The Kadomstev-Petviashvili (KP) equation ion acoustic waves in weakly relativistic plasma with nonthermal electron, positron and warm ion

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Abstract: The Kadomstev-Petviashvili equation is derived for propagating of ion acoustic waves in collisionless weakly relativistic plasma containing nonthermal electron, positron and warm ion. The amplitude and width of soliton are influenced with effect relativistic. The effects of relative density, nonthermal parameter and temperature on the waves are studied.

1. Introduction
The ion acoustic solitary waves in different plasma systems have been studied. Electron-positron plasmas are found in pulsar magnetosphere, solar atmosphere, magnetospheres of neutron stars and astronomy plasmas. Positrons are used in tokamaks and are found in most of these plasmas, in which ions can be present too. Thus, three component plasmas (electron-positron-ion) exist in such environments. In recent years, some authors have studied e-p-i plasmas [7-11]. However, most of them have investigated nonrelativistic plasmas. We know that the ion acoustic solitary wave’s propagation is modified when the ion velocity approaches that of light [1-7]. When ion velocity approaches the speed of light, relativistic effect becomes dominant and in these cases amplitude, width and energy of wave are changed. Using the Korteweg-de Vries (KdV) [1-5] and Kadomstev–petviashvili (KP) equations, the behavior of these waves for one-dimensional and two-dimensional propagation has been described [6, 7]. The investigation of relativistic nonlinear waves in physical systems such as laser-plasma interaction, space plasma phenomena and nonlinear interactions of circularly polarized waves has been very interesting. As a result, ion acoustic relativistic plasmas have been studied extensively. For example, Nejoh studied two-dimensional solitary waves and one-dimensional wave propagation in cold and warm relativistic plasmas containing electron and ion [1]. Honzawa and Singh derived KP equation in warm relativistic plasma containing electron and ion in two dimensions in 1993. The effects of density gradient and electron inertia in relativistic plasmas containing electrons and ions have been investigated by Malik. Gill and et.al studied relativistic plasmas containing electron, positron and ion in 2007 [5]. Propagation of two-dimensional ion acoustic waves in relativistic plasmas with positron, nonthermal electron and warm ion has not been studied yet. So this kind of propagation is studied in this paper. We study ion acoustic solitary waves in two-dimensions for weakly relativistic plasmas containing nonthermal electrons, positrons with Boltzmann distribution
and warm ions. We derive KP equation by using the reductive perturbation theory. Then, we study 
effects of relativistic factor, relative density and ion temperature on solitary waves numerically.

2. Basic equations
We consider collisionless and unmagnetized plasmas consisting nonthermal electrons, positrons and 
warm ions. Charge neutrality at equilibrium gives \( n_i = n_{\text{ep}} + n_{\text{e}} \), where \( n_i, n_{\text{ep}} \) and \( n_{\text{e}} \) are the unperturbed ion, electron and positron number densities, respectively. We assume the ion acoustic wave propagates in the x-direction; however, there are higher order transverse perturbations in the y-
direction. The two-dimensional equations of continuity, motion for adiabatic ions and Poisson are 
given by

\[
\begin{align*}
\frac{\partial n}{\partial t} + \frac{\partial (nu)}{\partial x} + \frac{\partial (nv)}{\partial y} &= 0 \\
\frac{\partial (\gamma u)}{\partial t} + u \frac{\partial (\gamma u)}{\partial x} + v \frac{\partial (\gamma u)}{\partial y} + \frac{\sigma}{n} \frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\sigma}{n} \frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial y} &= 0 \\
\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + 3p \left( \frac{\partial (\gamma u)}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0 \\
\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= n_e - n - n_p
\end{align*}
\] (1)

here \( n, n_e \) and \( n_p \) are the ion, electron and positron densities, respectively and are normalized by 
the \( n_i, u \) and \( v \) are the velocity components of ion fluid. \( \sigma = T_i / T_{\text{eff}} \), where \( T_i \) is the temperature of
ions and \( T_{\text{eff}} \) is the effective temperature \( T_{\text{eff}} = \frac{n_i}{\left( \frac{n_{\text{ep}}}{T_i} + \frac{n_{\text{ep}}}{T_p} \right)} \), which \( T_e \) and \( T_p \) are temperature of
electron and positron. \( \gamma \) is relativistic factor and for weakly relativistic plasma is approximated by its 
expansion up to second term

