Slowly decaying resonances of charged massive scalar fields in the Reissner-Nordström black-hole spacetime

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We determine the characteristic timescales associated with the linearized relaxation dynamics of the composed Reissner-Nordström-black-hole-charged-massive-scalar-field system. To that end, the quasinormal resonant frequencies \( \{ \omega_n(\mu, q, M, Q) \}_{n=0}^{\infty} \) which characterize the dynamics of a charged scalar field of mass \( \mu \) and charge coupling constant \( q \) in the charged Reissner-Nordström black-hole spacetime of mass \( M \) and electric charge \( Q \) are determined analytically in the eikonal regime \( 1 \ll M \mu < qQ \). Interestingly, we find that, for a given value of the dimensionless black-hole electric charge \( Q/M \), the imaginary part of the resonant oscillation frequency is a monotonically decreasing function of the dimensionless ratio \( \mu/q \). In particular, it is shown that the quasinormal resonance spectrum is characterized by the asymptotic behavior \( 3 \omega \to 0 \) in the limiting case \( M \mu \to qQ \).

This intriguing finding implies that the composed Reissner-Nordström-black-hole-charged-massive-scalar-field system is characterized by extremely long relaxation times \( \tau_{relax} \equiv 1/3 \omega \to \infty \) in the \( M \mu/qQ \to 1^- \) limit.

I. INTRODUCTION

Wheeler’s ‘no-hair’ conjecture \[1,2\] presents a simple physical picture according to which asymptotically flat black holes in Einstein’s theory of gravitation cannot support static matter fields outside their horizons. Interestingly, various no-hair theorems \[1–8\] provide strong support for the general validity of Wheeler’s famous conjecture. It is therefore expected that fundamental matter fields that propagate in a static black-hole spacetime would eventually be absorbed by the black hole or be scattered away to infinity. For rotating black-hole spacetimes, a third possibility also exists \[9,10\]: thanks to the intriguing physical mechanism of superradiant scattering of bosonic fields in spinning black-hole spacetimes, these black holes can support stationary (rather than static) linearized bound-state massive scalar configurations in their external regions \[9\]. Moreover, as explicitly shown in \[10\], rotating black holes can also support genuine bosonic hair (that is, non-linear stationary bosonic field configurations) in their external regions. These rotating hairy black-hole-bosonic-field configurations \[9,10\] provide explicit counterexamples to the no-hair conjecture in asymptotically flat non-static spacetimes.

It is important to stress the fact that the elegant no-hair theorems \[1–8\], which are valid for asymptotically flat static black-hole configurations, say nothing about the timescale associated with the dynamical process of black-hole hair shedding \[11\]. This characteristic relaxation timescale, \( \tau_{relax} \), will be the main focus of the present study.

The dynamics of fundamental matter and radiation fields in black-hole spacetimes are characterized by damped quasinormal oscillations of the form \( e^{-\omega t} \). These exponentially decaying oscillations are characterized by complex quasinormal resonant frequencies \( \{ \omega_n \}_{n=0}^{\infty} \) whose values depend on the physical parameters (such as mass, charge, angular momentum, and intrinsic spin) of the composed black-hole field system. In accord with Wheeler’s no-hair conjecture \[1,2\], these characteristic damped oscillations reflect the gradual decay of the fields in the external regions of the black-hole spacetimes. In particular, the characteristic timescale associated with the relaxation dynamics of an external field in a black-hole spacetime is determined by the imaginary part of the fundamental (least damped) quasinormal resonant frequency which characterizes the composed black-hole-field system:

\[ \tau_{relax} \equiv 1/3 \omega_0 . \] (1)

The main goal of the present paper is to determine the characteristic relaxation timescales, \( \tau_{relax} \), associated with the relaxation dynamics of charged massive scalar fields in the charged Reissner-Nordström (RN) black-hole spacetime. To that end, we shall explore below the quasinormal resonance spectrum which characterizes the linearized relaxation dynamics \[12,16\] of the composed RN-black-hole-charged-massive-scalar-field system. As we shall show below, the characteristic quasinormal resonances of this composed black-hole-field system can be studied analytically in the eikonal regime \[17\].

\[ 1 \ll M \mu < qQ , \] (2)

where \( \{ q, \mu \} \) are respectively the charge coupling constant and proper mass of the field, and \( \{ M, Q \} \) are respectively the mass and electric charge of the RN black hole. In particular, below we shall reveal the interesting fact that this
composed RN-black-hole-charged-massive-scalar-field system is characterized by extremely long dynamical relaxation times, \( \tau_{\text{relax}} \equiv 1/3\omega_0 \to \infty \), in the limiting case \( M\mu/qQ \to 1^- \).  

