Fractional photon-assisted tunneling for Bose-Einstein condensates in a double well

Niklas Teichmann, Martin Esmann, and Christoph Weiss
Institut für Physik, Carl von Ossietzky Universität, D-26111 Oldenburg, Germany
(Dated: May 19, 2009)

Half-integer photon-resonances in a periodically shaken double well are investigated on the level of the $N$-particle quantum dynamics. Contrary to non-linear mean-field equations, the linear $N$-particle Schrödinger equation does not contain any non-linearity which could be the origin of such resonances. Nevertheless, analytic calculations on the $N$-particle level explain why such resonances can be observed even for particle numbers as low as $N = 2$. These calculations also demonstrate why fractional photon resonances are not restricted to half-integer values.

PACS numbers: 03.75.Lm, 74.50.+r, 03.65.Xp
Keywords: double-well potential, photon-assisted tunneling, Bose-Einstein condensation

I. INTRODUCTION

Tunneling control of ultra-cold atoms via time-periodic shaking [11, 2, 3, 4] of potentials is currently established as an experimental method both on the single particle level [5] and on the level of Bose-Einstein condensates (BECs) [6]. An interesting effect is an analog of photon-assisted tunneling in periodically shaken systems of ultra-cold atoms. It was predicted theoretically both for the case that the driving frequency matches the potential difference between neighboring wells [3, 7] and for the case that the driving frequency is resonant with the interaction energy [5]. The $n$-photon resonances essentially are a single-particle effect which survives interactions; one- and two-photon resonances have been observed experimentally for BECs in periodically shaken lattices [9]. The “photons” are time-periodic potential modulations in the kilo-Hertz-regime.

However, photon-assisted tunneling is not restricted to integer photon resonances. Also half-integer Shapiro-like [10] resonances have been predicted numerically both on the mean-field (Gross-Pitaevskii) level and on the level of the multi-particle quantum dynamics (down to $N = 2$ particles) [3]. While the occurrence of higher or lower harmonics in non-linear equations is easy to understand qualitatively, it is not clear a priori how these resonances should occur in the linear $N$-particle Schrödinger equation. Thus, analytic calculations which can explain the occurrence of such resonances within the linear quantum dynamics will explain how effective non-linearities can arise from linear dynamics even for small particle numbers. Realistic experimental values for the number of atoms in a double well can be of the order of 1000 atoms [11] for BECs and down to less than 6 atoms [12] for few-atoms experiments.

Often, Floquet-states [13] help to understand the physics of BECs in periodically driven systems [14, 15, 16, 17]. The focus of the present paper, lies on a different approach: analytic calculations on the $N$-particle level developed in Ref. [18] (cf. Ref. [19]). By assuming the experimentally realistic initial condition of all particles being in one well [11], the calculations are done analogously to the time-dependent perturbation theory.

The paper is organized as follows: after introducing the two-mode model for a BEC in a double well (Sec. II), we develop the technique to calculate half-integer resonances in Sec. III. A crucial test is to show that the analytic result vanishes in the limit of non-interacting particles (Sec. IV). Other fractional resonances are discussed in Sec. V.

II. THE MODEL: A BEC IN A DOUBLE WELL

Bose-Einstein condensates in double-well potentials are interesting both experimentally and theoretically [11, 20, 21, 22, 23, 24, 25, 26]. In order to describe a BEC in a double well, we use a model originally developed in nuclear physics [27]: a multi-particle Hamiltonian in two-mode approximation [28].

$$\hat{H} = -\frac{\hbar \Omega}{2} \left( \hat{c}_1 \hat{c}_2 + \hat{c}_1^\dagger \hat{c}_2^\dagger \right) + \hbar \kappa \left( \hat{c}_1^\dagger \hat{c}_1 \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_2 \hat{c}_1 + \hat{c}_1^\dagger \hat{c}_2^\dagger \hat{c}_1 \right) + \hbar \left( \mu_0 + \mu_1 \sin(\omega t) \right) \left( \hat{c}_2^\dagger \hat{c}_2 - \hat{c}_1^\dagger \hat{c}_1 \right), \quad (1)$$

where the operator $\hat{c}_j^{(1)}$ annihilates (creates) a boson in well $j$; $\hbar \Omega$ is the tunneling splitting, $\hbar \mu_0$ is the tilt between well 1 and well 2 and $\hbar \mu_1$ is the driving amplitude. The interaction between a pair of particles in the same well is denoted by $2\hbar \kappa$.

