Group contractions and its consequences upon representations of different spatial symmetry groups.

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\textbf{Abstract}

We investigate the group contraction method for various space-time groups, including $[SO_3 \to E_2]$, $[SO_{3,1} \to G_3]$, $[SO_{3,-h,h} \to P_{3,1} (h = 1, 2)]$, and its consequences for representations of these groups. Following strictly quantum mechanical procedures we specifically pay attention in the asymptotic limiting procedure employed in the contraction $[G \to G']$, not only to the respective algebras but to their representation spaces spanned by the eigenvectors of the Cartan subalgebra and the eigenvalues labelling these representation spaces. Where appropriate a physical interpretation is given to the contraction procedure.

1 Introduction

The group contraction method was initially introduced by Inönü and Wigner in 1953. The main interest resulted from the study of the transition of relativistic to nonrelativistic quantum mechanics in the asymptotic limit when velocities are small compared to the velocity of light. Under this limit the Lorentz group becomes the Homogeneous Galilei group. The group contraction method continues to be a subject of active research in particularly as applied to quantum groups. In general this method determines under a change of parameter scale a limiting process on the group generators and its algebra producing as a consequence a new group with corresponding algebra. The main difficulty in a general approach is the construction of the sequence of limiting representation spaces. In this paper we focus on this aspect of the problem by reviewing some well-know applications to space-time symmetry groups. We employ a direct approach by which we apply the limiting procedure to the standard quantum mechanical construction of representation spaces, i.e., as an eigensystem problem of the generators of the algebra, \textit{à la} $SO_3$ of angular momentum theory.

Since a Lie group is uniquely determined by its algebra about the identity, it is possible and easier to discuss contraction with regard to Lie algebra. From this point of view, group contraction is defined as follows:

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**Definition:** Inönü-Wigner Contraction (Hermann [4]): Given a Lie algebra $L$ associated with $G$, and a basis $X_a (a = 1, 2, \ldots, n)$ of the vector space of $L$, satisfying

$$[X_a, X_b] = \sum_{c=1}^{n} D_{ab}^{\epsilon} X_c \quad a, b = 1, \ldots, n.$$ 

where $D_{ab}^{\epsilon}$ are the structure constants of $L$ with respect to this basis, the Jacobi identity imposes the following condition on the structure constants

$$\sum_{c=1}^{n} (D_{bd}^{\epsilon} D_{ac}^{\epsilon} + D_{dc}^{\epsilon} D_{ba}^{\epsilon} + D_{ab}^{\epsilon} D_{cd}^{\epsilon}) = 0. \quad (1)$$

For the basis of $L$, we suppose that with a continuous, directed, contraction parameter $\lambda$

- the infinite sequence $[X_a]^\lambda$ and the corresponding structure constants $[D_{ab}^{\epsilon}]^\lambda$ are known,
- the $\lim_{\lambda \to \infty} [D_{ab}^{\epsilon}]^\lambda = [D_{ab}^{\epsilon}]^\infty$ exists for all $a, b, c$, and
- eqn. (1) is consistent under the limit.

Then as a consequence $[D_{ab}^{\epsilon}]^\infty$ generates a new algebra $L'$ called the contraction of $L$.

With this as a starting point we investigate in the following sections the consequences of the group contraction method for representation spaces of various well-known spacetimes symmetry groups, including the de Sitter groups contracted to the Poincaré group.

## 2 Contraction of $SO_3 \to E_2$

### 2.1 The Algebras:

With the main purpose of establishing our procedure we begin with the algebra contraction $so_3 \to e_2$. The rotation group $SO_3$ in $R^3$ has three generators $\{J_1, J_2, J_3\}$ corresponding to infinitesimal rotations and satisfying the associated algebra

$$[J_r, J_s] = i \epsilon_{rst} J_t \quad r, s, t = 1, 2, 3, \quad (2)$$

where $\epsilon_{rst}$ is an rank-(2,1) tensor antisymmetric in the lower indices with $\epsilon_{12}^3 = 1$. To initiate the contraction, we introduce a positive real parameter $R$ and construct the following sequence of elements defined on the $so_3$ algebra, with $J_s R \equiv J_s (R)$ (See Appendix). In principle we don’t assume the linear relation $J = R \wedge P$ of classical mechanics because it actually is a consequence of the following contraction process.

$$J_0 \equiv J_3, \quad \Pi_s \equiv \frac{J_{s R}}{R} \quad K_s \equiv \lim_{R \to \infty} \Pi_s \quad s = 1, 2, \quad (3)$$

for which the resulting algebra takes the form

$$[\Pi_1, \Pi_2] = i R^2 J_0, \quad [\Pi_2, J_0] = i \Pi_1, \quad [J_0, \Pi_1] = i \Pi_2. \quad (4)$$

