Global Phase Diagram of a One-Dimensional Driven Lattice Gas

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We investigate the non-equilibrium stationary state of a translationally invariant one-dimensional driven lattice gas with short-range interactions. The phase diagram is found to exhibit a line of continuous transitions from a disordered phase to a phase with spontaneous symmetry breaking. At the phase transition the correlation length is infinite and density correlations decay algebraically. Depending on the parameters which define the dynamics, the transition either belongs to the universality class of directed percolation or to a universality class of a growth model which preserves the local minimal height. Consequences of some mappings to other models, including a parity-conserving branching-annihilation process are briefly discussed.

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The interplay of external driving fields and internal repulsive forces between particles can lead to interesting and unexpected phase transitions in the steady states of one-dimensional driven diffusive systems even if the interactions are only short-ranged \textsuperscript{[1]}. Generically, the presence of boundaries or single defects in driven systems leads to shock waves and mutual blocking mechanisms which result in a breakdown of homogeneous particle flow. Thus localized static inhomogeneities are responsible for a variety of phenomena including first- and second-order phase transitions \textsuperscript{[2]} or spontaneous symmetry breaking \textsuperscript{[3]}. These observations are of practical importance for the qualitative understanding of many-body systems in which the dynamic degrees of freedom reduce to effective one dimension as e.g. in traffic flow \textsuperscript{[4]}, kinetics of protein synthesis \textsuperscript{[5]}, gel-electrophoresis \textsuperscript{[6]}, or interface growth of thin films \textsuperscript{[7]}. Whether continuous phase transitions can occur also in spatially homogeneous non-equilibrium systems in one dimension is less well-understood \textsuperscript{[8]}. In particular, there is a long-standing conjecture \textsuperscript{[9]} that in systems with local interactions the steady states have rapidly decaying correlations and, like in 1-d equilibrium models, no phase transition accompanied by algebraically decaying correlations takes place. On the other hand, recent studies of more complicated driven systems of three or more species of particles in 1d have demonstrated that phase separation may take place in these models \textsuperscript{[10]}, thus proving the possibility of long-range order, but leaving open the issue of continuous phase transitions with algebraic decay of correlations. In the absence of a general framework for studying non-equilibrium phase transitions, analyzing specific models could provide useful insight in these complex phenomena.

In this context, several translationally invariant one-dimensional growth models with local interactions which exhibit roughening transitions have recently been introduced. A common feature of these models is that one of the local transition rates which govern their dynamics is set to zero. The resulting roughening transition in one class of models belongs to the universality class of directed percolation \textsuperscript{[11]}. In another class of growth models which preserve the local minimal height, the transition is found to belong to a different universality class \textsuperscript{[12,13]}. It would be of great interest to put these classes of models within a unifying framework, so that the various types of transitions, the associated crossover phenomena and the global phase diagram could be studied.

In this Letter we introduce a simple homogeneous driven 1d lattice gas model with local dynamics. It exhibits a phase transition where correlations decay algebraically and which is accompanied by spontaneous symmetry breaking. The model can be mapped onto a growth model where the transition becomes a roughening transition. By varying the parameters which define its dynamics, some types of the transitions discussed above can be realized. The various transitions and the global phase diagram are studied.

We consider a lattice gas which is an asymmetric exclusion process with next-nearest-neighbour interaction. Each lattice-site \(i \in \{1, 2, \ldots, L\} \) of a periodic chain may be either empty (\(\emptyset\)) or occupied by one particle of a single species, labeled \(A\). The model evolves by random sequential updating. Particles hop to the right with constant attempt rate \(r(q)\) if the right nearest neighbour site is vacant and the nearest neighbour site at the left is occupied (empty). Unlike in the KLS-models \textsuperscript{[14]}, the left-hopping mechanism is different: A particle hops to the left with rate \(p = 1 - q - r\) only if the next-nearest-neighbour site is empty as well. The model is therefore defined by the transitions

\[
\begin{align*}
A \ A \ \emptyset & \rightarrow A \ \emptyset \ A \quad \text{with rate } r, \\
\emptyset \ A \ \emptyset & \rightarrow \emptyset \ \emptyset \ A \quad \text{with rate } q, \\
\emptyset \ \emptyset \ A & \rightarrow \emptyset \ A \ \emptyset \quad \text{with rate } p.
\end{align*}
\]

By identifying vacancies with up-spins and particles with down-spins, these dynamics may be interpreted as a non-equilibrium spin-relaxation process. The choice \(p = 0\) is a special case of the kinetic Ising models of Ref. \textsuperscript{[14]}, with
A new mapping relates the two-dimensional Ising model to a one-dimensional interface growth process. The dynamics are described by a growth model that is conserved. As in the related mapping breaking (and associated ergodicity breaking in the system size), this is a signature for spontaneous symmetry breaking. The stationary density (the 'staggered magnetization' in spin language) is that of a one-dimensional Ising model. Because of ergodicity, the stationary value of the order parameter in a large system vanishes by symmetry. However, as a signature of spontaneously broken symmetry in the thermodynamic limit, one expects an initial decay to some quasi-stationary value $\Delta$, before $\Delta$ eventually approaches zero for very long times (exponentially large in system size). On the other hand, in the disordered phase one expects an initially ordered state would lead to a transition between the two degenerate stationary states with $\Delta = \pm 1$.

