Maximal lattice free bodies, test sets and the Frobenius problem

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Abstract

Maximal lattice free bodies are maximal polytopes without interior integral points. Scarf initiated the study of maximal lattice free bodies relative to the facet normals in a fixed matrix. In this paper we give an efficient algorithm for computing the maximal lattice free bodies of an integral \((d+1) \times d\) matrix \(A\). An important ingredient is a test set for a certain integer program associated with \(A\). This test set may be computed using algebraic methods.

As an application we generalize the Scarf-Shallcross algorithm for the three-dimensional Frobenius problem to arbitrary dimension. In this context our method is inspired by the novel algorithm by Einstein, Lichtblau, Strzebonski and Wagon and the Gröbner basis approach by Roune.

1 Introduction

We will introduce this paper by relating it to the Frobenius problem. Let \(\mathbb{N}\) denote the natural numbers and \(\mathbb{Z}\) the integers. For \(a_1, \ldots, a_n \in \mathbb{N}\) with \(\mathbb{Z}a_1 + \cdots + \mathbb{Z}a_n = \mathbb{Z}\), the complement of the semigroup \(\mathbb{N}a_1 + \cdots + \mathbb{N}a_n\) in \(\mathbb{N}\) is finite. Its maximum is denoted \(g(a_1, \ldots, a_n)\) and called the Frobenius number of \(a_1, \ldots, a_n\). Consider as an example, the semigroup \(S\) generated by 6, 10 and 15 in \(\mathbb{N}\). The complement of \(S\) is

\[\{1, 2, 3, 4, 5, 7, 8, 9, 11, 13, 14, 17, 19, 23, 27, 29\}.\]
and $g(6, 10, 15) = 29$. Computing the Frobenius number is in general an NP-hard problem, which is not known to be in NP (see [11, Appendix A]). This phenomenon is perhaps related to the naive method of computing the Frobenius number by finding the first consecutive sequence of $a_1$ natural numbers in the semigroup. The search for a consecutive sequence can be simplified through the following classical result due to Brauer and Shockley [5]: let $r_f$ be the smallest natural number of the form
\[x_2a_2 + \cdots + x_na_n\]
congruent to a given integer $f$ modulo $a_1$, where $x_2, \ldots, x_n \in \mathbb{N}$. Then
\[g(a_1, \ldots, a_n) = -a_1 + \max\{r_f \mid 0 < f < a_1\}\]  
(1)
The reader may find it useful to deduce the classical result
\[g(a_1, a_2) = a_1a_2 - a_1 - a_2\]
due to Sylvester (1884) as a special case.

In the language of optimization, (1) amounts to solving $a_1 - 1$ group programs\(^1\). These group programs may be interpreted as finding shortest paths from 0 in a graph with vertices $\mathbb{Z}/a_1\mathbb{Z}$ and edges suitably weighted by $a_2, \ldots, a_n$. This is basically the graph algorithm in [10] for computing the Frobenius number (see also [4]). The graph algorithms have the obvious draw-back that they are exponential in the input size in fixed dimension.

Surprisingly there are polynomial algorithms in fixed dimension related to the theory of integral points in convex polytopes. The first such algorithm was found by Kannan [8]. Later a remarkable algorithm related to rational generating functions was discovered by Barvinok and Woods [3]. Both of these algorithms involve deep mathematical insights, but use quite time consuming operations of polynomial complexity in fixed dimension.

In this paper we present an algorithm for enumerating maximal lattice free bodies with facet normals in the rows of a fixed integral matrix. A maximal lattice free body is a polytope maximal with respect to having no interior integral points. As an application we generalize the Scarf-Shallcross algorithm [13] for computing the Frobenius number for three natural numbers.

\(^1\)The notion of group programs and (graph) algorithms for solving them go back to Gomory [7].
Let us briefly recall the beautiful relation between maximal lattice free bodies and the Frobenius number: given the coprime natural numbers \( a = (a_1, \ldots, a_n)\), Scarf and Shallcross introduced the polytopes

\[
K_b = \{ x \in \mathbb{R}^{n-1} \mid Ax \leq b \}
\]

for \( b \in \mathbb{Z}^n \), where \( A \) is an \( n \times (n - 1) \) matrix with columns forming a basis of the lattice \( \{ v \in \mathbb{Z}^n \mid av = 0 \} \). The Frobenius number is then given by

\[
g(a_1, \ldots, a_n) = \max\{ ab \mid b \in \mathbb{Z}^n, K_b \text{ maximal lattice free body} \} - \sum a_i.
\]

For \( 3 \times 2 \) matrices Scarf and Shallcross gave a very efficient algorithm for computing the maximal lattice free bodies up to integral translation and thereby the Frobenius number for \( n = 3 \).

