EXAMPLES OF PARA-COCYCLIC OBJECTS INDUCED BY $BD$-LAWS

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Dedicated to Freddy Van Oystaeyen on the occasion of his 60th birthday.

Abstract. In a recent paper [BS], we gave a general construction of a para-cocyclic structure on a cosimplicial object, associated to a so called admissible septuple – consisting of two categories, three functors and two natural transformations, subject to compatibility relations. The main examples of such admissible septuples were induced by algebra homomorphisms. In this note we provide more general examples coming from appropriate ('locally braided') morphisms of monads.

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INTRODUCTION

History of cyclic homology started in the early eighties of the last century. The seminal works of the pioneers A Connes, B Tsygan, D Quillen and J-L Loday were motivated by looking for non-commutative generalizations of de Rham cohomology on one hand, and Lie algebra homology of matrices on the other hand.

In the subsequent decades cyclic homology has been extensively studied and became an important tool in diverse areas of mathematics, such as homological algebra, algebraic topology, Lie algebras, algebraic K-theory and so non-commutative differential geometry. Thus by most various motivations, lots of examples have been constructed. In order to study general features of the examples, and also to be able to construct new ones, it was desirable to find a unifying general description. A fundamental first step in this direction was made by A Kaygun in [Kay], who gave a construction of para-(co)cyclic objects in symmetric monoidal categories in terms of (co)monoids. In particular, in this way he managed to describe in a universal form all examples arising from Hopf cyclic theory (upto cyclic duality, cf. [KR]). Motivated by a generalization to bialgebroids (over non-commutative rings, in which case the underlying bimodule categories are not symmetric), in [BS] we made a further step of generalization and constructed para-(co)cyclic objects in arbitrary categories, in terms of (co)monads. Kaygun's construction can be recovered as a particular case when the (co)monads in question are induced by (co)monoids.

Mean examples of Kaygun's construction are induced by algebras over a commutative ring. By analogy, in this paper we show that appropriate monad morphisms (which are 'locally braided' in a sense to be described) induce examples of para-cocyclic objects in [BS].

The paper is organized as follows. In Section 1 we recall some facts about monads and $BD$-laws that are used in the paper. In Section 2 we introduce the notion of a locally braid preserving morphism of monads, generalizing a homomorphism of algebras, and we investigate their basic properties that are needed to state and prove our main result. In Section 3 we show that any such morphism determines an 'admissible septuple' in the sense of [BS], hence can be used to construct para-cocyclic objects. Here we also illustrate how this construction works in the example of a morphism of algebras in a braided monoidal category (i.e. when there is a global braiding). Particular examples will be provided by appropriate homomorphisms of (co)module algebras of a (co)quasitriangular Hopf algebra.

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1. Monads and the category of their (bi)modules

Throughout the paper, we use the notations introduced in [BS]. That is, in the 2-category CAT horizontal composition (of functors) is denoted by juxtaposition, while \( \circ \) is used for vertical composition (of natural transformations). For example, for two functors \( F : \mathcal{C} \to \mathcal{C}' \), \( G : \mathcal{C}' \to \mathcal{C}'' \) and an object \( X \) in \( \mathcal{C} \), instead of \( G(F(X)) \) we write \( GF X \). For two natural transformations \( \mu : F \to F' \) and \( \nu : G \to G' \) we write \( G'\mu X \circ \nu F X : GF X \to G'F' X \) instead of \( G'(\mu_X) \circ \nu_{F(X)} \). In equalities of natural transformations we shall omit the object \( X \) in our formulae.

We shall also use a graphical representation of morphisms in a category. For functors \( F_1, \ldots, F_n, \) \( G_1, \ldots, G_m \), which can be composed to \( F_1 F_2 \ldots F_n : D_1 \to \mathcal{C} \) and \( G_1 G_2 \ldots G_m : D_2 \to \mathcal{C} \), and objects \( X \) in \( D_1 \) and \( Y \) in \( D_2 \), a morphism \( f : F_1 F_2 \ldots F_n X \to G_1 G_2 \ldots G_m Y \) will be represented vertically, with the domain up, as in Figure 1(a). Furthermore, for a functor \( T : \mathcal{C} \to \mathcal{C}' \), the morphism \( Tf \) will be drawn as in (b). Keeping the notation from the first paragraph of this section, the picture representing \( \mu GX \) is shown in diagram (c). The composition \( g \circ f \) of the morphisms \( f : X \to Y \) and \( g : Y \to Z \) will be represented as in diagram (d). For the multiplication \( t \) and the unit \( \tau \) of a monad \( T \) on \( \mathcal{C} \) (see Definition 1.1), and an object \( X \) in \( \mathcal{C} \), to draw \( tX \) and \( \tau X \) we shall use the diagrams (e) and (f), while for a distributive law \( \iota : RT \to TR \) (see Definition 1.9) \( \iota X \) will be drawn as in the picture (g). If \( \iota \) is invertible, the representation of \( \iota^{-1} X \) is shown in diagram (h).

![Figure 1. Diagrammatic representation of morphisms in a category.](image)

For simplifying the diagrams containing only natural transformations, we shall always omit the last string, that corresponds to an object in the category.

**Definition 1.1.** A monad on a category \( \mathcal{C} \) is a triple \((R, r, \rho)\), where \( R : \mathcal{C} \to \mathcal{C} \) is a functor, \( r : R^2 \to R \) and \( \rho : \text{Id}_\mathcal{C} \to R \) are natural transformations such that the first two diagrams in Figure 2, expressing associativity and unitality, are commutative. We call \( r \) and \( \rho \) the multiplication and the unit of the monad \( R \), respectively.

![Figure 2. Monads and morphisms of monads.](image)

For two monads \((R, r, \rho)\) and \((T, t, \tau)\) on \( \mathcal{C} \), we say that a natural transformation \( \varphi : R \to T \) is a morphism of monads if the last two diagrams in Figure 2 are commutative.

**Example 1.2.** Let \((\mathcal{C}, \otimes, \alpha, \iota_l, \iota_r, 1)\) be a monoidal category with tensor product \( \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \) and unit object \( 1 \in \text{OB}(\mathcal{C}) \). Recall that the associativity constraint \( \alpha_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z) \)
and the unit constraints $\iota_X : X \otimes 1 \to X$ and $\iota_X : 1 \otimes X \to X$ are natural isomorphisms that obey Pentagon Axiom and Triangle Axioms, cf. [Kas, Chapter XI].

Algebras in monoidal categories can be defined as in the classical case, of algebras over a commutative ring, see [AMS, §1]. To such an algebra $R$ in $C$, with multiplication $r : R \otimes R \to R$ and unit $\rho : 1 \to R$, one associates a monad $R := R \otimes (-)$ on $C$. Its multiplication and unit are respectively defined by the morphisms $rX := (r \otimes X) \circ \alpha_{R,R,X}^1$ and $\rho X := \rho \otimes X$, where $X$ is an arbitrary object in $C$. An algebra homomorphism $\varphi : R \to T$ in $C$ induces a monad morphism $\varphi \otimes (-) : R \otimes (-) \to T \otimes (-)$ that will be denoted by $\varphi$.

**Definition 1.3.** For a monad $(R, r, \rho)$ on $C$, a pair $(X, x)$ is said to be an $R$-module if $X$ is an object in $C$ and $x : RX \to X$ is a morphism such that the first two diagrams in Figure 3, expressing associativity and unitality, are commutative. A *morphisms of $R$-modules* from $(X, x)$ to $(Y, y)$ is a morphism $f : X \to Y$ in $C$ such that the third diagram in Figure 3 is commutative. The category of $R$-modules is denoted by $\mathcal{R}C$.

![Diagram of modules over a monad and morphisms of modules.](image)

**Example 1.4.** For a monad $R = R \otimes (-)$, induced by an algebra $R$ in a monoidal category $C$ as in Example 1.2, $\mathcal{R}C$ is isomorphic to the category $R$-Mod of left modules for the algebra $R$. For the definition of modules over an algebra in a monoidal category, see [AMS, §1].

For a monad $(R, r, \rho)$ on a category $C$, there is a faithful forgetful functor $\rho^* : \mathcal{R}C \to C$. It is given by the object map $(X, x) \mapsto X$ and it acts on the morphisms as the identity map. The forgetful functor possesses a left adjoint, the free functor $\rho_* : C \to \mathcal{R}C$. It has the object map $X \mapsto (RX, rX)$ and it acts on the morphisms by $f \mapsto Rf$.

Assuming that coequalizers exist in $C$, it is natural to ask if the category of modules over a monad $T$ on $C$ has the same property. In what follows we will frequently use the standard result that this question has a positive answer if $T$ preserves coequalizers in $C$. Recall that a functor $F : C \to D$ preserves coequalizers if, for any coequalizer $(Z, \pi)$ in $C$

$$X \xrightarrow{f} Y \xrightarrow{\pi} Z,$$

$(FZ, F\pi)$ is the coequalizer of $(Ff, Fg)$. Note that in this case the canonical morphisms $\pi$ and $F\pi$ are epimorphisms (while for an arbitrary epimorphism $p$ in $C$, the morphism $Fp$ is not necessarily an epimorphism in $D$).

**Proposition 1.5.** Let $(T, t, \tau)$ be a monad on a category $C$ that preserves coequalizers. Assume that $(Z, \pi)$ is the coequalizer of a parallel pair of morphisms $(f, g)$ in $C$, as in (1.1). If, in addition, $X$ and $Y$ are $T$-modules such that $f$ and $g$ are morphisms of $T$-modules, then there is a unique $T$-module structure on $Z$ such that $(Z, \pi)$ is the coequalizer of $(f, g)$ in $\tau C$. In particular, if any pair of parallel morphisms in $C$ has a coequalizer then any pair of parallel morphisms in $\tau C$ has a coequalizer, too.

**Proof.** Let $x$ and $y$ denote the actions of $T$ on $X$ and $Y$, respectively. Since $(TZ, T\pi)$ is the coequalizer of $(Tf, Tg)$ in $C$, and $\pi \circ y$ coequalizes $(Tf, Tg)$, it follows that there is a unique morphism $z : TZ \to Z$ such that

$$z \circ T\pi = \pi \circ y.$$

(1.2)
Our aim is to prove that \((Z, z)\) is a \(T\)-module. Since \(T\) is a functor, in view of (1.2), associativity of \(y\) and of the fact that \(t\) is natural, we get
\[
z \circ Tz \circ TT\pi = z \circ T\pi \circ Ty = \pi \circ y \circ Ty = \pi \circ y \circ tY = z \circ T\pi \circ tY = z \circ tZ \circ TT\pi.
\]
Since \(T\) preserves coequalizers, \(TT\pi\) is an epimorphism, so the associativity condition in the definition of \(T\)-modules is verified. In a similar way one can show that \(\tau Z\) is a right inverse of \(z\). Thus \((Z, z)\) is a \(T\)-module and, in view of (1.2), \(\pi\) is a morphism of \(T\)-modules.

Let \(h\) be a morphism in \(\tau\mathcal{C}\) that coequalizes \((f, g)\). As \((Z, \pi)\) is the coequalizer of \((f, g)\) in \(\mathcal{C}\), there is a unique morphism \(\overline{h} \circ \pi = h\). Obviously, \(\overline{h}\) is a morphism of \(T\)-modules, so the proposition is proved. \(\square\)

Let \(\varphi : R \to T\) be a morphism of monads on \(\mathcal{C}\). We show that one can associate to \(\varphi\) “forgetful” and “free” functors connecting the categories of modules over \(R\) and \(T\). The construction of \(\varphi^* : \tau\mathcal{C} \to R\mathcal{C}\) is quite obvious, hence the proof is left to the reader.

**Proposition 1.6.** There exists a functor \(\varphi^* : \tau\mathcal{C} \to R\mathcal{C}\) such that
\[
\varphi^*(X, x) := (X, x \circ \varphi X)
\]
and \(\varphi^* f = f\), for every \(T\)-module \((X, x)\) and every \(T\)-module morphism \(f\).

**Proposition 1.7.** Let \(\mathcal{C}\) be a category with coequalizers. If \((R, r, \rho)\) and \((T, t, \tau)\) are monads on \(\mathcal{C}\), such that \(T\) that preserves coequalizers, and \(\varphi : R \to T\) is a morphism of monads then the functor \(\varphi^*\) in Proposition 1.6 has a left adjoint.

