Mappings by the complex exponential function

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Abstract. In mathematics, engineering, and physics, some problems can be solved through complex functions; in many cases, with geometric inconveniences or complicated domains. Conformal mappings are essential to transform a complicated analytic domain onto a simple domain. Physical approaches to visualization of complex functions can be used to represent conformal mappings, here we use the transformation of regions of the complex plane. This paper provides a graphical overview of the transformation of a set of regions by the complex exponential function.

1. Introduction
A complex function \( f \) is a rule that assigns to \( z \in \mathbb{C} \) an image complex number \( w = f(z) \). Let \( f \) be a function defined in \( \zeta \in \mathbb{C} \), \( f \) is holomorphic or analytic at \( \zeta \) if it is differentiable in a neighborhood of \( \zeta \). \( f \) is a holomorphic function, or analytic, in \( D \) if it is holomorphic at each point at \( D \), cf. [1].

Complex numbers can be plotted on the complex plane, also called Argand diagram [2], where the complex number \( z = a + bi \) is associated with the point \( (a, b) \). Thus, \( \text{Re}(z) = a \) is associated with points on the \( x \)-axis and \( \text{Im}(z) = b \) correspond to points on the \( y \)-axis; in this context, \( x \)-axis and \( y \)-axis are called real and imaginary axis respectively.

Complex functions can be considered as mappings, or transformations, between subsets of two complex planes which transplant objects from one plane to other [3]. A conformal mapping, also called a conformal transformation, is a transformation \( w = f(z) \) that preserves angles; these maps are given by holomorphic functions with nowhere vanishing derivatives [4]. The archetype of conformal maps are the Mobius transformations, they are given by the Equation (1).

\[
T(z) = \frac{az + b}{cz + d},
\]

with \( a, b, c, d \in \mathbb{C} \) and \( ad - bc \neq 0 \). Mapping properties of Mobius transformations were studied in [5]. The complex exponential \( f(z) = e^z \) can be defined through the property \( \frac{df}{dz} = f(z) \) cf. [6], or as the extension of the usual exponential function satisfying the properties \( f(z_1 + z_2) = f(z_1)f(z_2) \) with \( z_1, z_2 \in \mathbb{C} \) and \( f(x) = e^x \) for all \( x \in \mathbb{R} \) cf. [7], or by power series [8], etc. Here the complex exponential function is defined by the Equation (2).

\[
f(z) = e^z = e^{x+iy} = e^xe^{iy},
\]
where \( e^{iy} = \cos(y) + i \sin(y) \), Equation (2) is called Euler’s formula. In addition, the complex exponential function \( f(z) = e^z \) is an entire function [9], i.e. it is an holomorphic function in any considered domain, \( f'(z) = e^z \) and \( f'(z) \neq 0 \) for each \( z \in \mathbb{C} \). Thus, \( f(z) = e^z \) is a conformal map. Applications of conformal mappings in physics can be found in [10].

Physical concepts are useful to represent complex functions, the usual methods is a transformation of a domain in the complex plane. Therefore, it is necessary to consider special functions, in order to analyze the transformation of domains under complex functions. However, an interesting physical approach is to visualize complex functions as vector fields [11]. In complex analysis, the vast majority of authors present a few examples of mappings because mappings require two complex planes.

In this paper, we consider some two-dimensional graphics, henceforth called domains, to plot their images under the complex exponential function \( f(z) = e^z \) into the complex plane. In section 2, lines and domains obtained as combinations of lines are mapped; in section 3, the graph of functions in the form \( y = az^n \) and \( y = \frac{1}{z^n} \) are considered as domains; in section 4, we transform the graph of ellipses by the complex exponential function.

2. Lines and rectangular domains

For the horizontal line segment whose endpoints are \((x_1, y_1)\) and \((x_2, y_2)\), the image under \( f(z) = e^z \) is \( e^{x_1} \leq \rho \leq e^{x_2} \) with \( \phi_n = y_1 \), i.e., if \( w \) is a point in the image it is described by Equation (3).

\[
w = e^x(\cos(y_1) + i \sin(y_1)).
\]

Therefore, the image describes a ray of slope \( \tan(\phi) \) excluding the origin. On the other hand, for the vertical line segment whose endpoints are \((x_1, y_1)\) and \((x_1, y_2)\) the image under \( f(z) = e^z \) is \( \rho_v = e^{x_1} \) with \( y_1 \leq \phi_v \leq y_2 \).

