Linear Network Coding, Linear Index Coding and Representable Discrete Polymatroids

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Abstract—Discrete polymatroids, introduced by Herzog and Hibi, are the multi-set analogue of matroids. In this paper, we explore the connections between linear network coding, linear index coding and representable discrete polymatroids.

First, we consider the vector linear solvability of networks over a field \( \mathbb{F}_q \). It is well known that a scalar linear solution over \( \mathbb{F}_q \) exists for a network if and only if the network is matroidal with respect to a matroid representable over \( \mathbb{F}_q \). We define a discrete polymatroidal network and show that a vector linear solution over a field \( \mathbb{F}_q \) exists for a network if and only if the network is discrete polymatroidal with respect to a discrete polymatroid representable over \( \mathbb{F}_q \). An algorithm to construct networks starting from certain class of discrete polymatroids is provided. Every representation over \( \mathbb{F}_q \) for the discrete polymatroid, results in a vector linear solution over \( \mathbb{F}_q \) for the constructed network. Using the construction procedure, we obtain some new examples of networks which admit vector linear solution, but no scalar linear solution over \( \mathbb{F}_q \). We generalize the results on the connection between discrete polymatroids and vector linear solutions over \( \mathbb{F}_q \) to linear fractional network coding solutions over \( \mathbb{F}_q \) in which the message dimensions need not necessarily be the same and need not be the same as the edge vector dimension.

Next, we consider the index coding problem, which involves a sender which generates a set of messages \( X = \{x_1, x_2, \ldots, x_k\} \) and a set of receivers \( \mathcal{R} \) which demand messages. A receiver \( R \in \mathcal{R} \) is specified by the tuple \( (x, H) \) where \( x \in X \) is the message demanded by \( R \) and \( H \subseteq X \setminus \{x\} \) is the side information possessed by \( R \). We first show that a linear solution for an index coding problem exists if and only if there exists a representable discrete polymatroid satisfying certain conditions which are determined by the index coding problem considered. El Rouayheb et al. showed that the problem of finding a multi-linear representation for a matroid can be reduced to finding a perfect linear index coding solution for an index coding problem obtained from that matroid. Multi-linear representation of a matroid can be viewed as a special case of representation of an appropriate discrete polymatroid. We generalize the result of El Rouayheb et. al. by showing that the problem finding a representation for a discrete polymatroid can be reduced to finding a perfect linear index coding solution for an index coding problem obtained from that discrete polymatroid.

I. BACKGROUND AND RELATED WORK

The concept of network coding, originally introduced by Ahlswede et. al. in [1], helps towards providing more throughput in a communication network than what pure routing solutions provide. For multicast networks, it was shown in [2] that linear solutions exist for sufficiently large field size. An algebraic framework for finding linear solutions in networks was introduced in [3].

The connection between matroids and network coding was studied by Dougherty et. al. in [4]. In [4], the notion of matroidal network was introduced and it was shown that if a scalar linear solution over \( \mathbb{F}_q \) exists for a network, then the network is matroidal with respect to a representable matroid.

The converse that a scalar linear solution exists for a network if the network is matroidal with respect to a representable matroid was shown in [6].

A construction procedure was given in [4] to obtain networks from matroids, in which the resulting network admits a scalar linear solution over \( \mathbb{F}_q \), if the matroid is representable over \( \mathbb{F}_q \). Using the networks constructed using the construction procedure given in [4], it was shown in [5] that there exists networks which do not admit any scalar and vector linear solution, but admit a non-linear solution.

Extending the notion of matroidal network to networks which admit error correction, it was shown in [7] that a network admits a scalar linear error correcting network code if and only if it is a matroidal error correcting network associated with a representable matroid. Constructions of networks from matroids with error correction capability were provided in [7].

In scalar and vector network coding, it is inherently assumed that the dimensions of the message vectors are the same and also the same as the dimensions of the vectors carried in the edges of the network. It is possible that a network does not admit any scalar or vector solution, but admits a solution if all the dimensions of the message vectors are not equal to the edge vector dimension. Such network coding solutions, called Fractional Network Coding (FNC) solutions have been considered in [9] – [11]. The work in [9] primarily focusses on fractional routing, which is a special case of FNC. In [10], algorithms were provided to compute the capacity region for a network, which was defined to be the closure of all rates achievable using FNC. In [11], achievable rate regions for certain specific networks were found and it was shown that achievable rate regions using linear FNC need not be convex.

An index coding problem \( I(X, \mathcal{R}) \), which is a special case of the general network coding problem, involves a sender which generates a set of messages \( X = \{x_1, x_2, \ldots, x_k\} \) and a set of receivers \( \mathcal{R} \) which demand messages [12] – [14]. A receiver \( R \in \mathcal{R} \) is specified by the tuple \( (x, H) \) where \( x \in X \) is the message demanded by \( R \) and \( H \subseteq X \setminus \{x\} \) is the side information possessed by \( R \). In [14], El Rouayheb, Sprinston and Georghiades analyzed the connection between network coding, index coding and multi-linear representations of matroids. In [14], it was shown that the problem of finding a linear solution for a network coding problem can be reduced to the problem of finding a perfect linear index code (for a formal definition see Section III-B) for an index coding.
problem, which was obtained from the network considered. Also, it was shown in [14] that the problem of finding a multi-linear representation for a matroid can be reduced to finding a perfect linear index code for an index coding problem obtained from that matroid.

Discrete polymatroids introduced by Herzog and Hibi, are the multi-set analogue of matroids [17], [18]. Linear and multi-linear representations of matroids can be viewed equivalently as representations of appropriate discrete polymatroids. Representable discrete polymatroids have been used in the context of secret sharing in cryptography [19]–[22]. In this paper, for the first time to the best of our knowledge, we explore the connections between linear network coding, linear index coding and representable discrete polymatroids.

The organization of this paper is as follows: An overview of of matroids and discrete polymatroids is presented in Section II. Section III deals with the preliminaries related to network coding and index coding. Section IV deals with the connection between vector linear network coding and representable discrete polymatroids. The results in Section IV are generalized for linear FNC in Section V. The connection between linear index coding and representable discrete polymatroids is explored in Section VI.

The main contributions of this paper are as follows:

- Discrete polymatroids can be viewed as the generalization of matroids. It is known that the vectors which belong to a discrete polymatroid, can be viewed as the generalization of the notion of independent sets of a matroid and the basis vectors of a discrete polymatroid can be viewed as the generalization of the notion of basis sets of a matroid (Section II-C). To the best of our knowledge, the notion of circuits of matroids has not been generalized to discrete polymatroids. In Section II-D we introduce the notion of minimal excluded vector for a discrete polymatroid, which can be viewed as the generalization of the notion of circuits of a matroid. In the later sections, this notion of minimal excluded vector is extensively used to construct networks from discrete polymatroids, which admit vector linear and linear FNC solutions, as well as to construct index coding problems which admit perfect linear index coding solutions.

- In [8], Kim and Medard made the following comment: “... Unfortunately, the results presented in this paper do not seem to generalize to vector-linear network coding or more general network coding schemes. The difficulty is that the matroid structure requires that a subset of the ground set of a matroid is either independent or dependent, but what this corresponds to in vector-linear codes, for instance, is not clear...” In this paper, we establish that there is a fundamental connection between discrete polymatroids and vector linear network coding. Towards, establishing that connection, the notion of discrete polymatroidal network is introduced, which can be viewed as a generalization of the notion of matroidal network introduced in [4]. In Section IV-A, it is shown that a vector linear solution exists for a network over a field \( \mathbb{F}_q \) if and only if the network is discrete polymatroidal with respect to a discrete polymatroid representable over \( \mathbb{F}_q \).

- A construction algorithm to obtain networks from a class of discrete polymatroids is provided in Section IV-B. Starting from a discrete polymatroid which is representable over \( \mathbb{F}_q \), the resulting networks admit a vector linear solution over \( \mathbb{F}_q \). The construction algorithm provided is used to obtain some new examples of networks which admit a vector linear solution over \( \mathbb{F}_q \) but no scalar linear solution over \( \mathbb{F}_q \).

- The connection between discrete polymatroids and vector linear network coding established in Section IV-A is generalised to linear FNC in Section V-A. In Section V-B, the algorithm provided in Section IV-B is generalized to construct networks from a more general class of discrete polymatroids. Starting from a discrete polymatroid which is representable over \( \mathbb{F}_q \), the resulting networks admit an FNC solution over \( \mathbb{F}_q \). Using this construction, we obtain some new networks, which do not admit any scalar and vector solution over \( \mathbb{F}_q \), and for which FNC solutions with the message dimensions being different provide a larger throughput than FNC solutions for which the message dimensions are the same.

- In Section VI-A, it is shown that a linear solution to an index coding problem exists if and only if there exists a representable discrete polymatroid satisfying certain conditions which are determined by the index coding problem considered. In Section VI-B, we provide a construction of an index coding problem, starting from a discrete polymatroid. It is shown that a perfect linear index coding solution exists for this index coding problem, if and only if the discrete polymatroid from which the index coding problem was constructed is representable. In this way, the problem of finding a representation for a discrete polymatroid reduces to the problem of finding a perfect linear solution for an index coding problem constructed from the discrete polymatroid.

Fig. 1 shows a pictorial depiction of the main results in this paper.

The main differences between the work in this paper and the related work in [14] are as follows:

- The work in [14] considers multi-linear representations of matroids. In this paper, we consider discrete polymatroids, which are more general than matroids. With every matroid we can associate a unique discrete polymatroid, but not vice versa. All multi-linear representations of matroids can be viewed equivalently as representations of appropriate discrete polymatroids, but the converse is not true. There exists discrete polymatroids whose representation cannot be viewed equivalently as the multi-linear representation of any matroid. For more details on this, see Section II-C. We show that all vector linear solutions cannot be characterized using multi-linear representations of matroids, whereas they can be characterized using representations of discrete polymatroids.

- The relationship between multi-linear representation of matroids, linear index coding and linear network coding established in [14] is as follows: Starting from a matroid \( \mathcal{M} \), an index coding problem \( \mathcal{I}_\mathcal{M}(\mathcal{Z}, \mathcal{R}) \) was constructed...
and it was shown that a perfect-linear index coding solution exists for the index coding problem $\mathcal{I}_M(Z, R)$ if and only if the matroid $M$ has a multi-linear representation. Also, a network coding problem $\mathcal{N}(I_M)$ was obtained from $\mathcal{I}_M(Z, R)$, which has a vector linear solution if and only if the matroid $M$ has a multi-linear representation. This relationship between matroid multi-linear representation and network (index) coding problem constructed from $M$ and not for an arbitrary network (index) coding problem. The connections established in this paper between discrete polymatroids and linear network coding in Section V-A and Section V-A and between discrete polymatroids and linear index coding in Section VI-A, are valid for arbitrary networks and index coding problems respectively.

- The construction of networks and index coding problems presented in [14] are from matroids, where as the constructions provided in this paper are from discrete polymatroids, which are more general than matroids. The construction of index coding problem from discrete polymatroids provided in Section VI-A in this paper is a generalization of the construction from matroids in [14].

**Notations:** The set $\{1, 2, \ldots, r\}$ is denoted as $[r]$. $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integer. For a vector $v$ of length $r$ and $A \subseteq [r]$, $v(A)$ is the vector obtained by taking only the components of $v$ indexed by the elements of $A$. The vector of length $r$ whose $i^{th}$ component is one and all other components are zeros is denoted as $\mathbf{1}_r$. For $u, v \in \mathbb{Z}_r^r$, $u \preceq v$ if all the components of $v - u$ are non-negative and, $u < v$ if $u \preceq v$ and $u \neq v$. For $u, v \in \mathbb{Z}_{\geq 0}^r$, $u \vee v$ is the vector whose $i^{th}$ component is the maximum of the $i^{th}$ components of $u$ and $v$. A vector $u \in \mathbb{Z}_{\geq 0}^r$ is called an integral sub-vector of $v \in \mathbb{R}^r$ if $u \preceq v$. For a set $A$, $|A|$ denotes its cardinality and for a vector $v \in \mathbb{Z}_{\geq 0}^r$, $|v|$ denotes the sum of the components of $v$. For a vector $u \in \mathbb{Z}_{\geq 0}^r$, $|u|_2$ denotes the set of indices corresponding to the non-zero components of $u$.

## II. Matroids and Discrete Polymatroids

In Section II-A and Section II-B the basic definitions and notations related to matroids and discrete polymatroids are provided. In Section II-C how a matroid can be viewed as a special case of a discrete polymatroid is explained. In Section II-D the notion of minimal excluded vectors for a discrete polymatroid is introduced, which can be viewed as a generalization of the notion of circuits for matroids.

### A. Matroids

In this subsection, a brief overview of matroids is presented. For a comprehensive treatment, the readers are referred to [15].

**Definition 1** ([15]): A matroid contains a ground set $[r]$ and a set $I$ which is a collection of subsets of $[r]$ satisfying the following three axioms:

- $\emptyset \in I$,
- If $X \in I$ and $Y \subseteq X$, then $Y \in I$.
- If $U, V$ are members of $I$ with $|U| = |V| + 1$ there exists $x \in U \setminus V$ such that $V \cup x \in I$.

The sets which belong to $I$ are called the independent sets of the matroid and those which do not belong to $I$ are called the dependent sets. A maximal independent set is a basis set and a minimal dependent set is called a circuit.

The rank function of a matroid $\Upsilon : 2^{[r]} \to \mathbb{Z}_{\geq 0}$ is defined as $\Upsilon(A) = \max\{|X| : X \subseteq A, X \in I\}$, where $A \subseteq [r]$. The rank of the matroid $M$, denoted by $\text{rank}(M)$ is equal to $\Upsilon([r])$.

A matroid can be equivalently defined in terms of the set of basis sets, the set of circuits and the rank function [15].

**Example 1:** Consider the matroid on the ground set $[4]$ with the set of independent sets given by $\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$. This matroid is referred to as the uniform matroid $U_{2,4}$. The rank function for this matroid is given by, $\Upsilon(X) = \min\{|X|, 2\}, X \subseteq [4]$ and the rank of this matroid is equal to $2$.

A matroid $M$ is said to be representable over $\mathbb{F}_q$ if there exists one-dimensional vector subspaces $V_1, V_2, \ldots, V_r$ of a vector space $E$ such that $\dim(\sum_{i \in X} V_i) = \Upsilon(X), \forall X \subseteq [r]$ and the set of vector subspaces $V_i, i \in [r]$, is said to form a representation of $M$. The one-dimensional vector subspaces $V_i, i \in [r]$, can be described by a matrix $A$ over $\mathbb{F}_q$ whose $i^{th}$ column spans $V_i$.

**Example 2:** Continuing with Example 1 let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ be a matrix over $\mathbb{F}_3$. Let $V_i, i \in [4]$, denote the span of $i^{th}$ column of $A$ over $\mathbb{F}_3$. It can be verified that the vector subspaces $V_1, V_2, V_3$ and $V_4$ form a representation of $U_{2,4}$ over $\mathbb{F}_3$.

The notion of multi-linear representation of matroids was introduced in [23], [24]. A matroid $M$ on the ground set $[r]$ is said to be multi-linearly representable of dimension $n$ over $\mathbb{F}_q$ if there exists vector subspaces $V_1, V_2, \ldots, V_r$ of a vector space $E$ over $\mathbb{F}_q$ such that $\dim(\sum_{i \in X} V_i) = n\Upsilon(X), \forall X \subseteq [r]$. This relationship between matroid multi-linear representation and network (index) coding established in [14] is also a network coding problem from discrete polymatroids, which are more general than matroids.
The vector subspaces $V_i, i \in [r]$, are said to form a multi-linear representation of dimension $n$ over $\mathbb{F}_q$ for the matroid $M$. For $n = 1$, the notion of multi-linear representation reduces to the notion of representation of matroids.

**Example 3:** Continuing with Example 1 let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}$ be a matrix over $\mathbb{F}_2$. Let $V_i$ denote the column span of $A_i, i \in [4]$. It can be verified that the vector subspaces $V_1, V_2$ and $V_3$ and $V_4$ form a multi-linear representation of dimension 2 for $U_{2,4}$ over $\mathbb{F}_2$.

