On the Recognition of Fan-Planar and Maximal Outer-Fan-Planar Graphs *

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Abstract. Fan-planar graphs were recently introduced as a generalization of 1-planar graphs. A graph is fan-planar if it can be embedded in the plane, such that each edge that is crossed more than once, is crossed by a bundle of two or more edges incident to a common vertex. A graph is outer-fan-planar if it has a fan-planar embedding in which every vertex is on the outer face. If, in addition, the insertion of an edge destroys its outer-fan-planarity, then it is maximal outer-fan-planar.

In this paper, we present a polynomial-time algorithm to test whether a given graph is maximal outer-fan-planar. The algorithm can also be employed to produce an outer-fan-planar embedding, if one exists. On the negative side, we show that testing fan-planarity of a graph is NP-hard, for the case where the rotation system (i.e., the cyclic order of the edges around each vertex) is given.

1 Introduction

A simple drawing of a graph is a representation of a graph in the plane, where each vertex is represented by a point and each edge is a Jordan curve connecting its endpoints such that no edge contains a vertex in its interior, no two edges incident to a common end-vertex cross, no edge crosses itself, no two edges meet tangentially, and no two edges cross more than once.

An important subclass of drawn graphs is the class of planar graphs, in which there exist no crossings between edges. Although planarity is one of the most desirable properties when drawing a graph, many real-world graphs are in fact non-planar.

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On the other hand, it is accepted that edge crossings have negative impact on the human understanding of a graph drawing [22] and simultaneously it is NP-complete to find drawings with minimum number of edge crossings [14]. This motivated the study of “almost planar” graphs which may contain crossings as long as they do not violate some prescribed forbidden crossing patterns. Typical examples include \( k \)-planar graphs [23], \( k \)-quasi planar graphs [2], RAC graphs [9] and fan-crossing free graphs [7].

Fan-planar graphs were recently introduced in the same context [18]. A fan-planar drawing of graph \( G = (V, E) \) is a simple drawing which allows for more than one crossing on an edge \( e \in E \) iff the edges that cross \( e \) are incident to a common vertex on the same side of \( e \). Such a crossing is called fan-crossing. An equivalent definition can be stated by means of forbidden crossing patterns; see Fig. 1. A graph is fan-planar if it admits a fan-planar drawing. So, the class of fan-planar graphs is in a sense the complement of the class of fan-crossing free graphs [7], which simply forbid fan-crossings.

Kaufmann and Ueckerdt [18] showed that a fan-planar graph on \( n \) vertices cannot have more than \( 5n - 10 \) edges; a tight bound. An outer-fan-planar drawing is a fan-planar drawing in which all vertices are on the outer-face. A graph is outer-fan-planar if it admits an outer-fan-planar drawing. An outer-fan-planar graph is maximal outer-fan-planar if adding any edge to it yields a graph that is not outer-fan-planar. Note that the forbidden pattern II is irrelevant for outer-fan-planarity.

Our main contribution is a polynomial time algorithm for the recognition of maximal outer-fan-planar graphs (Section 2). We also prove that the general fan-planar problem is NP-hard, for the case where the rotation system (i.e., the circular order of the edges around each vertex) is given (Section 3). Due to space restrictions, some proofs are omitted or only sketched in the text; full proofs for all results can be found in [5].

**Related Work:** As already stated, \( k \)-planar graphs [23], \( k \)-quasi planar graphs [2], RAC graphs [9] and fan-crossing free graphs [7] are closely related to the class of graphs that we study. A graph is \( k \)-planar, if it can be embedded in the plane with at most \( k \) crossings per edge. Obviously, 1-planar graphs are also fan-planar. A 1-planar graph with \( n \) vertices has at most \( 4n - 8 \) edges and this bound is tight [21]. Grigoriev and Bodlaender [15], and, independently Kohrzi and Mohar [19] proved that the problem
of determining whether a graph is 1-planar is NP-hard and remains NP-hard, even if the deletion of an edge makes the input graph planar [6]. On the positive side, Eades et al. [10] presented a linear time algorithm for testing maximal 1-planarity of graphs with a given rotation system. Testing outer-1-planarity of a graph can be solved in linear time, as shown independently by Auer et al. [4] and Hong et al. [17]. In addition, every outer-1-planar graph admits an outer-1-planar straight-line drawing [12]. Note that an outer-1-planar graph is always planar [4], while this is not true in general for outer-fan-planar graphs. Indeed, the complete graph $K_5$ is outer-fan-planar, but not planar.

