Crossing probabilities for Voronoi percolation

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Abstract

We prove that the standard Russo-Seymour-Welsh theory is valid for Voronoi percolation. This implies that at criticality the crossing probabilities for rectangles are bounded by constants depending only on their aspect ratio. This result has many consequences, such as the polynomial decay of the one-arm event at criticality.

Introduction

The Russo-Seymour-Welsh (RSW) theory is one of the most important tools in the study of planar percolation. A RSW-result generally refers to an inequality that provides a bound on the probability to cross rectangles in the long direction, assuming a bound on the probability to cross squares (or a rectangle in the short direction). Heuristically, this inequality is obtained by “gluing” together square-crossings in order to obtain a crossing in a long rectangle.

Such results were first obtained for Bernoulli percolation on a lattice with a symmetry assumption [Rus78], [SW78], [Rus81], [Kes82]. For continuum percolation in the plane, a RSW-result has been proved in [Roy90] for open crossing events, and in [Ale96] for closed crossing events. A RSW-theory has been recently developed for FK-percolation, see e.g. [BDC12, DCHN11, DCST14b]. For Voronoi percolation and for Bernoulli percolation on a lattice without symmetry, weaker versions of the standard RSW-result have been proved in [BR06] and [BR10], respectively. Some RSW-techniques have also been recently developed for Bernoulli percolation on quasi-planar graphs, called slabs. The case of a thin slab is treated in [DNS12]. The study of thick slabs in [DCST14a] involves methods similar to those in the present paper.

At criticality, RSW-results imply the following statement, called the box-crossing property: the crossing probability for any rectangle remains bounded between $c$ and $1 - c$, where $c > 0$ is a constant depending only the aspect ratio of the rectangle (in particular it is independent of the scale). For the terminology, we follow [GM13] where the box-crossing is established for Bernoulli percolation on isoradial graphs.

For Bernoulli percolation, the original proof of the Russo-Seymour-Welsh theorem relies on the spatial Markov property and independence: assuming that a left-right crossing exists in a square, one can first find the lowest one by exploring the region below it. Then, the configuration can be sampled independently in the unexplored region (above
the path). This argument does not apply directly to models with spatial dependence. In this paper, we provide a new argument for RSW which does not rely on exploration; this allows us to prove RSW bounds for a larger class of models. We focus on Voronoi percolation, since it is a standard example for which the “lowest path” argument does not apply, due to local dependencies (see beginning of Section 1.2).

First introduced in the context of first passage percolation [VAW93], planar Voronoi percolation has been an active area of research, see for example [BR06, BS98, Aiz98, BBQ05]. It can be defined by the following two-step procedure. (A more detailed definition will be given in Section 1.) First, construct the Voronoi tiling associated to a Poisson point process in $\mathbb{R}^2$ with intensity 1. Then, color independently each tile black with probability $p$ and white with probability $1-p$. The self-duality of the model for $p = 1/2$ suggests that the critical value is $p_c = 1/2$. The first proof of this, given by [BR06], required some RSW-like bounds. Instead of a standard formulation, that paper gave the following, weaker version of the theorem: for $\rho \geq 1$ and $s \geq 1$, let $f_s(\rho)$ be the probability that there exists a left-right black crossing in the rectangle $[0, \rho s] \times [0, s]$. For fixed $0 < p < 1$, they proved that $\inf_{s \geq 0} f_s(1) > 0$ implies that $\limsup_{s \to \infty} f_s(\rho) > 0$ for all $\rho \geq 1$. In other words, a RSW-result has been obtained for arbitrarily large scale, but not for all scales. This result was strengthened in [vdBBV08]: they proved that the condition $\limsup_{s \to \infty} f_s(\rho)$ for some $\rho > 0$ suffices to imply that $\limsup_{s \to \infty} f_s(\rho) > 0$ for every $\rho > 0$.

Our main result is to prove a standard RSW for Voronoi percolation.

**Theorem 1.** Let $0 < p < 1$ be fixed. If $\inf_{s \geq 1} f_s(1) > 0$, then we have for all $\rho \geq 1$ $\inf_{s \geq 1} f_s(\rho) > 0$.

Our work also proves the “high-probability”-version of RSW, stated in Theorem 2. As we will see in Section 4, this second result can be derived from Theorem 1.

