ON AHLFORS CURRENTS

DINH TUAN HUYNH AND SONG-YAN XIE

ABSTRACT. We answer a basic question in Nevanlinna theory that Ahlfors currents associated to the same entire curve may be nonunique. Indeed, we will construct one exotic entire curve $f: \mathbb{C} \to X$ which produces infinitely many cohomologically different Ahlfors currents. Moreover, concerning Siu’s decomposition, for an arbitrary $k \in \mathbb{Z}_+ \cup \{\infty\}$, some of the obtained Ahlfors currents have singular parts supported on $k$ irreducible curves. In addition, they can have nonzero diffuse parts as well. Lastly, we provide new examples of diffuse Ahlfors currents on the product of two elliptic curves and on $\mathbb{P}^2(\mathbb{C})$, and we show cohomologically elaborate Ahlfors currents on blow-ups of $X$.

1. Introduction

Let $X$ be a compact complex manifold equipped with an area form $\omega$. Let $f: \mathbb{C} \to X$ be a nonconstant entire holomorphic curve. An associated Ahlfors current of $f$ is a positive closed current of bidimension $(1, 1)$ obtained as the weak limit of a certain sequence of positive currents of bounded masses

$$\left\{ \frac{[f(\mathbb{D}_{r_n})]}{\text{Area}_\omega f(\mathbb{D}_{r_n})} \right\}_{n \geq 1},$$

where $\mathbb{D}_{r_n}$ are discs of increasing radii $r_n \to \infty$ centered at the origin. Here, to ensure that such a limit current is closed, the sequence $\{r_n\}$ is chosen in such a way that the lengths of boundaries of the discs are asymptotically negligible compared with their areas, namely

$$\lim_{n \to \infty} \frac{\text{Length}_\omega (f(\partial \mathbb{D}_{r_n}))}{\text{Area}_\omega (f(\mathbb{D}_{r_n}))} = 0.$$

By Ahlfors’ lemma (c.f. [Bru99, Nev70]), for each positive number $\epsilon > 0$, the set

$$\left\{ r > 0 : \frac{\text{Length}_\omega (f(\partial \mathbb{D}_r))}{\text{Area}_\omega (f(\mathbb{D}_r))} \geq \epsilon \right\}$$

is of finite measure with respect to $\frac{dr}{r}$. Hence the above “length-area” condition is satisfied for most choices of increasing radii. Moreover, given a sequence of radii $\{r_n\}_{n \geq 1}$ with $r_n \to \infty$, after some small perturbation by scaling and extracting a subsequence, one can always obtain an Ahlfors current for $f$.

Ahlfors currents and their analogs obtained by taking the logarithmic average $\int \frac{dt}{t} (\cdot)$, called Nevanlinna currents, are fundamental tools in studying complex hyperbolicity, value distribution theory and complex dynamical systems. Notably, they played a crucial role in the work McQuillan [McQ98] on Green-Griffiths’ conjecture for algebraic surfaces of general type having positive Segre class (see also [Bru99] for a simplified proof by Brunella). By employing Ahlfors currents, Duval [Duv08] gave a quantitative version of the classical Brody’s Lemma and obtained a characterization of complex hyperbolicity in terms of linear isoperimetric inequality for holomorphic discs. Using such currents, some geometric refinement of the classical Cartan’s Second Main Theorem [DH18], as well as the high dimensional Weierstrass-Casorati Theorem [HVar] were obtained. The reader is also referred to [DS18] for recent key applications in complex dynamical systems.

Since Ahlfors currents and Nevanlinna currents encode geometric information of their original entire curves, several results in value distribution theory can be presented in terms of intersections of corresponding cohomology classes. For example, the First Main Theorem of Nevanlinna theory can be expressed as an inequality between the algebraic intersection and the geometric intersection (c.f. [DH18]).

---

2010 Mathematics Subject Classification. 32H30, 32Q45, 32U40, 32C30, 30D35.

Key words and phrases. Nevanlinna theory, entire curves, Ahlfors currents, Weierstrass canonical product, Jensen formula, Siu’s decomposition.
Note that in certain specific situations, Ahlfors currents (or Nevanlinna currents) from some holomorphic curve are unique [DS14, DH18, DS18, DV20], which subsequently leads to several interesting results. Therefore, it is natural and fundamental to ask generally

**Question 1.1.** Are all Ahlfors currents associated to the same entire curve cohomologically equivalent?

The study of such currents is itself of independent interest. By Siu’s decomposition Theorem [Siu74], an Ahlfors current \( T \) can be written as the sum \( T = T_{\text{Sing}} + T_{\text{Diff}} \), where the singular part \( T_{\text{Sing}} = \sum_{\ell \in J} c_{\ell} \cdot |C_{\ell}| \) is some positive linear combination \( (c_{\ell} > 0; I \subset \mathbb{Z}^{+}) \) of currents of integration on irreducible algebraic curves \( C_{\ell} \), and where the diffuse part \( T_{\text{Diff}} \) is a positive closed \((1, 1)\)-current having zero Lelong number along any algebraic curve. If the singular part \( T_{\text{Sing}} \) is nontrivial, Duval [Duv06] showed that any irreducible curve \( C_{\ell} \) above must be rational or elliptic (see also [Duv17b] for a local version). In [dC13], da Costa gave an example of entire curve in the projective plane whose associated Ahlfors current is supported in some line. This construction can be modified to produce Ahlfors currents supported on a rational or an elliptic curve [Huy16, Theorem 2.6.1]. On the other hand, we would like to mention the following unsolved question, which had been considered by Brunella [Bru99, page 200].

**Question 1.2.** Is there any Ahlfors current from an entire curve such that both of its singular part and diffuse part are nontrivial?

In this paper, we answer the above two questions by constructing explicit examples.

**Theorem 1.3.** There exists an entire curve producing cohomologically different Ahlfors currents.

By Siu’s decomposition, Ahlfors currents with nontrivial singular parts can be distinguished as different types by the data \( (|I| \in \mathbb{Z}^{+} \cup \{\infty\}, T_{\text{Diff}} \text{ is trivial / nontrivial}) \).

**Theorem 1.4.** There exists an entire curve producing all types of Ahlfors currents with nontrivial singular parts.

**Remark 1.5.** The above two results also hold true for Nevanlinna currents, see Subsection 7.2.

Lastly, it is natural to seek Ahlfors (Nevanlinna) currents with trivial singular part. Examples of such currents are known to exist on \( \mathbb{P}^{2}(\mathbb{C}) \), by looking at the Levi-flat real hypersurface in \((\mathbb{C}^{*})^{2}\) defined by the equation \(|x| = |y|^\alpha\), where \( \alpha \) is an irrational real number (c.f. [BD01, page 262]). Indeed, this real hypersurface is foliated by entire curves, while its closure in \( \mathbb{P}^{2}(\mathbb{C}) \) contains no algebraic curve. In [DS14] there are more examples of holomorphic curves whose associated Ahlfors (Nevanlinna) currents are diffuse and unique. In Section 7, we show new examples of diffuse Ahlfors currents on the product of two elliptic curves and on \( \mathbb{P}^{2}(\mathbb{C}) \), see Propositions 7.1, 7.2.

We now outline the ideas and the structure of this paper. As a matter of fact, our source of inspiration is an example of da Costa [dC13] about a nondegenerate entire curve clustering to a line in \( \mathbb{P}^{2}(\mathbb{C}) \) (see also [DH18] for more discussions). In Section 2, we start with an elliptic curve \( \mathcal{C} = \mathbb{C}/\Gamma \) equipped with a negative line bundle \( \mathcal{L} \). For some large integer \( m \gg 1 \), we construct a section \( s_{m} \) of \( \pi_{0}^{*}\mathcal{L}^{m} \) having large exponential growth of order 2, where \( \pi_{0} : \mathbb{C} \rightarrow \mathcal{C} \) is the canonical projection. The surface \( X \) is obtained by taking the geometric projectivization \( \mathbb{P}(\mathcal{L}^{m} \oplus \mathbb{C}) =: X \) of the vector bundle \( \mathcal{L}^{m} \oplus \mathbb{C} \) on \( \mathcal{C} \). Thus the section \( s_{m} \) induces a holomorphic map \( f_{0} : \mathbb{C} \rightarrow X \) clustering to the curve \( C_{\infty} \) corresponding to the “infinity section” of \( \pi_{0}^{*}\mathcal{L}^{m} \). To generate Ahlfors currents with larger singular supports, we hence modify the original section \( s_{m} \) by multiplying it with a Weierstrass canonical product \( \psi(z) = \prod_{\lambda \in \Lambda} (1 - \frac{z}{\lambda}) e^{\frac{z}{\lambda} + \frac{2\pi i}{\lambda^{2}}} \), whose zero locus \( \Lambda \) is distributed in a delicate pattern, to make sure that the new section \( \psi \cdot s_{m} \) induces an entire curve \( f : \mathbb{C} \rightarrow X \) producing Ahlfors currents with more singularities. Indeed, for every \( \lambda \in \Lambda \), since \( \psi \cdot s_{m}(\lambda) = 0 \), \( f(\lambda) \) touches the curve \( C_{0} := \mathcal{C} \times \{0 \oplus 1\} \), defined by the zero section of \( \pi_{0}^{*}\mathcal{L}^{m} \), at \( ([\lambda], [0 \oplus 1]) \).

The idea is that, the image of a small neighborhood of \( \lambda \in \Lambda \subset \mathbb{C} \) by \( f \) shall contribute moderate area \( O(1) \) near the fiber \( \mathbb{P}^{1}_{[\lambda]} \subset X \) over \( [\lambda] \in \mathcal{C} \), and once there are sufficiently many \( \lambda' \in \Lambda \) mapping to the same class \( [\lambda] \) by \( \pi_{0} \), the area of the image of \( f \) should spend a positive portion about \( \mathbb{P}^{1}_{[\lambda]} \), hence the Ahlfors currents should charge positive mass there. See the picture below for illustration.
Nevertheless, to make sense of this idea, we need to show, first of all, that the growth of $\psi$ is neither too rapid nor too slow, which will be accomplished in Section 3, by means of the Stirling formula as well as the symmetry of the lattice $\Gamma$. Consequently, in Section 4, we can manipulate Jensen’s formula to evaluate various areas, which distinguish the singularities of the Ahlfors currents. In Section 5, we present an algorithm for constructing the zero locus $\Lambda$, which is designed for the proofs of the main theorems in Section 6. In Section 7, we provide new examples of diffuse Ahlfors currents. Moreover, we show cohomologically elaborate Ahlfors currents on surfaces obtained by blowing-up $X$.

Convention: Throughout this paper, $K$ denotes positive numbers which are uniformly bounded from both sides $0 < K_1 < K < K_2 < \infty$. Further, notation $K_{n_1, n_2, n_3}$ indicates dependence on parameters $n_1, n_2, n_3$. The notation $\mathbb{D}(a, r)$ means the disc centered at $a$, in $\mathbb{C}$ or in the elliptic curve $\mathbb{C}/\Gamma$, with the radius $r$. When $a = 0 \in \mathbb{C}$, we write $\mathbb{D}_r$ instead of $\mathbb{D}(0, r)$. The differential operator $d^c$ is short for $\sqrt{-1}/4\pi (\overline{\partial} - \partial)$.

Acknowledgements. We would like to address our profound gratitude to Professor Julien Duval for introducing us to the subject and for many fruitful discussions from which we learned rich ideas. We are grateful to Professor Sibony for sharing the reference [DS14] and for suggesting related problems. We thank Professor Yusaku Tiba for his question about cohomology classes of Ahlfors currents. We thank Professor Joël Merker, Dr. Duc-Viet Vu and Dr. Ruiran Sun for valuable comments and suggestions. S.-Y. Xie is partially supported by the NSFC Grant No. 11688101. Part of this paper was written during his visit to Tianyuan Mathematical Center in Southeast China / Xiamen University invited by Professor Chunhui Qiu. He is grateful to the colleagues there for hospitality and excellent working conditions. D. T. Huynh is grateful to Academy of Mathematics and Systems Science in Beijing for enhanced scientific ambience. He also wants to acknowledge partial support from the Core Research Program of Hue University, Grant No. NCM.DHH.2020.15.

