STABILITY OF THE OPTIMAL VALUES UNDER SMALL PERTURBATIONS OF THE CONSTRAINT SET

DANIEL REEM, SIMEON REICH, AND ALVARO DE PIERRO

Abstract. This paper presents a general and useful stability principle which, roughly speaking, says that given a uniformly continuous function defined on an arbitrary metric space, if we slightly change the constraint set over which the optimal (extreme) values of the function are sought, then these values vary slightly. Actually, this apparently new principle holds in a much more general setting than a metric space, since the distance function may be asymmetric, may attain negative and even infinite values, and so on. Our stability principle leads to applications in parametric optimization, mixed linear-nonlinear programming and analysis of Lipschitz continuity, as well as to a general scheme for tackling a wide class of non-convex and non-smooth optimization problems. We also discuss the issue of stability when the objective function is merely continuous, and the stability of the sets of minimizers and maximizers.

1. Introduction

The issue of stability in optimization problems has both theoretical and practical importance. More precisely, suppose that we are given an optimization problem consisting of a space \( X \), an objective function (a target function) \( f : X \to [-\infty, \infty] \), a constraint set \( \emptyset \neq A \subseteq X \), and various parameters which influence the problem (for example, the parameters may influence the constraint set \( A \), namely \( A = A(t) \) for some fixed parameter \( t \) in a parameter space; another example: parameters which define \( f \), namely there is some function \( g \) of two variables such that \( f(x) = g(x,t) \), \( x \in A \), where \( t \) is a parameter in a parameter space; in both cases one minimizes with respect to \( x \)). A natural question is what happens to the optimal values of the objective function \( f \), as well as to its sets of minimizers and maximizers, when we slightly perturb some of the elements involved in the formulation of the problem. If the optimal values (and perhaps also the sets of minimizers and maximizers) change slightly as a result of the change in the elements of the problem, then the problem exhibits a certain kind of stability. Of course, the type of stability depends on the way in which we measure all the pertinent changes.

Stability is a desired property which ensures that if one is able to control various types of imprecision which are inherent in many optimization problems (such as noise, inexact measurements, representation errors due to real parameters/expressions which are approximated by a finite decimal representation of them, and so on), then the resulting optimal values will not change a lot.

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There is a very long chain of research works that contain results related to the issue of stability in optimization problems, as can be seen by looking at the following (far from being exhaustive) list of references and the references therein: [2, 6, 8, 9, 11, 12, 15–17, 22, 25, 26, 28, 29, 36, 38, 39, 43, 44, 46, 51–54]. These works consider a variety of optimization problems, in various settings (often Euclidean spaces, but sometimes also in infinite-dimensional Banach spaces or other spaces). They establish certain properties related to either the optimal values of the objective function or its sets of minimizers and maximizers, under suitable assumptions on the objective function and the structure of the set of constraints, such as convexity, linear or piecewise linear structure, quadratic structure, compactness, and so on. Frequently the established properties are semicontinuity (lower or upper) and closedness; occasionally (usually in finite-dimensional Euclidean spaces) stronger properties are established, such as continuity, Lipschitz continuity, convexity, and differentiability. The aforementioned results are interpreted as being stability results.

In the above-mentioned cases the perturbations occur in the space of parameters which define either the set of constraints or the objective function. This is one of the reasons why the corresponding research field to which these works belong is traditionally called “parametric optimization” (but they can also be regarded as belonging to “variational analysis” [27, 40]). The parameters themselves are frequently real numbers or vectors in a linear space endowed with a norm or a topology, but sometimes they belong to a different entity. For instance, in [4, 41] the parameter is a probability measure; in [37] the parameter is a pair consisting of a point in a topological space and a probability measure; in [48, 49] the parameter is a sample belonging to a probability space and one discusses various probabilistic types of convergence, as well as convergence in the sense of inner or outer limits (limits which are related to the Kuratowski-Painlevé notion of convergence). For related but somewhat different types of stability, see, for example, [5, 14, 30, 33, 47].

In this paper we discuss a different type of stability, which, in our opinion, is not less natural than other types of stability discussed in the literature. More precisely, after some preliminaries (Section 2) we present in Section 3 our stability principle which, roughly speaking, says the following: given a uniformly continuous function defined on an arbitrary metric space, if we slightly change the constraint set over which the optimal values of the function are sought, where the change is measured with respect to the Hausdorff distance, then the optimal values vary slightly.

One can think of this principle as a continuous dependence result, where the variable is the constraint set, and the resulting continuous functions are the infimum and the supremum of the objective function over the constraint set. This is how we actually formulate the stability principle (Theorem 3.1). The advantage of this point of view is that it enables one to establish other properties of the infimum and supremum functions, such as Lipschitz continuity. Such properties may not have a very intuitive interpretation. Theorem 3.1, which is new to the best of our knowledge, has a potential to extend, and perhaps to re-prove, some of the stability results mentioned earlier (for an illustration, see Section 6). It seems to be especially promising for finite-dimensional normed spaces, manifolds, and the like, since, as is well known, a continuous function defined on a closed and bounded (hence compact) subset of such spaces is uniformly continuous on this subset. Another relevant example is linear...
functions in finite-dimensional normed spaces, or, more generally, continuous linear functionals in arbitrary normed spaces (since, as is well known, they are automatically Lipschitz continuous).

Interestingly, the stability principle holds in a much more general setting than metric spaces, since we assume essentially nothing on the distance function (in particular, it can be negative, infinite, asymmetric, and so on). Thus this “topology-free” stability principle holds a promise to be relevant to many scenarios in the literature in which the distance function is not a metric, as we illustrate below. First example: the distance function is the well-known Bregman divergence/distance \[10\] (see also \[35\] for a recent semi-survey and an extensive re-examination) or one of its generalizations \[31\]; second example: distortion measures, divergences, and distance functions used in information theory, data analysis, data processing, machine learning and the like \[3,7,18,23,24\]; third example: Finsler quasi-metric spaces \[45\]; fourth example: general distance functions which appear in fixed point theory \[19–21,32\]; fifth example: numerous distance functions which appear in many scientific and technological areas \[13\].

The assumptions made in the formulation of Theorem 3.1 are essential, as we illustrate by using some counterexamples. Despite this, we are able to formulate a counterpart to Theorem 3.1, namely Theorem 3.2, in which the uniform continuity can be replaced by mere continuity, provided the optimal values are not finite. In Section 4 we discuss the issue of stability of the sets of minimizers and maximizers, and show that although instability holds in general, a weak stability principle can still be formulated (Theorem 4.1) under some assumptions. In Sections 5–8 we present several applications of Theorem 3.1. One application (Section 5) is in parametric optimization, where we prove a general continuity result related to the so-called “optimal value function”. A second application appears in Section 6, where we apply the result of Section 5 to mixed linear-nonlinear programming. A third application is presented in Section 7, where we obtain a sequence of Lipschitz constants related to certain functions. This latter application has been used recently in \[34\] in the context of estimating the rate of convergence of a certain first order proximal gradient method. A fourth application (Section 8) is a general scheme for tackling a wide class of non-convex and non-smooth optimization problems.

2. Preliminaries

Our setting is an arbitrary nonempty set \(X\) endowed with an arbitrary function \(d : X^2 \to [-\infty, \infty]\). We refer to the set \(X\) as the “space”, to \(d\) as the “distance function” or the “pseudo-distance”, and to \((X, d)\) as the “pseudo-distance space”. Note that \(d\) is, in general, not a metric, for example because it can attain negative and even infinite values, it may be asymmetric, and so on, and hence we use the notion “pseudo-distance space”. We denote by \(2^X\) the set of all subsets of \(X\). Given \(\psi : X \to [-\infty, \infty]\), its effective domain is the set \(\text{dom}(\psi) := \{x \in X : \psi(x) \in \mathbb{R}\}\). Given a point \(x \in X\) and a nonempty subset \(A\) of \(X\), the distance between \(x\) and \(A\) is defined by \(d(x, A) := \inf\{d(x, a) : a \in A\}\). Given \(r > 0\), we denote \(B[A, r] := \{x \in X : d(x, A) \leq r\}\) and refer to this set as the “closed ball of radius \(r\) around \(A\)” (this “ball” always contains \(A\) if \(d\) is a metric, but in general it may be empty). Given another \(\emptyset \neq A' \subseteq X\), we denote by \(d(A, A') := \inf\{d(a, a') : a \in A, a' \in A'\}\) the
standard distance between $A$ and $A'$. We define the asymmetric Hausdorff distance between $A$ and $A'$ by

$$D_{asyH}(A, A') := \sup\{d(a, A') : a \in A\}, \quad (1)$$

and the (symmetric) Hausdorff distance (also called the “Pompeiu-Hausdorff distance”) between $A$ and $A'$ by

$$D_H(A, A') := \max\{D_{asyH}(A, A'), D_{asyH}(A', A)\}. \quad (2)$$

It is immediate from (1) and (2) that both $D_{asyH}(A, A')$ and $D_H(A, A')$ belong to $[-\infty, \infty]$ and that $D_{asyH}(A, A') \leq D_H(A, A')$. Roughly speaking, the Hausdorff distance quantifies the “distance similarity” between $A$ and $A'$. More precisely, if (as it happens, for example, in the real world where $d$ is the usual Euclidean distance) there is a resolution parameter $r > 0$, namely a positive number $r$ with the property that we cannot distinguish, by using distance measurements, between two points $x$ and $y$ in $X$ if $d(x, y) < r$, and if $D_H(A, A') < r$, then we cannot distinguish between $A$ and $A'$ by using distance measurements. Indeed, the inequality $D_H(A, A') < r$ implies that for each $a \in A$, there is $a' \in A'$ such that $d(a, a') < r$ (namely, no point in $A$ can be distinguished from a point in $A'$) and for each $a' \in A'$, there is $a \in A$ such that $d(a', a) < r$ (namely, no point in $A'$ can be distinguished from a point in $A$). We note that in general (if, say, $d(x, y) = \infty$ for all $(x, y) \in X^2$), given $\delta > 0$ and $\emptyset \neq A \subseteq X$, there may not be any $\emptyset \neq A' \subseteq X$ such that $D_H(A, A') < \delta$; if, however, $d$ is a metric, then we always have $0 = D_H(A, A) < \delta$.

