BOUND ON VAN DER WAERDEN NUMBERS
AND SOME RELATED FUNCTIONS

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Abstract

For positive integers $s$ and $k_1, k_2, \ldots, k_s$, let $w(k_1, k_2, \ldots, k_s)$ be the minimum integer $n$ such that any $s$-coloring $\{1, 2, \ldots, n\} \rightarrow \{1, 2, \ldots, s\}$ admits a $k_i$-term arithmetic progression of color $i$ for some $i$, $1 \leq i \leq s$. In the case when $k_1 = k_2 = \cdots = k_s = k$ we simply write $w(k; s)$. That such a minimum integer exists follows from van der Waerden’s theorem on arithmetic progressions. In the present paper we give a lower bound for $w(k, m)$ for each fixed $m$. We include a table with values of $w(k, 3)$ which match this lower bound closely for $5 \leq k \leq 16$. We also give an upper bound for $w(k, 4)$, an upper bound for $w(4; s)$, and a lower bound for $w(k; s)$ for an arbitrary fixed $k$. We discuss a number of other functions that are closely related to the van der Waerden function.

1. Introduction

Two fundamental theorems in combinatorics are van der Waerden’s Theorem [19] and Ramsey’s Theorem [15]. The theorem of van der Waerden says, in particular, that for any two given positive integers $k$ and $m$, there exists a least positive integer $n = w(k, m)$ such that whenever the integers in $[1, n] = \{1, 2, \ldots, n\}$ are colored with two colors (i.e., partitioned into two sets), there is either a $k$-term arithmetic progression of the first color (i.e., contained in the first set) or an $m$-term arithmetic progression of the second color (i.e., contained in the second set).

Similarly, Ramsey’s Theorem has an associated “threshold” function $R(k, m)$ (which we will not define here). This function satisfies the inequality

$$R(k, m) \leq R(k - 1, m) + R(k, m - 1),$$

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which leads to an upper bound on $R(k, m)$ that is not so much larger than the best known lower bounds on $R(k, m)$ obtained by other means. Furthermore, the order of magnitude of $R(k, 3)$ is known to be $\frac{k^2}{\log k}$ [10].

For van der Waerden’s function $w(k, m)$ there is no corresponding recursive inequality known, and the order of magnitude of $w(k, 3)$ is not known. The best known lower and upper bounds on $w(k, k)$ are

$$(k - 1)2^{(k-1)} \leq w(k, k) < 2^{2^{2^{k+1}}}$$

the lower bound known only when $k - 1$ is prime. The lower bound is due to Berlekamp [2] and the upper bound to Gowers [6]. Narrowing this gap is a fundamental problem in Ramsey theory. Ron Graham, who had a long-standing offer of 1000 USD for a proof or disproof of $w(k, k) < 2^{2^{2^{2^{k-1}}}}$, a tower of $k$ 2s, paid S. Shelah 500 USD for Shelah’s improvement [17] of the bound obtainable from van der Waerden’s original proof, and paid T. Gowers 1000 USD for Gowers’s upper bound. Graham currently offers 1000 USD [3] for a proof or disproof of $w(k, k) < 2^{k^2}$.

Recently, there have been two breakthroughs in the study of the van der Waerden function $w(k, m)$. The first was the elegant proof by Graham [7] that if one defines $w_1(3, s)$ to be the least $n$ such that every 2-coloring of $[1, n]$ gives either a 3-term arithmetic progression in the first color or $s$ consecutive numbers in the second color, then

$$s^{\log s} < w_1(3, s) < s^{2^{s^2}},$$

for suitable constants $c, d > 0$. Of course this immediately gives $w(k, 3) < k^{dk^2}$ since we trivially have $w(k, 3) = w(3, k) \leq w_1(3, k)$. The second was the amazing (computer) calculation $w(6, 6) = 1132$ by Kouril [11], extending the list of previously known values $w(3, 3) = 9, w(4, 4) = 35, \text{ and } w(5, 5) = 178$. A list of other known exact values of $w(k, m)$ appears in [13]. In view of Graham’s bounds on $w_1(3, s)$, it would be extremely desirable to obtain improved bounds on $w(k, 3)$. Of particular interest is the question of whether or not there is a non-polynomial lower bound for $w(k, 3)$.

