On Construction of weighted orthogonal matrices over finite field and its application in cryptography

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Abstract

In this article, we propose a method to construct self orthogonal matrix, orthogonal matrix and anti orthogonal matrix over the finite field. Orthogonal matrices has numerous applications in cryptography, so here we demonstrate the application of weighted orthogonal matrix into cryptography. Using the proposed method of construction we see that it is very easy to transmit the private key and can easily convert the encrypted message into original message and at the same time it will be difficult to get the key matrix for intruder.

Keywords: Orthogonal matrix, Self Orthogonal matrix, Anti Orthogonal matrix, Finite field, Encryption scheme
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1 Introduction:

A square matrix $A$ is said to be Orthogonal matrix of order $n$ if

$$A^T A = AA^T = I_n.$$ 

This leads that the inverse of an Orthogonal matrix is its transpose. P.G. Farrell introduced a very interesting concept which is ”anti-code” [13]. Generally a

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"code" is designed to have a large minimum distance between its code words. As opposite of "large minimum distance" is "small maximum distance". Farrell used the term "anti-code" to describe a set of n-tuple designed to have small maximum distance between its code words [4]. To get inspired by Farrel’s "anti code" James L. Massey [4] introduced anti-orthogonal matrix, as opposite of I is \(-I\). If \(BB^T = -I\) then \(B\) is known as anti-orthogonal matrix. If each row of a matrix is orthogonal to every row including itself is self orthogonal matrix i.e. if \(CC^T = 0\), then \(C\) is a self-orthogonal matrix.

Cryptography is the study of techniques through which a secured communication of message or information executed i.e. communicated message/information cant be understood by anyone who is not the intended receiver. The method to obtain the cipher text from plain text is encryption and the reverse method is decryption. There are several techniques for encryption scheme [12, 9, 6, 1]. In 1929 Lester S. Hill [6] introduced the Hill cipher in which an invertible matrix is used as a private key and inverse of that matrix is used to decrypt. It is very difficult to obtain the inverse of a matrix for large size. Later on orthogonal matrices are used as a private key to disposed off the calculation of inverse of the matrix. As inverse of an orthogonal matrix is its transpose. C. Koukouvinos and D.E. Simos [1] also developed an encryption scheme using circulant Hadamard core in which the first row of the Hadamard core required to be transmitted as a private key.

In this paper we present a method to construct the weighted orthogonal matrices over a finite field \(GF(q)\), where \(q = p^\alpha\) such that \(p\) is a prime and \(\alpha\) is a positive integer. Also we develop an encryption scheme using the weighted orthogonal matrices over finite field \(GF(q)\). Our main aim is to protect the information about the elements of the key matrix. In this encryption scheme there is no need to transmit the matrix or first row or any element of the matrix to get the private key. Along with the encrypted message we need to transmit three numbers which include the size of the key matrix and additionally a primitive polynomial in case of \(\alpha > 1\). So for intruder it will be difficult to find the private key and the numbers make an easy transmission.

The paper is organized as follows:

In section 2, the required definitions and information have been discussed as preliminaries. In section 3, we present the methods to construct orthogonal, self-orthogonal and anti-orthogonal matrices of order \(q\) over the finite field \(GF(q)\). In section 4, we present the application of orthogonal matrix into cryptography. Here encryption scheme and analysis of time complexity of the scheme have been discussed. In section 5, we provide the example of the encryption scheme defined in section 4 and in section 6 we conclude the results.
2 Preliminaries

Definition 2.1. Weighted orthogonal matrix over a field
A square matrix $A = [a_{ij}]_{n \times n}$, where $a_{ij} \in GF(q)$ is said to be an orthogonal matrix with weight $k$ if
\[ AA^T = kI_n \] (1)
over $GF(q)$. In particular if $k = 0, 1, -1$, then $A$ is a self orthogonal matrix, orthogonal matrix and anti orthogonal matrix of order $n$ respectively over $GF(q)$.

Remark 2.1. If $q = 3$, then the matrix defined over $GF(3)$ which satisfies the equation (1) is a weighing matrix of order 3.

Definition 2.2. Primitive root
Let $GF(q)$ be a finite field of order $q$. A primitive root is an element of $GF(q)$ which is a generator of the multiplicative group of the field $GF(q)$.

