On one physical interpretation of generalized conditions Cauchy-Riemann

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Abstract. The message shows that in variables, the generalized Cauchy-Riemann conditions (CRC) coincide with the Maxwell system for the electromagnetic field, and the possible existence of magnetic charges and currents is assumed. This makes it possible to give a physical interpretation to all 8 components of the solution of generalized CRC. When the generalized CRC is taken as a basis, two scalar fields necessarily appear. They are interpreted as some fields of spatially distributed electric and magnetic charges and currents. The introduction of the attached generalized CRC allows us to give a method for determining the potentials of the electromagnetic field under the assumptions made and to determine the gauge transformations more General than the well-known ones. Of course, having made certain restrictions, all the results coincide with those accepted in classical electrodynamics. The author points out that he does not make any physical statements and all constructions are purely mathematical in nature.

1. Introduction

The theory of functions of a complex variable is one of the most important branches of mathematics, which contains both very deep theoretical results and important advances in solving problems of physics and technology. Therefore, the development of her methods in the field of functions of many variables is of great interest.

The article discusses the use of quaternion algebra to study the properties of the electromagnetic field. The issue of studying the properties of the electromagnetic field in the quaternionic space was first presented in the work of CJ Maxwell [1], but at present, many works are devoted to this issue [2-6].

The generalized Cauchy - Riemann conditions for the case of an eight-dimensional space of two sets of variables \( x_i, y_i \) for \( i = 0,3 \) were given in the author's paper [2] as a generalization of previously known results [7,8] and have the form formally very close to the Cauchy-Riemann system, namely

\[
\begin{aligned}
D_1 x - \psi D_2 &= 0, \\
\bar{D}_2 x + \bar{D}_1 \psi &= 0.
\end{aligned}
\]  

(1.1)
The minus sign is specially preserved to emphasize the similarity with the Cauchy-Riemann system. Here \( \chi, \psi \) quaternion functions of variables \( x_i, y_i \), of the form

\[
\chi = \chi_0(x,y) + \sum_{i=1}^{3} \chi_i(x,y)e_i, \quad \psi = \psi_0(x,y) + \sum_{i=1}^{3} \psi_i(x,y)e_i.
\]  

(1.2)

\( X, y \) are abbreviations for sets of variables, and \( e_i \) are units of the quaternion system in these records \[9\]. The part \( \psi_0 \) in (1.2) is usually called the scalar part based on the coordinate transformation rules, and the vector part \( \tilde{\psi} \).

Operators \( D_1, D_2 \) are fairly well known in mathematics \[8\], we can write as

\[
\frac{D_1}{D_1} = \frac{d}{dx_0} e_0 \pm \sum_{i=1}^{3} \frac{d}{dx_i} e_i, \quad \frac{D_2}{D_2} = \frac{d}{dy_0} e_0 \pm \sum_{i=1}^{3} \frac{d}{dy_i} e_i.
\]  

(1.3)

The dash indicates the quaternion conjugation operation. The operators \( D_1 \bar{D}_1, D_2 \bar{D}_2 \) coincide with the Laplace operators, therefore

\[
D_1 \bar{D}_1 = D_1 D_1 = \sum_{i=0}^{3} \frac{d^2}{dx_i^2} = \Delta_x (4), \quad D_2 \bar{D}_2 = D_2 D_2 = \sum_{i=0}^{3} \frac{d^2}{dy_i^2} = \Delta_y (4).
\]  

(1.4)

The operator of the wave equation, which will be needed below, will be denoted by the symbol

\[
[4] = \Delta_x (3) - \frac{1}{c^2} \frac{d^2}{dt^2}.
\]  

(1.5)

The quaternion function, the components of which the harmonic function of the corresponding variables will be called, perhaps not very well, the harmonic quaternion.

