Expanded evasion of the black hole no-hair theorem in dilatonic Einstein-Gauss-Bonnet theory

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We study a hairy black hole solution in the dilatonic Einstein-Gauss-Bonnet theory of gravitation, in which the Gauss-Bonnet term is non-minimally coupled to the dilaton field. Hairy black holes with spherical symmetry seem to be easily constructed with a positive Gauss-Bonnet coefficient $\alpha$ within the coupling function, $f(\phi) = \alpha e^{\gamma \phi}$, in an asymptotically flat spacetime, i.e., no-hair theorem seems to be easily evaded in this theory. Therefore, it is natural to ask whether this construction can be expanded into the case with the negative coefficient $\alpha$. In this paper, we present numerically the dilaton black hole solutions with a negative $\alpha$ and analyze the properties of GB term through the aspects of the black hole mass. We construct the new integral constraint allowing the existence of the hairy solutions with the negative $\alpha$. Through this procedure, we expand the evasion of the no-hair theorem for hairy black hole solutions.

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I. INTRODUCTION

The first astrophysical black hole is Cygnus A, which was recognized as a black hole later [1]. The black hole is the real thing, the most fascinating object and worth exploring more deeply in the universe. It was extensively investigated both observationally and theoretically. At the same time, various theories of gravitation inspired by string theory or astrophysics were also developed. Based on these backgrounds, a variety of black hole solutions such as a dilaton black hole [2–4] and Gauss-Bonnet (GB) black hole [5, 6] have been studied. Furthermore, recent observations of gravitational wave coming from the mergers of compact binary sources [7, 8] have opened the new horizons in astrophysics as well as cosmology, in which it could be very interesting to test which theory of gravitation describes our universe and the existence of the hairy black hole [9, 10].

The existence of the hairy black hole solution with GB term has been constructed and extensively studied over the past two decades [11–14], in which the black hole has an exponentially decaying dilaton hair. Due to the motivation coming from the string theory in general [15–18], the GB coefficient $\alpha$ is related to the Regge slope $\alpha'(= 16\sigma)$ [12, 19] and it is always treated to be a positive constant on those works. From the extensive studies, we noticed that there is the lower bound for a black hole mass, and that mass of a black hole increases when the dilaton coupling $\gamma$ increases [20, 21]. We thought that the GB term seems to provide a repulsive property, which makes the formation of the dilaton black hole harder and the lower bound increases as a result. Then, what would happen when we change the sign of GB coefficient? In this perspective, it is interesting to consider as a kind of the modified theory of gravitation, even though the motivation from the string theory would not valid. For this reason, we investigate more deeply on the dilaton black hole with the negative GB coefficient.

The no-hair theorem for black hole solutions was conjectured [22] to summarize the progress in black hole physics [23–28] and developed [29, 30] in Einstein-Maxwell theory, in which the solutions are associated with the Gauss law. In [29], the author used an integral constraint obtained from the equation of motion for the scalar field. Later, it has been further developed into the novel no-hair theorem through the analysis of energy-momentum tensor, especially $T_{r'}^r$ component [30]. If a black hole has the dilaton hair in the dilatonic Einstein-Gauss-Bonnet (dEGB) theory, the no-hair theorem should be avoided. Recently, it has shown that the no-hair theorems are evaded for the black hole solutions with a dilaton hair in dEGB theory [31, 32], by presenting both the old no-hair theorem is easily evaded and the novel no-hair theorem is not applicable for dEGB theory. However, the GB coupling functions were positive definite in their analysis. In this paper, we present numerically the dilaton black hole solutions with a negative $\alpha$ and analyze the properties of GB term through the aspects of the lower bound for a black hole mass in more detail. The purpose of this paper is to provide the expanded evasion of the no-hair theorem for hairy black hole solutions by constructing the new integral constraint to allow the existence of the dilaton black hole solution with arbitrary GB coefficients.

The paper is organized as follows: In section II, we review and calculate the numerical set-up. We analyze the components of the energy momentum tensor and construct the new integral constraint. In section III, we
present a dilaton black hole solution with the negative GB coefficient and analyze the black hole properties with respect to the dilaton coupling with each sign of GB coefficient. In section IV, we summarize our results and discuss about the role of GB term with the difference between the both cases.

