The Astrophysical Journal, 899:31 (11pp), 2020 August 10
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The Power Spectra of Polarized, Dusty Filaments
Kevin M. Huffenberger1, Aditya Rotti2, and David C. Collins1
1 Department of Physics, Florida State University, Tallahassee, FL 32306, USA
2 Jodrell Bank Centre for Astrophysics, School of Physics and Astronomy, The University of Manchester, Manchester M13 9PL, UK
Received 2019 August 13; revised 2020 May 29; accepted 2020 June 16; published 2020 August 10

Abstract

We develop an analytic model for the power spectra of polarized filamentary structures as a way to study the Galactic polarization foreground to the cosmic microwave background. Our approach is akin to the cosmological halo-model framework, and reproduces the main features of the Planck 353 GHz power spectra. We model the foreground as randomly oriented, three-dimensional, spheroidal filaments, accounting for their projection onto the sky. The main tunable parameters are the distribution of filament sizes, the physical aspect ratio of filaments, and the dispersion of the filament axis around the local magnetic field direction. The abundance and properties of filaments as a function of size determine the slopes of the foreground power spectra, as we show via scaling arguments. The filament aspect ratio determines the ratio of $B$-mode power to $E$-mode power, and specifically reproduces the Planck-observed dust ratio of one-half when the short axis is roughly one-fourth the length of the long axis. Filament misalignment to the local magnetic field determines the $TE$ cross-correlation, and to reproduce Planck measurements we need a (three-dimensional) misalignment angle with an rms dispersion of about 50$^\circ$. These parameters are not sensitive to the particular filament density profile. By artificially skewing the distribution of the misalignment angle, this model can reproduce the Planck-observed (and parity-violating) $TB$ correlation. The skewing of the misalignment angle necessary to explain $TB$ will cause a yet-unobserved, positive $EB$ dust correlation, a possible target for future experiments.

Unified Astronomy Thesaurus concepts: Cosmic microwave background radiation (322); Interstellar medium (847)

1. Introduction

Polarized Galactic microwave emission poses a challenge to the search for primordial $B$-mode polarization in the cosmic microwave background (CMB). The $B$-mode signal could provide direct constraints on the energy scale of inflation, but the Milky Way foregrounds may outshine it at all frequencies, everywhere on the sky (Abazajian et al. 2016; Hervías-Caimapo et al. 2016; Planck Collaboration et al. 2016a; Thorne et al. 2017; Remazeilles et al. 2018; Hanany et al. 2019). We have a limited understanding of this foreground, and we must learn more to ensure the reliability of future $B$-mode measurements.

The foreground emission involves the turbulent interplay of gas, dust, and magnetic fields in the Galaxy’s interstellar medium (ISM). The magnetic fields organize the flow and control the orientation of grains that give rise to the polarized dust signal. Although Planck’s 353 GHz polarization channel has given us a first look, several features of the dust polarization remain without physical explanations.

For example, the amplitude of dust polarization $B$-mode power is approximately half of $E$-mode power, $A_{BB}/A_{EE} = 0.53 \pm 0.01$, when fit on a large portion of the sky (for $f_{sky} = 0.52-0.71$). Smaller patches also show the same mean value $A_{BB}/A_{EE} = 0.51$, with small patch-to-patch dispersion $\sigma_{BB}/\sigma_{EE} = 0.18$ (Planck Collaboration et al. 2016a, 2018). This observation defied pre-Planck expectations. Random polarization orientations, and coherent orientations overlying random polarization intensity fluctuations, both yield equal amounts of $E$ and $B$ (Zaldarriaga 2001; Kamionkowski & Kovetz 2014).

We have some understanding of dust physics and its relationship to polarization modes. The amplitude and orientation of the dust signal are set by the integrated column density and magnetic field orientation. For $E$ to have more power than $B$ means qualitatively that density fluctuations (structures in the ISM density field) must prefer orientations parallel or perpendicular to the local magnetic field (Rotti & Huffenberger 2019). This picture is borne out by measurements of the magnetic field orientation in individual, bright, filamentary structures in the Planck 353 GHz data (Planck Collaboration et al. 2016b). This is further validated by the observations that linear structures in neutral hydrogen emission, highlighted by a rolling Hough transformation, also correlate with the magnetic field direction indicated by Planck dust polarization (Clark et al. 2014, 2015).

Other aspects of the dust polarization also need physical explanations. Both $E$-mode and $B$-mode spectra follow power laws ($C_l \propto l^\alpha$), with approximately the same slope, $\alpha_{BB} = -2.42 \pm 0.02$ and $\alpha_{EE} = -2.45 \pm 0.03$ (Planck Collaboration et al. 2016a, 2018). There is a positive correlation between dust intensity and $E$-mode polarization (noted by Caldwell et al. 2017), with correlation coefficient $r_{TE} = 0.357 \pm 0.003$ (Planck Collaboration et al. 2018) and significant scatter depending on the sky area but little evidence for scale dependence. Perhaps more intriguing is a parity-violating, positive $TB$ correlation (Planck Collaboration et al. 2018). Finally, the amplitude of dust polarization power correlates to intensity in patches, roughly as $(I)^{0.9}$ patch, for both $E$ and $B$ (Planck Collaboration et al. 2016a).

A few works have already tried to address these observations. Caldwell et al. (2017) examined the dust polarization power spectra of slow, fast, and Alfven MHD waves in terms of two parameters: the ratio of gas to magnetic pressure, and the anisotropy of the MHD modes around the background field direction. They found two regions of parameter space that can account for the $E$ to $B$ ratio and positive $TE$ correlation but judged that these scenarios are unlikely due to the uniformity of...
the polarization power spectrum across the sky, and instead suggested that Planck may be seeing large-scale displacements that are driving the turbulent ISM, rather than the turbulence itself.

