General time interval multidimensional BSDEs with generators satisfying a weak stochastic-monotonicity condition

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Abstract This paper establishes an existence and uniqueness result for the adapted solution of a general time interval multidimensional backward stochastic differential equation (BSDE), where the generator $g$ satisfies a weak stochastic-monotonicity condition and a general growth condition in the state variable $y$, and a stochastic-Lipschitz condition in the state variable $z$. This unifies and strengthens some known works. In order to prove this result, we develop some ideas and techniques employed in Xiao and Fan [25] and Liu et al. [15]. In particular, we put forward and prove a stochastic Gronwall-type inequality and a stochastic Bihari-type inequality, which generalize the classical ones and may be useful in other applications. The martingale representation theorem, Itô’s formula, and the BMO martingale tool are used to prove these two inequalities.

Keywords Backward stochastic differential equation, General time interval, Weak stochastic-monotonicity condition, Existence and uniqueness, Stochastic Gronwall-type inequality, Stochastic Bihari-type inequality

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1. Introduction

Let us fix two positive integers $k$ and $d$, a finite or infinite terminal time $T$ satisfying $0 < T \leq \infty$, and a $d$-dimensional standard Brownian motion $(B_t)_{t \in [0,T]}$ on a completed and filtrated probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \in [0,T]}$ is the natural, right-continuous, and completed $\sigma$-algebra filtration generated by $B$. We consider the following multidimensional backward stochastic differential equations (BSDEs):

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dB_s, \quad t \in [0, T],$$

where $\xi$ is an $\mathcal{F}_T$-measurable $\mathbb{R}^k$-valued random vector called the terminal condition. The stochastic function

$$g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$$

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is \((\mathcal{F}_t)\)-progressively measurable for each \((y, z)\), called the generator, and the pair of processes 
\((y_t, z_t)_{t \in [0, T]} \in \mathbb{R}^k \times \mathbb{R}^{k \times d}\) is \((\mathcal{F}_t)\)-progressively measurable, called the solution of equation (1), generally denoted by BSDE \((\xi, T, g)\).

It is well known that linear BSDEs were initially introduced by Bismut [2] for solving the optimal control problem, and nonlinear BSDEs were first investigated by Pardoux and Peng [20]. An existence and uniqueness result for the solution of a finite time interval multidimensional BSDE was initially established in [20] under a uniform Lipschitz condition of the generator \(g\) in the state variables \((y, z)\). Hereafter, unless otherwise noted, uniform means that the constant in the (Lipschitz) condition is uniform in the two state variables \(t\) and \(\omega\) of the generator \(g\), i.e., the constant is independent of \((t, \omega)\). Furthermore, Mao [17], Pardoux [19], and Fan et al. [13], respectively, weakened the uniform Lipschitz condition in \(y\) to a uniform non-Lipschitz condition, a uniform monotonicity condition, and a uniform Osgood condition. Fan and Jiang [12] (see also Fan [10]) unified these conditions and established an existence and uniqueness result for the solution of a finite time interval multidimensional BSDE, where the generator \(g\) satisfies a uniform weak monotonicity condition with a general growth condition in \(y\) and the uniform Lipschitz condition in \(z\). Up to now, BSDEs have been successfully applied to various areas, such as PDEs, mathematical finance, optimal control, etc., see, for example, El Karoui et al. [9], Morlais [18], and Peng [22] for details.

To the best of our knowledge, Chen and Wang [5] first investigated infinite time interval BSDEs, put forward a non-uniform (in \(t\)) Lipschitz condition of the generator \(g\) in \((y, z)\), and established the existence and uniqueness for solutions of the BSDEs. Recently, Morlais [18] and Xiao and Fan [25], respectively, relaxed the non-uniform (in \(t\)) Lipschitz condition of \(g\) in \(y\) to a non-uniform (in \(t\)) monotonicity condition and a non-uniform (in \(t\)) weak monotonicity condition, see also Fan [11] for more details. Very recently, Liu et al. [15] established an existence and uniqueness result for the solution of a general time interval multidimensional BSDE under a non-uniform (in both \(t\) and \(\omega\)) Lipschitz condition of the generator \(g\) in \((y, z)\), called the stochastic-Lipschitz condition. The stochastic-Lipschitz condition of \(g\) in \(y\) was further weakened to a non-uniform (in both \(t\) and \(\omega\)) monotonicity condition (called the stochastic-monotonicity condition) in Luo and Fan [16] for one-dimensional BSDEs. In particular, we would like to mention that some deep results on BSDEs with stochastic Lipschitz generators associated with the BMO (bounded in mean oscillation) martingale can be found in Briand and Confortola [3] and Delaen and Tang [6]. Readers are also referred to El Karoui and Huang [8], Bender and Kohlmann [1], and Wang et al. [23] for other kinds of stochastic conditions on the generator \(g\), in which some stronger integrability assumptions on the generator and the terminal condition as well as the solutions are required.

In this paper, we prove an existence and uniqueness result for the solution of a general time interval multidimensional BSDE, where the generator \(g\) satisfies a weak stochastic-monotonicity condition with a general growth condition in \(y\), and the stochastic-Lipschitz condition in \(z\), see Theorem 4.2 in section 4 for details. Since the assumptions (H2) and (H4) used in Theorem 4.2 are more general than those in the existing works (see Remark 4.1 in section 4), it strengthens some corresponding works mentioned in the last two paragraphs, including Theorem 3.1 in Liu et al. [15] and Theorem 6 in Xiao and Fan [25] for the case of the finite variation process \(V \equiv 0\), and some new and intrinsic difficulties arise naturally when proving it, see Remark 4.5 in section 4 for more details. In order to prove Theorem 4.2, we put forward and prove a stochastic Gronwall-type inequality and a stochastic Bihari-type inequality by virtue of the martingale representation theorem, Itô’s formula, and the BMO martingale tool, see Propositions 3.1 and 3.3 in section 3 for more details. These two stochastic inequalities generalize the classical ones, and may be useful elsewhere. Based on these
two inequalities and similar computations employed in [25] and [15], by dividing the time interval \([0, T]\) into a finite number of subintervals with stopping time ends, we successfully overcome the arising difficulties in our framework and give the proof of Theorem 4.2. As a corollary of Theorem 4.2, we also prove a general existence and uniqueness result for the solution of a multidimensional BSDE with general stopping time interval, see Corollary 4.4 in section 4 for details. Finally, we give an example (see Example 4.6 in section 4) in which Theorem 4.2 and Corollary 4.4 can be applied, but other known results cannot be applied.

The rest of the paper is organized as follows. In the next section, we introduce some notations. The stochastic Gronwall- and Bihari-type inequalities are stated and proved in section 3. Then, the existence and uniqueness results on the BSDEs are stated and proved in section 4 and section 5, respectively.

2. Notation

In this section, we introduce some notation used later. First, denote the interval \([0, +\infty)\) by \(\mathbb{R}_+\), the Euclidean norm of \(y \in \mathbb{R}^n\) by \(|y|\) for each \(n \geq 1\), and the indicator function of \(A\) by \(1_A\) for each set \(A\). Then, let \(L^2(\mathcal{F}_T; \mathbb{R}^k)\) be the set of all \(\mathcal{F}_T\)-measurable \(\mathbb{R}^k\)-valued random vectors \(\xi\) satisfying \(\mathbb{E}[|\xi|^2] < +\infty\), \(\mathcal{S}^2(0, T; \mathbb{R}^k)\) the set of all \((\mathcal{F}_t)\)-progressively measurable and continuous \(\mathbb{R}^k\)-valued processes \((Y_t)_{t \in [0, T]}\) such that

\[
\|Y\|_{\mathcal{S}^2} := \left( \mathbb{E} \left[ \sup_{t \in [0, T]} |Y_t|^2 \right] \right)^{1/2} < +\infty,
\]

and \(M^2(0, T; \mathbb{R}^{k \times d})\) the set of all \((\mathcal{F}_t)\)-progressively measurable \(\mathbb{R}^{k \times d}\)-valued processes \((Z_t)_{t \in [0, T]}\) satisfying

\[
\|Z\|_{M^2} := \left\{ \mathbb{E} \left[ \int_0^T |Z_t|^2 dt \right] \right\}^{1/2} < +\infty,
\]

and \(\mathcal{H}^2(0, T; \mathbb{R}^k)\) the set of all \((\mathcal{F}_t)\)-progressively measurable \(\mathbb{R}^k\)-valued processes \((X_t)_{t \in [0, T]}\) such that

\[
\|X\|_{\mathcal{H}^2} := \left\{ \mathbb{E} \left[ \left( \int_0^T |X_t| dt \right)^2 \right] \right\}^{1/2} < +\infty.
\]

