Gaugeon formalism in the framework of generalized BRST symmetry

Sudhaker Upadhyay¹,* and Bhabani Prasad Mandal²,*

¹S. N. Bose National Centre for Basic Sciences, Block JD, Sector III, Salt Lake, Kolkata-700098, India
²Department of Physics, Banaras Hindu University, Varanasi-221005, India
*E-mail: sudhakerupadhyay@gmail.com, sudhaker@boson.bose.res.in, bhabani.mandal@gmail.com

Received January 16, 2014; Revised February 28, 2014; Accepted March 15, 2014; Published May 2, 2014

We consider gaugeon formulations that relate to the quantum gauge freedom covariantly in the framework of the generalized BRST transformation for the Yang–Mills theory as well as the BRST-invariant Higgs model. We generalize the BRST symmetries of both the Yang–Mills theory and the Higgs model by making the transformation parameter finite and field-dependent. Remarkably, we observe that the gaugeon Lagrangian that describes the quantum gauge freedom appears automatically in the effective theories, along with a natural shift in gauge parameters under specific finite field-dependent parameters.

1. Introduction

In the usual canonical quantization of gauge theories, there exists no gauge freedom at the quantum level, as quantum theory is defined only after gauge fixing. The gaugeon formulation provides a wider framework in which one can consider quantum gauge transformation among a family of linear covariant gauges [1–7]. In this formulation the so-called gaugeon fields were introduced as the quantum gauge freedom; such a formulation was originally proposed to restore the problem of gauge parameter renormalization. The shift of gauge parameter, which arises through renormalization [8], is naturally derived in this formulation by connecting theories in two different gauges within the same family by a $q$-number gauge transformation [1]. The main drawback of this formulation is the unphysical gaugeon fields that do not contribute in the physical process. Thus, it is necessary to remove the gaugeon modes by imposing subsidiary conditions. Initially this was done by imposing a Gupta–Bleuler type of restriction, which has its own limitations. The gaugeon formulation was improved in certain cases by further extending the configuration space to incorporate the BRST quartet for gaugeon fields [9,10], where the Gupta–Bleuler-type subsidiary condition is replaced by Kugo–Ojima-type restrictions [11,12]. The gaugeon formalism with and without BRST symmetry has been studied in many different contexts in quantum field theory [9,10,13–18] as well as in perturbative gravity [19].

In the present work we would like to consider the gaugeon formulation in the light of the generalized BRST transformation [20]. The generalized BRST symmetry of making the infinitesimal parameter finite and field-dependent is known as the finite field-dependent BRST (FFBRST) transformation and has many implications in gauge theories [20–36]. This provides us with sufficient motivations to analyze the gaugeon formulation through generalized BRST symmetry. For this purpose, we consider
two different models, (i) the Yang–Mills theory [37], the backbone of all frontier high-energy physics, and (ii) the Higgs model [38,39], which provides a general framework to explain the observed masses of gauge bosons by means of charged and neutral Goldstone bosons that end up as the longitudinal components of gauge bosons, to describe the quantum gauge symmetry in the framework of gaugeon formalism.

We extend the effective action by introducing two gaugeon fields in both models. Such an extended theory possesses quantum gauge symmetry, under which the Lagrangian remains form invariant. These gaugeon fields do not contribute in physical processes and therefore lead to unphysical gaugeon modes. To remove the unphysical modes, we put a Gupta–Bleuler-type condition on the gaugeon fields, which finds certain limitations. This situation is further improved in the Higgs model by extending the action by introducing Faddeev–Popov ghosts, associated with gaugeon fields. Such an action remains invariant under both the extended BRST symmetry and the extended quantum gauge symmetry. Now, we generalized the full BRST symmetries of the theories by allowing the infinitesimal parameter to be finite and field-dependent with continuous interpolation of an arbitrary parameter \( \kappa \). Further, we calculate the Jacobian of the path integral measure in each case for a specific finite field-dependent parameter and show that the Jacobian produces the exact gaugeon part to the effective action with renormalized gauge parameters. Therefore, we claim that the FFBRST transformation with an appropriate transformation parameter produces the gaugeon effective action with an accurate shift in gauge parameter to describe the quantum gauge freedom. Even though we establish these results with the help of two different but typical models, namely, Yang–Mills theory and the Higgs model, these results hold good for any arbitrary models in gaugeon formulation.

The paper is organized as follows. In Section 2, we discuss the preliminaries of the FFBRST transformation. Section 3 is devoted to a study of the gaugeon formalism of both the Yang–Mills theory and the Higgs model. Within this section, we also investigate the FFBRST transformation and emergence of gaugeon mode through the Jacobian of the path integral measure. Conclusions are drawn in the last section.

2. The generalized BRST transformation: preliminaries

In this section, we recapitulate the generalized BRST transformation with the finite field-dependent parameter, which is also known as the FFBRST transformation [20]. For this purpose, let us begin with the usual BRST transformation defined by

\[
\delta_b \phi = s_b \phi \eta,
\]

(1)

where \( s_b \phi \) is the Slavnov variation of a generic field \( \phi \).

The properties of the usual BRST transformation do not depend on whether the transformation parameter \( \eta \) is finite or infinitesimal and field-dependent; however, it must be anticommuting and space-time independent. These requirements give the freedom to generalize the BRST transformation by making the parameter \( \eta \) finite and field-dependent without affecting its properties. To generalize the BRST transformation, we start by making the infinitesimal parameter field-dependent with the introduction of an arbitrary parameter \( \kappa \) (\( 0 \leq \kappa \leq 1 \)). We allow the fields \( \phi(x, \kappa) \) to depend on \( \kappa \) in such a way that \( \phi(x, \kappa = 0) = \phi(x) \), and \( \phi(x, \kappa = 1) = \phi'(x) \) is the transformed field. Furthermore, the usual infinitesimal BRST transformation is defined generically as [20]

\[
d\phi(x, \kappa) = s_b[\phi(x)]\Theta'[\phi(x, \kappa)]d\kappa,
\]

