A GLOBAL $Tb$ THEOREM FOR COMPACTNESS AND BOUNDEDNESS

PACO VILLARROYA

Abstract. We prove a $Tb$ Theorem that characterizes all Calderón-Zygmund operators that extend compactly on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. The result, whose proof does not require the property of accretivity, can be used to prove compactness of the Double Layer Potential operator on a wide class of domains.

The study also provides conditions for boundedness of singular integral operators by means of non-accretive testing functions.

1. Introduction

The seminal $T1$ Theory [5] was soon extended to a $Tb$ Theory in which boundedness of singular integral operators is tested through their action over functions $b$ more general than the function 1. A. McIntosh and I. Meyer [12] obtained a $Tb$ Theorem in the special case $Tb_1 = T^*b_2 = 0$ and, in an independent work, G. David, J. L. Journé and S. Semmes [6] solved the general case. Whereas the $T1$ Theorem proved boundedness of the Cauchy integral over graphs with small Lipschitz constant, the $Tb$ result established this result in full generality, and also boundedness of the Double Layer Potential operator. More on the early developments of the theory can be found in [4], [10].

These results generated an intense flow of research, still active nowadays, producing a variety of $Tb$ Theorems that apply to different settings: from singular integrals on non-homogeneous spaces [13], to operators with vector-valued Calderón-Zygmund kernels [7] or singular integral operators between weighted spaces [8].

The recent paper [14] introduced a $T1$ Theorem to characterize compactness of Calderón-Zygmund operators. Now, following the classical line of progress, we present in the current paper a compact $Tb$ result,
that is, a criterion of compactness relying on the action of the operator over testing functions \( b \) as general as possible (Theorem 4.1).

In the classical theory, the testing functions used to check boundedness satisfy a non-degeneracy property called accretivity, which essentially implies the existence of lower bounds for the testing functions or for their averages (see [4]). In the setting of compact operators, we show that the hypothesis of accretivity can be relaxed to a large extent. The reason, speaking quite broadly, is that compact singular integral operators exhibit an extra decay to zero (see [13]). Then one can use this decay to allow the averages of the testing functions to tend to zero as long as their inverses grow slower than the operator extra decay tends to zero. As a result, compactness can be checked over a larger class of testing functions. The class varies with the operator under study: the faster its bounds decay, the larger the class can be. This allows the existence of global but well localized testing functions and so, it justifies the development of a global \( Tb \) Theorem before studying the corresponding local result.

The main result in the paper is Theorem 4.1, which proves compactness of the Double Layer Potential operator for a large class of domains (see [10]). Classically, compactness is proved after verifying, by means of \( Tb \) Theorem, that the operator is bounded. Since the latter result requires the testing function being accretive, this imposes non necessary hypotheses on the regularity of the boundary of the domain. The new results weaken these hypotheses by allowing the use of non-accretive testing functions.

Furthermore, since the proof of compactness is based on deeper investigations on boundedness, in Corollary 4.2 we extend the classical \( Tb \) Theorem to a criterion of boundedness which does not require accretive testing functions.

In sections 2, 3 we introduce some notation and definitions, while in section 4 we state the main results. Sections 5, 6 and 7 are devoted to study the auxiliary functions used to characterize compactness, develop estimates for the dual pair over functions with adjacent supports, and define \( T b_1 \) and \( T^*b_2 \). In the following four sections we prove sufficiency of the hypotheses of Theorem 4.1 leaving their necessity for section 12.

I express my appreciation to Christoph Thiele and Diogo Oliveira e Silva for the organization of the Summer School 'T(1) and \( T(b) \) Theorems and Applications' and to all its participants. The meeting was a very exciting event and a great source of inspiration for this project. I also thank the support from 김지영 in Sunnyvale, USA, where most of this research was developed.
2. Notation and definitions

2.1. Notation. We denote by $\mathcal{C}$, $\mathcal{D}$ the families of cubes $I = \prod_{i=1}^{n}[a_i, b_i]$ and dyadic cubes $I = 2^j \prod_{i=1}^{n}[k_i, k_i + 1)$ for $j, k_i \in \mathbb{Z}$, respectively. Given a measurable set $\Omega \subset \mathbb{R}^n$, we denote by $\mathcal{D}(\Omega)$ the family of all $I \in \mathcal{D}$ such that $I \subset \Omega$.

For $I \in \mathcal{C}$, we denote its centre by $c(I)$, its side length by $\ell(I)$ and its volume by $|I|$. For $\lambda > 0$, we denote by $\lambda I$, the unique cube such that $c(\lambda I) = c(I)$ and $\ell(\lambda I) = \lambda \ell(I)$. We write $\mathbb{B} = [-1/2, 1/2]^n$ and $\mathbb{B}_\lambda = \lambda \mathbb{B}$. We denote by $|\cdot|_\infty$ the $\ell^\infty$-norm in $\mathbb{R}^n$ and by $|\cdot|$ the modulus of a complex number.

Given two cubes $I, J \in \mathcal{C}$, if $\ell(J) \leq \ell(I)$ we denote $I \wedge J = J$, $I \vee J = I$; while if $\ell(I) < \ell(J)$ we write $I \wedge J = I$, $I \vee J = J$. We define $\langle I, J \rangle$ as the unique cube containing $I \cup J$ with the smallest possible side length and such that $\sum_{i=1}^{n} c(I)_i$ is minimum. We denote its side length by $\text{diam}(I \cup J)$. Note the equivalence:

$$\text{diam}(I \cup J) \approx \ell(I)/2 + |c(I) - c(J)|_\infty + \ell(J)/2.$$ 

We also define the eccentricity and relative distance of $I$ and $J$ as

$$\text{ec}(I, J) = \frac{\ell(I \wedge J)}{\ell(I \vee J)}, \quad \text{rdist}(I, J) = \frac{\ell(\langle I, J \rangle)}{\ell(I \vee J)}.$$ 

Note that

$$\text{rdist}(I, J) \approx 1 + \frac{|c(I) - c(J)|_\infty}{\ell(I \vee J)} \approx 1 + \frac{\text{dist}(I, J)}{\ell(I \vee J)},$$

where $\text{dist}(I, J)$ is the set distance between $I$ and $J$ in the norm $|\cdot|_\infty$.

Given $I \in \mathcal{D}$, we denote by $\partial I$ the boundary of $I$ and by $\text{ch}(I)$ the family of dyadic cubes $I' \subset I$ such that $\ell(I') = \ell(I)/2$. Given $I \in \mathcal{D}$, we denote by $I_p$ the parent cube of $I$, that is, the only dyadic cube such that $I \subset \text{ch}(I_p)$. We define the inner boundary of $I$ as $\mathcal{D}_I = \cup_{I' \in \text{ch}(I)} \partial I'$. When $J \subset 3I$, we define the inner relative distance of $J$ and $I$ by

$$\text{inrdist}(I, J) = 1 + \frac{\text{dist}(J, \mathcal{D}_I)}{\ell(J)}.$$ 

2.2. Compact Calderón-Zygmund kernel.

**Definition 2.1.** For every $M \in \mathbb{N}$, let $\mathcal{C}_M$ be the family of cubes in $\mathbb{R}^n$ such that $2^{-M} \leq \ell(I) \leq 2^M$ and $\text{rdist}(I, \mathbb{B}_{2^M}) \leq M$. We define $\mathcal{D}_M = \mathcal{D} \cap \mathcal{C}_M$ and $\mathcal{D}_M(\Omega) = \mathcal{D}(\Omega) \cap \mathcal{D}_M$.

**Notation 2.2.** To study compactness of singular operators, we use three bounded functions $L, S, D : [0, \infty) \to [0, \infty)$ satisfying

$$\lim_{x \to \infty} L(x) = \lim_{x \to 0} S(x) = \lim_{x \to \infty} D(x) = 0.$$
Whenever formulation of a compact Calderón-Zygmund kernel: $F$ with $L(I) = L(\ell(I))$, $S(I) = S(\ell(I))$ and $D(I) = D(\text{rdist}(I, \mathbb{B}))$. Given three cubes $I_1, I_2, I_3$, we define $F(I_1, I_2, I_3) = L(I_1)S(I_2)D(I_3)$ and $F(I) = F(I, I, I)$.

For $\delta > 0$, we denote
\[
\tilde{L}(I) = \sum_{k \geq 0} 2^{-kn}L(2^{-k}\ell(I)), \quad \tilde{D}(I) = \sum_{k \geq 0} 2^{-k\delta}D(\text{rdist}(2^k I, \mathbb{B})),
\]
and write $\tilde{F}(I_1, I_2, I_3) = \tilde{L}(I_1)S(I_2)\tilde{D}(I_3)$ and $\tilde{F}(I) = \tilde{F}(I, I, I)$.

Since the dilation of a function satisfying one of the limits in (1) satisfies the same limit, namely $D_\lambda L(a) = L(\lambda^{-1} a)$ satisfies the first limit, we often omit universal constants appearing in the argument of these functions. We note that, by Lebesgue’s Dominated Convergence Theorem, $\tilde{L}$ and $\tilde{D}$ satisfy the corresponding limits in (1).

**Definition 2.3.** A measurable function $K : (\mathbb{R}^n \times \mathbb{R}^n) \setminus \{(t, x) \in \mathbb{R}^n \times \mathbb{R}^n : t = x\} \to \mathbb{C}$ is a compact Calderón-Zygmund kernel if it is bounded on compact sets of its domain and there exist $0 < \delta \leq 1$ and functions $L, S, D$ satisfying Definition 2.2 such that
\[
|K(t, x) - K(t', x')| \lesssim \frac{(|t - t'|_\infty + |x - x'|_\infty)^\delta}{|t - x|_\infty^{n+\delta}} F_K(t, x),
\]
whenever $2(|t - t'|_\infty + |x - x'|_\infty) < |t - x|_\infty$ with
\[
F_K(t, x) = L(|t - x|_\infty)S(|t - x|_\infty)D(|t + x|_\infty).
\]

For technical reasons, we will mostly use the following alternative formulation of a compact Calderón-Zygmund kernel:
\[
|K(t, x) - K(t', x')| \lesssim \frac{(|t - t'|_\infty + |x - x'|_\infty)^\delta}{|t - x|_\infty^{n+\delta}} F_K(t, t', x'),
\]
whenever $2(|t - t'|_\infty + |x - x'|_\infty) < |t - x|_\infty$, with $0 < \delta < 1$ and
\[
F_K(t, t', x') = L_1(|t - x|_\infty)S_1(|t - t'|_\infty + |x - x'|_\infty)D_1\left(1 + \frac{|t + x|_\infty}{1 + |t - x|_\infty}\right),
\]
where $L_1, S_1, D_1$ satisfy the limits in (1). As it is explained in (14), condition (3) can be obtained from (2).

In (14), we proved in the one-dimensional case that the smoothness condition (2) essentially implies the pointwise decay condition
\[
|K(t, x)| \lesssim \frac{F_K(t, x)}{|t - x|_\infty^n}
\]
with $F_K(t, x) = L(|t - x|_\infty)S(|t - x|_\infty)D(1 + \frac{|t + x|_\infty}{1 + |t - x|_\infty})$. 

For technical reasons, we will mostly use the following alternative formulation of a compact Calderón-Zygmund kernel:
2.3. Operator with a compact Calderón-Zygmund kernel.

Definition 2.4. Let $T$ be a linear operator bounded on $L^2(\mathbb{R}^n)$. Let $b_1, b_2$ be locally integrable functions.

$T$ is associated with a compact Calderón-Zygmund kernel if there exists a function $K$ satisfying Definition 2.3 such that for all $f, g$ with disjoint compact supports, the following integral representation holds:

$$\langle T(b_1 f), b_2 g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(t)g(x)K(t,x)b_1(t)b_2(x) \, dt \, dx.$$ 

Boundedness of the operator is assumed to provide the integral representation, but we will only use this hypothesis qualitatively. We will work to obtain bounds that only depend on the implicit constant of the compact Calderón-Zygmund kernel and the conditions of next section: the weak compactness condition and the BMO norm of $Tb_1, T^*b_2$.

Notation 2.5. Given an operator $T$ and $b_1, b_2$ measurable functions, we write $Tb = M^*b_2 \circ T \circ Mb_1$, with $Mb_i(f) = b_if$ the pointwise multiplication operator.

3. The weak compactness and the cancellation conditions

We introduce the hypotheses for compactness of singular integral operators: the weak compactness condition and the membership of $Tb_1, T^*b_2$ to $\text{CMO}_b(\mathbb{R}^n)$.

Definition 3.1. Given $b$ a locally integrable function from $\mathbb{R}^n$ to $\mathbb{C}$ and $1 \leq q \leq \infty$, we denote for every cube $I \in \mathcal{D}$

$$\langle b \rangle_I = \frac{1}{|I|} \int_I b(x) \, dx, \quad [b]_{I,q} = \left( \frac{1}{|I|} \int_I |b(x)|^q \, dx \right)^{\frac{1}{q}}.$$

Then the maximal function can be written as $Mqb(x) = \sup_{x \in I \in \mathcal{C}} [b]_{I,q}$.

3.1. The weak compactness condition.

Definition 3.2. Let $1 \leq q_1, q_2 \leq \infty$, and $b_1, b_2$ be locally integrable functions. A linear operator $T$ satisfies the weak compactness condition if there exists a bounded function $F_W$ satisfying Definition 2.2 such that for every $\epsilon > 0$ there exists $M_0 \in \mathbb{N}$ so that

$$|\langle T(b_1^\alpha \chi_I), b_2^\beta \chi_I \rangle| \lesssim |I| [b^\alpha_1]_{I,q_1} [b^\beta_2]_{I,q_2} (F_W(I; M) + \epsilon)$$

for all $I \in \mathcal{D}, M > M_0$ and $\alpha, \beta \in \{0, 1\}$. We note that $b_1^0 \equiv 1, b_1^1 \equiv b_1$.

We say that $T$ satisfies the weak boundedness condition if (6) holds with a function $F_W$ for which some of the limits in (6) may not hold.
Due to the presence of the exponents \( \alpha, \beta \), this definition is more restrictive than the classical concept of weak boundedness. But for most operators the same calculations used to check the standard inequality of weak boundedness (or compactness) suffice to establish (6). The need for this particular formulation originates in Lemma 9.14.

The factor \( F_W(I; M) + \epsilon \) is justified by the result in Proposition 12.1.

In [14], some other alternative definitions of this property are discussed.

3.2. Characterization of compactness.

**Definition 3.3.** Let \( E \) be a Banach space of functions with domain in \( \mathbb{R}^n \). Let \( (\psi_I)_{I \in \mathcal{D}} \) be a wavelet system in \( E \) and \((\tilde{\psi}_I)_{I} \) be the dual system. Then for \( M \in \mathbb{N} \) we define the lagom projection operator by

\[
P_M f = \sum_{I \in \mathcal{D}_M} \langle f, \tilde{\psi}_I \rangle \psi_I,
\]

where \( \langle f, \tilde{\psi}_I \rangle = \int_{\mathbb{R}^n} f(x) \tilde{\psi}_I(x) \, dx \). Note that \( P^*_M f = \sum_{I \in \mathcal{D}_M} \langle f, \psi_I \rangle \tilde{\psi}_I \).

We also define \( P^\perp_M f = f - P_M^* f \).

**Remark 3.4.** The unusual definition of \( \langle, \rangle \) in Definition 3.3 is a customary license in the literature on \( Tb \) theorems.

To prove compactness of an operator on \( L^2(\mathbb{R}^n) \) it is enough to show that \( \langle P_M^* \perp TP_M^\perp f, g \rangle \) tends to zero uniformly for all \( f, g \) in the unit ball of \( L^2(\mathbb{R}^n) \). The reason for this is the following decomposition:

\[
Tf = P_M^* Tf + P_M^* \perp TP_M f + P_M^* \perp TP_M^\perp f.
\]

The first term is a finite rank operator and thus, compact on \( L^2(\mathbb{R}^n) \). The adjoint of the second term, that is \( P_M^* T^* P_M^\perp \), is of finite rank and so, the second term is also compact on \( L^2(\mathbb{R}^n) \). Therefore, we only need to prove that the operator norm of the third term tends to zero.

3.3. The cancellation condition. The spaces \( \text{CMO}_b(\mathbb{R}^n) \) and \( H^1_b(\mathbb{R}^n) \). We now provide the definition of the space to which the functions \( Tb_1, T^* b_2 \) must belong when \( T \) is compact.

**Notation 3.5.** A locally integrable function \( b \) has non-zero dyadic averages if \( \langle b \rangle_I \neq 0 \) for all \( I \in \mathcal{D} \). Then, for \( I \in \mathcal{D} \) and \( 1 \leq q \leq \infty \), we write \( C^b_I = \frac{1}{|\langle b \rangle_I|} + \frac{1}{|\langle b \rangle_I^q|} \) and \( B^b_I, q = \frac{|\langle b \rangle_I^q|}{|\langle b \rangle_I|} + \frac{|\langle b \rangle_I^q|}{|\langle b \rangle_I^q|} \), where \( I \in \text{ch}(I_p) \).

**Definition 3.6.** Let \( b_1, b_2 \) be locally integrable functions with nonzero dyadic averages. Let \( (\psi_I^{b_2})_I \) be the wavelet system of Definition 9.2. We define \( \text{BMO}_b(\mathbb{R}^n) \) as the space of locally integrable functions \( f \) such that

\[
\sup_{I \in \mathcal{I}} \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} |\alpha_{I,J}|^2 |\langle f, \psi_J^{b_2} \rangle|^2 \right)^{1/2} < \infty
\]
with \( \alpha_{I,J} = (B^b_{I,q_2})^2 \left( 1 + \frac{1}{|b_1|} \right) \frac{|b_2|}{|b_1|} \) and similarly changing the roles of \( b_1 \) and \( b_2 \).

We define \( \text{CMO}_b(\mathbb{R}^n) \) as the closure in \( \text{BMO}_b(\mathbb{R}^n) \) of the space of continuous functions vanishing at infinity.

Since \( \alpha_{I,J} \gtrsim 1 \), we have that \( \text{BMO}_b(\mathbb{R}^n) \subseteq \text{BMO}(\mathbb{R}^n) \). When \( b_i \) are bounded and accretive, we have \( \text{BMO}_b(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n) \) and \( \text{CMO}_b(\mathbb{R}^n) = \text{CMO}(\mathbb{R}^n) \). Next lemma gives a characterization of \( \text{CMO}_b(\mathbb{R}^n) \) in terms of wavelet decompositions:

**Lemma 3.7.** The following statements are equivalent:

i) \( f \in \text{CMO}_b(\mathbb{R}^n) \),

ii) \( f \in \text{BMO}_b(\mathbb{R}^n) \) with \( \lim_{M \to \infty} \sup_{I \in \mathcal{I}} \frac{1}{|M|} \left( \sum_{J \in \mathcal{D}_M(I)} |\alpha_{I,J}|^2 |\langle f, \tilde{\psi}^{b_2}_J \rangle|^2 \right)^{1/2} = 0 \),

and similarly changing the roles of \( b_1 \) and \( b_2 \).

**Definition 3.8.** A locally integrable function \( a \) is called a \( p \)-atom with respect to \( b \) and \( I \in \mathcal{D} \) if it is compactly supported on \( I \), it has mean zero with respect to \( b \) and \( \|a_I b\|_{L^p(\mathbb{R}^n)} \leq B^b_{I,p} |I|^{-1/p'} \).

Then \( H^1_b(\mathbb{R}^n) \) is the space of functions \( f = \sum_{I \in \mathcal{D}} \lambda_I a_I \) where \( \lambda_I \in \mathbb{C} \) with \( \sum_{I \in \mathcal{D}} B^b_{I,p} |\lambda_I| < \infty \) and \( a_I \) is a \( p \)-atom with respect to \( b \) and \( I \).

We denote by \( \|f\|_{H^1_b(\mathbb{R}^n)} \) the infimum of \( \sum_{I \in \mathcal{D}} B^b_{I,p} |\lambda_I| \) among all possible atomic decompositions.

### 3.4. Compatibility instead of accretivity

We define the class of non-accretive testing functions available to characterize compactness.

**Notation 3.9.** Let \( T \) be a linear operator with compact Calderón-Zygmund kernel and corresponding function \( F_K \). Let \( b_1, b_2 \) be locally integrable functions with non-zero dyadic averages such that \( T \) satisfies the weak compactness condition with function \( F_W \).

Let \( BF : \mathcal{D} \times \mathcal{D} \to (0, \infty) \) be defined by \( BF = B \cdot F \) with

\[
B(I, J) = C^{b_1}_I C^{b_2}_J \left( \langle M_{q_1} b_1 \rangle_I \langle M_{q_2} b_2 \rangle_J + \langle M_{q_1} (b_1 \chi_I) \rangle_{I \wedge J} \langle M_{q_2} (b_2 \chi_J) \rangle_{I \wedge J} \cdot \chi_{\text{dist}(I \wedge J, I \vee J) \leq 3} \right)
\]

\[
F(I, J) = \tilde{F}_K(I \wedge J, I \wedge J, \langle I, J \rangle) + F_W(I; M_{T,c}) \chi_{I = J},
\]

where \( C^{b_i}_I \) are defined in Notation 3.3.

**Definition 3.10** (Compatible testing functions). With previous notation, we say that \( b_1, b_2 \) are testing functions compatible with \( T \) if

\[
\sum_{I \in \mathcal{D}} b_1(I) b_2(I) = 1.
\]
\[(8) \sup_{I, J \in \mathcal{D}} BF(I, J) \left( \frac{[b_1]_{I,q_1}}{\langle b_1 \rangle_I} \right)^2 \left( \frac{[b_2]_{J,q_2}}{\langle b_2 \rangle_J} \right)^2 < \infty,\]

\[(9) \lim_{M \to \infty} \sup_{I, J \in \mathcal{D}_M} BF(I, J) \left( \frac{[b_1]_{I,q_1}}{\langle b_1 \rangle_I} \right)^2 \left( \frac{[b_2]_{J,q_2}}{\langle b_2 \rangle_J} \right)^2 = 0,\]

with \(q_1^{-1} + q_2^{-1} < 1\), and the supremum in \((9)\) is calculated over the family \(\mathcal{F}_M\) of ordered pairs of cubes \(I, J \in \mathcal{D}_M\) such that either \(\ell(I \cap J) > 2^M\), \(\ell(I \cap J) < 2^{-M}\), or \(\text{rdist}(\langle I, J \rangle, \mathbb{B}) > M^\theta\) for some \(\theta \in (0, 1)\).

