Multivariate Subjective Fiducial Inference

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Abstract: The aim of this paper is to firmly establish subjective fiducial inference as a rival to the more conventional schools of statistical inference, and to show that Fisher’s intuition concerning the importance of the fiducial argument was correct. In this regard, a methodology outlined in an earlier paper is modified, enhanced and extended to deal with general inferential problems in which various parameters are unknown. As part of this, an analytical method or the Gibbs sampler is used to construct the joint fiducial distribution of all the parameters of the model concerned on the basis of first determining the full conditional fiducial distributions for these parameters. Although the resulting theory is classified as being ‘subjective’, it is maintained that this is simply due to the argument that all probability statements made about fixed but unknown parameters must be inherently subjective. In particular, a systematic framework is used to reason that, in general, there is no need to place a great emphasis on the difference between the fiducial probabilities that can be derived using this theory and objective probabilities. Some important examples of the application of this theory are presented.

Keywords: Data generating algorithm; Fiducial statistic; Full conditional fiducial distributions; Gibbs sampler; Primary random variable; Strength of probabilities.
1. Introduction

R. A. Fisher is one of the greatest, if not the greatest, statistician that has ever lived. Many of his contributions to statistical theory were considered to be revolutionary, but one concept that he developed and discussed at length, namely the fiducial argument, has as yet failed to gain many advocates. It is clear from his writings on the subject (see for example Fisher 1930, 1935, 1956) that he regarded this concept as being the foundation of a school of inference to rival the other two main schools of inference that still flourish to this day, that is Neyman-Pearson and Bayesian inference. However, since Fisher’s death in 1962, few have attempted to develop the fiducial argument into a separate school of inference, notable exceptions being the work of D. A. S. Fraser on structural inference, see Fraser (1966, 1972), and the theory contained in Wilkinson (1977).

There has been more activity, on the other hand, in attempting to use the fiducial argument to support other schools of inference. In particular it has been used to support Dempster-Shafer theory, see Dempster (1968) and Shafer (1976), and the Neyman-Pearson school of inference by means of both generalized fiducial inference, see Hannig (2009) and Hannig et al. (2016), and confidence distribution theory, see Xie and Singh (2013). Also it can be viewed as supporting the theories of Dempster-Shafer and Neyman-Pearson simultaneously as part of the recently developed theory of inferential models, see Martin and Liu (2015).

Nevertheless, for those who respect the intellect and intuition of Fisher, it may be disappointing to see that one of his most cherished theories has been reduced to only a subsidiary role in theories that have quite a distinct aim from what he had in mind. Furthermore, the same people may be surprised by the fact that, even to this day, it is often argued that fiducial inference is so closely related to Bayesian inference that if, as in many cases, the fiducial distribution is equal to the posterior distribution for some choice of the prior distribution then the two theories are indistinguishable, see Lecoutre
and Poitevineau (2014) and Liu and Martin (2015) and the work of many others. In summary, it could be rather colourfully said that, in recent years, fiducial inference has been like the wreck of a vintage car, which finds itself parked in a backstreet, sprayed with graffiti by youths who do not appreciate its uniqueness and inner beauty, and robbed by opportunists for spare parts to use in vehicles considered to be more commercially viable.

Following on from Bowater (2017b), the aim of the present paper is to attempt to address this sorry state of affairs. In particular, the theory of subjective fiducial inference presented in Bowater (2017b) will be modified, enhanced and extended to deal with general inferential problems in which various parameters are unknown.

Let us briefly outline the structure of the paper. A motivation for the theory is given in the following section and the concept of probability that will be used is explained in Section 3. After introducing some key definitions and proposing some useful procedures in Section 4, the resulting methodology is applied to various examples in Section 5. The specification of the conceptual framework is then completed in Section 6. The final two sections of the paper clarify the merits of the theory and discuss some open issues.

2. Motivation

The need for an alternative theory of inference is motivated by the inadequacies of established theories in addressing the issue of how to make inferences on the basis of data when nothing or little was known about the parameters of interest before the data was observed. Here we will briefly review the inadequacies of two such theories with regard to how they tackle this issue, namely objective Bayesian inference and frequentist post-data inference.

Objective Bayesian inference is a form of Bayesian inference that is based on prior distributions that have the property that the information contained within them has in some way been minimized, either explicitly or implicitly, compared to the information.
that is expected to be contained in the data. Advocates of this type of inference would argue that it offers a collection of methods that attempt to standardize the way in which a prior distribution can be formed that, through the posterior distribution, allows the data to ‘speak for themselves’. However, objective Bayesian inference faces the following severe criticisms:

1) If an objective Bayesian analysis is to avoid that inferences are dependent on the way the sampling model is parameterized, which was a drawback of the classical Bayesian methods based on the principle of insufficient reason that were proposed by both T. Bayes and P. S. Laplace, then the choice of prior will need to depend on the sampling model, as is the case in the methods that can be found for example in Jeffreys (1961), Kass and Wasserman (1996) and Ghosh (2011). However, as highlighted by many (see for example Seidenfeld 1979 and Lindley 1997), this completely breaks the logic of Bayesian theory as our state of knowledge about a parameter will depend on how we intend to go about collecting more information about the parameter.

2) Priors derived using objective Bayesian methods are very often improper. Such priors break the standard rules of probability without any special permission for doing so, and are therefore purely mathematical creations that have no direct real-world meaning.

3) Even taking into account the principle of stable estimation (see Edwards, Lindman and Savage 1963) and even when the sample size is large, posterior distributions will be generally very sensitive to differences in the priors that are derived by different objective Bayesian methods. Therefore, it is vital that there is a consensus on which objective Bayesian method is the best one to use, but such a consensus does not exist. Some methods even lead to different priors depending on what is the parameter of main interest, e.g. the method outlined in Bernardo (1979) and Berger and Bernardo (1992).

Let us now consider frequentist post-data inference, which we will assume refers
to the type of conditional frequentist approaches to inference outlined in Goutis and Casella (1995). The motivation for these methods clearly stems from the difficulties that arose from Fisher’s attempt to justify the fiducial argument in terms of frequentist probability. The reason that he chose to do this would seem to have come from a desire to place the fiducial argument on an objective footing and, in terms of quantifying uncertainty, objectivity for Fisher meant frequentist probability. To be more specific, the crux of Fisher’s line of reasoning (as it is presented in Fisher 1956) was that a fiducial interval for a parameter can only be valid in a frequentist sense if the sample space contains no subsets that are recognizable with respect to this interval. However, Buehler and Feddersen (1963) showed that even in one of the simplest and most common problems of inference, that of making inferences about the mean $\mu$ of a normal distribution when its variance $\sigma^2$ is unknown, recognizable subsets exist with respect to the standard fiducial interval for $\mu$. To this day, the ubiquity of recognizable subsets in proposed solutions to problems of inference represents a major obstacle to the further development of the theory of frequentist post-data inference.

3. A note about probability

Let us begin by establishing the concept of probability that will be used in this paper. We observe that it would be difficult to argue that subjective probability is a meaningless concept. The fact that a meteorological expert can say that he believes the probability of rain tomorrow is 0.3 and others find this information useful would seem to satisfactorily refute such an argument. Some would argue that subjective probability is the only concept of probability that is required, such that distinctions between different types of probability is essentially pointless. Such a view is common amongst advocates of the Bayesian paradigm, see for example, de Finetti (1974, 1975) and Savage (1954).

Perhaps the most standard position to take on whether probabilities are of different
types is to contend that two types of probability exist, one being subjective probability based on some kind of elicitation method, and the other being frequentist probability based on calculating the long run proportion of times a repeatable experiment produces a given outcome. In this viewpoint, it would appear that the value that is assigned as the probability of any given event is not sufficient to fully define the probability concerned as we also need to know whether the probability is subjective or frequentist. Also, the prevalence of this viewpoint naturally gives importance to the Bayesian frequentist controversy, which arises due to the fact that, according to given criteria, inferences made using the Bayesian approach often conflict with inferences made using the Neyman-Pearson (frequentist) approach.

