On the highest transcendentality in $\mathcal{N} = 4$ SUSY

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Abstract

We investigate the Eden-Staudacher equation for the anomalous dimension of the twist-2 operators at the large spin $s$ in the $N = 4$ super-symmetric gauge theory. This equation is reduced to a set of linear algebraic equations with the kernel calculated analytically. We prove that in perturbation theory the anomalous dimension is a sum of products of the Euler functions $\zeta(k)$ having the property of the maximal transcendentality with the coefficients being integer numbers. The radius of convergency of the perturbation theory is found. It is shown, that at $g = \infty$ the kernel has an essential singularity. The analytic properties of the solution of the Eden-Staudacher equation are investigated. In particular for the case of the strong coupling constant the solution has an essential singularity on the second sheet of the variable $j$ appearing in its Laplace transformation. Similar results are derived also for the Beisert-Eden-Staudacher equation which includes the contribution from the phase related to the crossing symmetry of the underlying $S$-matrix. We show, that its singular solution at large coupling constants reproduces the anomalous dimension predicted from the string side of the AdS/CFT correspondence.

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1 Introduction

The anomalous dimension matrices $\gamma$ and $\tilde{\gamma}$ of the twist-2 Wilson operators in Quantum Chromodynamics (QCD) govern the Bjorken scaling violation for structure functions respectively for the non-polarized and polarized cases. These quantities calculated in terms of the perturbative expansion in $a_s = \alpha_s/(4\pi)$ are related with the Mellin transformation

$$\gamma_{ab}(j) = \int_0^1 dx \ x^{j-1} W_{b \rightarrow a}(x) = \gamma_{ab}^{(0)}(j) a_s + \gamma_{ab}^{(1)}(j) a_s^2 + \gamma_{ab}^{(2)}(j) a_s^3 + O(a_s^4),$$

$$\tilde{\gamma}_{ab}(j) = \int_0^1 dx \ x^{j-1} \tilde{W}_{b \rightarrow a}(x) = \tilde{\gamma}_{ab}^{(0)}(j) a_s + \tilde{\gamma}_{ab}^{(1)}(j) a_s^2 + \tilde{\gamma}_{ab}^{(2)}(j) a_s^3 + O(a_s^4) \quad (1)$$

to the splitting kernels $W_{b \rightarrow a}(x)$ and $\tilde{W}_{b \rightarrow a}(x)$ for the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equation [1] describing the evolution of the parton densities $f_a(x, Q^2)$ and $\tilde{f}_a(x, Q^2)$ (hereafter $a = \lambda, g, \phi$ for the spinor, vector and scalar particles)

$$\frac{d}{d \ln Q^2} f_a(x, Q^2) = \int_x^1 \frac{dy}{y} \sum_b W_{b \rightarrow a}(x/y) f_b(y, Q^2),$$

$$\frac{d}{d \ln Q^2} \tilde{f}_a(x, Q^2) = \int_x^1 \frac{dy}{y} \sum_b \tilde{W}_{b \rightarrow a}(x/y) \tilde{f}_b(y, Q^2). \quad (2)$$

The scalar particles appear in the supersymmetric models but in the spin-dependent case their contribution is absent $a = \lambda, g$. The anomalous dimensions and splitting kernels in QCD are known up to the next-to-next-to-leading order (NN LO) of the perturbation theory [2, 3].

The QCD expressions for anomalous dimensions can be transformed to the case of the $\mathcal{N}$-extended Supersymmetric Yang-Mills Models (SYM) if one will use for the Casimir operators $C_A, C_F, T_f$ the following values $C_A = C_F = N_c, T_f n_f = \mathcal{N} N_c/2$. For $\mathcal{N} = 2$ and $\mathcal{N} = 4$ the anomalous dimensions of the Wilson operators get also additional contributions coming from diagrams with scalar particles [4]. These anomalous dimensions were calculated in the next-to-leading order (NLO) [4] for $\mathcal{N} = 4$ SYM.

It turns out, that the expressions for eigenvalues of the anomalous dimension matrix in $\mathcal{N} = 4$ SYM can be derived directly from the QCD anomalous dimensions without tedious calculations by using a number of plausible arguments. The method elaborated in Ref. [4] for this purpose (called the maximal transcendentality principle) is based on special properties of the integral kernel for the Balitsky-Fadin-Kuraev-Lipatov (BFKL) equation [6]-[8] and on an interesting relation between the BFKL and DGLAP equations in this model (see [4]). Really it was assumed, that the coefficients of the perturbation theory for the BFKL kernel and for the universal anomalous dimension should be linear combinations of the most complicated special functions which could appear in each given order. The results [4] for the anomalous dimension obtained with the use of the maximal transcendentality hypothesis were checked by direct calculations in Ref. [5]. Using the three-loop expressions for anomalous dimensions in QCD [3] and this hypothesis the eigenvalues of the anomalous dimension matrix for the $\mathcal{N} = 4$ SYM in the NNLO approximation were derived [9]. The direct verification of the obtained result for the case
of the Konishi operator was done in Ref. [10]. Also the independent calculation of the anomalous dimension for the twist-2 operator at the large spin $s$ gives the same correction in three loops [11]. Recently the four-loop contribution at $s \to \infty$ was computed and the KLV procedure [5] was used for the resummation of the perturbation theory [12]. The perturbative results are important for the verification of the AdS/CFT correspondence [13] and for checking the integrability of this model. Historically the integrability of the Yang-Mills theory at large energies appeared in the context of the solution of the Bartels-Kwiecinski-Prascalowich equation [14] in the multi-colour limit [15]. The effective hamiltonian of this equation coincides with the local hamiltonian of the Heisenberg spin model [16]. Later it was shown, that the equations for the anomalous dimensions of the so-called quasi-partonic operators [17] in the leading logarithmic approximation are also integrable and related to another Heisenberg spin model, but only in $\mathcal{N} = 4$ multi-colour SUSY [18]. Partly these remarkable properties remain also in QCD for a restricted class of quasi-partonic operators [19]. After the discovery of the AdS/CFT correspondence the integrability of the $\mathcal{N} = 4$ SYM was established in many loops in the weak and strong coupling regimes (see, for example, Ref. [20] and references therein). Several months ago in an interesting paper Eden and Staudacher (ES) derived an integral equation for the function which governs the behavior of the anomalous dimension for the twist-2 operators at large spins $s$ in all orders of the perturbation theory [21] (see also [22]). Its modification, which takes into account in the S-matrix an additional phase factor restoring the correspondence between the conformal field theory and the superstring model, was suggested recently by Beisert, Eden and Staudacher (BES) [23]. Our paper is devoted to an analytic solution of the ES and BES equations. Partly our results were presented by one of authors (L.L.) on the Potsdam Workshop [24]. The numerical solutions of the BES equation for all coupling constants were discussed recently in the papers of Bern with collaborators [12] and by Klebanov et al. [25] in the framework of the equivalent set of linear algebraic equations discussed by one of us in the Potsdam talk [24].

Our paper is organized as follows. In Section 2 the ES equation is simplified with the use of the inverse Laplace representation. Section 3 contains the derivation of a set of linear algebraic equations for the coefficients $a_{n,\epsilon}$ appearing in the expansion of its solution in terms of some special functions. The kernel for these equations is calculated in an explicit form and its analytic properties in the coupling constant plane are investigated. In Section 4 we study the anomalous dimension in the perturbation theory, verify its maximal transcendentality property and prove the Eden-Staudacher hypothesis [21] about the integer coefficients in the sum of products of the corresponding $\zeta$-functions. Section 5 is devoted to the investigation of analytical properties of the solution $\phi(j)$ of the ES equation in the Laplace variable $j$ for arbitrary complex constants. The obtained results give us a possibility to find the functional relations for $\phi(j)$ equivalent to the ES equation and to calculate the asymptotic behavior of the coefficients $a_{n,\epsilon}$. In Section 6 we study analytic properties of the solution of the ES equation at large coupling constants. In particular its behavior near an essential singularity on the second sheet of the $j$-plane is investigated. Using some plausible arguments we calculate from this result the anomalous dimension at large coupling constants. It oscillates around the value predicted from the string theory. Sections 7 and 8 contain a similar analysis for the Beisert-Eden-Staudacher
equation. It turns out, that in this case the behavior of the singular anomalous dimension at large coupling constants is stabilized and coincides with the result predicted from the string side of the AdS/CFT correspondence. In Conclusion we discuss obtained results and unsolved problems.

In Appendix A the duality relation for the dressing phase factor, proposed in [23] and entering in the BES equation, is proved. Appendix B contains an independent derivation of the linear sets of equations, obtained in Sections 3 and 7. In Appendices C and D their simplified derivation is done for an important case of large couplings.

2 ES equation in the inverse Laplace representation

One can write the anomalous dimension $\Delta - s$ of the twist-2 operators in the $N = 4$ super-symmetric gauge theory at the large Lorentz spin $s \to \infty$ in the form

$$\Delta - s = \gamma \ln s,$$

(3)

where the coefficient $\gamma$ in the t’Hooft limit depends only of the coupling constant $g$. Eden and Staudacher obtained for it the expression [21]

$$\gamma = 8 g^2 \sigma(0) = 4 g \sqrt{2} f(0)$$

(4)

in terms of the function

$$\sigma(t) = \epsilon f(x), \ t = \epsilon x, \ \epsilon = \frac{1}{g \sqrt{2}}.$$ (5)

This function satisfies the ES equation [21]

$$\epsilon f(x) = \frac{t}{e^t - 1} \left( \frac{J_1(x)}{x} - \int_0^\infty dx' K(x, x') f(x') \right)$$

(6)

with the integral kernel

$$K(x, y) = \frac{J_1(x) J_0(y) - J_1(y) J_0(x)}{x - y},$$

(7)

where $J_n(x)$ are the Bessel functions

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k + n)!} \left( \frac{x}{2} \right)^{2k+n}, \ J_1(x) = -J_0'(x).$$

(8)

Using the recurrent relation

$$J_{n-1}(x) = -J_{n+1}(x) + 2 n x^{-1} J_n(x),$$

(9)
we can write the kernel in a simpler way

\[ K(x, y) = \frac{2}{xy} \sum_{n=1}^{\infty} n J_n(x) J_n(y). \]  

(10)

Let us present the solution in the form of the Laplace integral

\[ f(x) = \int_{-i\infty}^{i\infty} \frac{d j}{2\pi i} e^{xj} \phi(j), \]  

(11)

where \( \phi(j) \) is the function analytic in the right semi-plane of the \( j \)-plane and decreasing at \( j \to \infty \) as follows

\[ \lim_{j \to \infty} \phi(j) = \frac{c(\epsilon)}{j}. \]  

(12)

The residue \( c(\epsilon) \) is related directly to the anomalous dimension

\[ \gamma = \frac{4}{\epsilon} c(\epsilon), \]  

(13)

Using the Laplace transformation

\[ \phi(j) = \int_0^{\infty} dx e^{-xj} f(x) \]  

(14)

and the following relations

\[ \int_0^{\infty} dx e^{-xj} J_n(x) = \frac{1}{n} \left( \frac{j^2 + 1}{j^2 + 1} - j \right)^n = \frac{1}{n} \left( \frac{j^2 + 1}{j^2 + 1} + j \right)^{-n}, \]  

(15)

\[ \int_0^{\infty} \frac{dx}{x} e^{-xj} J_n(x) = \frac{1}{n} \left( \frac{j^2 + 1}{j^2 + 1} - j \right)^n = \frac{1}{n} \left( \frac{j^2 + 1}{j^2 + 1} + j \right)^{-n}, \]  

(16)

\[ \int_0^{\infty} dx e^{-xj} (e^{x} - 1) f(x) = \phi(j - \epsilon) - \phi(j), \]  

(17)

we can write the ES equation for \( \phi(j) \) in the form

\[ \frac{\phi(j - \epsilon) - \phi(j)}{(j^2 + 1)^{-1/2}} = \frac{1}{(j^2 + 1)^{1/2} + j} - 2 \int_{-i\infty}^{i\infty} \frac{dj'}{2\pi i} \sum_{n=1}^{\infty} \phi(j') \left( \frac{-j'^2 + 1}{(j^2 + 1)^{1/2} + j} \right)^n. \]  

(18)

Here the integration contour in \( j' \) is assumed to be to the left from the cut of the function \(-\sqrt{j'^2 + 1} + j'\) at \(-i < j' < i\) and to the right from the singularities of the function \( \phi(j') \). The branch of \( \sqrt{j'^2 + 1} \) is chosen in such way to make the expression \(-\sqrt{j'^2 + 1} + j'\) decreasing at large \( j' \).
3 Set of linear algebraic equations

We search the solution of the ES equation in the form [24]

\[ \phi(j) = \sum_{n=1}^{\infty} \phi_{n,\epsilon}(j) \left( \delta_{n,1} - a_{n,\epsilon} \right), \]  

where \( \delta_{n,1} \) is the Kronecker symbol and

\[ \phi_{n,\epsilon}(j) = \sum_{s=1}^{\infty} \frac{\left( \sqrt{(j+s\epsilon)^2 + 1} + j + s\epsilon \right)^{-n}}{\sqrt{(j+s\epsilon)^2 + 1}}. \]  

The anomalous dimension can be expressed in terms of \( a_{1,\epsilon} \) as follows

\[ \gamma = \frac{2}{\epsilon^2} (1 - a_{1,\epsilon}) \]  

The coefficients in the linear combination of the functions \( \phi_{n,\epsilon}(j) \) satisfy the set of equations [24]

\[ a_{n,\epsilon} = \sum_{n'=1}^{\infty} K_{n,n'}(\epsilon) \left( \delta_{n',1} - a_{n',\epsilon} \right), \]

where

\[ K_{n,n'}(\epsilon) = 2 \int_{-i\infty}^{i\infty} \frac{dj'}{2\pi i} \phi_{n',\epsilon}(j') \left( -(j'^2 + 1)^{1/2} + j' \right)^n. \]  

The matrix elements \( K_{n,n'}(\epsilon) \) can be expressed in terms of the generalized hypergeometric function [24]. To show it, we begin with the following representation for \( a_{n,\epsilon} \)

\[ a_{n,\epsilon} = 2 \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{dj'}{2\pi i} \phi(j') \left( -(j'^2 + 1)^{1/2} + j' \right)^n = -2 \sum_{k=n}^{\infty} b_n(k) \frac{\phi^{(k-1)}(0)}{(k-1)!}, \]

where \( 0 > \sigma > -\epsilon \) and \( \phi^{(n-1)}(0) \) is the value of the \((n-1)\)-derivative of the function \( \phi(j) \) at \( j = 0 \). The coefficients \( b_n(k) \) are defined in terms of the large-\( j \) expansion

\[ \left( -(j^2 + 1)^{1/2} + j \right)^n = \sum_{k=n}^{\infty} b_n(k) j^{-k}. \]

These coefficients can be written as follows

\[ b_n(k) = \int_L \frac{dj}{2\pi i} \left( -(j^2 + 1)^{1/2} + j \right)^n j^{k-1} = (-1)^n \int_L \frac{dl}{2\pi i} 2^k (1 + l^2) \frac{(-l^2)^{k-1}}{l^{n+1}}, \]

where the integration contour \( L \) goes along a circle around the singularities of the integrand in a anti-clock-wise direction and we introduced the new integration variable

\[ l = (j^2 + 1)^{1/2} - j. \]
The integral over $l$ can be calculated by residues
\[ b_n(k) = (-1)^{(k+n)/2} 2^{-k} \frac{n \Gamma(k)}{\Gamma\left(\frac{k-n}{2} + 1\right) \Gamma\left(\frac{k+n}{2} + 1\right)}, \tag{27} \]
where
\[ \frac{k-n}{2} = r \tag{28} \]
is an integer number (for half-integer values of $r$ one obtains $b_n(k) = 0$).

