Thermodynamics of strongly disordered spin ladders

Eddy Yusuf and Kun Yang

National High Magnetic Field Laboratory and Department of Physics,
Florida State University, Tallahassee, Florida 32306

We study antiferromagnetic two-leg spin-1/2 ladders with strong bond randomness, using the real space renormalization group method. We find the low-temperature spin susceptibility of the system follows non-universal power laws, and the ground state spin-spin correlation is short-ranged. Our results suggest that there is no phase transition when the bond randomness increases from zero; for strong enough randomness the system is in a Griffith region with divergent spin susceptibility and short-range spin-spin correlation.

I. INTRODUCTION

One-dimensional (1D) spin systems are of interest to physicists since the early days of quantum mechanics. Considerable effort has been devoted to the theoretical study of antiferromagnetic (AF) spin chains, where some of the very few exact solutions of interacting Hamiltonians in physics were obtained and remarkably rich low-energy physics were uncovered using various non-perturbative methods. Interest in these systems were also enhanced by the recent experimental realization of such model systems due to technological advances. More recently, considerable attention has focused on another class of 1D spin systems, namely AF spin ladders. These systems are made of two or more coupled spin chains. The physics of such systems are closely related to, but even richer than the spin chain systems. Further motivation for study of these systems comes from the similarity in structure between these systems and undoped cuprates, and the discovery of superconductivity in them once charge carriers are introduced via doping.

The ubiquitous randomness is known to have particularly strong effects in low-dimensional systems. Recently, there has been rather extensive theoretical studies of effects of disorder in spin chains. Most of these studies are based on the celebrated real space renormalization group (RSRG) method introduced by Ma, Dasgupta and Hu (MDH) in the study of AF spin-1/2 chain with bond randomness, and Bhatt and Lee in the study of magnetic properties of doped semiconductors. This method was elaborated and extended in great detail by Fisher and applied (often with nontrivial extensions) by a number of other groups on various disordered spin chain models. A variety of disorder-dominated phases have been found, whose low-energy physics is qualitatively different from their disorder free counterparts. While the quantitative accuracy of the RSRG relies on the presence of strong randomness, it has been shown that even if the strength of randomness is weak, it tends to grow as the RSRG proceeds to lower and lower energy scales, thus giving qualitatively correct (and sometimes asymptotically exact) low-energy behavior. Indeed, many predictions of RSRG have been confirmed by complementary analytical and numerical studies using other methods.

Comparatively speaking, relatively few studies have focused on effects of randomness on spin ladders. Effects of doping by non-magnetic impurities (or site dilution) have been studied using quantum Monte Carlo and mapping to Dirac fermions with random mass. The stability of the pure ladders against various types of weak randomness has been studied by Orignac and Giamarchi using bosonization. In the present paper we study a two-leg AF spin-1/2 ladder with strong bond randomness, using the RSRG. We believe our work is complementary to the previous studies, as the effects of bond randomness and site dilution are quite different, and the RSRG is particularly suitable for studies of systems with strong randomness.

While the present work was being completed, a preprint appeared on the cond-mat archive, in which the authors used the RSRG as well as the density matrix renormalization group to study various disordered ladder models. While our work certainly overlaps with theirs, there exist two major differences. (i) Ref. focuses mainly on the distribution of the gap separating the ground and first excited states in finite clusters, while we study mainly thermodynamic properties and spin-spin correlation functions. (ii) Ref. has studied finite-size ladders with length up to 512. In our work we have studied ladders with length up to 20,000, nearly a factor of 40 bigger. The larger size is crucial to us for obtaining low-temperature, large distance behavior of the thermodynamic quantities and spin-spin correlation functions respectively. We will compare our results with those of Ref. and previous studies whenever appropriate.

Our main results are summarized as follows. We find the thermodynamics of the two-leg spin ladder remains non-universal, and the spin-spin correlation remains short-ranged, even in the strong-randomness limit. This is very different from the random AF spin-1/2 or spin-1 chain, where weak (for spin-1/2) or sufficiently strong (for spin-1) randomness drive the system into the Random Singlet (RS) phase with universal thermodynamics and power-law spin-spin correlation. For sufficiently strong randomness, the spin susceptibility of the ladders exhibits power-law divergence as the temperature $T \to 0$: $\chi(T) \sim T^{-\beta}$, where $\beta$ varies continuously with randomness strength. Combined with short-range spin-spin correlation, this is characteristic of quantum
Griffith behavior. Such behavior appears to persist even when the interchain coupling of the ladder is made very weak, suggesting weak interchain coupling immediately destabilizes the RS phase that controls the low-energy, long-distance physics of the decoupled chains.

The remainder of the paper is arranged as follows. In section II we introduce the model Hamiltonian we study, briefly review the RSRG method and its application to random spin chains, and discuss the necessary extensions we need to make in order to apply it to the ladder system. In section III we present our numerical results. In section IV we discuss the implications of our results, make contact with related theoretical and experimental work, and state our conclusions.

