Spectroscopy of all charm tetraquark states

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Abstract: The mass-spectra of all-charm tetraquark states [cc][cc] with the diquark-antidiquark configuration are investigated in this study. To determine the fitting parameters for the calculation of masses of all-charm tetraquarks states [cc][cc], we first calculate the mass-spectra of charmonia [cc] and its decay constants (f_{J/P}) . We estimated the masses of the tetraquark states in their ground state along with their radially and orbitally excited states. The spin-spin, spin-orbital, and tensor interactions are perturbatively added to the central potential to produce the mass-spectra of tetraquark states. The charmonia findings obtained in this study are reasonably consistent with earlier experimental and theoretical expectations, whereas the masses of the tetraquark states are in accordance with prior theoretical predictions. The mass discrepancy in the present study are having nearly 100 MeV for 3S-wave (0^+ + and 2^+ +) and nearly 50 MeV for 2P-wave (0^− +, 1^− + and 2^− +) states from experimental data . We propose that the X(6900) state, which has all charm [cc][cc] in its constituent and was recently detected by LHCb, could be classified as either radially (3S-wave) or orbitally (2P-wave) excited states.

Keywords: Hadronic molecule; Exotic hadrons; Pentaquark; Potential model; Hadron mass-spectra

1. Introduction

In recent decades, there has been a growing interest among hadron physicists in the search for evidence of the existence of multiquark states [1]. In the family of hadrons, multiquark states that contain more than three quarks are termed as exotic states and are of particular significance because they represent a novel form of matter that extends beyond the traditional mesons (qq) and baryons (qqq) [2]. The investigation of multiquark states has entered a new phase, particularly as a result of the observations of a variety of XYZ states that are similar to charmonium (cc). Exotic states like as tetraquarks, pentaquarks, and hexaquarks were postulated in 1964 [3], but their first detection occurred in 2003 [4, 5]. Hadronic physics has made significant progress over the last two decades as a result of enhancements to experimental facilities worldwide, including BaBar, Belle, BESIII, CLEO, and LHCb [6–8]. As a result of the experimental development, non-conventional states with quantum numbers and decay properties other than baryons and mesons were identified. Exotic states, such as X(3872), are characterised as states that challenge the traditional theory of hadrons [4, 9–12].

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The underlying nature of these unusual states demands a detailed research to rule out the possibility of a molecular tetraquark state [13] or a compact tetraquark [14–20].

The Belle collaboration [4] reported \(X_c(3872)\) as a narrow charmonium-like exotic state generated solely in the decay process \(B^+ \rightarrow K^{\pm }\pi^\mp J / \psi\), with a mass very close to the \(M_{D^0} + M_{D^0}\) threshold mass in 2003. Several research collaborations in various decay modes afterwards established the existence of \(X_c(3872)\) [21–23]. Since then, several theoretical models and experimental investigations have been developed to better understand the internal mechanism of tetraquark states [24–27].

Numerous well-established models account for the behaviour of tetraquark states, including lattice computations [28], QCD sum rules [29–31], coupled channel effects, and non-relativistic effective field theories (e.g., see [25], and references therein). The author addresses the findings on heavy tetraquarks in a relativistic quark model in Ref. [32]. The precise nature of tetraquarks’ internal structure may be understood using diquark-antidiquark [qq – \(\bar{q}\bar{q}\)] pairings [33, 34], and it has been shown to be a useful tool for explaining the tetraquarks described in Refs. [34–38]. A diquark [qq] is a bound pair of quarks, while an antidiquark [\(\bar{q}\bar{q}\)] is a bound pair of antiquarks. Subsequently, it is believed that the diquark-antidiquark pair is in
a non-singlet colour state, resulting in a color-singlet tetraquark state [39, 40]. Theoretically, several phenomenological models have been developed to examine the completely heavy tetraquark states in Refs. [41–47]. The majority of these research confirmed the existence of stable states containing four heavy quarks and predicted the ground states of completely heavy tetraquarks below X(6900).

Iwasaki published the first study on all-charm tetraquarks in 1975, [48]. Later in the diquark-antidiquark model with orbital excitation, Chao considered the possibility of all charm tetraquarks and predicted their production in the $e^−e^+$ annihilation [49]. The LHCb Collaboration [50] recently reported the observation of a fully heavy charm tetraquark, $[cc][c\bar{c}]$, in the $J/\psi$-pair spectrum with a mass of about 6.9 GeV, however the quantum numbers ($J^{PC}$) have not yet been confirmed. It is possible for a tetraquark state $T_{4c}$ to decay into two states: $J/\psi$ mesons or $J/\psi$ with a heavier charmonium state, which decays into $J/\psi$ [50, 51]. The masses of all-charm tetraquarks are anticipated to be in the range of 5.5−7 GeV in the majority of theoretical and experimental predictions, which is more than the predicted masses of charmonium $[cc]$ states [52, 53]. The plausible predictions of all-charm tetraquarks in the $J/\psi$ productions at the LHCb, CMS, and ATLAS collaborations are very promising [54–56]. Alternatively, double $cc$ production at Belle [57] and differential production cross sections for $J/\psi$ pairs at LHCb [54] around 6.0 to 8.0 GeV are possible.

Naturally, the most obvious possibility is a totally charmed compact tetraquark resonance. However, the majority of theoretical investigations suggest that the $cc\bar{c}\bar{c}$ ground state should have a mass less than 6.9 GeV. Furthermore, the energy gap of 700 MeV between the double $J/\psi$ threshold and 6.9 GeV is greater than the normal energy gap between the ground and radially or orbitally excited states. Lower states should occur if a $cc\bar{c}\bar{c}$ resonance with a mass of roughly 6.9 GeV exists. Because the phase space is smaller, such lighter states are predicted to have lower widths. However, no noticeable smaller peaks can be seen in the stated double $J/\psi$ spectrum [44].

