SUMS OF COMMUTATORS IN FREE PROBABILITY

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Abstract. We study the linear span of commutators of free random variables and show that these are the only quadratic forms which satisfy the following equivalent properties:

- preservation free infinite divisibility,
- free and strong cancellation of odd cumulants,
- symmetric distribution for any free family.

The main combinatorial tool is an involution on non-crossing partitions.

1. Introduction
Free probability was introduced by Voiculescu 30 years ago [26, 24] in order to solve some problems in von Neumann algebras of free groups. It has developed into a whole new field with numerous connections to different branches of mathematics like classical probability, combinatorics and analysis, in particular random matrices [27], noncrossing partitions [20] and operator algebras. Free probability is considered the most developed branch of noncommutative probability and during its development far-reaching analogies between classical and free probability emerged. For example, there is a central limit theorem with the famous Wigner semicircle law appearing in the limit, a corresponding Brownian motion, and more generally, one of the most striking features is the existence of the Bercovici-Pata bijection [4] between infinitely divisible distributions in the classical and the free world.

In the present article we continue our investigation of the distribution of quadratic forms [10]. The main result is a characterization of quadratic forms which preserve free infinite divisibility. It was shown in [2] that the free commutator of freely infinitely divisible random variables is also freely infinitely divisible and the authors ask whether there are other noncommutative polynomials which preserve free infinite divisibility. In [10] we showed that any quadratic form in free random variables which exhibits the phenomenon of cancellation of odd cumulants, i.e., whose distribution does not depend on the odd cumulants of the distributions of the original variables, preserves free infinite divisibility. Examples are the free commutator [21] and the free sample variance [10]. Note that the cancellation phenomenon for the latter only holds for free identically distributed families, while in the former arbitrary free random variables can be inserted.

In the present paper we give a unified proof of these results. In addition we record the observation that the cancellation phenomenon for the commutator also occurs without the freeness assumption, i.e., the remarkable phenomenon that the mixed odd cumulants cancel for sums of commutators of arbitrary noncommutative random variables. Using these results we introduce generalized tetilla laws.

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2. Preliminaries

2.1. Basic Notation and Terminology. A tracial noncommutative probability space is a pair \((\mathcal{A}, \tau)\) where \(\mathcal{A}\) is a von Neumann algebra, and \(\tau : \mathcal{A} \to \mathbb{C}\) is a normal, faithful, tracial state, i.e., \(\tau\) is linear and continuous in the weak* topology, \(\tau(XY) = \tau(YX), \tau(I) = 1, \tau(XX^*) \geq 0\) and \(\tau(XX^*) = 0\) implies \(X = 0\) for all \(X,Y \in \mathcal{A}\). The basic example of a noncommutative probability space is the algebra of complex \(N \times N\) matrices \(M_N(\mathbb{C})\). The unique tracial state is the normalized trace \(\tau_N(A) = \frac{1}{N} \text{Tr}(A) = \frac{1}{N} \sum A_{ii}\).

The elements \(X \in \mathcal{A}_{sa}\) are called (noncommutative) random variables; in the present paper all random variables are assumed to be self-adjoint. Given a noncommutative random variable \(X \in \mathcal{A}_{sa}\), the spectral theorem provides a unique probability measure \(\mu_X\) on \(\mathbb{R}\) which encodes the distribution of \(X\) in the state \(\tau\), i.e., \(\tau(f(X)) = \int_{\mathbb{R}} f(\lambda) d\mu_X(\lambda)\) for any bounded Borel function \(f\) on \(\mathbb{R}\).

2.2. Free Independence. A family of von Neumann subalgebras \((\mathcal{A}_i)_{i \in I}\) of \(\mathcal{A}\) is called free if \(\tau(X_1 \ldots X_n) = 0\) whenever \(\tau(X_j) = 0\) for all \(j = 1, \ldots, n\) and \(X_j \in \mathcal{A}_{(j)}\) for some indices \(i(1) \neq i(2) \neq \cdots \neq i(n)\). Random variables \(X_1, \ldots, X_n\) are freely independent (free) if the subalgebras they generate are free. Free random variables can be constructed using the reduced free product of von Neumann algebras [25].

2.3. Free Convolution and the Cauchy-Stieltjes Transform. It can be shown that the joint distribution of free random variables \(X_i\) is uniquely determined by the distributions of the individual random variables \(X_i\) and therefore the operation of free convolution is well defined: Let \(\mu\) and \(\nu\) be probability measures on \(\mathbb{R}\), and \(X,Y\) self-adjoint free random variables with respective distributions \(\mu\) and \(\nu\). The distribution of \(X+Y\) is called the free additive convolution of \(\mu\) and \(\nu\) and is denoted by \(\mu \boxplus \nu\). For more details about free convolutions and free probability theory we refer the reader to the standard sources [24, 22, 19].

The analytic approach to free convolution is based on the Cauchy transform

\[ (2.1) \quad G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z-y} \mu(dy) \]

of a probability measure \(\mu\). The Cauchy transform is analytic on the upper half plane \(\mathbb{C}^+ = \{x + iy | x, y \in \mathbb{R}, y > 0\}\) and takes values in the closed lower half plane \(\mathbb{C}^- \cup \mathbb{R}\). For measures with compact support the Cauchy transform is analytic at infinity and related to the moment generating function \(M_X\) as follows:

\[ (2.2) \quad M_X(z) = \sum_{n=0}^{\infty} \tau(X^n) z^n = \frac{1}{z} G_X(1/z). \]

Moreover the Cauchy transform has an inverse in some neighbourhood of infinity which has the form

\[ G_\mu^{-1}(z) = \frac{1}{z} + R_\mu(z), \]

where \(R_\mu(z)\) is analytic in a neighbourhood of zero and is called \(R\)-transform. The coefficients of its series expansion

\[ (2.3) \quad R_X(z) = \sum_{n=0}^{\infty} K_{n+1}(X) z^n \]

are called free cumulants of the random variable \(X\), see Section 2.7 below. As a formal generating series it will be convenient to consider instead the shift \(C_X(z) := z R_X(z)\) which is called free cumulant transform or free cumulant generating function. The free convolution can now be computed via the identity

\[ (2.4) \quad R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z), \]

see [26].
In order to accommodate for measures with noncompact support, the following reformulation is useful [5]. Let \( F_{\mu}(z) = 1/G_{\mu}(z) \) be the reciprocal Cauchy transform. Then \( F_{\mu}(z) \) has an analytic right compositional inverse \( F_{\mu}^{-1} \) on a region

\[
\Gamma_{\eta,M} = \{ z \in \mathbb{C} \mid |\text{Re} z| < \eta |\text{Im} z|, \text{ Im} z > M \};
\]

the Voiculescu transform is defined as the function

\[
\phi_{\mu}(z) = F_{\mu}^{-1}(z) - z
\]

which turns out to be \( \phi_{\mu}(z) = R_{\mu}(1/z) \).

2.4. Free infinite divisibility. In analogy with classical probability, a probability measure \( \mu \) on \( \mathbb{R} \) is said to be freely infinitely divisible (or FID for short) if for each \( n \in \{1,2,3,\ldots\} \) there exists a probability measure \( \mu_n \) such that \( \mu = \mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n \) (n-fold convolution).

Free infinite divisibility of a measure \( \mu \) is characterized by the property that its Voiculescu transform has a Nevanlinna-Pick representation [5]

\[
(2.5) \quad \phi_{\mu}(z) = \gamma + \int_{\mathbb{R}} \frac{1 + xz}{z - x} \rho(dx) = \gamma + \int_{\mathbb{R}} \left( \frac{1}{z - x} + \frac{x}{1 + x^2} \right)(1 + x^2)\rho(dx)
\]

for some \( \gamma \in \mathbb{R} \) and some nonnegative finite measure \( \rho \).

Combinatorially, the characterization (2.5) is equivalent to the statement that the sequence of free cumulants is conditionally positive definite, i.e., for all \( n \in \mathbb{N} \) and all vectors \( \xi \in \mathbb{C}^n \)

\[
(2.6) \quad \sum_{i,j=1}^{n} \xi_i \bar{\xi}_j K_{i+j}(X) \geq 0.
\]

Equivalently, the Hankel determinants \( [K_{i+j}(X)]_{i,j=1,2,\ldots,n} \) are positive for all \( n \geq 1 \), see [22, Lecture 13].

2.5. Some probability distributions. Let us now recall the basic properties of some specific probability distributions which play prominent roles in the present paper.

2.5.1. Wigner semicircle law. The Wigner semicircle law has density

\[
(2.7) \quad d\mu(x) = \frac{1}{2\pi} \sqrt{4 - x^2} dx
\]

on \(-2 \leq x \leq 2\). Its Cauchy-Stieltjes transform is given by the formula

\[
(2.8) \quad G_{\mu}(z) = \frac{z - \sqrt{z^2 - 4}}{2},
\]

where \( |z| \) is big enough and where the branch of the analytic square root is determined by the condition that \( \text{Im}(z) > 0 \Rightarrow \text{Im}(G_{\mu}(z)) \leq 0 \) (see [23]).

A non-commutative random variable \( X \) with semicircle law is called semicircular or free gaussian random variable. The reason for the latter is the fact that its free cumulants \( K_r = 0 \) for \( r > 2 \) and it appears in the free version of the central limit theorem.

