Higher-Derivative Supersymmetry
and the Witten Index\(^1\)

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Abstract

We propose higher-derivative generalization of the supersymmetric quantum mechanics. It is formally based on the standard superalgebra but supercharges involve differential operators of the order \(n\). As a result, their anticommutator entails polynomial of a Hamiltonian. The Witten index does not characterize spontaneous SUSY breaking in such models. The construction naturally arises after truncation of the order \(n\) parasupersymmetric quantum mechanics which in turn is built by glueing of \(n\) ordinary supersymmetric systems.

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1. Introduction

Supersymmetric quantum mechanics (SQM) is used for the description of hidden symmetries of various atomic and nuclear physical systems [1]. Besides, it provides a theoretical laboratory for investigation of algebraic and dynamical problems in SUSY field theory. The simplified setting of SUSY helps to analyze the difficult problem of dynamical SUSY breaking at full length and to examine the validity of the Witten index criterion [2].

Let us remind basic principles of the standard one-dimensional SQM. We consider only the simplest case when there are two conserved supercharges $Q^\pm$ obeying the algebra

$$\{Q^+, Q^-\} = H, \quad [Q^\pm, H] = 0, \quad (Q^\pm)^2 = 0, \quad Q^+ = (Q^-)^\dagger,$$  \hspace{1cm} (1)

where $H$ is the Hamiltonian of a system. For the sake of simplicity we assume that $H$ is self-adjoint operator with purely discrete spectrum and that $Q^\pm$ are well defined on all eigenstates of $H$. The discussion of the realizations when $H$ and $Q^\pm$ have different domains of definition can be found in Refs.[3,4].

The direct consequence of (1) is that all eigenvalues of $H$ are non-negative, $E_n \geq 0$. Furthermore, the positive energy levels prove to be double degenerate belonging to “boson”, or “fermion” sector specified by grading operator $\tau = (-1)^{\hat{n}_f}$, where $\hat{n}_f$ stands for a fermion number. Existence of zero-energy states depends on a particular topology of superpotential $W(x)$ entering the matrix realization of superalgebra (1),

$$Q^+ = \begin{pmatrix} 0 & 0 \\ q^- & 0 \end{pmatrix}, \quad Q^- = \begin{pmatrix} 0 & q^+ \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} h_B & 0 \\ 0 & h_F \end{pmatrix},$$

$$q^\pm = \mp \partial + W(x), \quad H = -\partial^2 + W^2 - W'\sigma_3.$$  \hspace{1cm} (2)

Note that $Q^\pm$ and $H$ are the first and the second order differential operators respectively. The grading in this representation is performed by the Pauli matrix $\sigma_3$. Zero-energy states can arise either in the boson, $q^-\psi_B = 0$, or in the fermion sector, $q^+\psi_F = 0$, for appropriate superpotentials $W(x)$. Let us denote by $N_B$ ($N_F$) the number of (normalizable) zero modes of $Q^+ (Q^-)$. The explicit form of possible solutions,

$$\psi_{B,F} = C \exp \left( \mp \int_a^x W(y)dy \right),$$  \hspace{1cm} (3)

shows that depending on the asymptotic behavior of $W(x)$ there might be three types of the vacuum. First two configurations $N_B = 0, N_F = 1$ and $N_B = 1, N_F = 0$ describe exact SUSY and the non-degenerate ground state. The third possibility corresponds to $N_B = N_F = 0$, when SUSY is spontaneously broken, i.e. ground state is not annihilated by $Q^\pm$. In the latter case vacuum energy is positive and degenerate.

The difference between unbroken and spontaneously broken SUSY can be indicated by means of the Witten index,

$$\Delta_W = \text{Tr} (-1)^{\hat{n}_f} = N_B - N_F = \text{dim ker } q^- - \text{dim ker } q^+ = 0; \pm 1.$$  \hspace{1cm} (4)
Evidently, in the case of ordinary SQM $\Delta_W = 0$ unambiguously characterizes the models with spontaneously broken SUSY. Some complications with the definition of $\Delta_W$ appear when continuum spectrum extends down to zero eigenvalue [5], we already neglected such cases.

