An elementary proof of the Vigdergauz equations for a class of square symmetric structures

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Abstract

For a periodically perforated structure, for which homogenization takes place in the linear theory of elasticity, the components of the effective elasticity tensor depend in general on the geometry of the holes as well as on the local elastic properties. These dependencies were shown by Vigdergauz in [7] to be separated in an elementary way for one particular class of structures. The original proof of this relation made use of the lattice approach to describe periodic functions using complex variables. In this paper we present a proof of the so-called Vigdergauz equations for a related class of square symmetric structures. Our proof relies solely on the fundamental theorem of real variable calculus. The differences between the two mentioned classes of structures are nontrivial which makes our result a partial generalization as well.

1 Introduction

In [7] Vigdergauz showed an explicit relation between the local and the effective elastic properties for a special class of periodic structures in the linear theory of elasticity. The result relates the components of the effective elasticity tensor to the local elastic properties as elementary functions involving one geometric constant for each distinct component of the effective tensor. For a subclass of the square symmetric planar structures with local elastic moduli \((K, G)\), the relation to the corresponding effective properties \((K^*, G^*, G_{45}^*)\) can be written

\[
\frac{1}{K^*} = \frac{1}{K} + A_1 \left( \frac{1}{K} + \frac{1}{G} \right),
\]

\[
\frac{1}{G^*} = \frac{1}{G} + A_2 \left( \frac{1}{K} + \frac{1}{G} \right),
\]

\[
\frac{1}{G_{45}^*} = \frac{1}{G} + A_3 \left( \frac{1}{K} + \frac{1}{G} \right),
\]

where \((A_1, A_2, A_3)\) are geometric constants that depend solely on the domain of a periodicity cell of the structure.

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The main restrictions defining the class of structures studies in [7] were that the periodicity cell should contain only one hole which is centered strictly inside the cell. In addition the boundary of the cell was supposed to be smooth. An example of such a structure is shown in Figure 1(a).

In this paper we will consider the class obtained by relaxing the assumptions on the number of holes and the smoothness of the boundary. More precisely, we consider the structures with positive definite square symmetric effective tensor, which have a cell with Lipschitz continuous boundary. Moreover, we suppose that the cell can be transversed by some lines parallel to the coordinate axes. The last requirement enables us to give a particularly simple proof of the explicit relations, which we call the Vigdergauz equations.

An example of a structure of the kind we will consider is shown in Figure 1(b). This is not of the type studied in [7] both because of the lack of smoothness and due to the number of holes in any periodicity cell. Conversely, the structure shown in Figure 1(a) is not in the class we will consider because no periodicity cell can be transversed by line segments. Hence the classes have nontrivial differences and so our result gives a partial generalization of the Vigdergauz equations.

As an illustration we consider now the first two of the above equations. To emphasize on the idea of proof we will make formal calculations and leave the precise definitions and arguments to the later sections.

Let \( C \) be the domain of one period of a structure in \( \mathbb{R}^2 \) and \( Y \) the corresponding cell, which we assume is a rectangle. We consider the case of linear elasticity and use the planar bulk modulus \( K \) and shear modulus \( G \) as the local material parameters. The stress \( \sigma \) and the strain \( \varepsilon \) are then linearly related via the local version of the Hooke law:

\[
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\gamma_{12}
\end{pmatrix} = \begin{pmatrix}
\frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) & \frac{1}{4} \left( \frac{1}{K} - \frac{1}{G} \right) & 0 \\
\frac{1}{4} \left( \frac{1}{K} - \frac{1}{G} \right) & \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) & 0 \\
0 & 0 & \frac{1}{G}
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{pmatrix}, \tag{1}
\]

here written in Voigt notation and \( \gamma_{12} \) is the engineering shear strain. The matrix on the right hand side is the inverse of the representation of the elasticity tensor. We write the two first equations explicitly as

\[
\varepsilon_{11} = \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) \sigma_{11} + \frac{1}{4} \left( \frac{1}{K} - \frac{1}{G} \right) \sigma_{22},
\]

\[
\varepsilon_{22} = \frac{1}{4} \left( \frac{1}{K} - \frac{1}{G} \right) \sigma_{11} + \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) \sigma_{22}.
\]

