On $k$-String Tensions and Domain Walls in $\mathcal{N} = 1$ Gluodynamics

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Abstract

We discuss the $k$ dependence of the $k$-string tension $\sigma_k$ in SU($N$) supersymmetric gluodynamics. As well known, at large $N$ the $k$-string consists, to leading order, of $k$ noninteracting fundamental strings, so that $\sigma_k = k \sigma_1$. We argue, both from field-theory and string-theory side, that subleading corrections to this formula run in powers of $1/N^2$ rather than $1/N$, thus excluding the Casimir scaling. We suggest a heuristic model allowing one to relate the $k$-string tension in four-dimensional gluodynamics with the tension of the BPS domain walls ($k$-walls). In this model the domain walls are made of a net of strings connected to each other by baryon vertices. The relation emerging in this way leads to the sine formula $\sigma_k \sim \Lambda^2 N \sin \pi k/N$. We discuss possible corrections to the sine law, and present arguments that they are suppressed by $1/k$ factors. We explain why the sine law does not hold in two dimensions. Finally, we discuss the applicability of the sine formula for non-supersymmetric orientifold field theories.

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1 Introduction

In confining theories, such as the Yang-Mills theory, non-supersymmetric or supersymmetric ($\mathcal{N} = 1$ gluodynamics), heavy probe quarks are connected by color flux tubes. These “QCD strings” are the fundamental strings of the “old” string theory of hadrons. A major improvement in our understanding of dynamics of the QCD strings, due to the AdS/CFT correspondence (Refs. [1, 2], for earlier ideas see Ref. [3]), involves a description in curved space of at least five dimensions.

In this work we will consider confining gauge theories in four dimensions. In what follows the gauge group is assumed to be SU($N$), and we will consider the limit of large $N$. Figure 1a schematically shows a flux tube connecting an infinitely heavy quark in the fundamental representation with its antiquark. If $k$ probe quarks in the fundamental representation are placed close to each other, a flux tube which connects them with $k$ antiquarks is called a $k$-string, see Fig. 1b displaying the $k = 2$ example.

![Diagram]

Figure 1: A flux tube for $k$-strings. On the left (1a) is the “fundamental” tube. On the right (1b) is the 2-string tube.

The tension of the $k$ string, $\sigma_k$, may be viewed as one of key parameters of the confinement dynamics. It is under intense scrutiny since mid-1980’s. Most frequently discussed are two competing hypotheses: (a) the Casimir scaling and (b) the Douglas-Shenker sine formula (for extensive reviews and
representative list of references see e.g. [4, 5, 6]). The Casimir scaling hypothesis reads that the $k$ dependence of $\sigma_k$ is

$$\sigma_k = \Lambda^2 k \left(1 - \frac{k - 1}{N - 1}\right), \quad k = 1, 2, ..., N,$$

where $\Lambda$ is the dynamical scale parameter. The sine formula can be presented as follows:

$$\sigma_k = N \Lambda^2 \sin \left(\frac{\pi k}{N}\right), \quad k = 1, 2, ..., N.$$  

Both exhibit the symmetry under the replacement $k \to N - k$, which corresponds to replacing quarks by anti-quarks. Moreover, both expressions imply that at $N = \infty$

$$\sigma_k = k \sigma_1,$$

where $\sigma_1$ is the tension of the fundamental string (i.e. $k = 1$). The difference is in the subleading in $1/N$ corrections. In the first case they run in powers of $1/N$ while in the second in powers of $1/N^2$.

The purpose of this paper is two-fold. First, we will argue\footnote{Strictly speaking the statement below refers to $d > 2$. Two-dimensional gauge theories must be discussed separately, see Sect. 6. In two-dimensional QCD with \textit{massive matter} the sine in Eq. (2) is replaced [7] by sine squared, namely, $\sigma_k \sim \text{mesin}^2 \pi k/N$.}, both from field theory and string theory side, that corrections to Eq. (3) run in powers of $1/N^2$. This then leaves no space to the Casimir scaling in four dimensions. Second, we suggest a heuristic picture, based on $k$-walls in $\mathcal{N} = 1$ gluodynamics, from which Eq. (2) naturally follows.

