SIGN-CHANGING SOLUTIONS FOR NON-LOCAL ELLIPTIC EQUATIONS WITH ASYMPTOTICALLY LINEAR TERM

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Abstract. In this article, we study the existence of sign-changing solutions for a problem driven by a non-local integrodifferential operator with homogeneous Dirichlet boundary condition

\[
\begin{align*}
-\mathcal{L}_K u &= f(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \mathbb{R}^n \setminus \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n (n \geq 2) \) is a bounded, smooth domain and \( f(x, u) \) is asymptotically linear at infinity with respect to \( u \). By introducing some new ideas and combining constraint variational method with the quantitative deformation lemma, we prove that there exists a sign-changing solution of problem (1).

1. Introduction. In recent years, the fractional and non-local operators of elliptic type have been widely investigated. This type of operators arise in several areas such as anomalous diffusion, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, quasi-geostrophic flows, multiple scattering and materials science. One can see [4, 9, 10, 11, 16, 17, 24, 25, 26] and their references.

On the other hand, many papers [3, 5, 6, 7, 8, 13, 15, 18, 21, 22, 23] are devoted to the study of the existence of sign-changing solutions of the nonlinear problem

\[
\begin{align*}
-\Delta u &= f(x, u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{in } \partial \Omega,
\end{align*}
\]

where \( \Omega \subset \mathbb{R}^n (n \geq 2) \) is a bounded domain with smooth boundary \( \partial \Omega \), \( f \in C(\Omega \times \mathbb{R}, \mathbb{R}) \). There have been several methods developed in studying sign-changing solutions of problem (2), such as the invariant sets of descending flow method developed by Liu and Sun [5, 18, 23], the minimax method which is established by

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Berestycki and Lions in the classical paper [8], Brouwer’s degree theory and variational method in [2, 15, 29]. When \( f(x, u) \) is asymptotically linear at infinity with respect to \( u \), the existence of solutions for problems like (2) has been studied in some papers, see Tang [15, 19, 27, 31].

We think the first natural question is whether these methods can be adapted to the fractional analogue of problem (2) under asymptotically linear assumption on \( f \), namely

\[
\begin{cases}
(-\Delta)^s u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

where \( f(x, u) \) is asymptotically linear at infinity with respect to \( u \), \((-\Delta)^s(0 < s < 1)\) is the fractional Laplacian operator, which (up to normalization factors) may be defined as

\[-(-\Delta)^s u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{n+2s}} dy.
\]

We should remark that the Dirichlet datum is given in \( \mathbb{R}^n \setminus \Omega \) and not simply on \( \partial \Omega \), consistently with the non-local character of the operator \((-\Delta)^s\).

Furthermore, the second natural problem is the existence of sign-changing solutions for the non-local elliptic problem

\[
\begin{cases}
-\mathcal{L}_K u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^n \setminus \Omega,
\end{cases}
\]

where the non-local integrodifferential operator \( \mathcal{L}_K \) is defined as follows

\[\mathcal{L}_K u(x) = \frac{1}{2} \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K(y) dy, \ x \in \mathbb{R}^n,\]

and \( K : \mathbb{R}^n \setminus \{0\} \mapsto (0, +\infty) \) is a function with the properties that

(\(K_1\)) \( \gamma K \in L^1(\mathbb{R}^n) \), where \( \gamma(x) = \min\{|x|^2, 1|\}; \)
(\(K_2\)) there exists \( \lambda > 0 \) such that \( K(x) \geq \lambda|x|^{-(n+2s)} \) for any \( x \in \mathbb{R}^n \setminus \{0\} \).

A typical example for \( K \) is given by \( K(x) = |x|^{-(n+2s)} \). In this case, \( \mathcal{L}_K \) is the fractional Laplace operator \(-(-\Delta)^s\), see (3).

In this article, the work space \( X \) introduced by Raffaella Servadei and Enrico Valdinoci [24, 25] is defined as a linear space of Lebesgue measurable functions from \( \mathbb{R}^n \) to \( \mathbb{R} \), such that, any function \( u \) restricted in \( X \) belongs to \( L^2(\Omega) \) and the map \((x, y) \mapsto (u(x) - u(y)) \sqrt{K(x-y)} \) is in \( L^2(\mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c), dxdy) \), where \( \Omega^c := \mathbb{R}^n \setminus \Omega \).