**Theorem 2.** Let $0 < p < 1$ be fixed. If $\lim_{s \to \infty} f_s(1) = 1$, then we have for all $\rho \geq 1$ $\lim_{s \to \infty} f_s(\rho) = 1$.

At criticality (when $p = 1/2$), it is known that $f_s(1) = 1/2$ for all $s$, and Theorem 1 above implies the following new results.

**Theorem 3.** Consider Voronoi percolation at $p = 1/2$. Then the following holds.

1. **[Box crossing property]** For all $\rho > 0$, there exists $c(\rho) > 0$ such that
   $$c(\rho) < f_s(\rho) < 1 - c(\rho), \quad \text{for all } s \geq 1.$$  

2. **[Polynomial decay of the 1-arm event]** Let $\pi_1(s,t)$ be the probability that there exists a black path from $[-s, s]^2$ to the boundary of $[-t, t]^2$. There exists $\eta > 0$, such that, for every $1 \leq s < t$,
   $$\pi_1(s,t) \leq \left(\frac{s}{t}\right)^{\eta}.$$
Remark 1. Theorem 3 is merely one potential application of Theorem 1. In the case of Bernoulli percolation, RSW bounds have many consequences. These include Kesten’s scaling relations, the computation of the universal exponents, and tightness of the interfaces in the study of scaling limits, to name a few. We expect that similar results can be derived for Voronoi percolation using Theorem 1.

Remark 2. Our proof is not restricted to Voronoi percolation, and Theorem 1 extends to a large class of planar percolation models. In order to help the reader interested in applying the technique of the present paper in a different context, we isolate in the framework of Voronoi percolation the three sufficient properties that we use (see Section 1 for the main definitions):

(i) Positive association. If $A, B$ are two (black-)increasing events, we have $\mathbb{P}[A \cap B] \geq \mathbb{P}[A] \mathbb{P}[B]$.

(ii) Invariance properties. The measure is invariant under translation, $\pi/2$-rotation and horizontal reflection.

(iii) Quasi-independence. We have

$$\lim_{s \to \infty} \sup_{A \in \sigma(A_{s,t}) \cap \sigma(R^2 \setminus A_{s,t})} |\mathbb{P}[A \cap B] - \mathbb{P}[A] \mathbb{P}[B]| = 0,$$

where we write $A_{s,t} = [-t, t]^2 \setminus [-s, s]^2$, $0 \leq s \leq t < \infty$, and $\sigma(S)$ the sigma-algebra defined by the events measurable with respect to the coloring in $S$, $S \subset \mathbb{R}^2$.

1 Voronoi percolation

1.1 Definitions and notation

General notation. The Lebesgue measure of a measurable set $A \subset \mathbb{R}^2$ is denoted by $\text{vol}(A)$. The cardinality for a set $S$ is denoted by $|S|$ (with $|S| = +\infty$ if $S$ is infinite). We write $d(u, v)$ the Euclidean distance between two points $u, v \in \mathbb{R}^2$. Finally, for $0 \leq s \leq t < \infty$, we consider the square $B_s = [-s, s]^2$ and the annulus $A_{s,t} = B_t \setminus B_s$.

Voronoi tilings. Let $\Omega$ be the set of all subsets $\omega$ of $\mathbb{R}^2$ such that the intersection of $\omega$ with any bounded set is finite. Equip $\Omega$ with the sigma-algebra generated by the functions $\omega \mapsto |\omega \cap A|$, $A \subset \mathbb{R}^2$. To each $\omega \in \Omega$ corresponds a Voronoi tiling, defined as follows. For every $z \in \omega$, let $V_z$ be the Voronoi cell of $z$, defined as the set of all points $v \in \mathbb{R}^2$ such that $d(v, z) \leq d(v, z')$ for all $z' \in \omega$. The family $(V_z)_{z \in \omega}$ of all the cells forms a tiling of the plane.