Domain walls are BPS objects in $\mathcal{N} = 1$ gluodynamics. They interpolate between distinct vacua of the theory, say vacuum $n$ and vacuum $n + k$ (altogether there are $N$ vacua). The domain walls interpolating between neighboring vacua are called “elementary,” while at $k > 1$ we deal with the $k$-walls which can be viewed as bound states of $k$ elementary walls.

The tension of $k$-walls is known \textit{exactly} [8],

$$T_k \equiv T_{n,n+k} = \frac{N}{8\pi^2} \left| \langle \lambda \lambda \rangle_{n+k} - \langle \lambda \lambda \rangle_n \right|$$

$$= N^2 \Lambda^3 \sin \left(\frac{\pi k}{N}\right), \quad k = 1, 2, ..., N.$$  

\footnote{Strictly speaking the statement below refers to $d > 2$. Two-dimensional gauge theories must be discussed separately, see Sect. 6. In two-dimensional QCD with \textit{massive matter} the sine in Eq. (2) is replaced [7] by sine squared, namely, $\sigma_k \sim \text{mesin}^2 \pi k/N$.}
(note that $T_{n,n+k}$ is $n$-independent). The similarity between Eqs. (2) and (4) is apparent. This becomes more than a mere similarity if one accepts our qualitative picture: we suggest that $k$-walls can be thought of as being “composed” of fluctuating $k$-strings connected by baryon vertices (junctions of $N$ strings). This picture originated from the observation that the wall tension scales as $N$, at large $N$. Moreover, it is supported by the recent finding [9] that a level $N$ Chern-Simons theory lives on the wall.

Indeed, if our picture is correct, one can easily show that

$$T_k \sim \sigma_k (\Lambda N) + \text{subleading corrections},$$

which leads directly to Eq. (2) for the $k$-string tension. We can also reverse the logic: if Eq. (2) is indeed a good approximation to the $k$-string tension of $\mathcal{N} = 1$ gluodynamics — a $k$-wall can be well approximated by a net of $k$-strings.

The organization of this paper is as follows: in Sect. 2 we explain why SU($N$) gauge dynamics leads to $1/N^2$ corrections to the leading $k$-string tension. In Sect. 3 we present our model and provide supporting arguments both from field theory and string theory. Section 4 is devoted to analysis of the $k$ dependence of binding forces versus $N$ dependence. A remark concerning non-supersymmetric Yang-Mills theories is presented in Sect. 5. In Sect. 6 we explain why the two dimensional case and the strong coupling lattice expansion are special. Section 7 is devoted to concluding remarks.

## 2 Corrections: $1/N$ or $1/N^2$?

At $N = \infty$ both $k$-walls and $k$-strings present ensembles of $k$ noninteracting constituents — elementary walls and fundamental strings, respectively. The binding emerges as a subleading in $1/N$ effect. The Casimir scaling predicts it at level $1/N$ while the sine formula at level $1/N^2$. In both cases the binding is weak at large $N$.

The Casimir scaling was abstracted from various models in which confinement is due to a one-gluon exchange\(^2\) (perhaps, appropriately modified in

\(^2\)The quadratic Casimir coefficient $C_R$ is defined as $T^a T^a = C_R \times 1_R$ where $T^a$ stands for the SU($N$) generators in the representation $R$ while $1_R$ is the unit matrix in the same representation. The color structure of the one-gluon exchange is proportional to $C_R$.\)
the infrared domain). In Sect. 6 we discuss an example — two-dimensional Yang-Mills theory. It enjoyed considerable numerical support from lattice simulations [10]. It is also obtained in the lattice strong coupling \(1/g^2\) expansion.

The sine law was motivated by various investigations of the Seiberg-Witten model and, almost simultaneously, from the string theory side. Equation (2) was first derived by Douglas and Shenker [11] as the QCD string tension in the softly broken \(\mathcal{N} = 2\) Yang-Mills theory\(^3\). Then it was obtained in the context of MQCD [13] and, more recently, in the AdS/CFT framework [14]. In the latter case, the sine formula (2) was found to be exact for the Maldacena-Nuñez background [15] and valid to a few percent accuracy in the Klebanov-Strassler background [16]. Recent lattice simulations suggest [17, 18] that the sine formula fits the \(k\)-string tension in pure Yang-Mills theory in four dimensions better than the Casimir formula.

