Analytic results for the massive sunrise integral in the context of an alternative perturbative calculational method

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ABSTRACT: An explicit investigation about the equal-mass two-loop sunrise Feynman graph is performed. Such perturbative amplitude is related with many important physical process treated in the standard model context. The background of this investigation is an alternative strategy to handle with the divergences typical of perturbative solutions of quantum field theory. Since its proposition, the mentioned method was exhaustively used to calculate and manipulate one-loop Feynman integrals with a great success. However, the great advances in precision of experimental data collected in particle physics colliders have pushed up theoretical physicists to improve their predictions through multi-loops calculations. In the present job, we describe the main steps required to perform two-loops calculations within the context of the referred method. We show that the same rules used for one-loop calculations are enough to deal with two-loops graphs as well. Analytic results for the sunrise graph are obtained in terms of elliptic multiple polylogarithms as well as a numerical analysis is provided.

KEYWORDS: Perturbative calculations, Two-loop sunrise graph, Elliptic multiple polylogarithms

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1 Introduction

Many years have passed since the creation of quantum field theories (QFTs). The amount of the experimental data produced in particle physics colliders, as well as their accuracy, has been incredible increased along this time. The newest and remarkable experimental machine, the larger hadron collider (LHC), the most powerful one ever built, is the main responsible for this. From a lot of data extracted from the performed experiments, it was possible to confirm the existence of the Higgs particle (predicted many years before), the last crucial ingredient of the standard model. It is expected that the experiments are not restricted to confirm well-known theoretical predictions but also to open new frontiers like occurred recently when the interpretation of experimental results seems to give support, at least in hypothesis level, for the existence of a new fundamental force of nature, fact that, if confirmed, will represent a true revolution in the conception of the fundamental particles and their interaction which will requires new theoretical developments. This increasingly higher experimental precision has forced theoretical particle physicists to improve its predictions as well. Since the majority of predictions extracted from the theoretical models, at high energy, are made within the framework of perturbative calculations, this, in practice, imply to go beyond one-loop approximation in calculations involving Feynman loop diagrams. This means to deal with the challenging goal of calculating Feynman integrals with two or more loops.

As it is well known, perhaps the major issue involved in this kind of calculation, is due to the fact that, in general, the Feynman integrals are divergent mathematical objects. Then, in order to give a meaning for such objects, one need to define a tool to deal with
such kind of undefined quantities. The most used techniques are based on regularization methods. Essentially, a regularization consist in introducing some kind of modification in the Feynman integrals, resulting in its convergence. However, the results obtained, unfortunately, are regularization dependent.

Throughout QFT’s history, many regularization methods were proposed and used mainly to perform one-loop calculations. Among all these methods, only the dimensional regularization (DR) \([1–3]\) showed potentiality to be used in calculations beyond one-loop and the reason is more of practical order than conceptual. Usually, in such regularizations, like Pauli-Villars \([4]\), the number of parameters increase fast with the number of loops, which often makes the integrals calculation unworkable or produces results that are difficult to use or interpret.

By using DR as the background framework, many powerful and interesting techniques has been developed in order to solve multi-loop/multi-legs integrals, divergent or not. In spite that, even today there are open problems about this issue, such that this a very active field of research (for some reviews see refs. \([5–9]\)). In the recent literature, the most popular and successful technique used to deal with multi-loop Feynman diagrams consist, as a first step, in to reduce an arbitrary scalar integral to a linear combination of a finite set of basis integrals, the so called master integrals. This is done by applying the integration-by-parts method \([10, 11]\), Lorentz-invariance identities, and symmetry properties of these integrals. In its turn, the master integrals are usually solved in two main ways. One of such approaches, which is widely used, is given by the well-known differential equations method \([12–17]\). In this method, one can derive a system of differential equations in the kinematic invariants satisfied by a set of master integrals, belonging to a certain topology, and then try to solve it in order to obtain a solution for such integrals without need a direct integration. Another possible technique start by introducing a parametric transformation in the Feynman integrals, in order to allow the integration over the loop momenta, for then to perform a direct integration over the parameters of such transformation.

The results of Feynman integrals calculations are commonly written in terms of special transcendental functions. For example, it was realized, many years ago, that all the one-loop Feynman integrals can be solved in terms of multiple polylogarithms (MPLs) functions \([62–65]\). However, for two-loop and beyond this cannot be done for all topologies. The two-loop equal mass sunrise integral is the simplest Feynman integral which is known that cannot be expressed in terms of MPLs since it involves integrals of elliptic type. Thus, it was necessary to consider generalizations of ordinary MPLs, called usually elliptic multiple polylogarithms (eMPLs) \([66–73]\). Along the last decade, various slightly different definitions of such transcendental functions have been introduced in the literature. Some recent definitions of eMPLs have been proposed in Refs. \([69, 70]\), where the authors present a rigorous and detailed discussion of these special functions, as well as give examples of applications such as the calculation of the sunset diagram. The present paper is in the same line of reasoning as these works and intends to add some contribution to this problem.

With the help of the cited methods, among others, much progress has been done along the last decades with respect to the Feynman integrals calculations. In spite of the success achieved so far, it would be desirable and useful, for many reasons, to have at our disposal
an alternative method to DR to deal with multi-loop calculations in a consistent way and which can be easily applicable. In fact, such a method already exist and was proposed in the early 20’s by one of the present authors in his doctoral thesis [18]. Since then, this alternative strategy was applied to many problems involving one-loop calculations with very quite successful results (some examples can be seen in the Refs. [19–27]). In such a papers, many subtle aspects of the perturbative calculation were raised and then solved in a clear way. The strategy is based on a very simple idea, which is to avoid, as much as possible, to perform the integration of a purely divergent integral. Instead, the method suggest to extract the physical content of a divergent integral by rewritten its integrand in a such way that it can be splitted into a sum of finite terms and purely divergent objects, which are the ones that do not have physical parameters inside them. One way that this can be done is by adopting a convenient representation for the propagators, as we will show in this paper. The finite integrals can be integrated out without restrictions and the set of divergent ones is reorganized into scalar objects and tensor surface terms. By following this approach, the final results obtained retain all the original properties of the integrals, making possible to make more general analyses of the pertinent physical process. In many situations, this represents an advantage with respect to the traditional regularization methods.

However, the potentiality of the method was little explored in scenarios involving perturbative corrections at two-loops order or more. The few jobs published in this line have just focused in some particular aspects, such as massless diagrams [28–31], and so they did not exploited the full potentiality of the method in the context multi-loop calculations. In this paper, we will start to fulfill this gap by using the method to calculate the well-known massive sunrise graph. This Feynman two-loop diagram is very important, since it appears in many physical process treated in the standard model context, and, therefore, have been studied in several papers by means of different approaches [32–61]. By handling this problem, we will discuss the main steps in order to perform two-loop computations within the method. In particular, we will see that the same recipe formulated to perform one-loop calculations works for multi-loop calculations as well. The issue of divergences present in subtopologies is also solved in a natural way.

In order to realize our program we organize the work as follows. In the Section II, we briefly review the aforementioned method, highlighting the main steps used to calculate some standard scalar one-loop integrals. The calculations of the equal mass sunrise integral, through the method, are explicitly shown in the Section III. As it will be shown in the prescription stated in the Section II, the mass parameter used to define what we call basic divergent objects is arbitrary. Thus, in Section IV, we obtain useful relations which connect typical two-loop basic divergent objects defined at two different mass scales. In the Section V, we perform explicitly the integration over the Feynman parameters and write the results in terms of eMPLs. A numerical analysis of the results obtained in the Section V is discussed in the Section VI and, finally, the final comments and remarks are given in the Section VII.
2 The method to handle (divergent) Feynman integrals - a brief review

All perturbative amplitudes, defining typical physical processes (scattering and particle decays) in the framework of QFTs, are, in fact, reduced to a combination of Feynman integrals. Unfortunately, many of them are divergent structures, which require much care with their manipulations. In this section we briefly describe the method that we adopt to treat divergent Feynman integrals by reviewing how it works in calculations involving one-loop integrals. The same prescription will be used to treat the two-loop sunrise integral in the next section. Much of the material succinctly presented in this section can be found extensively discussed in the Refs. [22, 74].

