Complexity volumes of splittable groups

Mihalis Sykiotis

Department of Mathematics, National and Kapodistrian University of Athens, Panepistimioupolis, GR-157 84, Athens, Greece

Abstract

Using graph of groups decompositions of finitely generated groups, we define Euler characteristic type invariants which are non-zero in many interesting classes of hyperbolic, limit and CSA groups, including elementarily free groups and one-ended torsion-free hyperbolic groups whose JSJ decomposition contains a maximal hanging Fuchsian vertex group.

Keywords: Groups acting on trees, Euler characteristics, Hyperbolic groups, Limit groups, CSA groups

2010 MSC: 20F65, 20E08, 20F67, 20E36

1. Introduction

Following [8], a generalised Euler characteristic (or volume) on a class of groups $C$, closed under subgroups of finite index, is a function $\chi : C \to R$, where $R$ is a commutative ring, satisfying the following condition: If $G \in C$ and $H$ is a subgroup of finite index in $G$, then $\chi(H) = [G : H] \chi(G)$.

The existence of such an invariant, non-vanishing on a group $G$ has two immediate, but important, consequences. First, isomorphic subgroups of finite index of $G$ have the same index. Second, every monomorphism from $G$ to $G$ with image of finite index is an automorphism.

Euler characteristics have been studied by Bass, Brown, Chi swell, Serre, Hattori and Stallings (see [8] and the references contained therein) on classes of groups satisfying certain homological finiteness properties.

The idea of defining generalised Euler characteristics on groups using a limiting process and subgroups of finite index goes back to Wall [25, Problem E10, p. 385]. More precisely, given a real valued function $f$ on a class of groups such that $f(H) \leq [G : H] f(G)$, for any subgroup $H$ of finite index in $G$, Wall defined

$$\tilde{f}(G) = \inf \left\{ \frac{f(H)}{[G : H]} : H \text{ subgroup of finite index in } G \right\}$$

and remarked that $\tilde{f}$ is a generalised Euler characteristic. He also gave two examples of functions that satisfy the inequality above: the minimum number of generators for a group.
and the minimum (over all presentations) of total of lengths of relators. Examples of invariants constructed using limits and subgroups of finite index can be found in [24], [33], [20], [7], [34].

In this paper, we follow the approach initiated in [33] to define invariants using graph of groups decompositions of groups. Let \( G \) be a group acting on a tree \( X \) (without inversions). By Bass-Serre theory, for which we refer to [9, 32], this is equivalent to saying that \( G \) is the fundamental group of the corresponding graph of groups \((\mathcal{G}, X/G)\). We also say that the pair \((\mathcal{G}, X/G)\) is a splitting of \( G \).

A vertex \( v \) of \( X/G \) is called degenerate if there is an edge \( e \) incident to \( v \) with \( G_v = G_e \).

We denote by \( V_{\text{ndeg}}(X/G) \) the set of non-degenerate vertices of \( X/G \), and by \( r(X/G) \) the rank of the free group \( \pi_1(X/G) \). The complexity \( C_X(G) \) (or simply \( C(G) \) when \( X \) is clear from context) of \( G \) with respect to the above action is defined to be the sum \( C_X(G) = r(X/G) + |V_{\text{ndeg}}(X/G)| \), if \( G \) contains hyperbolic elements, and 1 otherwise.

In the case where \( G \) is finitely generated the above sum is finite. To see this, let \( X_G \) be the minimal \( G \)-invariant subtree of \( X \), which is the union of the axes of all hyperbolic elements of \( G \). The fact that \( G \) is finitely generated implies that the quotient graph \( X_G/G \) is finite. On the other hand, by [33, Prop. 2.2], the subtree \( X_G \) is a “core” for the action of \( G \) on \( X \) in the sense that \( r(X/G) + |V_{\text{ndeg}}(X/G)| = r(X_G/G) + |V_{\text{ndeg}}(X_G/G)| \). It follows that \( C_X(G) \) is finite as claimed.

The above definition of complexity differs slightly from that in [33] in what the complexity of an elliptic action (i.e. each group element is elliptic) is now defined to be 1 instead of 0. It seems that 1 is more appropriate for our purposes. Moreover, in the case of an elliptic action of a finitely generated group \( G \), any minimal \( G \)-invariant subtree consists of a single vertex.

Suppose now that vertex groups are contained in disjoint classes \( \mathcal{C}_i \) of groups each of which is endowed with a volume \( \varphi_i \). To simplify notation we will denote each \( \varphi_i \) with \( \varphi \).

Suppose first that the group \( G \) acting on \( X \) contains hyperbolic elements. As usual, we denote by \( G_v \) the stabilizer of the vertex \( v \). In this case, we define the weighted complexity \( C^\varphi(G) \) of \( G \) with respect to the given action on \( X \) by the formula

\[
C^\varphi(G) = r(X/G) + \sum [1 + \varphi(G_v)],
\]

where the sum is over all non-degenerate vertices of \( X/G \), provided that the rank of \( X/G \) is finite and the sum converges (this is the case if \( G \) is finitely generated). As before if each element of \( G \) is elliptic, then the complexity is defined to be 1. In the case where \( \varphi \) is the trivial volume (i.e. \( \varphi \) is identically zero) the above formula defines the usual complexity \( C_X(G) \) of \( G \).

Let \( \mathcal{S} \) be a class of small groups (in the sense of Bestvina and Feighn [1]) closed under subgroups and let \( \mathcal{V} \) be any class of groups closed under subgroups of finite index. The
volume $V(G)$ of a finitely presented group (or almost finitely presented) $G$ with respect to $S$ and $V$ is defined to be the upper limit

$$\lim_{H \in N_G} \frac{C_{\text{max}}(H)}{[G : H]} \in [0, \infty],$$

where $N_G$ is the set of all normal subgroups of finite index in $G$ and $C_{\text{max}}(H)$ is the maximal complexity of $H$ over all reduced splittings of $H$ with vertex groups in $V$ and edge groups in $S$. Bestvina-Feighn’s accessibility theorem \[2\] ensures that $V(G)$ is finite (see Proposition 2.7).

Using splittings over finite subgroups, we obtain the volume $V_{\text{fin}}$ which is strictly positive on any finitely presented (or accessible) group with infinitely many ends and provides us with a basic tool for constructing volumes on one-ended groups.

In Section 3, we consider acylindrical splittings of finitely generated, one-ended groups over finitely generated, free abelian groups of rank at most $n$. We use any “suitable” volume $\varphi$ on the vertex groups (see Def. 3.1) and work with the corresponding weighted complexity. In this way, we define volumes $V_{\varphi}^n$ on finitely generated, one-ended groups, using reduced acylindrical splittings as above, of maximal weighted complexity. In this case, finiteness of the volume follows from acylindrical accessibility \[28, 35\] and an analogue of Grushko’s theorem for acylindrical splittings \[36\].

Our main results are the following.

**Theorem 3.7.** Suppose that $G$ is a finitely generated one-ended group which admits an acylindrical splitting $(G, Y)$ over free abelian groups of bounded rank, say by a positive integer $n$. If there is a vertex group $G_v$ with $\varphi(G_v) > 0$ (where $\varphi$ is as above), then $V_{\varphi}^n(G) > 0$.

**Theorem 3.9.** Let $G$ be a one-ended finitely generated group such that $V_{\varphi}^n(G) > 0$. If $f$ is an endomorphism of $G$ whose image is of finite index, then the decreasing sequence of images $f^k(G)$, $k \in \mathbb{N}$, is eventually constant. Moreover, if $G$ is torsion-free and every subgroup of finite index is Hopfian (e.g. $G$ residually finite or hyperbolic), then $f$ is an automorphism.

We also prove an analogous result for groups with infinitely many ends, which generalizes a result of Hirshon \[16\].

**Proposition 3.11.** Let $G$ be a finitely generated group with infinitely many ends. Suppose that $G$ splits as an amalgam or a HNN extension over a malnormal subgroup and that each subgroup of finite index of $G$ is Hopfian. Then each endomorphism of $G$ with image of finite index is an automorphism.

Groups with the property that every endomorphism with image of finite index is an automorphism are called cofinitely Hopfian in \[1\].
In Sections 4, 5 and 6, we use the volume $V_{\text{fin}}$ on vertex groups with infinitely many ends, and show how the above results can be applied in torsion-free, one-ended, hyperbolic groups, limit groups and, more generally, CSA groups (a group $G$ is CSA if maximal abelian subgroups are malnormal). In each of these cases, the existence of “surface type” vertex groups in such a splitting implies that its fundamental group has positive volume.

More precisely, in the case of torsion-free, one-ended hyperbolic groups, we consider cyclic splittings (i.e. splittings where each edge group is infinite cyclic) and denote the corresponding volume by $V_{\text{hyp}}^1$.

**Proposition 4.1.** Let $G$ be a torsion-free, one-ended hyperbolic group which admits a cyclic splitting with a free non-abelian vertex group. Then $V_{\text{hyp}}^1(G) > 0$.