The function space \( X \) is equipped with the following norm

\[\|u\|_X = \left( \|u\|_{L^2(\Omega)}^2 + \int_Q |u(x) - u(y)|^2 K(x-y) dxdy \right)^{\frac{1}{2}},\]

where \( Q = \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c) \). And the function space \( X_0 \) is defined as

\[X_0 = \{u \in X : u = 0 \ a.e. \ in \ \mathbb{R}^n \setminus \Omega\}\]

endowed with the norm

\[\|u\|_{X_0} = \left( \int_Q |u(x) - u(y)|^2 K(x-y) dxdy \right)^{\frac{1}{2}},\]

which is equivalent to the usual one defined in (5). Throughout this paper, we also denote the norm \( \| \cdot \|_X \) by \( \| \cdot \| \).
Recall that the fractional Sobolev space $H^s(\Omega)$ is defined as

$$H^s(\Omega) = \left\{ u \in L^2(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^s} \in L^2(\Omega \times \Omega) \right\},$$

endowed with the natural norm

$$\|u\|_{H^s(\Omega)} := \left( \int_\Omega |u|^2 dx + \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}} dxdy \right)^{1/2}.$$  \hspace{1em} (6)

For the basic properties of fractional Sobolev spaces, we refer the interested readers to [12, 14, 16, 26].

Note that even in the model case, where $K(x) = |x|^{-(n+2s)}$, the norms in (5) and (6) are not the same, because $\Omega \times \Omega$ is strictly contained in $Q$. This leads that the classical fractional Sobolev space approach is not sufficient for studying problem (3).

To the best of our knowledge, there are no works concerning the least energy sign-changing solutions for problems (3) and (4) with asymptotically linear case at infinity.

As is explained in [8, Remark 9.2], the minimax method of Berestycki and Lions strongly depends on a kind of nodal structure associated with problem (2), which is unknown for problems (3) and (4). The variational methods used in [5, 21] heavily rely on the following decomposition, for $u \in H^1_0(\Omega)$,

$$\langle \Phi'(u), u^+ \rangle = \langle \Phi'(u^+), u^+ \rangle, \quad \langle \Phi'(u), u^- \rangle = \langle \Phi'(u^-), u^- \rangle,$$

$$\Phi(u) = \Phi(u^+) + \Phi(u^-),$$  \hspace{1em} (7)

where

$$u^+ := \max\{u, 0\}, \quad u^- := \min\{u, 0\},$$

and $\Phi$ is the energy functional of problem (2) given by

$$\Phi(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \int_\Omega F(x, u) dx.$$

But for problem (4) and $u \in X_0$, we have

$$I(u) = I(u^+) + I(u^-) - \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy,$$

$$\langle I'(u), u^+ \rangle = \langle I'(u^+), u^+ \rangle - \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy,$$

where $I$ is the energy functional of problem (4) given by

$$I(u) = \frac{1}{2} \int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)dxdy - \int_\Omega F(x, u) dx.$$

In what follows, we denote

$$B(u) = -\int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)dxdy.$$

It is obvious that $B(u) \geq 0$. Clearly, the functional $I$ does no longer satisfy the decomposition (7). In present paper, motivated by [1, 3, 5, 6, 21, 28], we try to get the sign-changing solution for problem (4) by seeking the minimizer of the energy functional $I$ over the following constraint:

$$\mathcal{M} := \{ u \in X_0 : u^+ \neq 0, \langle I'(u), u^+ \rangle = \langle I'(u), u^- \rangle = 0 \}.$$  \hspace{1em} (8)
To obtain our result, we suppose that the function \( f: \bar{\Omega} \times \mathbb{R}^n \rightarrow \mathbb{R} \) verifying the following conditions:

\[(f_1)\ f \in C^1(\bar{\Omega} \times \mathbb{R}^n), \quad F(x, \tau) = \int_{x_0}^x f(x, t)dt \geq 0, \quad \text{and} \lim_{\tau \to 0} \frac{F(x, \tau)}{\tau} = 0, \quad \text{uniformly in} \ x \in \bar{\Omega};\]
\[(f_2)\ f(x, \tau) = V_\infty(x)\tau + f_\infty(x, \tau), \quad V_\infty \in C(\bar{\Omega}), \quad \text{and} \quad \lim_{\tau \to \infty} \frac{f_\infty(x, \tau)}{\tau} = 0 \quad \text{uniformly in} \ x \in \bar{\Omega};\]
\[(f_3)\ \text{for every} \ x \in \bar{\Omega} \ \text{the function} \ \tau \mapsto \frac{f(x, \tau)}{\tau} \ \text{is strictly increasing for all} \ |\tau| > 0;\]
\[(f_4)\ \inf_{x \in \Omega} V_\infty(x) > \mu := \inf_{u \in \Pi} \max\{\|u^+\|^2, \|u^-\|^2\}, \quad \text{where}\]
\[\Pi := \{u \in X_0 : u^\pm \neq 0, \int_{\Omega} |u^\pm|^2 dx = 1\};\]
\[(f_5)\ \tilde{F}(x, \tau) := \frac{1}{2}f(x, \tau)\tau - F(x, \tau) \rightarrow +\infty \ \text{as} \ \tau \rightarrow +\infty \ \text{uniformly in} \ x \in \bar{\Omega}.
\]

Choose a piecewise constant function in \( \Pi \), we can give a upper bound of \( \mu \):

\[\mu \leq \frac{2\int_Q K(x-y)dxdy}{|\Omega|}.
\]

Now, we give an example to illustrate the feasibility of assumptions \((f_1)-(f_5)\). Let

\[F(x, \tau) = \frac{V_\infty(x)}{2} \tau^2(1 - \frac{1}{1 + |\tau|^\alpha}), \quad \forall x \in \bar{\Omega}, \tau \in \mathbb{R},\]

where \( \alpha \in (0, 2), V_\infty \in C(\Omega), \inf_{\Omega} V_\infty > \mu \). By elementary computations, we can get that \( f \) satisfies \((f_1)-(f_5)\).

