**L-PACKETS AND FORMAL DEGREES FOR SL₂(K) WITH K A LOCAL FUNCTION FIELD OF CHARACTERISTIC 2**

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Abstract. Let \( \mathcal{G} = \text{SL}_2(K) \) with \( K \) a local function field of characteristic 2, we review Artin-Schreier theory for the field \( K \), and show that this leads to a parametrization of \( L \)-packets in the smooth dual of \( \mathcal{G} \). We relate this to a recent geometric conjecture. The \( L \)-packets in the principal series are parametrized by quadratic extensions, and the supercuspidal \( L \)-packets by biquadratic extensions. We compute the formal degrees of the elements in the supercuspidal packets.

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1. Introduction

In this article we consider a local function field \( K \) of characteristic 2, namely \( K = F_q((\varpi)) \), the field of Laurent series with coefficients in \( F_q \), with \( q = 2^f \). This example is particularly interesting because there are countably many quadratic extensions of \( F_q((\varpi)) \).

We consider \( \mathcal{G} = \text{SL}_2(K) \). Drawing on the accounts in [5, 16, 17], we review Artin-Schreier theory, adapted to the local function field \( F_q((\varpi)) \). This leads to a parametrization of \( L \)-packets in the smooth dual of \( \mathcal{G} \). In this article, we reserve the term \( L \)-packets for the ones which are not singletons.
The $L$-packets in the principal series are parametrized by quadratic extensions, and the supercuspidal $L$-packets by biquadratic extensions. There are countably many supercuspidal packets.

By canonical formal degree we shall mean formal degree with respect to the Euler-Poincaré measure on $G$, as in [12]. We compute the canonical formal degrees of the elements in the supercuspidal packets, relying on the Formal Degree component of the local Langlands correspondence, see [12, §6]. The canonical formal degrees are all dyadic rationals, in fact they are integer powers of 2. They depend on the residue degree $f$, and on the breaks in the lower ramification filtration of the Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

The commutative triangle in Theorem 4.3, and the bijective maps in §5, amount to a proof, for $G$, of the tempered version of the geometric conjecture in [1].

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2. Artin-Schreier theory

Let $K$ be a local field with positive characteristic $p$. The cyclic extensions of $K$ whose degree $n$ is coprime with $p$ are described by Kummer theory. It is well known that any cyclic extension $L/K$ of degree $n$, $(n,p) = 1$, is generated by a root $\alpha$ of an irreducible polynomial $x^n - a \in K[x]$. If $\alpha \in K^s$ is a root of $x^n - a$ then $K(\alpha)/K$ is a cyclic extension of degree $n$ and is called a Kummer extension of $K$.

Artin-Schreier theory aims to describe cyclic extensions of degree equal to or divisible by $\text{ch}(K) = p$. It is therefore an analogue of Kummer theory, where the role of the polynomial $x^n - a$ is played by $x^p - x - a$. Essentially, every cyclic extension of $K$ with degree $p = \text{ch}(K)$ is generated by a root $\alpha$ of $x^p - x - a \in K[x]$.

We fix an algebraic closure $\overline{K}$ of $K$ and a separable closure $K^s$ of $K$ in $\overline{K}$. Let $\phi$ denote the Artin-Schreier endomorphism of the additive group $K^s$ [9]:

$$\phi : K^s \to K^s, \quad x \mapsto x^p - x.$$ 

Given $a \in K$ denote by $K(\phi^{-1}(a))$ the extension $K(\alpha)$, where $\phi(\alpha) = a$ and $\alpha \in K^s$. We have the following characterization of finite cyclic Artin-Schreier extensions of degree $p$:

**Theorem 2.1.**

(i) Given $a \in K$, either $\phi(x) - a \in K[x]$ has one root in $K$ in which case it has all the $p$ roots are in $K$, or is irreducible.

(ii) If $\phi(x) - a \in K[x]$ is irreducible then $K(\phi^{-1}(a))/K$ is a cyclic extension of degree $p$, with $\phi^{-1}(a) \subset K^s$. 


(iii) If $L/K$ be a finite cyclic extension of degree $p$, then $L = K(\phi^{-1}(a))$, for some $a \in K$.

(See [16, p.34] for more details.)

We fix now some notation. $K$ is a local field with characteristic $p > 1$ with finite residue field $k$. The field of constants $k = \mathbb{F}_q$ is a finite extension of $\mathbb{F}_p$, with degree $[k : \mathbb{F}_p] = f$ and $q = p^f$.

Let $\mathfrak{o}$ be the ring of integers in $K$ and denote by $\mathfrak{p} \subset \mathfrak{o}$ the (unique) maximal ideal of $\mathfrak{o}$. This ideal is principal and any generator of $\mathfrak{p}$ is called a uniformizer. A choice of uniformizer $\varpi \in \mathfrak{o}$ determines isomorphisms $K \cong \mathbb{F}_q((\varpi))$, $\mathfrak{o} \cong \mathbb{F}_q[[\varpi]]$ and $\mathfrak{p} = \varpi \mathfrak{o} \cong \varpi \mathbb{F}_q[[\varpi]]$.

A normalized valuation on $K$ will be denoted by $\nu$, so that $\nu(\varpi) = 1$ and $\nu(K) = \mathbb{Z}$. The group of units is denoted by $\mathfrak{o}^\times$.

2.1. The Artin-Schreier symbol. Let $L/K$ be a finite Galois extension. Let $N_{L/K}$ be the norm map and denote $G_{L/K}^{ab} = \text{Gal}(L/K)^{ab}$ the abelianization of $\text{Gal}(L/K)$. The reciprocity map is a group isomorphism

\[
K^\times / N_{L/K} L^\times \xrightarrow{\cong} G_{L/K}^{ab}.
\]

The Artin symbol is obtained by composing the reciprocity map with the canonical morphism $K^\times \to K^\times / N_{L/K} L^\times$

\[
b \in K^\times \mapsto (b, L/K) \in G_{L/K}^{ab}.
\]

From the Artin symbol we obtain a pairing

\[
K \times K^\times \to \mathbb{Z}/p\mathbb{Z}, (a, b) \mapsto (b, L/K)(\alpha) - \alpha,
\]

where $\varphi(\alpha) = a$, $\alpha \in K^\times$ and $L = K(\alpha)$.

**Definition 2.2.** Given $a \in K$ and $b \in K^\times$, the Artin-Schreier symbol is defined to be

\[
[a, b] = (b, L/K)(\alpha) - \alpha.
\]

We summarize some important properties of the Artin-Schreier symbol.

