The asymptotic behaviour of the weights and the degrees in an $N$-interactions random graph model

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Abstract

A random graph evolution based on the interactions of $N$ vertices is studied. During the evolution both the preferential attachment method and the uniform choice of vertices are allowed. The weight of a vertex means the number of its interactions. The asymptotic behaviour of the weight and the degree of a fixed vertex, moreover the limit of the maximal weight and the maximal degree are described. The proofs are based on martingale methods.

1 Introduction

Network theory is one of the most popular research topics. During the last two decades, many types of networks were investigated. Empirical studies show that several real-world networks have certain similar features. For an overview of random graph models and their properties see [1] or [6]. It is known that many real life networks (the WWW, biological and social networks) are scale-free, see [1]. To describe the evolution of such networks, in [2] the preferential attachment model was suggested.

The scale-free property means that the asymptotic degree distribution follows a power law. Besides the degree distribution, other characteristics are worth to study. The degree of a fixed vertex and the maximal degree in preferential attachment models were investigated (see [6]). In [10], the maximum degree in a general random tree model was examined, which includes the Barabási-Albert random tree as a special case. In [11] and [12], the asymptotic behaviour of the degree of a given vertex and the maximal degree was studied in a 2-parameter scale-free random graph model. A well-known technique to analyze the growth of the maximal degree is Móri’s martingale method ([1], [9]).

There are several modifications of the preferential attachment model (see [6], [1], [5]). A random graph model based on the interactions of three vertices was introduced and power law

_\textbf{Key words and phrases:} Random graph, preferential attachment, scale-free, power law, submartingale, Doob-Meyer decomposition._

_\textbf{Mathematics Subject Classification:} 05C80, 60G42._

István Fazekas was supported by the TÁMOP-4.2.2.C-11/1/KONV-2012-0001 project. The project has been supported by the European Union, co-financed by the European Social Fund.

Bettina Porvázsnik was supported by the European Union and the State of Hungary, co-financed by the European Social Fund in the framework of TÁMOP 4.2.4.A/2-11-1-2012-0001 ‘National Excellence Program’.
degree distribution in that model was proved in [7] and [8]. Instead of the three-interactions model, the interactions of \( N \) vertices (\( N \geq 3 \) fixed) were studied in [18]. Scale-free properties for these generalized models were obtained in [17] and [18].

In this paper we extend some results of [7] and [8] to the \( N \)-interactions model. Our aim is to study the asymptotic behaviour of the weight and the degree of a fixed vertex. Moreover, we shall consider the limiting properties of the maximal weight and the maximal degree, as well. In our proofs we follow the lines of [7] and [8].

The \( N \)-interactions model

In this paper we study the following \( N \)-interactions model (see [18]). A complete graph with \( m \) vertices we call an \( m \)-clique, for short. We denote an \( m \)-clique by the symbol \( K_m \). At time \( n = 0 \) we start with a \( K_N \). The initial weight of this graph and the initial weight all of its cliques are one. After the initial step we start to increase the size of the graph. At each step, the evolution of the graph is based on the interaction of \( N \) vertices. More precisely, at each step \( n = 1, 2, \ldots \) we consider \( N \) vertices and draw all non-existing edges between these vertices. So we obtain a \( K_N \). The weight of this graph \( K_N \) and the weights of all cliques in \( K_N \) are increased by 1. The choice of the \( N \) interacting vertices is the following.

There are two possibilities at each step. With probability \( p \) we add a new vertex that interacts with \( N - 1 \) old vertices, on the other hand, with probability \((1 - p)\), \( N \) old vertices interact. Here \( 0 < p \leq 1 \) is fixed.

When we add a new vertex, then we choose \( N - 1 \) old vertices and they together will form an \( N \)-clique. However, to choose the \( N - 1 \) old vertices we have two possibilities. With probability \( r \) we choose an \((N - 1)\)-clique from the existing \((N - 1)\)-cliques according to the weights of the \((N - 1)\)-cliques. It means that an \((N - 1)\)-clique of weight \( w_i \) is chosen with probability \( w_i / \sum_h w_h \). On the other hand, with probability \( 1 - r \), we choose among the existing vertices uniformly, that is all \( N - 1 \) vertices have the same chance.

At a step when we do not add a new vertex, then \( N \) old vertices interact. As in the previous case, we have two possibilities. With probability \( q \), we choose one \( K_N \) of the existing \( N \)-cliques according to their weights. It means that an \( N \)-clique of weight \( w_i \) is chosen with probability \( w_i / \sum_h w_h \). On the other hand, with probability \( 1 - q \), we choose among the existing vertices uniformly, that is all subsets consisting of \( N \) vertices have the same chance.