Lemmata 10.1 and 9.9 justify the feasibility of Definition 3.10, in particular, equality \((9)\).

**Definition 3.11** (Compatible testing functions 2). We say that \(b_1, b_2\) are testing functions compatible with \(T_b\) if

\[(10) \sup_{I, J \in \mathcal{D}} BF(I, J) \left( \frac{[b_1]_{I,p,q_1}}{\langle b_1 \rangle_I} \right)^2 + \left( \frac{[b_2]_{J,p,q_2}}{\langle b_2 \rangle_J} \right)^2 < \infty,\]

\[(11) \lim_{M \to \infty} \sup_{I, J \in \mathcal{D}_M} BF(I, J) \left( \frac{[b_1]_{I,p,q_1}}{\langle b_1 \rangle_I} \right)^2 + \left( \frac{[b_2]_{J,p,q_2}}{\langle b_2 \rangle_J} \right)^2 = 0,\]

4. **Statement of the main results**

4.1. **Main result on compactness.**

**Theorem 4.1.** Let \(1 < p < \infty\) and \(T\) be a linear operator associated with a standard Calderón-Zygmund kernel \(K\).

Then \(T\) extends compactly on \(L^p(\mathbb{R}^n)\) if and only if \(K\) is a compact Calderón-Zygmund kernel and there exist functions \(b_1, b_2\) compatible with \(T\) so that \(T\) satisfies the weak compactness condition and \(Tb_1, T^*b_2 \in \text{CMO}_b(\mathbb{R}^n)\).

If we assume that \(b_1, b_2 \in L^\infty(\mathbb{R}^n)\) and that they are compatible with \(T_b\), then the same three conditions characterize compactness of \(T_b\).

Given a domain \(\Omega \subset \mathbb{R}^{d+1}\) and \(S = \partial \Omega\), the Double Layer Potential operator \(\mathcal{K}\) is defined as follows

\[
\mathcal{K}(f)(x) = \lim_{\epsilon \to 0} \int_{S \cap B(x, \epsilon)} f(y) \left( \nu(y), \frac{x - y}{|x - y|^d} \right) d\sigma(y)
\]

where \(\nu\) is the exterior unit normal vector to the surface \(S\) and \(\sigma\) is the surface measure on \(S\). Then Theorem 4.1 proves compactness of \(\mathcal{K}\) for a large class of domains \(\Omega\).
4.2. The result on boundedness. Theorem 4.1 can also characterize bounded operators with a standard Calderón-Zygmund kernel, just omitting the considerations of limits in \( (1) \) going to zero. For example, we can consider a kernel \( K \) satisfying inequality \( (2) \) with auxiliary function \( F_K \) defined only by the function \( S \), without \( L \) and \( D \). Despite the associated operator cannot be compact, it might be bounded and, in that case, the testing functions used to check its boundedness do not need to satisfy the accretivity condition for small cubes. In this line, Corollary 4.2 describes when boundedness of singular integral operators can be checked by means of non-accretive testing functions.

The following result holds:

**Corollary 4.2.** Let \( 1 < p < \infty \) and \( T \) be a continuous linear operator associated with a standard Calderón-Zygmund kernel.

Then \( T \) extends boundedly on \( L^p(\mathbb{R}^n) \) if and only if there exist functions \( b_1, b_2 \) compatible with \( T \) and such that \( T \) satisfies the weak boundedness condition and \( Tb_1, T^*b_2 \in \text{BMO}\_b(\mathbb{R}^n) \).

If \( b_1, b_2 \in L^\infty(\mathbb{R}^n) \) and they are compatible with \( T_b \), then the same two conditions characterize boundedness of \( T_b \).

Corollary 4.2 can be applied to prove boundedness of the double and single layer potential operators associated with boundary value problems for degenerate elliptic equations in divergence form, \( \text{div}(A\nabla u) - V \cdot u = 0 \), with appropriate non-negative potentials. In [3] it is shown that the Riesz potentials associated with these equations have kernels that decay for large cubes or cubes that are away from the origin, but not for small cubes. This would correspond to the case in which the functions \( L \) and \( D \) tend to zero, but not the function \( S \).

5. On compact Calderón-Zygmund kernels

In this section we describe some properties of the auxiliary functions \( L, S, D, \) and \( F \) of Definition 2.2.

We first note that, without loss of generality, \( L \) and \( D \) can be assumed to be non-creasing while \( S \) can be assumed to be non-decreasing. These assumptions imply analog properties to \( F \) and \( \tilde{F} \).

Regarding the equivalent formulation \( (3) \) given after Definition 2.3 we note that in the next lemma and forthcoming results we will often consider the particular case when \( t' = t \) and \( x' = c(J) \):

\[
F_K(t, x, c(J)) = L(|t - c(J)|_\infty)S(|x - c(J)|_\infty)D(1 + \frac{|t + c(J)|_\infty}{1 + |t - c(J)|_\infty}).
\]
Lemma 5.1. For $I, J \in \mathcal{C}$, we write $\Delta_{I,J} = \{ t \in \mathbb{R}^n : \ell((I,J))/2 < |t - c(J)|_{\infty} \leq \ell((I,J)) \}$. Then, for $t \in I \cap \Delta_{I,J}$ and $x \in J$, we have

$$F_K(t, x, c(J)) \lesssim F_K((I,J), J, (I,J))$$

with $F_K((I,J), J, (I,J)) = L(\ell((I,J)))S(\ell(J))D(\text{rdist}(I,J), \mathbb{B}))$.

Proof. Since $L$ is non-creasing and $S$ is non-decreasing, $|t - c(J)|_{\infty} > \ell((I,J))/2$, $|x - c(J)|_{\infty} \leq \ell((J))/2$, we only need to bound the factor $D$.

For all $t \in I$, we have $|t - c(J)|_{\infty} \leq \text{diam}(I \cup J) = \ell((I,J))$. Then, using $|c(J)|_{\infty} \leq (|t - c(J)|_{\infty} + |t + c(J)|_{\infty})/2$, we get

$$1 + \frac{|c(J)|_{\infty}}{1 + \ell((I,J))} \leq 1 + \frac{|c(J)|_{\infty}}{1 + |t - c(J)|_{\infty}} \leq 1 + \frac{3}{2} \left( 1 + \frac{|t + c(J)|_{\infty}}{1 + |t - c(J)|_{\infty}} \right).$$

Now, since $|c(I)|_{\infty} - |c(J)|_{\infty} \leq |c(I) - c(J)|_{\infty} \leq \ell((I, J))$, we bound below the numerator in the left hand side of (12) as follows:

$$1 + \ell((I,J)) + |c(J)|_{\infty} \geq 1 + \frac{\ell((I,J))}{2} + \frac{|c(I)|_{\infty} - |c(J)|_{\infty}}{2} + |c(J)|_{\infty} \geq 1 + \frac{1}{2} \left( 1 + \ell((I,J)) + \frac{1}{2} |c(I)|_{\infty} + c(J)|_{\infty} \right).$$

Therefore,

$$1 + \frac{|c(J)|_{\infty}}{1 + \ell((I,J))} \geq \frac{1}{3} \left( \frac{3}{2} + \frac{|c(I)|_{\infty} + c(J)|_{\infty}/2}{1 + \ell((I,J))} \right).$$

Now, since $(c(I) + c(J))/2 \in (I, J)$, we have $|c(I) + c(J))/2 - c((I,J))|_{\infty} \leq \ell((I,J))/2$ and so, we can bound below previous expression by

$$\frac{1}{3} \left( \frac{3}{2} + \frac{|c((I,J))|_{\infty}}{1 + \ell((I,J))} - \frac{1}{2} \right) \geq \frac{1}{3} \left( 1 + \frac{|c((I,J))|_{\infty}}{2 \max(\ell((I,J)), 1)} \right) \geq \text{rdist}(I,J), \mathbb{B}).$$

Then, omitting constants and using that $D$ is non-creasing, we get

$$F_K(t, x, c(J)) \lesssim L(\ell((I,J)))S(\ell(J))D(\text{rdist}(I,J), \mathbb{B}).$$

6. Estimates near the diagonal

In Lemma 6.2, we prove a Hardy’s inequality for compact operators.

Lemma 6.1. Let $1 \leq q_1, q_2 \leq \infty$ such that $\frac{1}{q_1} + \frac{1}{q_2} < 1$, and let $K$ be a compact Calderón-Zygmund kernel. For every $I \in \mathcal{D}$ and every bounded functions $f, g,$

$$|\langle T_b(f\chi_{(3I\setminus I)}, g\chi_I) \rangle| \lesssim |I|[fb_{1}\chi_{3I\setminus I}][gb_{2}]_{q_1,q_2}\tilde{F}_K(I)$$

where $\tilde{F}_K$ is given in Definition 2.2.

Previous lemma follows after proving the following result:
Lemma 6.2. With the same hypotheses, let $I, I' \in \mathcal{D}$ be such that $\ell(I) = \ell(I')$ and $\text{dist}(I, I') = 0$. Let $f, g$ be integrable and compactly supported on $I$ and $I'$ respectively. Then

$$\tag{14} |\langle T_b f, g \rangle| \lesssim |I| [f b_1]_{\ell'_q} [g b_2]_{\ell'_r,\ell'_q} \bar{F}_K(I).$$

Proof (of Lemma 6.2). We first assume $n q_1 + \frac{1}{q_2} < 1$. We define the kernels $K_\ell(t, x) = K(t, x)$ if $|t - x|_\infty > \epsilon$ and zero otherwise. Then

$$\tag{15} \|\langle T_b f, g \rangle\| \leq \sup_{m > 0} \int_{I'} \int_I |K_m^\ell(t, x)||f(t)b_1(t)||g(x)b_2(x)|dt \, dx = \sup_{m > 0} A_m.$$

Let $\theta \in (0, 1)$ such that $S(\theta) \leq \bar{F}(I)$. We denote $I_{\theta,0} = \theta I$, $I_{\theta,1} = I \setminus I_{\theta}$ and similar for $I'$. Then

$$A_m \leq \sum_{i, j \in \{0, 1\}} \int_{I_{\theta,1}} \int_{I_{\theta,0}} |K_m^\ell(t, x)||f(t)b_1(t)||g(x)b_2(x)|dt \, dx$$

$$\leq |I|^{\frac{1}{q_1} + \frac{1}{q_2}} [f b_1]_{\ell'_q} [g b_2]_{\ell'_r,\ell'_q} \sum_{i, j \in \{0, 1\}} \|K_m^\ell \chi_{I_{\theta,i} \times I'_{\theta,j}}\|_{L^q_{\ell'} L^{q'}_{\ell'}(\mathbb{R}^n)}$$

From the kernel decay condition (14), we have

$$\tag{15} \|K_m^\ell \chi_{I_{\theta,i} \times I'_{\theta,j}}\|_{L^q_{\ell'} L^{q'}_{\ell'}(\mathbb{R}^n)} \lesssim \left( \int_{I'_{\theta,j}} \left( \int_{I_{\theta,i}} \frac{F_K(t, x)^{q'_i}}{|t - x|^{q'_i}_{\infty}} \, dx \right)^{\frac{q'}{q'_i}} \, dt \right)^{\frac{1}{q'_i}}$$

with $F_K(t, x) = L(|t - x|_\infty)S(|t - x|_\infty)D(1 + |t + x|_\infty)$.

When $i = j = 1$, we have $0 < |t - x|_\infty \leq \theta$ in the domain of integration and so, by the reasoning in the proof of Lemma 5.1, $F_K(t, x) \lesssim S(\theta)$. Then the left hand side of (15) can be bounded by

$$\tag{16} S(\theta) \left( \int_{I'} \left( \int_{I} \frac{1}{|t - x|^{q'_i}_{\infty}} \, dx \right)^{\frac{q'}{q'_i}} \, dt \right)^{\frac{1}{q'_i}}.$$

In all remains cases, we have $\theta < |t - x|_\infty \lesssim \ell(I)$, $|x - c(I)|_\infty \lesssim \ell(I)$ and $(t + x)/2 - c(I)|_\infty \lesssim \ell(I)$ in the domain of integration. Then, by the proof of Lemma 5.1, we have $F_K(t, x) \leq L(|t - x|_\infty)S(\ell(I))D(\text{rdist}(I, \mathbb{B}))$. With this and using Hölder’s inequality for $q'_i = (1 + \epsilon)q_i$ with $\epsilon > 0$ sufficiently small, the left hand side of (15) can be bounded by

$$\tag{17} L(\ell(I))S(\ell(I))D(\text{rdist}(I, \mathbb{B})) \left( \int_{I'} \left( \int_{I} \frac{1}{|t - x|^{q'_i}_{\infty}} \, dx \right)^{\frac{q'}{q'_i}} \, dt \right)^{\frac{1}{q'_i}}$$

where

$$\bar{L}(\ell(I)) = \left( \int_{I'} \left( \int_{I} L(|t - x|_\infty)^{q'_i n(1 + \epsilon^{-1})} \, dt \right)^{\frac{q'}{q'_i}} \, dx \right)^{\frac{1}{q'_i n(1 + \epsilon^{-1})}}.$$
still satisfies the limit properties of (1). Now we work to bound the double integral in (16), being the integral in (17) very similar.

For $\text{dist}_\infty(x, I) \leq \rho \leq \text{dist}_\infty(x, I) + \ell(I)$, we denote $J(x, \rho) = \{ t \in I_0 : |t - x|_\infty = \rho \}$, which satisfies $|J(x, \rho)| \lesssim 2^n \rho^{n-1}$. Then, the double integral in (16) can be bounded by

$$
\int_I \left( \int_{\text{dist}_\infty(x, I) + \ell(I)}^{\text{dist}_\infty(x, I) + \ell(I)} \frac{1}{\rho^{q_1 n} d\rho} \right)^{\frac{2}{q_2}} dx
\lesssim \int_I \left( \int_{\text{dist}_\infty(x, I) + \ell(I)}^{\text{dist}_\infty(x, I) + \ell(I)} \frac{1}{\rho^{(q_1 - 1) n + 1} d\rho} \right)^{\frac{2}{q_2}} dx.
$$

(18)

We consider first $1 < q_1 < \infty$, for which (18) can be bounded by

$$
\left( \int_{\text{dist}_\infty(x, I)}^{\ell(I)} dx \right)^{\frac{1}{q_2}}.
$$

For $0 \leq \rho \leq \ell(I)$, we denote $J_\rho' = \{ x \in I' : \text{dist}_\infty(x, I) = \rho \}$, which satisfies $|J_\rho'| \lesssim \ell(I)^{n-1}$. Then, since $n^{q_2'} < 1$, we bound the last expression by a constant times

$$
\left( \ell(I)^{n-1} \int_0^{\ell(I)} \frac{1}{\rho^{n^{q_2'} n} d\rho} \right)^{\frac{1}{q_2}} \lesssim \left( \ell(I)^{n-1} \ell(I)^{1-n^{q_2'}} \right)^{\frac{1}{q_2}} = |I|^{\frac{1}{q_2} + \frac{1}{q_1} - 1}.
$$

Finally, when $q_1 = \infty$ and $q_2 > 1$, we use the same notation to bound the expression prior to (18) by

$$
\left( \left( \int_{\text{dist}_\infty(x, I)}^{\ell(I)} \frac{\ell(I)}{d\rho} \right)^{q_2'} \right)^{\frac{1}{q_2}}
\leq \ell(I)^{n-1} \int_0^{\ell(I)} \left( \log \left( 1 + \frac{\ell(I)}{\rho} \right) \right)^{q_2'} \rho^{q_2} d\rho
\leq |I|^{\frac{1}{q_2}} \left( \int_1^{\infty} \frac{1}{d\rho} \right)^{\frac{2}{q_2}} \lesssim |I|^{\frac{1}{q_2} + \frac{1}{q_1} - 1}
$$

With all this and the choice of $\theta$, we get

$$
A_m \lesssim |I|^{\frac{1}{q_1} + \frac{1}{q_2}} |Fb_{I,q_1} g_{b_2}|_{q_2} |F_{q_2} (S(\theta) + \tilde{F}_K(I))
\lesssim |I|^{\frac{1}{q_1} + \frac{1}{q_2}} + \tilde{F}_K(I)
$$

By symmetry, we have the same result under the assumption that $\frac{1}{q_1} + \frac{1}{q_2} < 1$. Now, we can interpolate between the cases $q_{1,0} > 1$, $q_{2,0} = \infty$ and $q_{1,0} = \infty$, $q_{2,0} > 1$. This way we obtain the result for $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_{1,0}} + \frac{1}{q_{2,0}} < 1$. 

7. The definition of $Tb_i$

Lemma 7.1 below defines $Tb_1, T^*b_2$ for locally integrable testing functions $b_1, b_2$ as functionals in the dual of the subspace of $C_0(\mathbb{R}^n)$ of functions with mean zero with respect to $b_2$ or $b_1$ respectively.

Then the hypothesis that $Tb_1 \in \text{BMO}_b(\mathbb{R}^n)$ means that $|\langle Tb_1, f \rangle| = |\langle Tb_1, b_2 f \rangle| \leq C$ holds for a dense subset of the unit ball of $H^1_b(\mathbb{R}^n)$. Furthermore, the hypothesis $Tb_1 \in \text{CMO}_b(\mathbb{R}^n)$ means that

$$\lim_{M \to \infty} |\langle P_M^1 Tb_1, f \rangle| = 0$$

holds uniformly in a dense subset of the unit ball of $H^1_b(\mathbb{R}^n)$. In particular, we verify this estimate for $f \in C_0(\mathbb{R}^n)$ with mean zero with respect to $b_2$. The necessity of $Tb_1 \in \text{CMO}_b(\mathbb{R}^n)$ when $T$ is a compact operator appears in Proposition 12.3.

**Lemma 7.1.** Let $T$ be a linear operator associated with a compact Calderón-Zygmund kernel $K$ with parameter $0 < \delta < 1$. Let $b_1, b_2$ be test functions compatible with $T$.

Let $J \in \mathcal{C}$, $f$ locally integrable with support on $J$ and mean zero with respect to $b_2$. Then the limit $L_b(f) = \lim_{k \to \infty} \langle Tb_1, f \rangle$ exists.

Moreover, for all $k \geq 2$

$$\langle L_b(f) - \langle Tb_1, f \rangle, \psi \rangle \leq 2^{-k\delta} |J| |\langle M_{q_2} b_1 \rangle_{2, k} [b_2, f]_s_{q_2} F_K(I, J, I)\rangle.$$  \hspace{1cm} (19)

**Proof.** For $k \geq 2$, we denote $\Delta_k = (2^{k+1} J \setminus 2^k J) = \{ t \in \mathbb{R}^n : 2^{k-1} \ell(J) \leq |t - c(J)| \leq 2^k \ell(J) \}$ and $\Psi_k = \chi_{\Delta_k}$. We aim to estimate $|\langle Tb \Psi_k, f \rangle|$. For $t \in \text{sup} \, \Psi_k = \Delta_k$ and $x \in \text{sup} \, f \subset J$, we have

$$|x - c(J)| \leq \ell(J)/2 < \ell(J) 2^{k-2} \leq 2^{-1} |t - c(J)|.$$  \hspace{1cm} (20)

Then $t$ and $x$ cannot be equal, which implies that the supports of $\Psi_k$ and $f$ are disjoint. Therefore, we can use the kernel representation and the zero mean of $f$ with respect to $b_2$ to write

$$\langle Tb \Psi_k, f \rangle = \int_J \int_{\Delta_k} \Psi_k(t) f(x) (K(t, x) - K(t, c(J))) b_1(t) b_2(x) dtdx.$$  \hspace{1cm}

Whence, $|\langle Tb \Psi_k, f \rangle|$ can be bounded by

$$\|b_1 \chi_I\|_{L^q(2^{k+1} J)} \|b_2\|_{L^q_2(J)} \left( \int_J \left( \int_{\Delta_k} |K(t, x) - K(t, c(J))|^{q_1} dt \right)^{q_2} dx \right)^{1/q_2}.$$  \hspace{1cm}

We denote the last factor by Int. By (20) and the smoothness condition of a compact Calderón-Zygmund kernel, we have

$$\text{Int} \lesssim \left( \int_J \left( \int_{\Delta_k} \frac{|x - c(J)|^q \delta}{|t - c(J)|^{n+q \delta}} \ F_K(t, x, c(J))^{q_1} dt \right)^{q_2} dx \right)^{1/q_2}.$$  \hspace{1cm}
with \( F_K(t, x, c(J)) = L(|t - c(J)|_\infty) S(|x - c(J)|_\infty) D(1 + \frac{|t + c(J)|_\infty}{1 + |t - c(J)|_\infty}) \).

By Lemma 5.1, \( F_K(t, x, c(J)) \leq F_K(2^k J, 2^k J) \) and so,

\[
\text{Int} \lesssim \frac{\ell(J)^\delta}{(2^k \ell(J))^n+\delta} F_K(2^k J, J, 2^k J) |\Delta_k|^\frac{1}{q_1} |J|^\frac{1}{q_2}.
\]

With this and \(|\Delta_k| \lesssim 2^{(k+1)n} |J| \), we have

\[
|\langle T_b \Psi, f \rangle| \lesssim [b_1]_{2k+1} |2^{k+1} J| \sum_{J,J,q} |J|^\frac{1}{q_1} |J|^\frac{1}{q_2} \frac{|\Delta_k|^{\frac{1}{q_1}} |J|^{\frac{1}{q_2}}}{2^{k(n+\delta)} |J|} F_K(2^k J, J, 2^k J)
\]

\[
\lesssim 2^{-k\delta} |J| [b_1]_{2k+1} |f_{b_2}, J, q|_{2k, J} F_K(2^k J, J, 2^k J) \leq 2^{-k\delta} |J|
\]

by hypothesis [8]. The right hand side of previous inequality tends to zero when \( k \) tends to infinity, proving that the sequence \( \langle T_b \chi_{2^k J}, f \rangle \rangle \) is Cauchy and thus, the existence of the limit, which we write as \( L_b(f) \).