On taking the limit $R \to \infty$ in eqn. (3), we obtain the algebra

$$[K_1, K_2] = 0, \quad [K_2, J_0] = i K_1, \quad [J_0, K_1] = i K_2.$$

These relations define the Euclidean algebra of ‘translations’ and rotations, denoted as $e_2 \equiv i_2 \oplus so_2$. Here the generator $J_3$ associated with rotations about the z-axis remains unchanged, while the generators $J_1, J_2$, determining rotations about the $x$ and $y$ axes respectively, are transformed under the limit into generators of ‘translation’ in directions $y$, $x$. Interpreting this one requires the symmetry $SO_3$ about the centre $P_{N/S} = (0, 0, \pm R)$ of the sphere of radius $R$ is locally seen as an approximate symmetry $T_2 \times SO_2$ of the tangent plane at the pole or the plane at infinity when $R \to \infty$ (see [5] for a generalization to $O_{n+1} \to E_n$).
Taking the Casimir of $SO_3$

$$J^rJ_r = J_1^2 + J_2^2 + J_3^2 = J.J,$$

and applying eqn. (3), we obtain

$$J_R, J_R = R^2(\Pi_1^2 + \Pi_2^2) + J_0^2.$$  

On taking the contraction limit define the Casimir for $E_2$ as

$$C_{E_2} \equiv \lim_{R \to \infty} \frac{J_R, J_R}{R^2} = \lim_{R \to \infty} (\Pi_1^2 + \Pi_2^2 + J_0^2/R^2) = K_1^2 + K_2^2 = K.K.$$  

Physically these Casimirs reflect the conservation laws of angular momentum and linear momentum, the one being the limit of the other under contraction.

In the standard treatment of $SO_3$ it is usual to transform to the spherical tensor basis of the $so_3$ algebra

$$J_+ \equiv J_1 + iJ_2, \quad J_- \equiv J_1 - iJ_2, \quad J_\pm^\dagger = J_\mp,$$

where the generators satisfy the algebra

$$[J_+, J_+] = +J_+, \quad [J_+, J_-] = 2J_3, \quad [J_-, J_-] = -J_-,$$

and the Casimir is now expressed as

$$JJ = \frac{1}{2}(J_+J_- + J_-J_+) + J_3^2.$$  

Applying the contraction to $J_\pm$, we obtain $K_\pm$ in $E_2$ defined as

$$K_+ \equiv K_1 + iK_2, \quad K_- \equiv K_1 - iK_2, \quad K_\pm^\dagger = K_\mp, \quad K_\pm = \lim_{R \to \infty} \frac{J_\pm R}{R},$$

with commutators

$$[K_+, K_] = 0, \quad [J_0, K_] = -K_-, \quad [J_0, K_+] = +K_+,$$  

where the associated Casimir takes the form

$$K.K = K_+K_- = \frac{1}{2} \{K_+, K_-\} = K_-K_+.$$  

2.2 The Representation Spaces:

We now investigate the effect of contraction on the representations space of $SO_3$. The bases of the representation spaces, labelled by half-integers $j$ and $m$, are spanned by the eigenvectors $|jm\rangle$ for $J.J$ and $J_3$ (see (4))

$$J.J|jm\rangle = |jm\rangle j(j+1), \quad J_3|jm\rangle = |jm\rangle m, \quad \text{with} \quad -j \leq m \leq j$$

$$J_+|jm\rangle = |j, m+1\rangle a_{jm}, \quad \|a_{jm}\|^2 = j(j+1) - m(m+1)$$

$$J_-|jm\rangle = |j, m-1\rangle b_{jm}, \quad \|b_{jm}\|^2 = j(j+1) - m(m-1).$$

The eigenvectors and corresponding eigenvalues of $K.K$, $J_0$ in $E_2$ are given as

$$K.K|km\rangle = |km\rangle k^2, \quad J_0|km\rangle = |km\rangle m.$$  

(7)

From eqns. (3) and (4) we require

$$J_0K_+|km\rangle = K_+|km\rangle (m+1) \Rightarrow K_+|km\rangle = |k, m+1\rangle a_{jm}$$

$$J_0K_-|km\rangle = K_-|km\rangle (m-1) \Rightarrow K_-|km\rangle = |k, m-1\rangle b_{jm}.$$  

(8)

(9)

Evaluating $a_{jm}, b_{jm}$ using eqns. (3) and (4) we obtain

$$K.K|km\rangle = K_+K_-|km\rangle = |km\rangle b_{jm}a_{jm} = |km\rangle k^2 \Rightarrow b_{jm}a_{jm} = k^2.$$  