As a special case we have:

**Corollary 4.4.** Let $G$ be a torsion-free, one-ended hyperbolic group. If the JSJ decomposition of $G$ contains a (MHF) subgroup, then $V_{\text{hyp}}^1(G) > 0$. In particular, if the outer automorphism group $\text{Out}(G)$ of $G$ is not virtually finitely generated free abelian, then $V_{\text{hyp}}^1(G) > 0$.

It should be noted that there are hyperbolic groups as above with strictly positive volume and trivial JSJ decomposition. For example, if $G$ is the fundamental group of a closed orientable surface $S_g$ of genus $g \geq 2$, then $V_{\text{hyp}}^1(G) \geq 2(g - 1)$ (see Example 4.2).

In the case of one-ended limit groups, we consider cyclic 2-acylindrical splittings, which arise naturally from the characterization of limit groups as finitely generated subgroups of $\omega$-residually free tower groups. Let $V_{\text{lim}}^1$ denote the corresponding volume.

**Proposition 5.2.** Let $H$ be a one-ended limit group of height $h \geq 1$ and $X_h$ an $\omega$-rft space such that the fundamental group of $X_h$ contains a copy of $H$. If the final block (in the construction of $X_h$) is quadratic, then $V_{\text{lim}}^1(H) > 0$.

In particular, if $G$ is a one-ended elementarily free group, then $V_{\text{lim}}^1(G) > 0$.

In order to generalize the above results to the case of torsion-free, one-ended CSA groups, we use the work of Guirardel and Levitt in [14], where a method is given for constructing an acylindrical $G$-tree $T_c$ (the tree of cylinders) from any $G$-tree $T$, for any finitely generated group $G$. If $G$ is a torsion-free, one-ended CSA group and the $G$-tree $T$ has finitely generated abelian edge stabilizers, then the edge stabilizers of $T_c$ are non-trivial abelian and $T_c$ is 2-acylindrical. Moreover, $T$ and $T_c$ have the same non-abelian stabilizers which are divided into two types: rigid and flexible. Non-abelian flexible vertex stabilizers are fundamental groups of compact surfaces with boundary. Let $V_n^{\text{CSA}}$ denote the volume defined using acylindrical splittings over abelian subgroups of bounded rank $n$, of one-ended, torsion-free CSA groups.
Proposition 6.2. Let $G$ be a finitely generated, torsion-free, one-ended CSA group such that abelian subgroups of $G$ are finitely generated of bounded rank. Suppose that $G$ admits a splitting over abelian subgroups such that the associated tree of cylinders $T_c$ has a non-abelian flexible vertex stabilizer $G_v$. Then there exists $n$ such that $V_n^{\text{CSA}}(G) > 0$.

Beside torsion-free hyperbolic groups and limit groups, other examples of Hopfian, CSA groups are $\Gamma$-limit groups, where $\Gamma$ is a torsion-free group relatively hyperbolic to a finite family of finitely generated abelian subgroups \cite{12, 13}.

Proposition 6.4. Let $\Gamma$ be a torsion-free group which is hyperbolic relative to a finite collection of free abelian subgroups and let $G$ be a one-ended $\Gamma$-limit group. Suppose that $G$ admits a splitting over abelian subgroups which has a non-abelian vertex group isomorphic to the fundamental group of a compact surface with boundary. Then there exists a positive integer $n = n(G)$ such that $V_n^{\text{CSA}}(G) > 0$.

2. Splittings and Volumes

A finitely generated group $G$ is small if it does not admit a minimal action on a tree for which there are two hyperbolic elements whose axes intersect in a compact set. Each finitely generated group not containing the free group of rank 2 is small. A splitting of $G$ as fundamental group of a graph of groups is reduced if for each edge $e$ not a loop, the edge group $G_e$ is contained properly in the vertex groups of its endpoints. This means that for each degenerated vertex the corresponding isomorphism is induced by an edge which is a loop. A $G$-tree is reduced if the corresponding graph of groups is reduced. In \cite{2} the term means something weaker: the vertex group of each vertex of valence two properly contains the corresponding edge groups.

Let $S$ be a class of small groups closed under subgroups and $V$ a class of groups closed under subgroups of finite index. For each finitely presented group $\Gamma$, let $C_{\text{max}}(\Gamma)$ denote the maximal complexity of $\Gamma$ over the complexities of all reduced splittings of $\Gamma$ over $S$ with vertex groups in $V$. Since the complexity of a splitting can be computed using the corresponding minimal invariant subtree, we may suppose that all splittings are minimal.

The main result in \cite{2} insures that $C_{\text{max}}(\Gamma) < \infty$. More specifically, Bestvina and Feighn proved that for each finitely presented group, there is an integer $\gamma(\Gamma)$ such that for each reduced $\Gamma$-tree $X$ with small edge stabilizers, the number of vertices in the quotient graph $X/\Gamma$ is bounded above by $\gamma(\Gamma)$. Since the rank of $X/\Gamma$ is bounded above by the minimal number of generators of $\Gamma$, it follows that $C_{\text{max}}(\Gamma) < \infty$.

We denote by $N_G$ the set of all normal subgroups of finite index in $G$, partially ordered by inverse inclusion.
**Definition 2.1.** Let $G$ be a finitely presented group. The volume $V(G)$ of $G$ with respect to $S$ and $\mathcal{V}$ is the upper limit \( \lim_{H \in N_G} \frac{C_{\text{max}}(H)}{|G:H|} \in [0, \infty] \), i.e. the supremum of the set $L_{N_G} = \{ \ell \in \mathbb{R} : \left( \frac{C_{\text{max}}(K)}{|G:K|} \right)_{K \in \mathcal{K}} \to \ell \}$ for some cofinal subset $\mathcal{K}$ of $N_G$.

It is not difficult to show that $L_{N_G} \neq \emptyset$. We will see in Proposition 2.7 that $V(G)$ is always finite.

**Remark 2.2.** If $H$ is a finite index subgroup of $G$, then $H$ is finitely presented and therefore $C_{\text{max}}(H) < \infty$.

**Remark 2.3.** We could also use any cofinal subset of the collection $\Lambda_G$ of all subgroups of finite index in $G$ (partially ordered by inverse inclusion) instead of $N_G$ (or other limit point of the net).

But, calculations are often simplified by working with normal subgroups.

**Remark 2.4.** In the case where $G$ contains finitely many subgroups of finite index, then $N_G$ has a minimum element, say $H_0$, and $V(G) = \frac{C_{\text{max}}(H_0)}{|G:H_0|}$. In particular, if $G$ is finite, then $V(G) = \frac{1}{|G|}$.

**Remark 2.5.** If $G$ contains infinitely many subgroups of finite index and $\mathcal{K}$ is a cofinal subset of $N_G$ with $\left( \frac{C_{\text{max}}(K)}{|G:K|} \right)_{K \in \mathcal{K}} \to \ell$, then there exists a sequence $K_n$ in $\mathcal{K}$ such that $\left( \frac{C_{\text{max}}(K_n)}{|G:K_n|} \right)_{n \in \mathbb{N}} \to \ell$ and $|G:K_n| \to \infty$. Indeed, we consider the sequence $(H_n)$, where $H_n$ is the intersection of all subgroups in $G$ of index at most $n$. Since $G$ is finitely generated and contains infinitely many subgroups of finite index, it follows that $|G:H_n| \to \infty$. The cofinality of $\mathcal{K}$ implies that we can choose for each $n$ a subgroup $K_n \in \mathcal{K}$ such that $K_n \subset H_n$ and thus $|G:K_n| \to \infty$. On the other, this sequence is a cofinal subset of $\mathcal{K}$ and therefore $\left( \frac{C_{\text{max}}(K_n)}{|G:K_n|} \right)_{n \in \mathbb{N}} \to \ell$.

**Remark 2.6.** Finally, if the vertex groups are partitioned in classes (closed under subgroups of finite index) in each of which is already defined a volume, then we can use in the above definition the weighted complexity. In this case, in order to insure that the corresponding volume $V^\varphi(G)$ is finite (with respect to $S$ and $\mathcal{V}$), it suffices to impose the following additional hypothesis:

(H) For each subgroup $H$ of finite index in $G$ and each vertex group $H_v$ in any splitting of $H$, with edge groups in $\mathcal{S}$ and vertex groups in $\mathcal{V}$, which attains $C_{\text{max}}^e(H)$, there are constants $A_1, A_2, A_3$ and $A_4$, independent of $H$, such that

$$\sum_{v \in V(\mathcal{Y})} r(H_v) \leq A_1 \cdot r(H) + A_2 \quad \text{and} \quad \varphi(H_v) \leq A_3 \cdot r(H_v) + A_4.$$

We will see in section 3 that hypothesis (H) is satisfied in many interesting cases.

For a group $\Gamma$, we denote by $r(\Gamma)$ the minimum number of generators of $\Gamma$.

