THE FRACTIONAL $p$-LAPLACIAN ON HYPERBOLIC SPACES

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Abstract. We present three equivalent definitions of the fractional $p$-Laplacian $(-\Delta_{\mathbb{H}^n})^s_p$, $0 < s < 1$, $p > 1$, with normalizing constants, on the hyperbolic spaces. The explicit values of the constants enable us to study the convergence of the fractional $p$-Laplacian to the $p$-Laplacian as $s \to 1^-$. 

1. Introduction

Operators of fractional-order have been studied extensively not only on the Euclidean spaces [16] but also on various spaces such as Riemannian manifolds [1, 2, 9, 14, 20, 21, 25], metric measure spaces [8, 11, 19, 22, 23], discrete models [12], Lie groups [7, 13, 17, 18], Wiener spaces [7], and so on. On the Euclidean spaces, there are several equivalent definitions of the fractional Laplacian [29] due to the simple structure of the spaces. In contrast to the case of Euclidean spaces, not all definitions are equivalent on general spaces. For instance, one can study a regional-type operator [25] or a spectral-type operator [31] on Riemannian manifolds. Moreover, some definitions, such as the one using the Fourier transform, do not even work on general Riemannian manifolds and metric measure spaces. However, several representations for the fractional Laplacians on some Riemannian manifolds, such as hyperbolic spaces and spheres, have been established [2, 14] by means of rich structures of the spaces.

The aim of this paper is two-fold. We first generalize representation formulas in [2] to the nonlinear regime on the hyperbolic spaces. Precisely, we define the fractional $p$-Laplacian $(-\Delta_{\mathbb{H}^n})^s_p$ for $n \in \mathbb{N}$, $0 < s < 1$, and $p > 1$ by using the heat semigroup and establish the singular integral representation and the Caffarelli–Silvestre extension. Note that the definition via the Fourier transform is not available because of the nonlinearity of the operator. We next study the pointwise convergence of $(-\Delta_{\mathbb{H}^n})^s_p u(x)$ as $s \to 1^-$ using the singular integral representation. For this purpose, we compute the explicit values of the normalizing constants in the singular integral representation. This explicit value was available only when $n = 3$ and $p = 2$, see [28].

Let us define the fractional $p$-Laplacian on the hyperbolic spaces. We adopt the definition proposed in [15, Section 8.2], which is a nonlinear extension of the Bochner’s definition [3]. See also [31]. To this end, let $\{e^{t\Delta_{\mathbb{H}^n}}\}_{t \geq 0}$ be the heat semigroup generated by the Laplacian $\Delta_{\mathbb{H}^n}$ on hyperbolic spaces. That is, for a given function $f : \mathbb{H}^n \to \mathbb{R}$ we denote by $e^{t\Delta_{\mathbb{H}^n}}[f](x)$

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the solution \( w(x, t) \) of a Cauchy problem
\[
\begin{cases}
\partial_t w(x, t) - \Delta_{\mathbb{H}^n} w(x, t) = 0, & x \in \mathbb{H}^n, t > 0, \\
w(x, 0) = f(x), & x \in \mathbb{H}^n.
\end{cases}
\] (1.1)

We define \( C^2_0(\mathbb{H}^n) \) by the space of bounded \( C^2 \) functions on \( \mathbb{H}^n \).

**Definition 1.1.** Let \( n \in \mathbb{N}, s \in (0, 1), \) and \( p > 1. \) Let \( u \in C^2_0(\mathbb{H}^n) \) and \( x \in \mathbb{H}^n. \) If \( p \in (1, \frac{2}{2-s}] \), assume in addition \( \nabla u(x) \neq 0. \) The fractional \( p \)-Laplacian on \( \mathbb{H}^n \) is defined by
\[
(-\Delta_{\mathbb{H}^n})^s_p u(x) = C_1 \int_0^\infty e^{t\Delta_{\mathbb{H}^n}} [\Phi_p(u(x) - u(\cdot))](x) \frac{dt}{t^{1+\frac{sp}{2}}},
\]
where
\[
C_1 = \frac{p}{\Gamma(\frac{n}{2})} \frac{2^{\frac{n}{2}(2-p)}}{\Gamma(-s)}
\]
and \( \Phi_p(r) = |r|^{s-p-r}. \)

The constant \( C_1 \) in (1.2) is chosen so that the pointwise convergence \( \lim_{s \to 1} (-\Delta_{\mathbb{H}^n})^s_p u(x) = (-\Delta_{\mathbb{H}^n})_p u(x) \) holds, see Theorem 1.4. The same constant is used in the case of Euclidean spaces [15]. Moreover, this choice is in accordance with the constant in the case \( p = 2, \) see [2].

The first result is the pointwise integral representation of the fractional \( p \)-Laplacian with singular kernels. Note that the hyperbolic geometry is distinguished from the Euclidean geometry only when \( n \geq 2. \)

**Theorem 1.2.** Let \( n \geq 2, s \in (0, 1), \) and \( p > 1. \) Let \( u \in C^2_0(\mathbb{H}^n) \) and \( x \in \mathbb{H}^n. \) If \( p \in (1, \frac{2}{2-s}] \), assume in addition \( \nabla u(x) \neq 0. \) Then, the fractional \( p \)-Laplacian on \( \mathbb{H}^n \) has the pointwise representation
\[
(-\Delta_{\mathbb{H}^n})^s_p u(x) = c_{n,s,p} \text{P.V.} \int_{\mathbb{H}^n} |u(x) - u(\xi)|^{p-2} (u(x) - u(\xi))K_{n,s,p}(d(x, \xi)) \, d\xi
\]
with the kernel \( K_{n,s,p} \) given by
\[
K_{n,s,p}(\rho) = C_2 \left( \frac{-\partial_r}{\sinh \rho} \right)^{\frac{n-1}{2}} \left( \rho^{-\frac{n+sp}{2}} K_{\frac{n+sp}{2}} \left( \frac{n-1}{2} \rho \right) \right)
\]
when \( n \geq 3 \) is odd and
\[
K_{n,s,p}(\rho) = C_2 \int_{\rho}^\infty \frac{\sinh r}{\sqrt{\pi} \cosh r - \cosh \rho} \left( \frac{-\partial_r}{\sinh \rho} \right)^{\frac{n-1}{2}} \left( r^{-\frac{n+sp}{2}} K_{\frac{n+sp}{2}} \left( \frac{n-1}{2} r \right) \right) \, dr
\]
when \( n \geq 2 \) is even, where
\[
c_{n,s,p} = \frac{p}{2} \frac{\sqrt{\pi/2}}{\Gamma(\frac{n+sp}{2})} \frac{2^{2s}}{\Gamma(\frac{n+sp}{2})} \frac{1}{\pi \sqrt{\pi} \Gamma(-s)}, \quad C_2 = \frac{1}{2^{n+sp}} \frac{\Gamma(\frac{n+sp}{2})}{\Gamma(-s)} \left( \frac{n-1}{2} \right)^{\frac{n+sp}{2}},
\]
and \( K_\nu \) is the modified Bessel function of the second kind. Moreover, the kernel \( K_{n,s,p} \) is positive and has the asymptotic behavior
\[
K_{n,s,p}(\rho) \sim \rho^{-n-sp}
\]
as \( \rho \to 0^+ \) and
\[
K_{n,s,p}(\rho) \sim \rho^{-1-\frac{sp}{2}} e^{-(n-1)\rho}
\]
as $\rho \to +\infty$, up to constants.

In the linear case $p = 2$, the pointwise integral representation with singular kernel is provided in [2, Theorem 2.4 and 2.5] without constants. The novelty of Theorem 1.2 is that it generalizes the representation formula to the nonlinear regime with the normalizing constant. We emphasize that the normalizing constant plays a crucial role in some contexts. For instance, it is used in the convergence result (Theorem 1.4) and the robust regularity theory (see [6, 28]).

The main tool in [2, 28] is the Fourier transform on the hyperbolic spaces. Since the Fourier transform is not available in the nonlinear setting, we use the heat kernel for the Laplace operator $\Delta_{\mathbb{H}^n}$ on hyperbolic spaces to prove Theorem 1.2. The explicit formula for the heat kernel with a normalizing constant given in [24] enables us to obtain the exact values of the constants in Theorem 1.2.

Let us proceed to another representation for the fractional $p$-Laplacian on $\mathbb{H}^n$. We recall that the fractional Laplacian on $\mathbb{R}^n$ is obtained by a Dirichlet-to-Neumann map via the Caffarelli–Silvestre extension [5]. Later, the article [31] relates the heat semigroup to this extension. Moreover, this relation is extended to the nonlinear framework [15] in $\mathbb{R}^n$. We further investigate this relation on the hyperbolic spaces. Let us consider the extension problem

$$
\begin{align*}
\Delta_x U(x, y) + \frac{1 - sp}{y} U_y(x, y) + U_{yy}(x, y) &= 0, \quad x \in \mathbb{H}^n, y > 0, \\
U(x, 0) &= f(x), \quad x \in \mathbb{H}^n,
\end{align*}
$$

(1.4)

and define an extension operator $E_{s,p}$ by $E_{s,p}[f] := U$. The following theorem is our next main result.

**Theorem 1.3.** Let $n \in \mathbb{N}$, $s \in (0, 1)$, and $p > 1$. Let $u \in C_0^2(\mathbb{H}^n)$ and $x \in \mathbb{H}^n$. If $p \in \left(1, \frac{2}{2-s}\right]$, assume $\nabla u(x) \neq 0$ additionally. Then

$$
(-\Delta_{\mathbb{H}^n})_p^s u(x) = C_3 \lim_{y \to 0} \frac{E_{s,p}[(\Phi_p(u(x) - u(\cdot)))](x, y)}{y^{sp}}
$$

(1.5)

$$
= C_3 \lim_{y \to 0} y^{-sp} \partial_y \left( E_{s,p}[(\Phi_p(u(x) - u(\cdot)))](x, y),
$$

where

$$
C_3 = \frac{p \sqrt{\pi} / 2}{2 \Gamma \left(\frac{2s}{p-1}\right) |\Gamma(-s)|}.
$$

To prove Theorem 1.3, we represent the solution $U$ of the extension problem (1.4) by using the heat semigroup. Then, the formula for the heat kernel [24] leads to the Poisson formula for $U$ and the representation (1.5).

The last result is the pointwise convergence of the fractional $p$-Laplacian on $\mathbb{H}^n$ as $s \to 1^-$. As one can expect, the fractional $p$-Laplacian converges to the $p$-Laplacian as a limit. Recall that the $p$-Laplacian on $\mathbb{H}^n$ is defined by $(-\Delta_{\mathbb{H}^n})_p u(x) = -\text{div}(|\nabla u(x)|^{p-2} \nabla u(x))$.