Now, the stated rate of convergence follows by summing a geometric series. For every \( k' \geq 2 \), we have

\[
|L_b(f) - \langle T_b \chi_{2^k J}, f \rangle| \leq \lim_{m \to \infty} |L_b(f) - \langle T_b \chi_{2^m J}, f \rangle| \leq \sum_{k' = k}^{\infty} |\langle T_b \Psi_{k'}, f \rangle|
\]

\[
\lesssim |J| [f_{b_2}, J, q] \sum_{k' = k}^{\infty} 2^{-k'\delta} [b_1]_{2k'+1} |f_{b_2}, J, 2^{k'} J|
\]

\[
\lesssim |J| [f_{b_2}, J, q] L(2^k J) S(J) \sum_{k' = k}^{\infty} 2^{-k'\delta} [b_1]_{2k'+1} D(2^{k'} J),
\]

using \( \ell(2^k J) \leq \ell(2^{k'} J) \) and that \( L \) is non-decreasing. For \( k' \geq k \) we have \([b_1]_{2k'+1, J, q} \leq \inf_{x \in 2^{k'} J} (M_{q_1} b_1)(x) \leq \langle M_{q_1} b_1 \rangle_{2^k J} \) and so, we obtain

\[
|L_b(f) - \langle T_b \chi_{2^k J}, f \rangle| \lesssim 2^{-k\delta} |J| \langle M_{q_1} b_1 \rangle_{2^k J} |f_{b_2}, J, q|
\]

\[
L(2^k J) S(J) \sum_{k' = 0}^{\infty} 2^{-k'\delta} D(2^{k'} (2^k J)).
\]

8. The Operator Acting on Bump Functions

In this section, we develop estimates of the dual pair \( \langle T_b h_I, h_I \rangle \) in terms of the space and frequency location of the bump functions \( h_I, h_J \).

Proposition 8.2 is an improvement of the analog result in [14]. Although the new proof is influenced by the works [4], [13], [9], we follow a different approach: we modify the proof in [14] by implementing all the necessary changes to deal with non-continuous bump functions.

**Definition 8.1.** Let \( b \) be locally integrable with \( \langle b \rangle_I \neq 0 \) for all \( I \in \mathcal{D} \).

We write \( h_{\mathcal{I}_I}^b = |I|^\frac{1}{2} \left( \frac{1}{|I|} \chi_I - \frac{1}{|I_p|} \chi_{I_p} \right) \), where \( I \in \mathcal{C}(I_p) \).
We note that \( \|h_t^b\|_{L^q(\mathbb{R}^n)} \lesssim C_b^q|I|^{\frac{1}{q} - \frac{1}{2}} \) and \( \|h_t^b b\|_{L^q(\mathbb{R}^n)} \lesssim B_{I,q} b|I|^{\frac{1}{q} - \frac{1}{2}} \), with constants defined in \( \mathcal{F} \).

**Proposition 8.2.** Let \( T \) be a linear operator with a compact C-Z kernel \( K \) and parameter \( 0 < \delta < 1 \). Let \( 1 < q_i \leq \infty \) with \( q_1^{-1} + q_2^{-1} < 1 \) and \( b_1, b_2 \) be functions compatible with \( T \). We assume that \( T \) satisfies the weak compactness condition and \( Tb_1 = T^*b_2 = 0 \).

Let \( I, J \in \mathcal{D} \) and \( h_I = h_I^{b_1}, h_J = h_J^{b_2} \) as in Definition 8.1.

1) When \( \text{rdist}(I_p, J_p) > 3 \),

\[ \langle T_b(h_I), h_J \rangle \lesssim \frac{ec(I, J)^\frac{n}{2} + \delta}{\text{rdist}(I, J)^{n+\delta}} B_1(I, J) F_1(I, J), \]

with \( B_1(I, J) = B_{I,q_1}^{b_1} B_{J,q_2}^{b_2} \) and \( F_1(I, J) = F_K(\langle I, J \rangle, I \wedge J, \langle I, J \rangle) \).

2) When \( \text{rdist}(I_p, J_p) \leq 3 \) and \( \text{inrdist}(I_p, J_p) > 1 \),

\[ \langle T_b(h_I), h_J \rangle \lesssim \frac{ec(I, J)^\frac{n}{2}}{\text{inrdist}(I, J)^{n}} B_2(I, J) F_2(I, J), \]

where now,

\[ B_2(I, J) = \alpha \sum_{R \in \{I, J\}} \frac{\langle M_{q_1} b_1 \rangle_R}{\langle b_1 \rangle_R} \sum_{R \in \{I, J\}} \frac{\langle M_{q_2} b_2 \rangle_R}{\langle b_2 \rangle_R} \]

\[ + C_{b_1}^{b_1} C_{b_2}^{b_2} \langle M_{q_1} (b_1 \chi_I) \rangle_{I \wedge J} \langle M_{q_2} (b_2 \chi_J) \rangle_{I \wedge J}, \]

\[ F_2(I, J) = \theta_K(\langle I, J \rangle, I \wedge J, \langle I, J \rangle) + F_K(I \wedge J, I \wedge J, \langle I, J \rangle) \]

with \( \alpha = 1 \) if \( I \wedge J \subseteq I \cap J \), \( \alpha = 0 \) otherwise, and \( \theta_K \) as in def. \( \mathcal{F} \).

3) When \( \text{rdist}(I_p, J_p) \leq 3 \) and \( \text{inrdist}(I_p, J_p) = 1 \),

\[ \langle T_b(h_I), h_J \rangle \lesssim ec(I, J)^\frac{n}{2} (B_2(I, J) F_2(I, J) + B_3(I, J) F_3(I, J)), \]

where \( B_2, F_2 \) are as before and, when \( I \neq J \),

\[ B_3(I, J) = C_{I}^{b_1} C_{J}^{b_2} \]

while when \( I = J \),

\[ B_3(I, J) = \sum_{I', I'' \in \mathcal{I}(I_p)} C_{I'}^{b_1} C_{I''}^{b_2} \]

for every \( \epsilon > 0 \) with the value \( M_{T, \epsilon} \) given in Definition 3.2.

**Remark 8.3.** We note that BF in Definition 3.10 of compatible testing function dominates all terms \( B_t F_t \) in the statement of Proposition 8.2.

In fact, \( F_1(I, J) \lesssim \theta_K(I \wedge J, I \wedge J, \langle I, J \rangle) + F_W(I; M_{T, \epsilon}) \). Moreover, \( [b_1]_{I, q_1} \lesssim \inf I M_{q_1} b_1 \lesssim \langle M_{q_1} b_1 \rangle_I \).
We note that $2 \text{rdist}(I_p, J_p) - 1 \leq \text{rdist}(I, J) \leq 2 \text{rdist}(I_p, J_p) + 1$ and $2 \text{inrdist}(I_p, J_p) \leq \text{inrdist}(I, J) \leq 2 \text{inrdist}(I_p, J_p) + 1$.

Proof. By symmetry, we assume $\ell(J) \leq \ell(I)$. Let $\psi(t, x) = h_I(t) h_J(x)$, which is supported on $I_p \times J_p$ and has mean zero in the variable $x$ with respect to $b_2$.

a) When $3\ell(I_p) < \text{diam}(I_p \cup J_p)$, we have that $(5I_p) \cap J_p = \emptyset$ and so, we can use the kernel representation and the zero mean of $\psi$ to write

$$\langle T_b h_I, h_J \rangle = \int_{I_p} \int_{I_p} \psi(t, x)(K(t, x) - K(t, c(J_p)))b_1(t)b_2(x) \, dt \, dx.$$}

Now, $(5I_p) \cap J_p = \emptyset$ and $\ell(J) \leq \ell(I_p)$ imply $\text{diam}(I_p \cup J_p) \leq \ell(I_p) + |c(I_p) - c(J_p)|_\infty$. With this and $|t - c(I_p)|_\infty \leq \ell(I_p)/2$, we prove:

$$|t - c(I_p)|_\infty \geq |c(I_p) - c(J_p)|_\infty - |t - c(I_p)|_\infty$$

$$\geq \text{diam}(I_p \cup J_p) - 3\ell(I_p)/2 > \text{diam}(I_p \cup J_p)/2,$$

$$|t - c(J_p)|_\infty \leq |c(I_p) - c(J_p)|_\infty + \ell(I_p)/2 \leq \text{diam}(I_p \cup J_p).$$

Then,

$$(21) \quad \left( \int_{I_p} \left( \int_{I \cap \Delta_{I_p, J_p}} |K(t, x) - K(t, c(J_p)|^{q_1'} dt \right)^{\frac{q_1}{q_1'}} dx \right)^{\frac{1}{q_2}},$$

where $\Delta_{I_p, J_p} = \{ t \in \mathbb{R}^n : \ell((I_p, J_p))/2 < |t - c(J_p)|_\infty \leq \ell((I_p, J_p)) \}$.

We denote by Int the integral in (21). From $3\ell(I_p) < \text{diam}(I_p \cup J_p) \leq \ell(I_p) + |c(I_p) - c(J_p)|_\infty$, we get $2\ell(I_p) < |c(I_p) - c(J_p)|_\infty$. This inequality and $|x - c(J_p)|_\infty \leq \ell(J_p)/2$ imply

$$|t - c(J_p)|_\infty \geq 2\ell(I_p) - \ell(I_p)/2 \geq 3\ell(J_p)/2 \geq 3|x - c(J_p)|_\infty.$$

Then, by the smoothness condition of a compact C-Z kernel,

$$\text{Int} \lesssim \left( \int_{I_p} \left( \int_{I \cap \Delta_{I_p, J_p}} \frac{|x - c(J_p)|_\infty^{\delta q_1'}}{|t - c(J_p)|^{(\delta + \delta)q_1}} F_K(t, x, c(J_p))^{q_1'} dt \right)^{\frac{q_1}{q_1'}} dx \right)^{\frac{1}{q_2}},$$

with $F_K(t, x, c(J_p)) = L(|t - c(J)|_\infty) S(|x - c(J_p)|_\infty) D\left(1 + \frac{|t + c(J_p)|_\infty}{1 + |t - c(J)|_\infty} \right)$.

By Lemma 5.1 $F_K(t, x, c(J_p)) \lesssim F_K((I, J), J, (I, J))$ and so,

$$\text{Int} \lesssim |I|^{\frac{1}{q_1'}} |J|^{\frac{1}{q_2}} \frac{\ell(J)^{\delta}}{\ell((I, J))^{n + \delta}} F_K((I, J), J, (I, J)).$$
We then continue the bound in (21) as

\[
|\langle Tb_I, h_J \rangle| \lesssim B_{I,q_1}|I|^{-\frac{1}{2}+\delta} B_{J,q_2}|J|^{-\frac{1}{2}+\frac{\delta}{2}}
\]

\[
|I|^{\frac{1}{3}}|J|^{\frac{1}{4}} \frac{\ell(J)^d}{\ell(I)^d \ell(I,J)} F_K(\langle I, J \rangle, J, \langle I, J \rangle)
\]

\[
= \left( \frac{\ell(J)}{\ell(I)} \right)^{\frac{1}{2}+\delta} \left( \frac{\ell(I)}{\ell(I,J)} \right)^{\frac{1}{2}+\frac{\delta}{2}} B_1(I, J) F_1(I, J).
\]

This is the result corresponding to the case 1) in the statement.

b) When \( \text{diam}(I_p \cup J_p) \leq 3\ell(I_p) \), we have \( J_p \subset 5I_p \).

We denote by \( \tilde{I}_p = \frac{\ell(I)}{\ell(J)} J_p \), the cube with \( c(\tilde{I}_p) = c(J_p) = c(I) \).

Let \( e \in \mathbb{N} \) such that \( 2^e = \frac{\ell(I)}{\ell(J)} \geq 1 \). We write \( \varphi_R = \frac{|R|^{1/2}}{|R|^{1/2}} \chi_R \) with \( R \in \{ I, I_p \} \) and define \( \tilde{h}_I(t) = \varphi_I(c(J_p)) \chi_{I_p \cap \tilde{I}_p}(t) - \varphi_I(c(J_p)) \chi_{I_p \cap I_p}(t) \).

Then we perform the decomposition

\[
\psi = \psi_0 + \psi_1,
\]

\[
\psi_1(t, x) = \tilde{h}_I(t) h_J(x).
\]

We work first with the term \( \psi_1 \). We denote by \( \psi_R(x) = \varphi_R(c(J_p)) h_J(x) \), which satisfies \( \psi_I \equiv 0 \), \( \psi_{I_p} \equiv 0 \) when \( J_p \subset (5I_p) \setminus I_p \) and

\[
\| \psi_R b_2 \|_{L^2(R^n)} \lesssim |\langle b_1 \rangle R|^{-1}|R|^{-\frac{1}{2}} B_{J,q_2} |J|^{\frac{1}{2}+\frac{\delta}{2}}.
\]

- When \( \text{indist}(I_p, J_p) > 1 \), we have either \( J_p \subset (5I_p) \setminus I_p \) with \( \ell(J) \leq \ell(I) \), or \( J_p \subset I_p \) with \( \ell(J) \leq \ell(I)/2 \). In the former case \( \tilde{h}_I \equiv 0 \) and so, we have that \( e \geq 3 \). Then, by the special cancellation condition \( Tb_1 = 0 \), equalities \( \ell(\tilde{I}_p) = 2^e \ell(J_p) \) and \( |I| \leq |R| \), \( h_J \) being supported on \( J_p \) with mean zero with respect to \( b_2 \) and the error estimate (19) of Lemma 7.1, with the selected \( e \geq 3 \), we can bound the contribution of \( \psi_1 \) by

\[
\sum_{R \in \{ I, I_p \}} |\langle T_b \chi_{I_p \cap I_p}, \psi_R \rangle| = \sum_{R \in \{ I, I_p \}} |\langle T_b \chi_{R \cap I_p}, \psi_R \rangle - \langle T_{b_1}, b_2 \psi_R \rangle| 
\]

\[
\lesssim 2^{-e \delta} |J| \sum_{R \in \{ I, I_p \}} \inf_{x \in R} \inf_{2^{e+1} J} M_{q_1} b_1(x)[\psi_R b_2]_{J,q_2} F_K(2^e J, J, 2^e J)
\]

\[
\lesssim \left( \frac{\ell(J)}{\ell(I)} \right)^{\delta} |J| B_{J,q_2} |J|^{-\frac{1}{2}} |I|^{-\frac{1}{2}} \sum_{R \in \{ I, I_p \}} \frac{\inf_{x \in R} M_{q_1} b_1(x)}{|\langle b_1 \rangle R|} F_K(\tilde{I}, J, \tilde{I})
\]

(22) \[\lesssim \left( \frac{\ell(J)}{\ell(I)} \right)^{\frac{1}{2}+\delta} \sum_{R \in \{ I, I_p \}} \frac{\langle M_{q_1} b_1 \rangle_R}{|\langle b_1 \rangle R|} B_{J,q_2} F_K(I, J, I).\]
The last two inequalities are due to the facts that \( I \subset I_p \subset 2\tilde{I}_p \) and that 
\[
\ell(\tilde{I}) = \ell(I), \quad |c(I_p) - c(\tilde{I}_p)|_{\infty} \lesssim \ell(I) \text{ imply } rdist(\tilde{I}, B) \approx rdist(I, B).
\]
Since \( J_p \subset 5I_p \) implies \( \ell(J_p) + \text{dist}_{\infty}(J_p, D_{I_p}) \leq 2\ell(I_p) \), we have that 
\[
\text{ec}(I, J) = \frac{\ell(J_p)}{\ell(I_p)} \lesssim \frac{\ell(J_p)}{\ell(I_p) + \text{dist}_{\infty}(J_p, D_{I_p})} \lesssim \text{irdist}(I_p, J_p)^{-1}
\]
and so, (22) is smaller than the first term of case 2) in the statement.

- When \( \text{irdist}(I_p, J_p) = 1 \), if \( e \geq 2 \) we can proceed exactly in the same way. The cases \( e = 0 \) or \( e = 1 \) will be treated at the end.

- Now, we work with the term \( \psi_0(t, x) = (h_I(t) - \tilde{h}_I(t))h_J(x) \), which we further decompose as follows:

\[
\psi_0 = \psi_{\text{out}} + \psi_{\text{in}},
\]
\[
\psi_{\text{in}}(t, x) = \psi_0(t, x)(1 - \chi_{3J_p}(t)).
\]

**c.1** We work first with

\[
\psi_{\text{out}}(t, x) = (h_I(t) - \tilde{h}_I(t))(1 - \chi_{3J_p}(t))h_J(x)
\]

and divide the study in two parts:

- When \( J_p \subset (5I_p) \setminus I_p \), we have \( \varphi_K(c(J_p)) = 0 \) for \( K \in \{I, I_p\} \). Then \( \tilde{h}_I(t) = 0 \) and \( \psi_{\text{out}}(t, x) = h_I(t)(1 - \chi_{3J_p}(t))h_J(x) \). Consequently, \( \psi_{\text{out}}(t, x) \neq 0 \) implies \( t \in I_p \cap (3J_p)^c \) and

\[
|t - c(J_p)|_{\infty} \geq \frac{\ell(J_p)}{2} + \text{dist}_{\infty}(I_p, J_p) = \frac{\ell(J_p)}{2} + \text{dist}_{\infty}(J_p, D_{I_p}).
\]

- When \( J_p \subset I_p \), we further divide in two more cases:

  - When \( J_p = I_p \), we have \( \tilde{I}_p = I_p \). Then \( h_I = \tilde{h}_I \) and so,

  \[
  \psi_{\text{out}}(t, x) \equiv 0.
  \]

  - When \( J_p \subsetneq I_p \), we have that \( J_p \subset I' \) for some \( I' \in \chi(I_p) \). Then \( h_I(t) = \varphi_I(c(J_p)) - \varphi_{I_p}(c(J_p)) \) for all \( t \in I' \) and, from (23), we get \( \psi_{\text{out}}(t, x) = 0 \). That is, \( \psi_{\text{out}}(t, x) \neq 0 \) implies \( t \in (I_p \setminus I') \cap (3J_p)^c \), getting again

\[
|t - c(J_p)|_{\infty} \geq \frac{\ell(J_p)}{2} + \text{dist}_{\infty}(I \setminus I', J_p) = \frac{\ell(J_p)}{2} + \text{dist}_{\infty}(J_p, D_{I_p}).
\]

In both cases then, \( |t - c(J_p)|_{\infty} \geq \frac{1}{2} \text{irdist}(I_p, J_p)\ell(J_p) \). Also in both cases, \( |t - c(J_p)|_{\infty} \leq \text{diam}(I_p \cup J_p) \leq 3\ell(I_p) \) and \( |t - c(J_p)|_{\infty} \geq 3\ell(J_p)/2 > \ell(J_p) \). The latter inequality and \( |x - c(J_p)|_{\infty} \leq \ell(J_p)/2 \) imply \( 2|x - c(J_p)|_{\infty} < |t - c(J_p)|_{\infty} \). Then we can use the kernel representation and the zero mean of \( \psi_{\text{out}} \) with respect to \( b_2 \) to write

\[
\langle T_b((h_I - \tilde{h}_I)(1 - \chi_{3J_p})), h_J \rangle
\]

\[
= \int_{J_p} \int_{I_p \cap \tilde{I}} \psi_{\text{out}}(t, x)(K(t, x) - K(t, c(J_p)))b_1(t)b_2(x) \, dt \, dx,
\]
where \( \bar{J} = \{ t \in \mathbb{R}^n : \ell(J_p)/2 + \text{dist}_\infty(J_p, \mathcal{D}_{I_p}) < |t - c(J_p)|_\infty \leq 3\ell(I_p) \} \).

Now, we decompose \( \bar{J} \subset \bigcup_{m=m_0}^{m_1} J^m \), where

\[
J^m = \{ t \in I_p : 2^m \ell(J_p) < |t - c(J_p)|_\infty \leq 2^{m+1} \ell(J_p) \},
\]

with \( m_0 = \log(\text{inrdist}(I_p, J_p)/4) \gtrsim \log(\text{inrdist}(I, J)) \) and \( m_1 = \log(3\ell(I)/\ell(J)) \).

Since \( J^m \) is the difference of two concentric cubes with diameters \( 2^{m+1} \ell(J_p) \) and \( 2^{m+2} \ell(J_p) \), with abuse of notation we write \( \ell(J^m) = 2^{m+2} \ell(J_p) \) and \( c(J^m) = c(J_p) \). This way, the modulus of (24) can be bounded by

\[
\sum_{m=m_0}^{m_1} \| \psi_{out} \|_{L^\infty(J^m) \times L^\infty(I_p)} \| b_1 \chi_I \|_{L^2(J^m)} \| b_2 \|_{L^{q2}(I_p)}
\]

\[
\left( \int_{I_p} \left( \int_{J^m} |K(t, x) - K(t, c(J_p))|^{q_1/2} \, dt \right)^{q_2} dx \right)^{1/q_2}.
\]

We note that \( \| \psi_{out} \|_{L^\infty(J^m) \times L^\infty(I_p)} \lesssim C_I |I|^{-1/4} C_{J_p}^{1/2} |J|^{-1/4} \). By the smoothness property (2), we estimate the double integral, denoted again \( \text{Int} \):

\[
\text{Int} \leq \left( \int_{I_p} \left( \int_{J^m} \frac{|x - c(J_p)|^{q_1\delta}}{|t - c(J_p)|^{q_1(n+\delta)}} F_K(t, x, c(J_p))^{q_1/2} dt \right)^{q_2} dx \right)^{1/q_2}
\]

with \( F_K(t, x, c(J_p)) = L(|t-c(J_p)|_\infty) S(|x-c(J_p)|_\infty) D \left( 1 + \frac{|t+c(J_p)|_\infty}{1+|t-c(J_p)|_\infty} \right) \).