3 Construction of Orthogonal Matrix

In this section we discuss some method to construct the orthogonal matrices of order $q$ over $GF(q)$.
Let $q = p^\alpha$, where $p$ is a prime and $\alpha$ is any positive integer,
\[ GF(q) = \{0 = a_0, a_1, a_2, \cdots, a_{q-1}\} \]

Lemma 3.1. Let $q = p^\alpha$, where $p$ is a prime and $\alpha$ is any positive integer. Then for any $1 \leq k < q - 1$
\[ \sum_{i=0}^{q-1} a_i^k \equiv 0 \pmod{p}. \] (2)

Proof. Let $g$ be a primitive root of $GF(q)$ so for every non-zero element $a_i \in GF(q)$, $a_i$ can be written as
$\quad a_i = g^\beta; \quad 1 \leq \beta \leq q - 1$
Now

\[
\sum_{i=0}^{q-1} a_i^k = 0 + \sum_{\beta=1}^{q-1} (g^\beta)^k
\]

\[
= g^k + g^{2k} + \cdots + g^{(q-1)k}
\]

\[
= g^k \{1 + g^k + g^{2k} + \cdots + g^{(q-2)k}\}
\]

\[
= g^k \left\{ \frac{(g^k)^{(q-1)} - 1}{g^k - 1} \right\}
\]

\[
= g^k \left\{ \frac{(g^{q-1})^k - 1}{g^k - 1} \right\}
\]

As \( g \) is a primitive root of \( GF(q) \) and \( 1 \leq k < q - 1 \) so \( g^{(q-1)} \equiv 1(\text{mod } p) \) and \( g^k \neq 1 \). Thus

\[
\sum_{i=0}^{q-1} a_i^k \equiv 0(\text{mod } p)
\]

\[\square\]

**Theorem 3.2.** Let \( q = p^\alpha \), where \( p \) is a prime and \( A \) is a square matrix of order \( q \) with entries \( [a_{ij}] \equiv (a_j^t - a_i^t)(\text{mod } p) \) where \( 0 \leq i, j < q - 1 \) and \( 1 \leq t < \frac{q-1}{2} \) and \( a_i, a_j \in GF(q) \). Then \( A \) is a self orthogonal skew symmetric matrix over \( GF(q) \).

**Proof.** Let \( A \) be a square matrix of order \( q \) with entries \( [a_{ij}] \equiv (a_j^t - a_i^t)(\text{mod } p) \), where \( 0 \leq i, j < q - 1 \) and \( 1 \leq t < \frac{q-1}{2} \).

So, each element of \( u^{th} \) row can be expressed as

\[ a_{ij} = (a_j^t - a_i^t)(\text{mod } p) \quad \text{for } 0 \leq j \leq q - 1 \]

Now, inner product of \( u^{th} \) row with itself is

\[
\langle R_u \cdot R_u \rangle \equiv \sum_{j=0}^{q-1} (a_j^t - a_i^t) \cdot (a_j^t - a_i^t)(\text{mod } p)
\]

\[
\equiv \sum_{j=0}^{q-1} \left\{ a_j^{2t} - a_j^t a_i^t + a_i^t a_j^t + a_i^t a_i^t \right\} (\text{mod } p)
\]

\[
\equiv \left\{ \sum_{j=0}^{q-1} a_j^{2t} - a_j^t a_i^t \sum_{j=0}^{q-1} a_j^t - a_i^t \sum_{j=0}^{q-1} a_i^t + \sum_{j=0}^{q-1} a_i^{2t} \right\} (\text{mod } p)
\]

\[
\equiv \left\{ \sum_{j=0}^{q-1} a_j^{2t} - a_i^t \sum_{j=0}^{q-1} a_j^t - a_i^t \sum_{j=0}^{q-1} a_j^t + qa_i^{2t} \right\} (\text{mod } p)
\]
Since $1 \leq t < \frac{q-1}{2} \Rightarrow 2t < q - 1$, so using lemma (3.1), each sum of right hand side is zero. So $\langle R_u \cdot R_u \rangle \equiv 0 \pmod{p}$.

