Martingales on manifolds with time-dependent connection

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Abstract
We define martingales on manifolds with time-dependent connection, extending in this way the theory of stochastic processes on manifolds with time-changing geometry initiated by Arnaudon, Coulibaly and Thalmaier (2008). We show that some, but not all properties of martingales on manifolds with a fixed connection extend to this more general setting.

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AMS subject classification: 53C44, 58J65, 60G44, 60G48

1 Introduction
Stochastic analysis on manifolds with a fixed connection or a fixed Riemannian metric has been studied for a long time, see e.g. the books by Hackenbroch and Thalmaier [6] and Hsu [7]. Motivated by Perelman’s proof of the geometrization and hence the Poincaré conjecture using Ricci flow [11,12,13], Arnaudon, Coulibaly and Thalmaier [1] introduced Brownian motion on a manifold with a time-dependent Riemannian metric. Thanks to the subsequent papers by Coulibaly-Pasquier [4], Kuwada and Philipowski [8,9] and Paeng [10], Brownian motion in such a time-dependent framework is now well understood.

Stochastic analysis on manifolds, however, is not restricted to the study of Brownian motion. Another important topic is martingale theory, which in the case of a fixed connection is treated in depth in e.g. [5,6,7], but which has not yet been studied in the case of a time-dependent connection. The aim of the present paper is to fill this gap.

2 Horizontal lift, stochastic parallel transport and stochastic development on manifolds with time-dependent connection
Let \( M \) be a \( d \)-dimensional differentiable manifold, \( \pi : \mathcal{F}(M) \to M \) the frame bundle and \((\nabla(t))_{t \geq 0} \) a family of linear connections on \( M \) depending smoothly on \( t \). Let \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \geq 0})\) be a filtered probability space. Throughout the whole paper, the notions of martingale, semimartingale, etc. are understood with respect to this filtration. Moreover, all processes are tacitly assumed to be continuous.

Definition 2.1 (cf. [6, Definition 7.135] for the case of a fixed connection). An \( \mathcal{F}(M) \)-valued semimartingale \( U \) is said to be \((\nabla(t))_{t \geq 0}\)-horizontal if

\[ \omega_t(\circ dU_t) = 0, \quad (2.1) \]

where \( \omega_t \) is the \( \mathbb{R}^{d \times d} \)-valued connection form with respect to \( \nabla(t) \).

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Proposition 2.2 (cf. [6] Satz 7.141 for the case of a fixed connection). Let X be an M-valued semimartingale and $U_0$ an $\mathcal{F}_0$-measurable $\mathcal{F}(M)$-valued random variable with $\pi U_0 = X_0$. Then there exists a unique $(\nabla(t))_{t \geq 0}$-horizontal lift $U$ of X starting at $U_0$, i.e. an $\mathcal{F}(M)$-valued semimartingale satisfying (2.2) and $\pi U = X$. Moreover, starting with an arbitrary lift $\tilde{U}$ of $X$ satisfying $\tilde{U}_0 = U_0$, the horizontal lift $U$ can be constructed in the following way: Let

$$\gamma_t := \int_0^t \omega_s( \circ d\tilde{U}_s),$$

(2.2)

and $G$ the solution of the $GL_d(\mathbb{R})$-valued SDE

$$dG_t = - \sum_{\alpha, \beta=1}^d E_{\alpha\beta} G_t \circ d\gamma^\beta_t, \quad G_0 = I,$$

(2.3)

where $E_{\alpha\beta} \in \mathbb{R}^{d \times d}$ is the matrix whose $(ij)$-entry is 1 if $i = \alpha$ and $j = \beta$, and 0 otherwise. Then

$$U_t = \tilde{U}_t G_t.$$  

(2.4)

Proof. We first show that the process $U$ defined by (2.4) is indeed $(\nabla(t))_{t \geq 0}$-horizontal. Letting $\Phi : \mathcal{F}(M) \times GL_d(\mathbb{R}) \to \mathcal{F}(M)$ defined by $\Phi(u, g) := ug$, we have

$$(\Phi^* \omega_t)(u, g) = (R^*_g \omega_t)_u + \theta_g,$$

where $\theta_g := dL_g^{-1} (L_g$ and $R_g$ denoting left resp. right multiplication with g). Since moreover by [6] Bemerkung 7.128 (ii), $R^*_g \omega_t = \text{Ad}(g^{-1}) \omega_t$, we obtain

$$\omega_t( \circ dU_t) = \omega_t( \circ d\Phi(\tilde{U}_t, G_t))$$

$$= (\Phi^* \omega_t)( \circ d(\tilde{U}_t, G_t))$$

$$= (R^*_G \omega_t)( \circ d\tilde{U}_t) + \theta( \circ dG_t)$$

$$= \text{Ad}(G_t^{-1}) \omega_t( \circ d\tilde{U}_t) + dL_{G_t}^{-1} ( \circ dG_t)$$

$$= \text{Ad}(G_t^{-1}) \circ d\gamma_t - \sum_{\alpha, \beta=1}^d G_t^{-1} E_{\alpha\beta} G_t \circ d\gamma^\beta_t = 0.$$

To show uniqueness assume that $U'$ is another $(\nabla(t))_{t \geq 0}$-horizontal lift of $X$ with $U'_0 = U_0$. Then $U = U' G$ with a $GL^d(\mathbb{R})$-valued semimartingale $G = (G_t)_{t \geq 0}$ starting at $I$. The above computation yields $dL_{G_t}^{-1} ( \circ dG_t) = 0$ and hence $dG_t = 0$, so that $G_t = I$ for all $t \geq 0$. \hfill $\square$

Definition 2.3 (cf. [6] Definition 7.144 for the case of a fixed connection). The $(\nabla(t))_{t \geq 0}$-parallel transport along an $M$-valued semimartingale $X$ is the family of isomorphisms $\parallel_{s,t} : T_{X_s} M \to T_{X_t} M \ (0 \leq s \leq t)$ defined by

$$\parallel_{s,t} := U_t U_s^{-1},$$

where $U$ is an arbitrary $(\nabla(t))_{t \geq 0}$-horizontal lift of $X$. (As in the case of a fixed connection, the result does not depend on the choice of the horizontal lift.)

Definition 2.4 (cf. [6] Definition 7.136 for the case of a fixed connection). Let $X$ be an $M$-valued semimartingale, $U_0$ an $\mathcal{F}_0$-measurable $\mathcal{F}(M)$-valued random variable with $\pi U_0 = X_0$, and $U$ the unique $(\nabla(t))_{t \geq 0}$-horizontal lift of $X$ starting at $U_0$. The $\mathbb{R}^d$-valued process

$$Z_t := \int_0^t \partial( \circ dU_s)$$

is called the generator of $U$. \hfill $\square$
is called the \((\nabla(t))_{t \geq 0}\)-antidevelopment of X (or U) with initial frame \(U_0\); here \(\vartheta\) is the canonical \(\mathbb{R}^d\)-valued 1-form on \(\mathcal{F}(M)\),
\[
\vartheta_u(w) = u^{-1}(d\pi w), \quad w \in T_u\mathcal{F}(M).
\]

**Remark 2.5.** A \((\nabla(t))_{t \geq 0}\)-horizontal semimartingale \(U\) can be recovered from its \((\nabla(t))_{t \geq 0}\)-antidevelopment \(Z\) and its initial value \(U_0\) as the solution of the SDE
\[
dU_t = \sum_{i=1}^{d} H^\nabla(t)_i(U_t) \circ dZ^i_t,
\]
where \((H^\nabla(t)_i)_{i=1}^{d}\) are the standard \(\nabla(t)\)-horizontal vector fields on \(\mathcal{F}(M)\), i.e.,
\[
H^\nabla(t)_i(u) = h^\nabla(t)_i(ue_i), \quad u \in \mathcal{F}(M),
\]
with \(h^\nabla(t)_i : T_{\pi(u)}M \rightarrow T_u\mathcal{F}(M)\) of the connection \(\nabla(t)\).