A $d$-dimensional one-loop scalar $N$-point integral, with arbitrary internal (loop) momenta and masses, can be define as

$$J_N^{(d)} (\{k_i\}, \{m_i\}) = \int \frac{d^d k}{(2\pi)^d} I_N (k_1, \ldots, k_N; m_1, \ldots, m_N),$$

(2.1)

$$I_N (k_1, \ldots, k_N; m_1, \ldots, m_N) = \prod_{i=1}^{N} \frac{1}{P(k + k_i, m_i)},$$

(2.2)

where $P(k + k_i, m_i) = (k + k_i)^2 - m_i^2$. The arbitrary internal momenta $k_i$ are related, in physical amplitudes, to the external ones through their differences. At a specific space-time dimension, some of the structures defined above may present a divergent character. Then, for their explicit evaluation, we have to specify some prescription to deal with the mathematical objects which are not well-defined. Usually the calculations become reliable only after adopting a regularization technique. Such a procedure invariably modify the integrand in order to get a convergent integral. Unfortunately, as is well known, the final results are, in general, regularization dependent. This usually means that it is not possible to specify, in a clear way, what are the particular effects of the adopted regularization for the results or, in other words, to know precisely in what extension the expression obtained is dependent in the used technique. Beside that, there is an ambiguity relative to the kind of regularization prescription which is chosen. Two different choices for the regularization can lead to different results for the calculated integrals. These kinds of issues, associated with regularizations prescriptions, are very well-known in the corresponding literature.

On the other hand, constructed to be an alternative to the standard regularization techniques, the method that we adopt in this paper aims to avoid, as much as possible, specific choices in intermediary steps of calculations, in such a way that all the possibilities still remain contained in the final results. This goal can be accomplished by following a simple sequence of steps. First, before introducing the integration sign, which can be thought representing the last Feynman rule, we make a power counting of loop momentum in order to get the superficial degree of divergence of the integral, focusing on a particular spacetime dimension. After that, we can rewrite the integrand $I_N (k_1, \ldots, k_N; m_1, \ldots, m_N)$
by using an alternative representation for \( P^{-1}(k + k_i, m_i) \), say
\[
\frac{1}{P(k + k_i, m_i)} = \sum_{j=0}^{N} \frac{(-1)^j (k_i^2 + 2k_i \cdot k + \lambda^2 - m_i^2)^j}{(k^2 - \lambda^2)^{j+1}} + \frac{(-1)^{N+1} (k_i^2 + 2k_i \cdot k + \lambda^2 - m_i^2)^{N+1}}{(k^2 - \lambda^2)^{N+1} [(k + k_i)^2 - m_i^2]},
\]
(2.3)
where the summation variable \( N \) is taken as equal or major than the superficial degree of divergence found. The arbitrary \( \lambda \) parameter has dimension of mass and plays the role of a mass scale. As would be expected, the expression above is an identity and the expression on the right hand side is really independent of the \( \lambda \). After this reorganization of the integrand, we can take the integration over the loop momentum \( k \). As a result, we have rewritten the original integral as a sum of finite as well as divergent new integrals. The unique assumption is that the linearity in the integration operation is a valid property for Feynman integrals. An important point about this reorganization is that the internal momenta dependent parts of the integrals are located only in finite ones. These finite integrals can be evaluated without restrictions and the divergent ones are just rewritten in terms of standard objects, conveniently defined. In this sense, the above identity is just one among many others which could play the same role. The procedure can be better visualized through a few examples, which we describe next.

Let us consider first the one-point integral in two dimensions which is given by (from now on we will hide the arguments of \( J_N^{(2)} \) always that they are not essential)
\[
J_1^{(2)} = \int \frac{d^2k}{(2\pi)^2} I_1 (k_1, m_1),
\]
(2.4)
which has a logarithmic degree of divergence. For this degree of divergence, it is enough to take \( N = 0 \) in Eq. (2.3), such that an alternative representation for the integrand \( I_1 (k_1, m_1) \) can be written as
\[
I_1 (k_1, m_1) = \frac{1}{[P(k, \lambda)]} - \frac{(k_1^2 + 2k_1 \cdot k + \lambda^2 - m_1^2)}{[P(k, \lambda)] [P(k + k_1, m_1)]},
\]
(2.5)
which, after substitution in (2.4), gives
\[
J_1^{(2)} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{[P(k, \lambda)]} - \int \frac{d^2k}{(2\pi)^2} \frac{(k_1^2 + 2k_1 \cdot k + \lambda^2 - m_1^2)}{[P(k, \lambda)] [P(k + k_1, m_1)]}.
\]
(2.6)
The last integral is finite and its integration can be easily done. We get
\[
J_1^{(2)} = \int \frac{d^2k}{(2\pi)^2} \frac{1}{[P(k, \lambda)]} - \frac{i}{(4\pi)} \ln \left( \frac{m_1^2}{\lambda^2} \right).
\]
(2.7)
The first term, in the equation above, is a representative of a divergent integral belonging to a class of Feynman integrals which we denominate basic divergent integrals. Such class of integrals are characterized by the absence of physical parameter in its integrands such
that, in this sense, carry no physical content. Following the main philosophy of the method, which means does not perform the integration operation of these divergent structures, we keep this object untouched and, for a better systematization of the results, we define a \(d\)-dimensional basic logarithmically divergent object, given by

\[
I_{\log}^{(d)} (\lambda^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{[P (k, \lambda)]^{\frac{d}{2}}} .
\]  

(2.8)

Given this definition we can write

\[
J_1^{(2)} = \left[ I_{\log}^{(2)} (\lambda^2) \right] - \frac{i}{(4\pi)} \ln \left( \frac{m_1^2}{\lambda^2} \right) .
\]  

(2.9)

The same integral, treated in four dimensions, namely \(J_1^{(4)}\), has a quadratic degree of divergence. This require, at least, to take \(N = 2\) in (2.3), which gives

\[
I_1 (k_1, m_1) = \frac{1}{[P (k, \lambda)]} - \frac{(k_1^2 + \lambda^2 - m^2)}{[P (k, \lambda)]^2} + 4k_1^2 k_1^\beta \frac{k_1 k_\beta}{[P (k, \lambda)]^3}
\]

\[
+ \frac{(k_1^2 + \lambda^2 - m^2)^2}{[P (k, \lambda)]^3} - \frac{(k_1^2 + \lambda^2 - m^2 + 2k \cdot k_1)^3}{[P (k, \lambda)]^4 [P (k + k_1, m_1)]} .
\]

(2.10)

After introducing the integration over the loop momentum, we note that the last two terms gives finite integrals. The final expression can be put in the form

\[
J_1^{(4)} = k_1^\alpha k_1^\beta \left[ \Delta^{(4)}_{\alpha\beta} (\lambda^2) \right] + \left[ I_{quad}^{(4)} (\lambda^2) \right] + (m_1^2 - \lambda^2) \left[ I_{\log}^{(4)} (\lambda^2) \right]
\]

\[
+ \frac{i}{(4\pi)^{d-1}} \left[ m_1^2 - \lambda^2 - m_1^2 \ln \left( \frac{m_1^2}{\lambda^2} \right) \right] ,
\]

(2.11)

where we have defined a \(d\)-dimensional basic quadratically divergent object,

\[
I_{quad}^{(d)} (\lambda^2) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{[P (k, \lambda)]^{\frac{d}{2}}} ,
\]