2. Construction

Fix a smooth elliptic curve $\mathcal{C} = \mathbb{C}/\Gamma$, where the lattice $\Gamma := \mathbb{Z} \oplus \mathbb{Z}\sqrt{-1}$ is chosen to affiliate the arguments later. We can find a negative line bundle $\mathcal{L}$ on $\mathcal{C}$ equipped with some hermitian metric $h'$ having strictly negative curvature. Now comparing with the Kähler form $d^c \left( |z|^2 \right)$ on $\mathcal{C}$ descending from the canonical projection $\pi_0 : \mathbb{C} \longrightarrow \mathbb{C}/\Gamma$, the curvature of $h'$ is cohomologous to $-2\alpha \, d^c \left( |z|^2 \right)$ for some positive constant $\alpha$, namely their difference is of the form $d^c \varphi$ for some smooth real function $\varphi$ on $\mathbb{C}/\Gamma$. Therefore, replacing the initial metric $h'$ by $h'e^\varphi =: h$, the curvature becomes $\Theta_h = -2\alpha \, d^c \left( |z|^2 \right)$. Noting that the line bundle $\pi_0^* \mathcal{L}$ on $\mathbb{C}$ is holomorphically trivial, it has a nowhere vanishing holomorphic section $k$, which by Lelong-Poincaré equation satisfies that $d^c \left( \log \|k\|^2_{\pi_0^* h} \right) = 2\alpha \, d^c \left( |z|^2 \right)$. Hence $\log \|k\|^2_{\pi_0^* h} - 2\alpha |z|^2$ is a harmonic function on $\mathbb{C}$, hence can be written as the real part of some holomorphic function $g$. Therefore, the modified section $s := e^{-g/2} \, k$ of $\pi_0^* \mathcal{L}$ has exponential growth of order two $\|s\|_{\pi_0^* h} = \|e^{-g/2} \, k\|_{\pi_0^* h} = e^{\alpha |z|^2}$. The above construction is based on an idea of [dC13].

We now amplify the negativity of $\mathcal{L}$, by introducing $\mathcal{L}^m := \mathcal{L}^\otimes m$ for some big multiplicity $m \geq 1$ to be determined. For the metric $h_m := h^\otimes m$ of $\mathcal{L}^m$, the section $s_m := s^\otimes m$ has large exponential growth

\begin{equation}
\|s_m\|_{h_m} = e^{m\alpha |z|^2}.
\end{equation}
Now we introduce the complex surface $X := \mathbb{P}(\mathcal{L}_m \oplus \mathbb{C})$ obtained by the geometric projectivization of the rank 2 vector bundle $\mathcal{L}_m \oplus \mathbb{C}$ over $\mathcal{C}$. Denote by $\pi_1 : X \to \mathcal{C}$ the canonical projection. By the fiberwise identification $\mathcal{L}_m \cong \mathcal{L}_m \oplus 1 \subset \mathbb{P}(\mathcal{L}_m \oplus \mathbb{C})$, the tautological space of $\mathcal{L}_m$ can be embedded into $X$ as an open subset, whose complement is the elliptic curve $\mathcal{C}_\infty := \mathcal{C} \times \{1 \oplus 0\} \subset \mathbb{P}(\mathcal{L}_m \oplus \mathbb{C})$ at “infinity”.

Next, we introduce an auxiliary holomorphic function

$$\psi(z) := \prod_{\lambda \in \Lambda} \left(1 - \frac{z}{\lambda}\right)e^{\frac{z^2}{2\lambda^2}}$$

obtained by Weierstrass canonical product, where the zero locus $\Lambda$ will be chosen carefully by the following sophisticated reasoning, to make sure that the global section $\psi \cdot s_m$ of $\pi_1^* \mathcal{L}_m$ together with the inclusion $i : \mathcal{L}_m \hookrightarrow X$ induce an entire curve $f : \mathbb{C} \to X$ producing complicated Ahlfors currents.

First of all, we would like to have the estimate $\log|\psi(z)| \leq O(|z|^2)$, at least for $|z|$ around $r_i$ for some specific radii $r_i \nearrow \infty$, in order to bound the area of $f(\mathbb{D}_{r_i})$ by $O(r_i^2)$.

Secondly, we require that the cardinality $|\Lambda \cap \mathbb{D}_{r_i}| = O(r_i^2)$, so that the image $f(\mathbb{D}_{r_i})$ intersects the curve $\mathcal{C}_0 := \mathcal{C} \times \{0 \oplus 1\} \subset X$ defined by the zero section of $\mathcal{L}_m$ sufficiently enough.

Lastly, we require that each time when the image of the entire curve $f$ intersects $\mathcal{C}_0$ for $\lambda \in \Lambda$ with $|\lambda| \gg 1$, it contributes $O(1)$ area near the fiber $\mathbb{P}_1^1[\lambda] := \pi_1^{-1}(\lambda)$.

Thus we declare that

(i) near each annulus $A_{r_i} := \{z \in \mathbb{C} : \frac{r_i}{2} \leq |z| \leq r_i\}$, the zero locus $\Lambda$ is a mild perturbation of $A_{r_i} \cap c\Gamma$, where $c \geq 5$ is some positive integer to be determined. More precisely

$$\Lambda = \bigcup_{i \geq 1} B_{r_i} \quad \text{where} \quad B_{r_i} := \bigcup_{\mu \in A_{r_i} \cap c\Gamma} \{\mu + x_\mu\}.$$

Here at the moment we only tell that all $x_\mu$’s take values in the fundamental domain $D := \{x + y\sqrt{-1} : 0 \leq x, y < 1\}$, and later in Section 5, we will elaborate on the choices of $x_\mu$’s for delicate reasons.

(ii) $\{r_i\}_{i \geq 1}$ grow very rapidly, say

$$r_1 \geq 200 \cdot c, \quad r_{i+1} \geq r_i^4 \quad (\forall i \geq 1). \tag{2.2}$$

3. Preparations

**Lemma 3.1.** One has a uniform estimate $|\sum_{\lambda \in B_{r_i}} \frac{1}{\lambda}| \leq K/c^2$ for all $i = 1, 2, \ldots$.

**Proof.** In the special case that all $x_\mu = 0$, by the symmetry of $\Gamma$ that $(-1) \cdot \Gamma = \Gamma$ and that of $A_{r_i}$, the sum $\sum_{\mu \in A_{r_i} \cap c\Gamma} \frac{1}{\mu + 0}$ is always 0.

In general, for every $\mu \in A_{r_i} \cap c\Gamma$, one has the estimate $|\frac{1}{\mu + x_\mu} - \frac{1}{\mu + 0}| \leq K/r_i^2$. Noting that the cardinality $|B_{r_i}| \leq K \cdot (r_i/c)^2$, we conclude that $|\sum_{\lambda \in B_{r_i}} \frac{1}{\lambda}| \leq K \cdot (r_i/c)^2 \cdot K/r_i^2 = K/c^2$.  

**Lemma 3.2.** One has a uniform estimate $|\sum_{\lambda \in B_{r_i}} \frac{1}{\lambda^2}| \leq K/(c^2 r_i)$ for all $i = 1, 2, \ldots$.

**Proof.** The argument goes much the same way as the preceding one, by using the rotational symmetry of $\Gamma$ that $\sqrt{-1} \cdot \Gamma = \Gamma$. Indeed, we have the identity $\sum_{\mu \in A_{r_i} \cap c\Gamma} \frac{1}{(\mu + 0)^2} = 0$. Moreover, for every $\mu \in A_{r_i} \cap c\Gamma$, we have $|\frac{1}{(\mu + x_\mu)^2} - \frac{1}{(\mu + 0)^2}| \leq K/r_i^2$. The remaining argument is clear.

We make a convention that $\log 0 = -\infty$.

**Proposition 3.3.** For every $i \geq 2$ and for $r_i/3 \leq |z| \leq 3r_i$, one has

$$\log|\psi(z)| \leq K \cdot r_i^2/c^2. \tag{3.1}$$

To bound the area of $f(\mathbb{D}_{r_i})$ by $K \cdot r_i^2$, it is crucial to have the above estimate. In fact, by classical complex analysis (c.f. [Lev96, Chapter 4]), we can check that $\psi$ is well-defined and that the infinite product is uniformly convergent in bounded domains, and that the exponential growth order of $\psi$ is, by applying Borel’s formula [Lev96, page 30, Theorem 3], exactly 2, i.e. $\log|\psi(z)| \leq K_\epsilon \cdot |z|^{2+\epsilon}$ for any $\epsilon > 0$ and for large $|z|$. Nevertheless, for the critical case that $\epsilon = 0$ we need more effort.
Thus the remaining problem is to show that
\[ \log x \leq (1 + 3r_i) \lambda r_i/c^2 \] concerning smaller annuli compared with \( A_{r_i} \).

Note that \( \prod_{\ell=1}^{i-1} \prod_{\lambda \in B_{r_i}} (1 - \frac{\lambda}{r_i}) \leq (1 + 3r_i)^{\lambda r_i/c^2} \). Hence by Lemmas 3.1, 3.2, we receive that \( \log |I| \leq \log (1 + 3r_i) \lambda r_i/c^2 + \sum_{\ell=1}^{i-1} \left( (\sum_{\lambda \in B_{r_i}} \frac{1}{\lambda} ||z|| + |\sum_{\lambda \in B_{r_i}} \frac{1}{\lambda^2} ||z^2/2|| | \right) \leq \mathcal{K} \cdot r_i^2/c^2.

Secondly, we observe \( II := \prod_{\lambda \in B_{r_i}} (1 - \frac{\lambda}{r_i}) e^{\frac{z}{r_i}} \) concerning the annulus \( A_{r_i} \). Note that each term \( |1 - \frac{z}{r_i}| \leq \mathcal{K} \) by our construction, and that \( |B_{r_i}| \leq \mathcal{K} \cdot (r_i/c)^2 \). Now using Lemmas 3.1, 3.2, we receive that \( \log |II| \leq \mathcal{K} \cdot r_i^2/c^2 \).

Lastly, we analyze \( III := \prod_{\ell > i} \prod_{\lambda \in B_{r_i}} (1 - \frac{\lambda}{r_i}) e^{\frac{z}{r_i}} \) concerning larger annuli compared with \( A_{r_i} \). Now the key point is that, for each \( \lambda \in B_{r_i} \), one has \( |\lambda| \leq \frac{3r_i}{r_i/3} \ll 1 \). Therefore we can apply the Taylor expansion of \( \log (1 - \frac{z}{r_i}) \) to achieve desired estimates. Indeed, noting that \( \log \left( (1 - \frac{z}{r_i}) e^{\frac{z}{r_i}} \right) = -\sum_{n \geq 3} \frac{1}{n} \left( \frac{z}{r_i} \right)^n \), hence

\[ \left| \log \prod_{\lambda \in B_{r_i}} (1 - \frac{\lambda}{r_i}) e^{\frac{z}{r_i}} \right| \leq \sum_{\lambda \in B_{r_i}} \sum_{n \geq 3} \frac{1}{n} \left| \frac{z}{r_i} \right|^n \leq \mathcal{K} \cdot r_i^2/c^2 \cdot \sum_{n \geq 3} \frac{1}{n} \left| \frac{3r_i}{r_i/3} \right|^n \leq \mathcal{K} \cdot c^2 \cdot \frac{r_i^3}{r_i} \cdot \mathcal{K}.
\]

Since the sequence \( \{r_i\}_i \) grows very rapidly by our construction (2.2), there holds \( \sum_{i \geq 1} \frac{r_i^3}{r_i} < \mathcal{K} \). Thus the above estimate yields \( \log |III| \leq \mathcal{K}/c^2 \).

Combining all the above estimates about \( I, II, III \), we receive the desired inequality (3.1).

By our construction of \( \Lambda \), it intersects each disc \( \mathbb{D}(z, 1) \) at most once. Therefore we introduce

\[ \psi_1(z) := \prod_{\lambda \in \Lambda \setminus \mathbb{D}(z, 1)} \left( 1 - \frac{\lambda}{r_i} \right) e^{\frac{z}{r_i}} \]

to capture the asymptotic behavior of \( \psi \) away from its zero locus \( \Lambda \).