In the sequel we consider various continuity notions of real functions. These notions are word for word as in the usual (metric) case, although in the general case their interpretation is less intuitive. More precisely, let $f : X \to \mathbb{R}$ and $\emptyset \neq A \subseteq X$ be given. We say that $f$ is continuous at $x \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $y \in X$ satisfying $d(x, y) < \delta$, we have $|f(x) - f(y)| < \epsilon$. We say that $f$ is continuous on $A$ if it is continuous at each $x \in A$. We say that $f$ is uniformly continuous on $A$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $(x, y) \in A^2$ satisfying $d(x, y) < \delta$, we have $|f(x) - f(y)| < \epsilon$. We say that $f$ is Lipschitz continuous on $A$ with a constant $\Lambda \geq 0$ (or, briefly, that $f$ is $\Lambda$-Lipschitz continuous on $A$) if $|f(x) - f(y)| \leq \Lambda d(x, y)$ for all $(x, y) \in A^2$. Given another pseudo-distance space $(I, d_I)$, a family $(A_t)_{t \in I}$ of nonempty subsets of $X$ and $t_0 \in I$, we say that $\limsup_{t \to t_0} D_{H, X}(A_{t_0}, A_t) \leq 0$ if for every $\epsilon > 0$, there exists $\delta > 0$ such for all $t \in I$ which satisfies $d_I(t_0, t) < \delta$, we have $D_{H, X}(A_{t_0}, A_t) < \epsilon$ (where, for the sake of clarity, we denoted $D_H$ by $D_{H, X}$, namely it is the Hausdorff distance induced by $d$ and not by $d_I$).

We finish this section with a few remarks. First, the fact that the pseudo-distance $d$ may be asymmetric, means that the above-mentioned types of continuity, as well as other notions (such as the distance $d(x, A)$ between some $x \in X$ and $\emptyset \neq A \subseteq X$, as well as the “ball” $B[A, r]$, $r > 0$) will be different if, in the corresponding definitions, we change the order in which we measure the distance between two given points. For example, in the definition of continuity, instead of assuming the inequality $d(x, y) < \delta$ we could have assumed that $d(y, x) < \delta$. Nevertheless, in either case the relevant proofs follow essentially the same reasoning. Second, we could extend our setting and some of the results even further (for example, by assuming that $d : X^2 \to L$, where
$L \neq \emptyset$ is an arbitrary linearly/totally/simply ordered set), but we refrain from doing this here.

3. Stability of the optimal values

In this section we present the stability principle in both its finite and infinite versions. The counterexamples which come afterward show that the assumptions imposed in the formulation of the principle are essential.

**Theorem 3.1. (Stability of the optimal values: the finite case)** Let $(X, d)$ be a pseudo-distance space and let $f : X \to \mathbb{R}$ be given. Let $\text{SUP}_f : 2^X \to [-\infty, \infty]$ and $\text{INF}_f : 2^X \to [-\infty, \infty]$ be defined by $\text{SUP}_f(A) := \sup\{f(a) : a \in A\}$ and $\text{INF}_f(A) := \inf\{f(a) : a \in A\}$ for each $A \in 2^X$, respectively, where we use the usual convention that $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. If $f$ is uniformly continuous in the sense of Section 2, then there is $r > 0$ such that for each $A \in \text{dom}(\text{SUP}_f)$, either $B[A, r] = \emptyset$ or $A' \in \text{dom}(\text{SUP}_f)$ for every $\emptyset \neq A' \subseteq B[A, r]$, and for each $A \in \text{dom}(\text{INF}_f)$, either $B[A, r] = \emptyset$ or $A' \in \text{dom}(\text{INF}_f)$ for every $\emptyset \neq A' \subseteq B[A, r]$. Moreover, $\text{SUP}_f$ is uniformly continuous on the pseudo-distance space $(\text{dom}(\text{SUP}_f), D_H)$, and $\text{INF}_f$ is uniformly continuous on the pseudo-distance space $(\text{dom}(\text{INF}_f), D_H)$. If $f$ is $\Lambda$-Lipschitz continuous for some $\Lambda > 0$ (in the sense of Section 2), then $\text{SUP}_f$ is $\Lambda$-Lipschitz continuous on $(\text{dom}(\text{SUP}_f), D_H)$ and $\text{INF}_f$ is $\Lambda$-Lipschitz continuous on $(\text{dom}(\text{INF}_f), D_H)$.

**Proof.** We only consider the assertions regarding $\text{SUP}_f$, because the assertions regarding $\text{INF}_f$ can be proved in a similar manner (or can be deduced from the assertions regarding $\text{SUP}_f$ by working with $-f$ instead of $f$).

We assume that $f$ is uniformly continuous. Then, given an arbitrary $\epsilon > 0$, there is $\delta > 0$ such that for all $(x, y) \in X^2$ which satisfy $d(x, y) < \delta$, we have $|f(x) - f(y)| < 0.5\epsilon$. Let $A \in \text{dom}(\text{SUP}_f)$ be given. In particular, $A \neq \emptyset$. Let $r$ be any positive number smaller than $\delta$. If $B[A, r] = \emptyset$, then we are done. Otherwise, we need to show that $\text{SUP}_f(A') \in \mathbb{R}$ for all $\emptyset \neq A' \subseteq B[A, r]$. Consider first the case where $A' = B[A, r]$ and suppose to the contrary that $\text{SUP}_f(A') \notin \mathbb{R}$. Let $a' \in A'$ be arbitrary. Since $\text{SUP}_f(A') \geq f(a') > -\infty$ and $\text{SUP}_f(A') \notin \mathbb{R}$, we have $\text{SUP}_f(A') = \infty$. As $\text{SUP}_f(A) \in \mathbb{R}$, it follows that $\text{SUP}_f(A) + 0.5\epsilon < \text{SUP}_f(A')$. The definition of $\text{SUP}_f(A')$ and the fact that $\text{SUP}_f(A') = \infty$ imply that there is some $c' \in A'$ such that $\text{SUP}_f(A) + 0.5\epsilon < f(c')$. The uniform continuity of $f$ implies, in particular, that for each $x \in X$ which satisfies $d(c', x) < \delta$, one has $|f(c') - f(x)| < 0.5\epsilon$. Since $d(c', A) \leq r < \delta$, as follows from the definition of $B[A, r]$, there is a point $x \in A$ such that $d(c', x) < \delta$. By combining the previous inequalities with the obvious inequality $f(x) \leq \text{SUP}_f(A)$, we arrive at the inequality $\text{SUP}_f(A) + 0.5\epsilon < f(c') < f(x) + 0.5\epsilon \leq \text{SUP}_f(A) + 0.5\epsilon$, namely $\text{SUP}_f(A) < \text{SUP}_f(A)$, a contradiction. Therefore we indeed have $\text{SUP}_f(A') \in \mathbb{R}$, that is, $B[A, r] \in \text{dom}(\text{SUP}_f)$. Now let $\emptyset \neq A' \subseteq B[A, r]$ be arbitrary. Fix some $a' \in A'$. As $\text{SUP}(B[A, r]) \in \mathbb{R}$, $A' \subseteq B[A, r]$ and $f(X) \subseteq \mathbb{R}$, we have $-\infty < f(a') \leq \text{SUP}_f(A') \leq \text{SUP}_f(B[A, r]) < \infty$, as required.