We select a set of variables \( x_1, x_2, x_3, y_0 \) and assume that all functions \( \chi, \psi \) depend only on these variables. Then the operators \( D_1, \bar{D}_1, D_2, \bar{D}_2 \) can be written

\[
\begin{align*}
\frac{D_1}{\bar{D}_1} &= \frac{d}{dy_0} q_0 \pm \text{grad}q_0 \pm \text{rot}q, \\
\frac{\bar{D}_1}{D_1} &= \frac{d}{dy_0} q_0 + \frac{d\tilde{q}}{dy_0}.
\end{align*}
\]  

(1.6)

The basic system (1.1) in this case, according to (1.6), has the explicit form

\[
\begin{align*}
- \text{div}(x)\tilde{\chi} + \text{grad}(x)\chi_0 + \text{rot}(x)\tilde{x} - \frac{d\psi_0}{dy_0} - \frac{d\tilde{\psi}}{dy_0} &= 0, \\
\frac{d\chi_0}{dy_0} + \frac{d\tilde{\chi}}{dy_0} + \text{div}(x)\tilde{\psi} - \text{grad}(x)\psi_0 - \text{rot}(x)\tilde{\psi} &= 0.
\end{align*}
\]

In this case, after the operators of vector analysis, in parentheses, it is indicated which variables are differentiated.

After separating the scalar and vector parts, we have
The first two equations are scalar, and the other two are vector. Already at this stage, a suspicion arises about the similarity of this system with the equations of Maxwell’s electrodynamics. It is somewhat more difficult to find in the system (1.7) the Dirac equation for particles with $m = 0$.

For further considerations related to the potentials of the electromagnetic field, we write down the associated system.

\[
\begin{align*}
- \text{div} \tilde{\chi} - \frac{d \psi_0}{d y_0} &= 0, \\
\text{div} \tilde{\psi} + \frac{d \tilde{\chi}_0}{d y_0} &= 0, \\
\text{rot} \tilde{\chi} - \frac{d \tilde{\psi}}{d y_0} + \text{grad} \tilde{\chi}_0 &= 0, \\
- \text{rot} \tilde{\psi} + \frac{d \tilde{\chi}}{d y_0} - \text{grad} \psi_0 &= 0.
\end{align*}
\tag{1.7}
\]

The choice of the main variables $x_{i1}, x_{i2}, x_{i3}, y_0$ is not unambiguous. You can choose set $y_{i1}, y_{i2}, y_{i3}, y_0$ with equal success.

2. The system of generalized CRC as a system of equations of classical electrodynamics

We assume that the coordinate $y_0$ is the so-called rotated time [11] (Wick time) as is customary in physics

\[y_0 = i \text{ct}.\tag{2.1}\]

Of course, this changes the quaternion system and causes zero divisors to appear. To justify it, we note that the division operation below is not used anywhere.
The first stage consists in the identification of the components of the quaternions \( \mathcal{q}, \mathcal{q} \), included in the generalized \( SCSC \), with the corresponding electrodynamics parameters. These are the strengths of the magnetic \( \mathcal{H} \) and electric \( \mathcal{E} \) fields.

This is easy to do based on the presence, both in the Maxwell equation and in expressions (1.7), of the same constructions of the operators \( \text{div} \) and \( \text{rot} \).

As further considerations confirm, one should choose

\[
\mathcal{q} = \mathcal{H}_0 + \mathcal{E}, \quad \mathcal{q} = \mathcal{H}_0 \mathcal{E}^*. \tag{2.2}
\]

Consider the most general case of the presence of an electric charge distributed with a density \( \rho \) and a current density \( j \), and a magnetic charge with corresponding values of \( m, j_m \).

For this, we take in system (1.9)

\[
\mu = 4\pi n + \frac{4\pi}{c} j, \quad \nu = 4\pi \rho + \frac{4\pi}{c} i j_m. \tag{2.3}
\]

The resulting system is presented below

\[
\begin{align*}
\text{div}\mathcal{E} &= 4\pi \rho + \frac{1}{c} \frac{d\mathcal{H}}{dt}, \\
\text{div}\mathcal{H} &= 4\pi n - \frac{1}{c} \frac{d\mathcal{E}}{dt}, \\
\text{rot}\mathcal{E} &= -\frac{1}{c} \frac{d\mathcal{H}}{dt} \text{grad} \mathcal{E}_0 + \frac{4\pi}{c} j_m, \\
\text{rot}\mathcal{H} &= \frac{1}{c} \frac{d\mathcal{E}}{dt} \text{grad} \mathcal{H}_0 + \frac{4\pi}{c} j. 
\end{align*} \tag{2.4}
\]

If we exclude magnetic charges and currents, putting \( \mathcal{E}_0 = \mathcal{H}_0 = 0, m = 0, j_m = 0 \), then we come to the usual form of Maxwell's equations. We take the modified name for system (2.4).