In this paper we elaborate non-standard realizations of SUSY in one-dimensional quantum mechanics which employ higher-order differential operators for supercharges. The Witten index criterion is not valid for such systems, i.e. this index does not select out models with spontaneously broken SUSY. In Sect.2 we construct a SUSY model where supercharges and a “quasihamiltonian” are polynomials in derivative of the second and fourth degrees respectively. In Sect.3 we connect a quasihamiltonian with the canonical Schrödinger operator and thereby introduce the higher-derivative SUSY quantum mechanics (HSQM) characterized by a polynomial relations between supercharges and a Hamiltonian. The connection between a topology of superpotentials and the ground-state space is investigated. Possible generalizations of HSQM and its extension on parasupersymmetric (PSQM) systems [6] are outlined in Sect.4.

2. Supercharges with second derivatives

Let us construct a representation of the formal algebra (1) using operators of higher-order in derivative. In this section we restrict ourselves to the order two,

$$ q^\pm = \partial^2 \pm \{ f(x), \partial \} + \varphi(x), \quad (5) $$

where $f, \varphi$ are nonsingular functions.

The quasihamiltonian,

$$ K = \{ Q^+, Q^- \} = \begin{pmatrix} q^+q^- & 0 \\ 0 & q^-q^+ \end{pmatrix} \equiv \begin{pmatrix} k_B & 0 \\ 0 & k_F \end{pmatrix}, \quad (6) $$

is now an operator of fourth order in derivative. It commutes with supercharges and possesses non-negative eigenvalues as before. Zero-mode states of $K$ can be revealed as solutions of equations

$$ q^\pm \psi_{B,F} = 0, \quad \psi_{B,F} = u_{B,F}(x) \exp \left\{ \pm \int_a^x f(y)dy \right\} \quad (7) $$

where $u_{B,F}$ obey the equation,

$$ -u'' + (f^2 - \varphi)u = 0. \quad (8) $$

In general, the number of vacuum states may be $N_{B,F} = 0; 1; 2$ and respectively the Witten index may take values $\Delta_W = 0; \pm 1; \pm 2$ depending on the asymptotic behavior of $f(x), \varphi(x)$. However, now one can see that zero value of the index, $\Delta_W = 0$, does not necessarily mean the absence of zero-mode states and thereby the spontaneous breaking of SUSY. This value describes two possible configurations $N_B = N_F = 0$ and $N_B = N_F = 1$.

We thus have built up a model where the formal SUSY algebra holds and nevertheless the Witten’s criterion does not take place. However, physical meaning of the quasihamiltonian $K$ is not clear, it is not a differential operator of the Schrödinger type. Eventually one can be interested in exploration of higher-derivative SUSY in the ordinary quantum mechanics which we are going to discuss further on.
3. Higher-derivative SUSY with Schrödinger Hamiltonians

Let us perform the factorization of supercharge ingredients (5) into two operators linear in derivatives,
\[ q^+ = (q^-)^\dagger = q_1^+ q_2^+ = (-\partial + W_1)(-\partial + W_2), \] \[ W_1 = W(x) - f(x), \quad W_2 = -W(x) - f(x), \quad W^2 - W' = f^2 - \varphi. \]
The components of the quasihamiltonian are factorized respectively,
\[ K = \begin{pmatrix} q_1^+ q_2^+ q_1^- 0 \\ 0 q_2^- q_1^- q_2^+ \end{pmatrix} \] \[ (10) \]
In these denotations the general solution of Eq.(7) reads,
\[ \psi_{B,F} = A_{B,F} \exp\left(-\int_a^x (W \pm f)dy\right) \left[ 1 + D_{B,F} \int_a^x \exp\left(2\int_a^y Wdz\right)dy\right], \] \[ (11) \]
where \(A_{B,F}, D_{B,F}\) are constants. The factorization makes it evident the correspondence between zero-modes of the quasihamiltonian \(K\) and operators \(q_{1,2}^\pm\).