This paper will rely on the definition of probability originally presented in Bowater (2017a), where it was referred to as type B probability, which was subsequently extended to probability distributions in Bowater (2017b). Under this definition, probability comprises of two components, namely a probability value, which is the sole recognized component in conventional definitions of probability, and the strength assigned to this probability value. Therefore, probabilities can be big and weak, small and strong, big and strong, small and weak etc. The strength component allows an ordered classification of probability types, and therefore is more sophisticated than the standard dichotomous system of classifying probabilities as simply being subjective or frequentist. For a full definition and explanation of this concept of probability, the reader is referred to the two aforementioned papers. Nevertheless, to summarize how this definition of probability can be used to determine a probability value and its strength for a single event, a modified version of an example that appears in Bowater (2017a) will now be presented.

Let us suppose that an individual wishes to determine his probability for the event of a first-term US president being reelected in three years’ time, which will be referred to as the event \( A \). From the earlier papers, it can be seen that we must first decide upon a
reference set of events \( R = \{ R_1, R_2, \ldots, R_k \} \). Taking into account the likely precision by which he may be able to determine his probability value for the event \( A \), let us imagine that the individual decides that the events \( R_i \) correspond to each of the outcomes of drawing a ball from an urn of 20 distinctly labelled balls. With the event \( R(\lambda) \) defined by substituting \( k = 20 \) into the general definition of this event, i.e.

\[
R(\lambda) = \begin{cases} 
    R_1 \cup R_2 \cup \cdots \cup R_{\lambda k} & \text{if } \lambda \in \{\Lambda, 1\} \\
    \emptyset & \text{if } \lambda = 0
\end{cases}
\]

where \( \Lambda = \{1/k, 2/k, \ldots, (k-1)/k\} \), his probability value for the event \( A \) is then defined as being the unique value of \( \lambda \in \{0, 0.05, 0.1, \ldots, 1\} \) that maximizes the similarity \( S(A, R(\lambda)) \), i.e. the similarity between his conviction that the event \( A \) will occur and his conviction that the event \( R(\lambda) \) will occur. Let us assume that this value is 0.7. Therefore, it is being assumed that the individual is capable of asserting that, in his opinion, the similarities \( S(A, R(0.65)) \) and \( S(A, R(0.75)) \) are less than the similarity \( S(A, R(0.7)) \), which seems a reasonable assumption to make.

Now let us consider an event associated with spinning what is known as a probability wheel (see Spetzler and Stael von Holstein 1975) which consists essentially of a rotatable disc with a fixed pointer in its centre. Assuming that the area of the disc is divided into a red sector and a blue sector, let the event of interest be the event of the pointer coming to rest in the red sector, which will be referred to as the event \( B \). If the proportion of the area of the disc that is red is 0.7, then using the definition of probability being considered it would not be at all surprising if, with respect to the aforementioned reference set \( R \), the individual assigned a probability value of 0.7 to the event \( B \).

However, although a probability value of 0.7 has been assigned to both the events \( A \) and \( B \), the strength that is associated with this probability value when it is assigned to event \( A \) is likely to be different from when it is assigned to event \( B \), even under the assumption that the probability value for the event \( A \) has been determined as precisely as possible. In particular, it is likely that the similarity \( S(A, R(0.7)) \) will be considered
to be substantially less than the similarity $S(B, R(0.7))$, which is equivalent to asserting that the probability of 0.7 is a much weaker probability for the event $A$ than for the event $B$. The reason for this should be fairly evident, since the nature of the uncertainty about whether event $A$ will occur is clearly different from the nature of the uncertainty about both whether event $R(0.7)$ will occur and whether event $B$ will occur. More specifically, the factors that can influence whether or not event $A$ will occur are likely to be considered vague and difficult to weigh up, while events $R(0.7)$ and $B$, on the other hand, are the outcomes of two standard types of physical experiment.

The idea that a probability comprises of both a probability value and its strength is supported by the need to explain the expression of ambiguity aversion in decision making, which is an issue that has been long debated in microeconomic theory (see Ellsberg 1961, Gilboa and Schmeidler 1989 and Alary, Gollier and Treich 2013). In this regard, the definition of probability being considered has been used to undermine the independence axiom upon which the foundations of Bayesian theory depend, since it facilitates a rational explanation of the paradox associated with Ellsberg’s three colour example (see Ellsberg 1961 for the example and Bowater 2017a for the explanation). Therefore, this counters the popular argument that any measure of the uncertainty of an event that is not solely the probability value assigned to the event must be invalid due to the measure not being compatible with Bayesian theory.

The concept of strength can be applied not just to individual probabilities but also to entire probability density functions in the sense that, under additional criteria, one density function can be classified as being weaker or stronger than another density function. Loosely speaking, a probability density $f_X(x)$ is defined as being stronger than another density $g_Y(y)$ at the level of resolution $\alpha$, if probabilities equal to $\alpha$ derived by integrating $f_X(x)$ over subspaces of $x$ are considered to be at least as strong as, and sometimes stronger than, probabilities equal to $\alpha$ derived by integrating $f_Y(y)$ over subspaces of $y$. 


A more detailed definition of this property can be found in Bowater (2017b).

Although in the sections that immediately follow attention will be focused on the determination of probability values and densities rather than on the determination of their strengths, this latter issue needs to be borne in mind. We will explicitly return to the task of completing probability definitions by assigning strengths to probability values and densities in Section 6.

4. Subjective fiducial inference

4.1. Sampling model

In general, it will be assumed that a sampling model that depends on one or various unknown parameters $\theta_1, \theta_2, \ldots, \theta_k$ generates the data $x$. Let the joint density of the data given the true values of $\theta_1, \theta_2, \ldots, \theta_k$ be denoted as $g(x | \theta_1, \theta_2, \ldots, \theta_k)$.

4.2. Univariate case

For the moment, we will assume that the only unknown parameter in the model is $\theta_1$, either because there are no other parameters in the model, or because the true values of the parameters $\theta_2, \ldots, \theta_k$ are known.

Definition 1: Fiducial statistics

Given this assumption, a fiducial statistic $Q(x)$ will be defined as being a one-dimensional sufficient statistic for $\theta_1$ if such a statistic exists, otherwise it may be assumed to be any one-to-one function of a unique maximum likelihood estimator of $\theta_1$.

Assumption 1: Data generating algorithm

Independent of the way in which the data were actually generated, it will be assumed
that the data set \( x \) was generated by the following algorithm:

1) Generate a value \( \gamma \) for a continuous one-dimensional random variable \( \Gamma \), which has a probability density function \( f_{\Gamma}(\gamma) \) that does not depend on the parameter \( \theta_1 \).

2) Determine a value \( q(x) \) for a fiducial statistic \( Q(x) \) by setting \( \Gamma \) equal to \( \gamma \) and \( q(x) \) equal to \( Q(x) \) in the following definition of the distribution of \( Q(x) \):

\[
Q(x) = \varphi(\Gamma, \theta_1)
\]  

(2)

where the function \( \varphi(\Gamma, \theta_1) \) is defined so that it satisfies the following conditions:

**Assumption 1.1: Conditions on the function \( \varphi(\Gamma, \theta_1) \)**

a) The distribution of \( Q(x) \) as defined by equation (2) is equal to what it would have been if \( Q(x) \) had been determined on the basis of the data set \( x \).

b) The only random variable upon which \( \varphi(\Gamma, \theta_1) \) depends is the variable \( \Gamma \).

c) Let \( G = \{ \gamma : f_{\Gamma}(\gamma) > 0 \} \), and let \( H_1 \) be the set of all possible values of \( \theta_1 \) as specified before any information about the data \( x \) has been obtained. If it is assumed that a value for \( Q(x) \) has been generated, but both its corresponding value \( \gamma \) for the variable \( \Gamma \) and the parameter \( \theta_1 \) are unknown, then substituting \( Q(x) \) in equation (2) by whatever value is taken by \( Q(x) \) would imply that this equation would define an injective mapping from the set \( G \) to the set \( H_1 \).

3) Generate the data set \( x \) by conditioning the sampling density \( g(x \mid \theta_1, \theta_2, \ldots, \theta_k) \) on the already generated value for \( Q(x) \).