II. DESCRIPTION OF THE SYSTEM

We shall analyze the quasinormal resonance spectrum which characterizes the linearized relaxation dynamics of a scalar field \( \Psi \) of mass \( \mu \) and charge coupling constant \( \varrho \) \([20]\) in the spacetime of a Reissner-Nordström black hole of mass \( M \) and electric charged \( Q \) \([21]\). The curved black-hole spacetime is described by the line element \([22]\)

\[
ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) ,
\]

(3)

where

\[
f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} .
\]

(4)

The zeros of the radial function \( f(r) \),

\[
r_\pm = M \pm (M^2 - Q^2)^{1/2} ,
\]

(5)
determine the horizon radii of the charged RN black hole.

The familiar Klein-Gordon wave equation \([23–29]\)

\[
[(\nabla_\nu - i\varrho A_\nu)(\nabla_\nu - i\varrho A_\nu) - \mu^2]\Psi = 0
\]

(6)
determines the linearized dynamics of the charged massive scalar field in the curved RN black-hole spacetime, where \( A_\nu = -\delta_\nu^0 Q/r \) is the electromagnetic potential of the charged black hole. Substituting the field decomposition \([30]\)

\[
\Psi(t, r, \theta, \phi) = \int \sum_{lm} e^{im\phi} S_{lm}(\theta) R_{lm}(r; \omega)e^{-i\omega t} d\omega ,
\]

(7)

into the Klein-Gordon wave equation (6), and using the black-hole metric function \([11]\), one obtains \([23–25]\) two ordinary differential equations of the confluent Heun type \([31, 32]\) for the eigenfunctions \( R \) and \( S \) which respectively describe the radial and angular behaviors of the charged massive scalar field in the curved black-hole spacetime.

The ordinary differential equation which determines the spatial behavior of the radial eigenfunction \( R(r) \) is given by \([23–25]\)

\[
\Delta \frac{d}{dr} \left( \Delta \frac{dR}{dr} \right) + UR = 0 ,
\]

(8)

where \( \Delta = r^2 f(r) \), and

\[
U = (\omega r^2 - qQr)^2 - \Delta (\mu^2 r^2 + K_l) .
\]

(9)

Here \( K_l = l(l + 1) \) (with \( l \geq |m| \)) is the characteristic eigenvalue of the angular eigenfunction \( S_{lm}(\theta) \) \([31, 33]\).

Defining the “tortoise” radial coordinate \( y \) by the differential relation

\[
dy = \frac{dr}{f(r)} ,
\]

(10)

and using the new radial eigenfunction

\[
\psi = rR ,
\]

(11)

one can transform the radial equation (8) into the more familiar form

\[
d^2\psi/dy^2 + V\psi = 0
\]

(12)
of a Schrödinger-like ordinary differential equation, where the effective radial potential in (12) is given by

$$V = V(r; M, Q, \omega, q, \mu, l) = \left( \omega - \frac{qQ}{r} \right)^2 - f(r)H(r)$$

(13)

with

$$H(r; M, Q, \mu, l) = \mu^2 + \frac{l(l + 1)}{r^2} + \frac{2M}{r^3} - \frac{2Q^2}{r^4}.$$  

(14)

