Exact solutions to the time-dependent supersymmetric multiphoton Jaynes-Cummings model and the Chiao-Wu model

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By using both the Lewis-Riesenfeld invariant theory and the invariant-related unitary transformation formulation, the present paper obtains the exact solutions to the time-dependent supersymmetric two-level multiphoton Jaynes-Cummings model and the Chiao-Wu model that describes the propagation of a photon inside the optical fiber. On the basis of the fact that the two-level multiphoton Jaynes-Cummings model possesses the supersymmetric structure, an invariant is constructed in terms of the supersymmetric generators by working in the sub-Hilbert-space corresponding to a particular eigenvalue of the conserved supersymmetric generators (i.e., the time-independent invariant). By constructing the effective Hamiltonian that describes the interaction of the photon with the medium of the optical fiber, it is further verified that the particular solution to the Schrödinger equation is the eigenfunction of the second-quantized momentum operator of photons field. This, therefore, means that the explicit expression (rather than the hidden form that involves the chronological product) for the time-evolution operator of wave function is obtained by means of the invariant theories.

Keywords: exact solutions, supersymmetric Jaynes-Cummings model, Chiao-Wu model, invariant theory

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I. INTRODUCTION

The model that describes the interaction between a two-level atom and a quantized single-mode electromagnetic field is termed the Jaynes-Cummings (J-C) model [1], which can be applied to the study of many quantum effects such as the quantum collapses and revivals of the atomic inversion, photon antibunching, squeezing of the radiation field, inversionless light amplification, electromagnetic induced transparency [2–5], etc. By making use of the generalized invariant theory [6], we can obtain exact solutions and geometric phase factor of the two-level J-C model whose parameters of the Hamiltonian are totally time-dependent. Additionally, there exists another type of J-C model (so-called two-level multiphoton Jaynes-Cummings model) which possesses supersymmetric Lie-algebraic structure. In this generalization of the J-C model, the atomic transitions are mediated by \( k \) photons [7,8]. Singh has shown that this model can be used to study the multiple atom scattering of radiation and the multiphoton emission, absorption, and the laser processes [9]. Some authors introduced a supersymmetric unitary transformation to diagonalize the Hamiltonian of this J-C model and obtain the eigenfunctions of the stationary Schrödinger equation [10,11]. In the present paper, we generalize this method and obtain the exact solutions and the expression for the geometric phase factor of the totally time-dependent two-level multiphoton Jaynes-Cummings (TLMJ-C) model through the invariant-related unitary transformation formulation.

The exact solutions and the geometric phase factor of the time-dependent spin model have been extensively investigated by many authors. Bouchiat and Gibbons discussed the geometric phase for the spin-1 system [12]. Datta et al. found the exact solution for the spin-\( \frac{1}{2} \) system [13] by means of the classical Lewis-Riesenfeld theory and Mizrahi calculated A-A phase for the spin-\( \frac{1}{2} \) system [14] in a time-dependent magnetic field. The more systematic approach to obtaining the formally exact solution for the spin-\( j \) system was proposed by Gao et al. [15] who made use of the Lewis-Riesenfeld quantum theory [16]. In this work, the generators of the Hamiltonian form the \( SU(2) \) algebra.

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The dynamical algebraic structure of a type of general case of spin model such as the time-dependent $L - S$ coupled system has been investigated and a set of $SU(N)$ generators was constructed to linearize the Hamiltonian by Cen et al. [17]. Investigation in the direction of many-spin system has been completed by F. Yan et al. [18] who used a unitary transformation formulation which originated in the work of Gao in 1991 [6]. They discussed the time evolution of the Heisenberg spin system and obtained the formally exact solutions of the Schrödinger equation in a time-dependent magnetic field. The relationship between the unitary transformations of the dynamics of quantum systems with time-dependent Hamiltonians and the gauge theories has been found by Montesinos et al. [19]. In particular, they showed that the nonrelativistic dynamics of spin- particles in a magnetic field $B^i(t)$ can be formulated in a natural way as $SU(2)$ gauge theory, with the magnetic field $B^i(t)$ playing the role of the gauge potential. This geometric interpretation provides a powerful method to find the exact solutions of the Schrödinger equation. Although many investigations have been done in the direction of the time-dependent spin model, the subjects of all these works were the first-quantized cases. In this paper, we consider another time-dependent second-quantized spin model, namely, the generalized Chiao-Wu model [20], which describes the propagation of the photon inside the optical fiber.

In 1984, Berry showed that the wavefunction in quantum adiabatic process would give rise to the geometric phase factor [21]. Afterwards, Chiao and Wu suggested their model, which is the first physical realization of Berry’s geometric phase. It is well known that the geometric phase factor appears only in the systems with the time-dependent Hamiltonian or the Hamiltonian involving the evolution parameters. Both the classical geometric phase factor and the quantum geometric phase factor that is associated with vacuum, exist in the systems of the second-quantized field theory. In 1980’s, there was an argument concerning whether the geometric phase in the fiber experiment performed by Tomita and Chiao [22] belongs to the quantum or classical level [23–27]. Chiao-Wu’s theory [20] which is concerned with the polarization of the light propagating inside the noncoplanar optical fiber has no expressions for the Hamiltonian; moreover, their theory is first-quantization. From the point of view of us, it is consequently not applicable to investigating the geometric phase factor of quantum level. Only the system, of which whose Hamiltonian is second-quantized will present the geometric phase factor of quantum level. In what follows we show that the propagation of the photon in the fiber is essentially a second-quantized problem, i.e., the quantum-theory problem whose Hamiltonian is time-dependent.

Note, however, that Berry’s theory of the geometric phase proposed in 1984 is applicable only to the case of adiabatic approximation [21]. The invariant theory that is appropriate to treat the time-dependent systems was first proposed by Lewis and Riesenfeld (L-R) [16] in 1969. In 1991, Gao et al. generalized the L-R invariant theory and proposed the invariant-related unitary transformation formulation [6,28]. One of the advantages of this unitary transformation method is that it can transform the hidden form (which is related to the chronological product) of the time-evolution operator $U(t)$ into the explicit expression. Many works have shown that the invariant-related unitary transformation approach is a powerful tool for investigating time-dependent systems and geometric phase factor [29–31].