II. DEGB BLACK HOLE

Let us consider the action with the GB term:

\[ S = \int d^3x \sqrt{-g} \left[ \frac{R}{2} - \frac{1}{2} \nabla_{\mu} \phi \nabla_{\mu} \phi + f(\phi)R_{GB}^2 \right] + S_b, \tag{1} \]

where \( g = \det g_{\mu\nu} \), the coupling function with the GB term is given by \( f(\phi) = \alpha e^{\gamma \phi} \), and \( \phi \) is a dilaton field. The scalar curvature of the spacetime is denoted by \( R \) and the GB curvature term is given by \( R_{GB}^2 = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \). In this work, the boundary term \( S_b \), \([33-36]\) is not important and so be abbreviated. The Einstein constant \( \kappa = 8\pi G \) is set to be unity for simplicity. The dilaton field equation is

\[ \frac{1}{\sqrt{-g}} \partial_{\mu}(\sqrt{-g} \partial^{\mu} \phi) + f(\phi)R_{GB}^2 = 0, \tag{2} \]

where the dot notation denotes the derivative with respect to \( \phi \) and Einstein’s equation is

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = T_{\mu\nu} = \partial_{\mu} \phi \partial^{\nu} \phi + \frac{1}{2} g_{\mu\nu} \partial_{\rho} \phi \partial^{\rho} \phi + T_{GB}^{\mu\nu}, \tag{3} \]

where \( T_{GB}^{\mu\nu} \) is the energy momentum tensor contributed from the GB term \([37]\) as follows:

\[ T_{GB}^{\mu\nu} = 4(\nabla_{\mu} \nabla_{\nu} f(\phi)) R - 4g_{\mu\nu}(\nabla^2 f(\phi)) R - 8(\nabla_{\mu} \nabla_{\nu} f(\phi)) R_{\mu\rho}^{\nu} + 8(\nabla^2 f(\phi)) R_{\mu\nu} + 8g_{\mu\nu}(\nabla_{\rho} \nabla_{\sigma} f(\phi)) R^{\rho\sigma}, \tag{4} \]

The equation only have the derivative terms of \( f(\phi) \) because of the minimally coupled terms in four-dimensions are cancelled identically \([38]\) and the contribution of GB term to the equations of motion is coming from the non-minimally coupled terms only.

Let us consider the spherically symmetric static metric in an asymptotically flat spacetime as follows:

\[ ds^2 = -e^{X(r)} dt^2 + e^{Y(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{5} \]

where the metric functions \( X \) and \( Y \) depend only on \( r \). Then, the dilaton field and Einstein’s equations turn out to be \([11]\)

\[ 0 = \phi'' + \phi' \left( \frac{X' + \frac{Y'}{2}}{2} + \frac{2}{r} \right) - \frac{4 \dot{f}}{r^2} \left( X'Y' + \left( 1 - e^{-Y} \right) \left( X'' + \frac{X'}{2} (X' - Y') \right) \right), \tag{6a} \]

\[ 0 = \frac{r}{2} \phi'^2 + \frac{1 - e^{-Y}}{r} - Y' \left( 1 + \frac{4 \dot{f}}{r} \left( 1 - 3e^{-Y} \right) \right) + \frac{8 \ddot{f}}{r} \left( \phi'' + \frac{\dot{f}}{f} \phi'^2 \right) \left( 1 - e^{-Y} \right), \tag{6b} \]

\[ 0 = \frac{r}{2} \phi'^2 + \frac{1 - e^{-Y}}{r} - X' \left( 1 + \frac{4 \dot{f}}{r} \left( 1 - 3e^{-Y} \right) \right), \tag{6c} \]

\[ 0 = X'' + \left( \frac{X'}{2} + \frac{1}{2} \right) (X' - Y') + \phi'^2 - \frac{8 \dot{f} e^{-Y}}{r} \left( \phi' X'' + \left( \phi'' + \frac{\dot{f}}{f} \phi'^2 \right) X' + \phi' X' - \frac{1}{2} (X' - 3Y') \right), \tag{6d} \]

where the prime notation denotes the derivatives with respect to \( r \). The equation \((6c)\) can be solved in terms of \( Y \) as follows:

\[ e^{Y(r)} = \frac{1}{4} \left( -r^2 \phi'^2 + 2rX' + 8\dot{f} X'\phi' + 2 \right) \]

\[ \pm \sqrt{(-r^2 \phi'^2 + 2rX' + 8\dot{f} X'\phi' + 2)^2 - 192 \dot{f} X' \phi'}. \tag{7} \]