Here we would like to address an aspect crucial for discriminating between the Casimir and sine laws, namely the \(N\) dependence of the binding energy of \(k\) fundamental strings. For simplicity we will use the example with \(k = 2\), although the argument is general and applicable for any \(k\).

Start from two well-separated fundamental quark-antiquark pairs, as it is shown in Fig. 2. When the separation \(l\) between two \(Q\)'s (or, which is the same, between two \(\bar{Q}\)'s) is large enough it is obvious that the tension of this configuration is just twice the fundamental tension \(\sigma_1\). As we adiabatically move the pair on the left towards the pair on the right, the attraction between the flux tubes switches on, and at \(l < L/N\) \((L\) is the length of the string, see Fig. 1b) the 2-string configuration of Fig. 1b becomes energetically favorable, see Appendix for a detailed derivation. When the separation \(l\) becomes \(< \Lambda^{-1}\) all remnants of 1-strings disappear. A similar process, described via supergravity, is given in [19]. The quarks \(Q^i\), \(Q^j\) (as well as \(\bar{Q}_k\), \(\bar{Q}_\ell\)) are in the mixed color state, symmetric plus antisymmetric, but this is unimportant because the \(k\)-string tension is supposed to depend only on the \(N\)-ality. It is

\(^3\)The original Douglas-Shenker formula is \(\sigma_k = N m \Lambda \sin(\pi k/N)\). Corrections of the order \(O(m^2)\) to this expression were studied in Ref. [12]. Note that the adjoint mass parameter \(m\) in Ref. [11] differs from that in Ref. [12] by a factor of \(N\). The large \(N\) scaling in softly broken \(\mathcal{N} = 2\) is not straightforward since the lightest “\(W\) boson” masses are not \(N\) independent; in fact, they are of the order \(O(\Lambda/N^2)\). At \(N \to \infty\) the description of the low-energy physics based on the \(U(1)^{N-1}\) limit inherent to the Seiberg-Witten solution, becomes invalid. The condition of applicability of Eq. (23) in Ref. [12] is \(m N \ll \Lambda\).
perfectly consistent to consider the *reducible* two-index representation.

![Figure 2: Two free “fundamental” flux tubes.](image)

Now, we can proceed to analysis of the graphs responsible for the attraction of two 1-strings. First of all, it will be convenient to formally define the 2-string tension (the definition for $k$-string is similar). Consider two rectangular Wilson contours, $C_1$ and $C_2$, both lying in the $\{x t\}$ plane and separated by distance $l$ in the $y$ direction ($z = 0$). The size of both contours is $L \times T \to \infty$, and they are parallel to each other. Each contour gives rise to the Wilson loop operator

$$W[C_i] = \text{tr} \exp \left( i \int_{C_i} A_\mu dx_\mu \right), \quad i = 1, 2,$$

(6)

describing the time evolution of 1-string of length $L$. Two 1-strings are parallel, oriented in the $x$ direction and separated by an interval $l$ in the $y$ direction; $z$ is set to zero. Then we define

$$\langle W[C_1], W[C_2] \rangle = \exp \left( -\Sigma_2(l) T L \right).$$

(7)

At $l \gg \Lambda^{-1}$

$$\Sigma_2(l) \to 2 \sigma_1,$$

while at $l \ll \Lambda^{-1}$

$$\Sigma_2(l) \to \sigma_2.$$
In field theory language the fundamental QCD string evolving in time can be viewed as a “fishnet” (Fig. 3). The attraction of the flux tubes is due to gluon exchanges connecting two planar “fishnets” (Fig. 4a). Note that the minimal number of gluons that are exchanged between the worldsheets is two, since one cannot transfer color between the flux tubes. It is rather obvious that the gluon exchanges are suppressed by $1/N^2$ since this is a non-planar contribution. Note that even gluons coupled to probe quarks do not produce $1/N$. Correspondingly, the sum of two Casimir coefficients, $C_{\square} + C_{\Box}$, has no $1/N$. The factor $1/N$ occurs at the stage of division of the reducible representation into two irreducible, $\Box \otimes \Box = \square \oplus \Box$. This separation is irrelevant for our purposes. The 1-string attraction (per unit length) is determined by exchanges that are localized in space on the scale $\Lambda^{-1}$. How the middle parts of the long 1-strings interact is independent on details of endpoints which are separated from the middle part by distance $L \gg \Lambda^{-1}$ — whether the quarks are symmetrized or antisymmetrized with respect to color, or none, does not matter. It must depend only on $N$-ality. In the general case of the $k$-string the quadratic Casimir is

