Instability of compact stars with a nonminimal scalar-derivative coupling

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Abstract. For a theory in which a scalar field $\phi$ has a nonminimal derivative coupling to the Einstein tensor $G_{\mu\nu}$ of the form $\phi G_{\mu\nu} \nabla^\mu \nabla^\nu \phi$, it is known that there exists a branch of static and spherically-symmetric relativistic stars endowed with a scalar hair in their interiors. We study the stability of such hairy solutions with a radial field dependence $\phi(r)$ against odd- and even-parity perturbations. We show that, for the star compactness $C$ smaller than 1/3, they are prone to Laplacian instabilities of the even-parity perturbation associated with the scalar-field propagation along an angular direction. Even for $C > 1/3$, the hairy star solutions are subject to ghost instabilities. We also find that even the other branch with a vanishing background field derivative is unstable for a positive perfect-fluid pressure, due to nonstandard propagation of the field perturbation $\delta \phi$ inside the star. Thus, there are no stable star configurations in derivative coupling theory without a standard kinetic term, including both relativistic and nonrelativistic compact objects.

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There have been many attempts for the construction of gravitational theories beyond General Relativity (GR) [1–3]. This is mostly motivated by the firm observational evidence of inflation, dark energy, and dark matter [4]. Usually, new degrees of freedom (DOFs) are taken into account to address these problems. One of the candidates for such new DOFs is a scalar field with associated potential and kinetic energies. If the scalar field is present in today’s Universe, it can affect the configuration of compact objects. In particular, after the dawn of gravitational-wave astronomy [5, 6], it is a great concern to search for the signature of new DOFs beyond GR around strong gravitational objects like black holes (BHs) and neutron stars (NSs) [7, 8].

For the canonical scalar field minimally coupled to gravity, the asymptotically flat and stationary BH solutions are characterized by only three physical quantities — mass, electric charge, and angular momentum [9–14]. This “no-hair” BH theorem also holds for scalar-tensor theories in which the scalar field $\phi$ is coupled to the Ricci scalar $R$ of the form $F(\phi)R$ [15–17]. The no-hair property of BHs does not persist in theories with nonminimal scalar derivative couplings to the Ricci scalar and Einstein tensor [18–23]. The most general scalar-tensor theories with second-order equations of motion accommodating such couplings are known as Horndeski theories [24–27].

In shift-symmetric Horndeski theories, there are conditions for the absence of static, spherically-symmetric and asymptotically flat BH solutions with a nonvanishing radial scalar derivative $\phi'(r)$. This is associated with the conservation of the scalar-field current $J^\mu_\phi$ such that $\nabla_\mu J^\mu_\phi = 0$ [28], where $\nabla_\mu$ is the covariant derivative operator. The radial current component $J^r_\phi$ can be expressed in the form $J^r_\phi = \phi' g^{rr} F(\phi'; g, g', g'')$, where $g^{rr}$ is the $rr$ component of metric tensor $g^{\mu\nu}$ and $F$ contains $\phi'$ and derivatives of $g^{\mu\nu}$. Provided that the scalar product $g_{\mu\nu} J^\mu_\phi J^\nu_\phi$ is regular, the regularity of $J^r_\phi$ on the BH horizon requires that $J^r_\phi = 0$ everywhere. If $F$ neither vanishes nor contains negative powers of $\phi'$, the allowed field profile consistent with $J^r_\phi = 0$ is the no-hair solution with $\phi'(r) = 0$. One counter example is a linearly time-dependent scalar field $\phi(t, r) = qt + \chi(r)$ [29], in which case $F = 0$ from the field equation of motion. This leads to hairy BH solutions with a static metric [30–32], including the stealth BH solution [29]. A negative function $F$ can be realized by a Gauss-Bonnet term linearly coupled to $\phi$, in which case the hairy BH is also present [21, 22].
For NSs in scalar-tensor theories, the existence of matter can give rise to nontrivial static and spherically-symmetric solutions which do not have an analogy with BHs. For instance, the nonminimal coupling $F(\phi)R$ allows the presence of hairy solutions where the mass and radius of NSs are modified from those in GR. This is the case for Brans-Dicke theory and $f(R)$ gravity [33–40], in which the solutions with nonvanishing $\phi(r)$ are present. For the nonminimal coupling containing the even power of $\phi$, there exists nonvanishing NS solutions with $\phi(r) \neq 0$ besides the GR branch with $\phi(r) = 0$ [41, 42] (see refs. [43–45] for rotating solutions). If one considers the function $F(\phi) = e^{-\beta \phi^2}$ with a negative coupling constant $\beta$, the GR branch can trigger a tachyonic instability to reach the other nontrivial branch [46–49], whose phenomenon is dubbed spontaneous scalarization. In such theories, the scalarized NS solutions are stable against odd- and even-parity perturbations on the static and spherically-symmetric background [50].

In shift-symmetric Horndeski theories with matter minimally coupled to gravity, there is a no-hair theorem for stars [51] generalizing the BH case. The theorem states that, under the following three conditions, the allowed solution is only the trivial branch with $\phi'(r) = 0$: (i) $\phi$ and $g_{\mu\nu}$ are regular everywhere, static and spherically symmetric, (ii) spacetime is asymptotically flat with $\phi' \to 0$ as $r \to \infty$, and (iii) there is a canonical kinetic term $X = -(1/2)g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$ in the action, where the action is analytic around a trivial scalar-field configuration. To realize hairy NS solutions, we need to break at least one of these conditions.

If we break the condition (iii), the shift-symmetric Lagrangian $\mathcal{L} = G_4(X)R + G_{4X}(X) \left[ (\Box \phi)^2 - (\nabla_\mu \phi)(\nabla^\mu \phi) \right]$ with $G_4(X) = 1/(16\pi G_N) + \eta X/2$ and $G_{4X} = dG_4/dX = \eta/2$, where $G_N$ is the Newton gravitational constant and $\eta$ is a constant of the derivative coupling, can give rise to hairy NS solutions [52–54]. This theory is equivalent to the nonminimal derivative coupling (NDC) $G_5(\phi) = -\eta \phi/2$ with the Lagrangian $\mathcal{L} = R/(16\pi G_N) + G_{5}(\phi)G_{\mu\nu} \nabla^\mu \nabla^\nu \phi$, which belongs to a subclass of Horndeski theories. The NS solutions in NDC theory have an interesting property that the scalar hair is present only inside the star with an external vacuum [52]. At the background level the mass-radius relation in NDC theory does not significantly differ from that in GR, but the quasi-normal mode of odd-parity perturbations exhibits notable difference between the two theories [55].

In this paper, we study the stability of hairy NS solutions in NDC theory with a radial-dependent field profile $\phi(r)$ by considering odd- and even-parity perturbations on the static and spherically-symmetric background. We do not consider a time-dependent scalar-field configuration like $\phi(t, r) = qt + \chi(r)$, by reflecting the fact that the BH solutions with $q \neq 0$ are generally prone to instabilities against odd-parity perturbations [56, 57]. We deal with baryonic matter inside the star as a perfect fluid described by a Schutz-Sorkin action [58–60].

The hairy NS solutions in NDC theory are stable against odd-parity perturbations with a superluminal radial propagation speed. However, we show that the angular propagation speed squared of scalar-field perturbation $\delta \phi$ in the even-parity sector is negative around the surface of NSs for $\mathcal{C} < 1/3$, where $\mathcal{C}$ is the star compactness. This leads to Laplacian instabilities for the perturbations with large multipoles $l$ in the angular direction. Even for some specific EOSs which give the compactness $\mathcal{C} > 1/3$, there is a ghost instability of even-parity perturbations. Thus, the compact star solutions in NDC theory with $\phi'(r) \neq 0$ are always unstable. We also show that, as long as the coupling $G_5(\phi) = -\eta \phi/2$ is present, the other branch satisfying $\phi'(r) = 0$ inside the star is prone to Laplacian instabilities. These properties are mostly related to the nonstandard propagation of $\delta \phi$ induced by the absence of standard kinetic term $X$ in the action.
2 Hairy relativistic stars with nonminimal derivative coupling

We consider the action of NDC theory given by

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{16\pi G_N} R + G_5(\phi) G_{\mu\nu} \nabla^\mu \nabla^\nu \phi \right] + S_m, \quad (2.1) \]

where \( g \) is the determinant of metric tensor \( g_{\mu\nu} \), and

\[ G_5(\phi) = -\frac{1}{2} \eta \phi, \quad (2.2) \]

where \( \eta \) is a constant. After the integration by parts, this theory is equivalent to the quartic-order nonminimal derivative coupling \( G_4(X) = 1/(16\pi G_N) + \eta X/2 \) in Horndeski theories [26]. In other words, the action (2.1) with the quintic coupling (2.2) belongs to a subclass of shift-symmetric Horndeski theories invariant under the shift \( \phi \to \phi + c \).

