FRW cosmological model inside an isolated Schwarzschild black hole

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Using the canonical quantum theory of spherically symmetric pure gravitational systems, we present a direct correspondence between the Friedmann-Robertson-Walker (FRW) cosmological model in the interior of a Schwarzschild black hole and the \(n\)th energy eigenstate of a linear harmonic oscillator. Such type of universe has a quantized mass of the order of the Planck mass and harmonic oscillator wave functions.

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I. INTRODUCTION

In the absence of a fundamental understanding of physics at very high energies and, in particular, in the absence of a consistent quantum theory of gravity, there is no hope, at present, to meet an understanding of the quantum origin of the Universe in a definitive way. However, in order to come nearer to this seemingly unattainable goal, it appears desirable to develop highly simplified, but consistent models, which contain as many as possible of those features which are believed to be present in a future complete quantum theory of gravity. It is believed that black holes may play an important role in our attempts to shed some light on the nature of a quantum theory of gravity.

Thus, in the framework of quantum gravity, black holes must be treated as quantum objects. As such, they are characterized by quantum numbers like mass, electric charge and angular momentum. For neutral, non-rotating Schwarzschild black hole, the only quantum number which is left is the mass \(M\). Classically, it is related to the area \(A\) of the black hole horizon by the relation \(A = \frac{16\pi G^2 M^2}{c^4}\), where \(G\) is the Newtonian gravitational constant. Important questions in black hole physics are what the spectrum of \(A\) looks like and what the degeneracies of states are for a given values of \(A\).

The quantization of black holes was first proposed by Bekenstein some years ago \([1]\). The fundamental idea of his work is the remarkable observation that the horizon area of a non-extremal black hole behaves as a classical adiabatic invariant. But in the spirit of Ehrenfest principle \([2]\), any classical adiabatic invariant should correspond to a quantum entity with discrete spectrum. Bekenstein conjectured that the horizon area of a quantum black hole should have a discrete spectrum with uniformly spaced eigenvalues of the form

\[
A_n = \gamma l_{pl}^2 n, \quad n = 1, 2, 3, ...
\]

where \(\gamma\) is a dimensionless constant to be determined, and \(l_{pl} = (\frac{\hbar}{c})^{1/2}\) is the Planck length.
Bekenstein’s proposal implies that the energy eigenvalues corresponding to the stationary states of the black holes are

\[ E_n = \sigma \sqrt{n E_{pl}}, \quad n = 1, 2, \ldots, \quad E_{pl} = \sqrt{\frac{\hbar c^5}{G}}, \]

where \( \sigma = \sqrt{\frac{2}{16\pi}} \) is of the order of unity.

Using a combination of thermodynamics and statistical physics arguments it was found in [3] that the dimensionless constant \( \gamma \) should be of the form \( \gamma = \frac{2}{n} \ln \alpha \), where \( \alpha \) is the degeneracy factor of the \( n \)th area level. Recently, Hod [4] employed Bohr’s correspondence principle and found evidence in favor of the value \( \alpha = 3^n \).

On the other hand, Rosen [5] put the equations of General Relativity for the case of a closed homogeneous isotropic universe in the form of a Schrödinger equation for the s-state of a hydrogen-like atom, and was able to obtain the relation \( m_n = \sqrt{n\pi}M_{pl} \) for the quantization of the mass spectrum.

The main goal of this work is to obtain a time independent Schrödinger equation for the case of the closed FRW model inside a black hole horizon. This is done following the canonical quantization procedure by means of which, we show that our system looks like a linear quantum oscillator. Besides, the area and energy quantum spectra are obtained, as well as, the wave function of the system.

II. THE CANONICAL HAMILTONIAN

We will start with the classical closed FRW universe inside a Schwarzschild black hole horizon, which is the simplest model which exhibits the collapse phenomenon. It must be filled either with gravitational radiation or with some other form of energy. Classically, the only black hole observable parameter for an asymptotic external observer at rest is the black hole mass \( M \). It is well known, that this mass \( M \) measured by this kind of observer coincides with the mass \( M \) in the usual expression of the Schwarzschild metric. Then, the observer can define the concept of energy \( E \) of the black hole as \( E = M c^2 \) [6,7], where \( c \) is the velocity of light in vacuum.

To know the energy spectrum of the black hole, the observer should derive a quantum equation for the system. In this work we will consider a time-dependent metric inside a black hole horizon.

In the interior region, we write the line element as in [8] (using comoving hyperspherical coordinates \( (\chi, \theta, \phi) \) for the star’s interior and putting the origin of coordinates at the star’s center)

\[ ds^2 = -N^2(t)c^2dt^2 + R^2(t) \left[ d\chi^2 + \sin^2 \chi \left( d^2 \theta + \sin^2 \theta d^2 \phi \right) \right], \]

where \( N(t) \) and \( R(t) \) are the lapse function and the scale factor, respectively.