**Proof.** Let \(V\) be the solution of the SDE
\[
dV_t = \sum_{i=1}^{d} H^\nabla(t)_i(V_t) \circ dZ^i_t, \quad V_0 = U_0.
\]
Then \(V\) is a \((\nabla(t))_{t \geq 0}\)-horizontal lift of \(\pi U\) with the same initial value as \(U\), hence \(V = U\). \(\square\)

**Corollary 2.6.** Let \(U\) be a \((\nabla(t))_{t \geq 0}\)-horizontal semimartingale, and \(X := \pi U\). Then we have the following Itô formulas:

1. For all smooth functions \(f\) on \(\mathbb{R}_+ \times \mathcal{F}(M)\) we have
\[
df(t, U_t) = \frac{\partial f}{\partial t}(t, U_t) dt + \sum_{i=1}^{d} H^\nabla(t)_i f(t, U_t) \circ dZ^i_t
\]
   \[= \frac{\partial f}{\partial t}(t, U_t) dt + \sum_{i=1}^{d} H^\nabla(t)_i f(t, U_t) dZ^i_t + \frac{1}{2} \sum_{i,j=1}^{d} H^\nabla(t)_i H^\nabla(t)_j f(t, U_t) d\langle Z^i, Z^j \rangle_t.\]

2. For all smooth functions \(f\) on \(\mathbb{R}_+ \times M\) we have
\[
df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \sum_{i=1}^{d} (U_t e_i) f(t, X_t) \circ dZ^i_t
\]
   \[= \frac{\partial f}{\partial t}(t, X_t) dt + \sum_{i=1}^{d} (U_t e_i) f(t, X_t) dZ^i_t + \frac{1}{2} \sum_{i,j=1}^{d} \text{Hess}^\nabla(t)(U_t e_i, U_t e_j) d\langle Z^i, Z^j \rangle_t.\]

**Remark 2.7.** In the situation of Proposition 2.2 let \(\tilde{Z}_t := \int_0^t \vartheta(\circ d\tilde{U}_s)\). Then
\[
dZ_t = G_t^{-1} \circ d\tilde{Z}_t. \tag{2.5}
\]

**Proof.** Since \(\pi U_t = \pi \tilde{U}_t\), we have
\[
dZ_t = \vartheta(\circ dU_t) = U_t^{-1} \circ \pi_* dU_t = G_t^{-1} \tilde{U}_t^{-1} \circ \pi_* d\tilde{U}_t = G_t^{-1} \vartheta(\circ d\tilde{U}_t) = G_t^{-1} \circ d\tilde{Z}_t. \quad \square\
3 Alternative definition of horizontality in the Riemannian case

In this section we assume that for each \( t \geq 0 \) the connection \( \nabla(t) \) is the Levi-Civita connection of a Riemannian metric \( g(t) \) depending smoothly on \( t \) (we call this the Riemannian case). In this situation it seems natural to require that each \( U_t \) takes values in the \( g(t) \)-orthonormal frames of \( M \), i.e. \( U_t \in \mathcal{O}_{g(t)}(M) \) for all \( t \geq 0 \). To ensure this, one has to add a correction term to (2.1).

**Definition 3.1.** An \( \mathcal{F}(M) \)-valued semimartingale \( U \) is said to be \((g(t))_{t \geq 0}\)-**Riemann-horizontal** if \( U_0 \in \mathcal{O}_{g(0)}(M) \) and

\[
\omega_t(\circ dU_t) = -\frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}(t, U_t e_\alpha, U_t e_\beta) E_{\alpha \beta} dt. \tag{3.6}
\]

In Proposition 3.2 below we will show that any \((g(t))_{t \geq 0}\)-Riemann-horizontal semimartingale \( U \) satisfies indeed \( U_t \in \mathcal{O}_{g(t)}(M) \) for all \( t \geq 0 \). Before doing so we show that the results of the previous section carry over to \((g(t))_{t \geq 0}\)-Riemann-horizontal processes with appropriate modifications:

**Proposition 3.2.** Let \( X \) be an \( M \)-valued semimartingale and \( U_0 \) an \( \mathcal{F}_0 \)-measurable \( \mathcal{O}_{g(0)}(M) \)-valued random variable with \( \pi U_0 = X_0 \). Then there exists a unique \((g(t))_{t \geq 0}\)-**Riemann-horizontal** lift \( U \) of \( X \) starting at \( U_0 \), i.e. an \( \mathcal{F}(M) \)-valued semimartingale satisfying (3.6) and \( \pi U = X \). Moreover, starting with an arbitrary lift \( \tilde{U} \) of \( X \) satisfying \( \tilde{U}_0 = U_0 \), the \((g(t))_{t \geq 0}\)-Riemann-horizontal lift \( U \) can be constructed in the following way: Let

\[
\gamma_t := \int_0^t \omega_s(\circ d\tilde{U}_s), \tag{3.7}
\]

and \( G \) the solution of the \( \text{GL}_d(\mathbb{R}) \)-valued SDE

\[
dG_t = - \sum_{\alpha,\beta=1}^{d} E_{\alpha \beta} G_t \circ d\gamma_t^\alpha - \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}(t, \tilde{U}_t G_t e_\alpha, \tilde{U}_t G_t e_\beta) G_t E_{\alpha \beta} dt, \quad G_0 = I. \tag{3.8}
\]

Then

\[
U_t = \tilde{U}_t G_t. \tag{3.9}
\]

**Proof.** We first show that the process \( U \) defined by (3.9) is indeed \((g(t))_{t \geq 0}\)-Riemann-horizontal. As in the proof of Proposition 2.2 we obtain

\[
\omega_t(\circ dU_t) = (R^*_{G_t} \omega_t)(\circ d\tilde{U}_t) + \theta(\circ dG_t)
\]

\[
= \text{Ad}(G_t^{-1}) \omega_t(\circ d\tilde{U}_t) + dL_{G_t}^{-1}(\circ dG_t)
\]

\[
= \text{Ad}(G_t^{-1}) \circ d\gamma_t - \sum_{\alpha,\beta=1}^{d} G_t^{-1} E_{\alpha \beta} G_t \circ d\gamma_t^\alpha - \frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}(t, \tilde{U}_t G_t e_\alpha, \tilde{U}_t G_t e_\beta) E_{\alpha \beta} dt
\]

\[
= -\frac{1}{2} \sum_{\alpha,\beta=1}^{d} \frac{\partial g}{\partial t}(t, U_t e_\alpha, U_t e_\beta) E_{\alpha \beta} dt.
\]

Uniqueness of \( U \) can be proved in the same way as in Proposition 2.2. \( \square \)

**Definition 3.3.** The \((g(t))_{t \geq 0}\)-**Riemann-parallel transport** along an \( M \)-valued semimartingale \( X \) is the family of isomorphisms \( //_{s,t} : T_{X_s} M \to T_{X_t} M \) \((0 \leq s \leq t)\) defined by

\[
//_{s,t} := U_t U_s^{-1},
\]

where \( U \) is an arbitrary \((g(t))_{t \geq 0}\)-Riemann-horizontal lift of \( X \).
**Definition 3.4.** Let $X$ be an $M$-valued semimartingale, $U_0$ an $\mathcal{F}_0$-measurable $O_{g(0)}(M)$-valued random variable with $\pi U_0 = X_0$, and $U$ the unique $(g(t))_{t \geq 0}$-Riemann-horizontal lift of $X$ starting at $U_0$. The $\mathbb{R}^d$-valued process

$$Z_t := \int_0^t \vartheta(\circ dU_s)$$

is called the $(g(t))_{t \geq 0}$-Riemann-antidevelopment of $X$ (or $U$) with initial frame $U_0$.

