Vortices and sound waves in superfluids

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We consider the dynamics of vortex strings and sound waves in superfluids in the phenomenological Landau-Ginzburg equation. We first derive the vortex equation where the velocity of a vortex is determined by the average fluid velocity and the density gradient near the vortex. We then derive the effective action for vortex strings and sound waves by the dual formulation. The effective action might be useful in calculating the emission rate of sound waves by moving vortex strings.

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Recently there was some new interest in understanding the dynamics of vortex strings and sound waves in superfluids or superconductors [1,2]. While many aspects of the vortex dynamics has been analyzed extensively and used to many applications, we feel that some aspects of the basic vortex dynamics, like the vortex equation or the interaction between vortices and sound waves, are not well understood as one hoped. Based on recent progress in rigorous understanding of the vortex dynamics in relativistic field theoretic models [3], where the Magnus force on the vortex string is important, we derive hopefully a new vortex equation. While this is not a ‘dynamical’ equation of motion, it leads to some insight into the vortex dynamics. The model we use to describe superfluids and superconductors is the nonlinear Schrödinger equation studied before [4]. In addition, following the dual formulation of the theory of a complex scalar field [5], we derive an effective action for vortices interacting with sound waves. While many aspects of our goal have been studied before, there still seems to be a room for improvements from our point of view.

In classical hydrodynamics a vortex moves with the local velocity of the fluid. Such a view has been applied to vortices in superfluids [1]. The quantization of the vortex strings has been also investigated [6]. However, the vortex equation we will derive has an additional term which depending on the spatial dependence of the fluid density and has an interesting physical interpretation. The vortex string equation we derive can be regarded as a nonrelativistic limit of the relativistic string equation. The relativistic vortex string equation has been useful in understanding the interaction between vortex strings very close to each other [3].

Since the work by Kalb and Ramond, there has been many works relating the antisymmetric tensor field coupled to vortex strings to vortex strings in superfluids and superconductors. However, the antisymmetric tensor field have usually described massless Goldstone bosons moving at speed of light rather than sound waves of speed less that unity [7]. The Kalb-Ramond antisymmetric tensor field is generated by the dual formulation of the massless Goldstone field. The nonzero fluid density appears as a uniform ‘magnetic’ field for vortex strings and give rise to the Magnus force on strings. Here we use the dual formulation to get the effective action for vortex strings and sound waves, extending the two dimensional result in the first paper of Ref. [5]. Recently, there has been also some interesting progress
in precise understanding of relativistic vortex dynamics where the Magnus force is important [8].

We start with the nonrelativistic Lagrangian for a complex scalar field, which may be coupled to the electromagnetic field,

\[ \mathcal{L} = \frac{i}{2} (\Psi^* \dot{\Psi} - \dot{\Psi}^* \Psi) - \frac{1}{2m} |(\nabla + ieA)\Psi|^2 - \frac{g}{2} (|\Psi|^2 - \rho_0)^2 + \ldots \]  

(1)

where the dots denote terms not important for our discussions like \( eA_0 |\Psi|^2 \). The average charge density is given by a constant \( \rho_0 \). The field equation is then

\[ i\dot{\Psi} = -\frac{1}{2m} (\nabla + ieA)^2 \Psi + \ldots \]  

(2)

There is a global Abelian symmetry whose charge density and current are \( \rho = |\Psi|^2 \) and

\[ J = -\frac{i}{2m} \left[ \Psi^* (\nabla + ieA) \Psi - (\nabla + ieA) \Psi^* \Psi \right] \]  

(3)

The conservation law would be \( \dot{\rho} + \nabla \cdot J = 0 \).

