BOUNDNESS OF VARIATION OPERATORS
ASSOCIATED WITH THE HEAT SEMIGROUP
GENERATED BY HIGH ORDER SCHRÖDINGER
TYPE OPERATORS

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Abstract In this article, we derive the $L^p$-boundedness of the variation operators associated with the heat semigroup which is generated by the high order Schrödinger type operator $(-\Delta)^2 + V^2$ in $\mathbb{R}^n (n \geq 5)$ with $V$ being a nonnegative potential satisfying the reverse Hölder inequality. Furthermore, we prove the boundedness of the variation operators on associated Morrey spaces. In the proof of the main results, we always make use of the variation inequalities associated with the heat semigroup generated by the biharmonic operator $(-\Delta)^2$.

Key words Variation operators; high order Schrödinger type operators; heat semigroup

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1 Introduction

Variation inequalities have been the subject of many recent articles in probability, ergodic theory, and harmonic analysis. The first variation inequality was proved by Lépingle [16] in martingale theory. In [3], Bourgain proved the variation inequality for the ergodic averages of a dynamic system. This work inaugurated a new research direction in ergodic theory and harmonic analysis. After that, Campbell, Jones, Reinhold, and Wierdl [5] proved the variation inequalities for the Hilbert transform. Since then, many other publications has come along to enrich the literature on this subject in harmonic analysis (see [4, 6, 9, 11–13, 19] and so on).

Let $\{T_t\}_{t>0}$ be a family of operators such that the limit $\lim_{t \to 0} T_t f(x)$ exists in some sense. A classical method of measuring the speed of convergence of the family $\{T_t\}_{t>0}$ is to consider the
“square function” of the type \( \left( \sum_{i=1}^{\infty} |T_{i+1}f - T_{i}f|^2 \right)^{1/2} \), where \( t_i \searrow 0 \), or more generally the variation operator \( V_\rho(T_i) \), where \( \rho > 2 \), is given by

\[
V_\rho(T_i)(f)(x) := \sup_{t_i \searrow 0} \left( \sum_{i=1}^{\infty} |T_{i+1}f(x) - T_{i}f(x)|^\rho \right)^{1/\rho},
\]

where the supremum is taken over all the positive decreasing sequences \( \{t_j\}_{j \in \mathbb{N}} \) which converge to 0. We denote with \( E_\rho \) the space including all the functions \( w : (0, \infty) \to \mathbb{R} \) such that

\[
\|w\|_{E_\rho} := \sup_{t_i \searrow 0} \left( \sum_{i=0}^{\infty} |w(t_i) - w(t_{i+1})|^\rho \right)^{1/\rho} < \infty.
\]

\( \|w\|_{E_\rho} \) is a seminorm on \( E_\rho \); it can be written as

\[
V_\rho(T_i)(f) = \|T_i f\|_{E_\rho}.
\]

In this article, we mainly focus on the variation operators associated with the high order Schrödinger type operators \( \mathcal{L} = (-\Delta)^2 + V^2 \) in \( \mathbb{R}^n \) with \( n \geq 5 \), where the nonnegative potential \( V \) belongs to the reverse Hölder class \( RH_q \) for some \( q > n/2 \); that is, there exists \( C > 0 \) such that

\[
\left( \frac{1}{|B|} \int_B V(x)^q \, dx \right)^{1/q} \leq \frac{C}{|B|} \int_B V(x) \, dx
\]

for every ball \( B \) in \( \mathbb{R}^n \). Some results related to \( (-\Delta)^2 + V^2 \) were first considered by Zhong in [29].

In [25], Sugano proved the estimation of the fundamental solution and the \( L^p \)-boundedness of some operators related to this operator. For more results related to this operator, see [7, 17, 18].

The heat semigroup \( e^{-t\mathcal{L}} \) generated by the operator \( \mathcal{L} \) can be written as

\[
e^{-t\mathcal{L}} f(x) = \int_{\mathbb{R}^n} B_t(x, y) f(y) \, dy, \quad \text{for } f \in L^2(\mathbb{R}^n), t > 0.
\]

The kernel of the heat semigroup \( e^{-t\mathcal{L}} \) satisfies the estimate

\[
|B_t(x, y)| \leq C t^{-\frac{n}{2}} e^{-\frac{d(x, y)^2}{Ct}};
\]

for more details see [1].

We recall the definition of the function \( \gamma(x) \), which plays an important role in the theory of operators associated with \( \mathcal{L} \):

\[
\gamma(x) = \sup \left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x, r)} V(x) \, dx \leq 1 \right\}, \quad x \in \mathbb{R}^n.
\]

This was introduced by Shen [21].

For the Schrödinger operator \( L = -\Delta + V \), Betancor et al. established the \( L^p \)-boundedness properties of the variation operators related to the heat semigroup \( \{e^{-t\mathcal{L}}\}_{t > 0} \) in [2]. It is a natural and interesting question as to whether we can establish the boundedness properties of the variation operators associated with \( \{e^{-t\mathcal{L}}\}_{t > 0} \) on \( L^p(\mathbb{R}^n) \). Our main result is as follows.

**Theorem 1.1** Assume that \( V \in RH_{q_0}(\mathbb{R}^n) \), where \( q_0 \in (n/2, \infty) \) and \( n \geq 5 \). For \( \rho > 2 \), there exists a constant \( C > 0 \) such that

\[
\|V_\rho(e^{-t\mathcal{L}})(f)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}, \quad 1 < p < \infty.
\]
We should note that our results are not contained in the article of Bui [4], because the estimates of the heat kernel are not the same.

On the other hand, Zhang and Wu [28] studied the boundedness of variation operators associated with the heat semigroup \(\{e^{-tL}\}_{t>0}\) on Morrey spaces related to the non-negative potential \(V\). Tang and Dong [26] introduced Morrey spaces related to non-negative potential \(V\) for extending the boundedness of Schrödinger type operators in Lebesgue spaces.

**Definition 1.2** Let \(1 \leq p < \infty, \alpha \in \mathbb{R},\) and \(0 \leq \lambda < n\). For \(f \in L^p_{\text{loc}}(\mathbb{R}^n)\) and \(V \in RH_q(q > 1)\), we say that \(f \in L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)\) if
\[
\|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} = \sup_{B(x_0,r) \subset \mathbb{R}^n} \left(1 + \frac{r}{\gamma(x_0)}\right)^{\alpha} r^{-\lambda} \int_{B(x_0,r)} |f(x)|^p \, dx < \infty,
\]
where \(B(x_0,r)\) denotes a ball centered at \(x_0\) and with radius \(r\) and \(\gamma(x_0)\) is defined as in (1.2).