**Remark 3.5.** A $(g(t))_{t \geq 0}$-Riemann-horizontal process $U$ can be recovered from its $(g(t))_{t \geq 0}$-antidevelopment $X$ and its initial value $U_0$ as the solution of the SDE

$$dU_t = \sum_{i=1}^d H_i^{(t)}(U_t) \circ dZ_t^i - \frac{1}{2} \sum_{\alpha, \beta = 1}^d \frac{\partial g}{\partial t}(t, U_t e_\alpha, U_t e_\beta) V_{\alpha \beta}(U_t) dt,$$

where $(V_{\alpha \beta})_{\alpha, \beta = 1}^d$ are the canonical vertical vector fields defined as

$$V^{\alpha \beta} f(u) = \frac{d}{ds}\bigg|_{s=0} f(u(I + sE_{\alpha \beta}))$$

($I$ denoting the identity matrix).

**Proof.** Noting that $\omega_t(V_{\alpha \beta}) = E_{\alpha \beta}$ (by the definition of $\omega_t$), this can be proved in the same way as Remark 2.5. \qed

**Corollary 3.6.** Let $U$ be a $(g(t))_{t \geq 0}$-Riemann-horizontal semimartingale, and $X := \pi U$. Then we have the following Itô formulas:

1. For all smooth functions $f$ on $\mathbb{R}_+ \times \mathcal{F}(M)$ we have

$$df(t, U_t) = \frac{\partial f}{\partial t}(t, U_t) dt + \sum_{i=1}^d H_i^{(t)} f(t, U_t) \circ dZ_t^i - \frac{1}{2} \sum_{\alpha, \beta = 1}^d \frac{\partial g}{\partial t}(t, U_t e_\alpha, U_t e_\beta) V_{\alpha \beta} f(t, U_t) dt$$

$$= \frac{\partial f}{\partial t}(t, U_t) dt + \sum_{i=1}^d H_i^{(t)} f(t, U_t) dZ_t^i + \frac{1}{2} \sum_{i,j=1}^d H_i^{(t)} H_j^{(t)} f(t, U_t) d\langle Z^i, Z^j \rangle_t$$

$$- \frac{1}{2} \sum_{\alpha, \beta = 1}^d \frac{\partial g}{\partial t}(t, U_t e_\alpha, U_t e_\beta) V_{\alpha \beta} f(t, U_t) dt. \quad (3.10)$$

2. For all smooth functions $f$ on $\mathbb{R}_+ \times M$ we have

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \sum_{i=1}^d (U_t e_i) f(t, X_t) \circ dZ_t^i$$

$$= \frac{\partial f}{\partial t}(t, X_t) dt + \sum_{i=1}^d (U_t e_i) f(t, X_t) dZ_t^i + \frac{1}{2} \sum_{i,j=1}^d \text{Hess}^{(t)} f(U_t e_i, U_t e_j) d\langle Z^i, Z^j \rangle_t.$$
Proof. We have to show that \( \langle U_i e_i, U_j e_j \rangle_{g(t)} \) is constant for all \( i, j \in \{1, \ldots, d\} \). To do so we fix \( i, j \in \{1, \ldots, d\} \) and apply Itô's formula \(3.10\) to the function \( f(t, u) := \langle u e_i, u e_j \rangle_{g(t)} \). Obviously,

\[
\frac{\partial f}{\partial t}(t, u) = \frac{\partial g}{\partial t}(u e_i, u e_j).
\]

Since \( f \) is constant along horizontal curves in \( \mathcal{F}(M) \), we have

\[
H_i^{\nabla(t)} f = H_i^{\nabla(t)} H_j^{\nabla(t)} f = 0.
\]

Finally, for \( u \in \mathcal{O}_{g(t)}(M) \),

\[
V^\alpha\beta f(t, u) = \frac{d}{ds} \bigg|_{s=0} f(t, u(I + s E_{\alpha\beta}))
= \frac{d}{ds} \bigg|_{s=0} \langle u(I + s E_{\alpha\beta}) e_i, u(I + s E_{\alpha\beta}) e_j \rangle_{g(t)}
= \frac{d}{ds} \bigg|_{s=0} \langle (I + s E_{\alpha\beta}) e_i, (I + s E_{\alpha\beta}) e_j \rangle_{\mathbb{R}^d}
= \langle E_{\alpha\beta} e_i, e_j \rangle_{\mathbb{R}^d} + \langle e_i, E_{\alpha\beta} e_j \rangle_{\mathbb{R}^d}
\begin{cases}
2 & \text{if } \alpha = \beta = i = j, \\
1 & \text{if } i \neq j \text{ and } (\alpha = i, \beta = j \text{ or } \alpha = j, \beta = i), \\
0 & \text{otherwise},
\end{cases}
\]

so that

\[
\frac{1}{2} \sum_{\alpha, \beta=1}^{d} \frac{\partial g}{\partial t}(t, u e_\alpha, u e_\beta) V^\alpha\beta f(t, u) = \frac{\partial g}{\partial t}(u e_i, u e_j) = -\frac{\partial f}{\partial t}(t, u).
\]

\[\square\]

Remark 3.8. In the situation of Proposition \(3.2\) let \( \tilde{Z}_t := \int_0^t \vartheta(\circ d\tilde{U}_s) \). Then

\[
d\tilde{Z}_t = G_t^{-1} \circ d\tilde{Z}_t. \tag{3.11}
\]

Proof. This can be proved in the same way as Remark \(2.7\) \[\square\]

Remark 3.9. Let \( X \) be an \( M \)-valued semimartingale and \( U_0 \) an \( \mathcal{F}_0 \)-measurable \( \mathcal{O}_{g(0)}(M) \)-valued random variable with \( \pi U_0 = X_0 \). Then \( X \) has on the one hand a unique \( (\nabla(t))_{t \geq 0} \)-horizontal lift \( U \) starting at \( U_0 \), \( (\nabla(t))_{t \geq 0} \)-parallel transports \( //_{s, t} \) \( (0 \leq s \leq t) \) and a \( (\nabla(t))_{t \geq 0} \)-antidevelopment \( Z \), and on the other hand a unique \( (g(t))_{t \geq 0} \)-Riemann-horizontal lift \( U_{\text{Riem}} \) starting at \( U_0 \), \( (g(t))_{t \geq 0} \)-Riemann-parallel transports \( //_{s, t}^{\text{Riem}} \) \( (0 \leq s \leq t) \) and a \( (g(t))_{t \geq 0} \)-Riemann-antidevelopment \( Z_{\text{Riem}} \). Proposition \(3.2\) implies that

\[
d(U_t^{-1} U_t^{\text{Riem}}) = -\frac{1}{2} U_t^{-1} \left( \frac{\partial g}{\partial t} \right)^\# U_t^{\text{Riem}} dt
\]

and

\[
d\left(//_{0, t}^{-1} //_{0, t}^{\text{Riem}}\right) = -\frac{1}{2} //_{0, t}^{-1} \left( \frac{\partial g}{\partial t} \right)^\# //_{0, t}^{\text{Riem}} dt.
\]

Moreover in this case the process \( \gamma \) defined in \(2.2\) resp. \(3.7\) and therefore also the process \( G \) defined in \(2.3\) resp. \(3.8\) is of finite variation, so that the Stratonovich differential appearing in \(2.5\) resp. \(3.11\) may be replaced by an Itô differential.
4 Quadratic variation and integration of 1-forms

