The gravitational analog of the Aharonov-Bohm electric effect

by

Doron Ludwin

Department of Physics, Technion, Haifa 32000, Israel

Abstract

The electric Aharonov-Bohm effect is a special case of the general Ab effect. However, when inserting a gravitational potential in the place of the time dependent potential, a different understanding of the phase shift could be gained. The usual topological phase is replaced by a phase with origin in the red shift of the particle at one of the paths taken relative to the other path. In this case, the change in the geometrical measure is the source of the phase shift, which therefore has a local interpretation along with the non-local topological explanation.
1 Introduction

The AB (Aharonov-Bohm) effect, is thought of as the main example to the idea that the electromagnetic potential has physical significance [1]. The wavefunction of a charged particle which travels through an AB potential, will receive a change in phase which could be observed through the shift of the interference fringes in a Young double slit type experiment.

This phenomena is usually interpreted as a non-local quantum phenomena, connected with the non-trivial topology of the AB potential. This interpretation is associated with the understanding that the charged particle wavefunction receives this phase change although there is no ”local” interaction with the electromagnetic field. Normally, such an interaction would have caused momentum transfer to the particle which would result with a change in the particle’s velocity, but to our best knowledge, such a change in velocity hasn’t been observed. Furthermore, it has been theoretically predicted that apart from a phase change there is no other physical phenomenon.

In further work [2, 3, 4, 5], generalizing the AB effect to the gravitational potential, different gravitational fields have been taken to show, that the AB phase difference may exist between two paths taken around a gravitational analog to the AB solenoid (such as a cosmic string), although the particles travel on a flat curvature. This is a gravitational analogue to the AB magnetic effect.

In this work, however, we develop an interesting gravitational analog to the electric AB effect [6, 7], which might teach us something further concerning the AB potentials in the gravitational case.

We show that when inserting a constant gravitational potential in the place of the time dependent potential in the electric AB effect, the usual topological phase could be replaced by a phase due to the red shift of the particle on one of the path taken relative to the other path. In this case the change of the geometrical measure induces the same phase shift as given by the topological argument.

In this work, we also show that when splitting the wave function into two parts, and expect to interfere the parts again, the role of proper time must be taken carefully into consideration.
2 The Electric AB effect

2.1 The ordinary Electric AB effect

Let us look at a region in space where a particle moves under the influence of a time dependent potential, \( U(t) \), with zero gradient over space.

This particle’s Lagrangian is given by:

\[
\mathcal{L} = \frac{1}{2} m \dot{x}^2 - U(t)
\]  

(1)

Using the Euler-Lagrange Equation:

\[
\frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{x}} \right) = \frac{\partial \mathcal{L}}{\partial x}
\]

(2)

we get:

\[
m \ddot{x} = 0
\]  

(3)

which means there is no net force acting upon the particle, although the potential the particle senses changes through time. There is naturally a change in the total particle’s energy, but this additional energy, constant over this region in space, acts as a constant energy shift added to the system, and will not be noticed by any measurement of a physical local parameter.

On the other hand, in QM, the Hamiltonian of the particle is given by:

\[
H = H_o + U(t)
\]  

(4)

where \( H_o \) is the Hamiltonian of a free particle.

Taking \( \Psi_o \) as the wavefunction of the free particle, and trying a solution of the type:

\[
\Psi = \Psi_o e^{-iS/\hbar},
\]  

(5)

we get that the schrödinger equation takes the form:

\[
i\hbar \frac{\partial \Psi}{\partial t} = (i\hbar \frac{\partial \Psi_o}{\partial t} + \Psi_o \frac{\partial S}{\partial t})e^{-iS/\hbar} = [H_o + U(t)]\Psi = H\Psi
\]

(6)

which gives the straightforward solution:

\[
S = \int U(t) dt
\]  

(7)
According to this, in QM the wave function of the particle changes when this time varying potential is introduced, only the change is added as a phase factor, and therefore does not affect any classical result. However, in an interference experiment, this phase could be noticed as a phase difference in an interference picture.

