On Some Topological Properties of Fourier Transforms of Regular Holonomic $\mathcal{D}$-Modules

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Abstract

We study Fourier transforms of regular holonomic $\mathcal{D}$-modules. In particular we show that their solution complexes are monodromic. An application to direct images of some irregular holonomic $\mathcal{D}$-modules will be given. Moreover we improve the classical theorem of Brylinski by showing its converse.

1 Introduction

First of all we recall Fourier transforms of algebraic $\mathcal{D}$-modules. Let $X = \mathbb{C}^N_z$ be a complex vector space and $Y = \mathbb{C}^N_w$ its dual. We regard them as algebraic varieties and use the notations $\mathcal{D}_X$ and $\mathcal{D}_Y$ for the rings of “algebraic” differential operators on them. Denote by $\text{Mod}_{\text{coh}}(\mathcal{D}_X)$ (resp. $\text{Mod}_{\text{hol}}(\mathcal{D}_X)$) the category of coherent (resp. holonomic) $\mathcal{D}_X$-modules. Let $W_N := \mathbb{C}[z, \partial_z] \simeq \Gamma(X; \mathcal{D}_X)$ and $W_N^* := \mathbb{C}[w, \partial_w] \simeq \Gamma(Y; \mathcal{D}_Y)$ be the Weyl algebras over $X$ and $Y$, respectively. Then by the ring isomorphism

$$W_N \xrightarrow{\sim} W_N^* \quad (z_i \mapsto -\partial_{w_i}, \ \partial_z \mapsto w_i)$$

we can endow a left $W_N$-module $M$ with a structure of a left $W_N^*$-module. We call it the Fourier transform of $M$ and denote it by $M^\wedge$. For a ring $R$ we denote by $\text{Mod}_f(R)$ the category of finitely generated $R$-modules. Recall that for the affine algebraic varieties $X$ and $Y$ we have the equivalences of categories

$$\text{Mod}_{\text{coh}}(\mathcal{D}_X) \simeq \text{Mod}_f(\Gamma(X; \mathcal{D}_X)) = \text{Mod}_f(W_N),$$

$$\text{Mod}_{\text{coh}}(\mathcal{D}_Y) \simeq \text{Mod}_f(\Gamma(Y; \mathcal{D}_Y)) = \text{Mod}_f(W_N^*)$$

(see e.g. [HTT08, Propositions 1.4.4 and 1.4.13]). For a coherent $\mathcal{D}_X$-module $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X)$ we thus can define its Fourier transform $\mathcal{M}^\wedge \in \text{Mod}_{\text{coh}}(\mathcal{D}_Y)$. It follows that we obtain an equivalence of categories

$$(\cdot)^\wedge : \text{Mod}_{\text{hol}}(\mathcal{D}_X) \xrightarrow{\sim} \text{Mod}_{\text{hol}}(\mathcal{D}_Y)$$

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between the categories of holonomic $\mathcal{D}$-modules. However the Fourier transform $\mathcal{M}^\wedge$ of a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ is not necessarily regular. For the regularity of $\mathcal{M}^\wedge$ we need some strong condition on $\mathcal{M}$. Recall that a constructible sheaf $\mathcal{F} \in \mathbf{D}_{\mathbb{C}^- \mathbb{C}}^b(\mathbb{C}_X)$ on $X = \mathbb{C}^N$ is called monodromic if its cohomology sheaves are locally constant on each $\mathbb{C}^*$-orbit in $X = \mathbb{C}^N$. Then the following beautiful theorem is due to Brylinski [Bry86].

**Theorem 1.1** (Brylinski [Bry86]). Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Assume that its solution complex $\text{Sol}_X(\mathcal{M})$ is monodromic. Then its Fourier transform $\mathcal{M}^\wedge$ is regular and $\text{Sol}_Y(\mathcal{M}^\wedge)$ is monodromic.

Recently in [IT18] the authors studied the Fourier transforms of general regular holonomic $\mathcal{D}$-modules very precisely by using the Riemann-Hilbert correspondence for irregular holonomic $\mathcal{D}$-modules established by D’Agnolo and Kashiwara [DK16] and the Fourier-Sato transforms for enhanced ind-sheaves developed by Kashiwara and Schapira [KS16a]. In this process we found that for a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ the enhanced solution complex of its Fourier transform $\mathcal{M}^\wedge$ satisfies some condition (see Section 3). From this we obtain the following result.

**Theorem 1.2.** Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then $\text{Sol}_Y(\mathcal{M}^\wedge)$ is monodromic.

It seems that this result is already implicit in the main theorem of Daia [Dai00]. Indeed, for regular holonomic $\mathcal{M} \in \text{Mod}_{\mathbb{C}^{-}}(\mathcal{D}_X)$ it implies that $\text{Sol}_Y(\mathcal{M}^\wedge)$ is $\mathbb{R}_+^{-}$-conic. Moreover the monodromicity of $\text{Sol}_Y(\mathcal{M}^\wedge)$ follows from its $\mathbb{C}$-constructibility and the $\mathbb{R}_+^{-}$-conicness (see Lemma 2.1). In this paper we prove Theorem 1.2 by using the theory of enhanced ind-sheaves and our results in [IT18]. In this way, we can also improve Brylinski’s Theorem 1.1 as follows.

**Theorem 1.3.** Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then $\mathcal{M}^\wedge$ is regular if and only if $\mathcal{M}$ is monodromic.

Namely we prove the converse of Brylinski’s theorem. Moreover, as a simple application of Theorem 1.2 we obtain the following result which may be of independent interest.

**Theorem 1.4.** Let $\rho : X = \mathbb{C}^N \to Z = \mathbb{C}^n$ be a surjective linear map of codimension one and $\mathcal{M}$ an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. For the dual $L \simeq \mathbb{C}^{N-1}$ of $Z$ let $\iota : L \hookrightarrow Y = \mathbb{C}^N$ be the injective linear map induced by $\rho$. Then for any point $a \in Y \setminus \iota(L)$ the direct image $\mathbf{D}_{\rho_*}(\mathcal{M} \overline{\otimes} \mathcal{O}_X e^{-\langle z, a \rangle}) \in \mathbf{D}_{\mathbb{R}_+}(\mathcal{D}_Z)$ is concentrated in degree 0.

### 2 Preliminary Notions and Results

In this section, we briefly recall some basic notions and results which will be used in this paper. We assume here that the reader is familiar with the theory of sheaves and functors in the framework of derived categories. For them we follow the terminologies in [KS90] etc. For a topological space $M$ denote by $\mathbf{D}^b(\mathbb{C}_M)$ the derived category consisting of bounded complexes of sheaves of $\mathbb{C}$-vector spaces on it. The following lemma will be used in the proofs of Theorems 3.2 and 3.7.
**Lemma 2.1.** Assume that a \( \mathbb{C} \)-constructible sheaf \( \mathcal{G} \in \mathbf{D}^b_{\mathbb{C} - c}(\mathbb{C}_\mathbb{C}) \) on \( \mathbb{C}^N \) is \( \mathbb{R}_+ \)-conic. Then it is monodromic.

