THE ŠILOV BOUNDARY FOR OPERATOR SPACES

EVGENIOS T.A. KAKARIADIS

Abstract. Motivated by the recent interest in the examination of unital completely positive maps and their effects in $C^\ast$-theory, we revisit an older result concerning the existence of the Šilov ideal. The direct proof of Hamana’s theorem for the existence of an injective envelope for a unital operator subspace $X$ of some $B(H)$ that we provide implies that the Šilov ideal is the intersection of $C^\ast(X)$ with any maximal boundary operator subsystem in $B(H)$. As an immediate consequence we deduce that the Šilov ideal is the biggest boundary operator subsystem for $X$ in $C^\ast(X)$.

The new proof of the existence of the Šilov ideal that we give does not use the existence of maximal dilations, provided by Dritschel and McCullough, and so it is independent of the one given by Arveson. As a consequence, the Šilov ideal can be seen as the set that contains the abnormalities in a $C^\ast$-cover $(C,\iota)$ of $X$ for all the extensions of the identity map $\iota_{\iota(X)}$. The interpretation of our results in terms of ucp maps characterizes the maximal boundary subsystems of $X$ in $B(H)$ as kernels of $X$-projections that induce completely minimal $X$-seminorms; equivalently, $X$-minimal projections with range being an injective envelope, that we view from now on as the Šilov boundary for $X$.

1. Introduction

Let $X,Y$ be linear spaces and $\varphi: X \to Y$ a linear map. We define $\varphi_\nu := \text{id}_\nu \otimes \varphi: M_\nu(X) \to B(H_\nu)$ by $\varphi_\nu([a_{ij}]) = [\varphi(a_{ij})]$.

An (abstract) operator space is a pair $(X,\{\| \cdot \|_\nu\}_{\nu \geq 1})$, consisting of a vector space, and a norm on $M_\nu(X)$ for all $\nu \in \mathbb{N}$, such that there exists a linear map $u: X \to B(H)$ (where $H$ is a Hilbert space) such that every $u_\nu$ is an isometry; equivalently, Ruan’s axioms hold for the sequence of norms. In this case we call the sequence $\{\| \cdot \|_\nu\}_{\nu \geq 1}$ an operator space structure on the vector space $X$. Throughout this paper $X$ is assumed unital, i.e., there is an element $e \in X$ such that $u(e) = I_H$, and the morphisms will always be unital.

If $X$ is a linear subspace of a $C^\ast$-algebra $C$, then $X$ is an operator space with the matrix norm structure inherited by a faithful representation of $C$.

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By Ruan’s theorem we can always assume that an operator space sits inside a C*-algebra.

An operator system is a selfadjoint linear subspace \( S \) of a unital C*-algebra, that contains the unit. (There is an alternative way of defining abstract operator systems, which we will not make use of.)

Let \( X \) be a unital operator space and \( \varphi: X \to B(H) \) be a linear map. We define \( \varphi_\nu := \text{id}_\nu \otimes \varphi: M_\nu(X) \to B(H^\nu) \) by \( \varphi_\nu([a_{ij}]) = [\varphi(a_{ij})] \). We call \( \varphi \) unital completely positive (ucp), unital completely contractive (ucc) or unital completely isometric (ucis) if \( \varphi_\nu \) is respectively positive, contractive or an isometry, for every \( \nu \in \mathbb{N} \).

We can use the decomposition of an element \( x \in S_{\text{sa}} \) in two positive elements in \( S \), i.e., \( x = (1\|x\|^2 + x)/2 - (1\|x\|^2 - x)/2 \), to prove that a linear map \( \varphi: S \to B(H) \) is ucp if, and only if, it is ucc. Also, if \( X \) is a unital operator space sitting in a C*-algebra and \( \varphi: X \to B(H) \) is a ucc map, then \( X + X^* \) is an operator system and there is a unique ucp extension \( \tilde{\varphi}: X + X^* \to B(H) \) of \( \varphi \).

Given two ucc maps \( \varphi_k: X \to B(H_k) \), \( k = 1, 2 \), we write \( \varphi_1 \leq \varphi_2 \) if \( H_1 \subseteq H_2 \) and \( P_{H_1} \varphi_2(x)|_{H_1} = \varphi_1(x), x \in X \); \( \varphi_2 \) is called a dilation of \( \varphi_1 \) and \( \varphi_1 \) is called a compression of \( \varphi_2 \). In general, we can have the following scheme that gives the existence of dilations of a ucc map. Let \( X \subseteq B(H) \) be a unital operator space and \( \varphi: X \to B(K) \) a ucc map. Arveson’s Extension Theorem implies that there is a ucp (thus ucc) map \( \psi: B(H) \to B(K) \) extending \( \varphi \). By applying Stinespring’s Dilation Theorem on \( \psi \), there is a Hilbert space \( W \supseteq K \) and a unital representation \( \pi: B(H) \to B(W) \) such that \( \psi(c) = P_K \pi(c)|_K \), for every \( c \in B(H) \). Hence, \( \pi|_X \) is a dilation of \( \varphi \).

Given an operator space \( X \), a natural question to ask is which is the smallest (in some sense) C*-algebra \( C \) for which there is a ucis map \( \varphi: X \to C \), i.e., the C*-envelope of \( X \). A C*-cover for \( X \) is a pair \((C, \iota)\) where \( \iota: X \to C \) is a ucis map amd \( C = C^*(\iota(X)) \).

**Definition 1.1.** Let \( X \) be a unital operator space. The C*-envelope of \( X \) is the C*-cover \((C^*_{\text{env}}(X), \iota)\) with the following (universal) property: for every C*-cover \((C, j)\) there exists a unique *-epimorphism \( \Phi: C \to C^*_{\text{env}}(X) \), such that \( \Phi(j(x)) = \iota(x) \), for every \( x \in X \).

The existence of the C*-envelope was first proved by Arveson, in the case where there were enough boundary representations. The first proof for the general case was given by Hamana in [6]. Twenty five years later Dritschell and McCullough gave an independent proof in [5] for the existence of the C*-envelope. We should remark here that the original versions of these theorems were stated in terms of operator algebras or operator systems. A moment of clarity shows that one can easily reformulate these theorems for (unital) operator spaces \( X \), simply by mimicking the simplified proof in [1].

