STABILITY OF PICARD SHEAVES
FOR VECTOR BUNDLES ON CURVES

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Abstract. We show that for any stable sheaf $E$ of slope $\mu(E) > 2g - 1$ on a smooth, projective curve of genus $g$, the associated Picard sheaf $\hat{E}$ on the Picard variety of the curve is stable. We introduce a homological tool for testing semistability of Picard sheaves.

Introduction

Throughout, $X$ is a smooth, projective genus $g$ curve over an algebraically closed field $k$. Let $\text{Pic} := \text{Pic}^0(X)$ be the Picard variety of $X$ and $\mathcal{P}$ the Poincaré line bundle on $X \times \text{Pic}$.

For a vector bundle $E \in \text{Coh}(X)$, its Picard complex is the Fourier–Mukai (or integral) transform $\hat{E} := \text{FM}_\mathcal{P}(E)$, an object of $\text{D}^b(\text{Pic})$. We denote its two cohomology sheaves by $\hat{E}^0$ and $\hat{E}^1$ and call these the Picard sheaves of $E$. Our goal is to show that $\hat{E}$ is (semi)stable on $\text{Pic}$ for general, (semi)stable bundles $E$ on $X$ for certain slopes. In fact, we prove this by showing that $\hat{E}$ is semistable when restricted to curves $i: X \hookrightarrow \text{Pic}$. We have the following result; see Corollaries 1.11, 2.3, 3.1, and 3.4:

Theorem A. If $E$ is a stable bundle on $X$ of slope $\mu(E) > 2g - 1$, then the Picard sheaf $\hat{E}^0$ is stable on $\text{Pic}$. Dually, if $E$ is stable of slope $\mu(E) < -1$, then the Picard sheaf $\hat{E}^1$ is stable. The analogous statements hold for semistability, using the non-strict inequalities.

Using our concept of orthogonality, we obtain results for Picard sheaves for generic semistable bundles of slope $\mu \in [g - 2, g]$ unless $\mu = g - 1$; see Proposition 3.7 and Corollary 3.8.

Theorem B. For $\mu \in (g - 1, g]$, there exists a semistable bundle $E$ on $X$ of slope $\mu$ such that its Picard sheaf $\hat{E}^0$ is semistable. Dually, for $\mu \in [g - 2, g - 1)$, there exists a semistable bundle $E$ on $X$ of slope $\mu$ such that its Picard sheaf $\hat{E}^1$ is semistable.

In order to show Theorem A, we generalise Clifford’s theorem about estimating global sections, from divisors to not necessarily semistable vector bundles. If $E = L_1 \oplus \cdots \oplus L_r$ is a direct sum of line bundles with all $\deg(L_i) \in [0, 2g - 2]$, then $h^0(L_i) - 1 \leq \deg(L_i)/2$ by the classical Clifford theorem. This sums up to $h^0(E) - r \leq \deg(E)/2$. Therefore, the best generalisation one can hope for is the following result; see Proposition 4.1, where we also give precise information about the equality case.

Proposition C. Let $E$ be a vector bundle of rank $r$ and degree $d$ on the smooth projective curve $X$ of genus $g$. If $\mu_{\text{max}}(E) \leq 2g - 2$ and $\mu_{\text{min}}(E) \geq 0$, then we have the estimate

$$h^0(E) - r \leq \frac{d}{2}.$$ 

The special case of semistable vector bundles of slope $\mu \in [0, 2g - 2]$ was already proved in [2, Theorem 2.1]. If the slope of a semistable bundle $E$ is not in this interval, then either $H^0(E) = 0$ or $H^1(E) = 0$, and the dimension of the remaining cohomology group is computed by the Riemann–Roch theorem.

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Known results. Classically, Picard sheaves are pushforwards of Poincaré bundles along the projection $X \times \text{Pic}^d(X) \to \text{Pic}^d(X)$, see [8, 12, 9]. Later on, this notion was extended to pushforwards of universal bundles onto the moduli space.

Kempf [6] has shown that the Picard bundle on $\text{Pic}^d(X)$ is stable for $d = 2g - 1$, the smallest degree where the Picard complex $\mathcal{O}_X = (\mathcal{O}_X)^0$ is a vector bundle. Here and later on, stability on the Picard variety is meant with respect to the polarisation by the theta divisor. In [3], Ein and Lazarsfeld proved the stability of Picard bundles on $\text{Pic}^d(X)$ for $d \geq 2g$. They do this by restricting the Picard bundles to the canonical curves $X \subset \text{Pic}^d(X)$ and $(-X) \subset \text{Pic}^d(X)$.

Li [7] considers the moduli space $\mathcal{M}_{r,d}$ of stable bundles on $X$ of rank $r$ and degree $d$ with $d > 2r(g - 1)$ and $(d, r) = 1$. If $\mathcal{H}$ denotes the universal bundle on $X \times \mathcal{M}_{r,d}$, then Li shows that the Picard bundle $\text{pr}_2^*\mathcal{H}$ on $\mathcal{M}_{r,d}$ is stable if $d > 2gr$.

In [1], the authors consider the same question for the moduli space of stable bundles of rank $r$ and fixed determinant $L$. Then the associated Picard bundle on the moduli space is stable (with respect to the unique polarisation) if $\deg(L) > 2r(g - 1)$ and $(r, \deg(L)) = 1$.

Here, we go back to the case of Jacobians, but now we consider the Poincaré bundle twisted by a vector bundle pulled back from the curve. In other words, we study preservation of (semi)stability for the Fourier–Mukai transform along the Poincaré bundle, taking bundles on the curve to sheaves on its Picard variety.

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Conventions. $X$ is always a smooth, projective curve of genus $g$ over a fixed algebraically closed field. The slope of a vector bundle is denoted $\mu(E) = \deg(E)/\text{rk}(E)$. Note $\mu(E \otimes F) = \mu(E) + \mu(F)$. We repeatedly use the Riemann–Roch formula, always in the form $\chi(E) = \text{rk}(E)(\mu(E) + 1 - g)$. We write $h^0(E) = \dim H^0(E)$ and $h^1(E) = \dim H^1(E)$.

We denote projections by $\text{pr}_X, \text{pr}_Y : X \times Y \to X$ or by $\text{pr}_1, \text{pr}_2 : X \times X \to X$. Given sheaves $E$ and $F$ on $X$, then as usual we write $E \boxtimes F = \text{pr}_1^*E \otimes \text{pr}_2^*F$. Sometimes, we follow standard usage and pack two statements into one, using (semi)stability and $(\leq)$.