2.5.2. Free Poisson law. The Marchenko-Pastur distribution or free Poisson distribution of rate \( \lambda \) has \( R \)-transform

\[
R(z) = \frac{\lambda}{1 - z}.
\]

Let \( \rho \) be a probability measure on the real line. The compound free Poisson distribution with parameters \((\lambda,\rho)\) has \( R \)-transform

\[
R(z) = \lambda(M_{\rho}(z) - 1),
\]

i.e., the free cumulants are \( K_n = \lambda m_n(\rho) \).
2.5.3. Tetilla law. If \( X \) and \( Y \) are two free semicircular random variables with variance one then the law \( \mu \) of the commutator \( i(XY - YX) \) is supported on the interval \( |x| < \sqrt{11 + 5\sqrt{5}} \) and is absolutely continuous with density 

\[
\mu(dx) = \frac{1}{2\sqrt{3\pi}|x|} \left[ \sqrt{1 + 18x^2 + 3\sqrt{3x^2 + 33x^4 - 6x^6}} - \sqrt{1 + 18x^2 - 3\sqrt{3x^2 + 33x^4 - 6x^6}} \right] \, dx.
\]

The above density is rescaled from [8, equation (2.8)]. The name tetilla law has its origin in the similarity of its density with the tetilla cheese from Galicia, see [8].

2.5.4. Compound free Poisson distribution. In particular the case when the free cumulants form a moment sequence, i.e., \( K_n(\mu) = \lambda \mu_n(\nu) \) for some \( \lambda > 0 \) and some probability measure \( \nu \). In this case \( \mu \) is called a compound free Poisson distribution of rate \( \lambda \) with jump distribution \( \nu \).

2.5.5. Even elements. We call an element \( X \in \mathcal{A} \) even if all its odd moments vanish, i.e., \( \tau(X^{2i+1}) = 0 \) for all \( i \geq 0 \). It is immediate that the vanishing of all odd moments is equivalent to the vanishing of all odd cumulants, i.e., \( K_{2i+1}(X) = 0 \) and thus the even cumulants contain the complete information about the distribution of an even element. The sequence \( \alpha_n = K_{2n}(X) \) of even cumulants is called the determining sequence of \( X \).

2.6. Noncrossing Partitions. We recall some facts about noncrossing partitions. For details and proofs see the lecture notes [22, Lecture 9]. Let \( S \subset \mathbb{N} \) be a finite subset. A partition of \( S \) is a set of mutually disjoint subsets (also called blocks) \( B_1, B_2, \ldots, B_k \subset S \) whose union is \( S \). The size of \( \pi \) is the number of blocks and will be denoted by \( |\pi| \). Any partition \( \pi \) defines an equivalence relation on \( S \), denoted by \( \sim_\pi \), such that the equivalence classes are the blocks \( \pi \). That is, \( i \sim_\pi j \) if \( i \) and \( j \) belong to the same block of \( \pi \). A partition \( \pi \) is called noncrossing if different blocks do not interlace, i.e., there is no quadruple of elements \( i < j < k < l \) such that \( i \sim_\pi k \) and \( j \sim_\pi l \) but \( i \not{\sim}_\pi j \).

The set of non-crossing partitions of \( S \) is denoted by \( NC(S) \), in the case where \( S = [n] := \{1, \ldots, n\} \) we write \( NC(n) := NC([n]) \). \( NC(n) \) is a lattice under refinement order, where the relation \( \pi \preceq \rho \) holds if every block of \( \pi \) is contained in a block of \( \rho \). The subclass of noncrossing pair partitions (i.e., noncrossing complete matchings) is denoted by \( NC_2(n) \).

The maximal element of \( NC(n) \) under this order is the partition consisting of only one block and it is denoted by \( 1_n \). On the other hand the minimal element \( 0_n \) is the unique partition where every block is a singleton. Sometimes it is convenient to visualize partitions as diagrams, for example \( 1_n = \prod_{l=1}^{n} \) and \( 0_n = 1 \ldots 1 \).

We will be concerned with the parity of block sizes. A block of a partition is called even (resp. odd) according to the parity of its cardinality. A partition is called even if each of its blocks has even cardinality. For even \( n \in \mathbb{N} \) we denote by \( NC^n(n) \) the subset of even noncrossing partitions and we will call odd noncrossing partitions the elements of the complement \( NC^c(n) \), i.e., those which have at least one odd block.

We will apply the product formula (2.13) below only in the case of pairwise products of random variables and in this case two specific pair partitions and their complements will play a particularly important role, namely the standard matching \( \hat{1}_n^a = \{n_1, \ldots, n\} \in NC(2n) \) and its shift \( \nu_{n} = \{n, \ldots, n_1\} \in NC(2n) \). The blocks \( (2k - 1, 2k) \), \( k \in [n] \) of the standard matching \( \hat{1}_n^a \), are called standard braces.

The action of the symmetric group \( \mathfrak{S}_n \) on the set \([n]\) naturally induces an action on set partitions, namely

\[
(2.9) \quad \sigma \cdot \pi = \{\sigma(B) \mid B \in \pi\}.
\]

2.7. Free Cumulants. Given a noncommutative probability space \((\mathcal{A}, \tau)\) the free cumulants are multilinear maps \( K_n : \mathcal{A}^n \rightarrow \mathbb{C} \) defined implicitly in terms of the mixed moments by the
relation

\begin{equation}
\tau(X_1X_2 \ldots X_n) = \sum_{\pi \in NC(n)} K_\pi(X_1, X_2, \ldots, X_n),
\end{equation}

where

\begin{equation}
K_\pi(X_1, X_2, \ldots, X_n) := \Pi_{B \in \pi} K_{|B|}(X_i : i \in B).
\end{equation}

Sometimes we will abbreviate univariate cumulants as $K_n(X_i) = K_n(X, \ldots, X)$. The action \((2.9)\) of a permutation on noncrossing partitions may introduce crossings. This is however not the case for cyclic permutations and mirror permutations. We record their effect on tracial cumulants in the following lemma, which follows directly from the corresponding properties of the trace.

**Lemma 2.1.** Let $X_1, X_2, \ldots, X_n \in \mathcal{A}$ be random variables in a tracial probability space, then

(i) $K_n(X_1, X_2, \ldots, X_n) = \overline{K_n(X_n^*, X_{n-1}^*, \ldots, X_1^*)}$

(ii) $K_n(X_2, X_3, \ldots, X_n, X_1) = K_n(X_1, X_2, \ldots, X_n)$.

Free cumulants provide a powerful technical tool to investigate free random variables. This is due to the basic property of vanishing of mixed cumulants. By this we mean the property that

$K_n(X_1, X_2, \ldots, X_n) = 0$

for any family of random variables $X_1, X_2, \ldots, X_n$ which can be partitioned into two mutually free subsets.

For free sequences this can be reformulated as follows. Let $(X_i)_{i \in \mathbb{N}}$ be a sequence of free random variables and $h : [r] \to \mathbb{N}$ a map. We denote by $\ker h$ the set partition which is induced by the equivalence relation

\begin{equation}
i \sim_{\ker h} j \iff h(i) = h(j).
\end{equation}

In this notation, vanishing of mixed cumulants implies that

\begin{equation}
K_\pi(X_{h(1)}, X_{h(2)}, \ldots, X_{h(n)}) = 0 \quad \text{unless} \quad h \geq \pi.
\end{equation}

Our main technical tool is the free version, due to Krawczyk and Speicher [16] (see also [22, Theorem 11.12]), of the classical formula of James and Leonov/Shiryaev [15, 18] which expresses cumulants of products in terms of individual cumulants.

**Theorem 2.2.** Let $r, n \in \mathbb{N}$ and $i_1 < i_2 < \cdots < i_r = n$ be given and let

\begin{equation}
\rho = \{(1, 2, \ldots, i_1), (i_1 + 1, i_1 + 2, \ldots, i_2), \ldots, (i_{r-1} + 1, i_{r-1} + 2, \ldots, i_r)\} \in NC(n)
\end{equation}

be the induced interval partition. Consider now random variables $X_1, \ldots, X_n \in \mathcal{A}$. Then the free cumulants of the products can be expanded as follows:

\begin{equation}
K_\rho(X_1 \ldots X_{i_1}, \ldots, X_{i_{r-1}+1} \ldots X_n) = \sum_{\pi \in NC(n)} K_\pi(X_1, \ldots, X_n).
\end{equation}

2.8. **Cumulants of quadratic forms.** Kreweras [17] discovered an interesting antiisomorphism of the lattice $NC(n)$, now called the Kreweras complementation map, of which we will need two variants. Given a noncrossing partition $\pi$ of $\{1, 2, \ldots, n\}$, the left Kreweras complement $\overline{\pi}$ is the maximal noncrossing partition of the ordered set $\{\overline{1}, \overline{2}, \ldots, \overline{n}\}$ such that $\pi \cup \overline{\pi}$ is a noncrossing partition of the interlaced set $\{\overline{1}, 1, 2, \ldots, \overline{n}, n\}$. Similarly, the right Kreweras complement $\overline{\pi}$ is the maximal noncrossing partition of the ordered set $\{1, \overline{2}, \ldots, \overline{n}\}$ such that $\pi \cup \overline{\pi}$ is a noncrossing partition of the interlaced set $\{1, \overline{1}, 2, \ldots, n, \overline{n}\}$. The two maps are inverse to each other and it can be shown that the sizes are related by the identity

\begin{equation}
|\pi| = |\overline{\pi}| = n + 1 - |\pi|.
\end{equation}

This motivates the following definition.
Definition 2.3 ([22, Ch. 17]). Let
\[ f(z_1, \ldots, z_m) = \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{m} a_{z_1, \ldots, z_n} z_{i_1} \ldots z_{i_n} \text{ and } g(z_1, \ldots, z_m) = \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{m} b_{z_1, \ldots, z_n} z_{i_1} \ldots z_{i_n}, \]
be two formal noncommutative power series. Their boxed convolution is defined as the coefficient of order \((i_1, \ldots, i_n)\) of the formal power series \(f \boxwedge g\) which is defined as
\[ \text{Cf}_{(i_1, \ldots, i_n)}(f \boxwedge g) = \sum_{\pi \in \text{NC}(n)} \text{Cf}_{(i_1, \ldots, i_n)}(f) \text{Cf}_{(i_1, \ldots, i_n)}(g). \]