**Proposition 2.7.** Under the above hypotheses, $V^\varphi(G) < \infty$. In particular $V(G) < \infty$. 

6
Proof. Let $H$ be a subgroup of finite index in $G$, $(H, Y)$ a reduced splitting of $H$ over small groups of maximal complexity and $X$ the corresponding universal tree. It is proved in [8] that the number of vertices of $Y$ is bounded above by $94\delta(H) + 233\beta_1(H) + 6\dim H^1(H; \mathbb{Z}_2) - 139$, where $\beta_1(H)$ is the (torsion-free) rank of the abelianization $H_{ab}$ of $H$ and $\delta(H)$ is a constant (see [10, 9]) defined as follows. Let $\Gamma$ be an almost finitely presented group. For each 2-dimensional complex $K$ such that $H^1(K; \mathbb{Z}_2) = 0$, on which $\Gamma$ acts freely with finite quotient $L$, we define $\delta(L) = 2 \dim H^1(L; \mathbb{Z}_2) + a_0 + a_2$, where $a_i$ is the number of $i$-simplices of $L$. The minimum of all possible $\delta(L)$, is denoted by $\delta(\Gamma)$.

We first note that $\beta_1(H) \leq r(H) \leq [G : H](r(G) - 1) + 1$.

To bound the number $\dim H^1(H; \mathbb{Z}_2)$, we use the isomorphisms

$$H^1(H; \mathbb{Z}_2) \cong \text{Hom}(H_1(H), \mathbb{Z}_2) = \text{Hom}(H_{ab}, \mathbb{Z}_2) \cong \bigoplus_{i=1}^k \text{Hom}(C_i, \mathbb{Z}_2),$$

where $C_i$ are cyclic groups whose direct sum is $H_{ab}$, i.e. $H_{ab} = C_1 \oplus \cdots \oplus C_k$. Since $\text{Hom}(\mathbb{Z}, \mathbb{Z}_2) = \mathbb{Z}_2$ and $\text{Hom}(\mathbb{Z}_n, \mathbb{Z}_2) = \mathbb{Z}_2[n] = \{ g \in \mathbb{Z}_2 : ng = 0 \}$, it follows that $H^1(H; \mathbb{Z}_2)$ is the direct sum of at most $k$ copies of $\mathbb{Z}_2$ and thus $\dim H^1(H; \mathbb{Z}_2) \leq k$. The well-known proof of the existence of the above decomposition of $H_{ab}$ (with induction on its rank) as direct sums of cyclic groups, actually shows that the number of summands is equal to its rank. Hence $\dim H^1(H; \mathbb{Z}_2) \leq k = r(H_{ab}) \leq r(H) \leq [G : H](r(G) - 1) + 1$.

Fix a finite 2-dimensional complex $L$ with fundamental group $G$. Let $a_i$ denote the number of $i$-simplices of $L$. The group $G$ acts freely on the universal cover $K$ of $L$ with finite quotient. Moreover, $H^1(K; \mathbb{Z}_2) = 0$. Since $H$ is of finite index in $G$, it acts freely on $K$ with finite quotient $L_1$. If $b_i$ is the number of $i$-simplices of $L_1$, then $b_i = [G : H]a_i$.

Therefore,

$$\delta(H) \leq \delta_H(L_1) = 2 \dim H^1(L_1; \mathbb{Z}_2) + b_0 + b_2 = 2 \dim \text{Hom}(H_1(L_1); \mathbb{Z}_2) + b_0 + b_2$$

$$= 2 \dim \text{Hom}(H_{ab}; \mathbb{Z}_2) + b_0 + b_2 \leq 2[G : H](r(G) - 1) + 1 + [G : H]a_0 + [G : H]a_2.$$

From the analysis above, we see that there are constants $A$, $B$ and $C$, such that

$$|V(Y)| \leq A \cdot [G : H](r(G) - 1) + B + C \cdot [G : H](a_0 + a_2).$$

Now,

$$\sum_{v \in V(Y)} (1 + \varphi(H_v)) \leq \sum_{v \in V(Y)} (1 + A_3 \cdot r(H_v) + A_4) \leq A_3(A_1 \cdot r(H) + A_2) + |V(Y)| \cdot (A_4 + 1).$$

Since $r(Y) \leq r(H)$, we conclude that there are constants $A'$, $B'$ and $C'$, such that

$$C'_{\max}^\varphi(H) \leq r(Y) + \sum_{v \in V(Y)} (1 + \varphi(H_v)) \leq A' \cdot [G : H](r(G) - 1) + B' + C' \cdot [G : H](a_0 + a_2).$$

It follows that the net $\left( \frac{C'_{\max}^\varphi(H)}{[G : H]} \right)_{H \in NG}$ is bounded above by $A' \cdot (r(G) - 1) + \frac{B'}{[G : H]} + C' \cdot (a_0 + a_2)$ and hence the volume $V^\varphi(G)$ of $G$ is finite. \qed
The volume of a finitely presented group (or even of an almost finitely presented group) with respect to a family of splittings as above, is a generalized Euler characteristic in the following sense:

**Proposition 2.8.** Let $H$ be a subgroup of finite index in a finitely presented group $G$. Then $V^\varphi(H) = [G : H]V^\varphi(G)$. In particular, if $V^\varphi(G) \neq 0$, then isomorphic subgroups of finite index of $G$ have the same index and every monomorphism from $G$ to $G$ with image of finite index is an automorphism.

**Proof.**

\[
V^\varphi(H) = \lim_{K \in \mathcal{N}_H} \max C(K) = [G : H] \lim_{K \in \mathcal{N}_H} C_{\max}(K) = [G : H]V^\varphi(G),
\]

where the first equality follows from the definition of $V^\varphi$ and the last one from the fact that for each subgroup $K$ of finite index in $G$, there is a normal subgroup $M$ of $G$ of finite index such that $M \subseteq K \cap H$.

We end this section by adapting the definition of complexity volume for finitely presented groups with infinitely many ends from [33] to the previous setting (which is essentially the same). It is the volume that is used mainly to define the weighted complexity of splittings of one-ended groups. We shall also present some calculations which lead to explicit formulae for the volume of finitely presented virtually torsion-free groups. These formulae show that our approach gives a well-behaved generalised Euler characteristic.

We consider reduced splittings of a finitely presented group over finite subgroups (without restrictions on vertex groups) and denote by $V_{\text{fin}}(G)$ the corresponding volume. It is worth noting that the maximal complexity among such reduced splittings of a finitely generated, torsion-free group $G$ is equal to the number of factors in the Grushko decomposition of $G$ (i.e. each factor is freely indecomposable).

**Remark 2.9.** It is not difficult to see that $V_{\text{fin}}(G)$ can be also defined using splittings over finite subgroups and vertex groups with at most one end, and that it is a finite number if one assumes only that $G$ is accessible (this follows from [25, Lemma 7.6]). In particular, by Linnell’s accessibility theorem $V_{\text{fin}}(G)$ is finite for each finitely generated virtually torsion free group $G$. The case of free products, i.e. splittings over the trivial group, is studied in more detail in [34].

**Examples 2.10.** (i) Let $G = G_1 \ast \cdots \ast G_n$ be a free product of torsion-free, freely indecomposable groups, satisfying the condition in Remark 2.5. The assumption on the free factors implies that the splitting of any finite index subgroup $H$ of $G$ inherited from the given one of $G$, is of maximal complexity. By [33, Prop. 3.2], $C_{\max}(H) - 1 = [G : H](n-1)$ from which it follows that $V_{\text{fin}}(G) = n - 1$. In particular, if $F_n$ is the free group of rank $n$, then $V_{\text{fin}}(F_n) = n - 1$.
(ii) Let $G$ be a finitely presented group with infinitely many ends, let $(\mathcal{G}, Y)$ be a non-trivial finite graph of groups decomposition of $G$ with finite edge groups such that each vertex group has at most one end, and let $X$ be the corresponding universal tree. Suppose that $G$ contains a normal, torsion-free subgroup $H$ of finite index, satisfying the condition in Remark 2.5 and that $v_1 \ldots, v_m$ are the vertices of $Y$ with finite vertex group. The subgroup $H$ acts edge freely on $X$ and each non-trivial vertex stabilizer is infinite and freely indecomposable being of finite index in the corresponding $G$-stabilizer. Therefore $V_{\text{fin}}(H) = n - 1$, where $n$ is the number of free factors of $H$ in the free product decomposition associated to the above action, i.e.

\[
\begin{align*}
    n - 1 & = C_X(H) - 1 = r(X/H) + |V_{\text{ndeg}}(X/H)| - 1 \\
    & = |E_+(X/H)| - |V(X/H)| + 1 + |V_{\text{ndeg}}(X/H)| - 1 \\
    & = |E_+(X/H)| - |V_{\text{ndeg}}(X/H)| = \sum_{e \in E_+(Y)} |H \backslash G / G_e| - \sum_{i=1}^m |H \backslash G / G_{v_i}|
\end{align*}
\]

It follows that

\[
V_{\text{fin}}(G) = \frac{V_{\text{fin}}(H)}{[G : H]} = \sum_{e \in E_+(Y)} \frac{1}{|G_e|} - \sum_{i=1}^m \frac{1}{|G_{v_i}|}
\]

(compare with [31, Cor. 3.4] and [4, Rem. IV.1.11(vii)]).