Since \( 2^{m+2} \ell(J) \geq 2^{m+1} \ell(J_p) \geq |t - c(J_p)|_\infty > 2^m \ell(J_p) \geq \ell(J) \) and \( |x - c(J_p)|_\infty \leq \ell(J) \), by the proof of Lemma 5.1, we have

\[
F_K(t, x, c(J_p)) \leq L(\ell(J)) S(\ell(J)) D \left( 1 + \frac{|c(J)|_\infty}{1+2^{m+2}\ell(J)} \right).
\]

Moreover, since \( J^m \subset 10I_p \), we get

\[
1 + \frac{|c(J)|_\infty}{1+2^{m+2}\ell(J)} \geq 1 + \frac{|c(J^m)|_\infty}{1+2^{m+3}\ell(J)} \gtrsim \text{rdist}(J^m, \mathbb{B})
\]

\[
\gtrsim \text{rdist}(10I_p, \mathbb{B}) \gtrsim \text{rdist}(I, \mathbb{B}),
\]

with clear meaning of \( \text{rdist}(J^m, \mathbb{B}) \) despite \( J^m \) is not a cube. Then

\[
F_K(t, x, c(J_p)) \leq L(\ell(J)) S(\ell(J)) D(\text{rdist}(I, \mathbb{B})) = F_K(J, J, I).
\]

With this and \( |J^m| \approx 2^{m}|J| \), we continue the bound in (26) as

\[
\text{Int} \lesssim F_K(J, J, I) \frac{\ell(J)^{\delta}}{(2^{m}\ell(J))^{n+\delta}} |J^m|^{1/q_1} |J|^{1/q_2}
\]

\[
\lesssim F_K(J, J, I) 2^{-m\delta} |J^m|^{-\frac{1}{q_1}} |J|^{\frac{1}{q_2}}.
\]
Therefore, we can estimate (23) by

\[
\sum_{m=m_0}^{m_1} C^m_I |I|^{\frac{1}{m}} C^m_J |J|^{\frac{1}{m}} |[b_1 \chi_I]_{J^m,q_1}| J^m |[b_2]_{J,q_2} |J|^{\frac{1}{m}} F_{K}(J,J,I) 2^{-m\delta} \]

\[
\lesssim \left( \frac{|J|}{|I|} \right)^{\frac{1}{2}} [b_2]_{J,q_2} F_{K}(J,J,I) \sum_{m=m_0}^{m_1} 2^{-m\delta} |[b_1 \chi_I]_{J^m,q_1}|.
\]

Now, since \( J^m \subset 2^{m+2} J_p \subset 14 I \), we have

\[
[b_1 \chi_I]_{J^m,q_1} \lesssim \frac{1}{2^m |I|} \int_{2^{m+2} J_p \cap I} |b_1(x)|^q \, dx \lesssim [b_1 \chi_I]_{2^{m+2} J_p,q_1}^{q_1}
\]

Moreover \([b_1 \chi_I]_{2^{m+2} J_p,q_1} \leq \inf_{x \in J} M_{q_1}(b_1 \chi_I)(x) \leq \langle M_{q_1}(b_1 \chi_I) \rangle_J \) and so,

\[
|\langle T_{b_i}(h_I - \tilde{h}_I)(1 - \chi_{3 J_p}), h_J \rangle| \]

\[
\lesssim ec(I,J)^{\frac{3}{2}} \langle M_{q_1}(b_1 \chi_I) \rangle_J [b_2]_{J,q_2} F_{K}(J,J,I) \sum_{m \geq \log(\text{inrdist}(I,J))} 2^{-m\delta}
\]

\[
\lesssim \frac{ec(I,J)^{\frac{3}{2}}}{\text{inrdist}(I,J)^{\delta}} \langle M_{q_1}(b_1 \chi_I) \rangle_J [b_2]_{J,q_2} F_{K}(J,J,I),
\]

smaller than the second term of case 2) and the first term of case 3).

**c.2)** We now work with

(27) \[
\psi_{\text{in}}(t,x) = (h_I(t) - \tilde{h}_I(t))\chi_{3 J_p}(t) h_J(x).
\]

**c.2.1** We first consider the case \( \ell(J) \leq \ell(I) \).

We start by showing that when inrdist\((J_p, I_p) > 1\), we have \( \psi_{\text{in}} \equiv 0 \) and so, this term does not appear in case 2) in the statement. As said before, the cubes for which inrdist\((J_p, I_p) > 1\) satisfy either \( J_p \subset (5I_p) \setminus I_p \) with \( I_p \cap 3J_p = \emptyset \) or \( 3J_p \subset I_p \) with \( \ell(J) \leq \ell(I)/8 \). In the former case, we have \( h_I(t)\chi_{3 J_p}(t) = \tilde{h}_I(t) = 0 \) and so, \( \psi_{\text{in}} \equiv 0 \). In the latter case, we get \( 3J_p \subset I' \) for some \( I' \in \text{ch}(I_p) \) and \( 3J_p \subset I_p \). Therefore, \( h_I(t)\chi_{3 J_p}(t) = \tilde{h}_I(t)\chi_{3 J_p}(t) \) and \( \psi_{\text{in}} \equiv 0 \) again.

We now consider those cubes \( J \) such that inrdist\((J_p, I_p) = 1\). The cardinality of this family of cubes is at most \( c^n(\ell(I)/\ell(J))^{n-1} \) for some constant \( c > 1 \). As before, we divide in two cases:

- When \( J_p \subset (5I_p) \setminus I_p \), we have \( \varphi_K(c(J_p)) = 0 \) for \( K \in \{I, I_p\} \) and (27) reduces to

\[
\psi_{\text{in}}(t,x) = h_I(t)\chi_{3 J_p}(t) h_J(x) = h_I(t)\chi_{(3 J_p) \setminus J_p}(t) h_J(x),
\]

- When \( J_p \cap 3J_p = \emptyset \) and \( \ell(J) \leq \ell(I)/8 \), we have \( \varphi_K(c(J_p)) = 0 \) for all \( K \in \{I, I_p\} \), and (27) reduces to

\[
\psi_{\text{in}}(t,x) = h_I(t)\chi_{3 J_p}(t) h_J(x) = h_I(t)\chi_{3 J_p}(t) h_J(x).
\]
since \(I_p \cap J_p = \emptyset\). We also note that in this case, \(h_I\) and \(h_J\) have disjoint compact support.

- When \(J_p \subset I_p\), we have \(J_p \subset I'\) for some \(I' \in \text{ch}(I_p)\) and so, \(c(J_p) \subset I'\). Then we decompose as \(3J_p = ((3J_p) \cap I') \cup ((3J_p) \setminus I')\).

For all \(t \in (3J_p) \cap I'\), we have \(t, c(J_p) \in I'\), which implies \(|t - c(J_p)|_{\infty} \leq \ell(I_p)/2\) and so, \(t \in \tilde{I}_p\). Then \(h_I(t) = \tilde{h}_I(t)\) and, from (27), we get \(\psi_{in}(t, x) = 0\).

On the other hand, for all \(t \in (3J_p) \setminus I'\), we have

\[
\psi_{in}(t, x) = (h_I(t) - \tilde{h}_I(t)) \chi_{(3J_p) \setminus I'}(t) h_J(x),
\]

where \((h_I - \tilde{h}_I) \chi_{(3J_p) \setminus I'}\) is disjoint with \(h_J\) since \(J_p \subset I'\). Moreover, \(\chi_{(3J_p) \setminus I'} \leq \chi_{3(J_p) \setminus I_p}\).

Then we can write in both cases \(\langle T_b h_I, h_J \rangle = \langle T_b (h_I \chi_{(3J_p) \setminus I_p}), h_J \chi_{I_p} \rangle\) and, by Lemma 6.1, have

\[
|\langle T_b h_I, h_J \rangle| \leq |J_p|[\alpha_{b_1} \beta_{b_2} |h_I| |h_J| F_3(J) \leq \left(\frac{\ell(J)}{\ell(I)}\right)^{\frac{\ell}{2}} (B_3(I, J) F_3(J)).
\]

This is the second term of case 3) in the statement when \(\ell(J) < \ell(I)\).

**c.2.2** Finally, we consider \(\ell(J) = \ell(I)\). For this case, which implies \(\text{inrdist}(I_p, J_p) = 1\), we recover the original notation \(h_I^{b_1}, h_J^{b_2}\) indicating the dependence of the bump functions. We note that \(\tilde{I}_p = I_p\) and so,

\[
\psi_{in}(t, x) = (h_I^{b_1}(t) - \tilde{h}_I^{b_1}(t)) h_J^{b_2}(x).
\]

If \(J_p \subset (5I_p) \setminus I_p\), we have \(\tilde{h}_I^{b_1} \equiv 0\). We apply Lemma 6.1 (as in subcase c.2.1) to obtain the second term in 3) for \(I \neq J\) and \(\ell(I) = \ell(J)\).

We are left with the case \(J_p = I_p\), for which \(J \in \text{ch}(I_p)\). For the first term of \(\psi_{in}\) in (28), we have

\[
\psi_{in}(t, x) = \sum_{I' \in \text{ch}(I_p)} \alpha_{I'} \chi_{I'}(t) \sum_{I'' \in \text{ch}(I_p)} \beta_{I''} \chi_{I''}(x)
\]

with \(\alpha_I = |I|^{\frac{1}{2}} \frac{1}{|I_p||b_1|} - \frac{1}{|I_p||b_2| I_p}\), \(\alpha_{I'} = -|I'|^{\frac{1}{2}} \frac{1}{|I_p||b_1| I_p}\) for \(I' \neq I\) and the same for \(\beta_{I''}\) just changing \(b_1\) and \(I\) by \(b_2\) and \(J\). This implies

\[
\langle T_b h_I^{b_1}, h_J^{b_2} \rangle = \sum_{I' \in \text{ch}(I_p)} \sum_{I'' \in \text{ch}(I_p)} \alpha_{I'} \beta_{I''} \langle T_b \chi_{I'}, \chi_{I''} \rangle.
\]

The same reasoning applied to the second term of \(\psi_{in}\) in (28) gives

\[
\langle T_b h_I^{b_1}, h_J^{b_2} \rangle = \alpha_{I'} \sum_{I'' \in \text{ch}(I_p)} \alpha_{I'} \beta_{I''} \langle T_b \chi_{I'}, \chi_{I''} \rangle,
\]

with \(I'\) such that \(c(J_p) \subset I'\). Thus, we can study both cases together.
For $I' \neq I''$, since $\text{dist}(I', I'') = 0$, we can proceed as in c.2.1): from (13) in Lemma 6.1 (or even (14) in Lemma 6.2) we get
\[ |\langle T_{b} \chi_{I'}, \chi_{I''} \rangle| \lesssim |I| |[b_{1}]_{r_{1}}[b_{2}]_{r_{2}} \tilde{F}_{K}(I)\]
For $I' = I''$, the weak compactness condition of Definition 3.2 gives
\[ |\langle T_{b} \chi_{I'}, \chi_{I'} \rangle| \lesssim |I| |[b_{1}]_{r_{1}}[b_{2}]_{r_{2}} (F_{W}(I; M) + \epsilon).\]
From $|\alpha_{I}| \lesssim C_{I}^{b_{1}} |I|^{-\frac{1}{2}}$, $|\beta_{I'}| \lesssim C_{I'}^{b_{2}} |I'|^{-\frac{1}{2}}$ we have $|I| |\alpha_{I}| |\beta_{I'}| \lesssim C_{I}^{b_{1}} C_{I'}^{b_{2}}$.
With this,
\[ |\langle T_{b} h_{I}^{b_{1}}, h_{I}^{b_{2}} \rangle| \lesssim \sum_{I', I'' \in \text{ch}(I_{p})} C_{I'}^{b_{1}} |[b_{1}]_{r_{1}} C_{I''}^{b_{2}} |[b_{2}]_{r_{2}} (\tilde{F}_{K}(I) + F_{W}(I; M_{T, \epsilon}) + \epsilon)\]
This is the second term of case 3) in the statement when $I = J$.

There is still one case to end the proof: the term left undone at the end of case b), that is, the bound for $|\langle T_{b} \chi_{I}, \tilde{\psi} \rangle|$ when $\text{inrdist}(I_{p}, J_{p}) = 1$ and $\text{ec}(I, J) \in \{0, 1\}$. But it now is clear that this expression can be bounded in the same way we did in case c.2.1) with the use of Lemma 6.1 and case c.2.2) using the weak compactness condition. This provides the first term of case 3) in the statement. □

9. The adapted wavelet system

Definition 9.1. Let $b$ be locally integrable with $\langle b \rangle \neq 0$ for $I \in \mathcal{D}$. Following [13], we define the expectation associated with $b$ $E_{Q} b f = \langle f \rangle_{Q} b \chi_{Q}$ for every locally integrable function $f$. And for every $k \in \mathbb{Z}$,
\[ E_{k}^{b} f = \sum_{Q \in \mathcal{D} \atop \ell(Q) = 2^{-k}} E_{Q}^{b} f.\]
We also define their corresponding difference operators
\[ \Delta_{k}^{b} f = E_{k}^{b} f - E_{k-1}^{b} f = \sum_{Q \in \mathcal{D} \atop \ell(Q) = 2^{-(k-1)}} \Delta_{Q}^{b} f,\]
where
\[ \Delta_{Q}^{b} f = \left( \sum_{I \in \text{ch}(Q)} E_{I}^{b} f \right) - E_{Q}^{b} f = \sum_{I \in \text{ch}(Q)} \left( \frac{\langle f \rangle_{I}}{\langle b \rangle_{I}} - \frac{\langle f \rangle_{Q}}{\langle b \rangle_{Q}} \right) b \chi_{I}.\]

Definition 9.2 (Adapted Haar wavelets). Let $b$ be a locally integrable function with non-zero dyadic averages. For $I \in \mathcal{D}$, we remind the functions given in Definition 8.1, $h_{I}^{b} = |I|^{\frac{1}{2}} \left( \frac{1}{|I||b_{I}} \chi_{I} - \frac{1}{|I||b_{I}||p_{I}} \chi_{p} \right)$. 

\[ \text{compute}\]
Then for $I \in \mathcal{D}$ we define the Haar wavelets adapted to $b$ and their corresponding dual wavelets as

$$\psi^b_I = h^b_I, \quad \tilde{\psi}^b_I = h^b_I \langle b \rangle_I.$$ 

We have the following result:

**Lemma 9.3.** For every locally integrable function $f$,

$$\Delta^b_Q f = \sum_{I \in \text{ch}(Q)} \langle f, \tilde{\psi}^b_I \rangle \psi^b_I,$$

where $\langle f, g \rangle = \int_{\mathbb{R}^n} f(x)g(x)dx$.

**Proof.** A direct computation starting at (29) shows that

$$\Delta^b_Q f = \sum_{I \in \text{ch}(Q)} \langle f \rangle_I \left( \frac{1}{\langle b \rangle_I} \chi_I - \frac{|I|}{|Q|} \frac{1}{\langle b \rangle_Q} \chi_Q \right) b = \sum_{I \in \text{ch}(Q)} \langle f \rangle_I |I|^{\frac{1}{2}} \psi^b_I.$$

Also from (29), we have

$$\langle \Delta^b_Q f \rangle_I = \langle f \rangle_I - \frac{\langle b \rangle_I}{\langle b \rangle_Q} \langle f \rangle_Q$$

and since

$$\sum_{I \in \text{ch}(Q)} \frac{\langle b \rangle_I}{\langle b \rangle_Q} \langle f \rangle_I |I|^{\frac{1}{2}} \psi^b_I = 0$$

we get

$$\Delta^b_Q f = \sum_{I \in \text{ch}(Q)} \langle \Delta^b_Q f \rangle_I |I|^{\frac{1}{2}} \psi^b_I.$$

Now, we use (30) to compute the coefficients and get:

$$|I|^{\frac{1}{2}} \langle \Delta^b_Q f \rangle_I = |I|^{\frac{1}{2}} \int f(x) \left( \frac{\chi_I(x)}{|I|} - \frac{\langle b \rangle_I}{\langle b \rangle_Q} \frac{\chi_Q(x)}{|Q|} \right) dx = \langle f, \tilde{\psi}^b_I \rangle.$$

**Remark 9.4.** From (31) or the dual equality

$$\sum_{I \in \text{ch}(Q)} \tilde{\psi}^b_I = 0,$$

we see that this wavelet system is not linearly independent.

**Corollary 9.5.** For $(\Delta^b_Q)^* f = \sum_{I \in \text{ch}(Q)} \left( \frac{\langle fb \rangle_I}{\langle b \rangle_I} - \frac{\langle fb \rangle_Q}{\langle b \rangle_Q} \right) \chi_I$, we have

$$(\Delta^b_Q)^* f = \sum_{I \in \text{ch}(Q)} \langle f, \psi^b_I \rangle \tilde{\psi}^b_I.$$
Next lemma states the orthogonality properties of the adapted Haar wavelets. The proof follows from direct calculations.

**Lemma 9.6.** For $I, J \in \mathcal{D}$, $\int \psi^b_I(x) \, dx = \int \tilde{\psi}^b_J(x) \, dx = 0$. Moreover,
\begin{equation}
\langle \psi^b_I, \tilde{\psi}^b_J \rangle = 0
\end{equation}
when $I_p \neq J_p$, while for $I_p = J_p$, we have
\begin{equation}
\langle \psi^b_I, \tilde{\psi}^b_J \rangle = \delta(I - J) - \frac{|J| \langle b \rangle_J}{|I_p| \langle b \rangle_{I_p}}.
\end{equation}
where $\delta(I - J) = 1$ if $I = J$ and zero otherwise.

Finally, $\| \tilde{\psi}^b_I \|_{L^q(\mathbb{R}^n)} \lesssim C^b_I \| |I| \|_{L^q(\mathbb{R}^n)}$ and $\| \psi^b_I \|_{L^q(\mathbb{R}^n)} \lesssim B^b_{I,q} |I|^{\frac{1}{q} - \frac{1}{2}}$.

The next result, which generalizes the classical Carleson’s Embedding Theorem, is used in Lemma 9.9 and Section 11. The proof follows from a direct adaptation of the demonstration included in [2].

**Lemma 9.7 (Carleson Embedding Theorem).** Let $(a_I)_{I \in \mathcal{D}}$ a collection of non-negative numbers such that for all $I \in \mathcal{D}$,
\begin{equation}
\sum_{J \in \mathcal{D}(I)} a_J \lesssim [b]_{I,2}^2 |I|.
\end{equation}
Then for every $f \in L^2(\mathbb{R}^n)$, $\sum_{I \in \mathcal{D}} [b]_{I,2}^{-2} a_I |\langle f \rangle_I|^2 \lesssim \| f \|^2_{L^2(\mathbb{R}^n)}$.

**Remark 9.8.** For $(a_I)_{I \in \mathcal{D}}$, $(b_I)_{I \in \mathcal{D}}$ with $a_I, b_I$ non-negative,
\begin{equation}
\sum_{I \in \mathcal{D}} a_I b_I |\langle f \rangle_I|^2 \lesssim \sup_{I \in \mathbb{R}^n} \left( \frac{b_I}{|I|} \sum_{J \in \mathcal{D}(I)} a_J \right) \| f \|^2_{L^2(\mathbb{R}^n)}.
\end{equation}

**Lemma 9.9.** Let $b$ be a locally integrable function compatible with an operator and let $BF$ as stated in Definition 3.10. Then,
\begin{equation}
\sum_{I \in \mathcal{D}} BF(I, J) |\langle f, \tilde{\psi}^b_I \rangle|^2 \lesssim \| f \|^2_{L^2(\mathbb{R}^n)}
\end{equation}
for every locally integrable function $f$ and every $J \in \mathcal{D}$.

Moreover, for $\epsilon > 0$, there is $M_0 \in \mathbb{N}$, such that for all $M > M_0$,
\begin{equation}
\sum_{I \in \mathcal{D}_M} \sup_{J \in \mathcal{D}_M} BF(I, J) |\langle f, \tilde{\psi}^b_I \rangle|^2 \lesssim \epsilon \| f \|^2_{L^2(\mathbb{R}^n)}
\end{equation}
for every locally integrable function $f$, where $\mathcal{F}_M$ is given after condition (9) of Definition 3.10.
Remark 9.10. The proof shows that the following inequality also holds
\[
\sum_{I \in \mathcal{D}} \left( \frac{[b]_{I_p,2}}{|\langle b \rangle_{I_p}|} \right)^{-2} |\langle f, \tilde{\psi}_I^b \rangle|^2 \lesssim \|f\|^2_{L^2(\mathbb{R}^n)}.
\]

Proof. On the one hand, for \(I \in \text{ch}(I_p)\), by the definition of \(\tilde{\psi}_I^b\) we have
\[
|\langle f, \tilde{\psi}_I^b \rangle| = |I|^{\frac{1}{2}} |\langle b \rangle_I| \left| \frac{\langle f \rangle_I}{\langle b \rangle_I} - \frac{\langle f \rangle_{I_p}}{\langle b \rangle_{I_p}} \right|
= |I|^{\frac{1}{2}} |\langle f \rangle_I - \langle f \rangle_{I_p} - \frac{1}{\langle b \rangle_{I_p}} \langle f \rangle_{I_p} (\langle b \rangle_I - \langle b \rangle_{I_p})| \\
\leq \frac{[b]_{I_p,2}}{|\langle b \rangle_{I_p}|} \left( |I|^{\frac{1}{2}} |\langle f \rangle_I - \langle f \rangle_{I_p}| + |I|^{\frac{1}{2}} [b]_{I_p,2} |\langle b \rangle_I - \langle b \rangle_{I_p}| |\langle f \rangle_{I_p}| \right),
\]

since \(|\langle b \rangle_{I_p}| \leq [b]_{I_p,2}\). Now, by conditions (8), (9) of Definition 3.10 of a compatible testing function, we have that \(BF(I, J) [b]_{I_p,2}^2 / |\langle b \rangle_I|^2 \lesssim C\) for all \(I, J \in \mathcal{D}\) and that given \(\epsilon > 0\), there is \(M_0 \in \mathbb{N}\) satisfying \(BF(I, J)[b]_{I_p,2}^2 / |\langle b \rangle_I|^2 \lesssim \epsilon\) for all \(M > M_0\) and \(I, J \in \mathcal{D}_M\) such that \((I, J) \in \mathcal{F}_M\). With this, we obtain
\[
\sum_{I \in \mathcal{D}} \sup_{J \in \mathcal{D}_M} BF(I, J) |\langle f, \tilde{\psi}_I^b \rangle|^2 \lesssim \epsilon \left( \sum_{I \in \mathcal{D}} |\langle f \rangle_I - \langle f \rangle_{I_p}|^2 |I| + \sum_{I \in \mathcal{D}} [b]_{I_p,2}^{-2} |\langle b \rangle_I - \langle b \rangle_{I_p}|^2 |\langle f \rangle_{I_p}|^2 \right)
\]
and the last expression is bounded by a constant times \(\epsilon \|f\|_{L^2(\mathbb{R}^n)}\) as we briefly indicate. The first term follows by the standard square function estimate. Moreover, the same square function estimate shows that
\[
\sum_{J \in \mathcal{D}(I)} |\langle b \rangle_J - \langle b \rangle_{I_p}|^2 |J| \leq \|b\|_{L^2(\mathbb{R}^n)} = [b]_{I_p,2} |I_p|,
\]
which proves that \(\left( \sum_{I \in \text{ch}(Q)} |\langle b \rangle_I - \langle b \rangle_Q|^2 |I| \right)_{Q \in \mathcal{D}}\) satisfies hypothesis (3.5) of Lemma 9.7. Then
\[
\sum_{Q \in \mathcal{D}} \sum_{I \in \text{ch}(Q)} [b]_Q^{-2} |\langle b \rangle_I - \langle b \rangle_Q|^2 |I| |\langle f \rangle_Q|^2 \lesssim \|f\|_2.
\]

Corollary 9.11. The following dual statement also holds:
\[
\sum_{I \in \mathcal{D}} \left( \frac{[b]_{I_p,2}}{|\langle b \rangle_I ||\langle b \rangle_{I_p}|} \right)^{-2} |\langle f, \tilde{\psi}_I^b \rangle|^2 \lesssim \|f b\|_{L^2(\mathbb{R}^n)}.
\]

Proof. Since \(\langle f, \tilde{\psi}_I^b \rangle = |\langle b \rangle_I|^{-1} \langle f b, \tilde{\psi}_I^b \rangle\), we can apply Remark 9.10.
Lemma 9.12. Let $b$ be a locally integrable function. Then the equality
\begin{equation}
    f = \sum_{I \in \mathcal{D}} \langle f, \tilde{\psi}^b_I \rangle \psi^b_I
\end{equation}
holds pointwise a.e. almost everywhere for $f$ integrable, compactly supported and with mean zero.

