Bounds on the Discrete Spectrum of Lattice Schrödinger Operators

V. Bach, W. de Siqueira Pedra, S.N. Lakaev

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Abstract

We discuss the validity of the Weyl asymptotics – in the sense of two-sided bounds – for the size of the discrete spectrum of (discrete) Schrödinger operators on the $d$–dimensional, $d \geq 1$, cubic lattice $\mathbb{Z}^d$ at large couplings. We show that the Weyl asymptotics can be violated in any spatial dimension $d \geq 1$ – even if the semi-classical number of bound states is finite. Furthermore, we prove for all dimensions $d \geq 1$ that, for potentials well-behaved at infinity and fulfilling suitable decay conditions, the Weyl asymptotics always hold. These decay conditions are mild in the case $d \geq 3$, while stronger for $d = 1, 2$. It is well-known that the semi-classical number of bound states is – up to a constant – always an upper bound on the size of the discrete spectrum of Schrödinger operators if $d \geq 3$. We show here how to construct general upper bounds on the number of bound states of Schrödinger operators on $\mathbb{Z}^d$ from semi-classical quantities in all space dimensions $d \geq 1$ and independently of the positivity-improving property of the free Hamiltonian.

Key words. Schrödinger operator on the lattice, Weyl asymptotics, semi-classical bounds.

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1 Introduction

Let $V \in L^{d/2}(\mathbb{R}^d, \mathbb{R}_+)$ be a non-negative potential in the $d$–dimensional space with $d \geq 3$. From standard results of spectral theory [11] it follows that the negative spectrum $\sigma[-\Delta - \lambda V(x)] \cap \mathbb{R}^-$ of the corresponding self-adjoint Schrödinger operator

$$-\Delta_{\mathbb{R}^d} - \lambda V(x)$$

on $L^2(\mathbb{R}^d)$ is purely discrete, i.e., consists only of isolated eigenvalues of finite multiplicity. Here, $\Delta_{\mathbb{R}^d} = \sum_{i=1}^d \partial_{x_i}^2$ is the Laplacian on $\mathbb{R}^d$ and $V$ acts as a multiplication operator, $[V \varphi](x) = V(x) \varphi(x)$. By a well-known theorem – first established by Weyl [20, 21] for the case of Dirichlet Laplacian in a bounded domain – the number
\(N_{\text{cont}}[\lambda V]\) of negative eigenvalues of \(-\Delta_{R^d} - \lambda V\) (counting multiplicities) is asymptotically
\[
N_{\text{cont}}[\lambda V] \doteq \text{Tr} \left\{ 1 \left[ -\Delta_{R^d} - \lambda V < 0 \right] \right\} \sim N_{\text{sc}}^{\text{cont}}[\lambda V]
\]  
(2)
as \(\lambda \to \infty\). The right side of (2) is the volume
\[
N_{\text{sc}}^{\text{cont}}[V] \doteq \int 1[p^2 - V(x) < 0] \frac{d^d x \, dp}{(2\pi)^d}
\]  
(3)
these bound states occupy in phase space \(\mathbb{R}^d \times (\mathbb{R}^*)^d = \mathbb{R}^{2d}\) according to semi-classical analysis. This so-called Weyl asymptotics (2) is complemented by the celebrated non-asymptotic bound of Rozenblum [12], Lieb [9], and Cwikel [2] on the number \(N_{\text{cont}}[V]\) of bound states of \(-\Delta_{R^d} - V\) of the form
\[
N_{\text{cont}}[V] \leq C_{\text{CLR}}(d) N_{\text{sc}}^{\text{cont}}[V]
\]  
(4)
for some \(C_{\text{CLR}}(d) \geq 1\). Lieb [8, Eq. (4.5)] has shown that the optimal choice for \(C_{\text{CLR}}(3)\) is smaller than 6.9. Note that
\[
N_{\text{sc}}^{\text{cont}}[V] = |S^{d-1}| (2\pi)^d \int V^{d/2}(x) \, d^d x,
\]  
(5)
where \(|S^{d-1}|\) is the volume of the \((d-1)\)-dimensional sphere.

In the present paper, we replace the Euclidean \(d\)-dimensional space \(\mathbb{R}^d\) by the \(d\)-dimensional hypercubic lattice \(\Gamma = \mathbb{Z}^d\) and study the discrete analogues of the Weyl asymptotics (2) and the Cwikel-Lieb-Rozenblum (CLR) bound (4). For a given potential \(V \in L^\infty(\Gamma, \mathbb{R}_0^+ )\), the discrete Schrödinger operator corresponding to (1) is
\[
-\Delta_\Gamma - \lambda V(x),
\]  
(6)
where \(V\) acts again as a multiplication operator and \(\Delta_\Gamma\) is the discrete Laplacian defined by
\[
[\Delta_\Gamma \varphi](x) = \sum_{|v|=1} \{ \varphi(x) - \varphi(x + v) \}.
\]  
(7)
More generally, we assume to be given a Morse function \(e \in C^2(\Gamma^*, \mathbb{R})\) on the \(d\)-dimensional torus (Brillouin zone) \(\Gamma^* = (\mathbb{R}/2\pi \mathbb{Z})^d = [-\pi, \pi)^d\), the dual group of \(\Gamma\). Given such a function \(e\), we consider the self-adjoint operator
\[
H(e, V) \doteq h(e) - V(x),
\]  
(8)
on \(\ell^2(\Gamma)\), where \(h(e) \in B(\ell^2(\Gamma))\) is the hopping matrix (convolution operator) corresponding to the dispersion relation \(e\), i.e.,
\[
[F^*(h(e)\varphi)](p) = e(p) [F^*(\varphi)](p),
\]  
(9)
for all \(\varphi \in L^2(\Gamma^*)\) and all \(p \in \Gamma^*\). Here,
\[
F^* : \ell^2(\Gamma) \to L^2(\Gamma^*), \quad [F^*(\varphi)](p) \doteq \sum_{x \in \Gamma} e^{-i(p,x)} \varphi(x)
\]  
(10)
is the usual discrete Fourier transformation with inverse
\[
\mathcal{F} : L^2(\Gamma^*) \to \ell^2(\Gamma), \quad [\mathcal{F}(\psi)](x) = \int_{\Gamma^*} e^{i\langle p, x \rangle} \psi(p) \, d\mu^*(p),
\]
where $\mu^*$ is the (normalized) Haar measure on the torus, $d\mu^*(p) = \frac{dp}{(2\pi)^d}$. Put differently, $h(\epsilon) = \mathcal{F} \epsilon \mathcal{F}^*$ is the Fourier multiplier corresponding to $\epsilon$. We assume w.l.o.g. that the minimum of $\epsilon$ is 0, so
\[
\epsilon(\Gamma^*) = [0, \epsilon_{\text{max}}(\epsilon)],
\]
and we call a Morse function $\epsilon \in C^2(\Gamma^*, \mathbb{R})$ obeying (12) an admissible dispersion relation. Note that $-\Delta_{\Gamma} = h(\epsilon_{\text{Lapl}})$, with
\[
\epsilon_{\text{Lapl}}(p) = \sum_{i=1}^d \left( 1 - \cos(p_i) \right), \quad \epsilon_{\text{Lapl}}(\Gamma^*) = [0, 2d],
\]
being admissible. We require that $V$ decays at infinity,
\[
V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+) \doteq \left\{ V : \Gamma \to \mathbb{R}_0^+ \mid \lim_{|x| \to \infty} V(x) = 0 \right\},
\]
or sometimes even that $V$ has bounded support. Note that $V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+)$ is compact as a multiplication operator on $\ell^2(\Gamma)$ and by another theorem of Weyl,
\[
\sigma_{\text{ess}}[H(\epsilon, V)] = \sigma_{\text{ess}}[H(\epsilon, 0)] = [0, \epsilon_{\text{max}}],
\]
where $\epsilon_{\text{max}} \equiv \epsilon_{\text{max}}(\epsilon)$. From the positivity of $V$ and the min-max principle we further obtain that all isolated eigenvalues of finite multiplicity lie below 0,
\[
\sigma_{\text{disc}}[H(\epsilon, V)] \subseteq \mathbb{R}^- \doteq (-\infty, 0).
\]
We note in passing that – different to Schrödinger operators on $\mathbb{R}^d$ – discrete Schrödinger operators possibly have positive eigenvalues when changing the sign of the potential. Counting the number of positive eigenvalues, however, can be traced back to the case treated here by replacing $\epsilon(p)$ by $\epsilon_{\text{max}} - \epsilon(p)$.

Our goal in this paper is to give – in all dimensions – both asymptotic and non-asymptotic bounds on the number
\[
N[\epsilon, V] \doteq \text{Tr} \left\{ 1[H(\epsilon, V) < 0] \right\}
\]
of negative eigenvalues of $H(\epsilon, V)$. Criteria for $N[\epsilon, V]$ to be finite or to be infinite were given in [1] [7] [14]. The main focus lies on the physically most relevant case $d \geq 3$. For $d = 1$ the situation is well-understood [3][4], and even asymptotics for the accumulation of eigenvalues near zero are known [5]. The case $d = 2$ is particularly difficult, see [18] for the best results currently known.
In the present paper, we aim at bounds on $N[\epsilon, V]$ in terms of the corresponding semi-classical quantity

$$N_{sc}[\epsilon, V] \doteq \sum_{x \in \Gamma^*} \int_{\Gamma^*} 1[\epsilon(p) - V(x) < 0] \, d\mu^*(p)$$

where the sizes $L_{V}[\alpha] \in \mathbb{N}_0$ of the level sets of $V$ are defined by

$$L_{V}[\alpha] \doteq \# \{ x \in \Gamma \mid V(x) \geq \alpha \}$$

for $\alpha > 0$. Note that $L_{V}[\alpha]$ is independent of the localization properties of $V$. This lets us introduce the notion of rearrangements of $V$. Given $V, \tilde{V} \in \ell_\infty^d(\Gamma, \mathbb{R}_0^+)$, we say that $\tilde{V}$ is a rearrangement of $V$, whenever $\forall \alpha > 0 : L_{\tilde{V}}[\alpha] = L_{V}[\alpha]$.

In other words, the supports of $\tilde{V}$ and $V$ have the same cardinality, and $\tilde{V}|_{\text{supp} \ \tilde{V}} = V \circ J$ for some bijection $J : \text{supp} \ \tilde{V} \rightarrow \text{supp} \ V$. Obviously, being rearrangements of each other defines an equivalence relation on $\ell_\infty^d(\Gamma, \mathbb{R}_0^+)$. The importance of the growth of the sizes $L_{V}[\alpha]$ of the level sets of $V$, as $\alpha \downarrow 0$, is also realized in [14, 15].

We emphasize that in most other studies of $N[\epsilon, V]$ and notably in [14, 15, 16, 17], the generator of the kinetic energy is assumed to be Markovian. By contrast, we use CLR bounds recently derived in [6] that do not require such an assumption and the only essential property of the dispersion $\epsilon$ we need in our proofs is its Morse property.

### 1.1 Non-Asymptotic Semi-classical Bounds

We first formulate our non-asymptotic results which correspond to the CLR bound (4) in the continuum case.

**Theorem 1.1** (Non-asymptotic upper bound, $d \geq 3$). *Let $d \geq 3$ and $\epsilon$ an admissible dispersion. Then there exists a constant $C_{11}(d, \epsilon) \in [1, \infty)$ such that*

$$N[\epsilon, V] \leq C_{11}(d, \epsilon) \, N_{sc}[\epsilon, V] < \infty$$

*for all $V \in \ell_\infty^{d/2}(\Gamma, \mathbb{R}_0^+)$.*

If $d < 3$, the following weighted version of the non-asymptotic bound on $N[\epsilon, V]$ still holds:

**Theorem 1.2** (Non-asymptotic upper bound, $d < 3$). *Let $d \in \{1, 2\}$ and assume that $\epsilon \in C^3(\Gamma^*, \mathbb{R}_0^+)$. Then there is a constant $C_{12}(d, \epsilon) < \infty$ such that, for any potential $V \in \ell_\infty^\infty(\Gamma, \mathbb{R}_0^+)$, *

$$N[\epsilon, V] \leq C_{12}(d, \epsilon) \left( 1 + N_{sc} [\epsilon, |x|^{d+5}V] \right).$$
Our results show that the right quantity to compare the number of eigenvalues to is the phase space volume $N_{ac}[\varepsilon, V]$ of the set $\{(p, x) \mid \varepsilon(p) - V(x) < 0\}$ and not (the $d^{th}$ power of) the $\ell^{d/2}$-norm of $V$. In the case of Schrödinger operators on $\mathbb{R}^d$, these quantities agree up to a multiplicative constant. See (5). While it is possible to bound $N_{ac}[\varepsilon, V]$ and hence also $N[\varepsilon, V]$ by a multiple of $|V|_{d/2}^{d/2} = \sum_x V^{d/2}(x)$, this bound grossly overestimates the number of eigenvalues in the limit of large couplings. For example, if $\Lambda \subset \Gamma$ is a finite subset then

$$N_{ac}[\varepsilon, \Lambda 1_{\Lambda}] = |\Lambda| \prec \lambda^{d/2}|\Lambda| = |\lambda 1_{\Lambda}|^{d/2}$$

for sufficiently large $\lambda > 0$.

In Sect. 3.2 we prove the optimality of Theorem 1.1 with respect to the class $\ell^{d/2}(\Gamma, \mathbb{R}^+_* \eta \varepsilon$ be an admissible dispersion for which $|h(\varepsilon)_{0,x}| \leq \text{const} \langle x \rangle^{-(d+1)}$ for some const $< \infty$. Then, for any $\varepsilon > 0$, there exists a potential $V_\varepsilon \in \ell^{d/2}(\Gamma, \mathbb{R}^+_\varepsilon) \setminus \ell^{d/2}(\Gamma, \mathbb{R}^+)$ such that $N[\varepsilon, V_\varepsilon] = N_{ac}[\varepsilon, V_\varepsilon] = \infty$.

Here, $h(\varepsilon)_{x,y} \equiv \langle \delta_x \varepsilon \delta_y \rangle$ are the matrix elements w.r.t. to the canonical basis of $\ell^2(\Gamma)$ of the hopping matrix $h(\varepsilon)$ of the dispersion relation $\varepsilon$, and $\langle x \rangle \equiv 1 + |x|$.