In this paper, the mass-spectra of the all-charm tetraquark state $T_{4c}$, which is categorized in the two body diquark-antidiquark $[cc][c\bar{c}]$ system, were determined using a non-relativistic model with a relativistic correction term to the central potential. The diquark and antidiquark elements of the colour singlet tetraquark state $T_{4c}$ are chosen to be in the colour antitriplet and triplet representations, respectively. It is crucial to investigate both the long- and short-range behaviour of QCD in order to study the dynamics of tetraquark states, and Cornell-like potential has proved especially useful in this respect. A Cornell-like potential and the relativistic mass correction $O(\frac{1}{m})$ will be used to interact between a diquark and antidiquark pair. Spin-dependent factors (spin-spin, spin-orbit, and tensor) have been incorporated as perturbative corrections to account for the degeneracy between the radial and orbital states.

After a brief introduction in Sect. 1, we will go through the theoretical model utilising the diquark-antidiquark approach, i.e. $[cc][c\bar{c}]$ in Sect. 2 in particular. A discussion of charmonium and S and P-wave tetraquark states, as well as findings for mass-spectra, can be found in Sect. 3 of the study. Section 4 is ended with the conclusion of the work.

2. Theoretical model

The spectroscopy of a hadronic bound state made up of four charm quarks may be studied using a non-relativistic framework in the diquark-antidiquark model [58, 59]. In order to determine the mass-spectra of the all-charm tetraquark $T_{4c}$, we used a numerical technique based on the fourth-order Runge–Kutta (RK4) method [60–63], which was first developed by W. Lucha et al. The four-body system may be categorized into two parts: diquarks and antidiquarks, each of which are made up of two quarks (antiquarks) that combine to produce the colour antitriplet (triplet) state. The interaction between the two components, diquark and antidiquark, forms a colour singlet structure of tetraquark. The mass-spectra of well-known charmonium $[cc]$ states has increased the study’s dependability and accuracy, as it assists in the choice of fitting parameters for obtaining the masses of $T_{4c}$.

In a non-relativistic framework [58], static potentials could be a plausible approximation for the rest mass energy of heavy quarks, which is greater than the kinetic energy of the constituent quarks. Time-independent Schrödinger equations are used to solve the two-body problem.

$$\left[ \frac{1}{2\mu} \left( -\frac{d^2}{dr^2} \right) + V^{(0)}(r) \right] \gamma(r) = Ey(r)$$

It is more convenient to work in the center-of-mass frame (CM), which incorporates two body problems in central potential [64]. Spherical harmonics can separate the angular and radial terms of a wave function. The kinetic energy of the system can be written as $\mu = \frac{m_1 m_2}{m_1 + m_2}$, where $m_1$ and $m_2$ are the masses of charm(anticharm) quarks in charmonium and diquarks, respectively, and the same input notation may be used for diquark masses in $T_{4c}$. In the spectroscopic study of the heavy-quarkonium system, the Cornell-like potential model of zeroth-order $V^{(0)}(r)$ has been used. In the Cornell-like potential model $V^{(0)}(r)$, a coulomb term ($V_C^{(0)}$) is responsible for one gluonic interaction.
between quarks and antiquarks, while a linear term \( V_L^{(0)} \) is responsible for quark confinement.

\[
V_{C+L}^{0}(r) = \frac{k_s z_s}{r} + br
\]  

(2)

where, \( z_s \) is known as the QCD running coupling constant, \( k_s \) is color factor, \( b \) is string tension. Because we are working with the charm quark, which according to PDG [65] is believed to be in the heavy-light mass limit, we have added the relativistic mass correction term \( V^1(r) \), which was initially derived by Y. Koma et al. [66], into the central potential. The final form of central potential is given by:

\[
V^0(r) = V_{C+L}^{0}(r) + V^1(r) \left( \frac{1}{m_1} + \frac{1}{m_2} \right) + \mathcal{O} \left( \frac{1}{m^2} \right)
\]  

(3)

The non-perturbative form of relativistic mass correction term \( V^1(r) \) is not yet known but leading order perturbation theory yields

\[
V^1(r) = -\frac{C_F C_A z_s^2}{4} \frac{r^2}{r^2}
\]  

(4)

In this equation, \( C_F = \frac{4}{3} \) and \( C_A = 3 \) are the Casimir charges of the fundamental and adjoint representations, respectively, as defined by [66]. When applied to the charmonium, it is discovered that the relativistic mass correction is identical to the coulombic component of the static potential, and that it is one-fourth of the coulombic term when applied to the bottomonium [67, 68]. We have looked the effects of relativistic mass correction term in all bottom and hidden bottom tetraquarks state which is just of a few MeV [69]. So that, we have removed the relativistic effects from tetra quarks mass-spectra and can conclude that its effect is found to be negligible in heavier systems.

In addition to the central interaction potential \( V^0(r) \), we have incorporated spin-dependent interactions (spin-spin \( V_{SS} \), spin-orbit \( V_{LS} \), and tensor \( V_T \)) into our model. All of these spin-dependent components are included in the model perturbatively.

