Expanding the simple pendulum’s rotation solution in action-angle variables

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Abstract
Integration of Hamiltonian systems by reduction to action-angle variables has proven to be a successful approach. However, when the solution depends on elliptic functions, the transformation to action-angle variables may need to remain in implicit form. This is exactly the case of the simple pendulum, where it is shown that in order to make explicit the transformation to action-angle variables, one needs to resort to nontrivial expansions of special functions and series reversion.

Keywords: simple pendulum, Hamilton–Jacobi method, series expansion, series reversion, action-angle variables, elliptic functions

(Some figures may appear in colour only in the online journal)

1. Introduction

The simple gravity pendulum, a massless rod with a fixed end and a mass attached to the free end, is one of the simplest integrable models. However, it provides a useful didactic system that may be available in most students’ laboratories to be used in different levels of physics [1]. In addition, this physical model serves as the basis for investigating many different phenomena which exhibit a variety of motions including chaos [2], and has the basic form that arises in resonance problems [3]. The case of small oscillations about the stable equilibrium position is customarily studied with linearized dynamics, and was for a long time the basis for implementing traditional timekeeping devices.
The dynamical system is only of one degree of freedom, but the motion may evolve in different regimes, and one must resort to the use of special functions to express its general solution in closed form [4]. In fact, this dynamical model is commonly used to introduce the Jacobi elliptic functions [5].

The traditional integration provides the period of the motion as a function of the pendulum’s length and the initial angle [6]. Alternatively, the solution can be computed by Hamiltonian reduction, a case in which action-angle variables play a relevant role [7]. In particular, they are customarily accepted as the correct variables for finding approximate solutions of almost integrable problems by perturbation methods [8].

When the closed-form solution of an integrable problem is expressed in terms of standard functions, the transformation to action-angle variables can be made explicitly in closed form, as, for instance, in the case of the harmonic oscillator. However, if the solution relies on special functions, whose evaluation will depend on one or more parameters in addition to the function’s argument, the action-angle variables approach may provide the closed-form solution in an implicit form. This fact does not cause trouble when evaluating the solution of the integrable problem, but may deprive this solution of some physical insight. In addition, in usual perturbation methods the disturbing function must be expressed in the action-angle variables of the integrable problem, thus making it necessary to expand the (implicit) transformation to action-angle variables as a Fourier series in the argument of the special functions. These kinds of expansions are not trivial at all, and finding them may be regarded as a notable achievement [9–11].

In the case of elliptic functions, the normal way of proceeding is to replace them by their definitions in terms of Jacobi theta functions, which in turn are replaced by their usual Fourier series expansion in trigonometric functions of the elliptic argument, with coefficients that are powers of the elliptic nome [12, 13]. This laborious procedure is greatly complicated when the modulus of the elliptic function remains as an implicit function of the action-angle variables, a case that requires an additional expansion and the series reversion of the resulting power series. The whole procedure is illustrated here for the rotation regime of the simple pendulum, a physical model for which efforts have been repeatedly undertaken to provide insightful solutions in more familiar functions than the elliptic ones, although limited to the oscillatory regime [14–17].

2. Hamiltonian reduction of the simple pendulum

The Lagrangian of a simple pendulum of mass $m$ and length $l$ under the only action of the gravity acceleration $g$ is written $L = T - V$ where, noting $\theta$, the angle with respect to the vertical direction, $V = mgl(1 - \cos \theta)$ is the potential energy and $T = (1/2)l\dot{\omega}^2$ is the kinetic energy, where $I = ml^2$ is the pendulum’s moment of inertia and $\omega = \dot{\theta}$, where the over dot means derivation with respect to time.

The conjugate momentum to $\theta$ is given by

$$\Theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta}$$

That is, $\Theta = l\omega$ is the angular momentum.