(10)
In addition using the adjoint property of the operators

\[(km|K_-K_+|km) = (km|km)\tilde{a}_{jm}\tilde{a}_{jm} \geq 0 \quad \Rightarrow k^2 = \tilde{a}_{jm}\tilde{a}_{jm} > 0\]

From this we deduce that

\[\tilde{b}_{jm} = \tilde{a}_{jm}, \quad |\tilde{a}_{jm}| = k > 0.\]

In order to establish a connection under group contraction between the two groups we need to correlate their respective group actions. Since \(J_0\) is unchanged by contraction its group action also remains unchanged. The Casimirs provide a different situation and therefore the eigenvalues \(j\), and hence eigenvectors, are dependent on \(R\) in such a way that \(\lim_{R \rightarrow \infty} \frac{j_R}{R} = k\) (see \(\text{[12, 13, 14]}\)). Thus we introduce \(j_R\) and eigenvectors \(\{j_R m\}\) with action

\[\Pi_+\{j_R m\} = |j_R, m + 1\rangle a_{j_R m} \quad \|a_{j_R m}\|^2 \equiv j_R(j_R + 1) - m(m + 1)\]

\[\Pi_-\{j_R m\} = |j_R, m - 1\rangle b_{j_R m} \quad \|b_{j_R m}\|^2 \equiv j_R(j_R + 1) - m(m - 1),\]

and satisfying the conditions

\[\lim_{R \rightarrow \infty} \frac{j_R}{R} = k \quad \lim_{R \rightarrow \infty} |j_R m\rangle = |km\rangle. \quad (10)\]

Therefore we obtain that

\[K.K|km\rangle = \lim_{R \rightarrow \infty} \frac{j_R J_R}{R^2} |j_R m\rangle\]

\[\Rightarrow |km\rangle k^2 = \lim_{R \rightarrow \infty} |j_R m\rangle \frac{j_R(j_R + 1) - m(m - 1)}{R^2},\]

We then see that from eqns.\(\text{[4]}\) and \(\text{[6]}\)

\[K_+K_-|km\rangle = \lim_{R \rightarrow \infty} \frac{1}{2} \{-\Pi_+, \Pi_-\}|j_R m\rangle\]

\[= \lim_{R \rightarrow \infty} |j_R m\rangle \frac{j_R(j_R + 1) - m(m - 1)}{R^2}\]

\[= |km\rangle k^2,\]

as required. From eqns.\(\text{[4]}\) and \(\text{[6]}\) we can infer that \(j_R \sim Rk \Rightarrow h j_R \sim R(hk) = R\ell\) in analogy to the standard classical angular momentum definition \(\vec{j} = \vec{r} \times \vec{p}\). Note that as \(R\) increases, \(j_R\) takes large values corresponding to the regime of classical physics.

### 3 Contraction of \(SO_{3,1} \rightarrow G_3\)

#### 3.1 The Algebras:

As our second application let us consider the homogeneous Lorentz group \(SO_{3,1}\), for which we have six generators \(J_r, B_r\) \((r = 1, 2, 3)\) obeying the following algebra

\[[B_r, B_s] = -ie_{rs} J_t, \quad [J_r, B_s] = ie_{rs} B_s, \quad [J_r, J_s] = ie_{rs} J_t, \quad (11)\]

where \(B_r\) represents the Lorentz boosts, and \(J_r\) spatial rotations which clearly form a subgroup of \(SO_{3,1}\). The Casimirs of the proper Lorentz group are found to be

\[C_{2,3}^{1,3} = J^r J_r - B^r B_r = J.J - B.B, \quad C_4^{1,3} = J^r B_r + B^r J_r = 2J.B. \quad (12)\]

In order to perform the contraction in eqn.\(\text{[4]}\), we define the following sequence with \(B_{r,c} \equiv B_r(c)\)

\[J_r, \quad \Omega_r \equiv \frac{B_{r,c}}{kc}, \quad G_r \equiv \lim_{c \rightarrow \infty} \Omega_r, \quad r = 1, 2, 3, \quad (13)\]
where we introduce the velocity of light $c$ to parametrize the contraction procedure. For convenience the imaginary unit appears solely to obtain positive valued Galilei group Casimirs defined below. In taking the limit $c \to \infty$, the relations given by eqn. (11) take the form

$$[G_r, G_s] = 0, \quad [J_r, G_s] = i\epsilon^s_{rs} G_t, \quad [J_r, J_s] = i\epsilon^t_{rs} J_t.$$  \hfill (14)