**Theorem 1.4.** Let $n \in \mathbb{N}$, $p > 1$, and $u \in C_0^2(\mathbb{H}^n)$. For $x \in \mathbb{H}^n$ such that $\nabla u(x) \neq 0$,

$$
\lim_{s \to 1} (-\Delta_{\mathbb{H}^n})_p^s u(x) = (-\Delta_{\mathbb{H}^n})_p u(x).
$$
The pointwise convergence of the fractional $p$-Laplacian on the Euclidean spaces is well known [4, 16, 26]. Recall that the proof uses Taylor’s theorem and the following computations:

$$\begin{align*}
\int_{\mathbb{S}^{n-1}} \int_{R} K(\rho) \rho^{n-1} \, d\rho \, d\omega &= \frac{|\mathbb{S}^{n-1}|}{sp} R^{-sp}, \\
\int_{\mathbb{S}^{n-1}} \int_{0}^{R} K(\rho) \rho^{p+n-1} \, d\rho \, d\omega &= \frac{|\mathbb{S}^{n-1}|}{p(1-s)} R^{p(1-s)}, \\
\int_{\mathbb{S}^{n-1}} \int_{0}^{R} K(\rho) \rho^{\beta+p+n-1} \, d\rho \, d\omega &= \frac{|\mathbb{S}^{n-1}|}{\beta + p(1-s)} R^{\beta+p(1-s)},
\end{align*}$$

where $\beta > 0$ and $K(\rho) = \rho^{-n-sp}$ is the kernel for the fractional $p$-Laplacian on $\mathbb{R}^n$. However, in our framework we need the integrals in (1.6) with the kernel $K$ and the volume element $\rho^{n-1} \, d\rho \, d\omega$ replaced by $K_{n,s,p}$ and $\sinh^{n-1} \, \rho \, d\rho \, d\omega$, respectively. These integrals do not seem to be of a form that is easily computed. Instead, we compute the limits of these integrals as $s \to 1^-$, which are sufficient to establish Theorem 1.4. This is still not straightforward, but can be obtained by using the asymptotic behavior of modified Bessel functions.

The paper is organized as follows. In Section 2 we recall the hyperboloid model and study the modified Bessel function and its properties. Section 3 is devoted to the proof of Theorem 1.2, which provides the pointwise integral representation of the fractional $p$-Laplacian with singular kernels. In Section 4, we relate the heat semigroup to the extension problem (1.4) and find the Poisson formula. Using the Poisson formula and the representation of the fractional $p$-Laplacian, we prove Theorem 1.3. Finally, we prove the pointwise convergence result, Theorem 1.4, in Section 5. An auxiliary result can be found in Appendix A.

2. Preliminaries

In this section, we recall the basics of the hyperbolic spaces and collect some facts about the modified Bessel function.

2.1. The hyperbolic space. There are several models for the hyperbolic spaces, but let us focus on the hyperboloid model in this paper. The hyperboloid model is given by

$$\mathbb{H}^n = \{ (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 - x_1^2 - \cdots - x_n^2 = 1, x_0 > 0 \}$$

with the Lorentzian metric $-dx_0^2 + dx_1^2 + \cdots + dx_n^2$ in $\mathbb{R}^{n+1}$. The Lorentzian metric induces the natural internal product

$$[x, \xi] = x_0 \xi_0 - x_1 \xi_1 - \cdots - x_n \xi_n$$

on $\mathbb{H}^n$. Moreover, the distance between two points $x$ and $\xi$ is given by

$$d(x, \xi) = \cosh^{-1}([x, \xi]).$$

Using the polar coordinates, $\mathbb{H}^n$ can also be realized as

$$\mathbb{H}^n = \{ x = (\cosh r, \sinh r \omega) \in \mathbb{R}^{n+1} : r \geq 0, \omega \in \mathbb{S}^{n-1} \}.$$ 

Then, the metric and the volume element are given by $dr^2 + \sinh^2 r \, d\omega^2$ and $\sinh^{n-1} \, r \, dr \, d\omega$, respectively.
2.2. The modified Bessel function. The modified Bessel functions naturally appear in the study of hyperbolic geometry. In this paper, they are used to describe the kernel of the fractional $p$-Laplacian and the Poisson kernel. For this purpose, we recall the definition and some properties of the modified Bessel functions.

We call the ordinary differential equation
\[ \rho^2 \frac{d^2y}{d\rho^2} + \rho \frac{dy}{d\rho} - (\rho^2 + \nu^2)y = 0 \]
the modified Bessel equation. The solutions are given by
\[ I_{\nu}(\rho) = \sum_{j=0}^{\infty} \frac{1}{j! \Gamma(\nu + j + 1)} \left( \frac{\rho}{2} \right)^{2j+\nu} \]
and
\[ K_{\nu}(\rho) = \frac{\pi}{2} I_{-\nu}(\rho) - I_{\nu}(\rho), \]
and they are called the modified Bessel functions of the first and the second kind, respectively. Since only $K_{\nu}$ appears in this work, we focus on the properties of $K_{\nu}$. This function has the following integral representation (see [30, 10.32.10]):
\[ K_{\nu}(\rho) = \frac{1}{2} \left( \frac{1}{2} \rho \right)^{\nu} \int_0^\infty e^{-\frac{\rho^2}{4t} t^{-\nu-1}} dt. \]

The asymptotic behavior of $K_{\nu}$ is given by
\[ K_{\nu}(\rho) \sim \frac{1}{2} \Gamma(\nu) \left( \frac{\rho}{2} \right)^{-\nu} \quad \text{as} \quad \rho \to 0^+, \quad \text{for} \quad \nu > 0, \quad \text{and} \]
\[ K_{\nu}(\rho) \sim \sqrt{\frac{\pi}{2\rho}} e^{-\rho} \quad \text{as} \quad \rho \to +\infty. \]
Moreover, $K_{\nu}$ satisfies the following recurrence relations:
\[ K_{\nu}' = -\frac{\nu}{R} K_{\nu-1} \quad \text{and} \quad K_{\nu}' = -\frac{\nu}{R} K_{\nu+1}. \]
We also recall that $K_{\nu}$ is increasing with respect to $\nu > 0$. For further properties of the modified Bessel functions, the reader may consult the handbook [30].

In the sequel, functions of the form $\rho^{-\nu}K_{\nu}(a\rho)$ with $\nu \in \mathbb{R}$ and $a > 0$ will appear frequently. For notational convenience, we define
\[ \mathcal{X}_{\nu,a}(\rho) := \rho^{-\nu}K_{\nu}(a\rho). \]
Then, it follows from (2.3)
\[ -\partial_\rho(\mathcal{X}_{\nu,a}(f(\rho))) = a f'(\rho)f(\rho) \mathcal{X}_{\nu+1,a}(f(\rho)) \]
for any differentiable function $f : (0, +\infty) \to (0, +\infty)$. 

3. Pointwise representation with singular kernel

The singular kernel $\rho^{-n-sp}$ for the fractional $p$-Laplacian on the Euclidean space $\mathbb{R}^n$ is homogeneous of degree $-n - sp$. This is a natural property coming from the scale invariance of the operator. However, this cannot be expected in the case of hyperbolic spaces because the hyperbolic geometry comes into play. Indeed, we will see that the kernel $\mathcal{K}_{n,s,p}(\rho)$ behaves like $\rho^{-n-sp}$ near $\rho = 0$ whereas it decays like $\rho^{-1} e^{-(n-1)\rho}$ as $\rho \to +\infty$, up to constants, by providing the explicit form of the kernel $\mathcal{K}_{n,s,p}$. Moreover, we investigate the pointwise integral representation of the fractional $p$-Laplacian on $\mathbb{H}^n$. 
It is well known that the Cauchy problem (1.1) has the unique solution
\[
w(x,t) = \int_{\mathbb{R}^n} p(t, d(x, \xi)) f(\xi) \, d\xi,
\]
provided that \( f \) is a bounded continuous function, where the heat kernel \( p(t,\rho) \) is given [24] by
\[
p(t,\rho) = \frac{1}{(2\pi)^m (4\pi t)^{1/2}} \left( \frac{-\partial_p}{\sinh \rho} \right)^m e^{-\frac{\rho^2}{4t}}
\]
when \( n = 2m + 1 \geq 1 \) is odd and
\[
p(t,\rho) = \frac{1}{2(2\pi)^{m+1/2}} t^{-3/2} e^{-\frac{(2m+1)^2}{4} t} \left( \frac{-\partial_p}{\sinh \rho} \right)^{m-1} \int_\rho^\infty \frac{re^{-\frac{\rho^2}{4}}}{\cosh r - \cosh \rho} \, dr
\]
when \( n = 2m \geq 2 \) is even. We use these explicit formulas for the heat kernels not only in the computation of the singular kernels \( K_{n,s,p} \) but also in the next sections. For this purpose, we first prove the following lemma, which is useful especially in the even dimensional case.

**Lemma 3.1.** Let \( \nu > 1/2, \, a \geq 1/2, \) and \( y \geq 0. \) For \( m \in \mathbb{N} \cup \{0\} \) define
\[
F_m(r) := \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left( \frac{-\partial_r}{\sinh r} \right)^m \mathcal{K}_{\nu,a} \left( \sqrt{r^2 + y^2} \right),
\]
where \( \mathcal{K}_{\nu,a} \) is the function given in (2.4). Then, \( F_m \) is integrable on \( (\rho, +\infty) \) and satisfies
\[
(\frac{-\partial_r}{\sinh r}) \int_\rho^\infty F_m(r) \, dr = \int_\rho^\infty F_{m+1}(r) \, dr
\]
for all \( m \in \mathbb{N} \cup \{0\} \).

**Proof.** Note that for any \( j \geq 1 \)
\[
-\partial_r \left( \frac{(e^r + e^{-r})^{j-1}}{(e^r - e^{-r})^j} \right) = j \frac{(e^r + e^{-r})^j}{(e^r - e^{-r})^{j+1}} - (j-1) \frac{(e^r + e^{-r})^{j-2}}{(e^r - e^{-r})^{j-1}}.
\]
Therefore, all derivatives of \( \frac{1}{\sinh r} \) (and \( r e^{-r} \)) have the same asymptotic behavior as \( e^{-r} \) (and \( re^{-r} \), respectively) as \( r \to +\infty. \) Hence, \( F_m(r) \sim r^{-\nu-1/2} e^{(1/2-m-a)r} \) as \( r \to +\infty, \) which shows that the function \( F_m \) is integrable.

Using the integration by parts, we have
\[
\int_\rho^\infty F_m(r) \, dr = \int_\rho^\infty 2\partial_r \left( \frac{\sinh r}{\cosh r - \cosh \rho} \right) \left( \frac{-\partial_r}{\sinh r} \right)^m \mathcal{K}_{\nu,a} \left( \sqrt{r^2 + y^2} \right) \, dr
\]
\[
= \int_\rho^\infty 2 \sinh r \sqrt{\cosh r - \cosh \rho} \left( \frac{-\partial_r}{\sinh r} \right)^{m+1} \mathcal{K}_{\nu,a} \left( \sqrt{r^2 + y^2} \right) \, dr.
\]
Thus, the recurrence relation (3.3) follows by applying the Leibniz integral rule. \( \square \)

Let us now prove Theorem 1.2 using the heat kernel and Lemma 3.1.

