A four-component Camassa-Holm type hierarchy

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Abstract
We consider a 3×3 spectral problem which generates four-component CH type systems. The bi-Hamiltonian structure and infinitely many conserved quantities are constructed for the associated hierarchy. Some possible reductions are also studied.

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1. Introduction
The Camassa-Holm (CH) equation

\[ u_t - u_{xxt} + 3uu_x = 2u_xu_{xx} + uu_{xxx}, \] (1)

has been a subject of steadily growing literature since it was derived from the incompressible Euler equation to model long waves in shallow water by Camassa and Holm in 1993 [1]. It is a completely integrable system, which possesses Lax representation and is a bi-Hamiltonian system and especially admits peakon solutions [1][2].

CH equation is a system with quadratic nonlinearity, and another such system was proposed in 1999 by Degasperis and Procesi [3], which reads as

\[ m_t + um_x + 3u_xm = 0, \quad m = u - u_{xx}, \] (2)

now known as called DP equation whose peakon solutions were studied in [4][14]. The following third order spectral problem for the DP equation [2] was found by Degasperis, Holm and Hone [4]

\[ \psi_{xxx} = \psi_x - \lambda \psi \] (3)
or equivalently it may be rewritten in the matrix form as

\[ \varphi_x = \begin{pmatrix} 0 & 0 & 1 \\ -\lambda m & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \varphi. \] (4)
Olver and Rosenau suggested so-called tri-Hamiltoninan duality approach to construct the 
CH type equations and many examples were worked out \[17\] (see also \[5\]\[6\]\[7\]). In particular,
they obtained a CH type equation with cubic nonlinearity
\[
m_t + [m(u^2 - u_x^2)]_x = 0, \quad m = u - u_{xx}
\] (5)
which, sometimes is referred as Qiao equation, was studied by Qiao \[19\].
Recently, Song, Qu and Qiao \[21\] proposed a two-component generalization of (5)
\[
\begin{align*}
m_t - [m(u_x v_x - uv + w v_x - u_x v)]_x &= 0, \quad m = u - u_{xx}, \\
n_t - [n(u_x v_x - uv + w v_x - u_x v)]_x &= 0, \quad n = v - v_{xx}
\end{align*}
\] (6)
based on the following spectral problem
\[
\varphi_x = \begin{pmatrix} \frac{1}{2} & \lambda m \\ \lambda n & -\frac{1}{2} \end{pmatrix} \varphi.
\] (7)
This system \[7\] is shown to be bi-Hamiltonian and is associated with WKI equation \[22\].
Moreover Xia, Qiao and Zhou \[23\] generalized previous system to the integrable two-component 
CH type equation
\[
\begin{align*}
m_t &= F + F_x - \frac{1}{2} m(u v - u_x v_x + w v_x u_x v), \\
n_t &= -G + G_x + \frac{1}{2} n(u v - u_x v_x + w v_x - u_x v), \\
m &= u - u_{xx}, \\
n &= v - v_{xx}
\end{align*}
\] (8)
where F and G are two arbitrary functions.
Working on symmetry classification of nonlocal partial differential equations, Novikov \[15\]
found another CH type equation with cubic nonlinearity. It reads as
\[
m_t + u^2 m_x + 3 u u_x m = 0, \quad m = u - u_{xx}.
\] (9)
Subsequently, Hone and Wang proposed a Lax representation for \[9\] and showed that it is 
associated to a negative flow in Sawada-Kotera hierarchy \[10\]. Furthermore, infinitely many 
conserved quantities and bi-Hamiltonian structure are also presented \[11\].
A two-component generalization of the Novikov equation was constructed by Geng and Xue \[8\].
\[
\begin{align*}
m_t + 3 u x v m + u v m_x &= 0, \quad m = u - u_{xx} \\
n_t + 3 v x u n + u v n_x &= 0, \quad n = v - v_{xx}
\end{align*}
\] (10)
with a spectral problem
\[
\varphi_x = \begin{pmatrix} 0 & \lambda m & 1 \\
0 & 0 & \lambda n \\
1 & 0 & 0 \end{pmatrix} \varphi.
\] (11)
which reduces to the spectral Novikov’s system as \( m = n \). They also calculated the \( N \)-peakons and conserved quantities and found a Hamiltonian structure.

In this paper we will discuss the properties of the equations which follow from the following generalized spectral problem

\[
Φ_x = UΦ, \quad U = \begin{pmatrix}
0 & λm_1 & 1 \\
λn_1 & 0 & λm_2 \\
1 & λn_2 & 0
\end{pmatrix}.
\]

This spectral problem was proposed by one of us (ZP) recently \[18\] and interestingly the resulted nonlinear systems can involve an arbitrary function. This freedom allows us to recover many known CH type equations from reductions. As we will show this spectral problem takes the spectral problem considered by Geng and Xue for three-component system \[9\], one and two-component Novikov’s equation \[11\]\[8\], and one or two component Song-Qu-Qiao equation \[21\] as special cases. In this sense, almost all known \( 3 \times 3 \) spectral problems for the CH type equations are contained in this case, so it is interesting to study this spectral problem.