In Figure 1, horizontal lines are mapped into horizontal lines due to \( \phi = 0 \) and \( \phi = \pi \). The graphical representation is a part of the circumference with radius \( e^{x_1} \) centered at the origin, if \( y_2 - y_1 < 2\pi \) the mapping is injective. Considering a rectangular grid as domain we may observe the most general case of transformation of horizontal and vertical lines by the complex exponential function. In Figure 2(a) it is easy to check that horizontal and vertical lines are orthogonal; since \( f(z) = e^z \) is a conformal mapping, their images still being orthogonal as we can verify in Figure 2(b).

![Figure 1](image1.png)

**Figure 1.** (a) Rectangle \([-1, 1] \times [0, \pi]\); (b) mapping of the rectangle.

![Figure 2](image2.png)

**Figure 2.** (a) Grid domain; (b) map of a grid domain.
Considering domains given by the graph of lines in the form $y = mx + b$ with $m \neq 0$ we obtain $e^y = e^{mx+b} = e^b e^{mx}$. Let $\rho_1$ and $\phi$ such that $e^{mx} = \rho_1 e^{\phi i}$, if we take $\rho = e^b \rho_1$ then $e^y = \rho e^{\phi i}$, whose graph is a spiral. Left-handed spirals and right-handed spirals can be obtained mapping the graph of $y = mx$ with $m < 0$ and $m > 0$ respectively.

In Figure 3, we perform the transformation of graph linear domains, considered with $-1 \leq x \leq 1$; following Figure 3(a) and Figure 3(b), if $|m| > |n|$ the mapping of $y = mx$ is a spiral with more turns than the mapping of $y = nx$.

### 3. Graphs mapping

In this section we map domains given by the graph of functions in the form $y = ax^n$, *i.e.* we map the set of points $(x, ax^n)$, where $n \geq 1$. The image of these points is given by $w = \rho e^{\phi i}$ where $\rho = e^x$ and $\phi = ax^n$, therefore we get a spiral. The number of turns of spirals generated through mapping of the graph of $y = ax^n$ is higher than the spirals presented in section 2, for $n > 1$.

In Figure 4, we verify that linear interpolation of the points into the spiral shows the fast change on the values of the mapping, therefore we need to take more points for the map plotting in order to avoid linear segments in the graph due to the fast increase of the $\phi$ value. In Figure 4, we will study the effect that the number of points taken in the domain can have on the mapping, we consider several parabolas with $-4 \leq x \leq 4$ as domains.

Left-handed spirals and right-handed spirals can be obtained with $a < 0$ and $a > 0$ respectively. In the Figure 5, we consider the mapping of the graph $y = 10x^n$ with $-1 \leq x \leq 1$.
for several values of \( n \); following Figure 5; if \( n \) is odd, the spiral stops and we get an almost constant part due to the small values of \( x \), i.e. \( |x| < \varepsilon \) with \( \varepsilon \) positive close to zero. Thus for \( x > \varepsilon \) the spiral continues to form. If \( n \) is even the spiral stops similar to the case \( n \) odd; however, the spiral continues to form and changes direction. If \( m > n \) then \( x^m < x^n \) for \( |x| < 1 \), hence the constant part is longer for the mapping \( y = x^m \) than the map of \( y = x^n \).

![Figure 5](image)

Figure 5. (a) Mapping of \( y = 10x^2 \); (b) mapping of \( y = 10x^3 \); (c) mapping of \( y = 10x^4 \); (d) mapping of \( y = 10x^5 \).

In the limit case for the mapping of \( y = 10x^n \) with \( |x| \leq 1 \) we get two concentric circumferences and a horizontal line between them, see Figure 6. For domains given by the graph of \( f(x) = \frac{a}{x^n} \), since \( x^n \) is closer to 0 we get great values for \( \frac{1}{x^n} \); from a graphical point of view, we obtain more turns in the spiral. In Figure 7 we present the mapping of the graph \( y = \frac{1}{x^n} \).

![Figure 6](image)

Figure 6. Mapping \( y = 10x^n \) with \(-1 \leq x \leq 1 \) for a great value \( n \).

![Figure 7](image)

Figure 7. (a) Mapping of \( y = \frac{1}{x} \). (b) Mapping of \( y = \frac{1}{x^2} \).
If \( m > n \) then \( \frac{1}{x^m} < \frac{1}{x^n} \) for \(|x| < 1\), thus we have more turns in the mapping of the graph of \( y = \frac{1}{x^m} \). For larger negative values of \( x \) the mappings do not follow a spiral form. In Figure 8, we study the effect of mapping negative and positive values for \( x \) in the domain \( y = \frac{1}{x^2} \).