In this note we give a lower bound of $k^{(2-o(1))} < w(k, 3)$. This seems weak, although we have $w(k, 3) < k^2$, for $5 \leq k \leq 16$ (see Table 1). It is our hope that others will find the question of obtaining improved bounds for $w(k, 3)$ to be interesting. Ultimately, one would like to find the true order of growth of the functions $w(k, 3), w(k, 4), \ldots, w(k, k)$. Perhaps this will be accomplished in this century, perhaps not! Our quest for a lower bound for $w(k, 3)$ turned (quite naturally) into a lower bound for $w(k, m)$ for an arbitrary fixed $m$. We also present an upper bound for $w(k, 4)$, an upper bound for $w(4; s)$, and a lower bound for $w(k; s)$ for an arbitrary fixed $k$. (The function $w(k; s)$ is defined below, in Section 2). Section 2 contains the just-mentioned bounds. In Section 3 we define several other related functions and discuss some relationships among these various functions. We also provide a table of values of these functions for small values of $s$ when $k = 3$. 

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Note that we use $c$ and $d$ repeatedly to stand for positive constants, but that these constants generally differ from paragraph to paragraph. The context will always make clear the meaning of a particular constant.

2. Upper and Lower Bounds for Certain van der Waerden Functions

We shall need several definitions, which we collect here.

For positive integers $k$ and $n$,

$$r_k(n) = \max_{S \subseteq [1,n]} \{|S| : S \text{ contains no } k\text{-term arithmetic progression}\}.$$

For positive integers $k$ and $m$, denote by $\chi_k(m)$ the minimum number of colors required to color $[1,m]$ so that there is no monochromatic $k$-term arithmetic progression.

The function $w_1(3,s)$ has been defined in Section 1. Similarly, we define $w_1(k,s)$ to be the least $n$ such that every 2-coloring of $[1,n]$ admits either a $k$-term arithmetic progression of the first color or $s$ consecutive integers of the second color.

Lastly, for positive integers $k$ and $s$, we denote the least positive integer $n$ such that every $s$-coloring of $[1,n]$ admits a monochromatic $k$-term arithmetic progression by $w(k,s)$.

We begin with an upper bound for $w_1(4,s)$. The proof is essentially the same as the proof given by Graham [7] of an upper bound for $w_1(3,s)$. For completeness, we include the proof here. We will make use of a recent result of Green and Tao [9], who showed that for some constant $c > 0$,

$$r_4(n) < ne^{-c\sqrt{\log \log n}},$$

for all $n \geq 3$.

**Proposition 2.1** There exists a constant $c > 0$ such that $w_1(4,s) < e^{cs\log s}$ for all $s \geq 2$.

**Proof.** Suppose we have a 2-coloring of $[1,n]$ (assume $n \geq 4$) with no 4-term arithmetic progression of the first color and no $s$ consecutive integers of the second color. Let $t_1 < t_2 < \cdots < t_m$ be the integers of the first color. Hence, $m < r_4(n)$. Let us define $t_0 = 0$ and $t_{m+1} = n$. Then there must be some $i, 1 \leq i \leq m$, such that

$$t_{i+1} - t_i > \frac{n}{2r_4(n)}.$$

(Otherwise, using $r_4(n) \geq 3$, we would have $n = \sum_{i=0}^{m}(t_{i+1} - t_i) \leq \frac{n(m+1)}{2r_4(n)} \leq \frac{n(r_4(n)+1)}{2r_4(n)} \leq \frac{n}{2} + \frac{n}{6}$.)

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Using (1), we now have an $i$ with
\[ t_{i+1} - t_i > \frac{n}{2r_4(n)} > \frac{1}{2} e^{c \sqrt{\log \log n}}. \]

If $n \geq e^{d \log s}$, $d = c^{-2}$, then $\frac{1}{2} e^{c \sqrt{\log \log n}} \geq s$ and we have $s$ consecutive integers of the second color, a contradiction. Hence, $n < e^{d \log s}$ and we are done. \qed

Clearly $w(4, s) \leq w_1(4, s)$. Consequently, we have the following result.

