On the different formulations of the E11 equations of motion

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Abstract

The non-linear realisation of the semi-direct product of $E_{11}$ with its vector representation leads to equation of motions for the fields graviton, three form, six form, dual graviton and the level four fields which correctly describe the degree of freedom of eleven dimensional supergravity at the linearised level. The equations with one derivative generically hold as equivalence relations and are often duality relations. From these equations, by taking derivatives, one can derive equations that are equations of motion of the familiar kind. The entire hierarchy of equations is $E_{11}$ invariant and the construction does not require any steps beyond $E_{11}$. We review these past developments with an emphasis on the features that have been overlooked in hep-th:1703.01305, whose alternative approach we also comment on.
1. Introduction

We consider the non-linear realisation of the semi-direct product of \( E_{11} \) and its vector representation, denoted \( E_{11} \otimes s l_1 \) [1,2]. For a review of this subject see reference [3]. In this paper we will focus in the eleven dimensional theory which arises when one takes the decomposition of \( E_{11} \) into the GL(11) subalgebra that arises from deleting the node usually labelled as eleven in the \( E_{11} \) Dynkin diagram. One finds that this theory contains the fields [1,4]

\[
\begin{align*}
    h_{a}^{b}, & \quad A_{a_{1}a_{2}a_{3}}; \quad A_{a_{1}...a_{6}}; \quad h_{a_{1}...a_{8},b}; \quad A_{a_{1}...a_{9},b_{1}b_{2}b_{3}}; \quad A_{a_{1}...a_{10},b_{1}b_{2}}; \quad A_{a_{1}...a_{11},b_{1}}; \quad \ldots
\end{align*}
\]

(1.1)

The first four fields correspond to the graviton, the three form, the six form and the dual graviton field at levels zero, one, two and three respectively. For these fields the indices in a given block are antisymmetric and the dual graviton field obeys the constraint [1,4]

\[
h_{[a_{1}...a_{8},b]} = 0
\]

(1.2)

The last three fields explicitly listed are those at level four and we will give their constraints later in the paper. The non-linear realisation contains a spacetime with the coordinates [2]

\[
    x^{a}, \quad x_{ab}, \quad x_{a_{1}...a_{5}}, \quad x_{a_{1}...a_{7},b}, \quad x_{a_{1}...a_{8}}, \quad \ldots
\]

(1.3)

and the fields depend on these coordinates.

The equations of motion are determined by the symmetries of the non-linear realisation whose procedure can be found in many \( E_{11} \) papers, see for example [3,6]. While partial results can be found in the early \( E_{11} \) papers it is only relatively recently that a systematic study of the equations of motion have been carried out using the higher level \( E_{11} \) symmetries and a better understanding of some of the more intricate details of how the non-linear realisation works in practice. It was shown [5,6] that the non-linear equations of motion of the eleven dimensional theory are uniquely determined at low levels and they are precisely those of eleven dimensional supergravity of Cremmer, Julia and Scherk when suitably truncated. The analogous calculations have also been carried out in five dimensions [5,6] with the same result. It is inevitable that similar results apply in all the other dimensions. This essentially confirms the \( E_{11} \) conjecture, namely that the low energy effective action of strings and branes has an \( E_{11} \) symmetry. These results were obtained at the full non-linear level and they included all the non-linear effects in the corresponding supergravity theories.

The results were extended to a higher level to include the level four fields, albeit at the linearised level [7]. The \( E_{11} \) invariant equations of motion of these fields were found. One finds that the equations correctly account for all the degrees of freedom of eleven dimensional supergravity through a series of duality relations as well as giving the eleven dimensional origin of Romans theory [7,8].

In a very recent paper [9] the authors start from the \( E_{11} \) formalism and also compute the equations of motion, at the linearised level, along the lines of some of the earlier \( E_{11} \) papers [1,2]. However, they have overlooked an important aspect of how the duality relations work and have concluded that the trace of the spin connection in missing and as a
result the theory does not describe Einstein’s theory. In this paper we will review the work on $E_{11}$ of references [5,6,7] emphasising the points that has been overlooked in [9], namely that the equations that have low numbers of derivatives generically hold as equivalence relations. While the equations of motion with the highest number of derivatives, for a given field, hold in the usual sense and they are the correct equations of motion in a familiar form [10,5,6,7].

2. Equations of motion including level three

The construction of the equations of motion from the $E_{11} \otimes s l_1$ non-linear realisation uses the Cartan forms which transform only under the action of the Cartan involution invariant subalgebra of $E_{11}$, denoted $I_c(E_{11})$, once one converts its one world index to a tangent index. The set of equations is essentially unique, at least at low levels, and is invariant under the symmetries of the non-linear realisation. One finds a hierarchy of equations and the results up to and including level three fields are given in the table below [5,6,7]. The extension to level four will be reviewed later in this paper. As one moves down the table the number of derivatives increases, indeed the superscript in brackets denotes the number of derivatives, and as one moves to the right the level of the fields involved increases. The horizontal arrows denote the effect of $I_c(E_{11})$ transformations.

Table 1. The $E_{11}$ variations of the equations of motion up to level three

\begin{align*}
E^{(1)}_{a_1a_2a_3a_4} = 0 & \iff E^{(1)}_a b_1 b_2 = 0 \\
\downarrow & \downarrow \\
E^{(2)}_{a_1 a_2 a_3} = 0 & \iff E^{(2)}_a b = 0 \\
& \iff \\
E^{(2)}_{a_1 \ldots a_6} = 0 & \iff E^{(2)}_{a_1 \ldots a_8, b} = 0
\end{align*}

