Reconstruction of a source domain from the Cauchy data: II. Three-dimensional case

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Abstract
This paper is concerned with reconstruction issue of some typical inverse problems and consists of three parts. First a framework of the enclosure method for an inverse source problem governed by the Helmholtz equation at a fixed wave number in three dimensions is introduced. It is based on the nonvanishing of the coefficient of the leading profile of an oscillatory integral over a domain having a conical singularity. Second an explicit formula of the coefficient for a domain having a circular cone singularity and its implication under the framework are given. Third, an application under the framework to an inverse obstacle problem governed by an inhomogeneous Helmholtz equation at a fixed wave number in three dimensions is given.

Keywords: exponentially growing solutions, enclosure method, inverse source problem, inverse obstacle problem, Helmholtz equation, circular cone singularity

1. Introduction

More than twenty years ago, in [4] the author obtained the extraction formula of the support function of an unknown polygonal source domain in an inverse source problem governed by the Helmholtz equation and polygonal penetrable obstacle in an inverse obstacle problem governed by an inhomogeneous Helmholtz equation. All the problems considered therein are in two dimensions and employ only a single set of Cauchy data of a solution of the governing equation

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at a fixed wave number in a bounded domain. Those results can be considered as the first application of a single measurement version of the enclosure method introduced in [5].

Succeeding to [4], in [6] the author found another unexpected application of the enclosure method out to the Cauchy problem for the stationary Schrödinger equation

$$-\Delta u + V(x)u = 0$$

(1.1)
in a bounded domain $\Omega$ of $\mathbb{R}^n$, $n = 2, 3$. Here $V \in L^\infty(\Omega)$ and both $u$ and $V$ can be complex valued functions. We established an explicit representation or computation formula for an arbitrary solution $u \in H^2(\Omega)$ in terms of its Cauchy data on a part of $\partial \Omega$. See also [11] for its numerical implementation. Note also that the idea in [6] has been applied to an inverse source problem governed by the heat equation together with an inverse heat conduction problem in [8, 9], respectively.

The idea introduced therein is to make use of the complex geometrical optics solutions (CGO) with a large parameter $\tau$ for the modified equation instead of (1.1):

$$-\Delta v + V(x)v = \chi_{D_y}(x)v, \quad x \in \Omega,$$

where $y$ is a given point in $\Omega$, $D_y \subset \subset \Omega$ is the inside of a triangle, tetrahedron for $n = 2, 3$, respectively with a vertex at $y$ and $\chi_{D_y}(x)$ is the characteristic function of $D_y$. The solution is the same type as constructed one in [13] for $n = 2$, [14] for $n = 3$ and has the following form as $\tau \to \infty$:

$$v \sim e^{i\zeta},$$

where $\zeta = \tau(\omega + iv\theta)$ and both $\omega$ and $\theta$ are unit vectors perpendicular to each other. This right-hand side is just the complex plane wave used in the Calderón method [1].

Note that, in [7] another simpler idea to make use of the CGO solutions of another modified equation described below is presented:

$$-\Delta v + V(x)v = \chi_{D_y}(x)e^{i\zeta}, \quad x \in \Omega.$$

Using integration by parts we reduced the problem of computing the value of $u$ at given point $y$, essentially, to clarifying the leading profile of the following oscillatory integral as $\tau \to \infty$:

$$\int_{D_y} e^{i\zeta \cdot \rho(x)} dx,$$

where $\rho(x)$ is uniformly Hölder continuous on $\overline{D_y}$. Note that the asymptotic behaviour of this type of oscillatory integral in two dimensions is the key point of the enclosure method developed in [4].

In [6] we clarified the leading profile in more general setting as follows. Given a pair $(p, \omega) \in \mathbb{R}^n \times S^{n-1}$ and $\delta > 0$ let $Q$ be an arbitrary non empty bounded open subset of the plane $x \cdot \omega = p \cdot \omega - \delta$ with respect to the relative topology from $\mathbb{R}^n$. Define the bounded open subset of $\mathbb{R}^n$ by the formula

$$D_{(p,\omega)}(\delta, Q) = \bigcup_{0 < r < \delta} \left\{ p + \frac{r}{\delta}(z - p), |z| \in Q \right\}. \quad (1.2)$$

This is a cone with the base $Q$ and apex $p$, and lying in the slab $\{ x \in \mathbb{R}^n | p \cdot \omega - \delta < x \cdot \omega < p \cdot \omega \}$. Note that $\delta = \text{dist}(\{p\}, Q)$ is called the height. If $Q$ is given by the inside of a polygon,
The cone \(1.2\) is called a *solid pyramid*. In particular, if \(Q\) is given by the inside of a triangle, cone \(1.2\) becomes a tetrahedron.

On \((2.2)\) in \([6]\) we introduced a special complex constant associated with the domain \((1.2)\) which is given by

\[
C_{(\rho,\omega)}(\delta, Q, \theta) = 2\pi \int_{\partial \Omega} \frac{dS_2}{(s - i(z - p) \cdot \theta)^n},
\]

where \(i = \sqrt{-1}\), \(0 < s < \delta\) and \(Q = D_{(\rho,\omega)}(\delta, Q) \cap \{ x \in \mathbb{R}^n | \omega = p \cdot \omega - s \}\) and the direction \(\theta \in S^{n-1}\) is perpendicular to \(\omega\). Note that in \([6]\) complex constant \(C_{(\rho,\omega)}(\delta, Q, \theta)\) is simply written as \(C_{\rho}(\omega, \omega^-)\) with \(\omega^+ = \theta\). As pointed therein out this quantity is independent of the choice \(s \in ]0, \delta[\). Because of the one-to-one correspondence between \(z \in Q\) and \(z' \in Q'\) by the formula

\[
\begin{align*}
\begin{cases}
z' = p + \frac{s'}{s}(z - p), \\
ds_2' = \left( \frac{s'}{s} \right)^{n-1} dS_2.
\end{cases}
\end{align*}
\]

The following lemma describes the relationship between complex constant \(C_{(\rho,\omega)}(\delta, Q, \theta)\) and an integral over \((1.2)\).

**Proposition 1.1 (Lemma 2 in [6]).** Let \(n = 2, 3\). Let \(D = D_{(\rho,\omega)}(\delta, Q)\) and \(\rho \in C^{0,\alpha}(\overline{D})\) with \(0 < \alpha \leq 1\). It holds that, for all \(\tau > 0\)

\[
\left| e^{-\tau p \cdot (\omega + i \theta)} \int_D \rho(x) e^{\tau x \cdot (\omega + i \theta)} \, dx \right| = \frac{n - 1}{2\pi^n} |\rho(p)| C_{(\rho,\omega)}(\delta, Q, \theta)
\]

\[
\leq |\rho(p)| \frac{|Q|}{\delta^{n-1}} \left( (\tau \delta + 1)^n + n - 2 \right) \frac{e^{-\tau \delta}}{\tau^n} + \|\rho\|_{C^{0,\alpha}(\overline{D})} \frac{|Q|}{\delta^{n-1}} \left( \frac{\text{diam } D}{\delta} \right)^\alpha C_{n,\alpha}
\]

where \(\|\rho\|_{C^{0,\alpha}(\overline{D})} = \sup_{x \in \overline{D}, \omega \neq 0} |\rho(x)| / |x - y|^\alpha\) and

\[
C_{n,\alpha} = \int_0^\infty s^{n-1+\alpha} e^{-s} \, ds.
\]

Thus we have, as \(\tau \to \infty\)

\[
e^{-\tau p \cdot (\omega + i \theta)} \int_{D_{(\rho,\omega)}(\delta, Q)} \rho(x) e^{\tau x \cdot (\omega + i \theta)} \, dx = \frac{n - 1}{2\pi^n} |\rho(p)| C_{(\rho,\omega)}(\delta, Q, \theta) + O(\tau^{-n+\alpha}).
\]

This is the meaning of complex constant \(C_{(\rho,\omega)}(\delta, Q, \theta)\). Note that the remainder estimate \(O(\tau^{-n+\alpha})\) is uniform with respect to \(\theta\). And also as a direct corollary, instead of \((1.3)\) we have another representation of \(C_{(\rho,\omega)}(\delta, Q, \theta)\):

\[
C_{(\rho,\omega)}(\delta, Q, \theta) = \frac{2}{n-1} \lim_{\tau \to \infty} \tau^n e^{-\tau p \cdot (\omega + i \theta)} \int_{D_{(\rho,\omega)}(\delta, Q)} e^{\tau x \cdot (\omega + i \theta)} \, dx.
\]

The convergence is uniform with respect to \(\theta\).

Proposition 1.1 is the one of two key points in \([6]\) and gives the role of the Hölder continuity of \(\rho\). Another one is the *non-vanishing* of \(C_{(\rho,\omega)}(\delta, Q, \theta)\) as a part of the leading coefficient of
the integral in proposition 1.1 as \( \tau \to \infty \). This is not trivial, in particular, in three-dimensional case. For this we have shown therein the following fact.

**Proposition 1.2 (Theorem 2 in [6]).**

- If \( n = 2 \) and \( Q \) is given by the inside of an arbitrary line segment, then for all \( \vartheta \) perpendicular to \( \omega \) we have \( C_{p,\omega}(\delta, Q, \vartheta) \neq 0 \).
- If \( n = 3 \) and \( Q \) is given by the inside of an arbitrary triangle, then for all \( \vartheta \) perpendicular to \( \omega \) we have \( C_{p,\omega}(\delta, Q, \vartheta) \neq 0 \).

The nonvanishing of complex constant \( C_{p,\omega}(\delta, Q, \vartheta) \) in case \( n = 2 \) has been shown in the proof of lemma 2.1 in [4]. The proof therein employs a local expression of the corner around apex as a graph of a function on the line \( x \cdot \omega = x \cdot p \) and so the proof by viewing \( D_{p,\omega}(\delta, Q) \) as a cone in [6] is not developed.

Note that, in the survey paper [7] on the enclosure method it is pointed out that ‘the Helmholtz version’ of proposition 1.1 is also valid. That is, roughly speaking, we have

\[
e^{-\rho(\tau_2 + i\sqrt{\tau_2^2 + \rho^2})} \int_{D_{p,\omega}(\delta, Q)} \rho(x) e^{i(\tau_2 + \sqrt{\tau_2^2 + \rho^2} x \cdot \omega)} \, dx = \frac{n - 1}{2\pi} \rho(p) e^{i(\tau_2 + \sqrt{\tau_2^2 + \rho^2} p \cdot \omega)} + O(\tau^{-1(n + \alpha)})
\]

with the same constant \( C_{p,\omega}(\delta, Q, \vartheta) \), where \( k \geq 0 \). See lemma 3.2 therein. The proof can be done by using the same argument as that of proposition 1.1. Note that the function \( v = e^{i(\tau_2 + \sqrt{\tau_2^2 + \rho^2} \omega)} \) satisfies the Helmholtz equation \( \Delta v + k^2 v = 0 \) in \( \mathbb{R}^3 \).

11. Role of nonvanishing in an inverse source problem

As an application of the nonvanishing of the complex constant \( C_{p,\omega}(\delta, Q, \vartheta) \), we present here its direct application to the inverse source problem considered in [4], however, in three dimensions.

