Some Characterizations of Focal Surfaces of A Tubular Surface in $\mathbb{E}^3$

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Abstract
Here, we focus on focal surfaces of a tubular surface in Euclidean 3—space $\mathbb{E}^3$. Firstly, we give the tubular surfaces with respect to Frenet, Bishop and Darboux frames. Then, we define focal surfaces of these tubular surfaces. We get some results for these types of surfaces to become flat and we show that there is no minimal focal surface of a tubular surface in $\mathbb{E}^3$. We give some examples for these type surfaces. Further, we show that $u$-parameter curves cannot be asymptotic curves and we obtain some results about $v$-parameter curves of the focal surface $M^*$.  

Keywords: Focal surface, tubular surface, Bishop frame, Darboux frame.  
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1 Introduction

Focal surfaces are known as line congruences. The concept of line congruences is defined for the first time in visualization in 1991 by Hagen and Pottman [9].

Let $M : X(u,v)$ be a surface defined as a real-valued function and $N (u,v)$ be a unit normal vector on the surface. The line congruence is defined as

$$ C (u,v,z) = X (u,v) + zE (u,v) ,$$

where $E (u,v)$ is the set of unit vectors. For each $(u,v)$, the equation (1) indicates a line congruence and called generatrix. Here, the parameter $z$ is a marked distance. In addition, there exist two special points (real, imaginary or unit) on the generatrix of $C$. These points are called as focal points which are the osculator points with generatrix. Therefore, focal surfaces are defined as a geometric locus of focal points. In general, there exist two focal surfaces. If
Let \( \gamma = \gamma(s) : I \subseteq \mathbb{R} \rightarrow \mathbb{E}^3 \) be a unit speed curve in the Euclidean space \( \mathbb{E}^3 \). Then the derivatives of the Frenet frame \( \{T, N_1, N_2\} \) of \( \gamma \) (Frenet-Serret
formula);

\[
\begin{bmatrix}
T' \\
N'_1 \\
N'_2
\end{bmatrix}
= \begin{bmatrix}
0 & \kappa & 0 \\
-k & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N_1 \\
N_2
\end{bmatrix},
\]

where \(\tau, \kappa\) are the torsion and curvature of the curve \(\gamma\), respectively \[4\].

The parallel transport frame or Bishop frame is an alternative frame which can be defined when second derivative of the curve is zero. The Bishop (frame) formulas are expressed as

\[
\begin{bmatrix}
T' \\
M'_1 \\
M'_2
\end{bmatrix}
= \begin{bmatrix}
0 & k_1 & k_2 \\
-k_1 & 0 & 0 \\
-k_2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
T \\
M_1 \\
M_2
\end{bmatrix},
\]

where \(\{T, M_1, M_2\}\) is the Bishop Frame and \(k_1, k_2\) are called first and second Bishop curvatures, respectively \[1\].

Frenet frame and Bishop frame has a relation as follows:

\[
\begin{bmatrix}
T' \\
Y' \\
N'
\end{bmatrix}
= \begin{bmatrix}
0 & k_g & k_n \\
-k_g & 0 & \tau_g \\
-k_n & -\tau_g & 0
\end{bmatrix}
\begin{bmatrix}
T \\
Y \\
N
\end{bmatrix},
\]

where \(k_g\) is the geodesic curvature, \(k_n\) is the normal curvature and \(\tau_g\) is the geodesic torsion of the curve \(\gamma\).

The relation between the Darboux frame and the Frenet frame is given as follows:

\[
\begin{bmatrix}
T' \\
Y' \\
N'
\end{bmatrix}
= \begin{bmatrix}
0 & \kappa & 0 \\
-k & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix}
\begin{bmatrix}
T \\
N_1 \\
N_2
\end{bmatrix},
\]

where \(\theta\) is the angle between the vectors \(Y\) and \(N_1\). Here Darboux curvatures are defined by \(k_g = \kappa \cos \theta, k_n = \kappa \sin \theta\) and \(\tau_g = \tau - \theta'\).

Let \(M: X(u, v)\) be a regular surface in \(\mathbb{E}^3\). The tangent space of \(M\) is spanned by the vectors \(X_u\) and \(X_v\) at a point \(p = X(u, v)\). The coefficients of the first fundamental form of \(M\) are defined as

\[
E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle,
\]

\[2\]
where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. We set $W^2 = EG - F^2 \neq 0$.

Then the unit normal vector field of $M$ is defined as
\[ N = \frac{X_u \times X_v}{\|X_u \times X_v\|} \quad \text{(3)} \]

The coefficients of the second fundamental form are as follow:
\[ l = \langle X_{uu}, N \rangle, \quad m = \langle X_{uv}, N \rangle, \quad n = \langle X_{vv}, N \rangle \quad \text{(4)} \]

The shape operator matrix of a surface is defined as
\[ A_N = \begin{bmatrix} l & \frac{l}{\|E\|} \left( m - \frac{F}{E} l \right) \\ \frac{1}{\|E\|} \left( m - \frac{F}{E} l \right) & \frac{1}{\|E\|} \left( En - 2Fm + \frac{F^2}{E} l \right) \end{bmatrix} \quad \text{(4)} \]

The Gaussian and the mean curvatures of $M$ are given as
\[ K = \frac{ln - m^2}{EG - F^2}, \quad \text{(5)} \]
\[ H = \frac{En + Gl - 2Fm}{2(EG - F^2)}, \quad \text{(6)} \]
respectively \[4][10].