In particular, if $\Gamma = \Gamma_1 \ast_A \Gamma_2$, where $\Gamma_1$ and $\Gamma_2$ are finite groups, then $V_{\text{fin}}(\Gamma) = \frac{1}{|A|} - \frac{1}{|\Gamma_1|} - \frac{1}{|\Gamma_2|}$. Also, if $G = G_1 \ast \cdots \ast G_n \ast G_{n+1} \cdots \ast G_{n+m}$ is a free product of freely indecomposable groups, satisfying the condition in Remark 2.5, then

\[
V_{\text{fin}}(G) = n + m - 1 - \sum_{i=n+1}^{n+m} \frac{1}{|G_i|}.
\]

(iii) Let $G = A \ast_H B$ or $A \ast_H$, where $A$, $B$ are finitely presented virtually torsion-free groups satisfying the condition in Remark 2.5 and $H$ is a finite group. Following the proof of [28, Theorem 7.3], it is easy to show that $G$ is virtually torsion-free and therefore the above formula can be used. If $(\mathcal{G}_A, Y_A)$ and $(\mathcal{G}_B, Y_B)$ are finite graph of group decompositions of $A$ and $B$, respectively, then we can construct a graph of groups decomposition of $G$ from these by attaching one edge $e$, with associated edge group $G_e$ such that $|G_e| = |H|$. Now, the formula in the above example implies that $V_{\text{fin}}(A \ast_H B) = V_{\text{fin}}(A) + V_{\text{fin}}(B) + \frac{1}{|H|}$ and $V_{\text{fin}}(A \ast_H) = V_{\text{fin}}(A) + \frac{1}{|H|}$. Both formulas (up to a sign) have been obtained recently in [28], in a more general context, using covering space arguments.

Given a positive integer $m$, we denote by $\mathcal{S}_m$ the class of finite groups of order at most $m$ and by $V_{\text{fin},m}(G)$ the volume of a group $G$ with respect to $\mathcal{S}_m$ (i.e. using splittings of $G$ over $\mathcal{S}_m$). It is immediate from the definition that $V_{\text{fin}}(G) \geq \cdots \geq V_{\text{fin},m+1}(G) \geq V_{\text{fin},m}(G)$.

For later use we need the following lemma.

9
Lemma 2.11. Let $G$ be a finitely generated group. Then $V_{\text{fin},m}(G) \leq m \cdot r(G) + 1$. If, moreover, there is a bound $C$ on the orders of the finite subgroups of $G$, then $V_{\text{fin}}(G) \leq C \cdot r(G) + 1$.

Proof. Let $H$ be a subgroup of $G$ of finite index and $(\mathcal{H}, Y)$ a reduced splitting of $H$ over $S_m$ of maximal complexity. Then, by [11, Theorem 2.3] (see also [22]),

$$\sum_{e \in E_+(Y)} |H^e| \leq r(H)$$

and thus $|E_+(Y)| \leq m \cdot r(H)$. It follows that $C_{\text{max}}(H) \leq r(Y) + |V(Y)| = |E_+(Y)| + 1 \leq m \cdot r(H) + 1 \leq [G : H]m \cdot (r(G) - 1) + m + 1$. By dividing by $[G : H]$, we see that

$$\frac{C_{\text{max}}(H)}{[G : H]} \leq m \cdot (r(G) - 1) + \frac{m + 1}{[G : H]} \leq m \cdot r(G) + 1.$$

Since the net $\left(\frac{C_{\text{max}}(H)}{[G : H]}\right)_{H \in \mathcal{N}_G}$ is bounded above by $m \cdot r(G) + 1$, the first claim follows. The second claim is proved in the same way. \qed

3. The case of one-ended groups-Acylindrical splittings

Following Sela [28], an action of a group $G$ on a tree $T$ is said to be $k$-acylindrical if the stabilizer of any path of length greater than $k + 1$ is trivial. A graph of groups decomposition of $G$ is $k$-acylindrical if the action of $G$ on the corresponding universal tree is $k$-acylindrical. For example, canonical JSJ decompositions of hyperbolic groups and cyclic decompositions of limit groups are acylindrical (in fact, 2-acylindrical).

In this section, we consider reduced $k$-acylindrical splittings of one-ended finitely generated groups over finitely generated free abelian subgroups of rank at most $n$, and use weighted complexity.

Definition 3.1. Let $\mathcal{C}$ be a class of groups (closed under subgroups of finite index) and let $\varphi_0$ be a generalized Euler characteristic on $\mathcal{C}$ satisfying $\varphi_0(\Gamma) \leq A_3 \cdot r(\Gamma) + A_4$ for some constants $A_3$ and $A_4$. Let $G$ be a finitely generated one-ended group and $(G, Y)$ a reduced $k$-acylindrical splitting of $G$ such that each edge group $G_e$ is free abelian with $r(G_e) \leq n$, for a given positive integer $n$. If $G_v$ is a vertex group of $(G, Y)$, then we define $\varphi(G_v) = \varphi_0(G_v)$ if $G_v$ is in $\mathcal{C}$ and $\varphi(G_v) = 0$, otherwise.

Let $V^\varphi_{n}(G) = \lim_{H \in \mathcal{N}_G} \frac{C_{\text{max}}(H)}{[G : H]} \in [0, \infty]$ be the associated volume.

Remark 3.2. The inequality in the above definition is satisfied by $V_{\text{fin}}$ when restricted to finitely generated groups with a uniform bound on the order of finite subgroups (see Lemma 2.11) as well as by the deficiency and rank volumes defined in [24] (with $A_3 = 1$ and $A_4 = -1$).

Suppose for a moment that $G$ is finitely presented. Then, in order to have that $V^\varphi_{n}(G) < \infty$, it suffices, by Proposition 2.7, to show that the sum of ranks of non-degenerate vertex groups in any $k$-acylindrical splitting is bounded above by a linear
expression of the rank of the fundamental group. For this we need the following Theorem from [36].

**Theorem 3.3** ([36], Theorem 0.1). Let \((G, Y)\) be a \(k\)-acylindrical, minimal and without trivial edge groups splitting of a finitely generated group \(G\). Then

\[
r(G) \geq \frac{1}{2k + 1} \left( \sum_{v \in V(Y)} r(G_v) - \sum_{e \in E_+(Y)} r(G_e) + 2|E_+(Y)| + r(Y) + 1 + 3k - \lfloor k/2 \rfloor \right),
\]

where \(|k/2|\) is the largest integer less than or equal to \(k/2\).

Moreover, using acylindrical splittings, we can obtain finiteness of \(V_n^\phi(G)\) for a finitely generated group \(G\), bypassing the proof of Proposition 2.7, because of the following accessibility result.

**Theorem 3.4** ([28, 35]). Let \(G\) be a non-cyclic, freely indecomposable finitely generated group which admits a minimal and \(k\)-acylindrical action on a tree \(T\). Then the quotient graph \(T/G\) has at most \(2k(r(G) - 1) + 1\) vertices.

Now, combining acylindrical accessibility and Theorem 3.3, we are in position to prove the finiteness of \(V_n^\phi(G)\).

**Proposition 3.5.** Let \(G\) be a one-ended finitely generated group. Then \(V_n^\phi(G) < \infty\).

**Proof.** Let \(H\) be a finite index subgroup of \(G\) and \((\mathcal{H}, Y)\) a reduced \(k\)-acylindrical splitting of \(H\) over free abelian groups of rank at most \(n\). By [36, Prop. 2.2] we may assume that \((\mathcal{H}, Y)\) is minimal.

By Theorem 3.3, we have

\[
\sum_{v \in V(Y)} r(H_v) \leq (2k + 1)r(H) + \sum_{e \in E_+(Y)} r(H_e) - 2|E_+(Y)| - r(Y) \\
\leq (2k + 1)r(H) + n|E_+(Y)| - 2|E_+(Y)| - r(Y) \\
\leq (2k + 1)r(H) + (n - 2)(r(Y) + |V(Y)| - 1) + r(Y) \\
\leq (2k + 1)(r(H) + (n - 1)r(Y) + (n - 2)(|V(Y)| - 1) \\
\leq (2k + n)r(H) + (n - 2)(2k(r(H) - 1))
\]

where the last inequality follows from Theorem 3.4 (and the well-known fact that the rank of \(H\) bounds the rank of \(Y\)). It follows that there are constants \(A_1\) and \(A_2\), depending only on \(k\) and \(n\), such that \(\sum_{v \in V(Y)} r(H_v) \leq A_1 \cdot r(H) + A_2\). Hence

\[
C^\phi_{max}(H) \leq r(Y) + \sum_{v \in V(Y)} (1 + \phi(H_v)) \\
\leq r(Y) + \sum_{v \in V(Y)} \left(A_3 \cdot r(H_v) + A_4 + 1\right) \\
\leq r(Y) + A_3 \sum_{v \in V(Y)} r(H_v) + (A_4 + 1)|V(Y)| \\
\leq r(H) + A_3(A_1 \cdot r(H) + A_2) + (A_4 + 1)(2k(r(H) - 1) + 1)
\]
The above inequality implies that there are constants $A$ and $B$, depending only on $k$ and $n$, such that

$$C_{\text{max}}(\phi) \leq A \cdot r(H) + B \leq A \cdot [(G : H)(r(G) - 1) + 1] + B$$

and therefore

$$\frac{C_{\text{max}}(\phi)}{[G : H]} \leq A \cdot (r(G) - 1) + \frac{A + B}{[G : H]}$$

which proves the proposition.

