Generation of maximally entangled mixed states of two atoms via on-resonance asymmetric atom-cavity couplings

Shang-Bin Li

Shanghai research center of Amertron-global, Zhangjiang High-Tech Park, 299 Lane, Bisheng Road, No. 3, Suite 202, Shanghai, 201204, P.R. China

A scheme for generating the maximally entangled mixed state of two atoms on-resonance asymmetrically coupled to a single mode optical cavity field is presented. The part frontier of both maximally entangled mixed states and maximal Bell violating mixed states can be approximately reached by the evolving reduced density matrix of two atoms if the ratio of coupling strengths of two atoms is appropriately controlled. It is also shown that exchange symmetry of global maximal concurrence is broken if and only if coupling strength ratio lies between $\sqrt{3}$ and $\sqrt{3}$ for the case of one-particle excitation and asymmetric coupling, while this partial symmetry-breaking cannot be verified by detecting maximal Bell violation.

PACS numbers: 03.67.-a, 03.65.Ud

Quantum entanglement plays a crucial role in quantum information processes [1]. In the last three years, much attention has been paid to the characterization and preparation of the maximally entangled mixed state [2-6]. Maximally entangled mixed states (MEMS) are those states that, for a given mixedness, achieve the greatest possible entanglement. For two-qubit systems and for various combinations of entanglement and mixedness measures, the form of the corresponding MEMS has been analyzed [6]. At present, most of the experimental verifications of the MEMS are based on polarized photons. The cavity QED systems have been recognized as an important candidate for implementing various kinds of quantum information processes, and the precise control of the coupling of individual atoms to a high-finesse optical cavity have been demonstrated experimentally [6, 7]. Current laboratory technologies have demonstrated the feasibility of generating maximally entangled state of two two-level atoms [8-10]. For generating the MEMS, the theoretical schemes based on the cavity QED or collective interaction with environment have been proposed [11, 12, 13]. For entangling atoms, on-resonance asymmetric coupling of individual atoms and cavity field has exhibited certain advantage over the large-detuning symmetric coupling because resonant cavity QED offers faster entanglement schemes [11, 12]. This motivates us to propose a feasible scheme for preparing the MEMS of two two-level atoms on-resonance asymmetrically coupling to single mode high-finesse optical cavity. It is found that the part frontier of both MEMS and maximal Bell violating mixed states (MBVMS) can be approximately reached by the evolving reduced density matrix of two atoms if the ratio of coupling strengths of two atoms is appropriately controlled. It is also shown that exchange symmetry of global maximal concurrence is broken if and only if coupling strength ratio lies between $\sqrt{3}$ and $\sqrt{3}$ for the case of one-particle excitation and asymmetric coupling. However, the global maximum of maximal Bell violation keep invariant under the exchange of two atoms, which implies this partial symmetry-breaking can not be verified by detecting maximal Bell violation. In the case of two-particle excitation, the maximally entangled state of two atom can also be generated when the coupling strength ratio is near 0.18. As the ratio of coupling strengths tends to 1, the critical-phenomenon-like behaviors of the global maximal entanglement or Bell violation can be found.

The system discussed here consists of two two-level atoms confined in a linear trap which has been surrounding by an optical cavity [16]. We refer to atom 1 and atom 2. The Hamiltonian describing the system is given by [17] ($\hbar = 1$)

$$H = \frac{\omega_1}{2}\sigma_1^{(1)} + \frac{\omega_2}{2}\sigma_2^{(2)} + \omega_a a^\dagger a$$

$$+ \lambda_1 \theta_1(t) (a^\dagger \sigma_1^{(1)} + a \sigma_1^{(1)}) + \lambda_2 \theta_2(t) (a^\dagger \sigma_2^{(2)} + a \sigma_2^{(2)})$$

where $a$ and $a^\dagger$ denote the annihilation and creation operators for the single mode cavity field, and $\sigma_1^{(1)} = |e_i\rangle\langle e_i| - |g_i\rangle\langle g_i|$, $\sigma_2^{(2)} = |e_i\rangle\langle g_i|$, $\sigma_2^{(1)} = |g_i\rangle\langle e_i|$ ($i = 1, 2$) are the atomic operators. $\omega_1$ and $\omega_2$ are the atomic transition frequencies of atom 1 and atom 2, respectively. $\lambda_1$ and $\lambda_2$ are the coupling strengths between the cavity field and atom 1, atom 2, respectively. $\theta_i(t)$ ($i = 1, 2$) represent the time dependence of the atom-cavity coupling. Plenio et al. have discussed the generation of maximally entangled states in such a system [17], in which the cavity decay is continuously monitored. Here we investigate two two-level atoms resonantly coupling with one mode optical cavity, i.e. $\omega_1 = \omega_2 = \omega_a = \omega$ and $\theta_1(t) = \theta_2(t) = \theta(t)$.