**Theorem 1.1.** Suppose that \((f_1)-(f_5)\) hold. Then problem (4) has a sign-changing solution.

**Remark 1.** In fact, the sign-changing solution \( u \in X_0 \) of problem (4) given by the above Theorem 1.1 has the least energy among all sign-changing solutions. For problem (2), we can follow the argument of [5] to show that the least energy sign-changing solution has exactly two nodal domains. But in problem (4), it seems not easy to get the same result.

The rest of the paper is organized as follows. In Section 2, we prove some lemmas, which are crucial to investigate our main result. The proof of Theorem 1.1 is given in Section 3.

2. Preliminaries. Firstly, we collect some useful information in the paper. We will denote \( o(1) \) by the infinitesimal as \( j \rightarrow +\infty \). For the sake of simplicity, the norm \( \| \cdot \|_{L_2(\Omega)} \) will be often written \( \| \cdot \|_2 \).

For the reader’s convenience, we review the main embedding results for the space \( X_0 \).

**Lemma 2.1 ([9, 24, 25]).** The embedding \( X_0 \hookrightarrow L^r(\Omega) \) is continuous for any \( r \in [2, 2^*_s] \), and compact for any \( r \in [2, 2^*_s) \), where \( 2^*_s = \frac{2n}{n-2s} \).

Now, we state some preliminary lemmas which will be used in the last section to prove our main result.

**Lemma 2.2.** Assume \( f \) satisfies \((f_1)-(f_2)\). Let \( u_j \) be a sequence such that \( u_j \rightharpoonup u \) in \( X_0 \), then, up to a subsequence,

\[(i)\ \lim_{j \to \infty} \int_\Omega f(x, u_j)u_jdx = \int_\Omega f(x, u)udx,
\]
Proof. (i) By the compact embedding \(X_0 \hookrightarrow L^p(\Omega)(2 \leq p < 2^*_n)\), taking if necessary a subsequence, we have \(u_j \to u\) in \(L^p(\Omega)\) and \(u_j(x) \to u(x)\) a.e. on \(\mathbb{R}^n\). By a standard discussion, there exists a function \(g \in L^p(\Omega)\) such that

\[
|u(x)|, |u_j(x)| \leq g(x).
\]

By \((f_2)\) and \(u \in L^p(\Omega)\), we have

\[
|f(x, u)|^{\frac{p}{p-1}} \leq C \frac{\lambda}{p-1} (1 + |u|^{p-1}) \frac{p}{p-1} \leq C \frac{\lambda}{1-\frac{2n}{p-1}} (1 + |u|^p) \in L^1(\Omega),
\]

it follows that \(f(\cdot, u) \in L^{\frac{p}{1-\frac{2n}{p-1}}}(\Omega)\). Since

\[
|f(x, u_j) - f(x, u)|^{\frac{p}{p-1}} \leq 2^\frac{p}{p-1} C \frac{\lambda}{1-\frac{2n}{p-1}} (1 + |g|^{p-1}) \frac{p}{p-1} \in L^1(\Omega),
\]

it follows from Lebesgue dominated convergence theorem that

\[
\lim_{j \to \infty} \int_{\Omega} |f(x, u_j) - f(x, u)|^{\frac{p}{p-1}} dx \to 0, \text{ as } j \to \infty.
\]

By the Hölder inequality, we have

\[
\int_{\Omega} (f(x, u_j) - f(x, u)) u_j dx
\leq \left( \int_{\Omega} |f(x, u_j) - f(x, u)|^{\frac{p}{p-1}} dx \right)^{\frac{p-1}{p}} \left( \int_{\Omega} |u_j|^p dx \right)^{\frac{1}{p}} \to 0,
\]

as \(j \to \infty\). Thus

\[
\lim_{j \to \infty} \int_{\Omega} f(x, u_j) u_j dx = \lim_{n \to \infty} \int_{\Omega} f(x, u) u_j dx = \int_{\Omega} f(x, u) u dx.
\]

(ii) By the mean value theorem, there exists \(\lambda \in [0, 1]\) such that

\[
\left| \int_{\Omega} (F(x, u_j) - F(x, u)) dx \right|
= \left| \int_{\Omega} f(x, u + \lambda (u_j - u)) (u_j - u) dx \right|
\leq \int_{\Omega} C (1 + |u + \lambda (u_j - u)|^{p-1}) |u_j - u| dx
\leq C \int_{\Omega} |u_j - u| dx + 2^pC \int_{\Omega} |u_j - u|^{p-1} dx + 2^p C \int_{\Omega} |u_j - u|^p dx
\leq C \|u_j - u\|_1 + 2^p C \|u_j - u\|_p^p \to 0.
\]