The objects that appear in the above table with one derivative are given by

\begin{equation}
E^{(1)}_{a_1 \ldots a_4} \equiv \partial_{[a_1} A_{a_2 a_3 a_4]} - \frac{1}{48} \epsilon_{a_1 a_2 a_3 a_4} b_1 \ldots b_7 \partial_{b_1} A_{b_2 \ldots b_7} = 0
\tag{2.1}
\end{equation}

\begin{equation}
E^{(1)}_{a, b_1 b_2} \equiv \omega_{a, b_1 b_2} - \frac{1}{4} \epsilon_{b_1 b_2 c_1 \ldots c_9} \partial_{c_1} h_{c_2 \ldots c_9, a} = 0
\tag{2.2}
\end{equation}

where

\begin{equation}
\omega_{a, b c} = -\partial_c h_{(ca)} + \partial_c h_{(bc) + \partial_a h_{[bc]}}
\tag{2.3}
\end{equation}

While the equations with two derivatives are given by

\begin{equation}
E^{(2)}_{a_1 a_2 a_3} \equiv \partial_b E^{(1)}_{b a_1 a_2 a_3} = \partial_b \partial^{[b} A_{a_1 a_2 a_3]} = 0
\tag{2.4}
\end{equation}
\[ E^{(2)a_{1}...a_{6}} = \frac{2}{7!} \partial_{b} e^{b a_{1}...a_{6} c_{1}...c_{4}} E^{(1)}_{c_{1}...c_{4}} = \partial_{b} \partial^{[b} A^{a_{1}...a_{6}]} = 0 \] \hspace{1cm} (2.5)

\[ E^{(2)}_{a_{1}...a_{6} b} = \partial_{a} \omega_{c_{1}...c_{4}} - \partial_{c_{1}...c_{4}} \omega_{a_{1}...a_{6}} \equiv R_{a_{1}...a_{6} b} = 0 \] \hspace{1cm} (2.6)

and

\[ E^{(2)}_{a_{1}...a_{8}, b} = - \frac{1}{4} \partial^{[d} \partial_{[d} h_{a_{1}...a_{8}], b]} = 0 \] \hspace{1cm} (2.7)

In the above objects we have only included the parts that contain terms with derivatives with respect to the usual spacetime coordinates, the terms with derivatives with respect to the level one coordinates have been found in [7]. However, it is important to realise that the latter terms are absolutely crucial for the invariance of the equations and we have omitted them here in order to make the presentation uncluttered and clearer. In the previous papers [5,6,7], the equations of motion were formulated in terms of the Cartan forms, to recover these results one makes the substitution \( \partial_{a} A_{\ast} \rightarrow G_{a_{1}...a_{6}\ast} \) which is valid at the linearised level.

In the above table the equations with only one derivative are duality relations, that is they express the derivative of one field in terms of the derivative of another field and they appear in the first row. One can take a derivative of these equations with one spacetime derivative and find equations that are second order in derivatives. If one does this in a precise way one can eliminate one of the two fields that occur in the equation with one spacetime derivative and find an equation of motion for only one of the fields that is second order in derivatives. These are the equations in the second row. The down arrow (\( \Downarrow \)) in the table correspond to this projection using the derivative. These later equations are the equations of motion for the degrees of freedom of eleven dimensional supergravity. However, they appear in a duality symmetric formulation in that one find equations in their usual form, that is for the graviton (2.6) and three form (2.4) as well as equivalent equations for the dual fields, that is for the six form (2.5) and the dual graviton (2.7). It is important to realise that one can not discard the first order equations as they ensure that the degrees of freedom are not duplicated.

We note that in general the precise form of the equations of motion for a given particle are determined by the requirement that they describe the corresponding irreducible representations of the Poincare group. The one exception being that one can use fields of different Lorentz character to describe the same particle. We note that the correct degrees of freedom of gravity are described by a dual graviton field \( h_{a_{1}...a_{8}, b} \) that has the equation of motion of equation (2.7) and also that obeys the constraint of equation (1.2). As one would expect, the equation of motion for the dual graviton obeys the same constraint, that is, the equation obeys the condition \( E^{(2)}_{a_{1}...a_{8}, b} = 0 \) [7]. Thus gravity can be formulated in terms of a dual field that satisfies equation (1.2) and the field equation has the corresponding constraint. Put differently it does not also need a field that has nine indices which are totally antisymmetric.

Equation (2.1) is the well known duality relation between the three form and the six form. While equation (2.2) relates the graviton field to the dual graviton field. This equation was given in the original \( E_{11} \) paper [1] with the difference that the constraint of equation (1.2) was not enforced as the equation was not derived from \( E_{11} \) but rather constructed by hand in order to show that the field \( h_{a_{1}...a_{8}, b} \) could correctly describe the
degree of freedom of gravity. In fact that such a field did describe gravity was first realised in five dimensions in reference [11] and such a field was suggested in a general dimension in reference [12], although only partial light cone based arguments were given to show that it really did describe gravity. It was shown in reference [22] that the equations of reference [1] were equivalent to the formulation of reference [11] in five dimensions. Thus the equation in reference [1] included a field $h_{a_1...a_9}$, that is, the completely antisymmetric part of $h_{a_1...a_9b}$. As equation (1.2) makes clear this totally antisymmetric field is not contained in the $E_{11}$ non-linear realisation. It was realised in reference [4] that a version of the gravity-dual gravity equation (2.2), which includes the field $h_{a_1...a_9}$ and had terms with the trace of the spin connection (see equation (4.19) of reference [1] or equation (4.3) of reference [4]) was Lorentz invariant if the usual Lorentz transformation of the graviton was compensated for by a transformation of the nine form indeed

$$
\delta h_{ab} = -\Lambda_{ab}, \quad \delta h_{a_1...a_9} = \frac{1}{4.7!} \varepsilon_{a_1...a_9c_1c_2} \Lambda^{c_1c_2} (2.8)
$$

This transformation follows in an obvious way once one takes into account the well known transformation of the spin connection which transforms by the inhomogeneous term, $\delta \omega_{\lambda, \mu_1 \mu_2} = \partial_\lambda \Lambda_{\mu_1 \mu_2} + \ldots$ where $\ldots$ indicate the homogeneous terms.