We assume that the initial state of the system (1) is described by the density matrix $\rho(0) = |0\rangle\langle 0| \otimes |eg\rangle\langle eg|$, i.e., atom 1 is in the excited state, atom 2 is in the

*Electronic address: stephenli74@yahoo.com.cn
ground state and the cavity field is in vacuum state, respectively. Substituting $\rho(0)$ into the Schrödinger equation, and tracing out the degree of freedom of the cavity field, we could obtain the reduced density matrix $\rho_a(t)$ describing the time evolution of two atoms.

$$
\rho_a(t) = \frac{\lambda^2}{2\lambda^2}(1 - \cos 2\Theta(t))|gg\rangle\langle gg|
+ \frac{\lambda^2}{2\lambda^2}(1 + \cos 2\Theta(t))|B_2\rangle\langle B_2|
+ \frac{\lambda^2}{\lambda^2}|B_1\rangle\langle B_1|
- \frac{\lambda_1\lambda_2}{\lambda^2}\cos\Theta(t)(|B_1\rangle\langle B_2| + |B_2\rangle\langle B_1|),
$$

where, $|B_1\rangle = \frac{1}{\sqrt{2}}(\lambda_1|ge\rangle - \lambda_2|eg\rangle)$, $|B_2\rangle = \frac{1}{\sqrt{2}}(\lambda_1|eg\rangle + \lambda_2|ge\rangle)$, and $\lambda = \sqrt{\lambda_1^2 + \lambda_2^2}$ and $\Theta(t) = \lambda \int_0^t \theta(\tau)d\tau$.

In order to quantify the degree of entanglement, we adopt the concurrence to calculate the entanglement between two atoms. The concurrence related to the density operator $\rho$ of a mixed state is defined by \[18\]

$$
C(\rho) = \max\{|\delta_1 - \delta_2 - \delta_3 - \delta_4, 0\},
$$

where the $\delta_i$ $(i = 1, 2, 3, 4$) are the square roots of the eigenvalues in decreasing order of magnitude of the "spin-flipped" density operator $R$

$$
R = \rho(\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y),
$$

where the asterisk indicates complex conjugation.

The explicit expression of the concurrence $C(t)$ characterizing the entanglement of two atoms can be found to be

$$
C(t) = \left| \frac{\lambda_1^2\lambda_2 - 2\lambda_1\lambda_2^3}{\lambda^4}
+ \frac{2\lambda_1\lambda_2(\lambda_2^2 - \lambda_1^2)}{\lambda^4}\cos\Theta(t)
+ \frac{\lambda_1^3\lambda_2}{\lambda^4}\cos 2\Theta(t) \right|,
$$

where $|x|$ gives the absolute value of $x$.

Here, we confine our consideration in the MEMS in linear entropy-concurrence plane. For the reduced density matrix $\rho_a$ in Eq. (2), the linear entropy is defined by $M = \frac{1}{3}(1 - Tr\rho_a^2)$ which can be used to quantify the mixedness of two atoms. In Fig. 1, the concurrence versus mixedness of two atoms are depicted for different values of the coupling strength ratio. In each trajectory labelled by different values of $\Theta(t)$, the data changes with $\gamma_2/\gamma_1$. In this case, part of the frontier of the concurrence versus linear entropy can be reached by the evolving reduced density matrix of two atoms. It means that the MEMS can be generated in the cavity QED system (1) if the ratio of coupling strengths of two atoms is appropriately controlled. Not only pure entangled states possessing any desired degree of entanglement can be determinedistically generated, but also the mixed states possessing any possible degree of entanglement can be controlled prepared if the desired linear entropy does not exceed a threshold value near 0.65.