Furthermore, for a local \(\mathbb{R}^k\)-valued or real-valued martingale \(\int_0^T z_s dB_s\), we say that it is a martingale of bounded mean oscillation, or BMO-martingale, means that

\[
\sup_{\tau \in \Sigma_T} \left\| \mathbb{E} \left[ \int_\tau^T |z_s|^2 ds \right| \mathcal{F}_\tau \right\| \mathcal{F}_\tau \| \mathbb{R}_+ < +\infty.
\]

Hereafter, \(\Sigma_T\) denotes the set of all \((\mathcal{F}_t)\)-stopping times \(\tau\) valued in \([0, T]\), and \(\|\xi\|_{\infty}\) the infinity norm of the essentially bounded real-valued random variable \(\xi\), i.e.,

\[
\|\xi\|_{\infty} := \sup\{x \in \mathbb{R}_+ : \mathbb{P}(|\xi| > x) > 0\}.
\]

We use \(L^\infty(\Omega; L^1([0, T]; \mathbb{R}_+))\) and \(L^\infty(\Omega; L^2([0, T]; \mathbb{R}_+))\), respectively, to denote the set of all \((\mathcal{F}_t)\)-progressively measurable nonnegative real-valued processes

\[
u_t(\omega), \quad v_t(\omega) : \Omega \times [0, T] \mapsto \mathbb{R}_+
\]

satisfying

\[
\mathbb{E}\int_0^T |\xi|^2 dt < +\infty.
\]
Finally, denote by $S$ the set of nondecreasing continuous functions $\rho(x) : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying the following conditions:

(i) $\rho(0) = 0$, $\rho(x) > 0$ for each $x > 0$, and $\int_{0^+}^{x} \frac{dx}{\rho(x)} := \lim_{\epsilon \to 0^+} \int_0^\epsilon \frac{dx}{\rho(x)} = +\infty$;

(ii) There exists a constant $A \geq 1$ such that $0 \leq \rho(x) \leq A(1 + x)$ for each $x \geq 0$;

(iii) For each real $c > 0$, the derivative function of $\rho$ on interval $[c, +\infty)$ is locally bounded, i.e., there exists a constant $M_c > 0$ depending only on $c$ such that $0 \leq \rho'(x) \leq M_c$ for each $x \in [c, +\infty)$.

We remark that if $\rho(x) : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, concave, and differentiable function satisfying condition (i), then $\rho(\cdot) \in S$ since conditions (ii) and (iii) hold automatically in this case. However, we also note that a function $\rho(\cdot)$ belonging to $S$ is not necessarily concave.

3. Stochastic Gronwall-type and Bihari-type inequalities

In this section, we put forward and prove a stochastic Gronwall-type inequality and a stochastic Bihari-type inequality, which respectively generalize the classical ones. Classical proof methods seem not to be applicable in the stochastic framework, and it is interesting that the martingale theorem, Itô’s formula and the BMO martingale tool play a crucial role in our proof of these two inequalities. These two inequalities are employed in section 5 to prove the existence and uniqueness result for Itô’s formula and the BMO martingale tool play a crucial role in our proof of these two inequalities. Not to be applicable in the stochastic framework, it is interesting that the martingale theorem, Bihari-type inequality, which respectively generalize the classical ones. Classical proof methods seem not to be applicable in the stochastic framework.

The following proposition extends the classical Gronwall inequality to a stochastic version. It also generalizes Theorem 1 in Wang and Fan [24], which states that in the case of $f \equiv 0$ and $\eta \equiv c$ for a constant $c \geq 0$, if (2) is satisfied for each $t \in [0, T]$, then (3) holds for each $t \in [0, T]$.

**Proposition 3.1** (Stochastic Gronwall-type inequality) Assume that $0 < T \leq \infty$, $\mu$ is an $(\mathcal{F}_t)$-progressively measurable, continuous, and nonnegative real-valued process satisfying

$$E \left[ \sup_{t \in [0, T]} \mu_t(\omega) \right] < +\infty,$$

$\eta$ is an $\mathcal{F}_T$-measurable nonnegative real-valued random variable satisfying $E[\eta] < +\infty$, $\beta$ is a process belonging to $L^\infty(\Omega; L^1([0, T]; \mathbb{R}_+))$, and both $f$ and $h$ are $(\mathcal{F}_t)$-progressively measurable nonnegative real-valued processes satisfying

$$E \left[ \int_0^T [f_t(\omega) + h_t(\omega)] dt \right] < +\infty.$$

If for each $t \in [0, T]$, it holds that

$$\mu_t \leq E \left[ \eta + \int_t^T (\beta_s \mu_s + f_s) ds \bigg| \mathcal{F}_t \right], \quad \mathbb{P}\mbox{-}a.s.,$$

(2)

then for each $t \in [0, T]$, we have

$$\mu_t \leq e^{\int_t^T \beta_s ds} E \left[ \eta + \int_t^T f_s ds \bigg| \mathcal{F}_t \right], \quad \mathbb{P}\mbox{-}a.s..$$

(3)
Moreover, if for each \( t \in [0, T] \), it holds that
\[
\mathbb{E} \left[ \sup_{s \in [t, T]} \mu_s + \int_t^T h_s \, ds \bigg| \mathcal{F}_t \right] \leq \mathbb{E} \left[ \eta + \int_t^T (\beta_s \mu_s + f_s) \, ds \bigg| \mathcal{F}_t \right], \quad \mathbb{P}\text{-a.s.,}
\]
then for each \( t \in [0, T] \), we have
\[
\mu_t \leq \mathbb{E} \left[ \sup_{s \in [t, T]} \mu_s + \int_t^T h_s \, ds \bigg| \mathcal{F}_t \right] \leq e^{\int_t^T \beta_s \, ds} \mathbb{E} \left[ \eta + \int_t^T f_s \, ds \bigg| \mathcal{F}_t \right], \quad \mathbb{P}\text{-a.s.}
\]

**Remark 3.2** For simplicity of notation, here and hereafter the random processes \( \mu_t(\omega), \beta_t(\omega), \) and \( f_t(\omega) \) are sometimes abbreviated as \( \mu_t, \beta_t, \) and \( f_t, \) respectively, and the \( \mathbb{P}\text{-a.s.} \) is usually omitted without causing confusion. We will also adopt similar notation for other processes.