**Proposition 4.1.** Let $X$ be an $M$-valued semimartingale, $U$ a $(\nabla(t))_{t \geq 0}$-horizontal or $(g(t))_{t \geq 0}$-Riemann-horizontal lift of $X$, and $Z_t := \int_0^t \vartheta(\circ dU_s)$ the corresponding $(\nabla(t))_{t \geq 0}$-antidevelopment resp. $(g(t))_{t \geq 0}$-Riemann-antidevelopment. Then for every adapted $T^*M \otimes T^*M$-valued process $B$ above $X$ (i.e. $B_t \in T^*_X M \otimes T^*_X M$ for all $t \geq 0$) we have

$$\int_0^t B_s(dX_s, dX_s) = \sum_{i,j=1}^d \int_0^t B_s(U_s e_i, U_s e_j) d\langle Z^i, Z^j \rangle_s.$$  

**Proof.** By [6] Lemma 7.56 (iv)] there exist $\ell \in \mathbb{N}$, real-valued adapted processes $(B^{\mu\nu})^\ell_{\mu,\nu=1}$ and functions $h_1, \ldots, h_\ell \in C^\infty(M)$ such that $B_t = \sum_{\mu,\nu=1}^\ell B^{\mu\nu}_t (dh_\mu \otimes dh_\nu)(X_t)$ for all $t \geq 0$. It follows that

$$\int_0^t B_s(dX_s, dX_s) = \sum_{\mu,\nu=1}^\ell \int_0^t (B^{\mu\nu}_s dh_\mu \otimes dh_\nu)(dX_s, dX_s)$$

$$= \sum_{\mu,\nu=1}^\ell \int_0^t B^{\mu\nu}_s d\langle h_\mu(X), h_\nu(X) \rangle_s.$$  

Since by Itô’s formula (Corollary 2.6 resp. Corollary 3.6)

$$d\langle h_\mu(X), h_\nu(X) \rangle_s = \sum_{i,j=1}^d (U_s e_i) h_\mu(X_s)(U_s e_j) h_\nu(X_s) d\langle Z^i, Z^j \rangle_s$$

$$= \sum_{i,j=1}^d (dh_\mu \otimes dh_\nu)(U_s e_i, U_s e_j) d\langle Z^i, Z^j \rangle_s,$$

the claim follows. 

By choosing $B_s = \text{Hess} \nabla^s f(X_s)$ or (in the Riemannian case) $B_s = g(s, X_s)$ we obtain the following two corollaries:

**Corollary 4.2.** For all smooth functions $f$ on $\mathbb{R}_+ \times M$ we have

$$df(t, X_t) = \frac{\partial f}{\partial t}(t, X_t) dt + \sum_{i=1}^d (U_t e_i) f(t, X_t) dZ^i_t + \frac{1}{2} \text{Hess} \nabla^t f(dX_t, dX_t).$$

**Corollary 4.3.** In the Riemannian case,

$$\int_0^t g(s)(dX_s, dX_s) = \sum_{i=1}^d \langle Z^i, Z^i \rangle_t.$$  

**Proposition 4.4.** Let $X$ be an $M$-valued semimartingale, $U$ a $(\nabla(t))_{t \geq 0}$-horizontal or $(g(t))_{t \geq 0}$-Riemann-horizontal lift of $X$, and $Z_t := \int_0^t \vartheta(\circ dU_s)$ the corresponding $(\nabla(t))_{t \geq 0}$-antidevelopment resp. $(g(t))_{t \geq 0}$-Riemann-antidevelopment. Then for every adapted $T^*M$-valued process $\Psi$ above $X$ (i.e. $\Psi_t \in T^*_X M$ for all $t \geq 0$) we have

$$\int_0^t \Psi_s( \circ dX_s) = \sum_{i=1}^d \int_0^t \Psi_s(U_s e_i) \circ dZ^i_s.$$  

Proof. By [6, Lemma 7.56 (v)] there exist \( \ell \in \mathbb{N} \), real-valued adapted processes \( \Psi_1, \ldots, \Psi_\ell \) and functions \( h_1, \ldots, h_\ell \in C^\infty(M) \) such that
\[
\Psi_t = \sum_{\nu=1}^\ell \Psi_\nu^t \, dh_\nu(X_t)
\]
for all \( t \geq 0 \). It follows that
\[
\int_0^t \Psi_s \circ dX_s = \sum_{\nu=1}^\ell \int_0^t (\Psi_\nu^s \, dh_\nu)(\circ dX_s)
= \sum_{\nu=1}^\ell \int_0^t \Psi_\nu^s \circ dh_\nu(X_s).
\]
Since by Itô’s formula (Corollary 2.6 resp. Corollary 3.6)
\[
\int_0^t \Psi_s \circ dX_s = \sum_{i=1}^d \int_0^t \Psi_s^i \circ dh_\nu(U_s e_i) \circ dZ_s^i,
\]
the claim follows.

5 Martingales on manifolds with time-dependent connection

Proposition 5.1 (cf. [6, Satz 7.147 (i)] for the case of a fixed connection). Let \( X \) be an \( M \)-valued semimartingale. Then the following conditions are equivalent:

1. The \( (\nabla(t))_{t \geq 0} \)-antidevelopment of \( X \) is an \( \mathbb{R}^d \)-valued local martingale.

2. For all smooth \( f : M \to \mathbb{R} \) the process
\[
f(X_t) - f(X_0) - \frac{1}{2} \int_0^t \text{Hess}^{(s)} f(dX_s, dX_s)
\]
is a real-valued local martingale.

Moreover, in the Riemannian case, these conditions are equivalent to the condition that the \( (g(t))_{t \geq 0} \)-Riemann-antidevelopment of \( X \) is an \( \mathbb{R}^d \)-valued local martingale.

Definition 5.2. \( X \) is called a \( (\nabla(t))_{t \geq 0} \)-martingale if the equivalent conditions of Proposition 5.1 are satisfied.

Proof of Proposition 5.1. Let \( Z \) be the \( (\nabla(t))_{t \geq 0} \)-antidevelopment or \( (g(t))_{t \geq 0} \)-Riemann-antidevelopment of \( X \), and \( f \in C^\infty(M) \). Then by Corollary 4.2
\[
f(X_t) - f(X_0) - \frac{1}{2} \int_0^t \text{Hess}^{(s)} f(dX_s, dX_s) = \sum_{i=1}^d \int_0^t (U_s e_i) f(X_s) dZ_s^i.
\]
This is a local martingale for all \( f \in C^\infty(M) \) if and only if \( Z \) is an \( \mathbb{R}^d \)-valued local martingale.

Proposition 5.3 (Local expression). A semimartingale \( X \) is a \( (\nabla(t))_{t \geq 0} \)-martingale if and only if in local coordinates
\[
dX_t^i = -\frac{1}{2} \sum_{jk} \Gamma_{jk}^i(t, X_t) d(X^j, X^k)_t
\]
up to the differential of a local martingale.

Proof. This can be proved in the same way as in the case of a fixed connection (see e.g. [5, Proposition 3.7]).
Example 5.4. Let $M = \mathbb{R}$ equipped with the standard metric $g_0$, and let $u$ be a strictly positive smooth function on $\mathbb{R}_+ \times \mathbb{R}$. Define the metric $g(t, \cdot)$ by $g(t, x) = u(t, x)g_0(x)$, and let $\nabla(t)$ be its Levi-Civita connection. Let $b$ and $\sigma$ be smooth functions on $\mathbb{R}_+ \times \mathbb{R}$, and $X$ the solution of the SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t,$$

where $W$ is a standard one-dimensional Brownian motion. Then $X$ is a $(\nabla(t))_{t \geq 0}$-martingale if and only if

$$b = -\frac{u' \sigma^2}{4u}$$
on $\{(t, X_t) | t \geq 0\}$ (the prime denotes differentiation with respect to $x$).