(2.12)

and a tensorial object which can be viewed as a surface term because it can be written as a total derivative

\[
\Delta_{\mu\nu}^{(d)} (\lambda^2) = \int \frac{d^d k}{(2\pi)^d} \frac{\partial}{\partial k_\mu} \left( \frac{k_\nu}{[P (k, \lambda)]^{\frac{d}{2}}} \right) ,
\]

\[
= \int \frac{d^d k}{(2\pi)^d} \left\{ \frac{d}{[P (k, \lambda)]^{\frac{d}{2}+1}} - \frac{g_{\mu\nu}}{[P (k, \lambda)]^d} \right\} .
\]

(2.13)

We should emphasize that neither of the above objects carry any physical parameter. In six or more dimensions, where the divergences grow, the procedure follows the same sequence of steps describe above, but require, if one want to keep the described systematization, definitions of new basic divergent objects similar to that showed above. In odd dimensions, the procedure works in the same line, without additional hypothesis.
The scalar two-point function in two dimensions is finite but, in four dimensions,
\[ J_2^{(4)} = \int \frac{d^4k}{(2\pi)^4} I_2 (k_1, m_1, k_2, m_2) , \] (2.14)
it has a logarithmic divergence. Here we can apply identity (2.3) for each \( P (k + k_i, m_i) \) separately or for both simultaneously. The last approach gives
\[ I_2 (k_1, m_1, k_2, m_2) = \frac{1}{[P (k, \lambda)]^2} \left( \frac{k_1^2 + 2k_1 \cdot k + \lambda^2 - m_1^2}{[P (k, \lambda)]^2} - \frac{(k_2^2 + 2k_2 \cdot k + \lambda^2 - m_2^2)}{[P (k, \lambda)]^2} \right) + \frac{(k_1^2 + 2k_1 \cdot k + \lambda^2 - m_1^2)(k_2^2 + 2k_2 \cdot k + \lambda^2 - m_2^2)}{[P (k, \lambda)]^2 [P (k + k_1, m_1)][P (k + k_2, m_2)]} . \] (2.15)
Only the first term will gives a divergent integral when the integration sign is inserted. The remain ones are finite and, after their integration, we obtain
\[ J_2^{(4)} = \left[ I_{\log}^{(4)} (\lambda^2) \right] - \frac{i}{(4\pi)^2} \left[ \xi_0^{(1)} (m_1^2; p^2, m_2^2; \lambda^2) \right] , \] (2.16)
with the finite part systematized by a set of functions defined in terms of an integral over a Feynman parameter
\[ \xi_k^{(n)} (m_1^2; p^2, m_2^2; \lambda^2) = \int_0^1 dx \ x^k \left\{ \frac{|Q|^n}{n!} \left[ \ln \left( \frac{Q}{\lambda^2} \right) - \psi (n + 1) + \gamma \right] \right\} , \] (2.17)
with \( \gamma \) being the Euler-Mascheroni constant and \( n = 0, 1, 2, \ldots \). The integration over the Feynman parameter \( x \) can be easily done and some explicit expressions can be viewed in Ref. [74].

In its turn, this two-point integral in six dimensions has a quadratic degree of divergence. Applying identity (2.3) for \( N = 2 \) and integrating the finite integrals allow us to write
\[ J_2^{(6)} = -\frac{1}{3} \left[ (k_1)_\alpha (k_2)_\beta - 2 (k_2 + k_1)_\alpha (k_2 + k_1)_\beta \right] \left[ \Delta^{(6)}_{\alpha\beta} (\lambda^2) \right] + \left[ I_{\text{quad}}^{(6)} (\lambda^2) \right] - \frac{1}{6} \left[ p^2 + 3 (\lambda^2 - m_1^2) + 3 (\lambda^2 - m_2^2) \right] \left[ I_{\log}^{(6)} (\lambda^2) \right] + \frac{i}{(4\pi)^2} \left[ \xi_0^{(1)} (p^2, m_1^2, m_2^2; \lambda^2) \right] . \] (2.18)
The method’s systematic is now clear. Given an arbitrary Feynman integral, we can write an alternative representation of its integrand in such a way that, when the sign integration is inserted, we find a sum of others Feynman integrals. While the finite integrals can be performed and organized through a set of well-defined functions conveniently defined (for one-loop divergent integrals it is enough the set defined in Eq. (2.17)), the remaining (divergent) part is composed by the following set of integrals
\[ \int \frac{d^d k}{(2\pi)^d} \left\{ k_\mu k_\nu k_\rho k_\sigma k_\lambda k_\beta \ldots \right\} . \] (2.19)
which are external momenta and masses independent. In this set, the tensor integrals are reduced to a combination of scalar ones plus (tensorial) surface terms. In the solution obtained, these integrals appear as coefficient of a polynomial in the masses, external momenta and (arbitrary) internal routing momenta $k_i$, as demonstrated in the cases treated above.

At this point one can ask: why such line of reasoning, for calculation and manipulation of Feynman integrals, can be considered convenient as well as useful? First of all, divergent mathematical structures are undefined quantities, which are usually fixed by some choices necessarily made when a regularization is adopted. In our approach, however, such indefiniteness are still present at the final results through the basic divergent objects and surface terms. In fact, the basic divergent objects do not need to be evaluated since their coefficients, in a physical amplitude, are polynomials in the external momenta such that they will invariably be absorbed in the renormalization process. In a nonrenormalizable model, on the other hand, they could be parameterized in order to fit the physical observables \[23\]. In its turn, the coefficients of the surface terms are potentially ambiguous if a generic choice for the labels of the internal lines momenta are made, as one can see in the above expression obtained for $J_2^{(6)}$. A series of investigations, made by the present authors and others, has revealed that the relevant dependence of a perturbative calculation with a regularization resides in the value attributed to these surface terms. In the DR, for instance, these ambiguities are eliminated since such surface terms are equal to zero. This is, in principle, an attractive possibility, but there are others possible ones. The discussion of what are the consistency values that should be attributed to such surface terms, if any exist, is long and is out of the scope of the present paper. The point to be highlighted is that the Feynman integrals, when evaluated through the method presented above, gives results which preserve the arbitrariness involved because no choices are made in the intermediary steps of the calculations. One obvious advantage of these approach is that, when these results are used to calculate physical processes, within a framework of a theory or model, such arbitrariness can be fixed through choices guided by consistency requirements such as symmetries maintenance and universality of calculations.

In the next section we will show how this approach can be used to calculate multi-loop Feynman integrals by evaluating the two-loop sunrise integral in details.

3 The equal-mass sunrise integral

In this paper we are interested in the calculation of the well-known 4-dimensional equal mass sunrise diagram, which can be represented schematically as in Fig. (1).

It has an integral representation given by

$$J_{SS} (p,m) = \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \left[ I_{SS} (k,\ell,p,m) \right],$$

(3.1)

$$I_{SS} (k,\ell,p,m) = \frac{1}{[P (\ell,m)] [P (k,m)] [P (k+\ell+p,m)]}.$$  

(3.2)
Following the strategy discussed in the last section, the integrand $I_{SS}$ can be rewritten by applying recursively the identity (2.3), but now written as

$$
\frac{1}{P(k + \ell + p, m)} = \sum_{j=0}^{N} (-1)^j \left[ \frac{p^2 + 2(\ell + k) \cdot p}{[P(k + \ell, m)]^{j+1}} \right]
$$

$$
+ \left[ \frac{(-1)^{N+1} [p^2 + 2(\ell + k) \cdot p]^{N+1}}{[P(k + \ell, m)]^{N+1}[P(k + \ell + p, m)]} \right].
$$

(3.3)

