Massive Gauge Fields and the Planck Scale

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The present work is devoted to massive gauge fields in special relativity with two fundamental constants-the velocity of light, and the Planck length, so called doubly special relativity (DSR). The two invariant scales are accounted for by properly modified boost parameters. Within above framework we construct the vector potential as the \((1/2, 0) \otimes (0, 1/2)\) direct product, build the associated field strength tensor together with the Dirac spinors and use them to calculate various observables as functions of the Planck length.

Key words: special relativity, massive gauge fields, fundamental constants

1. INTRODUCTION

In the standard theory of special relativity, Lorentz transformations preserve the energy-momentum dispersion relation of a particle observed from two different inertial frames according to

\[
\frac{E^2}{c^2} - p_x^2 - p_y^2 - p_z^2 = \left(\frac{E'}{c}\right)^2 - p'_x^2 - p'_y^2 - p'_z^2 = m^2 c^2.
\]  

(1)

Lorentz transformations are covered by rotations in the three space-like planes \((p_x, p_y, p_z)\), \((p_y, p_z, p_x)\), and \((p_z, p_x, p_y)\), on the one side, and by pseudo-rotations (i.e. rotations by an imaginary angle) in the \((E/c, p_x)\), \((E/c, p_y)\), and \((E/c, p_z)\) planes, on the other side. To be specific, for the text-book example of \(p'_y = p_y\), and \(p'_z = p_z\), the boost parametrizes as

\[
\begin{pmatrix}
E' \\
\frac{p'}{c}_x \\
\frac{p'}{c}_y \\
\frac{p'}{c}_z
\end{pmatrix} =
\begin{pmatrix}
cosh \phi & \sinh \phi & 0 & 0 \\
\sinh \phi & \cosh \phi & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
E \\
p_x \\
p_y \\
p_z
\end{pmatrix}.
\]
\[
cosh \phi = \frac{E}{mc^2}, \quad \sinh \phi = \frac{\vec{p}}{mc}.
\] (2)

Ordinary special relativity, to be referred to as SR in the following, shows up here through the dependence of the boost parameters on the velocity of light. Usually, velocities are given in units of \(c\), and one sets \(c = 1\).

In recent years, certain phenomena of Ultra High Energy Cosmic Rays (UHECR) seem to indicate that in the vicinity of the Planck length, ordinary special relativity may need an extension that accounts for the constancy of the Planck scale. To be more specific, in effect of collisions with the soft photons from the cosmic microwave background radiation, one expects cosmic ray protons, and cosmic gamma rays to slow down to energies below \(E_p < 5 \times 10^{19}\) eV, and \(E_\gamma < 20\) TeV, respectively. The UHECR protons slow down basically because of pion-photo production, while the ultra high energy gamma rays loose energy due to electron-positron pair production. Yet, cosmic protons with energies \(E_p > 5 \times 10^{19}\) eV (so called Greisen-Zatsepin-Kusmin (GZK) threshold value) as well as cosmic gamma rays with \(E_\gamma > 20\) TeV still arrive at earth, and the GZK limit seems too low in comparison to data \([1]\).

A possible solution to this so called cosmic ray problem has been advocated in Refs. \([2]\), \([3]\). According to Amelino-Camelia, quantum gravity effects may force deformations upon the energy-momentum dispersion relation as
\[
E^2 \approx c^2 p^2 + c^4 m^2 + \lambda_p E p^2 + \mathcal{O}(\lambda_p^2).
\] (3)

Modifications of this type can allow for a higher value of the GZK limit \([4]\), \([5]\), \([6]\). Special relativities with two invariant scales are known as Doubly Special Relativity (DSR), a notion due to Ref. \([2]\). The major idea of such theories of space-time is to replace the linear parametrization of the boost by a non-linear function of the Planck length without changing the algebra of the Lorentz group \([7]\). Amelino-Camelia’s deformed energy-momentum dispersion relation results from the following non-linear boost parametrization (to be referred to as DSRa in the following)
\[
cosh \xi = \frac{e^{\lambda_p E} - \cosh(\lambda_p m)}{\sinh(\lambda_p m)}, \quad \sinh \xi = \frac{\lambda_p |\vec{p}| e^{\lambda_p E}}{\sinh(\lambda_p m)}.
\] (4)

