Diagonalization of compact operators in Hilbert modules over $C^*$-algebras of real rank zero

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22 January 1995

Abstract

It is known that the classical Hilbert–Schmidt theorem can be generalized to the case of compact operators in Hilbert $A$-modules $H^*_A$ over a $W^*$-algebra of finite type, i.e. compact operators in $H^*_A$ under slight restrictions can be diagonalized over $A$. We show that if $B$ is a weakly dense $C^*$-subalgebra of real rank zero in $A$ with some additional property then the natural extension of a compact operator from $H_B$ to $H^*_A \supset H_B$ can be diagonalized with diagonal entries being from the $C^*$-algebra $B$.

1 Introduction

Let $A$ be a $C^*$-algebra. We consider Hilbert $A$-modules over $A$, i.e. (right) $A$-modules $M$ together with an $A$-valued inner product $\langle \cdot, \cdot \rangle : M \times M \rightarrow A$ satisfying the following conditions:

1. $\langle x, x \rangle \geq 0$ for every $x \in M$ and $\langle x, x \rangle = 0$ iff $x = 0$,
2. $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in M$,
3. $\langle \cdot, \cdot \rangle$ is $A$-linear in the second argument,
4. $M$ is complete with respect to the norm $\|x\|^2 = \|\langle x, x \rangle\|_A$.

By $M^* = \text{Hom}_A(M; A)$ we denote the $A$-module dual to $M$. Let $H_A$ be a right Hilbert $A$-module of sequences $a = (a_k), a_k \in A, k \in \mathbb{N}$ such that the series $\sum a_k^* a_k$ converges in $A$ in norm with the standard basis $\{e_k\}$ and let $L_n(A) \subset H_A$ be a submodule generated by the elements $e_1, \ldots, e_n$ of the basis. An inner $A$-valued product on module $H_A$ is given by $\langle x, y \rangle = \sum x_k^* y_k$ for $x, y \in A$. A bounded operator $K : H_A \rightarrow H_A$ is called compact if it possesses an adjoint operator and lies in the norm closure of the linear span of operators of the form $\theta_{x,y}, \theta_{x,y}(z) = x \langle y, z \rangle, x, y, z \in H_A$. From now on we
suppose that the compact operator $K$ is strictly positive, i.e. operator $\langle Kx, x \rangle$ is positive in $A$ and $\text{Ker} \ K = 0$. It is known [4] that in the case when $A$ is a $W^*$-algebra the inner product can be naturally prolonged to the dual module $\mathcal{H}_A^*$.

Definition 1.1. Let $A$ be a $W^*$-algebra. We call an operator $K$ diagonalizable if there exist a set $\{x_i\}$ of elements in $\mathcal{H}_A^*$ and a set of operators $\lambda \in A$ such that

i) $\{x_i\}$ is orthonormal, $\langle x_i, x_j \rangle = \delta_{ij}$,

ii) $\mathcal{H}_A^*$ coincides with the $A$-module $\mathcal{M}^*$ dual to the module $\mathcal{M}$ generated by the set $\{x_i\}$,

iii) $Kx_i = x_i \lambda_i$,

iv) for any unitaries $u_i, u_{i+1} \in A$ we have an operator inequality

$$u_i^* \lambda_i u_i \geq u_{i+1}^* \lambda_{i+1} u_{i+1}. \quad (1.1)$$

We call the elements $x_i$ “eigenvectors” and the operators $\lambda_i$ “eigenvalues” for the operator $K$. It must be noticed that the “eigenvectors” and “eigenvalues” are defined not uniquely.

The problem of diagonalizing operators in Hilbert modules was initiated by R. V. Kadison in [6] and was studied in different settings in [5],[11],[4],[15] etc. In [9],[10] we have proved the following

Theorem 1.2. If $A$ is a finite $\sigma$-finite $W^*$-algebra then a compact strictly positive operator $K$ can be diagonalized and its “eigenvalues” are defined uniquely up to unitary equivalence.