This does not, however, imply that $N[\varepsilon, V] = \infty$ whenever $N_{ac}[\varepsilon, V] = \infty$. For instance, if $V(x) = \langle x \rangle^{-2}(\log |x|)^{-\eta}$ for some $\eta \in (0, 2/d)$ then $N[\varepsilon_{\text{lapl}}, V] < \infty$ but $N_{ac}[\varepsilon_{\text{lapl}}, V] = \infty$. See the example in [15, Section 5.2].

We complement the non-asymptotic upper bounds by corresponding lower bounds:

**Theorem 1.1 (Non-Asymptotic Lower Bound).** Let $d \geq 1$ and $\varepsilon$ be an admissible dispersion. Then, for any potential $V \in \ell_{\infty}^0(\Gamma, \mathbb{R}^+)$ and all $c > \varepsilon_{\text{max}}$, \n
$$N[\varepsilon, V] \geq \mathcal{L}_V[c] = \# \{x \in \Gamma \mid V(x) \geq c\}. \quad (25)$$

From Theorems 1.1 and 1.4 emerges the interesting question, whether $N_{ac}[\varepsilon, V]$ or $\mathcal{L}_V[\varepsilon_{\text{max}}]$ (or both) are saturated in certain limits. It turns out that, for sparse potentials $V$, the number $N[\varepsilon, V]$ bound states is correctly described by $\mathcal{L}_V[\eta(\varepsilon)]$, where $0 \leq \eta(\varepsilon) < \varepsilon_{\text{max}}$ is defined by

$$\frac{1}{\eta(\varepsilon)} \equiv \int_{\Gamma^*} \frac{\text{d} \mu^*(p)}{\varepsilon(p)}, \quad (26)$$

for $d \geq 3$, and $\eta(\varepsilon) \equiv 0$ for $d \in \{1, 2\}$ — see, for instance, Lemma 3 and the proof of Corollary 4.4. Observe that, as $\eta(\varepsilon) \leq \varepsilon_{\text{max}}$, $\mathcal{L}_V[\varepsilon_{\text{max}}] \leq \mathcal{L}_V[\eta(\varepsilon)]$. Since for any $\delta > 0$ there is a potential $V \in \ell_{\infty}^0(\Gamma, \mathbb{R}^+)$ for which $\mathcal{L}_V[\eta(\varepsilon)]/\mathcal{L}_V[\varepsilon_{\text{max}}] < 1 + \delta$, the following theorem implies the optimality of the lower bound in Theorem 1.4 with respect to rearrangements:

**Theorem 1.5 (Optimality of Thm. 1.4 under rearrangements).** Let $d \geq 3$, $\varepsilon$ be an admissible dispersion. Given $\varepsilon \in (0, 1)$ and a potential $V \in \ell_{\infty}^0(\Gamma, \mathbb{R}^+)$, there exists a rearrangement $\widehat{V} \in \ell_{\infty}^0(\Gamma, \mathbb{R}^+)$ of $V$ such that

$$N[\varepsilon, \widehat{V}] \leq \mathcal{L}_V[(1 - \varepsilon)\eta(\varepsilon)] = \# \{x \in \Gamma \mid V(x) \geq (1 - \varepsilon)\eta(\varepsilon)\}. \quad (27)$$
In general, the semi-classical number of bound states $N_{sc}[\epsilon, \lambda V]$ is not a lower bound on $N[\epsilon, \lambda V]$ -- not even up to prefactors. This is illustrated by the following theorem.

**Theorem 1.6.** Let $d \geq 3$ and $\epsilon$ be an admissible dispersion. Then there exists a potential $V \notin \bigcup_{p \geq 1} \ell^p(\Gamma)$ with $N[\epsilon, V] = 0$.

### 1.2 (Weyl-)Asymptotic Semi-classical Bounds

The Weyl asymptotics (2) states that, for all fixed potentials $V \in L^{d/2}(\mathbb{R}^d)$,

$$
\lim_{\lambda \to \infty} \frac{N_{cont}[\lambda V]}{N_{cont}^\ell[\lambda V]} = 1,
$$

and that $N_{cont}^\ell[\lambda V] = \lambda^{d/2} N_{cont}[V]$. For discrete Schrödinger operators, only weaker statements hold true, as is illustrated by the following lemma.

**Lemma 1.7.** Assume $d \geq 3$ and $V \in \ell^{d/2}(\Gamma, \mathbb{R}^+_0)$. Then

$$
\lim_{\lambda \to \infty} \left\{ \lambda^{-d/2} N[\epsilon, \lambda V] \right\} = \lim_{\lambda \to \infty} \left\{ \lambda^{-d/2} N_{sc}[\epsilon, \lambda V] \right\} = 0. \tag{29}
$$

For a precise formulation of our asymptotic bounds, we introduce the numbers

$$
g_+(V) \doteq \sup_{r > 0} \limsup_{\ell \to \infty} \frac{2}{d r} \left( \ln \mathcal{L}_V [e^{-\ell r}] - \ln \mathcal{L}_V [e^{-\ell}] \right), \quad g_-(V) \doteq \inf_{r > 0} \liminf_{\ell \to \infty} \frac{2}{d r} \left( \ln \mathcal{L}_V [e^{-\ell r}] - \ln \mathcal{L}_V [e^{-\ell}] \right),
$$

built from the level sets of $V$. While the significance of $g_-(V)$ is made clear in Section 4.1, $g_+(V)$ directly enters the following theorem.

**Theorem 1.8** (Asymptotic bounds, $d \geq 3$). Assume $d \geq 3$ and $V \in \ell^{d/2}(\Gamma, \mathbb{R}^+_0)$. Then there are constants $0 < C_{1.8}(d, \epsilon) \leq C_{1.8}(d, \epsilon) < \infty$ such that

$$
(1 - g_+(V)) C_{1.8}(d, \epsilon) \leq \liminf_{\lambda \to \infty} \left\{ \frac{N[\epsilon, \lambda V]}{N_{sc}[\epsilon, \lambda V]} \right\} \leq \limsup_{\lambda \to \infty} \left\{ \frac{N[\epsilon, \lambda V]}{N_{sc}[\epsilon, \lambda V]} \right\} \leq C_{1.8}(d, \epsilon). \tag{32}
$$

A somewhat weaker form of Theorem 1.8 still holds in case $d < 3$.

**Theorem 1.9** (Asymptotic Bounds, $d < 3$). Assume that $d \in \{1, 2\}$ and $V \in \ell^{d/2}(\Gamma, \mathbb{R}^+_0)$. Then there are constants $0 < C_{1.9}(d, \epsilon) \leq C_{1.9}(d, \epsilon) < \infty$ such that

$$
(1 - g_+(V)) C_{1.9}(d, \epsilon) \leq \liminf_{\lambda \to \infty} \left\{ \frac{N[\epsilon, \lambda V]}{N_{sc}[\epsilon, \lambda V]} \right\}, \quad \limsup_{\lambda \to \infty} \left\{ \frac{N[\epsilon, \lambda V]}{1 + N_{sc}[\epsilon, \lambda V]} \right\} \leq C_{1.9}(d, \epsilon). \tag{33}
$$
We remark that if \( V \) is rapidly decaying then, typically, \( g_+(V) = 0 \). For instance, if
\[
c_1 e^{-\alpha_1 |x|} \leq V(x) \leq c_2 e^{-\alpha_2 |x|},
\]
for some constants \( c_1, \alpha_1, \alpha_2 > 0, c_2 < \infty \), and all \( x \in \Gamma \), then \( g_+(V) = 0 \). Moreover, by the bounds proven here, in this case the usual Weyl semi-classical asymptotics hold true in all dimensions \( d \geq 1 \) and for all admissible dispersion relations, in the sense that
\[
\lim_{\lambda \to \infty} \frac{N(\varepsilon, \lambda V)}{N_{\text{sc}}(\varepsilon, \lambda V)} = \lim_{\lambda \to \infty} \frac{N(\varepsilon, \lambda V)}{1 + N_{\text{sc}}(\varepsilon, \lambda |x|^q V)} = 1.
\]
(35)
We further remark that if \( V \) behaves at infinity like an inverse power of \( |x| \), i.e., if the limit
\[
\lim_{|x| \to \infty} \left\{ -\frac{\log |V(x)|}{\log |x|} \right\} = \beta
\]
exists, then \( g_+(V) = g_-(V) = 2\beta/d \). In particular, in this case \( g_+(V) < 1 \) implies \( V \in \ell^{d/2}(\Gamma, \mathbb{R}_0^+) \), and \( g_-(V) > 1 \) implies \( V \notin \ell^{d/2}(\Gamma, \mathbb{R}_0^+) \).

In contrast to the continuum case, the boundedness of \( V \) in \( \ell^{d/2} \) alone does not suffice to ensure the semi-classical asymptotic behavior of \( N(\varepsilon, \lambda V) \), but details of the behavior of \( V \) at infinity enter, too, as is illustrated by the following theorem.

**Theorem 1.10.** Let \( d \geq 3 \) and \( \varepsilon \) be an admissible dispersion. Then there exists a potential \( V \) with \( N_{\text{sc}}(\varepsilon, \lambda V) < \infty \) for all \( \lambda > 0 \) for which
\[
\lim \inf_{\lambda \to \infty} \frac{N_{\text{sc}}(\varepsilon, \lambda V)}{N(\varepsilon, \lambda V)} < \infty \quad \text{and} \quad \lim \sup_{\lambda \to \infty} \frac{N_{\text{sc}}(\varepsilon, \lambda V)}{N(\varepsilon, \lambda V)} = \infty.
\]

In fact, potentials on the lattice can be so peculiar that their eigenvalue asymptotics assumes any prescribed behavior in the sense of the following theorem.

**Theorem 1.11.** Let \( d \geq 3 \) and \( \varepsilon \) be any admissible dispersion. Let further \( F : [1, \infty) \to \mathbb{N} \) be an arbitrary monotonically increasing, positive, integer-valued, right-continuous function. Then, for any \( \varepsilon \in (0, 1/2) \), there exists a potential \( V_{F, \varepsilon} \in \ell_0^\infty(\Gamma) \) such that
\[
\forall \lambda \geq 2 : \quad F((1 - \varepsilon)\lambda) \leq N(\varepsilon, \lambda V_{F, \varepsilon}) \leq F((1 + \varepsilon)\lambda).
\]
(37)
Results similar to Theorems 1.8, 1.10, and 1.11 have been obtained in [14, 15], where the property \( g_+(V) < 1 \) has been characterized by \( V \in \ell_q, w(\mathbb{Z}^d) \) belonging to a weak \( \ell_q \)-space, for some \( q > d/2 \). The latter ensures that \( \mathcal{L}_V[\alpha] \leq C \cdot \alpha^{-q} \). To prove the analogue of Theorem 1.11 a different notion of sparsity of potentials is used in [14, 15]. In [16, 17], the results are generalized to fairly arbitrary graphs. Here the interesting observation is made that the global dimension \( D \) defined by the decay \( (e^{-tK}(x, x)) \leq C \cdot t^{-D/2} \) of the diagonal elements of the semigroup generated by the kinetic energy is the quantity that replaces the spatial dimension \( d \) of the hypercubic lattice \( \mathbb{Z}^d \).

We give an overview on where to find the proofs of the theorems above:
2 Birman-Schwinger Principle and the CLR-Bound

In the sequel, we use the Birman-Schwinger principle in the following form:

**Lemma 2.1 (Birman-Schwinger principle).** Let \( d \geq 1 \), \( \epsilon \) be an admissible dispersion relation and \( V \in \ell_0^\infty(\Gamma, \mathbb{R}_+^d) \). For any \( \rho > 0 \), define the compact, self-adjoint, non-negative Birman-Schwinger operator by

\[
B(\rho) = B(\rho, \epsilon, V) = \frac{1}{2} \left( \frac{\rho + h(\epsilon)}{2} \right)^{-1} V^{1/2}. \tag{38}
\]

Then the following assertions (i)–(iv) hold true.

(i) If \( \varphi \in \ell^2(\Gamma) \) solves \( H(\epsilon, V)\varphi = -\rho \varphi \) then \( \psi \equiv V^{1/2} \varphi \in \ell^2(\Gamma) \) solves \( \psi = B(\rho)\psi \).

(ii) If \( \psi \in \ell^2(\Gamma) \) solves \( \psi = B(\rho)\psi \) then \( \varphi = [\rho + h(\epsilon)]^{-1/2} \psi \in \ell^2(\Gamma) \) solves \( H(\epsilon, V)\varphi = -\rho \varphi \).

(iii) \(-\rho\) is an eigenvalue of \( H(\epsilon, V) \) of multiplicity \( M \) if and only if 1 is an eigenvalue of \( B(\rho) \) of multiplicity \( M \).

(iv) Counting multiplicities, the number of eigenvalues of \( H(\epsilon, V) \) less or equal than \(-\rho\) equals the number of eigenvalues of \( B(\rho) \) greater or equal than 1.

This result is well-know and its proof is given in Appendix A.1 for completeness. The estimate on the number of negative eigenvalues of \( H(\epsilon, V) \), stated below, is the celebrated CLR bound, which is generally derived from (some convenient form of) the Birman-Schwinger principle.

**Theorem 2.2 (CLR bound).** Let \( d \geq 3 \) and \( \epsilon \) be any admissible dispersion. Then, for some constant \( C_{d, \epsilon} < \infty \),

\[
N[\epsilon, V] \leq C_{d, \epsilon} |V|^{d/2}. \tag{39}
\]
This kind of estimate was proven the first time by Rozenblum [12], Lieb [9], and Cwikel [2] by three different methods, in the continuous case. See also [10, Theorem XIII.12] or [19, Theorem 9.3]. It was then shown by Rozenblum and Solomyak [13, 14] that the CLR bound is not only true for Schrödinger Operators of the form (1), but also for a very large class of operators including, in particular, discrete Schrödinger operators. Note that, when applied to the discrete Schrödinger operators of the form (8), most of the known methods to derive CLR bounds would need the hopping matrix \( h(e) \) to be positivity preserving. We use instead a beautiful recent observation made by Frank [6, Theorem 3.2] on the discrete spectrum of a class of selfadjoint operators, which implies the CLR bound for \( N[e, V] \) when \( d \geq 3 \), merely assuming that \( e \) is a Morse function (i.e., it is “admissible” in the sense defined above). For completeness, in Appendix, Section A.1, we reproduce Frank’s estimate and derive from it the CLR bound of Theorem 2.2.