Now we prove that $\text{SUP}_f$ is uniformly continuous on its effective domain. In fact, we show that the $\delta$ from the previous paragraph can be associated with $\epsilon$ in the definition of the uniform continuity of $\text{SUP}_f$. Let $A, A' \in \text{dom}(\text{SUP}_f)$ be arbitrary such that $D_H(A, A') < \delta$ (if no such sets exist, then the proof is complete, vacuously).
The definition of \( \text{SUP}_f(A) \) implies that for some \( a \in A \),

\[
\text{SUP}_f(A) - 0.5\varepsilon < f(a).
\]  

(3)

From (2) we have \( d(a, a') \leq D_H(A, A') < \delta \). Hence there is some \( a' \in A' \) such that \( d(a, a') < \delta \). The uniform continuity of \( f \) and (3) imply that \( \text{SUP}_f(A) - 0.5\varepsilon < f(a') + 0.5\varepsilon \). Since obviously \( f(a') \leq \text{SUP}_f(A') \), we obtain \( \text{SUP}_f(A) - \text{SUP}_f(A') < \varepsilon \).

We still need to show that \(-\varepsilon < \text{SUP}_f(A) - \text{SUP}_f(A')\). As \( \text{SUP}_f(A') \in \mathbb{R} \), there is \( a' \in A' \) which satisfies \( \text{SUP}_f(A') - 0.5\varepsilon < f(a') \). Since \( d(a', A) \leq D_H(A, A') < \delta \), there exists some \( a \in A \) such that \( d(a, a') < \delta \). Hence the uniform continuity of \( f \) and previous inequalities imply that \( \text{SUP}_f(A') - 0.5\varepsilon < f(a') < f(a) + 0.5\varepsilon \leq \text{SUP}_f(A) + 0.5\varepsilon \), namely \( \text{SUP}_f(A') < \text{SUP}_f(A) + \varepsilon \). In other words, \(-\varepsilon < \text{SUP}_f(A) - \text{SUP}_f(A')\).

Finally, assume that \( f \) is \( \Lambda \)-Lipschitz continuous for some \( \Lambda > 0 \). Fix some arbitrary \( A, A' \in \text{dom}(\text{SUP}_f) \). We need to show that \( |\text{SUP}_f(A) - \text{SUP}_f(A')| \leq \Lambda D_H(A, A') \). The assertion is obvious if \( D_H(A, A') = \infty \), and so from now on we can assume that \( D_H(A, A') < \infty \). We observe that the case \( D_H(A, A') = -\infty \) is impossible since the fact that \( f \) is \( \Lambda \)-Lipschitz continuous implies that \( 0 \leq |f(x) - f(y)| \leq \Lambda d(x, y) \) for all \( (x, y) \in X^2 \), and so \( d \) and hence \( D_H \) are nonnegative. Thus we can actually assume that \( D_H(A, A') \in [0, \infty) \). Now let \( \varepsilon > 0 \) be arbitrary. The definition of \( \text{SUP}_f(A) \) implies that (3) holds for some \( a \in A \). Since from (2) we have \( d(a, A') < D_H(A, A') + 0.5\varepsilon \), there is some \( a' \in A' \) such that \( d(a, a') < D_H(A, A') + 0.5\varepsilon \). We conclude from (3) and the \( \Lambda \)-Lipschitz continuity of \( f \) that \( \text{SUP}_f(A) - 0.5\varepsilon < f(a) \leq f(a') + \Lambda d(a, a') < f(a') + \Lambda D_H(A, A') + 0.5\varepsilon \). As obviously \( f(a') \leq \text{SUP}_f(A') \), it follows that \( \text{SUP}_f(A) - \text{SUP}_f(A') < \Lambda D_H(A, A') + 0.5\varepsilon \). Since \( \varepsilon \) can be arbitrarily small, we have \( \text{SUP}_f(A) - \text{SUP}_f(A') \leq \Lambda D_H(A, A') \).

It remains to show that \(-\Lambda D_H(A, A') < \text{SUP}_f(A) - \text{SUP}_f(A')\). Because \( \text{SUP}_f(A') \) is a real number, there is \( a' \in A' \) which satisfies \( \text{SUP}_f(A') - 0.5\varepsilon < f(a') \). Since \( d(a', A) < D_H(A, A') + 0.5\varepsilon \), there exists some \( a \in A \) such that \( d(a', a) < D_H(A, A') + 0.5\varepsilon \). Hence the \( \Lambda \)-Lipschitz continuity of \( f \) and previous inequalities imply that

\[
\text{SUP}_f(A') - 0.5\varepsilon < f(a') \leq f(a) + \Lambda d(a', a)
\]

\[
< f(a) + \Lambda (D_H(A, A') + 0.5\varepsilon) \leq \text{SUP}_f(A) + \Lambda D_H(A, A') + 0.5\varepsilon.
\]

As \( \varepsilon \) can be arbitrarily small, we indeed have \(-\Lambda D_H(A, A') \leq \text{SUP}_f(A) - \text{SUP}_f(A')\). 

\( \square \)

Here is an intuitive interpretation of Theorem 3.1. Suppose that we are given a real function \( f \) which is defined on an arbitrary pseudo-distance space \( (X, d) \), and suppose that we know that \( f \) is uniformly continuous on \( X \); given a nonempty subset \( A \) of \( X \), suppose that we perturb \( A \) slightly, where the perturbation is measured with respect to the Hausdorff distance; let \( A' \) be the new subset which is obtained from the original subset \( A \); then the infimum and supremum of \( f \) over \( A \) are, respectively, almost equal to the infimum and supremum of \( f \) over \( A' \); moreover, if \( f \) is known to be Lipschitz continuous, then the perturbation of the optimal values “behaves better”. Note that this result holds even though we assumed essentially nothing regarding \( d \), and hence, in particular, \( d \) does not have to satisfy any type of continuity (but \( f \) should be continuous with respect to \( d \)).
Theorem 3.2. (Stability of the optimal values: the infinite case) Let \((X,d)\) be a pseudo-distance space. Suppose that \(f : X \to \mathbb{R}\) is continuous in the sense of Section 2. Given an arbitrary \(\emptyset \neq A \subseteq X\), if \(\text{SUP}_f(A) := \sup\{f(a) : a \in A\} = \infty\), then for all \(M \in \mathbb{R}\), there exists \(\delta > 0\) such that for each nonempty subset \(A' \subseteq X\) satisfying \(D_{\text{asyH}}(A, A') < \delta\) (in particular, for each \(\emptyset \neq A' \subseteq X\) satisfying \(D_H(A, A') < \delta\)), the following inequality holds:

\[ M < \text{SUP}_f(A'). \tag{4} \]

Similarly, if \(\text{INF}_f(A) := \inf\{f(a) : a \in A\} = -\infty\), then for all \(M \in \mathbb{R}\), there exists \(\delta > 0\) such that for each nonempty subset \(A' \subseteq X\) satisfying \(D_{\text{asyH}}(A, A') < \delta\) (in particular, for each \(\emptyset \neq A' \subseteq X\) satisfying \(D_H(A, A') < \delta\)), one has

\[ \text{INF}_f(A') < M. \tag{5} \]

Proof. We only consider the case of where \(\text{SUP}_f(A) = \infty\), because the proof in the case where \(\text{INF}_f(A) = -\infty\) employs similar arguments (or, alternatively, can be deduced from the first case by taking \(-f\) instead of \(f\)). Let \(M \in \mathbb{R}\) and \(\epsilon > 0\) be arbitrary. The definition of \(\text{SUP}_f(A)\) and the fact that \(\text{SUP}_f(A) = \infty\) imply that there exists a point \(a \in A\) such that

\[ M + \epsilon < f(a). \tag{6} \]

Since \(f\) is continuous on \(X\), it is continuous at \(a\). Thus for the given \(\epsilon\), there exists \(\delta > 0\) such that for every \(x \in X\) satisfying \(d(a, x) < \delta\), we have

\[ f(a) < f(x) + \epsilon. \tag{7} \]

Let \( \emptyset \neq A' \subseteq X\) be arbitrary such that \(D_{\text{asyH}}(A, A') < \delta\). If no such subset \(A'\) exists, then the proof is complete (the assertion holds vacuously). Otherwise, the inequality \(D_{\text{asyH}}(A, A') < \delta\) and (1) imply that \(d(a, A') \leq D_{\text{asyH}}(A, A') < \delta\). Therefore there is a point \(a' \in A'\) such that \(d(a, a') < \delta\). We conclude from (6) and (7) (with \(x := a'\)) that \(M + \epsilon < f(a) \leq f(a') + \epsilon\), namely \(M < f(a')\). Since obviously \(f(a') \leq \text{SUP}_f(A')\), we have \(M < \text{SUP}_f(A')\), that is, (4) holds.