Hence, the usual construction of the Hamiltonian $H = \Theta \dot{\Theta}(\Theta) - L(\theta, \dot{\theta}(\Theta))$ gives

$$H = \frac{1}{2} \frac{\Theta^2}{ml^2} + mgl(1 - \cos \theta), \qquad (1)$$
which represents the total energy for given initial conditions \( \mathcal{H}(\theta_0, \dot{\theta}_0) = E \). Depending on the energy value, the pendulum may evolve in three different regimes:

- \( 0 \leq E < 2mgl \), the oscillation regime, with a fixed point of the elliptic type at \( E = 0 \) \( (\theta = 0, \dot{\theta} = 0) \);
- \( E = 2mgl \Rightarrow \theta = \pm 2ml^2 \sqrt{g/l} \cos(\theta/2) \), the separatrix, with fixed points of the hyperbolic type \( \theta = 0, \dot{\theta} = \pm \pi \); or
- \( E > 2mgl \), the rotation regime.

2.1. Phase space

For each energy manifold \( \mathcal{H} = E \), the Hamiltonian (1) has a geometric interpretation as a parabolic cylinder \( P_E = \{ (x, y, z) \mid z^2 - 2y = \text{const.} \} \), with

\[
\begin{align*}
x &= l \sin \theta, \\
y &= l \cos \theta, \\
z &= \frac{\Theta}{mlg^{1/2}},
\end{align*}
\]

and

\[
h = \frac{2E}{mg} - 2l = \text{const}.
\]

In addition, the constraint \( x^2 + y^2 = l^2 \) means that the phase space of the simple pendulum is realized by the intersection of parabolic cylinders, given by the different energy levels of the Hamiltonian (1), with the surface of the cylinder \( C = \{ (x, y, z) \mid x^2 + y^2 = l^2 \} \) of radius \( l \). This geometric interpretation of the phase space is illustrated in figure 1.
Alternatively, typical trajectories on the cylinder are displayed by means of simple contour plots of the Hamiltonian (1), as illustrated in figure 2, where the trajectories travel from left to right for positive heights ($\Theta > 0$) and from right to left for negative heights ($\Theta < 0$).

The traditional solution to the motion of the pendulum is approached by the direct integration of Hamilton equations

$$
\dot{\theta} = \frac{\partial H}{\partial \Theta} = \frac{\Theta}{ml^2}, \quad \dot{\Theta} = -\frac{\partial H}{\partial \theta} = -mlg \sin \theta,
$$

which are usually written as a single, second-order differential equation

$$
\ddot{\theta} + \frac{g}{l} \sin \theta = 0.
$$

Note that the summand $mlg$ in equation (1) is a constant term that does not affect the dynamics derived from Hamilton equations. Hence, this term is sometimes neglected, which implies a trivial displacement of the energy by a constant level. In addition, the Hamiltonian can be scaled by the pendulum’s moment of inertia to show that it depends only on a relevant parameter. Nevertheless, the dimensional parameters are maintained in following derivations, because, in addition to the physical insight that they provide, the simple and immediate test of checking dimensions is very useful in verifying the correctness of the mathematical developments.

2.2. Hamiltonian reduction

Alternatively to the classical integration of equation (2), the flow can be integrated by Hamiltonian reduction, finding a transformation of variables

$$
T: (\theta', \Theta') \rightarrow (\theta, \Theta),
$$

such that the Hamiltonian (1), when expressed in the new variables, is only a function of the new momentum, namely,
The solution of Hamilton equations in the new variables
\[ \dot{\Theta} = \frac{\partial \Phi}{\partial \Theta'}, \quad \dot{\Theta'} = -\frac{\partial \Phi}{\partial \Theta} = 0, \]
is trivial
\[ \Theta' = \text{const}, \quad \Theta' = \Theta_0 + \omega't, \]
where the frequency \( \omega' = \frac{\partial \Phi}{\partial \Theta'} \) depends only on \( \Theta' \) and, therefore, is constant. Plugging equation (5) into the equations of the transformation defined by \( T \) in equation (3) will give the time solution in the original phase space \((\theta, \Theta)\).

The Hamiltonian reduction in equation (4) can be achieved by the Hamilton–Jacobi method [19], in which the canonical transformation \( T \) is derived from a generating function \( S = S(\theta, \Theta') \) in mixed variables, the ‘old’ coordinate and the ‘new’ momentum, such that the transformation is given by
\[ \Theta' = \frac{\partial S}{\partial \Theta'}, \quad \Theta = \frac{\partial S}{\partial \theta}. \]

The functional expression of \( \Theta, \) given by the last equation in equation (6), is replaced in the Hamiltonian (1) to give the partial differential equation
\[ \frac{1}{2} ml^2 \left( \frac{\partial S}{\partial \theta} \right)^2 + 2mgl \sin^2(\theta/2) = \Phi(\Theta'), \]
where the trigonometric identity \( 1 - \cos \theta = 2 \sin^2(\theta/2) \) has been used.