**Example 4 (\[23\]):** Consider the matroid whose geometric representation is shown in Fig. 2 which is popularly known as the non-Pappus matroid. The rank function $r$ of the non-Pappus matroid can be described as follows: all subsets $X$ of $[9]$ of cardinality less than or equal to two have rank equal to $|X|$ and those whose cardinality is greater than or equal to four have rank 3. Among those subsets with cardinality 3, if all its elements lie on a line in the geometric representation shown in Fig. 2, the rank is two; otherwise the rank is three. The non-Pappus matroid is not representable over any field $\mathbb{F}_p$, but it has a multi-linear representation of dimension 2 over $\mathbb{F}_3$ as discussed below. Let $A = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$ be a matrix over $\mathbb{F}_3$. The vector spaces given by the column span of the matrices $A_i, i \in [9]$, form a multi-linear representation of dimension 2 over $\mathbb{F}_3$ for the non-Pappus matroid.

**B. Discrete Polymatroids**

In this subsection, an overview of discrete polymatroids is given. For more details on discrete polymatroids, interested readers are referred to \[17\], \[18\].

A discrete polymatroid $\mathbb{D}$ is defined as follows:

**Definition 2 (\[17\]):** A discrete polymatroid $\mathbb{D}$ on the ground set $[r]$ is a non-empty finite set of vectors in $\mathbb{Z}_{\geq 0}^r$ satisfying the following conditions:

- For all $u, v \in \mathbb{D}$ with $|u| < |v|$, there exists $w \in \mathbb{D}$ such that $u < w \leq u \lor v$.
- For a discrete polymatroid $\mathbb{D}$, the rank function $\rho^D : 2^r \rightarrow \mathbb{Z}_{\geq 0}$ is defined as $\rho^D(A) = \max \{|u(A) |, u \in \mathbb{D}\}$, where $\emptyset \neq A \subseteq [r]$ and $\rho^D(\emptyset) = 0$. Alternatively, a discrete polymatroid $\mathbb{D}$ can be written in terms of its rank function as $\mathbb{D} = \{ x \in \mathbb{Z}_{\geq 0}^r : |x(A)| \leq \rho^D(A), \forall A \subseteq [r]\}$. In the rest of the paper, for simplicity, the superscript $D$ in $\rho^D$ is dropped.

A function $\rho : 2^r \rightarrow \mathbb{Z}_{\geq 0}$ is the rank function of a discrete polymatroid if and only if it satisfies the following conditions:

\begin{align*}
\text{(D1)} & \quad \text{For } A \subseteq B \subseteq [r], \rho(A) \leq \rho(B). \\
\text{(D2)} & \quad \forall A, B \subseteq [r], \rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B). \\
\text{(D3)} & \quad \rho(\emptyset) = 0.
\end{align*}

A vector $u \in \mathbb{D}$ for which there does not exist $v \in \mathbb{D}$ such that $u < v$, is called a basis vector of $\mathbb{D}$. Let $B(\mathbb{D})$ denote the set of basis vectors of $\mathbb{D}$. The sum of the components of a basis vector of $\mathbb{D}$ is referred to as the rank of $\mathbb{D}$, denoted by $rank(\mathbb{D})$. Note that for all the basis vectors, sum of the components will be equal \[18\]. A discrete polymatroid is nothing but the set of all integral subvectors of its basis vectors.

**Example 5:** Let $\rho : 2^{[2]} \rightarrow \mathbb{Z}_{\geq 0}$ be defined as follows: $\rho(\{1\}) = 3$ and $\rho(\{2\}) = \rho(\{1, 2\}) = 5$. It can be seen that $\rho$ satisfies (D1)–(D3) and hence $\rho$ is the rank function of a discrete polymatroid. The vectors which belong to this discrete polymatroid are the points marked by ‘o’ in Fig. 3.

The set of basis vectors for this discrete polymatroid is given by $\{(0, 5), (1, 4), (2, 3), (3, 2)\}$.

**Example 6 (\[18\]):** Let $\rho : 2^{[3]} \rightarrow \mathbb{Z}_{\geq 0}$ be defined as follows: $\rho(\emptyset) = 0$, $\rho(\{1\}) = 1$, $\rho(\{2\}) = \rho(\{3\}) = \rho(\{1, 3\}) = 2$, $\rho(\{1, 2\}) = 3$, $\rho(\{2, 3\}) = \rho(\{1, 2, 3\}) = 4$. It can be verified that $\rho$ satisfies (D1)–(D3) and hence is the rank function of the discrete polymatroid given by,

$\{(0, 0, 0), (0, 1, 0), (0, 0, 1), (0, 1, 1), (1, 0, 0), (1, 0, 1), (0, 2, 0), (0, 2, 1), (0, 2, 2), (0, 2, 3), (1, 2, 1)\}$.

The set of basis vectors of this discrete polymatroid is $B(\mathbb{D}) = \{(0, 2, 2), (1, 2, 1)\}$. 

![Fig. 3. An example of a discrete polymatroid](image-url)
Example 7: Consider the discrete polymatroid $\mathbb{D}$ on the ground set $[3]$ with the rank function $\rho$ given by $\rho(\{1\}) = \rho(\{2\}) = \rho(\{3\}) = 2$, $\rho(\{1, 2\}) = \rho(\{1, 3\}) = \rho(\{2, 3\}) = 3$. Note that the function $\rho$ satisfies the conditions (D1), (D2) and (D3). The set of basis vectors for this discrete polymatroid is given by $B(\mathbb{D}) = \{(1, 1, 1), (1, 2, 0), (2, 0, 1), (2, 1, 0)\}$.

Let $V_1, V_2, \ldots, V_r$ be vector subspaces of a finite dimensional vector space $E$. Define the mapping $\rho: 2^{|r|} \rightarrow \mathbb{Z}_{\geq 0}$ as $\rho(X) = \dim(\sum_{i \in X} V_i)$, $X \subseteq [|r|]$. The mapping $\rho$ satisfies (D1)–(D3) and is the rank function of a discrete polymatroid, which we denote by $\mathbb{D}(V_1, V_2, \ldots, V_r)$. Note that $\rho$ remains the same even if we replace the vector space $E$ by the sum of the vector subspaces $V_1, V_2, \ldots, V_r$. In the rest of the paper, the vector space $E$ is taken to be the sum of the vector subspaces $V_1, V_2, \ldots, V_r$ considered. The vector subspaces $V_1, V_2, \ldots, V_r$ can be described by a matrix $A = [A_1 \ A_2 \ \ldots \ A_r]$, where $A_i, i \in [r]$, is a matrix whose columns span $V_i$.

Definition 3 ([19]): A discrete polymatroid $\mathbb{D}$ is said to be representable over $\mathbb{F}_q$ if there exists vector subspaces $V_1, V_2, \ldots, V_r$ of a vector space $E$ over $\mathbb{F}_q$ such that $\dim(\sum_{i \in X} V_i) = \rho(X)$, $\forall X \subseteq [|r|]$. The set of vector subspaces $V_i, i \in [r]$, is said to form a representation of $\mathbb{D}$. A discrete polymatroid is said to be representable over some field.

Example 8: Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ be matrices over $\mathbb{F}_2$. Let $V_i, i \in [3]$, denote the column span of $A_i$. It can be verified that the vector subspaces $V_1, V_2$ and $V_3$ form a representation over $\mathbb{F}_2$ of the discrete polymatroid given in Example 2.

Example 9: Let $A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_A$ be a matrix over $\mathbb{F}_q$. Let $V_i$ denote the column span of $A_i, i \in [4]$. The rank function $\rho$ of the discrete polymatroid $\mathbb{D}(V_1, V_2, V_3, V_4)$ is as follows: $\rho(X) = 1$, if $X \in \{\{1\}, \{2\}, \{3\}\}$; $\rho(X) = 2$, if $X \in \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{4\}\}$ and $\rho(X) = 3$ otherwise. The set of basis vectors for this discrete polymatroid is given by $\{0, 0, 1, 2, 0, 0, 1, 1, 1\}, (0, 1, 0, 1, 0, 1, 1, 1)$. For every independent set $I$ of the matroid $\mathbb{M}$, there exists a unique vector belonging to $\Delta(\mathbb{M})$ whose components indexed by the elements of $I$ take the value one and all other components are zeros. In other words, in terms of the set of independent sets $\mathcal{I}$ of $\mathbb{M}$, the discrete polymatroid $\Delta(\mathbb{M})$ can be written as $\Delta(\mathbb{M}) = \{\sum_{i \in I} e_i : I \in \mathcal{I}\}$. Conversely, the set of independent sets $\mathcal{I}$ of $\mathbb{M}$ is given by $\mathcal{I} = \{ (u)_i : u \in \Delta(\mathbb{M}) \}$.

Similarly, for a basis set $B$ of a matroid $\mathbb{M}$, the vector $\sum_{i \in B} e_i$ is a basis vector of $\Delta(\mathbb{M})$ and conversely, for a basis vector $b$ of $\Delta(\mathbb{M})$, the set $(b)_i$ is a basis set of $\mathbb{M}$.

Example 11: For the matroid $U_{2,4}$ considered in Example 1, the discrete polymatroid $\Delta(U_{2,4})$ is given by $\Delta(U_{2,4}) = \{ (0, 0, 0, 0), (0, 0, 1, 0), (0, 1, 0, 0), (1, 0, 0, 0)\}$. For every independent set $I$ of $\mathbb{M}$, $\Delta(\mathbb{M})$ contains a vector whose components indexed by the elements of $I$ are ones and all other components are zeros. A set of vector subspaces $V_i, i \in [r]$, forms a representation of $\mathbb{M}$ if and only if it forms a representation of $\Delta(\mathbb{M})$. In this way, the representability of the matroid $\mathbb{M}$ over $\mathbb{F}_q$ can be viewed equivalently as the representability of the discrete polymatroid $\Delta(\mathbb{M})$ over $\mathbb{F}_q$.

Example 12: The vector subspaces $V_i, i \in [4]$, provided in Example 2, which form a representation over $\mathbb{F}_q$ for $U_{2,4}$, form a representation over $\mathbb{F}_2$ for the discrete polymatroid $\Delta(U_{2,4})$ provided in Example 1.

For a discrete polymatroid $\mathbb{D}$ with rank function $\rho$, let $n\mathbb{D}$ denote the discrete polymatroid whose rank function $\rho^\prime(X) = n\rho(X)$, $\forall X \subseteq [|r|]$. Note that the function $\rho^\prime$ satisfies the conditions (D1)–(D3).

Example 13: For the uniform matroid $U_{2,4}$, the discrete polymatroid $2\Delta(U_{2,4})$ has the rank function $\rho^\prime$ given by $\rho^\prime(X) = \min(2|X|, 4)$, $X \subseteq [|4|]$. The set of basis vectors for this discrete polymatroid is given by.
\{(0, 0, 2, 2), (0, 1, 1, 2), (0, 1, 2, 1), (0, 2, 0, 2), (0, 2, 1, 1), (0, 2, 2, 0), (1, 0, 1, 2), (1, 0, 2, 1), (1, 1, 0, 2), (1, 1, 1, 1), (1, 1, 2, 0), (1, 2, 0, 1), (1, 2, 1, 0), (2, 0, 0, 2), (2, 0, 1, 1), (2, 0, 2, 0), (2, 1, 0, 1), (2, 1, 1, 0), (2, 2, 0, 0)\}.

It is straightforward to see that a matroid has a multi-linear representation of dimension \(n\) over \(F_q\) if and only if the discrete polymatroid \(nD(M)\) is representable over \(F_q\). In this way, the notion of multi-linear representation of dimension \(n\) of a matroid \(M\) can be viewed equivalently in terms of the notion of representation of the discrete polymatroid \(nD(M)\).

**Example 8:** The vector subspaces \(V_1, V_2, V_3\) in Example 8 which form a multi-linear representation of dimension 2 over \(F_2\) for the matroid \(U_{2,4}\), form a representation for the discrete polymatroid \(2D(U_{2,4})\) over \(F_2\).

While the multi-linear representation of any matroid can be viewed equivalently in terms of the representation of an appropriate discrete polymatroid, the converse is not true. For example, consider the representable discrete polymatroid \(D\) given in Example 8. The vector subspaces \(V_1, V_2, V_3\) in Example 8 which form a representation for \(D\) cannot form a multi-linear representation for any matroid. The reason for this is as follows: For vector subspaces \(V_1, V_2, V_3\) to form a multi-linear representation of a matroid, \(\dim(\sum_{i \in X} V_i)\) should be a multiple of \(n\), for some integer \(n\), for all \(X \subseteq [3]\). Since \(\dim(V_3) = 1\), the only possibility for \(n\) is 1. In that case, the matroid for which \(V_1, V_2, V_3\) form a multi-linear representation of dimension 1 should have a rank function \(\Upsilon\) which satisfies \(\Upsilon(\{1\}) = 2\), which is not possible since \(\Upsilon(\{1\}) \leq 1\).

**D. Excluded and Minimal Excluded vectors for a Discrete Polymatroid**

As explained in the previous subsection, the vectors which belong to a discrete polymatroid can be viewed as the generalization of independent sets of matroid and the basis vectors of a discrete polymatroid can be viewed as the generalization of basis sets of a matroid. To the best of our knowledge, the notions of dependent sets and circuits of a matroid have not been generalized to discrete polymatroids. In this subsection, we introduce the notions of excluded and minimal excluded vectors for discrete polymatroids, which when specialized to a matroid reduce to the well known notions of dependent vectors and circuits respectively. These notions are useful towards constructing networks and index coding problems from discrete polymatroids in Section \(\text{IV-B}\) and Section \(\text{V-B}\).

We define an excluded vector for a discrete polymatroid \(D\) as follows:

**Definition 4:** For a discrete polymatroid \(D\) on the ground set \([r]\), a vector \(u \in \mathbb{Z}_{\geq 0}^{r}\) is said to be an excluded vector if the \(i\)th component of \(u\) is less than or equal to \(\rho(i)\), \(\forall i \in [r]\), and \(u \notin D\).

Let \(D(D)\) denote the set of excluded vectors for the discrete polymatroid \(D\).

**Example 15:** For the discrete polymatroid considered in Example 5, the excluded vectors are the points indicated by ‘x’ in Fig. 4.

**Example 16:** For the discrete polymatroid considered in Example 7, the set of excluded vectors is given by \(\{(0, 2, 1), (1, 2, 1), (2, 1, 1), (2, 2, 0), (2, 2, 1)\}\).

The notion of excluded vectors for discrete polymatroids can be viewed as the generalization of the notion of dependent sets for matroids. For a matroid \(M\), the set of excluded vectors for \(D(M)\) uniquely determines the set of dependent sets for \(M\). The set of dependent sets for \(M\) is given by \(\{(u)_{>0} : u \in D(D(M))\}\). Conversely, for a dependent set \(D\) for \(M\), the vector \(\sum_{i \in D} \rho_i\) is an excluded vector for \(D(M)\).

**Example 17:** For the uniform matroid \(U_{2,4}\) considered in Example 1, the set of dependent sets is given by \(\{(0, 0, 1, 0), (1, 0, 0, 1), (0, 0, 1, 1), (1, 0, 1, 1)\}\). The set of excluded vectors for \(D(U_{2,4})\) is given by \(\{(1, 1, 1, 0), (1, 1, 0, 1), (0, 1, 1, 1)\}\).

We define a minimal excluded vector for a discrete polymatroid \(D\) as follows:

**Definition 5:** An excluded vector \(u \in D(D)\) is said to be a minimal excluded vector, if there does not exist \(v \in D(D)\) for which \(v < u\).

Let \(C(D)\) denote the set of minimal excluded vectors for the discrete polymatroid \(D\).

**Example 18:** For the discrete polymatroid considered in Example 5, the minimal excluded vectors are \((1, 5), (2, 4)\) and \((3, 3)\).

**Example 19:** For the discrete polymatroid considered in Example 7, the set of minimal excluded vectors is given by \(\{(0, 2, 1), (2, 1, 1), (2, 2, 0)\}\).

The notion of minimal excluded vectors for discrete polymatroids can be viewed as the generalization of the notion of circuits for matroids. The set of minimal excluded vectors for the discrete polymatroid \(D(M)\) uniquely determines the set of circuits for the matroid \(M\). The set of circuits of \(M\) is given by \(\{(u)_{>0} : u \in C(D(M))\}\). Conversely, for a circuit \(C\) for \(M\), the vector \(\sum_{i \in C} \rho_i\) is a minimal excluded vector for \(D(M)\).
Example 20: For the uniform matroid $U_{2,4}$ considered in Example 1, the set of circuits is given by \{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}. The set of minimal excluded vectors for the discrete polymatroid $\mathcal{D}(U_{2,4})$ is \{\{(1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1)\}\}.

III. NETWORK CODING AND INDEX CODING - PRELIMINARIES

In Section III-A the basic definitions and notations related to networks and their solvability are defined. In Section III-B the preliminaries related to the index coding problem are provided.

