Exact propagators for complex SUSY partners of real potentials

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Abstract

A method for calculating exact propagators for those complex potentials with a real spectrum which are SUSY partners of real potentials is presented. It is illustrated by examples of propagators for some complex SUSY partners of the harmonic oscillator and zero potentials.

Recently a considerable attention has been paid to investigating different properties of non-Hermitian Hamiltonians (see e.g. [1]). One of the reasons of that is an attempt to generalize the quantum mechanics by accepting non-Hermitian Hamiltonians with purely real spectrum (see e.g. [1, 2] and references therein). The propagator being the coordinate representation of the evolution operator is one of the important objects in quantum mechanics since it permits to describe the evolution of a quantum system for an arbitrary initial state. On the other hand the method of supersymmetric quantum mechanics (SUSY QM) is one of the main methods for getting complex exactly solvable Hamiltonians [3, 4]. Moreover, because of its nice property to convert a non-diagonalizable Hamiltonian into diagonalizable forms and delete spectral singularities from the continuous spectrum of a non-Hermitian Hamiltonian [5] it was presumed [6] that SUSY QM may become an essential ingredient of the infant complex quantum mechanics. The aim of this Letter is to show that SUSY QM may be useful for finding propagators for those complex potentials which are SUSY partners of real potentials for which both the propagator and the Green’s function are known thus giving an additional argument in favor of the above thesis. For simplicity we will consider only time-independent potentials although corresponding time-dependent technique is also available [7].

We consider the one-dimensional Schrödinger equation with a complex-valued potential $V_c(x)$

$$
\left[ i \frac{\partial}{\partial t} - h_c \right] \Phi(x, t) = 0, \tag{1}
$$

$$
h_c = -\frac{\partial^2}{\partial x^2} + V_c(x), \quad x \in \mathbb{R}. \tag{2}
$$

Since the potential $V_c(x)$ is time independent (stationary) solutions to equation (1) are expressed in terms of solutions of the stationary equation

$$
h_c \varphi_E(x) = E \varphi_E(x) \tag{3}
$$
in the usual way, $\Phi(x, t) = e^{-iEt} \varphi_E(x)$. But if initially the quantum system is prepared in the state $\varphi_0(x)$ which is not a state with the definite value of the energy we need to know the propagator
for describing the evolution of a quantum system. We use the usual definition of the propagator $K_c(x, y, t)$ as a solution to equation (1) with respect to variables $x$ and $t$ with the delta-like initial condition

$$K_c(x, y; 0) = \delta(x - y).$$

(4)

If $K_c(x, y, t)$ is given the function

$$\Phi(x, t) = \int_{-\infty}^{\infty} K_c(x, y; t)\varphi_0(y)dy$$

(5)

is a solution to equation (1) with the initial condition $\Phi(x, 0) = \varphi_0(x)$.

First of all we note that to be able to associate the function $\Phi(x, t)$ with a state of a quantum system the integral in (5) should converge and both the function $\varphi_0(x)$ and $\Phi(x, t)$ should belong to a certain class of functions. To present our method in its simplest form we will make several assumptions which shall simplify essentially our presentation keeping at the same time the essence of the method. Since in (5) the usual (Lebesgue) integration is involved it is natural to suppose that both $\varphi_0(x)$ and $\Phi(x, t)$ are square integrable. This means that the Hilbert space, where the operator (non-Hermitian Hamiltonian) $h_c$ associated with the differential expression $-\partial^2x/\partial x^2 + V_c(x)$ 'lives', is the usual space $L_2(\mathbb{R})$ and the equation (3) creates an eigenvalue problem for $h_c$ which is defined on a dense domain from $L_2(\mathbb{R})$. Similar eigenvalue problems were under an intensive study by mathematicians in the Soviet Union in the period between 50th and 70th of the previous century. Results of these investigations are mainly summarized in books [8, 9] to which we refer the interested reader where he can, in particular, find the strict definition of the spectrum, eigenfunctions, associated functions, domains of definition of operators created by non-Hermitian differential expressions and related non-selfadjoint operators. Here we would like to mention that the first essential result in this field was obtained by Keldysh [10] who proved the completeness of the set of eigenfunctions and associated functions for a non-selfadjoint operator and results by Lidskiy [11] having the direct relation to the current paper. In particular, Lidskiy made a deep analysis of conditions on the potential $V_c$ leading to an operator $h_c$ which is uniquely defined by its closure and has a purely discrete spectrum with a complete set of eigenfunctions and associated functions.