The boxed convolution \(\boxwedge\) is most frequently used with the so called the Zeta-series and the Möbius-series, which are defined as
\[ \zeta_m(z_1, \ldots, z_m) = \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n=1}^{m} z_{i_1} \ldots z_{i_n} \]
\[ \text{Möb}_m(z_1, \ldots, z_m) = \sum_{r=0}^{\infty} \frac{(2n-2)!}{(n-1)!n!} (z_1 + \ldots + z_m)^n. \]
The functions \(\zeta_m\) and \(\text{Möb}_m\) are inverse to each other with respect to \(\boxwedge\). In order to compute cumulants of quadratic forms we use the following results from our previous paper [10].

Lemma 2.4 ([10, Lemma 2.14]). Let \(r \in \mathbb{N}\) and \(\pi \in \text{NCE}(2r)\), then \(\pi \lor \hat{1}_r = \hat{1}_{2r}\) if and only if \(\pi \geq \nu_{0r} = \left[ \begin{array}{c} 1 \\ 1 \\ \vdots \end{array} \right] \), i.e., \(1\) and \(2r\) lie in the same block of \(\pi\) and elements \(2i\) and \(2i+1\) also lie in the same block of \(\pi\) for \(i \in [r-1]\). Consequently,
\[ \{ \pi : \pi \lor \hat{1}_r = \hat{1}_{2r} \} \cap \text{NCE}(2r) = [\nu_{0r}, \hat{1}_{2r}], \]
is a lattice isomorphic to \(\text{NC}(r)\).

We will use the following result from [10] to express cumulants of quadratic forms in even random variables in terms of the diagonal map of matrices. This is the conditional expectation \(E^D\) which annihilates all off-diagonal entries of a matrix, i.e., if we denote by \(E_i\) the projection matrix onto the subspace spanned by the \(i\)-th unit vector, then
\[ E^D : M_n(\mathbb{C}) \to M_n(\mathbb{C}) \]
\[ A \mapsto \sum_{i=1}^{n} E_i A E_i. \]

Proposition 2.5 ([10, Proposition 4.5]). Let \(X_1, X_2, \ldots, X_n \in A\) be a free family of even random variables, \(X = [X_i X_j]_{i,j=1}^{n}\), \(A = [a_{i,j}]_{i,j=1}^{n} \in M_n(\mathbb{C})\) a scalar matrix and \(Q_n = \sum a_{i,j} X_i X_j\) a quadratic form.

(i) The cumulants of \(Q_n\) are given by
\[ K_r(Q_n) = \sum_{i_1, \ldots, i_r \in [n]} \text{Tr}(AE_{i_1}AE_{i_2} \ldots AE_{i_r}) \sum_{\pi \in \text{NCE}(2r)} K_{r}(X_{i_r}, X_{i_1}, X_{i_2}, \ldots, X_{i_{r-1}}, X_{i_r}). \]
(ii) If we assume in addition that \(X_i\) are identically distributed then the previous formula simplifies to the following convolution-like expression
\[ K_r(Q_n) = \sum_{\pi \in \text{NC}(r)} \text{Tr}(E_\pi^2(A)) \prod_{B \in \pi} K_{2|B|}(X). \]

Remark 2.6. In the case of a free standard semicircular family formula (2.17) has only one contributing term and takes the particularly simple form
\[ K_r(Q_n) = \text{Tr}(A_r^n). \]
2.9. Special notations and definitions for noncrossing partitions.

Definition 2.7. A lattice \((L, \leq)\) is called bounded if it has a unique minimal and a unique maximal element, usually denoted \(\hat{0}\) and \(\hat{1}\), respectively. Let \(a \in L\). An element \(b \in L\) is called a complement of \(a\) if \(a \land b = \hat{0}\) and \(a \lor b = \hat{1}\). We will need the weaker notion of upper complements, i.e., the set
\[
\{b \in L \mid a \lor b = \hat{1}\}.
\]
We denote the set of upper complements of \(\hat{1}\) in \(NC(2n)\) by
\[
C_{2n} = \{\pi \in NC(2n) \mid \pi \lor n \land n \cdots n = \hat{1}_{2n}\}.
\]
Among these we single out the even ones
\[
C_{2n}^e = C_{2n} \cap NCE(2n),
\]
and the remaining ones
\[
C_{2n}^o = C_{2n} \setminus NCE(2n),
\]
which have at least one odd block. Our aim is to show that under certain conditions the contributions of \(C_{2n}^o\) in the expansion \((2.13)\) cancel each other. To this end we will define an involution on \(C_{2n}^o\) in Section 3. This involution is based on the concept of inner odd blocks, which we present next.

Definition 2.8. Let \(\pi\) be a noncrossing partition and \(B, B' \in \pi\) two distinct blocks of \(\pi\).

1. We denote by \(\alpha(B) = \min B\) and \(\omega(B) = \max B\) its extreme points. The interval \(I(B) = [\alpha(B), \alpha(B) + 1, \ldots, \omega(B)]\) is called the padding interval of \(B\).

2. Given another block \(B' \in \pi\) we say that \(B'\) is nested inside \(B\) if \(I(B') \subseteq I(B)\), i.e., if \(\alpha(B) < \alpha(B') \leq \omega(B') < \omega(B)\).

3. An inner odd block of \(\pi\) is a block \(B \in \pi\) such that no other odd block of \(\pi\) is nested inside \(B\).

In particular, every singleton is an inner odd block. Let us emphasize that for our purposes we allow even blocks to be nested inside inner odd blocks; see Figure 1 for examples.

![Figure 1. Some inner odd blocks](image)

Before proceeding with further definitions we record in the next lemma some preliminary facts about inner odd blocks.

Lemma 2.9. Let \(\pi \in NCE^c(2n)\), then

(i) \(\pi\) has at least one inner odd block.

(ii) If \(B \in \pi\) is an inner odd block, then its padding interval \(I(B)\) is odd, the end points \(\alpha(B)\) and \(\omega(B)\) have the same parity and the complement \([2n]\setminus B\) is a union of intervals, out of which exactly one is odd.

Proof. (i) The first part is obvious if \(\pi\) has a singleton, otherwise pick any odd block \(B \in \pi\). If it does not cover another odd block, we are done. Otherwise we choose any odd block nested inside \(B\) and continue the procedure recursively until an inner odd block is reached.

(ii) Let \(I(B)\) be an interval. Then the complement of \(I(B)\) is a union of intervals and by definition those intervals which are covered by \(B\) are even. It follows that the padding interval \(I(B)\) is the union of even blocks and exactly one odd block and therefore odd. Consequently \(\alpha(B)\) and \(\omega(B)\) have the same parity and exactly one of the “outer” intervals \([1, \alpha(B) - 1]\) and \([\omega(B) + 1, 2n]\) is odd (if \(I(B) = [1, 2, \ldots, 2k + 1]\), then \([1, \alpha(B) - 1]\) is empty set).

□

Definition 2.10. Let \(\pi\) be a noncrossing partition and \(B, B' \in \pi\) two distinct blocks of \(\pi\).
1. Given a block $C \in \rho$ from another partition $\rho$, we say that $C$ connects $B$ and $B'$ if both $B \cap C \neq \emptyset$ and $B' \cap C \neq \emptyset$.

2. Let $\pi \in C_{2n}$ and $B$ its leftmost inner odd block. By Lemma 2.9 the padding interval $I(B)$ has odd length and therefore there is a unique standard brace $\pi \in 1^n$ such that $I(B) \cap \pi$ contains exactly one element. We call $\pi$ the pivot brace of $\pi$. The unique point in the intersection of the pivot brace and the leftmost inner odd block is called the pivot element. In our figures the pivot brace will be highlighted by thick lines. Associated to the pivot brace we call the two unique blocks $\pi, \pi'$ such that $\pi \cap \pi' \neq \emptyset$ and $\pi' \cap \pi' \neq \emptyset$ the left and right pivot blocks of $\pi$.

3. For the pivot blocks we define the essentially nested blocks, namely if $A \subset X,Y$ then $N(A) := \{ B \mid B \in \pi \text{ and } B \text{ nest inside } A \setminus \pi \}$, where the notation $A \setminus \pi$ means that we remove those elements of $A$ which are included in the pivot block $\pi$, see Figure 2.

$$
\begin{align*}
\lambda_{\pi} & : \begin{array}{|c|c|c|}
\hline 
\pi & \pi & \pi \\
\hline 
\pi & \pi & \pi \\
\hline
\end{array} \quad \quad \lambda_{\pi} & : \begin{array}{|c|c|c|}
\hline 
\pi & \pi & \pi \\
\hline 
\pi & \pi & \pi \\
\hline
\end{array} \\
N(\lambda_{\pi}) &= \{(2,3),(6,7)\} & N(\lambda_{\pi}) &= \{(4,5)\}\\
N(\lambda_{\pi}) &= \{(12,13)\} & N(\lambda_{\pi}) &= \{(8,9),(12,13)\}.
\end{align*}
$$

Figure 2. Examples of pivot blocks $\lambda_{\pi}$, $\lambda_{\pi}$, pivot braces $\pi$ and essentially nested blocks.