\[
\gamma = (1 - \frac{u^2}{c^2})^{\frac{1}{2}} \approx 1 + \frac{u^2}{2c^2}
\] (2)

where, \( c \) is the velocity of light. All of the velocities are normalized by the \( kT_{\text{eff}} / m \), where \( k \) is Boltzmann’s constant and \( m \) is ion mass. \( \phi \) and \( P \) are the electrical potential and pressure of ion which are normalized by \( kT_{\text{eff}} / e \) and \( n_i kT_i \), respectively, where \( e \) is electron charge. The time \( t \) and 
space \( (x,y) \) are normalized by the ion-plasma period \( \omega_p^{-1} = \left( m / 4\pi n_i e^2 \right)^{1/2} \) and by electron Debye length 
\( \lambda_p = (kT_{\text{eff}} / n_i e^2)^{1/2} \), respectively. The electrons and positrons densities are governed by Boltzmann distribution

\[
\begin{align*}
n_e &= \frac{1}{1-\mu} \left( 1 - \beta \phi + \beta \phi^2 \right) e^\phi \\
n_p &= \frac{\mu}{1-\mu} e^{-\phi}
\end{align*}
\] (3)

where, \( s = \frac{T_e}{T_p}, \mu = \frac{n_{\text{ep}}}{n_{\text{e}}}, \) and \( \beta = \frac{4\alpha}{1+3\alpha} \) in which \( \alpha \) is a parameter determining the number of fast
(nonthermal) electrons.
3. Derivation of KP equation

In order to obtain KP equation we use the reductive perturbation method. The stretched coordinates are defined by

\[ \xi = e(x - \lambda t), \quad \eta = e^2 y, \quad \tau = e^3 t \]  

(4)

We expand the quantities \( n, u, v, P \) and \( \phi \) as follows

\[
\begin{align*}
  n &= 1 + \varepsilon^2 n_1 + \varepsilon^4 n_2 + 
  u &= u_0 + \varepsilon^2 u_1 + \varepsilon^4 u_2 + 
  v &= \varepsilon^3 v_1 + \varepsilon^5 v_2 + 
  p &= 1 + \varepsilon^2 p_1 + \varepsilon^4 p_2 + 
  \phi &= \varepsilon^2 \phi_1 + \varepsilon^4 \phi_2 + 
\end{align*}
\]

(5)

where \( \varepsilon \) is a small dimensionless expansion parameter which characterizes the strength of nonlinearity and \( \lambda \) is the phase velocity of the wave. Substituting (5) into (1) and collecting the terms in different powers of \( \varepsilon \), in the lowest order of \( \varepsilon \), we have

\[
\begin{align*}
  (\lambda - u_0)^2 - 3\sigma \gamma_1 &= \left( \frac{1 - \mu}{1 - \beta + \mu s} \right) 
  \frac{\partial^2 \phi_1}{\partial \xi^2} &= \left( \frac{1 - \mu}{1 - \beta + \mu s} \right) 
\end{align*}
\]

(6)

Where \( \gamma_1 = 1 + \frac{3u_0^2}{2c^2} \). The next order of \( \varepsilon \)

\[
-(\lambda - u_0) \frac{\partial v_1}{\partial \xi} + \frac{\partial}{\partial \eta} (\phi_1 + \sigma p_1) = 0 
\]

(7)

and at the higher order of \( \varepsilon \) we have the following set of equations