In the context of the above algorithm, the variable \( \Gamma \) will be referred to as a primary random variable (primary r.v.). However, if the above algorithm was rewritten so that the value \( \gamma \) of the variable \( \Gamma \) was generated by setting it equal to a deterministic function of an already generated value for \( Q(x) \) and the parameter \( \theta_1 \), then \( \Gamma \) would not be a
primary r.v. In relation to Neyman-Pearson theory, a primary r.v. could be classified as a type of pivot that is distinguished in terms of the way it is generated.

**Definition 2: Univariate subjective fiducial distributions**

Given a value \( q(x) \) for a fiducial statistic \( Q(x) \), the subjective fiducial distribution of the parameter \( \theta_1 \) conditional on the parameters \( \theta_2, \theta_3, \ldots, \theta_k \) being known is defined by setting \( Q(x) \) equal to \( q(x) \) in equation (2), and then treating the value \( \theta_1 \) as being a realization of the random variable \( \Theta_1 \), to give the expression:

\[
q(x) = \varphi(\Gamma, \Theta_1)
\]

where \( \Gamma \) has the density function \( f_\Gamma(\gamma) \) defined in step 1 of the data generating algorithm in Assumption 1. This equation implies a valid probability distribution for the parameter \( \theta_1 \) under condition (c) of Assumption 1.1. Also, it can be easily shown that this distribution for \( \theta_1 \) does not depend on the choice made for the fiducial statistic \( Q(x) \).

The classical fiducial argument can be seen through the fact that the distribution of the primary r.v. \( \Gamma \) is the same both before and after the fiducial statistic \( q(x) \) is observed.

### 4.3. Multivariate case

We will now consider the case where all the parameters \( \theta_1, \theta_2, \ldots, \theta_k \) in the sampling model are unknown.

For any given data set \( x \), let us assume that the method outlined in the previous section allows us to define the fiducial density of the parameter \( \theta_j \) conditional on all other parameters \( \theta_{-j} \) for all \( j = 1, 2, \ldots, k \). We will denote this set of full conditional fiducial densities as

\[
f(\theta_j | \theta_{-j}, x) \quad \text{for} \ j = 1, 2, \ldots, k
\]

We know that if these conditional densities determine a joint distribution for the param-
eters $\theta_1, \theta_2, \ldots, \theta_k$ then this distribution must be unique.

**Definition 3: Multivariate subjective fiducial distributions**

Under the assumption that the conditional densities in equation (4) determine a joint distribution for the parameters $\theta_1, \theta_2, \ldots, \theta_k$, we will define this latter distribution as being the joint subjective fiducial distribution of these parameters. Let us denote the corresponding joint density function as $f(\theta_1, \theta_2, \ldots, \theta_k | x)$. It will not be assumed though that this joint density or the set of conditional densities in equation (4) can necessarily be expressed in analytic form.

It is clear that there are two principal difficulties with the application of the method just described. First, we do not possess a universal theorem that states that the full conditional densities in equation (4) always determine a joint distribution for $\theta_1, \theta_2, \ldots, \theta_k$. Therefore, at the very least, we need some reassurance that it is appropriate to make this assumption.

Second, it may be difficult to calculate the expected value of any given function of interest $h(\theta_1, \theta_2, \ldots, \theta_k)$ with respect to the joint fiducial density of $\theta_1, \theta_2, \ldots, \theta_k$, i.e. calculate

$$
E(h(\theta_1, \theta_2, \ldots, \theta_k) | x) = \int_{\theta_1} \int_{\theta_2} \cdots \int_{\theta_k} h(\theta_1, \theta_2, \ldots, \theta_k) f(\theta_1, \theta_2, \ldots, \theta_k | x) d\theta_1 d\theta_2 \ldots d\theta_k
$$

(5)

While this difficulty also arises with respect to posterior densities of various parameters obtained using Bayesian inference, it would seem to be more acute in the case of subjective fiducial inference. This is because such posterior densities can usually be expressed in analytic form at least up to a normalizing constant, whereas by contrast, in many of the cases where this can be done, it will not be possible to obtain analytic expressions for the corresponding joint fiducial densities up to a normalizing constant.

In the sections that immediately follow we will propose two distinct ways of tackling
the aforementioned difficulties.

4.4. An analytical method

Given the set of full conditional densities in equation (4), a way of establishing whether they determine a joint distribution for $\theta_1, \theta_2, \ldots, \theta_k$ is simply to propose such a distribution, calculate the full conditional distributions for the proposed joint distribution and see if they match the full conditionals in equation (4). If they do, then this proves that a joint fiducial distribution exists and it is equal to the proposed joint distribution. Since joint fiducial densities are sometimes identical to joint posterior densities for a given choice of joint prior for $\theta_1, \theta_2, \ldots, \theta_k$, a good proposal, up to a normalizing constant, for the joint fiducial density may often be found by multiplying the likelihood function by a convenient mathematical choice for the joint prior density.

If a joint density for $\theta_1, \theta_2, \ldots, \theta_k$ can be found that has full conditional densities that match those in equation (4), the expected value in equation (5) could be determined using analytic manipulation, or approximated using numerical integration or Monte Carlo methods. Many methods that could be used would be identical or very similar to standard methods that are used to calculate or approximate expected values of functions over joint posterior densities.

4.5. The Gibbs sampler and convergence analysis

If we simply made the assumption that the full conditional densities in equation (4) determine a joint density for $\theta_1, \theta_2, \ldots, \theta_k$, then a natural way to try to generate random values from this joint fiducial density, especially if the number of parameters $k$ is large, would be to look at applying the Gibbs sampler (Geman and Geman 1984 and Gelfand and Smith 1990) to its already specified full conditional densities. As is the case in general with this simulation method, it would work especially well if it is easy to sample from each
of these full conditional densities and the autocorrelation between successively generated samples is low. Once a sufficiently large number $m$ of random samples have been obtained from the joint fiducial density of $\theta_1, \theta_2, \ldots, \theta_k$, the expectation in equation (5) could be approximated using the Monte Carlo estimator

$$\frac{1}{(m - b)} \sum_{i=b+1}^{m} h(\theta_1^{(i)}, \theta_2^{(i)}, \ldots, \theta_k^{(i)})$$

where $\theta_1^{(i)}, \theta_2^{(i)}, \ldots, \theta_k^{(i)}$ is the $i$th sample of parameter values, and $b$ is the number of initial samples that have been classified as belonging to the burn-in phase of the sampler.

As is usual when applying Markov chain Monte Carlo methods, it is necessary to study diagnostics of the samples produced by a Gibbs sampler to try to ascertain whether the sampler converges to its unique stationary distribution before all its $m$ cycles have been completed. If it is felt that this is not the case then the standard alternative strategies are available, e.g. the sampler could simply be run over a larger number of cycles or different starting points for the sampler could be tried. It is well known that these kind of strategies will be more effective in some situations than in others.

If the assumption that the full conditional densities in equation (4) determine a joint distribution for $\theta_1, \theta_2, \ldots, \theta_k$ is removed, then really only one additional complication is introduced. This is that, since the Gibbs sampler may not in fact have a stationary distribution, it may not converge even over an infinite number of cycles. To compensate for not being able to make the assumption that this stationary distribution exists, a generally greater emphasis therefore needs to be placed on checking for convergence of the sampler than when this assumption is beyond doubt, e.g. when sampling from a posterior distribution that is known to be proper. The convergence checks and tests described in Gelman and Rubin (1992), Cowles and Carlin (1996), and Brooks and Gelman (1998) and variants on these methods would seem to be particularly appropriate for this purpose.
5. Applications to multivariate cases

This section will present applications of the methodology detailed in previous sections to cases where more than one parameter is unknown.

5.1. Inference about a normal mean with variance unknown

To begin with, let us consider the standard problem of making inferences about the mean $\mu$ of a normal distribution, when its variance $\sigma^2$ is unknown, on the basis of a sample $x$ of size $n$, i.e. $x = (x_1, x_2, \ldots, x_n)$, drawn from the distribution concerned. Although a solution to this problem using subjective fiducial inference was put forward in Bowater (2017b), it should become clear when the conceptual framework is completed in Section 6 that the approach outlined in the present paper provides a more elegant way in which it can be resolved.