3 Non-Asymptotic Semi-Classical Bounds

3.1 Derivation of Non-Asymptotic Bounds

Now we are in a position to use Theorem 2.2 to yield a semi-classical bound, i.e., a bound on \( N[e, V] \) by multiples of \( N_{sc}[e, V] \). The following lemma is a standard estimate on the size of the discrete spectrum of a sum of self-adjoint operators. Its proof is given in Appendix A.2 for completeness.

**Lemma 3.1.** Let \( A = A^* \), \( B = B^* \in \mathcal{B}[\mathcal{H}] \) be two bounded self-adjoint operators on a separable Hilbert space \( \mathcal{H} \). Then

\[
N[A + B] \leq N[A] + N[B],
\]

where \( N[Q] \doteq \text{Tr} \{1_{\{Q < 0\}}\} \) denotes the number of negative eigenvalues, counting multiplicities, of a bounded self-adjoint operator \( Q \in \mathcal{B}[\mathcal{H}] \). We set \( N[Q] \doteq \infty \) if \( \sigma_{\text{ess}}(Q) \cap \mathbb{R}^- \neq \emptyset \).

A simple application of Lemma 3.1 with \( A = H(e, V_1) \), \( B = V_2 \), and \( A + B = H(e, V_1 + V_2) \), is the following corollary.

**Corollary 3.2.** Let \( d \geq 1 \), \( e \) be an admissible dispersion relation, and \( V_1, V_2 \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+) \) be two potentials. Then

\[
N[e, V_1 + V_2] \leq N[e, V_1] + \# \text{supp}\{V_2\}. \tag{41}
\]

In order to compare the contributions \( N[e, V_1] \) and \( \# \text{supp}\{V_2\} \) on the right-hand side of (41) to \( N_{sc}[e, V] \), we use the following definition.

**Definition 3.3.** Let \( d \geq 1 \). Given a dispersion relation \( e \) and a potential \( V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+) \), we define:

\[
N_{sc}^>[e, V] \doteq \# \{x \in \Gamma \mid V(x) \geq e_{\text{max}}\},
\]

\[
N_{sc}^<[e, V] \doteq \sum_{x \in \Gamma} 1\{V(x) < e_{\text{max}}\} V^{d/2}(x). \tag{43}
\]
Observe that, because dispersion relations are Morse functions, there are constants $0 < c_1(e) \leq c_2(e) < \infty$ such that for any potential $V \geq 0$,

$$c_1(e) \left( N_{sc}^\geq[e, V] + N_{sc}^\leq[e, V] \right) \leq N_{sc}[e, V] \leq c_2(e) \left( N_{sc}^\geq[e, V] + N_{sc}^\leq[e, V] \right). \quad (44)$$

Corollary 3.2 (44), and the CLR bound immediately yield Theorem 1.1:

**Theorem 3.4** (Thm. 1.1). Let $d \geq 3$ and $e$ be an admissible dispersion. Then there exists a constant $C_{3.4}(d, e) \in [1, \infty)$ such that

$$N[e, V] \leq C_{3.4}(d, e) N_{sc}[e, V] < \infty \quad (45)$$

for all $V \in \ell^{d/2}(\Gamma, \mathbb{R}^+)$.

**Proof:** We apply Corollary 3.2 to $V = V_1 + V_2$, with $V_1(x) \doteq V(x)1[V(x) < \epsilon_{\text{max}}]$ and $V_2(x) \doteq V(x)1[V(x) \geq \epsilon_{\text{max}}]$, and then Theorem 2.2 to $N[e, V_1]$. This gives

$$N[e, V] \leq N[e, V_1] + \# \text{supp } V_2 \quad (46)$$

and

$$\leq C_{2.2}(d) N_{sc}^\leq[e, V] + N_{sc}^\geq[e, V] \leq \frac{C_{2.2} + 1}{c_1(e)} N_{sc}[e, V]. \quad (47)$$

3.2 Saturation of the Non-Asymptotic Semi-classical Bounds

Below, we discuss the optimality of the bound in Theorem 1.1 in three different situations: For slowly decaying potentials, for strong and finitely supported potentials, and for weak potentials which are slowly varying in space.

We first show that if $V$ decays slower than $|x|^{-2}$ then 0 is an accumulation point of the discrete spectrum of $H(e, V)$ and, in particular, $H(e, V)$ has infinitely many negative eigenvalues, i.e., $N[e, V] = N_{sc}[e, V] = \infty$. To formulate the statement, we recall that $h_{x,y} = h(e)_{x,y} = \langle \delta_x | h(e) \delta_y \rangle$ denotes matrix elements of $h(e)$.

**Theorem 3.5** ($N[e, V] = \infty$ for slowly decaying potentials). Let $e$ be an admissible dispersion relation with hopping matrix $h(e)$ and $V \in \ell_0^\infty(\Gamma, \mathbb{R}^+)$. Assume that there are constants $\text{const} < \infty$ and $\text{const}'' > 0$ with $\alpha < \min\{\alpha', 2\}$ such that, for all $x \in \Gamma \backslash \{0\}$,

$$V(x) \geq \text{const}' |x|^{-\alpha}, \quad |h_{x,x}| \leq \text{const} \cdot |x|^{-(2d+\alpha')}. \quad (48)$$

Then $H(e, V)$ has infinitely many eigenvalues below 0.

The proof of this theorem is a bit lengthy and is given in Appendix A.2. For the case $e = e_{\text{Lapl}}$ and $d = 1$. See also [3].
Note that – assuming \( \alpha' \geq 2 \) – Theorem 3.5 together with the bound (39) implies that the case \( V(x) \sim |x|^{-2} \) is critical in dimension \( d \geq 3 \) in the sense that

\[
\exists \epsilon > 0 : \sup_{x \in \Gamma} \left\{ \frac{V(x)}{|x|^{2+\epsilon}} \right\} < \infty \quad \Rightarrow \quad N[\epsilon, V], N_{ac}[\epsilon, V] < \infty, \tag{49}
\]

\[
\exists \epsilon > 0 : \inf_{x \in \Gamma} \left\{ \frac{V(x)}{|x|^{2-\epsilon}} \right\} > 0 \quad \Rightarrow \quad N[\epsilon, V] = N_{ac}[\epsilon, V] = \infty. \tag{50}
\]

Observe also that Theorem 1.3 follows from Theorem 3.5.

\[\text{Lemma 3.6 (Lower bound on } N[\epsilon, V] \text{ without ?? and for } d \geq 1) \text{. Let } d \geq 1 \text{ and } \epsilon \text{ be an admissible dispersion relation. Furthermore let } V \in \ell_{0}^{\infty}(\Gamma, \mathbb{R}_0^+) \text{ be a potential decaying at } \infty. \text{ Then, for all } c > \epsilon_{\text{max}},\]

\[N[\epsilon, V] \geq L_{V_c}[c] = \{ x \in \Gamma \mid V(x) \geq c \}. \tag{51}\]

\[\text{Proof: For all } \rho > 0,\]

\[B(\rho) = V^{1/2} \frac{1}{\rho + h(\epsilon)} V^{1/2} \geq \frac{1}{\epsilon_{\text{max}}} V. \tag{52}\]

By the min-max principle and Lemma 2.1 (Birman-Schwinger principle), we hence obtain that

\[N[\epsilon, V] \geq L_{V}[c], \tag{53}\]

for all \( c > \epsilon_{\text{max}} \).

The following (stronger) result holds for sparse potentials:

\[\text{Lemma 3.7 (Lower bound on } N[\epsilon, V] \text{ for sparse potentials). Let } d \geq 3 \text{ and } \epsilon \text{ be an admissible dispersion relation. Let } 0 < \eta(\epsilon) < \epsilon_{\text{max}} \text{ be defined by}\]

\[\frac{1}{\eta(\epsilon)} = \int [\epsilon(p)]^{-1} d\mu^{*}(p).\]

\[\text{Furthermore, let } V \in \ell_{0}^{\infty}(\Gamma, \mathbb{R}_0^+) \text{ be a potential which is sparse in the sense that}\]

\[\eta(\epsilon) \cdot \sup_{\rho > 0} \left\{ \sup_{x \in \text{supp } V} \left( \sum_{p \in \text{supp } V \setminus \{ x \}} \left| \langle \delta_x | [\rho + h(\epsilon)]^{-1} \delta_p \rangle \right| \right) \right\} < \frac{\epsilon}{1 + \epsilon} < 1,\]

for some \( 0 < \epsilon < \infty \). Then

\[N[\epsilon, V] \geq L_{V}[(1 + \epsilon)\eta(\epsilon)] = \{ x \in \Gamma \mid V(x) \geq (1 + \epsilon)\eta(\epsilon) \}. \tag{54}\]

\[\text{Proof: Observe that } N[\epsilon, V] \geq N[\epsilon, V'] \text{ with } V'(x) = \max\{V(x), (1 + \epsilon)\eta(\epsilon)\}. \text{ Let } \rho > 0 \text{ and } x \in \Gamma. \text{ Similarly to (??), we have}\]

\[\langle \delta_x | B(\rho, \epsilon, V') \delta_x \rangle = V'(x) \left( \int_{\Gamma^{+}} \frac{d\mu^{*}(p)}{\rho + \epsilon(p)} \right). \tag{55}\]
Observe that, by the assumption on $V$ and the Schur bound, for all $\psi \in \ell^2(\Gamma)$,

$$
\sup_{\rho > 0} \langle \psi | B(\rho, e, V') \psi \rangle > \left( \sum_{x \in \mathcal{L}} |\psi_x|^2 (1 + \varepsilon) \right) - \varepsilon.
$$

where the summation runs over $x \in \mathcal{L} \doteq \mathcal{L}_V[(1 + \varepsilon)\eta(e)] = \{ x \in \Gamma | V(x) \geq (1 + \varepsilon)\eta(e) \}$. By Lemma 2.1 (Birman-Schwinger principle) and the min-max principle, we hence obtain that

$$
N[e, V'] \geq \mathcal{L}_V[(1 + \varepsilon)\eta(e)]. \tag{56}
$$

□

Note that Lemma 3.6, together with Corollary 3.2 and $N[e, 0] = 0$, implies that, for finitely supported potentials $V$, we have

$$
\lim_{\lambda \to \infty} N[e, \lambda V] = \lim_{\lambda \to \infty} N_{sc}[e, \lambda V] = \text{supp} V, \tag{57}
$$

and thus the semi-classical upper bound on $N[e, \lambda V]$ saturates when $\lambda \to \infty$.

Observe further that, on one hand, Theorem 3.9 below implies that the lower bound on $N[e, V]$ given in Lemma 3.6 strongly underestimates the size of the discrete spectrum of $H(e, V)$ in the case where $V$ is slowly varying in space. $N_{sc}[e, V]$ describes – in this precise case – the behavior of $N[e, V]$ more correctly. On the other hand, it seems that there is no other simple candidate for a lower bound on $N[e, V]$ holding in general and based on quantities like $N_{sc}[e, V]$ or $|V|^p$. See Corollary 4.5 and remark thereafter.

For any continuous function $f : \mathbb{R}^d \to \mathbb{R}^+$ define for all $M \in \mathbb{N}_0$ the step functions $f_{\lambda}^{(M)} : \mathbb{R}^d \to \mathbb{R}^+_0$ by:

$$
f_{\lambda}^{(M)}(x) = \sum_{X \in \mathbb{Z}^d} \mathbf{1}[x \in 2^{-M}X + [0, 2^{-M})^d] \min\{ f(x') | x' \in 2^{-M}X + [0, 2^{-M})^d \}. \tag{58}
$$

**Lemma 3.8.** Let $v \in C_0(\mathbb{R}^d, \mathbb{R}^d_0)$ be compactly supported. For all $L > 0$ define the potential $V_L : \Gamma \to \mathbb{R}^+_0$ by:

$$
V_L(x) = L^{-2} v(L^{-1}x). \tag{59}
$$

Let $e$ be any admissible dispersion relation from $C^3(\Gamma^*, \mathbb{R})$. Assume, moreover, that for some $D < \infty$ and some $\alpha > 2$, for all $x \in \Gamma$,

$$
|h(e)_{0,x}| \leq D \langle x \rangle^{2d+\alpha}. \tag{60}
$$

Then there are constants $\text{const}' > 0$, $\text{const} < \infty$, depending only on $e$ such that for all $M \in \mathbb{N}_0$,

$$
\liminf_{L \to \infty} N[e, V_L] \geq \text{const}' \int_{\mathbb{R}^d} v_{\lambda}^{(M)}(x)^{d/2} \mathbf{1}[v_{\lambda}^{(M)}(x) > \text{const}^{2M}]^d dx. \tag{61}
$$
We prove this by standard arguments using coherent states. See Appendix A.2. The following result is an immediate consequence of the lemma above.

**Theorem 3.9.** Let $\epsilon$ be any admissible dispersion relation from $C^3(\Gamma^*, \mathbb{R})$ and $v \in C_0(\mathbb{R}^d, \mathbb{R}^d_0)$ be compactly supported. Let the potentials $V_L = V_L(v)$ be defined as above. Then, for some constant $\text{const} > 0$ depending only on $\epsilon$,

$$\lim_{\lambda \to \infty} \liminf_{L \to \infty} N[\epsilon, \lambda V_L] \geq \text{const} \frac{\lambda^{d/2}}{d^d} \int_{\mathbb{R}^d} v(x)^{d/2} d^d x. \quad (62)$$

Observe, moreover, that from Theorem 3.9:

$${\tilde{N}}[\epsilon, \lambda V_L] \geq \text{const} N_{\text{sc}}[\epsilon, \lambda V_L]$$

for some $\text{const} > 0$ and sufficiently large $\lambda > 0$ and $L > 0$. Thus, as expected, like in the continuous case: $N[\epsilon, \lambda V_L] \sim N_{\text{sc}}[\epsilon, \lambda V_L]$ at large $\lambda > 0$ and $L > 0$.

