On Spaces of connected Graphs I

Properties of Ladders

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Abstract

We examine spaces of connected tri-/univalent graphs subject to local relations which are motivated by the theory of Vassiliev invariants. It is shown that the behaviour of ladder-like subgraphs is strongly related to the parity of the number of rungs: there are similar relations for ladders of even and odd lengths, respectively. Moreover, we prove that - under certain conditions - an even number of rungs may be transferred from one ladder to another.

1 Introduction

This note is the first in a series of three papers and its main goal is to provide relations that will be employed by the following ones. Nevertheless, we have the feeling that the results are interesting enough to stand alone. The objects we deal with are combinatorial multigraphs in which all vertices have valency 1 or 3, equipped with some additional data: at each trivalent vertex a cyclic ordering of the three incoming edges is specified and every univalent vertex carries a colour. In pictures we assume a counter-clockwise ordering at every trivalent vertex and indicate the colour of univalent vertices by integers. The graphs have to be connected and must contain at least one trivalent vertex, but we do not insist that there are univalent vertices. For simplicity, such graphs will be called diagrams from now on.

The motivation is given by the combinatorial access to Vassiliev invariants. We briefly mention the most relevant facts about this comprehensive family of link invariants. Vassiliev invariants with values in a ring $R$ form a filtered algebra. A deep theorem of Kontsevich (see [4] and [1]) states that for $\mathbb{Q} \subset R$ the filtration quotients are isomorphic to the graded dual of the Hopf-algebra of chord diagrams. By the structure theorem of Hopf-algebras it suffices to examine the primitive elements.

The algebra of chord diagrams is rationally isomorphic ([1]) to an algebra of tri-/univalent graphs (sometimes called Chinese characters), defined by relations named (AS) and (IHX) (see Definition 2.1). The coproduct is given by distributing the components into two groups, so the primitives are spanned by connected graphs. The colours of the univalent vertices represent the link-components and half the total number of vertices corresponds to the degree of Vassiliev invariants. The number of univalent vertices allows a second grading that corresponds to the eigenspace-decomposition of primitive Vassiliev invariants with respect to the cabling operation.

The most important open question in Vassiliev theory is whether all invariants of finite degree taken together form a complete invariant of knots and links. For knots there is a weaker (but equally essential) question:

Question 1.1 Are there any (rational valued) Vassiliev invariants that are able to detect non-invertibility of knots?
There is a weak hope that this question might be settled in the combinatorial setting, where it translates into the question whether all diagrams with an odd number of univalent vertices vanish.

A progress in the combinatorics of connected graphs ([6]) was initiated by the observation that replacing a trivalent vertex by the graph that is shown in relation (x) in the next section is a well-defined operation (i.e. independent of the choice of a trivalent vertex and the orientation). This operation called $x_n$ is an element of an algebra called $\Lambda$ that acts on spaces of connected graphs. One purpose of our investigations (and in fact the initial one) was to obtain a family of relations in $\Lambda$. The result is presented in [2].

Our philosophy is to declare diagrams that are coming from lower degrees by the action of $\Lambda$ as uninteresting, which explains why we factor by the relation (x) in $\hat{B}^u$ (see Definition 2.1). This can be justified in two ways: First, note that Question 1.1 is equivalent to asking whether $\hat{B}^u = 0$ for all odd $u$. Second, any good upper bound for the dimensions of $\hat{B}^u$ will lead to good upper bounds for $\dim B^u$ as we demonstrate in [3].

2 Results

First let us introduce notations for certain subgraphs of diagrams. An edge that connects a trivalent and a univalent vertex is called leg. For $n \geq 2$, a subgraph consisting of $3n + 2$ edges of the following type

\[
\begin{array}{c}
\hspace{1cm}
\end{array}
\]

is called a $n$-ladder. The two uppermost and the two lowermost edges are called ends of the ladder. They may be connected to univalent or trivalent vertices in the rest of the diagram. The $3n - 2$ other edges are called interior of the ladder. A ladder is said to be odd or even according to the parity of $n$. A 2-ladder is also called square. Finally, for a partition $u = (u_1, \ldots, u_n)$, $u_1 \leq u_2 \leq \cdots u_n$, we say a diagram is $u$-coloured if its univalent vertices carry the colours 1 to $n$ and there are exactly $u_i$ univalent vertices of colour $i$ for $1 \leq i \leq n$.

**Definition 2.1** For any partition $u$ let

\[
B(u) := Q\langle \text{u-coloured diagrams} \rangle / (\text{AS}), (\text{IHX}) \quad \text{and} \quad \hat{B}(u) := B(u) / (x),
\]

where (AS), (IHX) and (x) are the following local relations:

\[
\begin{array}{c}
\begin{array}{c}
\hspace{1cm}
\end{array}
\end{array} = - \begin{array}{c}
\begin{array}{c}
\hspace{1cm}
\end{array}
\end{array} \quad (\text{AS})
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hspace{1cm}
\end{array}
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hspace{1cm}
\end{array}
\end{array}
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hspace{1cm}
\end{array}
\end{array}
\end{array}
\end{array} = 0 \quad (\text{IHX})
\]

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\hspace{1cm}
\end{array}
\end{array}
\end{array}
\end{array} = 0 \quad (x)
\]

The extremal partitions $u_1 = \cdots = u_n = 1$ and $n = 1$ are most important, so they get their own names: $F(n) := B((1, \cdots, 1))$, $\hat{F}(n) := \hat{B}((1, \cdots, 1))$ and $B^u := B((u))$, $\hat{B}^u := \hat{B}((u))$.

**Remark 2.2** The careful reader will notice that in our calculations we have to divide by 2 and 3 only, so all statements remain valid if one works with $Z[1/6]$-modules instead of $Q$-vectorspaces.
Let us mention an important consequence of the relations (AS), (IHX) and (x):

\[ = + = - = 0 \quad (t) \]

By (IHX) and (t) the ends of a square may be permuted:

\[ = + = \]

This obviously implies that neighbouring ends of any \( n \)-ladder can be interchanged. But it is a little surprise that for all even ladders any permutation of the ends yields the same element, i.e. for \( (2m) \)-ladders \((m \geq 1)\) we have the following relation called LS (“ladder symmetry”):

\[ = = \quad (LS) \]

Here and later on, when making statement about \( n \)-ladders, we draw the corresponding picture for a generic but small value of \( n \). This should not lead to confusion.

For any \( m \geq 1 \), we have the following relation of the IHX-type, involving odd ladders of length \( 2m + 1 \):

\[ + + = 0 \quad (LIHX) \]

We present a further relation named LI, in which an edge has been glued to non-neighbouring ends of an odd ladder:

\[ 2 + = 0 \quad (LI) \]

Finally, we have the following relation LL that replaces two parallel ladders of lengths \( 2n + 1 \) and \( 2m + 1 \) by a single \((2n + 2m + 2)\)-ladder:

\[ = \quad (LL) \]

**Theorem 1** The relations (LS), (LIHX), (LI) and (LL) are valid in \( \hat{B}(u) \) for any partition \( u \).

**Definition 2.3** Suppose we have a diagram \( D \) containing two ladders \( L_1, L_2 \) which have disjoint interiors. If we remove the interiors of \( L_1 \) and \( L_2 \) from \( D \) we get a – possibly disconnected – graph \( D' \). Let \( D'_1, \ldots, D'_k \) denote the components of \( D' \) that contain at least one end of \( L_1 \) and at least one end of \( L_2 \). If \( k = 1 \) and \( D'_1 \) is a tree and \( D'_1 \) contains exactly one end of \( L_1 \) and exactly one end of \( L_2 \), then we say that \( L_1 \) and \( L_2 \) are weakly connected. Otherwise we call \( L_1 \) and \( L_2 \) strongly connected. If the intersection of the interiors of \( L_1 \) and \( L_2 \) is not empty (which implies that \( L_1 \) and \( L_2 \) are sub-ladders of a single longer ladder) then \( L_1 \) and \( L_2 \) are also called strongly connected.

Example: In the following diagrams the 3-ladders and 4-ladders are strongly connected to the square but weakly connected to each other (so being strongly connected is not a transitive relation):
For the rest of this section let us assume that $D$ is a $u$-coloured diagram with two specified 3-ladders $L_1$ and $L_2$. For $a, b \geq 2$ let $D_{a,b}$ denote the diagram that is obtained by replacing $L_1$ and $L_2$ by two ladders (in the same orientation) of length $a$ and $b$, respectively.

**Theorem 2 (Square-Tunnelling relation)** If $L_1$ and $L_2$ are strongly connected in $D$ then $D_{2,4} = D_{4,2}$ in $B(u)$.

If either $L_1$ or $L_2$ is subladder of a longer ladder, then $L_1$ or $L_2$ are automatically strongly connected, which allows to make a nice statement:

**Corollary 2.4** For any $a \geq 2, b \geq 4$ with $a + b \geq 7$ we have $D_{a,b} = D_{a+2,b-2}$ in $B(u)$.

One might ask whether the strong connectivity condition is essential in Theorem 2, in other words:

**Question 2.5** Are there any $u$-coloured diagrams $D$ with weakly connected ladders such that $D_{2,4} \neq D_{4,2}$?

At least for the case length($u$) = 1, we may give the answer:

**Theorem 3 (Square-Tunnelling relation in $\hat{B}^n$)** Let $D$ be a diagram of $B^n$ (i.e. all univalent vertices carry the same colour), then $D_{2,4} = D_{4,2}$.

Compared to the other relations, which are local, the square-tunnelling relation has a completely different character: it relates ladders that might be located arbitrarily far apart in a diagram. We already mentioned in the introduction a situation where a global structure (the action of $\Lambda$ on $B(u)$) emerges from local relations ((IHX) and (AS)). There, it is easy to understand how the subgraphs $x_n$ (shown in relation (x)) move around, because they can go from one vertex to a neighbouring one. But here, we are not able to trace the way of the square from the ladder in which it disappears to the other ladder. It is just like the quantum-mechanical effect where an electron tunnels through a classically impenetrable potential barrier: one can calculate that it is able to go from one place to the other, but we cannot tell how it actually does it.