**Proposition 3.4.** For every \( z \in \mathbb{C} \) with large \( |z| \), one has

\[ \log |\psi_1(z)| \geq -\mathcal{K} \cdot |z|^2/c^2. \]

**Proof.** Fix a positive small number \( \eta = \frac{1}{100} \). For every \( i \geq 1 \), we introduce the slightly larger annulus \( \tilde{A}_{r_i} := \{ x \in \mathbb{C} : (1 - \eta)r_i/2 \leq |x| \leq (1 + \eta)r_i \} \supseteq A_{r_i} \), to make sure that \( \tilde{A}_{r_i} \supseteq B_i \).

Case (i): \( |z| \) large with \( z \notin \bigcup_{i \geq 1} \tilde{A}_{r_i} \). Then \( z \) lies between some two consequent annuli \( \tilde{A}_{r_i} \) and \( \tilde{A}_{r_{i+1}} \), i.e., \( (1 + \eta)r_j < |z| < (1 - \eta)r_{j+1}/2 \), and it is clear that \( \psi_1(z) = \psi_1(z) \). Firstly, for each \( \lambda \in \bigcup_{i \geq 1} B_{r_i} \), we have \( |1 - \frac{z}{r_i}| \geq |\lambda| \geq 1 \geq \eta^j := \eta/2 \). Thus \( \prod_{i=1}^{j} \prod_{\lambda \in B_{r_i}} (1 - \frac{z}{r_i}) \geq \eta^j \prod_{i=1}^{j} |B_{r_i}| \geq \eta^j \mathcal{K} r_i^2/c^2 \geq \eta^j \mathcal{K} |z|^2/c^2 \).

Next, thanks to Lemmas 3.1, 3.2, we have \( \prod_{i=1}^{j} \prod_{\lambda \in B_{r_i}} e^{\frac{z}{r_i}} \geq e^{-|z| \sum_{i=1}^{j} |B_{r_i}|} \geq e^{-|z| \sum_{i=1}^{j} \mathcal{K} r_i^2/c^2} \geq e^{-\mathcal{K} |z|^2/c^2} \).

Lastly, by mimicking the estimate of \( III \) in the preceding proof, we receive that \( \log \left( \prod_{i=1}^{j} \prod_{\lambda \in B_{r_i}} (1 - \frac{z}{r_i}) e^{\frac{z}{r_i}} \right) \geq -\sum_{i=1}^{j} \sum_{\lambda \in B_{r_i}} \sum_{n \geq 3} \frac{1}{n} \left| \frac{z}{r_i} \right|^n \geq -\mathcal{K} \cdot |z|^2/c^2 \).

Summarizing the above estimates, we conclude that \( \log |\psi_1(z)| = \log |\psi(z)| \geq -\mathcal{K} \cdot |z|^2/c^2 \).

Case (ii): \( |z| \) large with \( z \in \tilde{A}_{r_i} \) for some \( j \). By repeating the same arguments as above, we can show that, first of all, \( \log \left( \prod_{i=1}^{j-1} \prod_{\lambda \in B_{r_i}} (1 - \frac{z}{r_i}) e^{\frac{z}{r_i}} \right) \geq -\mathcal{K} \cdot |z|^2/c^2 \), and secondly, \( \log \left( \prod_{i=j+1}^{j} \prod_{\lambda \in B_{r_i}} (1 - \frac{z}{r_i}) e^{\frac{z}{r_i}} \right) \geq -\mathcal{K} \cdot |z|^2/c^2 \).

By Lemmas 3.1, 3.2, we receive \( \log \left| \prod_{\lambda \in B_{r_i} \setminus \mathbb{D}(z, 1)} e^{\frac{z}{r_i}} \right| \geq -\mathcal{K} \cdot |z|^2/c^2 \).

Thus the remaining problem is to show that \( \log \left| \prod_{\lambda \in B_{r_i}} (1 - \frac{z}{r_i}) \right| \geq -\mathcal{K} \cdot |z|^2/c^2 \).

To start with, we find a point \( \mu_0 \in c \Gamma \) having the least Euclidean distance to \( z \). Then for every \( \lambda = \mu + x \mu \in B_{r_i} \), we have \( |\lambda - z| \geq |\mu - z| - |x \mu| \geq \frac{1}{2} (|\mu - z| + |\mu_0 - z|) - \sqrt{2} \geq \frac{1}{2} |\mu - \mu_0| - \sqrt{2} \). If
moreover assume that \( \mu \neq \mu_0 \), then we can continue to estimate \(|\lambda - z| \geq \frac{1}{2}|\mu - \mu_0| - \sqrt{2} \geq \frac{1}{4}|\mu - \mu_0| \), whence \(|1 - \frac{z}{\lambda}| = \frac{|\lambda - z|}{|\lambda|} \geq \frac{1}{8} \frac{|\mu - \mu_0|}{|\mu|} \). Since \((\frac{8}{\mu})^{8|\mu|} \geq \exp(-K \cdot |\mu|^2/c^2)\), we only need to show that

\[
\log \prod_{\mu_0 \neq \mu \in \mathbf{A}_{rz} \cap c\Gamma} \frac{|\mu - \mu_0|}{|\mu|} \geq -K \cdot |\mu_0| \geq -K \cdot \text{abs}(\mu_0) \geq -K \cdot |\mu_0| \quad (3.4)
\]

For any positive number \( r' \), denote by \( \Gamma_{\leq r'} \subset \Gamma \) the subset of points whose real and imaginary parts have absolute value \( \leq r' \). Note that, for every \( \mu \in \mathbf{A}_{rz} \cap c\Gamma \setminus (\mu_0 + \Gamma_{\leq r_0}) \) far away from \( \mu_0 \), we have

\[
\left| \frac{\mu - \mu_0}{\mu} \right| > \eta r_j = \eta.
\]

Thus these \( \mu \)'s, having cardinality \( \mathbf{A}_{rz} \cap c\Gamma \setminus (\mu_0 + \Gamma_{\leq r_0}) \), cause no trouble for (3.4).

Lastly, we handle \( \mu \in \mathbf{A}_{rz} \cap c\Gamma \setminus (\mu_0 + \Gamma_{\leq r_0}) \) simultaneously. Note that \( c\Gamma \cap \Gamma_{\leq r_0} \setminus \{0\} \) can be decomposed into 2 horizontal parts consisting of \( \pm \{ \ell \cdot e + 0 \cdot \sqrt{-1}\} \}_{\ell=1}^{2|\mu|} \), plus the remaining \( 4|\mu| + 2 \) vertical parts consisting of \( \pm \{ i \cdot e + \ell \cdot \sqrt{-1}\} \}_{\ell=1}^{2|\mu|} \) for \( i = 0, \pm 1, \pm 2, \ldots, \pm |\mu| \). Each part contains consequential \( \frac{|\mu|}{2} \) points having absolute values \( \geq 1 \cdot e, 2 \cdot e, \ldots, \frac{|\mu|}{2} \) respectively. Hence

\[
\prod_{0 \neq \mu' \in c\Gamma \cap \Gamma_{\leq r_0}} |\mu'| \geq (\left( \frac{\eta}{e} \right)^{2|\mu|} \cdot e^{2|\mu|})^{2|\mu|/2} = e^{\left( \frac{\eta}{e} \right)^{2|\mu|} \cdot e^{2|\mu|}}.
\]

Now it is time to apply the Stirling formula that for every positive integer \( n \), one has

\[
n! = n^e e^{-n} \sqrt{2\pi n} e^{\eta/12} \quad (3.6)
\]

for some \( |\rho_n| \leq 1 \). A straightforward computation yields

\[
\log \prod_{\mu_0 \neq \mu \in c\Gamma \setminus (\mu_0 + \Gamma_{\leq r_0})} \frac{|\mu - \mu_0|}{|\mu|} \geq \log \prod_{\mu_0 \neq \mu \in c\Gamma \cap \Gamma_{\leq r_0}} \frac{|\mu'|}{2|\mu|} \geq \log \left( \left( \frac{\eta}{e} \right)^{2|\mu|} \cdot e^{2|\mu|} \right)^{2|\mu|/2} = \log \left( \left( \frac{\eta}{e} \right)^{2|\mu|} \cdot e^{2|\mu|} \right) \geq \log \left( \left( \frac{\eta}{e} \right)^{2|\mu|} \cdot e^{2|\mu|} \right) + 2|\mu|/2
\]

(3.7) [by the Stirling formula]

\[
\geq -K \cdot |\mu_0| \geq -K \cdot |\mu_0| \geq -K \cdot |\mu_0| \quad (3.7)
\]

Now the remaining problem is that \( c\Gamma \cap (\mu_0 + \Gamma_{\leq r_0}) \) might exceed \( \mathbf{A}_{rz} \). Let us decompose \( c\Gamma \cap (\mu_0 + \Gamma_{\leq r_0}) \) with respect to \( \mathbf{A}_{rz} \) into two parts \( (\mu_0 + P_{in}) \cup (\mu_0 + P_{out}) \), where the first (resp. second) part lies entirely in (resp. outside) \( \mathbf{A}_{rz} \). Since \( \eta \) is small, we can find some point \( y \in \mathbf{A}_{rz} \cap c\Gamma \) such that \( y + P_{out} \subset \mathbf{A}_{rz} \) stays away from \( \mu_0 + \Gamma_{\leq r_0} \). See the picture below for illustration.

![Diagram of decomposition](image.png)

Lastly, we decompose \( \mathbf{A}_{rz} \cap c\Gamma \) into 3 disjoint parts, \( \mu_0 + P_{in}, y + P_{out} \) and \( R \) the remaining. Note that

\[
\prod_{\mu \in y + P_{out}} \frac{|\mu - \mu_0|}{|\mu|} \geq \prod_{\mu \in \mu_0 + P_{out}} \frac{|\mu - \mu_0|}{|\mu|}, \text{ because each factor on the left-hand-side } \geq \eta r_j = 8 \eta, \text{ while each factor on the right-hand-side } \leq 4 \sqrt{2} \eta. \text{ Thus } \prod_{\mu_0 \neq \mu \in \mathbf{A}_{rz} \cap c\Gamma} \frac{|\mu - \mu_0|}{|\mu|} \geq \prod_{\mu_0 \neq \mu \in c\Gamma \setminus (\mu_0 + \Gamma_{\leq r_0})} \frac{|\mu - \mu_0|}{|\mu|}.
\]
\[ \prod_{\mu \in R} \frac{|\mu - m_0|}{|\mu|}. \] By the estimates (3.5), (3.7) and that the cardinality \(|R| \leq K \cdot r_j^2/c^2\), we conclude the proof.

4. Estimates

4.1. \( f(z) \) is close to \( C_\infty \) unless \( z \) is near \( \Lambda \). Recalling (3.2), we first rewrite

\[ ||\psi \cdot s_m(z)||_{h_m} = ||\psi_1 \cdot s_m(z)||_{h_m} \cdot ||\psi_m - \psi_0|| \]

to concentrate positivity to the first factor, where if \( z \in \mathbb{D}(\lambda, 1)\) for certain \( \lambda \in \Lambda \) then \( \psi = (1 - \frac{z}{\lambda})e^{z^2/2} \), otherwise \( \psi = 1 \). Now thanks to (2.1), (3.3), the left part satisfies that

\[ ||\psi_1 \cdot s_m(z)||_{h_m} \geq \exp\left( (m - \alpha - K/c^2) \cdot |z|^2 \right). \]

A key trick in this paper is that we choose \( m \) and \( c \) such that

\[ m \cdot \alpha - K/c^2 > 0. \]

Thus for any small \( \epsilon > 0 \), for all sufficiently large \( |z| \gg 1 \) with \( \text{dist}(z, \Lambda) \geq \epsilon \), there holds

\[ ||\psi \cdot s_m(z)||_{h_m} \geq 1, \]

i.e., \( f(z) \) is very close to \( C_\infty \). Indeed, if \( \text{dist}(d, \Lambda) \geq 1 \), then \( \psi = 1 \), and there is nothing to prove; otherwise \( \epsilon \leq |z - \lambda| < 1 \) for some \( \lambda \in \Lambda \), hence \( \psi \geq \frac{|z - \lambda|}{2|\lambda|^2} \geq \exp\left( (m - \alpha - K/c^2) \cdot |z|^2 \right) \), thus

\[ ||\psi \cdot s(z)||_{h_m} = ||\psi_1 \cdot s(z)||_{h_m} \cdot ||\psi_m - \psi_0|| \]

is very large when \( |z| \gg 1 \).