4 Asymptotics of $N[\epsilon, \lambda V]$ for large $\lambda$

In this section we investigate the question whether the semi-classical number of bound states $N_{\text{sc}}[\epsilon, \lambda V]$ describes $N[\epsilon, \lambda V]$ correctly in the limit $\lambda \to \infty$ or not. This leads us to the proof of Theorems 1.8 and 1.9.

Equally interesting, however, is the observation made in this section that an asymptotic comparison of $N[\epsilon, \lambda V]$ to $N_{\text{sc}}[\epsilon, \lambda V]$ does not always make much sense. Namely, in Theorem 4.7 below, we prove that $\lambda \mapsto N[\epsilon, \lambda V]$ may approximate any given continuous and monotonically increasing function $F(\lambda)$ of $\lambda$. More precisely, given $F$, we can always find a potential $V_F$ such that $N[\epsilon, \lambda V_F] = F(\lambda)$ up to a small error.

4.1 Potentials with Semi-classical Asymptotic Behavior of $N[\epsilon, \lambda V]$ at large $\lambda$

This subsection is devoted to the proof of Theorems 1.8 and 1.9. To this end, we recall that

$$g_+(V) \doteq \sup_{r > 0} \limsup_{\ell \to \infty} \frac{2}{dr} \left( \ln \mathcal{L}_V \left[ e^{-\ell-r} \right] - \ln \mathcal{L}_V \left[ e^{-\ell} \right] \right), \quad (63)$$

$$g_-(V) \doteq \inf_{r > 0} \liminf_{\ell \to \infty} \frac{2}{dr} \left( \ln \mathcal{L}_V \left[ e^{-\ell-r} \right] - \ln \mathcal{L}_V \left[ e^{-\ell} \right] \right). \quad (64)$$

The following lemma illustrates that, for potentials with $g_+(V) < 1$, the main contribution to $N_{\text{sc}}[\epsilon, \lambda V]$ is given by $\# \{ \lambda V \geq \epsilon_{\text{max}} \}$, and that this actually defines a borderline in the sense that if $g_-(V) \geq 1$ then this assertion is reversed.

**Lemma 4.1.** Assume $d \geq 1$ and $V \in L^\infty_0(\Gamma, \mathbb{R}^d_0)$.

(i) Then there is a constant $C_{4,1}(d, \epsilon) > 0$ such that

$$\liminf_{\lambda \to \infty} \left\{ \frac{N_{\text{sc}}[\epsilon, \lambda V]}{N_{\text{sc}}[\epsilon, \lambda V]} \right\} \geq (1 - g_+(V)) C_{4,1}(d, \epsilon). \quad (65)$$
(ii) Conversely, if \( g_-(V) \geq 1 \) then

\[
\lim_{\lambda \to \infty} \frac{N^\infty_{sc}[\epsilon, \lambda V]}{N_{sc}[\epsilon, \lambda V]} = 0,
\]

where \( N^\infty_{sc}[\epsilon, V] = \mathcal{L}_V[\epsilon_{\text{max}}] = \# \{ V \geq \epsilon_{\text{max}} \} \) is defined in Definition 3.3.

**Proof:** We first fix \( x \in \Gamma \), set \( \rho_x = \min \{ 1, \lambda V(x)/\epsilon_{\text{max}} \} \), and observe that

\[
c_1 \rho_x^{\ell_x/2} \leq \int_{\Gamma^+} 1[\epsilon(p) < \lambda V(x)]d\mu^*(p) \leq C_1 \rho_x^{\ell_x/2},
\]

for some \( 0 < c_1 \equiv c_1(d, \epsilon) < C_1 \equiv C_1(d, \epsilon) < \infty \), since \( \mu(p) \) is a Morse function. Furthermore, we have that

\[
N_{sc}[\epsilon, \lambda V] = \sum_{x \in \Gamma} \int_{\Gamma^+} 1[\epsilon(p) < \lambda V(x)]d\mu^*(p)
\]

\[
= \sum_{x \in \Gamma} \int_{\Gamma^+} 1[\epsilon(p) < \lambda V(x) \leq \epsilon_{\text{max}}]d\mu^*(p) + \mathcal{L}_V[\lambda^{-1}\epsilon_{\text{max}}].
\]

Using that

\[
\rho_x^{\ell_x/2} = \frac{d}{2} \int_0^{\infty} 1[e^{-r} < \rho_x] e^{-dr/2} dr
\]

and \( \ell_x = \log(\lambda) - \log(\epsilon_{\text{max}}) \), we hence obtain

\[
N_{sc}[\epsilon, \lambda V] - \mathcal{L}_V[e^{-\ell_x}]
\]

\[
= \sum_{x \in \Gamma} \int_{\Gamma^+} 1[\epsilon(p) < \lambda V(x) < \epsilon_{\text{max}}]d\mu^*(p)
\]

\[
\leq \frac{dC_1}{2} \sum_{x \in \Gamma} \int_0^{\infty} \{ 1[e^{-r} \leq \lambda \epsilon_{\text{max}} V(x)] - 1[1 \leq \lambda \epsilon_{\text{max}} V(x)] \} e^{-dr/2} dr,
\]

\[
= \frac{dC_1}{2} \int_0^{\infty} \left\{ \mathcal{L}_V[e^{-\ell_x - r}] - \mathcal{L}_V[e^{-\ell_x}] \right\} e^{-dr/2} dr,
\]

\[
= \frac{dC_1}{2} \mathcal{L}_V[e^{-\ell_x}] \int_0^{\infty} \left\{ \frac{\mathcal{L}_V[e^{-\ell_x - r}]}{\mathcal{L}_V[e^{-\ell_x}]} \right\} e^{-dr/2} dr - C_1 \mathcal{L}_V[e^{-\ell_x}],
\]

and similarly

\[
N_{sc}[\epsilon, \lambda V] - \mathcal{L}_V[e^{-\ell_x}]
\]

\[
\geq \frac{dC_1}{2} \mathcal{L}_V[e^{-\ell_x}] \int_0^{\infty} \left\{ \frac{\mathcal{L}_V[e^{-\ell_x - r}]}{\mathcal{L}_V[e^{-\ell_x}]} \right\} e^{-dr/2} dr - C_1 \mathcal{L}_V[e^{-\ell_x}].
\]

Defining

\[
g_\epsilon(r) \doteq \frac{2}{dr} \left( \ln \mathcal{L}_V[e^{-\ell_x - r}] - \ln \mathcal{L}_V[e^{-\ell_x}] \right),
\]
we hence have
\[ \frac{dC_1}{2} \int_0^\infty \exp \left( -\left[ 1 - g_\ell (r) \right] \frac{d}{2} r \right) \, dr \geq \frac{N_{sc}[e, \lambda V]}{\mathcal{L}_V[e^{-\ell x}]} - 1 + C_1, \]  
\[ \frac{d c_1}{2} \int_0^\infty \exp \left( -\left[ 1 - g_\ell (r) \right] \frac{d}{2} r \right) \, dr \leq \frac{N_{sc}[e, \lambda V]}{\mathcal{L}_V[e^{-\ell x}]} - 1 + c_1, \]

Now, an application of Fatou’s Lemma yields
\[ \limsup_{\lambda \to \infty} \frac{N_{sc}[e, \lambda V]}{\mathcal{L}_V[e^{-\ell x}]} \leq 1 - C_1 + \frac{dC_1}{2} \int_0^\infty \exp \left( -\left[ 1 - g_+(V) \right] \right) \, rdr \]
\[ = 1 - C_1 + \frac{dC_1}{\left[ 1 - g_+(V) \right]}, \]
which implies (i). Assertion (ii) is similar, for if \( g_-(V) \geq 1 \) then another application of Fatou’s Lemma gives
\[ \liminf_{\lambda \to \infty} \frac{N_{sc}[e, \lambda V]}{\mathcal{L}_V[e^{-\ell x}]} \geq 1 - c_1 + \frac{d c_1}{2} \int_0^\infty \exp \left( \left[ g_-(V) - 1 \right] \frac{d}{2} r \right) \, dr = \infty. \]  

**Proof of Theorems 1.8 and 1.9**

By Theorem 1.4 and Definition 3.3, we have
\[ \frac{N[e, \lambda V]}{N_{sc}[e, \lambda V]} \geq \frac{\mathcal{L}_V[\lambda^{-1} \epsilon_{\text{max}}]}{N_{sc}[e, \lambda V]} = \frac{N_{\geq [e, \lambda V]}}{N_{sc}[e, \lambda V]}, \]  

Now, the left-hand inequality in (32) and the first inequality in (33) follow directly from Lemma 4.1 (i). The right-hand inequality in (32) follows from Theorem 1.1, while the second inequality in (33) is a consequence of Theorem 1.2.

**4.2 Failure of Semi-classical Asymptotic Behavior of \( N[e, \lambda V] \) at large \( \lambda \)**

For the continuum Schrödinger operator \(-\Delta - \lambda V(x)\) on \( \mathbb{R}^d \), the number of negative eigenvalues is asymptotically homogeneous of degree \( d/2 \) in \( \lambda \), i.e., \( N_{\text{cont}}[\lambda V] = \lambda^{d/2} N_{sc}[\lambda V] \). For discrete Schrödinger operators, only weaker statements hold true, as is illustrated by the following lemma. See also [15, Section 5.2].

**Lemma 4.2 (Lemma 1.7).** Assume \( d \geq 3 \) and \( V \in \ell^{d/2}(\Gamma, \mathbb{R}^+_0) \). Then
\[ \lim_{\lambda \to \infty} \left\{ \lambda^{-d/2} N[e, \lambda V] \right\} = \lim_{\lambda \to \infty} \left\{ \lambda^{-d/2} N_{sc}[e, \lambda V] \right\} = 0. \]  

**Proof:** It suffices to prove the second equality, since \( N[e, \lambda V] \leq C_{2,2}(d, \epsilon) N_{sc}[e, \lambda V] \), by Theorem 1.1. By (44), we have that
\[ \lambda^{-d/2} N[e, \lambda V] \leq c_2(\epsilon) \lambda^{-d/2} \left( N_{\geq [e, \lambda V]} + N_{< [e, \lambda V]} \right), \]  

where
\[ N_{\geq [e, \lambda V]} \leq \frac{\mathcal{L}_V[\lambda^{-1} \epsilon_{\text{max}}]}{N_{sc}[e, \lambda V]} \quad \text{and} \quad N_{< [e, \lambda V]} \leq \frac{\mathcal{L}_V[\lambda^{-1} \epsilon_{\text{max}}]}{N_{sc}[e, \lambda V]}. \]
and
\[
\lambda^{-d/2} \left( N_{s_c}^c[\epsilon, \lambda V] + N_{s_c}^c[\epsilon, \lambda V] \right) \geq \sum_{x \in \Gamma} \min \{ \epsilon_{\max}, \lambda^{d/2} V^{d/2}(x) \} = \sum_{x \in \Gamma} \min \{ \epsilon_{\max}, V^{d/2}(x) \}.
\]
(81)

Since
\[
\lim_{\lambda \to \infty} \min \{ \lambda^{-d/2} \epsilon_{\max}, V^{d/2}(x) \} = 0
\]
for every \( x \in \Gamma \) and \( \min \{ \lambda^{-d/2} \epsilon_{\max}, V^{d/2} \} \) is dominated by \( V^{d/2} \in \ell^1(\Gamma) \), the assertion follows from the dominated convergence theorem.

**Lemma 4.3.** Let \( d \geq 3 \) and \( \epsilon \) be an admissible dispersion relation. Then there is a constant \( C_{\text{dom}}(d, \epsilon) < \infty \) such that, for all \( \rho \in (0, 1] \) and all \( x, y \in \Gamma \), \( x \neq y \),
\[
|\langle \delta_x, (\rho + h(\epsilon))^{-1} \delta_y \rangle| \leq C_{\text{dom}}(d, \epsilon) |x - y|^{d/2}.
\]
(82)

**Proof:** Let \( \text{Min}(\epsilon) = \{ \xi \in \Gamma^* | \epsilon(\xi) = 0 \} \) be the set of points in \( \Gamma^* \) for which \( \epsilon \) is minimal. We construct a partition of unity localizing on the Voronoi cells
\[
\mathcal{V}(\xi) = \{ p \in \Gamma^* | \gamma(p, \xi) = \min_{\xi \in \text{Min}(\epsilon)} \gamma(p, \hat{\xi}) \},
\]
(83)

where \( \xi \in \text{Min}(\epsilon) \) and \( \gamma : \Gamma^* \times \Gamma^* \to \mathbb{R}_0^+ \) is the natural metric on \( \Gamma^* = (\mathbb{R}/2\pi\mathbb{Z})^d \).