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^3 \) with \( \partial \Omega \subset C^2 \). We denote by \( \nu \) the normal unit outward vector field on \( \partial \Omega \). Let \( k \geq 0 \). Let \( u \in H^1(\Omega) \) be an arbitrary weak solution of the Helmholtz equation in \( \Omega \) at the wave number \( k \):

\[
\Delta u + k^2 u = F(x), \quad x \in \Omega,
\]

where \( F(x) \) is an unknown source term such that \( \text{supp} \, F \subset \Omega \). Both \( u \) and \( F \) can be complex-valued functions. See [4] for the meaning of the solution and the formulation of the Cauchy data on \( \partial \Omega \) in the weak sense.

It is well known that, in general, one cannot obtain the uniqueness of the source term \( F \) itself from the Cauchy data of \( u \) on \( \partial \Omega \). In fact, given \( \varphi \in C^\infty_0(\Omega) \) let \( G = F + \Delta \varphi + k^2 \varphi \). We have \( \text{supp} \, G \subset \Omega \) and the function \( \tilde{u} = u + \varphi \) satisfies

\[
\Delta \tilde{u} + k^2 \tilde{u} = G(x), \quad x \in \Omega.
\]

Both \( u \) and \( \tilde{u} \) have the same Cauchy data on \( \partial \Omega \). It should be pointed out that, however, \( F \) and \( G \) coincide each other modulo \( C^\infty \). This means that the singularity of \( F \) and \( G \) coincides each other. This suggests a possibility of extracting some information about a singularity of \( F \) or its support from the Cauchy data of \( u \) on \( \partial \Omega \).
As done in [4] in two dimensions, we introduce the special form of the unknown source $F$:

$$
F(x) = F_{\rho,D}(x) = \begin{cases} 
0, & \text{if } x \in \Omega \setminus D, \\
\rho(x), & \text{if } x \in D.
\end{cases}
\tag{1.7}
$$

Here $D$ is an unknown nonempty open subset of $\Omega$ satisfying $D \subset \Omega$ and $\rho \in L^2(D)$ also unknown. We call $D$ the source domain, however, we do not assume the connectedness of not only $D$ but also $\Omega \setminus D$. The $\rho$ is called the strength of the source.

We are interested in the following problem.

**Problem 1.** Extract information about a singularity of the source domain $D$ of $F$ having form (1.7) from the Cauchy data $(u(x), \frac{\partial u}{\partial \nu}(x))$ for all $x \in \partial \Omega$.

Note that we are seeking a concrete procedure of the extraction.

Here we recall the notion of the regularity of a direction introduced in the enclosure method [4]. The function $h_D(\omega) = \sup_{x \in D} |x \cdot \omega|$, $\omega \in S^2$ is called the support function of $D$. It belongs to $C(S^2, \mathbb{R})$ because of the trivial estimate $|h_D(\omega_1) - h_D(\omega_2)| \leq \sup_{x \in D} |x| \cdot |\omega_1 - \omega_2|$ for all $\omega_1, \omega_2 \in S^2$. Given $\omega \in S^2$, it is easy to see that the set

$$
H_\omega(D) \equiv \{ x \in \partial \Omega | x \cdot \omega = h_D(\omega) \}
$$

is nonempty and contained in $\partial D$. We say that $\omega$ is regular with respect to $D$ if the set $H_\omega(D)$ consists of only a single point. We denote the point by $p(\omega)$.

We introduce a concept of a singularity of $D$ in (1.7).

**Definition 1.1.** Let $\omega \in S^2$ be regular with respect to $D$. We say that $D$ has a conical singularity from direction $\omega$ if there exists a positive number $\delta$, an open set $Q$ of the plane $x \cdot \omega = h_D(\omega) - \delta$ with respect to the relative topology from $\mathbb{R}^3$ such that

$$
D \cap \{ x \in \mathbb{R}^3 | h_D(\omega) - \delta < x \cdot \omega < h_D(\omega) \} = D_{\rho(h_\omega)(\delta, Q)}.
$$

Second we introduce a concept of an activity of the source term.

**Definition 1.2.** Given a point $p \in \partial D$ we say that the source $F = F_{\rho,D}$ given by (1.7) is active at $p$ if there exist an open ball $B_\eta(p)$ centered at $p$ with radius $\eta$, $0 < \eta \leq 1$ and a function $\tilde{\rho} \in C_0^\infty(B_\eta(p))$ such that $\rho(x) = \tilde{\rho}(x)$ for almost all $x \in B_\eta(p) \cap D$ and $\tilde{\rho}(p) \neq 0$. Note that $\rho$ together with $\tilde{\rho}$ can be a complex-valued function.

Now let $u \in H^1(\Omega)$ satisfies the equation (1.6) in the weak sense with $F = F_{\rho,D}$ given by (1.7). Given a unit vector $\omega \in S^2$ define $S(\omega) = \{ \vartheta \in S^2 | \vartheta \cdot \omega = 0 \}$.

Using the Cauchy data of $u$ on $\partial \Omega$, we define the indicator function as [4]

$$
I_{\omega,\vartheta}(\tau) = \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) dS,
$$

where $\vartheta \in S(\omega)$ and

$$
v = e^{i(\tau \vartheta + \sqrt{\tau^2 + k^2} \vartheta)}, \quad \tau > 0.
$$

And also its derivative with respect to $\tau$

$$
I_{\omega,\vartheta}^\prime(\tau) = \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v}{\partial \nu} u_\tau \right) dS,
$$

where $u_\tau$ is the normal derivative of $u$ with respect to $\partial \Omega$.
where
\[ v_\tau = \frac{\partial}{\partial \tau} v = \left\{ x \cdot \left( \omega + i \frac{\tau}{\sqrt{\tau^2 + k^2}} \right) \right\} v. \]

The following theorem clarifies the role of the complex constant \( C(p(\omega), (\delta, Q, \vartheta)) \) in the asymptotic behaviour of the indicator function together with its derivative as \( \tau \to \infty \).

**Theorem 1.1.** Let \( \omega \) be regular with respect to \( D \) and assume that \( D \) has a conical singularity from direction \( \omega \). Assume that \( F = F_\rho, D \) given by (1.7) is active at \( p(\omega) \). Then, we have
\[
\tau^3 e^{-\tau h_D(\omega)} e^{-i \sqrt{\tau^2 + k^2} p(\omega) \cdot \vartheta} I_{\omega, \vartheta}(\tau) = \tilde{\rho}(\omega) C_{p(\omega), \omega}(\delta, Q, \vartheta) + O(\tau^{-\alpha}) \tag{1.8}
\]
and
\[
\tau^3 e^{-\tau h_D(\omega)} e^{-i \sqrt{\tau^2 + k^2} p(\omega) \cdot \vartheta} I'_{\omega, \vartheta}(\tau) = \tilde{\rho}(\omega)(h_D(\omega) + i p(\omega) \cdot \vartheta) C_{p(\omega), \omega}(\delta, Q, \vartheta) + O(\tau^{-\alpha}). \tag{1.9}
\]
The remainder \( O(\tau^{-\alpha}) \) is uniform with respect to \( \vartheta \in S(\omega) \).

**Proof.** Integration by parts yields
\[
I_{\omega, \vartheta}(\tau) = \int_D \rho(x)v \, dx
\]
and thus
\[
I'_{\omega, \vartheta}(\tau) = \int_D \rho(x)v_\tau \, dx.
\]
Recalling definition 1.1, one has the decomposition
\[
D = D_{p(\omega), \omega}(\delta, Q) \cup D', \tag{1.10}
\]
where
\[
D' = D \setminus D_{p(\omega), \omega}(\delta, Q) \subset \{ x \in \mathbb{R}^3 | x \cdot \omega \leq h_D(\omega) - \delta \}. \tag{1.11}
\]
Besides, choosing \( \delta \) smaller if necessary, one may assume that \( D_{p(\omega), \omega}(\delta, Q) \subset B_{\eta}(p(\omega)) \), where \( \eta \) and \( B_{\eta}(p(\omega)) \) are same as those of definition 1.2.

Hereafter we set \( p = p(\omega) \) for simplicity of description. According to the decomposition (1.10), we have the decomposition of both \( I_{\omega, \vartheta}(\tau) \) and \( I'_{\omega, \vartheta}(\tau) \) as follows:
\[
e^{-\tau h_D(\omega)} e^{-i \sqrt{\tau^2 + k^2} p(\omega) \cdot \vartheta} I_{\omega, \vartheta}(\tau) = e^{-\tau h_D(\omega)} e^{-i \sqrt{\tau^2 + k^2} p(\omega) \cdot \vartheta} \int_{D_{p(\omega), \omega}(\delta, Q)} \tilde{\rho}(x)v \, dx
+ e^{-\tau h_D(\omega)} e^{-i \sqrt{\tau^2 + k^2} p(\omega) \cdot \vartheta} \int_{D'} \rho v \, dx \tag{1.12}
\]
and
\[
e^{-\tau k_\delta(\omega)} e^{-i\sqrt{\tau^2 + k^2} p\theta} I_{\omega,\theta}(\tau) = e^{-\tau k_\delta(\omega)} e^{-i\sqrt{\tau^2 + k^2} p\theta} \int_{D_{\delta,\theta}(\delta,\theta)} \tilde{\rho}(x)v_\tau \, dx
+ e^{-\tau k_\delta(\omega)} e^{-i\sqrt{\tau^2 + k^2} p\theta} \int_{\partial D_{\delta,\theta}(\delta,\theta)} \rho v_\tau \, dx,
\]
(1.13)
where \( p = p(\omega) \).

By (1.11), we see that the second terms on the right-hand sides of (1.12) and (1.13) have the common bound \( O(e^{-\tau\delta}) \). Thus from (1.5) and (1.12) we obtain (1.8) with the remainder \( O(\tau^{-\alpha}) \) which is uniform with respect to \( \delta \in S(\omega) \).

For (1.13) we write
\[
\int_{D_{\delta,\theta}(\delta,\theta)} \tilde{\rho}(x)v_\tau \, dx
= \int_{D_{\delta,\theta}(\delta,\theta)} \tilde{\rho}(x) \left\{ x \cdot \left( \omega + i\frac{\tau}{\sqrt{\tau^2 + k^2} \theta} \right) \right\} v \, dx
= \int_{D_{\delta,\theta}(\delta,\theta)} \tilde{\rho}(x)x \cdot \omega v \, dx + i\frac{\tau}{\sqrt{\tau^2 + k^2}} \int_{D_{\delta,\theta}(\delta,\theta)} \tilde{\rho}(x) \cdot \theta v \, dx.
\]
Thus applying (1.5) to each of the last terms and using (1.13), we obtain (1.9) with the remainder \( O(\tau^{-\alpha}) \) which is uniform with respect to \( \theta \in S(\omega) \).

Thus under the same assumptions as theorem 1.1, for each \( \delta \in S(\omega) \) one can calculate
\[
I(\omega, \theta) \equiv \tilde{\rho}(p(\omega)) C_{(p(\omega),\omega)}(\delta, Q, \theta)
\]
via the formula
\[
I(\omega, \theta) = \lim_{\tau \to \infty} \tau^3 e^{-\tau k_\delta(\omega)} e^{-i\sqrt{\tau^2 + k^2} p(\omega) \theta} I_{\omega,\theta}(\tau)
\]
(1.14)
by using the Cauchy data of \( u \) on \( \partial\Omega \) if \( p(\omega) \) is known.

