On Classical Teleportation and Classical Nonlocality

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Abstract

An interesting protocol for classical teleportation of an unknown classical state was recently suggested by Cohen, and by Gour and Meyer. In that protocol, Bob can sample from a probability distribution $\mathcal{P}$ that is given to Alice, even if Alice has absolutely no knowledge about $\mathcal{P}$. Pursuing a similar line of thought, we suggest here a limited form of nonlocality — “classical nonlocality”. Our nonlocality is the (somewhat limited) classical analogue of the Hughston-Jozsa-Wootters (HJW) quantum nonlocality. The HJW nonlocality (also known as “quantum remote steering”) tells us how, for a given density matrix $\rho$, Alice can generate any $\rho$-ensemble on the North Star. This is done using surprisingly few resources — one shared entangled state (prepared in advance), one generalized quantum measurement, and no communication. Similarly, our classical nonlocality (which we call “classical remote steering”) presents how, for a given probability distribution $\mathcal{P}$, Alice can generate any $\mathcal{P}$-ensemble on the North Star, using only one correlated state (prepared in advance), one (generalized) classical measurement, and no communication.

It is important to clarify that while the classical teleportation and the classical non-locality protocols are probably rather insignificant from a classical information processing point of view, they significantly contribute to our understanding of what exactly is quantum in their well established and highly famous quantum analogues.

1 Introduction and Notations

Processing information using quantum two-level systems (qubits), instead of classical bits, has led to many surprising results, such as quantum algorithms that are exponentially faster than the best known classical algorithm (e.g., for factoring large numbers), teleportation of unknown states [1], and quantum cryptography. Another cornerstone of quantum foundations and quantum information processing is the Einstein Podolski Rosen (EPR) paradox [2], or better stated — the EPR nonlocality, which was later on generalized [3,4] by Gisin and by Hughston, Jozsa and Wootters (HJW). Are all of these effects purely quantum, or do we need to look more carefully into the details of each effect to see where the “quantumness” plays a role?

A quantum state $\rho$ (whether pure or mixed), once measured, yields a classical probability distribution over some possible classical outcomes: $\mathcal{P} = \{p_i, i\}$, namely, the result $i$ is obtained with probability $p_i$, and $\sum_i p_i = 1$. If measured in another basis, the same quantum state yields another probability distribution $\mathcal{P'} = \{p'_i, i'\}$ (with $\sum_{i'} p'_i = 1$). A measurement of a quantum state in a particular basis is like sampling once from the appropriate classical probability distribution.

Here we compare quantum states to classical probability distributions, in order to improve our understanding of “quantumness” versus “classicality”. We show that classical probability distributions present some interesting phenomena that are closely related to teleportation and nonlocality.

“Classical states” notations: Let us first recall that for any (nonpure) density matrix $\rho$, one can write $\rho = \sum_i p_i \rho_i$ in infinitely many ways. Each of these ways is described by the ensemble of states $\{p_i, \rho_i\}$, and is conventionally called a “$\rho$-ensemble” of the density matrix $\rho$. Even if the states $\rho_i$ are all pure states
\( \rho_i = |\psi_i \rangle \langle \psi_i | \) there are still infinitely many different ensembles of the type \( \{ p_i, \rho_i \} \) that describe the same density matrix \( \rho \). For instance, \( \frac{1}{2}|0\rangle \langle 0| + |1\rangle \langle 1| = \frac{1}{2}(|+\rangle \langle +| + |\rangle \langle -|), \) with \( |\pm \rangle = (|0\rangle \pm |1\rangle)/\sqrt{2} \).

We denote a classical state of a bit, “0” and “1”, by the notation \( |0\rangle \) and \( |1\rangle \), and when we mix such states we use the notation \( |0\rangle \langle 0| \) and \( |1\rangle \langle 1| \), for consistency with quantum states notations. Let us refer to an arbitrary classical probability distribution by the name “classical state”. For instance, the classical (mixed) state of an honest coin is \( \mathcal{P}_{\text{coin}} = p|0\rangle \langle 0| + (1-p)|1\rangle \langle 1| \). Similarly, the state of an honest die is \( \mathcal{P}_{\text{honest-die}} = \sum_{i=1}^6 (1/6)|i\rangle \langle i| \), and the state of an arbitrary die is \( \mathcal{P}_{\text{die}} = \sum_{i=1}^6 p_i|i\rangle \langle i| \) with \( \sum_{i} p_i = 1 \). In this classical case — a measurement (e.g., a single sampling) yields one of the outcomes \( |i\rangle \) with the appropriate probability \( p_i \). This is similar to the case of a special quantum state which is diagonal in the computation basis, and is measured in that basis. We use the term “generalized coin” to specify a coin, or a die, or any other system whose classical state is a given probability distribution.