Voronoi percolation. Given a parameter $p \in [0, 1]$, define the Voronoi percolation process as follows. Let $X$ be a Poisson point process in $\mathbb{R}^2$ with density 1; for completeness, we recall that $X$ is defined as a random variable in $\Omega$ characterized by the following two
properties. For every measurable set \( A \) (with finite measure), \( X \cap A \) contains exactly \( k \) points with probability
\[
\frac{\text{vol}(A)^k}{k!} \exp(-\text{vol}(A)),
\]
and the random variables \( |X \cap A_1|, \ldots, |X \cap A_n| \) are independent whenever \( A_1, \ldots, A_n \) are disjoint measurable sets. Declare each point \( z \in X \) to be open with probability \( p \), and closed with probability \( 1 - p \), independently of each other and of the variable \( X \). Define then \( X_o \) and \( X_c \) to be respectively the set of open and closed points in \( X \). Notice that we could have equivalently defined \( X_o \) and \( X_c \) as two independent Poisson processes with density \( p \) and \( 1 - p \), and then formed \( X = X_o \cup X_c \). Through this paper we write \( \mathbb{P} \) the measure defining the random variable \((X_o, X_c)\) in the space \( \Omega^2 \). The definition of the model strongly depends on the value of \( p \). Nevertheless, in all the proofs, the value of \( p \) will be fixed, and we do not mention the dependence in the underlying \( p \) in our notation.

In Voronoi percolation, we consider the Voronoi tiling \((V_z)_{z \in X}\) associated to \( X \), and we are interested in the random coloring of the plane obtained by coloring black the points in the cells corresponding to open points, and white the points in the cells corresponding to closed points. Otherwise saying, the set of black points is the union of the cells \( V_z \), \( z \in X_o \), and the set of white points is the union of the cells \( V_z \), \( z \in X_c \). The points at the boundary between two cells of different colors are both black and white.

**Crossing events.** In our study, events will be simpler to define in terms of the colors of the points in \( \mathbb{R}^2 \). Let us introduce some notation that allows to do so. Given some set \( U \subset \mathbb{R}^2 \), write \( \mathcal{E}_U \) for the event that all the points in \( U \) are black, one can verify that this event is measurable. For \( S \subset \mathbb{R}^2 \), we say that an event is \( S \)-measurable if it lies in the sigma-algebra generated by the events \((\mathcal{E}_U)_{U \subset S}\). For a fixed subset \( V \) of \( S \), one can verify that the event that all the points in \( V \) are white is \( S \)-measurable. Roughly, an event is \( S \)-measurable if it is defined in terms of the colors in \( S \).

Let \( A, B \) and \( S \) be three subsets of \( \mathbb{R}^2 \) such that \( A, B \subset S \). We call black path from \( A \) to \( B \) in \( S \) an injective continuous map \( \gamma : [0, 1] \to S \) such that \( \gamma(0) \in A \), \( \gamma(1) \in B \), and all the points in the Jordan arc \( \gamma([0, 1]) \) are black. One can verify that the existence of a path from \( A \) to \( B \) in \( S \) is an \( S \)-measurable event. (Up to some negligeable set, it is equivalent to look only at \( \gamma \) obtained by connecting linearly a finite number of rational points.) In the same way, we define a black circuit in the annulus \( A_{s,t} \), \( s < t \) as a Jordan curve included in \( A_{s,t} \) such that the origin 0 is in its interior, and all its point are black. White paths and white circuits are defined analogously. Then, we define the circuit event by
\[
\mathcal{A}_s = \{ \text{there exists a black circuit in the annulus } A_{s,2s} \}.
\]
Finally, for \( \rho > 0 \) and \( s > 0 \), we introduce the crossing probability
\[
f_s(\rho) = \mathbb{P} \left[ \text{there exists a black path from } \{0\} \times [0, s] \text{ to } \{\rho s\} \times [0, s] \text{ in the rectangle } [0, \rho s] \times [0, s] \right].
\]
1.2 External ingredients

Independence properties. One main difficulty in Voronoi percolation is the spatial dependency between the colors of the points: given two fixed points in the plane, there is a positive probability for them to lie on the same tile, thus (for $0 < p < 1$) the probability that they are both black is larger than $p^2$. Due to these correlations, we cannot use the standard “lowest path” argument discussed in the introduction. Nevertheless, the spatial dependencies are only local and the color of a given point is determined with high probability by the process restricted to a neighbourhood of it. More precisely, Lemma 3.2. in [BR06] states that the color of the points in the box $B_s$ are determined with high probability by the process $(X_o, X_c)$ restricted to $B_{s + 2\sqrt{\log s}}$. In our approach, this property is stronger than what we really need, and the following lemma is sufficient. We consider the event

$$F_s = \{\text{for every } z \in A_{2s, 4s}, \text{there exists some point } x \in X \text{ at distance } d(z, x) < s\}.$$

Lemma 1.1. We have $\lim_{s \to \infty} P[F_s] = 1$ and, for any $A_{2s, 4s}$-measurable event $E$, the event $E \cap F_s$ is measurable with respect to the restriction of $(X_o, X_c)$ to $A_{s, 5s}$.