![Figure 8](image)

**Figure 8.** (a) Mapping with \( 0 \leq x \leq 3 \); (b) mapping with \(-3 \leq x \leq -0.2\).

### 4. Mapping of elliptical domains

The standard form of an ellipse in Cartesian coordinates is given by Equation (4).

\[
\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,
\]

where \((h,k)\) is the center of the ellipse, \(a\) and \(b\) are the semi axes. If \(a = b\) then we obtain the equation of a circle with center in \((h,k)\) and radius \(r = a = b\). In order to get the mapping of elliptical domains, we use the ellipse in the polar form \((x,y) = (a\cos(\theta), b\sin(\theta))\). Hence, by applying \( f(z) = e^z \) we get \( e^{a\cos(\theta)+ib\sin(\theta)} = e^{a\cos(\theta)}e^{ib\sin(\theta)} = \rho e^{i\phi} \), thus \( \rho = e^{a\cos(\theta)} \) and \( \phi = b\sin(\theta) \).

In Figure 9, we show the mapping of an elliptical domain with \(a = b = 30\), for \(0 \leq \theta \leq \pi\) (black line graph) and \(\pi \leq \theta \leq 2\pi\) (red line graph) respectively. Hence, for \(0 \leq \theta \leq \pi\) we get a right-handed spiral, similarly for \(\pi \leq \theta \leq 2\pi\) we have a left-handed spiral; thus in Figure 9, cardioid-like regions are generated by means of two spirals.

![Figure 9](image)

**Figure 9.** Mapping an elliptical domain with \(a = b = 30\).
In Figure 10, we consider the transformation $f(z) = e^z$ over elliptical domains with $a = 10, b = 50$ and $a = 50, b = 10$, centered at the origin. Since $f(z) = e^z$ is an entire function, if $f(z) = e^z$ is bounded by Liouville’s theorem $f(z) = e^z$ must be constant [12], that is a contradiction; hence $f(z) = e^z$ is not bounded. As consequence, the image of an ellipse is not bounded. In Figure 10, we verify that if $a_1 > a_2$ cardioid-like graph obtained mapping the ellipse with $a_1$ is larger than the image of the ellipse with $a_2$, for $a = 50$ we have values with magnitude $10^{21}$, see Figure 10(a), and for $a = 10$ we have a magnitude of $10^4$, see Figure 10(b).

5. Conclusion
Conformal mappings have various applications in physics; for this reason, it is essential to know how a particular mapping behaves in different domains. In this paper, we use the complex exponential function; similar works can be performed with other complex functions of interest.

Physical approaches to the visualization of complex functions can be used to analyze conformal mappings; here, we use the transformation of regions of the complex plane. However, complex functions can also be represented via vector fields. New visualization strategies can be obtained by considering other physical concepts.

References
[1] Apelian C 2010 Real and Complex Analysis (New York: Oxford University Press) chapter 4 p 291
[2] Howie J 2003 Complex Analysis (London: Springer) chapter 2 p 23
[3] Wegert E 2012 Visual Complex Functions (Basel: Springer) chapter 6 p 253
[4] Rodriguez R, Kra I, Gilman J 2012 Complex Analysis: In the Spirit of Lipman Bers (New York: Springer) chapter 6 p 110
[5] Osgur N 2009 On some mapping properties of Mobius transformations Aust. J. Math. Anal. Appl. 6(1) article 13:1-8
[6] Freitag E, Busam R 2009 Complex Analysis (Berlin: Springer) chapter 1 p 66
[7] Bak J, Newman D 2010 Complex Analysis (New York: Springer) chapter 3 p 40
[8] Sinha R 2018 Real and Complex Analysis vol 1 (Singapore: Springer) chapter 1 p 1
[9] Kreyszig E 2011 Advanced Engineering Mathematics (Hoboken: Wiley) chapter 13 p 630
[10] Schinzinger R, Laura P 2003 Conformal Mapping: Methods and Applications (New York: Dover Publications) chapter 3 p 33
[11] Needham T 1997 Visual Complex Analysis (New York: Oxford University Press) chapter 10 p 450
[12] Ward J, Churchill R 2014 Complex Variables and its Applications (New York: Mc Graw Hill) chapter 2 p 172