For scalar $a, b, c \in \mathbb{C}$ we denote by $[\begin{smallmatrix} c & a \\ b & c \end{smallmatrix}]_n$ the element in $M_n(\mathbb{C})$, where the diagonal elements are equal to $c$ and the upper-triangular entries are equal to $a$ and lower-triangular elements are $b$.

3. An involution on $C_{2n}$

We illustrate the idea of the proof of the cancellation phenomenon on the simplest example which is the commutator $XY - YX$. We expand the cumulant $K_r(XY - YX)$ multilinearly, apply the product formula (2.13) and obtain a sum

$$
(3.1) \quad \sum_{\pi \in NC(2r)} K_\pi(X_1, X_2, \ldots, X_n)
$$

where for $X_1, X_2, \ldots, X_n \in \{X, Y\}$; our involution will then provide a matching of equal terms with opposite signs roughly by shifting the endpoint of leftmost inner odd block according to the pattern shown in Figure 9.

However the definition of the involution is not as straightforward as it seems at a first glance. The complication arises from the fact that for certain partitions the leftmost inner odd block loses its property of being leftmost after the shift, see example (3.5) below. For this reason these partitions must be treated differently, by “flipping” certain intervals. Therefore we will call them flip partitions and their description is the content of the next subsection.

3.1. Flip partitions.

Lemma 3.1. Let $\pi \in C_{2n}$. If $\pi$ has two inner odd blocks which are connected by a standard brace then all other blocks of $\pi$ are even.
Proof. If two inner odd blocks $B$ and $B'$ are connected, they must lie adjacent to each other. Let $B$ be to the left of $B'$, then $\omega(B)$ is odd and $\alpha(B')$ is even, see Figure 3. But then the interval $J = I(B) \cup I(B')$ is even and the blocks of $1^2$ which are contained in $J$ are not connected to those contained in the complement of $J$. Since $\pi \in \mathcal{C}_{2n}^o$, it follows that $J = [2n]$ and since no odd block is nested neither inside $B$ nor inside $B'$, $\pi$ has no other odd blocks. \hfill \Box

Figure 3. Two adjacent inner odd blocks.

Definition 3.2. An element $\pi \in \mathcal{C}_{2n}^o$ is called a flip partition if it has exactly two odd blocks and the pivot brace connects these two blocks at their endpoints.

Examples of flip partitions are shown in Figure 4.

Type I

Type IIa

Type IIb

Figure 4. Examples of flip partitions

The next lemma provides us with a classification of flip partitions which will be essential for the definition of the involution.

Lemma 3.3. Let $\pi \in \mathcal{C}_{2n}^o$ be a flip partition. Then either

I. $\pi$ has two inner odd blocks;

II. $\pi$ has exactly one inner odd block and the pivot brace is either

(a) $\cap_\pi = (1,2)$

(b) $\cap_\pi = (2n-1,2n)$.

We refer to flip partitions of type I, IIa and IIb according to this scheme, see Figure 4.

Proof. Suppose that $\pi$ is a flip partition and condition (I) is not satisfied. Then $\pi$ has one inner odd block $B$ and one outer odd block $B'$. The padding interval $I(B')$ of the outer odd block is the union of $B \cup B'$ and some even blocks and therefore has an even number of elements. It follows that $\alpha(B')$ and $\omega(B')$ have different parity. Now by assumption $B$ and $B'$ are connected by a standard brace at their endpoints; either they are connected at their left endpoints, or at their right endpoints and in either case we conclude that $\alpha(B')$ is odd and $\omega(B')$ is even. This implies that all standard braces outside $I(B')$ are separated from the rest, and since $\pi \in \mathcal{C}_{2n}^o$ it follows that $I(B')$ is the full interval $[2n]$, i.e., we have indeed type IIa or type IIb. \hfill \Box

As a corollary we obtain the following decomposition of flip partitions which plays a major role in the involution to be defined below.

Corollary 3.4. Any flip partition $\pi$ can be decomposed as a disjoint union

$$\pi = N(\prec_\pi) \cup N(\succeq_\pi) \cup \{\prec_\pi\} \cup \{\succeq_\pi\},$$

The remaining odd partitions make up the last type.

Definition 3.5. A partition $\pi \in \mathcal{C}_{2n}^o$ which is not a flip partition is called type III. More specifically, it is type IIIa if the smallest element of the leftmost inner odd block is even, i.e., the left end point is the pivot element. It is type IIIb if the smallest element of the leftmost inner odd block is odd, i.e., the right end point is the pivot element.
3.2. Definition of the involution. We have now everything in place to define a sign-inverting involution on $\mathcal{C}_n^2$, which simultaneously switches the sign of the corresponding term in the expansion (3.1). The involution acts on each type separately and follows the patterns laid out in Figures 5, 7 and 9.

Types I and II are flip partitions and the two odd blocks are flipped in such a way that the decomposition (I) and (II) of Lemma 3.3, respectively, is preserved. In type III the pivot element is moved from one end of the leftmost inner odd block to the other. The braces are preserved except on the pivot brace, which is reversed. Hereby the types are preserved, more precisely:

(I) A partition of type I is mapped to type I, see Figures 5.

\[ \begin{align*}
\text{Type I} & \quad \text{Type I} \\
\begin{array}{c}
\pm XY \\
\end{array} & \quad \begin{array}{c}
\mp YX \\
\end{array}
\end{align*} \]

**Figure 5.** Involution of partitions of type I.

The length of the padding interval of the leftmost inner odd block (marked red in the diagram) is an odd number, say $2k + 1$. Then the length of the padding interval of the other odd block is $2k' + 1$, where $k' = r - k - 1$. Then the intervals $[1, 2, \ldots, 2k]$ and $[2k + 3, 2k + 4, \ldots, 2r]$ are flipped and the points $2k + 1$ and $2k + 2$ are exchanged; more precisely, the entries are mapped according to the action of the following permutation:

\[
\sigma_{I,r,k} : i \mapsto \begin{cases} 
 i + 2k' + 2 & \text{for } 1 \leq i \leq 2k \\
 2k' + 2 & \text{for } i = 2k + 1 \\
 2k' + 1 & \text{for } i = 2k + 2 \\
i - 2k - 2 & \text{for } 2k + 3 \leq i \leq 2r
\end{cases}
\]

see Figure 6 for a specific example. It is easy to see that $\sigma_{I,r,k'}$ is the inverse of $\sigma_{I,r,k}$.

\[ \begin{align*}
\text{Type I} & \quad \text{Type I} \\
\begin{array}{c}
\pm \\
\end{array} & \quad \begin{array}{c}
\mp \\
\end{array}
\end{align*} \]

**Figure 6.** An example of the involution of partitions of type I.

(II) Type IIa is mapped to type IIb and vice versa, see Figure 7.

\[ \begin{align*}
\text{Type IIa} & \quad \text{Type IIb} \\
\begin{array}{c}
\pm XY \\
\end{array} & \quad \begin{array}{c}
\mp YX \\
\end{array}
\end{align*} \]

**Figure 7.** Involution of partitions of type II.

The length of the padding interval of the leftmost inner odd block (marked red in the diagram) is an odd number, say $2k + 1$ and let $k' = r - k - 1$.

In the case of type IIa the interval $[3, 4, \ldots, 2k]$ is flipped with the interval $[2k + 1, 2k + 2, \ldots, 2r]$ and the pair $(1, 2)$ is mapped to the pair $(2r, 2r - 1)$ (notice the change of order); more precisely, the entries are mapped according to the action of the following permutation:
permutation:

\[
\sigma_{I_{IIa,r,k}} : i \mapsto \begin{cases} 
2r & \text{for } i = 1 \\
2r - 1 & \text{for } i = 2 \\
i + 2k' & \text{for } 3 \leq i \leq 2k \\
i - 2k - 2 & \text{for } 2k + 3 \leq i \leq 2r
\end{cases}
\]

see Figure 8 for a specific example.

In the case of type IIb we reverse the above process. Now the interval \([1, 2, \ldots, 2k]\) is flipped with the interval \([2k + 1, 2k + 2, \ldots, 2r - 2]\) and the pair \((2r, 2r - 1)\) is mapped to the pair \((1, 2)\); more precisely, the entries are mapped according to the action of the following permutation:

\[
\sigma_{I_{IIb,r,k}} : i \mapsto \begin{cases} 
i + 2k + 2 & \text{for } 1 \leq i \leq 2k \\
i - 2k + 2 & \text{for } 2k + 1 \leq i \leq 2r - 2 \\
2 & \text{for } i = 2r - 1 \\
1 & \text{for } i = 2r
\end{cases}
\]

(III) type IIIa is mapped to type IIIb and vice versa, see Figure 9.