We should take positive sign from the above equation to validate the near horizon limit. In terms of the above equation, \( Y \) and \( Y' \) can be eliminated from the equations of motion. Thus, we use the equations \((6a)\) and \((6d)\) mainly for numerical calculation and the remaining one for constraint. For later use, it is better to calculate the GB term \( R_{GB}^2 \), which is given by

\[ R_{GB}^2 = \frac{2e^{-Y}}{r^2} \left[ (1 - 3e^{-Y})X'Y' - (1 - e^{-Y})(X'^2 + 2X'') \right]. \tag{8} \]

To perform the numerical computation, we impose the boundary conditions at the black hole horizon \( r_h \) and asymptotically flat region \( r \gg 1 \). At the black hole horizon, the metric components should be zero such as \( g_{\theta\theta}(r_h) = e^{X(r_h)} = 0 \) and \( g_{rr}(r_h) = e^{-Y}(r_h) = 0 \). Then, the metric components and the dilaton field can be expanded in the near horizon limit by using the length pa-
parameter from the horizon, \( \delta r = r - r_h \) as follows:
\[
e^{-Y(r)} = 0 + x_1 \delta r + x_2 \delta r^2 + \mathcal{O}(\delta r^3), \tag{9a}
\]
\[
e^{-Y(r)} = 0 + y_1 \delta r + y_2 \delta r^2 + \mathcal{O}(\delta r^3), \tag{9b}
\]
\[
\phi(r) = \phi_h + \phi'_h \delta r + \phi''_h \delta r^2 + \mathcal{O}(\delta r^3). \tag{9c}
\]

The above equations provide the boundary conditions for \( X \) and \( \phi \), but the value \( \phi_h \) is not determined yet. To do so, we expand the equation (7) in the near horizon limit:
\[
e^{-Y(r) = (r + 4f \phi')X'}
+ \frac{2r - r^2 \phi'^2}{2(r + 4f \phi')} + \mathcal{O}\left(\frac{1}{X^2}\right). \tag{10}
\]

Now, differentiate the equation with respect to \( r \) and substitute the result with the equation (10) into the equations (6a) and (6d) as we discussed earlier. It eliminates \( Y' \) and we only need to solve \( X \) and \( \phi \) of the equations (6a) and (6b) as we discussed earlier. It eliminates \( Y' \) and we only need to solve \( X \) and \( \phi \) but the result is not simple [11, 20, 21]. Finally, expansion of the result in the near horizon limit give the results
\[
X''(r) = -\frac{r^4 + 8r^3 f \phi' + 16r^2 f^2 \phi'^2 - 48f^2}{r^4 + 4r^3 f \phi' - 96f^2}X'^2 + \mathcal{O}(X'), \tag{11}
\]
\[
\phi''(r) = \frac{-(r + 4f \phi')(r^3 \phi' + 4r^2 f \phi'^2 + 12f)}{r^4 + 4r^3 f \phi' - 96f^2}X' + \mathcal{O}(1). \tag{11}
\]

One might notice that \( \phi'' \) will diverge at the horizon because of \( X' \) diverges. Thus, the numerator should be zero, e.g. \( r^3 \phi' + 4r^2 f \phi'^2 + 12f = 0 \) not the other factor to be consistent with \( e^Y(r) \), which indicates the value of \( \phi'_h \) that is
\[
\phi'_h = -\frac{r^2_h \pm \sqrt{r^4_h - 192f^2_h}}{8f_h}. \tag{12}
\]