For more information about the Morrey spaces associated with differential operators, see [10, 23, 27].

We can now also obtain the boundedness of the variation operators associated to the heat semigroup \(\{e^{-tL}\}_{t>0}\) on Morrey spaces.

**Theorem 1.3** Let \(V \in RH_{q_0}(\mathbb{R}^n)\) for \(q_0 \in (n/2, \infty), n \geq 5,\) and \(\rho > 2\). Assume that \(\alpha \in \mathbb{R}\) and \(\lambda \in (0,n)\). There exists a constant \(C > 0\) such that
\[
\|V^\rho(e^{-tL})(f)\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)} \leq C \|f\|_{L^{p,\lambda}_{\alpha,V}(\mathbb{R}^n)}, \quad 1 < p < \infty.
\]

The organization of the article is as follows: Section 2 is devoted to giving the proof of Theorem 1.1. In order to prove this theorem, we should study the strong \(L^p\)-boundedness of the variation operators associated with \(\{e^{-t\Delta^2}\}_{t>0}\) first. We will give the proof of Theorem 1.3 in Section 3. We also obtain the strong \(L^p(\mathbb{R}^n)\) estimates \((p > 1)\) of the generalized Poisson operators \(P^\rho_{t,\mathcal{L}}\) on \(L^p\) spaces as well as Morrey spaces related to the non-negative potential \(V\), in Sections 2 and 3, respectively.

Throughout this article, the symbol \(C\) in an inequality always denotes a constant which may depend on some indices, but never on the functions \(f\) under consideration.

### 2 Variation Inequalities Related to \(\{e^{-tL}\}_{t>0}\) on \(L^p\) Spaces

In this section, we first recall some properties of the biharmonic heat kernel. With these kernel estimates, we will give the proof of \(L^p\)-boundedness properties of the variation operators related to \(\{e^{-t\Delta^2}\}_{t>0}\), which is crucial in the proof of Theorem 1.1.

#### 2.1 Biharmonic heat kernel

Consider the following Cauchy problem for the biharmonic heat equation:
\[
\begin{cases}
(\partial_t + \Delta^2)u(x,t) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\
u(x,0) = f(x) & \text{in } \mathbb{R}^n.
\end{cases}
\]
Its solution is given by
\[
u(x,t) = e^{-t\Delta^2} f(x) = \int_{\mathbb{R}^n} b(x-y,t) f(y) \, dy,
\]
where
\[ b(x,t) = t^{-\frac{n}{4}} g(\eta), \]
where \( \eta = xt^{-\frac{1}{4}} \) and
\[
g(\eta) = (2\pi)^{-\frac{n}{4}} \int_{\mathbb{R}^n} e^{i\eta k} \, dk = \alpha_n |\eta| \int_0^{\infty} e^{-s^4} J_{(n-2)/2}(s) \, ds, \quad \eta \in \mathbb{R}^n, \quad (2.1)
\]
where \( J_v \) denotes the \( v \)-th Bessel function and \( \alpha_n > 0 \) is a normalization constant such that
\[
\int_{\mathbb{R}^n} g(\eta) \, d\eta = 1,
\]
and \( g(\eta) \) satisfies the following estimates
\[
|g(\eta)| \leq C \left( 1 + |\eta| \right)^{-\frac{n}{4}} e^{-A_1 |\eta|^{\frac{4}{3}}},
\]
\[
\frac{d^m g}{d\eta^m}(\eta) \leq C_m \left( 1 + |\eta| \right)^{-\frac{n-m}{4}} e^{-A_1 |\eta|^{\frac{4}{3}}}, \quad m \in \mathbb{N}; \quad (2.2)
\]
see [14]. Then, by classical analysis, we have the following results (for details, see [24]):

(a) If \( f \in L^p(\mathbb{R}^n), 1 \leq p \leq \infty \), then
\[
\lim_{t \to 0} u(x,t) = f(x) \quad \text{a.e.} \quad x \in \mathbb{R}^n,
\]
and
\[
\|u(\cdot,t)\|_{L^p(\mathbb{R}^n)} \leq C \|f\|_{L^p(\mathbb{R}^n)}.
\]

(b) If \( 1 \leq p < \infty \), then
\[
\|u(\cdot,t) - f\|_{L^p(\mathbb{R}^n)} \to 0, \quad \text{when} \quad t \to 0^+.
\]

We should note that the heat semigroup \( e^{-t\Delta^2} \) does not have the positive preserving property; that is, when \( f \geq 0 \), \( e^{-t\Delta^2} f \geq 0 \) may not to be established. Thus, the boundedness of the variation operators associated with \( \{e^{-t\Delta^2}\}_{t>0} \) cannot be deduced by the results in [11].

For the heat kernel \( b(x,t) \) of the semigroup \( e^{-t\Delta^2} \), we have the following estimates:

**Lemma 2.1** For every \( t > 0 \) and \( \mathbb{R}^n \), we have
\[
|b(x,t)| \leq Ct^{-\frac{n}{4}} e^{-A_1 |xt^{-\frac{1}{4}}|^{\frac{4}{3}}}, \quad (2.3)
\]
\[
|\partial^k \nabla^l b(x,t)| \leq C \left( t^{\frac{n}{4}} + |x| \right)^{-n-k-l}, \quad \forall k, l \geq 1, \quad (2.4)
\]
\[
|\partial_t b(x,t)| \leq Ct^{-\frac{n}{4} - 1} \left( 1 + |xt^{-\frac{1}{4}}| \right)^{-\frac{n}{4} - \frac{1}{4}} e^{-A_1 |xt^{-\frac{1}{4}}|^{\frac{4}{3}}}, \quad (2.5)
\]
\[
|\nabla_x b(x,t)| \leq Ct^{-\frac{n+1}{4}} \left( 1 + |xt^{-\frac{1}{4}}| \right)^{-\frac{n+1}{4} - \frac{1}{4}} e^{-A_1 |xt^{-\frac{1}{4}}|^{\frac{4}{3}}}, \quad (2.6)
\]
where \( A_1 = \frac{321/3}{16} \).