**Proof.** By restrictions we may assume that \( N = 1 \). By the \( \mathbb{C} \)-constructibility of \( \mathcal{G} \) there exists a finite subset \( \{P_1, P_2, \ldots, P_k\} \subset \mathbb{C} \) of \( \mathbb{C} \simeq \mathbb{R}^2 \) such that \( (H^j\mathcal{G})|_{\mathbb{C}\setminus\{P_1, P_2, \ldots, P_k\}} \) is a local system for any \( j \in \mathbb{Z} \). For \( 1 \leq i \leq k \) such that \( P_i \neq 0 \) let \( \ell_i = \mathbb{R}_+ P_i \simeq \mathbb{R}_+ \) be the real half line in \( \mathbb{C} \simeq \mathbb{R}^2 \) passing through the point \( P_i \). Then by our assumption \( (H^j\mathcal{G})|_{\ell_i} \) is a constant sheaf for any \( j \in \mathbb{Z} \). This implies that for the function \( h_i : \mathbb{C} \to \mathbb{C} \), \( h_i(x) = x - P_i \) such that \( h^{-1}_i(0) = \{P_i\} \subset \mathbb{C} \) we have \( \phi_{h_i}(\mathcal{G}) \simeq 0 \), where

\[
\phi_{h_i} : \mathbf{D}^b_{\mathbb{C} - c}(\mathbb{C}_\mathbb{C}) \to \mathbf{D}^b_{\mathbb{C} - c}(\mathbb{C}_{h^{-1}_i(0)})
\]

is Deligne’s vanishing cycle functor. From now we shall use an argument in Sabbah [Sab06]. For \( j \in \mathbb{Z} \) let \( pH^j(\mathcal{G}) \in \text{Perv}(\mathbb{C}) \) be the \( j \)-th perverse cohomology sheaf of \( \mathcal{G} \). Recall that \( \phi_{h_i}(pH^j(\mathcal{G})) \) is concentrated in only degree 0 for any \( j \in \mathbb{Z} \). Hence there exists an isomorphism \( H^j\phi_{h_i}(\mathcal{G}) \simeq H^0\phi_{h_i}(pH^j(\mathcal{G})) \) for any \( j \in \mathbb{Z} \). We thus obtain \( \phi_{h_i}(pH^j(\mathcal{G})) \simeq 0 \) for any \( 1 \leq i \leq k \) and \( j \in \mathbb{Z} \). This shows that \( H^j(pH^j(\mathcal{G}))|_{\mathbb{C}^*} \) is a local system on \( \mathbb{C}^* \) for any \( j, \ell \in \mathbb{Z} \). Then the assertion immediately follows. \( \square \)

### 2.1 Ind-sheaves

We recall some basic notions and results on ind-sheaves. References are made to Kashiwara-Schapira [KS01] and [KS06]. Let \( M \) be a good topological space (which is locally compact, Hausdorff, countable at infinity and has finite soft dimension). We denote by \( \text{Mod}(\mathbb{C}_M) \) the abelian category of sheaves of \( \mathbb{C} \)-vector spaces on it and by \( \mathcal{I}_M \) that of ind-sheaves. Then there exists a natural exact embedding \( \iota_M : \text{Mod}(\mathbb{C}_M) \to \mathcal{I}_M \) of categories. We sometimes omit it. It has an exact left adjoint \( \alpha_M \), that has in turn an exact fully faithful left adjoint functor \( \beta_M \):

\[
\text{Mod}(\mathbb{C}_M) \xrightarrow{\iota_M} \mathcal{I}_M \xrightarrow{\beta_M} \mathcal{I}_M .
\]

The category \( \mathcal{I}_M \) does not have enough injectives. Nevertheless, we can construct the derived category \( \mathbf{D}^b(\mathcal{I}_M) \) for ind-sheaves and the Grothendieck six operations among them. We denote by \( \otimes \) and \( \mathbf{R}\text{Hom} \) the operations of tensor products and internal homs respectively. If \( f : M \to N \) be a continuous map, we denote by \( f^{-1}, \text{RF}_*, f! \) and \( \text{R}f! \) the operations of inverse images, direct images, proper inverse images and proper direct images respectively. We set also \( \mathbf{R}\text{Hom} := \alpha_M \circ \mathbf{R}\text{Hom} \). Note that \( (f^{-1}, \text{RF}_*) \) and \( (\text{R}f! , f!) \) are pairs of adjoint functors. We may summarize the commutativity of the various functors we have introduced in the table below. Here, “\( \circ \)” means that the functors commute, and “\( \times \)” they do not.

### 2.2 Ind-sheaves on Bordered Spaces

For the results in this subsection, we refer to D’Agnolo-Kashiwara [DK16]. A bordered space is a pair \( M_\infty = (\check{M}, \check{\check{M}}) \) of a good topological space \( \check{M} \) and an open subset \( M \subset \check{\check{M}} \). A morphism \( f : (M, \check{M}) \to (N, \check{N}) \) of bordered spaces is a continuous map \( f : M \to N \)

3
such that the first projection $\tilde{M} \times \tilde{N} \to \tilde{M}$ is proper on the closure $\overline{\Gamma_f}$ of the graph $\Gamma_f$ of $f$ in $\tilde{M} \times \tilde{N}$. The category of good topological spaces embeds into that of bordered spaces by the identification $M = (M, M)$. We define the triangulated category of ind-sheaves on $M_\infty = (M, \tilde{M})$ by

$$D^b(I\mathcal{C}_{M_\infty}) := \frac{D^b(I\mathcal{C}_M)}{D^b(I\mathcal{C}_{M\setminus M})}.$$ 

Let

$$q : D^b(I\mathcal{C}_M) \to D^b(I\mathcal{C}_{M_\infty})$$

be the quotient functor. For a morphism $f : M_\infty \to N_\infty$ of bordered spaces, the Grothendieck's operations

$$\otimes : D^b(I\mathcal{C}_{M_\infty}) \times D^b(I\mathcal{C}_{M_\infty}) \to D^b(I\mathcal{C}_{M_\infty}),$$

$$R\mathcal{I}hom : D^b(I\mathcal{C}_{M_\infty})^{\text{op}} \times D^b(I\mathcal{C}_{M_\infty}) \to D^b(I\mathcal{C}_{M_\infty}),$$

$$Rf_* : D^b(I\mathcal{C}_{M_\infty}) \to D^b(I\mathcal{C}_{N_\infty}),$$

$$f^{-1} : D^b(I\mathcal{C}_{N_\infty}) \to D^b(I\mathcal{C}_{M_\infty}),$$

$$Rf! : D^b(I\mathcal{C}_{M_\infty}) \to D^b(I\mathcal{C}_{N_\infty}),$$

$$f^! : D^b(I\mathcal{C}_{N_\infty}) \to D^b(I\mathcal{C}_{M_\infty}),$$

are defined by

$$q(F) \otimes q(G) := q(F \otimes G),$$

$$R\mathcal{I}hom(q(F), q(G)) := q(R\mathcal{I}hom(F, G)),$$

$$Rf_*(q(F)) := q(R\mathcal{I}hom(F, G)),$$

$$f^{-1}(q(G)) := q(R\mathcal{I}hom(F, G)),$$

$$Rf!(q(F)) := q(R\mathcal{I}hom(F, G)),$$

$$f^!(q(G)) := q(R\mathcal{I}hom(F, G))$$

respectively, where $pr_1 : \tilde{M} \times \tilde{N} \to \tilde{M}$ and $pr_2 : \tilde{M} \times \tilde{N} \to \tilde{N}$ are the projections.
2.3 Enhanced Sheaves