The energy momentum dispersion relation following from Eq. (4) reads
\[
2 \cosh \lambda_p E - \lambda_p^2 |\vec{p}|^2 e^{\lambda_p E} = 2 \cosh \lambda_p m.
\] (5)

In a similar spirit, Smolin and Magueijo proposed in Ref. \([8]\) a different boost parametrization (to be referred to as DSRb) as
\[
cosh \xi = \frac{E (1 - \lambda_p m)}{m (1 - \lambda_p E)}, \quad \sinh \xi = \frac{|\vec{p}| (1 - \lambda_p m)}{m (1 - \lambda_p E)}.
\] (6)

Both parameterizations reduce to Eq. (2) at energies significantly low compared to the Planck scale. It is important to notice that the \(\lambda_p\) parameter is real
and positive [9]. For a broader discussion of various DSR aspects the interested reader may wish to consult Refs. [10]-[13].

The goal of the present paper is to obtain massive Abelian gauge fields in special relativity with two invariant scales. The results easily extend to the non-Abelian case. Massive gauge bosons are especially interesting in all field theories with spontaneously broken local gauge symmetries such as the electroweak theory. Moreover, they are the basic building blocks of massive gravitinos, spin-3/2 gauge fermions that appear in supersymmetric theories and which can have a significant impact onto models of inflationary universe [14], [15], and dark matter [16].

The presentation is organized as follows. In the next Section we discuss differences between special and doubly special relativities. In Section 3 we build up the massive gauge field in DSR. In Section 4 we construct the associated field strength tensor and calculate various observables as a function of \( \lambda P \). The paper closes with a brief summary and perspectives.

2. CPT VIOLATION IN DSR

The DSRb energy-momentum dispersion relation resulting from Eq. (6) can be cast into the form

\[
E^2 \frac{(1 - \lambda P m)^2}{(1 - \lambda P E)^2} - \vec{p}^2 \frac{(1 - \lambda P m)^2}{(1 - \lambda P E)^2} = m^2. \tag{7}
\]

The main message from Eq. (7) is the energy cut off at \( E = 1/\lambda P \). Above this value (and for \( m \neq 1/\lambda P \)) Eq. (7) is equivalent to

\[
E^2 - \vec{p}^2 = f^2(E) m^2, \quad f(E) = \frac{1 - \lambda P E}{1 - \lambda P m}. \tag{8}
\]

In this way, the Planck length shows up only in the energy dependent “mass” form factor.

Equation (8) describes two hyperboloids in the \((E, \vec{p})\) space that differ from the ordinary SR hyperboloids in two aspects. First, \((E^2 - \vec{p}^2)\) is no longer a constant, and second, at rest, instead of the symmetric SR result, \( E = \pm m \), one finds the two asymmetric solutions

\[
E_1 = m, \quad E_2 = -\frac{m}{1 - 2\lambda P m}, \tag{9}
\]

respectively. The consequence of Eq. (9) is CPT violation in Doubly Special Relativity, an effect that reflects the non-locality introduced by the energy cut off. In DSRa, where the presence of \( \lambda P \) in the boost parameters does not result in discretization of the phase space, one finds the symmetric solution \( E = \pm m \) at \( \vec{p} = 0 \).
At this place a comment onto distinguishability between special- and doubly special relativity seems to be in order. From the formal point of view, one may redefine \( \cosh \xi \) and \( \sinh \xi \) as

\[
\cosh \xi = \frac{\epsilon}{\mu}, \quad \sinh \xi = \frac{|\vec{\pi}|}{\mu},
\]

and consider the quantities \( \epsilon, \vec{\pi}, \) and \( \mu \) as the physical observables. As long as these new kinematic quantities satisfy the \( \epsilon^2 - \vec{\pi}^2 = \mu^2 \) dispersion relation of special relativity, one may feel tempted to claim “indistinguishability” between SR and DSR. Notice, however, that the \((\epsilon, \vec{\pi})\) space is CPT conserving because if rest is defined as \( \vec{\pi} = 0 \) one finds the symmetric \( \epsilon = \pm\mu \) solution. The reason for this seeming contradiction is that one forgets that contrary to \( E \) and \( \vec{p} \) in SR, \( \epsilon \) and \( \vec{\pi} \) are discontinuous variables. In effect, CPT distinguishes between SR and DSR.

