Asymptotic Control Theory for a Closed String

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Abstract. We develop an asymptotical control theory for one of the simplest distributed oscillating systems, namely, for a closed string under a bounded load applied to a single distinguished point. We find exact classes of string states that admit complete damping and an asymptotically exact value of the required time. By using approximate reachable sets instead of exact ones, we design a dry-friction like feedback control, which turns out to be asymptotically optimal. We prove the existence of motion under the control using a rather explicit solution of a nonlinear wave equation. Remarkably, the solution is determined via purely algebraic operations. The main result is a proof of asymptotic optimality of the control thus constructed.

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1. INTRODUCTION

As is well known, the limit capabilities of a control system can be described in terms of reachable sets, i.e., sets of states reachable from a given state at a given time. In minimum-time problems, it is convenient to pass to the backward time. By passing to the backward time, we can consider sets reachable from the terminal manifold. Geometrically, the minimum time control is organized as follows. Any state corresponds to a reachable set such that its boundary passes through this state. The minimum-time control at the current state forces our system to move in the direction normal to the boundary of the corresponding reachable set.

Unfortunately, the reachable sets are difficult to study. However, sometimes we possess an approximation of the reachable sets. In particular, for linear systems, there is a systematic asymptotic theory of reachable sets as time goes to infinity [1]. By substituting an approximate reachable set for the exact reachable set, we can design a control. Alongside its simplicity, this control might be asymptotically optimal. This type of control can be interpreted as an action of a generalized dry friction. The behavior of any system of finite number of linear oscillators under the dry-friction control has been studied in [2, 3].

1.1. Problem Statement

In this paper, we apply this technique to control a simple distributed system, the closed string under an impulsive force applied at a fixed point in the string. The phase space $S$ of the system consists of pairs $f = (f_0, f_1)$ of distributions on the one-dimensional torus $\mathcal{T}$, and the motion is governed by the string equation

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} + u\delta, \quad |u| \leq 1.$$  \hspace{1cm} (1)

Here $x \in [0, 2\pi]$ is the angle coordinate on the torus, $t$ is time, $f_0 = f$, $f_1 = \frac{\partial f}{\partial t}$, $\delta$ is the Dirac $\delta$-function. In other words, $S$ is the space of initial data for Eq. (1). It is clear that any solution of Eq. (1) with zero initial data is even. This is why we assume that $S$ consists of pairs $f = (f_0, f_1)$ of even distributions. The equation describes oscillations of a closed homogeneous string under a bounded load applied at a fixed point (zero). We note that a mechanical model of such system can be not only a string, but also, e.g., a toroidal acoustic resonator.
Our goal is to design an easily implementable feedback control for damping the oscillations. This means that we do not necessarily want to stop the motion of the string as a whole, so that our target manifold $\mathcal{C}$ consists of pairs of constants

$$\mathcal{C} = \{(c_0, c_1)^* \in \mathbb{R}^2 \subset S\}. \quad (2)$$

Another useful point of view is to take the quotient space $\bar{S} = S/\mathcal{C}$ as the phase space of our system, and try to reach zero in this space. This is reasonable, because the target space $\mathcal{C}$ is invariant under the natural flow associated with the string equation. In what follows, we deal with a class of problems of minimum-time steering from the initial state to a terminal manifold $\mathcal{C}$ consisting of pairs of constants (2). More specifically, we study three problems:

1. complete stop at a given point: $\mathcal{C} = 0$;
2. stop moving: $\mathcal{C} = \mathbb{R} \times 0$;
3. oscillation damping: $\mathcal{C} = \mathbb{R}^2$.

The main result of the present paper is the construction of a generalized dry-friction control and the proof of its asymptotic optimality in the stop moving problem. A summary of our results was presented in [4].

There are interesting related problems of stabilization of solutions to linear differential equations in partial derivatives. The goal of stabilization is damping again, but it should be reached at infinite time, and there is no performance index. We mention a few interesting and relatively recent papers on the subject [5–7]. Moreover, problems of control for oscillating distributed systems are important for a wide spectrum of technological objects [8, 9].

\subsection*{1.2. Structure}

The paper is organized as follows. In Section 2, we discuss a mechanical model of the considered system. In Section 3, we find an explicit form of the support functions of the reachable sets. We then study the asymptotic behavior of the reachable sets as time goes to infinity using the language of shapes in Section 4. In Section 5, on the basis of this study, we design a generalized dry-friction by means of a rather explicit solution of the nonlinear equation. Remarkably, it is made via purely algebraic operations. In Section 6, we demonstrate that the suggested control is asymptotically optimal, and describe specific features of this control in Section 10. Finally, we conclude in Section 11. Appendices I–III contain a number of auxiliary results.