Let $\gamma : I \rightarrow M$ be a unit speed curve on the surface $M$. Then $\gamma$ is an asymptotic curve for which the normal curvature vanishes in the direction $\gamma'$. Recall that $\gamma$ is asymptotic if and only if $\gamma'$ is perpendicular to the normal vector $N$ of the surface $M$. Furthermore, $\gamma$ is a geodesic curve on $M$ if the tangential component $(\gamma'')^T$ of the acceleration of $\gamma$ vanishes \[4].

### 3 Focal surface of a tubular surface with Frenet frame

Tubular surface with Frenet frame was studied by Öztürk et al., in \[21\]. In this section, we handle the tubular surface with Frenet frame and give the focal surface of this surface in $E^3$.

Let $\gamma (u) = (\gamma_1(u), \gamma_2(u), 0) \subset E^3$ be a curve parametrized by arclength. Then the famous Frenet formulas become
\[ \gamma' = T, \quad T' = \kappa N_1, \quad N_1' = -\kappa T, \quad N_2' = 0. \]

The tubular surface with respect to Frenet frame has the parametrization:
\[ M : X(u,v) = \gamma(u) + r \left( \cos \nu N_1(u) + \sin \nu N_2 \right), \quad \text{(7)} \]
where \( r = \text{const.} \) is the radius of the spheres. The tangent space of \( M \) at a point \( p = X(u, v) \) is spanned by

\[
X_u = (1 - \kappa(u)r \cos v) T,
X_v = -r \sin v N_1 + r \cos v N_2. \tag{8}
\]

Then coefficients of the first fundamental form are

\[
E = (1 - \kappa(u)r \cos v)^2, \quad F = 0, \quad G = r^2, \tag{9}
\]

where \( W^2 = EG - F^2 = (1 - \kappa(u)r \cos v)^2 r^2 \) \[21\].

**Proposition 1** \[21\] \( X(u, v) \) is a regular tubular surface patch if and only if \( 1 - \kappa(u)r \cos v \neq 0 \).

The unit normal vector field and the second partial derivatives of \( M \) are obtained as

\[
N = -\cos v N_1 - \sin v N_2
\]

and

\[
X_{uu} = -\kappa'(u)r \cos v T + \kappa(u)(1 - \kappa(u)r \cos v) N_1,  \nonumber \\
X_{uv} = \kappa(u) r \sin v T,  \nonumber \\
X_{vv} = -r \cos v N_1 - r \sin v N_2, \tag{10}
\]

respectively.

Then the coefficients of the second fundamental form become

\[
l = -\kappa(1 - \kappa(u)r \cos v) \cos v, \quad m = 0, \quad n = r. \tag{11}
\]

Thus, from the equations (9) and (11), the Gaussian and mean curvature functions of \( M \) are calculated as

\[
K = \frac{-\kappa(u) \cos v}{r (1 - \kappa(u)r \cos v)},
\]

and

\[
H = \frac{1 - 2 \kappa(u)r \cos v}{2r (1 - \kappa(u)r \cos v)},
\]

respectively.

The shape operator matrix of the surface \( M \) is as follows:

\[
A_N = \begin{bmatrix}
-\kappa(u) \cos v & 0 \\
\frac{1}{1 - \kappa(u)r \cos v} & \frac{1}{r}
\end{bmatrix} \tag{12}
\]

\[21\].

From now on, we can give the parametrization of the focal surface \( M^* \) of \( M \) by obtaining the principal curvature functions of \( M \).
Using (12), we get the principal curvature functions as
\[
\kappa_1 = \frac{1}{r}, \quad \kappa_2 = \frac{-\kappa(u) \cos v}{1 - \kappa(u) r \cos v},
\]
(13)

From the definition of the focal surface of a given surface and using the equation (13), we obtain the focal surface \(M^*\) of \(M\) as
\[
X^*(u, v) = \gamma(u) + \frac{1}{\kappa(u)} \cos v (\cos v N_1(u) + \sin v N_2).
\]
(14)

The tangent space of the focal surface \(M^*\) is spanned by
\[
(X^*)_u = \frac{-\kappa'(u)}{\kappa^2(u)} (N_1 + \tan v N_2),
\]
(15)
and
\[
(X^*)_v = \frac{1}{\kappa(u) \cos^2 v} N_2.
\]

Thus from (15), the coefficients of the first fundamental form are obtained as
\[
E^* = \frac{(\kappa'(u))^2}{\kappa^4(u) \cos^2 v}, \quad F^* = \frac{-\kappa'(u) \sin v}{\kappa^3(u) \cos^3 v}, \quad G^* = \frac{1}{\kappa^2(u) \cos^4 v},
\]
(16)

where \((W^*)^2 = \frac{(\kappa'(u))^2}{\kappa^6(u) \cos^4 v} \neq 0\). Further, from (3) we find the normal vector of \(M^*\)
\[
N^* = T.
\]
(17)