Our next essential assumptions is that $h_c$ has a purely discrete spectrum, its set of associated functions is empty, it is diagonalizable, and its set of eigenfunctions $\phi_n(x)$, $n = 0, 1, \ldots$ is complete in the space $L_2(\mathbb{R})$. If $h_c^+$ is the adjoint differential expression it creates in the Hilbert space the adjoint operator with the eigenfunctions $\tilde{\phi}_k(x)$ which also form a complete set in $L_2(\mathbb{R})$. Moreover, if $E_n$ is an eigenvalue of $h_c$ then $E_n^*$ (asterisk means the complex conjugation) is an eigenvalue of $h_c^+$ so that $h_c^+\tilde{\phi}_n = E_n^*\tilde{\phi}_n$. Note that neither $\{\phi_n\}$ nor $\{\tilde{\phi}_n\}$, $n = 0, 1, \ldots$ form orthogonal systems but functions $\tilde{\phi}_k$ are biorthogonal with $\phi_n$ and they can always be normalized such that (see e.g. [9])

$$\int_{-\infty}^{\infty} \tilde{\phi}_k^*(x)\phi_n(x)dx = \delta_{nk}.$$

(6)

The completeness of the set of eigenfunctions of $h_c$ means that any $\phi \in L_2(\mathbb{R})$ can be developed into the Fourier series over the set $\{\phi_n\}$, $\phi(x) = \sum_{n=0}^{\infty} c_n\phi_n(x)$. Using the biorthonormality relation (6) we can find the coefficients $c_n$ in the usual way and put them back into the same relation thus
obtaining the symbolical form of the completeness condition of the set of eigenfunctions of $h_c$

$$
\sum_{n=0}^{\infty} \tilde{\phi}_n^*(x)\phi_n(y) = \delta(x - y) .
$$

(7)

Next we are assuming that the spectrum $h_c$ is real. Therefore the adjoint eigenvalue problem coincides with the complex conjugate form of equation (3) so that $\tilde{\phi}_n(x) = \phi_n^*(x)$. Under these assumptions equations (6) and (7) become (cf. with [12])

$$
\int_{-\infty}^{\infty} \phi_n(x)\phi_k(x)dx = \delta_{nk} ,
$$

(8)

$$
\sum_{n=0}^{\infty} \phi_n(x)\phi_n(y) = \delta(x - y) .
$$

(9)

From here it follows the Fourier series expansion of the propagator in terms of the basis functions $\phi_n$:

$$
K_c(x, y; t) = \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y)e^{-iE_nt} .
$$

(10)

Indeed, just like in the conventional Hermitian case the initial condition $\Phi(x, 0) = \varphi_0(x)$ for function (5) with $K_c$ of form (10) follows from (9) and the fact that $\Phi(x, t)$ (5) satisfies equation (1) follows from the property of the functions $\phi_n$ to be eigenfunctions of $h_c$ with the eigenvalues $E_n$.

Especial role between all non-selfadjoint operators is played by pseudo-Hermitian operators first introduced by Dirac and Pauli and latter used by Lee and Wick [13] to overcome some difficulties related with using Hilbert spaces with an indefinite metric and their recent generalization (weak pseudo-Hermiticity) by Solombrino and Scolarici [14] since there are strict indications that these operators are the most appropriate candidates for replacing selfadjoint operators while generalizing the conventional quantum mechanics by accepting non-Hermitian operators [14, 15].