In this paper we describe the asymptotic behaviour of the weight and the degree of a fixed vertex (Theorems 2.1 and 2.2), moreover we find the limit of the maximal weight and the maximal degree (Theorems 2.3 and 2.4). The theorems are listed in Section 2. All the proofs and some important auxiliary results are presented in Section 3.
2 Main results

Let us introduce the following notations.

\[
\alpha_1 = (1 - p)q, \quad \alpha_2 = \frac{N - 1}{N}pr, \quad \alpha = \alpha_1 + \alpha_2,
\]

\[
\beta_1 = (N - 1)(1 - r), \quad \beta_2 = \frac{N(1-p)(1-q)}{p}, \quad \beta = \beta_1 + \beta_2.
\]

At time \( n = 0 \) the initial complete graph on \( N \) vertices is symmetric in the sense that all of its \( N \) vertices have the same weight. Let one of these \( N \) vertices be labelled by 0. The other \( N - 1 \) vertices are not labelled. When a new vertex is born, let it be labelled by 1, 2, \ldots according the order in which they are added. Let \( j \geq 0 \) be fixed integer. Assume that the \( j \)th vertex exists after \( l \) steps, where \( 0 \leq j \leq l \leq n \). Let us denote by \( W[n, j] \) the weight of the \( j \)th vertex after the \( n \)th step. Let us denote by \( D[n, j] \) the degree of the \( j \)th vertex after \( n \) steps. (If \( n < j \), then \( W[n, j] = D[n, j] = 0 \).)

2.1 The weight and the degree of a fixed vertex

The following theorem describes the asymptotic behaviour of the weight of a fixed vertex. It is an extension of Theorem 4.1 in [7].

**Theorem 2.1.** Let \( j \geq 0 \) be fixed and let \( \alpha > 0 \). Then

\[
W[n, j] \sim \frac{1}{\Gamma(1+\alpha)} \gamma_j n^\alpha
\]

almost surely as \( n \to \infty \), where \( \gamma_j \) is a positive random variable.

We turn to the asymptotic behaviour of the degree of a fixed vertex. The following theorem is an extension of Theorem 5.2 in [8].

**Theorem 2.2.** Let \( j \geq 0 \) be fixed and let \( \alpha > 0 \). Then we have

\[
D[n, j] \sim \frac{1}{\Gamma(1+\alpha)} \frac{\alpha_2}{\alpha} \gamma_j n^\alpha
\]

almost surely as \( n \to \infty \), where the \( \gamma_j \) positive random variable is defined in (2.1).

2.2 The maximal weight and the maximal degree

In this subsection we use the following notations. At time \( n = 0 \), the vertices of the initial complete graph on \( N \) vertices are labelled by 0, \(-1, \ldots, -(N-1)\). Let us denote by \( W_n \) the maximum of the weights of the vertices after \( n \) steps, that is

\[
W_n = \max\{W[n, j] : -(N-1) \leq j \leq n\}.
\]

The following theorem is an extension of Theorem 5.1 in [8] for the \( N \)-interactions model.
Theorem 2.3. Let $\alpha > 0$. Then we have
\[ W_n \sim \frac{1}{\Gamma(1 + \alpha)} \mu n^\alpha \text{ almost surely, as } n \to \infty, \] (2.4)
where $\mu = \sup\{\gamma_j : j \geq -(N - 1)\}$ is a finite positive random variable with $\gamma_j$ defined in Theorem 2.1.

Let us denote by $D_n$ the maximal degree after $n$ steps, that is
\[ D_n = \max\{D[n, j] : -(N - 1) \leq j \leq n\}. \] (2.5)

The following theorem is an extension of Theorem 5.3 in [8].

Theorem 2.4. Let $\alpha > 0$. Then we have
\[ D_n \sim \frac{1}{\Gamma(1 + \alpha)} \frac{\alpha_2}{\alpha} \mu n^\alpha \text{ almost surely as } n \to \infty, \] (2.6)
where $\mu = \sup\{\gamma_j : j \geq -(N - 1)\}$ is the positive random variable defined in Theorem 2.3.

3 Proofs and auxiliary lemmas

Introduce the following notations. Let $F_{n-1}$ denote the $\sigma$-algebra of observable events after the $(n - 1)$th step. Let $j \geq 0$ be a fixed integer. $W[n, j]$ is the weight of the $j$th vertex after the $n$th step. (If $n < j$, then $W[n, j] = 0$.) Let $I[n, j]$ be the indicator of the event that the $j$th vertex exists after $n$ steps, that is
\[ I[n, j] = \begin{cases} 1, & \text{if } W[n, j] > 0, \\ 0, & \text{if } W[n, j] = 0. \end{cases} \]

Let $J[n, j]$ be the indicator of the event that the $j$th vertex is born at the $n$th step. Then $J[n, j] = I[n, j] - I[n - 1, j]$. For all $j, k, l$, $0 \leq j \leq l$, $1 \leq k$, fixed positive integers, we consider the following sequences:
\[ b[n, k] = \prod_{i=1}^{n} \left(1 + \frac{\alpha k}{i}\right)^{-1}, \] (3.7)
\[ d[n, k, j] = -\sum_{i=1}^{n-1} b[i + 1, k] \frac{\beta p}{V_i} \left(W[i, j] + k - 1\right), \] (3.8)
\[ e_n = \prod_{i=1}^{n} \left(1 - \frac{\alpha}{i}\right)^{-1}. \] (3.9)