Proof. Taking into account that the unique Christoffel symbol of $\nabla(t)$ equals $u'/2u$, the claim follows immediately from Proposition 5.3.

6 Convergence of martingales

6.1 Local convergence

Proposition 6.1 (cf. [6, Lemma 7.187] or [7, Theorem 2.5.6] for the case of a fixed connection).

Let $U \subseteq M$ be an open subset with the following property: There exists a smooth function $\varphi = (\varphi^1, \ldots, \varphi^d) : M \to \mathbb{R}^d$ such that

- $\varphi|_U$ is bounded,
- $\varphi|_U$ is a diffeomorphism onto its image, and
- $\text{Hess}\nabla(s)\varphi^i(x) \geq 0$ for all $i \in \{1, \ldots, d\}$, all $x \in U$ and all $t \geq 0$.

Then each $(\nabla(t))_{t \geq 0}$-martingale $X$ converges almost surely on the set $\Omega_0 := \{X$ lies eventually in $U\}$.

Remark 6.2. In the case of a fixed connection each point $x \in M$ has a neighbourhood $U$ with that property (see e.g. [6, Lemma 7.187] or [7, Theorem 2.5.6]).

Proof of Proposition 6.1. By Definition 5.2 for each $i \in \{1, \ldots, d\}$ there exists a real-valued local martingale $M^i$ such that

$$\varphi^i(X_t) = \varphi^i(X_0) + M^i_t + A^i_t,$$

where $A^i_t := \int_0^t \text{Hess}\nabla(s)\varphi^i(dX_s, dX_s)$.

Since $\text{Hess}\nabla(s)\varphi^i \geq 0$ on $U$, the process $A$ is eventually non-decreasing and in particular bounded from below on $\Omega_0$. Since $\varphi^i|_U$ is bounded, it follows that the local martingale $M^i$ is bounded from above and hence convergent on $\Omega_0$ (because it is a time-changed Brownian motion). This implies that the process $A^i$ is bounded and hence convergent on $\Omega_0$ (since it is eventually non-decreasing). Consequently, the process $\varphi^i(X)$ converges on $\Omega_0$, and, since $\varphi^i|_U$ is a diffeomorphism onto its image, so does the process $X$.

6.2 Darling-Zheng

An important result of martingale theory in the case of a fixed connection is the convergence theorem of Darling and Zheng (see e.g. [6, Satz 7.190]): Let $X$ be an $M$-valued martingale with respect to a fixed connection $\nabla$, and $g_0$ an arbitrary Riemannian metric on $M$. Then

$$\{X_\infty \text{ exists in } M\} \subset \{\int_0^\infty g_0(dX_s, dX_s) < \infty\} \subset \{X_\infty \text{ exists in } \tilde{M}\},$$
where $\tilde{M}$ is the Alexandrov compactification of $M$. In the case of a time-dependent connection at least the second inclusion does not hold. To see this consider the following example:

**Example 6.3.** In the situation of Example 5.4 take $u(t, x) = \exp(a(t)x)$, $\sigma(t, x) = \sigma(t)$ and $b(t, x) = -\frac{1}{4}a(t)\sigma^2(t)$ with smooth functions $a, \sigma : \mathbb{R}_+ \to \mathbb{R}$. Then $X$ is a $(\nabla(t))_{t \geq 0}$-martingale, and

$$X_t = X_0 - \frac{1}{4} \int_0^t a(s)\sigma(s)^2 ds + \int_0^t \sigma(s)dW_s,$$

so that $\int_0^t g_0(dX_s, dX_s) = \int_0^t \sigma(s)^2 ds$. If $\sigma$ is chosen in such a way that $\int_0^\infty \sigma(s)^2 ds < \infty$, then $\int_0^\infty g_0(dX_s, dX_s) < \infty$, but the function $a$ (being arbitrary) can be chosen in such a way that $X$ does not converge in $\mathbb{R}$.

In the Riemannian case one might hope that $\{\int_0^\infty g(s)(dX_s, dX_s) < \infty\} \subset \{X_\infty \text{ exists in } \tilde{M}\}$. This however turns out to be wrong as well:

**Example 6.4.** In the situation of Example 5.4 take $u(t, x) = u(t)$, $\sigma(t, x) \equiv 1$ and $b(t, x) \equiv 0$. Then $X$ is a $(\nabla(t))_{t \geq 0}$-martingale, and

$$X_t = X_0 + W_t,$$

so that $\int_0^t g(s)(dX_s, dX_s) = \int_0^t u(s)ds$. If $u$ is chosen in such a way that $\int_0^\infty u(s)ds < \infty$, then $\int_0^\infty g(s)(dX_s, dX_s) < \infty$, but obviously $X$ does not converge in $\mathbb{R}$.

7 Uniqueness of martingales with given terminal value

**Proposition 7.1** (cf. [6] Lemma 7.204 for the case of a fixed connection). Let $M_0$ be a submanifold of $M$ which is totally geodesic in $M$ with respect to $\nabla(t)$ for all $t$. Then for each $x_0 \in M_0$ and each $T \geq 0$ there exist an open neighbourhood $V$ of $x_0$ in $M$ and a non-negative function $f \in C^\infty(V)$ satisfying

$$f(x) = 0 \iff x \in M_0$$

and

$$\text{Hess}_{\nabla(s)} f(x) \geq 0 \quad (7.12)$$

for all $s \in [0, T]$ and all $x \in V$.

**Proof.** Let $d_0 := \dim M_0$. Choose coordinates $x_1, \ldots, x_d$ for $M$ on a neighbourhood $O$ of $x_0$ in such a way that

$$O \cap M_0 = O \cap \{x_{d_0+1} = \ldots = x_d = 0\}. \quad (7.13)$$

We will show that for sufficiently small $c > 0$ the function

$$f(x) := \frac{1}{2}(c + |\hat{x}|^2)|\hat{x}|^2,$$

where $\hat{x} := (x_1, \ldots, x_{d_0})$ and $\hat{x} := (x_{d_0+1}, \ldots, x_d)$, does the job on a possibly smaller neighbourhood $V$ of $x_0$. All we have to show is that (7.12) holds provided one chooses $c$ and $V$ small enough.