**Proof** For (2.3) and (2.4), see Lemma 2.4 in [14]. From (2.1), (2.2), and some simple calculations, we can derive (2.5) and (2.6). \( \square \)
2.2 Variation inequalities related to \( \{ e^{-t\Delta^2} \}_{t>0} \)

By Lemma 2.1 in Section 2.1, we know that the operator \( e^{-t\Delta^2} \) is a contraction on \( L^1(\mathbb{R}^n) \) and \( L^\infty(\mathbb{R}^n) \). Thus, \( e^{-t\Delta^2} \) is contractively regular. Then, by [15, Corollary 3.4], we have the following theorem (for more details, see [15]):

**Theorem 2.2** For \( \rho > 2 \), there exists a constant \( C > 0 \) such that

\[
\| \mathcal{V}_\rho(e^{-t\Delta^2})(f) \|_{L^p(\mathbb{R}^n)} \leq C \| f \|_{L^p(\mathbb{R}^n)}, \quad \forall f \in L^p(\mathbb{R}^n), \quad 1 < p < \infty.
\]

2.3 Variation inequalities related to \( \{ e^{-t\mathcal{L}} \}_{t>0} \)

First, we recall some properties of the auxiliary function \( \gamma(x) \), which will be used later.

**Lemma 2.3** ([21]) Let \( V \in RH_\frac{n}{2}(\mathbb{R}^n) \). Then there exist \( C \) and \( k_0 > 1 \) such that for all \( x, y \in \mathbb{R}^n \),

\[
\frac{1}{C} \gamma(x) \left(1 + \frac{|x-y|}{\gamma(x)}\right)^{-k_0} \leq \gamma(y) \leq C \gamma(x) \left(1 + \frac{|x-y|}{\gamma(x)}\right)^{k_0}.\]

In particular, \( \gamma(x) \sim \gamma(y) \) if \( |x-y| < C \gamma(x) \).

**Lemma 2.4** (Lemma 2.7 in [7]) Let \( V \in RH_{\frac{n}{2}}(\mathbb{R}^n) \) and \( \delta = 2 - n/q_0 \), where \( q_0 \in (n/2, \infty) \) and \( n \geq 5 \). Then there exists a positive constant \( C \) such that for all \( x, y \in \mathbb{R}^n \) and \( t \in (0, \gamma^4(x)) \),

\[
\int_{\mathbb{R}^n} \frac{V^2(y)}{t^{n/4}} e^{-A_4 \frac{|x-y|^{4/3}}{t^{1/3}}} dy \leq C t^{-1} \left( \frac{1}{\gamma(x)} \right)^{2\delta}.
\]

where \( A_4 = \min\{A, A_1\} \) and \( A, A_1 \) are constants as in (1.1) and (2.3), respectively.

Now we can prove the following kernel estimates of \( e^{-t\mathcal{L}} \):

**Lemma 2.5** For every \( N \in \mathbb{N} \), there exist positive constants \( C_2, A_2, \) and \( A_3 \) such that for all \( x, y \in \mathbb{R}^n \) and \( 0 < t < \infty \), we have that

(i) \( |B_t(x, y)| \leq Ct^{-\frac{n}{4}} \left(1 + \frac{\sqrt{t}}{\gamma^2(x)} + \frac{\sqrt{t}}{\gamma^2(y)}\right)^{-N} e^{-A_2 \frac{|x-y|^{4/3}}{t^{1/3}}} \),

(ii) \( \left| \frac{\partial}{\partial t} B_t(x, y) \right| \leq Ct^{-\frac{n+4}{4}} \left(1 + \frac{\sqrt{t}}{\gamma^2(x)} + \frac{\sqrt{t}}{\gamma^2(y)}\right)^{-N} e^{-A_3 \frac{|x-y|^{4/3}}{t^{1/3}}} \),

where \( A_2 = A_1/2, \) and \( A_3 < A_2 \).

**Proof** For (i), see Theorem 2.5 of [7].

Now we give the proof of (ii). As \( \mathcal{L} = (-\Delta)^2 + V^2 \) is a nonnegative self-adjoint operator, we can extend the semigroup \( \{ e^{-t\mathcal{L}} \} \) to a holomorphic semigroup \( \{ T_\xi \}_{\xi \in \mathbb{R}^{n/4}} \) uniquely. By a similar argument as to that in [8, Corollary 6.4], the kernel \( B_\xi(x, y) \) of \( T_\xi \) satisfies

\[
|B_\xi(x, y)| \leq C_N(\mathbb{R}^n)^{-n/4} \left(1 + \frac{\sqrt{\mathbb{R}^n}}{\gamma^2(x)} + \frac{\sqrt{\mathbb{R}^n}}{\gamma^2(y)}\right)^{-N} e^{-C \frac{|x-y|^{4/3}}{(\mathbb{R}^n)^{1/3}}}. \tag{2.7}
\]

The Cauchy integral formula combined with (2.7) gives

\[
\left| \frac{\partial}{\partial t} B_t(x, y) \right| = \frac{1}{2\pi} \int_{|t|=1/10} B_t(x, y) d\xi \leq C_N \left(1 + \frac{\sqrt{\mathbb{R}^n}}{\gamma^2(x)} + \frac{\sqrt{\mathbb{R}^n}}{\gamma^2(y)}\right)^{-N} e^{-C \frac{|x-y|^{4/3}}{(\mathbb{R}^n)^{1/3}}}.
\]

Thus, we complete the proof.

With the estimates above, we can give the proof of Theorem 1.1.