**Proof of Proposition 3.1** We develop the martingale representation method employed in Wang and Fan [24] to prove this proposition. Set
\[
\bar{\eta} := \eta + \int_0^T (\beta_s \mu_s + f_s) \, ds.
\]
From the assumptions of \( \eta, \beta, \mu, \) and \( f, \) it is clear that \( \mathbb{E}[\bar{\eta}] < +\infty. \) Then, by the martingale representation theorem (see Theorem 2.46 in Pardoux and Răsăcanu [21]), there exists an \( (\mathcal{F}_t) \)-progressively measurable \( \mathbb{R}^{1 \times d} \)-valued process \((z_t)_{t \in [0, T]}\) such that
\[
\mathbb{E}[\bar{\eta} | \mathcal{F}_t] = \mathbb{E}[\bar{\eta}] + \int_0^t z_s dB_s, \quad t \in [0, T].
\]
Now, let
\[
\bar{\mu}_t := \mathbb{E} \left[ \eta + \int_t^T (\beta_s \mu_s + f_s) \, ds \bigg| \mathcal{F}_t \right] = \mathbb{E} \left[ \bar{\eta} \bigg| \mathcal{F}_t \right] - \int_0^t (\beta_s \mu_s + f_s) \, ds, \quad t \in [0, T].
\]
Then, \( \bar{\mu} \) is \( (\mathcal{F}_t) \)-progressively measurable and, in view of (6),
\[
\bar{\mu}_t = \mathbb{E}[\bar{\eta}] - \int_0^t (\beta_s \mu_s + f_s) \, ds + \int_0^t z_s dB_s, \quad t \in [0, T].
\]
It follows from Itô's formula together with the fact of \( \mu \leq \bar{\mu} \) due to (2) that
\[
d \left( \bar{\mu}_r e^{\int_0^r \beta_s \, ds} \right) = e^{\int_0^r \beta_s \, ds} \left[-(\beta_r \mu_r + f_r) \, dr + z_r dB_r + \beta_r \mu_r \, dr \right] \\
geq e^{\int_0^r \beta_s \, ds} (-f_r \, dr + z_r dB_r), \quad r \in [0, T].
\]
(7)

Note that the process
\[
\left( \int_0^t e^{\int_0^r \beta_s \, ds} z_r dB_r \right)_{t \in [0, T]}
\]
is an \( (\mathcal{F}_t) \)-martingale. Integrating on the interval \([t, T]\) and taking the conditional expectation with respect to \( \mathcal{F}_t \) on both sides of (7), we obtain that
\[
\mathbb{E} \left[ \eta e^{\int_0^T \beta_s \, ds} - \bar{\mu}_t e^{\int_0^t \beta_s \, ds} \bigg| \mathcal{F}_t \right] \geq -\mathbb{E} \left[ \int_t^T e^{\int_0^r \beta_s \, ds} f_r \, dr \bigg| \mathcal{F}_t \right], \quad t \in [0, T].
\]
Consequently, in view of the fact that \( \bar{\mu}_t e^{\int_0^t \beta_s \, ds} \) is \( \mathcal{F}_t \)-measurable,
\[
\tilde{\mu}_t B_{0; t} \leq E \left[ \eta e^{\int_0^T \beta_s ds} + \int_t^T e^{\int_0^r \beta_s ds} f_s dr \bigg| \mathcal{F}_t \right], \quad t \in [0, T]. \tag{8}
\]

Then, since \( \mu \leq \tilde{\mu} \), the desired inequality (3) follows immediately from (8).

Moreover, if (4) is satisfied for each \( t \in [0, T] \), it is clear that (2) holds for each \( t \in [0, T] \). So, (8) also holds. Thus, the desired inequality (5) follows from inequalities (8) and (4) together with the definition of \( \tilde{\mu} \). The proof is then complete. \( \square \)

The following Proposition 3.3 generalizes the classical Bihari inequality to a stochastic version. We note that another stochastic Bihari-type inequality was established in Proposition 4.6 of Ding and Wu [7], but its form and proof differ from ours. In particular, Proposition 4.6 in [7] cannot be employed to prove our existence and uniqueness result on the BSDEs in section 5, and Proposition 3.3 cannot be derived from it.

**Proposition 3.3 (Stochastic Bihari-type inequality)** Assume that \( c > 0, \ 0 < T \leq \infty, \ \beta \in L^\infty(\Omega; L^1([0, T]; \mathbb{R}_+)), \) and \( \rho(x) \in S \). Let \( \mu \) be an \( (\mathcal{F}_t) \)-progressively measurable, continuous, and nonnegative real-valued process satisfying

\[
E \left[ \sup_{t \in [0, T]} \mu_t(\omega) \right] < +\infty.
\]

If for each \( t \in [0, T] \), it holds that

\[
\mu_t \leq c + E \left[ \int_t^T \beta_s \rho(\mu_s) ds \bigg| \mathcal{F}_t \right], \quad \mathbb{P}\text{-a.s.,} \tag{9}
\]

then for each \( t \in [0, T] \), we have

\[
\mu_t \leq \Theta^{-1} \left( \Theta(c) + \left\| \int_t^T \beta_s ds \right\|_{\infty} \right), \quad \mathbb{P}\text{-a.s.,} \tag{10}
\]

where

\[
\Theta(x) := \int_1^x \frac{1}{\rho(u)} \ du, \quad x > 0,
\]

is a strictly increasing function valued in \( \mathbb{R} \), and \( \Theta^{-1} \) is the inverse function of \( \Theta \).

Moreover, if (9) holds for \( c = 0 \), then \( \mu_t = 0 \) for each \( t \in [0, T] \).

**Proof** Note first that \( \rho(x) \leq A(1 + x) \) for each \( x \in \mathbb{R}_+ \) and that \( \beta \in L^\infty(\Omega; L^1([0, T]; \mathbb{R}_+)) \).

It follows from Proposition 3.1 that for each \( t \in [0, T] \),

\[
0 \leq \mu_t \leq e^{A\left\| \int_0^t \beta_s ds \right\|_\infty} \left( c + A \left\| \int_0^T \beta_s ds \right\|_\infty \right) =: C.
\]

Then, letting \( \eta := \int_0^T \beta_s \rho(\mu_s) ds \), we have

\[
0 \leq \eta \leq A(1 + C) \left\| \int_0^T \beta_s ds \right\|_\infty =: M. \tag{11}
\]

On the other hand, by the classical martingale representation theorem, there exists an \( (\mathcal{F}_t) \)-progressively measurable and square-integrable \( \mathbb{R}^{1 \times d} \)-valued process \((z_t)_{t \in [0, T]}\) such that
\[
E[\eta|\mathcal{F}_t] = E[\eta] + \int_0^t z_s dB_s, \quad t \in [0, T].
\] (12)

Since the stochastic integral in (12) is essentially bounded, it is a BMO-martingale, i.e.,
\[
\sup_{\tau \in \mathbb{L}_T} \left\| E\left[ \int_{\tau}^T |z_s|^2 ds \right| \mathcal{F}_\tau \right\|_\infty < +\infty.
\] (13)

Next, set
\[
\tilde{\mu}_t := c + E\left[ \int_t^T \beta_s \rho(\mu_s) ds \right| \mathcal{F}_t] = c + E[\eta|\mathcal{F}_t] - \int_0^t \beta_s \rho(\mu_s) ds, \quad t \in [0, T].
\]
Then, \( \tilde{\mu} \) is \( (\mathcal{F}_t) \)-progressively measurable, \( c \leq \tilde{\mu} \leq c + M \) due to (11), and in view of (12),
\[
\tilde{\mu}_t = c + E[\eta] - \int_0^t \beta_s \rho(\mu_s) ds + \int_0^t z_s dB_s, \quad t \in [0, T].
\]
It follows from Itô’s formula that, in view of the monotonicity of \( \rho(\cdot) \) together with the facts that \( \tilde{\mu} \geq c \) and \( \mu \leq \tilde{\mu} \) due to (9),
\[
d\Theta(\tilde{\mu}_s) = \frac{1}{\rho(\tilde{\mu}_s)} \left[ \beta_s \rho(\mu_s) ds + z_s dB_s \right] - \frac{1}{2} \rho'(\tilde{\mu}_s) \left| z_s \right|^2 ds
\]
\[
\geq -\beta_s ds + \frac{1}{\rho(\tilde{\mu}_s)} z_s \left[ d\tilde{B}_s - \frac{1}{2} \rho'(\tilde{\mu}_s) z_s^* ds \right], \quad s \in [0, T],
\] (14)
where herein \( z^* \) denotes the transpose of the matrix \( z \).