4.2. Bound the area \( \int_{\mathbb{D}_{2r_j}} f^* \omega_X \) from above. Fix a Kähler form \( \omega_C = d\overline{d}|z|^2 \) on \( C \). The metric \( h_m = h^{\otimes m} = \mathcal{L}_m \otimes \mathbb{C} \) and therefore it induces a metric on the tautological line bundle \( \mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \) on \( \mathbb{P}(\mathcal{E}) = X \). Restricting to any fiber of \( \pi_1 : X \to C \), the curvature form \( \Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} \) is strictly negative due to the property of the Fubini-Study metric for the tautological line bundle \( \mathcal{O}_{\mathbb{P}^1}(-1) \). Therefore, by standard compactness argument, for sufficiently small \( \epsilon_1 > 0 \), we receive a Kähler form on \( X \) of the shape

\[ \omega_X : = \pi_1^* \omega_C - \epsilon_1 \Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)}. \]

We can identify the tautological space \( \mathcal{L}_m = \{(z, \xi) : \xi \in \mathcal{L}_m | z \} \) with an open set of \( \mathbb{P}(\mathcal{L}_m \otimes \mathbb{C}) \), by mapping \((z, \xi) \mapsto (z, |\xi \otimes 1|)\). Thus in the local coordinates \((z, \xi)\), the curvature

\[ \Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} = -d \overline{d} \left( \log (\|\xi\|^2_{h_m} + 1) \right) \]

is of Fubini-Study shape. For \( r \) lies in \([\frac{1}{2}r_j, 3r_j]\) for some \( i \geq 1 \), by Jensen’s formula we receive

\[ \int_1^r \frac{dt}{t} \int_{\mathbb{D}_t} f^* \Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} = -\frac{1}{4\pi} \int_0^{2\pi} \log (\|\psi \cdot s_m\|^2_{h_m} + 1) (ue^{i\theta}) d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log (\|\psi \cdot s_m\|^2_{h_m} + 1) (ue^{i\theta}) d\theta \]

whence the Nevanlinna order function satisfies the estimate

\[ T_{f,r}(\omega_X) = \int_1^r \frac{dt}{t} \int_{\mathbb{D}_t} f^* \omega_X \geq \int_1^r \frac{dt}{t} \int_{\mathbb{D}_t} f^* \pi_1^* \omega_C - \epsilon_1 \int_1^r \frac{dt}{t} \int_{\mathbb{D}_t} f^* \Theta_{\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)} \geq K \cdot r_i^2. \]

Here is a useful observation

\[ T_{f,3r_j}(\omega_X) = \int_1^{3r_j} \frac{dt}{t} \int_{\mathbb{D}_t} f^* \omega_X \geq \int_{2r_j}^{3r_j} \frac{dt}{t} \int_{\mathbb{D}_{2r_j}} f^* \omega_X = \ln(3/2) \cdot \int_{\mathbb{D}_{2r_j}} f^* \omega_X. \]

Combining the two estimates above, we conclude that

\[ \int_{\mathbb{D}_{2r_j}} f^* \omega_X \leq K \cdot r_i^2. \]
4.3. **Bound the area** $\int_{D_{r_i/3}} f^* \omega_X$ **from below.** By Jensen’s formula, we have

$$\int_{r_i/4}^{r_i/3} \frac{dt}{t} \int_{D_t} f^*\Theta_{\varphi(t)}(-1) = -\frac{1}{4\pi} \int_0^{2\pi} \log (\|\psi \cdot s_m\|^2_{h_m} + 1) \frac{r_i}{3} e^{i\theta} \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log (\|\psi \cdot s_m\|^2_{h_m} + 1) \frac{r_i}{4} e^{i\theta} \, d\theta$$

(4.7) \hspace{1cm} \text{[for } r_i \gg 1] \hspace{1cm} = -\frac{1}{4\pi} \int_0^{2\pi} \log (\|\psi \cdot s_m\|^2_{h_m}) \frac{r_i}{3} e^{i\theta} \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log (\|\psi \cdot s_m\|^2_{h_m}) \frac{r_i}{4} e^{i\theta} \, d\theta + o(1).

Since the holomorphic function $\psi$ is nowhere vanishing on $\overline{D}_{r_i/3} \setminus D_{r_i/4}$ by our construction, $\log |\psi|^2$ is harmonic on $\overline{D}_{r_i/3} \setminus D_{r_i/4}$. Thus

$$-\frac{1}{4\pi} \int_0^{2\pi} \log |\psi|^2 \frac{r_i}{3} e^{i\theta} \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log |\psi|^2 \frac{r_i}{4} e^{i\theta} \, d\theta = 0.$$

Hence we can simplify (4.7) as

$$\int_{r_i/4}^{r_i/3} \frac{dt}{t} \int_{D_t} f^*\Theta_{\varphi(t)}(-1) = -\frac{1}{4\pi} \int_0^{2\pi} \log (\|s_m\|^2_{h_m}) \frac{r_i}{3} e^{i\theta} \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log (\|s_m\|^2_{h_m}) \frac{r_i}{4} e^{i\theta} \, d\theta + o(1)$$

\hspace{1cm} \text{[by (2.1)]} \hspace{1cm} = -\frac{1}{4\pi} \int_0^{2\pi} 2m\alpha \frac{r_i}{3} + \frac{1}{4\pi} \int_0^{2\pi} 2m\alpha \frac{r_i}{4} + o(1)

$$= -\frac{7m\alpha}{144} r_i^2 + o(1).$$

Hence

$$\int_{r_i/4}^{r_i/3} \frac{dt}{t} \int_{D_t} f^*\omega_X = \int_{r_i/4}^{r_i/3} \frac{dt}{t} \int_{D_t} f^*\pi^*_1\omega_C - \epsilon_1 \int_{r_i/4}^{r_i/3} \frac{dt}{t} \int_{D_t} f^*\Theta_{\varphi(t)}(-1) = \frac{7}{144} \left( \frac{\pi}{2} + \epsilon_1 m\alpha \right) r_i^2 + o(1).$$

Noting that

$$\int_{r_i/4}^{r_i/3} \frac{dt}{t} \int_{D_t} f^*\omega_X \leq \int_{r_i/4}^{r_i/3} \frac{dt}{t} \int_{D_{r_i/3}} f^*\omega_X = \log (4/3) \cdot \int_{D_{r_i/3}} f^*\omega_X,$$

we conclude that for $i \gg 1$ there holds

$$\int_{D_{r_i/3}} f^*\omega_X \geq K \cdot r_i^2.$$

4.4. **Estimates of** $\int_{D(\lambda,\epsilon)} f^*\omega_X$. Mark the curve $C_0 \subset X$ induced by the zero section of $L_m$. Note that $C_0$ and $C_\infty$ are disjoint. Contrasting to the observation in Subsection 4.1, for every $\lambda \in \Lambda$, since $\psi(\lambda) = 0$, $f(\lambda)$ must lies in $C_0$, which keeps certain positive distance to $C_\infty$.

Let $\epsilon$ be a small positive radius. Denote by $D(|\lambda|, \epsilon) \subset C$ the disc centered at $[\lambda]$ with the radius $\epsilon$. Then the image of $f(D(\lambda, \epsilon))$ is contained in the neighborhood $U_{[\lambda], \epsilon} := \pi^{-1}_{1}(D([\lambda], \epsilon))$ of $\pi^{-1}_{1}([\lambda])$. Recall that the metric $\omega_X$ on $X$ is given in (4.3). Firstly, since $\pi_1 \circ f = \pi_0$, where $\pi_0: \mathbb{C} \to \mathbb{C}/\Gamma$ is the canonical projection, we have $\int_{D(\lambda, \epsilon)} f^*\pi^*_1\omega_C = \pi^2$.

Next, we estimate $\int_{D(\lambda, \epsilon)} f^*\Theta_{\varphi(t)}(-1)$. Recalling (3.2), we can rewrite $\psi(z) = (\psi_1 \cdot e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda}}) \cdot (1 - \frac{z}{\lambda})$, where the factor $\psi_2 := \psi_1 \cdot e^{\frac{z}{\lambda} + \frac{z^2}{2\lambda}}$ is nonvanishing for $z \in D(\lambda, \epsilon)$. Thus $\log |\psi_2|$ is a harmonic function on $D(\lambda, \epsilon)$. Now, using Jensen’s formula, for $|\lambda| \gg 1$ large, we can estimate

$$\int_0^{2\pi} \frac{d\theta}{t} \int_{D(\lambda, \epsilon)} f^*\Theta_{\varphi(t)}(-1) = -\frac{1}{4\pi} \int_0^{2\pi} \log (\|s_m\|^2_{h_m} + 1)(\lambda + 2e^{i\theta}) \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log (\|s_m\|^2_{h_m} + 1)(\lambda + e^{i\theta}) \, d\theta$$

\hspace{1cm} \text{[by (4.2), as } |\lambda| \gg 1] \hspace{1cm} = -\frac{1}{4\pi} \int_0^{2\pi} \log (\|s_m\|^2_{h_m})(\lambda + 2e^{i\theta}) \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log (\|s_m\|^2_{h_m})(\lambda + e^{i\theta}) \, d\theta + o(1)

(4.10)$$

$$= \left[ -\frac{1}{4\pi} \int_0^{2\pi} \log (\|s_m\|^2_{h_m})(\lambda + 2e^{i\theta}) \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log (\|s_m\|^2_{h_m})(\lambda + e^{i\theta}) \, d\theta \right] + o(1).$$
By (2.1), the first bracket \( \cdots \) in (4.10) can be computed as

\[
(4.11) \quad -\frac{1}{4\pi} \int_0^{2\pi} 2m\alpha|\lambda + 2e^{i\theta}|^2 \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} 2m\alpha|\lambda + e^{i\theta}|^2 \, d\theta = \frac{-3m\alpha e^2}{2\pi}.
\]

Now we separate \( \log |\psi|'(s) = \log |\psi_2| \cdot \log |1 - \frac{s}{\lambda}| \). Thanks to the harmonicity of \( \log |\psi_2| \), we have

\[
-\frac{1}{4\pi} \int_0^{2\pi} \log (|\psi_2|^2)(\lambda + 2e^{i\theta}) \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log (|\psi_2|^2)(\lambda + e^{i\theta}) \, d\theta = 0.
\]

Thus we can calculate the second bracket \( \cdots \) of (4.10) as

\[
(4.12) \quad -\frac{1}{4\pi} \int_0^{2\pi} \log \left| 1 - \frac{\lambda + 2e^{i\theta}}{\lambda} \right|^2 \, d\theta + \frac{1}{4\pi} \int_0^{2\pi} \log \left| 1 - \frac{\lambda + e^{i\theta}}{\lambda} \right|^2 \, d\theta = -\log 2.
\]

Hence it follows from (4.10), (4.11), (4.12) that

\[
\int_\epsilon^\delta \frac{dt}{t} \int_{D(\lambda,t)} f^*\Theta_{\mathcal{O}(\epsilon)}(-1) = -\frac{3m\alpha e^2}{2\pi} - \log 2 + o(1) \quad \text{(for } |\lambda| > 1).\]

Therefore

\[
(4.13) \quad \int_\epsilon^\delta \frac{dt}{t} \int_{D(\lambda,t)} f^*\omega_X = \pi e^2 + \epsilon_1 \left( \frac{3m\alpha e^2}{2\pi} + \log 2 + o(1) \right).
\]