The Gross-Pitaevskii dynamics can be mapped to that of a nonrigid pendulum [20]. Including the term describing the periodic shaking, the Hamilton function is given by:

$$H_{mf} = \frac{N \kappa}{\Omega} z^2 - \sqrt{1 - z^2} \cos(\phi) - 2 z \left( \frac{\mu_0}{\Omega} + \frac{\mu_1}{\Omega} \sin \left( \frac{\pi}{\tau} \right) \right), \quad \tau = t \Omega, \quad (2)$$

where $\phi$ and $z$ are canonically conjugate variables. The quantity $z/2$ is the population imbalance with $z/2 = 0.5$. 

*Electronic address: christoph.weiss@uni-oldenburg.de
(z/2 = -0.5) referring to the situation with all particles in well 1 (well 2). The corresponding observable on the N-particle level is given by

$$J_z(t) = \frac{\langle \Psi(t) | \hat{c}_1^\dagger \hat{c}_1 - \hat{c}_2^\dagger \hat{c}_2 | \Psi(t) \rangle}{2N}. \tag{3}$$

For integer photon-assisted tunneling, the potential difference between both wells, 2ℏµ0, has to be bridged by an integer number of photons:

$$2\hbar \mu_0 = n \hbar \omega, \quad n = 1, 2, \ldots \tag{4}$$

The 1/2-integer resonance occurs for

$$2\hbar \mu_0 = \frac{1}{2} \hbar \omega; \tag{5}$$

for an interacting bose gas, these resonances are furthermore shifted [3].

For some parameter regimes (especially for interactions comparable to the onset of the self-trapping transition [11]), the differences between mean-field (Gross-Pitaevskii) dynamics and the N-particle quantum dynamics can be quite remarkable [29]. However, when concentrating on the (experimentally measurable [11]) time-averaged population imbalance,

$$\langle J_z \rangle_T/N = \frac{1}{T} \int_0^T \frac{J_z(t)}{N} dt, \tag{6}$$

for low interactions, the qualitative agreement between mean-field and N-particle dynamics for the occurrence of both integer and half-integer photon-assisted tunneling is excellent [3].

Photon-assisted tunneling is clearly visible in the experimentally measurable time-averaged population imbalance [6]. Figure 1 shows integer resonances [4], namely the one-photon peak with ω ≈ 3Ω and the two-photon peak at ω ≈ 1.5Ω. Furthermore, there are pronounced fractional-integer resonances at ω ≈ 6Ω, ω ≈ 2Ω and ω ≈ 1.2Ω corresponding to the 1/2, 3/2 and 5/2-photon peaks. While some of the resonances disappear [4] for specific choices of the driving amplitude, the initial phase of the periodic driving (cf. [4]) does not influence the occurrence of resonances in the situation investigated in this manuscript.

Figure 2 shows that it is not essential to start with all particles in one well in order to observe photon-assisted tunneling. Both for the ground-state of the untilted, undriven system (for which the initial population imbalance is zero) and for the ground-state of the tilted system with an initial population imbalance of ≈ 0.467, the main resonances of Fig. 1 where all particles were initially in well 1, can easily be identified.

Figure 3 displays the half-integer resonance for N = 2 particles. Contrary to what was observed for both larger particle numbers and for mean-field, the position of the 1/2 photon resonance does not shift with increasing energy. A first test of our analytic calculations towards the end of the next section will thus be to explain this feature.

