A GAUSS-KUZMIN THEOREM FOR SOME CONTINUED FRACTION EXPANSIONS

Dan Lascu
Mircea cel Batran Naval Academy, 1 Fulgerului, 900218 Constanta, Romania
Katsunori Kawamura†
College of Science and Engineering, Ritsumeikan University,
1-1-1 Noji Higashi, Kusatsu, Shiga 525-8577, Japan

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Abstract
We consider a family of continued fraction expansions of any number in the unit closed interval [0, 1] whose digits are differences of consecutive non-positive integer powers of an integer \( m \geq 2 \). For this expansion, we apply the method of Rockett and Szusz from \cite{6} and obtained the solution of its Gauss-Kuzmin type problem.

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1 Introduction
The purpose of this paper is to prove a Gauss-Kuzmin type problem for non-regular continued fraction expansions introduced by Chan \cite{1}.

1.1 Gauss’ Problem
One of the first and still one of the most important results in the metrical theory of continued fractions is so-called Gauss-Kuzmin theorem. Write \( x \in [0, 1) \) as a regular continued fraction

\[
x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} := [a_1, a_2, a_3, \ldots],
\]

*e-mail: lascudan@gmail.com.
†e-mail: kawamura@kurims.kyoto-u.ac.jp.
where \(a_n \in \mathbb{N}_+ := \{1, 2, 3, \ldots\}\). The metrical theory of continued fractions started on 25th October 1800, with a note by Gauss in his mathematical diary. Gauss wrote that (in modern notation)

\[
\lim_{n \to \infty} \lambda(\tau^n \leq x) = \frac{\log(1 + x)}{\log 2}, \quad x \in I := [0, 1].
\]

Here \(\lambda\) is Lebesgue measure and the map \(\tau : [0, 1) \to [0, 1)\), the so-called regular continued fraction (or Gauss) transformation, is defined by

\[
\tau(x) := \frac{1}{x} - \left[\frac{1}{x}\right], \quad x \neq 0; \quad \tau(0) := 0,
\]

where \([\cdot]\) denotes the floor (or entire) function. Gauss’ proof (if any) has never been found. A little more than 11 years later, in a letter dated 30 January 1812, Gauss asked Laplace to estimate the error

\[
e_n(x) := \lambda(\tau^{-n}[0, x]) - \frac{\log(1 + x)}{\log 2}, \quad n \geq 1, \quad x \in I.
\]

This has been called Gauss’ Problem. It received a first solution more than a century later, when R.O. Kuzmin (see [3]) showed in 1928 that \(e_n(x) = O(q^\sqrt{n})\) as \(n \to \infty\), uniformly in \(x\) with some (unspecified) \(0 < q < 1\). One year later, using a different method, Paul Lévy (see [5]) improved Kuzmin’s result by showing that \(|e_n(x)| \leq q^n\), \(n \in \mathbb{N}_+, \quad x \in I\), with \(q = 3.5 - 2\sqrt{2} = 0.67157\ldots\). The Gauss-Kuzmin-Lévy theorem is the first basic result in the rich metrical theory of continued fractions.

### 1.2 A non-regular continued fraction expansion

In this paper, we consider a generalization of the Gauss transformation and prove an analogous result. This transformation was studied in detail by Chan in [1] and Lascu and Kawamura in [4].

In [1], Chan shows that any \(x \in [0, 1)\) can be written in the form

\[
x = \frac{m^{-a_1(x)}}{1 + \frac{(m-1)m^{-a_2(x)}}{1 + \frac{(m-1)m^{-a_3(x)}}{1 + \cdots}}} := [a_1(x), a_2(x), a_3(x), \ldots]_m, \quad (1.1)
\]

where \(m \in \mathbb{N}_+, \quad m \geq 2\) and \(a_n(x)\)’s are non-negative integers.

