An approximation algorithm for shortest path based on the hierarchy networks

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Abstract
It is a critical issue to compute the shortest paths between nodes in networks. Exact algorithms for shortest paths are usually inapplicable for large scale networks due to the high computational complexity. In this paper, we propose a novel algorithm that is applicable for large networks with high efficiency and accuracy. The basic idea of our algorithm is to iteratively construct higher level hierarchy networks by condensing the central nodes and their neighbors into super nodes until the scale of the top level network is very small. Then the algorithm approximates the distances of the shortest paths in the original network with the help of super nodes in the higher level hierarchy networks. The experiment results show that our algorithm achieves both good efficiency and high accuracy compared with other algorithms.

Keywords
Complex network, hierarchy network, approximation algorithm, shortest path

1. Introduction
With the rapid development of the Internet, the research of complex network has attracted more and more attentions in many fields of society[1], such as communication[2], recommendation, analysis of terrorist attack[3], and key nodes mining[4]. Particularly, computing the shortest path between any two nodes efficiently is one of the most important issues. Exact algorithms such as Dijkstra's and Floyd-Warshall's algorithms can't be applied efficiently to large scale networks due to the high computational complexity. It is necessary to design an approximation algorithm to calculate the distances of the shortest paths. Chow[6] presented a heuristic
algorithm for searching the shortest paths on a connected, undirected network. However, that algorithm relies on a heuristic function whose quality affects its efficiency and accuracy largely. Rattigan et al\textsuperscript{[8]} designed a network structure index (NSI) algorithm to estimate the shortest path quickly by storing data in a structure, but construction of the structure consumes too much time and space. Recently, Tang et al\textsuperscript{[9]} presented an algorithm, CDZ, based on the local centrality and existed paths though the central nodes (10\% of all nodes). The algorithm approximates the distances by means of the shortest paths between the central nodes computed by Dijkstra's algorithm. Although CDZ can achieve high accuracy on some social networks within acceptable time, it is not suitable for the large scale networks due to a large number of central nodes required computing. Tretyakov et al\textsuperscript{[10]} proposed two algorithms, LCA and LBFS, based on the landmark selection and shortest-path trees (SPTs). Although these algorithms perform well in practice, they do not provide strong theoretical guarantees on approximation quality. LCA computes the shortest paths with the help of the lowest common ancestors derived from SPTs and the landmarks. LBFS adopts SPTs to collect all paths from nodes to landmarks selected by the method of best-coverage, and splits the network into some sub networks. Based on the usual BFS traversal in these sub networks, LBFS can approximate the shortest paths.

Even there exists a large variety of algorithms to calculate the shortest paths, there are few approximation algorithms based on the hierarchy networks so far. Therefore we present a novel approximation algorithm based on the hierarchy networks, which is able to scale up to large networks with high efficiency and accuracy. Our algorithm first condenses the central nodes and their neighbors into super nodes to build a higher level network iteratively until the scale of the network is reduced to a threshold scale. Then the distances of the shortest paths in the original network can be calculated by means of the central nodes in the hierarchy networks. The algorithm proposed is tested on four real networks, and the experimental results show that the runtime per query is only a few milliseconds on large networks, while the accuracy is still maintained.

2. The construction of the hierarchy networks

Let $G = (V, E)$ be an undirected and unweighted network with $n = |V|$ nodes and $e = |E|$ edges. A path $P_{s,t}$ between two nodes $s, t \in V$ is a sequence $(s, u_1, u_2, \ldots, u_{l-1}, t)$, where $\{s, u_1, u_2, \ldots, u_{l-1}, t\} \subseteq V$ and $\{(s, u_1), (u_1, u_2), \ldots, (u_{l-1}, t)\} \subseteq E$. $d(s, t)$ is defined as the length of the shortest path between $s$ and $t$. 
Based on $G$, we first construct a series of hierarchy networks. The hierarchy networks except $G$ are undirected and weighted networks with different scales. In the construction of the hierarchy networks, the original network is taken as the bottom level or the level 0 network. For each level of the construction from bottom to top, we iteratively do the following steps: a normal node with the largest degree is selected as a central node and condenses with its normal neighbors (other than super nodes) into a super node; the edges between normal nodes are redirected to the corresponding super nodes. The condensing of current level network is stopped when all nodes are merged into super nodes which are regarded as normal nodes in the next level network. Edges between two super nodes lead to a single link for two normal nodes in the next level network. The weight of an edge between two adjacent nodes in the next level network denotes the approximate distance between the nodes. When the number of the nodes in the next level network is below a given threshold $t$, the construction of hierarchy networks is stopped and the top level network is obtained.