(2)

where \( \Theta'[\phi(x, \kappa)]d\kappa \) is the infinitesimal but field-dependent parameter. The FFBRST transformation with the finite field-dependent parameter then can be constructed by integrating such an infinitesimal...
transformation from $\kappa = 0$ to $\kappa = 1$, to obtain

$$\phi' \equiv \phi(x, \kappa = 1) = \phi(x, \kappa = 0) + s_b[\phi(x)]\Theta[\phi(x)],$$

(3)

where

$$\Theta[\phi(x)] = \int_0^1 d\kappa' \Theta'[\phi(x, \kappa')]$$

(4)

is the finite field-dependent parameter [20]. This FFBRST transformation is the symmetry of the effective action. However, being transformation-parameter field-dependent, the path integral measure is no longer invariant under such a transformation. The Jacobian of the path integral measure changes non-trivially under the FFBRST transformation. To estimate the Jacobian $J(\kappa)$ of the path integral measure $(D\phi)$ under FFBRST transformations for a particular choice of the finite field-dependent parameter $\Theta[\phi(x)]$, we first calculate the infinitesimal change in the Jacobian using Taylor expansion, as follows [20]:

$$\frac{1}{J} \frac{dJ}{d\kappa} = -\int d^4y \left[ \pm s_b \phi(y, \kappa) \frac{\partial \Theta'[\phi(y, \kappa)]}{\partial \phi(y, \kappa)} \right],$$

(5)

where $\pm$ refers to whether $\phi$ is a bosonic or a fermionic field.

Further, the Jacobian $J(\kappa)$ can be replaced (within the functional integral) by

$$J(\kappa) \rightarrow \exp[iS_1[\phi(x, \kappa)]]$$

(6)

without changing the theory defined by the action $S_{\text{eff}}$ numerically if and only if the following essential condition is satisfied [19]:

$$\int (D\phi)(x) \left[ \frac{1}{J} \frac{dJ}{d\kappa} - i \frac{dS_1[\phi(x, \kappa)]}{d\kappa} \right] \exp \left[ i(S_{\text{eff}} + S_1) \right] = 0,$$

(7)

where $S_1[\phi]$ refers to some local functional of fields.

Consequently, the functional $S_1$ within the functional integral accumulate to give the effective action $S_{\text{eff}}$ and, therefore, the effective action modifies to $S_{\text{eff}} + S_1$, which becomes an extended effective action. Hence, FFBRST transformation with an appropriate parameter $\Theta$ extends the effective action of the theory. We utilize this fact to show that the gaugeon modes in the effective theory that describes the quantum gauge freedom are generated through FFBRST transformation. To produce the extra part $S_1$ in the effective action with some extra fields through Jacobian calculation, we first insert a well defined path integral measure corresponding to those extra fields in the functional integral by hand before performing the FFBRST transformation; thereafter, the Jacobian factor compensates the divergence factor. However, if the extra part $S_1$ contains only original fields, we do not need any extra path integral measure before performing the FFBRST transformation.

3. Gaugeon formalism and its emergence through generalized BRST symmetry

In this section, we review the Yokoyama gaugeon formalism to discuss the quantum gauge freedom for the Yang–Mills theory as well as for the Higgs model.
3.1. BRST-symmetric Yokoyama–Yang–Mills theory

To analyze the gaugeon formalism for Yang–Mills theory, let us start with the effective Lagrangian density for 4D Yang–Mills theory in the Landau gauge:

\[ \mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - A_\mu^a \partial^\mu B^a + i \partial^\mu c^a_\mu D_{\mu}^{ab} c^b, \]  

(8)

where \( A_\mu^a, B^a, c^a, \) and \( c^a_\mu \) are gauge field, multiplier field, ghost field, and anti-ghost field respectively. Here, the field-strength tensor \( (F_{\mu\nu}^a) \) and covariant derivative \( (D_{\mu}^{ab}) \) are defined by

\[ F_{\mu\nu}^a = \partial^\mu A^a_\nu - \partial^\nu A^a_\mu + g f^{abc} A^c_\mu A^b_\nu, \]
\[ D_{\mu}^{ab} = \partial^\mu \delta^{ab} - g f^{abc} A^c_\mu, \]

(9)

with coupling constant \( g \). The Lagrangian density (8) is invariant under the following nilpotent BRST transformations:

\[ \delta_b A^a_\mu = -D_{\mu}^{ab} c^b \eta, \quad \delta_b c^a = -\frac{g}{2} f^{abc} c^b c^c \eta, \]
\[ \delta_b c^a_\mu = -i B^a \eta, \quad \delta_b B^a = 0, \]

(10)

where \( \eta \) is an infinitesimal, anticommuting, and global parameter. Now, by introducing the gaugeon field \( Y \) and its associated field \( Y^\star \) subject to the Bose–Einstein statistics, the Yokoyama Lagrangian density for the Yang–Mills theory is demonstrated as [2]

\[ \mathcal{L}_Y(\phi, \alpha) = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} - A_\mu^a \nabla^\mu B^a - \partial_\mu Y_\star \partial^\mu Y + \frac{\epsilon}{2} (Y_\star + \alpha^a B^a)^2 + i \nabla^\mu c^a_\star D_{\mu}^{ab} c^b, \]

(11)

where \( \alpha^a \) is the group-vector-valued gauge-fixing parameter and \( \epsilon(\pm) \) is the sign factor. Here \( \nabla_\mu \) refers to the form-covariant derivative defined as

\[ \nabla_\mu V^a = \partial_\mu V^a + g f^{abc} \alpha^b V^c \partial_\mu Y, \quad (V^a = B^a, c^a_\star). \]

(12)

The Lagrangian density (11) remains invariant under the following BRST transformation:

\[ \delta_b A^a_\mu = -D_{\mu}^{ab} c^b \eta, \quad \delta_b c^a = -\frac{g}{2} f^{abc} c^b c^c \eta, \]
\[ \delta_b c^a_\mu = -i B^a \eta, \quad \delta_b B^a = 0, \quad \delta_b Y_\star = 0. \]

(13)