It would be interesting to check consistency of Eq. (9) with the measured proton-anti-proton mass difference for which the The Particle Data Group reports in Ref. [17] the following value:

\[
\frac{|m_p - m_{\bar{p}}|}{m_p} < 6 \times 10^{-8}. \tag{11}
\]

Incorporating Eq. (9) into (11) and using \( m_p = 938.271998 \text{ MeV} \) for the proton mass amounts to

\[
\lambda_P < 0.6309 \times 10^{-23} \text{ m}. \tag{12}
\]

This inequality is clearly satisfied by the theoretical value of \( \lambda_P \sim 10^{-35} \text{ m} \).

### 3. THE VECTOR POTENTIAL AT THE PLANCK SCALE

#### 3.1 Spinor and Co-Spinor Representations at the Planck scale

As long as one maintains the algebra of the Lorentz group intact, the group representations remain unaltered. We here are especially interested in the spinor \((1/2, 0)\)–, and co-spinor \((0, 1/2)\), representations, respectively. In what follows we shall use the notation \( \zeta_h(\vec{p}) \) and \( \hat{\zeta}_h(\vec{p}) \) (with \( h \) standing for spin-projection) for spinor \((1/2, 0)\), and co-spinor \((0, 1/2)\), respectively. In the Cartesian frame,

\[
\zeta^\uparrow(\vec{0}) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \zeta^\downarrow(\vec{0}) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \hat{\zeta}^\uparrow(\vec{0}) = \zeta^\uparrow(\vec{0}), \quad \hat{\zeta}^\downarrow(\vec{0}) = \zeta^\downarrow(\vec{0}). \tag{13}
\]

According to standard rules [18] spinors and co-spinors are boosted as

\[
\zeta_h(\vec{p}) = \exp \left( \pm \frac{\vec{\sigma} \cdot \vec{\xi}}{2} \right) \zeta_h(\vec{0}), \quad \hat{\zeta}_h(\vec{p}) = \exp \left( - \frac{\vec{\sigma} \cdot \vec{\xi}}{2} \right) \hat{\zeta}_h(\vec{0}), \quad h = \uparrow \downarrow, \tag{14}
\]

\[
\zeta_h(\vec{0}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \hat{\zeta}_h(\vec{0}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad h = \uparrow \downarrow. \tag{15}
\]
where $\sigma_i, i = 1, 2, 3$, are the standard Pauli matrices.

In writing down the exponentials in Eq. (14) and (15) as

$$\exp(\pm \frac{\vec{\sigma} \cdot \vec{\xi}}{2}) = I_2 \cosh \frac{\xi}{2} \pm (\vec{\sigma} \cdot \hat{\vec{\xi}}) \sinh \frac{\xi}{2},$$

one finds

$$\exp(\pm \frac{\vec{\sigma} \cdot \vec{\xi}}{2}) = l[I_2 B \pm (\vec{\sigma} \cdot \hat{\vec{\xi}}) \Gamma].$$

Here, $l = (B^2 - \Gamma^2)^{-1/2}$. The previous equation is of great importance for the present work, as the new observer independent scale, the Planck length, appears for the first time in the building blocks of the vector potential via

$$B^2 = e^{\lambda P} e^{-\lambda P m}, \quad \Gamma^2 = e^{\lambda P} e^{-\lambda P m} \quad \text{in DRSa},$$

$$B^2 = E + m - 2mE\lambda P, \quad \Gamma^2 = E - m. \quad \text{in DSRb}.$$}

### 3.2 The Vector Potential as $(1/2, 0) \otimes (0, 1/2)$

At rest, and in the Cartesian frame, the basis vectors of the product space are given by

$$a_{\uparrow\downarrow}(\vec{0}) = m \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad a_{\downarrow\uparrow}(\vec{0}) = m \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$a_{\uparrow\downarrow}(\vec{0}) = m \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad a_{\downarrow\uparrow}(\vec{0}) = m \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$ (20)