III. ANALYTIC CALCULATIONS

In order to analytically describe the time-evolution of the interacting system, the Fock basis |ν⟩ ≡ |N − ν, ν⟩ is used. The label ν = 0...N refers to a state with N − ν particles in well 1, and ν particles in well 2. The Hamiltonian [1] now is the sum of two (N + 1) × (N + 1)-matrices,

$$H = H_0(t) + H_1. \tag{7}$$
potential as a function of both driving frequency $\omega$ was derived [18] which is mathematically equivalent to the ansatz turned out to be useful [18].

Within this framework, a set of differential equations was derived [18] which is mathematically equivalent to the $N$-particle Schrödinger equation governed by the Hamiltonian [11]:

$$i\hbar \frac{\partial}{\partial t} \langle \nu \rangle(t) = \langle \nu | H_1 | \nu + 1 \rangle \langle \nu | H_0(t) | \nu + 1 \rangle + \langle \nu | H_1 | \nu - 1 \rangle \langle \nu | H_0(t) | \nu - 1 \rangle.$$  (10)

In Eq. (10), the notation $a_{\nu}(t) \equiv a_{N+1}(t) = 0$, was used; the phase factors are given by:

$$h_{\nu}(t) = \exp \left(i \left[2(N - 1 - 2\nu)\kappa t + 2\mu_0 t - 2\mu_1 \cos(\omega t)/\omega \right] \right).$$  (11)

with $-\cos(\omega t)/\omega = \int_{t'}^{t} \sin(\omega t') dt'$. To simplify the expression for subsequent integrals, one can use the expansion in terms of Bessel functions [30]

$$e^{i\sigma(\omega t)} = \sum_{k=-\infty}^{\infty} J_k(z)e^{ik\omega t}.$$  (12)

Equation (10) furthermore needs

$$\langle \nu | H_1 | n \rangle = -\frac{\hbar\Omega}{2} \delta_{\nu,n+1} \sqrt{N-n} \sqrt{n+1} - \frac{\hbar\Omega}{2} \delta_{\nu,n-1} \sqrt{N-n} + 1 \sqrt{n},$$  (13)

where $\delta_{n,m}$ is the Kronecker delta (which is zero except for $n = m$ where $\delta_{n,n} = 1$). The idea is to proceed along the lines of time-dependent perturbation theory [31]. Starting with a typical experimental initial condition such that all particles are in the first well [11], one has in zeroth order perturbation theory:

$$a_{0}^{(0)}(t) = 1, \quad a_{1}^{(0)}(t) = a_{2}^{(0)}(t) = \ldots = 0,$$  (14)

where $a_{\nu} = \sum_{k=0}^{\infty} a_{\nu}^{(k)}$. In first order perturbation theory, one gets:

$$a_{\nu}^{(1)}(t) = 0$$  (15)

if $\nu \neq 1$ and

$$a_{1}^{(1)}(t) = i \frac{\Omega}{2} \sqrt{N} \int_{t'}^{t} h_{0}(t) a_{0}^{(0)}(t').$$  (16)

Using Eqs. (11) and (12) one thus has

$$a_{1}^{(1)}(t) = i \frac{\Omega}{2} \sqrt{N} \sum_{k=-\infty}^{\infty} i^{k} J_{k}(2\mu_{1}/\omega) \int_{0}^{t} \exp(i\sigma_{k}t') dt' \quad$$  (17)

with

$$\sigma_{k} \equiv k\omega - 2\mu_{0} - 2(N-1)\kappa.$$  (18)

Therefore, after solving the integral, Eq. (17) is a sum of time-periodic functions except for the special case with $\sigma_{k} = 0$ which recovers the integer photon resonances of Eq. (4) investigated in Refs. [3, 9]. While for a double well as in Ref. [3], the population imbalance is ideal to investigate photon-assisted tunneling, the experiment [9]...
was performed in an optical lattice. The signatures of photon-assisted tunneling were seen in the width of the BEC after expansion in the shaken lattice. Surprisingly, a \(J_n^2\) dependence was measured. While this might be interpreted as being an indication for transition from ballistic to diffusive transport \([9]\), the present experiments cannot exclude other explanations. The \(J_n^2\)-dependence could either be an interaction-induced effect \([32]\) or the result of an effective average over the precise instant within the cycle at which the current is measured \([33]\).