The second partial derivatives of \(X^*(u, v)\) are as in the following:
\[
(X^*)_{uu} = \frac{\kappa'(u)}{\kappa(u)} T + \frac{-\kappa''(u) \kappa(u) + 2 (\kappa'(u))^2}{\kappa^3(u)} N_1
\]
\[
+ \frac{-\kappa''(u) \kappa(u) + 2 (\kappa'(u))^2}{\kappa^3(u)} \tan v N_2
\]
\[
(X^*)_{uv} = -\frac{\kappa'(u)}{\kappa^2(u)} (1 + \tan^2 v) N_2,
\]
(18)
\[
(X^*)_{vv} = \frac{2 \sin v}{\kappa(u) \cos^3 v} N_2.
\]

Hence, from (17) and (18), we get the coefficients of the second fundamental form of the focal surface \(M^*\) as
\[
l^* = \frac{\kappa'(u)}{\kappa(u)}, \quad m^* = 0, \quad n^* = 0.
\]
(19)

Thus, we can give the following results:
Theorem 2 Let $M$ be a tubular surface with Frenet frame given with the parametrization (7) and $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$. Then the Gaussian curvature of $M^*$ vanishes, so the focal surface is flat.

Proof. Let $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$. Using the equations (5), (16) and (19), we get $K^* = 0$, which completes the proof. □

Theorem 3 Let $M$ be a tubular surface with Frenet frame given with the parametrization (7) and $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$. Then the mean curvature of $M^*$ is

$$H^* = \frac{\kappa^3(u)}{2\kappa'(u)}. \quad (20)$$

Proof. Let $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$. Using the equations (6), (16) and (19), we get the result. □

Theorem 4 There is no minimal focal surface $M^*$ of the tubular surface $M$.

Proof. Assume that the focal surface $M^*$ of the surface $M$ is minimal. From the equation (20), the curvature function $\kappa$ according to Frenet frame vanishes identically, this is a contradiction. Thus, there is no minimal focal surface of the tubular surface $M$. □

Theorem 5 Let $M$ be a tubular surface with Frenet frame given with the parametrization (7) and $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$. Then the focal surface $M^*$ has constant mean curvature if and only if the curvature function $\kappa(u)$ satisfies

$$\kappa(u) = \pm \frac{\sqrt{(-u + c_1) c}}{-u + c_1 c}, \quad (21)$$

where $c, c_1$ are real constants.

Proof. Let $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$. From the equation (20), we get the differential equation

$$\kappa^3(u) - 2\kappa'(u)c = 0,$$

which has a non-trivial solution (21). □

Example 6 Let us consider the unit speed planar curve with the parameterization

$$\gamma(u) = \left( \frac{u}{\sqrt{2}} + 1 \right) \cos \left( \ln \left( \frac{u}{\sqrt{2}} + 1 \right) \right), \left( \frac{u}{\sqrt{2}} + 1 \right) \sin \left( \ln \left( \frac{u}{\sqrt{2}} + 1 \right) \right), 0 \right).$$
The Frenet apparatus of this curve are determined by

\[ T(u) = \gamma'(u) = \frac{1}{\sqrt[2]{2}} \left( \cos \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) - \sin \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right), \sin \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) + \cos \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right), 0 \right), \]

\[ N_1(u) = \frac{1}{\sqrt[2]{2}} \left( \sin \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) - \cos \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right), \cos \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) - \sin \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right), 0 \right), \]

\[ N_2(u) = (0, 0, 1), \]

\[ \kappa(u) = \frac{1}{u + \sqrt[2]{2}}, \quad \tau(u) = 0. \]

Hence, the parameterization of tubular surface around the curve \( \gamma(u) \) can be written with the Frenet frame as

\[ X(u, v) = \left( \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \cos \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) - \sqrt[2]{2} \cos v \left( \sin \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) + \cos \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) \right), \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \sin \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) + \sqrt[2]{2} \cos v \left( \cos \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) - \sin \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) \right), \right. \]

\[ \left. r \sin v \right). \]

Also, the parameterization of the focal surface of this surface is obtained as

\[ X^*(u, v) = \left( \left( \frac{u^2 + 2\sqrt[2]{2}u + 1}{\sqrt[2]{2}(u + \sqrt[2]{2})} \right) \cos \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) - \frac{1}{\sqrt[2]{2}(u + \sqrt[2]{2})} \sin \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right), \left( \frac{u^2 + 2\sqrt[2]{2}u + 1}{\sqrt[2]{2}(u + \sqrt[2]{2})} \right) \sin \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right) + \frac{1}{\sqrt[2]{2}(u + \sqrt[2]{2})} \cos \left( \ln \left( \frac{u}{\sqrt[2]{2}} + 1 \right) \right), \right. \]

\[ \left. \frac{1}{u + \sqrt[2]{2}} \tan v \right). \]

Further, by taking radius \( r = \sqrt[2]{2} \), we plot the tubular surface and its focal surface in \( E^3 \):
Figure 1: Tubular surface $M$ and the focal surface $M^*$

Now, we obtain the following results for parameter curves on the focal surface $M^*$.