In order to obtain rest frame basis of good spin, one needs to subject the direct-product basis to the following transformation:

$$\epsilon_{JM}(\vec{0}) = SS_1 a_{h,h'}(\vec{0}), \quad S_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$ (21)

Here, $\epsilon_{0,0}(\vec{0}) = SS_1 a_{\uparrow\downarrow}(\vec{0})$, is a scalar, while $\epsilon_{1,0}(\vec{0}) = SS_1 a_{\downarrow\uparrow}(\vec{0}), \epsilon_{1,1}(\vec{0}) = SS_1 a_{\uparrow\downarrow}(\vec{0}), \text{and } \epsilon_{1,-1}(\vec{0}) = SS_1 a_{\downarrow\uparrow}(\vec{0})$ constitute the spin-triplet. The latter three vectors describe in turn longitudinal–, right–, and left-handed circularly polarized gauge bosons.

The boost in the product space is obtained as the direct product of the spinor- and co-spinor boosts and reads:

$$\kappa(\vec{p}) = \exp\left(\frac{1}{2} \vec{\sigma} \cdot \vec{\xi} \right) \otimes \exp\left(-\frac{1}{2} \vec{\sigma} \cdot \vec{\xi} \right).$$ (22)
where,  

\[
\hat{\kappa}'(\vec{p}) = \kappa' \left( \vec{p} \right) = \mathcal{S} \mathcal{S}_1 \kappa(\vec{p}) \left( \mathcal{S}_1 \right)^{-1}.
\]

As long as neither \( J^2 \), nor \( J_3 \) commute with \( \kappa'(\vec{p}) \), the boosted vectors \( \kappa'(\vec{p}) \epsilon_{JM}(\vec{0}) \) can not any longer be characterized by the \( J \), and \( M \) quantum numbers but have to be labeled differently. Below we introduce following notations:

\[
\mathcal{A}_1^\mu(\vec{p}) := \kappa'(\vec{p}) \epsilon_{1,1}(\vec{0}) = \frac{-i}{\sqrt{2}} \begin{pmatrix}
2\mathcal{B}(\vec{p}^2 - \mathcal{G}^2) & -\mathcal{B}(\vec{p}_x + \mathcal{G} \vec{p}_y) \\
\mathcal{B}(\vec{p}_x + \mathcal{G} \vec{p}_y) & \mathcal{B}^2 - \mathcal{G}^2
\end{pmatrix}
\]

\[
\mathcal{A}_2^\mu(\vec{p}) := \kappa'(\vec{p}) \epsilon_{1,0}(\vec{0}) = m^2 \begin{pmatrix}
2\mathcal{B}\vec{p}_x \\
2\mathcal{G}\vec{p}_y
\end{pmatrix}
\]

\[
\mathcal{A}_3^\mu(\vec{p}) := \kappa'(\vec{p}) \epsilon_{1,-1}(\vec{0}) = \frac{-i}{\sqrt{2}} \begin{pmatrix}
2\mathcal{B}(\vec{p}_x - \mathcal{G} \vec{p}_y) & -\mathcal{B}(\vec{p}_x - \mathcal{G} \vec{p}_y) \\
\mathcal{B}(\vec{p}_x - \mathcal{G} \vec{p}_y) & \mathcal{B}^2 - \mathcal{G}^2
\end{pmatrix}
\]

\[
\mathcal{A}_4^\mu(\vec{p}) := \kappa'(\vec{p}) \epsilon_{0,0}(\vec{0}) = m^2 \begin{pmatrix}
2\mathcal{B}\vec{p}_x \\
2\mathcal{B}\vec{p}_y
\end{pmatrix}
\]

The massive gauge vector-potential is now completely described by the last expressions, which are the first result of the present study. Knowing the basis in \((1/2, 0) \otimes (0, 1/2)\) in DSR allows for the construction of the DSR field strength tensor, a topic considered in the next Section.