A drawn graph is $k$-quasi planar if it has no $k$ mutually crossing edges. It is conjectured that the number of edges of a $k$-quasi planar graph is linear in the number of its vertices. Pach et al. [20] and Ackerman [1] affirmatively answered this conjecture for 3- and 4-quasi planar graphs. Fox and Pach [13] showed that a $k$-quasi-planar $n$-vertex graph has at most $O(n \log^{1+o(1)} n)$ edges. Fan planar graphs are 3-quasi planar [18].

A different forbidden crossing pattern arises in RAC drawings where two edges are allowed to cross, if the crossings edges form right angles. Graphs that admit such drawings (with straight-line edges) are called RAC graphs. Didimo et al. [9] showed that a RAC graph with $n$ vertices has no more than $4n - 10$ edges; a tight bound. RAC graphs are quasi planar [9], while maximally dense RAC graphs (i.e., RAC graphs with $n$ vertices and exactly $4n - 10$ edges) are 1-planar [11]. Testing whether a graph is RAC is NP-hard [3]. Dekhordi and Eades [8] proved that every outer-1-plane graph has a straight-line RAC drawing, at the cost of exponential area.

**Preliminaries:** Unless otherwise specified, we consider finite, undirected, simple graphs. We also assume basic familiarity with SPQR-trees [16] (a short introduction is given in [5]). The rotation system of a drawing is the counterclockwise order of the incident edges around each vertex. The embedding of a drawn graph consists of its rotation system and for each edge the sequence of edges crossing it. For a graph $G$ and a vertex $v \in V[G]$, we denote by $G - \{v\}$ the graph that results from $G$ by removing $v$.

**Lemma 1.** A biconnected graph $G$ is outer-fan-planar if and only if it admits a straight-line outer-fan-planar drawing in which the vertices of $G$ are restricted on a circle $C$.

**Sketch of Proof.** Let $G$ be an outer-fan-planar graph and let $\Gamma$ be an outer-fan-planar drawing of $G$. We will only show that $G$ has a straight-line outer-fan-planar drawing whose vertices lie on a circle $C$ (the other direction is trivial). The order of the vertices along the outer face of $\Gamma$ completely determines whether two edges cross, as in a simple drawing no two incident edges can cross and any two edges can cross at most once. Now, assume that two edges cross another edge in $\Gamma$. Then, both edges have to be incident to the same vertex; hence, cannot cross each other. So, the order of the crossings on an edge is also determined by the order of the vertices on the outer face. Therefore, we can construct a drawing $\Gamma_C$ by placing the vertices of $G$ on a circle $C$ preserving their order in the outer face of $\Gamma$ and draw the edges as straight-line segments. $\square$
2 Recognizing and Drawing Maximal Outer-Fan-Planar Graphs

In this section, we prove that given a graph \( G = (V, E) \) on \( n \) vertices, there is a polynomial time algorithm to decide whether \( G \) is maximal outer-fan-planar and if so a corresponding straight-line drawing can be computed in linear time. By Lemma 1, we only have to check, whether \( G \) has a straight-line drawing on a circle \( C \) that is fan-planar. Note that such a drawing is determined by the cyclic order of the vertices on \( C \). Since fan-planar graphs with \( n \) vertices have at most \( 5n - 10 \) edges \([18]\), we may assume that the number of edges is linear in the number of vertices. We first consider the case that \( G \) is 3-connected and then using SPQR-trees we show how the problem can be solved for biconnected graphs. Observe that biconnectivity is a necessary condition for maximal outer-fan-planarity. Indeed, if an outer-fan-planar drawing has a cut-vertex \( c \), it is easy to see that it is always possible to draw an edge connecting two neighbors of \( c \) while preserving the outer-fan-planarity.

The 3-Connected Case: Assume that a straight-line drawing of a 3-connected graph \( G \) with \( n \) vertices on a circle \( C \) is given. Let \( v_1, \ldots, v_n \) be the order of the vertices around \( C \). An edge \( \{v_i, v_j\} \) is an outer edge, if \( i - j \equiv \pm 1 \pmod{n} \), a 2-hop, if \( i - j \equiv \pm 2 \pmod{n} \), and a long edge otherwise. \( G \) is a complete 2-hop graph, if there are all outer edges and all 2-hops, but no long edges. Two crossing long edges are a scissor if their end-points form two consecutive pairs of vertices on \( C \). We say that a triangle is an outer triangle if two of its three edges are outer edges. We call an outer-fan-planar drawing maximal, if adding any edge to it yields a drawing that is not outer-fan-planar.