For the matter sector, we consider a perfect fluid minimally coupled to gravity, which is described by the Schutz-Sorkin action [58–60],

\[ S_m = -\int d^4x \left[ \sqrt{-g} \rho(n) + J^\mu (\partial_\mu \ell + A_i \partial_\mu B^i) \right]. \quad (2.3) \]

Here, the matter density \( \rho \) is a function of the fluid number density \( n \), which is related to a vector current field \( J^\mu \) in the action (2.3), as

\[ n = \sqrt{\frac{g_{\mu\nu} J^\mu J^\nu}{g}}, \quad (2.4) \]

The fluid four-velocity \( u_\mu \) is given by

\[ u_\mu = \frac{J_\mu}{n \sqrt{-g}}, \quad (2.5) \]

which satisfies \( u^\mu u_\mu = -1 \). The scalar quantity \( \ell \) in eq. (2.3) is a Lagrange multiplier with the notation \( \partial_\mu \ell \equiv \partial \ell/\partial x^\mu \), whereas the spatial vectors \( A_i \) and \( B^i \) (with \( i = 1, 2, 3 \)) are the Lagrange multiplier and Lagrangian coordinates of the fluid, respectively. Both \( A_i \) and \( B^i \) are nondynamical intrinsic vector modes, so that they affect the evolution of dynamical fields through constraint equations. Varying the action (2.3) with respect to \( \ell \), it follows that

\[ \partial_\mu J^\mu = 0, \quad (2.6) \]

which corresponds to the current conservation of the perfect fluid.

Since \( \partial n/\partial J^\mu = -u_\mu/\sqrt{-g} \), the variation of eq. (2.3) with respect to \( J^\mu \) gives

\[ \partial_\mu \ell = \rho \rho_n u_\mu - A_i \partial_\mu B^i, \quad (2.7) \]

where \( \rho_n \equiv \partial \rho/\partial n \). Varying the Lagrangian \( L_m = -[\sqrt{-g} \rho(n) + J^\mu (\partial_\mu \ell + A_i \partial_\mu B^i)] \) in eq. (2.3) with respect to \( g^{\mu\nu} \), we can derive the matter energy-momentum tensor \( T_{\mu\nu} \). On using eq. (2.7) and the relation \( \delta n = (n/2) (g_{\mu\nu} - u_\mu u_\nu) \delta g^{\mu\nu} \), we obtain

\[ T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \delta L_m = (\rho + P) u_\mu u_\nu +Pg_{\mu\nu}, \quad (2.8) \]
where $P$ is the matter pressure defined by
\[ P \equiv n \rho_n - \rho . \]  

(2.9)

Thus, the matter action (2.3) leads to the standard form of perfect-fluid energy-momentum tensor (2.8). This obeys the continuity equation,
\[ \nabla^\mu T_{\mu \nu} = 0 . \]  

(2.10)

The current conservation (2.6) is equivalent to the equation $u^\nu \nabla_{\mu} T_{\mu \nu} = 0$ following from eq. (2.10) [50, 61].

We consider a static and spherically-symmetric background given by the line element,
\[ ds^2 = -f(r)dt^2 + h(r)^{-1}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \]  

(2.11)

where $f(r)$ and $h(r)$ are functions of $r$. On this background the four velocity in the fluid rest frame is given by $u^\mu = (-f(r)^{1/2}, 0, 0, 0)$, so that the energy-momentum tensor (2.8) reduces to
\[ T^\mu_\nu = \text{diag} (-\rho(r), P(r), P(r), P(r)) . \]  

(2.12)

The matter continuity eq. (2.10) yields
\[ P' + \frac{f'}{2f} (\rho + P) = 0 , \]  

(2.13)

where a prime represents the derivative with respect to $r$. From eq. (2.5), the vector field $J^\mu$ is expressed as
\[ J^\mu = \left( \sqrt{-g} n(r) f^{-1/2}(r), 0, 0, 0 \right) , \]  

(2.14)

where $n$ depends on $r$ alone, and $\sqrt{-g} = f^{1/2} h^{-1/2} r^2 \sin \theta$. Since $A_i = 0$ on the background (2.11), eq. (2.7) gives
\[ \partial_\mu \ell = \left( -\rho, n(r) f(r)^{1/2}, 0, 0, 0 \right) . \]  

(2.15)

For the background scalar field, we consider the configuration,
\[ \phi = \phi(r) . \]  

(2.16)

Varying the action (2.1) with respect to $f$ and $h$, respectively, we obtain
\[ h' = \frac{1 - h - 4\pi G_N [2Pr^2 + \eta h \phi' \{4hr\phi'' + (h + 1)\phi'\}]}{r(1 + 12\pi G_N \eta h \phi'^2)} , \]  

(2.17)

\[ f' = \frac{f}{h} \frac{1 - h + 4\pi G_N [2Pr^2 - \eta h \phi'^2(3h - 1)]}{r(1 + 12\pi G_N \eta h \phi'^2)} . \]  

(2.18)

Variation of (2.1) with respect to $\phi$ leads to the scalar-field equation $J_{\phi} = 0$, where
\[ J_{\phi} = \eta \sqrt{\frac{h}{f}} \phi' \left[ f(1 - h) - rf'h \right] . \]  

(2.19)
The conservation of $J_\phi$ arises from the fact that NDC theory with the coupling (2.2) belongs to a subclass of shift-symmetric Horndeski theories. Thus, the equation for $\phi$ reduces to $J_\phi = C$, where $C$ is an integration constant. To satisfy the boundary conditions $f \to 1$, $rf' \to 0$, $h \to 1$, and $\phi' \to 0$ at spatial infinity ($r \to \infty$), we require that $C = 0$. Then, we obtain

$$\eta \sqrt{\frac{h}{f}} \frac{\phi'}{f} \left[ f(1-h) - rf'h \right] = 0,$$

which means that there are two branches of solutions. The first one is the trivial branch with a vanishing field derivative, i.e., $\phi'(r) = 0$ at any distance $r$. The second one is the nontrivial branch satisfying

$$f' = \frac{f(1-h)}{hr}.$$

This is different from the corresponding equation $f' = f(1-h)/(hr) + 8\pi G N Prf/h$ in GR. In other words, the metric component $f$ does not feel the matter pressure even inside a star.

The main problem to be addressed in this paper is to elucidate whether the branch (2.21) is stable or not against perturbations on the background (2.11). We also discuss the stability of the other branch $\phi' = 0$ at the end.

Substituting eq. (2.21) into eq. (2.18), it follows that

$$\eta h \phi'^2 = Pr^2.$$

For this branch, the field derivative $\phi'$ is related to the fluid pressure $P$. Provided $P > 0$, the coupling constant $\eta$ is in the range,

$$\eta > 0.$$

We define the star radius $r_s$ at which the pressure vanishes, i.e., $P(r_s) = 0$. Outside the star ($r > r_s$), the field derivative $\phi'$ is 0 from eq. (2.22).

Taking the $r$ derivative of eq. (2.22) on account of eq. (2.13), both $\phi''$ and $\phi'$ can be expressed in terms of $\rho$ and $P$. Then, eq. (2.17) reduces to

$$h' = \frac{1 - h - 4\pi G N r^2[(1+h)\rho + 6hP]}{r(1 + 4\pi G N r^2 P)}.$$

Inside the star, eq. (2.24) differs from the corresponding GR equation $h' = (1 - h - 8\pi G N r^2 \rho)/r$. Outside the star we have $\rho = P = 0$, so eqs. (2.21) and (2.24) are equivalent to the differential equations of $f$ and $h$ in GR respectively. This means that the effect of NDC on the background spacetime appears only in the star interior. We define the mass function $M(r)$, as

$$h(r) = 1 - \frac{2G_N M(r)}{r},$$

as well as the mass of star, $M_s \equiv M(r_s)$. Since $\phi'(r) = 0$ outside the star, we have $M(r) = M_s$ even at spatial infinity, i.e., $M_s$ corresponds to the ADM mass. For $r > r_s$, the metric components are given by $f(r) = h(r) = 1 - 2G_N M_s/r$.