2.1. Spin-dependent terms

In order to gain a better understanding of the splitting between orbital and radial excitations for different combinations of quantum numbers of \( cc \) and \( T_{\ell \ell} \), it is necessary to incorporate the contributions from different spin-dependent terms, namely spin-spin \( V_{SS} \), spin-orbit \( V_{LS} \), and tensor \( V_T \), which all contribute significantly, especially in excited states [70]. All the three spin dependent terms are inspired with the Breit-Fermi Hamiltonian for one-gluon exchange [71, 72], and yields;

\[
V_{SS}(r) = C_{SS}(r) S_1 \cdot S_2,
\]  

(5)

\[
V_{LS}(r) = C_{LS}(r) L \cdot S,
\]  

(6)

\[
V_T(r) = C_T(r) S_{12}
\]  

(7)

The matrix element of operator \( S_1 \cdot S_2 \) acts on wave function and it generates a constant factor, still the \( V_{SS} \) is a function of \( r \) only and the expectation value of operator \( \langle S_1 \cdot S_2 \rangle \) can be obtained by using quantum - mechanical formula.

\[
\langle S_1 \cdot S_2 \rangle = \left\langle \frac{1}{2} (S^2 - S_1^2 - S_2^2) \right\rangle
\]  

(8)

where \( S_1 \) and \( S_2 \) are the spins of constituent quarks in case of charmonium and diquarks in case of tetraquark respectively. \( C_{SS}(r) \) may be defined by:

\[
C_{SS}(r) = \frac{2}{3m^2} \nabla^2 V_{SS}(r) = -\frac{8k_s z_s \pi}{3m^2} \delta^3(r),
\]  

(9)

In heavy quarkonium spectroscopy a good agreement between theoretically predicted states and experimental data available for \( cc \) can be obtained by introducing a new parameter \( \sigma \) Gaussian function in place of the Dirac delta. So now \( V_{SS} \) can be redefined as:

\[
V_{SS}(r) = -\frac{8\pi k_s z_s}{3m^2} \frac{\sigma}{\sqrt{\pi}} \exp^{-\sigma^2 r^2} S_1 \cdot S_2,
\]  

(10)

The expectation value of operator \( \langle L \cdot S \rangle \) is mainly dependent on the total angular momentum \( J \) which consists of \( J = L + S \), and can be obtained by using the formula,

\[
\langle L \cdot S \rangle = \left\langle \frac{1}{2} (J^2 - L^2 - S^2) \right\rangle
\]  

(11)

where \( L \) is the total orbital angular momentum of quarks and diquarks in the case of charmonium and tetraquark respectively. \( C_{LS}(r) \) can be calculated by using relation given below:

\[
C_{LS}(r) = -\frac{3k_s z_s \pi}{2m^2} \frac{1}{r^2} - \frac{b}{2m^2} \frac{1}{r}
\]  

(12)

In the spin-orbit interaction the second term is known as Thomas precession which is proportional to scalar term and it is assumed that confining interaction arises from the Lorentz scalar structure. The contribution of spin-tensor becomes very crucial in higher excited states which requires a bit of algebra and can be calculated by;

\[
V_T(r) = C_T(r) \left( \frac{(S_1 \cdot r)(S_2 \cdot r)}{r^2} - \frac{1}{3} (S_1 \cdot S_2) \right)
\]  

(13)
The results of \( S_1 \cdot S_2 \) can be obtained by solving the diagonal matrix elements for the spin \( \frac{1}{2} \) and spin-1 particles, for more one can see in the following Refs. [72, 73]. The simplified formulation can be expressed to solve the tensor interaction:

\[
S_{12} = 12 \left( \frac{(S_1 \cdot r)(S_2 \cdot r)}{r^2} - \frac{1}{3} (S_1 \cdot S_2) \right)
\]  

and which can be redefined as:

\[
S_{12} = 4 \left[ 3(S_1 \cdot \hat{r})(S_2 \cdot \hat{r}) - (S_1 \cdot S_2) \right]
\]

The results of \( S_{12} \) term can be obtained with the help of Pauli matrices and spherical harmonics with their corresponding eigenvalues. The following results are valid for charmonium and diquarks are obtained as follows can be found more on [73];

\[
\langle S_{12} \rangle_{\frac{1}{2}, \frac{1}{2}, S=1, l \neq 0} = -\frac{2l}{2l+3}, \text{ for } J = l + 1,
\]

\[
= -\frac{2(l+1)}{(2l-1)}, \text{ for } J = l - 1,
\]

\[
= +2, \text{ for } J = l
\]

For \( l = 0 \) and \( S = 0 \) the \( \langle S_{12} \rangle \) always vanishes but it gives non-zero values for excited states in charmonium states. Eq. (171819) can be generalised as \( \langle S_{12} \rangle = -\frac{2}{3}, +2, -4 \) for \( J = 2, 1, 0 \) respectively. These values are valid only in the case of spin-half particles specially used in charmonium and diquarks, but in the case of tetraquarks where spin-1 diquarks are involved and it requires tedious algebra which we will not discuss here in detail rather we use those results presented in Refs. [70, 73].

The tensor interaction of \( T_{dc} \) will be obtained by the same formula which is used in case of charmonium except that the wavefunction obtained here will be of spin 1 (anti)diquark.

\[
S_{d-d} = 12 \left( \frac{(S_d \cdot r)(S_d \cdot r)}{r^2} - \frac{1}{3} (S_d \cdot S_d) \right)
\]

\[
= S_{14} + S_{13} + S_{24} + S_{23}
\]

where \( S_d \) is the total spin of the diquark, \( S_d \) is the total spin of the antiquark. The interaction between the two quarks within the diquark (whose indices are (1) and (2)) is taken into consideration while solving the 2-body problem to derive the masses of the diquark (and since we consider diquarks only in S-wave state, only the spin-spin interaction is relevant; the spin-orbit and tensor are identically zero, as we have already discussed). Inside the antiquark, the interaction between the two antiquarks (with indices of 3 and 4) is believed to be identical. Because the tetraquark radial wavefunction is produced by treating the diquark and antiquidiquark as two-body systems similar to mesons, it is plausible to infer that the tensor term’s radial dependency equals the sum of four tensor interactions between each quark-antiquark pair \( [q - \bar{q}] \), and may be obtained using the radial wavefunction with Eq. (21).