As a direct corollary of formulae (1.8) and (1.9), we obtain a partial answer to problem 1 and the starting point of the main purpose in this paper.

**Theorem 1.2.** Let \( \omega \) be regular with respect to \( D \). Assume that \( D \) has a conical singularity from direction \( \omega \), \( F_{p,\delta} \) is active at \( p = p(\omega) \) and that direction \( \theta \in S(\omega) \) satisfies the condition
\[
C_{(p(\omega),\omega)}(\delta, Q, \theta) \neq 0.
\]
(1.15)
Then, there exists a positive number \( \tau_0 \) such that, for all \( \tau \geq \tau_0 \), \( I_{\omega,\theta}(\tau) \) is uniform with respect to \( \tau \) and we have the following three asymptotic formulae. The first formula is
\[
\lim_{\tau \to \infty} \frac{\log |I_{\omega,\theta}(\tau)|}{\tau} = h_D(\omega)
\]
(1.16)
and second one
\[
\lim_{\tau \to \infty} \frac{I'_{\omega,\theta}(\tau)}{I_{\omega,\theta}(\tau)} = h_D(\omega) + ip(\omega) \cdot \theta.
\]
(1.17)
The third one is the so-called $0$-$\infty$ criterion:

$$
\lim_{\tau \to \infty} e^{-\tau t} |I_{\omega, \vartheta}(\tau)| = \begin{cases} 0, & \text{if } t \geq h_D(\omega), \\ \infty, & \text{if } t < h_D(\omega). \end{cases}
$$

(1.18)

This provides us the framework of the approach using the enclosure method for the source domain with a conical singularity from a direction.

Some remarks are in order.

- In two dimensions, by proposition 1.2 the condition (1.15) is redundant and we have the same conclusion as theorem 1.2.
- The formula (1.17) is an application of the idea ‘taking the logarithmic derivative of the indicator function’ introduced in [10]. Therein inverse obstacle scattering problems at a fixed frequency in two dimensions are considered. Needless to say, formula (1.17) is not derived in [4].

The condition (1.15) is stable with respect to the perturbation of $\vartheta \in S(\omega)$ since from the expression (1.3) we see that the function $S(\omega) \ni \vartheta \mapsto C_{(p(\omega), \omega)}(\delta, Q, \vartheta)$ is continuous, where the topology of $S(\omega)$ is the relative one from $\mathbb{R}^3$. This fact yields a corollary as follows.

Corollary 1.1. Let $\omega$ be regular with respect to $D$. Under the same assumptions as those in Theorem 1.2 the point $p(\omega)$ is uniquely determined by the Cauchy data of $u$ on $\partial \Omega$.

Proof. From (1.16) one has $h_D(\omega) = p(\omega) \cdot \omega$. Choose $\vartheta' \in S(\omega)$ sufficiently near $\vartheta$ in such a way that $C_{(p(\omega), \omega)}(\delta, Q, \vartheta') \neq 0$. Then from the formula (1.17) for two linearly independent directions $\vartheta$ and $\vartheta'$ one gets $p(\omega) \cdot \vartheta$ and $p(\omega) \cdot \vartheta'$.

□

As another direct corollary of theorem 1.2 and proposition 1.2 in the case $n = 3$ we have the following result.

Corollary 1.2. Assume that $D$ is given by the inside of a convex polyhedron and in a neighbourhood of each vertex $p$ of $D$, the $D$ coincides with the inside of a tetrahedron with apex $p$ and that the source $F = F_{p, D}$ given by (1.7) is active at $p$. Then, we have all the formulae (1.16)--(1.18) for all $\omega$ regular with respect to $D$ and $\vartheta \in S(\omega)$.

Proof. We have: $D$ has a conical singularity from the direction $\omega$ that is regular with respect to $D$ with a triangle $Q$ at each $p(\omega)$. Thus (1.15) is valid for all $\omega$ regular with respect to $D$ and $\vartheta \in S(\omega)$. Therefore, we have all the formulae (1.16)--(1.18) for all $\omega$ regular with respect to $D$ and $\vartheta \in S(\omega)$.

□

Remark 1.1. Under the same assumptions as corollary 1.2 one gets a uniqueness theorem: the Cauchy data of $u$ on $\partial \Omega$ uniquely determines $D$. The proof is as follows. From (1.16) one gets $h_D(\omega)$ for all $\omega$ regular with respect to $D$. The set of all $\omega$ that are not regular with respect to $D$ consists of a set of finite points and arcs on $S^2$. This yields the set of all $\omega$ that are regular with respect to $D$ is dense and thus one gets $h_D(\omega)$ for all $\omega \in S^2$ because of the continuity of $h_D$. Therefore one obtains the convex hull of $D$ and thus $D$ itself by the convexity assumption. This proof is remarkable and unique since we never make use of the traditional contradiction argument. ‘Suppose we have two different source domains $D_1$ and $D_2$ which yields the same Cauchy data, . . . ’; any unique continuation argument of the solution of the governing equation. One can see such two arguments in [12] in the case when $k = 0$ for an inverse problem for detecting a source of gravity anomaly.
Some of typical examples of $D$ covered by corollary 1.2 are tetrahedron, regular hexahedron (cube), regular dodecahedron.

So now the central problem in applying theorem 1.2 to problem 1 for the source with various source domain under our framework is to clarify the condition (1.15) for general $Q$. In contrast to proposition 1.2, when $Q$ is general, we do not know whether there exists a unit vector $\hat{\vartheta} \in S(\omega)$ such that (1.15) is valid or not. Going back to (1.3), we have an explicit vector equation for the constant $C_{(p, \omega)}(\delta, Q, \hat{\vartheta})$, if $Q$ is given by the inside of a polygon. See proposition 4 in [6]. However, comparing with the case when $Q$ is given by the inside of a triangle, it seems difficult to deduce the non-vanishing $C_{(p, \omega)}(\delta, Q, \hat{\vartheta})$ for all $\hat{\vartheta} \in S(\omega)$ from the equation directly. This is an open problem.

1.2. Explicit formula and its implication

In this paper, instead of considering general $Q$, we consider another special $Q$. It is the case when $Q$ is given by the section of the inside of a circular cone by a plane.

Given $p \in \mathbb{R}^3$, $n \in S^2$ and $\theta \in ]0, \pi/2[$ let $V_p(-n, \theta)$ denote the inside of the circular cone with apex at $p$ and the opening angle $\theta$ around the direction $-n$, that is

$$V_p(-n, \theta) = \left\{ x \in \mathbb{R}^3 | (x - p) \cdot (-n) > |x - p| \cos \theta \right\}.
$$

Given $\omega \in S^2$ set

$$Q = V_p(-n, \theta) \cap \left\{ x \in \mathbb{R}^3 | x \cdot \omega = p \cdot \omega - \delta \right\}. \quad (1.19)$$

To ensure that $Q$ is non empty and bounded, we impose the restriction between $\omega$ and $n$ as follows:

$$\omega \cdot n > \cos(\pi/2 - \theta) = \sin \theta > 0. \quad (1.20)$$

This means that the angle between $\omega$ and $n$ has to be less than $\pi/2 - \theta$. Then it is known that $Q$ is the inside of an ellipse and we have

$$D_{(p, \omega)}(\delta, Q) = V_p(-n, \theta) \cap \left\{ x \in \mathbb{R}^3 | x \cdot \omega > p \cdot \omega - \delta \right\}. \quad (1.21)$$

The problem here is to compute the complex constant $C_{(p, \omega)}(\delta, Q, \hat{\vartheta})$ with all $\hat{\vartheta} \in S(\omega)$ for this domain $D_{(p, \omega)}(\delta, Q)$ with $Q$ given by (1.19).

Instead of (1.3) we employ the formula (1.4) with $D = D_{(p, \omega)}(\delta, Q)$ and $n = 3$:

$$C_{(p, \omega)}(\delta, Q, \hat{\vartheta}) = \lim_{\tau \to \infty} \tau^3 e^{-\tau p (\omega + i \hat{\vartheta})} \int_{D_{(p, \omega)}(\delta, Q)} e^{	au x (\omega + i \hat{\vartheta})} dx. \quad (1.22)$$

Here we rewrite this formula. Choosing sufficiently small positive numbers $\delta'$ and $\delta''$ with $\delta'' < \delta'$, we see that the set

$$D_{(p, \omega)}(\delta, Q) \cap \left\{ x \in \mathbb{R}^3 | x \cdot n < p \cdot n - \delta' \right\}$$

is contained in the half-space $x \cdot \omega < p \cdot \omega - \delta''$.  

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This yields
\[ e^{-\tau p(\omega+i\theta)} \int_{D(p,\omega)(\delta, Q)} e^{x \cdot (\omega+i\theta)} \, dx = e^{-\tau p(\omega+i\theta)} \int_{V} e^{x \cdot (\omega+i\theta)} \, dx + O(e^{-\tau p}), \]
where
\[ V = V_p(-n, \theta) \cap \{ x \in \mathbb{R}^3 \mid x \cdot n > p \cdot n - \delta' \} . \]

Thus from (1.22) we obtain a more convenient expression
\[ C_{(p,\omega)}(\delta, Q, \vartheta) = \lim_{\tau \to +\infty} \tau^3 e^{-\tau p(\omega+i\theta)} \int_{V} e^{x \cdot (\omega+i\theta)} \, dx. \tag{1.23} \]

Using this expression we have the following explicit formula of \( C_{(p,\omega)}(\delta, Q, \vartheta) \) for \( D_{(p,\omega)}(\delta, Q) \) given by (1.21).

**Proposition 1.3.** We have
\[ C_{(p,\omega)}(\delta, Q, \vartheta) = 6V(\theta)(n \cdot (\omega+i\theta))^{-3}, \tag{1.24} \]
where
\[ V(\theta) = \frac{\pi}{3} \cos \theta \sin^2 \theta. \]

Note that the value \( V(\theta) \) coincides with the volume of the circular cone with the height \( \cos \theta \) and the opening angle \( \theta \). This function of \( \theta \in ]0, \frac{\pi}{2}[ \) is monotone increasing in \( ]0, \tan^{-1}\sqrt{2}[ \) and decreasing in \( ]\tan^{-1}\sqrt{2}, \frac{\pi}{2}[ \); takes the maximum value \( \frac{\pi}{2\sqrt{2}} \) at \( \theta = \tan^{-1}\sqrt{2} \).

Now we describe an application to problem 1. First we introduce a singularity of a circular cone type for the source domain.

**Definition 1.3.** Let \( D \) be a non empty bounded open set of \( \mathbb{R}^3 \). Let \( p \in \partial D \). We say that \( D \) has a circular cone singularity at \( p \) if there exist a positive number \( \epsilon \), unit vector \( n \) and number \( \theta \in ]0, \frac{\pi}{2}[ \) such that
\[ D \cap B_{\epsilon}(p) = V_p(-n, \theta) \cap B_{\epsilon}(p). \]

It is easy to see that notion of the circular cone singularity is a special case of that of the conical one in the following sense.