Counterexample 3.3. The uniform continuity assumption on \(f\) in Theorem 3.1 is essential. Indeed, let \(X := \mathbb{R}\) and let \(d\) be the standard absolute value metric; let \(A := \bigcup_{k=2}^\infty [2k, 2k+1]\). For each \(1 < j \in \mathbb{N}\) and \(1 < k \in \mathbb{N}\), let \(J_{j,k}\) be defined by \(J_{j,k} := [2k, 2k + 1 + (1/k)]\) if \(j = k\) and \(J_{j,k} := [2k, 2k + 1]\) if \(j \neq k\). For every \(1 < j \in \mathbb{N}\), let \(A_j := \bigcup_{k=2}^\infty J_{j,k}\). Let \(f : X \to \mathbb{R}\) be defined as follows for each \(t \in X\):

\[
 f(t) := \begin{cases} 
 0, & \text{if } t \in (-\infty, 1], \\
 0, & \text{if } t \in [2k, 2k+1] \text{ for some } k \in \mathbb{N}\setminus\{1\}, \\
 -2kt + 2k(2k+1), & \text{if } t \in [2k+1, 2k+1 + \frac{1}{2k}] \text{ for some } k \in \mathbb{N}\setminus\{1\}, \\
 4kt - 4k(2k+1) - 3, & \text{if } t \in [2k+1 + \frac{1}{2k}, 2k+1 + \frac{1}{k}] \text{ for some } k \in \mathbb{N}\setminus\{1\}, \\
 \frac{kt}{1-k} - \frac{k(2k+2)}{1-k}, & \text{if } t \in [2k+1 + \frac{1}{k}, 2k+2] \text{ for some } k \in \mathbb{N}\setminus\{1\}. 
\end{cases}
\]

Then \(f(t)\) is defined for all \(t \in X\) and \(f\) is continuous, but not uniformly continuous on \(X\). In addition, for each \(1 < j \in \mathbb{N}\), we have \(\text{INF}_f(A_j) = -1\) and \(\text{SUP}_f(A_j) = 1\). Since \(\text{INF}_f(A) = 0 = \text{SUP}_f(A)\) and \(\lim_{j \to \infty} D_H(A, A_j) = 0\), we conclude that neither \(\text{INF}_f\) nor \(\text{SUP}_f\) are continuous at \(A\).
Counterexample 3.4. Here is another counterexample related to Theorem 3.1, where now \( X \) is bounded. For each \( k \in \mathbb{N} \), let \( e_k \in \ell_2 \) be the vector having 1 in its \( k \)-th component and 0 in the other components, and let \([0, e_k] \) be the line segment which connects the origin 0 of \( \ell_2 \) with \( e_k \). For each \( j, k \in \mathbb{N} \), define \( J_{j,k} := [0, ((k + 1)/k)e_k] \) (half-open line segment) if \( j = k \) and \( J_{j,k} := [0, e_k] \) if \( j \neq k \). Let \( X \) be defined by \( X := \cup_{k=1}^{\infty} [0, ((k + 1)/k)e_k] \) and \( d \) be the metric induced by the \( \ell_2 \) norm. Let \( A := \cup_{k=1}^{\infty} [0, e_k] \) and for all \( j \in \mathbb{N} \), let \( A_j := \cup_{k=1}^{\infty} J_{j,k} \). Then \( X, A \) and \( A_j, j \in \mathbb{N} \), are bounded, and we have \( A \subset X \) and also \( A_j \subset X \) for each \( j \in \mathbb{N} \). For each \( k \in \mathbb{N} \), let \( f_k : [0, (k + 1)/k) \to \mathbb{R} \) be defined by \( f_k(t) := 0 \) for every \( t \in [0, 1] \) and \( f_k(t) := \sin(2\pi/(1 + (1/k) - t)) \) for all \( t \in [1, (k + 1)/k) \). Now let \( f : X \to \mathbb{R} \) be defined as follows: given \( 0 \neq x = (x_i)_{i=1}^{\infty} \in X \), there exists a unique \( k \in \mathbb{N} \) such that \( x \in [0, ((k + 1)/k)e_k] \), namely \( x_i = 0 \) if \( i \neq k \) and \( x_i \in [0, (k + 1)/k) \) if \( i = k \), where \( i, k \in \mathbb{N} \); in this case, define \( f(x) := f_k(x_k) \); for \( x = 0 \), let \( f(x) := 0 \). Then \( f \) is continuous on \( X \), but it is not uniformly continuous there. Moreover, \( \text{INF}_f(A) = 0 = \text{SUP}_f(A) \). However, for each \( j \in \mathbb{N} \), we have \( \text{INF}_f(A_j) = -1 \) and \( \text{SUP}_f(A_j) = 1 \), and, in addition, \( \lim_{j \to \infty} D_H(A, A_j) = 0 \). Thus neither \( \text{INF}_f \) nor \( \text{SUP}_f \) are continuous at \( A \).

4. Stability of the sets of minimizers and maximizers

Theorem 3.1 raises the corresponding question regarding stability of the sets of minimizers and maximizers of the function under consideration, that is, not only the stability of its optimal values. In general, stability does not hold as is shown in Counterexample 4.2 below. However, under further assumptions a weak stability principle related to the minimizers and maximizers can be formulated, as we show in Theorem 4.1 below. Roughly speaking, Theorem 4.1, which is new to the best of our knowledge, can be summarized as follows: given a continuous function \( f : X \to \mathbb{R} \), where \((X, d)\) is a compact metric space, and given some nonempty and closed subset \( A \) of \( X \), if one slightly perturbs \( A \) to a new nonempty and closed subset \( A' \), then the set of minimizers of \( f \) over \( A \) is “slightly” perturbed too in the sense that its standard distance to the new set of minimizers (of \( f \) over \( A' \)) is small; an analogous assertion holds regarding the original and new sets of maximizers.

Theorem 4.1. (Weak stability of the sets of minimizers and maximizers)
Let \((X, d)\) be a compact metric space. Assume that \( f : X \to \mathbb{R} \) is continuous and \( \emptyset \neq A \subseteq X \) is closed. Denote by \( \text{MINZER}_f(A) \) and \( \text{MAXZER}_f(A) \) the set of all minimizers and maximizers, respectively, of \( f \) on \( A \). Then for every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for each nonempty and closed subset \( A' \) of \( X \) which satisfies \( D_H(A, A') < \delta \), the following statements hold: the sets \( \text{MINZER}_f(A), \text{MINZER}_f(A'), \text{MAXZER}_f(A) \) and \( \text{MAXZER}_f(A') \) are nonempty; \( d(\text{MINZER}_f(A), \text{MINZER}_f(A')) < \epsilon \), and, in addition, \( d(\text{MAXZER}_f(A), \text{MAXZER}_f(A')) < \epsilon \).

Proof. Consider the case of minimizers of \( f \); the case of maximizers is proved in a similar way. We first observe that the classical Extreme Value Theorem ensures that \( \text{MINZER}_f(S) \neq \emptyset \) whenever \( S \subseteq X \) is nonempty and closed, because \( f \) is continuous on \( X \) and \( S \) is compact since it is a closed subset of the compact space \( X \). Now suppose to the contrary that the assertion regarding the weak stability of the minimizers does not hold. Then there exist \( \epsilon > 0 \) and a sequence \((A'_k)_{k=1}^{\infty}\) of nonempty and compact
For each \( k \in \mathbb{N} \), let \( a'_k \in \text{MINZER}_f(A'_k) \) be arbitrary. The compactness of \( X \) ensures that there exist an infinite subset \( N_1 \) of \( \mathbb{N} \) and \( a'_\infty \in X \) such that \( \lim_{k \to \infty, k \in N_1} a'_k = a'_\infty \). Since \( a'_k \in A'_k \), we have \( d(a'_k, A) \leq D_H(A, A'_k) < 1/k \). This inequality and the continuity of the distance function imply that \( d(a'_\infty, A) = 0 \). Since \( A \) is closed, we have \( a'_\infty \in A \).

We claim that \( a'_\infty \) is a minimizer of \( f \) on \( A \). Indeed, since \( X \) is compact and \( f \) is continuous, a well-known classical theorem implies that \( f \) is uniformly continuous on \( X \). Hence Theorem 3.1 ensures that for each \( \rho > 0 \), there exists \( \eta > 0 \) such that for every \( \emptyset \neq A' \subseteq X \) satisfying \( D_H(A, A') < \eta \), one has \( \text{INF}_f(A') := \inf \{ f(a') : a' \in A' \} < \rho + \text{INF}_f(A) \). Let \( k \in N_1 \) be large enough so that \( D_H(A, A'_k) < 1/k < \eta \). Then the previous lines and the assumption that \( a'_k \in \text{MINZER}_f(A'_k) \) guarantee that \( f(a'_k) \leq \rho + \text{INF}_f(A) \). This inequality is true for all \( k \in N_1 \) sufficiently large, and hence, by passing to the limit \( k \to \infty \), \( k \in N_1 \), we conclude that \( f(a'_\infty) \leq \rho + \text{INF}_f(A) \). Since \( \rho \) was an arbitrary positive number, we have \( f(a'_\infty) \leq \text{INF}_f(A) \). But \( a'_\infty \in A \) and so obviously \( \text{INF}_f(A) \leq f(a'_\infty) \). Thus \( f(a'_\infty) = \text{INF}_f(A) \), that is, \( a'_\infty \in \text{MINZER}_f(A) \).