Thus

\[
\lim_{j \to \infty} \int_{\Omega} (F(x, u_j) - F(x, u)) dx = 0.
\]

(iii) Recall that \(u_j(x) \to u(x)\) a.e. on \(\mathbb{R}^n\), then, by Fatou’s Lemma, we have

\[
\liminf_{j \to \infty} B(u_j) = \liminf_{j \to \infty} \int_{\mathbb{R}^n} (-u_j^-(x) u_j^+(y) - u_j^+(y) u_j^+(x)) K(x-y) dx dy
\geq \int_{\mathbb{R}^n} (-u^-(x) u^+(y) - u^+(y) u^+(x)) K(x-y) dx dy = B(u).
\]
Lemma 2.3. Under assumptions \((f_1)-(f_3)\), for any \(u \in X_0\) with \(u^\pm \neq 0\), \(s,t \geq 0\) and \((s-1)^2+(t-1)^2 \neq 0\), we have
\[
I(u) > I(tu^+ + su^-) + \frac{1-t^2}{2} (I'(u), u^+) + \frac{1-s^2}{2} (I'(u), u^-) + B(u)(t-s)^2.
\]
Proof. For \(\tau \neq 0\), \((f_3)\) yields
\[
f(x,s) < \frac{f(x,\tau)}{|\tau|} |s|, \quad |s| < |\tau|;
\]
\[
f(x,s) > \frac{f(x,\tau)}{|\tau|} |s|, \quad |s| > |\tau|.
\]
It follows that
\[
\frac{1-\theta^2}{2} \tau f(x,\tau) > F(x,\tau) - F(x,\theta \tau), \quad \forall x \in \Omega, \tau \neq 0, \theta \geq 0 \text{ and } \theta \neq 1.
\]
So, we have
\[
I(u) - I(tu^+ + su^-) = \frac{1-t^2}{2} (I'(u), u^+) + \frac{1-s^2}{2} (I'(u), u^-) + (t-s)^2 B(u)
\]
\[
+ \int_\Omega \left[ \frac{1-t^2}{2} f(x,u^+)(u^+ - F(x,u^+)) - F(x,tu^+) \right] dx
\]
\[
+ \int_\Omega \left[ \frac{1-s^2}{2} f(x,u^-)(u^- - F(x,u^-)) - F(x,su^-) \right] dx
\]
\[
> \frac{1-t^2}{2} (I'(u), u^+) + \frac{1-s^2}{2} (I'(u), u^-) + (t-s)^2 B(u),
\]
for all \(s,t \geq 0\) and \((s-1)^2+(t-1)^2 \neq 0\). \(\square\)

Corollary 1. Under assumptions \((f_1)-(f_3)\), we have
\[
I(u) \geq I(tu^+ + su^-), \quad \forall u \in M, \quad s,t \geq 0.
\]
Now, we define the set \(E_0\) as follows
\[
E_0 := \{ u \in X_0 : \|u^\pm\|^2 - \int_\Omega V_\infty|u^\pm|^2 dx < 0 \}.
\]

Lemma 2.4. Under assumptions \((f_1)-(f_4)\), we have \(E_0 \neq \emptyset\) and \(M \subset E_0\).

Proof. It follows from \((f_1)-(f_4)\) that
\[
f_\infty(x,\tau) = o(|\tau|) \text{ as } |\tau| \to 0,
\]
\[
\tau \mapsto \frac{f_\infty(x,\tau)}{|\tau|} \text{ is strictly increasing on } (-\infty,0) \cup (0,\infty), \text{ which, together with } f_\infty(x,\tau) = o(|\tau|) \text{ as } |\tau| \to \infty \text{ uniformly } x \in \Omega, \text{ yields}
\]
\[
\tau f_\infty(x,\tau) < 0, \quad \forall \tau \neq 0.
\]
In view of \((f_4)\) and the definition of \(\mu\), we deduce that there exists \(V \in \Pi\) such that
\[
\max\{\|v^+\|^2, \|v^-\|^2\} \leq \frac{\mu + \inf_\Pi V_\infty}{2}.
\]
It follows that
\[
\|v^+\|^2 - \int_\Omega V_\infty |v^+|^2 dx \leq \max\{\|v^+\|^2, \|v^-\|^2\} - \inf_\Omega V_\infty \leq \frac{\mu - \inf_\Omega V_\infty}{2} < 0.
\]
Hence, we have \(v \in E_0\). This shows that \(E_0 \neq \emptyset\) because of \((f_4)\). Moreover, we can easily verify that for any \(u \in \mathcal{M}\),
\[
\|u^+\|^2 - \int_\Omega V_\infty |u^+|^2 dx = \int_\Omega f_\infty(x, u^+)u^+ dx - B(u) < 0.
\]
This shows that \(\mathcal{M} \subset E_0\). □

**Lemma 2.5.** Suppose that \((f_1)-(f_4)\) are satisfied. If \(u \in E_0\), then there exist \(s_u, t_u > 0\) such that
\[
\langle I'(t_u u^+ + s_u u^-), u^+ \rangle = 0 \quad \text{and} \quad \langle I'(t_u u^+ + s_u u^-), u^- \rangle = 0.
\]
As a consequence, \(t_u u^+ + s_u u^- \in \mathcal{M}\).