**Proof of Theorem 1.2.** Let \( \varepsilon > 0 \) and define \( g_\varepsilon(x,\xi) = \Phi_p(u(x) - u(\xi)) \chi_{d(x,\xi) > \varepsilon}. \) The heat semigroup associated to \( g_\varepsilon(x,\cdot) \) is given by
\[
e^{t\Delta_{\mathbb{R}^n}} [g_\varepsilon(x,\cdot)](x) = \int_{\mathbb{R}^n} \frac{1}{(2\pi)^m (4\pi t)^{1/2}} \left( \frac{-\partial_p}{\sinh \rho} \right)^m e^{-\frac{\rho^2}{4t}} g_\varepsilon(x,\xi) \, d\xi
\]
when \( n = 2m + 1 \geq 3 \) is odd and
\[
e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x, \cdot)](x) = \int_{\mathbb{H}^n} t^{-3/2} e^{-\frac{(2m+1)^2}{4} t} \left( -\partial_\rho \right)^{m-1} \int_0^\infty r e^{-\frac{r^2}{4\pi}} \frac{dr}{\sqrt{\cosh r - \cosh \rho}} g_\varepsilon(x, \xi) \, d\xi
\]
when \( n = 2m \geq 2 \) is even, where \( \rho = d(x, \xi) \). We will prove
\[
C_1 \int_0^\infty e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1+\frac{m}{2}}} = c_{n, s, p} \int_{d(x, \xi) > \varepsilon} \Phi_p(u(x) - u(\xi)) K_{n, s, p}(d(x, \xi)) \, d\xi
\]
in both cases.

Let us first consider the odd dimensional case. We fix \( \delta > 0 \) and integrate the heat semigroup with respect to the singular measure \( t^{1-\frac{m}{2}} \, dt \) over the interval \( (\delta, \infty) \) to obtain
\[
C_1 \int_{\delta}^\infty e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1+\frac{m}{2}}}
\]
\[
= C_1 \int_{\delta}^\infty \int_{\mathbb{H}^n} \frac{1}{(2\pi)^{\frac{m}{2}}} \frac{1}{(4\pi t)^{1/2}} \left( \frac{-\partial_\rho}{\sinh \rho} \right)^m e^{-m^2 t - \frac{2}{4\pi} t} g_\varepsilon(x, \xi) \, d\xi \frac{dt}{t^{1+\frac{m}{2}}}
\]
\[
= \frac{C_1}{(2\pi)^m (4\pi)^{1/2}} \int_{\mathbb{H}^n} \left( \frac{-\partial_\rho}{\sinh \rho} \right)^m \left( \int_{\delta}^\infty e^{-m^2 t - \frac{2}{4\pi} t} t^{-\frac{3+s+p}{2}} \, dt \right) g_\varepsilon(x, \xi) \, d\xi.
\]
Note that the function \( e^{-m^2 t - \frac{2}{4\pi} t} t^{-\frac{3+s+p}{2}} \) is integrable on \((0, \infty)\). Indeed, the formula (2.1) and the change of variables show
\[
\int_0^\infty e^{-m^2 t - \frac{2}{4\pi} t} t^{-\frac{3+s+p}{2}} \, dt = m^{1+s+p} \int_0^\infty e^{-t-(m^2)^{\frac{2}{4\pi}} t^{-\frac{3+s+p}{2}}} \, dt = 2(2m)^{1+s+p} \mathcal{K}_{1+s+p, m}(\rho).
\]
Thus, (3.4) in the odd dimensional case follows by combining (3.5)–(3.6) and passing the limit \( \delta \to 0 \).

We next consider the even dimensional case. Similarly as in the odd dimensional case, we obtain
\[
C_1 \int_{\delta}^\infty e^{t\Delta_{\mathbb{H}^n}}[g_\varepsilon(x, \cdot)](x) \frac{dt}{t^{1+\frac{m}{2}}}
\]
\[
= C_1 \int_{\delta}^\infty \int_{\mathbb{H}^n} \frac{t^{-3/2} \left( \frac{(2m+1)^2}{4} t \right)^m}{(2\pi)^{m+1/2}} \left( \frac{-\partial_\rho}{\sinh \rho} \right)^m \int_0^\infty r e^{-\frac{r^2}{4\pi}} \frac{dr}{\sqrt{\cosh r - \cosh \rho}} g_\varepsilon(x, \xi) \, d\xi \frac{dt}{t^{1+\frac{m}{2}}}
\]
\[
= \frac{C_1}{2(2\pi)^{m+1/2}} \int_{\mathbb{H}^n} \left( \frac{-\partial_\rho}{\sinh \rho} \right)^m \left( \int_{\delta}^\infty e^{-\frac{(2m+1)^2}{4} t - \frac{2}{4\pi} t} \left( \hat{\rho} - \frac{1}{2} \right) \right) \left( \frac{r}{\sqrt{\cosh r - \cosh \rho}} \right) \, d\xi \frac{dt}{t^{1+\frac{m}{2}}}
\]
Moreover, we have from (2.1) and (2.3)
\[
\int_0^\infty e^{-\frac{(2m+1)^2}{4} t - \frac{2}{4\pi} t} \left( \frac{r}{\sqrt{\cosh r - \cosh \rho}} \right) \, dt = 2(2m - 1)^{1+s+p} \mathcal{K}_{1+s+p, 2m-1}(\rho)
\]
\[
= 4(2m - 1)^{1+s+p} \left( \frac{-\partial_\rho}{\rho} \right) \mathcal{K}_{1+s+p, 2m-1}(\rho).
\]
Thus, we deduce
\[ C_1 \int_0^\infty e^{t\Delta_{\mathbb{H}^n}} [g_\varepsilon(x, \cdot)](x) \, \frac{dt}{t^{1+\frac{1}{2}}} \]
\[ = c_{n,s,p} C_2 \int_{\mathbb{R}^n} \left( \frac{-\partial_r}{\sinh r} \right)^{m-1} \int_\rho^\infty \left( \frac{-\partial_r}{\sqrt{\pi} \cosh r - \cosh \rho} \right) \, dr \, g_\varepsilon(x, \xi) \, d\xi \]
\[ = c_{n,s,p} C_2 \int_{\mathbb{R}^n} \int_0^\infty \frac{\sinh r}{\sqrt{\pi} \cosh r - \cosh \rho} \left( \frac{-\partial_r}{\sinh r} \right)^m \mathcal{K}_{1+sp, 2m-1}(r) \, dr \, g_\varepsilon(x, \xi) \, d\xi, \]
where we used Lemma 3.1 with \( \nu = \frac{1+sp}{2} \), \( a = \frac{2m-1}{2} \), and \( y = 0 \) in the last equality. This proves (3.4) in the even dimensional case.

On the one hand, the integral in the right-hand side of (3.4) converges to the Cauchy principal value
\[ \text{P.V.} \int_{\mathbb{H}^n} \Phi_p(u(x) - u(\xi)) K_{n,s,p}(d(x, \xi)) \, d\xi \]
as \( \varepsilon \searrow 0 \). For the left-hand side of (3.4), on the other hand, we need to estimate
\[ A := \int_0^\infty e^{t\Delta_{\mathbb{H}^n}} [\Phi_p(u(x) - u(\cdot))](x) \, \frac{dt}{t^{1+\frac{1}{2}}} - \int_0^\infty e^{t\Delta_{\mathbb{H}^n}} [g_\varepsilon(x, \cdot)](x) \, \frac{dt}{t^{1+\frac{1}{2}}}. \]
Proceeding as above, we have
\[ |A| \lesssim \left| \text{P.V.} \int_{d(x, \xi) \leq \varepsilon} \Phi_p(u(x) - u(\xi)) K_{n,s,p}(d(x, \xi)) \, d\xi \right|. \]
Thus, applying Lemma A.1 to \( K = K_{n,s,p} \) yields
\[ |A| \lesssim \int_{d(x, \xi) \leq \varepsilon} \rho^{\alpha} K_{n,s,p}(\rho) \, dy \lesssim \int_0^{\varepsilon} \rho^{\alpha} K_{n,s,p}(\rho) \sinh^{n-1} \rho \, d\rho, \]
where \( \alpha = 2p - 2 \) when \( p \in (\frac{2-s}{2}, 2) \) and \( \alpha = p \) when \( p \in (1, \frac{2-s}{2}] \cup [2, \infty) \). Note that \( K_{n,s,p} \) is positive, which will be proved later in Corollary 5.2. Since \( K_{n,s,p} \sim \rho^{-n-sp} \) as \( \rho \to 0^+ \) up to constants, the function \( \rho^{\alpha} K_{n,s,p}(\rho) \sinh^{n-1} \rho \) is integrable near zero and hence the right-hand side of (3.7) converges to zero as \( \varepsilon \searrow 0 \). Therefore, the left-hand side of (3.4) converges to that of (1.3) as \( \varepsilon \searrow 0 \). \hfill \square

4. Extension problem

In this section, we prove Theorem 1.3, which provides another representation of the fractional \( p \)-Laplacian on the hyperbolic spaces. We first relate the heat semigroup to the extension problem (1.4) and find the Poisson formula.

**Lemma 4.1.** Let \( n \geq 2 \), \( s \in (0, 1) \), and \( p > 1 \). If \( f \in C_b(\mathbb{H}^n) \), then the solution \( U = E_{s,p}[f] \) of the extension problem (1.4) is given by
\[ U(x, y) = \frac{y^{sp}}{2p \Gamma \left( \frac{sp}{2} \right)} \int_0^\infty e^{t\Delta_{\mathbb{H}^n}} [f](x) e^{-\sqrt{t} \frac{y}{\sqrt{r}}} \, \frac{dt}{t^{1+\frac{1}{2}}} . \]
Moreover, the solution can be represented by using the Poisson kernel:
\[ U(x, y) = \int_{\mathbb{H}^n} P(d(x, \xi), y) f(\xi) \, d\xi. \]
The Poisson kernel \( P(\rho, y) \) is given by
\[
P(\rho, y) = C_4 y^{sp} \left( -\frac{\partial_{\rho}}{\sinh \rho} \right)^{\frac{n-1}{2}} \mathcal{H}_{\frac{n+sp}{2}, \frac{n-1}{2}} \left( \sqrt{\rho^2 + y^2} \right)
\]
when \( n \geq 3 \) odd and
\[
P(\rho, y) = C_4 y^{sp} \int_{\rho}^{\infty} \frac{\sinh r}{\sqrt{\pi} \sqrt{\cosh r - \cosh \rho}} \left( -\frac{\partial_r}{\sinh r} \right)^{\frac{n}{2}} \mathcal{H}_{\frac{n+sp}{2}, \frac{n-1}{2}} \left( \sqrt{r^2 + y^2} \right) dr
\]
when \( n \geq 2 \) even, where
\[
C_4 = \frac{1}{2^{\frac{2n}{4}} \pi^\frac{n}{2} \Gamma(\frac{np}{2})} \left( \frac{n-1}{4} \right)^{1+sp/4}
\]
and \( \mathcal{H}_{\nu, \alpha} \) is the function given in (2.4).

