ \sum_{j \in A_2} f_j + h_j, \sum_{j \in A_2} f_j - h_j \right\}$$

$$= N^{-1} \max \left\{ \sum_{j \in E} f_j \pm \left( \sum_{j \in E} h_j + \sum_{j \in A_1} h_j \right) \right\}$$

$$\geq \max \left\{ 1 \pm N^{-1} \left( \sum_{j \in E} h_j + \sum_{j \in A_1} h_j \right) \right\}$$

$$\geq 1 - \varepsilon + 1 - \varepsilon - 1/(2k - 1),$$

which implies that $\varepsilon \geq 2/9 > 1/3N$, a contradiction.

Case (b). This is similar to Case (c).

Case (c). Either $|h| \geq 1$ or $|h_m| \geq 1$. Assume $|h| \geq 1$. Put $E_0 = E \cup \{1\}$ and let $E_0 \subset N \setminus E_t$ such that $A := E_t \cup E_0$ satisfies $|A| =
\( kN \) and \( N^{-1} \sum_{j \in E_0} |h_j| < \varepsilon \). (This is possible as \( h \in c_0 \)). For \( x := N^{-1} \sum_{j \in A} e_j^* \in A_N \) we get

\[
1 + \varepsilon \geq \|f \pm h\|
\geq (f \pm h)(x)
\]

\[
= N^{-1} \max \left\{ \sum_{j \in A} f_j + h_j, \sum_{j \in A} f_j - h_j \right\}
\]

\[
= N^{-1} \max \left\{ \sum_{j \in E} f_j \pm \left\{ \sum_{j \in E} h_j + h_l + \sum_{j \in E_0} h_j \right\} \right\}
\]

\[
= N^{-1} \max \left\{ N \pm \left( \sum_{j \in E} h_j + h_l + \sum_{j \in E_0} h_j \right) \right\}
\]

\[
= \max \left\{ 1 \pm N^{-1} \left( \sum_{j \in E} h_j + h_l + \sum_{j \in E_0} h_j \right) \right\}
\]

\[
\geq 1 - \varepsilon + N^{-1} - \varepsilon = 1 + N^{-1} - 2\varepsilon,
\]

which implies that \( \varepsilon \geq 1/3N \), a contradiction. \( \square \)

Let \( X := c_0(X_N) \) where the \( N \)s are even numbers. Then \( X^{**} = \ell_\infty(X_N^*) \). Let \( P\{N\} : X^{**} \to X^{**} \) be the natural projection onto \( X_N^{**} \). Now we can finally prove our main result.

**Theorem 3.4.** There exists a bidual renorming of \( \ell_\infty \) which is ASQ.

**Proof.** The space \( X^{**} \) satisfies the demands. Indeed, for the ASQ part let \( \varepsilon > 0 \) and \( (f^i)_{i=1}^K \subset S_{X^{**}} \). Choose \( M > 1/\varepsilon \). From Lemma 3.3 there exists for all \( i = 1, \ldots, K \) an \( h \in S_{X_M} \) such that \( \|P\{M\}f^i \pm h\|_M \leq 1 + 1/M \). Thus

\[
\|f^i \pm h\| = \max \left\{ \sup_{N \neq M} \|P\{N\}f^i\|_N, \|P\{M\}f^i \pm h\|_M \right\} \leq 1 + \frac{1}{M} < 1 + \varepsilon,
\]

as wanted.

That \( X^{**} \) is isomorphic to \( \ell_\infty \), follows since \( X \) is isomorphic to \( c_0 \). Indeed, define an operator \( T : X \to c_0 \) in the following way. For \( x = (x_N) \in X \) let

\[
T(x) = (x_1(1), x_2(1), x_1(2), x_2(2), x_3(1), \ldots).
\]

Clearly \( T \) is a linear isomorphism since all the \( \|\cdot\|_N \) norms are equivalent with the same constants. \( \square \)

**Remark 3.5.** From the construction we see that there exists a bidual ASQ Banach space which is isomorphic to \( \ell_\infty \) such that \( k \) in (10) is \( k = 2 \). Whether this constant is best possible we do not know.