As the aim of the present paper is to understand the fractional photon peaks like the interaction induced half-integer resonances of Ref. \([3]\), we can discard the integer-photon resonances characterized via \(\sigma_k = 0\) \([33]\) and thus write

\[
a^{(1)}_1(t) = i\frac{\Omega}{2} \sqrt{N} \sum_{k=-\infty}^{\infty} \frac{i^k J_k(2\mu_1/\omega)}{i\sigma_k} e^{i\omega_k^t} - 1
\]

(19)

In second order perturbation theory one has (see the appendix \(A\)):

\[
a^{(2)}_2(t) = \left(\frac{\Omega}{2}\right)^2 \sqrt{N}\sqrt{N-1}\frac{1}{\sqrt{2}}
\]

\[
\times \sum_{k=-\infty}^{\infty} \sum_{\ell=-\infty}^{\infty} i^{k+\ell} J_k(2\mu_1/\omega) J_{\ell}(2\mu_1/\omega)
\]

\[
\times \int_0^\infty \frac{\exp(i\sigma_k t') - 1}{i\sigma_k} \exp(i\sigma_\ell t') dt'
\]

with

\[
\sigma_\ell = \ell\omega - 2\mu_0 - 2(N-3)\kappa.
\]

Again, \(\sigma_\ell = 0\) can be discarded because it corresponds to integer photon resonances. However, if \(\sigma_\ell + \sigma_k = 0\) then \(a^{(2)}_2\) does have parts which increase linearly in time. In order to see that this indeed corresponds to a half-integer resonance, we choose \(N = 2\) and \(\omega/2 = 2\mu_0\). This implies \(\sigma_k = k\omega - \omega/2 - 2\kappa\) and \(\sigma_\ell = \ell\omega - \omega/2 + 2\kappa\) and the condition

\[
\sigma_\ell + \sigma_k = 0
\]

(22)

thus becomes independent of the interaction; it results in the simple equation

\[
k + \ell = 1.
\]

(23)

The above reasoning explains why we observe no shift of the resonance with increasing interaction in the numerics displayed in Fig. \(3\). The amplitude to find both particles in well two is given by (appendix \(A\)):

\[
a^{(2)}_2(t) = (a^{(2)}_2(t))_{oscil}
\]

(24)

\[
-\frac{\Omega^2}{2} \sum_{k=-\infty}^{\infty} J_k(2\mu_1/\omega) J_{1-k}(2\mu_1/\omega) \frac{t}{\sigma_k}
\]

\[
-\frac{\Omega^2}{2} \sum_{k=-\infty}^{\infty} J_k(2\mu_1/\omega) J_{1-k}(2\mu_1/\omega) \frac{e^{-i\sigma_k^t} - 1}{i\sigma_k},
\]

where the expression \((a^{(2)}_2(t))_{oscil}\) contains oscillatory terms which can be found in Eq. \((A.7)\). The convergence of this sum is ensured both by the scaling of \(\sigma_k\) on \(k\) and the behavior of Bessel functions with increasing \(k\) \([30]\).

\[
J_k(z) \sim \frac{1}{\sqrt{2\pi k}} \left(\frac{z \exp(1)}{2k}\right)^k, \quad k \to \infty
\]

(25)

combined with the fact that \(J_{-k}(x) = (-1)^k J_k(x)\) (for integer \(k\)). Figure \(4\) shows good qualitative agreement between the analytic and numeric calculations for the time-averaged probability that both particles, which initially have been in the first well, have tunneled to the second well. Already perturbation theory in the first order, in which the half-integer resonance becomes visible, correctly describes the occurrence of maxima and minima in the probability for both particles to occupy the second well.