For example, consider a single electron when the wave function splits into two parts due to some kind of barrier, and each part of the wavefunction is allowed to enter a long cylindrical metal tube, as shown in fig. 1.

After the parts of the wave function pass through the tubes, they are combined to interfere coherently at $F$. The electric potential, $U(t) = e\phi(t)$, in each tube is determined by a time delay mechanism in such a way that the potential is zero in region $I$ (until the electron’s wave packet is well inside the tubes). The potential then grows as a function of time, but differently in each tube. Finally it falls back to zero, before the electron comes near the other edge of the tube. Thus, the potential is nonzero only while the electrons are well inside the tube (region $II$). The purpose of this arrangement is to ensure that the electron is in a time-varying potential without ever being in a field. This is assured by the fact that the electron never senses a space gradient in the potential, but merely a time gradient.

Now, let $\Psi = \Psi_1^0(x,t) + \Psi_2^0(x,t)$ be the wavefunction before the electron enters the tubes. According to (5) and (7), the wavefunction, after passing through the tubes, is altered only by a phase factor as follows:

$$\Psi = \Psi_1^0 e^{-iS_1/\hbar} + \Psi_2^0 e^{-iS_2/\hbar}$$

(8)
where

\[ S_1 = e \int \varphi_1(t) dt, \quad S_2 = e \int \varphi_2(t) dt \]  

(9)

It is evident that the interference of the two parts at F will depend on the phase difference \((S_1 - S_2)/\hbar\). Thus, there is a measurable physical effect of the potentials even though no force is ever actually exerted on the electron. The effect is quantum mechanical in nature since it arises in the phenomenon of interference.

The phase difference \((S_1 - S_2)/\hbar\), can also be expressed as an integral around a closed circuit in time:

\[ \Delta \Phi = \frac{e}{\hbar} \oint \varphi dt \]  

(10)

The covariant statement of the above conclusion, points that there should be a similar result involving the space part of the four vector \(A_\mu\), meaning the vector potential, \(A\). That gives the more well known effect known as the magnetic AB effect, which we shall not deal with here.

### 2.2 An Electric AB effect in an elevator

We begin, as before, with an electron with wavefunction split into two parts, each going into a different elevator in which they continue to travel at the same speed in the \(x\) direction. The elevators can travel up and down in a static electric field, caused for instance by a great flat capacitor, which has a constant electric field \(E\), with a positive potential gradient in the \(z\) direction, but no force acting in the \(x\) direction. After the particle’s wave function is well inside the elevators, the elevators climb (against the electric field force) to a height \(h\). The elevator 1 comes back down right away, while elevator 2 stays at that height for a time \(\Delta t\) and then comes back down. The scenario is arranged this way to ensure that the only difference between both paths will be the time \(\Delta t\), in which elevator 1 returns down, making sure the particle’s wave function has seen a constant electric potential \(V_1\), while the other elevator stays at height \(h\) assuring the particle’s wave function sees a potential \(V_2\).

Although the same exact forces act upon the two possible electron paths, the time \(\Delta t\) in which the particles were under different electric potentials, with no force acting upon them, gives rise to an additional different phase
shift for each of them:

\[ \Psi_1 = \Psi_{\text{total}} e^{\frac{ie}{\hbar} V_1 \Delta t} \]  
\[ \Psi_2 = \Psi_{\text{total}} e^{\frac{ie}{\hbar} V_2 \Delta t} \]  

(11)

The phase shift is therefore:

\[ \Delta \phi = \frac{e}{\hbar} (V_1 - V_2) \Delta t \]  

(12)

This is a very simple example in which the electric potential, even if it stays constant over the path, has an influence on a phase shift, which might be encountered. The reason we have discussed the elevator is in preparation for the gravitational analog given in the next section.