Now, we define the following quantum gauge transformation under which the Lagrangian density (11) remains form invariant [2]:

\[ A^a_\mu \rightarrow \hat{A}^a_\mu = A^a_\mu + \tau (\alpha^a \partial_\mu Y + g f^{abc} A^b_\mu c^c_\star Y), \]
\[ B^a \rightarrow \hat{B}^a = B^a + \tau g f^{abc} B^b \alpha^c Y, \]
\[ Y_\star \rightarrow \hat{Y}_\star = Y_\star - \tau \alpha^a B^a, \]
\[ Y \rightarrow \hat{Y} = Y, \]
\[ c^a \rightarrow \hat{c}^a = c^a + \tau g f^{abc} \alpha^b c^c_\star Y, \]
\[ c^a_\mu \rightarrow \hat{c}^a_\mu = c^a_\mu + \tau g f^{abc} c^b_\mu \alpha^c Y, \]

(14)
parameter shift:

\[ \alpha^a \rightarrow \hat{\alpha}^a = \alpha^a + \tau \alpha^a. \] (15)

Further, to remove the unphysical modes of the theory and to define physical states, we impose two subsidiary conditions [2]:

\[ Q_b|\text{phys}\rangle = 0, \]
\[ (Y^\star + \alpha^a B^a)^{(+)}|\text{phys}\rangle = 0, \] (16)

where \( Q_b \) is the BRST charge calculated as

\[ Q = \int d^3x \left[ -F^{0\nu a}D_\nu c^b - i\frac{g}{2}f^{abc}c^a_b c^c - D^{0ab}c^b B^a \right]. \] (17)

The Kugo–Ojima-type subsidiary condition (first of Eq. (16)) is subjected to removal of the unphysical gauge field modes from the total Fock space. However, the second Gupta–Bleuler-type condition guarantees that no gaugeon appears in the physical states. The second subsidiary condition is well defined when the combination \( (Y^\star + \alpha^a B^a) \) satisfies the following free equation [2]:

\[ \partial_\mu \partial^\mu (Y^\star + \alpha^a B^a) = 0. \] (18)

The above free equation assures the decomposition of \( (Y^\star + \alpha^a B^a) \) in positive and negative frequency parts. The subsidiary conditions (16) guarantee the metric of our physical state-vector space to be positive semi-definite:

\[ \langle \text{phys}|\text{phys}\rangle \geq 0, \] (19)

and, consequently, we have a desirable physical subspace on which our unitary physical \( S \)-matrix exists.

Now, we analyze the emergence of gaugeon mode in the effective Yang–Mills theory by calculating the Jacobian of the path integral measure under FFBRST transformation. First of all, we construct the FFBRST transformation by making the infinitesimal parameter \( \eta \) of Eq. (13) finite and field-dependent (in the same fashion as discussed in an earlier section) as follows:

\[ \delta b A^\mu = -D^{ab}_\mu c^b \Theta[\phi], \quad \delta b c^a = -\frac{g}{2}f^{abc}c^b c^c \Theta[\phi], \] \[ \delta b c^a_\star = -iB^a \Theta[\phi], \quad \delta b Y^a = 0, \quad \delta b Y^a_\star = 0, \] (20)

where \( \Theta[\phi] \) is an arbitrary finite field-dependent parameter with ghost number \(-1\). Now, we choose the following infinitesimal field-dependent parameter:

\[ \Theta'[\phi] = \int d^4y \left[ g f^{abc}c^a_b c^c A^\mu A^\mu - \varepsilon c^a_\star A^\mu Y - \varepsilon \hat{\alpha}^a c^a_\star \left( \frac{1}{2} \hat{\alpha}^b B^b + Y^\star \right) + c^a_\star B^a (B^b)^{-2} \partial_\mu Y^\star \partial^\mu Y \right. \] \[ \left. - \frac{\varepsilon}{2} c^a_\star B^a (B^b)^{-2} Y^\star \right], \] (21)

to construct the specific \( \Theta[\phi] \) using relation (4), where \( \hat{\alpha} \) denotes the shifted gauge parameter as defined in (15). Now, exploiting the relation (5), the infinitesimal change in Jacobian for the above
\[ \Theta' [\phi] \text{ yields} \]
\[ \frac{1}{J(\kappa)} \frac{dJ(\kappa)}{d\kappa} = - \int d^4x \left[ g f^{abc} \hat{\alpha}^b (i B^c) A_\mu^a \partial^\mu Y - g f^{abc} \hat{\alpha}^b (D_\mu c)^a \hat{\epsilon}^c \partial^\mu Y - \epsilon \hat{\alpha}^a (i B^a) \right] \]
\[ + \left( \frac{1}{2} \hat{\alpha}^b B^b + Y_\ast \right) + i \partial_\mu Y_\ast \partial^\mu Y - i \frac{\epsilon}{2} Y^2_\ast \right]. \]
\[ = - \int d^4x \left[ i g f^{abc} \hat{\alpha}^b B^c A_\mu^a \partial^\mu Y + g f^{abc} \hat{\alpha}^b \hat{\epsilon}^c \partial^\mu Y (D_\mu c)^a - i \frac{\epsilon}{2} (\hat{\alpha}^a B^a)^2 \right. \]
\[ + i \epsilon \hat{\alpha}^a B^a Y_\ast + i \partial_\mu Y_\ast \partial^\mu Y - i \frac{\epsilon}{2} Y^2_\ast \right] . \tag{22} \]

Now, the consistency condition (7), together with Eqs. (22) and (25), leads to the following constraint:
\[ \xi_i (\kappa) = 0. \tag{24} \]

At the physical ground, the theory remains unaltered when the above \( S_1 \) and change in Jacobian given in (22) satisfy the crucial condition (7). To check this consistency, we first calculate the infinitesimal difference in \( S_1 \) with respect to parameter \( \kappa \) with the help of (2) as follows:
\[ \frac{S_1 [\phi(x, \kappa), \kappa]}{d\kappa} = \int d^4x \left[ \xi_1 (\kappa) g f^{abc} \hat{\alpha}^b B^c A_\mu^a \partial^\mu Y + \xi_2 (\kappa) g f^{abc} \hat{\alpha}^b \hat{\epsilon}^c \partial^\mu Y (D_\mu c)^a + \xi_3 (\kappa) (\hat{\alpha}^a B^a)^2 \right. \]
\[ + \xi_4 (\kappa) \hat{\alpha}^a B^a Y_\ast + \xi_5 (\kappa) \partial_\mu Y_\ast \partial^\mu Y + \xi_6 (\kappa) Y^2_\ast - \xi_1 (\kappa) g f^{abc} \hat{\alpha}^b B^c (D_\mu c)^a \partial^\mu Y \Theta' \]
\[ \left. + \xi_2 (\kappa) g f^{abc} \hat{\alpha}^b (i B^c) \Theta' \partial^\mu Y (D_\mu c)^a \right]. \tag{25} \]