Since the tetraquark is treated as a two-body system, the expectation value of the radial wavefunction between every \( qq \) pair is the same and may be factorised. The following functional expression for spin-\( \frac{1}{2} \) particles relies on generic angular momentum elementary theory [74] instead of any particular relation or eigenvalues. Within this approximation, a total of four tensor interactions between four quark-antiquark pairs may be considered, as shown in Eqs. 20 and 21. In Ref. [73], has a comprehensive treatment of tensor interaction, which can be referred. To observe the separate contributions, all spin-dependent (spin-spin, spin-orbital, and tensor) interactions have been addressed perturbatively in the model.

2.2. Decay constant

The square modulus of the wave function at the origin \( |\psi(0)|^2 \) is an essential quantity; to determine the decay widths, wavefunction or derivative of wavefunction at the origin may be utilised [70]. The centrifugal term in the Schrödinger equation causes a “centrifugal barrier” in quarkonium models, and as a result, the wavefunction at the origin can only be determined for \( l = 0 \), i.e. S-states, and it disappears in the excited states for \( l \neq 0 \).

\[
|\psi(0)|^2 = |Y_{l}^{m} (\theta, \phi)R_{nl}(0)|^2 = \frac{|R_{nl}(0)|^2}{4\pi}
\]

Numerical calculations can be used to determine the square modulus of a radial wavefunction at the origin \( |R_{nl}(0)|^2 \). The values acquired for pseudoscalar and vector charmonium states will be important in determining the decay constants, which is another crucial component in understanding the decay characteristics. The author of Ref. [75] looked into the decay constants of light-heavy mesons using a relativistic approach. Using the radial wave function for pseudoscalar and vector charmonium states, we determine the decay constants in a non-relativistic framework. The formula is given by Van-Royen-Weisskopf [76, 77];

\[
\Gamma_{\text{rad}}^{2} = \frac{3|M_{nP/V}(0)|^2}{\pi M_{nP/V}}
\]

where \( |M_{nP/V}(0)|^2 \) and \( M_{nP/V} \) are wave-function at the origin and mass of the pseudoscalar and vector states respectively.
3. Results and discussion

3.1. Charmonium

To compute the mass-spectra of diquarks and tetraquarks, we would first calculate the mass-spectra of the charmonium states $c\bar{c}$, the findings of which are shown in Table 1. The model’s reliability and consistency were evaluated by deriving mass-spectra and decay constants for the charmonium states, which were found to be extremely similar to experimental data in a recent PDG [65]. The SU(3) colour symmetry enables only colourless quark combinations |$q\bar{q}$| to produce any colour singlet state [34, 78], as in our instance, $c\bar{c}$ is a meson and exhibits |$q\bar{q}$| : $3 \otimes \bar{3} = 1 \oplus 8$ representation, resulting in a colour factor $k_c = -\frac{4}{3}$ [73, 78]. The Schrödinger equation can be used to compute the masses of certain $c\bar{c}$ states. Therefore:

$$M_{c\bar{c}} = 2m_c + \langle E^{(0)}_{c\bar{c}} \rangle + \langle V^1(r) \rangle$$

(24)

The final mass $M_f$ obtained from the following calculation includes spin-dependent components (spin-spin, spin-orbital, and spin-tensor), as well as the relativistic-mass correction term. Table 1 additionally includes $M^{exp}$ for comparing the current model mass with experimental data [65].

Parameters (adopted from PDG [65]) used in the present work are $m_c = 1.4459$ GeV, $\alpha_s = 0.5202$, $\beta = 0.1463$ GeV$^2$, and $\sigma = 1.0831$ GeV. The quantum numbers $j_{PC}$ of $q\bar{q}$ states i.e. parity and charge conjugation can be calculated by: $P = (-1)^{L+1}$ and $C = (-1)^{L+S}$ respectively, where