**Corollary 2.2** There exists a constant $d > 0$ such that $w(k, 4) < e^{k \log k}$ for all $k \geq 2$.

Using Green and Tao’s result, it is not too difficult to obtain an upper bound for $w(4; s)$.

**Proposition 2.3** There exists a constant $d > 0$ such that $w(4; s) < e^{d \log s}$ for all $s \geq 2$.

**Proof.** Consider a $\chi_4(m)$-coloring of $[1, m]$ for which there is no monochromatic 4-term arithmetic progression. Some color must be used at least $\frac{m}{\chi_4(m)}$ times, and hence $\frac{m}{\chi_4(m)} \leq r_4(m)$ so that $m \leq \chi_4(m)$. Let $c > 0$ be such that (1) holds for all $n \geq 3$, and let $m = e^{d \log s}$, where $d = c^{-2}$. Then $\chi_4(m) \geq \frac{m}{r_4(m)} > e^{c \sqrt{\log \log m}} = s$. This means that every $s$-coloring of $[1, m]$ admits a monochromatic 4-term arithmetic progression. Since $m = e^{d \log s}$, the proof is complete. \qed

It is interesting that the bounds in Corollary 2.2 and Proposition 2.3 have the same form.

The following theorem is deduced without too much difficulty from the Symmetric Hypergraph Theorem as it appears in [8], combined with an old result of Rankin [16]. To the best of our knowledge it has not appeared in print before, even though it is better, for large $s$, than the standard bound $\frac{e^k}{k} (1 + o(1))$ (see [8]), the bound $s^{k+1} - \sqrt{c(k+1) \log(k+1)}$ by Erdős and Rado [4], and the bound $\frac{k^3}{\log(k+1)^5}$ by Everts [5]. We give the proof in some detail. The proof makes use of the following facts:

\[ \chi_k(n) < \frac{2n \log n}{r_k(n)} (1 + o(1)), \quad (2) \]

which appears in [8] as a consequence of the Symmetric Hypergraph Theorem, and

\[ r_k(n) > n e^{-c \log n \frac{1}{1+t}}, \quad (3) \]

which, for some constant $c > 0$, holds for all $n \geq 3$ (this appears in [16]).

**Theorem 2.4** Let $k \geq 3$ be fixed, and let $z = \lfloor \log_2 k \rfloor$. There exists a constant $d > 0$ such that $w(k; s) > s^{d \log s} z$ for all sufficiently large $s$. 4
Proof. Fix $k \geq 3$ and let $z = \lfloor \log_2 k \rfloor$. Note that for positive integers $s$ and $m$,

\[ [s \geq \chi_k(m)] \Rightarrow [w(k; s) > m]. \]

This observation, which can be verified by unraveling the definitions, is an essential ingredient of the proof.

For large enough $m$, (2) gives

\[
\chi_k(m) < \frac{2m \log m}{r_k(m)} \left(1 + \frac{1}{2}\right) = \frac{3m \log m}{r_k(m)}.
\]

(4)

Now let $d = \left(\frac{1}{2e}\right)^{z+1}$, and let $m = s^{d(log s)^z}$, where $s$ is large enough so that (1) holds. By (3), noting that $\log m = d(log s)^z+1 = \left(\frac{\log s}{2}\right)^{z+1}$, we have

\[
\frac{m}{r_k(m)} < e^{c(log n)^{z+1}} = e^{c \log s} = \sqrt{s}.
\]

Therefore,

\[
\frac{3m \log m}{r_k(m)} < 3d \sqrt{s}(log s)^{z+1} < s
\]

for sufficiently large $s$. Thus, for sufficiently large $s$,

\[
\chi_k(m) < \frac{3m \log m}{r_k(m)} < s.
\]

According to the observation at the beginning of the proof, this implies that $w(k; s) > m = s^{d(log s)^z}$, as required.

We now give a lower bound on $w(k, m)$. We make use of the László Local Lemma (see [8] for a proof), which will be implicitly stated in the proof.

