Inclusion of a perfect fluid term into the Einstein-Hilbert action

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Abstract

I introduce a method to obtain the stress-energy tensor of the perfect fluid by adding a suitable term to the Einstein-Hilbert action. Variation should be understood with respect to the metric.

1 Introduction

Since the gravitational side of the Einstein equations is variational one would like to obtain the right-hand said from the variation of a matter Lagrangian. This raises the problem as to whether the dynamical equations for the most straightforward form of matter, the perfect fluid, are variational in character. These equations, namely the continuity and the Euler equations have been given variational formulation following different approaches [25, 12, 23, 3, 20, 17].

The most common strategy [25, 5, 15, 16, 6, 14, 7, 9, 13, 11, 10, 24, 1] considers a submersion $\xi: M \to B$ to the body frame. The fibers $\xi^{-1}(b)$ represent the flow lines. The action depends on the flow lines namely on the map $\xi$, i.e. the coordinate functions $\xi^A$, and on their spacetime derivatives $\xi^A_\mu$. In this fashion the dynamics of the continua admits a field theoretical formulation. Different type of continua are described by different geometric structures placed on $B$, for instance, in a fluid $B$ would be endowed with a volume form $r(\xi) d\xi^1 \cdots d\xi^n$ (telling us the amount of matter in a portion of the continua), while in an elastic material $B$ would be endowed with a metric $\gamma$ (telling us the distance between particles in their rest state). In all cases $B$ inherits a time dependent contravariant metric (telling us the distance between particles on spacetime) by push forward of the contravariant spacetime metric $G^{-1} = \xi^* g^{-1}$, i.e. $G^{AB} = g^{\mu\nu} \xi^A_\mu \xi^B_\nu$.

The density can then be shown to be $\rho = r \sqrt{\det{G^{AB}}}$, and natural Lagrangians can be constructed as functions of $G^{AB}$ or, in the case of a perfect fluid, of $\rho$, $L = F(\rho)$. In fact one can show that variation with respect to the metric returns the stress-energy tensor of the perfect fluid.

This approach is natural but somewhat elaborated. It is necessary to introduce the projection $\xi$ and to write an action dependent on the derivatives of

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such projection, though one is really interested on dynamical equations which
do not involve these variables. This drawback has motivated some authors to
look for alternative approaches [19, 2].

In this little note I show that the perfect fluid stress-energy tensor can be
obtained variationally in a more elementary and direct way. The derivation is
really easy and has turned out to be essentially the same of Schutz and Schmid
[22, 21]. Though at present this work is not meant for publication, it could
still be useful as an introduction to the topic. As with some other references
[5, 9, 24], I take the view that the variation should be taken with respect to the
metric, and I shall not consider variation with respect to the flow lines.

There are three reasons for this choice. Firstly this is the type of variation
needed in the variational formulation of gravity coupled with matter. Secondly,
while there is evidence that at the fundamental level matter is composed by
particles described by vector fields which obey variational equations, there is no
reason of principle to believe that continua should have a dynamical variational
description. Indeed, the process of averaging needed to obtain the continua
description might lead to ‘averaged’ equations which, though coming from vari-
ational equations, might not be themselves variational. Finally, it is known that
variation with respect to \( u \) and other thermodynamic quantities cannot give the
correct dynamical equations without the introduction of constraints [23]. On
the contrary, we shall not need to introduce neither constrains nor Lagrange
multipliers.

1.1 Completing the Einstein-Hilbert action

Let us denote for short \( \sqrt{-g} \, dx^4 \rightarrow dx \), and let us adopt the conventions of [18]
(metric signature \((-+, +, +, +)\), units chosen so that \( c = G = 1 \)). The Einstein
equations are

\[
G_{\alpha\beta} + \Lambda g_{\mu\nu} = 8\pi T_{\alpha\beta},
\]

where if variational the stress-energy tensor \( T_{\alpha\beta} \) is identified with

\[
T_{\alpha\beta} = g_{\alpha\beta} L_{(m)} - \frac{1}{2} \frac{\partial L_{(m)}}{\partial g^{\alpha\beta}},
\]

where \( L_{(m)} \) is the matter Lagrangian. In this case the variational principle is

\[
S = \int \left( \frac{1}{16\pi} (R - 2\Lambda) + L_{(m)} \right) dx,
\]

indeed the variation gives

\[
\delta S = \int \left( -\frac{1}{16\pi} (G^{\mu\nu} + \Lambda g^{\mu\nu}) + \frac{1}{2} L_{(m)} g^{\mu\nu} - \frac{\partial L_{(m)}}{\partial g^{\alpha\beta}} g^{\alpha\mu} g^{\beta\nu} \right) \delta g_{\mu\nu} dx,
\]

where we used \( \delta dx = \frac{1}{2} g^{\alpha\beta} \delta g_{\alpha\beta} dx \). By definition a perfect fluid is a continua
which admits a stress-energy tensor of the form

\[
T_{\alpha\beta} = \rho u_\alpha u_\beta + p (g_{\alpha\beta} + u_\alpha u_\beta),
\]

(1)
where $u$ is the normalized velocity, $g_{\alpha\beta}u^\alpha u^\beta = -1$, and for barotropic fluids the constitutive relation, $\rho = f(p)$, establishes a functional constraint between the density $\rho$ and the pressure $p$.