For the results in this subsection, see Kashiwara-Schapira [KS16a] and D’Agnolo-Kashiwara [DK17]. Let $M$ be a good topological space. We consider the maps

$$M \times \mathbb{R}^2 \xrightarrow{p_1,p_2,\mu} M \times \mathbb{R} \xrightarrow{\pi} M$$

where $p_1, p_2$ are the first and the second projections and we set $\pi(x,t) := x$ and $\mu(x,t_1,t_2) := (x,t_1 + t_2)$. Then the convolution functors for sheaves on $M \times \mathbb{R}$ are defined by

$$F_1 \otimes^+ F_2 := R\mu(p_1^{-1}F_1 \otimes p_2^{-1}F_2),$$

$$R\text{Hom}^+(F_1, F_2) := Rp_1^*R\text{Hom}(p_2^{-1}F_1, \mu^!F_2).$$

We define the triangulated category of enhanced sheaves on $M$ by

$$E^b(C_M) := D^b(C_M \times \mathbb{R})/\pi^{-1}D^b(C_M).$$

Let

$$Q : D^b(C_M \times \mathbb{R}) \to E^b(C_M)$$

be the quotient functor. The convolution functors are defined also for enhanced sheaves. We denote them by the same symbols $\otimes^+, R\text{Hom}^+$. For a continuous map $f : M \to N$, we can define naturally the operations $E f^{-1}$, $E f_*$, $E f^!$, $E f_!$ for enhanced sheaves. We have also a natural embedding $\varepsilon : D^b(C_M) \to E^b(C_M)$ defined by

$$\varepsilon(F) := Q(C_{\{t \geq 0\}} \otimes \pi^{-1}F).$$

For a continuous function $\varphi : U \to \mathbb{R}$ defined on an open subset $U \subset M$ of $M$ we define the exponential enhanced sheaf by

$$E^\varphi_{U|M} := Q(C_{\{t + \varphi \geq 0\}},$$

where $\{t + \varphi \geq 0\}$ stands for $\{(x,t) \in M \times \mathbb{R} \mid x \in U, t + \varphi(x) \geq 0\}$.

2.4 Enhanced Ind-sheaves

We recall some basic notions and results on enhanced ind-sheaves. References are made to D’Agnolo-Kashiwara [DK16] and Kashiwara-Schapira [KS16b]. Let $M$ be a good topological space. Set $\mathbb{R}_\infty := (\mathbb{R}, \mathbb{R})$ for $\mathbb{R} := \mathbb{R} \cup \{-\infty, +\infty\}$, and let $t \in \mathbb{R}$ be the affine coordinate. Then we define the triangulated category of enhanced ind-sheaves on $M$ by

$$E^b(\mathcal{I}C_M) := D^b(\mathcal{I}C_M \times \mathbb{R}_\infty)/\pi^{-1}D^b(\mathcal{I}C_M),$$

where $\pi : M \times \mathbb{R}_\infty \to M$ is a morphism of bordered spaces induced by the first projection $M \times \mathbb{R} \to M$. The quotient functor

$$Q : D^b(\mathcal{I}C_M \times \mathbb{R}_\infty) \to E^b(\mathcal{I}C_M)$$
has fully faithful left and right adjoints \( L^E, R^E \) defined by

\[
L^E(QK) := (\mathbb{C}_{t \geq 0} \oplus \mathbb{C}_{t \leq 0})^+ \otimes K, \quad R^E(QK) := R\mathbb{I}hom^+(\mathbb{C}_{t \geq 0} \oplus \mathbb{C}_{t \leq 0}, K),
\]

where \( \{t \geq 0\} \) stands for \( \{(x, t) \in M \times \mathbb{R} \mid t \in \mathbb{R}, t \geq 0\} \) and \( \{t \leq 0\} \) is defined similarly.

We consider the maps

\[
M \times \mathbb{R}_\infty^2 \xrightarrow{p_1,p_2,\mu} M \times \mathbb{R}_\infty \xrightarrow{\pi} M
\]

where \( p_1, p_2 \) and \( \pi \) are morphisms of bordered spaces induced by the projections. And \( \mu \) is a morphism of bordered spaces induced by the map \( M \times \mathbb{R}_\infty \ni (x, t_1, t_2) \mapsto (x, t_1 + t_2) \in M \times \mathbb{R} \). Then the convolution functors for ind-sheaves on \( M \times \mathbb{R}_\infty \) are defined by

\[
F_1 \ast F_2 := R\mu!(p_1^{-1}F_1 \otimes p_2^{-1}F_2), \quad R\mathbb{I}hom^+(F_1, F_2) := Rp_1^*R\mathbb{I}hom(p_2^{-1}F_1, \mu_1^*F_2).
\]

The convolution functors are defined also for enhanced ind-sheaves. We denote them by the same symbols \( \ast, R\mathbb{I}hom^+ \). For a continuous map \( f : M \rightarrow N \), we can define also the operations \( E\!f^{-1}, Ef_* , E\!f^! , Ef_! \) for enhanced ind-sheaves. For example, by the natural morphism \( f : M \times \mathbb{R}_\infty \rightarrow N \times \mathbb{R}_\infty \) of bordered spaces associated to \( f \) we set

\[
Ef_* (QK) = R\!f_*(K).
\]

The other operations are defined similarly. We thus obtain the six operations \( \otimes, R\mathbb{I}hom^+, E\!f^{-1}, Ef_* , E\!f^!, Ef_! \) for enhanced ind-sheaves. Set \( \mathbb{C}^E_M := Q" \lim_{a \rightarrow +\infty}^\leftarrow \mathbb{C}_{(t \geq 0)} \in \mathbb{E}^b(I\!C_M) \). Then we have natural embeddings \( \varepsilon, e : D^b(I\!C_M) \rightarrow \mathbb{E}^b(I\!C_M) \) defined by

\[
\varepsilon(F) := Q(\mathbb{C}_{(t \geq 0)} \otimes \pi^{-1}F),
\]

\[
e(F) := \mathbb{C}^E_M \otimes \pi^{-1}F \simeq \mathbb{C}^E_M \ast \varepsilon(F).
\]

For a continuous function \( \varphi : U \rightarrow \mathbb{R} \) defined on an open subset \( U \subset M \) of \( M \) we define the exponential enhanced ind-sheaf by

\[
E^b_{U|M} := \mathbb{C}^E_M \ast E^b_{U|M} = \mathbb{C}^E_M \ast QC_{(t + \varphi \geq 0)},
\]

where \( \{t + \varphi \geq 0\} \) stands for \( \{(x, t) \in M \times \mathbb{R} \mid t \in \mathbb{R}, x \in U, t + \varphi(x) \geq 0\} \).