Proof. By Lemma 9.3 we have $\Delta^b_Q f = \sum_{I \in \text{ch}(Q)} \langle f, \tilde{\psi}^b_I \rangle \psi^b_I$. Then the right hand side of (37) is understood as
\[
\lim_{M \to \infty} \sum_{I \in \mathcal{D}} \langle f, \tilde{\psi}^b_I \rangle \psi^b_I = \lim_{M \to \infty} \sum_{-M < k \leq M} \Delta^b_k f.
\]
We choose $R \in \mathcal{D}$ with $\text{supp} f \subset R$, and $M \in \mathbb{N}$ with $2^{-M} < \ell(R) < 2^M$. For every $x \in R$, we select $I, J \in \mathcal{D}$ such that $x \in J \subset I$, $\ell(J) = 2^{-M}$ and $\ell(I) = 2^M$. Since $R \subseteq I$ and $f$ has zero mean, then $\langle f \rangle_I = 0$. With this, by summing a telescopic series, we have
\begin{equation}
\sum_{-M < k \leq M} \Delta^b_k f(x) = E^b_M f(x) - E^b_{-M} f(x) = \frac{\langle f \rangle_J}{\langle b \rangle_J} \chi_J(x) b(x).
\end{equation}
Now, since $f$ and $b$ are both locally integrable, by Lebesgue’s Differentiation Theorem, the right hand side of (38) tends to $f(x)$ pointwise almost everywhere when $M$ tends to infinity.

By a similar reasoning, we can prove the following dual result:

Lemma 9.13. Let $b$ be a locally integrable function. Then the equality
\begin{equation}
    f = \sum_{I \in \mathcal{D}} \langle f, \psi^b_I \rangle \tilde{\psi}^b_I
\end{equation}
holds pointwise almost everywhere for $f$ integrable, compactly supported and with mean zero with respect to $b$.

Lemma 9.14. Let $T$ be a bounded operator on $L^2(\mathbb{R}^n)$ with compact Calderón-Zygmund kernel $K$. Let $b_i$ be two locally integrable functions compatible with $T$ and $(\psi^b_I)_{I \in \mathcal{D}}$ be the wavelet systems of Definition 9.2. Then for $f, g$ locally integrable,
\begin{equation}
    \langle T f, g \rangle = \sum_{I, J \in \mathcal{D}} \langle f, \tilde{\psi}^b_I \rangle \langle g, \tilde{\psi}^b_J \rangle \langle T \psi^b_I, \psi^b_J \rangle,
\end{equation}
\begin{equation}
    \langle T_b f, g \rangle = \sum_{I, J \in \mathcal{D}} \langle f, \psi^b_I \rangle \langle g, \psi^b_J \rangle \langle T_b \psi^b_I, \psi^b_J \rangle.
\end{equation}
We note that the lack of accretivity is the reason for the unusual definition of weak compactness (Definition 3.2) and the extra work required to prove Lemma 9.14. When the testing functions are accretive, the lemma follows directly from convergence of a wavelet frame on $L^2(\mathbb{R}^n)$ and the continuity of $T$. However, without accretivity the chosen wavelet system does not converge on $L^2(\mathbb{R}^n)$. Moreover, one cannot use the classical $Tb$ Theorem to deduce that $T$ is already known to be bounded because in general the testing functions are not accretive.

**Proof (of Lemma 9.14).** We only show (40). Let $(h_1^i)_{i \in I}$ be the Haar-wavelet system. By Lemma 9.12 applied to the accretive functions $b_i = 1$ and the continuity of $T$ on $L^2(\mathbb{R}^n)$,

$$\langle Tf, g \rangle = \sum_{I,J \in \mathcal{D}} \langle f, h_1^I \rangle \langle g, h_1^J \rangle \langle Th_1^I, h_1^J \rangle = \lim_{N \to \infty} \langle Tf_N, g_N \rangle$$

with $f_N = \sum_{I \in \mathcal{D}_N} \langle f, h_1^I \rangle h_1^I$ and similar for $g_N$. Then, we can assume $f, g$ to be in the unit ball of $L^2(\mathbb{R}^n)$, supported on $Q \in \mathcal{D}$ with $\ell(Q) > 1$, constant on dyadic cubes of side length $\ell \leq 1$ and with mean zero.

As shown in the proof of Lemma 9.12, we have

$$\sum_{I \in \mathcal{D}} \langle f, \tilde{\psi}_I^b \rangle \psi_I = \sum_{-M < k \leq M} \Delta_k f = E_M^b f,$$

and similar expression for $g$. We then need to prove that $|\langle Tf, g \rangle - \langle T(E_M^b f), E_M^b g \rangle|$ tends to zero when $M$ tends to infinity uniformly for functions $f, g$ in the unit ball of $L^2(\mathbb{R}^n)$. We bound this difference by

$$|\langle Tf - E_M^{b_1} f, g \rangle| + |\langle T(E_M^{b_1} f), g - E_M^{b_2} g \rangle| = S_1 + S_2$$

and we only work to estimate the second term. As explained before, $g - E_M^{b_2} g$ does not necessarily tend to zero on $L^2(\mathbb{R}^n)$.

By condition (3) of Definition 3.10, given $\epsilon > 0$ there exists $M_0 \in \mathbb{N}$ such that $2^{-M_0} < \ell$ and for all $M > M_0$,

$$\sup_{I, J \in \mathcal{D}} BF(I, J) \leq \sup_{(I, J) \in \mathcal{F}_M} BF(I, J) \left( \frac{[b_1]_{I, \mathbb{R}^2}}{||\langle b_1 \rangle_I||^2} + \frac{[b_2]_{J, \mathbb{R}^2}}{||\langle b_2 \rangle_J||^2} \right) < \epsilon,$$

where $\mathcal{F}_M$ is the family of ordered pairs of cubes $I, J \in \mathcal{D}_M^c$, with either $\ell(I \wedge J) > 2^M$, $\ell(I \wedge J) < 2^{-M}$ or $\text{rdist}(\langle I, J \rangle, \mathbb{R}) > M^\theta$ for $\theta \in (0, 1)$.

We fix now $M > M_0$. Let $(I_i)_{i \in \mathbb{Z}_M^n} \subset \mathcal{D}$ a partition of $Q$ with $Z_M^n = \{ i \in \mathbb{N}^n : ||i||_\infty \leq 2^M \ell(Q) \}$ such that $\ell(I_i) = 2^{-M}$ and the cubes
\[ I_i \text{ are enumerated so that } \text{dist}(I_i, I_j) = \max(\|i - j\|_\infty - 1, 0) 2^{-M}. \text{ Then,} \]
\[ S_2 \leq \sum_{i,j \in \mathbb{Z}_T^M} |\langle T(E_M^b f \chi_{I_i}, (g - E_M^b g) \chi_{I_j}) \rangle| = \sum_{i,j \in \mathbb{Z}_T^M} T_{i,j} \]

We consider the cases \( \text{dist}(I_i, I_j) > 0 \) or \( \text{dist}(I_i, I_j) = 0 \). In the first one, by the integral representation of \( T \) and the mean zero of \( (g - E_M^b g) \chi_{I_i} \),
\[ T_{i,j} = \left| \int_{I_i} \int_{I_j} \left( \frac{\langle f \rangle_{I_i}}{\langle b_1 \rangle_{I_i}} b_1(t)(g(x) - \left\langle \frac{\langle g \rangle_{I_j}}{\langle b_2 \rangle_{I_j}} \right\rangle \langle b_2 \rangle_{I_j}) (K(t,x) - K(t, c(I_j))) dt dx \right| \]

Since \( \text{dist}(I_i, I_j) > 0 \) and \( \ell(I) = \ell(I_j) \), we have that \( \text{dist}(I_i, I_j) \geq \ell(I) \). Then, for all \( t \in I_i \) and \( x \in I_j \) we have \( 2|x - c(I_j)|_\infty \leq \ell(I_j) \leq \text{dist}(I_i, I_j) < |t - c(I_j)|_\infty \). With this,
\[ |K(t, x) - K(t, c(I_j)))| \leq \frac{|x - c(I_j)|^\delta}{|t - c(I_j)|^{n+\delta}} F_K(t, x, c(I_j)) \]

with \( F_K(t, x, c(I_j)) = L(|t - c(I_j)|_\infty) S(|x - c(I_j)|_\infty) D \left( 1 + \frac{|x - c(I_j)|_\infty}{|t - c(I_j)|_\infty} \right) \). Now, the properties of \( I_i, I_j \) also imply \( \text{dist}(I_i, I_j) \geq \ell((I_i, I_j))/3 \). With this, we get the inequalities: \( |t - c(I_j)|_\infty > \ell((I_i, I_j))/3 \), \( |x - c(I_j)|_\infty \leq \ell(I_j)/2 \) and \( |c(I_i) - c((I_i, I_j))|_\infty \leq \ell((I_i, I_j))/2 \). Then, by the proof of Lemma 5.1 we have in the domain of integration
\[ F_K(t, x, c(J)) \lesssim L(\ell((I_i, I_j))) S(\ell(I_j)) D \left( 1 + \frac{|c(I_j)|_\infty}{1 + \ell((I_i, I_j))} \right) \]
\[ \lesssim F_K((I_i, I_j), I_j, \langle I_i, I_j \rangle) =: F_K(i, j). \]

Since \( g \) is constant on \( I_j \), we have \( g \chi_{I_j} = \langle g \rangle_{I_j} \chi_{I_j} \) and so, for all \( x \in I_j \),
\[ g(x) - \left\langle \frac{\langle g \rangle_{I_j}}{\langle b_2 \rangle_{I_j}} \right\rangle \langle b_2 \rangle_{I_j} = \frac{\langle g \rangle_{I_j}}{\langle b_2 \rangle_{I_j}} \langle b_2 \rangle_{I_j} - b_2(x) \]

Moreover, since \( \frac{\ell((I_i, I_j))}{\ell(I_j)} = \text{rdist}(I_i, I_j) = \frac{\ell(I_i) + \text{dist}(I_i, I_j)}{\ell(I_i)} = \|i - j\|_\infty \),
\[ T_{i,j} \leq |\langle f \rangle_{I_i}||g \rangle_{I_j}||I_i||_{L^1(I_i)} \frac{\ell(I_j)}{\ell((I_i, I_j))^{n+\delta}} \left\| \frac{\langle b_1 \rangle_{I_i}}{\langle b_2 \rangle_{I_j}} \right\| F_K(i, j) \]
\[ \lesssim |\langle f \rangle_{I_i}||g \rangle_{I_j}||I_i||_{L^1(I_i)} \frac{\ell(I_j)}{\ell((I_i, I_j))^{n+\delta}} \left\| \frac{\langle b_1 \rangle_{I_i}}{\langle b_2 \rangle_{I_j}} \right\| F_K(i, j) \]
\[ \lesssim \epsilon |\langle f \rangle_{I_i}||g \rangle_{I_j}||I_i||_{L^1(I_i)} \|i - j\|_\infty^{-(n+\delta)} \]

The last inequality is due to \( \ell(I_j) = 2^{-M} \), Definition 3.10 and (42):
\[ \frac{|b_1|_{I_i,1}}{|b_1|_{I_i}^{n+\delta}} + \frac{|b_2|_{I_j,1}^{n+\delta}}{|b_2|_{I_j}^{n+\delta}} \frac{F_K(i, j) - BF(I_i, I_j)}{BF(I_i, I_j) \leq \sup_{I, J \in \mathcal{D}} BF(I, J) \leq \epsilon} \]
\[ \epsilon(I, J) < 2^{-(M-1)} \]
Therefore, the corresponding sum can be bounded as follows:

\[
\sum_{i,j : \|i-j\|_\infty \geq 1} T_{i,j} \lesssim \epsilon \sum_{i,j : \|i-j\|_\infty \geq 1} |\langle f \rangle_{I_i}||\langle g \rangle_{I_j}||i-j||_\infty^{-(n+\delta)} \\
\leq \epsilon \left( \sum_{i \in \mathbb{Z}_M^d} |\langle f \rangle_{I_i}|^2 |I_i| \sum_{j : \|i-j\|_\infty \geq 1} ||i-j||_\infty^{-(n+\delta)} \right)^{\frac{1}{2}} \\
\leq \epsilon \left( \frac{1}{\|b_1\|_{I_i}} \sum_{i : \|i-j\|_\infty \geq 1} ||I_i||_1 \right)^{\frac{1}{2}} \leq \epsilon \ 
\]

On the other hand, when \( \text{dist}(I_i, I_j) = 0, I_i \neq I_j \), we write

\[
T_{i,j} = \left| \langle T(\frac{\langle f \rangle_{I_i} b_1 \chi_{I_i} \rangle}{\langle b_1 \rangle_{I_i}}, \frac{|\langle g \rangle_{I_j}|}{\langle b_2 \rangle_{I_j}}((b_2)_{I_j} - b_2) \chi_{I_j} \rangle \right| \\
\leq |\langle f \rangle_{I_i}||\langle g \rangle_{I_j}||I_i||I_j| \\
\leq \epsilon \left( \sum_{i \in \mathbb{Z}_M^d} |\langle f \rangle_{I_i}|^2 |I_i| \right)^{\frac{1}{2}} \left( \sum_{j \in \mathbb{Z}_M^d} |\langle g \rangle_{I_j}|^2 |I_j| \right)^{\frac{1}{2}} \leq \epsilon 
\]

By Lemma 6.2 we have that the terms inside the parentheses can be bounded by a constant times

\[
|I_i| |b_1|_{I_{i,q_1}} \left( 1 + \left[ \frac{|b_2|_{I_{j,q_2}}}{\|b_2\|_{I_j}} \right] \right) F_K(I_i) \lesssim |I_i| BF(I_i, I_j) \lesssim |I_i|, 
\]

where the last inequality is due to (12) and the fact that \( \ell(I_i) = 2^{-M} \).

Since for each fixed index \( i \) there are only \( 3^n - 1 \) indexes \( j \) such that \( \text{dist}(I_i, I_j) = 0, I_i \neq I_j \), the corresponding sum can be bounded by

\[
\sum_{i,j : \|i-j\|_\infty = 1} T_{i,j} \lesssim \epsilon \sum_{i,j : \|i-j\|_\infty = 1} |\langle f \rangle_{I_i}||\langle g \rangle_{I_j}||I_j| \\
\lesssim \epsilon \left( \sum_{i \in \mathbb{Z}_M^d} |\langle f \rangle_{I_i}|^2 |I_i| \right)^{\frac{1}{2}} \left( \sum_{j \in \mathbb{Z}_M^d} |\langle g \rangle_{I_j}|^2 |I_j| \right)^{\frac{1}{2}} \leq \epsilon 
\]

Finally, when \( I_i = I_j \), we have similarly as before:

\[
T_{i,i} = \frac{|\langle f \rangle_{I_i}||\langle g \rangle_{I_i}|}{|\langle b_1 \rangle_{I_i}|^2 |\langle b_2 \rangle_{I_i}|} |\langle T(\chi_{I_i}, b_1 \chi_{I_i}), ((b_2)_i, - b_2) \chi_{I_i} \rangle| \\
\]

By weak compactness, \( |\langle T(\chi_{I_i}, b_2 \chi_{I_i}) \rangle| \lesssim |I_i| |b_1|_{I_{i,q_1}} |b_2|_{I_{i,q_2}} F_W(i, i) \) and \( |\langle T(\chi_{I_i}, \chi_{I_i}) \rangle| \lesssim |I_i| |b_1|_{I_{i,q_1}} F_W(i, i) \), where now we write \( F_W(i, i) = \)
\[ F_W(I_i; M) + \epsilon. \] Then, by (12) and \( \ell(I_i) = 2^{-M} \), we get
\[
\sum_{i \in \mathbb{Z}_M^n} T_{i,i} \lesssim \sum_{i \in \mathbb{Z}_M^n} \|f\|_I \|g\|_I \left| \frac{[b_1]_{I \cup I}}{|(b_1)_I|} \frac{[b_2]_{I \cup I}}{|(b_2)_I|} F_W(i, i) \right|
\lesssim \sum_{i \in \mathbb{Z}_M^n} \|f\|_I \|g\|_I |B_F(I_i, I_i) \lesssim \epsilon
\]

**Corollary 9.15.** With the same hypotheses of Lemma 9.14, let \( P_{1,M} \), \( P_{2,M} \) be the projections related to each system. Then
\[
\langle (P_{2,M}^*)^\perp T \perp f, g \rangle = \sum_{I,J \in \mathcal{D}_M^c} \langle f, \tilde{\psi}_I^{b_1} \rangle \langle g, \tilde{\psi}_J^{b_2} \rangle \langle T \psi_I^{b_1}, \psi_J^{b_2} \rangle.
\]

The dual representation for \( T_b \) also holds.

**Proof.** We have that
\[
\langle (P_{2,M}^*)^\perp T \perp f, g \rangle = \langle T \perp f, P_{2,M}^* g \rangle
= \langle T f, g \rangle - \langle T f, P_{2,M} g \rangle - \langle T \perp f, g \rangle + \langle T \perp f, P_{2,M} g \rangle
\]
and by (10) the last expression coincides with the statement.

10. \( L^p \) Compactness

We start this section with the following technical result:

**Lemma 10.1.** Let \( \tilde{L}, S, \tilde{D} \) be the functions of Definition 2.2 and let \( F(I, J) = \tilde{F}_K(I \wedge J, (I, J)) + F_W(I; M_{T,v}) \chi_{I=J} \). Given \( \epsilon > 0 \), let \( M > 0 \) be large enough so that \( \tilde{L}(2^M) + S(2^{-M}) + \tilde{D}(M^{4/3}) < \epsilon \).

Then for all \( I \in \mathcal{D}_M^c \) and \( J \in \mathcal{D}_M^c \) we have that: either \( F(I, J) < \epsilon \), \( |\log(ec(I, J))| \gtrsim \log M \), or \( \text{rdist}(I, J) \gtrsim M^{4/3} \).

**Remark 10.2.** As showed in the proof, \( F(I, J) < \epsilon \) holds when either \( \ell(I \wedge J) > 2^M \), or \( \ell(I \wedge J) < 2^{-M} \), or \( \text{rdist}(I, J, \mathbb{B}) > M^{1/8} \). For this reason, in Definition 2.10 we denote by \( \mathcal{F}_M \) the family of ordered pairs \((I, J)\) with \( I, J \in \mathcal{D}_M^c \) satisfying some of these three inequalities.

**Proof.** We start with \( F_W(I; M_{T,v}) \chi_{I=J} \), for which the proof is simpler. Since \( I \vee J = I \wedge J = (I, J) = I = J \in \mathcal{D}_M^c \), we study three cases:

a) When \( \ell(I) < 2^{-M} \), we have \( F_W(I; M_{T,v}) \lesssim S(\ell(I)) \leq S(2^{-M}) < \epsilon \).

b) When \( \ell(I) > 2^M \), we get \( F_W(I; M_{T,v}) \lesssim \tilde{L}(\ell(I)) \leq \tilde{L}(2^M) < \epsilon \).

c) When \( 2^{-M} \leq \ell(I) \leq 2^M \) with \( \text{rdist}(I, \mathbb{B}_{2^M}) > 2M \), we finally get \( F_W(I; M_{T,v}) \lesssim \tilde{D}(\text{rdist}(I, \mathbb{B}_{2^M})) \leq \tilde{D}(2^M) < \epsilon \).