$$C_R = kN + \sum_i r_i^2 - \sum_i c_i^2 - \frac{k^2}{N},$$

where $r_i$ and $c_i$ are the lengths of the rows and the columns, respectively,
of the Young tableau of the representation $R$. As we have argued above for the 2-string, in the general case the string tension should depend only on the $N$-ality and not on the specific representation — hence, it cannot depend on $r_i$ or $c_i$ which lead to potential $1/N$ dependences.

When the strings are separated by a large distance, $l \gg \Lambda^{-1}$, it makes no sense to speak of the gluon exchanges between the fundamental strings. The force between well-separated 1-strings is controlled by the lightest glueball exchange (“scalar dilaton”), which is certainly $1/N^2$ effect.

In the AdS/CFT framework, the Wilson loop is described by a minimal surface [2] (the string worldsheet) that extends “inside” the AdS space,

$$\langle W \rangle = \exp (-S_{NG}),$$

(9)

where $S_{NG}$ is the Nambu-Goto action. The $k$-string is described by $k$ coincident elementary worldsheets [19]. Clearly, the string tension will acquire a factor $k$ at the level of free strings, $\sigma_k = k\sigma_1$. The interaction between string worldsheets in the bulk AdS is via an exchange of closed strings (see Fig. 4b) and this process is obviously proportional to $g_{st}^2$ which translates into $1/N^2$. In fact, string theory predicts not only the right power ($1/N^2$), but also the right sign (minus): the flux tubes will attract due to an exchange of NS-NS fields.

Figure 4: The interaction of two “fundamental” flux tubes: (a) Field-theory picture — two-gluon exchange; (b) String-theory picture — exchange of a closed string between two worldsheets.
3 The model

In this section we outline our model or, better to say, a heuristic picture which has been mentioned in the introduction. We use both field theory and string theory languages.

![Figure 5: The domain wall in supersymmetric gluodynamics as a net of strings joined via baryon vertices. The example above refers to SU(6). One should imagine an irregular lattice of nodes connected by lines in a chaotic (but planar) way, so that each node either receives or emits \( N \) lines. The net fluctuates in the quantum-mechanical sense, and in no way represents a regular structure similar to fullerenes.]

3.1 Field theory picture

The elementary wall tension in \( \mathcal{N} = 1 \) gluodynamics scales at large \( N \) as \( NA^3 \). One may ask how the factor \( N \) appears in the theory where there are no fields in the fundamental representation of SU(\( N \)), only in the adjoint.
Ordinary solitons scale as $N^2$. This led Witten [20] to suggest that the BPS domain walls at hand are in fact QCD D-branes. A D-brane signature of the $\mathcal{N} = 1$ domain walls is that QCD strings can end on these domain walls [20] (see also [21, 22, 23]) and parallel walls can exchange glueballs (closed strings) between them [24].

This is no explanation on the microscopic field-theoretic level, however. In a bid to explain the linear $N$ dependence of the wall tension in a “natural way” we will assume that the wall presents a (fluctuating) network of interconnected $N$-string junctions, as in Fig. 5. It is known since long that $N$ fundamental strings can join each other in a baryon vertex. Let us assume that these nodes form a flat two-dimensional structure. Every node either emits or absorbs $N$ fundamental strings. The strings intertwine the nodes randomly (but in a planar way) creating an elementary domain wall. If the density of nodes is of order one per area $\Lambda^{-2}$ of the wall surface, then the tension will naturally scale as $N$.

Since the wall is made of $N$-string junctions we can relate the tension of the wall to the tension of the string. Indeed, on the one hand, the energy of the square with area $\Lambda^{-2}$ on the elementary wall is $T_1 \Lambda^{-2}$, by definition. On the other hand, it is of the order of $\sigma_1 N \Lambda^{-1}$. Thus, $\sigma_1 \sim T_1 (N \Lambda)^{-1}$. The axial charge of the wall is carried by the baryonic junctions (we discuss this issue in more details in the next subsection).