A. Network Coding

A communication network consists of a directed, acyclic graph with the set of vertices denoted by $\mathcal{V}$ and the set of edges denoted by $\mathcal{E}$. For an edge $e$ directed from $x$ to $y$, $x$ is called the head vertex of $e$ denoted by $head(e)$ and $y$ is called the tail vertex of $e$ denoted by $tail(e)$. The in-degree of an edge $e$ is the in-degree of its head vertex and out-degree of $e$ is the out-degree of its tail vertex. The messages in the network are generated at nodes with in-degree zero, which are called the input edges of the network and let $S \subseteq \mathcal{E}$ denote the set of input edges with $|S| = m$. Let $x_i, i \in [m]$, denote the row vector of length $k_i$ generated at the $i^{th}$ input edge of the network. Let $x = [x_1, x_2, \ldots, x_m]$. An edge which is not an input edge is referred to as an intermediate edge. All the intermediate edges in the network are assumed to carry a vector of dimension $n$ over $\mathbb{F}_q$. A vertex $v \in \mathcal{V}$ demands the set of messages generated at the input edges given by $\delta(v) \subseteq S$, where $\delta$ is called the demand function of the network. $In(v)$ denotes the set of incoming edges of a vertex $v$ ($In(v)$ includes the intermediate edges as well as the input edges which are incoming edges at node $v$) and $Out(v)$ denotes the union of the set of intermediate edges originating from $v$ and $\delta(v)$.

A $(k_1, k_2, \ldots, k_m; n)$-FNC solution over $\mathbb{F}_q$ is a collection of functions $\{\psi_e : \mathbb{F}_q^{\sum_{i=1}^{m} k_i} \rightarrow \mathbb{F}_q^{k_i}, e \in S\} \cup \{\psi_e : \mathbb{F}_q^{\sum_{i=1}^{m} k_i} \rightarrow \mathbb{F}_q, e \in \mathcal{E} \setminus S\}$, where the function $\psi_e$ is called the global encoding function associated with edge $e$. The global encoding functions satisfy the following conditions:

1. $\psi_e(x) = [x_i], \forall i \in S$,
2. For every $v \in \mathcal{V}$, for all $j \in \delta(v)$, there exists a function $\chi_{v,j} : \mathbb{F}_q^{\sum_{i=1}^{m} k_i} \rightarrow \mathbb{F}_q$ called the decoding function for message $j$ at node $v$ which satisfies $\chi_{v,j}(\psi_{i_1}(x), \psi_{i_2}(x), \ldots, \psi_{i_x}(x)) = x_j$, where $In(v) = \{i_1, i_2, \ldots, i_x\}$.

3. For all $i \in \mathcal{E} \setminus S$, there exists $\phi_i : \mathbb{F}_q^{\sum_{i=1}^{m} k_i} \rightarrow \mathbb{F}_q$ such that $\psi_e(x) = \phi_i(\psi_{i_1}(x), \psi_{i_2}(x), \ldots, \psi_{i_x}(x))$, where $In(head(i)) = \{i_1, i_2, \ldots, i_x\}$. The function $\phi_i$ is called the local encoding function associated with edge $i$.

Note that the dimension of the $i^{th}$ message vector $k_i$ need not necessarily be less than the edge vector dimension $n$. For example, for the network considered in Fig. 5 in which nodes 4 and 5 respectively demand messages $x_1$ and $x_2$, a $(2,2;1)$-FNC solution exists which is in fact a fractional routing solution.

An FNC solution with $k_1 = k_2 = \ldots = k_m = n = 1$ is called a scalar solution and an FNC solution for which $k_1 = k_2 = \ldots = k_m = n = k$ is called a vector solution of dimension $k$. A solution for which all the local encoding functions and hence the global encoding functions are linear is said to be a linear solution. For a linear $(k_1, k_2, \ldots, k_m; n)$-FNC solution, the global encoding function $\psi_e, e \in \mathcal{E} \setminus S$, is of the form $\psi_e(x) = xM_i$, where $M_i$ is an $\sum_{i=1}^{m} k_i \times n$ matrix over $\mathbb{F}_q$ called the global encoding matrix associated with edge $i$.

If a network admits a $(k_1, k_2, \ldots, k_m; n)$-FNC solution, then $(k_1/n, k_2/n, \ldots, k_m/n)$ is said to be an achievable rate vector and the scalar $\frac{1}{m} \sum_{i=1}^{m} \frac{k_i}{n}$ is said to be an achievable average rate [11]. The closure of the set of all achievable rate vectors is said to be the achievable rate region of the network and the supremum of all achievable average rates is said to be the average coding capacity of the network [11]. A $(k, k, \ldots, k; n)$-FNC solution is said to be a uniform FNC solution and the scalar $k/n$ is called a uniform achievable rate. The supremum of all uniform achievable rates is defined to be the uniform coding capacity of the network [11].

B. Index Coding

Most of the definitions and notations in this subsection have been adapted from [14].

An index coding problem $\mathcal{I}(X, \mathcal{R})$ includes

- a set of messages $X = \{x_1, x_2, \ldots, x_m\}$
- a set of receiver nodes $\mathcal{R} \subseteq \{\{x,H\}; x \in X, H \subseteq X \setminus \{x\}\}$.

For a receiver node $R = (x,H) \in \mathcal{R}$, $x$ denotes the message demanded by $R$ and $H$ denotes the side information possessed by $R$. Each one of the messages $x_i, i \in \{1, 2, \ldots, m\}$, are assumed to be row vectors of length $n$, over an alphabet set, which in this paper is assumed to be a finite field $\mathbb{F}_q$ of size $q$. Let $y = [x_1 x_2 \ldots x_m]$ denote the row vector of length $nm$ obtained by the concatenation of the $m$ message vectors.

An index coding solution (also referred to as an index code) over $\mathbb{F}_q$ of length $c$ and dimension $n$ for the index coding problem $\mathcal{I}(X, \mathcal{R})$ is a function $f : \mathbb{F}_q^{nm} \rightarrow \mathbb{F}_q^c$, an integer, which satisfies the following condition: For every
$R = (x, H)$ $\in R_c$ there exists a function $\psi_R : F_q^{n(H) + c} \rightarrow F_q^n$ such that $\psi_R((x_i)_{i \in H}, f(y)) = x, \forall y \in F_q^{m,n}$. The function $\psi_R$ is referred to as the decoding function at receiver $R$.

An index coding solution for which $n = 1$ is called a scalar solution; otherwise it is called a vector solution. An index coding solution is said to be linear if the functions $f$ and $\psi_R$ are linear.

For an index coding problem $I(X, R)$, define $M(I(X, R)) = \max_{Y \subseteq X} \{|R = (x, H) \in R : H = Y\}$. The length $c$ and dimension $n$ of an index coding solution for the index coding problem $I(X, R)$ satisfy the condition $c/n \geq M(I(X, R))$ [4].

Definition 6 (|4|): An index coding solution for which $c/n = M(I(X, R))$ is said to be a perfect index coding solution.

IV. VECTOR LINEAR NETWORK CODING AND REPRESENTABLE DISCRETE POLYMATROIDS

In this section, the focus is on the vector linear solvability of networks. In Section IV-A the notion of a discrete polymatroidal network is defined and it is shown that a network is vector linearly solvable over $F_q$ if and only if it is discrete polymatroidal with respect to a discrete polymatroidal representable over $F_q$. In Section IV-B an algorithm to construct networks from a class of discrete polymatroids is provided. If the discrete polymatroid from which the network is constructed is representable over $F_q$, then the constructed network has a vector linear solution over $F_q$. The results presented in this section for vector linear solutions are generalized to FNC solutions in Section V.

A. Vector Linear Solvability of Networks and Representation of Discrete Polymatroids

The notion of a matroidal network was introduced by Dougherty et. al. in [4]. In [4], it was shown that if a scalar linear solution exists for a network, then the network is matroidal with respect to a representable matroid. The converse that a scalar linear solution exists for a network if the network is matroidal with respect to a representable matroid was shown in [6]. In this section, we generalize this result to vector linear solvable networks.

For a discrete polymatroid $\mathbb{D}$, let $\rho_{\text{max}}(\mathbb{D}) = \max_{\{i \in [r]\}} \rho(\{i\})$.

We define a discrete polymatroidal network as follows:

Definition 7: A network is said to be discrete polymatroidal with respect to a discrete polymatroid $\mathbb{D}$, if there exists a map $f : E \rightarrow [r]$ which satisfies the following conditions:

(DN1): $f$ is one-one on the elements of $S$.

(DN2): $\sum_{i \in f(E)} \rho_{\text{max}}(\mathbb{D})_{i,r} \in \mathbb{D}$.

(DN3): $\rho(f(\text{In}(x))) = \rho(f(\text{In}(x) \cup \text{Out}(x))), \forall x \in V$.

It can be verified that a network is matroidal with respect to a matroid $\mathbb{M}$ if and only if it is discrete polymatroidal with respect to $\mathbb{D}(\mathbb{M})$. In this way, for a discrete polymatroid $\mathbb{D}(\mathbb{M})$, the notion of a discrete polymatroidal network with respect to $\mathbb{D}(\mathbb{M})$ is equivalent to the notion of a matroidal network with respect to $\mathbb{M}$.

The connection between the vector linear solvability over $F_q$ for a network and the network being discrete polymatroidal with respect to a discrete polymatroid representable over $F_q$ is established in the following theorem.

Theorem 1: A network has a $k$ dimensional vector linear solution over $F_q$ if and only if it is discrete polymatroidal with respect to a discrete polymatroid $\mathbb{D}$ representable over $F_q$ with $\rho_{\text{max}}(\mathbb{D}) = k$.

Proof: Assume the edge set $E$ to be $[l]$, with the set of input edges $S = [m]$. Assume the set of intermediate edges to be $\{m + 1, m + 2, \ldots, l\}$, with the edges in the set arranged in the ancestral ordering, which exists since the networks considered in the paper are acyclic.

First we prove the if part. Consider a network which is discrete polymatroidal with a respect to a representable discrete polymatroid $\mathbb{D}(V_1, V_2, \ldots, V_r)$ which is denoted as $\mathbb{D}$ for brevity, with $\rho_{\text{max}}(\mathbb{D}) = \max_{i \in [r]} \dim(V_i) = k$. Let $f$ be the mapping from edge set of the network $E$ to the ground set $[r]$ of the discrete polymatroid which satisfies (DN1)-(DN3). Since the map $f$ is one-one on the elements of $S$, assume $f(S) = [m]$. Let $u = \sum_{i \in [m]} e_i, r$. From (DN2) it follows that

$$k|u(A)| \leq \dim \left( \sum_{s \in A} V_s \right), \forall A \subseteq [r].$$

It is claimed that without loss of generality, we can take $[r]$ to be the image of $f$. Otherwise, let the image of $f$ be the set $\{i_1, i_2, \ldots, i_l\} \subseteq [r]$. We show that the network is discrete polymatroidal with respect to the discrete polymatroid $D' = D(V_{i_1}, V_{i_2}, \ldots, V_{i_l})$, with $f$ as the network discrete polymatroid mapping. (DN1) and (DN3) follow from the fact that the network is discrete polymatroidal with respect to $D$ with $f$ as the network discrete polymatroid mapping. Let $u = k \sum_{i \in [m]} e_{i,t}$. To prove that (DN2) is also satisfied, it needs to be shown that $k|u(A')| \leq \dim \left( \sum_{s \in A'} V_s \right), \forall A' \subseteq [l]$, which follows from (3) and from the facts that $\{i_o : o \in A'\} \subseteq [r]$ and $u(A') = v\{i_o : o \in A'\}$.

It is claimed that $\dim(\sum_{i \in [r]} V_i) = km$. The proof of the claim is as follows: Define $s_0 = [m]$. Let $s_1 = s_0 \cup \{f(m + 1)\}$. Since the edges in the set $\{m + 1, m + 2, \ldots, l\}$ are arranged in ancestral ordering, $\text{In}(\text{head}(m + 1))$ is contained in $s_0$. Hence, from (DN3) we have $\rho(s_1) = \dim(\sum_{i \in s_0} V_i + V_{f(m + 1)}) = \dim(\sum_{i \in s_0} V_i) = \rho(s_0)$. Iteratively, defining $s_{i+1} = s_i \cup \{f(m+i+1)\}$, using a similar argument, we have $\rho(s_{i+1}) = \rho(s_0)$. Hence, we have $\rho(s_{l-m}) = \rho(s_0)$. But $s_{l-m} = [r]$, since the image of $f$ is $[r]$. Hence, we have, $\rho([r]) = \rho([m])$. Since the network is discrete polymatroidal, $\sum_{i \in [m]} k_{i,r} \in D$ and from the definition of $D$, it follows that $km \leq \dim \left( \sum_{i \in [m]} V_i \right)$. But from (D2) we have, $\dim(\sum_{i \in [m]} V_i) = \rho([m]) \leq \rho(1) + \rho(2, 3, \ldots, m) \leq \sum_{i \in [2]} \rho(i) + \rho(3, 4, \ldots, m) \leq \cdots \leq \sum_{i \in [m]} \rho(i) \leq km$. Hence, $\rho([r]) = \rho([m]) = km$. Hence $\dim(\sum_{i \in [r]} V_i) = km$.

The vector subspace $V_{i,r} \in [r]$, can be written as the column span of a matrix $A_i$ of size $km \times k$. Also, since $\dim(\sum_{i \in [m]} V_i) = km$, the matrix $B = [A_1 A_2 \ldots A_m]$ is invertible and hence can be taken to be the $km \times km$ identity
matrix (Otherwise, it is possible to choose \( A'_i = B^{-1} A_i \), and \( V'_i \) to be the column span of \( A'_i \), so that \( \mathbb{D}(V'_1, V'_2, \ldots, V'_r) = \mathbb{D}(V_1, V_2, \ldots, V_r) \) and \([A'_1, A'_2, \ldots, A'_m]\) is the identity matrix).

Taking the global encoding matrices \( M_i \) to be \( A_{f(i)} \), we get a \( k \)-dimensional network coding solution over \( \mathbb{F}_q \) for the network. Since \([A_1, A_2, \ldots, A_m]\) is the identity matrix, the network is characterized using representations of discrete polymatroids.

The solution \( A_{f(i)} \) can be written as \( \sum_{j=1}^{t} A_{f(i)}(j) W_p \), where \( W_p \in \mathbb{F}_q^{k \times k} \), which shows that (N2) and (N3) are satisfied.

To prove the only if part, consider a network which has a \( k \)-dimensional vector linear solution over \( \mathbb{F}_q \). Take the vector subspace \( V_i \) to be the column span of the global encoding matrix \( M_i, i \in [t] \). Consider the discrete polymatroid \( \mathbb{D}(V_1, V_2, \ldots, V_i) \). The edge \( i \in \mathcal{E} \) is mapped by the function \( f \) to the element \( i \) in the ground set \( [t] \) of the discrete polymatroid. It can be easily seen that (DN1)-(DN3) are satisfied and hence the network is discrete polymatroidal with respect to \( \mathbb{D}(V_1, V_2, \ldots, V_i) \). Also, we have \( \rho_{\text{max}}(\mathbb{D}(V_1, V_2, \ldots, V_i)) = \max_{i \in [t]} \text{rank}(V_i) = k \).

Specializing for \( k = 1 \), from Theorem 1 we obtain the following corollary:

**Corollary 1:** A scalar linear solution exists for a network over \( \mathbb{F}_q \) if and only if the network is matroidal with respect to a matroid representable over \( \mathbb{F}_q \).

Note that the statement in Corollary 1 is more general than the statement of Theorem 13 in [6] stated as follows: “A network is scalar-linearly solvable over a finite field of characteristic \( p \) if and only if the network is a matroidal network associated with a representable matroid over a finite field of characteristic \( p \).” For a network which is matroidal with respect to a matroid representable over a field \( \mathbb{F}_q \), Theorem 13 in [6] implies that a scalar linear solution exists for the network over a sufficiently large field whose characteristic is the same as that of \( \mathbb{F}_q \). In contrast, the result in Corollary 1 above implies that such a scalar linear solution exits over the filed \( \mathbb{F}_q \) itself, and there is no need to look for solutions over larger fields.

It is important to note that the discrete polymatroid \( \mathbb{D} \) in Theorem 1 need not be unique. A network can admit more than one \( k \)-dimensional vector linear solution over \( \mathbb{F}_q \) and from these solutions it may be possible to obtain multiple discrete polymatroids with respect to which the network is discrete polymatroidal, as illustrated in Example 21 below.