The Hamilton–Jacobi equation (7) is a partial differential equation that can be solved by quadrature
\[ S = \sqrt{2} ml^2 \int \sqrt{\Phi(\Theta') - 2mgl \sin^2(\theta/2)} \, d\theta. \]

But, in fact, the generating function \( S \) does not need to be computed because the transformation (6) requires only the partial derivatives of equation (8). Thus,
\[ \Theta' = \sqrt{2} ml^2 \frac{\partial \Phi}{\partial \Theta'} \int_0^\theta \frac{d\theta}{2 \sqrt{\Phi(\Theta') - 2mgl \sin^2(\theta/2)}}, \]
\[ \Theta = \sqrt{2} ml^2 \sqrt{\Phi(\Theta') - 2mgl \sin^2(\theta/2)}, \]
where \( \theta = 0 \) corresponds to the upper limit of the square root in equation (9).

Remarkably, the quadrature in equation (9) can be solved without having to choose the form of the new Hamiltonian in advance [18, 20].

2.3. Rotation regime: \( \Phi(\Theta') > 2mgl \)

Equation (9) is reorganized as
\[ \Theta' = \sqrt{\frac{T}{g}} k \frac{\partial \Phi}{\partial \Theta'} F(\phi, k^2), \]
where the partial differentiation has been replaced by the total derivative in view of the single dependency of \( \Phi \) on \( \Theta' \), and
\[ F(\phi, k^2) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}. \]

is the incomplete elliptic integral of the first kind of amplitude
\[ \phi = \theta/2, \quad (11) \]
and elliptic modulus
\[ k = \sqrt{\frac{2mg}{\Phi(\theta')}} < 1. \quad (12) \]

Therefore, the rotational motion of the simple pendulum is solved by the transformation in mixed variables
\[ \theta' = \sqrt{\frac{T}{g}} k \frac{d\phi}{d\Theta'} F(\phi, k^2), \quad (13) \]
\[ \Theta = \sqrt{\frac{2m}{l^2}} \left[ \frac{g}{l} \right] k \frac{1}{k} \sqrt{1 - k^2 \sin^2 \phi}, \quad (14) \]
which is, in fact, a whole family of transformations parameterized by \( \Phi \). For instance, a 'simplifying' option could be choosing \( \phi \equiv \Theta'^2/(8 m l^2) \), which results in
\[ \sin \frac{\theta}{2} = \sin(\theta', k^2), \quad (15) \]
\[ \frac{\Theta}{2 \sqrt{m^2 g l^3}} = \frac{1}{k} \text{dn}(\theta', k^2), \quad (16) \]
where \( k = \sqrt{\frac{m^2 g l^3}{\Theta'}} \), from equation (12). Note that \( \phi = \text{am}(\theta', k^2) \), where the Jacobi amplitude \( \text{am} \) is the inverse function of the elliptic integral of the first kind, and \( \text{sn} \) and \( \text{dn} \) are the Jacobi sine amplitude and delta amplitude functions, respectively.

From the definition of \( k \) in equation (12), one may express the reduced Hamiltonian in the standard form
\[ \Phi = 2mgl \frac{1}{k^2}. \quad (17) \]
Then, by simple differentiation,
\[ \frac{d\Phi}{d\Theta'} = -4mgl \frac{1}{k^3} \frac{dk}{d\Theta'}, \quad (18) \]
which turns equation (13) into
\[ \theta' = -4mgl \sqrt{\frac{T}{g}} k^2 \frac{dk}{d\Theta'} F(\phi, k^2). \quad (19) \]

3. Action-angle variables

Note that, in general, the new variables \((\theta', \Theta')\) defined by the mixed transformation in equations (13) and (14) may be of a different nature than the original variables \((\theta, \Theta)\): an angle and angular momentum, respectively. Indeed, the new Hamiltonian choice leading to equations (15)–(16) assigns \( \theta' \) the dimension of time. However, choosing a transformation that preserves the dimensions of the new variables \((\theta', \Theta')\) as angle and angular momentum—
the so-called transformation to action-angle variables—may provide a deeper geometrical insight to the solution and, furthermore, is especially useful in perturbation theory. Alternative derivations to the one given below can be found in the literature [3].