**Proposition 2.3.** The Artin-Schreier symbol is a bilinear map satisfying the following properties:

1. $[a_1 + a_2, b] = [a_1, b] + [a_2, b]$;
2. $[a, b_1 b_2] = [a, b_1] + [a, b_2]$;
3. $[a, b] = 0$, $\forall a \in K \iff b \in N_{L/K} L^\times$, $L = K(\alpha)$ and $\varphi(\alpha) = a$;
4. $[a, b] = 0$, $\forall b \in K^\times \iff a \in \varphi(K)$.

(See [16, p.341])
2.2. The groups $K/\wp(K)$ and $K^\times/K^{\times p}$. In this section we recall some properties of the groups $K/\wp(K)$ and $K^\times/K^{\times p}$ and use them to redefine the pairing (2.3).

Consider the additive group $K$. The index of $\wp(K)$ in $K$ is infinite \cite[p.146]{6}. Hence, $K/\wp(K)$ is infinite.

**Proposition 2.4.** $K/\wp(K)$ is a discrete abelian torsion group.

**Proof.** The ring of integers decomposes as a (direct) sum

$$\mathcal{o} = \mathbb{F}_q + \mathfrak{p}$$

and we have

$$\wp(\mathcal{o}) = \wp(\mathbb{F}_q) + \wp(\mathfrak{p}).$$

The restriction $\wp : \mathfrak{p} \to \mathfrak{p}$ is an isomorphism, see \cite[Lemma 8]{5}. Hence,

$$\wp(\mathcal{o}) = \wp(\mathbb{F}_q) + \mathfrak{p}$$

and $\mathfrak{p} \subset \wp(K)$. It follows that $\wp(K)$ is an open subgroup of $K$ and $K/\wp(K)$ is discrete. Since $\wp(K)$ is annihilated by $p$, $K/\wp(K)$ is a torsion group. \hfill \square

Now we concentrate on the multiplicative group $K^\times$. The subgroup $K^{\times p}$ is not open in $K^{\times}$ and the index $[K^{\times} : K^{\times p}]$ is infinite \cite[Lemma p.115]{6}. Hence, $K^\times/K^{\times p}$ is infinite. The next result gives a characterization of the topological group $K^\times/K^{\times p}$.

**Proposition 2.5.** $K^\times/K^{\times p}$ is a profinite abelian $p$-torsion group.

**Proof.** There is a canonical isomorphism $K^\times \cong \mathbb{Z} \times \mathcal{o}^\times$. By \cite[p.25]{8}, the group of units $\mathcal{o}^\times$ is a direct product of countable many copies of the ring of $p$-adic integers

$$\mathcal{o}^\times \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p \times \ldots = \prod_{\mathbb{N}} \mathbb{Z}_p.$$ 

Give $\mathbb{Z}$ the discrete topology and $\mathbb{Z}_p$ the $p$-adic topology. Then, for the product topology, $K^\times = \mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p$ is a topological group, locally compact, Hausdorff and totally disconnected.

$K^{\times p}$ decomposes as a product of countable many components

$$K^{\times p} \cong p\mathbb{Z} \times p\mathbb{Z}_p \times p\mathbb{Z}_p \times \ldots = p\mathbb{Z} \times \prod_{\mathbb{N}} p\mathbb{Z}_p.$$ 

Denote by $y = \prod_n y_n$ and element of $\prod_{\mathbb{N}} \mathbb{Z}_p$, where $y_n = \sum_{i=0}^{\infty} a_{i,n} p^i \in \mathbb{Z}_p$, for every $n$.

The map

$$\varphi : \mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}_p \to \mathbb{Z}/p\mathbb{Z} \times \prod_{\mathbb{N}} \mathbb{Z}/p\mathbb{Z}, (x,y) \mapsto (x(mod p), \prod_n \text{pr}_0(y_n))$$

where $\text{pr}_0(y_n) = a_{0,n}$ is the projection, is clearly a group homomorphism.
Now, $\mathbb{Z}/p\mathbb{Z} \times \prod N \mathbb{Z}/p\mathbb{Z} = \prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a topological group for the product topology, where each component $\mathbb{Z}/p\mathbb{Z}$ has the discrete topology. Moreover, it is compact by Tychonoff Theorem, Hausdorff and totally disconnected [2, TGI.84, Prop. 10]. Therefore, $\prod_{n=0}^{\infty} \mathbb{Z}/p\mathbb{Z}$ is a profinite group.

Since $\ker \varphi = p\mathbb{Z} \times \prod N p\mathbb{Z} p$, it follows that there is an isomorphism of topological groups

$$K^* / K^{xp} \cong \prod N p\mathbb{Z},$$

where $K^* / K^{xp}$ is given the quotient topology. Therefore, $K^* / K^{xp}$ is profinite. □

From Propositions 6.1 and 2.5 $K/\varphi(K)$ is a discrete abelian group and $K/K^{xp}$ is an abelian profinite group, both annihilated by $p = ch(K)$. Therefore, Pontryagin duality coincides with $Hom(-, \mathbb{Z}/p\mathbb{Z})$ on both of these groups, see [17]. See also [13] for more details on Pontryagin duality. The pairing (2.3) restricts to a pairing

$$(2.1) \quad [.,.] : K/\varphi(K) \times K^* / K^{xp} \to \mathbb{Z}/p\mathbb{Z},$$

which we refer from now on to the Artin-Schreier pairing. It follows from (iii) and (iv) of Proposition 2.3 the pairing is nondegenerate (see also [17, Proposition 3.1]). The next result shows that the pairing is perfect.

**Proposition 2.6.** The Artin-Schreier symbol induces isomorphisms of topological groups

$$K^* / K^{xp} \xrightarrow{\sim} Hom(K/\varphi(K), \mathbb{Z}/p\mathbb{Z}), bK^{xp} \mapsto (a + \varphi(K) \mapsto [a, b])$$

and

$$K/\varphi(K) \xrightarrow{\sim} Hom(K^* / K^{xp}, \mathbb{Z}/p\mathbb{Z}), a + \varphi(K) \mapsto (bK^{xp} \mapsto [a, b])$$

**Proof.** The result follows by taking $n = 1$ in Proposition 5.1 of [17], and from the fact that Pontryagin duality for the groups $K/\varphi(K)$ and $K^* / K^{xp}$ coincide with $Hom(-, \mathbb{Z}/p\mathbb{Z})$ duality. Hence, there is an isomorphism of topological groups between each such group and its bidual. □

Let $B$ be a subgroup of the additive group of $K$ with finite index such that $\varphi(K) \subseteq B \subseteq K$. The composite of two finite abelian Galois extensions of exponent $p$ is again a finite abelian Galois extension of exponent $p$. Therefore, the composite

$$K_B = K(\varphi^{-1}(B)) = \prod_{a \in B} K(\varphi^{-1}(a))$$
is a finite abelian Galois extension of exponent $p$. On the other hand, if $L/K$ is a finite abelian Galois extension of exponent $p$, then $L = K_B$ for some subgroup $\varphi(K) \subseteq B \subseteq K$ with finite index.