It is well known that in the commutative case, i.e. for $C = C(X)$ being a commutative $C^*$-algebra, compact operators cannot be diagonalized inside $\mathcal{H}_C$ but it becomes possible if we pass to a bigger module over a bigger $W^*$-algebra $L^\infty(X) \supset C$. It leads us to the following

Definition 1.3. Let $C$ be a $C^*$-algebra admitting a weakly dense inclusion in a finite $\sigma$-finite $W^*$-algebra $A$ and let $K$ be a compact strictly positive operator in $\mathcal{H}_C$. We can naturally extend $K$ to the bigger module $H_A^*$ where it will remain compact and strictly positive and by the theorem 1.2 it can be diagonalized in this module. We call a $C^*$-algebra $C$ admitting weak diagonalization if the diagonal entries for any $K$ in $\mathcal{H}_A^*$ can be taken from $C$ instead of $A$.

Problem. Describe the class of $C^*$-algebras admitting weak diagonalization.

Throughout this paper we denote by $A$ a finite $\sigma$-finite $W^*$-algebra. Denote by $Z = C(Z)$ the center of $A$ and by $T$ the standard exact center-valued trace
defined on $A$, $T(1) = 1$. Suppose that for a $C^*$-subalgebra $B$ of $A$ the following condition holds:

(*) for any two projections $p, q \in B$ there exist in $B$ equivalent (in $B$) projections $r_p \sim r_q$, $r_p \leq p$, $r_q \leq q$ such that $T(r_p) = T(r_q) = \min\{T(p)(z), T(q)(z)\}$, $z \in \mathbb{Z}$.

The purpose of this paper is to show that the class of $C^*$-algebras admitting weak diagonalization contains real rank zero weakly dense $C^*$-subalgebras of finite $\sigma$-finite $W^*$-algebras with the property (*). Recall that real rank zero ($\text{RR}(B) = 0$) means [2] that every selfadjoint operator in $B$ can be approximated by operators with finite spectrum, i.e. having the form $\sum \alpha_i p_i$, where $p_i \in B$ are selfadjoint mutually orthogonal projections and $\alpha_i \in \mathbb{R}$. By [2] we have in this case also $\text{RR}(\text{End}_B(L_n(B))) = 0$.

2 Continuity of “eigenvalues”

For the further we need to establish some continuity properties of the “eigenvalues” of compact operators in modules over $W^*$-algebras.

**Lemma 2.1.** Let $K_1 = \sum \alpha_i^{(1)} P_i^{(1)}$, $K_2 = \sum \alpha_i^{(2)} P_i^{(2)}$ be strictly positive operators in $L_n(A)$ with finite spectrum and let $\|K_1 - K_2\| < \varepsilon$. Then

i) one can find a unitary $U$ in $L_n(A)$ such that it maps the “eigenvectors” of $K_2$ to the “eigenvectors” of $K_1$ and $\|U^* K_1 U - K_2\| < \varepsilon$,

ii) “eigenvalues” $\{\lambda_i^{(r)}\}$ of operators $K_r$ ($r = 1, 2$) can be chosen in such a way that $\|\lambda_i^{(1)} - \lambda_i^{(2)}\| < \varepsilon$.

**Proof.** As the algebra $A$ can be decomposed into a direct integral of finite factors, so it is sufficient to prove the lemma for the case when $A$ is a type $\Pi_1$ factor (for type $\text{I}_n$ factors lemma is trivial). Denote by $E_\mathcal{K}(\lambda)$ the spectral projection for the operator $\mathcal{K}$ corresponding to the set $(-\infty, \lambda)$. If $\tau$ is an exact finite trace on $A$, it can be prolonged to the (infinite) trace $\bar{\tau} = \text{tr} \otimes \tau$ on the algebra $\text{End}_A(H_A^*)$ and to the finite trace on a lesser algebra $\text{End}_A(L_n(A))$ where we have $\bar{\tau}(1) = n$. Put

$$\varepsilon_\mathcal{K}(\alpha) = \inf_{\tau(E_\mathcal{K}(\lambda)) \geq \alpha} \lambda, \quad 0 \leq \alpha \leq n.$$  