For integral \( n \times (n - 1) \) matrices our algorithm generalizes the Scarf-Shallcross algorithm. In the context of the Frobenius problem, an algebraic version of our algorithm has led to record breaking computations [12]. It is possible to extend our algorithm to compute the maximal lattice free bodies for arbitrary integral matrices by working with ideal points on facets as in [1]. We only treat the simplicial case in this paper.

This work was originally prompted by an attempt to understand the novel algorithm of Einstein, Lichtblau, Strzebonski and Wagon [6] through the geometric language in [13]. We thank Daniel Lichtblau and Stan Wagon for several discussions related to [6].

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\section{Preliminaries}

We begin by recalling a few concepts and a bit of notation from polyhedral geometry. First we define the \( m \)-dimensional vector \( \max(b_1, \ldots, b_r) \) as the
coordinate-wise maximum of the vectors $b_1, \ldots, b_r \in \mathbb{R}^m$. For example

$$\max \left( \begin{pmatrix} -1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix}, \begin{pmatrix} -5 \\ -3 \\ 6 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 4 \\ 8 \end{pmatrix}.$$ 

2.1 Polyhedra

A polyhedron $P$ in $\mathbb{R}^n$ is the set of solutions to a finite number of linear inequalities in $n$ variables: $a_i^t x \leq b_i$ for $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$, where $i = 1, \ldots, m$. We will use the notation $P_A(b) := P = \{x \in \mathbb{R}^n \mid Ax \leq b\}$ collecting the normal vectors $a^t_1, \ldots, a^t_m$ in the rows of an $m \times n$ matrix $A$ and the bounds $b_i$ in a vector $b \in \mathbb{R}^m$. Notice that some of the inequalities may be redundant in defining $P_A(b)$. If $a^t_i x \leq b_i$ is redundant we call it an inactive facet.

A polyhedron of the form $P_A(0)$ is called a polyhedral cone. A polyhedral cone is called pointed if it does not contain a line i.e. if Ker($A$) = 0. A polyhedron, $P_A(b)$, is called rational if the entries of $A$ and $b$ are rational numbers.

Suppose $A$ is fixed. Then the smallest polyhedron containing $v_1, \ldots, v_r \in \mathbb{R}^n$ is given by

$$\langle v_1, \ldots, v_r \rangle_A := P_A(\max(Av_1, \ldots, Av_r)).$$

3 Integer programs and test sets

Consider a cost vector $c \in \mathbb{R}^n$. We let IP$_{A,c}(b)$ denote the integer linear program

$$\min \{ c^t x \mid x \in P_A(b) \cap \mathbb{Z}^n \} = \min \{ c^t x \mid Ax \leq b, x \in \mathbb{Z}^n \},$$

where $b \in \mathbb{R}^m$ and $A \in \mathbb{R}^{m \times n}$.

Informally a test set for an integer program is a finite set of integral (test) vectors, such that a feasible solution is optimal if it cannot be improved by moving in the direction of any of the test vectors. Here is the precise definition.

**Definition 3.1.** A test set for the family IP$_{A,c} = \{\text{IP}_{A,c}(b) \mid b \in \mathbb{R}^m\}$ of integer programs is a finite set $T \subset \mathbb{Z}^n$ such that
(a) \( c'v < 0 \) for every \( v \in T \).

(b) \( v_0 \in P_A(b) \cap \mathbb{Z}^n \) is optimal if and only if \( v_0 + v \notin P_A(b) \) for every \( v \in T \).

Test sets always exist when \( A \) is an integral (or rational) matrix. This follows for example from the following result due to Cook, Gerards, Schrijver and Tardos.

**Theorem 3.2 ([14, §17.3]).** Let \( A \) be an integral \( m \times n \) matrix, with all subdeterminants at most \( \Delta \) in absolute value, let \( b \) be a column \( m \)-vector and \( c \) a row \( n \)-vector. Let \( z \) be a feasible, but not optimal, solution of \( \max \{ cx | Ax \leq b; x \text{ integral} \} \). Then there exists a feasible solution \( z' \) such that \( cz' > cz \) and \( \| z - z' \|_\infty < n\Delta \).