We assume that the strain components \( \varepsilon_{ij} \), hence also the stress components \( \sigma_{ij} \), are \( Y \)-periodic, where \( Y = [0, l_1] \times [0, l_2] \), for some positive real numbers \( l_1 \) and \( l_2 \). We also assume global quadratic symmetry. Hence, we have the corresponding average relations

\[
\langle \varepsilon_{11} \rangle = \frac{1}{4} \left( \frac{1}{K^*} + \frac{1}{G^*} \right) \langle \sigma_{11} \rangle + \frac{1}{4} \left( \frac{1}{K^*} - \frac{1}{G^*} \right) \langle \sigma_{22} \rangle,
\]

\[
\langle \varepsilon_{22} \rangle = \frac{1}{4} \left( \frac{1}{K^*} - \frac{1}{G^*} \right) \langle \sigma_{11} \rangle + \frac{1}{4} \left( \frac{1}{K^*} + \frac{1}{G^*} \right) \langle \sigma_{22} \rangle,
\]

where \( \langle \cdot \rangle \) represents an average over \( C \) or \( Y \). Here \( K^* \) denotes the effective bulk modulus, and \( G^* \) one of the two effective (transverse) shear moduli.
We assume that the possible holes in $C$ are such that there exists both horizontal and vertical line segments contained in $C$, joining opposite faces of $Y$. Then we can integrate along a horizontal line in the $Y$–cell, and find the value $\int \sigma_{22} \, dx$. It is possible to show that this integral is independent of $y$. Similarly, we find, by integrating vertically, that $\int \sigma_{11} \, dy$ is independent of $x$. Thus, we find that

$$\langle \sigma_{22} \rangle = \frac{1}{l_1} \int \sigma_{22} \, dx, \quad \langle \sigma_{11} \rangle = \frac{1}{l_2} \int \sigma_{11} \, dy.$$ 

Due to the periodicity of the deformed structure, $\int \varepsilon_{11} \, dx = u_1(l_1, y) - u_1(0, y)$ and $\int \varepsilon_{22} \, dy = u_2(x, l_2) - u_2(x, 0)$, are independent of $x$ and $y$. Hence,

$$\langle \varepsilon_{22} \rangle = \frac{1}{l_2} \int \varepsilon_{22} \, dy, \quad \langle \varepsilon_{11} \rangle = \frac{1}{l_1} \int \varepsilon_{11} \, dx. \quad (2)$$

Let us now assume that the average vertical forces is zero, that is $\langle \sigma_{22} \rangle = 0$. Then, integrating along lines without holes we obtain

$$\langle \varepsilon_{11} \rangle = \frac{1}{l_1} \int \varepsilon_{11} \, dx = \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) \frac{1}{l_1} \int \sigma_{11} \, dx,$$

$$\langle \varepsilon_{22} \rangle = \frac{1}{l_2} \int \varepsilon_{22} \, dy = \frac{1}{4} \left( \frac{1}{K} - \frac{1}{G} \right) \frac{1}{l_2} \int \sigma_{11} \, dy + \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) \frac{1}{l_2} \int \sigma_{22} \, dy.$$

Let $a$ and $b$ be defined by

$$(1 + a) \langle \sigma_{11} \rangle = \frac{1}{l_1} \int \sigma_{11} \, dx, \quad b \langle \sigma_{11} \rangle = \frac{1}{l_2} \int \sigma_{22} \, dy.$$ 

Then, by the equations in (2) we obtain that

$$\langle \varepsilon_{11} \rangle = \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) (1 + a) \langle \sigma_{11} \rangle,$$

$$\langle \varepsilon_{22} \rangle = \frac{1}{4} \left( \frac{1}{K} - \frac{1}{G} \right) \langle \sigma_{11} \rangle + \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) b \langle \sigma_{11} \rangle.$$ 

Combined with the reduced average relations,

$$\langle \varepsilon_{11} \rangle = \frac{1}{4} \left( \frac{1}{K^*} + \frac{1}{G^*} \right) \langle \sigma_{11} \rangle, \quad \langle \varepsilon_{22} \rangle = \frac{1}{4} \left( \frac{1}{K^*} - \frac{1}{G^*} \right) \langle \sigma_{11} \rangle,$$

we find that

$$\left( \frac{1}{K} + \frac{1}{G} \right) (1 + a) = \frac{1}{K^*} + \frac{1}{G^*},$$

$$\left( \frac{1}{K} - \frac{1}{G} \right) + \left( \frac{1}{K} + \frac{1}{G} \right) b = \frac{1}{K^*} - \frac{1}{G^*}.$$ 

