Exactly unsolved problems of interacting 1D fermions

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Applications of the integrable system techniques to the non-equilibrium transport problems are discussed. We describe one-dimensional electrons tunneling through a point-like defect either by the s-d exchange (Kondo) mechanism, or via the resonanse level (Anderson) mechanism. These models are potential candidates to be solved exactly in the presence of arbitrary external bias. We draw attention also to several mesoscopic systems which can be tackled by the massless form-factor approach, as perturbations of integrable models. The basic unperturbed model is the massless sine-Gordon model with the interaction (cosine) term restricted to one point, which is integrable. It is being perturbed by the second interaction term, which destroys integrability. Quasi-exact results can be obtained by making use of the basis of massless quasiparticles of the sine-Gordon model.

INTRODUCTION

Bethe ansatz technique (BA) is a powerful tool for solving strongly interacting systems. Among its most interesting applications are the exact solutions of Kondo and Anderson models of impurities in the metals, sin-Gordon/Thirring model, etc. The central issue of the Bethe ansatz is a phenomenon of factorized scattering which allows for solving the theory completely starting from the two-particle scattering matrices as the only input data. Factorized scattering approach turned out to be more fruitful than thought earlier in the traditional frame of BA. Withing the standard scope of BA, as we define it, lies solving the bare Hamiltonian for its excitation spectrum, the distribution functions and, eventually, the partition function, which contains the full information on the equilibrium properties. Factorized scattering concept, however, led further to the development of new methods, providing us with additional important physical quantities: the series expansion of the correlation functions that follows from the form-factor approach and the finite-size ground state energy of Zamolodchikov’s thermodynamic BA. The attitude which is involved here (as opposed to the traditional BA) is similar in spirit to the Landau theory of Fermi liquids: the fundamental objects which characterize the behavior of the system are the dressed quasiparticles, rather than the bare particles of the free problem. Free particles, after the interactions have been turned on, will acquire the screening cloud of virtual pairs that will follow it, resulting

1 The author was supported by Kreitman and by Kaufmann fellowships, and partially funded by the Israel Science Foundation, grant No 3/96-1. These notes were inspired by the discussions with Natan Andrei, Yuval Geffen and Yigal Meir.
in the formation of new objects – quasiparticles that substitute the free electrons in the non-interacting system. The spectrum, density and interactions of quasiparticles completely determine the physical properties of the system. In practice it is hardly possible to determine the properties of quasiparticles exactly, starting from the non-interacting system and switching on interactions, except for the special integrable cases, when the Bethe ansatz is applicable. Integrable quasiparticles are, however, very special as far as their interactions are concerned: they are stable, meaning that the conservation laws force the momentum conservation of each individual quasiparticle, and their scattering is factorized into the two-particle processes (in some more exotic integrable models there can in principle exist quasiparticles with imaginary energy - finite lifetime, so-called monstrons, but we do not consider them here). Adopting integrable quasiparticle basis, one can do the best of it in describing also non-integrable models as perturbations of the integrable ones. This direction was considered in the works of Delfino, Mussardo and Simonetti [Nucl.Phys. B473 (1996) 469]. The quasiparticles approach turned out also to be effective in the exact description of non-equilibrium properties of integrable systems as shown in the works of N.Andrei [Phys.Lett. A87 (1982) 299], Fendley, Ludwig and Saleur [Phys.Rev. B52 (1995) 8934], Peres, Sacramento and Carmelo [cond-mat/9709144]. In the present notes we are trying to pursue further this approach and sketch a few possible applications of integrable quasiparticle techniques to the tunneling transport problems, limiting ourselves to the technically simplest cases that naturally lead to more realistic and complicated models.

I. EXACT NON-LINEAR TUNNELING CURRENT

In this section we describe two 1D models of electrons tunneling through a point-like defect either by the s-d exchange (Kondo) mechanism, or via the resonance level (Anderson) mechanism. These models are potential candidates to be solved exactly in the presence of arbitrary external bias by means of the Bethe-ansatz technique.

1. s-d exchange model of zero-bias tunneling anomalies.

The model we focus on dates back to the original work of J.Appelbaum [Phys.Rev.Lett. 17 (1966) 91]. Imagine two pieces of metal with different chemical potentials, separated by an insulating material. Tunneling of electrons is made possible at one special point, as if the two metals were connected to each other adiabatically by a “weak link.” Suppose that, by integrating out transversal modes, one can reduce the problem to effectively one-dimensional, with a point-like impurity at $z = 0$. The precise nature of the transversal modes is not essential as far as only the low-energy physics is concerned and the transitions between different modes can be neglected. Consider first the right hand side metal. In the low-energy limit one can linearize the electron spectrum near the Fermi energy, leading to the Hamiltonian $H = \sum_\epsilon \epsilon a_{Re}^+ a_{Re} + \epsilon a_{Le}^+ a_{Le}$. It is important that the states responsible for the low-energy physics can be labeled by a single parameter, $\epsilon$. It is convenient to introduce