In the case of type III we apply a rotation to the padding interval of the leftmost inner odd block augmented by missing element from the pivot brace. More precisely, if in type IIIa the leftmost inner odd block starts at \(2k\) and ends at \(2l\) then the permutation is the square of the cycle spanned by its padding interval together with the pivot brace:

\[
\sigma_{I_{IIIa,r,k,l}} = (2l, 2l - 1, \ldots, 2k - 1)^2 = (2l, 2l - 2, \ldots, 2k) \circ (2l - 1, 2l - 3, \ldots, 2k - 1)
\]

conversely, if in type IIIb the leftmost inner odd block starts at \(2k + 1\) and ends at \(2l - 1\) then the permutation is the square of the corresponding cycle

\[
\sigma_{I_{IIIb,r,k,l}} = (2k + 1, 2k + 2, \ldots, 2l)^2 = (2k, 2k + 2, \ldots, 2l) \circ (2k + 1, 2k + 3, \ldots, 2l - 1)
\]

see Figure 10 for specific examples.

**Proposition 3.6.** (i) The previously constructed permutations are inverse to each other, more precisely:

\[
\sigma_{I,r,k} = \sigma_{I,r,-k-1}^{-1} \\
\sigma_{I_{IIa,r,k}} = \sigma_{I_{IIb,r,-k-1}}^{-1} \\
\sigma_{I_{IIIa,k,l}} = \sigma_{I_{IIIb,l,k}}^{-1}
\]
(ii) For a flip partition \( \pi \in C_{2r}^o \), let us denote by \( \sigma_\pi \) the permutation constructed above. Then the map

\[
\psi : C_{2r}^o \rightarrow C_{2r}^o \\
\pi \mapsto \sigma_\pi \cdot \pi
\]

is an involution.

Proof. Part (i) is immediate.

To see part (ii) we first observe that each type is mapped onto itself. In type I the map is obviously involutive; as for type II and III, the role of the innermost odd block is left invariant and thus we have indeed an involution. \( \square \)

Remark 3.7. (1) In type I and II we flip blocks and the pivot brace \( \pi \) is reversed, which will imply a change of sign and thus a cancellation in the formulas below; the odd blocks are flipped appropriately and the remaining blocks in this decomposition are shifted but the tracial structure is unchanged.

In type III we shift and flip the pivot brace \( \pi \) only; the remaining blocks stay in place. During this procedure we also rotate appropriately the two points of the blocks \( \wedge_\pi \) and \( \vee_\pi \) which are joined by \( \cap_\pi \). Otherwise the structure of this block is not changed.

(2) If \( \sigma : \pi \rightarrow \pi' \), then \( \pi \) and \( \pi' \) have the same block structure i.e., for every \( 1 \leq m \leq n \), the two partitions \( \pi \) and \( \pi' \) have the same number of blocks with \( m \) elements. Indeed observe that our condition just say that we remove one point and add one point or shift corresponding blocks. During this procedure we can not change framework of block.

(3) The following example shows that we cannot apply the rules of type III universally without losing the involutive property. Namely, applying the rule of type III to the following type II partition twice we obtain:

\[
\begin{align*}
\{(1,2,6),(3,4,5)\} & \quad \rightarrow \quad \{(1,2,3),(4,5,6)\} \\
\{(1,2,3),(4,5,6)\} & \quad \rightarrow \quad \{(2,3,4),(1,5,6)\}
\end{align*}
\]

4. Preservation of infinite divisibility and cancellation of odd cumulants

4.1. The main result. In our previous paper we observed that the phenomenon of cancellation of odd cumulants to be defined below is related to preservation of free infinite divisibility.

Theorem 4.1 ([10, Corollary 4.14]). Let \( X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa} \) be a free family of \( \boxplus \)-infinitely divisible random variables. Let \( P \) be a selfadjoint polynomial of degree 2 in noncommuting variables which exhibits cancellation of odd cumulants. Then the distribution of \( Y \) is \( \boxplus \)-infinitely divisible as well.

In [10, Conjecture 5.2] we also conjectured that the cancellation phenomenon is actually equivalent to preservation of infinite divisibility. In this section we confirm this conjecture for quadratic forms and it turns out that the equivalence can be extended to several other properties.
Definition 4.2. Let \( Q_n = \sum_{i,j=1}^{n} a_{i,j} X_i X_j \) be a quadratic form in noncommuting variables \( X_1, X_2, \ldots, X_n \) with system matrix \( A = [a_{i,j}]_{i,j=1}^{n} \in M_n(\mathbb{C}) \), which is assumed to be selfadjoint. When the formal variables \( X_i \) are replaced by (noncommutative) selfadjoint random variables, then \( Q_n \) becomes a selfadjoint random variable as well. With the aid of the product formula (2.13) a universal formula for its free cumulants can be computed, involving the coefficients \( a_{i,j} \) and the joint cumulants of the random variables \( X_i \).

(i) We say that \( Q_n \) exhibits \textit{free cancellation of odd cumulants} if for any free family of selfadjoint noncommutative random variables \( X_1, X_2, \ldots, X_n \) the odd cumulants of the \( X_i \) do not contribute to the universal formula for the cumulants of \( Q_n \).

(ii) We say that \( Q_n \) exhibits \textit{strong cancellation of odd cumulants} if for any family of selfadjoint noncommutative random variables \( X_1, X_2, \ldots, X_n \) in a tracial noncommutative probability space the odd joint cumulants of the \( X_i \) do not contribute to the universal formula for the cumulants of \( Q_n \).

(iii) We say that \( Q_n \) \textit{preserves infinite divisibility} if for any free family of selfadjoint noncommutative random variables \( X_1, X_2, \ldots, X_n \) with freely infinitely divisible laws, the law of \( Q_n \) is also freely infinitely divisible.

Remark 4.3. 1. We would like to emphasize that freeness is not assumed in condition (ii), which asserts that in addition to the univariate odd cumulants also the mixed odd cumulants cancel.

2. Free and strong cancellation of odd cumulants are not equivalent for higher order polynomials. For example, it is immediate that the iterated commutator \( [[X_1, X_2], X_3] \) exhibits free cancellation of odd cumulants, but if \( X_1 \) and \( X_2 \) are identically distributed with free cumulants \( r_1, r_2, r_3, \ldots \) and \( X_3 \) is replaced by \( X_1 \), a calculation shows that the third cumulant

\[
K_3([[X_1, X_2], X_1]) = -6r_2r_3r_4 + 6r_3^2 - 6r_2^3
\]

depends on the third cumulant \( r_3 \).

3. Concerning item (i) we remark that any free family of selfadjoint variables can be realized in a tracial probability space [24, Proposition 2.5.3.] and therefore the traciality condition is implicitly satisfied.

We call a selfadjoint matrix \( A \in M_n(\mathbb{C}) \) \textit{skew symmetric} if \( A = -A^T \); that is, \( A = i\tilde{A} \), where \( \tilde{A} \) is a real skew symmetric matrix (this is in order to distinguish between skew-Hermitian matrix).

Theorem 4.4. The following properties are equivalent for a quadratic form \( T_n = \sum_{i,j=1}^{n} a_{i,j} X_i X_j \) with selfadjoint system matrix \( A = [a_{i,j}]_{i,j=1}^{n} \in M_n(\mathbb{C}) \).

(i) \( T_n \) exhibits strong cancellation of odd cumulants.

(ii) \( T_n \) exhibits cancellation of odd cumulants.

(iii) \( T_n \) preserves free infinite divisibility.

(iv) \( A \) is skew symmetric or equivalently, \( T_n = \sum_{k<l} a_{k,l}(X_kX_l - X_lX_k) \) is a sum of commutators.

(v) The distribution of \( T_n \) is symmetric for any free family of selfadjoint random variables \( X_1, X_2, \ldots, X_n \).

The crucial steps are the implications (ii)\(\implies\) (iii) , (iv)\(\implies\) (i) and (ii)+(iv)\(\implies\) (v). The former is the content of Theorem 4.1 and for the latter we will apply the involution of the previous section in combination with the following lemma.

Lemma 4.5. (i) Let \( \pi \) be a partition of type I, II or III and \( \sigma \) the corresponding permutation constructed in section 3.2. Then for elements \( X_1, X_2, \ldots, X_n \) of a tracial probability space the cumulant is invariant:

\[
K_\pi(X_{i_1}, X_{i_2}, \ldots, X_{i_2r}) = K_{\sigma \cdot \pi}(X_{i_{\sigma(1)}}, X_{i_{\sigma(2)}}, \ldots, X_{i_{\sigma(2r)}})
\]

(ii) Let \( A \) be a skew-symmetric matrix and \( \sigma \) a permutation of type I, II or III as above. Then

\[
a_{i_{\sigma(1)},i_{\sigma(2)}}a_{i_{\sigma(3)},i_{\sigma(4)}}\cdots a_{i_{\sigma(2r-1)},i_{\sigma(2r)}} = -a_{i_1,i_2}a_{i_3,i_4}\cdots a_{i_{2r-1},i_{2r}}\]
Proof. Both claims are easily verified for each type separately by inspecting the diagrams in Figures 5, 7 and 9. On the one hand, the permutations act tracially on the blocks of $\pi$ and on the other hand, braces are preserved and keep their order with the unique exception of the pivot brace which is reversed. \hfill \square

Proof of Theorem 4.4. We will first prove the equivalence of (i)–(iv); then we show that (ii) together with (iv) implies (v) and finally that (v) implies (iv).

(i)$$\implies$$(ii) is obvious.

(ii)$$\implies$$(iii) follows directly from Theorem 4.1.