### 3 The local ladder relations

Let us start with two well-known and frequently-used implications of (IHX), (AS) and (t). We assume that we are given a diagram $D$ of $\hat{F}(n)$ together with an arbitrary grouping of the legs of $D$ into two classes called “entries” and “exits”. Say there are $p$ entries and $q$ exits (thus $p + q = n$). For $1 \leq i \leq p$ ($1 \leq j \leq q$) let $D_i$ ($D^j$) denote the elements of $\hat{F}(n + 1)$ one obtains when the $(n + 1)$-th leg is glued to the $i$-th entry (the $j$-th exit) of $D$. Furthermore for $1 \leq i < j \leq p$ ($1 \leq k < l \leq q$) let $D_{ij}$ ($D^{kl}$) denote the elements of $\hat{F}(n)$ having an additional edge between the $i$-th and $j$-th entry (the $k$-th and $l$-th exit) of $D$. $D_i, D^j, D_{ij}, D^{kl}$ look typically like this:

\[
\begin{align*}
\text{Diagram 1} & \quad \text{Diagram 2} & \quad \text{Diagram 3} & \quad \text{Diagram 4}
\end{align*}
\]

**Lemma 3.1** For any $D \in \hat{F}(n)$ with $p$ entries and $q$ exits, we have

\[ a) \quad \sum_{i=1}^{p} D_i = \sum_{j=1}^{q} D^j \]
\[ b) \sum_{1 \leq i < j \leq p} D_{ij} = \sum_{1 \leq k < l \leq q} D_{kl}. \]

**Proof** We cut the box into little slices, such that each slice is of one of the following types:

\[
\begin{array}{cccc}
\text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} \\
\text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} \\
\text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} \\
\end{array}
\]

It is easy to verify that a) and b) are valid for slices of these types. \(\square\)

Let us introduce a notation for certain elements of \(\hat{\mathcal{F}}(4)\). For a word \(w\) in the letters \(c_1, c_2, c_3\), let \(\langle w \rangle\) be a diagram of \(\hat{\mathcal{F}}(4)\) that is constructed as follows: Take three strings, put edges between the strings according to \(w\), and then glue string 1 and string 2 together on the right side. \(c_1, c_2\) and \(c_3\) correspond to 1-2, 2-3 and 1-3 edges, respectively. For example \(\langle c_2c_3c_1^2c_3 \rangle\), \(\langle c_2^3 \rangle\), \(\langle c_3^4 \rangle\) are the following diagrams:

\[
\begin{array}{cccc}
\text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} \\
\text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} \\
\text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} \\
\end{array}
\]

**Proposition 3.2** For \(m \geq 1\) and any word \(u\) in the letters \(c_1, c_2, c_3\), we have \(\langle uc_2^m c_3^m \rangle = \langle uc_3^m \rangle\).

**Proof** The statement is implicitly hidden in section 5 of [6], but to the readers convenience we will give a simple inductive proof of the statement. We know already that it is true for \(m = 1\).

We have the following equalities for arbitrary words \(u, v\):

\[
\begin{align*}
\langle uc_1v \rangle + \langle uc_2v \rangle + \langle uc_3v \rangle &= 0 \quad \text{(i)} \\
\langle uc_1c_2^n \rangle &= \langle uc_1c_3^n \rangle \quad \text{(ii)} \\
\langle uc_2c_1c_3^n \rangle &= \langle uc_2c_3c_1^n \rangle \quad \text{for } n \geq 2 \quad \text{(iii)} \\
\langle uc_3c_1c_2^n \rangle &= \langle uc_3c_2c_1^n \rangle \quad \text{for } n \geq 2 \quad \text{(iv)}
\end{align*}
\]

(i) is due to Lemma 3.1 b). (ii) and (iii) are implications of (IHX) and (x):

\[
\begin{array}{cccc}
\text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} \\
\text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} \\
\text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} & \text{\phantom{xxx}} \\
\end{array}
\]

(iv) is shown similarly to (iii), interchanging string 1 with string 2. (i), (iii) and (iv) imply

\[
2\langle uc_2c_3c_1^n \rangle = -\langle uc_2^{n+2} \rangle \quad \text{and} \quad 2\langle uc_3c_2c_3^n \rangle = -\langle uc_3^{n+2} \rangle \quad \text{for } n \geq 2. \quad \text{(v)}
\]

Assuming \(m \geq 2\) and \(\langle uc_2^{2m-2} \rangle = \langle uc_3^{2m-2} \rangle\) by induction hypothesis, we obtain

\[
\begin{align*}
\langle uc_2^{2m} \rangle - \langle uc_3^{2m} \rangle &\overset{(v)}{=} -\langle uc_2^{2m-1} \rangle - \langle uc_1c_2^{2m-1} \rangle + \langle uc_2c_3^{2m-1} \rangle + \langle uc_1c_3^{2m-1} \rangle \\
&\overset{(v)}{=} -\langle uc_3^{2m-1} \rangle + \langle uc_2c_3^{2m-1} \rangle \\
&\overset{1.H.}{=} -\langle uc_3c_2^{2m-2} \rangle + \langle uc_2c_3c_2^{2m-2} \rangle \overset{(v)}{=} \frac{1}{2}\langle uc_3^{2m} \rangle - \frac{1}{2}\langle uc_2^{2m} \rangle.
\end{align*}
\]

\(\square\)

**Proof of Theorem 1** The first equality of the (LS)-relation is true by Proposition 3.2. To prove the second equality, we simply swap the upper ends of the ladder and use the first equality of (LS):
If we take three ends of a \((2m)\)-ladder as entries and the forth as exit, apply Lemma 3.1 b) and (LS), we get the \((\text{LIHX})\)-relation:

\[
0 = \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array} = \begin{array}{c}
\text{Diagram 4} \\
\text{Diagram 5} \\
\text{Diagram 6}
\end{array}
\]

Relation \((\text{LI})\) is obtained when we append an edge between the ends on the right side of \((\text{LIHX})\) and use \((\text{IHX})\) and \((\chi)\):

\[
0 = \begin{array}{c}
\text{Diagram 7} \\
\text{Diagram 8} \\
\text{Diagram 9}
\end{array} = \begin{array}{c}
2 \text{Diagram 7} - \text{Diagram 10} + \text{Diagram 11}
\end{array}
\]

\((\text{LL})\) is the hardest one; we apply \((\text{IHX})\) and use Lemma 3.1 a):

\[
\begin{array}{c}
\text{Diagram 12} = \text{Diagram 13} - \text{Diagram 14} = \text{Diagram 15} - \text{Diagram 16}
\end{array}
\]

\[
= \begin{array}{c}
2 \text{Diagram 17} - \text{Diagram 18} + \text{Diagram 19}
\end{array}
\]

To each of the three resulting diagrams we apply the following identity

\[
2 \text{Diagram 20} = \text{Diagram 21} + \text{Diagram 22} = \text{Diagram 23} + \text{Diagram 24} = \text{Diagram 25} + \text{Diagram 26}
\]

to obtain by \((\tau)\), \((\text{IHX})\), \((\chi)\), \((\text{LS})\) and \((\text{LI})\) the promised result:

\[
\text{Diagram 27} = \text{Diagram 28} + \text{Diagram 29} - \frac{1}{2} \text{Diagram 30} - \frac{1}{2} \text{Diagram 31} - \frac{1}{2} \text{Diagram 32} = \frac{1}{4} \text{Diagram 33}
\]

\[\square\]

4 Relations in \(\hat{F}(6)\)

The key to Theorem 1 is a certain equation in \(\hat{F}(6)\). Written in the notation we will introduce in this section, it appears misleadingly simple:

\[
[y]^3 = 0.
\]

Nevertheless, it requires a lot of calculation. We have spent a considerable amount of time trying to take the computation into a bearable form. We hope — for the readers sake — that this has not been a vain endeavour.

4.1 The algebra \(\Xi\)

We do not like to draw pictures all the time, so we introduce an algebra that enables us to write down sufficiently many elements of \(\hat{F}(6)\) in terms of a few number of symbols.
Let $\Xi$ be the space of tri-/univalent graphs with exactly six numbered univalent vertices quotiented by the (AS), (IHX) and $(x)$. The only difference between $\Xi$ and $F(6)$ is that in $\Xi$ we do not require the graphs to be connected. Also we do not insist that there actually are any trivalent vertices.

All graphs in an (AS)- or (IHX)-relation have same number of components. This implies that $\Xi = F(6) \oplus \Delta$ where $\Delta$ is the subspace of $\Xi$ that is spanned by disconnected graphs. So we can project any relation that we find in $\Xi$ to $\Delta$.

Let $\Xi$ be the space of tri-/univalent graphs with exactly six numbered univalent vertices. Furthermore let $z$ denote the elements of $\Xi$ that are represented by the following graphs:

$$
\begin{align*}
&\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array} \quad \begin{array}{c}
\text{4} \\
\text{5} \\
\text{6}
\end{array} \\
\text{2} & \quad 3 \\
\text{5} & \quad 4
\end{align*}
$$

Let $c_1, c_2, c_3, s_1, s_2, d_1, d_2, y$ denote the elements of $\Xi$ that are represented by the following graphs:

Furthermore let $l_1, l_2, l_3, l_4, \ldots$ be the family of elements of $\Xi$ that is represented by the following graphs ($l_n$ contains a $n$-ladder for $n \geq 2$ and $l_1 = c_1c_2$):

Let $z := c_1 + c_2 + c_3$, then by Lemma 3.1 a), $z$ is central in $\Xi$. We will continually substitute $c_3$ by $z - c_1 - c_2$ and collect all $z$’s at the beginnings of words.