Figure 1 shows the process of constructing a hierarchy network. Figure 1(a) is the current level network whose black nodes are central nodes. Central nodes condense with their neighbors (white nodes) into super nodes respectively. And then, the next level network is obtained, as shown in Fig. 1(b). All super nodes in the previous level network, which are marked with dash squares in Fig. 1(a), are considered as normal nodes, i.e., one black and two white nodes in Fig. 1(b). Again, in the higher level network as shown in Fig. 1(b), the black node will be selected as the central node, and condense with its neighbors into a super node.

![Figure 1 Illustration of constructing a hierarchy network.](image)

3. Algorithm based on the hierarchy networks

After transforming the original network into a number of hierarchy networks, the distance of the shortest path between any two nodes in the lower level network will be
Let $\hat{d}_i(s,t)$ be the approximate distance between nodes $s$ and $t$ in the level $i$ network. The distance of shortest path between nodes $s$ and $t$ in the original network is approximated by $\hat{d}_0(s,t)$. In general, $\hat{d}_i(s,t)$ is iteratively computed by

$$\hat{d}_i(s,t) = \begin{cases} 
\hat{d}_i(s,c_s) + \hat{d}_i(t,c_t) & c_s = c_t, \ i \geq 0, \\
\hat{d}_i(s,c_s) + \hat{d}_{i+1}(c_s,c_i) + \hat{d}_i(c_t,t) & c_s \neq c_t,
\end{cases}$$

where $c_s$ and $c_t$ are the central nodes of nodes $s$ and $t$ respectively. Fig. 2 gives an example of the shortest path approximation using Eq. 1. $\hat{d}_i(s,c_s)$ and $\hat{d}_i(t,c_t)$ are the approximate distances from nodes $s_i$ and $t_i$ to their central nodes $c_s$ and $c_t$ in the level $i$ network. $\hat{d}_{i+1}(s,c_i)$ and $\hat{d}_{i+1}(c,t_i)$ are the distances from nodes $s_{i+1}$ and $t_{i+1}$ to their common central nodes in the level $i+1$ network respectively.

![Diagram](image_url)

**Figure 2 Illustration of the iterative approximation**

We define the longest distance from the sub nodes to the central node as the radius of the super node. Suppose normal nodes in the level $i$ have the same radius $r$, the radiiuses of super nodes will range from $r$ to $3r + 1$. Considering the limited memory space and running time, we should not record and compute all radiiuses of super nodes in hierarchy networks. Thus, we define the **appr radius** for all super nodes in the level $i$ network by approximating the distance between two adjacent normal nodes in the level $i$ network. As shown in Fig.3, the **appr radiuses** of super nodes in the level $i+1$ network are the approximate distances between the centers of two adjacent normal
nodes (represented as hollow circles). The normal nodes in the level $i+1$ network are derived from the super nodes in the level $i$ network, their appr radius are the same. In general, the appr radius of normal nodes in level $i$ network is defined by

$$r_i = \begin{cases} 
0, & i = 0 \\
2r_{i-1} + 1, & 0 < i < k 
\end{cases}$$

(2)

where $k$ is the number of hierarchy networks with different scales including the original network. In addition, Eq. 2 can be transformed to $r_i = 2^i - 1$ when $0 \leq i < k$. We approximate the distance from node $s$ to its central node $c$ in the level $i$ network by:

$$\hat{d}_i(s, c) = 2r_i + 1.$$  
(3)

The approximated distance between adjacent nodes in the level $i$ network is also $2r_i + 1$. Plugging Eq. 2 and Eq. 3 into Eq. 1, the upper bound of $\hat{d}_i(s, t)$ can be calculated by

$$\hat{d}_i(s, t) = \begin{cases} 
2 & i = 0, c_s = c_t \\
2^{i+2} - 2 & 0 < i < k - 1, c_s = c_t \\
2^{i+2} - 2 + \hat{d}_{i+1}(c_s, c_t) & 0 < i < k - 1, c_s \neq c_t \\
d_{k-1}(s, t) & i = k - 1 
\end{cases}$$

(4)

The distance between any adjacent nodes in the top level network is approximated as $2r_{i+1} + 1$, and $d_{i-1}(s, t)$ in Eq. 4 is the length of shortest path between the nodes $s$ and $t$ in the top level network computed by Dijkstra's algorithm.
The construction of each level hierarchy network is described in Algorithm 1. Algorithm 1 consumes $O(n_i \log n_i)$ time to rank the $n_i$ nodes in the level $i$ network and $O(n_i + e_i)$ time to generate the level $i+1$ network, where $e_i$ is the number of edges on the level $i$ network. The space complexity of Algorithm 1 is $O(n + e)$.