In the above, we wrote each classical state as a mixture of pure classical states \( |i\rangle \). This presentation is unique, as there is no choice of basis as in the quantum case. Note that, unlike an unknown quantum state that cannot be cloned, a classical state is defined in only one basis, and therefore can be cloned if sampling with replacement is allowed.

This presentation of the classical state (via the \( |i\rangle \) basis) does not describe the most general way of mixing classical states. For instance, the totally-mixed state describing an honest coin can be made by mixing (via an equal mixture) two dishonest coins, one with probability \( p \) of being \( |0\rangle \) and the other with probability \( 1-p \) of being \( |1\rangle \). In general, any classical state \( \mathcal{P} \) can be written as being made of an ensemble of classical probability distributions (namely a mixture of classical states) \( \mathcal{P} = \sum_j p_j \mathcal{P}_j \). As in the quantum analogue, one can write \( \mathcal{P} \) in infinitely many ways, and each of these ways is described by an ensemble of classical states \( \{ p_j, \mathcal{P}_j \} \). Following the quantum case, we call each of these ways a “\( \mathcal{P} \)-ensemble” of the classical probability distribution (namely, the classical state) \( \mathcal{P} \).

The structure of this paper is as follows: In Section 2 we present a recent result regarding teleportation: In the original teleportation scheme, an unknown quantum state can be teleported, via a shared maximally-entangled state, and two bits of classical communication [1]. In a more limited sense, an unknown classical state can also be teleported, and this is done via a shared classical state, and one bit [5, 6].

In Section 3 we present the EPR [2] and the HJW [4] nonlocalities. In brief, let Alice (who is in Haifa) and Bob (who is far away, say on the North Star) be two parties. For any density matrix \( \rho \), one can write \( \rho = \sum_i p_i \rho_i \) in many ways using various \( \rho \)-ensembles. The HJW nonlocality tells us the following: If Alice and Bob share a pure entangled state such that the reduced density matrix in Bob’s hands is \( \rho_{\text{bob}} \), then Alice can generate for Bob (nonlocally) any ensemble of quantum states \( \{ p_i; \rho_i \} \) (where \( p_i \) is the probability of \( \rho_i \)) as long as \( \sum_i p_i \rho_i = \rho_{\text{bob}} \). This generation of any desired \( \rho \)-ensemble is done without any communication between Alice and Bob; all Alice needs to do is to perform an appropriate (generalized) measurement on her part of the shared entangled state. The EPR nonlocality can be viewed as a special case in which the shared state is a singlet and Alice chooses a standard measurement in the \( z \) basis or the \( x \) basis.

Then, we present in Section 4 a new type of nonlocality — a “classical nonlocality”. As was already mentioned, any classical state, \( \mathcal{P} = \sum_i q_i|i\rangle \langle i| \), can also be written as a mixture of classical states \( \mathcal{P} = \sum_j p_j \mathcal{P}_j \), in many ways using various \( \mathcal{P} \)-ensembles, \( \{ p_j, \mathcal{P}_j \} \). Our “classical nonlocality” argument is as follows: If Alice and Bob share a correlated classical state such that the resulting state in Bob’s hands (namely, Bob’s marginal probability distribution) is \( \mathcal{P}_{\text{bob}} \), then Alice can generate for Bob (nonlocally) any ensemble of classical states \( \{ p_j; \mathcal{P}_j \} \) (where \( p_j \) is the probability of \( \mathcal{P}_j \)) as long as \( \sum_j p_j \mathcal{P}_j = \mathcal{P}_{\text{bob}} \). This generation of any desired \( \mathcal{P} \)-ensemble is done without any communication between Alice and Bob; all Alice needs to do is to perform an appropriate (generalized) measurement on her part of the shared correlated state.