In Example 21 below, we consider the popular example of M-network introduced in [5], which was shown to have a 2-dimensional vector linear solution, which is in fact a vector routing solution, but does not admit scalar linear solution over any field. It was shown in [4] that the M-network is not matroidal with respect to any representable matroid. But since the M-network admits a vector linear solution, from Theorem 1, it follows that the M-network is discrete polymatroidal with respect to a representable discrete polymatroid, as discussed in the following example.

**Example 21:** Consider the M-network shown in Fig. 6. We consider two possible solutions for the M-network, from which it is possible to obtain two different discrete polymatroids with respect to which the M-network is discrete polymatroidal.

**Solution 1:** Assume the global encoding matrix of edge \( i, i' \in [12] \), to be the matrix \( A_i \) given in [10] at the top of this page. Take \( A_5 \) to be the global encoding matrix of the edges 13, 14, 15, 16 and \( A_9 \) to be that of 17, 18, 19, 20. The solution thus obtained for the M-network is as shown in Fig. 6(b). Let the network discrete polymatroid mapping \( f_1 \) be defined as follows:

\[
 f_1(i) = \begin{cases} 
i : i \in \{1, 2, \ldots, 12\} \\
5 : i \in \{13, 14, 15, 16\} \\
8 : i \in \{17, 18, 19, 20\} 
\end{cases}
\]

Define \( V_i \) to be the column span of \( A_i, i \in [12] \). It can be verified that the M-network is discrete polymatroidal with respect to \( \mathbb{D}(V_1, V_2, \ldots, V_{12}) \), with \( f_1 \) being the network discrete polymatroid mapping.
that the vector subspaces (excluding the trivial zero vector subspaces) which form a multi-linear representation of dimension $k$ for a matroid should be $k$-dimensional. Note that the vector subspaces $V_i, i \in [12]$, have dimension 2 and they form a representation for the discrete polymatroid $D(V_1, V_2, \ldots, V_{12})$. Despite having their dimensions to be equal, the vector subspaces $V_i, i \in [12]$, cannot form a multi-linear representation of dimension 2 for any matroid. The reason for this is that $\dim(V_i + V_5) = 3$ which is not a multiple of 2.

**Solution 2:** Assume the global encoding matrices of edge $i, i \in [20]$, to be the matrix $A_i'$ (defined in (2) at the top of this page). The solution thus obtained for the M-network is as shown in Fig. 6(b). Let the network discrete polymatroid mapping $f_2(i) = i, i \in [20]$. Define $V_i'$ to be the column span of $A_i'$, $i \in [20]$. It can be verified that the M-network is discrete polymatroidal with respect to $D(V_1', V_2', \ldots, V_{20}')$, with $f_2$ being the network discrete polymatroid mapping.

Note that all the vector subspaces $V_i, i \in [12]$, in Solution 1 have the same dimension 2. In contrast, in Solution 2, the vector subspaces $V_1', V_2', \ldots, V_{12}'$ have dimension 2, while the vector subspaces $V_{13}', V_{14}', \ldots, V_{20}'$ have dimension 1. The M-network is discrete polymatroidal with respect to two different discrete polymatroids $D(V_1, V_2, \ldots, V_{12})$ and $D(V_1', V_2', \ldots, V_{20}')$.

As shown in Theorem 1 and illustrated in the previous example, there is a fundamental connection between vector linear solvability of networks and representations of discrete polymatroids, whereas such a connection does not exist with multi-linear representations of matroids. In the next subsection, some more examples of networks obtained from discrete polymatroids, which admit a vector linear solution over $\mathbb{F}_q$ but no scalar linear solution over $\mathbb{F}_q$ are provided.

### B. Construction of Vector Linear Solvable Networks from Discrete Polymatroids

In this section, an algorithm to construct networks from a class of discrete polymatroids is provided. If the discrete polymatroid $D$ from which the network is constructed is representable over $\mathbb{F}_q$ with $\rho_{\text{max}}(D) = k$, the constructed network has a $k$-dimensional vector linear solution over $\mathbb{F}_q$.

From the proof of Theorem 1, it follows that from every vector linear solution for a network, it is possible to obtain a representable discrete polymatroid $D$ satisfying the following conditions: (i) $\rho_{\text{max}}(D) = k$ and (ii) there exists a basis vector $b \in B(D)$ for which all the non-zero components are equal to $k$.

In this subsection, we restrict ourselves to the only class of discrete polymatroids which satisfy the above two conditions and provide an algorithm to construct networks from discrete polymatroids which belong to this class. The construction algorithm provided is generalized to a more general class of discrete polymatroids in the Section 4.2.8

Before providing the construction algorithm, we provide some useful definitions.

Let $C_i'(D)$ denote the set of vectors $u$ in $C_i'(D)$ which satisfy the condition that there does not exist $v \in C_i'(D), v \neq u$, for which $(v)_{>0} \subset (u)_{>0}$. The elements of $C_i'(D)$ are referred to as reduced i-unit minimal excluded vectors.

**Example 22:** For the discrete polymatroid $2D(U_{2,4})$ provided in Example 13 the set of minimal excluded vectors $C(2D(U_{2,4}))$ is given by

\[
\{(0, 1, 2, 2), (0, 2, 1, 2), (0, 2, 2, 1), (1, 0, 2, 2), (1, 1, 1, 2), (1, 1, 2, 1), (1, 2, 2, 0), (1, 2, 0, 2), (1, 2, 1, 1), (2, 0, 2, 1), (2, 0, 2, 1), (2, 1, 0, 2), (2, 1, 1, 1), (2, 1, 2, 0), (2, 2, 0, 1), (2, 2, 1, 0)\}\.

The set of i-unit minimal excluded vectors, $i \in [4]$ are given by,
\[
C'_1(2D(U_2,4)) = \{(1,0,2,2),(1,1,1,2),(1,1,2,1),(1,2,0,2), \\
(1,2,1,1),(1,2,2,0)\},
\]
\[
C'_2(2D(U_2,4)) = \{(0,1,2,2),(1,1,1,2),(1,1,2,1),(2,1,0,2),
(2,1,1,1),(2,1,2,0)\},
\]
\[
C'_3(2D(U_2,4)) = \{(0,2,1,2),(1,1,1,2),(1,2,1,1),(2,0,1,2),
(2,1,1,1),(2,2,1,0)\},
\]
\[
C'_4(2D(U_2,4)) = \{(0,2,2,1),(1,1,2,1),(1,2,1,1),(2,0,2,1),
(2,1,1,1),(2,2,1,0)\}.
\]

The set of reduced \(i\)-unit minimal excluded vectors, \(i \in \{4\}\) are given by,
\[
C_1(2D(U_2,4)) = \{(1,0,2,2),(1,2,0,2),(1,2,2,0)\},
\]
\[
C_2(2D(U_2,4)) = \{(0,1,2,2),(2,1,0,2),(2,1,2,0)\},
\]
\[
C_3(2D(U_2,4)) = \{(0,2,1,2),(2,0,1,2),(2,2,1,0)\},
\]
\[
C_4(2D(U_2,4)) = \{(0,2,2,1),(2,0,2,1),(2,2,1,0)\}.
\]

Now we proceed to give the construction algorithm.

**ALGORITHM 1.**

**Step 1.** From the basis of vectors \(B(D)\), choose a vector \(v\) for which all the non-zero components are equal to \(\rho_{\max}(D)\).

For every \(i \in \langle v \rangle\), add a node \(i\) to the network with an input edge \(e_i\) which generates the message \(x_i\). Let \(f(e_i) = i\).

Define \(M = T = \langle v \rangle\).

**Step 2:** For \(i \in \langle r \rangle \notin T\), find a vector \(u \in C_i(D)\), for which \((u - e_i)_{>0} \subseteq T\). Add a new node \(i'\) to the network with incoming edges from all the nodes which belong to \((u - e_i)_{>0}\). Also, add a node \(i\) with a single incoming edge from \(i'\), denoted as \(v_{i,i'}\). Define \(f(e) = head(e)\), \(\forall e \in In(i)\) and \(f(e_{i,i'}) = i\). Let \(T = T \cup \{i\}\). Repeat step 2 until it is no longer possible.

**Step 3:** For \(i \in M\), choose a vector \(u \in C_i(D)\) for which \((u)_{>0} \subseteq T\). Add a new node \(j\) to the network which demands message \(x_i\) and which has connections from the nodes in \((u - e_i)_{>0}\). Define \(f(e) = head(e)\), \(\forall e \in In(j)\). Repeat this step as many number of times as desired.

**Step 4:** For a basis vector \(b \in B(D)\), add a node \(j\) which has incoming edges from the nodes which belong to \((b)_{>0}\) and demands all the messages. Define \(f(e) = head(e)\), \(\forall e \in In(j)\). Repeat this step as many number of times as desired.

For a matroid \(M\), when specialized to the discrete polymatroid \(\mathbb{D}[M]\), **ALGORITHM 1** reduces to the construction from matroids provided in Section V-B in [4].

As shown in the following theorem, the network constructed using **ALGORITHM 1** is discrete polymatroidal with respect to \(D\) with the network discrete polymatroid mapping \(f\) defined in the algorithm. Hence, if \(D\) is representable over \(\mathbb{F}_q\), then the constructed network admits a vector linear solution over \(\mathbb{F}_q\).

**Theorem 2:** A network constructed using **ALGORITHM 1** from a discrete polymatroid \(D\) which is representable over \(\mathbb{F}_q\) with \(\rho_{\max}(D) = k\), admits a vector linear solution of dimension \(k\) over \(\mathbb{F}_q\).

**Proof:** The theorem is proved by showing that the constructed network is discrete polymatroidal with respect to \(D\) with the network discrete polymatroid mapping \(f\) defined in **ALGORITHM 1**. Clearly, **Step 1** of **ALGORITHM 1** ensures that (DN1) and (DN2) are satisfied.

The nodes in the network constructed using **ALGORITHM 1** can be classified into five kinds: (i) nodes added in **Step 1** which belong to the set \(M\), (ii) nodes added in **Step 2** which are labelled \(i', i \in \langle r \rangle\), (iii) nodes added in **Step 2** which are labelled \(i, i \in \langle r \rangle\), (iv) nodes added in **Step 3** which demand messages and (v) nodes added in **Step 4** which demand messages. For a node \(x\) of kind (i) or of kind (iii), since the in-degree is one and all the outgoing edges are mapped by \(f\) to the same element in \(\langle r \rangle\), \(f(In(x)) = f(In(x) \cup Out(x)) = f(In(x) \cup Out(x)) = f(In(x))\),

Consider a node \(i' \in \langle r \rangle\) of kind (ii). Let \(v_{i',i}\) denote the edge connecting \(i'\) and \(i\). Let \(u' \in C_i(D)\) denote the vector which was used in **Step 2** while adding the node \(i\) and \(i'\) to the network. Since \(f(e_{i',i}) = i\), we need to show that \(\rho(f(In(i'))) = \rho(f(In(i')) \cup \{i\})\). Since \(f(In(i')) = (u' - e_i)_{>0}\), it needs to be shown that \(\rho((u' - e_i)_{>0}) = \rho((u')_{>0})\), i.e., \(\dim \left( \sum_{j \in \langle u' \rangle_{>0}} V_j \right) < \dim \left( \sum_{j \in A} V_j \right)\). Let \(a' = (u' - e_i)\). Since \(u'\) is a minimal excluded vector, \(a' \in D\) and hence for all \(A \subseteq \langle r \rangle\), we have,

\[
|a'(A)| \leq \dim \left( \sum_{j \in A} V_j \right),
\]

Since \(u' \notin D\), we have,

\[
\dim \left( \sum_{j \in A'} V_j \right) = \dim \left( \sum_{j \in \langle u' \rangle_{>0}} V_j \right) = \dim \left( \sum_{j \in \langle u' \rangle_{>0} \cap A'\} V_j \right) = \dim \left( \sum_{j \in \langle u' \rangle_{>0} \cap A'\} V_j \right) = |a'(A')|. \]

\[\text{dim} \left( \sum_{j \in \langle u' \rangle_{>0} \cap A'\} V_j \right) = |a'(A')|. \]

Fig. 7. Pictorial depiction of the sets \(\langle r \rangle\), \((a')_{>0}\) and \(A'\) used in the proof of **Theorem 2**.
\[ |a^i(A^i)| = |a^i((a_i)_{i > 0} \cap A^i)| \leq \dim \left( \sum_{j \in (a_i)_{i > 0} \cap A^i} V_j \right). \]

Hence, \( \dim \left( \sum_{j \in (a_i)_{i > 0} \cap A^i} V_j \right) = \dim \left( \sum_{j \in A^i} V_j \right) \). Since \( i \in A^i \), it follows that \( \dim \left( \sum_{j \in (a_i)_{i > 0} \cap A^i} V_j + V_i \right) = \dim \left( \sum_{j \in (a_i)_{i > 0} \cap A^i} V_j \right) \). As a result, we have,

\[ \dim \left( \sum_{j \in (a_i)_{i > 0} \cap A^i} V_j \right) = \dim \left( \sum_{j \in \{a_i, a\}_{i > 0} \cap A^i} V_j \right), \]

i.e., \( \dim \left( \sum_{j \in \{a_i, a\}_{i > 0}} V_j \right) = \dim \left( \sum_{j \in \{a_i, a\}_{i > 0}} V_j \right) \).

Following a procedure exactly similar to the one used for a node kind \( \text{(ii)} \), it can be shown that \( \rho(f(\text{Out}(x))) = \rho(f(\text{In}(x) \cup \text{Out}(x))) \) for a node \( x \) of kind \( \text{(iv)} \).

To show that \( \text{(DN3)} \) is satisfied for a node kind \( \text{(v)} \), it needs to be shown that for every \( b \in B(\mathbb{D}) \),

\[ \rho \left( (b)_{i > 0} \cup f(\{i\}) \right) = \rho \left( (b)_{i > 0} \right), \forall i \in M. \]

It needs to be shown that \( \dim \left( \sum_{j \in (b)_{i > 0}} V_j + V_f(\{i\}) \right) = \dim \left( \sum_{j \in (b)_{i > 0}} V_j \right), i \in M \), which is true since \( b \) is a basis vector for \( \mathbb{D} \). This completes the proof of Theorem 2.

The construction procedure provided in ALGORITHM 1 is illustrated using the following examples.

**Example 23:** Continuing with Example 22 for simplicity, let \( \mathbb{D} \) denote the discrete polymatroid \( 2\mathbb{D}(U_{2,4}) \). The construction procedure for the discrete polymatroid considered in Example 22 is summarized in Table 1. The steps involved in the construction are illustrated in Fig. 8.

\[ \text{TABLE I} \]

| Step | Vector chosen | Node added | Dimension from \( \text{In}(x) \) and \( \text{Out}(x) \) |
|------|---------------|------------|---------------------------------------------|
| Step 1 | \( s = (2, 2, 0, 0) \in \mathbb{D}(\mathbb{D}) \) \( s = (\text{In}(x) \cup \text{Out}(x)) \) | 1 \( \text{In} \) edge (message \( x_{11} \)) \( T = \{1, 2\} \) | \( M = \{1, 2\} \) |
| Step 2 | \( u = (2, 2, 1, 0) \in \mathbb{D}(\mathbb{D}) \) \( u - t_{\{1, 2\}} \) | 3 \( 1 \) and 2 \( T = \{1, 2\} \) | \( T = \{1, 2, 3\} \) |
| Step 3 | \( v = (2, 2, 0, 1) \in \mathbb{D}(\mathbb{D}) \) \( v - t_{\{1, 2\}} \) | 4 \( 1 \) and 2 \( T = \{1, 2, 3\} \) | \( T = \{1, 2, 3, 4\} \) |

The network thus obtained is given in Fig. 8. A vector linear solution of dimension 2 over \( \mathbb{F}_2 \) shown in Fig. 8 is obtained by taking the global encoding matrices for the edges \( 3' \to 3 \) and \( 4' \to 4 \) to be the matrices \( A_3 \) and \( A_4 \) given in Example 3. All the outgoing edges of a node which has in-degree one carry the same vector as that of the incoming edge. The network in Fig. 8 does not admit a scalar linear solution over \( \mathbb{F}_2 \) as shown in the following lemma.

**Lemma 1:** The network given in Fig. 8 does not admit a scalar linear solution over \( \mathbb{F}_2 \).

**Proof:** Observe that node 5 demands \( x_2 \) and the only path from 2 to 5 is via the edge \( 3' \to 3 \). Also, node 6 demands \( x_1 \) and the only path from 1 to 6 is via the edge \( 3' \to 3 \). To satisfy these demands, the edge \( 3' \to 3 \) needs to carry \( x_1 + x_2 \). By a similar reasoning, to satisfy the demands of nodes 7 and 8, the edge \( 4' \to 4 \) needs to carry \( x_1 + x_2 \). But if the edges \( 3' \to 3 \) and \( 4' \to 4 \) carry \( x_1 + x_2 \), the demands of nodes 9 and 10 cannot be satisfied.