II. THE IN Variant Theory AND THE IN Variant-related Unitary Transformation FORMULATION

For the sake of illustrating the L-R invariant theory [16] easily, we consider a one-dimensional system whose Hamiltonian $H(t)$ is time-dependent. According to the L-R invariant theory, a Hermitian operator $I(t)$ is called invariant if it satisfies the following invariant equation (in the unit $\hbar = 1$)

$$\frac{\partial I(t)}{\partial t} + \frac{1}{i}[I(t), H(t)] = 0. \quad (2.1)$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$I(t) |\lambda_n, t\rangle = \lambda_n |\lambda_n, t\rangle, \quad (2.2)$$

where

$$\frac{\partial \lambda_n}{\partial t} = 0. \quad (2.3)$$

The time-dependent Schrödinger equation (in the unit $\hbar = 1$) for the system is

$$i \frac{\partial |\Psi(t)\rangle_s}{\partial t} = H(t) |\Psi(t)\rangle_s. \quad (2.4)$$
In terms of the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of Eq. (2.4) differs from the eigenfunction $|\lambda_n, t\rangle$ of the invariant $I(t)$ only by a phase factor $\exp\left[\frac{i}{\hbar}\phi_n(t)\right]$, then the general solution of the Schrödinger equation (2.4) can be written as

$$\Psi(t) = \sum_n C_n \exp\left[\frac{i}{\hbar}\phi_n(t)\right]|\lambda_n, t\rangle,$$

(2.5)

where

$$\phi_n(t) = \int_0^t \left\langle \lambda_n, t' \left| H(t') - i \frac{\partial}{\partial t'} \right| \lambda_n, t' \right\rangle dt'.$$

(2.6)

$|\lambda_n, t\rangle_s = \exp\left[\frac{i}{\hbar}\phi_n(t)\right]|\lambda_n, t\rangle$ $(n = 1, 2, \cdots)$ are said to form a complete set of the solutions of Eq. (2.4). The statement outlined above is the basic content of the L-R invariant theory.

A time-dependent unitary transformation operator can be constructed to transform $I(t)$ into a time-independent invariant $I_V \equiv V(t)I(t)V(t)$ [6,28] with

$$I_V |\lambda_n\rangle = \lambda_n |\lambda_n\rangle,$$

(2.7)

$$|\lambda_n\rangle = V(t)|\lambda_n, t\rangle.$$

(2.8)

Under the unitary transformation $V(t)$, the Hamiltonian $H(t)$ is correspondingly changed into $H_V(t)$

$$H_V(t) = V(t)H(t)V(t) - V(t)i\frac{\partial V(t)}{\partial t}. $$

(2.9)

In accordance with this unitary transformation method [6], it is very easy to verify that the particular solution $|\lambda_n, t\rangle_{s0}$ of the time-dependent Schrödinger equation associated with $H_V(t)$

$$i\frac{\partial |\lambda_n, t\rangle_{s0}}{\partial t} = H_V(t)|\lambda_n, t\rangle_{s0},$$

(2.10)

is different from the eigenfunction $|\lambda_n\rangle$ of $I_V$ only by the same phase factor $\exp\left[\frac{i}{\hbar}\phi_n(t)\right]$ as that in Eq. (2.5), i.e.,

$$|\lambda_n, t\rangle_{s0} = \exp\left[\frac{i}{\hbar}\phi_n(t)\right]|\lambda_n\rangle.$$

(2.11)

Substitution of $|\lambda_n, t\rangle_{s0}$ of Eq. (2.10) into Eq. (2.11) yields

$$\dot{\phi}(t) |\lambda_n\rangle = H_V(t)|\lambda_n\rangle,$$

(2.12)

which means that $H_V(t)$ differs from $I_V(t)$ only by a time-dependent multiplying c-number factor. It can be seen from Eq. (2.10) that the particular solution of Eq. (2.10) can be easily obtained by calculating the phase from Eq. (2.12). Thus, one is led to the conclusion that if the $V(t)$, $I_V$, $H_V(t)$ and the eigenfunction $|\lambda_n\rangle$ of $I_V$ have been found, the problem of solving the complicated time-dependent Schrödinger equation (2.4) reduces to that of solving the much simplified equation (2.10). This paper obtains the exact solutions of the time-dependent Schrödinger equation describing TLMJ-C model and expression for its geometric phase factor by making use of this invariant-related unitary transformation method.

III. THE EXACT SOLUTIONS OF THE TIME-DEPENDENT TLMJ-C MODEL

The time-dependent Schrödinger equation and the Hamiltonian of the time-dependent TLMJ-C model under the rotating wave approximation are given by

$$H(t)|\Psi(t)\rangle = i\frac{\partial}{\partial t}|\Psi(t)\rangle,$$

$$H(t)|\Psi(t)\rangle = i\frac{\partial}{\partial t}|\Psi(t)\rangle,$$
where \(a^\dagger\) and \(a\) are the creation and annihilation operators for the electromagnetic field, and obey the commutation relation \([a, a^\dagger] = 1\); \(\sigma_z\) and \(\sigma_\pm\) denote the two-level atom operators which satisfy the commutation relation \([\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm\); \(g(t)\) and \(g^*(t)\) are the coupling coefficients and \(k\) is the photon number in each atom transition process; \(\omega_0(t)\) and \(\omega(t)\) are respectively the transition frequency and the mode frequency. All the parameters in \(H(t)\) are time-dependent.

The supersymmetric structure can be found in the TLMJ-C model by defining the following supersymmetric transformation generators [10,11]:

\[
N = a^\dagger a + \frac{k-1}{2}\sigma_z + \frac{1}{2} = \left( \begin{array}{cc} a^\dagger a + \frac{k}{2} & 0 \\ 0 & a a^\dagger - \frac{k}{2} \end{array} \right), \quad N' = \left( \begin{array}{cc} a^k(a^\dagger)^k & 0 \\ 0 & (a^\dagger)^k a^k \end{array} \right);
\]
\[
Q = (a^\dagger)^k \sigma_- = \left( \begin{array}{cc} 0 & (a^\dagger)^k \\ (a^\dagger)^k & 0 \end{array} \right), \quad Q' = a^k \sigma_+ = \left( \begin{array}{cc} 0 & a^k \\ a^k & 0 \end{array} \right).
\]

It is easily verified that \((N, N', Q, Q')\) form supersymmetric generators and have supersymmetric Lie algebra properties, i.e.,

\[
Q^2 = (Q')^2 = 0, \quad [Q, Q] = N' \sigma_z, \quad [N, N'] = 0, \quad [N, Q] = Q,
\]
\[
[N, Q'] = -Q', \quad \{Q^\dagger, Q\} = N', \quad \{Q, \sigma_z\} = \{Q^\dagger, \sigma_z\} = 0,
\]
\[
[Q, \sigma_z] = 2Q, \quad [Q^\dagger, \sigma_z] = -2Q^\dagger, \quad (Q^\dagger - Q)^2 = -N'.
\]

where \(\{\}\) denotes the anticommuting bracket. By the aid of (3.2) and (3.3), the Hamiltonian (3.1) of the TLMJ-C model can be rewritten as

\[
H(t) = \omega(t)N + \frac{\omega(t) - \delta(t)}{2}\sigma_z + g(t)Q + g^*(t)Q^\dagger - \frac{\omega(t)}{2}
\]

with \(\delta(t) = k\omega(t) - \omega_0(t)\).