We continue with \( \tilde{F}_K \). Since \( I \in \mathcal{D}_M^c \), we consider three cases:
a) When $\ell(I) < 2^{-M}$, we have $\ell(I \wedge J) < 2^{-M}$ and so, we get
\[ F_K(I \wedge J, I \wedge J, (I, J)) \lesssim S(\ell(I \wedge J)) \lesssim S(2^{-M}) < \epsilon. \]

b) When $\ell(I) > 2^{2M}$, since $J \in \mathcal{D}_M^\ell$, we distinguish two cases:

b.1) When $\ell(J) > 2^M$, we get $\ell(I \wedge J) > 2^M$ and so, we obtain
\[ F_K(I \wedge J, I \wedge J, (I, J)) \lesssim L(\ell(I \wedge J)) < L(2^M) < \epsilon. \]

b.2) When $\ell(J) \leq 2^M$, we have that
\[ \text{ec}(I, J) = \frac{\ell(I \wedge J)}{\ell(I \vee J)} = \frac{\ell(J)}{\ell(I)} < \frac{2^M}{2^{2M}} = 2^{-M}. \]

c) When $2^{-2M} \leq \ell(I) \leq 2^{2M}$ with $\text{rdist}(I, \mathbb{B}_{2^{2M}}) > 2M$, we have $|c(I)|_\infty > (2M - 1)2^{2M}$. We fix $\alpha = \frac{1}{8}$, $\beta = \frac{1}{4}$. Then,

c.1) When $\ell(J) > (2M)^\alpha 2^{2M}$, since $\alpha > 0$ we have
\[ \text{ec}(I, J) = \frac{\ell(I)}{\ell(J)} < \frac{2^M}{(2M)^\alpha 2^{2M}} \lesssim M^{-\frac{1}{\alpha}}. \]

c.2) When $\ell(J) \leq (2M)^\alpha 2^{2M}$, we have $\ell(I \vee J) < (2M)^\alpha 2^{2M}$. Now:

c.2.1) When $\text{rdist}((I, J), \mathbb{B}) > (2M)^\beta$, we obtain
\[ F_K(I \wedge J, I \wedge J, (I, J)) \lesssim \tilde{D}(\text{rdist}((I, J), \mathbb{B})) < \tilde{D}(M^\frac{\beta}{2}) < \epsilon. \]

c.2.2) When $\text{rdist}((I, J), \mathbb{B}_{2^{(2M)^\beta}}) \leq (2M)^\beta$, we get $|c((I, J))|_\infty \leq (2M)^\beta (1 + \ell((I, J)))$. Then, we examine the last two cases:

- When $\ell((I, J)) > (2M)^\gamma 2^{2M}$, we get
\[ \text{rdist}(I, J) = \frac{\ell((I, J))}{\ell(I \vee J)} > \frac{(2M)^\gamma 2^{2M}}{(2M)^\alpha 2^{2M}} \gtrsim M^{\gamma - \alpha} = M^\frac{\gamma}{2}. \]

- When $\ell((I, J)) \leq (2M)^\gamma 2^{2M}$, we have instead
\[ |c(I) - c(J)|_\infty > |c(I)|_\infty - |c((I, J)) - c(J)|_\infty - |c((I, J))|_\infty \geq |c(I)|_\infty - 2^{-1} \ell((I, J)) - (2M)^\beta (1 + \ell((I, J))) \geq (2M - 1)2^{2M} - (2M)^\gamma 2^{2M} - (2M)^\beta (1 + (2M)^\gamma 2^{2M}) \gtrsim (M^{-M^\gamma - M^\beta} - M^{\beta + \gamma})2^{2M} \gtrsim (M - 3M^\frac{\gamma}{2})2^{2M} \geq 2^{-1}M2^{2M} \]
for $M \geq 36$. Whence,
\[ \text{rdist}(I, J) \geq \frac{|c(I) - c(J)|_\infty}{\ell(I \vee J)} \gtrsim \frac{M2^{2M}}{(2M)^\alpha 2^{2M}} \gtrsim M^{1 - \alpha} = M^\frac{\gamma}{2}. \]

\[ \square \]

We now demonstrate our main result on compactness of singular integral operators when the special cancellation conditions hold.
Theorem 10.3. Let $T$ be a linear operator bounded on $L^2(\mathbb{R}^n)$ with a compact Calderón-Zygmund kernel. Let $b_1, b_2$ be locally integrable functions compatible with $T$. We assume that $T$ satisfies the weak compactness condition and $Tb_1 = T^*b_2 = 0$.

Then $T$ can be extended to a compact operator on $L^2(\mathbb{R}^n)$.

Proof. Let $(\psi_I^{b_i})_{i \in \mathcal{D}}$ be the Haar wavelet systems of Definition 9.2 and $P_{i,M}$ be the projections associated with each system. By the comments after Remark 3.4, to prove compactness of $T$ on $L^2(\mathbb{R}^n)$ it is enough to show that $\langle (P_{2,M}^*)^\perp TP_{1,M}^\perp f, g \rangle$ tends to zero uniformly for $f, g$ in the unit ball of $L^2(\mathbb{R}^n)$. From now we write both projections as $P_{i,M}^\perp, (P_{i,M}^*)^\perp$.

By Lemma 10.1, given $\epsilon > 0$, there exists $M_0 \in \mathbb{N}$ so that $F(I, J) < \epsilon$, $|\log(\epsilon c(I, J))| \geq \log M$, or $\text{rdist}(I, J) \geq M^{\frac{4}{5}}$.

Let now $BF : \mathcal{D} \times \mathcal{D} \to [0, \infty]$ be as given in Definition 3.10. By Lemma 9.9 there exists $M_1 \in \mathbb{N}, M_1 > M_0$ so that $M_1 > 3^8$ and

$$\sum_{I \in \mathcal{D}} BF(I, J) |\langle f, \tilde{\psi}_I^{b_1} \rangle|^2 \lesssim \|f\|_{L^2(\mathbb{R}^n)}$$

for all $J \in \mathcal{D}$ and

$$\sum_{I \in \mathcal{D}_M} \sup_{J \in \mathcal{D}_M} BF(I, J) |\langle f, \tilde{\psi}_I^{b_1} \rangle|^2 \lesssim \epsilon \|f\|_{L^2(\mathbb{R}^n)}$$

for all $M > M_1$. Similarly for $b_2$ and $g$.

Now, for fixed $\epsilon > 0$ and the chosen $M_1 \in \mathbb{N}$, we prove that for $M > M_1$ such that $2M^{-\frac{\delta}{8}} + M^{-\frac{\delta}{4+2\delta}} < \epsilon$, we have

$$|\langle (P_{M}^*)^\perp TP_{2M}^\perp f, g \rangle| \lesssim \epsilon.$$

By Corollary 9.15

$$\langle (P_{M}^*)^\perp TP_{2M}^\perp f, g \rangle = \sum_{I \in \mathcal{D}_{2M}} \sum_{J \in \mathcal{D}_M} \langle f, \tilde{\psi}_I^{b_1} \rangle \langle g, \tilde{\psi}_J^{b_2} \rangle \langle T\psi_I^{b_1} \rangle \langle \psi_J^{b_2} \rangle.$$

According to Proposition 8.2, we parametrize the sums by eccentricity, relative distance and inner relative distance of the cubes $I, J$ as follows. For fixed $e \in \mathbb{Z}, m \in \mathbb{N}$ and $J \in \mathcal{D}$, we define the family

$$J_{e,m} = \{I \in \mathcal{D} : \ell(I) = 2^e \ell(J), m \leq \text{rdist}(I, J) < m + 1\}.$$

When $m \leq 3$, we define for every $1 \leq k \leq 2^{\min(e,0)},$

$$J_{e,m,k} = J_{e,m} \cap \{I \in \mathcal{D} : k \leq \text{irdist}(I, J) < k + 1\}.$$

The cardinality of $J_{e,m}$ is comparable to $2^{-\min(e,0)n}n(2m)^{n-1}$. On the other hand, when $m \leq 3$, the cardinality of $J_{e,m,k}$ is comparable to $n(2^{-\min(e,0)} - k)^{n-1}$. Moreover, by symmetry, the family $\{(I, J) :$
The cardinality of $J_{e,m}$ is comparable to $2^{-\min(e,0)n}m^{-1}$ and so, the cardinality of $I_{-e,m}$ is comparable to $2^{\max(e,0)n}m^{-1}$. Then we bound the previous expression by a constant comparable to $2^n$ multiplied by
\[
\sum_{e \in \mathbb{Z}} \sum_{m \geq 4} 2^{-|e| \left( \frac{n}{2} + \delta \right)} \left( 2^{\max(e,0)n} m^{n+1} - \sum_{I \in \mathcal{D}_M^e} \sup_{J \in \mathcal{D}_M^e} BF(I, J) \left| \langle f, \tilde{\psi}_I^b \rangle \right|^2 \right) \frac{1}{2}
\]

\[
\left( 2^{-\min(e,0)n} m^{n+1} - \sum_{I \in \mathcal{D}_M^e} \sup_{J \in \mathcal{D}_M^e} BF(I, J) \left| \langle g, \tilde{\psi}_J^b \rangle \right|^2 \right) \frac{1}{2}
\]

\[
\lesssim \epsilon \left( \sum_{e \in \mathbb{Z}} 2^{-|e| \delta} \sum_{m \geq 4} m^{-(1+\delta)} \right) \| f \|_{L^2(\mathbb{R}^n)} \| g \|_{L^2(\mathbb{R}^n)} \lesssim \epsilon
\]

where in the first inequality we used (44) and \(2^{\max(e,0)}2^{\min(e,0)} = 2|e|\).

1.2) When \(m \leq 3\), we assume \(\ell(J) \leq \ell(I)\) to simplify notation. This choice implies that \(\epsilon \geq 0\). Then, as before, we bound the terms in (46) corresponding to this case by a constant times the following quantity:

\[
\sum_{e \geq 0} 2^{-e^2/2} \left( \sum_{J \in \mathcal{D}_M^e} \sup_{(I, J) \in \mathcal{F}_M} BF(I, J) \left| \langle f, \tilde{\psi}_I^b \rangle \right|^2 \right) \sum_{k=1}^{2^{-\min(-e,0)}} \sum_{J \in \mathcal{J}_{-e,m,k}} k^{-2\delta} \right) \frac{1}{2}
\]

\[
\left( \sum_{J \in \mathcal{D}_M^e} \sum_{k=1}^{2^{-\min(e,0)}} \sum_{I \in \mathcal{J}_{e,m,k}} \sup_{J \in \mathcal{D}_M^e} BF(I, J) \left| \langle g, \tilde{\psi}_J^b \rangle \right|^2 \right) \frac{1}{2}
\]

(47) \[
\lesssim \epsilon \sum_{e \geq 0} 2^{-e^2/2} \| f \|_{L^2(\mathbb{R}^n)} \left( 2^e(n-1) \sum_{k=1}^{2^e} k^{-2\delta} \right) \frac{1}{2} \| g \|_{L^2(\mathbb{R}^n)},
\]

using (44) and the facts that the cardinality of \(I_{-e,m,k}\) is comparable to \(n(2^e-k)^{(n-1)}\) while the cardinality of \(J_{e,m,k}\) is comparable to \(n\).

Let \(0 < \theta < 1\) to be chosen later. Using \(k \geq 1\) and \(\delta > 0\), we have

\[
2^e \sum_{k=1}^{2^e} k^{-2\delta} = 2^{e\theta} \sum_{k=1}^{2^e} k^{-2\delta} + \sum_{k=2^{e\theta}+1}^{2^e} k^{-2\delta} \lesssim 2^{e\theta} + 2^{-2e\theta} 2^e.
\]

Then, expression (47) is bounded by a constant multiplied by

\[
\epsilon \sum_{e \geq 0} 2^{-e^2/2} \left( 2^{e\frac{n-1+\theta}{2}} + 2^{e\frac{n-2\theta}{2}} \right) \lesssim \epsilon \left( \sum_{e \geq 0} 2^{-e^2/2} + \sum_{e \geq 0} 2^{-e\theta\delta} \right) \lesssim \epsilon,
\]

since \(0 < \theta < 1\). This finishes the first case.

2) We now study the case when \(I \in \mathcal{D}_M^e, J \in \mathcal{D}_M^e\) are such that \(F(I, J) \geq \epsilon\). By Lemma [10.1] we have that \(|\log(ec(I, J))| \geq \log M\), or \(\text{rdist}(I, J) \geq M^{\frac{\theta}{2}}\). Therefore, instead the smallness of \(BF\), in this case
we use that the size and location of the cubes $I$ and $J$ are such that either their eccentricity or their relative distance are extreme.

We fix $e_M \in \{0, \log M\}$, $m_M \in \{M^{\frac{1}{2}}, 1\}$ such that $e_M = 0$ implies $m_M = M^{\frac{1}{2}}$. When $m > 3$, using (43) and the calculations developed in the case 1.1), we bound the relevant part of (46) by a constant times

$$\sum_{|e| \geq e_M} \sum_{m \geq m_M} \frac{2^{-|e|}e^{m+\delta}}{m^{n+\delta}} \sum_{J \in \mathcal{D}_{2M}^M} \sum_{I \in J, m, \cap \mathcal{D}_{2M}^M} |\langle f, \tilde{\psi}_I^{b_1} \rangle| |\langle g, \tilde{\psi}_J^{b_2} \rangle| BF(I, J)$$

$$\lesssim \left( \sum_{|e| \geq e_M} \sum_{m \geq m_M} 2^{-|e|}m^{-(1+\delta)} \right) \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$$

$$\lesssim 2^{-e_M \delta} m^{-\delta}_M \lesssim (M^{-\delta} + M^{-\frac{\delta}{2}}) < \epsilon,$$

by the choice of $M$.

When $m \leq 3$, we have that $m_M \leq m \leq 3 < M^{\frac{1}{2}}$ implies $m_M = 1$ and so, $e_M = \log M$. Then the calculations of case 1.2) show that, using (43), the relevant part of (46) can be bounded by a constant times

$$\sum_{e \geq e_M} \sum_{m=1}^{3} \sum_{k=1}^{2} 2^{-e_M} \sum_{J \in \mathcal{D}_{2M}^M} \sum_{I \in J, m, \cap \mathcal{D}_{2M}^M} |\langle f, \tilde{\psi}_I^{b_1} \rangle| |\langle g, \tilde{\psi}_J^{b_2} \rangle| BF(I, J)$$

$$\lesssim \sum_{e \geq e_M} \left( 2^{-e_M} + 2^{-e_M \delta} \right) \|f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}$$

$$\lesssim \sum_{e \geq \log M} 2^{-e\min(\theta^2, 1-\delta)} \lesssim M^{-\frac{\delta}{1+2\theta}} \leq \epsilon,$$

by the choice of $M$ and $\theta = \frac{1}{2+2\theta} \in (0, 1)$. This completely finishes the proof of compactness on $L^2(\mathbb{R}^n)$.

**Corollary 10.4.** With the hypotheses of Theorem 10.3 plus the extra condition $b_1, b_2 \in L^\infty(\mathbb{R}^n)$, we obtain compactness of $T_b$ on $L^2(\mathbb{R}^n)$.

**Proof.** To prove compactness of $T_b$, we use the dual representation of Corollary 9.15 (or, equivalently, equality (11)),

$$\langle (P_M^{*})^{-1} T_b P_{2M}^{1} f, g \rangle = \sum_{I \in \mathcal{D}_{2M}^M} \sum_{J \in \mathcal{D}_{2M}^M} \langle f, \psi_I^{b_1} \rangle \langle g, \psi_J^{b_2} \rangle \langle T_b \psi_I^{b_1}, \psi_J^{b_2} \rangle$$

$$= \sum_{I \in \mathcal{D}_{2M}^M} \sum_{J \in \mathcal{D}_{2M}^M} \langle f, \psi_I^{b_1} \rangle \langle g, \psi_J^{b_2} \rangle \langle b_1 \rangle_I \langle b_2 \rangle_J \langle T_b b_1^{b_1}, h_2^{b_2} \rangle.$$ 

By Lemma 9.11 we have

$$\sum_{I \in \mathcal{D}} \left( \frac{[b_I]}{\langle (b_1) I \rangle \langle (b_1) I_p \rangle} \right)^{-2} |\langle f, \psi_I^{b_1} \rangle |^2 \lesssim \|f b\|_{L^2(\mathbb{R}^n)} \leq \|b\|_{L^\infty(\mathbb{R}^n)} \|f\|_{L^2(\mathbb{R}^n)}$$
and the same for $g$ and $b_2$. This implies the following two inequalities, which are similar to (13) and (14):

$$\left( \sum_{I \in \mathcal{D}} BF(I, J) \langle f, \psi_I^{b_1} \rangle^2 |[b_1]|^2 \right)^{1/2} \lesssim \|f\|_{L^2(\mathbb{R}^n)},$$

for $J \in \mathcal{D}$; and given $\epsilon > 0$, there exists $M_0 \in \mathbb{N}$ with

$$\left( \sum_{I \in \mathcal{D}_M} \sup_{J \in \mathcal{D}_M} BF(I, J) \langle f, \psi_I^{b_1} \rangle^2 |[b_1]|^2 \right)^{1/2} \lesssim \epsilon \|f\|_{L^2(\mathbb{R}^n)}$$

for all $M > M_0$ and $f \in C_0(\mathbb{R}^n)$. We have analog inequalities for $b_2, g$. From here we can proceed as in the proof of Theorem 10.3.

As in [14], we deduce compactness on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$ by interpolation between compactness on $L^2(\mathbb{R}^n)$ and boundedness $L^p(\mathbb{R}^n)$. We refer to the classical Krasnosel’skii’s Theorem, whose proof in a more general setting can be found in [11].

**Theorem 10.5.** Let $1 \leq p_1, r_1, p_2, r_2 \leq \infty$ be a set of indices with $r_1 < \infty$. Let $T$ be a given linear operator which is continuous simultaneously as a mapping from $L^{p_1}(\mathbb{R}^n)$ to $L^{r_1}(\mathbb{R}^n)$ and from $L^{p_2}(\mathbb{R}^n)$ to $L^{r_2}(\mathbb{R}^n)$. Assume in addition that $T$ is compact as a mapping from $L^{p_1}(\mathbb{R}^n)$ to $L^{r_1}(\mathbb{R}^n)$. Then $T$ is compact as a mapping from $L^p(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$, where $1/p = t/p_1 + (1 - t)/p_2$, $1/r = t/r_1 + (1 - t)/r_2$, $0 < t < 1$.

### 11. Compact Paraproducts

For general $Tb_1, T^*b_2 \in CMO_b(\mathbb{R}^n)$, we construct paraproducts $\Pi_{Tb_1}, \Pi_{T^*b_2}$ with compact C-Z kernels such that $\Pi_{Tb_1}(b_1) = Tb_1$, $\Pi_{T^*b_2}(b_1) = 0$ while $\Pi_{Tb_1}(b_2) = 0$, $\Pi_{T^*b_2}(b_2) = T^*b_2$. This way, the operator

$$\tilde{T} = T - \Pi_{Tb_1} - \Pi_{T^*b_2}$$

satisfies the hypotheses of Theorem 10.3 and so, $\tilde{T}$ is compact on $L^p(\mathbb{R}^n)$. Since the paraproducts $\Pi_{Tb_1}$ and $\Pi_{T^*b_2}$ are compact by construction, we deduce that the operator $T$ is also compact on $L^p(\mathbb{R}^n)$.

We start with two technical lemmata. The first one describes the $BMO_b(\mathbb{R}^n) - H^1_b(\mathbb{R}^n)$ duality. Since this result is well known for bounded accretive functions $b$, we just sketch its proof to show the validity of the result. Some considerations regarding the use of finite decompositions should be added to obtain a rigorous demonstration (see [11]). However, since we only use the estimates starting at the right hand side of (18), the calculations in the paper are not affected by these issues.
Lemma 11.1. Let \( b \) be a locally integrable function with non-zero dyadic averages. Then for all \( f \in \text{BMO}_b(\mathbb{R}^n) \), \( g \in H^1_b(\mathbb{R}^n) \)

\[
\left| \int_{\mathbb{R}^n} f(x)g(x)b(x)dx \right| \leq \|f\|_{\text{BMO}_b(\mathbb{R}^n)} \|g\|_{H^1_b(\mathbb{R}^n)}
\]

Proof. We assume that \( g \in C_0(\mathbb{R}^n) \) with support in \( Q \in \mathcal{D} \). By Definition 3.8 of \( H^1_b(\mathbb{R}^n) \), there exists a decomposition \( g = \sum_{I \in \mathcal{D}} \lambda_I a_I \) with \( a_I \) \( L^2 \)-atoms supported on \( I_p \in \mathcal{D}(Q) \) and \( \|a_I b\|_{L^2(I)} \lesssim B^b_{I,2} |I|^{-\frac{1}{2}} \), such that \( \sum_{I \in \mathcal{D}} B^b_{I,2} |\lambda_I| \leq 2 \|g\|_{H^1_b(\mathbb{R}^n)} \). By Lemma 9.12 we have that

\[
a_I b = \sum_{J \in \mathcal{D}(I)} \langle a_I b, \tilde{\psi}^b_J \rangle \psi^b_J
\]

with a.e. convergence and \( J_p \subseteq I_p \) since otherwise \( \langle a_I, \tilde{\psi}^b_J \rangle = 0 \). This is trivial when \( I_p \cap J_p = \emptyset \). When \( I_p \subseteq J_p \), is due to \( a_I b \) having mean zero and \( \tilde{\psi}^b_J \) being constant on the support of \( a_I b \). Moreover,

\[
\left\| \sum_{I \in \mathcal{D}} \lambda_I a_I b \right\|_{L^1(Q)} \leq \sum_{I \in \mathcal{D}} |\lambda_I| \|a_I b\|_{L^2(I)} |I|^\frac{1}{2} \leq \sum_{I \in \mathcal{D}} |\lambda_I| B^b_{I,2} \lesssim \|g\|_{H^1_b(\mathbb{R}^n)}
\]

and so, by Vitali’s Dominated Convergence Theorem,

\[
\int f(x)g(x)b(x)dx = \sum_{I \in \mathcal{D}} \lambda_I \sum_{J \in \mathcal{D}(I)} \langle f, \psi^b_J \rangle \langle a_I b, \tilde{\psi}^b_J \rangle.
\]

Then, by Cauchy-Schwarz inequality,

\[
\left| \int f(x)g(x)b(x)dx \right| \leq \sum_{I \in \mathcal{D}} |\lambda_I| \left( \sum_{J \in \mathcal{D}(I)} \left( \frac{[b]_{I_p,2}}{|\langle b \rangle_{I_p}|} \right)^2 |\langle f, \psi^b_J \rangle|^2 \right)^{\frac{1}{2}} \left( \sum_{J \in \mathcal{D}(I)} \left( \frac{[b]_{I_p,2}}{|\langle b \rangle_{I_p}|} \right)^2 |\langle a_I b, \tilde{\psi}^b_J \rangle|^2 \right)^{\frac{1}{2}}
\]

By Remark 9.10 the last factor is bounded by \( \|a_I b\|_{L^2(I)} \lesssim B^b_{I,2} |I|^{-\frac{1}{2}} \). Then, by Definitions 3.6 and 3.8

\[
\left| \int f(x)g(x)dx \right| \leq \sum_{I \in \mathcal{D}} |\lambda_I| B^b_{I,2} \left( \frac{1}{|I|} \sum_{J \in \mathcal{D}(I)} \left( \frac{[b]_{I_p,2}}{|\langle b \rangle_{I_p}|} \right)^2 |\langle f, \psi^b_J \rangle|^2 \right)^{\frac{1}{2}} \leq \sum_{I \in \mathcal{D}} |\lambda_I| B^b_{I,2} \|f\|_{\text{BMO}_b(\mathbb{R}^n)} \|g\|_{H^1_b(\mathbb{R}^n)}.
\]

Although the wavelet system \( (\psi_I)_{I \in \mathcal{D}} \) is not orthogonal, we have the following lemma, which is a direct consequence of (33), (34) and (32).

Lemma 11.2. Let \( \langle \alpha_I \rangle_{I \in \mathcal{D}} \) a sequence of complex numbers and let \( f = \sum_{J \in \mathcal{D}} \alpha_I \psi^b_J \). Then \( \langle f, \tilde{\psi}^b_J \rangle = \alpha_I \).
Now we state and prove the main result of this section.