The key step of proving the existence of the C*-envelope in [5] was the proof of the existence of a maximal representation for \( X \). The following definitions are equivalent.
Definitions 1.2. (i) A ucc map $\varphi: X \to \mathcal{B}(H)$ is said to be maximal if it has no nontrivial dilations, i.e. $\varphi' \geq \varphi \Rightarrow \varphi' = \varphi \oplus \psi$, for some ucc map $\psi$.

(ii) A ucc map $\pi: X \to \mathcal{B}(H)$ is said to have the unique extension property if

1. $\pi$ has a unique completely positive extension $\tilde{\pi}: C^*(X) \to \mathcal{B}(H)$, and
2. $\tilde{\pi}: C^*(X) \to \mathcal{B}(H)$ is a representation of $C^*(X)$ on $H$.

(iii) A ucc map $\varphi: X \to \mathcal{B}(H)$ is called a $\partial$-representation if for any dilation $\nu \geq \varphi$ the Hilbert space $H$ is $\nu(X)$-reducing.

Theorem 1.3. (Dritschel-McCullough) Let $X$ be an operator space in a $\mathcal{B}(H)$. Then the identity map $id: X \to \mathcal{B}(H)$ has a dilation $\nu: X \to \mathcal{B}(K)$ that is maximal. Hence, $C^*(\nu(X)) \simeq C^*_{\text{env}}(X)$.

In contrast, the direction for the proof in [6] is completely different and has an algebraic flavor. It is based on proving the existence of an injective envelope by using the notion of minimal $X$-seminorms.

Definition 1.4. An operator space $E$ is called injective if for any pair $Z,Y$ of operator spaces such that $Z \subseteq Y$, and every ucc map $\varphi: Z \to E$, there exists a ucc map $\psi: Y \to E$ that extends $\varphi$.

If $\iota: X \to E$ is a ucis in an operator space $E$, then the pair $(E,\iota)$ is called an extension of $X$. We say that an extension $(E,\iota)$ is rigid if $id_E$ is the only ucc map $E \to E$ that extends the identity map on $\iota(X)$. We say that an extension $(E,\iota)$ is essential if whenever $\varphi: E \to Z$ is a ucc map into another operator space $Z$ such that $\varphi \circ \iota$ is a complete isometry, then $\varphi$ is a complete isometry. We say that $(E,\iota)$ is an injective envelope of $X$ if $E$ is injective and there is no injective subspace of $E$ containing $\iota(X)$. One can prove that an injective extension $(E,\iota)$ is an envelope of $X$ if and only if it is rigid if and only if it is essential. Also, if $(E,\iota)$ is an injective envelope and $\varphi: E \to \mathcal{B}(K)$ is a ucc map such that the restriction of $\varphi$ to $\iota(X)$ is ucis, then $\varphi$ is a ucis map and $(\varphi(E),\varphi \circ \iota)$ is an injective envelope for $X$ in $\mathcal{B}(K)$.

Theorem 1.5. (Hamana) Let $X$ be an operator space in a $\mathcal{B}(H)$. Then there is an injective envelope $(E,\iota)$ of $X$. Thus $C^*(\iota(X)) \simeq C^*_{\text{env}}(X)$.

We pinpoint two lemmas concerning injective envelopes of Hamana’s theory that we are going to use in the following sections.

Lemma 1.6. Let $(S_k,\iota_k)$, $k = 1,2$, be injective envelopes for an operator space $X$. Then the mapping $\iota_1(x) \to \iota_2(x)$ extends to a ucc map $\varphi: S_1 \to S_2$ which is a necessarily unique ucis onto map.

Proof. By injectivity of $S_2$ there is a ucc extension $\varphi: S_1 \to S_2$ that fixes $X$ elementwise, i.e., $\varphi(\iota_1(x)) = \iota_2(x)$, for all $x \in X$. Since $S_1$ is also an essential envelope $\varphi$ is a ucis map and $\varphi(S_1)$ is an injective envelope in $S_2$. Therefore $\varphi$ is also onto $S_2$. Now let $\psi: S_1 \to S_2$ be a ucis map such that $\psi(\iota_1(x)) = \iota_2(x)$, for all $x \in X$. Then $\psi$ is also a ucis map onto $S_2$. 

Moreover the restriction of \( \psi^{-1} \circ \varphi \) to \( \iota_1(X) \) is the identity mapping, hence \( \psi^{-1} \circ \varphi = \text{id} \) on \( S_1 \), by rigidity of \( S_1 \). Therefore \( \psi = \varphi \).

**Lemma 1.7.** Let \((E, j)\) be an injective extension and \((S, \iota)\) be an injective envelope for an operator space \( X \subseteq B(H) \). Then any uc\(c\) map \( \varphi : E \rightarrow S \) such that \( \varphi(j(x)) = \iota(x) \), for all \( x \in X \), is onto \( S \). Moreover, if \( E = B(H) \) and \((S, \text{id}_X)\) is the injective envelope, then \( \varphi \) is a projection.

**Proof.** Let such a map \( \varphi : E \rightarrow S \) and \( \sigma : S \rightarrow E \) be an extension of the mapping \( \iota(x) \mapsto j(x) \), for all \( x \in X \). Then \( \varphi \circ \sigma(\iota(x)) = \iota(x) \) for all \( x \in X \) and the range of \( \varphi \circ \sigma \) is in \( S \). Thus \( \varphi \circ \sigma = \text{id} \) by rigidity of \( S \), and the proof of the first statement is complete.

If \( S \subseteq B(H) \), then the restriction of \( \varphi : S \equiv \varphi(B(H)) \rightarrow X = \text{id}(X) = \varphi(X) \) is the identity map, therefore \( \varphi^2(a) = \varphi(\varphi(a)) = \text{id}(\varphi(a)) = \varphi(a) \), for all \( a \in B(H) \), by rigidity of \( S \).

The approaches of [6] and [5] gave independently the existence of the \( C^* \)-envelope, thus the existence of a second object, the Šilov ideal. As a result the Šilov ideal is described as the kernel of a necessarily unique \( \ast \)-epimorphism.