From now on, we will hide the mass dependence in our notation, unless it become necessary for the sake of clarity. Since the total degree of divergence of $J_{SS}$ is quadratic, it is enough to choose $N = 2$ in the above identity. Thus

$$
I_{SS} = \frac{1}{P(k)}\frac{1}{P(\ell)}\frac{1}{P(k + \ell)} - \frac{p^2}{[P(k)][P(\ell)][P(k + \ell)]^2}
$$

$$
+ 4p^\alpha p^\beta \frac{[P(k)][P(\ell)][P(k + \ell)]^3}{[P(k)][P(\ell)][P(k + \ell)]^3}
$$

$$
- \frac{[p^2 + 2(\ell + k) \cdot p]^3}{[P(k)][P(\ell)][P(k + \ell + p)]},
$$

(3.4)

where we have discarded odd terms. By power counting, the last two terms above are convergent, but still hide a subdivergence. If one integrate them over the loop momenta, a divergence will emerge in the Feynman parameters integrals. This a characteristic associated to certain types of two-loop Feynman integrals. A careful analysis, however, reveals that there is a mutual cancellation of such subdivergences coming from both referred terms. In order to see that, it is convenient to first perform a shift ($\ell' = \ell + k$) and after use an identity similar to (2.3), i.e.,

$$
\frac{1}{P(k + \ell)} = \frac{1}{P(k)} - \frac{[\ell^2 - 2(\ell \cdot k)]}{[P(k)][P(k + \ell)]},
$$

(3.5)
to write

\[
J_{SS} = \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \frac{1}{[P(k)][P(\ell)][P(k + \ell)]} \\
+ p^\alpha p^\beta \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{4(\ell + k)\alpha(\ell + k)\beta}{[P(k)][P(\ell)][P(k + \ell)]^3} - \frac{g_{\alpha\beta}}{[P(k)][P(\ell)][P(k + \ell)]^2} \right\} \\
+ \frac{1}{2} \frac{p^4}{(4\pi)^4 m^2} \int_0^1 dx \frac{x(1-2x)}{x(1-x)-1} \left\{ \frac{1}{2} - \frac{1}{x(1-x)-1} + \frac{\ln[x(1-x)]}{[x(1-x)-1]^2} \right\} \\
- \frac{p^2}{(4\pi)^4} \int_0^1 dx \int_0^1 dy \frac{(1-2x)(1-2y)}{(1-x)[x(1-x)-1]} \ln \left[ \frac{p^2y(1-y) + (m^2 - \mu^2)y - m^2}{(m^2 - \mu^2)y - m^2} \right],
\]

(3.6)

But, it is straightforward to see that

\[
\int \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{p^4}{[P(\ell)]^3} - \frac{[p^2 + 2\ell \cdot p]^3}{[P(\ell)]^3[P(\ell + p)]} \right\} = 0,
\]

(3.7)

such that the referred cancellation occurs.

The integration over the loop momenta in the last term in Eq. (3.6) can easily be done and the result can be written as

\[
J_{SS} = \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \frac{1}{[P(k)][P(\ell)][P(k + \ell)]} \\
+ p^\alpha p^\beta \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \left\{ \frac{4(\ell + k)\alpha(\ell + k)\beta}{[P(k)][P(\ell)][P(k + \ell)]^3} - \frac{g_{\alpha\beta}}{[P(k)][P(\ell)][P(k + \ell)]^2} \right\} \\
- \frac{1}{2} \frac{2p^4}{(4\pi)^4 m^2} \int_0^1 dx \frac{x(1-2x)}{x(1-x)-1} \left\{ \frac{1}{2} - \frac{1}{x(1-x)-1} + \frac{\ln[x(1-x)]}{[x(1-x)-1]^2} \right\} \\
- \frac{p^2}{(4\pi)^4} \int_0^1 dx \int_0^1 dy \frac{(1-2x)(1-2y)}{(1-x)[x(1-x)-1]} \ln \left[ \frac{p^2y(1-y) + (m^2 - \mu^2)y - m^2}{(m^2 - \mu^2)y - m^2} \right],
\]

(3.8)

with the definition \( \mu^2 = \frac{m^2}{x(1-x)}. \) In the multi-loop calculations, the integration over the Feynman parameters, in a closed form, may represents a new challenge to be overcome. In one-loop calculations, it is well-known that the results can be written in terms of MPLs [62–65]. On the other hand, for integrals beyond one-loop, this is not more possible for all of them. In our case, the integrals over the Feynman parameters showed above can be performed both numerically or analytically by using eMPLs [66–72]. We will done this task in a separate section ahead.

As a main feature of the method, the remaining divergences (the two first lines in the equation above) have been isolated into integrals that do not contain physical momenta \( p \) inside them, for such a reason called basic divergent ones. The physical mass \( m \), at this stage of calculation, still inside of these integrands, can be replaced by an arbitrary mass scale \( \lambda \) by using the so called scale relations [21], which will be discussed in the Section (4).

Such objects are not well-defined and could not be integrated out without a regularization
or similar prescription. In the one-loop integrals, we shown that such terms can be reduced to scalar objects plus tensorial surface terms. As we will show, in the case of two-loop sunrise graph the same can be done.

The prescription applied above to calculate $J_{SS}$ is exactly the same one used to calculate the one-loop integrals, in the Section (2). However, the method is flexible enough to allows us to use different approaches too. In order to see that, let us recover the result (3.8) but now following another possible path to deal with the $J_{SS}$ integral within the method. As we will show soon, both results will be equivalent, but having a slightly different representation for the finite integrals, which will be an advantage when the Feynman parameters integrals are performed in the Section (5). In this second approach, before applying the identity (3.3), we first rewrite the integrand by using some identities constructed through the method of integration-by-parts, which is a standard tool used in Feynman loop calculations. First let us consider two trivial total derivatives

$$
\frac{\partial}{\partial k_\mu} \left\{ \frac{k_\mu}{[P(k)][P(\ell)][P(k+\ell+p)]} \right\} = 4 \left[ \frac{1}{[P(k)][P(\ell)][P(k+\ell+p)]} - 2k^2 \frac{[P(k)][P(\ell)][P(k+\ell+p)]}{[P(k)][P(\ell)][P(k+\ell+p)]^2} \right],
$$

(3.9)

$$
\frac{\partial}{\partial \ell_\mu} \left\{ \frac{\ell_\mu}{[P(k)][P(\ell)][P(k+\ell+p)]} \right\} = 4 \left[ \frac{1}{[P(k)][P(\ell)][P(k+\ell+p)]} - 2\ell^2 \frac{[P(k)][P(\ell)][P(k+\ell+p)]}{[P(k)][P(\ell)][P(k+\ell+p)]^2} \right].
$$

(3.10)

Summing up the above expressions and reducing the bilinears in the numerator gives

$$
I_{SS} = \Delta I_{SS} + \frac{3m^2}{[P(k)][P(\ell)][P(k+\ell+p)]^2} - \frac{p \cdot (k+\ell+p)}{[P(k)][P(\ell)][P(k+\ell+p)]^2},
$$

(3.11)

with the definition

$$
\Delta I_{SS} = \frac{1}{2} \frac{\partial}{\partial k_\mu} \left\{ \frac{k_\mu}{[P(k)][P(\ell)][P(k+\ell+p)]} \right\} + \frac{1}{2} \frac{\partial}{\partial \ell_\mu} \left\{ \frac{\ell_\mu}{[P(k)][P(\ell)][P(k+\ell+p)]} \right\}.
$$

(3.12)

Through this step, we have reduced the integrand into surface terms plus two others terms which have logarithmic and linear degree of divergence, respectively. Next, let us apply the
identity (3.3) in order to rewrite these two terms, in the same spirit of the discussion above. We get

\[
\frac{1}{[P(k)][P(\ell)][P(k+\ell+p)]^2} = \frac{1}{[P(k)][P(\ell)][P(k+\ell)]^2} - \frac{p \cdot (k+\ell+p)}{[P(k)][P(\ell)][P(k+\ell+p)]^2} \cdot (3.13)
\]

\[
\frac{2(p \cdot k)[p \cdot (\ell + k)]}{[P(\ell)][P(k)]^2[P(k+\ell)]^2} - \frac{(p \cdot k)[p^2 + 2(\ell + k) \cdot p]}{[P(\ell)][P(k)]^2[P(k+\ell+p)]^2} \cdot (3.14)
\]

Then

\[
J_{SS} = \Delta J_{SS} + 3m^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \frac{1}{[P(k)][P(\ell)][P(k+\ell)]^2} - 2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \frac{p \cdot (k+\ell+p)}{[P(k)][P(\ell)][P(k+\ell+p)]^2} - 3m^2 \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \frac{1}{[P(\ell)][P(k)]^2[P(k+\ell)]^2} \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{[P(k)][P(\ell)][P(k+\ell)]^2} + p^\mu \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[P(\ell)][P(k+\ell+p)]^2} \int \frac{d^4k}{(2\pi)^4} \frac{\ell_\mu}{[P(k)][P(\ell)][P(k+\ell)]^2} \cdot (3.15)
\]

where \( \Delta J_{SS} = \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \Delta I_{SS} \). Now, the last two integrals are convergent and can be easily done.