Now let \( k \in N_1 \) be large enough so that \( d(a'_\infty, a'_k) < \epsilon \). On the one hand, there exists such an index \( k \) since \( a'_\infty = \lim_{k \to \infty, k \in N_1} a'_k \). On the other hand, this existence is impossible since \( a'_k \in \text{MINZER}_f(A'_k) \) and hence, according to (8), we have \( \epsilon \leq d(\text{MINZER}_f(A), \text{MINZER}_f(A'_k)) \leq d(a'_\infty, a'_k) < \epsilon \), a contradiction. Consequently, the assertion formulated in the theorem does hold, as claimed.

Counterexample 4.2. Consider the case where \( X := [-20, 20] \) with the absolute value metric. Let \( A := [0, \pi] \) and \( A'_\epsilon := [-\epsilon, \pi - \epsilon] \) for every \( \epsilon \in (0, 1/2) \). In addition, let \( f : X \to \mathbb{R} \) be defined by \( f(t) := |\sin(t)|, t \in X \). Then \( f \) is uniformly continuous on \( X \) and \( \text{MINZER}_f(A) = \{0, \pi\} \). However, no matter how small \( \epsilon \) is, \( \text{MINZER}_f(A'_\epsilon) = \{0\} \). Thus \( D_H(\text{MINZER}_f(A), \text{MINZER}_f(A'_\epsilon)) = \pi \) despite the fact that \( D_H(A, A'_\epsilon) = \epsilon \to 0 \) as \( \epsilon \) tends to 0. On the other hand, Theorem 4.1 does hold because \( d(\text{MINZER}_f(A), \text{MINZER}_f(A'_\epsilon)) = 0 \).

As another example, let \( X := \mathbb{R} \) with the absolute value metric, let \( A := X \) and let \( A'_\epsilon := (-\infty, -\epsilon] \cup [\epsilon, \infty) \) for each \( \epsilon > 0 \). Let \( f : X \to \mathbb{R} \) be defined by \( f(t) := 1 - |t| \) when \( t \in [-1, 1] \) and \( f(t) := (2/\pi) \arctan(|t| - 1) \) when \( |t| \geq 1 \). Then \( f \) is uniformly continuous on \( X \) and \( \text{MAXZER}_f(A) = \{0\} \). However, \( \text{MAXZER}_f(A'_\epsilon) = \emptyset \) for each \( \epsilon > 0 \) despite the fact that \( \lim_{\epsilon \to 0} D_H(A, A'_\epsilon) = 0 \). There is no contradiction to Theorem 4.1 because \( X \) is not compact.

5. Application 1: Parametric Optimization I (a general statement)

In this section we show how Theorem 3.1 can be used to prove that the so-called “optimal value function” (or “marginal function”, or “inf-projection”) from parametric optimization is continuous under certain assumptions.

Corollary 5.1. Let \( (X, d_X) \) and \( (I, d_I) \) be two arbitrary pseudo-distance spaces. Suppose that \( \mathcal{C} \) is a nonempty set of nonempty subsets of \( X \) having the property that for each \( A \in \mathcal{C} \), there is \( r_A > 0 \) such that \( B[A, r_A] \in \mathcal{C} \). Suppose also that \( f : X \to \mathbb{R} \) is uniformly continuous on each \( A \in \mathcal{C} \). Assume that \( \{A_t\}_{t \in I} \) is a family of subsets in \( \mathcal{C} \) having the property that for each \( t_0 \in I \), one has \( \limsup_{t \to t_0} D_{H,X}(A_{t_0}, A_t) \leq 0 \).
in the sense of Section 2 (here \(D_{H,X}\) is the Hausdorff distance induced by \(d_X\) and not by \(d_I\)). Define \(\phi^*: I \rightarrow [-\infty, \infty]\) by \(\phi^*(t) := \sup \{ f(t) : x \in A_t \}, t \in I\). If \(A_t \in \text{dom}(\sup_f)\) for all \(t \in I\), then \(\phi^*\) is a continuous function from \(I\) to \(\mathbb{R}\). Similarly, if \(\phi_*(t) := \inf \{ f(x) : x \in A_t \}\) and \(A_t \in \text{dom}(\inf_f)\) for every \(t \in I\), then \(\phi_*\) is a continuous function from \(I\) to \(\mathbb{R}\).

**Proof.** Let \(t_0 \in I\) and \(\epsilon > 0\) be arbitrary. Since we assume that \(B[A_{t_0}, r_0] \in \mathcal{C}\) for some \(r_0 > 0\), it follows, in particular, that \(B[A_{t_0}, r_0] \neq \emptyset\). Moreover, if \(\emptyset \neq A' \subseteq X\) satisfies \(D_{H,X}(A_{t_0}, A') < r_0\), then \(A' \subseteq B[A_{t_0}, r_0]\), as follows from the definition of \(B[A_{t_0}, r_0]\) and (2). Since \(f\) is uniformly continuous on any subset which belongs to \(\mathcal{C}\), it is uniformly continuous on \(B[A_{t_0}, r_0]\). Moreover, by our assumption, \(A_{t_0} \in \text{dom}(\sup_f)\). Thus, we can apply Theorem 3.1, where the space there is \(B[A_{t_0}, r_0]\) and the pseudo-distance is the restriction of \(d_X\) to \(B[A_{t_0}, r_0]^2\). This theorem implies, in particular, that for our \(\epsilon\), there exists \(\delta_0 \in (0, r_0]\) such that for all \(\emptyset \neq A' \subseteq B[A_{t_0}, r_0]\) satisfying \(D_{H,X}(A_{t_0}, A') < \delta_0\), we have \(\sup_f(A') \in \mathbb{R}\) and

\[
|\phi^*(t_0) - \sup_f(A')| < \epsilon. \tag{9}
\]

Since we assume that \(\limsup_{t \to t_0} D_{H,X}(A_{t_0}, A_t) \leq 0\), it follows that for the positive number \(\delta_0\), there exists \(\delta > 0\) such that for each \(t \in I\) which satisfies \(d_I(t_0, t) < \delta\), we have \(D_{H,X}(A_{t_0}, A_t) < \delta_0\). Hence we can substitute \(A' := A_t\) in (9) and this inequality implies that \(|\phi^*(t_0) - \phi^*(t)| < \epsilon\) for each \(t \in I\) satisfying \(d_I(t_0, t) < \delta\). In other words, \(\phi^*\) is continuous at \(t_0\). The proof of the claim regarding \(\phi_*\) is similar. \(\square\)

**Remark 5.2.** An example of a set \(\mathcal{C}\) having the property mentioned in the formulation of Corollary 5.1 is provided by the set of all nonempty and bounded subsets of a metric space: in Corollary 7.1 we use this example. A second example is the set of all nonempty subsets of a metric space, or, more generally, of a pseudo-distance space \((X, d)\) having the property that \(0 = d(x, x) \leq d(x, y)\) for every \((x, y) \in X^2\) (for instance, this happens if \((X, d)\) is the effective domain of a Bregman function and \(d\) is the associated Bregman divergence [10,35]); in Section 6 below we use this set. A third example is as follows: again, we consider a pseudo-distance space \((X, d)\) having the property that \(0 = d(x, x) \leq d(x, y)\) for every \((x, y) \in X^2\), and are also given a uniformly continuous function \(f : X \rightarrow \mathbb{R}\); then Theorem 3.1 implies that we can take \(\mathcal{C}\) to be either \(\text{dom}(\sup_f)\) or \(\text{dom}(\inf_f)\) (our assumption on \((X, d)\) and the fact that \(\sup_f(A) \in \mathbb{R}\) whenever \(A \in \text{dom}(\sup_f)\) ensure that \(\emptyset \neq A \subseteq B[A, r]\) and hence \(B[A, r] \neq \emptyset\) for all \(r > 0\)).

6. **APPLICATION 2: Parametric Optimization II (mixed linear-nonlinear programming)**

In this section we consider two mixed linear-nonlinear programming problems and establish continuity properties of the corresponding optimal value functions. In the first case (Example 6.2) the objective function is, in general, nonlinear but the constraints are linear, and in the second case (Example 6.3) the function is nonlinear and the constraints are partly linear and partly nonlinear. Our results extend partly, but significantly, a theory which was developed in previous works: see Remark 6.4 below for more details.
Before presenting our results, we need a lemma. It is related to, but different from [50, Theorem 1] (the differences are both in the formulations and in the methods of proof).

**Lemma 6.1.** Let \( X \neq \{0\} \) be an arbitrary normed space with some arbitrary norm \( \| \cdot \|_X \) and let \( d \) be the distance induced by \( \| \cdot \|_X \). Let \( (Y, \| \cdot \|_Y) \) be another normed space and suppose that \( 0 \neq L : X \to Y \) is a linear operator with a finite-dimensional image \( I := L(X) \). For all \( t \in I \), denote \( A_t := \{ x \in X : Lx = t \} \). Then there exists some \( \alpha > 0 \) such that \( D_H(A_s, A_t) \leq \alpha \| s - t \|_Y \) for all \( s, t \in I \).