### 2.5 \( \mathcal{D} \)-Modules

In this subsection we recall some basic notions and results on \( \mathcal{D} \)-modules. References are made to [HTT08], [KS01, §7], [DK16, §8, 9] and [KS16b, §3, 4, 7]. For a complex manifold \( X \) we denote by \( dX \) its complex dimension. Denote by \( \mathcal{O}_X, \Omega_X \) and \( \mathcal{D}_X \) the sheaves of holomorphic functions, holomorphic differential forms of top degree and holomorphic differential operators, respectively. Let \( D^b(\mathcal{D}_X) \) be the bounded derived category of left \( \mathcal{D}_X \)-modules and \( D^b_\text{coh}(\mathcal{D}_X) \) be that of right \( \mathcal{D}_X \)-modules. Moreover we denote by \( D^b_\text{coho}(\mathcal{D}_X), D^b_\text{good}(\mathcal{D}_X), D^b_\text{hol}(\mathcal{D}_X) \) and \( D^b_\text{rh}(\mathcal{D}_X) \) the full triangulated subcategories of \( D^b(\mathcal{D}_X) \) consisting of objects with coherent, good, holonomic and regular holonomic cohomologies, respectively. For a morphism \( f : X \rightarrow Y \) of complex manifolds, denote
Note that we have isomorphisms

\[ D_\mathcal{D}(\mathcal{D}_X) \cong \mathcal{D}_\mathcal{O}(\mathcal{D}_X) \]

Moreover, we set \( \Omega \) and \( \Omega \)

Then the tempered de Rham and solution functors are defined by

\[ DR_X : \mathcal{D}_\mathcal{O}(\mathcal{D}_X) \to \mathcal{D}_b(\mathcal{C}_X), \quad \mathcal{M} \mapsto \Omega_X \otimes_{\mathcal{D}_X} \mathcal{M}, \]

\[ \text{Sol}_X : \mathcal{D}_\mathcal{O}(\mathcal{D}_X)^{\text{op}} \to \mathcal{D}_b(\mathcal{C}_X), \quad \mathcal{M} \mapsto \text{RHom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X). \]

For a closed hypersurface \( D \subset X \) in \( X \) we denote by \( \mathcal{O}_X(*D) \) the sheaf of meromorphic functions on \( X \) with poles in \( D \). Then for \( \mathcal{M} \in \mathcal{D}_b(\mathcal{D}_X) \) we set \( \mathcal{M}(*D) := \mathcal{M} \otimes \mathcal{O}_X(*D) \).

For \( f \in \mathcal{D}_X(*D) \) and \( U := X \setminus D \), set

\[ D_X e^f := \mathcal{D}_X\{ P \in \mathcal{D}_X \mid P e^f|_U = 0 \}, \]

\[ \epsilon^f_{U|X} := D_X e^f(*D). \]

Note that \( \epsilon^f_{U|X} \) is holonomic and there exists an isomorphism

\[ \mathcal{D}_X(\epsilon^f_{U|X})(*D) \simeq \epsilon^{-f}_{U|X}. \]

Namely \( \epsilon^f_{U|X} \) is a meromorphic connection associated to \( d + df \).

One defines the ind-sheaf \( \mathcal{O}_X^t \) of tempered holomorphic functions as the Dolbeault complex with coefficients in the ind-sheaf of tempered distributions. More precisely, denoting by \( X^c \) the complex conjugate manifold to \( X \) and by \( X_R \) the underlying real analytic manifold of \( X \), we set

\[ \mathcal{O}_X^t := \mathcal{RIt}_{\mathcal{D}_X}(\mathcal{O}_X^c, \mathcal{D}_b^t_{X^c}), \]

where \( \mathcal{D}_b^t_{X^c} \) is the ind-sheaf of tempered distributions on \( X_R \) (for the definition see [KS01, Definition 7.2.5]). Moreover, we set

\[ \Omega_X^t := \beta_X \Omega_X \otimes_{\beta_X \mathcal{O}_X} \mathcal{O}_X^t. \]

Then the tempered de Rham and solution functors are defined by

\[ DR_X^t : \mathcal{D}_\mathcal{O}(\mathcal{D}_X) \to \mathcal{D}_b(\mathcal{C}_X), \quad \mathcal{M} \mapsto \Omega_X^t \otimes_{\mathcal{D}_X} \mathcal{M}, \]

\[ \text{Sol}_X^t : \mathcal{D}_\mathcal{O}(\mathcal{D}_X)^{\text{op}} \to \mathcal{D}_b(\mathcal{C}_X), \quad \mathcal{M} \mapsto \text{RIt}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{O}_X^t). \]

Note that we have isomorphisms

\[ \text{Sol}_X(\mathcal{M}) \simeq a_X \text{Sol}_X^t(\mathcal{M}), \]

\[ DR_X(\mathcal{M}) \simeq a_X DR^t_X(\mathcal{M}), \]

\[ \text{Sol}_X^t(\mathcal{M})[d_X] \simeq DR^t_X(\mathcal{D}_X, \mathcal{M}). \]

Let \( i : X \times \mathbb{R}_\infty \to X \times \mathbb{P} \) be the natural morphism of bordered spaces and \( \tau \in \mathbb{C} \subset \mathbb{P} \) the affine coordinate such that \( \tau|_{\mathbb{R}} \) is that of \( \mathbb{R} \). We then define objects \( \mathcal{O}_X^E \in \mathcal{E}_b(\mathcal{ID}_X) \) and \( \Omega_X^E \in \mathcal{E}_b(\mathcal{ID}_X^{\mathbb{P}}) \) by

\[ \mathcal{O}_X^E := \text{RIt}_{\mathcal{D}_X}(\mathcal{O}_X^c, \mathcal{D}_b^t_{X^c}) \simeq i^! \text{RIt}_{\mathcal{D}_X}(\mathcal{E}_c^\tau, \mathcal{O}_X^t)[2], \]

\[ \Omega_X^E := \Omega_X^t \otimes_{\mathcal{O}_X} \mathcal{O}_X^E \simeq i^!(\Omega_X^t \otimes_{\mathcal{D}_X} \mathcal{E}_c^\tau)[1], \]
where $\mathcal{D}b^{\mathbb{E}}_X$ stand for the enhanced ind-sheaf of tempered distributions on $X$ (for the definition see [DK16, Definition 8.1.1]). We call $\mathcal{O}^{\mathbb{E}}_X$ the enhanced ind-sheaf of tempered holomorphic functions. Note that there exists an isomorphism

$$i_0^! \mathbb{R}^{\mathbb{E}} \mathcal{O}^{\mathbb{E}}_X \simeq \mathcal{O}^{\mathbb{E}}_X,$$

where $i_0 : X \to X \times \mathbb{R}_\infty$ is the inclusion map of bordered spaces induced by $x \mapsto (x, 0)$. The enhanced de Rham and solution functors are defined by

$$DR_X^{\mathbb{E}} : \mathcal{D}b^{\mathbb{coh}}(\mathcal{D}X) \to \mathcal{E}b(I\mathcal{C}_X), \quad \mathcal{M} \mapsto \Omega_X^L \otimes_{\mathcal{D}X} \mathcal{M},$$

$$Sol_X^{\mathbb{E}} : \mathcal{D}b^{\mathbb{coh}}(\mathcal{D}X)^{\mathit{op}} \to \mathcal{E}b(I\mathcal{C}_X), \quad \mathcal{M} \mapsto RI\mathcal{H}om_{\mathcal{D}X}(\mathcal{M}, \mathcal{O}^{\mathbb{E}}_X).$$

Then for $\mathcal{M} \in \mathcal{D}b^{\mathbb{coh}}(\mathcal{D}X)$ we have isomorphism $Sol_X^{\mathbb{E}}(\mathcal{M})[d_X] \simeq DR_X^{\mathbb{E}}(\mathcal{D}X\mathcal{M})$ and $Sol_X^{\mathbb{E}}(\mathcal{M}) \simeq i_0^! \mathbb{R}^{\mathbb{E}} Sol_X^{\mathbb{E}}(\mathcal{M})$. Finally, we recall the following theorem of [DK16].