**Proof.** By \((f_1)\) and \((f_2)\), for any \(\varepsilon > 0\), there exists \(C_\varepsilon > 0\) such that
\[
|uf(x, u)| \leq \varepsilon |u|^2 + C_\varepsilon |u|^p, \quad p \in (2, 2^*_\ast).
\]
By \((9)\), \(X_0 \hookrightarrow L^2(\Omega)\) and \(X_0 \hookrightarrow L^p(\Omega)\), for all \(s, t > 0\) we have
\[
\langle I'(t_u u^+ u^-), tu^+ \rangle = t^2\|u^+\|^2 + stB(u) - \int_\Omega tu^+ f(x, tu^+) dx
\]
\[
\geq t^2\|u^+\|^2 - \int_\Omega tu^+ f(x, tu^+) dx \geq t^2\|u^+\|^2 - \varepsilon t^2\|u^+\|^2 - C\varepsilon t^2\|u^+\|^2
\]
\[
\geq (1 - \varepsilon \gamma_2) t^2\|u^+\|^2 - C\varepsilon t^2\|u^+\|^p.
\]
We choose \(\varepsilon = \frac{1}{272}\), then there exists \(r > 0\) small enough such that \(\langle I'(ru^+ + su^-), ru^+ \rangle > 0\) for all \(s > 0\), and similarly there exists \(\tilde{r} > 0\) small enough such that \(\langle I'(tu^+ + \tilde{r}u^-), \tilde{r}u^- \rangle > 0\) for all \(t > 0\).

By Lemma 2.4 and \((f_2)\), for any fixed \(s > 0\) we have
\[
\langle I'(tu^+ + su^-), tu^+ \rangle = t^2\left(\|u^+\|^2 - \int_\Omega V_\infty |u^+|^2 dx\right) + t(sB(u)) - \int_\Omega tu^+ f_\infty(x, tu^+) dx \to -\infty,
\]
as \(t \to +\infty\). So, there exists \(R > 0\) sufficiently large such that \(\langle I'(Ru^+ + su^-), Ru^- \rangle < 0\) and similarly we can find \(\tilde{R} > 0\) such that \(\langle I'(tu^+ + \tilde{R}u^-), \tilde{R}u^- \rangle < 0\) for any fixed \(t > 0\). As a consequence, we have proved the existence of suitable \(0 < r < R\) such that, for all \(t, s \in [r, R]\) it holds
\[
\langle I'(ru^+ + su^-), ru^+ \rangle > 0, \quad \langle I'(tu^+ + \tilde{r}u^-), \tilde{r}u^- \rangle > 0,
\]
\[
\langle I'(Ru^+ + su^-), Ru^- \rangle < 0, \quad \langle I'(tu^+ + \tilde{R}u^-), \tilde{R}u^- \rangle < 0.
\]
By applying Miranda’s theorem [20] we can conclude. □

**Lemma 2.6.** Suppose that \((f_1)-(f_5)\) are satisfied. Let \(\{w_j\} \subset \mathcal{M}\), such that
\[
I(w_j) \to \inf_{v \in \mathcal{M}} I(v) = c_0 > 0.
\]
Then \(\{w_j\}\) is bounded in \(X_0\).
Proof. Arguing by contradiction, suppose that \( \|w_j\| \to \infty \). Let \( v_j = \frac{w_j}{\|w_j\|} \), then \( \|v_j\| = 1 \). By Lemma 2.1, passing to a subsequence, we may assume that there exists \( v \in X_0 \) such that \( v_j \rightharpoonup v \) weakly in \( X_0 \), \( v_j \to v \) strongly in \( L^p(\Omega), 2 \leq p < 2^* \). If \( v = 0 \), then \( v_j \to 0 \) in \( L^p(\Omega) \). By \((f_1),(f_2)\), for any \( \varepsilon > 0 \), there exists \( C_\varepsilon > 0 \) such that

\[
F(x, \tau) \leq \varepsilon |\tau|^2 + C_\varepsilon |\tau|^p, \quad 2 < p < 2^*.
\]

(11)

Fix \( R > [2(1 + c_0)]^{\frac{1}{p}} \), then we have

\[
\limsup_{j \to +\infty} \int_{\Omega} F(x, Rv_j)dx \leq \varepsilon R^2 \lim_{j \to +\infty} |v_j|^2 + C_\varepsilon R^p \lim_{j \to +\infty} |v_j|^p = 0.
\]

(12)