We take the negative sign to get the appropriate solution. It is easy to see that the positive sign will not give a solution what we want [11]. To obtain the real value, there exists a restriction for \( \phi'_h \) from the square root term of \( \phi'_h \), however it is highly model- dependent. Let us assume that \( \phi_h \) is an arbitrary constant but it satisfies \( r^4 \geq 192f^2_h \). Substitution of \( \phi'_h \) into the equations (11) at the horizon reduces the equations as follows:
\[
\phi''(r) \approx 0, \quad \text{and} \quad X''(r) \approx -X'^2. \tag{13}
\]

Then, the derivative of metric function \( X' \) is obtained
\[
X'(r) = \frac{1}{\delta r} + \mathcal{O}(1), \tag{14}
\]
which recovers the originally assumed near horizon limit, the equation (9a). As a result, the GB term at the horizon is also obtained by
\[
R^2_{GB} \approx \frac{4e^{-2Y}}{r^2} X'^2. \tag{15}
\]

We also can expand the metric components and dilaton field in the asymptotically flat region in terms of the Arnowitt-Deser-Misner (ADM) mass \( M \) [39, 40] and dilaton charge \( D \) as follows:
\[
e^{-Y(r)} = 1 - \frac{2M}{r} + \mathcal{O}(r^{-2}), \tag{16a}
\]
\[
e^{-Y(r)} = 1 + \frac{2M}{r} + \mathcal{O}(r^{-2}), \tag{16b}
\]
\[
\phi(r) = \phi_h + \frac{D}{r} + \mathcal{O}(r^{-2}). \tag{16c}
\]

The GB term in the asymptotically flat region is then
\[
R^2_{GB} = \frac{48M^2}{r^6} + \mathcal{O}(r^{-7}). \tag{17}
\]

In order to focus on the numerical analysis, we will not show the relation between the parameters in the near horizon limit and asymptotically flat region in more detail, see the references [31, 32]. The ADM mass is represented as follows:
\[
M = M(r_h) + M_{\text{hair}}, \tag{18}
\]
where the first term is the mass inside the horizon, \( M(r_h) = r_h^2/2 \) and the second term is the mass of the dilaton hair. Once the metric is obtained numerically by shooting method from the horizon, it is possible to obtain the mass and charge of black hole by matching the behavior of metric at the asymptotically flat region.

Since the dilaton black hole mass has the contribution coming from the existence of a scalar hair, it seems to evade the no-hair theorem. For this reason, we analyze whether or not there is a contradiction in the equations of motion with the energy-momentum tensor as in [31]. This is an important procedure to both compare and evade the novel no-hair theorem in [30].

The \((tt)\) and \((rr)\) components of the energy-momentum tensor are given by
\[
T_t^t = -\frac{e^{-Y}}{2r^2}\left(\frac{r^2 \phi'^2}{2} - 8f \phi' Y'(1 - 3e^{-Y})
+ 16(f \phi'' + f \phi') (1 - e^{-Y})\right), \tag{19a}
\]
\[
T_r^r = \frac{e^{-Y}}{2r^2}\left(\frac{r^2 \phi'^2}{2} - 8f \phi' X'(1 - 3e^{-Y})\right), \tag{19b}
\]
\[
(T_r^r)' = \frac{e^{-Y}}{2r^2}\left(2r^2 \phi' \phi'' - r^2 \phi'^2 Y' - 8f \phi' X''(1 - 3e^{-Y})
- 8(\phi'' + f \phi') X'(1 - 3e^{-Y})
+ 8f \phi' X' Y'(1 - 6e^{-Y}) + \frac{16}{r} f \phi' X'(1 - 3e^{-Y})\right). \tag{19c}
\]

In the near horizon limit, \( T_t^t, T_r^r \) and \((T_r^r)' \) are reduced
\[ T_t^t = \frac{4r^3 \dot{f} \delta'}{r^2(r^4 + 4r^3 f \delta' - 96f^2)} + \mathcal{O}(\delta r), \quad (20a) \]
\[ T_r^r = -\frac{4 \dot{f} \delta'}{r^2(r + 4f \delta')} + \mathcal{O}(\delta r), \quad (20b) \]
\[ (T_r^r)' = 0 \times X' + \mathcal{O}(1) + \mathcal{O}(\delta r). \quad (20c) \]