(iii)$$\implies$$(iv). Fix $i \in \{1, 2, \ldots, n\}$, let $X_i$ be a semicircular element with mean and variance 1 and put $X_j = 0$ for $j \neq i$. Then $T_n = a_{i,i}X_i^2$ but $X_i^2$ is not infinitely divisible (see [9]), and we conclude that $a_{i,i} = 0$.

Thus we have shown that the diagonal entries vanish. To cope with the off-diagonal entries, it is sufficient to prove that $a_{i,1} = -a_{2,1}$, the proof for the remaining entries being analogous.

Let now $X_1$ and $X_2$ have semicircular distribution with mean and variance one and $a_{1,2} = \alpha + i\beta$.

Then a computer calculation (using FriCas [1], code available on request) shows that the free cumulants of the quadratic form $T_2 = a_{1,2}X_1X_2 + a_{2,1}X_2X_1$ are

\begin{align*}
K_1(T_2) &= 2\alpha \\
K_2(T_2) &= 2\beta^2 + 10\alpha^2 \\
K_3(T_2) &= 24\alpha^3 \\
K_4(T_2) &= 2\beta^4 + 4\alpha^2\beta^2 + 66\alpha^4 \\
K_5(T_2) &= 160\alpha^5 \\
K_6(T_2) &= 2\beta^6 + 6\alpha^2\beta^4 + 6\alpha^4\beta^2 + 386\alpha^6 \\
K_7(T_2) &= 896\alpha^7 \\
K_8(T_2) &= 2\beta^8 + 8\alpha^2\beta^6 + 12\alpha^4\beta^4 + 8\alpha^6\beta^2 + 2050\alpha^8.
\end{align*}

To show that this law is not infinitely divisible, it suffices to disprove conditional positive definiteness of the cumulant sequence (2.6). To this end we compute a few Hankel determinants

$$(4.1) h_n = \det [K_{i+j}(T_2)]_{1 \leq i, j \leq n}$$

and obtain

\begin{align*}
h_2 &= 4(\beta^6 + 7\alpha^2\beta^4 + 43\alpha^4\beta^2 + 21\alpha^6) \\
h_3 &= 32\alpha^2(\beta^2 + \alpha^2)(\beta^8 - 12\alpha^2\beta^6 + 2\alpha^4\beta^4 - 52\alpha^6\beta^2 - 131\alpha^8) \\
h_4 &= -256\alpha^6(\beta^2 - 3\alpha^2)^4(\beta^2 + \alpha^2)^3
\end{align*}

Thus the fourth determinant is negative unless $\beta = \pm \alpha\sqrt{3}$; in that case however $h_3 = -65536\alpha^{12}$ and we conclude that $\text{Re} a_{1,2} = \text{Re} a_{2,1} = \alpha = 0$ and consequently $\text{Im} a_{1,2} = \text{Im} a_{2,1} = \beta = 0$ as well.

(iv)$$\implies$$(i). Suppose that $A$ is skew-symmetric. We apply again the product formula from Theorem 2.2 and obtain

\begin{align*}
K_r(T_n) &= \sum_{i_1, i_2, \ldots, i_r \in [n]} K_r(a_{i_1, i_2}X_{i_1}X_{i_2}, a_{i_3, i_4}X_{i_3}X_{i_4}, \ldots, a_{i_{2r-1}, i_{2r}}X_{i_{2r-1}}X_{i_{2r}}) \\
&= \sum_{i_1, i_2, \ldots, i_r \in [n]} \sum_{\pi \in NC(2r)} a_{i_1, i_2}a_{i_3, i_4} \cdot \cdots \cdot a_{i_{2r-1}, i_{2r}}K_\pi(X_{i_1}, X_{i_2}, \ldots, X_{i_{2r}}) \\
&= \sum_{i_1, i_2, \ldots, i_r \in [n]} \sum_{\pi \in NC(2r)} a_{i_1, i_2}a_{i_3, i_4} \cdot \cdots \cdot a_{i_{2r-1}, i_{2r}}K_\pi(X_{i_1}, X_{i_2}, \ldots, X_{i_{2r}}) \\
&= \sum_{\pi \in \mathbb{C}_{2r}^*_1} \sum_{i_1, i_2, \ldots, i_r \in [n]} a_{i_1, i_2}a_{i_3, i_4} \cdot \cdots \cdot a_{i_{2r-1}, i_{2r}}K_\pi(X_{i_1}, X_{i_2}, \ldots, X_{i_{2r}}) \\
&+ \sum_{\pi \in \mathbb{C}_{2r}^*_1} \sum_{i_1, i_2, \ldots, i_r \in [n]} a_{i_1, i_2}a_{i_3, i_4} \cdot \cdots \cdot a_{i_{2r-1}, i_{2r}}K_\pi(X_{i_1}, X_{i_2}, \ldots, X_{i_{2r}}).
\end{align*}
We claim that in this decomposition the second sum cancels. To see this, we observe that the involution constructed in section 3.2 extends to an involution

$$[n]^{2r} \times C_{2r}^0 \rightarrow [n]^{2r} \times C_{2r}^0$$

$$(i_1, i_2, \ldots, i_{2r}; \pi) \mapsto (i_{\sigma_\pi(1)}, i_{\sigma_\pi(2)}, \ldots, i_{\sigma_\pi(2r)}; \sigma_\pi \cdot \pi)$$

where $\sigma_\pi$ is the permutation constructed according to the type I/II/III of $\pi$. Therefore we have

$$\sum_{\pi \in C_{2r}^0} \sum_{i_1, i_2, \ldots, i_{2r} \in [n]} a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{2r-1}, i_{2r}} K_\pi(X_{i_1}, X_{i_2}, \ldots, X_{i_{2r}})$$

$$= \sum_{\pi \in C_{2r}^0} \sum_{i_1, i_2, \ldots, i_{2r} \in [n]} a_{i_{\sigma_\pi(1)}, i_{\sigma_\pi(2)}, \ldots, i_{\sigma_\pi(2r)}; \pi} K_{\pi} X_{i_{\sigma_\pi(1)}}, X_{i_{\sigma_\pi(2)}}, \ldots, X_{i_{\sigma_\pi(2r)}}$$

Now by Lemma 4.5 the effect on a term is

$$a_{i_{\sigma_\pi(1)}, i_{\sigma_\pi(2)}, \ldots, i_{\sigma_\pi(2r)}} K_{\pi} X_{i_{\sigma_\pi(1)}}, X_{i_{\sigma_\pi(2)}}, \ldots, X_{i_{\sigma_\pi(2r)}}$$

and therefore the sum vanishes.

This concludes the first circle of implications; let us now turn to the remaining ones.

(ii)+(iv)$$\Rightarrow$$ (v). We expand the product formula (2.13) and obtain

$$K_r(T_n) = \sum_{i_1, i_2, \ldots, i_{2r} \in [n]} a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{2r-1}, i_{2r}} K_r(X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_{2r-1}}, X_{i_{2r}})$$

$$= \sum_{i_1, i_2, \ldots, i_{2r} \in [n]} \sum_{\pi \in NC(2r)} a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{2r-1}, i_{2r}} K_\pi(X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_{2r-1}}, X_{i_{2r}})$$

now by assumption (ii) we may omit all odd cumulants from this formula, i.e., we can restrict the sum to even partitions

$$= \sum_{i_1, i_2, \ldots, i_{2r} \in [n]} \sum_{\pi \in NC(2r)} a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{2r-1}, i_{2r}} K_\pi(X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_{2r-1}}, X_{i_{2r}})$$

and by Lemma 2.4 these partitions have a special shape

$$= \sum_{i_1, i_2, \ldots, i_{2r} \in [n]} \sum_{\pi \in NC(2r)} a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{2r-1}, i_{2r}} K_\pi(X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_{2r-1}}, X_{i_{2r}})$$

and by freeness we can impose the condition $i_{2k} = i_{2k+1}$ on the indices

$$= \sum_{i_1, i_2, \ldots, i_{2r} \in [n]} \sum_{\pi \in NC(2r)} a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{2r-1}, i_{2r}} K_\pi(X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_{2r-1}}, X_{i_{2r}})$$

next we apply the mirror permutation $i_j \rightarrow i_{r+1-j}$ to the indices; this fixes $i_1$ and mirrors the remaining ones

$$= \sum_{i_1, i_2, \ldots, i_{2r} \in [n]} \sum_{\pi \in NC(2r)} a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{2r-1}, i_{2r}} K_\pi(X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_{2r-1}}, X_{i_{2r}})$$

$$= \sum_{i_1, i_2, \ldots, i_{2r} \in [n]} (-1)^r a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{2r-1}, i_{2r}} K_\pi(X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_{2r-1}}, X_{i_{2r}})$$
where we used assumption (iv) that the matrix $A$ is skew-symmetric. Now we apply Lemma 2.1 (note that the random variables $X_i$ are free and self-adjoint and therefore the cumulants are real valued) and obtain

$$= (-1)^r \sum_{i_1, i_2, \ldots, i_r \in [n]} a_{i_1, i_2} a_{i_3, i_4} \cdots a_{i_{r-1}, i_r} K_r(X_{i_1} X_{i_2} X_{i_3} \cdots, X_i, X_i)$$

which implies $K_r(T_n) = 0$ for odd $r$.