Next we discuss the Bell violation of two atoms in this system. The most commonly discussed Bell inequality is the CHSH inequality \[13\, 20\]. The CHSH operator reads

$$
\hat{B} = \hat{a} \cdot \hat{\sigma} \otimes (\hat{b} + \hat{b}') \cdot \hat{\sigma} + \hat{a}' \cdot \hat{\sigma} \otimes (\hat{b} - \hat{b}') \cdot \hat{\sigma},
$$

where $\hat{a}, \hat{a}', \hat{b}, \hat{b}'$ are unit vectors. In the above notation, the Bell inequality reads

$$
|\langle \hat{B} \rangle| \leq 2.
$$

The maximal amount of Bell violation of a state $\rho$ is given by \[21\]

$$
|\mathcal{B}|_{\text{max}} = 2\sqrt{\kappa + \tilde{\kappa}},
$$

where $\kappa$ and $\tilde{\kappa}$ are the two largest eigenvalues of $T_\rho^T T_\rho$. The matrix $T_\rho$ is determined completely by the correlation functions being a $3 \times 3$ matrix whose elements are $(T_\rho)_{nm} = Tr(\rho \sigma_n \otimes \sigma_m)$. Here, $\sigma_1 \equiv \sigma_x$, $\sigma_2 \equiv \sigma_y$, and $\sigma_3 \equiv \sigma_z$ denote the usual Pauli matrices. We call the quantity $|\mathcal{B}|_{\text{max}}$ the maximal violation measure, which indicates the Bell violation when $|\mathcal{B}|_{\text{max}} > 2$ and the maximal violation when $|\mathcal{B}|_{\text{max}} = 2\sqrt{2}$.

For the density operator $\rho_a$ in Eq. (2), $\kappa + \tilde{\kappa}$ can be written as follows

$$
\kappa + \tilde{\kappa} = C^2 + \max[C^2, (1 - \lambda_1^2(1 - \cos 2\Theta(t)))^2],
$$

In Ref. \[6\], the analytical form of the mixed states which possesses the maximal value of $|\mathcal{B}|_{\text{max}}$ of two qubits for a given linear entropy has been derived. Now we show that part of the frontier of the maximal Bell violation versus
FIG. 2: The maximal Bell violation $|B|_{\text{max}}$ versus the mixedness of two atoms is displayed. Different trajectories are chosen from $\frac{\lambda_2}{\lambda_1} \in [0, 0.4]$ for different values of $\Theta(t) \in [0, 1, 3.0]$. The solid square data points represent the frontier of maximal Bell violation versus the linear entropy, namely, for a given linear entropy, the maximal value of $|B|_{\text{max}}$ of two atoms can not exceed the solid square data points.

FIG. 3: (a) The maximal value of the concurrence of two atoms during the whole evolution is plotted as the function of the rate of $\lambda_2/\lambda_1$. At $\lambda_2/\lambda_1 = \sqrt{2} - 1$ or $\lambda_2/\lambda_1 = \sqrt{2} + 1$, two atoms can achieve the maximally entangled state. It reflects the certain kind of partial structural symmetry of the initial state. However, we can also observer that the local minimum of the global maximal concurrence is achieved at $\lambda_2/\lambda_1 = \sqrt{32} - 5$ but not at $\lambda_2/\lambda_1 = 1$, which is different with the case of global maximum of $|B|_{\text{max}}$. (b) The maximal value of $|B|_{\text{max}}$ of two atoms during the whole evolution is plotted as the function of the rate of $\lambda_2/\lambda_1$.

The linear entropy can be approximately approached by two atoms. In Fig. 2, the Bell violation and linear entropy of the atom 1 and 2 are depicted. It can be observed that reduced density matrix of two atoms can evolve into the vicinity of the frontier of $|B|_{\text{max}}$ versus $M$ if appropriate ratio of coupling strengths and the time $t$ are chosen. Though the gap between the MBVMS and the possible nearest reduced density matrix becomes large when the linear entropy exceeds beyond 0.5, the MBVMS with small mixedness can be approximately generated with very high fidelity.

In ref. [22, 23], it has been shown that entanglement and Bell violation of two qubits may be inconsistent with each other in certain kinds of dynamical processes. Here, through investigating the global maximal values of concurrence and Bell violation of two atoms initially in $|eg\rangle$, we obtain more information about their dynamical discrepancy. From the Eq. (5), one can easily derive the maximal value of the concurrence of two atoms possibly achieved during the whole evolution as follows:

$$C_M \equiv \max_{t \in [0, \infty)} C(t) = \frac{4|\chi - \chi^3|}{(1 + \chi^2)^2}, \quad (10)$$

when $\Theta(t) = \pi$ if $0 < \chi < \sqrt{32} - 5$ or $\chi > \sqrt{3}$. Otherwise,