**Proof of Theorem 1.1** For \( f \in L^p(\mathbb{R}^n), \) \( 1 < p < \infty \), we consider the local operators

\[
e^{t\mathcal{L}} f(x) = \int_{|x-y| < \gamma(x)} B_t(x, y) f(y) dy, \quad x \in \mathbb{R}^n,
\]

\[ \square \] Springer.
and
\[
e^{-t\Delta^2} f(x) = \int_{|x-y| < \gamma(x)} b_t(x-y) f(y) dy, \quad x \in \mathbb{R}^n.
\]

Then, we have
\[
V_\rho(e^{-t\mathcal{L}})(f) \leq V_\rho(e^{-t\mathcal{L}} - e^{-t\Delta^2})(f) + V_\rho(e^{-t\Delta^2})(f) + V_\rho(e^{-t\mathcal{L}} - e^{-t\mathcal{L}})(f) =: J_1 + J_2 + J_3.
\]

Let us analyze term \( J_2 \) first:
\[
J_2 \leq \sup_{t_j / 2 < 0} \left( \sum_{j=0}^{\infty} \left| e^{-t_j \Delta^2} f(x) - e^{-t_{j+1} \Delta^2} f(x) \right|^p \right)^{1/p} + \sup_{t_j / 2 < 0} \left( \sum_{j=0}^{\infty} \left| \int_{|x-y| > \gamma(x)} b(x-y, t_j) - b(x-y, t_{j+1}) dy \right|^p \right)^{1/p} \\
\leq V_\rho(e^{-t\Delta^2})(f) + \sup_{\varepsilon > 0} \left\| \int_{|x-y| > \varepsilon} b(x-y, t) f(y) \right\|_{E_\rho}.
\]

Now, we consider the operator defined by
\[
T : L^2(\mathbb{R}^n) \rightarrow L^2_{E_\rho}(\mathbb{R}^n)
\]
\[
f \rightarrow Tf(x) = \int_{\mathbb{R}^n} b(x-y, t) f(y) dy,
\]
which is bounded from \( L^2(\mathbb{R}^n) \) into \( L^2_{E_\rho}(\mathbb{R}^n) \) according to Theorem 2.2. Moreover, \( T \) is a Calderón-Zygmund operator with the \( E_\rho \)-valued kernel \( b(x-y, t) \). In fact, the kernel \( b(x-y, t) \) has the following two properties:

(A) By (2.5), we have
\[
\left\| b(x-y, \cdot) \right\|_{E_\rho} \leq \sup_{t_j / 2 < 0} \sum_{j=0}^{\infty} \int_{t_j}^{t_{j+1}} \left| \frac{\partial}{\partial t} b(x-y, t) \right| dt \\
\leq C \int_0^{\infty} \left| \frac{\partial}{\partial t} b(x-y, t) \right| dt \\
\leq C \int_0^{\frac{1}{|x-y|^4}} \left| \frac{\partial}{\partial t} b(x-y, t) \right| dt + C \int_0^{\infty} \left| \frac{\partial}{\partial t} b(x-y, t) \right| dt \\
\leq C \int_0^{\frac{1}{|x-y|^4}} t^{-\frac{n}{4} - 1} \left( \frac{t^{1/3}}{|x-y|^{4/3}} \right)^{(n+4)/4} dt + \int_0^{\infty} t^{-\frac{n}{4} - 1} dt \\
\leq C|x-y|^{-n}, \quad x, y \in \mathbb{R}^n.
\]

(B) Proceeding a similar way, together with (2.4), we have
\[
\left\| \frac{\partial}{\partial x} b(x-y, \cdot) \right\|_{E_\rho} + \left\| \frac{\partial}{\partial y} b(x-y, \cdot) \right\|_{E_\rho} \leq C|x-y|^{-n-1}, \quad x, y \in \mathbb{R}^n.
\]

Thus, by proceeding as in the proof of [22, Proposition 2 in p.34 and Corollary 2 in p.36], we can prove that the maximal operator \( T^* \) defined by
\[
T^* = \sup_{\varepsilon > 0} \left\| \int_{|x-y| > \varepsilon} b(x-y, t) f(y) \right\|_{E_\rho}
\]
is bounded on \( L^p(\mathbb{R}^n) \) for every \( 1 < p < \infty \). Combining this with Theorem 2.2, we conclude that \( V_\rho(e^{-t\Delta^2}) \) is bounded from \( L^p(\mathbb{R}^n) \) into itself for every \( 1 < p < \infty \).
Next, we consider term $J_3$:

$$J_3 = \sup_{t_j > 0} \left( \sum_{j=0}^{\infty} \int_{|x-y| > \gamma(x)} |(B_{t_j}(x, y) - B_{t_{j+1}}(x, y))f(y)|^p dy \right)^{\frac{1}{p}}$$

$$\leq \sup_{t_j > 0} \sum_{j=0}^{\infty} \int_{|x-y| > \gamma(x)} |f(y)| \int_{t_j}^{t_{j+1}} \left| \frac{\partial}{\partial t} B_t(x, y) \right| dt dy$$

$$\leq \int_{|x-y| > \gamma(x)} |f(y)| \int_0^\infty \left| \frac{\partial}{\partial t} B_t(x, y) \right| dt dy$$

$$\leq \int_{|x-y| > \gamma(x)} |f(y)| \left( \int_0^{\gamma(x)} \left| \frac{\partial}{\partial t} B_t(x, y) \right| dt + \int_0^{\gamma(x)} \left| \frac{\partial}{\partial t} B_t(x, y) \right| dt \right) dy$$

$$:= J_{31} + J_{32}.$$

To estimate $J_{31}$, by Lemma 2.5 with $N = n + 2$ and changing variables, we have

$$J_{31} \leq C \int_{|x-y| > \gamma(x)} |f(y)| \int_0^{\gamma(x)} t^{-\frac{n+2}{2}} \left( 1 + \frac{\sqrt{t}}{\gamma(x)} \right)^{-n-2} e^{-A_3 \frac{t}{\sqrt{t}}} dt dy$$

$$\leq C \int_{|x-y| > \gamma(x)} |f(y)| \int_0^{\gamma(x)} \frac{1}{\gamma_n(x)} t^{-\frac{n+2}{2}} (1 + u)^{n+2} e^{-\frac{A_3}{\sqrt{u}}} du dy$$

$$\leq C \frac{1}{\gamma_n(x)} \int_{|x-y| > \gamma(x)} |f(y)| \left( \frac{\gamma(x)}{|x-y|} \right)^{n+2} dy$$

$$\leq C \frac{1}{\gamma_n(x)} \int_{|x-y| > \gamma(x)} |f(y)| \left( \frac{\gamma(x)}{|x-y|} \right)^{n+2} dy$$

$$\leq C \frac{1}{\gamma_n(x)} \int_{|x-y| > \gamma(x)} |f(y)| \left( \frac{\gamma(x)}{|x-y|} \right)^{n+2} dy$$