The form of system (2.4) emphasizes that the derivative \( \frac{1}{c} \frac{d\mathcal{H}}{dt} \) can be regarded as a certain distributed density of the electric charge, and the derivative \( \frac{1}{c} \frac{d\mathcal{E}}{dt} \) as the density of the magnetic charge.

Let us consider expressions \( \text{grad}\mathcal{H}_0, \text{grad}\mathcal{E}_0 \) as current densities.

These densities of charges and currents are determined by the system and are not external data, in contrast to \( \rho, m, j, j_m \).

Only \( \rho, j \) are actually taken into account and exist.

3. Plane electromagnetic wave

Let us present the simplest solution of system (2.4) in the form of a plane electromagnetic monochromatic wave for the case when \( \mathcal{H}_0, \mathcal{E}_0 \) are nonzero \( (\mathcal{H}_0, \mathcal{E}_0 \neq 0) \). To do this, we write down the wave solution in complex form as is done in the general case [10],

\[
\begin{align*}
H_1 &= H_0 e^{i\omega}, \\
E_1 &= E_0 e^{i\omega}.
\end{align*} \tag{3.1}
\]
where phase $\varphi$ is

$$\varphi = k \vec{r} - \omega t. \quad (3.2)$$

It is assumed here that $k = \frac{2\pi}{\lambda}$ is the wave vector and $\omega$ is the cyclic frequency $\omega = 2\pi \nu$, and $H_0, E_0$ is the amplitude values of the magnetic and electric scalar fields.

We take the following relations, which we substitute in (3.1)

$$\vec{X} = \vec{H}, \quad \vec{\psi} = \vec{E}, \quad y_0 = i ct, \quad \vec{X}_0 = H_0, \quad \psi_0 = E_0. \quad (3.3)$$

In this case, we find for the scalar parts [10]

$$\text{div}\vec{H} + \frac{1}{c} \frac{dE_0}{dt} = 0 \quad \text{and} \quad \text{div}\vec{E} - \frac{1}{c} \frac{dH_0}{dt} = 0. \quad (3.5)$$

Let us proceed to consider the question of a plane electromagnetic wave under the assumptions made that $H_0, E_0 \neq 0$.

Let us first turn to the first pair of equations (3.5). Substituting (3.1) into (3.5), we find

$$\left(\vec{H}, \vec{k}\right) = \frac{\omega}{c} E_0, \quad \left(\vec{E}, \vec{k}\right) = -\frac{\omega}{c} H_0. \quad (3.6)$$

If $H_0, E_0 \neq 0$ (nonzero) from (3.6) it follows that the vectors $\vec{H}, \vec{E}$ are not orthogonal to the wave vector $\vec{k}$. This means the presence of some longitudinal electromagnetic wave.

Recall that $H_0, E_0$ are the scalar components of quaternions $\vec{X}, \vec{\psi}$.

Tensions $\vec{H}, \vec{E}$ are considered at a given point at a certain point in time. Let us draw planes through the pairs $\vec{E}, \vec{k}$ and $\vec{H}, \vec{k}$ (figure 1).

Let us introduce unit vectors $\vec{s}, \vec{e}, \vec{m}$, where $\vec{s}$ is defined as $\vec{s} = \frac{\vec{k}}{k}$ and therefore indicates the direction of wave propagation. The vectors $\vec{e}, \vec{m}$ are orthogonal to $\vec{s}$, but not generally orthogonal to each other (figure 2).

We write the expansion of the vectors $\vec{E}$ and $\vec{H}$ in the system $\vec{s}, \vec{e}$ and $\vec{s}, \vec{m}$

$$\vec{E} = E_s \vec{s} + E_e \vec{e}, \quad \vec{H} = H_s \vec{s} + H_m \vec{m}. \quad (3.7)$$

Substituting expansion (3.7) into the first pair of equations, we have the following equations for the amplitudes
It follows from (3.8) that $\vec{E}$ and $\vec{H}$ are generally not orthogonal to $\vec{s}$, and substituting (3.7) we find

$$\begin{align*}
E_s + H_0 &= 0, \\
H_s - E_0 &= 0.
\end{align*}$$

(3.9)