Obviously these relations define locally the homogeneous Galilei group $\mathbf{G}_3 \equiv K_3 \times SO_3$, which contains the transformations associated with spatial rotations $J_r$ and a change of inertial reference system $G_r$. Clearly this group is isomorphic with the Euclidean group in 3-Dim $\mathbf{E}_3 \equiv T_3 \times SO_3$ but its physical content is distinct — one is the symmetry of 3-space and the other the symmetry group of the laws of Newtonian motion. We find on using the substitutions given by eqn. (13) in the expressions for the Lorentz Casimirs given by eqn. (12), redefining the Casimirs and then taking the limit

$$C_2^3 \equiv \lim_{c \to \infty} \frac{C_{1,3}^3}{c^2} = \lim_{c \to \infty} (J^r J_r / c^2 + \Omega^r \Omega_r) = G^r G_r = G.G,$$

$$\lim_{c \to \infty} \frac{C_{1,3}^3}{ic} = \lim_{c \to \infty} 2(J^r \Omega_r) = 2(J^r G_r) = 2J.G.$$  \hfill (16)

But while the former remains a Casimir for the Galilei group, the latter is not as $[J, J.G] \not= 0$. Naturally, in the nonrelativistic limit $c \to \infty$ (or physically small velocities $v \to 0$) the homogeneous Lorentz group is transformed into the homogeneous Galilei group.

The usual analysis of the Lorentz group employs the isomorphism between its algebra and that of $\mathbf{SL}_2 \times \mathbf{SL}_2$ where the overscore signifies that the two $\mathbf{SL}_2$ are strictly related by complex conjugation. We redefine the generators of the Lorentz group as

$$A_r^- \equiv \frac{1}{2}(J_r - iB_r), \quad A_r^+ \equiv \frac{1}{2}(J_r + iB_r),$$

with $A_r^\pm = A_r^x, J_r = J_r$ and $B_r = -B_r$, and we obtain

$$[A_r^-, A_r^+] = i\epsilon_{rs} A_r^s, \quad [A_r^+, A_r^-] = 0, \quad [A_r^+, A_r^+] = i\epsilon_{rs} A_r^s.$$  \hfill (17)

The Casimirs of each $sl_2$ algebra are now

$$A^- . A^- = A^- \cdot A^- , \quad A^+ . A^+ = A^+ \cdot A^+ ,$$

while the Lorentz Casimirs can be expressed as

$$C_{1,3}^3 = 2(A^-. A^- + A^+. A^+), \quad C_{1,3}^4 = 2i(A^- . A^- - A^+ . A^+).$$

As the operators $\{A^-\}$ and $\{A^+\}$ form two sets of independent generators of $\mathbf{SL}_2$ (see [10]), and therefore can be separately redefined in terms of spherical vector operators $A^{\pm}_\mu$ and $A^{\pm}_\mu$ with $\mu = (+, 0, -)$, as in section 3.2,

$$A_+^\pm \equiv A_+^x + iA_+^y, \quad A_-^\pm \equiv A_-^x - iA_-^y, \quad A_0^\pm \equiv A_0^x.$$  \hfill (18)

3.2 The Finite Representations:

Before developing the representation theory of the Lorentz and Galilei groups, we can rewrite the Lorentz generators as $A_r^- \equiv A_r \otimes I$ and $A_r^+ \equiv I \otimes A_r$ to emphasis the fact that we are dealing with two commuting $\mathbf{SL}_2$ algebras. These generators act on the direct product space $V_{j_1} \otimes V_{j_2}$ and correspond to two independent vector spaces under transformations of Lorentz group. Since $\mathbf{SL}_2 \sim SO_3$ locally, we can use the results of the previous section, and as a consequence the representations of the Lorentz group can be labeled as $[j_1 \otimes j_2]$ with dimension $(2j_1 + 1)(2j_2 + 1)$.

The two $\mathbf{SL}_2$ Casimirs $A^+, A^\mp$ have eigenspectra $j_1(j_1 + 1)$ and $j_2(j_2 + 1)$ respectively, which establishes the eigenspectra of the two Lorentz Casimirs as

$$C_{1,3}^3 : 2[j_1(j_1 + 1) + j_2(j_2 + 1)] = 2(j_1 + j_2 + 1)(j_1 + j_2) - 4j_1j_2.$$  \hfill (19)
The basis of the space $V_{i_1} \otimes V_{i_2}$ is constructed as a direct product of vector bases of $SO_3$ with vectors $|j_1m_1, j_2m_2\rangle$. The eigenspectra of $\hat{A}_0$ and $A_0$ are $m_1$ and $m_2$ respectively, while we also get

$$\hat{A}_\pm |j_1m_1, j_2m_2\rangle = |j_1m_1, j_2m_2 \pm 1\rangle \sqrt{j_1(j_1+1) - m_1(m_1 \pm 1)},$$

$$A_\pm |j_1m_1, j_2m_2\rangle = |j_1m_1, j_2m_2 \pm 1\rangle \sqrt{j_2(j_2+1) - m_2(m_2 \pm 1)}.$$ 