$$\leq C \int_{|x-y| > \gamma(x)} |f(y)| \left( \frac{\gamma(x)}{|x-y|} \right)^{n+2} dy$$

$$\leq C \int_{|x-y| > \gamma(x)} |f(y)| \left( \frac{\gamma(x)}{|x-y|} \right)^{n+2} dy$$

$$= CM(f)(x),$$

where $M(f)$ is the Hardy-Littlewood maximal function of $f$. For $J_{32}$, by Lemma 2.5, we have

$$J_{32} \leq C \int_{|x-y| > \gamma(x)} |f(y)| \int_0^{\gamma(x)} t^{-\frac{n+2}{2}} e^{-A_3 \frac{t}{\sqrt{t}}} dt dy$$

$$\leq C \int_0^{\gamma(x)} \frac{e^{A_3 \frac{x}{\sqrt{t}}}}{t} \int_{\mathbb{R}^n} e^{-\frac{A_3}{\sqrt{t}} |x-y|} |f(y)| dy dt$$

$$\leq C \sup_{t>0} t^{-\frac{n+2}{2}} \int_{\mathbb{R}^n} e^{-\frac{A_3}{\sqrt{t}} |x-y|} |f(y)| dy \leq CM(f)(x).$$

Thus, from the estimates $J_{31}$ and $J_{32}$, we have $J_3 \leq CM(f)(x)$, which implies that the operator $V_\rho(e^{-t\mathcal{L}} - e^{-t_0\mathcal{L}})(f)$ is bounded from $L^p(\mathbb{R}^n)$ into itself for every $1 < p < \infty$.

Finally, we consider the term $J_1$:

$$J_1 = \sup_{t_j > 0} \left( \sum_{j=0}^{\infty} \int_{|x-y| < \gamma(x)} \left| (B_{t_j}(x, y) - b(x - y, t_j)) - (B_{t_{j+1}}(x, y) - b(x - y, t_{j+1})) \right| f(y) dy \right)^{\frac{1}{p}}$$

$$\leq \sup_{t_j > 0} \left( \sum_{j=0}^{\infty} \int_{|x-y| < \gamma(x)} \left| (B_{t_j}(x, y) - b(x - y, t_j)) - (B_{t_{j+1}}(x, y) - b(x - y, t_{j+1})) \right| f(y) dy \right)^{\frac{1}{p}}.$$
The formula (2.7) in [7] implies that

Then we have

Using (2.3), and Lemmas 2.5 and 2.4, we obtain

Applying Lemma 2.1 and Lemma 2.5, we have

The formula (2.7) in [7] implies that

Then we have

We rewrite $J_{12}$ as

Using (2.3), and Lemmas 2.5 and 2.4, we obtain

\[ \sum_{m=1}^{3} T_m f(x). \]
As a consequence,
\[
|T_1(f)(x)| \leq C \int_{|x-y|<\gamma(x)} |f(y)| \int_0^{\gamma(x)} t^{-1/2} e^{-A_2 \frac{|x-y|^{4/3}}{r^{1/3}}} \left( t^{1/4} \gamma(x)^{25} \right) dt dy
\]
\[
\leq C \int_0^{\gamma(x)} t^{-1+\delta/2} \frac{1}{\gamma(x)^{25}} \int_{\mathbb{R}^n} |f(y)| e^{-A_2 \frac{|x-y|^{4/3}}{r^{1/3}}} \, dy dt
\]
\[
\leq C \sup_{t>0} \frac{1}{t^{\frac{4}{3}}} \int_{\mathbb{R}^n} |f(y)| e^{-A_2 \frac{|x-y|^{4/3}}{r^{1/3}}} \, dy \leq CM(f)(x).
\]

Next, we note that when \(0 < s < t/2, t - s \sim t\). By (2.5), and Lemmas 2.5 and 2.4, we have
\[
\int_0^{\gamma(x)} |K_2(x, y, t)| dt
\]
\[
\leq C \int_0^{\gamma(x)} t^{1/2} \int_0^t \int_{\mathbb{R}^n} V^2(z) \frac{1}{(t-s)^{1/2+1}} e^{-A_1 \frac{|x-z|^{4/3}}{t^{1/3}}} \left( \frac{1}{s^{1/2}} \right) e^{-A_2 \frac{|y-z|^{4/3}}{t^{1/3}}} \, dz ds dt
\]
\[
\leq C \int_0^{\gamma(x)} t^{1/2} \int_0^t \int_{\mathbb{R}^n} e^{-A_1 \frac{|x-z|^{4/3}}{t^{1/3}}} \left( \frac{1}{s^{1/2}} \right) V^2(z) e^{-A_2 \frac{|y-z|^{4/3}}{t^{1/3}}} \, dz ds dt
\]
\[
\leq C \int_0^{\gamma(x)} t^{1/2} \int_0^t \int_{\mathbb{R}^n} e^{-A_1 \frac{|x-z|^{4/3}}{t^{1/3}}} \left( \frac{1}{s^{1/2}} \right) \frac{1}{t^{1/3}} V^2(z) e^{-A_2 \frac{|y-z|^{4/3}}{t^{1/3}}} \, dz ds dt
\]
\[
\leq C \int_0^{\gamma(x)} t^{1/2} \int_0^t \int_{\mathbb{R}^n} e^{-A_1 \frac{|x-y|^{4/3}}{t^{1/3}}} \left( \frac{1}{s^{1/2}} \right) \frac{1}{t^{1/3}} V^2(z) e^{-A_3 \frac{|y-z|^{4/3}}{s^{1/3}}} \, dz ds dt
\]
\[
\leq C \frac{1}{\gamma(x)^{25}} \int_0^{\gamma(x)} \frac{1}{t^{1/2}} e^{-A_1 \frac{|x-y|^{4/3}}{t^{1/3}}} \, dt.
\]