Now, inner product of $u^{th}$ row $v^{th}$ row of the matrix $A$

$$\langle R_u \cdot R_v \rangle \equiv \sum_{j=0}^{q-1} (a_j^t - a_u^t) \cdot (a_j^t - a_v^t) \pmod{p}$$

$$\equiv \sum_{j=0}^{q-1} \{ a_j^{2t} - a_u^t a_v^t - a_u^t a_j^t + a_u^t a_v^t \} \pmod{p}$$

$$\equiv \left\{ \sum_{j=0}^{q-1} a_j^{2t} - a_u^t \sum_{j=0}^{q-1} a_j^t - a_u^t \sum_{j=0}^{q-1} a_j^t + \sum_{j=0}^{q-1} a_u^t a_v^t \right\} \pmod{p}$$

$$\equiv \left\{ \sum_{j=0}^{q-1} a_j^{2t} - a_u^t \sum_{j=0}^{q-1} a_j^t - a_u^t \sum_{j=0}^{q-1} a_j^t + q \cdot a_u^t a_v^t \right\} \pmod{p}$$

Since $1 \leq t < \frac{q-1}{2} \Rightarrow 2t < q - 1$, so using lemma (3.1)

$$\sum_{j=0}^{q-1} a_j^{2t} \equiv (\pmod{p}) \quad \text{and} \quad \sum_{j=0}^{q-1} a_j^t \equiv 0(\pmod{p})$$

So,

$$\langle R_u \cdot R_v \rangle \equiv 0(\pmod{p})$$

So, we have $AA^T = 0 \pmod{p}$ i.e. $A$ is a self orthogonal matrix of order $q$.

Now, to show that $A$ is a skew symmetric matrix it is enough to show that $a_{ij} = -a_{ji}$.

$$a_{ij} \equiv (a_j^t - a_i^t) \pmod{p}$$

$$\equiv -(a_i^t - a_j^t) \pmod{p}$$

$$= -a_{ji}$$

So, $A$ is a skew symmetric matrix of order $q$. \hfill \square$

In particular, if matrix $A$ has same sequence of leading rows and columns then it can be observed that all diagonal entries of $A$ are 0.

**Theorem 3.3.** If $B$ is a self orthogonal skew symmetric matrix over $GF(q)$ with diagonal 0 then $B + rI$, where $r \in GF(q)$, is an orthogonal matrix with weight $r^2$, a quadratic residue in $GF(q)$. 

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Proof. Here
\[
(B + rI)(B + rI)^T = (B + rI)(B^T + rI) \\
= BB^T + rB + rB^T + r^2 I
\]
Since $B$ is a self orthogonal skew symmetric matrix so $BB^T = 0$ and $B^T = -B$
Hence,
\[
(B + rI)(B + rI)^T = r^2 I
\]
So $B + rI$ is an orthogonal matrix with weight $r^2$. \qed

Remark 3.1. For any $q = p^\alpha$, 1 is a quadratic residue in $GF(q)$. So $B + rI$ is an orthogonal matrix of order $q$ for $r^2 \equiv 1(\mod p)$. However, for $q = p^\alpha$ with $p \equiv 1(\mod 4)$, $-1$ is a quadratic residue in $GF(q)$ implies that $B + rI$ is an anti-orthogonal matrix of order $q$ for $r^2 \equiv -1(\mod p)$.

From the above discussion we observed that the weight of the orthogonal matrices is a quadratic residue of $GF(q)$. It is know that each element of $GF(2^\alpha)$ is a quadratic residue in $GF(2^\alpha)$ \cite{Sh}. So for each $k \in GF(2^\alpha)$ there exist an orthogonal matrix of order $2^\alpha$ with weight $k$.

We are interested to find orthogonal matrices of any weight $k \in GF(q)$ over $GF(q)$, where $q$ is an odd prime power.

It is known that every element of a finite field $GF(q)$ can be written as sum of two squares \cite{Sh}. Using this result the orthogonal matrices of order $2q$ of weight $k \in GF(q)$ may be obtained in next results.

Theorem 3.4. If $A$ and $B$ are two self orthogonal matrices of order $q$ obtained by theorem (3.2) then $AB \equiv 0(\mod p)$.