**Definition 1.8.** If \( \iota : X \rightarrow C \) is a uc\(c\) map and \( C = C^*(\iota(X)) \), then an ideal \( I \) of \( C^*(\iota(X)) \) is called boundary if the restriction of the natural \( \ast \)-epimorphism \( q_I : C \rightarrow C/I \) to \( \iota(X) \) is a uc\(c\) map. The biggest boundary ideal in \( C^*(\iota(X)) \) is called the Šilov ideal of \( \iota(X) \) in \( C \).

It appears that the Šilov ideal is a very tractable tool for finding the \( C^* \)-envelope in recent papers (and I will avoid making any advertisement here, as it is irrelevant to our subject). The crucial remark used in some of these cases is that the \( C^* \)-envelope contains no non-trivial boundary ideals. Indeed, if \( I \) is the Šilov ideal in a \( C^* \)-cover \((C, \iota)\), then \( C/I \simeq C^*_{\text{env}}(X) \). Let us show here how the existence of the \( C^* \)-envelope implies the existence of the Šilov ideal.

Note that by definition the Šilov ideal is unique.

**Proposition 1.9.** If there exists a \( C^* \)-cover for an operator space \( X \) that has the universal property of the \( C^* \)-envelope, then the Šilov ideal exists.

**Proof.** Assume that \((C^*_{\text{env}}(X), \iota)\) has the universal property and let \((C, j)\) be a \( C^* \)-cover for \( X \). Then there is a unique \( \ast \)-epimorphism \( \Phi : C \rightarrow C^*_{\text{env}}(X) \), such that \( \Phi(j(x)) = \iota(x) \) for all \( x \in X \). Then, by the first theorem for \( \ast \)-isomorphisms \( C/\ker \Phi \simeq C^*_{\text{env}}(X) \), via the \( \ast \)-isomorphism

\[
\hat{\Phi}(c + \ker \Phi) = \Phi(c), \text{ for all } c \in C.
\]

Since \( \hat{\Phi} \) is a \( \ast \)-isomorphism, hence a uc\(c\) map, the ideal \( \ker \Phi \) is boundary.

Let \( J \) be any boundary ideal in \( C \). Then \((C/J, q_J \circ j)\), where \( q_J : C \rightarrow C/J \) is the canonical \( \ast \)-epimorphism, is a \( C^* \)-cover for \( X \). Therefore, by the universal property of \( C^*_{\text{env}}(X) \), there is a unique \( \ast \)-epimorphism \( \Pi : C/J \rightarrow C^*_{\text{env}}(X) \), such that \( \Pi(q_J(j(x))) = \iota(x) \) for all \( x \in X \). Then \( \Pi \circ q_J(j(x)) = \iota(x) \) for all \( x \in X \), as desired.
\[ \Phi(j(x)) \text{ for all } x \in X, \text{ thus } \Pi \circ q_J = \Phi, \text{ since } \Pi, q_J \text{ and } \Phi \text{ are } *-\text{homomorphisms} \]
\[ C = C^* (j(X)). \]
Hence,
\[ \ker \Phi = \ker (\Pi \circ q_J) \supseteq \ker q_J = J. \]
Therefore, the ideal \( \ker \Phi \) contains all the boundary ideals in \( C \). So it is the Šilov ideal.

It is interesting that there is not a known proof of the converse without the additional use of the existence of maximal representations proved by Dritschell and McCullough [5], except from that given by Arveson in [1], or the use of the existence of an injective envelope. Yet, the advantage of a direct proof of the existence of the Šilov ideal provides additional information; the proof provided by Arveson in [1] characterizes the Šilov ideal as the kernel of (the unique extension of) a maximal representation. The proof, that we provide in what follows, gives additional characterizations of the Šilov ideal in terms of maximal boundary subsystems and/or kernels of minimal \( X \)-maps.

2. The proof

Let us fix a Hilbert space \( H \) such that \( X \subseteq B(H) \) (completely isometrically). If \( S \subseteq B(H) \) is an operator system that contains \( X \), we say that a selfadjoint subspace \( \mathcal{I} \) of \( S \) is a boundary subsystem for \( X \), if the restriction to \( X \) of the quotient linear map \( q_{\mathcal{I}}: S \to S/\mathcal{I} \subseteq B(H)/\mathcal{I} \) is completely isometric. The matrix norms in \( B(H)/\mathcal{I} \) come from the identification of \( M_\nu(B(H)/\mathcal{I}) \cong M_\nu(B(H))/M_\nu(\mathcal{I}) \), i.e.,
\[ \|[x_{ij}] + M_\nu(\mathcal{I})\|_\nu = \|[x_{ij}] + \mathcal{I}\|_\nu = \inf \{ \|[x_{ij}] + y_{ij}\|_\nu : y_{ij} \in \mathcal{I} \} \]
It is trivial to see that boundary subsystems are never unital, since \( X \) is unital. The translation of the invariance principle in our context is the following.

**Proposition 2.1.** If \( \mathcal{I} \) is a boundary subsystem for \( X \) in a \( B(H) \) and \( V/\mathcal{I} \) is a boundary subsystem of \( q_{\mathcal{I}}(X) \) in \( B(H)/\mathcal{I} \), then \( V \) is a boundary subsystem for \( X \) in \( B(H) \).