**Proof.** Let \((X, x)\) be an \(R\)-module. It is not difficult to see that \((TRX, tRX)\) and \((TX, tx)\) are \(T\)-modules, and that \(tx \circ T\varphi X\) and \(Tx\) are morphism of \(T\)-modules. Let
\[
(\varphi_*(X, x), \pi X) := \text{Coeq}(tX \circ T\varphi X, Tx).
\]
By Proposition 1.5, \(\varphi_*(X, x)\) is a \(T\)-module such that \(\pi X : (TX, tx) \to \varphi_*(X, x)\) is a morphism of \(T\)-modules. By the universal property of coequalizers, for any \(R\)-module morphism \(f : X \to Y\), there is a \(T\)-module morphism \(\varphi_*, f\) such that
\[
\varphi_*, f \circ \pi X = \pi Y \circ Tf.
\]
In conclusion, we have constructed a functor \(\varphi_*\) from \(R\mathcal{C}\) to \(\tau\mathcal{C}\). Moreover, one can compose the forgetful functor \(\rho^* : R\mathcal{C} \to \mathcal{C}\) with the free functor \(\tau_\mathcal{C} : \mathcal{C} \to \tau\mathcal{C}\) to obtain a functor from \(R\mathcal{C}\) to \(\tau\mathcal{C}\), given by \((X, x) \mapsto (TX, tX)\) and \(f \mapsto Tf\). Therefore, (1.5) means that \((\pi X)_{X \in \text{Ob} \, R\mathcal{C}}\) defines a natural transformation \(\pi : \tau_\mathcal{C} \circ \rho^* \to \varphi_*\).

It remains to show that \(\varphi_*\) is a left adjoint of \(\varphi^*\). For objects \((X, x)\) in \(R\mathcal{C}\) and \((Y, y)\) in \(\tau\mathcal{C}\) we define
\[
\Phi_{X, Y} : \text{Hom}_{R\mathcal{C}}(\varphi_*(X, x), (Y, y)) \to \text{Hom}_{\tau\mathcal{C}}((X, x), \varphi^*(Y, y)), \quad \Phi_{X, Y}(f) = f \circ \pi X \circ \tau X.
\]
In order to prove that \(\Phi_{X, Y}(f)\) is a morphism of \(R\)-modules, recall that, by the proof of Proposition 1.5, the module structure on \(\varphi_*(X, x)\) is given by the unique morphism \(\pi\) satisfying
\[
\pi \circ T \pi X = \pi X \circ tX.
\]
As \(\pi X \circ tX \circ T \varphi X = \pi X \circ tX\), by a simple but tedious computation, one can show that
\[
y \circ \varphi Y \circ R \Phi_{X, Y}(f) = \Phi_{X, Y}(f) \circ x,
\]
i.e. \(\Phi_{X, Y}(f)\) is a morphism of \(R\)-modules from \((X, x)\) to \(\varphi^*(Y, y)\).

We are going to construct an inverse of \(\Phi_{X, Y}\). Let \(g\) be a morphism of \(R\)-modules from \((X, x)\) to \(\varphi^*(Y, y)\). Then \(y \circ T g\) is a morphism of \(T\)-modules and coequalizes \((T X, tX \circ T \varphi X)\). Since \(\varphi_*(X, x)\) is the coequalizer in \(\tau\mathcal{C}\) of this pair, there exists a unique \(T\)-module morphism \(\Theta_{X, Y}(g)\) such that
\[
\Theta_{X, Y}(g) \circ \pi X = y \circ T g.
\]
Furthermore, by construction of \(\Phi_{X, Y}\) and \(\Theta_{X, Y}\), and the facts that \(\tau\) is natural and \(y\) is unital, we get
\[
\Phi_{X, Y} \left(\Theta_{X, Y}(g)\right) = \Theta_{X, Y}(g) \circ \pi X \circ \tau X = y \circ T g \circ \tau X = y \circ \tau Y \circ g = g.
\]
Thus we deduce that \( \Theta_{X,Y} \) is a right inverse of \( \Phi_{X,Y} \). Let \( f : \varphi_*(X,x) \to (Y,y) \) be a morphism of \( T \)-modules. To conclude, we must prove

\[
\Theta_{X,Y}(\Phi_{X,Y}(f)) \circ \pi X = f \circ \pi X.
\]

The left hand side of this relation can be rewritten as

\[
y \circ T \Phi_{X,Y}(f) = y \circ T f \circ T \pi X \circ T \tau X = f \circ \pi X \circ T \tau X = f \circ \pi X \circ t X \circ T \tau X = f \circ \pi X,
\]

where for the first two equalities we used the definitions of \( \Phi_{X,Y} \) and \( \Theta_{X,Y} \). Since \( f \) and \( \pi X \) are morphisms of \( T \)-modules we obtained the third and the fourth relations, while for the last one we used the definition of monads.

\( \square \)

**Example 1.8.** Let \((C, \otimes, \alpha, \mu, \tau, 1)\) be an abelian monoidal category, that is \( C \) be abelian, and \( X \otimes (-) : C \to C \) and \((-) \otimes X : C \to C\) be additive and right exact functors, for any \( X \in C \). For details on abelian monoidal categories, the reader is referred to [AMS, §1.1]. Following [AMS, (1.11)], for a right \( R \)-module \( \mu_X : X \otimes R \to X \) and a left \( R \)-module \( \mu_Y : R \otimes Y \to Y \) one defines \( X \otimes_R Y \) by

\[
X \otimes_R Y := \text{Coker}(\mu_X \otimes Y - X \otimes \mu_Y).
\]

Take another algebra \( T \) in \( C \). Proceeding as in [AMS, (1.4)] one defines the category \( T \text{-Mod-}R \) of \( T \)-\( R \) bimodules in \( C \). By definition, \((X, \mu'_X, \mu_X^*)\) is an \( T \)-\( R \) bimodule if \( \mu'_X : T \otimes X \to X \) and \( \mu_X^* : X \otimes R \to X \) define compatible left and right module structures such that

\[
\mu'_X \circ (T \otimes \mu_X^*) = \mu_X^* \circ (\mu'_X \otimes R) \circ \alpha_{T,X,R}^{-1}.
\]

If \((X, \mu'_X, \mu_X^*)\) is a \( T \)-\( R \) bimodule and \((Y, \mu'_Y, \mu_Y^*)\) is an \( R \)-\( S \) bimodule, then \( X \otimes_R Y \) is a \( T \)-\( S \) bimodule, for any algebra \( S \) in \( C \). Its left \( T \)-action and right \( S \)-action are respectively induced by \( \mu'_X \otimes Y \) and \( X \otimes \mu_Y^* \). As in [AMS, Theorem 1.12], one can prove that \( T \text{-Mod-}R \) is abelian. Moreover, \( R \text{-Mod-}R \) is monoidal with respect to \((-) \otimes_R (-)\), see [AMS, Theorem 1.12].

For a monad morphism \( \varphi : R \to T \), the functors in Propositions 1.6 and 1.7 are \( \varphi^* : T \text{-Mod} \to R \text{-Mod} \), the ‘restriction of scalars’, and \( \varphi^* = T \otimes_R (-) : R \text{-Mod} \to T \text{-Mod} \), the ‘extension of scalars’.

Distributive laws were introduced by J. Beck [Be]. As we shall see later, they give a way to compose two monads in order to obtain a monad.

**Definition 1.9.** A distributive law between two monads \((R, r, \rho)\) and \((T, t, \tau)\) is a natural transformation \( \iota : RT \to TR \) satisfying the four conditions in Figure 4.

![Figure 4](image)

**Figure 4.** The definition of distributive laws.

**Remark 1.10.** Since we are using for the first time the diagrammatic representation of morphisms, we note that the first and the third relations in Figure 4 can be explicitly written as follows:

\[
\iota \circ rT = Tr \circ \iota R \circ Rl, \quad \iota \circ \rho T = T \rho.
\]
Example 1.11. Let $R$ and $T$ be two algebras in a braided monoidal category $(C, \otimes, \alpha, \psi, \delta, 1, \chi)$, with braiding $\chi : C \times C \rightarrow C \times C$. Let $R = R \otimes (-)$ and $T = T \otimes (-)$ be the monads induced by $R$ and $T$, as in Example 1.2. For an object $X$ in $C$, we put

$$tX := \alpha_{T,R,X} \circ (\chi_{R,T} \otimes X) \circ \alpha_{R,T,X}^{-1}.$$

$$rX := \alpha_{R,R,X} \circ (\chi_{R,R} \otimes X) \circ \alpha_{R,R,X}^{-1}.$$

$$tX := \alpha_{T,T,X} \circ (\chi_{T,T} \otimes X) \circ \alpha_{T,T,X}^{-1}.$$

One can see easily that if $t : RT \rightarrow TR$, $r : RR \rightarrow RR$ and $t : TT \rightarrow TT$ are distributive laws.

**Definition 1.12.** Consider a monad $(T, t, \tau)$ on a category $C$. A distributive law $t : TT \rightarrow TT$ is said to be a $BD$-law if it satisfies the $YB$-equation:

$$tT \circ Tt \circ tT = Tt \circ tT \circ tT.$$

For a diagrammatic representation of the $YB$-equation see the first picture in Figure 5.

The distributive laws $t$ and $r$ in Example 1.11 are, in fact, $BD$-laws.

2. **Braided pairs of monads and their morphisms**

Our starting point of a construction of para-cocyclic objects will be the following notion.

**Definition 2.1.** The sextuple $S := (C, T, R, t, t, \tau)$ is said to be a braided pair of monads if and only if the following conditions are satisfied:

- $C$ is a category, in which any pair of parallel morphisms has a coequalizer.
- $(R, r, \rho)$ and $(T, t, \tau)$ are monads on $C$, both of which preserve coequalizers.
- $t : RT \rightarrow TR$ is an invertible distributive law.
- $t : TT \rightarrow TT$ and $r : RR \rightarrow RR$ are invertible $BD$-laws such that conditions (A 1) and (A 2) in Figure 5 hold.

A morphism of monads $\varphi : R \rightarrow T$ is said to be braid preserving if it satisfies conditions (A 3) and (A 4) in Figure 5.

**Remark 2.2.** In the proof of several results we do not need all assumptions from the definition of a braid preserving monad morphism in Definition 2.1. Still, for reader’s convenience, we prefer to state all of them at the same time.

**Example 2.3.** Let $\varphi : (R, r, \rho) \rightarrow (T, t, \tau)$ be a morphism of algebras in a braided monoidal category $(C, \otimes, \alpha, \psi, \delta, 1, \chi)$. We take the monads $R$ and $T$ as in Example 1.2, and the distributivity laws $t, r$ and $t$ as in Example 1.11. Clearly, $(C, R, T, t, r, t)$ is a braided pair of monads. Moreover, by pushing $\varphi$ over $t$ in the second diagram in (A 3) and under $t$ in the second picture in (A 4), it follows that $\varphi := \varphi \otimes (-)$ is a braid preserving monad morphism if, and only if,

$$(\varphi \otimes T) \circ \chi_{T,R} = (\varphi \otimes T) \circ \chi_{R,T}^{-1} \quad \text{and} \quad \chi_{R,T} \circ (R \otimes \varphi) = \chi_{T,R}^{-1} \circ (R \otimes \varphi). \quad (2.1)$$

**Figure 5.** $YB$-equation and conditions (A 1)–(A 4).
For example, if $\chi_{C,R} = \chi_{R,T}^{-1}$, then $\varphi$ is such a morphism. In a similar manner one can show that $\varphi$ is a braid preserving monad morphism, provided that $\chi_{R,R} = \chi_{R,R}^{-1}$ and $\chi_{C,T} = \chi_{C,T}^{-1}$. For, it is enough to rewrite the second equalities in (A 3) and (A 4) by pushing $\varphi$ under $\Gamma^{-1}$.