Here we can see that the sequences $b[n, k]$ and $e_n$ are deterministic, while $d[n, k, j]$ is a sequence of $F_{n-1}$-measurable random variables for any $k$ and $j$. Using the definition of $b[n, k]$ and the Stirling-formula for the Gamma function, we can show that
\[ b[n, k] \sim b_k n^{-k\alpha}, \quad \text{as } n \to \infty, \] (3.10)
where $b_k = \Gamma (1 + \alpha k) > 0$, $k$ is fixed. Moreover, we can easily see that

$$e_n \sim \Gamma (1 - \alpha) n^\alpha, \quad \text{as } n \to \infty.$$ (3.11)

In the following lemma we introduce a martingale which will play important role in the proofs. This lemma is an analogue of Lemma 4.1 in [7] and Lemma 5.1 in [8].

**Lemma 3.1.** Let $j, k, l$, $0 \leq j \leq l$ be fixed nonnegative integers and let

$$Z[n, k, j] = \left( b[n, k] \left( \frac{W[n, j] + k - 1}{k} \right) + d[n, k, j] \right) I[l, j].$$ (3.12)

Then $(Z[n, k, j], \mathcal{F}_n)$ is a martingale for $n \geq l$.

**Proof.** At each step, the weight of a fixed vertex is increased by 1 if and only if it takes part in an interaction. As in [7], [17] and in [18], it is easy to show that the conditional probability that the $j$th vertex takes part in an interaction at step $(n + 1)$ is

$$\frac{W[n, j]}{n + 1} \alpha + \frac{p}{V_n} \beta,$$ (3.13)

provided that $W[n, j] > 0$. Using this, we can see for $n \geq l$

$$\mathbb{E} \left\{ \left( \frac{W[n + 1, j] + k - 1}{k} \right) I[l, j] | \mathcal{F}_n \right\} =$$

$$= I[l, j] \left( 1 - \left( \frac{W[n, j]}{n + 1} \alpha + \frac{p}{V_n} \beta \right) \right) \left( \frac{W[n, j] + k - 1}{k} \right) + I[l, j] \left( \frac{W[n, j]}{n + 1} \alpha + \frac{p}{V_n} \beta \right) \left( \frac{W[n, j] + k}{k} \right) =$$

$$= I[l, j] \frac{p}{V_n} \beta \left( \frac{W[n, j] + k - 1}{k - 1} \right) + I[l, j] \left( 1 + \alpha \frac{k}{n + 1} \right) \left( \frac{W[n, j] + k - 1}{k} \right).$$

Multiplying both sides by $b[n + 1, k]$, we obtain

$$\mathbb{E} \left\{ b[n + 1, k] \left( \frac{W[n + 1, j] + k - 1}{k} \right) I[l, j] | \mathcal{F}_n \right\} =$$

$$= I[l, j] \left( \left( \frac{W[n, j] + k - 1}{k} \right) b[n, k] + d[n, k, j] - d[n + 1, k, j] \right).$$ (3.14)

Using that $d[n + 1, k, j]$ is $\mathcal{F}_n$-measurable, we obtain the desired result. \hfill \Box

**Lemma 3.2.**

$$\left( \frac{e_n I[k, j]}{W[n, j] - 1}, \mathcal{F}_n \right)$$

is a nonnegative supermartingale, where $n = j, j + 1, \ldots$. (3.15)
Proof. In a similar way as in the proof of Lemma 3.1, we have for \( n \geq k \)

\[
\mathbb{E} \left\{ \frac{I[k,j]}{W[n+1,j] - 1 | \mathcal{F}_n} \right\} = 
\]

\[
= \left( \frac{W[n,j]}{n+1} \alpha + \frac{p}{V_n^\beta} \right) \frac{I[k,j]}{W[n,j] + 1} + \left( 1 - \left( \frac{W[n,j]}{n+1} \alpha + \frac{p}{V_n^\beta} \right) \right) \frac{I[k,j]}{W[n,j] - 1} 
\]

\[
= \left( \frac{W[n,j]}{n+1} \alpha + \frac{p}{V_n^\beta} \right) \left( \frac{I[k,j]}{W[n,j]} - \frac{I[k,j]}{W[n,j] - 1} \right) + \frac{I[k,j]}{W[n,j] - 1} 
\]

\[
\leq - \frac{\alpha I[k,j]}{(n+1)(W[n,j] - 1)} + \frac{I[k,j]}{W[n,j] - 1} = \frac{I[k,j]}{W[n,j] - 1} \left( 1 - \frac{\alpha}{n+1} \right). \quad (3.16)
\]

Here we used that

\[
\left( \frac{W[n,j]}{n+1} \alpha + \frac{p}{V_n^\beta} \right) \frac{1}{W[n,j] - 1} \leq - \frac{\alpha I[k,j]}{(n+1)(W[n,j] - 1)}. \]

Multiplying both sides of (3.16) by \( e_{n+1} \), we obtain the result. \( \square \)

Proof of Theorem 2.1. Following the method presented in [7], the proof contains two parts. First, we will show that the above result is valid with nonnegative \( \gamma_j \). In the second part of the proof we will show that \( \gamma_j \) is positive with probability 1.