If $\sigma^2$ is known, a sufficient statistic for $\mu$ is the sample mean $\bar{x}$, which can therefore be treated as being the fiducial statistic $q(x)$. Defining the primary r.v. $\Gamma$ as having a standard normal distribution implies that equation (2) can be expressed as

$$\bar{x} = \varphi(\Gamma, \mu) = \mu + (\sigma/\sqrt{n})\Gamma$$

meaning that, according to Definition 2, the fiducial distribution of $\mu$ is defined by

$$\mu | \sigma^2, x \sim N(\bar{x}, \sigma^2/n) \quad (6)$$

which is the standard fiducial distribution for $\mu$ in this problem. On the other hand, if $\mu$ is known, a sufficient statistic for $\sigma^2$ is the variance estimator

$$\hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2 \quad (7)$$

which will be treated as being $q(x)$. Defining $\Gamma$ as having a $\chi^2$ distribution with $n$ degrees of freedom implies that equation (2) can be expressed as

$$\hat{\sigma}^2 = \varphi(\Gamma, \sigma^2) = (\sigma^2/n)\Gamma$$
meaning that the fiducial distribution of $\sigma^2$ is defined by

$$\sigma^2 \mid \mu, x \sim \text{Scale-inv-} \chi^2(n, \hat{\sigma}^2)$$

(8)

i.e. it is a scaled inverse $\chi^2$ distribution with $n$ degrees of freedom and scaling parameter equal to $\hat{\sigma}^2$, which again would be generally accepted as the standard fiducial distribution for the problem concerned.

To verify that the full conditional distributions in equations (6) and (8) determine a joint distribution for $\mu$ and $\sigma^2$ we can use the analytical method outlined in Section 4.4. In particular, the full conditional distributions of the joint proper posterior distribution of $\mu$ and $\sigma^2$ that corresponds to choosing the prior density of $\mu$ and $\sigma^2$ to be the improper density $p(\mu, \sigma^2) \propto 1/\sigma^2$ are identical to the full conditionals in equations (6) and (8). Therefore, the conditional distributions in these equations determine a joint fiducial distribution for $\mu$ and $\sigma^2$ and by integrating over this joint distribution, it can be established that the marginal fiducial distribution for $\mu$ is defined by

$$\mu \mid x \sim t_{n-1}(\bar{x}, s/\sqrt{n})$$

where $s$ is the sample standard deviation, i.e. it is the familiar non-standardised Student $t$ distribution with $n - 1$ degrees of freedom, location parameter equal to $\bar{x}$ and scaling parameter equal to $s/\sqrt{n}$, which again is the standard fiducial distribution for the case in question.

5.2. Inference about both parameters of a Pareto distribution

Let us now consider the problem of making inferences about both the shape parameter $\alpha$ and the scale parameter $\beta$ of a Pareto distribution on the basis of a sample $x$ from the density function concerned, i.e. the function

$$f(y \mid \alpha, \beta) = \begin{cases} \alpha \beta^\alpha y^{-(\alpha+1)} & \text{if } y \geq \beta \\ 0 & \text{otherwise} \end{cases}$$
If $\beta$ is known, a sufficient statistic for $\alpha$ is $\sum_{i=1}^{n} \log x_i$, which will be treated as being the fiducial statistic $q(x)$. Defining the primary r.v. $\Gamma$ as having a Gamma($n$, 1) distribution, i.e. a gamma distribution with shape $n$ and rate 1, implies that equation (2) can be expressed as

$$\sum_{i=1}^{n} \log x_i = \varphi(\Gamma, \alpha) = (\Gamma/\alpha) + n \log \beta$$

meaning that, according to Definition 2, the fiducial distribution of $\alpha$ is defined by

$$\alpha \mid \beta, x \sim \Gamma \left( n, \sum_{i=1}^{n} (\log x_i - \log \beta) \right) \quad (9)$$

On the other hand, if $\alpha$ is known, a sufficient statistic for $\beta$ is the minimum value of the sample, i.e. $\min(x)$, which will be treated as being $q(x)$. Defining $\Gamma$ as having an exponential distribution with rate equal to 1, implies that equation (2) can be expressed as

$$\min(x) = \varphi(\Gamma, \beta) = \exp((\Gamma/n \alpha) + \log \beta)$$

meaning that the fiducial density of $\beta$ is

$$f(\beta \mid \alpha, x) = \begin{cases} \frac{n \alpha}{\beta} \exp(-n \alpha (\log(\min(x)) - \log \beta)) & \text{if } \beta > 0 \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

The full conditional distributions of the joint proper posterior distribution of $\alpha$ and $\beta$ that corresponds to choosing the prior density of $\alpha$ and $\beta$ to be the improper density $p(\alpha, \beta) \propto (\alpha \beta)^{-1}$ are identical to the full conditionals in equations (9) and (10). Therefore, the conditional distributions in these equations determine a joint fiducial density for $\alpha$ and $\beta$, which is defined by

$$f(\alpha, \beta \mid x) \propto \alpha^{n-1} \beta^{n\alpha - 1} \prod_{i=1}^{n} x_i^{-(\alpha+1)}$$

5.3. **Inference about all parameters of a normal quadratic regression model**

To show how the approach outlined in Section 4 can be applied to normal polynomial regression models, let us consider the example of applying this approach to the problem
of making inferences about all the parameters $\beta_0$, $\beta_1$, $\beta_2$ and $\sigma^2$ of a normal quadratic regression model, i.e.

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon \quad \varepsilon \sim N(0, \sigma^2)$$
on the basis of a sample $y_+ = \{(x_i, y_i) : i = 1, 2, \ldots, n\}$ from the model concerned.

Sufficient statistics for each of the parameters $\beta_0$, $\beta_1$, $\beta_2$ and $\sigma^2$ conditional on all parameters except the parameter itself being known are respectively

$$n \sum_{i=1}^n y_i, \quad n \sum_{i=1}^n x_i y_i, \quad n \sum_{i=1}^n x_i^2 y_i \quad \text{and} \quad n \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2$$

which would therefore be suitable fiducial statistics $q(y_+)$. By applying the methodology described in Section 4 to each of these fiducial statistics, it can be shown that the full conditional fiducial distributions for this problem are defined by:

$$\beta_0 \mid \beta_1, \beta_2, \sigma^2, y_+ \sim N \left( \frac{n}{2} \sum_{i=1}^n y_i/n - \beta_1 \sum_{i=1}^n x_i/n - \beta_2 \sum_{i=1}^n x_i^2/n, \frac{\sigma^2}{n} \right) \quad (11)$$

$$\beta_1 \mid \beta_0, \beta_2, \sigma^2, y_+ \sim N \left( \frac{1}{2} \sum_{i=1}^n x_i y_i/n - \beta_0 \sum_{i=1}^n x_i/n - \beta_2 \sum_{i=1}^n x_i^2/n - \beta_1 \sum_{i=1}^n x_i^4/n, \frac{\sigma^2}{n} \right) \quad (12)$$

$$\beta_2 \mid \beta_0, \beta_1, \sigma^2, y_+ \sim N \left( \frac{1}{2} \sum_{i=1}^n x_i^2 y_i/n - \beta_0 \sum_{i=1}^n x_i^2/n - \beta_1 \sum_{i=1}^n x_i^4/n, \frac{\sigma^2}{n} \right) \quad (13)$$

$$\sigma^2 \mid \beta_0, \beta_1, \beta_2, y_+ \sim \text{Scale-inv-\chi}^2 \left( n, \frac{1}{n} \sum_{i=1}^n (y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2 \right) \quad (14)$$

The full conditional distributions of the joint proper posterior distribution of $\beta_0$, $\beta_1$, $\beta_2$ and $\sigma^2$ that corresponds to choosing the joint prior density of these parameters to be the improper density $p(\beta_0, \beta_1, \beta_2, \sigma^2) \propto 1/\sigma^2$ are identical to the full conditionals in equations (11) to (14). Therefore, the conditional distributions in these equations determine a joint fiducial density for $\beta_0$, $\beta_1$, $\beta_2$ and $\sigma^2$, which is defined by

$$f(\beta_0, \beta_1, \beta_2, \sigma^2 \mid y_+) \propto \sigma^{-(n+2)} \exp \left( -\frac{1}{2} \sum_{i=1}^n \frac{(y_i - \beta_0 - \beta_1 x_i - \beta_2 x_i^2)^2}{\sigma} \right)$$
5.4. Inference about both parameters of a gamma distribution

We will now consider the problem of making inferences about both the shape parameter \( \alpha \) and the rate parameter \( \beta \) of a gamma distribution on the basis of a sample \( x \) from the density function concerned, i.e. the function

\[
f(y | \alpha, \beta) = \frac{\beta^\alpha y^{\alpha-1} \exp(-y\beta)}{G(\alpha)} \quad \text{if } y \geq 0, \text{ and 0 otherwise}
\]

where \( G(\alpha) \) is the gamma function evaluated at \( \alpha \).