### 1.2 Barotropic fluids

Let us show that there is indeed a Lagrangian $L_{(m)}(g_{\alpha\beta})$ which returns the perfect fluid stress-energy tensor with a chosen-in-advance functional dependence between $\rho$ and $p$. We claim that the action with this property is

$$
S[g_{\alpha\beta}] = \int \left( \frac{1}{16\pi} (R - 2\Lambda) + P \left( \sqrt{-g^{\alpha\beta}v_\alpha v_\beta} \right) \right) dx, \tag{2}
$$

where function $P(x)$ is determined by the differential equation

$$
xP' - P = f(P). \tag{3}
$$

Observe that the left-hand side is $P^*(P'(x))$ if the Legendre transform $P^*$ exists.

Equation (3) can be easily integrated since the variables can be separated

$$
x = \exp \left( \int \frac{P}{f(P) + P} d\tilde{P} \right). 
$$

At the stationary point the integral curves of $v^\alpha := g^{\alpha\beta}v_\beta$ are physically interpreted as the flow lines of the fluid, the pressure is $p := P$ and $\rho = f(p)$. The variable $x$ is called index of the fluid and has been proved very useful in the study of perfect fluids [8], for it is basically the logarithmic acceleration potential (see Euler equation (11) below). The starting point of the integration is arbitrary, as a consequence $x$ can be redefined up to a factor. It turns out that at the stationary point $v^\alpha = xu^\alpha$, namely $v$ is the dynamical velocity of the fluid, again a very useful quantity in the study of perfect fluids [8].

In (2) $v_\alpha$ is a future directed timelike 1-form field but it is not a dynamical field with respect to which we need to take a variation (and we stress that the data is the covariant object $v_\alpha$, not $v^\alpha$). Furthermore, it could be normalized with respect to $g$ but that does not imply that the variations of $g$ respect the normalization. This is the key observation which gives room for an interesting variational principle, for otherwise we would have to replace $g^{\alpha\beta}v_\alpha v_\beta$ with $-1$ in Eq. (2), obtaining something uninteresting. The main idea is that some terms might be trivial ‘on shell’ but variationally non-trivial in general. This fact helps to explain why this action passed unnoticed.

Let us prove the claims. Taking the variation and setting $x = (-g^{\alpha\beta}v_\alpha v_\beta)^{1/2}$

$$
T_{\alpha\beta} = \frac{1}{x} P'(x) v_\alpha v_\beta + g_{\alpha\beta} P(x).
$$
On the stationary point let us set

\[ u_\mu := \frac{v_\mu}{\sqrt{-g^{\alpha\beta} v_\alpha v_\beta}}, \]
\[ \rho := xP'(x) - P(x), \]
\[ p := P(x), \]

where \( u^\alpha \) is interpreted as the covariant velocity of the continua and \( \rho \) and \( p \) as density and pressure, respectively. With these definitions \( T_{\alpha\beta} \) takes the form \( \Box \) and by Eq. (3), \( \rho = f(p) \) as desired.

**Remark 1.1.** It is natural to ask if a similar result could be obtained given as data a vector field \( v^\alpha \). Indeed, it can be done using as action

\[ S[g_{\alpha\beta}] = \int \left( \frac{1}{16\pi} (R - 2\Lambda) + P \left( \frac{1}{\sqrt{-g_{\alpha\beta} v^\alpha v^\beta}} \right) \right) dx, \]

where \( P(x) \) is related to \( f \) as before, at the stationary point \( x \) is still the index of the fluid, and \( \rho \) and \( p \) depend on \( x \) as before. However, it is not true that \( v^\alpha \) is the dynamical velocity of the fluid, which is why we presented the theory in the 1-form version.

### 1.3 General perfect fluids

Let us denote with \( n \) the density of baryons, with \( T \) the temperature, with \( s \) the entropy per baryon (so \( v := 1/n \) is the specific volume and \( u := \rho/n \) is the energy per baryon). The first law of thermodynamics is \( \Box \) Sect. 22

\[ d \left( \frac{\rho}{n} \right) = -p d \left( \frac{1}{n} \right) + T ds \]

which can also be rewritten

\[ d\rho = \frac{\rho + p}{n} dn + nT ds. \]

The enthalpy per baryon \( h(p, s) \) is the thermodynamic potential defined by \( h := u + pv \), namely

\[ h = \frac{\rho + p}{n}. \]

From Eq. (5)

\[ dh = \frac{1}{n} dp + T ds. \]

We can invert \( h(p, s) \) so obtaining the function \( p = P(h, s) \) which satisfies \( \Box \)

\[ dp = ndh - nT ds, \]

thus \( p_n \) and \( p_s \) satisfy \( -nT \). Using Eq. (4) we get \( \rho = h p_n - h \).
Given the function \( P \) let

\[
S[g_{\alpha\beta}] = \int \left( \frac{1}{16\pi} (R - 2\Lambda) + P \left( \sqrt{-g} v_\alpha v_\beta, s \right) \right) \, dx,
\]

(7)

then by the already presented calculations we obtain that variation with respect to \( g_{\alpha\beta} \) gives the stress-energy tensor of the fluid. It can be observed that the variable \( x \) this time is the enthalpy and at the stationary point \( v^\alpha = h u^\alpha \) which again is the dynamical velocity (Taub current) in the general case [8].