Hence, the requirement that \( \theta' = \theta'(\theta, \Theta) \) be an angle, given by the condition \( \oint \theta' = 2\pi \) [8], is imposed on equation (13). That is, when \( \theta \) completes a period along an energy manifold \( H(\theta, \Theta) = E \), then \( \theta' \) must vary between \( \theta_0 \) and \( \theta_0 + 2\pi \).

Note in equation (11) that when \( \theta \) evolves between \( 0 \) and \( 2\pi \), \( \phi' \) evolves between \( 0 \) and \( \pi \). Therefore, using equation (13), the angle condition reads

\[
2\pi = \int \frac{d\phi}{k} \left[ F(\pi, k^2) - F(0, k^2) \right],
\]

where

\[
F(0, k^2) = 0, \quad F(\pi, k^2) = 2F(\pi/2, k^2) = 2K(k^2).
\]

Hence,

\[
\frac{d\Phi}{d\Theta'} = \sqrt{\frac{k}{l}} \frac{\pi}{K(k^2)},
\]

which is replaced in equation (13) to give

\[
\theta' = \frac{\pi}{K(k^2)} F(\phi, k^2).
\]  

On the other hand, eliminating \( d\Phi/d\Theta' \) between equations (18) and (20) results in separation of variables. Then, \( \Theta' \) is solved by quadrature to give

\[
\Theta' = \frac{4}{\pi} m l^2 \sqrt{\frac{k}{l}} \frac{1}{k} E(k^2),
\]

where \( E(k^2) = E(\pi/2, k^2) \) is the complete elliptic integral of the second kind, and

\[
E(\phi, k^2) = \int_0^\phi \sqrt{1 - k^2 \sin^2 \theta} \, d\theta
\]

is the incomplete elliptic integral of the second kind.

It is worth noting that the elliptic modulus cannot be solved explicitly from equation (22), and hence the reduced Hamiltonian \( \Phi \) must remain as an implicit function of \( \Theta' \) in the standard form of equation (17). However, the rotation frequency of \( \theta' \) is trivially derived from Hamilton equations by using equation (20), namely

\[
n_{\theta'} = \frac{d\theta'}{dr} = \frac{d\Phi}{d\Theta'} = \sqrt{\frac{k}{l}} \frac{\pi}{K(k^2)}.
\]

In summary, starting from \( (\theta, \Theta) \), corresponding action-angle variables \( (\theta', \Theta') \) are computed from the algorithm as follows:

1. compute \( E = H(\theta, \Theta) \) from equation (1);
2. compute \( k \) from equation (12) with \( \Phi = E \);
3. compute \( \Theta' \) from equation (22), and \( \theta' \) from equation (21), where \( \phi = \theta/2 \).

On the other hand, starting from \( (\theta', \Theta') \), the inverse transformation is computed as follows:

1. solve the implicit equation (22) for \( k \);
2. invert the elliptic integral of the first kind in equation (21) to compute
\[ \phi = \text{am}\left( \pi^{-1}\text{K}(k^2)\theta', k^2 \right); \]

3. \( \theta = 2\phi \), and \( \Theta \) is evaluated from equation (14).

4. Series expansions

In spite of the fact that evaluation of elliptic functions is standard these days, using approximate expressions based on trigonometric functions will provide a better insight into the nature of the solution and may be accurate enough for different applications.

Thus, equations (21) and (14) are written
\[ \theta = 2 \text{am}(u, k^2), \]  \hspace{1cm} (25)
\[ \Theta = 2 ml^2 \frac{2}{k} \text{dn}(u, k^3), \]  \hspace{1cm} (26)
which is formally the same transformation as the one in equations (15) and (16), except for the argument
\[ u \equiv F(\phi, k^2) = \frac{\text{K}(k^2)}{\pi} \theta', \]
which now fulfills the requirement that \( \theta' \) be an angle.