The equation which governs the time evolution of the TLMJ-C model is the time-dependent Schrödinger equation (2.4). We will show the solvability of Eq. (2.4) by using the generalized invariant formulation in what follows.

According to the invariant theory, we should first construct an invariant \(I(t)\). From the invariant equation (2.1) one can see that \(I(t)\) is the linear combination of \(N, \sigma_z, Q\) and \(Q^\dagger\). However, it should be pointed out that the generalized invariant theory can only be applied to the study of the system for which there exists the quasi-algebra defined in Ref. [14]. It is easily seen from (3.3) that there is no such quasi-algebra for the TLMJ-C model. We generalize the method which has been used for finding the dynamical algebra \(O(4)\) of the hydrogen atom to treat this type of time-dependent models. In the case of hydrogen, the dynamical algebra \(O(4)\) was found by working in the sub-Hilbert-space corresponding to a particular eigenvalue of the Hamiltonian [32]. In this paper, we will show that a generalized quasi-algebra can also be found by working in a sub-Hilbert-space corresponding to a particular eigenvalue of \(N\) in the time-dependent TLMJ-C model. This generalized quasi-algebra enables one to obtain the complete set of exact solutions for the Schrödinger equation. It is easily verified that \(N'\) commutes with \(H(t)\) and is a time-independent invariant according to Eq. (2.1).

Use is made of \(a^k(a^\dagger)^k |m\rangle = \frac{(m+k)!}{m!} |m\rangle\) and \((a^\dagger)^k a^k |m + k\rangle = \frac{(m+k)!}{m!} |m + k\rangle\), then one can arrive at

\[
N' \left( \begin{array}{c} |m\rangle \\ |m + k\rangle \end{array} \right) = \lambda_m \left( \begin{array}{c} |m\rangle \\ |m + k\rangle \end{array} \right)
\]

with \(\lambda_m = \frac{(m+k)!}{m!}\). Thus we obtain the supersymmetric quasi-algebra \((N, Q, Q^\dagger, \sigma_z)\) in the sub-Hilbert-space corresponding to the particular eigenvalue \(\lambda_m\) of \(N'\), by replacing the generator \(N'\) with \(\lambda_m\) in the commutation relations in (3.3), namely,

\[
[Q^\dagger, Q] = \lambda_m \sigma_z, \quad \{Q^\dagger, Q\} = \lambda_m, \quad (Q^\dagger - Q)^2 = -\lambda_m.
\]

In accordance with the invariant theory, the invariant \(I(t)\) is often of the form

\[
I(t) = c(t)Q^\dagger + c^*(t)Q + b(t)\sigma_z
\]
Thus, the invariant $I$ convenience, and the results are that, by the complicated and lengthy computations, if $\alpha$ in the meanwhile, under the unitary transformation (3.9), the Hamiltonian (3.4) can be transformed into where we use the Baker-Campbell-Hausdorff formula [33]

$$I(t) = \exp[\alpha(t)Q - \alpha^*(t)Q^\dagger]$$  \hspace{1cm} (3.9)$$

with $\alpha^*(t)$ being the complex conjugation of $\alpha(t)$. With the help of the commutation relations (3.3), it can be found that, by the complicated and lengthy computations, if $\alpha(t)$ and $\alpha^*(t)$ satisfy the following equations

$$\sin(4\alpha^*\lambda_m)^\dagger = \frac{\lambda_m(\alpha^*c + c^*\alpha)}{(4\alpha^*\lambda_m)^\dagger}, \quad \cos(4\alpha^*\lambda_m)^\dagger = b,$$  \hspace{1cm} (3.10)$$
a time-independent invariant can be obtained as follows

$$I_V \equiv V^\dagger(t)I(t)V(t) = \sigma_z.$$  \hspace{1cm} (3.11)$$

From Eq. (3.10), we substitute the time-dependent parameters $\lambda$ and $\gamma$ for $c, c^*$ and $b$ in $I(t)$ for simplicity and convenience, and the results are

$$\alpha = \frac{i}{2} \exp(i\gamma), \quad \alpha^* = \frac{i}{2} \exp(-i\gamma),$$
$$c = \frac{\sin \lambda \exp(i\gamma)}{\lambda_{m}}, \quad c^* = \frac{\sin \lambda \exp(-i\gamma)}{\lambda_{m}},$$  \hspace{1cm} (3.12)$$

Thus, the invariant $I(t)$ in (3.7) can be rewritten

$$I(t) = \frac{\sin \lambda}{\lambda_{m}}[\exp(i\gamma)Q + \exp(-i\gamma)Q^\dagger] + \cos \lambda \sigma_z.$$  \hspace{1cm} (3.13)$$

In the meanwhile, under the unitary transformation (3.9), the Hamiltonian (3.4) can be transformed into

$$H_V(t) \equiv V^\dagger(t)H(t)V(t) - V^\dagger(t)\frac{\partial}{\partial t}V(t)$$
$$= \omega N - \frac{\omega}{2} + \left\{ \frac{\omega}{2}(1 - \cos \lambda) + \frac{1}{2} \lambda_m^\dagger [g \exp(-i\gamma) + g^* \exp(i\gamma)] \sin \lambda +$$
$$+ \frac{\omega - \delta}{2} \cos \lambda + \frac{\gamma}{2}(1 - \cos \lambda) \right\} \sigma_z,$$  \hspace{1cm} (3.14)$$

where we use the Baker-Campbell-Hausdorff formula [33]

$$V^\dagger(t)\frac{\partial}{\partial t}V(t) = \frac{\partial}{\partial t}L + \frac{1}{2!}[\frac{\partial}{\partial t}L, L] + \frac{1}{3!}[\frac{\partial}{\partial t}L, [L, L]] + \frac{1}{4!}[[\frac{\partial}{\partial t}L, [L, L]], L] + \cdots.$$  \hspace{1cm} (3.15)$$

with $V(t) = \exp[L(t)]$. The eigenstates of $\sigma_z$ corresponding to the eigenvalue $\sigma = +1$ and $\sigma = -1$ are $|1\rangle$ and $|0\rangle$, and the eigenstate of $N'$ is $|m\rangle$ in terms of (3.5). From Eq. (2.6), (2.11), (2.12), we obtain two particular solutions of the time-dependent Schrödinger equation of the time-dependent TLMJ-C model which are written in the forms

$$|\Psi_{m, \sigma=+1}(t)\rangle = \exp\left\{ \frac{1}{4} \int_0^t [\dot{\phi}_d, \sigma=+1(t') + \dot{\phi}_g, \sigma=+1(t')] dt' \right\} V(t)|m\rangle$$  \hspace{1cm} (3.16)$$
with
\[ \dot{\phi}_{d,\sigma=+1}(t') = (m + \frac{k}{2})\omega(t') + \frac{1}{2}\lambda_m \{ g(t') \exp[-i\gamma(t')] + g^*(t') \exp[i\gamma(t')] \} \sin \lambda(t') - \frac{\delta(t')}{2} \cos \lambda(t') \] (3.17)