Furthermore, if we consider the equation obtained varying \( v^\alpha \) through exact forms (notice that \( v^\alpha \) is not necessarily closed), namely \( v^\alpha \rightarrow v^\alpha + \partial^\alpha \varphi \), we get

\[- \int P \frac{g^{\alpha\beta} v_\beta \partial_\beta \varphi}{\sqrt{-g} v^\alpha v_\beta} \, dx = 0\]

which after integration by parts and using \( p_h \) gives

\[
\nabla_\alpha (n u^\alpha) = 0,
\]

(8)

which is the conservation of baryons. This approach is essentially that of [22] (see also [21, 4]). We shall see in the next section that the Einstein equations imply \( \nabla u^\rho + (\rho + p) \nabla \cdot u = 0 \). Since the first principle built in the function \( P \) implies \( \nabla u^\rho = h \nabla u + n T \nabla u s \), we have

\[ h \nabla_\alpha (n u^\alpha) + n T \nabla u s = 0, \]

(9)

and since we have baryon conservation we have also entropy conservation along the flow lines. The idea is that entropy cannot increase if we don’t have neither heat flow nor creation of particles. However, in my opinion, it could be incorrect to impose stationarity under variation of \( v \). If done one should hope to get the same equations implied by stress-energy conservation or more, not just different ones. In this way we could consider the matter Lagrangian not in pair with the gravitational one.

Schutz goes on to consider a variation of the form \( v_\alpha \rightarrow v_\alpha + \partial_\alpha \varphi + \theta \partial_\alpha s \) where \( s \) is the entropy per baryon. The variation with respect to \( \varphi \) gives again (8). The variation with respect to \( \theta \) gives

\[ \nabla u s = 0 \]

while variation with respect to \( s \) gives

\[ \nabla_\alpha (\theta n u^\alpha) - nT = 0 \Rightarrow \nabla u \theta = T. \]

The variable \( \theta = \int T \, d\tau + \text{cnst} \) is called thermasy. Actually, Schutz rather than considering these restricted variations of \( v \) claims that \( v \) can be parametrized using potentials of which \( s, \varphi, \theta \) are a subset. However, it is strange that \( s \) appears twice, also outside \( v \) in the Lagrangian, and furthermore, if the potentials parametrize any \( v \) the variation with respect to the potentials should imply the equation obtained through the variation of \( v \) namely \( n = 0 \), which is clearly untenable. I am therefore not entirely convinced that it could be meaningful to vary with respect to the potentials. If one allows for other forms of variations then the option \( v_\alpha \rightarrow v_\alpha + \partial_\alpha \varphi + \varphi \partial_\alpha s \) is interesting since it gives directly (9).
1.4 The equations of motion

We have shown that it is possible to obtain the stress-energy tensor of the perfect fluid variationally. Now, since the left-hand side of the Einstein equation is divergence free (by the naturality of the gravitational action, see [12, Sect. 3.3]), so is the right-hand side, namely \( T^{\mu\nu}_{(m);\nu} = 0 \). From here there follow the conservation of mass-energy (continuity equation/first law of thermodynamics)

\[
\nabla_u \rho + (\rho + p) \nabla \cdot u = 0,
\]

and the dynamical equation for the continua [18, p. 563] (Euler’s equation)

\[
(\rho + p) a^\alpha = - h^{\alpha\beta} \nabla_\beta p,
\]

where \( a^\alpha = u^\alpha_{\beta\gamma} u^\beta \) is the acceleration and \( h^{\alpha\beta} = \delta^{\alpha\beta} + u^\alpha u_\beta \) is the projection on the subspace orthogonal to \( u \). Since these calculations are well known they will not be repeated here.

Example 1.2.

For the linear constitutive relation, \( f(y) = ky \), we have \( P = C x^{1+k} \). Observe that for the vacuum equation of state, namely for \( k = -1 \), we obtain \( P = C \) namely a contribution to the cosmological constant. For a gas of radiation \( k = 3 \) and \( P = C x^4 \).

We conclude that the dynamical equations for perfect fluid continua do admit a simple variational formulation. It is interesting to observe the mechanisms for obtaining these equations requires the Einstein-Hilbert term, namely, the fluid moves as expected but with respect to a spacetime geometry which reacts to the motion of the fluid. Mathematically it could be seen as a drawback since the spacetime geometry is not held fixed, say to the Minkowski form. Physically, however, this could be a satisfactory behavior for the variational formulation seems admissible precisely under physically reasonable assumptions. Still one could perhaps fix the geometry in various ways, either introducing constraints or more naturally, taking the limit in which the gravitational constant goes to zero after the variation, so as to make the influence of matter on geometry negligible.

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