CONTRA \( \alpha g\mu \)-CONTINUOUS FUNCTIONS

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ABSTRACT

In this paper we introduce and discuss some basic properties of contra \( \alpha g\mu \)-continuous functions.

Keywords: \( \alpha g\mu \)-open, contra \( \alpha g\mu \)-continuous

1. INTRODUCTION

The notion \( \alpha g\mu \)-closed sets in topological spaces was introduced by R. Devi, V. Vijayalakshmi and V. Kokilavani. The concept of Contra continuous mappings was introduced and investigated by J. Dontchev. In this paper we introduce the notion of contra \( \alpha g\mu \)-continuous functions and discuss their basic properties.

2. PRELIMINARIES

2.1. Definition

A subset \( A \) of space \( (X, \tau) \) is called

1. a generalized closed (briefly \( g \)-closed) set (Njastad, 1965) if \( \text{cl}(A) \subseteq U \) and \( U \) is open in \( (X, \tau) \); the complement of a \( g \)-closed set is called a \( g \)-open set,

2. an \( \alpha \)-generalized closed (briefly \( g\alpha \)-closed) set (Maki et al., 1994) if \( \text{acl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is open in \( (X, \tau) \),

3. a \( \mu \)-closed set (Veera Kumar, 2005) if \( \text{cl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( g\alpha \)-open in \( (X, \tau) \),

4. an \( \alpha g\mu \)-closed set (Devi et al., 2007) if \( \text{acl}(A) \subseteq U \) whenever \( A \subseteq U \) and \( U \) is \( \mu \)-open in \( (X, \tau) \).

2.2. Definition

1. A space \( (X, \tau) \) is said to be \( T_{\alpha g\mu} \) (Devi et al., 2007) if every \( \alpha g\mu \)-closed set is closed in \( X \).

2. A space \( (X, \tau) \) is said to be locally indiscrete (Atick, 1997) if every open subset of \( X \) is closed.

3. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be regular set connected (Atick, 1997) if \( f^{-1}(V) \) is clopen in \( (X, \tau) \) for every regular open set \( V \) of \( (Y, \sigma) \).

4. A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be perfectly continuous (Atick, 1997) if \( f^{-1}(V) \) is clopen in \( X \) for every open set \( V \) of \( Y \).

2.3. Definition

A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be contra continuous (Dontchev, 1996) if for every open set in \( (Y, \sigma) \) there exist a closed set in \( (X, \tau) \).

2.4. Definition

A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be \( \alpha g\mu \)-continuous (Devi et al., 2007) if for every open set in \( (Y, \sigma) \) there exist an \( \alpha g\mu \)-open set in \( (X, \tau) \).

3. CONTRA \( \alpha G\mu \)-CONTINUOUS FUNCTION

3.1. Definition

A function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is said to be contra \( \alpha G\mu \)-continuous if for every open set in \( (Y, \sigma) \) there exist an \( \alpha G\mu \)-closed set in \( (X, \tau) \).

3.2. Theorem

Every contra continuous function is contra \( \alpha G\mu \)-continuous.

Proof. Let \( V \) be open in \( (Y, \sigma) \). Since \( f : (X, \tau) \rightarrow (Y, \sigma) \) is contra continuous, \( f^{-1}(V) \) is closed in \( (X, \tau) \) and hence \( \alpha g\mu \)-closed By (Devi et al., 2007). Thus, \( f \) is contra \( \alpha g\mu \)-continuous.

Converse of the above theorem need not be true by the following example.

3.3. Example

Let \( X = Y = \{a, b, c\} \), \( \tau = \{\phi, X, \{a\}, \{b, c\}\} \) and \( \sigma = \{\phi, Y, \{a\}, \{a, b\}\} \).

Define \( f : (X, \tau) \rightarrow (Y, \sigma) \) by \( f(a) = b, f(b) = c \) and \( f(c) = a \). The \( \alpha g\mu \)-closed sets of \( X \) are \( \phi, X, \{a\}, \{b, c\}, \{a, b\}, \{b, c\}, \{a, c\} \). Here \( \{a, b\} \) is an open set of \( (Y, \sigma) \) but \( f^{-1}(\{a, b\}) = \{b, c\} \) is not a closed set of \( (X, \tau) \). Hence \( f \) is contra \( \alpha g\mu \)-continuous but not contra continuous.