\section*{2. STRING AS A MECHANICAL SYSTEM}

The equations of motion of the free string (1) are that of the following Lagrangian system, where $q$ is an even function such that $\frac{\partial q}{\partial x} \in L^2(T)$, and the Lagrangian

$$L(q, \dot{q}) = \frac{1}{2} \int_T |\dot{q}|^2(x) \, dx - \frac{1}{2} \int_T \left| \frac{\partial q}{\partial x} \right|^2(x) \, dx - uq(0), \quad (3)$$

so that $\frac{1}{2} \int_T |\dot{q}|^2 \, dx$ is the kinetic energy, and $\frac{1}{2} \int_T \left| \frac{\partial q}{\partial x} \right|^2 \, dx$ is the potential energy of the system. The Lagrangian corresponds to Hooke’s law: the strain (deformation) is proportional to the applied stress. In terms of the Lagrangian, the stress at a point $x$ is $\frac{\delta L}{\delta q}(x) = \frac{\partial q}{\partial x}(x)$, and the strain at $x$ is $\frac{\partial q}{\partial x}(x)$, so that the coefficient of proportionality is 1.

The string also admits a Hamiltonian description. The phase space is then the set of pairs $(p, q)$, where $p$ is an (even) function in $L^2(T)$, $q$ is an (even) function in the space $N$ of functions such that $\frac{\partial q}{\partial x} \in L^2(T)$, and the Hamiltonian is

$$H(p, q) = \frac{1}{2} \int_T |p|^2(x) \, dx + \frac{1}{2} \int_T \left| \frac{\partial q}{\partial x} \right|^2(x) \, dx + uq(0). \quad (4)$$
The canonical symplectic structure $\omega = dp \wedge dq$ is given by $\omega((X, X'), (Y, Y')) = \langle X, Y' \rangle - \langle X', Y \rangle$. Here $X, Y \in L_2(\mathcal{T})$, $X', Y' \in N$, and the angle brackets stand for the inner product in $L_2(\mathcal{T})$.

Finally, the Pontryagin Hamiltonian $H^\text{pont}$ in the “coordinates” $\tilde{f} = (f_0, f_1)$ takes the form
\begin{equation}
H^\text{pont}(\tilde{f}, \tilde{\xi}) = \langle f_1, \xi_0 \rangle + \langle \Delta f_0, \xi_1 \rangle + |\xi_1(0)|,
\end{equation}
with $\tilde{\xi} = (\xi_0, \xi_1)$ being the adjoint variables.

### 3. THE SUPPORT FUNCTION OF REACHABLE SETS

The first issue we deal with is that of controllability of the system considered above. We approach it by computing the support function. The approach has much in common with that of [10].

To make a comparison with the finite-dimensional case clear, we rewrite the governing equation in the form of first-order system
\begin{equation}
\frac{\partial \tilde{f}}{\partial t} = Af + Bu, \quad |u| \leq 1,
\end{equation}
\begin{equation}
A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}, \quad \Delta = \frac{\partial^2}{\partial x^2}, \quad B = \begin{pmatrix} 0 \\ \delta \end{pmatrix}.
\end{equation}

We begin with computing the support function $H = H_{D(T)}(\xi)$ of the reachable set $D(T)$, $T \geq 0$, of system (6) with zero initial condition. In other words, we have to find $H = \sup_u \langle \tilde{f}(T), \tilde{\xi} \rangle$, where $\xi \in S^*$ is a dual vector, and the sup is taken over admissible controls. Towards this end, we extend $\xi = (\xi_0, \xi_1)$ to the solution of the Cauchy problem
\begin{equation}
\frac{\partial \xi}{\partial t} = -A^* \xi, \quad \xi(T) = \xi, \quad A^* = \begin{pmatrix} 0 & \Delta \\ 1 & 0 \end{pmatrix},
\end{equation}
where $A^*$ is the adjoint operator to $A$. In particular, note that $\xi_1$ satisfies the wave equation. These equations are exactly the equations for the adjoint variables of the Pontryagin maximum principle.