**Theorem 2.2 ([DK16, Theorem 9.5.3 (Irregular Riemann-Hilbert Correspondence)])**. The enhanced solution functor induces a fully faithful one

$$Sol_X^{\mathbb{E}} : \mathcal{D}b^{\mathbb{hol}}(\mathcal{D}X)^{\mathit{op}} \to \mathcal{E}b(I\mathcal{C}_X).$$

### 3 Fourier Transforms of Regular Holonomic $\mathcal{D}$-modules

In this section we inherit the situation and the notations in Section 1. Let $X \leftarrow X \times Y \rightarrow Y$ be the projections. Then by Katz-Laumon [KL85], for an algebraic holonomic $\mathcal{D}_X$-module $\mathcal{M} \in \mathcal{M}od_{\mathbb{hol}}(\mathcal{D}_X)$ we have an isomorphism

$$\mathcal{M}^\wedge \simeq \mathcal{D}p_* (\mathcal{D}p^* \mathcal{M} \otimes_{\mathcal{O}_{\mathcal{X} \times \mathcal{Y}}} e^{-z,w}),$$

where $\mathcal{D}p^*$, $\mathcal{D}q_*$, $\otimes$ are the operations for algebraic $\mathcal{D}$-modules and $\mathcal{O}_{\mathcal{X} \times \mathcal{Y}} e^{-z,w}$ is the integral connection of rank one on $\mathcal{X} \times \mathcal{Y}$ associated to the canonical paring $\langle , \rangle : \mathcal{X} \times \mathcal{Y} \to \mathbb{C}$. In particular the right hand side is concentrated in degree zero. Let $\overline{X} \simeq \mathbb{P}^N$ (resp. $\overline{Y} \simeq \mathbb{P}^N$) be the projective compactification of $X$ (resp. $Y$). By the inclusion map $i_X : X = \mathbb{C}^N \hookrightarrow \overline{X} = \mathbb{P}^N$ we extend a holonomic $\mathcal{D}_X$-module $\mathcal{M} \in \mathcal{M}od_{\mathbb{hol}}(\mathcal{D}_X)$ on $X$ to the one $\tilde{\mathcal{M}} := i_{X*} \mathcal{M} \simeq \mathcal{D}i_{X*} \mathcal{M}$ on $\overline{X}$. Denote by $\overline{X}^{\mathbb{an}}$ the underlying complex manifold of $\overline{X}$ and define the analytification $\widetilde{\mathcal{M}}^{\mathbb{an}} \in \mathcal{M}od_{\mathbb{hol}}(\mathcal{D}_{\overline{X}^{\mathbb{an}}})$ of $\tilde{\mathcal{M}}$ by $\widetilde{\mathcal{M}}^{\mathbb{an}} := \mathcal{O}_{\overline{X}^{\mathbb{an}}} \otimes_{\mathcal{O}_{\overline{X}}} \tilde{\mathcal{M}}$. Then we set

$$Sol_X^{\mathbb{E}}(\tilde{\mathcal{M}}) := Sol_X^{\mathbb{E}}(\widetilde{\mathcal{M}}^{\mathbb{an}}) \in \mathcal{E}b(I\mathcal{C}_{\overline{X}^{\mathbb{an}}}).$$

Similarly for the Fourier transform $\mathcal{M}^\wedge \in \mathcal{M}od_{\mathbb{hol}}(\mathcal{D}_Y)$ we define $Sol_Y^{\mathbb{E}}(\mathcal{M}^\wedge) \in \mathcal{E}b(I\mathcal{C}_{\overline{Y}^{\mathbb{an}}})$. Let

$$\overline{Y}^{\mathbb{an}} \leftarrow \overline{X}^{\mathbb{an}} \times \overline{Y}^{\mathbb{an}} \rightarrow \overline{Y}^{\mathbb{an}},$$

8
be the projections. Then the following theorem is essentially due to Kashiwara-Schapira [KS16a] and D’Agnolo-Kashiwara [DK17]. For $F \in \mathcal{E}^b(\mathbb{I}C^a)$ we set

$$L \tilde{F} := \mathcal{E}_{\mathcal{I}}^*(\mathcal{E}_{\mathcal{I}}^{-1} \tilde{F} \hat{\otimes} \mathcal{E}_{X \times Y}^{-\text{Re}(z,w)}[N]) \in \mathcal{E}^b(\mathbb{I}C^a)$$

(here we denote $X^a \times Y^a$ etc. by $X \times Y$ etc. for short) and call it the Fourier-Sato (Fourier-Laplace) transform of $F$.

**Theorem 3.1.** For $\mathcal{M} \in \text{Mod}_{\text{hol}}(\mathcal{D}_X)$ there exists an isomorphism

$$\text{Sol}_\mathcal{E}^E(\widetilde{\mathcal{M}}^\wedge) \simeq L \text{Sol}_\mathcal{E}^E(\widetilde{\mathcal{M}}).$$

Form now on, we focus our attention on Fourier transforms of regular holonomic $\mathcal{D}_X$-modules. For such a $\mathcal{D}_X$-module $\mathcal{M}$, by [HTT08, Theorem 7.1.1] we have an isomorphism

$$\text{Sol}_\mathcal{E}^E(\widetilde{\mathcal{M}}) \simeq i_X! \text{Sol}_\mathcal{E}^E(\mathcal{M}),$$

where the right hand side $i_X! \text{Sol}_\mathcal{E}^E(\mathcal{M}) \in \mathcal{D}^b(\mathbb{C}^a)$ is the extension by zero of the classical solution complex of $\mathcal{M}$ to $\mathbb{C}^a$. Moreover by [DK16, Proposition 9.1.3 and Corollary 9.4.9] there exists an isomorphism

$$\text{Sol}_\mathcal{E}^E(\widetilde{\mathcal{M}}) \simeq \mathcal{E}

For an enhanced sheaf $F \in \mathcal{E}^b(\mathbb{C}^a)$ on $\mathbb{C}^a$ we define its Fourier-Sato (Fourier-Laplace) transform $L \tilde{F} \in \mathcal{E}^b(\mathbb{C}^a)$ by

$$L \tilde{F} := \mathcal{E}_{\mathcal{I}}^*(\mathcal{E}_{\mathcal{I}}^{-1} \tilde{F} \hat{\otimes} \mathcal{E}_{X \times Y}^{-\text{Re}(z,w)}[N]) \in \mathcal{E}^b(\mathbb{C}^a).$$

Since we have

$$L(\mathcal{E}^b(\mathbb{C}^a) \otimes (\cdot)) \simeq \mathcal{E}^b(\mathbb{C}^a) \otimes L(\cdot)$$

it suffices to study the Fourier-Sato transform of the enhanced sheaf $\mathcal{E}(i_X! \text{Sol}_\mathcal{E}^E(\mathcal{M})) \in \mathcal{E}^b(\mathbb{C}^a)$ on $\overline{X}^a$. The following theorem is due to Brylinski [Bry86]. Here we give another proof to it.