On the other hand, Kandel et al. (2017) argued with a similar analysis that the observed $E/B$ power ratio can be realized in an analytic MHD model, as long as the turbulent flow is sub-Alfvénic. Kandel et al. (2018) extended this analysis to examine the $TE$ correlation and synchrotron emission. Such an MHD analysis assumes that the density is passively advected by the turbulent motions, an assumption that may be violated for the cold phase of the ISM. The formation of filaments in the multiphase ISM (per Xu et al. 2019) could also affect this MHD analysis.

Other works approach the problem using MHD simulations. For the most part, the ISM is filled with transonic and supersonic flows, which are nonlinear (e.g., Elmegreen & Scalo 2004; Burkhardt et al. 2010). Both Kritsuk et al. (2018) and Kim et al. (2019) made MHD simulations of the ISM, and modeled the dust polarization signals. Both works find slopes and power ratios that are reasonably close to the observed values, but the slopes are especially sensitive to the masking procedure. What MHD simulations do not provide is a straightforward and direct way to understand why these polarization properties arise.

Here we seek to gain physical intuition with very simple models of polarized filaments. We do not yet know the degree to which filamentary structures contribute to the polarization foreground. There is certainly evidence for filamentary structure in H i and CMB data (Clark et al. 2014; Planck Collaboration et al. 2016b) as well as in simulations of the interstellar medium (de Avillez & Breitschwerdt 2005; Hennebelle 2013). Using the H i morphology alone, Clark & Hensley (2019) were able to reproduce the filament signal and the statistics of the polarized sky, which indicates that filaments may be a significant contribution. However, apparent filaments found in H i velocity channel maps may not be coherent density structures (physical filaments). Instead they could be coherent velocity structures without the need for associated density enhancements (Yuen et al. 2019).

Considering this second viewpoint, Hu et al. (2020) used a combination of velocity gradients and principal component analysis to predict polarization properties of the ISM, and also reproduced the microwave sky to high correlation.

Filamentary structure is a natural consequence of strongly magnetized turbulence (Goldreich & Sridhar 1995; Kritsuk & Norman 2004; Heiles & Havercorn 2012; Hennebelle 2013; Micelotta et al. 2018; Beattie et al. 2019; Ossenkopf-Okada & Stepanov 2019; Xu et al. 2019). The distribution of filament properties is a non-trivial function of the state of the gas. The properties of velocity and magnetic field fluctuations in MHD turbulence are better understood (e.g., Brandenburg & Lazarian 2013) than the statistics of density, particularly for turbulence with high sonic Mach number. For subsonic and transonic gas, eddies are elongated in the direction of the local mean magnetic field (Goldreich & Sridhar 1995; Xu et al. 2019), while for supersonic gas, shocks form perpendicular to the local mean field (Soler et al. 2013; Beattie et al. 2019). The nearby ISM is certainly a mixture of components (McKee & Ostriker 1977; Vazquez-Semadeni 2009; Heiles & Havercorn 2012), but typically seems to have low velocity relative to the speed of sound for the majority of the nearby high-latitude gas (Redfield & Linsky 2004; McClure-Griffiths et al. 2006; Kalberla et al. 2016; Clark & Hensley 2019; Skalidis & Pelgrims 2019). Our model depends on the distribution of filament properties and their alignment with the field, and thus will depend on the properties of the kinematics of the gas. The exact details of this relationship are the focus of ongoing research.

The purpose of this paper is to explore what the observed power spectra can tell us about their physical properties. We compute the temperature and polarization power spectra of filaments using a method akin to the cosmological halo model (e.g., Seljak 2000; Cooray & Sheth 2002). However, instead of spherical halos, we use magnetized, prolate-spheroidal filaments as the basic ingredients, and integrate over their population. Because turbulence is not well understood in the multiphase ISM, we do not start with many prior constraints on the statistics of the filament parameters.

We organize this paper so that in Section 2 we describe our formalism for characterizing the filament signal and for computing the power spectra. In Section 3, we show the power spectra and discuss how the parameters of the filament population affect them. In Section 4, we conclude and discuss the implications and possible future directions. An Appendix describes how the distributions of filament and magnetic field orientations in three dimensions appear when projected onto the plane of the sky.

2. Method

We define a projected filament profile, $f(x)$, upon which we paint the temperature (i.e., intensity) and polarization signals. Thus the temperature profile is

$$T(x) = T_0 f(x),$$

for sky position $x$. We model the polarization with an overall polarization fraction and polarization direction. In terms of the Stokes parameters, the polarization for a filament is

$$X(x) = (Q + iU)(x) = f_{\text{pol}} \exp(2i\psi_{\text{pol}}) T_0 f(x).$$

In the HEALPix polarization convention ( Görski et al. 2005), the $+x$-axis points south and the polarization angle $\psi_{\text{pol}}$ increases east of south. Because we will integrate over angles in our computation of the power spectrum (and because $E$ and $B$ fields are coordinate-independent) we can analyze a filament that has its long axis aligned (in projection) with the $x$-axis without loss of generality. For simplicity, we assume that the intrinsic, microphysical contribution to $f_{\text{pol}}$ is common to all filaments, although we will account for geometrical and projection effects in this work. If the long axis of that filament were aligned with the local magnetic field, the precession of the dust grains would cause the angle to be $\psi_{\text{pol}} = 90^\circ$, perpendicular to the filament axis.

Working in the flat sky approximation, the Fourier components of the scalar polarized modes are

$$(E + iB)(\ell) = \exp(-2i\phi)[f_{\text{pol}}] \times X(\ell).$$

Figure 1 shows the Stokes $T$, $Q$, $U$ and scalar $E$, $B$ quantities on the sky for sample north–south filaments with $\psi_{\text{pol}} = 90^\circ$, so the magnetic field is parallel to the filament direction and the
polarization is perpendicular. (Our choice of coordinates implies that Stokes $U$ is zero in these cases.) Rotti & Huffenberger (2019) pointed out that, when the magnetic field aligns with the filament direction, the real-space kernels for the $E/B$ signals show immediately that the $E$-type polarization is positive along the filament, regardless of its orientation. Since the temperature signal is also strong there, such filaments naturally yield a strong and positive $TE$ cross-correlation, as observed in the Planck data. The same work showed that the $B$ signal is concentrated at the ends of the filament, so filaments with long and thin aspect ratios will have less $B$ power relative to $E$ power than more squat ones.