Proof. Let us consider an absolute constant $C > 0$ such that, for every $s \geq 1$, there exists a covering of $A_{2s, 4s}$ by $C$ Euclidean balls of diameter $s$. Fix $s \geq 1$ and a covering of $A_{2s, 4s}$ by $C$ Euclidean balls of diameter $s$. Consider the event that each of these balls contains at least one point of the Poisson process $X$. Using that it is a sub-event of $F_s$, we obtain

$$P[F_s] \geq 1 - Ce^{-\pi s^2/4}.$$

For the second part of the lemma, observe that the color of a point in $A_{2s, 4s}$ is determined by the color of its closest point of the process $X$. When $F_s$ holds, this point lies in $A_{s, 5s}$. Thus, for any $U \subset A_{2s, 4s}$, the event $E_U$ is measurable with respect to $(X_o \cap A_{s, 5s}, X_c \cap A_{s, 5s})$.

FKG inequality. The FKG inequality is an important tool allowing to “glue” black paths. Its proof can be found in [BR10]. Before stating it, we need to define increasing events in the context of Voronoi percolation. An event $E$ is black-increasing if for any configurations $\omega = (\omega_o, \omega_c)$ and $\omega' = (\omega'_o, \omega'_c)$, we have

$$\begin{align*}
\omega \in E \\
\omega_o \subset \omega'_o \text{ and } \omega_c \supset \omega'_c
\end{align*} \Rightarrow \omega' \in E.$$

Proposition 1.2 (FKG inequality). Let $E$ and $F$ be two black-increasing events, then

$$P[E \cap F] \geq P[E]P[F].$$

The following standard inequalities can be easily derived from Proposition 1.2.

Corollary 1.3. Let $s \geq 1$.

1. $f_s(1 + i\kappa) \geq f_s(1 + \kappa)^i f_s(1)^{i-1}$ for any $\kappa > 0$ and any $i \geq 1$,
2. $P[A_s] \geq f_s(3)^4$,
3. $f_s(2) \geq P[A_s]^2$. 5

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1.3 Organization of the proof of Theorem 1

We fix \(0 < p < 1\), and assume that there exists a constant \(c_0 > 0\) such that for all \(s \geq 1\),

\[ f_s(1) \geq c_0. \tag{1} \]

Our goal is to prove that \(\inf_{s \geq 1} P [\mathcal{A}_s] > 0\). Rather than studying only the sequence \((P [\mathcal{A}_s])_{s \geq 1}\), we introduce at each scale \(s\) a real value \(\alpha_s\) and study the pair \((P [\mathcal{A}_s], \alpha_s)_{s \geq 1}\) altogether. (The quantity \(\alpha_s\) is defined at the beginning of Section 2.)

**Step 1: definition of good scales.** In Section 2, a geometric construction valid only when \(\alpha_s \leq 2\alpha_{2s/3}\) provides a RSW-result at scale \(s\). We will refer to such scale as a “good scale”.

**Step 2: renormalization.** In Section 3, we use the independence properties of the model to show that the good scales are close to each other. More precisely, we construct an infinite sequence \(s_1, s_2, \ldots\) of good scales such that \(2s_i \leq s_{i+1} \leq Cs_i\).

Throughout the proof, we will work with constants. By convention, they are elements of \((0, \infty)\), and they do not depend on any parameter of the model. In particular, they never depend on the scale parameter \(s\). These constants will generally be denoted by \(c_0, c_1, \ldots\) or \(C_0, C_1, \ldots\) (depending on whether they have to be thought small or large).