By the same trick as (4.5), we have

\[
\int_\epsilon^\delta \frac{dt}{t} \int_{D(\lambda,t)} f^*\omega_X \geq \int_\epsilon^\delta \frac{dt}{t} \int_{D(\lambda,\epsilon)} f^*\omega_X \geq \log 2 \cdot \int_{D(\lambda,\epsilon)} f^*\omega_X.
\]

Combining the above two estimates, we conclude

\[
(4.14) \quad \int_{D(\lambda,\epsilon)} f^*\omega_X \leq \frac{1}{\log 2} \left( \pi e^2 + \epsilon_1 \left( \frac{3m\alpha e^2}{2\pi} + \log 2 + o(1) \right) \right) \quad \text{(for } |\lambda| > 1).\]

Next, we provide an lower bound for \( \int_{D(\lambda,\epsilon)} f^*\omega_X \). Substituting \( \epsilon \) by \( \frac{\epsilon}{2} \) in (4.13), we receive that

\[
(4.15) \quad \int_{\frac{\epsilon}{2}}^\delta \frac{dt}{t} \int_{D(\lambda,t)} f^*\Theta_{\mathcal{O}(\epsilon)}(-1) = \frac{\pi e^2}{4} + \epsilon_1 \left( \frac{3m\alpha e^2}{8\pi} + \log 2 + o(1) \right).
\]

Note that

\[
\int_{\frac{\epsilon}{2}}^\delta \frac{dt}{t} \int_{D(\lambda,t)} f^*\omega_X \leq \int_{\frac{\epsilon}{2}}^\delta \frac{dt}{t} \int_{D(\lambda,\epsilon)} f^*\omega_X = \log 2 \cdot \int_{D(\lambda,\epsilon)} f^*\omega_X.
\]

Hence it follows from the above two estimates that

\[
(4.16) \quad \int_{D(\lambda,\epsilon)} f^*\omega_X \geq \frac{1}{\log 2} \left( \frac{\pi e^2}{4} + \epsilon_1 \left( \frac{3m\alpha e^2}{8\pi} + \log 2 + o(1) \right) \right) \quad \text{(for } |\lambda| > 1).\]

4.5. **Area estimates of** \( f(\mathbb{C}) \) **near horizontal curves.** An irreducible algebraic curve \( D \subset X \) is said to be vertical if \( \pi_1(D) \) is a point; otherwise it is called horizontal, in the sense that \( \pi_1(D) = \mathcal{C} \).

Firstly, for a vertical curve \( \mathbb{P}^1_{[y]} = \pi_1^{-1}([y]) \), by the estimates (4.14) and (4.16), the area of \( f(\mathbb{D}_r) \) near \( \mathbb{P}^1_{[y]} \), as \( r \to \infty \), is mostly determined by asymptotic growth of \( |D_r \cap \Lambda| \).

Next, for the horizontal curve \( D = \mathcal{C}_\infty \), by Subsection 4.1, \( f(\mathbb{D}_r) \) shall concentrate a large portion of area near \( \mathcal{C}_\infty \) as \( r \to \infty \).

Lastly, for any other irreducible horizontal curve \( D \neq \mathcal{C}_\infty \), we devote this subsection to prove that, roughly speaking, every time when \( f(\mathbb{D}_r) \) intersects with \( D \), it contributes negligible area about there.

To start with, we take a general point \( d_0 \in D \setminus \mathcal{C}_\infty \) such that \( \pi_1|D \) is regular at \( d_0 \), i.e., some small open neighborhood \( U \subset D \) of \( d_0 \) is a graph over \( \pi_1(U) \) containing \( \pi_1(d_0) =: c_0 \). By shrinking \( U \) we may assume that \( U \) stays away from \( \mathcal{C}_\infty \), that \( \pi_1(U) \) is a small disc \( \mathbb{D}(c_0, 3\delta) \) for some \( \delta > 0 \), and that the line bundle \( \mathcal{L}_m \) locally has a trivialization \( \mathcal{L}_m|_{\mathbb{D}(c_0, 3\delta)} \cong \mathbb{D}(c_0, 3\delta) \times \mathbb{C} \), which extends to an identification

\[
(4.17) \quad \vartheta : \pi_1^{-1}(\mathbb{D}(c_0, 3\delta)) \xrightarrow{\cong} \mathbb{D}(c_0, 3\delta) \times \mathbb{P}^1(\mathbb{C})
\]
by fiberwise compactification $\mathbb{C} \hookrightarrow \mathbb{P}^1(\mathbb{C})$ sending $z \mapsto [z:1]$. Hence we can read the coordinates of $U$ in the chart $\mathbb{D}(c_0, 3\delta) \times \mathbb{C}$ as the graph of some holomorphic map $u : \mathbb{D}(c_0, 3\delta) \to \mathbb{C}$.

Let $p_1, p_2$ be the projections of $\mathbb{D}(c_0, 3\delta) \times \mathbb{P}^1(\mathbb{C})$ to the two factors. Let $\omega_{FS}$ be the Fubini-Study form on $\mathbb{P}^1(\mathbb{C})$. By compactness argument, the metric $p_1^*\omega_c + p_2^*\omega_{FS}$ is comparable with $(\vartheta^{-1})^*\omega_X$ on $\mathbb{D}(c_0, \frac{5\delta}{2}) \times \mathbb{P}^1(\mathbb{C})$, namely

$$K_{c_0,\delta,\vartheta}^{-1} \cdot (p_1^*\omega_c + p_2^*\omega_{FS}) \leq (\vartheta^{-1})^*\omega_X \leq K_{c_0,\delta,\vartheta} \cdot (p_1^*\omega_c + p_2^*\omega_{FS}).$$

Fix a positive number $\varepsilon \ll \delta$. Then the neighborhood $\pi_1^{-1}(\mathbb{D}(c_0, \delta)) \cap D$ of $d_0$ in the coordinates reads as

$$U_1 := \{(z, w) : z \in \mathbb{D}(c_0, \delta), w = u(z)\},$$

whose $\varepsilon$--open neighborhood is

$$U_1^\varepsilon := \{(z, w) : z \in \mathbb{D}(c_0, \delta + \varepsilon), |w - u(z)| < \varepsilon\}.$$

Fix a positive small number $\delta' < \delta/2$. By Subsection 4.1, for $|z| \gg 1$ large with $\text{dist}(z, \Lambda) > \delta'$, one sees that $f(z)$ is very close to $C_\infty$, hence it is outside $U_1^\varepsilon$. Thus for bounding the area of $f(\mathbb{D}_r) \cap U_1^\varepsilon$ from above by $o(1) \cdot r^2$, we only need to show that, as $\lambda \in \Lambda$ with $|\lambda| \gg 1$, the area $f(\mathbb{D}(\lambda, \delta')) \cap U_1^\varepsilon$ is negligible $o(1)$.

Observation 4.1. Put $f_2 := p_2 \circ \vartheta \circ f$. For $\lambda \in \Lambda$ with $|\lambda| \gg 1$ and $|\lambda| \in \mathbb{D}(c_0, \delta + \delta')$, one has

$$(\vartheta \circ f)^{-1}(U_1^\varepsilon) \cap \mathbb{D}(\lambda, \delta') \subset f_2^{-1}\left(\mathbb{D}(\mathbb{D}(\lambda, \delta'), 2\varepsilon)\right) \cap \mathbb{D}(\lambda, \delta').$$

Proof. By continuity of $u$ and by compactness of $\mathbb{D}(c_0, \frac{5}{2}\delta)$, there exists some positive number $\delta_\varepsilon < \delta'$ such that, for any two points $x_1, x_2 \in \mathbb{D}(c_0, \frac{5}{2}\delta)$ with $|x_1 - x_2| < \delta_\varepsilon$, there holds $|u(x_1) - u(x_2)| < \varepsilon$. By Subsection 4.1, for all $\lambda \in \Lambda$ with $|\lambda| \gg 1$, the image of $\mathbb{D}(\lambda, \delta') \setminus \mathbb{D}(\lambda, \delta_\varepsilon)$ under $\vartheta \circ f$ is outside $U_1^\varepsilon$, therefore

$$(\vartheta \circ f)^{-1}(U_1^\varepsilon) \cap \mathbb{D}(\lambda, \delta') \subset (\vartheta \circ f)^{-1}(U_1^\varepsilon) \cap \mathbb{D}(\lambda, \delta_\varepsilon).$$

By definition, every element $z$ in the right-hand-side satisfies that $|f_2(z) - u([\lambda])| < \varepsilon$ and $|z - \lambda| < \delta_\varepsilon$. Thus $|f_2(z) - u([\lambda])| \leq |f_2(z) - u([\lambda])| + |u([\lambda]) - u([\lambda])| < \varepsilon + \varepsilon = 2\varepsilon$, which concludes the proof. \qed

Now for every $v \in \mathbb{C}$ having absolute value $|v| \leq R := \max\{|u(z)| + \varepsilon : z \in \mathbb{D}(c_0, \frac{5}{2}\delta)\} < \infty$, for $\lambda \in \Lambda$ with $|\lambda| \gg 1$ and $|\lambda| \in \mathbb{D}(c_0, \delta + \delta')$, consider the restricted holomorphic function

$$f_2 : \mathbb{D}(\lambda, \delta') \longrightarrow \mathbb{C}.$$ 

Since $|f_2| \gg 1$ (in particular $|f_2| > R$) on $\partial\mathbb{D}(\lambda, \delta')$ by Subsection 4.1, by the Argument Principle, the number of solutions of the equation $f_2(y) = v$ on the disc $\mathbb{D}(\lambda, \delta')$, counting multiplicities, equals to

$$\frac{1}{2\pi\sqrt{-1}} \int_{z \in \partial\mathbb{D}(\lambda, \delta')} \frac{(f_2 - v)'(z)}{f_2 - v}(z) \, dz.$$ 

Noting that the above quantity takes integer value, and that it varies continuously with respect to $v \in \mathbb{D}_R$, it must be a constant for every $v \in \mathbb{D}_R$. Now checking the special value $v = 0$, we see that the number of solution is just 1. Thus Observation 4.1 implies that, for $\lambda \in \Lambda$ with $|\lambda|$ sufficiently large and $|\lambda| \in \mathbb{D}(c_0, \delta + \delta')$, we have

$$(4.19) \quad \text{Area} \left((\vartheta \circ f)^{-1}(U_1^\varepsilon) \cap \mathbb{D}(\lambda, \delta')\right)_{f_2^*\omega_{FS}} \leq \text{Area} \left(\mathbb{D}(\mathbb{D}(\lambda, \delta'), 2\varepsilon)\right)_{\omega_{FS}} \leq K \cdot \varepsilon^2.$$

Also, by Subsection 4.1, for every positive number $\epsilon' > 0$, for $\lambda \in \Lambda$ with sufficiently large $|\lambda|$, we have

$$(4.20) \quad (\vartheta \circ f)^{-1}(U_1^\varepsilon) \cap \mathbb{D}(\lambda, \delta') \subset \mathbb{D}(\lambda, \epsilon').$$

Therefore

$$(4.21) \quad \text{Area} \left((\vartheta \circ f)^{-1}(U_1^\varepsilon) \cap \mathbb{D}(\lambda, \delta')\right)_{\pi_1^*\omega_c} \leq \text{Area} \left(\mathbb{D}(\lambda, \epsilon')\right)_{\pi_1^*\omega_c} = \pi \cdot \epsilon'^2.$$ 

Summarizing (4.18), (4.19), (4.20), for $\lambda \in \Lambda$ with $|\lambda| \gg 1$ and $|\lambda| \in \mathbb{D}(c_0, \delta + \delta')$, we have

$$\text{Area} \left((\vartheta \circ f)^{-1}(U_1^\varepsilon) \cap \mathbb{D}(\lambda, \delta')\right)_{f_2^*\omega_X} \leq K_{c_0,\delta,\vartheta} \cdot (K \varepsilon^2 + \pi \epsilon'^2).$$
4.6. Area of $f$ near $\lambda \in \Lambda$ revisit. An alternative way to interpret (4.16) is the following.