Now let us consider the gravity-dual gravity equation (2.2) which appears in the $E_{11}$ non-linear realisation where the dual graviton satisfies equation (1.2). Clearly this equation is not invariant under local Lorentz transformations. The resolution of this dilemma is to think of equation (2.2) as being valid only modulo local Lorentz transformations. In other words it holds modulo the transformations [10,5,6,7]

$$
E^{(1)}_{a,b_1b_2} \sim E^{(1)}_{a,b_1b_2} + \partial_a \Lambda_{b_1b_2} + \ldots (2.9)
$$

where $\ldots$ indicate the homogeneous Lorentz transformations of $E^{(1)}_{\lambda, \mu_1 \mu_2}$. The use of the symbol $\sim$ in equation (2.2) implies that the equation only holds modulo the local Lorentz transformations as just discussed.

To put it in a more mathematical sense; we are regarding equation (2.2) to belong to an equivalence class, the equivalence relation being that of equation (2.9). Physicists are familiar to such relations, for example in the definition of physical states in the context of the BRST formalism. This way of proceeding is a completely correct and rigorous. As one takes the derivatives to find the second order equations one eliminates the transformations that the equation is modulo, indeed the projection is chosen in just such a way as to do this, and one finds equations which hold in the usual sense, that is, are not modulo any transformations.

We note that taking equation (2.2) to hold as an equivalence relation has the same effect as adding a field $h_{a_1...a_9}$ and then realising that the equation is invariant under the Lorentz transformations of equation (2.8). In effect the Lorentz transformation is just the field $h_{a_1...a_9}$.

It was realised in references [13] that if one naively takes the trace of equation (2.2) then the trace of the spin connection vanished due to the condition of equation (1.2) and we find that

$$
E^{(1)}_{c,cb} = \omega_{c,cb} = \partial^b h^c_c - \partial_c h^{cb} = 0 (2.10)
$$
Clearly this is not consistent with Einstein’s equation. However, as we have discussed just above, equation (2.2) is to be thought of as an equivalence relation and we have to take this into account. Following reference [10], and its implementation in references [5,6,7], one find that if we take the trace of equation (2.2) we find, instead of equation (2.10), the result

\[ E^{(1)}_{c,cb} = \partial^d h^c_{(d)} - \partial_c h^{(bc)} + \partial_c k^{cb} = 0 \]  

(2.11)

where \( k_{ab} = (h_{[ab]} - \Lambda_{ab}) \) and \( \Lambda_{ab} \) is the Lorentz transformation the equation is subject to.

Taking the derivative of equation (2.2) we find that

\[ \partial^c E^{(1)}_{a,cb} = \partial_b \partial^c h_{(ca)} - \partial^2 h_{(ba)} + \partial_a \partial^c k_{cb} = 0 \]  

(2.12)

where \( \partial^2 = \partial^c \partial_c \).

To eliminate the local Lorentz transformation we can act on equation (2.11) with \( \partial_a \) and subtract equation (2.12), to find an equation that is independent of the dual graviton and the local Lorentz transformation and is given by

\[ \partial^c E^{(1)}_{a,cb} - \partial_a E^{(1)c,cb} = \partial_a \partial^c h_{(ca)} + \partial_b \partial^c h_{(b)} - \partial^2 h_{(ab)} - \partial_b \partial_a h^c = 0 \]  

(2.13)

This is indeed the linearised Einstein equation.

We observe that acting with the derivative \( \partial_b \) on equation (2.11) we can also find an equation that is independent of the local Lorentz transformation

\[ \partial^2 h^b_b - \partial^a \partial^b h_{ab} = 0 \]  

(2.14)

which is indeed the linearised version of the trace of the Einstein equation.

The authors of reference [9] have studied reference [13] and so are aware of the difficulties on taking the trace of equation (2.2) and the contradiction embodied in equation (2.10). However, they have not realised the corresponding resolution of the difficulties [10, 5,6,7] and so come to the conclusion that one must add fields. Unfortunately reference [13] did previously realise that the gravity-dual gravity equation was an equivalence relation but then this paper incorrectly concluded that this did not solve the problem whereas, as we have just seen, it does [10,5,6,7].

To summarise, the non-linear realisation of \( E_{11} \otimes s l_1 \) leads to equations that do not miss the trace of the spin connection as the equation in which it occurs is an equivalence relation. From this equation one finds the correct equation for the degrees of freedom of the gravity whether encoded in the usual graviton field, or the dual gravity field.

The above story is the general pattern; the equations of motion with low numbers of derivatives are equivalence relations and from these one can derive, by taking derivatives, equations that hold in the traditional sense and these are the familiar equations of motion for the fields involved. The full system of equations is invariant under the symmetries of the non-linear realisation once one takes account of the fact that some of the equations are equivalence relations. In the \( E_{11} \otimes s l_1 \) non-linear realisation we find fields with more and more blocks of indices as the level increases and one can expect that these should obey the traditional equations of motion that have one derivative for every block of indices that they contain. Hence as one considers higher and higher level fields in \( E_{11} \) one expects to
find traditional equations with higher and higher numbers of derivatives. The way $E_{11}$ handles this situation is to have a hierarchy of equations containing increasing numbers of derivatives. The equations with the lower number of derivatives are equivalence relations, that is, only hold modulo certain transformations. The very first duality relation of equation (2.1) is unusual in that it already possess one derivative for each block of indices that the field carries, namely one. This pattern becomes particularly apparent when one considers level four fields as we do in the next section.

Reference [14] established a no-go theorem for the dual graviton and this is often quoted as an obstacle to the $E_{11}$ programme. As we have seen above there is no obstacle at the linear level and one finds a perfectly correct theory. These is also no obstacle at the non-linear level as reference [14] investigates if one can find a theory that involves the dual graviton field alone. However, this is not the path chosen by $E_{11}$ which involves in the non-linear dual graviton equation both the gravity and the dual gravity fields. This equation is under construction and will be published elsewhere [15].