All such extensions lie in the maximal abelian extension of exponent $p$, which we denote by $K_p = K(\varphi^{-1}(K))$. The extension $K_p/K$ is infinite and Galois. The corresponding Galois group $G_p = Gal(K_p/K)$ is an infinite profinite group and may be identified, under class field theory, with $K^\times/K^{\times p}$, see [17, Proposition 5.1]. The case $ch(K) = 2$ leads to $G_2 \cong K^\times/K^{\times 2}$ and will play a fundamental role in the sequel.

3. Quadratic characters

From now on we take $K$ to be a local function field with $ch(K) = 2$. Therefore, $K$ is of the form $\mathbb{F}_q((\varpi))$ with $q = 2^f$.

Recall that a character of $K^\times$ is a group homomorphism

$$\chi : K^\times \to \mathbb{T}$$

where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ is the unit circle. Denote by $\widehat{K^\times}$ the group of characters of $K^\times$. There is a canonical isomorphism

$$\widehat{K^\times} \cong \mathbb{Z} \times \mathfrak{o}^\times \cong \mathbb{T} \times \mathfrak{o}^\times.$$

Therefore, given a character $\chi \in \widehat{K^\times}$, we may write $\chi = z^\nu \chi_0$, where $z \in \mathbb{T}$, $\nu$ is the valuation and $\chi_0 \in \mathfrak{o}^\times$. If $\chi_0 \equiv 1$ we say that $\chi$ is unramified. A character $\chi$ of $K^\times$ is called quadratic if $\chi^2 = 1$. Since the unique quadratic character of $\mathbb{Z}$ is $n \mapsto (-1)^n$, a nontrivial quadratic character has the form $\chi = (-1)^\nu \chi_0$, with $\chi_0^2 = 1$.

When $K = \mathbb{F}_q((\varpi))$, we have, according to [8, p.25],

$$\mathfrak{o}^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \ldots = \prod_{N} \mathbb{Z}_2$$

with countably infinite many copies of $\mathbb{Z}_2$, the ring of 2-adic integers.

Artin-Schreier theory provides a way to parametrize all the quadratic extensions of $K = \mathbb{F}_q((\varpi))$. By Proposition 2.5 there is a bijection between the set of quadratic extensions of $\mathbb{F}_q((\varpi))$ and the group

$$\mathbb{F}_q((\varpi))^\times/\mathbb{F}_q((\varpi))^{\times 2} \cong \prod_{N} 2\mathbb{Z}_2 = G_2$$

where $G_2$ is the Galois group of the maximal abelian extension of exponent 2. Since $G_2$ is an infinite profinite group, there are countably many quadratic extensions.

To each quadratic extension $K(\alpha)/K$, with $\alpha^2 - \alpha = a$, we associate the Artin-Schreier symbol

$$[a, .) : K^\times/K^{\times 2} \to \mathbb{Z}/2\mathbb{Z}.$$
Now, let $\varphi$ denote the isomorphism $\mathbb{Z}/2\mathbb{Z} \cong \mu_2(\mathbb{C}) = \{\pm 1\}$ with the group of roots of unity. We obtain, by composing with the Artin-Schreier symbol, a unique multiplicative quadratic character $\chi_a = \varphi([a,.]):$

$$
\chi_a : K^\times \rightarrow \mathbb{C}^\times.
$$

Proposition 2.6 shows that every quadratic character of $F_q((\wp))$ arises in this way.

Example 3.1. The unramified quadratic extension of $K$ is $K(\wp^{-1}(\mathfrak{o}))$, see [5] proposition 12. According to Dalawat, the group $K/\wp(K)$ may be regarded as an $F_2$-space and the image of $\mathfrak{o}$ under the canonical surjection $K \rightarrow K/\wp(K)$ is an $F_2$-line, i.e., isomorphic to $F_2$. Since $\varphi|_p : p \rightarrow p$ is an isomorphism, the image of $p$ in $K/\wp(K)$ is $\{0\}$, see lemma 8 in [5]. Now, choose any $a_0 \in \mathfrak{o}\setminus p$. The quadratic character $\chi_{a_0} = \varphi([a_0,.])$ associated with $K(\wp^{-1}(\mathfrak{o}))$ via class field theory is precisely the unramified character $(n \mapsto (-1)^n)$ from above. Note that any other choice $b_0 \in \mathfrak{o}\setminus p$, with $a_0 \neq b_0$, gives the same unique unramified character, since there is only one nontrivial coset $a_0 + \wp(K)$ for $a_0 \in \mathfrak{o}\setminus p$.

Let $G$ denote $SL_2(K)$, let $B$ be the standard Borel subgroup of $G$, let $T$ be the diagonal subgroup of $G$. Let $\chi$ be a character of $T$. Then, $\chi$ inflates to a character of $B$. Denote by $\pi_\chi$ the (unitarily) induced representation $\text{Ind}_{B}^{G}(\chi)$. The representation space of $V_\chi$ of $\pi_\chi$ consists of locally constant complex valued functions $f : G \rightarrow \mathbb{C}$ such that, for every $a \in K^\times$, $b \in K$ and $g \in G$, we have

$$
f\left(\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}\right) = |a|\chi(a)f(g)
$$

The action of $G$ on $V_\chi$ is by right translation. The representations $(\pi_\chi, V_\chi)$ are called (unitary) principal series of $G = SL_2(K)$.

Let $\chi$ be a quadratic character of $K^\times$. The reducibility of the induced representation $\text{Ind}_{B}^{G}(\chi)$ is well known in zero characteristic. W. Casselman proved that the same result holds in characteristic 2 and any other positive characteristic $p$.

Theorem 3.2. [3, 4] The representation $\pi_\chi = \text{Ind}_{B}^{G}(\chi)$ is reducible if, and only if, $\chi$ is either $|.|^\pm$ or a nontrivial quadratic character of $K^\times$.

For a proof see [3, Theorems 1.7, 1.9] and [4, §9].

From now on, $\chi$ will be a quadratic character. It is a classical result that the unitary principal series for $GL_2$ are irreducible. For a representation of $GL_2$ parabolically induced by $1 \otimes \chi$, Clifford theory tells
us that the dimension of the intertwining algebra of its restriction to $\text{SL}_2$ is 2. This is exactly the induced representation of $\text{SL}_2$ by $\chi$:

$$\text{Ind}^{\text{GL}_2(K)}_{\tilde{B}}(1 \otimes \chi)_{|\text{SL}_2(K)} \xrightarrow{\sim} \text{Ind}^{\text{SL}_2(K)}_{B}(\chi)$$

where $\tilde{B}$ denotes the standard Borel subgroup of $\text{GL}_2(K)$. This leads to reducibility of the induced representation $\text{Ind}^G_B(\chi)$ into two inequivalent constituents. Thanks to M. Tadic for helpful comments at this point.