Our algorithm is based on the observation that if a graph is 3-connected maximal outer-fan-planar, then it is a complete 2-hop graph, or we can repeatedly remove any degree-3 vertex from any 4-clique until only a triangle is left. In a second step, we reinsert the vertices maintaining outer-fan-planarity (if possible). It turns out that we have to check a constant number of possible embeddings. In the following, we prove some necessary properties. The first three lemmas are used in the proof of Lemma 5. Their proofs are based on the 3-connectivity of the input graph; see Fig. 2a, 2b and 2c.

**Lemma 2.** Let \( G \) be a 3-connected outer-fan-planar graph embedded on a circle \( C \). If two long edges cross, then two of its end-points are consecutive on \( C \).

**Lemma 3.** Let \( G \) be a 3-connected outer-fan-planar graph embedded on a circle \( C \). If there are two long crossing edges, then there is a scissor, as well.

**Lemma 4.** Let \( G \) be a 3-connected graph embedded on a circle \( C \) with a maximal outer-fan-planar drawing. If \( G \) contains a scissor, then its end vertices induce a \( K_4 \).

**Lemma 5.** Let \( G \) be a 3-connected graph with a maximal outer-fan-planar drawing and assume that the drawing contains at least one long edge. Then, \( G \) contains a \( K_4 \) with all four vertices drawn consecutively on the circle.

**Proof.** First consider the case where the graph contains at least two crossing long edges and, thus, by Lemma 3 a scissor. Removing the vertices of a scissor, splits \( G \) into two connected components. Assume that we have chosen the scissor such that the smaller
of the two components is as small as possible (thus, scissor-free) and that the vertices around $C$ are labeled such that this scissor is $\{v_1, v_i\}$, $\{v_i, v_n\}$ with $i \leq n - 1$, i.e., the component induced by $v_2, \ldots, v_{i-1}$ is the smaller one. Recall that by Lemma 4 a scissor induces a $K_4$.

If $i = 3$, i.e., if $\{v_1, v_3\}$ is a 2-hop, then $G$ should contain either $\{v_2, v_n\}$ or $\{v_2, v_4\}$, as otherwise $v_1$ and $v_3$ is a separation pair; see Fig. 2d. Say w.l.o.g. $\{v_2, v_n\}$. Then, $v_1, v_2, v_3$ together with $v_4$ induce a $K_4$ with all vertices consecutive on circle $C$.

If $i > 3$, let $\{v_k, v_{\ell}\}$, $1 \leq k < \ell \leq i$ be a long edge such that there is no long edge $\{v_{k'}, v_{\ell'}\} \neq \{v_k, v_{\ell}\}$ with $k \leq k' < \ell' \leq \ell$; see Fig. 2e. Then, no long edge is crossing the edge $\{v_k, v_{\ell}\}$, as otherwise by Lemma 5 such a crossing would yield a new scissor, contradicting the choice of $\{v_1, v_{i+1}\}$ and $\{v_i, v_n\}$. Since $\{v_k, v_{\ell}\}$ is not crossed by a long edge, it must be crossed by exactly one 2-hop, say $\{v_{k-1}, v_{k+1}\}$. Now, $\ell - k > 3$ is not possible, since we could add the edge $\{v_{k-1}, v_{k+1}\}$, which is long. Hence, $\ell - k = 3$ and by maximality of the outer-fan-planar drawing, $v_k, v_{k+1}, v_{k+2}, v_{\ell}$ induces a $K_4$ with all vertices consecutive on $C$. Finally, if $G$ contains no two crossing long edges, let $\{v_k, v_{\ell}\}, 1 \leq k < \ell \leq n$ be a long edge such that there is no long edge $\{v_{k'}, v_{\ell'}\} \neq \{v_k, v_{\ell}\}$ with $k \leq k' < \ell' \leq \ell$. By the same argumentation as above, we obtain that $v_k, v_{k+1}, v_{k+2}, v_{\ell}$ induces a $K_4$ with all vertices consecutive on $C$. \hfill $\Box$

**Lemma 6.** Let $G$ be a 3-connected outer-fan-planar graph with at least six vertices. If $G$ contains a $K_4$ with all vertices drawn consecutively on circle $C$, then this $K_4$ contains exactly one vertex of degree three and this vertex is neither the first nor the last of the four vertices.