Distributive laws, as we have already remarked, were introduced by Beck in order to construct a monad on the composite of two monads. For two monads $(R, r, \rho)$ and $(T, t, \tau)$, a distributive law $1: RT \to TR$ induces a monad $TR$ on $C$, with multiplication and unit

$$Tr \circ tR^2 \circ T\mathcal{R} = tR \circ T^2R \circ T\mathcal{R}: TRTR \to TR \quad \text{and} \quad T\rho \circ \tau = \tau R \circ \rho : \text{Id}_C \to TR.$$  

Therefore, it makes sense to speak about $T\mathcal{R}C$, the category $T\mathcal{R}$-modules. Its objects can be described equivalently as follows. Let $(X, x)$ be an object in $T\mathcal{R}C$. We set $x_T := x \circ T\rho X$ and $x_R := x \circ \tau RX$. One can prove that $(X, x_R)$ is an $R$-module, $(X, x_T)$ is a $T$-module and these structures commute in the sense that

$$x_T \circ T x_R \circ x = x_R \circ Rx_T.$$  \hspace{1cm} (2.2)

Conversely, if $(X, x_R)$ is an $R$-module and $(X, x_T)$ is a $T$-module that obey relation (2.2) then $X$ is an $T\mathcal{R}$-module with respect to $x := x_T \circ T x_R$. Moreover, a morphism in $\mathcal{C}$ is a morphism of $T\mathcal{R}$-modules if, and only if, it is a morphism of $R$-modules and $T$-modules with respect to the above defined structures. Thus, $T\mathcal{R}C$ is isomorphic to the category of triples $(X, x_R, x_T)$ such that $x_R$ and $x_T$ define commuting module structures over $R$ and $T$, respectively. For details see [KLV].

For any $BD$-law $\tau : RR \to RR$, one can deform the monad $(R, r, \rho)$ to obtain a new monad

$$R^\tau := R, \quad r^\tau := r \circ \tau, \quad \rho^\tau := \rho.$$  

For a braided pair of monads $S := (C, T, R, l, t, \tau)$ and a braid preserving monad morphism $\varphi : R \to T$, it follows by the definition of distributive laws and (A 1) that $t : R^\tau T \to TR^\tau$ is a distributive law too. Hence $TR^\tau$ admits a monad structure. By the above considerations, the category of $TR^\tau$-modules is isomorphic to the category of triples $(X, x_R^0, x_T)$, where $(X, x_R^0)$ is an $R$-module and $(X, x_T)$ is a $T$-module such that the compatibility condition

$$x_T \circ T x_R^0 \circ x = x_R^0 \circ R x_T$$  \hspace{1cm} (2.3)

holds. Their morphisms are both $T$-module, and $R^\tau$-module morphisms. We call the triple $(X, x_R^0, x_T)$ a $(T, R)$-bimodule and denote their category by $T \mathcal{R} C$.

**Remark 2.4.** For the above considerations, in particular, for any monad $R$ and $BD$-law $\tau : RR \to RR$, also $\tau : R^\tau R \to RR^\tau$ is a distributive law. Hence, as above, $R^\tau := RR^\tau$ is a monad, that is called the *enveloping monad of $R$*. The category $R \mathcal{C}$ is isomorphic to the category of $(R, R)$-bimodules. Recall that an $(R, R)$-bimodule is an object $X$ in $\mathcal{C}$, together with two morphisms $x : RX \to X$ and $x^0 : R^\tau X \to X$ such that $(X, x)$ is an $R$-module, $(X, x^0)$ is an $R^\tau$-module and these structures commute in the sense that

$$x \circ R x^0 \circ x X = x^0 \circ R^\tau x.$$  \hspace{1cm} (2.4)

The category of $(R, R)$-bimodules will be denoted by $R \mathcal{C} R$. In order to simplify notations, the forgetful functor $U : r \mathcal{C} R \to \mathcal{C}$ (with object map $(X, x, x^0) \mapsto X$) will be omitted in our formulae whenever it causes no danger.

For an invertible $BD$-law $\tau : RR \to RR$, also $\tau^{-1} : R^\tau R \to RR^\tau$ is an (invertible) $BD$-law. Hence the previous construction can be repeated with $\tau : RR \to RR$ replaced by $\tau^{-1} : R^\tau R \to RR^\tau$. The objects of the resulting category $R \mathcal{C} R^\tau$ are triples $(X, x^0, x)$, where $(X, x^0)$ and $(X, x)$ are $R^\tau$ and $R = (R^\tau)^{-1}$-modules, respectively, such that

$$x^0 \circ R^\tau x \circ \tau^{-1} X = x \circ Rx^0.$$  

Hence the object map $(X, x, x^0) \mapsto (X, x^0, x)$ induces an isomorphism $r \mathcal{C} R \to r \mathcal{C} R^\tau$.

**Example 2.5.** Let $R$ and $T$ be the monads coming from algebras $R$ and $T$ in a braided monoidal $\mathcal{C}$, as in Example 1.2. We denote by $l$, $t$ and $\tau$ the distributive laws defined in Example 1.11. In this case, the category $T \mathcal{C} R$ is isomorphic to the category $T \text{-Mod-} R$, defined in Example 1.8. Indeed,
an object in $\tau \mathcal{C}_R$ is a triple $(X, x, x^0)$, such that $(X, x)$ is a $T$-module, $(X, x^0)$ is an $R^s$-module and these structures satisfy relation (2.3). We define a functor

$$F : \tau \mathcal{C}_R \to \mathbf{T}\text{-Mod-}R, \quad F(X, x, x^0) := (X, x, x^0 \circ \chi_{1,R,X}^{-1}).$$

Let us first prove that $F$ is well defined, that is $(X, x, x^0 \circ \chi_{1,R,X}^{-1})$ is an object in $\mathbf{T}\text{-Mod-}R$. We have to check that $(X, x^0 \circ \chi_{1,R,X}^{-1})$ is a right $R$-module and that the structure maps obey (1.7). The graphical proofs of these properties are given in Figure 6, where for simplicity we use the notation $\chi_1 := \chi_{R,X}, \chi_2 := \chi_{R,R}$ and $\chi_3 := \chi_{R,T}$. In the first frame we prove that $X$ is a right $R$-module. The first and the fourth equalities follow by the fact that the braiding $\chi$ is a natural transformation. The second identity is trivial, while for the third one we used that $(X, x^0)$ is an $R^s$-module. In the second frame we show that the module structures of $X$ satisfy condition (1.7). The first relation in that frame is obvious. The second one follows by (2.3) and the last one is a consequence of naturality of the braiding. Obviously, $F^{-1} : \mathbf{T}\text{-Mod-}R \to \tau \mathcal{C}_R$ is given by $F^{-1}(X, \mu_X^r, \mu_X^l) := (X, \mu_X^r \circ \chi_{1,R,X}).$

In view of this example, from now on, we shall always regard an object in $\tau \mathcal{C}_R$ as a $\mathbf{T}\text{-R}$ bimodule.

**Proposition 2.6.** Let $(R, r, \rho)$ be a monad and $\tau : RR \to RR$ be a BD-law. For an $(R, R)$-bimodule $(X, x, x^0)$, also the triple $(RX, Rx \circ \tau X, R x^0 \circ \tau X)$ is an $R$-bimodule. This construction defines a lifting $\tilde{R} : R\mathcal{C}_R \to R\mathcal{C}_R$ of $R$.

**Proof.** Let us denote the actions of $R$ and $R^s$ on $RX$ by $y$ and $y^0$, respectively. In the first frame of Figure 7 we prove that $RX$ is an $R$-module. The first equality follows by the definition of $y$. The second equality is a consequence of the definition of distributive laws, cf. Figure 4, while the third one results by the definition of $R$-modules. For the last relation we use the definition of $y$. In the second frame we prove that $RX$ is an $R^s$-module too. We proceed as above using, in addition, $YB$-equation. The fact that $y$ and $y^0$ commute, i.e. they satisfy relation (2.4), is proved in the third frame, where for the first relation we use the definition of the actions and $YB$-equation. Since $x$ and $x^0$ commute, we obtain the second relation. We conclude the proof of the proposition by applying the definition of $y$ and $y^0$. □
Remark 2.7. If \( \tau : RR \to RR \) is an invertible BD-law, Proposition 2.6 can be applied to the BD-law \( \tau^{-1} : R^*R^*\to R^*R^* \) and the \( R^* \)-bimodule \( (X,x^0,x) \). Using the isomorphism \( R_C R \cong R_C R \) in Remark 2.4, we obtain a functor \( \tilde{R}^0 : R_C R \to R_C R \).

Example 2.8. For a monad \( R = R \otimes (-) \), induced by an algebra \( R \) as in Example 1.2, the functor \( \tilde{R} : R_{-}\text{-Mod} \to R_{-}\text{-Mod} \), maps an \( R \)-bimodule \( (X,\mu_X,\mu'_X) \) to \( R \otimes X \), with \( R \)-actions

\[
\mu_{i,RX} := (R \otimes \mu_X) \circ \alpha_{R,R,X} \circ (\chi_{R,R} \otimes X) \circ \alpha_{R,R,X}^{-1}, \quad \mu_{R \otimes X} := (R \otimes \mu_X) \circ \alpha_{R,R,R}. \tag{2.5}
\]

Remark 2.9. Note that, for a braided pair of monads \( S := (C,T,R,l,t) \) and any braided preserving monad morphism \( \varphi : R \to T \) is braided preserving also for the braided pair of monads \( S^0 := (C,T^{-1},R^*,l,t^{-1},r^{-1}) \).

Proposition 2.10. Consider a braided pair of monads \( S := (C,T,R,l,t) \) and a braided preserving monad morphism \( \varphi : R \to T \). For an \( R \)-module \( (X,x) \) and an \( R^* \)-module \( (Y,y) \), the following assertions hold.

1. The triple \( (TX,x_i,x^0_i) \) defines an \((R,R)\)-bimodule, where
   \[
   x_i := TX \circ iX \quad \text{and} \quad x^0_i := tX \circ \varphi X \circ tX. \tag{2.6}
   \]
   It is called the inside structure on \( TX \). This construction defines a functor \( R_C \to R_C R \).

2. The triple \( (TY,y_o,y^0_o) \) defines an \((R,R)\)-bimodule, where
   \[
   y_o := tY \circ \varphi TY \quad \text{and} \quad y^0_o := Ty^0 \circ tY. \tag{2.7}
   \]
   It is called the outside structure on \( TY \). This construction defines a functor \( R_C \to R_C R \).

Proof. Claim (1) is verified by straightforward computation, which is left to the reader. Part (2) is obtained from (1) by replacing the braided pair of monads \( S \) with \( S^0 \) in Remark 2.9, and replacing the \( R \)-module \( (X,x) \) with the \( R^* \)-module \( (Y,y) \). \( \square \)

Example 2.11. Consider the braided preserving monad morphism coming from an algebra homomorphism \( \varphi : R \to T \) in a braided monoidal category, as in Example 2.3. For a left \( R \)-module \( (X,\mu) \), the inner actions on \( T \otimes X \) in Proposition 2.10 (1) come out as

\[
\mu_i := (T \otimes \mu) \circ \alpha_{T,R,X} \circ (\chi_{R,T} \otimes X) \circ \alpha_{R,T,X}^{-1}, \quad \mu'_i := (t \circ (T \otimes \varphi) \circ X) \circ \alpha_{T,R,X}^{-1} \circ (T \otimes \chi_{R,X}^{-1}) \circ \alpha_{T,X,R}. \]

For a right \( R \)-module \( (Y,\nu) \), the outer actions on \( T \otimes Y \) in Proposition 2.10 (2) come out as

\[
\nu'_o := (t \circ (\varphi \otimes T) \circ Y) \circ \alpha_{R,T,Y}^{-1}, \quad \nu_o := (T \otimes \nu) \circ \alpha_{T,Y,R}. \]

Composition of the forgetful functor \( R_C R \to R_C \) with the functor in Proposition 2.10 (1) yields a functor \( T_i : R_C R \to R_C R \). Symmetrically, composition of the forgetful functor \( R_C R \to R_C \) with the functor in Proposition 2.10 (2) yields a functor \( T_o : R_C R \to R_C R \). In the following proposition some natural transformations between various composites of these endofunctors on \( R_C R \), and the functors \( \tilde{R} \) and \( \tilde{R}^0 \) in Proposition 2.6, are studied.