The two irreducible constituents

$$(3.2) \quad \pi_\chi = \text{Ind}^G_B(\chi) = \pi^+_\chi \oplus \pi^-_\chi$$

define an $L$-packet $\{\pi^+_\chi, \pi^-_\chi\}$ for $\text{SL}_2$.

4. A COMMUTATIVE TRIANGLE

In this section we confirm part of the geometric conjecture in [1] for $\text{SL}_2(\mathbb{F}_q((\varpi)))$. We begin by recalling the underlying ideas of the conjecture.

Let $\mathcal{G}$ be the group of $K$-points of a connected reductive group over a nonarchimedean local field $K$. We have the Bernstein decomposition

$$\text{Irr}(\mathcal{G}) = \bigsqcup \text{Irr}(\mathcal{G})^s$$

over all points $s \in \mathcal{B}(\mathcal{G})$ the Bernstein spectrum of $\mathcal{G}$, see [14].

Let $\chi_{a_0} = \varphi([a_0],.)$ denote the unramified character of $K^\times$ associated with the unramified quadratic extension $K(a_0) = K(\varphi^{-1}(a))$ as in example 3.1. Fix a quadratic character $\chi_a = \varphi([a,])$ associated via class field theory with the quadratic extension $K(\alpha)$ (in a fixed algebraic closure $K$), where $\alpha^2 - \alpha = a$.

**Proposition 4.1.** There is a unique quadratic extension $K(\beta)$ with associated character $\chi_{a_0+a}$. Moreover, $\chi_{a_0+a} = \chi_{a_0}\chi_a$.

**Proof.** The compositum $K(\alpha)K(a_0)$ is Galoisian over $K$, with Galois group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Therefore, contains three subfields, which are quadratic extensions of $K$, namely $K(a_0)$, $K(\alpha)$ and, say, $K(\beta)$. The extension $K(\beta)$ is such that $\beta^2 - \beta = a_0 + a$, and has an associated quadratic character given by $\chi_{a_0+a}$. Hence

$$\chi_{a_0+a} = \varphi([a_0+a,]) = \varphi([a_0,]+[a,]) = \varphi([a_0,.])\varphi([a,]) = \chi_{a_0}\chi_a.$$

By theorem 3.2, the induced representations

$$\pi_{a_0} = \text{Ind}^G_B(\chi_{a_0}) \quad \pi_a = \text{Ind}^G_B(\chi_a) \quad \text{and} \quad \pi_{a_0+a} = \text{Ind}^G_B(\chi_{a_0+a})$$

are reducible and split into a direct sum of two irreducible component.

Central to the geometric conjecture is the concept of extended quotient of the second kind, which we now define.
Let $W$ be a finite group and let $X$ be a complex affine algebraic variety. Suppose that $W$ is acting on $X$ as automorphisms of $X$. Define

$$\widetilde{X}_2 := \{(x, \tau) : \tau \in \text{Irr}(W_x)\}.$$ 

Then $W$ acts on $\widetilde{X}_2$:

$$\alpha(x, \tau) = (\alpha \cdot x, \alpha \ast \tau).$$

**Definition 4.2.** The extended quotient of the second kind is defined as

$$(X//W)_2 := \widetilde{X}_2/W.$$ 

Thus the extended quotient of the second kind is the ordinary quotient for the action of $W$ on $\widetilde{X}_2$.

**Theorem 4.3.** Let $G = \text{SL}_2(K)$ with $K = \mathbb{F}_q((\varpi))$. Let $s = [T, \chi]_G$ be a point in the Bernstein spectrum for the principal series of $G$. Let $\text{Irr}(G)^s$ be the corresponding Bernstein component in $\text{Irr}(G)$. Then the conjecture $[1]$ is valid for $\text{Irr}(G)^s$ i.e. there is a commutative triangle of natural bijections

$$\begin{array}{ccc}
(T^s//W^s)_2 & \longrightarrow & \mathcal{L}(G)^s \\
\text{Irr}(G)^s & \downarrow & \\
& \text{Irr}(G)^s & \end{array}$$

where $\mathcal{L}(G)^s$ denotes the equivalence classes of enhanced parameters attached to $s$.

**Proof.** We recall that $(G, T)$ are the complex dual groups of $(G, T)$. Let $W_K$ denote the Weil group of $K$. If $\varphi$ is an $L$-parameter

$$W_K \times \text{SL}_2(\mathbb{C}) \to G$$

then $S_\varphi$ is defined as follows:

$$S_\varphi := \pi_0 C_G(\text{im} \varphi).$$

By an enhanced Langlands parameter, we shall mean a pair $(\varphi, \rho)$ where $\varphi$ is a parameter and $\rho \in \text{Irr}(S_\varphi)$. Following Reeder [12], we shall denote an enhanced Langlands parameter by $\varphi(\rho)$.

**Case 1.** Let $\chi$ be a quadratic character of $T$: $\chi^2 = 1, \chi \neq 1$. Let $L/K$ be the quadratic extension determined by $\chi$. Now $G$ contains a unique (up to conjugacy) subgroup $H \simeq \mathbb{Z}/2\mathbb{Z}$. Each quadratic extension $L/K$ creates a parameter

$$\varphi_L : W_K \to \text{Gal}(L/K) \to G.$$ 

The map $\text{Gal}(L/K) \to H$ factors through $K^x/N_{L/K}L^x$:

$$\varphi_L : W_K \to \text{Gal}(L/K) \simeq K^x/N_{L/K}L^x \to H \to G.$$ 

which shows that $\varphi_L$ is the parameter attached to the packet $\pi_\chi$. 
To compute $S_{\varphi_L}$, let $1, w$ be representatives of the Weyl group $W = W(G)$. Then we have

$$C_G(\text{im } \varphi_L) = T \sqcup wT$$

So $\varphi$ is a non-discrete parameter, and we have

$$S_{\varphi_L} \simeq \mathbb{Z}/2\mathbb{Z}.$$  

We have two enhanced Langlands parameters, namely $\varphi_L(\text{triv})$ and $\varphi_L(\rho)$ where $\rho$ is the nontrivial character of $S_{\varphi_L}$.

Now define

$$\chi(\varpi) = \chi \begin{pmatrix} \varpi & 0 \\ 0 & \varpi^{-1} \end{pmatrix}$$

where $\varpi$ is a uniformizer in $K$.