Another useful discussion is that the form (10) for the propagator may be interpreted as the coordinate representation of the abstract evolution operator.\textsuperscript{1} To show this we introduce ket-vectors (kets) $|\phi_n\rangle$ as eigenvectors of $h_c$ and bra-vectors (bras) $\langle\tilde{\phi}_n|$ as functionals acting in the space of kets according to\textsuperscript{2}

$$
\langle\tilde{\phi}_n|\phi_k\rangle = \delta_{nk} .
$$

(11)

Kets corresponding to the previous bras are just properly normalized eigenvectors of $h_c^+$ which is defined by the adjoint eigenvalue problem where $V_c(x)$ is replaced by its complex conjugate $V_c^*(x)$ so that (11) is nothing but the same biorthogonality condition (8) written in the abstract representation. As usual the coordinate representation of the above abstract eigenvectors are

\textsuperscript{1}This property has been demonstrated in the report of the anonymous referee.

\textsuperscript{2}Without going into details we note that using the system $|\phi_n\rangle$ one can construct the Hilbert space $H$ so that the set of all finite linear combinations of $\phi_n$ is dense in $H$ and formula (11) uniquely defines a functional in $H$, see e.g. [16].
\( \phi_n(x) = \langle x | \phi_n \rangle = \langle \tilde{\phi}_n | x \rangle \) where \( |x\rangle \) is an eigenvector of the coordinate operator. The completeness condition in the abstract form now reads
\[
\sum_{n=0}^{\infty} |\phi_n\rangle \langle \tilde{\phi}_n| = 1 
\]
(12)
and the formula
\[
h_c = \sum_{n=0}^{\infty} |\phi_n\rangle E_n \langle \tilde{\phi}_n| 
\]
(13)
presents the spectral decomposition of the Hamiltonian \( h_c \). Now in the known way [14, 15] one can introduce an automorphism \( \eta \) to establish the property that \( h_c \) is (weakly) pseudo-Hermitian and construct a basis in which \( h_c \) takes a real form. We will not go into further details of the known properties of (weakly) pseudo-Hermitian operators since this is not the aim of this paper. The interested reader can consult papers [13, 14, 15] and the recent preprint [17] where biorthogonal systems are widely used in the study of different properties of non-Hermitian Hamiltonians. Our last comment here is that the abstract evolution operator given by its spectral decomposition
\[
U(t) = \sum_{n=0}^{\infty} |\phi_n\rangle e^{-iE_nt} \langle \tilde{\phi}_n| 
\]
(14)
written in the coordinate representation \( K_c(x,y,t) = \langle x | U(t) | y \rangle \) is just the propagator (10).

We would like to emphasis that conditions (8) and (9) have almost the usual form, only the complex conjugation is absent. Therefore they coincide with the corresponding equations for the Hermitian Hamiltonians in case when their eigenfunctions are real.

The final assumption we make is that the Hamiltonian \( h_c \) is a SUSY partner of a Hermitian Hamiltonian \( h_0 \) with a purely discrete spectrum and a complete set of eigenfunctions \( \psi_n \) which always can be chosen real
\[
h_0 = -\frac{\partial^2}{\partial x^2} + V_0(x), \quad h_0 \psi_n(x) = E_n \psi_n(x), \quad \psi_n(x) = \psi_n^*(x), \quad n = 0, 1, \ldots
\]
so that both the completeness and normalization conditions are given by equations (9) and (8) respectively with the replacement \( \phi_n \rightarrow \psi_n \).