Proof. For each \( x \in \mathbb{H}^n \) and \( y > 0 \), we define \( V(x, y) \) by the function given in the right-hand side of (4.1). Then, we have
\[
V(x, y) = \frac{y^{sp}}{2^{sp} \Gamma(\frac{np}{2})} \int_0^\infty \int_{\mathbb{H}^n} p(t, \rho) f(\xi) d\xi e^{-\frac{y^2}{4t}} dt \frac{1}{t^{1+sp/2}},
\]
where \( \rho = d(x, \xi) \). Recalling the expression (3.1) for the heat kernel \( p(t, \rho) \) and using (2.1), we obtain
\[
V(x, y) = \frac{y^{sp}}{2^{sp} \Gamma(\frac{np}{2})} \int_0^\infty \int_{\mathbb{H}^n} \frac{1}{(2\pi)^m (4\pi t)^{1/2}} \left( \left( -\frac{\partial_{\rho}}{\sinh \rho} \right)^m e^{-\frac{m^2 t - \frac{sp}{4} y^2}{4t}} \right) f(\xi) d\xi dt
\]
\[
= \int_{\mathbb{H}^n} \frac{y^{sp}}{2^{sp} \Gamma(\frac{np}{2})} \frac{1}{(2\pi)^m (4\pi t)^{1/2}} \left( -\frac{\partial_{\rho}}{\sinh \rho} \right)^m \left( \int_0^\infty e^{-\frac{m^2 t - \frac{sp}{4} y^2}{4t}} dt \right) f(\xi) d\xi
\]
when \( n = 2m + 1 \) is odd. If \( n = 2m \) is even, then we use (3.2) instead of (3.1) to have
\[
V(x, y) = \frac{y^{sp}}{2^{sp} \Gamma(\frac{np}{2})} \int_0^\infty \int_{\mathbb{H}^n} \frac{1}{2^{(2m-1)^2 t}} \left( -\frac{\partial_{\rho}}{\sinh \rho} \right)^{m-1} \int_\rho^\infty \frac{e^{-\frac{(2m-1)^2 t}{4}}}{\sqrt{\cosh r - \cosh \rho}} \frac{r e^{-\frac{r^2 + y^2}{4t}}}{(2\pi)^m} dr f(\xi) dt
\]
\[
= \frac{y^{sp}}{2^{sp} \Gamma(\frac{np}{2})} \int_{\mathbb{H}^n} \left( -\frac{\partial_{\rho}}{\sinh \rho} \right)^{m-1} \int_\rho^\infty \frac{e^{-\frac{(2m-1)^2 t}{4}}}{\sqrt{\cosh r - \cosh \rho}} \frac{r e^{-\frac{r^2 + y^2}{4t}}}{(2\pi)^m} dr f(\xi) d\xi.
\]
Moreover, using (2.1) we compute
\[
\int_0^\infty e^{-\frac{(2m-1)^2 t}{4} - \frac{2}{4} t - \frac{2}{4} t} dt = 2(2m - 1)^{\frac{1+sp}{2}} \mathcal{H}_{\frac{3+sp}{2}, \frac{2m-1}{2}} \left( \sqrt{t^2 + y^2} \right)
\]
\[
= 4(2m - 1)^{\frac{1+sp}{2}} \left( -\frac{\partial_{\rho}}{r} \right)^{\frac{3+sp}{2}, \frac{2m-1}{2}} \left( \sqrt{r^2 + y^2} \right).
\]
Therefore, we obtain
\[
V(x, y) = C_4 y^{sp} \int_{\mathbb{H}^n} \left( -\frac{\partial_{\rho}}{\sinh \rho} \right)^{m-1} \int_\rho^\infty \left( -\frac{\partial_r}{\sinh \rho} \right)^{\frac{2m-1}{2}} \left( \sqrt{r^2 + y^2} \right) dr f(\xi) d\xi
\]
\[
= \int_{\mathbb{H}^n} P(d(x, \xi), y) f(\xi) d\xi
\]
in the even dimensional case as well, where we used Lemma 3.1 in the last equality.

It only remains to prove the equality in (4.1) to conclude lemma. Note that (4.2) will follow from (4.1) and the representations of $V$ above. To prove the equality in (4.1), we check that the function $V$ solves the extension problem (1.4). Since the heat semigroup $e^{t\Delta_H}\left[f\right]$ solves (1.1), $V$ satisfies

$$\Delta_x V = \frac{ysp}{2sp\Gamma\left(\frac{sp}{2}\right)} \int_0^\infty \partial_t \left(e^{t\Delta_H}\left[f\right](x)\right) e^{-\frac{y^2}{4}t^{-1-\frac{sp}{4}}} dt.$$

Using the integration by parts and the fact that $|e^{t\Delta_H}\left[f\right](x)| \leq \|f\|_{L^\infty}$, we obtain

$$\Delta_x V = \frac{ysp}{2sp\Gamma\left(\frac{sp}{2}\right)} \left[ e^{t\Delta_H}\left[f\right](x) e^{-\frac{y^2}{4}t^{-1-\frac{sp}{4}}} \right]_0^\infty - \int_0^\infty e^{t\Delta_H}\left[f\right](x) \partial_t \left(e^{-\frac{y^2}{4}t^{-1-\frac{sp}{4}}} \right) dt$$

$$= -\frac{ysp}{2sp\Gamma\left(\frac{sp}{2}\right)} \int_0^\infty e^{t\Delta_H}\left[f\right](x) \left(\frac{y^2}{4}e^{-\frac{y^2}{4}t^{-3-\frac{sp}{2}}} - \left(1 + \frac{sp}{2}\right) e^{-\frac{y^2}{4}t^{-2-\frac{sp}{2}}} \right) dt.$$

Since

$$V_y = \frac{1}{2sp\Gamma\left(\frac{sp}{2}\right)} \int_0^\infty e^{t\Delta_H}\left[f\right](x) e^{-\frac{y^2}{4}t^{-1-\frac{sp}{4}}} dt$$

$$- \frac{ysp+1}{2sp+1\Gamma\left(\frac{sp}{2}\right)} \int_0^\infty e^{t\Delta_H}\left[f\right](x) e^{-\frac{y^2}{4}t^{-2-\frac{sp}{4}}} dt$$

and

$$V_{yy} = \frac{sp(sp - 1)y^{sp-2}}{2sp\Gamma\left(\frac{sp}{2}\right)} \int_0^\infty e^{t\Delta_H}\left[f\right](x) e^{-\frac{y^2}{4}t^{-1-\frac{sp}{4}}} dt$$

$$- \frac{2sp + 1}{2sp+1\Gamma\left(\frac{sp}{2}\right)} \frac{ysp}{2sp\Gamma\left(\frac{sp}{2}\right)} \int_0^\infty e^{t\Delta_H}\left[f\right](x) e^{-\frac{y^2}{4}t^{-2-\frac{sp}{4}}} dt$$

$$+ \frac{ysp+2}{2sp+1\Gamma\left(\frac{sp}{2}\right)} \int_0^\infty e^{t\Delta_H}\left[f\right](x) e^{-\frac{y^2}{4}t^{-3-\frac{sp}{4}}} dt,$$

one can easily compute

$$\Delta_x V(x, y) + \frac{1 - sp}{y} V_y(x, y) + V_{yy}(x, y) = 0.$$  

Finally, we prove $V(x, 0) = f(x)$. Indeed, we have $P(\rho, y) \to 0$ as $\gamma \to 0$ if $\rho \neq 0$ by definition. Moreover, since the heat kernel $p(t, \rho)$ satisfies

$$\int_{H^n} p(t, d(x, \xi)) d\xi = 1,$$

we obtain

$$\int_{H^n} P(d(x, \xi), y) d\xi = \frac{ysp}{2sp\Gamma\left(\frac{sp}{2}\right)} \int_0^\infty \left(\int_{H^n} p(t, d(x, \xi)) d\xi\right) e^{-\frac{y^2}{4}t^{-1-\frac{sp}{4}}} dt$$

$$= \frac{ysp}{2sp\Gamma\left(\frac{sp}{2}\right)} \int_0^\infty e^{-\frac{y^2}{4}t^{-1+\frac{sp}{4}}} dt = 1.$$  

This concludes that $V$ solves the extension problem (1.4). \(\square\)

Let us now prove Theorem 1.3 by using the Poisson formula in Lemma 4.1.
Proof of Theorem 1.3. We have the kernel representation of \((-\Delta_{\mathbb{H}^n})_s u(x)\) from Theorem 1.2 and the Poisson kernel representation of \(E_{s,p}[\Phi_p(u(x) - u(\cdot))] (x, y)\) from Lemma 4.1. Since \(c_{n,s,p}C_2 = C_3C_4\), it is enough to show
\[
\left| \int_{\mathbb{H}^n} \Phi_p(u(x) - u(\xi)) K(d(x, \xi)) \, d\xi \right| \to 0
\]
as \(y \searrow 0\), where
\[
K(\rho) = \left( \frac{-\partial_{\rho}}{\sinh \rho} \right) \frac{n-1}{2} \left( K_{1+sp, \frac{n-1}{2}}(\rho) - K_{1+sp, \frac{n-1}{2}}(\sqrt{\rho^2 + y^2}) \right)
\]
when \(n\) is odd and
\[
K(\rho) = \int_{\rho}^{\infty} \frac{\sinh r}{\sqrt{\pi} \sqrt{\cosh r - \cosh \rho}} \left( \frac{-\partial_r}{\sinh r} \right)^{\frac{n}{2}} \left( K_{1+sp, \frac{n-1}{2}}(r) - K_{1+sp, \frac{n-1}{2}}(\sqrt{r^2 + y^2}) \right) \, dr
\]
when \(n\) is even.