II. MODEL AND THE RENORMALIZATION-GROUP SCHEME

Consider an antiferromagnetic nearest-neighbor Heisenberg spin-1/2 ladder. The Hamiltonian for a two-leg ladder is given by:

$$H = \sum_{i=1}^{N} \sum_{j=1,2} J_{i,j} S_{i,j} \cdot S_{i+1,j} + \sum_{i=1}^{N} K_{i} S_{i,1} \cdot S_{i,2},$$  \hspace{1cm} (1)$$

where $N$ is the number of spins on a single chain, $S_{i,j}$ is a spin-1/2 operator, and the positive coupling constants $J_{i,j}$ (couplings along the chains, or legs of the ladder) and $K_{i}$ (couplings between the chains, or along the rungs of the ladder) are distributed randomly according to some probability distributions $P_{\parallel}(J_{i,j})$ and $P_{\perp}(K_{i}).$

In this work we use the real space renormalization group (RSRG) method to study the Hamiltonian (1). We begin by briefly reviewing its application to the random AF spin-1/2 chains to illustrate the basic ideas behind it. In this approach one identifies the strongest coupling of the system, say, $J_{3}$ that couples spins 2 and 3, and the two neighboring spins that are coupled to this strongly coupled pair. The Hamiltonian of this four-spin cluster is given by

$$H = H_{0} + H_{I},$$  \hspace{1cm} (2)$$

where

$$H_{0} = J_{2} S_{2} \cdot S_{3},$$  
$$H_{I} = J_{1} S_{1} \cdot S_{2} + J_{3} S_{3} \cdot S_{4}. $$  \hspace{1cm} (3)$$

In the presence of strong randomness, $J_{2}$, being the strongest coupling in the system, is likely to be much stronger than other two couplings $J_{1}$ and $J_{3}$. In this case to a very good approximation spins 2 and 3 form a singlet pair in the low-energy states of the entire system and become inert. The weak perturbation $H_{I}$ to this pair induces virtual transitions to the excited (or triplet) states of $H_{0}$; the main effect of such virtual transitions is to induce an effective coupling between spins 1 and 4 of the form:

$$H_{eff} = \tilde{J} S_{1} \cdot S_{4};$$  \hspace{1cm} (4)$$

to second order in $H_{I}$, $\tilde{J}$ is given by:

$$\tilde{J} = \frac{J_{1} J_{3}}{2 J_{2}} > 0.$$  \hspace{1cm} (5)$$

In essence, the RSRG procedure replaces the four spin cluster by spins 1 and 4, which are the active degrees of freedom at low energies, with a new effective AF bond $\tilde{J}$ that couples them. $\tilde{J}$ is typically much weaker than the original bonds $\{J_{1}, J_{2}, J_{3}\}$ so the bond distribution broadens and the energy scale is lowered. The decimation does not change the lattice structure of the chain, as after spins 2 and 3 are decimated, spins 1 and 4 become nearest neighbors, thus the new effective Hamiltonian still describes an AF spin-1/2 chain with nearest-neighbor interactions, and this procedure can be repeated. The renormalization scheme is depicted in Fig. I.

When we apply this technique to the ladder systems, some new features not encountered before appear, and corresponding modifications to the spin decimation procedure described above need to be made. Firstly, the structure of the ladder gets distorted as soon as RSRG is applied, in contrast to the chain case. This situation requires us to keep track of the structure of the system as we decimate the spins and bonds. Secondly, when one decimates strongly coupled spin pairs, both AF and ferromagnetic (F) effective bonds are generated; these F bonds can lead to effective spins higher than spin-1/2 at low energies. The initial renormalization step for the ladder is illustrated in Fig. 1b, from which we can see how the ladder structure is distorted and ferromagnetic interactions are generated. These generated F bonds are much weaker than the original bonds that get decimated. However as we move on, more and more spins get decimated and the energy scale is lowered so at some point the generated ferromagnetic bonds can become important because it might be the strongest bond in the system. This spin pair, instead of forming a singlet, form a triplet or an effective spin-1 object at low-energy. Clearly larger spins can also be generated by RG as energy scale gets lowered. This situation is illustrated in Fig. 1c. In short, we need to keep track of both the lattice structure, and size of the spins, together with the coupling constants (which can now be either AF or F) in our RG procedure.

Now we turn the discussion to some technical details involved in the application of RG scheme to ladder. First consider two spins connected by a strong AF bond. These two spins are also coupled to the other 4 spins as shown in Fig. 1b. We will make a slight change of notation for our discussion here just for simplicity. We label the spins participating in the process by number 1 - 6. The Hamiltonian for the 6-spin cluster is given by