There are quantized vortex strings in the system. With \( \Psi = \sqrt{\rho} e^{i\theta} \), the phase \( \theta \) changes by \( 2\pi \) around each vortex string. The natural core size of vortex strings would be given by the coherence length \( \xi = 1/\sqrt{gm\rho_0} \). To understand the vortex string dynamics, let us first consider a single vortex string whose position is given by \( R(t, \sigma) \) with string coordinate \( \sigma \). The tangent vector along the string would be then \( R'(t, \sigma) = \partial R/\partial \sigma \). We choose the \( \sigma \) sign so that the \( \theta \) field increases with the right-hand rule. We introduce two unit vectors \( N_1(t, \sigma), N_2(t, \sigma) \) perpendicular to \( R' \) at the point on the vortex line, \( R', N_1, N_2 \) would form a right-handed orthogonal basis for three space. There is a new coordinate system \( (\sigma, \zeta^1, \zeta^2) \) on this basis near that point such that

\[ x = R(t, \sigma(t, x)) + \sum_{A=1,2} \zeta^A(t, \bar{x}) N_A(t, \sigma(t, x)) \]  

(4)

Near the vortex, the relation between \( x \) and \( (\sigma, \zeta^1, \zeta^2) \) would be invertable.
By evaluating the time derivative of Eq.(4), we get that on the point on the vortex

$$\dot{\zeta}_A(t, \mathbf{R}) = -\dot{\mathbf{R}} \cdot \mathbf{N}_A$$  \quad (5)$$

Similarly, the space derivative $\partial/\partial x^j$ on Eq.(4) leads to

$$\nabla \sigma = \frac{\mathbf{R}'}{|\mathbf{R}'|^2}$$
$$\nabla \zeta^A = \mathbf{N}_A$$  \quad (6)$$
on the point. Two spatial derivative $\nabla^2$ of Eq.(4) leads to

$$\nabla^2 \zeta^A = -\frac{\mathbf{R}'' \cdot \mathbf{N}_A}{|\mathbf{R}'|^2}$$  \quad (7)$$

To obtain a vortex equation from Eq.(1), we first notice that near a vortex string, the scalar field would be $\Psi \sim \psi_s \equiv (\zeta^1 + i\zeta^2)$. In general the scalar field can be written as $\Psi = \psi_s \Phi$, where $\Phi$ would not in general vanish on the vortex string. Let us evaluate Eq.(1) on a point on the string, to get

$$i\Phi \dot{\mathbf{R}} \cdot \nabla \psi_s = -\frac{1}{2m} \left[ \Phi \nabla^2 \psi_s + 2\nabla \Phi \cdot \nabla \psi_s + 2ie\Phi A \cdot \nabla \psi_s \right]$$  \quad (8)$$
The rest of terms vanishes on the vortex string. After dividing Eq.(8) by $\Phi$ and using Eqs.(5), (6) and (7), we get

$$-i(N_1 + iN_2) \cdot \dot{\mathbf{R}} = -\frac{1}{2m} (N_1 + iN_2) \cdot \left\{ -\frac{\mathbf{R}''}{|\mathbf{R}'|^2} + 2\nabla \ln |\Phi| + 2i(\nabla \text{Arg} \Phi + eA) \right\}$$  \quad (9)$$
Identifying the real and imaginary parts separately, we get the equation of motion for vortex strings,

$$\dot{\mathbf{R}} = \alpha \mathbf{R}' - \frac{1}{m|\mathbf{R}'|^2} \times \left( \mathbf{R}' \times (\nabla \text{Arg} \Phi + eA) \right) - \frac{1}{2m|\mathbf{R}'|^2} \times \left( -\frac{\mathbf{R}''}{|\mathbf{R}'|^2} + 2\nabla \ln |\Phi| \right)$$  \quad (10)$$
where $\alpha$ is undetermined. One convenient parameterization is such that $\dot{\sigma} = 0$ and so $\dot{\mathbf{R}} \cdot \mathbf{R}' = 0$, in which case $\alpha$ vanishes. This is the equation which gives
the velocity of a vortex string once the field configuration \( \Phi \) is known at a given moment. However, it is not a dynamical equation of motion in usual sense because we do not in general know the field \( \Phi \) in terms of a given configuration of vortices. Sometimes, we can make a good guess of \( \Phi \) and will get some insight into the vortex dynamics. Furthermore, the above equation might be a good start for understanding two vortices crossing each other. This equation would be a sort of nonrelativistic limit of the vortex string equation in the relativistic field theory [3], as the term proportional to \( \dddot{R} \times \dot{R}' / m \) is negligible in that limit.