By parity symmetry, the $TB$ and $EB$ cross-correlations are zero when the polarization is perpendicular to the filament (i.e., $\psi_{\text{pol}} = 90^\circ$). In Figure 2 we show how the $E$ and $B$ patterns transform into each other (and change sign) as $\psi_{\text{pol}}$ varies away from $90^\circ$. In these cases, the $TB$ and $EB$ correlation can be non-zero for individual filaments, but as long as the average $\langle \psi_{\text{pol}} \rangle = 90^\circ$, there will be no overall cross-correlation for the whole population.

### 2.1. Projection on the Sky

We next discuss the projection of a three-dimensional filament onto the plane of the sky. Many important quantities depend on the angle to the line of sight of (1) the long axis of the filament ($\theta_L$) and (2) the magnetic field vector ($\theta_H$).

Another important quantity is the plane-of-sky projection of the angle between these vectors ($\psi_{LH}$), which controls the polarization angle and the amounts of $E/B$ polarization present. We depict these angles in Figure 3. If the magnetic field direction aligns somewhat with the filament direction, as is the case in strong-field MHD, all these angles will be correlated.

We assume that, on average, the filaments align with the local magnetic field. In the Appendix, we use simple geometry to compute the distribution of the magnetic field projection angle $\theta_H$ and relative orientation angle $\psi_{LH}$ as functions of $\theta_L$. We base the distribution on the assumption of a Gaussian distribution for the angle $\langle \theta_{LH} \rangle$ between the filament and the magnetic field in three dimensions, characterized by the dispersion $\text{rms}(\theta_{LH})$.

The field angle $\theta_H$ is correlated with $\psi_{LH}$, so our numerical procedure yields the tabulated joint distribution,

$$p(\psi_{LH}, \theta_H | \theta_L).$$

This distribution centers on aligned filaments ($\psi_{LH} = 0^\circ$, $\theta_H = \theta_L$), and the distribution for $\psi_{LH}$ broadens for filaments nearly along the line of sight. Its precise form is not vital for this discussion and is plotted in the Appendix in Figure A1.

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*Elsewhere in the ISM literature, the angles are often given with reference to the plane of the sky, e.g., $\gamma = 90^\circ - \theta_H$ and so on.*
On the other hand, the probability distribution for the line-of-sight angle of randomly oriented filaments is determined purely by geometry,
\[ p(\theta_L) = \sin \theta_L, \]
for \( \theta_L \in [0^\circ, 180^\circ] \).

These quantities relate immediately to the polarization. Although the dust polarization fraction depends on the microphysical details of the emission, it has a geometric dependence similar to \( f_{\text{pol}} \propto \sin^2 \theta_L \) (Fiege & Pudritz 2000). Meanwhile, the polarization angle for a filament projected along the x-axis is \( \psi_{\text{pol}} = \psi_{\text{LH}} + 90^\circ \).

We model the filament as a prolate spheroid, and label the major axis as \( L_a \) and the minor axis as \( L_b \). The axis ratio is thus \( \epsilon = L_b/L_a < 1 \). The column density (and therefore the surface brightness and ultimately the observed temperature perturbation) is proportional to the density and the line-of-sight distance through the filament, and so (approximately)
\[ T_0 \propto \rho_0 (L_a^2 \cos^2 \theta_L + L_b^2 \sin^2 \theta_L)^{1/2} \]
\[ \propto \rho_0 L_a (\cos^2 \theta_L + \epsilon^2 \sin^2 \theta_L)^{1/2}, \]
where \( \rho_0 \) is a characteristic density for the filament. So all else being equal, a filament that lies along the line of sight will have the greatest column density and the brightest temperature signal. On the other hand, \( f_{\text{pol}} \propto \sin^2 \theta_L \), so if the magnetic field lies along the line of sight, there is no polarization. The polarization fraction is maximum when the magnetic field is perpendicular to the line of sight.

Figure 4 describes the relationship between the line-of-sight angle, column density, and polarization fraction. It also shows that since the polarized amplitude depends on the product of these two, the filaments with the brightest polarization are inclined, but not perpendicular, to the line of sight. The polarization maximum depends on axis ratio through its impact on the column density. Nearly round filaments have the polarization maximum when oriented near 90° to the line of sight, while in the limit of thin filaments (\( \epsilon \to 0 \)) the orientation of polarization maximum approaches \( \theta_L = 45^\circ \) for perfect magnetic field alignment. If there is significant misalignment of the magnetic field and filament directions, the situation can become more complicated, depending on the particular combination of axis ratio and misalignment dispersion. In such cases, filaments along the line of sight can have significant polarization. Still, the typical line-of-sight orientation angle for maximum polarization, averaging over the magnetic field directions, is around \( \theta_L = 45^\circ \).

We compute the filament’s projected angular sizes along its two axes as if it were a cylinder. These depend on its distance \( R \) and are
\[
\Theta_a = (L_a^2 \sin^2 \theta_L + L_b^2 \cos^2 \theta_L)^{1/2}/R = (\sin^2 \theta_L + \epsilon^2 \cos^2 \theta_L)^{1/2} L_a/R,
\]
\[
\Theta_b = L_b/R = \epsilon L_a/R.
\]
Thus the projected axis ratio is
\[
\epsilon a = \Theta_b/\Theta_a = \frac{\epsilon}{\sin^2 \theta_L + \epsilon^2 \cos^2 \theta_L}^{1/2},
\]
which goes to unity for filaments along the line of sight, and to the true value (\( \epsilon \)) for filaments perpendicular to the line of sight.