**Theorem 2.5** Let $m \geq 3$ be fixed. Then for all sufficiently large $k$,

\[
w(k, m) > k^{m-1 - \frac{1}{2 \log \log k}}.
\]

**Proof.** Given $m$, choose $k > m$ large enough so that

\[
k^{\frac{1}{2m \log \log k}} > \left(m - \frac{1}{2 \log \log k}\right) \log k
\]

(5)

and

\[
6 < \frac{\log k}{\log \log k}.
\]

(6)
Next, let \( n = \lfloor k^{m - \frac{1}{2m \log \log k}} \rfloor \). To prove the theorem, we will show that there exists a (red, blue)-coloring of \([1, n] \) for which there is no red \( k \)-term arithmetic progression and no blue \( m \)-term arithmetic progression.

For the purpose of using the L\'ovasz Local Lemma, randomly color \([1, n] \) in the following way. For each \( i \in [1, n] \), color \( i \) red with probability \( p = 1 - k^{\alpha - 1} \) where

\[
\alpha \overset{\text{def}}{=} \frac{1}{2m \log \log k},
\]

and color it blue with probability \( 1 - p \).

Let \( \mathcal{P} \) be any \( k \)-term arithmetic progression. Then, since \( 1 + x \leq e^x \), the probability that \( \mathcal{P} \) is red is

\[
p^k = \left(1 - k^{\alpha - 1}\right)^k \leq (e^{-k^{\alpha - 1}})^k = e^{-k^\alpha}.
\]

Hence, applying (5), we have

\[
p^k < \left(\frac{1}{e}\right)^{\left(m - \frac{1}{2m \log \log k}\right) \log k} = \frac{1}{k^{m - \frac{1}{2m \log \log k}}}.
\]

Also, for \( \mathcal{Q} \) any \( m \)-term arithmetic progression, the probability that \( \mathcal{Q} \) is blue is

\[
(1 - p)^m = (k^{\alpha - 1})^m = \frac{1}{k^{m - \frac{1}{2m \log \log k}}}.
\]

Now let \( \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t \) be all of the arithmetic progressions in \([1, n] \) with length \( k \) or \( m \). So that we may apply the Lovasz Local Lemma, we form the “dependency graph” \( G \) by setting \( V(G) = \{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_t\} \) and \( E(G) = \{\{\mathcal{P}_i, \mathcal{P}_j\} : i \neq j, \mathcal{P}_i \cap \mathcal{P}_j \neq \emptyset\} \). For each \( \mathcal{P}_i \in V(G) \), let \( d(\mathcal{P}_i) \) denote the degree of the vertex \( \mathcal{P}_i \) in \( G \), i.e., \( |\{e \in E(G) : \mathcal{P}_i \in e\}| \). We now estimate \( d(\mathcal{P}_i) \) from above. Let \( x \in [1, n] \). The number of \( k \)-term arithmetic progressions \( \mathcal{P} \) in \([1, n] \) that contain \( x \) is bounded above by \( k \cdot \frac{n}{k - 1} \), since there are \( k \) positions that \( x \) may occupy in \( \mathcal{P} \) and since the gap size of \( \mathcal{P} \) cannot exceed \( \frac{n}{k - 1} \). Similarly, the number of \( m \)-term arithmetic progressions \( \mathcal{Q} \) in \([1, n] \) that contain \( x \) is bounded above by \( m \cdot \frac{n}{m - 1} \).

Let \( \mathcal{P}_i \) be any \( k \)-term arithmetic progression contained in \([1, n] \). The total number of \( k \)-term arithmetic progressions \( \mathcal{P} \) and \( m \)-term arithmetic progressions \( \mathcal{Q} \) in \([1, n] \) that can intersect \( \mathcal{P}_i \) is bounded above by

\[
k \left( k \cdot \frac{n}{k - 1} + m \cdot \frac{n}{m - 1} \right) < kn \left(2 + \frac{2}{m - 1}\right),
\]

since \( k > m \). Thus, \( d(\mathcal{P}_i) < kn \left(2 + \frac{2}{m - 1}\right) \) when \( |\mathcal{P}_i| = k \). Likewise, \( d(\mathcal{P}_i) < mn \left(2 + \frac{2}{m - 1}\right) \) when \( |\mathcal{P}_i| = m \). Thus, for all vertices \( \mathcal{P}_i \) of \( G \), we have \( d(\mathcal{P}_i) < kn \left(2 + \frac{2}{m - 1}\right) \).
To finish setting up the hypotheses for the L"{o}v{a}sz Local Lemma, we let $X_i$ denote the event that the arithmetic progression $P_i$ is
\[
\begin{cases}
\text{red} & \text{if } |P_i| = k \\
\text{blue} & \text{if } |P_i| = m.
\end{cases}
\]