Outside the black hole horizon the line element is

\[ ds^2 = -N^2(t) \left( 1 - \frac{R_s}{r} \right) c^2dt^2 + \frac{1}{1 - \frac{R_s}{r}} dr^2 + r^2 \left( d^2 \theta + \sin^2 \theta d^2 \phi \right), \]
where $R_s = \frac{2MG}{c^2}$ is the Schwarzschild radius. The connection between the two metrics (3) and (4) is made by the matching conditions on the star surface [8].

The spatial volume for the metric (3) is

$$V = R^3 \int_0^{\chi_0} d\chi \int_0^\pi d\theta \int_0^{2\pi} d\phi \sin^2 \chi \sin \theta = R^3 \pi^2,$$

(5)

where $\chi_0$ is determined by the matching conditions, which for the case of black hole give $\chi_0 = \frac{\pi}{2}$.

We consider a pure gravity system with Einstein action and the corresponding term for the total energy [6]

$$S = \int \left[ -\frac{c^2}{2GN} R \dot{R}^2 + \frac{c^4}{2G} NR - NE_s \right] dt,$$

(6)

where $E_s = Mc^2$ is the ADM energy. Here, the last term in the action (6) corresponds to the surface integral at spatial infinity, which leads to the Schwarzchild black hole mass [6].

The action (6) preserves the invariance under time reparametrization

$$\delta t = a(t),$$

(7)

if the transformations of $N(t)$ and $R(t)$ are defined as

$$\delta N = \frac{d(aN)}{dt}, \quad \delta R = a \frac{dR}{dt}.$$

(8)

Note that if we take the lapse function as

$$N(t) = \tilde{N}(t)R(t) \frac{c^2}{M_{pl}G},$$

(9)

and substituting into (6), we have the following invariant action

$$S = \int \left[ -\frac{M_{pl}}{2N} \dot{R}^2 + \frac{c^6}{2M_{pl}G^2} \tilde{N}R^2 - \tilde{N} \frac{M_{pl}^4}{G} \tilde{R}^2 \right] dt.$$

(10)

Using the relations (8,9), it is easy to show that $\tilde{N}(t)$ transforms as

$$\delta \tilde{N} = \frac{d(a\tilde{N})}{dt}.$$

(11)

Proceeding with the Hamiltonian analysis we define the usual canonical momentum conjugate to the $R(t)$ coordinate, $P_R = \frac{\partial L}{\partial \dot{R}}$ and performing the Lagrange transformation, we can obtain the following canonical Hamiltonian

$$H_{can} = \tilde{N} \left[ -\frac{P_R^2}{2M_{pl}} - \frac{c^6}{2M_{pl}G^2} R^2 + \frac{M_{pl}^4}{2G} \tilde{R}^2 \right].$$

(12)
where $\omega_0 = \frac{c^3}{M_{pl}G}$ is the fundamental frequency of the system. This form of the canonical Hamiltonian explains the fact, that the lapse function $\tilde{N}(t)$ is a Lagrange multiplier, which enforces the only first class constraint $H = 0$. The latter manifests the invariance of the action under reparametrization transformations (7,8). According to the Dirac’s constraint quantization procedure, the wave function must be annihilated by the operator version of the classical constraint.

We transform Eq. (12) by defining $\xi = R - \frac{MG}{c^2}$, thus its momentum conjugate becomes $P_\xi = P_R$ and the constraint at the classical level reads as follows

$$H_{\text{can}} = \tilde{N}H = \tilde{N} \left[ -\frac{P_\xi^2}{2M_{pl}} - \frac{M_{pl}^2\omega_0^2\xi^2}{2} + \frac{M}{2M_{pl}}Mc^2 \right] = 0, \quad (13)$$

that can be also rewritten as

$$\frac{P_\xi^2}{2M_{pl}} + \frac{M_{pl}^2\omega_0^2\xi^2}{2} = \frac{M}{M_{pl}} \frac{M}{2} \frac{E_s}{M_{pl}} = \frac{M}{M_{pl}} \frac{E_s}{2}, \quad (14)$$

### III. HARMONIC OSCILLATOR EQUATION AND QUANTIZATION RULES

Making the usual realization of the operator $\frac{P_\xi^2}{2M_{pl}} = -\hbar^2 \frac{d^2}{2M_{pl} d\xi^2}$ and applying it to the wave-function $\psi$, we get the following linear harmonic oscillator equation

$$\left[ -\hbar^2 \frac{d^2}{2M_{pl} d\xi^2} + \frac{M_{pl}^2\omega_0^2\xi^2}{2} \right] \psi = \frac{M}{M_{pl}} \left( \frac{E_s}{2} \right) \psi. \quad (15)$$