The transformations of the equations of motion given in table 1 were given in reference [7] and for completeness we give the results here. The horizontal arrows correspond to the effect of varying under $I_c(E_{11})$ which is the non-trivial symmetry which acts on the Cartan forms which contain the above objects. The transformations have a parameter $\Lambda_{b_1b_2b_3}$, for an explanation see the earlier papers [3,4,5,6,7,13]. We begin with the transformations of the duality relations with only one derivative:

$$\delta E^{(1)}_{a_1...a_4} = \frac{1}{4!} \epsilon_{a_1...a_4} b_{1...7} \Lambda_{b_1b_2b_3} E^{(1)}_{b_4...b_7} + 3 E^{(1)}_{c[a_1a_2} \Lambda^{c}_{a_3a_4]}$$

(2.15)

$$\delta E^{(1)}_{\lambda,\mu_1\mu_2} = \frac{7}{12} \epsilon_{\mu_1\mu_2} \nu_1...\nu_6 \sigma_1 \sigma_2 \sigma_3 E^{(1)}_{\lambda\nu_1...\nu_6} \Lambda_{\sigma_1\sigma_2\sigma_3} + \frac{1}{2} \epsilon_{\mu_1\mu_2} \nu_1...\nu_7 \sigma_1 \sigma_2 E^{(1)}_{\nu_1...\nu_7} \Lambda_{\sigma_1\sigma_2\lambda}$$

$$+ \frac{55}{2} \Lambda_{\sigma_1\sigma_2[\mu_1} \epsilon_{\mu_2]} \nu_1...\nu_{10} E^{(1)}_{\nu_1...\nu_{10}\lambda} \sigma_1 \sigma_2 - \frac{55}{18} \Lambda_{\sigma_1\sigma_2\sigma_3} \eta_{\lambda[\mu_1} \epsilon_{\mu_2]} \nu_1...\nu_{10} E^{(1)}_{\nu_1...\nu_{10},\sigma_1 \sigma_2 \sigma_3}$$

$$+ \frac{3}{4} \Lambda_{\mu_1\mu_2} \sigma \epsilon_{\rho_1...\rho_{11}} E^{(1)}_{\rho_1...\rho_{11},\sigma \lambda} + \partial_{\lambda} \tilde{\Lambda}_{\mu_1\mu_2}$$

(2.16)

where

$$\partial_{\lambda} \tilde{\Lambda}_{\mu_1\mu_2} = - \epsilon_{\mu_1\mu_2} \nu_1...\nu_{9} \left[ \frac{1}{12} \partial_{\lambda} A_{\nu_1...\nu_{6}} \Lambda_{\nu_7\nu_8\nu_9} ight.$$  

$$+ \frac{55}{36} \partial_{\lambda} A_{\nu_1...\nu_{9}, \sigma_1 \sigma_2 \sigma_3} \Lambda^{\sigma_1 \sigma_2 \sigma_3} + \frac{55}{16} \partial_{\lambda} A_{\sigma_1 \sigma_2 \sigma_3 \nu_1...\nu_9 \sigma_1 \sigma_2 \sigma_3} \right],$$

(2.17)

The transformations include the contributions from level four fields which we will discuss in the next section. We observe that the variation of the gravity-dual gravity relations involves a local Lorentz transformation consistent with the fact that this relation is an equivalence relation. The symbol $\mathcal{E}$ is equal to $E$ plus terms that contain the derivatives with respect to the higher level coordinates. The precise form of the $\mathcal{E}$’s can be found in reference [7] and we note that without these the result would not hold.
The variations of the equations of motion with two spacetime derivatives are given by

\[
\delta E^{(2) a_1 a_2 a_3} = \frac{3}{2} E^{(2) b [a_1 | a_2 a_3]} - \frac{1}{24} \epsilon^{a_1 a_2 a_3 \nu c_1 ... c_4 b_1 b_2 b_3} \partial_{\nu} E^{(1) c_1 ... c_4 | a_1 b_2 b_3} = \frac{3}{2} E^{(2) b [a_1 | a_2 a_3]} + 3.57 E^{(2) a_1 a_2 a_3 b_1 b_2 b_3} \Lambda_{b_1 b_2 b_3} \tag{2.18}
\]

\[
\delta E^{(2) a_1 ... a_6} = \frac{8}{7} \Lambda_{a_1 a_2 a_3} E^{(2) a_4 a_5 a_6} - 27.64 E^{(2) a_1 ... a_6 c_1 c_2 c_3} \Lambda^{c_1 c_2 c_3} \tag{2.19}
\]

\[
\delta E^{(2) a b} = -36 \Lambda^{d_1 d_2} E^{(2) b d_1 d_2} - 36 \Lambda^{d_1 d_2} E^{(2) a d_1 d_2} + 8 \eta_{a b} E^{(2) d_1 d_2 d_3} \Lambda^{d_1 d_2 d_3} \tag{2.20}
\]

\[
\delta E^{(2) \rho_1 ... \rho_8, \lambda} = -\frac{7}{4} E^{(2) \sigma [\rho_1 ... \rho_5} \Lambda^{\sigma \rho_6 \rho_7} \eta_{\rho_8] \lambda} + 275 \left( E^{(2) \rho_1 ... \rho_8 \sigma_1, \sigma_2 \sigma_3 \lambda} - \frac{1}{9} E^{(2) \rho_1 ... \rho_8 \lambda, \sigma_1 \sigma_2 \sigma_3} \right) \Lambda^{\sigma_1 \sigma_2 \sigma_3} + \frac{1}{4.7!} \left( \epsilon^{\rho_1 ... \rho_8 \sigma_1 \tau_1 \tau_2} \partial^{\tau_1} E^{(1) \tau_2 \sigma_2 \sigma_3 \lambda} - \frac{1}{9} \epsilon^{\rho_1 ... \rho_8 \lambda \tau_1 \tau_2} \partial^{\tau_1} E^{(1) \tau_2 \sigma_1 \sigma_2 \sigma_3} \right) + \frac{165}{8} \left( E^{(2) \nu \rho_1 ... \rho_8 \sigma_1 \sigma_2, \nu \lambda, \sigma_3} - \frac{1}{9} E^{(2) \sigma_2 \rho_1 ... \rho_8 \lambda \nu, \sigma_1 \sigma_3, \nu} \right) \Lambda^{\sigma_1 \sigma_2 \sigma_3} \tag{2.21}
\]

In the above we have included the level four terms whose definitions are given in the next section. We observe that the equations do rotate into each other under the symmetries of the non-linear realisation provided we take account of the fact that one of them is an equivalence relation.