Proposition 2.12. Let \( S := (C,T,R,l,t) \) be a braided pair of monads and \( \varphi : R \to T \) be a braided preserving monad morphism. Consider the above endofunctors \( \tilde{R}^0, T_i \) and \( T_o \) on \( R_C R \). Then the following hold.

1. The mappings \( \text{Ob}(R_C R) \to \text{Mor}(R_C R), (X,x,x^0) \mapsto x_i \) and \( (X,x,x^0) \mapsto x^0_i \), defined in terms of the inner actions in Proposition 2.10 (2), determine natural transformations \( \tilde{R}T_o \to T_o \).

2. The mappings \( \text{Ob}(R_C R) \to \text{Mor}(R_C R), (X,x,x^0) \mapsto x_o \) and \( (X,x,x^0) \mapsto x^0_o \), defined in terms of the outer actions in Proposition 2.10 (3), determine natural transformations \( R^0 T_i \to T_i \).

3. The BD-law \( t \) defines a natural transformation \( T_i T_o \to T_o T_i \).

4. The BD-law \( t \) defines a natural transformations \( \tilde{R}T_o \to T_o \tilde{R}^0 \) and \( \tilde{R}^0 T_i \to T_o \tilde{R}^0 \).
Proof. Note that part (2) is obtained from part (1) by replacing the braided pair of monads $S$ with $\mathcal{S}^0$ in Remark 2.9, and using the isomorphism $R\mathcal{C}_R \cong \rho R\mathcal{C}_R$.

Claims (1), (3) and (4) are proven by straightforward but somewhat lengthy computations. We illustrate the main steps on the example of part (3). We use the graphical representation of morphisms in a category again. We denote the morphisms defining the $(R, R)$-bimodule structure of $T, T \alpha X$ by $x_{\alpha i}$ and $x_{\alpha o}^0$, respectively. Similarly, for the actions of $R$ and $R^0$ on $T, T \alpha X$ we use the notation $x_{\alpha i}$ and $x_{\alpha o}^0$. The diagrammatic representation of these morphisms is given in the first frame of Figure 8. In the second frame we prove that $tX$ is a morphism of $R$-modules. Note that for the first equality we used that $i$ is a distributive law and $\varphi$ satisfies (A 4). For the graphical proof of the fact $tX$ is a morphism of $R^0$-modules see the third frame of Figure 8. Note that, besides the properties of $i$ and $\varphi$ that we already used, we also need condition (A 2) in Definition 2.1. □

![Figure 8. $tX$ is a morphism of $(R, R)$-bimodules.](image)

**Proposition 2.13.** For a braided pair of monads $S := (C, T, R, \iota, \iota, \tau)$ and a braid preserving monad morphism $\varphi : R \to T$, consider an $(R, R)$-bimodule $(X, x, x^0)$ and a $(T, R)$-bimodule $(Y, y, y^0)$. Then the following assertions hold.

1. The actions $x^0$ and $y^0$ induce $R^0$-module structures respectively on $\varphi^+(X, x)$ and $\varphi^+(Y, y)$.
2. With respect to the above actions, $\varphi_*$ in Proposition 1.7 and $\varphi^*$ in Proposition 1.6 can be regarded as functors $\varphi_* : R\mathcal{C}_R \to \tau \mathcal{C}_R$ and $\varphi^* : \tau \mathcal{C}_R \to R\mathcal{C}_R$.
3. The functors constructed in part (2) define a pair of adjoint functors $(\varphi_*, \varphi^*)$.

Proof. (1) and (2). Obviously, $y^0$ induces an $R^0$-action on $\varphi^+(Y, y) = Y$. By construction, $\varphi^+(Y, y) = (Y, y \circ \varphi Y)$. We claim that $(Y, y \circ \varphi Y, y^0)$ is an $(R, R)$-bimodule. We have to check that the module structures commute. Since $\varphi$ is a natural morphism, condition (A 3) holds true and $y$ and $y^0$ commute, we get

$$y \circ \varphi Y \circ R y^0 \circ t Y = y \circ T y^0 \circ \varphi R^0 Y \circ t Y = y \circ T y^0 \circ t Y \circ R^0 \varphi Y = y^0 \circ R^0 y \circ R^0 \varphi Y.$$ 

Hence $y \circ \varphi Y$ and $y^0$ commute too. Consequently, $\varphi^*$ can be seen as a functor from $\tau \mathcal{C}_R$ to $R\mathcal{C}_R$.

Recall from the proof of Proposition 1.7 that the $T$-module $(\varphi_* X, \overline{\pi})$ is the coequalizer of $(T, tX \circ t \varphi X)$. In terms of the inner actions in (2.6), the $T$-module morphisms $T x$ and $t X \circ t \varphi X$ are equal to $x \circ \iota^{-1} X$ and $x^0 \circ t^{-1} X$, respectively. Hence they are $R^0$-module morphisms $T_o \overline{R} X \to T_o X$ by Proposition 2.12 (1) and (4). Thus $T x$ and $t X \circ t \varphi X$ are morphisms of $(T, R)$-bimodules from $(T, R \alpha x \circ T R \alpha x \circ t \varphi x \circ T R \alpha x)$ to $(T, tX, x^0)$. Hence, by Proposition 1.5, there is an action $\overline{\pi}^0 : R^0 \varphi_* X \to \varphi_* X$ such that $(\varphi_* X, \overline{\pi}, \overline{\pi}^0)$ is a $(T, R)$-bimodule and $\pi X$ is a morphism of $(T, R)$-bimodules form $(T, T, x^0)$ to $(\varphi_* X, \overline{\pi}, \overline{\pi}^0)$.

Let $f : (X, x, x^0) \to (Z, z, z^0)$ be a morphism of $(R, R)$-bimodules. We already know that $\varphi_* f$ is a morphism of $T$-modules. By Proposition 2.10 (2), $T_o f$ is an $R^0$-module morphism. Hence so is $\varphi_* f$ by Proposition 1.5. Thus $\varphi_*$ can be regarded as a functor from $R\mathcal{C}_R$ to $\tau \mathcal{C}_R$.

(3) Let $(X, x, x^0)$ be an $(R, R)$-bimodule and let $(Y, y, y^0)$ be a $(T, R)$-bimodule. It is sufficient to show that $\Phi_{X,Y}(f)$ and $\Theta_{X,Y}(g)$ are morphisms of $R^0$-modules, for any $f : \varphi_* X \to Y$ in $\rho R\mathcal{C}_R$. 


and any \( g : X \to \varphi^* Y \) in \( R\mathcal{C}_R \) (for the definition of \( \Phi_{X,Y} \) and \( \Theta_{X,Y} \) see the proof of Proposition 1.7). The fact that \( \Phi_{X,Y}(f) \) is a morphism of \( R^t \)-modules is proved in the following computation:

\[
y^0 \circ Rf \circ R\pi X \circ R^t \tau X = f \circ \tau^\circ \circ R\pi X \circ R\tau X = f \circ \pi X \circ T x^0 \circ \iota X \circ R\tau X = f \circ \pi X \circ \tau X \circ x^0.
\]

Since \( R\pi X \) is an epimorphism, the relation meaning that \( \Theta_{X,Y}(g) \) is a morphism of \( R^t \)-modules follows from the computation below:

\[
y^0 \circ R^t \Theta_{X,Y}(g) \circ R^t \pi X = y^0 \circ R^t y \circ R^t T g = y \circ Ty^0 \circ Ty \circ R^T T g = y \circ Tg \circ Tx^0 \circ \iota X = \Theta_{X,Y}(g) \circ \pi X \circ T x^0 \circ \iota X \circ \Theta_{X,Y}(g) \circ \pi^0 \circ R^t \pi X.
\]

To get the equalities (A), (B) and (C) above we used the commutation relation between the module structures of \( T \) and \( \tau \) are morphisms of \( R^t \)-modules.

**Corollary 2.14.** For a braided pair of monads \( S := (\mathcal{C}, T, R, t, \tau) \) and a braided preserving monad morphism \( \varphi : R \to T \), consider the adjoint functors \( (\varphi^*, \varphi^\circ) \) in Proposition 2.13.

1. The unit of the adjunction \( \sigma : \text{Id}_{R\mathcal{C}_R} \to \varphi^* \varphi^\circ \) is given, for any \((R, R)\)-bimodule \((X, x, x^0)\), by

\[
\sigma X := \pi X \circ \tau X.
\]

2. The counit of the adjunction \( \xi : \varphi^* \varphi^\circ \to \text{Id}_{R\mathcal{C}_R} \) satisfies the following relation for any \((T, R)\)-bimodule \((Y, y, y^0)\).

\[
\xi Y \circ \pi \varphi^*(Y, y, y^0) = y.
\]

**Proof.** By definition, \( \sigma X := \Phi_{X, \varphi^*(X, x, x^0)}(\text{Id}_{\varphi^*(X, x, x^0)}) \) and \( \xi Y = \Theta_{\varphi^*(Y, y, y^0), Y}(\text{Id}_{\varphi^*(Y, y, y^0)}) \). □

**Example 2.15.** For the braided preserving monad morphism, coming from an algebra homomorphism \( \varphi : R \to T \) in a braided monoidal category as in Example 2.3, \( T \) is a \( T\mathcal{R} \) bimodule via the left regular \( T \)-action and the right \( R \)-action induced by \( \varphi \). If (as in Example 1.8) the functors \( X \otimes (\cdot) \) and \( (\cdot) \otimes X \) are right exact, for any object \( X \in \mathcal{C} \), then the pair of adjoint functors in Proposition 2.13 (3) consists of the ‘restriction of scalars’ functor \( \varphi^* : \text{Mod-}R \to \text{Mod-}R \) and the induction functor \( \varphi_* = T \otimes_R (\cdot) : \text{Mod-}R \to \text{Mod-}R \).

### 3. The para-cocyclic object associated to a braided preserving homomorphism

In this section we show, using the main result in [BS], that to every braided preserving monad morphism there corresponds an admissible septuple, and hence a certain para-cocyclic object.

Recall that, for every pair \((F, G)\) of adjoint functors, with \( F : \mathcal{C} \to \mathcal{D} \) and \( G : \mathcal{D} \to \mathcal{C} \), the triple \((GF, G\xi F, \sigma)\) is a monad on \( \mathcal{C} \), where \( \xi \) and \( \sigma \) are respectively the counit and the unit of the adjunction. For details the reader is referred to [We, Main Application 8.6.2, p. 280]. In particular, the pair of adjoint functors \((\varphi_*, \varphi^*)\), constructed in Proposition 2.13 (3), determines a monad structure on \( \varphi^* \varphi_* : R\mathcal{C}_R \to R\mathcal{C}_R \) that will be denoted by \((\overline{T_0}, \overline{t_0}, \overline{\pi_0})\).

Since \( I \) is invertible, the object in \( \mathcal{C} \), underlying the \((R, R)\)-bimodule \( \overline{T_0}(X, x, x^0) \), is the coequalizer of the morphisms \( x_i \) and \( x_i^0 \) in (2.6). Since \( x_i \) and \( x_i^0 \) are \((R, R)\)-bimodule morphisms by Proposition 2.12 (1), it follows by Proposition 1.5 that

\[
\overline{T_0}(X, x, x^0) = \text{Coeq}(x_i, x_i^0)
\]

in \( R\mathcal{C}_R \). That is, \( \varphi^* \) takes the canonical epimorphism \( \pi X : (TX, tX, x^0) \to \varphi^*(X, x, x^0) \) to a canonical \((R, R)\)-bimodule epimorphism \( \pi_0 x : \overline{T_0}(X, x, x^0) \to \overline{T_0}(X, x, x^0) \). In other words, the respective actions \( \pi_0 \) and \( \pi_0^0 \) of \( R \) and \( R^t \) on \( \overline{T_0}X \) are uniquely determined such that \( \pi_0 x : \overline{T_0}X \to \overline{T_0}X \) is a morphism of \((R, R)\)-bimodules, in the sense that

\[
\pi_0 \circ R\pi_0 X = \pi_0 X \circ x_o \quad \text{and} \quad \pi_0^0 \circ R^t \pi_0 X = \pi_0 X \circ x^0_o.
\]

The unit of \( \overline{T_0} \) is

\[
\overline{t_0} = \pi_0 \circ \iota.
\]

In order to compute the multiplication \( \overline{t_0} \) of \( \overline{T_0} \), substitute \( Y = \varphi^*(X, x, x^0) \) in (2.9), apply \( \varphi^* \) on it, and compose the resulting relation on the right by \( T\pi_0 X \). Since \( T\pi_0 X \) is a coequalizer and the
module structure $\mathfrak{T}$ satisfies (1.6), we deduce that $\tilde{t}_oX$ is the unique morphism in $R\mathcal{C}_R$ satisfying relation
\[ \tilde{t}_oX \circ \pi_o \mathfrak{T}_oX \circ T \pi_oX = \pi_oX \circ tX. \] (3.3)

Summarizing, we prove the following proposition.