**Proposition 4.1** The following relations are fulfilled in $\Xi$:

$$
\begin{align*}
& s_1^2 = s_2^2 = (s_1s_2)^3 = 1 \\
& d_ic_i = c_id_i = 0 \quad \text{for } i = 1, 2 \\
& s_ic_i = c_is_i \quad \text{for } i = 1, 2 \\
& s_ic_j = c_ks_k \quad \text{for } \{i, j, k\} = \{1, 2, 3\} \text{ and } i \not= 3 \\
& d_i = c_i(1 - s_i) \quad \text{for } i = 1, 2 \\
& c_1c_2 = c_2c_1 \\
& d_2c_1^{2n+1}d_2 = d_2c_1c_2^{2n-1}c_1d_2 = zd_2c_1^{2n}d_2 = -\frac{1}{2}z^{2n+2}d_2 \quad \text{for } n \geq 1 \\
& l_n(1 + s_1) = d_1c_2c_1^{n-1} + c_3c_1^n \quad \text{for } n \geq 2 \\
& s_2l_n = -l_n + c_2c_1^{n-1}d_1d_2 + c_2c_3 \quad \text{for } n \geq 2 \\
& c_1yc_1 = s_1c_1yc_1 = zd_2c_1 - zc_1s_1s_2d_1c_2 \\
& z^2d_2c_1 = z^2s_1s_2d_1c_2 \\
\end{align*}
$$

**Proof** The first six equalities are obvious: $d_ic_i$ and $c_id_i$ contain triangles, $s_ic_i = c_is_i$ is a double application of (AS) and $d_i = c_i - s_ic_i$ is just (IHX). $c_1d_2c_1$ and $c_2d_1c_2$ are identical graphs:

$$
\begin{align*}
&\begin{array}{c}
\text{1} \\
\text{2} \\
\text{3}
\end{array} \quad \begin{array}{c}
\text{4} \\
\text{5} \\
\text{6}
\end{array} \\
\text{2} & \quad 3 \\
\text{5} & \quad 4
\end{align*}
$$

We will use in the sequel that due to Lemma 3.1 b), $z^kd_2$ is represented by the following graph:
Then all equalities in (4) are simple consequences of (LI) and (LS):

\[
\begin{align*}
-2 \begin{array}{c}
\text{Diagram 1}
\end{array} &= \begin{array}{c}
\text{Diagram 2}
\end{array} &= \begin{array}{c}
\text{Diagram 3}
\end{array} \\
-2 \begin{array}{c}
\text{Diagram 4}
\end{array} &= -2 \begin{array}{c}
\text{Diagram 5}
\end{array} &= \begin{array}{c}
\text{Diagram 6}
\end{array} &= \begin{array}{c}
\text{Diagram 7}
\end{array} \\
\end{align*}
\]

To show (5) and (6), we have to use Lemma 3.1 a) (for (6) simply rotate the picture by \(\pi\)):

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 8}
\end{array} + \begin{array}{c}
\text{Diagram 9}
\end{array} &= \begin{array}{c}
\text{Diagram 10}
\end{array} + \begin{array}{c}
\text{Diagram 11}
\end{array} &= \begin{array}{c}
\text{Diagram 12}
\end{array} + \begin{array}{c}
\text{Diagram 13}
\end{array} \\
\end{align*}
\]

Finally we use (t) and (IHX) to show (7) and (8):

\[
\begin{align*}
\begin{array}{c}
\text{Diagram 14}
\end{array} &= \begin{array}{c}
\text{Diagram 15}
\end{array} &= \begin{array}{c}
\text{Diagram 16}
\end{array} - \begin{array}{c}
\text{Diagram 17}
\end{array} \\
\end{align*}
\]

Let us elaborate some consequences of the relations, which we will use later. (Note that the application of a previous equation is indicated by stacking its number atop the equality sign but we do not mention the use of equations without number.)

\[
\begin{align*}
s_i d_i &= d_is_i &\overset{(2)}{=} s_i(1-s_i)c_i = (s_i-1)c_i &\overset{(2)}{=} -d_i \\
d_i^2 &= d_i c_i - d_i s_i c_i = 2c_id_i = 0 \\
d_2 c_1 d_2 &= \overset{(9)}{=} \frac{1}{2}d_1c_2d_3 + \frac{1}{2}d_2c_1s_2d_2 = \frac{1}{2}d_2(c_1 + c_3)d_2 = \frac{1}{2}d_2(z-c_2)d_2 &\overset{(10)}{=} 0 \\
d_2 d_1c_2 &= \overset{(11)}{=}(1-s_2)c_2d_1c_2 = (1-s_2)c_1d_2c_1 &\overset{(12)}{=} c_1d_2c_1 + c_3d_2c_1 = zd_2c_1 \\
d_2d_1d_2 &= \overset{(13)}{=} d_2d_1c_2(1-s_2) &\overset{(14)}{=} zd_2c_1(1-s_2) &\overset{(15)}{=} zd_2(c_1 + c_3) = z^2d_2 \\
\end{align*}
\]
\[
(c_2 - d_2)c_1^2 d_2 \overset{(2)}{=} c_2 s_2 c_1^2 d_2 \overset{(19)}{=} c_2 c_3^2 d_2 = -c_2(z - c_1 - c_2)^2 d_2 = 2z c_2 c_1 d_2 - c_2^2 c_1 d_2 - c_2 c_1^2 d_2
\]

(16)

4.2 The Subspace \([Ξ_0]\)

The symmetric group \(S_6\) operates on \(Ξ\) by permutation of the univalent vertices. This allows us to regard \(Ξ\) as a \(\mathbb{Z}[S_6]\)-module. To prevent confusion, we use a dot to indicate this operation (so \(σ \in \mathbb{Z}[S_6]\) applied to \(ξ \in Ξ\) will be written as \(σ · ξ\)). Note that in general \(σ · ξ_1 ξ_2 \neq (σ · ξ_1)ξ_2\).

The elementary transpositions in \(S_6\) will be named \(τ_i := (i \ i+1)\). Obviously, \(τ_i · ξ = s_i ξ, τ_2 · ξ = s_2 ξ, τ_4 · ξ = s_8 ξ, \) and \(τ_5 · ξ = ξ s_1\). Furthermore let \(μ := (1 \ 6)(2 \ 5)(3 \ 4)\), then \(μ\) operates by mirroring along the \(y\)-axis (there is always an even number of trivalent vertices, so \(A S\) causes no change of sign). For abbreviation we introduce the symbol \([s]_*^i\):

\[ [ξ]^i_1 := (1 - μ)(1 + τ_1)(1 + τ_3)(1 + τ_5) · (c_1 ξ c_1^i) \quad \text{for any } ξ \in Ξ \text{ and } i, j \geq 0. \]

The maps \(ξ \to [ξ]^i_j\) are well-defined \(Q\)-endomorphisms of \(Ξ\). Obviously \([c_1 ξ]^i_j = [ξ]^i_{j+1}\) and \([ξ^j_1]^i_j^1 = [ξ]^j_{i+1}\). A useful feature of this notation is that whenever \(μ · w = w\) then \([z^k w]^i_j = 0\). For instance, this is the case if \(w\) is a palindrom in the letters \(s_*, c_*, d_*\). Other pleasing properties of \([s]_*^i\) are given in the following proposition.

Proposition 4.2 For any \(ξ \in Ξ, i, j \geq 0\) we have

\[
[ξ]^i_1 = -[μ · ξ]^i_j = [s_1 ξ]^i_1 = [τ_3 · ξ]^i_1 = [ξ s_1]^i_1
\]

(17)

\[
c_1 [ξ]^i_1 c_1 = [ξ]^i_{j+1}
\]

(18)

\[
c_1 [ξ]^i_j c_1 + [ξ]^i_{j+1} c_1 = [ξ]^i_{j+1} + [ξ]^i_{j+1}. \quad \text{(19)}
\]

Moreover, if \(μ · ξ = ξ\) and \([ξ]^0_0 = 0\), then \([ξ]^i_0 = 0\) for all \(i, j \geq 0\).

Proof The last three factors of \(π := (1 - μ)(1 + τ_1)(1 + τ_3)(1 + τ_5)\) commute with each other and their product commutes with \((1 - μ)\). Since \(τ_1^2 = μ^2 = 1\), we have \(π = -π μ = π τ_1 = π τ_3 = π τ_5\), which implies (17). Equality (18) becomes clear if one realises that for \(i \in \{1, 3, 5\}\)

\[
c_1 (τ_i · ξ) = τ_i c_1 ξ, \quad (τ_i · ξ)c_1 = τ_i ξ c_1
\]

\[
c_1 (μ · ξ) = μ · ξ c_1, \quad (μ · ξ)c_1 = μ · c_1 ξ.
\]

Let \(x := (1 + τ_1)(1 + τ_3)(1 + τ_5) · c_1 ξ c_1^i\), then \([ξ]^i_j = x - μ · x\) and

\[
c_1 [ξ]^i_j c_1 = c_1 x - c_1 (μ · x) + xc_1 - (μ · x)c_1 = c_1 x - μ · xc_1 + xc_1 - μ · c_1 x = (1 - μ) · c_1 x + (1 - μ) · xc_1 = [ξ]^i_{j+1} + [ξ]^i_{j+1}.
\]

Finally, the following identity shows by induction that \([ξ]^0_0 = [ξ]^1_0 = 0\) implies \([ξ]^0_0 = 0\) for \(j \geq 2\):

\[
[ξ]^i_0 = [ξ]^0_0 + [ξ]^j_{i-1} - [ξ]^j_{i-1} \overset{(10, 18)}{=} c_1 [ξ]^j_{i-1} - [ξ]^j_{i-1} c_1 - c_1 [ξ]^j_{i-2} c_1
\]

By (18) we obtain \([ξ]^i_j = c_1 [ξ]^j_{i-1} c_1 = 0\) for \(i ≤ j\). If \(i > j\) we have \([ξ]^i_j \overset{(17)}{=} -μ · ξ]^i_j = [ξ]^i_j = 0\).

Definition 4.3 Let \(Ξ_0\) denote the subalgebra of \(Ξ\) that is generated by \{\(c_1, c_2, c_3, d_1, d_2\}\). Let \([Ξ_0]\) denote the subspace of \(Ξ\) that is spanned by \{\([ξ]^i_j \mid ξ \in Ξ_0; i, j ≥ 0\}\).