$$H = H_{0} + H_{I},$$  \hspace{1cm} (6)$$
where
\[
H_0 = J_{23} S_2 \cdot S_3,
\]
\[
H_I = J_{12} S_1 \cdot S_2 + J_{34} S_3 \cdot S_4 + J_{25} S_2 \cdot S_5 + J_{36} S_3 \cdot S_6,
\]
(7)
where \( J_{ij} \) is the bond between \( S_i \) and \( S_j \). This 6-spin problem is quite complicated to solve but it can be simplified using the fact that we can treat \( H_I \) as a perturbation to \( H_0 \), especially when the randomness is strong. It is easy to see that to second-order, \( H_I \) only generates pairwise interaction among the spins. It is thus only necessary to include a pair of spins coupled to the two spins connected by the strongest bond, when we consider the effective interaction between them. This fact simplifies the calculation as we can now reduce a 6-spin cluster problem to six 4-spin clusters which can be classified into three different types of 4-spin clusters as represented by: spin 1, 2, 3, and 4 (1234), (1235), and (2356). The Hamiltonians for these clusters are given by:
\[
H_1 = J_{23} S_2 \cdot S_3 + J_{12} S_1 \cdot S_2 + J_{34} S_3 \cdot S_4,
\]
\[
H_2 = J_{23} S_2 \cdot S_3 + J_{12} S_1 \cdot S_2 + J_{25} S_2 \cdot S_5,
\]
\[
H_3 = J_{23} S_2 \cdot S_3 + J_{25} S_2 \cdot S_5 + J_{36} S_3 \cdot S_6,
\]
(8)
\( H_1 \) has the same form as Eqs. (3) and (4) which lead to the recursion relation Eq. (5). Cluster 1235, given by the Hamiltonian \( H_2 \), is a new cluster structure not encountered in the chain case. Second order perturbation calculation shows that there is a new negative effective interaction between \( S_1 \) and \( S_5 \) given by:
\[
\tilde{J}_{15} = -\frac{J_{12} J_{25}}{2 J_{23}} < 0,
\]
(9)
i.e. we have a ferromagnetic interaction. Physically this is due to the fact that \( S_1 \) and \( S_5 \) are both coupled antiferromagnetically to \( S_2 \); this makes it favorable to have \( S_1 \) and \( S_5 \) parallel to each other, thus an effective ferromagnetic bond is generated. Cluster 2356, given by the Hamiltonian \( H_3 \), looks almost the same as cluster 1234 except that spin \( S_5 \) and \( S_6 \) are already connected by an original bond \( J_{56} \). This original bond will be renormalized when \( J_{23} \) is decimated together with \( J_{25} \) and \( J_{36} \). The renormalized bond is then given by:
\[
\tilde{J}_{56} = J_{56} + \frac{J_{25} J_{36}}{2 J_{23}}.
\]
(10)
The generated interaction between \( S_5 \) and \( S_6 \) is antiferromagnetic because they are sitting on the opposite sub-lattices.

As discussed earlier, the effective F bonds generated by RSRG can lead to formations of effective spins with size larger than 1/2. We thus need to incorporate this possibility in our scheme, and generalize the Hamiltonian in Eq. (2) and (3) by giving arbitrary sizes to the spin operators in the Hamiltonian and by having either sign for the couplings. We treat \( H_I \) as a perturbation to \( H_0 \) as before. In the space of degenerate ground states of \( H_0 \), the spins \( S_1 \) and \( S_2 \) form a state of maximum total spin \( S = S_2 + S_3 \) for ferromagnetic \(( J_2 < 0) \) or of minimum total \( S = |S_2 - S_3| \) for antiferromagnetic \(( J_2 > 0) \) while the spins \( S_1 \) and \( S_4 \) can point in any direction. The degenerate ground states span the Hilbert space \( \mathcal{H} \) which is the product space of the spin spaces \( S_1, S, \) and \( S_4 \). \( H_I \) will partially lift the degeneracy in \( \mathcal{H} \) and induce an effective Hamiltonian in \( \mathcal{H} \). The effective Hamiltonian can be calculated using the projection theorem:
\[
H_{eff} = PHP,
\]
(11)
where \( P \) is the projection operator that projects the full Hamiltonian \( H \) into the subspace where \( S \) is maximum(minimum). The detail of this calculation is available in Ref. [11]. Here we just give the final result. After the strong bond is decimated, we can write down the effective Hamiltonian \( H_{eff} \) as:
\[
H_{eff} = \tilde{J}_1 S_1 \cdot S + \tilde{J}_3 S_3 \cdot S_4 + \text{constant},
\]
(12)
where
\[
\tilde{J}_1 = \frac{S(S + 1) + S_2 (S_2 + 1) - S_3 (S_3 + 1)}{2S(S + 1)} J_1,
\]
\[
\tilde{J}_3 = \frac{S(S + 1) + S_2 (S_2 + 1) - S_3 (S_3 + 1)}{2S(S + 1)} J_3,
\]
(13)
where \( S = |S_2 \pm S_3| \) depending on the sign of \( J_2 \).

In the case where \( J_2 > 0 \) and \( S_2 = S_3 \), the ground state of the strong bond is a singlet and there is no effective spin left after decimation. Second-order perturbation expansion yields a non-zero interaction between \( S_1 \) and \( S_4 \):
\[
\tilde{J} = \frac{2}{3} S_2 (S_2 + 1) \frac{J_1 J_3}{J_2}.
\]
(14)
It can be shown that the cases discussed above exhaust all possible situations we may encounter when applying the RSRG to a spin ladder.