**Observation 4.2.** Let $\delta' > 0$ be a small positive number. Let $U$ be an open neighborhood of $C_\infty$ such that its closure $\overline{U}$ stays away from $C_0$. Then one has the estimate

$$\text{Area} \left((X \setminus U) \cap f(\mathbb{D}(\lambda, \delta'))\right)_\omega_X \geq K_U \quad (\forall \lambda \in \Lambda \text{ with } |\lambda| \geq 1).$$

This strengthens (4.16) in a qualitative sense, and will be helpful for discussing diffuse parts later. Before going to the proof of the above result, recall the following special case of Wirtinger’s inequality.

**Proposition 4.3.** (c.f. [Duv17a, page 7]) Let $C$ be a proper holomorphic curve in the ball $B(0, \epsilon) \subset \mathbb{C}^n$ passing through 0. Then with the standard Euclidean metric, one has $\text{Area}(C) \geq \pi \epsilon^2$. \hfill $\Box$

**Proof of Observation 4.2.** By compactness, $C_0$ can be covered by finitely many open neighborhoods $U_i$, being disjoint with $U$, with charts $\vartheta_i : U_i \to V_i \subset \mathbb{C}^2$. By shrinking $U_i$’s if necessary, we can assume that every pull-back by $\vartheta_i$ of the standard Euclidean metric on $\mathbb{C}^2$ is comparable with $\omega_X$. Again by the compactness of $C_0$, for every point $c \in C_0$, certain chart $V_i$ of $U_i$ contains a sufficiently large ball centered at $\vartheta_i(c)$ with a uniform radius $r > 0$. Now, by Subsection 4.1, for $\lambda \in \Lambda$ with $|\lambda| \geq 1$, for $c = f(\lambda) \in C_0$, in the chart $V_i$ we see that $\vartheta_i^{-1}(f(\mathbb{D}(\lambda, \delta'))) \cap U_i$ contains a proper holomorphic curve in the ball $B(\vartheta_i(c), r)$, having positive area $\geq \pi r^2$ by Proposition 4.3. The desired conclusion then follows from the comparability of metrics. \hfill $\Box$

5. Algorithm

First of all, we require that $m, c$ satisfy the condition (4.1).

Next, we choose distinct points in a strip of $D$

$$\{y_i\}_{i \in \mathbb{Z}_+} \subset \{x + y\sqrt{-1} : 1/6 \leq x < 1/3, 0 \leq y < 1\}.$$  

A collection of $N \geq 1$ points

$$b_1, \ldots, b_N \in D_R := \{x + y\sqrt{-1} : 1/2 \leq x < 1, 0 \leq y < 1\},$$

located in the right-half of $D$, is said to be distributed sparsely, if for any disc $\mathbb{D}(a, r)$, the following cardinality estimate holds

$$|\mathbb{D}(a, r) \cap \{b_1, \ldots, b_N\}| \leq \max\{1, K \cdot r^2 N\}.$$  

For instance, this can be reached by choosing distinct points

$$b_1, \ldots, b_N \in \left\{\left[\frac{\sqrt{N}}{2} + 1 + \frac{\ell_1}{\sqrt{N}}\right] + \frac{\ell_2}{\sqrt{N}} : 0 \leq \ell_1, \ell_2 \leq \sqrt{N}\right\}.$$  

Let $\mathcal{S} = \{I \subset \mathbb{Z}_+ : I \text{ is a finite nonempty set, or } I = \emptyset, \text{ or } I = \mathbb{Z}_+\}$. Then $\mathcal{S}$ is countable, i.e., there exists some bijection $\sigma : \mathcal{S} \to \mathbb{Z}_+$. On the other hand, we can decompose $\mathbb{Z}_+$ into some infinite disjoint union $\bigcup_{i \in \mathbb{Z}_+} Z_i$, where each component $Z_i$ contains infinitely many integers. For every $I \in \mathcal{S}$, write all the elements of $Z_{\sigma(I)}$ in the increasing order as $Z_{I^1} < Z_{I^2} < Z_{I^3} < \cdots$. Thus we can rearrange

$$\mathbb{Z}_+ = \bigcup_{I \in \mathcal{S}} Z_{\sigma(I)} = \bigcup_{I \in \mathcal{S}} \bigcup_{j \geq 1} \{Z_{I^j}\}.$$  

For every positive integer $i = Z_{I^j}$, we now choose all the $x_\mu \in D$ for $\mu \in A_{r_i} \cap \ell \Gamma$ as follows.

- **Case I:** $I = \emptyset$.
  We require that all the $x_{\mu}$’s are distributed sparsely in $D_R$.

- **Case II:** $I = \{i_1, \ldots, i_k\}$ is some finite set of $k \geq 1$ elements, and $j \geq 1$ is an odd integer.
  We choose all $x_\mu$ from $\{y_{i_1}, \ldots, y_{i_k}\}$, such that, for every $\ell = 1, \ldots, k$, $x_\mu = y_{i_\ell}$ for at least $\left\lfloor \frac{[A_{r_i} \cap \ell \Gamma]}{k} \right\rfloor$ times.

- **Case II’:** $I = \{i_1, \ldots, i_k\}$ is some finite set of $k \geq 1$ elements, and $j \geq 1$ is an even integer.
  We choose some $x_\mu = y_{i_\ell}$ for $\left\lfloor \frac{[A_{r_i} \cap \ell \Gamma]}{2k} \right\rfloor$ times, where $\ell = 1, \ldots, k$, and we require the remaining $x_\mu$’s to be distributed sparsely in $D_R$.  

• Case III: $I = \mathbb{Z}_+$, and $j \geq 1$ is odd.

   Fix a sequence of positive numbers $\{\alpha_j\}_{j=1}^\infty$ with $\sum_{j=1}^\infty \alpha_j = 1$. We choose all $x_\mu$ from $\{y_\ell\}_{\ell=1}^\infty$, such that, for every $\ell \geq 1$, $x_\mu = y_\ell$ for at least $[\alpha_\ell \cdot |A_{r_\ell} \cap c\Gamma|]$ times.

• Case III’ $I = \mathbb{Z}_+$, and $j \geq 1$ is even.

   For every $\ell \geq 1$, we choose some $x_\mu = y_\ell$ for $[\frac{\alpha_\ell}{2} \cdot |A_{r_\ell} \cap c\Gamma|]$ times; and we choose the remaining $x_\mu$’s to be distributed sparsely in $D_R$.

6. Proofs

We are now in position to prove the main results. Recall that from a given sequence of discs of increasing radii $r_\ell \nearrow \infty$, after a perturbation and passing to some subsequence, we can always receive an Ahlfors current for $f$.

Observation 6.1. From the sequence of radii $\{r_\ell\}_{\ell \geq 1}$, one receives a singular Ahlfors current $T$ of the shape

$$T = c_\infty \cdot [C_\infty].$$

Proof. Note that all points in $\mathbb{D}_{r_\ell/3} \setminus \mathbb{D}_{r_{\ell-1}+2}$ keep positive distance $\geq 2 - \sqrt{2}$ to $\Lambda$. Thus for any small open neighborhood $U$ of $C_\infty$, by Subsection 4.1, for $\ell \geq 1$, every $z \in \mathbb{D}_{r_\ell/3} \setminus \mathbb{D}_{r_{\ell-1}+2}$, we have $f(z) \in U$. Not that for $i \gg 1$, by (4.6) and (4.9), one see that the area of $f(\mathbb{D}_{r_{\ell-1}+2})$ is negligible comparing with that of $f(\mathbb{D}_{r_{\ell/3}})$, namely

$$\int_{\mathbb{D}_{r_{\ell-1}+2}} f^*\omega_X \leq K \cdot r_{\ell-1}^2 = o(1) \cdot r_{\ell-1}^2 \leq o(1) \cdot \int_{\mathbb{D}_{r_{\ell/3}}} f^*\omega_X.$$ 

Thus $T$ charges zero mass outside $U$. Since this holds true for any open neighborhood $U \supset C_\infty$, we conclude that $T$ must be supported on $C_\infty$. \qed

Observation 6.2. From the sequence of radii $\{r_j^{Z_j}\}_{j \geq 1}$, one gets an Ahlfors current $T$ having the shape

$$T = a_\infty \cdot [C_\infty] + T_{\text{Diff}},$$

where $a_\infty$ is some positive number and $T_{\text{Diff}}$ is a nontrivial diffuse part.

Proof. Step 1: $T$ charges positive mass along $C_\infty$.

For any open neighborhood $U$ of $C_\infty$, it follows from the preceding proof and the estimate (4.9) that, for $j \gg 1$ and for $i = Z_j$, we have $f(\mathbb{D}_{r_j/3} \setminus \mathbb{D}_{r_{j-1}+2}) \subset U$ and $\int_{\mathbb{D}_{r_j/3} \setminus \mathbb{D}_{r_{j-1}+2}} f^*\omega_X \geq K \cdot r_{j-1}^2$. On the other hand, by (4.6) we know that $\int_{\mathbb{D}_{r_j}} f^*\omega_X \leq K \cdot r_j^2$. Thus $T$ charges $U$ by positive mass $\geq K > 0$. Since this holds true for any $U$, we conclude that $T$ charges positive mass along $C_\infty$.

Step 2: $T$ does not charge any other algebraic curve.

If $D \neq C_\infty$ is an irreducible horizontal curve, using the same notations as that of Subsection 4.5, by the estimate (4.21), and by choosing $\epsilon' \leq \epsilon$, we know that $T$ charges the neighborhood $\vartheta^{-1}(U_j')$ by a small mass $\leq K_{\alpha,\delta,\delta} \cdot \epsilon^2$. Letting $\epsilon \to 0$, we receive that $T$ charges no mass on $U_1 \subset D$. Thus $T$ cannot charge positive mass on $D$.

If $D = \mathbb{P}_a^1$ is an irreducible vertical curve, for an open neighborhood $U$ of $C_\infty$, we claim that $T$ charges no mass on $D \setminus U$. Indeed, for any small $\epsilon > 0$, by Subsection 4.1, for $|z| \gg 1$ with $f(z) \in \pi_1^{-1}(\mathbb{D}(a, \epsilon)) \setminus U$, there must be some $\lambda \in \Lambda$ such that $z \in \mathbb{D}(\lambda, \epsilon)$. Note also that $\pi_1(z) = [z]$, we get $|\lambda| \in \mathbb{D}(a, 2\epsilon)$. By (5.2), we have

$$|\pi_0^{-1}(\mathbb{D}(a, 2\epsilon)) \cap \Lambda \setminus \mathbb{D}_{r_{Z_j}}| \leq K \epsilon^2 \cdot r_{Z_j}^2.$$ 

Hence by the estimates (4.14) and (4.9), such points $z \in \mathbb{D}_{r_{Z_j}}$ with $f(z) \in \pi_1^{-1}(\mathbb{D}(a, \epsilon)) \setminus U$ constitute only small portion of area measured by $f^*\omega_X$, thus $T$ charges mass $\leq K \cdot \epsilon^2$ over $\pi_1^{-1}(\mathbb{D}(a, \epsilon)) \setminus U$. Letting $\epsilon \to 0$, we conclude the claim. Since this holds for any open neighborhood $U$, we receive that $T$ does not charge $\mathbb{P}_a^1$, which finishes this step.

Step 3: $T$ has positive mass outside $C_\infty$. 

Take any small open neighborhood $U$ of $C_\infty$ such that $\overline{U} \cap C_0$ is empty. Note that for every $\lambda \in \Lambda \cap A_{r_{z_j}^2}$ where $j \gg 1$, by (4.22), the image of $f$ about $\lambda$ contributes $\geq K$ area outside $U$. Moreover, by our construction $|\Lambda \cap A_{r_{z_j}^2}| \geq K \cdot r_{z_j}^2$, thus the total area of $f(\mathbb{D}_{r_{z_j}}) \setminus U$ is $\geq K \cdot r_{z_j}^2$. Lastly, by (4.6), we conclude that $T$ charges positive mass outside $U$. \hfill \square

**Observation 6.3.** Fix any finite subset $I = \{i_1, \ldots, i_k\} \subset \mathbb{Z}_+$ having cardinality $k \geq 1$. Then from the sequence of radii $\{r_{z_j}^I\}_{j \geq 1}$, one receives an Ahlfors current $T$ having the shape

$$T = a_\infty \cdot |C_\infty| + \sum_{\ell=1}^k a_i \cdot [\mathbb{P}^1_{\{y_{i\ell}\}}],$$

where $a_\infty, a_{i1}, \ldots, a_{ik}$ are some positive numbers.