Adding and subracting these two equations yield, respectively,

$$\frac{1}{K} + \left( \frac{1}{K} + \frac{1}{G} \right) \frac{a + b}{2} = \frac{1}{K^*},$$

$$\frac{1}{G} + \left( \frac{1}{K} + \frac{1}{G} \right) \frac{a - b}{2} = \frac{1}{G^*}.$$
In summary, we have found the following representations of the Vigdergauz constants:

\[ A_1 = \frac{1}{2} \int \sigma_{11} \, dx + \frac{1}{2} \int \sigma_{22} \, dy - \frac{1}{2}, \]

\[ A_2 = \frac{1}{2} \int \sigma_{11} \, dx - \frac{1}{2} \int \sigma_{22} \, dy - \frac{1}{2}. \]

The original proof of the above mentioned relations made use of some properties of the Weierstrass zeta-function and the Kolosov-Muskhelishvili potentials. In this paper we present a proof of the Vigdergauz equations for a class of square symmetric structures. Our proof relies solely on the Newton-Leibniz and the Green formulas, and a lemma of Michell type. The Michell lemma can be seen as a consequence of the theorem of Stokes in the current setting.

The rest of this paper is organized as follows. In the next section we give the precise definition of the class of structures that will be studied and set the notation. In Section 3 and 4 we give some representations of the average stress and the quasiperiods for the cell problem in the definition of the effective tensor. These results are used in the proof of the Vigdergauz equations in Section 5.

Figure 1: Two periodic structures (a) and (b) with square symmetric effective elasticity tensors. Periodicity cells are marked with dashed polygons.

2 Setting of the problem

We will consider a periodic structure in the planar linear theory of elasticity. Let \( \Omega \) denote a period of the structure. The associated elasticity tensor is assumed to be homogeneous and isotropic. The local constitutive relation between the stress \( \sigma \) and the strain \( \varepsilon \) will be the standard Hooke law which we write using matrix notation as

\[
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{pmatrix} = \begin{pmatrix}
K + G & K - G & 0 \\
K - G & K + G & 0 \\
0 & 0 & 2G
\end{pmatrix} \begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
2\varepsilon_{12}
\end{pmatrix}.
\] (3)
The two parameters $K$ and $G$ above are the planar bulk modulus and the shear modulus, which here are assumed to be positive real numbers so that the matrix is positive definite. Some additional assumptions will be made on the domain $\Omega$.

For a displacement field $u = (u_1, u_2)$ defined on $\Omega$, with values in $\mathbb{R}^2$, the gradient will be decomposed into its symmetric and antisymmetric parts:

$$\nabla u = \varepsilon(u) + \omega(u).$$

Explicitly, the components of these are

$$\varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad \omega_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right).$$

The stress $\sigma(u)$ is then defined by equation (3). When the displacement field is clear from the context, it will be dropped from the strain, the rotation, and the stress, writing just $\varepsilon$, $\omega$, and $\sigma$.

The domain $\Omega$ is assumed to be a bounded and connected open set in $\mathbb{R}^2$ with Lipschitz continuous boundary. Moreover, we assume that there correspond periods $l_i > 0$ such that the periodic extension $\Lambda$ of $\Omega$, is positive definite. Some additional assumptions will be made on the domain $\Omega$, which here are assumed to be positive real numbers so that the matrix $\sigma$ is clear from the context, it will be dropped from the strain, the rotation, and the stress, writing just $\varepsilon$, $\omega$, and $\sigma$.

Apart from the above standard hypotheses on the periodicity cell, we will restrict our study to the following special type of structure. We assume that $G \in \mathbb{R}^l$ of the cell $\Omega$ inside a cell $Y$ respectively, are subsets of $\Omega$ for some $\alpha$ and $\beta$. Illustrations of a perforated cell $\Omega$ inside a cell $Y$ with line segments $\gamma_1$ and $\gamma_2$ are shown in Figure 1(b) and 2(a).

We denote by $H^1_{\text{per}}(\Omega)$ the closed subspace of $H^1(\Omega)$ of elements with equal traces on the opposite faces of $\Gamma$. Let $S$ denote the set of symmetric $2 \times 2$ matrices with real entries. We say that $u \in H^1(\Omega)^2$ is quasiperiodic if $u - \xi x \in H^1(\Omega)^2$ for some $\xi \in S$, and $\xi$ is here called a quasiperiod.

Let $f$ be a function defined on some bounded domain $X$. The average value of $f$ will be denoted by $\int_X f \, dx = |X|^{-1} \int_X f \, dx$. Any $f$ on $X$ will be considered as a function on $\mathbb{R}^2$ by periodic extension followed by extension by zero. The relevant example of usage is with $g$ defined on $\Omega$, we will write $\int_Y g \, dx$ for $|Y|^{-1} \int_{\Omega} g \, dx$.