**Remark 3.6.** In particular, the proof shows that there are constants $C$ and $D$ depending only on $k, n$ such that $V_{\text{nc}}^\phi(G) \leq C \cdot r(G) + D$.

Our first main result is the following.

**Theorem 3.7.** Suppose that $G$ is a finitely generated one-ended group which admits an acylindrical splitting $(\mathcal{G}, Y)$ over free abelian groups of bounded rank, say by a positive integer $n$. If there is a vertex group $G_v$ with $\phi(G_v) > 0$ (where $\phi$ is as above), then $V_{\text{nc}}^\phi(G) > 0$.

**Proof.** Since $\phi(G_v) > 0$ and each edge group is isomorphic to a proper subgroup of finite index, it follows that $v$ is a non-degenerate vertex. Let $X$ be the corresponding tree and $v_0$ a lift of $v$. Then

$$V(X) = \coprod_{u \in V(Y)} G/G_u \quad \text{and} \quad E_+(X) = \coprod_{e \in E_+(Y)} G/G_e.$$  

By replacing $X$ with the minimal $G$-invariant subtree $X_G$ of $X$, we may assume that the action is minimal. Note that $v \in X_G/G$ by \cite{X} Prop. 2.2. We also note that the collapse of an edge does not increase the “degree” of acylindricity. Thus, by collapsing any edge which is not a loop having degenerate endpoint, we can suppose further that $(\mathcal{G}, Y)$ is reduced.

Let $H$ be a normal subgroup of $G$ of finite index. If the action of $G$ on $X$ is $k$-acylindrical, then the action of $H$ on $X$ is $k$-acylindrical as well and each edge $H$-stabilizer is free abelian of rank at most $n$. Since $H$ is of finite index in $G$, it follows that $H_u$ is of finite index in $G_u$ as well, for each vertex $u$ of $X$. We note that the vertices and edges of the quotient graph $X/H$ have the following form:

$$V(X/H) = \coprod_{u \in V(Y)} H \backslash G/G_u \quad \text{and} \quad E_+(X/H) = \coprod_{e \in E_+(Y)} H \backslash G/G_e.$$  

Let $D_H$ be the set of $H$-degenerate vertices of $X$. Since $H$ is normal in $G$, the set $D_H$ is invariant under the action of $G$. If $D_H/G = \{v_1, \ldots, v_m\}$, then all vertices of type $H \backslash G/G_{v_i}$ for $i = 1, \ldots, m$ of $X/H$ are degenerate. For each $i \in \{1, \ldots, m\}$, we choose an edge $\psi(v_i) \in E_+(Y)$ such that $H_{i(\psi(v_i))} = H_{v_i}$ or $H_{ii(\psi(v_i))} = H_{v_i}$, and $H_{v_i} =$
$G_v \cap H = G_{\psi(v_1)} \cap H = H_{\psi(v_1)}$. In this way we get a map $\psi : \{v_1, \ldots, v_m\} \to E_+(Y)$ such that the inverse image of $\psi(v_i)$ under $\psi$ consists of at most two vertices. Now, if $\text{Im}(\psi) = \{e_1, \ldots, e_k\}$, then $\{v_1, \ldots, v_m\} = \psi^{-1}(e_1) \sqcup \cdots \sqcup \psi^{-1}(e_k)$.

It is easy to check that the collapse of a degenerate edge not a loop (i.e. the operation which makes a graph of groups reduced) can go up complexity by one. Therefore, if we denote the sum $\sum \varphi(H_u)$ over all non-degenerate vertices of $X/H$ by $\Phi$, then

$$C_{\max}^\varphi(H) \geq C^\varphi(\text{of a reduced obtained from } X/H)$$

$$\geq C^\varphi(X/H) = r(X/H) + |V_{\text{ndeg}}(X/H)| + \Phi$$

$$= |E_+(X/H)| - |V(X/H)| + 1 + |V_{\text{ndeg}}(X/H)| + \Phi$$

$$= \sum_{e \in E_+(Y)} |H \setminus G/G_e| - \sum_{i=1}^m |H \setminus G/G_{v_i}| + 1 + \Phi$$

$$= \sum_{e \in E_+(Y)} \frac{|G : H|}{|G_e : G_e \cap H|} - \sum_{i=1}^m \frac{|G : H|}{|G_{v_i} : G_{v_i} \cap H|} + 1 + \Phi,$$

where the last equality follows from the normality of $H$ in $G$. Hence

$$\frac{C_{\max}^\varphi(H)}{|G : H|} \geq \sum_{e \in E_+(Y)} \frac{1}{|G_e : G_e \cap H|} - \sum_{i=1}^m \frac{1}{|G_{v_i} : G_{v_i} \cap H|} + 1 + \Phi$$

$$= \sum_{e \notin \text{Im}(\psi)} \frac{1}{|G_e : G_e \cap H|} + \sum_{e \in \text{Im}(\psi)} \frac{1}{|G_e : G_e \cap H|} - \sum_{i=1}^m \frac{1}{|G_{v_i} : G_{v_i} \cap H|} + 1 + \Phi$$

$$= \sum_{e \notin \text{Im}(\psi)} \frac{1}{|G_e : G_e \cap H|} + \sum_{e \in \text{Im}(\psi)} \left( \frac{1}{|G_e : G_e \cap H|} - \sum_{i=1}^m \frac{1}{|G_{v_i} : G_{v_i} \cap H|} \right) + 1 + \Phi$$

$$= \sum_{e \notin \text{Im}(\psi)} \frac{1}{|G_e : G_e \cap H|} + \sum_{e \in \text{Im}(\psi)} \frac{1}{|G_e : G_e \cap H|} - \sum_{i=1}^m \frac{1}{|G_{v_i} : G_{v_i} \cap H|} + \frac{1}{|G : H|}.$$
Finally, $V_h^2(G) \geq \varphi(G_v) > 0$. \hfill \qed

Remark 3.8. Let $S$ be a class (closed under subgroups) consisting of small groups $G$ such that the set of indices of all finite index subgroups of $G$ is bounded above. It is not difficult to see that there are infinite groups satisfying this property by considering, for example, direct products $A \times F$, where $A$ is a (finitely generated) periodic simple group (such as a Tarski monster) and $F$ is a finite group. Let us denote by $V_h$ the volume defined using splittings of finitely presented groups over $S$. The same arguments as above (with $\varphi = 0$) can be used to show the following:

Theorem. Let $G$ be a finitely presented group which admits a nontrivial splitting over $S$ such that at least one vertex group has no subgroup of finite index contained in $S$. Then $V_h(G) > 0$.

The hypothesis that $G$ is finitely presented is only needed to insure that $V_h(G) < \infty$. In particular, when $S$ is the class of finite groups this gives another proof of [33, Theorem 4.7]: Let $G$ be a finitely presented group (or more generally an accessible group) with infinitely many ends. Then $V_{fin}(G) > 0$.

We close this section by proving two results concerning a form of strong Hopficity for finitely generated groups with positive volume. Recall that a group $G$ is said to be Hopfian if every surjective endomorphism of $G$ is necessarily an automorphism. A well-known result of Malcev states that every finitely generated residually finite group is Hopfian. A deep result of Sela [26] states that any torsion-free hyperbolic group is Hopfian. Note that both classes of groups (i.e. residually finite and hyperbolic) are closed under subgroups of finite index.

The following proposition can be thought of as an algebraic analogue of Theorem 2 in [24] (for the case of infinitely many ends see Proposition 3.11).

**Theorem 3.9.** Let $G$ be a one-ended finitely generated group such that $V_h^2(G) > 0$. If $f$ is an endomorphism of $G$ whose image is of finite index, then the decreasing sequence of images $f^k(G)$, $k \in \mathbb{N}$, is eventually constant. Moreover, if $G$ is torsion-free and every subgroup of finite index is Hopfian (e.g. $G$ residually finite or hyperbolic), then $f$ is an automorphism.

**Proof.** Since each subgroup $f^k(G)$ is of finite index in $G$, we have $[G : f^k(G)] = \frac{V_h^2(f^k(G))}{V_h^2(G)}$. By Remark 3.6 the numerator in this expression is bounded above by $C \cdot r(f^k(G)) + D$, and therefore by $C \cdot r(G) + D$. It follows that the set of indices $[G : f^k(G)]$, $k \in \mathbb{N}$ is bounded. On the other hand, $G$ (being finitely generated) has finitely many subgroups of bounded finite index, and the first part of the result follows. In particular, there is $k$ such that the restriction $f : f^k(G) \to f^k(G)$ is onto. Now, the Hopficity of $f^k(G)$ implies
that $f$ is an automorphism on $f^k(G)$. Therefore the intersection $\ker(f) \cap f^k(G)$ is trivial and thus $\ker(f)$ is finite. Since $G$ is torsion-free, we have that $\ker(f) = 1$ and $f$ is a monomorphism. Finally, $[G : f(G)] = \frac{V^\infty_f(f(G))}{V^\infty_n(G)} = 1$, which shows that $f$ is onto, hence it is an automorphism.