Now, the consistency condition (7), together with Eqs. (22) and (25), leads to
\[ \int d^4x \left[ \left( \frac{d\xi_1}{d\kappa} + 1 \right) g f^{abc} \hat{\alpha}^b B^c A_\mu^a \partial^\mu Y + \left( \frac{d\xi_2}{d\kappa} - i \right) g f^{abc} \hat{\alpha}^b \hat{\epsilon}^c \partial^\mu Y (D_\mu c)^a \right. \]
\[ + \left( \frac{d\xi_3}{d\kappa} - 1 \right) (\hat{\alpha}^a B^a)^2 + \left( \frac{d\xi_4}{d\kappa} - \epsilon \right) \hat{\alpha}^a B^a Y_\ast + \left( \frac{d\xi_5}{d\kappa} + 1 \right) \partial_\mu Y_\ast \partial^\mu Y \]
\[ \left. + \left( \frac{d\xi_6}{d\kappa} - \frac{\epsilon}{2} \right) Y^2_\ast - (\xi_1 - i \xi_2) g f^{abc} \hat{\alpha}^b B^c (D_\mu c)^a \partial^\mu Y \Theta' \right] = 0, \tag{26} \]

where the non-local (\( \Theta' \)-dependent) term vanishes, leading to the following constraint:
\[ \xi_1 (\kappa) - i \xi_2 (\kappa) = 0. \tag{27} \]

However, the disappearance of local terms from the LHS of expression (26) leads to the following exactly solvable linear differential equations:
\[ \frac{d\xi_1}{d\kappa} + 1 = 0, \quad \frac{d\xi_2}{d\kappa} - i = 0, \]
\[ \frac{d\xi_3}{d\kappa} - 1 = 0, \quad \frac{d\xi_4}{d\kappa} - \epsilon = 0, \]
\[ \frac{d\xi_5}{d\kappa} + 1 = 0, \quad \frac{d\xi_6}{d\kappa} - \frac{\epsilon}{2} = 0. \tag{28} \]
The solutions of the above equations satisfying the initial boundary conditions (24) are
\[\xi_1(\kappa) = -\kappa, \quad \xi_2(\kappa) = i\kappa, \quad \xi_3(\kappa) = +\kappa, \quad \xi_4(\kappa) = +\varepsilon \kappa, \quad \xi_5(\kappa) = -\kappa, \quad \xi_6(\kappa) = \frac{\varepsilon}{2}\kappa. \tag{29}\]

With these solutions, expression (23) at \(\kappa = 1\) takes the following form:
\[
S_1[\phi(x, 1), 1] = \int d^4x \left[ -gf^{abc} \hat{\alpha}^b B^c A^a_\mu \partial_\mu Y + ig f^{abc} \hat{\alpha}^b c^c \partial_\mu Y (D_\mu c)^a + (\hat{\alpha}^a B^a)^2 \right. \\
\left. + \varepsilon \hat{\alpha}^a B^a Y_* - \partial_\mu Y_* \partial_\mu Y + \frac{\varepsilon}{2} Y_*^2 \right]. \tag{30}\]

Now, by adding \(S_1[\phi(x, 1), 1]\) to the effective action corresponding to (8), we get
\[
\int d^4x \mathcal{L}_{YM} + S_1[\phi(x, 1), 1] = \int d^4x \left[ -\frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a - A^a_\mu \nabla^\mu B^a - \partial_\mu Y_* \partial_\mu Y \right. \\
\left. + \frac{\varepsilon}{2} (Y_* + \hat{\alpha}^a B^a)^2 + i \nabla_\mu c^a D^a_\mu \right], \tag{31}\]
which is nothing but the gaugeon action for Yang–Mills theory with shifted gauge parameter \(\hat{\alpha}^a = \alpha^a(1 + \tau)\). Hence, we end this subsection with the following remark: under a specific generalized BRST transformation, the gaugeon modes in the effective Yang–Mills action appear manifestly.

### 3.2. BRST-symmetric Higgs model

To describe the gaugeon formulation of the Higgs model in the framework of FFBRST transformation, we begin with the classical Lagrangian density of the Higgs model defined by
\[
\mathcal{L}_H = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + (D_\mu \varphi)^\dagger (D^\mu \varphi) + \mu^2 \varphi^\dagger \varphi - \frac{\lambda}{2} (\varphi^\dagger \varphi)^2, \tag{32}\]
where \(\varphi\) is the complex scalar field, and \(\mu^2\) and \(\lambda\) are positive constants. The field-strength tensor and covariant derivative are defined, respectively, by
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \\
D_\mu = \partial_\mu - eA_\mu. \tag{33}\]

Here the complex scalar field \(\varphi\) has the following vacuum expectation value:
\[
\langle 0 | \varphi | 0 \rangle = \frac{v}{\sqrt{2}} = \frac{\mu}{\sqrt{\lambda}}. \tag{34}\]

It is well known that the Lagrangian density (32) is gauge invariant. Therefore, to quantize it correctly, we need to break the local gauge invariance by fixing a suitable gauge. There are many choices for the gauge-fixing condition. For example, the gauge-fixed Lagrangian density corresponding to the \(R_\xi\) gauge condition introduced by Fujikawa, Lee, and Sanda [40] is given by
\[
\mathcal{L}_{gf} = \frac{1}{2\xi} B^2 + B \left( \partial_\mu A^\mu + \frac{1}{\xi} M \chi \right), \tag{35}\]
where \(B\) is the multiplier field and \(\xi\) is a numerical gauge-fixing parameter. Here \(M = ev\) is the mass of \(A_\mu\) acquired through spontaneous symmetry breaking and the Hermitian field \(\chi\) is the Goldstone...
mode defined along with the physical Higgs mode $\psi$ as

$$\varphi = \frac{1}{\sqrt{2}} (v + \psi + i \chi). \quad (36)$$

Now, the Faddeev–Popov ghost term corresponding to the above gauge-fixing term is constructed as

$$L_{gh} = -i \partial_\mu c_\ast \partial^\mu c + i c_\ast \frac{M^2}{\xi} c, \quad (37)$$

where $c$ and $c_\ast$ are the ghost and anti-ghost fields respectively.