3. Equations of motion at level four

The fields of the non-linear realisation given in equation (1.1) include those of level four and for these fields all blocks of indices are antisymmetrised except for the second block of the field \( A_{a_1 ... a_{10}, b_1 b_2} \) which is symmetric, that is, \( A_{a_1 ... a_{10}, b_1 b_2} = A_{a_1 ... a_{10}, b_2 b_1} \). The fields also obey the usual SL(11) irreducibility conditions, that is,

\[
A_{[a_1 ... a_9, b_1] b_2 b_3} = 0 = A_{[a_1 ... a_{10}, b_1] b_2} \tag{3.1}
\]

The equations of motion, including the above fields, are given in the table 2 below, so extending the results of the previous section which were given in table 1.

Table 2. The \( E_{11} \) equations of motion including level four fields [7].
\[ E^{(1)}_{a_1 a_2 a_3 a_4} = 0 \Leftrightarrow E^{(1)}_{a_1 b_1 b_2} = 0 \Leftrightarrow E^{(1)}_{a_1 \ldots a_{10}, b_1 b_2} \Leftrightarrow E^{(1)}_{a_1 \ldots a_{11}, b_1 b_2} \]

\[ E^{(2)}_{a_1 a_2 a_3} = 0 \Leftrightarrow E^{(2)}_{a} = 0 \Leftrightarrow E^{(2)}_{a_1 \ldots a_9, b_1 b_2 b_3} = 0 \]

\[ E^{(2)}_{a_1 \ldots a_6} = 0 \Leftrightarrow E^{(2)}_{a_1 \ldots a_8, b} = 0 \Leftrightarrow E^{(2)}_{a_1 \ldots a_{11}, b_1 b_2 c} = 0 \]

\[ E^{(3)}_{a_1 \ldots a_{11}, b_1 b_2 c_1 c_2} = 0 \]

In the above table 2 the objects not contained in table 1 with one derivative are given by

\[ E^{(1)}_{\mu_1 \ldots \mu_{10}, \sigma_1 \sigma_2 \sigma_3} \equiv \partial_{[\mu_1 A_{\ldots \mu_{10}], \sigma_1 \sigma_2 \sigma_3} - \frac{1}{5.5.117!} \epsilon_{\mu_1 \ldots \mu_{10}} \partial_{[\tau A_{\sigma_1 \sigma_2 \sigma_3}]} \]

\[ E^{(1)}_{\rho_1 \rho_2 \ldots \rho_{11}, \sigma_4} \equiv \partial_{[\rho_1 A_{\rho_2 \ldots \rho_{11}], \sigma_4} \]

and those with two derivatives by

\[ E^{(2)}_{\mu_1 \ldots \mu_{10}, \sigma_1 \sigma_2 \sigma_3} \equiv E^{(2)}_{\nu \rho_1 \ldots \rho_9, \nu \sigma_1 \sigma_2 \sigma_3} \]

where

\[ E^{(2)}_{\mu_1 \ldots \mu_{10}, \sigma_1 \ldots \sigma_4} \equiv \partial_{[\sigma_1]} E^{(1)}_{\mu_1 \ldots \mu_{10}, \sigma_2 \sigma_3 \sigma_4} \]

and

\[ E^{(2)}_{\nu_1 \nu_2 \ldots \nu_{11}, \kappa \tau, \rho} \equiv \partial_{\tau} E^{(1)}_{\nu_1 \nu_2 \ldots \nu_{11}, \rho \kappa} - \partial_{\kappa} E^{(1)}_{\nu_1 \nu_2 \ldots \nu_{11}, \rho \tau} \]

The one object with three derivatives is defined to be

\[ E^{(3)}_{c_1 \ldots c_{11}, a_1 a_2, b_1 b_2} = -\frac{1}{2} (\partial_{a_1} E^{(2)}_{c_1 \ldots c_{11}, b_1 b_2, a_2} - \partial_{a_2} E^{(2)}_{c_1 \ldots c_{11}, b_1 b_2, a_1}) \]

\[ = 2 \partial_{[a_1} \partial_{[b_1} \partial_{[c_1} A_{c_2 \ldots c_{11}], a_2 b_2]} = 0, \]

As before the down arrow means we take a derivative to find a new equation, the precise projection is given in the above definitions. The horizontal arrows in table 2 correspond to the \( I_c(E_{11}) \) variations which are discussed below.

As before the presence of the symbol \( \dot{=} \) indicates that the equation should be viewed as an equivalence relation, that is equations (3.2), (3.3) and equation (3.6). We now comment on the general procedure for finding what are the transformations that are required in the
equivalence relations. These can be found in two ways. Either by integrating up the exact equation in the hierarchy which has the largest number of derivatives, or by carrying out an $I_c(E_{11})$ transformation on the equation and finding out what additional transformations in addition to the previously found equations arise. Of course having found the results using one method one can check the results using the other method. We now illustrate the two methods in the context of the $A_{a_1...a_9,b_1b_2b_3}$ field and the $A_{a_1...a_{10},b_1b_2}$ field respectively.