$$\Psi = \Psi_L + \Psi_R, \quad z > 0$$

(1)
where $\Psi_{L,R}$ are the one-dimensional left and right moving electron fields, build by the wavefunctions of effective 1D problem: $\Psi_{L,R} = \pm \sum e^{\pm i\epsilon z} e^{i\epsilon t} a_\epsilon$. These fields obey the usual anti-commutation relations. With the boundary condition imposed,

$$\Psi(0) = \Psi_L(0) + \Psi_R(0) = 0,$$

situation corresponding to the full reflection. One can map one of the fields, say, right-moving, to the negative semi-axis preserving the boundary condition at zero as follows

$$\tilde{\Psi}_L(-z) = -\Psi_R(z) \quad (3)$$

$$\Psi_L(0) = \tilde{\Psi}_L(0) \quad (4)$$

The same “unfolding” procedure can be applied to the left piece of metal to map the problem eventually to the full axis. We end up with two species of free Dirac electrons on the line, to which we assign the flavor index $m = 1, 2$. The Hamiltonian describing this problem is:

$$H_0 = iv_F \sum_{\sigma,m=1,2} \int_{-\infty}^{\infty} \Psi_{\sigma m}^+ \partial_z \Psi_{\sigma m} dz + \sum_{\sigma,m=1,2} \mu_m \int_{-\infty}^{\infty} \Psi_{\sigma m}^+ \Psi_{\sigma m} \quad (5)$$

where we introduced the bias term which accounts for the fact that the left and right metals are at different chemical potentials $\mu_{1,2}$. We set $\mu_1 = -\mu_2 = \mu$ and rewrite the last term as

$$Y_0 = \mu \sum_{\sigma} \int_{-\infty}^{\infty} (\Psi_{1,\sigma}^+ \Psi_{1,\sigma} - \Psi_{2,\sigma}^+ \Psi_{2,\sigma}) \quad (6)$$

The Hamiltonian (5) possesses $SU(2)^{flavor}$ and $SU(2)^{spin}$ symmetries, as well as $U(1)^{charge}$ symmetry, and the total number of type 1 and type 2 particles is conserved separately. Now we introduce a weak tunneling between the metals via an impurity spin $S = 1/2$ at the point $z = 0$:

$$H_{\text{int}} = \frac{3}{2} \sum_{\lambda=1}^{3} \sum_{ab} J_{\lambda}^{ab} \Psi_{\lambda a}^+(0) \sigma^\lambda_{\sigma\sigma'} \Psi_{\lambda b}(0) S^\lambda \quad (7)$$

where $\sigma^\lambda$ are Pauli matrices and $J_{\lambda}^{ab}$ are coupling constants. We make a natural assumption that $J_x = J_y$ and $J_{12} = J_{21}$, $J_{11} = J_{22}$. The full Hamiltonian $H_0 + H_{\text{int}}$ resembles the two-channel Kondo model, with however additional intra-channel interaction. The interaction term $J_{12}$ breaks the $SU(2)^{flavor}$ symmetry down to its $U(1)^{flavor}$ subgroup. As a result, the number of type 1 and type 2 particles is no longer conserved separately; however, the total number of type 1 and type 2 particles is still a good quantum number:

$$Q^{\text{charge}} = \int \Psi_1^+ \Psi_1 + \Psi_2^+ \Psi_2 \quad (8)$$

The $S_x^{flavor}$ charge corresponding to $U(1)^{flavor}$ subgroup also remains conserved:

$$S_x^{flavor} = \sum_{ab} \int \Psi_a^+ \Psi_b = \int \Psi_1^+ \Psi_2 + \Psi_2^+ \Psi_1 \quad (9)$$

Here $\tau_x$ is the first Pauli matrix acting in the flavor space, and the summation over the spin index is implicitly assumed.
We are not going to discuss the experimental relevance of this particular model, i.e. what physical systems it actually describes. We believe that it is one of the simplest and yet non-trivial theoretical models, and therefore it is instructive to learn as much as possible from it. We shall mention, however, that more complicated and realistic models, such as the one introduced by Vladar and Zawadowski [Phys.Rev. B28 (1983) 1564], are its direct generalizations as far as the methodology is concerned. The reader interested in the relevant experimental advances can see the papers of Ralph and Buhrman [Phys.Rev.Lett. 69 (1992) 2118, Phys.Rev. B51 (1995) 3554]. Some exact theoretical results in this field, based on the conformal field theory and the Hershfield’s formalism, include the exact scaling functions found by Delft, Ludwig and Ambegaokar [cond-mat/9702049, submitted to Annals of Physics]. It is worth mentioning that the one-dimensional approximation that we adopted may not describe correctly certain microscopic quantities, such as the local density of states in the realistic point contacts, as was noticed by Ulreich and Zwerger [cond-mat/9710174, submitted to Europhys.Lett.]. However, it does work extremely well for the linear conductance at zero temperature [Phys.Rev. B50 (1994) 17320].