We can write $a_{n,\varepsilon}$ as follows
\[ a_{n,\varepsilon} = \sum_{r=0}^{\infty} B_n(r) \phi^{(2r+n-1)}(0), \tag{29} \]
where
\[ B_n(r) = -2 \frac{b_n(2r+n)}{(2r+n-1)!} = (-1)^{r+n+1} 2^{-2r-n+1} \frac{n}{r!(r+n)!}. \tag{30} \]
Therefore one obtains
\[ (-j^2 + 1)^{1/2} + j)^n = -\sum_{r=0}^{\infty} \frac{(2r+n-1)!}{2} B_n(r) j^{-2r-n}. \tag{31} \]

To calculate the kernel $K_{n,n'}$ appearing in the set of equations for $a_{n,\varepsilon}$ let us use the expansion
\[ \frac{(j^2 + 1)^{1/2} - j)^n}{(j^2 + 1)^{1/2}} = -\frac{1}{n} \frac{\partial}{\partial j} ((j^2 + 1)^{1/2} - j)^n = -(-1)^n \sum_{r=0}^{\infty} \frac{(2r+n)!}{2 n^{2r+n+1}} B_n(r) j^{-2r-n-1}. \tag{32} \]

It gives a possibility to express $\phi_{n,\varepsilon}(j)$ in terms of simpler functions
\[ \phi_{n,\varepsilon}(j) = (-1)^{n+1} \sum_{r=0}^{\infty} \frac{(2r+n)!}{2 n} B_n(r) \chi_{2r+n+1,\varepsilon}(j), \quad \chi_{k,\varepsilon}(j) = \sum_{s=1}^{\infty} (j + \varepsilon s)^{-k}. \tag{33} \]

In particular,
\[ \phi_{n',\varepsilon}^{(2r+n-1)}(0) = (-1)^{n+n'} \sum_{r'=0}^{\infty} B_n'(r') \frac{(2r+n+2r'+n'-1)!}{2 n' \varepsilon^{2r+n+2r'+n'}} \zeta(2r+n+2r'+n'), \tag{34} \]
where the Euler $\zeta$-function is defined as follows
\[ \zeta(k) = \sum_{s=1}^{\infty} s^{-k}. \tag{35} \]

Using above relations we can present the kernel $K_{n,n'}$ as follows
\[ K_{n,n'}(\varepsilon) \]
\[ = (-1)^{n+n'} \sum_{r=0}^\infty \frac{B_n(r) \sum_{r'=0}^\infty B_{n'}(r') (2r + n + 2r' + n' - 1)!}{2^n \epsilon^{2r+n+2r'+n'}} \zeta(2r + n + 2r' + n'). \]  

It can be written in the form

\[ K_{n,n'}(\epsilon) = 2n \sum_{R=0}^\infty \frac{(2R + n + n' - 1)!}{\epsilon^{2R+n+n'}} \zeta(2R + n + n') C_{n,n',R}, \]

where \( C_{n,n',R} \) is given below

\[ C_{n,n',R} = \frac{(-1)^{n+n'} R}{4nn'} \sum_{r=0}^\infty B_n(r) B_{n'}(R-r) = (-1)^R \frac{F(-R, -R-n'; n+1; 1)}{2^{2R+n+n'} n! R! (R+n')!}. \]

Here \( F(a, b; c; x) \) is the hypergeometric function which can be calculated at \( x = 1 \). Therefore one obtains

\[ C_{n,n',R} = (-1)^R \frac{2^{-2R-n-n'} (2R + n + n')!}{R! (R+n)! (R+n')! (R+n+n')!}. \]

As a result, we have the following expression for the kernel \([24]\)

\[ K_{n,n'}(\epsilon) = 2n \sum_{R=0}^\infty (-1)^R \frac{2^{-2R-n-n'} \epsilon^{2R+n+n'} \zeta(2R + n + n') (2R + n + n' - 1)! (2R + n + n')!}{R! (R+n)! (R+n')! (R+n+n')!}. \]

The perturbation series for \( K_{n,n'}(\epsilon) \) has a finite radius of convergence due to the singularity at \( \epsilon^{-2} = -1/4 \). But it can be analytically continued around this singularity with the use of the contour integral representation and known relations for the \( \Gamma \)-functions

\[ K_{n,n'}(\epsilon) = \frac{n}{\pi} \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \left( \frac{2}{\epsilon} \right)^{n+n'-2s} \zeta(n+n'-2s) \frac{\Gamma(s) \Gamma^2 \left( \frac{n+n'+1}{2} - s \right) \Gamma \left( \frac{n+n'}{2} - s + 1 \right)}{\Gamma(n+1-s) \Gamma(n'+1-s) \Gamma(n+n'+1-s)}, \]

where the integration contour is to the right of the pole at \( s = 0 \) and to the left of the pole of \( \zeta \)-function at \( s = (n+n'-1)/2 \). The integral is rapidly convergent as \( \sim \int ds / \sin \pi s \) at \( s \to \pm i\infty \). For small positive \( \epsilon \) the contour of integration should be enclosed around singularities situated to the left of it. Note, that the use of the Mellin-Barnes representation \([11]\) for sum \([10]\) corresponds to chosing a definite receipt of resummation of the divergent series. Another possibility is to apply to it the KLV approach \([5]\). This approach was very successful for resummation of anomalous dimensions obtained in three \([9]\) and four \([12]\) loops. One can also solve the set of linear equations \([22]\) approximately reducing it to a finite number of them with the kernel, calculated for all coupling constants numerically \([25]\). Two above methods give similar predictions. The shortage of these approaches is that they do not lead to analytic results.

One can express the kernel in terms of the sum of generalized hypergeometric function \( 4F3 \) \([24]\)

\[ K_{n,n'}(\epsilon) = \frac{\Gamma^2 \left( \frac{n+n'+1}{2} \right) \Gamma \left( \frac{n+n'}{2} + 1 \right) \Gamma \left( \frac{n+n'}{2} \right)}{\pi \Gamma(n) \Gamma(n'+1) \Gamma(n+n'+1)} \sum_{k=1}^\infty \left( \frac{2}{k \epsilon} \right)^{n+n'} F_{n,n'} \left( \frac{-4}{k^2\epsilon^2} \right), \]
where
\[ F_{n,n'}(-4k^2\epsilon^2) \equiv F_3\left(\frac{n + n' + 1}{2}, \frac{n + n' - 1}{2}; n + 1, n' + 1; -4k^2\epsilon^2\right). \] (43)

4 Iterative solution of the SE equation

The formal solution of the ES equation for the coefficients \(a_{n,\epsilon}\) can be written as a matrix element of the geometrical progression constructed from the operator \(\hat{K}(\epsilon)\) corresponding to the kernel \(K_{n,n'}(\epsilon)\)

\[ a_{n,\epsilon} = \sum_{r=1}^{\infty} (-1)^{r+1} \left(\hat{K}^r(\epsilon)\right)_{n,1}. \] (44)

In particular, for the anomalous dimension we have

\[ \gamma(\epsilon) = \frac{2}{\epsilon^2} \rho(\epsilon), \quad \rho(\epsilon) = \sum_{r=0}^{\infty} (-1)^r \left(\hat{K}^r(\epsilon)\right)_{1,1}. \] (45)

In two first orders one can obtain

\[ \rho(\epsilon) = 1 - K_{1,1}(\epsilon) + ..., \] (46)

where

\[ K_{1,1}(\epsilon) = \frac{1}{\pi} \int_{i\infty}^{-i\infty} \frac{ds}{2\pi i} \left(\frac{2}{\epsilon}\right)^{2-2s} \frac{\Gamma(s) \Gamma^2\left(\frac{3}{2} - s\right)}{(1-s) \Gamma(3-s)}. \] (47)

The correction \(K_{1,1}(\epsilon)\) increases with decreasing \(\epsilon\). For example, \(K_{1,1}(\epsilon) \approx 8/(3\epsilon)\) for \(\epsilon \to 0\). Therefore one can obtain the solution by the iteration in \(\hat{K}(\epsilon)\) only for sufficiently large \(\epsilon\) (small \(g\)).

In the usual perturbation theory \(1/\epsilon \to 0\) the iteration of the solution in \(\hat{K}(\epsilon)\) leads to small corrections to \(\gamma\)

\[ \gamma(\epsilon) = \frac{2}{\epsilon^2} - \frac{1}{\epsilon^4} \zeta(2) + \frac{1}{2\epsilon^6} \left(3\zeta(4) + \zeta^2(2)\right) \]
\[ - \frac{1}{8\epsilon^8} \left(25\zeta(6) + 12\zeta(2)\zeta(4) + 2\zeta^3(2) - 2\zeta^2(3)\right) + O(\epsilon^{-10}). \] (48)

According to the Eden-Staudacher hypothesis [21], in the series

\[ \gamma(\epsilon) = -8 \sum_{k=1}^{\infty} \left(\frac{1}{4\epsilon^2}\right)^k c_k \] (49)

the quantities \(c_k\) are products of \(\zeta\)-functions satisfying the transcendentality principle with integer coefficients. For example,

\[ c_1 = 1, \quad c_2 = 2\zeta(2), \quad c_3 = 4 \left(3\zeta(4) + \zeta^2(2)\right), \] (50)

\[ c_4 = 8 \left(5\zeta(6) - 3\zeta(4)\right), \quad c_5 = 16 \left(7\zeta(8) - 6\zeta(6)\right), \] (51)

\[ c_6 = 32 \left(11\zeta(10) - 10\zeta(8)\right), \quad c_7 = 64 \left(15\zeta(12) - 12\zeta(10)\right), \] (52)

\[ c_8 = 128 \left(19\zeta(14) - 16\zeta(12)\right). \] (53)
\[ c_4 = 4 \left( 25 \zeta(6) + 12 \zeta(2) \zeta(4) + 2 \zeta^3(2) - 2 \zeta^2(3) \right). \]  

(51)

One can prove easily this hypothesis if the factor entering in the expression for \( K_{n,n'}(\epsilon) \)

\[ T_{n,n';R} = 2^n \frac{(2R + n + n' - 1)! \cdot (2R + n + n')!}{R! \cdot (R + n)! \cdot (R + n')! \cdot (R + n + n')!}. \]  

(52)

is an integer number.

Indeed, the coefficients \( c_k \) can be presented as follows

\[ c_k = \sum_{r=0}^{\infty} S_k^{(r)}, \]  

(53)

where in the expression for \( S_k^{(r)} \)

\[ S_k^{(r)} = \sum_{s_1=2}^{\infty} \sum_{s_2=2}^{\infty} \cdots \sum_{s_r=2}^{\infty} (-1)^{N_{s_1,s_2,\ldots,s_r}} U_{s_1,s_2,\ldots,s_r} \prod_{i=1}^{r} \zeta(s_i). \]  

(54)

the sum is performed over the \( \zeta \)-function arguments \( s_t \) satisfying the maximal transcendentality condition

\[ \sum_{t=1}^{r} s_t = 2k - 2. \]  

(55)

It is obviously, that the number of the \( \zeta \)-functions with odd arguments \( s_t = 2m + 1 \) should be even.

The factor \( U_{s_1,\ldots,s_r} \) is given below

\[ U_{s_1,\ldots,s_r} = \sum_{n_1=1}^{\infty} \cdots \sum_{n_{r-1}=1}^{\infty} T_{n_1,s_1-1-n_1} T_{n_2,s_2-n_2-n_3} \cdots T_{n_{r-1},s_{r-2}-n_{r-2}-1}. \]  

(56)

Note, that the summation is performed only over such \( n_t \), for which

\[ R_t = \frac{s_t - n_{t-1} - n_t}{2} \]  

(57)

is integer. The quantities \( U_{s_1,\ldots,s_r} \) are positive integer numbers providing that \( T_{n,n';R} \) are integer.

As for the quantity \( N_{s_1,s_2,\ldots,s_r} \) entering in the phase factor, it can be written as follows (see (40))

\[ N_{s_1,s_2,\ldots,s_r} = \sum_{t=1}^{r} R_t + r - (k - 1) = r - 1 - \sum_{t=1}^{r-1} n_t, \]  

(58)

where we used the relation

\[ \sum_{t=1}^{r} R_t = \frac{1}{2} \sum_{t=1}^{r} s_t - 1 - \sum_{s=1}^{r-1} n_s = k - 2 - \sum_{s=1}^{r-1} n_s. \]  

(59)
The numbers $N_{s_1, s_2, \ldots, s_r}$ are defined modulo 2, which means, that one can add to them an arbitrary even number. Because

$$2R_t = s_t - (n_{t-1} + n_t)$$

are even numbers, one can express $N_{s_1, s_2, \ldots, s_r}$ only in terms of $s_t$

$$N_{s_1, s_2, \ldots, s_r} = \sum_{l=1}^{r} (r - l) s_l . \quad (60)$$

In particular, when all arguments $s_l$ of $\zeta$-functions are even, the phase factor $(-1)^N$ is absent. Generally it is enough to sum only over $l$ for which $s_l$ are odd numbers. Moreover, because such $s_l$ equal 1 modulo 2 one can substitute $s_l$ by 1

$$N_{s_1, s_2, \ldots, s_r} = \sum_{l} l , \quad (s_l = \text{odd}) , \quad (61)$$

where we took into account, that the number of odd $s_l$ is even.

Returning to the Eden-Staudacher hypothesis, it is helpful to present $T_{n, n'}; R$ as the product of three factors

$$T_{n, n'; R} = C_{2R+n+n'}^R C_{2R+n+n'}^R \frac{2n}{2R + n + n'} . \quad (62)$$

Here the numbers of distributions

$$C_{2R+n+n'}^R = \frac{(2R + n + n')!}{t!(2R + n + n' - t)!} \quad (63)$$

are integers. The product of all three factors is also an integer number. To begin with, in the case when $2R + n + n'$ is a prime number $p$ the first two factors are proportional to $2R + n + n'$ and can be divided on $2R + n + n'$. If $2R + n + n'$ is a product of a prime number $p$ and an integer number $k$, than the product of two first factors does not contain the multiplier $p$ only if both $n$ and $n'$ are proportional to $p$, but in the last case the multiplier $p$ is contained in the numerator of the third factor. It proves the Eden-Staudacher hypothesis.