Let us study the implications of Eq.(10) on the vortex dynamics. The field \( \Phi \) would not change much over the scale \( \xi \) near a vortex and would be determined at a point near a vortex by the shape of the vortex and other vortices and by other factors. The local fluid velocity near the vortex would be a sum of the average induced velocity and the current which goes around the vortex string. Naturally, the vortex motion would be determined by the average induced velocity,

\[
\mathbf{u} = \frac{1}{m}(\nabla \text{Arg} \Phi + e\mathbf{A}) \tag{11}
\]

In Eq.(10), the second term in the right hand side would lead to the velocity of the vortex string at given point is \( \dot{\mathbf{R}} = \mathbf{u} \), if we neglect the last term. This is the standard equation for vortices in incompressible fluids. The above expression for the average velocity near the vortex is clearly better than the naive expression, \( (\nabla \theta + e\mathbf{A}) / m \) which is very singular at the vortex position.

The term proportional to \( \mathbf{R}' \times \mathbf{R}'' \) is interesting. The self-induction contribution to the fluid velocity (11) near the vortex, as we will see, has such a term with much larger coefficient. Thus we can regard the term proportional to \( \mathbf{R}' \times \mathbf{R}'' \) as a small correction to the induced velocity near the vortex.

The last term in Eq.(10) is relevant when the charge density has a spatial dependence, which is possible because the fluid is compressible. To understand its physics concretely, let us consider a vortex effectively moving on a two dimensional \((x,y)\) plane where the charge density is forced to decrease along a direction, say along the \(+x\) direction. We are imagining here, for example, a three dimensional fluid on a container whose bottom is not leveled and so the effective two dimensional density decreases along the \(+x\) axis, and we assume a vortex is straight along the
vertical direction. Eq.(10) tells us that it would move along the +y direction. The physics behind this motion is clear. The fluid rotation around the vortex would introduce more fluid to the −y direction from the −x direction and takes out more fluid from +y direction to the −x direction. Thus, the fluid pressure difference would push the vortex to the +y direction. However, we can expect the net fluid velocity is zero far way from vortex as the above phenomena is simply a continuous exchange of position among the vortex and the neighboring fluid. This effect can in principle be tested in an experiment of a rotating superfluid in a bucket whose bottom is tilted. This phenomena looks somewhat similar to an ascending oil drop in water by buoyancy.

Since our fluid is compressible, there are sound waves due to the fluctuations, \( \Psi = \sqrt{\rho_0 + \delta \rho e^{i\theta}} \). Linearized equation for the fluctuation implies that \( \delta \rho = -\delta \dot{\theta} / g \) and the sound speed is

\[
v_s = \sqrt{\frac{g \rho_0}{m}}
\]

From Eq.(10), we see the coupling of sound waves to vortex strings is primarily due the fluid current fluctuations \( \delta \theta \) in the long-wave length limit. The coupling via the density fluctuation term would become weaker by inverse of the wave length.

To understand the interaction between sound wave and vortices, it is best to use the dual formulation of the original theory. Since the dual formulation has been derived many times in past [5], we will summerize briefly. We start with variables \( \rho, \theta \) with the measure \([d\rho d\theta]\). We introduce an auxilliary vector field, \( I \) such that

\[
\int [dI] \exp \left\{ i \int d^4x \left( \frac{m}{2\rho} I^2 - I \cdot \nabla \theta \right) \right\} = \rho^{3/2} \exp \left\{ i \int d^4x \frac{\rho}{2m} (\nabla \theta)^2 \right\}
\]

where irrelevant numerical factors are dropped. The phase variable is given as a sum of the singular part \( \bar{\theta} \) and the nonsingular single-valued part \( \eta \). Because \([d\theta] = [d\bar{\theta} d\eta]\), we can now integrate over the single valued \( \eta \), getting the charge conservation constraint \( \dot{\rho} + \nabla \cdot I = 0 \). This can be solved by introducing the anti-symmetric tensor field \( B_{\mu\nu} \) such that \( (\rho, I)^\mu = \epsilon^{\mu\nu\rho\sigma} \partial_\nu B_{\rho\sigma} / 2 \). The dual Lagrangian would be then written in terms of \( B_{\mu\nu} \) and \( \bar{\theta} \).
The coupling between vortices and $B_{\mu\nu}$ would be then $\varepsilon^{\mu\nu\rho\sigma} \partial_{\mu}\tilde{\theta}\partial_{\nu}B_{\rho\sigma}/2$. This can be written better by introducing the topologically conserved antisymmetric tensor current for vortices,