**Corollary 3.6.** The space \( X \) is ASQ, but not M-embedded.
Proof. That $X$ is ASQ follows from the proof of Theorem 3.4 (or one can simply use that a $c_0$ sum of Banach spaces is always ASQ). Moreover, all subspaces of an $M$-embedded space are $M$-embedded [HWW93, III.1. Theorem 1.6]. Since by Lemma 3.3 not all component spaces $X_N$ are LASQ, and thus not $M$-embedded [ALL16, Theorem 4.2], $X$ is not $M$-embedded. □

Let us also note that a similar argument to the one given in the proof of Lemma 3.3 shows that the first part of the conclusion of that lemma also holds for any given $N \in \mathbb{N}$ and any $(f_i)_{i=1}^K \subset S_{F_N}$ where $F_N := (\ell_\infty^m, \| \cdot \|_N)$, provided the dimension $m = m(N)$ of $F_N$ is sufficiently big. Note that every space $F_N$ is isomorphic to $\ell_\infty^m$ with fixed constants $k^{-1}$ and $k$ (see (10)). Now if we let $Y = c_0(F_N)$ and use that the property of being $M$-embedded is preserved by taking $c_0$-sums [HWW93, III.1. Theorem 1.6], and otherwise argue as in the proof of Theorem 3.4 we get that $Y$ satisfies the demands in

**Theorem 3.7.** There exists a renorming of $c_0$ which is $M$-embedded and whose bidual is ASQ.

As consequences we now get the following renorming results.

**Theorem 3.8.** A dual Banach space admits a dual ASQ norm if and only if it contains an isomorphic copy of $c_0$.

Proof. If $(Y^*, \| \cdot \|)$ contains $c_0$, then $Y$ contains a complemented subspace $Z$ isomorphic to $\ell_1$ which in turn is isomorphic to $X^*$ where $X = c_0(X_N)$. Let $W$ be the complement of $Z$ in $Y$. Then $Y$ is isomorphic to $W \oplus_1 X^*$. Hence $Y^*$ is isomorphic to $(W \oplus_1 X^*)^* = W^* \oplus_\infty X^{**}$ which clearly is ASQ as $X^{**}$ is. □

This improves [BGLPRZ16, Theorem 2.5] which says that the result above holds for ASQ norms (not necessarily dual norms).

**Theorem 3.9.** Every separable Banach space which contains $c_0$ admits an equivalent norm such that its bidual (and thus the space itself) is ASQ.

Proof. Let $Y$ be a separable Banach space which contains $c_0$. By Sobczyk’s theorem there exists a complemented subspace $V$ of $Y$ which is isomorphic to $c_0$. Let $Z$ be the complement of $V$. Then $Y = Z \oplus V \simeq Z \oplus_\infty c_0 \simeq Z \oplus_\infty X$, where $X = c_0(X_N)$. Thus $X^{**} \simeq (Y \oplus_\infty X)^{**} = Y^{**} \oplus_\infty X^{**}$, which is ASQ. □

**Remark 3.10.** From Theorem 3.9 we have in particular that every separable ASQ space admits an equivalent norm for which the bidual is ASQ. This result is in analogy with the recent result from [LLP] (partly
answering a question of Godefroy) that every separable octahedral Banach spaces admits an equivalent norm for which the bidual is octahedral (see Section 4 for the definition of an octahedral space).

4. A characterization of separable real ASQ spaces

Let us end the paper with a characterization of separable real ASQ spaces. The characterization is in the same vein as following result of Godefroy from [God89, p. 12].

**Theorem 4.1.** Let $X$ be a separable Banach space. Then the following are equivalent.

(a) $X$ is octahedral.

(b) There exists $h \in X^{**} \setminus \{0\}$ such that for all $x \in X$ we have

$$\|x + h\| = \|x\| + \|h\|.$$ 

Recall that a Banach space is octahedral if for every for $\varepsilon > 0$ and every finite set $(x_i)_{i=1}^n \subset S_X$ there is $h \in S_X$ such that $\|x_i \pm h\| \geq 2 - \varepsilon$ for every $i = 1, \ldots, n$.