$$C_M = \frac{\chi}{2}, \quad (11)$$

when $\Theta(t) = \arccos(\frac{\chi}{2})$ if $\sqrt{32} - 5 \leq \chi \leq \sqrt{3}$. In the above two equation, $\chi \equiv \frac{\lambda_2}{\lambda_1}$. In deriving the above or following maximal value, $\Theta(t)$ is assumed to be large enough as $t \to \infty$. The maximum of the concurrence of two atoms initially in $|eg\rangle$ completely depends on the rate of $\chi$. Two atoms can become the maximally entangled if $\chi = \sqrt{2} + 1$ or $\chi = \sqrt{2} - 1$ [14]. When two atoms symmetrically couples to the cavity mode, the maximum of the concurrence is $\frac{\chi}{4}$. In addition, in the cases with $1 < \chi < \sqrt{3}$ or $\frac{\chi}{2} < \chi < 1$, the maximum of the concurrence can not keep invariant under the transformation of $\chi \leftrightarrow \frac{1}{\chi}$, which implies that, in those asymmetrically resonant coupled systems with $1 < \lambda_2/\lambda_1 < \sqrt{3}$, two kinds of initial states $|eg\rangle \otimes |0\rangle$ and $|ge\rangle \otimes |0\rangle$ result in the different maximum of the concurrence, namely, the exchange symmetry is destroyed in this situation. However, for other cases with $\chi \geq \sqrt{3}$ or $\chi \leq \frac{\sqrt{3}}{2}$, the maximum of the concurrence can keep invariant under the transformation of $\chi \leftrightarrow \frac{1}{\chi}$ or the exchange of $|eg\rangle \leftrightarrow |ge\rangle$. In Fig. 3(a), we plot the maximum of the concurrence as the function of the rate $\lambda_2/\lambda_1$. It is shown that the maximum of the concurrence $C_M$ of two atoms firstly increases from zero to 1 with the increase of $\lambda_2/\lambda_1$ from zero to $\sqrt{2} - 1$, then decreases from 1 to $\sqrt{32} - 5$ with the increase of $\lambda_2/\lambda_1$ from $\sqrt{2} - 1$ to $\sqrt{32} - 5$. Furthermore, it increases from $\sqrt{32} - 5$ to $1 + \sqrt{2}$, and then $C_M$ decreases with the further increase of $\lambda_2/\lambda_1$. Further calculation shows $\frac{\partial^2 C_M}{\partial \lambda_1^2}$ is discontinuous at $\chi = \sqrt{3}$.

Then, we turn to investigate the global maximal value of Bell violation $|B|_{\text{max}}$ of two atoms during the whole evolution. From the Eqs. (8, 9), we can easily derive the analytical expression for the maximal value of Bell violation of two atoms,

$$|B|_{\text{max}}^{(M)} \equiv \max_{t \in [0, \infty)} |B|_{\text{max}} = 2\sqrt{1 + \frac{(4\chi - 4\chi^3)^2}{(1 + \chi^2)^4}}, \quad (12)$$

which is achieved by two atoms at the time $\Theta(t) = \pi$. Surprisingly, the expression in Eq. (12) keeps invariant
under the transformation of $\chi \leftrightarrow \frac{1}{\lambda}$ for the whole range of the parameter $\chi$. In Fig.3(b), the global maximum of Bell violation is plotted as the function of the rate $\lambda_2/\lambda_1$. It is shown that $|B|^{(M)}_{\text{max}}$ exhibits the similar behavior to the $C_M$ in most of the parameter $\chi$ except for $\sqrt{32} - 5 < \chi < 1$, in which range $C_M$ increases but $|B|^{(M)}_{\text{max}}$ decreases.

The above analysis mainly focuses on the specific initial states, i.e. the $|eg\rangle \otimes |0\rangle$ or $|ge\rangle \otimes |0\rangle$. In what follows, we further consider the case that two atoms are initially in the excited state $|ee\rangle$ and the cavity field is still in the vacuum state $|0\rangle$. In Fig.4(a), the maximal value of pairwise concurrence of two atoms during the whole evolution is plotted as the function of the coupling coefficients rate $\lambda_2/\lambda_1$. It is shown that the maximal value of pairwise concurrence firstly increases with $\lambda_2/\lambda_1$ and achieves 1 at $\lambda_2/\lambda_1 \approx 0.18$, then decreases with $\lambda_2/\lambda_1$. An abrupt decline from 0.5 to 1 can be observed as $\lambda_2/\lambda_1 \rightarrow 1$. In Fig.4(b), the global maximal value of $|B|_{\text{max}}^{(M)}$ of two atoms during the whole evolution is plotted as the function of the coupling coefficients ratio $\lambda_2/\lambda_1$. $|B|_{\text{max}}^{(M)}$ exhibits similar dependence on $\lambda_2/\lambda_1$ with the entanglement among all range of $\lambda_2/\lambda_1$ in two-particle excitation case. A interesting point is that we can simultaneously obtain pure single-photon state in cavity field and a maximally entangled state of two atoms in the situation of $\lambda_2/\lambda_1 \approx 0.18$. Both the single-photon state of the cavity field and the maximally entangled state of two atoms are the valuable resource in quantum information processes. Therefore, it is hopeful that this scheme may have very significant applications. One may conjecture that the influence of spontaneous emission and cavity decay on the generation of maximally entangled state in two-particle excitation case is more significant than the one-particle excitation case. For fixing this potential weakness, the timing single-photon detection may be a good choice for purifying the entangled state of two atoms. The details will be discussed elsewhere.