Hence,
\[
|T_2(f)(x)| \leq C \frac{1}{\gamma(x)^{25}} \int_{|x-y|<\gamma(x)} |f(y)| \int_0^{\gamma(x)} \frac{1}{t^{1/2+1+\frac{\delta}{2}}} e^{-A_1 \frac{|x-y|^{4/3}}{t^{1/3}}} \, dt dy
\]
\[
\leq C \frac{1}{\gamma(x)^{25}} \int_0^{\gamma(x)} \frac{1}{t^{1+\frac{\delta}{2}}} \int_{\mathbb{R}^n} e^{-A_1 \frac{|x-y|^{4/3}}{t^{1/3}}} \, dy dt
\]
\[
\leq C \sup_{t>0} \frac{1}{t^{\frac{4}{3}}} \int_{\mathbb{R}^n} |f(y)| e^{-A_1 \frac{|x-y|^{4/3}}{t^{1/3}}} \, dy \leq CM(f)(x).
\]

As in the previous proof, proceeding with a similar computation, we can also obtain
\[
|T_3(f)(x)| \leq CM(f)(x).
\]

Owing to the above estimates, we know that \(J_{12} \leq CM(f)(x)\). Consequently, we have \(J_1 \leq CM(f)(x)\). As \(M(f)\) is bounded from \(L^p(\mathbb{R}^n)\) into itself for every \(1 < p < \infty\), the proof of Theorem 1.1 is complete. \(\Box\)

### 2.4 The generalized Poisson operators \(\mathcal{P}_{t,L}^\sigma\)

For \(0 < \sigma < 1\), the generalized Poisson operator \(\mathcal{P}_{t,L}^\sigma\) associated with \(L\) is defined as
\[
\mathcal{P}_{t,L}^\sigma f(x) = \frac{t^{2\sigma}}{4^\sigma \Gamma(\sigma)} \int_0^\infty \frac{e^{-\frac{r}{4}}}{4^\sigma \Gamma(\sigma)} e^{-t L f(x)} \frac{dr}{r^{1+\sigma}} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-\frac{r}{4}} e^{-t L f(x)} \frac{dr}{r^{1-\sigma}}.
\]
We should note that, when \( \sigma = 1/2 \), \( P_i^\sigma = P_i^{1/2} \) is just the Poisson semigroup.

For the variation operator associated with the generalized Poisson operators \( \{P_i^\sigma\}_{i>0} \), we have the following theorem:

**Theorem 2.6** Assume that \( V \in RH_{q_0}(\mathbb{R}^n) \), where \( q_0 \in (n/2, \infty) \) and \( n \geq 5 \). For \( p > 2 \), there exists a constant \( C > 0 \) such that \( \|V_p(P_i^\sigma)\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)} \), \( 1 < p < \infty \).

**Proof** We note that

\[
V_p(P_i^\sigma)(f)(x) = \|P_i^\sigma f\|_{E_p} = \frac{1}{\Gamma(\sigma)} \left\| \int_0^\infty e^{-t} e^{-\frac{2t}{n}} f(x) \frac{dt}{r^{1-\sigma}} \right\|_{E_p} \\
\leq \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t} \left\| e^{-\frac{2t}{n}} f(x) \right\|_{E_p} \frac{dt}{r^{1-\sigma}}.
\]

Then, for \( 1 < p < \infty \), by Theorem 1.1 we have

\[
\|V_p(P_i^\sigma)f\|_{L^p(\mathbb{R}^n)} \leq \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t} \left\| e^{-\frac{2t}{n}} f(x) \right\|_{L^p(\mathbb{R}^n)} \frac{dt}{r^{1-\sigma}} \\
\leq \frac{C}{\Gamma(\sigma)} \int_0^\infty e^{-t} \|f\|_{L^p(\mathbb{R}^n)} \frac{dt}{r^{1-\sigma}} \leq C\|f\|_{L^p(\mathbb{R}^n)}.
\]

\( \square \)

### 3 Variation Inequalities on Morrey Spaces

In this section, we will give the proof of Theorem 1.3. For convenience, we first recall the definition of classical Morrey spaces \( L^{p,\lambda}(\mathbb{R}^n) \), which were introduced by Morrey [20] in 1938.

**Definition 3.1** Let \( 1 \leq p < \infty \), \( 0 \leq \lambda < n \). For \( f \in L^p_{loc}(\mathbb{R}^n) \), we say that \( f \in L^{p,\lambda}(\mathbb{R}^n) \) provided that

\[
\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{B(x_0,r) \subset \mathbb{R}^n} r^{-\lambda} \int_{B(x_0,r)} |f(x)|^p dx < \infty,
\]

where \( B(x_0,r) \) denotes a ball centered at \( x_0 \) and with radius \( r \).

In fact, when \( \alpha = 0 \) or \( V = 0 \) and \( 0 < \lambda < n \), the spaces \( L^{p,\lambda}(\mathbb{R}^n) \), which were defined in Definition 1.2, are the classical Morrey spaces \( L^{p,\lambda}(\mathbb{R}^n) \).

We establish the \( L^{p,\lambda}(\mathbb{R}^n) \)-boundedness of the variation operators related to \( \{e^{-t\Delta^2}\}_{t>0} \) as follows:

**Theorem 3.2** Let \( \rho > 2 \) and \( 0 < \lambda < n \). If \( 1 < p < \infty \), then

\[
\|V_p(e^{-t\Delta^2})(f)\|_{L^{p,\lambda}(\mathbb{R}^n)} \leq C\|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.
\]

**Proof** For any fixed \( x_0 \in \mathbb{R}^n \) and \( r > 0 \), we write \( f(x) = f_0(x) + \sum_{i=1}^\infty f_i(x) \), where \( f_0 = f\chi_{B(x_0,2r)}; f_i = f\chi_{B(x_0,2^{i+1}r) \setminus B(x_0,2^ir)} \) for \( i \geq 1 \). Then

\[
\left( \int_{B(x_0,r)} |V_p(e^{-t\Delta^2})(f)(x)|^p dx \right)^{\frac{1}{p}} \\
\leq C \left( \int_{B(x_0,r)} |V_p(e^{-t\Delta^2})(f_0)(x)|^p dx \right)^{\frac{1}{p}} + C \sum_{i=1}^\infty \left( \int_{B(x_0,r)} |V_p(e^{-t\Delta^2})(f_i)(x)|^p dx \right)^{\frac{1}{p}} \\
=: I + II.
\]
For $I$, by Theorem 2.2, we have

$$I = \int_{B(x_0, r)} |\mathcal{V}_\rho(e^{-t\Delta^2})(f_0)(x)|^p dx \leq C \int_{B(x_0, 2r)} |f(x)|^p dx \leq C r^\lambda \|f\|^p_{L^p, \lambda(\mathbb{R}^n)}.$$ 