To determine the action of $J_\nu$ and $B_\nu$ on $|j_1m_1, j_2m_2\rangle$ we follow the development for the $so_3$ algebra in section 2.2. We have

$$J_\nu = A^- \nu + A^+ \nu \quad B_\nu = i(A^- \nu - A^+ \nu).$$

and therefore

$$J_\nu |j_1m_1, j_2m_2\rangle = A^- \nu |j_1m_1, j_2m_2\rangle + A^+ \nu |j_1m_1, j_2m_2\rangle,$$

$$B_\nu |j_1m_1, j_2m_2\rangle = i(A^- \nu |j_1m_1, j_2m_2\rangle - A^+ \nu |j_1m_1, j_2m_2\rangle).$$

In particular for $J_3$ and $B_3$, we have

$$J_3 |j_1m_1, j_2m_2\rangle = |j_1m_1, j_2m_2\rangle (m_1 - m_2),$$

$$B_3 |j_1m_1, j_2m_2\rangle = (j_1m_1, j_2m_2)i(m_1 - m_2).$$

Exploiting the various isomorphisms, all the analysis for $SO_{3,1}$ is similar to the above analysis for $SO_3$.

A separate analysis of $G_3$, which allows us to construct an eigenbasis appropriate to $G.G$, $J_3$ and $G_3$

$$G_3 |g, m, s\rangle = |g, m, s\rangle s, \quad J_3 |g, m, s\rangle = |g, m, s\rangle m,$$

$$G.G |g, m, s\rangle = |g, m, s\rangle g^2.$$ (21)

The action of the other generators can be obtained using the commutator relations but as we do not need them here we shall omit them.

With these results we now establish the connection between the two groups under the group contraction $SO_{3,1} \to G_3$. In analogy to $SO_3$ we begin with the eigenvalues of the respective Casimirs and we introduce eigenvalues $j_c$ and $m_c$ with eigenvectors $|j_1c m_1c, j_2c m_2c\rangle$. Using eqns.(13) and (15)

$$G.G |g, m, s\rangle = \lim_{c \to \infty} \frac{J_3 - B_3 \nu}{c^2} |j_1c m_1c, j_2c m_2c\rangle$$

which yields

$$|g, m, s\rangle g^2 = 2 \lim_{c \to \infty} |j_1c m_1c, j_2c m_2c\rangle \frac{j_1c^2 + j_2c^2 + j_1c + j_2c}{c^2}.$$ 

However using the quartic Casimir of eqns.(14) and (16) we find

$$2J_\nu G |g, m, s\rangle = \lim_{c \to \infty} \frac{2J_\nu B_\nu}{c^2} |j_1c m_1c, j_2c m_2c\rangle$$

$$= \lim_{c \to \infty} |j_1c m_1c, j_2c m_2c\rangle \frac{2i(j_1c + j_2c + 1)(j_1c - j_2c)}{c^2}$$

$$= \lim_{c \to \infty} |j_1c m_1c, j_2c m_2c\rangle \frac{2j_1c^2 + j_2c^2 + j_1c + j_2c}{c^2}$$

Here we have rewritten the quartic Casimir eigenvalue as

$$\frac{2i(j_1c + j_2c + 1)(j_1c - j_2c)}{c^2} = 2c \left( \frac{j_1c^2 + j_2c^2 + j_1c + j_2c}{c^2} - \frac{j_2c^2 + j_2c}{c^2} \right)$$

which yields

$$|g, m, s\rangle g^2 = 2 \lim_{c \to \infty} |j_1c m_1c, j_2c m_2c\rangle \frac{j_1c^2 + j_2c^2 + j_1c + j_2c}{c^2}.$$
for which the first term on the right yields in the limit the eigenvalue of the quadratic Galilei Casimir \( g^2 \). To avoid the appearance of infinities in the limit we must then have

\[
\lim_{c \to \infty} \frac{j_1, c}{c} = \lim_{c \to \infty} \frac{j_2, c}{c} = \frac{g}{2}.
\] (22)