Note the trivial special case in which Alice measures the shared state in the pure-state basis, just by looking at her part. Then, with probability \( p_i \), she obtains the result \( i \) and she knows with certainty that Bob’s resulting pure state is \( |i\rangle \). As this presentation (via pure classical states) is unique, there is no classical analogue to the EPR nonlocality.
2 Classical teleportation of classical states

Consider a classical coin in an unknown classical state, namely an unknown probability distribution \( P_{\text{coin}} = \rho |0\rangle \langle 0| + (1 - \rho) |1\rangle \langle 1| \). The probability of a “head” is \( \rho \). Charley gives this coin to Alice (on earth) and she would like to teleport it to Bob. Here we mean that Bob will be able to flip the coin once and obtain a result according to the correct probability distribution. Namely, Bob can sample the probability distribution \textit{once}.

We would like to avoid the trivial solution in which Alice samples the probability distribution \textit{once} and tells Bob the outcome. Note that, in this solution, by hearing Alice’s bit Bob is forced to sample the probability distribution whenever he wants. Furthermore, note that if Alice samples the state and tells Bob the outcome, then Alice, and also any other receiver of that data, will be in the same position as Bob. Namely, they will all share Bob’s data. Charley would like to see a scenario in which only Bob can sample the coin he (Charley) gave to Alice, and that Bob can do so (once only) whenever he wishes to. [This situation then resembles quantum teleportation after which only Bob holds the state and can measure it whenever he wishes to.]

What are the minimal resources required for that operation? Will it help Alice and Bob if some data, unrelated to \( \rho \), was shared in advance? As found by [5, 6], Alice can indeed classically teleport the state of the coin to Bob with infinite accuracy by sharing one bit with Bob in advance, and sending one bit: Let us assume that Alice and Bob share one classical bit, which is 0 with probability half, and 1 with probability half, but they do not know its value. Namely, they share the state

\[
P_{\text{correlated}} = \sum_{i=0}^{1} \frac{1}{2} |i\rangle_{\text{Alice}} \otimes |i\rangle_{\text{Bob}}.
\]

Let Alice flip the coin she got from Charley. Alice then measures the eXclusive OR (XOR) of the values of the coin given by Charley, and her part of the correlated state \( P_{\text{correlated}} \) (but without learning each one separately). For instance, she could give Charley her part of the correlated state; Charley then would flip his coin (instead of Alice) to obtain one bit, look at Alice’s secret bit, and tell her just one bit of information — the parity of the two bits. This parity is then sent to Bob. If it is ZERO, Bob does nothing; if it is ONE he flips his part of the shared correlated state. In both cases, he now holds the state \( P_{\text{coin}} \) of Alice’s coin. The probability of viewing 0 or 1 is identical to the probability determined by the coin Charley gave to Alice.

It is important to mention that the main difference between this and the case where Alice flips the coin and tells Bob the result is that here only Bob (in addition to Charley, of course) has the correct distribution, and no other person will. Not even Alice. In addition, here, even Bob did not yet (at the end of the protocol) sample the coin. Such a protocol is interesting since it emphasizes the quantumness of the quantum teleportation by showing that not all its aspects are quantum. In particular, it helps in demystifying quantum teleportation, showing that the ability to teleport an \textit{unknown} state has (in some sense) a classical analogue. This protocol also emphasizes the differences between \textit{having} a state and \textit{sampling} (measuring) a state, in both the classical and quantum domains.

Is this classical teleportation protocol interesting from a classical information processing point of view? Probably not. The interplay between Alice, Bob, and Charley as described above is very similar to a trivial classical analogue — the “one-time pad”. In the one-time pad protocol Alice and Bob share one secret bit, and when Alice wishes to send Bob a bit (potentially, a secret bit given to her by Charley), she sends Bob the parity of that new bit and the shared bit. This way, only Bob can learn Alice’s bit so it is a secret to others. This conventional description of the one-time pad ignores the possibility (which is at the heart of the classical teleportation protocol) that Alice’s new bit is not fixed by Charley, but instead, it is sampled at random from some fixed probability distribution.