By comparing the two mappings, we find that the dynamics satisfies detailed balance with respect to an energy functional which is proportional to the area under the interface. The point $p = q = 1/2$ (corresponding to a change in the sign of the energy $E$) marks the transition from an antiferromagnetic state to a state where complete phase ordering takes place and translational invariance is spontaneously broken. This transition is analogous to the wetting transition of Ref. [3].

This summary of exact results demonstrates the rich behavior that even rather simple homogeneous lattice gases may show and also indicates a certain degree of universality of these phenomena in 1D non-equilibrium systems. Here, we want to discuss the behavior of the system as it crosses the phase transition line between the broken symmetry phase $I$ and the disordered phase $II$. We shall focus on the line $r = q$ with the limiting cases $r = q = 1/2$ (usual right hopping TASEP with uncorrelated disordered stationary state) and $r = q = 0$ (left hopping TASEP with next-nearest-neighbour repulsion and fully ordered stationary states).

We study the quantity

$$\Delta(t) = \frac{1}{J} \int_0^t dt \, 2 \sum_{i=1}^{L} (-1)^i \langle n_i(t) \rangle. \quad (2)$$

where $n_i = 0$ corresponds to an empty site $i$ and $n_i = 1$ to an occupied one. In the limit $t \to \infty$, it corresponds to the non-conserved order parameter $2/L \sum_i (-1)^i \langle n_i \rangle$, which is the stationary difference in sublattice particle densities (the 'staggered magnetization' in spin language). Because of ergodicity, the stationary value of the order parameter in a finite system vanishes by symmetry. However, as a signature of spontaneously broken symmetry in the thermodynamic limit, one expects an initial decay to some quasi-stationary value $\Delta_0$, before $\Delta$ eventually approaches zero for very long times (exponentially large in system size). On the other hand, in the disordered phase one expects an initially ordered state with $\Delta = 1$ to rapidly disorder, i.e. one expects $\Delta$ to decay quickly to zero.

A second quantity of interest is the stationary particle current which, according to the definition (1) of the process, on the line $r = q = (1 - p)/2$ is given by $j(q) = q \langle n_i(1 - n_{i+1}) \rangle - (1 - 2q) \langle (1 - n_{i-1})(1 - n_i) \rangle$. Clearly, $j(0) = 0$ and $j(1/2) = 1/8$, up to a small finite-size correction of order $1/L$. The presence of spontaneous symmetry breaking suggests $j = 0$ for all $q \leq q_c$. (up to exponentially small corrections in system size), since any finite current would lead to a transition between the two degenerate stationary states with $\Delta = \pm \Delta_0$ within a finite time.
This intuitive picture is well-supported by our Monte Carlo simulations (Fig. 3). The current $j$ vanishes in phase $I$ and the order parameter $\Delta_0$ vanishes in phase $II$. We find a phase transition point $q_\text{c} = 0.1515 \pm 0.0005$ for $r = q$, above which the current decays with a power law

$$j \sim (q - q_\text{c})^y$$

with $y = 1.7 \pm 0.1$. Approaching the critical point $q_\text{c}$ from below, $\Delta_0$ decays with a power law

$$\Delta_0 \sim (q_\text{c} - q)^\theta$$

with $\theta \approx 0.54 \pm 0.04$. To investigate whether this continuous bulk phase transition is accompanied by spatial long-range order—as one would expect in an equilibrium system—we examine the stationary density correlation function $C(k) = 4\langle (n_i n_{i+k}) - (n_i)\langle n_{i+k} \rangle \rangle$ which turns out to decay to a non-zero value below $q_\text{c}$. At the critical point, correlations decay algebraically

$$C(k) \sim k^{-\gamma},$$

where $\gamma \approx 1.0 \pm 0.1$ (Fig. 3) [10].