$$K^{\mu\nu} = \varepsilon^{\mu\nu\rho\sigma} \partial_{\rho}\tilde{\theta}$$

$$= 2\pi \sum_{a} \int d\sigma (\dot{R}^{\mu}_{a} R_{a}^\nu - R_{a}^{\mu} \dot{R}^{\nu}_{a}) \delta^{3}(x - R_{a}(t, \sigma))$$

(14)

where $R_{a}^{\mu} = (t, R_{a}(t, \sigma))$ is the position for the a-th vortex. This current is conserved identically, $\partial_{\mu}K^{\mu\nu} = 0$ and leads to a conservation charge $\int_{S} d\Sigma^{i} \cdot K^{0i}$ on any surface $S$ without boundary, which counts the net number of vortices piercing the surface. Since the charge density is near constant outside the vortex string core region of the scale $\xi$, the current conservation becomes $\partial^{2}_{i}\theta = 0$ in that outside region. Then, we can use Eq.(14) to get the Biot-Savart law

$$\partial_{i}\theta = \frac{1}{2} \sum_{a} \oint dR_{a} \times \frac{(x - R_{a})}{|x - R_{a}|^{3}}$$

(15)

We emphasize that the solution is valid only outside the vortex core region. From this one can get $\text{Arg } \Phi$ near a vortex and can calculate the induced speed $u$. It is well known [9] that the above expression for a single vortex can be approximated by

$$\partial_{i}\theta(t, \zeta^{A}, \sigma) = \frac{R' \times \zeta^{A} N_{A}}{|R'|} B_{\zeta}^{B} + \frac{1}{2} \frac{R' \times R''}{|R'|^{3}} \ln(L/\xi)$$

(16)

near a vortex position. Here the large length cut off is given as $L$ and the current is calculated at the point $\zeta^{A} \zeta^{A} \approx \xi^{2}$. The first term in the right-hand side would be the current around the vortex string. The second term is $\partial_{i}\text{Arg } \Phi$ and is fixed by the local self-induced current. The second term is in general much bigger than the $R' \times R''$ term in Eq.(10). Thus Eq.(16) gives us the $\Phi$ field outside the vortex core region. Then Eq.(10) allows us to calculate the vortex velocity. This is how the cutoff $\xi$ comes in the calcuration naturally. If we have considered magnetic flux vortices in an extremal type II superconductor, the natural value for the cutoff $L$ would be the penetration length $\lambda$ which is much longer than the coherence length $\xi$. 

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The field strength of $B_{\mu\nu}$ is $H_{\nu\rho\sigma} = \partial_\nu B_{\rho\sigma} + \partial_\rho B_{\sigma\nu} + \partial_\sigma B_{\nu\rho}$. The dual Lagrangian becomes

$$\mathcal{L}_D = \frac{mH_{0ij}^2}{4H_{123}} - \frac{\nabla H_{123}}{8mH_{123}} - \frac{g}{2}(H_{123} - \rho_0)^2 + \frac{1}{2}B_{\mu\nu}K^{\mu\nu}$$

(17)

The measure for the path integral changes from $[d\rho d\theta]$ to $[H_{123}^{-3/2} dB_{\mu\nu} d\mathbf{R}_a]$. There will be also the gauge fixing term for the antisymmetric tensor field. (If there is a gauge coupling, we can integrate over the gauge field $A_\mu$, getting additional terms in the dual Lagrangian.) The above expression is exact. Due to the second term in Eq.(17), the energy of a vortex does not diverge at the vortex position only if the charge density vanishes at the position at the vortex. Naively, there seems that a vortex equation would arise from the above dual Lagrangian under the variation of the vortex position $\delta \mathbf{R}$. However, the coefficient $H_{\mu\nu\rho}$ of the variation vanishes at the vortex position and so there is no additional equation. The reason behind is of course that the dynamics of vortices is completely determined by the field dynamics around vortices. Classically, the antisymmetric tensor field is related to the original current by

$$J^\mu = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} H_{\nu\rho\sigma}$$