Furthermore, in view of the assumptions of \( \rho(\cdot) \) and the fact that \( c \leq \tilde{\mu} \leq c + M \), we know that
\[
0 \leq \rho'(\tilde{\mu}_t)/\rho(\tilde{\mu}_t) \leq M_c/\rho(c)
\]
for each \( t \in [0, T] \). It then follows from (13) that the process
\[
H_t := \frac{1}{2} \int_0^t \rho'(\tilde{\mu}_s) z_s dB_s, \quad t \in [0, T]
\]
is a BMO-martingale under probability measure \( \mathbb{P} \). Then by Theorem 2.3 in Kazamaki [14], the stochastic exponential of \( H \),
\[
\mathcal{E}(H)_t = \exp \left( H_t - \frac{1}{2} (H)_t \right), \quad t \in [0, T]
\]
with \( (H) \) being the quadratic variation process of \( H \), is a uniformly integrable martingale under \( \mathbb{P} \). We then denote a probability measure \( \mathbb{Q} \) on \( (\Omega, \mathcal{F}_T) \) by \( \frac{d\mathbb{Q}}{d\mathbb{P}} := \mathcal{E}(H)_T \). Thus, note that, in view of (13) and the fact of \( 0 \leq 1/\rho(\tilde{\mu}_t) \leq 1/\rho(c) \) for each \( t \in [0, T] \),
\[
\mathcal{V}_t := \int_0^t \frac{1}{\rho(\tilde{\mu}_s)} z_s dB_s, \quad t \in [0, T]
\]
is also a BMO-martingale under \( \mathbb{P} \). It follows that the Girsanov transform of \( \mathcal{V} \),
\[
\int_0^t \frac{1}{\rho(\tilde{\mu}_s)} z_s \left[ d\tilde{B}_s - \frac{1}{2} \rho'(\tilde{\mu}_s) z_s^* ds \right], \quad t \in [0, T]
\]
is a BMO-martingale under the probability measure \( \mathbb{Q} \).

In the following, integrating on the interval \([t, T]\) and taking the conditional expectation with respect to \( \mathcal{F}_t \) under \( \mathbb{Q} \) on both sides of (14), we obtain that for each \( t \in [0, T] \),
\[ \Theta(c) - \Theta(\mu_t) = \mathbb{E}_Q \left[ \Theta(c) - \Theta(\mu_t) \bigg| \mathcal{F}_t \right] \geq -\mathbb{E}_Q \left[ \int_t^T \beta_s ds \bigg| \mathcal{F}_t \right] \geq -\| \int_t^T \beta_s ds \|_{\infty}, \quad \mathbb{P}\text{-a.s.,} \]

where \( \mathbb{E}_Q[X|\mathcal{F}_t] \) denotes the conditional expectation of random variable \( X \) with respect to \( \mathcal{F}_t \) under \( Q \). Then, in view of (9) and the definition of \( \mu_t \), the desired inequality (10) follows immediately from the last inequality and the strict monotonicity of the function \( \Theta(\cdot) \).

Finally, if (9) holds for \( c = 0 \), then for each \( n \geq 1 \), we have

\[ 0 \leq \mu_t \leq \Theta^{-1}\left( \Theta\left(\frac{1}{n}\right) + \left\| \int_0^T \beta_s ds \right\|_{\infty} \right). \]

The last desired assertion follows by sending \( n \to \infty \) in the previous inequality. \( \square \)

**Remark 3.4** Let \( \overline{\mathbb{P}} \) be an equivalent probability measure to \( \mathbb{P} \) defined on the space \((\Omega, \mathcal{F}_T)\). From the above proof, it is not difficult to verify that the conclusions in Proposition 3.3 still hold if the expectation and the conditional expectation appearing in the assumptions are taken under \( \overline{\mathbb{P}} \) rather than \( \mathbb{P} \). Consequently, Proposition 3.3 can be compared with Lemma 2.1 in Fan [11], where the function \( \rho(\cdot) \) does not need to satisfy condition (iii) in the definition of set \( S \), but the process \( \beta_t \) has to be deterministic, namely, it is independent of the variable \( \omega \). In addition, we also mention that due to the randomness of \( \beta_t \), the ODE method used to prove Lemma 2.1 in Fan [11] cannot be applied to prove Proposition 3.3.

### 4. Statement of the existence and uniqueness result

Before stating the existence and uniqueness result, we introduce the following assumptions on the generator \( g \):

- **(H1)** \( \mathbb{d}\mathbb{P} \times \mathbb{d}t\text{-a.e.}, \ g(\omega, t, \cdot, z) \) is continuous for each \( z \in \mathbb{R}^{k \times d} \).
- **(H2)** \( g \) satisfies a weak stochastic-monotonicity condition in \( y \), i.e., there exists a function \( \rho(\cdot) \in S \) and a process \( u. \in L^\infty(\Omega; L^1([0, T]; \mathbb{R}_+)) \) such that \( \mathbb{d}\mathbb{P} \times \mathbb{d}t\text{-a.e.} \), for each \( y_1, y_2 \in \mathbb{R}^k \) and \( z \in \mathbb{R}^{k \times d} \), we have
  \[ \langle y_1 - y_2, g(\omega, t, y_1, z) - g(\omega, t, y_2, z) \rangle \leq u_t(\omega)\rho(|y_1 - y_2|^2). \]
- **(H3)** \( g \) has a general growth in \( y \), i.e., for each \( r \in \mathbb{R}_+ \), it holds that
  \[ \mathbb{E} \left[ \int_0^T \psi_r(\omega, t)dt \right] < +\infty \]
  with
  \[ \psi_r(\omega, t) := \sup_{|y| \leq r} |g(\omega, t, y, 0) - g(\omega, t, 0, 0)|. \]

And, \( g(\omega, t, 0, 0) \in \mathcal{H}^2(0, T; \mathbb{R}^k) \).

- **(H4)** \( g \) satisfies a stochastic-Lipschitz condition in \( z \), i.e., there exists a \( v. \in L^\infty(\Omega; L^2([0, T]; \mathbb{R}_+)) \) such that \( \mathbb{d}\mathbb{P} \times \mathbb{d}t\text{-a.e.} \), for each \( y \in \mathbb{R}^k \) and \( z_1, z_2 \in \mathbb{R}^{k \times d} \),
  \[ |g(\omega, t, y, z_1) - g(\omega, t, y, z_2)| \leq v_t(\omega)|z_1 - z_2|. \]

**Remark 4.1** Assumption \((H2)\) is strictly weaker than both the non-uniform \((in \ t)\) weak monotonicity condition \(i.e., \mu \text{ is independent of } \omega\) and the stochastic-monotonicity condition.
(i.e., $\rho(x) = x$) of the generator $g$ in $\gamma$ employed, respectively, in Fan [11], Xiao and Fan [25], and Luo and Fan [16]. Also, assumption (H4) is strictly weaker than the non-uniform (in $t$) Lipschitz condition of $g$ in $z$ used in Chen and Wang [5], Morlais [18], Fan [11], and Xiao and Fan [25].

The following existence and uniqueness theorem is one of the main results of this paper.

**Theorem 4.2** Let $0 < T \leq \infty$ and $g$ satisfy assumptions (H1)–(H4). Then, for each $\xi \in L^2(F_T; \mathbb{R}^k)$, BSDE $(\xi, T, g)$ admits a unique solution $(y_t, z_t)_{t \in [0, T]} \in S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})$.

**Remark 4.3** In view of Remark 4.1, Theorem 4.2 strengthens Theorem 3.1 in Liu et al. [15] and Theorem 6 in Xiao and Fan [25] for the case of the finite variation process $V \equiv 0$ together with some corresponding existence and uniqueness results obtained, for example, in Pardoux and Peng [20], Mao [17], Chen and Wang [5], Briand et al. [4], and Fan et al. [13].

The following corollary follows from Theorem 4.2, which gives a general existence and uniqueness result for the solution of a multidimensional BSDE with a stopping time interval.