The first negative flow corresponding to the spectral problem (12) is

\[
\begin{align*}
m_{1t} + n_2g_1g_2 + m_1(f_2g_2 + 2f_1g_1) &= 0, \\
m_{2t} - n_1g_1g_2 - m_2(f_1g_1 + 2f_2g_2) &= 0, \\
n_{1t} - m_2f_1f_2 - n_1(f_2g_2 + 2f_1g_1) &= 0, \\
n_{2t} + m_1f_1f_2 + n_2(f_1g_1 + 2f_2g_2) &= 0, \\
m_i &= u_i - u_{ixx}, \\nq_i &= v_i - v_{ixx}, \quad i = 1, 2,
\end{align*}
\]

where

\[
\begin{align*}
f_1 &= u_2 - v_{1x}, \\f_2 &= u_1 + v_{2x}, \\
g_1 &= v_2 + u_{1x}, \\g_2 &= v_1 - u_{2x},
\end{align*}
\]

which will be shown to be a bi-Hamiltonian system.

The paper is organized as follows. In the section 2, we will construct bi-Hamiltonian operators related to the spectral problem (12) and present a bi-Hamiltonian representation for the system \[13\]. In the section 3, we construct infinitely many conserved quantities for the integrable hierarchy of (12). In the section 4, we consider the special reductions of our spectral problem. The last section contains concluding remarks.

2. Bi-Hamiltonian structures in general

Let us consider the following Lax pair

\[
Φ_x = UΦ, \quad Φ_t = VΦ,
\]

where \( U \) is defined by (12) and \( V = (V_{ij})_{3x3} \) at the moment is an arbitrary matrix. The compatibility condition of (14) or the zero-curvature representation

\[
U_t - V_x + [U, V] = 0,
\]

\[
(15)
\]
is equivalent to
\[
\begin{aligned}
\lambda m_{1t} &= V_{12x} - V_{32} + \lambda (m_1 V_{11} + n_2 V_{13} - m_1 V_{22}), \\
\lambda m_{2t} &= V_{23x} + V_{21} + \lambda (m_2 V_{22} - m_2 V_{33} - n_1 V_{13}), \\
\lambda n_{1t} &= V_{21x} + V_{23} + \lambda (n_1 V_{22} - m_2 V_{31} - n_1 V_{11}), \\
\lambda n_{2t} &= V_{32x} - V_{12} + \lambda (n_2 V_{33} + m_1 V_{31} - n_2 V_{22}),
\end{aligned}
\]
with
\[
\begin{aligned}
V_{11} &= V_{31x} + V_{33} - \lambda n_2 V_{21} + \lambda n_1 V_{32}, \\
V_{13} &= V_{33x} + V_{31} + \lambda m_2 V_{32} - \lambda n_2 V_{23}, \\
V_{22x} &= \lambda (n_1 V_{12} + m_2 V_{32} - m_1 V_{21} - n_2 V_{23}), \\
2V_{31x} + V_{33x} &= \lambda ((\partial n_2 + m_1) V_{23} - (\partial m_2 + n_1) V_{32} - m_2 V_{12} + n_2 V_{21}), \\
2V_{33x} + V_{31x} &= \lambda ((\partial n_2 + m_1) V_{21} - (\partial n_1 + m_2) V_{32} - n_1 V_{12} + n_2 V_{23}).
\end{aligned}
\]
Taking account of (17) and through a tedious calculation, the system (16) yields
\[
\begin{pmatrix}
m_1 \\
m_2 \\
n_1 \\
n_2
\end{pmatrix}
_{t} = (\lambda^{-1} \mathcal{K} + \lambda \mathcal{L})
\begin{pmatrix}
V_{21} \\
V_{32} \\
V_{12} \\
V_{23}
\end{pmatrix},
\]
with
\[
\mathcal{K} =
\begin{pmatrix}
0 & -1 & \partial & 0 \\
1 & 0 & 0 & \partial \\
\partial & 0 & 0 & 1 \\
0 & \partial & -1 & 0
\end{pmatrix}, \quad \mathcal{L} = \mathcal{J} + \mathcal{F},
\]
and
\[
\mathcal{J} =
\begin{pmatrix}
2m_1 \partial^{-1} m_1 & -m_1 \partial^{-1} m_2 & J_{13} & J_{14} \\
-m_2 \partial^{-1} m_1 & 2m_2 \partial^{-1} m_2 & J_{23} & J_{24} \\
-J_{13}^* & -J_{23}^* & 2n_1 \partial^{-1} n_1 & -n_1 \partial^{-1} n_2 \\
-J_{14}^* & -J_{24}^* & -n_2 \partial^{-1} n_1 & 2n_2 \partial^{-1} n_2
\end{pmatrix},
\]
\[
\mathcal{F} = (2P + S \partial)(\partial^3 - 4 \partial)^{-1}P^T - (2S + P \partial)(\partial^3 - 4 \partial)^{-1}S^T,
\]
where
\[
J_{13} = -2m_1 \partial^{-1} n_1 - n_2 \partial^{-1} m_2, \quad J_{14} = m_1 \partial^{-1} n_2 + n_2 \partial^{-1} m_1, \\
J_{23} = m_2 \partial^{-1} n_1 + n_1 \partial^{-1} m_2, \quad J_{24} = -2m_2 \partial^{-1} n_2 - n_1 \partial^{-1} m_1, \\
P = (m_1, m_2, -n_1, n_2)^T, \quad S = (-n_2, n_1, -m_2, m_1)^T.
\]
By specifying the matrix \( V \) properly, we can obtain the hierarchy of equations associated with the Lax pair (14). For example, if we assume
\[
V =
\begin{pmatrix}
-f_1 g_1 & \frac{g_1}{\lambda} & -g_1 g_2 \\
\frac{f_2}{\lambda} & -\frac{1}{\lambda} + f_1 g_1 + f_2 g_2 & \frac{g_2}{\lambda} \\
-f_1 f_2 & \frac{f_2}{\lambda} & -f_2 g_2
\end{pmatrix},
\]