For irrational matrices the existence of test sets is more subtle. This issue has been addressed in Scarf’s theory of neighbors [2]. The following simple lemma will be applied in §6.2.

**Lemma 3.3.** Let \( U \in \text{GL}_m(\mathbb{Z}) \). Then \( T \subset \mathbb{Z}^m \) is a test set for \( \text{IP}_{A,c}(b) \) if and only if \( U^{-1}T \) is a test set for \( \text{IP}_{AU,cU}(b) \).

### 3.1 Conversion to group programs

Suppose that \( A \) is an integral \( m \times n \) matrix and \( b \) an integral (column) \( m \)-vector. The integer program \( \text{IP}_{A,c}(b) \) may be transcribed in the following way. A feasible solution \( x \in \mathbb{Z}^n \) to \( \text{IP}_{A,c}(b) \) corresponds to a vector \( u \in \mathbb{N}^m \) with \( Ax + u = b \). If we let \( \mathcal{L} \) denote the subgroup of \( \mathbb{Z}^m \) generated by the columns of \( A \), this means that \( \text{IP}_{A,c}(b) \) may be formulated as the group program \( \text{Grp}_{\mathcal{L},c'}(b) \):

\[
\begin{align*}
\min & \quad (c')^t u \\
\text{subject to} & \quad u \equiv b \pmod{\mathcal{L}} \\
& \quad u \in \mathbb{N}^n,
\end{align*}
\]

where \( c' \in \mathbb{R}^m \) is some cost vector corresponding to \( c \in \mathbb{R}^n \) (recovering \( x \) from \( Ax + u = b \)). In complete analogy with \( \text{IP}_{A,c}(b) \) we have the following definition.

**Definition 3.4.** A test set for the family \( \{ \text{Grp}_{\mathcal{L},c'}(b) \mid b \in \mathbb{Z}^m \} \) of group programs is a finite set \( T \subset \mathcal{L} \) such that...
(a) \((c')^t v < 0\) for every \(v \in T\).

(b) A vector \(u_0 \in \mathbb{N}^n\) with \(u_0 \equiv b \mod \mathcal{L}\) is optimal if and only if

\[ u_0 + v \not\in \mathbb{N}^n \]

for every \(v \in T\).

We use the term group program (as opposed to lattice program) since the optimization problem (2) concerns optimizing a function over certain representatives of a coset in the group \(\mathbb{Z}^n/\mathcal{L}\).

In the following we recall how test sets for group programs may be constructed using Hilbert bases for rational cones. This gives in principle an algorithm for computing the test set alluded to in Theorem 3.2.

### 3.2 Hilbert bases and test sets for group programs

A pointed polyhedral cone \(C \subset \mathbb{R}^n\) carries a natural partial order \(<\) given by

\[ x < y \iff y - x \in C \]

for \(x, y \in C\). If \(C\) is rational and \(\mathcal{L}\) is a subgroup of \(\mathbb{Z}^n\), then the semigroup \(\mathcal{L} \cap C\) has finitely many \(<\)-minimal elements (see [14, §16.4]). These elements are called the Hilbert basis of \(\mathcal{L} \cap C\) and denoted \(\mathcal{H}(\mathcal{L} \cap C)\). Every element of \(\mathcal{L} \cap C\) is a finite non-negative integral linear combination of the elements in \(\mathcal{H}(\mathcal{L} \cap C)\).

The Euclidean space \(\mathbb{R}^m\) decomposes into the \(2^m\) orthants \(\{O_j\}_{j=1}^{2^m}\), which are pointed rational cones. For a lattice \(\mathcal{L} \subset \mathbb{Z}^m\) we put

\[ \text{Gr}(\mathcal{L}) = \bigcup_{j=1}^{2^m} \mathcal{H}(\mathcal{L} \cap O_j). \]

This finite set of vectors is called the Graver basis of \(\mathcal{L}\). It is not too difficult to prove that \(\{v \in \text{Gr}(\mathcal{L}) \mid c^tv < 0\}\) is a test for the family \(\{\text{Grp}_{\mathcal{L},c}(b) \mid b \in \mathbb{Z}^m\}\) of group programs. The full Graver basis is a test set for the bigger family \(\{\text{Grp}_{\mathcal{L},c}(b) \mid c \in \mathbb{R}^m, b \in \mathbb{Z}^m\}\) of group programs, where both \(b\) and \(c\) are allowed to vary.
3.3 Generic cost vectors and Gröbner bases

Sufficiently generic cost vectors $c \in \mathbb{R}^m$ may be viewed as term orders in computational algebra. The term sufficiently generic includes the case of linearly independent entries over $\mathbb{Q}$ in $c$. In practice we want $c^t v \neq 0$ for $v \in \mathcal{L}$ inside a sufficiently big ball centered at 0. Usually the cost vector is made generic by breaking ties with a term order $\prec$. We will not make this precise here but refer the reader to the reference given at the end of this subsection.