### 2.2. Filaments in Fourier Space

For several terms in our power spectrum calculation, we need the Fourier transform of the projected filament profile:
\[ f(\ell) = \int d^2x \ f(x) \exp(-i\ell \cdot x). \]

Rather than project rays through a three-dimensional model to obtain the filament profile, we make a simplifying assumption for computational efficiency. From the size and orientation of a filament, we take the angular dimensions and compute under the assumption that the profile is a distortion from an axisymmetric function \( g \):
\[
f(x, y) = g(x/\Theta_a, y/\Theta_b) = g(x^\ast, y^\ast) = g(r)
\]
where \( \Theta_a, \Theta_b \) are the semimajor and semiminor axes of the elliptical distortion. As we stated, by convention and without loss of generality, we orient the long axis of the filament along the x-axis.

Then the transform of \( f \) is simply related to the transform of \( g \):
\[
f(\ell) = \Theta_a \Theta_b \int d^2x^\ast \ g(x^\ast) \exp(-i(\Theta_a \ell_x x^\ast + \Theta_b \ell_y y^\ast))
\]
\[= \Theta_a \Theta_b \ g(\ell^\ast)
\]
where \( \ell^\ast(\ell) = (\Theta_a^2 \ell_x^2 + \Theta_b^2 \ell_y^2)^{1/2} \) and
\[
g(\ell^\ast) = \int d^2x^\ast \ g(x^\ast) \exp(i\ell^\ast \cdot x^\ast)
\]
\[= 2\pi \int dr \ r \ g(r) J_0(\ell^\ast r) .
\]
The input profile \( g(x^*) \) is real and even, and so the Fourier transform is too. The power spectra we find are not very sensitive to the profile that we use. In this work we have used an exponential for the basic filament profile \( g(r) = \exp(-r) \), but we have checked our best-fitting power spectrum model with a Gaussian profile \( g(r) = \exp(-r^2/2) \) and a Plummer profile \( g(r) = (1 + r^2)^{-5/2} \), and find the same results.

### 2.3. Parameters and One-filament Term

For a parameterized set of filament properties,

\[
\alpha = (L_a, L_b, \psi_{LL}, \theta_L, \theta_H, R, \ldots),
\]

we can write the number density distribution \( n(\alpha) \), so that the average number of filaments in a realization of the sky is

\[
\langle N \rangle = \int d\alpha \, d\alpha \, n(\alpha)
\]

where the integral is over

\[
d\alpha = dL_a dL_b d\psi_{LL} d\theta_H d\theta_L dR.
\]

Expressed another way, \( n(\alpha) = \langle N \rangle p(\alpha) \), where the normalized probability distribution of the filament population is

\[
p(\alpha) = p(L_a, L_b) p(\psi_{LL}, \theta_L) p(\theta_H) p(R).
\]

This integral over the population is at least six-dimensional. For a screen at a distance \( R \), it is a five-dimensional integral. Since the angular power spectrum for foregrounds is a power law, if we can reproduce it on a single screen, putting that screen at different distances will maintain the same power spectrum. If we further fix the physical aspect ratio of the filaments, it is a four-dimensional integral, over \( L_a, \psi_{LL}, \theta_L, \theta_H \). (The projected aspect ratio will still vary with the line-of-sight angle \( \theta_L \).)

The contributions to the power spectrum from filaments correlated with themselves are

\[
C_{TT}^T = \frac{1}{2\pi} \int d\phi \int d\alpha \, n(\alpha) \left| T(\ell, \alpha) \right|^2,
\]

\[
C_{EE}^E = \frac{1}{2\pi} \int d\phi \int d\alpha \, n(\alpha) \left| E(\ell, \alpha) \right|^2,
\]

\[
C_{BB}^B = \frac{1}{2\pi} \int d\phi \int d\alpha \, n(\alpha) \left| B(\ell, \alpha) \right|^2,
\]

\[
C_{TE}^T = \frac{1}{2\pi} \int d\phi \int d\alpha \, n(\alpha) \left| T(\ell, \alpha) E(\ell, \alpha) \right|.
\]

Similar expressions hold for the other cross-correlations, but these vanish if the orientations of the filaments are random. These computations of power spectra are directly analogous to the one-halo term in the cosmological halo model (Seljak 2000).

### 3. Results

There are clear relationships between the physical properties of the filaments and the temperature and polarization power spectra that they produce. The slopes of the power spectra are determined primarily by the size distribution of filaments, with other effects responsible for the smaller differences between the components. The ratio of \( BB/EE \) power is determined mostly by the aspect ratio of the filaments and somewhat by the misalignment of the filament directions to the background magnetic field. These same factors also determine the cross-correlation \( TT \), but here misalignment is much more important. They also affect the \( TE/EE \) power ratio, but this quantity is more directly affected by the overall polarization fraction.

#### 3.1. Power Spectrum Shape

We can relate the slope of a power-law spectrum to scaling relations for parameters in the filament profiles. This allows us to place constraints on the distribution of filament sizes and the scaling of other parameters. For a generic parameter \( \alpha_0 \), if the filament’s contribution to the power spectrum scales as

\[
C_\ell \propto \int d\alpha_0 \, n(\alpha_0) \times \alpha_0^q F(\alpha_0 \ell)
\]

for any function \( F \), and furthermore if the weighting distribution for the parameter is a power law, \( n(\alpha_0) \propto \alpha_0^p \), then we can rescale the integration with a straightforward substitution, \( u = \alpha_0 \ell^{p+q+1}/r \);

\[
C_\ell \propto \ell^{-(p+q+1)/r} \times \int d\ell \, (\alpha_0 \ell^{p+1}/r) \times (\alpha_0 \ell^{p+1}/r)^q \times \left( (\alpha_0 \ell^{p+1}/r)^q \right)
\]

\[
\propto \ell^{-(p+q+1)/r} \times \int du \, u^{p+q} F(u) du.
\]

The integral no longer has any multipole dependence and evaluates to some constant value, whatever the details of \( F \). Thus we are left with a power-law power spectrum with

\[
C_\ell \propto \ell^{-(p+q+1)/r}.
\]

This argument holds not just for filaments, but for any signal with a power-law power spectrum that is built from a set of objects that are similarly related to each other, so long as they are weighted by power-law scalings and distributions. So if we observe a power-law spectrum with \( C_\ell \propto \ell^p \), it implies that the parameter distribution’s index is \( p = -r s - q - 1 \), regardless of the objects’ profiles.