**Theorem 7** Let $M$ be a tubular surface with Frenet frame given with the parametrization (7) and $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$. Then,

i) $u$-parameter curves of the focal surface $M^*$ cannot be asymptotic curves.

ii) $v$-parameter curves of the focal surface $M^*$ are asymptotic curves.

**Proof.** Let $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$.

i) From the definition of an asymptotic curve, using (17) and (18), we get

$$\langle (X^*)_u^u, N^* \rangle = \kappa'(u)/\kappa(u) = 0$$

if and only if $\kappa$ is constant which contradicts the regularity of $M^*$. Thus, $u$-parameter curves of the focal surface $M^*$ cannot be asymptotic curves.

ii) Again using the same equations, we obtain

$$\langle (X^*)_v^v, N^* \rangle = 0$$

which means $v$-parameter curves of the focal surface $M^*$ are asymptotic curves.

**Theorem 8** Let $M$ be a tubular surface with Frenet frame given with the parametrization (7) and $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$. Then,

i) $u$-parameter curves of the focal surface $M^*$ are geodesic curves if and only if the equation

$$\kappa(u) = -\frac{1}{c_1 u + c_2},$$

(22)

held for the curvature of $\gamma$, where $c_1, c_2$ are real constants.

ii) $u$-parameter curves of the focal surface $M^*$ are geodesic curves if and only if $v = t\pi, t \in \mathbb{Z}$. 

Proof. i) From (17) and (18), we get \((X^*)_{uu} \wedge N^* = 0\) if and only if 
\[-\kappa'' + 2(\kappa')^2 = 0,\]
which has a solution (22) for real constants \(c_1\) and \(c_2\).

ii) Again using the same equations, we obtain \((X^*)_{vv} \wedge N^* = 0\) if and only if \(\sin v = 0\) which completes the proof.

4 Focal surface of a tubular surface with Bishop frame

Tubular surface with the Bishop frame was studied by Doğan and Yaylı in [5].

Here, we handle the tubular surface according to the Bishop frame and give the focal surface of this surface in \(\mathbb{E}^3\).

Let \(\gamma(u) = (\gamma_1(u), \gamma_2(u), \gamma_3(u))\) be a unit speed curve in \(\mathbb{E}^3\). The tubular surface with respect to Bishop frame has the parametrization:

\[M : X(u,v) = \gamma(u) + r (\cos v M_1(u) + \sin v M_2(u)),\]  
(23)

where \(r = \text{const.}\) is the radius of the spheres. The tangent space of \(M\) is spanned by

\[X_u = (1 - fr) T,\]
\[X_v = -r \sin v M_1 + r \cos v M_2,\]
(24)

where

\[f(u,v) = k_1(u) \cos v + k_2(u) \sin v.\]

Then coefficients of the first fundamental form are

\[E = (1 - fr)^2, \quad F = 0, \quad G = r^2,\]
(25)

where \(W^2 = EG - F^2 = (1 - fr)^2 r^2\) [5].

Proposition 9 [5] \(X(u,v)\) is a regular tubular surface patch if and only if \(f \neq \frac{1}{r}\).

The unit normal vector field and the second partial derivatives of \(M\) are obtained as

\[N = -\cos v M_1 - \sin v M_2.\]
(26)

and

\[X_{uu} = (-k_1'r \cos v - k_2'r \sin v) T + (1 - fr) k_1 M_1 + (1 - fr) k_2 M_2,\]
\[X_{uv} = (k_1 \sin v - k_2 r \cos v) T,\]
\[X_{vv} = -r \cos v M_1 - r \sin v M_2,\]
(27)

respectively.

Then the coefficients of the second fundamental form become

\[l = - (1 - fr) f, \quad m = 0, \quad n = r.\]
(28)

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Thus, from the equations (25) and (28), the Gaussian and mean curvature functions of $M$ are calculated as

\[ K = \frac{f}{r(fr - 1)}, \]  

(29)

and

\[ H = rK - \frac{K}{2f} \]

respectively.

The shape operator matrix of the surface $M$ is as follows:

\[ \begin{bmatrix} -f & f \\ \frac{-fr}{1(fr - 1)} & 0 \\ 0 & \frac{1}{r} \end{bmatrix} \]  

(30)

From now on, we can give the parametrization of the focal surface $M^*$ of $M$ by obtaining the principal curvature functions of $M$.

Using (30), we get the principal curvature functions as

\[ \kappa_1 = \frac{1}{r}, \quad \kappa_2 = \frac{f}{fr - 1}. \]  

(31)

From the definition of the focal surface of a given surface and using the equation (31), we obtain the focal surface $M^*$ of $M$ as

\[ X^*(u, v) = \gamma(u) + \frac{1}{f} (\cos vM_1 + \sin vM_2). \]  

(32)

The tangent space of the focal surface $M^*$ is spanned by

\[ (X^*)_u = -\frac{f_u}{f^2} \cos vM_1 - \frac{f_u}{f^2} \sin vM_2, \]  

(33)

and

\[ (X^*)_v = -\frac{k_2}{f^2} M_1 + \frac{k_1}{f^2} M_2. \]