Since $\chi^2 = 1$, there is a point of reducibility. We have, at the level of elements,

$$\{\chi(\varpi), \text{triv}, \chi(\varpi), \rho\} \rightarrow \{\varphi_0(\text{triv}), \varphi_1(\text{triv})\}$$

**Case 2.** Let $\chi = 1$. The principal parameter is the composite map $\varphi_0 : W_K \times \text{SL}_2(\mathbb{C}) \rightarrow \text{PSL}(2, \mathbb{C})$.

defined by extending the principal homomorphism $\text{SL}_2(\mathbb{C}) \rightarrow \text{PSL}_2(\mathbb{C})$ trivially on $W_K$, is a canonical discrete parameter for which $S_{\varphi_0} = 1$.

In the local Langlands correspondence for $G$, the enhanced parameter $\varphi_0(\text{triv})$ corresponds to the Steinberg representation $\text{St}_G$, see [12, 6.1.8].

Let $\varphi_1$ be the unique parameter for which $\varphi_1(W_K \times \text{SL}_2(\mathbb{C})) = 1$.

We have

$$\text{im } \varphi_1 = 1, \quad C_G(\text{im } \varphi_1) = G, \quad S_{\varphi_1} = 1.$$  

There is a unique enhanced parameter, namely $\varphi_1(\text{triv})$. We have, at the level of elements, the commutative triangle

$$\{\varphi_0(\text{triv}), \varphi_1(\text{triv})\} \rightarrow \{\varphi_0(\text{triv}), \varphi_1(\text{triv})\}$$

**Case 3.** $\chi^2 \neq 1$. There are no points of reducibility, and we have a commutative triangle of sets, each with one element:
Corollary 4.4. Let $L/K$ be a quadratic extension of $K$. The $L$-parameters $\varphi_L$ serve as parameters for the $L$-packets in the principal series of $\text{SL}_2(K)$.

It follows from §3 that there are countably many $L$-packets in the principal series of $\text{SL}_2(K)$.

5. The tempered dual

The following picture

\[ \text{\includegraphics[scale=0.5]{diagram.png}} \]

serves two purposes. First, it is an accurate portrayal of the extended quotient of the second kind

\[ (\mathbb{T}//W)_2 \]

where $\mathbb{T} = \{ z \in \mathbb{C} : |z| = 1 \}$ and the generator of $W = \mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{T}$ sending $z$ to $z^{-1}$. Secondly, it is (conjecturally) an accurate portrayal of a connected component in the tempered dual of $\mathcal{G}$.

The topology on $(\mathbb{T}//W)_2$ comes about as follows. Let

\[ \textbf{Prim}(C(\mathbb{T}) \rtimes W) \]

denote the primitive ideal space of the noncommutative $C^*$-algebra $C(\mathbb{T}) \rtimes W$. By the classical Mackey theory for semidirect products, we have a canonical bijection

\[ (5.1) \quad \textbf{Prim}(C(\mathbb{T}) \rtimes W) \simeq (\mathbb{T}//W)_2. \]

The primitive ideal space on the left-hand side of (5.1) admits the Jacobson topology. So the right-hand side of (5.1) acquires, by transport of structure, a compact non-Hausdorff topology. The picture at the beginning of this section is intended to portray this topology. We shall see that the Langlands parameters respect this topology. The double-points in the picture arise precisely when the corresponding induced representation has length 2.

The Plancherel Theorem of Harish-Chandra is valid for any local non-archimedean field, see Waldspurger [18]. This implies that, in the case at hand, the discrete series and the unitary principal series enter into the Plancherel formula. That is, the tempered dual of $\mathcal{G}$
comprises the discrete series and the irreducible constituents in the unitary principal series.

We now focus on the case of induced elements.

Suppose \( \chi^2 \neq 1 \), with \( s = [T, \chi]_G \). Let \( \psi \) be an unramified unitary character of \( T \). Then we have a natural bijection

\[
\text{Irr}^\text{temp}(G)^s \cong T, \quad \text{Ind} \pi_{\psi \chi} \mapsto \psi(\varpi).
\]

Suppose \( \chi^2 = 1, \chi \neq 1 \), with \( s = [T, \chi]_G \). Let \( W = \mathbb{Z}/2\mathbb{Z} \). Then we have a bijective map

\[
\text{Irr}^\text{temp}(G)^s \cong (T//W)_2.
\]

This map is defined as follows. Let \( \rho \) denote the nontrivial character of \( W \).

- If \( \psi^2 \neq 1 \), send \( \text{Ind} \pi_{\psi \chi} \) to \( \psi(\varpi) \).
- If \( \psi = 1 \), send the pair of irreducible constituents \( \pi^+_{\chi}, \pi^-_{\chi} \) to the pair of points \( (1, \text{triv}), (1, \rho) \in (T//W)_2 \).
- If \( \psi = \epsilon \) the unique unramified quadratic character of \( T \), send the pair of irreducible constituents \( \pi^+_{\epsilon \chi}, \pi^-_{\epsilon \chi} \) to the pair of points \( (-1, \text{triv}), (-1, \rho) \in (T//W)_2 \).

Suppose \( \chi = 1 \) and let \( s_0 = [T, 1]_G \). Then we have a continuous bijection which is not a homeomorphism:

\[
\text{Irr}^\text{temp}(G)^s \to (T//W)_2.
\]

- If \( \psi^2 \neq 1 \), send \( \text{Ind} \pi_{\psi \chi} \) to \( \psi(\varpi) \).
- If \( \psi = 1 \), send the irreducible representations \( \text{triv}_G, \text{St}_G \) to the pair of points \( (1, \text{triv}), (1, \rho) \in (T//W)_2 \).
- If \( \psi = \epsilon \) send the pair of irreducible constituents \( \pi^+_{\epsilon}, \pi^-_{\epsilon} \) to the pair of points \( (-1, \text{triv}), (-1, \rho) \in (T//W)_2 \).

By proposition 4.1 and the above argument, we may represent that part of the tempered dual \( \text{Irr}^\text{temp}(\text{SL}_2(p(\varpi))) \) which corresponds to the unitary principal series in a diagram along the lines of [11, p.418].

\[
\pi_0 \quad \pi_1 \quad \pi_a \quad \pi_{a_0} \quad \pi_{a_1} \quad \ldots
\]

The first double point represent the \( L \)-packet \( \{\pi^+_{a_0}, \pi^-_{a_0}\} \). The second and third double-points represent, respectively, the \( L \)-packets \( \{\pi^+_a, \pi^-_a\} \) and \( \{\pi^+_{a_0+a}, \pi^-_{a_0+a}\} \). The second half-circle is repeated countably many times, and is parametrized by \( L \)-parameters \( \{\varphi_a\}_{a \in p(K)} \), see theorem 4.1.