It is important to note that Cohen [5] uses very different notations, and also defines the classical states in a way that makes them unclonable, due to sampling a probability distribution \textit{without replacement}.
3 Quantum Nonlocality (Quantum Remote Steering)

We first describe the EPR (Bohm’s version) nonlocality. Alice and Bob share a fully entangled state of two qubits, say $|00⟩ + |11⟩/\sqrt{2}$, and Alice measures her qubit in one basis ($z$) or a conjugate one ($x$). Whatever result she obtains, she now knows Bob’s state. This is true even if Alice and Bob are space-like separated, so that no information can go from one to the other. The “paradox” is obtained if one assumes that quantum states are physical realities that cannot change faster than light, since then it seems that Bob’s state must be well defined in both the $z$ basis and the $x$ basis, contradicting the uncertainty principle. Once one is willing to accept that quantum states are not local physical realities, the paradox is changed into a nonlocality argument: In the case in which Alice measures in the $z$ basis she determines from far away that Bob’s state (which is the completely mixed state, as far as Bob can test on his own) will be made of the states $|0⟩$ and $|1⟩$, and she then also knows which of the two states it is. Alternatively, in the case in which Alice measures in the $x$ basis she determines from afar that Bob’s state will be composed of $|+⟩$ and $|−⟩$. Thus, Alice can determine the ensemble of states from which Bob’s density matrix will be constructed, and furthermore, Alice can tell which specific state will appear at a particular instance of running the protocol. Obviously, Alice could have made any standard projection measurement as well on her qubit, and thus enforce the appropriate distribution on Bob’s state. This type of nonlocality does not contradict Einstein’s causality principle, in the sense that information is not transmitted faster than light.

The HJW generalization is done by having Alice and Bob share any pure entangled state, and by letting Alice perform any generalized measurement (POVM) \cite{7,8}. Generalized measurements are equivalent to standard projection measurements when the latter are performed on an enlarged system containing the original system plus an ancilla in a known initial state \cite{8}. Now, the HJW nonlocality argument is the following: Let Alice and Bob share a pure entangled state $|ψ_{AB}⟩$, such that the reduced density matrix in Bob’s hands is $ρ_{Bob}$, namely,

$$ρ_{Bob} = Tr_{Alice}[|ψ_{AB}⟩⟨ψ_{AB}|].$$

Alice can generate for Bob (nonlocally) any ensemble of quantum states $\{ρ_i; p_i\}$ (where $p_i$ is the probability of $ρ_i$) such that $\sum_i p_i ρ_i = ρ_{Bob}$. Alice can generate this ensemble in Bob’s hands in the sense that she chooses which ensemble to generate, and after she prepares her desired $ρ$-ensemble, she can tell us (or tell Bob) which of the states $ρ_i$ in the ensemble he has in his hand. The preparation of the desired $ρ$-ensemble is done by choosing the appropriate generalized measurement and performing it in her lab (on earth).

Quantum teleportation can be thought of as a special case of HJW nonlocality in which the shared state is a singlet and the ensemble is built from the teleported qubit-state ($|α⟩$) and its rotations around the three axes \cite{9,10}. In that case Alice also tells Bob which rotation to perform in order to get the state ($|α⟩$), while both of them do not need to know $α$ and $β$ (e.g., if the qubit is provided to Alice by Charley).

4 Classical Nonlocality (Classical Remote Steering)

In order to search for the classical analogue of the above nonlocality, let us note that, for pure entangled states, we are always promised that a basis exists such that the state can be written as $|ψ_{AB}⟩ = \sum_i α_i |i⟩_{Alice} ⊗ |i⟩_{Bob}$ (with $α_i$ the normalized amplitudes) via the Schmidt decomposition (see, for instance \cite{8}). If this state decoheres so that nondiagonal terms become zero, the density matrix $|ψ_{AB}⟩⟨ψ_{AB}|$ changes to $\sum_i P_i |i⟩_Alice ⟨i|_Alice ⊗ |i⟩_{Bob} ⟨i|_{Bob}$ with $p_i = |α_i|^2$.