Proof. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two self orthogonal matrices over $GF(q)$. Then using theorem (3.2)
\[
a_{ij} = (a^t_j - a^s_i)(\mod p)
\]
and
\[
b_{ij} = (a^t_j - a^s_i)(\mod p)
\]
where $1 \leq t, s < \frac{q-1}{2}$ and $a_i \in GF(q)$ for $0 \leq i, j \leq q - 1$
Now,

\[ AB = \left\{ \sum_{j=0}^{q-1} a_{ij} b_{jk} \right\} \pmod{p} \]

\[ = \left\{ \sum_{j=0}^{q-1} \left( a_j^t - a_i^t \right) \left( a_k^s - a_j^s \right) \right\} \pmod{p} \]

\[ = \left\{ \sum_{j=0}^{q-1} a_k^s a_j^t - a_j^t a_j^s - a_i^t a_k^s + a_i^t a_j^s \right\} \pmod{p} \]

\[ = \left\{ a_k^s \sum_{j=0}^{q-1} a_j^t - \sum_{j=0}^{q-1} a_j^{t+s} - \sum_{j=0}^{q-1} a_i^t a_k^s + a_i^t \sum_{j=0}^{q-1} a_j^s \right\} \pmod{p} \]

\[ = \left\{ a_k^s \sum_{j=0}^{q-1} a_j^t - \sum_{j=0}^{q-1} a_j^{t+s} - q \cdot a_i^t a_k^s + a_i^t \sum_{j=0}^{q-1} a_j^s \right\} \pmod{p} \]

As, \( 1 \leq t, s < \frac{q-1}{2} \), so \( t + s < q - 1 \). Now using lemma (3.1)

\[ \sum_{j=0}^{q-1} a_j^t \equiv 0 \pmod{p}, \sum_{j=0}^{q-1} a_j^s \equiv 0 \pmod{p} \text{ and } \sum_{j=0}^{q-1} a_j^{t+s} \equiv 0 \pmod{p} \]

Hence

\[ AB \equiv 0 \pmod{p} \]

\[ \square \]

**Theorem 3.5.** If \( A \) and \( B \) are two skew symmetric self orthogonal matrices of order \( q = p^\alpha \), where \( p \) is a prime, then there is a weighted orthogonal matrix of order \( 2q \) with any weight \( k \in GF(q) \).

**Proof.** Since \( A \) and \( B \) are skew symmetric self orthogonal matrices of order \( q \). Since each element of \( GF(q) \) can be expressed as a sum of two squares, consider \( k = r^2 + s^2 \) for some \( r, s \in GF(q) \).

Consider

\[ A' = \begin{bmatrix} A + rI & B + sI \\ -(B + sI)^T & (A + rI)^T \end{bmatrix} \] (3)
Then

\[
A^T A' = \begin{bmatrix}
A + rI & B + sI \\
-(B + sI)^T & (A + rI)^T
\end{bmatrix}
\begin{bmatrix}
A + rI & B + sI \\
-(B + sI)^T & (A + rI)^T
\end{bmatrix}^T
\]

\[
= \begin{bmatrix}
A + rI & B + sI \\
-(B + sI)^T & (A + rI)^T
\end{bmatrix}
\begin{bmatrix}
A^T + rI & -B - sI \\
B^T + sI & A + rI
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(r^2 + s^2)I & -AB + BA \\
-B^T A^T + A^T B^T & (r^2 + s^2)I
\end{bmatrix}
\]

\[
= \begin{bmatrix}
(r^2 + s^2)I & 0 \\
0 & (r^2 + s^2)I
\end{bmatrix}
\]

since \(AB = BA = 0 \mod p\)

\[
= (r^2 + s^2)I \otimes I_2
\]

\[
= kI_{2q}
\]

Therefore \(A'\) is a weighted orthogonal matrix of order \(2q\) with weight \(k\). \(\Box\)

In particular, if \(r = 1\) and \(s = 0\), \(A'\) is an orthogonal matrix of order \(2q\).