Let \( B_{n+1} = \{ W[n+1,j] = W[n,j] + 1 \} \). Consider the event that the \( j \)th vertex exists after \( n \) steps. On this event, by (3.13), \( \mathbb{P}(B_{n+1} | \mathcal{F}_n) \geq \frac{\alpha}{n+1} \).

(3.17)

\[
\text{The sequence } (B_n, n \in \mathbb{N}) \text{ is adapted to the sequence } (\mathcal{F}_n, n \in \mathbb{N}) \text{ of } \sigma \text{-algebras. Using Corollary VII-2-6 of [13] and (3.17), we have}
\]

\[
W[n,j] \rightarrow \infty \quad \text{a.s. as } n \rightarrow \infty. \quad (3.18)
\]

Consider the martingale \((Z[n,k,j], \mathcal{F}_n)\) in Lemma 3.1 and let \( k = 1 \). Then

\[
Z[n,1,j] = (b[n,1]W[n,j] + d[n,1,j]) I[l,j]. \quad (3.19)
\]

Applying the Marcinkiewicz strong law of large numbers to the number of vertices, we have

\[
V_n = pn + o \left( n^{1/2+\varepsilon} \right) \quad (3.20)
\]

almost surely, for any \( \varepsilon > 0 \). By this and (3.10), we obtain that

\[
d[n,1,j] = - \sum_{i=1}^{n-1} b[i+1,j] \beta p \frac{p}{V_i} \sim -\beta \Gamma(1 + \alpha) \sum_{i=1}^{n-1} \frac{p}{p_i + o \left( i^{1/2+\varepsilon} \right)^{-\alpha}} \]

\[-\beta \Gamma (1 + \alpha) \sum_{i=1}^{n-1} i^{-(1+\alpha)} (1 + (\alpha (1)) .\]

Using that \(\alpha > 0\), we see that \(d[n,1,j]\) converges, as \(n \to \infty\), and therefore the martingale \(Z[n,1,j]\) is bounded from below. Moreover, we shall see that the martingale \(Z[n,1,j]\) has bounded differences. The sequence \(b[n,1]\) is monotonically decreasing, hence

\[Z[n+1,1,j] - Z[n,1,j] \leq b[n,1] (W[n+1,j] - W[n,j]) \leq 1.\]

It is also easy to compute that

\[Z[n,1,j] - Z[n+1,1,j] \leq (b[n,1] - b[n+1,1]) W[n,j] + (d[n,1,j] - d[n+1,1,j]) = (b[n,1] - b[n+1,1]) W[n,j] + b[n+1,1] \frac{p}{V^n} \beta \leq b[n+1,1] \left( \alpha + \frac{p}{N} \beta \right) \leq \alpha + \frac{p}{N} \beta.\]

As the martingale \(Z[n,1,j]\) is bounded from below and it has bounded differences, by Proposition VII-3-9 of [13], it is convergent almost surely, as \(n \to \infty\). By the definition of \(Z[n,1,j]\), we see that \(b[n,1]W[n,j]\) also converges almost surely on the event \(\{W[l,j] > 0\}\). This, (3.18), and (3.10) implies that (2.1) is true with nonnegative \(\gamma_j\).

Now we will show that \(\gamma_j\) is positive with probability 1. Consider the supermartingale \(\left(e_n I[k,j], W[n,j] - 1, F_n\right), n \geq j,\) in Lemma 3.2. This supermartingale is nonnegative, therefore according to the submartingale-convergence theorem, it converges almost surely.

\[\lim_{l \to \infty} I[l,j] = 1\]

almost surely, hence \(\frac{e_n}{W[n,j] - 1}\) also converges almost surely, as \(n \to \infty\). This and (3.11) imply that \(\gamma_j\) is positive almost surely.

Assume that the \(j\)th vertex exists after \(l\) steps, where \(0 \leq j \leq l \leq n\). We denote by \(D[n,j]\) the degree of the \(j\)th vertex after \(n\) steps (\(j \geq 0\) is fixed).

Remark 3.1. Contrary to the weights, at each step the degree of a fixed vertex can grow by 0, 1, \ldots, \(N - 1\). Hence \(0 \leq D[n,j] - D[n-1,j] \leq N - 1\) for all fixed \(j \geq 0\). Moreover, the degree of a fixed vertex does not change in such interactions when we do not add a new vertex and the choice is done according to the preferential attachment rule.