Around the center of star, we impose the regular boundary conditions $f(0) = f_c$, $h(0) = 1$, $\rho(0) = \rho_c$, $P(0) = P_c$ and $f'(0) = h'(0) = \rho'(0) = P'(0) = 0$. For the hairy branch satisfy-
The scalar field satisfies the regular boundary condition \( \varphi' (0) = 0 \). Note that we can set \( f_c = 1 \) by virtue of the time-rescaling invariance of eq. (2.11). In GR the expansion of \( h \) is given by \( h(r) = 1 - 8 \pi G N \rho_o r^2 / 3 + \mathcal{O}(r^4) \), while the forms of \( f(r) \) and \( P(r) \) are the same as eqs. (2.26) and (2.28) respectively.

For a given equation of state (EOS) \( P = P(\rho) \), the values of \( f(r) \), \( h(r) \), \( P(r) \), and \( \varphi'(r) \) inside the star are known by integrating eqs. (2.13), (2.21), and (2.24) with eq. (2.22). In eq. (2.22), we choose the branch with \( \varphi'(r) > 0 \) without loss of generality. As an example, we consider the SLy EOS of NSs, whose analytic representation is given in ref. [62]. In the left panel of figure 1, we plot the mass function \( M \), field derivative \( \varphi' \), pressure \( P \), ratio \( P/\rho \) as a function of \( r \) for the central density \( \rho_c = 10 \rho_0 \) and the coupling \( \eta = r_s^2 \), where \( \rho_0 = 1.6749 \times 10^{14} \text{ g/cm}^3 \). Since \( \eta \) does not appear in the differential eqs. (2.13), (2.21), and (2.24), the star configuration is independent of the coupling strength. As we see in eq. (2.22), for smaller \( \eta \), \( \varphi' \) gets larger. Up to the star surface the function \( M(r) \) continuously grows to the ADM mass \( M_s \), whereas the pressure \( P(r) \) monotonically decreases toward 0. In the left panel of figure 1, the perfect fluid is in a relativistic region with \( P/\rho \simeq 0.315 \) around the center of star. We observe that, as \( r \) approaches \( r_s \), the ratio \( P/\rho \) decreases. Around the surface of star, the EOS is in a nonrelativistic region characterized by \( P/\rho \ll 1 \). From eq. (2.29), the field derivative grows as \( \varphi' \propto r \) around \( r = 0 \). Since \( \varphi'(r_s) = 0 \) from eq. (2.22), there is a point inside the star at which \( \varphi'(r) \) reaches a maximum. This can be confirmed in the numerical simulation of figure 1.

In the right panel of figure 1, we plot \( M_s \) (normalized by the solar mass \( M_\odot \)) versus \( r_s \) for the SLy EOS in NDC theory with the branch satisfying (2.21). For the increasing central density \( \rho_c \), the radius \( r_s \) tends to decrease, together with the growth of \( M_s \) up to the density \( \rho_c = 14.4 \rho_0 \). After \( M_s \) reaches the maximum value \( 1.93 M_\odot \), it starts to decrease for \( \rho_c > 14.4 \rho_0 \). In comparison to the case of GR (plotted as a dashed line), the nonvanishing scalar field in NDC theory works to reduce both \( r_s \) and \( M_s \). Independent of the coupling constant \( \eta > 0 \), the mass and radius in NDC theory are uniquely fixed for a given EOS. The mass-radius relation is not significantly different from that in GR, but the relativistic star is endowed with a scalar hair inside the body. The difference between NDC theory and GR manifests itself for the propagation of perturbations, as we will study in the subsequent sections.

3 Odd-parity perturbations

On the static and spherically-symmetric background (2.11), the metric perturbations \( h_{\mu\nu} \) can be decomposed into odd- and even-parity modes with respect to a rotation in the two-dimensional plane \((\theta, \varphi)\) [63–65]. In this section, we consider the propagation of odd-parity perturbations for NDC theory given by the action (2.1). We express the perturbations in
Figure 1. (Left) $M$, $\phi'$, $\ln P$, and $\ln(P/\rho)$ versus $r/r_s$ in NDC theory with the coupling $\eta = r_s^2$. We choose the SLy EOS with the central density $\rho_c = 10\rho_0$, where $\rho_0 = 1.6749 \times 10^{14}$ g/cm$^3$. The quantities $M$, $\phi'$, and $P$ are normalized by $M_\odot$, $(r_s\sqrt{8\pi G N})^{-1}$, and $\rho_0$, respectively. (Right) Mass-radius relations for the SLy EOS in NDC theory (solid) and in GR (dashed).

terms of the expansion of spherical harmonics $Y_{lm}(\theta, \varphi)$. For the multipoles $l \geq 2$ we choose the so-called Regge-Wheeler gauge [63] in which the components $h_{ab}$, where $a, b$ is either $\theta$ or $\varphi$, vanish. The nonvanishing odd-parity metric perturbation components are then given by

$$h_{ta} = \sum_{l,m} Q(t,r) E_{ab} \nabla^b Y_{lm}(\theta, \varphi), \quad h_{ra} = \sum_{l,m} W(t,r) E_{ab} \nabla^b Y_{lm}(\theta, \varphi),$$

(3.1)

where $Q$ and $W$ are functions of $t$ and $r$. The tensor $E_{ab}$ is defined by $E_{ab} = \sqrt{\gamma} \varepsilon_{ab}$, where $\gamma$ is the determinant of two-dimensional metric $\gamma_{ab}$ and $\varepsilon_{ab}$ is the anti-symmetric symbol with $\varepsilon_{\theta\varphi} = 1$.

For the odd-parity sector of the perfect fluid, the components of $J_\mu$ in the action (2.3) can be expressed as [50]

$$J_t = -n(r) \sqrt{f(r)} \sqrt{-g}, \quad J_r = 0, \quad J_a = \sum_{l,m} \sqrt{-g} \delta j(t,r) E_{ab} \nabla^b Y_{lm}(\theta, \varphi),$$

(3.2)

where $\sqrt{-g} = \sqrt{f(r)}/h(r) r^2 \sin \theta$ is the background value, and the components $J_a$ have a perturbation $\delta j(t, r)$. The intrinsic vectors $A_i$ and $B^i$ can be chosen in the forms,

$$A_i = \delta A_i, \quad B^i = x^i + \delta B^i,$$

(3.3)

with the odd-parity perturbations,

$$\delta A_r = 0, \quad \delta A_a = \sum_{l,m} \delta A(t,r) E_{ab} \nabla^b Y_{lm}(\theta, \varphi),$$

(3.4)

$$\delta B^r = 0, \quad \delta B^a = \sum_{l,m} \delta B(t,r) E^{a}_{b} \nabla^b Y_{lm}(\theta, \varphi).$$

(3.5)
The Lagrange multiplier $\ell$ in eq. (2.7) is not affected by the odd-parity perturbation, so that its explicit form without containing the even-parity perturbation is given by

$$\ell = -\rho_n(r)\sqrt{f(r)}t.$$ (3.6)

From eq. (2.4), the perturbation of fluid number density $n$ is expressed as

$$\delta n = \rho - \rho_n n(r)\left(\frac{hW^2}{f} - \frac{Q^2}{f} + \frac{2}{n\sqrt{f}}Q\delta j - \frac{\delta j^2}{n^2}\right)\left[(\partial_\theta Y_{lm})^2 + \frac{(\partial_\phi Y_{lm})^2}{\sin^2 \theta}\right] + \mathcal{O}(\epsilon^4),$$ (3.7)

with $\rho = \rho(r) + \rho_n\delta n + \mathcal{O}(\epsilon^4)$, where $\epsilon^i$ represents the $i$-th order of perturbations. Expanding the Schutz-Sorkin action (2.3) up to the order of $\epsilon^2$, the resulting second-order action contains a term proportional to $\delta A$. The variation of this term with respect to $\delta A$ leads to

$$\dot{\delta B} = \frac{nQ - \sqrt{f} \delta j}{nr^2},$$ (3.8)

where a dot represents the derivative with respect to $t$. Substituting this relation into $S_m$, the resulting quadratic-order action contains the Lagrangian,

$$\mathcal{L}_{\delta j} = \frac{\sqrt{f}(\rho + P)[(\partial_\theta Y_{lm})^2 \sin^2 \theta + (\partial_\phi Y_{lm})^2]}{2n^2\sqrt{h}\sin \theta} \delta j^2.$$ (3.9)

Varying $\mathcal{L}_{\delta j}$ with respect to $\delta j$, we obtain

$$\delta j = 0,$$ (3.10)

which means that the perturbations of $J_{(i)}$ in the odd-parity sector vanish. On using eq. (3.10), the matter perturbations arising from the Schutz-Sorkin action are integrated out from the total action (2.1). This is analogous to the case of cosmological perturbations in the presence of a perfect-fluid action (see, e.g., ref. [69]). On the cosmological background in scalar-tensor theories, the matter perturbation arising from the Schutz-Sorkin action does not contribute to tensor perturbations, but it only modifies the dynamics of scalar perturbations. On the static and spherically-symmetric background the scalar perturbation corresponds to the even-parity mode, so the odd-parity perturbation $\delta j$ in the Schutz-Sorkin action vanishes.