We then have
\begin{equation}
\frac{d}{dt} \langle f(t), \xi(t) \rangle = \langle Af + Bu, \xi \rangle - \langle \tilde{f}, A^* \xi \rangle = \langle Bu, \xi \rangle = u(t)\xi_1(0, t).
\end{equation}

A standard formal computation shows that
\begin{equation}
H = H_{D(T)}(\xi) = \sup_{|u| \leq 1} \int_0^T u(t)\xi_1(0, t)dt = \int_0^T |\xi_1(0, t)|dt.
\end{equation}

The reachable sets $D(T)$, $T \geq 0$, are closed in the standard topology of distributions, and they are certainly convex. Therefore, they are uniquely defined by their support functions.

Now we can characterize the vectors $\tilde{f}$ reachable from zero at time $T$ as follows:
\begin{equation}
\tilde{f} \in D(T) \iff \langle \tilde{f}, \xi \rangle \leq \int_0^T |\xi_1(0, t)|dt \quad \text{for any } \xi \in S^*.
\end{equation}

Here the function $x, t \mapsto \xi_1(x, t)$ is defined via the solution of the Cauchy problem (7).

In particular, the space $\mathcal{D}$ generated by the vectors $\tilde{f} \in \bigcup_{T \geq 0} D(T)$ reachable from zero in an arbitrary time $T \geq 0$ is the dual space to the Fréchet space of vectors $\xi$ with finite norms
\begin{equation}
\|\xi\| = \|\xi\|_T = \int_0^T |\xi_1(0, t)|dt.
\end{equation}
for any \( T > 0 \). This space \( D \) coincides with the set of vectors reachable from zero at an arbitrary time \( T \geq 0 \) by means of a control which is bounded (not necessarily by 1).

It is not difficult to compute \( \xi_1(0, t) \) in terms of the Fourier coefficients of the functions \( \psi = \xi_1 \) and \( \phi = \xi_0 \). Suppose that

\[
\psi(x, t) = \sum_{n=-\infty}^{\infty} \psi_n(t) e^{inx}
\]

is the Fourier expansion of \( \xi_1 \). Since \( \psi \) is an even and real distribution, the coefficients \( \psi_n \) are real, and \( \psi_n = \psi_{-n} \), so that Eq. (12) is, in fact, the cosine-expansion:

\[
\psi(x, t) = \sum_{n=0}^{\infty} \psi_n(t) \cos nx.
\]

The quantity we want to compute is

\[
\|\xi\| = \int_0^T |\xi_1(0, t)| dt = \int_0^T \left| \sum_{n=0}^{\infty} \psi_n(t) \right| dt.
\]

From Eq. (7), we immediately conclude that, for \( n \neq 0 \),

\[
\psi_n(t) = e^{int} a_n + e^{-int} b_n,
\]

where \( a_n \) and \( b_n \) are constants. For \( n = 0 \), we have \( \psi_0(t) = a_0 + b_0 t \). It is clear that, for \( n \neq 0 \),

\[
a_n = \frac{1}{2} \left( \psi_n + \frac{\phi_n}{n} \right), \quad b_n = \frac{1}{2} \left( \psi_n - \frac{\phi_n}{n} \right),
\]

where \( \phi_n \) is the \( n \)th Fourier coefficient of \( \phi \), and

\[
a_0 = \psi_0, \quad b_0 = \phi_0.
\]

The Fourier coefficients of \( \phi \), like those of \( \psi \), are real and even with respect to \( n \).

It follows from the above equations that

\[
\xi_1(0, t) = \sum_{n \neq 0} \left( \psi_n \cos nt + \frac{\phi_n}{n} \sin nt \right) + \psi_0 + \phi_0 t,
\]

in agreement with the d’Alembert formula

\[
\xi_1(x, t) = \frac{1}{2} (\xi_1(x, t, 0) + \xi_1(x, t, 0)) - \frac{1}{2} \int_{-t}^{t} \xi_0(y, 0) dy
\]

for the solution \( \xi_1(x, t) \) of the wave equation.