**Remark 3.10.** In the case of 0-acylindrical splittings (i.e. free products), each finite normal subgroup is trivial and therefore from the argument in the above proof follows Theorem 2 of [18] which can be stated as follows: Let $G$ be a finitely generated group with infinitely many ends (in order to have positive volume) which splits as a free product and whose each subgroup of finite index is Hopfian. If $f$ is an endomorphism of $G$ with image of finite index, then $f$ is an automorphism.

Our argument also leads to the following generalization of Hirshon’s result:

**Proposition 3.11.** Let $G$ be a finitely generated group with infinitely many ends. Suppose that $G$ splits as an amalgam or a HNN extension over a malnormal subgroup and that each subgroup of finite index of $G$ is Hopfian. Then each endomorphism of $G$ with image of finite index is an automorphism.

**Proof.** We first note that the malnormality assumption implies that each finite normal subgroup of $G$ is trivial. If $G$ is virtually torsion-free, then there is a bound $C$ on the orders of its finite subgroups. It follows, by Lemma 2.11, that $V_{\text{fin}}(H) \leq C \cdot r(H) + 1$ for each subgroup of $G$ of finite index. Furthermore, since $G$ has infinitely many ends, we have $V_{\text{fin}}(G) > 0$.

Suppose now that $G$ is not virtually torsion-free. Then in each splitting of $G$ over finite subgroups, there is at least one infinite vertex group. Thus, by Remark 3.8, there is a positive integer $m$ such that $V_{\text{fin},m}(G) > 0$ (in this case $S = S_m$ and $V_{\text{bi}}(G) = V_{\text{fin},m}(G)$). Again by Lemma 2.11, we have $V_{\text{fin},m}(H) \leq m \cdot r(H) + 1$ for each subgroup of $G$ of finite index. Thus, in either case, the argument in the proof of Theorem 3.9 applies.

4. Cyclic splittings of one-ended hyperbolic groups

We apply the results of the previous section to cyclic splittings (i.e. each edge group is infinite cyclic) of one-ended, torsion-free hyperbolic groups using weighted complexity with $\varphi_0 = V_{\text{fin}}$ for finitely presented, torsion-free vertex groups (see Lemma 2.11 and Definition 3.1). We write $V^\text{hyp}_1$ to denote the corresponding volume. Since such a splitting is acylindrical, Theorem 3.7 immediately implies:

**Proposition 4.1.** Let $G$ be a torsion-free, one-ended hyperbolic group which admits a cyclic splitting with a free non-abelian vertex group. Then $V^\text{hyp}_1(G) > 0$. 

15
Example 4.2. Let \((G,Y)\) be a cyclic splitting of a one-ended, torsion-free hyperbolic group such that each vertex group \(G_v\) is a non-abelian free group of rank \(n_i\). Suppose that \(E_+(Y) = \{e_1, \ldots, e_k\}\) and that \(V(Y) = \{v_1, \ldots, v_m\}\). We denote the edge group \(G_{e_i}\) by \(H_i\) and the vertex group \(G_{v_i}\) by \(G_i\). Let \(X\) be the universal tree and \(H\) a normal subgroup of \(G\) of finite index. Then each vertex of \(X\) is non-degenerate under the action of \(H\) and we have

\[
C^\phi(X/H) \geq r(X/H) + \sum_{v \in V_{\text{adj}}(X/H)} \left( 1 + V_{\text{fin}}(H_v) \right)
\]

where

\[
C^\phi(X/H) = |E_+(X/H)| + 1 + \sum_{v \in X/H} V_{\text{fin}}(H_v)
\]

\[
= \sum_{i=1}^k |H \setminus G/H|_i + \sum_{i=1}^m |H \setminus G/G_i| \cdot V_{\text{fin}}(H \cap G_i) + 1
\]

\[
= \sum_{i=1}^k \frac{|G:H|}{|G_i:H_i|} + \sum_{i=1}^m \frac{|G:H|}{|G_i:H_i|} \cdot V_{\text{fin}}(H \cap G_i) + 1
\]

\[
= \sum_{i=1}^k \frac{|G:H|}{|G_i:H_i|} + \sum_{i=1}^m |G : H| \cdot (r(G_i) - 1) + 1,
\]

where the last equality follows from the formula \(V_{\text{fin}}(H \cap G_i) = |G_i : H \cap G_i| \cdot V_{\text{fin}}(G_i)\) and the fact that \(V_{\text{fin}}(F) = r(F) - 1\), if \(F\) is a free group. Thus

\[
\frac{C^\phi(X/H)}{|G : H|} \geq \sum_{i=1}^m (r(G_i) - 1) + \sum_{i=1}^k \frac{1}{|H_i : H \cap H_i|} + \frac{1}{|G : H|},
\]

and hence \(V^{\text{hyp}}_1(G) \geq \sum_{i=1}^m (r(G_i) - 1)\).

In particular, if \(G\) is the fundamental group of a closed orientable surface \(S_g\) of genus \(g \geq 2\), then the pair of pants decomposition of \(S_g\) gives a cyclic splitting for \(G\) with \(2g - 2\) vertices and \(3g - 3\) edges such that each vertex group is free of rank two. Therefore \(V^{\text{hyp}}_1(G) \geq 2(g - 1) = -\chi(S_g)\), where \(\chi(S_g)\) denotes the (usual) Euler characteristic of \(S_g\).

A particularly important class of cyclic splittings of one-ended torsion-free hyperbolic groups are JSJ decompositions. There are two approaches, almost equivalent, both appropriate for our purpose. The first is due to Sela [27] and the other due to Bowditch [3]. We record some of the properties of Bowditch’s JSJ decomposition which is completely canonical.

Let \(G\) be a torsion-free, one-ended hyperbolic group. Then there is a cyclic splitting of \(G\), called the JSJ decomposition of \(G\), such that each vertex group is either a maximal infinite cyclic subgroup of \(G\), a maximal hanging Fuchsian subgroup, or a rigid vertex group. These three types of vertices are mutually exclusive. A maximal hanging Fuchsian (MHF) subgroup is a non-cyclic group which is isomorphic to the fundamental group of a compact surface with boundary and therefore a non-abelian free group. Edge groups incident to a (MHF) vertex group are precisely the peripheral subgroups associated to the boundary components. Rigid vertex groups do not admit splittings over cyclic subgroups relative to the incident edge groups.

Remark 4.3. Note that although the JSJ decomposition of the fundamental group of a closed orientable surface \(S_g\) of genus \(g \geq 2\) is trivial, by the example above we have that its volume is positive.
Sela [27] (see also [21]) used the JSJ decomposition of a one-ended hyperbolic group $G$ to obtain a description of its outer automorphism group $\text{Out}(G)$, from which one can deduce the existence of (MHF) vertices in the case where $\text{Out}(G)$ is not virtually finitely generated free abelian.

Since a (MHF) subgroup is a non-abelian free group, by Proposition 4.1 we obtain:

**Corollary 4.4.** Let $G$ be a torsion-free, one-ended hyperbolic group. If the JSJ decomposition of $G$ contains a (MHF) subgroup, then $V_1^{hyp}(G) > 0$. In particular, if the outer automorphism group $\text{Out}(G)$ of $G$ is not virtually finitely generated free abelian, then $V_1^{hyp}(G) > 0$.

### 5. One-ended limit groups

A group $G$ is fully residually free (or $\omega$-residually free) if, for each finite subset $X$ of $G$, there exists a homomorphism from $G$ to a free group $F$ which is injective on $X$. A limit group is a finitely generated fully residually free group. Fully residually free groups have been extensively studied by Kharlampovich and Myasnikov [17, 18], and by Remeslennikov [23] (under the name $\exists$-free groups).

The term “limit group” was introduced by Sela [29] and reflects the fact that they are obtained from limits of sequences of homomorphisms of an arbitrary finitely generated group to a free group.

Examples of limit groups include finitely generated free groups, finitely generated free abelian groups and fundamental groups of closed hyperbolic surfaces with Euler characteristic less than -1.

Here we use the characterization of limit groups as finitely generated subgroups of $\omega$-residually free tower groups [31, 38] (see also [1] for a proof).

**Definition 5.1 ([29]).** An $\omega$-rft space of height 0 is the wedge of finitely many circles, $n$-dimensional tori $\mathbb{T}^n$ ($n \geq 2$) and closed hyperbolic surfaces of Euler characteristic less than -1.