**Lemma 1.1.** Let \( \omega \in S^2 \) be regular with respect to \( D \). Assume that \( D \) has a circular cone singularity at \( p(\omega) \). Then, \( D \) has a conical singularity from direction \( \omega \) at \( p(\omega) \). More precisely, for a sufficiently small \( \delta \) we have the expression
\[ D \cap \{ x \in \mathbb{R}^3 \mid h_D(\omega) - \delta < x \cdot \omega < h_D(\omega) \} = D_{(p(\omega),\omega)}(\delta, Q), \]
where \( Q \) is given by (1.19) with \( V_p(-n, \theta) \) at \( p = p(\omega) \) in the definition 1.3 satisfying (1.20).

As a direct corollary of theorems 1.1 and 1.2, Proposition 1.3 and lemma 1.1 we immediately obtain all the results in theorem 1.2 without the condition (1.15). We summarize one of the result as corollary 1.3 as follows.

**Corollary 1.3 (Detecting the point \( p(\omega) \)).** Let \( u \in H^1(\Omega) \) be an arbitrary solution of (1.6) with the source \( F = F_{\mu,D} \) given by (1.7). Let \( \omega \in S^2 \) be regular with respect to \( D \). Assume that: \( D \) has a circular cone singularity at \( p = p(\omega) \); the source \( F \) is active at \( p(\omega) \). Choose
two linearly independent vectors \( \vartheta = \vartheta_1 \) and \( \vartheta_2 \) in \( S(\omega) \). Then, the point \( p(\omega) \) itself and thus \( h_D(\omega) = p(\omega) \cdot \omega \) can be extracted from the Cauchy data of \( u \) on \( \partial \Omega \) by using the formula

\[
p(\omega) \cdot \omega + i p(\omega) \cdot \vartheta_j = \lim_{\tau \to \infty} \frac{F_{\omega,\vartheta_j}(\tau)}{I_{\omega,\vartheta_j}(\tau)}, \quad j = 1, 2.
\]

(1.25)

By virtue of the formula (1.24), the function \( I(\omega, \cdot) \) has the expression

\[
I(\omega, \vartheta) = 6 \tilde{\rho}(p(\omega)) V(\vartheta) (n \cdot (\omega + i \vartheta))^{-3}.
\]

(1.26)

Formula (1.26) yields the following results.

**Corollary 1.4.** Let \( u \in H^1(\Omega) \) be a solution of (1.6) with the source \( F = F_{p,D} \) given by (1.7). Let \( \omega \in S^2 \) be regular with respect to \( D \). Assume that: \( D \) has a circular cone singularity at \( p(\omega) \) such as \( D \cap B_\epsilon(p(\omega)) = V_{p,\omega}(-n, \vartheta) \cap B_\epsilon(p(\omega)) \) with \( \epsilon > 0 \).

(a) Assume that \( F \) is active at \( p(\omega) \). The vector \( \omega \) coincides with \( n \) if and only if the function \( I(\omega, \cdot) \) is a constant function.

(b) The vector \( n \) and \( \vartheta \) of \( V_{p,\omega}(-n, \vartheta) \) and the source strength \( \tilde{\rho}(p(\omega)) \) satisfy the following two equations:

\[
6 |\tilde{\rho}(p(\omega))| V(\vartheta) = (n \cdot \omega)^3 \max_{\vartheta \in S(\omega)} |I(\omega, \vartheta)|;
\]

(1.27)

\[
6 \tilde{\rho}(p(\omega)) V(\vartheta) (3(n \cdot \omega)^2 - 1) = \frac{1}{\pi} \int_{S(\omega)} I(\omega, \vartheta) d\sigma(\vartheta).
\]

(1.28)

Using the equations (1.26)–(1.28) one gets the following corollary.

**Corollary 1.5.** Let \( u \in H^1(\Omega) \) be a solution of (1.6) with the source \( F = F_{p,D} \) given by (1.7). Let \( \omega \in S^2 \) be regular with respect to \( D \). Assume that: \( D \) has a circular cone singularity at \( p(\omega) \) such as \( D \cap B_\epsilon(p(\omega)) = V_{p,\omega}(-n, \vartheta) \cap B_\epsilon(p(\omega)) \) with \( \epsilon > 0 \). Assume that \( F \) is active at \( p(\omega) \) and that \( \omega \approx n \) in the sense that

\[
n \cdot \omega > \frac{1}{\sqrt{3}}.
\]

(1.29)

Then, the value \( \gamma = n \cdot \omega \) is the unique solution of the following quintic equation in \( \left[ \frac{1}{\sqrt{3}}, 1 \right] \):

\[
\gamma^3 (3\gamma^2 - 1) = \frac{\left| \int_{S(\omega)} I(\omega, \vartheta) d\sigma(\vartheta) \right|}{\pi \max_{\vartheta \in S(\omega)} |I(\omega, \vartheta)|}.
\]

(1.30)

Besides, for an arbitrary \( \vartheta \in S(\omega) \) the value \( \mu = n \cdot \vartheta \) is given by the formulae

\[
\mu^2 = \frac{\gamma^3 - \text{Re} T(\omega, \vartheta)}{3\gamma}
\]

(1.31)

and

\[
\mu = \frac{\text{Im} T(\omega, \vartheta)}{3\gamma^2 - \mu^2}.
\]

(1.32)
where
\[
T(\omega, \vartheta) = \frac{\int_{S(\omega)} I(\omega, \vartheta) \, ds(\vartheta)}{\pi(3\gamma^2 - 1)I(\omega, \vartheta)},
\]
(1.33)

The condition (1.29) is equivalent to the statement: the angle between \(\omega\) and \(n\) is less than \(\tan^{-1}\sqrt{2}\). Thus it is not so strict a condition. The denominator of (1.32) is not zero because of \(3\gamma^2 - \mu^2 \geq 3\gamma^2 - 1\) and (1.29).

Under the same assumptions as corollary 1.5, one can finally calculate the quantity
\[
\tilde{\rho}(p(\omega))V(\theta)
\]
(1.34)
and \(n\) from the Cauchy data of \(u\) on \(\partial \Omega\). This is the final conclusion.

The procedure is as follows.

**Step 1.** Calculate \(p(\omega)\) via the formula (1.25).

**Step 2.** Calculate \(I(\omega, \vartheta)\) via the formula (1.14) and the computed \(p(\omega)\) in step 1.

**Step 3.** If \(I(\omega, \vartheta)\) looks like a constant function, decide \(\omega \approx n\) in the sense (1.29). If not so, search another \(\omega\) around the original one in such a way that \(\omega \approx n\) as above by try and error and finally fix it.

**Step 4.** Find the value \(\gamma = n \cdot \omega\) by solving the quintic equation (1.30).

**Step 5.** Find the value (1.34) via the formulae (1.28) with the computed \(n \cdot \omega\) in step 4.

**Step 6.** Choose linearly independent vectors \(\vartheta_1, \vartheta_2 \in S(\omega)\) and calculate \(T(\omega, \vartheta_j), j = 1, 2\) via the formula (1.33) using the computed value \(\gamma\) in step 4.

**Step 7.** Find \(\mu = \mu_j = n \cdot \vartheta_j\) by solving (1.31) and (1.32) using the computed \(T(\omega, \vartheta_j)\) in step 6.

**Step 8.** Find \(n\) by solving \(n \cdot \omega = \gamma, n \cdot \vartheta_j = \mu_j, j = 1, 2\).

Note that, in addition, if the opening angle \(\theta\) (the source strength \(\tilde{\rho}(p(\omega))\) is known, then one obtains the value of \(\tilde{\rho}(p(\omega))/\)the volume \(V(\theta)\) via the computed value (1.34) in step 5.

This paper is organized as follows. In the next section we give a proof of proposition 1.3. It is based on the integral representation (2.8) of the complex constant \(C(p,\omega)(\delta, Q, \vartheta)\) and the residue calculus. Proofs of corollaries 1.4 and 1.5 are given in section 3. In section 4, an inverse obstacle problem for a penetrable obstacle in three dimensions is considered. The corresponding results in this case are given and in section 5 a possible direction of the extension of all the results in this paper is commented. Appendix is devoted to an example covered by the results in section 4.

### 2. Proof of proposition 1.3

In order to compute the right-hand side on (1.23), we choose two unit vectors \(l\) and \(m\) perpendicular to each other in such a way that \(n = l \times m\).

We see that the intersection of \(\partial V_p(-n, \vartheta)\) with the plane \((x - p) \cdot n = -1/(\tan \theta)\) coincides with the circle with radius 1 centered at the point \(p - (1/\tan \theta)n\) on the plane. The pointing vector of an arbitrary point on the circle with respect to point \(p\) has the expression
\[
\vartheta(w) = \cos w l + \sin w m - \frac{1}{\tan \theta} n
\]
(2.1)

\(^2\)We have
\[
\frac{3\pi}{10} > \frac{\pi}{100} > \tan^{-1}\sqrt{2} > \frac{3\pi}{10}.
\]
with a parameter $w \in [0, 2\pi]$. Besides, from the geometrical meaning of $\vartheta(w)$, we have
\[
\max_{w \in [0, 2\pi]} \vartheta(w) \cdot \omega < 0. \tag{2.2}
\]

**Lemma 2.1.** We have the expression
\[
(\omega + i\vartheta)C_{p,w}(\delta, Q, \vartheta) = \frac{1}{\tan \theta} \int_0^{2\pi} \cos w l + \sin w m + \tan \theta n \frac{\vartheta(w)}{(\vartheta(w) + (\omega + i\vartheta))^2} dw. \tag{2.3}
\]

**Proof.** Let $a$ be an arbitrarily three-dimensional complex vector. We have
\[
\int_V \nabla \cdot (e^{\tau(w+i\vartheta)}a) \, dx = \tau(w + i\vartheta) \cdot a \int_V e^{\tau(w+i\vartheta)}a \cdot \nu \, dS(x), \tag{2.4}
\]
where $\nu$ denotes the outer unit normal vector to $\partial V$.

Decompose $\partial V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, where
\[
V_1 = \{ x | (x - p) \cdot n = |x - p| \cos \theta, -\delta' < (x - p) \cdot n < 0 \},
\]
\[
V_2 = \{ x | |x - (p - \delta'n)| < \delta' \tan \theta, (x - p) \cdot n = -\delta' \}.
\]

To compute the surface integral over $V_1$, we make use of the change of variables as follows:
\[
x = (p - \delta'n) + r(\cos w l + \sin w m) + \left(\delta' - \frac{r}{\tan \theta} \right) n
\]
\[
= p + r\vartheta(w), \tag{2.5}
\]
where $(r, w) \in [0, \delta' \tan \theta] \times [0, 2\pi]$ and $\vartheta(w)$ is given by (2.1). Then the surface element has the expression
\[
dS(x) = \frac{r}{\sin \theta} \, dr \, dw
\]
and outer unit normal $\nu$ to $V_1$ takes the form
\[
\nu = \sin \theta \left( n + \frac{\cos w l + \sin w m}{\tan \theta} \right).
\]

Now from (2.4) and the decomposition $\partial V = V_1 \cup V_2$, we have
\[
e^{-\tau p(w+i\vartheta)}(\omega + i\vartheta) \cdot a \int_V v \, dx
\]
\[
e^{-\tau p(w+i\vartheta)} \tau^{-1} \int_{V_1} \nu a \cdot \nu \, dS(x) - e^{-\tau p(w+i\vartheta)} \tau^{-1} \int_{V_2} \nu a \cdot n \, dS(x)
\]
\[
\equiv I + II, \tag{2.6}
\]
where $v = e^{\tau x(w+i\vartheta)}$.