we obtain

\[ V = -\lambda \begin{pmatrix} 0 & m_1 \Gamma & 0 \\ n_1 \Gamma & 0 & m_2 \Gamma \\ 0 & n_2 \Gamma & 0 \end{pmatrix}. \] (20)

we obtain

\[
\begin{aligned}
    m_{1t} + (\Gamma m_1)_x - n_2 \Gamma &= 0, \\
    m_{2t} + (\Gamma m_2)_x + n_1 \Gamma &= 0, \\
    n_{1t} + (\Gamma n_1)_x + m_2 \Gamma &= 0, \\
    n_{2t} + (\Gamma n_2)_x - m_1 \Gamma &= 0,
\end{aligned}
\] (21)

where \( \Gamma \) is an arbitrary function.

Now the combinations of \( V \) and \( \tilde{V} \) leads us to the following system of equations

\[
\begin{aligned}
    m_{1t} + (\Gamma m_1)_x + n_2 (g_1 g_2 - \Gamma) + m_1 (f_2 g_2 + 2 f_1 g_1) &= 0, \\
    m_{2t} + (\Gamma m_2)_x - n_1 (g_1 g_2 - \Gamma) - m_2 (f_1 g_1 + 2 f_2 g_2) &= 0, \\
    n_{1t} + (\Gamma n_1)_x - m_2 (f_1 f_2 - \Gamma) - n_1 (f_2 g_2 + 2 f_1 g_1) &= 0, \\
    n_{2t} + (\Gamma n_2)_x + m_1 (f_1 f_2 - \Gamma) + n_2 (f_1 g_1 + 2 f_2 g_2) &= 0,
\end{aligned}
\] (22)

which has the Lax pair

\[ \Phi_x = U \Phi, \quad \Phi_t = (V + \tilde{V}) \Phi. \]

It is obvious that the operator \( \mathcal{K} \) given by (19) is a Hamiltonian operator. Indeed we have the following theorem

**Theorem 1.** The operators \( \mathcal{K} \) and \( \mathcal{L} \) defined by (19) constitute a pair of compatible Hamiltonian operators. In particular, the four-component system (13) is a bi-Hamiltonian system, namely it can be written as

\[
\begin{pmatrix} m_1 \\ m_2 \\ n_1 \\ n_2 \end{pmatrix}_t = \mathcal{K} \begin{pmatrix} \frac{\delta H_0}{\delta m_1} \\ \frac{\delta H_0}{\delta m_2} \\ \frac{\delta H_0}{\delta n_1} \\ \frac{\delta H_0}{\delta n_2} \end{pmatrix} = \mathcal{L} \begin{pmatrix} \frac{\delta H_1}{\delta m_1} \\ \frac{\delta H_1}{\delta m_2} \\ \frac{\delta H_1}{\delta n_1} \\ \frac{\delta H_1}{\delta n_2} \end{pmatrix}
\] (23)

where

\[ H_0 = \int (f_1 g_1 + f_2 g_2)(m_2 f_2 + n_1 g_1)dx, \quad H_1 = \int (m_2 f_2 + n_1 g_1)dx. \] (24)

**Proof:** It is easy to check that \( \mathcal{L} \) is a skew-symmetric operator. Thus, what we need to do is to verify the Jacobi identity for \( \mathcal{L} \) and the compatibility of two operators \( \mathcal{K} \) and \( \mathcal{L} \). To this end, we follow Olver and use his multivector approach (16). Let us introduce

\[
\Theta_{\mathcal{K}} = \frac{1}{2} \int \Theta \wedge \mathcal{K} \Theta dx, \quad \Theta_{\mathcal{L}} = \frac{1}{2} \int \Theta \wedge \mathcal{L} \Theta dx
\] (25)

\[ 5 \]
with $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$. A tedious but direct calculation shows that
\[ pr v_{L\theta}(\Theta_L) = 0, \quad pr v_{K\theta}(\Theta_L) + pr v_{L\theta}(\Theta_K) = 0 \tag{26} \]
hold. The details of this calculation are postponed to the Appendix.

Since we have a bi-Hamiltonian pair $K$ and $L$, we may formulate an integrable hierarchy of the corresponding nonlinear evolutions equations through recursion operator. While the system (21) possesses a Lax pair, we could not expect that it is a bi-Hamiltonian system for any $\Gamma$ and the constraint of $\Gamma$ will appear if it is requested to commute with the negative flow.