1. Orthogonality and Stability

1.1. Definition of orthogonality and first properties. We first recall a classical result of Faltings [4, Theorem 1.2], which expresses semistability of a vector bundle on a curve as an orthogonality condition on $X$:

Theorem 1.1 (Faltings 1993). A vector bundle $E$ on $X$ is semistable if and only if there exists a vector bundle $0 \neq F$ such that $H^*(E \otimes F) = 0$.

Proof. For the convenience of reader, we prove the easy implication of this statement. Assume $H^*(E \otimes F) = 0$, and let $0 \neq G \subset E$ be a destabilising subsheaf, i.e. $\mu(G) > \mu(E)$.

Then $G \otimes F \subset E \otimes F$ with $\mu(G \otimes F) > \mu(E \otimes F)$. Since $H^*(E \otimes F) = 0$, Riemann–Roch gives $\chi(G \otimes F) > \chi(E \otimes F) = 0$. Hence $h^0(G \otimes F) > 0$, which contradicts $H^0(G \otimes F) \subset H^0(E \otimes F) = 0$. \qed

Here, we introduce two other orthogonality notions, on $X \times X$, which work well with Picard sheaves:

Definition. Two coherent sheaves $E$ and $F$ on $X$ are called orthogonal with respect to $-\Delta$ if the sheaf $\text{pr}_1^*E \otimes \mathcal{O}_{X \times X}(-\Delta) \otimes \text{pr}_2^*F$ on $X \times X$ has vanishing cohomology.
Analogously, $E$ and $F$ are orthogonal with respect to $\Delta$ if the cohomology groups of $\text{pr}^*_1 E \otimes \mathcal{O}_{X \times X}(\Delta) \otimes \text{pr}^*_2 F$ vanish. In short, we write

$$E \perp F \iff H^*(E \boxtimes F(-\Delta)) = 0,$$

$$E \perp F \iff H^*(E \boxtimes F(\Delta)) = 0.$$

Orthogonality has the following simple numerical description:

**Lemma 1.2.** Let $E$ and $F$ be coherent sheaves on $X$.

(i) If $H^*(E \otimes F) = 0$, then $\mu(F) = -\mu(E) + g - 1$.

(ii) If $E \perp F$, then $\mu(F) = g + \frac{g}{\mu(E) - g}$.

**Proof.** (i) follows readily from Riemann–Roch.

(ii) The cohomology of the exact sequence $0 \to E \boxtimes F(-\Delta) \to E \boxtimes F \to E \boxtimes F|\Delta \to 0$,

using $E \boxtimes F|\Delta \cong E \otimes F$ and $H^*(E \boxtimes F(-\Delta)) = 0$ from $E \perp F$ gives: $H^*(E \otimes F) \cong H^*(E) \otimes H^*(F)$, the latter isomorphism from the Künneth formula. Hence $\chi(E \otimes F) = \chi(E) \chi(F)$, or $(\mu(E) + \mu(F) + 1 - g) = (\mu(E) + 1 - g)(\mu(F) + 1 - g)$ by Riemann–Roch. Manipulating this equation yields the claimed formula. \(\square\)

We collect the following statements for referability; the proofs are immediate:

**Lemma 1.3.** For vector bundles $E$ and $F$ on $X$, we have the three equivalences:

(i) $E \perp F \iff F \perp E$,

(ii) $E \perp F \iff F \perp E$,

(iii) $E \perp F \iff (\omega_X \otimes E^\vee)^{\perp} \perp (\omega_X \otimes F^\vee)$ (Serre duality).

**Definition.** We define two functors $F_+, F_- : \text{Coh}(X) \to \text{D}^b(X)$ by

$$F_+(E) = \mathbb{R}\text{pr}_{2*}(\mathcal{O}_{X \times X}(\Delta) \otimes \text{pr}^*_1 E),$$

$$F_-(E) = \mathbb{R}\text{pr}_{2*}(\mathcal{O}_{X \times X}(-\Delta) \otimes \text{pr}^*_1 E).$$

We denote the cohomology sheaves by

$$F^+_i = \mathbb{R}^i \text{pr}_{2*}(\mathcal{O}_{X \times X}(\Delta) \otimes \text{pr}^*_1 E)$$

and similarly for $F^-_i$. Since the fibres are 1-dimensional and objects in $\text{D}^b(X)$ decompose into their cohomology sheaves, we have $F_+(E) = F^0_+(E) \oplus F^1_+(E)[-1]$, and $F_-(E) = F^0_-(E) \oplus F^1_-(E)[-1]$.

**Lemma 1.4.** For vector bundles $E$ and $F$ on $X$, we have the two equivalences

(i) $E \perp F \iff H^*(F^0_+(E) \otimes F) = 0$, and $H^*(F^1_+(E) \otimes F) = 0$ and

(ii) $E \perp F \iff H^*(F^0_-(E) \otimes F) = 0$, and $H^*(F^1_-(E) \otimes F) = 0$.

**Proof.** We only show (i), as the proof of (ii) works analogously.

Let $G = \text{pr}^*_1 E \otimes \mathcal{O}_{X \times X}(-\Delta) \otimes \text{pr}^*_2 F$. We compute the cohomology of $G$ using the Leray spectral sequence for $\text{pr}_2$. For dimension reasons, the spectral sequence degenerates, thus

$$H^0(G) = H^0(\text{pr}_{2*} G), \quad H^2(G) = H^1(\mathbb{R}^1 \text{pr}_{2*} G),$$

and a short exact sequence

$$0 \to H^1(\text{pr}_{2*} G) \to H^1(G) \to H^0(\mathbb{R}^1 \text{pr}_{2*} G) \to 0.$$
Therefore, we get $R^i\text{pr}_2^*(G) = F^i(E) \otimes F$, using the projection formula, together with the definitions of $G$ and $F^i(E)$. This proves both implications of the assertion, noting that $E \otimes F$ is tantamount to $H^*(G) = 0$. \qed

**Proposition 1.5.** For a coherent sheaf $E$ on $X$, the following conditions are equivalent:

1. There exists a coherent $F \neq 0$ such that $E \otimes F$.
2. $F^0(E)$ and $F^1(E)$ are semistable sheaves of the same slope.