**Corollary 4.4** Let $0 < T \leq \infty$ and $\tau$ be any $(\mathcal{F}_t)$-stopping time valued in $[0, T]$. If the generator $g$ satisfies assumptions (H1)–(H4), then for each $\mathcal{F}_\tau$-measurable $\mathbb{R}^k$-valued random vector $\xi$ satisfying $\mathbb{E}[|\xi|^2] < +\infty$, BSDE $(\xi, \tau, g)$ admits a unique solution $(y_t, z_t)_{t \in [0, T]}$ in the space $S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})$ in the sense that $(y_t, z_t)_{t \in [0, T]} \in S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})$, $d\mathbb{P} \times dt$-a.e., $z_{1_{t \geq \tau}} = 0$ and $\mathbb{P}$-a.s., the following equation holds:

$$y_t = \xi + \int_{t \wedge \tau}^\tau g(s, y_s, z_s)ds - \int_{t \wedge \tau}^\tau z_s dB_s, \quad t \in [0, T].$$

**Remark 4.5** Since the assumptions used in Theorem 4.2 and Corollary 4.4 are more general than those in the existing works, some new and intrinsic difficulties naturally arise when proving them. In particular, due to the presence of the function $\rho(\cdot)$ in assumption (H2), it seems impossible to obtain a contraction by virtue of the weighted norms employed in El Karoui and Huang [8], Bender and Kohlmann [1], and Wang et al. [23]. And, due to the randomness of the processes $\mu$ and $\nu$, in assumptions (H2) and (H4), it also seems impossible to obtain a contraction by slicing the whole time interval $[0, T]$ into a finite number of deterministic subintervals, which is employed in Chen and Wang [5] and Xiao and Fan [25]. In addition, also due to the randomness of the processes $\mu$ and $\nu$, the usual (deterministic) Gronwall inequality and Bihari inequality which play a crucial role in Xiao and Fan [25] are not applicable any longer, and then we have to extend them to a stochastic version.

The following example shows that Theorem 4.2 and Corollary 4.4 are not covered by any known results.

**Example 4.6** Let $k = 2$, $0 < T \leq \infty$ and $M > 0$. Define the following $(\mathcal{F}_t)$-stopping times

$$\tau_1(\omega) := \inf \left\{ t \in [0, T] : \int_0^t |B_s(\omega)| ds \geq \frac{M}{2} \right\} \wedge T$$

and

$$\tau_2(\omega) := \inf \left\{ t \in [0, T] : \int_0^t |B_s(\omega)|^2 ds \geq \frac{M}{2} \right\} \wedge T$$

Results.
with the convention \( \inf \emptyset = +\infty \), and define the following two processes:

\[
\tilde{u}_t(\omega) := |B_t(\omega)|1_{t \leq \tau_1(\omega)} \quad \text{and} \quad \tilde{v}_t(\omega) := |B_t(\omega)|1_{t \leq \tau_2(\omega)}, \quad (\omega, t) \in \Omega \times [0, T].
\]

It is clear that \( \tilde{u} \in L^\infty(\Omega; L^1([0, T]; \mathbb{R}_+)) \) and \( \tilde{v} \in L^\infty(\Omega; L^2([0, T]; \mathbb{R}_+)) \).

For \( i = 1, 2 \), let \( y_i \) and \( z_i \), respectively, represent the \( i \)-th component of the vector \( y \) and the \( i \)-th row of the matrix \( z \). Consider the following generator: for \( (\omega, t, y, z) \in \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \),

\[
g(\omega, t, y, z) = \tilde{u}_t(\omega) \left[ \frac{h(|y_2|) - e^{y_1}}{h(|y_1|) - e^{y_2}} \right] + \tilde{v}_t(\omega) \left[ \frac{|z_2|}{|z_1|} \right] + \left[ |B_t(\omega)| \right],
\]

where \( h(x) := (-x \ln x)1_{0 \leq x \leq \delta} + (h'(\delta) - (x - \delta) + h(\delta))1_{x > \delta} \) with small enough \( \delta \).

It is not very hard to verify that \( g \) satisfies assumptions (H1) –(H4) with \( u_t(\omega) = \tilde{u}_t(\omega), v_t(\omega) = \tilde{v}_t(\omega), \) and \( \rho(x) = h(x) \). It then follows from Theorem 4.2 that for each \( \xi \in L^2(F_T; \mathbb{R}^k) \), BSDE \((\xi, T, g)\) admits a unique solution in the space \( S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d}) \). Furthermore, for each \((F_i)\)-stopping time \( \tau \) valued in \([0, T]\) and each \( F_\tau \)-measurable \( \mathbb{R}^k \)-valued random vector \( \eta \) satisfying \( \mathbb{E}[|\eta|^2] < +\infty \), it follows from Corollary 4.4 that BSDE \((\eta, \tau, g)\) also admits a unique solution in the space \( S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d}) \).

We emphasize that the above conclusions cannot be obtained by any existing results since this generator \( g \) fails to fulfil their assumptions due to the facts that \( e^x \) has a general growth in \( x \), \( h(\cdot) \) is not a linear function, and \( \tilde{u}_t(\omega) \) and \( \tilde{v}_t(\omega) \) cannot be, respectively, dominated by two deterministic nonnegative functions \( \tilde{u}_t \) and \( \tilde{v}_t \) defined on \([0, T]\) satisfying

\[
\int_0^T \tilde{u}_t \, dt < +\infty \quad \text{and} \quad \int_0^T \tilde{v}_t^2 \, dt < +\infty. \tag{15}
\]

In what follows, we show the last assertion. Indeed, we show that inequality (15) fails to hold if there exists two functions \( \tilde{u}_t, \tilde{v}_t : [0, T] \to \mathbb{R}^+ \) such that \( d\mathbb{P} \times dt \)-a.e.,

\[
\tilde{u}_t(\omega) \leq \tilde{u}_t \quad \text{and} \quad \tilde{v}_t(\omega) \leq \tilde{v}_t. \tag{16}
\]

Observe that for each \( t \in (0, T] \),

\[
\{ \omega \in \Omega : \tilde{u}_t(\omega) > \tilde{u}_t \} = \left\{ \omega \in \Omega : \int_0^t |B_s(\omega)| \, ds \leq \frac{M}{2} \quad \text{and} \quad |B_t(\omega)| > \tilde{u}_t \right\}
\]

\[
\sup_{s \in [0, t]} |B_s(\omega)| \leq \frac{M}{2t} \quad \text{and} \quad |B_t(\omega)| > \tilde{u}_t \right\}
\]

\[
\sup_{s \in [0, t]} |B_s(\omega)| = B_t(\omega) \quad \text{and} \quad \tilde{u}_t < |B_t(\omega)| \leq \frac{M}{2t} \right\}.
\]

It is simple to verify that for each \( t \in (0, T] \), if \( \tilde{u}_t < \frac{M}{2t} \), then the set in the last line has a positive probability and then \( \mathbb{P}(\{ \omega \in \Omega : \tilde{u}_t(\omega) > \tilde{u}_t \}) > 0 \). Consequently, if (16) holds, then

\[
\tilde{u}_t \geq \frac{M}{2t} \quad \text{dt-a.e. on } [0, T]
\]

and

\[
\tilde{v}_t \geq \sqrt{\frac{M}{2t}} \quad \text{dt-a.e. on } [0, T],
\]
which means that

\[ \int_0^T \dot{u}_t \, dt \geq \int_0^T \frac{M}{2t} \, dt = +\infty \quad \text{and} \quad \int_0^T \dot{v}_t^2 \, dt \geq \int_0^T \frac{M}{2t} \, dt = +\infty. \]

The desired assertion is then proved.

5. Proof of the existence and uniqueness result

In this section, we give the proof of Theorem 4.2 and Corollary 4.4. First, by virtue of Proposition 3.1, we prove an important a priori estimate for the solutions of multidimensional BSDEs — Proposition 5.1. The following assumption on the generator \( g \) will be used.