Let us work out the surface term \( \Delta J_{SS} \) by using again the identity (3.3). After this, we see that it is, in fact, independent of the momentum \( p \), i.e.,

\[
\Delta J_{SS} = \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \frac{\partial k_\mu}{[P(k)][P(\ell)][P(k+\ell)]^2} + \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \frac{d^4\ell}{(2\pi)^4} \frac{\partial \ell_\mu}{[P(k)][P(\ell)][P(k+\ell)]^2} \cdot (3.16)
\]

By taking the derivatives and eliminating the bilinears which appears in the numerator we get

\[
\Delta J_{SS} = \int \frac{d^4\ell}{(2\pi)^4} \frac{d^4k}{(2\pi)^4} \left\{ \left[ \frac{1}{[P(k)][P(\ell)][P(k+\ell)]^2} - \frac{3m^2}{[P(k)][P(\ell)][P(k+\ell)]^2} \right] \right\} \cdot (3.17)
\]
and

\[ J_{SS} = \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 \ell}{(2\pi)^4} \frac{1}{[P(k)] [P(\ell)] [P(k + \ell)]} \]

\[-2 p^\mu p^\nu \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 \ell}{(2\pi)^4} \frac{k_\mu (\ell + k)_\nu}{[P(k)] [P(\ell)] [P(k + \ell)]^2} \]

\[ + \frac{1}{(4\pi)^4} \int_0^1 dy \int_0^1 dx \left[ \frac{3m^2}{x} - p^2 (1 - y) \right] \ln \left[ \frac{p^2 y (1 - y) + (m^2 - \mu^2) y - m^2}{(m^2 - \mu^2) y - m^2} \right]. \]

(3.18)

Finally, by noting the obvious identity

\[ \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 \ell}{(2\pi)^4} \frac{2k_\mu (\ell + k)_\nu}{[P(k)] [P(\ell)] [P(k + \ell)]^2} \]

\[ = - \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 \ell}{(2\pi)^4} \left\{ \frac{4(\ell + k)_\mu (\ell + k)_\nu}{[P(k)] [P(\ell)] [P(k + \ell)]^3} - \frac{g_{\mu\nu}}{[P(k)] [P(\ell)] [P(k + \ell)]^2} \right\}, \quad (3.19) \]

one can see that the above formula for \( J_{SS} \) is, in fact, similar to the Eq. (3.8), but with the finite part written in a slightly different form. In this form, the task of write the result of \( J_{SS} \) in terms of eMPLs, in the Sec. (5), will become simpler.

Performing the (finite) integration over momentum \( \ell \), in the integral on the right hand side of the above equation, allow us to find that its divergent part can be written as one-loop basic objects, i.e.,

\[ \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 \ell}{(2\pi)^4} \frac{2k_\mu (\ell + k)_\nu}{[P(k)] [P(\ell)] [P(k + \ell)]^2} \]

\[ = \frac{i}{32\pi^2} \left\{ g_{\mu\nu} \left[ \frac{I^{(4)}_{log}(m^2)}{2} + \frac{\Delta^{(4)}_{\mu\nu}(m^2)}{2} \right] \right\} \]

\[ - \frac{1}{(4\pi)^4} \frac{g_{\mu\nu}}{4} \int_0^1 dx \left\{ 1 - \frac{2}{x (1 - x) - 1} + \frac{2 \ln \left[ x (1 - x) \right]}{[x (1 - x) - 1]^2} \right\}, \quad (3.20) \]

such that

\[ J_{SS} = \left[ I_{(4)}^{quad}(m^2) \right] - \frac{i}{32\pi^2} \left\{ p^2 \left[ I_{log}^{(4)}(m^2) \right] + p^\mu p^\nu \left[ \Delta^{(4)}_{\mu\nu}(m^2) \right] \right\} \]

\[ + \frac{1}{(4\pi)^4} \frac{p^2}{4} \int_0^1 dx \left\{ 1 - \frac{2}{x (1 - x) - 1} + \frac{2 \ln \left[ x (1 - x) \right]}{[x (1 - x) - 1]^2} \right\} \]

\[ + \frac{1}{(4\pi)^4} \int_0^1 dy \int_0^1 dx \left[ \frac{3m^2}{x} - p^2 (1 - y) \right] \ln \left[ \frac{p^2 y (1 - y) + (m^2 - \mu^2) y - m^2}{(m^2 - \mu^2) y - m^2} \right]. \]

(3.21)

where we defined a typical 4-dimensional two-loop basic object with quadratic degree of divergence

\[ I_{(4)}^{quad}(m^2) = \int \frac{d^4 k}{(2\pi)^4} \frac{d^4 \ell}{(2\pi)^4} \frac{1}{[P(k)] [P(\ell)] [P(k + \ell)]}. \quad (3.22) \]

In the previous formula for \( J_{SS} \), one can see that the undefined quantities are coefficient of a polynomial in momentum \( p \), which allows us to absorb them in a renormalization
procedure, if this sunrise integral would part of a physical process prediction made within a renormalizable theory. In this sense, such undefined objects do not require any explicit calculation and, then, for practical purposes no regularization is required.

In order to complete the calculation of $J_{SS}$, we need to perform the integration over the Feynman parameters and also its numerical analysis. We will perform this task in the section (5). Before that, let us discuss the scale properties of the two-loop basic divergent objects.

4 Scale relations for two-loops scalar basic divergent objects

In the last section, we have defined the basic divergent objects using the physical mass $m$. However, the mass parameter used to define the basic divergent objects is arbitrary because it can be chosen freely in the separation process made through the identity (3.3). Thus, it is possible to find relations connecting such objects defined at different mass scales, which are called scale relations. For objects typical of one-loop, such relations are well-known and they are discussed in Refs. [21, 23]. In this section we will show the scale relations for two typical two-loop scalar divergent objects, having logarithm and quadratic degree of divergence, respectively.

4.1 Logarithmic divergence

Let us start with the logarithmic one defined by

$$I_{\log}^{(2L)} (\lambda^2) = \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{[P(k, \lambda)] [P(\ell, \lambda)] [P(\ell + k, \lambda)]^2}, \quad (4.1)$$

where $\lambda$ is an arbitrary mass which may play a role of a mass scale. It does not appears in the $J_{SS}$ result, but it will be present in other two-loops Feynman diagrams. Differentiation with respect to $\lambda^2$ gives

$$\frac{\partial}{\partial \lambda^2} \left[ I_{\log}^{(2L)} (\lambda^2) \right] = \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 \ell}{(2\pi)^4} \frac{2}{(k^2 - \lambda^2)^2 (\ell^2 - \lambda^2) \left[ (\ell + k)^2 - \lambda^2 \right]^2} + \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 \ell}{(2\pi)^4} \frac{2}{(k^2 - \lambda^2) (\ell^2 - \lambda^2) \left[ (\ell + k)^2 - \lambda^2 \right]^3}. \quad (4.2)$$