4. FIELD STRENGTH TENSOR IN DSR

The field strength tensor, \( \mathcal{F}^{\mu\nu}(\vec{p}, \vec{z}) \), (with \( \vec{z} = 0, 1, 2, 3 \)) is constructed from \( \mathcal{A}_\nu(\vec{p}) \) as a solution of the massive Proca equation

\[
p_{\nu} \mathcal{F}^{\mu\nu}(\vec{p}, \vec{z}) = m^2 \epsilon^\mu_{\nu}(\vec{p}).
\]
Here, we defined

\[ e_0(\vec{p}) = A_4(\vec{p}), \quad e_1(\vec{p}) = \frac{1}{\sqrt{2}} (A_3(\vec{p}) - A_1(\vec{p})), \]

\[ e_3(\vec{p}) = A_2(\vec{p}), \quad e_2(\vec{p}) = \frac{i}{\sqrt{2}} (A_3(\vec{p}) + A_1(\vec{p})). \] (30)

Equation (29) translates into a matrix form as

\[ \mathcal{F}(\vec{p}, \zeta) G e_0(\vec{p}) = m^2 e_{\zeta}(\vec{p}), \] (31)

where the \( G \) matrix stands for the metric tensor \( G = \text{diag}(1, -1, -1, -1) \). As a solution of Eq. (31) one finds (see Refs. [20], [21] for more details)

\[ \mathcal{F}(\vec{p}, \zeta) = e_{\zeta}(\vec{p}) e_0^1(\vec{p}) e_0^1(\vec{p}). \] (32)

In DSR, the explicit expressions for \( e_{\zeta}(\vec{p}) \) are given by

\[
\begin{aligned}
e_1(\vec{p}) &= ml^2 \begin{pmatrix}
2\hat{B}\hat{G}\hat{p}_x \\
B^2 - \Gamma^2(1 - 2\hat{p}_y^2) \\
2\hat{G}\hat{p}_y \\
2\hat{G}\hat{p}_z
\end{pmatrix}, \\
e_2(\vec{p}) &= ml^2 \begin{pmatrix}
2\hat{B}\hat{G}\hat{p}_y \\
B^2 - \Gamma^2(1 - 2\hat{p}_y^2) \\
2\hat{G}\hat{p}_y \\
2\hat{G}\hat{p}_z
\end{pmatrix}, \\
e_3(\vec{p}) &= ml^2 \begin{pmatrix}
2\hat{G}\hat{p}_z \\
2\hat{G}\hat{p}_y \\
B^2 - \Gamma^2(1 - 2\hat{p}_y^2) \\
2\hat{G}\hat{p}_z
\end{pmatrix}, \\
e_0(\vec{p}) &= ml^2 \begin{pmatrix}
2\hat{B}\hat{G}\hat{p}_x \\
2\hat{B}\hat{G}\hat{p}_y \\
2\hat{B}\hat{G}\hat{p}_y \\
2\hat{B}\hat{G}\hat{p}_z
\end{pmatrix},
\end{aligned}
\] (33)

with \( l^2 = B^2 - \Gamma^2 \) from Eq. (19). Equation (32) allows to construct the massive field strength tensor in momentum space. In so doing one finds:

\[
\begin{aligned}
\mathcal{F}(\vec{p}, x) &= m^2 l^2 \begin{pmatrix}
0 & -(B^2 + \Gamma^2) & 2\hat{G}\hat{p}_x & 2\hat{G}\hat{p}_x \\
(B^2 + \Gamma^2) - 2\hat{G}^2\hat{p}_x^2 & 0 & 2\hat{G}\hat{p}_y & 2\hat{G}\hat{p}_z \\
-2\hat{G}\hat{p}_y & -2\hat{G}\hat{p}_y & 0 & 0 \\
-2\hat{G}\hat{p}_x & -2\hat{G}\hat{p}_x & 0 & 0
\end{pmatrix}, \\
\mathcal{F}(\vec{p}, y) &= m^2 l^2 \begin{pmatrix}
0 & 2\hat{G}\hat{p}_y & -(B^2 + \Gamma^2) & 2\hat{G}\hat{p}_y \\
-2\hat{G}\hat{p}_y & 0 & -2\hat{G}\hat{p}_x & 0 \\
(B^2 + \Gamma^2) - 2\hat{G}^2\hat{p}_y^2 & 2\hat{G}\hat{p}_x & 0 & 0 \\
-2\hat{G}\hat{p}_y & 0 & -2\hat{G}\hat{p}_z & 0
\end{pmatrix}, \\
\mathcal{F}(\vec{p}, z) &= m^2 l^2 \begin{pmatrix}
0 & 2\hat{G}\hat{p}_z & 2\hat{G}\hat{p}_y & -(B^2 + \Gamma^2) + 2\hat{G}^2\hat{p}_z^2 \\
-2\hat{G}\hat{p}_z & 0 & 0 & -2\hat{G}\hat{p}_x \\
-2\hat{G}\hat{p}_y & 0 & 0 & -2\hat{G}\hat{p}_y \\
(B^2 + \Gamma^2) - 2\hat{G}^2\hat{p}_z^2 & 2\hat{G}\hat{p}_x & 2\hat{G}\hat{p}_y & 0
\end{pmatrix}.
\] (34)
The correspondence between the field strength tensor components and the vector and axial vector fields, in turn denoted by $\vec{E}(\vec{p}, \zeta)$, and $\vec{B}(\vec{p}, \zeta)$, is standard and reads

$$
\mathcal{F}^{\mu \nu}(\vec{p}, \zeta) = \begin{pmatrix}
0 & E_x(\vec{p}, \zeta) & E_y(\vec{p}, \zeta) & E_z(\vec{p}, \zeta) \\
-E_x(\vec{p}, \zeta) & 0 & -B_z(\vec{p}, \zeta) & B_y(\vec{p}, \zeta) \\
-E_y(\vec{p}, \zeta) & B_z(\vec{p}, \zeta) & 0 & -B_x(\vec{p}, \zeta) \\
-E_z(\vec{p}, \zeta) & -B_y(\vec{p}, \zeta) & B_x(\vec{p}, \zeta) & 0
\end{pmatrix}. \quad (35)
$$

Equations (34) represent a basis for any massive Abelian field strength tensor near Planck energy.

### 4.1 The $\lambda_P \to 0$ Limit of the Field Strength Tensor

Before proceeding further it is necessary to verify that Eqs. (34)–(35) also qualify for the description of massless gauge fields in special relativity, i.e. for $\lambda_P \to 0$, and $m \to 0$ they have to reduce to the standard electromagnetic field strength tensors. In other words, one needs to confirm several established relations of standard electrodynamics such as covariant normalization of $F_{\mu \nu}(\vec{p}, \zeta)$, the divergences of the electric, and magnetic fields, and the continuity conditions. In so doing we find

$$
-\frac{1}{4} F^{\mu \nu}(\vec{p}, \zeta) F_{\mu \nu}(\vec{p}, \zeta) = \frac{1}{2} \left( \vec{E}^2(\vec{p}, \zeta) - \vec{B}^2(\vec{p}, \zeta) \right) = \frac{E^2 - p^2}{2} \xrightarrow{m \to 0} 0,
$$

$$
\text{div}\vec{B}(\vec{p}, \zeta) = 0, \quad \text{div}\vec{E}(\vec{p}, \zeta) = -m^2 p_\zeta \xrightarrow{m \to 0} 0. \quad (36)
$$

Notice that as long as our vector potentials are in the Lorentz gauge, the massive electric field is not necessarily divergence-less. Moreover, one can calculate the (classical) energy density of electromagnetic field, $U(\vec{p}, \zeta)$, together with the Poynting vector, $\vec{S}(\vec{p}, \zeta)$, and check the continuity condition,

$$
-U(\vec{p}, \zeta) = \frac{1}{2m} \left( \vec{E}(\vec{p}, \zeta)^2 + \vec{B}(\vec{p}, \zeta)^2 \right) = \frac{1}{2m} \left( 2E^2 - m^2 - \frac{2E^2 - m^2}{p^2} \zeta^2 \right),
$$

$$
\vec{S}(\vec{p}, \zeta) = \frac{1}{m} \vec{E}(\vec{p}, \zeta) \times \vec{B}(\vec{p}, \zeta). \quad (38)
$$

By means of Eq. (38) one calculates the continuity equation as

$$
E U(\vec{p}, \zeta) - \vec{p} \cdot \vec{S}(\vec{p}, \zeta) = \frac{E}{m} \xrightarrow{m \to 0} 0. \quad (39)
$$

Therefore, our construct for the DSR field strength tensor reproduces correctly the massless electrodynamics limit.