According to the general scheme of SUSY QM (see e.g. [19]) operators \( h_0 \) and \( h_c \) are (1-)SUSY partners if and only if there exists a first order differential operator \( L \) such that
\[
L h_0 = h_c L . 
\]
(15)
Operator \( L \) of the form
\[
L = -u'(x)/u(x) + \partial/\partial x , 
\]
(16)
where the prime means the derivative with respect to \( x \) and the function \( u(x) \) is a solution to equation
\[
h_0 u(x) = \alpha u(x) , 
\]
(17)
exists for any \( h_0 \) and a rather restricted set of operators \( h_c \), the potentials \( V_c \) of which are defined by
\[
V_c = V_0 - 2[\log u(x)]'' . 
\]
The spectrum of \( h_c \) may either (i) coincide with the spectrum of \( h_0 \) or (ii) may differ from it by one (real) level which is absent in the spectrum of \( h_0 \). The case (i) may be realized only with a complex parameter \( \alpha \) which is called the factorization constant. This statement is a consequence of a proposition proved in [4]. The case (ii) may be realized only for a real factorization constant since \( E = \alpha \) is just the discrete level of \( h_c \) missing in the spectrum of \( h_0 \) and we want that \( h_c \) has a real spectrum. Therefore in this case one has to choose \( u(x) \) as a linear combination of two real linearly independent solutions to equation (17).

Together with operator \( L \) (16) we need also its 'transposed form' which we define as follows:

\[
L^t = -u'(x)/u(x) - \partial/\partial x.
\]  
(18)

Just like in the usual SUSY QM the following factorizations take place:

\[
L^tL = h_0 - \alpha, \quad LL^t = h_c - \alpha
\]  
(19)

which can easily be checked by the direct calculation.

One of the main features of the method is that for the most physically interesting Hamiltonians \( h_0 \) operator (16) has the property \( L\psi_E(\pm \infty) = 0 \) provided \( \psi_E(\pm \infty) \). As a result the set of functions

\[
\phi_n = N_n L\psi_n, \quad n = 0, 1, \ldots
\]  
(20)

is complete in the space \( L^2(\mathbb{R}) \) in case (i). In case (ii) we have to add to this set the function \( \phi_\alpha = N_\alpha / u \). The normalization coefficients \( N_n \) may be found by integration by parts in equation (8) and with the help of factorization property (19) which yields

\[
N_n = (E_n - \alpha)^{-1/2}.
\]  
(21)

The main result of the present Letter is given by the following

\textbf{Theorem 1.} The propagator \( K_c(x,y;t) \) of the Schrödinger equation with the Hamiltonian \( h_c \) related with \( h_0 \) by a SUSY transformation is expressed in terms of the propagator \( K_0(x,y;t) \) of the same equation with the Hamiltonian \( h_0 \) and the Green’s function \( G_0(x,y;E) \) of the stationary equation with the same Hamiltonian as follows:

in case (i) \( K_c(x,y,t) = K_L(x,y,t) \)

in case (ii) \( K_c(x,y,t) = K_L(x,y,t) + \phi_\alpha(x)\phi_\alpha(y)e^{-i\alpha t} \)

where \( K_L(x,y,t) \) is the 'transformed' propagator

\[
K_L(x,y,t) = L_x L_y \int_{-\infty}^{\infty} K_0(x,z,t) G_0(z,y,\alpha) dz.
\]  
(22)

Here \( L_x \) is defined by (16) and \( L_y \) is the same operator where \( x \) is replaced by \( y \).

\textbf{Remark 1.} We use the definition of the Green’s function, which we denote \( G_0(x,y,E) \) where \( E \) belongs to the resolvent set of \( h_0 \), as a coordinate representation of the resolvent \( (h_0 - E)^{-1} \) (see e.g. [20]). The Green’s function is a kernel of an integral operator acting in the space \( L^2(\mathbb{R}) \).

There exists two equivalent representations for the Green’s function:

(a) In terms of the basis functions \( \psi_n(x) \) with eigenvalues \( E_n, n = 0, 1, \ldots \)

\[
G_0(x,y,E) = \sum_{m=0}^{\infty} \frac{\psi_m(x)\psi_m(y)}{E_m - E}
\]  
(23)
provided $\psi_n(x)$ are real and
(b) in terms of two special linearly independent solutions to the Schrödinger equation with the
given value of $E$ (see an example below).

Proof. We prove the theorem only for case (ii) since case (i) does not have essentially new
moments.