**Proof.** Let the vertices around circle $C$ be labeled so that $v_1, v_2, v_3, v_4$ induce a $K_4$. Since $v_3$ and $v_4$ are not a separation pair, there is an edge between $v_2$ or $v_3$ and a vertex, say $v_k$, among $v_5, \ldots, v_n$. Hence, three out of the four vertices $v_1, v_2, v_3$ and $v_4$ have
degree at least four; see Fig. 21. If \( v_3 \) had a neighbor in \( v_5, \ldots, v_n \), then this could only be \( v_k \), as otherwise \( \{v_1, v_4\} \) would be crossed by two independent edges. Since \( G \) has at least 6 vertices, we assume w.l.o.g. that \( k > 5 \). Since \( v_4 \) and \( v_k \) is not a separation pair, there has to be an edge \( \{v_5, v_m\} \) for some \( 4 < \ell < k \) and \( j \notin \{4, \ldots, k\} \). But such an edge would not be possible in an outer-fan-planar drawing. \( \square \)

**Lemma 7.** Let \( G \) be a 3-connected outer-fan-planar graph with at least six vertices. If \( G \) contains a \( K_4 \) with a vertex of degree 3, then this \( K_4 \) has to be drawn consecutively on circle \( C \) in any outer-fan-planar drawing of \( G \).

**Proof.** Observe that any outer-fan-planar drawing of a \( K_4 \) contains exactly one pair of crossing edges. If two 2-hops cross, then all vertices of the \( K_4 \) are consecutive. If the \( K_4 \) contains two crossing long edges, then each of the vertices of the \( K_4 \) is incident to an outer edge not contained in the \( K_4 \); thus, has degree at least four. If a long edge and a 2-hop cross, assume that the vertices around \( C \) are labeled such that \( v_1, v_2, v_3, v_4 \) induce a \( K_4 \) for some \( 5 < k < n \); see Fig. 2g. Since \( v_1, v_2 \) and \( v_k \) are incident to an outer edge not contained in the \( K_4 \), they have degree at least four. We claim that \( v_2 \) has degree at least four. Since \( v_3 \) and \( v_k \) is not a separation pair, there is an edge between a vertex among \( v_4, \ldots, v_k - 1 \) and \( v_2 \) or \( v_1 \) and an edge between a vertex among \( v_k + 1, \ldots, v_n \) and \( v_2 \) or \( v_3 \). Choosing \( v_1 \) and \( v_3 \) in the first and second case respectively, yields two independent edges crossing \( \{v_2, v_k\} \). So, \( v_2 \) is connected to a vertex outside \( K_4 \). \( \square \)

**Lemma 8.** Let \( G \) be a 3-connected graph with \( n \geq 5 \) vertices and let \( v \in V[G] \) be a vertex of degree three that is contained in a \( K_4 \). Then, \( G - \{v\} \) is 3-connected.

**Proof.** Let \( a, b, c \) and \( d \) be four arbitrary vertices of \( G - \{v\} \). Since \( G \) was 3-connected, there was a path \( P \) from \( a \) to \( b \) in \( G - \{c, d\} \). Assume that \( P \) contains \( v \). Since \( v \) is only connected to vertices that are connected to each other, there is also another path from \( a \) to \( b \) in \( G - \{c, d\} \) not containing \( v \). Hence, \( a \) and \( b \) cannot be a separation pair in \( G - \{v\} \). Since \( a \) and \( b \) were arbitrarily selected, \( G - \{v\} \) is 3-connected. \( \square \)

**Lemma 9.** Let \( G \) be a 3-connected graph with \( n > 6 \) vertices, let \( v_1, v_2, v_3 \) and \( v_4 \) be four vertices that induce a \( K_4 \), such that the degree of \( v_k \) is three. Then, \( G - \{v_3\} \) has a maximal outer-fan-planar drawing if \( G \) has a maximal outer-fan-planar drawing.

**Proof.** Consider a maximal outer-fan-planar drawing of \( G \) on a circle \( C \) and let \( v_1, v_2, v_3, \ldots, v_n \) be the order of the vertices on \( C \) (recall Lemma 7). Assume to the contrary that after removing \( v_3 \), we could add an edge \( e \) to the drawing; see Fig. 25. By Lemma 8 \( \{v_3, v_1\} \) is the only edge incident to \( v_2 \) that crosses some edges of \( G - \{v_3\} \). Hence, there must be an edge \( e' \) that is crossed by \( e \) and \( \{v_3, v_1\} \). Since \( \{v_3, v_1\} \) crosses only edges incident to \( v_2 \) that also cross \( \{v_1, v_4\} \), it follows that \( e' \) has to be incident to \( v_2 \). Further, since \( G - \{v_3\} \) plus \( e \) is outer-fan-planar it follows that \( e \) is incident to \( v_1 \) or \( v_4 \). Moreover, since \( G + e \) is not outer-fan-planar it follows that \( e \) is incident to \( v_4 \).