If \( \alpha \) is known, a sufficient statistic for \( \beta \) is \( \sum_{i=1}^{n} x_i \), which will be treated as being the fiducial statistic \( q(x) \). Defining the primary r.v. \( \Gamma \) as having a Gamma\((n\alpha, 1)\) distribution, implies that equation (2) can be expressed as:

\[
\sum_{i=1}^{n} x_i = \varphi(\Gamma, \beta) = \Gamma / \beta
\]

meaning that the fiducial distribution of \( \beta \) is defined by

\[
\beta | \alpha, x \sim \text{Gamma} \left( n\alpha, \sum_{i=1}^{n} x_i \right) \quad (15)
\]

On the other hand, if \( \beta \) is known, a sufficient statistic for \( \alpha \) is \( \sum_{i=1}^{n} \log x_i \), which will be treated as being \( q(x) \). However, defining the primary r.v. \( \Gamma \) and the required function \( \varphi(\Gamma, \alpha) \) in equation (2) is not straightforward in this case. This is due to the cumulative density function of \( \sum_{i=1}^{n} \log x_i \) not being mathematically very tractable. A rudimentary way of approximating the distribution of \( \sum \log x_i \) is to use the central limit theorem. This is the approximation method that will be adopted here.

It can be shown that the mean and variance of \( \sum \log x_i \) are \( n(\psi(\alpha) - \log \beta) \) and \( n\psi'(\alpha) \) respectively, where \( \psi(\alpha) \) and \( \psi'(\alpha) \) are respectively the digamma and trigamma functions evaluated at \( \alpha \). Therefore, assuming that \( \sum \log x_i \) is approximately normally distributed, equation (2) can be approximated by

\[
\sum_{i=1}^{n} \log x_i = \varphi(\Gamma, \alpha) = n(\psi(\alpha) - \log \beta) + \Gamma \sqrt{n\psi'(\alpha)} \quad (16)
\]
where $\Gamma$ is defined as having a $N(0, 1)$ distribution. If $n$ is sufficiently large then, given a fixed value of $\sum \log x_i$, this equation defines an injective mapping from subsets of values $\gamma$ for the variable $\Gamma$ to the space of $\alpha$ except when these subsets of $\gamma$ values contain extremely positive or negative numbers. Therefore, the function $\varphi(\Gamma, \alpha)$ in this equation approximately satisfies condition (c) of Assumption 1.1, and as a result, under Definition 2, this equation approximately defines the fiducial distribution of $\alpha$ conditional on $\beta$ being known.

Figure 1 shows the results of running a Gibbs sampler on the basis of the conditional fiducial distributions for $\beta$ given $\alpha$ and for $\alpha$ given $\beta$ defined by equations (15) and (16) respectively. The data set $x$ was a random sample of $n = 20$ values from a gamma distribution with $\alpha = 2$ and $\beta = 0.5$. To generate each value from the fiducial distribution for $\alpha$ defined by equation (16), a random value $\gamma$ of the variable $\Gamma$ was first drawn from a $N(0, 1)$ distribution truncated to lie in the interval $[-5, 5]$ and then equation (16) was numerically solved to find the corresponding value of $\alpha$. Truncating the distribution of $\Gamma$ in the way just described meant that there was always an injective mapping from the space of possible values for $\gamma$ to the space of $\alpha$, i.e. condition (c) of Assumption 1.1 was always satisfied.

Figures 1(a) and 1(c) show the progression of one run of 100,000 cycles of the Gibbs sampler in terms of the parameters $\alpha$ and $\beta$ respectively. The histograms in Figures 1(b) and 1(d) were formed on the basis of all samples of $\alpha$ and $\beta$ after the initial 500 samples, which were classified as belonging to the burn-in phase of the sampler, had been removed.

The Gibbs sampler was run from various starting points, and a careful study of appropriate diagnostics corresponding to these runs provided no evidence to suggest that the sampler was failing to converge, or was getting trapped in just one mode of a proper or improper multimodal distribution. With reference to the comments made in Section 4.5, it would therefore seem reasonably safe to assume that the conditional distributions de-
Figure 1: Gibbs sampling of the joint fiducial distribution of the parameters of a gamma distribution

fined by equations (15) and (16) determine a joint fiducial distribution for $\alpha$ and $\beta$, and that we have succeeded in obtaining a series of random samples from this distribution.

5.5. Inference about both parameters of a beta distribution

The next problem we will consider is that of making inferences about both parameters $\alpha$ and $\beta$ of a beta distribution on the basis of a sample $x$ from the density function concerned, i.e. the function

$$f(y \mid \alpha, \beta) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)} \quad \text{if } 0 \leq y \leq 1, \text{ and } 0 \text{ otherwise}$$

where $B(\alpha, \beta)$ is the beta function evaluated at $\alpha$ and $\beta$.

If $\beta$ is known, a sufficient statistic for $\alpha$ is $\sum_{i=1}^{n} \log x_i$, which will be treated as being the fiducial statistic $q(x)$. However, the cumulative density function of $q(x)$ in this case is again not mathematically very tractable. Therefore, similar to what was done in the
previous example, the central limit theorem will be used to approximate the distribution of \( q(x) \).

Using the same procedure as in Section 5.4, it follows that if \( \sum_{i=1}^{n} \log x_i \) is approximately normally distributed then equation (2) can be approximated by

\[
\sum_{i=1}^{n} \log x_i = \varphi(\Gamma, \alpha) = n(\psi(\alpha) - \psi(\alpha + \beta)) + n^{1/2}(\psi' (\alpha) - \psi' (\alpha + \beta))^{1/2} \Gamma
\]  

(17)

where \( \Gamma \sim N(0, 1) \). If \( n \) is sufficiently large, then the function \( \varphi(\Gamma, \alpha) \) in this equation satisfies condition (c) of Assumption 1.1 under the restriction that values of \( \gamma \) that are extremely positive or negative are excluded from the set \( G \), and as a result, under Definition 2, this equation approximately defines the fiducial distribution of \( \alpha \) conditional on \( \beta \) being known.

On the other hand, if \( \alpha \) is known, a sufficient statistic for \( \beta \) is \( \sum_{i=1}^{n} \log(1 - x_i) \), which will be treated as being \( q(x) \). Since the cumulative density function of \( q(x) \) is again not mathematically very tractable, the distribution of \( q(x) \) will be approximated on the basis of the central limit theorem using the same procedure as was just described.

By proceeding in this way, it follows that if \( \sum_{i=1}^{n} \log(1 - x_i) \) is approximately normally distributed then equation (2) can be approximated by

\[
\sum_{i=1}^{n} \log(1 - x_i) = \varphi(\Gamma, \beta) = n(\psi(\beta) - \psi(\alpha + \beta)) + n^{1/2}(\psi' (\beta) - \psi' (\alpha + \beta))^{1/2} \Gamma
\]  

(18)

where again \( \Gamma \sim N(0, 1) \). As was the case before, the function \( \varphi(\Gamma, \beta) \) in this equation approximately satisfies condition (c) of Assumption 1.1 and, as a result, under Definition 2, this equation approximately defines the fiducial distribution of \( \beta \) conditional on \( \alpha \) being known.

Figure 2 shows the results of running a Gibbs sampler on the basis of the full conditional fiducial distributions defined by equations (17) and (18). This figure has been constructed on the basis of one run of 100,000 cycles of this algorithm using the same criteria as were used to construct Figure 1, with the data set \( x \) in question being a random
Figure 2: Gibbs sampling of the joint fiducial distribution of the parameters of a beta distribution

sample of $n = 50$ values from a beta distribution with $\alpha = 8$ and $\beta = 3$. Random values were generated from the fiducial distributions for $\alpha$ and $\beta$ defined by equations (17) and (18) using the same numerical method that was used in Section 5.4 to generate values from the fiducial distribution defined by equation (16).