One can bosonize the fermions according to the standard procedure and express the Hamiltonian in terms of four left-moving boson fields:

\[ \Psi^+_{\sigma m} = e^{i\varphi_{\sigma m}}. \]

Following Emery and Kivelson [Phys.Rev.B47 (1992) 10812], introduce linear combinations of the bosons as follows:

\[ \phi_c = \frac{1}{2}(\varphi_{\uparrow 1} + \varphi_{\downarrow 1} + \varphi_{\uparrow 2} + \varphi_{\downarrow 2}) \]

\[ \phi_s = \frac{1}{2}(\varphi_{\uparrow 1} - \varphi_{\downarrow 1} + \varphi_{\uparrow 2} - \varphi_{\downarrow 2}) \]

\[ \phi_f = \frac{1}{2}(\varphi_{\uparrow 1} + \varphi_{\downarrow 1} - \varphi_{\uparrow 2} - \varphi_{\downarrow 2}) \]

\[ \phi_{sf} = \frac{1}{2}(\varphi_{\uparrow 1} - \varphi_{\downarrow 1} - \varphi_{\uparrow 2} + \varphi_{\downarrow 2}) \]

The field \( \phi_c \) represents the total charge degree of freedom in two channels and decouples from the Hamiltonian. The resulting Hamiltonian depends on the remaining three fields as follows:

\[ H_0 = \frac{v_F}{2\pi} \sum_{m=s,f,sf} \int (\partial_z \phi_m)^2 + \frac{\mu}{\pi} \int \partial_z \phi_f \]

\[ H_{int} = 4[J_{x}^{12} \cos \phi_f(0) + J_{x}^{11} \cos \phi_{sf}(0)] \cdot [\cos \phi_s(0)S^x + \sin \phi_s(0)S^y] \]

\[ -4J_{x}^{12} \sin \phi_f(0) \sin \phi_{sf}(0)S^z + \frac{J_{z}^{11}}{\pi} \partial_z \phi_s(0)S^z \]

Interaction term (16) resembles the one-channel Kondo model where the coupling constant \( J_x \) became a dynamical degree of freedom, \( J_x \rightarrow [J_{x}^{12} \cos \phi_f(0) + J_{x}^{11} \cos \phi_{sf}(0)] \). There exists a special point in the space of couplings \( (J_{x}^{11} = 2\pi v_F \) and \( J_{x}^{12} = 0) \), an analogue of the Toulouse point in the Kondo model, where the full Hamiltonian \( H = H_0 + H_{int} \) becomes quadratic in terms of new fermion operators and can be solved exactly. This was noticed by
Schiller and Hershfield [Phys.Rev.B51 (1995) 12896], who found the non-equilibrium current at this “Toulouse point.”

In the absence of bias, the full interacting Hamiltonian (3)-(7) is integrable. To see this, let us make a canonical transformation to another basis of fermions:

\[ a_\sigma = \frac{1}{\sqrt{2}}(\Psi_{1\sigma} + \Psi_{2\sigma}) \]  
\[ b_\sigma = \frac{1}{\sqrt{2}}(-\Psi_{1\sigma} + \Psi_{2\sigma}) \]

In this new basis the matrix of couplings \( J^{ab} \) becomes diagonal with the eigenvalues \( J^\pm = J_{11}^\pm \pm J_{12} \). The resulting Hamiltonian is the channel-anisotropic two-channel Kondo model, which was solved by means of Bethe ansatz by N.Andrei and A.Jerez [Phys.Rev.Lett.74 (1995) 4507]:

\[ H = i v_F \sum_\sigma \int \left( a_\sigma^+ \partial_x a_\sigma + b_\sigma^+ \partial_x b_\sigma \right) + \sum_\lambda \left[ J^+_\lambda a_\sigma^+(0)\sigma^\lambda_{\sigma\sigma'} a_{\sigma'}(0) + J^-_\lambda b_\sigma^+(0)\sigma^\lambda_{\sigma\sigma'} b_{\sigma'}(0) \right] S^\lambda \]

\[ Y_0 = \mu \sum_\sigma \int a_\sigma^+ b_\sigma + b_\sigma^+ a_\sigma \]

The advantage of this new basis is that the number of a-particles as well as b-particles is a conserved quantity, and the Bethe wave functions can be constructed in the sector of fixed quantum numbers \( N_a \) and \( N_b \). These conserved charges correspond to the total particle charge and \( SU(2) \) flavor symmetries. Operator \( Y_0 \), being a generator of the \( SU(2) \) flavor symmetry, does not commute with the Hamiltonian, except for the channel-isotropic point \( J^+ = J^- \), corresponding to \( J_{12} = 0 \), where the \( SU(2) \) flavor symmetry is restored (note that in general \( J^+ \geq J^- \)). The non-conservation of \( SU(2) \) flavor charge, in particular of its component \( Y_0 \), is responsible for the transport properties we are interested in.