For any \(m \in \mathbb{N}_+\) with \(m \geq 2\), define the transformation \(\tau_m\) on \(I\) by

\[
\tau_m(x) = \begin{cases} 
\frac{m \left\{\log x^{-1}\right\}}{\log m} - 1, & \text{if } x \neq 0 \\
0, & \text{if } x = 0,
\end{cases} \quad (1.2)
\]
where \{\cdot\} stands for fractionary part. It is easy to see that \(\tau_m\) maps the set \(\Omega\) of irrationals in \(I\) into itself. For any \(x \in (0, 1)\) put
\[
a_n(x) = a_1(\tau_m^{n-1}(x)), \quad n \in \mathbb{N}_+,
\]
with \(\tau_m^0(x) = x\) and
\[
a_1(x) = \begin{cases} \lfloor \log x^{-1}/\log m \rfloor, & \text{if } x \neq 0 \\ \infty, & \text{if } x = 0. \end{cases}
\]

Transformation \(\tau_m\) which generates the continued fraction expansion (1.1) is ergodic with respect to an invariant probability measure, \(\gamma_m\), where
\[
\gamma_m(A) = k_m \int_A \frac{dx}{((m-1)x+1)((m-1)x+m)}, \quad A \in \mathcal{B}_I,
\]
with \(k_m = \frac{(m-1)^2}{\log(m^2/(2m-1))}\) and \(\mathcal{B}_I\) is the \(\sigma\)-algebra of Borel subsets of \(I\) (which, by definition, is the smallest \(\sigma\)-algebra containing intervals).

Next, we briefly present the results of Lascu and Kawamura in [4]. Were given some basic metric properties of the continued fraction expansion in (1.1). Hence, it was found the Brodén-Borel-Lévy formula used to determine the probability structure of \((a_n)_{n \in \mathbb{N}_+}\) under \(\lambda\) (=the Lebesgue measure).

Also, it was given the so-called natural extension \(\tau_m\), and were defined and studied the extended incomplete quotients \(a_l\), \(l \in \mathbb{Z}\). The associated Perron-Frobenius operator under different probability measures on \(\mathcal{B}_I\) was derived. The Perron-Frobenius operator of \(\tau_m\) under the invariant measure \(\gamma_m\) induced by the limit distribution function was studied and it was derived the asymptotic behaviour of this operator. Also, the Perron-Frobenius operator was restricted to the linear space of all complex-valued functions of bounded variation and to the space of all bounded measurable complex-valued functions.

The main result in [4] is the solving of Gauss-Kuzmin type problem using the method of random systems with complete connections by Iosifescu [2].

1.3 Main theorem

We show our main theorem in this subsection. For this purpose let \(\mu\) be a non-atomic probability measure on \(\mathcal{B}_I\) and define
\[
F_n(x) = \mu(\tau_m^n < x), \quad x \in I, \quad n \in \mathbb{N},
\]
\[
F(x) = \lim_{n \to \infty} F_n(x), \quad x \in I,
\]
with \(F_0(x) = \mu([0, x])\). Our main result is the following theorem.

Theorem 1.1. Let \(k_m = \frac{(m-1)^2}{\log(m^2/(2m-1))}\). Then there exists
\[
F_n(x) = \frac{k_m}{(m-1)^2} \log \frac{m((m-1)x+1)}{(m-1)x+m} + \mathcal{O}(q_m^n),
\]
where \(0 < q_m < 1\).
2 Proof of Theorem 1.1

First, we show that \( \{F_n\} \), defined in (1.4), satisfy a Gauss-Kuzmin-type equation, i.e., the following holds.

**Proposition 2.1.** If \( \{F_n\} \) are the functions defined in (1.4), then \( F_n \) satisfies the following Gauss-Kuzmin-type equation

\[
F_{n+1}(x) = \sum_{i \in \mathbb{N}} \left( F_n \left( \alpha^i \right) - F_n \left( \frac{\alpha^i}{1 + (m-1)x} \right) \right), \quad x \in I, \ n \in \mathbb{N}. \tag{2.1}
\]

**Proof.** Let \( \alpha = 1/m \). Since \( \tau^n_m(x) = \frac{m^{-a_{n+1}(x)}}{1 + (m-1)\tau^{n+1}_m(x)} \) it follows that

\[
F_{n+1}(x) = \mu \left( \tau^{n+1}_m > x \right) = \sum_{i \in \mathbb{N}} \mu \left( \frac{\alpha^i}{1 + (m-1)x} < \tau^n_m < \alpha^i \right)
= \sum_{i \in \mathbb{N}} \left( F_n \left( \alpha^i \right) - F_n \left( \frac{\alpha^i}{1 + (m-1)x} \right) \right).
\]