The classical analogue of the HJW quantum nonlocality is based on Alice and Bob sharing the classical analogue of the decohered state

$$ρ_{AB} = \sum_i p_i |i⟩_Alice ⟨i| ⊗ |i⟩_Bob ⟨i|.$$

The classical state can then be called “fully correlated” because each party can measure his part of the

\footnote{These measurements are known as complete measurements, or von-Neumann measurements.}
shared state and learn with probability 1 the state of the other party. The state is written as
\[ P_{\text{fully-correl}} = \sum p_i \ket{i}_\text{Alice} \otimes \ket{i}_\text{Bob}. \]
Bob’s “reduced state” (Bob’s marginal probabilities) is then given by
\[ P_{\text{Bob}} = \text{Tr}_{\text{Alice}}[P_{\text{fully-correl}}] = \sum p_i \ket{i}_\text{Bob}. \]

In order to study what Alice can do with the shared state, we start the analysis by considering the EPR nonlocality. This nonlocality has no nontrivial classical analogue. The reason is that in a quantum world, there is more than one basis, while classically there is only one. Thus, if Alice is only allowed to perform measurements on the system (without an ancilla), the quantum case is very interesting as the possibility for various measurements leads to the EPR nonlocality, while the classical case is trivial: Alice and Bob will share the fully correlated state, \( P_{\text{fully-correl}} \), with \( p_i \) equal half (for \( i = 0 \) and \( i = 1 \)). Alice will measure in the classical basis, which is the only allowed basis, and thus will know \( i \).

A slightly more complicated scenario is obtained if we let Alice choose between doing nothing with probability \( q \), or measuring her part of the shared state, with probability \( 1 - q \). Then, the following ensemble is created, \( P_\text{fl} = qP_{\text{totally-mixed}} + [(1-q)/2]00 + [(1-q)/2]11 \), with \( P_{\text{totally-mixed}} = 1/2\ket{00} + 1/2\ket{11} \). Note that Alice will know which of the three possible states is now held by Bob. Note also that the choice that Alice made of whether to measure or do nothing can be mimicked by flipping an appropriate dishonest coin that has a probability \( q \) for a “head”. Note also that Alice could actually view all cases, and with probability \( q \) take an obtained result and “forget” it. This way she also obtains three sets as before.

Once we allow Alice to add another system, even though it is a classical one, the situation becomes less trivial, and we can make a similar connection between nonlocality and teleportation as was made in \([9, 10]\) for the quantum case. Let Alice and Bob share the state \( P_{\text{fully-correl}} \). What can Alice do if she wants Bob to hold a dishonest coin with probability \( p \not= 1/2 \) of yielding one result and probability \( 1-p \) of yielding the other? Namely, she wants him to sample (once) from a probability distribution \( P_0 = p\ket{0} + (1-p)\ket{1} \), or from a probability distribution \( P_1 = (1 - p)\ket{0} + p\ket{1} \), and to know which probability distribution he sampled from. Alice will take such a coin (with probability distribution \( P_0 \)), flip it, and observe the parity of this coin and her part of the shared state without looking at each bit separately. Bob’s probability distribution, conditioned on Alice observing 0, is \( P_0 = p\ket{0} + (1-p)\ket{1} \). On the other hand, Bob’s probability distribution, conditioned on Alice observing 1, is \( P_1 = p\ket{1} + (1-p)\ket{0} \). As in the previous example, also here Alice could actually measure both her crooked coin and her part of the shared bit, but she must forget the results and remember just the parity when she splits the outcomes into two sets.

Her success in predicting Bob’s resulting state can, of course, be verified: If Charley gave Alice the coin, and only he knows \( p \), and they repeat the experiment many times, it is easy for Alice to convince Charley that she controls Bob’s probability distribution, that it is sometimes \( P_0 \) and sometimes \( P_1 \), and that she knows when is which. Note that this example of a “classical nonlocality protocol” requires no communication between Alice and Bob.

Let us look at the case in which communication is added. Alice tells Bob the parity bit, Bob rotates his part of the shared state if the parity bit he learns equals 1 (or does nothing if that parity equals 0), and they then recover the classical teleportation protocol of Section 2. Note that Alice and Bob could apply the protocol even if Alice does not know the classical state.

In the more general case, Alice can use a larger space (e.g., flip several coins together with the shared bit, or throw a die together with her part of the shared state) and predict one out of many (instead of one out of two) probability distributions.