We walk through this scaling argument for a simple (and unrealistic) case—with plane-of-sky filaments with identical surface brightnesses \( (T_0) \) is the same for all filaments) and a constant projected axis ratio \( (\Theta_a = \Theta_a) \) and analyze the distribution for \( \Theta_a \), the angular size of filaments. For the filament Fourier transform, we have \( f(\ell) \propto \Theta_a^2 g(\ell^2) \) with \( \ell^2 \propto \Theta_a \). The contribution to the power spectrum is proportional to \( f^2 \), so comparing the scaling for angular size parameter \( \Theta_a \) (Equation (17)), we find \( q = 4 \) and \( r = 1 \). In the polarization case, to reproduce \( C_\ell \propto \ell^{-2.4} \) (meaning \( s = -2.4 \)), the number density distribution of such objects on the sky must approximately scale like \( n(\Theta_a) \propto \Theta_a^p \) where \( p = 2.4 - 4 - 1 = -2.6 \). Indeed this yields the desired slope of the power spectrum when calculated in our model.

In a more realistic case, with three-dimensional filaments, we can make a similar argument to deduce the distribution of filament lengths. Surface brightness depends on column density, which is proportional to length (after integrating out any distribution of axis ratios—compare Equation (6)—and assuming a density normalization independent of length). The solid angle scales like length squared. After squaring those...
three powers during the computation of the power spectrum, the overall scaling is $q = 6$. The multipole $\ell$ scaling should also go like length, so $r = 1$, the same as the plane-of-sky case above. So with no other dependence on length, we should have distribution of lengths $n(L_o) \propto L_o^p$ where $p = 2.4 - 6 - 1 = -4.6$.

We have verified that this distribution produces the proper slope in Figure 5. All the temperature and polarization spectra have the specified slope in common. The complications of the modeling of the three-dimensional orientation are not important to the slope, only the weighting and distribution of filament size.

Other than the slope, there are no clear features in the Planck-measured spectra. We note that features in the distribution of filament sizes would break the power-law behavior of the resulting spectra. For example, if we impose a maximum filament size (semi-transparent lines in Figure 5), it causes the low-$\ell$ behavior of $C_\ell$ to deviate: at scales much larger than the filament, the temperature spectrum adopts the flat, white, Poisson spectrum of point sources. The $EE$ and $BB$ spectra flatten the same way, and on scales large compared to the filaments, the aspect ratios of the filaments become unimportant and the powers in $EE$ and $BB$ equalize. The $TE$ cross-correlation falls off at large scales, possibly because the positive and negative contributions to $E$ are being averaged over. If, on the other hand, we impose a minimum filament size, it causes the high-$\ell$ behavior to deviate: the spectrum will drop off with increasing $\ell$, where there is no more contribution to the power.

In yet more realistic cases, we can make the slopes of all the temperature and polarization spectra differ. For example, this can happen if there is another effect that changes the size scaling of the polarization relative to the temperature. For example, to make polarization slope shallower than the temperature slope, one could make smaller objects more polarized than large ones, or the aspect ratio of filaments could change as a function of size.

Considering the Planck data, it is not immediately clear what conclusion to draw. Planck Collaboration et al. (2018) quote $EE$, $BB$, $TE$ slopes for the dust foreground, but not the $TT$ slope. Using our own tools, we have computed the $TT$ power spectrum based on the Planck data, and find a $TT$ slope that is about $-2.6$, somewhat steeper than the polarization spectra at $-2.4$ to $-2.5$. Like Planck Collaboration et al. (2018), for the mask we used the LR71 polarization mask supplemented with a point-source mask (based on intensity maps), resulting in a mask with effective $f_{sky} \approx 0.6$. We can approximately reproduce a temperature slope of $-2.6$ and a polarization slope of $-2.4$ with $n(L_o) \propto L_o^{-4.4}$ and $f_{pol} \propto L_o^{-0.08}$. This argues that in the unmasked region, the polarization is higher in smaller filaments.

However, this conclusion may not be correct because it depends sensitively on the mask. We reasoned that small filaments, oriented along the line of sight, might look like a point source and be included in the masked area. These end-on filaments could also have low polarization (note Figure 4), and excluding them might not have much effect on the polarization results. Thus for comparison, we recomputed the $TT$ spectrum with different masks. When we use only the polarization LR71 mask without removing the additional point sources from the intensity map, we get a shallower $TT$ spectrum with slope $-2.5$. When we use a mask that keeps the same large-scale features but does not mask any point sources (Planck’s publicly available GAL70 mask) we find a $TT$ slope of $-2.1$, notably shallower than the polarization spectra. The polarization spectra change somewhat between these masks, but the changes in the polarization slopes are small compared to the change in the $TT$ slopes. Some of the masked sources are extragalactic, so this slope with all point sources unmasked is probably too shallow to describe the ISM component, but can serves as a bound. The upshot is that we are not certain whether the spectrum for all filaments is steeper or shallower in temperature than polarization, and so it is difficult to draw conclusions on the size dependence of the polarization fraction.