For finite $1/\epsilon$ the corrections will be insignificant only if the maximal eigenvalue $\lambda(\epsilon)$ of the matrix $\hat{K}(\epsilon)$ will be sufficiently small. In an opposite case we should use a non-perturbative approach. For this purpose it is helpful to know the expansion of $K_{n, n'}(\epsilon)$ near singular points $\epsilon^2 = -4$ and $\epsilon = 0$, which can be obtained from its integral representation.

Note, that for $\epsilon^2 \to -4$ the large values of the summation variable $R$ are essential and therefore $\zeta(n + n' + 2R)$ can be substituted by unity. With the use of the Stirling formulas for the $\Gamma$-functions we obtain the singularity of $K_{n, n'}(\epsilon)$ at $\epsilon^2 = -4$

$$K_{n, n'}(\epsilon)|_{\epsilon^2 \to -4} = \frac{n}{\pi} \left( \frac{2}{\epsilon} \right)^{n+n'} \sum_{R=1}^{\infty} \frac{(-1)^R}{R^2} \left( \frac{4}{\epsilon^2} \right)^R \approx \frac{n}{\pi} \left( \frac{2}{\epsilon} \right)^{n+n'} (1 + \frac{4}{\epsilon^2}) \ln(1 + \frac{4}{\epsilon^2}) . \quad (64)$$
Using the iterative procedure near this singularity one can obtain the following expression for the anomalous dimension

\[ \gamma(\epsilon) = \frac{2/\epsilon^2}{1 + \frac{1+4/\epsilon^2}{(1-4/\epsilon^2)^2} \ln (1 + 4/\epsilon^2)}. \]  

(65)

To calculate \( \gamma \) at \( \epsilon^2 \to -4 \) with a better accuracy we should know the regular part of \( K_{n,n'} \) at \( \epsilon^2 = -4 \)

\[ K_{n,n'}(\pm 2i) = \frac{n}{\pi} (-i)^{n+n'} \sum_{R=0}^{\infty} \zeta(n+n'+2R) \frac{\Gamma^2(\frac{n+n'+1}{2} + R) \Gamma(\frac{n+n'}{2} + R)}{R! (R+n)! (R+n')! (R+n+n')!}. \]  

(66)

The sum in this expression is convergent \( \approx \sum R^{-2} \). One can assume, that the result does not depend essentially on \( n \) and \( n' \). Then the above expression for \( \gamma \) could be approximately valid at \( \epsilon^2 \to -4 \) even after taking into account the regular contribution to \( K_{n,n'} \).

Another possibility to estimate the behavior of the kernel \( K_{n,n'}(\epsilon) \) at \( \epsilon^2 \to -4 \) is to use the known expansion for the generalized hypergeometric function \( F_{n,n'}(x) \) \( \text{[43]} \) for \( x = -4/(k^2 \epsilon^2) \) near the singular point \( x = 1 \). The summation over \( k \) leads then to the singularities of \( K_{n,n'}(\epsilon) \) \( \text{[23]} \) in the points \( \epsilon^2 = -4/k^2 \) for \( k = 1, 2, \ldots \). Therefore at \( \epsilon = 0 \) the kernel has an essential singularity \( \text{[24]} \). The singularities at \( \epsilon^2 = -4/k^2 \) appear because in the initial expression for \( K_{n,n'}(\epsilon) \) \( \text{[23]} \) written as an integral over \( j' \) the singularities of \( \phi_{n,\epsilon}(j) \) situated at \( j' = -k \epsilon \pm i \) pinch the integration contour \( L \) together with the square-root singularities at \( j' = \pm i \).

On the other hand, we can attempt to find the behavior of \( K_{n,n'}(\epsilon) \) for \( \epsilon \to +0 \) directly from its integral representation. In this limit the integration contour should be shifted to the right up to the first poles of \( \zeta \)- and \( \Gamma \)- functions situated at \( s = (n+n'-1)/2 \) and \( s = (n+n')/2 \), respectively. The contributions to \( K_{n,n'} \) from the corresponding residues are

\[ K_{n,n'}(\epsilon) = \frac{2n/\epsilon}{(n+n')^2 - 1} \frac{1}{\Gamma\left(\frac{n-n'+3}{2}\right) \Gamma\left(\frac{n'-n+3}{2}\right)} - \frac{n}{n+n'} \frac{1}{\Gamma\left(\frac{n-n'+2}{2}\right) \Gamma\left(\frac{n'-n+2}{2}\right)}. \]  

(67)

The solution of the set of linear equations for \( a_{n,\epsilon} \) can be written in the strong coupling limit \( \epsilon \to 0 \) as follows

\[ a_{n,\epsilon} = \delta_{n,1} + \epsilon \Delta a_{n,\epsilon}, \]  

(68)

where \( \Delta a_{n,\epsilon} \) satisfies the following inhomogeneous equation

\[ \epsilon \Delta a_{n,\epsilon} = -\delta_{n,1} - \sum_{n'=1}^{\infty} \left( \frac{2n}{(n+n')^2 - 1} \frac{1}{\Gamma\left(\frac{n-n'+3}{2}\right) \Gamma\left(\frac{n'-n+3}{2}\right)} - \frac{\epsilon n}{n+n'} \frac{1}{\Gamma\left(\frac{n-n'+2}{2}\right) \Gamma\left(\frac{n'-n+2}{2}\right)} \right) \Delta a_{n,\epsilon}. \]  

(69)

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This set of linear equations with the leading term for \( K_{n,n'}(\epsilon) \) at \( \epsilon \to 0 \) corresponds to the ES equation in the form
\[
-\epsilon \sqrt{j^2 + 1} \frac{\partial}{\partial j} \phi(j) = \frac{1}{(j^2 + 1)^{1/2} + j} - 2 \int_{-\infty}^{\infty} \frac{dj'}{2\pi i} \sum_{n=1}^{\infty} \phi(j') \left( \frac{-(j'^2 + 1)^{1/2} + j'}{(j^2 + 1)^{1/2} + j} \right)^n.
\]

The next corrections \( \sim \epsilon^k \) to the kernel \( K_{n,n'}(\epsilon) \) can be obtained from its integral representation by taking residues in the poles situated at \( s = (n + n' + k)/2 \) for \( k = 1, 2, \ldots \). Another possibility is to use the known expansion of the generalized hypergeometric function \( F_{n,n'}(x) \) (43), where \( x = -4/(k^2 \epsilon^2) \), for the large argument \( x \).

### 5 Analytic properties of the solution of the ES equation

By summing the geometrical progression we can rewrite the SE equation in the form
\[
\phi(j - \epsilon) - \phi(j) = \frac{1}{(j^2 + 1)^{1/2} + j} - 2 \int_{-\infty}^{\infty} \frac{dj'}{2\pi i} \frac{\phi(j') \left( -(j'^2 + 1)^{1/2} + j' \right)}{(j^2 + 1)^{1/2} + j + (j'^2 + 1)^{1/2} - j'}. \tag{71}
\]

The integration contour can be enclosed around the cut situated to the right of it. To calculate the discontinuity one should preliminary anti-symmetrize the integral kernel extracting from it the square-root singularity as a factor
\[
-\frac{2 \left( -(j'^2 + 1)^{1/2} + j' \right)}{(j^2 + 1)^{1/2} + j + (j'^2 + 1)^{1/2} - j'} \to \frac{-(j'^2 + 1)^{1/2} + j'}{(j^2 + 1)^{1/2} + j + (j'^2 + 1)^{1/2} - j'} - \frac{(j'^2 + 1)^{1/2} + j'}{(j^2 + 1)^{1/2} + j - (j'^2 + 1)^{1/2} - j'}. \tag{72}
\]

As a result, we can write the equation in the form
\[
(\phi(j - \epsilon) - \phi(j)) \sqrt{j^2 + 1} = \frac{1}{\sqrt{j^2 + 1} + j} + 2 \int_{-1}^{1} \frac{dj'}{2\pi i} \phi(j') \left( \frac{-\sqrt{j'^2 + 1} + j'}{\sqrt{j'^2 + 1} + j + \sqrt{j'^2 + 1} - j} - \frac{\sqrt{j'^2 + 1} + j'}{\sqrt{j'^2 + 1} + j - \sqrt{j'^2 + 1} - j'} \right). \tag{73}
\]

It is convenient to introduce the new variable
\[
z = (j^2 + 1)^{1/2} + j, \tag{74}
\]
with the inverse relation
\[
j = \frac{z^2 - 1}{2z}. \tag{75}
\]
and the new function

\[ \phi(j) = \chi(z). \tag{76} \]

The transformation \( z = z(j) \) performs the conformal mapping of two sheets of the Riemann surface for \( \phi(j) \) on one sheet for the function \( \chi(z) \). The physical sheet in the \( j \)-plane corresponds to the region \(|z| > 1\). The transition to the second sheet is realized by the transformation \( z \rightarrow -z^{-1} \).

In the new variables the SE equation takes the form

\[
(\chi(z_{\epsilon}) - \chi(z)) \frac{z^2 + 1}{2z} = \frac{1}{z} - \int_{-i}^{i} \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z'} \frac{\chi(z')}{z - z'} \left( \frac{z'}{z} + \frac{1/z'}{z + 1/z'} \right), \tag{77}
\]

where the integration over \( z' \) is done along the unit circle in the anti-clock direction from \(-i\) to \(i\). We used the notation

\[ z_{\epsilon} = \hat{R}_{\epsilon} z = \left( \left( \frac{z^2 - 1}{2z} - \epsilon \right)^2 + 1 \right)^{1/2} + \frac{z^2 - 1}{2z} - \epsilon. \tag{78} \]

On the other hand, one can write the ES equation as follows

\[
(\chi(z_{\epsilon}) - \chi(z)) \frac{z^2 + 1}{2z} = \frac{1}{z} - \int_{L} \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z'} \frac{\chi(z')}{z - z'}, \tag{79}
\]

where the integration contour \( L \) chosen in a form of the unit circle is passed in an anti-clock direction. We used the variable \( \tilde{z} \) equal to \( z \) on the right part of the circle and to \(-1/z\) on its left part

\[ \tilde{z}_{Re \geq 0} = z, \quad \tilde{z}_{Re < 0} = -z^{-1}. \tag{80} \]

The substitution \( z \rightarrow -1/z \) in the argument of the function \( \chi(z) \) means an analytic continuation of the function \( \phi(j) \) to the second sheet of the \( j \)-plane with the substitution \( \sqrt{j^2 + 1} \rightarrow -\sqrt{j^2 + 1} \). But according to the representation \([19]\) of \( \phi(j) \) as a sum of the functions \( \phi_{n,\epsilon}(j) \) it has the square-root singularities only in the points \( j = -s\epsilon \pm i \) \((s=1,2,...)\) being analytic at \( j = \pm i \). Therefore

\[ \chi(\tilde{z}) = \chi(z') \tag{81} \]

and we can write the equation in the simple form

\[
(\chi(z_{\epsilon}) - \chi(z)) \frac{z^2 + 1}{2z} = \frac{1}{z} - \int_{L} \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z'} \frac{\chi(z')}{z - z'}, \tag{82}
\]

where the contour \( L \) is a unit circle passed in an anti-clock direction. The pole at \( z' = z \) is situated outside it. The singularities of the integral appear when this pole and inner singularities of \( \chi(z') \) pinch the contour. For the singular part of \( \chi(z) \) we obtain the relation

\[
(\chi(z_{\epsilon}) - \chi(z))_{\text{sing}} \frac{z^2 + 1}{2z} = \frac{1}{z} - \frac{z^2 + 1}{z} \chi(z)_{\text{sing}}. \tag{83}
\]
Let us anti-symmetrize the right and left hand sides of the above ES equation to the substitution $z \rightarrow -1/z$ with the analytic continuation of the added term from $1/|z| < 1$ to $|z| = 1/|z| = 1$

$$\left(\chi(z) - \chi(z) + \chi((-1/z)\epsilon) - \chi(-1/z)\right) \frac{z^2 + 1}{2z} = \frac{z^2 + 1}{z} - \frac{z^2 + 1}{z} \chi(z), \quad (84)$$

where the integral contribution in its right hand side was simplified with the use of the relations

$$\int_{L} \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z'} = \int_{L} \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z'} \chi(z') \left(\frac{1}{z - z'} - \frac{z/z'}{z - z'}\right). \quad (85)$$

It is implied here, that in the first and second terms one takes $|z| \rightarrow 1 + 0$ and $|z| \rightarrow 1 - 0$, respectively, which gives a possibility to calculate the integral by residues taking also into account, that

$$\int_{L} \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z'^2} \chi(z') = 0 \quad (86)$$

due to the symmetry of $\chi(z')$ under the substitution $z' \rightarrow -1/z'$.

As a result, we obtain the equation

$$\frac{1}{2} \left(\chi(z) - \chi(z) + \chi((-1/z)\epsilon) - \chi(-1/z)\right) = 1 - \chi(z) \quad (87)$$

or in a simpler form \[24\]

$$\frac{1}{2} \left(\chi(z) + \chi((-1/z)\epsilon)\right) = 1. \quad (88)$$

This equation is important, because it relates the function $\phi(j)$ on two sheets of the Riemann surface

$$\frac{1}{2} \left(\phi\left(j - \epsilon + \sqrt{(j - \epsilon)^2 + 1}\right) + \phi\left(j - \epsilon - \sqrt{(j - \epsilon)^2 + 1}\right)\right) = 1, \quad (89)$$

where $j$ takes pure imaginary values in the interval $[-i, i]$. In the $z$-representation the above relation allows one to find the large-$z$ asymptotics of the function $\chi(z)$ providing that its behavior at $z = 0$ is known.

It is convenient to introduce the function

$$\xi(z) = \frac{z^2 + 1}{2z} \left(\chi(z) - \chi(z)\right) \quad (90)$$

with the inverse relation

$$\chi(z) = 2 \sum_{s=1}^{\infty} z^{-s} \frac{\xi(z - s\epsilon)}{\xi(z - s\epsilon + 1)} = \phi(j) = \sum_{s=1}^{\infty} \frac{\xi\left(j + s\epsilon + \sqrt{(j + s\epsilon)^2 + 1}\right)}{\sqrt{(j + s\epsilon)^2 + 1}}. \quad (91)$$

The function $\xi(z)$ has the following simple expansion in terms of coefficients $a_{n,\epsilon}$ entering in ansatz \[13\] for $\phi(j)$

$$\xi(z) = \sum_{n=1}^{\infty} z^{-n} \left(\delta_{n,1} - a_{n,\epsilon}\right) \quad (92)$$
and it is analytic at \(|z| > 1 - \delta\), where \(\delta > 0\). Moreover, this expression satisfies the dispersion-type relation
\[
\xi(z) = \int_{L} \frac{dz'}{2\pi i} \frac{\xi(z') - \xi(-1/z')}{z - z'}
\]
without any subtraction terms.