\section{IV. Half-integer resonances disappear in the limit of low interactions}

Despite the agreement displayed in Fig. \(4\) at the first glance Eq. \((24)\) seems to contain a flaw: numerically, we observe that the half-integer resonance disappears for zero interaction. However, there seems to be a sum of non-zero terms proportional to \(t\) even for \(\kappa = 0\). As it is not obvious that these terms cancel, the next step will be to demonstrate that \(a^{(2)}_2\) indeed approaches zero for vanishing interaction.

As shown in the appendix, the terms proportional to \(t\) in \(a^{(2)}_2\) are due to situations such that Eq. \((22)\) is fulfilled. In the limit \(\kappa \to 0\) this results again in the condition \((23)\), independent of the particle number. The part of \(a^{(2)}_2\)
which increases linearly in time is thus proportional to

\[ A \equiv \sum_{k=-\infty}^{\infty} J_k(2\mu_1/\omega)J_{1-k}(2\mu_1/\omega) \frac{1}{\sigma_k}. \]  

(26)

Dividing the sum into two parts \( \left( \sum_{k=1}^{\infty} \cdots + \sum_{\ell=-\infty}^{0} \cdots \right) \) and then setting \( 1 - \ell = k \), one obtains

\[ A = \sum_{k=1}^{\infty} J_k(2\mu_1/\omega)J_{1-k}(2\mu_1/\omega) \left( \frac{1}{\sigma_k} + \frac{1}{\sigma_{1-k}} \right). \]  

(27)

In the limit \( \kappa \to 0 \), the position of the half-integer resonance approaches the value for \( N = 2 \) particles. Therefore, one has \( \sigma_k = k\omega - \omega/2 \) and thus \( \sigma_{1-k} = -\sigma_k \) which implies

\[ A = 0. \]  

(28)

Thus, in agreement with the numerics, the half-integer photon peak disappears with vanishing interactions.

V. FRACTIONAL INTEGER RESONANCES

Fractional integer resonances are not, however, restricted to the half-integer resonances investigated numerically in Ref. [3] and analytically in Sec. IV. For \( N = 3 \) particles and a driving frequency such that \( \omega/3 = 2\mu_0 \), the condition

\[ \sigma_k + \tilde{\sigma}_\ell + \tilde{\sigma}_m = 0 \]  

(29)

with

\[ \tilde{\sigma}_m = m\omega - 2\mu_0 - 2(N - 5)\kappa \]  

(30)

(throughout this section: \( N = 3 \)) is fulfilled for

\[ k + \ell + m = 1. \]  

(31)

The amplitude to find three particles in well 2 again contains oscillatory terms, the term which becomes the leading-order term for large \( t \) can be obtained by a calculation analogously to the half-integer resonance in appendix A

\[ \left( d_3^{(3)} \right)_{\text{linear}} = -\frac{3\Omega^3}{4} \sum_k \sum_\ell J_k(2\mu_1/\omega) \]  

\[ \times J_\ell(2\mu_1/\omega)J_{1-\ell-k}(2\mu_1/\omega) \frac{t}{\sigma_k(\sigma_k + \tilde{\sigma}_\ell)}. \]  

(32)

This one-third photon resonance can indeed be observed in the numerics (see Fig. 5). As this resonance only occurs in third order perturbation theory (rather than second order for the half-integer resonances), the amplitudes would be rather small for interactions as in Fig. 3. However, choosing an also realistic value of \( N\kappa/\Omega = 1.5 \) leads to a time-averaged population imbalance with a peak-height of the same order of magnitude as in Fig. 3.

In a similar manner, smaller fractions could be treated in higher order perturbation theory. As the resonances thus are a higher order effect, they will tend to decrease.