Denote by \( r > 0 \) the largest radius, such that \( B_r(\xi, 2r) \subseteq \mathcal{V}(\xi) \), for all \( \xi \in \text{Min}(\epsilon) \), and choose \( j \in C_0^\infty(\mathbb{R}^d, \mathbb{R}_0^+) \) such that \( \text{supp } j \subseteq B(0, 1) \) and \( \int_{\mathbb{R}^d} j(p) d^d p = 1 \). We then set \( j_r(p) = r^{-d} j(p/r) \) for \( p \in \Gamma^* \) (which makes sense because \( r > 0 \) is sufficiently small), and
\[
\chi_\xi = j_r \ast 1_{\mathcal{V}(\xi)}.
\]
(84)

We list a few properties of this partition in combination with the dispersion \( \epsilon \) deriving from the fact that \( \epsilon \) is a Morse function.
\[
\forall p \in \Gamma^*: \sum_{\xi \in \text{Min}(\epsilon)} \chi_\xi(p) = 1,
\]
(85)
\[
\forall p \in \Gamma^* \forall \xi, \hat{\xi} \in \text{Min}(\epsilon), \xi \neq \hat{\xi} : \chi_\xi(p) > 0 \implies \gamma(p, \hat{\xi}) > r,
\]
\[
\exists c_1 > 0 \forall p \in \Gamma^* \forall \xi \in \text{Min}(\epsilon) : \nabla_p \chi_\xi(p) > 0 \implies \epsilon(p) \geq c_1,
\]
\[
\exists c_2 > 0 \forall p \in \Gamma^* \forall \xi \in \text{Min}(\epsilon) : \chi_{\hat{\xi}}(p) > 0 \implies \epsilon(p) \geq c_2(p - \xi)^2,
\]
\[
\exists c_3 < \infty \forall p \in \Gamma^* \forall \xi \in \text{Min}(\epsilon) : \chi_\xi(p) > 0 \implies |\nabla \epsilon(p)| \leq c_3 |p - \xi|.
\]

By translation invariance, it suffices to prove (82) for \( y = 0 \) and \( x \neq 0 \). We observe
that

\[ |x|^2 \left| \left\langle \delta_x \left| (\rho + h(\epsilon))^{-1} \delta_0 \right. \right\rangle \right| = \left| \int_{\Gamma^*} \frac{x \cdot \nabla_p \left( e^{i\rho x} \right) \, d\mu^* (p)}{\rho + \epsilon (p)} \right| \]  
\[ = \left| \sum_{\xi \in \text{Min}(e)} \int_{\Gamma^*} x \cdot \nabla_p \left( e^{i(\rho - \xi)x} - 1 \right) \frac{\chi\xi (p) \, d\mu^* (p)}{\rho + \epsilon (p)} \right| \]
\[ = \left| \sum_{\xi \in \text{Min}(e)} \int_{\Gamma^*} \left( e^{i(\rho - \xi)x} - 1 \right) \left\{ \frac{x \cdot \nabla_p \chi\xi (p)}{\rho + \epsilon (p)} - \frac{\chi\xi (p) x \cdot \nabla_p \epsilon (p)}{|\rho + \epsilon (p)|^2} \right\} \, d\mu^* (p) \right|. \]

Now we use \( \left| e^{i(\rho - \xi)x} - 1 \right| \leq 2 \), and \( \left| e^{i(\rho - \xi)x} - 1 \right| \leq 2 |x|^\beta |p - \xi|^\beta \) to obtain

\[ |x|^{1/2} \left| \langle \delta_x \left| (\rho + h(\epsilon))^{-1} \delta_0 \right. \right\rangle \right| \leq \sum_{\xi \in \text{Min}(e)} \int_{\Gamma^*} \left\{ \frac{2 |\nabla_p \chi\xi (p)|}{c_1} + \frac{\chi\xi (p) c_1}{c_2 |\rho - \xi|^5/2} \right\} \, d\mu^* (p) \leq C_4, \]

for some constant \( C_4 < \infty \), since \( |p - \xi|^{-5/2} \) is locally integrable for \( d \geq 3 \). We remark that we may have improved this estimate to \( \mathcal{O}(|x|^{\beta - 1}) \), for any \( \beta > 0 \), by using \( \left| e^{i(\rho - \xi)x} - 1 \right| \leq 2 |x|^{\beta} |p - \xi|^{\beta} \).

\( \Box \)

**Lemma 4.4.** Let \( d \geq 3 \) and \( \epsilon \) be an admissible dispersion. Let \( \mathcal{r} \doteq (r_k)_{k=0}^{\infty} \) be an increasing sequence of positive integers with \( 9r_k \leq r_{k+1} \) for all \( k \geq 0 \), and define \( \omega(\mathcal{r}) \doteq \{ x_0, x_1, x_2, \ldots \} \subseteq \Gamma \) by

\[ x_k \doteq (r_k, 0, \ldots, 0). \]

If \( V \in \ell^\infty(\Gamma) \) with \( \text{supp} \, V \subseteq \omega(\mathcal{r}) \) and

\[ |V|_{\infty} < \eta(\epsilon) - \frac{1}{4} C_{4.3} (d, \epsilon) \eta(\epsilon)^2 r_0^{-1/2}, \]

then \( N[\epsilon, V] = 0 \).

**Proof:** For any normalized \( \psi = (\psi_x)_{x \in \Gamma} \in \ell^2(\Gamma) \) and all \( \rho > 0 \), we have that

\[ \left\langle \psi \left| V^{1/2} \left( (\rho + h(\epsilon))^{-1} V^{1/2} \right) \psi \right. \right\rangle \]
\[ \leq \frac{1}{\eta(\epsilon)} \sum_{x \in \omega(\mathcal{r})} V(x) |\psi_x|^2 \]
\[ + \sum_{x, y \in \omega(\mathcal{r}), x \neq y} \bar{\psi}_x \psi_y |V(x)V(y)|^{1/2} \left| \langle \delta_x \left| (\rho + h(\epsilon))^{-1} \delta_y \right. \right\rangle \right| \]
\[ \leq |V|_{\infty} \left( \frac{1}{\eta(\epsilon)} + \sup_{x \in \omega(\mathcal{r})} \left\{ \sum_{y \in \omega(\mathcal{r}) \setminus \{x\}} |\langle \delta_x \left| (\rho + h(\epsilon))^{-1} \delta_y \right. \right\rangle \right| \right). \]
by the Schur bound. From Lemma 4.3, it follows that
\[ \sup_{x \in \omega(\Omega)} \left\{ \sum_{y \in \omega(\Omega) \setminus \{x\}} |\langle \delta_x, (\rho + h(\epsilon))^{-1} \delta_y \rangle| \right\} \leq C_{4.3}(d, \epsilon) \sup_{k \geq 0} \{X_k + Y_k\}, \]
where
\[ X_k = \sum_{\ell=0}^{k-1} r_k - r_\ell \] and \[ Y_k = \sum_{\ell=k+1}^{\infty} |r_k - r_\ell|^{-1/2}. \] (92)

For \( \ell < k \), we have that \( |r_k - r_\ell| \geq 8r_k \geq 8 \cdot 9^k r_0 \) and hence
\[ X_k \leq \frac{k 3^{-k}}{\sqrt{8} r_0}. \] (94)

Similarly, we have that \( |r_k - r_\ell| \geq 8r_\ell \geq 8 \cdot 9^\ell r_0 \) for \( \ell > k \), and thus
\[ Y_k \leq \frac{3^{-k}}{3 (1 - \frac{1}{3}) \sqrt{8} r_0} = \frac{3^{-k}}{2 \sqrt{8} r_0}. \] (95)

We hence conclude that
\[ \sup_{x \in \omega(\Omega)} \left\{ \sum_{y \in \omega(\Omega) \setminus \{x\}} |\langle \delta_x, (\rho + h(\epsilon))^{-1} \delta_y \rangle| \right\} \leq \frac{C_{4.3}(d, \epsilon)}{2 \sqrt{8} r_0}. \] (96)

Thus, the operator norm of the Birman-Schwinger operator is strictly smaller than one,
\[ \|V^{1/2} (\rho + h(\epsilon))^{-1} V^{1/2}\| \leq \|V\|_\infty \left( \frac{1}{\eta(\epsilon)} + \frac{C_{4.3}(d, \epsilon)}{2 \sqrt{8} r_0} \right) < 1, \] (97)
for all \( \rho > 0 \), which implies that \( N[\epsilon, V] = 0 \). \( \square \)

The last lemma has the following immediate consequences.

**Corollary 4.5** (Thm. 1.6). Let \( d \geq 3 \) and \( \epsilon \) be an admissible dispersion. Then there exists a potential \( V \notin \bigcup_{p \geq 1} \ell^p(\Gamma) \) with \( N[\epsilon, V] = 0 \).

**Proof:** Fix \( r_0 \in \mathbb{N} \), choose \( r_k = 9^k r_0 \) and \( x_k = (r_k, 0, \ldots, 0) \), and set
\[ V(x) = \sum_{j=0}^{\infty} 1_{\{x_j\}} \left( x \right) \frac{\eta(\epsilon)}{\ln(4 + j)}. \] (98)

Note that \( V \in \ell^\infty_0(\Gamma) \) but that, for all \( p \geq 1 \), the \( p \)-norm of \( V \) diverges, \( \|V\|_p = \eta(\epsilon) \sum_{j=0}^{\infty} \left[ \ln(4 + j) \right]^{-p} = \infty \). Moreover, \( \|V\|_\infty = \frac{\eta(\epsilon)}{\ln(4)} < \eta(\epsilon) \), and Lemma 4.4 implies that \( N[\epsilon, V] = 0 \) provided \( r_0 \in \mathbb{N} \) is chosen sufficiently large such that \( C_{4.3}(d, \epsilon) \eta(\epsilon) r_0^{-1/2} < 4 \left( 1 - \frac{1}{\ln(4)} \right) \). \( \square \)

We remark that \( N_{sc}[\epsilon, V] = \infty \) in Corollary 4.5 since \( V \notin \bigcup_{p \geq 1} \ell^p(\Gamma) \). Thus, a lower bound on \( N[\epsilon, V] \) in terms of \( \ell^p \)-norms or in multiples of \( N_{sc}[\epsilon, V] \) cannot, in general, hold true. See also [15] Eq. (1.8)].
Corollary 4.6 (Thm. 1.5). Let \( d \geq 3, \epsilon \) be an admissible dispersion. Given \( \epsilon \in (0, 1) \) and a potential \( V \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+) \), there exists a rearrangement \( \tilde{V} \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+) \) of \( V \) such that
\[
N[\epsilon, \tilde{V}] \leq \mathcal{L}_V[(1 - \epsilon)\eta(\epsilon)] = \# \{ x \in \Gamma \mid V(x) \geq (1 - \epsilon)\eta(\epsilon) \}.
\] (99)

Proof: We write \( V = V^{(>} + V^{(<)} \) with
\[
V^{(>} = V \cdot 1 \{ V \geq (1 - \epsilon)\eta(\epsilon) \} \quad \text{and} \quad V^{(<)} = V \cdot 1 \{ V < (1 - \epsilon)\eta(\epsilon) \}.
\] (100)
Note that \( V^{(>} \) has bounded support. Thus, choosing \( \tilde{V}^{(<)} \) to be a rearrangement of \( V^{(<)} \) with
\[
\text{supp } \tilde{V}^{(<)} \subset \{(r_k, 0, \ldots, 0) \mid r_k \leq g_k r_0, \ k \in \mathbb{N}_0\}
\] (101)
and \( r_0 \in \mathbb{N} \) chosen sufficiently large, we find that
\[
\| \tilde{V}^{(<)} \|_{\infty} = (1 - \epsilon)\eta(\epsilon) < \eta(\epsilon) - \frac{1}{4} C_{4.3}(d, \epsilon) \eta(\epsilon)^2 r_0^{-1/2},
\] (102)
and Lemma 4.4 implies that \( N[\epsilon, \tilde{V}^{(<)}] = 0 \). Hence, defining \( \tilde{V} = V^{(>} + \tilde{V}^{(<)} \), we have for sufficiently large \( r_0 \in \mathbb{N} \) that \( \text{supp } V^{(>} \cap \text{supp } \tilde{V}^{(<)} = \emptyset, \tilde{V} \) is a rearrangement of \( V \), and
\[
N[\epsilon, \tilde{V}] \leq \# \text{supp } V^{(>} + N[\epsilon, \tilde{V}^{(<)}]
\] (103)
\[
= \# \text{supp } V^{(>} \leq \mathcal{L}_V[(1 - \epsilon)\eta(\epsilon)],
\] (104)
by Corollary 3.2. \( \square \)

The next theorem illustrates for \( d \geq 3 \) that – opposed to the continuum case – the asymptotics of \( N[\epsilon, \lambda V] \) as \( \lambda \to \infty \) can be prescribed arbitrarily.

Theorem 4.7 (Thm. 1.11). Let \( d \geq 3 \) and \( \epsilon \) be any admissible dispersion. Let further \( F : [1, \infty) \to \mathbb{N} \) be an arbitrary monotonically increasing, positively integer-valued, right-continuous function. Then, for any \( \epsilon \in (0, 1/2) \), there exists a potential \( V_{F,\epsilon} \in \ell_0^\infty(\Gamma, \mathbb{R}_0^+) \) such that
\[
\forall \lambda \geq 2 : F((1 - \epsilon)\lambda) \leq N[\epsilon, \lambda V_{F}] \leq F((1 + \epsilon)\lambda).
\] (105)

Proof: For the proof, we abbreviate \( \eta \doteq \eta(\epsilon) \). Since \( F : [1, \infty) \to \mathbb{N} \) is monotonically increasing and right-continuous, there is a monotonically increasing sequence \( 1 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots \) such that
\[
F(\lambda) = \sum_{j=1}^{\infty} 1[\lambda_j \leq \lambda].
\] (106)
Note that the monotonicity of \( F \) is not necessarily strict, and possibly \( \lambda_j = \lambda_{j+1} \). For a sequence \( \xi = (r_k)_{k=0}^{\infty} \) of positive integers, with \( 9r_k \leq r_{k+1} \), to be further specified later, and \( x_k = (r_k, 0, \ldots, 0) \in \Gamma, \) we set
\[
V_{F,\epsilon}(x) \doteq \sum_{j=1}^{\infty} \eta \frac{1}{\lambda_j} 1_{\{x_j\}}(x).
\] (107)
Let \( \varepsilon' > 0 \) be such that \((1 + \varepsilon')^{-1} > 1 - \varepsilon\). Choosing \( r_0 > 0 \) large enough such that

\[
\eta \sup_{x \in \text{supp} \ V_{F, \varepsilon}} \sup_{y \in \text{supp} \ V_{F, \varepsilon} \setminus \{x\}} \sum_{y \in \text{supp} \ V_{F, \varepsilon}} \frac{|(\delta_x | (\rho + h(x))^{-1} \delta_y)|}{1 + \varepsilon'} < \frac{\varepsilon'}{1 + \varepsilon'}
\]

we observe that

\[
\mathcal{L}_{\lambda V_{F, \varepsilon}}((1 + \varepsilon') \eta) = \mathcal{L}_{(1 + \varepsilon') \eta / \lambda}
\]

we have thus established the lower bound on \( N[e, \lambda V_{F, \varepsilon}] \) in (105).

\[
F((1 - \varepsilon) \lambda) \leq F((1 + \varepsilon')^{-1} \lambda) \leq N[e, \lambda V_{F, \varepsilon}]
\]

for all \( \lambda \geq 2 \). Choose now \( \varepsilon' > 0 \) such that \((1 - \varepsilon')^{-1} < 1 + \varepsilon \). For the proof of the upper bound in (105) we write \( \lambda V_{F, \varepsilon} = V_{F, \varepsilon}^{(>)} + V_{F, \varepsilon}^{(<)} \), where

\[
V_{F, \varepsilon}^{(>)}(x) = \lambda V_{F, \varepsilon} \mathbf{1}[V_{F, \varepsilon}(x) \geq (1 - \varepsilon') \frac{\eta}{\lambda}]
\]

and it remains to fix the sequence \( r \) so that

\[
N[e, V_{F, \varepsilon}^{(<)}] = 0,
\]

for all \( \lambda \geq 1 \). To this end, we first note that

\[
\|V_{F, \varepsilon}^{(<)}\|_{\infty} \leq \eta (1 - \varepsilon').
\]
From Lemma 4.4 holds by choosing \( r_0 > 0 \) large enough and the right-hand inequality in (105) follows. \( \square \)

A similar result in proven in [15, Section 6]. Observe, however, that, in contrast to [15], we do not assume that \( \lambda_j/\lambda_{j+1} \rightarrow 1 \), as \( j \rightarrow \infty \), for the asymptotics of eigenvalues. Moreover, the positivity preserving property of the hopping matrix \( h(\epsilon) \) is not needed.