and
\[ \dot{\phi}_{g,\sigma=+1}(t') = \frac{\dot{\gamma}(t')}{2} [1 - \cos \lambda(t')]; \] (3.18)

and
\[ |\Psi_{m,\sigma=-1}(t)\rangle = \exp\left\{ \frac{1}{t} \int_0^t [\dot{\phi}_{d,\sigma=-1}(t') + \dot{\phi}_{g,\sigma=-1}(t')] dt' \right\} V(t) \left( \frac{0}{|m + k|} \right) \] (3.19)

with
\[ \dot{\phi}_{d,\sigma=-1}(t') = (m + \frac{k}{2})\omega(t') + \frac{1}{2}\lambda_m \{ g(t') \exp[-i\gamma(t')] + g^*(t') \exp[i\gamma(t')] \} \sin \lambda(t') + \frac{\delta(t')}{2} \cos \lambda(t') \] (3.20)

and
\[ \dot{\phi}_{g,\sigma=-1}(t') = -\frac{\dot{\gamma}(t')}{2} [1 - \cos \lambda(t')]. \] (3.21)

These two particular solutions of the Schrödinger equation (2.4) contain corresponding dynamical phase factor \( \exp\left[ \frac{i}{\hbar} \int_0^t \dot{\phi}_{d,\sigma}(t') dt' \right] \) and the geometric phase factor \( \exp\left[ \frac{i}{\hbar} \int_0^t \dot{\phi}_{g,\sigma}(t') dt' \right] \) with \( \sigma = \pm 1 \). It should be noted that when the parameter \( \lambda \) is taken to be a constant in the expression (3.18) and (3.21), the geometric phases in a cycle in the parameter space of the invariant \( I(t) \) can be rewritten as
\[ \phi_\sigma(T) = \frac{\sigma}{2} 2\pi (1 - \cos \lambda), \] (3.22)

where \( 2\pi (1 - \cos \lambda) \) is the solid angle over the parameter space of the invariant \( I(t) \) which represents the geometric meanings of the phase factor. Apparently, it can be seen that the former is dependent on the transition frequency \( \omega_0(t) \) and the mode frequency \( \omega(t) \), and the coupling coefficients \( g(t) \) and \( g^*(t) \) as well, whereas the latter is immediately independent of these frequency parameters and the coupling coefficients.

**IV. THE SECOND-QUANTIZED SPIN MODEL AND THE SOLUTION TO THE TIME-DEPENDENT SCHRODINGER EQUATION**

The exact solutions of the time-dependent spin model can be obtained by using the invariant-related unitary transformation method. The Hamiltonian of the spin model is generally of the form
\[ H(t) = \vec{c}(t) \cdot \vec{J}, \] (4.1)

where \( \vec{c}(t) \) is the time-dependent arbitrary vector parameters and can be taken
\[ \vec{c}(t) = c_0(t)[\sin \theta(t) \cos \varphi(t), \sin \theta(t) \sin \varphi(t), \cos \theta(t)]. \] (4.2)

With \( J_\pm = J_1 \pm iJ_2 \) satisfies the commuting relations \( [J_3, J_\pm] = \pm J_\pm, [J_+, J_-] = 2J_3 \). Then the expression (4.1) for \( H(t) \) can be rewritten
\[ H(t) = c_0(t)\left\{ \frac{1}{2} \sin \theta(t) \exp[-i\varphi(t)] J_+ + \frac{1}{2} \sin \theta(t) \exp[i\varphi(t)] J_- + \cos \theta(t) J_3 \right\}. \] (4.3)

Note that the analytic solution to the time-dependent Schrödinger equation governing the spin model is easily obtained from the previous solutions to the time-dependent supersymmetric two-level multiphoton Jaynes-Cummings.
It follows from (3.1) and (3.3) that if when \(\omega(t)\) and \(N'\) are chosen to be 0 and 1, respectively, and the supersymmetric operators, \(Q^\dagger\), \(Q\), and \(\frac{1}{2}\sigma_z\), are respectively replaced with \(J_+\), \(J_-\) and \(J_3\), then the above supersymmetric Jaynes-Cummings is formally changed into the spin model, and it is therefore convenient to solve this time-dependent spin model. In other word, we can introduce an invariant-related unitary transformation operator \(V(t)\) as follows

\[
V(t) = \exp[\beta(t)J_+ - \beta^*(t)J_-]
\]

where the time-dependent parameters are taken to be

\[
\beta(t) = -\frac{\lambda(t)}{2} \exp[-i\gamma(t)], \quad \beta^*(t) = -\frac{\lambda(t)}{2} \exp[i\gamma(t)].
\]

By making use of the Baker-Campbell-Hausdorff formula [33] and \(V(t)\) in expression (4.4), or immediately from Eq. (3.14), one can obtain \(H_V(t)\) from \(H(t)\)

\[
H_V(t) = V^\dagger(t)H(t)V(t) - V^\dagger(t)i\frac{\partial V(t)}{\partial t} = \{c_0[\cos\lambda \cos\theta + \sin\lambda \sin\theta \cos(\gamma - \varphi)] + \dot{\gamma}(1 - \cos\lambda)\}J_3.
\]

Analogous to Eq. (3.11), the invariant

\[
I(t) = \frac{1}{2} \sin\lambda(t) \exp[-i\gamma(t)]J_+ + \frac{1}{2} \sin\lambda(t) \exp[i\gamma(t)]J_- + \cos\lambda(t)J_3
\]

is changed into the time-independent operator, \(J_3\), i.e.,

\[
I_V(t) = V^\dagger(t)I(t)V(t) = J_3,
\]

under this unitary transformation \(V(t)\). From the two expressions (4.6) and (4.8), one can see that \(H_V(t)\) differs from \(I_V\) only by a time-dependent c-number factor. Thus with the help of (2.10), (2.11) and (2.12), it is easy to get the general solution of the time-dependent Schrödinger equation (2.4)

\[
|\Psi(t)\rangle_s = \sum_mC_m \exp[\frac{1}{i}\phi_m(t)]V(t)|m\rangle
\]

with the coefficients \(C_m = \langle m, t = 0|\Psi(0)\rangle_s\). The phase \(\phi_m(t) = \phi_{d,m}(t) + \phi_{g,m}(t)\) includes the dynamical phase

\[
\phi_{d,m}(t) = \int_0^t \langle m|V^\dagger(t')H(t')V(t')|m\rangle \, dt'
= m \int_0^t c_0(t')\{\cos\lambda(t') \cos\theta(t') + \sin\lambda(t') \sin\theta(t') \cos(\gamma(t') - \varphi(t'))\} \, dt'
\]

and the geometric phase

\[
\phi_{g,m}(t) = \int_0^t \langle m| - V^\dagger(t')i\frac{\partial V(t')}{\partial t'}|m\rangle \, dt' = m \int_0^t \dot{\gamma}(t')[1 - \cos\lambda(t')] \, dt'.
\]