Similarly there exists such an equivalence for orthogonality with respect to $+\Delta$.

1' There exists a coherent $F \neq 0$ such that $E \otimes F$.
2' $F^0(E)$ and $F^1(E)$ are semistable sheaves of the same slope.

**Proof.** We start with (1) $\implies$ (2). Assume $E \otimes F$ for some $F \neq 0$. From Lemma 1.4 we conclude $H^i(F^i(E) \otimes F) = 0$ for $i \in \{0, 1\}$. By the easy direction of Theorem 1.1, $F^i(E)$ is semistable. Moreover, we get $\mu(F^i(E)) = -\mu(F) + g - 1$ from Lemma 1.2(ii).

(2) $\implies$ (1). With $F^0(E)$ and $F^1(E)$ semistable of the same slope, their direct sum $F^0(E) \oplus F^1(E)$ is semistable as well. Thus by Faltings’ Theorem 1.1, there exists a sheaf $F \neq 0$ such that $H^i((F^0(E) \oplus F^1(E)) \otimes F) = 0$. By Lemma 1.4 we are done. \qed

**Remark 1.6.** Whenever we have an orthogonal pair $E \otimes F$ with non-zero sheaves $E$ and $F$, then we conclude that the six sheaves $E$, $F^0(E)$, $F^1(E)$, $F$, $F^0(F)$, and $F^1(F)$ are semistable.

However, in most situations we consider here, one of the two sheaves $F^0(E)$ or $F^1(E)$ will be zero. Anyway, they cannot be both zero, as the following argument shows: Set $r = \text{rk}(E)$, $R = \text{rk}(F^0(E)) - \text{rk}(F^1(E))$, $d = \text{deg}(E)$, and $D = \text{deg}(F^0(E)) - \text{deg}(F^1(E))$. A short Riemann–Roch computation along the lines of Lemma 1.2(ii) gives

$$
\begin{pmatrix}
R \\
D
\end{pmatrix} =
\begin{pmatrix}
-g & 1 \\
0 & -1
\end{pmatrix}
\begin{pmatrix}
r \\
d
\end{pmatrix}.
$$

Thus, we can deduce the pair $(r, d)$ from $(R, D)$ unless $g = 0$.

1.2. Picard sheaves, and embedding $X$ into the Picard scheme. We denote by $\text{Pic} = \text{Pic}^0(X)$ the Picard scheme of the smooth curve $X$ and by $\mathcal{P}$ the Poincaré bundle on $X \times \text{Pic}$. Fixing a point $P_0 \in X(k)$, we normalise the Poincaré bundle by the additional assumption that $\mathcal{P}|_{\{P_0\} \times \text{Pic}} \cong \mathcal{O}_{\text{Pic}}$. The projections from $X \times \text{Pic}$ will be denoted

$$
X \xleftarrow{\text{pr}_X} X \times \text{Pic} \xrightarrow{\text{pr}_\mathcal{P}} \text{Pic}.
$$

For a coherent sheaf $E$ on $X$, we define its Picard complex to be the object

$$
\mathcal{E} := \mathcal{F}M_{\mathcal{P}}(E) = \mathcal{R}\text{pr}_{\mathcal{P}}(\mathcal{P} \otimes \text{pr}_X^* E)
$$

in the derived category $\mathcal{D}^b(\text{Pic})$. Since $\mathcal{P}$ is $\text{pr}_1$-flat, and $\text{pr}_2$ is of dimension 1, we have only two cohomology sheaves of our complex, and we call these the Picard sheaves of $E$:

$$
\mathcal{E}^0 := \mathcal{F}M_{\mathcal{P}}^0(E) = \text{pr}_{\mathcal{P}}(\mathcal{P} \otimes \text{pr}_X^* E), \quad \text{and}
$$

$$
\mathcal{E}^1 := \mathcal{F}M_{\mathcal{P}}^1(E) = \mathcal{R}\text{pr}_{\mathcal{P}}(\mathcal{P} \otimes \text{pr}_X^* E).
$$

We are interested mainly in the case when one of $\mathcal{E}^0$ or $\mathcal{E}^1$ is zero. To study their semistability, we will restrict them to the curves $(X)_M$ and $(-X)_N$, to be defined next.

For any line bundle $M$ of degree 1, we have an embedding of $\iota_M: X \rightarrow \text{Pic}$ given by $P \mapsto M(-P)$. The image of $\iota_M$ is denoted by $(-X)_M$. In the same way, any line bundle $N$ of degree $-1$ defines an embedding $\iota_N: X \rightarrow \text{Pic}$ by $\iota_N(P) = N(P)$ with image $(X)_N$. The next proposition gives the restriction of $\mathcal{E}^i$ to the curves $(X)_N$ and $(-X)_M$. 

\label{1.7}

Proposition 1.7. Let $E$ be a coherent sheaf on $X$.
For arbitrary line bundles $N \in \text{Pic}^{-1}(X)$ and $M \in \text{Pic}^{1}(X)$, there are isomorphisms

\[\text{FM}_P(E) \otimes^L \mathcal{O}_{(X)} = \mathcal{E}_\tau(E \otimes N) \otimes \mathcal{O}_X(-P_0),\]
\[\text{FM}_P(E) \otimes^L \mathcal{O}_{(-X)} = \mathcal{E}_\tau(E \otimes M) \otimes \mathcal{O}_X(P_0)\]

in the derived category of $(X)_N \cong X \cong (-X)_M$. Moreover, for general line bundles $N \in \text{Pic}^{-1}(X)$ and $M \in \text{Pic}^{1}(X)$, there are isomorphisms of sheaves

\[\hat{E}^0|_{(X)_N} \cong \mathcal{F}_0(E \otimes N) \otimes \mathcal{O}_X(-P_0),\]
\[\hat{E}^1|_{(X)_N} \cong \mathcal{F}_1(E \otimes N) \otimes \mathcal{O}_X(-P_0)\]
\[\hat{E}^0|_{(-X)_M} \cong \mathcal{F}_0(E \otimes M) \otimes \mathcal{O}_X(P_0),\]
\[\hat{E}^1|_{(-X)_M} \cong \mathcal{F}_1(E \otimes M) \otimes \mathcal{O}_X(P_0)\].