**Theorem 3.2.** Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Assume that $\text{Sol}_\mathcal{E}^E(\mathcal{M})$ is monodromic. Then $\mathcal{M}^\wedge$ is also a regular holonomic $\mathcal{D}_Y$-module and $\text{Sol}_\mathcal{E}^E(\mathcal{M}^\wedge)$ is monodromic.

**Proof.** By the above argument we have isomorphisms

$$\text{Sol}_\mathcal{E}^E(\widetilde{\mathcal{M}}^\wedge) \simeq L \text{Sol}_\mathcal{E}^E(\widetilde{\mathcal{M}}) \simeq L\left(\mathcal{E}^E_{\mathbb{C}^a} \hat{\otimes} \mathcal{E}(i_X! \text{Sol}_\mathcal{E}^E(\mathcal{M}))\right) \simeq \mathcal{E}^E_{\mathbb{C}^a} \hat{\otimes} L(\mathcal{E}(i_X! \text{Sol}_\mathcal{E}^E(\mathcal{M}))),$$

where $(\cdot)^\wedge$ stands for the Fourier-Sato transform for $\mathbb{R}_+\text{-conic sheaves}$ (see [KS90]) and in the last isomorphism we applied [KS16a, Theorem 5.7] to the $\mathbb{R}_+\text{-conic sheaf}$ $\text{Sol}_\mathcal{E}^E(\mathcal{M})$. Note that $\text{Sol}_\mathcal{E}^E(\mathcal{M})^\wedge$ is not only $\mathbb{R}_+\text{-conic}$ but also $\mathbb{C}$-constructible by [KS90, Proposition
10.3.18]. Hence it is monodromic by Lemma 2.1. Moreover by applying the functor $i^!_Y R E$ to the isomorphism $\text{Sol}^E_Y(\mathcal{M}^\wedge) \simeq \mathbb{C}_{\overline{\mathcal{Y}}}^E \otimes \varepsilon(i_Y \text{Sol}_X(\mathcal{M})^\wedge)$ we obtain an isomorphism

$$\text{Sol}_Y(\mathcal{M}^\wedge) \simeq i_Y \text{Sol}_X(\mathcal{M})^\wedge.$$ 

This implies that $i_Y \text{Sol}_X(\mathcal{M})^\wedge$ is an (algebraic) constructible sheaf on the algebraic variety $\overline{Y}$. By [HTT08, Corollary 7.2.4] we can take a regular holonomic $\mathcal{D}$-module $\mathcal{N} \in \text{Mod}_{\text{rh}}(\mathcal{D}_{\mathcal{Y}})$ on $\overline{Y}$ such that $\text{Sol}_Y(\mathcal{N}) \simeq i_Y \text{Sol}_X(\mathcal{M})^\wedge$. Then we have isomorphisms

$$\text{Sol}^E_Y(\mathcal{M}^\wedge) \simeq \mathbb{C}_{\overline{\mathcal{Y}}}^E \otimes \varepsilon(i_Y \text{Sol}_X(\mathcal{M})^\wedge)$$

$$\simeq \mathbb{C}_{\overline{\mathcal{Y}}}^E \otimes \varepsilon(\text{Sol}_Y(\mathcal{N}))$$

$$\simeq \text{Sol}^E_Y(\mathcal{N}).$$

By Theorem 2.2 we thus obtain an isomorphism

$$(\mathcal{M}^\wedge)^{an} \simeq \mathcal{N}^{an} \in \text{Mod}_{\text{rh}}(\mathcal{D}^{an})$$

of analytic $\mathcal{D}$-modules on $\overline{Y}^{an}$. Then the assertion follows from Lemma 3.3 of Brylinski [Bry86, Théorème 7.1] below. □

**Lemma 3.3** (Brylinski [Bry86, Théorème 7.1]). Let $Z$ be a smooth projective variety. Then the analytification functor

$$(\cdot)^{an} : \mathbf{D}^b_{\text{rh}}(\mathcal{D}_Z) \to \mathbf{D}^b_{\text{rh}}(\mathcal{D}_Z^{an})$$

is an equivalence of categories.

**Proof.** This result is due to Brylinski [Bry86, Théorème 7.1]. We shall give a new proof to it. Let $\mathbf{D}^b_{\mathcal{C}}(\mathcal{Z})$ (resp. $\mathbf{D}^b_{\mathcal{C}}(\mathcal{Z}^{an})$) be the derived category of $\mathbb{C}$-constructible sheaves on the algebraic variety $\mathcal{Z}$ (resp. the complex manifold $\mathcal{Z}^{an}$). Then we have a commutative diagram of functors

$$\begin{array}{ccc}
\mathbf{D}^b_{\text{rh}}(\mathcal{D}_Z) & \xrightarrow{\sim} & \mathbf{D}^b_{\mathcal{C}}(\mathcal{Z}) \\
(\cdot)^{an} \downarrow & & \downarrow \\
\mathbf{D}^b_{\text{rh}}(\mathcal{D}_Z^{an}) & \xrightarrow{\sim} & \mathbf{D}^b_{\mathcal{C}}(\mathcal{Z}^{an}),
\end{array}$$

where the horizontal arrows are the Riemann-Hilbert correspondences of algebraic and analytic $\mathcal{D}$-modules respectively (see e.g [HTT08, Theorem 7.2.2]). By Chow’s theorem the right vertical arrow

$$\mathbf{D}^b_{\mathcal{C}}(\mathcal{Z}) \to \mathbf{D}^b_{\mathcal{C}}(\mathcal{Z}^{an})$$

is also an equivalence of categories. Then the assertion immediately follows. □

Recall that for $\mathcal{F} \in \mathbf{D}^b(\mathcal{C}_X)$ we set

$$\varepsilon_X(\mathcal{F}) = \mathbb{C}_{\{t \geq 0\}} \otimes \pi^{-1} \mathcal{F} \in \mathbf{E}^b(\mathcal{C}_X).$$

For $s \in \mathbb{R}_+$ let

$$m_s : Y = \mathbb{C}^N \xrightarrow{\sim} Y = \mathbb{C}^N, \quad w \mapsto sw$$
be the multiplication by \( s \). We shall use also the morphism \( \ell_s : Y \times \mathbb{R}_\infty \to Y \times \mathbb{R}_\infty \) on the bordered space \( Y \times \mathbb{R}_\infty \) induced by the diagonal action

\[
Y \times \mathbb{R} \xrightarrow{\sim} Y \times \mathbb{R}, \quad (w, t) \mapsto (sw, st).
\]

Let \( f : X \times Y \times \mathbb{R}_\infty \to X, \, g : X \times Y \times \mathbb{R}_\infty \to Y \times \mathbb{R}_\infty \) be the projection. Then the following lemma was obtained in (the proof) of Ito-Takeuchi [IT18, Theorem 4.4].