### 4.2 Covariant Norm and Energy Density of Massive Gauge Fields in DSR
Equipped with the confidence of having created a construct consistent with standard electrodynamics, we are now in a position to calculate DSR corrections to the classical field energy density at the Planck scale as:

$$\frac{1}{2m} \left( \vec{E}^2(\vec{p}, \zeta) + \vec{B}^2(\vec{p}, \zeta) \right) = \frac{\mathcal{B}^4 + 6\mathcal{B}^2\mathcal{G}^2 + 8\mathcal{B}^2\mathcal{G}^2 \hat{\mathcal{R}}}{2(\mathcal{B}^2 - \mathcal{G}^2)^2}, \quad (40)$$

for each one of the tensors (34). This expression makes manifest the modification of the electromagnetic field energy density at the Planck scale. To first order on $\lambda_P$, Eq. (40) takes the form

$$\frac{1}{2m} \left( \vec{E}^2(\vec{p}, \zeta) + \vec{B}^2(\vec{p}, \zeta) \right) = \frac{1}{2m} \left( 2E^2 - m^2 - 2(E^2 - m^2) \frac{p_\perp^2}{p^2} + \delta_\zeta \lambda_P \right), \quad (41)$$

with:

$$\delta_\zeta = \begin{cases} 2E^2 - m^2)(1 - \frac{p_\perp^2}{p^2}) \text{ in DRSa,} \\ 4E^2(E - m)(1 - \frac{p_\perp^2}{p^2}) \text{ in DSRb.} \end{cases} \quad (42)$$

In the expressions above $\zeta$ takes the values $\zeta = 1, 2, 3$. Equation (41) is valid for low energy Abelian gauge fields with masses comparable to $\lambda_P^{-1}$ as they can appear in supersymmetric theories.

The DSR counterpart of Eq. (36) is

$$-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2} m^2. \quad (43)$$

The normalization of the field strength tensor, in being brought about by the normalization of the $(1/2,1/2)$ basis vectors [20] in Eq. (20)), comes out same in DSR and SR though in the former case we did not have to make use of the corresponding energy-momentum dispersion relation. That in the rhs in Eq. (43) one finds $m^2$, the SR mass, and not $\mu^2$, the DSR mass counterpart, is one more argument in favor of distinguishability between SR and DSR.

### 4.3 Energy Dependent Charge in DSR

Another interesting observable is the vector Noether charge probed by the massive gauge field under consideration. In order to calculate it, one first needs to construct the DSR matter fields. We here construct the Dirac spinors in DSR and calculate the time component of the vector current. To do so we first notice equality of rest-frame spinor and co-spinor $\zeta^\uparrow(\vec{0}) = \zeta^\downarrow(\vec{0})$, and introduce the Dirac rest frame spinors, $u_\uparrow(\vec{0})$, in the standard way as the direct sum of a spinor–and co-spinor spaces [18]:

$$u_\uparrow(\vec{p}) = k(\vec{p}) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad u_\downarrow(\vec{p}) = k(\vec{p}) \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \quad (44)$$