The Gibbs sampler was run from various starting points and the results provided no evidence to suggest that the sampler was failing to converge or was getting trapped in just one mode of a proper or improper multimodal distribution. Hence, it would seem reasonable to draw similar conclusions to that which were made in Section 5.4 and thereby assume that the conditional distributions defined by equations (17) and (18) determine a joint fiducial distribution for $\alpha$ and $\beta$. 

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5.6. The Behrens-Fisher problem

As a prelude to examining the general problem of making inferences about all the parameters of a bivariate normal distribution based on a data set consisting of realizations of the two variables $X$ and $Y$ described by this distribution, let us consider the special case of this problem in which the covariance of $X$ and $Y$ is assumed to be zero. The data set will be assumed to comprise of a sample $x$ of $n_x$ realizations of the variable $X$ and a sample $y$ of $n_y$ realizations of the variable $Y$. Let $\mu_x$ and $\mu_y$ denote the means of the variables $X$ and $Y$ respectively, and let the variances of $X$ and $Y$ be denoted by $\sigma^2_x$ and $\sigma^2_y$ respectively.

On the basis of the results presented in Section 5.1, it is clear that the full conditional fiducial distributions for this problem are defined by first substituting $\mu_x$ for $\mu$, $\sigma^2_x$ for $\sigma^2$ and $n_x$ for $n$ into equations (6), (7) and (8) to obtain the conditional fiducial distributions $f(\mu_x | \mu_y, \sigma^2_x, \sigma^2_y, x, y)$ and $f(\sigma^2_x | \mu_x, \mu_y, \sigma^2_y, x, y)$, and then by substituting $\mu_y$ for $\mu$, $\sigma^2_y$ for $\sigma^2$, $n_y$ for $n$ and the sample $y$ for the sample $x$ into the same equations to get the conditional fiducial distributions $f(\mu_y | \mu_x, \sigma^2_x, \sigma^2_y, x, y)$ and $f(\sigma^2_y | \mu_x, \mu_y, \sigma^2_x, x, y)$.

The full conditional distributions of the joint proper posterior distribution of $\mu_x$, $\mu_y$, $\sigma^2_x$, $\sigma^2_y$ that corresponds to choosing the prior density of $\mu_x$, $\mu_y$, $\sigma^2_x$, $\sigma^2_y$ to be the improper density $p(\mu_x, \mu_y, \sigma^2_x, \sigma^2_y) \propto 1/\sigma^2_x \sigma^2_y$ are identical to the full conditional fiducial distributions for this problem. Therefore, these latter conditional distributions determine a joint fiducial distribution for $\mu_x$, $\mu_y$, $\sigma^2_x$ and $\sigma^2_y$, and on the basis of this joint distribution, it can be established that the marginal fiducial distribution for $\mu_x - \mu_y$ is defined by

$$\mu_x - \mu_y = \bar{x} - \bar{y} + B \sqrt{\frac{s^2_x}{n_x} + \frac{s^2_y}{n_y}}$$

where $s^2_x$ and $s^2_y$ are the observed variances of the samples $x$ and $y$ respectively, and $B$ is a random variable that has a Behrens-Fisher distribution with degrees of freedom $n_x - 1$ and $n_y - 1$, and with angle parameter equal to $\tan^{-1}((s_x \sqrt{n_y})/(s_y \sqrt{n_x}))$. The
distribution for $\mu_x - \mu_y$ defined by equation (19) is the fiducial distribution for $\mu_x - \mu_y$ that was advocated by R. A. Fisher for this problem.

5.7. Inference about all parameters of a bivariate normal distribution

The final problem we will consider in this section is that of making inferences about all the parameters of a bivariate normal distribution, i.e. the two means $\mu_x$ and $\mu_y$, the two variances $\sigma^2_x$ and $\sigma^2_y$ and the correlation coefficient $\rho$, on the basis of a sample $(x, y) = \{(x_i, y_i) : i = 1, 2, \ldots, n\}$ from the distribution concerned.

If all parameters except $\mu_x$ are known, a sufficient statistic for $\mu_x$ is

$$\sum_{i=1}^{n} x_i - \rho \left( \frac{\sigma_x}{\sigma_y} \right) \sum_{i=1}^{n} y_i$$

which will be treated as being the fiducial statistic in this case. Defining the primary r.v. $\Gamma$ as having a $N(0, 1)$ distribution allows equation (2) to be expressed as:

$$\sum_{i=1}^{n} x_i - \rho \left( \frac{\sigma_x}{\sigma_y} \right) \sum_{i=1}^{n} y_i = \varphi(\Gamma, \mu_x) = n\mu_x - n\rho \left( \frac{\sigma_x}{\sigma_y} \right) \mu_y + (n\sigma^2_x(1 - \rho^2))^{\frac{1}{2}} \Gamma$$

meaning that the fiducial distribution of $\mu_x$ is defined by

$$\mu_x | \mu_y, \sigma^2_x, \sigma^2_y, \rho, (x, y) \sim N \left( \bar{x} + \rho \left( \frac{\sigma_x}{\sigma_y} \right) (\mu_y - \bar{y}), \frac{\sigma^2_x(1 - \rho^2)}{n} \right)$$

(20)

Due to the symmetry of the bivariate normal distribution, if all parameters except $\mu_y$ are known, the fiducial distribution of $\mu_y$ is defined by

$$\mu_y | \mu_x, \sigma^2_x, \sigma^2_y, \rho, (x, y) \sim N \left( \bar{y} + \rho \left( \frac{\sigma_y}{\sigma_x} \right) (\mu_x - \bar{x}), \frac{\sigma^2_y(1 - \rho^2)}{n} \right)$$

(21)

If all parameters except $\sigma^2_x$ are known, then no single sufficient statistic for $\sigma^2_x$ exists and therefore, in agreement with Definition 1, we define the fiducial statistic for $\sigma^2_x$ to be the unique maximum likelihood estimator of $\sigma^2_x$, which is obtained by solving the following quadratic equation for $\hat{\sigma}^2_x$:

$$n(1 - \rho^2)\hat{\sigma}^2_x + \rho(\hat{\sigma}_x/\sigma_y) \sum_{i=1}^{n} x'_i y'_i = 0$$

(22)
where \( x'_i = x_i - \mu_x \) and \( y'_i = y_i - \mu_y \). It is well known that a maximum likelihood estimator of a parameter is usually asymptotically normally distributed with mean equal to the true value of the parameter and variance equal to the inverse of Fisher information. Since it can be shown that Fisher information of the likelihood function in question with respect to \( \sigma_x \) is given by

\[
I(\sigma_x) = \frac{-n(2 - \rho^2)}{\sigma_x^2(1 - \rho^2)}
\]

equation (2) can be approximated by

\[
\hat{\sigma}_x = \sqrt{\varphi(\Gamma, \sigma_x^2)} = \sigma_x + \Gamma \sigma_x \left( \frac{(1 - \rho^2)}{n(2 - \rho^2)} \right)^{\frac{1}{2}}
\]

where \( \hat{\sigma}_x \) is the maximum likelihood estimator of \( \sigma_x \) defined by equation (22) and \( \Gamma \) is defined as having a \( N(0,1) \) distribution. Solving this equation for \( \sigma_x^2 \) leads to the following approximate definition of the fiducial distribution for \( \sigma_x^2 \):

\[
\sigma_x^2 = \hat{\sigma}_x^2 \left( \Gamma \left( \frac{(1 - \rho^2)}{n(2 - \rho^2)} \right)^{\frac{1}{2}} + 1 \right)^{-2} \tag{23}
\]

Again due to the symmetry of the bivariate normal distribution, if all parameters except \( \sigma_y^2 \) are known, the fiducial distribution of \( \sigma_y^2 \) is approximately defined by equation (23) with \( \sigma_x^2 \) and \( \hat{\sigma}_x^2 \) substituted by \( \sigma_y^2 \) and \( \hat{\sigma}_y^2 \) respectively, where \( \hat{\sigma}_y^2 \) is the maximum likelihood estimator of \( \sigma_y^2 \).