An $\omega$-rft space $X_h$ of height $h$ is obtained from an $\omega$-rft space $X_{h-1}$ of height $h-1$ by attaching a block of one of the two following types:

1. **Abelian block.** $X_h$ is the quotient space $X_{h-1} \sqcup \mathbb{T}^n/\sim$, where a coordinate circle in $\mathbb{T}^n$ is identified with a nontrivial loop in $X_{h-1}$ that generates a maximal abelian subgroup in $\pi_1(X_{h-1})$ (in fact infinite cyclic).

2. **Quadratic block.** $X_h = X_{h-1} \sqcup \Sigma/\sim$, where $\Sigma$ is a connected compact hyperbolic surface of Euler characteristic at most -2 or a punctured torus whose each boundary component is identified with a homotopically non-trivial loop in $X_{h-1}$. It is also required that there exists a retraction $r : X_h \to X_{h-1}$ with $r_*(\pi_1\Sigma)$ non-abelian.
Applying the Seifert-van Kampen theorem to the decomposition defined by the final block, we see that the fundamental group of an \(\omega\)-rft space \(X_h\) of height \(h \geq 1\) admits a cyclic splitting as a 2-vertex graph of groups where one of the vertex groups is \(\pi_1(X_{h-1})\) and the other free or free abelian of finite rank at least 2. Moreover, edge groups are maximal infinite cyclic subgroups in the second type vertex groups. The malnormality of peripheral subgroups in \(\Sigma\) as well as the malnormality of maximal abelian subgroups in limit groups imply that the above splitting is 2-acylindrical (see [4, Lemma 1.4]).

The height of a limit group \(G\) is the minimal number \(h\) for which there is an \(\omega\)-rft space \(X_h\) of height \(h\) such that \(\pi_1(X_h)\) contains a subgroup isomorphic to \(G\).

We consider cyclic 2-acylindrical splittings of one-ended limit groups using weighted complexity with \(\varphi_0 = V_{\text{fin}}\) for finitely generated vertex groups with infinitely many ends. Let us denote by \(V_{\text{lim}}^1\) the associated volume.

Let \(G\) be a one-ended limit group which is isomorphic to the fundamental group of an \(\omega\)-rft space \(X_h\) such that the final block added is quadratic. Then the 2-acylindrical splitting of \(G\), defined as above by the final block, contains a free non-abelian group and thus, by Theorem 3.7, we have \(V_{\text{lim}}^1(G) > 0\). The following shows that the same is true for any one-ended finitely generated subgroup of height \(h\) of \(G\).

**Proposition 5.2.** Let \(H\) be a one-ended limit group of height \(h \geq 1\) and \(X_h\) an \(\omega\)-rft space such that the fundamental group of \(X_h\) contains a copy of \(H\). If the final block (in the construction of \(X_h\)) is quadratic, then \(V_{\text{lim}}^1(H) > 0\).

**Proof.** In this case it follows as in the proof of [4, Lemma 1.4], that \(H\) admits a two-acylindrical splitting as before which has a non-abelian free group and hence, by Theorem 3.7, \(V_{\text{lim}}^1(H) > 0\). \(\square\)

In [31] (see also [19]), Sela proved that a finitely generated group has the same elementary theory as a non-abelian free group if and only if it is the fundamental group of an \(\omega\)-rft space \(X_h\), whose construction involves only quadratic blocks. Such groups are called elementarily free.

**Corollary 5.3.** If \(G\) is a one-ended elementarily free group, then \(V_{\text{lim}}^1(G) > 0\).

**Remark 5.4.** As before, we consider cyclic 2-acylindrical splittings of one-ended limit groups of height \(h\). If we could show that the height of a limit group is equal to the height of any finite index subgroup, then we could define a volume \(V_{\text{lim}}^{1,h}\) inductively as follows:

If \(G\) is a one-ended limit group of height 0, then \(G\) is either a free abelian group of finite rank and thus \(V_{\text{lim}}^{1,0}(G) = 0\), or a hyperbolic surface group and we define \(V_{\text{lim}}^{1,0}(G) = V_{1}^{\text{hyp}}(G) > 0\).

If \(G\) is a one-ended limit group of height \(h > 0\), then we use for any cyclic 2-acylindrical splitting \((G, Y)\) weighted complexity defined by \(\varphi(G_v) = V_{\text{fin}}(G_v) > 0\), if \(G_v\) is a vertex.
group with infinitely many ends, and \( \varphi(G_v) = V_{1}^{l,k}(G_v) \), if \( G_v \) is a one-ended limit group of height \( k \leq h - 1 \). We denote by \( V_{1}^{l,h}(G) \) the associated volume.

Note that, by Lemma 2.11 and Remark 3.7, hypothesis 2.6 is satisfied in each step. This means that \( V_{1}^{l,h}(G) < \infty \).

By [4, Lemma 1.4], each one-ended limit group \( G \) of height \( h \) admits a cyclic (acylindrical) splitting such that at least one vertex group is a non-abelian limit group of height \( \leq h - 1 \). Thus it would follow (inductively) from Theorem 3.7 that \( V_{1}^{l,h}(G) > 0 \) for any non-abelian, one-ended limit group.

It is worth noting that if \( H \) is a finite index subgroup of a one-ended, limit group \( G \) of height 1, then the height of \( H \) is also 1. Indeed, this follows immediately by a well-known theorem of Eckmann and Müller which states that each virtual surface group is a surface group. Therefore, the above discussion shows the following:

**Corollary 5.5.** Let \( G \) be a one-ended, non-abelian limit group of height 1. Then \( V_{1}^{l,1}(G) > 0 \).

### 6. Acylindrical splittings of CSA groups over abelian subgroups

In this section, we generalize the preceding results to the case of one-ended CSA groups. Recall that a group \( G \) is CSA (or conjugately separated abelian) if the maximal abelian subgroups of \( G \) are malnormal. If \( \Gamma \) is a torsion-free group which is hyperbolic relative to a (finite) family of finitely generated free abelian subgroups, then each \( \Gamma \)-limit group (or equivalently, by Theorem [13, Theorem 5.2], each finitely generated and fully residually \( \Gamma \)) is CSA (see [12, Lemma 6.9]). We refer to [12, 13] for the basic definitions and properties concerning \( \Gamma \)-limit groups.

In particular, limit groups, torsion-free hyperbolic groups and more generally finitely generated subgroups of torsion-free groups hyperbolic relative to a collection of free abelian subgroups, are CSA.

Let \( G \) be a finitely generated group. In [14], Guirardel and Levitt gave a method for constructing an acylindrical \( G \)-tree \( T_c \), called the *tree of cylinders*, from any \( G \)-tree \( T \) with edge stabilizers in a class \( \mathcal{A} \) closed under conjugation. Using trees of cylinders they obtain \( \text{Out}(G) \)-invariant splittings which in many cases give a lot of information about the structure of \( \text{Out}(G) \) (in the same way as in the case of torsion-free hyperbolic groups). The vertex stabilizers of \( T_c \) are divided into two types: rigid and flexible. A vertex stabilizer \( G_v \) is rigid if, whenever \( G_v \) acts on a tree with edge stabilizers in \( \mathcal{A} \), then it stabilizes a vertex, and flexible otherwise (see [12] for more details and for a description of flexible vertex stabilizers when \( \mathcal{A} \) consists of slender groups).

If we restrict to the case of splittings of one-ended CSA groups over abelian groups, we have the following result.
Theorem 6.1. ([1], Prop. 6.3], [2], Theorem 9.5]) Let $G$ be a finitely generated, one-ended and torsion-free CSA group, let $T$ be a $G$-tree with finitely generated abelian edge stabilizers, and let $T_c$ be the associated tree of cylinders.

1. Edge stabilizers of $T_c$ are non-trivial and abelian.
2. $T$ and $T_c$ have the same non-abelian vertex stabilizers; non-abelian flexible vertex stabilizers are fundamental groups of compact surfaces with boundary.
3. $T_c$ is 2-acylindrical.

Let $G$ be a finitely generated, torsion-free, one-ended CSA group such that abelian subgroups of $G$ are finitely generated of bounded rank. For each acylindrical splitting of such a group over abelian subgroups of bounded rank $n$, we use weighted complexity with $\varphi_0 = V_{\text{fin}}$ (see Definition 3.1) and we denote the associated volume by $V_n^{\text{CSA}}$.

Proposition 6.2. Let $G$ be a finitely generated, torsion-free, one-ended CSA group such that abelian subgroups of $G$ are finitely generated of bounded rank. Suppose that $G$ admits a splitting over abelian subgroups such that the associated tree of cylinders $T_c$ has a non-abelian flexible vertex stabilizer $G_v$. Then there exists $n$ such that $V_n^{\text{CSA}}(G) > 0$.

Proof. Let $n = n(G)$ be a positive integer that bounds the ranks of the abelian subgroups of $G$. In particular, $n$ bounds the ranks of the abelian subgroups of each subgroup $H$ of $G$. The splitting $(G, T_c/G)$ of $G$ corresponding to the action on $T_c$ is acylindrical, $\varphi_0(G_v) = r(G_v) - 1 > 0$ ($G_v$ being non-abelian free), so Theorem 3.7 applies.