To understand the appearance of the arbitrary function occurred in the present case, we now calculate the Casimir functions of the Hamiltonian operator $L$. Let
\[ L(A, B, C, D)^T = 0, \tag{27} \]
and define
\[ K_1 = m_1A - n_1C, \quad K_2 = m_2B - n_2D, \tag{28} \]
\[ K_3 = m_2C - m_1D, \quad K_4 = n_2A - n_1B. \tag{29} \]
The system (27) consists of four equations of similar type. For example the one of them is
\[ m_1(\partial^{-1}(2K_1 - K_2) + (\partial^3 - 4\partial)^{-1}(2(K_1 + K_2) + (K_3 + K_4)_x) = n_2(\partial^{-1}K_3 + \partial^3 - 4\partial)^{-1}(2(K_3 + K_4) + (K_1 + K_2)_x)). \]
Solving these equations we found
\[ K_1 = (m_2n_2\Lambda)_x + (n_1n_2 - m_1m_2)\Lambda, \]
\[ K_2 = -(m_1n_1)\Lambda_x + (n_1n_2 - m_1m_2)\Lambda, \]
\[ K_3 = (m_1m_2)\Lambda_x + (m_1n_1 - m_2n_2)\Lambda, \]
\[ K_4 = -(n_2n_2)\Lambda_x + (m_1n_1 - m_2n_2)\Lambda, \]
where $\Lambda = \frac{k}{m_1n_1 + m_2n_2}$ and $k$ is an arbitrary number. Substituting above expressions for $K_i$ into (28)-(29) and solving the resulted linear equations leads to
\[ A = -n_1\Gamma + \frac{n_1}{m_1m_2}K_3 + \frac{1}{m_1}K_1, \]
\[ B = -n_2\Gamma + \frac{1}{m_2}K_2, \quad C = -m_1\Gamma + \frac{1}{m_2}K_3, \quad D = -m_2\Gamma, \]
where $\Gamma$ is an arbitrary function. This implies that $L$ is a degenerate Hamiltonian operator. For the special case $k = 0$ and $\Gamma = m_1n_1 + m_2n_2$, the Casimir is $H_c = -\frac{1}{2} \int \Gamma^2 dx$ and hence the system (21) in this case is a Hamiltonian system
\[
\begin{pmatrix}
m_1 \\
m_2 \\
n_1 \\
n_2
\end{pmatrix}_t = K
\begin{pmatrix}
\frac{\delta H_c}{\delta m_1} \\
\frac{\delta H_c}{\delta m_2} \\
\frac{\delta H_c}{\delta n_1} \\
\frac{\delta H_c}{\delta n_2}
\end{pmatrix}.
\]
3. Conserved quantities

An integrable system normally possesses infinity number of concerned quantities and such property has been taken as one of the defining properties for integrability. In this section, we show that infinitely many conserved quantities can be constructed for the nonlinear evolution equations related with the spectral problem (12). Indeed, we may derive two sequences of conserved quantities utilizing the projective coordinates in the spectral problem. We can introduce these coordinates in three different manners as

I.) \[ a = \frac{\varphi_1}{\varphi_2}, \quad b = \frac{\varphi_3}{\varphi_2}, \]
II.) \[ \sigma = \frac{\varphi_2}{\varphi_1}, \quad \tau = \frac{\varphi_3}{\varphi_1}, \]
III.) \[ \alpha = \frac{\varphi_1}{\varphi_3}, \quad \beta = \frac{\varphi_2}{\varphi_3}. \]

Case 1: In these coordinates we obtain that

\[ \rho = (\ln \varphi_2)_{x} = \lambda n_1 a + \lambda m_2 b, \] (30)

is conserved quantity with \( a, b \) satisfy

\[ a_x = \lambda m_1 + b - a \rho, \quad b_x = a + \lambda n_2 - b \rho. \] (31)

Substituting the Laurent series expansions in \( \lambda \) of \( a \) and \( b \) into (31)

\[ a = \sum_{i \geq 0} a_i \lambda^i, \quad b = \sum_{j \geq 0} b_j \lambda^j, \]

then we find

\[ a_0 = 0, \quad a_1 = -v_2 - u_{1x} = -g_1, \quad a_2 = 0, \]
\[ b_0 = 0, \quad b_1 = -u_1 - v_{2x} = -f_2, \quad b_2 = 0, \]

and

\[ a_{k,x} = b_k - \sum_{i+j=k-1} (n_1 a_i a_j + m_2 a_i b_j), \]
\[ b_{k,x} = a_k - \sum_{i+j=k-1} (n_1 a_i b_j + m_2 b_i b_j), \quad (k \geq 3). \]

With the aid of \( a_1, b_1 \), we obtain a simple conserved quantity

\[ \rho_1 = - \int (n_1 g_1 + m_2 f_2) dx. \] (32)