As it is shown in [12] (the continuous minimax principle) one has

$$\varepsilon_\mathcal{K}(\alpha) = \inf_{P \in \mathcal{P}, \bar{\tau}(P) \geq \alpha} \left\{ \sup_{\xi \in \text{Im} P, \|\xi\| = 1} (\mathcal{K}\xi, \xi) \right\}, \quad (2.1)$$
where \((\cdot, \cdot)\) denotes an inner product in a Hilbert space where the algebra \(\text{End}_A(L_n(A))\) is represented and \(P\) denotes the set of projections in \(\text{End}_A(L_n(A))\). It follows from (2.1) that if \(\|K_1 - K_2\| < \varepsilon\), then

\[ |\varepsilon_{K_1}(\alpha) - \varepsilon_{K_2}(\alpha)| < \varepsilon. \]  

(2.2)

Let \(Q_i^{(r)}\) be projections on the “eigenvectors” \(x_i^{(r)}\) of the operators \(K_r\), corresponding to the maximal “eigenvalues” \(\lambda_i^{(r)}\), \(\bar{\tau}(Q_i^{(r)}) = 1\). For two divisions \(\{P_i^{(1)}, Q_i^{(1)}\}\) and \(\{P_i^{(2)}, Q_i^{(2)}\}\) of unity given by decompositions of \(K_1\) and \(K_2\) we can construct a finer division of unity. By [10] there exist sets of mutually orthogonal projections \(R_i^{(r)} \in \text{End}_A(L_n(A))\) such that

\[ i) \bigoplus_m R_m^{(r)} = 1, \]

\[ ii) \bar{\tau}(R_m^{(1)}) = \bar{\tau}(R_m^{(2)}), \]

\[ iii) \text{for every } m \text{ we have } R_m^{(r)} \leq Q_i^{(r)} \text{ or } R_m^{(r)} \leq P_j^{(r)} \text{ for some } i \text{ or } j. \]

Then (after renumbering) one can write the operators \(K_r\) in the form \(K_r = \sum \alpha_m^{(r)} R_m^{(r)}\) with \(\alpha_1^{(r)} \leq \alpha_2^{(r)} \leq \ldots, \alpha_m^{(r)} \in \mathbb{R}\). It makes possible to define a unitary \(U : L_n(A) \rightarrow L_n(A)\) such that

\[ U(\text{Im} R_m^{(2)}) = \text{Im} R_m^{(1)}, \]  

(2.3)

hence \(U(\text{Im} Q_i^{(2)}) = \text{Im} Q_i^{(1)}\) so \(U\) maps the \(A\)-modules generated by the “eigenvectors” \(x_i^{(1)}\) into the modules generated by \(x_i^{(2)}\), hence \(U x_i^{(2)} = x_i^{(1)} \cdot u_i = \bar{x}_i^{(1)}\) for some unitaries \(u_i \in A\). Put

\[ n(\alpha) = \min\{n | \bar{\tau}(\bigoplus_{m \geq n} R_m^{(r)}) \geq \alpha\}. \]

Then \(\varepsilon_{K(\alpha)} = \alpha_n^{(r)}\) and it follows from (2.2) that \(|\alpha_{n(\alpha)}^{(1)} - \alpha_{n(\alpha)}^{(2)}| < \varepsilon\). But changing \(\alpha\) we obtain that

\[ |\alpha_m^{(1)} - \alpha_m^{(2)}| < \varepsilon \]  

(2.4)

for all \(m\). Taking \(\alpha = 1\) (then \(i = 1\)) we have

\[ K_r|_{\text{Im} Q_1^{(r)}} = \Lambda_1^{(r)} = \sum_{m \geq n(1)} \alpha_m^{(r)} P_m^{(r)}. \]

From (2.3) and (2.4) we conclude that

\[ \|U^* A_1^{(1)} U - A_1^{(2)}\| = \| \sum_{m \geq n(1)} (\alpha_m^{(1)} - \alpha_m^{(2)}) P_m^{(2)} \| \leq \varepsilon \| \bigoplus_{m \geq n(1)} P_m^{(2)} \| = \varepsilon. \]  

(2.5)
Choosing appropriate $\lambda_1^{(r)}$ to satisfy the conditions $\Lambda_1^{(1)} \bar{x}_1^{(1)} = \bar{x}_1^{(1)} \lambda_1^{(1)}$ and $\Lambda_1^{(2)} x_1^{(2)} = x_1^{(2)} \lambda_1^{(2)}$ we obtain the estimate
\[
\|\lambda_1^{(1)} - \lambda_1^{(2)}\| < \varepsilon. \tag{2.6}
\]
By the same way estimates (2.5), (2.6) can be obtained for all $i$ and it proves the lemma.