Using equations (10), (20) and (21) we get

\[ K_c(x, y, t) = L_x L_y \sum_{m=0}^{\infty} \frac{\psi_m(x)\psi_m(y)}{E_m - \alpha} e^{-iE_m t} + \phi_\alpha(x)\phi_\alpha(y)e^{-iat}. \]  

Because of the equality

\[ \psi_m(x)e^{-iE_m t} = \int_{-\infty}^{\infty} K_0(x, z, t)\psi_m(z)dz \]

equation (24) assumes the form

\[ K_c(x, y, t) = L_x L_y \int_{-\infty}^{\infty} K_0(x, z, t) \sum_{m=0}^{\infty} \frac{\psi_m(z)\psi_m(y)}{E_m - \alpha} dz + \phi_\alpha(x)\phi_\alpha(y)e^{-iat}. \]  

(25)

Here under the summation sign we recognize the Green’s function (23) which ends the proof. □

We have to note that both in case (i) and in case (ii) $\alpha$ belongs to the resolvent set of $h_0$. Hence,
the Green’s function $G_0(x, y, \alpha)$ entering in (22) and (25) is regular. Another useful comment is
that almost the same proof is valid for the case when $h_0$ (and hence $h_c$) has a continuous spectrum
so that the theorem is valid for this case also provided $h_c$ has no spectral singularities in the
continuous part of the spectrum.

To show how the theorem works we will consider a couple of typical examples.

Consider first the harmonic oscillator Hamiltonian $h_0(x) = -\partial^2/\partial x^2 + x^2/4$ for which the
propagator is well-known [21]

\[ K_0(x, y, t) = \frac{1}{\sqrt{4\pi i \sin t}} e^{i\left[\frac{x^2+y^2}{\sin t} - 2xy\right]}. \]

To apply our theorem we need also the Green’s function of the oscillator Hamiltonian at $E = \alpha$.
To find it we use the definition of the Green’s function in terms of two special solutions $f_i(x, E)$
and $f_r(x, E)$ of the equation

\[ (h_0 - E)f_{i,r}(x, E) = 0, \quad f_i(-\infty, E) = 0, \quad f_r(\infty, E) = 0 \]

which is

\[ G(x, y, E) = [f_i(x, E)f_r(y, E)\Theta(y-x) + f_i(y, E)f_r(x, E)\Theta(x-y)]/W(f_i, f_r) \]  

(26)

where $\Theta$ is the Heaviside step function and $W$ stands for the Wronskian.

In the simplest case we can choose $\alpha = -1/2$ and

\[ u(x) = e^{x^2/4}(C + \text{erf}(x/\sqrt{2})), \quad \text{Im} C \neq 0 \]  

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Functions $f_l$ and $f_r$ from (26) at $E = \alpha = -1/2$ read

$$f_l(x, -1/2) = \sqrt{\pi/2} e^{x^2/4} (1 + \text{erf} (x/\sqrt{2})), \quad f_r(x, -1/2) = \sqrt{\pi/2} e^{x^2/4} (1 - \text{erf} (x/\sqrt{2})).$$ \hspace{1cm} (27)

The spectrum of the complex-valued transformed potential

$$V_c(x) = \frac{x^2}{4} - 1 + 2xQ_1(x) e^{-x^2/2} + 2Q_1^2 e^{-x^2}, \quad Q_1(x) = \sqrt{\frac{\pi}{2} [C + \text{erf} (x/\sqrt{2})]}$$ \hspace{1cm} (28)

consists of all oscillator energies $E_n = n + 1/2$, $n = 0, 1, \ldots$ and one additional level $E_{-1/2} = -1/2$ with the eigenfunction

$$\phi_{-1/2}(x) = (2\pi)^{-1/4} \sqrt{C^2 - 1} u^{-1}(x).$$ \hspace{1cm} (29)

It is not difficult to check by the direct calculation that

$$\int_{-\infty}^{\infty} \phi_{-1/2}^2(x) dx = 1.$$ \hspace{1cm} (30)