The quasinormal resonant frequencies \(\{\omega_n(M, Q, \mu, q, l)\}_{n=0}^{\infty}\), which characterize the linearized relaxation dynamics of the charged scalar field in the charged black-hole spacetime, are determined by imposing on the Schrödinger-like wave equation (12) the physically motivated boundary conditions of purely ingoing waves at the black-hole horizon and purely outgoing waves at spatial infinity \[34\]. That is,

$$\psi \sim \begin{cases} e^{-i(\omega - qQ/r_+)y} & \text{as } r \to r_+ \ (y \to -\infty) ; \\ y^{-iqQe^{i\sqrt{\omega^2 - \mu^2}y}} & \text{as } r \to \infty \ (y \to \infty). \end{cases}$$  

(15)

In the next section we shall study analytically the quasinormal resonance spectrum \(\{\omega_n(M, Q, \mu, q, l)\}_{n=0}^{\infty}\) which characterizes the relaxation dynamics of the composed Reissner-Nordström-black-hole-charged-massive-scalar-field system in the eikonal large-mass regime \[2\].

**III. THE QUASINORMAL RESONANCE SPECTRUM OF THE COMPOSED REISSNER-NORDSTRÖM-BLACK-HOLE-CHARGED-MASSIVE-SCALAR-FIELD SYSTEM**

In the present section we shall perform a WKB analysis in order to determine the complex resonant frequencies which characterize the composed Reissner-Nordström-black-hole-charged-massive-scalar-field system in the large-mass regime

$$l + 1 \ll M\mu.$$  

(16)

In the eikonal large-mass regime (16), the radial potential (13), which characterizes the dynamics of the charged massive scalar field in the charged RN black-hole spacetime, can be approximated by

$$V(r) = \left( \omega - \frac{qQ}{r} \right)^2 - \left( 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \right)\mu^2 \cdot \{1 + O[(M\mu)^{-2}]\}.$$  

(17)

This radial potential has the form of an effective potential barrier whose maximum \(r_0\) is located at

$$r_0 = \frac{Q^2(q^2 - \mu^2)}{qQ\omega - M\mu^2}.$$  

(18)

As we shall now show, the fundamental complex resonances associated with the effective scattering potential (13) can be determined analytically in the large-mass regime (16) using standard WKB methods \[35-38\]. In particular, as shown in \[35, 36\], the WKB resonance condition which characterizes the complex scattering resonances (the quasinormal frequencies) of the Schrödinger-like radial equation (12) in the eikonal large-frequency regime is given by

$$\frac{iV_0}{\sqrt{2V_0^{(2)}}} = n + \frac{1}{2},$$  

(19)

where the various derivatives \(V_0^{(k)} \equiv d^kV/dy^k\) (with \(k \geq 0\)) that appear in the WKB resonance equation (19) are evaluated at the maximum point \(y = y_0(r_0)\) [see Eq. (18)] which characterizes the effective scattering potential \(V(y)\).

Substituting Eqs. (17) and (18) into the WKB resonance equation (19) and using the differential relation (10), one finds the characteristic resonance condition

$$\frac{\left( \omega - \frac{qQ}{r_0} \right)^2 - \left( 1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} \right)\mu^2}{2\left( 1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} \right)\sqrt{\frac{qQ\omega - M\mu^2}{r_0}}} = -i\left( n + \frac{1}{2} \right).$$  

(20)
for the quasinormal resonant frequencies of the composed Reissner-Nordström-black-hole-charged-massive-scalar-field system. As we shall now show, the rather complicated resonance equation (20) can be solved analytically in the regime

\[ \omega_R \gg \omega_I. \]  

This strong inequality, which characterizes the fundamental quasinormal frequencies of the composed black-hole-field system in the eikonal regime [2] [see Eqs. (25) and (29) below], enables one to decouple the real and imaginary parts of the WKB resonance equation (20). In particular, one finds

\[ \left( \omega_R - \frac{qQ}{r_0} \right)^2 - \left( 1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} \right) \mu^2 = 0 \]  

for the real part of the resonance equation (20), and

\[ \left( \omega_R - \frac{qQ}{r_0} \right) \omega_I = \left( 1 - \frac{2M}{r_0} + \frac{Q^2}{r_0^2} \right) \sqrt{\frac{qQ\omega_R - M\mu^2}{r_0^3}} \cdot \left( n + \frac{1}{2} \right) \]  

for the imaginary part of the resonance equation (20).