Since the set $V_2$ is contained in the half-space $x \cdot \omega \leq p \cdot \omega - \delta''$, one gets
\[
II = O(\tau^{-1} e^{-\tau \delta''}), \tag{2.7}
\]
\[
\]
On $I$, using the change of variables given by (2.5), one has
\[ x \cdot \omega = p \cdot \omega + r \theta(w) \cdot \omega, \]
\[ x \cdot \vartheta = p \cdot \vartheta + r \theta(w) \cdot \vartheta. \]

And also noting (2.2), one gets
\[ \text{Inverse Problems} \]

Proof.
Set
\[ C = \int_0^{2\pi} dw \int_0^{\tau} dr \, e^{i r \theta(w)} \left( n + \frac{\cos (w l + \sin w m)}{\tan \theta} \right) \cdot a \]
\[ = \frac{1}{\tau^3} \int_0^{2\pi} dw \int_0^{\tau} dr \, s \, ds \, e^{i r \theta(w)} \left( n + \frac{\cos (w l + \sin w m)}{\tan \theta} \right) \cdot a + O(\tau^{-3}). \]

Here one can apply the following formula to this right-hand side:
\[ \int_0^{\infty} s e^{w} e^{i w \tau} \, ds = \frac{1}{(a + ib)^{\tau}}, \quad a < 0. \]

Then one gets
\[ I = \frac{1}{\tau^3} \tan \theta \int_0^{2\pi} \frac{\cos (w l + \sin w m + \tan \theta n)}{(\theta(w) \cdot (\omega + i \vartheta))^2} \, dw + O(\tau^{-5}). \]

Now this together with (1.23), (2.6) and (2.7) yields the desired formula.

Now from (1.20) and (2.3) we have the integral representation of $C_{(p,\omega)}(\delta, Q, \vartheta)$:
\[ C_{(p,\omega)}(\delta, Q, \vartheta) = \frac{1}{n \cdot (\omega + i \vartheta)} \int_0^{2\pi} \frac{dw}{\{\theta(w) \cdot (\omega + i \vartheta))^2}. \quad (2.8) \]

This formula shows that the constant $C_{(p,\omega)}(\delta, Q, \vartheta)$ is independent of $p$ and $\delta$ when $Q$ is given by (1.19).

By computing the integral of the right-hand side on (2.8) we obtain the explicit value of $C_{(p,\omega)}(\delta, Q, \vartheta)$.

Lemma 2.2. We have: $C_{(p,\omega)}(\delta, Q, \vartheta) \neq 0$ if and only if
\[ \frac{\sin \theta}{1 + \cos \theta} < \frac{|n \cdot (\omega + i \vartheta)|}{(l - im) \cdot (\omega + i \vartheta)} < \frac{1 + \cos \theta}{\sin \theta} \]  \hspace{1cm} (2.9)

and then
\[ C_{(p,\omega)}(\delta, Q, \vartheta) = 2\pi \cos \theta \sin^2 \theta (n \cdot (\omega + i \vartheta))^{-3}. \]

Proof. Set
\[ A = l \cdot (\omega + i \vartheta), B = m \cdot (\omega + i \vartheta), \quad C = -\frac{1}{\tan \theta} n \cdot (\omega + i \vartheta) \]
and $z = e^{iw}$. One can write
\[
\vartheta(w) \cdot (\omega + i\vartheta) = A \cos w + B \sin w + C = A \frac{1}{2}(z + z^{-1}) - B \frac{1}{2}(z - z^{-1}) + C = \frac{1}{2z}(A - iB)z^2 + 2Cz + (A + iB).
\]

Here we claim
\[
A - iB \equiv (l - im) \cdot (\omega + i\vartheta) \neq 0.
\] (2.11)

Assume contrary that $A - iB = 0$. Since we have
\[
A - iB = l \cdot \omega + m \cdot \vartheta + i(l \cdot \vartheta - m \cdot \omega),
\]
it must hold that
\[
l \cdot \omega = -m \cdot \vartheta, \quad m \cdot \omega = l \cdot \vartheta.
\] (2.12)

Then we have
\[
(n \cdot \vartheta)^2 = |\vartheta|^2 - (l \cdot \vartheta)^2 - (m \cdot \vartheta)^2 = |\omega|^2 - (l \cdot \omega)^2 - (m \cdot \omega)^2 = (n \cdot \omega)^2.
\] (2.13)

On the other hand, we have
\[
0 = \omega \cdot \vartheta = (l \cdot \omega)(l \cdot \vartheta) + (m \cdot \omega)(m \cdot \vartheta) + (n \cdot \omega)(n \cdot \vartheta).
\]

Here by (2.12) one has $(l \cdot \omega)(l \cdot \vartheta) + (m \cdot \omega)(m \cdot \vartheta) = 0$. Thus one obtains
\[
0 = (n \cdot \omega)(n \cdot \vartheta).
\]

Now a combination of this and (2.13) yields $n \cdot \omega = 0$. However, by (1.20) this is impossible. Therefore we obtain the expression
\[
\vartheta(w) \cdot (\omega + i\vartheta) \equiv \frac{A - iB}{2z} f(z)|_{z = e^{iw}},
\] (2.14)

where
\[
f(z) = \left(\frac{z}{A - iB} + \frac{C}{A - iB} \right)^2 = \frac{C^2 - (A^2 + B^2)}{(A - iB)^2}.
\]
Here we write
\[
C^2 - (A^2 + B^2) = \frac{1}{\tan^2 \theta} (n \cdot (\omega + i \vartheta))^2 - \{(l \cdot (\omega + i \vartheta))^2 + (m \cdot (\omega + i \vartheta))^2\}
\]
\[
= \frac{1}{\tan^2 \theta} \left\{ (n \cdot \omega)^2 - (n \cdot \vartheta)^2 + 2i (n \cdot \omega) (n \cdot \vartheta) \right\}
- \left\{ (l \cdot \omega)^2 + (m \cdot \omega)^2 - (l \cdot \vartheta)^2 - (m \cdot \vartheta)^2 + 2i (l \cdot \omega) (l \cdot \vartheta)
+ 2i (m \cdot \omega) (m \cdot \vartheta) \right\}
\]
\[
= \left( \frac{1}{\tan^2 \theta} + 1 \right) (n \cdot \omega)^2 - \left( \frac{1}{\tan^2 \theta} + 1 \right) (n \cdot \vartheta)^2 + \frac{1}{\tan^2 \theta} 2i (n \cdot \omega) (n \cdot \vartheta)
- 2i \{ (l \cdot \omega) (l \cdot \vartheta) + (m \cdot \omega) (m \cdot \vartheta) \}
\]
\[
= \frac{1}{\sin^2 \theta} \left\{ (n \cdot \omega)^2 - (n \cdot \vartheta)^2 \right\} + \frac{1}{\sin^2 \theta} 2i (n \cdot \omega) (n \cdot \vartheta)
- 2i \{ (l \cdot \omega) (l \cdot \vartheta) + (m \cdot \omega) (m \cdot \vartheta) + (n \cdot \omega) (n \cdot \vartheta) \}
\]
\[
= \frac{1}{\sin^2 \theta} \left\{ (n \cdot \omega)^2 - (n \cdot \vartheta)^2 \right\} + \frac{1}{\sin^2 \theta} 2i (n \cdot \omega) (n \cdot \vartheta)
- 2i \omega \cdot \vartheta.
\]

Since \( \omega \cdot \vartheta = 0 \), we finally obtain
\[
C^2 - (A^2 + B^2) = \left( \frac{n \cdot (\omega + i \vartheta)}{\sin \theta} \right)^2.
\]

Now set
\[
z_\pm = \frac{(\cos \theta \pm 1) n \cdot (\omega + i \vartheta)}{\sin \theta (l - im) \cdot (\omega + i \vartheta)}.
\]

Then one gets the factorization
\[
f(z) = (z - z_+) (z - z_-).
\]

By (2.15) we have \( |z_+| > |z_-| \). Besides, from (2.2), (2.11) and (2.14) we have \( f(e^{i\omega}) \neq 0 \) for all \( w \in [0, 2\pi] \). This ensures that the complex numbers \( z_+ \) and \( z_- \) are not on the circle \( |z| = 1 \).

Thus from (2.14) one gets
\[
\int_0^{2\pi} \frac{d\omega}{(v(\omega) \cdot (\omega + i \vartheta))^2} = \frac{4}{i(A - iB)^2} \int_{|z|=1} \frac{dz}{(z - z_+)^2(z - z_-)^2}.
\]

(2.16)
The residue calculus yields

\[
\int_{|z| = 1} \frac{z \, dz}{(z - z_+)^2(z - z_-)^2} = \begin{cases} 
0 & \text{if } |z_-| > 1, \\
0 & \text{if } |z_-| < 1 \text{ and } |z_+| < 1, \\
2\pi i \left( \frac{z_+ + z_-}{(z_+ - z_-)^3} \right) & \neq 0 \text{ if } |z_-| < 1 < |z_+|.
\end{cases}
\]

And also (2.15) gives

\[
2\pi i \left( \frac{z_+ + z_-}{(z_+ - z_-)^3} \right) = 2\pi i \cdot 2 \cos \theta \cdot \frac{n \cdot (\omega + i\vartheta)}{\sin \theta (l - im) \cdot (\omega + i\vartheta)} \left( \frac{\sin \theta}{2} \right)^3 \left\{ \frac{(l - im) \cdot (\omega + i\vartheta)}{n \cdot (\omega + i\vartheta)} \right\}^3
\]

\[
= \frac{\pi i}{2} \cos \theta \sin^2 \theta \left\{ \frac{(l - im) \cdot (\omega + i\vartheta)}{n \cdot (\omega + i\vartheta)} \right\}^2
\]

\[
= \frac{\pi i}{2} \cos \theta \sin^2 \theta \left\{ \frac{\theta}{n \cdot (\omega + i\vartheta)} \right\}^2.
\]

Thus (2.16) yields

\[
\int_{0}^{2\pi} \frac{d\omega}{|\vartheta(n \cdot (\omega + i\vartheta))|^2} = 2\pi \cos \theta \sin^2 \theta \left\{ \frac{1}{n \cdot (\omega + i\vartheta)} \right\}^2
\]

provided \(|z_-| < 1 < |z_+|\).