**Corollary 2.2.** Let $K_r : \mathcal{H}_A \rightarrow \mathcal{H}_A$, $r = 1, 2$ be compact strictly positive operators and let $\|K_1 - K_2\| < \varepsilon$. Then

i) one can find a unitary $U$ in $\mathcal{H}_A^*$ such that it maps the “eigenvectors” of $K_2$ to the “eigenvectors” of $K_1$ and $\|U^* K_1 U - K_2\| < \varepsilon$,

ii) “eigenvalues” $\{\lambda_i^{(r)}\}$ of operators $K_r$ ($r = 1, 2$) can be chosen in such a way that $\|\lambda_i^{(1)} - \lambda_i^{(2)}\| < \varepsilon$.

**Proof.** Let $L_n^{(r)}(A) \in \mathcal{H}_A^*$ denotes the Hilbert submodule generated by the first $n$ “eigenvectors” of the operator $K_r$, $L_n^{(r)}(A) \cong L_n(A)$. It was shown in [11] that the orthogonal complement to such submodule is isomorphic to $\mathcal{H}_A^*$ and the norm of restriction of compact operator $K_n$ on the orthogonal complement to $L_n^{(r)}(A)$ in $\mathcal{H}_A^*$ tends to zero, henceforth it is sufficient to consider only the case of operators in $L_n(A)$ and there one can approximate these operators by operators with finite spectrum.

### 3 Case of $RR(B) = 0$

In this section we show that $C^*$-algebras of real rank zero with the property $(\ast)$ admit weak diagonalization.

**Theorem 3.1.** Let $B$ be a weakly dense $C^*$-subalgebra in $A$ with the property $(\ast)$ and let $RR(B) = 0$. If $K$ is a compact strictly positive operator in the $B$-module $\mathcal{H}_B$ then the “eigenvalues” $\{\lambda_i\}$ of diagonalization of the natural prolongation of $K$ to the $A$-module $\mathcal{H}_A^*$ can be chosen in a way that $\lambda_i \in B$ would hold.

**Proof** is based on the results of S. Zhang [17]. By [2, 17] the operator $K$ can be approximated by operators $K_n \in \text{End}_B(L_n(B))$ with finite spectrum. By [17], corollary 3.5 there exist such unitaries $U_n \in \text{End}_B(L_n(B))$ that the operators
\[
U_n^* K_n U_n = \begin{pmatrix}
\lambda_1^{(n)} & 0 & \cdots \\
0 & \ddots & \vdots \\
0 & \cdots & \lambda_n^{(n)}
\end{pmatrix}
\]
are diagonal and \( \lambda_i^{(n)} \in \mathcal{B} \) are operators with finite spectrum. Show that due to the property (\( \ast \)) by an appropriate choice of such \( U_n \) one can make the condition (\[3.1\]) valid for “eigenvalues” \( \{\lambda_i^{(n)}\} \). Let \( \lambda_a = \sum_{k} \alpha_k q_k, \lambda_b = \sum_{l} \beta_l r_l \) where \( q_k, r_l \in \mathcal{B} \) are projections and suppose that \( a < b \) but for some \( m \) and \( n \) inequality \( \beta_m > \alpha_n \) holds. Using the possibility to diagonalize projections \[3.1\] we can find projections \( s_l \in \mathcal{B} \) equivalent to \( r_l \) and such that \( s_l = \oplus_k s_k^{(l)} \) and \( s_k^{(l)} \leq q_k \). Then put

\[
\lambda'_a = \sum_{k \neq n} \alpha_k q_k \oplus \sum_{l \neq m} \alpha_n s_n^{(l)} \oplus \beta_n s_n^{(m)},
\]

\[
\lambda'_b = \sum_{l \neq m} \beta_l s_l \oplus \sum_{k \neq n} \beta_k s_k^{(m)} \oplus \alpha_n s_n^{(m)}
\]

and notice that the operators

\[
\begin{pmatrix}
\lambda_1 & 0 & \cdots \\
0 & \ddots & 0 \\
\cdots & \cdots & \lambda_n
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
\lambda'_1 & 0 & \cdots \\
0 & \ddots & 0 \\
\cdots & \cdots & \lambda'_n
\end{pmatrix}
\]