These expressions show that the longitudinal stresses $E_s, H_s$ coincide with $E_0, H_0$, but $E_s, H_0$ are opposite in sign determined by the choice (3.5). After substituting (3.1) into the second pair of equations (2.4), we have for the amplitudes $\vec{E}, \vec{H}$ the system of algebraic equations

$$\begin{align*}
+ \frac{\omega}{c} \vec{E} + \left[ \vec{k} \cdot \vec{H} \right] + \vec{k}H_0 &= 0, \\
- \frac{\omega}{c} \vec{H} + \left[ \vec{k} \cdot \vec{E} \right] + \vec{k}E_0 &= 0.
\end{align*}$$

(3.10)

(3.11)

The vectors $\left[ \vec{k}H \right], \left[ \vec{k}E \right]$ are normal to $\vec{k}$.

We substitute expansion (3.7) into equation (3.11) and, taking into account the property of the vector product, we obtain

$$\begin{align*}
\frac{\omega}{c} \left( E_s \vec{s} + E_0 \vec{e} \right) + kH_0 \left[ \vec{s} \cdot \vec{m} \right] + k\vec{s}H_0 &= 0, \\
- \frac{\omega}{c} \left( H_s \vec{s} + H_0 \vec{m} \right) + kE_0 \left[ \vec{s} \cdot \vec{e} \right] + k\vec{s}E_0 &= 0.
\end{align*}$$

(3.12)

(3.13)

Taking into account the above remark and the orthogonality of $\vec{s}, \vec{e}$ and $\vec{s}, \vec{m}$, we have
\[\begin{align*}
\frac{\omega}{c} E_s + kH_0 &= 0, \\
-\frac{\omega}{c} H_s + kE_0 &= 0.
\end{align*}\]

The second relations (3.12), (3.13) can be satisfied only if \(\vec{e}, \vec{m}\) are orthogonal.

\[\begin{align*}
\frac{\omega}{c} E_v + kH_m (-1) &= 0, \\
-\frac{\omega}{c} H_m + kE_v &= 0.
\end{align*}\] (3.14)

It follows from these relations that the vectors \(E_v, E_m\) transverse components of the electric and magnetic fields are orthogonal.

The first two relations (3.14) have already been obtained earlier. The second pair for \(E_s, H_m\) is the usual relationship for a shear wave.

Let us find the angle \(\alpha\) formed by the strengths of the electric \(\vec{E}\) and magnetic \(\vec{H}\) fields

\[\cos \alpha = \frac{(\vec{E}, \vec{H})}{|\vec{E}| |\vec{H}|}.\] (3.15)

If the projections \(H_s, E_s\) disappear, then we return to the complete mutual orthogonality of the intensities \(\vec{E}, \vec{H}\) and the vector \(\vec{s}\).

If \(H_s, E_s\) are small compared to \(H_m, E_m\), then

\[\cos \alpha = \frac{H_s E_s}{H_m E_v}.\] (3.16)

4. Introduction of the potentials of the electromagnetic field

The process of introducing potentials from the point of view of the quaternion formalism looks relatively simple. We introduce the quaternion potentials \(\alpha, \beta\) using the operator of the associated system (1.10), namely

\[\begin{align*}
\chi &= D_1 \alpha + \beta D_2, \\
\psi &= -\alpha D_2 + D_1 \beta.
\end{align*}\] (4.1)

Substitution of (4.1) into the basic system (1.1) leads to the fact that the quaternion potentials \(\alpha, \beta\), in the absence of an external specification of charges and currents (both electric and magnetic (i.e., system (1.1) is homogeneous)), are harmonic quaternions, since

\[\Delta(\alpha) = 0, \quad \Delta(\beta) = 0.\] (4.2)

Indeed, we have

\[D_1 \chi - \psi D_2 = D_1 (D_1 \alpha + \beta D_2) - (\alpha D_2 + D_1 \psi) D_2 = D_1 D_1 \alpha + D_1 \beta D_2 + \alpha D_2 D_2 - D_1 \psi D_2 = 0.\]
After reducing similar ones, we have
\[ D_1 \overline{D}_1 \alpha + D_2 \overline{D}_2 \alpha = 0. \quad (4.3) \]

The second relation (4.2) is proved similarly.
Recall, since it was previously accepted that all functions \( \chi, \psi, \alpha, \beta \) depend only on \( x_i, y_0 \). Hence, in fact
\[ \Delta(8)\alpha = 0, \quad \Delta(8)\beta = 0. \]