While the network in Fig. 8 does not admit a scalar linear solution over \( \mathbb{F}_2 \), it has a scalar linear solution over all fields of size greater than two, as shown in the following lemma.

**Lemma 2:** The network in Fig. 8 admits a scalar linear solution over all fields of size greater than two.

**Proof:** It can be verified that the network shown in Fig. 8 is matroidal with respect to the uniform matroid \( U_{2,4} \) with the mapping \( f \) from the edge set to the ground set \( [4] \) of the matroid defined as follows: for \( i \in [4] \), all the elements of \( \text{In}(i) \) are mapped to \( \text{head}(i') \), the elements of \( \text{out}(i) \) and the edge joining \( i' \) and \( i \) are mapped to \( i \). Since \( U_{2,4} \) is representable over all fields of size greater than or equal to three (follows from Proposition 6.5.2, Page 203, [16]), the network in Fig. 8 admits a scalar linear solution over all fields of size greater than two.

The network constructed in the previous example turned out to be matroidal with respect to a matroid representable over all fields other than \( \mathbb{F}_2 \) and as a result it admitted scalar linear...
solutions over all $\mathbb{F}_p$ other than $\mathbb{F}_2$. In the following example, the constructed network is discrete polymatroidal with respect to a representable discrete polymatroid whereas it cannot be matroidal with respect to any representable matroid. Hence it is not scalar linearly solvable over any field, but is vector linear solvable.

**Example 24:** Let $V_i, i \in [12]$, denote the column span of the matrix $A_i$ shown in (1). Let $\mathbb{D}$ denote the discrete polymatroid $\mathbb{D}(V_1, V_2, \ldots, V_{12})$. The steps involved in the construction of a network from this discrete polymatroid is summarized in Table I. The network thus constructed is shown in Fig. 9. The vector linear dimension of dimension 2, which is in fact a vector routing solution, is obtained by choosing the global encoding matrix of the edge $i^* \to i, i \in [12]$, to be $A_i$, as shown in Fig. 9. All the outgoing edges of a node which has in-degree one carry the same vector as that of the incoming edge.

| Step 1 | Vector chosen $x = (2, 2, 2, 0, 0, 0, 0, 0, 0, 0, 0, 0) \in \mathbb{D}(\mathbb{D})$ | Network $N$ | Edge routing matrix $N$ |
|--------|-------------------------------------------------|-------------|-----------------|
| 1      | Input edge message $x$ | $T = (1, 2, 3, 4)$ | $M = (1, 2, 3, 4)$ |
| 2      | Input edge message $x$ | $T = (1, 2, 3, 4)$ | $M = (1, 2, 3, 4)$ |
| 3      | Input edge message $x$ | $T = (1, 2, 3, 4)$ | $M = (1, 2, 3, 4)$ |
| 4      | Input edge message $x$ | $T = (1, 2, 3, 4)$ | $M = (1, 2, 3, 4)$ |

**Table II**

**Steps Involved in the Construction of the Network in Example 24**

The following lemma shows that the network in Fig. 9 is not scalar linearly solvable.

**Lemma 3:** The network in Fig. 9 is not scalar linearly solvable.

**Proof:** To prove the lemma, it is shown that the network cannot be matroidal with respect to a representable matroid. The ideas used in the proof are similar to the ones used in the proof of Theorem V.8 in [4].

On the contrary, assume that the network is matroidal with respect to a representable matroid $\mathbb{M}$ on the ground set $[r]$ and let $f$ be the network-matroid mapping. Let the set of one dimensional vector spaces $V_i, i \in [r]$ form a representation of $\mathbb{M}$. All the outgoing edges of a node which has in-degree one carry the same vector as that of the incoming edge. For simplicity, let $i$ denote the incoming edge of node $i$, where $i \in [12]$. Let $\rho$ denote the rank function of $\mathbb{D}(\mathbb{M})$. Let $g(x) = \rho(f(x)), x \subseteq [12]$.

We have $g(\{1, 2\}) \leq 2$. From (DN2), it follows that $\sum_{i \in [12]} f(i) \leq 2 \leq 2$. Hence, we have $g(\{1, 2\}) = 2$. Similarly, we also have $g(\{3, 4\}) = 2$.

It is claimed that $g(\{5\}) = 1$. Otherwise, $g(\{5\})$ has to be 0. In that case, since the nodes 13 and 16 demand $x_1$ and $x_2$ respectively, from (DN3) it follows that $\dim(V_{f(1)} + V_{f(6)}) = \dim(V_{f(6)})$ and $\dim(V_{f(2)}) = \dim(V_{f(6)})$. This will force $V_{f(1)} = V_{f(2)}$ which is not possible. Hence $g(\{5\})$ has to be 1. Similarly, it can be shown that $g(\{8\}) = 1$.

We have,

$$g(\{3, 8\}) + g(\{4, 8\}) \geq g(\{8\}) + g(\{3, 4, 8\}) \geq 1 + g(\{3, 4\}) = 3,$$

where (6) holds since $g(\{3, 4, 8\}) = g(\{4, 8\})$ (follows from (DN3)) and (7) follows from the facts that $g(\{8\}) = 1$ and $g(\{3, 4\}) = 2$.

Similarly, it can be shown that

$$g(\{1, 5\}) + g(\{2, 5\}) \geq 3.$$

Also, we have,

$$g(\{2, 5\}) + g(\{3, 8\}) = g(\{2, 5, 3, 8\}) \leq g(\{2, 5, 3, 8, 11\}) \leq g(\{2, 5, 11\}) + g(\{3, 8, 11\}) - g(\{11\}) = g(\{5, 11\}) + g(\{8, 11\}) - 1 \leq 3,$$

where (9) follows from the fact that $\dim(V_{f(2)} + V_{f(5)}) +
We first prove the ‘if’ part of the theorem. Assume that the network considered is \((k_1, k_2, \ldots, k_m; n)\)-discrete polymatroidal with respect to a representable discrete polymatroid \(D(V_1, V_2, \ldots, V_r)\) on the ground set \([r]\). For brevity, the discrete polymatroid \(D(V_1, V_2, \ldots, V_r)\) is denoted as \(D\). Let \(f\) denote the network-discrete polymatroid mapping. Since, \(f\) is one-to-one on the elements of \(S\), assume \(f(S)\) to be \([m]\).

Using the same procedure used in the proof of Theorem 1 it can be shown that without loss of generality, the set \([r]\) can be taken to be the image of the map \(f\).

Next it will be shown that \(\text{dim}(\sum_{i=1}^{l} V_i) = \sum_{i=1}^{m} k_i\). Using similar ideas as in the proof of Theorem 1, it can be shown that \(\rho([r]) = \rho([m])\). From (GDN2), we have \(\sum_{i\in[m]} k_{i}, i, r, i, r \in D\). Hence from the definition of a discrete polymatroid, we have \(\sum_{i=1}^{m} k_i \leq \rho([m])\). From (D2), we have \(\rho([m]) \leq \rho([1]) + \rho([2, 3, \ldots, m]) \leq \sum_{i=1}^{m} \rho([i])\). We have \(\rho([m]) \leq \sum_{i=1}^{m} k_i\), since from (GDN3) \(\rho([i]) = k_i\) for \(i \in f(S)\). As a result \(\text{dim}(\sum_{i=1}^{r} V_i) = \rho([r]) = \rho([m]) = \sum_{i=1}^{m} k_i\).

The vector subspace \(V_i, i \in [r], i \not\in [m]\) can be described by a matrix \(A_i\) of size \(\sum_{i=1}^{m} k_i \times n\) whose columns span \(A_i\). For \(i \in [m]\), the vector subspace \(V_i\) can be written as the column span of a matrix \(A_i\) of size \(\sum_{i=1}^{m} k_i \times k_i\). Let \(B = [A_1, A_2, \ldots, A_m]\). Since \(\text{dim}(\sum_{i=1}^{m} V_i) = \sum_{i=1}^{m} k_i\), \(B\) is invertible and can be taken to be the \(\sum_{i=1}^{m} k_i \times \sum_{i=1}^{m} k_i\) identity matrix (Otherwise, it is possible to define \(A'_i = B^{-1}A_i\) and \(V'_i\) to be the column span of \(A'_i\) so that \(D(V'_1, V'_2, \ldots, V'_r) = D(V_1, V_2, \ldots, V_r)\)).

The claim is that taking the global encoding matrix of edge \(i\) to be \(A_{f(i)}\) forms a \((k_1, k_2, \ldots, k_m; n)\)-FNC solution for the network. The proof of the claim is as follows: Since \(B\) is an identity matrix, \(A_i x = x_i\) for \(i \in [m]\) and hence (N1) is satisfied. For any node \(v\) in the network, from (GDN4) it follows that \(\text{dim}(\sum_{i\in[m]} \text{Out}(v) V_i) = \text{dim}(\sum_{i\in[m]} \text{In}(v) V_i)\). Hence, \(v\) is a 1-FNC solution, with \(A_{f(v)}\) can be written as \(\sum_{i\in[m]} V_i A_{f(i)}\). Hence, (N2) and (N3) are satisfied. This completes the ‘if’ part of the proof.

For the ‘only if’ part of the proof, assume that the network considered admits a \((k_1, k_2, \ldots, k_m; n)\)-FNC solution, with \(A_{i}, i \in [l]\), being the global encoding matrix associated with edge \(i\). Consider the discrete polymatroid \(D(V_1, V_2, \ldots, V_r)\), where \(V_i\) denotes the column span of \(A_i\). Let \(f(i) = i, i \in [l]\) be the mapping from the edge set of the network to the ground set of the discrete set of the discrete polymatroid. It can be verified that the network is \((k_1, k_2, \ldots, n)\)-discrete polymatroidal with respect to \(D(V_1, V_2, \ldots, V_r)\).

The result in Theorem 1 is illustrated in the following two examples.

Example 25: Consider the network given in Fig. 10. This network admits a linear \((1, 1, 1; 2)\)-FNC solution shown in Fig. 10. Consider the representable discrete polymatroid \(D(V_1, V_2, V_3, V_4)\) defined in Example 9 as explained below, the network shown in Fig. 10 is \((1, 1, 1; 2)\)-discrete polymatroidal with respect to the discrete polymatroid \(D(V_1, V_2, V_3, V_4)\), with the network-discrete polymatroid mapping \(f\) defined as follows: all the incoming and outgoing edges of node \(i\) are mapped to \([m]\). The network considered is \((k_1, k_2, \ldots, k_m; n)\)-discrete polymatroidal with respect to a representable discrete polymatroid \(D(V_1, V_2, \ldots, V_r)\) on the ground set \([r]\). For brevity, the discrete polymatroid \(D(V_1, V_2, \ldots, V_r)\) is denoted as \(D\). Let \(f\) denote the network-discrete polymatroid mapping. Since, \(f\) is one-to-one on the elements of \(S\), assume \(f(S)\) to be \([m]\).

Clearly, \(f\) is one-to-one on the elements of \(S\) and hence
an FNC solution exists

Fig. 10. A network for which scalar and vector solutions do not exist but an FNC solution exists

Lemma 4: The network shown in Fig. 10 has the properties listed in the following lemma.

1) The network shown in Fig. 10 does not admit any scalar or vector solution.

2) For the network in Fig. 10, there does not exist an achievable rate tuple \((k_1/n, k_2/n, k_3/n)\) for which \((k_1/n, k_2/n, k_3/n) > (1/2, 1/2, 1/2)\). Note that the rate tuple achieved by the \((2,1,1;2)\)-FNC solution provided in Fig. 10 is \((1/2, 1/2, 1/2)\).

3) The uniform coding capacity of the network shown in Fig. 10 is equal to \(1/2\). Hence, the \((1,1,1;2)\)-FNC solution provided in Fig. 10 which is a uniform FNC solution, achieves the uniform coding capacity.

Proof: 1) To satisfy the demand of node 5, the edge from 4 to 4 has to carry \(x_1\), which would mean that the demands of the nodes 6 and 7 cannot be met. Hence, the network shown in Fig. 10 does not admit any scalar and vector solution.

2) Consider a \((k_1, k_2, k_3; n)\)-FNC solution for which \(k_i/n \geq 1/2, \forall i \in [3]\). To satisfy the demand at node 5, \(k_1\) out of \(n\) dimensions of the edge joining 4 and 4 should carry \(x_1\). Hence, to satisfy the demands of node 6 and 7, the conditions \((n-k_1) \geq k_2\) and \((n-k_1) \geq k_3\) needs to be satisfied. Since, \(k_i/n \geq 1/2, \forall i \in [3]\), we have \(k_1 + k_2 = k_1 + k_3 = n\), from which it follows that \(k_i/n = 1/2, \forall i \in [3]\).

3) Every \((k, k, k; n)\)-FNC solution for this network should satisfy the condition that \(\frac{k}{n} \leq \frac{1}{2}\). The reason for this is as follows: \(k\) out of \(n\) dimensions of the vector flowing in the edge joining 4 and 4 should carry \(x_1\) to satisfy the demand of node 5. The demands of node 6 and node 7 should be met by what is carried in the remaining \(n-k\) dimensions. Hence, \(n-k\) should be at least \(k\) to be able to satisfy the demands of nodes 6 and 7.

In Example 25 a uniform FNC solution was provided. In Example 26 we provide a network with a non-uniform FNC solution and for which the average rate achieved by the FNC solution provided is greater than the uniform coding capacity.

Example 26: Consider the network given in Fig. 11. A linear \((2,1,1;2)\)-FNC solution for this network is shown in Fig. 11. All the outgoing edges of a node which has only one incoming edge, are assumed to carry the same vector as that of the incoming edge. Consider the discrete polymatroid \(\mathcal{D}(V_1, V_2, V_3, V_4, V_5)\) defined in Example 10. It can be verified that the network shown in Fig. 11 is \((2,1,1;2)\)-discrete polymatroidal with respect to the discrete polymatroid \(\mathcal{D}(V_1, V_2, V_3, V_4, V_5)\) with the network-discrete polymatroid mapping \(f\) defined as follows: all the incoming and outgoing edges of node \(i, i \in \{1, 2, 3, 4, 5\}\) are mapped on to the ground set element \(i\) of the discrete polymatroid \(\mathcal{D}(V_1, V_2, V_3, V_4, V_5)\).

Lemma 5 below lists some of the properties of the network.
Lemma 5: The network given in Fig. 11 has the following properties:

1) The network in Fig. 11 does not admit any scalar or vector solution.
2) The average coding capacity of the network in Fig. 11 is 2/3. Hence, the solution provided in Fig. 11 achieves the average coding capacity.
3) The uniform capacity of the network in Fig. 11 is 1/2. Hence the (2,1,1,2)-FNC solution provided in Fig. 11 achieves an average rate of 2/3 which is strictly greater than the maximum average rate of 1/2 achievable using uniform FNC.

Proof: 1) To deliver message $x_3$ to node 12, the edge connecting nodes 5' and 5 needs to carry $x_3$. In that case, message $x_2$ cannot be delivered to node 8, since the only path from node 2 which generates $x_2$ to node 8 contains the edge joining 5' and 5. Hence, the network in Fig. 11 does not admit any scalar or vector solution.

2) To prove that the average coding capacity is 2/3, it needs to be shown that for all $(k_1,k_2,k_3;n)$-FNC solutions, $k_1 + k_2 + k_3 \leq 2n$. First note that $k_1 \leq n$. This follows from the fact that node 11 demands $x_1$ and there is only one path connecting the nodes 1 and 11. Hence, it can be assumed that the edges $1 \to 6$ and $1 \to 7$ carry $x_1$. Since the nodes 6 and 7 demand $x_2$ and $x_3$ respectively, given the vectors carried in the edges $1 \to 7$ and $4' \to 4$, one must be able to determine $x_1$, $x_2$, and $x_3$. Hence, $k_1 + k_2 + k_3 \leq 2n$. Hence the average coding capacity is upper bounded by 2/3. Since the solution provided Fig. 11 has an average achievable rate of 2/3, the average coding capacity is 2/3.

3) For any $(k,k,k;n)$-FNC solution, $k$ cannot exceed $\frac{1}{2}$. The reason is as follows: $k$ dimensions of the vector transmitted from 5' to 5 should carry $x_3$ and to ensure that node 8 gets $x_2$, $n-k$ should be at least $k$, i.e., $\frac{k}{n} \leq \frac{1}{2}$. A uniform rate of $k/n = 1/2$ can be achieved by choosing $x_{12}$ to be always zero in the FNC solution provided in Fig. 11.