Another feature of the Planck data is the differing slopes in $E$ and $B$. We can reproduce this feature by varying the aspect ratio as a function of filament size. For the LR71 mask in Planck Collaboration et al. (2018), the slopes for $(BB, EE, TE)$ are roughly $(-2.5, -2.4, -2.5)$ and we found a $TT$ slope of $-2.6$. So $BB$ is steeper than $EE$, which should happen if smaller filaments are proportionally thinner than longer ones. Modifying the aspect ratio in this way also affects the $TT$ slope, breaking the simple relation that we saw earlier in this section. By trial and error, we found that this set of slopes are approximately reproduced with the following parameter dependence: $e \propto L_o^{-0.1}$, $n(L_o) \propto L_o^{-4.45}$, and $f_{pol} \propto L_o^{-0.1}$. Here we are simply exploring what is possible, but the relationship between the measured slopes and these filament parameters should be made more systematic and quantitative.

In light of these complications and uncertainties, in what follows we keep a common slope of $-2.4$ for all the temperature and polarization components while we explore their other parameter dependencies.

### 3.2. BB/EE Power Ratio

The aspect ratio of the filaments is the major factor determining the ratio of $B$-mode power to $E$-mode power. In Figure 6, we plot
the power ratio against the aspect ratio for varying degrees of filament–magnetic field misalignment. To reproduce the Planck-observed ratio of \(\sim 0.5\), filaments need to have an aspect ratio \(\epsilon\) slightly less than 0.26, so filaments must be slightly less than four times longer than they are wide. If the model deviates too much from this ratio, the required magnetic field misalignment is made so large that the model has trouble fitting the TE correlation.

It is difficult to compare this result quantitatively to the aspect ratios of observed filaments from the literature without making a detailed accounting of the filament selection function. Projection effects will also tend to lower observed aspect ratios. The stacked filaments in Figure 7 of Planck Collaboration et al. (2016b) appear to have axis ratios not so far from what we are finding here. The filaments identified by the rolling Hough transformation in Clark et al. (2014) on H I maps tend to be longer and thinner than this.

### 3.3. TE Cross-correlation

In the context of the filament model, we find that the level of the TE cross-correlation implies that filaments cannot be precisely aligned to their local magnetic field direction. The correlation coefficient is defined as

\[
r_{\ell}^{TE} = \frac{C_{\ell}^{TE}}{\sqrt{C_{\ell}^{EE} C_{\ell}^{EE}}},
\]

and perfect alignment of the filaments and the fields causes far too much TE correlation compared to the Planck observations.

The Planck dust data show \(r_{\ell}^{TE} \approx 0.35\) with little scale dependence (Planck Collaboration et al. 2018). Figure 7 shows that to match this, the field misalignment angle \(\theta_{LH}\) must have an rms dispersion of nearly 50°, while maintaining the axis ratio \(\epsilon \approx 0.26\) needed to reproduce the BB\(\rightarrow\)EE power ratio. If the misalignment dispersion is independent of filament size, as in our modeling, it causes no scale dependence; \(r_{\ell}^{TE}\) is constant.

Projection effects cause the distribution of the projected angle \(\psi_{LH}\) to have a positive kurtosis (see Appendix, Figure A3), and so we can describe its dispersion in a few ways. For the case with \(\mathrm{rms}(\theta_{LH}) = 50°\), 68\% of the probability is bounded by \(|\psi_{LH}| < 45°\). Alternatively, \(|\text{Var}(\psi_{LH})|^{1/2} = 48°\). For comparison, Planck Collaboration et al. (2016b) fit a Gaussian with 19° dispersion (1\() to the projected field–projected filament histogram of relative orientations for the filaments they found. Again, this comparison is not direct because of selection effects. Their Hessian-based selection of filaments would disfavor those with small projected aspect ratios (close to the line of sight), and such filaments can have the largest differences in the projected orientation.

The overall level of \(C_{\ell}^{TE}\) (and the polarization spectra) depends on the polarization fraction. The relative power ratio has a dependence according to

\[
C_{\ell}^{TE} / C_{\ell}^{EE} \propto \langle f_{\text{pol}} \rangle / \langle f_{\text{pol}}^2 \rangle,
\]

while for the cross-correlation it is

\[
r_{\ell}^{TE} \propto \langle f_{\text{pol}} \rangle / \langle f_{\text{pol}}^2 \rangle^{1/2}.
\]

Thus a purely multiplicative rescaling of the polarization fraction affects the ratios of the powers in TT\(\rightarrow\)TE/EE but not the correlation coefficient \(r_{\ell}^{TE}\).

To reproduce the Planck-measured ratio \(C_{\ell}^{TE} / C_{\ell}^{EE} \sim 2.7\) (in our case that already fits \(C_{\ell}^{BB} / C_{\ell}^{EE}\) and \(C_{\ell}^{TE}\)) requires \(f_{\text{pol}} = 0.15 \sin^2 \theta_H\). We have only modeled the amplitude of polarization fraction and the geometric dependence on the magnetic field orientation, but in addition, the polarization fraction depends on grain geometry and small-scale turbulence (Fiege & Pudritz 2000), and filaments need not in reality all have the same intrinsic polarization fraction.

Our other tests have shown that the TE correlations differ in their sensitivity to intrinsic dispersion in the polarization fraction. For example, the correlation \(r_{\ell}^{TE}\) is not very sensitive to the maximum polarization fraction, but the power ratio is very sensitive to it: decreasing the maximum polarization fraction decreases \(C_{\ell}^{TE}\) but decreases the denominator \(C_{\ell}^{EE}\) more, and so raises the ratio.
3.4. Parity Violation: TB and EB

One surprising finding in the dust spectra of Planck Collaboration et al. (2018) is a non-zero TB correlation, with $r_{TE}^{TB} \approx 0.05$. Like $r_{TE}$, the observed $r_{TB}$ correlation has little scale dependence (up to multipoles of several hundred).

Because non-zero TB and EB are parity-violating correlations, our model cannot reproduce them for randomly oriented filaments. To get a positive TB correlation we would need to favor polarization angles in the range $\psi_{Tb} \in [90^\circ, 180^\circ]$ relative to the filament direction (compare Figure 2).