By (12) and Corollary 1, one has

\[
c_0 = I(w_j) + o(1) \geq I(\frac{R}{\|w_j\|}w_j) + o(1)
\]

\[
= \frac{R^2}{2} - \int_{\Omega} F(x, Rv_j)dx + o(1) = \frac{R^2}{2} + o(1)
\]

\[
> c_0 + 1 + o(1),
\]

which is a contradiction. Thus \( v \neq 0 \). Denote \( A = \{x \in \Omega : v(x) \neq 0\} \). Then for \( x \in A \), we have \( \lim_{j \to +\infty} |w_j(x)| = +\infty \). By \((f_3)\) and Fatou’s Lemma, it follows that

\[
c_0 + 1 \geq \lim_{j \to +\infty} [I(w_j) - \frac{1}{2}\langle I'(w_j), w_j \rangle] \geq \liminf_{j \to +\infty} \int_A \tilde{F}(x, w_j)dx = +\infty.
\]

This contradiction shows that \( \{w_j\} \) is bounded in \( X_0 \). \( \square \)

**Definition 2.7.** For each \( v \in X_0 \) with \( v^\pm \neq 0 \), let us consider the function \( h^v : [0, +\infty) \times [0, +\infty) \to \mathbb{R} \) given by

\[
h^v(t, s) = I(tv^+ + sv^-)
\]

and its gradient \( \Phi^v : [0, +\infty) \times [0, +\infty) \to \mathbb{R}^2 \) defined by

\[
\Phi^v(t, s) = (\Phi_1^v(t, s), \Phi_2^v(t, s)) = \left( \frac{\partial h^v}{\partial t}(t, s), \frac{\partial h^v}{\partial s}(t, s) \right)
\]

(13)

\[
= \langle I'(tv^+ + sv^-), v^+ \rangle, \langle I'(tv^+ + sv^-), v^- \rangle \rangle.
\]

Furthermore, we consider the Jacobian matrix of \( \Phi^v \):

\[
(\Phi^v)'(t, s) = \begin{pmatrix}
\frac{\partial \Phi_1^v}{\partial t}(t, s) & \frac{\partial \Phi_1^v}{\partial s}(t, s) \\
\frac{\partial \Phi_2^v}{\partial t}(t, s) & \frac{\partial \Phi_2^v}{\partial s}(t, s)
\end{pmatrix}.
\]

And we have the following Lemma.

**Lemma 2.8.** Suppose that \((f_1) - (f_3)\) are satisfied. Then,

(i) there exists \( w \in M \) such that \( I(w) = c_0 > 0 \);

(ii) \( \det(\Phi^w)'(1, 1) > 0 \).
Thus, we have $\langle I'(w_j), w_j^+ \rangle = 0$. By Lemma 2.1 and the Hölder inequality, for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|w_j^\pm\|^2 = \int_\Omega w_j^\pm f(x, w_j^\pm)dx - B(w_j) \leq \int_\Omega w_j^\pm f(x, w_j^\pm)dx$$

$$\leq \varepsilon \|w_j^\pm\|_2^2 + C_\varepsilon \int_\Omega |w_j^\pm|^2|w_j^\pm|^{p-2}dx$$

$$\leq \varepsilon \gamma_2 \|w_j^\pm\|^2 + C_\varepsilon \gamma_p \|w_j^\pm\|^2 \|w_j^\pm\|_p^{p-2},$$

where $\gamma_2, \gamma_p > 0$ are the embedding constants. Let $\varepsilon = \frac{1}{2\gamma_2}$, it follows that

$$\|w_j^\pm\|_p^p \geq \frac{1}{2\gamma_2}.\varepsilon$$

By the compactness of the embedding $X_0 \hookrightarrow L^p(\Omega)$ for $2 \leq p < 2^*_s$, we obtain

$$\|w_j^\pm\|_p^{p-2} = \lim_{j \to \infty} \|w_j^\pm\|_p^{p-2} \geq \frac{1}{2\gamma_2}.\varepsilon.$$

Thus, $w^\pm \neq 0$, so $w = w^+ + w^-$ is sign-changing. By Lemma 2.5, there exist $s_w, t_w > 0$ such that

$$\langle I'(t_w w^+ + s_w w^-), w^\pm \rangle = 0, \quad \langle I'(t_w w^+ + s_w w^-), w^- \rangle = 0 \quad (14)$$

and $t_w w^+ + s_w w^- \in \mathcal{M}$. Now, we prove that $s_w, t_w \leq 1$. Since $w_j \in \mathcal{M}$, we have $\langle I'(w_j), w_j^\pm \rangle = 0$ or equivalently

$$\|w_j^\pm\|^2 + B(w_j) - \int_\Omega w_j^\pm f(x, w_j^\pm)dx = 0. \quad (15)$$

The weak lower semicontinuity of the norm $\| \cdot \|$ in $X_0$ yields

$$\|w_j^\pm\|^2 \leq \liminf_{n \to \infty} \|w_j^\pm\|^2. \quad (16)$$

By using (15), (16) and Lemma 2.2, we get

$$\langle I'(w), w^\pm \rangle \leq 0 \quad \text{and} \quad \langle I'(w), w^- \rangle \leq 0. \quad (17)$$