From these together with (2.8) we obtain the desired conclusion. \(\square\)

Note that (2.10) is nothing but (1.24). Since (2.9) looks like a condition depending on the choice of \(l\) and \(m\) we further rewrite the number

\[
K(\vartheta; \omega, n) = \left| \frac{n \cdot (\omega + i\vartheta)}{(l - im) \cdot (\omega + i\vartheta)} \right|.
\]

We have

\[
|\omega \times \vartheta| = l \cdot \omega \cdot \vartheta - l \cdot \vartheta \cdot m \cdot \omega.
\]

Here we see that

\[
n \cdot (\omega \times \vartheta) = l \cdot \omega \cdot \vartheta - l \cdot \vartheta \cdot m \cdot \omega.
\]

Thus one has

\[
|\omega \times \vartheta| = l \cdot \omega \cdot \vartheta - l \cdot \vartheta \cdot m \cdot \omega.
\]

\[
= 2 - (n \cdot \omega)^2 - (n \cdot \vartheta)^2 + 2n \cdot (\omega \times \vartheta).
\]

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Therefore we obtain

\[ K(\vartheta; \omega, n) = \frac{\sqrt{(n \cdot \omega)^2 + (n \cdot \vartheta)^2}}{\sqrt{2 - (n \cdot \omega)^2 - (n \cdot \vartheta)^2 + 2n \cdot (\omega \times \vartheta)}}. \]

Besides, we have

\[ \frac{1 - \cos \theta}{\sin \theta} = \tan \frac{\theta}{2} \]

and

\[ \frac{1 + \cos \theta}{\sin \theta} = \frac{1}{\tan \frac{\theta}{2}}. \]

Thus (2.9) is equivalent to the condition

\[ \tan \frac{\theta}{2} < K(\vartheta; \omega, n) < \frac{1}{\tan \frac{\theta}{2}}. \tag{2.17} \]

Here consider the case \( \omega \times n \neq 0 \). Choose

\[ \vartheta = \frac{\omega \times n}{|\omega \times n|}. \]

We have \( \vartheta \cdot \omega = \vartheta \cdot n = 0 \) and \( \vartheta \in S^2 \).

Since we have

\[ n \cdot (\omega \times \vartheta) = -|\omega \times n| \]

and

\[ 1 = (n \cdot \omega)^2 + |\omega \times n|^2, \]

one gets

\[
2 - (n \cdot \omega)^2 - (n \cdot \vartheta)^2 + 2n \cdot (\omega \times \vartheta) \\
= 1 + |\omega \times n|^2 - 2|\omega \times n| \\
= (1 - |\omega \times n|)^2.
\]

Therefore, we obtain

\[ K(\omega \times n; \omega, n) = \frac{\omega \cdot n}{1 - |\omega \times n|}. \]
Note that we are considering $\omega$ satisfying (1.20). Let $\varphi$ denote the angle between $\omega$ and $n$. Under the condition $\omega \times n \neq 0$, we see that (1.20) is equivalent to the condition

$$0 < \varphi < \frac{\pi}{2} - \theta. \quad (2.18)$$

Then one can write

$$K(\omega \times n; \omega, n) = \frac{\cos \varphi}{1 - \sin \varphi} = \frac{1 + \sin \varphi}{\cos \varphi} = \frac{1 + \cos \left( \frac{\pi}{2} - \varphi \right)}{\sin \left( \frac{\pi}{2} - \varphi \right)} = \frac{1}{\tan \frac{\theta}{2} \left( \frac{\pi}{2} - \varphi \right)}.$$ 

Thus (2.18) gives

$$1 < K(\omega \times n; \omega, n) < \frac{1}{\tan \frac{\pi}{2}}. \quad (2.19)$$

Since we have $\tan \frac{\theta}{2} < 1$ for all $\theta \in \left( 0, \frac{\pi}{2} \right]$, (2.19) yields the validity of (2.17).

Next consider the case $\omega \times n = 0$. By (1.20) we have $\omega = n$. Then, for all $\vartheta$ perpendicular to $n$ satisfies

$$K(\vartheta; n, n) = 1.$$ 

This yields that (2.17) is valid for all $\theta \in \left( 0, \frac{\pi}{2} \right]$.

The results above are summarized as follows. Given $\omega \in S^2$ with (1.20) define the subset of $S^2$

$$K(\omega; n) = \left\{ \vartheta \in S^2 \mid \vartheta \cdot \omega = 0, \ K(\vartheta; \omega, n) \text{ satisfies (2.17)} \right\}.$$ 

Then, we have

- If $\omega \neq n$, then $\omega \times n \in K(\omega; n, \theta)$.
- If $\omega = n$, then $K(\omega; n, n) = \{ \vartheta \in S^2 \mid \vartheta \cdot \omega = 0 \} \equiv S(\omega)$.

Thus, any way the set $K(\omega; n, \theta)$ is not empty and clearly open with respect to the topology of the set $S(\omega)$ which is the relative topology of $S^2$. Besides, we can say more about $K(\omega; n, \theta)$. We claim set $K(\omega; n, \theta)$ is closed. For this, it suffices to show that if a sequence $\{ \vartheta_n \}$ of $K(\omega; n, \theta)$ converges to a point $\vartheta \in S(\omega)$, then $\vartheta \in K(\omega; n, \theta)$. This is proved as follows. By assumption, each $\vartheta_n$ satisfies

$$\tan \frac{\theta}{2} < K(\vartheta_n; \omega, n) < \frac{1}{\tan \frac{\pi}{2}}.$$ 

Taking the limit, we have

$$\tan \frac{\theta}{2} \leq K(\vartheta; \omega, n) \leq \frac{1}{\tan \frac{\pi}{2}}.$$
By (2.15) this is equivalent to $|z_+| \geq 1$ and $|z_-| \leq 1$. However, in the proof of lemma 2.2 we know that $|z_+| \neq 1$ and $|z_-| \neq 1$. Thus we have $|z_+| > 1$ and $|z_-| < 1$. This is equivalent to $\vartheta \in {\cal K}(\omega; \mathbf{n}, \theta)$.

Since $S(\omega)$ is connected, $\mathcal{K}(\omega; \mathbf{n}, \theta)$ is not empty, open and closed we conclude $\mathcal{K}(\omega; \mathbf{n}, \theta) = S(\omega)$.

This completes the proof of proposition 1.3.

3. Proof of corollaries 1.4 and 1.5

Note that $\omega$ satisfies (1.20).

3.1. On corollary 1.4

From (1.26) we have, if $\omega = \mathbf{n}$, then for all $\vartheta \in S(\omega)$

$$I(\omega, \vartheta) = 6 \tilde{\rho}(p(\omega))V(\vartheta)(\mathbf{n} \cdot \omega)^{-3}.$$  

On the other hand, if $\omega \neq \mathbf{n}$, then we have $\omega \times \mathbf{n} \neq 0$ (under the condition (1.20)) and

$$S(\omega) \cap S(\mathbf{n}) = \left\{ \pm \frac{\omega \times \mathbf{n}}{|\omega \times \mathbf{n}|} \right\} \neq \emptyset.$$  

Thus from (1.26) one sees that the function $I(\omega, \cdot)$ attains the value $6 \tilde{\rho}(p(\omega))V(\vartheta)(\mathbf{n} \cdot \omega)^{-3}$ and its absolute value coincides with $\max_{\vartheta \in S(\omega)} |I(\omega, \vartheta)|$. Besides, for $\vartheta = \pm \frac{\omega \times \mathbf{n}}{|\omega \times \mathbf{n}|} \in S(\omega)$ we have

$$I(\omega, \vartheta) = 6 \tilde{\rho}(p(\omega))V(\vartheta)\left(\mathbf{n} \cdot \omega \mp i \frac{|\omega \times \mathbf{n}|^2}{|\omega \times (\omega \times \mathbf{n})|}\right)^{-3}.$$  

This implies that function $I(\omega, \cdot)$ is not a constant one.

Thus one gets the assertion (a) and (1.27) in (b). For (1.28) it suffices to prove the following fact.

**Lemma 3.1.** Let the unit vectors $\omega$ and $\mathbf{n}$ satisfy $\omega \cdot \mathbf{n} \neq 0$. We have

$$\int_{S(\omega)} \frac{d\vartheta(\vartheta)}{(\mathbf{n} \cdot (\omega + i\vartheta))^3} = \pi(3(\mathbf{n} \cdot \omega)^2 - 1). \quad (3.1)$$

**Proof.** The right-hand side on (3.1) is invariant with respect to the change $\omega \rightarrow -\omega$, it is easy to see that the case $\omega \cdot \mathbf{n} < 0$ can be derived from the result in the case $\omega \cdot \mathbf{n} > 0$. Thus, hereafter we show the validity of (3.1) only for this case.

If $\mathbf{n} \cdot \omega = 1$, then $\omega = \mathbf{n}$. Thus $S(\omega) = S(\mathbf{n})$. Then for all $\vartheta \in S(\omega)$ we have $\mathbf{n} \cdot (\omega + i\vartheta) = 1$. This yields

$$\int_{S(\omega)} \frac{d\vartheta(\vartheta)}{(\mathbf{n} \cdot (\omega + i\vartheta))^3} = 2\pi.$$  

Thus the problem is the case when $\mathbf{n} \cdot \omega \neq 1$. Choose an orthogonal $3 \times 3$-matrix $A$ such that $A^T \omega = \mathbf{e}_1$. Introduce the change of variables $\vartheta = A\vartheta'$. We have $\vartheta \in S(\omega)$ if and only if $\vartheta' \in S(\mathbf{e}_1)$ and

$$\mathbf{n} \cdot (\omega + iA\vartheta') = \mathbf{n}' \cdot (\mathbf{e}_1 + i\vartheta'),$$

where $\mathbf{n}' = \mathbf{n} A$. Thus

$$\int_{S(\omega)} \frac{d\vartheta(\vartheta)}{(\mathbf{n} \cdot (\omega + i\vartheta))^3} = \int_{S(\mathbf{e}_1)} \frac{d\vartheta'(\vartheta')}{(\mathbf{n}' \cdot (\mathbf{e}_1 + i\vartheta'))^3}.$$  

By the invariance of (3.1) under the change $\mathbf{n} \rightarrow \mathbf{n}'$, we have

$$\int_{S(\mathbf{e}_1)} \frac{d\vartheta'(\vartheta')}{(\mathbf{n}' \cdot (\mathbf{e}_1 + i\vartheta'))^3} = \frac{1}{3}.\pi(3(\mathbf{n}' \cdot \mathbf{e}_1)^2 - 1).$$  

Since $\mathbf{n}' \cdot \mathbf{e}_1 = 1$, we obtain

$$\int_{S(\omega)} \frac{d\vartheta(\vartheta)}{(\mathbf{n} \cdot (\omega + i\vartheta))^3} = \pi(3(\mathbf{n} \cdot \omega)^2 - 1).$$
where $n' = A^T n \in S^2$.

Here we introduce the polar coordinates for $\vartheta' \in S(e_3)$:

$$\vartheta' = (\cos \varphi, \sin \varphi, 0)^T, \quad \varphi \in [0, 2\pi].$$

Then, we have

$$I \equiv \int_{S(\omega)} \frac{d\vartheta}{(n \cdot (\omega + i\vartheta))^3} = \int_0^{2\pi} \frac{d\varphi}{(n' \cdot (i \cos \varphi, i \sin \varphi, 1)^T)^3}$$

$$= \frac{1}{i} \int_0^{2\pi} \frac{d\varphi}{(n' \cdot (\cos \varphi, \sin \varphi, -i)^T)^3}$$

$$= \frac{1}{-i} \int_0^{2\pi} \frac{d\varphi}{(n' \cdot (\cos \varphi + b \sin \varphi - ic)^T)},$$

(3.2)

where $n' = (a, b, c)^T$. The numbers $a, b, c$ satisfy $a^2 + b^2 + c^2 = 1$ and $0 < c < 1$ since we have $c = n' \cdot e_3 = n \cdot \omega$. Thus $a^2 + b^2 \neq 0$. To compute the integral on the right-hand side of (3.2) we make use of the residue calculus.