Thus from (33), the coefficients of the first fundamental form is obtained as follows:

\[ E^* = \frac{f_u^2}{f^4}, \quad F^* = \frac{f_uk_2}{f^4} \cos v - \frac{f_uk_1}{f^4} \sin v, \quad G^* = \frac{k_1^2 + k_2^2}{f^4}, \]  

(34)

where $(W^*)^2 = \frac{f^2}{f^2} \neq 0$. Using the first partial derivatives of $X^*(u, v)$, we see that

\[ N^* = -T. \]  

(35)
The second partial derivatives of \( X^* (u, v) \) are

\[
(X^*)_{uu} = \frac{f_u T - f_{uu} f - 2 f_u^2}{f^3} \cos v M_1 - \frac{f_{uu} f - 2 f_u^2}{f^3} \sin v M_2, \\
(X^*)_{uv} = -\frac{f_{uv} f + 2 f_u f_v}{f^3} \sin v M_2 - \frac{f_u}{f^2} \cos v M_2, \\
(X^*)_{vv} = \frac{2 k_2 f_v}{f^3} M_1 + \frac{-2 k_1 f_v}{f^3} M_2.
\]

(36)

Hence, from (35) and (36), we get the coefficients of the second fundamental form of the focal surface \( M^* \) as

\[
i^* = -\frac{f_u}{f}, \quad m^* = 0, \quad n^* = 0.
\]

(37)

Thus, we can give the following results:

**Theorem 10** Let \( M \) be a tubular surface with respect to Bishop frame given by the parametrization (23) and \( M^* \) be the focal surface of \( M \) with the parametrization (32) in \( \mathbb{E}^3 \). Then the Gaussian curvature of \( M^* \) vanishes, so the focal surface is flat.

**Proof.** Let \( M^* \) be the focal surface of \( M \) with the parametrization (32) in \( \mathbb{E}^3 \). Using the equations (34), (37) and (5), we get \( K^* = 0 \), which completes the proof.

**Theorem 11** Let \( M \) be a tubular surface with respect to Bishop frame given with the parametrization (23) and \( M^* \) be the focal surface of \( M \) with the parametrization (32) in \( \mathbb{E}^3 \). Then the mean curvature of \( M^* \) is

\[
H^* = -\frac{(k_1^2 + k_2^2)}{2} \left( k_1 \cos v + k_2 \sin v \right)
\]

(38)

**Proof.** Let \( M^* \) be the focal surface of \( M \) with the parametrization (32) in \( \mathbb{E}^3 \). Using the equations (6), (34) and (37), we get the result.

**Theorem 12** There is no minimal focal surface \( M^* \) of the tubular surface \( M \).

**Proof.** Assume that the focal surface \( M^* \) of the surface \( M \) is minimal. From the equation (38), the curvature functions \( k_1, k_2 \) according to Bishop frame vanishes identically, which is a contradiction. Thus, there is no minimal focal surface of the tubular surface \( M \).

**Example 13** Let us consider the unit speed curve with the parameterization

\[
\gamma(u) = \left( \cos \frac{u}{\sqrt{2}}, \sin \frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}} \right).
\]
The Bishop frame and curvatures $k_1, k_2$ of this curve are determined by

$$T(u) = \gamma'(u) = \frac{1}{\sqrt{2}} \left(-\sin \frac{u}{\sqrt{2}}, \cos \frac{u}{\sqrt{2}}, 1\right),$$

$$M_1(u) = \frac{1}{\sqrt{2}} \left(-\cos \frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}} \sin \frac{u}{\sqrt{2}}, -\sin \frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cos \frac{u}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$M_2(u) = \frac{1}{\sqrt{2}} \left(-\cos \frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}} \sin \frac{u}{\sqrt{2}}, -\sin \frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cos \frac{u}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right),$$

$$k_1(u) = \frac{1}{2\sqrt{2}}, \quad k_2(u) = \frac{1}{2\sqrt{2}}.$$

Hence, the parameterization of tubular surface around the curve $\gamma(u)$ can be written according to Bishop frame as

$$X(u, v) = \left(\cos \frac{u}{\sqrt{2}} - \cos v \cos \frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cos v \sin \frac{u}{\sqrt{2}} - \sin v \cos \frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}} \sin v \sin \frac{u}{\sqrt{2}}, \sin \frac{u}{\sqrt{2}} - \cos v \sin \frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cos v \cos \frac{u}{\sqrt{2}} - \sin v \sin \frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}} \sin v \cos \frac{u}{\sqrt{2}}, \frac{u - \cos v \sin \frac{u}{\sqrt{2}}}{\sqrt{2}}\right).$$

Also, the parameterization of the focal surface of this surface is obtained as

$$X^*(u, v) = \left(\cos \frac{u}{\sqrt{2}} + \frac{2 \cos v}{\cos v + \sin v} \left(-\cos \frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}} \sin \frac{u}{\sqrt{2}}\right) + \frac{2 \sin v}{\cos v + \sin v} \left(-\cos \frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}} \sin \frac{u}{\sqrt{2}}\right), \sin \frac{u}{\sqrt{2}} + \frac{2 \cos v}{\cos v + \sin v} \left(-\sin \frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cos \frac{u}{\sqrt{2}} + \frac{2 \sin v}{\cos v + \sin v} \left(-\sin \frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cos \frac{u}{\sqrt{2}}\right)\right), \frac{u - \cos v \sin \frac{u}{\sqrt{2}}}{\cos v + \sin v}\right).$$