The effective elasticity tensor is assumed to be square symmetric and positive definite. This means additional restrictions on $\Omega$ and which can be explicitly stated as follows. The effective tensor, described with parameters $K^*, G^*$, and $G_{45}^*$, defined for $\xi \in S$ by the equation

$$\begin{pmatrix} f_Y \sigma_{11} \, dx \\ f_Y \sigma_{22} \, dx \\ f_Y \sigma_{12} \, dx \end{pmatrix} = \begin{pmatrix} K^* + G^* & K^* - G^* & 0 \\ 0 & K^* + G^* & K^* - G^* \\ 0 & 0 & G_{45}^* \end{pmatrix} \begin{pmatrix} \xi_{11} \\ \xi_{22} \\ 2\xi_{12} \end{pmatrix},$$

with $u - \xi x \in H^1_{\text{per}}(\Omega)^2$ being a minimizer of the elastic energy $\int_{\Omega} \varepsilon(u) \cdot \sigma(u) \, dx$. The assumption of
Figure 2: A cell (a) and its extension together with line segments (b). The periodicity cell $\Omega$ in (a) corresponds to the dashed rectangle $Y$ in Figure 1(b). The line segments marked with arrows in (a) are examples of $\gamma_1$ and $\gamma_2$. In (b) the cell shown in (a) is marked as a colored region and the dashed rectangle shows the translated cell $Y'$. The line segments marked with arrows in (b) are the translates $\gamma'_1$ and $\gamma'_2$ of $\gamma_1$ and $\gamma_2$, respectively, from (a) as defined in the proof of Lemma 1.

Positive definiteness is equivalent to $K^*, G^*, G_{45}^* > 0$, which are the scaled eigenvalues of the matrix on the right hand side in equation (4). The matrix is in particular nonsingular.

The existence of a solution $u$ to the above minimization problem is a direct consequence of the Riesz representation theorem and the Korn inequality. The displacement field $u$ is by periodicity unique up to translation, which shows that the corresponding stress $\sigma$ is unique. Moreover, since the data are smooth, $u$ is smooth in some neighborhoods of $\gamma_1$ and $\gamma_2$. A proof of the regularity of $u$ can be found in [2] and we refer for concreteness also to [6, Chapter 6].

The relation between the local parameter $K$ and $G$, and the components $c_{ijkl}$ of the elasticity compliance tensor in the standard notation is given by

$$c_{1111} = c_{2222} = \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right), \quad c_{1122} = \frac{1}{4} \left( \frac{1}{K} - \frac{1}{G} \right), \quad c_{1212} = \frac{1}{4G},$$

where the rest of the components are zero, up to the standard symmetry. The parameters in relation (4) are called the effective planar bulk modulus $K^*$, and the effective shear moduli $G^*$ and $G_{45}^*$. The relation to the effective compliance tensor is, similarly to the above description, given by

$$c_{1111}^* = c_{2222}^* = \frac{1}{4} \left( \frac{1}{K^*} + \frac{1}{G^*} \right), \quad c_{1122}^* = \frac{1}{4} \left( \frac{1}{K^*} - \frac{1}{G^*} \right), \quad c_{1212}^* = \frac{1}{4G_{45}^*}.$$ 

The rest of the components are determined by symmetry as in the local relation.

For a stress field $\sigma$, the divergence and the trace will be denoted as follows:

$$\text{div } v = \left( \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2}, \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} \right), \quad \text{Tr } \sigma = \sigma_{11} + \sigma_{22}.$$ 

1. The inverse of the elasticity tensor.
2. For all indices, $c_{ijkl} = c_{jikl} = c_{ijlk}$. 

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The outward unit normal of at a Lipschitz continuous boundary will be denoted by \( \nu \), and the normal traction of the stress by \( \sigma \nu \). The stress and the strain are sometimes represented by the vectors \( \sigma = (\sigma_{11}, \sigma_{22}, \sigma_{12}) \) and \( \varepsilon = (\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}) \), as in equation (3) and in a sense also in equation (4). The product is then the standard inner product on \( \mathbb{R}^3 \), which coincides with the Frobenius inner product of the tensor fields. The relevant example here is the elastic energy above.

### 3 Representation of average stress

In this section we give a representation of the average stress over the periodicity cell in terms of the average stress over line segments in the definition of the effective elasticity tensor. See Figure 2(a) for an illustration of the line segments \( \gamma_1 \) and \( \gamma_2 \). In Figure 2(b), the other line segments used in the proof of the lemma below are illustrated.