Since \( \dot{f} \delta' \) is negative definite from the equation (12) at the horizon, \( T_r^r \) is positive definite in the near horizon limit. However for \( (T_r^r)' \), the first order which depend on \( X' \) is identically zero, and so we should consider the next order. But, it is very hard to find and complex to express, we calculate the sign by putting whole horizon values into the second order and obtain the negative value. The sign is same as shown in [31], even though we were not able to reproduce the results what they obtained. Similar to the case in the near horizon limit, those are given by in the asymptotically flat region

\[-T_t^t = T_r^r = \frac{1}{2} \phi'' + \mathcal{O}(r^{-5}), \quad \text{and} \]
\[(T_r^r)' = \phi \phi'' + \mathcal{O}(r^{-6}) = -\frac{2}{r} \phi'^2 + \mathcal{O}(r^{-6}), \quad (21)\]

where we used the asymptotic relation \( \phi'' = -(2/r) \phi' + \mathcal{O}(r^{-4}) \) from the equation (6a). As a result, the tendency of \( T_r^r \) and \( (T_r^r)' \) are summarized as in table I. We also plot the numerical results in section III A which correspond with our description. Thus, there is no contradiction in the equations of motion with the energy-momentum tensor and we argue that the dilaton black hole evades the novel no-hair theorem even for the negative \( \alpha \).

| Near horizon region | Asymptotically flat region |
|---------------------|--------------------------|
| \( T_t^t \)        | > 0                      | < 0                      |
| \( T_r^r \)        | > 0                      | > 0                      |
| \( (T_r^r)' \)     | < 0                      | < 0                      |

**TABLE I.** Behavior summary of \( T_r^r \) and \( (T_r^r)' \)

To make sure our result, we also checked the old no-hair theorem as shown in [31, 32], in which they developed the integral constraint equation with the positive definite coupling function \( f(\phi) \) to show the evasion of the no-hair theorem. We construct the new integral constraint allowing the existence of the hairy solutions with arbitrary coupling functions. Starting with the equation (2), it is possible to obtain the integral constraint,

\[
\int e^{\gamma f(\phi)} \left( \nabla^2 \phi + \dot{f}(\phi)R_{\text{GB}}^2 \right) = -\int \dot{f}(\phi) \left( \phi'^2 - f(\phi)R_{\text{GB}}^2 \right) = 0, \quad (22)\]

where they used the integration by part only for the first term. The boundary term vanishes at the horizon and infinity due to the exponential factor of the metric and the derivative of dilaton field, respectively. [31]. Simply \( \phi'^2 \) is positive definite. The GB term is positive definite both on the horizon and in the asymptotically flat region. Thus one can guess that the GB term is positive definite for all region and monotonically decreasing with respect to the radial length. Indeed, it is really happen and we will show the result in the section III A. In order to avoid the no-hair theorem, the only condition for \( f(\phi) \) is positive definite. In our study, we consider the negative \( \alpha \) which makes the coupling function negative definite, in which the no-hair theorem seems to valid in this analysis. Therefore, we should find another way of treating the integral constraint and expand that, which covers all sign definite cases of \( f(\phi) \). As a result, we construct the integral constraint equation as follows:

\[
\int e^{\gamma f(\phi)} \left( \nabla^2 \phi + \dot{f}(\phi)R_{\text{GB}}^2 \right) = -\int \dot{f}(\phi) \left( \phi'^2 - f(\phi)R_{\text{GB}}^2 \right) = 0, \quad (23)\]

where we also used the integration by part only for the first term. In the above equation, \( \phi'^2 \) and \( R_{\text{GB}}^2 \) are positive definite and the coupling function \( f(\phi) \) can be arbitrary. Thus, it is shown that the dilaton black hole solutions with arbitrary coupling functions evade the old no-hair theorem.