In implementing the RSRG procedure outlined above, one finds that each spin is coupled to more other spins as more and more spins are decimated, and the couplings can be either F or AF. There is, however, one major simplification due to the bipartite nature of the original lattice, which is also of physical importance as we discuss below. In the beginning we have a lattice structure which can be divided into two sublattices (A and B) in which spins in sublattice A get coupled only to spins in sublattice B. As we run our RG procedure this is no longer true. Not only spins from the same sublattice can get coupled together but also the sizes of the spins are no longer the same. It becomes a relevant question to ask where to put an effective spins formed by two spins with different sizes and what the types of interactions are between this effective spin with the rest of the lattice. We apply the majority rule in our RG scheme to incorporate this situation. The idea of this rule is to put the effective spin formed
by two spins with different sizes connected by AF/F coupling on the sublattice where the larger spin is. Using this rule we are able to show that two spins sitting on opposite sublattices will always have AF couplings while those sitting on the same sublattice will always have F couplings. This is clearly true in the beginning; we will show below the RG procedure combined with the majority rule preserves this structure. Physically this simply reflects the fact that the nearest neighbor AF couplings on a bipartite lattice have no frustration; they prefer the spins in the same sublattice to be parallel, and in opposite sublattices to be antiparallel.

Let us elaborate this idea in more detail to better understand the majority rule. We have seen in our discussion above that there are three different cases which exhaust all the possible combinations encountered in our RG procedure. First, we have two spins with the same size connected by AF coupling. Second, two spins with the same or different sizes connected by F coupling and third, two spins with different sizes connected by AF coupling. These three cases are shown in Fig. 1b and c. In the first case we do not have to worry about applying the majority rule because there is no effective spin formed.

The configuration is shown as cluster 1234 on Fig. 1b. We just use the recursion relation derived in Eq. (5) to determine the type of interactions between the spins which were the third nearest neighbors. Some of the possible sublattice combinations for this case is shown in Table (I). Here it is clearly shown that two spins sitting on opposite sublattices will have AF interactions and those sitting on the same sublattice have F interactions.

The configuration for the second case is shown on Fig. 1c. We have already seen from Table (I) that for two spins to have a ferromagnetic coupling, they must be sitting on the same sublattice. In this case there is no ambiguity where to put the effective spin. We can choose the effective spin to be located on the site where either $S_2$ or $S_3$ is used to be located. We can figure out the sign of the renormalized couplings in the same way as it is done in Table (I). The renormalized coupling is given by:

$$\tilde{J}_{12} = \frac{S_2}{S_2 + S_3}J_{12}. \quad (15)$$

With this recursion relation and majority rule, we can determine the sign of the renormalized couplings for all combinations possible. This is shown in Table (II). The conclusion that AF coupling is always on opposite sublattices and F coupling is always on the same sublattice remains valid.

The last case is when $S_2 \neq S_3$ and $J_{23} > 0$. The majority rule tells us to put the effective spin on the sublattice of the spin with bigger size. If $S_2 > S_3$, we put the effective spin on the sublattice in which $S_2$ is sitting and vice versa. The recursion relations for the couplings are given by the equation:

$$\tilde{J}_{12} = J_{12} \frac{S_2 + 1}{S_2 - S_3 + 1}. \quad (16)$$

The type of interaction between the effective spin and the rest of the lattice is shown in Table (III), where we take an example $S_2 > S_3$. The result is the same as the two previous cases where AF coupling is always on opposite sublattices and F coupling is always on the same sublattice. Should we change $S_2 < S_3$, the result would remain valid. Table (III) shows the configurations where $S_2 > S_3$.

We have thus shown that the application of the majority rule will preserve the type of interactions between spins sitting on opposite sublattices or the same sublattice. If the spins are sitting on the opposite sublattices, the interaction is always antiferromagnetic and if on the same sublattice, the interaction is always ferromagnetic. This conclusion can be generalized to higher dimensions as long as the original AF interactions couple only spins sitting on opposite sublattices.

### III. NUMERICAL RESULTS

We have carried out the renormalization scheme for the ladder as described in section II numerically, with length of the ladders up to 20,000. In the decimation process, we pick up the strongest bond as defined by the absolute value of the bond strength, except if we are not sure about what to do with that bond, we decimate it, and calculate the renormalized couplings to the neighboring spins. This procedure is iterated until the number of spins in the ladder is about 3% of the original number of spins. The initial distributions are taken to be in power-

| 1 2 3 4 | $J_{23}$ | $J_{12}$ | $\tilde{J}_{12}$ |
|---|---|---|---|
| A A B A | + - + - |
| A A B B | + - - + |
| B A B A | + + + - |
| B A B B | + + - - |

**TABLE I: Some possible sublattice combinations for $S_2 = S_3$ and $J_{23} > 0$.**

| 1 2 3 4 | $S_{eff}$ | $J_{23}$ | $J_{12}$ | $\tilde{J}_{12}$ |
|---|---|---|---|---|
| A B A A | - + - - |
| B B B A | - - + + |