| Table 1 Mass-Spectra of charmonium states $c\bar{c}$ along with the relativistic mass correction. (Units are in MeV) |
|---------------------------------------|-------|-------|-------|-------|-------|-------|-------|
| $N^{2S+1}L_J$ | $J^{PC}$ | $(K.E.)$ | $(E^{(0)}_{c\bar{c}})$ | $(V^{(0)}_{t_{c\bar{c}}})$ | $(V^{(0)}_{\psi})$ | $(V^{(1)}_{F})$ | $(V^{(1)}_{F})$ | $M_f$ | $M^{exp}$ [65] | Meson |
| 1$^1S_0$ | 0$^+$ | 470.3 | 134.9 | -537.9 | 271.1 | -69.4 | 0 | 0 | -4.6 | 2983 | 2983.4±0.5 | $\eta_c$ (1S) |
| 1$^3S_1$ | 1$^−$ | 377.1 | 134.2 | -537.9 | 271.1 | 23.1 | 0 | 0 | -4.7 | 3075 | 3096.9±0.006 | $J/\psi$ (1S) |
| 2$^1S_0$ | 0$^+$ | 464.5 | 734.5 | -290.7 | 589.8 | -29.0 | 0 | 0 | -2.4 | 3623 | 3639.2±1.2 | $\eta_c$ (2S) |
| 2$^3S_1$ | 1$^−$ | 428.1 | 736.8 | -290.7 | 589.7 | 9.6 | 0 | 0 | -2.5 | 3664 | 3686.097±0.025 | $J/\psi$ (2S) |
| 3$^1S_0$ | 0$^+$ | 548.4 | 1148.2 | -220.7 | 840.7 | -20.1 | 0 | 0 | -1.7 | 4046 | - | |
| 3$^3S_1$ | 1$^−$ | 522.2 | 1148.9 | -220.7 | 840.7 | 6.7 | 0 | 0 | -1.6 | 4073 | 4039±1 | $\psi$ (4040) |
| 4$^1S_0$ | 0$^+$ | 635.7 | 1493.3 | -184.7 | 1058.2 | -15.9 | 0 | 0 | -1.2 | 4395 | - | |
| 4$^3S_1$ | 1$^−$ | 614.8 | 1493.7 | -184.7 | 1058.3 | 5.3 | 0 | 0 | -1.3 | 4417 | 4421±4 | $\psi$ (4415) |
| 1$^3P_0$ | 0$^+$ | 362.6 | 591.3 | -254.4 | 480.8 | 2.2 | -69.4 | -31.4 | -2.9 | 3410 | 3414.75±0.31 | $\zeta_0$ (1P) |
| 1$^3P_1$ | 1$^+$ | 362.5 | 591.2 | -254.4 | 480.8 | 2.2 | -34.7 | 15.7 | -3.1 | 3492 | 3510.66±0.07 | $\zeta_1$ (1P) |
| 1$^3P_2$ | 2$^+$ | 362.6 | 591.3 | -254.4 | 480.8 | 2.2 | 34.7 | -3.1 | -2.9 | 3543 | 3556.20±0.09 | $\zeta_2$ (1P) |
| 2$^3P_0$ | 0$^+$ | 463.3 | 1019.0 | -191.8 | 745.2 | 2.3 | -64.8 | -27.9 | -1.6 | 3846 | - | |
| 2$^3P_1$ | 1$^+$ | 464.5 | 1020.3 | -191.8 | 745.2 | 2.3 | -32.4 | 13.9 | -2.0 | 3922 | - | |
| 2$^3P_2$ | 2$^+$ | 473.6 | 1019.9 | -191.8 | 745.2 | -7.0 | 0 | 0 | -2.0 | 3930 | - | |
| 3$^3P_0$ | 0$^+$ | 561.2 | 1374.3 | -160.2 | 971.0 | 2.3 | -62.9 | -26.2 | -1.4 | 4205 | - | |
| 3$^3P_1$ | 1$^+$ | 561.8 | 1374.9 | -160.2 | 971.0 | 2.3 | -31.4 | 13.1 | -1.3 | 4276 | - | |
| 3$^3P_2$ | 2$^+$ | 571.1 | 1375.0 | -160.2 | 971.0 | -6.9 | 0 | 0 | -1.4 | 4286 | - | |
| 1$^1D_1$ | 1$^−$ | 409.3 | 873.7 | -179.7 | 643.9 | 0.2 | -9.7 | -4.0 | -2.3 | 3778 | 3773±13.35 | $\psi$ (3770) |
| 1$^3D_1$ | 2$^−$ | 409.2 | 873.6 | -179.7 | 643.9 | 0.2 | -3.2 | 4.0 | -2.3 | 3792 | - | |
| 1$^1D_2$ | 2$^+$ | 410.1 | 873.7 | -179.7 | 643.9 | -0.06 | 0 | 0 | -2.3 | 3791 | - | |
| 1$^3D_3$ | 3$^−$ | 409.2 | 873.7 | -179.7 | 643.9 | 0.2 | 6.5 | -1.1 | -2.3 | 3797 | - | |
| 2$^1D_1$ | 1$^−$ | 512.2 | 1242.3 | -149.4 | 879.1 | 0.3 | -12.6 | -3.8 | -1.6 | 4144 | 4191±5 | $\psi$ (4160) |
| 2$^3D_1$ | 2$^−$ | 512.2 | 1242.3 | -149.4 | 879.1 | 0.3 | -4.2 | 3.8 | -1.5 | 4160 | - | |
| 2$^1D_2$ | 2$^+$ | 513.7 | 1242.4 | -149.4 | 879.1 | -0.1 | 0 | 0 | -1.6 | 4159 | - | |
| 2$^3D_3$ | 3$−$ | 512.2 | 1242.3 | -149.4 | 879.1 | 0.3 | 8.4 | -0.1 | -1.6 | 4167 | - | |
L is the orbital angular momentum and S being the total spin of quarks.

Table 1 shows the masses of charmonium S-wave states, which are in excellent agreement with experimental data. The final mass $M_f$ is influenced by all three spin-dependent terms (spin-spin, spin-orbital, and tensor), as well as the relativistic correction term. The coulomb term $\langle V_C(0) \rangle$ has a stronger attractive strength, indicating that one gluon exchange (OGE) prevails over other interactions, suppressing the masses of these states. The spin-spin interaction is attractive for pseudoscalar states and repulsive for vector states in all radially excited states; however, it diminishes in higher orbitally excited states. The spin-spin interaction $\langle V_S(0) \rangle$ and the relativistic $\langle V(1) \rangle$ correction provide the highest attractive strength at the lowest S-wave states, leading in a mass very comparable to the empirically observed $\eta_c(1S)$ meson. The spin-tensor and spin-orbital interactions simply vanish in S-wave states, as well as higher orbital excited states, when the total spin of the quark and antiquark is zero, e.g. $1^1P_1$, $1^3D_2$, $1^1F_3$, and so on. The $J/\psi(1S)$ meson’s mass in our model is 20 MeV lower than the observed mass [65]. Similarly, the masses of radially excited S-wave states $\eta_c(2S)$, $\psi(2S)$, $\psi(4040)$, and $\psi(4415)$ match the measured mass in [65]. The masses of $\chi_{c0}(1P)$, $\chi_{c1}(1P)$, $h_{c}(1P)$, and $\chi_{c2}(1P)$ in higher orbital excited states, notably in (1P) wave, vary by around 10 - 15 MeV, which is due to the distinct set of parameters we used in our model. Our model’s masses for two D-wave states, $\psi(3770)$ and $\psi(4160)$, match experimentally observed mesons.