Assume that for a given potential \( V \in \ell^\infty(\Gamma, \mathbb{R}^+_0) \), \( N[\epsilon, \lambda V] \sim N_{sc}[\epsilon, \lambda V] < \infty \) at large \( \lambda > 0 \), i.e., that \( N[\epsilon, \lambda V] \) is finite and obeys the Weyl asymptotics at large \( \lambda \). Then it would follow that \( N[\epsilon, \lambda V] = O(\lambda^{d/2}) \). By the last theorem, for any \( \alpha > 0 \), there are potentials \( V_\alpha \in \ell^\infty(\Gamma, \mathbb{R}^+_0) \) such that \( N[\epsilon, \lambda V] \) behaves like \( \lambda^\alpha \) as \( \lambda \rightarrow \infty \). In particular, the semi-classical asymptotics cannot hold for \( \alpha > d/2 \).

Define the potentials \( V_{\alpha} \): \( \Gamma \rightarrow \mathbb{R}^+_0 \) by

\[
V_{\alpha}(x) = \frac{1}{\langle x \rangle^2 \ln \langle x \rangle}, \quad V_{2}(x) = e^{-|x|}.
\]

Clearly, \( g_-(V_1) = 1 \) and \( g_+(V_2) = 0 \). By Lemma 4.1

\[
\lim_{\lambda \rightarrow \infty} \frac{N_{sc}[\epsilon, \lambda V_1]}{N_{sc}[\epsilon, \lambda V]} = 0, \quad \lim_{\lambda \rightarrow \infty} \frac{N_{sc}[\epsilon, \lambda V_2]}{N_{sc}[\epsilon, \lambda V]} > 0.
\]

For any monotonically increasing sequence \( \alpha = (\alpha_n)_{n \in \mathbb{N}} \) of positive real numbers define \( \beta_\alpha : \Gamma \rightarrow \{0, 1\} \) by \( \beta_\alpha(x) = 1 \) if \( \alpha_{1+2n} \leq |x| \leq \alpha_{2+2n} \) for some \( n \in \mathbb{N} \), and \( \beta_\alpha(x) = 0 \) else. Consider potentials of the form \( \tilde{V} = V_\alpha = \beta_\alpha(V_1 - V_2) + V_2 \geq 0 \). By (117), there exists a sequence \( \alpha \) such that:

\[
\lim_{\lambda \rightarrow \infty} \frac{N_{sc}[\epsilon, \lambda \tilde{V}]}{N_{sc}[\epsilon, \lambda V]} < \infty, \quad \lim_{\lambda \rightarrow \infty} \frac{N_{sc}[\epsilon, \lambda \tilde{V}]}{N_{sc}[\epsilon, \lambda \tilde{V}]} = \infty.
\]

By (118) and Lemma 3.6, for any rearrangement \( \tilde{V} \) of \( V \),

\[
\lim_{\lambda \rightarrow \infty} \frac{N_{sc}[\epsilon, \lambda V]}{N[\epsilon, \lambda V]} < \infty.
\]
Observe that, by Corollary 3.2 and Lemma 4.4, there is a rearrangement $V$ of $\tilde{V}$ such that
\[
\limsup_{\lambda \to \infty} \frac{N_{sc}[e, \lambda V]}{N[e, \lambda V]} > 1.
\] (119)
To conclude the proof use that for some $1 < C < \infty$,
\[
C^{-1} N_{sc}[e, \lambda V] \leq N_{sc}[e, \lambda (2e_{\max}/\eta(\epsilon))V] \leq C N_{sc}[e, \lambda V]
\] (120)
for all $\lambda > 0$. This together with (118) and (119) imply
\[
\limsup_{\lambda \to \infty} \frac{N_{sc}[e, \lambda V]}{N[e, \lambda V]} = \infty.
\]
Note that we have used above the invariance of the semi-classical quantities $N_{sc}[e, \tilde{V}]$ and $N_{sc}[e, \tilde{V}]$ w.r.t. rearrangements of $\tilde{V}$. □

5 One and Two Dimensions

We start this section by showing (Corollary 5.3) that the semi-classical upper bound, as stated in Theorem 1.1 for instance, cannot be valid in less than three dimensions.

Lemma 5.1. Let $d \in \{1, 2\}$, $\epsilon$ be an admissible dispersion relation, and $V \geq 0$ be a potential with finite support. For all $\rho > 0$ and all rearrangements $\tilde{V}$ of $V$ define the compact self-adjoint operator
\[
K(\rho, \tilde{V}) = P_{\text{Ran} \tilde{V}} \tilde{V}^{1/2} (\rho + h(\epsilon))^{-1} \tilde{V}^{1/2} P_{\text{Ran} \tilde{V}} - P_{\text{Ran} \tilde{V}}. 
\] (121)
Then there exist $\rho > 0$ and a rearrangement $\tilde{V}$ of $V$ such that $K(\rho, \tilde{V}) > 0$.

Proof: If $\text{supp} V = \emptyset$ there is nothing to prove, so we assume that $V \neq 0$. Let $\tilde{V} \geq 0$ be a rearrangement of $V$. Then for all $\rho > 0$ and all $\psi = (\psi_x)_{x \in \Gamma} \in \text{Ran} \tilde{V}$,
\[
\langle \psi | K(\rho, \tilde{V}) | \psi \rangle = -|\psi|^2 + \sum_{x \in \text{supp} \tilde{V}} \tilde{V}(x)|\psi_x|^2 \int_{\Gamma^*} \frac{d\mu^*(\rho)}{\rho + \epsilon(p)}
\]
\[
+ \sum_{x,y \in \text{supp} \tilde{V}, x \neq y} [\tilde{V}(x)\tilde{V}(y)]^{1/2} \langle \delta_x | (\rho + h(\epsilon))^{-1} \delta_y \rangle \bar{\psi}_x \psi_y,
\]
and thus
\[
K(\rho, \tilde{V}) \geq -1 + \min_{x \in \text{supp} V} V(x) \int_{\Gamma^*} \frac{d\mu^*(\rho)}{\rho + \epsilon(p)}
\]
\[
- |V|_\infty \sup_{\psi \in \text{Ran} \tilde{V}, |\psi|^2 = 1} \sum_{x,y \in \text{supp} \tilde{V}, x \neq y} |\langle \delta_x | (\rho + h(\epsilon))^{-1} \delta_y \rangle \bar{\psi}_x \psi_y|.
\]
Choose $\rho > 0$ such that
\[
\min_{x \in \text{supp} V} V(x) \int_{\Gamma^*} \frac{d\mu^*(\rho)}{\rho + \epsilon(p)} > 2. 
\] (124)
This is always possible since $d \leq 2$. For any fixed $\rho > 0$, we have that
\[
\langle \delta_x | (\rho + h(\epsilon))^{-1} \delta_y \rangle \rightarrow 0
\]
as $|x - y| \rightarrow \infty$. This follows from the Riemann-Lebesgue Lemma since $\langle \delta_x | (\rho + h(\epsilon))^{-1} \delta_y \rangle$ is the Fourier transform of the integrable function $(\rho + \epsilon)^{-1} \in L^1(\Gamma^s)$. In particular, there is a rearrangement $\tilde{V}$ of $V$ such that
\[
\sup_{\psi \in \text{Ran} \tilde{V}, \|\psi\|_2 = 1} \sum_{x, y \in \text{supp} \tilde{V}, \ x \neq y} |\langle \delta_x | (\rho + h(\epsilon))^{-1} \delta_y \rangle \tilde{\psi}_x \tilde{\psi}_y| \leq 1. \tag{125}
\]
For such $\rho > 0$ and $\tilde{V}$ we hence have that $K(\rho, \tilde{V}) > 0$. \hfill \Box

**Theorem 5.2.** Let $d \in \{1, 2\}$ and $\epsilon$ be an admissible dispersion relation. Then, for any finitely supported potential $V$, there is a rearrangement $\tilde{V}$ of $V$ such that
\[
N[\epsilon, \tilde{V}] = \# \text{supp} \tilde{V} = \# \text{supp} V. \tag{126}
\]

**Proof:** Clearly, for any rearrangement $\tilde{V}$ of $V$, we have $N[\epsilon, \tilde{V}] \leq \# \text{supp} V$, as follows, e.g., from Corollary 3.2 and the fact that $N[\epsilon, 0] = 0$. Let $\rho > 0$ and the rearrangement $\tilde{V}$ of $V$ be as in the lemma above. Then, by the min-max principle and the bound $K(\rho, \tilde{V}) > 0$, the compact operator $(\tilde{V})^{1/2}(\rho + h(\epsilon))^{-1}(\tilde{V})^{1/2}$ has at least $\dim \text{Ran} \tilde{V} = |\text{supp} \tilde{V}|$ discrete eigenvalues above 1. By Lemma 2.1 it follows from this that $N[\epsilon, \tilde{V}] \geq \# \text{supp} \tilde{V}$. \hfill \Box

Observing that the semi-classical number of bound states $N_{sc}[\epsilon, V]$ is invariant w.r.t. rearrangements of the potential $V$, the following corollary follows immediately:

**Corollary 5.3** (Breakdown of the semi-classical upper bound in $d = 1, 2$). Let $d \in \{1, 2\}$ and $\epsilon$ be any admissible dispersion. Then, for all $\epsilon > 0$,
\[
\sup \left\{ \frac{N[\epsilon, V]}{N_{sc}[\epsilon, V]} \right\} = \infty. \tag{127}
\]

The last corollary implies in one or two dimensions that multiples of $N_{sc}[\epsilon, V]$ cannot be, in general, an upper bound on $N[\epsilon, V]$. The discussion above shows, more precisely, that const $N_{sc}[\epsilon, V]$ fails to be such an upper bound in the case of sparse potentials, i.e. in the situation where the distance between points in the support of the potential $V$ is large. Hence, any quantity $Q(V)$ which is supposed to be an upper bound on $N[\epsilon, V]$ should keep track of the behavior of $V$ in space. This motivates the use of the weighted semi-classical quantities $N_{sc}[\epsilon, \tilde{V}(V)]$ – as stated in Theorem 1.2 – as upper bounds on $N[\epsilon, V]$ in one and two dimensions.

For any $p > 0$, $m \geq 0$, and any function $V : \Gamma \to \mathbb{R}^d_+$ define
\[
|V|_{p,m} = \left( \sum_{x \in \Gamma} V^p(x) \langle x \rangle^m \right)^{1/p}. \tag{128}
\]
Observe that $| \cdot |_{p,m}$ is not a norm, for $p \in (0, 1)$, but only a homogeneous functional of degree one. For any function $\epsilon \in C^m(\Gamma^*, \mathbb{C})$ and $m \in \mathbb{N}_0$, define the $C^m$-(semi)norms by

$$
\| \epsilon \|_{C^m} \doteq \max_{n \in \mathbb{N}_0, \| \cdot \|_n=m} \max_{p \in \Gamma} | \partial^{2n} \epsilon (p) |. \quad (129)
$$

Let $\epsilon$ be an admissible dispersion relation. We denote the set of all critical points of $\epsilon$ by

$$
\text{Crit}(\epsilon) \doteq \{ p \in \Gamma^* \mid \nabla \epsilon (p) = 0 \}. \quad (130)
$$

Recall that, as $\Gamma^*$ is compact, dispersion relations have at most finitely many critical points. $\text{Min}(\epsilon) \subset \text{Crit}(\epsilon)$ denotes the set of points on which the minimum of $\epsilon$ is taken.

Let $\epsilon''(p)$ be the Hessian matrix of $\epsilon$ at $p \in \text{Crit}(\epsilon)$. Define the minimal curvature of (the graph of) $\epsilon$ at $p \in \text{Crit}(\epsilon)$ by

$$
K(\epsilon, p) \doteq \min \{| \lambda |^{1/2} \mid \lambda \in \sigma(\epsilon''(p))\} > 0. \quad (131)
$$

Define also the minimal (critical) curvature of $\epsilon$ by

$$
K(\epsilon) \doteq \min \{ K(\epsilon, p) \mid p \in \text{Crit}(\epsilon) \} > 0. \quad (132)
$$

**Lemma 5.4** (A priori upper bound on $N(\epsilon, V)$, $d = 1, 2$). Let $\epsilon$ be any dispersion relation from $C^3(\Gamma^*, \mathbb{R})$. Let $C < \infty$ and $K > 0$ be such that $\| \epsilon \|_{C^3} < C$ and $K(\epsilon) > K$. Define $\delta \doteq \min \{ \epsilon(p) \mid p \in \text{Crit}(\epsilon) \text{ \setminus } \text{Min}(\epsilon) \} > 0$.

(i) There is a constant $C^{(5,4)}_{(i)} < \infty$ depending only on $\epsilon, C, K, \# \text{Min}(\epsilon)$, and $\delta$ such that $N[\epsilon, V] \leq \# \text{Min}(\epsilon)$ whenever $| V |_{1/2,1} < C^{(5,4)}_{(i)}$.