(A) There exist two processes \( \mu \in L^\infty(\Omega; L^1([0, T]; \mathbb{R}^k)) \) and \( \lambda \in L^\infty(\Omega; L^2([0, T]; \mathbb{R}^k)) \) as well as a function \( \kappa(\cdot) \in \mathcal{S} \) such that \( d\mathbb{P} \times dt \)-a.e., for each \( (y, z) \in \mathbb{R}^k \times \mathbb{R}^{k \times d} \),

\[ \langle y, g(\omega, t, y, z) \rangle \leq \mu_t(\omega)\kappa(||y||^2) + \lambda_t(\omega)||y||z|| + f_t(\omega)||y||, \]

where \( f \) is an \( (\mathcal{F}_t) \)-progressively measurable nonnegative real-valued process with

\[ \mathbb{E}\left[ \left( \int_0^T f_t \, dt \right)^2 \right] < +\infty. \]

**Proposition 5.1** Let \( 0 < T \leq \infty \), \( \xi \in L^2(\mathcal{F}_T; \mathbb{R}^k) \), the generator \( g \) satisfies assumption (A), and \( (y_t, z_t)_{t \in [0, T]} \) be a solution of BSDE \((\xi, T, g)\) in the space \( \mathcal{S}^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d}) \). Then for each \( t \in [0, T] \), we have

\[ \mathbb{E}\left[ \sup_{s \leq t} |y_s|^2 + \int_t^T |z_s|^2 \, ds \bigg| \mathcal{F}_t \right] \leq C_1^1 \mathbb{E}\left[ |\xi|^2 + \int_t^T \mu_s \, ds + \left( \int_t^T f_s \, ds \right)^2 \bigg| \mathcal{F}_t \right] \] (17)

and

\[ \mathbb{E}\left[ \sup_{s \leq t} |y_s|^2 + \int_t^T |z_s|^2 \, ds \bigg| \mathcal{F}_t \right] \leq C_2^1 \mathbb{E}\left[ |\xi|^2 + \int_t^T \mu_s \kappa(||y_s||^2) \, ds + \left( \int_t^T f_s \, ds \right)^2 \bigg| \mathcal{F}_t \right], \] (18)

where

\[ C_1^1 := 4c^2 A^2 e^{2cA} \| f_t^1 (\mu_s + \lambda_s^2) \|_\infty \quad \text{and} \quad C_2^1 := 4c^2 e^{2c} \| f_t^2 \lambda_s^2 \|_\infty \]

with \( c \geq 1 \) being a universal constant and \( A \geq 1 \) being the constant in the definition of set \( \mathcal{S} \).

**Proof** In view of assumption (A), using Itô’s formula, the Burkholder–Davis–Gundy inequality, and the basic inequality

\[ 2\alpha\beta \leq \varepsilon \alpha^2 + \frac{1}{\varepsilon} \beta^2 \quad \text{for each} \quad \alpha, \beta, \varepsilon > 0 \] (19)

together with a standard computation, we deduce the existence of a constant \( c \geq 1 \) such that

\[ \mathbb{E}\left[ \sup_{s \leq t} |y_s|^2 + \int_t^T |z_s|^2 \, ds \bigg| \mathcal{F}_t \right] \leq c \mathbb{E}\left[ |\xi|^2 + \int_t^T [\mu_s \kappa(||y_s||^2) + \lambda_s^2 |y_s|^2 + f_s |y_s|] \, ds \bigg| \mathcal{F}_t \right], \quad t \in [0, T]. \]
Then, in view of the assumptions of $\xi, \mu, \lambda, f_\ast, y_\ast$, and $z$, together with the fact that $\kappa(x) \leq A(1 + x)$ with $A \geq 1$ for $x \geq 0$, it follows from Proposition 3.1 that for each $t \in [0, T)$,

$$
\mathbb{E}\left[ \sup_{s \in [t, T]} |y_s|^2 + \int_t^T |z_s|^2 ds \bigg| \mathcal{F}_t \right] 
\leq c A e^{c A} \left\| f_t^T (\mu_s + \lambda_s^2) ds \right\| \infty \mathbb{E}\left[ |\xi|^2 + \int_t^T |\mu_s + f_s y_s| ds \bigg| \mathcal{F}_t \right]
$$

and

$$
\mathbb{E}\left[ \sup_{s \in [t, T]} |\tilde{y}_s|^2 + \int_t^T |\tilde{z}_s|^2 ds \bigg| \mathcal{F}_t \right] 
\leq c e^{2c} \left\| f_t^T \lambda_s^2 ds \right\| \infty \mathbb{E}\left[ |\xi|^2 + \int_t^T [\mu_s \kappa(|y_s|^2) + f_s y_s] ds \bigg| \mathcal{F}_t \right].
$$

Finally, by using the basic inequality (19) again, the desired inequalities (17) and (18) follow immediately from the last two inequalities. The proposition is then proved.

**Remark 5.2** From the above proof it is not difficult to see that under the assumptions of Proposition 5.1, for each $t \in [0, T)$ and any ($\mathcal{F}_t$)-stopping times $\sigma, \tau$ satisfying $0 \leq \sigma \leq \tau \leq T$, it holds that

$$
\mathbb{E}\left[ \sup_{s \in [t, T]} |\tilde{y}_s|^2 + \int_t^T |\tilde{z}_s|^2 ds \bigg| \mathcal{F}_t \right] 
\leq C_1 \mathbb{E}\left[ |\tilde{y}_\tau|^2 \bigg| \mathcal{F}_t \right] + \mathbb{E}\left[ \left( \int_0^T f_s ds \right)^2 \bigg| \mathcal{F}_t \right]
$$

and

$$
\mathbb{E}\left[ \sup_{s \in [t, T]} |\tilde{y}_s|^2 + \int_t^T |\tilde{z}_s|^2 ds \bigg| \mathcal{F}_t \right] 
\leq C_2 \mathbb{E}\left[ |\tilde{y}_\tau|^2 + \int_t^T \mu_s \kappa(|y_s|^2) ds + \left( \int_0^T f_s ds \right)^2 \bigg| \mathcal{F}_t \right],
$$

where $\tilde{y}_s := 1_{\sigma \leq s \leq \tau} y_s$, $\tilde{z}_s := 1_{\sigma \leq s \leq \tau} z_s$, $\tilde{f}_s := 1_{\sigma \leq s \leq \tau} f_s$,

$$
C_1 := 4 c^2 A e^{2cA} \left\| f_t^T (\mu_s + \lambda_s^2) ds \right\| \infty \text{ and } C_2 := 4 c^2 e^{2c} \left\| f_t^T \lambda_s^2 ds \right\| \infty.
$$

And, from the above proof we also observe that if there exists a constant $\gamma > 0$ such that $|y_t| \leq \gamma$ for each $t \in [0, T]$, then $(f_t)_{t \in [0, T]}$ in assumption (A) must only satisfy

$$
\mathbb{E}\left[ \int_0^T f_t dt \right] < +\infty,
$$

and the estimates in (17) and (18) still hold with $(\int_t^T f_s ds)^2$ being replaced with $\int_t^T \gamma f_s ds$.

The following proposition can be derived from Propositions 3.1 and 3.3.

**Proposition 5.3** Let $0 < T \leq +\infty$, $\beta \in L^\infty(\Omega; L^1([0, T]; \mathbb{R}_+))$, and $\rho(\cdot) \in S$. Assume that for each positive integer $n \geq 1$, $\hat{\eta}^n$ is an $\mathcal{F}_t$-measurable, integrable, and non-negative real-valued random variable, $\hat{Y}^n \in S^2(0, T; \mathbb{R}^k)$ and $\hat{Z}^n \in M^2(0, T; \mathbb{R}^{k \times d})$. If $\lim_{n \to \infty} \mathbb{E}[\hat{\eta}^n] = 0$ and

$$
\mathbb{E}\left[ \sup_{s \in [t, T]} |\hat{Y}_s^n|^2 + \int_t^T |\hat{Z}_s^n|^2 ds \bigg| \mathcal{F}_t \right] 
\leq \mathbb{E} \left[ \hat{\eta}^n + \int_t^T \beta_s \rho(|\hat{Y}_s^n|^2) ds \bigg| \mathcal{F}_t \right], \quad t \in [0, T],
$$

then
\[ \lim_{n \to \infty} \mathbb{E} \left[ \sup_{s \in [0,T]} |Y^n_s|^2 \right] = 0. \] (21)