**Topology on the tempered dual.** Let \( G = \text{SL}_2(p(\varpi)) \). The tempered dual of \( G \) is the disjoint union \( X = X_G \) of the discrete series and the irreducible constituents in the principal series. We equip
X with the following topology \( \mathfrak{T} \): The topology \( \mathfrak{T} \) must induce the standard topologies on each point, each copy of \( \mathbb{T} \), and each copy (except one) of \( (\mathbb{T}//\mathcal{W})_2 \), all of which (except one) must become \( \mathfrak{T} \)-open sets. On the exceptional copy of \( (\mathbb{T}//\mathcal{W})_2 \) the Steinberg point \( \text{St}_G \) must be \( \mathfrak{T} \)-isolated. Then \( \mathfrak{T} \) is a locally compact topology on \( X \). It is not Hausdorff.

In the space \( X \), each \( L \)-packet in the unitary principal series will feature as a \( \mathfrak{T} \)-double-point.

There will be countably many double-points, one for each quadratic extension \( K(\alpha); \text{ cf. the diagram in } [11] \) for the tempered dual of \( \text{SL}_2(\mathbb{Q}_p) \) with \( p > 2 \). In that diagram, there are just three double-points. For \( \text{SL}_2(\mathbb{Q}_2) \) there would be seven double-points.

Each supercuspidal \( L \)-packet will feature as four \( \mathfrak{T} \)-isolated points in \( X \).

We conjecture that \( \mathfrak{T} \) coincides with the Jacobson topology on the primitive ideal space of the reduced \( C^* \)-algebra of \( \mathcal{G} \).

6. Biquadratic extensions of \( \mathbb{F}_q((\varpi)) \)

Biquadratic extensions \( L/K \) are obtained by adjoining an \( \mathbb{F}_2 \)-line \( D \subset K/\varphi(K) \). Therefore, \( L = K(\varphi^{-1}(D)) = \hat{K}(\alpha) \) where \( D = \text{span}(a + \varphi(K)) \). In particular, if \( a_0 \) is integer, the \( \mathbb{F}_2 \)-line \( V_0 = \text{span}(a_0 + \varphi(K)) \) contains all the cosets \( a_i + \varphi(K) \) where \( a_i \) is an integer and so \( K(\varphi^{-1}(a)) = K(\varphi^{-1}(V_0)) = K(a_0) \) where \( a_0^2 - a_0 = a_0 \) gives the unramified quadratic extension.

Biquadratic extensions are computed the same way, by considering planes \( W = \text{span}(a + \varphi(K), b + \varphi(K)) \subset K/\varphi(K) \). Therefore, if \( a + \varphi(K) \) and \( b + \varphi(K) \) are \( \mathbb{F}_2 \)-linearly independent then \( K(\varphi^{-1}(W)) := K(\alpha, \beta) \) is biquadratic, where \( \alpha^2 - \alpha = a \) and \( \beta^2 - \beta = b \), \( \alpha, \beta \in K^s \). Therefore, \( K(\alpha, \beta)/K \) is biquadratic if \( b - a \not\in \varphi(K) \).

A biquadratic extension containing the line \( V_0 \) is of the form \( K(\alpha_0, \beta)/K \). There are countably many quadratic extensions \( L_0/K \) containing the unramified quadratic extension. They have ramification index \( e(L_0/K) = 2 \). And there are countably many biquadratic extensions \( L/K \) which do not contain the unramified quadratic extension. They have ramification index \( e(L/K) = 4 \).

So, there is a plentiful supply of biquadratic extensions \( K(\alpha, \beta)/K \).

6.1. Ramification. The space \( K/\varphi(K) \) comes with a filtration

\[
(6.1) \quad 0 \subset V_0 \subset f V_1 = V_2 \subset f V_3 = V_4 \subset f ... \subset K/\varphi(K)
\]

where \( V_0 \) is the image of \( o_K \) and \( V_i \) \( (i > 0) \) is the image of \( p^{-i} \) under the canonical surjection \( K \rightarrow K/\varphi(K) \). For \( K = \mathbb{F}_q((\varpi)) \) and \( i > 0 \), each inclusion \( V_{2i} \subset f V_{2i+1} \) is a sub-\( \mathbb{F}_2 \)-space of codimension \( f \). The
\[ \dim_{\mathbb{F}_2} V_n = 1 + \lceil n/2 \rceil, \]

where \( \lceil x \rceil \) is the smallest integer not less than \( x \).

Let \( L/K \) denote a Galois extension with Galois group \( G \). For each \( i \geq -1 \) we define the \( i \)-th ramification subgroup of \( G \) (in the lower numbering) to be:

\[ G_i = \{ \sigma \in G : \sigma(x) - x \in p^{i+1}_L, \forall x \in \mathfrak{o}_L \}. \]

An integer \( t \) is a break for the filtration \( \{ G_i \}_{i \geq -1} \) if \( G_t \neq G_{t+1} \). The study of ramification groups \( \{ G_i \}_{i \geq -1} \) is equivalent to the study of breaks of the filtration.

There is another decreasing filtration with upper numbering \( \{ G^u \}_{i \geq -1} \) and defined by the Hasse-Herbrand function \( \psi = \psi_{L/K} \):

\[ G^u = G_{\psi(u)}. \]

In particular, \( G^{-1} = G_{-1} = G \) and \( G^0 = G_0 \), since \( \psi(0) = 0 \).

Let \( G_2 = \text{Gal}(K_2/K) \) be the Galois group of the maximal abelian extension of exponent 2, \( K_2 = K(\varphi^{-1}(K)) \). Since \( G_2 \cong K^x/K^{x^2} \) (Proposition 2.5), the pairing \( K^x/K^{x^2} \times K/\varphi(K) \to \mathbb{Z}/2\mathbb{Z} \) from (2.1) coincides with the pairing \( G_2 \times K/\varphi(K) \to \mathbb{Z}/2\mathbb{Z} \).

The profinite group \( G_2 \) comes equipped with a ramification filtration \( \{ G^u \}_{u \geq -1} \) in the upper numbering, see [5, p.409]. For \( u \geq 0 \), we have an orthogonal relation [5, Proposition 17]

\[ (G^u_2)^\perp = \mathfrak{p}^{-|u|+1} = V_{[u]-1} \]

under the pairing \( G_2 \times K/\varphi(K) \to \mathbb{Z}/2\mathbb{Z} \).

Since the upper filtration is more suitable for quotients, we will first compute the upper breaks and then use the Hasse-Herbrand function to compute the lower breaks in order to obtain the lower ramification filtration.

According to [5, Proposition 17], the positive breaks in the filtration \( (G^v)_v \) occur precisely at integers prime to \( p \). So, for \( ch(K) = 2 \), the positive breaks will occur at odd integers. The lower numbering breaks are also integers. If \( G \) is cyclic of prime order, then there is a unique break for any decreasing filtration \( (G^v)_v \) (see [5], Proposition 14). In general, the number of breaks depends on the possible filtration of the Galois group.