To analyze the gaugeon formalism for the $R_\xi$ gauge avoiding non-polynomial terms in the Lagrangian density, we use the following parametrization [15]:

$$\varphi(x) = (v + \rho(x)) e^{i \pi(x)/\sqrt{\lambda}}, \quad (38)$$

instead of (36). Here fields $\rho$ and $\pi$ show a resemblance to fields $\psi$ and $\chi$ of Eq. (36). In terms of the parametrization, the Lagrangian density given in Eq. (32) is expressed as

$$L_H = -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} M^2 \left( 1 + \frac{e}{M} \rho \right)^2 \left( A_\mu - \frac{1}{M} \partial_\mu \pi \right)^2$$

$$+ \frac{1}{2} \left( \partial_\mu \rho \partial^\mu \rho - \lambda v^2 \rho^2 \right) - \frac{1}{2} \lambda v \rho^3 - \frac{\lambda}{8} \rho^4 + \frac{1}{8} \lambda v^4, \quad (39)$$

where $\sqrt{\lambda} v$ is the mass of the Higgs boson $\rho$. The above Lagrangian density is invariant under the following classical gauge transformations:

$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x),$$

$$\pi(x) \longrightarrow \pi'(x) = \pi(x) + M \Lambda(x),$$

$$\rho(x) \longrightarrow \rho'(x) = \rho(x), \quad (40)$$

where $\Lambda(x)$ is an arbitrary local parameter of transformation. Now, we recast the gauge-fixing and ghost terms given in (35) and (37) in accordance with parametrization (38) as follows:

$$L_{gf} = \frac{1}{2} \alpha_1 B^2 + B \left( \partial_\mu A^\mu + \beta_1 M \pi \right),$$

$$L_{gh} = -i \partial_\mu c_\ast \partial^\mu c + i \beta_1 c_\ast M^2 c, \quad (41)$$

where $\alpha_1$ and $\beta_1$ are gauge parameters.

Now, the effective Lagrangian density for the Higgs model,

$$L_{eff} = L_H + L_{gf} + L_{gh}, \quad (42)$$

possesses the following nilpotent BRST transformation:

$$\delta_b A_\mu = -\partial_\mu c \eta, \quad \delta_b \pi = -M c \eta,$$

$$\delta_b \rho = 0, \quad \delta_b c = 0,$$

$$\delta_b c_\ast = i B \eta, \quad \delta_b B = 0, \quad (43)$$

where $\eta$ is an anticommuting global parameter.
Further, to analyze the quantum gauge freedom of the Higgs model, we extend the effective Lagrangian density (42) to a most general gaugeon Lagrangian density by introducing the gaugeon field $Y$ and its associated field $Y_*$ as well as the corresponding ghost fields $K$ and $K_*$ as

$$
\mathcal{L}_{YH} = \mathcal{L}_H + B \partial_{\mu} A^\mu - \partial_{\mu} Y_* \partial^\mu Y + (\beta_1 B + \beta_3 Y_*) M \pi \\
+ (\beta_2 B + \beta_4 Y_*) M^2 Y + \frac{1}{2} \alpha_1 B^2 + \alpha_2 B Y_* + \frac{1}{2} \alpha_3 Y_*^3 \\
- i \partial_{\mu} c_* \partial^\mu c - i \partial_{\mu} K_* \partial^\mu K + i (\beta_1 c_* + \beta_3 K_*) M^2 c \\
+ i (\beta_2 c_* + \beta_4 K_*) M^2 K.
$$

(44)

where $\alpha_i$ ($i = 2, 3$) and $\beta_i$ ($i = 2, 3, 4$) are constant gauge parameters. Now, the gaugeon fields and respective ghost fields vary under the BRST transformation as follows:

$$
\delta_b Y = - K \eta, \quad \delta_b K = 0, \\
\delta_b K_* = i Y_* \eta, \quad \delta_b Y_* = 0,
$$

(45)

and form the BRST quartet. The gaugeon Lagrangian density (44) is invariant under the effect of the combined BRST transformations (43) and (45). Consequently, the corresponding BRST charge $Q_b$ annihilates the physical subspace of $\mathcal{V}_{\text{phys}}$ of total Hilbert space, i.e.

$$
Q_b |_{\text{phys}} = 0.
$$

(46)

This single subsidiary condition of Kugo–Ojima type removes both the unphysical gauge modes as well as the unphysical gaugeon modes.

The gaugeon Lagrangian density (44) also admits the following quantum gauge transformations:

$$
A_\mu \rightarrow \hat{A}_\mu = A_\mu + \tau \partial_\mu Y, \\
\pi_\mu \rightarrow \hat{\pi} = \pi + \tau M Y, \\
Y_* \rightarrow \hat{Y}_* = Y_* - \tau B, \\
B \rightarrow \hat{B} = B, \\
Y \rightarrow \hat{Y} = Y, \\
c \rightarrow \hat{c} = c + \tau K, \\
K_* \rightarrow \hat{K}_* = K_* - \tau c_* , \\
c_* \rightarrow \hat{c}_* = c_* , \\
K \rightarrow \hat{K} = K.
$$

(47)

Under the above quantum gauge transformation, $\mathcal{L}_{YH}$ remains form invariant, leading to the following shift in gauge parameters:

$$
\alpha_1 \rightarrow \hat{\alpha}_1 = \alpha_1 + 2 \alpha_2 \tau + \alpha_3 \tau^2, \\
\alpha_2 \rightarrow \hat{\alpha}_2 = \alpha_2 + \alpha_3 \tau, \\
\alpha_3 \rightarrow \hat{\alpha}_3 = \alpha_3.
$$
\[\beta_1 \rightarrow \hat{\beta}_1 = \beta_1 + \beta_3 \tau,\]
\[\beta_2 \rightarrow \hat{\beta}_2 = \beta_2 + (\beta_4 - \beta_1) \tau - \beta_3 \tau^2,\]
\[\beta_3 \rightarrow \hat{\beta}_3 = \beta_3,\]
\[\beta_4 \rightarrow \hat{\beta}_4 = \beta_4 - \beta_3 \tau.\]

(48)

We observe that the quantum gauge transformations (47) commute with the BRST transformations mentioned in (45). Consequently, it is confirmed that the Hilbert space resulting from physical states annihilated by the BRST charge is also invariant under the quantum gauge transformations.