For $II$, we first analyze $\mathcal{V}_\rho(e^{-t\Delta^2})(f_i)(x)$. For every $i \geq 1$,

$$\mathcal{V}_\rho(e^{-t\Delta^2})(f_i)(x) = \sup_{t_j \to 0} \left( \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} (b(x - y, t_j) - b(x - y, t_{j+1})) f_i(y) dy \right)^{\frac{1}{p}} \leq C \sup_{t_j \to 0} \int_{\mathbb{R}^n} |f_i(y)| \int_{t_{j+1}}^{t_j} \left| \frac{\partial}{\partial t} b(x - y, t) \right| dt dy \leq C \int_{B(x_0, 2^{i+1} r) \setminus B(x_0, 2^i r)} |f(y)| \int_0^\infty \left| \frac{\partial}{\partial t} b(x - y, t) \right| dt dy.$$ 

Note that for $x \in B(x_0, r)$ and $y \in \mathbb{R}^n \setminus B(x_0, 2r)$, we know that $|x - y| > \frac{1}{2} |x_0 - y|$. By using (2.5), we have

$$\int_0^\infty \left| \frac{\partial}{\partial t} b(x - y, t) \right| dt = \int_0^{|x_0 - y|^4} \frac{\partial}{\partial t} b(x - y, t) dt + \int_{|x_0 - y|^4}^\infty \frac{\partial}{\partial t} b(x - y, t) dt \leq C \int_0^{|x_0 - y|^4} t^{-\frac{4}{3}} e^{-A_1|x_0 - y|^4 t^{-\frac{1}{4}}} dt + C \int_{|x_0 - y|^4}^\infty t^{-\frac{4}{3} - 1} dt \leq C |x_0 - y|^{-n} \int_1^\infty u^{\frac{4}{3} - 1} e^{-A_1 u} du + C |x_0 - y|^{-n} \leq C |x_0 - y|^{-n}.$$ 

Thus,

$$\mathcal{V}_\rho(e^{-t\Delta^2})(f_i)(x) \leq C \int_{B(x_0, 2^{i+1} r) \setminus B(x_0, 2^i r)} |f(y)||x_0 - y|^{-n} dy \leq C \left( \int_{B(x_0, 2^{i+1} r) \setminus B(x_0, 2^i r)} |f(y)|^p dy \right)^{\frac{1}{p}} \left( \int_{B(x_0, 2^{i+1} r) \setminus B(x_0, 2^i r)} \frac{1}{|x_0 - y|^n} dy \right)^{\frac{1}{p}} \leq C (2^i r)^{-\frac{3}{p}} \left( \int_{B(x_0, 2^{i+1} r)} |f(y)|^p dy \right)^{\frac{1}{p}}.$$ 

Therefore, we have

$$II \leq C \sum_{i=1}^\infty \left( 2^{-in} \int_{B(x_0, 2^{i+1} r)} |f(y)|^p dy \right)^{\frac{1}{p}} \leq C \sum_{i=1}^\infty \left( 2^{-i(n-\lambda)} \|f\|^p_{L^p, \lambda(\mathbb{R}^n)} \right)^{\frac{1}{p}} \leq C r^{\lambda} \|f\|^p_{L^p, \lambda(\mathbb{R}^n)}.$$ 

Consequently,

$$\|\mathcal{V}_\rho(e^{-t\Delta^2})(f)\|_{L^p, \lambda(\mathbb{R}^n)} \leq C \|f\|_{L^p, \lambda(\mathbb{R}^n)}.$$ 

The proof of this theorem is complete. \qed

The following is devoted to the proof of Theorem 1.3.

**Proof of Theorem 1.3** Without loss of generality, we may assume that $\alpha < 0$. Fixing any $x_0 \in \mathbb{R}^n$ and $r > 0$, we write

$$f(x) = f_0(x) + \sum_{i=1}^\infty f_i(x),$$
where \( f_0 = f \chi_{B(x_0, 2r)} \), \( f_i = f \chi_{B(x_0, 2^{i+1}r) \setminus B(x_0, 2^i r)} \) for \( i \geq 1 \). Then

\[
\left( \int_{B(x_0, r)} |\mathcal{V}_\rho(e^{-t\mathcal{L}})(f)(x)|^p \, dx \right)^{\frac{1}{p}} 
\leq C \left( \int_{B(x_0, r)} |\mathcal{V}_\rho(e^{-t\mathcal{L}})(f_0)(x)|^p \, dx \right)^{\frac{1}{p}} + C \sum_{i=1}^{\infty} \left( \int_{B(x_0, r)} |\mathcal{V}_\rho(e^{-t\mathcal{L}})(f_i)(x)|^p \, dx \right)^{\frac{1}{p}} 
=: I + II.
\]

From (i) of Theorem 1.1, we have

\[
I \leq C \int_{B(x_0, 2r)} |f(x)|^p \, dx \leq Cr^\alpha \left( 1 + \frac{r}{\gamma(x_0)} \right)^{-\alpha} \|f\|_{L_{\gamma(x_0)\gamma}^p}^p.
\]

For \( II \), we first analyze \( \mathcal{V}_\rho(e^{-t\mathcal{L}})(f_i)(x) \). For every \( i \geq 1 \),

\[
\mathcal{V}_\rho(e^{-t\mathcal{L}})(f_i)(x) = \sup_{t_j < 0} \left( \sum_{j=0}^{\infty} \left| \int_{\mathbb{R}^n} (B_{t_j}(x, y) - B_{t_{j+1}}(x, y)) f_i(y) \, dy \right| \right)^{\frac{1}{p}}.
\]

\[
\leq C \sup_{t_j < 0} \sum_{j=0}^{\infty} \int_{\mathbb{R}^n} |f_i(y)| \left| \frac{\partial}{\partial t} B_t(x, y) \right| \, dy \, dt \, dy
\]

\[
\leq C \int_{B(x_0, 2^{i+1}r) \setminus B(x_0, 2r)} |f(y)| \left| \frac{\partial}{\partial t} B_t(x, y) \right| \, dy \, dt.
\]