### 3 The gravitational analog of the electric AB effect

The gravitational analog to the AB effect has been discussed widely in the literature [2, 3, 4, 5]. However, the analogy was given to the magnetic effect, where different metrics have been considered which have topological singularities although the particle travels in a locally flat space with no curvature, while the analogy to a locally curved space has not been discussed. These authors have shown that there is an analogy between the electromagnetic potential \( A_\mu \) and the gravitational vector \( h_\mu = (\frac{1}{2}h_{00}, h_{10}, h_{20}, h_{30}) \) for the low velocity limit on a weak gravitational field, where \( h_{\mu\nu} \) is a small curvature disturbance over the flat Minkowski space, such that:

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \]  

(13)

It was shown that the expected change in phase of the wave function of a particle of mass \( m \), when taken around a curve \( V_1 \) is:

\[ \delta \beta = \frac{m}{\hbar} \int_{V_1} h_\mu dx^\mu \]  

(14)

in complete analogy with the electromagnetic case.

The magnetic analog, involving the analogy between the spacial parts of the four vectors \( A \sim h \), has been studied in the literature. However, the
analogy to the electric effect, which yields the analogy $\phi \sim h_0$, hasn’t been discussed. Perhaps that is because it seems less "magical" since the particle travels in an ordinary curved space, where gravitational forces may act on it. However, trying to follow the influence of local features and not merely topological features of the space-time, the gravitational analog to the electric effect is particularly interesting. This way, the contribution of the change in the space-time measure caused by the curved space to the phase shift could be quantified.

3.1 The Newtonian analog of the effect

We shall start by replacing the electric potential given in the last section with a Newtonian gravitational potential, which is given by:

$$\Phi = -\frac{GM}{R}$$

where $M$ is the mass of earth and $R$ is the distance from the center of earth. The Newtonian gravitational constant $G$ is taken to be 1 from now on.

Taking the particle again through the choice of two different elevators that start at an altitude $R_1$. Both elevators climb to an altitude $R_2$ through a similar path, where at the end, the first elevator travels straight down, and the second elevator stays at the altitude $R_2$ for a time $\Delta t$ and then goes down along a path similar to the first one. At the whole time, the particle didn’t change it’s vertical velocity $v$. Following exactly the same steps as before, the only difference in phase is caused by the time $\Delta t$ when the particle travels in the elevators at the same speed but at different altitudes.

At that part of the particle’s path, the Hamiltonian of the particle is:

$$H = H_o + m\Phi$$

where $H_o$ is the Hamiltonian of a free particle, and $m\Phi$ is constant over the time $\Delta t$.

We then get the straightforward solution:

$$\Psi = \Psi_o e^{im/\hbar \Phi \Delta t}$$

So although there is no force acting in the direction of motion upon the
particle at that time, the difference in the constant gravitational potential 
the wave function encounters causes the change in phase:

$$\Delta \phi = \frac{mM}{\hbar} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \Delta t$$  \hspace{1cm} (18)

This is the expected phase shift when introducing the gravitational po-
tential instead of the electric potential. Moving on to general relativity, we 
shall first show that we get the Newtonian result also for the weak gravita-
tional field approximation (in the low velocity limit), and then we shall see 
that the phase could be explained by the difference in the space-time measure 
between both paths of the particle.

### 3.2 The gravitational effect in a weak Schwarzschild field

#### 3.2.1 The straightforward approach

In the gravitational analog, we have an analogy between $\phi$ and $\frac{1}{2}h_{00}$, where 
for the weak Schwarzschild field:

$$h_{00} = g_{00} + 1 = \frac{2M}{R}$$  \hspace{1cm} (19)

According to this, the electric phase shift:

$$\Delta \phi = \frac{e}{\hbar} \oint \Phi(t) dt$$  \hspace{1cm} (20)

is replaced by:

$$\Delta \phi = \frac{m}{\hbar} \oint \frac{1}{2} h_{00}(t) dt = -\frac{m}{\hbar} \oint \frac{M}{R(t)} dt$$  \hspace{1cm} (21)