In order to find the dilaton black hole solution, we used the Dormand-Prince method [41] which is the one of Runge-Kutta method with specific parameters. Since the metric function diverges at the horizon, we starts our calculation at \( \delta r = \epsilon = 10^{-8} \) and also set the infinity as \( r_{\text{max}} = 10^5 \). Let us define the subscript \( h \) and \( \infty \) by means of the value at the initial point \( r_h + \epsilon \) and the final point \( r_{\text{max}} \), respectively. Then, the initial conditions of the metric functions and field with the given coupling function \( f(\phi) = \alpha e^{\gamma \phi} \) are

\[
X_h = \log(x_1\epsilon), \quad X_h' = \frac{1}{\epsilon}, \quad \phi_h \leq \frac{1}{2\gamma} \log \left( \frac{r_h^\gamma}{192h^2\gamma^2} \right),
\]
\[\text{and} \quad \dot{\phi}_h = -\frac{r_h^\gamma - \sqrt{r_h^\gamma - 192h^2\dot{f}_h^2}}{8r_h\dot{f}_h}. \quad (24)\]

One can see that the equations of motion are invariant under the shift of a dilaton field \( \phi \rightarrow \phi + \phi_0 \) for a constant \( \phi_0 \) with the rescaling of \( r \rightarrow r e^{\gamma \phi_0/2} \) [11]. Thus, we fix \( r_h = 1 \), vary \( \phi_h \) to get a different black hole solution
as a free parameter and rescale the result. Until now, the parameter $x_1$ is arbitrary. On the other side, the boundary conditions at the infinity are given by

$$X_\infty = 0, \quad X'_\infty = 0, \quad \phi = 0, \quad \text{and} \quad \phi'_\infty = 0. \quad (25)$$

To make $X_\infty = 0$, $x_1$ should be chosen wisely because the equations of motion depend only on $X'$ and so there can exist the non-zero remaining constant $X_\infty$.

In our calculation, we obtain the value by setting $x_1 = 1$ and do the same procedure again with $\log x_1 = -X_\infty$ which is the way of fixing the parameter $x_1$. We also want to make the dilaton field vanish at the asymptotically flat region. For the dilaton field, there also exist the value of $\phi_\infty$. The value can be absorbed by using the symmetry between $r$ and $\phi$ such as the rescaling of $r_h$ as $r_h \rightarrow r_h e^{-\gamma \phi_\infty/2}$. Thus, the numerical calculation starts with the same $r_h$, but it can vary with the dilaton field value $\phi_h$ and the solutions form an one parameter family. Finally, we obtain $X$ and $\phi$ from the equations of motion and it is possible to obtain $Y$ by using the equation (7).

The ADM mass $M$ and dilaton charge $D$ are obtained by fitting the equation at the asymptotically flat region.

### III. RESULTS

In this section, we present a hairy black hole solution in dEGB theory. We set the dilaton coupling function $f(\phi) = \alpha e^{\gamma \phi}$ where the GB coefficient $\alpha$ has the negative value, not positive one as usual. Since the dEGB theory has the rescaling invariance under the $r \rightarrow r/\sqrt{\alpha}$, we choose $\alpha = -1$ for all our data. Furthermore, one can see that the theory is invariant under the changes of $\gamma \rightarrow -\gamma$ and $\phi \rightarrow -\phi$. Thus, we always choose the positive dilaton coupling $\gamma$, that is enough to obtain the solutions. Even in this unusual negative coefficient set up, we obtained the dilaton black hole solution and the different tendency of a minimum black hole mass depending on the $\gamma$.

#### A. Dilaton black hole

This is an example of a hairy black hole solution with the negative $\alpha$. In order to get and present the dilaton black hole in this section, we set $\gamma = 1$ and $\phi_h = \log(r_h^4/192\alpha^2 \gamma^2)/2\gamma$ which is the maximum value of the given range in the initial condition, equation (24).