Following Andrei and Jerez, let us study one and two-particle sectors of the first-quantized analogue of Hamiltonian (20). Naively, the scattering matrix on impurity would be

\[ R^a = \exp(iJ^+ \vec{S}) \quad R^b = \exp(iJ^- \vec{S}) \quad R^b = 0 \]

where the scattering is obviously diagonal in the flavor space. Respectively, the scattering matrix on impurity dictates us what the scattering matrix in the bulk would be, since the basis of the wave-functions in the bulk is not unique and must be chosen in the consistent way with the impurity. Let us clarify this point. Consider one of the possible representations of the two-particle wave function for free 1D fermions with the linear spectrum:

\[ \psi(x_1, x_2) = \theta(x_1 - x_2)(A_{\sigma_1\sigma_2} e^{i k_{1x_1} + i k_{2x_2}} + B_{\sigma_1\sigma_2} e^{i k_{1x_2} + i k_{2x_1}}) + \theta(x_2 - x_1)\left((SA)_{\sigma_1\sigma_2} e^{i k_{1x_1} + i k_{2x_2}} + (SB)_{\sigma_1\sigma_2} e^{i k_{1x_2} + i k_{2x_1}}\right) \]

Here \( A \) and \( B \) are spinors, \( S \) is for the time being arbitrary matrix acting on the spinors, and \( \theta(x) \) is a step function. It is easy to check that \( \psi(x_1, x_2) \) is a solution of Schrödinger equation

\[ (-i\partial_{x_1} - i\partial_{x_2})\psi = E\psi \]
with $E = k_1 + k_2$. In addition, one must require $\psi(x_1, x_2)$ to be antisymmetric with respect to the simultaneous permutation of spin and coordinate indexes. This leads to the relation between $A$ and $B$,

$$B = -PSA$$

and the constraint for $S$:

$$S_{12}S_{21} = 1 \quad (22)$$

where $P$ is the permutation operator acting on spinors and $S_{21} = PS_{12}P$. The condition (22) leaves some freedom for us to choose the matrix $S$ (for example, $S = I$ and $S = P$ are valid solutions to (22)). The consistency between impurity and bulk scattering is represented by the boundary Yang-Baxter equation,

$$S_{12}R_2R_1 = R_1R_2S_{12}$$

which imposes a further constraint on $S$, or in other words selects bulk scattering in a consistent way with the impurity. With the impurity scattering matrix given above, the Yang-Baxter equation would be solved by the bulk S-matrix

$$S_{12} = p^{\text{spin}}P^{\text{flavor}}$$

where $P$ are exchange operators acting in spin or flavor spaces of two particles. As noticed by Wiegmann and Tsvelik, as well as by Andrei and Destri, such a choice of scattering matrices leads to wrong results. The Hamiltonian must be regularized appropriately from the very beginning, which modifies S-matrices in such a way that they become momentum-dependent (it was not possible before regularization because $J$ is a dimensionless coupling and there is no scale in the problem; after the regularization the model acquires a scale which enters S-matrices in the dimensionless combination with momentum). Andrei and Destri [Phys.Rev.Lett 52 (1984) 364] suggested an elegant regularization compatible with the integrability: one introduces the second-order derivative term coupled to a mass scale $\Lambda$ in the first-quantized Hamiltonian, $\Lambda^{-1}\partial^2$, thus “delinearizing” the spectrum. After such a regularization the bare scattering matrices become:

$$R_{a\alpha}(k) = \frac{k/\Lambda + 1 - iJ^+(\vec{\sigma}\vec{S} + 1/2)}{k/\Lambda + 1 - iJ^+}$$

$$R_{b\beta}(k) = \frac{k/\Lambda + 1 - iJ^-(\vec{\sigma}\vec{S} + 1/2)}{k/\Lambda + 1 - iJ^-} \quad (24)$$

(impurity scattering is still flavor-diagonal)

$$S_{12}^{\text{spin}}(k_1, k_2) = \frac{(k_1/\Lambda + 1)/J^+ - (k_2/\Lambda + 1)/J^x - iP^{\text{spin}}}{(k_1/\Lambda + 1)/J^+ - (k_2/\Lambda + 1)/J^x - i} \quad (25)$$

where $J^+(J^-)$ is chosen depending on the flavor of the first and second particles,

$$S_{12}^{\text{flavor}}(k_1 - k_2) = \frac{i}{2 \sinh[\kappa(k_1 - k_2)/\Lambda + iv]} [\tau_x \otimes \tau_x + \tau_y \otimes \tau_y] + \frac{1}{2} [1 + \tau_z \otimes \tau_z]$$

$$+ \frac{1}{2 \sinh[\kappa(k_1 - k_2)/\Lambda + iv]} [1 - \tau_z \otimes \tau_z] \quad (26)$$
Here \( \tau \) are the Pauli matrices acting on the flavor space, \( \kappa \) and \( \nu \) are functions of \( J^\pm \). In the scaling limit we have \( \nu / \kappa = J^- \). The complete bulk scattering matrix is a tensor product of spin and flavor terms, \( S_{12} = S_{12}^{\text{spin}} \otimes S_{12}^{\text{flavor}} \). The \( S_{12}^{\text{spin}} \) term reflects the \( SU(2)^{\text{spin}} \) invariance, while the flavor component \( S_{12}^{\text{flavor}} \) reflects the breaking of the \( SU(2)^{\text{flavor}} \) symmetry to \( U(1)^{\text{flavor}} \) subgroup in the presence of anisotropy.