The networks in Example 10 and Example 11 have been constructed from discrete polymatroids using ALGORITHM 2 provided in the next subsection.

B. Construction of Networks with FNC solutions from Discrete Polymatroids

In this subsection, we generalize the algorithm provided in Section IV-B to construct networks from certain classes of discrete polymatroids to a more general class of discrete polymatroids. The network constructed admits a linear FNC solution over $\mathbb{F}_q$, if the discrete polymatroid from which it was constructed is representable over $\mathbb{F}_q$.

Consider a discrete polymatroid $\mathcal{D}$ for which there exists a basis vector $b = \sum_{i \in [r]} k_i e_i \in B(\mathcal{D})$ which satisfies the condition that $\forall i \in (b)_{>0}$, $\rho(i) = k_i$. In this subsection, we only consider those discrete polymatroids which satisfy this condition. Note that the class of discrete polymatroids considered in Section IV-B are a special case of the class of discrete polymatroids considered in this subsection.

ALGORITHM 2 provided below is a generalization of ALGORITHM 1 provided in Section IV-B.

ALGORITHM 2

Step 1: Choose a basis vector $b \in B(\mathcal{D})$ given by $\sum_{i \in [r]} k_i e_i$, which satisfies the condition that $\rho(i) = k_i, \forall i \in (b)_{>0}$. For every $i \in (b)_{>0}$, add a node $i$ to the network with an input edge $e_i$ which generates the message $x_i$. Let $f(e_i) = i$. Define $M = T = (b)_{>0}$.

Step 2: For $i \in [r] \setminus T$, find a vector $u \in C_i(\mathcal{D})$, for which $(u - e_i)_{>0} \subseteq T$. Add a new node $i'$ to the network with incoming edges from all the nodes which belong to $(u - e_i)_{>0}$. Also, add a node $i$ with a single incoming edge from $i'$, denoted as $e_{i',i}$. Define $f(e) = head(e), \forall e \in In(i)$ and $f(e_{i',i}) = i$. Let $T = T \cup \{i\}$. Repeat step 2 until it is no longer possible.

Step 3: For $i \in M$, choose a vector $u$ from $C_i(\mathcal{D})$ for which $(u)_{>0} \subseteq T$. Add a new node $h$ to the network which demands message $x_i$ and which has connections from the nodes in $(u - e_i)_{>0}$. Define $f(e) = head(e), \forall e \in In(h)$. Repeat this step as many number of times as desired.

Step 4: For a basis vector $b \in B(\mathcal{D})$, add a node $j$ which has incoming edges from the nodes which belong to $(b)_{>0}$ and demands all the messages. Define $f(e) = head(e), \forall e \in In(j)$. Repeat this step as many number of times as desired.

Theorem 4 below establishes the connection between the network constructed using ALGORITHM 2 and the discrete polymatroid from which the network was constructed, for a discrete polymatroid representable over $\mathbb{F}_q$.

Theorem 4: A network constructed using ALGORITHM 2 from a discrete polymatroid $\mathcal{D}$ which is representable over $\mathbb{F}_q$, with the basis vector $b$ given by $\sum_{i \in [r]} k_i e_i$, chosen in Step 1, admits a linear $(k_1,k_2,\ldots,k_m;n)$-FNC solution over $\mathbb{F}_q$, where $n = \rho_{max}(\mathcal{D})$.

Proof: The proof of the theorem is given by showing that the constructed network is $(k_1,k_2,\ldots,k_m;n)$-discrete polymatroidal with respect to the representable discrete polymatroid $\mathcal{D}$ from which it is constructed. The satisfaction of (GDN1) is ensured by step 1 of the construction procedure. Since the vector $b = \sum_{i \in [r]} k_i e_i \in B(\mathcal{D})$, it belongs to $\mathbb{D}$ as well and hence (GDN2) is satisfied. Also, since $\rho(i) = k_i, \forall i \in (b)_{>0}$ and $n = \rho_{max}(\mathcal{D})$, (GDN3) is satisfied.

The nodes in the network constructed using ALGORITHM 2 are of five kinds (i) node $i, i \in M$, which are added in step 1, (ii) node $i', i \in [r] \setminus M$, added in Step 2, (iii) node $i, i \in [r] \setminus M$, added in Step 2, (iv) nodes added in Step 3 which demand messages and (v) nodes added in Step 4 which demand messages. Following a similar approach as in the proof of Theorem 2 it can be shown that for a node $x$ which belongs to any one of the five kinds, $\rho(f(In(x))) = \rho(f(In(x) \cup Out(x)))$ and hence (GDN4) is satisfied.

The following examples illustrate the construction procedure provided in ALGORITHM 2.

Example 27: For the discrete polymatroid considered in Example 2 the set of $i$-unit minimal excluded vectors $C_i(\mathcal{D}), i \in [4]$, are given by $C_1(\mathcal{D}) = \{1,0,0,2\}$, $C_2(\mathcal{D}) = \{0,1,1,2\}, C_3(\mathcal{D}) = \{0,1,1,2\}$ and $C_4(\mathcal{D}) = \{1,1,1,1\}$. The construction procedure for the
discrete polymatroid considered in Example 9 is summarized in Table III. The different steps involved in the construction are depicted in Fig. 12. Since, in step 1, the basis vector \( b = (1, 1, 1, 0) \) is used and \( \rho_{\text{max}}(\mathcal{D}) = \rho(\{4\}) = 2 \), the constructed network admits a linear \((1, 1, 1, 2)\)-FNC solution. The linear \((1, 1, 1, 2)\)-FNC solution shown in Fig. 12 is obtained by taking the global encoding matrix of the edge joining \( 4' \) and 4 to be the matrix \( A_4 \) given in Example 10 and that of the edge joining \( 5' \) and 5 to be the matrix \( A_5 \) given in Example 10.

### Table III

| Step | Vector chosen | Node added | Incoming edges from |
|------|---------------|------------|---------------------|
| Step 1 | \( b = (1, 1, 1, 0) \) | \( 1 \) | Input edge (message \( x_1 \)) |
| | \( \rho(\{1\}) = \rho(\{2\}) \) | \( 2 \) | Input edge (message \( x_2 \)) |
| | \( (b)_{>0} = \{1, 2, 3\} \) | \( 3 \) | Input edge (message \( x_3 \)) |
| Step 2 | \( u = (1, 1, 1, 1) \) \( \in \mathcal{C}_4(\mathcal{D}) \) | \( 4' \) | 1, 2 and 3 |
| | \( (u - c_{4,3})_{>0} = \{1, 2, 3\} \) \( \subseteq T = \{1, 2, 3\} \) | \( 4 \) | 4' |
| Step 3 | \( (0, 0, 0, 2) \) \( \in \mathcal{C}_1(\mathcal{D}) \) | 5 | 4 demands \( x_1 \) |
| | \( (0, 1, 1, 2) \) \( \in \mathcal{C}_2(\mathcal{D}) \) | 6 | 3 and 4 demands \( x_2 \) |
| | \( (0, 1, 1, 2) \) \( \in \mathcal{C}_3(\mathcal{D}) \) | 7 | 2 and 4 demands \( x_3 \) |

### Table IV

| Step | Vector chosen | Node added | Incoming edges from |
|------|---------------|------------|---------------------|
| Step 1 | \( b = (2, 1, 1, 0, 0) \) | 1 | Input edge (message \( x_1 \)) |
| | \( \rho(\{1\}) = 2 \) | 2 | Input edge (message \( x_2 \)) |
| | \( \rho(\{2\}) = \rho(\{3\}) = 1 \) | 3 | Input edge (message \( x_3 \)) |
| Step 2 | \( u = (2, 1, 1, 1, 0) \) \( \in \mathcal{C}_4(\mathcal{D}) \) | 4' | 1, 2 and 3 |
| | \( (u - c_{4,3})_{>0} = \{1, 2, 3\} \) \( \subseteq T = \{1, 2, 3\} \) | 4 | 4' |
| Step 3 | \( (2, 1, 0, 0, 0) \) \( \in \mathcal{C}_1(\mathcal{D}) \) | 5 | 1 and 4 demands \( x_2 \) |
| | \( (2, 0, 0, 0, 0) \) \( \in \mathcal{C}_2(\mathcal{D}) \) | 7 | 1 and 4 demands \( x_2 \) |
| | \( (1, 1, 0, 0, 0) \) \( \in \mathcal{C}_1(\mathcal{D}) \) | 9 | 4 and 5 demands \( x_2 \) |
| | \( (1, 1, 0, 0, 0) \) \( \in \mathcal{C}_1(\mathcal{D}) \) | 11 | 2, 3 and 4 demands \( x_3 \) |

Fig. 12. Diagram showing the steps involved in the construction of a network from the discrete polymatroid in Example 10.

**Example 28:** For the discrete polymatroid considered in Example 10 it can be verified that the sets of \( i \)-unit minimal excluded vectors \( \mathcal{C}_i(\mathcal{D}) \), \( i \in [5] \) are given by, \( \mathcal{C}_1(\mathcal{D}) = \{(1, 0, 0, 2, 2), (1, 1, 1, 2, 0)\} \), \( \mathcal{C}_2(\mathcal{D}) = \{(0, 1, 0, 2, 2), (2, 1, 0, 0, 2), (2, 1, 0, 2, 0)\} \), \( \mathcal{C}_3(\mathcal{D}) = \{(2, 0, 1, 2, 0), (0, 0, 1, 0, 2)\} \), \( \mathcal{C}_4(\mathcal{D}) = \{(0, 0, 1, 1, 2), (2, 1, 1, 1, 0)\} \) and \( \mathcal{C}_5(\mathcal{D}) = \{(0, 0, 1, 2, 1), (2, 0, 0, 2, 1), (2, 1, 1, 0, 1)\} \). The construction procedure for the discrete polymatroid considered in Example 10 is summarized in Table IV. The different steps involved in the construction are depicted in Fig. 13 Since, in step 1, the basis vector \( b = (2, 1, 1, 1, 0) \) is used and \( \rho_{\text{max}}(\mathcal{D}) = \rho(\{5\}) = 2 \), the constructed network admits a linear \((2, 1, 1, 2)\)-FNC solution. The linear \((2, 1, 1, 2)\)-FNC solution shown in Fig. 13 is obtained by taking the global encoding matrix of the edge joining \( 4' \) and 4 to be the matrix \( A_4 \) given in Example 10 and that of the edge joining \( 5' \) and 5 to be the matrix \( A_5 \) given in Example 10.

### VI. Linear Index Coding and Discrete Polymatroid Representation

In this section, we explore the connections between linear index coding and representable discrete polymatroids. In Section VI-A it is shown that existence of a linear solution for an index coding problem is connected to the existence of a representable discrete polymatroid which satisfies certain conditions determined by the index coding problem considered. In Section VI-B a construction of an index coding
A linear index code over $\mathbb{F}_q$ of length $c$ and dimension $n$ exists for an index coding problem $\mathcal{I}(X, R)$, if and only if there exists a discrete polymatroid $\mathcal{D}$ representable over $\mathbb{F}_q$ on the ground set $[m + 1]$ with $\rho(\mathcal{D}) = nm$ satisfying the following conditions:

(C1): $\rho(\{i\}) = n, \forall i \in [m], \rho(\{m\}) = nm$ and $\rho(\{m + 1\}) = c$.

(C2): $\forall (x_i, H) \in R$, where $H = \{x_{j_1}, x_{j_2}, \ldots, x_{j_l}\}$, $\rho(\{i\} \cup \{j_1, j_2, \ldots, j_l\} \cup \{m + 1\}) = \rho(\{j_1, j_2, \ldots, j_l\} \cup \{m + 1\})$.

Proof: To prove the ‘if’ part, assume that there exists a discrete polymatroid $\mathcal{D}$ of rank $nm$ representable over $\mathbb{F}_q$ which satisfies (C1) and (C2). Let $V_1, V_2, \ldots, V_m, V_{m+1}$ denote the vector subspaces over $\mathbb{F}_q$, which form a representation for $\mathcal{D}$. From (C1), it follows that the vector subspaces $V_i, i \in [m]$, can be written as the column span of $nm \times n$ matrices $A_i$ over $\mathbb{F}_q$, with $\text{rank}(A_i) = n$ and $\text{rank}([A_1, A_2, \ldots, A_m]) = nm$. Also, the vector subspace $V_{m+1}$ can be written as the column span of a $mn \times c$ matrix $A_{m+1}$ over $\mathbb{F}_q$ which has a rank $c$. Let $B$ denote the invertible $nm \times nm$ matrix given by $[A_1 A_2 \ldots A_m]$. Define $A'_1 = B^{-1}A_i, i \in [m + 1]$. The claim is that the map $f : \mathbb{F}_{nm}^m \rightarrow \mathbb{F}_q^m$ given by $f(y) = yA'_{m+1}$ forms an index code of length $c$ and dimension $n$ over $\mathbb{F}_q$. Consider the receiver node $(x_i, H) \in R$, where $H = \{x_{j_1}, x_{j_2}, \ldots, x_{j_l}\}$. Let $b_H = [x_{j_1}, x_{j_2}, \ldots, x_{j_l}]$. From (C2), it follows that the matrix $A_i$ can be written as $[A_{j_1} A_{j_2} \ldots A_{j_l} A_{m+1}] M_i$, where $M_i$ is of size $(n|H| + c) \times n$. Hence, $A'_1 = [A'_{j_1} A'_{j_2} \ldots A'_{j_l} A'_{m+1}] M_i$. We have, $b_H f(y) = y[A'_{j_1} A'_{j_2} \ldots A'_{j_l} A'_{m+1}] M_i$. The function $\psi_R$ defined as $\psi_R(H, f(y)) = [b_H f(y)] M_i$ forms a valid decoding function at $R$ since $[b_H f(y)] M_i = y[A'_{j_1} A'_{j_2} \ldots A'_{j_l} A'_{m+1}] M_i = y A'_i = x_i$.

To prove the ‘only if’ part, assume that there exists a vector linear index code $f$ of length $c$ and dimension $n$ for the index coding problem $\mathcal{I}(X, R)$. Define $A_i$ to be the $nm \times n$ matrix with $(i, j)^{th}$ entries being one for $i = (l - 1)n + t, j = t$, where $t \in [n]$ and all other entries being zeros. The function $f$ can be written as $f(y) = yA_{m+1}$ where $A_{m+1}$ is a matrix of size $nm \times c$ matrix over $\mathbb{F}_q$. Define $V_i$ to be the column span of $A_i$. It can be verified that the discrete polymatroid $\mathbb{D}(V_1, V_2, \ldots, V_{m+1})$ satisfies the condition (C1) and (C2).

Example 29: Consider the index coding problem presented in [14], with the message set $X = \{x_1, x_2, x_3\}$ and the set of receivers $R = \{(x_3, \{x_1\}), (x_3, \{x_2, x_4\}), (x_1, \{x_2, x_3\}), (x_2, \{x_1, x_3\})\}$. The function $f(X) = [x_1 + x_2 + x_3, x_1 + x_4]$ forms a valid linear index code of length 2 and dimension 1 for this index coding problem over any field $\mathbb{F}_q$. Let $A_i, i \in [4]$, denote the column vector of length 4 whose $i^{th}$ component is 1 and all other components are zeros. Let $A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$. Let $V_i, i \in [5]$ denote the column span of $A_i$ over $\mathbb{F}_q$. It can be verified that the discrete polymatroid $\mathbb{D}(V_1, V_2, V_3, V_4, V_5)$ on the ground set $[5]$ has a rank 4 and it satisfies the conditions (C1) and (C2) given in Theorem 5.
Example 30: Consider the discrete polymatroid provided in Example 30:

\[ \sum_{i \in [r], j \in \rho(\{i\})} \]

For the index coding problem \( \mathcal{I}(\mathbb{Z}, \mathcal{R}) \) defined above, we have 
\[ \mathcal{M}(\mathcal{I}(\mathbb{Z}, \mathcal{R})) = \sum_{i \in [r]} \rho(\{i\}) \]

For a matroid \( \mathbb{M} \), the index coding problem \( \mathcal{I}(\mathbb{M}) \) reduces to the index coding problem \( \mathcal{I}(\mathbb{M}) \) provided in Section IV-B in [14].

Example 30: Consider the discrete polymatroid provided in Example 30. We have 
\[ \text{rank}(\mathbb{D}) = k = 3 \]

(i) The set of messages \( Z = X \cup Y \), where \( X = \{x_1, x_2, x_3\} \) and \( Y = \{y_1, y_2, y_3, y_4, y_5\} \).