The scalar field $\phi$ does not have the odd-parity perturbation. We integrate the total second-order action with respect to $\theta$ by setting $m = 0$ without loss of generality. After the integration by parts with respect to $t$ and $r$, the quadratic-order action of odd-parity perturbations reduces to

$$S^{(2)}_{\text{odd}} = \sum_l \int dt dr \left[\frac{L}{4r^2} \mathcal{H} \left(\dot{W} - Q' + \frac{2Q}{r}\right)^2 - \frac{L(L - 2)\sqrt{\mathcal{H}}}{4r^2}GW^2 + \frac{L(L - 2)}{4\sqrt{\mathcal{H}r^2}} \mathcal{F}Q^2\right],$$ (3.11)

where

$$L = l(l + 1),$$ (3.12)

and

$$\mathcal{H} = \mathcal{G} = \frac{1}{8\pi G_N} + \frac{1}{2} \eta h \phi'^2, \quad \mathcal{F} = \frac{1}{8\pi G_N} - \frac{1}{2} \eta h \phi'^2.$$ (3.13)

For the derivation of eq. (3.11) we did not choose the branch of either $\phi'(r) = 0$ or $\phi'(r) \neq 0$, so the result is valid for both cases. The second-order action (3.11) does not contain any matter
perturbations arising from the Schutz-Sorkin action (2.3). The existence of a perfect fluid inside the star affects the dynamics of odd-parity perturbations only through the background metric components \(f\) and \(h\). Since the matter perturbations are fully integrated out from the odd-parity action, the result (3.11) coincides with the second-order action of ref. [66] derived in the context of BH perturbations for full Horndeski theories.

Although there are two metric perturbations \(W\) and \(Q\) in eq. (3.11), the system can be described by a single dynamical perturbation [67, 68],

\[
\chi = \dot{W} - Q' + \frac{2Q}{r}. \tag{3.14}
\]

To see this property, we express the action (3.11) in the form,

\[
S^{(2)}_{\text{odd}} = \sum_l \int dt dr \left\{ \frac{L}{4} \sqrt{\frac{h}{f}} \mathcal{H} \left[ 2\chi \left( \dot{W} - Q' + \frac{2Q}{r} \right) - \chi^2 \right] - \frac{L(L-2)\sqrt{\mathcal{H}}}{4r^2} GW^2 + \frac{L(L-2)}{4\sqrt{\mathcal{H}r^2}} FQ^2 \right\}. \tag{3.15}
\]

Varying eq. (3.15) with respect to \(W\) and \(Q\), it follows that both \(W\) and \(Q\) can be expressed in terms of \(\chi\) and its \(t\) and \(r\) derivatives. Substituting these relations into eq. (3.15) and integrating it by parts, the second-order action for \(l \geq 2\) reduces to

\[
S^{(2)}_{\text{odd}} = \sum_l \int dt dr \left( C_1 \dot{\chi}^2 + C_2 \chi^2 + C_3 \chi^2 \right), \tag{3.16}
\]

where

\[
C_1 = \frac{L\sqrt{\mathcal{H}} r^2 \mathcal{H}^2}{4f^{3/2}(L-2)G}, \quad C_2 = -\frac{Lh^{3/2}r^2 \mathcal{H}^2}{4\sqrt{f}(L-2)F}, \quad C_3 = -\frac{L\sqrt{\mathcal{H}}}{8(L-2)f^{5/2}F} \tilde{C}_3, \tag{3.17}
\]

and

\[
\tilde{C}_3 = 2(L-2)f^2 F + f^2 r F'[ (h'h' + 2h h') r + 4h \mathcal{H} ] - 4f^2 F' [ (h'h + h h') r - h \mathcal{H} ] - f^2 h r^2 F' H - f^2 r^2 F' (2h H'' + 3h'h + h'' h) + f r^2 [ F' (f'' h H + f' h' H + f h H') - f h F' H ] . \tag{3.18}
\]

In the following, we focus on the hairy branch satisfying the condition (2.22) by the end of this section. From eq. (3.16) the ghost is absent under the condition \(C_1 > 0\), which translates to

\[
\mathcal{G} = \frac{1}{8\pi G_N} + \frac{1}{2} Pr^2 > 0. \tag{3.19}
\]

For \(P > 0\), this condition is automatically satisfied.

The dispersion relation in the radial direction follows by substituting the solution \(\chi = e^{i(\omega t - kr)}\) into eq. (3.16) and taking the limits of large \(\omega\) and \(k\), such that \(\omega^2 C_1 + k^2 C_2 = 0\). The associated propagation speed squared in proper time is given by \(c_i^2 = (\omega^2/k^2)/(fh) = -(C_2/C_1)/(fh)\), i.e.,

\[
c_i^2 = \frac{\mathcal{G}}{\mathcal{F}} = \frac{1 + 4\pi G_N Pr^2}{1 - 4\pi G_N Pr^2}. \tag{3.20}
\]

The Laplacian instability in the radial direction is absent for

\[
4\pi G_N Pr^2 < 1. \tag{3.21}
\]
Provided that $P \lesssim \rho$, the quantity $4\pi G N Pr^2$ is at most of the order of the star compactness $C = G N M_s / r_s$. Hence the condition (3.21) is well satisfied for NSs with $C \lesssim 0.3$. Indeed, in the numerical simulation of figure 1, we confirmed that the condition (3.21) holds inside the star. From eq. (3.20), the radial propagation of odd-parity perturbations is superluminal ($c_r^2 > 1$) for $0 < r < r_s$. Around $r = 0$ the pressure $P$ is given by eq. (2.28), so that $c_r^2 = 1$ at $r = 0$. For increasing $r$ inside the star, $c_r^2$ first increases toward the superluminal region and then it starts to decrease at some radius to approach the value $c_r^2 = 1$ as $r \to r_s$.

The speed of propagation of metric perturbations in the angular direction follows by plugging the solution of the form $\chi = e^{i(\omega t - \theta)}$ into eq. (3.16) and taking the limit $l \to \infty$. In this case the dispersion relation yields $\omega^2 C_1 + C_3 = 0$, with $C_1 \approx \sqrt{h^2} / (4 f^3/2 G)$ and $C_3 \approx -l^2 \sqrt{h} / (4 \sqrt{f})$. In proper time, the propagation speed squared is given by $c^2_\Omega = (\omega^2 r^2/l^2) / f = -C_3 r^2 / (C_1 l^2 f)$, so that

$$c^2_\Omega = \frac{G}{\mathcal{H}} = 1. \quad (3.22)$$

Hence the Laplacian instability is absent along the angular direction, with $c_\Omega$ equivalent to that of light.

As in ref. [50], the perturbation (3.14) for $l = 1$ does not propagate as a dynamical degree of freedom, so there are no additional stability conditions arising from the dipole mode.

4 Even-parity perturbations

Let us proceed to the stability analysis against even-parity perturbations for the multipoles $l \geq 2$. We choose the gauge in which the components $h_{ta}$ and $h_{ab}$ of metric perturbations $h_{\mu\nu}$ vanish. Then, the nonvanishing components of $h_{\mu\nu}$ are given by

$$h_{tt} = f(r) \sum_{l,m} H_0(t, r) Y_{lm}(\theta, \varphi), \quad h_{tr} = h_{rt} = \sum_{l,m} H_1(t, r) Y_{lm}(\theta, \varphi),$$

$$h_{rr} = h(r) \sum_{l,m} H_2(t, r) Y_{lm}(\theta, \varphi), \quad h_{ra} = h_{ar} = \sum_{l,m} \alpha(t, r) \nabla_a Y_{lm}(\theta, \varphi), \quad (4.1)$$

where $H_0$, $H_1$, $H_2$, $\alpha$ are arbitrary quantities. We also expand the scalar field in the form,

$$\phi = \phi(r) + \sum_{l,m} \delta \phi(t, r) Y_{lm}(\theta, \varphi), \quad (4.2)$$

where $\phi(r)$ is the background value, and $\delta \phi$ is the scalar perturbation.