Let \( i \) be maximal so that there is an edge \( \{v_2, v_i\} \). If \( i \neq n \), then \( v_1 \) and \( v_2 \) is a separation pair: Any edge connecting \( \{v_4, \ldots, v_{n-1}\} \) to \( \{v_2, v_3, \ldots, v_{i-1}\} \) and not being incident to \( v_2 \) crosses \( \{v_2, v_i\} \). But edges crossing \( \{v_2, v_i\} \) can only be incident to \( v_1 \), a contradiction. Now, let \( j > 4 \) be minimum such that there is an edge \( \{v_2, v_j\} \).
We claim that \( j = 5 \). If this is not the case, then similarly to the previous case \( v_4 \) and \( v_j \) would be a separation pair in \( G - \{ v_3 \} \) plus \( e \), which is not possible due to Lemma 8.

It follows that \( G \) has to contain edge \( \{ v_1, v_3 \} \): Since \( G \) is outer-fan-planar, in \( G \) there cannot be an edge \( \{ v_4, v_k \} \) for some \( k = 6, \ldots, n \), since it would cross \( \{ v_2, v_5 \} \) which is crossed by \( \{ v_3, v_1 \} \). So, \( \{ v_1, v_5 \} \) crosses only edges incident to \( v_2 \) that are already crossed by \( \{ v_3, v_1 \} \) and \( \{ v_4, v_1 \} \). Hence, \( \{ v_1, v_5 \} \) could be added to \( G \) without violating outer-fan-planarity; a clear contradiction. Since \( e \) and \( \{ v_2, v_n \} \) both cross \( \{ v_1, v_5 \} \) it follows that \( e = \{ v_4, v_n \} \). But now, \( v_5 \) and \( v_n \) has to be a separation pair. \( \square \)

**Remark 1.** Let \( G \) be a graph with 6 vertices containing a vertex \( v \) of degree three. Then \( G \) is maximal outer-fan-planar if and only if \( G - \{ v \} \) is a \( K_5 \) missing one of the edges that connects a neighbor of \( v \) to one of the other two vertices.

**Lemma 10.** It can be tested in linear time whether a graph is a complete 2-hop graph. Moreover, if a graph is a complete 2-hop graph, then it has a constant number of outer-fan-planar embeddings and these can be constructed in linear time.

**Proof.** Let \( G \) be an \( n \)-vertex graph. We test whether \( G \) is a complete 2-hop as follows. If \( n \in \{ 4, 5 \} \), then \( G \) is either \( K_4 \) or \( K_5 \). Otherwise, check first whether all vertices have degree four. If so, pick one vertex as \( v_1 \), choose a neighbor as \( v_2 \) and a common neighbor of \( v_1 \) and \( v_2 \) as \( v_3 \) (if no such common neighbor exists then \( G \) is not a complete 2-hop). Assume now that we have already fixed \( v_1, \ldots, v_i \), \( 3 \leq i < n \). Test whether there is a unique vertex \( v \in V \setminus \{ v_1, \ldots, v_i \} \) that is adjacent to \( v_i \) and \( v_{i-1} \). If so, set \( v_{i+1} = v \). Otherwise reject. If we have fixed the order of all vertices check whether there are only outer edges and 2-hops. Do this for any possible choices of \( v_2 \) and \( v_3 \), i.e., for totally at most 6 choices. \( \square \)

**Remark 2.** No degree 3 vertex can be added to an \( n \)-vertex complete 2-hop with \( n \geq 5 \).

We are now ready to describe our algorithm. If the graph is not a complete 2-hop graph, recursively try to remove a vertex of degree 3 which is contained in a \( K_4 \). If \( G \) is maximal outer-fan-planar, Lemmas 5 and 8 guarantee that such a vertex always exists in the beginning. Remark 2 guarantees that also in subsequent steps there is a long edge and, thus, Lemmas 8 and 9 guarantee that also in subsequent steps, we can apply Lemma 5 as long as we have at least six vertices. Remark 1 guarantees that we can also remove two more vertices of degree 3 ending with a triangle.