Proof. For the formula for $\text{FM}_P(E) \otimes^L \mathcal{O}_{(X)_N}$, we compute

\[\text{FM}_P(E) \otimes^L \mathcal{O}_{(X)_N} = \text{Rpr}_{\text{pr}_*}(\text{pr}_X^*E \otimes \mathcal{P}) \otimes^L \mathcal{O}_{(X)_N}\]
\[\cong \text{Rpr}_{\text{pr}_*}(\text{pr}_X^*E \otimes \mathcal{P} \otimes \text{pr}_X^*\mathcal{O}_{(X)_N})\]
\[\cong \text{Rpr}_{\text{pr}_*}(\text{pr}_X^*E \otimes \mathcal{P}|_{\text{pr}_X(X)_N})\]
\[\cong \text{Rpr}_{\text{pr}_*}(\text{pr}_X^*(E \otimes N) \otimes \mathcal{O}_{X \times X}(\Delta) \otimes \text{pr}_X^*\mathcal{O}_X(-P_0))\]
\[\cong \text{Rpr}_{\text{pr}_*}(\text{pr}_X^*(E \otimes N) \otimes \mathcal{O}_{X \times X}(\Delta)) \otimes \mathcal{O}_X(-P_0)\]
\[= \mathcal{F}_\tau(E \otimes N) \otimes \mathcal{O}_X(-P_0)\]

where in $(\ast)$, we identify $(X)_N$ with $X$, so the universal family $\mathcal{P}$ restricted to $X \times (X)_N$ is the line bundle $\text{pr}_1^*N \otimes \mathcal{O}_{X \times X}(\Delta) \otimes \text{pr}_2^*\mathcal{O}_X(-P_0)$ on $X \times X$.

Choosing a nice resolution $0 \rightarrow E_1 \rightarrow E_0 \rightarrow E \rightarrow 0$ where the $E_i$ are vector bundles with $\mu_{\text{max}}(E_i) < 0$, we see that $\text{FM}_P(E)$ can be represented by the complex of vector bundles $E_1 \rightarrow E_0^\vee$ on $\text{Pic}$. For a general $N$, the curve $(X)_N$ does not contain any of the associated components of the cohomologies of that complex. Thus tensoring with $\mathcal{O}_{(X)_N}$ is exact. Similarly for $M$.

\[\square\]

1.3. Orthogonality proofs of semistability of Picard sheaves.

Proposition 1.8. For a coherent sheaf $E$ on $X$, the following implications hold

\[(1) \iff (2) \implies (3) \iff (2') \iff (1')\]

among these five conditions:

(1) $(E \otimes M) \not\subseteq F$ for some line bundle $M$ of degree $1$, and a sheaf $F \neq 0$.

(2) For a general line bundle $M$ of degree $1$, the restrictions of the Picard sheaves $\hat{E}^0$ and $\hat{E}^1$ to the curve $(X)_N$ are both semistable of the same slope.

(3) The sheaves $\hat{E}^0$ and $\hat{E}^1$ are $\mu$-semistable with respect to the theta divisor on $\text{Pic}$.  

(1') $(E \otimes N) \not\subseteq F$ for some line bundle $N$ of degree $-1$, and a sheaf $F \neq 0$.

(2') For a general line bundle $N$ of degree $-1$, the restrictions of the Picard sheaves $\hat{E}^0$ and $\hat{E}^1$ to the curve $(X)_N$ are both semistable of the same slope.

Proof. The proof of the equivalence $(1) \iff (2)$ follows from Propositions 1.5 and 1.7, having in mind that twisting with the line bundle $\mathcal{O}_X(P_0)$ does not affect semistability. The implication $(2) \implies (3)$ is standard: given a sheaf $F$ on a variety $Y$ such that $F|_C$ is semistable, where $C \subset Y$ is a curve cut out by divisors in the linear system $|H|$ of the polarisation $H$, then $F$ is $\mu_H$-semistable on $Y$ — a destabilising subsheaf $F' \subset F$ would induce a destabilising subsheaf $F'|_C \subset F|_C$.

\[\square\]

Remark 1.9. Indeed, the implications $(2) \implies (3) \iff (2')$ were also used in [3] as a main tool. Using orthogonality we can not detect whether a semistable sheaf is stable.
1.4. A first example.

**Lemma 1.10.** Let $E$ be a semistable coherent sheaf on $X$ of degree 0. Then there exists a coherent sheaf $F \neq 0$ such that $E \nmid F$.

*Proof.* By Faltings’ Theorem 1.1, there exists a vector bundle $F$ with $H^*(E \otimes F) = 0$. Since $F$ can be taken in an open subset in the moduli space of rank $R$ and degree $R(g - 1)$ vector bundles, we may furthermore assume that $H^*(F) = 0$. Tensoring the short exact ideal sheaf sequence of $\Delta$ on $X \times X$ with $E \boxtimes F = \text{pr}_1^* E \otimes \text{pr}_2^* F$, we obtain

$$0 \rightarrow E \boxtimes F(-\Delta) \rightarrow E \boxtimes F \rightarrow E \boxtimes F|_\Delta \rightarrow 0.$$  

We have $H^*(E \boxtimes F) = H^*(E) \otimes H^*(F) = 0$ by the K"unneth formula, and $H^*(E \boxtimes F|_\Delta) = H^*(E \otimes F) = 0$ by our assumption. Hence $H^*(E \boxtimes F(-\Delta)) = 0$, i.e. $E \nmid F$. \hfill $\square$

**Corollary 1.11.** Let $E$ be a semistable vector bundle on $X$ with $\mu(E) = -1$. Then $\hat{E}^0 = 0$ and $\hat{E}^1$ is a vector bundle of rank $g \cdot \text{rk}(E)$ which is also semistable. Moreover, the restriction of $\hat{E}^1$ to any curve $(-X)_M$ is semistable.

*Proof.* Since $E$ is semistable of negative degree, we have $\hat{E}^0 = 0$, and it follows that $\hat{E}^1$ is a vector bundle of the given rank. Now for any line bundle $M$ of degree 1, the tensor product $E \otimes M$ is semistable of degree 0. Thus by Lemma 1.10, there exists a sheaf $F \neq 0$ such that $(E \otimes M) \nmid F$, and Proposition 1.8 shows the semistability of $\hat{E}^1$.