The lattice ideal $I_{\mathcal{L}}$ associated to $\mathcal{L}$ is defined as the ideal generated by the binomials

$$\{u^{v^+} - u^{v^-} \mid v \in \mathcal{L}\} \subset \mathbb{Q}[u_1, \ldots, u_m]$$

where $v \in \mathbb{Z}^m$ is decomposed into vectors $v^+, v^- \in \mathbb{N}^m$ with disjoint support such that $v = v^+ - v^-$ and $u^w$ denotes the monomial $u_1^{w_1} \cdots u_m^{w_m}$ for $w = (w_1, \ldots, w_n) \in \mathbb{N}^m$. In this context we have the following result.

**Theorem 3.5.** Let $c$ be a sufficiently generic vector in $\mathbb{R}^m$. Then a minimal Gröbner basis of $I_{\mathcal{L}}$ with respect to the term order given by $-c$ is

$$\{u^{v^+} - u^{v^-} \mid v \in T\},$$

where $T \subset \mathcal{L}$ is a test set for the family $\{\text{Grp}_{\mathcal{L},c}(b) \mid b \in \mathbb{Z}^m\}$ of group programs.

This result is so far the most efficient way of computing test sets for integer programs: the algebraic viewpoint makes it possible to apply highly optimized algorithms for computing Gröbner bases of lattice ideals. A state of the art implementation is in the program 4ti2 [http://www.4ti2.de], which also contains functions for computing and manipulating test sets for integer programs.

For further information on the relation between Gröbner bases and test sets we refer the reader to [15].

4 Maximal lattice free bodies

A convex body is a compact convex subset of $\mathbb{R}^n$ with non-empty interior. We call a convex body lattice free if its interior does not contain any integral points. For a beautiful exposition of lattice free bodies we refer to §3 in the survey by Lovasz [9].
It is known that a maximal lattice free convex body $B$ is a polytope [9, Proposition 3.2]. A surprising result due to Bell and Scarf says that $B$ has at most $2^n$ facets [14, §16.5]. A very useful characterization of maximal lattice free bodies is contained in the following result.

**Proposition 4.1.** A polytope is maximal lattice free if and only if its interior does not contain any integral points and every facet contains an integral point in its relative interior.

Clearly, $\mathbb{Z}^n$ acts on the set of maximal lattice free bodies in $\mathbb{R}^n$ by translation. We let $\mathcal{M}(A)$ denote the set of maximal lattice free bodies of the form $P_A(b)$, where $A$ is an $m \times n$ matrix and $b \in \mathbb{R}^m$ is the right hand side. These maximal lattice free bodies are the ones with facet normals in the rows of $A$. For varying $b$, several of the rows in $A$ may define inactive facets. The $\mathbb{Z}^n$-action on maximal lattice free bodies restricts to an action on $\mathcal{M}(A)$ by

$$z + P_A(b) = P_A(b + Az).$$

It is reasonable to identify maximal lattice free bodies which differ by an integral translation. With this convention we have the following result.

**Theorem 4.2.** Let $A$ be an integral $m \times n$ matrix. Up to integral translation, $A$ has finitely many maximal lattice free bodies i.e.

$$\mathcal{M}(A)/\mathbb{Z}^n$$

is a finite set.

**Proof.** Let $P_A(b) \in \mathcal{M}(A)$. We may assume without loss of generality that all facets of $P_A(b)$ are active. Let $a_1, \ldots, a_m$ denote the rows of $A$. By Proposition 4.1 and integral translation we may also assume that the facet corresponding to $a_1$ contains 0 in its relative interior. With these reductions, we may assume that $b = (b_1, \ldots, b_m)^t$ with $b_1 = 0$ and $b_i > 0$ for $i > 1$. We now prove that there can only be finitely many maximal lattice free bodies of the form $P_A(b)$.