We observe that in total spin-1 states, the uncertainty in the masses of $c\bar{c}$ states is larger (although only by a few MeV) than in total spin-0 quark states. The radial wavefunction at the origin, as well as the decay constants for pseudoscalar and vector charmonium states $c\bar{c}$, are shown in Table 2. We can observe that the wavefunction values for pseudoscalar and vector states vary only slightly due to the relativistic correction component contained in the central potential. Table 2 also includes the findings of our model’s decay constants for $c\bar{c}$ states, as well as experimental data and other prior works for comparison. Our non-relativistic results for pseudoscalar and vector state decay constants are quite similar to those obtained using a full relativistic method. Some experimentally observed S-wave mesons, such as the $[79]$, have masses in the range of 3–4 GeV, resulting in identical quantum numbers $1^{--}$, which has caused a lot of misunderstanding. We’ve obtained the decay constants since decay characteristics are the instrument that enables us to distinguish states.

3.2. Diquarks

A (anti)diquark is a confined state in which two (anti)quarks interact by one gluon exchange [82, 83]. Using the same approach as for charmonium, we would determine the mass-spectra of (anti)diquarks by solving the Schrödinger equation. According to QCD colour symmetry, when two quarks are combined in the basic (3) representation, we obtain $3 \otimes 3 = 3 \oplus 6$. In the 3 form, antiquarks are also combined, resulting in antidiquark in the (3) representation [64, 78]. These QCD colour symmetry produces a colour
factor $k_s = -\frac{2}{3}$ in the antitriplet state, which makes the short distance component of the interaction ($\frac{2}{3}$) attractive [73]. The diquark’s colour wave function is antisymmetric since we choose to deal with an attractive antitriplet colour state. The diquarks’ spin should be 1 to obey the Pauli exclusion principle, leading in an antisymmetric diquark [64] wavefunction. To get the most compact diquark, the ground state ($1^3S_1$) diquark [cc] with no orbital or radial excitations will be employed. (For additional information on diquarks, see [48, 84] and the references therein.) Note that everything done for quark-quark interactions is identical to what is done for antiquark-antiquark interactions; the only difference is that colours are replaced by anti-colours. Because of the SU(3) colour symmetry rule of combination, the resultant antiquark-antiquark configurations are complementary to their quark-quark counterparts. The diquark-antidiquark structure be used to create a colour singlet if treated as a bound structure (quark-quark as diquark and antiquark-antiquark as antidiquark). As a result, the diquark-antidiquark interaction (which is considered as a 2-body problem) is analogous to the quark-antiquark interaction, and the colour factor required to produce the colour singlet is the same as that required to produce a conventional meson. To summarise, there are two particularly noteworthy situations involving the four quarks we have seen: the singlet colour factor (which is utilised in the ordinary meson potential and will be used in our diquark-antidiquark interaction), and the antitriplet colour factor, which is the attractive (and dominating) colour interaction between quark-quark and antiquark-antiquark which we will use to construct the diquarks and antidiquarks [33, 38]. The masses of diquarks are calculated in the same way as the mass spectrum of charmonium is calculated as mentioned in Table 3.

3.3. Tetraquark

A $T_{dc}$ colour singlet state is comprised of two body system diquark-antidiquark pairs in antitriplet and triplet colour configurations, yielding a colour factor of $k_s = -\frac{4}{3}$ [34, 78]. These spin-1 diquark-antidiquark combined together to form colour singlet tetraquark state [82] and that can be represented as, $|O\bar{O}^3 \otimes \bar{O}O^3\rangle = 1 \oplus 8$. To compute the masses of the four body system, we utilised the masses of diquark-antidiquark ($m_{cc} = m_{\bar{c}\bar{c}}$) in the $1^3S_1$ state as two inputs. The mass-spectra of all charm tetraquarks ($cc - \bar{c}\bar{c}$) were derived using the same formulation as the mass-spectra of charmonium $cc$ and (anti)diquark.

$$M_{(T_{dc})} = m_{cc} + m_{\bar{c}\bar{c}} + \langle E_{(cc)\bar{(c\bar{c})}} \rangle$$  

(25)

Where $m_{cc}$ and $m_{\bar{c}\bar{c}}$ are the masses of diquark and antidiquarks respectively, whereas $\langle E_{(cc)\bar{(c\bar{c})}} \rangle$ represents the binding energy of the tetraquarks. The masses of $T_{dc}$ obtained by using above expression which is influenced by the Cornell potential. The contributions of all spin-dependent interactions (spin-spin, spin-orbital, and spin-tensor) were calculated separately, but their contribution are included in the final mass. All spin-dependent interactions have been computed for spin-1 diquarks and spin-1 antidiquarks that couple to form a colour singlet tetraquark with spin $S_T = 0, 1, 2$. The mass-spectrum of radial and orbital excitation is obtained by coupling the total spin $S_T$ and total orbital angular momentum $L_T$, which gives rise to the total angular momentum $J_T$. To calculate the tetraquark’s total spin $S_T$ and quantum number $J^{PC}$, we’ll use diquark spin $S_d$ and antidiquark spin $S_{\bar{d}}$ notations respectively. By coupling $S_T$ with the orbital angular momentum $L_T$, the colour singlet state $S_T \otimes L_T$ is generated.

$$|T_{dq}\rangle = |S_d, S_{\bar{d}}, S_T, L_T\rangle_{J_T}$$  

(26)

where $J_T$ is the total angular momentum of the tetraquark, which is obtained by $S_T \otimes L_T$. To find out the $J^{PC}$ quantum numbers of the tetraquark states, we will obtain the results of charge-conjugation and parity from the given relations [64]:

$$C_T = (-1)^{L_T+S_T}, P_T = (-1)^{L_T}.$$  

(27)