**Theorem 3.6.** Let \(A + rI\) be a weighted orthogonal matrix of order \(p\) with weight \(r^2\). If \(H_{4n}\) is a Hadamard matrix of order \(4n\) then the kronecker product of \((A + rI)\) and \(H_{4n}\) is a weighted orthogonal matrix with weight \(4nr^2\).

**Proof.** Let \(A + rI\) be a weighted orthogonal matrix of order \(p\) with weight \(r^2\). So,

\[
(A + rI)(A + rI)^T = r^2 I
\]

Since \(H_n\) is a Hadamard matrix of order \(4n\) so by the definition of Hadamard matrix

\[
H_{4n}H_{4n}^T = 4nI_{4n}
\]

Now

\[
\{H_{4n} \otimes (A + rI)\} \{H_{4n} \otimes (A + rI)\}^T
\]

\[
= \{H_{4n} \otimes (A + rI)\} \{H_{4n}^T \otimes (A + rI)^T\}
\]

\[
= H_{4n} H_{4n}^T \otimes (A + rI)(A + rI)^T
\]

\[
= 4nI_{4n} \otimes r^2 I_p
\]

\[
= 4nr^2 I_{4np}
\]

This shows that kronecker product of Hadamard matrix and weighted orthogonal matrix is again a weighted orthogonal matrix. \(\Box\)

**Note:** If \(\gcd(n, p) \neq 1\) then \(H_{4n} \otimes (A + rI)\) is a self orthogonal matrix of order \(4np\).
4 Encryption Scheme

Orthogonal matrix is widely used in cryptography. In this section we have used the weighted orthogonal matrix, discussed above, in encryption scheme. Suppose the language, in which message or information is written, contains $q = p^\alpha$ ($p$ is a prime) distinct characters. Let $M$ be the numeric plain text of the corresponding message which has to be transmitted securely. Here weighted orthogonal matrix of order $q$ is used to encrypt the message. The encrypted message $C$ is obtained from the plain text $M$ using the transformation

$$C \equiv WM \pmod{p} \tag{4}$$

where $W$ is a weighted orthogonal matrix of order $q$ with weight $r^2$. $W$ is defined as

$$W = A + rI \tag{5}$$

where $A$ is a self orthogonal matrix of order $q$.

Using theorems (3.2) and (3.3) the key matrix $W$ of order $q = p^\alpha$ over a field $GF(q)$ may be constructed. For $\alpha = 1$ the field may be considered as $\mathbb{Z}_q$ but for $\alpha > 1$ it requires a primitive polynomial $P(p^\alpha)$ to construct the required field $GF(q)$. Thus formation of private key for $\alpha = 1$ it requires the numbers $(p, t, r)$ however for $\alpha > 1$ it requires $P(p^\alpha)$ as an additional information.

Here algorithm has been discussed to develop the key matrix

The encryption algorithm for the scheme is given by:

In general, to obtain the original message it requires $W^{-1}$. Since $W$ is a weighted orthogonal matrix, so $W^{-1} = lW^T$, where $l$ is a solution of $r^2x \equiv 1 \pmod{p}$. The intended receiver has to decrypt the message using the transformation

$$M \equiv lW^TC \pmod{p} \tag{6}$$

Along with the security, the main objective of the cryptography is to decrypt the message uniquely.

**Theorem 4.1.** If $C$ is an encrypted message with the encryption algorithm then the decrypted message

$$D \equiv lW^TC \pmod{p}$$

is uniquely determined and is equal to $M$, where $l$ is a solution of $r^2x \equiv 1 \pmod{p}$