Let $T$ be the test set in Theorem 3.2 associated with the matrix $A$. Consider the integer program

$$\min \{a_1 x \mid Ax \leq b, x \in \mathbb{Z}^n\}, \quad (3)$$

Let $\epsilon > 0$ be sufficiently small. Since $P_A(b)$ has no interior integral points, the integer program given by changing the right hand side in (3) according to
\( b_2 := b_2 - \epsilon, \ldots, b_m := b_m - \epsilon \) has 0 as optimal solution. On the other hand 0 is a feasible but not optimal solution for the integer program in (3) when the right hand side is changed according to \( b_2 := b_2 - \epsilon, \ldots, b_j := b_j, \ldots, b_m := b_m - \epsilon \), since \( P_A(b) \) contains an integral point in the relative interior of the \( a_j \)-facet. By definition of a test set, there exists \( z_j \in T \) with \( a_1 z_j < 0, a_i z_j < b_i \) for \( j \neq i \) and \( a_j z_j = b_j \). This shows that a certain subset \( \{ z_2, \ldots, z_r \} \) with \( r - 1 \) elements of \( T \) uniquely defines \( P_A(b) \) up to integral translation. Since there are finitely many such subsets, this proves the claim.

**Example 4.3.** The \( 4 \times 2 \) matrix

\[
A = \begin{pmatrix}
1 & 0 \\
-1 & 0 \\
0 & 1 \\
0 & -1
\end{pmatrix}
\]

has \( M(A) = \emptyset \).

An interesting question is when \( M(A) \neq \emptyset \)? This is known to hold if \( A \) has full rank and every \( m \times m \) minor is non-vanishing. This condition obviously fails in Example 4.3.

### 4.1 The simplicial case

The proof of Theorem 4.2 shows that \( M(A) / \mathbb{Z}^n \) can be computed using a test set for the family

\[
\{ \text{IP}_{A, a_1}(b) \mid b \in \mathbb{R}^m \}
\]

of integer programs, where \( a_1 \) denotes the first row of \( A \). We will describe an algorithm for doing this in the more tractable case related to the Frobenius problem: assume that \( A \) is an integral \( (d + 1) \times d \) matrix of full rank such that \( yA = 0 \), for some \( y \in \mathbb{N}_{>0}^{d+1} \). If \( a_0, \ldots, a_d \) are the rows of \( A \) and \( y = (y_0, \ldots, y_d)^t \), then \( y_a a_0 + \cdots + y_d a_d = 0 \).

These assumptions imply that every \( d \times d \) minor of \( A \) is non-zero and every \( K_b := P_A(b) \) is either empty, containing one point or a full dimensional simplex in \( \mathbb{R}^d \). The advantage of being in this setting, is that all facets in \( A \) are active in a maximal lattice free body: let

\[
F_i(b) = \{ x \in K_b \mid a_i x = b_i \}
\]
denote the facets of a maximal lattice free body $K_b$, where $i = 0, \ldots, d$. By Proposition 4.1 we may find an integral point in the relative interior of $F_0(b)$. We may therefore assume by translation that 0 is in the relative interior of $F_0(b)$. Unless the matrix is sufficiently generic, there may be several choices for this translation as the following example shows.

**Example 4.4.** The $\mathbb{Z}^2$-orbit of the maximal lattice free triangle

$$\left\{ \begin{array}{c} x + 2y \leq 0 \\ x - 3y \leq 1 \\ 2x - y \leq 5 \end{array} \right\}$$

has two representatives containing 0 in the relative interior of $F_0(b)$:

![Graph](image)

### 4.2 Test sets and maximal lattice free bodies

We are interested in computing a well defined representative of a maximal lattice free body in $\mathbb{M}(A)/\mathbb{Z}^d$. This means, for example, that we have to choose between the two possible candidates in Example 4.4. To make a well defined choice of a representative we use the following result.
Lemma 4.5. Every maximal lattice free body is an integral translation of a unique maximal lattice free body $K_b$ with the following two properties:

(a) $b = (0, b_1, \ldots, b_d)^t \in \mathbb{N}^{d+1}$, where $b_1, \ldots, b_d > 0$.

(b) 0 is the optimal solution of the integer program

$$\min \{a_0'z \mid z \in \mathbb{Z}^d, a_1z \leq b_1 - 1, \ldots, a_dz \leq b_d - 1\},$$

where

$$a_0' := a_0 + \epsilon a_1 + \cdots + \epsilon^d a_d$$

for $\epsilon > 0$ small, is a perturbation of the first row vector $a_0$ in $A$.