We first split the integral as follows:
\[
\left| \int_{\mathbb{H}^n} \Phi_p(u(x) - u(\xi)) K(d(x, \xi)) \, d\xi \right| \leq \left| \int_{d(x, \xi) \leq 1} \Phi_p(u(x) - u(\xi)) K(d(x, \xi)) \, d\xi \right| + \left| \int_{d(x, \xi) > 1} \Phi_p(u(x) - u(\xi)) K(d(x, \xi)) \, d\xi \right| = J_1 + J_2.
\]
For \(J_1\), we apply Lemma A.1 to \(K\) to obtain
\[
J_1 \lesssim \int_{d(x, \xi) \leq 1} d(x, \xi)^{\alpha} |K(d(x, \xi))| \, d\xi,
\]
where \(\alpha = 2p - 2\) when \(p \in \left(\frac{2}{2-s}, 2\right)\) and \(\alpha = p\) when \(p \in (1, \frac{2}{2-s}] \cup [2, \infty)\). For \(J_2\), we have
\[
J_2 \lesssim \|u\|_{L^\infty(\mathbb{H}^n)}^{p-1} \int_{d(x, \xi) > 1} |K(d(x, \xi))| \, d\xi.
\]
By the dominated convergence theorem, we conclude \(J_1 + J_2 \to 0\) as \(y \searrow 0\). \(\square\)

5. Pointwise Convergence

This section is devoted to the proof of Theorem 1.4, which uses the pointwise representation (1.3) of the fractional \(p\)-Laplacian on \(\mathbb{H}^n\). As mentioned in Section 1, the limits of the integrals
\[
c_{n,s,p} \int_{R}^{\infty} K_{n,s,p}(\rho) \sinh^{n-1} \rho \, d\rho, \quad c_{n,s,p} \int_{0}^{R} \rho^p K_{n,s,p}(\rho) \sinh^{n-1} \rho \, d\rho,
\]
and
\[
c_{n,s,p} \int_{0}^{R} \rho^{\beta+p} K_{n,s,p} \sinh^{n-1} \rho \, d\rho, \quad \beta > 0,
\]
as \(s \to 1^-\), play a key role in the proof of Theorem 1.4. Let us begin with the following lemma.
Lemma 5.1. Let $\nu > 1/2$, $a \geq 1/2$, and $m \in \mathbb{N} \cup \{0\}$. Then, the function
\[
\rho \mapsto \left(\frac{-\partial_{\rho}}{\sinh \rho}\right)^m \mathcal{K}_{\nu,a}(\rho)
\]
is positive, where $\mathcal{K}_{\nu,a}$ is the function given in (2.4).

Proof. Using the formula (2.1) and change of variables, we have
\[
\mathcal{K}_{\nu,a}(\rho) = \frac{a^{\nu}}{2^{\nu+1}} \int_0^{\infty} e^{t - \frac{a^{2\nu}}{4t} t^{-\nu-1}} dt = \frac{1}{2(2a)^\nu} \int_0^{\infty} \frac{1}{t^{1/2}} e^{-a^2 t - \frac{c^2}{4t}} \frac{dt}{t^{\nu+1/2}}.
\]
Thus, recalling the expression of the heat kernel (3.1) for odd dimensional case, we obtain
\[
(5.3) \quad \left(\frac{-\partial_{\rho}}{\sinh \rho}\right)^m \mathcal{K}_{\nu,a}(\rho) = \int_0^{\infty} e^{(m^2 - a^2)\rho} p(t, \rho) \frac{dt}{t^{\nu+1/2}}.
\]
It is known [10, Lemma 2.3] that the heat kernel $p(t, \rho)$ is strictly decreasing with respect to $\rho$. Since $p(t, \rho) \to 0$ as $\rho \to \infty$, we deduce that $p(t, \rho)$ is positive. Therefore, the conclusion follows from (5.3).

As a consequence of Lemma 5.1, we obtain the positivity of the kernel $K_{n,s,p}$.

Corollary 5.2. Let $n \in \mathbb{N}$, $s \in (0, 1)$, and $p > 1$. The kernel $K_{n,s,p}$ is positive.

In the following series of lemmas, we compute limits of the integrals in (5.1) and (5.2) with the help of Lemma 5.1.

Lemma 5.3. Let $n \geq 2$ and $p > 1$. For any $R > 0$,
\[
\lim_{s \searrow 1} c_{n,s,p} \int_R^{\infty} K_{n,s,p}(\rho) \sinh^{n-1} \rho d\rho = 0.
\]

Proof. Let us first consider the case $n = 2m + 1$ with $m \geq 1$. Since $c_{n,s,p} C_2 \leq C(1 - s)$ for some $C = C(n, p) > 0$, by using Lemma 5.1 we have
\[
0 \leq c_{n,s,p} \int_R^{\infty} K_{n,s,p}(\rho) \sinh^{n-1} \rho d\rho \leq (1 - s) \int_R^{\infty} \sinh^{2m} \rho \left(\frac{-\partial_{\rho}}{\sinh \rho}\right)^m \mathcal{K}_{\frac{1}{2} + p, 2m}(\rho) d\rho.
\]
Thus, it is enough to show that the right-hand side of (5.4) converges to zero as $s \to 1^-$. We actually prove the following stronger statement:
\[
(5.5) \quad \lim_{s \searrow 1} (1 - s) \int_R^{\infty} \sinh^{m+a} \rho \left(\frac{-\partial_{\rho}}{\sinh \rho}\right)^m \mathcal{K}_{\frac{1}{2} + p, 2m}(\rho) d\rho = 0 \quad \text{for each } a > 0.
\]

We use the induction on $m$. When $m = 1$, using (2.3) and the fact that $K_{\nu}$ is increasing with respect to $\nu > 0$, we have
\[
\int_R^{\infty} \sinh^{1+a} \rho \left(\frac{-\partial_{\rho}}{\sinh \rho}\right) \mathcal{K}_{\frac{1}{2} + p, a}(\rho) d\rho = a \int_R^{\infty} (\sinh^{\rho}) \rho^{-\frac{1 + p}{2}} K_{\frac{3}{2} + p}(\rho) d\rho \leq a \int_R^{\infty} (\sinh^{\rho}) \rho^{-\frac{1 + p}{2}} K_{\frac{3}{2} + p}(\rho) d\rho.
\]
By (2.2), there exists $M = M(p) > 1$ such that
\[
(5.6) \quad K_{\frac{3}{2} + p}(\rho) \leq \sqrt{\pi} e^{-\rho} \quad \text{for } \rho > M.
\]
The inequalities $\rho^{-\frac{1+p}{2}} \leq \max\{\rho^{-\frac{1}{2}}, \rho^{-\frac{1+sp}{2}}\}$ and $\sinh \rho < e^\rho$, together with (5.6), yield
\[
\int_R^\infty (\sinh^a \rho) \rho^{-\frac{1+sp}{2}} K_{\frac{1+sp}{2}}(a \rho) \, d\rho \\
\leq \int_R^{M/a} (\sinh^a \rho) \max\{\rho^{-\frac{1}{2}}, \rho^{-\frac{1+sp}{2}}\} K_{\frac{1+sp}{2}}(a \rho) \, d\rho + \sqrt{\pi \alpha} \int_{M/a}^\infty \rho^{-1-\frac{sp}{2}} \, d\rho.
\]
Note that the first integral in the right-hand side of the inequality above is a constant depending on $a$, $p$, and $R$ only. For the second integral, we estimate
\[
\sqrt{\pi \alpha} \int_{M/a}^\infty \rho^{-1-\frac{sp}{2}} \, d\rho = \frac{2}{sp} \sqrt{\pi \alpha} \left(\frac{a}{M}\right)^{\frac{sp}{2}} \leq \frac{2}{sp} \sqrt{\pi \alpha} \max\left\{\left(\frac{a}{M}\right)^{\frac{sp}{2}}, 1\right\}.
\]
Thus, we arrive at
\[
\lim_{s/a} (1-s) \int_R^\infty \sinh^{1+a} \rho \left(-\frac{\partial \rho}{\sinh \rho}\right) K_{\frac{1+sp}{2}}(a \rho) \, d\rho = 0,
\]
which proves (5.5) for $m = 1$.

Assume now that (5.5) is true for $m$ and prove it for $m + 1$. Using integration by parts, we have
\[
\lim_{s/a} (1-s) \int_R^\infty \sinh^{m+1+a} \rho \left(-\frac{\partial \rho}{\sinh \rho}\right) K_{\frac{1+sp}{2}}(a \rho) \, d\rho \\
= \lim_{s/a} (1-s)(m + a) \int_R^\infty \sinh^{m+a-1} \rho \cosh \rho \left(-\frac{\partial \rho}{\sinh \rho}\right) m K_{\frac{1+sp}{2}}(a \rho) \, d\rho.
\]
Thus, by an inequality
\[
\cosh \rho \leq \coth R \sinh \rho \quad \text{for} \quad \rho \geq R,
\]
Lemma 5.1, and the induction hypothesis, we conclude
\[
\lim_{s/a} (1-s) \int_R^\infty \sinh^{m+1+a} \rho \left(-\frac{\partial \rho}{\sinh \rho}\right) m K_{\frac{1+sp}{2}}(a \rho) \, d\rho \\
\leq (m + a)(\coth R) \lim_{s/a} (1-s) \int_R^\infty \sinh^{m+a} \rho \left(-\frac{\partial \rho}{\sinh \rho}\right) m K_{\frac{1+sp}{2}}(a \rho) \, d\rho = 0.
\]
This finishes the proof of the lemma in the odd dimensional case.

Let us next consider the even dimensional cases $n = 2m$ with $m \geq 1$. Similarly as in the odd dimensional case, since
\[
0 \leq \int_R^\infty K_{n,s,p}(\rho) \sinh^{n-1} \rho \, d\rho \\
\leq (1-s) \int_R^\infty \sinh^{2m-1} \rho \int_\rho^\infty \frac{\sinh r}{\cosh r - \cosh \rho} \left(-\frac{\partial r}{\sinh r}\right) m K_{\frac{1+sp}{2}, 2m-1}(r) \, dr \, d\rho,
\]
the desired result will follow once we prove the following:
\[
\lim_{s/a} (1-s) \int_R^\infty \sinh^{2m-1+a} \rho \int_\rho^\infty \frac{\sinh r}{\cosh r - \cosh \rho} \left(-\frac{\partial r}{\sinh r}\right) m K_{\frac{1+sp}{2}, 2m-1}(r) \, dr \, d\rho = 0
\]
for each $a \geq 1/2$. 

If $m = 1$, then
\[
\int_R^\infty \sinh^{1+a} \rho \int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left( \frac{-\partial_r}{\sinh r} \right) \mathcal{K}_{1+s^p a}(r) \, dr \, d\rho \\
\leq a \int_R^{M/a} \sinh^{1+a} \rho \int_\rho^\infty \frac{1}{\sqrt{\cosh r - \cosh \rho}} \rho^{-\frac{1+s^p}{2}} K_{\frac{3+s^p}{2}} (ar) \, dr \, d\rho \\
+ a \int_{M/a}^\infty \sinh^{1+a} \rho \int_\rho^\infty \frac{1}{\sqrt{\cosh r - \cosh \rho}} \rho^{-\frac{1+s^p}{2}} K_{\frac{3+s^p}{2}} (ar) \, dr \, d\rho =: J_1 + J_2.
\]

For $J_2$, we use (5.6) to obtain
\[
J_2 \leq \sqrt{\frac{\pi}{\pi a}} \int_{M/a}^\infty \sinh^{1+a} \rho \frac{1}{\rho^{\frac{3}{2}} e^{a \rho}} \int_\rho^\infty \frac{1}{\sqrt{\cosh r - \cosh \rho}} \, dr \, d\rho.
\]

Since
\[
\int_\rho^\infty \frac{1}{\sqrt{\cosh r - \cosh \rho}} \, dr = \frac{1}{\sqrt{2}} \int_\rho^\infty \frac{1}{\sqrt{\frac{r^2 + \rho^2 - r \rho}{2}}} \, dr \\
\leq \frac{1}{\sqrt{2} \sinh \rho} \int_\rho^\infty \frac{1}{\sqrt{\frac{r^2 - \rho^2}{2}}} \, dr \\
= \frac{1}{\sqrt{2} \sinh \rho} \int_0^\infty \frac{1}{\sqrt{\sinh \frac{r^2}{2}}} \, dr = \frac{\Gamma(1/4)}{\Gamma(3/4)} \sqrt{\frac{\pi}{\sinh \rho}}
\]

and $\sinh^a \rho \leq e^{a \rho}$, we have
\[
J_2 \leq \frac{\Gamma(1/4)}{\Gamma(3/4)} \sqrt{\pi a} \int_{M/a}^\infty \rho^{-\frac{1+s^p}{2}} \, d\rho = \frac{\Gamma(1/4)}{\Gamma(3/4)} \frac{\pi \sqrt{a}}{sp} \frac{\rho^{sp}}{M^{sp}}.
\]

On the other hand, for $J_1$ we observe
\[
J_1 \leq a \int_R^{M/a} \sinh^{1+a} \rho \int_\rho^\infty \max \{r^{-\frac{1+s^p}{2}}, r^{-\frac{3}{2}} \} K_{\frac{3+s^p}{2}} (ar) \, dr \, d\rho.
\]

Since the inner integral is continuous and integrable on $[R, M/a]$, $J_1$ is controlled by some constant $C = C(a, p, R) > 0$. Therefore, we conclude $\lim_{s \to 1} (1 - s) (J_1 + J_2) = 0$, which proves (5.8) for $m = 1$.