We can obtain the following relations $E_s = \frac{c^4}{G} R_s$, $E_{pl} = \frac{c^4}{G} \ell_{pl}^2$ (this relation is equivalent to that appearing in (2)), and making the transformation $\frac{\xi}{\ell_{pl}} = x$, one can rewrite (15) as

$$\frac{1}{2} \left[ x^2 - \frac{d^2}{dx^2} \right] \psi = \frac{1}{4} \frac{R_s E_s}{\ell_{pl} E_{pl}} \psi. \quad (16)$$

Using the creation-annihilation representation,

$$a = \frac{1}{\sqrt{2}} \left( x + \frac{d}{dx} \right), \quad (17)$$

$$a^\dagger = \frac{1}{\sqrt{2}} \left( x - \frac{d}{dx} \right), \quad (18)$$

with the usual algebra between them, $[a,a^\dagger] = 1$, we can rewrite Eq. (16) as

$$a^\dagger a \psi = \frac{1}{2} \left[ x^2 - \frac{d^2}{dx^2} \right] \psi - \frac{1}{2} \psi = \left( -\frac{1}{2} + \frac{1}{4} \frac{R_s E_s}{\ell_{pl} E_{pl}} \right) \psi = n\psi, \quad n = 0, 1, 2, \cdots. \quad (19)$$

In this way, we have the following useful relations
\[ R_s E_s = 4(n + \frac{1}{2})\ell_{pl} E_{pl} \]
\[ = 4(n + \frac{1}{2})\hbar c, \quad (20) \]
\[ E_s^2 = 2(n + \frac{1}{2})E_{pl}^2, \quad (21) \]
\[ \frac{E_s}{2} = (n + \frac{1}{2})\hbar \omega_0. \quad (22) \]

It is clear that the system, even in its lowest energy state \( n = 0 \), has a finite, minimal energy. Eq. (20) implies the following quantization mass rule
\[ M_n = \sqrt{2n + 1} M_{pl}, \quad (23) \]
and Eq. (21) is the equivalent relation of (2), considering the FRW cosmological model as the metric inside the Schwarzschild black hole. Thus, the universe of this type has a quantized mass of the order of the Planck mass \( M_{pl} = 2.18 \times 10^{-8}\text{Kg} \). These results are similar to those obtained by other methods [5,9–12].

If one goes over from mass to energy units one finds the following figure for the ground state energy
\[ M_0 c^2 = 1.22 \times 10^{28}\text{eV}, \quad (24) \]
and for the excitation to the next state the energy required will be
\[ (M_1 - M_0) c^2 = 0.9 \times 10^{28}\text{eV}. \quad (25) \]

On the other hand, we can see that Eq. (20) remains invariant under the dual symmetries,
\[ E_s \rightarrow \frac{c^4}{2G} R_s, \quad R_s \rightarrow \frac{2G}{c^4} E_s, \quad (26) \]
in analogy with the case of magnetic and electric charges [13].

Let us write the equation (16) in the following form
\[ \frac{d^2 \psi}{dx^2} + (\alpha_n^2 - x^2)\psi = 0, \quad (27) \]
where \( \alpha_n \) is parameter associated with the energy of the nth eigenstate
\[ \alpha_n^2 = \frac{1}{2} \frac{R_s E_s}{\ell_{pl} E_{pl}} = 2(n + \frac{1}{2}), \quad \text{thus,} \quad \alpha_n = \frac{E_s}{E_{pl}}, \quad (28) \]
and the quantum solution is similar to the harmonic oscillator case
\[ \psi_n(x) = \left(\frac{1}{\sqrt{\pi n!2^n}}\right)^{\frac{1}{2}} H_n(x) e^{-\frac{x^2}{2}}, \quad (29) \]
with \( H_n(x) \) the Hermite polynomials.
The area parameter $A = 4\pi R_s^2$ has corrections depending on the $n$th eigenstate (see Eq. (1)),

$$A = 2\pi \left[ R_s^2 + \left( \frac{2G}{c^4} \right)^2 E_s^2 \right] = 32\pi (n + \frac{1}{2})\ell_{pl}^2. \quad (30)$$

The area of the event horizon of a black hole can take only discrete values, such that, the quanta of the area are of the same order of magnitude as the Planck area. It is easy to check that this parameter is invariant under the transformation (26) and is similar to Eq. (1).

On the other hand, the calculation of the partition function can be done along the lines of [14], leading to similar results.

**IV. CONCLUSIONS**

In this paper using the canonical quantization, a time-independent Schrödinger equation for the case of FRW model inside a Schwarzschild black hole has been obtained.

This system looks like a quantum linear harmonic oscillator, and using the creation-annihilation representation we find interesting relations between the quantities $R_s, E_s, E_{pl}$ and $\ell_{pl}$, (see (20,21,22)), in terms of the discrete parameter $n$. With these relations, we obtain the discrete mass spectrum for this type of Planck scale closed universe (23). A generalization of this procedure to the supersymmetric quantum black hole will be reported elsewhere.

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