**TABLE II: Possible sublattice combinations for $S_2 > S_3$ and $J_{23} < 0$. $S_{eff}$ gives us the sublattice where we put the effective spin.**

| 1 2 3 4 | $S_{eff}$ | $J_{23}$ | $J_{12}$ | $\tilde{J}_{12}$ |
|---|---|---|---|---|
| A B A A | + - + - |
| B A B A | + + + - |

**TABLE III: Possible sublattice combinations for $S_2 > S_3$ and $J_{23} < 0$. $S_{eff}$ gives us the sublattice where we put the effective spin.**
law form:

\[ P_{\parallel}(J_{i,j}) = (1 - \alpha)J_{i,j}^{-\alpha}, 0 < J_{i,j} < 1; \]

\[ P_{\perp}(K_i) = \frac{1 - \alpha}{\Lambda^{1 - \alpha}}K_i^{-\alpha}, 0 < K_i < \Lambda. \]  

(17)

Here \( 0 \leq \alpha < 1 \) is the measure of disorder (the bigger \( \alpha \), the stronger the randomness strength), and \( 0 < \Lambda \leq 1 \) is the anisotropy parameter; in the limit \( \Lambda \to 0 \) the two chains decouple. We use a power-law form for our initial distributions because in the case of random spin chains, fixed point distributions at low energies typically have a power law form; we can thus hope to be able to approach the low-energy fixed points faster by starting with a power law distribution.

As discussed earlier, due to the presence of F bonds generated by RSRG, effective spins with size bigger than 1/2 appear at low energies. One might think that such larger spins may proliferate, and the typical size of the spins may grow indefinitely, leading to a phase dominated by weakly coupled large spins. This was found to be the case in spin chains with random AF and F couplings studied by Westerberg et al.\(^{4}\) We find, however, this is not the case in the present problem. We address the issue of proliferation of F bonds and large spins in Fig. 2, where data for \( \alpha = 0 \) and \( \alpha = 0.6 \) (both with \( \Lambda = 1 \)) are shown. We plot the ratio of the numbers of AF bonds and F bonds as a function of bond strength cutoff \( \Omega \) in (a). At the early stages of RG the system consists of a large fraction of AF bonds and a small percentage of F bonds generated by the decimation process. As the energy scale is lowered more F bonds are generated and more AF bonds are removed so the ratio of the number goes down. In the low energy limit, we find the number of F bonds is very close to the number of AF bonds. This can be seen more clearly at the insets in Fig. 2a. Even though the numbers of AF and F bonds are almost equal, the strengths of AF and F bonds behave completely differently in this limit. AF bonds always dominate the system. In (b) we plot the ratio of the average strength of AF and F bonds. When the bond cutoff \( \Omega \) goes below 0.2, where the numbers of AF and F bonds are almost equal, the ratio of the averages grows rapidly which means the AF bonds are much stronger than the F bonds in the low energy limit. In (c) we plot the difference of the averaged logarithms of AF and F bonds; the exponential of this quantity reflects the ratio between typical AF and F bonds. Similar to (b), here we see the difference grows very fast, again showing the dominance of AF bonds over the F bonds. In (d) we plot the sample averaged ratio of the number of spins larger than 1/2 to the total number of spins. Here we plot that while larger spins do appear, their percentage remains small, and the percentage decreases with the cutoff \( \Omega \) going down in the low-energy limit. Another piece of information that is not included in the figure is that most of the larger spins are spin-1’s, with a very small percentage of spin 3/2 and spin 2. We have not found any trace of spins larger than 2 in our simulations. We find qualitatively similar behavior in all initial distributions we have looked at, indicating this is generic.

Physically, such behavior has its origin in the bipartite nature of the lattice structure of the 2-leg ladder. As we have shown earlier, the effective couplings generated by the RSRG is always AF between spins of opposite sublattices, and F between spins of the same sublattice. Since the numbers of spins in the two sublattices are the same, the number of F and AF bonds become very close in the low-energy limit. On the other hand spins in opposite sublattices tend to be closer to each other, leading to the fact that AF bonds dominate F bonds in strength. This in turn suppresses formation of large spins.

Our most important results are presented in Figs. 3-6, where we plot the temperature dependence of the spin susceptibility, and the ground state spin-spin correlation function. The susceptibility is calculated as the following. We proceed with the RSRG until the bond cutoff \( \Omega \) is equal to the temperature \( T \). We neglect contributions of spins that have already been decimated, and treat the remaining spins as free spins, thus their contribution to the susceptibility is just the Curie susceptibility. This is a good approximation as long as the bond distribution is broad. The total susceptibility is thus given by:

\[ \chi_{\text{tot}} = \frac{g\mu_B}{3k_B T} \sum_s N_s s(s + 1), \]