The elliptic functions in equations (25) and (26) are expanded using standard relations\(^4\),
\[ \text{dn}(u, k^2) = \frac{\pi}{\text{K}(k^2)} \left[ \frac{1}{2} + 2 \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \cos(n\theta') \right], \]  \hspace{1cm} (27)
\[ \text{am}(u, k^2) = \frac{1}{2} \theta' + 2 \sum_{n=1}^{\infty} \frac{q^n}{n(1 + q^{2n})} \sin(n\theta'), \]  \hspace{1cm} (28)
where the \textit{elliptic nome} \( q \) is defined as
\[ q = \exp\left[ -\pi \frac{1}{\text{K}(1 - k^2)/\text{K}(k^2)} \right], \]  \hspace{1cm} (29)
and \( k \) must be written explicitly in terms of \( \Theta' \). This is done by expanding \( \Theta' \) as given by equation (22) in the power series of \( k \), viz.
\[ \Theta' = \frac{2\sqrt{g} m^2}{k} \left[ 1 - \frac{1}{4} k^2 - \frac{3}{64} k^4 - \frac{5}{256} k^6 - \frac{175}{16384} k^8 \right. \]
\[ - \frac{441}{65536} k^{10} - \frac{4851}{1048576} k^{12} + O(k^{13}) \].

\(^4\) See http://dlmf.nist.gov/22.11E3 and http://dlmf.nist.gov/22.16.E9 for equations (27) and (28), respectively.
Then, by series reversion,

\[
k = 2\sqrt{\epsilon} \left[ 1 - \epsilon + \frac{5}{4} \epsilon^2 - \frac{7}{4} \epsilon^3 + \frac{161}{64} \epsilon^4 - \frac{239}{64} \epsilon^5 + O(\epsilon^6) \right], \tag{30}\]

where the notation

\[
e = \frac{m^2 g l^3}{\Theta^2} \tag{31}\]

has been introduced. Needless to say, \(\epsilon\) must be small so that the expansions make sense.

Once the required operations have been carried out, the transformation to action-angle variables given by equations (25) and (26) is written as the expansion

\[
\Theta = \Theta' \left[ 1 - \epsilon + \frac{1}{4} \epsilon^2 - \frac{1}{4} \epsilon^3 + \left( 1 + \frac{1}{4} \epsilon^2 + \frac{3}{16} \epsilon^4 \right) \cos 2\Theta' \right.
\]

\[
+ \frac{1}{4} \epsilon^2 \cos 2\Theta' + \left( \frac{1}{2} + \frac{1}{4} \epsilon + \frac{3}{32} \epsilon^3 \right) \cos 30' + O(\epsilon^6), \tag{32}\]

\[
\theta = \theta' + \left( 1 + \frac{11}{16} \epsilon^2 + \frac{247}{256} \epsilon^4 \right) \sin \theta' \]

\[
+ \left( \frac{1}{8} + \frac{3}{16} \epsilon^2 \right) \sin 2\theta' + \left( \frac{1}{48} + \frac{3}{64} \epsilon^2 \right) \sin 3\theta' \]

\[
+ \frac{1}{256} \epsilon \sin 4\theta' + \frac{1}{1280} \epsilon^5 \sin 5\theta' + O(\epsilon^6), \tag{33}\]

Proceeding analogously, the standard Hamiltonian in equation (17) is expanded like as

\[
\Phi = \frac{\Theta^2}{2 l^2 m} \left( 1 + \frac{1}{2} \epsilon^2 + \frac{5}{32} \epsilon^4 + \frac{9}{64} \epsilon^6 + \ldots \right). \tag{34}\]

5. Conclusion

Action-angle variables are widely accepted to be a very useful set of canonical coordinates in the solution of integrable problems. In the case of the simple pendulum, the solution of the equations of motion in action-angle variables can be achieved in closed form by standard application of the Hamilton–Jacobi method. However, the fact that this solution relies on the use of special functions, which besides may need to solve implicit functions to evaluate their arguments, makes it desirable to find explicit approximations to the solution that are accurate enough.

In particular, the Fourier series expansion of the transformation to action-angle variables of the simple pendulum solution in the rotation regime is provided here up to a higher order. In spite of the fact that expanding elliptic integrals and functions is a cumbersome task, one may find non-negligible help in using corresponding formulas that are available in the literature. In addition, when higher orders of the expansions are required, the use of algebraic manipulators can provide fundamental assistance in deriving and simplifying the transformation equations.
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