Now, we analyze the emergence of gaugeon mode in the effective action for the Higgs model by calculating the Jacobian of the path integral measure under FFBRST transformation. To achieve this goal, we construct the FFBRST transformation by making the infinitesimal parameter \(\eta\) of (43) and (45) finite and field-dependent such that

\[\delta b A_\mu = -\partial_\mu c \Theta[\phi], \quad \delta b \pi = -Mc \Theta[\phi],\]
\[\delta b \rho = 0, \quad \delta b c = 0,\]
\[\delta b c_* = iB \Theta[\phi], \quad \delta b B = 0,\]
\[\delta b Y = -K \Theta[\phi], \quad \delta b K = 0,\]
\[\delta b K_* = iY_* \Theta[\phi], \quad \delta b Y_* = 0,\]

(49)

where \(\Theta[\phi]\) is a finite field-dependent parameter constructed from the following infinitesimal field-dependent parameter:

\[\Theta'[\phi] = \int d^4x \left[ \partial_\mu K_* \partial^\mu Y - \hat{\beta}_2 c_* M^2 Y - \hat{\beta}_3 K_* M \pi - \hat{\beta}_4 K_* M^2 Y \right.\]
\[\left. - \frac{1}{2} c_* (\hat{\alpha}B + \hat{\alpha}_2 Y) - \frac{1}{2} K_* (\hat{\alpha}_2 B + \hat{\alpha}_3 Y) \right].\]

(50)

Here \(\hat{\alpha}_2, \hat{\alpha}_3, \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3,\) and \(\hat{\beta}_4\) are shifted gauge parameters and have the same definitions as in (48). However, the parameter \(\hat{\alpha}\) is defined in terms of \(\tau\) explicitly as

\[\hat{\alpha} = 2\alpha_2 \tau + \alpha_3 \tau^2.\]

(51)

We again calculate the infinitesimal change in Jacobian for the path integral measure under FFBRST transformation in the same way as calculated in the last subsection:

\[\frac{1}{J(\kappa)} \frac{d J(\kappa)}{d\kappa} = \int d^4x \left[ -i\partial_\mu Y_* \partial^\mu Y + K \partial_\mu \partial^\mu K_* + i\hat{\beta}_2 BM^2 Y + \hat{\beta}_2 c_* M^2 + i\hat{\beta}_3 Y_* M \pi 
\right.
\[+ \hat{\beta}_3 M^2 c K_* + i\hat{\beta}_4 Y_* M^2 Y + \hat{\beta}_4 K_* M^2 + \frac{1}{2} i B (\hat{\alpha} B + \hat{\alpha}_2 Y_*)
\[\left. + \frac{1}{2} i Y_* (\hat{\alpha}_2 B + \hat{\alpha}_3 Y_*) \right].\]

(52)

Dropping the total derivative terms, the above expression reduces to

\[\frac{1}{J(\kappa)} \frac{d J(\kappa)}{d\kappa} = \int d^4x \left[ -i\partial_\mu Y_* \partial^\mu Y + \partial_\mu K_* \partial^\mu K + i\hat{\beta}_2 BM^2 Y - \hat{\beta}_2 c_* M^2 K + i\hat{\beta}_3 Y_* M \pi 
\right.
\[+ \hat{\beta}_3 M^2 c K_* + i\hat{\beta}_4 Y_* M^2 Y - \hat{\beta}_4 K_* M^2 + \frac{1}{2} i B (\hat{\alpha} B + \hat{\alpha}_2 Y_*)
\[\left. + \frac{1}{2} i Y_* (\hat{\alpha}_2 B + \hat{\alpha}_3 Y_*) \right].\]

(53)
Further, by considering all the terms appearing in the above expression, we postulate the functional $S_1$ to have the following form:

$$S_1[\phi(x, \kappa), \kappa] = \int d^4x \left[ \xi_1(\kappa) \partial_\mu Y_+ \partial^\mu Y + \xi_2(\kappa) \partial_\mu K_+ \partial^\mu K + \xi_3(\kappa) \hat{\beta}_2 BM^2 Y + \xi_4(\kappa) \hat{\beta}_2 c_+ M^2 K + \xi_5(\kappa) \hat{\beta}_3 Y_+ M \pi + \xi_6(\kappa) \hat{\beta}_3 K_+ M^2 c + \xi_7(\kappa) \hat{\beta}_4 Y_+ M^2 Y + \xi_8(\kappa) \hat{\beta}_4 K_+ M^2 K + \xi_9(\kappa) B(\hat{\alpha} B + \hat{\alpha}_2 Y_+) + \xi_{10}(\kappa) Y_+ (\hat{\alpha}_2 B + \hat{\alpha}_3 Y_+) \right],$$

where all $\kappa$-dependent constant parameters ($\xi_i, i = 1, 2, \ldots, 10$) are prescribed to satisfy the following initial boundary conditions:

$$\xi_i(\kappa = 0) = 0.$$ 

Now, the infinitesimal change in $S_1$ is evaluated as

$$\frac{dS_1}{d\kappa} = \int d^4x \left[ \frac{d\xi_1}{d\kappa} \partial_\mu Y_+ \partial^\mu Y + \frac{d\xi_2}{d\kappa} \partial_\mu K_+ \partial^\mu K + \frac{d\xi_3}{d\kappa} \hat{\beta}_2 BM^2 Y + \frac{d\xi_4}{d\kappa} \hat{\beta}_2 c_+ M^2 K + \frac{d\xi_5}{d\kappa} \hat{\beta}_3 Y_+ M \pi + \frac{d\xi_6}{d\kappa} \hat{\beta}_3 K_+ M^2 c + \frac{d\xi_7}{d\kappa} \hat{\beta}_4 Y_+ M^2 Y + \frac{d\xi_8}{d\kappa} \hat{\beta}_4 K_+ M^2 K + \frac{d\xi_9}{d\kappa} B(\hat{\alpha} B + \hat{\alpha}_2 Y_+) + \frac{d\xi_{10}}{d\kappa} (\kappa) Y_+ (\hat{\alpha}_2 B + \hat{\alpha}_3 Y_+) \right].$$