The equations that involve the three form and the $A_{a_1...a_9,b_1b_2b_3}$ field are those of equations (3.2), (3.4) and (3.5). The equation $E^{(2)}~\rho_1...\rho_9,\sigma_1\sigma_2\sigma_3 = 0$ arises from the $I_c(E_{11})$ variation of the dual graviton equation of motion (2.21). It is a duality relations between the three form $A_{a_1a_2a_3}$ and the field $A_{a_1...a_9,b_1b_2b_3}$. However, in order to eliminate the field $A_{a_1a_2a_3}$ in equation (3.4) we must take the triple trace to find the equation

$$E^{(2)}~\rho_1...\rho_9,\nu_1...\nu_4, \nu_1...\nu_4 = \partial[\nu_1 \partial[\rho_1 A_{\rho_2...\rho_9,\nu_1...\nu_4},\nu_2...\nu_4] = 0$$ (3.8)

This is indeed the correct equation of motion for the $A_{a_1...a_9,b_1b_2b_3}$ to describe the same degrees of freedom which are usually encoded in the three form [11]. In fact from the first order duality relation, equation (3.2) one can deduce a stronger condition, namely $E^{(2)}~\rho_1...\rho_{10},\sigma_1...\sigma_4 = 0$, rather than equation (3.4) even taking account of the modulo transformations. This later equation may be contained in the non-linear realisation as it could also result from varying the equations of motion of the level five fields.

Equation (3.4), which has two derivatives, can be integrated up to find a first order equation, namely equation (3.2), that involves the same fields. The result is equation (3.2) which involves the object $E^{(1)}_{a_1...a_{10},b_1b_2b_3}$ which is first order in derivatives. This equation is an equivalence relation which can be written as

$$E^{(1)}_{\mu_1...\mu_{10},\sigma_1\sigma_2\sigma_3} \sim E^{(1)}_{\mu_1...\mu_{10},\sigma_1\sigma_2\sigma_3} + \partial[\sigma_1] \partial[\mu_1] A_{\mu_2...\mu_{10},|\sigma_2\sigma_3]}$$ (3.9)

How to integrate up the equation was discussed in reference [16], on page 26. One would expect in this process to find a single derivative acting on a new function, however, this new function can in turn be written as the derivative of another function $A_{\mu_2...\mu_{10},\sigma_2\sigma_3}$ as a result of the fact that the field $A_{a_1...a_9,b_1b_2b_3}$ is an irreducible representation of SL(11). We refer the reader to reference [16] for the complete discussion. One can also recover equation (3.2) from the $I_c(E_{11})$ variation of the gravity-dual gravity relations, see equation (2.16), thus verifying the result of integrating up.

We observe that equation (3.4) is gauge invariant under the expected gauge transformation for the field $A_{a_1...a_9,b_1b_2b_3}$ namely[16]

$$\delta A_{a_1...a_9,b_1b_2b_3} = 9 \partial[a_1 A_{a_2...a_9},b_1b_2b_3] + 3 (\partial[b_1] A_{a_2...a_9},[b_1,b_2b_3] + \frac{9}{7} \partial[a_1 A_{a_2...a_9}],b_1b_2b_3)]$$ (3.10)

It is important to note that this gauge invariance is not a requirement but instead one finds that the equation (3.4) which emerges from the $E_{11} \otimes l_1$ non-linear realisation possess this symmetry. We also observe that the transformation that occurs in equation (3.9) can be interpreted as a gauge transformation.

The general procedure for finding the modulo transformations that occur in the equivalence relations by integrating up expect to be as follows. For a field that has $r$ blocks of
indices we can find an equation with \( r \) derivatives that varies under \( I_c(E_{11}) \) transformation into the equations of motion we have already found at lower levels plus higher level terms. Given this equation, which holds exactly, one can integrate it up to find an equation with a lower number of derivatives and in doing so one finds some arbitrary functions which define the equivalence relation. The level four field \( A_{a_1 \ldots a_9, b_1 b_2 b_3} \) has two blocks of indices and equation (3.4) which is the equation of motion has two derivatives and holds exactly. We can also apply this strategy to the Einstein equation \( R_{ab} = 0 \) to find the gravity-dual gravity relation given above. It is inevitable that the equations derived in this way will be equivalence relations. It would be instructive to carry out the calculations when all the local symmetries of the non-linear realisation are fixed and in particular when the local Lorentz symmetry is used to make the graviton symmetric. We note however that the gravity-dual gravity duality equation would still be an equivalence relation with a transformation of the dual graviton field; see equation (5.14) of reference [7]. Indeed the effect of taking this modulo transformation is the same in terms of determining the higher derivative equations and it might be best to regard this modulo transformation as being the fundamental one.

We now illustrate the second method of ascertaining what is the precise form of the equivalence relations by carrying out the \( I_c(E_{11}) \) variations in the context of the \( A_{a_1 \ldots a_{10}, b_1 b_2} \) field which occurs in equations (3.3), (3.6) and (3.7). From the variation of the gravity-dual gravity relation of equation (2.16) we find equation (3.3) for the field \( A_{a_1 \ldots a_{10}, b_1 b_2} \) and has only one derivative. Varying this latter equation under \( I_c(E_{11}) \) we find that

\[
\delta \mathcal{E}^{(1)}_{c_1 \ldots c_{11}, a_1 a_2} = - \frac{60}{11 \cdot 11!} \varepsilon_{c_1 \ldots c_{11}} E^{(1)}_{(a_1|, d_1 d_2} \Lambda_{|a_2)} d_1 d_2 - \varepsilon_{c_1 \ldots c_{11}} \partial_{(a_1} \tilde{\Lambda}_{a_2)}, \tag{3.11}
\]

where

\[
\partial_{(a_1} \tilde{\Lambda}_{a_2)} = - \frac{60}{11 \cdot 11!} \left( \partial_{(a_1} h_{|d_1 d_2|} \Lambda_{|a_2)} d_1 d_2 + \frac{1}{20} \varepsilon^{d_1 \ldots d_{11}} \partial_{(a_1} h_{|d_1 \ldots d_8|, a_2)} \Lambda_{d_9 d_{10} d_{11}} \right). \tag{3.12}
\]