### 3.1. Natural Norm in the Dual Space

In view of (18), we conclude that

\[
\|\xi\| = \|\xi\|_T = \int_0^T \left| \sum_{n \neq 0} \left( \psi_n \cos nt + \frac{\phi_n}{n} \sin nt \right) + \psi_0 + \phi_0 t \right| dt.
\]
Suppose that $T \geq 2\pi$. Then the Banach norm $\|\xi\|$ is equivalent to the following more familiar Sobolev-type norm

$$\|\xi\| = \|\xi_1\|_1 + \|\eta\|_1.$$  

(21)

Here $\|g\|_1 = \int_{-T/2}^{T/2} |g|\, dt$ is the usual $L_1$-norm, and $\eta(t) = \int_0^t \xi_0(x)\, dx$. Indeed, denote by $f$ the integrand

$$f(t) = \sum_{n \neq 0} \left( \psi_n \cos nt + \phi_n \sin nt \right) + \psi_0 + \phi_0 t.$$  

(22)

The norm $\|\xi\|$ is equivalent to $\|f\|_1 = \int_{-T/2}^{T/2} |f|\, dt = \int_{-T/2}^{T/2} |f^+ + f^-|\, dt$, where

$$f^+(t) = \sum_{n \neq 0} \psi_n \cos nt + \psi_0 = \xi_1(t), \quad f^-(t) = \sum_{n \neq 0} \phi_n \sin nt + \phi_0 t = \eta(t)$$  

(23)

are even and odd parts of the function $f$, respectively.

Indeed, put $g(t) = f(t) - \phi_0 t$. Then $\|\xi\|$ is equivalent to $\int_0^T |g|\, dt + |\phi_0|$, while $\|f\|_1$ is equivalent to $\int_{-T/2}^{T/2} |g|\, dt + |\phi_0|$. Since the function $g$ is $2\pi$-periodic, and the intervals of integration have the length $T \geq 2\pi$, both the integrals $\int_0^T |g|\, dt$ and $\int_{-T/2}^{T/2} |g|\, dt$ are equivalent to $\int_0^{2\pi} |g|\, dt$.

The $L_1$-norms of functions $f^\pm$ can be estimated via the $L_1$-norm of $f$:

$$\|f^\pm\|_1 \leq \|f\|_1.$$  

(24)

Therefore, $\|\xi\| = \|f^+\|_1 + \|f^-\|_1 \leq 2\|f\|_1$. On the other hand, it is obvious that

$$\|f\|_1 \leq \|f^+\|_1 + \|f^-\|_1 = \|\xi_1\|_1 + \|\eta\|_1 = \|\xi\|'.$$  

(25)

We conclude that the norms $\|\xi\|$ and $\|\xi\|$ are equivalent indeed. Therefore, if $T \geq 2\pi$, then the dual space to the Banach space with norm $\|\xi\|$ coincides with the space of pairs $\tilde{f} = (f_0, f_1)$, where $\frac{\partial f_0}{\partial x} \in L_\infty$ and $f_1 \in L_\infty$. Thus, it is possible to damp the string, where the initial state $\tilde{f} = (f_0, f_1)$ has these properties, by a bounded load applied to a fixed point. Here by damping we mean the complete stop, when not only the oscillations, but also the displacement of the string as a whole is forbidden.

**Remark.** Our arguments show that the equivalence class of the norm $\|\xi\|_T$ does not depend on $T$ provided that $T \geq 2\pi$. Theorem 1 and Theorem 2 give a more quantitative statement of this independence of $T$.

### 3.2. Damping the Oscillations

In order to deal with damping oscillations only, it suffices to make an analog of previous computations in the quotient-space $S/C$. The corresponding support functions are almost the same as those previously found. We just have to assume that the zero-mode coefficients $\phi_0, \psi_0$ of the dual vectors $\xi$ are zero. Then, the formula for support functions of the corresponding reachable sets $\overline{D}(T)$ takes the following form:

$$H_{\overline{D}(T)}(\xi) = \int_0^T \left| \sum \left( \psi_n \cos nt + \phi_n \sin nt \right) \right| \, dt = \int_0^T \left| \xi_1(y) + \int_0^y \xi_0(x)\, dx \right| \, dy.$$  

(26)

Basically the same (but simpler) arguments that were used in the previous subsection, Subsection 3.1 prove that the states $\tilde{f} = (f_0, f_1)$, where $\frac{\partial f_0}{\partial x} \in L_\infty$ and $f_1 \in L_\infty$, are exactly those that can be damped.
4. THE SHAPE OF THE REACHABLE SET $D(T)$

Following [1], one can easily find an asymptotic formula for the above support function (20). For $T > 0$, define a linear isomorphism $C(T) : S → S$ by $C(T)f = \frac{1}{T}(f_0, f^T_0)^*$, where

$$g^T(t) = \sum_{n \neq 0} g_n \cos nt + \frac{1}{T}g_0, \quad \text{if } g(t) = \sum_{n \neq 0} g_n \cos nt + g_0.$$  \hfill (27)

It is clear that

$$H_{C(T)D(T)}(\xi) = \frac{1}{T} \int_0^T \left| \sum_{n \neq 0} \left( \psi_n \cos nt + \frac{\phi_n}{n} \sin nt \right) + \psi_0 + \frac{\phi_0}{T} t \right| dt,$$  \hfill (28)

where $\xi$ is the pair $(\xi_0, \xi_1)$, and $\xi_0(x) = \sum \phi_n \cos nx$, $\xi_1(x) = \sum \psi_n \cos nx$.