If all parameters except the correlation coefficient \( \rho \) are known, then no single sufficient statistic for \( \rho \) exists and therefore, similar to the case just discussed, we define the fiducial statistic for \( \rho \) to be the unique maximum likelihood estimator of \( \rho \), which is obtained by solving the following cubic equation for \( \hat{\rho} \):

\[
-n\hat{\rho}^3 + \left( \frac{\sum_{i=1}^{n} x'_i y'_i}{\sigma_x \sigma_y} \right) \hat{\rho}^2 + \left( n - \frac{\sum_{i=1}^{n} (x'_i)^2}{\sigma_x^2} - \frac{\sum_{i=1}^{n} (y'_i)^2}{\sigma_y^2} \right) \hat{\rho} + \frac{\sum_{i=1}^{n} x'_i y'_i}{\sigma_x \sigma_y} = 0
\]

The distribution of this estimator will be approximated in the same way as the distributions of the estimators \( \hat{\sigma}_x^2 \) and \( \hat{\sigma}_y^2 \) were approximated. In particular, since it can be
shown that Fisher information of the likelihood function with respect to \( \rho \) is given by

\[
I(\rho) = \frac{n(1 + \rho^2)}{(1 - \rho^2)^2}
\]
equation (2) can be approximated by

\[
\hat{\rho} = \varphi(\Gamma, \rho) = \rho + \frac{(1 - \rho^2)\Gamma}{\sqrt{n(1 + \rho^2)}}
\]

where again \( \Gamma \sim N(0, 1) \). Under Definition 2, this equation defines the fiducial distribution for \( \rho \). We observe that if a random value of \( \Gamma \) is substituted into equation (24) then the value of \( \rho \) that solves this equation will be unique and will be a random value of \( \rho \) from its fiducial distribution.

Figure 3 shows the results of running a Gibbs sampler on the basis of the full conditional fiducial distributions for all five parameters of a bivariate normal distribution defined by equations (20) and (21), equation (23) in terms of both \( \sigma_x^2 \) and \( \sigma_y^2 \), and equation (24). This figure has been constructed on the basis of one run of 100,000 cycles of this algorithm using the same criteria as were used to construct Figures 1 and 2, with the data set \((x, y)\) in question being a random sample of \( n = 200 \) values from a bivariate normal distribution with \( \mu_x = \mu_y = 0, \sigma_x^2 = \sigma_y^2 = 1 \) and \( \rho = 0.8 \).

Running the Gibbs sampler from various starting points provided no evidence to suggest that the sampler was failing to converge or was getting trapped in just one mode of a proper or improper multimodal distribution. Hence, it would seem reasonable to draw similar conclusions to that which were made in Sections 5.4 and 5.5, and thereby assume that the full conditional fiducial distributions for the five parameters concerned determine a joint fiducial distribution for these parameters.

6. Determining the strengths of subjective fiducial probabilities

We will now fulfill the undertaking made at the end of Section 3 and complete the definition of the fiducial densities that have been derived by drawing some general conclusions
Figure 3: Gibbs sampling of the joint fiducial distribution of the parameters of a bivariate normal distribution
about what should be the strengths that are assigned to the probability values that are obtained by integrating over these densities.

Using again the terminology proposed in Bowater (2017a), let the reference set \( R \) be the balls in urn reference set defined in Section 3 but with \( k \) instead of 20 balls in the urn, which we will assume are numbered from 1 to \( k \). Observe that if, within the methodology of this earlier paper, it is quite reasonably assumed that the similarity \( S \) between any given event \( A \) and another event is maximized when the other event is also \( A \), then the event \( R(\lambda) \) defined by equation (1) must have a probability with respect to the set \( R \) that equals \( \lambda \).

To make a comparison with this event, let us consider the event of the primary r.v. \( \Gamma \) being less than \( \gamma(\lambda) \), where \( \gamma(\lambda) \) is defined by

\[
\int_{-\infty}^{\gamma(\lambda)} f_\Gamma(\gamma) d\gamma = \lambda
\]

and \( \lambda \) is a member of the set \( \Lambda \) defined below equation (1). The probability \( \lambda \) that would be assigned both to this latter event when we are in step 1 of the data generating algorithm in Assumption 1, and to the event \( R(\lambda) \) before the ball is drawn out of the urn, would usually be classified as an objective probability. As a consequence it would usually be the case that the similarity

\[
S(R(\lambda), \{ \Gamma < \gamma(\lambda) \})
\]

is regarded as being very high, and hence, the probability \( \lambda \) of the event \( \{ \Gamma < \gamma(\lambda) \} \) is considered as being very strong.

In the definition of a univariate subjective fiducial distribution, i.e. Definition 2, the probability of the event \( \{ \Gamma < \gamma(\lambda) \} \) is effectively treated as being \( \lambda \) after the data has been observed. To determine what strength ought to be assigned to this probability, let us consider a modified version of one of the abstract scenarios that were outlined in Bowater (2017a).
In particular, suppose that someone, who will be referred to as the selector, randomly
draws a ball out of the urn that is associated with the set \( R \) and then, without looking
at the ball, hands it to an assistant. The assistant, by contrast, looks at the ball, but
conceals it from the selector, and then places it under a cup. The selector believes that
the assistant smiled when he looked at the ball.

Under these conditions, the selector is asked to assign a probability to the event of the
number on the ball being less than or equal to \( k\lambda \), where generally \( \lambda \) can be any given
value in \( \Lambda \), but it may be helpful to assume that \( \lambda \) is not too close to 0 or 1. Let this
event be denoted as \( R^* (\lambda) \). Finally, we will assume that it was known from the outset
that the aim of this exercise was for the selector to assign a probability to this particular
event.

Clearly in this scenario, a smile by the assistant would in general need to be taken
into account, since it could imply that it is less likely or more likely that the event \( R^* (\lambda) \)
has taken place. However, the selector may feel that, if the assistant had indeed smiled,
he would not have understood the smile’s meaning. For this reason, he may decide that
the probability that the number on the ball is less than or equal to \( k\lambda \) is what it was
before the ball was drawn out of the urn, i.e. it is \( \lambda \).

It would seem undeniable that this probability is a subjective probability as it depends
on a subjective judgement regarding the meaning of a supposed smile. However, given
his lack of understanding about this meaning, the selector may feel that the similarity
\( S(R^*(\lambda), R(\lambda)) \) is very high, and hence that the probability \( \lambda \) of the event \( R^*(\lambda) \) is very
strong or, to put it another way, that this probability can be regarded in a certain sense
as being almost like a physical probability.

In addressing the main issue of what strength should be assigned after the data has
been observed to the probability \( \lambda \) of the event \( \{ \Gamma < \gamma(\lambda) \} \), an analogy can be drawn
between the supposed smile of the assistant in this abstract scenario and the event of
observing the data $x$. In particular, under the assumptions of Section 4.2, if little or
nothing was known about the parameter $\theta_1$ before the data was observed, the event of
observing the data should have little or no meaning in terms of how it should affect the
probability of the event $\{\Gamma < \gamma(\lambda)\}$. Therefore, although the similarity in equation (25)
when judged after the data has been observed may be considered as being marginally
less than what it was before the data was observed, it still ought to be considered as
being very high, and hence, the probability $\lambda$ of the event $\{\Gamma < \gamma(\lambda)\}$ still ought to be
regarded as being very strong.

Obviously, if there were strong beliefs about $\theta_1$ before the data was observed, then it
may be quite clear how observing the data $x$ should affect the probability of the event
$\{\Gamma < \gamma(\lambda)\}$. For example, if such beliefs are strong enough so that they can be adequately
summarized by placing a probability distribution over $\theta_1$, then the probability of this
event after the data has been observed could be determined by using Bayes’ theorem.
Nevertheless, the fiducial argument is traditionally applied under the assumption that
little or nothing was known about the parameters of interest before the data was observed,
and the present work will not deviate from this tradition.