Now we show that $s_w, t_w \leq 1$. Since $t_w w^+ + s_w w^- \in \mathcal{M}$, we get

$$t_w^2 \|w^+\|^2 + t_w s_w B(w) = \int_\Omega t_w w^+ f(x, t_w w^+)dx,$$

$$s_w^2 \|w^-\|^2 + t_w s_w B(w) = \int_\Omega s_w w^- f(x, s_w w^-)dx.$$

Assume that $t_w \geq s_w$, since $B(w) \geq 0$, we have

$$t_w^2 \|w^+\|^2 + t_w^2 B(w) \geq \int_\Omega t_w w^+ f(x, t_w w^+)dx. \quad (18)$$
Since $\langle I'(w), w^+ \rangle = 0 (w \in \mathcal{M})$, we deduce that

$$\|w^+\|^2 + B(w) = \int_{\Omega} w^+ f(x, w^+) dx$$

which together with (18) gives

$$\int_{\Omega} \left[ \frac{f(x, tw^+)}{tw^+} - \frac{f(x, w^+)}{w^+} \right] |w^+|^2 dx \leq 0.$$ 

By (f3) we can infer that $t_w \leq 1$. Then, $s_w, t_w \leq 1$.

Next, we show that $I(t_w w^+ + s_w w^-) = c_0$ and $t_w = s_w = 1$. It follows from the condition (f3) that for every $x \in \Omega$

$$\tau \mapsto \frac{1}{2} \tau f(x, \tau) - F(x, \tau)$$

is strictly increasing for all $\tau > 0$, and

$$\tau \mapsto \frac{1}{2} \tau f(x, \tau) - F(x, \tau)$$

is strictly decreasing for all $\tau < 0$. (19)

By using $t_w w^+ + s_w w^- \in \mathcal{M}, w_j \in \mathcal{M}, (19), (10), s_w, t_w \in (0, 1]$ and Lemma 2.2 we can see

$$c_0 \leq I(t_w w^+ + s_w w^-) = I(t_w w^+ + s_w w^-) - \frac{1}{2} (I'(t_w w^+ + s_w w^-), t_w w^+ + s_w w^-)$$

$$= \int_{\Omega} \left[ \frac{1}{2} f(x, t_w w^+ + s_w w^-)(t_w w^+ + s_w w^-) - F(x, t_w w^+ + s_w w^-) \right] dx$$

$$= \int_{\Omega \cap \{w \geq 0\}} \left[ \frac{1}{2} t_w w^+ f(x, t_w w^+) - F(x, t_w w^+) \right] dx$$

$$+ \int_{\Omega \cap \{w \leq 0\}} \left[ \frac{1}{2} s_w w^- f(x, s_w w^-) - F(x, s_w w^-) \right] dx$$

$$\leq \int_{\Omega} \left[ \frac{1}{2} w^+ f(x, w^+) - F(x, w^+) \right] dx + \int_{\Omega} \left[ \frac{1}{2} w^- f(x, w^-) - F(x, w^-) \right] dx$$

$$= \lim_{j \to \infty} \int_{\Omega} \left[ \frac{1}{2} w_j^+ f(x, w_j^+) - F(x, w_j^+) \right] dx + \int_{\Omega} \left[ \frac{1}{2} w_j^- f(x, w_j^-) - F(x, w_j^-) \right] dx$$

$$= \lim_{j \to \infty} \left[ I(w_j) - \frac{1}{2} \langle I'(w_j), w_j \rangle \right]$$

$$= \lim_{j \to \infty} I(w_j) = c_0.$$ 

Then, by the above calculation, we can infer that $I(t_w w^+ + s_w w^-) = c_0$ and $t_w = s_w = 1$.

(ii) Firstly, let us observe that

$$\frac{\partial \Phi_1^w}{\partial t}(t, s) = \|w^+\|^2 - \int_{\Omega} f'(x, tw^+)(w^+)^2 dx$$

$$\frac{\partial \Phi_1^w}{\partial s}(t, s) = \|w^-\|^2 - \int_{\Omega} f'(x, tw^-)(w^-)^2 dx$$

$$\frac{\partial \Phi_1^w}{\partial t}(t, s) = \frac{\partial \Phi_2^w}{\partial t}(t, s) = -\int_{\mathbb{R}^n} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x - y)dxdy.$$ 

(20)
By \((f_3)\), it is easy to see that for every \(x \in \bar{\Omega}\)
\[\tau^2 f'(x, \tau) - \tau f(x, \tau) > 0\]
for all \(\tau \neq 0\). \hfill (21)

