Remarks on unsolved basic problems of the Navier–Stokes equations

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Abstract

There is renewed interest in the question of whether the Navier–Stokes equations (NSE), one of the fundamental models of classical physics and widely used in engineering applications, are actually self-consistent. After recalling the essential physical assumptions inherent in the NSE, the notion of weak solutions, possible implications for the energy conservation law, as well as existence and uniqueness in the incompressible case are discussed. Emphasis will be placed on the possibility of finite time singularities and their consequences for length scales which should be consistent with the continuum hypothesis.

1 Introduction

As computational fluid dynamics makes progress towards the simulation of realistic three-dimensional flows, the validity of the Navier–Stokes equations (NSE) can be tested in a more and more refined way. To put it from an applied point of view: Before experiments in wind tunnels are substituted by computer simulations, one should make sure that the underlying theory is at least self-consistent. As a matter of fact, after the classical mathematical work by Leray [1], Hopf [2], Ladyzhenskaya [3], Serrein [4], Temam [5], to refer to important contributions in the field, there is renewed interest in the fundamentals of the NSE, see for instance the monograph of Doering and Gibbon [6], or a series of papers by Lions [7] and references therein.

This contribution focuses on the question of self-consistency which arises, when one of the assumptions inherent to the NSE, namely the continuum hypothesis, is confronted with the length scales emerging from solutions of the deterministic NSE. First, the NSE will be briefly derived from physical principles with due attention paid to the continuum hypothesis. After recalling the notion of weak solutions, the state of the art of mathematical existence and uniqueness proofs will be indicated. The implication of weak solutions upon energy conservation will be discussed. The possibility of finite time singularities will be related to length scales and thus to the problem of self-consistency.

2 Derivation of the NSE

The NSE are based on the conservation of mass and on Newton’s second law. In addition, the more special assumption of a so-called Newtonian fluid is adopted, which is justified in a great many cases of hydrodynamic flows. To formulate the conservation laws it is customary to pick out a connected cluster of molecules contained in volume $V_t$ which is deformed in time and translated according to the local velocity $v(x,t)$ of the flow. Time derivatives of corresponding magnitudes are conveniently evaluated by means of the Reynolds transport theorem

\[ \int_{V_t} dV \left( \frac{df}{dt} \right) := \int_{V_t} dV \left[ \frac{\partial f}{\partial t} + \text{div}(f v) \right] \]

where $f$ is an arbitrary scalar function and $v$ the local velocity of the flow. The integral over the volume $V_t$ at time $t$ is denoted by $\int_{V_t}$. The time derivative of $f$ is denoted by $\frac{df}{dt}$.

\[ \frac{d}{dt} \int_{V_t} dV f(x,t) \equiv \int_{V_t} dV \left( \frac{df}{dt} \right) \]