(ii) The set of receivers \( R = R_1 \cup R_2 \cup R_3 \) where \( R_1, R_2 \) and \( R_3 \) are as given below:

(a) As mentioned in Example 30, the set of basis vectors \( \mathbb{B}(\mathbb{D}) = \{(1, 1, 1), (1, 2, 0), (2, 0, 1), (2, 1, 0)\} \).

We have:

\[ S_1((1, 1, 1)) = \left\{ (x_1, \{y_1^1, y_2^2, y_3^3\}) : i \in [3], j, k \in [2] \right\} \]

\[ S_1((1, 2, 0)) = \left\{ (x_1, \{y_1^1, y_2^2, y_3^3\}) : i \in [3], j \in [2] \right\} \]

\[ S_1((2, 1, 0)) = \left\{ (x_1, \{y_1^1, y_2^2, y_3^3\}) : i \in [3], j \in [2] \right\} \]

(b) From Example 22 it follows that the minimal excluded vectors for \( \mathbb{D} \) are \( c_1 = (0, 2, 1), c_2 = (2, 1, 1) \) and \( c_3 = (2, 2, 0) \). We have:

\[ S_2(c_1, 2, 1) = \{ (y_2^1, \{y_3^3, y_4^4\}) \} \]

\[ S_2(c_1, 2, 2) = \{ (y_2^1, \{y_3^3\}) \} \]

\[ S_2(c_3, 3, 1) = \{ (y_3^3, \{y_1^1\}) \} \]

\[ S_2(c_3, 1, 1) = \{ (y_1^1, \{y_2^2, y_3^3\}) : i \in [2] \} \]

(c) The set \( R_3 \) is given by:

\[ R_3 = \{(y_1^1, X), (y_2^2, X), (y_3^3, X), (y_4^4, X)\} \]

For the index coding problem constructed in this example, we have 
\[ \mathcal{M}(\mathcal{I}(\mathbb{Z}, \mathcal{R})) = 5 \]

In the following theorem, it is shown that existence of a perfect linear index coding solution of dimension \( n \) over \( \mathbb{F}_q \) for \( \mathcal{I}(\mathbb{Z}, \mathcal{R}) \) implies the existence of a representation for the discrete polymatroid \( n \mathbb{D} \) over \( \mathbb{F}_q \).

Theorem 6: If a perfect linear index coding solution of dimension \( n \) over \( \mathbb{F}_q \) exists for the index coding problem \( \mathcal{I}(\mathbb{Z}, \mathcal{R}) \), then the discrete polymatroid \( n \mathbb{D} \) is representable over \( \mathbb{F}_q \).

Proof: Let \( t = (k + \sum_{i=1}^{3} \rho(\{i\})) \) denote the number of messages in the index coding problem \( \mathcal{I}(\mathbb{Z}, \mathcal{R}) \). If a perfect linear index coding solution of dimension \( n \) over \( \mathbb{F}_q \) exists for the index coding problem \( \mathcal{I}(\mathbb{Z}, \mathcal{R}) \) over \( \mathbb{F}_q \), then from Theorem 1, there exists a discrete polymatroid \( \mathbb{D}^t \) representable over \( \mathbb{F}_q \) of rank \( nt \) on the ground set \( [t+1] \) satisfying conditions (C1) and (C2). Let \( V_1, V_2, \ldots, V_{t+1} \) denote the vector subspaces over \( \mathbb{F}_q \) which form a representation for \( \mathbb{D}^t \). From (C1), it follows that 
\[ \text{dim}(V_i) = n, i \in [t] \] 
and 
\[ \text{dim}(V_{t+1}) = n \sum_{i=1}^{t} \rho(\{i\}) \] 
Let \( A_i, i \in [t] \), denote an \( nt \times n \) matrix whose columns span \( V_i \) and let \( A_{t+1} \) denote an \( nt \times n \sum_{i=1}^{t} \rho(\{i\}) \) matrix whose columns span \( V_{t+1} \). From (C1), it follows that 
\[ \text{rank}([A_1 A_2 \ldots A_t]) = nt \] 
Since the matrix \( B = [A_1 A_2 \ldots A_t] \) is invertible, it can be taken to be the identity matrix of size \( nt \). Otherwise, define \( A_i' = B^{-1}A_i, i \in [t+1] \) and vector subspaces by \( V_i \) which are given by the columns of \( A_i' \) will also form a representation for \( \mathbb{D}^t \).

Let \( A_{t+1} = [C^T \mathbb{D}^T]T \), where \( C \) and \( D \) are matrices of size \( nk \times n \sum_{i=1}^{t} \rho(\{i\}) \) and \( n \sum_{i=1}^{t} \rho(\{i\}) \times n \sum_{i=1}^{t} \rho(\{i\}) \) respectively. The matrix \( D \) has to be full rank, since (C2) needs to be satisfied for receivers \( R \subseteq R_3 \). We can assume \( D \) to be identity matrix, otherwise we can define \( A'_i = A_iD^{-1} \), so that the column spans of \( A_{t+1} \) and \( A_{t+1}' \) are the same. Let \( C_i, i \in [r] \), denote the matrix obtained by taking only the \( n \sum_{i=1}^{t} \rho(\{i\}) \) th columns of \( C_i \). Let \( C_{i,j}, j \in \rho(\{i\}) \) denote the \( nk \times n \) matrix obtained by taking the \( (j-1)n + 1 \) th to \( jn \) th columns of \( C_i \).

Let \( V_i' \) denote the column span of \( C_i \) and \( V_{i,j} \) denote the column span of \( C_{i,j} \). It is claimed that the vector subspaces \( V_i, i \in [r] \), form a representation for the discrete polymatroid \( n \mathbb{D} \). To prove the claim, it needs to be shown that for all \( S \subseteq [r] \), 
\[ \text{dim}(\sum_{i \in S} V_i') = n \rho(S) \]

We have 
\[ \rho(S) = \max_{b \in \mathbb{D}} |b(S)| \]

For \( S \subseteq [r] \), let \( b^S = \arg \max_{b \in \mathbb{D}} |b(S)| \). Let \( b^S \) denote the \( i \) th component of \( b^S \). The vector \( b^S \) should be a basis vector for \( \mathbb{D} \), otherwise there should exist a basis vector \( b^S \) of \( \mathbb{D} \) for which \( b^S < b^S \) and \( |b^S(S)| \leq |b^S(S)| \). Choose \( b^S \) vector subspaces from the set \( \mathcal{V}_i = \{ V_1 : j \in \rho(\{i\}) \} \), denoted as \( V_{i,0}, V_{i,1} \), \ldots, \( V_{i,\rho(i)} \) for every \( i \in [r] \). Let \( V_i = \sum_{j \in [\rho(i)]} V_{i,j} \). From the fact that (C2) needs to be satisfied for the receivers which belong to \( S_1(b^S) \), it follows that 
\[ \text{dim}(\sum_{i \in [r]} V_i) = n |b^S| = n \text{rank}(\mathbb{D}) \]

As a result, we have 
\[ \text{dim}(\sum_{i \in S} V_i) = n |b^S(S)| \]
plete the proof, it needs to be shown that \( \dim(\sum_{i \in S} V_i) \leq n|b^S(S)| \).

Let \( S = \{s_1, s_2, \ldots, s_m\} \cup \{s_{m+1}, s_{m+2}, \ldots, s_l\} \), where \( b^S_{s_i} < \rho(\{s_i\}) \), for \( i \in [m] \) and \( b^S_{s_i} = \rho(s_i) \), for \( i \in \{m+1, m+2, \ldots, l\} \).

Consider the vector \( u = (b^S_{s_i} + 1) \epsilon_{s_i} + \sum_{j \in S \setminus \{s_i\}} b^S_{s_j} \epsilon_{s_j} \). The vector \( u \) is an excluded vector.

Otherwise, the choice of \( b^S, b^S = \arg \max_{b \in \mathbb{D}} |b(S)| \) is contradicted, since \( |u(S)| = |b^S(S)| + 1 \). Let \( u_m \) be a minimum excluded vector for which \( u_m \leq u \). The \( s_1 \)th component of \( u_m \) has to be \( b^S_{s_1} + 1 \), otherwise \( u_m \) satisfies \( u_m < b^S \) and hence cannot be an excluded vector. The vector \( u_m \) can be written as \( (b^S_{s_1} + 1) \epsilon_{s_1} + \sum_{j \in S \setminus \{s_1\}} c^S_{s_j} \epsilon_{s_j} \), where \( c^S_{s_j} \leq b^S_{s_j} \). From the fact that (C2) needs to be satisfied for the receivers which belong to the set \( S_2(u_m, s_1, p) \), \( \forall p \in \{0,1,2, \ldots, 2^{b^S_{s_1}}\} \), it follows that,

\[
\sum_{p \in \rho(\{s_1\})} V'_{s_1, p} \subseteq \sum_{i \in (s_1, m) > 0} V_i + \sum_{j \in [s_1, s_m]} V'_{s_1, a_j}.
\]

From the above equation it follows that \( \sum_{p \in \rho(\{s_1\})} V'_{s_1, p} \subseteq \sum_{i \in (s_1, m) > 0} V_i \cup \sum_{j \in [s_1, s_m]} V'_{s_1, a_j} \). By a similar reasoning, \( V'_{s_j, p} \subseteq \sum_{i \in S \setminus \{s_j\}} V_i, V_j \subseteq [m] \).

Hence, \( \dim(\sum_{i \in S} V_i) \leq \dim(\sum_{i \in S} V_i) = n|b^S(S)| \). This completes the proof.

For a basis vector \( b \in B(\mathbb{D}) \), let \( b_i \) denote the \( i \)th component of \( b \). Define

\[
N(\mathbb{D}) = 1 + \max_{i \in [r]} \sum_{b \in B(\mathbb{D}), b_i > 0} \left( \frac{\rho(i)}{b_i - 1} \right) \prod_{j \in (b_i, 0 \setminus \{i\})} \left( \frac{\rho(j)}{b_j} \right).
\]

The following theorem shows that the converse of Theorem \( \text{[6]} \) holds for fields of sufficiently large size.

**Theorem 7:** If the discrete polymatroid \( n\mathbb{D} \) is representable over \( \mathbb{F}_q \), then a perfect linear solution of dimension \( n \) exists for the index coding problem \( \mathcal{I}(Z, R) \) over \( \mathbb{F}_q \'), where \( \mathbb{F}_q \) is an extension field of \( \mathbb{F}_q \) with size \( q' > N(\mathbb{D}) \).

**Proof:** See Appendix A

From Theorem \( \text{[7]} \) it follows that for \( q > N(\mathbb{D}) \), if the discrete polymatroid \( n\mathbb{D} \) is representable over \( \mathbb{F}_q \), then there exists a perfect linear solution of dimension \( n \) for the index coding problem \( \mathcal{I}(Z, R) \) over \( \mathbb{F}_q \). Combining the results in Theorem \( \text{[6]} \) and Theorem \( \text{[7]} \) we have the following theorem.

**Theorem 8:** For \( q > N(\mathbb{D}) \), a perfect linear solution of dimension \( n \) over \( \mathbb{F}_q \) exists for the index coding problem \( \mathcal{I}(Z, R) \), if and only if the discrete polymatroid \( n\mathbb{D} \) is representable over \( \mathbb{F}_q \).

When specialized to the discrete polymatroid \( \mathbb{D}(\mathbb{M}) \), where \( \mathbb{M} \) is a matroid, the statement of Theorem \( \text{[8]} \) reduces to the following statement: For \( q > N(\mathbb{D}(\mathbb{M})) \), a perfect linear solution of dimension \( n \) over \( \mathbb{F}_q \) exists for the index coding problem \( \mathcal{I}(\mathbb{M})(Z, R) \), if and only if the matroid \( \mathbb{M} \) has a multi-linear representation of dimension \( n \) over \( \mathbb{F}_q \). Note that this is the same as the statement of Theorem 12 in \( \text{[14]} \), with the additional restriction on the field size \( q \). As remarked in Remark 1 in the proof of Theorem \( \text{[1]} \) for the discrete polymatroid \( \mathbb{D}(\mathbb{M}) \), this restriction on the field size is unnecessary and the converse of Theorem \( \text{[8]} \) holds for all \( \mathbb{F}_q \).

It follows from Theorem \( \text{[8]} \) that a perfect linear solution of dimension \( n \) exists over a sufficiently large field for the index coding problem \( \mathcal{I}(Z, R) \), if and only if the discrete polymatroid \( n\mathbb{D} \) is representable and it is stated as the following corollary.

**Corollary 2:** A perfect linear solution of dimension \( n \) exists for the index coding problem \( \mathcal{I}(Z, R) \), if and only if the discrete polymatroid \( n\mathbb{D} \) is representable.

Specializing Corollary \( \text{[2]} \) for the case \( n = 1 \), we have the following corollary.

**Corollary 3:** A scalar perfect linear solution exists for the index coding problem \( \mathcal{I}(\mathbb{Z}, R) \), if and only if the discrete polymatroid \( \mathbb{D} \) is representable over \( \mathbb{F}_q \).

Note that in Theorem \( \text{[7]} \), the condition that the field size \( q' \) should be greater than \( N(\mathbb{D}) \) is only a sufficient condition. Even for a field size less than or equal to \( N(\mathbb{D}) \), a perfect linear solution of dimension \( n \) might exist for the index coding problem \( \mathcal{I}(Z, R) \). This is illustrated in the following example.

**Example 31:** Consider the index coding problem \( \mathcal{I}(Z, R) \), provided in Example \( \text{[30]} \). For this case, we have \( N(\mathbb{D}) = 9 \). Even though the discrete polymatroid \( \mathbb{D} \) has a representation over \( \mathbb{F}_2 \), given in Example \( \text{[8]} \), it is shown in Lemma \( \text{[6]} \) below that the index coding problem \( \mathcal{I}(Z, R) \) does not admit a scalar perfect linear index code over \( \mathbb{F}_2 \). This illustrates the fact that the converse of Theorem \( \text{[8]} \) need not hold when the field size is not sufficiently large. For a field of size greater than 9, a perfect linear solution of dimension \( n \) is guaranteed to exist for \( \mathcal{I}(Z, R) \), provided the discrete polymatroid \( n\mathbb{D} \) is representable over that field. In this example, we provide a perfect linear solution of dimension 1 for \( \mathcal{I}(Z, R) \) over the finite field \( \mathbb{F}_4 = \{0, 1, \alpha, 1 + \alpha\} \) of size 4, where \( \alpha \) is a root of the irreducible polynomial \( x^2 + x + 1 = 0 \) over \( \mathbb{F}_2 \). It can be verified that the function \( f \) given by,

\[
f(Z) = [y_1^2 \ y_1^2 \ y_1^2 \ y_1^2] + [x_1 \ x_2 \ x_3] \begin{bmatrix} 1 & 0 & 1 & 1 \ 0 & 1 & 1 & 0 \ 0 & 0 & 1 + \alpha & 1 \ \end{bmatrix},
\]

forms a scalar perfect linear index code for \( \mathcal{I}(Z, R) \) over \( \mathbb{F}_4 \). Let \( V_1 \) denote the span of the first two columns of \( A \) over \( \mathbb{F}_4 \). Also, let \( V_2 \) denote the span of the third and fourth columns of \( A \), and let \( V_3 \) denote the span of the last column of \( A \) over \( \mathbb{F}_4 \). The vector subspaces \( V_1, V_2 \) and \( V_3 \) form a representation over \( \mathbb{F}_4 \) for the discrete polymatroid \( \mathbb{D} \).