Let us search for a particular sort of superpotentials \(W, f\) that allows to express the quasihamiltonian as a function of a Schrödinger-type hamiltonian and thereby to apply our higher-derivative SUSY to ordinary quantum systems. We find the connection between \(W\) and \(f\) when imposing the following condition,
\[ q_1^- q_1^+ = q_2^- q_2^+ \equiv h, \] \[ (12) \]
which leads to
\[ 2Wf + f' = 0. \] \[ (13) \]
Under such a condition the quasihamiltonian (10) can be related to the Schrödinger-type operator \(H\),
\[ K = \begin{pmatrix} (q_1^+ q_1^-)^2 & 0 \\ 0 & (q_2^- q_2^+)^2 \end{pmatrix} \equiv \begin{pmatrix} (h_1)^2 & 0 \\ 0 & (h_2)^2 \end{pmatrix} = H^2. \] \[ (14) \]
We remark that in fact the Hamiltonian \(H\) is prepared from two ordinary SQM Hamiltonians,
\[ H^{(1)} = \begin{pmatrix} h_1 & 0 \\ 0 & h \end{pmatrix}, \quad H^{(2)} = \begin{pmatrix} h & 0 \\ 0 & h_2 \end{pmatrix} \] \[ (15) \]
in denotations of Eqs.(12), (14). At first one should glue these two systems into a \(3 \times 3\) (second order) PSQM Hamiltonian applying Eq.(12) (see Ref.[6,7,3]) and then delete the intermediate component \(h\) together with a corresponding Hilbert subspace. Thereby we have built a quantum system which possesses the conserved supercharges \(Q^\pm\). However the SUSY now is characterized by the non-linear algebra,
\[ [H, Q^\pm] = 0, \quad (Q^\pm)^2 = 0, \quad \{Q^+, Q^-\} = H^2. \] \[ (16) \]
This higher-derivative SQM (HSQM) obviously yields the double degeneracy of positive part of energy spectrum.

Normalizability of zero-modes (7) determines the ground-state structure of $H$. Substituting Eq.(13) into Eq.(11) we have,

$$\psi_{B,F}(x) = A_{B,F}|f|^{1/2}\exp \left( \mp \int_{a}^{x} f(y)dy \right) \left[ 1 + D_{B,F} \int_{a}^{x} \frac{dy}{f(y)} \right].$$

(17)

Insofar as ground-state functions are nodeless, $f(x)$ should be chosen of a definite sign on the entire axis. The number and specification of zero-modes $N_B, N_F$ is dictated by asymptotics of $f(x)$ at the infinity. Respectively one can find that normalizable solutions (17) arise for $D_{B,F} = 0$ only.

The typical options to obtain zero-modes are described by different configurations of $f(x)$.

1. Let

$$f(x) \rightarrow +\infty \quad f(x) \rightarrow 0$$

so that

$$0 > \int_{-\infty}^{x} f(y)dy > -\infty, \quad \int_{a}^{x} f(y)dy \rightarrow +\infty, \quad x \rightarrow +\infty.$$

Evidently in this case $N_B = 1, N_F = 0$.

2. In a similar way the case $N_B = 0, N_F = 1$ can be derived for configurations with

$$f(x) \rightarrow 0 \quad f(x) \rightarrow +\infty$$

$$x \rightarrow +\infty, \quad x \rightarrow -\infty.$$

3. The most interesting situation arises for configurations with

$$f(x) \rightarrow 0$$

$$x \rightarrow \pm \infty, \quad \int_{-\infty}^{+\infty} |f(y)|dy < \infty.$$

One has now both a bosonic and a fermionic zero modes $N_B = N_F = 1$. The normalizability is provided by the factor $|f|^{1/2}$ in (17). The explicit connection between ground-state wave functions reads

$$\psi_B(x) = \psi_F(x) \left[ 1 + const \cdot \int_{a}^{x} (\psi_F(y))^2dy \right]^{-1}.$$  

(18)

From this analysis we conclude that the Witten’s proposition can not be applied to HSQM. Namely, the last configuration has $\Delta_W = 0$ though it does not reveal spontaneous breaking of SUSY due to the existence of zero modes (moreover, one has double degeneracy of the zero-energy eigenvalue).