**Proposition 3.1.** Associated to a braided pair of monads $\mathcal{S} := (\mathcal{C}, T, R, l, t, r)$ and a braided preserving monad morphism $\varphi : R \to T$, there is a monad $(\mathfrak{T}_o, \tilde{t}_o, \mathfrak{T}_o)$ on $R\mathcal{C}_R$ such that, for every $(R, R)$-bimodule $(X, x, x^0)$,
\[ (\mathfrak{T}_oX, \pi_oX) = \text{Coeq}(x_0, x^0), \]
(cf. Proposition 2.10 (1)). That is, the $R$, and $R^\tau$-actions $\mathfrak{T}_o$ and $\mathfrak{T}_o^\tau$ on $\mathfrak{T}_oX$ are uniquely defined such that $\pi_oX : T_oX \to \mathfrak{T}_oX$ is a morphism of $(R, R)$-bimodules. The multiplication $\mathfrak{T}_o : \mathfrak{T}_o \mathfrak{T}_o \to \mathfrak{T}_o$ satisfies (3.3) and the unit $\mathfrak{T}_o : R\mathcal{C}_R \to \mathfrak{T}_o$ satisfies (3.2).

A symmetrical version of Proposition 3.1 is obtained by replacing the braided pair of monads $\mathcal{S}$ with $\mathcal{S}^0$, introduced in Remark 2.9.

**Proposition 3.2.** Associated to a braided pair of monads $\mathcal{S} := (\mathcal{C}, T, R, l, t, r)$ and a braided preserving monad morphism $\varphi : R \to T$, there is a monad $(\mathfrak{T}_i, \tilde{t}_i, \mathfrak{T}_i)$ on $R\mathcal{C}_R$ such that, for every $(R, R)$-bimodule $(X, x, x^0)$,
\[ (\mathfrak{T}_iX, \pi_iX) = \text{Coeq}(x_0, x^0), \]
(cf. Proposition 2.10 (2)). That is, the actions $\mathfrak{T}_i$ and $\mathfrak{T}_i^\tau$ on $\mathfrak{T}_iX$ are uniquely defined such that $\pi_iX : T_iX \to \mathfrak{T}_iX$ is a morphism of $(R, R)$-bimodules, i.e.
\[ \pi_i \circ R \pi_iX = \pi_iX \circ x_i \quad \text{and} \quad \mathfrak{T}_i^\tau \circ R \pi_iX = \pi_iX \circ x^0. \] (3.5)

The unit $\mathfrak{T}_i$ and the multiplication $\tilde{t}_i$ satisfy
\[ \mathfrak{T}_i = \pi_i \circ \tau \quad \text{and} \quad \tilde{t}_i \circ \pi_i \mathfrak{T}_i \circ T \pi_i = \pi_i \circ t \circ t^{-1}. \] (3.6)

**Example 3.3.** Let $\mathcal{C}$ be a braided monoidal category. For the braided preserving monad morphism coming from an algebra homomorphism $\varphi : R \to T$ as in Example 2.3, $T$ has a natural $R$-bimodule structure via $\varphi$. If (as in Example 1.8) the functors $X \otimes (-)$ and $(-) \otimes X$ are right exact, for any object $X$ in $\mathcal{C}$, then the functor underlying the monad in Proposition 3.1 is induced by the $R$-bimodule $T$, i.e. $\mathfrak{T}_o = T \otimes_R (-) : R\mathcal{Mod}_R \to R\mathcal{Mod}_R$. Multiplication is given by $(\tilde{t}_o \otimes_R (-)) \circ \varpi_{TT}(\cdot, \cdot)$, where $\varpi$ is the associator isomorphism in $R\mathcal{Mod}_R$ and the $R$-bimodule morphism $\mathfrak{T}_o : T \otimes_R T \to T$ is defined as the projection of $t : T \otimes T \to T$. The unit is $\varphi \otimes_R (-)$ (where we used that $R \otimes_R (-)$ is naturally isomorphic to the identity functor on $R\mathcal{Mod}_R$).

Symmetrically, on an $R$-bimodule $(X, \mu^X, \mu^X)$, the functor $\mathfrak{T}_i$ is defined as the coequalizer of $x_o = (t \otimes X) \circ [(\varphi \otimes T) \otimes X] \circ \alpha_{T,X}^{-1}$ and $x^0_o = (T \otimes \mu^X) \circ \alpha_{T,X,R} \circ \chi_{R,T,X}. \otimes X$. Equivalently, composing both $x_o$ and $x^0_o$ by the isomorphism $\alpha_{R,T,X} \circ \chi^{-1}_{R,T} \otimes X$, as a coequalizer
\[
\begin{align*}
(T \otimes R) \otimes X & \xrightarrow{[T \otimes (\mu^X \otimes_X \varphi)] \circ \alpha_{T,R,X}} T \otimes X \xrightarrow{T \otimes \pi_1} \mathfrak{T}_oX.
\end{align*}
\]

In order to obtain the form of the morphism corresponding to the lower one of the parallel arrows, we used the first identity in (2.1). Note that $\mu^X \circ \chi_{R,X}$ is a left action, and $t \circ \chi_{T,T} \circ (T \otimes \varphi)$ is a right action for the algebra $R^X := (R, r \circ \chi_{R,R,\rho})$. Therefore, $\mathfrak{T}_iX = T \otimes_R X$. Since $\varphi$ can be regarded as an algebra homomorphism $R^X \to T^{-1}$, the unit of the monad $\mathfrak{T}_i(-) = T \otimes_R (-)$ is given by $\varphi \otimes_R (-)$. Multiplication is induced by the projection $\tilde{t}_i : T \otimes_R T \to T$ of the $R^X$-bimodule morphism $t \circ \chi_{T,T}^{-1} : T \otimes T \to T$. Note that, if $\chi$ is a symmetry, then $R^X = R^{op}$, hence $\mathfrak{T}_i = T^{op} \otimes_R T^{op} (-) \cong (-) \otimes_R T$.

Next we introduce a functor $\Pi : \mu \mathcal{C}_R \to \mathcal{C}$, that will be used to construct a para-cocyclic module associated to a braided preserving monad morphism.
**Definition 3.4.** For a BD-law \( \tau : RR \to RR \), the functor \( \Pi : R\mathcal{C}_R \to \mathcal{C} \) is defined, for an \((R, R)\)-bimodule \((X, x, x^0)\), by
\[
(\Pi X, pX) := \text{Coeq}(x, x^0).
\]
For every morphism \( f : X \to Y \) of \((R, R)\)-bimodules, \( \Pi f \) is the unique morphism in \( \mathcal{C} \) such that \( pY \circ f = \Pi f \circ pX \). Hence \( p \) can be interpreted as a natural epimorphism from the forgetful functor \( U : R\mathcal{C}_R \to \mathcal{C} \) to \( \Pi \).

In the graphical notation we do not denote the forgetful functor \( U : R\mathcal{C}_R \to \mathcal{C} \). That is, a box representing the natural transformation \( p \) in Definition 3.4 has only a lower leg, corresponding to the functor \( \Pi \).

**Remark 3.5.** In the context of Definition 3.4, the functor \( \Pi T_i : R\mathcal{C}_R \to \mathcal{C} \) is equal to the composite of \( \mathcal{T}_o : R\mathcal{C}_R \to R\mathcal{C}_R \) and the forgetful functor \( U : R\mathcal{C}_R \to \mathcal{C} \). Symmetrically, \( \Pi T_o = U T_i \).

**Example 3.6.** For the braided preserving monad morphism coming from an algebra homomorphism \( \varphi : R \to T \) as in Example 2.3, the functor \( \Pi \) maps an \( R \)-bimodule \((X, \mu^R_X, \mu_X^R)\) to
\[
\Pi X = \text{Coker}(\mu^R_X - \mu^R_X \circ \chi R, x).
\]
For two \( R \)-bimodules \( X \) and \( Y \) we shall use the notation \( X \hat{\otimes}_R Y := \Pi(X \otimes_R Y) \) and we shall say that this object in \( \mathcal{C} \) is the *braided cyclic tensor product* of \( X \) and \( Y \).

In the next theorem we prove that the BD-law \( \tau \), in a braided pair of monads \((\mathcal{C}, T, R, l, t, \tau)\), lifts to a distributive law of the monads \( \mathcal{T}_o \) and \( T_i \) in Propositions 3.1 and 3.2.

**Theorem 3.7.** Let \( \mathcal{S} = (\mathcal{C}, T, R, l, t, \tau) \) be a braided pair of monads and \( \varphi : R \to T \) be a braided preserving monad morphism. Consider the associated functors \( T_i, T_o \) and \( \mathcal{T}_i, \mathcal{T}_o \), and the natural epimorphisms \( \pi_i : T_i \to T_i \) and \( \pi_o : T_o \to T_o \) in Propositions 3.1 and 3.2.

1. There is a natural transformation \( \theta^i : T_i\mathcal{T}_o \to T_o T_i \), between endofunctors on the category of \((R, R)\)-bimodules, such that
\[
\theta^i \circ T_i \pi_o = \pi_o T_i \theta^i.
\]
2. There is a natural transformation \( \theta^o : T_o \mathcal{T}_o \to T_o T_o \), between endofunctors on the category of \((R, R)\)-bimodules, such that
\[
\pi_o T_i \theta^o \circ T_o \pi_i = \theta^o \circ \pi_i T_o \theta^o \circ T_o \pi_o.
\]
3. \( \theta^i : T_i\mathcal{T}_o \to T_o T_i \) is a distributive law.

**Proof.** (1). Let \((X, x, x^0)\) be a given \((R, R)\)-bimodule. Since \( T \) preserves coequalizers, the sequence in the top row of the diagram in Figure 9 is a coequalizer. Since \( t \) is invertible, the sequence in the bottom row is a coequalizer by the definition of \( \pi_o \). By Proposition 2.12 (3) the square whose horizontal edges are \( g_1 \) and \( g_2 \) is commutative. Using that \( t \) is a BD-law and taking into account condition (A 4), it results that the square with horizontal arrows \( f_1 \) and \( f_2 \) is also commutative. Hence \( \pi_o T_i X \circ t X \) coequalizes \((f_1, g_1)\). Furthermore, by Proposition 2.12 (3), \( t X : T_i T_o X \to T_o T_i X \) is a morphism of \((R, R)\)-bimodules. By construction of \( \mathcal{T}_o \), the canonical epimorphism \( \pi_o X : T_o X \to \mathcal{T}_o X \) is a morphism of \((R, R)\)-bimodules. Hence \( \pi_o T_i X : T_o T_i X \to \mathcal{T}_o T_i X \) and

![Figure 9. Construction of \( \theta^i \).](image-url)
$T_i \pi_o X : T_i T_o X \to T_i \overline{T}_o X$ are also morphisms in $\mathcal{C}_R$. Moreover, $tX \circ T \varphi X$ and $Tx$ are $(R, R)$-bimodule morphisms by Proposition 2.12 (1) and (4). Hence so are $f_1$ and $q_1$ by Proposition 2.10 (1). In conclusion, we can apply Proposition 1.5 in the category $\mathcal{C}_R$ to show that there exists an $(R, R)$-bimodule morphism $t'X$, rendering commutative the right hand side square on Figure 9. By construction, $t'$ is natural.