where we have utilized the relation (2). The essential condition (7), which validates the functional $S_1$ together with Eqs. (53) and (56), yields

$$\int d^4x \left[ i \left( \frac{d\xi_1}{d\kappa} + 1 \right) \partial_\mu Y_+ \partial^\mu Y + i \left( \frac{d\xi_2}{d\kappa} + 1 \right) \partial_\mu K_+ \partial^\mu K + i \left( \frac{d\xi_3}{d\kappa} + 1 \right) \hat{\beta}_2 BM^2 Y + i \left( \frac{d\xi_4}{d\kappa} + 1 \right) \hat{\beta}_2 c_+ M^2 K + \left( \frac{d\xi_5}{d\kappa} + 1 \right) \hat{\beta}_3 Y_+ M \pi + \left( \frac{d\xi_6}{d\kappa} + 1 \right) \hat{\beta}_3 K_+ M^2 c + \left( \frac{d\xi_7}{d\kappa} + 1 \right) \hat{\beta}_4 Y_+ M^2 Y + \left( \frac{d\xi_8}{d\kappa} + 1 \right) \hat{\beta}_4 K_+ M^2 K + iB(\hat{\alpha} B + \hat{\alpha}_2 Y_+) \right] \left[ \left( \frac{d\xi_9}{d\kappa} + \frac{d\xi_{10}}{d\kappa} \right) (\kappa) Y_+ (\hat{\alpha}_2 B + \hat{\alpha}_3 Y_+) \right] = 0.$$ 

Comparing the coefficients of the various terms present in the above expression from the LHS to the RHS, we get the following differential equations:

$$\frac{d\xi_1}{d\kappa} + 1 = 0, \quad \frac{d\xi_2}{d\kappa} + 1 = 0, \quad \frac{d\xi_3}{d\kappa} + 1 = 0, \quad \frac{d\xi_4}{d\kappa} + 1 = 0, \quad \frac{d\xi_5}{d\kappa} + 1 = 0, \quad \frac{d\xi_6}{d\kappa} + 1 = 0, \quad \frac{d\xi_7}{d\kappa} + 1 = 0, \quad \frac{d\xi_8}{d\kappa} + 1 = 0, \quad \frac{d\xi_9}{d\kappa} + \frac{d\xi_{10}}{d\kappa} \bigg|_{\frac{1}{2}} = 0, \quad \frac{d\xi_{10}}{d\kappa} \bigg|_{\frac{1}{2}} = 0.$$
together with
\[\begin{align*}
\xi_1 + i \xi_2 &= 0, \quad \xi_3 + i \xi_4 = 0, \\
\xi_5 + i \xi_6 &= 0, \quad \xi_7 + i \xi_8 = 0.
\end{align*}\] (59)

The solutions of the above equations satisfying the initial conditions (55) are
\[\begin{align*}
\xi_1 &= -\kappa, \quad \xi_2 = -i \kappa, \quad \xi_3 = \kappa, \\
\xi_4 &= i \kappa, \quad \xi_5 = \kappa, \quad \xi_6 = i \kappa, \\
\xi_7 &= \kappa, \quad \xi_8 = i \kappa, \quad \xi_9 = \frac{1}{2} \kappa, \\
\xi_{10} &= \frac{1}{2} \kappa.
\end{align*}\] (60)

With these identifications of constant parameters \(\xi_i\), the exact form of \(S_1\) is given by
\[S_1[\phi(x, \kappa), \kappa] = \int d^4x \left[ -\kappa \partial_\mu Y_\star \partial^\mu Y - i \kappa \partial_\mu K_\star \partial^\mu K + \kappa \hat{\beta}_2 B M^2 Y + i \kappa \hat{\beta}_2 c_\star M^2 K \\
+ \kappa \hat{\beta}_3 Y_\star M \pi + i \kappa \hat{\beta}_3 K_\star M^2 c + \kappa \hat{\beta}_4 Y_\star M^2 Y + i \kappa \hat{\beta}_4 K_\star M^2 K \\
+ \frac{\kappa}{2} B (\hat{\alpha} B + \hat{\alpha}_2 Y_\star) + \frac{\kappa}{2} Y_\star (\hat{\alpha}_2 B + \hat{\alpha}_3 Y_\star) \right],\] (61)

which vanishes at \(\kappa = 0\). However, the functional \(S_1\) at \(\kappa = 1\) (under FFBRST transformation) accumulates to the effective action (42) within the functional integral as
\[\int d^4x \, L_{\text{eff}} + S_1[\phi(x, 1), 1] = \int d^4x \left[ L_H + B \partial_\mu A^\mu - \partial_\mu Y_\star \partial^\mu Y + (\hat{\beta}_1 B + \hat{\beta}_3 Y_\star) M \pi \\
+ (\hat{\beta}_2 B + \hat{\beta}_3 Y_\star) M^2 Y + \frac{1}{2} \hat{\alpha}_1 B^2 + \hat{\alpha}_2 B Y_\star + \frac{1}{2} \hat{\alpha}_3 Y_\star^3 \\
- i \partial_\mu c_\star \partial^\mu c - i \partial_\mu K_\star \partial^\mu K + i (\hat{\beta}_1 c_\star + \hat{\beta}_3 K_\star) M^2 c \\
+ i (\hat{\beta}_2 c_\star + \hat{\beta}_4 K_\star) M^2 K \right],\] (62)

which is nothing but the BRST-invariant effective action for the gaugeon Higgs model with shifted gauge parameters. Therefore, we conclude that, under generalized BRST transformations with an appropriate finite field-dependent parameter, the gaugeon modes to describe quantum gauge freedom in the Higgs model appear naturally in the effective action.