**Proof.** For any small open neighborhood $U$ of $C_\infty \cup \mathbb{P}^1_{\{y_{i1}\}} \cup \cdots \cup \mathbb{P}^1_{\{y_{ik}\}}$, by Subsection 4.1, for $j \gg 1$, for all points $z \in \mathbb{D}_{r_{z_j}} \setminus \mathbb{D}_{r_{z_{j-1}}} + 1$, we have $f(z) \in U$. Indeed, choose a very small $\epsilon > 0$ such that $U$ contains $\pi^{-1}_1(\mathbb{D}([y_{i\ell}], \epsilon))$ for every $\ell = 1, \ldots, k$. Then if $\text{dist}(z, \Lambda) \geq \epsilon$, we know that $f(z)$ is very close to $C_\infty$, whence $f(z) \in U$; otherwise $\text{dist}(z, \Lambda) < \epsilon$, that is $z \in \mathbb{D}(\Lambda, \epsilon)$ for some $\lambda \in \pi_0^{-1}([y_{i\ell}])$ ($\ell = 1, \ldots, k$) by our construction of $\Lambda$, hence $f(z) \in \pi^{-1}_1(\mathbb{D}([y_{i\ell}], \epsilon)) \subset U$.

Therefore, by the same argument as the Step 1 of Observation 6.1, one sees that $T$ is supported in $C_\infty \cup \mathbb{P}^1_{\{y_{i1}\}} \cup \cdots \cup \mathbb{P}^1_{\{y_{ik}\}}$. It remains to check that $T$ charges positive mass in each of these components.

Indeed, first of all, our algorithm guarantees that

$$|\Lambda \cap A_{r_{z_j}} \cap \pi_0^{-1}([y_{i\ell}])| \geq K \cdot r_{z_j}^2 \quad (\ell = 1, \ldots, k).$$

By (4.16), for any fixed small $\epsilon > 0$, for large $j \gg 1$ and $i = r_{z_j}^I$, for any $\lambda \in \Lambda \cap A_{r_{z_j}} \cap \pi_0^{-1}([y_{i\ell}])$, the holomorphic disc $f(\mathbb{D}(\lambda, \epsilon))$ is contained in $\pi^{-1}_1(\mathbb{D}([y_{i\ell}], \epsilon))$ with area $\int_{\mathbb{D}(\lambda, \epsilon)} f^* \omega_X \geq K$ bounded from below by some uniformly positively constant independent of $\epsilon$. Thus the total area of such discs is $\geq K \cdot r_{z_j}^2$. Noting that (4.6) implies $\int_{\mathbb{D}(r_{z_j})} \leq K \cdot r_{z_j}^2$, thus $T$ charges mass $\geq K$ on $\pi^{-1}_1(\mathbb{D}([y_{i\ell}], \epsilon))$. Letting $\epsilon \to 0$, we conclude that $T$ charges positive mass on $\mathbb{P}^1_{\{y_{i\ell}\}}$. Lastly, by the same argument as the Step 1 of Observation 6.2, we see that $T$ charges positive mass on $C_\infty$. Thus we conclude the proof. \hfill \square

**Observation 6.4.** Fix any finite subset $I = \{i_1, \ldots, i_k\} \subset \mathbb{Z}_+$ having cardinality $k \geq 1$. Then from the sequence of radii $\{r_{z_j}^I\}_{j \geq 1}$, one receives an Ahlfors current $T$ having the shape

$$T = a_\infty \cdot |C_\infty| + \sum_{\ell=1}^k a_{i\ell} \cdot [\mathbb{P}^1_{\{y_{i\ell}\}}] + T_{\text{Diff}},$$

where $a_\infty, a_{i\ell}$ ($1 \leq \ell \leq k$) are some positive constants and where $T_{\text{Diff}}$ is a nontrivial diffuse part.

**Proof.** **Step 1:** $T$ charges positive mass along $C_\infty, \mathbb{P}^1_{\{y_{i1}\}}, \cdots, \mathbb{P}^1_{\{y_{ik}\}}$.

This follows from the same arguments as in the preceding proof.

**Step 2:** $T$ does not charge any other algebraic curve.

We can check it by using the same arguments as in the Step 2 of Observation 6.2.

**Step 3:** $T$ has positive mass outside $C_\infty \cup \mathbb{P}^1_{\{y_{i1}\}} \cup \cdots \cup \mathbb{P}^1_{\{y_{ik}\}}$.

The argument is similar to the Step 3 of Observation 6.2. The key point is that, by our algorithm,

$$|(\mathcal{D}_R + \Gamma) \cap \Lambda \cap \mathbb{D}_{r_{z_j}}^2| \geq K \cdot r_{z_j}^2,$$

and for every $\lambda \in (\mathcal{D}_R + \Gamma) \cap \Lambda \cap \mathbb{D}_{r_{z_j}}^2 \cdot [\lambda]$ keeps uniform distances $\geq \epsilon > 0$ to $[y_{i1}], \ldots, [y_{ik}]$. Assuming moreover that $j \gg 1$, in the same notation as Observation 4.2 we receive

$$\text{Area} \left( (X \setminus U) \cap f(\mathbb{D}(\lambda, \epsilon/2)) \right) \omega_X \geq K_U.$$
Note that \( f(\mathbb{D}(\lambda, \epsilon/2)) \) stays away from \( \pi_1^{-1}(\mathbb{D}([y_\ell], \epsilon/2)) \) for all \( \ell = 1, \ldots, k \). Thus by the same argument as the preceding proof, we see that \( T \) charges positive mass outside \( U \cup (\bigcup_{\ell=1}^k \pi_1^{-1}(\mathbb{D}([y_\ell], \epsilon/2))) \).

By much the same arguments, we have the following two results.

**Observation 6.5.** From the sequence of radii \( \{r_{z_{2j-1}}\}_{j \geq 1} \), one receives an Ahlfors current \( T \) having the shape

\[
T = a_\infty \cdot [C_\infty] + \sum_{\ell=1}^\infty a_\ell \cdot [P_{[y_\ell]}],
\]

where \( a_\infty, a_\ell (\ell \geq 1) \) are positive numbers.

**Observation 6.6.** From the sequence of radii \( \{r_{z_{2j}}\}_{j \geq 1} \), one receives an Ahlfors current \( T \) having the shape

\[
T = a_\infty \cdot [C_\infty] + \sum_{\ell=1}^\infty a_\ell \cdot [P_{[y_\ell]}] + T_{\text{Diff}},
\]

where \( a_\infty, a_\ell (\ell \geq 1) \) are positive numbers, and where \( T_{\text{Diff}} \) is a nontrivial diffuse part.

Thus we prove Theorems 1.3, 1.4.

### 7. Examples

#### 7.1. Diffuse Ahlfors currents.

Let \( \mathcal{A} = \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda \) be the surface obtained as the product of two elliptic curves where \( \Lambda \) is a lattice. Fix a reference metric \( \omega_\Lambda := d\bar{c} |z_1|^2 + d\bar{c} |z_2|^2 \) on \( \mathcal{A} \). Choose an irrational number \( \lambda \in \mathbb{R} \setminus \mathbb{Q} \). Consider the holomorphic curve \( f: \mathbb{C} \to \mathcal{A} \) given by \( f(z) = ([z], [\lambda z]) \).

**Proposition 7.1.** Any Ahlfors current \( T \) of \( f \) is diffuse.

**Proof.** Since there is no nonconstant holomorphic map from \( \mathbb{P}^1(\mathbb{C}) \) to an elliptic curve, \( \mathcal{A} \) contains no rational curve. Hence by a theorem of Duval [Duv06], it suffices to check that \( T \) charges zero mass along any elliptic curve in \( \mathcal{A} \).

**Fact** (c.f. [DS05, Prop. 1.3.2]). Let \( \Phi : \mathbb{C}/\Gamma_1 \to \mathbb{C}/\Gamma_2 \) be a holomorphic map between complex tori. Then there exist complex numbers \( m, b \) with \( m\Gamma_1 \subset \Gamma_2 \), such that \( \Phi([z]) = [mz + b] \).

Therefore, any nonconstant holomorphic map \( \iota : \mathbb{C}/\Gamma_3 \to \mathbb{C}/\Lambda \times \mathbb{C}/\Lambda \) from an elliptic curve to \( \mathcal{A} \) can be written explicitly as \( \iota([z]) = ([m_1 z + b_1], [m_2 z + b_2]) \) for some complex numbers \( m_1, m_2, b_1, b_2 \), such that \( (m_1, m_2) \neq (0, 0) \) and \( m_1 \Gamma_3, m_2 \Gamma_3 \subset \Gamma \). Hence \( m_2 - \lambda m_1 \neq 0 \). We claim that the intersection numbers

\[
|\iota(\mathbb{C}/\Gamma_3) \cap f(\mathbb{D}_r)| \leq K \cdot r^2
\]

for \( r \gg 1 \). Indeed, we can find a large disc \( \mathbb{D}_R \) containing a fundamental domain of \( \Gamma_3 \). Then for \( z \in \mathbb{D}_R, y \in \mathbb{D}_R \) with \( ([z], [\lambda z]) = ([m_1 y + b_1], [m_2 y + b_2]) \), we receive that

\[
z - (m_1 y + b_1) = \lambda_1, \quad \lambda z - (m_2 y + b_2) = \lambda_2
\]

for some \( \lambda_1, \lambda_2 \in \Lambda \) having absolute values less than \( r + K, |\lambda| \cdot r + K \) respectively.

Since \( m_2 - \lambda m_1 \neq 0 \), we can solve the linear equation (7.2) as

\[
z = \frac{m_2 (\lambda_1 + b_1) - m_1 (\lambda_2 + b_2)}{m_2 - \lambda m_1}, \quad y = \frac{\lambda (\lambda_1 + b_1) - (\lambda_2 + b_2)}{m_2 - \lambda m_1},
\]

Noting that \( y \in \mathbb{D}_R \), for any fixed \( \lambda_1 \), the cardinality of possible choices of

\[
\lambda_2 \in \left( (-m_2 + \lambda m_1) \cdot \mathbb{D}_R + \lambda (\lambda_1 + b_1) - b_2 \right) \cap \Lambda
\]

is \( \leq K \). Thus the cardinality of possible choices of such \( (\lambda_1, \lambda_2) \in \Lambda \times \Lambda \) is \( \leq K \cdot (r + K)^2 \cdot K \leq K \cdot r^2 \). Hence the estimate (7.1) is proved.

By the compactness of \( \iota(\mathbb{C}/\Gamma_3) \), and by shrinking neighborhood \( U \) of \( \iota(\mathbb{C}/\Gamma_3) \) if necessary, each intersection point corresponds to a small area \( o(1) \) component of \( f(\mathbb{D}_r) \cap U \), thus the total area of \( f(\mathbb{D}_r) \cap U \) is \( \leq o(1)K \cdot r^2 \). However, the area growth of \( f(\mathbb{D}_r) \) is \( K\lambda \cdot r^2 \). Thus any obtained Ahlfors current of \( f \) charges mass \( \leq o(1)K \) on \( U \). By shrinking \( U \), we know that \( T \) charges zero mass along \( \iota(\mathbb{C}/\Gamma_3) \). Hence we conclude the proof.