**Lemma 1.** Let \( u \) be a quasiperiodic minimizer of \( \int_{\Omega} \varepsilon(u) \cdot \sigma(u) \, dx \). Then

\[
\int_{Y'} \sigma_{11} \, dx = \int_{\gamma_2} \sigma_{11} \, dx,
\]

\[
\int_{Y'} \sigma_{22} \, dx = \int_{\gamma_1} \sigma_{22} \, dx,
\]

\[
\int_{Y'} \sigma_{12} \, dx = \int_{\gamma_2} \sigma_{12} \, dx = \int_{\gamma_1} \sigma_{12} \, dx.
\]

**Proof.** Since \( u \) minimizes the elastic energy, the stress \( \sigma = \sigma(u) \) satisfies the equilibrium equation \( \text{div} \, \sigma = 0 \) in the sense of distributions with vanishing normal stress \( \sigma \nu \) at the boundaries of any possible hole in the global structure. The quasiperiodicity of \( u \) implies the periodicity of \( \varepsilon \) and hence of \( \sigma \). We therefore have

\[
\text{div} \, \sigma = 0 \text{ in } \Lambda,
\]

\[
\sigma \nu = 0 \text{ on } \partial \Lambda.
\]

Let \( \gamma_1' \) be the translate of \( \gamma_1 \) such that the left endpoint of \( \gamma_1' \) is the common point of \( \gamma_1 \) and \( \gamma_2 \). Let \( \gamma_2' \) be the translate of \( \gamma_2 \) such that its lower endpoint is the left endpoint of \( \gamma_1' \). Let \( Y' \) be a translate of \( Y \) such that \( \gamma_1' \) and \( \gamma_2' \) are the lower and left sides of the boundary of \( Y' \cap \Lambda \), respectively. Let \( \Omega' = Y' \cap \Lambda \).

Since \( \sigma \) and \( \text{div} \, \sigma \) have components in \( L^2(\Omega') \), the following Green formula holds for all \( \varphi \in H^1(\Omega')^2 \):

\[
\int_{\Omega'} \sigma \cdot \nabla \varphi \, dx + \int_{\partial \Omega'} \text{div} \, \sigma \cdot \varphi \, dS = \langle \sigma \nu, \varphi \rangle,
\]

where \( \langle \cdot, \cdot \rangle \) denotes the pairing of \( H^{1/2}(\partial \Omega')^2 \) and its continuous dual. Since \( \text{dist}(\partial Y', \partial \Lambda) > 0 \) and \( \sigma \nu = 0 \) on \( \partial \Lambda \) it follows from the regularity of \( u \) that \( \sigma \nu \in L^2(\partial Y')^2 \). Hence

\[
\langle \sigma \nu, \varphi \rangle = \int_{\partial Y'} \sigma \nu \cdot \varphi \, dS = \int_{\partial Y'} \sigma \nu \cdot \varphi \, dx.
\]
Let $\eta \in \mathbb{R}$ be such that $\varphi = (x_1 + \eta, 0)$ vanishes on $\gamma'_2$. Then by periodicity and the Green formula,

$$\int_{\Omega} \sigma_{11} \, dx = \int_{\Omega'} \sigma_{11} \, dx = \int_{\partial Y'} \sigma \cdot (l_1, 0) \, dx = l_1 \int_{\gamma'_2} \sigma_{11} \, dx = l_2 \int_{\gamma_2} \sigma_{11} \, dx.$$ 

A division by $|Y|$ yields the first equation in the statement of the lemma.

The other three equations follow by the same argument when using test functions of the types $(0, x_2 + \eta)$, $(x_2 + \eta, 0)$, and $(0, x_1 + \eta)$, respectively, for suitable $\eta$.

In the proof of the above lemma used the Green formula which relies on the existence of a normal trace operator for elements in $L^2(\Omega)^n$ with divergence in $L^2(\Omega)$. We refer the reader to [3, Chapter 2] (and [4]) for its construction.

4 Representation of quasiperiods

In this section we give a representation of the quasiperiods of the displacement fields in the definition of the effective elasticity tensor in terms the average stress over the line segments and the periodicity cell. The idea of the proof is to write $\nabla = \epsilon - \omega$ and then use the Newton-Leibniz formula and that the local elasticity tensor is homogeneous by supposition. Hence it is essentially an application of the Cesàro formula.

Let $\gamma'_1$ and $\gamma'_2$ denote the translations of $\gamma_1$ and $\gamma_2$, respectively, that were considered in the proof of Lemma 1. For illustrations see Figure 2(a) and 2(b).