**Proof** Note that \( \rho(x) \leq A(1 + x) \) for \( x \geq 0 \) by the definition of \( S \). In view of (20), it follows from Proposition 3.1 that for each \( n \geq 1 \),
\[ |\hat{Y}^n_t|^2 \leq e^{A\|I^n_t \beta_s ds\|_\infty} \mathbb{E} \left[ \hat{\eta}^n + \int_t^T A \beta_s ds \bigg| \mathcal{F}_t \right] \]
\[ \leq e^{A\|I^n_0 \beta_s ds\|_\infty} \left( \mathbb{E} [\hat{\eta}^n | \mathcal{F}_t] + A \left( \int_0^T \beta_s ds \right)_\infty \right), \ t \in [0,T]. \] (22)

Set
\[ \mu_t := \lim_{n \to \infty} |\hat{Y}^n_t|^2, \ t \in [0,T]. \]

Since \( \lim_{n \to \infty} \mathbb{E}[\hat{\eta}^n] = 0 \), it follows from (22) that
\[ 0 \leq \mu_t \leq Ae^{A\|I^n_0 \beta_s ds\|_\infty} \left( \int_0^T \beta_s ds \right)_\infty < +\infty, \ t \in [0,T]. \] (23)

In view of the continuity and monotonicity of \( \rho(\cdot) \), sending \( n \) to infinity and using Fatou’s lemma in (20) yields that
\[ \mu_t \leq \mathbb{E} \left[ \int_t^T \beta_s \rho(\mu_s) ds \bigg| \mathcal{F}_t \right], \ t \in [0,T]. \]

Then, in view of (23), it follows from Proposition 3.3 that \( \mu_t = 0 \) for each \( t \in [0,T] \). Finally, in view of the definition of \( \mu \), Fatou’s lemma and the continuity and monotonicity of \( \rho(\cdot) \) together with the fact of \( \rho(0) = 0 \), the desired conclusion (21) follows from (20) by letting \( t = 0 \) and then sending \( n \to \infty \). The proposition is then proved. \( \square \)

The following proposition considers a special case of Theorem 4.2.

**Proposition 5.4** Let \( 0 < T \leq \infty \) and the generator \( g \) satisfy assumptions (H1)–(H4). If the generator \( g \) is independent of the state variable \( z \), then for each \( \xi \in \mathbb{L}^2(\mathcal{F}_T; \mathbb{R}^k) \), BSDE \( \xi, T, g \) admits a unique solution \( (y_t, z_t)_{t \in [0,T]} \) in the space \( S^2(0,T; \mathbb{R}^k) \times M^2(0,T; \mathbb{R}^{k \times d}) \).

**Proof** Thanks to Propositions 5.1 and 5.3 together with Remark 5.2, closely following the proof procedure of Proposition 10 in Xiao and Fan [25], we deduce the desired conclusion. The detailed proof is omitted here. \( \square \)

Now, we give the proof of Theorem 4.2.

**Proof of Theorem 4.2** In view of Propositions 3.3 and 5.1, by a similar argument as that in the proof of the uniqueness part of Theorem 6 in [25], we prove the uniqueness part.

We now show the existence part. We let \( (y^n_0, z^n_0):=(0,0) \) and use Picard’s iteration method. First, since \( g \) satisfies assumptions (H1)–(H4) and \( \xi \in \mathbb{L}^2(\mathcal{F}_T; \mathbb{R}^k) \), it is not very hard to verify that for each \( n \geq 1 \) and \( z^n \in M^2(0,T; \mathbb{R}^{k \times d}) \), the generator \( g(t, y, z^{n-1}) \) satisfies all assumptions in Proposition 5.4, see [25, p.801] for details. It then follows from Proposition 5.4 that we define recursively
by the unique solution of the following BSDEs:

\[
y^n_t = \xi + \int_t^T g(s, y^n_s, z^{n-1}_s)ds - \int_t^T z^n_s dB_s, \quad t \in [0, T].
\]  

Next, we show that the sequence of processes \((y^n, z^n)_{n \geq 1}\) is Cauchy in the whole space \(S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})\) by dividing the time interval \([0, T]\) into a finite number of subintervals with stopping time ends. For each \(n, i \geq 1\), set

\[
y^{n,i}_t := y^{n+i}_t - y^n \quad \text{and} \quad z^{n,i}_t := z^{n+i}_t - z^n.
\]

Then, \((\tilde{y}^{n,i}_t, \tilde{z}^{n,i}_t)_{t \in [0, T]}\) solves the following BSDE:

\[
\tilde{y}^{n,i}_t = \int_t^T \tilde{g}^{n,i}(s, \tilde{y}^{n,i}_s)ds - \int_t^T \tilde{z}^{n,i}_s dB_s, \quad t \in [0, T],
\]

where, for each \(y \in \mathbb{R}^k\),

\[
\tilde{g}^{n,i}(t, y) := g(t, y + y^n_t, z^{n+i-1}_t) - g(t, y^n_t, z^{n-1}_t).
\]

Furthermore, from assumptions (H2) and (H4) it is not difficult to verify that the generator \(\tilde{g}^{n,i}\) satisfies assumption (A) with

\[
\mu = u, \quad \kappa(\cdot) = \rho(\cdot), \quad \lambda \equiv 0, \quad \text{and} \quad f = |\tilde{z}^{n-1,i}|v.
\]

It then follows from Proposition 5.1 and Remark 5.2 together with Hölder’s inequality that for each \(n, i \geq 1\), \(t \in [0, T]\) and any \((F_t)\)-stopping times \(\sigma, \tau\) satisfying \(0 \leq \sigma \leq \tau \leq T\), we have

\[
\mathbb{E} \left[ \sup_{s \in [\sigma, \tau]} |\tilde{y}^{n,i}_s|^2 + \int_{\sigma}^{\tau} |\tilde{z}^{n,i}_s|^2 ds \middle| F_\tau \right] 
\]

\[
\leq C \left( \mathbb{E} \left[ |\tilde{y}^{n,i}_\tau|^2 + \int_0^\tau \tilde{v}^2_s ds \int_{\sigma}^{\tau} |\tilde{z}^{n-1,i}_s|^2 ds \middle| F_\tau \right] + \left\| \int_0^{\tau} u_s ds \right\|_\infty \right),
\]

and

\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |\tilde{y}^{n,i}_s|^2 + \int_t^T |\tilde{z}^{n,i}_s|^2 ds \middle| F_t \right] 
\]

\[
\leq C \mathbb{E} \left[ |\tilde{y}^{n,i}_T|^2 + \int_t^T u_s \rho(|\tilde{y}^{n,i}_s|^2) ds + \int_t^T \tilde{v}^2_s ds \int_t^T |\tilde{z}^{n-1,i}_s|^2 ds \middle| F_t \right],
\]

where \(\tilde{y}^{n,i}_s := 1_{\sigma \leq s \leq \tau} y^{n,i}_s\), \(\tilde{z}^{n,i}_s := 1_{\sigma \leq s \leq \tau} z^{n,i}_s\), \(\tilde{v}_s := 1_{\sigma \leq s \leq \tau} v_s\) and

\[
C := 4c^2 A^2 e^{2cA} \left\| \int_t^T u_s ds \right\|_\infty
\]

with \(c \geq 1\) being a universal constant and \(A \geq 1\) being the constant in the definition of set \(S\).