and the divergence of the vector field $f v$ is denoted by $\text{div}(f v)$.
where \( f \) is a scalar function. If \( \rho(x,t) \) denotes the mass density, then conservation of mass, namely
\[
\frac{d}{dt} \int_{V_t} dV \rho = 0
\]
gives rise to the continuity equation
\[
\frac{d \rho}{dt} \equiv \frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0,
\]
which if \( \rho = \rho_0 \) is constant, leads to the incompressibility condition
\[
\text{div}(\mathbf{v}) = 0.
\]
Newton’s second law implies that any change in momentum is caused by external forces which in continuum physics are described by the volume force density \( f \) (e.g. gravity) and by a tensorial force \( \Pi \). This tensor reflects the influence of the adjacent fluid on a given fluid particle. The momentum balance reads
\[
\frac{d}{dt} \int_{V_t} dV \rho \mathbf{v} = \int_{V_t} dV \rho \mathbf{f} + \oint_{\partial V_t} dS \Pi \circ \mathbf{n}
\]
where \( \mathbf{n} dS \) is the oriented surface element of the volume \( V_t \). It is convenient to separate in \( \Pi \) an isotropic part, the pressure \( p \), which is present also in the hydrostatic case, from the so-called stress tensor \( T \)
\[
\Pi_{ik} = -p \delta_{ik} + T_{ik}, \quad i, k = 1, 2, 3.
\]
The Newtonian fluid assumption now amounts to the following linear relations between \( T \) and the strain (rate) tensor \( S \):
\[
T_{ik} = \sum_{m,n=1}^{3} C_{ikmn} S_{mn}
\]
with
\[
S_{mn} = \frac{1}{2} \left( \frac{\partial v_m}{\partial x_n} + \frac{\partial v_n}{\partial x_m} \right).
\]
The 4th rank tensor \( C \) is constant and describes the effect of viscosity. In the isotropic case, \( C \) is of the form
\[
C_{ikmn} = \nu \delta_{ik} \delta_{mn} + \mu (\delta_{im} \delta_{kn} + \delta_{in} \delta_{km})
\]
where \( \mu \) and \( \nu \) are macroscopic viscosity parameters. In the incompressible case, \( \nu \) drops out, and after making use of mass conservation we can write down the momentum balance as follows
\[
\int_{V_t} dV \left[ \rho_0 \frac{\partial \mathbf{v}}{\partial t} + \rho_0 (\mathbf{v} \cdot \nabla) \mathbf{v} + \text{grad}(p) - \mu \Delta \mathbf{v} - \rho_0 \mathbf{f} \right] = 0.
\]
Here \( V_t \) is an arbitrary local space volume. To be sure of the existence of the above integral, one may adopt the sufficient conditions that the following fields are locally square integrable
\[
\mathbf{v}, \frac{\partial}{\partial t} \mathbf{v}, \frac{\partial}{\partial x_i} \mathbf{v}, \frac{\partial^2}{\partial x_i \partial x_k} \mathbf{v}, \frac{\partial}{\partial x_i} p, \mathbf{f}.
\]
This can be easily seen with the aid of the Schwarz inequality. For instance, if \( \mathbf{n}_i \) is a cartesian unit vector, then we can write
\[
\left| \int_{V_t} dV \frac{\partial v_i}{\partial t} \right|^2 \leq \int_{V_t} dV \left( \frac{\partial v_i}{\partial t} \right)^2 \leq V_t \int_{V_t} dV \frac{\partial v_i}{\partial t} \frac{\partial v_i}{\partial t}.
\]
As will be discussed later on, there may arise difficulties with the conservation laws when certain weak conditions on the velocity field \( \mathbf{v} \) are adopted as is customary in the frame of functional analysis. From eq.(10), the following standard NSE in the form of partial differential equations are inferred
\[
\rho_0 \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\text{grad}(p) + \mu \Delta \mathbf{v} + \rho_0 \mathbf{f}
\]
where \( p \) is determined through the incompressibility condition \( \text{div}(\mathbf{v}) = 0 \).

3 Continuum assumptions and length scales

The NSE describe macroscopic physical quantities which constitute mean values with respect to the underlying atomic degrees of freedom. The density \( \rho(x,t) \) at the space point \( x \), for instance, has to be understood as an average over some volume \( \Delta V \) centered at \( x \). If \( \Delta V \) is chosen too small, a
single measurement of $\rho$ may largely deviate from its mean value due to molecular fluctuations. An estimate for a physically reasonable lower bound of $\Delta V$ can be deduced from the mean thermal density fluctuation $\Delta \rho$ as given in standard textbooks of thermodynamics: \[ \frac{\Delta \rho}{\rho} = \sqrt{\frac{kT\kappa}{\Delta V}} \] (14)

where $k$ is the Boltzmann constant, $T$ the absolute temperature and $\kappa$ the compressibility. If we require the relative fluctuation $\Delta \rho/\rho$ to be smaller than, say $10^{-3}$, at $T = 300^\circ$ Kelvin, then we find that the diameter $d$ of the volume $\Delta V$ should be $d \geq 3 \cdot 10^{-7}$m for air, or $d \geq 10^{-8}$m for water.