What is the form of the above S-matrices in the original basis of \( \Psi \)-fields? Since our transformation (18)-(19) mixes up the flavors (and not the spins), we will list here only two of the above terms mapped to the original basis:

\[
\tilde{R}_1^1 = \tilde{R}_2^2 = \frac{1}{2}(R_a^a + R_b^b) \tag{27}
\]

\[
\tilde{R}_1^2 = \tilde{R}_2^1 = \frac{1}{2}(R_a^a - R_b^b) \tag{28}
\]

The impurity scattering is obviously non-diagonal, allowing the processes of flavor violation \( 1 \to 2 \), which accounts for the non-trivial transport through impurity from the left to the right metal. Further, the bulk scattering becomes

\[
\tilde{S}_{12}^{\text{flavor}} = \frac{i}{2 \sinh[\kappa(k_1 - k_2)/\Lambda + i\nu]} [\tau_z \otimes \tau_z + \tau_y \otimes \tau_y] + \frac{1}{2}[1 + \tau_x \otimes \tau_x] \tag{29}
\]

The non-conservation of flavor is also explicitly present in this term: the in-state \( |1\rangle \otimes |1\rangle \) will be mapped to the superposition of \( |1\rangle \otimes |1\rangle \) and \( |2\rangle \otimes |2\rangle \). So, the scattering of particles of first kind into particles of the second kind occurs not only on the impurity, but also in the bulk.

2. Non-equilibrium transport through a resonance level.

Anderson proposed a more generic model than Kondo-type model described in the previous section. All the arguments of the previous section up to formula (7) should be recalled as they are unchanged here. Following Anderson, one assumes the existence of localized states (either impurity or surface states) on the interface between two metals. Such states, or for our purposes just one state is introduced formally as \( d^+_\sigma |0\rangle \), and the coupling to bulk electrons takes a form (instead of equation (7))

\[
H_{\text{int}} = V \sum_{\sigma,m} [\Psi_{\sigma m}^+(0) d_\sigma + d_\sigma^+ \Psi_{\sigma m}(0)] \tag{30}
\]

Besides, one adds to Hamiltonian the part which describes the localized level:

\[
H_d = \epsilon_d \sum_{\sigma = \pm} d^+_\sigma d_\sigma + U \sum_{\sigma = \pm} d^+_\sigma d_\sigma d^+_\sigma^* d^-_\sigma \tag{31}
\]

The stationary one-particle eigenstate of \( H_0 + H_{\text{int}} + H_d \) is a superposition of three terms,

\[
|\psi_\sigma\rangle = \int dx [g_1(x) \Psi_{1\sigma}^+(x) + g_2(x) \Psi_{2\sigma}^+(x)] |0\rangle + e d^+_\sigma |0\rangle \tag{32}
\]
(the particle can be either on the localized level or in the bulk in one of two flavors with spin preserved). From it we infer the impurity scattering matrix

\[ R_1^1 = R_2^2 = R_1^2 = R_2^1 = e^{i\Delta(E)} \] (33)

\[ \Delta(E) = -2 \tan^{-1} \frac{V^2}{E - \epsilon_d} \] (34)

The Anderson model has $U(1)^{\text{flavor}}$ symmetry, like the Kondo model of the previous section. Notice, however, that the flavor $SU(2)$-symmetric two-channel Anderson model can be used to get the above model (30)-(31). The $SU(2)$-symmetric Anderson model differs by the presence of two localized levels, $d_{\uparrow\sigma}\ket{0}$ and $d_{\downarrow\sigma}\ket{0}$, each one coupled to its channel,

\[ H_{\text{int}} = V \sum_{\sigma, m} \left[ \Psi_{\sigma m}^+(0)d_{\sigma m} + d_{\sigma m}^+\Psi_{\sigma m}(0) \right] \] (35)

\[ H_d = \epsilon_d \sum_{m, d} d_{m\sigma}^+d_{m\sigma} - \frac{U}{2} \sum_{m, \sigma, \sigma'} d_{m\sigma}^+d_{m\sigma'}^+d_{m\sigma'}d_{m\sigma} \] (36)