(18)

where \( N_s \) is the number of spins left at energy scale \( \Omega = T \) for a given spin size \( s \) and the summation runs over all possible spin sizes. In Fig. 3 we plot the susceptibility per spin for different samples as a function of temperature for different disorder strength \( \alpha \), all with isotropic coupling (\( \Lambda = 1 \)). In all cases we find the low-\( T \) susceptibilities can be fit quite well to power-law dependence on \( T \); \( \chi \sim T^{-\beta} \); the power-law exponent \( \beta \), which we obtain from a least-square fit to the low-\( T \) part of the data, is non-universal; it can describe both divergent (\( \beta > 0 \)) \( \chi \) for stronger randomness (larger \( \alpha \)), or vanishing \( \chi \) (\( \beta < 0 \)) for weaker randomness (smaller \( \alpha \)), as \( T \to 0 \). It is worth noting that for \( \Lambda = 1 \), we always have \( \beta < \alpha \), and such behavior persists for very strong disorder like \( \alpha = 0.9 \). Such behavior is very different from random AF spin-1/2 chain with any amount of randomness, or random AF spin-1 chain with sufficiently strong randomness, where the system flows to the so-called random singlet (RS) fixed point, in which the bond distribution is infinitely broad, the spin-spin correlation follows a universal power-law, and the susceptibility diverges in a universal manner.

\[ \chi \sim 1/(T \ln^2(\Omega_0/T)). \]

(19)

Instead, the fact that we find power-law exponent \( \beta \) to be always less than 1 indicates the width of the bond distribution is finite. Of course, in principle we cannot completely rule out the possibility that our system size (and correspondingly, temperature range) is not wide enough for us to approach the true low-\( T \) asymptotic behavior of \( \chi \), which for strong enough randomness may be con-
trolled by a fixed point with infinitely broad bond distribution and universal. We believe, however, this is highly unlikely for the following reasons. (i) Our power-law fit already extends to a very wide range in $T$. In particular, for $\alpha = 0.9$, a single power-law fits all the data very well that is over eleven orders of magnitudes in $T$, with no indication of crossover to other behavior at low $T$. (ii) As we will see later, the spin-spin correlation function appears to be short-ranged, indicating that the long-distance, low-temperature physics is not controlled by a single scale-invariant fixed point.

In the absence of interchain coupling, the ladder becomes two decoupled random AF spin-1/2 chains, where the long-distance, low-temperature physics is controlled by the RS fixed point and universal. To address how the system crosses over from one behavior to another we have studied how the susceptibility varies with the anisotropy parameter $\Lambda$. In Fig. 4 the susceptibility per spin for different values of $\Lambda$ is presented, for a fixed $\alpha = 0.6$. Again, we find non-universal behavior here. As we vary $\Lambda$ from 1 to 0, the power-law exponent of the susceptibility increases continuously. In the case of $\Lambda = 0$ we have decoupled chains and the susceptibility is expected to follow Eq. (19). While for a finite range of $T$ it can be fit reasonably well to a power-law with $\beta$ very close to 1, the small upward curvature of the data indicates $\beta$ would increase as one goes to lower $T$, consistent with Eq. (14). On the other hand a very weak interchain coupling (e.g., $\Lambda = 0.025$) leads to a significant change it $\beta$, and there is no longer obvious upward curvature in the data. This suggests that a weak interchain coupling immediately destabilizes the RS fixed point.

We now turn the discussion to the ground state spin-spin correlation function along the chain:

$$g(|i - j|) = (-1)^{|i - j|} \langle \langle S_{i,k} \cdot S_{j,k} \rangle \rangle$$

where $\langle \langle \rangle \rangle$ stands for both quantum and disorder average. We calculate $g(|i - j|)$ in the following way. We run the RSRG until all spins are decimated, and then simply count the number of singlet pairs formed for a given distance $|i - j|$, divide this number by the total number of pairs and multiply the result by $3/4$. In the RS phase, $g(|i - j|) \sim |i - j|^{-2}$. In Fig. 5a we study how the interchain interaction affects the correlation along the chain by varying the anisotropy parameter $\Lambda$. Fitting the data to a power-law dependence: $g(|i - j|) \sim |i - j|^{-\nu}$, we obtain $\nu = 1.97$ for $\Lambda = 0$ (decoupled chain case), which is very close to the analytical result $\nu = 2$. For nonzero $\Lambda$, the correlation decays much faster than that of the chain. Even a small amount of interchain interactions (say, $\Lambda = 0.001$) change the behavior of the correlation considerably. We can see a downward curvature in the data, which is particularly obvious for $\Lambda = 1$ and 0.5, indicating the short range (decaying faster than any power-law) behavior of the correlation. If we try to fit the ground state correlation for non-zero $\Lambda$ to a power-law, we would get considerably larger power-law exponent $\nu$, even for $\Lambda$ as small as 0.001. This strongly suggests that introduction of interchain interactions immediately destabilizes the RS phase that controls the low-energy of the decoupled chains, and leads to short-range spin-spin correlation in the ground state. We have also calculated how the correlation changes as we vary $\alpha$ for fixed $\Lambda = 1$, in Fig. 5a. Here we find while stronger randomness (larger $\alpha$) tends to enhance correlation at large distances, the correlation is still short-ranged for very strong randomness ($\alpha = 0.9$) as evidenced by the downward curvature of the data.