(ii) There is a constant $C^{(5,4)}_{(ii)} < \infty$ depending only on $\epsilon, C, K, \# \text{Min}(\epsilon)$, and $\delta$ such that

$$
N[\epsilon, V] \leq C^{(5,4)}_{(ii)} | V |_{1/2,2} + \# \text{Min}(\epsilon). \quad (133)
$$

**Proof:**

Let $C^1(\Gamma^*)$ be the Banach space of all continuously differentiable functions $\Gamma^* \to \mathbb{C}$ with norm $\| \cdot \|_{C^1}$. Observe that if $| V |_{1/2,1}$ is finite $\mathcal{F}^* \circ V^{1/2}$ defines a continuous linear map $\ell^2(\Gamma) \to C^1(\Gamma^*)$ with

$$
\| \mathcal{F}^* \circ V^{1/2} \|_{\mathcal{B}(\ell^2(\Gamma), C^1(\Gamma^*))} \leq | V |_{1/2,1}. \quad (134)
$$

Let $\text{Min}(\epsilon) = \{ p^{(1)}, \ldots, p^{(m)} \}$, $m = \# \text{Min}(\epsilon)$, and define the linear functionals $\zeta_i$, $i = 1, 2, \ldots, m$, on $\ell^2(\Gamma)$ by $\zeta_i(\varphi) \doteq \mathcal{F}^* \circ V^{1/2}(\varphi)(p^{(i)})$. By (134), the functionals $\zeta_i$ are continuous. Let $X = \bigcap_{i=1}^m \ker \zeta_i$. Assume that $H(\epsilon, V)$ has more than $m$ eigenvalues (counting multiplicities) below 0. Then, by Lemma 2.4 and the min-max principle, there is some $\rho > 0$ and some $(m + 1)$-dimensional subspace $S \subset \ell^2(\Gamma)$ with

$$
\min_{\varphi \in S, \| \varphi \|_2=1} \langle \varphi \mid V^{1/2}(\rho + h(\epsilon))^{-1}V^{1/2}\varphi \rangle > 1. \quad (135)
$$

Observe that for all $\varphi \in \ell^2(\Gamma)$,

$$
\langle \varphi \mid V^{1/2}(\rho + h(\epsilon))^{-1}V^{1/2}\varphi \rangle = \frac{\mathcal{F}^* \circ V^{1/2}(\varphi)(p)^2}{\rho + \epsilon(p)}d\mu^*(p). \quad (136)
$$
As the dimension of $S$ is larger than $m$, there is a vector $\tilde{\varphi} \in S \cap X$, $|\tilde{\varphi}|_2 = 1$. Notice that in this case there is a constant $\text{const} < \infty$ depending only on $C$ and $m$ such that for all $p \in \Gamma^*$,

$$|\mathcal{F}^* \circ V^{1/2}(\tilde{\varphi})(p)|^2 \leq \text{const} \, |V|_{1/2,1} \prod_{i=1}^{m} (1 - \cos(p - p^{(i)})),$$

where for each $q = (q_1, \ldots, q_d) \in \Gamma^*_d$,

$$\cos(q) = d^{-1}(\cos(q_1) + \ldots + \cos(q_d)).$$

It means that

$$1 < \text{const} \, |V|_{1/2,1} \int_{\Gamma^*} \prod_{i=1}^{m} (1 - \cos(p - p^{(i)})) \, \frac{1}{\rho + \epsilon(p)} \, d\mu^*(p). \quad (138)$$

Observing that the integral on the right-hand side of (138) is bounded by a constant depending only on $C$, $K$ and $m$ this concludes the proof of (i).

Now we prove (ii). For any $q \in \Gamma^*$ define the linear maps $\zeta^\ell_q : \ell^2(\Gamma) \to C \times C^d$ by

$$\zeta^\ell_q(\varphi) = \big((\mathcal{F}^* \circ V^{1/2})(\varphi)(q), (\nabla \mathcal{F}^* \circ V^{1/2})(\varphi)(q)\big). \quad (139)$$

By $|V|_{1/2,1} \leq |V|_{1/2,2} < \infty$ it follows that $\zeta^\ell_q$ is continuous.

There is a constant $\text{const} < \infty$ such that, for any fixed $\mu > 0$ small enough, there is a set of points $\{q_1, \ldots, q_{n(\mu)}\}$ from $\Gamma^*$ containing $\text{Min}(\epsilon)$ with the property that $n(\mu) \leq \mu^{-1}$ and, for all $q \in \Gamma^*$, $\min_{i=1,2,\ldots,n(\mu)} |q - q_i| \leq \text{const} \mu^{1/d}$. If the subspace $S \subset \ell^2(\Gamma)$ has dimension larger than $(d+1)\mu^{-1}$ then there is a vector $\tilde{\varphi} \in S$ with $|\tilde{\varphi}|_2 = 1$ and

$$\tilde{\varphi} \in \bigcap_{j=1}^{n(\mu)} \ker \zeta^\ell_{q_j}. \quad (140)$$

By Taylor expansions, for such a vector $\tilde{\varphi}$ we have, similarly as in the proof of (i), that for some constant $\text{const} < \infty$ and all $p \in \Gamma^*$:

$$|\mathcal{F}^* \circ V^{1/2}(\tilde{\varphi})(p)| \leq \text{const} \, |V|_{1/2,1} \prod_{i=1}^{m} (1 - \cos(p - p^{(i)})), \quad (141)$$

$$|\mathcal{F}^* \circ V^{1/2}(\tilde{\varphi})(p)| \leq \text{const} \mu \, |V|_{1/2,2}^{1/2}. \quad (142)$$

Using the last two inequalities we get

$$|\langle \tilde{\varphi}, V^{1/2} h(\epsilon)^{-1} V^{1/2} \tilde{\varphi} \rangle|$$

$$\leq |\mathcal{F}^* \circ V^{1/2}(\tilde{\varphi})| \int_{\Gamma^*} \frac{|\mathcal{F}^* \circ V^{1/2}(\tilde{\varphi})(p)|}{\epsilon(p)} \, d\mu^*(p)$$

$$\leq \text{const} \mu |V|_{1/2,2}. \quad (143)$$

Thus, by (i), Lemma 2.1 and the min-max principle, for some $\text{const} < \infty$, $H(\epsilon, V)$ has at most $(\text{const} \, |V|_{1/2,2} + m)$ eigenvalues below 0. \hfill \Box
Corollary 5.5 (Semi-classical upper bound on $N[\varepsilon, V]$ for $d = 1, 2$). Let $d \in \{1, 2\}$ and $\varepsilon$ be any admissible dispersion relation from $C^3(\Gamma^*)$. Then there is a constant $c(\varepsilon) < \infty$ such that for all potentials $V \geq 0$,

$$N[\varepsilon, V] \leq c(\varepsilon)(1 + N_{sc}[\varepsilon, \tilde{V}]),$$

where the effective potential $\tilde{V}(x) = V(x)|x|^{d+5}$.

Proof: From Lemma 5.4 and Corollary 3.2

$$N[\varepsilon, V] \leq |\{x \in \Gamma | \langle x \rangle^{d+5}V(x) \geq \varepsilon_{\text{max}}\}| + \#\text{Min}(\varepsilon)
+ \sum_{x \in \Gamma, \langle x \rangle^{d+5}V(x) < \varepsilon_{\text{max}}} \langle x \rangle^{-d+1} \langle \langle x \rangle^{d+1} \langle x \rangle^4 V(x) \rangle^{1/2}.$$ 

Thus, by the Cauchy-Schwarz inequality:

$$N[\varepsilon, V] \leq |\{x \in \Gamma | \langle x \rangle^{d+5}V(x) \geq \varepsilon_{\text{max}}\}| + \#\text{Min}(\varepsilon)
+ \sum_{x \in \Gamma, \langle x \rangle^{d+5}V(x) < \varepsilon_{\text{max}}} \langle x \rangle^{d+5}V(x).$$

As $\varepsilon$ is a Morse function this implies (144) in the case $d = 2$. Observing that $\langle x \rangle^{d+5}V(x) \leq \varepsilon_{\text{max}} \langle x \rangle^{d+5}V(x) \rangle^{1/2}$, whenever $\langle x \rangle^{d+5}V(x) \leq \varepsilon_{\text{max}}$, the case $d = 1$ follows from the last inequality as well.

A Appendix

A.1 Proof of Lemma 2.1 and Theorem 2.2

Proof of Lemma 2.1: We recall that, due to the compactness of $V$, the Birman-Schwinger operator $B(\rho)$ is compact and has only discrete spectrum above 0. Similarly, the spectrum of $H(\varepsilon, V)$ below 0 is discrete because $-V = H(\varepsilon, V) - H(\varepsilon, 0)$ is compact.

Suppose that $-\rho < 0$ is an eigenvalue of $H(\varepsilon, V)$ of multiplicity $M \in \mathbb{N}$ and let $\{\varphi_1, \ldots, \varphi_M\} \subseteq \ell^2(\Gamma)$ be an ONB of the corresponding eigenspace. Set

$$\psi_1 = V^{1/2}\varphi_1, \ldots, \psi_M = V^{1/2}\varphi_M.$$ (145)

Then $\psi_m \in \ell^2(\Gamma)$ since $V \in \ell^\infty(\Gamma)$. Moreover,

$$\varphi_m = [\rho + h(\varepsilon)]^{-1}V\varphi_m = [\rho + h(\varepsilon)]^{-1}V^{1/2}\psi_m,$$ (146)

and the boundedness of $[\rho + h(\varepsilon)]^{-1}V^{1/2}$ implies that $\{\psi_1, \ldots, \psi_M\} \subseteq \ell^2(\Gamma)$ is linearly independent. Clearly, (145) and (146) also yield

$$B(\rho)\psi_m = V^{1/2}[\rho + h(\varepsilon)]^{-1}V^{1/2}\psi_m = \psi_m.$$ (147)
and hence the eigenspace of $B(\rho)$ corresponding to the eigenvalue 1 has at least dimension $M$.

Conversely, if $\{\psi_1, \ldots, \psi_L\} \subseteq \ell^2(\Gamma)$ is an ONB of the eigenspace of $B(\rho)$ corresponding to the eigenvalue 1 then we set

$$\varphi_1 \doteq [\rho + h(\epsilon)]^{-1/2}\psi_1, \ldots, \varphi_L \doteq [\rho + h(\epsilon)]^{-1/2}\psi_L.$$  \hfill (148)

Since $[\rho + h(\epsilon)]^{-1/2}$ is bounded, $\varphi_\ell \in \ell^2(\Gamma)$. Moreover,

$$\psi_\ell = B(\rho)\psi_\ell = V^{1/2}\psi_\ell,$$  \hfill (149)

and the boundedness of $V^{1/2}$ implies that $\{\varphi_1, \ldots, \varphi_L\} \subseteq \ell^2(\Gamma)$ is linearly independent. Clearly, (148) and (149) also yield

$$H(\epsilon, V)\varphi_\ell = -\rho\varphi_\ell,$$  \hfill (150)

and hence the eigenspace of $H(\epsilon, V)$ corresponding to the eigenvalue $-\rho$ has at least dimension $L$.

These arguments prove (i) and (ii) and, furthermore, $M = L$ and thus (iii), i.e.,

$$\forall \rho > 0 : \dim \ker \left[ H(\epsilon, V) + \rho \right] = \dim \ker \left[ B(\rho) - 1 \right].$$  \hfill (151)

Observe that for all $\rho', \rho$ with $\rho' \geq \rho > 0$: $B(\rho') \leq B(\rho)$. As the map $\rho \mapsto B(\rho)$ is norm continuous on $\mathbb{R}^+$ and $\lim_{\rho \to \infty} B(\rho) = 0$, by the min-max principle, if $z_k > 1$ is the $k$-th eigenvalue of $B(\rho)$ counting from above with multiplicities, then there is a $\rho_k > \rho$ such that 1 is the $k$-th eigenvalue of $B(\rho_k)$ (counting from above with multiplicities). Clearly, $\rho_{k'} \leq \rho_k$, whenever $k' \geq k$. By (iii), this implies that $H(\epsilon, V)$ has at least as many eigenvalues less or equal $-\rho$ as $B(\rho)$ has eigenvalues greater or equal 1. By similar arguments, $B(\rho)$ has at least as many eigenvalues greater or equal 1 as $H(\epsilon, V)$ has eigenvalues less or equal $-\rho$. 

\[\square\]

To prove Theorem 2.2 we use the following estimate derived in [6]:

**Proposition A.1** (Frank). Let $(X, \mu)$ be any $\sigma$-finite measure space and $T$ a positive selfadjoint operator on $L^2(X, \mathbb{C})$ whose kernel is trivial. Assume that there are given constants $\nu > 2$ and $\nu_1 \in \mathbb{R}^+$, such that, for all $E > 0$ and any measurable set $\Omega \subset X$,

$$\text{tr} \left( \chi_\Omega T^{-1} 1_{[T \leq E]} \chi_\Omega \right) \leq \nu_1 \mu(\Omega) E^{-\nu},$$

where $\chi_\Omega$ is the multiplication operator with the characteristic function of $\Omega$. Let $V$ be any bounded positive-valued measurable function and denote by $N(T, V)$ the number of discrete negative eigenvalues, counting multiplicities, of the selfadjoint operator $T - V$. Then

$$N(T, V) \leq \nu_1 \nu \nu \nu_2 \left( \frac{\nu}{\nu - 2} \right)^{\nu - 2} \int_X V(x)^{\frac{\nu}{2}} d\mu(x).$$
Observe that the above proposition is only a special case of [6, Theorem 3.2].