**Lemma 2.** Let $u$ be a quasiperiodic minimizer of $\int_{\Omega} \varepsilon(u) \cdot \sigma(u) \, dx$. Then the quasiperiod $\xi \in S$ of $u$ satisfies

$$\xi_{11} = \frac{1}{4K} \left( \int_{\gamma_1} \sigma_{11} \, dx + \int_{Y} \sigma_{22} \, dx \right) + \frac{1}{4G} \left( \int_{\gamma_1} \sigma_{11} \, dx - \int_{Y} \sigma_{22} \, dx \right),$$

$$\xi_{22} = \frac{1}{4K} \left( \int_{\gamma_2} \sigma_{22} \, dx + \int_{Y} \sigma_{11} \, dx \right) + \frac{1}{4G} \left( \int_{\gamma_2} \sigma_{22} \, dx - \int_{Y} \sigma_{11} \, dx \right),$$

$$\xi_{12} = \frac{1}{2G} \int_{Y} \sigma_{12} \, dx + \frac{1}{8} \left( \frac{1}{K} + \frac{1}{G} \right) \left( \int_{\gamma_1} \frac{\partial \text{Tr} \sigma}{\partial x_1} \, dx + \int_{\gamma_2} \frac{\partial \text{Tr} \sigma}{\partial x_2} \, dx \right).$$

**Proof.** Since $u$ is quasiperiodic we have by the Newton-Leibniz formula and the local elastic relation (3),

$$\xi_{11} = u_1(b) - u_1(a) = \int_{\gamma_1} \frac{\partial u_1}{\partial x_1} \, dx = \int_{\gamma_1} \varepsilon_{11} \, dx$$

$$= \frac{1}{4K} \left( \int_{\gamma_1} \sigma_{11} \, dx + \int_{\gamma_1} \sigma_{22} \, dx \right) + \frac{1}{4G} \left( \int_{\gamma_1} \sigma_{11} \, dx - \int_{\gamma_1} \sigma_{22} \, dx \right),$$

where $a$ and $b$ denote the endpoints of $\gamma_1$. By Lemma 1

$$\xi_{11} = \frac{1}{4K} \left( \int_{\gamma_1} \sigma_{11} \, dx + \frac{1}{l_1} \int_{Y} \sigma_{22} \, dx \right) + \frac{1}{4G} \left( \int_{\gamma_1} \sigma_{11} \, dx - \frac{1}{l_2} \int_{Y} \sigma_{22} \, dx \right).$$

A division by $l_1$ gives the first representation and the second follows by an interchange of the indices.
Denote by $a$ the common point of $\gamma_1$ and $\gamma_2$. With $b$ being the other endpoint of $\gamma'_1$, we have by periodicity,

$$\xi_{12} = u_2(b) - u_2(a) = \int_{\gamma'_1} \frac{\partial u_2}{\partial x_1} \, dx = \int_{\gamma'_1} \varepsilon_{12} \, dx + \int_{\gamma'_1} \omega_{21} \, dx$$

$$= \frac{1}{2G} \int_{\gamma'_1} \sigma_{12} \, dx + \int_{\gamma'_1} \omega_{21} \, dx = \frac{1}{2G} \int_{\gamma_1} \sigma_{12} \, dx + \int_{\gamma_1} \omega_{21} \, dx.$$ 

An integration by parts gives by periodicity,

$$\int_{\gamma'_1} \omega_{21} \, dx = l_1 \omega_{21}(a) - \int_{\gamma'_1} \frac{\partial \omega_{21}}{\partial x_1} (x_1 - b_1) \, dx = l_1 \omega_{21}(a) - \int_{\gamma'_1} \frac{\partial \omega_{21}}{\partial x_1} x_1 \, dx.$$ 

By the local relation (3),

$$\int_{\gamma'_1} \frac{\partial \omega_{21}}{\partial x_1} x_1 \, dx = \int_{\gamma'_1} \left( \frac{\partial \varepsilon_{12}}{\partial x_1} - \frac{\partial \varepsilon_{11}}{\partial x_2} \right) x_1 \, dx$$

$$= \frac{1}{2G} \int_{\gamma'_1} \frac{\partial \sigma_{12}}{\partial x_1} x_1 \, dx - \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) \int_{\gamma'_1} \frac{\partial \sigma_{11}}{\partial x_2} x_1 \, dx$$

$$- \frac{1}{4} \left( \frac{1}{K} - \frac{1}{G} \right) \int_{\gamma'_1} \frac{\partial \sigma_{22}}{\partial x_2} x_1 \, dx.$$ 