Figure 1 represents the metric functions and the profile of the dilaton field for a black hole solution with respect to $r$. In figure 1 (a), the black and red lines indicate the metric components $-g_{tt}(r)$ and $g_{rr}(r)$ which are converges or diverges at the horizon, respectively. Both metric components converge to unity at infinity. In figure 1 (b), the dilaton field $\phi(r)$ always has negative value, which means that the derivative of dilaton field has positive value at all. Both quantities also become zero at infinity. The blue dashed line in each figure indicates the value of horizon radius, which is not unity. Originally, we set the horizon radius $r_h = 1$ but it is modified by the factor $e^{-\gamma \phi_\infty/2}$, as we explained in the previous section.

Figure 2 shows the positivity of the $(rr)$ component of the energy-momentum tensor, $T_{rr}(r)$, and the negative value of its derivative, $-(T_{rr})'(r)$ with respect to $r$. Those quantities have positive values at the horizon and diminish when $r$ goes to infinity, but the signs never change. Therefore, there is no contradiction in the equations of motion with the energy-momentum tensor which show that the novel no-hair theorem is not applicable and finally is not valid for the dEGB theory, as we claimed before.

Figure 3 (a) illustrates the GB term $R^2_{\text{GB}}(r)$ with respect to $r$. The GB term is positive definite on all regions of $r$ and shows the monotonically decreasing behavior when $r$ increases. The result well corresponds with our expectation about the old no-hair theorem and so the no-hair theorem is again evaded. In figure 3 (b), the black and red lines depict the $(tt)$ and $(rr)$ components of the energy-momentum tensor, $-T_{tt}(r)$ and $T_{rr}(r)$, respectively. The energy density $-T_{tt}(r)$ has the negative value only for the near horizon region and the positive
value for all the other region of $r$. One of key assumptions in the novel no-hair theorem is related to the energy condition. The energy density is non-negative everywhere for any timelike observer. Thus, the existence of the negative value in some region shows the violation of the key assumptions to be satisfied in the novel no-hair theorem.

**B. Spectrum of dilaton black holes**

Now, we have the dilaton black hole solutions with the negative GB coefficient $\alpha$. In order to investigate the properties of the dilaton black holes with negative $\alpha$, we obtain the dilaton black hole solutions with same boundary conditions and compare them, but just change the sign of $\alpha$ with respect to $\gamma$.

Figure 4 represents the lower bound for the dilaton black hole mass with respect to $\alpha$ for several selected values of $\gamma$. The $\phi_h$ is also chosen by the maximum value. It clearly shows that there exists the $\sqrt{|\alpha|}$ dependency of the black hole mass. The $\alpha$ dependency can be absorbed by the radial coordinate transformation, $r \rightarrow r/\sqrt{|\alpha|}$ as we discussed earlier. Therefore, we focused on the $\gamma$ dependency of the black hole mass for each sign of $\alpha$. The lower bound is increased when $\alpha$ has the positive value, but the lower bound is increased up to some specific $\gamma$ and decreased when $\alpha$ has the negative value, as $\gamma$ is increased. Therefore, we expect that there exist some maximum $\gamma$ value, which restrict the dilaton black hole for the negative $\alpha$ and this is the most different behavior.
between the dilaton black holes with different signs of $\alpha$.

Figure 5, 6 and 7 illustrate the mass of dilaton black holes with respect to $\gamma$ for each $\alpha$ signs. The black line denotes the black hole mass with the maximum value of $\phi_h$. The red dashed line indicates the maximum $\gamma$ which can have the dilaton black hole when $\alpha$ is negative. The blue dashed line represents the dilaton black hole which have the minimum mass under the variations of $\phi_h$ when $\alpha$ is positive. It is already well known result that the maximum $\phi_h$ is not always giving the minimum mass of the dilaton black hole [13, 14, 20, 21]. The black and blue lines tell about the dilaton black hole with maximum $\phi_h$ or minimum mass. By changing $\phi_h$, we obtain the dilaton black holes with more higher mass than the one represented by black or blue line. Therefore, there also exist the dilaton black holes over the black or blue line but not under the lines.