As an implication, if solutions of the deterministic NSE turn out to vary on a space scale much smaller than the above lower bounds, then we are outside of the validity domain of these equations. Here is the point where the self-consistency problem arises.

In the turbulent regime, length scales decrease with increasing Reynolds number $R$. As is listed e.g. in [2], the Kolmogorov length $\delta_K$ below which eddies are destroyed by dissipation, is given by $\delta_K = L/R^{3/4}$ where $L$ is a typical external length, e.g. the diameter of the containment. As another example the thickness $\delta_B$ of a turbulent boundary layer scales as $\delta_B \sim L/(R \log R)$. If $L = 1$ cm, then $\delta_K$ and $\delta_B$ reach the continuum limit at $R \approx 10^6$.

4 Weak solutions and energy balance

Since Leray’s pioneering work [1], one has been looking for generalized solutions $\mathbf{v}(\mathbf{x}, t)$ of the incompressible NSE in the space time domain $\Omega_T := \Omega \times [0, \tau]$ with the following properties:

\[ \mathbf{v}(\mathbf{x}, t = 0) = \mathbf{\alpha}(\mathbf{x}), \quad (15) \]
\[ \mathbf{v}|_{\partial \Omega} = 0, \quad (16) \]
\[ \text{div}(\mathbf{v}) = 0. \quad (17) \]

The above equations correspond to the initial condition, no-slip boundary condition and incompressibility, respectively. To establish weak solutions, test vector fields $\Phi \in \mathbf{S}$ are introduced with the following properties

\[ \mathbf{D} := \{ \Phi \mid \Phi \in D(\Omega); \text{div}(\Phi) = 0 \} \] (18)

where $D(\Omega)$ is the Schwartz space ($C^\infty$ and compact support in $\Omega$). Now $\mathbf{v}(\mathbf{x}, t)$ is called a weak solution if it is locally square integrable and if the following projections of the NSE and the continuity equation hold for every $\Phi \in \mathbf{D}$ and for every $C^1$ scalar function $\phi$ with compact support in $\Omega$, respectively:

\[ \int_0^\tau \int_\Omega \mathbf{D} dV [\Phi_k \frac{\partial \mathbf{v}_k}{\partial t} - \nu_i \mathbf{v}_k \frac{\partial \Phi_k}{\partial x_i} - \nu_0 \mathbf{v}_k \Delta \Phi_k - \Phi_k \mathbf{f}_k] = 0, \quad (19) \]
\[ \int_\Omega dV \mathbf{v} \cdot \text{grad}(\phi) = 0 \] (20)

where $\nu_0 = \mu/\rho_0$ denotes the kinematic viscosity and summation convention is adopted. The pressure term dropped out in (19) due to the solenoidal property of $\Phi$. A typical theorem reads [3]:

**Theorem:** A unique weak solution exists, at least in the time interval $t \in [0, \tau_1]$ with $\tau_1 \leq \tau$, provided the initial velocity field $\mathbf{\alpha}(\mathbf{x}) \in W^2_2$ and the external force density $\mathbf{f}$ obeys the condition

\[ \int_0^\tau \int_\Omega dV \left[ f^2 + \left( \frac{\partial f}{\partial t} \right)^2 \right]^{1/2} < \infty \] (21)

where $W^2_2$ denotes the Sobolev space with the second space derivatives being square integrable.

As should be noticed, even if the condition (21) on the external field $\mathbf{f}$ holds for arbitrarily large $\tau$, uniqueness can be guaranteed by the above theorem only for the smaller time interval $t \in [0, \tau_1]$. While this is typical in space dimension three, one has $\tau_1 = \tau$ in the case of two-dimensional flows.