Equivalently, this corresponds to projected field angles in the range $\psi_{LH} \in [0^\circ, 90^\circ]$. Such an effect may be due to some large-scale feature in the Galaxy’s magnetic field or differential gas flow (e.g., Planck Collaboration et al. 2018; Bracco et al. 2019).

We can determine how far away from random this correlation is by artificially weighting the distribution of the projected misalignment angles, favoring the $\psi_{LH} > 0$ portion of the distribution of $p(\psi_{LH}, \theta_l \theta_z)$ over the $\psi_{LH} < 0$ portion, while keeping the same functional form. We find that we can approximately reproduce the Planck-measured TB correlation by giving the preferred $\psi_{LH}$ directions about 55% of the total weight, rather than the 50% that comes naturally from randomly oriented filaments. Similar to the $r_{TE}$ correlation (also set by field–filament misalignment), this effect is not scale-dependent, and so $r_{TB}$ is constant to high $\ell$ in this model.

Our modeling comes from the one-halo term only, and shows that the TB correlation can be explained if filament orientations in projection are slightly twisted counterclockwise from the projected local magnetic field. Our model does not address the structure of that underlying field, but we may speculate that some differential, shearing hydrodynamic forcing could preferentially twist the filaments, according to our point of view, from the global mean field direction of the Milky Way.

Bracco et al. (2019) argue that the observed TE and TB correlations may be features of the large-scale structure of the Galactic magnetic field. They show that a helical component can create such correlations, but in their modeling the correlations show a strong scale dependence, with $r_{TE,TB}$ falling substantially already by $\ell = 22$. The Planck spectra have much flatter $r_{TE,TB}$ correlations, consistent with the filament modeling here.

Planck did not detect an EB correlation, but since $E$ and $B$ both have a factor of the polarization fraction, we would naturally expect this correlation to be smaller. It may be there, hidden in the noise. In the presence of a positive TB correlation, in the context of the filament modeling, we would expect a positive EB correlation too (including at high $\ell$), and it should be a target for future experiments. Both TB and EB dust correlations can potentially interfere with sky-calibration of the polarization angles of CMB instruments (Abitbol et al. 2016) or with CMB lensing reconstruction (e.g., Fantaye et al. 2012; Challinor et al. 2018).

4. Conclusions

We do not know how much of the dusty microwave polarization foreground is due to filamentary structure, but if it is a substantial portion, we can discern details of the filament population from the foreground power spectra. We showed that the slopes of the power spectra relate to the distribution of lengths. We showed that the BB/EE power ratio relates to the filament axis ratio. We showed that the TE cross-correlation relates to the axis ratio and the rms misalignment of filaments to the magnetic field. We showed that TB correlations could be caused by a slight preference for one handedness in the misalignment between the magnetic field and the filament orientation.

Despite its relative success in reproducing the features of the dust polarization power spectrum, this formalism lacks some essential features for modeling the real sky. Foremost, it begins with the assumption of a population of filaments, which is not itself on a firm physical basis, and which may not be a unique way to explain the features in the foreground polarization power spectra. Even in the context of filaments, this formalism includes only the one-filament term in the power spectra. This is obviously an approximation, for the Planck data have shown that the Galaxy’s projected magnetic field has coherent, large-scale features, and the HI-identified filaments are clearly correlated with it and with measurements of starlight polarization (Clark et al. 2015). On the other hand, in the halo model, the transition from one-halo-dominated to two-halo-dominated scales often leaves a mark on the power spectrum. Since there are no clear features in the dust polarization spectra, such as a break in the slope, we may speculate that the two-halo component may not be necessary to describe the main properties of the power spectra. Inclusion of a proper two-halo formalism is complicated by the correlated direction dependence of the filaments. We may be able to import some of the techniques developed to describe intrinsic alignments of galaxies (e.g., Schneider & Bridle 2010), since the mathematical description of the problem is similar.

We have not tried to systematically probe the parameter degeneracies or place proper uncertainties on any of the parameters of this filament model. We can do this straightforwardly by interfacing the model with a Monte Carlo Markov Chain and developing a likelihood based on the Planck dust spectra. We plan to pursue this in further work.

By looking at observations and simulations of the filamentary ISM, we could attempt to verify some of the population statistics for filaments. For example, we could compare the length distribution of actual or simulated filaments to that implied by the slope of the power spectra.

Because this filament model is non-Gaussian, we may be able to use it to design novel diagnostics to probe for residual foregrounds in surveys that aim for the primordial B-modes (in the spirit of Kamionkowski & Kovetz 2014; Rotti & Huffenberger 2016; Philcox et al. 2018; Coulton & Spergel 2019). Similarly, we could use this model to compute the four-point contributions to polarized CMB lensing estimators. This could help place constraints on potential foreground contamination. Such statistics may be sensitive to the internal density structure of the filaments in a way that the power spectrum is not.

Due to its flexibility, its ability to model the Planck dust polarization data, its ease of computation, and its straightforward interpretation, this filament model may become a useful tool in the study of CMB polarization foregrounds.

K.M.H., A.R., and D.C.C. acknowledge support from the NASA ATP program under grant NNX17AF87G. K.M.H. acknowledges support from the NSF AAG program under grant 1815887. A.R. acknowledges support from the European
Research Council (ERC) under the European Union’s Horizon 2020 research and innovation program (grant agreement No 725456, CMBSPEC). We thank J. Colin Hill, Susan Clark, Marc Kamionkowski, and Carlos Hervias-Caimapo for useful conversations. We thank François Boulanger for providing access to masks used in the analysis of Planck Collaboration et al. (2018). The anonymous referee provided useful suggestions for placing this work in the context of the turbulent ISM.