(18)

In the absence of vortices, the energy of the system would be minimum when the expectation of $H_{123}$ is the background charge density $\rho_0$. The small fluctuations around this background are sound waves and described by a simple Lagrangian

$$\mathcal{L}_s = \frac{m}{4\rho_0}H_{0ij}^2 - \frac{g}{2}H_{123}^2$$

(19)

in the long wave length limit. Here we have shifted the antisymmetric tensor field

When we consider the small perturbations on the field configuration of moving vortices, we can regard the charge density $H_{123} = \rho$ to be almost constant outside the vortex string core region of the transverse radius $\xi$. With the cutoff near vortices of length scale $\xi$, the dual Lagrangian can be written as

$$\mathcal{L}_D = \frac{m}{4\rho_0}H_{0ij}^2 - \frac{g}{2}(H_{123} - \rho_0)^2 - \frac{1}{2}B_{\mu\nu}K^{\mu\nu}$$

(20)

Here we neglect the terms higher order in spatial derivatives or density fluctuations.
This is the effective Lagrangian for vortex strings and the sound wave. We can go further by choosing the gauge for $B_{\mu\nu}$ so that $\partial^\rho B_{\rho i} = 0$ and $\partial_i B_{0i} = 0$. First, there will be decoupling between the $B_{0i}$ and $B_{ij}$ in the Lagrangian. We can now integrate over the auxiliary variables $B_{0i}$. The variation of $B_{0i}$ leads to the Gauss law

$$\frac{m}{\rho_0} \partial_j^2 B_{0i} = 2\pi \sum_a dR^i_a \delta(x - R_a)$$

(21)

We substitute $B_{0i}$ satisfying Eq.(21) to Eq.(20). We shift the spatial components of the antisymmetric tensor field as

$$B_{ij} = \frac{1}{3} \rho_0 \epsilon_{ijk} x^k + \sqrt{\frac{\rho_0}{m}} \epsilon_{ijk} C^k$$

(22)

where $C$ satisfies the gauge $\nabla \times C = 0$ and can be regarded as the fluctuation around the uniform charge density.

After that we end up with the effective Lagrangian for vortices and sound waves with a cut off scale $\xi$

$$L_{\text{eff}} = \frac{2\pi \rho_0}{3} \sum_a \oint dR_a \cdot R_a \times \dot{R}_a - \sum_{a,b} \frac{\pi \rho_0}{m} \int \int \frac{dR_a \cdot dR_b}{|R_a - R_b|}$$

$$+ \int d^3 x \left\{ \frac{1}{2} \dot{C}^2 - \frac{v_s^2}{2} (\nabla \cdot C)^2 \right\} + 2\pi \sqrt{\frac{\rho_0}{m}} \sum_a \oint dR_a \cdot C(t, R_a) \times \dot{R}_a$$

(23)

where the gauge fixing condition $\nabla \times C = 0$ is assumed. The first two terms has been studied before [6]. The above action is the effective Lagrangian for vortices interacting with sound waves. It has been known that the vortex equation has many interesting time dependent solutions like, Kelvin waves, Hasimoto solitons, and smoke rings. Perhaps, the above action will allow us to calculate the emission rate of sound waves by these solutions. This effective action might be also useful in understanding the superfluid turbulence. We can quantized the reduced effective action by the path integral formalism or the canonical formalism. That could allow us to calculate the energy levels of the excited vortex string loops and the transition amplitudes between the energy levels by the phonon emission.
Since superfluid is compressible, there could be supersonic shock waves which interact with vortex strings. In addition, vortex strings could move faster than the sound speed, still being nonrelativistic. It would be interesting to understand this supersonic phenomena. The dual action (17) might be useful there.

Finally, it would be interesting to get an effective action for magnetic flux vortex strings in extreme type II superconductors, interacting with vector gauge bosons of small mass. This effective action probably approaches the effective action (23) in the limit where the electromagnetic coupling constant $e$ goes to zero.

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