Further, by taking radius $r = \sqrt{2}$, we plot the tubular surface and its focal surface in $\mathbb{E}^3$:

Figure 2: Tubular surface $M$ and the focal surface $M^*$
\textbf{Theorem 14} Let $M$ be a tubular surface with respect to Bishop frame given with the parametrization (23) and $M^*$ be the focal surface of $M$ with the parametrization (32) in $E^3$. Then,

i) $u$-parameter curves of the focal surface $M^*$ cannot be asymptotic curves.

ii) $v$-parameter curves of the focal surface $M^*$ are asymptotic curves.

\textbf{Proof.} Let $M^*$ be the focal surface of $M$ with the parametrization (32) in $E^3$.

i) By the use of (36), (35) and with the help of the definition of an asymptotic curve, we obtain $(X^*)_{uu}^* N^* = f_u f = 0$ which contradicts the regularity of $M^*$. Thus, $u$-parameter curves of the focal surface $M^*$ cannot be asymptotic curves.

ii) Similarly, from the same equations, we get $(X^*)_{vv}^* N^* = 0$ which means $v$-parameter curves of the focal surface $M^*$ are asymptotic curves.

\textbf{Theorem 15} Let $M$ be a tubular surface with respect to Bishop frame given with the parametrization (4) and $M^*$ be the focal surface of $M$ with the parametrization (14) in $E^3$. Then,

i) $u$-parameter curves of the focal surface $M^*$ are geodesic curves if and only if

$$-f_{uu}f + 2(f_u)^2 = 0$$

holds.

ii) $v$-parameter curves of the focal surface $M^*$ cannot be geodesic curves.

\textbf{Proof.} i) By the use of (36), (35), we have $(X^*)_{uu}^* N^* = 0$ if and only if the equation (39) is hold.

ii) Using the same equations, we have $(X^*)_{vv}^* N^* = 0$ if and only if $k_1 f_v = 0$ and $k_2 f_v = 0$. Then, we get $k_1 = 0$ and $k_2 = 0$, and so $f$ vanishes identically, which contradicts the representation of the focal surface. Thus, $v$-parameter curves of the focal surface $M^*$ cannot be geodesic curves.

\section{Focal surface of a tubular surface with Darboux frame}

Tubular surface with the Darboux frame was studied by Doğan and Yaylı in [6]. In this part, we handle the tubular surface according to the Darboux frame and give the focal surface of this surface in $E^3$.

Let $\gamma(u) = (\gamma_1(u), \gamma_2(u), \gamma_3(u))$ be a unit speed curve on a surface $S$. The tubular surface with respect to Darboux frame has the following parametrization:

$$M : X(u,v) = \gamma(u) + r(\cos vY(u) + \sin vU(u)),$$

where $r = \text{const.}$ is the radius of the spheres and $U$ is the unit normal of the surface $S$ along the curve $\gamma$. Then the tangent space of $M$ is spanned by $X_u$ and $X_v$ at a point $p = X(u,v)$:

$$X_u = (1 - br)T - r\tau_g \sin vY + r\tau_g \cos vU,$$

$$X_v = -r \sin vY + r \cos vU,$$
where
\[ b(u, v) = k_g(u) \cos v + k_n(u) \sin v. \]  \tag{42}

Then coefficients of the first fundamental form become
\[ E = (1 - br)^2 + r^2 \tau_g^2, \quad F = r^2 \tau_g, \quad G = r^2, \]  \tag{43}

where \( W^2 = EG - F^2 = (1 - br)^2 r^2 \) \[6\].

Proposition 16 [6] \( X(u, v) \) is a regular tubular surface patch if and only if \( b \neq \frac{1}{r} \).

The unit normal vector field and the second partial derivatives of \( M \) are obtained as
\[ N = -\cos vY - \sin vU, \]  \tag{44}

and
\[ X_{uu} = (-b_u r - r \tau_g b_v) T + (k_g (1 - br) - r \tau'_g \sin v - r \tau_g^2 \cos v) Y \]  
\[ + (k_n (1 - br) + r \tau'_g \cos v - r \tau_g^2 \sin v) U, \]  
\[ X_{uv} = -b_v r T - r \tau_g \cos v Y - r \tau_g \sin v U, \]  
\[ X_{vv} = -r \cos v Y - r \sin v U, \]  \tag{45}

respectively.

Then the coefficients of the second fundamental form become
\[ l = - (1 - br) b + r \tau_g^2, \quad m = r \tau_g, \quad n = r. \]  \tag{46}

Thus, from the equations (43) and (46), the Gaussian and mean curvature functions of \( M \) are calculated as
\[ K = \frac{b}{r (br - 1)}, \]  \tag{47}

and
\[ H = \frac{1 - 2br}{2 (1 - br) r} \]  \tag{48}

respectively [6].

Now, we focus on the parametrization of the focal surface \( M^* \) of \( M \) by obtaining the principal curvature functions of \( M \).