Now, let us pass to $k$-strings and $k$-walls. It is established that $k$ elementary walls form a bound state, the $k$-wall [25, 26]. It is also well established that the multiplicity of the $k$-wall is [9, 26]

$$\nu_k = \frac{N!}{k!(N-k)!}. \quad (10)$$

This is also the multiplicity of the \textit{antisymmetric} $k$-index representation. On the other hand, as various arguments indicate, for $N$-ality $k$ the lowest energy configuration for two distant groups of $k$ fundamental quarks (antiquarks) corresponds to quarks in the \textit{antisymmetric} representation of SU($N$). This is suggestive that the $k$-wall is built of the $k$-strings in the same way as the elementary wall of fundamental strings. The $k$ wall can be viewed as a flat two-dimensional fluctuating web of nodes each of which emanates or absorbs $N$ $k$-strings. Here we assume the existence of the $k$-string junctions, which can be viewed as bound states of 1-string junctions.
If so, we arrive at the relation (5). Unlike the $k$-wall tension which is given by the sine formula exactly, see Eq. (4), we do not expect the sine formula to be exact for the $k$-strings: the wall is a BPS object, whereas the QCD string is not.

A general function that satisfies the requirements: (i) symmetry under $k \to N - k$; (ii) being even in powers of $N$; and (iii) $\sigma_k$ tending to $k\sigma_1$ at large $N$, can be written as follows:

$$NA^2 \sum_{\text{odd } l} C_l(N) \left( \sin \frac{\pi k}{N} \right)^l,$$

with $C_1 = 1$. Since the QCD string is non-BPS, apriori all $C_l$ are non-vanishing.

What can be said of corrections in the right-hand side of Eq. (5)? In the web of $k$ strings forming a $k$-wall, binding interactions in the direction transverse to the wall plane, which are $k$-dependent, are contaminated by interactions along the wall plane, which we expect to be $k$ independent. That is to say that we expect

$$\frac{\sigma_k}{k} - \Lambda^2 \left( \frac{N}{k} \right) \sin \frac{\pi k}{N} = O(1/k).$$

If so, the corrections in the right-hand side of Eq. (5) are of order $1/k$. At large $N$ and $k/N$ fixed, these corrections are then subleading.

We would like to mention a possible difficulty in our model, a question to be addressed in the future\textsuperscript{4}. The 1-wall thickness is $\sim 1/(\Lambda N)$ [23], whereas the flux-tube thickness is much larger, $\sim 1/\Lambda$. That is, the QCD 1-string hardly fits the 1-wall in the perpendicular direction. The thickness of $k$-walls and $k$-strings has not yet been established. It might well be that the thickness of both of them at $k/N$ fixed and $N \to \infty$ is the same. This is another reason why our picture may be successful at large $k$ but definitely fails at small $k$.

The arguments above certainly do not have the status of a direct field-theoretical proof of Eq. (2) in $\mathcal{N} = 1$ gluodynamics. They are nevertheless supplementary and independent of other arguments, such as MQCD-based

\textsuperscript{4}A.A. thanks J. Barbón for addressing both the problem and its possible solution.
derivation (it does not apply to $\mathcal{N} = 1$ gluodynamics, but, rather, to unknown theories in the same universality class), or the supergravity argument (which holds for theories having a different ultraviolet content than $\mathcal{N} = 1$ gluodynamics). The picture we have conjectured is entirely within field theory per se.

### 3.2 String theory picture

From the string theory point of view our model is rather natural. For example, in the Polchinski-Strassler supergravity dual [27], domain walls are represented by wrapped D5 branes that look like D2 branes from the 4d point of view. The baryons are wrapped D3 branes that look effectively like D0 branes. It was even conjectured by Polchinski and Strassler that “an assembly of baryons can be arranged into a spherical domain wall!”.

In the type IIA realization of $\mathcal{N} = 1$ gluodynamics by Acharya and Vafa [9], domain walls are represented by wrapped D4 branes and baryons as wrapped D2 branes.