9
The boost in the direct sum space is standard and given by

\[ k(\vec{p}) = \exp\left( + \frac{1}{2} \vec{\sigma} \cdot \vec{\xi} \right) \oplus \exp\left( - \frac{1}{2} \vec{\sigma} \cdot \vec{\xi} \right), \tag{45} \]

with the result

\[ k(\vec{p}) = \begin{pmatrix} \frac{B - \Gamma_{\hat{p}z} z \Gamma(\hat{p}_x - i\hat{p}_y)}{B + \Gamma_{\hat{p}z} z} & \frac{\Gamma(\hat{p}_x + i\hat{p}_y)}{B - \Gamma_{\hat{p}z} z} & 0 & 0 \\ \frac{\Gamma(\hat{p}_x - i\hat{p}_y)}{B - \Gamma_{\hat{p}z} z} & B - \Gamma_{\hat{p}z} z & 0 & 0 \\ 0 & 0 & B + \Gamma_{\hat{p}z} z & 0 \\ 0 & 0 & 0 & B - \Gamma_{\hat{p}z} z \end{pmatrix}. \tag{46} \]

In effect, one obtains the following expressions for the boosted Dirac particle spinors in DSR:

\[ u_\uparrow(\vec{p}) = \frac{l}{\sqrt{2}} \begin{pmatrix} B + \Gamma_{\hat{p}z} z \\ \Gamma(\hat{p}_x - i\hat{p}_y) \\ B - \Gamma_{\hat{p}z} z \\ -\Gamma(\hat{p}_x + i\hat{p}_y) \end{pmatrix}, \quad u_\downarrow(\vec{p}) = \frac{l}{\sqrt{2}} \begin{pmatrix} \Gamma(\hat{p}_x - i\hat{p}_y) \\ B - \Gamma_{\hat{p}z} z \\ -\Gamma(\hat{p}_x - i\hat{p}_y) \\ (B + \Gamma_{\hat{p}z} z) \end{pmatrix}. \tag{47} \]

In the \( \lambda_P \to 0 \) limit, Eqs. (47) reduce to ordinary Dirac spinors in special relativity. The antiparticle spinors, \( \bar{v}(\vec{p}) \) are then obtained as \( \bar{v}_h(\vec{p}) = \gamma_5 u_h(\vec{p}) \), with \( \gamma_5 = \text{diag}(1, 1, -1, -1) \).

We now exploit Eq. (47) to calculate the vector charge density which at zero three momentum transfer is given by

\[ Q(E) = \bar{u}_h(\vec{p}) \gamma_0 u_h(\vec{p}) = \cosh \xi, \quad \gamma_0 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}. \tag{48} \]

For the DSR/SR charge density ratio one finds

\[ \frac{Q_{\text{DSR}}(E)}{Q_{\text{SR}}(E)} = \begin{cases} \frac{1 - \lambda_P m}{1 - \lambda_P E} & \text{in DSRb,} \\ \frac{\cosh(\lambda_P m)}{\sinh(\lambda_P m)} & \text{in DSRa.} \end{cases} \tag{49} \]

In other words, compared to SR, the DSR Noether charge density appears anti-screened in DSRb.

5. SUMMARY AND PERSPECTIVES

In this work we elaborated matter–, and massive gauge fields in a space-time with two fundamental scales- the velocity of light, and the Planck length. Thereby we provide the basics of a quantum field theory within the scenario of Doubly Special Relativity. Once having Dirac and gauge spinors at our disposal, various boson-fermion couplings like

\[ \bar{u}_h(\vec{p}_f) \gamma_{\mu} u_h(\vec{p}_i) A_\mu(q), \quad \bar{u}_h(\vec{p}_f) \sigma_{\mu\nu} u_h(\vec{p}_i) F_{\mu\nu}(q), \quad q = p_i - p_f, \tag{50} \]
etc can be written down and exploited in calculations of processes like, say, scattering and bremsstrahlung. Moreover, also the construction of higher spins, be they matter- or gauge fields, is straightforward. For example, the massive spin-3/2 spinors in DSR, be they gravitinos, or matter particles, $u^\nu(p)$, are constructed as

$$u^\nu_M(p) = \sum_{\mu, h} \left( \frac{1}{2} \frac{1}{2} \frac{3}{2} \frac{M}{2} \right) \kappa^h(p) \epsilon^\nu_{\mu}(\vec{0}) \otimes u_h(p).$$

The physics of massive gravitino is important, among others, in inflationary models of the Universe [22]. Finally, the kinematic anti-screening effect encountered in DSR is especially intriguing because it has been known so far only in non-Abelian theories and attributed to self-interaction of the gauge field.

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