Finally, let us assume that (5.8) holds for $m$ and prove it for $m + 1$. By Lemma 3.1, we have
\[
\lim_{s \to 1} (1 - s) \int_R^\infty \sinh^{2m+1+a} \rho \int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left( \frac{-\partial_r}{\sinh r} \right) \mathcal{K}_{1+s^{p+1} a}(r) \, dr \, d\rho \\
= \lim_{s \to 1} (1 - s) \int_R^\infty \sinh^{2m+1+a} \rho \left( \frac{-\partial_r}{\sinh r} \right) \int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left( \frac{-\partial_r}{\sinh r} \right) \mathcal{K}_{1+s^{p+1} a}(r) \, dr \, d\rho.
\]

Using integration by parts, (5.7), and Lemma 5.1, we deduce
\[
\lim_{s \to 1} (1 - s) \int_R^\infty \sinh^{2m+1+a} \rho \int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left( \frac{-\partial_r}{\sinh r} \right) \mathcal{K}_{1+s^{p+1} a}(r) \, dr \, d\rho \\
\leq C \lim_{s \to 1} (1 - s) \int_R^\infty \sinh^{2m-1+a} \rho \int_\rho^\infty \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left( \frac{-\partial_r}{\sinh r} \right) \mathcal{K}_{1+s^{p+1} a}(r) \, dr \, d\rho
\]
for some $C = C(m, a, R)$. Therefore, the statement \((5.8)\) for $m + 1$ follows by the induction hypothesis. \hfill \square

**Lemma 5.4.** Let $n \geq 2$ and $p > 1$. For any $R > 0$, \begin{equation}
(5.10) \lim_{s \searrow 1} c_{n, s, p} \int_0^R \rho^p K_{n, s, p}(\rho) \sinh^{n-1} \rho \, d\rho = \frac{1}{\pi} \frac{\Gamma\left(\frac{p+2}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)}.
\end{equation}

The proof of **Lemma 5.4** for the even dimensional case needs the following lemma.

**Lemma 5.5.** Let $a > 0$ and $\nu > -1/2$. Then, \begin{equation*}
\int_0^\infty \frac{\rho^{-\nu} K_{\nu+1}(ar)}{\sqrt{\cosh r - \cosh \rho}} \, dr \sim \sqrt{\frac{\pi}{2}} \frac{\Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + \frac{3}{2})} K_{\nu+1}(a \rho) \rho^{\nu-1}.
\end{equation*}
as $\rho \to 0^+$. \hfill \square

**Proof.** By the change of variables $r = \rho t$, we have
\begin{equation*}
\int_0^\infty \frac{1}{\sqrt{\cosh r - \cosh \rho}} \rho^{-\nu} K_{\nu+1}(ar) \, dr = \int_1^\infty \frac{t^{-\nu} K_{\nu+1}(a t)}{\sqrt{\cosh(\rho t) - \cosh \rho}} \frac{K_{\nu+1}(a \rho)}{\rho} \, d\rho dt.
\end{equation*}

We define for each $\rho \in (0, 1)$ a function $f_\rho$ by \begin{equation*}
f_\rho(t) = \frac{t^{-\nu}}{\sqrt{\cosh(\rho t) - \cosh \rho}} \frac{K_{\nu+1}(a t)}{K_{\nu+1}(a \rho)} \rho
\end{equation*}
on $(1, \infty)$. Note that we have \begin{equation*}
\frac{\cosh(\rho t) - \cosh \rho}{\rho^2} \geq \frac{1}{2}(t^2 - 1).
\end{equation*}
Moreover, by [27, Equation (2.17)], we have \begin{equation*}
\frac{K_{\nu+1}(a t)}{K_{\nu+1}(a \rho)} \leq t^{-\nu-1}.
\end{equation*}
Thus, $f_\rho$ is bounded from above by a function \begin{equation*}
f(t) := \frac{t^{-2\nu-1}}{\sqrt{(t^2 - 1)/2}},
\end{equation*}
which is integrable on $(1, \infty)$. Indeed, by the change of variables $t^2 - 1 = \tau$, we obtain \begin{equation*}
\int_1^\infty f(t) \, dt = \frac{1}{\sqrt{2}} \int_1^\infty \frac{\tau^{-1/2}}{(1 + \tau)^{1+\nu}} \, d\tau = \frac{1}{\sqrt{2}} B\left(\frac{1}{2}, \nu + \frac{1}{2}\right) = \frac{\sqrt{\pi} \Gamma(\nu + \frac{1}{2})}{\Gamma(\nu + 1)} < +\infty,
\end{equation*}
where $B$ is Euler’s Beta Integral (see [30, Section 5.12]).

For fixed $t \in (1, \infty)$, we have \begin{equation*}
\frac{\cosh(\rho t) - \cosh \rho}{\rho^2} \to \frac{1}{2}(t^2 - 1) \quad \text{and} \quad \frac{K_{\nu+1}(a t)}{K_{\nu+1}(a \rho)} \to t^{-\nu-1}
\end{equation*}
as $\rho \to 0^+$. Hence, we obtain $\lim_{\rho \searrow 0} f_\rho(t) = f(t)$. Therefore, the Lebesgue dominated convergence theorem concludes the lemma. \hfill \square

We are in a position to prove **Lemma 5.4** by using **Lemma 5.5**.
**Proof of Lemma 5.4.** Let us first consider the odd dimensional case \( n = 2m + 1 \) with \( m \geq 1 \). One can easily check that (5.10) is equivalent to

\[
\lim_{s \nearrow 1} (1-s) \int_0^R \rho^p \sinh^{2m} \rho \left( \frac{-\partial \rho}{\sinh \rho} \right)^m \mathcal{K}_{\frac{1+sp}{2},m}(\rho) \, d\rho
\]

\[
= \frac{2^{m-1}}{p} \left( \frac{2}{m} \right) \frac{\Gamma}{2} \left( \frac{p+2m+1}{2} \right)
\]

by using \( \lim_{s \nearrow 1} (1-s)\Gamma(-s) = 1 \). Actually, we will prove the following statement, which is slightly stronger than (5.11):

\[
\lim_{s \nearrow 1} (1-s) \int_0^R \rho^p \sinh^{2m} \rho \left( \frac{-\partial \rho}{\sinh \rho} \right)^m \mathcal{K}_{\frac{1+sp}{2},a}(\rho) \, d\rho
\]

\[
= \frac{2^{m-1}}{p} \left( \frac{2}{a} \right)^{\frac{1+sp}{2}} \frac{\Gamma}{2} \left( \frac{p+2m+1}{2} \right) \quad \text{for each } a \geq 1.
\]

Let \( \varepsilon \in (0, 1) \), then there exists \( \delta_0 \in (0, 1) \) such that

\[
1 - \varepsilon \leq \frac{\sinh \rho}{\rho} \leq 1 + \varepsilon
\]

for all \( \rho \in (0, \delta_0) \). Moreover, using the asymptotic behavior (2.2) of the modified Bessel function, for each \( s \in [0, 1] \) we find \( \delta_0 > 0 \) such that

\[
\frac{1 - \varepsilon}{2} \frac{\Gamma}{2} \left( \frac{3 + sp}{2} \right) \left( \frac{\rho}{2} \right)^{-\frac{3+sp}{2}} \leq K_{\frac{1+sp}{2}}(\rho) \leq \frac{1 + \varepsilon}{2} \frac{\Gamma}{2} \left( \frac{3 + sp}{2} \right) \left( \frac{\rho}{2} \right)^{-\frac{3+sp}{2}}
\]

for all \( \rho \in (0, \delta_0) \). Furthermore, since \( K_{\nu} \) is uniformly continuous with respect to \( \nu \), we may assume that \( \delta_0 \) has been chosen continuously on \( s \). Let us take \( \delta = \delta_0 \wedge \min_{s \in [0, 1]} \delta_0 \wedge R \), then \( \delta = \delta(\varepsilon, p, R) > 0 \), and (5.13) and (5.14) hold for all \( \rho \in (0, \delta) \).

We fix \( a \geq 1 \) and denote by \( G_{s,p,m,a}(\rho) \) the integrand in the left-hand side of (5.12). Then, \( |G_{s,p,m,a}(\rho)| \) is bounded by the function \( \sup_{0 \leq s \leq 1} |G_{s,p,m,a}(\rho)| \), which is independent of \( s \) and bounded on a compact interval \([\delta/a, R]\). Thus, we have

\[
\lim_{s \nearrow 1} (1-s) \int_{\delta/a}^R G_{s,p,m,a}(\rho) \, d\rho = 0,
\]

and hence

\[
\lim_{s \nearrow 1} (1-s) \int_0^R G_{s,p,m,a}(\rho) \, d\rho = \lim_{s \nearrow 1} (1-s) \int_{\delta/a}^R G_{s,p,m,a}(\rho) \, d\rho.
\]

Let us now prove (5.12) by induction. When \( m = 1 \), we first use (2.3) to have

\[
G_{s,p,1,a}(\rho) = ap^{\frac{1+sp}{2}} K_{\frac{1+sp}{2}}(a \rho) \sinh \rho.
\]

If \( \rho < \delta/a \), then \( \rho \leq a \rho < \delta \) since \( a \geq 1 \). Thus, we utilize (5.13) and (5.14) to obtain

\[
(1-\varepsilon)^2 \left( \frac{2}{a} \right)^{\frac{1+sp}{2}} \frac{\Gamma}{2} \left( \frac{3 + sp}{2} \right) \rho^{p(1-s)-1} \leq G_{s,p,1,a}(\rho) \leq (1+\varepsilon)^2 \left( \frac{2}{a} \right)^{\frac{1+sp}{2}} \frac{\Gamma}{2} \left( \frac{3 + sp}{2} \right) \rho^{p(1-s)-1}.
\]
This leads us to the inequalities
\[
\lim_{s \nearrow 1} (1 - s) \int_0^R G_{s,p,1,a}(\rho) \, d\rho = \lim_{s \nearrow 1} (1 - s) \int_0^{\delta/a} G_{s,p,1,a}(\rho) \, d\rho \\
\leq \lim_{s \nearrow 1} (1 - s)(1 + \varepsilon)^2 \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{3 + sp}{2} \right) \int_0^{\delta/a} \rho^{p(1-s)-1} \, d\rho \\
= (1 + \varepsilon)^2 \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{p + 3}{2} \right)
\]
and
\[
\lim_{s \nearrow 1} (1 - s) \int_0^R G_{s,p,1,a}(\rho) \, d\rho \geq (1 - \varepsilon)^2 \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{p + 3}{2} \right).
\]
Therefore, the statement (5.12) for \( m = 1 \) follows by taking \( \varepsilon \to 0 \).