\[
\begin{align*}
  &-(\lambda - u_0) \frac{\partial^2 n_1}{\partial \xi^2} + \frac{\partial n_1}{\partial \tau} + \frac{\partial (u_0 u_1)}{\partial \xi} + \frac{\partial u_1}{\partial \eta} + \frac{\partial v_1}{\partial \eta} = 0 
  &-(\lambda - u_0) \frac{\partial u_1}{\partial \xi} - 2\gamma_1 (\lambda - u_0) u_1 - \frac{\partial u_1}{\partial \tau} + \gamma_1 \frac{\partial u_1}{\partial \xi} + \gamma_1 \frac{\partial u_1}{\partial \eta} - \sigma n_1 \frac{\partial P_1}{\partial \xi} + \sigma \frac{\partial P_1}{\partial \eta} = 0 
  &-(\lambda - u_0) \frac{\partial P_1}{\partial \xi} + \frac{\partial P_1}{\partial \tau} + u \frac{\partial P_1}{\partial \xi} + 3\gamma_1 \frac{\partial u_1}{\partial \xi} + 6\gamma_1 u_1 \frac{\partial u_1}{\partial \xi} + 3\gamma_1 \frac{\partial u_1}{\partial \eta} + u_1 \frac{\partial \phi_1}{\partial \xi} + \frac{3}{2} \frac{\partial \phi_1}{\partial \eta} = 0 
  &\frac{\partial^2 \phi_1}{\partial \xi^2} = \left( \frac{1 - \beta + \mu s}{1 - \mu} \right) \phi_1 + \frac{1}{2} \left( \frac{1 - \mu}{1 - \beta + \mu s} \right) n_2 
\end{align*}
\]

(8)

where \( \gamma_2 = 3u_0 \frac{2c^2}{c^2} \). By eliminating the second order quantities \( n_2, u_2, P_2 \) and \( \phi_2 \) from (8) and using (6) and (7) we obtain KP equation

\[
\frac{\partial}{\partial \xi} \left[ \frac{\partial \phi_1}{\partial \tau} + A \frac{\partial \phi_1}{\partial \xi} + B \frac{\partial^2 \phi_1}{\partial \xi^2} + C \frac{\partial^2 \phi_1}{\partial \eta^2} \right] = 0 
\]

(9)

where

\[
\begin{align*}
  A &= \frac{(\lambda - u_0)(1 - \beta + \mu s)}{2[3\sigma \gamma_1(1 - \beta + \mu s) + (1 - \mu)]} \left[ \frac{(1 - \beta + \mu s)}{(1 - \mu)} \right] \left[ \frac{2(1 - \gamma_2)(\lambda - u_0)}{\gamma_1} + \gamma_1(\lambda - u_0)^2 + 9\gamma_1^2 \right] - \frac{(1 - \mu \gamma_2^2)}{(1 - \beta + \mu s)} 
  B &= \frac{1}{2} \left[ \frac{(1 - \mu \gamma_2)(\lambda - u_0)}{(1 - \beta + \mu s) + 3\sigma \gamma_1(1 - \beta + \mu s)^2} \right] 
  C &= \frac{3\sigma(1 - \beta + \mu s) + (1 - \mu)}{3\sigma(1 - \beta + \mu s) + (1 - \mu)} 
\end{align*}
\]

(10)
Exact solutions of KP equation have been obtained by S Zhang [8]. Solitonic solutions for (9) is given by [9]

$$\phi_o = \phi_s \sec h^2 (\chi/W)$$

(11)

where \(\chi = \xi + \eta - U \tau\), that U is velocity of profile of soliton and soliton amplitude and width are

$$\phi_o = 3(U-C)/A \quad \quad W = 2\sqrt{B/(U-C)}$$

(12)

Relativistic effect is represented as

$$\eta = \frac{u_o}{c}.$$

4. Discussion

Some researchers have already studied the two-dimensional propagation of ion acoustic solitary waves in weakly relativistic plasma containing ion and electron [6,7] and the results of their studies are comparable with results of this paper. Also, we can compare the above results with the obtained results in one-dimensional propagation. When \(\beta = \sigma = 0\), that is there are no fast electrons and warm ions, the results obtained here are reduced to [5]. When \(\beta = \mu = 0\), that is there are no fast electrons and positrons, the above mentioned equations agree with those of Y N Nejoh [1]. In these two cases, there are only compressive solitons. In the absence of positrons and for cold plasma \((\sigma = 0)\), that is, when the ion temperature is approximately equal to zero, the extracted soliton solutions are reduced to [4]. For nonrelativistic plasma without positrons and fast electrons, that is, \(\eta = 0\) and \(\mu = \beta = 0\), the equations which have been obtained here are reduced to [2] and finally, for cold and nonrelativistic plasmas with Boltzmann distributed electrons but without positrons, the above results are reduced to [3]. Since \((\lambda - u_o)\) is positive and \(\mu, s\) and \(\sigma\) are less than 1 and also \(\gamma_2/\gamma_1 << 1\), one can show that A is positive, for \(\beta = 0\) [1,5]. In these cases there are only compressive solitary waves. Now we analyze the results obtained with numerical computation.