We then have the following precise statement.

**Theorem 1.** Consider problem (1) of the introduction that corresponds to the terminal manifold $C = 0$. Then, as $T → ∞$, we have the following limit formula:

$$\lim_{T → ∞} H_{C(T)D(T)}(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \left| \sum_{n \neq 0} \left( \psi_n \cos nt + \frac{\phi_n}{n} \sin nt \right) + \psi_0 + \phi_0 t \right| dt \, d\theta.$$  \hfill (29)

**Proof.** An easy proof follows the arguments used in the basic results of [1], which we reproduce here for the convenience of the reader. It suffices to consider $T$ of the form $T = 2\pi N$, $N ∈ \mathbb{Z}$. Then we can rewrite Eq. (28) in the following form:

$$H_{C(T)D(T)}(\xi) = \frac{1}{2\pi N} \sum_{i=0}^{N-1} \int_0^{2\pi} \left| g(t) + \phi_0 \left( \frac{i}{k} + \frac{t}{2\pi N} \right) \right| dt,$$  \hfill (30)

where

$$g(t) = \sum_{n \neq 0} \left( \psi_n \cos nt + \frac{\phi_n}{n} \sin nt \right) + \psi_0$$  \hfill (31)

is a $2\pi$-periodic function. We then note that the right-hand side of Eq. (30) equals

$$\frac{1}{2\pi N} \sum_{i=0}^{N-1} \int_0^{2\pi} \left| g(t) + \phi_0 \frac{i}{k} \right| dt + O\left( \frac{1}{N} \right),$$  \hfill (32)

which is the Riemann sum for the integral over $[0, 1]$ of the continuous function

$$\tau \mapsto \frac{1}{2\pi} \int_0^{2\pi} |g(t) + \phi_0 \tau| dt.$$  \hfill (33)

Since the Riemann sums converge to the integral

$$\frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |g(t) + \phi_0 \tau| \, dt \, d\tau \quad \text{as } \quad N → ∞,$$  \hfill (34)

the assertion is proved.

Recall that the shape $\text{Sh}\Omega$ of a set $Ω ⊂ S$ is the orbit of the group of linear (topological) isomorphisms of the space $S$ acting on $Ω$. In terms of shapes, one can say that the limit shape

$$\text{Sh}_∞ = \lim_{T → ∞} \text{Sh}D(T)$$  \hfill (35)

is related to the convex body $Ω$ corresponding to the support function

$$H_Ω(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 \left| \sum_{n \neq 0} \left( \psi_n \cos nt + \frac{\phi_n}{n} \sin nt \right) + \psi_0 + \phi_0 \tau \right| dt \, d\tau.$$  \hfill (36)

This means that the shape of the convex set with the support function (36) coincides with $\text{Sh}_∞$. Similarly, in the reduced space $\overline{S}$, we have the following precise result.
Theorem 2. Consider problems (2)–(3) in the introduction that correspond to the terminal manifolds \( C = \mathbb{R} \times 0 \) or \( C = \mathbb{R}^2 \). Then the following limit formula holds:

\[
\lim_{T \to \infty} \frac{1}{T} H_{D(T)}(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum \left( \psi_n \cos nt + \frac{\phi_n}{n} \sin nt \right) \right| dt = \frac{1}{2\pi} \int_0^{2\pi} |\zeta(t)| dt,
\]

where \( \zeta(t) = \xi_1(t) + \int_0^t \xi_0(x) \, dx \).

Note that the operator of multiplication by \( (T)^{-1} \) in the quotient space \( \mathcal{S} \) is induced by the operator \( C(T) \), and Eq. (37) describes the limit shape \( \lim_{T \to \infty} \text{Sh}_{\mathcal{S}}(T) \). Denote by \( \Omega \) the convex body such that its support function is given by the right-hand side of (37):

\[
H_{\Omega}(\xi) = \frac{1}{2\pi} \int_0^{2\pi} \left| \sum \left( \psi_n \cos nt + \frac{\phi_n}{n} \sin nt \right) \right| dt = \frac{1}{2\pi} \int_0^{2\pi} |\zeta(t)| dt.
\]

According to Theorem 2, the set \( T\Omega \) is an approximation of \( D(T) \) if \( T \) is large.