**Proof.** Since \( I \neq \{0\} \) and \( I \) is finite-dimensional, it has a basis \( (w_j)_{j=1}^\ell \), where \( \ell \in \mathbb{N} \) and for all \( j \in \{1, \ldots, \ell\} \) there is some \( v_j \in X \) such that \( w_j = L(v_j) \). Let \( V := \text{span}\{v_j : j \in \{1, \ldots, \ell\}\} \) and \( M : V \to I \) be the restriction of \( L \) to \( V \). Then \( M \) is invertible because \( (w_j)_{j=1}^\ell \) is a basis of \( I \) and \( M^{-1} \) is bounded since it is a linear mapping which acts between the finite-dimensional spaces \( I \) and \( V \). Now fix some \( s, t \in I \). Since \( M = L \) on \( V \), it follows that \( t = L(v) \), namely \( M^{-1}(t) = v \in A_t \). This fact, when combined with the well-known fact that the set of solutions \( x \) to the non-homogeneous equation \( Ax = t \) is the sum of the kernel \( A_0 \) of \( L \) with a particular solution to the equation, implies that \( A_t = M^{-1}(t) + A_0 \). Similarly, \( A_s = M^{-1}(s) + A_0 \). Let \( x \in A_s \). Then \( x = M^{-1}(s) + a_0x \) for some \( a_0x \in A_0 \). Let \( y := M^{-1}(t) + a_0x \). Then \( y \in M^{-1}(t) + A_0 \), namely, \( y \in A_t \). In addition, \( \|x - y\|_X = \|M^{-1}(s) - M^{-1}(t)\|_Y \leq \|M^{-1}\| \|s - t\|_Y \). Thus \( d(x, A_t) \leq \|M^{-1}\| \|s - t\|_Y \). This is true for each \( x \in A_s \) and hence \( D_{asy}(A_s, A_t) \leq \|M^{-1}\| \|s - t\|_Y \). Similarly, \( D_{asy}(A_t, A_s) \leq \|M^{-1}\| \|s - t\|_Y \) and so \( D_H(A_s, A_t) \leq \alpha \|s - t\|_Y \) for \( \alpha := \|M^{-1}\| \).

**Example 6.2. (Nonlinear function, linear constraints)** Consider the setting of Lemma 6.1. Assume that \( f : X \to \mathbb{R} \) is uniformly continuous and bounded from below. Denote by \( d_I \) the distance (on \( I \)) induced by \( \| \cdot \|_Y \). Let \( \phi_s : I \to \mathbb{R} \) be the optimal value function defined by \( \phi_s(t) := \text{INF}_f(A_t), t \in I \), namely

\[ \phi_s(t) = \inf\{ f(x) : x \in X, Lx = t \}, \quad t \in L(X). \]

Let \( \mathcal{C} \) be the set of all nonempty subsets of \( X \). Since \( f \) is bounded from below and \( A_t \neq \emptyset \) for every \( t \in I \), it follows that \( A_t \in \text{dom}(\text{INF}_f) \) for each \( t \in I \). Moreover, as we have shown in Lemma 6.1, there exists some \( \alpha > 0 \) such that \( D_H(A_s, A_t) \leq \alpha \|s - t\|_Y \) for all \( s, t \in I \). Hence \( \lim_{t \to s} D_H(A_s, A_t) = 0 \) for all \( s \in I \), and so Corollary 5.1 implies that \( \phi_s \) is continuous. As a matter of fact, if, in addition, \( f \) is \( \Lambda \)-Lipschitz continuous for some \( \Lambda > 0 \), then \( \phi_s \) is even Lipschitz continuous, as follows from Theorem 3.1:

\[ |\phi_s(s) - \phi_s(t)| = |\text{INF}_f(A_s) - \text{INF}_f(A_t)| \leq \Lambda D_H(A_s, A_t) \leq \alpha \Lambda \|s - t\|_Y, \quad \forall s, t \in I. \]

**Example 6.3. (Nonlinear function, mixed linear-nonlinear constraints)** Given \( n \in \mathbb{N} \), endow \( \mathbb{R}^n \) with some norm \( \| \cdot \| \) and suppose that \( C \) is compact, convex and has a nonempty interior \( \text{Int}(C) \). Assume that \( f : C \to \mathbb{R} \) is continuous. Given some \( m \in \mathbb{N} \), suppose that \( L : \mathbb{R}^n \to \mathbb{R}^m \) is a linear operator, where the norm on \( \mathbb{R}^m \) is \( \| \cdot \|_\mathbb{R}^m \) (just an arbitrary norm). For all \( t \) in the image \( L(\mathbb{R}^n) \) of \( \mathbb{R}^n \) by \( L \), denote \( E_t := \{ x \in \mathbb{R}^n : Lx = t \} \). Denote by \( I \) the set of all \( t \in L(\mathbb{R}^n) \) for which the following regularity condition holds: the affine subspace \( E_t \) intersects \( \text{Int}(C) \), namely there is some \( x \in \text{Int}(C) \) such that \( Lx = t \). Assume also that \( I \neq \emptyset \) (as shown in Remark
6.4(ii) below, this latter assumption actually implies that $I$ is convex and open in $L(\mathbb{R}^n)$. For all $t \in I$, let $A_t := E_t \cap C$ and let $\phi_s : I \to \mathbb{R}$ be the optimal value function defined by $\phi_s(t) := \text{INF}_j(A_t), t \in I$, that is,

$$
\phi_s(t) = \inf \{ f(x) : x \in C, Lx = t \}, \quad t \in I.
$$

Given $t \in I$, since $E_t \cap \text{Int}(C) \neq \emptyset$, we have $A_t \neq \emptyset$, and since $E_t$ is closed and $C$ is compact, $A_t$ is compact. Thus the Extreme Value Theorem implies that $A_t \in \text{dom}(\text{INF}_j)$ for each $t \in I$. We prove below that $\lim_{t \to s} D_H(A_s, A_t) = 0$ for all $s \in I$. This fact, when combined with the fact that the continuous function $f$ is actually uniformly continuous (since $C$ is compact), implies that we can use Corollary 5.1 (in which the first space is $C$ with the restriction of $\| \cdot \|$ to $C$ as the distance function, $d_f$ is the distance induced on $I$ by $\| \cdot \|_{\mathbb{R}^m}$, and $\phi$ is the set of all nonempty subsets of $C$), from which we conclude that $\phi_s$ is continuous.

To see that indeed $\lim_{t \to s} D_H(A_s, A_t) = 0$ for all $s \in I$, suppose to the contrary that this is not true. Thus there are $t \in I$, $\epsilon > 0$ and a sequence $(t_k)_{k=1}^{\infty}$ of elements in $I$ such that $\lim_{k \to \infty} t_k = s$ and $D_H(A_s, A_{t_k}) \geq \epsilon$ for each $k \in \mathbb{N}$. Therefore either $D_{asyH}(E_{t_k} \cap C, E_s \cap C) \geq \epsilon$ for all $k \in N_1$, where $N_1 \subseteq \mathbb{N}$ is an infinite set, or $D_{asyH}(E_s \cap C, E_{t_k} \cap C) \geq \epsilon$ for all $k \in N_2$, where $N_2 \subseteq \mathbb{N}$ is an infinite set.

Consider the first case. It implies that for each $k \in N_1$, there is some $x_k \in E_{t_k} \cap C$ such that $d(x_k, E_s \cap C) > 0.5\epsilon$. Since the sequence $(x_k)_{k \in N_1}$ is contained in the compact set $C$, there is an infinite set $N_{11} \subseteq N_1$ and $x_\infty \in C$ such that $\lim_{k \in N_{11}} x_k = x_\infty$. Lemma 6.1 and the assumption that $\lim_{k \to \infty} t_k = s$ imply that $\lim_{k \to \infty} D_H(E_{t_k}, E_s) = 0$, and so from (1) we conclude that $\lim_{k \to \infty, k \in N_{11}} d(x_k, E_s) = 0$. This fact and the continuity of the distance function imply that $d(x_\infty, E_s) = 0$. Thus (since $E_s$ is closed) $x_\infty \in E_s$. We conclude that $x_\infty \in E_s \cap C$. However, since $d(x_k, E_s \cap C) > 0.5\epsilon$ for each $k \in N_1$, by passing to the limit $k \to \infty$, $k \in N_{11}$, and using the continuity of the distance function, we have $d(x_\infty, E_s \cap C) \geq 0.5\epsilon$, a contradiction.