Lemma 3.3. Let \(j \geq 0\) be fixed. We have

\[
\mathbb{E}(I[k,j]D[n+1,j]|F_n) = I[k,j] \left( D[n,j] \left[ \left(1 - \frac{W[n,j]}{n+1}\right)^\alpha - \frac{p}{V^n} \beta \right] + \right.
\]

\[
+ (1 - p) \left( q \frac{W[n,j]}{n+1} + (1 - q) \left( \frac{D[n,j]}{V^{n-1}} \right) \right) \right) + \\
+ (D[n,j] + 1) \left[ p \left( r \frac{(N - 1) W[n,j]}{N (n+1)} + (1 - r) \left( \frac{D[n,j]}{V^{n-2}} \right) \right) \right] + \\
+ (D[n,j] + 1) \left[ p \left( r \frac{(N - 1) W[n,j]}{N (n+1)} + (1 - r) \left( \frac{D[n,j]}{V^{n-2}} \right) \right) \right] + \\
+ (D[n,j] + 1) \left[ p \left( r \frac{(N - 1) W[n,j]}{N (n+1)} + (1 - r) \left( \frac{D[n,j]}{V^{n-2}} \right) \right) \right].
\]
\[ + (1 - p) (1 - q) \left( \frac{D[n,j]}{V_n} \right) \left( V_n - D[n,j] - 1 \right) \left( \frac{V_n}{V_n} \right) \] + \cdots + \\
\[ + (D[n,j] + m) \left[ p \left( 1 - r \right) \frac{D[n,j]}{N-1} \right] \left( V_n - D[n,j] - 1 \right) \left( \frac{V_n}{V_n} \right) \] + \cdots + \\
\[ + (1 - p) (1 - q) \left( \frac{D[n,j]}{N-1} \right) \left( V_n - D[n,j] - 1 \right) \left( \frac{V_n}{V_n} \right) \] + \cdots + \\
\[ + (D[n,j] + (N - 1)) \left[ p \left( 1 - r \right) \frac{V_n-D[n,j]-1}{N-1} \right] \left( \frac{V_n}{V_n} \right) \] + \cdots + \\
\[ + (1 - p) (1 - q) \frac{V_n-D[n,j]-1}{N-1} \left( \frac{V_n}{V_n} \right) \] \]}

for \( n \geq k \).

**Proof.** As in [18], the probability that an old vertex of weight \( W[n,j] \) takes part in the interaction at step \( (n+1) \) is
\[
\frac{W[n,j]}{n+1} \alpha + \frac{p}{V_n} \beta.
\]

Consider a fixed vertex with weight \( W[n,j] \) and degree \( D[n,j] \). Using the basic properties of the model, we have the probability that in the \( (n+1) \)th step

- its degree \( D[n,j] \) does not change is
\[
1 - \left( \frac{W[n,j]}{n+1} \alpha + \frac{p}{V_n} \beta \right) + (1 - p) \left( q \frac{W[n,j]}{n+1} + (1 - q) \frac{D[n,j]}{V_n} \right);
\]

- its degree is increased by 1 is
\[
p \left( r \frac{(N - 1) W[n,j]}{N(n+1)} + (1 - r) \frac{D[n,j]}{V_n} \right) + (1 - p) (1 - q) \frac{D[n,j]}{N-1} \left( \frac{V_n}{V_n} \right) \]

- its degree is increased by \( m \) (\( 1 < m \leq N - 1 \)) is
\[
p \left( 1 - r \right) \frac{D[n,j]}{N-m-1} \left( \frac{V_n-D[n,j]-1}{V_n} \right) + (1 - p) (1 - q) \frac{D[n,j]}{N-m-1} \left( \frac{V_n-D[n,j]-1}{m} \right). \]

Using the above formulae, we obtain equation (3.21). \( \square \)
Corollary 3.1. Let $j \geq 0$ be fixed. For $n \geq k$, we have

$$0 \leq \mathbb{E}\{I[k,j]D[n+1,j]|\mathcal{F}_n\} = I[k,j]\left(D[n,j] + \alpha_2 \frac{W[n,j]}{n+1} + R_n\right), \tag{3.22}$$

where $0 \leq R_n \leq (N - 1)p\beta V_n$.