Assuming that for some \( p \in \mathbb{N} \) the derivative \( F'_p \) exists everywhere in \( I \) and is bounded, it is easy to see by induction that \( F'_{p+n} \) exists and is bounded for all \( n \in \mathbb{N}_+ \). This allows us to differentiate (1.4) term by term, obtaining

\[
F'_{n+1}(x) = \sum_{i \in \mathbb{N}} \frac{(m-1)\alpha^i}{1 + (m-1)x} F'_n \left( \frac{\alpha^i}{1 + (m-1)x} \right). \tag{2.2}
\]

Further, write \( f_n(x) = (1 + (m-1)x)(m + (m-1)x)F'_n(x), \ x \in I, \ n \in \mathbb{N}, \) then (2.2) is rewritten as

\[
f_{n+1}(x) = \sum_{i \in \mathbb{N}} P^i_m ((m-1)x) f_n \left( \frac{\alpha^i}{1 + (m-1)x} \right), \tag{2.3}
\]

where

\[
P^i_m(x) = \frac{(m-1)\alpha^{i+1}(x + 1)(x + m)}{(x + (m-1)\alpha^{i+1})(x + (m-1)\alpha^{i+1} + 1)}, \quad x \in I. \tag{2.4}
\]

Putting \( \Delta_i = \alpha^i - \alpha^{2i}, \ i \in \mathbb{N} \), we get

\[
P^i_m ((m-1)x) = (m-1) \left[ \alpha^{i+1} + \frac{\Delta_i}{(m-1)x + (m-1)\alpha^{i+1} + 1} \right] - \frac{\Delta_{i+1}}{(m-1)x + (m-1)\alpha^{i+1} + 1}.
\]

**Proof of Theorem 1.1** Introduce a function \( R_n(x) \) such that

\[
F_n(x) = \frac{k_m}{(m-1)^2} \log \frac{m((m-1)x+1)}{(m-1)x + m} + R_n \left( \frac{k_m}{(m-1)^2} \log \frac{m((m-1)x+1)}{(m-1)x + m} \right), \tag{2.5}
\]
where $k_m = \frac{(m-1)^2}{\log (m^2/(2m-1))}$.

Because $F_n(0) = 0$ and $F_n(1) = 1$, we have $R_n(0) = R_n(1) = 0$. To prove the theorem, we have to show that

$$R_n(x) = \mathcal{O}(q_m^n),$$

(2.6)

where $0 < q_m < 1$.

If we can show that $f_n(x) = k_m + \mathcal{O}(q_m^n)$, then its integration will show the equation (1.5).

To demonstrate that $f_n(x)$ has this desired form, it suffices to establish that $f_n'(x) = \mathcal{O}(q_m^n)$.

We have

$$(P_m^i((m-1)x))' = (m-1) \left[ \frac{(m-1)\Delta_{i+1}}{((m-1)x + (m-1)\alpha_i^{i+1} + 1)^2} - \frac{(m-1)\Delta_i}{((m-1)x + (m-1)\alpha_i^{i} + 1)^2} \right].$$

Now from (2.3), we have

$$f_{n+1}'(x) = \sum_{i \in \mathbb{N}} \left[ (P_m^i((m-1)x))' f_n \left( \frac{\alpha_i^{i+1}}{1 + (m-1)x} \right) f_n' \left( \frac{\alpha_i^i}{1 + (m-1)x} \right) - P_m^i((m-1)x) \frac{(m-1)\alpha_i^i}{((m-1)x + 1)^2} f_n' \left( \frac{\alpha_i^i}{1 + (m-1)x} \right) \right]$$

(2.7)

where

$$A_i = \frac{(m-1)\Delta_{i+1}}{((m-1)x + (m-1)\alpha_i^{i+1} + 1)^2} \times \left( f_n \left( \frac{\alpha_i^{i+1}}{1 + (m-1)x + 1} \right) - f_n \left( \frac{\alpha_i^i}{1 + (m-1)x + 1} \right) \right)$$

$$B_i = P_m^i((m-1)x) \frac{(m-1)\alpha_i^i}{((m-1)x + 1)^2} f_n' \left( \frac{\alpha_i^i}{1 + (m-1)x} \right).$$