The construction of all hierarchy networks is described in Algorithm 2. In Algorithm 2, Algorithm 1 is repeatedly executed until the top level network is achieved where the number of nodes is below a certain threshold (e.g. 100). The time complexity of building hierarchy networks is $O\left(\sum_{i=0}^{k-1} (n_i \log n_i + n_i + e_i)\right)$, and the space complexity is $O(kn + e)$. In the top level network, Dijkstra's algorithm requires $O(m^2)$ time to compute shortest paths between each pair of nodes and $O(m^2)$ space to store the distances, where $m$ is the number of nodes in the top level network. Hence, the algorithm requires $O(kn + e + m^2)$ to store the hierarchy networks and the distances in the top level network.
Finally, the approximate distances of the shortest paths in the original network can be calculated by Algorithm 3. Algorithm 3 requires at most \(O(kn^2)\) time to compute the shortest paths for all pair of nodes.

| Algorithm 3 CalculatesShortestPath |
|------------------------------------|
| **Require:** Network network=G(V, E), \(d[i][j]\) records shortest distances between nodes in the top level network got by Algorithm 2, \(k\) denotes the number of hierarchy networks constructed by Algorithm 2 including the original network, \(c_s\) and \(c_t\) are the central nodes of \(s\) and \(t\) respectively, supernode\((c_s)\) is the super node contains nodes \(c_s\) and \(s\). |

1: function IterativeApproximation(s, t, i)
2: if \(i < k-1\)
3: if \(c_s = c_t\) in the level \(i\) network // hierarchy networks
4: return \(d(s, c_s)+d(t, c_t)\) // use Eq. 4 to calculate \(d(s, c_s), d(t, c_t)\)
5: end if
6: else if \(c_s \neq c_t\) in the level \(i\) network
7: return \(d(s, c_s)+IterativeApproximation(\text{supernode}(c_s), \text{supernode}(c_t), i+1)+d(t, c_t)\) // Eq. 4
8: end else
9: end if
10: else
11: return \(d[s][t]\) // use the distances between nodes in the top network got by Dijsktra in Algorithm 2
12: end else
13: end function

14: function CalculateShortestPath
15: if \(s \in V\) is directly connected with \(t \in V\) then
16: result = 1;
17: end if
18: else
19: result = IterativeApproximation(s, t, 0) // iterative approximation for the \(d_0(s, t)\)
20: end else
21: return result
22: end function

For approximating the shortest distances in an undirected and unweighted network with \(n\) nodes and \(e\) edges, the time complexity of the algorithm is \(O(m^2 + \sum_{i=0}^{n-1} (n_i \log n_i + n_i + e_i) + kn^2)\), and the memory complexity is \(O(kn+e+m^2)\).

For comparison, the complexity of CDZ\[^9\] and LBFS\[^10\] is summarized here. CDZ algorithm selects \(c\) central areas in the network with \(n\) nodes and \(e\) edges and
computes the distances between $c$ central nodes by Dijkstra's algorithm. The time complexity of CDZ algorithm is $O(ed+n\log n+c^3)$. The number of central areas in CDZ algorithm is about 10% of the number of nodes. LBFS algorithm selects $M$ pairs of nodes from network to obtain $l$ landmarks by the best-coverage strategy in the network with $n$ nodes and $e$ edges. The time complexity of LBFS is $O(M^3+le+D^3)$, where $D$ is the average size of the sub network related to the landmarks. Sometimes the number of pairs of nodes in LBFS is too large to make a best coverage and selection. The advantage of our algorithm is that the number of the nodes in the top network is always very small ($m$ is below a certain threshold), which can significantly reduce the complexity.

4. Experimental results and discussions

Our algorithm is implemented by Java on a PC with 2.93GHZ CPU and 4GB RAM. The performance of the algorithm is evaluated by four real undirected and unweighted networks: Email-Enron$^{[11]}$, itdk0304_rlinks$^{[12]}$, DBLP$^{[13]}$ and roadNet. Email-Enron contains about half million email communications among users, the nodes are the email addresses of senders or receivers, while the edges are the communication relationships. Tdk0304_rlinks contains the visiting relationships among nodes in the router level where the nodes are users or websites and the edges are the relationships between them. DBLP network contains information of computer science publications, in which each node corresponds to an author and two authors are connected by an edge if they have co-authored at least one publication. The roadNet is a traffic network composed of roads and sites. The number of nodes $V$, the number of edges $E$, the diameter of network $D$, the average degree $<k>$, and the largest degree $k_{max}$ of these four networks are shown in Table 1.

|        | Email-Enron | itdk0304_rlinks | DBLP     | roadNet   |
|--------|-------------|-----------------|----------|-----------|
| $V$    | 36 692      | 190 914         | 511 163  | 1 405 790 |
| $E$    | 367 662     | 1 215 220       | 3 742 140| 23 442 590|
| $D$    | 12          | 24              | 25       | 30        |
| $<k>$  | 10.020      | 6.365           | 7.320    | 9.433     |
| $k_{max}$ | 1 383      | 1 071           | 976      | 1244      |

In this paper, Path Ratio $p$ is used to assess the accuracy of the algorithm. It is defined by
where \( pr \) is the total number of pairs of nodes, \( P_{fl} \) is the distances of the pairs of nodes computed by the approximate algorithm, and \( P_{oi} \) is the accurate distances computed by the Dijkstra’s algorithm. The value of \( p \) could not be below one because the approximate distance is always longer than that of accurate distance.