(v)$\Rightarrow$(iv) is easily verified. Fix $i$, put $X_i = I$ and $X_j = 0$ for $j \neq i$. Then $T_n = a_{i,i}^2 I$ and property (v) implies that $a_{i,i} = 0$. To cope with the off-diagonal terms, put $X_1 = X_2 = I$, then $T = a_{1,2} X_1 X_2 + a_{2,1} X_2 X_1 = 2 \text{Re} a_{1,2} I$ is odd and therefore $\text{Re} a_{1,2} = 0$. This implies that $A = -A^\top$.

\[\Box\]

**Remark 4.6.** The traces of the odd powers of a skew-symmetric matrix are zero. This fact can be generalized as follows. A selfadjoint matrix $A \in M_n(\mathbb{C})$ is skew-symmetric if and only if for every $\pi \in \text{NC}(r)$ where $r$ is odd we have $\text{Tr}(E_{\pi}^2 (A)) = 0$.

4.2. **Distributions of quadratic forms.** In [21, Theorem 1.2] the authors provide an analytic description of the $R$-transform of free commutators in terms of the combinatorial convolution \[\boxplus\] of the even cumulant transforms. To be specific, if $X$ is free from $Y$, then

$$C_i(XY - YX)(z) = 2(C_X^{(\text{even})} \boxplus C_Y^{(\text{even})} \boxplus \zeta)(z^2).$$

where by $C_Z^{(\text{even})}(z) := \sum_{n=1}^{\infty} K_{2n}(Z)z^n$ we denote the generating function of the even free cumulants of $Z \in \mathcal{A}_a$. An important ingredient in proof of the preceding results is the notion of $R$-diagonality. An $R$-diagonal pair is a pair of random variables $A$ and $B$ such that all cumulants vanish with the exception of the alternating ones, i.e., those of the form $K_{2r}(A, B, A, B, \ldots, A, B)$ and $K_{2r}(B, A, B, A, \ldots, B, A)$. It turns out that for free even elements $X$ and $Y$ the products $XY$ and $YX$ form an $R$-diagonal pair and therefore the moments of the commutator $i(XY - YX)$ are computable.

This observation however is specific to the commutator. The result below gives an alternative combinatorial description and holds for arbitrary quadratic forms in even elements. For this purpose we define for $A \in M_n(\mathbb{C})$ the generating function

$$f_A(z_1, \ldots, z_n) := \sum_{r=1}^{\infty} \sum_{i_1, \ldots, i_r = 1}^{n} \text{Tr}(AE_{i_1} AE_{i_2} \ldots AE_{i_r})z_{i_1} \ldots z_{i_r}.$$

and recall that the Hadamard product of two formal power series $f(z) = \sum a_n z^n$ and $g(z) = \sum b_n z^n$ is defined as

$$f(z) \odot g(z) = \sum a_n b_n z^n.$$

**Theorem 4.7.** Let $X_1, X_2, \ldots, X_n \in \mathcal{A}_a$ be a free family of even random variables. Then for any selfadjoint matrix $A = [a_{i,j}]_{i,j=1}^{n} \in M_n(\mathbb{C})$ the $C$-transform of the quadratic form $T_n = \sum_{i,j} a_{i,j} X_i X_j$ can be represented as the Hadamard product

$$C_{T_n}(z) = \left[ f_A \odot \left( (C_{X_1}^{(\text{even})}) + \cdots + C_{X_n}^{(\text{even})} \boxplus \zeta_n \right) \right](z, \ldots, z).$$

**Proof.** Using Proposition 2.5 (i), we have

$$K_r(T_n) = \sum_{i_1, \ldots, i_r \in [n]} \text{Tr}(AE_{i_1} AE_{i_2} \ldots AE_{i_r}) \sum_{\pi \in \text{NCE}(2r)} K_{\pi}(X_{i_r}, X_{i_1}, X_{i_2}, \ldots, X_{i_{r-1}}, X_{i_r}),$$
and by traciality of cumulants this is
\[
= \sum_{i_1, \ldots, i_r \in [n]} \text{Tr}(AE_{i_1}AE_{i_2} \cdots AE_{i_r}) \sum_{\pi \in \mathcal{NCE}(2r)} K_{\pi}(X_{i_1}, X_{i_2}, X_{i_3}, \ldots, X_{i_r}, X_{i_r}, X_{i_r}).
\]

Now let $\tilde{X}_i$ be the formal random variable obtained from $X_i$ by skipping all odd cumulants, i.e., $K_n(\tilde{X}_i) = K_{2n}(X_i)$, then we can use the isomorphism from Lemma 2.4 and continue
\[
= \sum_{i_1, \ldots, i_r \in [n]} \text{Tr}(AE_{i_1}AE_{i_2} \cdots AE_{i_r}) \tau(\tilde{X}_{i_1}\tilde{X}_{i_2} \cdots \tilde{X}_{i_r}).
\]

Finally by [22, Proposition 17.4], we can write this as
\[
\sum_{i_1, \ldots, i_r \in [n]} \text{Tr}(AE_{i_1}AE_{i_2} \cdots AE_{i_r}) \text{Cf}_{(i_1, \ldots, i_r)} \left( (C^{(\text{even})}_{X_{i_1}} + \cdots + C^{(\text{even})}_{X_{i_r}}) \right) \zeta_n
\]
which finishes the proof. \(\square\)

4.3. Preservation of free infinite divisibility for higher order polynomials. There are many higher order polynomials which cancel odd cumulants and preserve infinite divisibility, take for example higher free commutators like $[[X,Y], Z]$ or $[[X,Y], [A,B], Z]$. Similarly, take a skew-symmetric quadratic form $T_n = \sum_{i,j=1}^{n} a_{i,j}X_iX_j$ as in Theorem 4.4, then $T_n$ is symmetric and we conclude from [2, Theorem 2.2] that $T_n$ preserves free infinite divisibility.

The purpose of this subsection is to produce higher order polynomials which preserve free infinite divisibility but don’t exhibit the cancellation phenomenon; we do not know however whether the reverse implication is true.

For the concept of free regular distributions we refer to [2].

**Proposition 4.8.** Let $X, Y, Z$ be free random variables such that $X$ and $Y$ are freely infinitely divisible. Then the selfadjoint element $[X,Y]Z[X,Y]$ has compound free Poisson distribution of rate 1 with jump distribution $\mu_Z \boxtimes \sigma$ for some free regular distribution $\sigma$. Consequently it is freely infinitely divisible and the odd cumulants of $Z$ do not cancel.

**Proof.** First recall that [2, Theorem 2.2] asserts that if a random variable has even FID law $\mu$ then the law of its square can be decomposed $\mu^2 = m \boxtimes \sigma$ where $m$ is the free Poisson law of rate 1 and $\sigma$ is free regular.

Next recall that for any law $\nu$ the law $\nu \boxtimes m$ is the law of the free compression with a semicircular random variable and therefore free Poisson with jump distribution $\nu$.

Now let $\mu$ be the law of $i[X,Y]$. It follows from [21, Theorem 1.2] and [2, Corollary 6.5] that $\mu$ is both even and freely infinitely divisible. The law of $[X,Y]Z[X,Y]$ is $\mu_Z \boxtimes \mu^2 = \mu_Z \boxtimes \sigma \boxtimes m$ and therefore our random variable has compound free Poisson distribution with rate 1 and jump distribution $\mu_Z \boxtimes \sigma$. \(\square\)

4.4. The generalized tetilla law. Formulas (2.16) and (2.17) are hard to evaluate in general. There are however two settings for which the distribution can be computed explicitly. The first result is a kind of central limit theorem for sums of commutators which gives rise to the free tangent law and which will appear in a separate paper [11].

The second result is presented in this section and concerns sums of commutators of semicircular elements for which the sum (2.16) simplifies considerably.

Motivated by [8] we propose the following definition.

**Definition 4.9.** Let $X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa}$ with $n \geq 2$ be a free family of semicircular random variables of variance one. The law of the random variable $\sum_{k,j=1}^{n} i(X_kX_j - X_jX_k)$ is called the **generalized tetilla law** with $n$ degrees of freedom. We denote this distribution by $\mathcal{T}_n$. 

The tangent numbers

\begin{equation}
T_k = (-1)^{k+1} \frac{4^k (4^k - 1) B_{2k}}{2k}
\end{equation}

for \( k \in \mathbb{N} \) are the Taylor coefficients of the tangent function

\[ \tan z = \sum_{n=1}^{\infty} T_n \frac{z^n}{n!} = z + \frac{2}{3!} z^3 + \frac{16}{5!} z^5 + \frac{272}{7!} z^7 + \cdots , \]

see [13, Page 287]).

On the other hand let us denote by \( A_{n}^{(k)} \) the arctangent numbers (see [6, p. 260] or [7]) defined by their exponential generating function

\begin{equation}
\frac{(\arctan z)^k}{k!} = \sum_{n=k}^{\infty} \frac{A_{n}^{(k)}}{n!} z^n ;
\end{equation}

**Proposition 4.10.** The generalized tetilla law with \( n \) degrees of freedom has the following properties.

(i) It is \( \boxplus \)-infinitely divisible with discrete Lévy measure \( \nu = \delta_{\cot \frac{\pi}{2n}} + \cdots + \delta_{\cot \frac{\pi n}{2n}} \);

(ii) It is symmetric and its even cumulants are

\begin{equation}
K_{2m} = (-1)^m n + \frac{1}{(2m-1)!} \sum_{k=1}^{m} n^{2k} A_{2m}^{(2k)} T_{2k-1}
\end{equation}

where by \( T_k \) and \( A_{n}^{(k)} \) we denote the tangent and arctangent numbers, respectively.