Finally, $\hat{E}^1$ is semistable when restricted to any curve $(-X)_M$ because the base change argument from the proof of Proposition 1.7 simplifies, with $\text{FM}_P(E) = \hat{E}^1[-1]$ a shifted vector bundle. \hfill $\square$

1.5. A more subtle example.

**Lemma 1.12.** Assume that the genus $g \neq 1$. Let $E$ be a vector bundle on $X$ of slope $\mu(E) = g - 1$. Then the following three conditions are equivalent:

1. $H^*(E) = 0$.
2. $E \nmid \mathcal{O}_X$.
3. There exists a coherent sheaf $F \neq 0$ such that $E \nmid F$.

*Proof.* (1) $\implies$ (2): From $H^*(E) = 0$ we get $H^*(E \boxtimes \mathcal{O}_X) = H^*(E) \otimes H^*(\mathcal{O}_X) = 0$ and $H^*(E \otimes \mathcal{O}_X|_\Delta) = H^*(E) = 0$. Therefore, the long exact cohomology sequence of $0 \rightarrow E \boxtimes \mathcal{O}_X(-\Delta) \rightarrow E \boxtimes \mathcal{O}_X \rightarrow E \otimes \mathcal{O}_X|_\Delta \rightarrow 0$ gives $E \boxtimes \mathcal{O}_X(-\Delta) = 0$, i.e. $E \nmid \mathcal{O}_X$.

(2) $\implies$ (3): This implication is obvious.

(3) $\implies$ (1): If we have such an orthogonality, then $E$ and $F$ are semistable. Note that $\text{deg}(F) = 0$ by Lemma 1.2(ii). Moreover, $E$ is also orthogonal to the direct sums $F^{\otimes N}$ for all $N > 0$, which are semistable sheaves of rank $N \cdot \text{rk}(F)$ and degree 0. We may take $N$ big enough such that the theta divisor $\Theta_E$ in the moduli space of semistable degree 0 bundles on $X$ of rank $N \cdot \text{rk}(F)$ is effective. By semicontinuity, being orthogonal to $E$ is an open condition. Thus, we may assume there exists an $F'$ outside $\Theta_E$, i.e. $E \nmid F'$ and $H^*(E \otimes F') = 0$. Again we consider the long exact cohomology sequence of $0 \rightarrow E \boxtimes F'(-\Delta) \rightarrow E \boxtimes F' \rightarrow E \otimes F'|_\Delta \rightarrow 0$; here it yields $H^*(E) \otimes H^*(F') = 0$. We have $\chi(F') \neq 0$ from Riemann–Roch, if $g \neq 1$. So $H^*(F') \neq 0$, hence $H^*(E) = 0$. \hfill $\square$

**Corollary 1.13.** Let $E$ be a semistable sheaf of slope $g - 2$. If there exists a line bundle $M$ of degree 1 with $H^*(E \otimes M) = 0$, then $\hat{E}^0 = 0$, and $\hat{E}^1$ is semistable of rank $\text{rk}(E)$.

*Remark 1.14.* Raynaud proved in [11] the existence of stable sheaves $E$ having integral slope with the following property: $H^*(E \otimes M) \neq 0$ for all line bundles $M$. These base points of the theta divisor form a proper closed subset of the moduli space. Thus, we can only hope that the Picard sheaves $\hat{E}$ are semistable for general semistable sheaves $E$.
We take $E$ to be a globally generated vector bundle on $X$. Thus, we have

$$0 \to F^0(E) \to H^0(E) \otimes O_X \to E \to 0$$

and $F^1(E) \cong H^1(E) \otimes O_X$.

Note that a semistable bundle of slope $> 2g - 1$ is globally generated.

For a subsheaf $E' \subset E$ we obtain an injective map $H^0(E') \hookrightarrow H^0(E)$. From both morphisms we eventually obtain a subsheaf $F^0(E') \subset F^0(E)$. The next result tells us, that these subsheaves are enough to test (semi)stability.

**Lemma 2.1.** The sheaf $F^0(E)$ is (semi)stable if for all globally generated subsheaves $E' \subset E$ we have the inequality $\mu(F^0(E')) (\leq) \mu(F^0(E))$.

**Proof.** We have the short exact sequence $0 \to F^0(E) \to H^0(E) \otimes O_X \to E \to 0$, as $E$ is globally generated. Let $U \subset F^0(E)$ be a subbundle. The inclusion $U \hookrightarrow H^0(E) \otimes O_X$ induces a surjection $H^0(E') \otimes O_X \to H^0(U') \otimes O_X$. Denote by $V' \subset H^0(U')$ the image of the induced map on global sections, $H^0(E') \to H^0(V')$. The commutative diagram

$$H^0(E') \otimes O_X \\
V' \otimes O_X \to H^0(U') \otimes O_X \to U'$$

shows that $V' \otimes O_X \to U'$ is surjective. We get inclusions $U \hookrightarrow V \otimes O_X \hookrightarrow H^0(E) \otimes O_X$, which combine into a commutative diagram of short exact sequences

$$0 \to U \to V \otimes O_X \to E' \to 0$$

$$0 \to F^0(E) \to H^0(E) \otimes O_X \to E \to 0$$

where all vertical arrows are injective. Denote $v = \dim(V)$ and $r' = \text{rk}(E')$. Then

$$\mu(U) = -\frac{\text{deg}(E')}{v - r'} \leq -\frac{\text{deg}(E')}{h^0(E') - r'} = \mu(F^0(E')),$$

which shows that testing the (semi)stability condition only on subsheaves of the form $F^0(E')$ suffices to deduce it for arbitrary subsheaves. \hfill \Box

**Corollary 2.2.** If $E$ is a (semi)stable vector bundle of slope $\mu(E) > 2g$, then $F^0(E)$ is (semi)stable.

**Proof.** Since $\mu := \mu(E) > 2g - 2$, we have $h^1(E) = 0$ and $h^0(E) = \text{deg}(E) + (1 - g) \text{rk}(E)$. From the short exact sequence $0 \to F^0(E) \to H^0(E) \otimes O_X \to E \to 0$, we deduce

$$\mu(F^0(E)) = -\frac{\text{deg}(E)}{h^0(E) - \text{rk}(E)} = -\frac{\text{deg}(E)}{\chi(E) - \text{rk}(E)} = -\frac{\text{deg}(E) - g \cdot \text{rk}(E)}{\mu - g} = -\frac{\mu}{\mu - g}.$$