are unitarily equivalent. After repeating this procedure for all cases when \( \beta_l > \alpha_k \) we obtain validity of (\[3.1\]) for \( \lambda'_a \) and \( \lambda'_b \). By the same way we can order all “eigenvalues” of \( \mathcal{K}_n \) remaining in \( \mathcal{B} \). But by the property (\( \ast \)) if \( \|\mathcal{K}_n - \mathcal{K}_{n-1}\| < \varepsilon_n \) then one can find such unitaries \( u_{i,n} \) in \( \mathcal{B} \) that

\[
\| u_{i,n}^* \lambda_i^{(n)} u_{i,n} - \lambda_i^{(n-1)} \| < \varepsilon_n
\]

Then \( u_{i,n}^* \lambda_i^{(n)} u_{i,n} \in \mathcal{B} \). Taking a subsequence of \( \{\mathcal{K}_n\} \) if necessary we can take in (\[3.1\]) \( \varepsilon_n = \frac{1}{2^n} \). Then the sequence

\[
\tilde{\lambda}_i^{(1)} = \lambda_i^{(1)}, \quad \tilde{\lambda}_i^{(2)} = u_{i,2}^* \lambda_i^{(2)} u_{i,2}, \quad \tilde{\lambda}_i^{(3)} = u_{i,3}^* u_{i,2}^* \lambda_i^{(3)} u_{i,2} u_{i,3}, \ldots
\]

is fundamental in \( \mathcal{B} \). Denote its limit by \( \tilde{\lambda}_i \in \mathcal{B} \). By the corollary 2.2 for all \( \mathcal{K}_n \) we can find unitaries \( U_n \) which map the first \( n \) “eigenvectors” of \( \mathcal{K} \) to “eigenvectors” of \( \mathcal{K}_n \). Put \( \mathcal{K}'_n = U_n^* \mathcal{K}_n U_n \in \text{End}_A(\mathcal{H}'_n) \). Then we have

\[
\mathcal{K}_n x_i = x_i \tilde{\lambda}_i^{(n)} \quad \text{(3.2)}
\]

and \( \|\mathcal{K}'_n - \mathcal{K}\| \to 0 \). Taking limit in (\[3.2\]) we obtain \( \mathcal{K} x_i = x_i \tilde{\lambda}_i \), hence \( \tilde{\lambda}_i \) are “eigenvalues” of \( \mathcal{K} \). .

Notice that the condition (\( \ast \)) is necessary for a \( C^* \)-algebra to have the weak diagonalization property. Indeed if \( \mathcal{K} \) is a direct sum of two projections, \( \mathcal{K} = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \) then the “eigenvalues” of \( \mathcal{K} \) can be ordered only if the “common part” of \( 1 - p \) and \( q \) lies in \( \mathcal{B} \).

**Remark.** In the case of \( C^* \)-algebras \( A_\theta \) of irrational rotation one has \( RR(A_\theta) = 0 \) (cf [3]) and the property (\( \ast \)) is valid, so the theorem 3.1 gives
the answer to the problem of [1] where we have considered the Schrödinger operator in magnetic field with irrational magnetic flow. It is known that this operator can be viewed as an operator acting in a Hilbert $A_{\theta}$-module. As we can imbed $A_{\theta}$ in a type II$_1$ factor $\mathcal{A}$ as a weakly dense subalgebra [1] so we can diagonalize this operator in a Hilbert $\mathcal{A}$-module. The present paper shows that the “eigenvalues” of this operator can be chosen to be elements of $A_{\theta}$. So this situation is a noncommutative analogue of the case $\theta = 1$ when the corresponding operator can be diagonalized over $W^*$-algebra $L^\infty(T^2)$ but the diagonal elements lie in a lesser $C^*$-algebra $C(T^2)$. Notice that in case of rational $\theta$ this operator is also diagonalizable.

Acknowledgement. This work was partially supported by the Russian Foundation for Fundamental Research (grant N 94-01-00108-a) and the International Science Foundation (grant N MGM000). I am indebted to M. Frank, A. A. Irmatov, A. S. Mishchenko and E. V. Troitsky for helpful discussions.

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