Also, due to

\[ a_3 - a_{3xx} = n_1 f_2 g_1 + m_2 f_2^2 + (n_1 g_1^2 + m_2 f_2 g_1)_x, \]
\[ b_3 - b_{3xx} = n_1 g_1^2 + m_2 f_2 g_1 + (n_1 f_2 g_1 + m_2 f_2^2)_x, \]
we obtain the next conserved quantity by

\[ \rho_3 = \int n_1 a_3 + m_2 b_3 dx = \int (v_1 (a_3 - a_{3x}) + u_2 (b_3 - b_{3x})) dx = \int (n_1 g_1 + m_2 f_2)(f_1 g_1 + f_2 g_2) dx. \]  

(33)

In addition, we may consider alternative expansions of \(a, b\) in negative powers of \(\lambda\), namely

\[ a = \sum_{i \geq 0} \tilde{a}_i \lambda^{-i}, \quad b = \sum_{j \geq 0} \tilde{b}_j \lambda^{-j}. \]

As above, inserting these expansions into (31) we may find recursive relations for \(\tilde{a}_i, \tilde{b}_j\). The first two conserved quantities are

\[ \rho_0 = \int \sqrt{m_1 n_1 + m_2 n_2} dx, \]  

(34)

\[ \rho_{-1} = \int \frac{2m_1 m_2 + 2n_1 n_2 + m_1 n_1 - m_1 n_1}{4(m_1 n_1 + m_2 n_2)} dx. \]

**Case 2:** The quantity \(\bar{\rho}\) defined as

\[ \bar{\rho} = (\ln \varphi_1)_x = \lambda m_1 \sigma + \tau, \]

(35)

with \(\sigma, \tau\) satisfy

\[\sigma_x = \lambda n_1 + \lambda m_2 \tau - \sigma \bar{\rho}, \quad \tau_x = 1 + \lambda n_2 \sigma - \tau \bar{\rho}.\]

(36)

is conserved quantity. Expanding \(\sigma\) and \(\tau\) in Laurent series of \(\lambda\) then once again we may find the corresponding conserved quantities. For instance, in the case \(k \geq 0\), we get

\[ \bar{\rho}_2 = \frac{1}{2} \int (m_1 + n_2)(f_1 + g_2) dx, \]

while in the case \(k \leq 0\), we obtain

\[ \bar{\rho}_{-1} = \int \frac{2m_2 n_2^2 + 2m_1 n_1 n_2 - m_2 n_1 n_2 + 4m_1 m_2 n_2 - 3n_2 m_1 m_2 + m_1 m_1 n_1 - n_1 m_1^2}{4m_1 (m_1 n_1 + m_2 n_2)} dx. \]

**Case 3:** For this case the conserved quantity is defined as

\[ \hat{\rho} = (\ln \varphi_3)_x = \alpha + \lambda n_2 \beta, \]

(37)

with \(\alpha, \beta\) satisfy

\[\alpha_x = \lambda m_1 \beta + 1 - \alpha \hat{\rho}, \quad \beta_x = \lambda n_1 \alpha + \lambda m_2 - \beta \hat{\rho}.\]

(38)

Expanding \(\alpha\) and \(\beta\) in Laurent series of \(\lambda\) and substituting them into (38), we may obtain the conserved quantities and apart from those found in last two cases, we have

\[ \hat{\rho}_{-1} = \int \frac{2n_1 n_2^2 + 2m_1 m_2 n_2 - m_2 n_1 n_2 + 4n_2 m_1 n_1 - 3m_1 n_1 n_2 + n_2 m_2 n_2 - n_1 m_1 n_2}{4n_2 (m_1 n_1 + m_2 n_2)} dx. \]

Let us remark that these conserved quantities have been obtained from the \(x\)-part of the Lax pair representation only hence they are valid for the whole hierarchy. As we checked they are conserved for the system (13) as well as for the (22).
4. Reductions

We now consider the possible reductions of our four component spectral problem (14) and relate them to the spectral problems known in literatures.

4.1. A three component reduction

Assuming \( m_1 = u_1 = 0 \), we have

\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}_x = \begin{pmatrix}
0 & 0 & 1 \\
\lambda n_1 & 0 & \lambda m_2 \\
1 & \lambda n_2 & 0
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix},
\]

which, by identifying the variables as follows

\[
n_2 = u, \quad m_2 = v, \quad n_1 = w + (v/u)_x,
\]

may be reformulated as

\[
\begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{pmatrix}_x = \begin{pmatrix}
0 & 1 & 0 \\
1 + \lambda^2 v & 0 & u \\
\lambda^2 w & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\varphi_1 \\
\varphi_2 \\
\varphi_3
\end{pmatrix}.
\]