Parameters are $m_{cc} = 3.124$ GeV, $\alpha = 0.5304$, $b = 0.1480$ GeV$^2$, and $\sigma = 1.048$ GeV. In the case of charmonium ($cc$) and diquark ($cc$), there were only two spin ($S_T$) combinations, however in the case of tetraquark states $T_{dc}$, there are three spin combinations because spin-1 diquark-antidiquark are used.

$$|0^{++}\rangle_{T_{dc}} = |S_{cc} = 1, S_{\bar{c}\bar{c}} = 1, S_T = 0, L_T = 0\rangle_{J_T = 0};$$  

(28)

$$|1^{+-}\rangle_{T_{dc}} = |S_{cc} = 1, S_{\bar{c}\bar{c}} = 1, S_T = 1, L_T = 0\rangle_{J_T = 1};$$  

(29)

$$|2^{++}\rangle_{T_{dc}} = |S_{cc} = 1, S_{\bar{c}\bar{c}} = 1, S_T = 2, L_T = 0\rangle_{J_T = 2}.$$  

(30)

Results for all $T_{dc}$ masses identified in this study, as well as comparisons with the two meson thresholds, are summarised in Table 4. According to our observations, the coulombic interaction between two spin-1 (anti)diquarks dominates all other interactions. This is further evidence of the OGE mechanism’s dominance in explaining the strong diquark-antidiquark binding. $\langle V_C^{(0)} \rangle$ is much stronger in S-wave states, which suppresses mass below threshold by a significant margin.

The spin-spin interaction is reduced for orbital excited states owing to the increase in the root mean square radius and the presence of a regularised delta function in the spin-spin interaction. Because the regularisation is far from a
real delta function, the spin-spin contribution decreases as the radius increases. Despite the fact that the spin-spin interaction includes a momentum term, which implies that the strength of this interaction should decrease rather than increase in the case of the tetraquark $T_{4c}$, this finding is surprising. A tetraquark’s colour interaction leads the
diquark and antidiquark to interact significantly, unlike charmonia, where the charm and anticharm interact.

The \( T_{ac} \) states have been primarily characterised by the \( J/\psi J/\psi, \eta_c, \eta_c \), or the admixture of \( S \) and \( P \)-wave charmonium like states [34]. The mass-spectra obtained in the present study are also consistent with the two meson-thresholds \( J/\psi J/\psi, \eta_c, \eta_c \), etc., and also earlier described work in Table 5. In his QCD sum rule study [29], the author proposed that the mass of \( S \)-wave \([ cc][cc] \) would be about 6.5 GeV. The author of Ref. [85] indicates that the author proposed that the mass of \( S \)-wave \( JPC \).

Comparison of all charm tetraquark masses (MeV) with other prior works

Table 5

| \( J^{PC} \) | Ours [31] | [32] | [33] | [34] | [38] | [43] | [35] | [53] | [86] | [87] | [88] | [89] | [90] | [92] | [93] |
|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| 0++  | 5942  | 5990  | 6190  | 5960  | 5969  | 5883  | 6044  | 5960  | 6351  | 5966  | 6437  | 6518  | 6346  | 6466  | 6055^{+69}_{-74}  |
| 1+-  | 5989  | 6271  | 6009  | 6021  | 6120  | 6230  | 6009  | 6441  | 6051  | –     | 6500  | 6441  | 6494  | –     | 6050  |
| 2++  | 6082  | 6090  | 6367  | 6100  | 6115  | 6246  | 6287  | 6100  | 6471  | 6223  | –     | 6524  | 6574  | 6551  | 6090^{+62}_{-66}  |
| 1–   | 6555  | –     | 6631  | –     | 6577  | –     | –     | –     | 6718  | –     | –     | –     | –     | –     | –     |
| 0–   | 6462  | –     | 6628  | –     | 6480  | –     | –     | –     | 6718  | –     | –     | –     | –     | –     | –     |
| 1–   | 6556  | –     | 6634  | –     | 6577  | –     | –     | –     | –     | –     | –     | –     | –     | –     | –     |
| 2–   | 6589  | –     | 6644  | –     | 6609  | –     | –     | –     | –     | –     | –     | –     | –     | –     | –     |
| 3–   | 6625  | –     | 6664  | –     | 6641  | –     | –     | –     | –     | –     | –     | –     | –     | –     | –     |

Using the di-\( J/\psi \) invariant mass spectrum, the LHCb collaboration found a broad structure in the mass range of 6.2 GeV–6.8 GeV, including a narrow X(6900) state with a signal significance of more than 5\( \sigma \) [50]. Tetraquark X(6900), predicted in Ref. [92], is to exist in 3S-waves with quantum numbers \( (0^+, 2^+) \), along with another potential candidates of 2P-waves are permitted to be \( (0^+, 1^-) \), and \( (2^+) \). In our current model, 3S-wave states have a mass range of 100 MeV above X(6900) for quantum numbers \( 0^+, 1^- \), and \( 2^+ \), respectively. When compared to X(6900), the masses of the 2P wave states with quantum numbers \( 0^+, 1^-, 1^+ \) and \( 2^+ \) are have uncertainty of only 25 MeV to 50 MeV with X(6900) and also near to the two meson threshold, as shown in Table 4. As mass discrepancy is lower in P-waves than S-wave, the probability of it being in the 2P-wave is greater. The computed masses of 1S-wave are below di-\( J/\psi \) and di-\( \eta_c \) meson threshold whereas the masses of higher orbital and radial excited states are above the di-\( J/\psi \), di-\( \eta_c \) and other meson thresholds, which suggest that it their strong decays into these two channels, as well as numerous additional di-charmonia channels, are possible [29].
antidiquarks suggests that these states are likely to be compact resonance states. It is conceivable to describe the reported state X(6900) as a compact resonance state with \( J^P = 0^+ \) [44, 45]. In Ref. [95], the author predicts that the peak at 6.9 GeV corresponds to a \( 2^{++} \) resonance in terms of both production rate and overall breadth, but nothing can be determined until a direct spin-parity determination is obtained. If the latter is actually the 2S-wave state, it is envisaged that there will be further states below and above 6.9 GeV. Alternatively, it is possible that 1S-wave states are below threshold and that we are observing the spectrum of radial and orbital excitations.