One general concern in numerical calculations of the kind discussed here is finite-size effects. We show in Fig. 5b that the system sizes we use in this work are large enough that the finite-size effects are negligible. The sample averages of the susceptibility per spin do not show any noticeable fluctuations as the system sizes are varied from $N = 2,000$ to $N = 20,000$. The same is true for the ground state spin-spin correlation. At large separation there are some variations due to sample to sample fluctuations. We can thus safely conclude that the finite-size effect is negligible in our study.

### IV. SUMMARY AND DISCUSSIONS

In this work we have used the RSRG method to study an AF two-leg spin-1/2 ladder, with strong bond randomness. We find that the spin susceptibility is non-universal, and the ground state spin-spin correlation is short-ranged, for any randomness and interchain coupling strength. For sufficiently strong randomness or sufficiently weak interchain coupling, the spin susceptibility exhibit power law divergence as $T \rightarrow 0$, which is characteristic of quantum Griffith behavior.

Melin et al. used the RSRG method as well as density matrix RG to study the distribution of the energy gap separating the ground and first excited states in clusters (with length up to 512) of AF two-leg spin-1/2 ladders. They find that the dynamic exponent $z$ that characterizes this distribution is non universal and depends continuously on randomness strength. Based on this they conclude that the low-energy physics of the system is controlled by a fixed point with a finite width in the bond distribution function, and the system is in a quantum Griffith phase. Our results and conclusions agree with theirs.

It is by now well established that in the absence randomness, the two-leg AF spin-1/2 ladder supports a finite excitation gap, and the spin-spin correlation is short-ranged. It is generally true that randomness tends to introduce low-energy excitations, which can lead to divergent spin susceptibilities as found here. Our results indicate however, despite the low-energy excitations introduced, the phase with short range spin-spin correlation appears to be stable against any amount of randomness. This is certainly consistent with Ref. [21] where the authors find the pure ladder to be remarkably stable against various kinds of disorder. On the other hand this
is very different from the AF spin-1 chain, where sufficiently strong bond randomness drives the system from the Haldane phase to the random singlet phase with universal thermodynamics and power-law spin-spin correlation, through a second-order phase transition.

As discussed earlier, for the present system the bond distribution has a finite width in the low-energy limit, no matter how strong the randomness is initially. This indicates that the RSRG method is not asymptotically exact when applied to the present model. However this method is quantitatively accurate as long as the randomness is strong, we thus believe the qualitative conclusions we draw from our results are robust.

Acknowledgments

This work was supported by NSF grant No. DMR-9971541, and the Research Corporation. K.Y. was also supported in part by the A. P. Sloan Foundation.