2 Gluing at good scales

**Definition of \(\alpha_s\).** Let \(s \geq 1\) and \(-s/2 \leq \alpha \leq \beta \leq s/2\). Define \(\mathcal{H}_s(\alpha, \beta)\) to be the event that there exists a black path in the square \(B_{s/2}\), from the left side to \(\{s/2\} \times [\alpha, \beta]\) (see Fig. 1 for an illustration). Define also \(\mathcal{X}_s(\alpha)\) to be the event that there exists in \(B_{s/2}\)

- a black path \(\gamma_{-1}\) from \((-s/2) \times [-s/2, -\alpha]\) to \((-s/2) \times [\alpha, s/2]\),
- a black path \(\gamma_1\) from \((s/2) \times [-s/2, -\alpha]\) to \((s/2) \times [\alpha, s/2]\),
- a black path from \(\gamma_{-1}\) to \(\gamma_1\).

![Figure 1: The event \(\mathcal{H}_s(\alpha, \beta)\)](image1)

![Figure 2: The event \(\mathcal{X}_s(\alpha)\)](image2)
Lemma 2.1. For every $s \geq 1$, there exists $\alpha_s \in [0, s/4]$ such that the following two properties hold.

(P1) For all $\alpha \leq \alpha_s$, $P[\mathcal{X}_s(\alpha)] \geq c_1$.

(P2) If $\alpha < s/4$, then for all $\alpha \geq \alpha_s$, $P[\mathcal{H}_s(0, \alpha)] \geq c_0/4 + P[\mathcal{H}_s(\alpha, s/2)]$.

(The constant $c_0$ appearing above was defined in Equation (1.3).)

In the rest of the paper, we fix for every $s \geq 1$ a real number $\alpha_s \in [0, s/4]$ satisfying (P1) and (P2) above.

Proof. For $\alpha \in [0, s/2]$, define

$$\phi_s(\alpha) = P[\mathcal{H}_s(0, \alpha)] - P[\mathcal{H}_s(\alpha, s/2)].$$

For fixed $s$, one can verify that $\phi_s$ is continuous, increasing, and $\phi_s(0) \leq 0$. Define then

$$\alpha_s = \begin{cases} \phi_s^{-1}(c_0/4) & \text{if } \phi_s(s/4) > c_0/4 \\ s/4 & \text{otherwise.} \end{cases}$$

With this definition of $\alpha_s$, the property (P2) is clearly verified. We only need to show that (P1) holds. If $\alpha \leq \alpha_s$, our hypothesis (1.3) and symmetries imply that

$$c_0 \leq 2P[\mathcal{H}_s(0, s/2)]$$

$$\leq 2P[\mathcal{H}_s(0, \alpha)] + 2P[\mathcal{H}_s(\alpha, s/2)]$$

$$\leq 2\phi_s(\alpha) + 4P[\mathcal{H}_s(\alpha, s/2)]$$

$$\leq c_0/4 + 4P[\mathcal{H}_s(\alpha, s/2)].$$

We obtain, for every $\alpha \leq \alpha_s$,

$$P[\mathcal{H}(\alpha, s/2)] \geq c_0/8.$$
larger than $P[\mathcal{X}_s(s/4)]$, and when all of them occur, it implies a vertical black crossing in the rectangle $[0, s] \times [0, 2s]$. We conclude using FKG inequality that
\[ f_s(2) \geq c_1. \]

Now, let $s$ be such that $\alpha_s \leq 2\alpha_{2s/3}$ and $\alpha_s < s/4$. We use the event $\mathcal{X}_{2s/3}(\alpha_{2s/3})$ to connect at scale $2s/3$ two crossings at scale $s$. Consider the two squares $R = (-s/6, -\alpha_{2s/3}) + B_{s/2}$ and $R' = (s/6, -\alpha_{2s/3}) + B_{s/2}$. Notice that $B_{s/3} \subset R$ and $B_{s/3} \subset R'$ since $\alpha_{2s/3} \leq s/6$. Let $\mathcal{E}$ be the event that there exists a black path from left to $\{s/3\} \times [-\alpha_{2s/3}, \alpha_{2s/3}]$ in $R$. Similarly, define $\mathcal{E}'$ as the event that there exists a black path from $\{-s/3\} \times [-\alpha_{2s/3}, \alpha_{2s/3}]$ to right in $R'$. Since $\alpha_s \leq 2\alpha_{2s/3}$, (P2) in Lemma 2.1 ensures that both events $\mathcal{E}$ and $\mathcal{E}'$ occur with probabilities larger than $c_0/4$.