In summary, the generation of the maximally entangled mixed state of two atoms which are asymmetrically coupled to a single mode optical cavity field is analyzed. It is shown that two atoms can achieve the maximally entangled mixed state or the maximal Bell violating mixed states with high fidelity in the on-resonance asymmetric coupling case. It is found that, for those cases with $\frac{1}{3} < \lambda_1 < \sqrt{3}$ and $\lambda_1 \neq \lambda_2$, exchange symmetry of global maximal concurrence is broken when the initial state of two atoms interchanges between $|eg\rangle$ and $|ge\rangle$. However, the global maximum of maximal Bell violation keep invariant under the exchange of two atoms. In the case of two-particle excitation, the maximally entangled state of two atom can also be generated when the coupling strength ratio is near 0.18. As the ratio of coupling strengths tends to 1, the critical-phenomenon-like behaviors of the global maximal entanglement or Bell violation can be found. Though these results presented here can be directly generalized to other systems such as two dipolar-coupled quantum dots in photonic crystal microcavity, three-qubit Heisenberg spin chain, and trapped ions coupled to motional degree of freedom etc, full considerations of the effects of corresponding decoherence mechanisms on the preparation of MEMS or the partial exchange-symmetry-breaking of global maximal entanglement are nontrivial and very desirable.

The author thanks Prof. X.-B. Zou and Prof. J.-B. Xu for helpful discussion.

[1] C. Monroe, Nature 416, 238 (2002).
[2] N.A. Peters et al., Phys. Rev. Lett. 92, 133601 (2004).
[3] M. Barbieri et al., Phys. Rev. Lett. 92, 177901 (2004).
[4] S. Ishizaka and T. Hiroshima, Phys. Rev. A 62, 022310 (2000).
[5] F. Verstraete, K. Audenaert and B. De Moor, Phys. Rev. A 64, 012316 (2001).
[6] T.C. Wei et al., Phys. Rev. A 67, 022110 (2003).
[7] G.R. Guthöhrlein et al., Nature 414, 49 (2001).
[8] J. Escher et al., Nature 413, 495 (2001).
[9] S.-B. Zheng, and G.-C. Guo, Phys. Rev. Lett. 85, 2392 (2000).
[10] S. Osnaghi et al., Phys. Rev. Lett. 87, 037902 (2001).
[11] S.-B. Li and J.-B. Xu, eprint: quant-ph/0507072.
[12] S.G. Clark and A.S. Parkins, Phys. Rev. Lett. 90, 047905 (2003).
[13] S.-B. Li and J.-B. Xu, eprint: quant-ph/0506216.
[14] A. Olaya-Castro, N.F. Johnson, and L. Quiroga, Phys. Rev. A 70, 020301(R) (2004); J. Opt. B: Quantum Semiclass. Opt. 6, S730 (2004).
[15] L. Zhou et al., J. Opt. B: Quantum Semiclass. Opt. 6, 378 (2004).
[16] J.M. Raimond, M. Brune, and S. Haroche, Rev. Mod. Phys. 73, 565 (2001).
[17] M. Plenio, S.F. Huelga, A. Beige, and P.L. Knight, Phys. Rev. A 59, 2468 (1999).
[18] W.K. Wootters, Phys. Rev. Lett. 80, 2245 (1998).
[19] J.S. Bell, Physics (N. Y.) 1, 195 (1965).
[20] J.F. Clauser, M.A. Horne, A. Shimony, and R.A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[21] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Lett. A 200, 340 (1995).
[22] F. Verstraete and M.M. Wolf, Phys. Rev. Lett. 89, 170401 (2002).
[23] S.-B. Li and J.-B. Xu, Phys. Rev. A 72, 022332 (2005).