On a Quantum Action Principle

Natalya Gorobey and Alexander Lukyanenko

1Department of Experimental Physics, St. Petersburg State Polytechnical University, Polytekhnicheskaya 29, 195251, St. Petersburg, Russia

A quantum version of the action principle is formulated in terms of real parameters of a wave functional. The classical limit of the quantum action of a harmonic oscillator is obtained.

PACS numbers:

I. INTRODUCTION

In the work [1] a new form of quantum mechanics in terms of a quantum version of the action principle was proposed. The quantum action principle was formulated for a new object - a wave functional \( \Psi [x(t)] \), which, unlike a wave function \( \psi(x,t) \), describes the dynamics of a particle as the movement along a trajectory \( x(t) \). The wave functional has the meaning of a probability density in the space of trajectories, provided that a scalar product in the space of functionals is introduced and the normalization condition

\[
(\Psi, \Psi) \equiv \int_0^T dx(t)|\Psi [x(t)]|^2 = 1 \tag{1}
\]

is fulfilled. The new form of quantum mechanics is equivalent to Schrödinger wave mechanics on a set of multiplicative wave functionals that are connected with an \( \varepsilon \)-division of a time interval \([0,T]\) [1].

In the present paper the quantum action principle is formulated as an extremum principle on a set of analytical functionals, i.e. the functionals that are represented by a functional series of \( x(t) \) degrees. In this framework the dynamics of a particle is reduced to a system of differential equations for coefficients of the series. We consider quantum dynamics of a harmonic oscillator and obtain its quasi-classical limit.

II. QUANTUM ACTION PRINCIPLE

The quantum action principle is based on the following equation for eigenvalues of the action operator:

\[
\hat{I}\Psi \equiv \int_0^T dt \left[ \frac{\hbar}{i} \dot{x}(t) \frac{\delta \Psi}{\delta x(t)} + \frac{\hbar^2}{2m} \frac{\delta^2 \Psi}{\delta x^2(t)} - U(x(t),t) \right] \Psi = \lambda \Psi. \tag{2}
\]

The action operator \( \hat{I} \) is a quantum analog of the classical action represented in the canonical form:

\[
I = \int_0^T dt \left[ p(x,t) - \frac{p^2}{2m} - U(x,t) \right]. \tag{3}
\]

"Quantization" of the classical action \( I \) is performed by the replacement of the canonical momentum \( p(t) \) by the functional-differential operator

\[
\hat{p}(t) \equiv \frac{\hbar}{i} \delta \frac{\delta x(t)}{\delta x(t)}, \tag{4}
\]

where the constant \( \hbar \) has the dimensionality \( D_j \cdot s^2 \). On the set of multiplicative functionals, which corresponds to an \( \varepsilon \)-division of the time interval \([0,T]\), this constant is proportional to the "ordinary" Plank constant:

\[
\hbar = \hbar \cdot \varepsilon. \tag{5}
\]

For the set of analytical functionals considered here the constant \( \hbar \) has to be defined after a definition of "observables" and a comparison of the theory with an experiment. This will be the subject of our next work.

Let us suppose that a potential \( U(x,t) \) is an analytical function of a variable \( x \). For simplicity, in the present work we only consider a harmonic oscillator with a potential:

\[
U(x) = \frac{kx^2}{2}. \tag{6}
\]

Let us introduce an exponential representation for the wave functional:

\[
\Psi [x(t)] \equiv \exp \left( \frac{i}{\hbar} S [x(t)] + \sigma [x(t)] \right), \tag{7}
\]

where \( S [x(t)], \sigma [x(t)] \) are unknown analytical functionals. The operator \( \hat{I} \) is Hermitian with respect to the scalar product [1], therefore, its eigenvalues are real. From the equation \( 2 \), with account of \( 7 \), we obtain an eigenvalue:

\[
\lambda = \int_0^T \left[ \frac{\delta S}{\delta x} \frac{\delta \sigma}{\delta x} - \frac{1}{2m} \left( \frac{\delta \sigma}{\delta x} \right)^2 - U \right] dt + \frac{\hbar^2}{2m} \left( \frac{\delta \sigma}{\delta x} \right)^2 dt, \tag{8}
\]