Which price do we have to pay for accepting weak solutions? To discuss a possible implication for energy conservation, we recall the notion of weak and strong convergence of a sequence of real functions $a^{(1)}, a^{(2)}, \ldots, a^{(N)}, \ldots$. This sequence is called to converge weakly against the function $a^*$, if for any
square integrable function \( g \)
\[
\int_{\Omega} dV \ a^{(N)} a^{(N)} < \infty \quad \text{and} \\
\lim_{N \to \infty} \int_{\Omega} dV \ a^{(N)} g = \int_{\Omega} dV \ a^* g. \tag{22}
\]

It converges strongly, if
\[
\lim_{N \to \infty} \int_{\Omega} dV \ a^{(N)} a^{(N)} = \int_{\Omega} dV \ a^* a^*. \tag{23}
\]

In the case of weak convergence, we have the identity
\[
\lim_{N \to \infty} \left\{ \int_{\Omega} dV \ (a^{(N)} - a^*)^2 - \int_{\Omega} dV \ a^{(N)} a^{(N)} + \int_{\Omega} dV \ a^* a^* \right\} = 0 \tag{24}
\]
which is true because \( \int_{\Omega} dV \ a^{(N)} a^* \) converges (weakly) to \( \int_{\Omega} dV \ a^* a^* \) and the two non-converging terms \( \int_{\Omega} dV \ a^{(N)} a^{(N)} \) cancel each other identically. As a consequence one has in particular
\[
\lim_{N \to \infty} \int_{\Omega} dV \ a^{(N)} a^{(N)} \geq \int_{\Omega} dV \ a^* a^*. \tag{25}
\]

Here, the equality sign is guaranteed only in the case of strong convergence where simultaneously \( \lim \inf = \lim \sup \).

To derive the energy balance for a sequence of approximations \( v^{(N)} \) which converge weakly against a solution \( v^* \) of (19), we use basis functions \( \Phi^{(N)} \in D \) with the properties (18) as
\[
v^{(N)}(x, t) := \sum_{n=1}^{N} c^{(N)}(t) \Phi^{(N)}(x), \quad c^{(N)} \in \mathbb{R}. \tag{26}
\]

It is convenient to introduce the following abbreviation for the kinetic energy at time \( t \)
\[
E^{(N)}(t) := \frac{1}{2} \int_{\Omega} dV \ \dot{v}^{(N)}(x, t) \cdot \dot{v}^{(N)}(x, t). \tag{27}
\]

\( E^*(t) \) denotes the energy corresponding to the weak solution
\[
v^* \equiv v^{(N)} + r^{(N)} \tag{28}
\]
where \( r^{(N)} \) is the remainder to the approximate field \( v^{(N)} \). We now insert into (19) the above expression for the weak solution \( v^* \) together with the test field \( \Phi = v^{(N)} \in D \) and obtain
\[
E^{(N)}(\tau) - E^{(N)}(0) + \nu_0 \int_{0}^{\tau} dt \int_{\Omega} dV \ \frac{\partial v^{(N)}_k}{\partial x_i} \frac{\partial v^{(N)}_k}{\partial x_i} \\]
\[
- \int_{0}^{\tau} dt \int_{\Omega} dV \ v^{(N)}_k f_k = R^{(N)} \tag{29}
\]
with
\[
R^{(N)} = \int_{0}^{\tau} dt \int_{\Omega} dV \ \left[ -v_k^{(N)} \frac{\partial r_k^{(N)}}{\partial t} + v_k^{*} \frac{\partial r_k^{(N)}}{\partial x_i} \right] \]
\[
+ \nu_0 r_k^{(N)} \Delta v_k^{(N)} + r_k^{(N)} f_k \]. \tag{30}

Apart from partial integrations, we made use of the incompressibility condition (20) which implies the relation
\[
\int_{\Omega} dV \ \dot{v}_i^{(N)} v_k^{(N)} \frac{\partial v_k^{(N)}}{\partial x_i} = \frac{1}{2} \int_{\partial \Omega} dS \ (v^{(N)})^2 \hat{n} \cdot v^{(N)}. \tag{31}
\]

The above surface integral vanishes because \( v^{(N)} \in D \). It should be noticed that eq.(29) holds true for any cutoff \( N \); it follows strictly from the definition (19) of a weak solution; in particular, no approximate projection scheme was adopted as is common in Galerkin representations.