Remark 4.4 Equations (9) and (17) imply that \([z^k w]^i_j = 0\) if \(w\) begins or ends with \(d_1\). (3), (12), (13) and \(d_1 c_1 = c_1 d_1 = d_2^2 = 0\) allow to eliminate all \(d_1\)-s in the middle of words. Thus \([Ξ_0]\) is spanned by elements of the form \([z^k w]^i_j\) where \(w\) is a word in the letters \(c_1, c_2, d_2\).
Let us do some auxiliary calculations in $[\Xi_0]$ based on Proposition 4.2.

\[
[z^k c_2]_i^{(17)} = \frac{1}{2} [z^k c_2]_i^{(17)} + \frac{1}{2} [z^k s_1 c_2 s_1]_i^{(1)} = \frac{1}{2} [z^k (c_2 + c_3)]_i^{(1)} = \frac{1}{2} [z^k (z - c_1)]_i^{(1)} = 0
\]  

(20)

\[z^2 d_2 c_1 (1 + s_1) \overset{(15)}{=} z^3 d_2 (1 + s_1)\] implies $[z^k d_2]_i^1 = [z^{k+1} d_2]_i^0 = 0$, so by Proposition 4.2

\[
[z^k d_2]_i^j = 0 \quad \text{for } k \geq 2.
\]  

(21)

We can generalise this a little more:

\[
[z^k d_2 w]_i^j = 0 \quad \text{if } k \geq 2 \text{ and } w \text{ is a word in the letters } c_1 \text{ and } c_2.
\]  

(22)

This is shown by induction on the length of $w$. If $w$ is empty, then the statement is (21), otherwise if $w$ begins with $c_2$ then $d_2 w = 0$, so let us assume $w = c_1 u$.

\[
2[z^k d_2 c_1 u]_i^j \overset{(2)}{=} [z^k d_2 c_1 (1 + s_1) u]_i^j + [z^k d_2 d_1 u]_i^j \overset{(15)}{=} [z^{k+1} d_2 u]_i^j + [z^{k+1} d_2 s_1 u]_i^j + [z^{k+1} d_2 d_1 u]_i^j
\]

By (1) and (17), the second term is equal to $[z^{k+1} d_2 u']_i^j$ where $u' = s_1 u s_1$ is obtained by replacing in $u$ every $c_2$ by $z - c_1 - c_2$. If $u$ is empty or begins with $c_1$ then $[z^k d_2 d_1 u]_i^j = 0$, otherwise $u = c_2 v$ and $[z^k d_2 d_1 u]_i^j \overset{(12)}{=} [z^{k+1} d_2 c_1 v]_i^j$. So the induction hypothesis applies to all three terms on the right side of the equation above and we are done.

We are interested in $[\Xi_0]$ because it contains the element $[y]_i^3$ we want to calculate.

\[
[y]_i^{j+1} - [z d_2]_i^{j+1} = [c_1 y c_1 - z c_1 d_2 c_1]_i^j \overset{(17)}{=} -[z c_1 s_1 s_2 d_1 c_2]_i^j \overset{(1)}{=} -[z s_1 s_2 c_3 d_1 c_2]_i^j
\]

\[
\overset{(17)}{=} -[2 c_1 x c_1 - z c_1 d_2 c_1]_i^j - [z s_2 c_2 d_1 c_2]_i^j \overset{(8, 2)}{=} -[2 z d_2 c_1]_i^j - [z c_1 c_2 d_1 c_2]_i^j
\]

\[
\overset{(3, 12)}{=} -[2 d_2]_i^{j+1} + [z c_1 d_2 c_1]_i^j - [z^2 d_2 c_1]_i^j \overset{(21)}{=} [z d_2]_i^{j+1}.
\]

This allows us to express $[y]_i^j$ as element of $[\Xi_0]$:

\[
[y]_i^j = 2[z d_2]_i^j \quad \text{for } i, j \geq 1
\]  

(23)

Next, we show that for $i, j, k \geq 0$ and arbitrary $\xi \in \Xi$, we have

\[
2[\xi c_2]_i^j = [z \xi]_i^{j+2} - [\xi]_i^{j+3}
\]  

(24)

\[
2[\xi d_2 c_1]_i^j = 2[z \xi d_2]_i^{j+1} - [\xi d_2]_i^{j+2}
\]  

(25)

\[
2[\xi c_2 c_1]_i^j = [z \xi c_2]_i^{j+1} - [\xi c_2]_i^{j+2} + [\xi c_1 d_2]_i^{j+1}.
\]  

(26)

Equations (24)-(26) are shown simultaneously:

\[
[u(c_1 - d_1) c_2]_i^j \overset{(2)}{=} [u c_1 s_1 c_2]_i^j \overset{(17)}{=} [u c_1 c_3]_i^j = [u c_1 (z - c_1 - c_2)]_i^j = [z u]_i^{j+1} - [u]_i^{j+2} - [u c_1 c_2]_i^j
\]

This implies $2[u c_1 c_2]_i^j = [z u]_i^{j+1} - [u]_i^{j+2} + [u d_1 c_2]_i^j$. If we set $u = \xi c_1$, $u = \xi d_2$, $u = \xi c_2$ and apply $c_1 d_1 = 0$, $d_2 d_1 c_2 \overset{(12)}{=} z d_2 c_1$, $c_2 d_2 c_2 \overset{(3)}{=} c_1 d_2 c_1$ we obtain (24), (25), (26), respectively.

**Remark 4.5** Using the relations we have found so far, one can show that $[\Xi_0]$ is spanned by the elements of the following forms:

- $[z^k c_2]_i^{j+1}$ with $k \geq 1, i \geq 0$.
- $[z^k d_2]_i^j$ with $k \in \{0, 1\}$ and $0 \leq i < j$.
- $[d_2 c_2]_i^{j+1}$ with $0 \leq i < j$ and $n \geq 1$. 


We only give a few hints how this can be shown. Let \( \sigma_n := \frac{1}{2}(1 + s_1)c_1^n \), then \( \sigma_n \sigma_m = \sigma_{n+m} \) and \( c_1 \sigma_n = \sigma_n c_1 = \sigma_{n+1} \). By (2) and Remark 4.4, \([\Xi_0]\) then is spanned by elements of the form \([z^k \sigma_0 w_1 \sigma_n w_2 \cdots \sigma_{n-1} w_0 \sigma_n]\) with \( n_i \geq 1 \) and \( w_i = d_2 \) or \( w_i = c_2^j \); this will be called normal form from now on. The \( w_i \) are called segments and we define the simplicity of an element in normal form by: \( k + \text{number of } d_2\text{-segments} \). For a linear combination \( x \) of words of the same simplicity we say \( x \sim 0 \) if any element \([z^k u x w_0]\) in normal form can be expressed by a sum of elements with simpler normal forms (i.e. elements that contain more \( z\text{-s} \) or more \( d_2\text{-s} \)). Of course, by \( x \sim y \) we mean \( x - y \sim 0 \). It is not hard to show that

\[
\begin{align*}
\sigma_n c_2 \sigma_m & \sim -\frac{1}{2} \sigma_{n+m+1} \quad (a) \\
\sigma_n c_2^3 \sigma_m & \sim -2 \sigma_n c_2^1 \sigma c_2 \sigma_m \quad (a) \quad \sigma_n c_2^1 \sigma_{m+2} \quad (c) \\
\sigma_n c_2^1 \sigma_{m+1} + \sigma_n c_2^2 \sigma_m & \sim -\frac{5}{4} \sigma_{n+m+3} \\
\sigma_n c_2^2 \sigma_1 c_2 \sigma_m & \sim -\frac{1}{2} \sigma_n c_2^2 \sigma_m \quad (d) \\
\sigma_n c_2 \sigma_1 d_2 \sigma_m & \sim -2 \sigma_n c_2 \sigma d_2 \sigma_m \quad (e)
\end{align*}
\]

Using (a) and (b), we may eliminate all \( c_2^j\text{-segments} \) with \( j \neq 2 \). Due to (c), we can reduce the \( \sigma_n\text{-s} \) between two \( c_2^j\text{-segments} \) or between a \( c_2^j\text{-segment} \) and a \( d_2\text{-segment} \) to become \( \sigma_1 \) and then apply (d) or (e). Like this we are left only with two types of normal forms: the ones that contain only \( d_2\text{-segments} \) and those which consist of a single \( c_2^j\text{-segment} \). In the first case we may eliminate \( d_2 \sigma_{n-1} d_2 \) and \( z d_2 \sigma_{n-1} d_2 \) by (4) and (11). If a normal form contains \( d_2 \sigma_{n-1} d_2 \sigma_{n-2} d_2 \) then after rotating the even ladders with (LS), we may apply the relation (LL) to transform it into \( \frac{1}{2} z^{n+1} \). In the second case we may use (c) to put the \( c_2^j \) in the middle to get \([z^k c_2^j]\) or \([z^k c_2^j]\) at \( k = 0 \) we may apply equation (36). This shows that any element of \([\Xi_0]\) is a linear combination of elements of the types in Remark 4.5.

**Remark 4.6** Using the relations that will be found in the next section, the statement of Remark 4.5 still can be improved a little bit: \([\Xi_0]\) is spanned by \([z^k c_2^i]\), \([d_2]^i\), \([z d_2]^i\), \([z d_2^i]\), \([z d_2^i]\) with \( i, k \geq 1 \).