The change of variables $z = e^{i\varphi}$ gives

$$a \cos \varphi + b \sin \varphi - ic$$

$$= \frac{1}{2} \left\{ a \left( z + \frac{1}{z} \right) + \frac{b}{i} \left( z - \frac{1}{z} \right) - 2ic \right\}$$

$$= \frac{1}{2z} \left\{ (a - ib)z^2 - 2ic + (a + ib) \right\}$$

$$= \frac{a - ib}{2z} \left\{ \left( \frac{ic}{a - ib} \right)^2 - \left( \frac{i}{a - ib} \right)^2 \right\}$$

$$= \frac{a - ib}{2z} (z - \alpha)(z - \beta),$$

(3.3)

where

$$\alpha = \frac{i(c + 1)}{a - ib}, \quad \beta = \frac{i(c - 1)}{a - ib}.$$

Since $1 - c < 1 + c$ and $a \cos \varphi + b \sin \varphi - ic \neq 0$ for $z = e^{i\varphi}$, we have $|\beta| < 1 < |\alpha|$.

Substituting (3.3) into (3.2) and using $d\varphi = \frac{dz}{iz}$, we have

$$I = \frac{1}{i} \int_{|z|=1} \frac{2^3}{(a - ib)^3} \frac{z^3}{(z - \alpha)^3(z - \beta)^3} \cdot \frac{dz}{iz}$$

$$= \left( \frac{2}{a - ib} \right)^3 \int_{|z|=1} \frac{z^2 \cdot dz}{(z - \alpha)^3(z - \beta)^3}. $$

(3.4)
The residue calculus yields
\[
\int_{|z|=1} \frac{z^2 \, dz}{(z - \alpha)^3(z - \beta)^3} = 2\pi i \operatorname{Res}_{z=\beta} \left( \frac{z^2}{(z - \alpha)^3(z - \beta)^3} \right) = 2\pi i \cdot \frac{\alpha^2 + 4\alpha\beta + \beta^2}{(\beta - \alpha)^2}.
\]

Here we have the expression
\[
\alpha - \beta = \frac{2i}{a - ib}
\]
and
\[
\alpha^2 + 4\alpha\beta + \beta^2 = \frac{(c + 1)^2 + 4(c^2 - 1) + (c - 1)^2}{(a - ib)^2} = -\frac{2(3c^2 - 1)}{(a - ib)^2}.
\]

Thus from (3.4) and (3.5) we obtain
\[
I = -2\pi \left( \frac{a - ib}{2} \right)^2 (\alpha^2 + 4\alpha\beta + \beta^2) = \pi(3c^2 - 1).
\]

This completes the proof of (3.1).

\[\square\]

### 3.2. On Corollary 1.5

Let us explain the uniqueness of the solution of the quintic equation (1.30) in \( \left[ \frac{1}{\sqrt{3}}, 1 \right] \).

From (1.27)–(1.29) we have
\[
\left| \int_{S_\omega} I(\omega, \vartheta) \, d\vartheta \right| \leq \pi \max_{\vartheta \in S_\omega} |I(\omega, \vartheta)| = (n \cdot \omega)^3 (3(n \cdot \omega)^2 - 1)
\]
and thus
\[
0 < \left| \int_{S_\omega} I(\omega, \vartheta) \, d\vartheta \right| \leq 2.
\]

Since \( \left[ \frac{1}{\sqrt{3}}, 1 \right] \ni \gamma \mapsto \gamma^3 (3\gamma^2 - 1) \in [0, 2] \) is bijective, the solution of quintic equation (1.30) in \( \left[ \frac{1}{\sqrt{3}}, 1 \right] \) is unique and its solution is just \( \gamma = n \cdot \omega \).

The formulae (1.31) and (1.32) are derived as follows. A combination of (1.26) and (1.28) yields
\[
(n \cdot \omega + in \cdot \vartheta)^3 = T(\omega, \vartheta),
\]
where $T(\omega, \vartheta)$ is given by (1.33). By expanding the left-hand side, we obtain immediately the desired formulae.

4. Application to an inverse obstacle problem

As pointed out in [4] the enclosure method developed here can be applied also to an inverse obstacle problem in three dimensions governed by the equation

$$
\Delta u + k^2 n(x)u = 0, \quad x \in \Omega,
$$

where $k$ is a fixed positive number. We assume that $\partial \Omega \in C^\infty$, for simplicity. Both $u$ and $n$ can be complex-valued functions.

In this section we assume that $n(x)$ takes the form $n(x) = 1 + F(x)$, $x \in \Omega$, where $F = F_{\rho, D}(x)$ is given by (1.7). We assume that $\rho \in L^\infty(D)$ instead of $\rho \in L^2(D)$ and that $u \in H^2(\Omega)$ is an arbitrary non trivial solution of (4.1) at this stage. We never specify the boundary condition of $u$ on $\partial \Omega$. By the Sobolev imbedding theorem [3] one may assume that $u \in C^{\alpha, \alpha}(\Omega)$ with $0 < \alpha \leq \frac{1}{2}$.

In this section we consider.

Problem 2. Extract information about the singularity of $D$ from the Cauchy data of $u$ on $\partial \Omega$.

We encounter this type of problem, for example, $u$ is given by the restriction to $\Omega$ of the total wave defined in the whole space and generated by a point source located outside of $\Omega$ or a single plane wave coming from infinity. The surface where the measurements are taken is given by $\partial \Omega$ which encloses the penetrable obstacle $D$ with a different reflection index $1 + \rho$, $\rho \neq 0$. See [2] for detailed information about the direct problem itself. Any way we start with having the Cauchy data of an arbitrary (nontrivial) $H^2(\Omega)$ solution of (4.1).

Using the Cauchy data of $u$ on $\partial \Omega$, we introduce the indicator function

$$
I_{\omega, \vartheta}(\tau) = \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} v - \frac{\partial v}{\partial \nu} u \right) dS,
$$

where the function $v = v(x), x \in \mathbb{R}^3$ is given by

$$
v = e^{i(\omega + i\sqrt{\tau^2 + k^2})}, \quad \tau > 0
$$

and $\vartheta \in S(\omega)$. And also its derivative with respect to $\tau$ is given by the formula

$$
I'_{\omega, \vartheta}(\tau) = \int_{\partial \Omega} \left( \frac{\partial u}{\partial \nu} v_\tau - \frac{\partial v_\tau}{\partial \nu} u \right) dS,
$$

where

$$
v_\tau = \partial_\tau v = \left\{ x \cdot \left( \omega + i \frac{\tau}{\sqrt{\tau^2 + k^2}} \right) \right\} v.
$$
As done in the proof of theorem 1.1 integration by parts yields
\[ I_{\omega, \delta}(\tau) = -k^2 \int_D \frac{\rho(x)}{\omega} u(x) v(x) \, dx \]
and
\[ I'_{\omega, \delta}(\tau) = -k^2 \int_D \frac{\rho(x)}{\omega} (u(x) v'(x) - u'(x) v(x)) \, dx. \]

Thus this can be viewed as the case \( \rho(x) \) in problem 1 is given by \(-k^2 \rho(x) u(x) \) and \( \tilde{\rho}(x) \) in definition 1.2 by \(-k^2 \tilde{\rho}(x) u(x) \).

Thus we obtain

**Theorem 4.1.** Let \( \omega \) be regular with respect to \( D \) and assume that \( D \) has a conical singularity from direction \( \omega \). Assume that \( n(x) - 1 = F_{\rho, D}(x) \) is active at \( p(\omega) \) in the sense of definition 1.2. Then, we have
\[ \tau^3 e^{-\tau h_D(\omega)} e^{-i(\sqrt{\tau^2 + k^2} \rho(\omega)} \, \hat{I}_{\omega, \delta}(\tau) = -k^2 \tilde{\rho}(p(\omega)) u(p(\omega)) C_{(p(\omega), \omega)}(\delta, Q, \vartheta) + O(\tau^{-\alpha}) \]
and
\[ \tau^3 e^{-\tau h_D(\omega)} e^{-i(\sqrt{\tau^2 + k^2} \rho(\omega)} \, \hat{I}'_{\omega, \delta}(\tau) = -k^2 \tilde{\rho}(p(\omega)) u(p(\omega))(h_D(\omega) + i p(\omega) \cdot \vartheta) C_{(p(\omega), \omega)}(\delta, Q, \vartheta) + O(\tau^{-\alpha}). \]

The remainder \( O(\tau^{-\alpha}) \) is uniform with respect to \( \vartheta \in S(\omega) \).

Thus under the same assumptions as theorem 4.1, for each \( \vartheta \in S(\omega) \) one can calculate
\[ I(\omega, \vartheta) \equiv -k^2 \tilde{\rho}(p(\omega)) u(p(\omega)) C_{(p(\omega), \omega)}(\delta, Q) \]
via the formula
\[ I(\omega, \vartheta) = \lim_{\tau \to \infty} \tau^3 e^{-\tau h_D(\omega)} e^{-i(\sqrt{\tau^2 + k^2} \rho(\omega)} \, \hat{I}_{\omega, \delta}(\tau) \tag{4.4} \]
by using the Cauchy data of \( u \) on \( \partial \Omega \) if \( p(\omega) \) is known.

And also we have

**Theorem 4.2.** Let \( \omega \) be regular with respect to \( D \). Assume that \( D \) has a conical singularity from direction \( \omega \); \( n(x) - 1 = F_{\rho, D}(x) \) is active at \( p(\omega) \) in the sense of definition 1.2 and the value of \( u \) at \( p(\omega) \) satisfies
\[ u(p(\omega)) \neq 0. \tag{4.5} \]
If the direction \( \vartheta \in S(\omega) \) satisfies the condition (1.15), then all the formulae (1.16)–(1.18) for the indicator function defined by (4.2) together with its derivative (4.3) are valid.

Note that the assumption (4.5) ensures \( u \neq 0 \). See appendix A for an example of \( u \) satisfying (4.5).

The following corollaries corresponds to corollaries 1.1 and 1.2.

**Corollary 4.1.** Let \( \omega \) be regular with respect to \( D \). Under the same assumptions as those in theorem 4.2 the point \( p(\omega) \) is uniquely determined by the Cauchy data of \( u \) on \( \partial \Omega \).
Corollary 4.2. Let $u \in H^2(\Omega)$ be a solution of (4.1). Assume that $D$ is given by the inside of a convex polyhedron and that in a neighbourhood of each vertex $p$ of $D$, the $D$ coincides with the inside of a tetrahedron with apex $p$ and that $n - 1 = F_{\rho,D}$ given by (1.7) is active at $p$ and the value of $u$ at $p$ satisfies (4.5). Then, all the formulae (1.16)–(1.18) for the indicator function defined by (4.2) together with its derivative (4.3) are valid for all $\omega$ regular with respect to $D$ and $\vartheta \in S(\omega)$. Besides, the Cauchy data of $u$ on $\partial\Omega$ uniquely determines $D$.