Let $\Gamma_{ij}^k(s, x)$ be the Christoffel symbols with respect to $\nabla(s)$. Since $M_0$ is totally geodesic and because of (7.13) one has

$$\Gamma_{ij}^k(s, x) = 0, \quad i, j \leq d_0, \quad k \geq d_0 + 1$$

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for all $s \geq 0$ and all $x \in O \cap M$. By the compactness of $[0, T]$ this implies the existence of a constant $C < \infty$ such that

$$|\Gamma_{ij}^k(s, x)| \leq C|x|, \quad i, j \leq d_0, \quad k \geq d_0 + 1. \quad (7.14)$$

Since

$$\text{Hess}^{(s)} f(x) = \sum_{i,j=1}^d H_{ij}(s, x) \, dx_i \otimes dx_j,$$

where

$$H_{ij}(s, x) := \frac{\partial^2 f}{\partial x_i \partial x_j}(x) - \sum_{k=1}^d \Gamma_{ij}^k(s, x) \frac{\partial f}{\partial x_k}(x),$$

it suffices to show that the matrix $H(s, x)$ is positive definite for all $s \in [0, T]$ and all $x \in V \setminus M_0$, provided that $c$ and $V$ are chosen small enough. Using the decomposition of $\{1, \ldots, d\}$ into $I = \{1, \ldots, d_0\}$ and $J = \{d_0 + 1, \ldots, d\}$, this is true if and only if the same statement holds for the block matrix $H^*(s, x)$ defined by

$$H^*(s, x) := \begin{pmatrix} \frac{1}{|\hat{x}|^2}(H_{ij}(s, x))(i,j) & 1 \sum_{k=1}^d \Gamma_{ij}^k(s, x) \frac{\partial f}{\partial x_k}(x) \\ 1 \sum_{k=1}^d \Gamma_{ij}^k(s, x) \frac{\partial f}{\partial x_k}(x) & (H_{ij}(s, x))(i,j) \end{pmatrix}.$$ 

Since

$$\frac{\partial f}{\partial x_k}(x) = \begin{cases} x_k|\hat{x}|^2 & 1 \leq k \leq d_0, \\ x_k(c + |\hat{x}|^2) & d_0 + 1 \leq k \leq d, \end{cases}$$

and

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \begin{cases} \delta_{ij}|\hat{x}|^2 & 1 \leq i, j \leq d_0, \\ 2x_i x_j & 1 \leq i \leq m \text{ and } d_0 + 1 \leq j \leq n, \\ \delta_{ij}(c + |\hat{x}|^2) & m + 1 \leq i, j \leq n, \end{cases}$$

it is easy to see (using $(7.14)$) that on $[0, T] \times (V \setminus M)$ the matrix $H^*(s, x)$ is arbitrarily close to the (obviously positive definite) block matrix

$$\begin{pmatrix} I & 0 \\ 0 & cI \end{pmatrix}$$

provided that $c$ and $V$ are chosen small enough. \hfill \Box

**Corollary 7.2.** Let $M_0$ be submanifold of $M$ which is totally geodesic in $M$ with respect to $\nabla(t)$ for all $t$. Then given $T > 0$ each point $x_0 \in M_0$ has an open neighbourhood $V$ in $M$ with the following property: If $X$ is a $V$-valued $(\nabla(t))_{t \geq 0}$-martingale such that a.s. $X_T \in M_0$, then a.s. $X_t \in M_0$ for all $t \in [0, T]$.

**Proof.** Choose $V$ and $f$ as in Proposition 7.1. Then $f(X)$ is a non-negative submartingale with $f(X_T) = 0$ a.s., hence $f(X) \equiv 0$ a.s. on $[0, T]$. \hfill \Box

**Corollary 7.3** (Uniqueness of $(\nabla(t))_{t \geq 0}$-martingales with given terminal value). Given $T > 0$ each point $x \in M$ has an open neighbourhood $V$ with the following property: If $X$ and $Y$ are two $V$-valued $(\nabla(t))_{t \geq 0}$-martingales such that a.s. $X_T = Y_T$, then a.s. $X_t = Y_t$ for all $t \in [0, T]$.

**Proof.** Apply Corollary 7.2 to the diagonal embedding of $M$ into $M \times M$ equipped with the product connections $\nabla(t) \otimes \nabla(t)$. \hfill \Box
8 Behaviour of semimartingales under maps

Proposition 8.1 (cf. [6 Satz 7.156] for the case of fixed connections). Let \( N \) be another manifold, also equipped with a smooth family of connections \((\nabla(t))_{t \geq 0}\), and let \( f \in C^\infty(\mathbb{R}_+ \times M, N)\). Let \( X \) be a semimartingale on \( M \), \( U \) a \((\nabla(t))_{t \geq 0}\)-horizontal or \((g(t))_{t \geq 0}\)-Riemann-horizontal lift of \( X \), and \( Z \) the corresponding \((\nabla(t))_{t \geq 0}\)-antidevelopment or \((g(t))_{t \geq 0}\)-Riemann-antidevelopment. Moreover, let \( \bar{U} \) be a \((\nabla(t))_{t \geq 0}\)-horizontal or \((\bar{g}(t))_{t \geq 0}\)-Riemann-horizontal lift of the image process \( \bar{X}_t := f(t,X_t) \), and \( \bar{Z} \) the corresponding \((\nabla(t))_{t \geq 0}\)-antidevelopment or \((\bar{g}(t))_{t \geq 0}\)-Riemann-antidevelopment. Then the following formula holds:

\[
d\bar{Z}_t = \bar{U}_t^{-1} \frac{\partial f}{\partial t}(t,X_t)dt + \bar{U}_t^{-1}dU_t dZ_t + \frac{1}{2} \bar{U}_t^{-1} \text{Hess}^{\nabla(t)} \bar{\nabla}(t)f(t,X_t)(dX_t, dX_t). \tag{8.15}
\]

Proof. Let \( n := \dim N \) and \( \varphi \in C^\infty(N) \). Then by Corollary 4.2 and the pullback formula for the quadratic variation (see e.g. [6 Satz 7.61]),

\[
d\varphi(\bar{X}_t) = \sum_{k=1}^{n} (\bar{U}_t e_k) \varphi(\bar{X}_t) d\bar{Z}_t^k + \frac{1}{2} \text{Hess}^{\nabla(t)} \bar{\nabla}(t) \varphi(dX_t, d\bar{X}_t)
\]

\[
= \sum_{k=1}^{n} (\bar{U}_t e_k) \varphi(\bar{X}_t) d\bar{Z}_t^k + \frac{1}{2} (f^* \text{Hess}^{\nabla(t)} \varphi)(dX_t, dX_t). \tag{8.16}
\]

On the other hand, using the Hessian composition formula

\[
\text{Hess}^{\nabla(t)}(\varphi \circ f) = d\varphi \circ \text{Hess}^{\nabla(t)} \bar{\nabla}(t)f + f^* \text{Hess}^{\nabla(t)} \varphi
\]

(see e.g. [6 Satz 7.155]), we obtain

\[
d\varphi(\bar{X}_t) = d(\varphi \circ f)(t,X_t)
\]

\[
= \frac{\partial(\varphi \circ f)}{\partial t}(t,X_t) dt + \sum_{i=1}^{d} (U_t e_i)(\varphi \circ f)(t,X_t) dZ_t^i + \frac{1}{2} \sum_{i,j=1}^{d} \text{Hess}^{\nabla(t)}(\varphi \circ f)(dX_t, dX_t)
\]

\[
= \frac{\partial(\varphi \circ f)}{\partial t}(t,X_t) dt + \sum_{i=1}^{d} (U_t e_i)(\varphi \circ f)(t,X_t) dZ_t^i
\]

\[
+ \frac{1}{2} (d\varphi \circ \text{Hess}^{\nabla(t)} \bar{\nabla}(t)f)(dX_t, dX_t) + \frac{1}{2} (f^* \text{Hess}^{\nabla(t)} \varphi)(dX_t, dX_t). \tag{8.17}
\]

Combining (8.16) and (8.17) we obtain

\[
\sum_{k=1}^{n} (\bar{U}_t e_k) \varphi(\bar{X}_t) d\bar{Z}_t^k = \frac{\partial(\varphi \circ f)}{\partial t}(t,X_t) dt + \sum_{i=1}^{d} (U_t e_i)(\varphi \circ f)(t,X_t) dZ_t^i
\]

\[
+ \frac{1}{2} (d\varphi \circ \text{Hess}^{\nabla(t)} \bar{\nabla}(t)f)(dX_t, dX_t).
\]