One can present equation (88) in the form
\[
\xi(z) - \xi(-1/z) = \frac{z^2 + 1}{z} (1 - \chi(z)).
\]
Really the analyticity of \(\xi(z)\) corresponding to Eqs. (92) and (93) together with expression (91) for \(\chi(z)\) are equivalent to the initial ES equation. There is an analogy of these formulas with the dispersion representation for the scattering amplitude and the unitarity condition allowing us to express its imaginary part in terms of probabilities of various processes. In our case the expression for \(\xi(z) - \xi(-1/z)\) is a superposition of functions \(\xi(z)\) with the shifted arguments \(z \to z - s\epsilon\).

In the \(j\)-representation Eq. (94) has the form
\[
\xi\left(j + \sqrt{j^2 + 1}\right) - \xi\left(j - \sqrt{j^2 + 1}\right) = 1 - \sum_{s=1}^{\infty} \xi\left(j + s\epsilon + \sqrt{(j + s\epsilon)^2 + 1}\right) \frac{1}{\sqrt{(j + s\epsilon)^2 + 1}}
\]
and it is equivalent to the ES equation providing that the analyticity condition for \(\xi(z)\) at \(|z| > 1\) is fulfilled.

For the singularities of \(\phi(j)\) on the second sheet of the \(j\)-plane we obtain
\[
(\phi_{\text{sing}}(j - \epsilon) - \phi_{\text{sing}}(j)) (j^2 + 1)^{1/2} = \frac{1}{(j^2 + 1)^{1/2} + j} - 2 \phi_{\text{sing}}(j) (j^2 + 1)^{1/2}.
\]
Inserting in it our ansatz (19) for \(\phi(j)\) as a linear combination of \(\phi_{n,\epsilon}(j)\) we obtain near the corresponding singularities
\[
(j^2 + 1)^{-1/2} \sum_{n=1}^{\infty} \left(j - (j^2 + 1)^{1/2}\right)^n a_{n,\epsilon} = 2 \sum_{n' \neq 1} \phi_{n',\epsilon}(j) (\delta_{n',1} - a_{n',\epsilon}).
\]
The left hand side of this equality should be analytically continued to the second sheet. We do not discuss its singularities at \(j = \pm i\) because they are canceled exactly (see (95)). The singularities existing in its right hand side should appear also in the left hand side as a result of the divergence of the sum over \(n\). Thus, the analytic properties of the ES equation lead to additional relations among the coefficients \(a_{n,\epsilon}\).

In more details, the right hand side has the square-root singularities situated in the points
\[
j = -s\epsilon \pm i, \; s = 1, 2, \ldots
\]
They appear in each term of the sum over \(n'\), because approximately we have near these singularities
\[
\frac{(j + s\epsilon + ((j + s\epsilon)^2 + 1)^{1/2})^{-n'}}{(j + s\epsilon)^2 + 1)^{1/2}} \approx \frac{(\pm i)^{-n'}}{(\pm 2i)^{1/2} (j + s\epsilon \mp i)^{1/2}}.
\]
As a result, the condition for the appearance of the singularities of $\phi(j)$ on the second sheet of the $j$-plane at $j \to -s\epsilon \pm i$ has the form

$$
\sum_{n=1}^{\infty} \left( j + (j^2 + 1)^{1/2} \right)^n a_{n,\epsilon} = -\frac{2(s^2 \epsilon^2 \mp 2i s \epsilon)^{1/2}}{(\pm 2i)^{1/2} (j + s \epsilon \mp i)^{1/2}} C_\pm(\epsilon),
$$

where

$$
j + (j^2 + 1)^{1/2} = \left(-s\epsilon \pm i + (s^2 \epsilon^2 \mp 2i s \epsilon)^{1/2}\right) \exp\left(\frac{j + s \epsilon \mp i}{\sqrt{s^2 \epsilon^2 \mp 2i s \epsilon}}\right),
$$

and

$$
C_\pm(\epsilon) = \sum_{n' = 1}^{\infty} (\pm i)^{-n'} (\delta_{n',1} - a_{n',\epsilon}).
$$

Thus, to reproduce the singularity at $j \to -s\epsilon \pm i$ the large-$n$ asymptotics of the coefficients $a_{n,\epsilon}$ should contain the contribution

$$
\lim_{n \to \infty} a_{n,\epsilon} \approx -2\Re \left( \frac{-s\epsilon + i + (s^2 \epsilon^2 - 2i s \epsilon)^{1/2}}{\sqrt{n}} - 2(s^2 \epsilon^2 - 2i s \epsilon)^{1/4} C_+(\epsilon) \right).
$$

In above transformations we used the relations

$$
\lim_{x \to 1} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} x^n = \sqrt{\pi} (1 - x)^{-1/2}
$$

and

$$
\left(1 - \frac{j + (j^2 + 1)^{1/2}}{-s\epsilon \pm i + (s^2 \epsilon^2 \mp 2i s \epsilon)^{1/2}}\right)^{-1/2} \approx (s^2 \epsilon^2 \mp 2i s \epsilon)^{1/4} (-s\epsilon \pm i - j)^{-1/2}.
$$

The branch of the square-root in the expression $j + (j^2 + 1)^{1/2}$ is defined in such way, that it grows on the second sheet at large $j$ in all directions of the complex plane. Therefore in the perturbation theory $\epsilon \to \infty$ the sum over $n$ is convergent inside the large circle $|j| < |\epsilon|$ because for $s = 1$ we have

$$
\lim_{n \to \infty} a_{n,\epsilon} \approx -2\Re \left( \frac{-\epsilon + i - \epsilon(1 - 2i/\epsilon)^{1/2}}{\sqrt{n}} - 2(-\epsilon)^{1/2} (1 - 2i/\epsilon)^{1/4} C_+(\epsilon) \right).
$$

In an opposite limit of the strong coupling $\epsilon \to 0$ the nearest singularity is situated at $j = \epsilon \pm i \to \pm i$. Therefore the radius of convergence of the sum over $n$ tends to zero for $j \to \pm i$ and $\phi(j)$ has a singularity at $j \to \pm i$.

Note, that if one introduce the function

$$
\Phi(j) = \phi(j - \epsilon) - \phi(j) = (j^2 + 1)^{-1/2} \sum_{n=1}^{\infty} \left(j - (j^2 + 1)^{1/2}\right)^n (\delta_{n,1} - a_{n,\epsilon}),
$$

we obtain the following expression for its singular part on the second sheet

$$
\Phi_{\text{sing}}(j) = \frac{(j^2 + 1)^{-1/2}}{(j^2 + 1)^{1/2} + j} - 2 \sum_{s=1}^{\infty} \Phi(j + s \epsilon).
$$
6 Strong coupling limit of the ES equation

Let us consider now the strong coupling regime

\[ \epsilon \to 0. \] (107)

To find possible corrections at large coupling constants it is helpful to use the expansion of \( z_\epsilon \) (78) at small \( \epsilon \)

\[ z_\epsilon = z - 2 \frac{z^2}{1 + z^2} \epsilon + 4 \frac{z^3}{(1 + z^2)^3} \epsilon^2 + 4 \frac{z^2(1 - z^2)(1 + z^4)}{(1 + z^2)^5} \epsilon^3 + O(\epsilon^3). \] (108)

The SE equation for \( \epsilon \to 0 \) is simplified

\[ -\epsilon z \frac{\partial}{\partial z} \chi(z) = \frac{1}{z} - \int_L \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z'} \chi(z'), \] (109)

where the integration contour is the unit circle passing in the anti-clock direction and \( \tilde{z} \) is defined in terms of \( z \) in eq. (80).

The substitution \( z' \to \tilde{z}' \) in the argument of \( \chi \) is in agreement with the linear set of equations (22) for the coefficients of \( \chi(z) \) in its expansion at \( z \to \infty \) and small \( \epsilon \) (cf. (19))

\[ \chi(z) = \sum_{n=1}^{\infty} \frac{z^{-n}}{\epsilon n} (\delta_{n,1} - a_{n,\epsilon}). \] (110)

Indeed, \( a_{n,\epsilon} \) has the following representation

\[ a_{n,\epsilon} = \int_L \frac{dz'}{2\pi i} (z'^2 + 1) z'^{(n-2)} \chi(z') = \sum_{n'=1}^{\infty} K_{n,n'}(\epsilon) (\delta_{n',1} - a_{n',\epsilon}). \] (111)

Therefore the kernel in the strong coupling case \( \epsilon \to 0 \) is

\[ K_{n,n'}(\epsilon) = \frac{1}{\epsilon n'} \int_{-i}^{i} \frac{dz'}{2\pi i} (z'^2 + 1) \chi(z') = \sum_{n'=1}^{\infty} \frac{1}{\Gamma\left(\frac{n'+3}{2}\right) \Gamma\left(\frac{n'+3}{2}\right)} \left(\frac{n-n'+3}{2}\right)^{n'} \, \frac{n}{(n+n')^2 - 1} \frac{1}{(n-n')^2 - 1}, \] (112)

which is in an agreement with (67).

By an anti-symmetrization of Eq. (109) to the substitution \( z \to -1/z \) we can obtain the relation

\[ -\epsilon \frac{\partial}{\partial z} (\chi(z) + \chi(-1/z)) = \frac{1 + z^2}{z} - \int_L \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z'} \left(\frac{\chi(z')}{z - z'} - \frac{\chi((-1/z)')}{z - z'}\right), \] (113)

where at the integrand it is assumed, that \( |z| \to 1 \) in the first term from above and in the second term from below.
From relation (88) valid in a general case one could obtain for \( \epsilon \to 0 \)
\[- \epsilon \frac{z^2}{z^2 + 1} \frac{\partial}{\partial z} (\chi(z) + \chi(-1/z)) = 1 - \chi(z), \]  
(114)
but this relation is valid for \( \chi \) satisfying to a different equation in comparison with eq. (109). Namely, in it \( z' \) should be substituted by \( z' \).

The singular part of the homogeneous equation for \( \chi \) inside the circle \(|z| < 1\) satisfies the equation
\[- \epsilon z \frac{\partial}{\partial z} \chi_{\text{hom}}(z) = -\text{Sing} \left( \frac{z^2 + 1}{z} \chi_{\text{hom}}(z) \right), \]  
(115)
where the sign \( \text{Sing} \) means, that in the right hand side only singular terms \( \sim z^{-r} \) \((r = 1, 2, ...)\) are left.

Its solution is
\[ \chi_{\text{hom}}(z) = \int_L \frac{dz'}{2\pi i z'} \frac{z}{z'} \exp \frac{z^2 - 1}{z'} (z - z'). \]  
(116)

The additional factor \( z/z' \) in the integrand leads to the constant term \( d_0 \) in the large-\( z \) expansion of \( \chi_{\text{hom}} \)
\[ \chi_{\text{hom}}(z) = \sum_{k=0}^{\infty} \frac{d_k}{z^k}, \]  
(117)
appearing in an agreement with singularities of the homogeneous equation.

Using the following relation
\[ \exp \left( \frac{z - z^{-1}}{\epsilon} \right) = \sum_{n=-\infty}^{\infty} z^n J_n(2\epsilon^{-1}), \quad J_n(x) = (-1)^n J_n(x), \]  
(118)
where \( J_n(x) \) are the Bessel function, we can present \( \chi_{\text{hom}}(z) \) as follows
\[ \chi_{\text{hom}}(z) = \sum_{n=-\infty}^{\infty} (-z)^{-n} J_n(2\epsilon^{-1}). \]  
(119)

Let us attempt to find the singular part of the solution \( \chi(z) \) of the inhomogeneous ES equation in the form of the expansion
\[ \chi_{\text{sing}}(z) = \sum_{k=1}^{\infty} \frac{d_k}{z^k}, \]  
(120)
where the regular term with \( k = 0 \) is absent. One can expect just such behavior of \( \chi(z) \) at large \( z \) and large coupling constants if the string theory prediction for the anomalous dimension is right. We obtain for the coefficients \( d_k \) the relations
\[ \epsilon d_1 = 1 - d_2 \]  
(121)
and
\[ n\epsilon d_n = -d_{n-1} - d_{n+1}. \]  
(122)
for \( n = 2, 3, \ldots \).

They are recurrent relations for the Bessel functions \( J_n(2\epsilon - 1) \) and \( Y_n(2\epsilon - 1) \), where

\[
Y_n(x) = \lim_{\nu \to n} \pi^{-1} \left( \frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{J_\nu(x)}{\partial \nu} \right).
\]  

(123)

The last function has the logarithmic singularities in \( x = 2\epsilon - 1 \) due to the equality

\[
\pi Y_n(x) = 2 J_n(x) \ln(x^2) - \sum_{m=0}^{n-1} \frac{(x^2 - 1)!}{m!} - \sum_{l=0}^{\infty} \frac{(x^2 - 1)^{n+l+1}}{l!(n+l)!} \psi(n+l+1) + \psi(l+1) l! (n+l)!.
\]  

(124)

incompatible with the expansion over \( \epsilon^{-1} \) in the perturbation theory. Therefore we neglect this function and write the solution of recurrent relations (121), (122) only in terms of the simple Bessel function \( J_n(2\epsilon - 1) \) as follows

\[
d_k = (-1)^{k+1} \frac{J_k(2\epsilon - 1)}{J_0(2\epsilon - 1)}.
\]  

(125)

Thus, to remove the zero mode one should impose on the solution the constraint corresponding to the possibility of its perturbative expansion.

For the singular part of the solution at small \( z \) we obtain the explicit expression

\[
\chi_{\text{sing}}(z) = -\frac{1}{J_0(2\epsilon - 1)} \int_L \frac{dz'}{2\pi i} \frac{\exp\left(\frac{z'^2 - 1}{2\epsilon}ight)}{z - z'}.
\]  

(126)

Note, that \( \chi_{\text{sing}}(z) \) is the solution of the equation

\[
-\epsilon z \frac{\partial}{\partial z} \chi_{\text{sing}}(z) = \frac{1}{z} - \int_L \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z'} \chi_{\text{sing}}(z').
\]  

(127)

In comparison with eq. (109) here \( z' \) is substituted by \( z' \). In principle we can obtain the equation with such substitution starting from the ES equation (82) in the strong coupling limit \( \epsilon \to 0 \). Thus, there is an uncertainty in the limiting form of the ES equation. Presumably it is related to the fact, that the function \( \phi(j) \) (19) in the limit \( \epsilon \to 0 \) develops a singularity in the point \( j = \pm i \) and therefore relation (81) is violated. We have shown above, that from eq. (109) one can obtain the linear system of equations with the kernel (112) coinciding with expression (67), derived from the Mellin-Barnes representation (41). But this representation is only one of possibilities to sum the divergent series (41). Another way of its resummation could lead to the kernel corresponding to eq. (127). At least the results obtained from equations (109) and (127) are presumably close each to another.