We have seen above that for all $i$, $1 \leq i \leq t$, the probability of the event $X_i$ is less than
\[
q \overset{\text{def}}{=} \frac{1}{k^{m-1} \log \log k}.
\]

Let $d = \max_{1 \leq i \leq t} d(P_i)$. We showed above that
\[
d < 2kn \left(1 + \frac{1}{m-1}\right).
\]

We are now ready to apply the L"{o}v{a}sz Local Lemma, which says that in these circumstances, if the condition $eq(d+1) < 1$ is satisfied, then there is a (red, blue)-coloring of $[1, n]$ such that no event $X_i$ occurs, i.e., there is a (red, blue)-coloring of $[1, n]$ for which there is no red $k$-term arithmetic progression and no blue $m$-term arithmetic progression. This will imply
\[
w(k, m) > n = k^{m-1} \frac{1}{\log \log k},
\]
as desired. Thus, the proof will be complete when we verify that $eq(d+1) < 1$. Using $m \geq 3$, we have $d < 3kn$, so that $d+1 < 3kn+1 < e^2kn$. Hence, it is sufficient to verify that
\[
e^3qkn < 1.
\]
(7)

Since $q = \frac{1}{k^{m-1} \log \log k}$ and $n \leq k^{m-1} \frac{1}{\log \log k}$, inequality (7) may be reduced to (6), and the proof is now complete.

\[\square\]

Remark. For condition (5), it suffices to have $k^{\frac{1}{2m \log \log k}} > m \log k$, or $\log k > 2m \log m (\log \log k) + 2m (\log \log k)^2$. When $k \geq e^{e^3}$, this condition becomes $e^3 > 2m^4 \log m + 2m^7$. Since, for $m \geq 3$, we have $e^m > m^9 > 2m^4 \log m + 2m^7$, having $k > e^{e^3}$ is sufficient for both (5) and (6).

3. Some Related Functions

In this section we define some functions related to $w(k, m)$ and mention various bounds for, and relationships among, these. For reference, we define all functions used in this section (including those already defined).
The least positive integer $n$ such that every 2-coloring of $[1, n]$ admits either a $k$-term arithmetic progression of the first color or an $s$-term arithmetic progression of the second color.

The least positive integer $n$ such that every 2-coloring of $[1, n]$ admits either a $k$-term arithmetic progression of the first color or $s$ consecutive integers of the second color.

The least positive integer $n$ such that every $s$-coloring of $[1, n]$ admits a monochromatic $k$-term arithmetic progression.

The least positive integer $n$ such that for every set $S = \{x_1 < x_2 < \cdots < x_n\}$ with $x_i - x_{i-1} \leq s$, $2 \leq i \leq n$, $S$ contains a $k$-term arithmetic progression.

$G(k, s)$ denotes the least positive integer $n$ such that whenever $X = \{x_1, x_2, \ldots, x_n\}$ and $x_i \in [(i-1)s, is - 1]$, $1 \leq i \leq n$, there is a $k$-term arithmetic progression in $X$.

$w^*(k; s)$ denotes the least positive integer $n$ such that every $s$-coloring $\chi: [1, n] \to [1, s]$ admits either a monochromatic $k$-term arithmetic progression or a $k$-term arithmetic progression whose colors form an arithmetic progression (increasing or decreasing).

We start with the following inequalities involving $w_1(k, s)$.

**Proposition 3.1** For any positive integers $k$ and $s$, the following hold:

(i) $w_1(k, s) \leq sM(k, s)$;

(ii) $w_1(k, s) \leq sG(k, s)$;

(iii) $w_1(k, s) \leq w(k; s) + s$.

**Proof.** As the proofs of (i) and (ii) are quite similar, we include the proof of (i) and leave the other to the reader. Let $m = M(k, s)$ and let $n = sm$. Let $\chi$ be any (red, blue)-coloring of $[1, n]$. Assume there are no $s$ consecutive blue integers. So, for each $i$, $1 \leq i \leq m$, the interval $[(i-1)s + 1, is]$ contains a red element, say $a_i$. Then, by the definition of $M(k, s)$, there is a $k$-term arithmetic progression among the $a_i$’s.