Proof. If $K_b$ is a maximal lattice free body, then the optimal solutions to the integer program

$$\min \{a_0z \mid z \in \mathbb{Z}^d, a_1z \leq b_1 - 1, \ldots, a_dz \leq b_d - 1\}$$

are the integral points in the relative interior of $F_0(b)$. Therefore they all satisfy $a_0z = 0$. By perturbing $a_0$ into $a_0'$ for $\epsilon > 0$ small, we identify a unique integral solution $z_0 \in F_0(b)$ as the optimal solution to the perturbed integer program

$$\min \{(a_0')z \mid z \in \mathbb{Z}^d, a_1z \leq b_1 - 1, \ldots, a_dz \leq b_d - 1\}.$$

For each maximal lattice free body with 0 in the relative interior, $F_0$, of $F_0(b)$ this identifies a unique integral point in $F_0$. The perturbation of $a_0$ thereby identifies unique representatives of the cosets in $\mathcal{M}(A)/\mathbb{Z}^d$. \hfill $\square$

The key idea is to use the much smaller test set associated with the perturbation $a_0'$ in constructing representatives of the maximal lattice free bodies in $\mathcal{M}(A)$:

Theorem 4.6. Let $T$ be a test set for the family of integer programs given by

$$\min \{a_0'z \mid z \in \mathbb{Z}^d, a_1z \leq b_1, \ldots, a_dz \leq b_d\},$$

where $b_1, \ldots, b_d \in \mathbb{Z}$. If $K_b$ is a maximal lattice free body as in Lemma 4.5, then there exists a $d$-tuple $(z_1, \ldots, z_d)$ of elements in $T$ such that

(a) $a_i z_i > 0$
(b) \( a_iz_j < a_iz_i \) for \( j \neq i \)

(c) There exists no \( z \in T \) with \( a_kz < a_kz_k \) for all \( k = 1, \ldots, d \),

(d) \( a_0z_i \leq 0 \).

where \( i = 1, \ldots, d \).

**Proof.** This is a straightforward translation of the proof of Theorem 4.2 except for (d): we only have a test set with respect to the perturbed cost vector \( a'_0 \); i.e. we have \( a'_0z_i < 0 \) in (d) but not necessarily \( a_0z_i < 0 \).

\[ \square \]

### 4.3 Backtracking maximal lattice free bodies

In this section we outline a backtracking algorithm for enumerating the unique representatives of the maximal lattice free bodies \( K_b \) alluded to in Lemma 4.5. Recall that

\[ F_i(b) = \{ x \in K_b \mid a_ix = b_i \} \]

denotes the facet given by the \( i \)-th row \( a_i \) of \( A \).

By Theorem 4.6, \( K_b = \langle 0, v_1, \ldots, v_d \rangle_A \) for a certain \( d \)-tuple \( V = (v_1, \ldots, v_d) \) of \( T \), such that

\[ v_i \in F_i(b) \setminus \bigcup_{j=1,j \neq i}^d F_j(b). \]

Using this observation, there is a simple backtracking algorithm for generating these specific tuples discarding most of the \( d \)-tuples of \( T \): by definition, none of

\[ \langle 0, v_1 \rangle_A \subset \cdots \subset \langle 0, v_1, \ldots, v_d \rangle_A \]

contain interior points from \( T \). The basic geometric idea is to assign points from \( T \) to \( F_1(b), \ldots, F_d(b) \) keeping Proposition 4.1 in mind. Define for \( v \in T \) the set

\[ H_i(v) = \{ u \in T \mid a_iu < a_iv \}, \]

where \( 1 \leq i \leq d \). Assume inductively that we have constructed a (partial) tuple \( V' = (v_1, \ldots, v_i) \) with the property that

\[ v_i \in H_1(v_1) \cap \cdots \cap H_{i-1}(v_{i-1}) \]

for \( i \geq 2 \) and such that \( \langle 0, v_1, \ldots, v_i \rangle_A \) contains no interior points from \( T \). Then we add \( v \in T \) to \( V' \) if and only if
(a) $v \in H_1(v_1) \cap \cdots \cap H_i(v_i)$

(b) $(0, v_1, \ldots, v_i, v)_A$ contains no interior points from $T$.

If this is not possible, $V'$ can never be extended to define a maximal lattice free body and we backtrack to $V'' = (v_1, \ldots, v_{i-1})$ knowing that the extension of $V''$ by $v_i$ leads to a dead end.