The limiting process for the quadratic Casimir eigenvalues takes form

\[
g^2 = 2 \lim_{c \to \infty} \left[ \frac{j_1^2, c + j_2^2, c + j_1, c + j_2, c}{c^2} \right],
\]

while that of the quartic Casimir of eqn. (14) and (19) we obtain

\[
\lim_{c \to \infty} 2(\frac{j_1, c + j_2, c + 1}{c}) (\frac{j_1, c - j_2, c}{c}) = 0.
\]

and as a consequence \( 2JG(g, m, s) = 0 \). Similarly we obtain

\[
\lim_{c \to \infty} m_{1, c} + m_{2, c} = m,
\]

and

\[
|g, m, s\rangle = \lim_{c \to \infty} |j_{1, c} m_{1, c}, j_{2, c} m_{2, c}\rangle
\]

by observing that under the contraction limit

\[
J_3|g, m, s\rangle = \lim_{c \to \infty} J_3|j_{1, c} m_{1, c}, j_{2, c} m_{2, c}\rangle,
\]

\[
G_3|g, m, s\rangle = \lim_{c \to \infty} \frac{B_{3, c}}{1c} |j_{1, c} m_{1, c}, j_{2, c} m_{2, c}\rangle
\]

and hence we have

\[
|g, m, s\rangle m = \lim_{c \to \infty} |j_{1, c} m_{1, c}, j_{2, c} m_{2, c}\rangle (m_{1, c} + m_{2, c})
\]

\[
|g, m, s\rangle s = \lim_{c \to \infty} |j_{1, c} m_{1, c}, j_{2, c} m_{2, c}\rangle \frac{m_{1, c} - m_{2, c}}{c}.
\]

Introducing the linear expressions \( m_{1, c} = a_1 c + b_1, m_{2, c} = a_2 c + b_2, \) we find solutions

\[
\lim_{c \to \infty} (a_1 + a_2)c + (b_1 + b_2) = m,
\]

\[
\lim_{c \to \infty} \frac{(a_1 - a_2)c + (b_1 - b_2)}{c} = s,
\]

\[
\Rightarrow a_1 = -a_2 = \frac{s}{2}, \quad b_1 + b_2 = m,
\]

and this draws the connection between the labelling of the representation spaces of the Lorentz and Galilei groups.

The action of generators \( J_r \) and \( \Omega_r \) on these new eigenvectors is as for the Lorentz group action except for the substitution of \( j \) for \( j \), and \( m \) for \( m \) in all the matrix elements.

From all of the above we can pass over without difficulty to the contraction of the Poincaré group to the inhomogeneous Galilei group, ie. \( P_{3,1} \equiv T_{3,1} \times SO_{3,1} \rightarrow G_{3,1} \equiv T_3 \times G_3 \), given the fact that the contraction only affects the generators of Lorentz boosts, and thus contracts \( SO_{3,1} \rightarrow G_3 \) which has just been shown.

4 \( dS/AdS \) groups and Their Contraction to \( P_{3,1} \)

4.1 The Algebras

The two \( SO_{5-h,h} \) groups with \( h = 1 \) or 2 correspond to maximal group of symmetries in De Sitter/Anti-de Sitter \( (dS/AdS) \) spaces respectively, and represent uniform and curved space-time manifolds that are compatible with an expanding universe. The universe can be described
as a hypersurface inside a 5-dimensional spacetime of signature (4, 1) or (3, 2). If we use coordinates \(y^i\) with \(i = 1, \ldots, 5\), the hypersurface with curvature \(a\) is defined as

\[
(y^1)^2 + (y^2)^2 + (y^3)^2 + (y^4)^2 - (y^5)^2 = \eta_{ij} y^i y^j = -a^{-2} \quad \text{with} \quad a \equiv \frac{1}{R}.
\]

This hypersurface is invariant under linear transformations that preserve the metric \(\eta_{ij} = \text{diag}(+, +, +, +, -)\). These transformations comprise the Sitter/Anti-de Sitter groups \((SO_{h-h,h})\) whose algebra consists of \(\frac{1}{2}(5(5-1)) = 10\) generators, \(J_{ij}\) representing generalized rotations in \(E_5\). These generators satisfy the algebra

\[
[J_{ij}, J_{kl}] = i(J_{ik}\eta_{jl} - J_{il}\eta_{jk} + J_{lj}\eta_{ik} - J_{lk}\eta_{ij}),
\]

with \(i, j, k, l, m = 1, \ldots, 5\).

The two Casimirs of the de Sitter/Anti-de Sitter groups are

\[
C_2 \equiv \frac{1}{2} J_{ij} J^{ij}, \quad C_4 \equiv W_i W_i,
\]

where \(W_i\) represent a 5-dimensional vector defined as

\[
W_i \equiv \frac{1}{2} \epsilon_{ijklm} J^{jk} J^{lm}, \quad J^{ij} = J_{ik} J^{ik} \eta^{ij}.
\]

\(\epsilon\) is tensor totally antisymmetric with 5 index.