\( \square \)
Take a holomorphic surjective map $\pi_2: \mathcal{A} \to \mathbb{P}^2(\mathbb{C})$, which induces an entire curve
\[ f_2 := \pi_2 \circ f : \mathbb{C} \to \mathbb{P}^2(\mathbb{C}). \]
Since $\pi_2^*\omega_{FS} \geq 0$ is closed, by the geometry of $\mathcal{A}$, in the cohomology class $[\pi_2^*\omega_{FS}]$ we can find a harmonic representative
\[ \omega = a_1\sqrt{-1} \, dz_1 \wedge d\bar{z}_1 + a_2\sqrt{-1} \, dz_2 \wedge d\bar{z}_2 + a_3\sqrt{-1} \, dz_1 \wedge d\bar{z}_2 + a_4\sqrt{-1} \, dz_2 \wedge d\bar{z}_1 \geq 0 \]
for some constants $a_1, a_2, a_3, a_4$. Thus $f^*\omega = K\sqrt{-1} \, dz \wedge d\bar{z} \geq 0$ where
\[ K = a_1 + a_2\lambda^2 + \lambda(a_3 + a_4) \geq 0. \]
Since $a_1, a_2, a_3 + a_4$ cannot vanish simultaneously, at most one $\lambda$ in $\mathbb{R} \setminus \mathbb{Q}$ can make $K = 0$. Now we only choose $\lambda \in \mathbb{R} \setminus \mathbb{Q}$ such that $K > 0$.

**Proposition 7.2.** Any Ahlfors current $T_2$ of $f_2$ is diffuse.

It is interesting to see that $f_2$ is tangent to a multi-valued vector field induced by the push-forward of the constant vector field $(1, \lambda)$ on $\mathcal{A}$.

**Proof.** Assume that $T_2$ is obtained by an increasing radii $\{r_i\}_{i \geq 1} \nearrow \infty$. By our chosen metric $\omega_{\mathcal{A}}$, the “Length-Area” condition of Ahlfors’ lemma is automatically satisfied, thus by passing to some subsequence $\{r_{i_k}\}_{k \geq 1}$ we can receive an Ahlfors current $T$ of $f$.

Noting that $f^*\omega = K\sqrt{-1} \, dz \wedge d\bar{z} > 0$, by closeness of $T$ and by $\text{Area}(f(D_r))\omega_A = K \cdot r^2$, we receive
\[ T(\pi_2^*\omega_{FS}) = T(\omega) \geq K > 0. \]
By the construction of $T$, we receive that
\[ \text{Area}(f_2(D_{r_{i_k}}))_{FS} \geq K \cdot r_{i_k}^2 \quad (k \geq 1). \]
Fix some $K$ such that $\pi_2^*\omega_{FS} \leq K \cdot \omega_A$. For any irreducible curve $C \subset \mathbb{P}^2(\mathbb{C})$, for any open neighborhood $U$ of $C$, we have
\[ \text{Area}(f_2(D_{r_{i_k}}) \cap U)_{FS} = \text{Area}(f(D_{r_{i_k}}) \cap \pi_2^{-1}(U))_{FS} \leq K \cdot \text{Area}(f(D_{r_{i_k}}) \cap \pi_2^{-1}(U))_{\omega_A}. \]
Since $T$ charges no mass along $\pi_2^{-1}(C)$ by Proposition 7.1, by shrinking $U$, the right-hand-side above is $\leq o(1) \cdot r_{i_k}^2$. Thus $T_2$ charges no mass along $C$. Since $C$ is arbitrary, we conclude the proof. 

### 7.2. **Singular Nevanlinna currents on** $X$. Replacing “Ahlfors currents” by “Nevanlinna currents” in Observations 6.1 – 6.6, the same statements still hold true by much the same arguments. Indeed, every upper or lower bound about $\int_{D(\lambda, \epsilon)} f^*\omega_X$ or $\int_{D(r)} f^*\omega_X$ has a corresponding one about order function. A remaining technical detail we would like to mention is the following

**Observation 7.3.** For every $\ell \geq 1$, there exists some positive $\beta_\ell < 1$ such that, for $j \gg 1$ and $i = \mathbb{Z}^+_{\ell}$,
\[ |\Lambda \cap A_{\ell j} \cap D_{\beta(r)} \cap \pi_0^{-1}([y_j])| \geq K_\ell \cdot r_{j}^2. \]
It will be helpful to show that certain Nevanlinna currents of $f$ charge positive mass along $\mathbb{P}^1_{[y_j]}$.

**Proof.** Note that for $j \gg 1$ we have $|\Lambda \cap A_{\ell j} \cap \pi_0^{-1}([y_j])| > \frac{\alpha_\ell}{\ell} \cdot K \cdot r_{j}^2$. Moreover, for any fixed $\beta < 1$, for $j \gg 1$, we have $|\Lambda \cap A_{\ell j} \setminus D_{\beta(r)}| \leq K \cdot (1 - \beta) r_{j}^2$. By these two estimates, we can conclude the proof. 

Therefore we can replace “Ahlfors currents” by “Nevanlinna currents” in the statements of Theorems 1.3, 1.4. Also, by much the same proofs, Propositions 7.1, 7.2 also hold true for Nevanlinna currents.
7.3. Singular Ahlfors currents on blow-ups of \( X \). We sketch a construction of elaborate (in the sense of cohomology classes) singular Ahlfors currents, on the blow-ups of \( X \) having Picard numbers \( \geq 3 \).

For any given positive integer \( n \geq 1 \), recall the collection of points \( y_1, \ldots, y_n \) given in (5.1), let \( \mathcal{X} \) be the blow-up of \( X \) at these points with the corresponding exceptional divisors \( E_1, \ldots, E_n \). Let \( p : \mathcal{X} \to X \) be the projection. We now use the section \( \psi^2 \cdot s_m \) instead of \( \psi \cdot s_m \) to induce an entire curve \( f : \mathbb{C} \to X \).

By lifting we thus receive an entire curve \( \zeta : \mathbb{C} \to \mathcal{X} \). We strengthen our choices of \( m, c \) in (4.1) by the condition \( m \cdot c - 2K/c^2 > 0 \), to make sure that the same clustering phenomenon as in Subsection 4.1 holds true for \( f \) and \( C_\infty \). Let \( e_i \) be the intersection point of the strict transformation \( \tilde{C}_0 \) of \( C_0 \) with \( E_i \) \((i = 1, \ldots, n)\).

The purpose of using \( \psi^2 \) instead of \( \psi \) is to make sure that, for \( \lambda \in \Lambda \) with \( [\lambda] = [y_i] \), we have the certain value \( \zeta(\lambda) = e_i \).

It is well-known that, there exist some hermitian metrics \( h_i \) of the line bundles \( O(-E_i) \) and some small positive constant \( \epsilon_2 < 1 \) such that \( \omega_{\mathcal{X}} := p^* \omega_X + \epsilon_2 \sum_{\ell=1}^n \Theta_{h_{\ell}} \) is a Kähler form on \( \mathcal{X} \) (c.f. [Voi07, Proposition 3.24]). Moreover, comparing the lifting \( \zeta : \mathbb{C} \to \mathcal{X} \) with \( f : \mathbb{C} \to X \), we have

\[
T_{\zeta, f}(\omega_{\mathcal{X}}) := \int_1^r \frac{dt}{t} \int_{\mathbb{D}_t} \zeta^* \omega_{\mathcal{X}} \leq T_{f, \epsilon}(\omega_X) + O(1),
\]

(c.f. [Huy16, page 64, Observation 2.5.1]). Thus we can use the same arguments for (4.6) to conclude that

\[
\int_{\mathbb{D}_{2r_i}} \zeta^* \omega_{\mathcal{X}} \leq K \cdot r_i^2.
\]

For \( \lambda \in \Lambda \) with \( [\lambda] = [y_i] \), computing in local coordinates around \( e_i \), for any small \( \epsilon > 0 \), for any open neighborhood \( U \) of \( E_i \), assuming further that \( [\lambda] \gg 1 \), then the area of \( \zeta(\mathbb{D}(\lambda, \epsilon)) \cap U \) is uniformly positively bounded (independent of \( U \) and \( \epsilon \)) from below by using Propositions 3.4 and 4.3.

Therefore, as an analogue of Observation 6.3, for the finite subset \( I = \{1, \ldots, n\} \), from the sequence of radii \( \{r_{2j-1}\}_{j \geq 1} \), after a perturbation and passing to a subsequence, we can receive an Ahlfors current

\[
T = a_{\infty} \cdot \widetilde{C}_\infty + \sum_{\ell=1}^n a_{\ell} \cdot [\mathbb{C}[y_i]] + \sum_{\ell=1}^n b_\ell \cdot [E_i],
\]

where \( \widetilde{C}_\infty, [\mathbb{C}[y_i]] \) stand for the strict transformations of \( C_\infty, \mathbb{C}[y_i] \), and where \( a_{\infty}, a_{\ell}, b_\ell > 0 \) \((\ell = 1, \ldots, n)\) are some positive numbers.

Similarly, we have the counterparts of other Observations 6.1 – 6.6.

REFERENCES

[BD01] François Berteloot and Julien Duval. Sur l’hyperbolicité de certains complémentaires. Enseign. Math. (2), 47(3-4):253–267, 2001. ↑
[Bru99] Marco Brunella. Courbes entières et feuilletages holomorphes. L’Enseignement Mathématique, 45:195–216, 1999. ↑
[dcC13] Bernardo Freitas Paulo da Costa. Deux exemples sur la dimension moyenne d’un espace de courbes de Brody. Annales de l’Institut Fourier, 63:2223–2237, 2013. ↑
[DH18] Julien Duval and Dinh Tuan Huynh. A geometric second main theorem. Math. Ann., 370(3-4):1799–1804, 2018. ↑
[DS05] Fred Diamond and Jerry Shurman. A first course in modular forms, volume 228 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2005. ↑
[DS14] Tien-Cuong Dinh and Nessim Sibony. Rigidity of Julia sets for Hénon type maps. J. Mod. Dyn., 8(3-4):499–548, 2014. ↑
[DS18] Tien-Cuong Dinh and Nessim Sibony. Unique ergodicity for foliations in \( \mathbb{P}^2 \) with an invariant curve. Invent. Math., 211(1):1–38, 2018. ↑
[Duv06] Julien Duval. Singularités des courants d’Ahlfors. Ann. Sci. École Norm. Sup. (4), 39(3):527–533, 2006. ↑
[Duv08] Julien Duval. Sur le lemme de Brody. Invent. Math., 173(2):305–314, 2008. ↑
[Duv17a] Julien Duval. Around brody lemma. arXiv:1703.01850, 2017. ↑
[Duv17b] Julien Duval. Singularities of local Ahlfors currents. Internat. J. Math., 28(8):1750062, 5, 2017. ↑
[DV20] Tien-Cuong Dinh and Duc-Viet Vu. Algebraic flows on commutative complex Lie groups. Comment. Math. Helv., 95(3):421–460, 2020. ↑
[Huy16] Dinh Tuan Huynh. Sur le Second Théorème Principal. Ph.D. Thesis. Orsay, 2016. ↑
[HVar] Dinh Tuan Huynh and Duc-Viet Vu. On the set of divisors with zero geometric defect. J. Reine Angew. Math., To appear. ↑
[Lev96] B. Ya. Levin. *Lectures on entire functions*, volume 150 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. In collaboration with and with a preface by Yu. Lyubarskii, M. Sodin and V. Tkachenko, Translated from the Russian manuscript by Tkachenko. †

[McQ98] Michael McQuillan. Diophantine approximations and foliations. *Inst. Hautes Études Sci. Publ. Math.*, (87):121–174, 1998. †

[Nev70] Rolf Nevanlinna. *Analytic functions*. Translated from the second German edition by Phillip Emig. Die Grundlehren der mathematischen Wissenschaften, Band 162. Springer-Verlag, New York-Berlin, 1970. †

[Siu74] Yum Tong Siu. Analyticity of sets associated to Lelong numbers and the extension of closed positive currents. *Invent. Math.*, 27:53–156, 1974. †

[Voi07] Claire Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, english edition, 2007. Translated from the French by Leila Schneps. †

HUA LOO-KENG CENTER FOR MATHEMATICAL SCIENCES, ACADEMY OF MATHEMATICS AND SYSTEM SCIENCE, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA & DEPARTMENT OF MATHEMATICS, UNIVERSITY OF EDUCATION, HUE UNIVERSITY, 34 LE LOI ST., HUE CITY, VIETNAM

Email address: dinhtuanhuynh@hueuni.edu.vn

ACADEMY OF MATHEMATICS AND SYSTEM SCIENCE & HUA LOO-KENG KEY LABORATORY OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING 100190, CHINA

Email address: xiesongyan@amss.ac.cn