### 4.3 Additional Relations in \([\Xi_0]\)

Let us consider a family \((H_n)_{n \geq 1}\) of elements of \( \Xi \), where \( H_n \) is the sum of the four possible ways of attaching the lower ends of a \( n\)-ladder to the first or second strand of 1, and the upper ends to the third strand. For example, \( H_3 \) looks like this:

\[
\begin{array}{cccc}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
+ & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
+ & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
+ & \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Let \( R_n := [H_n]_0 \). Applying Lemma 3.1 a) twice, we see that \( H_n c_1 = c_1 H_n \). Obviously \( \mu \cdot H_n = H_n \), so \( (1 - \mu) \cdot H_n c_1 = H_n c_1 - c_1 H_n = 0 \), thus

\[
R_n = 0 \quad (27)
\]

We now present another way to write \( R_n \) as element of \([\Xi_0]\) in order to obtain new relations. Let \( H_{n, i} \) denote the \( i\)-th term of \( H_n \) in the order of the picture above, then \( H_{n, 1} = s_1 H_{n, 1} s_1 \) and \( H_{n, 3} = s_1 H_{n, 3} s_1 \). Thus by (17) we have

\[
R_n = 2[H_{n, 1}]_0 + 2[H_{n, 3}]_0.
\]

\( H_{n, 3} \) contains a \((n + 2)\)-ladder, so by (LS) and (LIHX)

\[
H_{n, 3} = \begin{cases} 
  c_2^{n+2} & \text{if } n \text{ is even} \\
  -c_2^{n+2} - \tau_3 \cdot c_2^{n+2} & \text{if } n \text{ is odd}
\end{cases}
\]
and by (17) we get
\[ 2[H_{n,3}]_0 = (3 \cdot (-1)^n - 1) [c_2^{n+2}]_0. \]

Lemma 3.1 a) implies
\[ H_{n,1} = -c_1 H_{n-1,3} + d_2 l_n + s_2 l_{n+1} \quad \text{for } n \geq 1. \quad (28) \]

Now \([c_2 H_{n-1,3}]_0 = [H_{n-1,3}]_0 = 0\) because of \(\mu \cdot H_{n-1,3} = H_{n-1,3}\). The second and third term of 2\([H_{n,1}]_0\) are according to (28)
\[
2\left[ d_2 l_n \right]_0 = [d_2 l_n (1 + s_1)]_0 = [d_2 d_1 c_2 c_1^{n-1} + d_2 c_3 c_1^n]_0 = [zd_2 c_1^3 + d_2(z - c_1 - c_2) c_1^2]_0 \\
2\left[ s_2 l_{n+1} \right]_0 = 2[ic_1 c_2 d_2 + z c_2^{n+1}]_0 = 2[c_2 c_1 d_2]_0 + 2[z c_2^{n+1}]_0 - 2[c_2^{n+1} - 2c_2^{n+2}]_0. 
\]

The first equation is valid only for \(n \geq 2\); for \(n = 1\) we use \(l_1 = c_1 c_2\). In the last equation we used \([l_n]_0 = 0\), which is a consequence of \(\tau_3 \cdot l_n = -l_n\) (relation (AS)). Summarising the calculations of this subsection so far, we may state that
\[
R_1 = 2[d_2 c_1 c_2^2]_0 + 2[c_2 c_1 d_2]_0 + 2[z c_2^2]_0 - 2[c_2^2]_0 - 6[c_2]_0 \quad \text{and} \quad (29) \\
R_2 = 2[z d_2]_0 + 2[z c_2^{n+1}]_0 - 2[c_2^{n+1}]_0 + 3((-1)^n - 1)[c_2^{n+2}]_0. \quad (30)
\]

with \(n \geq 2\) are trivial elements of \([\Xi_0]\). In the spirit of Remark 4.5 one is able to present \(R_n\) as linear combination of simple elements. We need this for \(n \leq 3\) which requires a lengthy computation that is done in the appendix. The results are
\[
R_1 = 3[d_2]^2 - 6[z c_2^2]_0 \quad (31) \\
R_2 = 2[d_2 c_1 d_2]_0 - 3[z d_2]^3 \quad (32) \\
R_3 = 4[d_2 c_1^3 d_2]_0 - 6[z d_2]^4. \quad (33)
\]

To simplify the calculation of \(R_2\) and \(R_3\), the following formula has been used
\[
[d_2]^i = 2[z c_2^2]_{i-1}^3 \quad \text{for } i, j \geq 1. \quad (34)
\]

Because of \([c_1 d_2 c_1 - 2z c_2^2]_0 = \frac{1}{3} R_1 \quad (35)\) Proposition 4.2 implies (34).

Now we finally may combine (32), (33) and (23) to obtain the desired result:
\[
0 = \frac{1}{3} R_3 - \frac{2}{3} [c_1 R_2 + R_2 c_1] = \frac{4}{3}[d_2 c_1^3 d_2]_0 - 2[z d_2]^4 - \frac{4}{3}[d_2 c_1 d_2]_0 + 2[z d_2]^3 \quad (36)
\]

4.4 Calculations for parts 2 and 3

Now, having the relation \([y]_1^3 = 0\) in hand, we may stop messing around in \(\Xi\) and proceed with the proof of the square-tunnelling relation. But now that we have gone through so much trouble, we first take profit out of our current knowledge to provide some more equations that are urgently needed in [2] and [3].

We start with the observation that \(\tau_3 \cdot c_2 s_2 c_1 c_2 = 0\):
We obtain for any $i, j \geq 0$

$$
[c_{2i}^4]_{i}^{j+1} = [c_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^j = [(1 + \tau_3)c_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{(17,2)} 2[(c_2 - d_2)c_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{(26,25)} [c_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{j+1} - [c_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{j+2} + [c_1d_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{j+1} - 2[zd_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{j+2} - [d_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{j+2} = [d_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{j+1} + [d_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{j+2} - 2[zd_{2i}^4c_{2i}^4c_{2i}^4c_{2i}^4]^{j+1}
$$

(36)

We use this for $i = 1$ and $j = 1$ to get an expression for $c_{1}^2c_{1}^2c_{1}^2$:

$$
0 = c_{1}(c_{1}^2 [d_{2i}^2] - [d_{2i}^2] - [d_{1}^2] + 2[zd_{2i}^2]^{(12)} \ c_{1}(c_{1}^2 [d_{2i}^2] - [d_{2i}^2] + [y]^{2}) = (1 + \tau_1)(1 + \tau_3)(1 + \tau_5) \ c_{1}^2c_{1}^2c_{1}^2 = \ c_{1}^2c_{1}^2c_{1}^2 = c_{1}^2c_{1}^2c_{1}^2, \text{ we obtain}
$$

$$
c_{1}^2c_{1}^2c_{1}^2 = \frac{1}{4}(1 + \tau_1)(1 + \tau_5) \ c_{1}^2c_{1}^2c_{1}^2c_{1}^2 + c_{1}^2c_{1}^2c_{1}^2c_{1}^2 - c_{1}^2c_{1}^2c_{1}^2c_{1}^2)
$$

(37)

Another relation that will be needed in [3] is the following: ($n \geq 1$)

$$
d_{1}H_{n,1} \overset{(28)}{=} -d_{1}c_{1}H_{n-1,1} + d_{1}d_{2}l_{n} + d_{1}s_{2}l_{n+1}
$$

(38)

Equations (18) and (35) imply $[y]^n + 2 = 0$ in $\hat{F}(6)$. In [2] we need a similar relation in $F(6)$. This means that the whole calculation should be repeated in $F(6)$, i.e. using the following relation instead of (x):

Interpreting $[y]^n$ as element of $F(6)$ we would then obtain a relation of the following form (note that only $x_n$-s with $n \leq 4$ can occur and that $x_1 = 2t, x_2 = t^2, x_4 = \frac{4}{3}t, x_3 = \frac{1}{3}t^4$, see [6]):

$$
[y]^n + 2 = \sum_{i} \lambda_i \xi_i \text{ with } \lambda_i \in \mathbb{Q}[t, x_3] - \mathbb{Q} \text{ and } \xi_i \in \hat{F}(6)
$$

For $n = 2$ we may even take the relation into a form in which all $\xi_i$ are of the form $[y]^3$ (which is not possible for $n = 1$):

$$
3[y]^3 = 9t[y]^3 + 3t[y]^3 - 9t^2[y]^3 + 6t^2[y]^3 + (4t^3 + 2x_3)[y]^3 - 18t^3[y]^3 + 4t(4t^3 - x_3)[y]^3 + 8t^2(x_3 - t^3)[y]^3
$$

(39)

4.5 The Square-Tunnelling Relation

First, we use $\tau_1 \ c_{1}yc_{1}^3 = s_1c_{1}yc_{1}^3 \overset{(2)}{=} c_{1}yc_{1}^3$ and $\tau_5 \ c_{1}yc_{1}^3 = c_{1}yc_{1}^3s_1 \overset{(2)}{=} c_{1}yc_{1}^3$ to reformulate equation (35):

$$
(1 + \tau_3) \ c_{1}yc_{1}^3 = \frac{1}{4}(1 - \mu)(1 + \tau_1)(1 + \tau_3)(1 + \tau_5) \ c_{1}yc_{1}^3 = \frac{1}{4}[y]^3 \overset{(2)}{=} 0
$$

(40)

We turn our attention towards $\hat{F}(n + 4)$. For $n \geq 0$ let $D_{n,1}, D_{n,1}', D_{n,2}$ and $D_{n,2}'$ denote the elements of $\hat{F}(n + 4)$ that are represented by:

Furthermore let $D_n := D_{n,1} - D_{n,2}$ and $D'_n := D'_{n,1} - D'_{n,2}$.
Lemma 4.7 $D_n = 0$ for all $n \geq 0$.

**Proof** The assertion is true for $n = 0$, since $D_{0,1} = D_{0,2}$. We proceed by induction and assume that $n \geq 1$ and $D_i = 0$ for $i < n$. We glue a line with $n$ legs with labels from 1 to $n$ between the ends 3 and 4 of $c_1yc_1^3$ and rename the ends 1, 2, 4 to $n+1$, $n+2$, $n+3$, $n+4$; the result is $D_{n,1}$:

If we do the same with $\tau_3 \cdot c_1yc_1^3$ and apply (AS) at the $n$ legs, we get $(-1)^n D'_{n,1}$.

A similar statement can be done for $c_1yc_1^3$, $\tau_3 \cdot c_1yc_1^3$ and $D_{n,2}$, $(-1)^n D'_{n,2}$, respectively, thus equation (40) implies:

$$D_n + (-1)^n D'_n = D_{n,1} - D_{n,2} + (-1)^n D'_{n,1} - (-1)^n D'_{n,2} = 0. \quad (41)$$

By permuting neighbouring ends of both ladders, we get the following representation of $D'_{n,1}$:

By Lemma 3.1 a) we can push all $n$ legs off the lower strand, one after another. Let us push them all to the right, then we get $3^n$ terms which have legs at the edges indicated by the letters A, B, C in the picture above. The coefficient of each term is $(-1)^n$ number of legs in position A.