When the three events $\mathcal{X}_{2s/3}(\alpha_{2s/3})$, $\mathcal{E}$ and $\mathcal{E}'$ occur, a black path must exist from left to right in the rectangle $R \cup R'$ (see Fig. 3). The rectangle $R \cup R'$ has aspect ratio $4/3$, and we can conclude using the FKG inequality that
\[ f_s(4/3) \geq P[\mathcal{X}_{2s/3}(\alpha_{2s/3}) \cap \mathcal{E} \cap \mathcal{E}'] \geq c_1 \left( \frac{c_0}{4} \right)^2. \]

![Figure 3: The simultaneous occurrence of $\mathcal{X}_{2s/3}(\alpha_{2s/3})$, $\mathcal{E}$ and $\mathcal{E}'$ implies the existence of a horizontal crossing in $R \cup R'$.](image)

\[ \square \]

### 3 All the scales are good

**Lemma 3.1.** There exists $c_3 > 0$ such that the following holds for every $s \geq 1$ and $t \geq 4s$.

If $P[A_s] \geq c_2$ and $\alpha_t \leq s$, then $P[A_t] \geq c_3$. 

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Proof. Let \( s \geq 1 \) and \( t \geq 4s \). Assume that \( \mathbb{P}[\mathcal{A}_s] \geq c_2 \) and \( \alpha_t \leq s \). Consider the event that there exist

- a black path from left to \( \{0\} \times [0,s] \) in the square \( [-t,0] \times [-t/2,t/2] \),
- a black path from \( \{0\} \times [0,s] \) to right in the square \( [0,t] \times [-t/2,t/2] \),
- and a black circuit in the annulus \( A_{s,2s} \).

Since \( \alpha_t \leq s \), Lemma 2.1 implies that each of the first two paths exists with probability larger than \( c_0/4 \). When the event depicted above occurs, it implies the existence of a horizontal black crossing in the rectangle \( [-t,t] \times [-t/2,t/2] \). Using the FKG inequality, we obtain

\[
 f_t(2) \geq \left(\frac{c_0}{4}\right)^2 c_2.
\]

The standard inequalities of Corollary 1.3 allow to conclude that

\[
 \mathbb{P}[\mathcal{A}_s] \geq c_3,
\]

for some constant \( c_3 > 0 \). \( \square \)

The next lemma is the part of the proof that uses the independence properties of the model. Before stating it, we invoke Lemma 1.1 and define \( s_0 \) such that \( \mathbb{P}[\mathcal{A}_s] \geq 1 - c_3/2 \) for all \( s \geq s_0 \).

**Lemma 3.2.** Define a constant \( C_1 \geq 4 \) large enough, so that

\[
 (1 - c_3/2)^{\log_5(C_1)} < c_0/2. \tag{2}
\]

Let \( s \geq s_0 \) such that \( \mathbb{P}[\mathcal{A}_s] \geq c_2 \), then there exists \( s' \in [4s,C_1s] \) such that \( \alpha_{s'} \geq s \).

**Proof.** Let \( s \geq s_0 \) such that \( \mathbb{P}[\mathcal{A}_s] \geq c_2 \). Assume for contradiction that \( \alpha_t \leq s \) for all \( 4s \leq t \leq C_1s \). Consider for \( i = 1, 2, \ldots \) the event \( \mathcal{E}_i = \mathcal{F}_{s_i} \cap \mathcal{A}_{s_i} \). By Lemma 1.1 these events are independent, and by Lemma 3.1 the probability of \( \mathcal{E}_i \) is larger than \( c_3/2 \) for all \( i \leq \log_5 C_1 \). Since \( \alpha_{C_1s} \leq s \), (P2) in Lemma 2.1 gives

\[
 c_0/2 \leq \mathbb{P}[\mathcal{H}(0,s) \setminus \mathcal{H}(s,C_1s)]
 \leq \mathbb{P}\left[\bigcap_{\log_5(C_1)}^{c} \mathcal{E}_i^c\right]
 \leq (1 - c_3/2)^{\log_5 C_1},
\]

which contradicts Equation (3.2). \( \square \)