Assume that $E$ is stable. Let $E' \subset E$ be a globally generated proper subsheaf of $E$. In Corollary 4.2 (Section 4 is independent of the rest of the article), we draw the following consequence from the generalised Clifford theorem: $h^0(E') - \text{rk}(E') < \frac{\mu - 2}{\mu} \text{deg}(E')$. Thus

$$\mu(F^0(E')) = -\frac{\text{deg}(E')}{h^0(E') - \text{rk}(E')} < -\frac{\mu}{\mu - g} = \mu(F^0(E)).$$

By Lemma 2.1, to show the stability of $F^0(E)$ it suffices to check this inequality for the subsheaves of type $F^0(E')$. The claim about semistability follows from this by the Jordan–Hölder filtration of $E$. \hfill \Box
Corollary 2.3. Let $E$ be a (semi)stable vector bundle of slope $\mu(E) > 2g - 1$. Then the restriction of $\hat{E}^0$ to any curve $(-X)_M$ is (semi)stable. In particular, $\hat{E}^0$ is (semi)stable.

Proof. We have $\hat{E}^0|_{(-X)_M} \cong F^0(E \otimes M) \otimes O_X(P_0)$ from Proposition 1.7. This holds for all curves $(-X)_M$ because $E$ semistable of slope $> 2g - 1$ implies that $\hat{E}^0$ is locally free. Moreover, $E \otimes M$ is a (semi)stable bundle of slope $\mu(E \otimes M) > 2g$, so Corollary 2.2 applies and yields the (semi)stability of $F^0(E \otimes M)$ and hence of $\hat{E}^0|_{(-X)_M}$. □

3. APPLICATION OF ORTHOGONALITY

There are two ways how to apply the orthogonality condition $E \perp F$, in order to deduce the semistability of the Picard bundle $\hat{F}$ from the semistability of another Picard sheaf $\hat{F}$. Either we use the symmetry of orthogonality, i.e. Lemma 1.3(i) and (ii), or we employ Serre duality, i.e. Lemma 1.3(iii). We start with the latter method.

Corollary 3.1. Let $E$ be a semistable vector bundle on $X$ with $\mu(E) = 2g - 1$. Then $\hat{E}^1 = 0$ and $\hat{E}^0$ is a vector bundle of rank $g \cdot \text{rk}(E)$ which is also semistable. Moreover, the restriction of $\hat{E}^0$ to any curve $(X)_N$ is semistable.

Proof. The vanishing of $\hat{E}^1$ follows from cohomology and base change, and for the same reason, $\hat{E}^0$ is a vector bundle of the given rank. As $\hat{E}$ is semistable, so is its dual $E^\vee$. Thus, $E^\vee \otimes \omega_X$ is semistable of degree $-1$. By Lemma 1.10, there exists a sheaf $F$ such that $(E^\vee \otimes \omega_X \otimes M)\perp F$ for any line bundle $M$ of degree 1. By Serre duality, Lemma 1.3(iii), this implies $(E \otimes M^\vee) \perp (F^\vee \otimes \omega_X)$. Now we proceed as in the proof of Corollary 1.11. □

Remark 3.2. Applying Corollary 3.1 to a degree $2g - 1$ line bundle $L$, we obtain the semistability of the Picard bundle $P_{2g-1} = \hat{L}^0$. Thus, the above result is a generalisation of Kempf’s result [6]. In fact, Kempf shows the stability of $P_{2g-1}$. The stability follows along the lines of Corollaries 2.2 and 4.2. Indeed, if $X$ is not hyperelliptic, and $L \otimes M \otimes \omega_X^\vee$ is not effective, then the restriction of $\hat{L}^0$ to $(-X)_M$ is stable.

Lemma 3.3. If $E$ is a stable vector bundle on $X$ with $\mu(E) < -2$, then $F^1(E)$ is stable.

Proof. Tensoring the short exact sequence $0 \rightarrow \mathcal{O}_{X \times X} \rightarrow \mathcal{O}_{X \times X}(\Delta) \rightarrow \mathcal{O}_X(\Delta) \rightarrow 0$ on $X \times X$ with $E \otimes \mathcal{O}_X$, and applying $\text{pr}_2$, yields the following short exact sequence on $X$:

$$0 \rightarrow E \otimes \omega_X^\vee \rightarrow H^1(E) \otimes \omega_X \rightarrow F^1(E) \rightarrow 0,$$

because $F^0(E) = 0$ from stability of $E$ with $\mu(E) < -2$. Dualising this sequence yields

$$0 \rightarrow (F^1(E))^\vee \rightarrow H^1(E)^\vee \otimes \omega_X \rightarrow E^\vee \otimes \omega_X \rightarrow 0.$$

Thus, by classical Serre duality, $F^1(E)$ is the dual of $F^0(E')$ for $E' = E^\vee \otimes \omega_X$. However $E'$ is also stable, of slope $\mu(E') = 2g - 2 - \mu(E) > 2g$. So by Corollary 2.2 the sheaf $F^0(E')$ is stable. This proves the lemma. □

Corollary 3.4. If the vector bundle $E$ on $X$ is (semi)stable of slope $\mu(E) < -1$, then the Picard sheaf $\hat{E}^1$ is (semi)stable when restricted to any curve $(X)_N$. In particular, $\hat{E}^1$ is (semi)stable.

Proof. For stability, this follows from Proposition 1.7 and Lemma 3.3.

The claim for semistable $E$ then follows using the Jordan–Hölder filtration of $E$. □

Next, we give examples for how to apply the symmetry property of orthogonality. As usual, for a rational number $x$ we denote by $\lceil x \rceil$ the round up of $x$. For any $r, h \in \mathbb{N}$,
Lemma 1.4, using Serre duality as in Lemma 1.2, we have that $\text{Corollary 2.2}$. Lemma 1.2
Lemma 3.6
(iii).
Proposition 3.7
Lemma 1.3
Lemma 1.4, if $\mathcal{X}$ and $\mathcal{Y}$ there exists a vector bundle $\mathcal{F}$ of rank $2$ such that $\mathcal{F}^0(\mathcal{F}) \oplus \mathcal{F}^1(\mathcal{F})$ is semistable.

Proof. We begin with the involution $\mathbb{Q} \to \mathbb{Q}, \mu \mapsto \mu^- := g + \frac{g}{\mu - g}$. It is decreasing on $\mathbb{Q}_{>g}$.