The last integral is finite by power counting but still contains a subdivergence. To see that, we first perform a shift $\ell + k = k'$ and apply the identity

$$\frac{1}{(k - \ell)^2 - \lambda^2} = \frac{1}{(\ell^2 - \lambda^2)} - \frac{[k^2 - 2 (k \cdot \ell)]}{(\ell^2 - \lambda^2) \left[ (k - \ell)^2 - \lambda^2 \right]}, \quad (4.3)$$

to get

$$\frac{\partial}{\partial \lambda^2} \left[ I_{\log}^{(2L)} (\lambda^2) \right] = -\frac{i}{(4\pi)^2} \frac{1}{\lambda^2} \left[ I_{\log}^{(4)} (\lambda^2) \right] + \frac{1}{(4\pi)^4} \frac{1}{\lambda^2}. \quad (4.4)$$
Using the scale relation for (one-loop) \( I^{(4)}_{\log} (\lambda^2) \) (see Ref. [21]), i.e.,

\[
I^{(4)}_{\log} (\lambda^2) = I^{(4)}_{\log} (\lambda_0^2) + \frac{i}{(4\pi)^2} \ln \left( \frac{\lambda^2}{\lambda_0^2} \right),
\]

where \( \lambda_0 \) is another arbitrary mass scale, and integrating on both sides gives

\[
I^{(2L)}_{\log} (\lambda^2) = -\frac{i}{(4\pi)^2} \left[ I^{(4)}_{\log} (\lambda_0^2) \right] \ln \lambda^2 + \frac{1}{(4\pi)^2} \left[ \ln (\lambda_0^2) \ln \lambda^2 - \frac{1}{2} \ln^2 (\lambda^2) - \ln \lambda^2 \right] + C_1,
\]

with \( C_1 \) being a constant independent of \( \lambda \). By choosing \( \lambda = \lambda_0 \) in the above expression

\[
I^{(2L)}_{\log} (\lambda_0^2) = -\frac{i}{(4\pi)^2} \left[ I^{(4)}_{\log} (\lambda_0^2) \right] \ln \lambda_0^2 + \frac{1}{(4\pi)^2} \left[ \ln (\lambda_0^2) \ln \lambda_0^2 - \frac{1}{2} \ln^2 (\lambda_0^2) - \ln \lambda_0^2 \right] + C_1,
\]

and subtracting the last two equations gives

\[
I^{(2L)}_{\log} (\lambda^2) = I^{(2L)}_{\log} (\lambda_0^2) - \frac{i}{(4\pi)^2} \left\{ I^{(4)}_{\log} (\lambda_0^2) - \frac{i}{(4\pi)^2} \left[ 1 + \frac{1}{2} \ln \left( \frac{\lambda^2}{\lambda_0^2} \right) \right] \right\} \ln \left( \frac{\lambda^2}{\lambda_0^2} \right),
\]

which is the scale relation searched for the object \( I^{(2L)}_{\log} (\lambda^2) \).

### 4.2 Quadratic divergence

Next let us consider the object

\[
I^{(2L)}_{quad} (\lambda^2) = \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 \ell}{(2\pi)^4} \frac{1}{(k^2 - \lambda^2)(\ell^2 - \lambda^2)(\ell + k)^2 - \lambda^2},
\]

which has emerged in (3.21). By following the same previously strategy, we first differentiate the above equation with respect to \( \lambda^2 \)

\[
\frac{\partial}{\partial \lambda^2} \left[ I^{(2L)}_{quad} (\lambda^2) \right] = 3 \left[ I^{(2L)}_{\log} (\lambda^2) \right],
\]

use the scale relation obtained for \( I^{(2L)}_{\log} (\lambda^2) \), Eq. (4.8), and after integrate on both sides to get

\[
I^{(2L)}_{quad} (\lambda^2) = 3 \lambda^2 \left[ I^{(2L)}_{\log} (\lambda_0^2) \right] + \frac{i}{(4\pi)^2} 3 \lambda^2 \left[ I^{(4)}_{\log} (\lambda_0^2) \right] \left[ 1 - \ln \left( \frac{\lambda^2}{\lambda_0^2} \right) \right]
\]

\[
- \frac{1}{(4\pi)^2} \frac{3}{2} \lambda^2 \ln^2 \left( \frac{\lambda^2}{\lambda_0^2} \right) + C_2,
\]

where \( C_2 \) is a constant independent of \( \lambda \). Taking \( \lambda = \lambda_0 \) in the above formula gives

\[
I^{(2L)}_{quad} (\lambda_0^2) = 3 \lambda_0^2 \left[ I^{(2L)}_{\log} (\lambda_0^2) \right] + \frac{i}{(4\pi)^2} 3 \lambda_0^2 \left[ I^{(4)}_{\log} (\lambda_0^2) \right] + C_2.
\]
Subtracting the two previous equations gives the scale relation for $I^{(2L)}_{quad}(\lambda^2)$,

$$I^{(2L)}_{quad}(\lambda^2) = I^{(2L)}_{quad}(\lambda^2) + 3(\lambda^2 - \lambda^2_0) \left[ I^{(2L)}_{log}(\lambda^2_0) \right] + i \left( \frac{4\pi}{2} \right)^2 3(\lambda^2 - \lambda^2_0) \left[ I^{(4)}_{log}(\lambda^2_0) \right]$$

$$- i \left( \frac{4\pi}{2} \right)^2 3\lambda^2 \left[ I^{(4)}_{log}(\lambda^2_0) \right] - i \left( \frac{4\pi}{2} \right)^2 \frac{1}{2} \ln \left( \frac{\lambda^2}{\lambda^2_0} \right) \ln \left( \frac{\lambda^2}{\lambda^2_0} \right). \quad (4.13)$$

With this scale relation one can change the physical mass $m$ inside the undefined object $I^{(2L)}_{quad}$ in the sunrise diagram $J_{SS}$, Eq.(3.21).

5 Evaluating the integrals over Feynman parameters

In the evaluation of $J_{SS}$ (at end of Section (3)), we left the finite part of the result (3.21) as being represented by integrals over Feynman parameters, which we now define as

$$\frac{(4\pi)^4}{m^2} I_F = \frac{k^2}{4} \int_0^1 dz \left\{ 1 - \frac{2}{z(1-z) - 1} + \frac{2 \ln[z (1 - z)]}{z (1 - z) - 1} \right\}$$

$$+ \int_0^1 dz \int_0^1 dx \left[ \frac{3}{z} - k^2 (1-z) \right] \ln \left[ \frac{p^2 z (1-z) + (m^2 - \mu^2) z - m^2}{(m^2 - \mu^2) z - m^2} \right], \quad (5.1)$$

where $k^2 \equiv \frac{p^2}{m}$. Integration over $x$, in the second integral above, followed by a trivial integrations-by-parts, gives an one-dimensional integral representation which we write as

$$\frac{(4\pi)^4}{m^2} I_F = I^{(A)}_F + I^{(B)}_F, \quad (5.2)$$

$$I^{(A)}_F = -\frac{k^2}{4} + \frac{k^2}{4} \int_0^1 dz \left\{ 1 - \frac{2}{z(1-z) - 1} + \frac{2 \ln[z (1 - z)]}{z (1 - z) - 1} \right\}$$

$$- \int_0^1 dz \frac{3 - k^2 (1-z) (3z+1)}{\sqrt{3z+1} (1-z)} \ln \left[ \frac{3z+1 + \sqrt{(3z+1)(1-z)}}{3z+1 - \sqrt{(3z+1)(1-z)}} \right], \quad (5.3)$$

$$I^{(B)}_F = -\frac{1}{2} \int_0^1 dz k^2 (k^2 + 3) z^2 - \frac{(k^4 - 4k^2 + 3) z + (k^2 - 5)}{k^2 \sqrt{T_4}} \ln \left[ \frac{(z - z_2) (z - z_3) - \sqrt{T_4}}{(z - z_2) (z - z_3) + \sqrt{T_4}} \right], \quad (5.4)$$

where $P_4 = (z - z_1) (z - z_2) (z - z_3) (z - z_4)$ is a quartic polynomial and $z_i$ are its roots,

$$\begin{cases} 
z_1 = \frac{1}{k^2}, & z_2 = \frac{(k^2 - 3)}{2k^2} - \frac{\sqrt{k^4 - 10k^2 + 9}}{2k^2}, \\
z_3 = \frac{(k^2 - 3)}{2k^2} + \frac{\sqrt{k^4 - 10k^2 + 9}}{2k^2}, & z_4 = 1. 
\end{cases} \quad (5.5)$$

The integral $I^{(A)}_F$ can be easily performed and written in terms of ordinary MPLs, yielding

$$I^{(A)}_F = \frac{k^2}{2} + (\xi_+ - \xi_-) \left\{ G \left( 0, \frac{1}{\xi_-}, 1 \right) - G \left( 0, \frac{1}{\xi_+}, 1 \right) \right\}, \quad (5.6)$$
with the standard definition for the MPLs

\[ G(a_1, \ldots, a_n; x) = \int_0^x \frac{dx'}{x' - a_1} G(a_2, \ldots, a_n; x'), \tag{5.7} \]

and \( \xi_{\pm} = \frac{1 \pm i \sqrt{3}}{2} \).