**Lemma 3.3.** There exist a constant \( C_3 \geq 4 \) and an infinite sequence \( s_1, s_2, \ldots \) of scales such that for all \( i \geq 1 \),

- \( 4s_i \leq s_{i+1} \leq C_3s_i \),
- \( \mathbb{P}[\mathcal{A}_{s_i}] > c_2 \).
Proof. Since $\alpha_s \leq s$, the sequence $\alpha_s$ cannot grow super-linearly, and there must exist $s_1 \geq s_0$ such that $\alpha_{s_1} \leq 2 \alpha_{2s_1/3}$. By Lemma 2.2 we obtain that $\mathbb{P}[A_{s_1}] > c_2$. Therefore, Lemma 3.2 implies the existence of $s'_1 \in [4s_1, C_1s_1]$ such that

$$\alpha_{s'_1} \geq s'_1/C_1.$$ 

Then, there must exist $s_2 \in [s'_1, C_1^{\log_{4/3}(3/2)} s'_1]$ such that $\alpha_{s_2} \leq 2 \alpha_{2s_2/3}$, otherwise the bound $\alpha_s \leq s$ would be contradicted. Define $C_3 = C_1^{1+\log_{4/3}(3/2)}$. We have $s_2 \in [4s_1, C_3s_1]$ and we find from Lemma 2.2 that $\mathbb{P}[A_{s_2}] \geq c_2$.

The constant $C_3$ is independent of the scale, we can thus iterate the construction above, and find by induction $s_3, s_4, \ldots$ Theorem 1 follows easily from Lemma 3.3 and the standard inequalities of Corollary 1.3.

4 Theorem 1 implies Theorem 2

In this section, we will need the following proposition, called the “square root trick”. It is a standard consequence of the FKG inequality (see e.g. [Gri99]).

**Proposition 4.1. (square root trick)** Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be increasing events, and write $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$. Then, the following inequality holds:

$$\max_{1 \leq i \leq k} \mathbb{P}_p[\mathcal{E}_i] \geq 1 - (1 - \mathbb{P}_p[\mathcal{E}])^{1/k}.$$ 

**Proof of Theorem 2.** We assume that $f_s(1)$ converges to 1 when $s$ tends to infinity. We prove that $f_s(4/3)$ converges also to 1. The more general statement of Theorem 2 can be then obtained by using the standard inequalities of Corollary 1.3

Fix $\varepsilon > 0$. By Theorem 1 there exists a constant $c > 0$ such that $\mathbb{P}[A_s] > c$ for all $s \geq 1$. With the same argument as in the proof of Lemma 3.2, we can use Lemma 1.1 to show the following. There exists $\eta > 0$, such that for every $s$ large enough,

$$\mathbb{P}[\text{There exists a black circuit surrounding } B_{\eta s} \text{ in } A_{\eta s, s/4}] > 1 - \varepsilon.$$ 

The right side of $B_{s/2}$ can be covered with less than $[1/\eta]$ segments of length 2$\eta s$. Thus, by the square root trick, there exists $y_s \in [-s/2, s/2]$ such that

$$\mathbb{P}[\mathcal{H}_s(y_s - \eta s, y_s + \eta s)] \geq 1 - (1 - f_s(1))^{1/\eta}.$$ 

Consider the event that there exist

- a black path from left to $\{s/2\} \times [y_s - \eta s, y_s + \eta s]$ in $B_{s/2}$,
- a black path from $\{s/2\} \times [y_s - \eta s, y_s + \eta s]$ to right in $(s, 0) + B_{s/2}$,
- a black circuit in the annulus $(s/2, y_s) + A_{\eta s, s/4}$. 


When this event occurs, it implies the existence of a left-right crossing in the rectangle \([-s/2, 3s/2] \times [-3s/4, 3s/4]\). By the FKG inequality, we obtain that for all \(s\) large enough
\[
f_{3s/2}(4/3) \geq (1 - (1 - f_s(1))^{1/\eta})^2 (1 - \epsilon).
\]
This implies that \(\lim \inf_{s \to \infty} f_s(4/3) \geq 1 - \epsilon\).

\[
\square
\]

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