Since this holds for all \( \varphi \in C^\infty(N) \), it follows that

\[
\sum_{k=1}^{n} (\bar{U}_t e_k) d\bar{Z}_t^k = \frac{\partial f}{\partial t}(t,X_t) dt + \sum_{i=1}^{d} (U_t e_i)f(t,X_t) dZ_t^i + \frac{1}{2} \text{Hess}^{\nabla(t)} \bar{\nabla}(t)f(dX_t, dX_t)
\]

and hence

\[
d\bar{Z}_t = \bar{U}_t^{-1} \frac{\partial f}{\partial t}(t,X_t) dt + \bar{U}_t^{-1}dU_t dZ_t + \frac{1}{2} \bar{U}_t^{-1} \text{Hess}^{\nabla(t)} \bar{\nabla}(t)f(t,X_t)(dX_t, dX_t). \tag*{\square}
\]
\textbf{Corollary 8.2.} If the connections $\nabla(t)$ are the Levi-Civita connections of Riemannian metrics $g(t)$ and if $X$ is a $(g(t))_{t \geq 0}$-Brownian motion (whose $(g(t))_{t \geq 0}$-Riemann-antidevelopment $W$ is a Euclidean Brownian motion), then
\begin{equation}
\frac{d\hat{Z}_t}{dt} = \hat{U}_t^{-1} \left( \frac{\partial f}{\partial t} + \frac{1}{2} \Delta_{g(t),\hat{\nabla}(t)} f \right) (t, X_t) dt + \hat{U}_t^{-1} df U_t dW_t,
\end{equation}
where $\Delta_{g(t),\hat{\nabla}(t)} u$ is the tension field of $u$ with respect to $g(t)$ and $\hat{\nabla}(t)$.

\textbf{Corollary 8.3.} The function $f$ maps $(g(t))_{t \geq 0}$-Brownian motions to $(\nabla(t))_{t \geq 0}$-martingales if and only if
\[ \frac{\partial f}{\partial t} + \frac{1}{2} \Delta_{g(t),\hat{\nabla}(t)} f = 0 \]
for all $t \geq 0$.

\textbf{Remark 8.4.} In the situation of Proposition 8.1 one may consider the “intrinsic” antidevelopments of $X$, respectively $\tilde{X}$, defined by
\[ \mathcal{A}_t := U_0 Z_t, \quad \text{respectively} \quad \tilde{\mathcal{A}}_t := U_0 \hat{Z}_t, \]
which take values in $T_{X_0} M$, respectively $T_{\tilde{X}_0} N$. Note that
\[ d\mathcal{A}_t = (//_{0,t}^{-1} \circ dX_t, \quad \text{respectively} \quad d\tilde{\mathcal{A}}_t = (//_{0,t}^{-1} \circ d\tilde{X}_t, \]
where $(//_{0,t} \equiv U_t U_0^{-1}$ and $(//_{0,t} \equiv \hat{U}_t \hat{U}_0^{-1}$ denote the parallel transports along $X$, respectively $\tilde{X}$. Then formula (8.18) reads more intrinsically as
\[ d\mathcal{A}_t = (//_{0,t}^{-1} \frac{\partial f}{\partial t} (t, X_t) dt + (//_{0,t}^{-1} df//(//_{0,t}^{-1} d\mathcal{A}_t + \frac{1}{2} (//_{0,t}^{-1} Hess_{\nabla(t),\hat{\nabla}(t)} f (t, X_t) dX_t, dX_t). \]

The same remark applies to formula (8.18) which then reads as
\[ d\tilde{\mathcal{A}}_t = (//_{0,t}^{-1} \left( \frac{\partial f}{\partial t} + \frac{1}{2} \Delta_{g(t),\hat{\nabla}(t)} f \right) (t, X_t) dt + (//_{0,t}^{-1} df//(//_{0,t}^{-1} d\mathcal{A}_t. \]

Recall that in this formula $\mathcal{A}_t = U_0 W_t$ is a Euclidean Brownian motion in $T_{X_0} M$.

\section{Derivative processes, martingales on the tangent bundle and applications to the nonlinear heat equation}

In this section we assume for simplicity that the connections $\nabla(t)$ are torsion-free. Let $\nabla^h(t)$ the complete and $\nabla^h(t)$ the horizontal lift of $\nabla(t)$ to the tangent bundle $TM$. In the same way as in [3] one can obtain the following results.

\textbf{Theorem 9.1} (cf. [3] Theorem 3.1) for the case of a fixed connection). Let $I$ be an open interval containing 0 and $(X_t(s))_{t \geq 0, s \in I}$ a $C^1$-family of continuous $M$-valued $(\nabla(t))_{t \geq 0}$-martingales. Then the $TM$-valued derivative process $(X'_t)_{t \geq 0}$ defined by
\[ X'_t := \frac{\partial}{\partial s} \bigg|_{s=0} X_t(s) \]
is a $(\nabla(t))_{t \geq 0}$-martingale.
**Theorem 9.2** (cf. [3] Corollary 4.4] for the case of a fixed connection). A $TM$-valued semimartingale $J$ is a $(\nabla^h(t))_{t\geq0}$-martingale if and only if

1. its projection $X$ to $M$ is a $(\nabla(t))_{t\geq0}$-martingale, and

2. $d(\|\nabla^{-1}_{0,t} J_t\|) = 0$.

**Theorem 9.3** (cf. [3] Theorem 4.12] for the case of a fixed connection). A $TM$-valued semimartingale $J$ is a $(\nabla'(t))_{t\geq0}$-martingale if and only if

1. its projection $X$ to $M$ is a $(\nabla(t))_{t\geq0}$-martingale, and

2. $d(\Theta_{0,t}^{-1} J_t) = 0$, where $\Theta_{0,t}: TX_0 M \to TX_t M$ denotes the damped parallel transport along $X$, defined by the covariant equation

$$d \left(\|\nabla^{-1}_{0,t} \Theta_{0,t}\|\right) = -\frac{1}{2} \|\nabla^{-1}_{0,t} R^h(t) (\Theta_{0,t}, dX_t) dX_t, \quad \Theta_{0,0} = \text{Id}_{TX_0 M}.$$ 

**Remark 9.4.** In the Riemannian case the condition $d(\|\nabla^{-1}_{0,t} J_t\|) = 0$ in Theorem 9.2 can also be expressed using the Riemann-parallel transport $\|\nabla^{Riem}_{0,t}\|$. Using Remark 3.9, one obtains that it is equivalent to

$$d \left(\|\nabla^{Riem}^{-1}_{0,t} J_t\|\right) = \frac{1}{2} \|\nabla^{Riem}^{-1}_{0,t} \left(\frac{\partial g}{\partial t}\right)\|^\# J_t dt.$$ 

Similarly, the equation defining the damped parallel transport is equivalent to

$$d \left(\|\nabla^{-1}_{0,t} \Theta_{0,t}\|\right) = \frac{1}{2} \|\nabla^{-1}_{0,t} \left(\frac{\partial g}{\partial t}\right)\|^\# J_t - \frac{1}{2} \|\nabla^{-1}_{0,t} R^g(t) (\Theta_{0,t}, dX_t) dX_t. \quad (9.19)$$

If $X$ is a $(g(t))_{t\geq0}$-Brownian motion, (9.19) simplifies to

$$d \left(\|\nabla^{-1}_{0,t} \Theta_{0,t}\|\right) = \frac{1}{2} \|\nabla^{-1}_{0,t} \left(\frac{\partial g}{\partial t}\right)\|^\# - (\text{Ric}(g(t)))^\# J_t dt, \quad (9.20)$$

which coincides with the expression given in [1] Definition 2.1 and [1] Definition 3.1.