Figure 5 (a) and (b) show the dilaton black hole mass increases and decreases when $\alpha$ is negative but it keep increases even for the black holes with minimum mass when $\alpha$ is positive by increasing $\gamma$, as we have shown in the figure 4. When $\alpha$ is negative, we cannot obtain the dilaton black hole solution with the $\gamma$ value which exceeds the red dashed line. In order to get the contribution coming from the GB term in more detail, we need to investigate the large $\gamma$ region, because the GB term is dependent on the $\gamma$ exponentially. In this region, it seems that the GB term decreases its own repulsive property and assists to make the dilaton black hole relatively easy. As a result, we argue that the minimum mass of the dilaton black hole decreases in large $\gamma$ region. However, when $\alpha$ is positive, the GB term looks giving the dispersive behavior and it disturb to make the dilaton black hole. Thus, we also argue that the minimum mass of the dilaton black hole increases depending on $\gamma$.

Figure 6 (a) and (b) show the hairy mass of the black hole and figure 7 (a) and (b) show the ratio between the hairy mass and the total mass with respect to $\gamma$. The hairy mass increases in general, except the near maximum $\gamma$ region of the case with the negative $\alpha$ and the minimum mass black hole of the case with the positive
The hairy mass ratio vs. $\gamma$ with $\alpha = -1$. 

![Graph](image)

The hairy mass ratio vs. $\gamma$ with $\alpha = 1$.

![Graph](image)

FIG. 7. Several mass figures with respect to $\gamma$.

$\alpha$. However, the ratio is always increased or decreased when $\alpha$ is negative or positive, respectively. Especially for the case with the negative $\alpha$, the ratio is increased even higher than $> 0.5$ near the maximum $\gamma$, the red line and the behavior are really strange. We are wondering whether the value will keep growing until it reaches to unity or not, which means that the black hole horizon would disappear and have the mass only as the dilaton hair. This is not happen when $\alpha$ is positive. The maximum and restricted value of $\gamma$ seems to be motivated from the reason. However, we cannot do the exact numerical calculation on the limit of maximum $\gamma$ due to the difficulties of error control, thus our argument is remained as an open question but it is reasonable.

IV. CONCLUSION

We have investigated the hairy black hole solutions in dEGB theory, especially with the negative GB coefficient $\alpha$. In Refs. [31, 32], the authors had shown that the no-hair theorems are easily evaded for the hairy black hole solutions in dEGB theory. They considered the black hole solutions with only the positive $\alpha$ and many scalar couplings, and constructed the integral constraint in the theory. In this paper, we tried to expand the description about the dilaton black hole into negative GB coefficients by changing the sign of $\alpha$. We constructed the new integral constraint equation allowing the existence of the hairy black hole solution with the arbitrary signature of $\alpha$. Through this procedure, we have expanded the evasion of the no-hair theorem for hairy black hole solutions.

As a consequence of our analysis, we have numerically obtained the dilaton black hole solutions with the negative $\alpha$. The dilaton black holes have more hair than the case for the positive $\alpha$ in general. We restricted our calculation into the dilaton black holes which have the maximum values of dilaton field at the horizon $\phi_h$ or minimum masses. It is enough to investigate the properties of the dilaton black hole. The minimum mass of a dilaton black hole with the positive $\alpha$ is obtained from the maximum $\phi_h$ in small dilaton coupling $\gamma$ region. When $\gamma$ is increased, the cases of dilaton black holes with the minimum mass and the maximum $\phi_h$ are divided. The mass of a dilaton black hole increases in both cases. We think that the GB term seems to provide the repulsive effect and it disturb to form the dilaton black hole harder. However, those two cases are not divided with the negative $\alpha$ and the minimum mass decreases for large $\gamma$. Since the minimum mass decreases, it seems that the GB term decreases its own repulsive property and the black hole forms relatively easy.

Furthermore, there exists a maximum value $\gamma$ which limits the existence of the dilaton black hole solution. Until the maximum $\gamma$, the minimum mass decreases but the hairy mass increases and decreases again. Interestingly, the hairy mass ratio to the total mass is always increased. The results give us an expectation that the dilaton black hole solution with the maximum $\gamma$ would have no horizon and the mass of the black hole is composed by the dilaton field only. Even though it is yet expectation due to the difficulties of the numerical calculation, it is worthwhile to investigate in more detail the properties and implications of such behaviors for large $\gamma$ and we postpone further analysis for the future work.

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