Thinking of our experiment as carried over a circular path of the particle, and 
after omitting the identical contributions to the phase and to the amplitude 
given by both parts of the path, the only non-vanishing contributions are 
given by:

$$\Delta \phi = -\frac{m}{\hbar} \int_{\Delta t} \frac{M}{R_1} dt - \frac{m}{\hbar} \int_{-\Delta t} \frac{M}{R_2} dt = \frac{mM}{\hbar} \left( \frac{1}{R_2} - \frac{1}{R_1} \right) \Delta t$$  \hspace{1cm} (22)
getting as expected the Newtonian result for the weak field approximation. In the next section, we get this result through a semi-covariant description of a wave equation, and see what are the conditions on the wave equation to get the correct result.

3.2.2 A semi-covariant description of the wave equation

Starting from the scalar product of the energy-momentum vector:

\[ g_{\mu\nu}p^\mu p^\nu = -m^2c^4 \] (23)

Separating the equation to the time and spatial parts, for a block separated metric tensor, we can get:

\[ -g_{tt}E^2 = g_{ij}p_ip_jc^2 + m^2c^4 \] (24)

which gives:

\[ \sqrt{-g_{tt}}E = \sqrt{g_{ij}p_ip_jc^2 + m^2c^4} = mc^2\sqrt{1 + \frac{g_{ij}p_ip_j}{m^2c^2}} \approx mc^2 + \frac{g_{ij}p_ip_j}{2m} \] (25)

Replacing \( \sqrt{-g_{tt}}E \) with the operator \( -i\hbar\sqrt{-g_{tt}}\frac{\partial}{\partial t} \), we get the wave equation:

\[ -i\hbar\frac{\partial \Psi}{\partial t} = \left[ \frac{mc^2}{\sqrt{-g_{tt}}} + \frac{g_{ij}p_ip_j}{2m\sqrt{-g_{tt}}} \right] \Psi \] (26)

Equation (26), is a good approximation for describing a geodesic motion of a free falling spinless massive particle, in the environment of a time-space block separated metric. The form of the equation is Schrodinger-like, and assumes causal evolution in the direction of the \( t \) axis (this is correct in the weak field approximation and far from metric singularities).

For the Schwarzschild metric, \( -g_{tt} = (1 - \frac{2M}{Rc^2})^{-1} \), and therefore, for the weak field approximation:

\[ (\approx 1 - \frac{M}{Rc^2}) \] (27)

Furthermore, in the particle’s path we are looking at, the particle travels at a constant altitude \( R \), and therefore, \( p_r = 0 \). Therefore for the Schwarzschild metric:
\[ g_{ij}p_i p_j = g_{rr} p_r^2 + p_\Omega^2 = p^2 \]  

(28)

Substituting (27) and (28) into (26), we obtain:

\[-i\hbar \frac{\partial \Psi}{\partial t} = \left[ mc^2 + \frac{p^2}{2m} - \frac{mM}{R} \left( 1 + \frac{p^2}{2m^2c^2} \right) \right] \Psi \]  

(29)

In the low velocity limit \( mc \gg p \), and therefore, the last term falls out leaving the wave equation:

\[-i\hbar \frac{\partial \Psi}{\partial t} = \left[ mc^2 + \frac{p^2}{2m} - \frac{mM}{R} \right] \Psi \]  

(30)

This result is consistent with the Newtonian approximation. Taking the gravitational field into account in a path at a constant height, induces an effective potential \( U(R) = \frac{mM}{R} \), which changes the energy of the system without changing the momentum. This is exactly the Newtonian potential which causes the effect at the gravitational analog of the AB electric effect. In the next section we shall see how we get this result using the role of proper time for the free-falling particle in the background of the Schwarzschild metric.

### 3.3 A Space-Time local explanation of the phase shift

We shall now try to get the effect, by the phase difference which is a result of the space-time self length difference between both paths.