Having the above arguments in mind, with the use of eqs (12), (13) we obtain for the anomalous dimension the following result [24]

\[
\gamma_{\text{sing}} = 2 \frac{J_1(2\epsilon - 1)}{\epsilon J_0(2\epsilon - 1)}.
\]  

(128)
In particularly in the weak coupling regime we have in an agreement with the perturbation theory
\[
\lim_{g \to 0} \tilde{\gamma} \approx \frac{2}{\epsilon^2} = 4 g^2.
\] (129)

In the strong coupling regime this expression is simplified as follows [24]
\[
\lim_{g \to \infty} \tilde{\gamma} \approx 2 \sqrt{2} g \frac{\cos(\frac{2}{\epsilon} - \frac{3\pi}{4})}{\cos(\frac{2}{\epsilon} - \frac{\pi}{4})} = 2 \sqrt{2} g \tan(\frac{2}{\epsilon} - \frac{\pi}{4}).
\] (130)

The last result should be compared with the prediction of Polyakov and collaborators [26]
\[
\gamma_{Pol} = 2 \sqrt{2} g.
\] (131)

Therefore from the ES equation we obtain the anomalous dimension which in the strong coupling limit rapidly oscillates around the string prediction.

Using the perturbative expansion of the Bessel functions we obtain in the weak coupling limit
\[
\gamma_{sing} = \frac{2}{\epsilon} \frac{1}{\epsilon} - \frac{1}{2} \frac{1}{\epsilon^2} + \frac{1}{2} \frac{1}{\epsilon^3} = \frac{2}{\epsilon^2} + \frac{1}{\epsilon^3} - \frac{4}{3} \frac{1}{\epsilon^6}.
\] (132)

It agrees in the Born approximation \(\sim 1/\epsilon^2\) with the solution of the SE equation, but in upper orders for the simplified version of the equation, where the factor \(t/(e^t - 1)\) is substituted by unity, there are corrections of the odd order in \(1/\epsilon\). Indeed, in the leading order
\[
\chi^{(1)}(z) = \frac{1}{\epsilon z}.
\]

The next-to-leading correction satisfies the equation
\[
-\epsilon z \frac{\partial}{\partial z} \chi^{(2)}(z) = -\frac{1}{\epsilon} \int_{-i}^{i} \frac{dz'}{2\pi i} \frac{z'^2 + 1}{z' - z} \left( \frac{z'}{z'} + \frac{1/z'}{z + 1/z'} \right)
\]
\[
= -\frac{1}{2\pi i \epsilon} \left( \frac{(1 + z^2)^2}{z^2} \ln \frac{z + i}{z - i} + \frac{\pi i}{z^2} + \frac{2i}{z} - 2iz \right).
\] (133)

Therefore
\[
\chi^{(2)}(z) = \frac{1}{2\pi i \epsilon^2} \int_{\infty}^{z} dz' \left( \frac{(1 + z'^2)^2}{z'^3} \ln \frac{z' + i}{z' - i} + \frac{\pi i}{z'^3} + \frac{2i}{z'^2} - 2i \right) =
\]
\[
\frac{1}{2\pi i \epsilon^2} \left( 2 \int_{\infty}^{z} \frac{dz'}{z'} \ln \frac{z' + i}{z' - i} + \frac{z^4 - 1}{2z^2 \ln \frac{z + i}{z - i} - \frac{\pi i}{2z^2} - i} \right).
\] (134)

In particular, one can obtain the following asymptotic behavior of \(\chi^{(2)}(z)\)
\[
\lim_{z \to \infty} \chi^{(2)}(z) = -\frac{16}{3\pi \epsilon^2} z = -\frac{8}{3\pi \epsilon^2} \frac{i}{z}.
\] (135)
which leads to the correction to the anomalous dimension
\[ \gamma^{(2)} = -\frac{32}{3\pi \epsilon^3}. \] (136)

In the real case, when we do not neglect the factor \( t/(e^t - 1) \) such corrections \( \sim 1/\epsilon^3 \) are absent.

Let us investigate now the singular behavior of \( \chi(z) \) at \( z \to \pm i \). For this purpose we calculate the discontinuity of the SE equation on the cut \(-i < z < i\)

\[ \epsilon z \frac{\partial}{\partial z} (\chi(z + 0) - \chi(z - 0)) = \frac{z^2 + 1}{z} (\chi(z + 0) - \chi(-1/z)) . \] (137)

On the other hand, the functions \( \chi(z) \) and \( \chi(-1/z) \) can be expanded near the points \( z = \pm i \) in the Taylor series. In two first orders we obtain

\[ \chi(z + 0) - \chi(-1/z) = 2\chi'(\pm i) \left( z \mp i \pm \frac{i}{2} (z \mp i)^2 \right) . \] (138)

Simplifying the right hand side of the equation near the points \( z = \pm i \) we obtain

\[ \epsilon \frac{\partial}{\partial z} (\chi(z + 0) - \chi(z - 0)) \approx \mp 4i \chi'(\pm i) (z \mp i)^2 \left( 1 \pm \frac{i}{2} (z \mp i) \right) . \] (139)

Therefore the singularities at the point \( z = \pm i \) are very soft

\[ \chi(z)_{z \to \pm i} = \frac{4}{3\epsilon} \chi'(\pm i) (z \mp i)^3 \left( 1 + \frac{3i}{4} (z \mp i) \right) . \] (140)

This formula is valid also in the perturbation theory \( \epsilon \to \infty \), where

\[ \chi'(\pm i) \approx \frac{1}{\epsilon} . \] (141)

The knowledge of the singularity of the function \( \chi(z) \) for \( z \to \pm i \) gives a possibility to find the asymptotic behavior of the coefficients \( a_{n,\epsilon} \) for \( n \to \infty \) in ansatz (110). Indeed, it is in an agreement with the expansion of this function at large \( z \) providing that the coefficients \( a_{n,\epsilon} \) behave at large \( n \) as follows

\[ \lim_{n \to \infty} a_{n,\epsilon} = \frac{8}{\pi} \Re \chi'(i) \frac{i^{n+3}}{n^3} , \] (142)

where we used the relation

\[ \lim_{x \to 1} \sum_{n=1}^{\infty} \frac{x^{-n}}{n^4} = \frac{(x-1)^3}{3!} \ln(x-1) , \quad x = \frac{z}{\pm i} \] (143)

for the singularity of the sum.

The value \( \chi'(\pm i) \) can be expressed in terms of coefficients \( a_{n,\epsilon} \) as follows

\[ \chi'(\pm i) = -\sum_{n=1}^{\infty} \frac{(\pm i)^{n-1}}{\epsilon} (\delta_{n,1} - a_{n,\epsilon}) . \] (144)
One can consider the ES equation on the \( z \)-plane, which includes the second sheet of the \( j \)-plane with the cuts going from the points \( z = i \) and \( z = -i \) to \( z = i \infty \) and \( z = -i \infty \), respectively. We can present the function on this plane through the dispersion integrals over these cuts and through the above singular contribution \( \chi_{\text{sing}}(z) \)

\[
\chi(z) = \int_{i}^{i \infty} \frac{dz'}{2\pi i} \frac{\chi(z') - \chi(-1/z')}{z' - z} + \int_{-i \infty}^{-i} \frac{dz'}{2\pi i} \frac{\chi(z') - \chi(-1/z')}{z' - z} + \chi_{\text{sing}}(z),
\]

where \( \chi(z') \) is defined as an analytic continuation from the left part of the cuts. The functions \( \chi(-1/z') \) have their argument on the line between \(-i\) and \(i\) where the cut is absent.

### 7 Beisert-Eden-Staudacher equation

Recently Beisert, Eden and Staudacher (BES) \[23\] derived a new equation for the anomalous dimension at large \( j \). In this equation the authors took into account the phase factor resulting from the necessity to have an agreement with calculations on the superstring side in the framework of the AdS/CFT correspondence \[27\]. In Appendix A we analytically continue this phase factor to the weak coupling regime and obtain for it the expression coinciding with that of Ref. \[23\]. Let us redefine the function \( f(x) \) satisfying the ES equation (6)

\[
f(x) = \frac{t}{e^t - 1} F(x), \quad t = \epsilon x
\]

and introduce the operators \( K_0 \) and \( K_1 \) acting on the new function

\[
K_r F(x) = \int_{0}^{\infty} dx' \frac{t'}{e^{t'} - 1} K_r(x, x') F(x'),
\]

where the integral kernels are (cf. (10))

\[
K_0(x, y) = \frac{2}{x y} \sum_{r=1}^{\infty} (2r - 1) J_{2r-1}(x) J_{2r-1}(y), \quad K_1(x, y) = \frac{2}{x y} \sum_{r=1}^{\infty} 2r J_{2r}(x) J_{2r}(x),
\]

\[
K(x, y) = K_0(x, y) + K_1(x, y).
\]

In particular the operators \( K_0 \) and \( K_1 \) act on the Dirac \( \delta \)-function as follows

\[
K_0 \delta(x) = \frac{J_1(x)}{x}, \quad K_1 \delta(x) = 0.
\]

The BES equation \[23\] in the operator form can be written as follows

\[
\epsilon F(x) = \left(1 + \frac{2}{\epsilon} K_1\right) K_0 \delta(x) - \left(K_0 + K_1 + \frac{2}{\epsilon} K_1 K_0\right) F(x).
\]
It is convenient to separate the solution in its symmetric and antisymmetric parts under the substitution $x \rightarrow -x$

$$F(x) = F_+(x) + F_-(x), \ F_\pm(-x) = \pm F_\pm(x). \quad (151)$$

Then the BES equation is equivalent to the set of two equations

$$\epsilon F_+(x) = K_0 \delta(x) - K_0 (F_+(x) + F_-(x)),$$
$$\epsilon F_-(x) = \frac{2}{\epsilon} K_1 K_0 \delta(x) - K_1 \left(1 + \frac{2}{\epsilon} K_0 \right) (F_+(x) + F_-(x)). \quad (152)$$

By finding the formal solution of the second equation for $F_-(x)$ and inserting it in the first equation for $F_+(x)$ we get

$$\left(1 + K_0 \frac{1}{\epsilon + K_1 \left(1 + \frac{2}{\epsilon} K_0 \right)} \right) F_+(x) = K_0 \frac{1}{\epsilon + K_1 \left(1 + \frac{2}{\epsilon} K_0 \right)} \delta(x). \quad (153)$$

Because $F_+(x)$ is even, we can present it as follows

$$F_+(x) = K_+ \phi(x), \quad (154)$$

where $\phi(x)$ is the function which does not have any symmetry under the substitution $x \rightarrow -x$. From (153) we obtain the following equation for $\phi(x)$

$$\left(\epsilon + K_0 + K_1 + \frac{2}{\epsilon} K_1 K_0 \right) \phi(x) = \delta(x), \quad (155)$$

where it was assumed, that the operator

$$O = K_0 \frac{1}{\epsilon + K_1 + \frac{2}{\epsilon} K_1 K_0}$$

does not have zero modes.

Let us show, that one can derive the same equation for $\phi(x)$ starting from a simpler set of equations

$$\epsilon F_+(x) = K_0 \delta(x) - K_0 (F_+(x) + iF_-(x)),$$
$$\epsilon F_-(x) = -K_1 (iF_+(x) + F_-(x)). \quad (156)$$

Indeed, by finding $F_-(x)$ from the second equation and inserting it in the first equation we have

$$\epsilon \left(1 + K_0 \frac{1}{\epsilon + K_1 \left(1 + \frac{2}{\epsilon} K_1 \right)} \right) F_+(x) = K_0 \delta(x). \quad (157)$$

Therefore by introducing the function $\phi(x)$ as above, we obtain for it the same equation. Of course, the function $F_-(x)$ will be different in two cases, but we need only the function $F_+(x)$, because the anomalous dimension is expressed only in terms of it

$$\gamma = \frac{4}{\epsilon} F_+(0). \quad (158)$$

23
Note, that the BES equation for $F(x)$ has the form similar to the equation for $\phi(x)$, but with another term in the right hand side. It is fulfilled, which can be verified by adding the equations for $F_{\pm}(x)$.

Similar to the case of the ES equation we can construct the set of linear algebraic equations equivalent to the system (156) if we write the solution as superpositions of the Bessel functions (cf. (19))

$$F_{+}(x) = \sum_{r=1}^{\infty} (2r - 1) \frac{J_{2r-1}(x)}{x} (\delta_{r,1} - \tilde{a}_{2r-1}) \ , \ F_{-}(x) = - \sum_{r=1}^{\infty} (2r) \frac{J_{2r}(x)}{x} \tilde{a}_{2r} .$$

Then the anomalous dimension is given by the same expression as earlier (cf. (21))

$$\gamma = \frac{2}{\epsilon^2} \left( 1 - \tilde{a}_1 \right) ,$$

but for the coefficients $\tilde{a}_k$ now we have another system of algebraic equations (cf. (22))

$$\tilde{a}_{2r-1} = \sum_{r'=1}^{\infty} K_{2r-1,2r'-1}(\epsilon) \left( \delta_{r',1} - \tilde{a}_{2r'-1} \right) - i \sum_{r'=1}^{\infty} K_{2r-1,2r'}(\epsilon) \tilde{a}_{2r'} ,$$

$$\tilde{a}_{2r} = - \sum_{r'=1}^{\infty} K_{2r,2r'}(\epsilon) \tilde{a}_{2r'} + i \sum_{r'=1}^{\infty} K_{2r,2r'-1}(\epsilon) \left( \delta_{r',1} - \tilde{a}_{2r'-1} \right) ,$$

where the kernel $K_{n,n'}(\epsilon)$ is given by expression (40). Therefore the coefficients of the perturbative expansion of $\gamma(\epsilon)$

$$\gamma(\epsilon) = -8 \sum_{k=1}^{\infty} \left( -\frac{1}{4\epsilon^2} \right)^k \tilde{c}_k$$

are calculated from the expression similar to (53)

$$\tilde{c}_k = \sum_{r=0}^{\infty} \tilde{S}_{k}^{r} .$$

The quantity $\tilde{S}_{k}^{r}$ is presented below (cf. (54))

$$\tilde{S}_{k}^{(r)} = \sum_{s_1=2}^{\infty} \sum_{s_2=2}^{\infty} \ldots \sum_{s_r=2}^{\infty} (-1)^{\tilde{N}_{s_1,s_2,\ldots,s_r}} U_{s_1,s_2,\ldots,s_r} \prod_{i=1}^{r} \zeta(s_i) ,$$

where $U_{s_1,s_2,\ldots,s_r}$ are defined by expression (56) and due to the factors $i$ in coefficients of Eqs. (161) we have an additional contribution to the integer number $N_{s_1,s_2,\ldots,s_r}$ presented in expression (61)

$$\tilde{N}_{s_1,s_2,\ldots,s_r} = \sum_{l} l + \frac{1}{2} \sum_{l} 1 \ , \ s_l = 2k + 1 .$$

The additional contribution is a half of the number of $s_l$ having odd values among $r$ of them. We remind, that the number of such $s_l$ is even.
We can generalize the ES equation (71) written for the function obtained by the Laplace transformation (11) to the BES case. We consider only the strong coupling regime, where the ES equation in the $z$-representation (74) is simplified (see (109)). For the case of the BES case one should introduce two functions satisfying the set of equations

\[
-\epsilon z \frac{\partial}{\partial z} \chi_+(z) = \frac{1}{z} - \frac{1}{2} \int_L \frac{d z'}{2\pi i} \frac{z'^2 + 1}{z'} \left( \frac{1}{z - z'} + \frac{1}{z + z'} \right) \left( \chi_+ (\tilde{z}') + i \chi_- (\tilde{z}') \right),
\]

\[
-\epsilon z \frac{\partial}{\partial z} \chi_-(z) = -\frac{1}{2} \int_L \frac{d z'}{2\pi i} \frac{z'^2 + 1}{z'} \left( \frac{1}{z - z'} - \frac{1}{z + z'} \right) \left( \chi_- (\tilde{z}') + i \chi_+ (\tilde{z}') \right). \tag{166}
\]

Here the integral is performed over the unit circle in the anti-clock direction. The variable $\tilde{z}'$ coincides with $z'$ on the right part of the contour $L$ and $\tilde{z}' = -z'^{-1}$ on its left part. The variable $z$ initially is outside the unit circle, but we analytically continue the integrals to the region $|z| < 1$. In the next section we discuss the solution of eq. (166).