Using (47), (48), and the equation \( \kappa_i = H \pm \sqrt{H^2 - K}, (i = 1, 2) \), we get the principal curvature functions as
\[ \kappa_1 = \frac{1}{r}, \quad \kappa_2 = \frac{b}{br - 1}. \]  \tag{49}

From the definition of the focal surface of a given surface and using the equation (49), we obtain the focal surface \( M^* \) of \( M \) as
\[ X^* (u, v) = \gamma (u) + \frac{1}{b(u, v)} (\cos v Y(u) + \sin v U(u)), \]  \tag{50}
where \( b(u, v) = k_g(u) \cos v + k_n(u) \sin v \).

The tangent space of the focal surface \( M^* \) is spanned by the vectors

\[
(X^*)_u = \left( -\frac{b_u}{b^2} \cos v - \frac{1}{b} \tau_g \sin v \right) T + \left( -\frac{b_u}{b^2} \sin v + \frac{1}{b} \tau_g \cos v \right) U, \tag{51}
\]

and

\[
(X^*)_v = \left( -\frac{b_v}{b^2} \cos v - \frac{1}{b} \sin v \right) T + \left( -\frac{b_v}{b^2} \sin v + \frac{1}{b} \cos v \right) U. \tag{52}
\]

Thus from (51) and (52), the coefficients of the first fundamental form are obtained as follows:

\[
E^* = \frac{b_u^2 + b^2 \tau_g^2}{b^4}, \quad F^* = \frac{b_u b_v + b^2 \tau_g}{b^4}, \quad G^* = \frac{b_v^2 + b^2}{b^4}, \tag{53}
\]

where \((W^*)^2 = \frac{1}{b^4} (b_u - b_v \tau_g)^2\).

**Proposition 17** \( X^*(u, v) \) is a regular tubular surface patch if and only if \((k'_g - k_n \tau_g) \cos v + (k'_n + k_g \tau_g) \sin v \neq 0 \).

**Proof.** Substituting the partial derivatives of the equation \([42]\) in \( W^* \), we get the result.

One can find the normal vector of the focal surface \( M^* \) as

\[
N^* = T. \tag{54}
\]

The second partial derivatives of \( X^*(u, v) \) are

\[
(X^*)_{uu} = \left( \frac{b_u - b_v \tau_g}{b} \right) T
+ \left( -\frac{b_u b^2 \cos v + 2b b_u^2 \cos v + 2b b_u b^2 \tau_g \sin v - b^3 \tau_g^2 \cos v - b^3 \tau_g^2 \sin v}{b^4} \right) Y
+ \left( -\frac{b_u^2 b \cos v + 2b b_u^2 \cos v - b_u b^2 \tau_g \cos v - b^3 \tau_g^2 \sin v + b^3 \tau_g^2 \cos v}{b^4} \right) U. \tag{55}
\]

\[
(X^*)_{uv} = \left( -\frac{b_u b^2 \cos v + 2b b_u b_v \cos v + b_u b^2 \sin v + b_u b^2 \tau_g \sin v - b^3 \tau_g \cos v}{b^4} \right) Y
+ \left( -\frac{b_u^2 b \sin v + 2b b_u b_v \sin v - b_u b^2 \cos v - b_u b^2 \tau_g \cos v - b^3 \tau_g \sin v}{b^4} \right) U;
\]

\[
(X^*)_{vv} = \left( -\frac{b_v b^2 \cos v + 2b b_v b_u \cos v + 2b b_v b^2 \sin v - b^3 \cos v}{b^4} \right) Y
+ \left( -\frac{b_v^2 b \sin v + 2b b_v b_u \sin v - 2b b_v b^2 \cos v - b^3 \cos v}{b^4} \right) U.
\]

Hence, from (53) and (55), we get the coefficients of the second fundamental form of the focal surface \( M^* \) as

\[
l^* = \frac{b_u - b_v \tau_g}{b}, \quad m^* = 0, \quad n^* = 0. \tag{56}
\]

Thus, we can give the following results:
Theorem 18 Let $M$ be a tubular surface according to Darboux frame given with the parametrization (40) and $M^*$ be the focal surface of $M$ with the parametrization (50) in $\mathbb{E}^3$. Then the Gaussian curvature of $M^*$ vanishes, so the focal surface is flat.

Proof. Let $M^*$ be the focal surface of $M$ with the parametrization (50) in $\mathbb{E}^3$. Using the equations (53), (56) and (5), we get $K^* = 0$, which completes the proof. ■

Theorem 19 Let $M$ be a tubular surface according to Darboux frame given with the parametrization (40) and $M^*$ be the focal surface of $M$ with the parametrization (50) in $\mathbb{E}^3$. Then the mean curvature of $M^*$ is

$$H^* = \frac{(b_v^2 + b_u^2) b}{2(b_u - b_u r_g)}.$$ (57)

Proof. Let $M^*$ be the focal surface of $M$ with the parametrization (50) in $\mathbb{E}^3$. Using the equations (6), (53) and (56), we get the result. ■

Theorem 20 There is no minimal focal surface of the tubular surface $M$.