(2). It follows by condition (A 4), naturality of $\varphi$ and the fact that $t$ is a distributive law that the morphisms $x_o$ and $x'_o$ in (2.7) satisfy

$$T_o x_o \circ iT_i X \circ R tX = tX \circ t T_o X \circ \varphi T_i T_o X \quad \text{and} \quad T_i x'_o \circ iT_i X \circ R tX = tX \circ T_i x'_o \circ iT_o X. \quad (3.9)$$

Taking into account (3.7), the identities in (3.9) imply

$$t'X \circ T_i \pi_o X \circ iT_o X \circ \varphi (t T_o X) = \pi_o T_i X \circ T_o x_o \circ iT_i X \circ R tX \quad \text{and} \quad t'X \circ T_i \pi_o X \circ T_i x'_o \circ iT_o X = \pi_o T_i X \circ T_o x'_o \circ iT_i X \circ R tX. \quad (3.10)$$

Using the naturality of $t$, $\varphi$ and $\pi_o$ together with the $(R, R)$-bimodule morphism property (3.1) of $\pi_o$, the equations in (3.10) can be seen to be equivalent to the commutativity of the left hand side squares on Figure 10. Since $\pi_i X$ coequalizes $(x_o, x'_o)$, the morphism $\overline{T}_o \pi_i X$ coequalizes $\overline{T}_o x_o \circ \overline{T}_o x'_o$.

![Figure 10. Construction of $t'$.](image)

$\pi_o \overline{R}^0 T_i X \circ iT_i X \circ \overline{R}^0 tX$ and $\overline{T}_o x'_o \circ \pi_o \overline{R}^0 T_i X \circ iT_i X \circ \overline{R}^0 tX$. Hence $\overline{T}_o \pi_i X \circ t'X$ coequalizes $t \overline{T}_o X \circ \varphi T_i \overline{T}_o X$ and $\overline{R}^0 T_i \pi_o X$ and $T_i x'_o \circ iT_o X \circ \overline{R}^0 T_i \pi_o X$. Since $\overline{R}^0 T_i \pi_o X$ is an epimorphism, this implies the existence of a morphism $tX$ in $\mathcal{C}$, rendering commutative the right hand side square on Figure 10. Furthermore, the parallel arrows $t \overline{T}_o X \circ \varphi T_i \overline{T}_o X$ and $T_i x'_o \circ iT_o X \circ \overline{R}^0 T_i \pi_o X$ on Figure 10 are morphisms of $(R, R)$-bimodules by Proposition 2.12 (2) and $t'X$ is a morphism of $(R, R)$-bimodules by part (1). By applying Proposition 1.5 in $\mathcal{C}_R$, we conclude that $\overline{t}X$ is an $(R, R)$-bimodule morphism. Naturality of $\overline{t}$ follows obviously from the naturality of $t'$.

(3). The first axiom in Definition 1.9 is proven in the left frame in Figure 11. The first and fourth equalities follow by (3.8) and the fact that $\pi_i$ and $\pi_o$ are natural transformations. The second and fifth equalities are consequences of the second identity in (3.6). For the third relation we used $YB$-equation on $t$ and that $t$ is a distributive law. Since $\pi_i \overline{T}_i \overline{T}_o X \circ T \pi_i \overline{T}_o X \circ TT \pi_o X$ is epi, this proves the first axiom in the definition of distributivity laws.

Similarly, taking into account the first identity in (3.6) we get the first and last equalities in the right frame of Figure 11. By (3.8) and the fact that $t$ is a distributive law we deduce the second and the third identities. Note that in the first, second and last equalities we also used naturality which, in this case, means that e.g. $\pi_o$ can be pushed up and down along a string. Since $\pi_o$ is an epimorphism, this proves that $\overline{t}$ satisfies the third axiom of a distributive law in Definition 1.9. The remaining two axioms are verified similarly. 

\[\text{Theorem 3.8.} \quad \text{Take a braided pair of monads} \quad \mathcal{S} = (\mathcal{C}, T, R, t, \varphi) \quad \text{and a braid preserving monad morphism} \quad \varphi : R \to T. \quad \text{Consider the functors} \quad \overline{T}_o \quad \text{and} \quad \overline{T}_i, \quad \text{constructed in Propositions 3.1 and 3.2, respectively, and the functor} \quad \Pi \quad \text{introduced in Definition 3.4.} \quad \text{For these data there are mutually} \]
The squares are commutative, for any \((\pi_0, \pi_1)\) maps are given, for any object \(R, R\), the squares are also commutative, as \(\pi_0: T_0 X \to T_1 X\) is a morphism of \((R, R)\)-bimodules. Hence \(p_{T_1} X \circ \pi_1 X\) coequalizes \((x_i, x_0^i)\). Then there is a morphism \(i'X: T_0 X \to \Pi T_1 X\) such that \(i'X \circ \pi_0 X = p_{T_1} X \circ \pi_1 X\). In the second diagram of Figure 12, the squares are also commutative, as \(\pi_0 X\) is a morphism of \((R, R)\)-bimodules. Therefore \(i'X\) coequalizes \((\pi_0, \pi_0^0)\), as \(\pi_1 X\) coequalizes \((x_0, x_0^0)\) and \(R\pi_0 X\) is an epimorphism. Thus there exists a morphism \(iX: \Pi T_0 X \to \Pi T_1 X\) satisfying the required relation.

To construct \(j\) one proceeds analogously. The natural morphisms \(i\) and \(j\) are mutual inverses, as \(p_{T_1} X \circ \pi_0 X\) and \(p_{T_0} X \circ \pi_0 X\) are epimorphisms. Naturality of \(i\) follows by (3.11) and naturality of \(p, \pi_o\) and \(\pi_i\).

**Example 3.9.** Let \(C\) be an abelian braided monoidal category such that the functors \(X \otimes (\cdot)\) and \((\cdot) \otimes X\) are right exact, for any object \(X\) in \(C\) (cf. Example 1.8). For the monad coming from an algebra homomorphism \(\phi: R \to T\) as in Example 2.3, the natural transformation \(\mathfrak{i}\) in Theorem 3.7 (2) is the projection of \(\alpha_{T,T,X} \circ (\chi_{T,T} \otimes X) \circ \alpha^{-1}_{T,T,X}\). That is, it is the unique morphism satisfying

\[
\pi_o(T \otimes_{RX} X) \circ (T \otimes \pi_i X) \circ \alpha_{T,T,X} \circ (\chi_{T,T} \otimes X) \circ \alpha^{-1}_{T,T,X} = \mathfrak{i}X \circ \pi_i (T \otimes_{RX} X) \circ (T \otimes \pi_o X),
\]

where notations in Example 3.3 are used.

For an \(R\)-bimodule \(X\), the natural transformation \(iX: T \otimes_{RX} X \to \Pi(T \otimes_{RX} X)\), constructed in the previous theorem, is the projection of the identity map \(T \otimes X \to T \otimes X\).

As it is explained in [We, page 281], for any monad \((T, \mu, \eta)\) on a category \(\mathcal{M}\) and an object \(X\) in \(\mathcal{M}\), there is an associated cosimplex in \(\mathcal{M}\), given at degree \(n\) by \(T^{n+1} X\). Coface and codegeneracy maps are given, for \(k = 0, \ldots, n\), by

\[
T^k \eta T^{n-k} X: T^n X \to T^{n+1} X \quad \text{and} \quad T^k \mu T^{n-k} X: T^{n+2} X \to T^{n+1} X,
\]

**Figure 11.** \(\mathfrak{i}\) is a distributivity law.
respectively. Clearly, application of any functor \( \Pi : \mathcal{M} \to \mathcal{C} \) yields a cosimplex in \( \mathcal{C} \). In [BS], para-cocyclic structures on the resulting cosimplex in \( \mathcal{C} \) were studied. Recall from [BS] the following construction.

**Definition 3.10.** [BS, Definitions 1.7 and 1.8] An admissible septuple \( \mathcal{A} \) consists of the data 
\( (\mathcal{M}, \mathcal{C}, T_o, T_t, \Pi, t, i) \), where

- \( \mathcal{C} \) and \( \mathcal{M} \) are categories,
- \( (T_o, \mu_o, \eta_o) \) and \( (T_t, \mu_t, \eta_t) \) are monads on \( \mathcal{M} \),
- \( \Pi \) is a functor \( \mathcal{M} \to \mathcal{C} \),
- \( t : T_t T_o \to T_o T_t \) is a distributive law,
- \( i : \Pi T_o \to \Pi T_t \) is a natural transformation,

subject to the conditions

\[
i \circ \Pi \eta_o = \Pi \eta_t \quad \text{and} \quad i \circ \Pi \mu_o = \Pi \mu_t \circ i T_t \circ \Pi t \circ i T_o. \tag{3.12}
\]

A transposition morphism for the admissible septuple \( \mathcal{A} \) is a pair \( (X, w) \), consisting of an object \( X \) and a morphism \( w : T_t T_o X \to T_o T_t X \) in \( \mathcal{M} \), satisfying

\[
w \circ \eta_t X = \eta_o X \quad \text{and} \quad w \circ \mu_t X = \mu_o X \circ T_o w \circ t X \circ T_t w. \tag{3.13}
\]

**Theorem 3.11.** [BS, Theorem 1.10] Consider an admissible septuple \( (\mathcal{M}, \mathcal{C}, T_o, T_t, \Pi, t, i) \) and a transposition morphism \( (X, w) \) for it. The associated cosimplex \( Z^* := \Pi T_o^{-1} X \) is para-cocyclic with para-cocyclic morphism

\[
w_n := \Pi T_o^n w \circ \Pi T_o^{-n-1} t X \circ \Pi T_o^{-n-2} t T_o X \circ \cdots \circ \Pi T_o t T_o^{-2} X \circ \Pi t T_o^{-1} X \circ i T_o^n X. \tag{3.14}
\]

The next theorem is our main result.

**Theorem 3.12.** Consider a braided pair of monads \( \mathcal{S} = (\mathcal{C}, T, R, l, t, \tau) \) and a braid preserving monad morphism \( \varphi : R \to T \). Then, data, consisting of

- the categories \( \mathcal{C} \) and \( \mathcal{M} := \mathcal{K}_R \),
- the monads \( \mathcal{T}_o \) in Proposition 3.1 and \( \mathcal{T}_t \) in Proposition 3.2,
- the functor \( \Pi : \mathcal{K}_R \to \mathcal{C} \) in Definition 3.4,
- the natural transformations \( \tau : T_t \mathcal{T}_o \to \mathcal{T}_o T_t \), constructed in Theorem 3.7 (2), and \( i : \mathcal{K}_T \mathcal{T}_o \to \mathcal{K}_T \mathcal{T}_t \), constructed in Theorem 3.8,

constitute an admissible septuple in the sense of Definition 3.10. Moreover, a transposition morphism for it consists of an \((R, R)\)-bimodule \( X \) and an \((R, R)\)-bimodule map \( w : T_t T_o X \to T_o T_t X \), satisfying

\[
\tilde{w} \circ \tau X = \tau_o X, \quad \text{and} \quad \tilde{w} \circ t X = \tilde{t}_o X \circ \tau X \circ T_t \tilde{w} \circ t X, \tag{3.15}
\]

where \( \tilde{w} \) is defined in terms of the natural morphism \( \pi_t \) in Proposition 3.2 as \( \tilde{w} := w \circ \pi_t X \) and the natural transformation \( \tilde{\tau} : T_t T_o \to T_o T_t \) is defined in Theorem 3.7 (1).

**Proof.** \( \mathcal{T}_o \) and \( \mathcal{T}_t \) are monads on \( \mathcal{K}_R \) by Proposition 3.1 and Proposition 3.2, respectively. \( \Pi \) is a functor \( \mathcal{K}_R \to \mathcal{C} \) by Definition 3.4. \( \tilde{\tau} \) is a distributive law \( \mathcal{T}_t \mathcal{T}_o \to \mathcal{T}_o \mathcal{T}_t \), by Theorem 3.7 (3). Thus in order to verify that they constitute an admissible septuple, we have to prove that the natural transformation \( i : \Pi \mathcal{T}_o \to \Pi \mathcal{T}_t \) in Theorem 3.8 satisfies conditions (3.12). In terms of the natural epimorphism \( p \) from the forgetful functor \( \mathcal{K}_R \to \mathcal{C} \) to \( \Pi \),

\[
i \circ \Pi \tau_o \circ p = i \circ \Pi \pi_o \circ \Pi \tau \circ \pi = i \circ \Pi \pi_o \circ p T \circ \tau = p \tilde{\tau}_t \circ \pi_t \circ \tau = p \tilde{\tau}_t \circ \tau = \Pi \tilde{\tau}_t \circ p. \tag{3.16}
\]

In the first equality we used (3.2) and in the penultimate equality we used the first identity in (3.6). The third equality is a consequence of the first identity in (3.11). The other equalities follow by naturality. Since \( p \) is epi, (3.16) proves the first condition in (3.12).