In what follows, when the time-dependent arbitrary vector parameters of the Hamiltonian is chosen to be a special form, the time-dependent second-quantized spin model is reduced to the generalized Chiao-Wu model that describes the photon propagating inside the fiber. Consider a noncoplanarly curved optical fiber that is wound smoothly on a large enough diameter [22], a photon propagating inside the fiber is along it at each point at arbitrary time, one can draw a conclusion that the eigenvalue of the helicity \(\frac{\ell(t)}{c}\cdot \vec{J}\) of the photon is conserved in motion and its helicity operator \(\frac{\ell(t)}{c}\cdot \vec{J}\) is an invariant. In terms of the invariant equation (2.1), we construct an effective Hamiltonian \(H_{eff}(t)\)
which represents the interaction between the photon field and medium of the fiber. The effective Hamiltonian $H_{eff}(t)$ is of the form\(^1\)

$$H_{eff}(t) = \frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2} \cdot \mathbf{J}, \quad (4.12)$$

where $\mathbf{J} = \mathbf{L} + \mathbf{S}$ is the total angular momentum operator of second-quantized electromagnetic field, which is given by

$$\mathbf{L} = -\frac{i}{2} \sum_{\lambda} \int d\mathbf{k} \{[(\mathbf{k} \times \nabla) a^\dagger(k, \lambda)]a(k, \lambda) - [(\mathbf{k} \times \nabla) a(k, \lambda)]a^\dagger(k, \lambda)\},$$

$$\mathbf{S} = -\frac{i}{2} \int d\mathbf{k} \mathbf{\hat{k}} [a_1^\dagger(k, 1)a(k, 2) - a^\dagger_1(k, 2)a(k, 1) + a(k, 2)a_1^\dagger(k, 1) - a(k, 1)a_1^\dagger(k, 2)]. \quad (4.13)$$

The infinitesimal rotation operator of motion of the photon in the fiber is given as follows

$$U_R = 1 - i\mathbf{\vec{\gamma}} \cdot \mathbf{J} \quad (4.14)$$

with

$$\mathbf{\vec{\gamma}} = \frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2} dt, \quad (4.15)$$

where dot denotes the time rate of change of $\mathbf{k}(t)$. It follows from Eq. (4.14) that the photon state $|\sigma, \mathbf{k}(t)\rangle$ ( $\sigma$ denotes the eigenvalue of helicity) satisfies the following time-evolution equation

$$i \frac{\partial |\sigma, \mathbf{k}(t)\rangle}{\partial t} = \frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2} \cdot \mathbf{J} |\sigma, \mathbf{k}(t)\rangle \quad (4.16)$$

with $H_{eff}(t) = \frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2} \cdot \mathbf{J}$ being the effective Hamiltonian of photon inside the fiber. Apparently, together with the invariant $I(t)$, the effective Hamiltonian $H_{eff}(t)$ agrees with the invariant equation (2.1). From the form of the effective Hamiltonian $H_{eff}(t)$ and the expression (4.13) for $\mathbf{J}$, one can see that the problem of the rotation of the polarization plane is actually in analogy with that of the time-dependent second-quantized spin model, and $\frac{\mathbf{k}(t) \times \dot{\mathbf{k}}(t)}{k^2}$ may be considered a general magnetic field.

Set the components of momentum of a photon

$$\frac{\mathbf{k}(t)}{k} = \frac{1}{2} (\sin \lambda(t) \cos \gamma(t), \sin \lambda(t) \sin \gamma(t), \cos \lambda(t)) \quad (4.17)$$

\(^1\)As far as we are concerned, the photon propagation problem can be ascribed to a time-dependent second-quantized spin model, where the effective (phenomenological) Hamiltonian is of a second-quantized form. Whereas in the previous researches [20–26], this problem was treated often by using classical Maxwell’s Equations and first-quantized Schrödinger equation (and Berry’s adiabatic geometric phase formula as well [21]). Although these investigations can be said to be somewhat outstandingly successful in both predicting and studying adiabatic geometric phases of photons in the fibre, here I still want to emphasize two points: for one thing, only by using the second-quantization formulation can we investigate the photon geometric phases at quantum level; for another, only when we consider the non-normal-product second-quantized Hamiltonian can it enable us to predict the existence of geometric phases at quantum-vacuum level. Tomita and Chiao may also agree to the above first point. They held the arguments [26] that although the geometric phases in the curved fibre can also be obtained by means of classical Maxwell’s electrodynamics, they preferred to think of this phenomenon as originating at the quantum level, but surviving the correspondence-principle limit into the classical level. However, this point is not the main subject in the present paper, which will be further discussed elsewhere. Here, instead, we concentrate only on the second point, i.e., the geometric phases at quantum-vacuum level resulting from the zero-point radiation fields of vacuum, which has not been investigated in previous researches.
where the time-dependent parameters $\lambda(t)$ and $\gamma(t)$ denote the angle displacement of $\vec{k}(t)$ in the spherical polar coordinate system. In terms of the above process for getting the solution of the spin model, we use the invariant-related unitary transformation operator $V(t) = \exp[\beta(t)J_+ - \beta^*(t)J_-]$, which enables one to transform the time-dependent invariant $I(t)$ into $I_V(t) = J_3$. Under the unitary transformation $V(t)$, one can obtain $H_V(t)$ which is written as (4.6) from $H(t)$ by making use of the Baker-Campbell-Hausdorff formula, where the time-dependent parameters $\theta$ and $\varphi$ represent the angle displacement of $\frac{\vec{k}(t) \times \vec{k}(t)}{k^2}$ in the spherical polar coordinate, namely,

$$\frac{\vec{k}(t) \times \vec{k}(t)}{k^2} = (\sin \varphi \cos \theta, \sin \theta \sin \varphi, \cos \theta). \quad (4.18)$$

By using Eq. (4.17), (4.18) and the invariant equation (2.1), two auxiliary equations can be derived

$$\dot{\gamma} \sin^2 \lambda = \cos \theta, \quad \dot{\lambda} \cos \gamma - \dot{\gamma} \cos \lambda \sin \sigma \sin \gamma = \sin \theta \sin \varphi. \quad (4.19)$$

From Eqs. (4.19), it is shown that in Eq. (4.6)

$$\cos \lambda \cos \theta + \sin \lambda \sin \theta \cos(\gamma - \varphi) = 0, \quad (4.20)$$

thus, the expression (4.6) can be rewritten as

$$H_V(t) = \dot{\gamma}(t)[1 - \cos \lambda(t)]J_3. \quad (4.21)$$

According to Eq. (4.11), the geometric phase of a photon whose eigenvalue of helicity is $\sigma$ can be expressed by

$$\phi_{\sigma}(t) = \{ \int_0^t \dot{\gamma}(t')[1 - \cos \lambda(t')]|d't' \} \langle \sigma | J_3 | \sigma \rangle. \quad (4.22)$$

When we consider the adiabatic case, where $\lambda$ is time-independent, $t$, the geometric phase $\phi_{\sigma}$ in a cycle over the parameter space of the invariant $I(t)$ may be written

$$\phi_{\sigma}(T) = 2\pi(1 - \cos \lambda), \quad (4.23)$$

where $\langle \sigma | J_3 | \sigma \rangle = 1$ when the eigenvalue of helicity of the photon is taken to be $\sigma = 1$. It should be noted that this result agrees with that derived from the Chiao-Wu theory [20].