In the case of strong solutions with \( v^{*} \in W_2^2 \) in the space time domain \( \Omega \times \tau \), one can show that \( R^{(N)} \to 0 \) in the limit \( N \to \infty \) so that we would have the physically plausible energy balance
\[
E^*(\tau) + \nu_0 \int_{0}^{\tau} dt \int_{\Omega} dV \ \frac{\partial v^{*}_k}{\partial x_i} \frac{\partial v^{*}_k}{\partial x_i} \]
\[
= E^*(0) + \int_{0}^{\tau} dt \int_{\Omega} dV \ v^{*}_k f_k, \tag{32}
\]
or in words: the kinetic energy at time \( \tau \) plus the energy dissipated up to \( \tau \) equals the initial kinetic energy plus the work done by the volume force \( f \) up to time \( \tau \).

However, if \( v^* \) is a weak solution, then we have only the property of boundedness of the integrals in (29) and (30), except for the \( f_k \)-integrals and the initial energy \( E^{(N)}(0) \) which converges under the assumptions specified in Theorem A. Making use of the inequality (25) we can write
\[
\lim \inf_{N \to \infty} \left[ E^{(N)}(\tau) + \nu_0 \int_{0}^{\tau} dt \int_{\Omega} dV \ \frac{\partial v^{(N)}_k}{\partial x_i} \frac{\partial v^{(N)}_k}{\partial x_i} \right] =
\[ E^*(\tau) + v_0 \int_0^\tau dt \int_{\Omega} dV \frac{\partial v^*_k}{\partial x_i} \frac{\partial v^*_i}{\partial x_k} + L_s \] (33)

where \( L_s \geq 0 \). With \( R_s \) denoting the limes inferior of \( R^{(N)} \), the energy balance \( (29) \) reads in the same limit

\[
E^*(\tau) + v_0 \int_0^\tau dt \int_{\Omega} dV \frac{\partial v^*_k}{\partial x_i} \frac{\partial v^*_i}{\partial x_k} = E^*(0) + \int_0^\tau dt \int_{\Omega} dV v^*_k f_k + R_s - L_s. \quad (34)
\]

Thus, in the case of weak solutions there may be unphysical sources or sinks (depending on the sign of \( R_s - L_s \)) of kinetic energy due to the presence of singularities. The latter are connected with the space gradients of the velocity field, since \( E^{(N)}(t), t \in (0, \tau) \) can be shown to converge under rather general assumptions, see also [3]. If \( R_s - L_s < 0 \), then the kinetic energy \( E^*(t) \) is smaller than physically expected; this is known as Leray inequality, see e.g. p. 104 of [3].

5 Uniqueness and finite time singularities

As already mentioned, one gets square integrable solutions \( \mathbf{v} \) under rather general assumptions on the external data. The main basic problem of the NSE is related to uniqueness which so far is tied to the existence of the following time integral, for a recent discussion see [6].

\[
I(\tau) := \int_0^\tau dt \| D\mathbf{v} \|_\infty \quad (35)
\]

with the supremum norm

\[
\| D\mathbf{v} \|_\infty := \max \max_{i,k} \left| \frac{\partial v_k}{\partial x_i} \right|. \quad (36)
\]

The origin of this integral will be indicated in the Appendix. Up to now, in three dimensions the existence of \( I(\tau) \) has been corroborated only for finite time intervals \( \tau \). If \( I(\tau) \) exists for arbitrarily large \( \tau \), then both uniqueness and existence of weak solutions can be established for arbitrarily large times under quite general conditions.