Let $C$ be an encrypted message with transformation

$$C \equiv WM \pmod{p}$$
Algorithm 1 Formation of key matrix

Require: \( p, \alpha, t, r, P(p^\alpha) \)
Ensure: \( 1 \leq t < \frac{q-1}{2} \)
1: if \( \alpha = 1 \) then
2: \( q \leftarrow p \)
3: Form an array \( Z_q \) and element of the field \( GF(q) \)
4: for \( i \leftarrow 0 \) to \( q-1 \) do
5: \( Z \leftarrow \{a^i \pmod{p} : a_i \in Z_q \text{ and } 0 \leq i \leq q-1\} \)
6: end for
7: else \( \{\alpha > 1\} \)
8: \( q \leftarrow p^\alpha \)
9: Form an array \( F \) and element of \( F \) is defined as:
\[ F = \{\sum_{i=0}^{a-1}{b_i x^i} : b_i \in Z_p\} \]
10: for \( i \leftarrow 0 \) to \( q-1 \) do
11: \( Z \leftarrow \{a_i^i \pmod{P(p^\alpha)} : a_i \in F\} \)
12: end for
13: end if
14: Construct a square matrix \( A = [e_{ij}] \) of order \( q \)
15: for \( i \leftarrow 0 \) to \( q-1 \) do
16: for \( j \leftarrow 0 \) to \( q-1 \) do
17: \( e_{ij} \leftarrow (a_j^i - a_i^i)(\pmod{p}) : a_j^i, a_i^i \in Z \)
18: end for
19: end for

Algorithm 2 Encryption Algorithm

Require: Require msg to encrypt
1: select \((p, \alpha, t, r, P(p^\alpha))\)
2: if \( \alpha = 1 \) then
3: \( k \leftarrow (p, t, r) \)
4: else \( \{\alpha > 1\} \)
5: \( k \leftarrow (p, t, r, P(p^\alpha)) \)
6: end if
7: \( W \leftarrow A + rI \)
8: \( M \leftarrow \text{convert(msg)} \)
9: \( C \leftarrow WM \pmod{p} \)
10: Transmit\((k, C)\)
Since

\[ D \equiv lW^T C \pmod{p} \]
\[ \equiv lW^T WM \pmod{p} \]
\[ \equiv lr^2IM \pmod{p} \]
\[ \equiv M \pmod{p} \]

as \( W \) is a weighted orthogonal matrix so \( W^TW = r^2I \)

\[ \equiv M \pmod{p} \]

as \( l \) is a solution of \( r^2x \equiv 1 \pmod{p} \)

It shows that message is uniquely decrypted. The decryption algorithm is defined below

### Algorithm 3 Decryption Algorithm

**Require:** Require obtained cipher text

1. receive \((C, p, t, r, P(p^α))\)
2. obtain the value of \( l \) /* using euclidean algorithm
3. construct matrix \( A \) /* using algorithm (1)
4. \( W \leftarrow A + rI \)
5. \( M \leftarrow lW^TC \pmod{p} \) /* decrypt the message

#### 4.1 Cryptanalysis of Encryption scheme

In the above discussed encryption scheme, sender transmits numbers \((p, r, t)\) and \( P(p^α) \) in either case as an additional information to form the private key. To get the plain text intended receiver has to use the transformation \[ (6) \]. So receiver has to find out \( W^T \) and \( l \) only. The matrix \( W^T \) and integer \( l \) may be obtained by using algorithm \( (1) \) and the Euclidean algorithm respectively. The time complexity to find \( W^T \) and the integer \( l \) is \( O(q^3) \) and \( O(\log q) \) the respectively.

The main aim of encryption scheme is to protect the information about the private key. But the intruder always tries all possible combinations to find the keys and checks which one of them return the plain text. Since in this encryption scheme numbers \((p, r, t)\) and \( P(p^α) \) in either case as an additional information will be transmitted as a private key. So it will be difficult to guess the size of the key matrix. And if any one could guess the order of a key matrix it is still difficult to obtain the key matrix \( W \). As matrix \( W \) consist the elements of \( GF(q) \). So the size of the key space, \( K(W) \), is \( q^{q^2} \). Thus the computational complexity to find the key matrix \( W \) is \( O(q^{q^2}) \) and it increases exponentially.

#### Observation

1. To form the private key numbers \((p, r, t, P(p^α))\) is used, so it is easy to transmit.
2. As orthogonal matrix is used to encrypt the message so it is easy to decrypt the message as inverse of an orthogonal matrix is its transpose.

3. For intruder after knowing the actual size still it is difficult to find the key matrix as the computational complexity to find the key matrix is $\mathcal{O}(q^2)$.