*Electronic address: alex.lukyan@rambler.ru
and, in addition, a condition of its reality:

$$
\int_0^T \left[ \frac{\delta \sigma}{\delta x} - \frac{1}{m} \frac{\delta S}{\delta \sigma} - \frac{1}{2m} \frac{\delta^2 S}{\delta x^2} \right] dt = 0. \tag{9}
$$

The representation (8) is not final, because eigenvalues must have no dependence on a trajectory except for boundary points \(x_0 = x(0)\) and \(x_T = x(T)\). Let us introduce a functional series:

$$
S[x(t)] = \int_0^T S_1(t) x(t) dt + \frac{1}{2} \int_0^T \int_0^T S_2(t, t') x(t) x(t') dt dt' + ..., \tag{10}
$$

$$
\sigma[x(t)] = \int_0^T \sigma_1(t) x(t) dt + \frac{1}{2} \int_0^T \int_0^T \sigma_2(t, t') x(t) x(t') dt dt' + .... \tag{11}
$$

Then one obtains the final representation for the eigenvalue:

$$
\lambda = S[T] - \frac{1}{2m} \int_0^T S_2^2(t) dt + \frac{\hbar^2}{2m} \int_0^T (\sigma_1^2(t) + \sigma_2(t, t)) dt, \tag{12}
$$

and, in addition to the reality condition (9), the condition of its independence on a trajectory \(x(t)\):

$$
\int_0^T dt \left[ \dot{S} + \frac{1}{2m} \left( \frac{\delta S}{\delta x} \right)^2 + U \right]
- \frac{\hbar^2}{2m} \left( \frac{\delta \sigma}{\delta x} \right)^2 + \\
- \frac{\hbar^2}{2m} \int_0^T (\sigma_1^2(t) + \sigma_2(t, t)) dt
- \frac{1}{2m} \int_0^T S_1^2(t) dt = 0. \tag{13}
$$

Here \(\dot{S}\) denotes series (10) in which the coefficients are replaced by their time derivatives.

Now, we are ready to formulate the quantum action principle. First of all, functional equations (10) and (11) must be solved. They are equivalent to a set of the first-order differential equations for coefficients of series (10) and (11). A solution of this system depends on initial values of the coefficients at the moment \(t = 0\). Therefore, the eigenvalue \(\lambda\), Eq. (12), of the action operator is a function of the initial data. It is this function that has to be extremal in the quantum action principle. In the next section this principle will be considered in the simplest case of a harmonic oscillator.

### III. Quantum Dynamics of Harmonic Oscillator

In the case of harmonic oscillator it is enough to consider quadratic terms in series (10) and (11), taking into account that they are local with respect to the time variable, i.e.,

$$
S_2(t, t') = S_2(t) \delta(t - t'), \quad \sigma_2(t, t') = \sigma_2(t) \delta(t - t'). \tag{14}
$$

In this case, functional equations (10) and (11) are equivalent to the following set of differential equations:

$$
\dot{\sigma}_1 + \frac{1}{m} (\sigma_1 S_2 + \sigma_2 S_1) = 0, \\
\dot{\sigma}_2 + \frac{1}{m} \sigma_2 S_2 = 0, \\
\dot{S}_1 + \frac{1}{m} S_1 S_2 - \frac{\hbar^2}{2m} \sigma_1 \sigma_2 = 0, \\
\dot{S}_2 + \frac{1}{m} S_2^2 + k - \frac{\hbar^2}{m} \sigma_2^2 = 0. \tag{15}
$$

Moreover there is the following algebraic equation for the initial data:

$$
\left( \frac{\sigma_1 x + \sigma_2 x^2}{2} \right)^T \left( \frac{\sigma_1 x + \sigma_2 x^2}{2} \right)_0 - \frac{1}{m} \int_0^T (\sigma_1 S_1 + 2S_2) dt = 0. \tag{16}
$$

A final representation of the action eigenvalue takes a form:

$$
\lambda = \left( S_1 x + S_2 x^2 \right)^T_0 - \frac{1}{2m} \int_0^T S_1^2 dt + \frac{\hbar^2}{2m} \int_0^T (\sigma_1^2 + \sigma_2) dt. \tag{17}
$$

Exponential representation (7) of the wave functional is the most suitable for the formulation of quasi-classical asymptotics.