We do the same trick for $D'_{n,2}$ and get a similar sum with the 2-ladder and the 4-ladder exchanged. If there are legs in position B or C, then there must be less than $n$ legs between the two ladders. By induction assumption all these terms in $D'_{n,1}$ are equal to those in $D'_{n,2}$. So in $D'_n = D'_{n,1} - D'_{n,2}$ only the two terms having all $n$ legs in position A remain. The process reverses the order of the legs, so the remaining two diagrams are $D_{n,1}$ and $D_{n,2}$:

$$D'_n = (-1)^n D_{n,1} - (-1)^n D_{n,2} = (-1)^n D_n \quad (42)$$

The equations (41) and (42) imply the assertion of the lemma.

**Proof of Theorem 2** We consider a graph containing two ladders with disjoint interiors. Let us remove the interiors of these ladders and call the components of the remaining graph that contain ends of both ladders joining. A sequence of different consecutive edges from one ladder to the other is called joining path. Remember that the condition of Theorem 2 is satisfied iff

- either there are two or more joining components, or
- there is a single joining component that contains two ends of one of the ladders or
- there is a single joining component that contains a circle.

The last case splits up into two subcases: either there are at least two different paths joining the ladders or there is a unique joining path. So to prove Theorem 2 we have to show that in the following four situations we may exchange the square and the 4-ladder:

a) ![Diagram](a)

b) ![Diagram](b)

c) ![Diagram](c)

d) ![Diagram](d)

In situation a), we choose a joining path $p$ and push all edges arriving at $p$ by Lemma 3.1 a) through one of the ladders, as indicated by the little arrow. We can do the same for the graph in
which both ladders have been exchanged. Both times we get a linear combination of diagrams in which \( p \) has become a single edge. The only difference is that in each term the ladders have been exchanged. By Lemma 4.7 the two expressions are identical.

In situation b) we have two joining paths \( p_1 \) and \( p_2 \) that meet each other and then coincide. We first push away all edges arriving at the common part of \( p_1 \) and \( p_2 \), to get diagrams in which \( p_1 \) and \( p_2 \) have only an edge \( e \) in common (\( e \) is the end of one of the ladders \( L \)). By Lemma 4.7 a) we may express each of these diagram by the three graphs we obtain if we cut off \( p_2 \) from \( e \) and glue it to one of the three other ends of \( L \). So we may express any graph in situation b) by a sum of graphs of situation a).

Exactly the same trick allows to reduce case c) to b). Finally in situation d), we have a joining path \( p_1 \), a circle \( l \) and a path \( p_2 \) connecting \( l \) and \( p_1 \). We assume that \( p_2 \) is a single edge (otherwise proceed as above to clean \( p_2 \) as indicated by the arrow) and apply (IHX) at \( p_2 \). The result is a difference of two graphs of type c). This completes the proof of Theorem 2.

Proof of Theorem 3  The univalent vertices in the following pictures are marked by bullets. If a graph in \( B^n \) has weakly connected ladders of lengths 2 and 4, it typically looks like this (the omitted parts of the graph are supposed to be located inside the boxes):

Since all univalent vertices carry the same colour, the (AS)-relation implies:

\[
\begin{array}{c}
\text{Lemma 3.1 a) yields} \\
= 0 \\
\end{array}
\]

if the box contains no univalent vertices. So we assume that there is at least one univalent vertex inside each box and consider diagrams that look like this:

As before, we push all disturbing edges one by one through the ladders to obtain graphs that are trivial because of (\( \ast \)), graphs that are of type b) or d) (in the proof of Theorem 2) and graphs that look like this:

In the latter case, we push away all legs between the ladders. Because of (\( \ast \)) we only have to consider the terms in which all these legs end up in the two boxes. Thus to prove the theorem, it remains to show that we may exchange ladders in graphs of this type:

This is done by gluing univalent vertices of the same colour to the ends 3 and 4 in equation (40) so that \( \xi \) and \( \tau_3 \cdot \xi \) become the same element.
So in the case of unicoloured univalent vertices we may drop the somehow artificial condition of strong connectivity and state that the square-tunnelling relation holds in all connected graphs. □

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Appendix

Here we give a detailed calculation that allows to express the elements $R_1$, $R_2$ and $R_3$ (given by (29) and (30) in section 4.3) as linear combination of basic elements (according to Remark 4.5). In every step, we use one of the following equations $(i,j)$ in (29) and (30) as linear combination of basic elements (according to Remark 4.5).

$$[u c_i^j v_i^j] = \frac{1}{2}[u v_i^2]_{i+2} - \frac{1}{2}[u v_i^1]_{i+3}$$  \hspace{1cm} (A)

$$[u d_i^j v_i^j] = [u d_i^1]_{i+1} - \frac{1}{2}[u d_i^2]_{i+2}$$  \hspace{1cm} (B)

$$[u c_i^j c_j^i] = \frac{1}{2}[u c_i^2]_{i+1} - \frac{1}{2}[u c_j^2]_{i+1} + \frac{1}{2}[u c_i^1]_{i+1}$$  \hspace{1cm} (C)

$$[u c_i^j v_i^j] = [u c_i^2]_{i+1} - 2[u c_i^1]_{i+1} + [u c_i^1]_{i+1}$$  \hspace{1cm} (D)

$$[u c_i^j c_j^i] = 2[u c_i^1]_{i+1} - 2[u c_i^1]_{i+1} + [u c_i^2]_{i+1}$$  \hspace{1cm} (E)

$$[u c_i^j v_i^j] = [u c_i^2]_{i+1} - 2[u c_i^1]_{i+1} + [u c_i^1]_{i+1}$$  \hspace{1cm} (F')

The equations (A)-(F) are (24), (25), (26), (14), (16), (34); (C') and (D') are equivalent to (C) and (D). Note that equation (F) is not used in the calculation of $R_1$, because the value of $R_1$ is an essential ingredient in the proof of (34).

Elements of the form $[z^k w_i^j]$ (where $w$ is a word in the letters $c_1, c_2, d_2$) are trivial in each of the following cases: (see (11), (20), (22), (4))

a) $w$ contains $d_2 c_2$ or $c_2 d_2$ or $d_2 c_1 d_1$ as subword,

b) $w = c_2$.

c) $k \geq 2$ and $w = d_2$ or $w = d_2 c_1 e_2^2$.

d) $w = d_2 c_1 c_2 d_2$.

e) $k \geq 1$ and $w = d_2 e_2^2 d_2$.

f) $w$ is a palindrome and $i = j$.

The basic elements are named $c_1, \ldots, e_k$. In the calculations each term $t$ of the left side of equations (A) to (F) has one of the following three properties:

- $t$ or $\mu \cdot t$ is one of the basic elements, or
- $t$ or $\mu \cdot t$ belongs to one of the classes a) to f) and is henceforth 0, or
- $t$ or $\mu \cdot t$ has been calculated in a previous equation.

We indicate the second case by writing $0_x$ (x one of the letters a to f), and the third case by the equation number in brackets. If $\mu \cdot t$ (the reversed word) instead of $t$ satisfies the condition, the sign of the coefficient of $t$ has to be changed according to (17):

$$[w_i^j] = -\mu \cdot [w_i^j] = -[\mu \cdot w_i^j].$$

In each step the result is a linear combination of basic elements which is written as $k$-dimensional row-vector.
Calculation of $R_1$

Basic elements: $e_1 = [z \, c_2^2]_1^1$, $e_2 = [d_2]_1^3$, $e_3 = [d_2]_1^2$, $e_4 = [z \, d_2]_1^0$.

$$[d_2^2 c_1^2 c_2^0]_0 = \frac{1}{2} [z \, d_2]_1^0 - \frac{1}{2} [d_2]_1^3$$

$$z \, d_2 c_1^2 c_2^0 = 0 - \frac{1}{2} e_4 = (0, -\frac{1}{2}, 0, \frac{1}{2})$$  \(1\)

$$[z \, d_2 c_1^2 d_2^0]_0 = \frac{1}{2} [z \, d_2]_1^0 - \frac{1}{2} [z \, d_2]_1^2$$

$$z \, d_2 c_1^2 d_2^0 = 0 - \frac{1}{2} e_4 = (0, 0, 0, -\frac{1}{2})$$  \(2\)

$$[c_2^2 c_1 d_2^0]_0 = 2 [z \, c_2^2 c_1 d_2^0] - 2 [c_2^2 c_2^2]_0 + [d_2^2 c_2^2]_0$$

$$c_2^2 c_1 d_2^0 = -2(2) + 2(1) + 0 = (0, -1, 0, 2)$$  \(3\)

$$[c_2^0 c_1 c_2^2]_1 = \frac{1}{2} [z \, c_2]_1^0 - \frac{1}{2} [c_2]_1^3 + \frac{1}{2} [d_2]_1^3$$

$$c_2^0 c_1 c_2^2 = 0 + 0 + \frac{1}{2} e_3 = (0, 0, \frac{1}{2}, 0)$$  \(4\)

$$[c_2^0 c_1 c_2^2]_1 = \frac{1}{2} [z \, c_2]_1^0 - \frac{1}{2} [c_2]_1^3 + \frac{1}{2} [d_2]_1^3$$

$$c_2^0 c_1 c_2^2 = 0 + 0 + \frac{1}{2} e_3 = (0, 0, \frac{1}{2}, 0)$$  \(5\)

$$[d_2 c_1 c_2^1]_1 = [z \, d_2]_1^1 - \frac{1}{2} [d_2]_1^3$$

$$d_2 c_1 c_2^1 = e_4 - \frac{1}{2} e_2 = (0, -\frac{1}{2}, 0)$$  \(6\)

$$[d_2 c_1 c_2^1]_0 = [z \, d_2]_1^1 - \frac{1}{2} [d_2]_1^3$$

$$d_2 c_1 c_2^1 = e_4 - \frac{1}{2} e_2 = (0, -\frac{1}{2}, 0)$$  \(7\)