**Proof.** By contractivity of \( q_{\mathcal{I}} \), for \( x \in X \) and \( v \in V \) we get that
\[ \|q_{\mathcal{I}}(x) + q_{\mathcal{I}}(v)\| = \|q_{\mathcal{I}}(x + v)\| \leq \|x + v\| \]
hence by taking the infimum over all \( v \in V \)
\[ \|x\| = \|q_{\mathcal{I}}(x)\| = \|q_{\mathcal{I}}(x) + V/\mathcal{I}\| = \inf \{ \|q_{\mathcal{I}}(x) + V/\mathcal{I}\| : v \in V \} \leq \inf \{ \|x + v\| : v \in V \} = \|x + V\| \leq \|x\|. \]
Thus \( \|x\| = \|x + V\| \), for all \( x \in X \), so the restriction of \( q_V \) on \( X \) is isometric. A similar argument for all matrix norms gives that \( V \) is a boundary subsystem for \( X \) in \( B(H) \).
Thus and Zorn’s Lemma applies. For each \((i,j)\) there exists a \(k\) such that for every \(\varepsilon > 0\) there are \(a_{ij} \in \bigcup_k I_k\) such that
\[
\|[x_{ij}] + M_\nu(J)\| = \inf\{\|[x_{ij}] + [a_{ij}]\| : a_{ij} \in J\}
\]
\[
= \inf\{\|[x_{ij}] + [a_{ij}]\| : a_{ij} \in \bigcup_k I_k\}.
\]
Thus for every \(\varepsilon > 0\) there are \(a_{ij} \in \bigcup_k I_k\) such that
\[
\|[x_{ij}] + M_\nu(J)\| \leq \|[x_{ij}] + [a_{ij}]\| \leq \|[x_{ij}] + M_\nu(J)\| + \varepsilon.
\]
For each \((i,j)\) \(\in \{1,2,\ldots,\nu\} \times \{1,2,\ldots,\nu\}\), let \(I_{k_{ij}} \in \{I_k\}\) such that \(a_{ij} \in I_{k_{ij}}\). Since \(\{I_k\}\) is a chain and \(I_{k_{ij}}\) are finite in number, there is an \(I \in \{I_k\}\) such that \(I_{k_{ij}} \subseteq I\), for every \(i,j = 1,\ldots,\nu\). But \(I\) is a boundary subsystem, therefore for every \(\varepsilon > 0\) we get,
\[
\|[x_{ij}]\| = \|[x_{ij} + I]\| = \inf\{\|[x_{ij} + b_{ij}]\| : b_{ij} \in I\}\}
\]
\[
\leq \|[x_{ij} + a_{ij}]\| \leq \|[x_{ij}] + M_\nu(J)\| + \varepsilon \leq \|[x_{ij}]\| + \varepsilon.
\]
Thus \(\|[x_{ij}]\| = \|[x_{ij}] + M_\nu(J)\|\), so \(J\) is an upper bound for the chain \(\{I_k\}_k\) and Zorn’s Lemma applies.

By Proposition [21] one can easily deduce the following.

**Corollary 2.3.** Let \(I\) be a maximal boundary subsystem for \(X\) in \(\mathcal{B}(H)\). Then \(\mathcal{B}(H)/I\) contains no non-trivial boundary subsystems for \(q_I(X)\).

From now on let us fix a maximal boundary subsystem \(I\) for \(X\) in \(\mathcal{B}(H)\). Our aim is to prove that \((\mathcal{B}(H)/I, q_I)\) is an injective envelope of \(X\). In order to do so we use the notion of averages of a ucc map. Let \(\varphi : \mathcal{B}(H) \to \mathcal{B}(H)\) be a ucc map. For a fixed \(k \geq 1\), we define the map
\[
m_k(\varphi) : \mathcal{B}(H) \to \mathcal{B}(H) : a \mapsto \frac{\varphi(a) + \cdots + \varphi^k(a)}{k}, \text{ for all } a \in \mathcal{B}(H).
\]
Since \(\varphi\) is linear, \(m_k(\varphi)\) is also linear. Moreover \(m_k(\varphi)\) is contractive because
\[
\|m_k(\varphi)(a)\| = \left\| \frac{\varphi(a) + \cdots + \varphi^k(a)}{k} \right\| \leq k \left\| \frac{\varphi(a)}{k} \right\| = \|\varphi(a)\| \leq \|a\|.
\]
For \(\nu \geq 1\), we get that \((m_k(\varphi))_\nu = m_k(\varphi)\nu\). Hence \(m_k(\varphi)\) is a ucc map for all \(k \in \mathbb{N}\).

Similarly, if \(S\) is an operator system (in \(\mathcal{B}(H)\)) and \(\varphi : S \to S\) is a ucc map, then \(m_k(\varphi) : S \to S\) is a well defined ucc map.
Lemma 2.4. Let $X$ be a unital operator space and $S$ be an operator system with $X \subseteq S$. If $\varphi : S \to S$ is a ucc map extending $\text{id}_X$ and $h$ is a self-adjoint element in $S$, then the subspace $\text{span}\{\varphi(h) - h\}$ of $S$ is a boundary subsystem for $X$.

Proof. It is immediate that $\text{span}\{\varphi(h) - h\}$ is a self-adjoint subspace of $S$. Fix a $k \in \mathbb{N}$ and let $m_k(\varphi)$ be the $k$-average of $\varphi$. Since $\varphi(x) = x$ for all $x \in X$, it is easy to check that $m_k(\varphi)(x) = x$ for all $x \in X$. For any $\lambda \in \mathbb{C}$ we will then have

\begin{equation}
\|x + \lambda(\varphi(h) - h)\| \geq \|m_k(\varphi)(x + \lambda(\varphi(h) - h))\| \\
= \|x + m_k(\varphi)(\lambda(\varphi(h) - h))\| \\
= \|x + \lambda m_k(\varphi)(\varphi(h) - h)\| \\
= \|x + \frac{\lambda}{k}(\varphi^{k+1}(h) - \varphi(h))\|.
\end{equation}

But,

$$\left\| \frac{\lambda}{k}(\varphi^{k+1}(h) - \varphi(h)) \right\| \leq \frac{|\lambda|}{k} \left( \|\varphi^{k+1}(h)\| + \|h\| \right) \leq \frac{2|\lambda|}{k} \|h\| \xrightarrow{k \to 0} 0.$$  

Hence, by taking the limit with respect to $k$ to the inequality (1) we have

$$\|x + \lambda(\varphi(h) - h)\| \geq \|x\|, \text{ for all } x \in X.$$  

Taking infimum over $\lambda \in \mathbb{C}$ we deduce that

$$\|x + \text{span}\{\varphi(h) - h\}\| \geq \|x\|, \text{ for all } x \in X,$$

and the last inequality becomes equality by noting that

$$\|x\| \geq \inf\{\|x + v\| : v \in \text{span}\{\varphi(h) - h\}\} = \|x + \text{span}\{\varphi(h) - h\}\|.$$  

The same arguments can be repeated for the matrix-norms, by substituting $x$, $m_k(\varphi)$ and $\lambda(\varphi(h) - h)$ with $[x_{ij}]$, $m_k(\varphi_{ij})$ and $[\lambda_{ij}(\varphi(h) - h)] = [\lambda_{ij}] \text{diag}\{\varphi(h) - h\}$, respectively. Hence $\text{span}\{\varphi(h) - h\}$ is a boundary subsystem for $X$ in $S$. ■

Lemma 2.5. If $\mathcal{I}$ is a maximal boundary subsystem of $X$ in $\mathcal{B}(H)$, then $(\mathcal{B}(H)/\mathcal{I}, q_\mathcal{I})$ is a rigid extension of $X$.