(iii) The \( R \)-transform is equal to

\[ R_{T_n}(z) = \frac{n \tan(n \arctan z) - nz}{1 + z^2} \]

**Proof.** Symmetry and \( \boxplus \)-infinite divisibility follow directly from Theorem 4.4. Since \( X_i \) are random variables of variance one, we can compute the cumulants by using formula (2.18) and evaluate to

\[ K_r \left( \sum_{k,j=1}^{n} (X_k X_j - X_j X_k) \right) = \text{Tr}(A_n^r) \] where \( A_n = \left[ \begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right] \).

The eigenvalues and characteristic polynomial of the matrix \( A_n \) were computed in [12, Lemma 2.1, \( \alpha = \pi/2 \)] and they are \( \lambda_k = \cot \left( \frac{-i n}{2n} + \frac{k n}{n} \right) \) for \( k \in \{0, \ldots, n-1\} \) (including repeated eigenvalues), and

\begin{equation}
\chi_n(\lambda) = \frac{(\lambda - i)^n + (\lambda + i)^n}{2}.
\end{equation}

Hence the odd cumulants vanish and the even cumulants are equal to the cotangent sums

\[ K_{2m} \left( \sum_{k,j=1}^{n} i(X_k X_j - X_j X_k) \right) = \sum_{k=0}^{n-1} \cot^{2m} \left( \frac{\pi}{2n} + \frac{k n}{2n} \right) \]

which were evaluated explicitly in [12, Corollary 6.4] and the result is (4.4).
Once having realized \( \lambda_k \) as roots of a polynomial, it is easy to write down the generating function as a logarithmic derivative. Indeed, let

\[
g_n(z) = \sum_{k=0}^{n-1} \frac{1}{z - \lambda_k}
= \frac{\lambda'_n(z)}{\lambda_n(z)}
= n \frac{(z - i)^{n-1} + (z + i)^{n-1}}{(z - i)^n + (z + i)^n}
\]

then the ordinary generating function is

\[
F_n(z) = \sum_{m=0}^{\infty} \sum_{k=0}^{n-1} \cot^m \frac{\pi}{2k\pi} \frac{1}{n} z^m
= \frac{1}{z g_n \left( \frac{1}{z} \right)}
= \frac{n}{1 + z^2} \frac{(1 - iz)^{n-1} + (1 + iz)^{n-1}}{(1 - iz)^n + (1 + iz)^n}
= \frac{n}{1 + z^2} \left( 1 + iz \frac{(1 - iz)^n - (1 + iz)^n}{(1 - iz)^n + (1 + iz)^n} \right)
= \frac{n(1 + z \tan(n \arctan z))}{1 + z^2}
\]

where in the last step we used the well known formula [3, item 16]

\[
\tan(n \arctan z) = i \frac{(1 - iz)^n - (1 + iz)^n}{(1 - iz)^n + (1 + iz)^n},
\]

Finally we use the relation \( R_{\tau_n}(z) = (F_n(z) - n)/z \), which gives \( R \)-transform.

\[\square\]

4.5. Free skew-symmetric laws.

**Definition 4.11.** Let \( X_1, X_2, \ldots, X_n \in \mathcal{A}_{sa} \) be a free family of semicircular random variables with variance one and \( A = [a_{i,j}]_{i,j=1}^n \in M_n(\mathbb{C}) \) be a selfadjoint matrix such that \( A = -A^T \). The law of the random variable \( \sum_{k,j=1}^n a_{k,j} (X_kX_j - X_jX_k) \) is called the free skew-symmetric distribution with matrix \( A \).

**Proposition 4.12.** A distribution \( \mu \) is free skew-symmetric with matrix \( A \) if and only if \( \mu \) can be decomposed as a free convolution of rescaled tetilla distributions

\[
\mu = D_{\lambda_1}(\mathcal{T}_2) \boxplus \cdots \boxplus D_{\lambda_{n/2}}(\mathcal{T}_2),
\]

where the scale parameters \( \lambda_i \) are the positive eigenvalues of \( A \) and the dilation \( D_r \) is defined as \( D_r(\nu)(A) = \nu(A/r) \) if \( r \neq 0 \) and \( D_0(\nu)(A) = \delta_0 \). \( \lfloor \cdot \rfloor \) is the floor function which rounds down to the nearest integer.

**Proof.** Let \( A \) be a selfadjoint skew-symmetric matrix and and \( \mu \) the corresponding distribution. Assume first that \( n \) is even. Recall that \( iA \) is a real skew-symmetric matrix so the nonzero eigenvalues of this matrix are \( \pm i\lambda_1, \ldots, \pm i\lambda_{n/2} \). It is possible to bring every skew-symmetric matrix to a block diagonal form by an orthogonal transformation, see for example [28]. To be specific, every \( n \times n \) real skew-symmetric matrix can be written in the form \( iA = Q\Sigma Q^T \) where

\[
Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & -Q_1^T \end{bmatrix},
\]

with \( Q_1 \) and \( Q_2 \) orthogonal matrices and \( \Sigma = \begin{bmatrix} \Lambda_1 & \Sigma_1 \\ \Sigma_1^T & -\Lambda_1 \end{bmatrix} \) where \( \Lambda_1 \) and \( \Sigma_1 \) are diagonal matrices.

\[
\Lambda_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},
\]

\[
\Sigma_1 = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix},
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_{n/2} \) are the eigenvalues and \( \sigma_1, \sigma_2, \ldots, \sigma_{n/2} \) are the singular values of \( A \). Now if \( \mu = \nu(A) \) is a free skew-symmetric law with matrix \( A \), then

\[
\mu = \nu(Q^T A Q) = \nu(Q^T \Sigma Q) = \nu(\Sigma_1^T \Sigma_1 Q_1 \oplus \Sigma_1^T \Sigma_2 Q_2 \oplus \Sigma_2^T \Sigma_1 Q_2 \oplus \Sigma_2^T \Sigma_2 Q_1).
\]

Thus, the corresponding distribution is a free convolution of rescaled tetilla distributions.

Conversely, if \( \mu = \nu(Q^T A Q) \), then

\[
\mu = \nu(Q^T \Sigma Q) = \nu(\Sigma_1^T \Sigma_1 Q_1 \oplus \Sigma_1^T \Sigma_2 Q_2 \oplus \Sigma_2^T \Sigma_1 Q_2 \oplus \Sigma_2^T \Sigma_2 Q_1).
\]

Therefore, \( \mu \) is a free skew-symmetric law with matrix \( A \).

\[\square\]
\( Q \) is orthogonal and

\[
\Sigma = \begin{bmatrix}
0 & \lambda_1 & \cdots & 0 & 0 \\
-\lambda_1 & 0 & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & \lambda_{n/2} \\
0 & 0 & \cdots & -\lambda_{n/2} & 0
\end{bmatrix},
\]

for real \( \lambda_k \). Let \( X_1, X_2, \ldots, X_n \in A_{sa} \) be as in Definition 4.11, then we have

\[
\sum_{k,j=1}^{n} a_{k,j} (X_k X_j - X_j X_k) = -i \left[ X_1 \ldots X_n \right] i A \left[ X_1 \ldots X_n \right]^T \\
= -i \left[ X_1 \ldots X_n \right] \Sigma \Sigma^T \left[ X_1 \ldots X_n \right]^T.
\]

Now by [14, Theorem 3.5] the vector \( \left[ Y_1 \ldots Y_n \right] := \left[ X_1 \ldots X_n \right] Q \) is a free family of semicircular random variables with variance one and we obtain a linear combination of free tetilla elements

\[
= \lambda_1 i(Y_2 Y_1 - Y_1 Y_2) + \lambda_2 i(Y_4 Y_3 - Y_3 Y_4) + \cdots + \lambda_{n/2} i(Y_n Y_{n-1} - Y_{n-1} Y_n).
\]

For the converse, just pick the matrix \( \Sigma \) from (4.7).

In the odd-dimensional case the same orthogonal decomposition is true but in this case \( \Sigma \) always has at least one row and column of zeros which does not contribute. \( \square \)

**Corollary 4.13.** 1. Every free skew-symmetric distribution with system matrix \( A \) of odd degree \( n \) can be represented as a skew-symmetric distribution of even degree \( n - 1 \). In particular the generalized tetilla law with 3 degrees of freedom can be obtained by a dilation from the tetilla law as

\( T_3 = D_3 \mathcal{S}(\mathcal{T}_2), \)

the corresponding eigenvalues being \( \cot(\frac{\pi}{6}) = \sqrt{3}, \cot(\frac{\pi}{2}) = 0 \) and \( \cot(\frac{5\pi}{6}) = -\sqrt{3}. \)

2. Every free skew-symmetric distribution \( \mu \) is a compound free Poisson distribution. Indeed, \( \mathcal{T}_2 \) has a compound free Poisson distribution with symmetric jump distribution \( \frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_1 \) and rate 2. From Proposition 4.12 (with the same designation) we infer that \( \mu \) has free compound Poisson distribution with rate \( n \) and symmetric jump distribution

\[
\frac{1}{n} \delta_{\lambda_1} + \frac{1}{n} \delta_{-\lambda_1} + \cdots + \frac{1}{n} \delta_{-\lambda_{n/2}} + \frac{1}{n} \delta_{\lambda_{n/2}}.
\]

Consequently every compound free Poisson variable with symmetric jump distribution supported on a finite set, with rate \( n \) and evenly distributed mass can be modeled as a linear combination of free commutators.

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