By the aid of (4.9) and (4.22), it is easy to get the general solution of the time-dependent Schrödinger equation in the fiber experiment

$$|\Psi(t)\rangle_s = \sum_{\sigma} C_{\sigma} \exp[\frac{1}{i}\phi_{\sigma}(t)]V(t) |\sigma\rangle \quad (4.24)$$

with the coefficients $C_{\sigma} = \langle \sigma, t = 0 | \Psi(0) \rangle_s$.

It can be seen from the expression (4.20) and (4.21) that, in the noncoplanar optical fiber, the dynamical phase of photon due to the effective Hamiltonian vanishes, and its geometric phase is expressed by the expression (4.22). Since it is a second-quantized spin model, we do not take the normal product for the third component of the angular momentum $\vec{J}$ of the linear polarized photon.

**V. THE EXPLICIT EXPRESSION FOR THE TIME-EVOLUTION OPERATOR**

One of the applications of the invariant-related unitary transformation formulation is that it can change the hidden form of the time-evolution operator $U(t) = P \exp\{\frac{1}{i} \int_0^t H_{eff}(t')dt'\}$ of wave functions into an explicit expression, where $P$ denotes the chronological product operator. Assume that the initial momentum $\vec{k}(t = 0)$ is parallel to the third component of the space coordinate, that is,

$$k_1 = 0, \quad k_2 = 0, \quad k_3 = k, \quad (5.1)$$

we have accordingly

$$\lambda(0) = 0, \quad V(0) = 1. \quad (5.2)$$
First we compute

\[
U(t) = V(t) \exp(f J_3),
\]

where

\[
f = \frac{1}{i} \int_0^t \gamma(t') [1 - \cos \lambda(t')] dt'.
\]

At \( t = 0 \), the eigenstates of \( I_V \) corresponding to the eigenvalues \( \sigma = \pm 1 \) are of the forms

\[
\begin{align*}
|\sigma = +1, \vec{k}(t = 0)\rangle &= \frac{1}{\sqrt{2}} [a^\dagger(k, 1) + ia^\dagger(k, 2)] |0\rangle \equiv a^\dagger_R(k) |0\rangle, \\
|\sigma = -1, \vec{k}(t = 0)\rangle &= \frac{1}{\sqrt{2}} [a^\dagger(k, 1) - ia^\dagger(k, 2)] |0\rangle \equiv a^\dagger_L(k) |0\rangle
\end{align*}
\]

where \( a^\dagger_R(k) \) and \( a^\dagger_L(k) \) are the creation operators of the right- and left- rotation single-mode polarized photons, and \( k \) denotes the momentum of photon, and \(|0\rangle\) represents the initial vacuum state at \( t = 0 \). Then \(|\sigma, \vec{k}(t)\rangle\) at arbitrary time is given

\[
|\sigma, \vec{k}(t)\rangle = U(t) |\sigma, \vec{k}(t = 0)\rangle = V(t) \exp(f J_3) a^\dagger_\sigma(k) |0\rangle.
\]

In the following, we show that \(|\sigma, \vec{k}(t)\rangle\) is the eigenstate of the momentum operator

\[
p_\mu = \sum_\lambda \int d\vec{k} k_\mu a^\dagger_\sigma(k, \lambda) a(k, \lambda).
\]

of photon field. Prior to this, some useful commuting relations are spread out as follows:

\[
\begin{align*}
[ip_2 + p_1, J_+] &= [ip_2 - p_1, J_-] = 0, \\
[ip_2 - p_1, J_+] &= [ip_2 + p_1, J_-] = 2p_3, \\
[J_3, p_\mu] &= i(\delta_1 p_2 - \delta_2 p_1), [p_3, J_\pm] = ip_2 \pm p_1, \\
[p_\mu, J_\pm] &= i\delta_3 p_2 - i\delta_2 p_3 \pm i(\delta_1 p_3 - i\delta_3 p_1).
\end{align*}
\]

Thus, the following formula can be calculated by using the above expressions

\[
p_\mu U(t) a^\dagger_\sigma(k) |0\rangle = V(t) [p_\mu, \exp(f J_3)] a^\dagger_\sigma(k) |0\rangle + V(t) \exp(f J_3) p_\mu a^\dagger_\sigma(k) |0\rangle + [p_\mu, V(t)] \exp(f J_3) a^\dagger_\sigma(k) |0\rangle.
\]

First we compute \([p_\mu, \exp(f J_3)]\). By using the formulae (5.8) one will arrive at

\[
\exp(-f J_3) p_\mu \exp(f J_3) = \frac{e^f - e^{-f}}{2} (\delta_{1\mu} p_1 - \delta_{1\mu} p_2) + \\
+ \frac{e^f + e^{-f} - 2}{2} (\delta_{2\mu} p_1 + \delta_{2\mu} p_2) + p_\mu,
\]

then

\[
[p_\mu, \exp(f J_3)] = \frac{e^f - e^{-f}}{2} \exp(f J_3) (\delta_{2\mu} p_1 - \delta_{1\mu} p_2) + \\
+ \frac{e^f + e^{-f} - 2}{2} \exp(f J_3) (\delta_{1\mu} p_1 + \delta_{2\mu} p_2).
\]

Use is made of the expressions (5.5), we get

\[
p_\mu = \sum_\lambda \int d\vec{k} k_\mu a^\dagger_\sigma(k, \lambda) a(k, \lambda) = \int d\vec{k} k_\mu \{a^\dagger_R(k) a_R(k) + a^\dagger_L(k) a_L(k)\},
\]

then \( p_\mu a^\dagger_\sigma(k) |0\rangle = k_\mu a^\dagger_\sigma(k) |0\rangle \). In terms of the initial conditions Eq.(5.2), the following eigenvalue equations of momentum can be obtained.
\[ p_1 a_\mu^\dagger(k) |0\rangle = k \sin \lambda(0) \cos \gamma(0) a_\mu^\dagger(k) |0\rangle = 0, \]
\[ p_2 a_\mu^\dagger(k) |0\rangle = k \sin \lambda(0) \sin \gamma(0) a_\mu^\dagger(k) |0\rangle = 0, \]
\[ p_3 a_\mu^\dagger(k) |0\rangle = k \cos \gamma(0) a_\mu^\dagger(k) |0\rangle = ka_\mu^\dagger(k) |0\rangle. \]  
(5.12)