In order to deal with the most general case of this classical nonlocality, let us rephrase the “classical teleportation” case we have just seen (without the communication step). Let Alice replace the calculation of the parity by a different method: she looks at the bit she shares with Bob, if it is 0 she flips the coin, and if is is 1 she switches the numbers “0” and “1” on the coin, and flips the coin. Now she looks at the result of the coin and “forgets” whether she had seen 0 or 1 on the shared bit. Clearly, if the result of the flipped coin is “0”, Bob’s probability distribution is \( P_0 \) and if the result of the flipped coin is “1”, Bob’s
probability distribution is \( \mathcal{P}_1 \). By generalizing this method we will now obtain the proof of the most general case of our classical nonlocality.

**A protocol presenting classical nonlocality:** Let Alice and Bob share the fully correlated state \( \mathcal{P}_{\text{fully-correl}} = \sum_i p_i |i\rangle_{\text{Alice}} \otimes |i\rangle_{\text{Bob}} \). Then Bob’s reduced state (Bob’s marginal probability distribution) is \( \mathcal{P}_{\text{Bob}} = \sum_i p_i |i\rangle_{\text{Bob}} \) when written as the (unique) mixture of pure classical states \( |i\rangle \). Alice can generate any \( \mathcal{P} \)-ensemble on the North Star, provided that the ensemble satisfies \( \sum_j p_j \mathcal{P}_j = \mathcal{P}_{\text{Bob}} \). Let \( I \) be the set of events \( i \): the fully correlated state is sampled and it provides the value \( i \), with probability \( p_i \). Let \( J \) be the set of events \( j \): with probability \( p_j \) Bob has the classical state \( \mathcal{P}_j \).

Let us write each of Bob’s states \( \mathcal{P}_j \) via the basis states as \( \mathcal{P}_j = \sum_i p(i,j) |i\rangle_{\text{Bob}} \) where, for each \( j \), \( \sum_i p(i,j) = 1 \), and \( \sum_j p(i,j) = p_i \). To create her desired \( \mathcal{P} \)-ensemble, Alice follows this protocol:

1. Alice looks at the shared state to find \( |i\rangle \).

2. For each result \( |i\rangle \) Alice throws a generalized coin\(^2\) distributed according to \( p_{(j|i)} \), where these probabilities are obtained from the known probability distributions \( p_i, p_j \), and \( p_{(ij)} \) using the Bayes rule, \( p_{(j|i)} = p_j p_{(ij)}/p_i \). The state of the \( i \)’th generalized coin can be written as \( \mathcal{Q}_i = \sum_j p(j|i) |j\rangle \langle j| \).

3. Alice views the result \( |j\rangle \) obtained by throwing her \( i \)’th generalized coin, so that now she knows both the result \( |i\rangle \) and the resulting \( |j\rangle \) (sampled by throwing her generalized coin).

4. Alice “forgets” \( i \), to be left with the result \( |j\rangle \). Namely, she “forgets” which generalized coin she used.

Let us assume that Alice is left with the result \( |j\rangle \). What can we now learn from this protocol? After the first step, Alice has the result \( |i\rangle \) with probability \( p_i \). After the third step, Alice has the result \( |j\rangle \) with probability \( p_{(j|i)} \). Actually, she knows both \( |i\rangle \) and \( |j\rangle \). The probability of this event is \( p_{ij} = p_i p_{(ij)} \). After “forgetting” the outcome \( i \), the probability of Alice being left with the result \( |j\rangle \) is given by \( \sum_i p_{ij} \).

- **CLAIM 1:** the probability that Alice is left with the result \( |j\rangle \) is \( p_j \) (note that this is exactly the probability \( p_j \) that she wanted to generate for Bob’s state \( \mathcal{P}_j \)).

  **PROOF:** As we said, the probability of Alice being left with the result \( |j\rangle \) is given by \( \sum_i p_{ij} \), which equals \( p_j \).

- **CLAIM 2:** The state remaining in Bob’s hands is \( \mathcal{P}_j \).

  **PROOF:** From the Bayes rule we get the probability of Bob’s result being \( |i\rangle \) given that Alice’s result is \( |j\rangle \): \( p_{(i|j)} = p_i p_{(ij)}/p_j \). However, this means that Bob’s state is \( \mathcal{P}_j \).

- **COROLLARY:** Alice succeeded to generate the desired \( \mathcal{P} \)-ensemble.