Using Theorem 1 and equations (26) and (27) we obtain the propagator for the Hamiltonian with potential (28)

$$K_c(x, y, t) = -\sqrt{\frac{\pi(C + 1)}{2u(y)}} L_x \int_{-\infty}^{y} K_0(x, z, t) e^{z^2/4} (1 + \text{erf} (z/\sqrt{2})) dz$$
$$+ \frac{\sqrt{\pi(C - 1)}}{2u(y)} L_x \int_{y}^{\infty} K_0(x, z, t) e^{z^2/4} (1 - \text{erf} (z/\sqrt{2})) dz + \phi_{-1/2}(x) \phi_{-1/2}(y) e^{it/2}.$$ \hspace{1cm} (31)

As the second example we consider a complex extension of the one-soliton potential

$$V_c(x) = \frac{-2a^2}{\cosh^2(ax + c)}.$$ \hspace{1cm} (32)

which is obtained from $V_0(x) = 0$ with the choice

$$u(x) = \cosh(ax + c), \quad \alpha = -a^2, \quad \text{Im} c \neq 0, \quad \text{Im} a = 0$$
as the transformation function. The Hamiltonian $h_c$ with potential (32) has a single discrete level $E_0 = -a^2$ with the eigenfunction

$$\phi_{-a^2}(x) = \sqrt{\frac{a}{2}} \frac{1}{u(x)}.$$ \hspace{1cm} (33)

Just like in the previous example this function is such that

$$\int_{-\infty}^{\infty} \phi_{-a^2}^2(x) dx = 1.$$ \hspace{1cm} (34)

To find the propagator $K_c$ we are using the known propagator and Green’s function for the free particle which are

$$K_0(x, y; t) = \frac{1}{\sqrt{4\pi it}} e^{i(x - y)^2/4t}, \quad G_0(x, y, E) = \frac{i}{2\kappa} e^{i\kappa|x - y|}, \quad \text{Im} \kappa > 0, \quad E = \kappa^2.$$
After some transformations similar to that described in details in [22] we get the propagator for the Hamiltonian \( h_c \)

\[
K_c(x, y; t) = \frac{1}{\sqrt{4\pi it}} e^{\frac{(x-y)^2}{4it}} + \frac{ae^{it}}{4u(x)u(y)} \left[ \text{erf}_+ + \text{erf}_- \right]
\]

(34)

where the notation \( \text{erf}_\pm = \text{erf} \left[ a\sqrt{it} \pm \frac{1}{2\sqrt{it}}(x - y) \right] \) is used and for \( c \) we take the value \( c = \text{arctanh} \frac{b^2-a^2}{2iab} \), \( \text{Im}(b) = 0 \).

We have to note that formula (34) is in the prefect agreement with the result obtained by Jauslin [23] where the replacement \( t \rightarrow it \) should be made since this author got the propagator for the heat equation with the one-soliton potential. Another comment we would like to make is that the Jauslin’s method is based on an integral formula which relates solutions of two Schrödinger equations whose Hamiltonians are SUSY partners. Unfortunately, integrals which need to be calculated when the method is applied to the Schrödinger equation become divergent. Therefore the author found the propagator for the heat equation with the one-soliton potential. The Schrödinger equation may be considered as the heat equation with the imaginary time. In this respect the following question arises: whether the Jauslin’s result after the replacement \( t \rightarrow it \) gives the propagator for the Schrödinger equation with the one-soliton potential? We want to stress that the answer to this question is not trivial since such a replacement at the level of the Jauslin’s integral transformation leads to divergent integrals and only the replacement in the final result gives the finite value for the propagator. Thus, our analysis shows that the answer to this question is positive.

In conclusion we note that in this letter we presented a method for finding propagators for those non-Hermitian Hamiltonians with a purely real spectrum, SUSY partners of which have the known both the propagator and Green’s function. The general theorem is illustrated by considering a complex anharmonic oscillator Hamiltonian and a complex extension of the one-soliton potential.

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