4. Conclusions

In this paper, we have first evoked the gaugeon formalism for both the Yang–Mills theory [13] and Higgs model [15]. Following Refs. [13,16], we have extended the configuration space by introducing the gaugeon field and its associated field in the effective actions of these models. Further, the quantum gauge transformation has been derived for such extended actions. Under quantum gauge transformation, the extended action remains form invariant along with a shift in gauge parameters. These natural shifts in gauge parameters show a resemblance to those that appear through proper renormalization [1]. Since these gaugeon fields are unphysical, one needs to remove them. For this purpose, we have inserted a subsidiary condition of Gupta–Bleuler type for Yang–Mills theory that removes the unphysical gaugeon modes. However, the Gupta–Bleuler-type restriction has certain limitations.
This situation is improved in the Higgs model, where we have enlarged the configuration space by incorporating ghost fields corresponding to the gaugeon fields in the effective action. Now, such an enlarged action possesses both BRST symmetry and quantum gauge symmetry. In this enlarged Higgs action, the unphysical gaugeon modes are removed by the more acceptable Kugo–Ojima-type condition.

In this work we have considered the Yang–Mills theory and Higgs model to investigate the quantum gauge freedom through the Yokoyama gaugeon formalism in the framework of the generalized BRST (FFBRST) transformation. We have generalized the BRST symmetry by making the infinitesimal transformation parameter finite and field-dependent. Such a generalized BRST transformation has symmetry of the action only, not of the generating functional of the Green’s functions. We have shown that, for a particular finite field-dependent parameter, the Jacobian of the path integral measure under the generalized BRST transformation generates the gaugeon mode in the effective action in a more rigorous way. We have established the results in both the Yang–Mills and Higgs theories with explicit calculations. Further implications and aspects of the present investigations in certain string theory, M-theory, and gravity theory would also be interesting.

Funding
Open Access funding: SCOAP³.

References
[1] K. Yokoyama, Prog. Theor. Phys. 51, 1956(1974).
[2] K. Yokoyama, Prog. Theor. Phys. 59, 1699 (1978).
[3] K. Yokoyama, Prog. Theor. Phys. 60, 1167 (1978).
[4] K. Yokoyama, Phys. Lett. B 79, 79 (1978).
[5] K. Yokoyama and R. Kubo, Prog. Theor. Phys. 52, 290 (1974).
[6] K. Yokoyama, M. Takeda, and M. Monda, Prog. Theor. Phys. 60, 927 (1978).
[7] K. Yokoyama, M. Takeda, and M. Monda, Prog. Theor. Phys. 64, 1412 (1980).
[8] M. Hayakawa and K Yokoyama, Prog. Theor. Phys. 44, 533 (1970).
[9] K. Izawa, Prog. Theor. Phys. 88, 759 (1992).
[10] M. Koseki, M. Sato, and R. Endo, Prog. Theor. Phys. 90, 1111 (1993).
[11] T. Kugo and I. Ojima, Prog. Theor. Phys. Suppl. 66, 1 (1979).
[12] T. Kugo and I. Ojima, Nucl. Phys. B 144, 234 (1978).
[13] M. Koseki, M. Sato, and R. Endo, Bull. Yamagata Univ., Nat. Sci. 14, 15 (1996).
[14] Y. Nakawaki, Prog. Theor. Phys. 98, 5 (1997).
[15] R. Endo and M. Koseki, Prog. Theor. Phys. 103, 3 (2000).
[16] H. Miura and R Endo, Prog. Theor. Phys. 117, 4 (2007).
[17] M. Faizal, Commun. Theor. Phys. 57, 637 (2012).
[18] M. Faizal, Mod. Phys. Lett. A 27, 1250147 (2012).
[19] S. Upadhyay, Ann. Phys. 344, 290 (2014); Eur. Phys. J. C. 74, 2737 (2014).
[20] S. D. Joglekar and B. P. Mandal, Phys. Rev. D 51, 1919 (1995).
[21] S. Upadhyay, Phys. Lett. B 727, 293 (2013) [arXiv:1310.2013 [hep-th]].
[22] R. Banerjee and S. Upadhyay, arXiv:1310.1168 [hep-th].
[23] S. D. Joglekar and B. P. Mandal, Int. J. Mod. Phys. A 17, 1279 (2002).
[24] R. Banerjee and B. P. Mandal, Phys. Lett. B 488, 27 (2000).
[25] S. Upadhyay, S. K. Rai, and B. P. Mandal, J. Math. Phys. 52, 22301 (2011).
[26] S. D. Joglekar and A. Misra, Int. J. Mod. Phys. A 15, 1453 (2000).
[27] S. Upadhyay and B. P. Mandal, Mod. Phys. Lett. A 25, 3347 (2010).
[28] S. Upadhyay and B. P. Mandal, Europhys. Lett. 93, 31001 (2011).
[29] S. Upadhyay and B. P. Mandal, Eur. Phys. J. C 72, 2065 (2012).
[30] S. Upadhyay and B. P. Mandal, Ann. Phys. 327, 2885 (2012).
[31] S. Upadhyay and B. P. Mandal, AIP Conf. Proc. 1444, 213 (2012).
[32] S. Upadhyay, M. K. Dwivedi, and B. P. Mandal, Int. J. Mod. Phys. A 28, 1350033 (2013).
[33] M. Faizal, B. P. Mandal, and S. Upadhyay, Phys. Lett. B 721, 159 (2013).
[34] B. P. Mandal, S. K. Rai, and S. Upadhyay, Europhys. Lett. 92, 21001 (2010).
[35] R. Banerjee, B. Paul, and S. Upadhyay, Phys. Rev. D 88, 065019 (2013).
[36] S. Upadhyay, Europhys. Lett. 104, 61001 (2013); Europhys. Lett. 105, 21001 (2014); Ann. Phys. 340, 110 (2014); arXiv:1404.2633.
[37] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).
[38] P. W. Higgs, Phys. Rev. 145, 1156 (1966).
[39] T. W. B. Kibble, Phys. Rev. 155, 1554 (1967).
[40] K. Fujikawa, B. W. Lee, and A. I. Sanda, Phys. Rev. D 6, 2923 (1972).