FIG. 5: The plot illustrates the 1/3-resonance for \( N = 3 \) particles and interaction parameter \( N\kappa/\Omega = 1.5 \). Initially, all three particles were in the lower well; the static tilt is again given by \( 2\mu_0/\Omega = 3 \). The time averaged probability (averaged over time \( T\Omega = 100 \)) to find all particles in the upper well as a function of the driving frequency \( \omega/\Omega \) and the driving amplitude \( 2\mu_1/\omega \) has a clear peak at \( \omega \approx 9\Omega \).

VI. CONCLUSION

Contrary to the integer-photon peaks [3], fractional-integer photon peaks cannot be explained by simply replacing the time-dependent Hamiltonian by a time-independent Hamiltonian with renormalized tunneling frequencies. As half-integer resonances already appear for two particles in a double well, this experimentally relevant case [12] was investigated both numerically and analytically. The perturbation calculations can explain for which parameters the non-integer resonances occur. As the fractional-integer resonances are only visible for finite interactions between the particles, they allow to investigate beyond single-particle effects for very small particle numbers. Experiments similar to Ref. [12] could thus verify fractional-integer peaks in photon assisted tunneling and thus help to understand the emergence of effects similar to the non-linearities of a mean-field approach well below the limit \( N \to \infty \).

Acknowledgments

We thank M. Holthaus for his continuous support. CW thanks A. Eckardt and A. L. Fetter for insightful discussions; NT and ME acknowledge funding by the Studienstiftung des deutschen Volkes.
APPENDIX A: SECOND ORDER PERTURBATION THEORY

When solving the integral

\[ I_{k,\ell} \equiv \int_0^t \frac{1}{i\sigma_k} \left[ \exp \left( i(\sigma_k + \tilde{\sigma}_\ell) t' \right) - \exp \left( i\tilde{\sigma}_\ell t' \right) \right] dt' \]  

(A1)

in Eq. (20), one can again assume \( \sigma_k \neq 0 \) and \( \tilde{\sigma}_\ell \neq 0 \) as \( \sigma_k = 0 \) and \( \tilde{\sigma}_\ell = 0 \) would correspond to the inter-photon resonances discarded here. It then remains to distinguish cases with

\[ \sigma_k + \tilde{\sigma}_\ell = 0 , \]  

(A2)

which turn out to be the origin of the half-integer resonance, from those for which this equation is not fulfilled. If Eq. (A2) is fulfilled, one has

\[ I_{k,\ell} = \frac{1}{i\sigma_k} \left[ t + \frac{1}{i\sigma_k} \left( \exp(-i\sigma_k t) - 1 \right) \right] \]  

(A3)

otherwise

\[ I_{k,\ell} = \frac{1}{i\sigma_k} \left[ \frac{\exp[i(\sigma_k + \tilde{\sigma}_\ell) t] - 1}{i(\sigma_k + \tilde{\sigma}_\ell)} - \frac{\exp(i\tilde{\sigma}_\ell t) - 1}{i\tilde{\sigma}_\ell} \right] . \]  

(A4)

Collecting all terms given by Eq. (A3), one has the leading-order contribution:

\[ \left( \frac{a_2^{(2)}(t)}{\sigma_0} \right)_{\text{leading-order}} = \]  

(A5)

\[ -\frac{\Omega^2}{2} \sum_{k=-\infty}^{\infty} J_k(x) J_{1-k}(x) \frac{t}{\sigma_k} \frac{\exp(-i\sigma_k t) - 1}{i\sigma_k} , \]

with

\[ x \equiv 2\mu_1/\omega , \]  

(A6)

and an oscillatory part

\[ \left( \frac{a_2^{(2)}(t)}{\sigma_0} \right)_{\text{oscil}} = \]  

(A7)

\[ -\frac{\Omega^2}{2} \sum_{k=-\infty}^{\infty} \sum_{\ell=-1}^{\infty} \frac{J_k(x) J_{1-k}(x) t^{k-1}}{\sigma_k} \frac{\exp(i(\sigma_k + \tilde{\sigma}_\ell) t) - 1}{(i(\sigma_k + \tilde{\sigma}_\ell))} \frac{\exp(i\tilde{\sigma}_\ell t) - 1}{i\tilde{\sigma}_\ell} . \]