Let us consider now an important generalization of the higher-derivative SUSY that is realized by modification of Eq.(12),

$$q_{1}^{\pm} q_{1}^{\pm} = q_{2}^{\pm} q_{2}^{\pm} + c, \quad 2Wf + f' + c/2 = 0,$$

(19)
where \( c \) is a constant. For definiteness we choose \( c > 0 \). The HSQM Hamiltonian is constructed again from two ordinary SQM hamiltonians by means of glueing and truncation (see Eq.(15) and Ref.[6]), but now

\[
H = \begin{pmatrix}
  h_1 & 0 \\
  0 & h_2 + c
\end{pmatrix},
\]

(20)

where \( h_{1,2} \) were defined in Eq.(14).

The related modification of HSQM algebra reads as follows,

\[
K = \{ Q^+, Q^- \} = H(H - c)
\]

(21)

It is convenient to parameterize zero modes (11) in terms of \( f(x) \) and \( c \),

\[
\psi_{B,F} = A_{B,F} \exp\left\{ \int_a^x (\mp f + \frac{2f' + c}{4f})dy \right\} + D_{B,F} \exp\left\{ \int_a^x (\mp f + \frac{2f' - c}{4f})dy \right\}
\]

(22)

Normalizability of these states is determined by the asymptotic behavior of \( f(x) \) near its zeros and at infinities. In any case, the operator \( K \) may have no more than two zero modes in the bosonic or fermionic sector that corresponds to the range of values of the Witten index \( \Delta_W = 0; \pm 1; \pm 2 \).

The most interesting situation, \( \Delta_W = 0, N_B = N_F = 1 \) arises for

\[
f(x) \to \mp 0, \quad x \to \pm \infty; \quad f(x) \big|_{x \sim x_0} = -\frac{1}{2}c(x - x_0) + o(x - x_0).
\]

(23)

The slope of \( f(x) \) in the vicinity of its zero is adjusted to compensate a singularity at \( x = x_0 \) in the exponent of (22) and to provide two nodeless normalizable solutions. One of solutions \( \psi_B \) then has the ground-state energy \( E_{0,B} = 0 \) for the Hamiltonian \( H \) (20) and another one, \( \psi_F \) has a (positive) ground-state energy \( E_{0,F} = c \). The limiting case \( c \to 0 \) is reproduced when simultaneously \( x_0 \to \infty \).

The breaking of the Witten criterion has the same character as for \( c = 0 \). However, there is a difference in the behavior of regularized Witten index which starts to depend on the temperature,

\[
\Delta_W^{reg} \equiv \text{Tr}\left[ (-1)^{\hat{\eta}} \exp(-\beta H) \right] = 1 - \exp(-\beta c)
\]

(24)

for a particular configuration \( N_B = N_F = 1 \). The limit \( \beta \to \infty \) does not reproduce correct index value. Such a dependence on regularizing parameter even in the purely discrete spectrum models is a typical one for \( c \neq 0 \). We thus see that for the higher-derivative SUSY the Witten index in any form does not characterize the spontaneous breaking of SUSY.