Now consider the second case which was mentioned two paragraphs earlier. It implies that for each $k \in N_2$, there is some $y_k \in E_s \cap C$ such that $d(y_k, E_{t_k} \cap C) > 0.5\epsilon$. Since the sequence $(y_k)_{k \in N_2}$ is contained in the compact set $E_s \cap C$, there is an infinite set $N_{22} \subseteq N_2$ and $y_\infty \in E_s \cap C$ such that $\lim_{k \in N_{22}} y_k = y_\infty$. According to our assumption, $E_s \cap \text{Int}(C) \neq \emptyset$. Let $z \in E_s \cap \text{Int}(C)$ and $r := \min \{0.1\epsilon, 0.5\|y_\infty - z\|\}$. If $y_\infty = z$, then $r = 0$. We let $w := z$ and observe that $w \in \text{Int}(C)$. Otherwise, let $w := y_\infty + r((z - y_\infty)/\|z - y_\infty\|)$. In this latter case $w$ belongs to the half-open line segment $(y_\infty, z)$. As is well known [38, Theorem 6.1, p. 45], since $z \in \text{Int}(C)$ and $y_\infty \in C$, the convexity of $C$ implies that $(y_\infty, z)$ is contained in $\text{Int}(I)$. Thus $w \in \text{Int}(C)$ again. Hence there is some $\rho \in (0, r)$ such that the ball of radius $\rho$ with center $w$ is contained in $C$. Moreover, $w \in E_s$ since $E_s$ is convex and $(y_\infty, z) \subseteq E_s$.

Lemma 6.1 implies that $\lim_{k \to \infty} D_H(E_{t_k}, E_s) = 0$. Hence $\lim_{k \to \infty} d(w, E_{t_k}) = 0$. Thus for all $k \in N$ sufficiently large there is some $u_k \in E_{t_k}$ such that $\|w - u_k\| < \rho$. Our choice of $\rho$ implies that $u_k \in C$ and the triangle inequality implies that $\|y_\infty - u_k\| \leq \|y_\infty - w\| + \|w - u_k\| < r + \rho < 2r \leq 0.2\epsilon$. Therefore

$$
d(y_\infty, E_{t_k} \cap C) \leq \|y_\infty - u_k\| < 0.25\epsilon
$$

(10) for all $k \in N$ sufficiently large and, in particular, for all $k \in N_{22}$ sufficiently large. On the other hand, the fact that $\lim_{k \in N_{22}} y_k = y_\infty$ implies that $\|y_\infty - y_k\| < 0.25\epsilon$ for all $k \in N_{22}$ sufficiently large. Since $d(y_k, E_{t_k} \cap C) > 0.5\epsilon$ for each $k \in N_2$, the
triangle inequality implies that $0.5\epsilon < d(y_k, E_{t_k} \cap C) \leq \|y_k - y_{\infty}\| + d(y_{\infty}, E_{t_k} \cap C) < 0.25\epsilon + d(y_{\infty}, E_{t_k} \cap C)$. Thus $0.25\epsilon < d(y_{\infty}, E_{t_k} \cap C)$ for all $k \in N_{\epsilon}$ sufficiently large. This inequality contradicts (10) and proves that the second case mentioned several paragraphs above cannot hold too. Thus we indeed have $\lim_{t \to s} D_H(A_s, A_t) = 0$ for all $s \in I$, as asserted.

Remark 6.4. Here are a few remarks regarding Examples 6.2 and 6.3.

(i) Examples 6.2 and 6.3 extend partly, but significantly, the stability theory developed in [52, pp. 279–280] and [51, Theorem 2] (for a related theory, see [52, p. 281] and [42, Lemma 4.1]). This theory has been applied to analyzing stochastic programs [51, Section 4], [42, Section 4]. The setting in [52, pp. 279–280] and [51, Theorem 2] is a finite-dimensional Euclidean space, a polyhedral set $C$, and an objective function $f$ which is Lipschitz continuous on a set which contains $C$; another requirement in [52, pp. 279–280] is that either the level-sets of $f$ are bounded or $C^\infty \cap A_0 = \{0\}$, where $A_0$ is the kernel of the linear operator $L$ and $C^\infty$ is the horizontal cone associated with $C$. It is proved in [52] that the associated optimal value function $\phi_*$ is Lipschitz continuous under these assumptions (note: the constraint set there is written as $\{x \in C : Lx = b - t\}$, where $b$ is a given vector and $t$ is the parameter; thus by a simple change of variable we can arrive at this formulation). It can be seen that Examples 6.2 and 6.3 extend this theory to the case of possibly general normed spaces, a constraint set $C$ which is either the entire space or a (usually non-polyhedral) convex body, a linear operator $L$ which should have a finite-dimensional image and should satisfy a mild regularity condition, and an objective function $f$ which is either Lipschitz continuous or merely uniformly continuous. We are still able to derive the continuity of $\phi_*$ under these conditions, and sometimes (Example 6.2) its Lipschitz continuity.

We believe that the theory developed in this section can be extended further, and, in particular, that it is possible to remove (at least in some interesting cases) the compactness and finite-dimensionality assumptions, as well as the above-mentioned regularity condition, and also to obtain corresponding Lipschitz continuity results.

(ii) The set $I$ mentioned in Example 6.3 is actually open and convex whenever it is nonempty. Indeed, given $s \in I$, let $x \in E_s \cap \text{Int}(C)$. In particular, $x \in \text{Int}(C)$ and hence there exists some $r > 0$ such that the ball of radius $r$ about $x$ is contained in $\text{Int}(C)$. Lemma 6.1 implies that there is some $\alpha > 0$ such that $D_H(E_s, E_t) \leq \alpha\|s - t\|_{\mathbb{R}^n}$ for all $t \in L(\mathbb{R}^n)$. Thus, if $\delta := r/\alpha$, then for each $t \in L(\mathbb{R}^n)$ which satisfies $\|s - t\|_{\mathbb{R}^n} < \delta$, we have $D_H(E_s, E_t) < r$. This inequality and the fact that $x$ also belongs to $E_s$ imply that there is some $y \in E_t$ such that $\|x - y\| < r$. Thus $y \in \text{Int}(C) \cap E_t$ and so $t \in I$ for all $t$ in the ball of radius $\delta$ about $s$. To see that $I$ is convex, let $s, t \in I$ and $\lambda \in [0, 1]$ be given. Then $Lx_s = s$ and $Lx_t = t$ for some $x_s, x_t \in \text{Int}(C)$. We have $\lambda x_s + (1 - \lambda)x_t \in \text{Int}(C)$ since $\text{Int}(C)$ is convex. In addition, $L(\lambda x_s + (1 - \lambda)x_t) = \lambda L(x_s) + (1 - \lambda)L(x_t) = \lambda s + (1 - \lambda)t$. Consequently, $\lambda s + (1 - \lambda)t \in I$, as required.
7. Application 3: a sequence of Lipschitz constants

In this section we use Corollary 5.1 in order to show that a rather general sequence of positive numbers can be a sequence of Lipschitz constants associated with a given function (each Lipschitz constant corresponds to a certain subset on which one measures the Lipschitz continuity of the function). Corollary 7.1 below has recently been applied in the analysis of a telescopic proximal gradient method \[34\].

**Corollary 7.1.** Suppose that \( f : U \to \mathbb{R} \) is a twice continuously (Fréchet) differentiable function defined on an open and convex subset \( U \) of some real normed space \((X, \| \cdot \|)\), \( X \neq \{0\} \). Suppose that \( C \) is a convex subset of \( X \) which has the property that \( C \cap U \neq \emptyset \). Assume that \( f'' \) is bounded and uniformly continuous on bounded subsets of \( C \cap U \). Fix an arbitrary \( y_0 \in C \cap U \), and let \( s := \sup\{\|f''(x)\| : x \in C \cap U\} \) and \( s_0 := \|f''(y_0)\| \). If \( s = \infty \), then for each strictly increasing sequence \( (\lambda_k)_{k=1}^\infty \) of positive numbers which satisfies \( \lambda_1 > s_0 \) and \( \lim_{k \to \infty} \lambda_k = \infty \), there exists an increasing sequence \( (S_k)_{k=1}^\infty \) of bounded and convex subsets of \( C \) (and also closed if \( C \) is closed) which satisfies the following properties: first, \( S_k \cap U \neq \emptyset \) for all \( k \in \mathbb{N} \), second, \( \bigcup_{k=1}^\infty S_k = C \), third, for each \( k \in \mathbb{N} \), the function \( f' \) is Lipschitz continuous on \( S_k \cap U \) with \( \lambda_k \) as a Lipschitz constant; moreover, if \( C \) contains more than one point, then also \( S_k \cap U \) contains more than one point for each \( k \in \mathbb{N} \). Finally, if \( s < \infty \), then \( f' \) is Lipschitz continuous on \( C \cap U \) with \( s \) as a Lipschitz constant.