Proof. By definition $(\mathcal{B}(H)/\mathcal{I}, q_\mathcal{I})$ is an extension of $X$. Let $\varphi : \mathcal{B}(H)/\mathcal{I} \to \mathcal{B}(H)/\mathcal{I}$ be a ucc map such that $\varphi(x + \mathcal{I}) = x + \mathcal{I}$ for all $x \in X$. Recall that $\mathcal{B}(H)/\mathcal{I}$ is spanned by its self-adjoint elements. Therefore if $\varphi \neq \text{id}$ then there is a self-adjoint element $h \in \mathcal{B}(H)/\mathcal{I}$ such that $\varphi(h) \neq h$. Hence by Lemma [2.3] the subspace $\text{span}\{\varphi(h) - h\}$ is a non-trivial subsystem for $X$ in $\mathcal{B}(H)/\mathcal{I}$. But this contradicts with Corollary [2.3]. ■

Lemma 2.6. If $\mathcal{I}$ is a maximal boundary subsystem for $X$ in $\mathcal{B}(H)$, then $(\mathcal{B}(H)/\mathcal{I}, q_\mathcal{I})$ is an injective envelope for $X$.  

Proof. We will show that \((B(H)/I, q_I)\) is an injective and rigid extension of \(X\). By Lemma 2.5 it suffices to show that \(B(H)/I\) is injective.

First, let us fix a ucc map \(\sigma: B(H)/I \to B(H)\) that extends the mapping \(x + I \mapsto x\). The existence of \(\sigma\) is implied by the injectivity of \(B(H)\). Note that \(q_I \circ \sigma|_X = id_X\), thus \(q_I \circ \sigma = id\), by Lemma 2.5.

To show that \(B(H)/I\) is injective, let \(Z \subseteq Y\) be operator spaces and \(\varphi: Z \to B(H)/I\) be a ucc map. Then the map \(\sigma \circ \varphi: Z \to B(H)\) is a ucc map, thus it extends to a ucc map \(\tilde{\sigma} \circ \varphi: Y \to B(H)\). Then the ucc map \(q_I \circ \tilde{\sigma} \circ \varphi: Z \to B(H)/I\) extends \(\varphi\), since

\[
q_I \circ \tilde{\sigma} \circ \varphi(z) = q_I \circ \sigma \circ \varphi(z) = id \circ \varphi(z) = \varphi(z), \text{ for all } z \in Z,
\]

and the proof is complete. \(\blacksquare\)

Remark 2.7. Once the existence of an injective envelope is proved, the existence of the C*-envelope is implied. The key elements are the Choi-Effros’ Theorem and Lemma 2.9 that will follow. Note that Lemma 2.6 gives a concrete picture of the injective envelope which we investigate in Section 4. Due to this fact, we can replace Choi-Effros’ Theorem with a simpler, yet similar, argument (see Remark 4.2).

Theorem 2.8. (Choi-Effros) Let \(S \subseteq B(H)\) be an injective operator system and let a ucp \(\varphi: B(H) \to S\) be a projection onto \(S\). Then setting \(a \circ b = \varphi(ab)\) defines a multiplication on \(S\) and \(S\) together with this multiplication and its usual *-operation is a C*-algebra.

For the next lemma, recall that the multiplicative domain of a ucc map \(\varphi: C \to B(H)\) is the C*-subalgebra of \(C\)

\[
C_{\varphi} := \{a \in C : \varphi(a)^* \varphi(a) = \varphi(a^* a) \text{ and } \varphi(a) \varphi(a)^* = \varphi(a a^*)\}.
\]

The restriction of \(\varphi\) to \(C_{\varphi}\) is a *-homomorphism.

Lemma 2.9. Let \((S, \iota)\) be an injective envelope for an operator space \(X \subseteq B(H)\). If \(\varphi: B(H) \to S\) is a ucc extension of the mapping \(x \mapsto \iota(x)\) then \(X\) is in the multiplicative domain of \(\varphi\), with respect to the C*-algebraic structure \((S, \circ)\) induced on \(S\) by Choi-Effros’ Theorem. Consequently, the restriction of \(\varphi\) to \(C^*(X)\) is a *-homomorphism.

Proof. Let \(\psi: S \to B(H)\) be a ucc extension of the mapping \(\iota(x) \mapsto x\). The rigidity of \(S\) implies that \(\varphi \circ \psi = id\).

By Schwarz inequality we obtain \(\psi(\iota(x))^* \psi(\iota(x)) \leq \psi(\iota(x)^* \circ \iota(x))\), hence applying the ucc map \(\varphi\) we get that

\[
\iota(x)^* \circ \iota(x) = \varphi(x)^* \circ \varphi(x) \leq \varphi(x^* x)
\]

\[
= \varphi(\psi(\iota(x))^* \psi(\iota(x))) \leq \varphi \circ \psi(\iota(x)^* \circ \iota(x))
\]

\[
= id(\iota(x)^* \circ \iota(x)) = \iota(x)^* \circ \iota(x),
\]

therefore \(\varphi(x)^* \circ \varphi(x) = \varphi(x^* x)\), for all \(x \in X\). A symmetric calculation shows also that \(\varphi(x) \circ \varphi(x)^* = \varphi(xx^*)\), for all \(x \in X\), which completes the proof. \(\blacksquare\)
**Theorem 2.10.** Let $X \subseteq C = C^*(X)$ be an operator space. Then the Šilov ideal $I$ exists and $C/I$ is the $C^*$-envelope of $X$.

**Proof.** Assume that $C \subseteq \mathcal{B}(H)$ and fix a maximal boundary subsystem $\mathcal{I}$ for $X$ in $\mathcal{B}(H)$. By Lemma 2.9, $X$ is in the multiplicative domain of the ucc map $q_\mathcal{I}: \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{I}$, hence the restriction of $q_\mathcal{I}$ to $C = C^*(X)$ is a $*$-homomorphism. Therefore, $\mathcal{I} = \ker(q_\mathcal{I}|_C)$ is an ideal in $C$. Moreover, $\mathcal{I} = \ker q_\mathcal{I} \cap C = \mathcal{I} \cap C$, and $C/I \simeq q_\mathcal{I}(C) = q_\mathcal{I}(C^*(X)) = C^*(q_\mathcal{I}(X))$.