At this stage, we already know that if the graph is outer-fan-planar, it is indeed maximal outer-fan-planar. Either, we started with a complete 2-hop graph or we iteratively removed vertices of degree three yielding a triangle. Note that in the latter case we must have started with \( 3n - 6 \) edges. On the other hand, if we apply the above procedure to an \( n \)-vertex 3-connected maximal outer-fan-planar graph, we get that the number of its edges is exactly \( 2n \) or \( 3n - 6 \).

Finally, we try to reinsert the vertices in the reversed order in which we have deleted them. By Lemma 7, we can insert the vertex of degree three only between its neighbor, that is, there are at most two possibilities where we could insert the vertex. Lemma 11 guarantees that in total, we have to check at most four possible drawings for \( G \).
Lemma 11. When reinserting a sequence of degree 3 vertices starting from a triangle, at most the first two vertices have two choices where they could be inserted.

Proof. Let $H$ be a outer-fan-planar graph and let three consecutive vertices $v_1, v_2, v_3$ induce a triangle. Assume, we want to insert a vertex $v$ adjacent to $v_1, v_2, v_3$. By Lemma 6 we have to insert $v$ between $v_1$ and $v_2$ or between $v_2$ and $v_3$. Note that the edges that are incident to $v_2$ and cross $\{v_1, v_3\}$ are also crossed by an edge $e$ incident to $v$. So, if there is an edge incident to $v_2$ that was already crossed twice before inserting $v$, this would uniquely determine whether $e$ is incident to $v_1$ or $v_3$ and, thus, where to insert $v$.

We will now show that after the first insertion each relevant vertex is incident to an edge that is crossed at least twice. When we insert the first vertex we create a $K_4$. From the second vertex on, whenever we insert a new vertex, it is incident to an edge that is crossed at least twice. Also, after inserting the second degree 3 vertex, three among the four vertices of the initial $K_4$ are also incident to an edge that is crossed at least twice. The forth vertex of the initial $K_4$ is not the middle vertex of a triangle consisting of three consecutive vertices. It can only become such a vertex if its incident inner edges are crossed by a 2-hop. But then these inner edges are all crossed at least twice.

Summarizing, we obtain the following theorem; in order to exploit this result in the biconnected case, it is also tested whether a prescribed subset (possibly empty) of edges can be drawn as outer edges.

Theorem 1. Given a 3-connected graph $G$ with a subset $E'$ of its edge set, it can be tested in linear time whether $G$ is maximal outer-fan-planar and has an outer-fan-planar drawing such that the edges in $E'$ are outer edges. Moreover if such a drawing exists, it can be constructed in linear time.

Sketch of Proof. Let $n$ be the number of vertices. By Lemma 10 a complete 2-hop graph has only a constant number of outer-fan-planar embeddings which can be computed in linear time. Whenever we remove a vertex from the graph, we append it to a queue. Any vertex that was removed from the queue will never be appended again. Hence, there are at most $n$ iterations.

To check whether the degree three vertices can be reinserted back in the graph, we only have to consider in total four different embeddings. Assume that we want to insert a vertex $v$ into an outer triangle $v_1, v_2, v_3$. Then we just have to check whether $v_1$ or $v_3$ are incident to edges other than the edge $\{v_1, v_3\}$ that cross an edge incident to $v_2$. This can be done in constant time by checking only two pairs of edges.

The Biconnected Case: We now sketch how to test outer-fan-planar maximality on a biconnected graph.

Lemma 12. Let $v_1, \ldots, v_n$ be the order of the vertices around the circle in an outer-fan-planar drawing of a 3-connected graph $G$. If we can add a vertex $v$ between $v_i$ and $v_n$ with an edge $\{v, v_i\}$ for some $i = 2, \ldots, n - 1$, then $i = 2$ or $i = n - 1$.

Proof. Otherwise, since $v_1, v_i$ cannot be a separation pair of $G$, there has to be an edge from a $v_k$ for some $k = 2, \ldots, i - 1$ that crosses $\{v, v_i\}$ and hence an edge $\{v_k, v_n\}$. Since $v_n, v_i$ cannot be a separation pair of $G$, there has to be an edge $\{v_1, v_\ell\}$ for some $\ell = i + 1, \ldots, n - 1$. But now there are three independent edges crossing.
We say that an outer edge $\{v_1, v_n\}$ is porous around $v_1$ if we could add a vertex $v$ between $v_1$ and $v_n$ and an edge $\{v, v_2\}$ maintaining outer-fan-planarity. Note that any edge of a simple cycle, i.e., of the skeleton of an S-node is porous around any of its end vertices. Any outer edge of a $K_4$ is porous around any of its end vertices; see Fig. 3.