Proof. Let $1 < m < N$ be an integer. Using the above notations we can rewrite (3.21) into the following form:

$$\mathbb{E}\{I[k,j]D[n+1,j]|\mathcal{F}_n\} = I[k,j]\left[D[n,j] + \alpha_2 \frac{W[n,j]}{n+1} + R_n\right],$$

where

$$R_n = \beta_1 p \left[-\frac{D[n,j]}{V_n} + (D[n,j]) \frac{1}{N} \frac{N-1}{(V_n/N)} + \cdots + \frac{1}{N} \frac{N-1}{(V_n/N)} \right] + \beta_2 p \left[-\frac{D[n,j]}{V_n} + (D[n,j]) \frac{1}{N} \frac{N-1}{(V_n/N)} + \cdots + \frac{1}{N} \frac{N-1}{(V_n/N)} \right] = \beta_1 p R^{(1)} + \beta_2 p R^{(2)}. \tag{3.23}$$

Now, we give upper bounds for $R^{(1)}$ and $R^{(2)}$ separately. It is easy to see that

$$R^{(1)} V_n = \left[-D[n,j] + (D[n,j] + 1) \frac{N-1}{(V_n/N)} + \cdots + \frac{1}{N} \frac{N-1}{(V_n/N)} \right] =$$

$$\cdots + (D[n,j] + m) \frac{1}{N} \frac{N-1}{(V_n/N)} + \cdots + (D[n,j] + (N - 1)) \frac{1}{N} \frac{N-1}{(V_n/N)}.$$


Using (3.23), (3.24) and (3.25), we have

\[ \left[ -D[n, j] + D[n, j] \frac{1}{\binom{N-1}{N-2}} \sum_{k=0}^{N-2} \binom{D[n, j]}{k} \left( V_n - D[n, j] - 1 \right) \right] + \frac{1}{\binom{N-1}{N-2}} \sum_{k=0}^{N-2} (N - 1 - k) \binom{D[n, j]}{k} \left( V_n - D[n, j] - 1 \right) \leq -D[n, j] + D[n, j] + (N - 1) = (N - 1). \]

Similarly, we have

\[ R_n^{(2)} = \left[ -D[n, j] + D[n, j] \frac{\binom{D[n,j]}{N-1}}{\binom{N-1}{N-2}} + (D[n, j] + 1) \frac{\binom{D[n,j]-1}{N-2}}{\binom{N-1}{N-2}} + \cdots + (D[n, j] + 1) \frac{\binom{V_n-D[n,j]-1}{N-1}}{\binom{N-1}{N-2}} \right] = \]

\[ = \left[ -D[n, j] + D[n, j] \frac{1}{\binom{N-1}{N-1}} \sum_{k=0}^{N-1} \binom{D[n, j]}{k} \left( V_n - D[n, j] - 1 \right) \right] + \frac{1}{\binom{N-1}{N-2}} \sum_{k=0}^{N-1} (N - 1 - k) \binom{D[n, j]}{k} \left( V_n - D[n, j] - 1 \right) \leq \]

\[ \leq -D[n, j] + D[n, j] + (N - 1) = (N - 1). \]

Using (3.23), (3.24) and (3.25), we have

\[ R_n = \beta_1 pR^{(1)} + \beta_2 pR^{(2)} \leq (N - 1) \frac{p \beta}{V_n}. \]

The proof is complete. \( \square \)

**Proof of Theorem 2.2.** Consider the following bounded random variable: \( \xi_n = \frac{I[k, j]}{N-1} (D[n, j] - D[n-1, j]) \). By Remark 3.1, we have \( 0 \leq \xi_n \leq 1 \). Applying an appropriate version of Corollary VII-2-6 of [13] (see Proposition 2.4 of [16]), then using Corollary 3.1 and (2.1), we have

\[ D[n, j] = (N - 1) \sum_{i=1}^{n} \xi_i \sim (N - 1) \sum_{i=1}^{n} \mathbb{E}(\xi_i | F_{i-1}) = \sum_{i=1}^{n} \left( \alpha_2 \frac{W[i-1, j]}{i} + R_{i-1} \right) \sim \]

\[ \sim \frac{1}{\Gamma(1 + \alpha)} \frac{\alpha_2}{\alpha} \gamma_j n^\alpha, \]

provided that \( j \)th vertex exists after \( k \) steps. As \( \lim_{k \to \infty} W[k, j] = \infty \) a.s., we obtain the statement. \( \square \)
The following lemma is an extension of Lemma 5.2 in [8]. This statement is useful when we consider the asymptotic behaviour of the maximal weight.

**Lemma 3.4.** For all \( k \geq 0, 1 \leq m \leq n \) fixed nonnegative integers, let

\[
S[m, n, k] = \sum_{j=m}^{n} \mathbb{E} \left( b[n, k] \left( \frac{W[n, j] + k - 1}{k} \right) I[n, j] \right).
\] (3.28)

Then

\[
S[m, n, k] \leq C_k \sum_{j=m}^{n} j^{-\alpha k}
\] (3.29)

with a positive constant \( C_k \).