1 H. A. Bethe, Z. Phys. 71, 265 (1931).
2 I. Affleck, T. Kennedy, E. H. Lieb and H. Tasaki, Phys. Rev. Lett. 59, 799 (1988).
3 F. D. M. Haldane, Phys. Lett. 93A, 464 (1983); Phys. Rev. Lett. 50, 1153 (1983).
4 For a review, see, e.g., Extended Linear Chain Compounds, Vol. 3, edited by J. S. Miller, Plenum Press (New York, 1983).
5 For a review, see, e.g., E. Dagotto and T. M. Rice, Science 271, 618 (1996).
6 S. K. Ma, C. Dasgupta and C.-K. Hu, Phys. Rev. Lett. 43, 1434 (1979); C. Dasgupta and S. K. Ma, Phys. Rev. B 22, 1305 (1980).
7 R. N. Bhatt and P. A. Lee, Phys. Rev. Lett. 48, 344 (1982).
8 D. S. Fisher, Phys. Rev. B 50, 3799 (1994).
9 D. S. Fisher, Phys. Rev. Lett. 69, 634 (1992); Phys. Rev. B 51, 6411 (1995).
10 E. Westerberg, A. Furusaki, M. Sigrist, and P. A. Lee, Phys. Rev. Lett. 75, 4302 (1995); Phys. Rev. B 55, 12578 (1997).
11 R. A. Hyman, K. Yang, R. N. Bhatt and S. M. Girvin, Phys. Rev. Lett. 76, 839 (1996); K. Yang, R. A. Hyman, R. N. Bhatt and S. M. Girvin, J. Appl. Phys. 79, 5096 (1996).
12 B. Boechat, A. Saguia, and M. A. Continentino, Solid State Commun. 98, 411 (1996); A. Saguia, B. Boechat, and M. A. Continentino, Phys. Rev. B 58, 58 (1998).
13 R. A. Hyman and K. Yang, Phys. Rev. Lett. 78, 1783 (1997).
14 C. Monthus, O. Golinelli, and Th. Jolicoeur, Phys. Rev. Lett. 79, 3254 (1997); Phys. Rev. B 58, 805 (1998).
15 K. Yang and R. N. Bhatt, Phys. Rev. Lett. 80, 4562 (1998); Physica B 259, 1048 (1999); R. N. Bhatt and K. Yang, J. Appl. Phys. 83, 7231 (1998).
16 K. Damle, O. Motrunich, and D. A. Huse, Phys. Rev. Lett. 84, 3434 (2000); O. Motrunich, K. Damle, and D. A. Huse, Phys. Rev. B 63 134424 (2001).
17 Y. Iino and M. Imada, J. Phys. Soc. Jpn. 65, 3728 (1996).
18 T. Miyazaki, M. Troyer, M. Ogata, K. Ueda and D. Yoshioka, J. Phys. Soc. Jpn. 66, 2580 (1997).
19 M. Steiner, M. Fabrizio, and A. O. Gogolin, Phys. Rev. B 57, 8290 (1998).
20 A. O. Gogolin, A. A. Nerseyan, A. M. Tsvelik, and L. Yu, Nucl. Phys. B 540, 705 (1999).
21 E. Orignac and T. Giamarchi, Phys. Rev. B 57, 5812 (1998).
22 R. Melin, Y.-C. Lin, P. Lajko, H. Reiger, and F. Igloi, cond-mat/0112261 (published in Phys. Rev. B 65, 104415 (2002)).
23 C. A. Doty and D. S. Fisher, Phys. Rev. B 45, 2167 (1992).
24 As spins and bonds get decimated, the number of spins coupled to a given spin (or its effective coordination number) gets bigger and bigger, which is typical when couplings beyond 1D are present (see, e.g., Olexei Motrunich, Siun-Chun Mau, David A. Huse, and Daniel S. Fisher, Phys. Rev. B 61, 1160 (2000)); thus the number of spins that are coupled to the strongest bond one plans to decimate also grows. However just like this case to second-order we only need to treat a 4-spin cluster at a given time.
25 See, e.g., J. J. Sakurai, Modern Quantum Mechanics (Addison Wesley, 1994), p. 241.
26 When the sizes of the spins are not uniform, in principle there can be different ways to define the strongest bond; see, e.g., Ref. 10. As we will see later in the present case the majority of the spins remain to be spin-1/2 in all stages of RG, we thus simply choose to define the strongest bond to be the bond with the biggest absolute value of the bond strength, regardless of the sizes of the spins that it couples.
FIG. 1: (a) The renormalization scheme for a four-spin problem with AF couplings, as encountered in random AF spin-1/2 chains. Here the strongest bond $J_2$ is decimated, together with the neighboring bonds $J_1$ and $J_3$, yielding an effective interaction $\tilde{J}$ between what were the third-nearest neighbors. (b) Schematic diagram for decimation in ladder. The dashed lines are the renormalized couplings. The thick dashed lines are the ferromagnetic couplings generated in the decimation process. (c) The renormalization scheme for a four spin problem where the strongest bond is ferromagnetic. The two spins connected by F coupling forms an effective spin object having renormalized interactions with its neighbors.
FIG. 2: Numerical simulation results of the proliferation of ferromagnetic bonds and larger spins, for two different initial bond distribution ($\Lambda = 1$ in both cases). (a) The ratio of numbers of AF and F bonds; (b) the ratio of average strength of AF and F bonds; (c) the difference of the averaged logarithm of AF and F bond strengths, and (d) the ratio of number of spins larger than $1/2$ and the total spins, all as functions of energy scale $\Omega$. All samples used have size $N = 20,000$ and in (a)-(c) different symbols represent different samples.
FIG. 3: The susceptibility per spin as a function of temperature for several disorder strengths $\alpha$. The anisotropy parameter $\Lambda = 1$ is fixed, and the system size is $N = 20,000$. The power law exponents, $\beta$, are calculated using the least square fit to the low-temperature data. The different symbols in the figure correspond to different samples. We do not take the sample average because the sample to sample variations are small.
FIG. 4: The susceptibility per spin as a function of temperature for a given $\alpha = 0.6$ with varying anisotropy parameter $\Lambda$. System size fixed to be $N = 20,000$. The power law exponent increases as we decrease the anisotropy parameter. For $\Lambda = 0$ we have decoupled chains. We plot all different samples in the figure without taking the sample average because the variations are very small.
FIG. 5: The sample-averaged spin-spin correlation along the chain (a) with varying $\Lambda$ and $\alpha = 0.6$, (b) with varying $\alpha$ and $\Lambda = 1$, for $N = 20,000$. The fit to power-law behavior yields the following power-law exponents: (a) $\nu = 1.97$ for $\Lambda = 0$, $\nu = 2.56$ for $\Lambda = 0.001$, $\nu = 2.87$ for $\Lambda = 0.5$, $\nu = 3.23$ for $\Lambda = 1$, and (b) $\nu = 3.45$ for $\alpha = 0.3$, $\nu = 3.23$ for $\alpha = 0.6$, $\nu = 2.89$ for $\Lambda = 0.9$.

FIG. 6: (a) The susceptibility per spin as a function of temperature, (b) the spin-spin correlation along the chain, for $\alpha = 0.6$ and $\Lambda = 1$ with varying system sizes. No significant variation in these two quantities as the system sizes are varied. Both (a) and (b) are sample averages.