Combining Theorems 9.1 and 9.3 one obtains

**Corollary 9.5.** Let $I$ be an open interval containing $0$, $(X_t(s))_{s\in I}$ a $C^1$-family of continuous $M$-valued martingales, $X_t := X_t(0)$, and $(X'_t)_{t\geq0}$ the $TM$-valued derivative process defined by $X'_t := \frac{d}{dt}|_{s=0} X_t(s)$. Then the process $(\Theta_{0,t}^{-1} X'_t)_{t\geq0}$ is a $TX_0 M$-valued local martingale.

Let now $N$ another differentiable manifold and $T_1 < T_2$. Let $(g(t))_{T_1 \leq t \leq T_2}$ be a smooth family of Riemannian metrics on $M$, $\nabla(t)$ the Levi-Civita connection of $g(t)$ and $(\nabla(t))_{T_1 \leq t \leq T_2}$ a smooth family of connections on $N$. Let $u: [T_1, T_2]^2 M \to N$ be a solution of the non-linear heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u(t, \nabla(t) u). \quad (9.21)$$

We fix $x \in M$, let $(X_t)_{0 \leq t \leq T_2 - T_1}$ be an $M$-valued $(g(T_2 - t))_{0 \leq t \leq T_2 - T_1}$-valued Brownian motion starting at $x$, and define

$$\tilde{X}_t := u(T_2 - t, X_t), \quad 0 \leq t \leq T_2 - T_1.$$
Proposition 9.6 (cf. [6, (5.22)]) for the case of fixed Riemannian metrics. Let

$$u : [T_1, T_2] \times M \to N$$

be a solution of Eq. (9.21). Let $\Theta_{0,t} : T_x M \to T_{X_t} M$ be the damped parallel transport along $X$, and $\Theta_{0,t} : T_{X_0} N \to T_{X_t} N$ the damped parallel transport along $X$, where $X$ and $\bar{X}$ are defined as above. Then for each $v \in T_x M$ the $T_{u(T,x)} N$-valued process

$$\Theta_{0,t}^{-1} du(T_2 - t, X_t) \Theta_{0,t} v, \quad 0 \leq t \leq T_2 - T_1,$$

is a local martingale.

Proof. Let $\gamma : \mathbb{R} \to M$ be a smooth curve with $\gamma(0) = x$ and $\dot{\gamma}(0) = v$. By [2, Theorem 3.1] there exists a smooth family $(X_t(s))_{0 \leq t \leq T_2 - T_1, s \in \mathbb{R}}$ of $M$-valued $(g(T_2 - t))_{0 \leq t \leq T_2 - T_1}$-Brownian motions satisfying $X_0(s) = \gamma(s)$ for all $s \in \mathbb{R}$, $X_t(0) = X_t$ for all $t \in [0, T_2 - T_1]$ and

$$\Theta_{0,t}^{-1} \frac{\partial}{\partial s} v = \Theta_{0,t} v.$$ 

Let

$$\bar{X}_t(s) := u(T_2 - t, X_t(s)).$$

By Corollary [5.3] the process $(\bar{X}_t(s))_{0 \leq t \leq T_2 - T_1}$ is an $\bar{N}$-valued $(\bar{V}(t))_{0 \leq t \leq T_2 - T_1}$-martingale for each $s \in \mathbb{R}$. Moreover,

$$\Theta_{0,T_2 - T_1}^{-1} \frac{\partial}{\partial s} \bar{X}_t(s) = du(T_2 - t, X_t) \Theta_{0,t} v.$$ 

Therefore the result follows immediately from Corollary [9.5].

Remark 9.7. If the local martingale in Proposition 9.6 is a true martingale, we obtain the stochastic representation formula

$$du(T_2, x) = E \left[ \Theta_{0,T_2 - T_1}^{-1} du(T_1, X_{T_2 - T_1}) \Theta_{0,T_2 - T_1} \right]. \quad (9.22)$$

Theorem 9.8. Let $M$ be a connected differentiable manifold equipped with a smooth family $(g(t))_{-\infty < t \leq T}$ of Riemannian metrics satisfying

$$\frac{\partial g}{\partial t} + \text{Ric}(g(t)) \geq K > 0 \quad (9.23)$$

(uniformly strict super Ricci flow), and let $(N, \bar{g})$ be a Riemannian manifold of nonpositive sectional curvature. Then every ancient solution $u : [0, T] \times M \to N$ of the non-linear heat equation

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta g(t, \bar{g}) u$$

whose differential is bounded is constant.

Proof of Theorem 9.8. The curvature conditions imply that $\|\Theta_{0,s}\| \leq e^{-Ks/2}$ and $\|\Theta_{0,s}^{-1}\| \leq 1$, so that the local martingale in Proposition 9.6 is bounded and hence a true martingale. The representation formula (9.22) then implies that

$$\|du(t, x)\|_{\bar{g}} \leq e^{-Ks/2} \sup_{y \in M} \|du(t - s, y)\|.$$ 

The claim now follows from letting $s \to \infty$. 

Remark 9.9. More refined representation formulas and Liouville theorems for the nonlinear heat equation in the spirit of [14] will be derived in a subsequent paper.
References

[1] M. Arnaudon, K. A. Coulibaly, A. Thalmaier, Brownian motion with respect to a metric depending on time: definition, existence and applications to Ricci flow. *C. R. Acad. Sci. Paris, Ser. I* 346 (2008), 773–778.

[2] M. Arnaudon, K. A. Coulibaly, A. Thalmaier, Horizontal diffusion in $C^1$ path space, in: C. Donati-Martin, A. Lejay, A. Rouault (Eds.), *Séminaire de Probabilités XLIII* (Springer-Verlag, Berlin, Heidelberg, 2011), 73–94.

[3] M. Arnaudon, A. Thalmaier, Complete lifts of connections and stochastic Jacobi fields. *J. Math. Pures Appl.* 77 (1998), 283–315.

[4] K. A. Coulibaly-Pasquier, Brownian motion with respect to time-changing Riemannian metrics, applications to Ricci flow. *Ann. Inst. H. Poincaré, Probab. Stat.* 47 (2011), 515–538.

[5] M. Émery, Martingales continues dans les variétés différentiables, in: P. Bernard (Ed.), *École d’Été de Probabilités de Saint-Flour XXVIII-1998* (Springer-Verlag, Berlin, Heidelberg, 2000), 1–84.

[6] W. Hackenbroch, A. Thalmaier, *Stochastische Analysis*. B. G. Teubner, Stuttgart, 1994.

[7] E. P. Hsu, *Stochastic Analysis on Manifolds*. American Mathematical Society, Providence, RI, 2002.

[8] K. Kuwada, R. Philipowski, Non-explosion of diffusion processes on manifolds with time-dependent metric. *Math. Z.* 268 (2011), 979–991.

[9] K. Kuwada, R. Philipowski, Coupling of Brownian motions and Perelman’s $L$-functional. *J. Funct. Anal.* 260 (2011), 2742–2766.

[10] S.-H. Paeng, Brownian motion on manifolds with time-dependent metrics and stochastic completeness. *J. Geom. Phys.* 61 (2011), 940–946. (Erratum in *J. Geom. Phys.* 61 (2011), 2417–2418.)

[11] G. Perelman, The entropy formula for the Ricci flow and its geometric applications. Preprint (2002), arXiv:math/0211159v1.

[12] G. Perelman, Ricci flow with surgery on three-manifolds. Preprint (2003), arXiv:math/0303109v1.

[13] G. Perelman, Finite extinction time for the solutions to the Ricci flow on certain three-manifolds. Preprint (2003), arXiv:math/0307245v1.

[14] A. Thalmaier, F.-Y. Wang, A stochastic approach to a priori estimates and Liouville theorems for harmonic maps. *Bull. Sci. Math.* 135 (2011), 816–843.