Appendix

Distributions of Angles

We describe the directions of (the long axis of) the filament and the magnetic field with the coordinates in Figure 3. The line of sight lies along the $z$-axis, while the $x$-axis is down and the $y$-axis is left. We assume that the filament ($\mathbf{L}$) lies in the $x$-$z$ plane, making an angle $\theta_L$ to the line of sight. Equivalently, the directions can be expressed as a rotation around the $y$-axis:

$$\hat{L} = R_y(\theta_L)\hat{\mathbf{x}}.$$  \hspace{1cm} (A1)

The magnetic field ($\mathbf{H}$) we describe with respect to the filament axis, using spherical polar coordinates ($\theta_{LH}$, $\phi_{LH}$):

$$\hat{H} = R_L(\phi_{LH})R_S(\theta_{LH})\hat{L}.$$  \hspace{1cm} (A2)

When we numerically generate realizations of these directions, we use the Rodrigues rotation formula to rotate the vectors around the proper axis. We use the distribution of $\theta_{LH}$ values to statistically characterize the misalignment in three dimensions between the filaments and their local magnetic fields.

An important quantity for computing the polarization of this filament is the angle that the magnetic field makes with the line of sight, expressed as

$$\cos \theta_H = \frac{\mathbf{H} \cdot \hat{\mathbf{z}}}. \hspace{1cm}$$ (A3)

When projected onto the plane of the sky, the filament and the field are separated by a misalignment angle $\psi_{LH}$, computed as

$$\tan \psi_{LH} = \frac{H_y}{H_x}.$$ \hspace{1cm} (A4)

in the proper quadrant. For dust, the polarization angle relative to the filament direction is $\psi_{pol} = \psi_{LH} + \pi/2$.

For a particular filament angle $\theta_L$ and field–filament misalignment $\theta_{LH}$, changing the angle $\phi_{LH}$ rotates the field $\mathbf{H}$ to sweep out a cone around the filament direction. We can use the fact that $p(\phi_{LH})$ is uniform on $[0, 2\pi]$ to numerically accumulate the joint distribution of the field angle and projected field–filament misalignment:

$$p(\theta_H, \psi_{LH}|\theta_L, \theta_{LH}).$$ \hspace{1cm} (A5)

Because of the projections, this makes a loop of probability in the ($\theta_H$, $\psi_{LH}$) parameter space. When the separation between the filament and field $\theta_{LH}$ is small, the loop centers tightly around $\theta_H = \theta_L$, $\psi_{LH} = 0$, and when $\theta_{LH}$ is larger, the loop is larger and more distorted.

If we make an assumption for the distribution of the field–filament misalignment angle $\theta_{LH}$, we can marginalize over it:

$$p(\theta_H, \psi_{LH}|\theta_L) = \int d\theta_{LH} p(\theta_H, \psi_{LH}|\theta_L, \theta_{LH})p(\theta_{LH}).$$ \hspace{1cm} (A6)

In Figure A1, we show this joint distribution for several filament directions, under the assumption that the misalignment angle is Gaussian distributed. For filaments along the line of sight (small $\theta_L$), the distribution in projected angle ($\psi_{LH}$) is broad, which makes sense because variations in the angle $\phi_{LH}$ cause a wide variety of $\psi_{LH}$ angles. In Figure A2, we also examine the marginal distributions $p(\psi_{LH}|\theta_L)$ and $p(\theta_H|\theta_L)$.

In Figure A3, we further marginalize over the filament angles to get the distribution of the projected field–filament misalignment:

$$p(\psi_{LH}) = \int d\theta_L d\theta_H p(\psi_{LH}|\theta_L)p(\theta_L).$$ \hspace{1cm} (A7)

These marginalized distributions have positive kurtosis and are more sharply peaked than the Gaussian distribution from which they are derived. It is important to note that the polarized flux in practice will depend on both the polarization fraction (dependent on $\sin^2 \theta_H$) and the optical depth (dependent on the filament’s physical size, aspect ratio, and orientation $\theta_L$), and so the observed distribution of misalignment above some cut in signal-to-noise ratio will differ, and must be computed from the multidimensional distribution accounting for survey characteristics.

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5 At the beginning we fixed the plane-of-sky orientation of the filament, but we could have favored the magnetic field instead, and by symmetry we should have $p(\theta_H|\theta_L) = p(\theta_L|\theta_H)$. 

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Figure A1. Joint distribution of the line-of-sight angle of the magnetic field ($\theta_H$) and the projected angle between the magnetic field and the long axis of the filament ($\psi_{LH}$), under the assumption that the magnetic field direction has a Gaussian random distribution around the filament direction. The contours mark lines of constant probability density, and the number records the integrated probability outside the contour. The peak of the distribution is at $\psi_{LH} = 0^\circ$, $\theta_H = \theta_L$, corresponding to a filament aligned with the local magnetic field. The distribution is symmetric in the projected separation so we only show the half with $\psi_{LH} > 0^\circ$. In the left column, the field and filament are more closely aligned ($\text{rms} (\theta_{LH}) = 10^\circ$) than in the right column ($\text{rms} (\theta_{LH}) = 50^\circ$). In the top row, the filament is perpendicular to the line of sight ($\theta_L = 90^\circ$) and so is in the plane of the sky. In the bottom row, the filament aligns nearly along the line of sight ($\theta_L = 10^\circ$). For filaments along the line of sight (small $\theta_L$), even a small misalignment with magnetic field can cause the projected angle ($\psi_{LH}$) to vary widely, and so the distribution of the projected angle is broad.

Figure A2. Left: distribution of the magnetic field angle, for three different filament angles, assuming a Gaussian distribution for the misalignment of the field and filament angle in three dimensions. Each distribution peaks at the filament direction (for aligned filaments). Right: distribution of the projected misalignment between the filament and the magnetic field. The distribution is symmetric about $\psi_{LH} = 0^\circ$. 
Figure A3. Distribution of the projected filament–field misalignment angle for randomly oriented filaments, under the assumption that the three-dimensional misalignment angle is Gaussian distributed, for various misalignment dispersions. The projected distribution is symmetric around $\psi_{LH} = 0$.

**ORCID iDs**

Kevin M. Huffenberger @ https://orcid.org/0000-0001-7109-0099

David C. Collins @ https://orcid.org/0000-0001-6661-2243

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