Substituting Eq. (18) into Eq. (22), one finds

\[ r_0 \frac{M}{\omega_I} = \left( 1 - \frac{\mu^2}{Q^2} + \sqrt{(1 - \mu^2)(1 - Q^2)} \right) \frac{1 - \mu}{1 - (\bar{\mu}/Q)^2} \]  

and

\[ \omega_R = q\bar{Q} \cdot \frac{1 + \left( \frac{\bar{Q}}{Q} \right)^2}{1 + \sqrt{1 - (\bar{\mu}/Q)^2}} \]  

where

\[ \bar{\mu} \equiv \frac{\mu}{q} ; \quad \bar{Q} \equiv \frac{Q}{M}. \]  

Note that the relations (24) and (25) are valid in the regime \( \bar{\mu} < \bar{Q} \leq 1 \) [40], which corresponds to

\[ \frac{\mu}{q} < \frac{Q}{M} \leq 1. \]  

Interestingly, one finds from (25) that, for a given value of the dimensionless black-hole electric charge \( \bar{Q} \), the real part of the resonant oscillation frequency, \( \omega_R \), is a monotonically increasing function of the dimensionless ratio \( \bar{\mu} \). In particular, one finds the limiting behaviors [see Eq. (25)]

\[ \{ \omega_R \to (qQ/r_0)^+ \quad \text{for} \quad \bar{\mu}/\bar{Q} \to 0 \} \quad \text{and} \quad \{ \omega_R \to (qQ/M)^- \quad \text{for} \quad \bar{\mu}/\bar{Q} \to 1^- \}. \]  

Substituting Eqs. (24) and (26) into Eq. (28), one finds

\[ M \omega_I = \frac{\sqrt{1 - Q^2}}{1 - \bar{\mu}^2 + \sqrt{(1 - \bar{\mu}^2)(1 - Q^2)}} \left( n + \frac{1}{2} \right) \]  

for the imaginary parts of the quasinormal resonances which characterize the composed RN-black-hole-charged-massive-scalar-field system in the regime [2]. Interestingly, one finds from (24) that, for a given value of the dimensionless black-hole electric charge \( \bar{Q} \), the imaginary part of the resonant oscillation frequency, \( \omega_I \), is a monotonically decreasing function of the dimensionless ratio \( \bar{\mu} \). In particular, one finds the limiting behaviors [see Eq. (29)]

\[ \{ M\omega_I \to \frac{\sqrt{1 - Q^2}}{1 + \sqrt{1 - Q^2}} \cdot (n + 1/2) \quad \text{for} \quad \bar{\mu}/\bar{Q} \to 0 \} \quad \text{and} \quad \{ M\omega_I \to 0^+ \quad \text{for} \quad \bar{\mu}/\bar{Q} \to 1^- \}. \]
Note that the expression (29) for the imaginary parts of the RN-black-hole-charged-massive-scalar-field resonances can be written in the compact form [see Eqs. (5) and (24)]

\[ \omega_I = 2\pi T_{BH} \left( \frac{r_+}{r_0} \right)^2 \left(n + \frac{1}{2}\right), \tag{31} \]

where

\[ T_{BH} = \frac{r_+ - r_-}{4\pi r_+^2} \tag{32} \]

is the Bekenstein-Hawking temperature of the RN black hole [41–43].

IV. THE REGIME OF VALIDITY OF THE WKB APPROXIMATION

It is important to emphasize that our WKB results (25) and (29) for the real and imaginary parts of the quasinormal resonant frequencies which characterize the relaxation dynamics of the charged massive scalar fields in the charged RN black-hole spacetime were derived under the assumption that higher-order correction terms that appear in the large-frequency WKB approximation can be neglected. In particular, as explicitly shown in [35–38], an extension of the WKB approximation to include higher-order derivatives of the effective scattering potential yields the correction term

\[ \Lambda(n) = \sqrt{\frac{2}{V_0^{(2)}}} \left[ \frac{1 + (2n + 1)^2}{32} \frac{V_0^{(4)}}{V_0^{(2)}} - \frac{28 + 60(2n + 1)^2}{1152} \left( \frac{V_0^{(3)}}{V_0^{(2)}} \right)^2 \right] \tag{33} \]

on the r.h.s of the resonance equation (19). Thus, the WKB resonance condition (19) is valid provided

\[ \frac{\Lambda(n)}{n + \frac{1}{2}} \ll 1 \tag{34} \]