5 Example

Suppose the plain text COVID–19 has to be transmit. Firstly we convert the plain text into its corresponding numerical value (ASCII Code)

| $C$ | $O$ | $V$ | $I$ | $D$ | $1$ | $9$ |
|-----|-----|-----|-----|-----|-----|-----|
| 67  | 79  | 86  | 73  | 68  | 45  | 49  |

Here highest value present in the plain text is 86. Consider $p = 89$ (we need to choose a prime number which is greater or equal to the highest numeric value present in plain text). In plain text we need to add ”space” 81 times. Thus

$$M = \begin{bmatrix} 67 & 79 & 86 & 73 & 68 & 45 & 49 & 57 & 32 & \cdots & 32 & 32 \\ & & & & & & & \overset{81 \text{ times}}{\cdots} & & & \end{bmatrix}^{T}_{1 \times 89}$$

Consider $r = 5, t = 2$ and construct the weighted orthogonal matrix $W$ of order 89 by applying algorithm (1). The cipher text $C$ is obtained by the transformation

$$C \equiv WM \pmod{89}$$

is

$$C = \begin{bmatrix} 56 & 26 & 58 & 77 & 45 & 10 & 19 & 46 & 84 & 67 & 48 & 27 & 4 & 68 & 41 & 12 & 70 & 37 & 2 & 54 & 15 & 63 \\ 20 & 64 & 17 & 57 & 6 & 42 & 76 & 19 & 49 & 77 & 14 & 38 & 60 & 80 & 9 & 25 & 39 & 51 & 61 & 69 & 75 \\ 79 & 81 & 81 & 79 & 75 & 69 & 61 & 51 & 39 & 25 & 9 & 80 & 60 & 38 & 14 & 77 & 49 & 19 & 76 & 42 \\ 6 & 57 & 17 & 64 & 20 & 63 & 15 & 54 & 2 & 37 & 70 & 12 & 41 & 68 & 4 & 27 & 48 & 67 & 84 \\ 10 & 23 & 34 & 43 & 50 & 55 & 58 \end{bmatrix}^{T}_{1 \times 89}$$

It is known that $0 – 31$ and 127 are non printable character. So for our convenience we define the corresponding numerical value (ASCII code)

0 → (0)*
1 → (1)*
2 → (2)*

:
Convert the numeric value of $C$ in their corresponding character

$$C = \begin{bmatrix}
8 & *(26) & : & M & - & (10)* & (19)* & . & T & C & 0 & (27)* & (4)* & D & \\
(12)* & F & % & (2)* & 6 & (15)* & ? & (20)* & @ & (17)* & 9 & (6)* & * & L \\
(19)* & 1 & M & (14)* & \& & < & P & (9)* & (25)* & ' & 3 & = & E & K \\
O & Q & O & O & K & E & = & 3 & ' & (25)* & (9)* & P & < & \& \\
(14)* & M & 1 & (19)* & L & * & (6)* & 9 & (17)* & @ & (20)* & ? & (15)* & 6 \\
(2)* & % & F & (12)* & ) & D & (4)* & (27)* & 0 & C & T & (10)* & (23)* & " \\
+ & 2 & 7 & : & 
\end{bmatrix}_{1\times 89}^{T}$$

We transmit $C, 89, 5, 2$.

To obtain the plain text intended receiver has to construct the weighted orthogonal matrix $W$ using algorithm [1]. Then by the help of transformation

$$M = lW^TC \pmod{p}$$

$l$ is the solution of $25^2x \equiv 1 \pmod{89}$ as $r = 5$ so in this case $l = 57$ and the plain text will be

$$M = \begin{bmatrix}
67 & 79 & 86 & 73 & 68 & 45 & 49 & 57 & 32 & \cdots & 32 & 32
\end{bmatrix}_{1\times 89}^{T}$$

i.e.

$$\begin{bmatrix}
C & O & V & I & D & - & 1 & 9 \\
67 & 79 & 86 & 73 & 68 & 45 & 49 & 57
\end{bmatrix}$$

6 Conclusion

In this paper we have developed a method to construct weighted orthogonal matrices over a finite field $GF(q)$. There is no non-zero self-orthogonal matrix over $\mathbb{R}$ however self orthogonal matrix has been developed over $GF(q)$ in modular arithmetic. Furthermore an encryption scheme has been developed using the obtained weighted orthogonal matrices. In this scheme it is easier to transmit a finite number of numbers which yields private key.

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