3.4. Lemma

The following properties hold for subsets \( A, B \) of a space \( X \):
(a) \( x \in \ker(A) \) if and only if \( A \cap F \neq \emptyset \) for any \( F \in C(X,x) \).

(b) \( A \subseteq \ker(A) \) and \( A = \ker(A) \) if \( A \) is open in \( X \).

(c) If \( A \subseteq B \), then \( \ker(A) \subseteq \ker(B) \).

3.5. Theorem

For a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) the following conditions are equivalent:

1) \( f \) is contra \( \alpha \mu \)-continuous;

2) for every closed subset \( F \) of \( Y \), \( f^{-1}(F) \in \alpha \mu O(X) \);

3) for each \( x \in X \) and each \( F \in C(Y, f(x)) \), there exists \( U \in \alpha \mu O(X,x) \) such that 
   \[ f(U) \subseteq F ; \]

4) \( \alpha \mu cl(A) \subseteq \ker(f(A)) \) for every subset \( A \) of \( X \);

5) \( \alpha \mu cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B)) \) for every subset \( B \) of \( Y \).

Proof.

1) \( \Rightarrow \) (2) Since \( f \) is contra \( \alpha \mu \)-continuous, inverse image of a closed subset \( F \) of \( Y \) is \( \alpha \mu O(X) \).

2) \( \Rightarrow \) (3) It is given \( F \) is closed subset of \( Y \) and \( f^{-1}(F) \) is \( \alpha \mu O(X) \). Hence for \( x \in X \), there exists \( U \in \alpha \mu O(X) \) such that 
   \[ f(U) \subseteq F ; \]

3) \( \Rightarrow \) (2) Let \( F \) be an arbitrary open set of \( Y \) containing \( f(x) \). Then \( f(x) \in F \) and there exists \( U \in \alpha \mu O(X,x) \) such that 
   \[ f(U) \subseteq F ; \]

4) \( \Rightarrow \) (5) Let \( B \) be any subset of \( Y \). By lemma 3.4, we have 
   \[ f(\alpha \mu cl(f^{-1}(B))) \subseteq \ker(f(f^{-1}(B))) \subseteq \ker(B) \] 
   thus \( \alpha \mu cl(f^{-1}(B)) \subseteq f^{-1}(\ker(B)) \).

5) \( \Rightarrow \) (1) Let \( V \) be any open set of \( Y \). Then by lemma 3.4, we have 
   \[ \alpha \mu cl(f^{-1}(V)) \subseteq f^{-1}(\ker(V)) = f^{-1}(V) \]
   and \( \alpha \mu cl(f^{-1}(V)) \subseteq f^{-1}(\ker(B)) \) which shows that \( f^{-1}(V) \) is \( \alpha \mu \)-closed in \( X \).

3.6. Theorem

If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is contra \( \alpha \mu \)-continuous and \( Y \) is regular, then \( f \) is \( \alpha \mu \)-continuous.

Proof. Let \( x \) be an arbitrary point of \( X \) and let \( V \) be an open set of \( Y \) containing \( f(x) \). Since \( Y \) is regular, there exists an open set \( W \) in \( Y \) containing \( f(x) \) such that \( cl(W) \subseteq V \). Since \( f \) is \( \alpha \mu \)-continuous, so by theorem 3.5, there exists \( U \in \alpha \mu O(X,x) \) such that 
   \[ f(U) \subseteq cl(W) ; \]

3.7. Corollary

If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is contra \( \alpha \mu \)-continuous and \( Y \) is regular, and then \( f \) is continuous.

We introduce the following definitions

3.8. Definition

1) A space \((X, \tau)\) is said to be locally \( \alpha \mu \)-indiscrete if every \( \alpha \mu \)-open set is closed.

2) A function \( f : X \rightarrow Y \) is called almost \( \alpha \mu \)-continuous if for each \( x \in X \) and each open set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in \alpha \mu O(X,x) \) such that 
   \[ f(U) \subseteq int(cl(V)) \].

3.9. Theorem

If a function \( f : (X, \tau) \rightarrow (Y, \sigma) \) is contra \( \alpha \mu \)-continuous and \( X \) is a \( T_{\alpha \mu} \)-space, then \( f \) is \( \alpha \mu \)-continuous.