By Theorem 4.6, we are sure that $B = \langle 0, v_1, \ldots, v_d \rangle_A$ is lattice free. However if some $(a_0)^t v_i = 0$, $B$ may be contained in a strictly larger lattice free body by moving the facet corresponding to $a_i$. So the backtracking algorithm above may generate a superset of the maximal lattice free bodies. Note that it is easy to decide from this superset $S$ whether a lattice free body is maximal by checking if it is maximal in $S$.

5 The reverse lexicographic term order

Consider the usual lexicographic term order $\prec_{\text{lex}}$ on vectors in $\mathbb{Z}^n$ starting from the left most coordinate and moving right. Then the reverse lexicographic term order (for vectors in the subgroup annihilated by a positive vector $y \in \mathbb{N}^n$) is given by $u \prec v \iff -u \prec_{\text{lex}} -v$.

The perturbation

$$a'_0 = a_0 + \epsilon a_1 + \cdots + \epsilon^d a_d$$

considered in Lemma 4.5 has very specific algebraic meaning. The (minimal) test set for this perturbation corresponds to a reverse lexicographic Gröbner basis for the ideal $I_L$, where $L$ is the lattice generated by the columns of $A$. This can be read off from the translation (see §3.1) of the integer program

$$\min \{ a'_0 z \mid Az \leq b \}$$

into the group program

$$\min \{ -u_0 - \epsilon u_1 - \cdots - \epsilon^d u_d \mid u \in \mathbb{N}^{d+1}, u \equiv b \pmod{L} \}.$$  

When $\epsilon$ is infinitesimally small, then

$$u \prec v \iff (-1, -\epsilon, -\epsilon^2, \ldots, -\epsilon^d)(v - u) > 0$$

for $u, v \in \mathbb{Z}^n$.  

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6 The Scarf-Shallcross algorithm

In this section we explain how our algorithm specializes to the Scarf-Shallcross algorithm for $3 \times 2$-matrices. First we review the connection between maximal lattice free bodies and the Frobenius problem given in [13].

Consider the Frobenius problem given by $a = (a_1, \ldots, a_n)$, where $a_1, \ldots, a_n$ are coprime positive integers. We collect a $\mathbb{Z}$-basis of $L = \{v \in \mathbb{Z}^n \mid av = 0\}$ in the columns of the $n \times (n - 1)$ matrix $A$. The key observation is the following.

**Theorem 6.1** (Scarf and Shallcross). Let $b \in \mathbb{Z}^n$. The integer $ab$ is representable as a non-negative integral linear combination of $a_1, \ldots, a_n$ if and only if $K_b \cap \mathbb{Z}^{n-1} = \emptyset$, where

$$K_b = \{x \in \mathbb{R}^{n-1} \mid Ax \leq b\}.$$

**Proof.** If $ab = au$ for $u \in \mathbb{N}^n$, then $b - u \in \mathcal{L}$. Therefore $b - u = Ay$ for some $y \in \mathbb{Z}^{n-1}$ and $K_b$ contains the integral point $y$. If $K_b$ contains an integral point $x$, then $u = b - Ax \in \mathbb{N}^n$ satisfies $au = ab$. \qed

Theorem 6.1 shows that

$$g(a_1, \ldots, a_n) = \max \{ab \mid b \in \mathbb{Z}^n \text{ and } K_b \cap \mathbb{Z}^{n-1} = \emptyset\}.$$

In terms of maximal lattice free bodies we have

$$g(a_1, \ldots, a_n) = \max \{ab \mid b \in \mathbb{Z}^n \text{ and } K_b \text{ is a maximal lattice free body } \} - \sum a_i.$$

6.1 Computing the Frobenius number

In the notation of the beginning of §6, note that if

$$K_c = z + K_b$$

for $z \in \mathbb{Z}^{n-1}$ and $b, c \in \mathbb{Z}^n$, then $ac = ab$. This implies that

$$g(a_1, \ldots, a_n) = \max \{ab \mid K_b + \mathbb{Z}^{n-1} \in \mathcal{M}(A)/\mathbb{Z}^{n-1}\} - \sum a_i.$$

This means that the algorithm in §4.3 may be used to compute the Frobenius number. The output from this algorithm may contain non-maximal lattice free bodies along with the representatives of the maximal ones. From the point of view of computing the Frobenius number, it is a computational advantage to simply compute the maximum above using all these bodies.
6.2 \( n = 3 \)

In this last subsection we explain how the Scarf-Shallcross algorithm for computing Frobenius numbers in dimension three may be viewed in the framework of this paper. We will argue that the reduction procedure in [13] actually computes a test set for a specific integer program. The Scarf-Shallcross algorithm is highlighted in Example 6.2 of this paper. To get a feeling for the algorithm the reader is encouraged to start with this example.