Note that for \( x \in B(x_0, r) \) and \( y \in \mathbb{R}^n \setminus B(x_0, 2r) \), we have \( |x - y| > \frac{1}{2} |x_0 - y| \). We discuss \( \int_0^\infty \left| \frac{\partial}{\partial t} B_t(x, y) \right| \, dt \) in two cases. For the one case, \( |x_0 - y| \leq \gamma(x_0) \), by (ii) of Lemma 2.5 we have

\[
\int_0^\infty \left| \frac{\partial}{\partial t} B_t(x, y) \right| \, dt = \int_0^{|x_0 - y|^4} \left| \frac{\partial}{\partial t} B_t(x, y) \right| \, dt + \int_{|x_0 - y|^4}^\infty \left| \frac{\partial}{\partial t} B_t(x, y) \right| \, dt
\]

\[
\leq C \int_0^{|x_0 - y|^4} t^{-\frac{n}{4} - 1} e^{A_1(|x_0 - y|t^{-\frac{1}{4}})} \, dt + C \int_{|x_0 - y|^4}^\infty t^{-\frac{n}{4} - 1} \, dt
\]

\[
\leq C|x_0 - y|^{-n} + C \int_0^{|x_0 - y|^4} t^{-\frac{n}{4} - 1} \left( \frac{t^{1/3}}{|x_0 - y|^{4/3}} \right)^{3(n+4)/4} \, dt
\]

\[
\leq C|x_0 - y|^{-n} \left( 1 + \frac{|x_0 - y|}{\gamma(x_0)} \right)^{-N}.
\]

(3.1)

For the other case, \( |x_0 - y| \geq \gamma(x_0) \), applying (ii) of Lemma 2.5 together with Lemma 2.3 we have

\[
\int_{|x_0 - y|^4}^\infty \left| \frac{\partial}{\partial t} B_t(x, y) \right| \, dt \leq C \int_{|x_0 - y|^4}^\infty t^{-\frac{n}{4} - 1} \left( 1 + \frac{\sqrt{t}}{\gamma^2(y)} \right)^{-k} e^{A_1(|x_0 - y|t^{-\frac{1}{4}})} \, dt
\]

\[
\leq C \left( 1 + \frac{|x_0 - y|^2}{\gamma^2(y)} \right)^{-k} |x_0 - y|^n
\]

\[
\leq C \left( 1 + \frac{\left( \frac{|x_0 - y|^2}{\gamma(x_0)} \right)^2}{c_0 (1 + \frac{|x_0 - y|^2}{\gamma(x_0)}))^{\frac{2n}{n+1}}} \right)^{-k} |x_0 - y|^n
\]

\[
\leq C |x_0 - y|^n \left( 1 + \frac{|x_0 - y|^2}{\gamma(x_0)} \right)^{-N},
\]

(3.2)

where we take \( N = \left\lceil \frac{k(k_0 - 1)}{k_n + 1} \right\rceil \) for any \( k \in \mathbb{N} \). And

\[
\int_0^{|x_0 - y|^4} \left| \frac{\partial}{\partial t} B_t(x, y) \right| \, dt
\]

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Combining (3.1), (3.2) and (3.3), we have

\[
\int_0^\gamma \frac{\partial}{\partial t} B_t(x, y) \, dt + \int_{\gamma^2(x_0)} |x_0 - y|^t \, dt \\
\leq C \int_0^\gamma t^{-\frac{n}{2} - 1} e^{-A_1 \left( |x_0 - y| t^{\frac{1}{2}} \right)^{4/3}} \, dt + C \int_{\gamma^2(x_0)} |x_0 - y|^t \, dt \\
\leq C \int_{|x_0 - y|^4/3} \int_0^\gamma |x_0 - y|^{-n} u_n \, du + C \gamma(x_0)^{-n} \int_{|x_0 - y|^4/3} |x_0 - y|^t \, dt \\
\leq C |x_0 - y|^{-n} \int_{|x_0 - y|^4/3} |x_0 - y|^{-n} \left( 1 + \frac{|x_0 - y|}{\gamma(x_0)} \right)^{-N} \\
\leq C |x_0 - y|^{-n} \left( 1 + \frac{|x_0 - y|}{\gamma(x_0)} \right)^{-N}. \tag{3.3}
\]

Thus, taking \( N = [\alpha] + 1, \) we obtain

\[
\int_{B(x_0, r)} \left| \mathcal{V}_\rho \left( e^{-t \mathcal{L}} \right) (f_t)(x) \right|^p \, dx \\
\leq C \int_{B(x_0, 2r)} \left| f(y) \right|^p \, dy \\
\leq C \int_{B(x_0, 2r)} \left| f(y) \right|^p \, dy \\
\leq C \left( 1 + \frac{2r}{\gamma(x_0)} \right)^{-N} \left( \int_{B(x_0, 2r)} \left| f(y) \right|^p \, dy \right)^{\frac{1}{p}}.
\]

Because \( \lambda < n, \) we have \( II \leq C \| f \|_{L^{p, \lambda}_{n, \lambda}(\mathbb{R}^n)}. \) Hence,

\[
\| \mathcal{V}_\rho \left( e^{-t \mathcal{L}} \right) (f) \|_{L^{p, \lambda}_{n, \lambda}(\mathbb{R}^n)} \leq C \| f \|_{L^{p, \lambda}_{n, \lambda}(\mathbb{R}^n)}.
\]

The proof of the theorem is completed. \( \square \)

Finally, we can give the boundedness of the variation operators related to generalized Poisson operators \( \mathcal{P}_{\mathcal{L}, \rho}^\alpha \) in the Morrey spaces as follows:

**Theorem 3.3** Let \( V \in RH_{q_0}(\mathbb{R}^n) \) for \( q_0 \in (n/2, \infty), \) \( n \geq 5, \) and \( \rho > 2. \) Assume that \( \alpha \in \mathbb{R} \) and \( \lambda \in (0, n). \) There exists a constant \( C > 0 \) such that

\[
\| \mathcal{V}_\rho (\mathcal{P}_{\mathcal{L}, \rho}^\alpha (f) \|_{L^{p, \lambda}_{n, \lambda}(\mathbb{R}^n)} \leq C \| f \|_{L^{p, \lambda}_{n, \lambda}(\mathbb{R}^n)}, \quad 1 < p < \infty.
\]

**Proof** We can prove this theorem by the same procedure used in the proof of Theorem 2.6. \( \square \)

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