If $\mathcal{J}$ is a second maximal boundary subsystem for $X$ in $\mathcal{B}(H)$ then $\mathcal{B}(H)/\mathcal{J}$ is also an injective envelope, hence there is a unique uci and onto map $\Phi: \mathcal{B}(H)/\mathcal{J} \to \mathcal{B}(H)/\mathcal{I}$ that fixes $X$, by Lemma 1.6. Therefore, the following diagram commutes:

$$
\begin{array}{ccc}
\mathcal{B}(H)/\mathcal{J} & \xrightarrow{\Phi} & \mathcal{B}(H)/\mathcal{I} \\
\downarrow{q_\mathcal{J}} & & \downarrow{q_\mathcal{I}} \\
C & \xrightarrow{q_\mathcal{I}} & \mathcal{B}(H)/\mathcal{I}
\end{array}
$$

since $\Phi(q_\mathcal{J}(x)) = q_\mathcal{I}(x)$ for all $x \in X$, $X$ is in the multiplicative domain of $q_\mathcal{J}$. $q_\mathcal{J}(X)$ is in the multiplicative domain of $\Phi$ and $C = C^*(X)$. Thus $\mathcal{I} = \ker q_\mathcal{I}|_C = \ker(\Phi \circ q_\mathcal{J}|_C) = \ker q_\mathcal{J}|_C = \mathcal{J} \cap C$. Hence, $\mathcal{I} = \mathcal{I} \cap C$ for any maximal boundary subsystem $\mathcal{I}$ for $X$ in $\mathcal{B}(H)$.

We will show that $\mathcal{I}$ is the Šilov ideal. Since $\mathcal{I} \subseteq \mathcal{I}$ then $\mathcal{I}$ is also a boundary (ideal) for $X$ in $C$. Let $J$ be a boundary ideal for $X$ in $C$ and let $\mathcal{J}$ be the maximal boundary subsystem that contains $J$. Then $\mathcal{J} \subseteq \mathcal{J} \cap C = I$.

The proof is complete by observing that the $C^*$-cover $(q_\mathcal{J}(C), q_\mathcal{I})$ has the universal property of the $C^*$-envelope. Indeed, let $(B, j)$ be a $C^*$-cover for $X$ with $B = C^*(j(X)) \subseteq \mathcal{B}(K)$. Then by Lemma 2.3 the map $j(x) \mapsto q_\mathcal{I}(x)$ extends uniquely to a $*$-epimorphism $\Phi: B \to C^*(q_\mathcal{I}(X)) = q_\mathcal{I}(C)$.

### 3. The Šilov ideal

The proof of Theorem 2.10 gives additional information for the Šilov ideal which we isolate in the next corollary.

**Corollary 3.1.** Let $X \subseteq C = C^*(X) \subseteq \mathcal{B}(H)$. If $\mathcal{I}$ is any maximal boundary subsystem for $X$ in $\mathcal{B}(H)$, then $\mathcal{I} \cap C^*(X) = I$ is the Šilov ideal.

By definition the Šilov ideal contains the boundary ideals for $X$ in $C$. But this does not ensure that the Šilov ideal contains also all the boundary subsystems for $X$ in $C$, as it is not obvious that the ideal generated by a boundary subsystem is in turn boundary. Nevertheless this is implied by the proof of the existence of the Šilov ideal provided here.

We say that an $a \in (\mathcal{B}(H))_m$ is a boundary element if the operator subsystem span$\{a\}$ is boundary for $X$.

**Corollary 3.2.** Let $X \subseteq C = C^*(X) \subseteq \mathcal{B}(H)$. Then...
The Šilov ideal contains all the selfadjoint boundary subsystems for $X$ in $C$. Thus it contains the (closed) linear span of selfadjoint boundary elements.

(2) The ideal generated by a boundary subsystem for $X$ in $C$ is also boundary.

**Proof.** Let $V \subseteq C$ be a selfadjoint boundary subsystem for $X$ in $C$ and $I$ be a maximal boundary subsystem that contains $V$. Then, by Corollary 3.3 $V \subseteq I \cap C = I$, where $I$ is the Šilov ideal for $X$. Moreover, since $I$ is a boundary ideal we get that the ideal $\langle V \rangle$, generated by $V$ in $C$, is also a boundary ideal in $I$.

**Corollary 3.3.** Let $X \subseteq C = C^*(X) \subseteq B(H)$. Then the Šilov ideal for $X$ in $C$ is the biggest boundary subsystem for $X$ contained in $C$.

Let $X \subseteq B(H)$ and $\varphi: C^*(X) \rightarrow B(H)$ be a ucc map that extends $id_X$. We define the set of abnormalities in $C^*(X)$ relevant to $\varphi$ as the set $P_\varphi = \{\varphi(c) - c \in C^*(X) | c \in C^*(X)\}$. It is clear that $P_\varphi$ is a selfadjoint linear subspace of $C^*(X)$, since $\varphi$ is a ucp map.

For simplicity, we denote the set $\{\varphi: C^*(X) \rightarrow B(H) | \varphi \text{ extends } id_X\}$ by $\text{ext}(id_X)$.

**Proposition 3.4.** Let $X \subseteq B(H)$. Then the set $\bigcup\{P_\varphi | \varphi \in \text{ext}(id)\}$ equals to the Šilov ideal, and consequently it is closed.

**Proof.** Let $I$ be the Šilov ideal of $X$ in $C^*(X)$ and $a \in I$. Then the ideal $\langle a \rangle$ that is generated by $a$ is in $I$, thus it is a boundary ideal. Hence the restriction of the quotient map $q_{\langle a \rangle}$ to $X$ is a ucis. Let $\psi: C/\langle a \rangle \rightarrow B(H)$ be a ucis map that extends the mapping $x + \langle a \rangle \mapsto x$. Then the mapping $\psi \circ q_{\langle a \rangle}$ is a ucis map that extends $id_X$ and $\psi \circ q_{\langle a \rangle}(a) = 0$. Hence $a = a - \psi \circ q_{\langle a \rangle}(a) \in P_{\psi q_{\langle a \rangle}}$. Thus $I \subseteq \bigcup\{P_\varphi | \varphi \in \text{ext}(id_X)\}$.