This spectral problem is nothing but the one proposed by Geng and Xue [9] and the associated integrable flows are also bi-Hamiltonian [13]. It is easy to see that (40) is an invertible transformation, so to find the bi-Hamiltonian structures of the flows resulted from the spectral problem (39) we may convert those for the Geng-Xue spectral problem into the present case. Direct calculations yield

\[
L_1 = \begin{pmatrix}
-\frac{m_2}{n_2} \partial - \partial \frac{m_2}{n_2} - S^* \partial S + S^* \partial + \partial \left( \frac{m_2}{n_2} \partial + \partial \frac{m_2}{n_2} \partial \right) - 1 - \partial^2 \\
-\partial \left( \frac{m_2}{n_2} \partial + \partial \frac{m_2}{n_2} \partial \right) - S^* \partial S + S^* \partial + \partial \left( \frac{m_2}{n_2} \partial + \partial \frac{m_2}{n_2} \partial \right) - 1 - \partial^2 \\
0
\end{pmatrix},
\]

\[
L_2 = -\frac{1}{2} \begin{pmatrix}
m_2 \partial + 2m_2x \\
m_2 \partial + 3n_1 \partial + 2n_1x \\
3n_2 \partial + 2n_2x
\end{pmatrix} \left( \partial^3 - 4\partial \right)^{-1} \begin{pmatrix}
m_2 \partial + 2m_2x \\
m_2 \partial + 3n_1 \partial + 2n_1x \\
3n_2 \partial + 2n_2x
\end{pmatrix}^*,
\]

where \( S = \frac{m_2}{n_2} (1 - \partial^2) \). We remark that it is not clear how to obtain above Hamiltonian pair from [19].
4.2. a two-component reduction

In this case, we assume
\[ n_1 = m_2, \quad n_2 = m_1 \]
or
\[
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix}_x =
\begin{pmatrix}
0 & \lambda m_1 & 1 \\
\lambda m_2 & 0 & \lambda m_2 \\
1 & \lambda m_1 & 0
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2 \\
\phi_3
\end{pmatrix},
\]
which yields
\[
\begin{pmatrix}
\phi_1 + \phi_3 \\
\phi_2
\end{pmatrix}_x =
\begin{pmatrix}
1 & 2\lambda m_1 \\
\lambda m_2 & 0
\end{pmatrix}
\begin{pmatrix}
\phi_1 + \phi_3 \\
\phi_2
\end{pmatrix}.
\] (41)

By the following change of variables
\[ \phi_1 + \phi_3 = e^{\frac{1}{2}x} \psi_1, \quad \phi_2 = e^{\frac{1}{2}x} \psi_2, \quad 2m_1 = m, \quad m_2 = n, \]
(41) gives
\[
\begin{pmatrix}
\psi_1 \\
\psi_2
\end{pmatrix}_x =
\begin{pmatrix}
\frac{1}{2} & \lambda m \\
\lambda n & -\frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
\phi_1 \\
\phi_2
\end{pmatrix},
\]
a spectral problem considered by Song, Qu and Qiao [21]. Therefore the bi-hamiltonian structure of Song-Qu-Qiao system (see [22]) is hidden in our bi-hamiltonian structure.

5. Concluding remarks

In this paper, started from a general 3×3 problem, we considered the related four-component CH type systems. We obtained the bi-Hamiltonian structure and suggested the way to construction of infinitely many conserved quantities for the integrable equations. Different reductions were also considered.

As noticed above, the positive flows allow for an arbitrary function \( \Gamma \) involved and such systems are interesting since different specifications of \( \Gamma \) lead to different CH type equations. Although the flow equations with arbitrary \( \Gamma \) do possess infinitely many conserved quantities, we do not expect they are (bi-) Hamiltonian systems in general case. Also, we explained the appearance of this arbitrary function by studying the kernel of one of the Hamiltonian operators and it seems that further study of such systems is needed.

A remarkable property of CH type equations is that it possess peakon solutions. One may find that the first negative flow, which does not depend on \( \Gamma \), only possess stationary peakons. The systems such as [22] may admit non-stationary peakon solutions. This and other related issues may be considered in further publication.

Appendix A.

We first prove that \( \mathcal{L} \) is also a Hamiltonian operator. For this purpose, we first define
\[
\Theta_J = \frac{1}{2} \int \theta \wedge J \theta dx, \quad \Theta_F = \frac{1}{2} \int \theta \wedge F \theta dx, \quad \mathcal{A} = (\partial^3 - 4\partial)^{-1}.
\]
By direct calculation, we have

\[\Theta_J = \int \left( (n_2 \theta_1 - n_1 \theta_2) \wedge \partial^{-1}(m_1 \theta_4 - m_2 \theta_3) + (m_1 \theta_1 - n_1 \theta_3) \wedge \partial^{-1}(n_2 \theta_4 - m_2 \theta_2) \\
+ (n_2 \theta_4 - m_2 \theta_2) \wedge \partial^{-1}(n_2 \theta_4 - m_2 \theta_2) + (m_1 \theta_1 - n_1 \theta_3) \wedge \partial^{-1}(m_1 \theta_1 - n_1 \theta_3) \right) dx,\]

and

\[\Theta_F = \frac{1}{2} \int \left( (2Q - R \partial) \wedge AQ + (Q \partial - 2R) \wedge AR \right) dx \]

\[= \int (Q \wedge AQ - R \wedge AR + Q \wedge \partial AR) dx,\]

where

\[Q = m_1 \theta_1 + m_2 \theta_2 - n_1 \theta_3 - n_2 \theta_4, \quad R = n_2 \theta_1 - n_1 \theta_2 + m_2 \theta_3 - m_1 \theta_4.\]