4. Generalizations and extensions

1. The natural generalization of HSQM constructed in the previous section is generated by glueing of several SUSY systems with \( c_i \neq 0 \) and by truncation of all intermediate components of
the resulting (parasupersymmetric) matrix hamiltonian [6,8]. Explicitly, instead of one relation (19) one has a chain of constraints upon the superpotentials \( W_i(x) \),

\[
W'_i(x) + W'_{i+1}(x) + W^2_i(x) - W^2_{i+1}(x) = c_i, \quad i = 1, 2, \ldots, n - 1.
\]  

(25)

The associated order \( n \) PSQM Hamiltonian is a diagonal \((n + 1) \times (n + 1)\)-matrix

\[
H^{PSQM}_{ij} = h_i \delta_{ij}, \quad h_k = q^+_k q^-_k + \lambda_k, \quad \lambda_k = \sum_{l=1}^{k-1} c_l, \quad k = 1, \ldots, n,
\]

(26)

\[
h_{n+1} = q^-_n q^+_n + \lambda_n, \quad q^+_k = \mp \partial + W_i(x),
\]

where \( \lambda_1 \equiv 0 \). Shrinking \( H^{PSQM} \) to a \( 2 \times 2 \) size by deleting of all internal columns and rows one gets an order \( n \) HSQM model

\[
\{Q^+, Q^-\} = P_n(H) = \prod_{k=1}^{n} (H - \lambda_k), \quad [H, Q^\pm] = (Q^\pm)^2 = 0,
\]

(27)

\[
Q^- = \begin{pmatrix}
0 & q_1^+ q_2^+ \cdots q_n^+ \\
0 & 0
\end{pmatrix}, \quad Q^+ = (Q^-)^\dagger,
\]

\[
H = \begin{pmatrix}
h_1 & 0 \\
0 & h_{n+1}
\end{pmatrix} = -\partial^2 + U(x) + B(x)\sigma_3,
\]

(28)

where potential \( U(x) \) and “magnetic field” \( B(x) \) are linear combinations of the potentials of \( h_1 \) and \( h_{n+1} \). Note that now supercharges and a quasihamiltonian \( P_n(H) \) are the differential operators of order \( n \) and \( 2n \) respectively. Obviously any conceivable polynomial can be produced with an appropriate set of shifting parameters \( c_i \) at the intermediate steps of truncation. The Witten criterion is invalid and zero mode states form a subspace of generally non-degenerate energy levels with dimension \( \leq n \).

2. When truncation is incomplete and not all of the intermediate components of Hamiltonian (26) are deleted, one derives the PSQM quasihamiltonian and charges of higher order in derivative. The typical situation is created by glueing ordinary SUSY Hamiltonian with the HSQM one (20) (set \( c = 0 \) for simplicity). In this way one gets

\[
H^{PSQM} = \begin{pmatrix}
h_1 & 0 & 0 \\
0 & h_2 & 0 \\
0 & 0 & h_3
\end{pmatrix}, \quad Q = \begin{pmatrix}
0 & q_1^+ & 0 \\
q_1^- & 0 & \gamma q_2^+ q_3^+ \\
0 & \gamma q_3^- q_2^- & 0
\end{pmatrix},
\]

(29)

\[
h_1 = q_1^+ q_1^- , \quad h_2 = q_2^+ q_1^- = q_2^+ q_2^- , \quad h_3 = q_3^- q_3^+ ,
\]

where \( Q \) is a hermitian charge and \( \gamma \) is a dimensional parameter. Under the auxiliary condition, \( q_2^- q_2^+ = q_3^- q_3^+ \), they satisfy the following non-linear PSQM algebra,

\[
Q^3 = Q(H + \gamma^2 H^2), \quad [H, Q] = 0.
\]

(30)

There are more conserved charges and trilinear algebraic relations but we shall not discuss them here.
A natural PSQM generalization of the Witten index has the form \[\Delta_n = \text{Tr} e^{2\pi i \hat{n}_{pf}/(n+1)},\] \[(31)\]

where \(\hat{n}_{pf}\) is a parafermion number operator, \((\hat{n}_{pf})_{ij} = (i - 1)\delta_{ij}, \ i,j = 1, \ldots, n + 1\). At \(n = 1\) one has \(\Delta_1 \equiv \Delta_W\), and for \(n > 1\) the index \(\Delta_n\) is a sum of roots of unity. It is intuitively clear that only zero value of \(\Delta_n\) describes spontaneously broken SUSY for all intermediate SQM Hamiltonians composing (26), i.e. only at \(\Delta_n = 0\) all operators \(q_i^\pm\) have not zero modes. In this sense \(\Delta_n\) provides proper description of the ground-state structure. However, after any (even partial) truncation of the PSQM there is no good index criterion and one has to refer to \(\Delta_n\) and full system (26) in order to characterize subspace of zero modes.