For the converse, let $a \in \bigcup\{P_\varphi | \varphi \in \text{ext}(id_X)\}$. Then there is a ucis map $\varphi \in \text{ext}(id_X)$ and $c \in C^*(X)$ such that $a = \varphi(c) - c$. For $c_1 = -\frac{\varphi(c)}{2}$ and $c_2 = \frac{c - \varphi(c)}{2}$, we get that $a = \varphi(c_1) - c_1 + i(\varphi(c_2) - c_2) \in \langle \varphi(c_1) - c_1 \rangle + \langle \varphi(c_2) - c_2 \rangle$. By Lemma 2.4, the elements $\varphi(c_1) - c_1$ and $\varphi(c_2) - c_2$ are boundary. Therefore the ideal $\langle \varphi(c_1) - c_1 \rangle + \langle \varphi(c_2) - c_2 \rangle$ is boundary as the sum of two boundary ideals, by Corollary 3.2. Thus $a \in I$.

**Remark 3.5.** The definition of the sets $P_\varphi$ is rather tricky. It refers to elements $c \in C^*(X)$ such that $\varphi(c) \in C^*(X)$ for some ucc extension $\varphi$ of $id_X$, and not to elements such that $\varphi(c) \in B(H)$. The reason to be careful is that a ucc extension $\varphi$ of $id_X$ may take values outside $C^*(X)$, even outside an original injective envelope $S$ that contains $X$ (even when $C^*(X)$ is considered as the $C^*_\text{env}(X)$). On the other hand this assumption seems reasonable as the Šilov ideal lies in $C^*(X)$. It would be of great interest a result similar to Proposition 3.4 for sets of the form $\{\varphi(c) - c | c \in C^*(X)\}$.
4. Maximal Boundary Subsystems

Corollary 3.1 associates the Šilov ideal with maximal boundary subsystems. In this section we investigate further these spaces.

Proposition 4.1. Let $X \subseteq \mathcal{B}(H)$. Then $\mathcal{I}$ is a maximal boundary subsystem for $X$ in $\mathcal{B}(H)$ if and only if $\mathcal{I} = \ker \varphi$ for some ucc map $\varphi: \mathcal{B}(H) \to S$, where $(S, \iota)$ is an injective envelope for $X$ and $\varphi(x) = \iota(x)$, for all $x \in X$.

Proof. If $\mathcal{I}$ is a maximal boundary subsystem then the appropriate $\varphi$ is $q_{\mathcal{I}}: \mathcal{B}(H) \to \mathcal{B}(H)/\mathcal{I}$. For the converse, let $S$ be an injective envelope and $\varphi: \mathcal{B}(H) \to S$ be a ucc map that fixes $X$ elementwise. By Lemma 1.7, the map $\varphi$ is onto $S$. Therefore $\mathcal{B}(H)/\ker \varphi \simeq S$ via the ucc map $\hat{\varphi}: x + \ker \varphi \mapsto \varphi(x)$. Thus $\ker \varphi$ is a boundary operator subsystem (because $\varphi$ is also positive) for $X$. Moreover, $\mathcal{B}(H)/\ker \varphi$ is also an injective envelope for $X$. Indeed, it suffices to prove that $(\hat{\varphi})^{-1}$ is ucc, because then $\hat{\varphi}$ is ucis. To this end let $\sigma: S \to \mathcal{B}(H)$ be the ucis map that extends the map $\iota(x) \mapsto x$. Then $\hat{\varphi} \circ q_{\ker \varphi} \circ \sigma: S \to S$ and

$$\hat{\varphi} \circ q_{\ker \varphi} \circ \sigma(\iota(x)) = \hat{\varphi}(x + \ker \varphi) = \varphi(x) = \iota(x), \text{ for all } x \in X.$$

Therefore by rigidity of $S$ we get that $\hat{\varphi} \circ q_{\ker \varphi} \circ \sigma = \text{id}_S$. Then

$$q_{\ker \varphi} \circ \sigma = \text{id}_{\mathcal{B}(H)/\ker \varphi} \circ q_{\ker \varphi} \circ \sigma = (\hat{\varphi})^{-1} \circ \hat{\varphi} \circ q_{\ker \varphi} \circ \sigma = (\hat{\varphi})^{-1} \circ \text{id}_S = (\hat{\varphi})^{-1}.$$

Thus the map $(\hat{\varphi})^{-1}$ is ucc, hence $\hat{\varphi}$ is ucis, therefore $\mathcal{B}(H)/\ker \varphi$ is an injective envelope.

If $\mathcal{I}$ is a maximal boundary operator subsystem that contains $\ker \varphi$, then $\mathcal{B}(H)/\mathcal{I}$ is also an injective envelope. Lemma 1.6 implies then that the canonical onto map

$$\mathcal{B}(H)/\ker \varphi \to \mathcal{B}(H)/\mathcal{I}: a + \ker \varphi \mapsto a + \mathcal{I},$$

is a (necessarily unique) ucis, therefore $\ker \varphi = \mathcal{I}$. 

Remark 4.2. We can always assume that there is an injective envelope $(S, \text{id})$ for $X \subseteq \mathcal{B}(H)$. (Indeed, if $(S, \iota)$ is an injective envelope and $\varphi: S \to \mathcal{B}(H)$ is any extension of the mapping $\iota(x) \mapsto x$, then $\varphi(S)$ is an injective envelope containing $\varphi(\iota(X)) = X$. Note that essentiality of $S$ guarantees that $\varphi(S) = \varphi(S)$.) Let $\mathcal{I}$ be a maximal boundary subsystem for $X$; there is a trivial (but isomorphic) way of inducing a $C^*$-algebraic structure on $\mathcal{B}(H)/\mathcal{I}$ avoiding the use of Choi-Effros’ Theorem. We view $\mathcal{B}(H)/\mathcal{I}$ as $\mathcal{B}(H)/\ker \varphi$, for a $\varphi$ as in Proposition 4.1 and for $a, b \in \mathcal{B}(H)$ we define

$$(a + \ker \varphi) \odot (b + \ker \varphi) := \varphi(a)\varphi(b) + \ker \varphi.$$