For the perfect fluid, the current $J_\mu$ in the even-parity sector contains three metric perturbations $\delta J_t(t, r)$, $\delta J_r(t, r)$, and $\delta J_a(t, r)$, as

$$J_t = J_t + \sum_{l,m} \sqrt{-g} \delta J_t(t, r) Y_{lm}(\theta, \varphi), \quad J_r = \sum_{l,m} \sqrt{-g} \delta J_r(t, r) Y_{lm}(\theta, \varphi),$$

$$J_a = \sum_{l,m} \sqrt{-g} \delta J_a(t, r) \nabla_a Y_{lm}(\theta, \varphi), \quad (4.3)$$

where $J_t = -n(r)(f(r) / \sqrt{h(r)}) r^2 \sin \theta$. The intrinsic spatial vector fields $\mathcal{A}_i$ and $\mathcal{B}_i$ are chosen as eq. (3.3) with the perturbed components,

$$\delta \mathcal{A}_t = \sum_{l,m} \delta \mathcal{A}_1(t, r) Y_{lm}(\theta, \varphi), \quad \delta \mathcal{A}_r = \sum_{l,m} \delta \mathcal{A}_2(t, r) \nabla_a Y_{lm}(\theta, \varphi), \quad (4.4)$$

$$\delta \mathcal{B}_t = \sum_{l,m} \delta \mathcal{B}_1(t, r) Y_{lm}(\theta, \varphi), \quad \delta \mathcal{B}_r = \sum_{l,m} \delta \mathcal{B}_2(t, r) \nabla_a Y_{lm}(\theta, \varphi). \quad (4.5)$$
The density $\rho$ in the action (2.3) is expanded as
\[
\rho = \rho(r) + \rho, n \delta n + \frac{\rho, n}{2 n} \epsilon_m^2 \delta n^2 + O(\epsilon^3),
\]
where $\rho(r)$ is the background value, and $\epsilon_m^2$ is the matter sound speed squared defined by
\[
\epsilon_m^2 = \frac{n \rho, mn}{\rho, n}.
\]
We define the matter density perturbation $\delta \rho(t, r)$ according to
\[
\delta n = \sum_{l, m} \frac{\delta \rho(t, r)}{\rho, n(r)} Y_{lm}(\theta, \varphi).
\]
The $\theta, \varphi$ components of rotational-free four velocity $u_\mu$ are related to the velocity potential $v(t, r)$, as
\[
u_a = \sum_{l, m} v(t, r) \nabla_a Y_{lm}(\theta, \varphi).
\]
From eq. (2.7), we have $\partial_\tau \ell = \rho, n u_a - \delta A_a$ up to first order in even-parity perturbations. Integrating this relation with respect to $\tau$ on account of eqs. (4.4) and (4.9), it follows that
\[
\ell = -\rho, n(r) \sqrt{f(t, r)} t + \sum_{l, m} \left[ \rho, n(r) v(t, r) - \delta A_2(t, r) \right] Y_{lm}(\theta, \varphi).
\]
For the expansion of the total action (2.1), we perform the integral with respect to $\theta$ by setting $m = 0$ without loss of generality. Following the same procedure as in ref. [50], the nondynamical variables $\delta A_1, \delta A_2, \delta B_1, \delta B_2, \delta J$, and $\delta J_\tau$ can be integrated out from the second-order action of even-parity perturbations. The resulting quadratic-order action is given by $S_{\text{even}}^{(2)} = \int dt dr \mathcal{L}$, with the Lagrangian
\[
\mathcal{L} = H_0 \left[ a_1 \delta \phi'' + a_2 \delta \phi' + a_3 H_2' + L_a \alpha' + (a_5 + L_a) \delta \phi + (a_7 + L_a) H_2 + L_a \alpha + a_{10} \delta \rho \right] + L_b H_2' + H_1 \left( b_2 \delta \phi' + b_3 \delta \phi + b_4 H_2 + L_b \alpha \right) + c_1 \delta \phi H_2 + H_2 \left[ c_2 \delta \phi' + (c_3 + L_c) \delta \phi + L_c \alpha + c_5 \delta \phi \right] + c_6 H_2' + L c_1 \alpha^2 + L \alpha \left( d_2 \delta \phi' + d_3 \delta \phi \right) + L d_4 \alpha^2 + e_1 \delta \phi' + e_2 \delta \phi^2 + (c_3 + L_c) \delta \phi^2 + L f_1 v^2 + f_2 \delta \rho^2 + f_3 \delta \rho \delta \phi \right].
\]
where the coefficients $a_1$ etc are given in appendix. Variations of the Lagrangian (4.11) with respect to $H_0, H_1$, and $v$ lead, respectively, to
\[
a_1 \delta \phi'' + a_2 \delta \phi' + a_3 H_2' + L_a \alpha' + (a_5 + L_a) \delta \phi + (a_7 + L_a) H_2 + L_a \alpha + a_{10} \delta \rho = 0, \quad (4.12)
2 L b_1 H_1 + b_2 \delta \phi' + b_3 \delta \phi + b_4 H_2 + L b \alpha = 0, \quad (4.13)
2 L f_1 v - f_3 \delta \rho - c_5 H_2 = 0. \quad (4.14)
\]
Besides the scalar-field perturbation $\delta \phi$, there are dynamical perturbations arising from the gravity and perfect-fluid sectors. To study the propagation of latter two perturbations, we introduce the combinations [50, 70],
\[
\psi \equiv a_3 H_2 + L_a \alpha + a_1 \delta \phi',
\]
\[
\delta \rho_m \equiv \delta \rho + \frac{2 \sqrt{f} h f_1}{f_3 \left[ a_3 (2 f - f') + L f a_4 \right]} \psi' - \frac{h f_1 r^2 \left[ a_1 (2 f - f') + f^2 \rho \right]}{\sqrt{f} f_3 \left[ a_3 (2 f - f') + L f a_4 \right]} \delta \phi'.
\]
Taking the \( r \) derivative of eq. (4.15), the derivatives \( a_1 \delta \phi'' \), \( a_3 H_2' \), and \( L a_4 \alpha' \) can be simultaneously eliminated from eq. (4.12), so that \( \alpha \) can be expressed in terms of \( \psi \), \( \delta \phi \), and \( \delta \rho_m \) and their first radial derivatives. We also differentiate eq. (4.15) with respect to \( \alpha \), and eliminate \( \dot{H}_2 \) in eqs. (4.13) and (4.14) to solve for \( H_1 \) and \( v \), respectively. Substituting these relations into eq. (4.11), the second-order action of even-parity perturbations reduces to the form,

\[
S^{(2)}_{\text{even}} = \sum_i \int dt dr \left( \ddot{\bar{X}}^i K \ddot{\bar{X}}^i + \dddot{\bar{X}}^i G \dddot{\bar{X}}^i + \dddot{\bar{X}}^i Q \dddot{\bar{X}}^i + \dddot{\bar{X}}^i M \dddot{\bar{X}}^i \right),
\]
with the dynamical perturbations,

\[
\bar{X}^i = (\delta \rho_m, \psi, \delta \phi).
\]

The components of \( 3 \times 3 \) matrices \( K \), \( G \), \( Q \), \( M \), which we denote \( K_{ij} \) etc, determine the stability of stars against even-mode perturbations.

In the following, we will consider the branch in NDC theory obeying eqs. (2.21)–(2.24). At the end of this section, we also study the stability of the other branch \( \phi'(r) = 0 \).