Proof. Let \( V \) be a closed set in \( Y \). Since \( f \) is contra \( \alpha \mu \)-continuous, \( f^{-1}(V) \) is \( \alpha \mu \)-open in \( X \). Since \( X \) is a \( \alpha \mu \)-space, \( f^{-1}(V) \) is open in \( X \). Hence \( f \) is contra-

3.10. Theorem

Let \( X \) be locally \( \alpha \mu \)-indiscrete. If \( f : (X, \tau) \rightarrow (Y, \sigma) \) is contra \( \alpha \mu \)-continuous, then \( f \) is continuous.

Proof. Let \( V \) be a closed set in \( Y \). Since \( f \) is contra \( \alpha \mu \)-continuous, \( f^{-1}(V) \) is \( \alpha \mu \)-open in \( X \). Since \( X \) is a \( T_{\alpha \mu} \)-space, \( f^{-1}(V) \) is open in \( X \). Hence \( f \) is continuous.

3.11. Theorem

A function \( f : X \rightarrow Y \) is almost \( \alpha \mu \)-continuous if and only if for each \( x \in X \) and each regular open set \( V \) of \( Y \) containing \( f(x) \), there exists \( U \in \alpha \mu O(X,x) \) such that 
   \[ f(U) \subseteq V \].

Proof. Let \( V \) be regular open set of \( Y \) containing \( f(x) \) for each \( x \in X \). Since every regular open set is open
(Njastad, 1965), V be an open set of Y containing f(x) for each x ∈ X.

Since f is almost αgμ-continuous, there exists U ∈ αgμO(X, x) such that f(U) ⊆ αgμint(cl(V)) ⊆ V. Therefore f(U) ⊆ V.

Conversely, if for each x ∈ X and each regular open set V of Y containing f(x), there exists U ∈ αgμO(X, x) such that f(U) ⊆ V. This implies V is an open set of Y containing f(x), such that f(U) ⊆ V = αgμint(cl(V)). Therefore f is almost αgμ-continuous.

3.12. Theorem

If a function f: (X, τ) → (Y, σ) is pre αgμ-open and contra αgμ-continuous, then f is almost αgμ-continuous.

Proof. Let x be any arbitrary point of X and V be an open set containing f(x). Since f is contra αgμ-continuous, then there exists U ∈ αgμO(X, x) such that f(U) ⊆ cl(V). Since f is pre αgμ-open, f(U) is pre αgμ-open in Y. Therefore, f(U) = αgμint(f(U)) ⊆ αgμint(cl(f(U))) ⊆ αgμint(cl(V)). This shows that f is almost αgμ-continuous.

3.13. Definition

The graph of a function f: X → Y is said to be contra αgμ-closed if for each

\[(x, y) ∈ (X × Y) − Gr(f), \text{there exists } U ∈ αgμO(X, x) \text{ and } V ∈ C(Y, y) \text{ such that } (U × V) ∩ Gr(f) = \emptyset.\]

3.14. Theorem

If f: X → Y is contra αgμ-continuous and Y is Urysohn, then f is Cαgμ-closed in the product space X × Y.

Proof. Let (x, y) ∈ (X × Y) − Gr(f). Then y ∈ f(x) and there exists open sets A and B such that f(x) ∈ A, y ∈ B and cl(A) ∩ cl(B) = φ. Then there exists V ∈ αgμO(X, x) such that f(V) ⊆ cl(A). Therefore, we obtain f(V) ∩ cl(B) = φ. This shows that f is Cαgμ-closed.

3.15. Theorem

If f: X → Y is contra αgμ-continuous with X as locally αgμ-indiscrete then f is continuous.

Proof. Let V be an open set in Y. Since f is contra αgμ-continuous, f −1(V) is αgμ-closed in X. Since X is locally αgμ-indiscrete every αgμ-closed set is open. Hence f −1(V) is open in X. Therefore f is continuous.

3.16. Theorem

If f: X → Y is contra αgμ-continuous and X is cT_αgμ-space, then f is contra-continuous.

Proof. Let V be open set in Y. Since f is contra αgμ-continuous, f −1(V) is αgμ-closed in X. Since X is a cT_αgμ-space, every αgμ-closed set is closed. Hence f −1(V) closed in X. Therefore f is contra-continuous.