$$[c_2^0]_0 = [z \, c_2]_1^1 - 2 [c_2^2 c_1 c_2^0] + [c_2^2 c_1 d_2^0]$$

$$c_2^0 = e_1 - 2(5) - (7) = (1, 1, \frac{1}{2}, -2)$$  \(8\)

$$[c_2^0]_0 = [z \, c_2]_1^1 - 2 [c_2^2 c_1 c_2^0] + [c_2^2 c_1 d_2^0]$$

$$c_2^0 = e_1 - 2(4) + (6) - (7) = (1, -\frac{1}{2}, \frac{1}{2}, 1)$$  \(9\)

$$R_1 = 2 [d_2 c_1 c_2^1]_1 + 2 [c_2^2 c_1 d_2^0] + 2 [z \, c_2]_1^1 - 2 [c_2^0]_0 - 6 [c_2^0]_0$$

$$R_1 = 2(6) - 2(7) + 2e_1 - 2(8) - 6(9) = (-6, 0, 3, 0)$$
Calculation of $R_2$

We use equation (F) at one place (line (9)). The basic elements are:

\[ e_1 = [z^2 c_2^1]_{10}, \quad e_2 = [d_2^4]_{10}, \quad e_3 = [z d_2^3]_{10}, \quad e_4 = [z d_2^2]_{11}, \quad e_5 = [d_2^4 c_2]_{10}. \]

\[
[d_2 c_1 c_2]_{10} \equiv \frac{1}{2} [z^2 c_2^1]_{10} - \frac{1}{2} [z d_2^3]_{10} = 0 e - \frac{1}{2} e_3 = (0, 0, -\frac{1}{2}, 0, 0) \tag{1}
\]

\[
[z c_2^2 c_1 d_2]_{10} \equiv \frac{1}{2} [z^2 c_2^1 c_2]_{10} - 2 [z c_2 c_1 d_2]_{10} + [z d_2 c_2^2]_{10} = 0 e + 2 (1) + 0 e = (0, 0, 0, -1, 0) \tag{2}
\]

\[
[z c_2^3 c_1 c_2]_{10} \equiv \frac{1}{2} [z^2 c_2^3]_{10} - \frac{1}{2} [z c_2^2 c_1 c_2]_{10} + \frac{1}{2} [z c_2 d_2 c_1 c_2]_{10} + \frac{1}{2} [z c_2^3 c_1 d_2]_{10} = 0 e + 0 a + \frac{1}{2} (2) = (0, 0, -\frac{1}{2}, 0, 0) \tag{3}
\]

\[
[d_2 c_1 c_2]_{10} \equiv [z d_2^1]_{2} - \frac{1}{2} [d_2]_{1}^2 = -e_4 + 0 e = (0, 0, 0, -1, 0) \tag{4}
\]

\[
[d_2 c_1 c_2]_{10} \equiv [z d_2^3]_{10} - \frac{1}{2} [d_2]_{1}^1 = e_3 - \frac{1}{2} e_2 = (0, 0, 0, 1, 0, 0) \tag{5}
\]

\[
[z d_2 c_1 c_2]_{10} \equiv \frac{1}{2} [z^2 d_2^3]_{10} - \frac{1}{2} [z d_2]_{1}^3 = 0 e - \frac{1}{2} e_3 = (0, 0, 0, -1, 0) \tag{6}
\]

\[
[z c_2 c_1 c_2]_{10} \equiv \frac{1}{2} [z^2 c_2^3]_{10} - \frac{1}{2} [z c_2 c_1 c_2]_{10} + \frac{1}{2} [z c_2 d_2 c_1 c_2]_{10} = 0 e + 0 b + \frac{1}{2} e_4 = (0, 0, 0, \frac{1}{2}, 0) \tag{7}
\]

\[
[z d_2 c_1 c_2]_{10} \equiv [z d_2^1]_{1} - \frac{1}{2} [z d_2]_{2}^1 = 0 e - \frac{1}{2} e_4 = (0, 0, 0, 0, -\frac{1}{2}, 0) \tag{8}
\]

\[
\frac{1}{2} [d_2]_{1}^1 \equiv [z c_2^3]_{10} \equiv \frac{1}{2} [z^2 c_2^1 c_2]_{10} - 2 [z c_2 c_1 d_2]_{10} + [z d_2 c_2 c_1]_{10} = e_1 - 2 (3) - (8) = (1, 0, 1, \frac{1}{2}, 0, 0) \tag{9}
\]

\[
[c_2 c_1 c_2]_{10} \equiv \frac{1}{2} [z c_2]_{1}^3 - \frac{1}{2} [c_2]_{1}^3 + \frac{1}{2} [d_2]_{1}^3 = 0 e + 0 b + (9) = (1, 0, 1, \frac{1}{2}, 0) \tag{10}
\]

\[
[d_2^2 c_1 c_2]_{10} \equiv \frac{1}{2} [z d_2]_{1}^3 - \frac{1}{2} [d_2]_{1}^3 = \frac{1}{2} e_4 - (9) = (-1, 0, -1, 0, 0) \tag{11}
\]

\[
[c_2^2 c_1 d_2]_{10} \equiv 2 [z c_2 c_1 d_2]_{1}^1 - 2 [c_2 c_1 d_2]_{1}^1 + [d_2 c_2^2 d_2]_{10} = -2 (8) + 2 (11) + e_5 = (-2, 0, -2, 1, 1) \tag{12}
\]

\[
[z c_2^3]_{10} \equiv [z^2 c_2]_{1}^3 - 2 [z c_2 c_1 c_2]_{10} + [z d_2 c_1 c_2]_{10} + [z c_2 c_1 d_2]_{10} = e_1 - 2 (7) + (6) - (8) = (1, 0, -1, 1, 0) \tag{13}
\]

\[
[c_2^3 c_1 d_2]_{10} \equiv \frac{1}{2} [z^2 c_2]_{1}^3 - \frac{1}{2} [c_2 c_1 c_2]_{10} + [d_2^2 c_1 c_2]_{10} + [c_2 c_1 d_2]_{10} = (9) - 2 (10) + (5) - (4) = (-1, -1, 0, \frac{1}{2}, 0) \tag{14}
\]

\[
R_2 = 2 [z d_2]_{10}^3 - [d_2]_{10}^3 + 2 [c_2^2 c_1 d_2]_{10} + 2 [z c_2^2]_{10} - 2 [c_2^3]_{10} = 2 e_3 - e_2 + 2 (12) + 2 (13) - 2 (14) = (0, 0, -3, 0, 2)
\]
Calculation of $R_3$

Relation (F) will be used in line (24); furthermore it allows to use the basic element $e_1$ in two different forms:

$$e_1 = [z c_2]_1^2 = \frac{1}{2}[d_2]_2^3, \quad e_2 = [z^3 c_2]_1^2, \quad e_3 = [d_2]_0^5,$$

$$e_4 = [z d_2]_1^4, \quad e_5 = [z d_2]_3^3, \quad e_6 = [d_2 c_1^2 d_2]_0^6.$$

$$[d_2 c_1 c_2]_0^3 \equiv [z d_2]_0^4 - \frac{1}{2}[d_2]_0^5 = e_4 - \frac{1}{2} e_3 = (0, 0, 0, 0, 0, 0) \quad (1)$$

$$[d_2 c_1 c_2]_2^1 \equiv [z d_2]_2^4 - \frac{1}{2}[d_2]_2^3 = 0 - e_1 = (-1, 0, 0, 0, 0, 0) \quad (2)$$

$$[c_2 c_1 c_2]_1^2 \equiv \frac{1}{2}[z c_2]_1^3 - \frac{1}{2}[c_2]_1^4 + \frac{1}{2}[d_2]_2^3$$

$$0_b + 0_a + e_1 = (1, 0, 0, 0, 0, 0) \quad (3)$$

$$[z^2 c_2 c_1 c_2]_0^0 \equiv \frac{1}{2}[z^3 c_2]_0^3 - \frac{1}{2}[z^2 c_2]_0^3 + \frac{1}{2}[z^2 c_2 d_2 c_1 c_2]_0^0 + \frac{1}{2}[z^2 c_2 c_1 d_2]_0^0$$

$$= 0 + 0 + 0 + 0 = (0, 0, 0, 0, 0, 0) \quad (4)$$

$$[z d_2 c_1^2 c_2]_1^0 \equiv \frac{1}{2}[z^2 c_2]_1^3 - \frac{1}{2}[z d_2]_1^3$$

$$0_c - \frac{1}{2} e_5 = (0, 0, 0, 0, 0) \quad (5)$$

$$[z^2 c_2 c_1 c_2]_1^1 \equiv \frac{1}{2}[z^3 c_2]_1^3 - \frac{1}{2}[z^2 c_2]_0^3 + \frac{1}{2}[z^2 d_2]_0^3$$

$$= 0 + 0 + 0 = (0, 0, 0, 0, 0, 0) \quad (6)$$

$$[z d_2 c_1 c_2]_1^1 \equiv [z^2 d_2]_1^3 - \frac{1}{2}[z d_2]_1^3$$

$$0_c - \frac{1}{2} e_5 = (0, 0, 0, 0, 0) \quad (7)$$

$$[z d_2 c_1 c_2]_1^0 \equiv [z^2 d_2]_1^3 - \frac{1}{2}[z d_2]_0^3$$

$$0_c - \frac{1}{2} e_4 = (0, 0, 0, 0, 0) \quad (8)$$

$$[z d_2 c_1 c_2]_2^0 \equiv [z^2 d_2]_2^3 - \frac{1}{2}[z d_2]_2^3$$

$$0_c + 0_t = (0, 0, 0, 0, 0, 0) \quad (9)$$

$$[z c_2 c_1 c_2]_0^2 \equiv \frac{1}{2}[z^2 c_2]_0^3 - \frac{1}{2}[z c_2]_0^3 + \frac{1}{2}[z d_2]_1^3$$