**Proof.** As in [8], we use induction on \( k \). Let \( k = 0 \). Then

\[
S[m, n, 0] = \sum_{j=m}^{n} \mathbb{E} (b[n, 0]I[n, j]) = \sum_{j=m}^{n} \mathbb{P} (W[n, j] > 0) \leq n - m + 1.
\]

Suppose that the statement is true for \( k - 1 \), that is

\[
S[m, n, k - 1] = \sum_{j=m}^{n} \mathbb{E} \left( b[n, k - 1] \left( \frac{W[n, j] + k - 2}{k - 1} \right) I[n, j] \right) \leq C_{k-1} \sum_{j=m}^{n} j^{-\alpha(k-1)}.
\] (3.30)

By Lemma 3.1, \( Z[n, k, j] \) is a martingale. The difference of two martingales is also a martingale. So, in the definition of \( Z[n, k, j] \) changing \( I[l, j] \) for \( J[l, j] \), we obtain again a martingale. Using the definitions of \( J[n, j] \) and \( Z[n, k, j] \), we have

\[
S[m, n, k] = \sum_{j=m}^{n} \mathbb{E} \left( \sum_{l=j}^{n} \left( b[n, k] \left( \frac{W[n, j] + k - 1}{k} \right) J[l, j] \right) \right) = 
\]

\[
= \sum_{j=m}^{n} \mathbb{E} \left( \sum_{l=j}^{n} (Z[n, k, j] - d[n, k, j]) J[l, j] \right) = \sum_{j=m}^{n} \mathbb{E} \left( \sum_{l=j}^{n} (Z[l, k, j] - d[n, k, j]) J[l, j] \right) = 
\]

\[
= \mathbb{E} \left( \sum_{j=m}^{n} \sum_{l=j}^{n} b[l, k] J[l, j] \right) + \mathbb{E} \left( \sum_{j=m}^{n} \sum_{l=j}^{n} (d[l, k, j] - d[n, k, j]) J[l, j] \right). \quad (3.31)
\]

In the last step we used that \( W(l, j) = 1 \) if \( J[l, j] = 1 \).

Now, we give an upper bound for the two terms in (3.31) separately. We have already seen that, for a fixed \( k \), the sequence \( b[n, k] \) is monotonically decreasing. Therefore, applying also (3.10),

\[
\mathbb{E} \left( \sum_{j=m}^{n} \sum_{l=j}^{n} b[l, k] J[l, j] \right) \leq \sum_{j=m}^{n} b[j, k] \mathbb{E} \left( \sum_{l=j}^{n} J[l, j] \right) \leq \sum_{j=m}^{n} b[j, k] \mathbb{E} I[n, j] \leq
\]
For the second term in (3.31), changing the order of summation and using that $I[i, j] = \sum_{l=j} J[l, j]$, we have

$$
\sum_{j=m}^{n} \sum_{l=j} b[l, k] \leq C_k \sum_{j=m}^{n} j^{-\alpha k}.
$$

(3.32)

In the last step we applied that $W[i, j] + k - 1 \leq k$ if $I[i, j] > 0$.

Now, following the line of the proof presented in [5], we give upper bound on the events

$$
\{ V_i < \frac{p_i}{2} \} \quad \text{and} \quad \{ V_i \geq \frac{p_i}{2} \}
$$

separately. Using the induction hypothesis (3.30), we have

$$
E \left( \frac{p}{V_i} \beta \prod_{V_i \geq \frac{p_i}{2}} \sum_{j=m}^{i} b[i, k-1] \left( \frac{W[i, j] + k - 2}{k - 1} \right) I[i, j] \right) \leq \frac{2 \beta}{i} S[m, i, k - 1] \leq \frac{2 \beta}{i} C_{k-1} \sum_{j=m}^{i} j^{-\alpha(k-1)}.
$$

(3.34)

(Here $I_A$ is the indicator of the set $A$.) On the other hand, by (3.10),

$$
E \left( \frac{p}{V_i} \beta \prod_{V_i < \frac{p_i}{2}} \sum_{j=m}^{i} b[i, k-1] \left( \frac{W[i, j] + k - 2}{k - 1} \right) I[i, j] \right) \leq \frac{p}{N} \beta P \left\{ V_i < \frac{p_i}{2} \right\} \sum_{j=m}^{i} b[i, k-1] \left( \frac{i + k - 2}{k - 1} \right) = O \left( e^{-\epsilon i} i^{-(k-1)\alpha} k^{-(k-1)} \right) = o \left( \frac{1}{i} \sum_{j=m}^{i} j^{-\alpha(k-1)} \right),
$$

(3.35)
as $i \to \infty$. In the above computation we used Hoeffding’s exponential inequality (Theorem 2 in [15]) to obtain the following upper bound: $P \left\{ V_i < \frac{p_i}{2} \right\} \leq e^{-\epsilon i}$, where $\epsilon$ only depends on $p$, (actually $\epsilon = \frac{p^2}{2}$). Therefore, by (3.33), (3.35) and (3.34), we have

$$
E \left( \sum_{j=m}^{n} \sum_{l=j}^{n} (d[l, k, j] - d[n, k, j]) J[l, j] \right) \leq \frac{1}{i} \sum_{j=m}^{i} j^{-\alpha(k-1)}.
$$
Above we applied that, by (3.10),
\[ b[i + 1, k] - b[i, k - 1] = O(i^{-\alpha}) \text{ as } i \to \infty. \]
Finally, by (3.31), (3.32) and (3.36), we have
\[ S[m, n, k] \leq C^{(1)}_k \sum_{j=m}^{n} j^{-\alpha} + C^{(3)}_k \sum_{j=m}^{n} j^{-\alpha k}. \]