**Lemma 3.4.** For \( \mathcal{F} \in D^b(\mathbb{C}_X) \) there exists an isomorphism

\[
L(\varepsilon_X \mathcal{F}) \simeq Rg(C_{\{t - \Re \langle z, w \rangle \geq 0\} \otimes f^{-1} \mathcal{F}})[N].
\]

**Proposition 3.5.** Let \( \mathcal{F} \in D^b(\mathbb{C}_X) \). Then for any \( s \in \mathbb{R}_+ \) we have an isomorphism

\[
\ell_s^{-1}(L(\varepsilon_X \mathcal{F})) \simeq L(\varepsilon_X \mathcal{F}).
\]

**Proof.** Consider the cartesian diagram

\[
\begin{array}{ccc}
X \times Y \times \mathbb{R}_\infty & \xrightarrow{id_X \times \ell_s} & X \times Y \times \mathbb{R}_\infty \\
\downarrow g & & \downarrow g \\
Y \times \mathbb{R}_\infty & \xrightarrow{\ell_s} & Y \times \mathbb{R}_\infty.
\end{array}
\]

Then we have isomorphisms

\[
\ell_s^{-1}(L(\varepsilon_X (\mathcal{F}))) \simeq \ell_s^{-1} Rg(C_{\{t - \Re \langle z, w \rangle \geq 0\} \otimes f^{-1} \mathcal{F}})[N]
\]

\[
\simeq Rg((id_X \times \ell_s)^{-1}(C_{\{t - \Re \langle z, w \rangle \geq 0\} \otimes f^{-1} \mathcal{F}})[N]
\]

\[
\simeq Rg(C_{\{t - \Re \langle z, w \rangle \geq 0\} \otimes f^{-1} \mathcal{F}})[N]
\]

\[
\simeq L(\varepsilon_X (\mathcal{F}))
\]

where in the third isomorphism we used

\[
st - \Re \langle z, sw \rangle \geq 0 \iff t - \Re \langle z, w \rangle \geq 0
\]

and \( f \circ (id_X \times \ell_s) = f \). \( \square \)

From now we shall consider the special case where \( \mathcal{F} = Sol_X(\mathcal{M}) \in D^b(\mathbb{C}_X) \) for \( \mathcal{M} \in \text{Mod}_{\text{rh}}(\mathcal{D}_X) \). Recall that there exist isomorphisms

\[
L^E Sol^E_{\mathcal{F}}(\widetilde{\mathcal{M}}^\wedge) \simeq L^E Sol^E_{\mathcal{F}}(\widetilde{\mathcal{M}})
\]

\[
\simeq L(\mathcal{C}_{\{t > 0\}} \otimes \varepsilon_X (Sol_X(\mathcal{M})))
\]

\[
\simeq C_{\{t > 0\}} \otimes L(\varepsilon_X \mathcal{F}).
\]

By Proposition 3.5, this implies that for \( s \in \mathbb{R}_+ \) we have isomorphisms

\[
\ell_s^{-1} L^E Sol^E_{\mathcal{F}}(\widetilde{\mathcal{M}}) \simeq \ell_s^{-1} (\mathcal{C}_{\{t > 0\}} \otimes L(\varepsilon_X \mathcal{F}))
\]

\[
\simeq C_{\{t > 0\}} \otimes \ell_s^{-1}(L(\varepsilon_X \mathcal{F}))
\]

\[
\simeq L^E Sol^E_{\mathcal{F}}(\mathcal{M}^\wedge).
\]
Proposition 3.6. Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then for any $s \in \mathbb{R}_+$ we have an isomorphism

$$m_s^{-1}S_{\text{oly}}(\mathcal{M}^\wedge) \simeq S_{\text{oly}}(\mathcal{M}^\wedge).$$

Proof. By (the proof) of Ito-Takeuchi [IT18, Lemma 3.12] there exist isomorphisms

$$S_{\text{oly}}(\mathcal{M}^\wedge) \simeq \alpha_Y i_0^! R^E(S_{\text{oly}}(\mathcal{M}^\wedge)) \simeq \alpha_Y R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, L^E S_{\text{oly}}^{\mathcal{F}}(\mathcal{M}^\wedge)).$$

Consider the commutative diagram

$$\begin{array}{ccc}
Y \times \mathbb{R}_\infty & \xrightarrow{\pi} & Y \\
\downarrow{\ell_s} & & \downarrow{m_s} \\
Y \times \mathbb{R}_\infty & \xrightarrow{\pi} & Y.
\end{array}$$

It is easy to see that it is Cartesian. Then we have isomorphisms

$$m_s^{-1}S_{\text{oly}}(\mathcal{M}^\wedge) \simeq m_s^{-1}\alpha_Y R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, L^E S_{\text{oly}}^{\mathcal{F}}(\mathcal{M}^\wedge))$$

$$\simeq \alpha_Y R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, L^E S_{\text{oly}}^{\mathcal{F}}(\mathcal{M}^\wedge))$$

$$\simeq \alpha_Y R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, L^E S_{\text{oly}}^{\mathcal{F}}(\mathcal{M}^\wedge))$$

$$\simeq \alpha_Y R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, L^E S_{\text{oly}}^{\mathcal{F}}(\mathcal{M}^\wedge))$$

$$\simeq \alpha_Y R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, L^E S_{\text{oly}}^{\mathcal{F}}(\mathcal{M}^\wedge))$$

$$\simeq \alpha_Y R\pi_* R\text{Ihom}(\mathbb{C}_{\{t \geq 0\}} \oplus \mathbb{C}_{\{t \leq 0\}}, L^E S_{\text{oly}}^{\mathcal{F}}(\mathcal{M}^\wedge))$$

$$\simeq S_{\text{oly}}(\mathcal{M}^\wedge).$$

\[ \square \]

Theorem 3.7. Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then $S_{\text{oly}}(\mathcal{M}^\wedge)$ is monodromic.

Proof. Since the Fourier transform $\mathcal{M}^\wedge$ of $\mathcal{M}$ is also holonomic, $S_{\text{oly}}(\mathcal{M}^\wedge)$ is $\mathbb{C}$-constructible. Moreover it is $\mathbb{R}_+$-conic by Proposition 3.6. Then the assertion follows from Lemma 2.1.\[ \square \]

By this theorem we can improve Brylinski’s Theorem 3.2 as follows.

Corollary 3.8. Let $\mathcal{M}$ be an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. Then $\mathcal{M}^\wedge$ is regular if and only if $\mathcal{M}$ is monodromic.

Proof. By Theorem 3.2 the Fourier transform $\mathcal{M}^\wedge$ is regular if $\mathcal{M}$ is monodromic. It suffices to show the converse. Assume that $\mathcal{M}^\wedge$ is regular. Then by Theorems 3.2 and 3.7 the original regular holonomic $\mathcal{D}_X$-module $\mathcal{M} \simeq (\mathcal{M}^\wedge)^\vee$ is monodromic.\[ \square \]
4 An Application to Direct Images of $\mathcal{D}$-Modules

In this section, we apply our results to direct images of some irregular holonomic $\mathcal{D}$-modules. We inherit the situation and the notations in Section 1. For a point $a \in Y = \mathbb{C}^N$, let $\tau_a : Y \xrightarrow{\sim} Y, w \mapsto w + a$ be the translation by it.