**Lemma 6:** The index coding problem \( \mathcal{I}(Z, R) \) provided in Example \( \text{[30]} \) does not admit a scalar perfect linear solution over \( \mathbb{F}_2 \).
Every scalar perfect linear solution for \( \mathcal{D}^b(Z, R) \) can be written as \( f(Z) = [y_1^1 y_2^2 y_3^3]A + [x_1 x_1 x_3][G_1 G_2 G_3] \), where \( A \) is a 5 \times 5 matrix over \( \mathbb{F}_2 \), \( G_1 \) and \( G_2 \) are 3 \times 2 matrices over \( \mathbb{F}_2 \), and \( G_3 \) is a 3 \times 1 matrix over \( \mathbb{F}_2 \). In order to ensure the existence of decoding matrices for the receivers which belong to the set \( R_3 \), \( A \) needs to be full rank. Hence, without loss of generality, we can assume \( A \) to be the identity matrix, and the only column of \( A \) that can be considered is the first column of \( G_2 \) can be assumed to be 
\[
\begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
\]
and the first column of \( G_2 \) can be assumed to be 
\[
\begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix}
\]. The reason for this is that if the matrix \( G \) can be obtained by the concatenation of \( G_1 \) and the first column of \( G_2 \) is not the identity matrix, taking \( G^{-1}G_i \), \( G_i \in [3] \), the function \( f^{i}(Z) = [y_1^1 y_2^2 y_3^3]A + [x_1 x_1 x_3][G_1 G_2 G_3] \) forms a valid scalar perfect linear index code. The second column of \( G_2 \) and the only column of \( G_3 \) need to be chosen. It is claimed that the only possibility for \( G_3 \) is \( G_3 = [1 1 1]^T \). \( G_3 \) cannot be \( [1 0 0] \), \( [0 1 0] \) and \( [1 1 0] \), since \( \dim(V_1 + V_3) = 3 \). The only other possibilities for \( G_3 \) are \([0 1 0]^T, [0 0 1]^T \) and \([1 1 1]^T \). If \( G_3 = [0 1 1]^T \), it will not be possible to find a decoding function for the receiver nodes \((x_i, \{y_1^1, y_2^2, y_3^3\}) \), \( i \in [3] \). Similarly, if \( G_3 = [1 0 1]^T \) \( (G_3 = [0 0 1]^T) \), it will not be possible to find decoding functions for the receiver nodes \((x_i, \{y_1^1, y_2^2, y_3^3\}) \), \( i \in [3] \). Since \( \dim(V_2 + V_3) = 2 \) and \( \dim(V_2) = 2 \), the only possibilities for the second column of \( G_2 \) are \([1 0 1]^T \) and \([1 1 1]^T \). If the second column of \( G_2 \) is equal to \([1 0 1]^T \) \( ([1 1 1]^T) \), then it will not be possible to find decoding functions for the receiver nodes \((x_i, \{y_1^1, y_2^2, y_3^3\}) \), \( (x_i, \{y_1^1, y_2^2, y_3^3\}) \). This shows that there cannot exist a scalar perfect linear solution over \( \mathbb{F}_2 \) for \( \mathcal{D}^b(Z, R) \).

### VII. Discussion

In this paper, the connections between linear network coding, linear index coding and representable discrete polymatroids were explored. The notion of a discrete polymatroidal network was introduced and it was shown that the existence of a linear solution for a network is connected to the network being discrete polymatroidal. Also, it was shown that a linear solution exists for an index coding problem if and only if there exists a representable discrete polymatroid satisfying certain conditions which are determined by the index coding problem considered. Also, constructions of networks and index coding problems from discrete polymatroids were provided, for which the existence of linear solutions depends on discrete polymatroid representability. This paper considers only representable discrete polymatroids. An interesting problem for future research is to investigate whether any connections exist between non-representable discrete polymatroids and non-linear network/index coding solutions.

### APPENDIX A

**PROOF OF THEOREM 7**

Before proving Theorem 7 some useful lemmas are stated.

**Lemma 7**: If \( b \) is a basis vector of a discrete polymatroid \( D \), then \( nb \) is a basis vector of the discrete polymatroid \( nD \).

**Proof**: Since \( b \in B(D) \), we have \( |b(X)| \leq \rho(X), \forall X \subseteq [r] \). Hence, we have \( |nb(X)| = n|b(X)| \leq n\rho(X) = \rho(nb)(X), \forall X \subseteq [r] \). Hence, it follows that \( nb \in nD \). To complete the proof, it needs to be shown that there does not exist \( u \in nD \) for which \( u > b \). On the contrary, assume that such a \( u \) exists. Then, we have, \( |u| > |nb| = n|b| = n \text{ rank}(D) = \text{rank}(nD) \), which means that \( u \notin nD \), a contradiction.

**Lemma 8**: Consider a representable discrete polymatroid \( D \), with vector subspaces \( V_1, V_2, \ldots, V_r \) forming a representation for \( D \). Let \( b \) be a basis vector of \( D \) and let \( b_t \) denote the \( ith \) component of \( b \). There exists vector subspaces \( V_i' \) of \( V_i \), \( i \in (b)_{>0} \), such that \( \dim(V_i') = b_t \) and \( \dim(\sum_{i \in (b)_{>0}} V_i') = \text{rank}(D) \).

**Proof**: Follows from Lemma 6.3 in [2].

Now we proceed to give the proof of Theorem 7.

**PROOF OF THEOREM 2**

Assume that the vector subspaces \( V_i, i \in [r] \), form a representation for the discrete polymatroid \( nD \) over \( \mathbb{F}_q \). Let \( A_i, i \in [r] \) denote a matrix over \( \mathbb{F}_q \) of size \( nk \times n \rho(i) \) whose columns span \( V_i \). Let \( A'_i = A_i \Gamma_i \), where \( \Gamma_i \) is a matrix of size \( n \rho(i) \times n \rho(i) \), whose entries are indeterminates. Let \( A'_j(j, j \in [\rho(i)]) \), denote the submatrix of \( A'_i \) of size \( nk \times n_b \) obtained by taking only the \((j-1)n + 1^{th} \) to \( j^{th} \) columns of \( A'_i \).

Let \( b \) be a basis vector of \( D \). Let \( b_t \) denote the \( ith \) element of \( b \). Let us define a set of polynomials with the entries of the matrices \( \Gamma_i, i \in [r] \) as the indeterminates as follows: Choose \( b_t \) integers from the set \( \{\rho(i)\}, \forall i \in [r] \), denoted by \( l_1, l_2, \ldots, l_{b_t} \). Consider the polynomial which is the determinant of the \( nk \times nk \) matrix obtained by the concatenation of all the matrices \( A'_j(l_j) \), where \( j \in [b_t] \) and \( i \in [r] \). Let \( P(b) \) denote the set of all polynomials obtainable using the procedure mentioned above, for a fixed basis vector \( b \).

Suppose we want to find an assignment for the indeterminates in the matrices \( \Gamma_i, i \in [r] \), from a field \( \mathbb{F}_{q'} \), such that the following conditions are satisfied:

1. (i) the determinant of all the matrices \( \Gamma_i \) evaluate to non-zero values and
2. (ii) for all the basis vectors \( b \in B(D) \), all the polynomials which belong to the set \( P(b) \) evaluate to non-zero values.

The claim is that from an extension field \( \mathbb{F}_{q'} \) of \( \mathbb{F}_q \) of size greater than \( |N(D)| \), it is possible to find an assignment for the indeterminates such that the above two conditions are satisfied.

**Remark 1**: If the discrete polymatroid \( D \) is of the form \( D(M) \), where \( M \) is a matroid, assigning \( \Gamma_i 's \) to be identity matrices, the two conditions given above are satisfied. There is no need to look for an extension field whose size is greater than \( q \) for this case.

Towards proving the claim, we first show that all the polynomials which belong to the set \( P(b) \) are non-zero polynomials, for all \( b \in B(D) \). To show this, it is enough to show that there exists an assignment of values for the indeterminates for each one of the polynomials which belong to \( P(b) \), possibly
different for different polynomials, such that the polynomials evaluate to non-zero values in $\mathbb{F}_q$.

From Lemma 7 it follows that for $b \in \mathbb{D}$, $nb \in B(n\mathbb{D})$. Since $nb \in B(n\mathbb{D})$, from Lemma 8 it follows that there exists vector subspaces $V_i'$ of $V_i$, $i \in (b)_0$, of dimension $nb_i$ such that $\dim(\sum_{i \in (b)_0} V_i') = nk$. Let $B_i$ denote a matrix whose columns span $V_i'$. Since the columns of the matrix $A_i$ form a basis for $V_i$ and $V_i'$ is a subspace of $V_i$, $B_i$ can be written as $A_i B_i$, where $A_i$ is an $n \rho(i) \times nb_i$ matrix over $\mathbb{F}_q$. The determinant of the $nk \times nk$ matrix obtained by the concatenation of the matrices $B_i, i \in (b)_0$ is non-zero. A polynomial which belongs to $P(b)$ is nothing but the determinant of a $nk \times nk$ matrix obtained by the concatenation of matrices obtained multiplying the matrix $A_i$ by $nb_i$ columns of $G_i$, for every $i \in (b)_0$. Assigning the $nb_i$ columns of $G_i$ to be the columns of $B_i$, the polynomials which belong to $P(b)$ evaluate to non-zero values and hence they are non-zero polynomials.

To find an assignment for the indeterminates in $G_i, i \in [r]$ such that the two conditions (i) and (ii) are satisfied, it suffices to find an assignment for the indeterminates such that the following polynomial evaluates to a non-zero value:

$$P(\Gamma_1, \Gamma_2, \ldots, \Gamma_r) = \left( \prod_{i \in [r]} \det(\Gamma_i) \right) \left( \prod_{b \in B(\mathbb{D})} \prod_{p \in P(b)} p \right).$$

If the field size $q'$ is greater than the degree of the above polynomial in every indeterminant, then an assignment for the indeterminates form $\mathbb{F}_{q'}$, for which the above polynomial evaluates to a non-zero value is guaranteed to exist (follows from Lemma 19.27 in Chapter 19. 20).

Consider an indeterminate which is an entry of the matrix $G_i$, which is denoted by $\gamma_i$. For a basis vector $b$ for which $b_i > 0$, there are $\left( \rho(\gamma_i) \prod_{j \in (b)_0 \setminus \{i\}} \rho(\gamma_j) \right)$ polynomials in $P(b)$ which involve $\gamma_i$ and in each one of these polynomials, the degree of $\gamma_i$ is one. Also, the degree of the polynomial $\det(\Gamma_i)$ in $\gamma_i$ is one. Hence, the degree of the polynomial $P(\Gamma_1, \Gamma_2, \ldots, \Gamma_r)$ in $\gamma_i$ is $1 + \left( \rho(\gamma_i) \prod_{j \in (b)_0 \setminus \{i\}} \rho(\gamma_j) \right)$. Maximizing over all $i \in [r]$, it follows that for $q' > N(\mathbb{D})$ there exists an assignment for $\Gamma_i, i \in [r]$ for which the polynomial $P(\Gamma_1, \Gamma_2, \ldots, \Gamma_r)$ evaluates to a non-zero value. Let $\Omega_i, i \in [r]$, denote one such assignment. Let $G_i = A_i \Omega_i$. Note that $G_i$ has a rank $\rho(\gamma_i)$ and the columns of $G_i$ span $V_i$. Let $\theta = [x_1, x_2 \ldots x_k]$. Define the function $f$ as,

$$f(Z) \triangleq [r_1^1 r_2^1 \ldots r_{\rho(\gamma_i)}^1 r_2^2 \ldots r_{\rho(\gamma_i)}^2 \ldots r_2^3 \ldots r_{\rho(\gamma_i)}^3 \ldots r_2^r \ldots r_{\rho(\gamma_i)}^r] = [y_1^1 y_2^1 \ldots y_{\rho(\gamma_i)}^1 y_2^2 \ldots y_{\rho(\gamma_i)}^2 \ldots y_2^3 \ldots y_{\rho(\gamma_i)}^3 \ldots y_2^r \ldots y_{\rho(\gamma_i)}^r] + \theta[G_1 G_2 \ldots G_r].$$

Let $G_i = [G_i(1) \ G_i(2) \ldots G_i(\rho(\gamma_i))], \theta_i G_i(j) \in [\rho(\gamma_i)]$ are $nk \times n$ matrices. It is shown below that $f$ forms a perfect linear index coding solution of dimension $n$ over $\mathbb{F}_q$ for the index coding problem $\mathcal{P}^q(Z, R)$.

For a receiver node $R = [y'_1, X]$ which belongs to $R_3$, the function $\Psi_R(f(Z), X) = \tau_i - \theta G_i(j)$ forms a valid decoding function.

Consider a receiver node $\{\tilde{e}, \bigcup_{l \in (\tilde{b})_0} \eta_l\}$ which belongs to the set $S_i(b)$, where $j \in [k], \eta_l \subseteq \tilde{e}, |\eta_l| = \tilde{b}_l$, and $\tilde{G} = \{y_1^1, y_2^1, \ldots y_{\rho(\gamma_i)}^i\}$. Consider the matrix $M$ of size $nk \times nk$ obtained by the concatenation of the matrices $G_i(t)$, where $l \in (\tilde{b})_0$ and $t$ is such that $|\eta_l| = \tilde{b}_l$. By virtue of the choice of $G_i$’s, the matrix $M$ is full rank. Let $\chi$ denote the vector obtained by the concatenation of the vectors which belong to $\bigcup_{l \in (\tilde{b})_0} \eta_l$. Let $\omega$ denote the concatenation of $\tau_i$’s for which $l \in (\tilde{b})_0$ and $t$ is such that $|\eta_l| = \tilde{b}_l$. The vector $\theta$ is given by $(\omega - \chi)M^{-1}$. Hence, decoding functions exist for receivers which belong to $R_1$.

Let $c$ be a minimal vector for vector $D$ and let $c_l$ denote the $l$th component of $c$. Consider a receiver node $(y'_i, \Gamma_1 \cup \Gamma_2)$ which belongs to $S_i(c, j, p)$, where $j \in (c)_0$, $p \in [\rho(\gamma_i)]$. Let $\Gamma_i = \bigcup_{l \in (c)_0 \setminus \{j\}} \eta_l, \bigcup_{l \in (c)_0 \setminus \{j\}} \eta_l = c_i, \Gamma_2 \subseteq c_j \setminus \{y_i^p \}, |\Gamma_2| = c_j - 1$. Let $M'$ denote the concatenation of the matrices $G_i(t)$, where $l \in (c)_0$ and $t$ is such that $|\eta_l| \in \Gamma_1 \cup \Gamma_2$. It is claimed that $\text{rank}([M' \ G_i(p)]) = \text{rank}(M')$. Since $c$ is a minimal excluded vector, the vector $u = \sum_{l \in (c)_0} c_i \eta_l (c_j - 1) \epsilon_i, \epsilon_i$ belongs to $D$. Hence, there exists a basis vector $b \in B(\mathbb{D})$ for which $\epsilon_i < b$. Note that $b_j = c_j - 1$, if $b_j > c_j - 1$, then $c < b$ and hence $b$ and $c$ respectively cannot be simultaneously basis and excluded vectors. Define the set of matrices $G_j = \{G_j(o) : y_i^p \in \Gamma_2\}$ and for $i \in (c)_0 \setminus \{j\}$, define $G_i = \{G_i(o) : y_i^p \in \Gamma_1\}$. Note that the matrix $M'$ is the concatenation of the matrices which belong to the sets $G_i, i \in (c)_0$. For $i \in (c)_0$, define $G_i'$ to be a set of matrices which is a subset of size $(b)_0 - (c)_0$ of the set $\{G_i(o) : o \in [\rho(\gamma_i)]\} \setminus G_i$. Note that $G_i'$ is the null set. Let $M''$ denote the $nk \times nk$ matrix obtained by the concatenation of matrices which belong to $G_i$ and $G_i', i \in (b)_0$. The choice of the matrices $G_i$’s ensures that $M''$ is full rank equal to $nk$. Since $M'$ is a submatrix of $M''$ of size $nk \times n(|c| - 1)$, $M'$ should be of rank $n(|c| - 1)$. Note the the vector subspace $V_i$ is the column span of $G_i$. Since, the vector subspaces $V_i, i \in [r]$, form a representation of $\mathbb{D}$, the rank of the matrix $M$ obtained by the concatenation of the matrices $G_i, i \in (c)_0$ should equal $\rho(c_0|c|_0)$, which is equal to $n(|c| - 1)$. Since $[M' \ G_i(p)]$ is a submatrix of $M$, we have,

$$n(|c| - 1) = \text{rank}(M') \leq \text{rank}([M' \ G_i(p)]) \leq \text{rank}(M) = n(|c| - 1).$$

Hence, we have $\text{rank}([M' \ G_i(p)]) = \text{rank}(M')$ and the matrix $G_j(p)$ can be written as $M'W$, where $W$ is of size $n(|c| - 1) \times n$. Let $r'$ denote the concatenation of the vectors which belong to the set $\{y_i^{r'} : y_i^{r'} \in \Gamma_1 \cup \Gamma_2\} and let y' denote the concatenation of the vectors which belong to the set $\Gamma_1 \cup \Gamma_2$. We have, $y_i^{r'} = \tau_i - \theta G_i(p) = \tau_i - \theta M'W = \tau_i - (r' - y')W$. Hence, decoding functions exist for the receivers which belong to $R_2$. This completes the proof of Theorem 7.

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