While from the variation of the dual graviton equation of motion (2.21) we find equation (3.6), which has two derivatives, and its \( I_c(E_{11}) \) variation is given by

\[
\delta \mathcal{E}^{(2)}_{c_1 \ldots c_{11}, a_1 b_1 b_2} = \frac{60}{11 \cdot 11!} \varepsilon_{c_1 \ldots c_{11}} \partial_{[b_1} \left( E^{(1)}_{[b_2|, d_1 d_2} \Lambda_{a} d_1 d_2 + E^{(1)}_{a, d_1 d_2} \Lambda_{|b_2]} d_1 d_2 \right)
\]

\[
+ \varepsilon_{c_1 \ldots c_{11}} \partial_a \partial_{[b_1} \tilde{\Lambda}_{b_2]}, \tag{3.13}
\]

where \( \partial_{[b_1} \tilde{\Lambda}_{b_2]} \) is given in equation (3.12), but with antisymmetrisation instead of symmetrisation.

We observe that these equation do not vary into the field equations that we already have and we can interpret the additional terms as those required in the equivalence relation. The result is that we define the equivalence relations

\[
\mathcal{E}^{(1)}_{c_1 \ldots c_{11}, a_1 a_2} = 0, \quad \text{meaning} \quad \mathcal{E}^{(1)}_{c_1 \ldots c_{11}, a_1 a_2} - \varepsilon_{c_1 \ldots c_{11}} \partial_{(a_1} \tilde{\Lambda}_{a_2)} = 0 \tag{3.14}
\]
\[ E^{(2)}_{c_1...c_{11}, a, b_1 b_2} = 0, \quad \text{meaning} \quad E^{(2)}_{c_1...c_{11}, a, b_1 b_2} + \varepsilon_{c_1...c_{11}} \partial_a \partial_{b_1} \hat{A}_{b_2} = 0 \quad (3.15) \]

We therefore search for an equation of motion that is independent of \( \hat{A}_b \) by taking one more derivative. The result is equation (3.7) whose \( I_c(E_{11}) \) variation is given by

\[ \delta E^{(3)}_{c_1...c_{11}, a_1 a_2, b_1 b_2} = -\frac{60}{11 \cdot 11!} \varepsilon_{c_1...c_{11}} \partial_{[a_1]} \partial_{b_1} \left( E^{(1)}_{[a_2], d_1 d_2} \Lambda_{|b_2|}^{d_1 d_2} + E^{(1)}_{[b_2], d_1 d_2} \Lambda_{|a_2|}^{d_1 d_2} \right), \quad (3.16) \]

As it varies into our previous equations of motion we conclude that this equation holds exactly and is a traditional equation rather than an equivalence relation. In carrying out the steps above we have added terms which have higher level derivatives to the equations of motion, namely \( \mathcal{E} \), but these we have not been shown but can be found in reference [7].

We observe that equation (3.7) is invariant under the gauge transformation [7]

\[ \delta A_{a_1...a_{10}, b_1 b_2} = \partial_{(b_1} \Lambda_{|a_1...a_{10}|, b_2)} - \frac{10}{11} \partial_{[a_1} \Lambda_{a_2...a_{10]}(b_1, b_2)} + \partial_{[a_1} \Lambda_{a_2...a_{10}], b_1 b_2} \quad (3.17) \]

Gauge transformations for the \( E_{11} \) theory were proposed in reference [17] and one can verify that the gauge transformations of equation (3.17) and equation (3.10) are precisely of this form. The modulo transformations of equations (3.9), (3.14) and (3.15) are closely related to these gauge transformations. As noted previously the same would be true for the gravity-dual gravity duality relation if one worked in a formalism which fixed completely the local Lorentz symmetry to have the graviton be a symmetric field. It is tempting to assume that the modulo transformations are just the local transformations given in reference [17]. One could then check if this was consistent with the \( I_c(E_{11}) \) transformations rather than take the path so far which has been to find the modulo transformations from \( I_c(E_{11}) \) variations and then see if they are gauge transformations.

The above procedures explain how to find the equivalence relations as one calculates the equations of motion of the fields level by level and it would be good to find a systematic procedure. Clearly the transformations that define the equivalence relations are very closely related to the gauge transformations proposed in reference [17] and it would seem likely that the modulo transformations are indeed just these gauge transformations. As suggested just above assuming this could lead to a systematic method for determining the hierarchy of equations of motion discussed in this paper. We note, however, as the modulo transformations occur in the \( E_{11} \) variations they are gauge transformations of a rather specific kind.

To summarise, the \( E_{11} \otimes_s l_1 \) non-linear realisation leads to equations of motion for the level four fields that are given above. The \( A_{a_1...a_9, b_1 b_2 b_3} \) field obeys a duality equation with the three form that leads to the equation of motion for the former field, equation (3.8), that is the one required to account for the degrees of freedom given by the irreducible representation of the Poincare group, that is a third rank tensor of the little group SO(9). The field \( A_{a_1...a_{10}, b_1 b_2} \) obeys equations that lead to no degrees of freedom but it is physical in that it gives the eleven dimensional origin [7,8] of Romans theory [18].
4. Discussion of hep-th:1703.01305.

The authors of reference [9] have followed the $E_{11}$ programme [1,2] and constructed the dynamics up to level three at the linearised level. While they have used the symmetries of the $E_{11} \otimes s \mathfrak{l}_1$ non-linear realisation they have also considered the gauge symmetries given in reference [17] as a tool to construct the dynamics. They encountered problems with describing the gravitational degrees of freedom related to the trace of the spin connection. These difficulties were previously observed in reference [13], however, as was explained in reference [10], and implemented in references [5,6,7], the equations of motion that follow from the non-linear realisation form a hierarchy with an increasing number of derivatives up to and including equations that have a number of derivatives that is equal to the number of blocks of indices on the fields concerned. The equations with less than this number are equivalence relations, that is, they only hold modulo certain transformations while the equations that have the same number of derivatives hold in a traditional sense and are the equations of motion that we are familiar with. Acting on the equivalence relations with derivatives in a precise way one finds the equations with higher number of derivatives and the entire system of equations is invariant under the $E_{11}$ symmetries. In this paper we have reviewed the derivation [5,6,7] of the equations of motion up to and including level four fields. The equations of motion do indeed correctly describe the degrees of freedom of eleven dimensional supergravity including those of the gravity.