Substituting Eqs. (10), (17), (24), and (25) into Eq. (34), one finds

\[ \Lambda(n) = \frac{1}{qQ\sqrt{1 - \bar{\mu}^2}} \times \left[ \frac{1}{8} \left( 1 - \frac{6M}{r_0} + \frac{6Q^2}{r_0^2} \right) - \frac{3}{8} \left( 1 - \frac{10M}{3r_0} + \frac{10Q^2}{3r_0^2} + \frac{4(1 - \bar{\mu}^2)Q^2}{3\bar{\mu}^2r_0^2} \right) \right] \cdot (2n + 1)^2 \tag{35} \]

which implies that the WKB resonance condition that we have used, Eq. (19), is valid in the large coupling (eikonal) regime [see Eq. (34)] [44, 45]

\[ qQ\sqrt{1 - \bar{\mu}^2} \gg n + \frac{1}{2} \tag{36} \]

V. SUMMARY AND DISCUSSION

We have determined the characteristic timescales associated with the relaxation dynamics of the composed Reissner-Nordström-black-hole-charged-massive-scalar-field system. To that end, the quasinormal resonance spectrum \( \{\omega_n(\mu, q, M, Q)\}_{n=0}^{\infty} \) which characterizes the dynamics of a linearized charged scalar field of mass \( \mu \) and charge coupling constant \( q \) in the charged RN black-hole spacetime of mass \( M \) and electric charge \( Q \) was studied analytically in the eikonal regime \( 1 \ll M\mu < qQ \) [see Eq. (2)]. In particular, we have derived the analytical expression [see Eqs. (25) and (29)]

\[ M\omega(M, Q, \mu, q; n) = qQ \cdot \frac{1 + \left( \frac{\bar{\mu}}{Q} \right)^2}{1 + \sqrt{\frac{1 - Q^2}{1 - \bar{\mu}^2}}} - i\sqrt{1 - Q^2} \frac{1 - (\bar{\mu}/Q)^2}{1 - \bar{\mu}^2 + \sqrt{(1 - \bar{\mu}^2)(1 - Q^2)}} \cdot \left(n + \frac{1}{2}\right) \tag{37} \]

for the quasinormal resonant frequencies which characterize the composed RN-black-hole-charged-massive-scalar-field system in the eikonal regime [2] [46, 47].
The characteristic timescale $\tau_{\text{relax}} \equiv 1/3\omega_0$ associated with the linearized relaxation dynamics of the composed RN-black-hole-charged-massive-scalar-field system is determined by its fundamental (least damped) quasinormal resonant frequency. In particular, from \cite{48, 49}, one finds the expression

$$\frac{\tau_{\text{relax}}}{M} = \frac{2}{\sqrt{1 - Q^2}} \left[ 1 - \bar{\mu}^2 + \sqrt{(1 - \bar{\mu}^2)(1 - Q^2)} \right]^2$$

\hspace{1cm} (38)

for the characteristic relaxation time of the composed black-hole-field system. Interestingly, one finds from (38) that the composed RN-black-hole-charged-massive-scalar-field system is characterized by extremely long relaxation times in the limiting case $\bar{\mu}/Q \to 1^-:$

$$\tau_{\text{relax}} \to \infty \text{ for } \frac{M\bar{\mu}}{qQ} \to 1^- .$$

\hspace{1cm} (39)

Thus, although Reissner-Nordström black holes cannot support static matter fields outside their horizons \cite{3, 15, 16, 18, 48, 49}, we conclude that these black-hole spacetimes may host extremely long-lived (exponentially decaying with long relaxation times, $\tau_{\text{relax}} \gg M$) charged massive scalar fields in their external regions.