**Proof.** Suppose first that \( s = \infty \). Let \( I := [0, \infty) \) and let \( d_t \) be the standard absolute value metric. For each \( t \in I \), define \( B_t \) to be the intersection of \( C \) with the closed ball of radius \( t \) and center \( y_0 \) (here \( B_0 := \{y_0\} \)). Then \( B_t \) is a bounded and convex subset of \( C \) for each \( t \in I \), and it is also closed if \( C \) is closed. Let \( A_t := B_t \cap U \) for each \( t \in I \). Then \( A_t \neq \emptyset \) (it contains \( y_0 \)), convex and bounded for all \( t \in I \). In addition, \( \bigcup_{t \in I} B_t = C \) and \( \bigcup_{t \in I} A_t = C \cap U \). An immediate verification shows that \( D_H(A_t, A_{t'}) \leq |t - t'| \) for all \( t, t' \in I \). Since \( f'' \) exists and is bounded on bounded subsets of \( C \cap U \), the function \( h \), which is defined by \( h(x) := \|f''(x)\|, x \in C \cap U \), is finite at each point, and it is also bounded on each bounded subset of \( C \cap U \). This implies that the function \( \phi^* : I \to (-\infty, \infty) \), which is defined by \( \phi^*(t) := \sup h(A_t) \) for each \( t \in I \), satisfies \( \phi^*(t) \in [0, \infty) \) for each \( t \in I \). In addition, since \( f'' \) is uniformly continuous on bounded subsets of \( C \cap U \), the triangle inequality shows that the function \( h \), too, is uniformly continuous on bounded subsets of \( C \cap U \). We conclude from the previous lines that the conditions needed in Corollary 5.1 hold (the first pseudo-distance space there is \( C \cap U \), where the pseudo-distance is the metric which is induced by the restriction of the norm of \( X \) to \( C \cap U \); in addition, the set \( C \) in Corollary 5.1 is the set of all nonempty and bounded subsets of \( C \cap U \)), and consequently, \( \phi^* \) is a continuous function on \( I \).

Since \( s = \sup\{\|f''(x)\| : x \in C \cap U\} = \infty \), for each \( \rho \geq \|f''(y_0)\| = s_0 \), there exists \( x \in C \cap U \) such that \( \|f''(x)\| > \rho \). Since \( \bigcup_{t \in I} A_t = C \cap U \), there exists \( t(x) \in I \) such that \( x \in A_{t(x)} \). As a result, from the definition of \( \phi^* \) we see that \( \phi^*(t(x)) \geq \|f''(x)\| > \rho \). By applying the classical Intermediate Value Theorem to the continuous function \( \phi^* \) on the interval \([0, t(x)]\), we conclude that each value between \( \phi^*(0) = s_0 \) and \( \phi^*(t(x)) \) is attained. In particular, \( \rho \) is attained. Since \( \rho \) was an arbitrary number which is greater than or equal to \( s_0 \) and since \( \phi^* \) is increasing, it follows that the image of \( I = [0, \infty) \) under \( \phi^* \) is the interval \([s_0, \infty)\). Therefore, given \( k \in \mathbb{N} \), since
\( \lambda_k \geq \lambda_1 > s_0 \), there exists \( t_k \in [0, \infty) \) such that \( \phi^* (t_k) = \lambda_k \), and this \( t_k \) must be positive, otherwise \( t_k = 0 \) and hence \( s_0 = \phi^* (0) = \phi^* (t_k) = \lambda_k \), a contradiction.

Let \( S_k := B_{t_k} \) for each \( k \in \mathbb{N} \). Then \( S_k \) is bounded and convex for each \( k \in \mathbb{N} \), and it is also closed if \( C \) is closed. In addition, \( S_k \cap U \) is nonempty (it contains \( y_0 \)), bounded and convex for every \( k \in \mathbb{N} \). Since \( \| f^n (x) \| \leq \sup \{ \| f^n (y) \| : y \in S_k \cap U \} = \phi^* (t_k) \) for all \( x \in S_k \cap U \), and since \( f' \) is continuously differentiable on \( U \) and hence on \( S_k \cap U \), the (generalized) Mean Value Theorem applied to \( f' \) (see [1, Theorem 1.8, p. 13, and also p. 23]; this theorem is formulated for Gâteaux differentiable functions acting between real Banach spaces, but it holds as well for Fréchet differentiable functions acting between real normed spaces, because no completeness assumption is needed in the proof, and the Fréchet and Gâteaux derivatives coincide in our case) implies that \( f' \) is Lipschitz continuous on \( S_k \cap U \) with \( \phi^* (t_k) \) as a Lipschitz constant, namely with \( \lambda_k \) as a Lipschitz constant.

Now we show that \( \cup_{k=1}^{\infty} S_k = C \). Indeed, since \( \phi^* \) is increasing and \( (\lambda_k)_{k=1}^{\infty} \) is strictly increasing, it follows that \( (t_k)_{k=1}^{\infty} \) is increasing. Hence \( \ell := \lim_{k \to \infty} t_k \) exists and it must be that \( \ell = \infty \), otherwise \( \lambda_k = \phi^* (t_k) \leq \phi^* (\ell) < \infty \) for all \( k \in \mathbb{N} \), a contradiction to the assumption \( \lim_{k \to \infty} \lambda_k = \infty \). Hence the union of the closed balls with common center \( y_0 \) and radii \( t_k \), \( k \in \mathbb{N} \), is \( X \). Thus the intersection of this union with \( C \) is \( C \). On the other hand, this intersection is \( \cup_{k=1}^{\infty} S_k \), as follows from the definition of the subsets \( S_k \), \( k \in \mathbb{N} \). In other words, \( \cup_{k=1}^{\infty} S_k = C \).

It remains to show that if \( C \) contains more than one point, then \( S_k \cap U \) also contains more than one point for every \( k \in \mathbb{N} \). Indeed, take some arbitrary \( w_0 \in C \) which satisfies \( w_0 \neq y_0 \). The line segment \([y_0, w_0]\) is contained in \( C \) because \( C \) is convex. Since \( U \) is open and \( y_0 \in U \), there is a sufficiently small closed ball \( B \) of center \( y_0 \) and positive radius \( r < \min \{ \| y_0 - w_0 \|, t_k \} \) such that \( B \subset U \). The intersection of \( B \) with \( C \) contains the segment \([y_0, y_0 + r \theta]\), where \( \theta := (w_0 - y_0) / \| w_0 - y_0 \| \). Since \( r < t_k \), it follows from the definition of \( S_k \) that \( B \cap C \subset B_{t_k} = S_k \). Hence \( S_k \) contains the nondegenerate segment \([y_0, y_0 + r \theta]\), namely it contains more than one point.

Finally, we need to consider the case where \( s < \infty \). In this case \( \| f^n (x) \| \leq s < \infty \) for every \( x \in C \cap U \). Since \( C \cap U \) is convex and \( f' \) is Fréchet (hence Gâteaux) differentiable on \( U \), the Mean Value Theorem applied to \( f' \) implies that \( f' \) is Lipschitz continuous on \( C \cap U \) with \( s \) as a Lipschitz constant.

8. Application 4: A general scheme for tackling a wide class of nonconvex and nonsmooth optimization problems

Given a pseudo-distance space \((X, d)\), consider the general optimization problem of minimizing (or maximizing) a given uniformly continuous function \( f : X \to \mathbb{R} \) over a nonempty subset \( A \subseteq X \). Theorem 3.1 suggests a general scheme for approximating both \( \inf f (A) \) and \( \sup f (A) \). Indeed, consider the case of approximating \( \inf f (A) \) (the case of approximating \( \sup f (A) \) follows a similar reasoning) and assume that it is known that \( \inf f (A) \in \mathbb{R} \). Assume also that we are able to approximate \( A \) by a sequence \((A_k)_{k=1}^{\infty} \) of subsets of \( X \) such that \( \lim_{k \to \infty} D_H (A, A_k) = 0 \) and \( \inf f (A_k) \in \mathbb{R} \) for all \( k \in \mathbb{N} \). Furthermore, assume that we are also able to compute an approximation \( \tilde{\sigma}_k \) to \( \inf f (A_k) \) so that \( \lim_{k \to \infty} |\tilde{\sigma}_k - \inf f (A_k)| = 0 \). Then Theorem 3.1 ensures that \( \lim_{k \to \infty} \tilde{\sigma}_k = \lim_{k \to \infty} (\hat{\sigma}_k - \inf f (A_k)) + \lim_{k \to \infty} \inf f (A_k) = \inf f (A) \). Consequently, the general scheme is nothing but to compute \( \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_3, \ldots \). In particular, one can
apply this method in the common scenario where $X$ is a finite-dimensional normed space or manifold, $A$ is any closed and bounded subset of $X$ and $h$ is continuous on $A$ (or in a neighborhood of $A$); in this latter case, in many instances one can let $A_k$ have a simple form, such as an increasing union of cubes (as in the case of integration), and it might be that because of this it is relatively easy to produce $(\tilde{\sigma}_k)_{k \in \mathbb{N}}$.

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Daniel Reem, Department of Mathematics, The Technion - Israel Institute of Technology, 3200003 Haifa, Israel.

E-mail address: dream@technion.ac.il

Simeon Reich, Department of Mathematics, The Technion - Israel Institute of Technology, 3200003 Haifa, Israel.

E-mail address: sreich@technion.ac.il

Alvaro De Pierro, CNPq, Brazil

E-mail address: depierro.alvaro@gmail.com