**Proof of Theorem 2.2 (CLR Bound):** Let \( d \geq 3 \) and take, in the above proposition, \( X \equiv \mathbb{Z}^d, \mu \) as being the counting measure, and \( T \equiv h(\varepsilon) \). Then, clearly,

\[
\text{Tr} \left( \chi_\Omega T^{-1} \mathbf{1}[T \in (0, E)] \chi_\Omega \right) \leq \mu(\Omega) \int_{\varepsilon^{-1}(0, E)} \frac{1}{\varepsilon(p)} \, d\mu^*(p),
\]

where we recall that \( \mu^* \) is the (normalized) Haar measure on the \( d \)-dimensional torus \( \Gamma^* \). If the dispersion \( \varepsilon \) is a Morse function then \( \varepsilon(p) = O(|p - p_0|^2) \) near momenta \( p_0 \in \Gamma^* \) minimizing \( \varepsilon \) and, hence,

\[
\int_{\varepsilon^{-1}(0, E)} \frac{1}{\varepsilon(p)} \, d\mu^*(p) \leq C_A E^{\frac{d-2}{2}}
\]

for some \( C_A \in \mathbb{R}^+ \) and all \( E > 0 \). Observe that this constant can be chosen uniformly w.r.t. \( \|\varepsilon\|_{C^3} \) and \( K(\varepsilon) \). The theorem directly follows from these two estimates combined with Proposition A.1. \( \square \)

**A.2 Proof of Lemma 3.1, Theorem 3.5 and Lemma 3.8**

**Proof of Lemma 3.1.** We assume that \( \mathcal{N}[B], \mathcal{N}[A] < \infty \), otherwise there is nothing to prove. As \( \mathcal{N}[A + B] \leq \mathcal{N}[A - B_-] \) and \( \mathcal{N}[B] = \mathcal{N}[-B_-] \), it suffices to show that

\[
\mathcal{N}[A - B_-] \leq \mathcal{N}[A] + \mathcal{N}[-B_-].
\]

Here, \( B_- \equiv |B| \mathbf{1}[B < 0] \). Let \( M \equiv \mathcal{N}[B] = \text{dim} \ \text{Ran}(B_-) \) and assume that \( A - B_- \) has at least \( \mathcal{N}[A] + M + 1 \) eigenvalues (counting multiplicities) below 0. Then, by the min-max principle, there is a subspace \( X \subset \mathcal{H}, \text{dim} X = \mathcal{N}[A] + M + 1 \), for which

\[
\sup_{\psi \in X, \|\psi\|_2 = 1} \langle \psi \mid (A - B_-)(\psi) \rangle < 0.
\]

Hence

\[
\sup_{\psi \in X \cap \ker(B_-), \|\psi\|_2 = 1} \langle \psi \mid (A - B_-)(\psi) \rangle = \sup_{\psi \in X \cap \ker(B_-), \|\psi\|_2 = 1} \langle \psi \mid A(\psi) \rangle < 0.
\]

\[
\text{dim} \ X \cap \ker(B_-) \geq \text{dim} \ X - \mathcal{N} = \mathcal{N}[A] + 1. \text{ Again by the min-max principle, this would then imply that } \mathcal{N}[A] \geq \mathcal{N}[A] + 1. \] \( \square \)

For any \( \chi \in C^\infty(\mathbb{R}^d, \mathbb{R}) \), define its Gevrey norms by:

\[
\|\chi\|_{s,R} \equiv \sum_{\beta \in \mathbb{N}_0^d} \frac{R(|\beta|)}{|\beta|!}^s \sup_{p \in \mathbb{R}^d} |\partial^\beta \chi(p)|, \quad s \geq 1, \ R > 0.
\]

(152)

The function \( \chi \) is called \( s \)-Gevrey if for some \( R > 0 \), \( \|\chi\|_{s,R} < \infty \).
**Lemma A.2.** Let $\chi \in C^\infty_0(\mathbb{R}^d, \mathbb{R})$. Then, for all $p \in \mathbb{R}^d$,

$$|\hat{\chi}(p)| \leq \|\chi\|_{R,s} |\text{supp} \chi| \exp\left(1 - (e^{-1} |p|)^\gamma\right).$$

Here, $|p| = \max\{|p_1|, |p_2|, \ldots, |p_d|\}$, $\hat{\chi}(p) = \int_{\mathbb{R}^d} e^{-ipx} \chi(x) \frac{d^d x}{(2\pi)^d}$ is the Fourier transform of $\chi$ on $\mathbb{R}^d$, and $|\text{supp} \chi|$ is the volume of the support of the function $\chi$.

**Proof:** The bound above is obvious if $e^{-1} |p| \leq 1$. Therefore, we only consider the case $e^{-1} |p| > 1$. By assumption, for all $n \in \mathbb{N}$:

$$|\hat{\chi}(p)| \leq \frac{(n!)^s}{(R \max\{|p_1|, |p_2|, \ldots, |p_d|\})^n} \|\chi\|_{R,s} |\text{supp} \chi| \leq \frac{n^{sn}}{R^n |p|^n} \|\chi\|_{R,s} |\text{supp} \chi|.$$

Now use that for all $r$ with $e^{-1} r > 1$

$$\min_{n \in \mathbb{N}} \left\{ \frac{n^{sn}}{r^n} \right\} \leq \max_{\xi \in [-1, 0] + (e^{-1} r)^\mathbb{Z}} \left\{ e^{\xi \log(\xi) - \log(r)}\right\}.$$

\[\square\]

**Lemma A.3** (Poisson summation formula). Let $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$ be smooth and assume that $\text{supp} \chi$ is compact. Define $\check{\chi} : \Gamma_+^d \rightarrow \mathbb{C}$ by

$$\check{\chi}([p]) = \sum_{x \in \mathbb{Z}^d} \chi(x) e^{ipx}.$$

Then, for all $p \in [-\pi, \pi]^d$,

$$\check{\chi}([p]) = (2\pi)^{d/2} \sum_{q \in (2\pi \mathbb{Z})^d} \hat{\chi}(p + q).$$

**Corollary A.4.** For all $p \in [-\pi, \pi]^d$, all $R > 1$, and all $s \geq 1$,

$$|\check{\chi}([p]) - (2\pi)^{d/2} \hat{\chi}(p)| \leq (2\pi)^{d/2} \|\chi\|_{R,s} |\text{supp} \chi| \exp \left[1 - R^{1/\gamma}\right] \sum_{p' \in \mathbb{Z}^d} \exp \left[-|p'|^{1/\gamma}\right]$$

$$\leq \text{const} \|\chi\|_{R,s} |\text{supp} \chi| \exp \left[-R^{1/\gamma}\right],$$

where const $< \infty$ is a constant depending only on $s$ and $d$.

**Proof of Theorem A.5.** For simplicity, we temporarily assume that the hopping matrix $h(\epsilon)$ has finite range. Let $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ be any Gevrey function with: $0 \leq \chi(x) \leq 1$ for all $x \in \mathbb{R}$; $\chi(x) = 1$ for all $x, |x| \leq 1$; and $\chi(x) = 0$ for all $x, |x| \geq 2$. Such a $s$-Gevrey function exists for any $s > 1$. For each $L, \Delta L > 0$ define the Gevrey function $\tilde{\Phi}_{L, \Delta L} : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$\tilde{\Phi}_{L, \Delta L}(x) = \chi((x_1 + L)/\Delta L) \chi(x_2/\Delta L) \cdots \chi(x_d/\Delta L).$$
If $\chi$ is a $s$-Gevrey function, by definition of the Gevrey norms, for some constant $< \infty$, some $\Delta L_0 > 0$, and all $L, \Delta L > 0$:

$$\|\Phi_L, \Delta L\|_{s, \Delta L/\Delta L_0} \leq \text{const.} \quad (155)$$

Let $p^{(0)} \in \text{Min}(\epsilon)$, i.e. $\epsilon(p^{(0)}) = 0$. Define for each $L, \Delta L > 0$, the vector $\Phi_L, \Delta L \in \ell^2(\Gamma)$,

$$\Phi_L, \Delta L(x) = e^{ip^{(0)}x} \Phi_L, \Delta L(x), \quad x \in \Gamma. \quad (156)$$

By (155), Lemma 6.3 and Corollary 6.4, for some constant $\text{const} > 0$ and all $L, \Delta L \geq 1$:

$$|\langle \Phi_L, \Delta L | h(\epsilon) \Phi_L, \Delta L \rangle| \leq \text{const} \ (\Delta L)^{-2} \ |\Phi_L, \Delta L|^2. \quad (157)$$

Observe that, by the assumption (43), for some constant $\text{const} > 0$ and all $L, \Delta L \geq 1$:

$$\langle \Phi_L, \Delta L | V \Phi_L, \Delta L \rangle \geq \text{const} \ (L + \Delta L)^{-\alpha} |\Phi_L, \Delta L|^2. \quad (158)$$

Let $R < \infty$ be the range of the hopping matrix $h(\epsilon)$. Notice that, for all $L, \Delta L > 0$ and all $L', \Delta L' > 0$ with $L + 2\Delta L + R < L' - 2\Delta L' - R$,

$$\langle \Phi_L, \Delta L | H(\epsilon, V) \Phi_L', \Delta L' \rangle = 0. \quad (159)$$

For any fixed $N \in \mathbb{N}$ and $L > 0$, define $L_k, \Delta L_k$, $k = 1, 2, \ldots, N$, by:

$$L_k = kL, \quad \Delta L_k = L/8. \quad (160)$$

Then, for $L$ sufficiently large, (159) is satisfied for all $(L, \Delta L) = (L_k, \Delta_k), (L', \Delta L') = (L_l, \Delta_l), k \neq l$. Furthermore, by (157) and (158), as $\alpha < 2$, for $L$ large enough:

$$\langle \Phi_L, \Delta L | H(\epsilon, V) \Phi_L, \Delta L \rangle < 0, \quad k = 1, 2, \ldots, N. \quad (161)$$

It follows by the min-max principle that for all $N \in \mathbb{N}, N[\epsilon, V] \geq N$.

Now assume that $h(\epsilon)$ is not necessarily finite range, but still satisfies the bound in (43). Then, for some constant $< \infty$ not depending on $L$ and all $k, l = 1, 2, \ldots, N, k \neq l$,

$$|\langle \Phi_L, \Delta L | H(\epsilon, V) \Phi_L, \Delta L \rangle| < \text{const} \ (\Delta L)^{-\alpha} |\Phi_L, \Delta L|^2 \ |\Phi_L, \Delta L|^2 \quad (162)$$

It follows from this bound, (157), and (158) that

$$\max_{\varphi \in \text{span} \{\Phi_{L_1, \Delta L_1}, \ldots, \Phi_{L_N, \Delta L_N} \}, \ |\varphi| = 1} \langle \varphi | H(\epsilon, V) \varphi \rangle \leq \text{const}' (\Delta L)^{-\alpha} - \text{const} L^{-\alpha}$$

for some constant $> 0$, constant $< \infty$ depending on $N$ but not on $L$. As, by assumption, $\alpha < \alpha'$, the right-hand side of the equation above is strictly negative for $L$ sufficiently large. Thus, by the min-max principle, for all $N \in \mathbb{N}, N[\epsilon, V] \geq N$. \hfill \Box

**Proof of Lemma 3.3:** Let $\chi : \mathbb{R} \to \mathbb{R}^+_0$ be a smooth function with $\chi(x) = 1$ if $|x - 1/2| \leq 1/2$, and $\chi(x) = 0$ if $|x - 1/2| \geq 3/4$. We will assume that $\chi$ is
a s–Gevrey function for some \( s > 1 \). For all \( M, m \in \mathbb{N}_0 \), all \( X \in \mathbb{Z}^d \), and all \( k \in \{0, 1, \ldots, 2^m - 1\}^d \) define the function \( \Phi(M, m | X, k) : \mathbb{R}^d \to \mathbb{R}_0^+ \) by

\[
\Phi(M, m | X, k)(y) = \prod_{i=1}^d \chi \left( 2^{M+m}(y_i - 2^{-M}X_i - 2^{-M-m}k_i) \right).
\]

Clearly, if \((X, k) \neq (X', k')\),

\[
dist(\text{supp} \, \Phi(M, m | X, k), \text{supp} \, \Phi(M, m | X', k')) \geq 2^{- (M + m + 2)}.
\]

Let \( p(0) \in \text{Min}(\epsilon) \) and let \( c_0 < \infty \) be some constant such that for some \( \epsilon > 0 \) and all \( p \in B(p_0, \epsilon), \epsilon(p) \leq c_0 |p - p(0)|^2 \). Let further \( c_1 \) be a constant with

\[
\int_{\mathbb{R}^d} |p|^2 |\hat{\Phi}(p)|^2 \, dp \leq c_1 \int_{\mathbb{R}^d} |\hat{\Phi}(p)|^2 \, dp,
\]

where \( \hat{\Phi} \) is the Fourier transform of \( \Phi(0, 0 | 0, 0) \).

Let \( X = \{X_1, \ldots, X_N\} \) be the set of points from \( \mathbb{Z}^d \) on which

\[
2c_0c_1[2^{M+m_n}]^2 < v_n^M(2^{-M}X_n) \text{ for some } m_n \geq 0.
\]

For all \( n \in \{1, \ldots, N\} \) let \( m_n \in \mathbb{N}_0 \) be the largest integer satisfying (166).

For all \( L > 0 \) define the functions \( \Phi^{(L)}_{n, k} \in \ell^2(\Gamma), n = \{1, 2, \ldots, N\}, k \in \{0, 1, \ldots, 2^{m_n} - 1\}^d \) by

\[
\Phi^{(L)}_{n, k}(x) = e^{ip_{0} \cdot x} \Phi(M, m_n | X_n, k)(L^{-1}x).
\]

Using Lemma A.3 we see that, by construction, for all \( n = \{1, 2, \ldots, N\} \) and all \( k \in \{0, 1, \ldots, 2^{m_n} - 1\}^d \),

\[
\langle \Phi^{(L)}_{n, k} | H(e, V_L)\Phi^{(L)}_{n, k} \rangle \leq \left[ -\frac{1}{2} L^{-2} v_n^M(2^{-M}X_n) + O(L^{-3}) \right] |\Phi^{(L)}_{n, k}|^2.
\]

Furthermore, for all \( (n, k), (n', k') \), \( n, n' \in \{1, 2, \ldots, N\}, k \in \{0, 1, \ldots, 2^{m_n} - 1\}^d \), \( k' \in \{0, 1, \ldots, 2^{m_n} - 1\}^d \) with \((n, k) \neq (n', k')\), we have, for some \( \text{const} < \infty \) not depending on \( L \), the following estimate:

\[
|\langle \Phi^{(L)}_{n, k} | H(e, V_L)\Phi^{(L)}_{n', k'} \rangle | \leq \text{const} |\Phi^{(L)}_{n, k}|^2 |\Phi^{(L)}_{n', k'}|^2.
\]

Finally, (61) follows by using the min-max principle and observing that, by the choice of the numbers \( m_n \), for some \( \text{const} > 0 \),

\[
2^{dM}2^{dm_n} \geq \text{const}' [v_n^M(2^{-M}X_n)]^{d/2}.
\]

\[\square\]

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