In the $SU(2)$-symmetric model the number of particles of each kind is conserved separately (particles in the bulk and on the localized level must be added together to obtain a conserved charge), and moreover, the $SU(2)$ symmetry in the flavor space is present. To break this symmetry down to $U(1)$ and to destroy the conservation of each kind of particles separately, thus reproducing the original Anderson model (30)-(31), we introduce an additional term, describing hopping between localized levels:

\[ H_{12} = -t \sum_{\sigma} (d_{1\sigma}^+d_{2\sigma} + d_{2\sigma}^+d_{1\sigma}) \] (37)

Then one must take the limit $t \to \infty$ to reproduce the model of interest (intuitively, this limit means identifying $d_1$ with $d_2$ and calling both of them by $d$).2

The same discussion about conserved charges as in the previous section applies to Anderson model. In particular, after the same canonical transformation of fields (18)-(19) we obtain the Anderson model with one kind of particles coupled to impurity, while another kind of particles being free. These two kinds of particles are, however, coupled to each other by the bias term:

\[ H_a = i\nu_F \sum_{\sigma} \int a_{\sigma}^+\partial_z a_{\sigma} + V \sum_{\sigma} \left[ a_{\sigma}^+(0)d_{\sigma} + d_{\sigma}^+a_{\sigma}(0) \right] + \epsilon_d \sum_{\sigma} d_{\sigma}^+d_{\sigma} + U \sum_{\sigma} d_{\sigma}^+d_{\sigma}d_{\sigma}^+d_{-\sigma} \] (38)

\[ H_b = i\nu_F \sum_{\sigma} \int b_{\sigma}^+\partial_z b_{\sigma} \] (39)

\[ Y_0 = \mu \sum_{\sigma} \int a_{\sigma}^+b_{\sigma} + b_{\sigma}^+a_{\sigma} \] (40)

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2When hopping amplitude $t$ is very large, it is worth to work in the basis of mixed states $e_1 = d_1 - d_2$, $e_2 = d_1 + d_2$, where the hopping and $\epsilon_d$-terms become diagonal: $(\epsilon_d + t)e_1^+e_1 + (\epsilon_d - t)e_2^+e_2$. In the limit $t \to \infty$ one of the mixed localized levels ($e_1$) becomes so high in energy that the transitions to this level are unlikely to take place and this level decouples. Then, after rescaling $\epsilon_d$ one can identify $e_2 = d$ (the spin index is assumed everywhere).
3. Open questions.

Computing the non-equilibrium current implies an evaluation of the average

\[ I = e \frac{d}{dt} \langle Y_0 \rangle = e \mu \frac{d}{dt} \sum_\sigma \int \langle a_\sigma^+ b_\sigma + b_\sigma^+ a_\sigma \rangle \]  

(41)
on the Bethe states. Fendley, Ludwig and Saleur [Phys.Rev. B52 (1995) 8934] suggested that at least for certain integrable models (in their case quantum Hall bar with a constriction) the following procedure can be used. First, based on the Bethe ansatz solution in the absence of bias, one identifies bulk excitations (physical quasiparticles) that carry a non-trivial charge/flavor. Further, using standard techniques, one calculates scattering matrices in the bulk and on the impurity for the physical quasiparticles and, in the absence of bias, in the presence of bias, densities of states. Then, based on the probabilistic arguments, one employs the Boltzmann rate equation to obtain the current:

\[ I \sim \int d\epsilon [n_1(\epsilon, \mu) - n_2(\epsilon, \mu)] |S_{12}(\epsilon)|^2 \]  

(42)

Note that this procedure has not been shown rigorously to coincide with (41), and remains as a conjecture [cond-mat/9708163, 9710205]. Working in the original Ψ-basis, we face the problem of non-diagonal bulk scattering – the scattering with the flavor violation \( 1 \to 2 \) happens everywhere, and not only on the impurity, as it was in the case of the Quantum Hall problem with a constriction. The flavor z-projection \( Y_0 \) is not diagonal on the Bethe ansatz states. This leads to some technical difficulties, as well as to the question of the applicability of (42) for such situations. Equivalently, the difficulty can be understood in the representation (20) of a and b-particles and Eq (41). Operator \( Y_0 \) (x-component of the flavor in this representation) strongly disturbs the Bethe state and does not allow us to work only with a few significant low-lying excitations. The state obtained after applying \( Y_0 \) to the Bethe vacuum state involves infinite amount of excitations in the thermodynamic limit. The situation is rather similar to the XXZ chain and the \( S_x \) operator, which perturbs spins on all sites simultaneously, while the Bethe states are constructed as local spin-flips with respect to some reference state. Thus, one says that \( S_x \) excites an infinite amount of elementary particles.

To summarize, the questions to be answered are: what are the physical excitations in the Bethe ansatz solution of (5)-(7) and their scattering matrices? What are the matrix elements of the fields between the excited states (form-factors for the two-channel Kondo model)? Can one apply Eq (42) to the model (5)-(7)?