By making use of the expressions (5.11) and (5.12), the first term on the right-hand side in (5.9) is evidently
\[ V(t)[p_\mu, \exp(fJ_3)]a_\mu^\dagger(k) |0\rangle = 0 \]  
(5.13)
and the second term is expressed by
\[ V(t) \exp(fJ_3)p_\mu a_\mu^\dagger(k) |0\rangle = k_\mu V(t) \exp(fJ_3)a_\mu^\dagger(k) |0\rangle. \]  
(5.14)

Next we calculate \([p_\mu, V(t)].\) Using the commuting relations (5.8), the complicated calculation yields
\[ V^\dagger p_\mu V = \exp\{-[\beta(t)J_+ - \beta^*(t)J_-]\} p_\mu \exp[\beta(t)J_+ - \beta^*(t)J_-]\]
\[ = p_\mu + \frac{\sin \lambda}{\lambda} (\Delta_\mu p_3 + \delta_{3\mu} C) + \frac{1}{\lambda^2} (1 - \cos \lambda)(\Delta_\mu C - 4\beta^* \delta_{3\mu} p_3) \]  
(5.15)
where
\[ \Delta_\mu = \beta^*(i\delta_{2\mu} - \delta_{1\mu}) - \beta(i\delta_{2\mu} + \delta_{1\mu}), \]
\[ C = \beta^*(p_1 - ip_2) + \beta(p_1 + ip_2). \]  
(5.16)
\[ [p_\mu, V(t)] \]  
can be derived from Eq. (5.15). Based on this, one can derive the third term on the right-hand side in (5.9), which may be rewritten as
\[ [p_\mu, V(t)] \exp(fJ_3)a_\sigma^\dagger(k) |0\rangle = \frac{\sin \lambda}{\lambda} \Delta_\mu V \exp(fJ_3)p_\sigma a_\sigma^\dagger(k) |0\rangle \]
\[ -(1 - \cos \lambda) \delta_{3\mu} V \exp(fJ_3)p_\sigma a_\sigma^\dagger(k) |0\rangle. \]  
(5.17)

According to Eq. (5.12) and (5.16), we can arrive at
\[ p_\mu U(t)a_\sigma^\dagger(k) |0\rangle = [k(\delta_{1\mu} \cos \gamma + \delta_{2\mu} \sin \gamma) \sin \lambda - k(1 - \cos \lambda) \delta_{3\mu} + k_\mu] U(t)a_\sigma^\dagger(k) |0\rangle. \]  
(5.18)

By combining Eq. (5.18) with (5.12), the eigenvalue equation of momentum of photons field can be rewritten in the form
\[ p_\mu U(t)a_\sigma^\dagger(k) |0\rangle = k_\mu(t) U(t)a_\sigma^\dagger(k) |0\rangle \]  
(5.19)
where \(k_1(t) = k \sin \lambda \cos \gamma, k_2(t) = k \sin \lambda \sin \gamma, k_3(t) = k \cos \lambda.\) We thus show that \(U(t)a_\sigma^\dagger(k) |0\rangle\) is truly the eigenstate of momentum operator of the second-quantization photons field, and the eigenvalue is the time-dependent momentum \(k_\mu(t)\) of the single-mode photon.

In accordance with the L-R invariant theory, the wave function \(|\sigma, \tilde{k}(t)\rangle = U(t)a_\sigma^\dagger(k) |0\rangle\) of the single-mode photon is also the eigenstate of the invariant \(I(t) = \frac{\lambda(t)}{k} \cdot \tilde{J}.\) The significance of the invariant-related unitary transformation method will be illustrated in the following, by using the explicit expression for the time-evolution operator. The applications of the following commuting relations
\[ [J_1, V] = \sin \lambda \cos \gamma VJ_3 + \frac{\cos \gamma(1 - \cos \lambda)}{\lambda} V(\beta^* J_- + \beta J_+), \]
\[ [J_2, V] = \sin \lambda \sin \gamma VJ_3 + \frac{\sin \gamma(1 - \cos \lambda)}{\lambda} V(\beta^* J_- + \beta J_+), \]
\[ [J_3, V] = (\cos \lambda - 1) VJ_3 + \frac{\sin \lambda}{\lambda} V(\beta^* J_- + \beta J_+) \]  
(5.20)
enable one to get
\[ \left[ \frac{\tilde{k}(t)}{k} \cdot \tilde{J}, V \right] \exp(fJ_3) = U(t)J_3 - V \frac{\tilde{k}(t)}{k} \cdot \tilde{J} \exp(fJ_3) \]
\[ = U(t)J_3 - V I(t) \exp(fJ_3). \]  
(5.21)
Since
\[
\frac{\vec{k}(t)}{k} \cdot \vec{J} U(t) a_{\sigma}^{\dagger}(k) |0\rangle = V \left[ \frac{\vec{k}(t)}{k} \cdot \vec{J}, \exp(f J_3) a_{\sigma}^{\dagger}(k) |0\rangle \right] + \left[ \frac{\vec{k}(t)}{k} \cdot \vec{J}, V \right] \exp(f J_3) a_{\sigma}^{\dagger}(k) |0\rangle + U(t) \frac{\vec{k}(t)}{k} \cdot \vec{J} a_{\sigma}^{\dagger}(k) |0\rangle ,
\]
we apply the initial conditions (5.2) and (5.12) to Eq.(5.22) and obtain
\[
\frac{\vec{k}(t)}{k} \cdot \vec{J} U(t) a_{\sigma}^{\dagger}(k) |0\rangle = \sigma U(t) a_{\sigma}^{\dagger}(k) |0\rangle 
\]
with the eigenvalue of \( \frac{\vec{k}(t)}{k} \cdot \vec{J} \) being \( \sigma = \pm 1 \) corresponding to the right- and left- rotation linear polarized photons, respectively.

From what has been discussed above, one can draw a conclusion that one of the advantages of the invariant-related unitary transformation method is transforming the evolution operator \( U(t) \) of hidden form to the explicit expression (5.3).

VI. CONCLUDING REMARKS

We construct an invariant in the sub-Hilbert-space corresponding to a particular eigenvalue of the time-independent invariant \( N' \) and get the exact solutions of the time-dependent TLMJ-C model by making use of the invariant-related unitary transformation formulation. In view of the above calculation, we can see that this unitary transformation formulation has some useful applications, for instance, it can solve the time-dependent systems and treat the geometric phase factor, and obtain the explicit expressions, instead of the hidden form, for the evolution operator of the wave functions.

Since the three-level two-mode Jaynes-Cummings model plays an important role in Quantum Optics, the supersymmetric structure and the exact solutions of the time-dependent three-level two-mode multiphoton J-C model deserves further investigations by the formalism suggested in the present paper.