The following result is an extension of theorem 4.1 in [4] to three-dimensional case.

Corollary 4.3. Let $u \in H^2(\Omega)$ be a solution of (4.1). Let $\omega \in S^2$ be regular with respect to $D$. Assume that: $D$ has a circular cone singularity at $p = p(\omega)$; $n(x) - 1 = F_{\rho,D}(x)$ is active at $p(\omega)$ in the sense of definition 1.2 and the value of $u$ at $p(\omega)$ satisfies (4.5). Choose two linearly independent vectors $\vartheta = \vartheta_1$ and $\vartheta_2$ in $S(\omega)$. Then, the point $p(\omega)$ itself and thus $h_2(\omega) = p(\omega) \cdot \omega$ can be extracted from the Cauchy data of $u$ on $\partial\Omega$ by using the formula

$$p(\omega) \cdot \omega + ip(\omega) \cdot \vartheta_j = \lim_{r \to \infty} \frac{F_{\rho(\omega),j}(\tau)}{L_{\omega(\omega),j}(\tau)}, \quad j = 1, 2. \quad (4.6)$$

By virtue of the formula (1.24), the function $I(\omega, \cdot)$ has the expression

$$I(\omega, \vartheta) = -6k^2 \tilde{\rho}(p(\omega))u(p(\omega))V(\vartheta)(n \cdot (\omega + i\vartheta))^{-3}. \quad (4.7)$$

Similarly to corollary 1.4 formula (4.7) yields immediately the following results.

Corollary 4.4. Let $u \in H^2(\Omega)$ be a solution of (4.1). Let $\omega \in S^2$ be regular with respect to $D$. Assume that: $D$ has a circular cone singularity at $p(\omega)$ such as $D \cap B_r(p(\omega)) = V_{\rho(\omega)}(-n, \vartheta_1) \cap B_r(p(\omega))$ with $\epsilon > 0$.

(a) Assume that $n(x) - 1 = F_{\rho,D}(x)$ is active at $p(\omega)$ in the sense of definition 1.2 and the value of $u$ at $p(\omega)$ satisfies (4.5). The vector $\omega$ coincides with $n$ if and only if the function $I(\omega, \cdot)$ is a constant function.

(b) The vector $n$ and $\vartheta$ of $V_{\rho(\omega)}(-n, \vartheta)$ and $\tilde{\rho}(p(\omega))u(p(\omega))$ satisfy the following two equations:

$$6k^2 \tilde{\rho}(p(\omega))u(p(\omega))V(\vartheta) = (n \cdot \omega)^3 \max_{\vartheta \in S(\omega)} |I(\omega, \vartheta)|; \quad (4.8)$$

$$-6k^2 \tilde{\rho}(p(\omega))u(p(\omega))V(\vartheta)(3(n \cdot \omega)^2 - 1) = \frac{1}{\pi} \int_{S(\omega)} I(\omega, \vartheta) ds(\vartheta). \quad (4.9)$$

Using the equations (4.7)–(4.9) one gets the following corollary.

Corollary 4.5. Let $u \in H^2(\Omega)$ be a solution of (4.1). Let $\omega \in S^2$ be regular with respect to $D$. Assume that: $D$ has a circular cone singularity at $p(\omega)$ such as $D \cap B_r(p(\omega)) = V_{\rho(\omega)}(-n, \vartheta) \cap B_r(p(\omega))$ with $\epsilon > 0$. Assume that $n(x) - 1 = F_{\rho,D}(x)$ is active at $p(\omega)$ in the sense of definition 1.2 and the value of $u$ at $p(\omega)$ satisfies (4.5). Assume also that $\omega \approx n$ in the sense that (1.29) holds. Then, we have the completely same statement and formulae as those of corollary 1.5.

Note that under the same assumptions as corollary 4.5, one can finally calculate the quantity

$$\tilde{\rho}(p(\omega))u(p(\omega))V(\vartheta) \quad (4.10)$$

and $n$ from the Cauchy data of $u$ on $\partial\Omega$. Since the steps for the calculation are similar to the steps presented in subsection 1.2 for the inverse source problem, we omit its description.
However, it should be noted that, in addition, if \( \tilde{\rho}(p(\omega)) \) is known to be a real number, then one can recover the phase of the complex number \( u(p(\omega)) \) modulo \( 2\pi n \), \( n = 0, \pm 1, \pm 2, \ldots \) from the computed value (4.10).

**Remark 4.1.** One can apply the result in [6] to the computation of the value \( u(p(\omega)) \) itself. For simplicity we assume that \( \Omega \) is convex like a case when \( \Omega = B_R(x_0) \) centered at a point \( x_0 \) with a large radius \( R \). From formula (4.6) we know the position of \( p(\omega) \) and thus the domain \( \Omega \cap \{ x \in \mathbb{R}^3 | x \cdot \omega > x \cdot p(\omega) \} \). Because of the continuity of \( u \) on \( \overline{\Omega} \), one has, for a sufficiently small \( \epsilon > 0 \)

\[
u(p(\omega)) \approx u(p(\omega) + \epsilon \omega).
\]

Since the point \( p(\omega) + \epsilon \omega \in \Omega \cap \{ x \in \mathbb{R}^3 | x \cdot \omega > x \cdot p(\omega) \} \) and therein \( u \in H^2 \) satisfies the Helmholtz equation \( \Delta u + k^2 u = 0 \), one can calculate the value \( u(p(\omega) + \epsilon \omega) \) itself from the Cauchy data of \( u \) on \( \partial \Omega \cap \{ x \in \mathbb{R}^3 | x \cdot \omega > x \cdot p(\omega) \} \) by using theorem 1 in [6].

5. **Final remark**

All the results in this paper can be extended also to the case when the governing equation of the background medium is given by a Helmholtz equation with a known coefficient \( n_0(x) \). It means that if one considers, instead of (1.6) and (4.1) the equations

\[
\Delta u + k^2 n_0(x) u = F_{\rho,D}(x), \quad x \in \Omega
\]

and

\[
\Delta u + k^2 (n_0(x) + F_{\rho,D}(x)) u = 0, \quad x \in \Omega,
\]

respectively, then one could obtain all the corresponding results.

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**Data availability statement**

No new data were created or analysed in this study.

**Appendix A. On condition (4.5)**

As suggested in [4] the condition (4.5) can be satisfied if \( k \) is sufficiently small under the situation when \( u \) is given by the restriction onto \( \Omega \) of the total field \( U \) in the whole space scattering problem generated by, for example, the point source located at a point \( z \) in \( \mathbb{R}^3 \setminus \overline{\Omega} \).

The \( U \) has the expression \( U = \Phi(x, z) + w_\zeta(x) \), where

\[
\Phi(x, z) = \frac{1}{4\pi} \frac{e^{i|z|}}{|x - z|}, \quad x \in \mathbb{R}^3 \setminus \{z\}
\]
and \( w_z \in H^2_{\text{local}}(\mathbb{R}^3) \) is the unique solution of the inhomogeneous Helmholtz equation

\[
\Delta w_z + k^2 w_z + k^2 F(x)(w_z + \Phi(x, z)) = 0, \quad x \in \mathbb{R}^3
\]

with the outgoing Sommerfeld radiation condition

\[
\lim_{r \to \infty} r \left( \frac{\partial}{\partial r} w_z(x) - ik w_z(x) \right) = 0,
\]

where \( r = |x| \) and \( F = F_{\rho, D} \) is given by (1.7). See [2] for the solvability.

Here we claim

**Proposition A.** Let \( 0 < R_1 < R_2 \) satisfy \( D \subset B_{R_2}(z) \backslash B_{R_1}(z) \). Let \( M > 0 \) and \( R > 0 \) satisfy \( |D| \leq M \) and \( \|\rho\|_{L^\infty(D)} \leq R \), respectively.

If \( k \) satisfies the system of inequalities

\[
C \equiv \frac{3k^2 R}{2} \left( \frac{M}{4\pi} \right)^{2/3} < 1
\]

and

\[
\frac{C}{1 - C} < \frac{R_1}{R_2},
\]

then, for all \( x \in \partial D \) we have

\[
|U(x)| \geq \frac{1}{4\pi} \left( \frac{1}{R_2} - \frac{C}{1 - C} \frac{1}{R_1} \right).
\]

**Proof.** Note that \( w_z \in C^{0,\alpha}(\Omega) \) with \( 0 < \alpha \leq \frac{1}{2} \) by the Sobolev imbedding theorem. It is well known that the function \( w_z \) satisfies the Lippman-Schwinger equation

\[
w_z(x) = k^2 \int_D \Phi(x, y) \rho(y) w_z(y) dy + k^2 \int_D \Phi(x, y) \Phi(y, z) \rho(y) dy
\]

and thus, for all \( x \in \partial D \) we have

\[
|w_z(x)| \leq \frac{k^2 R}{4\pi} \left( \|w_z\|_{L^\infty(D)} + \frac{1}{4\pi R_1} \right) \int_D \frac{dy}{|x - y|}.
\]

Let \( \epsilon > 0 \). We have

\[
\int_D \frac{dy}{|x - y|} = \int_{D \cap B_\epsilon(x)} \frac{dy}{|x - y|} + \int_{D \setminus B_\epsilon(x)} \frac{dy}{|x - y|}
\]

\[
\leq \int_{B_\epsilon(x)} \frac{dy}{|x - y|} + \frac{|D|}{\epsilon}
\]

\[
\leq 2\pi \epsilon^2 + \frac{M}{\epsilon}.
\]

Choose \( \epsilon \) in such a way that this right-hand side becomes minimum, that is,

\[
\epsilon = \left( \frac{M}{4\pi} \right)^{1/3}.
\]
Then one gets
\[
\int_{\mathcal{D}} \frac{dy}{|x - y|} \leq 6\pi \left( \frac{M}{4\pi} \right)^{2/3}.
\]
Thus this together with (A.4) yields
\[
(1 - C) \|w_C\|_{L^\infty(\mathcal{D})} \leq \frac{C}{4\pi R_1}.
\]
This together with the estimate
\[
|U(x)| \geq \frac{1}{4\pi R_2} - \|w_C\|_{L^\infty(\mathcal{D})}
\]
yields the desired estimate (A.3) under the assumptions (A.1) and (A.2).

Note that since \(R_2 > R_1\), the set of inequalities (A.1) and (A.2) are equivalent to the single inequality
\[
C < \frac{R_1}{R_1 + R_2}.
\]
(A.5)
Thus if we choose \(k^2\) sufficiently small in the sense of (A.5), then we have, for all \(x \in \mathcal{T}\)
\[
|u(x)| \geq \frac{1}{4\pi} \left( \frac{1}{R_2} - \frac{C}{1 - C R_1} \right) > 0.
\]
Thus the condition (4.5) for \(u = U|_{\Omega}\) is satisfied. The choice of \(k\) depends only on the a-priori information about \(\mathcal{D}\) and \(\rho\) described by \(R_1, R_2, M\) and \(R\).

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