We compare our algorithm with CDZ\cite{9} on preprocess time, average query time, and path ratio, as shown in Table 2. The threshold of the algorithm is 100. \( T_{init} \) represents the total time of preprocessing. \( T_q \) is the average runtime of 10000 random queries. From Table 2, it can be seen that our algorithm achieves an average query processing time of few milliseconds and maintains high accuracy on four networks. Our algorithm runs more than 10 times faster than CDZ especially on DBLP and roadNet. Moreover, our algorithm is more accurate than CDZ on Email-Enron and itdk0304_rlinks, and is comparable with CDZ on DBLP and roadNet.

Table 2 Runtime and accuracy of CDZ and our algorithm on four networks

|             | Our algorithm | CDZ              |
|-------------|---------------|------------------|
|             |   \( T_{init}\) (ms) | \( T_q\) (ms) | \( p \) | \( T_{init}\) (ms) | \( T_q\) (ms) | \( p \) |
| Email-Enron | 12845          | 1.38             | 1.022 | 112448              | 11.38         | 1.026 |
| itdk0304_rlinks | 54969      | 5.59             | 1.020 | 1092348             | 109.34        | 1.023 |
| DBLP        | 103126         | 10.41            | 1.020 | 8623459             | 862.44        | 1.020 |
| roadNet     | 196558         | 19.75            | 1.019 | 18824559            | 1882.55       | 1.019 |

We also compare our algorithm with the other two algorithms named LCA and LBFS based on the landmarks\cite{10} (see Table 3). Table 3 shows that our algorithm is better than LBFS on efficiency and accuracy. In DBLP and roadNet, our algorithm runs two times faster than LBFS. Comparing with LCA, the accuracy of our algorithm is much better although the runtime is little longer.

Table 3 Comparison about the performance of the algorithms

|             | Our algorithm | LCA   | LBFS   |
|-------------|---------------|-------|--------|
|             |   \( T_q\) (ms) |   p   | \( T_q\) (ms) | \( p \) | \( T_q\) (ms) | \( p \) |
| Email-Enron | 1.38          | 1.022 | 0.84   | 1.095 | 1.29          | 1.030 |
| itdk0304_rlinks | 5.59      | 1.020 | 3.57   | 1.083 | 5.60          | 1.028 |
| DBLP        | 10.41         | 1.020 | 6.35   | 1.072 | 15.99         | 1.025 |
In the experiment, we find that threshold $t$ affects the efficiency and accuracy of our algorithm. Table 4 shows the influence of thresholds $t$. From Table 4, it can be seen that runtime tends to increase and path ratio tends to decrease if threshold increases since more nodes in the top level network leads to more accurate and more time for Dijsktra's algorithm. When $t$ increases from 40 to 100, $T_q$ increases a little, but $p$ improves significantly. While the value of the threshold increases from 100 to 180, $T_q$ increases sharply, but $p$ remains nearly constant. So, we set the threshold as 100 in order to obtain a rather good trade-off.

|            | Threshold |
|------------|-----------|
| Email-Enron|           |
| $T_q$ (ms) | 0.54      |
| $p$        | 1.028     |
| itdk0304_rlinks |           |
| $T_q$ (ms) | 2.73      |
| $p$        | 1.030     |
| DBLP       |           |
| $T_q$ (ms) | 6.54      |
| $p$        | 1.026     |
| roadNet    |           |
| $T_q$ (ms) | 12.93     |
| $p$        | 1.026     |

5. Conclusion

Based on the hierarchy networks, we propose an approximate shortest path algorithm which obtains high efficiency and accuracy on large scale networks. The basic idea of our algorithm is condensing the central nodes and their neighbors into super nodes to iteratively construct higher level hierarchy networks until the scale of the top level network becomes very small. The algorithm approximates the distances of the shortest paths in the original network by means of super nodes in the higher level hierarchy networks. We test the performance of the proposed algorithm on four real networks. The results show that our algorithm can maintain high accuracy while achieve the runtime per query within a few milliseconds on large scale networks. Comparing with other algorithms, our algorithm is two times faster than LBFS and 10 times faster than CDZ. Furthermore, as the threshold could affect both the efficiency and the accuracy, a rather good trade-off can be obtained when the value of the threshold is set to 100 according to the experiment results.

The proposed algorithm is focused on undirected and unweighted networks. In the future, we will focus on directed and weighted networks by exploring the
approximate distance between a node and its central node based on the hierarchy networks. Moreover, we will consider an adaptive algorithm for different sorts of networks.

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