In addition, since it is around 700 MeV over the \( J/\psi \) threshold, it is unlikely to represent a ground state of totally charmed tetraquarks X(6900). In fact, numerous studies have anticipated that the ground state of a completely charmed tetraquark would be roughly 6.00 GeV or even below to this [35, 85, 88, 92, 96]. To explain X(6900), some of them assigned it as a radially or orbitally excited state, such as Refs. [85, 92]. The lack of bound states predicted by full quark-model calculations, which are constrained to very specific configurations. These findings hold true for both lattice QCD techniques [97, 98], and constituent models [99, 100], as long as no further constraints are imposed by Pauli. All-heavy tetraquark bound states are not found in these investigations, although resonances are not ruled out as a possibility either.

4. Conclusions

We examined a model for all charm tetraquarks \( T_{cc} \) that is considered to be compact and consists of diquark-antidiquark \([cc][c\bar{c}]\) pairings in an antitriplet \( (3\bar{c})\)-triplet \( (3c)\) colour configuration respectively. We calculated the mass-spectra of tetraquarks using the Cornell inspired potential. Spin-dependent terms (spin-spin, spin-orbital, and tensor) were included to examine the splitting between states with different quantum numbers. We begin by formulating the charmonia \([cc]\) model and obtained the most recent data for mesons, which we used to fit the model’s parameters. We next determined the masses of vector diquarks, which are the composite states of the tetraquark. The tetraquark masses derived from our model correlate well with those reported from prior studies mentioned in Table 5.

The LHCb Collaboration, recently discovered an all-charm tetraquark mass of roughly 6.9 GeV in the \( J/\psi \) invariant mass spectrum, indicating that more resonance structures in the 6.2–6.8 GeV range are possibly feasible [50]. In Ref. [91] Waang et al. describes all-charm tetraquark structures in the invariant mass spectrum of \( J/\psi(1S) - \psi(3686), J/\psi(1S) - \psi(3770), \psi(3686) - \psi(3686). \) Other all-charm tetraquark structures that may be identified in the channels include \( J/\psi(1S) - J/\psi(1S), \eta_c(1S) - \eta_c(1S) \) and \( J/\psi(1S) - \eta_c(1S), \) implying that alternate structures could be found. Similarly, the masses of 1P and 2P wave states are in the range of 6.5 to 6.9 GeV, and this mass range supports the good agreement between our results and experimentally observed states. The mass range of 1P wave (6.4–6.6 GeV) could also be possible due to admixture of S-wave and P-wave charmonium states like \( \eta_c(1S) - \eta_c(1S), \eta_c(1S) - \eta_c(1S), \eta_c(1S) - \eta_c(1S), J/\psi(1S) - \eta_c(1S), J/\psi(1S) - \eta_c(1S) \).

In Ref. [92], the X(6900) can be interpreted as a radially excited state (3S-wave) \( 0^{++} \) and \( 2^{++} \) or an orbitally excited states (2P-wave) state with quantum numbers \( 0^{-+}, 1^{-+}, 2^{-+} \). As described in the previous section, discrete combinations of parity (P) and charge conjugation (C) are suggested by the quantum numbers \( (J^P)^C \) predicted by various preceding publications and mentioned therein. We conclude that the foregoing theoretical predictions that X(6900), which has a mass range of 6.2–6.9 GeV, could be identified as radial excited 3S-wave states for \( 0^{++} \) and \( 2^{++} \) quantum states. But mass disparity from 3S-wave states are nearly 100 MeV than X(6900) reduces the possibility of S-wave states. Whereas, it may be described as orbitally excited state, i.e. 2P-wave state with mass-disparity of less than 50 MeV enhances the probability of being orbitally excited states for quantum numbers \( 0^{-+}, 1^{-+}, 2^{-+} \). Other possible structures of \([cc\bar{c}\bar{c}]\) state, could be accommodated in future as 2S-wave or 1P-wave states in the broad range of 6.2–6.9 GeV, predicted by LHCb.

All of these findings are quantitatively compatible with the LHCb’s experimental data for the broad structure, which ranges from 6.2–6.8 GeV for the broad structure and might support its internal structure as \([cc\bar{c}\bar{c}]\) tetraquark state, among other possibilities. It is anticipated that more experimental and theoretical studies will disclose more about the nature of exotic states in the future. If a tetraquark state’s energy is lower than all possible two-meson thresholds, it should be stable against the strong interaction. If the energy of the tetraquark is above its threshold, strong interactions can break it into two mesons. Strong decay into two mesons is forbidden if it falls below the threshold, hence decay must take place via weak or electromagnetic interactions. Our findings, along with those obtained in recent literature on the \( T_{cc} \) tetraquark, should motivate a comprehensive experimental search for these states at the LHCb [54] and Belle II [57]. Our findings point to a broad spectrum of \( T_{cc} \) states in the charm sector, which will require further investigation at different experimental facilities. Many other tetraquark possibilities with light quarks (u,d,s) configurations, including charm, will be
scanned at experimental facilities like PANDA at FAIR in the near future [101, 102].

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