$$= 0 + 0 + 0 = (0, 0, 0, 0, 0) \quad (10)$$

$$[z^2 c_2 c_1 c_2]_0^0 \equiv [z^3 c_2]_0^3 - 2[z^2 c_2 c_1 c_2]_0^0 + [z^2 c_2 c_4 d_2]_0^1$$

$$e_2 - 2(4) + 0_c = (0, 1, 0, 0, 0, 0) \quad (11)$$

$$[d_2 c_1^2 c_2]_0^1 \equiv \frac{1}{2}[z d_2]_0^5 - \frac{1}{2}[d_2]_0^5$$

$$\frac{1}{2} e_4 - \frac{1}{2} e_3 = (0, -\frac{1}{2}, \frac{1}{2}, 0, 0) \quad (12)$$
\[ [z \, d_2 \, c_1 \, c_2]_0^2 \quad \text{(B)} \quad [z^2 \, d_2^3]_0^2 - \frac{1}{2}[z \, d_2]_0^4 = 0_c - \frac{1}{2}e_4 = (0, 0, 0, \frac{1}{2}, 0, 0) \quad (13) \]

\[ [d_2 \, c_1 \, c_2 \, c_2]_0^2 \quad \text{(C)} \quad \frac{1}{2}[z \, d_2 \, c_1 \, c_2]_0^3 - \frac{1}{2}[d_2 \, c_1 \, c_2]_0^3 + \frac{1}{2}[d_2 \, c_2^3]_0^3 \]

\[ = \frac{1}{2}(13) - \frac{1}{2}(1) + \frac{1}{2}e_6 = (0, 0, 0, 0, \frac{1}{2}, 0, -1, 0) \quad (14) \]

\[ [c_2^2 \, c_1 \, d_2]_0^2 \quad \text{(B)} \quad 2[z \, c_2 \, c_1 \, d_2]_0^3 - 2[c_2 \, c_2^2 \, d_2]_0^2 + [d_2 \, c_2^2]_0^2 = -2(13) + 2(12) - e_6 = (0, 0, 0, 0, 0, 0) \quad (15) \]

\[ [z \, c_2^3]_0^2 \quad \text{(B)} \quad [z^2 \, c_2^3]_0^2 - 2[z \, c_2 \, c_1 \, c_2]_0^3 + [z \, d_2 \, c_1 \, c_2]_0^3 + [z \, c_2 \, c_1 \, d_2]_0^3 = (11) - 2(10) + (13) - (9) = (0, 0, 0, 0, 0, 0) \quad (16) \]

\[ [z \, c_2^3 \, c_1 \, d_2]_0^2 \quad \text{(B)} \quad 2[z^2 \, c_2 \, c_1 \, d_2]_0^3 - 2[z \, c_2 \, c_1^2 \, d_2]_0^3 + [z \, d_2 \, c_2^2 \, d_2]_0^3 = 0_c + 2(8) + 0_e = (0, 0, 0, 0, 0) \quad (17) \]

\[ [d_2 \, c_2 \, c_2]_0^2 \quad \text{(D)} \quad [z \, d_2 \, c_2 \, c_2]_0^3 - 2[d_2 \, c_1 \, c_2 \, c_2]_0^3 + [d_2 \, c_1 \, d_2 \, c_2]_0^3 + [d_2 \, c_1 \, c_2 \, c_1]_0^3 = -(17) - 2(14) + 0_c + 0_a = (0, 0, 0, 0, 0) \quad (18) \]

\[ [z \, c_2 \, c_1 \, c_2]_0^2 \quad \text{(C)} \quad \frac{1}{2}[z \, c_2^3]_0^2 - \frac{1}{2}[z \, d_2]_0^3 + \frac{1}{2}[z \, c_2 \, c_1]_0^2 = 0_c - \frac{1}{2}e_1 - \frac{1}{2}(7) = (-1, 0, 0, 0, 0) \quad (19) \]

\[ [z \, c_2^3]_0^2 \quad \text{(D)} \quad [z^2 \, c_2^3]_0^2 - 2[z \, c_2 \, c_2]_0^3 + [z \, d_2 \, c_2 \, c_2]_0^3 + [z \, c_2 \, c_2 \, c_2]_0^3 = e_2 - 2(6) + 0_c + e_6 = (0, 1, 0, 0, 0) \quad (20) \]

\[ [z \, c_2^3 \, c_1 \, d_2]_0^2 \quad \text{(B)} \quad 2[z^2 \, c_2 \, c_1 \, d_2]_0^3 - 2[z \, c_2 \, c_1^2 \, d_2]_0^3 + [z \, d_2 \, c_2^2 \, d_2]_0^3 = 0_c + 2(5) + 0_e = (0, 0, 0, 0, 0) \quad (21) \]

\[ [z \, c_2^4]_0^2 \quad \text{(D)} \quad [z^2 \, c_2^4]_0^2 - 2[z \, c_2 \, c_2^2]_0^3 + [z \, d_2 \, c_2 \, c_2]_0^3 + [z \, c_2 \, c_2 \, c_2]_0^3 = (20) + 2(19) - (17) + 0_a = (-1, 0, 0, 1, 0, 0) \quad (22) \]

\[ [z \, c_2^3 \, c_2 \, c_2]_0^2 \quad \text{(C)} \quad \frac{1}{2}[z \, c_2^3]_0^2 - \frac{1}{2}[z \, c_2^3]_0^2 + \frac{1}{2}[z \, c_2 \, d_2 \, c_1]_0^2 + \frac{1}{2}[z \, c_2^3 \, c_1]_0^2 \]

\[ = \frac{1}{2}(20) - \frac{1}{2}(22) + 0_a + \frac{1}{2}(21) = (\frac{1}{2}, 0, 0, 0, 0, 0) \quad (23) \]

\[ \frac{1}{2}[d_2^2]_0^2 \quad \text{(C)} \quad [z \, c_2^3]_0^2 - 2[z \, c_2 \, c_2]_0^3 + [z \, c_2 \, c_1]_0^2 = (11) - 2(23) - (9) = (-1, 1, 0, 0, 0) \quad (24) \]

\[ [d_2 \, c_2^3 \, c_2]_0^2 \quad \text{(A)} \quad \frac{1}{2}[z \, d_2]_0^3 - \frac{1}{2}[d_2]_0^4 = \frac{1}{2}e_5 - (24) = (1, -1, 0, 0, 0, 0) \quad (25) \]

\[ [d_2 \, c_2 \, c_2]_0^2 \quad \text{(B)} \quad [z \, d_2]_0^3 - \frac{1}{2}[d_2]_0^4 = e_5 - (24) = (1, -1, 0, 0, 0, 0) \quad (26) \]

\[ [c_2^2 \, c_1 \, d_2]_0^2 \quad \text{(B)} \quad 2[z \, c_2 \, c_1 \, d_2]_0^3 - 2[c_2 \, c_2^2 \, d_2]_0^3 + [d_2 \, c_2^2 \, d_2]_0^3 = -2(7) + 2(25) + 0_t = (2, -2, 0, 0, 0) \quad (27) \]
\[
[c_2^3]_1^2 = (\alpha^2) - 2[c_2c_1c_2]_1^2 + [d_2c_1c_2]_1^2 + [c_2c_1d_2]_1^2
\]
\[
= e_1 - 2(3) + (26) - (2) = (1, -1, 0, -1, -\frac{1}{2}, 0) \quad (28)
\]
\[
[c_2^3c_1c_2]_0^3 = (\alpha^3) - \frac{1}{2}[c_2^3]_1 + \frac{1}{2}[c_2^3c_1d_2]_1
\]
\[
= 0 - \frac{1}{2}(28) + \frac{1}{2}(27) = (\frac{1}{2}, \frac{1}{2}, 0, -\frac{1}{2}, 0, 0) \quad (29)
\]
\[
[c_2^3c_1c_2]_2^2 = (\alpha^2) - \frac{1}{2}[c_2^3]_1 + \frac{1}{2}[c_2^3c_1d_2]_1
\]
\[
= -\frac{1}{2}e_1 + 0t - \frac{1}{2}(26) = (-1, \frac{1}{2}, 0, \frac{1}{2}, 0, 0) \quad (30)
\]
\[
[d_2c_1c_2c_2]_1^1 = (\alpha^1) - \frac{1}{2}[d_2c_1c_2]_1 - \frac{1}{2}[d_2c_1c_2]_1^2 + \frac{1}{2}[d_2c_1c_2]_1^1
\]
\[
= \frac{1}{2}(7) - \frac{1}{2}(26) + 0t = (-\frac{1}{2}, \frac{1}{2}, 0, \frac{1}{2}, 0, 0) \quad (31)
\]
\[
[c_2^3c_1d_2]_h^0 = (\alpha^3) - 2[c_2c_1c_2c_2]_0 + [d_2c_1c_2c_2]_0 + [c_2c_1d_2c_2]_0
\]
\[
= (21) + 2(31) + 0d + 0a = (-1, 1, 0, 1, -1, 0) \quad (32)
\]
\[
[c_2^3]_h^0 = (\alpha^3) - 2[c_2c_1c_2]_0 + [d_2c_1c_2]_0 + [c_2c_1d_2c_2]_0
\]
\[
= (16) + 2(30) - (15) + 0a = (-2, 2, 1, -\frac{3}{2}, -\frac{1}{2}, 1) \quad (33)
\]
\[
[c_2^3]_0^0 = (\alpha^3) - 2[c_2c_1c_2]_0 + [d_2c_1c_2]_0 + [c_2c_1d_2c_2]_0
\]
\[
= (22) + 2(29) + (18) + 0a = (0, 0, -\frac{1}{2}, \frac{1}{2}, 0, -1) \quad (34)
\]
\[
R_3 = 2[d_2]_h^0 - [d_2]_h^0 + 2[c_2c_1c_2]_0 + 2[c_2^3]_h^0 - 2[c_2^3]_0^0 - 6[c_2^3]_0^0
\]
\[
= 2e_4 - e_3 + 2(32) + 2(22) + 2(33) - 6(34) = (0, 0, 0, -6, 0, 4)
\]