For a fully-detailed example: see Appendix[A]

Three important remarks are now in order. First, as in the quantum case, also here — given the prepared state, and Alice’s choice of the \( \mathcal{P} \)-ensemble, all four steps of the protocol can be combined into a single operation which we call a “generalized measurement”, analogous to the generalized quantum measurement (POVM). Second, the above analogy is based on the following special case of the quantum POVM: The case in which the POVM elements are of rank larger than one (this is the quantum analogue of “forgetting” the result \( i \)). Third, if we only use rank-1 POVM, the quantum case (which can make use of nonorthogonal states) is still very interesting, as the number of outcomes, namely \( |J| \), can be larger than the size of the original reduced state which is \( |I| \). In contrast, if we are only allowed to use states diagonal in the computation basis then using rank-1 POVM leads to the trivial case of just measuring \( |i\rangle \). In this respect, it is clear that the classical nonlocality is trivial unless the operation of “forgetting” is used (namely rank larger than one for the POVM).

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\(^2\) A coin with \( j \) outcomes.
5 Discussion

In this work we analyzed in detail a classical teleportation scheme suggested by Cohen [5], and by Meyer and Gour [6], and we proposed a classical nonlocality argument which resembles the quantum nonlocality arguments of EPR and HJW.

We think that this work sheds a new light on the question of what quantum nonlocality really means. While quantum nonlocality without entanglement was already discovered [11, 12] and became the topic of extensive research, the topic of nonlocality without any quantumness is offered here (following [5, 6]), probably for the first time.

We leave open the question of whether such a classical nonlocality protocol can lead to interesting and useful classical information processing protocols.

6 Acknowledgement

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Asher was always fascinated by the differences between quantum physics and classical physics, specifically by entanglement and nonlocality. I think that the only open question he had regarding quantum teleportation (that he co-authored, as is well known) was the following: “Does the fact that the teleported state is unknown tell us something new about quantum physics?” I believe that providing definitions of classical teleportation and classical nonlocality and looking at their connections to their quantum counterparts is a research direction which helps to demystify quantum teleportation and nonlocality, and hence is particularly appropriate for this special issue dedicated to Asher’s memory.

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This example is solved here to clarify the protocol: Let Alice and Bob’s state be

\[ \rho_{\text{fully-correl}} = \frac{1}{32} |0\rangle_\text{Alice} \otimes |0\rangle_\text{Bob} + \frac{21}{32} |1\rangle_\text{Alice} \otimes |1\rangle_\text{Bob} \]

so that Bob’s reduced state is \( \rho_\text{Bob} = \frac{11}{32} |0\rangle + \frac{21}{32} |1\rangle \). We write Bob’s reduced state using the basis states \(|i\rangle\) as a vector of norm 1,

\[ \rho_\text{Bob} = \left\{ \frac{11}{32}, \frac{21}{32} \right\} \, . \]

Let the desired probability distributions that Alice would like to generate at Bob’s hands be \( \rho_j = \left\{ \frac{1}{2}, \frac{1}{2} \right\} \) with probability \( p_j = \frac{1}{2} \), \( \rho_j = \left\{ \frac{1}{3}, \frac{2}{3} \right\} \) with probability \( p_j = \frac{1}{3} \), and \( \rho_j = \left\{ \frac{1}{4}, \frac{3}{4} \right\} \) with probability \( p_j = \frac{1}{4} \). The probabilities are chosen such that \( \rho_\text{Bob} = \sum_j p_j |j\rangle \langle j|_\text{Bob} \).

Then,

\[ p_0 = \frac{11}{32} + \frac{21}{32} = \frac{32}{32} = 1, \quad p_1 = \frac{11}{32} + \frac{21}{32} + \frac{21}{32} + \frac{21}{32} = \frac{84}{32} = \frac{21}{8} \, . \]

Now, depending on the state \(|i\rangle\) Alice observes at step 1 of the protocol, she chooses one of the two generalized coins \( Q_{i=0} = \left\{ \frac{8}{11}, \frac{3}{11}, \frac{1}{11} \right\} \) or \( Q_{i=1} = \left\{ \frac{8}{21}, \frac{6}{21}, \frac{7}{21} \right\} \), with \( p(j|i) \) calculated using Bayes rule. Then, after throwing the appropriate generalized coin, \( Q_i \), Alice obtains a result \(|j\rangle\).

By “forgetting” \(|i\rangle\), she can now predict Bob’s state to be \( \rho_j \), and that result is obtained with probability \( p_j \).