8 Strong coupling limit of the BES equation

By simplifying Eqs. (161) at $\epsilon \to 0$, we can substitute the set (121), (122) of equations for coefficients of the singular solution (120) of the ES equation by the corresponding equations for the BES case

\[
\chi_{\text{BES}}^{\text{sing}} (z) = \sum_{k=1}^{\infty} \hat{d}_k \frac{z^k}{z^k}, \tag{167}
\]

\[
\epsilon \hat{d}_1 = 1 - i \hat{d}_2 \tag{168}
\]

and

\[
n \epsilon \hat{d}_n = -i \hat{d}_{n-1} - i \hat{d}_{n+1}. \tag{169}
\]

for $n = 2, 3, \ldots$

They are recurrent relations for the Bessel functions $J_n(-i2\epsilon^{-1})$ and $Y_n(-i2\epsilon^{-1})$. But similar to the section 6, one can argue that the solution of the above recurrent relations should contain only the Bessel functions $J_n(i2\epsilon^{-1}) = i^n I_n(2\epsilon^{-1})$, because the function $Y_n(i2\epsilon^{-1})$ has the singularities incompatible with the perturbation theory expansion (see (124)). So, we obtain (cf. (125))

\[
\hat{d}_k = (-i)^{k-1} \frac{I_k(2\epsilon^{-1})}{I_0(2\epsilon^{-1})}. \tag{170}
\]

It corresponds to the singular solution of the BES equation in the form (cf. (126))

\[
\chi_{\text{sing}}^{\text{BES}} (z) = \frac{i}{I_0(2\epsilon^{-1})} \int_L \frac{d z'}{2\pi i} \exp \frac{1 - z'^2}{i \epsilon z'}. \tag{171}
\]

Again, as in the case of the ES equation we stress, that $\chi_{\text{sing}}^{\text{BES}} (z)$ can be considered as a solution of the equations obtained from the strong coupling BES equations (166) by the substitution $\tilde{z}' \to z'$ (cf. (82)). As earlier there is an uncertainty in the limiting
procedure also for the BES equation. However we believe, that the results obtained from the equations derived in the strong coupling limit from the different forms of the BES equation should be close each to others. Therefore assuming, that one can calculate the anomalous dimension from the large-$z$ asymptotics of the singular solution we obtain for the anomalous dimension of the BES equation the result (cf. (128))

$$
\gamma_{\text{BES}}^{\text{sing}} = \frac{2}{\epsilon} d_1 = \frac{2}{\epsilon} \frac{I_1(2\epsilon^{-1})}{I_0(2\epsilon^{-1})}.
$$

In the strong coupling regime $\epsilon \to 0$ the expression for $\gamma_{\text{BES}}^{\text{sing}}$ is significantly simplified

$$
\lim_{g \to \infty} \gamma_{\text{BES}}^{\text{sing}} \approx 2 - \frac{1}{2} - \frac{1}{16} \epsilon = 2 \sqrt{2} g - \frac{1}{2} - \frac{\sqrt{2}}{32 g},
$$

because the large-$t$ asymptotic of $I_n(t)$ is following

$$
I_n(t) = \frac{e^t}{\sqrt{2\pi t}} \left[ 1 - \frac{4n^2 - 1}{8t} + \frac{(4n^2 - 1)(4n^2 - 9)}{2!(8t)^2} + O\left(\frac{1}{t^3}\right) \right].
$$

The result (173) should be compared with the string prediction [26]

$$
\gamma_{\text{Pol}} = 2 \sqrt{2} g - \frac{3}{\pi} \ln 2 \approx 2 \sqrt{2} g - 0.661907.
$$

So, we reproduced exactly the leading behavior for $\gamma_{\text{Pol}}$ but the NLO Frolov-Zeitlin coefficient is obtained only with 30% accuracy. Note, however, that there are additional corrections to Eqs. (168) and (169) which should be responsible for the difference (see Appendix D and discussions therein).

9 Conclusion

In this paper we investigated the solutions of the ES and BES equations for the anomalous dimension at the large spin $s$. Each of these integral equations was presented as a set of linear algebraic equations (22), (161) with the kernel expressed in an explicit form (41), which gave us a possibility to find the convergence radius of the small coupling expansion of this kernel and to calculate its strong coupling asymptotics. In particular we proved the maximal transcendentality property of the perturbative expansion for the anomalous dimension combined with the the ES hypothesis about the integer coefficients in front of the product of the $\zeta$-functions. Performing the inverse Laplace transformation of the solution together with its subsequent conformal mapping to the $z$-plane we reduced the ES equation to the simple functional relations (91), (94) valid for all coupling constants. The similar relations are valid also for the solution of the BES equation. In the strong coupling limit the singular behavior of solutions of these two equations at $z \to 0$ was constructed (see (129) and (171)). Assuming, that this singular behavior is valid also for large $z$, we calculated the corresponding anomalous dimensions at large coupling.
constants (128), (172). The asymptotic behavior of the anomalous dimension obtained from the BES equation at $g \to \infty$ (173) is in the agreement with the string predictions (see [26]). The difference in corrections to these asymptotic expressions is presumably related to simplifications maid by us to get the exact singular solution (171)). Thus, we demonstrated above that the AdS/CFT correspondence together with the integrability and transcendentality incorporated in the Beisert-Eden-Staudacher equation allow one to construct the asymptotic behavior of the anomalous dimensions of twist-2 operators at $s \to \infty$ for $N = 4$ SUSY for all coupling constants. However, because we do not know how to find corrections to the singular solution (171)) of this equation in a regular way, the problem of finding next-to-leading terms $\epsilon \sim 1/g$ for expression (172) remains to be solved.

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Appendix A. Dressing phase

The dressing phase factor according to Refs. [23, 28] should have the following form

$$\exp(i\theta_{12}) = \exp \left[ i \sum_{r=2}^{\infty} \sum_{s=r+1}^{\infty} c_{r,s} \left( q_r \left( x_1^{\pm} \right) q_s \left( x_2^{\pm} \right) - q_s \left( x_1^{\pm} \right) q_r \left( x_2^{\pm} \right) \right) \right].$$  \hspace{1cm} (A1)

Here $q_r(x)$ are the conserved magnon charges

$$q_r \left( x^{\pm} \right) = \frac{i}{r-1} \left( \frac{1}{(x^+)^{r-1}} - \frac{1}{(x^+)^{r-1}} \right)$$  \hspace{1cm} (A2)

and $c_{r,s}$ are some coefficients depending on the coupling constant $g = \sqrt{\lambda}/4\pi$, where $\lambda$ is the ’t Hooft coupling.

The strong coupling expansion of $c_{r,s}$ was obtained on the string theory side [29, 30, 27]

$$c_{r,s} = \left( 1 - \left( -1 \right)^{r+s} \right) \frac{1}{2} (r-1)(s-1) \hat{c}_{r,s}, \quad \hat{c}_{r,s} = \sum_{n=0}^{\infty} \hat{c}_{r,s}^{(n)} g^{1-n}. \hspace{1cm} (A3)$$

Here the coefficients

$$\hat{c}_{r,s}^{(n)} = \frac{1}{(2\pi)^n} \frac{\zeta(n)}{\Gamma(n-1)} \Gamma \left[ \frac{1}{2} (s + r + n - 3) \right] \Gamma \left[ \frac{1}{2} (s - r + n - 1) \right]$$

\hspace{2cm} \Gamma \left[ \frac{1}{2} (s - r - n + 3) \right] \Gamma \left[ \frac{1}{2} (s + r - n + 1) \right]$$  \hspace{1cm} (A4)

were derived in [27] with the use of the crossing symmetry [31].

Recently the week coupling expression for $c_{r,s}$ was suggested in [23] in the form of the expansion

$$\hat{c}_{r,s} = - \sum_{n=0}^{\infty} \hat{c}_{r,s}^{(-n)} g^{1+n}, \hspace{1cm} (A5)$$

The purpose of this appendix is to prove expression (A5).

### A.1 Basic formulas

Eq. (A3) means that only the odd integer values of the sum $r + s$ are important, i.e.

$$r + s = 2m + 1, \hspace{1cm} (A6)$$

where $m \geq 2$, because $r \geq 2$ and $s \geq 3$.

Further, following to Ref. [23] it is useful to extract from the sum in (A2) the first two terms:

$$\hat{c}_{r,s} = \hat{c}_{r,s}^{(1)} g + \hat{c}_{r,s}^{(0)} + B_{r,m}, \hspace{1cm} (A7)$$
where
\[ c_{r,s}^{(0)} = \frac{\delta_{r+1,s}}{(r-1)r} = \frac{\delta_{r,m}}{(r-1)r}, \]  
\[ c_{r,s}^{(1)} = -\frac{2}{\pi} \frac{1}{(s+r-2)(s-r)} = -\frac{2}{\pi} \frac{1}{(2m-1)(2m-r+1)}, \]  
\[ B_{r,m} = \sum_{l=0}^{\infty} \hat{B}_{r,m}^{(l)} g^{-(l+1)} \]

and
\[ B_{r,m}^{(l)} = \frac{1}{(-2\pi)^{l+2}} \frac{\zeta(l+2)}{\Gamma(l+1)} \frac{\Gamma(m+l/2)\Gamma(m+1-r+l/2)}{\Gamma(m-l/2)\Gamma(m+1-r-l/2)} \]

Let us consider now eq. (A5). According to [23], \( \hat{c}_{r,s}^{(n)} \) can be presented using the relation
\[ \zeta(1-z) = 2(2\pi)^{-z} \cos \left( \frac{\pi z}{2} \right) \Gamma(z) \zeta(z), \quad \Gamma(1-z) = \frac{\pi}{\sin(\pi z)\Gamma(z)} \]  
in the form (for odd values of \( r+s \))
\[ \hat{c}_{r,s}^{(n)} = \frac{2(-1)^{s-1-n} \cos \left( \frac{\pi}{2} n \right) \zeta(1-n)\Gamma(2-n)\Gamma(1-n)}{\Gamma \left[ \frac{1}{2}(5-n-r-s) \right] \Gamma \left[ \frac{1}{2}(3-n+r+s) \right] \Gamma \left[ \frac{1}{2}(3-n-r+s) \right] \Gamma \left[ \frac{1}{2}(1-n+r+s) \right]} \]

For \( r+s = 2m+1 \) and \( n = 2k \) (\( \cos[\pi(k+1/2)] = 0 \)) and, thus, \( \hat{c}_{r,s}^{(-2k-1)} = 0 \) the eq. (A5) can be rewritten as follows
\[ \hat{c}_{r,s} = -2 \sum_{k=0}^{\infty} (-1)^{k+r} \frac{\zeta(1+2k)\Gamma(2+2k)\Gamma(1+2k)}{\Gamma(2-m+k)\Gamma(r+1-m+k)\Gamma(2+m-r+k)\Gamma(2+m-k)\Gamma(m+1+k)} g^{1+2k} \]  

For \( k \leq m-2 \) the coefficients are zero and, thus, one can substitute \( \sum_{k=0}^{\infty} \rightarrow \sum_{k=1}^{\infty} \) in the r.h.s. Then, eq. (A13) can be rewritten in the new variable \( p = k-m+1 \)
\[ \hat{c}_{r,s} = 2 \sum_{p=0}^{\infty} (-1)^{n+r+p} \frac{\zeta(2p+2m-1)\Gamma(2p+2m-1)\Gamma(2p+2m)}{\Gamma(p+1)\Gamma(p+r)\Gamma(p+2m+1-r)\Gamma(p+2m)} g^{2p+2m-1} \]
\[ = 2(-1)^{m+r} g^{2m-1} \hat{K}_{r-1,2m-r}, \]  
where
\[ \hat{K}_{r,s} = \sum_{p=0}^{\infty} (-g^2)^p \frac{\zeta(2p+r+s)(2p+r+s)!}{p!(p+r)!(p+r+s)!}, \]

Note that \( \hat{K}_{r,s} \) contributes to the perturbative results for the ES and BES equations (see [24, 23]), because
\[ \hat{K}_{r,s} = \int_{0}^{\infty} dz \frac{J_r(2gz)J_s(2gz)}{z(e^z-1)} \]  
and \( J_r(2gz) \) is the Bessel function.
A.2 Results

Let us write the Mellin-Barnes representation for the coefficients $B_{r,m}$

$$B_{r,m} = \int_{-i\infty}^{i\infty} \frac{dt}{2\pi i} \frac{1}{(2\pi)^2 - t} \Gamma(t)\zeta(2 - t) \frac{\Gamma(m - t/2)\Gamma(m + 1 - r - t/2)}{\Gamma(m + t/2)\Gamma(m + 1 - r + t/2)} g^{t - 1}, \quad (A17)$$

where the integration contour is to the right of the pole at $t = 0$ and to the left of the pole at $t = 1$. Closing the contour to the left, we can reconstruct the Eqs. (A10) and (A11) from the poles of $\Gamma(t)$ at $t \to -l$.

Using relation (A12) for $\zeta(1 - z)$ (A12) one can rewrite (A17) as follows

$$B_{r,m} = \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{dt}{2\pi i} \Gamma(t) \cos \left( \frac{\pi}{2} (t - 1) \right) \frac{\Gamma(t - 1)\zeta(t - 1)}{\Gamma(m - t/2 + \Delta)\Gamma(m + 1 - r - t/2 + \delta)} g^{t - 1}, \quad (A18)$$

where small quantities $\Delta$ and $\delta$ were added to the argument of the last above $\Gamma$-function to prevent contributions from coinciding poles.

Closing now the integration contour of eq.(A18) to the right, we have four terms coming from the poles of $\Gamma(t - 1)$, $\zeta(t - 1)$, $\Gamma(m - t/2 + \Delta)$ and $\Gamma(m + 1 - r - t/2 + \delta)$, respectively.

1. The pole of $\Gamma(t - 1)$ at $t \to 1$ gives at $\Delta = \delta = 0$ (with $\zeta(0) = -1/2$)

$$B_{r,m} = \frac{1}{\pi} \int_{-i\infty}^{i\infty} \frac{dt}{2\pi i} \Gamma(t) \cos \left( \frac{\pi}{2} (t - 1) \right) \frac{\Gamma(t - 1)\zeta(t - 1)}{\Gamma(m - t/2 + \Delta)\Gamma(m + 1 - r - t/2 + \delta)} g^{t - 1}, \quad (A19)$$

and cancels the term $c^{(1)}_{r,s}$ in eq. (A7).