**Proposition 11.3.** Let $b_1, b_2$ be locally integrable functions with non-zero dyadic averages and let $(\psi^b_I)_I \in D$ be the Haar wavelet system of Definition 9.2. We assume $Tb_1 \in \text{CMO}_b(\mathbb{R}^n)$. Then the operator

$$\sum_{I \in D} \langle Tb_1, \psi^b_I \rangle \frac{\langle f \rangle_I}{\langle b \rangle_I} \langle g, \bar{\psi}^b_I \rangle$$

has a compact Calderón-Zygmund kernel, it is compact on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$, and satisfies $\langle \Pi_{Tb_1} b_1, g \rangle = \langle Tb_1, g \rangle$ and $\langle \Pi_{Tb_1} b_2, g \rangle = 0$.

**Remark 11.4.** The proof shows that $\Pi_{Tb_1}$ is a perfect Calderón-Zygmund operator (see 2 for the definition). Moreover, writing $E_k^{b_1}(f) = \frac{\langle f \rangle_b}{\langle b \rangle_b} \chi_1$, we have $\Pi_{Tb_1} f = \sum_{k \in \mathbb{Z}} \Delta^{b_1}_k(Tb_1) E_k^{b_1}(f)$.

On the other hand, since $\langle Tb_1, \psi^b_I \rangle = \langle T b_1, h^b_I \rangle$ and $\bar{\psi}^b_I = \langle b \rangle_I h^b_I$, we have

$$\langle \Pi_{Tb_1} f, b \rangle = \sum_{I \in D} \langle Tb_1, \bar{\psi}^b_I \rangle \frac{\langle f \rangle_I}{\langle b \rangle_I} \langle g, \bar{\psi}^b_I \rangle =: \langle \Pi_{Tb_1} f, g \rangle.$$

This is the paraproduct needed to prove compactness of $Tb$. Moreover, $\Pi_{Tb_1} f = \sum_{k \in \mathbb{Z}} (\Delta^{b_1}_k)^*(Tb_1) E_k^{b_1}(f)$.

**Proof.** Formally, $\langle \Pi_{Tb_1} b_1, g \rangle = \langle Tb_1, \sum_{I \in D} \langle g, \bar{\psi}^b_I \rangle \psi^b_I \rangle = \langle Tb_1, g \rangle$. Moreover, by Lemma 9.6 we get $\langle b_2, \bar{\psi}^b_I \rangle = 0$ and so, $\langle \Pi_{Tb_1} f, b_2 \rangle = 0$.

To prove that $\Pi_{Tb_1}$ is compact on $L^2(\mathbb{R}^n)$, we verify that $(\langle P^\perp_{Tb_1} f, g \rangle, f, g \in L^2(\mathbb{R}^n))$ tends to zero uniformly for all $f, g$ in the unit ball of $L^2(\mathbb{R}^n)$, with $P_M$ the projection operator associated with $(\psi_I^b)_I$. We start by proving the equality $\langle P^\perp_{Tb_1} f, g \rangle = \langle \Pi_{P^\perp_{Tb_1} f}, g \rangle$.

Since $g \in L^2(\mathbb{R}^n)$, by Lemma 9.12 we have $P^\perp_M g = \sum_{J \in D_M^c} \langle g, \psi^b_J \rangle \psi^b_J$ with a.e. pointwise convergence. Moreover, by the orthogonality properties of Lemma 9.3, $\langle P^\perp_M g, \psi^b_I \rangle = \sum_{J \in \text{ch}(\mu)} \langle g, \psi^b_J \rangle \langle \psi^b_J, \psi^b_I \rangle$. Then

$$\langle P^\perp_{Tb_1} f, g \rangle = \sum_{I \in D} \langle Tb_1, \psi^b_I \rangle \frac{\langle f \rangle_I}{\langle b \rangle_I} \langle \psi^b_I, \psi^b_J \rangle \langle g, \bar{\psi}^b_I \rangle = \sum_{I \in D_M^c} \sum_{J \in \text{ch}(\mu)} \langle Tb_1, \psi^b_I \rangle \frac{\langle f \rangle_I}{\langle b \rangle_I} \langle g, \bar{\psi}^b_I \rangle \langle \psi^b_J, \bar{\psi}^b_I \rangle.$$

On the other hand, since $Tb_1 \in \text{CMO}_b(\mathbb{R}^n)$, we have $P^\perp_{Tb_1} = \sum_{I \in D_M^c} \langle Tb_1, \psi^b_I \rangle \bar{\psi}^b_I$ with a.e. pointwise convergence. By Lemma 9.6.
\[ \langle P_M^* Tb_1, \psi_{I,2}^{b_2} \rangle = \sum_{J \in \text{ch}(I_p)} \langle Tb_1, \psi_{I,2}^{b_2} \rangle \langle \tilde{\psi}_{I,2}^{b_2} \rangle. \] Then, since \( J_p = I_p \),

\[ \langle \Pi_{Tb_1} f, g \rangle = \sum_{I \in D} \langle P_M^* Tb_1, \psi_{I,2}^{b_2} \rangle \frac{\langle f \rangle_{I_p}}{\langle b_1 \rangle_{I_p}} \langle g, \tilde{\psi}_{I,2}^{b_2} \rangle 
\]

\[ = \sum_{J \in \text{ch}(I_p)} \sum_{I \in D} \langle Tb_1, \psi_{I,2}^{b_2} \rangle \langle \tilde{\psi}_{I,2}^{b_2} \rangle \frac{\langle f \rangle_{I_p}}{\langle b_1 \rangle_{I_p}} \langle g, \tilde{\psi}_{I,2}^{b_2} \rangle. \]

Symmetry of \( I, J \) in previous expressions proves the claimed equality.

Now, by Carleson’s Theorem (in particular (36) and remark 3.10)

\[ |\langle P_M^* \Pi_{Tb_1} f, g \rangle| = |\langle \Pi_{Tb_1} f, g \rangle| = \left| \sum_{I \in D} \langle P_M^* Tb_1, \psi_{I,2}^{b_2} \rangle \frac{\langle f \rangle_{I_p}}{\langle b_1 \rangle_{I_p}} \langle g, \tilde{\psi}_{I,2}^{b_2} \rangle \right| \]

\[ \lesssim \left( \sum_{I \in D} \left( \frac{[b_2]_{I_p,2}}{\langle b_1 \rangle_{I_p} \langle b_2 \rangle_{I_p}} \right)^2 \right)^{1/2} \left( \sum_{I \in D} \left( \frac{[b_2]_{I_p,2}}{\langle b_2 \rangle_{I_p}} \right)^{-2} \langle g, \tilde{\psi}_{I,2}^{b_2} \rangle^2 \right)^{1/2} \]

\[ \lesssim \sup_{I \in \mathcal{D}} \left( \frac{1}{|I|} \sum_{J \in \text{ch}(I_p)} \frac{[b_2]_{I_p,2}}{|\langle b_1 \rangle_{I_p} \langle b_2 \rangle_{I_p}|} \right)^2 \left( \langle P_M^* Tb_1, \psi_{I,2}^{b_2} \rangle^2 \right)^{1/2} \|f\|_2 \|g\|_2 \]

\[ \leq \|P_M^* Tb_1\|_{\text{BMO}_b(\mathbb{R}^n)}, \]

which tends to zero when \( M \) tends to infinity since \( Tb_1 \in \text{CMO}_b(\mathbb{R}^n) \).

To end the proof, we show that \( \Pi_{Tb_1} \) has a a compact Calderón-Zygmund kernel, namely, that the integral representation of Definition 2.4 holds. For \( f, g \) with disjoint support,

\[ \langle \Pi_{Tb_1} f, g \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} f(t) g(x) \sum_{I \in D} \langle Tb_1, \psi_{I,2}^{b_2} \rangle \frac{1}{\langle b_1 \rangle_{I_p}} \frac{\chi_{I_p}(t)}{|I_p|} \tilde{\psi}_{I,2}^{b_2}(x) dt dx. \]

As we will see later, the disjointness of the supports of \( f \) and \( g \) guarantees the convergence of the infinite sum. The kernel of \( \Pi_{Tb_1} \) is hence

\[ K(t, x) = \left( \frac{1}{\langle b_1 \rangle_{I_p}} \frac{\chi_{I_p}(t)}{|I_p|} \tilde{\psi}_{I,2}^{b_2}(x) \psi_{I,2}^{b_2} \right). \]

Due to the singularities of \( \chi_{I_p} \) and \( \tilde{\psi}_{I,2}^{b_2} \), this kernel does not satisfy Definition 2.3 of a compact Calderón-Zygmund kernel. However, a careful read of the proofs presented shows that all results hold if the kernel satisfies the following alternative inequality: given \( I, J \in \mathcal{D} \),

\[ |K(t, x) - K(t, x')| \lesssim \frac{\ell(J)^6}{|t - x|^{n+\sigma}} L(|t - x|_\infty) S(|t - x|_\infty) D(|t + x|_\infty) \]

for all $t \in I$ and $x, x' \in J$ with $2|x - x'|_\infty < |t - x|_\infty$. We will prove that the kernel of $\Pi_{b_1}$ satisfies this inequality with $\delta = 1$. This is equivalent to saying that $\Pi_{b_1}$ has a perfect Calderón-Zygmund kernel.

In fact, we will prove that $|K(t, x) - K(t, x')|$ can be estimated by $\ell(J)/|t - x|_\infty^{n+1}$ times a bounded function which tends to zero when $|t - x|_\infty \to 0$ or $|x - t|_\infty \to 0$ or $|t + x|_\infty \to \infty$. First of all, we have

$$K(t, x) - K(t, x') = \sum_{I \in \mathcal{D}} \langle Tb_1, \psi_I \rangle \frac{1}{\langle b_1 \rangle_{I_p}} \chi_{I_p}(t) (\tilde{\psi}_I(x) - \tilde{\psi}_I(x')).$$

To simplify notation, we write $\tilde{\psi}_I(x, x') = \tilde{\psi}_I(x) - \tilde{\psi}_I(x').$

We note that $\chi_I(t)\tilde{\psi}_I(x, x') \neq 0$ implies that for all cubes $I$ in the sum we have $t \in I_p$ and either $x \in I$ or $x' \in I$. Let $I_{t,x,x'}$, $I_{t,x}$, $I_{t,x'}$ and $I_{x,x'}$ be the smallest dyadic cubes containing the points in the subindexes. By hypothesis, $|t - x|_\infty \approx |t - x'|_\infty$. And by symmetry, we assume $|t - x|_\infty \leq |t - x'|_\infty$. Then all cubes $I$ in the sum satisfy $I_{t,x} \subset I_p$ and the previous expression can be written as

$$\sum_{I \in \mathcal{D}} \langle Tb_1, \psi_I \rangle \frac{\chi_{I_p}(t)}{\langle b_1 \rangle_{I_p}|I_p|} \tilde{\psi}_I(x, x').$$

Notice that $|t - x|_\infty \leq \ell(I_{t,x}) \leq \ell(I_{t,x,x'})$. We will see later that if $I_{t,x,x'} \subset I_{t,x} = I_{t,x,x'}$ then $K(t, x) - K(t', x) = 0$. That is, $K$ is a perfect Calderón-Zygmund kernel.

Since $Tb_1 \in \operatorname{CMO}_b(\mathbb{R}^n)$, for every $\epsilon > 0$ there is $M_0 \in \mathbb{N}$ such that $\|P^{s_1}Tb_1\|_{\operatorname{BMO}_b(\mathbb{R}^n)} < \epsilon$ and $2^{-M_0/2}(1 + \|Tb_1\|_{\operatorname{BMO}_b(\mathbb{R}^n)}) < \epsilon$ for all $M > M_0$. We are going to prove that

$$|K(t, x) - K(t, x')| \lesssim \epsilon \frac{\ell(I_{t,x,x'})}{|t - x|_\infty^{n+1}},$$

when either $|t - x|_\infty > 2^{M+1}$, or $|x + t|_\infty > M2^{M+2}$, or $|t - x|_\infty < 2^{2M}$.

1) When $|t - x|_\infty > 2^{M+1}$, all cubes $I \in \mathcal{D}$ in the sum satisfy $2\ell(I) = \ell(I_p) \geq \ell(I_{t,x}) \geq |x - t|_\infty > 2^{M+1}$ and so, $I \in \mathcal{D}_M^c$. We can then rewrite (50) as

$$\sum_{I \in \mathcal{D}} \langle P^s Tb_1, \psi_I \rangle \frac{1}{\langle b_1 \rangle_{I_p}|I_p|} \tilde{\psi}_I(x, x').$$

To be used in case 2), we note that this is the only instance when we use the actual inequality $|t - x|_\infty > 2^{M+1}$. From now, we will only
use that \( I \in \mathcal{D}_M \). By Lemma 11.2 we have
\[
\left\langle \sum_{J \in \mathcal{D}} \frac{\chi_{J_p}(t)}{|J_p|} \tilde{\Psi}_I^{b_2}(x, x') \psi_I^{b_2}, \tilde{\Psi}_I^{b_2} \right\rangle = \frac{\chi_{J_p}(t)}{|I_p|} \tilde{\Psi}_I^{b_2}(x, x')
\]
Then, (51), and thus \( K(t, x) - K(t, x') \), can be rewritten as
\[
\sum_{I \in \mathcal{D}} \sum_{J \in \mathcal{D}} \frac{1}{|b_1|_{I_p}} \left\langle P^\perp M T b_1, \psi_I^{b_2} \right\rangle \left\langle \frac{\chi_{J_p}(t)}{|J_p|^{\frac{1}{2}}} \tilde{\Psi}_I^{b_2}(x, x') \right\rangle (J_p) \frac{1}{|b_1|_{I_p}} \left\langle P^\perp M T b_1, \psi_I^{b_2} \right\rangle
\]
where we interchanged \( I, J \) and condition \( J \in \mathcal{D}(I_p) \) is due to (54). By Lemma 9.6, \( \| \psi_I^{b_2} \|_{L^2(\mathbb{R}^n)} \lesssim \| M_{b_2} \| \). Since (52) coincides with the right hand side of (48), by the proof of Lemma 11.1 (52) is bounded by
\[
\sum_{I \in \mathcal{D}} \frac{\chi_{J_p}(t)}{|I_p|^{\frac{1}{2}}} |\tilde{\Psi}_I^{b_2}(x, x')| \sup_{I \in \mathcal{D}} \left( \frac{b_2}{|I_p|} \right)^2 \sum_{J \in \mathcal{D}(I_p)} \frac{1}{|b_1|_{I_p}} \left\langle P^\perp M T b_1, \psi_I^{b_2} \right\rangle \left\langle \frac{1}{|b_1|_{I_p}} \left\langle P^\perp M T b_1, \psi_I^{b_2} \right\rangle \right\rangle
\]
where we used \( |I| \leq |I_p| \). We now work to bound
\[
\sum_{I \in \mathcal{D}} \frac{\chi_{J_p}(t)}{|I_p|^{\frac{1}{2}}} |\tilde{\Psi}_I^{b_2}(x) - \tilde{\Psi}_I^{b_2}(x')|.
\]
If the sum is non-zero then \( I_{t, x} \subseteq I_{x, x'} \): if \( I_{x, x'} \not\subseteq I_{t, x} \), then all cubes \( I \) such that \( I_{t, x} \subseteq I \) satisfy \( x, x' \in I' \) with \( I' \in \mathcal{D}(I_p) \); this implies \( \tilde{\Psi}_I^{b_2}(x) = \tilde{\Psi}_I^{b_2}(x') \) and so, the sum in (53) is zero. Moreover, if \( \tilde{\Psi}_I^{b_2}(x, x') \) is non-zero only then \( x, x' \) do not belong to the same child of \( I_p \). Then
\[
|\tilde{\Psi}_I^{b_2}(x) - \tilde{\Psi}_I^{b_2}(x')| \leq |(b_2)|_{I_p} |I|^{\frac{1}{2}} \frac{1}{|I_{b_2}|_{I_p}} = |I|^{-\frac{1}{2}}.
\]
Now, we parametrize the cubes in the sum by their side length: let \( (I^k)_{k \in \mathbb{N}} \) be the family of dyadic cubes such that \( I_{t, x} \subseteq I^k \) with \( \ell(I^k) = 2^k \ell(I_{t, x}) \). Then, the sum in (53) can be bounded by
\[
\sum_{k \geq 0} \frac{1}{|I^k|} = \sum_{k \geq 0} 2^k \ell(I_{t, x}) \lesssim \frac{\ell(I_{t, x})}{\ell(I_{t, x})} \ell(I_{t, x}) \leq \frac{\ell(I_{t, x})}{|t - x|^{n+1}}.
\]
With all this, we obtain by the choice of \( M \),
\[
|K(t, x) - K(t', x)| \lesssim \| P^\perp M T b_1 \|_{\text{BMO}(\mathbb{R}^n)} \frac{\ell(I_{t,x'})}{|t - x|^{n+1}} \leq \epsilon \frac{\ell(I_{t,x'})}{|t - x|^{n+1}}.
\]
We work the case \(|t + x|_\infty > M2^{M+2}\). Since every cube \(I\) in the sum (51) satisfies \(I_{t,x} \subseteq I_p\), we have \((t + x)/2 \in I_p\). Then \(|c(I_p) - (x + t)/2|_\infty < \ell(I_p)/2\) and so, \(|c(I_p)|_\infty \geq |t + x|_\infty / 2 - \ell(I_p)/2\). Now, if \(\ell(I_p) > 2^{M+1}\), we get as before that \(I \in \mathcal{D}_M^c\). If instead \(\ell(I) \leq 2^{M+1}\),

\[
\text{rdist}(I, \mathbb{B}_{2^M}) \geq \text{rdist}(I, \mathbb{B}_{2^M}) = \frac{\text{diam}(I_p \cup \mathbb{B}_{2^M})}{2^M} \gtrless \frac{|c(I_p)|_\infty + 2^{M-1} + \ell(I_p)/2}{2^M} \geq \frac{|t + x|_\infty}{2^{M+1}} + \frac{1}{2} > M,
\]

by the property of \(|t + x|_\infty\). Therefore, we also obtain \(I \in \mathcal{D}_M^c\) and with this, we conclude as in the previous case that

\[
|K(t, x) - K(t', x)| \lesssim \|P_M^{t,x} T b_1\|_{\text{BMO}_b(\mathbb{R}^n)} \frac{\ell(I_{x,x'})}{|t - x|^{n+1}} \leq \epsilon \frac{\ell(I_{x,x'})}{|t - x|^{n+1}}.
\]

3) The last case, \(|t - x|_\infty < 2^{-2M}\), is more involved. The cubes in the sum such that \(\ell(I) < 2^{-M}\) or \(\ell(I) > 2^M\) satisfy \(I \in \mathcal{D}_M^c\) and so, they may be taken care of as in the two previous cases.

However, those cubes such that \(2^{-M} \leq \ell(I) \leq 2^M\) may belong to \(\mathcal{D}_M\) and the previous argument can not be used. Instead, we reason as follows. The terms under consideration in (50) are given by those cubes \(I \in \mathcal{D}\) such that \(I_{t,x} \subseteq I_p\) and \(2^{-M} \leq \ell(I) \leq 2^M\). From the work in case 1), we know that \(I_{t,x} \subseteq I_{x,x'}\). Therefore, these cubes can be parametrized by their side length as \(\ell(I^k) = 2^k \ell(I_{t,x})\) with \(k_0 \leq k \leq k_1\) where \(k_0 = \max(-M - \log \ell(I_{t,x}), 0)\) and \(k_1 = M - \log \ell(I_{t,x})\). Then,

\[
K(t, x) - K(t, x') = \sum_{k_0 \leq k \leq k_1} \langle T b_1, \psi_{I_{p}}^{b_2} \rangle \frac{\chi_{I_{p}}^{b_2}(t)}{\langle b_1 \rangle I_{p}^{b_2}} (\tilde{\psi}_{I_{p}}^{b_2}(x) - \tilde{\psi}_{I_{p}}^{b_2}(x')).
\]

As in the first case, we bound the modulus of previous expression by

\[
\|T b_1\|_{\text{BMO}_b(\mathbb{R}^n)} \sum_{k_0 \leq k \leq k_1} \frac{\chi_{I_{p}}^{b_2}(t)}{|I_{p}^{b_2}|^{\frac{1}{2}}} |\tilde{\psi}_{I_{p}}^{b_2}(x) - \tilde{\psi}_{I_{p}}^{b_2}(x')|.
\]

By the same reasoning, we bound the last factor by a constant times

\[
(54) \quad \sum_{k_0 \leq k \leq k_1} \frac{1}{2^{kn} \ell(I_{t,x})^n}.
\]

We distinguish two cases depending whether \(|t - x|_\infty \leq 2^{-M/2} \ell(I_{t,x})\) or \(|t - x|_\infty > 2^{-M/2} \ell(I_{t,x})\). In the first case, (54) is bounded by

\[
\sum_{0 \leq k} \frac{1}{2^{kn} \ell(I_{t,x})^n} \lesssim \frac{\ell(I_{t,x})}{\ell(I_{t,x})^{n+1}} \leq 2^{-\frac{M}{2}(n+1)} \frac{\ell(I_{x,x'})}{|t - x|^{n+1}}.
\]
In the second case, we have that \( \ell(I,t,x) < 2^{M/2} |t-x|_{\infty} < 2^{-3M/2} \) and so, \( k_0 \geq -M - \log \ell(I,t,x) \geq M/2 \). Then, (54) is bounded by

\[
\sum_{k \geq \frac{M}{2}} \frac{1}{2^{kn}} \frac{1}{\ell(I,t,x)^n} \lesssim \frac{1}{2^{\frac{M}{2}n}} \frac{\ell(I,t,x)}{\ell(I,t,x')} \leq \frac{1}{2^{\frac{M}{2}n}} |t-x|_{\infty}^{n+1}.
\]

Therefore, in both cases we have by the choice of \( M \) again,

\[
|K(t,x) - K(t,x')| \lesssim \|Tb\|_{\text{BMO}\((\mathbb{R}^n)\)} \frac{1}{2^{\frac{M}{2}n}} \frac{\ell(I,t,x)}{|t-x|_{\infty}^{n+1}} \leq \frac{\ell(I,t,x)}{|t-x|_{\infty}^{n+1}}.
\]

Similar reasoning applies to \( K(t,x) - K(t',x) \) finishing the proof.