By applying the mean value theorem of calculus to the difference

$$f_n \left( \frac{\alpha_i^{i+1}}{1 + (m-1)x + 1} \right) - f_n \left( \frac{\alpha_i^i}{1 + (m-1)x + 1} \right),$$

we obtain

$$f_n \left( \frac{\alpha_i^{i+1}}{1 + (m-1)x + 1} \right) - f_n \left( \frac{\alpha_i^i}{1 + (m-1)x + 1} \right) = \frac{\alpha_i^{i+1} - \alpha_i^i}{(m-1)x + 1 - (m-1)x + 1} f_n'(\theta_i),$$

where $0 < \theta_i < 1$.
with \[
\frac{\alpha^{i+1}}{(m-1)x+1} < \theta_i < \frac{\alpha^i}{(m-1)x+1}.
\]
Thus, from (2.7), we have
\[
f'_{n+1}(x) = (m-1)^2 \sum_{i \in \mathbb{N}} \frac{\alpha^{i+1} \Delta_{i+1}}{((m-1)x+1)((m-1)x+(m-1)\alpha^{i+1}+1)^2} f_n'(\theta_i) \\
-(m-1) \sum_{i \in \mathbb{N}} P^i_m((m-1)x) \frac{\alpha^i}{((m-1)x+1)^2} f_n'\left(\frac{\alpha^i}{1+(m-1)x}\right). \tag{2.8}
\]

Let \(M_n\) be the maximum of \(|f'_{n}(x)|\) on \(I\), i.e., \(M_n = \max_{x \in I} |f'_{n}(x)|\). Then (2.8) implies
\[
M_{n+1} \leq M_n \cdot \max_{x \in I} \left| (m-1)^2 \sum_{i \in \mathbb{N}} \frac{\alpha^{i+1} \Delta_{i+1}}{((m-1)x+1)((m-1)x+(m-1)\alpha^{i+1}+1)^2} \\
+ (m-1) \sum_{i \in \mathbb{N}} P^i_m((m-1)x) \frac{\alpha^i}{((m-1)x+1)^2} \right|. \tag{2.9}
\]
We now must calculate the maximum value of the sums in this expression.

First, we note that
\[
\frac{\alpha^{i+1} \Delta_{i+1}}{(m-1)x+1)((m-1)x+(m-1)\alpha^{i+1}+1)^2} \leq \frac{\alpha^{2i+2}}{((m-1)\alpha^{i+1}+1)^2}. \tag{2.10}
\]
Note that, in the upper inequality, we have used that \(\Delta_i = \alpha^i - \alpha^{2i}\) and that \(0 \leq x \leq 1\).

Next, observe that the function
\[
h_m(x) := \frac{\alpha^i P^i_m((m-1)x)}{((m-1)x+1)^2}
\]
is decreasing for \(x \in I\) and \(i \in \mathbb{N}\). Hence, \(h_m(x) \leq h_m(0)\). This leads to
\[
\frac{\alpha^i P^i_m((m-1)x)}{((m-1)x+1)^2} \leq \frac{(m-1)\alpha^{2i}}{((m-1)\alpha^{i+1}+1)^2}. \tag{2.11}
\]
The relations (2.10) and (2.11) allow us to rewrite inequality (2.9) as
\[
M_{n+1} \leq q_m \cdot M_n, \tag{2.12}
\]
where
\[
q_m := (m-1)^2(m^2+1) \sum_{i \in \mathbb{N}} \frac{1}{(m^{i+1} + m - 1)^2}. \tag{2.13}
\]
Since, for \(i \geq 2\), we have
\[
\frac{1}{(m^{i+1} + m - 1)^2} \leq \frac{1}{m^2(m-1)^2(m^2+1)} \left(\frac{1}{m}\right)^i,
\]
therefore,

\[ q_m \leq (m - 1)^2(m^2 + 1) \]

\[ \times \left( \frac{1}{(2m - 1)^2} + \frac{1}{m^2 + m - 1} + \frac{1}{m^2(m - 1)^2 + 1} \sum_{i \geq 2} \left( \frac{1}{m} \right)^i \right) \]

\[ = (m - 1)^2(m^2 + 1) \left( \frac{1}{(2m - 1)^2} + \frac{1}{m^2 + m - 1} + \frac{1}{m^3(m - 1)^3(m^2 + 1)} \right) \]

\[ \leq 1, \]

for any \( m \in \mathbb{N}, m \geq 2. \) \( \Box \)

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