2. The pole of $\zeta(t - 1)$ at $t = 2 + 2\varepsilon$ produces at $\Delta = \delta = 0$

$$B_{r,m} = -\frac{1}{\pi} \cos \left( \frac{\pi}{2} (1 + 2\varepsilon) \right) \frac{\Gamma(2 + 2\varepsilon)\Gamma(1 + 2\varepsilon)}{\Gamma(m + 2 - r + \varepsilon)} \frac{\Gamma(m - 1 - \varepsilon)\Gamma(m - r - \varepsilon)}{\Gamma(m + 1 + \varepsilon)\Gamma(m + 2 - r + \varepsilon)} g^{1 + 2\varepsilon}.$$  \( (A20) \)

The result is zero if $r \neq m$, because

$$\cos \left( \frac{\pi}{2} (1 + 2\varepsilon) \right) = -\pi \varepsilon.$$  \( (A21) \)

For $r = m$ there is an additional pole at $\varepsilon \to 0$ in $\Gamma(-\varepsilon)$. So, finally we have

$$B_{r,m} = -\frac{\delta_{r,m}}{m(m - 1)} g,$$  \( (A21) \)

which cancels the term $c^{(0)}_{r,s} g$ in eq. (A7).
3. \( \Gamma(m - t/2 + \Delta) \) produces poles at \( t \to 2(m + k + \Delta) \). One can obtain at \( \delta = 0 \)

\[
3B_{r,m} = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \cos \left[ \pi \left( m + k + \Delta - \frac{1}{2} \right) \right] \Gamma(2(m + k + \Delta) - 1)
\]

\[
\zeta(2(m + k + \Delta) - 1) \frac{\Gamma(2(m + k + \Delta))\Gamma(1 - k - r - \Delta)}{\Gamma(2m + k + \Delta)\Gamma(2m + k + 1 - r + \Delta)} g^{2(m+k+\Delta)-1} \tag{A22}
\]

Because

\[
\cos \left[ \pi \left( m + k + \Delta - \frac{1}{2} \right) \right] = (-1)^{m+k+\Delta} + O(\Delta^2),
\]

\[
\Gamma(1 - k - r - \Delta) = (-1)^r \frac{\Gamma(1 - \Delta)\Gamma(\Delta)}{\Gamma(k + r + \Delta)}, \tag{A23}
\]

eq. [A22] can be rewritten as follows

\[
3B_{r,m} = \sum_{k=0}^{\infty} (-1)^{m+r+k} \frac{\Gamma(2(m + k))\Gamma(2(m + k) - 1)\zeta(2(m + k) - 1)}{k!\Gamma(k + r)\Gamma(k + 2m)\Gamma(k + 2m + 1 - r)} g^{2(m+k)-1}
\]

\[
= (-1)^{m+r} g^{2m-1} \tilde{K}_{r-1,2m-r}, \tag{A24}
\]

i.e. \( 3B_{r,m} \) is two times less than \( \hat{c}_{r,s} \) (see eq. [A14]).

4. \( \Gamma(m + 1 - r - t/2 + \delta) \) produces poles at \( t \to 2(m + 1 - r + k + \delta) \).

Similar to the previous subsection after simple algebraic transformations one can obtain the contribution \( 4B_{r,m} \), which coincides with \( 3B_{r,m} \), i.e.

\[
4B_{r,m} = 3B_{r,m} = (-1)^{m+r} g^{2m-1} \tilde{K}_{r-1,2m-r}. \tag{A25}
\]

So, the sum \( 3B_{r,m} + 4B_{r,m} \) reproduces the result [B15]

\[
3B_{r,m} + 4B_{r,m} = 2(-1)^{m+r} g^{2m-1} \tilde{K}_{r-1,2m-r} = \hat{c}_{r,s}. \tag{A26}
\]

Thus, we proved the duality between large and week expansions of of \( c_{r,s} \) at odd values of \( s + r \)

\[
\hat{c}_{r,s} = \sum_{n=0}^{\infty} c_{r,s}^{(n)} g^{1-n} = -\sum_{n=0}^{\infty} c_{r,s}^{(-n)} g^{1+n}, \quad (s + r = 2m + 1) \tag{A27}
\]

with \( c_{r,s}^{(n)} \) given by Eq. [A14].
Appendix B. Independent derivation of linear algebraic equations

Here we present the derivation of the set of linear equations (22) without using the Laplace transformation (11).

1. We start with the initial version (6) of the ES equation with the kernel (10) and write it as follows

\[ \Psi(x) = \frac{J_1(x)}{x} - 2 \sum_{k=1}^{\infty} k J_k(x) x \int_0^\infty \frac{dz}{e^{\epsilon z} - 1} J_k(z) \Psi(z), \]

(B1)

where the new function

\[ \Psi(x) = \frac{e^{\epsilon x} - 1}{x} f(x). \]

(B2)

is introduced.

One can search its solution in the form

\[ \Psi(x) = \sum_{n=1}^{\infty} \hat{a}_n \frac{J_n(x)}{x} \]

(B3)

By inserting the ansatz (B3) in Eq. (B1) and comparing the coefficients in the front of \( J_k(x) \) in the l.h.s. and r.h.s., we obtain

\[ \hat{a}_k = \delta_{k,1} - 2kb_k, \]

(B4)

where

\[ b_k = \int_0^\infty \frac{dz}{e^{\epsilon z} - 1} J_k(z) \Psi(z) \]

\[ = \frac{1}{\epsilon} \int_0^\infty \frac{dt}{e^{t} - 1} J_k(t/\epsilon) \Psi(t/\epsilon) = \sum_{n=1}^{\infty} \frac{1}{\epsilon} \int_0^\infty \frac{dt}{e^{t} - 1} J_k(t/\epsilon) d_n \frac{J_n(t/\epsilon)}{(t/\epsilon)}. \]

(B5)

Eqs. (B4) and (B5) are similar to (22) and (23) in the main text.

Expanding both Bessel functions in the r.h.s. of Eq. (B5) in series according to (8) and applying the integral representation for the Euler \( \zeta \)-function

\[ p!\zeta(p+1) = \int_0^\infty \frac{dt}{e^t - 1} t^p, \]

(B6)

we obtain the following expression for \( b_k \)

\[ b_k = \sum_{n=1}^{\infty} \hat{a}_n \sum_{s=0}^{\infty} \frac{1}{s!(s+k)!} \sum_{l=0}^{\infty} \frac{(-1)^{s+l}}{l!(l+n)!} \left( 2(s+l)+k+n-1 \right) \zeta\left( 2(s+l)+k+n \right) \bar{g}^{2(s+l)+k+n}, \]

(B7)

where \( \bar{g} = g/\sqrt{2} = 1/(2\epsilon) \).
The interchange of the summation order gives

\[ b_k = \sum_{n=1}^{\infty} \hat{a}_n g^{k+n} \sum_{p=0}^{\infty} (-g^2)^p \left( 2p+k+n-1 \right) \zeta(2p+k+n) \sum_{s=0}^{\infty} \frac{1}{s!(s+k)!(p-s)!(p-s+n)!} \sum_{s=0}^{\infty} \frac{1}{s!(s+k)!(p-s)!(p-s+n)!}. \]  

(B8)

The last term in the r.h.s. can be calculated exactly

\[ \sum_{s=0}^{\infty} \frac{1}{s!(s+k)!(p-s)!(p-s+n)!} = 2F_1(-p, -(p+n), k+1, 1) \]  

\[ s!(s+k)!(p-s)!(p-s+n)! = \frac{(2p+k+n)!}{p!(p+k)!(p+n+k)!} \]  

(B9)

where \( 2F_1 \) is the hypergeometric function.

Thus, we obtain

\[ b_k = \sum_{n=1}^{\infty} \hat{a}_n g^{k+n} \sum_{p=0}^{\infty} \frac{(2p+n+k)!}{p!(p+n+k)!} (-g^2)^p \]  

\[ \equiv \frac{1}{2k} \sum_{n=1}^{\infty} \hat{a}_n g^{k+n} K_{k,n}(\bar{\gamma}), \]  

(B10)

where \( K_{k,n} \) is given by Eq. (40).

So, the obtained set of equations is

\[ \hat{a}_k = \delta_{1,k} - \sum_{n=1}^{\infty} \hat{a}_n g^{k+n} K_{k,n}(\bar{\gamma}), \]  

(B11)

which coincides with Eqs (19) and (22), if \( \hat{a}_n \to a_{n,\e} \). The coincidence is not trivial, because \( \hat{a}_n \) and \( a_{n,\e} \) are coefficients of the expansions of the functions \( \Psi(x) \) and \( f(x) \), respectively.

2. Let us consider now the BES equation (see Eq. (B3) in [BES]) in the form similar to (B1)

\[ \Psi(x) = \frac{J_1(x)}{x} + 2 \sum_{k=1}^{\infty} (-1)^k c_{2k+1,2}(\bar{\gamma}) J_{2k}(x) \frac{J_k(x)}{x} - 2 \sum_{k=1}^{\infty} \frac{J_k(x)}{x} \int_0^{\infty} \frac{dz}{e^{\gamma z} - 1} J_k(z) \Psi(z) \]  

\[ - 4 \sum_{k=1}^{\infty} \frac{J_{2k}(x)}{x} \sum_{s=1}^{\infty} (-1)^{k+s+1} c_{2k+1,2s}(\bar{\gamma}) \int_0^{\infty} \frac{dz}{e^{\gamma z} - 1} J_{2s-1}(z) \Psi(z), \]  

(B12)

where

\[ c_{r,s}(\bar{\gamma}) = 2 \cos \left[ \frac{\pi}{2} (s-r-1) \right] (r-1)(s-1) \int_0^{\infty} \frac{dt}{t(e^{\gamma t} - 1)} J_{r-1}(\bar{\gamma}t) J_{s-1}(\bar{\gamma}t) \]  

\[ \equiv 2 \cos \left[ \frac{\pi}{2} (s-r-1) \right] (r-1)(s-1) \tilde{c}_{r,s}(\bar{\gamma}) \]  

(B13)

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One can introduce even and odd components of \( \Psi(x) \), such that \( \Psi_{\pm}(-x) = \pm \Psi_{\pm}(x) \):

\[
\Psi(x) = \Psi_{+}(x) + \Psi_{-}(x), \quad \Psi_{\pm}(x) = \frac{1}{2} \left( \Psi(x) \pm \Psi(-x) \right)
\]

Adding and subtracting Eq. (B12) and that with the replacement \( x \to -x \), we have

\[
\Psi_{+}(x) = \frac{J_{1}(x)}{x} - 2 \sum_{k=1}^{\infty} (2k-1) J_{2k-1}(x) \int_{0}^{\infty} \frac{dz}{e^{\epsilon z} - 1} J_{2k-1}(z) \Psi(z)
\]

\[
\Psi_{-}(x) = 2 \sum_{k=1}^{\infty} (-1)^{k} c_{2k+1,2}(g) \frac{J_{2k}(x)}{x} - 2 \sum_{k=1}^{\infty} (2k-1) J_{2k}(x) \int_{0}^{\infty} \frac{dz}{e^{\epsilon z} - 1} J_{2k}(z) \Psi(z)
\]

\[
-4 \sum_{k=1}^{\infty} J_{2k}(x) \sum_{s=1}^{\infty} (-1)^{k+s+1} c_{2k+1,2s}(g) \int_{0}^{\infty} \frac{dz}{e^{\epsilon z} - 1} J_{2s-1}(z) \Psi(z)
\]

The solutions for \( \Psi_{\pm}(x) \) can be found by analogy with the previous subsection. They have the form

\[
\hat{a}_{2k-1} = \delta_{k,1} - 2(2k-1)b_{2k-1},
\]

\[
\hat{a}_{2k} = 2(-1)^{k} c_{2k+1,2}(g) \delta_{k,1} - 4kb_{2k} - 4 \sum_{s=1}^{\infty} (-1)^{k+s+1} c_{2k+1,2s}(g) b_{2s-1}.
\]

Thus, for \( \hat{a}_{2k-1} \) we have

\[
\hat{a}_{2k-1} = \delta_{1,k} - \sum_{n=1}^{\infty} \hat{a}_{n} \eta^{2k+n-1} \zeta(-1)^{n} K_{2k-1,n}(g).
\]

To obtain the expression for \( \hat{a}_{2k} \) we should calculate \( c_{2k+1,2s}(g) \). Expanding both Bessel functions in the r.h.s. of Eq. (B13) in series according to (5) and applying the integral representation (B6) for the Euler \( \zeta \)-functions, we obtain

\[
\tilde{c}_{2k+1,2s}(g) = \sum_{m=0}^{\infty} \frac{1}{m!(m+2k)!} \sum_{l=0}^{\infty} \frac{(-1)^{m+l}}{l!(l+2s-1)!} \left( \frac{2(k+s+m+l)}{m+l} \right)!
\]

\[
\cdot \frac{2(k+s+m+l-1)}{g^{2(k+s+m+l-1)}}.
\]

Analogously to Eqs. (B8) and (B9) we interchange the summation order

\[
\tilde{c}_{2k+1,2s}(g) = \sum_{p=0}^{\infty} (-1)^{p} g^{2(p+k+s)-1} \frac{[(2p+s+k-1)!][2(p+s+k)-1]!}{p!(p+2s-1)!(p+2k)!(p+2s+2k-1)!}
\]

\[
\cdot \zeta(2p+k+s) \equiv \frac{1}{g^{2(k+s)-1}} K_{2k,2s-1}(g).
\]

and obtain

\[
c_{2k+1,2s}(g) = \frac{2s-1}{2} (-1)^{s+k} g^{2(k+s)-1} K_{2k,2s-1}(g),
\]
which is in agreement with eq. (A14).

Using Eqs. (B11) and (B22), one obtains

\[ \hat{a}_{2k} = g^{2k+1} K_{2k,1}(\bar{g}) - \sum_{n=1}^{\infty} \hat{a}_n g^{2k+n} K_{2k,n}(\bar{g}) + \sum_{n=1}^{\infty} \hat{a}_n g^{2k+n} \sum_{s=1}^{\infty} g^{2s-2} K_{2k,2s-1}(\bar{g}) K_{2s-1,n}(\bar{g}) \]  

(B23)

Note that in agreement with (161), the eqs. (B19) and (B23) can be rewritten in more simpler form

\[ \hat{a}_{2k-1} = \delta_{1,k} - \sum_{m=1}^{\infty} \hat{a}_{2m-1} g^{2k+2m-2} K_{2k-1,2m-1}(\bar{g}) - i \sum_{m=1}^{\infty} \hat{a}_{2m} g^{2k+2m-1} K_{2k-1,2m}(\bar{g}), \]

\[ \hat{a}_{2k} = -i \sum_{m=1}^{\infty} \hat{a}_{2m-1} g^{2k+2m-1} K_{2k,2m-1}(\bar{g}) - \sum_{m=1}^{\infty} \hat{a}_{2m} g^{2k+2m} K_{2k,2m}(\bar{g}), \]  

(B24)

which coincides with the one for ES equation with replacement \( K_{r,s} \rightarrow iK_{r,s} \) for the odd values of the sum \( r+s \).