We now show (iii). By definition, there exists a (red, blue)-coloring of $[1, w_1(k, s) - 1]$ with no red $k$-term arithmetic progression and no $s$ consecutive blue elements. Let the red elements under this coloring be $R = \{r_1 < r_2 < \cdots < r_t\}$. Note that $r_1 \leq s$ and $r_t \geq w_1(k, s) - s - 1$. Define the following $s$-coloring of $[r_1, w_1(k, s) - 1]$. Color all elements in $R$ with color 0. For $i = 1, 2, \ldots, s - 1$, in order, color all elements in $(R + i) \setminus \bigcup_{j=0}^{i-1}(R + j)$ with color $i$. This is well defined since $r_{x+1} - r_x \leq s$ for any $x$. Since $R$ contains no $k$-term arithmetic progression, none of $(R + i) \setminus \bigcup_{j=0}^{i-1}(R + j)$ contain a $k$-term arithmetic progression. Since $[r_1, w_1(k, s) - 1]$ contains at least $w_1(k, s) - s - 1$ elements and our $s$-coloring admits no monochromatic $k$-term arithmetic progression, we see that $w(k; s) \geq w_1(k, s) - s$. \qed

**Remark.** Using $w(4, k) \leq w_1(4, k)$ and part (iii) from the above proposition, we see that Proposition 2.1 and Corollary 2.2 follow from Proposition 2.3, without appealing to Graham’s argument.
In the next proposition, we give an alternate way of describing $M(3, s)$. Before doing so, we introduce some terminology. A 3-term $ap^+$ is an ordered triple of the form $x, x + d, x + 2d + 1$ where $x, d \geq 1$. A 3-term $ap^−$ is an ordered triple of the form $x, x + d, x + 2d − 1$, where $x \geq 1$ and $d \geq 2$. Note that, in either case, the three terms are distinct. Finally, an ordered triple $a, b, c$ is called an arithmetic progression (mod $s$) if $c − b \equiv b − a$ (mod $s$).

We make use of the following lemma.

**Lemma 3.2** Let $s \geq 2$. For $i \in \mathbb{Z}^+$, let $B_i = [(i − 1)s + 1, is]$. For $j \in \mathbb{Z}^+$, define $r_j$ to be the unique member of $\{1, 2, \ldots, s\}$ such that $j \equiv r_j \pmod{s}$. Let $1 \leq x < y < z$ with $x \in B_{i_1}$, $y \in B_{i_2}$, and $z \in B_{i_3}$. Then $x, y, z$ is an arithmetic progression if and only if one of the following holds:

1. A 3-term $ap$ is an ordered triple of the form $x, x + d, x + 3d + 1$.
2. For $i \geq 2$, $i \equiv i_1 \equiv i_2 \equiv i_3 \pmod{s}$.
3. For $i < 2$, $i \equiv i_1 \equiv i_2 \equiv i_3 \pmod{s}$.
4. For $i = 0$, $i \equiv i_1 \equiv i_2 \equiv i_3 \pmod{s}$.
5. For $i = 1$, $i \equiv i_1 \equiv i_2 \equiv i_3 \pmod{s}$.

**Proof.** If (i) holds, then for some $d$ satisfying $|d| \leq \lfloor \frac{s−1}{2} \rfloor$ we have

$$z − y = (i_3 − i_2)s + d = (i_2 − i_1)s + d = y − x.$$  

Now assume (ii) holds. Note that $r_y − r_x = s + r_z − r_y$. Thus, $z − y = (i_3 − i_2)s − r_y + r_z = (i_2 − i_1 + 1)s − r_y + r_z = (i_2 − i_1)s + (s − r_y + r_z) = (i_2 − i_1)s + r_y − r_z = y − x$.

Now assume (iii) holds. In this case $r_x − r_y = s + r_y − r_x$. Therefore,

$$z − y = (i_3 − i_2)s + r_z − r_y = (i_2 − i_1)s − s + r_z − r_y = (i_2 − i_1)s + r_y − r_x = y − x.$$  

For the converse, it suffices to consider three cases.