While Eq. (A5) includes the leading-order behavior for large times and most parameters, it vanishes in the limit \( \mu_1 \rightarrow 0 \). Thus to evaluate the analytic formula with the help of a computer algebra program, we include the only non-vanishing term for \( \mu_1 = 0 \) to obtain the data displayed in Fig. 4.

\[ \left( \frac{a_2^{(2)}(t)}{\sigma_0} \right)_{\text{approx}} = \left( \frac{a_2^{(2)}(t)}{\sigma_0} \right)_{\text{leading-order}} \]  

(A8)

\[ -\frac{\Omega^2}{2} \sum_{k=-\infty}^{\infty} \frac{J_k(x) J_{1-k}(x) t^{k-1}}{\sigma_k} \frac{\exp(i(\sigma_k + \tilde{\sigma}_0) t) - 1}{(i(\sigma_k + \tilde{\sigma}_0))} \frac{\exp(i\tilde{\sigma}_0 t) - 1}{i\tilde{\sigma}_0} . \]

[1] F. Grossmann, T. Dittrich, P. Jung, and P. Hänggi, Phys. Rev. Lett. 67, 516 (1991).
[2] M. Holthaus, Phys. Rev. Lett. 69, 1596 (1992).
[3] A. Eckardt, T. Jinasundera, C. Weiss, and M. Holthaus, Phys. Rev. Lett. 95, 200401 (2005).
[4] C. E. Creffield and F. Sols, Phys. Rev. Lett. 100, 250402 (2008).
[5] E. Kierig, U. Schnorrberger, A. Schietinger, J. Tomkovic, and M. K. Oberthaler, Phys. Rev. Lett. 100, 190405 (2008).
[6] A. Zenesini, H. Lignier, D. Ciampini, O. Morsch, and E. Arimondo, Phys. Rev. Lett. 102, 100403 (2009).
[7] S. Kohler and F. Sols, New J. Phys. 5, 94 (2003).
[8] C. E. Creffield and T. S. Monteiro, Phys. Rev. Lett. 96, 210403 (2006).
[9] C. Sias, H. Lignier, Y. P. Singh, A. Zenesini, D. Ciampini, O. Morsch, and E. Arimondo, Phys. Rev. Lett. 100, 040404 (2008).
[10] S. Shapiro, Phys. Rev. Lett. 11, 80 (1963).
[11] M. Albiez, R. Gati, J. Fölling, S. Hunsmann, M. Cristiani, and M. K. Oberthaler, Phys. Rev. Lett. 95, 010402 (2005).
[12] P. Cheinet, S. Trotzky, M. Feld, U. Schnorrberger, M. Moreno-Cardoner, S. Fölling, and I. Bloch, Phys. Rev. Lett. 101, 090404 (2008).
[13] J. H. Shirley, Phys. Rev. 138, B979 (1965).
[14] T. Jinasundera, C. Weiss, and M. Holthaus, Chem. Phys. 322, 118 (2006).
[15] A. Eckardt and M. Holthaus, Phys. Rev. Lett. 101, 245302 (2008).
[16] M. P. Strzys, E. M. Graefe, and H. J. Korsch, New J. Phys. 10, 013024 (2008).
[17] W. Hai, C. Lee, and Q. Zhu, J. Phys. B 41, 095301 (2008).
[18] C. Weiss and T. Jinasundera, Phys. Rev. A 72, 053626 (2005).
[19] G. Kalosakas, A. R. Bishop, and V. M. Kenkre, Phys. Rev. A 68, 023602 (2003).
[20] A. Smerzi, S. Fantoni, S. Giovannazzi, and S. R. Shenoy, Phys. Rev. Lett. 79, 4950 (1997).
For \( \sigma_k = 0 \) the amplitude \( a_1^{(1)} \) would contain a part which increases linearly in time. This both signifies the breakdown of our perturbation theory (for too large times) and the onset of photon-assisted tunneling.