If \( I(\tau) \) exists only up to some time \( \tau^* \), then \( \| D\mathbf{v} \|_\infty \) is singular at \( t = \tau^* \) in a way, that there is at least one space point \( x_0 \in \Omega \), where one of the components \( \partial v_i/\partial x_k \) diverges, for instance as follows

\[
\left| \frac{\partial v_i(x,t)}{\partial x_k} \right|_{x=x_0} \rightarrow \frac{\alpha^2}{(\tau^* - t)\gamma}, \quad t < \tau^*, \quad \gamma \geq 1. \quad (37)
\]

Since for \( t \nearrow \tau^* \) the behaviour \( (37) \) implies changes of the velocity field over arbitrarily small length scales, it is in conflict with the continuum assumption. The length scales are then small compared to the diameter of the volume \( \Delta V \) of a fluid particle with the consequence that microscopic molecular forces come into play and can no longer be neglected. In other words we are then out of the validity domain of the deterministic NSE and we would have to consider stochastic forces in addition to the deterministic external forces. It is therefore not yet settled, whether the phenomenon of hydrodynamic turbulence is a manifestation of deterministic chaos alone.

As should be noted, the problem of finite time singularities cannot be overcome by some averaging recipe, because the existence of \( I(\tau) \) is connected to the uniqueness of solutions as a sufficient condition, and it may turn out to be also necessary.

Similarly, in the case of compressible flows finite time singularities could not be excluded so far [6]. The proof or disproof of the existence of finite time singularities constitute one of the basic unsolved problems in the analysis of the NSE. In the inviscid case of the Euler equation, there is a general argument for possible finite time singularities, see for instance Frisch [10]. From a direct numerical simulation of the Euler equations, Grauer and Sideris [11] recently reported on evidence for a singularity of the type as given in \( (37) \) with \( \gamma = 1 \).

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5
Appendix

In the following it is sketched how the integral $I(\tau)$ shows up in uniqueness proofs, see [6]. At variance with [6] we do not adopt periodic boundary conditions. Let us assume there are two different solutions $v$ and $v'$ of the NSE (13). Then we define $u := v - v'$ and obtain after subtracting the NSE for $v$ and $v'$

$$\frac{\partial u}{\partial t} - u_i \frac{\partial u_k}{\partial x_i} + v_i \frac{\partial u_k}{\partial x_i} + u_i \frac{\partial v_k}{\partial x_i} = - \frac{\partial(p - p')}{\partial x_k} + \nu_0 \Delta u_k. \quad (38)$$

When this equation is scalarly multiplied by $u$ and integrated over the volume $\Omega$, then, apart from the pressure term, the second and third terms of the left hand side vanish by the same argument used before in (31). With the abbreviation

$$\|u\|^2 = \int_\Omega dV \ u \cdot u \quad (39)$$

we can write

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 = A + B,$$

$$A := -\nu_0 \int_\Omega dV \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_i}; \quad B := - \int_\Omega dV \ u_k \frac{\partial v_k}{\partial x_i} u_i. \quad (40)$$

Now the viscosity term is estimated by means of the Poincaré inequality [12]

$$-A \equiv |A| \geq \frac{2}{l^2} \|u\|^2 \quad (41)$$

where $l$ denotes the smallest distance between two parallel planes which just contain $\Omega$. The $B$ term is estimated by using the definition (36) and the Schwarz inequality as follows

$$|B| = | \int_\Omega dV \ u_k \frac{\partial v_k}{\partial x_i} u_i | \leq \|Dv\|_\infty \sum_{k,i=1}^3 \int_\Omega dV \ |u_k u_i|$$

$$\leq 9 \|Dv\|_\infty \|u\|^2. \quad (42)$$

One arrives at the ordinary differential inequality

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 \leq -\frac{2\nu_0}{l^2} + 9 \|Dv\|_\infty \|u\|^2, \quad (43)$$

which by Gronwall’s lemma can be integrated to the final inequality

$$\|u(t)\|^2 \leq \|u(0)\|^2 \exp \left[-\frac{4\nu_0}{l^2} t + 18I(t) \right] \quad (44)$$

This result tells that, since the two supposed solutions $v, v'$ possess the same initial conditions and therefore $u(0) = 0$, we have $u(t) = 0$ for times $t \in (0, \tau)$ for which $I(t)$ exists. This conclusion holds true also in the inviscid limit $\nu_0 \rightarrow 0$.

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