Proof. Assume that the focal surface $M^*$ of the surface $M$ is minimal. From the equation (57), the curvature functions $H^* = 0 \iff b = 0$, a contradiction. Thus, there is no minimal focal surface of the tubular surface $M$. ■

Example 21 Let us consider the unit speed curve with the parameterization

$$\gamma(u) = \left( \cos \frac{u}{\sqrt{2}}, \sin \frac{u}{\sqrt{2}}, \frac{u}{\sqrt{2}} \right).$$

The Darboux frame and curvatures $k_g$, $k_n$ of this curve are determined by

$$T(u) = \gamma'(u) = \frac{1}{\sqrt{2}} \left( -\sin \frac{u}{\sqrt{2}}, \cos \frac{u}{\sqrt{2}}, 1 \right),$$

$$Y(u) = \frac{1}{\sqrt{2}} \left( -\cos \frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}} \sin \frac{u}{\sqrt{2}}, -\sin \frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cos \frac{u}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),$$

$$U(u) = \frac{1}{\sqrt{2}} \left( \cos \frac{u}{\sqrt{2}} + \frac{1}{\sqrt{2}} \sin \frac{u}{\sqrt{2}}, \sin \frac{u}{\sqrt{2}} - \frac{1}{\sqrt{2}} \cos \frac{u}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right),$$

$$k_g(u) = \frac{1}{2\sqrt{2}}, \quad k_n(u) = \frac{1}{2\sqrt{2}}.$$

Hence, the parameterization of tubular surface around the curve $\gamma(u)$ can be written according to Darboux frame as

$$X(u, v) = \left( \cos \frac{u}{\sqrt{2}} (1 - \cos v + \sin v) + \frac{1}{\sqrt{2}} \sin \frac{u}{\sqrt{2}} (\cos v + \sin v), \sin \frac{u}{\sqrt{2}} (1 - \cos v + \sin v) - \frac{1}{\sqrt{2}} \cos \frac{u}{\sqrt{2}} (\cos v + \sin v), \frac{u + \cos v + \sin v}{\sqrt{2}} \right).$$
Also, the parameterization of the focal surface of this surface is obtained as

$$X^*(u, v) = \left( \cos \frac{u}{\sqrt{2}} \left( \frac{3}{2} \sin v - \cos v \right) + \sqrt{2} \sin \frac{u}{\sqrt{2}}, \sin \frac{u}{\sqrt{2}} \left( \frac{3}{2} \sin v - \cos v \right) - \sqrt{2} \cos \frac{u}{\sqrt{2}} \frac{u+2}{\sqrt{2}} \right).$$

Further, by taking radius $r = \sqrt{2}$, we plot the tubular surface and its focal surface in $E^3$:

![Image of tubular surface and focal surface]

Figure 3: Tubular surface $M$ and the focal surface $M^*$

**Theorem 22** Let $M$ be a tubular surface with Darboux frame given with the parametrization (40) and $M^*$ be the focal surface of $M$ with the parametrization (50) in $E^3$. Then,

i) $u$-parameter curves of the focal surface $M^*$ cannot be asymptotic curves.

ii) $v$-parameter curves of the focal surface $M^*$ are asymptotic curves.

**Proof.** Let $M^*$ be the focal surface of $M$ with the parametrization (50) in $E^3$.

i) Using the equations (54), (55) and the definition of asymptotic curve, $\langle (X^*)_u, N^* \rangle = 0$ if and only if $b_u - b_v \tau_g = 0$. But, this contradicts the regularity of $M^*$. Thus, $u$-parameter curves of the focal surface $M^*$ cannot be asymptotic curves.

ii) Again using the same equations, we obtain $\langle (X^*)_v, N^* \rangle = 0$ which means $v$-parameter curves of the focal surface $M^*$ are asymptotic curves. ■

**Theorem 23** Let $M$ be a tubular surface with Darboux frame given with the parametrization (40) and $M^*$ be the focal surface of $M$ with the parametrization (50) in $E^3$. Then,

i) $u$-parameter curves of the focal surface $M^*$ are geodesic curves if and only if

$$-2\tau_g \left( b_u \tau_g' + b_v \tau_g'' \right) + 4b \left( \tau_g' \right)^2 - 4b \tau_g^4 = 0$$

holds.

ii) $v$-parameter curves of the focal surface $M^*$ cannot be geodesic curves.
Proof. i) From (17) and (18), \((X^*)_{uu} \wedge N^* = 0\) if and only if

\[
b \left( -b_{uu}b + 2 \left( b_u \right)^2 - b^2 (\tau_g)^2 \right) = 0, \tag{59}
\]

\[
b^2 \left( 2 b_u \tau_g - b \tau'_g \right) = 0. \tag{60}
\]

Therefore, by the use of these relations, we obtain the desired result.

ii) Similarly, with the help of (17) and (18), \((X^*)_{vv} \wedge N^* = 0\) if and only if

\[
b \left( b_{vv}b - 2 \left( b_v \right)^2 + b^2 \right) = 0, \tag{61}
\]

\[
b_v = 0. \tag{62}
\]

Using these relations, we get \(b = 0\) which contradicts the representation of the focal surface. Hence, we get the result and this completes the proof. ■

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