The authors of reference [9] have added fields to the $E_{11} \otimes s \mathfrak{l}_1$ non-linear realisation in an $E_{11}$ covariant manner using a tensor hierarchy algebra [21] to account for the fields that they thought were missing since they had not realised that the equations with low numbers of derivatives were equivalence relations. In effect they wish to convert the equivalence relations to be of the usual type of equations of motion. For the gravity-dual gravity equations they wish to convert the first order gravity-dual gravity duality relation into a standard equation by adding a nine form field which is related to the trace of the spin connection. This alternative way of proceeding does not affect the equation of motion for a given field with the most number of derivatives and in particular it does not affect the equations of motion for the three form, six form, graviton or the dual graviton as well as the analogous level four equations which were given in reference [7]. Indeed it also does not affect the full non-linear equations for the fields, below level four, which were derived in references [5,6] and which were found, when suitably truncated, to be the equations of motion of eleven dimensional supergravity. The corresponding non-linear equation for the dual graviton will be given in a future publication [15].

While there is no need to add new fields to the $E_{11} \otimes s \mathfrak{l}_1$ non-linear realisation it could have been advantageous to have a more familiar system of equations and this may help to find a general method. On the other hand one is, by construction, adding field which do not appear in the final dynamical equations and so are unphysical. It will be interesting to see how easy is it to add the new fields of reference [9] in comparison with just working with equivalence relations and in particular to see the extension of the results of reference [9] to find the equations of motion at level four and at the non-linear level.

An alternative somewhat artificial procedure would be to add fields $\chi^A$ in the $l_1$ representation and then consider the fields $\tilde{A}_\alpha = A_\alpha + (D_\alpha)A^B \partial_B \chi^A$ where $A_\alpha$ are the $E_{11}$ fields. The new fields are obviously invariant under the gauge transformations $\delta A_\alpha = $
(D_d)_{A}^{B} \partial_B \Lambda^A \ [17] provided one also takes $\delta \chi^A = -\Lambda^A$. Since the modulo transformations of the equivalence relations are closely related to gauge transformations one then has to ensure that the fields $\chi^A$ drop out of the equations of motion with the highest number of derivatives. As a result although one now has standard equations the situation is very little different from using the equivalence relations and then removing the modulo transformations by taking derivatives.

The authors of reference [9] have concentrated on finding the first order duality equations rather than the hierarchy of equations considered in references [5,6,7]. In particular, as they did not take account of the equivalence nature of the first order equations, they had more freedom to derive second order equations and as a result concluded that the $A_{a_1...a_9,b_1b_2b_3}$ field obeys a single trace equation rather than the correct equation (3.8). Their approach did not include deriving equations of motion which are the second order in derivatives and higher, where required, using $I_c(E_{11})$ variations as was done in reference [7] and in reference [5,6] where the full non-linear results for the three form and graviton were given.

The authors of reference [9] noted that the three form equation of motion $\mathcal{E}_{a_1,a_2a_3} = 0$ transforms as

$$
\delta \mathcal{E}_{a_1,a_2a_3} \propto \partial^b \partial^{d_1} \partial_{d_1} \Lambda_{d_2a_1a_2a_3}
$$

(4.1)

under the linearised gauge transformations of reference [17]. As a result the authors of reference [9] proposed that the theory should satisfy a section condition. A section condition appeared in Siegel theory [19], more recently called doubled field theory. The one proposed for the $E_{11}$ theory in reference [9] was also discussed as a possible section condition in reference [20] using BPS arguments.

Gauge symmetries are usually essential for constructing invariant action using field strengths. For example Maxwell’s theory is unique at low energy if one demands that it is invariant under special relativity but also gauge invariant. However, the situation with $E_{11}$ is rather different; the equations of motion, at least at low levels, are determined essentially uniquely by the symmetries of the nonlinear realisation; indeed the result essentially follows from the $E_{11}$ Dynkin diagram. One finds that the equations of motion are automatically invariant under the local transformations needed for the physical consistency of the theory, such as diffeomorphisms and gauge symmetries whose parameters depend on the usual spacetime As such in the $E_{11}$ theory one does not apparently need to require any gauge transformations to find the equations of motion as they are already determined and have the required gauge transformations that depend on the usual spacetime. As such it would not seem to be required to impose the additional gauge transformation of reference [17] whose gauge parameters depend on the higher level coordinates. While the higher level coordinates are crucial for the invariance using the symmetries of the non-linear realisation the physical reason for their presence has yet to be clarified. The strategy of the $E_{11}$ programme to date has been to proceed in a conservative manner by only requiring that the equations of motion be invariant under the $E_{11}$ symmetries. We note that the modulo transformations used in the equivalence relations that occur in the $E_{11}$ variations are very closely related to gauge transformations but they are very specific field dependent gauge transformations that could well obey special conditions. We note that having to impose a section condition on the gauge transformations is not the same as imposing a section
condition on the fields. The authors of reference [9] have calculated to a higher level in the derivatives than previous. It would be interesting to see if the section condition arise when carrying out the $I_c(E_{11})$ variations to find the equations of motion.

The non-invariance of equation (4.1) has been know to the author of this paper for many years. Generalised field strengths in certain lower dimensions were constructed in reference [22] using the gauge transformations of reference [17]. One found that the analogous calculation to that of equation (4.1) did work, that is, the field strengths were invariant at the level considered. This raises the prospect that there may be an alternative way of resolving the dilemma raised by equation (4.1).

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