Lemma 4.1. For $\mathcal{M} \in \text{Mod}_{\text{coh}}(\mathcal{D}_X)$ and $a \in Y = \mathbb{C}^N$ we have an isomorphism

$$\text{Dr}^a_{\ast}(\mathcal{M}^\wedge) \simeq (\mathcal{M} \otimes \mathcal{O}_X e^{-(z,a)})^\wedge.$$

Proof. By Katz-Laumon [KL85] there exist isomorphisms

$$(\mathcal{M} \otimes \mathcal{O}_X e^{-(z,a)})^\wedge \simeq \text{Dp}_2^a((\mathcal{M} \otimes \mathcal{O}_X e^{-(z,a)}) \otimes \mathcal{O}_{X,Y} e^{-(z,w)}$$

$$\simeq \text{Dp}_2^a(\mathcal{M} \otimes \mathcal{O}_{X,Y} e^{-(z,w)})$$

$$\simeq \text{D} \tau_a^\ast(\mathcal{M}^\wedge).$$

$$\Box$$

Theorem 4.2. Let $\rho : X = \mathbb{C}^N \to Z = \mathbb{C}^n$ be a surjective linear map and $\mathcal{M}$ an algebraic regular holonomic $\mathcal{D}$-module on $X = \mathbb{C}^N$. For the dual $L \simeq \mathbb{C}^n$ of $Z$ let $\iota : L \hookrightarrow Y = \mathbb{C}^N$ be the injective linear map induced by $\rho$. Assume that for a point $a \in Y \setminus \iota(L)$ the affine linear subspace $K = \tau_a(\iota(L)) \subset Y = \mathbb{C}^N$ is non-characteristic for the Fourier transform $\mathcal{M}^\wedge \in \text{Mod}_{\text{hol}}(\mathcal{D}_Y)$ of $\mathcal{M}$. Then the direct image $\text{D} \rho_\ast(\mathcal{M} \otimes \mathcal{O}_X e^{-(z,a)}) \in \text{D}^b_{\text{hol}}(\mathcal{D}_Z)$ is concentrated in degree 0.

Proof. Let $i_L : L \hookrightarrow Y = \mathbb{C}^N$ and $i_K : K \hookrightarrow Y = \mathbb{C}^N$ be the inclusion maps. Then via the identification $L \simeq K$ obtained by the translation $\tau_a$, we have isomorphisms

$$\text{D} i_K^\ast(\mathcal{M}^\wedge) \simeq \text{D} i_L^\ast \text{D} \tau_a^{-1}(\mathcal{M}^\wedge)$$

$$\simeq \text{D} i_L^\ast(\mathcal{M} \otimes \mathcal{O}_X e^{-(z,a)})^\wedge$$

$$\simeq (\text{D} \rho_\ast(\mathcal{M} \otimes \mathcal{O}_X e^{-(z,a)}))^\wedge,$$

where in the second (resp. third) isomorphism we used Lemma 1.1 (resp. [HTT08, Proposition 3.2.6]). By our assumption, the left hand side $\text{D} i_K^\ast(\mathcal{M}^\wedge) \in \text{D}^b_{\text{hol}}(\mathcal{D}_K)$ is concentrated in degree 0. Then the assertion follows from the fact the Fourier transform is an exact functor.

$\Box$

Corollary 4.3. In the situation of Theorem 4.2 assume also that $n = N - 1$ i.e. the surjective linear map $\rho : X = \mathbb{C}^N \to Z = \mathbb{C}^n$ is of codimension one. Then for any point $a \in Y \setminus \iota(L)$ the direct image $\text{D} \rho_\ast(\mathcal{M} \otimes \mathcal{O}_X e^{-(z,a)}) \in \text{D}^b_{\text{hol}}(\mathcal{D}_Z)$ is concentrated in degree 0.

Proof. By Theorem 3.7 the Fourier transform $\mathcal{M}^\wedge$ of $\mathcal{M}$ is monodromic. Since the affine linear subspace $K = \tau_a(\iota(L)) \subset Y = \mathbb{C}^N$ does not contain the origin $0 \in Y = \mathbb{C}^N$, this implies that $K$ is non-characteristic for $\mathcal{M}^\wedge$. Then the assertion follows from Theorem 4.2.

$\Box$
References

[Bry86] Jean-Luc Brylinski, Transformations canoniques, dualité projective, théorie de Lefschetz, transformations de Fourier et sommes trigonométriques, Astérisque, (140-141):3–134, 251, 1986, Géométrie et analyse microlocales.

[Dai00] Liviu Daia, La transformation de Fourier pour les D-modules, Ann. Inst. Fourier (Grenoble), 50(6):1891–1944 (2001), 2000.

[DHMS17] Andrea D’Agnolo, Marco Hien, Giovanni Morando, and Claude Sabbah, Topological computation of some stokes phenomena on the affine line, arXiv:1705.07610v1, preprint.

[DK16] Andrea D’Agnolo and Masaki Kashiwara, Riemann-Hilbert correspondence for holonomic D-modules, Publ. Math. Inst. Hautes Études Sci., 123:69–197, 2016.

[DK17] Andrea D’Agnolo and Masaki Kashiwara, A microlocal approach to the enhanced fourier-sato transform in dimension one, arXiv:1709.03579v1, preprint.

[HTT08] Ryoshi Hotta, Kiyoshi Takeuchi, and Toshiyuki Tanisaki, D-modules, perverse sheaves, and representation theory, volume 236 of Progress in Mathematics, Birkhäuser Boston, 2008.

[IT18] Yohei Ito and Kiyoshi Takeuchi, On irregularities of Fourier transforms of regular holonomic D-Modules, arXiv:1801.07444v10, preprint.

[KS90] Masaki Kashiwara and Pierre Schapira, Sheaves on manifolds, volume 292 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, 1990.

[KS01] Masaki Kashiwara and Pierre Schapira, Ind-sheaves, Astérisque, (271):136, 2001.

[KS06] Masaki Kashiwara and Pierre Schapira, Categories and Sheaves, volume 332 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag Berlin Heidelberg, 2006.

[KS16a] Masaki Kashiwara and Pierre Schapira, Irregular holonomic kernels and Laplace transform, Selecta Math., 22(1):55–109, 2016.

[KS16b] Masaki Kashiwara and Pierre Schapira, Regular and irregular holonomic D-modules, volume 433 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2016.

[KL85] Nicholas M. Katz and Gérard Laumon, Transformation de Fourier et majoration de sommes exponentielles, Inst. Hautes Études Sci. Publ. Math., (62):361–418, 1985.

[Mal88] Bernard Malgrange, Transformation de Fourier géométrique, Astérisque, (161-162):Exp. No. 692, 4, 133–150 (1989), 1988, Séminaire Bourbaki, Vol. 1987/88.

[Moc10] Takuro Mochizuki, Note on the Stokes structure of Fourier transform, Acta Math. Vietnam., 35(1):107–158, 2010.
[Pre11] Luca Prelli, Conic sheaves on subanalytic sites and laplace transform, Rend. Sem. Mat. Univ. Padova, 125:173–206, 2011.

[Sab93] Claude Sabbah, Introduction to algebraic theory of linear systems of differential equations, In D-modules cohérents et holonomes, Éléments de la théorie des systèmes différentiels, volume 45 of Travaux en Cours, pages 1–80. Hermann, Paris, 1993.

[Sab06] Claude Sabbah, Hypergeometric periods for a tame polynomial, Port. Math., 63:173–226, 2006.

[Tam08] Dmitry Tamarkin, Microlocal condition for non-displaceability, arXiv:0809.1584v1.

[Ver83] J.-L. Verdier, Spécialisation de faisceaux et monodromie modérée, In Analysis and topology on singular spaces, II, III (Luminy, 1981), volume 101 of Astérisque, pages 332–364. Soc. Math. France, Paris, 1983.