In both realizations [27, 9] the wall looks like a D2 brane and the baryon-vertex as a D0 brane. We suggest that the D2 brane is made out of D0 branes and fundamental strings that connect them.

Indeed, according the Acharya-Vafa [9] there is a level $\mathcal{N}$ Chern-Simons (CS) term on the wall. The source of this term is the interaction term of the D4 with a RR bulk field. Since the D4 wraps an $S^2$ with a total RR flux of $\mathcal{N}$ units, we get a level $\mathcal{N}$ CS term in the 2+1 theory.

The CS term can be viewed as a baryon vertex, if the field strength on the domain wall is replaced by a delta function. This is exactly what we obtain by wrapping a D2 over $S^2$. So, the wrapped D4 can be thought of as made out of D2 branes: both serve as a source for $\mathcal{N}$ fundamental strings.

To conclude, we suggest that the wall (a D-brane) is made out of a net of lower dimensional D-branes and the strings that connect them. The tension of the wall is due to the tension of the strings (but also due to the tension of the D0 branes).

The axial charge of the wall, that is the RR charge of the D2 brane, is due to the RR charge of the D0 branes (the baryon vertices). This should explain why the order parameter $\langle \lambda \lambda \rangle$ changes its phase when one pierces the wall in passing from left to right or vice versa.
It is difficult to understand how the wall, which is a BPS object, is built from non-BPS constituents (recall that even the wrapped D2 is not BPS). It means that our construction is approximate. At large \(k\), however, the corrections are expected to be small.

The picture that we advocate in this section is obviously similar to Myers effect [28].

4 \(k\) dependence versus \(N\) dependence

In Sect. 2 we discussed in detail the structure of the \(1/N\) corrections responsible for the binding of \(k\) 1-strings into a \(k\)-string. The heuristic picture of “\(k\)-strings as building blocks for \(k\)-walls,” discussed in detail in Sect. 3, leads, at large \(k\), to the sine law, Eq. (2), for the \(k\)-wall tension, which implies not only a very special \(N\) dependence, but a special \(k\) dependence too. This \(k\) dependence may seem counterintuitive, at first sight. Our task is to invert the argument and see whether we can learn anything new from this analysis.

Let us start from the \(k\)-wall tension (4), which is the exact result of \(\mathcal{N} = 1\) gluodynamics. (In the remainder of this section, for simplicity, we put \(\Lambda = 1\).) Expanding the sine we get

\[
T_k = N \pi k - \frac{\pi^3}{6} \frac{k^3}{N} + \frac{\pi^5}{120} \frac{k^5}{N^3} - .... \tag{12}
\]

The first term represents \(k\) noninteracting 1-walls, while the second and higher terms are due to a binding force. Let us examine the second term, proportional to \(k^3/N\). The \(N\) dependence tells us that it is due to a binary interaction, since the domain walls are QCD D-branes, see Ref. [24]. Naively one would then require a coefficient proportional to \(k^2\) rather than \(k^3\), which is just a combinatorial factor. The exact formula tells us that this is impossible. What went wrong?

When we have an ensemble of \(k\) constituents with binary interactions (with a coupling \(g^2\)) bound in a “compound nucleus,” the binding energy scales as \(g^2 k^2\) only provided that the size of the system and the coupling \(g\) are \(k\)-independent. This is what happens, for instance, in Witten’s picture of baryons [29]. However, in the case of the domain walls, we have an ordered system along the line, with a non-universal coupling\(^5\). The coupling of a

\(^5\)We thank A. Ritz who pointed out to us the importance of the wall ordering and
wall to its $\ell$-th neighbor is $\ell/N$. The sum over all possible pairs yields the $k^3$ behavior. A similar $k^3$ behavior is expected in a one-dimensional system with size $\sim k$ and a linear potential between pairs. Other examples of one-dimensional ordered $k$-body systems with the ground state energy scaling as $k^3$ are known in the literature, see e.g. [30].