Appendix C. Linear algebraic equations for \( \epsilon \to 0 \)

Here we derive the set of linear algebraic equations for solution of the ES equation at large coupling constant limit \( g \to \infty \).

The equations in this limit can be obtained from the Mellin-Barnes representation (41) for the kernel \( K_{nn'} \). Nevertheless it is useful to derive them independently.

At \( \epsilon \to 0 \), Eq. (6) is simplified as follows

\[ \epsilon f(x) = \frac{J_1(x)}{x} - 2 \sum_{k=1}^{\infty} k \frac{J_k(x)}{x} \int_0^{\infty} \frac{dz}{z} J_k(z) f(z) \]  

(C1)

Similar to Appendix B one can introduce even and odd components of \( f(x) \), such that \( f_\pm(-x) = \pm f_\pm(x) \):

\[ f(x) = f_+(x) + f_-(x), \quad f_\pm(x) = \frac{1}{2} \left( f(x) \pm f(-x) \right) \]  

(C2)

Adding and subtracting Eq. (C1) and that with the replacement \( x \to -x \), we have

\[ \epsilon f_+(x) = \frac{J_1(x)}{x} - 2 \sum_{k=1}^{\infty} (2k-1) \frac{J_{2k-1}(x)}{x} \int_0^{\infty} \frac{dz}{z} J_{2k-1}(z) f(z) \]  

(C3)

\[ \epsilon f_-(x) = -4 \sum_{k=1}^{\infty} k \frac{J_{2k}(x)}{x} \int_0^{\infty} \frac{dz}{z} J_{2k}(z) f(z) \]  

(C4)
1. We can present the function \( f(x) \) in the form similar to (B3)

\[
f(x) = \sum_{n=0}^{\infty} \frac{a_{n+1} J_{n+1}(x)}{x}, \quad f_{+}(x) = \sum_{n=0}^{\infty} \frac{a_{2n+1} J_{2n+1}(x)}{x}, \quad f_{-}(x) = \sum_{n=0}^{\infty} \frac{a_{2n+2} J_{2n+2}(x)}{x}.
\]

The integrals appearing in the r.h.s. of (C3) and (C4) can be calculated as follows

\[
\begin{align*}
\int_{0}^{\infty} \frac{dz}{z^2} J_{2s}(z) J_{2l}(z) &= \frac{4}{\pi} \frac{(-1)^{s+l+1}}{[4(s + l)^2 - 1][4(s - l)^2 - 1]} \\
\int_{0}^{\infty} \frac{dz}{z^2} J_{2s+1}(z) J_{2l+1}(z) &= \frac{4}{\pi} \frac{(-1)^{s+l+1}}{[4(s + l + 1)^2 - 1][4(s - l)^2 - 1]} \\
\int_{0}^{\infty} \frac{dz}{z^2} J_{2s}(z) J_{2l+1}(z) &= \frac{1}{4\Gamma(s + l + 2)\Gamma(1 + s - l)\Gamma(l - s + 2)} \\
&\quad \times \begin{cases} \\
\frac{1}{\beta_{s}(s+1)}, & \text{if } s = l, \\
\frac{1}{s_{s}(s-1)}, & \text{if } s = l + 1.
\end{cases}
\end{align*}
\]

Therefore we obtain the following linear set of equations for coefficients \( a_k \)

\[
\epsilon a_{2s+1} = \delta_{0,s} - \frac{a_{2s}(1 - \delta_{0,s})}{4s} - \frac{a_{2(s+1)}}{4(s+1)} - \frac{8(2s + 1)}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+s+1} a_{2m+1}}{[4(m + s + 1)^2 - 1][4(m - s)^2 - 1]} \tag{C9}
\]

\[
\epsilon a_{2s+2} = -\frac{2a_{2s+1}}{2(2s + 1)} - \frac{a_{2s+3}}{2(2s + 3)} - \frac{16(s + 1)}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+s+1} a_{2m+2}}{[4(m + s + 2)^2 - 1][4(m - s)^2 - 1]} \tag{C10}
\]

The first and second equations came from Eqs. (C3) and (C4), respectively. These results can be derived also directly form Eqs. (111) and (112) with replacement \( a_{n,\ell} \rightarrow a_{2s+1} \), i.e. \( n \rightarrow 2s + 1 \).

For the singular form of ES equation the formulas can be obtained by the replacement \( a_p \rightarrow 2a_p \) \((p = 2s, 2s + 1, 2s + 2, 2s + 3)\) and \( a_{2m+1} = a_{2m+2} = 0 \) in the r.h.s. of (C9) and (C10). So, the equations are similar for odd and even part and can be rewritten in the simpler form

\[
\epsilon a_n = \delta_{n,1} - \frac{a_{n-1}(1 - \delta_{n,1})}{(n - 1)} - \frac{a_{n+1}}{(n + 1)}, \tag{C11}
\]

which coincides with the ones in (121) and (122) after the replacement \( a_n = nd_n \).

2. We can search the functions \( f(x), f_{+}(x) \) and \( f_{-}(x) \) in another form

\[
f(x) = \sum_{n=0}^{\infty} \tilde{d}_{n} J_{n}(x), \quad f_{+}(x) = \sum_{n=0}^{\infty} \tilde{d}_{2n} J_{2n}(x), \quad f_{-}(x) = \sum_{n=1}^{\infty} \tilde{d}_{2n-1} J_{2n-1}(x). \tag{C12}
\]

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By substituting this ansatz in Eqs. (C3) and (C4) and using the formulas
\[ \int_0^\infty \frac{dz}{z} J_{2s}(z) J_{2l}(z) = \frac{1}{4s} \delta_{s,l} \] (C13)
\[ \int_0^\infty \frac{dz}{z} J_{2s+1}(z) J_{2l+1}(z) = \frac{1}{2(2s-1)} \delta_{s,l} \] (C14)
\[ \int_0^\infty \frac{dz}{z} J_{2s}(z) J_{2l-1}(z) = \frac{2(-1)^{s+l}}{\pi [4s^2 - (2l - 1)^2]} \] (C15)
we obtain the following set of equations for coefficients \( d_k \)
\[
2(2l - 1) \epsilon \sum_{m=0}^{l-1} \tilde{d}_{2m} + O(\epsilon^2) = \delta_{l,1} - \tilde{d}_{2l-1} - \frac{4(2l - 1)}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+l} \tilde{d}_{2m}}{4m^2 - (2l - 1)^2} \] (C16)
\[
4l \epsilon \sum_{m=0}^{l-1} \tilde{d}_{2m+1} + O(\epsilon^2) = -\tilde{d}_{2l} - \frac{8l}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+l} \tilde{d}_{2m-1}}{4l^2 - (2m - 1)^2} \] (C17)
derived from the comparison of coefficients in the front of the functions \( J_{2l-1}(x)/x \) and \( J_{2l}(x)/x \), respectively. The equations (C16) and (C17) look simpler than ones (C9) and (C10).

3. Eqs. (C16) and (C17) can be simplified by considering as independent ones the coefficients \( \tilde{d}_{2l-1} \). Indeed, from Eq. (C17) one can express the coefficients \( \tilde{d}_{2l} \) through \( \tilde{d}_{2l-1} \) and insert the results in the l.h.s. of Eq. (C16).

Using some algebraic manipulations we obtain at \( \epsilon = 0 \)
\[
\tilde{d}_0 = \frac{\pi}{4} \delta_{l,1} - \frac{(-1)^l}{2} \tilde{d}_{2l-1} \left( \frac{1}{2} \Psi'(l+1/2) + \frac{2}{(2l - 1)^2} \right) - \frac{(2l - 1)^2}{2} \left[ \sum_{m=1}^{l-1} + \sum_{m=l+1}^{\infty} \right] (-1)^m \tilde{d}_{2m-1} R(l, m), \] (C18)
where
\[
R(l, m) = \frac{1}{2(l - m)(l + m - 1)} \left[ 2 \Psi(m + 1/2) - \pi t g(\pi m) - \Psi(l + 1/2) + \pi t g(\pi l) \right. \\
+ \left. \frac{4(l - m)}{(1 - 2m)(1 - 2l)} \right] = \frac{1}{(l - m)(l + m - 1)} \left[ \Psi(m + 1/2) - \Psi(l + 1/2) \right] \\
- \frac{2}{(1 - 2m)(1 - 2l)(l + m - 1)} \] (C19)
and \( \Psi \) and \( \Psi' \) are Riemann \( \Psi \)-function and its derivation. The r.h.s. result is correct for integer \( m \) and \( l \) values.

We see that now the number of equations is two times less in comparison with the Section B.2, but there are more complicated coefficients in the front of the numbers \( \tilde{d}_{2l-1} \). Note that if one would express all odd coefficients \( \tilde{d}_{2l-1} \) through even ones \( \tilde{d}_{2l} \), the formulas look even more complicated.
The linear equations (C9), (C11), (C16), (C17) and (C18) can be solved by the matrix equation technique (see [32] and references therein). We plan to return to the consideration of these equations in future.

Appendix D. Linear algebraic equations for BES equation at $\epsilon \to 0$

Here we derive the set of linear algebraic equations from the BES equation at large coupling constants $g \to \infty$.

Let us introduce even and odd components of $f(x)$ as in eq. (C2).

According to above considerations, the set of equations for $f_{\pm}(z)$ in the framework of the BES equation can be obtained directly from (C3) and (C4) by the replacement $f_{-}(z) \to if_{-}(z)$ in the r.h.s. of (C3) and $f_{+}(z) \to if_{+}(z)$ in the r.h.s. of (C4):

$$\epsilon f_{+}(x) = \frac{J_{1}(x)}{x} - 2 \sum_{k=1}^{\infty} (2k-1) \frac{J_{2k-1}(x)}{x} \int_{0}^{\infty} \frac{dz}{z} J_{2k-1}(z) \left( f_{+}(z) + if_{-}(z) \right) \quad (D1)$$

$$\epsilon f_{-}(x) = -4 \sum_{k=1}^{\infty} \frac{J_{2k}(x)}{x} \int_{0}^{\infty} \frac{dz}{z} J_{2k}(z) \left( if_{+}(z) + f_{-}(z) \right). \quad (D2)$$

1. Let start with the form (C12) for the functions $f(x)$, $f_{+}(x)$ and $f_{-}(x)$ considered in the subsection 2 of the Appendix C.

Following the above argument and in an agreement with (D1) and (D2), to have the set of equations for coefficients $\tilde{d}_{k}$ in the case of the BES equation we should replace $\tilde{d}_{2l-1} \to i\tilde{d}_{2l-1}$ and $\tilde{d}_{2l} \to i\tilde{d}_{2l}$ in Eqs. (C16) and (C17), respectively. So, one can obtain the following linear algebraic equations

$$2(2l-1)\epsilon \sum_{m=0}^{l-1} \tilde{d}_{2m} + O(\epsilon^2) = \delta_{l,1} - i\tilde{d}_{2l-1} - \frac{4(2l-1)}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+l}\tilde{d}_{2m}}{4m^2 - (2l-1)^2}, \quad (D3)$$

$$4\epsilon \sum_{m=0}^{l-1} \tilde{d}_{2m+1} + O(\epsilon^2) = -i\tilde{d}_{2l} - \frac{8l}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m+l}\tilde{d}_{2m-1}}{4l^2 - (2m-1)^2}, \quad (D4)$$

derived from the comparison of coefficients in the front of the functions $J_{2l-1}(x)/x$ and $J_{2l}(x)/x$, respectively.

2. We can present also the function $f(x)$ in the form (C5). Using integrals (C6)-(C8), one can obtain the following linear set of equations for coefficients $a_{k}$

$$\epsilon a_{2s+1} = \delta_{0,s} - i\frac{a_{2s}(1-\delta_{0,s})}{4s} - i\frac{a_{2(s+1)}}{4(s+1)} - \frac{8(2s+1)}{\pi} \sum_{m=0}^{\infty} \frac{16}{\pi} \frac{(-1)^{m+s+1}a_{2m+1}}{4(m+s+1)^2 - 1} \frac{1}{4(m-s)^2 - 1} \quad (D5)$$

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\[ \epsilon a_{2s+2} = -i \frac{a_{2s+1}}{2(2s+1)} - i \frac{a_{2s+3}}{2(2s + 3)} \]

\[ - \frac{16(s + 1)}{\pi} \sum_{m=0}^{\infty} \frac{16}{\pi} \frac{(-1)^{m+s+1} a_{2m+2}}{[4(m + s + 2)^2 - 1][4(m - s)^2 - 1]}, \] (D6)

which coincides with the one in Eqs. (C9) and (C10) with the replacement \( a_{2s} \to ia_{2s} \) and \( a_{2(s+1)} \to ia_{2(s+1)} \) in the r.h.s. of (C9) and \( a_{2s+1} \to ia_{2s+1} \) and \( a_{2s+3} \to ia_{2s+3} \) in the r.h.s. of (C10), respectively.

By analogy with the Appendix B we can show, that the results for the singular form of BES equation can be obtained by the replacement \( a_p \to 2a_p \) (\( p = 2s, 2s+1, 2s+2, 2s+3 \)) and \( a_{2m+1} = a_{2m+2} = 0 \) in the r.h.s of (D5) and (D6). So, the equations should be similar for odd and even part and can be rewritten in simpler form

\[ \epsilon a_n = \delta_{1,n} - i \frac{a_{n-1} - (1 - \delta_{1,n})}{(n-1)} - i \frac{a_{n+1}}{(n+1)}, \] (D7)

which coincides with the ones in (168) and (169) after the replacement \( a_n = n \hat{d}_n \). These equations have been already solved in the Section 8.

Note that the solution has corrections of two types. The first ones came from the last terms in the r.h.s of (D5) and (D6) and should be responsible for the difference between \( \epsilon \)-corrections in eq. (173) and the corresponding Frolov-Tseytlin correction (see (175)).

The second type of corrections to results for anomalous dimension in (173) comes from the corrections to the l.h.s of (D5) and (D6). Indeed, here we keep only the first term at \( \epsilon \to 0 \) of the factor \( (e^{ex} - 1)/x \cdot f(x) \). Expanding \( e^{ex} \), by analogy with (C16) and (C17) we obtain the additional contributions in the l.h.s of (D5) and (D6):

\[ \epsilon a_{2s+1} \to \epsilon a_{2s+1} + \epsilon^2 (2s + 1) (1 - \delta_{0,s}) \sum_{m=0}^{s-1} a_{2m+2} + O(\epsilon^3), \] (D8)

\[ \epsilon a_{2s+2} \to \epsilon a_{2s+2} + \epsilon^2 (2s + 2) \sum_{m=0}^{s} a_{2m+1} + O(\epsilon^3) \] (D9)

The study of the both type of the corrections is the subject of our future investigations.

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