As figures 1-6 show the amplitude and width of soliton change with changes of \(\mu, \sigma, s, \beta\) (or \(\alpha\)) and \(\eta = \frac{u_o}{c}\). Figures 1, 2 and 3 show the variation of the amplitude with respect to \(\eta\). It is clear that the amplitude decreases as \(\sigma\) and \(\mu\) increases. Also, we see that for different values of \(\alpha\), positive or negative values of the amplitude of soliton appear. Thus, the existence of the nonthermal (fast) electrons suggests that both compressive and rarefactive solitons exist.

**Figure 1.** The variation of amplitude with respect to \(\eta\) for \(\sigma = 1, s = 1, \mu = 0.001, u = 1.5\) and different values of \(\alpha\)

**Figure 2.** The variation of amplitude with respect to \(\eta\) for \(\alpha = 0.2, s = 1, \mu = 0.001, u = 1.5\) and different values of \(\sigma\)
Figures 3-6 show the variation of width of soliton with respect to $\eta$. These figures indicate that increasing nonthermal parameter and temperature of ion leads to increasing values of width of soliton. Also, the width decreases as density of positrons increases. It is clear that width is almost independent of relativistic effect ($\eta$) in figures 5 and 6. In cold plasma containing only electron and ion the amplitude of soliton increases as relativistic effect increases [5,6], but for warm plasma containing electron, positron and ion the amplitude decreases when relativistic effect increases (figure 3).
In figure 4 we see that for $\alpha = 0.586$ width of soliton increases when relativistic effect ($\eta$) increases and for other values soliton width is almost constant when $\eta$ increases. More detailed studies show the peak amplitude of solitons decreases with increasing in relative temperature ($s$) and these changes are more obvious for higher values of $s$.

5. Conclusion and remarks

The KP equation was obtained for propagating of ion acoustic waves in weakly relativistic plasma containing nonthermal electron, positron and warm ion. In this paper we showed that when the number of fast electron changes, both rarefactive and compressive solitary waves can be propagated. It was clear that amplitude (width) of soliton decreases (increases) when velocity of ion approaches that of light. We saw that the amplitude and width of solitons in three components (ion-positron-electron) are less than that of two components (electron-ion). Also, we discussed the effect of ion temperature and saw that the amplitude (width) of soliton in cold plasma is more (less) than that of warm plasma. The following situations are very interesting to be investigated further:

i) Since the parameter “$A$” can be positive or negative it can also be zero. But a solitonic solution can not be established when “A” is zero; therefore, “$A$” has a critical value. In this case modified KP equation is studied [10].

ii) In recent years, the study of dust-ion-acoustic in plasma has become one of the most important problems of plasma physics [11]. Thus we can study solitary and shock waves in these plasmas with relativistic ions.

iii) Electron and positron densities vary with coordinates, for example $\mu = \mu(x)$ [12].

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References

[1] Nejoh Y N 1987 J. Plasma Phys. 37 487
[2] Tagare S G 1973 J. Plasma Phys. 15 1247
[3] Washimi H, T. Taniuti T 1996 Phys. Rev. Lett. 17 996
[4] Das G C and Paul S N 1985 Phys. Fluids 28 823
[5] Gill T S, Singh A, Kaur H, Saini N S and Bala P 2007 Phys. Lett. A 361 346
[6] Nejoh Y N 1987 J. Plasma Phys. 38 439
[7] EL-Labany S K, Nafie H O and EL-Shrikh A 1996 J. Plasma Phys. 56 13
[8] Gill T S, Nareshpal S S and Harvinder K 2006 Chaos Soliton. Fract. 28 1106
[9] Zhang S 2007 Chaos Soliton. Fract. 32 1375
[10] Lin M M and Duan W S 2007 Chaos Soliton. Fract. 33 1189
[11] Waleed M M 2006 Chaos Soliton. Fract. 28 994
[12] Xue J K and Zhang L P 2007 Chaos Soliton. Fract. 32 592