As we pass to $\sigma_k$, the situation may be rather similar. In this case the sine law implies

$$\sigma_k = \pi k - \frac{\pi^3 k^3}{6 N^2} + \frac{\pi^5 k^5}{120 N^4} - \ldots$$

(13)

Again, the $N$ dependence of the term $k^3/N^2$ has the structure typical of binary interactions. A “natural” $k$ dependence in this case, as was mentioned, is $k^2$. This “natural” $k^2$ factor follows from a picture of a “compound state” with a typical size being $k$-independent. Basing on the lesson of $k$-walls we are inclined to think that a more complicated $k^3$ scaling takes place due to a $k$-dependence of a typical size of the cross-section of the $k$-string.

5 What can be said about non-supersymmetric gauge theories?

Above we presented a model yielding a relation between the $k$-wall and $k$-string tensions in $\mathcal{N} = 1$ gluodynamics. It is natural to ask now about non-supersymmetric gauge theories, in particular, the orientifold theory discussed in Ref. [31]. In this theory, the gluino adjoint field is replaced by a Dirac two-index field in the antisymmetric representation, which, obviously, makes it non-supersymmetric. The orientifold theory was shown [31] to be planar equivalent to $\mathcal{N} = 1$ gluodynamics in the mutually overlapping sectors. At large $N$ both have $N$ discrete vacua and domain walls of a very similar structure. This may suggest that the $k$-string tension in the non-supersymmetric orientifold theory is described by the same sine formula as in $\mathcal{N} = 1$ gluodynamics.

At first sight this suggestion looks heretical. Indeed, in $\mathcal{N} = 1$ gluodynamics any $k$-index source cannot be screened, and, therefore, develops a string. On the other hand, in the orientifold theory, any source with even $k$ can be non-universality of the interaction force.
completely screened by dynamical quarks. Any source with odd $k$ can be almost screened — only one fundamental index remains unscreened. Common wisdom prompts one that there is no place for such quantity as $\sigma_k$ in this theory.

Let us remember, however, that our argument in favor of the sine formula refers to large $k$, namely, $k \sim N$ at $N \to \infty$ (we can obtain no result otherwise). At large $k$, in order to screen the $k$-index source, one has to (pair)produce a large number of dynamical quarks, scaling as $N$, from the vacuum. This process is suppressed, presumably exponentially, as $e^{-Ck}$ where $C$ is a positive number (we are certain that it is at least power suppressed). Thus, at large $k$ there is confinement in the orientifold theory, $\sigma_k \neq 0$, and, according to our hypothesis, obeys the sine law.

6 Why two-dimensional Yang-Mills theory and strong coupling lattice expansion are special?

In two-dimensional Yang-Mills theory the “$k$-string tension” is known exactly, and it obeys the Casimir scaling, $\sigma = g^2 C_R$. A question which immediately comes to one’s mind is “what makes two-dimensional formula special?”

The answer is as follows. Genuine strings, with local interaction (such as those one deals with in four dimensions) never develop in two dimensions. The two-dimensional Yang-Mills theory (in the axial gauge) is free, Gaussian. Non-linearity is absent, and so are transverse dimensions. Confinement is due to the one-gluon exchange (instantaneous in time) which provides a linear potential with the coefficient proportional to the Casimir operator $C_R$. As a reflection of the absence of the bona fide strings in two dimensions, please, observe that the “$k$-string tension” depends in this case on the specific representation and not merely on the $N$-ality.

The same situation takes place in a variety of ad hoc four-dimensional models where (a modified) one-gluon exchange is postulated to be responsible for confinement. In these models the gluon propagator at small momenta is assumed to scale as $1/p^4$ rather than $1/p^2$. This certainly provides a linear confinement with the $C_R$ coefficient. This approach is totally inconsistent
with our arguments.

The situation in (four-dimensional) lattice strong coupling expansion [32] is similar to that in two-dimensional Yang-Mills theory. As well-known, see e.g. [4], the strong coupling Wilson expansion implies a linear confinement with the coefficient in front of the linear potential determined by $C_R$. The “1-strings” do not interact “in the bulk.” Therefore, we do not have a *bona fide* 1-string binding of the type we expect in actual four-dimensional Yang-Mills theory, where the scale parameter is dynamically generated, and the attraction of 1-strings (per unit length) is localized along the string.