In its turn, the integral \( I_F^{(B)} \) cannot be written in terms of these ordinary MPLs. This is due to the fact that the integral involved is of elliptic type. In order to circumvent this problem, in this last decade an elliptic generalization of the MPLs (named of eMPLs) was proposed and developed, in a slight different forms, in a series of papers. The aim of this section is to compute \( I_F^{(B)} \) and express the result in terms of such eMPLs. Here we follow closely some of the notations and definitions stated in Refs. [69, 70], where the authors define the eMPLs \( E_4 \) as iterated integrals,

\[ E_4 \left( \frac{n_1 \cdots n_k}{c_1 \cdots c_k}; z \right) = \int_z^z dz' \psi_{n_1} (c_1, z') E_4 \left( \frac{n_2 \cdots n_k}{c_2 \cdots c_k}; z' \right), \tag{5.8} \]

with the suitable integration kernels \( \psi_{n_1} (c_1, z) \) defined by

\[
\begin{align*}
\psi_0 (0, z) &= \frac{\sqrt{(z_1 - z_3)(z_2 - z_4)}}{2\sqrt{P_4}}, \quad \psi_1 (0, z) = \frac{1}{z}, \\
\psi_{-1} (0, z) &= \frac{z_1 z_4}{z\sqrt{P_4}} - \frac{1}{z}, \quad \psi_{-1} (\infty, z) = \frac{z}{\sqrt{P_4}}, \\
\psi_{-2} (\infty, z) &= \frac{z}{z\sqrt{P_4}} Z_4 (z) - \frac{2}{\sqrt{(z_1 - z_3)(z_2 - z_4)}}, \\
\psi_{-2} (0, z) &= \frac{z_1 z_4}{z\sqrt{P_4}} Z_4 (z).
\end{align*}
\]

In the kernels defined above we have

\[ Z_4 (z) = \int_{z_1}^z dz \left\{ \tilde{\Phi}_4 (z) + 2\sqrt{(z_1 - z_3)(z_2 - z_4)} \left( \frac{E (\lambda)}{K (\lambda)} - \frac{(2 - \lambda)}{3} \right) \frac{1}{\sqrt{P_4}} \right\}, \tag{5.9} \]

\[
\tilde{\Phi}_4 (z) = \frac{2}{\sqrt{(z_1 - z_3)(z_2 - z_4)}\sqrt{P_4}} \left\{ z^2 - \frac{1}{2} (z_1 + z_2 + z_3 + z_4) z \\
+ \frac{1}{6} (z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4) \right\}, \tag{5.10} \]

with \( K (\lambda) \) and \( E (\lambda) \) being two well-known elliptic integrals,

\[ K (\lambda) = \int_0^1 dt \frac{1}{\sqrt{(1 - t^2)(1 - \lambda t^2)}}, \tag{5.11} \]

\[ E (\lambda) = \int_0^1 dt \frac{1 - \lambda t^2}{\sqrt{(1 - t^2)(1 - \lambda t^2)}}, \tag{5.12} \]

\[ \lambda = \frac{(z_1 - z_4)(z_2 - z_3)}{(z_1 - z_3)(z_2 - z_4)}. \tag{5.13} \]
If we adopt the following integral representation for the logarithm showed in Eq. (5.4), as suggested in [70], i.e.,

\[
\ln\left(\frac{(z - z_2) (z - z_3) - \sqrt{P_1}}{(z - z_2) (z - z_3) + \sqrt{P_1}}\right) = E_4\left(\frac{-1}{\infty}; z\right) - E_4\left(\frac{-1}{0}; z\right) - E_4\left(\frac{1}{0}; z\right),
\]

we obtain, after some algebraic manipulations,

\[
J_3^{(B)} = \frac{(k^2 + 3)}{2}
+ C_1 \left\{ E_4\left(\frac{-2}{\infty}; 1\right) - E_4\left(\frac{-2}{0}; 1\right) - Z_4(1) \left[ E_4\left(\frac{-1}{\infty}; 1\right) - E_4\left(\frac{-1}{0}; 1\right) \right] \right\}
+ C_2 \left\{ E_4\left(\frac{-1}{\infty}; 1\right) - E_4\left(\frac{-1}{0}; 1\right) + E_4\left(\frac{1}{0}; 1\right) \right\}
+ C_3 \left\{ E_4\left(\frac{0}{\infty}; 1\right) - E_4\left(\frac{0}{0}; 1\right) + E_4\left(\frac{1}{0}; 1\right) \right\},
\]

(5.15)

with the coefficients

\[
C_1 = \frac{(k^2 + 3) \sqrt{(z_1 - z_3)(z_2 - z_4)}}{4},
\]

(5.16)

\[
C_2 = \frac{(k^4 - 4k^2 + 3) - (k^2 + 3)(k^2 - 1)}{2k^2},
\]

(5.17)

\[
C_3 = \frac{(k^2 + 3)(k^4 - 3) - 6k^2(k^2 - 5)}{6k^4 \sqrt{(z_1 - z_3)(z_2 - z_4)}}
+ \frac{(k^2 + 3)}{2} \left( \frac{E(\lambda)}{K(\lambda)} - \frac{(2 - \lambda)}{3} \right) \sqrt{(z_1 - z_3)(z_2 - z_4)}.
\]

(5.18)

In this way, the formula obtained for \( J_{SS} \) in terms of eMPLs is very simple. In order to complete the analysis of the \( J_{SS} \) diagram, in the next section we will show the numerical analysis of the obtained results.

### 6 The numerical analysis

Let us complete our study about the sunrise diagram by performing a brief numerical analysis of the obtained result. With respect to the finite part obtained for \( J_{SS} \) in Eq. (3.21), the double integral over Feynman parameters

\[
F = \int_{0}^{1} dz \int_{0}^{1} dx \left[ \frac{3}{k^2} \frac{x}{x - (1 - z)} \right] \ln \left[ \frac{[k^2(z - (1 - z) + z - 1)] x(1 - x - z)}{(z - 1) x(1 - x - z)} \right],
\]

(6.1)

deserves more attention. For numerical calculations it is convenient to write (after integration in \( x \))

\[
F = \int_{0}^{1} dz \left\{ -\frac{3}{2k^2} \ln^2 \left( \frac{1 + \sqrt{X}}{1 - \sqrt{X}} \right) + \frac{(1 - z)}{\sqrt{X}} \ln \left( \frac{1 + \sqrt{X}}{1 - \sqrt{X}} \right) \right\} \{ X \to X_0 \},
\]

(6.2)
with
\[ X = \frac{(z - z_1)(z - z_4)}{(z - z_2)(z - z_3)}, \quad X_0 = \frac{1 - z}{1 + 3z}. \] (6.3)

The second integral, involving the variable \( X_0 \), is straightforward. Let us focus on the first integral, which we from now on will denominate \( \tilde{F} \). In the following, we split up our analysis into three possible regions of \( k^2 \).