**Case 1.** $r_x \leq r_y \leq r_z$ or $r_x \geq r_y \geq r_z$. In this case, it is clear that $i_3 − i_2 = i_2 − i_1$.

**Case 2.** $r_x < r_y$ and $r_z \leq r_y$. In this case, $i_3 − i_2 > i_2 − i_1$. Furthermore, $i_3 − i_2 \geq i_2 − i_1 + 2$ is not possible, since then we would have $z − y \geq y − x + s + 1$. Hence $i_3 − i_2 = i_2 − i_1 + 1$, so that $i_1, i_2, i_3$ is an $ap^+$. Also,

$$z − y = (i_3 − i_2)s − (s − r_y + r_z),$$  

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and
\[ y - x = (i_2 - i_1)s - (r_y - r_x). \]
So \( r_y - r_x \equiv r_z - r_y \pmod{s} \).

**Case 3.** \( r_x > r_y \) and \( r_z \geq r_y \). The proof is almost the same as that for Case 2, and we leave it to the reader. \( \square \)

**Proposition 3.3** For all \( s \geq 2 \), \( M(3, s) \) is the least positive integer \( n \) such that every \( s \)-coloring \( \chi : [1, n] \to [1, s] \) admits a triple \( A = \{a < b < c\} \) satisfying one of the following:

(i) \( A \) is an arithmetic progression and \( \chi(b) - \chi(a) = \chi(c) - \chi(b) \) (possibly negative or 0);

(ii) \( A \) is a 3-term \( ap^+ \) and \( (\chi(a), \chi(b), \chi(c)) \), with \( \chi(a) < \chi(b) \), is an arithmetic progression \( \pmod{s} \), but not an arithmetic progression;

(iii) \( A \) is a 3-term \( ap^- \) and \( (\chi(a), \chi(b), \chi(c)) \), with \( \chi(a) > \chi(b) \), is an arithmetic progression \( \pmod{s} \), but not a (decreasing) arithmetic progression.

**Proof.** Let an \( s \)-coloring of \( [1, n] \) be given, using the colors 1, 2, \ldots, \( s \). We use this coloring to define a set \( \{x_1, x_2, \ldots, x_n\} \) as follows: For each \( i \), let \( x_i \) be that element of the block \( B_i = [(i - 1)s + 1, is] \) which is congruent to the color of \( i \). By Lemma 3.2, the minimum \( n \) such that any set \( \{x_1, x_2, \ldots, x_n\} \) constructed in this way must contain a 3-term arithmetic progression is \( M(3, s) \). \( \square \)

The following inequalities are proved by Nathanson [14].

**Theorem 3.4** (Nathanson) For all positive integers \( k \) and \( s \),

(i) \( G(k, s) \leq sM(k, s) \),

(ii) \( M(k, s) \leq G(k, 2s - 1) \),

(iii) \( G(k, s) \leq w(k; s) \),

(iv) \( M(k, s) \leq w(k; s) \),

(v) \( w(k; s) \leq M(s(k - 1) + 1, s) \), and

(vi) \( w(k; s) \leq G(s(k - 1) + 1, 2s - 1) \).

Investigating \( G(k, s) \), Alon and Zaks [1] have shown the following result. Evaluating their bound when \( s = 2 \) gives, to our eyes, a surprising result since we may view this (loosely speaking) as a 2-coloring of \( [1, 2^{k(1+o(1))}] \) with no monochromatic \( k \)-term arithmetic progression, where one of the color classes has no two consecutive integers. It is surprising that they proved a lower bound for such a restricted family of colorings that is almost as large as the best known lower bound for \( w(k, k) \).
Theorem 3.5 (Alon and Zaks) For every \( s \geq 2 \), there exists a constant \( c > 0 \) (dependent upon \( s \)) such that
\[
G(k, s) > s^{k-c\sqrt{k}}
\]
for all \( k \geq 3 \).

Note that, from part (vi) of Theorem 3.4, an upper bound for \( G(k, s) \) would give an upper bound for \( w(k; s) \). In particular, an upper bound on \( G(k, 3) \) for odd \( k \) would lead to an upper bound for \( w(k, 3) \).