The second condition in (3.12) is proven in Figure 13. The first, fourth and seventh equalities follow by the first identity in (3.11). The third equality is a consequence of (3.8). In the fifth equality we used the second condition in (3.6) and in the seventh one we used (3.3). The other equalities follow by naturality. Since \( p \tilde{T}_o T_o \circ \pi_o T_o \circ T \pi_o \) is epi, we have the second condition in (3.12) proven.
The last claim about transposition morphisms is proven by showing the equivalence of conditions (3.15) and (3.13). In view of the first condition in (3.6), the first conditions in (3.15) and (3.13) are obviously equivalent. In the second frame of Figure 13 it is shown that the other conditions in (3.15) and (3.13) are equivalent. Composition on the right of both sides of the second condition in (3.13) with the epimorphism \( \pi_t \) yields an equivalent condition, namely the first equality in the above mentioned picture. This is equivalent to the second equality, in view of (3.6) together with the fact that, by (3.7) and (3.8), \( T_o \pi_r \circ t' = t \circ \pi_t T_o \). To conclude the proof we use \( w = w \circ \pi_t X \).

Combining Theorem 3.12 with Theorem 3.11, we obtain the following

**Corollary 3.13.** Let \( S = (C, T, R, l, t, \pi) \) be a braided pair of monads and \( \varphi : R \to T \) be a braid preserving monad morphism. Consider the monads \( T_o \) on \( \mathcal{H} \) in Proposition 3.1 and \( \mathcal{T}_t \) in Proposition 3.2 and the functor \( \Pi : \mathcal{H} \to C \) in Definition 3.4. Then any \((R, R)\)-bimodule morphism \( w : T_o X \to T_o X \), satisfying (3.15), determines a para-cocyclic structure on the cosimplex \( \Pi T_o^{n+1} X \). The para-cocyclic morphism is given in terms of the natural transformation \( \iota \), constructed in Theorem 3.8, as

\[
\Pi T_o^n w \circ \Pi T_o^{n-1} t o \Pi T_o^{n-2} t T_o \pi \cdots \Pi T_o \pi T_o^{n-1} \circ \Pi T_o^{n-1} \circ \iota T_o \circ \Pi T_o^{n+1} X \to \Pi T_o^{n+1} X.
\]

An immediate example of the situation in Corollary 3.13 is induced by an algebra homomorphism in a braided monoidal category.

**Example 3.14.** In this example, in order to simplify formulae, we do not write the associativity constraints. Let \( \varphi : R \to T \) be a morphism of algebras in a braided monoidal category, in which the functors \( X \otimes (-) \) and \((-) \otimes X \) are right exact, for any object \( X \) in \( C \). Assume that \( \varphi \) satisfies (2.1) and consider the corresponding braid preserving monad morphism in Example 2.3. It determines an admissible septuple \( (\mathcal{C}, R\text{-Mod} R, T \otimes R (-), T \otimes R X (-), \Pi, \mathcal{T}, \iota) \), where the functors \( T \otimes R (-) \), \( T \otimes R X (-) \) and \( \Pi \) are described in Example 3.3 and Example 3.6, respectively, and the natural transformations \( \iota \) and \( \iota X \) can be found in Example 3.9. For any \( R \)-bimodule \( X \) there is a corresponding cosimplex, given at degree \( n \) by

\[
Z^n(X, \varphi) := T^\otimes R \oplus X,
\]
known as a (braided) cyclic \( R \)-module tensor product. A transposition morphism is an \( R \)-bimodule map \( w : T \otimes R X \to T \otimes R X \), satisfying the conditions

\[
w \circ (\varphi \otimes R X) = \varphi \otimes R X
\]

\[
w \circ (T_\iota \otimes R X) = (T_\iota \otimes R X) \circ (T \otimes R w) \circ \iota X \circ (T \otimes R X w),
\]

where the multiplication maps \( r : T \otimes R T \to T \) and \( T \otimes R X \) are induced by \( t : T \otimes T \to T \) and \( \iota X \otimes 1 \) respectively, (cf. Example 3.3). In degree \( n \), the para-cocyclic operator is given by

\[
w_n = (T^\otimes R w) \circ \Pi T^\iota X \circ \iota (T \otimes R X).
\]
where $\tilde{T}X$ is the projection of $\chi_{T,T\otimes R} = X$, i.e. the unique map $T \otimes_{R^e} (T \otimes_{R^e} X) \rightarrow T \otimes_{R^e} (T \otimes_{R^e} X)$, for which

$$P_{0i} \circ \left( \chi_{T,T\otimes R} \otimes X \right) = \tilde{T}X \circ P_{0i},$$

where $P_{0i} : T \otimes_{R^e} T \otimes X \rightarrow T \otimes_{R^e} (T \otimes_{R^e} X)$ and $P_{io} : T \otimes_{R^e} T \otimes X \rightarrow T \otimes_{R^e} (T \otimes_{R^e} T \otimes X)$ denote the canonical epimorphisms.

Examples of cosimplices in Hopf cyclic theory, corresponding to (co)module algebras of a bialgebroid (or in particular a bialgebra) over $R$, are of this kind [BS]. Further examples are presented in Example 3.15 and Example 3.16 below.

**Example 3.15.** Let $H$ be a Hopf algebra with comultiplication $\Delta : H \rightarrow H \otimes H$ and counit $\varepsilon : H \rightarrow \mathbb{K}$. For simplifying computations, we shall use the Sweedler-Heynemann $\Sigma$-notation

$$\varepsilon(h) = \sum h_{(1)} \otimes h_{(2)}, \quad \Delta(h) = \sum h_{(1)} \otimes h_{(2)}, \quad \forall h \in H.$$

An important class of braided monoidal categories is provided by the categories $\mathcal{M}^H$ of right comodules over a coquasitriangular Hopf algebra $H$. Cf. [Mo, Chapter 10], a Hopf algebra $H$ is *coquasitriangular* if there is an invertible (with respect to the convolution product) $\mathbb{K}$-linear map

$$\langle -,- \rangle : H \otimes H \rightarrow \mathbb{K}$$

such that, for $h,k,l \in H$,

$$\begin{align}
\sum \langle h_{(1)}, k_{(1)} \rangle k_{(2)} h_{(2)} &= \sum \langle h_{(1)} k_{(1)}, h_{(2)}, k_{(2)} \rangle, \quad (3.17) \\
\langle h, k l \rangle &= \sum \langle h_{(1)}, k \rangle \langle h_{(2)}, l \rangle, \quad (3.18) \\
\langle h k, l \rangle &= \sum \langle h, l_{(2)} \rangle \langle k, l_{(1)} \rangle, \quad (3.19)
\end{align}$$

If, in addition, the inverse in convolution of $\langle -,- \rangle$ is equal to $\langle -,- \rangle \circ \tau$, we shall say that $H$ is *cotriangular*. Here $\tau : H \otimes H \rightarrow H \otimes H$ denotes the usual flip map. By definition, $\langle -,- \rangle$ is called the coquasitriangular map of $H$.

We fix a coquasitriangular Hopf algebra $(H,\langle -,- \rangle)$ over a field $\mathbb{K}$. Let us briefly recall the braided monoidal structure of $\mathcal{M}^H$, that corresponds to $\langle -,- \rangle$. First, for $(M,\rho_M)$ and $(N,\rho_N)$ in $\mathcal{M}^H$, the tensor product of $M$ and $N$ in $\mathcal{M}^H$ is $M \otimes_{\mathbb{K}} N$, regarded as a comodule with respect to the right diagonal coaction

$$\rho(m \otimes n) = \sum m_{(0)} \otimes n_{(0)} \otimes m_{(1)} n_{(1)}.$$

In this formula, for a right comodule $(M,\rho_M)$, we used the $\Sigma$-notation $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$. The associativity and unity constraints in $\mathcal{M}^H$ are induced by the corresponding structures in the monoidal category of $\mathbb{K}$-linear spaces. The unit object is $\mathbb{K}$, regarded as a trivial $H$-comodule. To define the braiding in $\mathcal{M}^H$ we use the coquasitriangular map as follows. For two right $H$-comodules $M$ and $N$ we define the natural map

$$\chi_{M,N} : M \otimes_{\mathbb{K}} N \rightarrow N \otimes_{\mathbb{K}} M, \quad \chi_{M,N}(m \otimes n) = \sum \langle n_{(1)}, m_{(1)} \rangle n_{(0)} \otimes m_{(0)}.$$

It is well-known that $\mathcal{M}^H$ is a braided monoidal category with braiding $\chi$. Furthermore, $\mathcal{M}^H$ is a symmetric monoidal category with respect to $\chi$ (that is $\chi_{N,M} \circ \chi_{M,N} = \text{Id}_{M \otimes_{\mathbb{K}} N}$, for $N,M \in \mathcal{M}^H$), if and only if $H$ is *cotriangular*.

Commutative Hopf algebras are the simplest examples of cotriangular Hopf algebras. In this case the cotriangular structure $\langle -,- \rangle$ may be taken the trivial $\mathbb{K}$-linear map $\langle k,k \rangle := \varepsilon(k) \varepsilon(k)$, for any $h,k \in H$. Thus, the braiding is induced by $\tau$, the canonical flip map.

Group Hopf algebras are not necessarily coquasitriangular. Indeed, let $G$ denote a group and assume that $\langle -,- \rangle$ is a coquasitriangular map on $\mathbb{K}G$. Let $\gamma : G \times G \rightarrow \mathbb{K}$ be the map

$$\gamma(h,k) := \langle h,k \rangle, \quad \text{for } h,k \in G.$$

Since the coquasitriangular map is invertible in convolution, it follows easily that $\langle h,k \rangle \in \mathbb{K}^*$, for any $h,k \in G$. Hence $\gamma$ can be regarded as a map to $\mathbb{K}^*$. Clearly, then the condition (3.17) is equivalent to the fact that $G$ is abelian. On the other hand, relations (3.18) and (3.19) are equivalent to

$$\gamma(hk,g) = \gamma(h,g) \gamma(k,g) \quad \text{and} \quad \gamma(g,hk) = \gamma(g,h) \gamma(g,k), \quad \forall h,g,k \in G.$$
A map \( \gamma : G \times G \to K^* \), \( \gamma(h, k) = \langle h, k \rangle \) satisfying the above identities is called bi-character of \( G \). In conclusion, \( K \) is coquasitriangular if, and only if, \( G \) is abelian and there is a bi-character \( \gamma : G \times G \to K \) on \( G \). As a matter of fact, we have also proved that, for an abelian group \( G \), there is an one-to-one correspondence between the set of coquasitriangular structures on \( K \) and the set of bi-characters on \( G \).

The category \( \mathcal{M}^{KG} \) is equivalent to the category of \( G \)-graded vector spaces. Therefore, an object in \( \mathcal{M}^{KG} \) is a vector space \( V \) together with a decomposition \( V := \bigoplus_{g \in G} V_g \). The tensor product of \( V \) and \( W \) in \( \mathcal{M}^{KG} \) is \( V \otimes_K W \) on which we take the decomposition

\[
(V \otimes_K W)_g = \bigoplus_{h,k=g} V_h \otimes_K W_k.
\]

The braiding \( \chi_{V,W} : V \otimes_K W \to W \otimes_K V \), for \( v \in V_h \) and \( w \in W_k \), is given by

\[
\chi_{V,W}(v \otimes w) = \gamma(h,k)w \otimes v.
\]

Note that the bi-character \( \gamma : G \times G \to K^* \) defines a cotriangular structure on \( K \) if, and only if, \( \gamma \) is symmetric, that is, for \( h \in G \) and \( k \in G \),

\[
\gamma(h,k)^{-1} = \gamma(h,k).
\]

Recall that the category of super vector spaces can be seen as the symmetric monoidal category of \( \mathbb{Z}_2 \)-graded vector spaces, whose braiding is induced by the bi-character \( \gamma : \mathbb{Z}_2 \times \mathbb{Z}_2 \to K^* \), \( \gamma(g,h) = (-1)^{gh} \).