Since

\[\pr v_{\mathcal{E}_0}(\Theta_J) = \pr v_{\mathcal{E}_0}(\Theta_J) + \pr v_{\mathcal{E}_0}(\Theta_F),\]

we calculate \(\pr v_{\mathcal{E}_0}(\Theta_J)\) and \(\pr v_{\mathcal{E}_0}(\Theta_F)\). Indeed, we have

\[-\pr v_{\mathcal{E}_0}(\Theta_J) = \int \left[ \left( \partial^{-1}(n_2 \theta_1 - n_1 \theta_2) \wedge \partial^{-1}(m_1 \theta_4 - m_2 \theta_3) \right) \wedge \partial^{-1}Q \right] dx \]

\[+ (2m_1 \theta_1 + 2n_1 \theta_3 - n_2 \theta_4 - m_2 \theta_2) \wedge (AR_x + 2AQ) \wedge \partial^{-1}(m_1 \theta_1 - n_1 \theta_3) \]

\[+ (2m_2 \theta_3 - 2n_2 \theta_1 + m_1 \theta_4 - n_1 \theta_2) \wedge (AQ_x + 2AR) \wedge \partial^{-1}(m_1 \theta_1 - n_1 \theta_3) \]

\[+ (2m_1 \theta_4 - 2n_1 \theta_2 + m_2 \theta_3 - n_2 \theta_1) \wedge (AQ_x + 2AR) \wedge \partial^{-1}(m_1 \theta_1 - n_1 \theta_3) \]

\[+ (m_1 \theta_1 + n_1 \theta_3 - 2m_2 \theta_2 - 2n_2 \theta_4) \wedge (AR_x + 2AQ) \wedge \partial^{-1}(n_2 \theta_4 - m_2 \theta_2) \]

\[+ (m_1 \theta_4 - m_2 \theta_3) \wedge (AR_x + 2AQ) \wedge \partial^{-1}(n_2 \theta_1 - n_1 \theta_2) \]

\[- (n_1 \theta_3 + n_2 \theta_4) \wedge (AQ_x + 2AR) \wedge \partial^{-1}(n_2 \theta_1 - n_1 \theta_2) \]

\[+ (m_1 \theta_1 + m_2 \theta_2) \wedge (AQ_x + 2AR) \wedge \partial^{-1}(m_1 \theta_4 - m_2 \theta_3) \]

\[- (n_2 \theta_1 - n_1 \theta_2) \wedge (AR_x + 2AQ) \wedge \partial^{-1}(m_1 \theta_4 - m_2 \theta_3)] dx.\]

Next we consider \(\pr v_{\mathcal{E}_0}(\Theta_F)\). For simplicity, we denote

\[\Upsilon = 2m_1 \theta_1 - m_2 \theta_2 - 2n_1 \theta_3 + n_2 \theta_4, \quad \Omega = m_1 \theta_1 - 2m_2 \theta_2 - n_1 \theta_3 + 2n_2 \theta_4,\]
then a direct calculation shows that $pr\, v_{\mathcal{L}\theta}(\Theta_F)$ can be expressed as

$$-pr\, v_{\mathcal{L}\theta}(\Theta_F) = \int [(m_1\theta_1 + n_1\theta_2) \wedge \partial^{-1} Y \wedge (2AQ + AR_x)$$

$$+ 2(n_2\theta_1 - n_1\theta_2 - m_2\theta_3 + m_1\theta_4) \wedge (AR_x + 2AQ) \wedge (AQ_x + 2AR)$$

$$- (m_2\theta_2 + n_2\theta_4) \wedge \partial^{-1} \Omega \wedge (2AQ + AR_x)$$

$$+ (n_2\theta_1 - n_1\theta_2) \wedge \partial^{-1}(m_1\theta_1 - m_2\theta_3) \wedge (2AQ + AR_x)$$

$$- (m_2\theta_3 - m_2\theta_3) \wedge \partial^{-1}(n_2\theta_1 - n_1\theta_2) \wedge (2AQ + AR_x)$$

$$+ (m_2\theta_4 - n_1\theta_4) \wedge \partial^{-1} Y \wedge (2AR + AQ_x)$$

$$- (n_2\theta_1 - n_2\theta_2) \wedge \partial^{-1}\Omega \wedge (2AR + AQ_x)$$

$$+ (2AQ + AR_x) \wedge (AQ_x + 2AR + AQ_x)] \, dx.$$  

Letting $f = n_2\theta_1 - n_1\theta_2, g = m_1\theta_4 - m_2\theta_3$ and substituting above expansions into (??) lead to

$$-pr\, v_{\mathcal{L}\theta}(\Theta_L) = \int 2(n_2\theta_1 - n_1\theta_2 - m_2\theta_3 + m_1\theta_4) \wedge (AR_x + 2AQ) \wedge (AQ_x + 2AR)$$

$$+ 2((n_2\theta_1 - n_1\theta_2) \wedge \partial^{-1}(m_1\theta_1 - m_2\theta_3)$$

$$+ (m_2\theta_1 - m_1\theta_4) \wedge \partial^{-1}(n_2\theta_1 - n_1\theta_2) \wedge (2AQ + AR_x)$$

$$+ ((n_2\theta_1 - n_1\theta_2 - m_2\theta_3 + m_1\theta_4) \wedge \partial^{-1} Q$$