We now offer a few more inequalities involving the functions discussed in this section. We use Szemerédi’s result on arithmetic progressions [18] to prove part (iii) of Proposition 3.6, which improves (for large \( s \)) part (i) of Theorem 3.4.

Proposition 3.6 For all positive integers \( k \) and \( s \), the following hold.

(i) \( w(k, s) \leq w_1(k, s) \)

(ii) \( M(k, s) \leq w^*(k; s) \leq w(k; s) \)

(iii) Let \( c > 0 \) be constant. For \( k \) fixed and \( s \) sufficiently large, \( G(k, s) < cM(k, s) \).

(iv) \( w_1(k, 2s - 1) \geq s(M(k, s) - 1) + 1 \)

Proof. Parts (i) and (ii) are immediate from the definitions and Proposition 3.3. To prove (iii), we assume that \( cs \) is an integer and let \( t = csM(k, s) \). Let \( X = \{x_1 < x_2 < \cdots < x_t\} \) be a set such that \( x_i - x_{i-1} \leq s \) for \( i = 2, 3, \ldots, t \). We may assume \( x_1 \in [1, s] \). We must show that \( X \) contains a \( k \)-term arithmetic progression. Define \( X_i = \{x_{(i-1)cs+1}, \ldots, x_{ics}\} \) for \( i = 1, 2, \ldots, M(k, s) \). If for some \( i \) we have \( X_i \subseteq [(i-1)s + 1, is] \) then, for \( s \) sufficiently large, Szemerédi’s result on arithmetic progressions tells us that \( X_i \) must contain a \( k \)-term arithmetic progression. Assuming this is not the case, we must then have that each interval \([[(i-1)s + 1, is] \) for \( 1 \leq i \leq M(k, s) \) contains at least one element of \( X \). By the definition of \( M(k, s) \), we have our \( k \)-term arithmetic progression.

We now prove (iv). We will show that if \( s \) is even, then
\[
w_1(k, s - 1) \geq \frac{s}{2} \left( M \left( k, \frac{s}{2} \right) - 1 \right) + 1.
\]
The case when \( s \) is odd is similar and is left to the reader. Let \( n = M \left( k, \frac{s}{2} \right) \). Then there exists \( X = \{x_1, x_2, \ldots, x_{n-1}\} \) containing no \( k \)-term arithmetic progression and such that \( x_i \in \left[ \frac{(i-1)s}{2} + 1, \frac{is}{2} \right] \) for \( 1 \leq i \leq n - 1 \). Consider the following 2-coloring of \( \left[ 1, \frac{(n-1)s}{2} \right] \colon \chi(x) = 0 \) for each \( x_i \in X \), and \( \chi(x) = 1 \) otherwise. Clearly there is no \( k \)-term arithmetic progression with color 0. Also, since each interval \( \left[ \frac{(i-1)s}{2} + 1, \frac{is}{2} \right] \) contains an element with color 0, the longest
A string of consecutive elements with color 1 has length not exceeding $s - 2$, which implies the desired result. \hfill \Box

We end with a table of computed values. These were all computed with a standard backtrack algorithm except for $w(3, 14), w(3, 15)$, and $w(3, 16)$, which are due to Michal Kouril [12]. Based on the values in this table, we make the following conjecture (only the last inequality is known to hold).

**Conjecture** For all $s \geq 2$,

$$G(3, s) \leq w(3, s) \leq M(3, s) \leq w_1(3, s) \leq w^*(3, s) \leq w(3; s).$$

|    | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|----|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|
| $G(3, s)$ | 5 | 9 | 11 | 17 | 22 | 33 | 37 | 48 | ? | ? | ? | ? | ? | ? | ? |
| $w(3, s)$ | 6 | 9 | 18 | 22 | 32 | 46 | 58 | 77 | 97 | 114 | 135 | 160 | 186 | 218 | 238 |
| $M(3, s)$ | 7 | 11 | 18 | 29 | 37 | 48 | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| $w_1(3, s)$ | 9 | 23 | 34 | 73 | 113 | 193 | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| $w^*(3, s)$ | 9 | 23 | 40 | $\geq 75$ | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |
| $w(3; s)$ | 9 | 27 | 76 | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? | ? |

**Table 1: Small values of van der Waerden-like functions**

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