First of all, there are no ghosts under the following three conditions,

\[
K_{11} > 0, \quad K_{11} K_{22} - K_{12} K_{21} > 0, \quad \det K > 0.
\] (4.19) (4.20) (4.21)

The inequality (4.19) is satisfied for

\[
\rho + P > 0,
\]
which holds for standard baryonic matter. Under the condition (4.22), the inequality (4.20) translates to

\[
h(L - 2) \left( 1 + 20 \pi G_N P r^2 \right) + 4 \pi G_N L r^2 \left( 1 + 4 \pi G_N P r^2 \right) \left[ \rho(1 + 12 \pi G_N h P r^2 + P) + 32 P^2 \pi^2 G_N^2 (11 L - 3) h r^4 + 1152 P^3 \pi^3 G_N^3 (L + 1) h r^6 \right] > 0,
\]

which is trivially satisfied for \( l \geq 2 \). As long as the conditions (2.23) and (4.22) are satisfied, the third condition (4.21) amounts to

\[
K \equiv \rho(3h - 1) \left[ (L - 2) h + 4 \pi G_N L \rho r^2 \left( 1 + 4 \pi G_N P r^2 \right) \right] + P \left[ h(19h - 1)(L - 2) + 8 \pi G_N L h^2 \left( L(27h - 1) - 8h - 4 \pi G_N \rho r^2 \left( 3(L - 2) h^2 - L(27h - 2) - 6h \right) \right) \right] + 16 P^2 \pi^2 G_N^2 \left[ 27(L - 2) h^2 - L(24h - 1) - 6h \right] > 0.
\] (4.24)

On using the expansions (2.26)–(2.28) around \( r = 0 \), the dominant contribution to the right-hand side of eq. (4.24) is given by \( 2(L - 2)(\rho_c + 9 P_c) \), so the inequality (4.24) holds around the center of star. For increasing \( r \), the effect of pressure \( P \) tends to be negligible relative to the density \( \rho \), see the left panel of figure 1. Around \( r = r_s \), the dominant contribution to \( K \) in eq. (4.24) is the first term containing \( \rho \), i.e.,

\[
K_s \equiv \rho(3h - 1) \left[ (L - 2) h + 4 \pi G_N L \rho r^2 \right].
\] (4.25)

The positivity of \( K_s \) requires that

\[
h > \frac{1}{3}.
\] (4.26)
For the SLy EOS used in the numerical simulation of figure 1, we confirmed that the no-ghost condition (4.24) holds inside the star irrespective of the central density \( \rho_c \), with \( h > 1/3 \) around \( r = r_s \). This is also the case for other NS EOSs like FPS [62] and BSk19, 20, 21 [71].

Let us proceed to the discussion for the propagation of even-parity perturbations along the radial direction. In the small-scale limit, the speed \( c_r \) in proper time can be derived by solving

\[
\det \left| f \hbar c_r^2 K + G \right| = 0. \tag{4.27}
\]

From eq. (4.9) the matter velocity potential \( v \) is related to only the \( \theta \) and \( \varphi \) components of \( u_\mu \). Hence there is no propagation of matter perturbations \( \delta \rho_m \) in the radial direction. This property manifests itself as the vanishing matrix components \( G_{11}, G_{12}, \) and \( G_{13} \) of \( G \), so that the first solution to eq. (4.27) is given by

\[
c_{r1}^2 = 0. \tag{4.28}
\]

The other two propagation speed squares are

\[
c_{r\pm}^2 = \frac{\alpha_1 \pm \sqrt{\alpha_1^2 - \alpha_2 K}}{K}, \tag{4.29}
\]

where \( K \) is defined by eq. (4.24), and

\[
\alpha_1 = \frac{\rho}{2} (3h - 1) \left\{ (L - 2)h + 4\pi G N L c_m^2 \rho^2 (1 + 4\pi G N P r^2) \right\} + \frac{P}{2} \left\{ (L - 2)h (19h + 3) + \pi G N \rho^2 \{ 24(L - 2)h^2 + 8(13c_m^2 + 1)hL - 16(2c_m^2 + 1)h - 8L(c_m^2 - 2) \} \right\}
\]

\[
+ P^2 r^2 \pi G N (r^2 \{ 8(26c_m^2 + 3)hL + 16(4c_m^2 - 3)h - 16L(c_m^2 - 2) \} \pi G N \rho
\]

\[
+ 4h^3 (L - 2)(23 + 6\pi G N \rho r^2) + 2(23c_m^2 + 2)hL - 8(2c_m^2 + 1)h - 2L(c_m^2 - 4) \}
\]

\[
+ 8P^3 r^4 \pi^2 \pi G N [ 27(L - 2)h^2 + (23c_m^2 - 1)hL + 2(4c_m^2 + 1)h - L(c_m^2 - 4) ] \right) , \tag{4.30}
\]

\[
\alpha_2 = 4P (1 + 4\pi G N P r^2) \left\{ (L - 2)h (1 + 4\pi G N P r^2) + 4\pi G N c_m^2 (\rho + P) r^2 \{ (L - 2)h + L \} \right\}. \tag{4.31}
\]

Exploiting the background solutions (2.26)–(2.28) around \( r = 0 \), it follows that

\[
c_{r+}^2 = 1 + O(r^2), \tag{4.32}
\]

\[
c_{r-}^2 = \frac{2P}{\rho c + 9P} + O(r^2), \tag{4.33}
\]

whose leading-order terms are both positive.

Around \( r = r_s \), the perfect fluid is in the nonrelativistic regime characterized by \( P/\rho \ll 1 \). Then, we have

\[
c_{r+}^2 = \frac{(L - 2)h + 4\pi G N L c_m^2 \rho r^2}{(L - 2)h + 4\pi G N L c_m^2 \rho r^2} + O \left( \frac{P}{\rho} \right), \tag{4.34}
\]

\[
c_{r-}^2 = \frac{4(L - 2)h (1 + 4\pi G N c_m^2 \rho r^2) + 16\pi G N L c_m^2 \rho^2 P}{(3h - 1) [(L - 2)h + 4\pi G N L c_m^2 \rho r^2]} + O \left( \frac{P^2}{\rho^2} \right). \tag{4.35}
\]

In this region, \( c_{r+}^2 \) is subluminal for \( 0 \leq c_m^2 \leq 1 \) and approaches 1 as \( r \to r_s \). We note that \( c_{r+} \) corresponds to the propagation speed of the gravitational perturbation \( \psi \), which smoothly joins the external value 1 at \( r = r_s \). On the other hand, \( c_{r-} \) is the speed of the
Figure 2. The radial propagation speed squares $c_{r+}^2$ and $c_{r-}^2$ versus $r/r_s$ in NDC theory with the branch (2.21). We choose the SLy EOS with the central densities $\rho_c = 5\rho_0$ (left) and $\rho_c = 15\rho_0$ (right).

scalar-field perturbation $\delta\phi$. Provided that $h > 1/3$, $c_{r-}^2$ is positive around the surface of star and approaches +0 as $r \to r_s$.

In figure 2, we depict $c_{r+}^2$ and $c_{r-}^2$ versus $r/r_s$ for the SLy EOS with two different central densities $\rho_c$. As estimated from eq. (4.32), $c_{r+}^2$ is close to 1 in the region $r/r_s \ll 1$. For $\rho_c = 5\rho_0$, $c_{r+}^2$ is in the range $0 < c_{r+}^2 \leq 1$ inside the star and approaches 1 as $r \to r_s$. For $\rho_c = 15\rho_0$, $c_{r+}^2$ is superluminal at small radius, but it becomes subluminal around 0.41$r_s$ and then reaches the value 1 at $r = r_s$. The other propagation speed squared $c_{r-}^2$ is in good agreement with the analytic estimation (4.33) around $r = 0$. As $r$ increases from 0, $c_{r-}^2$ first grows and begins to decrease toward the value +0 at the surface. In figure 2, we can confirm that both $c_{r+}^2$ and $c_{r-}^2$ are positive throughout the star interior, so the Laplacian instabilities of perturbations $\psi$ and $\delta\phi$ are absent along the radial direction.