We construct an effective Hamiltonian in this paper since the helicity of the photon field is a pseudo scalar and cannot be regarded as the Hamiltonian. We transform the problem of motion of the photon in the optical fiber into that of the time-dependent quantum spin model. Effective Hamiltonian is obtained by making use of the invariant equation (2.1), rather than through analyzing the electromagnetic interaction between the photons and the medium of the optical fiber.

The invariant-related unitary transformation formulation is an effective method for treating the geometric phase factor \([34,35]\). This formulation replaces eigenstates of the time-dependent invariants by that of the time-independent invariants through the unitary transformation. It uses the invariant-related unitary transformation and obtain the explicit expression for the time-evolution operator, instead of the formal solution that is related to the chronological product.

Note added: In addition, we also treat the non-normal-order spin operators and consider the potential effects (e.g., quantum-vacuum geometric phases) of quantum fluctuation fields arising in a curved optical fibre. The quantum-vacuum geometric phase, which is of physical interest, can be deduced by using the operator normal product, and the doubt of validity and universality for the normal-normal procedure applied to time-dependent quantum systems is thus proposed. Our brief history of investigating photon geometric phases in the fibre is as follows: in April 2000, Gao and I began to consider the non-cyclic non-adiabatic geometric phases of photon fields in the curved fibre based on a second-quantized spin model. In May 2000, Gao first proposed the concept of quantum-vacuum geometric phases. The existence problem of quantum-vacuum geometric phases is strongly relevant to whether the second quantization in spin model is adopted or not. Since it is in connection with properties of quantum electromagnetic vacuum and, moreover, this geometric phase is related close to the topological and global features of time evolution of vacuum-fluctuation fields, we think this concept is of essential significance and therefore deserves detailed investigations. From then on, these problems gained our attention and we tried to investigate this topological phases at quantum-vacuum level.
Since quantum-vacuum geometric phases has an important connection with vacuum energies, these experimental realizations may be relevant to the validity problem of normal-product procedure in the time-dependent quantum field theory (TDQFT), i.e., we also aim to re-examine the normal-product procedure in some extensions. If the quantum-vacuum geometric phases is proved present experimentally, then it is reasonably believed that it is not suitable for us to remove vacuum fluctuation energies and infinite charge density just by using the old formulation, e.g., re-defining the vacuum background energies and electric charges by utilizing the normal-product procedure, since in this re-definition, some potential physically interesting vacuum effects may also be removed theoretically. We think only for the time-independent quantum field systems can we use safely the normal-product procedure without any fear of introducing any new problems other than those which quantum field theory had encountered before. However, for the time-dependent quantum field systems, (e.g., photon fields propagating inside a helically curved fibre and quantum fields in an expanding universe), the physically interesting vacuum effects will unfortunately be deducted by the second-quantization normal-order formulation. So, the normal-order technique may therefore not be applicable to the time-dependent quantum fields. To the best of our knowledge, in the literature, this normal order problem in the time-dependent quantum field theory gets less attention and interests than it deserves. To test our above theoretical viewpoints, we hope the quantum-vacuum geometric phases of photons in the curved fibre would be investigated experimentally in the near future.

Unfortunately, the left-handed polarized light due to vacuum fluctuation is often accompanied by the zero-point right-handed polarized light and their total quantum-vacuum geometric phases is therefore vanishing. So, it is not easy for physicists to investigate experimentally the quantum-vacuum geometric phases. This, therefore, means that our above theoretical remarks as to whether the normal-product procedure is valid or not for the time-dependent quantum field theory (TDQFT) cannot be examined experimentally. During the last three years, I tried my best but unfortunately failed to suggest an excellent idea of experimental realization of quantum-vacuum geometric phases. We conclude that it seems not quite satisfactory to test the quantum-vacuum geometric phases by using the optical fibre that is made of isotropic media, inhomogeneous media (e.g., photonic crystals\(^2\)), left-handed media (a kind of artificial composite metamaterial with negative refractive index), uniaxial (biaxial) crystals or chiral materials. Is it truly extremely difficult to realize such a goal? It is found finally that perhaps in the fibre composed of some anisotropic media (such as gyrotropic materials, including also gyroelectric and gyromagnetic media, where both electric permittivity and magnetic permeability are respectively the tensors) the quantum-vacuum geometric phases may be achieved test experimentally. In these gyrotropic media, only one of the LRH polarized lights can be propagated without being absorbed by media. This result holds also for the zero-point electromagnetic fields. It is well known that people can manipulate vacuum so as to alter the zero-point mode structures of vacuum, which has been illustrated in photonic crystals and Casimir’s effect (additionally, the space between two parallel mirrors, cavity in cavity QED, etc.). If, for example, in some certain gyrotropic media one of the LRH polarized lights, say, the left-handed polarized light, dissipates due to the medium absorption and only the right-handed light is allowed to be propagated (in the meanwhile the mode structure of vacuum in these anisotropic media also alters correspondingly), then the quantum-vacuum geometric phase of right-handed polarized light can be easily tested in the fibre fabricated from these gyrotropic media.

In order to illustrate our brief history (April, 2000 ∼ March, 2003) of investigating geometric phases of photons in the noncoplanarly curved fibre, an outline is given as follows:

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\(^2\)Photonic crystals are artificial materials patterned with a periodicity in dielectric constant, which can create a range of forbidden frequencies called a photonic band gap. Such dielectric structure of crystals offers the possibility of molding the flow of light (including the zero-point electromagnetic fields of vacuum). It is believed that, in the similar fashion, this effect (i.e., modifying the mode structures of vacuum electromagnetic fields) may also take place in gyrotropic media. This point will be applied to the discussion which follows.
Chiao-Wu’s model (photons moving inside a helically curved fibre)  
Berry’s phase formula

cyclic adiabatic geometric phase \( \phi^{(c)}(T) = 2\pi \sigma (1 - \cos \lambda) \)

Lewis-Riesenfeld invariant theory

non-cyclic non-adiabatic geometric phase \( \phi^{(n)}(T) = \sigma \left\{ \int_0^T \dot{\gamma}(t') [1 - \cos \lambda(t')] dt' \right\} \)

second-quantized spin model

quantal geometric phases \( \phi^{(q)}(t) = (n_R - n_L) \left\{ \int_0^t \dot{\gamma}(t') [1 - \cos \lambda(t')] dt' \right\} \)

under non-normal order

quantum-vacuum phases \( \phi^{(vacuum)}_{\sigma = \pm 1}(t) = \pm \frac{1}{2} \left\{ \int_0^t \dot{\gamma}(t') [1 - \cos \lambda(t')] dt' \right\} \)

validity problem of normal order in TDQFT

experimental test is required

unfortunately, \( \phi^{(vacuum)}_L(t) + \phi^{(vacuum)}_R(t) = 0 \) by using gyrotrropic-medium fibre

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