We use the SPQR-tree of a biconnected graph to characterize whether it is maximal outer-fan-planar; see [5] for a proof of this theorem.

**Theorem 2.** A biconnected graph is maximal outer-fan-planar iff the following hold:

1) The skeleton of any $R$-node is maximal outer-fan-planar and has an outer-fan-planar drawing in which all virtual edges are outer edges,
2) No $R$-node is adjacent to an $R$-node or an $S$-node,
3) All $S$-nodes have degree three,
4) All $P$-nodes have degree three and are adjacent to a $Q$-node, and
5) Let $G_1$ and $G_2$ be the skeleton of the two neighbors of a $P$-node other than the $Q$-node and let $\{s, t\}$ be the common virtual edge of $G_1$ and $G_2$. Then $G_i, i = 1, 2$ must not admit an outer-fan-planar drawing with $t_i, s, t, s_i$ being consecutive around the circle and
   (a) $\{s, t\}$ is porous in both $G_1$ and $G_2$ around the same vertex, or
   (b) edge $\{t_1, s\}$ ($\{s_2, t\}$) is real and porous around $s$ ($t$, resp.), or
   (c) edge $\{s_1, t\}$ ($\{t_2, s\}$) is real and porous around $t$ ($s$, resp.).

As the number of outer-fan-planar embeddings of a 3-connected graph is bounded by a constant, the conditions of Thm. 2 can be tested in polynomial time. If the conditions are fulfilled, then an outer-fan-planar drawing can be constructed in linear time.

**3 The NP-hardness of the Fan-Planarity with Fixed Rotation System Problem**

In this section, we study the Fan-Planarity with Fixed Rotation System problem (FP-FRS), that is, the problem of deciding whether a graph $G = (V, E)$ with a fixed rotation system $\mathcal{R}$ admits a fan-planar drawing preserving $\mathcal{R}$.

**Theorem 3.** Fan-Planarity with Fixed Rotation System is NP-hard.
Proof. We prove the statement by using a reduction from 3-PARTITION (3P). An instance of 3P is a multi-set $A = \{a_1, a_2, \ldots, a_{3m}\}$ of $3m$ positive integers in the range $(B/A, B/2)$, where $B$ is an integer such that $\sum_{i=1}^{3m} a_i = mB$. 3P asks whether $A$ can be partitioned into $m$ subsets $A_1, A_2, \ldots, A_m$, each of cardinality 3, such that the sum of the numbers in each subset is $B$. As 3P is strongly NP-hard, it is not restrictive to assume that $B$ is bounded by a polynomial in $m$.

Before describing our transformation, we need to introduce the concept of barrier gadget. An $n$-vertex barrier gadget is a graph consisting of a cycle of $n \geq 5$ vertices plus all its 2-hop edges; a barrier gadget is therefore a maximal outer-2-planar graph. We make use of barrier gadgets in order to constraint the routes of some specific paths of $G_A$. Indeed, in a fan-planar drawing of a biconnected graph containing an outer-2-planar drawing $\Gamma_b$ of a barrier gadget, no path can enter inside the boundary cycle of $\Gamma_b$ and cross a 2-hop edge. Also, if a path enters in $\Gamma_b$ without crossing any 2-hop edge, then it must immediately exit from $\Gamma_b$ forming a fan-crossing with an outer edge of $\Gamma_b$.

Now, we are ready to describe how to transform an instance $A$ of 3P into an instance $(G_A, R_A)$ of FP-FRS. We start from the construction of graph $G_A$, which will be always biconnected. First of all, we create a global ring barrier by attaching four barrier gadgets $G_t, G_r, G_b$ and $G_l$ as depicted in Fig. 4. $G_t$ is called the top beam and contains exactly $3mK$ vertices, where $K = \lceil B/2 \rceil + 1$. $G_r$ is the right wall and has only five vertices. $G_b$ and $G_l$ are called the bottom beam and the left wall, respectively, and they are defined in a specular way. Observe that $G_t, G_r, G_b$ and $G_l$ can be embedded so that all their vertices are linkable to points within the closed region delimited by the global ring barrier. Then, we connect the top and bottom beams by a set of $3m$ columns, see Fig. 4 for an illustration of the case $m = 3$. Each column consists of a stack of $2m-1$ cells; a cell consists of a set of pairwise disjoint edges, called the vertical edges of that cell. In particular, there are $m-1$ bottommost cells, one central cell and $m-1$ topmost cells. Cells of a same column are separated by $2m-2$ barrier gadgets, called floors. Central cells (that are $3m$ in total) have a number of vertical edges depending on the elements of $A$. Precisely, the central cell $C_i$ of the $i$-th column contains $a_i$ vertical edges connecting its delimiting floors ($i \in \{1, 2, \ldots, 3m\}$). Instead, all the remaining cells have, each one, $K$ vertical edges. Hence, a non-central cell contains more edges than any central cell. Further, the number of vertices of a floor is given by the number of its incident vertical edges minus two. Let $u$ and $v$ be the “central” vertices of the left and right walls, respectively (see also Fig. 4). We conclude the construction of graph $G_A$ by connecting vertices $u$ and $v$ with $m$ pairwise internally disjoint paths, called the transversal paths of $G_A$; each transversal path has exactly $(3m - 3)K + B$ edges.