Now, let us fix arbitrarily a positive integer \(N\) satisfying that

\[
\frac{M}{N} \leq \frac{1}{4C}
\]

with

\[
M := \left\| \int_0^T (u_s + \tilde{v}_s^2) ds \right\|_\infty,
\]

where

\[
1_{\sigma \leq s \leq \tau} \quad \text{for} \quad \sigma, \tau \in [0, T].
\]
and define the following $(\mathcal{F}_t)$-stopping times:

\[ T_0 = 0; \]

\[ T_1 = \inf \left\{ t \geq 0 : \int_0^t v_s^2 ds \geq \frac{M}{N} \right\} \land T; \]

\[ \vdots \]

\[ T_j = \inf \left\{ t \geq T_{j-1} : \int_0^t v_s^2 ds \geq \frac{jM}{N} \right\} \land T; \]

\[ \vdots \]

\[ T_N = \inf \left\{ t \geq T_{N-1} : \int_0^t v_s^2 ds \geq \frac{NM}{N} \right\} \land T = T. \]

Thus, we have subdivided the time interval $[0, T]$ into a finite number of stochastic intervals $[T_{j-1}, T_j], \ j = 1, \ldots, N$. And, for each $j = 1, \ldots, N$, we have

\[ \int_0^T (1_{T_{j-1} \leq s \leq T_j} v_s^2) ds \leq \frac{M}{N} \leq \frac{1}{4C}, \ \mathbb{P}\text{-a.s.} \]  

(27)

Furthermore, in view of (27) and the fact of $\hat{y}_{T_j}^{n,i} = 0$, letting $\sigma = T_{N-1}$ and $\tau = T_N = T$ in (25) and (26) yields that for each $t \in [0, T]$,

\[ \mathbb{E} \left[ \sup_{s \in [t, T]} |\hat{Y}_s^{n,i}|^2 + \int_t^T |\hat{Z}_s^{n,i}|^2 ds \left| \mathcal{F}_t \right. \right] \leq CM + \frac{1}{4} \mathbb{E} \left[ \int_t^T |\hat{Z}_s^{n-1,i}|^2 ds \left| \mathcal{F}_t \right. \right] \]  

(28)

and

\[ \mathbb{E} \left[ \sup_{s \in [t, T]} |\hat{Y}_s^{n,i}|^2 + \int_t^T |\hat{Z}_s^{n,i}|^2 ds \left| \mathcal{F}_t \right. \right] \leq C \mathbb{E} \left[ \int_t^T u_s \rho \left( |\hat{Y}_s^{n,i}|^2 \right) ds \left| \mathcal{F}_t \right. \right] + \frac{1}{4} \mathbb{E} \left[ \int_t^T |\hat{Z}_s^{n-1,i}|^2 ds \left| \mathcal{F}_t \right. \right], \]  

(29)

where

\[ \hat{Y}_s^{n,i} := \hat{y}_s^{n,i} 1_{T_{N-1} \leq s}, \quad \text{and} \quad \hat{Z}_s^{n,i} := \hat{z}_s^{n,i} 1_{T_{N-1} \leq s \leq T}, \ s \in [0, T]. \]

Thus, thanks to (28), using a similar induction analysis to that in [25, p.802–803], we derive that for each $n \geq 1$,

\[ \sup_{i \geq 1} \left( \mathbb{E} \left[ \sup_{s \in [t, T]} \left| \hat{Y}_s^{n,i} \right|^2 + \int_t^T \left| \hat{Z}_s^{n,i} \right|^2 ds \left| \mathcal{F}_t \right. \right] \right) \leq 2CM + \frac{1}{2} \mathbb{E} \left[ \int_0^T |z_s^1|^2 ds \left| \mathcal{F}_t \right. \right] =: \mathcal{M}_t < +\infty, \ t \in [0, T]. \]  

(30)

In view of (30), Fatou’s lemma, and the continuity and monotonicity of $\rho(\cdot)$, taking first the supremum with respect to $i$ and then the super limit with respect to $n$ on both sides of (29), we obtain that

\[ \mu_t \leq C \mathbb{E} \left[ \int_t^T u_s \rho(\mu_s) ds \left| \mathcal{F}_t \right. \right], \ t \in [0, T] \]

with
\[
\mu_t := \lim_{n \to \infty} \sup_{i \geq 1} |\hat{Y}^{n,i}_t|^2 < +\infty.
\]

Applying Proposition 3.3 to the last inequality yields that
\[
\mu_t = \lim_{n \to \infty} \sup_{i \geq 1} |\hat{Y}^{n,i}_t|^2 = 0, \quad t \in [0, T].
\]

Now, taking the supremum with respect to \( i \) and the super limit with respect to \( n \) on both sides of (29) again leads to that for each \( t \in [0, T] \),
\[
\lim_{n \to \infty} \sup_{i \geq 1} \mathbb{E} \left[ \sup_{s \in [t, T]} |\hat{Y}^{n,i}_s|^2 + \int_t^T |\hat{Z}^{n,i}_s|^2 ds \right] = 0. \tag{31}
\]

Finally, in view of (27) and (30), letting \( \sigma = T_{N-2} \) and \( \tau = T_{N-1} \) in (25) and (26) yields that for each \( t \in [0, T] \),
\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |\tilde{Y}^{n,i}_s|^2 + \int_t^T |\tilde{Z}^{n,i}_s|^2 ds \right] \leq C(M + M_t) + \frac{1}{4} \mathbb{E} \left[ \int_t^T |\tilde{Z}^{n-1,i}_s|^2 ds \right] \tag{32}
\]
and
\[
\mathbb{E} \left[ \sup_{s \in [t, T]} |\tilde{Y}^{n,i}_s|^2 + \int_t^T |\tilde{Z}^{n,i}_s|^2 ds \right] \leq C \mathbb{E} \left[ |\tilde{Y}^{n,i}_{T_{N-1}}|^2 \right] + \mathbb{E} \left[ \int_t^T u_s \rho \left( |\tilde{Y}^{n,i}_s|^2 \right) ds \right] + \frac{1}{4} \mathbb{E} \left[ \int_t^T |\tilde{Z}^{n-1,i}_s|^2 ds \right], \tag{33}
\]
where
\[
\tilde{Y}^{n,i}_s := \tilde{y}^{n,i}_{s \wedge T_{N-1}} 1_{T_{N-2} \leq s}, \quad \text{and} \quad \tilde{Z}^{n,i}_s := \tilde{z}^{n,i}_{s \wedge T_{N-2}} 1_{T_{N-3} \leq s \leq T_{N-1}}, \quad s \in [0, T].
\]

Then, thanks to (31), by a similar analysis as in the last paragraph, we use (32) and (33) to deduce that for each \( t \in [0, T] \),
\[
\lim_{n \to \infty} \sup_{i \geq 1} \mathbb{E} \left[ \sup_{s \in [t, T]} |\tilde{Y}^{n,i}_{s \wedge T_{N-2}} 1_{T_{N-2} \leq s}|^2 + \int_t^T |\tilde{Z}^{n,i}_{s \wedge T_{N-2}} 1_{T_{N-3} \leq s \leq T_{N-2}}|^2 ds \right] = 0. \tag{34}
\]

We advance the above procedure to derive that for each \( j = 3, \ldots, N \) and \( t \in [0, T] \),
\[
\lim_{n \to \infty} \sup_{i \geq 1} \mathbb{E} \left[ \sup_{s \in [t, T]} |\tilde{Y}^{n,i}_{s \wedge T_{N-j+1}} 1_{T_{N-j} \leq s}|^2 + \int_t^T |\tilde{Z}^{n,i}_{s \wedge T_{N-j}} 1_{T_{N-j} \leq s \leq T_{N-j+1}}|^2 ds \right] = 0. \tag{35}
\]

Thus, combining (31), (34), and (35) yields that
\[
\lim_{n \to \infty} \sup_{i \geq 1} \mathbb{E} \left[ \sup_{s \in [t, T]} |\tilde{Y}^{n,i}_s|^2 + \int_t^T |\tilde{Z}^{n,i}_s|^2 ds \right] = 0, \quad t \in [0, T], \tag{36}
\]
which means that \((y^n, z^n)_{n \geq 1}\) is a Cauchy sequence in the space \( S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})\).

We denote the limit process by \((y_t, z_t)_{t \in [0, T]}\) and take limit under the uniform convergence in probability in (24) to see, in view of (36) together with assumptions (H1), (H3), and (H4), that \((y_t, z_t)_{t \in [0, T]}\) is the desired solution to BSDE \((\xi, T, g)\) in the space \( S^2(0, T; \mathbb{R}^k) \times M^2(0, T; \mathbb{R}^{k \times d})\).

The proof is then completed. \(\square\)
Proof of Corollary 4.4 Thanks to Theorem 4.2, we can use a similar argument to that in Theorem 12 of [25] to obtain the desired conclusion. The details are omitted here. □

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