12. Necessity of the hypotheses

In this last section, we prove necessity of the three hypotheses of Theorem 4.1. Since in [14], we proved that Calderón-Zygmund operators compact on \( L^p(\mathbb{R}^n) \) have compact Calderón-Zygmund kernels, we focus on the other two hypotheses: the weak compactness condition and the membership of \( Tb_1 \) and \( T^*b_2 \) to \( \text{CMO}(\mathbb{R}^n) \).

12.1. The weak compactness condition.

**Proposition 12.1.** Let \( 1 < p < \infty \) and \( T \) be bounded on \( L^p(\mathbb{R}^n) \). Let \( 1 \leq q_i \leq \infty \) and \( b_i \) be two locally integrable functions. If either \( p \leq q_1 \) and \( p' \leq q_2 \), or \( b_1, b_2 \) are accretive then for every \( M \in \mathbb{N}, Q \in \mathcal{D} \),

\[
|\langle T_b \chi_Q, \chi_Q \rangle| \lesssim |Q| |b_1| \|b_2\|_{Q,q_2} \left[ \|P_M^*\chi_Q\|_{p,p} + \|P_M^*T\|_{p,p} \chi_{[0,1]} \left( \frac{\ell(Q)}{2M} \right) \left( 1 + \frac{2^{-M}}{\ell(Q)} \right)^{-\frac{n}{2}} \chi_{[0,1]} \left( \frac{\text{rdist}(Q, B_{2M})}{M} \right) \right].
\]

**Corollary 12.2.** Let \( 1 < p < \infty \) and \( T \) compact on \( L^p(\mathbb{R}^n) \). Let \( q_i \) and \( b_i \) as before. Then \( T \) satisfies the weak compactness condition.

**Proof.** We start with the decomposition

\[
|\langle T_{b_1} \chi_Q, \chi_Q \rangle| \leq |\langle P_M^* T_{b_1} \chi_Q, \chi_Q \rangle| + |\langle P_M T_{b_1} \chi_Q, \chi_Q \rangle|.
\]

Since \( \langle b_2 f, \widetilde{\psi}_I^b \rangle = b_2 \langle f, \psi_I^b \rangle \) for all \( I \in \mathcal{D} \), we have \( P_M(b_2 f) = b_2 P_M^* f \). With this and the hypothesis on \( q_i \) or \( b_i \), we get

\[
|\langle P_M^* T_{b_1} \chi_Q, \chi_Q \rangle| = |\langle (P_M^* \chi_Q) T(b_1 \chi_Q), b_2 \chi_Q \rangle| \leq \|P_M^* T(b_1 \chi_Q)\|_{p'} \|b_2\|_{L^{p'}(Q)} = \|P_M^* T\|_{p,p} |b_1|_{Q,q_1} |b_2|_{Q,q_2}.
\]

We deal now with the second term. If \( Q \in \mathcal{D}_M \), we have as before

\[
|\langle P_M T_{b_1} \chi_Q, \chi_Q \rangle| \lesssim \|P_M T\|_{p,p} |b_1|_{Q,q_1} |b_2|_{Q,q_2},
\]

which is compatible with the statement since \( 2^{-M} \leq \ell(Q) \leq 2^M \) and \( \text{rdist}(Q, B_{2M}) < M \).
If $Q \notin D_M$, we proceed in a different way. By Lemma 9.12

$$b_2\chi_Q = \sum_{J \in D, Q \subseteq J} \langle b_2\chi_Q, \tilde{\psi}_J^{b_2} \rangle \psi_J^{b_2}$$

with a.e. pointwise convergence. The constraint $Q \subseteq J$ is due to $\langle b_1\chi_Q, \tilde{\psi}_J^{b_1} \rangle = 0$ for $J \cap Q = \emptyset$ while, by Lemma 9.6, $\langle b_2\chi_Q, \tilde{\psi}_J^{b_2} \rangle = \langle b_2, \psi_J^{b_2} \rangle = 0$ for $J \subseteq Q$. Therefore,

$$(55) \quad P_M(b_2\chi_Q) = \sum_{J \in D_M, Q \subseteq J} \langle b_2\chi_Q, \tilde{\psi}_J^{b_2} \rangle \psi_J^{b_2},$$

where the sum is finite. With $P_M(b_2f) = b_2P^*_Mf$ and $P^2_M = P_M$, we have

$$\langle P_M T_b\chi_Q, \chi_Q \rangle = \langle P^*_M T(b_1\chi_Q), b_2P^*_M \chi_Q \rangle$$

$$(56) \quad = \langle P^*_M T(b_1\chi_Q), P_M(b_2\chi_Q) \rangle = \sum_{J \in D_M, Q \subseteq J} \langle P^*_M T(b_1\chi_Q), \psi_J^{b_2} \rangle \langle b_2\chi_Q, \tilde{\psi}_J^{b_2} \rangle.$$

Now, we separate into three cases: $\ell(Q) > 2^M; \ell(Q) < 2^{-M};$ and $2^{-M} \leq \ell(Q) < 2^M$ with $\text{rdist}(Q, B_{2^M}) > M$.

1) When $\ell(Q) > 2^M$, all cubes $J$ in the sum satisfy $\ell(J) \geq \ell(Q) > 2^M$, which is contradictory with $J \in D_M$. Then the sum in (56) is empty and $\langle P_M T_b\chi_Q, \chi_Q \rangle = 0$.

2) When $\ell(Q) < 2^{-M}$, since $J \in D_M$ we have that $\ell(Q) < 2^{-M} \leq \ell(J)$. By telescoping, the sum in (56) can be rewritten as $P_M(b_2\chi_Q) = \sum_{-M < k \leq M} \Delta_k^b(b_2\chi_Q) = E^b_M(b_2\chi_Q) - E^M_{-b}(b_2\chi_Q)$. Let $J^0, J^1 \in D_M$ such that with $Q \subseteq J^i$ and $\ell(J^i) = 2^{(-1)^iM}$. Then

$$\|E^b_M(b_2\chi_Q)\|_{L^p'(R^n)} = \|\left(\frac{b_2\chi_Q}{|b_2\chi_Q|}\right)\|_{L^p'(R^n)}$$

$$= \frac{|Q|}{|J^1|} \left|\frac{\langle b_2\chi_Q \rangle_{J^1}}{|\langle b_2\chi_Q \rangle_{J^1}}\right| \left|\frac{\langle b_2\chi_Q \rangle_{J^1}}{|\langle b_2\chi_Q \rangle_{J^1}}\right| \frac{2^M |Q|}{|\langle b_2\chi_Q \rangle_{J^1}|} \lesssim 2^M |Q|,$$

using the hypothesis on $q_i$ or $b_i$. Similarly, we get a smaller estimate for $\|E^b_M(b_2\chi_Q)\|_{L^p'(R^n)}$. Whence, from the first equality in (56), we have

$$|\langle P_M T_b\chi_Q, \chi_Q \rangle| \leq \|P^*_M T\|_{p,p} \|b_1\|_{L^p(Q)} \|P_M(b_2\chi_Q)\|_{L^p'(R^n)}$$

$$\lesssim \|P^*_M T\|_{p,p} |Q|^{\frac{1}{p'}} |\langle b_1\rangle\chi_Q, \langle b_2\chi_Q \rangle_{J^1}| 2^{-M} |Q|$$

$$= \|P^*_M T\|_{p,p} |Q| |\langle b_1\rangle\chi_Q, \langle b_2\chi_Q \rangle_{J^1}| (2^{-M} |Q|)^{\frac{1}{p'}}$$

with the hypothesis on $q_i$ or $b_i$. This ends the second case.
3) We consider the case $2^{-M} < \ell(Q) < 2^M$ and $\text{rdist}(Q, \mathbb{B}_{2M}) > M$. Since $2^{-M} \leq \ell(J) \leq 2^M$ and $Q \subset J$, we have that

$$\text{rdist}(Q, \mathbb{B}_{2M}) = \frac{\ell((Q, \mathbb{B}_{2M}))}{2^M} \leq \frac{\ell((J, \mathbb{B}_{2M}))}{2^M} = \text{rdist}(J, \mathbb{B}_{2M})$$

Then, $\text{rdist}(J, \mathbb{B}_{2M}) > M$, which is contradictory with $J \in D_M$. Therefore, the sum in (56) is again empty and $\langle P_M T_b \chi_Q, \chi_Q \rangle = 0$. 

\section{12.2. Membership in CMO$_b(\mathbb{R}^n)$.}

\begin{proposition}
Let $T$ be a linear operator with a standard Calderón-Zygmund kernel that extends compactly on $L^p(\mathbb{R}^n)$ for some $1 < p < \infty$. Then for $b_1, b_2$ locally integrable functions compatible with $T$ we have $Tb_1, T^*b_2 \in \text{CMO}_b(\mathbb{R}^n)$.

\end{proposition}

\begin{proof}
Since $T$ is bounded on $L^p(\mathbb{R}^n)$, by the classical theory, $T$ is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. Thus, by interpolation, $T$ turns out to be compact on all $L^p(\mathbb{R}^n)$ spaces.

To prove membership in BMO$_b(\mathbb{R}^n)$, we show first that $\mathcal{L}_b$ defined in Lemma 7.1 is a bounded linear functional on $H^1_b(\mathbb{R}^n)$. Since linearity is trivial, we prove its continuity on $H^1_b(\mathbb{R}^n)$.

By standard arguments, it is enough to prove the result for $p'$-atoms. Let $I \in \mathcal{D}$ be fixed and $f$ be an atom in $H^1_b(\mathbb{R}^n)$ supported on $I_p$ with mean zero with respect $b_2$ and $\|fb_2\|_{L^{p'}(\mathbb{R}^n)} \lesssim B_{I,p}^{b_2}\|I\|^{-1/p}$.

Let $\Psi_k = \chi_{2^{k+1}I_p} - \chi_{2^kI_p}$. For $k \in \mathbb{N}$, $k > 1$, we have

$$|\mathcal{L}_b(f)| \leq |\langle Tb\chi_{I_p}, f \rangle| + \sum_{k' = 0}^{k-1} |\langle Tb\Psi_{k'}, f \rangle| + |\mathcal{L}_b(f) - \langle Tb\chi_{2^kI_p}, f \rangle|.$$  

Using boundedness of $T$ on $L^p(\mathbb{R}^n)$ we estimate the first term by

$$\|T\|_{p,p'}\|b_1\chi_{I_p}\|_{L^p(\mathbb{R}^n)}\|b_2f\|_{L^{p'}(\mathbb{R}^n)} \lesssim [b_1]_{I,p}\|I_p\|^{\frac{1}{p}} B_{I,p}^{b_2}\|I\|^{-\frac{1}{p'}} = [b_1]_{I,p}B_{I,p}^{b_2}.$$  

Since $T$ is compact, its kernel $K$ is a compact Calderón-Zygmund kernel with parameter $\delta$. Then from the proof of Lemma 7.1 the second term is bounded by a constant times

$$\sum_{k' = 0}^{k-1} 2^{-k'\delta}\|I_p\|^{\frac{1}{p}} B_{I,q_2}^{b_2}\|b_2f\|_{L^2(\mathbb{R}^n)} \inf_{x \in 2^{k'}I} M_{q_1}b_1(x) \tilde{F}_K(2^{k'}I, I, 2^{k'}I) \lesssim \|I_p\|^{\frac{1}{p}} B_{I,q_2}^{b_2}\|b_2f\|_{L^2(\mathbb{R}^n)} \inf_{x \in I} M_{q_1}b_1(x) \lesssim \langle M_{q_1}b_1 \rangle_I B_{I,q_2}^{b_2},$$

where $\tilde{F}_K$ is defined in (52).
where we used boundedness of $\tilde{F}_K$. Finally, we apply the result of Lemma 7.1 to bound the last term by

$$2^{-k \delta} |I_p|^{1/2} \|b_2 f\|_{L^{q_2}(\mathbb{R}^n)} \inf_{x \in 2^k I} M_{q_1} b_1(x) \tilde{F}_K (2^k I, I, 2^k I) \lesssim B_{I_q;2}^b \langle M_{q_1} b_1 \rangle_I.$$

These estimates show $|L_b(f)| \lesssim 1$ for every atom $f$, proving that $L_b$ defines a bounded linear functional on $H^1_b(\mathbb{R}^n)$. Hence, by the $H^1_b(\mathbb{R}^n)$-BMO$_b(\mathbb{R}^n)$ duality of Lemma 11.1, the functional $L_b$ is represented by a function in BMO$_b(\mathbb{R}^n)$ denoted by $Tb_1$, that is, $L_b(f) = \langle Tb_1, b_2 f \rangle$.

In order to prove membership in CMO$_b(\mathbb{R}^n)$, we need to show that $\lim_{M \to \infty} \langle P_M^{\perp} Tb_1, b_2 f \rangle = 0$ uniformly for all $f$ in the unit ball of $H^1_b(\mathbb{R}^n)$. Let $I \in D$ and $f$ be an atom in $H^1_b(\mathbb{R}^n)$ supported on $I$ with zero mean with respect $b_2$ and $\|f b_2\|_{L^{p'}(\mathbb{R}^n)} \lesssim B_{I;2}^{b_2} |I|^{-1/p}$.

For $\epsilon > 0$, we fix $k \in \mathbb{N}$, $k > 1$, so that $2^{-k \delta} \langle M_{q_1} b_1 \rangle_I B_{I;2}^{b_2} < \epsilon$. Moreover, due to compactness of $T$, we can choose $M > 0$ such that $I \in D_M$ and $\|P_M^{\perp} T\|_{p,p} \lesssim 2^{k/p} [b_1]_2 k^{p} B_{I;2}^{b_2} < \epsilon$.

We decompose as follows:

$$\langle P_M^{\perp} Tb_1, b_2 f \rangle = \langle P_M^{\perp} T(b_1 \chi_{2^k I}), b_2 f \rangle + \sum_{k' = k}^{\infty} \langle T(b_1 \Psi_{k'}), P_M^{\perp} (b_2 f) \rangle. \tag{57}$$

Notice that $P_M(b_2 f) = \sum_{J \in D_M} |J|^{1/2} \langle b_2 f, \psi_{J-2\delta} \rangle |J|^{-1/2} \psi_{J-2\delta}^{b_2}$ is a finite linear combination of functions $|J|^{-1/2} \psi_{J-2\delta}^{b_2} = |J|^{-1/2} h_2^{b_2} I_{2^k I}$ with $|J|^{-1/2} h_2^{b_2}$ being $p'$-atoms and so, it belongs to $H^1_b(\mathbb{R}^n)$. Then we also get $P_M^{\perp} (b_2 f) \in H^1_b(\mathbb{R}^n)$, which justifies (57). We bound the first term in (57) as follows:

$$\|P_M^{\perp} T\|_{p,p} \|b_1 \chi_{2^k I}\|_{L^p(\mathbb{R}^n)} \|b_2 f\|_{L^{p'}(\mathbb{R}^n)} \lesssim \|P_M^{\perp} T\|_{p,p} \|b_1\|_{2^k I,p} \|2^k I\|_p B_{I;2}^{b_2} |I|^{-\frac{1}{p'}}$$

$$= \|P_M^{\perp} T\|_{p,p} \|b_1\|_{2^k I,p} 2^k B_{I;2}^{b_2} < \epsilon.$$

Applying the definition of $P_M^{\perp}$, we rewrite the second term as

$$\sum_{k' = k}^{\infty} \langle T(b_1 \Psi_{k'}), b_2 f \rangle - \sum_{k' = k}^{\infty} \langle P_M^{\perp} T(b_1 \Psi_{k'}), b_2 f \rangle = A + B.$$

For the new first term, we have from the proof of Lemma 7.1 that

$$|A| \lesssim \sum_{k' = k}^{\infty} 2^{-k \delta} |I_p|^{1/2} \|b_2 f\|_{L^{q_2}(\mathbb{R}^n)} \inf_{x \in 2^k I} M_{q_1} b_1(x) F_{K}^D (2^{k'} I, I, 2^{k'} I) \lesssim \sum_{k' = k}^{\infty} 2^{-k \delta} B_{I_q;2}^{b_2} \inf_{x \in I} M_{q_1} b_1(x) \lesssim 2^{-k \delta} B_{I_q;2}^{b_2} \langle M_{q_1} b_1 \rangle_I < \epsilon.$$
By definition, we rewrite each term of $B$ as

$$
(P_M^* T(b_1 \Psi_{k'}), b_2 f) = \sum_{J \in D_M(I)} \langle T(b_1 \Psi_{k'}), \psi^b_j \rangle \langle \tilde{\psi}^b_j, b_2 f \rangle
$$

Since $b_2 f$ and $\tilde{\psi}^b_j$ have compact support on $I_p$ and $J_p$ respectively, the non-null terms in the sum arise when $J_p \subset I_p$. This is obvious when $I_p \cap J_p = \emptyset$. For those cubes $J$ so that $I_p \subseteq J_p$, we have $\langle \tilde{\psi}^b_j, b_2 f \rangle = \langle b_2 \rangle_J \langle h^b_j, b_2 f \rangle = 0$ since $h^b_j$ is constant on $I_p$ and $b_2 f$ has mean zero.

Now, $b_1 \Psi_k$ is supported on $(2^k I_p)^c$ and $\psi^b_j$ is supported on $I_p$ and so, we can use the integral representation to rewrite (58) as

$$
\sum_{J \in D_M(I)} \int \int b_1(t) \Psi_{k'}(t) \psi^b_j(z) K(t, z) dz dt \int b_2(x) f(x) \tilde{\psi}^b_j(x) dx
$$

$$
= \int \int b_1(t) \Psi_{k'}(t) b_2(x) f(x) K(t, x) dt dx
$$

with $t \in (2^{k'+1} I_p) \setminus (2^k I_p)$, $K(t, x) = \sum_{J \in D_M(I)} \int \psi^b_j(z) K(t, z) dz \tilde{\psi}^b_j(x)$.

We now aim to estimate $K(t, x)$. By the mean zero of $\psi^b_j(z)$,

$$
\int \psi^b_j(z) K(t, z) dz = \int \psi^b_j(z) (K(t, z) - K(t, c(J))) dz.
$$

Since $t \in (2^k I_p)^c$, $z \in J_p \subset I_p$ and $k' > 1$, we get

$$
|t - z|_\infty \geq |t - c(I_p)|_\infty - |c(I_p) - z|_\infty \geq 2^{k'-1} \ell(I_p) - \ell(I_p)/2
$$
$$
> \ell(I_p) \geq \ell(J_p) \geq 2|z - c(J_p)|_\infty.
$$

Whence,

$$
\left| \int \psi^b_j(z) K(t, z) dz \right| \lesssim \int \left| \psi^b_j(z) \frac{|z - c(J_p)|_\infty}{|t - z|_\infty^{\delta + \delta}} F_K(t, z, c(J_p)) dz,
$$

with $F_K(t, z, c(J_p)) = L(|t - c(J_p)|_\infty) S(|z - c(J_p)|_\infty) D \left( 1 + \frac{|t + c(J_p)|_\infty}{1 + |t - c(J_p)|_\infty} \right)$.

By Lemma 5.1 $F_K(t, z, c(J_p)) \lesssim F_K(2^k I, J, 2^k I)$ and so,

$$
\left| \int \psi^b_j(z) K(t, z) dz \right| \leq \| \psi^b_j \|_{L^1(\mathbb{R}^n)} \frac{\ell(J)^\delta}{2^{k(n+\delta)} \ell(I)^{n+\delta}} F_K(2^k I, J, 2^k I)
$$

$$
\leq B^b_{k_1} |J|^{\frac{\delta}{2^k(n+\delta)}} \frac{\ell(J)^\delta}{2^{k(n+\delta)} \ell(I)^{n+\delta}} F_K(2^k I, J, 2^k I).
$$
using Lemma 9.6. With this and $\|\tilde{\psi}^b_k\|_{L^\infty(\mathbb{R}^n)} \lesssim C^b_k |\langle b_2 \rangle| |J|^{-\frac{\rho}{2}}$,

$$|\mathcal{K}(t, x)| \lesssim \sum_{J \in \mathcal{D}_M} \sum_{x \in J_p \subseteq I_p} (C^b_k)^2 |\langle b_2 \rangle| |\langle b_2 \rangle| F_K(2^{k'} I, J, 2^{k'} I) \frac{\ell(J)^{\delta}}{2^{k'(n+\delta)} \ell(I)^{n+\delta}}$$

$$\lesssim \frac{1}{2^{k'(n+\delta)} |I_p|} \sum_{J \in \mathcal{D}_M} BF(I, J) \left( \frac{\ell(J)}{\ell(I)} \right)^{\delta}.$$ 

Since the cubes in the sum satisfy $x \in J_p \in \mathcal{D}_M(I)$, they can be parametrized by their side length as $\ell(J^r) = 2^{-r} \ell(I_p)$ with $0 \leq r \leq M + \log \ell(I)$. With this and the bound $BF(I, J) \lesssim 1$,

$$|\mathcal{K}(t, x)| \lesssim \frac{1}{2^{k'(n+\delta)} |I_p|} \sum_{r \geq 0} 2^{-r \delta} \lesssim \frac{1}{2^{k'(n+\delta)} |I_p|}$$

With this estimate we have from (58) and (59)

$$|\langle P^*_M T(b_1 \Psi_{k'}, b_2 f) \rangle| \lesssim \int \int |b_1(t) \Psi_{k'}(t)| |b_2(x) f(x)| \frac{1}{2^{k'(n+\delta)} |I_p|} dtdx$$

$$\lesssim \frac{1}{2^{k'(n+\delta)} |I_p|} \|b_1\|_{L^1(2^{k'} I)} \|b_2 f\|_{L^1(I)}$$

$$\lesssim \frac{1}{2^{k'(n+\delta)} |I_p|} \|b_1\|_{L^1(2^{k'} I)} 2^{k'} |B^{b_2}_{1,1}| \lesssim \frac{1}{2^{k'(n+\delta)} |I_p|}.$$ 

Then, by the choice of $k$, we finally get

$$|B| \leq \sum_{k' = k}^{\infty} \langle P^*_M T(b_1 \Psi_{k'}, b_2 f) \rangle \lesssim \langle M_1 b_1 \rangle_{I} \langle B^{b_2}_{1,1} \rangle \sum_{k' = k}^{\infty} \frac{1}{2^{k' \delta}} < \epsilon$$

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\textsc{Department of Mathematics, University of Georgia, Athens, GA 30602, USA}
\textit{E-mail address: paco.villarroya@uga.edu}