Now let $\mu = \frac{d}{r} \in (g, g + 1]$, then $\mu^- \in [2g, \infty)$. Let $\mathcal{E}$ be a stable vector bundle on $X$ of rank $d - rg$ and degree $gd - g^2r + rg$, i.e. $\mu(\mathcal{E}) = \mu^-$. By Corollary 2.2 or Remark 3.2, we have that $\mathcal{F}^0(\mathcal{E})$ is semistable. Since $\mu(\mathcal{E}) > 2g - 1$, we also conclude $\mathcal{F}^1(\mathcal{E}) = 0$. So we can use the Riemann–Roch formula to compute $\text{rk}(\mathcal{F}^0(\mathcal{E})) = gr$, and $\deg(\mathcal{F}^0(\mathcal{E})) = gr - gd - g^2r$. Popa’s result Theorem 3.5 implies that for any $R = k \cdot r$ with $k$ as above there exists a vector bundle $\mathcal{F}$ of rank $R$ such that $\mathcal{H}^*(\mathcal{F}^0(\mathcal{E}) \otimes \mathcal{F}) = 0$.

By Lemma 1.4, this yields $\mathcal{E} \not\subseteq \mathcal{F}$. Symmetry, i.e. Lemma 1.3(i), then gives $\mathcal{F} \not\subseteq \mathcal{E}$. So, again by Lemma 1.4, it follows that $\mathcal{H}^*(\mathcal{F}^0(\mathcal{E}) \oplus \mathcal{F}^1(\mathcal{E})) \otimes \mathcal{E} = 0$. This implies the semistability of the direct sum $\mathcal{F}^0(\mathcal{E}) \oplus \mathcal{F}^1(\mathcal{F})$.

Proposition 3.7. Let $\mu = \frac{d}{r} \in \mathbb{Q}$ with $\mu \in (g - 1, g]$, let $k \geq \text{Proposition 4.1.}$. $\mathcal{E}$ such that $\mathcal{E}^0$ restricted to $(-\mathcal{X})_M$ is semistable.

Proof. It is enough to show the existence of some $\mathcal{F}$ such that $\mathcal{E}^0$ restricted to $(-\mathcal{X})_M$ is semistable. Let $\mu$ and $R$ be as in the proposition. Let $\mathcal{E}'$ be a vector bundle of slope $\mu + 1$ and rank $R$ such that $\mathcal{F}^0(\mathcal{E}) \oplus \mathcal{F}^1(\mathcal{F})$ is semistable, which exists by Lemma 3.6. Then $\mathcal{F}^0(\mathcal{E}') \otimes \mathcal{O}_X(\mathcal{P}_0)$ is also semistable.

We set $\mathcal{F} = \mathcal{E}' \otimes \mathcal{O}_X(\mathcal{P}_0)$. Then $\mathcal{F}^0(\mathcal{E}') \otimes \mathcal{O}_X(\mathcal{P}_0)$ is also semistable. By Proposition 1.7, the restriction of $\mathcal{F}^0$ to $(-\mathcal{X})_M$ is the semistable sheaf $\mathcal{F}^0(\mathcal{E}') \otimes \mathcal{O}_X(\mathcal{P}_0)$.

Corollary 3.8. Let $\mu = \frac{d}{r} \in \mathbb{Q}$ with $\mu \in [g - 2, g - 1)$, let $k \geq \text{Proposition 4.1.}$. $\mathcal{E}$ such that $\mathcal{E}^1$ is semistable.

Proof. This follows from Proposition 3.7, using Serre duality as in Lemma 1.3(iii).

4. Clifford’s theorem for vector bundles on a curve

Let us remind the reader that $\mu_{\text{max}}(\mathcal{E})$ denotes the maximum of all slopes of subbundles of $\mathcal{E}$, and $\mu_{\text{min}}(\mathcal{E})$ denotes the minimal slope of all quotient bundles of $\mathcal{E}$.

Proposition 4.1. Let $\mathcal{E}$ be a vector bundle of rank $r$ and degree $d$ on the smooth projective curve $X$ of genus $g$. If $\mu_{\text{max}}(\mathcal{E}) \leq 2g - 2$ and $\mu_{\text{min}}(\mathcal{E}) \geq 0$, then we have the estimate

$$h^0(\mathcal{E}) - r \leq \frac{d}{2}.$$ 

Moreover, if $\mu_{\text{max}}(\mathcal{E}) < 2g - 2$ and $\mu_{\text{min}}(\mathcal{E}) > 0$ and $h^0(\mathcal{E}) - r = \frac{d}{2}$, then $X$ is hyperelliptic, and the determinant line bundle $\det(\mathcal{E})$ is a multiple of the hyperelliptic line bundle $\mathcal{M}$, and $\mathcal{E}$ possesses a filtration with graded object $\text{gr}(\mathcal{E}) = \bigoplus_{i=1}^r \mathcal{M}^{a_i}$, with $0 < a_i < g - 1$. 

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Proof. We first prove the inequality \( h^0(E) - r \leq \frac{d}{2} \) by induction on \( r \).

If \( r = 1 \), then \( E = \mathcal{O}(D) \) is a line bundle associated to a divisor \( D \). In this case \( d = \mu_{\text{max}}(\mathcal{O}(D)) = \mu_{\text{min}}(\mathcal{O}(D)) \). If \( D \) is effective and special, then the claim is precisely the well known theorem of Clifford, see for example [5, Theorem IV.5.4]. If \( D \) is not effective, then \( h^0(\mathcal{O}(D)) = 0 \) and the statement is trivial. If \( D \) is non-special, then by Riemann–Roch \( h^0(\mathcal{O}(D)) - 1 = \chi(\mathcal{O}(D)) - 1 = d - g < \frac{d}{2} \), the inequality following from \( d = \mu_{\text{max}} \leq 2g - 2 \).