Concerning the choice of a rotation system $R_A$, we define a cyclic order of edges around each vertex that is compatible with the one depicted in Fig. 4. From what said, it is straightforward to see that an instance of 3P can be transformed into an instance of FP-FRS in polynomial time in $m$.

Let $A$ be a Yes-instance of 3P, we show that $(G_A, R_A)$ admits a fan-planar drawing $\Gamma_A$ preserving $R_A$. We observe that such a drawing is easy to compute if one omits all the transversal paths. It is essentially a drawing like that one depicted in Fig. 4, where columns are one next to the other within the closed region delimited by the global ring barrier. However, by exploiting a solution $\{A_1, A_2, \ldots, A_m\}$ of 3P for the
instance $A$, also the transversal paths can be easily embedded without violating the fan-planarity. The idea is to route these paths in such a way that: (R.1) they do not cross each other; (R.2) they do not cross any barrier; (R.3) each path passes through exactly $3$ central cells and $3m - 3$ non-central cells; (R.4) each cell is traversed by at most one path. Eventually, each transversal path crosses exactly $(3m - 3)K + B$ vertical edges, which is the same number of its edges. Therefore, it is possible to draw these paths by ensuring that each of their edges crosses exactly one vertical edge, which preserves the fan-planarity. Hence, eventually we get a fan-planar drawing $\Gamma_A$ preserving the rotation system $\mathcal{R}_A$.

We conclude the proof by showing that if $\langle G_A, \mathcal{R}_A \rangle$ is a Yes-instance of FP-FRS, then $A$ is a Yes-instance of 3P. Let $\Gamma_A$ be a fan-planar drawing of $G_A$ preserving the rotation system $\mathcal{R}_A$. We first observe that the top beam and the bottom beam are disjoint, otherwise there would be at least a 2-hope edge in one beam that is crossed by another edge of the other beam, thus violating the fan-planarity. We also note that columns can partially cross each other, but this does not actually affect the validity of the proof. Indeed, an edge $e$ of a column $L$ might cross an edge $e'$ of another column $L'$ only if $e$ is incident to a vertex in the rightmost (leftmost) side of $L$, $e'$ is a leftmost (rightmost) vertical edge of $L'$, and $L$ and $L'$ are two consecutive columns. With a similar argument, it is immediate to see that vertices $u$ and $v$ must be separated by all the columns. Therefore, every transversal path satisfies conditions R.1, R.2 and it must pass through at least three central cells, if not it would cross a number of pairwise disjoint edges that is greater than the number of its edges, hence $\Gamma_A$ would not be fan-planar. On the other hand, because of condition R.4, which is obviously satisfied, there cannot be any transversal path passing through more than three central cells. Otherwise, there would be some other transversal path that traverses a number of central cells that is strictly less than three. Hence, also condition R.3 is satisfied. In conclusion, every transversal path $\pi_j$ ($j \in \{1, 2, \ldots, m\}$) crosses $(3m - 3)K + B$ vertical edges and traverses exactly three central cells $C_{1j}$, $C_{2j}$ and $C_{3j}$. If $m(C_{1j})$, $m(C_{2j})$ and $m(C_{3j})$ denote the number of edges of these cells, then $m(C_{1j}) + m(C_{2j}) + m(C_{3j}) = B$, because each non-central cell has $K$ edges. Therefore, the partitioning of $A$ defined by $A_1, A_2, \ldots, A_m$, where $A_j = \{m(C_{1j}), m(C_{2j}), m(C_{3j})\}$, is a solution of 3P for the instance $A$. □
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