A division by $l_1$ yields

$$\xi_{12} = \omega_{21}(a) + \frac{1}{2G} \int_{\gamma_1} \sigma_{12} \, dx - \frac{1}{2G} \int_{\gamma'_1} \frac{\partial \sigma_{12}}{\partial x_1} x_1 \, dx$$

$$+ \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) \int_{\gamma'_1} \frac{\partial \sigma_{11}}{\partial x_2} x_1 \, dx + \frac{1}{4} \left( \frac{1}{K} - \frac{1}{G} \right) \int_{\gamma'_1} \frac{\partial \sigma_{22}}{\partial x_2} x_1 \, dx$$

$$= \omega_{21}(a) + \frac{1}{2G} \int_{Y} \sigma_{12} \, dx + \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) \int_{\gamma'_1} \frac{\partial \text{Tr} \sigma}{\partial x_2} x_1 \, dx,$$

where we in the last step used Lemma 1 and that $\text{div} \sigma = 0$. By periodicity $\omega_{12}(a) = \omega_{12}(b)$ and thus we get by an interchange of the indices,

$$\xi_{12} = \omega_{12}(a) + \frac{1}{2G} \int_{Y} \sigma_{12} \, dx + \frac{1}{4} \left( \frac{1}{K} + \frac{1}{G} \right) \int_{\gamma'_1} \frac{\partial \text{Tr} \sigma}{\partial x_1} x_2 \, dx.$$ 

By the antisymmetry of $\omega$, taking the arithmetic mean of these two representations of $\xi_{12}$ completes the proof.

5 The Vigdergauz equations

In this section we give a proof of the Vigdergauz equations for the considered class of structures. The proof is a straightforward application of the representation lemmas of the previous sections to the definition of the effective elasticity tensor. The independence of the local elastic properties $K$ and $G$ follow from the Michell lemma. We briefly recall the requirements on the periodicity cell. The

The point $a$ is also the common point of $\gamma'_1$ and $\gamma'_2$. 
cell $\Omega$ is assumed to be such that the effective elasticity tensor is positive definite and has square symmetry so that it can be described by three parameters $K^*$, $G^*$, and $G_{45}^*$. Moreover, it is assumed that there exist line segments in $\Omega$, as in Figure 2(a), connecting the opposite sides without touching the boundary of any possible hole in the periodic extension $\Lambda$ of $\Omega$. The Michell lemma states that for the pure traction problem in $\mathbb{R}^2$ with homogeneous and isotropic elasticity tensor on a bounded Lipschitz domain, the stress field is independent of the values of the components of the elasticity tensor. A reference is given in the end of this section.

**Theorem 1.** There exist nonnegative real numbers $A_i$ depending only on $\Omega$ such that

\[
\frac{1}{K^*} = \frac{1}{K} + A_1 \left( \frac{1}{K} + \frac{1}{G} \right), \\
\frac{1}{G^*} = \frac{1}{G} + A_2 \left( \frac{1}{K} + \frac{1}{G} \right), \\
\frac{1}{G_{45}^*} = \frac{1}{G} + A_3 \left( \frac{1}{K} + \frac{1}{G} \right).
\]

**Proof.** Let $K, G > 0$ be given. Let $\eta \in S$. Then by the nonsingularity of the effective tensor, there exists a quasiperiodic minimizer of $\int_{\Omega} \varepsilon(u) : \sigma(u) \, dx$ with unique quasiperiod $\xi \in S$ and with average stress, defined by the left hand side of equation (4), which is equal to $\eta$ represented by $(\eta_{11}, \eta_{22}, \eta_{12})$. The displacement field $u$ is unique up to translation. Therefore the stress $\sigma$ is unique. Thus $\sigma$ is the unique solution to the corresponding pure traction problem. Hence by the Michell lemma, the stress $\sigma$ is independent of both $K$ and $G$. We proceed in three steps.

Choose $\sigma$, by specifying $\eta$, such that

\[
\int_Y \sigma_{11} \, dx \neq 0, \quad \int_Y \sigma_{22} \, dx = \int_Y \sigma_{12} \, dx = 0.
\]

By equation (4) and Lemma 2

\[
\frac{1}{K^*} + \frac{1}{G^*} = \frac{4 \xi_{11}}{\int_Y \sigma_{11} \, dx} = \frac{\int_Y \sigma_{11} \, dx}{\int_Y \sigma_{11} \, dx} \left( \frac{1}{K} + \frac{1}{G} \right). \tag{5}
\]

Choose $\sigma$ such that

\[
\int_Y \sigma_{11} \, dx = \int_Y \sigma_{22} \, dx \neq 0, \quad \int_Y \sigma_{12} \, dx = 0.
\]