The contraction is defined by

\[
\Pi_\mu \equiv \frac{1}{R} J_{\mu, R}, \quad \lim_{R \to \infty} \Pi_\mu = K_\mu, \quad \text{with} \quad z = 1 \text{ or } 5.
\]

with

\[
J_{\mu, R} \equiv J_{\mu}(R).
\]

Using eqn.(28) we reexpress eqn.(23) in a 4-dimension spacetime notation as

\[
[J_{\mu \nu}, J_{\rho \sigma}] = i(J_{\mu \nu} \eta_{\rho \sigma} - J_{\mu \sigma} \eta_{\nu \rho} + J_{\nu \sigma} \eta_{\mu \rho} - J_{\nu \rho} \eta_{\mu \sigma}),
\]

with \(\mu, \nu, \rho, \sigma = 1, 2, 3, 4\) (see also [11]). Using eqn.(28) in eqns.(27,28) we obtain the following algebra

\[
[J_{\mu \nu}, J_{\rho \sigma}] = i(J_{\mu \rho} \eta_{\nu \sigma} - J_{\mu \sigma} \eta_{\nu \rho} + J_{\nu \sigma} \eta_{\mu \rho} - J_{\nu \rho} \eta_{\mu \sigma}),
\]

\[
[K_\mu, J_{\rho \sigma}] = i(K_\rho \eta_{\mu \sigma} - K_\sigma \eta_{\mu \rho}). \quad [K_\rho, K_\sigma] = 0.
\]

\(K_\mu\) denotes the translation operator in flat space-time, where a rotation on either the surface \((x_1, x_2)\) or \((x_3, x_5)\) transforms as a space-time translation in the limit of curvature zero \((R \to \infty)\). The generators \(K_\mu, J_{\rho \sigma}\) along with eqns.(23,27,28) define the Poincaré group in the contraction. Rewriting the 'de Sitter' Casimir invariants in (24) using eqn.(28) we have

\[
C_2 = \frac{1}{2} J_{ij} J^{ij} \rightarrow C_{2,R} = R^2 \Pi_\mu \Pi^\mu + \frac{1}{2} J_{\mu \nu} J^{\mu \nu}.
\]

We define the first Casimir invariant of Poincaré in the limit \(R \to \infty\) as

\[
C_{2,1}^2 \equiv \lim_{R \to \infty} \frac{C_{2,R}}{R^2} = K_\mu K^\mu.
\]

Using eqn.(28) in eqn.(25), the second 'de Sitter' Casimir invariant takes the form

\[
C_{4,R} = W_i W_i R = \frac{1}{4} R^2 \epsilon_{\lambda, \mu, \rho, \delta} \Pi^\mu J_{\mu \nu} \eta^{\lambda \rho} \eta^{\nu \delta} \Pi^\rho J_{\rho \omega} \eta^{\omega \mu} + \frac{1}{4} J_{\mu \nu} J_{\rho \omega} J_{\mu \nu} J_{\rho \omega}.
\]

The second term of the right-hand in eqn.(33) vanishes in the limit of zero curvature, and therefore the second Poincaré invariant takes the form

\[
C_{4,1}^1 \equiv \lim_{R \to \infty} \frac{C_{4,R}}{R^2} = J_{\mu \nu} J_{\rho \omega} K^\rho K_\omega - J_{\mu \nu} K_\rho J_{\rho \omega} K^\omega.
\]
4.2 AdS Unitary Representations

Following Nicolai [12], the generators $J_{ij} \in SO_{3,2}$ permit a spinorial representation in terms of the gamma-matrices $\langle \ldots | J_{ij} | \ldots \rangle = \Gamma_{ij}$, this matrix set is written as

$$\Gamma_{ij} = \Gamma_{ij}^\dagger \text{ for } J_{rs}, \ J_{45} \text{ with } r, s = 1, 2, 3,$$

$$\Gamma_{ij} = -\Gamma_{ij}^\dagger \text{ for } J_{4r}, \ J_{5r}. \tag{35}$$

We see that not all the $J_{ij} s$ are unitary, in accordance with the fact that finite-dimensional representations of a non-compact group are not unitary representations. To obtain a Hermitian representation of the generators $J_{ij} = J_{ij}^\dagger$, an infinite-dimensional representation is required. From the eqn.(35) we distinguish the compact generators $J_{45}$ and $J_{rs}$ from the non-compact generators $J_{4r}$ and $J_{5r}$. The operators $J_{rs}$ and $J_{45}$ generate a maximal compact subalgebra associated with $SO_3 \times SO_2$. We now define the following operators in $SO_{3,2}$ as

$$M^r_+ \equiv iJ_{4r} + J_{5r}, \ \ M^-_r \equiv iJ_{4r} - J_{5r}, \ \ \text{where } M^-_r = - (M^+_r)^\dagger, \tag{36}$$

with $r = 1, 2, 3$. The associated algebra is given as

$$[M^r_+, M^-_s] = 2(\delta_{rs} J_{45} + iJ_{rs}), \ \ [M^r_+, M^+_s] = [M^-_r, M^-_s] = 0, \ \ [J_{45}, M^-_s] = M^+_s, \ \ [J_{45}, M^+_r] = -M^-_r. \tag{37}$$