$$+ \partial^{-1}(n_2\theta_1 - n_1\theta_2 - m_2\theta_3 + m_1\theta_4) \wedge Q) \wedge (2AR + AQ_x) \, dx$$

$$= \int (f \wedge \partial^{-1} 2g + \partial^{-1} f \wedge 2g) \wedge (2AQ + AR_x)$$

$$+ 2(f + g) \wedge (AR_x + 2AQ) \wedge (AQ_x + 2AR)$$

$$+ ((f + g) \wedge \partial^{-1} Q + \partial^{-1}(f + g) \wedge Q) \wedge (2AR + AQ_x) \, dx$$

$$= \int \partial^{-1}(2g + R) \wedge (Q \wedge (AQ_x + 2AR) - R \wedge (AR_x + 2AQ))$$

$$+ (2g + R) \wedge (-\partial^{-1} R \wedge (AR_x + 2AQ) + \partial^{-1} Q \wedge (AQ_x + 2AR)$$

$$+ 2(AR_x + 2AQ)(AQ_x + 2AR) \, dx$$

$$= \int \partial^{-1}(2g + R) \wedge (\partial^{-1} R \wedge (AR_x + 2AQ_x) - \partial^{-1} Q \wedge (AQ_x + 2AR_x)$$

$$- 2(AR_x + 2AQ)(AQ_x + 2AR) = 0.$$  

where we use $f - g = R$ for short. Thus, $\mathcal{L}$ given by (??) is a Hamiltonian operator.

Finally we prove the compatibility of $\mathcal{K}$ and $\mathcal{L}$, which is equivalent to

$$pr\, v_{\mathcal{L}\theta}(\Theta_K) + pr\, v_{\mathcal{K}\theta}(\Theta_L) = pr\, v_{\mathcal{K}\theta}(\Theta_K) = pr\, v_{\mathcal{K}\theta}(\Theta_J) + pr\, v_{\mathcal{K}\theta}(\Theta_F) = 0. \quad (A2)$$

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To this end, we notice

\[-pr v_{K\theta}(\Theta_J) = \int (2(\theta_4 \wedge \theta_2)x + (\theta_1 \wedge \theta_3)x + \theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4) \wedge \partial^{-1}(n_2 \theta_4 - m_2 \theta_2)\]

\[+((\theta_4 \wedge \theta_2)x + 2(\theta_1 \wedge \theta_3)x - \theta_1 \wedge \theta_2 - \theta_3 \wedge \theta_4) \wedge \partial^{-1}(m_1 \theta_1 - n_1 \theta_3)\]

\[+((\theta_1 \wedge \theta_2)x - \theta_1 \wedge \theta_3 - \theta_2 \wedge \theta_4) \wedge \partial^{-1}(m_1 \theta_4 - m_2 \theta_3)\]

\[+((\theta_4 \wedge \theta_3)x + \theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4) \wedge \partial^{-1}(n_2 \theta_1 - n_1 \theta_2)dx\]

\[= \int -((\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4) \wedge \partial^{-1}Q + (\theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4) \wedge \partial^{-1}R\]

\[+((2\theta_4 \wedge \theta_2 + \theta_1 \wedge \theta_3) \wedge (n_2 \theta_4 - m_2 \theta_2) - (\theta_1 \wedge \theta_2) \wedge (m_1 \theta_4 - m_2 \theta_3)\]

\[-(\theta_4 \wedge \theta_2 + 2\theta_1 \wedge \theta_3) \wedge (m_1 \theta_1 - n_1 \theta_3) + (\theta_3 \wedge \theta_4) \wedge (n_2 \theta_1 - n_1 \theta_2)dx\]

\[= \int -((\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4) \wedge \partial^{-1}Q + (\theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4) \wedge \partial^{-1}Rdx.\]

and

\[-pr v_{K\theta}(\Theta_F) = \int (\theta_1 \wedge (\theta_3x - \theta_2) + \theta_2 \wedge (\theta_1 + \theta_4x) - \theta_3 \wedge (\theta_1x + \theta_4) - \theta_4 \wedge (\theta_2x - \theta_3))\]

\[\wedge(2AQ + \partial AR) + (AQ_2 + 2AR) \wedge ((\theta_2x - \theta_3) \wedge \theta_1 - (\theta_1x + \theta_4) \wedge \theta_2\]

\[+((\theta_1 + \theta_4x) \wedge \theta_3 - (\theta_3x - \theta_2) \wedge \theta_4)dx.\]

\[= \int ((\partial^2 - 4)(\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4)) \wedge AQ - ((\partial^2 - 4)(\theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4)) \wedge ARdx\]

\[= \int (\theta_1 \wedge \theta_2 + \theta_3 \wedge \theta_4) \wedge \partial^{-1}Q - (\theta_1 \wedge \theta_3 + \theta_2 \wedge \theta_4) \wedge \partial^{-1}Rdx.\]

Therefore we arrive at

\[pr v_{K\theta}(\Theta_L) = pr v_{K\theta}(\Theta_J + \Theta_F) = 0,\]

so $\mathcal{K}$ and $\mathcal{L}$ are two compatible Hamiltonian operators.

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