Let us consider the classical limit \((\hbar = 0)\). In this limit, from Eq. (15) one can obtain:

$$
S_1 = S_{10} \frac{\cos \omega_0 t_0}{\cos \omega_0 (t - t_0)}, \\
S_2 = -\sqrt{m} k \cos \omega_0 (t - t_0), \tag{18}
$$

where \(\omega_0 \equiv \sqrt{k/m}\) is the own frequency of the oscillator, \(S_{10}\) is an initial value of the function \(S_1(t)\). The constant \(t_0\) is related with the initial value of the function \(S_2(t)\):

$$
S_{20} = \sqrt{m} k \cos \omega_0 t_0. \tag{19}
$$
Substituting solution (18) into Eq. (17), in the classical limit we obtain:

\[
\lambda = S_{10} \left( x_T \frac{\cos \omega_0 t_0}{\cos \omega_0 (T - t_0)} - x_0 \right) \\
- \frac{1}{2} \sqrt{mk} \left( x_T^2 \tan \omega_0 (T - t_0) + x_0^2 \tan \omega_0 t_0 \right) \\
- S_{10}^2 \frac{\cos^2 \omega_0 t_0}{2\sqrt{mk}} (t \tan \omega_0 (T - t_0) + t \tan \omega_0 t_0). \tag{20}
\]

Now we must find an extremum of this function with respect to two variables, \( S_{10} \) and \( t_0 \). The extremum condition gives for \( S_{10} \):

\[
S_{10} = \sqrt{mk} \left( x_T \frac{\cos \omega_0 t_0}{\cos \omega_0 T} - x_0 \frac{\cos \omega_0 (T - t_0)}{\sin \omega_0 T} \right). \tag{21}
\]

However, the constant \( t_0 \) remains indefinite. Therefore, the eigenvalue \( \lambda \) is degenerate in the classical limit. As it must be in the classical limit, this eigenvalue is equal to the classical action of the harmonic oscillator (see, for example, [2]):

\[
\lambda = \sqrt{mk} \frac{x_T^2 + x_0^2}{2 \sin \omega_0 T} \cos \omega_0 T - 2x_T x_0. \tag{22}
\]

The corresponding eigenfunctional has a phase which is parameterized as follows:

\[
S[x(t)] = -\frac{\sqrt{mk}}{2} \int_0^T (x(t) - \bar{x}(t))^2 dt, \tag{23}
\]

where

\[
\bar{x}(t) \equiv \frac{x_T \cos \omega_0 t_0 - x_0 \cos \omega_0 (T - t_0)}{\sin \omega_0 T \sin \omega_0 (t - t_0)}. \tag{24}
\]

The parameter \( \sigma[x(t)] \) remains indefinite in the classical limit. It is necessary to outline that similar to ordinary quantum mechanics (see, for example, [3]), the classical limit in our theory is valid up to the first order in the constant \( \tilde{\hbar} \). The arbitrary parameter \( t_0 \) can be interpreted as a degree of the oscillator excitation. It plays a role of an energy parameter within our theory.

IV. CONCLUSIONS

We conclude that the new form of quantum mechanics has a proper classical limit. Quantum corrections to the quantum action can give essential predictions of the new quantum theory. In a further paper, we will give a relationship between a new \( \tilde{\hbar} \) and the ordinary \( \hbar \) Plank constants.

We are thanks A. V. Goltsev for useful discussions.

---

[1] N. N. Gorobey and A. S. Lukyanenko, ArXiv: 0807.3508 (July 2008).
[2] R. P. Feynman and A. R. Hibbs, Quantum mechanics and path integrals (N.Y., 1965).
[3] D. Bohm, Quantum Theory (N.Y., 1952).