The laddering operators $M^+_r$ and $M^-_r$ respectively raise and lower the energy eigenvalues in unit of one when applied to the eigenstates of the energy operator $J_{45}$. Choosing $J_{45}, J^r J_r,$ and $J_3$ as our set of mutually commuting operators we identify our eigensystem as follows:

$$J^r J_r |(\ldots)E_0, s, m\rangle = s(s+1)|(\ldots)E_0, s, m\rangle,$$

$$J_{45} |(\ldots)E_0, s, m\rangle = E_0 |(\ldots)E_0, s, m\rangle,$$

$$J_3 |(\ldots)E_0, s, m\rangle = m |(\ldots)E_0, s, m\rangle, \tag{38}$$

where $(\ldots)$ denote a non-specified set of labels.

The $C_2$ Casimir can be written as:

$$C_2 = (J_{45})^2 + \frac{1}{2} J^r J_r + J^4 J_4 + J^5 J_5$$

$$= (J_{45})^2 + J^r J_r + \frac{3}{2} \{M^+_r, M^-_s\}. \tag{39}$$

Note that $\frac{1}{2} J^r J_r = J^r J_r, J^4 J_4 = 3 J_5, J^5 J_5 = 3 J_4$ and $\{M^+_r, M^-_s\} = -2(\bar{J}^r + J^4)$. Taking the representations for which exists a state that annihilates $M^-_r$, where the energy spectrum has a lower bound. If we denote the lowest energy eigenvalue by $E_0$ and the angular momentum values by $s$, the ground state consists of $(2s + 1)$ states $(E_0, s) E_0, s, m$, $m = -s, -s+1, ..., s$. To evaluate $C_2$ on the ground state $|E_0, s\rangle$, we use

$$M^-_r \langle E_0, s | E_0, s, m \rangle = 0. \tag{40}$$

Replacing $\{M^+_r, M^-_r\}$ in eqn.(39) by $-\{M^+_r, M^-_r\}$ and using eqns.(27,38) and (39), we obtain

$$C_2\langle E_0, s | E_0, s, m \rangle = E_0(E_0 - 3) + s(s+1)(E_0, s) E_0, s, m.$$ 

Using eqns.(27,38) we introducing $E_{0,R}$ in the contraction and obtain

$$K_{\mu} K^{\mu} |k, \lambda\rangle = \lim_{R \to \infty} \frac{C_2}{R^2} \langle E_{0,R}, s | E_{0,R}, s, m \rangle$$

$$= \lim_{R \to \infty} \frac{E_{0,R}(E_{0,R} - 3)}{R^2} \langle E_{0,R}, s | E_{0,R}, s, m \rangle,$$

and using $E_{0,R} = Rk$ with $k = mc/\hbar$ ($m \Rightarrow$ rest-mass), the contraction is given as

$$K_{\mu} K^{\mu} |k, \lambda\rangle = \lim_{R \to \infty} \frac{E_{0,R}^2}{R^2} \langle E_{0,R}, s | E_{0,R}, s, m \rangle = k^2 |k, \lambda\rangle.$$
Appendix

Since the classic angular moment is defined as \( \vec{\jmath} = \vec{r} \wedge \vec{p} \), an infinitesimal rotation is generated when \( |\vec{r}| \to \infty \), with \( \vec{\theta} \) the angle and \( \vec{\ell} \) arc length, where

\[
\vec{\jmath} \vec{\theta} = \vec{r} \wedge \vec{p} \frac{\vec{\ell}}{|\vec{r}|} = (\vec{e} \wedge \vec{p}) \vec{\ell} \quad \text{with} \quad \vec{e} = \frac{\vec{r}}{|\vec{r}|}
\]

Without loss of majority, in the poles we obtain that

\[
\frac{\vec{\jmath}}{|\vec{r}|} = \frac{\vec{r} \wedge \vec{p} i}{|\vec{r}|} = \vec{k} \wedge \vec{p} i = (-p_y \vec{i} \quad \text{or} \quad p_x \vec{j})
\]

In the quantization process we obtain a \( R \)-dependent operator \( J = R \wedge P \), with \( R \) and \( P \) the position and momentum operators respectively, and where an infinitesimal rotation is rewritten as

\[
J \vec{\theta} = R \wedge P \frac{\vec{x}}{|R|} = (e \wedge P) \vec{x},
\]

with \( |R| \) the norm of the operator \( R \). In the limit \( |R| \to \infty \) we obtain 'the momentum operator'.

In the previous sections we defined \( K_i = e \wedge \kappa_i \), with \( P_i = h \kappa_i \).

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