If \( y_0 = i\alpha t \), then the operator \( \Delta(4) \) will be the wave operator, which was denoted
\[ [4] = \Delta(3) - \frac{1}{c^2} \frac{d^2}{dt^2}. \quad (4.4) \]

In the presence of magnetic charges and currents, in addition to the usual scalar \( \varphi \) and vector \( \vec{A} \) potentials, we have a scalar magnetic potential \( b \) and vector \( \vec{B} \). To match the results with the generally accepted form of recording, you must accept
\[ \alpha = -b - \vec{A}, \quad \beta = -i\varphi + i\vec{B}. \quad (4.5) \]

Let us assume that \( \varphi, b \) are scalar potentials of electric and magnetic fields, and \( \vec{A}, \vec{B} \) are the corresponding vector potentials. If we put \( b = 0, \vec{B} = 0 \), then we return to the usual proven assumption that magnetic charges and currents do not really exist.

The relation (4.1) of stresses and potentials in explicit vector form is expressed by the relations
\[
\begin{align*}
H_0 &= -\text{div} \vec{A} - \frac{1}{c} \frac{d\varphi}{dt}, \\
\vec{H} &= \text{rot} \vec{A} - \frac{1}{c} \frac{d\vec{B}}{dt} + \text{grad} b, \\
E_0 &= \frac{1}{c} \frac{db}{dt} - \text{div} \vec{B}, \\
\vec{E} &= -\text{grad} \varphi - \frac{1}{c} \frac{dA}{dt} - \text{rot} \vec{B}.
\end{align*}
\]

If \( b = 0, \vec{B} = 0 \), then we return to the usual case [10]. The first equation in (4.6) coincides with the Lorentz condition.
Using (2.4), formulas (4.6) can be verified directly in vector form.

5. Calibration transformations of electromagnetic potentials
Let us proceed to consider the question of gauge transformations of potentials in the case of taking into account the scalar components of the quaternions \( \chi, \psi \), that is, in the case \( E_0, H_0 \neq 0 \).

Let us also preserve the possibility of the existence of magnetic charges and currents.
Based on the adopted system of quaternionic potentials, it is easy to establish that the potentials \( \alpha, \beta \) are determined up to quaternionic functions \( \eta, \tau \).

\[
\alpha' = \alpha + \tau, \quad \beta' = \beta + \eta,
\]

which satisfy the associated system

\[
\begin{align*}
\overline{D}_1 \tau + \eta D_2 &= 0, \\
-\tau \overline{D}_2 + D_1 \eta &= 0.
\end{align*}
\]

(5.1)

Thus, when \( \alpha, \beta \) is replaced by \( \alpha', \beta' \), the functions \( \chi, \psi \) do not change.

The solution to system (5.2), as is known and easy to check, can be expressed as

\[
\tau = D_1 f, \quad \eta = f \overline{D}_2,
\]

(5.3)

where \( f \) is some harmonic quaternion.

Two different cases are possible. Let \( f \) be a scalar quaternion

\[
f = f_0.
\]

(5.4)

Then, according to (1.6), we find

\[
\tau = \text{grad} f_0, \quad \eta = -\frac{1}{ic} \frac{df_0}{dt}.
\]

This leads to the well-known relations

\[
\begin{align*}
\vec{A}' &= \vec{A} - \text{grad} f_0, \\
\phi' &= \phi + \frac{1}{ic} \frac{df_0}{dt}.
\end{align*}
\]

(5.5)

In this case, no new scalar fields appear. This is a common form of gauge transformations used not only in classical electrodynamics but also in quantum electrodynamics.

A somewhat different situation arises if \( f \) is the vector quaternion

\[
\tau = D_1 \vec{f}, \quad \eta = \vec{f} \overline{D}_2.
\]

(5.6)

In this case, by (1.6), we find

\[
\tau = -\text{div} \vec{f} + \text{rot} \vec{f}, \quad \eta = \frac{1}{ic} \frac{d\vec{f}}{dt}.
\]

(5.7)

In explicit vector form, this leads to the relations

\[
\begin{align*}
\vec{A}' &= \vec{A} - \text{rot} \vec{f}, \\
\vec{B}' &= \vec{B} - \frac{1}{c} \frac{d\vec{f}}{dt}, \\
\phi' &= \phi, \\
b' &= b + \text{div} \vec{f}.
\end{align*}
\]

(5.8)

Thus, in this case, \( b, \vec{B} \) are affected, so that if they were absent before the transformation, then after the transformation (5.8) they will inevitably arise.
If an assumption that \( f \) in (5.3) in variable \( x_i, y_0 \) is harmonic to reject, then it is possible to admit \( H_0 = 0 \) and Lorentz condition in (4.6) is satisfied.