Then by equation (4) and Lemma 2

\[
\frac{1}{K^*} = \frac{2 \xi_{11}}{\int_Y \sigma_{11} \, dx} = \frac{1}{K} + \frac{1}{2} \left( \frac{\int_Y \sigma_{11} \, dx}{\int_Y \sigma_{11} \, dx} - 1 \right) \left( \frac{1}{K} + \frac{1}{G} \right). \tag{6}
\]

\[\text{The Neumann problem in which only the normal traction } \sigma_{\nu} \text{ is specified on the boundary and no external body forces are present. In addition, the total force on each hole is required to vanish.}\]
From the equations (5) and (6), the two first equations in the statement of the lemma follow, since the stress is independent of the local elastic properties.

Choose \( \sigma \) such that
\[
-\int_Y \sigma_{11} \, dx = \int_Y \sigma_{22} \, dx = 0, \quad -\int_Y \sigma_{12} \, dx \neq 0.
\]

From equation (4) and Lemma 2 it follows that
\[
\frac{1}{G_{45}} = \frac{1}{G} + \frac{\int_Y x_1 \partial_T \sigma x_1 \, dx + \int_Y x_2 \partial_T \sigma x_2 \, dx}{\int_Y \sigma_{12} \, dx} \left( \frac{1}{K} + \frac{1}{G} \right),
\]
which completes the proof of the separation of variables.

Left to show is the nonnegativity of the geometric constants. This follows directly from the assumption of positive definite effective elasticity tensor. For the matrix is symmetric and thus has strictly positive real eigenvalues, so the same holds for its inverse. Using the equations (5)–(7) to write the components of the effective compliance matrix in terms of \( A_1, A_2, A_3, \) and \( K, G \), we find that the eigenvalues \( \lambda_i \) are
\[
\lambda_1 = \frac{(K + G)A_1 + K}{2KG}, \quad \lambda_2 = \frac{(K + G)A_2 + K}{2KG}, \quad \lambda_3 = \frac{(K + G)A_3 + K}{KG},
\]
for any choice of \( K, G > 0 \). Hence \( A_1, A_2, A_3 \geq 0 \).

**Remark 1.** There are natural alternatives to letting \( \sigma \) be such that its average value over the periodicity cell is some scaled element of the canonical basis of \( \mathbb{R}^3 \) as in the proof of Theorem 1. For example, one can choose \( \sigma \) such that the average stress vector varies over eigenvectors of the effective elasticity matrix in equation (4), which are clearly also the eigenvectors of the compliance matrix. In this way the deformations are connected to the physical intuition of what loads the bulk and shear moduli are supporting and in that sense arrive more directly to the Vigdergauz equations as exemplified in [7]. However, the proof given above would not be simplified in any way by making such modification.

**Remark 2.** The nonnegativity of the geometric constants \( A_1, A_2, A_3 \) in Theorem 1 can also be seen as a direct consequence of the Hashin-Shtrikman bounds, which for square symmetric tensors can be written
\[
\frac{1}{K^*} \geq \frac{1}{K} + \frac{1 - \rho}{\rho} \left( \frac{1}{K} + \frac{1}{G} \right),
\]
\[
\frac{1}{G^*} \geq \frac{1}{G} + \frac{1 - \rho}{\rho} \left( \frac{1}{K} + \frac{1}{G} \right),
\]
\[
\frac{1}{G_{45}^*} \geq \frac{1}{G} + \frac{1 - \rho}{\rho} \left( \frac{1}{K} + \frac{1}{G} \right),
\]
where \( \rho \) denotes the positive volume fraction \( \rho = |\Omega|/|Y| \leq 1 \).

A proof of the Michell lemma for smooth domains can be found in [5, Chapter 5]. For Lipschitz domains, the theorem of Stokes can be used. We refer the reader to [3, Chapter 2] for an appropriate version of the theorem, which should be applied to the simply connected components of a decomposition of \( \Omega \). The
Hashin-Shtrikman bounds in terms of $K^*$, $G^*$, and $G_{45}^*$, that were used in the above proof, can be found in [1].

Remark that if some lines at angles $\pi/4$ and $3\pi/4$ transverse the corresponding periodicity cell without touching the boundary of any possible hole, the proof for $G_{45}^*$ simplifies to that of $G^*$. This is because by the hypothesis of the symmetry of the effective tensor, $G^*$ and $G_{45}^*$ interchange roles upon a rotation of the coordinate system by $\pi/4$. The structure and cell illustrated in Figure 1(b) constitute an example of a structure for which no such lines exist.

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