7 Discussion and conclusions

The main thrust of this paper is on the model presenting the BPS $k$-wall in $\mathcal{N} = 1$ gluodynamics as a planar network of baryonic vertices (nodes) connected by $N$ $k$-strings. The network is irregular and fluctuating. The model is admittedly heuristic and qualitative. Taking it at its face value, however, one can get a relation between the $k$-wall tension which is exactly known, and the $k$-string tension.

Our relation implies the Douglas-Shenker sine formula. Irrespective of particular models is the result we have proven *en route*: the $1/N$ corrections to the free string limit, $\sigma_k = k\sigma_1$, run in powers of $1/N^2$ rather than $1/N$. This rules out the Casimir scaling. In general, models with linear confinement basically fall into two categories. The coefficient in front of the linear term in the potential between two probe objects in the given model may either depend on a particular color representation of the probe object, or only on the $N$-ality. In the first class of models corrections to $\sigma_k = k\sigma_1$ run in powers of $1/N$. In the second class, which includes four-dimensional Yang-Mills theory, corrections run in powers of $1/N^2$. Note that in the first class of models, it is wrong to label the string tension only by the $N$-ality $k$. One should indicate the corresponding Young tableau for the particular color representation of the probe object.

The central question in the issue of the $k$-string tension is the nature and size of corrections to the sine formula. We presented an argument showing that at $k/N$ fixed and $N \to \infty$ these corrections are suppressed by $1/k$ factors, i.e. suppressed parametrically. On the other hand, one of the backgrounds considered in Ref. [14] (the Klebanov-Strassler background [16])
leads to corrections to the sine formula which are strongly suppressed numerically, but not parametrically. Is it due to the fact that the ultraviolet behavior of the AdS/CFT models is different from that in $\mathcal{N} = 1$ gluodynamics? The question remains open. Moreover, recently it was argued [33] that in (1+2)-dimensional Yang-Mills theory corrections to the sine formula are so large that they, in fact, convert it into a (well) approximated Casimir formula for the background of Cvetić et al. [34], an analog of the Klebanov-Strassler background. It is curious that for the (1+2)-dimensional analog of the Maldacena-Nuñez supergravity background (see Ref. [35]) a sine formula is suggested in Ref. [33]. From our standpoint, the only difference between the (1+2)-dimensional and (1+3)-dimensional gauge theories is due to the fact that in 1+3 dimensions the mass scale is dynamically generated (via dimensional transmutation), while in 1+2 dimensions the mass scale is given by the square of the gauge coupling constant. In both cases non-linearities are well-developed, and the spacial string structure is “local.” In other words, the Casimir scaling behavior looks suspicious, and, in fact, we expect that in 1+2 dimensions large-$N$ corrections to $\sigma_k = k\sigma_1$ run in powers of $1/N^2$, much in the same way as in 1+3 dimensions.

It is important to note that our consideration is fully invertible. If in the future someone succeeds in deriving the Douglas-Shenker sine formula from other principles, this will simultaneously substantiate our picture of the BPS domain $k$-walls made of fluctuating networks of $k$-strings.

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Appendix

In this appendix we calculate the distance where two 1-strings merge into a 2-string (see Fig. 1b). Let us denote the angle between the two 1-strings by $2\alpha$. Then the forces on the junction yield

$$\sigma_2 = 2\sigma_1 \cos \alpha.$$  \hfill (A.1)

Assuming the sine formula (2) for the string tension, we obtain

$$\alpha = \frac{\pi}{N},$$

which is physically reasonable: in the large $N$ limit the two strings hardly interact, and the “merging angle” should vanish.

Comparing the energy of the two separate 1-strings (Fig. 2) with the energy of the configuration in Fig. 1b we arrive at

$$2\sigma_1 L = 4\sigma_1 \frac{l}{2 \sin \alpha} + \sigma_2 \left( L - \frac{l}{2 \tan \alpha} \right).$$  \hfill (A.2)

Now, by substituting the sine formula (2) and the angle $\alpha = \frac{\pi}{N}$ we obtain

$$L \left( 1 - \cos \frac{\pi}{N} \right) = l \left( \frac{2 - 2 \cos \frac{\pi}{N}}{2 \sin \frac{\pi}{N}} \right).$$  \hfill (A.3)

Thus, in the large $N$ limit, the critical distance is

$$l = L \frac{\pi}{2N}.$$  \hfill (A.4)

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