![FIG. 2. (a) Stationary current and (b) order parameter along the line $r = q$. Below: Correlation function $C(k)$ (c) for maximal distance $k = L/2$ as a function of $r$ and $q$ and (d) as a function of $k$ for $r = q = q_\text{c}$. The dashed line corresponds to a slope of $-1$. The measured data are connected by straight lines as a guide for the eye.](image)

We can gain further insight by considering the mapping to an interface model [13] which is described by height difference variables $1 - 2n_i$ and an additional stochastic variable $h$, representing the absolute height of the interface at some reference point. Each time a particle hops to the right, the local height increases by two units (deposition), whereas hopping to the left describes a height decrease (evaporation) (Fig. 3). Thus the current gives the stationary growth velocity of the interface, while the density correlation function measures height-gradient correlations. Growth occurs at local minima with rate $q$, independently of the precise nature of the immediate environment. However, evaporation of particles does not occur from a “flat” part of the interface: The corresponding process $A0 \rightarrow AA0$ is forbidden.

![FIG. 3. Mapping between lattice gas dynamics and interface growth in 1+1 dimensions. A positive (negative) unit slope belongs to a vacancy (particle). The interface flips (vertical arrows) correspond to particles hopping on the lattice (horizontal arrows).](image)

Here we find similar behavior which is most transparent in the two limiting cases $q = 0$ and $q = 1/2$, respectively. The limit $q \rightarrow 1/2$ corresponds to the TASEP (growing, rough interface), which indeed describes interface growth in the KPZ universality class [17]. In the limit $q \rightarrow 0$, there is no current and one has spontaneous symmetry breaking between (macroscopically) flat interfaces on an even or odd height level, respectively. We stress, however, that spontaneous symmetry breaking occurs already on the level of the particle description, i.e. without reference to the extra height variable. Assuming universality, one expects [11] the exponent $y$ to be given by the critical exponent $\nu_0 \approx 1.73$ of the DP coherence time [18] and also a logarithmic divergence of the interface width $w = [L^{-1} \sum_i \langle h_i - L^{-1} \sum_j h_j \rangle^2]^{1/2}$. This is in agreement with our results in Eq. (3) and Fig. 3. Also the value 1 of the order parameter exponent $\theta$ is consistent with the result $\theta = 0.55 \pm 0.05$ reported in Ref. [11], thus independently confirming universality. Results on the correlation exponent $\gamma$ have not been reported in earlier work.

The transition at $r = 0$ is of a different nature. Here the model satisfies detailed balance and the current vanishes both above and below the transition. At the phase transition point $q_\text{c} = 1/2$ the lattice gas is uncorrelated. Using the interface representation of the model one can show [13] that the interface width diverges algebraically with an exponent 1/3 as $q$ approaches 1/2 from below (Fig. 3).
The understanding of θ and of the new correlation exponent γ (which have no conventional interpretation within the framework of directed percolation), and the behavior of these two quantities at the transition at r = 0 have to be addressed in future work. Also the behavior of the system away from half-filling, where preliminary results suggest the disappearance of phase I, is an open issue. Returning to our original question we conclude at this point that the stationary states of homogeneous one-dimensional lattice gas models may exhibit continuous bulk phase transitions with an algebraic decay of correlations even if interactions are short-ranged. In our model, this transition results from dynamical constraints which—unlike in the KLS models—lead to a competition between a disordering dynamics (the right-hopping process) and processes forcing the system into either of two antiferromagnetically ordered states (the restricted left hopping process). For sufficiently strong ordering processes, the stationary current ceases to flow and spontaneous symmetry breaking sets in.

It is interesting to consider yet another mapping of our model, obtained by mapping particles into vacancies and vice versa on one (either even or odd) sublattice. The resulting dynamics are that of a new class of parity-conserving (PC) branching-annihilation processes θAA ⇀ 000 and A00 ⇀ AAA with no absorbing state. In addition to particle-parity conservation (particle number modulo 2), there is a U(1) symmetry which results from the particle number conservation of the original hopping process. Generically, one expects parity-conserving branching-annihilation processes not to be in the DP universality class, but in a distinct PC universality class [19]. From our results it appears that, in the presence of additional symmetries, the picture of phase transitions in 1D branching-annihilation processes is more complicated.

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FIG. 4. Interface width close to the phase transition points along (a) r = q, suggesting a logarithmic-like divergence, and (b) r = 0, respectively. The slope of the dashed line is −1/3.

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**Diagram (a):**
Current $j$ vs $q$.

**Diagram (b):**
$\Delta_0$ vs $q$.

**Diagram (c):**
Correlation $C(L/2)$.

**Diagram (d):**
$\log(C(k))$ vs $\log(\text{Distance } k)$.
Interface Width

\[ \log(q_c - q) \]

(a) \[ r=q \]

\[ \log(\frac{1}{2} - q) \]

(b) \[ r=0 \]