If \( k^2 \leq 1 \), we have \( 0 \leq X \leq 1 \) for the full integration interval \( (0 \leq z \leq 1) \), which makes the numerical integration of (6.2) easily. In the interval \( 1 < k^2 < 9 \), the roots \( z_2 \) and \( z_3 \) are complex conjugate of each other. In this case, it is convenient to split the interval of integration into two parts, since
\[
\left\{ \begin{array}{l}
0 \leq X \leq 1 \text{ for } 0 \leq z \leq z_1 , \\
X \leq 0 \text{ for } z_1 \leq z \leq 1 .
\end{array} \right.
\]

Then, we find out that numerical integration of \( \tilde{F} \), within the interval \( 1 < k^2 < 9 \), is most easily performed if written in the form
\[
\tilde{F} (1 < k^2 < 9) = \int_0^{z_1} dz \left\{ -\frac{3}{2k^2} \ln^2 \left( \frac{1 + \sqrt{X}}{1 - \sqrt{X}} \right) + \frac{(1 - z)}{\sqrt{X}} \ln \left( 1 + \sqrt{X} \right) \right\}
+ \int_{z_1}^1 dz \left\{ \frac{6}{k^2} \arctan^2 \left( \sqrt{|X|} \right) + 2 \frac{(1 - z)}{\sqrt{|X|}} \arctan \left( \sqrt{|X|} \right) \right\} . \quad (6.4)
\]

In its turn, for \( k^2 \geq 9 \) (the three massive particle cut) there is a threshold at \( k^2 = 9 \) and all the roots are real and ordered \( z_1 < z_2 < z_3 < z_4 \). In this region we find
\[
\left\{ \begin{array}{l}
0 \leq X \leq 1 \text{ for } 0 \leq z \leq z_1 , \\
X \leq 0 \text{ for } z_1 \leq z \leq z_2 \text{ and } z_3 \leq z \leq 1 , \\
X > 1 \text{ for } z_2 \leq z \leq z_3 .
\end{array} \right.
\]

For numerical calculations, we found it convenient split out this integration region into the above intervals, which gives
\[
\tilde{F} (k^2 \geq 9) = 3\pi^2 \sqrt{(k^2 - 9)(k^2 - 1)}
- \int_0^{z_1} dz \left\{ \frac{3}{2k^2} \ln^2 \left( \frac{1 + \sqrt{X}}{1 - \sqrt{X}} \right) - \frac{(1 - z)}{\sqrt{X}} \ln \left( 1 + \sqrt{X} \right) \right\}
+ \int_{z_1}^{z_2} dz \left\{ \frac{6}{k^2} \arctan^2 \left( \sqrt{|X|} \right) + 2 \frac{(1 - z)}{\sqrt{|X|}} \arctan \left( \sqrt{|X|} \right) \right\}
- \int_{z_2}^{z_3} dz \left\{ \frac{3}{2k^2} \ln^2 \left( \frac{1 + \sqrt{X}}{1 - \sqrt{X}} \right) - \frac{(1 - z)}{\sqrt{X}} \ln \left( 1 + \sqrt{X} \right) \right\}
+ \int_{z_3}^1 dz \left\{ \frac{6}{k^2} \arctan^2 \left( \sqrt{|X|} \right) + 2 \frac{(1 - z)}{\sqrt{|X|}} \arctan \left( \sqrt{|X|} \right) \right\}
+ i\pi \Theta (k^2 - 9) \int_{z_2}^{z_3} dz \left\{ \frac{(1 - z)}{\sqrt{X}} - \frac{3}{k^2} \ln \left( \frac{1 + \sqrt{X}}{1 - \sqrt{X}} \right) \right\} . \quad (6.5)
\]
With the formulae presented above, one can easily build up the graph of $F$ as a function of $k^2$ by using some standard numerical package to perform the integrals. It is showed in the Fig. (2).

![Graph](image)

**Figure 2.** Representation of the function $F$.

7 Final remarks and conclusions

In the present work we considered in details the explicit calculation of the well-known two-loop equal mass sunrise Feynman diagram. The investigation was based mainly on the ideas developed in the alternative method described in Section 2. This referred strategy gives us a consistent framework to promote investigations in perturbative calculations in situations where the traditional methods are not consistent or not applicable. It does not have limitations of applicability since it works equally well in even and odd space-time dimensions, in the presence of tensors and pseudo-tensors amplitudes in even dimensions and in the context of renormalizable and nonrenormalizable theories. In particular, in all situations where the DR applies, one could find, in principle, a map that puts the results in a precise correspondence. This represents an obvious advantage in calculations involving multi-loops, since that almost all results available on such issue are performed within the DR context and would be desirable to have at our disposal an alternative approach in order to be possible to compare the results. Many previously investigations made revealed the general and consistent character of this approach when applied to one-loop calculations. In the present paper we have shown that this method can be equally applicable for perturbative calculations involving two-loop graphs, without need to add any new rule, when compared with one-loop calculations. The procedure is simple to apply, consisting in rewrite the
integrand of divergent Feynman integrals, by means of an alternative representation of propagators, into a sum of terms splitted into two distinct classes.

The first of such classes is formed by terms which have denominators that are momenta and masses dependent, but are finite when integrated over the loop momenta. Therefore, these finite integrals are ready to be integrated over the loop momenta and the results are written through integrals over Feynman parameters. In the Section 3 we showed that, for the sunrise graph calculation, this class is represented by the following terms (see eq. (3.21))

\[
\frac{1}{(4\pi)^2} \frac{p^2}{4} \int_0^1 dx \left\{ 1 - \frac{2}{x(1-x)-1} + \frac{2\ln[x(1-x)]}{x(1-x)-1} \right\}
\]

\[
+ \frac{1}{(4\pi)^2} \int_0^1 dy \int_0^1 dx \left[ \frac{3m^2}{x} - p^2 (1-y) \right] \ln \left\{ \frac{p^2 y (1-y) + (m^2 - \mu^2) y - m^2}{(m^2 - \mu^2) y - m^2} \right\},
\] (7.1)

The first integral above can be done easily and the second one is more involving since it is of elliptic type, which is not a surprise at all. In the Section 5, we in particular provide the explicit representation of this integral in terms of eMPLs functions, which are, in some sense, generalizations of the well-known MPLs functions. The expression obtained has a simple algebraic form.

The second class contain divergent integrals which are momenta and masses (physical) independent. Such terms are not integrated out, since they are undefined quantities. Instead, they are organized through (tensor) surface terms and scalar basic divergent objects. The calculation of \( J_{SS} \) revealed that this class are composed by

\[
\left[ I^{(2L)}_{quad} (m^2) \right] - \frac{i}{32\pi^2} \left\{ p^2 I^{(4)}_{\log} (m^2) \right\} + p^\mu p^\nu \left[ \Delta^{(4)}_{\mu\nu} (m^2) \right],
\] (7.2)

where \( I^{(4)}_{\log} \) and \( \Delta^{(4)}_{\mu\nu} \) are typical quantities of one-loop integrals and \( I^{(2L)}_{quad} \) is a object which is present in two-loop calculations. One can note that, in the adopted strategy, such divergent quantities emerge having coefficients which are polynomial in the external momentum and, therefore, they could be naturally removed in a renormalization process without the requirement of evaluating the integral over the internal momenta, which is ill-defined. In this sense, one can say that no regularization is need in practice.

The procedure applied in this work can also be used to calculate others topologies of two-loop graphs or yet graphs of higher order in the perturbative expansion, with no restriction of applicability. Since it has simple and universal rules, independent of Feynman integral considered, its implementation is easy and systematic. Thus, it can be considered, at least, an alternative consistent method to DR for treat divergent multi-loop graph. Beside that, many techniques invented to deal with dimensionally regulated integrals can also be used together with the method, as we have given an example in Section 3, where we used the technique of integration-by-parts in order to obtain the identities (3.9) and (3.10).

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