Lemma 6.3. Let $\Gamma$ be as above. If $G$ is a $\Gamma$-limit group, then abelian subgroups of $G$ are finitely generated of bounded rank.

Proof. Abelian subgroups of $G$ are finitely generated by [3], Cor. 5.12]. Following the proof (and the terminology) of [3] Prop. 5.11], there is a sequence of shortening quotients of $G$:

$$G \to L_2 \to \cdots \to L_s$$

such that $L_s$ is a free product of a finitely generated free group and a finite collection of finitely generated subgroups of $\Gamma$. Moreover, $L_i$ is a free product of a finitely generated
free group and freely indecomposable $\Gamma$-limit groups such that each free factor of $L_i$ is either (i) a finitely generated free group; (ii) a finitely generated subgroup of $L_{i+1}$; or (iii) can be represented as the fundamental group of a finite graph of groups in which the vertex groups are finitely generated subgroups of $L_{i+1}$ and finitely generated free groups, and the edge groups are finitely generated abelian subgroups.

If $A$ is a non-cyclic abelian subgroup of $L_0$, then $A$ is contained in a free factor and therefore in $\Gamma$. It follows that its rank is bounded by the maximal rank of the parabolic subgroups, say $r(A_{i_0})$. If now $A$ is a non-cyclic abelian subgroup of $L_i$, then $A$ is contained in a free factor and thus is either a subgroup of $L_{i+1}$ or acts non-trivially on a tree with vertex groups finitely generated subgroups of $L_{i+1}$ and edge groups finitely generated abelian. In the latter case $A$ acts on the associated tree and the axis of any hyperbolic element of $A$ is an $A$-invariant line. Thus we have an homomorphism from $A$ to $\mathbb{Z}$ whose kernel is an abelian subgroup of $L_{i+1}$. It follows that, in each case, the rank of $A$ is bounded above by the rank of the kernel plus 1. Finally, the rank of an abelian subgroup of $G$ is bounded by $r(A_{i_0}) + s - 1$.

By combining the preceding lemma with Proposition 6.2 we obtain:

**Proposition 6.4.** Let $\Gamma$ be a torsion-free group which is hyperbolic relative to a finite collection of free abelian subgroups and let $G$ be a one-ended $\Gamma$-limit group. Suppose that $G$ admits a splitting over abelian subgroups which has a non-abelian vertex group isomorphic to the fundamental group of a compact surface with boundary. Then there exists a positive integer $n = n(G)$ such that $V_n^{CSA}(G) > 0$.

**Corollary 6.5.** Let $\Gamma$ and $G$ be as in the preceding proposition. Then each endomorphism of $G$ with image of finite index is an automorphism.

**Proof.** By [13, Cor. 5.5], each finitely generated subgroup of a $\Gamma$-limit group is Hopfian, and Theorem 3.3 applies.

It is proved in [3, Theorem 3.1] that each endomorphism of $\Gamma$ whose image is of finite index in $\Gamma$, is a monomorphism.

**Corollary 6.6.** Let $G$ be a one-ended group which is either (i) a hyperbolic group with $Out(G)$ not virtually finitely generated free abelian; (ii) an elementarily free group; or (iii) a non-abelian limit group of height 1. Then every endomorphism of $G$ with image of finite index is an automorphism.

**Proof.** It follows from Corollaries 4.4, 5.3 and 5.5, that in each case the (corresponding) volume of $G$ is positive.
7. Acknowledgements

The author thanks the referee for the careful reading of the manuscript and useful suggestions.

References

[1] E. Alibegović and M. Bestvina, Limit groups are $\text{CAT}(0)$, J. London Math. Soc. (2) 74 (2006), 259–272.

[2] M. Bestvina and M. Feighn, Bounding the complexity of simplicial group actions on trees, Invent. Math. 103 (1991), No.3, 449–469.

[3] B. Bowditch, Cut points and canonical splittings of hyperbolic groups. Acta Math., 180 (2) (1998), 145–186.

[4] M.R. Bridson and J. Howie, Normalisers in limit groups, Math. Ann. 337 (2007), 385–394.

[5] M.R. Bridson and J. Howie, Subgroups of direct products of elementarily free groups, Geom. Funct. Anal. 17 (2007), 385–403.

[6] M. R. Bridson, D. Groves, J. A. Hillman, and G. J. Martin, Cofinitely Hopfian groups, open mappings and knot complements, Groups Geom. Dyn. 4 (2010), no. 4, 693–707.

[7] M. R. Bridson and D. H. Kochloukova, Volume gradients and homology in towers of residually-free groups, Math. Ann. 367 (2017), no. 3-4, 1007–1045.

[8] I. M. Chiswell, Euler characteristics of discrete groups, in Groups: topological, combinatorial and arithmetic aspects (ed. T. W. Müller), London Math. Soc. Lecture Note Ser. 311, Cambridge Univ. Press (2004), 106–254.

[9] W. Dicks and M.J. Dunwoody, “Groups acting on graphs”, Cambridge University Press, 1989.

[10] M.J. Dunwoody, The accessibility of finitely presented groups, Invent. Math. 81 (1985), 449–457.

[11] M.J. Dunwoody, Folding sequences, in The Epstein birthday schrift, Geom. Topol. (Coventry) (1998), p. 139–158 (electronic).

[12] D. Groves, Limit groups for relatively hyperbolic groups. I. The basic tools, Algebr. Geom. Topol. 9 (2009), no. 3, 1423–1466.
[13] D. Groves, Limit groups for relatively hyperbolic groups, II: Makanin-Razborov diagrams, Geom. Topol. 9 (2005), 2319–2358.

[14] V. Guirardel and G. Levitt, Trees of cylinders and canonical splittings, Geom. Topol., 15 (2) (2011), 977–1012.

[15] V. Guirardel and G. Levitt, JSJ decompositions of groups, Astérisque No. 395 (2017), viii+165 pp. ISBN: 978-2-85629-870-1.

[16] R. Hirshon, Descending chains in free products, Math. Z. 164 (1979), 293–296.

[17] O. Kharlampovich and A. Myasnikov, Irreducible affine varieties over a free group. I. Irreducibility of quadratic equations and Nullstellensatz, J. Algebra 200 (1998), no. 2, 472–516.

[18] O. Kharlampovich and A. Myasnikov, Irreducible affine varieties over a free group. II. Systems in triangular quasi-quadratic form and description of residually free groups, J. Algebra 200 (1998), no. 2, 517–570.

[19] O. Kharlampovich and A. Myasnikov, Elementary theory of free nonabelian groups, J. Algebra 302 (2006), 451–552.

[20] M. Lackenby, Expanders, rank and graphs of groups, Israel J. Math. 146 (2005), 357–370.

[21] G. Levitt. Automorphisms of hyperbolic groups and graphs of groups. Geom. Dedicata, 114 (2005), 49–70.

[22] P. A. Linnell, On accessibility of groups. J. Pure Appl. Algebra, 30 (1) (1983), 39–46

[23] V. N. Remeslennikov, $\exists$-free groups, Sibirsk. Mat. Zh. 30 (1989), 193–197.

[24] A. Reznikov, Volumes of discrete groups and topological complexity of homology spheres , Math. Ann. 306 (1996), 547–554.

[25] P. Scott and C. T. C. Wall, Topological methods in group theory, in Homological group theory (ed. C. T. C. Wall), London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press (1979), 137–203.

[26] Z. Sela, Endomorphisms of hyperbolic groups: I, Topology 38 (1999), 301–321.

[27] Z. Sela, Structure and rigidity in (Gromov) hyperbolic groups and discrete groups in rank 1 Lie groups. II, Geom. Funct. Anal., 7 (3) (1997), 561–593.
[28] Z. Sela, Acylindrical accessibility for groups, Invent. Math. 129, no.3, (1997), 527–565.

[29] Z. Sela, Diophantine geometry over groups I: Makanin-Razborov diagrams. Publ. Math. Inst. Hautes Etudes Sci., 93 (2001), 31–105.

[30] Z. Sela, Diophantine geometry over groups II: Completions, closures and formal solutions, Israel J. Math. 134 (2003), 173–254.

[31] Z. Sela, Diophantine geometry over groups VI: The elementary theory of free groups, Geom. Funct. Anal. 16 (2006), 707–730.

[32] J.-P. Serre, “Trees”, Springer-Verlag, Berlin-Heidelberg-New York, 1980.

[33] M. Sykiotis, On subgroups of finite complexity in groups acting on trees, J. Pure Appl. Algebra 200 (2005), 1–23.

[34] K. Tsouvalas, Euler characteristics on virtually free products, http://front.math.ucdavis.edu/1511.09436

[35] R. Weidmann, The Nielsen method for groups acting on trees, Proc. Lond. Math. Soc. (3) 85, no. 1, (2002), 93–118.

[36] R. Weidmann, A rank formula for acylindrical splittings, Ann. Fac. Sci. Toulouse Math. (6) 24 (2015), no. 5, 1057–1078.

Department of Mathematics
National and Kapodistrian University of Athens
Panepistimioupolis, GR-157 84, Athens, Greece

e-mail: msykiot@math.uoa.gr