Given a plane \( W \subset K/\varphi(K) \), the filtration \( \{ V_i \} \) on \( K/\varphi(K) \) induces a filtration \( \{ W_i \} \) on \( W \), where \( W_i = W \cap V_i \). There are three possibilities for the filtration breaks on a plane and we will consider each case individually.
**Case 1:** $W$ contains the line $V_0$, i.e. $L_0 = K(\wp^{-1}(W))$ contains the unramified quadratic extension $K(\wp^{-1}(V_0)) = K(\alpha_0)$ of $K$. The extension has residue degree $f(L_0/K) = 2$ and ramification index $e(L_0/K) = 2$. In this case, there is an integer $t > 0$, necessarily odd, such that the filtration $(W_i)$ looks like

\[
0 \subset_1 W_0 = W_{t-1} \subset_1 W_t = W.
\]

By the orthogonality relation (6.3), the upper ramification filtration on $G = Gal(L_0/K)$ looks like

\[
\{1\} = \ldots = G^{t+1} \subset_1 G^t = \ldots = G^0 \subset_1 G^{-1} = G
\]

Therefore, the upper ramification breaks occur at $-1$ and $t$. The lower ramification breaks can be computed using the Hasse-Herbrand function. The table for the index of $G^u$ in $G^0$ is as follows:

| $u$ \in [0,t] \cup [t, +\infty] | $G^u$ | $(G^0 : G^u)$ |
|----------------|-------|----------------|
| $G^0$          | $G^0$ | $\{1\}$       |
| $(G^0 : G^u)$  | 1     | 2              |

We have, $\psi(t) = \int_0^t (G^0 : G^u) du = t$, and the lower ramifications breaks occur at $-1$ and $t$. It follows that the lower filtration is

(6.4) $G_{-1} = G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$; $G_0 = \ldots = G_t = \mathbb{Z}/2\mathbb{Z}$; $G_{t+1} = \{1\}$

The number of such $W$ is equal to the number of planes in $V_t$ containing the line $V_0$ but not contained in the subspace $V_{t-1}$. Note that this number can be computed and equals the number of biquadratic extensions of $K$ containing the unramified quadratic extensions and with a pair of upper ramification breaks $(-1, t)$, $t > 0$ and odd.

**Example 6.1.** The number of biquadratic extensions containing the unramified quadratic extension and with a pair of upper ramification breaks $(-1, 1)$ is equal to the number of planes in an $1 + f$-dimensional $\mathbb{F}_2$-space, containing the line $V_0$. There are precisely

\[
1 + 2 + 2^2 + \ldots + 2^{f-1} = \frac{1 - 2^f}{1 - 2} = q - 1
\]

of such biquadratic extensions.

**Case 2.1:** $W$ does not contain the line $V_0$ and the induced filtration on the plane $W$ looks like

\[
0 = W_{t-1} \subset_2 W_t = W
\]

for some integer $t$, necessarily odd.
The number of such $W$ is equal to the number of planes in $V_t$ whose intersection with $V_{t-1}$ is $\{0\}$. Note that, there are no such planes when $f = 1$. So, for $K = \mathbb{F}_2((x))$, case 2.1 does not occur.

Suppose $f > 1$. By the orthogonality relation, the upper ramification filtration on $G = \text{Gal}(L/K)$ looks like
\[
\{1\} = \ldots = G^{t+1} \subset_2 G^t = \ldots = G^{-1} = G
\]
Therefore, there is a single upper ramification break occur at $t > 0$ and necessarily odd. The lower ramification breaks occurs at the same $t$, since we have:
\[
\begin{align*}
  u &\in [0, t] \cup [t, +\infty] \\
  G^u &= G^0 \quad \{1\} \\
  (G^0 : G^u) &= 1 \quad 2^2
\end{align*}
\]
and so, $\psi(t) = \int_0^t (G^0 : G^u)du = t$, and the lower ramifications breaks occur at $-1$ and $t$. It follows that the lower filtration is
\[
(6.5) \quad G_{-1} = G = \ldots = G_t = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ; G_{t+1} = \{1\}
\]
For $f = 1$ there is no such biquadratic extension. For $f > 1$, the number of these biquadratic extensions equals the number of planes $W$ contained in an $\mathbb{F}_2$-space of dimension $1 + fi, t = 2i - 1$, which are transverse to a given codimension-$f \mathbb{F}_2$-space.

**Case 2.2**: $W$ does not contains the line $V_0$ and the induced filtration on the plane $W$ looks like
\[
0 = W_{t_1 - 1} \subset_1 W_{t_1} = W_{t_2 - 1} \subset_1 W_{t_2} = W
\]
for some integers $t_1$ and $t_2$, necessarily odd, with $0 < t_1 < t_2$.

The orthogonality relation for this case implies that the upper ramification filtration on $G = \text{Gal}(L/K)$ looks like
\[
\{1\} = \ldots = G^{t_2+1} \subset_1 G^{t_2} = \ldots = G^{t_1+1} \subset_1 G^{t_1} = \ldots = G
\]
The upper ramification breaks occur at odd integers $t_1$ and $t_2$.

Now, index of $G^u$ in $G^0$ is:
\[
\begin{align*}
  u &\in [0, t_1] \cup [t_1, t_2] \cup [t_2, +\infty] \\
  (G^0 : G^u) &= 1 \quad 2 \quad 2^2
\end{align*}
\]
The lower breaks occur at
\[
\psi(t_1) = \int_0^{t_1} (G^0 : G^u)du = t_1.
\]
and at
\[ \psi(t_2) = \int_0^{t_2} (G^0 : G^u) du = \int_0^{t_1} (G^0 : G^u) du + \int_{t_1}^{t_2} (G^0 : G^u) du = t_1 + 2(t_2 - t_1) = 2t_2 - t_1. \]
In this case, the lower breaks occur at \( t_1 \) and \( 2t_2 - t_1 \), with \( 0 < t_1 < t_2 \) the odd integers where the upper ramification breaks occur.

We conclude that the lower filtration is given by
\[ G = G_0 = \ldots = G_{t_1} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \]
\[ G_{t_1+1} = \ldots = G_{2t_2-t_1} = \mathbb{Z}/2\mathbb{Z}; \quad G_{2t_2-t_1+1} = \{1\} \]
(6.6) (6.7)
There is only a finite number of such bi-quadratic extensions, for a given pair of upper breaks (or lower breaks) \( (t_1, t_2) \).