II. COULOMB BLOCKADE AND THE INTERFERENCE

We describe several problems of interest to condensed matter physics which could be solved by the massless form-factors approach, as perturbations of integrable models. The basic unperturbed model is the massless sine-Gordon model with the interaction (cosine) term
restricted to one point, which is integrable. It is being perturbed by the second interaction term, $V(\phi)$, which destroys the integrability:

$$H = \frac{1}{2} \int_0^\infty \left[ \pi^2(x) + (\partial_x \phi)^2 \right] dx + \mu \cos \beta \phi(0) + \lambda V(\phi) \quad (43)$$

Quasi-exact results can be obtained by making use of the basis of massless quasiparticles of the sine-Gordon model. Namely, one expands the evolution operator, $S = T e^{i\lambda \int V(\phi)}$, in the various matrix elements of interest,

$$\langle T O_1 O_2 \ldots \rangle = \langle T O_1 O_2 \ldots \rangle_0 + i\lambda \langle T O_1 O_2 \ldots \int^t V(\phi) \rangle_0 + \ldots \quad (44)$$

to generate a perturbation series for the physical quantities of interest. Further, for the evaluation of averages in (44) one inserts the full set of intermediate states – massless integrable quasiparticles of the unperturbed sine-Gordon model:

$$\langle O_1 O_2 \ldots \int V(\phi) \rangle = \sum_{n,m,l} \langle 0 | O_1 | n \rangle \langle n | O_2 | m \rangle \ldots \langle l | \int^t V(\phi) | 0 \rangle \quad (45)$$

The form-factors $\langle n | O | m \rangle$ are known exactly. While generating the perturbation series, the following normalization conditions should be respected:

$$\langle 0 | S | 0 \rangle = \langle 0 | 0 \rangle \quad (46)$$
$$\langle 1 | S | 1' \rangle = \langle 1 | 1' \rangle \quad (47)$$

The first equation (46) takes into account the renormalization of the ground state energy (disconnected diagrams), while the second one (47) controls the norm of one-particle states.

1. **Quantum dot**

Matveev [Phys. Rev. B51 (1995) p.1743] argued that the physics of Coulomb blockade in the quantum dots can be adequately described by the following Hamiltonian, $H = H_0 + H_C + H'$:

$$H_0 = \frac{1}{2} \int_{-\infty}^{\infty} dx \left[ \pi^2(x) + (\partial_x u)^2 \right] \quad (48)$$

$$H_C = E_C [u(0) - N]^2 \quad (49)$$

$$H' = -V \cos[2\pi u(0)] \quad (50)$$

where $E_C = e^2 / 2C_0$ is the charging energy characterizing the dot, parameter $N$ is proportional to the gate voltage which is adjustable. Electron tunneling into the dot leads to the increase of energy by the amount $E_C$. Therefore, the tunneling conductance is suppressed. The gate voltage allows to control electrostatic energy of the dot:

$$E_Q = \frac{(Q - eN)^2}{2C_0} \quad (51)$$
where $Q$ is the total charge of the dot. At the values of the gate voltage corresponding to $N = n + 1/2$ the energies of states with charges $en$ and $e(n + 1)$ are equal, leading to the favorable conditions in the electron transport through the dot. Therefore, one observes periodic resonances of the conductance as a function of the gate voltage. The term $H'$ describes quantum tunneling (backscattering) effects, while the term $H_C$ describes electrostatic charging effects. The Coulomb blockade shows up in the oscillations of the ground state energy or the average charge $\langle Q \rangle$ as a function of $N$, which can be calculated. Experimentally the capacitance $C = \partial^2 E/\partial N^2$ can be measured. At the perfect transmission, $V = 0$, the ground state energy is $N$-independent, as can be seen from (48)-(49) by the change of variables $u \rightarrow u + N$. The Coulomb blockade is completely suppressed. On the contrary, for the perfect reflection, $V \rightarrow \infty$, the dependence on $N$ is the most drastic, for the field is pinned at zero by the value $n$ corresponding to one of the equivalent minima, and a departure of $N$ from integer $n$ leads to the increase of energy given by the term $H_C$, thus lifting the degeneracy. At the values $N = n + 1/2$ two neighboring minima still remain degenerate, leading to the drastic changes in the transport properties. Matveev in his work calculates perturbative corrections to the ground state energy. He treats the term $H'$ as a small perturbation and uses the exact eigenstates of $H_0 + H_C$. The form-factors of sine-Gordon model allow, on the contrary, treat exactly the $H'$ term, starting from the integrable model $H_0 + H'$ and to perturb it by $H_C$ term. The required form-factors of the field $u$, originating from the expression for the ground state energy correction, $\langle 0|T \exp(i \int dt H_C)|0 \rangle$, are available.