It is well defined and in order to prove the $C^*$-identity, we will use that $\varphi$ is a ucp projection on $S$ (by Lemma 1.7), that $\hat{\varphi}: \mathcal{B}(H)/\ker \varphi \to S$ is a ucis.
map (by Lemma 1.6), and the C*-identity on \( B(H) \). For \( a \in B(H) \) we get

\[
\|a + \ker \varphi\|^2 = \|\varphi(a)\|^2 = \|\varphi(a)^*\varphi(a)\| = \|\varphi^2(a)^*\varphi^2(a)\| \leq \|\varphi(\varphi(a)^*\varphi(a))\| = \|\varphi(a)^*\varphi(a) + \ker \varphi\| \\
= \|\varphi(\varphi(a)^*\varphi(a))\| \leq \|\varphi(a)^*\varphi(a)\| \\
= \|\varphi(a)\|^2 = \|a + \ker \varphi\|^2.
\]

Another way of characterizing maximal boundary subsystems for \( X \) in \( B(H) \) is by using minimal \( X \)-maps. A map \( \varphi: B(H) \to B(H) \) is called an \( X \)-map if \( \varphi \) is ucc and \( \varphi(x) = x \) for all \( x \in X \). Trivially, the kernel of an \( X \)-map is a boundary subsystem. For example,

\[
\|x\| \geq \|x + \ker \varphi\| \geq \|\varphi(x)\| = \|x\|, \text{ for all } x \in X.
\]

An \( X \)-map is called an \( X \)-projection, if it is a projection. We write \( \psi \prec \varphi \), if \( \psi \) is an \( X \)-projection such that \( \psi \circ \varphi = \psi = \varphi \circ \psi \).

For an \( X \)-map \( \varphi \) we can define an \( X \)-seminorm \( p_\varphi \) on \( B(H) \) such that \( p_\varphi(a) = \|\varphi(a)\| \), for all \( a \in B(H) \). Unlike in [S], we write \( p_\psi \leq_c p_\varphi \), if \( \psi \) is an \( X \)-map and \( p_{\varphi_\nu}([a_{ij}]) \leq p_{\varphi_\nu}([a_{ij}]) \) for all \( a_{ij} \in B(H) \) and \( \nu \in \mathbb{N} \).

**Theorem 4.3.** Let \( X \) be an operator space in \( B(H) \). Then the following are equivalent

1. \( \mathcal{I} \) is a maximal boundary subsystem for \( X \) in \( B(H) \),
2. \( \mathcal{I} = \ker \varphi \) for some \( \varphi: B(H) \to S \), where \( (S, \iota) \) is an injective envelope for \( X \) and \( \varphi(x) = \iota(x) \), for all \( x \in X \).
3. \( \mathcal{I} = \ker \varphi \) for some \( X \)-map \( \varphi \) such that \( p_\varphi \) is a \( \leq_c \)-minimal \( X \)-seminorm,
4. \( \mathcal{I} = \ker \varphi \) for some \( \prec \)-minimal \( X \)-projection \( \varphi \) and \( (\varphi(B(H)), \varphi |_X) \) is an injective envelope for \( X \).

**Proof.** The equivalence \([1] \iff [2]\) is Proposition 1.1 and the implications \([3] \Rightarrow [4]\) can be derived, for example, by the same arguments as in [S] proof of Theorem 15.4. Obviously \([4] \Rightarrow [2]\).

For \([2] \Rightarrow [3]\), we can assume that \( S \subseteq B(H) \), by Remark 4.2. Let \( \varphi: B(H) \to S \) be a map that fixes \( X \) pointwise. Hence it is an \( X \)-projection onto \( S \) and \( \ker \varphi \) is a maximal boundary subsystem for \( X \) in \( B(H) \). Let \( \psi: B(H) \to B(H) \) such that \( \|\psi_\nu([a_{ij}])\| \leq \|\varphi_\nu([a_{ij}])\| \) for all \( a_{ij} \in B(H) \) and \( \nu \in \mathbb{N} \). Then the mapping \( \sigma: S \to B(H) \), such that \( \sigma(\varphi(a)) = \psi(a) \), is a well defined ucc map onto \( \psi(B(H)) \) and fixes \( X \) elementwise. Moreover, \( \ker \psi = \ker \sigma \circ \varphi \supseteq \ker \varphi \). Since \( \ker \varphi \) is a maximal boundary subsystem and \( \ker \psi \) is a boundary subsystem (being the kernel of an \( X \)-map), this implies...
that $\ker \varphi = \ker \psi$. Therefore the following diagram is commutative

\[
\begin{array}{ccc}
B(H)/\ker \varphi & \xrightarrow{\hat{\varphi}} & S \\
\downarrow & & \downarrow \\
B(H)/\ker \psi & \xrightarrow{\hat{\psi}} & \psi(B(H))
\end{array}
\]

where the induced quotient maps $\hat{\varphi}$ and $\hat{\psi}$ are ucis. Indeed, this is implied by the fact that $S$ is an injective envelope, hence $B(H)/\ker \varphi = B(H)/\ker \psi$ is also an injective envelope. Thus, $\sigma$ is a ucis map. But then

$$
\|\psi_\nu([a_{ij}]\| = \|(\sigma \circ \varphi)_\nu([a_{ij}]\| = \|\sigma_\nu(\varphi_\nu([a_{ij}]\|) = \|\varphi_\nu([a_{ij}]\|,
$$

for all $a_{ij} \in B(H)$ and $\nu \in \mathbb{N}$; hence $p_{\varphi}$ is $\leq c$-minimal.

**Remark.** We remind that our arguments work for the category of unital operator spaces with ucis maps. If $X$ is an operator system this is the usual category, whereas if $X$ is a non-unital operator algebra, one can prove the analogous results by passing to the unitization of $X$ and by using Meyer’s Theorem [7]. However, in general there are results that suggest that the $C^*$-envelope of non-unital operator spaces may not be the natural object (see [2, 4, 9]). The author would like to thank M. Anoussis and A. Katavolos for bringing this to his attention.

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Pure Mathematics Department, University of Waterloo, On N2L3G1, Canada

*E-mail address*: ekakaria@uwaterloo.ca