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\[\text{[References]}\]
It is interesting to note that the relation (31) with $r_\text{Q}$ stems from the characteristic requirement $r_0 > r_+$ for the peak location of the effective scattering potential $\tilde{\mu}$ which reflect the backscattering of the fields by the effective curvature potential of the black-hole spacetime at large distances from the central black hole.

It is worth noting that, in asymptotically flat black-hole spacetimes, these exponentially damped quasinormal oscillations, which characterize the composed black-hole-field system, are followed by decaying power-law wave tails which reflect the backscattering of the fields by the effective curvature potential of the black-hole spacetime at large distances from the central black hole.

This characteristic inequality implies that the relaxation time $\tau_{\text{relax}} = 1/3\omega_0$ of the composed black-hole-field system is bounded from below by the relation $\tau_{\text{relax}} \times T_{\text{BH}} > 1/\pi$, in agreement with the conjectured time-times-temperature (TTT) inequality.

It is important to stress the fact that existing no-hair theorems (see [3] and references therein) rule out the existence of asymptotically flat static charged black-hole-scalar-field hairy configurations. In addition, it was proved in [16] that stationary (or exponentially growing in time) charged scalar fields (linearized bound-state charged scalar configurations) cannot be supported in the external spacetime regions of asymptotically flat charged Reissner-Nordström black holes. Thus, charged scalar fields which propagate outside the horizon of a spherically symmetric asymptotically flat Reissner-Nordström black hole are expected to be swallowed by the black hole or to be scattered away to infinity.

It is important to emphasize that the long-lived quasinormal resonances of neutral massive scalar fields were observed in former numerical studies of composed black-hole-field systems. To the best of our knowledge, our analysis (to be presented below) provides the first fully analytical study of these intriguing long-lived black-hole-field quasinormal resonances.
Note that ω_R > qQ/r whereas ω_I < 2πT_{BH} \cdot (n + 1/2) < (n + 1/2)/4M [see Eqs. (28) and (30)]. Thus, our assumption ω_R ≫ ω_I [see (21)] is valid in the large coupling (eikonal) regime qQ ≫ n + 1 [see Eq. (2)].

It is worth emphasizing the fact that vacuum polarization effects (the quantum discharge phenomenon of charged black holes caused by the Schwinger-type pair production mechanism [45]) restrict the physical parameters (masses and electric charges) of the composed RN-black-hole-charged-massive-scalar-field system to the quantum regime qQ ≪ µ^2 r^2 / h [27]. Thus, one finds \sqrt{1 - \bar{\mu}^2 qQ/\mu^2 r^2} < qQ/\mu^2 r^2 ≪ 1 for the last term in Eq. (35).

Note that, in the extremal (1 - \bar{Q}^2)/(1 - \bar{\mu}^2) → 0 (T_{BH} → 0) limit, one finds from Eq. (37) the leading-order behavior ω_n = qQ/r_{+} - i2πT_{BH} \cdot (n + 1/2) for the quasinormal resonant frequencies which characterize the composed near-extremal RN-black-hole-charged-massive-scalar-field system. This expression agrees with the results presented in [26].

Note that, in the small mass \bar{\mu}/\bar{Q} ≪ 1 limit, one finds from Eq. (37) the leading-order behavior ω_n = qQ/r_{+} - i2πT_{BH} \cdot (n + 1/2) for the quasinormal resonant frequencies which characterize the composed RN-black-hole-charged-scalar-field system. This expression agrees with the results presented in [27, 29].

It is important to note that, for the case of quasi-bound states, a similar behavior (that is, Mω_I → 0 as Mµ → qQ) was observed numerically in the interesting work of Sampaio et. al. [48]. It is worth emphasizing that the quasi-bound states studied in [48] are characterized by the relation ω_R < µ (qQ < Mµ), whereas the free oscillations (quasinormal resonances) analyzed in the present paper are characterized by the relation µ > ω_R (Mµ < qQ). In addition, it is worth noting that the interesting work of Sampaio et. al. [48] uses numerical techniques in order to study the quasi-bound states, whereas in the present work we use analytical techniques (the WKB approximation) in order to study the free oscillations of the charged massive fields.