6. Conclusion

The paper shows that the generalized system of Cauchy - Riemann equations can be considered as a mathematical model of the equations of classical electrodynamics. However, the following main conclusions should be pointed out:

1. There is a linear first-order partial differential operator with an algebraic structure defined by generalized CCR that transforms an ordered set \( V \) of two quaternions \( \chi, \psi \) into a set \( R \) of two quaternions \( \mu, \nu \)

\[
DV = R, \tag{6.1}
\]

It is assumed that all components of the \( \chi, \psi \) functions are continuously differentiable in some domain \( Q \).

1. With a certain choice of components of quaternions \( \chi, \psi \) and \( \mu, \nu \), system (6.1) coincides with the system of Maxwell's equations for the strengths of the electromagnetic field created by electric charges and currents, as well as hypothetical magnetic charges and currents. The case of nonzero scalar components of quaternions \( \chi, \psi \) is considered, which leads to the conclusion about the existence of some charges and currents determined by the Maxwell system itself.

3. It is pointed out that there exists an operator \( \tilde{D} \) (said to be associated with \( D \)) with the properties indicated in item 1, which takes the set \( P \) of quaternions \( (\alpha, \beta) \) into \( (\chi, \psi) \)

\[
\tilde{D}P = V, \tag{6.2}
\]

so

\[
\tilde{D}\tilde{D}P = R,
\]

where \( \Delta \) is the Laplace operator

\[
D\tilde{D} = \tilde{D}D = \Delta.
\]

It is assumed here that all components of \( P \) are twice continuously differentiable with respect to all independent variables in the domain \( Q \).

4. It is shown that after the appropriate identification this operation (4.2) corresponds to the usual procedure for introducing electromagnetic potentials.

5. Since \( P \) for a given \( V \) is not uniquely defined, namely

\[
P' = P + G, \tag{6.3}
\]

where the set of \( G \) quaternions \( (\tau, \eta) \) is the solution to the system

\[
\tilde{D}G = 0.
\]

References

[1] Maxwell J K 1952 Selected works on the theory of the electromagnetic field (Moscow: Gostekhizdat) p 688
[2] Gladyshev Yu A 2019 Quaternion methods in electrodynamics Proceedings of the Lobachevsky Mathematical Center vol 57 (Kazan: Kazan University Publishing House) pp 111-115

[3] Gladyshev Yu A 2019 On calibration transformations of electromagnetic potentials in quaternion form Proceedings of the international conference Voronezh spring math school Pontryagin readings-XXX Modern methods of the theory of boundary value problems (Voronezh: VSU Publishing House) pp 103-105

[4] Gladyshev Y A 2018 Generalization of the Cauchy-Riemann equations of complex variable theory in the area of quaternion functions Proceedings of XI international conference Modern methods of applied mathematics, control theory and computer technology (Voronezh: Scientific Book Publishing House) pp 48-50

[5] Christiano V, Smarandache F and Umniyati Y 2020 Towards realism interpretation of wave mechanics based on Maxwell equations in quaternion space and some implications, including Smarandache's hypothesis AIP Conference Proceedings 2234 040008

[6] Arbab A I and Alsaawi N 2019 Maxwell's equations of dual photon (Optik) 184 pp 499-507

[7] Moisil M G 1931 Sur los quaternions monogenes (Bull. Sc. Math. 55) pp 168-174

[8] Fueter R 1936 Zur Teorie der regularen Funktionen einer quaternion (Monatshefte of Math. and Phis. vol 43) pp 69-74

[9] Kostrikin A I 1980 Linear algebra and geometry (Moscow: Moscow State University Publishing House) p 319

[10] Landau L D 1973 Field theory (Moscow: Science) p 504

[11] Vainshtein A I 1981 Instanton alphabet (Moscow: ITEP) p 84