Let \( a = (a_1, a_2, a_3) \) be three coprime positive integers and put \( L = \{ v \in \mathbb{Z}^3 \mid av = 0 \} \). Then \( L \) has a particularly favorable \( \mathbb{Z} \)-basis (see [13]) given by

\[
L = \mathbb{Z}u + \mathbb{Z}v, \quad \text{where}
\]

\[
u = \begin{pmatrix} -\gamma \\ \lambda a_1 \\ -\mu a_1 \end{pmatrix}, \quad v = \begin{pmatrix} 0 \\ -a_3/\gamma \\ a_2/\gamma \end{pmatrix},
\]

for \( \gamma = \gcd(a_2, a_3) \) and \( \gamma = \lambda a_2 - \mu a_3 \), where \( \lambda, \mu \in \mathbb{Z} \) with \( 0 \leq \mu < a_2/\gamma \) and \( 0 < \lambda \leq a_3/\gamma \). By our algorithm we need to compute a test set for the family of integer programs given by

\[
\min \{-x \mid (\begin{pmatrix} \lambda a_1 & -a_3/\gamma \\ -\mu a_1 & a_2/\gamma \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}) \leq \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, x, y \in \mathbb{Z}\}, \quad (4)
\]

where \( b_1, b_2 \in \mathbb{Z} \), to find the maximal lattice free bodies \( P_{A}(b) \) for the \( 3 \times 2 \) matrix

\[
A = \begin{pmatrix} -\gamma & 0 \\ \lambda a_1 & -a_3/\gamma \\ -\mu a_1 & a_2/\gamma \end{pmatrix}.
\]

The Scarf-Shallcross algorithm ([13], §3) finds a unimodular matrix \( U \) (see Lemma 3.3) transforming the integer programming problem (4) into

\[
\min \{c^t v \mid Bv \leq b, v \in \mathbb{Z}^2\},
\]

where

\[
c_1 < 0, c_2 \leq 0 \\
B_{11} > 0, B_{12} < 0 \\
B_{21} \leq 0, B_{22} > 0
\]

with \( B_{11} + B_{12} \geq 0 \) and \( B_{21} + B_{22} > 0 \). For this integer programming problem, a test set is \( \{e_1, e_2, e_1+e_2\} \). According to Lemma 3.3, \( \{U^{-1}e_1, U^{-1}e_2, U^{-1}(e_1+e_2)\} \) is a test set for the original problem.
Example 6.2. Suppose we wish to compute \( g(12, 13, 17) \) using the Scarf-Shallcross algorithm. Then

\[
\mathcal{L} = \{(x, y, z) \mid 12x + 13y + 17z = 0\} = \mathbb{Z} \begin{pmatrix} -1 \\ 48 \\ -36 \end{pmatrix} + \mathbb{Z} \begin{pmatrix} 0 \\ -17 \\ 13 \end{pmatrix}.
\]

The following shows the unimodular transformation \( U \):

\[
\begin{pmatrix} -1 & 0 \\ 48 & -17 \\ -36 & 13 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 14 & -17 \\ -10 & 13 \end{pmatrix}, \begin{pmatrix} -1 & -1 \\ 14 & -3 \\ -10 & 3 \end{pmatrix}, \begin{pmatrix} -4 & -1 \\ 5 & -3 \\ -1 & 3 \end{pmatrix}.
\]

At each step we add a non-negative of one column to the other preserving the sign pattern for the matrix \( A \) we wish to reach. We stop when this is not possible (as in the last matrix in the example). In this particular example we see that every maximal lattice free body is an integral translation of \( K_{b_1} \) or \( K_{b_2} \), where

\[
b_1 = \max \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 5 \end{pmatrix}
\]

and

\[
b_2 = \max \left( \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.
\]

The Frobenius number is

\[
g(12, 13, 17) = \max(-12 + 4 \cdot 13 + 1 \cdot 17, -12 + 1 \cdot 13 + 2 \cdot 17) = 57.
\]

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