2. Quantum Hall interferometer.

Chamon et al [cond-mat/9607195] discussed the device based on the quantum Hall bar with two constrictions which allows observation of interesting interference effects. The underlying Hamiltonian is $H = H_0 + H_{tun}$:

$$H_0 = \frac{1}{8\pi} \int_{-\infty}^{\infty} dx \left[ \pi^2(x) + (\partial_x \phi)^2 \right]$$

$$H_{tun} = \gamma_1 \cos \theta(0) + \gamma_2 \cos[\theta(d) + \Phi_0]$$

where the two fields $\theta, \phi$ are related to the left-right moving fields $\phi_L, \phi_R$ as follows:

$$\sqrt{2}\phi = \phi_L + \phi_R$$

$$\sqrt{2}\theta = \phi_L - \phi_R$$

The quantity of interest is the tunnelling current $I_t(\Phi_0, d)$ as a function of flux $\Phi_0$ and distance $d$. For the current $I_t$ it is sufficient to take its linear response limit, given by Kubo’s formula. The current-current correlators can be obtained using form-factors method and treating one of the cosine terms in (53) as a perturbation of integrable sine-Gordon model.

3. Double barrier in the Luttinger liquid.
Kane and Fisher [Phys.Rev. B46 (1992) 15233] studied various tunneling effects through the barriers in Luttinger liquid. The double barrier action reads:

\[
S = S_0 + V \int d\tau \left\{ \cos[2\sqrt{\pi}\theta(\tau, x = 0)] + \cos[2\sqrt{\pi}\theta(\tau, x = d) + k_F d] \right\} \\
+ \frac{V_G}{\sqrt{\pi}} \int d\tau [\theta(\tau, x = d) - \theta(\tau, x = 0)]
\]

where \(S_0\) is the action of the pure Luttinger liquid. In the last term the total number of particles between two barriers is coupled to the chemical potential of the island, \(V_G\). In analogy with the problem of quantum dot above, one can adjust \(V_G\) to have the energy cost of adding another particle vanish. Then, one expects a resonant transmission. The mass scale \(M = k_F / gd\) measures the fluctuations of the charge on the island:

\[
\langle n \rangle = \frac{k_F d}{2\pi} + \frac{V_G}{M}
\]

The first term in the above formula is just the background density. The resonance is achieved when \(\langle n \rangle = \text{half-integer}\). In the strong barrier limit, \(V \gg M\), the total charge of the island prefers to be a particular integer times \(e\), since the discreteness of the electron charge is important. The Coulomb blockade suppresses the transport through the island. When \(\langle n \rangle\) is tuned to be a half-integer, an additional symmetry is present which leads to the degenerate states of different charge. In the opposite, weak barrier limit, \(V \ll M\), following Kane and Fisher one can integrate out small fluctuations of the charge around the value \(\langle n \rangle\) and obtain an effective action:

\[
S = S_0 + aV \int d\tau \cos[2\sqrt{\pi}\theta(0)] + \frac{bV^2}{2M} \int d\tau \cos[4\sqrt{\pi}\theta(0)]
\]

where \(a, b\) are dimensionless constants. The second cosine term in the last formula is irrelevant for \(g > 1/4\), while the first one is relevant. At small temperatures the effective barrier strength \(a\) grows, making the conductance to vanish at \(T = 0\). As for the effective coupling \(b\), it decreases, leading thus to the perfect conductance. Thus, by fine-tuning the single parameter \(a = 0\) (corresponding to the vanishing of the \(2k_F\) Fourier component of the double-barrier scattering potential \(V(x)\)), one achieves the resonance. Qualitatively, the physical meaning of the resonance is the following: forbidding the one-electron backscattering processes (\(\hat{V}(2k_F) = 0\), one creates favorable conditions for the coherent transport of electron pairs (\(\hat{V}(4k_F) \neq 0\)). It is interesting to find the conductance as a function of \(a, b\) by the form-factor method, treating one of the cosine terms in (58) as perturbation.

4. Potential difficulties.

In all the three problems one has to deal with IR divergences in the form-factor expansion, since the perturbing terms are either \(\cos \phi\) or field \(\phi\) itself. Such divergences are peculiar for the massless theories. Lesage and Saleur [cond-mat/9608112] have discussed already the situation. The partial resummation of the most divergent terms of the perturbation series is necessary.
In the first problem it is not clear that the perturbation theory in $E_C$ will describe correctly the interesting regime of large $E_C$, although the rapid convergence of the form-factors series leaves us some hope that one is allowed to reach large enough values of $E_C$ while the terms of the order of $E_C^2$ and higher still can be neglected.

In the second problem the usual trick of changing the basis of fields to the odd and even fields does not work with two constrictions. The odd and even fields are not decoupled any longer.

In the third problem one has to deal with the irrelevant perturbation, for which the form-factors are not quite known. One can in principle overcome this difficulty by an analytic continuation from the space of relevant perturbations. The breather form-factors can be analytically continued from the sinh-Gordon model, while kink and anti-kink form factors can be obtained by continuing analytically the recent results of Lukyanov [hep-th/9703190].