**Proposition 4.2.** Let $X$ be a separable real Banach space. Then the following are equivalent.

(a) $X$ is ASQ.

(b) There exists $h \in X^{****} \setminus \{0\}$ such that for all $x \in X$

$$\|x + h\| = \max\{\|x\|, \|h\|\}.$$ 

**Remark 4.3.** Statement (b) says that $X$ is an M-summand in the subspace $\text{span}(X, \{h\})$ in the fourth dual $X^{****}$ of $X$ (see [HWW93] for a definition of an M-summand). Under the assumption that $X$ is ASQ, it is generally not possible to find non-trivial subspaces in the second dual $X^{**}$ in which $X$ is an M-summand. Indeed, if this was possible it would follow that there exists $h \in X^{**} \setminus \{0\}$ such that $\|h + x\| = \|h - x\|$ for all $x \in X$. But this yields that $X$ contains $\ell_1$ by a result of Maurey [Mau83, Theorem]. Thus $X = c_0$ is a concrete example where $X$ is ASQ and where (b) with $h \in X^{**} \setminus \{0\}$ fails. The upshot of Proposition 4.2 is, however, that for a real separable ASQ space $X$, there is always enough room to find subspaces in the fourth dual in which $X$ is an M-summand.

To prove Proposition 4.2 we need a lemma.

**Lemma 4.4.** In a separable ASQ space there exists a sequence $(h_n) \subset S_X$ such that for all $x \in X$

$$\lim_n \|x + h_n\| = \max\{\|x\|, 1\}.$$
Proof. Let \((x_i)_{i=1}^{\infty}\) be a dense subset of \(S_X\) and \((\varepsilon_n)\) a decreasing 0 sequence of positive numbers. Since \(X\) is ASQ we can for each \(n \in \mathbb{N}\) find \(h_n \in S_X\) such that
\[
\|x_i \pm h_n\| - 1 < \varepsilon_n \text{ for every } i = 1, \ldots, n.
\]
Then for all \(x_i\) we have
\[
\|x_i \pm h_n\| \to 1.
\]
Now let \(x \in S_X\) arbitrarily and let \(\varepsilon > 0\). Find \(x_i\) such that \(\|x - x_i\| < \varepsilon/2\) and \(N \in \mathbb{N}\) (using (11)) such that \(\varepsilon_n < \varepsilon/2\) for all \(n \geq N\). Then for all \(n \geq N\) we have
\[
\|x \pm h_n\| \leq \|x - x_i\| + \|x_i - h_n\| < 1 + \varepsilon \text{ and }
\|x \pm h_n\| \geq \|x_i - h_n\| - \|x - x_i\| > 1 - \varepsilon,
\]
and we are done. \(\square\)

Proof of Proposition 4.2. (a) \(\Rightarrow\) (b). Let \((h_n) \subset S_X\) be the sequence from Proposition 4.4. Note that the function \(\tau : X \to \mathbb{R}\) given by
\[
\tau(x) = \lim_n \|x + h_n\|
\]
defines a type with the property that \(\tau(x) = \max\{\|x\|, 1\}\). Using this fact it is straightforward to show that \(\tau\) is a \(c_{0+}\)-type, i.e.
\[
\tau(x) = \lim_m \lim_n \|x + ah_m + bh_n\|
\]
whenever \(a, b \geq 0\) and \(\max\{|a|, |b|\} = 1\). (Actually it holds for any \(a, b \in \mathbb{R}\) with \(\max\{|a|, |b|\} = 1\) so \(\tau\) is an \(\ell_\infty\)-type.) Thus by [Far88, Proposition 2.10], there exists \(h \in X^{****}\) such that for all \(x \in X\)
\[
\tau(x) = \|x + h\|.
\]
It follows that for all \(x\) in \(X\) we have \(\|x + h\| = \max\{\|x\|, 1\}\).

(b) \(\Rightarrow\) (a). This is straightforward using the Principle of Local Reflexivity twice. \(\square\)

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