Assume now that (5.12) holds for \( m \geq 1 \). Then, a similar argument shows
\[
\lim_{s \nearrow 1} (1 - s) \int_0^R G_{s,p,m+1,a}(\rho) \, d\rho \\
= \lim_{s \nearrow 1} (1 - s) \int_0^{\delta/a} \rho^p \sinh^{2m+2} \rho \left( \frac{-\partial \rho}{\sinh \rho} \right)^{m+1} \mathcal{K}_{1+p,a}(\rho) \, d\rho \\
\leq \lim_{s \nearrow 1} (1 - s)(1 + \varepsilon)^{2m+1} \int_0^{\delta/a} \rho^{p+2m+1} \left( \frac{-\partial \rho}{\sinh \rho} \right)^{m} \mathcal{K}_{1+p,a}(\rho) \, d\rho,
\]
where nonnegativity of the integrands follows from Lemma 5.1. Using the integration by parts, (5.13), and the induction hypothesis, we arrive at
\[
\lim_{s \nearrow 1} (1 - s) \int_0^R G_{s,p,m+1,a}(\rho) \, d\rho \\
\leq (1 + \varepsilon)^{2m+1} (p + 2m + 1) \lim_{s \nearrow 1} (1 - s) \int_0^{\delta/a} \rho^{p+2m} \left( \frac{-\partial \rho}{\sinh \rho} \right)^{m} \mathcal{K}_{1+p,a}(\rho) \, d\rho \\
\leq \frac{(1 + \varepsilon)^{2m+1}}{(1 - \varepsilon)^{2m}} (p + 2m + 1) \lim_{s \nearrow 1} (1 - s) \int_0^{\delta/a} \rho^p \sinh^{2m} \rho \left( \frac{-\partial \rho}{\sinh \rho} \right)^{m} \mathcal{K}_{1+p,a}(\rho) \, d\rho \\
= \frac{(1 + \varepsilon)^{2m+1} 2^m}{(1 - \varepsilon)^{2m}} \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{p + 2m + 1}{2} \right) \\
= \frac{(1 + \varepsilon)^{2m+1} 2^m}{(1 - \varepsilon)^{2m}} \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{p + 2m + 3}{2} \right).
\]
Similarly, we obtain
\[
\lim_{s \nearrow 1} (1 - s) \int_0^R G_{s,p,m+1,a}(\rho) \, d\rho \geq \frac{(1 - \varepsilon)^{2m+1} 2^m}{(1 + \varepsilon)^{2m}} \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{p + 2m + 3}{2} \right),
\]
from which (5.12) for \( m + 1 \) follows by taking \( \varepsilon \to 0 \). The statement (5.12) has been proved for all \( m \in \mathbb{N} \), finishing the proof of (5.10) for the odd dimensional case.
Let us next consider the even dimensional case \( n = 2m \) with \( m \geq 1 \). In this case, (5.10) is equivalent to
\[
\lim_{s \to 1} (1 - s) \int_0^R \rho^p \sinh^{2m-1} \rho \int_0^\infty \frac{\sinh r}{\cosh r - \cosh \rho} \left( \frac{-\partial_r}{\sinh r} \right)^m K_{\frac{1+sp}{2}, \frac{2m-1}{2}}(r) \, dr \, d\rho
\]
\[
= \sqrt{\frac{\pi}{2}} \sqrt{\frac{2m-1}{p}} \left( \frac{2}{m - 1/2} \right)^{m+1/2} \Gamma \left( \frac{p + 2m}{2} \right).
\]
As in the odd dimensional case, we will prove a stronger statement:
\[
\lim_{s \to 1} (1 - s) \int_0^R \rho^p \sinh^{2m-1} \rho \int_0^\infty \frac{\sinh r}{\cosh r - \cosh \rho} \left( \frac{-\partial_r}{\sinh r} \right)^m K_{\frac{1+sp}{2}, \frac{2m-1}{2}}(r) \, dr \, d\rho
\]
\[
= \sqrt{\frac{\pi}{2}} \sqrt{\frac{2m-1}{p}} \left( \frac{2}{a} \right)^{m+1/2} \Gamma \left( \frac{p + 2m}{2} \right) \quad \text{for each } a \geq 1/2.
\]
Recall that we have taken \( \delta \) so that (5.13) and (5.14) hold for all \( \rho \in (0, \delta) \). Let us fix \( a \geq 1/2 \). By Lemma 5.5, for each \( s \in [0, 1] \) we find \( \delta_s > 0 \) such that
\[
(1 - \varepsilon) \sqrt{\frac{\pi}{2}} \sqrt{\frac{2m-1}{p}} \left( \frac{2}{a} \right)^{m+1/2} \Gamma \left( \frac{p + 2m}{2} \right) \leq \int_0^\infty \frac{r^{1+sp}}{\cosh r - \cosh \rho} K_{\frac{1+sp}{2}, \frac{2m-1}{2}}(r) \, dr
\]
\[
\leq (1 + \varepsilon) \sqrt{\frac{\pi}{2}} \sqrt{\frac{2m-1}{p}} \left( \frac{2}{a} \right)^{m+1/2} \Gamma \left( \frac{p + 2m}{2} \right) \rho^{1+sp} K_{\frac{1+sp}{2}, \frac{2m-1}{2}}(ar) \, dr
\]
for all \( \rho \in (0, \tilde{\delta}) \). Moreover, we may assume that \( \tilde{\delta} \) has been chosen continuously on \( s \). Let \( \tilde{\delta} = \delta \wedge \min_{s \in [0, 1]} \delta_s \), then \( \delta = \delta(\varepsilon, p, R, a) > 0 \) and (5.16) holds for all \( \rho \in (0, \tilde{\delta}) \).

We denote by \( H_{s,p,m,a}(\rho) \) the integrand in the left-hand side of (5.15). Then, the same argument as in the odd dimensional case shows
\[
\lim_{s \to 1} (1 - s) \int_0^R \rho^p H_{s,p,m,a}(\rho) \, d\rho = \lim_{s \to 1} (1 - s) \int_0^{2m} \rho^p H_{s,p,m,a}(\rho) \, d\rho.
\]
We argue by induction again to prove (5.15). If \( m = 1 \), then
\[
H_{s,p,1,a}(\rho) = a \rho^p \sinh \rho \int_0^\infty \frac{1}{\cosh r - \cosh \rho} \rho^{1+sp} K_{\frac{1+sp}{2}, \frac{2m-1}{2}}(ar) \, dr.
\]
Since \( a \geq 1/2 \), we have \( \rho < \delta \) and \( a \rho < \delta \) for \( \rho < \frac{\delta}{2a} \). Thus, by (5.13), (5.16), and (5.14), we obtain
\[
(1 - \varepsilon)^3 \sqrt{\frac{\pi}{2}} \left( \frac{2}{a} \right)^{1+sp} \Gamma \left( \frac{2 + sp}{2} \right) \rho^{1+sp} \leq H_{s,p,1,a}(\rho)
\]
\[
\leq (1 + \varepsilon)^3 \sqrt{\frac{\pi}{2}} \left( \frac{2}{a} \right)^{1+sp} \Gamma \left( \frac{2 + sp}{2} \right) \rho^{1+sp}.
\]
Therefore, we have
\[
(1 - \varepsilon)^3 \sqrt{\frac{\pi}{2p}} \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{p+3}{2} \right) \leq \lim_{s \to 1} (1 - s) \int_0^R H_{s,p,1,a}(\rho) \, d\rho
\]
\[
\leq (1 + \varepsilon)^3 \sqrt{\frac{\pi}{2p}} \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{p+3}{2} \right),
\]
from which we deduce (5.15) for \( m = 1 \) by taking \( \varepsilon \to 0 \).

Suppose that (5.15) is true for \( m \geq 1 \). Then, by (5.13), Lemma 3.1, and Lemma 5.1, we have
\[
\lim_{s \to 1} (1 - s) \int_0^R H_{s,p,m+1,a}(\rho) \, d\rho
\]
\[
= \lim_{s \to 1} (1 - s) \int_0^{\frac{\delta}{2a}} \rho^p \sinh^{2m+1} \rho \int_0^\infty \frac{\sinh r}{\cosh r - \cosh \rho} \left( -\frac{\partial_r}{\sinh r} \right)^m \mathcal{K}_{s,p,1/2}(r) \, dr \, d\rho
\]
\[
\leq \lim_{s \to 1} (1 - s)(1 + \varepsilon)^{2m} \int_0^{\frac{\delta}{2a}} \rho^{p+2m} (-\partial_r) \int_0^\infty \frac{\sinh r}{\cosh r - \cosh \rho} \left( -\frac{\partial_r}{\sinh r} \right)^m \mathcal{K}_{s,p,1/2}(r) \, dr \, d\rho.
\]
Using the integration by parts, (5.13), and the induction hypothesis, we arrive at
\[
\lim_{s \to 1} (1 - s) \int_0^R H_{s,p,m+1,a}(\rho) \, d\rho
\]
\[
= (1 + \varepsilon)^{2m}(p + 2m)
\]
\[
\times \lim_{s \to 1} (1 - s) \int_0^{\frac{\delta}{2a}} \rho^{p+2m-1} \int_0^\infty \frac{\sinh r}{\cosh r - \cosh \rho} \left( -\frac{\partial_r}{\sinh r} \right)^m \mathcal{K}_{s,p,1/2}(r) \, dr \, d\rho
\]
\[
\leq \frac{(1 + \varepsilon)^{2m}}{(1 - \varepsilon)^{2m-1}} (p + 2m) \lim_{s \to 1} (1 - s) \int_0^{\frac{\delta}{2a}} H_{s,p,m,a}(\rho) \, d\rho
\]
\[
= \frac{(1 + \varepsilon)^{2m}}{(1 - \varepsilon)^{2m-1}} \sqrt{\frac{\pi}{2p}} \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{p+3}{2} \right).
\]
The inequality
\[
\lim_{s \to 1} (1 - s) \int_0^R H_{s,p,m+1,a}(\rho) \, d\rho \geq \frac{(1 - \varepsilon)^{2m}}{(1 + \varepsilon)^{2m-1}} \sqrt{\frac{\pi}{2p}} \left( \frac{2}{a} \right)^{\frac{p+1}{2}} \Gamma \left( \frac{p+3}{2} \right)
\]
can be obtained in the same way. Thus, we conclude that (5.15) for \( m + 1 \) holds by taking \( \varepsilon \to 0 \). This finish the proof for the even dimensional case. \( \Box \)