3.17. Theorem

If f: X → Y is a surjective pre-closed contra αgμ-continuous with X as cT_αgμ-space, then Y is locally indiscrete.

Proof. Let V be an open subset in Y. Since f is contra αgμ-continuous, f −1(V) is αgμ-closed in X. Since X is cT_αgμ space f −1(V) is closed in X. Since f is pre-closed, V is pre-closed in Y. Now we have cl(V) = cl(int(V)) ⊆ V which implies cl(V) = V. This means V is closed in X and hence Y is locally indiscrete.

3.18. Definition

A space X is said to be αgμ-connected if X cannot be written as a disjoint union of two non-empty αgμ-open sets.

3.19. Theorem

A contra αgμ-continuous image of a αgμ-connected space is connected.

Proof. Let f: X → Y be a contra αgμ-continuous map of a αgμ-connected space X on to a topological space Y. If possible, let Y be disconnected. Let A and B form a disconnection of Y. Then A and B are clopen and Y = A ∪ B, where A ∩ B = φ. Since f is contra αgμ-continuous map, X = f −1(Y) = f −1(A ∪ B) = f −1(A) ∪ f −1(B) where f −1(A) and f −1(B) are non-empty αgμ-open sets in X. Also f −1(A) ∩ f −1(B) = φ. Hence X is not αgμ-connected. This is a contradiction. Therefore Y is connected.

3.20. Theorem

If f is contra αgμ-continuous map from a αgμ-connected space on to any space Y, then Y is not a discrete space.

Proof. Suppose that Y is discrete. Let A be a proper non-empty open and closed subset of Y. Since f is contra αgμ-continuous, f −1(A) is a proper non-empty αgμ-open and αgμ-closed subset of X, which is a contradiction to the fact that X is αgμ-connected space. Therefore Y is not a discrete space.

3.21. Theorem

If f: X → Y is αgμ-irresolute map with Y as locally αgμ-indiscrete space and g : Y → Z is contra
Let $A$ be any closed set in $Z$. Since $g: Y \to Z$ is contra $\alpha g\mu$-continuous, $g^{-1}(A)$ is $\alpha g\mu$-open set in $Y$. Since $Y$ is locally $\alpha g\mu$-indiscrete, $g^{-1}(A)$ is closed in $Y$. Hence $g^{-1}(A)$ is $\alpha g\mu$-closed set in $Y$. Since $f$ is $\alpha g\mu$- irresolute $(g_{\alpha} f)^{-1}(A)) = f^{-1}(g^{-1}(A))$ is $\alpha g\mu$-open in $X$. Therefore $g_{\alpha} f$ is $\alpha g\mu$-continuous.

To prove only if part, let $g_{\alpha} f: X \to Z$ is contra $\alpha g\mu$-continuous and let $A$ be a closed set in $Z$. Then $(g_{\alpha} f)^{-1}(A)$ is $\alpha g\mu$-open of $X$. That is $f^{-1}(g^{-1}(A))$ is an $\alpha g\mu$-open subset of $X$. Since $f$ is pre $\alpha g\mu$-open, $f(f^{-1}(g^{-1}(A)))$ is $\alpha g\mu$-open subset of $Y$. So, $g^{-1}(A)$ is an $\alpha g\mu$-open subset of $Y$. Hence $g$ is contra $\alpha g\mu$-continuous.

3.25. Theorem

If $f: X \to Y$ is contra $\alpha g\mu$-continuous, closed injection and $Y$ is ultra normal, then $X$ is $\alpha g\mu$-normal.

Proof. Let $A$ and $B$ be disjoint closed subsets of $X$. Since $f$ is closed injective, $f(A)$ and $f(B)$ are disjoint closed subsets of $Y$. Since $Y$ is ultra normal, $f(A)$ and $f(B)$ are separated by disjoint clopen sets $V$ and $W$ respectively. Hence $A \subseteq f^{-1}(V)$ and $B \subseteq f^{-1}(W),$ $f^{-1}(V)$ $\subseteq \alpha g\mu O(X)$ and $f^{-1}(W) \subseteq \alpha g\mu O(X)$. Also $f^{-1}(V) \cap f^{-1}(W) = \emptyset.$ Thus $X$ is $\alpha g\mu$-normal.

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