The propagation speed $c_{\Omega}$ in the angular direction is related to the matrix $M$ in eq. (4.17) as a coefficient containing the term $L$. This can be obtained by solving

$$\det \left| l^2 f c_{\Omega}^2 K + r^2 M \right| = 0, \quad (4.36)$$

with the limit $l \to \infty$. One of the solutions to eq. (4.36) is the matter sound speed squared, i.e.,

$$c_{\Omega1}^2 = c_m^2. \quad (4.37)$$

The other two propagation speed squares are given by

$$c_{\Omega\pm}^2 = \frac{\beta_2 \pm \sqrt{\beta_2^2 - 4\beta_1\beta_3}}{2\beta_1}, \quad (4.38)$$
where
\[ \beta_1 = h(3h - 1)\rho (4h - 1 + 16\pi G N \rho r^2) - h [\rho + P - h(3\rho + 19P)] \\
+ P h (76h^2 - 23h + 1) - 16P \pi G N h r^2 [2\rho + P - h(26 \rho + 8 (19h + 23))] \\
- 64P \pi G^3 h r^4 [3P(\rho + 9P)h^2 - (\rho + P)\{(3h - 1)\rho + (24h - 1) P\}]. \tag{4.39} \]
\[ \beta_2 = \rho [h(3h + 1)(5h - 3) + 12\pi G N (h^2 - 1)\rho r^2] + P h (83h^2 + 8h - 3) \\
+ 8P \pi G N r^2 [13h^2 + 6h - 3 + 6\pi G N (h - 1)^2\rho r^2] \\
+ 4P^2 \pi G N r^2 [16h + 39h^2 + 12h - 3 - 12\pi G N \rho r^2 (5h^3 - 10h^2 + 5h + 2)] \\
- 16P^3 \pi G^2 N^4 r^4 (99h^3 + h^2 - 15h + 3), \tag{4.40} \]
\[ \beta_3 = h [(3\rho + 7P)h^2 + 12Ph - 3(\rho + P)] - 16P^2 \pi G N h r^2 [\pi G N r^2 (3\rho (h^2 - 1) \\
+ P (27h^2 + 4h + 1)] + 4h(h - 1). \tag{4.41} \]

On using the background solutions (2.26)–(2.28) around \( r = 0 \), it follows that
\[ c^2_{\Omega+} = 1 + \mathcal{O}(r^2), \tag{4.42} \]
\[ c^2_{\Omega-} = \frac{2P_c}{\rho_c + 9P_c} + \mathcal{O}(r^2), \tag{4.43} \]
whose leading-order terms are the same as eqs. (4.32) and (4.33), respectively.

Around \( r = r_s \), applying the approximation \( P/\rho \ll 1 \) to eq. (4.38) gives
\[ c^2_{\Omega+} = \frac{h}{h + 4\pi G \rho r^2} + \mathcal{O}\left(\frac{P}{\rho}\right), \tag{4.44} \]
\[ c^2_{\Omega-} = -\frac{3(1 - h^2)}{4h(3h - 1)} + \mathcal{O}\left(\frac{P}{\rho}\right). \tag{4.45} \]

The propagation speed squared (4.44), which corresponds to that of the perturbation \( \psi \), is subluminal and approaches 1 as \( r \to r_s \). On the other hand, the leading-order term of eq. (4.45), which is associated with the propagation of \( \delta \phi \), is negative for \( 1/3 < h < 1 \). On the star surface, we have
\[ c^2_{\Omega-}(r_s) = -\frac{3C(1 - C)}{2(1 - 3C)(1 - 2C)}, \tag{4.46} \]
where \( C = G N M_s/r_s \) is the compactness. For compact objects satisfying the condition,
\[ 0 < C < \frac{1}{3}, \tag{4.47} \]
it follows that \( c^2_{\Omega-}(r_s) < 0 \). This means that, even though \( c^2_{\Omega-} > 0 \) around \( r = 0 \), the sign of \( c^2_{\Omega-} \) changes to negative at some radius inside the star.

In figure 3, we plot \( c^2_{\Omega+} \) and \( c^2_{\Omega-} \) versus \( r/r_s \) for the SLy EOS with \( \rho_c = 5\rho_0 \) (left) and the BSk19 EOS with \( \rho_c = 15\rho_0 \) (right). For the latter, we use analytic representations of the EOS given in ref. [71]. As estimated from eqs. (4.42) and (4.44), \( c^2_{\Omega+} \) approaches 1 in both limits \( r \to 0 \) and \( r \to r_s \), with the stability condition \( c^2_{\Omega+} > 0 \) satisfied inside the star. The value of \( c^2_{\Omega-} \) around \( r = 0 \) is given by \( c^2_{\Omega-} \approx 2P_c/(\rho_c + 9P_c) > 0 \), but it enters the region \( c^2_{\Omega-} < 0 \) at a distance \( r_s \). In the numerical simulation of figure 3, this critical distance is \( r_s = 0.479r_s \) (left) and \( r_s = 0.481r_s \) (right). In figure 3, we observe that \( c^2_{\Omega-} \) exhibits some small increase around the star surface, but \( c^2_{\Omega-} \) is negative in the region \( r_s < r < r_s \). The
star compactness in figure 3 is given by \( C = 0.148 \) (left) and \( C = 0.276 \) (right), in which cases the analytic estimation (4.46) gives \( c_{\Omega}^2(r_s) = -0.483 \) (left) and \( c_{\Omega}^2(r_s) = -3.89 \) (right), respectively. They are in good agreement with the numerical values of \( c_{\Omega}^2(r_s) \) at \( r = r_s \). Hence there are Laplacian instabilities of the field perturbation \( \delta \phi \) along the angular direction in the region close to the surface of star. This means that the background field profile \( \phi = \phi(r) \) satisfying the relation (2.22) is unstable against even-parity perturbations for large \( l \) modes.

In the left panel of figure 4, we depict the numerical values of \( c_{\Omega}^2(r_s) \) versus the compactness \( C \) for the SLy and BSk20 EOSs. They are obtained by choosing different central densities \( \rho_c \) in the range \( \rho_0 \leq \rho_c \leq 50\rho_0 \). We confirm that the analytic value of \( c_{\Omega}^2(r_s) \) in eq. (4.46), which depends on \( C \) alone, is in good agreement with the numerical results. For increasing \( \rho_c \), \( C \) first increases with the decrease of \( c_{\Omega}^2(r_s) \), but there is the saturation for the growth of \( C \). As we observe in the right panel of figure 4, the compactness corresponding to the SLy EOS reaches a maximum value \( C_{\text{max}} = 0.3159 \) around \( \rho_c = 34\rho_0 \). This situation is similar for other EOSs, with different maximum values of \( C \), e.g., the BSk20 EOS gives \( C_{\text{max}} = 0.3251 \). Since \( C_{\text{max}} \) is smaller than \( 1/3 \) for the five EOSs used in figure 4, we have \( c_{\Omega}^2(r_s) < 0 \) for any central density \( \rho_c \). Indeed, the left panel of figure 4 shows that, irrespective of the EOSs, \( c_{\Omega}^2(r_s) \) is solely determined by the compactness \( C \) and that the Laplacian instability is generally present for \( C < 1/3 \). This means that not only relativistic but also nonrelativistic stars (\( C \ll 1 \)) in NDC theory are prone to the Laplacian instability along the angular direction.

There may be some specific NS EOSs leading to \( C \) larger than \( 1/3 \). In this case, however, the metric component \( h \) is smaller than \( 1/3 \) at \( r = r_s \). Then, the term \( K_s \) of eq. (4.25), which is the dominant contribution to \( K \) in eq. (4.24) around \( r = r_s \), becomes negative and hence there is the ghost instability. This implies that, even for \( C > 1/3 \), the solutions are plagued by the appearance of ghosts. Thus, we showed that the hairy compact objects in NDC theory are generally subject to instabilities of even-parity perturbations.
Figure 4. (Left) The angular propagation speed squared $c_{\Omega r}^2$ at $r = r_s$ versus the star compactness $C$ in NDC theory with the branch (2.21). The solid black and dashed red lines correspond to the SLy and BSk20 EOSs, respectively. (Right) The compactness $C$ versus $\rho_c / \rho_0$ for the SLy, FPS, BSk19, BSk20, BSk21 EOSs. We also show the border $C = 1/3$ as a dashed line. For the five EOSs with any central density $\rho_c$, $C$ is smaller than $1/3$.

We also make a brief comment on the stability of above hairy solutions outside the compact object. Since $P = 0$ outside the star, eq. (2.22) shows that the background field derivative $\phi'(r)$ vanishes for $r > r_s$. On using the background eqs. (2.17) and (2.18), all the terms containing $\delta\phi$ and its derivatives in eq. (4.11) disappear for $r > r_s$, so the resulting second-order Lagrangian is the same as that in GR with no propagation of the scalar field. Hence the instabilities of hairy solutions are present only inside the star.