Then, by using the fact that \(w \in \mathcal{M}\), \((20)\) and \((21)\) we have
\[\det(\Phi^w)'(1, 1)
= \left[\|w^+\|^2 - \int_{\Omega} f'(x, w^+)(w^+)^2 dx \right] \left[\|w^-\|^2 - \int_{\Omega} f'(x, w^-)(w^-)^2 dx \right]
- \left[ \int_{\mathbb{R}^n} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x - y)dx dy \right]^2
= \left[ \int_{\Omega} ((w^+)^2 f'(x, w^+) - w^+ f(x, w^+))dx
- \int_{\mathbb{R}^n} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x - y)dx dy \right]
\times \left[ \int_{\Omega} ((w^-)^2 f'(x, w^-) - w^- f(x, w^-))dx
- \int_{\mathbb{R}^n} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x - y)dx dy \right]
- \left[ \int_{\mathbb{R}^n} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x - y)dx dy \right]^2 > 0.
\]

\[\square\]

3. Proof of main result.

Proof of Theorem 1.1. Let \(\{w_j\}\) be a sequence in \(\mathcal{M}\) such that
\[\lim_{j \to \infty} I(w_j) = c_0.\]

By Lemma 2.8-(i), we see that \(c_0 > 0\) can be achieved at \(w \in \mathcal{M}\). Next, we show that \(I(w) = 0\). Arguing by contradiction, we assume \(I'(w) \neq 0\). Therefore, there exist \(\delta > 0\) and \(\rho > 0\) such that
\[v \in X_0, \|v - w\| \leq 3\delta \Rightarrow \|I'(v)\| \geq \rho.\]

Let \(D := (\frac{1}{2}, \frac{3}{2}) \times (\frac{1}{2}, \frac{3}{2})\). It follows from Lemma 2.3 that
\[\bar{c}_0 := \max_{(t,s) \in \partial D} I(tw^+ + sw^-) < c_0. \hfill (22)\]

For \(\varepsilon := \min\{\delta \max_{\bar{\Omega}} - \delta, \frac{\delta}{2}, \frac{\delta}{2}\}\), \(B := \{v \in X_0 : \|v - w\| \leq \delta\}\). Lemma 2.3 in [30] yields a deformation \(\eta \in C([0, 1] \times X_0, X_0)\) such that
\[(a) \ \eta(1, v) = v \text{ if } I(v) < c_0 - 2\varepsilon \text{ or } I(v) > c_0 + 2\varepsilon;
(b) \ \eta(1, I^{c_0+\varepsilon} \cap B) \subset I^{c_0-\varepsilon};
(c) \ I(\eta(1, w)) \leq I(w), \text{ for all } w \in E_0,\]
where \(I^{c_0+\varepsilon} = \{v \in X_0 : I(v) \leq c_0 + \varepsilon\}\). By Corollary 1, \(I(tw^+ + sw^-) \leq I(w) = c_0\), for \(s, t \geq 0\), then from (b) it follows that
\[I(\eta(1, tw^+ + sw^-)) \leq c_0 - \varepsilon, \forall s, t \geq 0, |t - 1|^2 + |s - 1|^2 < \frac{\delta^2}{2\|w\|^2}. \hfill (23)\]

On the other hand, by (c) and Corollary 1, one has
\[I(\eta(1, tw^+ + sw^-)) \leq I(tw^+ + sw^-) < I(w) = c_0, \hfill (24)\]
for \( t, s \geq 0 \), \(|t-1|^2 + |s-1|^2 \geq \frac{\delta^2}{2|w|^2} \). Combining (23) and (24), we have
\[
\max_{(t,s)\in \tilde{D}} I(\eta(1, tw^+ + sw^-)) < c_0. \tag{25}
\]

We now prove that \( \eta(1, tw^+ + sw^-) \cap \mathcal{M} \neq \emptyset \) for some \((t, s) \in \tilde{D}\), which contradicts with the definition of \( c_0 \). Let
\[
\Psi_0(t, s) := \langle I'(tw^+ + sw^-), w^+ \rangle, \langle I'(tw^+ + sw^-), w^- \rangle
\]
and
\[
\Psi_1(t, s) := \langle I'(\eta(1, tw^+ + sw^-)), (\eta(1, tw^+ + sw^-))^+, \frac{t}{s} \rangle.
\]

By Lemma 2.8-(ii) and degree theory, we can derive that
\[
\deg(\Psi_0, D, 0) = \text{sgn}(\text{det}(\Phi^w)'(1, 1)) = 1.
\]

From (22) and (a) it follows that \( \eta(1, tw^+ + sw^-) = tw^+ + sw^- \) on \( \partial D \). Consequently,
\[
\deg(\Psi_1, D, 0) = \deg(\Psi_0, D, 0) = 1,
\]
and so, \( \Psi_1(\bar{t}, \bar{s}) = 0 \) for some \( \bar{t}, \bar{s} \in D \), that is \( \eta(1, \bar{t}w^+ + \bar{s}w^-) \in \mathcal{M} \), which contradicts (25). From this, we conclude that \( w \) is a critical point of \( I \).

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