Under this assumption and taking into account that the fiducial density of $\theta_1$, i.e. the
density $f(\theta_1 \mid x)$, is fully defined by the density of the primary r.v. $\Gamma$ and known constants,
the probability $\lambda$ of the event $\{\theta_1 < \theta_1(\lambda)\}$ after the data has been observed, where $\theta_1(\lambda)$
is defined by
\[
\int_{-\infty}^{\theta_1(\lambda)} f(\theta_1 \mid x) d\theta_1 = \lambda
\]
and $\lambda$ is any given member of $\Lambda$, ought to be regarded as being a very strong probability,
or in other words, in spite of this probability being a fiducial probability it ought to be
regarded in a certain sense as being almost like a physical probability. If probabilities
derived by integrating univariately over the full conditional density $f(\theta_j \mid \theta_{-j}, x)$ with
respect to the parameter $\theta_j$ are considered as being very strong for all $j \in \{1, 2, \ldots, n\}$,
then since the joint fiducial density of the parameters $\theta_1, \theta_2, \ldots, \theta_k$ is fully defined by its full conditional densities, it can be argued that the probabilities derived by integrating over this joint density ought to be treated as though they are almost equivalent to physical probabilities.

In conclusion, although the type of fiducial inference outlined in the present paper is classified as being ‘subjective’, this is in fact simply due to the argument that all probability statements made about fixed but unknown parameters must be inherently subjective, rather than due to a need to emphasize how different the fiducial probability densities that can be derived using this type of inference are from objective probability densities.

7. Comparing subjective fiducial inference to Bayesian inference

As mentioned in the Introduction, given that in many cases the standard fiducial distribution is equal to the posterior distribution obtained through Bayes’ theorem for a given choice of the prior distribution, it has become a convention to claim that, to a large extent, fiducial inference is indistinguishable from Bayesian inference. For this reason, it is worth comparing subjective fiducial inference to Bayesian inference. This comparison will be carried out using the definition of probability upon which the present work is based, i.e. the definition outlined in Bowater (2017a, 2017b). It is therefore necessary to apply this definition to the probabilities used in Bayes’ theorem.

The probabilities that enter into Bayes’ theorem are provided by the sampling density $g(x \mid \theta)$ and the prior density $p(\theta)$. Probabilities obtained by integrating over the sampling density will usually be physical probabilities and therefore, under the important assumption that the sampling model actually generated the observed data, these probabilities can usually be regarded as being very strong. If probabilities obtained by integrating over the prior density are also very strong, it would seem logical to regard
the probabilities produced by Bayes’ theorem, i.e. those obtained by integrating over the posterior density $p(\theta \mid x)$, as also being very strong. This would be the case, for example, if the prior distribution represents the uncertainty concerning the outcome of a well-understood physical experiment. Similarly, if the prior density is elicited on the basis of the subjective opinion of one or a number of scholars in the subject area concerned who have quite detailed knowledge about the relative plausibility of different values for the parameter $\theta$, then probabilities obtained by integrating over this prior density may well be regarded as being reasonably strong and, as a result, it is arguable that the posterior probabilities that correspond to the use of this prior should also be regarded as being reasonably strong.

However, to make a direct comparison with subjective fiducial inference, it needs to be assumed that little or nothing was known about $\theta$ before the data was observed. If $\theta$ is a single parameter unrestricted on the real line, it is common to try to represent this lack of knowledge by placing a diffuse symmetric prior density over $\theta$ centred at some given value $\theta^*(0.5)$. Assuming that this has been done, let us consider the similarity between the event $R(\lambda)$ as specified in Section 6 and the event of $\theta$ being less than the value $\theta^*(\lambda)$, i.e. the similarity $S(R(\lambda), \{\theta < \theta^*(\lambda)\})$, where $\theta^*(\lambda)$ is defined by the expression:

$$\int_{-\infty}^{\theta^*(\lambda)} p(\theta)d\theta = \lambda$$

in which $p(\theta)$ is the chosen prior density and $\lambda \in \Lambda$. Notice that no matter how diffuse the prior density is chosen to be, this similarity is likely to be regarded as being very low for any given value of $\lambda$. For example, if $\lambda = 0.5$ then $S(R(\lambda), \{\theta < \theta^*(\lambda)\})$ is effectively the similarity between the event of drawing a red ball out of an urn that contains an equal number of red balls and blue balls and the event of $\theta$ being less than $\theta^*(0.5)$, which is clearly going to be a very low similarity as the choice of the median $\theta^*(0.5)$ is completely arbitrary. This implies that for any value $\lambda \in \Lambda$, the probability $\lambda$ of the event $\{\theta < \theta^*(\lambda)\}$ is likely to be regarded as being very weak.
Since posterior probabilities are determined through Bayes’ theorem simply by reweighting prior probabilities, that is by normalizing prior probabilities after they have been multiplied by the likelihood function, it would seem difficult to argue that such probabilities should be considered as being generally stronger than prior probabilities, except of course if they could be justified in an alternative way, i.e. a non-Bayesian way. Therefore, if a proper prior density is used to try to represent a lack of knowledge about an unrestricted parameter $\theta$ before the data is observed, then it could be argued that it would be difficult to use Bayesian reasoning to make any kind of claim that probabilities not equalling zero or one that are obtained by integrating over the resulting posterior density ought to be regarded as being anything other than very weak probabilities. A similar argument could be presented in the case where the parameter $\theta$ is restricted on the real line.

This is a clear inadequacy of Bayesian inference. Also, notice that the severe criticisms of carrying out Bayesian inference using objective prior distributions that were highlighted in Section 2 do not apply to subjective fiducial inference. Therefore, a strong case has been made that subjective fiducial inference is superior to Bayesian inference when little or nothing was known about the parameter of interest before the data was observed.

8. Open issues

In this closing section, we will briefly discuss some open issues concerning subjective fiducial inference.

As pointed out in Lindley (1958), standard fiducial inference can be incoherent in the sense that conditioning in the Bayesian way a fiducial distribution corresponding to a data set $x^{(1)}$ on another independent data set $x^{(2)}$ does not lead to the fiducial distribution corresponding to the combined data set $\{x^{(1)}, x^{(2)}\}$. Much has been made of the existence
of this anomaly, however little attention has been given to its practical consequences. In particular, little research has been done into establishing in what situations fiducial inference fails to at least approximately satisfy this coherency condition, especially when either or both of the samples $x^{(1)}$ and $x^{(2)}$ are at least moderately sized.

Furthermore, in cases where there may be a substantial difference between the fiducial distribution corresponding to the data set $\{x^{(1)}, x^{(2)}\}$ and the fiducial distribution corresponding to the data set $x^{(1)}$ conditioned in the Bayesian way on the data set $x^{(2)}$, a sensible strategy exists for choosing between these two distributions. In particular, if the data set $x^{(1)}$ is large enough so that the fiducial distribution that corresponds to $x^{(1)}$ is considered to be a very strong distribution, according to the criteria given in Section 3 and in Bowater (2017b), then it would seem sensible to regard this distribution conditioned on the data set $x^{(2)}$ as providing the most appropriate inferences about the parameters of interest on the basis of the data set $\{x^{(1)}, x^{(2)}\}$. On the other hand, if the data set $x^{(1)}$ is so small that some doubts exist with regard to classifying the fiducial distribution that corresponds to $x^{(1)}$ as being a very strong distribution then, in general, the most appropriate inferences about the parameters concerned could be regarded as being provided by the fiducial distribution that corresponds to the combined data set $\{x^{(1)}, x^{(2)}\}$.

As was illustrated in Section 5.7 when applying subjective fiducial inference as defined in the present paper to the univariate case as specified in Section 4.2, it is not necessary that the fiducial statistic is a sufficient statistic for the unknown parameter $\theta_1$, since if a single sufficient statistic for $\theta_1$ does not exist, it can be defined to be any one-to-one function of a unique maximum likelihood estimator of $\theta_1$. This represents a departure from standard fiducial inference which may have some impact on the issue that has just been raised, but which nevertheless substantially opens up the range of applications of the methodology that has been discussed. It is left as an open issue as to whether and by
how much subjective fiducial inference could perform better in some cases if the fiducial statistic was allowed to be another type of non-sufficient statistic.

Finally we note that, in general, it would appear that subjective fiducial inference is more computationally demanding than Bayesian inference. However, similar to what has been seen in recent years in relation to Bayesian inference, it would be expected that advances with respect to the computational aspects of subjective fiducial inference will gradually extend its range of applications as well as improving the sophistication of the kind of numerical approximations upon which it is clearly going to depend.

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