So far, we have discussed the stability of hairy solutions satisfying eq. (2.21). There is also the other branch of eq. (2.20) characterized by

$$\phi'(r) = 0. \quad (4.48)$$

For this branch, the metric components $h$ and $f$ obey eqs. (2.17) and (2.18) with $\phi'(r) = \phi''(r) = 0$, so the background solutions are the same as those in GR without the scalar field. However, we need to caution that the scalar-field perturbation $\delta\phi$ still propagates inside the star. Substituting $\phi'(r) = \phi''(r) = 0$ into the second-order Lagrangian (4.11), we find that the perturbation $\delta\phi$ is decoupled from the other fields $\psi$ and $\delta\rho_m$. Although the second-order Lagrangian of $\psi$ and $\delta\rho_m$ is the same as that in GR, there is the additional scalar-field Lagrangian given by

$$\mathcal{L}_{\delta\phi} = e_1 \delta\phi^2 + e_2 \delta\phi'^2 + (e_3 + Le_4) \delta\phi^2, \quad (4.49)$$

where

$$e_1 = \frac{4\eta \pi G_N \rho r^2}{\sqrt{f h}}, \quad e_2 = 4\eta \sqrt{f h} \pi G_N P r^2, \quad e_3 = 0, \quad e_4 = 4\eta \sqrt{f h} \pi G_N P. \quad (4.50)$$
This shows that, for $\eta \neq 0$, the field perturbation $\delta \phi$ propagates inside the star. The propagation speed squares in the radial and angular directions are given, respectively, by

\[
(c_r^2)_{\delta \phi} = -\frac{e_2}{f h e_1} \frac{P}{\rho}, \quad (c_\Omega^2)_{\delta \phi} = -\frac{r^2 e_4}{f e_1} \frac{P}{\rho},
\]

where we have taken the limit $L \to \infty$ for the derivation of $(c_\Omega^2)_{\delta \phi}$. This means that, for a positive perfect-fluid pressure $P$, the perturbation $\delta \phi$ is subject to Laplacian instabilities inside the star along both radial and angular directions. Hence the scalar field does not maintain the background profile (4.48).

5 Conclusions

We studied the stability of relativistic stars in NDC theory given by the action (2.1) with the coupling (2.2). We dealt with baryonic matter inside the star as a perfect fluid described by the action (2.3). Besides the trivial branch characterized by $\phi'(r) = 0$, there is a nontrivial solution endowed with a scalar hair inside compact objects. Since the field derivative $\phi'$ is related to the matter pressure $P$ according to eq. (2.22), the radius $r_s$ and mass $M_s$ of star for the latter branch are determined by integrating eqs. (2.13), (2.21), and (2.24) for a given EOS. In comparison to GR, the mass-radius relation shifts to the region with smaller values of $r_s$ and $M_s$.

In section 3, we first discussed the propagation of odd-parity perturbations on the static and spherically-symmetric background. For the multipoles $l \geq 2$ the second-order action reduces to the form (3.15), with a single dynamical perturbation $\chi$ given by eq. (3.14). For the hairy branch in NDC theory, the stability against odd-parity perturbations requires that $4\pi G_N Pr^2 < 1$, which is well satisfied for NSs with the compactness $C \lesssim 1/3$. Under this condition the radial propagation speed $c_r$ in the odd-parity sector is superluminal, while the angular propagation speed $c_\Omega$ is equivalent to that of light.

In section 4, we analyzed the stability of hairy NS solutions in NDC theory against even-parity perturbations. In this sector, there are three dynamical degrees of freedom, i.e., matter perturbation $\delta \rho_m$, gravitational perturbation $\psi$, and scalar-field perturbation $\delta \phi$. Under the weak-energy condition $\rho + P > 0$, the ghost is absent for $K > 0$, where $K$ is given by eq. (4.24). The numerical simulations for several different EOSs showed that $K$ is positive throughout the star interior. For the radial propagation in the even-parity sector, the speed $c_{r+}$ associated with $\delta \rho_m$ vanishes by reflecting the fact that the matter velocity potential is related to the $\theta, \phi$ components of four velocity. The other two propagation speed squares $c_{r\pm}^2$ are both positive, so the Laplacian instabilities are absent for the radial propagation of perturbations $\psi$ and $\delta \phi$.

Along the angular direction, the matter perturbation propagates with the sound speed squared $c_m^2$ given by eq. (4.7). The gravitational perturbation $\psi$ has a positive propagation speed squared $c_{\Omega+}^2$. However, we showed that the speed squared $c_{\Omega-}^2$ of $\delta \phi$ in the angular direction becomes negative around the surface of star with the compactness $C$ smaller than $1/3$. In particular, the value of $c_{\Omega-}^2$ at $r = r_s$ is solely determined by the compactness $C$, see eq. (4.46). By choosing five different NS EOSs, we showed that the Laplacian instability associated with negative values of $c_{\Omega-}^2$ is always present, with $C < 1/3$. Even for some specific EOS leading to $C > 1/3$, the ghost in the even-parity sector appears around the surface of star. We also found that, in the presence of NDCs ($\eta \neq 0$), even the branch with $\phi' = 0$ is prone to Laplacian instabilities of field perturbation $\delta \phi$ inside the star.
These generic instabilities in NDC theory are mostly attributed to the nonstandard propagation of \( \delta \phi \). The propagation is modified by adding a canonical kinetic term \( X \) in the action (2.1), but in this case the no-hair theorem of ref. [51] states that there is only a trivial branch with \( \phi' = 0 \). Hence the hairy NS solutions disappear in the presence of the standard kinetic term. The other possibility for allowing the existence of a nontrivial branch with \( \phi' \neq 0 \) is to add noncanonical kinetic terms like \( X^2 \) or the cubic Galileon Lagrangian \( X \Box \phi \) to the action (2.1). It will be of interest to study whether stable hairy star solutions exist or not in such generalized theories. This is left for a future work.

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A Second-order action of even-parity perturbation

For NDC theory given by the action (2.1) with (2.2), the coefficients in the second-order action (4.11) are given by

\[
\begin{align*}
& a_1 = \eta \sqrt{\frac{h^{3/2} r}{f}} , \quad a_2 = \frac{\eta}{2} \sqrt{\frac{h}{f}} [2 h r \phi'' + (1 + h + 3 h') \phi'] , \quad a_3 = -r \sqrt{\frac{h}{f}} \left( \frac{1}{16 \pi G N} + \frac{3}{4} \eta h \phi'^2 \right) , \\
& a_4 = \sqrt{\frac{h}{f}} \left( \frac{1}{16 \pi G N} + \frac{3}{4} \eta h \phi'^2 \right) , \quad a_5 = 0 , \quad a_6 = -\frac{\eta}{4 \pi} \sqrt{\frac{f}{h}} [2 h r \phi'' + \phi' (r h' + 2 h)] , \\
& a_7 = a_3 \frac{r^2}{2} f_1 , \quad a_8 = -\frac{a_4}{2 h} , \quad a_9 = a_4 + \left( \frac{1}{r} - \frac{f'}{2 f} \right) a_4 , \quad a_{10} = \frac{r^2}{2} \sqrt{\frac{f}{h}} , \\
& b_1 = \frac{a_4}{4 f}, \quad b_2 = -\frac{2}{f} a_1 , \quad b_3 = \frac{1}{f} (a_1 - a_2) , \quad b_4 = -\frac{2}{f} a_3 , \quad b_5 = -2 b_1 , \\
& c_1 = -\frac{a_1}{f h} , \quad c_2 = -\frac{\eta}{2} \sqrt{\frac{h}{f}} \phi' [3 h r f' + (3 h - 1) f] , \quad c_3 = 0 , \quad c_4 = \frac{\eta}{4 \pi} \sqrt{\frac{h}{f}} \phi' \left( f' + \frac{2 f}{r} \right) , \\
& c_5 = -\left( \frac{f'}{2 f} + \frac{f}{r} \right) \sqrt{\frac{f}{h}} \left( \frac{1}{16 \pi G N} + \frac{3}{4} \eta h \phi'^2 \right) , \quad c_6 = \frac{r^2 f_1}{r f} , \quad c_7 = \frac{\sqrt{h \left( r f' + f \right)}}{3 \pi G N \sqrt{r}} , \\
& d_1 = b_1 , \quad d_2 = 2 h c_4 , \quad d_3 = -\frac{\eta}{2} \sqrt{\frac{h}{f}} \phi' (r f' + f) , \quad d_4 = \frac{a_4}{r} , \quad d_5 = -\frac{r h' + h - 1}{2 \sqrt{h}} , \\
& e_2 = \frac{\eta}{2} \sqrt{\frac{h}{f}} [h r f' + (h - 1) f] , \quad e_3 = 0 , \quad e_4 = \eta \frac{2 f (h r f'' + f h') - f' (h r f' - f r h' - 2 h)}{8 r f^2 \sqrt{h}} .
\end{align*}
\]

(A.1)

Since we did not specify the branches of background solutions, these coefficients are valid for both \( \phi'(r) = 0 \) and \( \phi'(r) \neq 0 \).

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