3. The rich class of potentials discussed in Refs.[10-13] is easily described within the HSQM context. Suppose that in (28) \(h_{n+1}\) is related to \(h_1\) by simple transformation. From the one hand this would mean that their spectra essentially coincide. On the other hand, supercharges \(Q^\pm\) map eigenstates of \(h_1\) and \(h_{n+1}\) onto each other. As a result, if at least one bosonic or fermionic state is known exactly, then one may expect that the whole spectrum is generated by SUSY. This is not the case if every state is mapped precisely onto itself which happens when \(B(x) = 0\) (or, \(h_1 = h_{n+1}\)). At \(B = 0\) SUSY is always realized trivially but only for \(n = 1; 2\) one has trivial (constant) potential \(U(x)\). Actually, in odd \(n\) cases, \(n = 2p + 1\), one gets finite-gap potentials with \(p\) being the number of finite permitted bands in the spectrum [10]. More complicated potentials, related to the Painlevé transcendents, arise if one puts \(B(x) = \text{const}\) (or, \(h_{n+1} = h_1 + \text{const}\)). In the latter case supercharges formally generate the whole (equidistant) spectrum (see [10]).

A peculiar self-similar potential defined by some mixed finite-difference-differential equation was described in Ref.[11]. It has purely exponential discrete spectrum which is generated by the \(q\)-deformed Heisenberg-Weyl algebra. A deformation of SQM, inspired by this model, was suggested in Ref.[12]. By repetition of basic steps one can deform HSQM and get the following algebra

\[Q^-Q^+ + q^{-2n}Q^+Q^- = \prod_{k=1}^{n}(\mathcal{H} - q^{\sigma_3 - 1} \lambda_k), \quad (Q^\pm)^2 = 0, \quad \mathcal{H}Q^\pm = q^{\mp 2}Q^\pm \mathcal{H},\] \[(32)\]

where \(q\) is a scaling parameter,

\[Q^+ = T_q^{-1}Q^+, \quad Q^- = Q^{-}T_q, \quad T_qf(x) = \sqrt{f(qx)}, \quad T_q^\dagger = T_q^{-1},\]

\[\mathcal{H} = \begin{pmatrix} h_1 & 0 \\ 0 & q^{-2}T_q^{-1}h_{n+1}T_q \end{pmatrix} = -\partial^2 + U(x,q) + B(x,q)\sigma_3.\] \[(33)\]

Within this “\(q\)-deformed” HSQM context, the general set of \(q\)-transcendental potentials of Ref.[13] corresponds to the very simple constraint \(B(x,q) = \text{const}\) in (33). At \(B \neq 0\) the discrete spectra of such systems formally comprise \(n\) independent geometric series.

4. To conclude, the higher-derivative generalization of SQM is natural in a sense that corresponding symmetry algebra is formally the same. One simply has to substitute instead of the Hamiltonian some polynomial combination of it. Since the essence of SUSY is preserved, all high-frequency modes are degenerate. However the vacuum structure has dramatically changed, e.g.
cancellation of zero energies of the bosonic and fermionic sectors in general does not take place. Higher-dimensional HSQM models can not be constructed in a straightforward manner because of the matrix character of standard SUSY transformations (intertwining relations for subhamiltonians) [14]. We hope to discuss separately arising difficulties in detail. Probably it will not be easy also to construct the field theory analog of HSQM. If the latter nevertheless exists, then corresponding high-energy behavior should be similar to that of ordinary SUSY models, the real difference occuring only in the low-energy region.

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