Our aim now is to specialize the construction in Example 3.14 to the case when \( \varphi : R \to T \) is a braid preserving homomorphism of algebras in \( \mathcal{M}^H \), for a coquasitriangular Hopf algebra \( H \). In this case, the conditions in (2.1) take the form

\[
\varphi(r) \otimes_K t = \sum \langle r(1)_{(1)}, t(1)(1) \rangle \varphi(r(1)_{(2)}). t(1)(2) \rangle \otimes_K t(0) \quad \text{and} \quad \varphi(r) \otimes_K r' = \sum \langle r(1)_{(1)}, r'(1)_{(1)} \rangle \varphi(r(1)_{(2)}). r'(1)_{(2)} \rangle \otimes_K r'(0),
\]

for \( r, r' \in R \), \( t \in T \).

Clearly, \( T_o \) is the functor \( T \otimes_R (-) \). The multiplication in \( R^X \) is defined, for \( r' \) and \( r'' \) in \( R \), by

\[
r' \cdot r'' := \sum \langle r''_{(1)}, r'(1) \rangle \varphi(r''(0)) r'(0).
\]

We have already noticed that \( R^X \) is an algebra in \( \mathcal{M}^H \) and that \( T \) is a right \( R^X \)-module with respect to the action

\[
t \cdot r := \sum \langle r(1), t(1) \rangle \varphi(t(0)) r(0).
\]

Furthermore, every \( R \)-bimodule \( X \) can be seen as a left \( R^X \)-module, with respect to the action

\[
r \cdot x := \sum \langle x(1), r(1) \rangle x(0) r(0),
\]

where \( x \in X \) and \( r \in R \). Hence \( T_i := T \otimes_{R^X} (-) \). For \( X \) as above, \( \Pi X \) is the quotient of \( X \) with respect to the vector space generated by the commutators

\[
[x, r] := rx - \sum \langle x(1), r(1) \rangle x(0) r(0),
\]

where \( r \) and \( x \) run arbitrarily in \( R \) and \( X \), respectively. For \( t', t'' \in T \) and \( x \in X \), the morphisms \( \Pi X : T \otimes_{R^X} (T \otimes_{R^X} X) \to T \otimes_{R^X} (T \otimes_{R^X} X) \) and \( iX : T \otimes_{R^X} X \to \Pi(T \otimes_{R^X} X) \) are given by the following formulae

\[
\Pi X(t' \otimes_{R^X} t'' \otimes_{R^X} x) = \sum \langle t''_{(1)}, t'(1) \rangle t''(0) \otimes_R t'(0) \otimes_{R^X} x,
\]

\[
iX(t \otimes_{R^X} x) = p(t \otimes_{R^X} x),
\]

where \( p : \text{Id}_{R_{-Mod-R}} \to \Pi \) is the canonical projection and \( \otimes_R \) denotes the braided cyclic tensor product introduced in Example 3.6.

Let \( X \) be an \( R \)-bimodule in \( \mathcal{M}^H \). For a transposition morphism \( w : T \otimes_{R^X} X \to T \otimes_{R^X} X \), we use the notation

\[
w(t \otimes_{R^X} x) = \sum t_w \otimes_R x_w.
\]
Using this notation, the defining conditions of a transposition morphism are equivalent to
\[ w(1 \otimes_R x) = 1 \otimes_R x, \quad \text{and} \quad \sum \left\langle t''_{(1)}, t'_{(1)} \right\rangle t''_{(0)} t'_{(0)} \otimes_R x_w = \sum \left\langle t''_{(1)}, t'_{(1)} \right\rangle (t''_{(0)} t'_{(0)}) \otimes_R (x_w x_w'). \]

In degree \( n \), the corresponding para-cocyclic object is given by \( Z^n(X, \varphi, w) := T^\otimes_R n+1 \otimes_R X \). Its para-cocyclic operator is
\[
w_n(t^0 \otimes_R \cdots \otimes_R t^n \otimes_R x) = \sum \left\langle t^0_{(1)}, t^0_{(n)} \right\rangle \cdots \left\langle t^n_{(1)}, t^n_{(1)} \right\rangle t^0_{(0)} \otimes_R \cdots \otimes_R t^n_{(0)} \otimes_R \left( t^0_{(0)} \otimes_R \cdots \otimes_R t^n_{(0)} \right) \otimes_R (x_w x_w').
\]

In the particular case when \( H := \mathbb{K}G \) and the braiding is defined by a bi-character \( \gamma \), conditions (3.20) reduce to
\[
\varphi(r) \otimes_{\mathbb{K}} t = \gamma(k, h) \gamma(h, k) \varphi(r) \otimes_{\mathbb{K}} t \quad \text{and} \quad \varphi(r) \otimes_{\mathbb{K}} r' = \gamma(k, h) \varphi(r) \otimes_{\mathbb{K}} r',
\]
for \( h, k \in G, r \in R_h, r' \in R_k \) and \( t \in T_k \). Note that, if some component \( R_h \) lies within the kernel of \( \varphi \), then these conditions may hold also for a non-trivial braiding between copies of \( R \) and \( T \).

The second condition in the definition of a transposition map becomes
\[
\sum \gamma(k, h) (t'' t') \otimes_R x_w = \sum (g, h) (t'' g) t' \otimes_R (x_w x_w'),
\]
where \( h, k, g \in G \) and \( t'' \in T_h, t' \in T_k \) and \( x \) belongs to the component \( X_g \) of the \( G \)-graded vector space \( X \). In this particular case, for \( t' \in T_g \), the para-cocyclic operator satisfies
\[
w_n(t^0 \otimes_R \cdots \otimes_R t^n \otimes_R x) = \sum \gamma(g_1 \cdots g_n) t^1 \otimes_R \cdots \otimes_R t^n \otimes_R (x_w x_w').
\]

**Example 3.16.** Dually, the category \( H \mathcal{M} \) of left modules over a \( \mathbb{K} \)-Hopf algebra \( H \) is braided monoidal if, and only if, \( H \) is quasitriangular. Cf. [Mo, Chapter 10], a Hopf algebra \( H \) is quasitriangular if there is an invertible element \( R := \sum a_i \otimes b_i \) in \( H \otimes H \) such that
\[
\sum h_{(2)} \otimes h_{(1)} = R(\sum h_{(1)} \otimes h_{(2)}) R^{-1},
\]
\[
(\Delta \otimes \text{Id}_H)(R) = R^{13} R^{23},
\]
\[
(\text{Id}_H \otimes \Delta)(R) = R^{13} R^{12},
\]
where \( R^{12} = \sum a_i \otimes b_i \otimes 1 \) and \( R^{13} \) and \( R^{23} \) are defined analogously. If \( R^{-1} := \sum b_i \otimes a_i \), then we say that \( H \) is triangular.

The braided monoidal structure that corresponds to a quasitriangular element \( R \) is defined as follows. For \( M \) and \( N \) in \( H \mathcal{M} \), the tensor product of \( M \) and \( N \) in \( H \mathcal{M} \) is \( M \otimes_{\mathbb{K}} N \), regarded as a module with respect to the left diagonal action
\[
h \triangleright (m \otimes n) = \sum h_{(1)} \triangleright m \otimes h_{(2)} \triangleright n.
\]

The associativity and unity constraints in \( H \mathcal{M} \) are induced by the corresponding structures in the monoidal category of \( \mathbb{K} \)-linear spaces. The unit object is \( \mathbb{K} \), regarded as a trivial \( H \) -module. The functorial morphism of left \( H \)-modules
\[
\chi_{M, N} : M \otimes_{\mathbb{K}} N \to N \otimes_{\mathbb{K}} M, \quad \chi_{M, N}(m \otimes n) = \sum_j (c_j \triangleright n) \otimes (d_j \triangleright m)
\]
is a braiding on \( H \mathcal{M} \), where \( R^{-1} = \sum c_j \otimes d_j \).

Let us specialize the para-cocyclic object constructed in Example 3.14 to a homomorphism of algebras \( \varphi : R \to T \) in the braided monoidal category \( H \mathcal{M} \). In this case, conditions (2.1) read as
\[
\varphi(r) \otimes_{\mathbb{K}} t = \sum_{j,k} (c_k d_j \triangleright \varphi(r)) \otimes_{\mathbb{K}} (d_k c_j \triangleright t) \quad \text{and} \quad \varphi(r) \otimes_{\mathbb{K}} r' = \sum_{j,k} (c_k d_j \triangleright \varphi(r)) \otimes_{\mathbb{K}} (d_k c_j \triangleright r'),
\]
for \( r, r' \in R \) and \( t \in T \), where \( \sum j c_j \otimes d_j = R^{-1} = \sum k c_k \otimes d_k \). Clearly, \( T_o \) is the functor \( T \otimes_R (\cdot) \).

Moreover, the multiplication in \( R^X \) is defined, for \( r' \) and \( r'' \) in \( R \), by
\[
r' \cdot r'' := \sum_j (c_j \triangleright r'')(d_j \triangleright r').
\]
We have already noticed that $R^X$ is an algebra in $\mathcal{HM}$ and $T$ is a right $R^X$-module with respect to the action
\[ t \cdot r := \sum (c_j \triangleright \varphi(r)) (d_j \triangleright t). \]

Furthermore, if $X$ is an $R$-bimodule then, for $x \in X$ and $r \in R$, the following formula
\[ r \cdot x := \sum_j (c_j \triangleright x) (d_j \triangleright r) \]
defines a left $R^X$-module structure. Hence $\mathcal{T}_n := T \otimes_{R^X} (-)$. For $X$ as above, $\Pi X$ is the quotient of $X$ with respect to the vector space generated by the commutators
\[ [x, r] := rx - \sum_j (c_j \triangleright x) (d_j \triangleright r), \]
where $r$ and $x$ run arbitrarily in $R$ and $X$, respectively. For $t'$, $t'' \in T$ and $x \in X$, the morphisms $\mathcal{I}X : T \otimes_{R^X} (T \otimes_{R^X} X) \to T \otimes_{R^X} (T \otimes_{R^X} X)$ and $iX : T \otimes_{R^X} X \to \Pi(T \otimes_{R^X} X)$ are given by the following formulae
\[ iX(t' \otimes_{R^X} t'' \otimes_{R^X} x) = \sum_j c_j \triangleright t'' d_j \triangleright t' \otimes_{R^X} x, \]
\[ iX(t \otimes_{R^X} x) = p(t \otimes_{R^X} x), \]
where $p : \text{Id}_{R^X} \to \Pi$ is the canonical projection and $\otimes_{R}$ denotes the braided cyclic tensor product introduced in Example 3.6.

Let $X$ be an $R$-bimodule in $\mathcal{HM}$. For a transposition morphism $w : T \otimes_{R^X} X \to T \otimes_{R^X} X$, we use the notation
\[ w(t \otimes_{R^X} x) = \sum t_w \otimes_{R} x_w. \]

Using this notation, the defining conditions of a transposition map are equivalent to
\[ w(1 \otimes_{R^X} x) = 1 \otimes_{R} x, \quad \text{and} \quad \sum_j [(c_j \triangleright t'')(d_j \triangleright t)]_w \otimes_{R} x_w = \sum_j (c_j \triangleright t'')(d_j \triangleright t')x_w \otimes_{R} x_w. \]

In degree $n$, the corresponding para-cocyclic object is given by $Z^n(X, \varphi, w) := \mathcal{T}^n \otimes_{R^X} \Pi X$ and its para-cocyclic operator is
\[ w_n(t_0 \otimes_{R^X} \cdots \otimes_{R^X} t_n \otimes_{R^X} x) = \sum_{j_1, \ldots, j_n} c^{(1)}_{j_1} \triangleright t_1 \otimes_{R^X} \cdots \otimes_{R^X} c^{(n)}_{j_n} \triangleright t_n \otimes_{R^X} \left[ d^{(1)}_{j_1} \triangleright \cdots \triangleright (d^{(n)}_{j_n} \triangleright t_0) \cdots \right]_w \otimes_{R^X} x_w, \]
where $\sum_j c^{(k)}_{j_k} \otimes d^{(k)}_{j_k} = R^{-1}$, for every $k = 1, \ldots, n$.

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