7. Formal degrees

In this section, we are influenced by the lecture notes of Reeder [12], and the preceding three talks in Washington, DC. For \( \mathcal{G} = \text{SL}_2(K) \), the dual group \( G = \text{SO}_3(\mathbb{C}) \) contains a unique (up to conjugacy) subgroup \( J \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), whose nontrivial elements are 180-degree rotations about three orthogonal axes. One can check that the centralizer and normalizer of \( J \) are given by
\[ C_G(J) = J, \quad N_G(J) = O \]
where \( O \cong S_4 \) is the rotation group of the octahedron whose vertices are the unit vectors on the given orthogonal axes. The quotient \( O/J \cong \text{GL}_2(\mathbb{Z}/2) \) is the full automorphism group of \( J \).

Each bi-quadratic extension \( L/K \) gives a surjective homomorphism
\[ \varphi_L : W_F \rightarrow J \]
which is a discrete parameter with \( S_{\varphi_L} = J \), since \( C_G(J) = J \), and whose conjugacy class depends only on \( L \), since \( O/J = \text{Aut}(J) \).

Since
\[ |S_{\varphi_L}| = 4 \]
the \( L \)-packet \( \Pi_{\varphi_L} \) has 4 constituents. There are countably many bi-quadratic extensions, therefore there are countably many \( L \)-packets with 4 constituents.

None of these packets contains the Steinberg representation \( \text{St}_G \) and so, according to Conjecture 6.1.4 in [12], these are all supercuspidal \( L \)-packets, each with 4 elements.

Consider the principal parameter:
\[ \text{Ad} \varphi_0 : W_K \times \text{SL}_2(\mathbb{C}) \rightarrow \text{SL}_2(\mathbb{C}) \rightarrow \text{PSL}_2(\mathbb{C}) \rightarrow \text{Aut}(\mathfrak{sl}_2(\mathbb{C})) \]
The adjoint gamma value is given by
\[ \gamma(\varphi_0) = \frac{q}{1 + q^{-1}} \]
where \( q = 2^f \).

Concerning the adjoint gamma value \( \gamma(\varphi) \) we have

\[
\text{Ad} \varphi : W_K \rightarrow J \rightarrow SO_3(\mathbb{C}) \rightarrow \text{Ad} \cdot \text{Aut}(so_3(\mathbb{C}))
\]

The adjoint representation of \( SO_3(\mathbb{C}) \) is equivalent to the standard representation of \( SO_3(\mathbb{C}) \) on \( \mathbb{C}^3 \) and so we replace the above sequence of maps by

\[
\text{Ad} \varphi : W_K \rightarrow J \rightarrow SO_3(\mathbb{C})
\]

For the \( L \)-function, we have

\[
L(\text{Ad} \varphi, s) = \frac{1}{1 + q^{-s}}
\]

and so we have

\[
\gamma(\varphi) = \frac{2}{1 + q^{-1}} \cdot \varepsilon(\varphi)
\]

where

\[
\varepsilon(\varphi) = \pm q^{\alpha(\varphi)/2}.
\]

Note that we have

\[
\frac{\gamma(\varphi)}{\gamma(\varphi_0)} = \frac{2}{q} \cdot \varepsilon(\varphi).
\]

Now \( \alpha(\varphi) \) is the Weil-Deligne version of the Artin conductor which is given here by

\[
\alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{D_i})}{[D_0 : D_i]}
\]

see [12], Reeder’s notation.

We have to take the cases separately, beginning with (6.4).

**Case 1:** We have

\[
G_{-1} = G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ; \ G_0 = ... = G_t = \mathbb{Z}/2\mathbb{Z} ; \ G_{t+1} = \{1\}
\]

We have

\[
\alpha(\varphi) = (1 + t)2
\]

According to Conjecture 6.1(1) in [14], we have

\[
\text{Deg}(\pi_{\rho_L}(\rho)) = \frac{1}{4} \cdot \left| \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \right| = \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| = 2^{t-f}
\]

the **canonical formal degree** of each supercuspidal constituent in the packet \( \Pi_{\rho_L} \), i.e. the formal degree w.r.t. the Euler-Poincaré measure on \( \mathcal{G} \). If we fix the field \( K \), then the formal degree tends to \( \infty \) as the break number \( t \) tends to \( \infty \).

The least allowed value of \( t \) is \( t = 1 \). When \( t = f = 1 \), the canonical formal degree of each element in the packet \( \Pi_{\rho_L} \) is equal to 1. The lower ramification filtration is

\[
G_{-1} = G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ; \ G_0 = G_1 = \mathbb{Z}/2\mathbb{Z} ; \ G_2 = \{1\}
\]
and so, according to 6.1(5 ) in [12], the elements in this packet are not of depth zero.

**Case 2.1:** The lower ramification filtration is

\[ G_{-1} = G = \ldots = G_t = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} ; \ G_{t+1} = \{1\} \]

We have

\[ \alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{D_i})}{[D_0 : D_i]} = (t + 1)3 \]

According to 6.1(1) in [12], we have

\[ \text{Deg}(\pi_{\varphi_L}(\rho)) = \frac{1}{4} \cdot \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \]
\[ = \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| \]
\[ = \frac{1}{2q} \cdot 2^{\alpha(\varphi)/2} \]
\[ = \frac{1}{2q} \cdot 2^{3(1+t)/2} \]
\[ = 2^{3(1+t)/2 - f - 1} \]

Note that \( t \) is odd, therefore the formal degree is a rational number.

**Case 2.2:** This case admits the following lower ramification filtration:

\[ G = G_0 = \ldots = G_{t_1} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \]
\[ G_{t_1 + 1} = \ldots = G_{2t_2 - t_1} = \mathbb{Z}/2\mathbb{Z} ; \ G_{2t_2 - t_1 + 1} = \{1\} \]

We have

\[ \alpha(\varphi) = \sum_{i \geq 0} \frac{\dim(\mathfrak{g}/\mathfrak{g}^{D_i})}{[D_0 : D_i]} = (t_1 + 1)3 + \frac{(2t_2)^2}{2} = 3 + 3t_1 + 2t_2 \]
and, according to 6.1(1) in [12], we have

\[
\text{Deg}(\pi_{\varphi_L}(\rho)) = \frac{1}{4} \cdot \frac{\gamma(\varphi)}{\gamma(\varphi_0)} \\
= \frac{1}{4} \cdot \frac{2}{q} \cdot |\varepsilon(\varphi)| \\
= \frac{1}{2q} \cdot 2^{\alpha(\varphi)/2} \\
= \frac{1}{2q} \cdot 2^{3(1+t_1)/2+t_2} \\
= 2^{3(1+t_1)/2+t_2-f-1}
\]

the canonical formal degree of each supercuspidal in the packet $\Pi_{\varphi_L}$.
If we fix $f$, then the formal degree tends to $\infty$ as the break numbers tend to $\infty$.

Note that $t_1$ is odd, therefore all the formal degrees are rational numbers, in conformity with the rationality of the gamma ratio [7, Prop. 4.1].

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