Now suppose that \( E \) is of rank \( r \geq 2 \), and the inequality holds for all vector bundles of rank smaller than \( r \) which meet the slope conditions. We consider two cases:

Case 1: \( E \) is not semistable. Take the subsheaf \( E_1 \) of \( E \) of slope \( \mu_{\text{max}}(E) \) and of maximal possible rank. This \( E_1 \) is the first sheaf appearing in the Harder–Narasimhan filtration of \( E \). We obtain a short exact sequence

\[
0 \to E_1 \to E \to E_2 \to 0.
\]

We have \( \mu_{\text{max}}(E_1) = \mu_{\text{min}}(E_1) = \mu_{\text{max}}(E) \), \( \mu_{\text{max}}(E) > \mu_{\text{max}}(E_2) \), and \( \mu_{\text{min}}(E) = \mu_{\text{min}}(E_2) \). In particular, we see that the induction hypothesis applies to the vector bundles \( E_1 \) and \( E_2 \). Hence \( h^0(E_i) - \text{rk}(E_i) \leq \frac{1}{2} \text{deg}(E_i) \) for \( i \in \{1, 2\} \). Taking global sections of the above short exact sequence, we conclude \( h^0(E) \leq h^0(E_1) + h^0(E_2) \). So we get

\[
h^0(E) - r \leq h^0(E_1) + h^0(E_2) - r = (h^0(E_1) - \text{rk}(E_1)) + (h^0(E_2) - \text{rk}(E_2)) \leq \frac{1}{2} \text{deg}(E_1) + \frac{1}{2} \text{deg}(E_2) = \frac{1}{2} \text{deg}(E).
\]

Case 2: \( E \) is semistable. Again, we distinguish two cases, by inspecting the slope of \( E \).

Case 2.1: \( \mu(E) \leq g - 1 \). We may assume \( h^0(E) > 0 \). Let \( E_1 \) be a line subbundle of \( E \) of maximal possible degree \( d_1 \). From \( h^0(E) > 0 \) we conclude that \( d_1 \geq 0 \). We obtain a short exact sequence

\[
0 \to E_1 \to E \to E_2 \to 0.
\]

Since any quotient of \( E_2 \) is also a quotient of \( E \) we conclude \( \mu_{\text{min}}(E_2) \geq \mu_{\text{min}}(E) \geq 0 \). We want to show that \( \mu_{\text{max}}(E_2) \leq 2g - 2 \). Assume the contrary. Then we have a subsheaf \( E_3 \subset E_2 \) of rank \( r_3 \) and slope \( \mu(E_3) \geq 2g - 2 \). The kernel \( K \) of the composition of surjections

\[
E \to E_2 \to E_2/E_3
\]

is of rank \( r_3 + 1 \) and of slope \( \mu(K) = \frac{d_1 + \mu(E_1)r_3}{r_3 + 1} \geq \frac{\mu(E_3)r_3}{r_3 + 1} = \frac{\mu(E_3)}{r_3 + 1} > \frac{2g - 2}{1 + 1/r_3} \geq g - 1 \). This contradicts the semistability of \( E \). Thus, for both sheaves \( E_1 \) and \( E_2 \) the induction hypothesis applies, and we can proceed like in Case 1.

Case 2.2: \( \mu(E_1) > g - 1 \). The Serre dual bundle \( E' = E^\vee \otimes \omega_X \) has slope \( \mu(E') = 2g - 2 - \mu(E) < g - 1 \). Therefore, as we have seen in Case 2.1

\[
h^0(E') \leq \frac{\text{deg}(E')}{2} = \frac{\text{rk}(E)(2g - 2) - \text{deg}(E)}{2}.
\]

By Serre duality \( h^0(E') = h^1(E) \). So when adding the Riemann–Roch formula \( h^0(E) - h^1(E) = \text{deg}(E) + \text{rk}(E)(1 - g) \) to the above inequality, we obtain the stated inequality.

The statement for the case \( h^0(E) - r = \frac{d}{2} \) follows along the same lines. Indeed, we must have this equality for \( E_1 \) and \( E_2 \) and can proceed by induction since \( \text{det}(E) \cong \text{det}(E_1) \otimes \text{det}(E_2) \). The passage from \( E \) to the Serre dual \( E' = E^\vee \otimes \omega_X \) sends a vector bundle \( E \) with \( \text{det}(E) = M \otimes^a \) to a bundle with \( \text{det}(E') = M \otimes^{(r(g-1)-a)} \) where \( M \) denotes the hyperelliptic line bundle. \( \square \)
Corollary 4.2. Let $E$ be a stable vector bundle of slope $\mu = \mu(E) > 2g$. For any globally generated subsheaf $E' \subsetneq E$ which is not a trivial bundle, we have the strict inequality

$$h^0(E') - \text{rk}(E') < \left(1 - \frac{g}{\mu}\right) \deg(E').$$

Proof. As $E' \subset E$ is globally generated, we have $\mu_{\min}(E') \geq 0$. For one sheaf $E'_1$ in the Harder–Narasimhan filtration of $E'$, we have $\mu_{\min}(E'_1) > 2g-2$, and $\mu_{\max}(E'/E'_1) \leq 2g-2$.

We set $E'_2 = E'/E'_1$. Now $E'_1$ is semistable with $\mu_{\min}(E'_1) > 2g-2$, hence $h^1(E'_1) = 0$. So

$$h^0(E'_1) - \text{rk}(E'_1) = \chi(E'_1) - \text{rk}(E'_1) = \deg(E'_1) - g \cdot \text{rk}(E'_1) = \left(1 - \frac{g}{\mu(E'_1)}\right) \deg(E'_1)$$

by Riemann–Roch. Since $\mu(E'_1) < \mu$, and the function $x \mapsto 1 - \frac{x}{2}$ is strictly increasing for $x > 0$, we deduce the inequality

$$h^0(E'_1) - \text{rk}(E'_1) < \left(1 - \frac{g}{\mu}\right) \deg(E'_1).$$

The sheaf $E'_2$ satisfies the assumptions of Proposition 4.1. Moreover, $E'_2$ is itself a globally generated sheaf, and we have $\mu_{\min}(E'_2) \geq \mu_{\min}(E') > 0$, the latter inequality from the assumption that $E'$ is not a trivial bundle. Hence $\deg(E'_2) > 0$ and so we have

$$h^0(E'_2) - \text{rk}(E'_2) \leq \frac{1}{2} \deg(E'_2) < \left(1 - \frac{g}{\mu}\right) \deg(E'_2).$$

This last inequality holds, because $\mu > 2g$ implies $\frac{1}{2} < 1 - g/\mu$. Adding the two inequalities for $h^0(E'_i) - \text{rk}(E'_i)$ for $i = 1, 2$, we obtain the statement of the corollary.

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