The proof is complete. \( \square \)

**Proof of Theorem 2.3.** To obtain (2.4), we can apply the method Theorem 5.1 in [8]. Let
\[ M[m, n] = \max \{W[n, j] : -(N-1) \leq j < m\}, \]
where \( 1 \leq m \leq n \) fixed. From Theorem 2.1, we have
\[ \Gamma (1 + \alpha) \lim_{n \to \infty} n^{-\alpha} M[m, n] = \max \{\gamma_j : -(N-1) \leq j < m\} \]
almost surely. Using (3.14), we can prove that the following process is a submartingale:
\[ b[n, k] \left( W[n, j] \right) = b[n, k] \left( W[n, j] + k - 1 \right) I[n, j], \quad n \geq j. \]
Let \( Q[m, n] = \max_{m \leq j \leq n} W[n, j] \). Then \( 0 \leq W_n - M[m, n] \leq Q[m, n] \). As the maximum of increasing numbers of submartingales is also a submartingale, therefore
\[ b[n, k] \left( Q[m, n] + k - 1 \right), \quad n \geq m \]
is also a submartingale. For nonnegative numbers the maximum is majorized by the sum. Therefore, and by Lemma 3.4, we obtain
\[ \mathbb{E} \left( b[n, k] \left( Q[m, n] + k - 1 \right) \right) \leq S[m, n, k] \leq C_k \sum_{j=m}^{n} j^{-\alpha k}. \]
Since
\[ 0 \leq (b[n, 1] Q[m, n])^k \leq \frac{b[n, 1]^k}{b[n, k]} b[n, k] \left( Q[m, n] + k - 1 \right), \]
we see that the submartingale \( b[n, 1] Q[m, n] \) is bounded in \( L^k \) for all \( k \alpha > 1 \). Hence, this submartingale converges almost surely and in \( L^k \) for every \( k > \frac{1}{\alpha} \). Moreover, by (3.10), (3.39) and (3.40), we have
\[ \mathbb{E} \left( \limsup_{n \to \infty} n^{-\alpha} Q[m, n] \right)^k \leq k! C_k \frac{1}{\Gamma(1 + \alpha k)} \sum_{j=m}^{\infty} j^{-\alpha k}. \]
Now, using the monoton convergence theorem, we have
\[ E \left( \lim_{m \to \infty} \limsup_{n \to \infty} (n^{-\alpha}Q[m, n])^k \right) = 0, \]
for \( k > \frac{1}{\alpha} \). As \( Q[m, n] \) is decreasing, as \( m \) increases, so
\[ \lim_{m \to \infty} \limsup_{n \to \infty} n^{-\alpha}Q[m, n] = 0 \quad \text{a.s.} \quad (3.42) \]
Therefore, as \( 0 \leq W_n - M[m, n] \leq Q[m, n] \),
\[ \lim_{m \to \infty} \limsup_{n \to \infty} \left( n^{-\alpha} (W_n - M[m, n]) \right) = 0 \quad \text{a.s.} \]
This relation and (3.38) imply (2.4). Using relation (3.41) we can show that \( \mu = \sup \{ \gamma_j : j \geq -(N-1) \} \) is a.s. finite.

**Proof of Theorem 2.4.** We follow the line of the proof of Theorem 5.3 in [8]. The evolution mechanism of the graph implies that \( D[n, j] \leq (N-1)W[n, j] \). Therefore we have
\[ \max \{ D[n, j] : -(N-1) \leq j < m \} \leq D_n \leq \max \{ D[n, j] : -(N-1) \leq j < m \} + \max \{ (N-1)W[n, j] : m \leq j \leq n \}. \]
Multiplying both sides by \( n^{-\alpha} \) and then considering the limit as \( n \to \infty \), Theorem 2.2 implies
\[ \liminf_{n \to \infty} D_n n^{-\alpha} \leq \limsup_{n \to \infty} D_n n^{-\alpha} \leq \max \left\{ \frac{1}{\Gamma(1+\alpha)} \frac{\alpha_2}{\alpha} \gamma_j : -(N-1) \leq j < m \right\} \]
\[ \leq \max \left\{ \frac{1}{\Gamma(1+\alpha)} \frac{\alpha_2}{\alpha} \gamma_j : -(N-1) \leq j < m \right\} + (N-1) \limsup_{n \to \infty} n^{-\alpha}Q[m, n] \]
as \( n \to \infty \). As \( m \to \infty \), by (3.42), we obtain the desired result. \( \square \)

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