For that we shall assume that each of the parts of the wave function, travels on a locally flat metric, and the Schwarzschild metric is noticed only when looking at the difference between both paths in interference. We also make sure that there is a clock which measures the proper time at each elevator to make sure the elevators are synchronized to one another.

The time \( \Delta t \) in which the first elevator stays at altitude \( R_2 \) could therefore be given as \( \Delta \tau \) (proper time) instead. If we now look at the wave equation which describes the evolution of each separate part of the wave-function, at different constant altitudes, we get the ordinary Schrodinger equation for a free particle, according to proper time \( \tau \) instead of \( t \):

\[-i\hbar \frac{\partial \Psi}{\partial \tau} = mc^2 + \frac{p^2}{2m} \]  

(31)
However, if we want the parts of the wave-function to interfere in the end, we must go back to the evolution according to the coordinate $t$.

The connection between $dt$ and proper time $d\tau$ is:

$$d\tau^2 = -ds^2 = -g_{tt}dt^2 - g_{rr}dr^2 - R^2d\Omega^2$$  \hspace{1cm} (32)

For the low velocity limit, where the advance in time $dt$ is much greater than the spatial advance:

$$d\tau^2 = -g_{tt}dt^2 \implies d\tau = \sqrt{-g_{tt}}dt$$  \hspace{1cm} (33)

which is the result for an ordinary red shift.

Substituting (33) into (31) we get the evolution for the variable $t$:

$$-i\hbar \frac{1}{\sqrt{-g_{tt}}} \frac{\partial \Psi}{\partial t} = mc^2 + \frac{p^2}{2m}$$  \hspace{1cm} (34)

or:

$$-i\hbar \frac{\partial \Psi}{\partial t} = \sqrt{-g_{tt}} \left( mc^2 + \frac{p^2}{2m} \right)$$  \hspace{1cm} (35)

As before, we take the Schwarzschild weak field approximation:

$$(-g_{tt})^{1/2} \approx 1 - \frac{M}{Rc^2}$$  \hspace{1cm} (36)

and obtain:

$$-i\hbar \frac{\partial \Psi}{\partial t} = \left[ mc^2 + \frac{p^2}{2m} - \frac{mM}{R} \left( 1 + \frac{p^2}{2m^2c^2} \right) \right] \Psi$$  \hspace{1cm} (37)

and as before, in the low velocity limit $mc \gg p$, and therefore, the last term falls out leaving the wave equation:

$$-i\hbar \frac{\partial \Psi}{\partial t} = \left[ mc^2 + \frac{p^2}{2m} - \frac{mM}{R} \right] \Psi$$  \hspace{1cm} (38)

This result is similar to (30). We can see that this equation will lead exactly to the analog of the electric AB effect, although it has a simple alternative local explanation to the phase shift in agreement with the topological analysis.
4 Summary

We have studied the gravitational analog of the electric AB effect and find that the topological argument leads to an effect that can be equivalently explained by a local change in effective path length, raising interesting questions for further study in the relativistic case.
References

[1] Y. Aharonov and D. Bohm, *Significance of Electromagnetic Potentials in the Quantum Theory*, Phys. Rev. 115 (1959), 485.

[2] B. S. de Witt, Phys. Rev. Lett. 16 (1966), 1092.

[3] J. S. Dowker, *A Gravitational Aharonov-Bohm Effect*, Nuov. Ciment. LII B (1967), 129.

[4] J. S. Dowker and J.A. Roche, *The gravitational analogues of magnetic monopoles*, Proc. Phys. 92 (1967), 1.

[5] G. Papini, *Particle Wave Functions in Weak Gravitational Fields*, Nuov. Ciment. LII B (1967), 137.

[6] Y. Aharonov and A. Casher, Phys. Rev. Lett. 54 (1983), 2469.

[7] Y. Aharonov, P. Pearle and L. Vaidman, Phys. Rev. A 37 (1988), 4052.