**Lemma 5.6.** Let \( n \geq 2 \) and \( p > 1 \). For any \( R > 0 \) and \( \beta > 0 \),
\[
(5.17) \quad \lim_{s \to 1} c_{n,s,p} \int_0^R \rho^{p+\beta} \mathcal{K}_{s,p,1}(\rho) \sinh^{n-1} \rho \, d\rho = 0.
\]

**Proof.** We proceed as in the previous lemma to prove (5.17). When \( n = 2m + 1 \) with \( m \geq 1 \), we show
\[
\lim_{s \to 1} \int_0^R \rho^{p+\beta} \sinh^{2m} \rho \left( -\frac{\partial_{\rho}}{\sinh \rho} \right)^m \mathcal{K}_{1/2,1/2}(\rho) \, d\rho = 0 \quad \text{for each} \ a \geq 1
\]
by induction. Indeed, for $\varepsilon \in (0, 1)$ let $\delta > 0$ be the constant given in the proof of Lemma 5.4. Then, by using (5.13) and (5.14) we prove

$$
\lim_{s \nearrow 1} (1 - s) \int_0^R \rho^\beta G_{s,p,1,a}(\rho) \, d\rho \\
\leq \lim_{s \nearrow 1} (1 - s)(1 + \varepsilon)^2 \left( \frac{2}{a} \right)^{\frac{1 + sp}{2}} \Gamma \left( \frac{3 + sp}{2} \right) \int_0^{\delta/a} \rho^{p(1-s)+\beta-1} \, d\rho \\
= \lim_{s \nearrow 1} (1 - s)(1 + \varepsilon)^2 \left( \frac{2}{a} \right)^{\frac{1 + sp}{2}} \Gamma \left( \frac{3 + sp}{2} \right) \frac{1}{p(1-s) + \beta} \left( \frac{\delta}{a} \right)^{p(1-s)+\beta} = 0
$$

for the case $m = 1$, where $G_{s,p,m,a}$ is the function defined in the proof of Lemma 5.4. Moreover, one can follow the steps in the proof of Lemma 5.4 to obtain

$$
\lim_{s \nearrow 1} (1 - s)(1 + \varepsilon)^2 \left( \frac{2}{a} \right)^{\frac{1 + sp}{2}} \Gamma \left( \frac{3 + sp}{2} \right) \frac{1}{p(1-s) + \beta} \left( \frac{\delta}{a} \right)^{p(1-s)+\beta} = 0
$$

for each $a \geq 1/2$. This can be proved by the induction as in the previous lemma, so we omit the proof. □

Let us provide the proof of Theorem 1.4 by using the pointwise representation (1.3) and Taylor’s theorem, and gathering pieces of limits in the preceding lemmas.

**Proof of Theorem 1.4.** Let $u \in C_0^2(\mathbb{H}^n)$ and let $x \in \mathbb{H}^n$ be such that $\nabla u(x) \neq 0$. Let $R > 0$, then by Lemma 5.3 we first have

$$
\left| c_{n,s,p} \int_{d(x,\xi) \geq R} \Phi_p(u(x) - u(\xi)) K_{n,s,p}(d(x,\xi)) \, d\xi \right| \lesssim c_{n,s,p} \int_{R}^{\infty} K_{n,s,p}(\rho) \sinh^{n-1} \rho \, d\rho \to 0
$$

as $s \to 1^-$. Thus, by the pointwise representation (1.3) of the fractional $p$-Laplacian, we obtain

$$
\lim_{s \nearrow 1} \left( -\Delta_{\mathbb{H}^n} \right)_p^s u(x) = \lim_{s \nearrow 1} c_{n,s,p} \text{P.V.} \int_{d(x,\xi) < R} \Phi_p(u(x) - u(\xi)) K_{n,s,p}(d(x,\xi)) \, d\xi.
$$
Let $v = \exp_{x}^{-1}\xi$ be a tangent vector in $T_{x}\mathbb{H}^{n}$ and denote by $T_{x}\xi$ the point $\exp_{x}(-v) \in \mathbb{H}^{n}$. Since $K_{n,s,p}(d(x,\xi)) = K_{n,s,p}(d(x,T_{x}\xi))$, we write

$$
\int_{d(x,\xi)<R} \Phi_{p}(u(x) - u(\xi))K_{n,s,p}(d(x,\xi)) \, d\xi
= \frac{1}{2} \int_{d(x,\xi)<R} |u(x) - u(\xi)|^{p-2}(2u(x) - u(\xi) - u(T_{x}\xi))K_{n,s,p}(d(x,\xi)) \, d\xi
+ \frac{1}{2} \int_{d(x,\xi)<R} (|u(x) - u(T_{x}\xi)|^{p-2} - |u(x) - u(\xi)|^{p-2}) (u(x) - u(T_{x}\xi))K_{n,s,p}(d(x,\xi)) \, d\xi
=: J_{1} + J_{2}.
$$

By Taylor’s theorem, we have

$$
u(x) - u(\xi) = -\langle \nabla u(x), v \rangle + O(|v|^2), \quad u(x) - u(T_{x}\xi) = \langle \nabla u(x), v \rangle + O(|v|^2),$$

and

$$2u(x) - u(\xi) - u(T_{x}\xi) = -\langle D^2 u(x)v, v \rangle + O(|v|^3).$$

If we write $\omega = v/|v|$, then

$$|u(x) - u(\xi)|^{p-2} = |v|^{p-2}|\langle \nabla u(x), \omega \rangle|^{p-2} + O(|v|^{p-1}).$$

Thus, we obtain

$$|u(x) - u(\xi)|^{p-2}(2u(x) - u(\xi) - u(T_{x}\xi)) = -|v|^{p-1}|\langle \nabla u(x), \omega \rangle|^{p-2} \langle D^2 u(x)\omega, \omega \rangle + O(|v|^{p+1}).$$

Therefore, we deduce

$$J_{1} = -\frac{1}{2} \int_{0}^{R} \int_{\mathbb{S}^{n-1}} \rho^{p-1}|\langle \nabla u(x), \omega \rangle|^{p-2} \langle D^2 u(x)\omega, \omega \rangle K_{n,s,p}(\rho) \sinh^{n-1}\rho \, d\omega \, d\rho
+ \frac{1}{2} \int_{d(x,\xi)<R} O(d(x,\xi)^{p+1})K_{n,s,p}(d(x,\xi)) \, d\xi.
(5.19)$$

For $J_{2}$, since

$$|u(T_{x}\xi) - u(x)|^{p-2} - |u(x) - u(\xi)|^{p-2}
= (p-2)|v|^{p-1}|\langle \nabla u(x), \omega \rangle|^{p-2} \langle D^2 u(x)\omega, \omega \rangle + O(|v|^p),$$

we have

$$Brackets:\langle u(T_{x}\xi) - u(x)|^{p-2} - |u(x) - u(\xi)|^{p-2} = (p-2)|v|^{p-1}|\langle \nabla u(x), \omega \rangle|^{p-2} \langle D^2 u(x)\omega, \omega \rangle + O(|v|^{p+1}).$$

Thus, we obtain

$$J_{2} = -\frac{p-2}{2} \int_{0}^{R} \int_{\mathbb{S}^{n-1}} \rho^{p-1}|\langle \nabla u(x), \omega \rangle|^{p-2} \langle D^2 u(x)\omega, \omega \rangle K_{n,s,p}(\rho) \sinh^{n-1}\rho \, d\omega \, d\rho
+ \frac{1}{2} \int_{d(x,\xi)<R} O(d(x,\xi)^{p+1})K_{n,s,p}(d(x,\xi)) \, d\xi.
(5.20)$$

Combining (5.18), (5.19), and (5.20), and using Lemma 5.4 and Lemma 5.6, we arrive at

$$\lim_{s,p \rightarrow 1} (-\Delta_{\mathbb{H}^{n}})^{p}_{u}(x) = -\frac{p-1}{2} \frac{1}{\pi \frac{n}{2}} \frac{\Gamma\left(\frac{p+n}{2}\right)}{\Gamma\left(\frac{p-1}{2}\right)} \int_{\mathbb{S}^{n-1}} |\langle \nabla u(x), \omega \rangle|^{p-2} \langle D^2 u(x)\omega, \omega \rangle \, d\omega.$$
The argument as in the proof of [4, Theorem 2.8] shows

\[ \int_{\mathbb{S}^{n-1}} |\langle \nabla u(x), \omega \rangle|^{p-2} \langle D^2 u(x) \omega, \omega \rangle \, d\omega = \gamma_p(\Delta_{\mathbb{H}^n}) \]

when \( \nabla u(x) \neq 0 \), where

\[ \gamma_p = \int_{\mathbb{S}^{n-1}} |\omega_n|^{p-2} \omega_1^2 \, d\omega = \pi^{\frac{n-1}{2}} \frac{\Gamma\left(\frac{p-1}{2}\right)}{\Gamma\left(\frac{p+n}{2}\right)}. \]

See [26, Lemma 2.1] for the computation of (5.21). This finishes the proof. \( \square \)

**Appendix A. Auxiliary result**

In this section, we recall an auxiliary result from [15] that helps proving Theorem 1.2 in Section 3.

**Lemma A.1.** Let \( p > 1, r > 0, u \in C_0^2(\mathbb{H}^n) \), and \( x \in \mathbb{H}^n \). If \( p \in (1, \frac{2}{p-2}] \), assume \( \nabla u(x) \neq 0 \) additionally. If \( K : \mathbb{H}^n \to \mathbb{R} \) is rotationally symmetric with respect to \( x \), that is, \( K(\xi) = K(d(x, \xi)) \) for all \( \xi \in \mathbb{H}^n \), then

\[ \left| \text{P.V.} \int_{d(x, \xi) < r} \Phi_p(u(x) - u(\xi)) K(d(x, \xi)) \, d\xi \right| \leq C \int_{d(x, \xi) < r} d(x, \xi)^\alpha |K(d(x, \xi))| \, d\xi \]

for some constant \( C = C(n, p, \|u\|_{C^2(\mathbb{H}^n)}) > 0 \), where \( \alpha = 2p - 2 \) when \( p \in (\frac{2}{p-2}, 2) \) and \( \alpha = p \) otherwise.

The cases \( p \in (1, \frac{2}{p-2}) \), \( p \in (\frac{2}{p-2}, 2) \), and \( p \in [2, \infty) \) are proved in [15, Lemma A.1, A2, and A3], respectively, for the case of Euclidean spaces. We omit the proof of Lemma A.1 because the same proofs work in our framework.

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