5. DRY-FRICTION CONTROL

Our control design is based on the following idea: The optimal control at a state \( f \) implements the steepest descent in the direction normal to boundaries of the reachable sets \( D(T) \). Our control implements the steepest descent in the direction normal to boundaries of the approximate reachable sets \( T\Omega \), where \( \Omega \) is defined via Eq. (38). This means that, in the notation of Eq. (37), we have

\[
u(f) = -\text{sgn}(B, \xi) = -\text{sgn} \xi_1(0) = -\text{sgn} \zeta(0),
\]

where the momentum \( \xi \) is to be found from the equation

\[
T^{-1}f = \frac{\partial H_{\Omega}}{\partial \xi}(\xi),
\]

or, equivalently, \( f = (f_0, f_1) \), where

\[
T^{-1}f_0(x) = -\int_0^x (\text{sgn} \zeta(y))^+ dy, \quad T^{-1}f_1(x) = (\text{sgn} \zeta(x))^+,
\]

where the notation \( f^\pm \) stands for even/odd part of the function \( f \):

\[
f^\pm(x) = \frac{1}{2}(f(x) \pm f(-x)).
\]

These identities are to be treated as inclusions, because the sgn-mapping is multivalued. Namely, their precise meaning is

\[
T^{-1}f_0(x) = -\int_0^x \phi(y)^- dy, \quad T^{-1}f_1(x) = \psi(x)^+,
\]

where \( \phi(y) \in \text{sgn} \zeta(y) \) and \( \psi(x) \in \text{sgn} \zeta(x) \).
5.1. Duality Transform

We invoke a general duality transformation related to Eq. (40). To this end, we denote the function $H_\Omega$ simply by $H = H(\xi)$, and the factor $T$ by $\rho(f)$. Then the relation between $H$ and $\rho$ is similar to the Legendre transformation:

$$\langle \xi, \xi \rangle = \rho(f)H(\xi), \quad \rho(f) = \max_{H(\xi) \leq 1} \langle f, \xi \rangle, \quad H(\xi) = \max_{\rho(f) \leq 1} \langle f, \xi \rangle,$$

(44)

where the correspondence $f \to \xi$ has the following form:

$$f = \rho(f)\frac{\partial H}{\partial \xi}(\xi),$$

(45)

$$\xi = H(\xi)\frac{\partial \rho}{\partial f}(f).$$

(46)

Here $\xi$ and $f$ are the points at which the maxima in (44) are attained. These relations make sense provided that $H$ and $\rho$ are norms, i.e., convex functions positive homogeneous of degree 1 such that the sublevel sets $\{H(\xi) \leq 1\}$ and $\{\rho(f) \leq 1\}$ are convex bodies. These sublevels are mutually polar to each other. In other words, if $\Omega = \{\rho(f) \leq 1\}$ and $\Omega^\circ = \{H(\xi) \leq 1\}$, then $\Omega^\circ = \{f : \langle f, \xi \rangle \leq 1, \xi \in \Omega^\circ\}$, and vice versa. In the language of Banach spaces, the normed spaces $(V, \rho)$ and $(V^*, H)$ are dual to each other. The derivatives in Eq. (46) should be treated as subgradients. If the functions $H$ and $\rho$ are differentiable, then Eq. (46) has the classical meaning. If one of the functions $H$ and $\rho$ is differentiable and strictly convex, then so is the other one.

In the cases at hand, we need to calculate the dual function $\rho$ for the function $H = H_\Omega$ from Eq. (38). We then arrive at the following result.

**Theorem 3.** Consider problems (2)–(3) in the introduction that correspond to the terminal manifolds $C = \mathbb{R} \times 0$ or $C = \mathbb{R}^2$.

1. If $C = \mathbb{R}^2$, then $\rho(f) = 2\pi \left| \frac{\partial f_0}{\partial x} + f_1 \right|$, where the norm $|\phi|_\infty$ of a function $\phi$ on the torus $T = \mathbb{R}/2\pi\mathbb{Z}$ is $\inf_{c \in \mathbb{R}} \sup_{x \in T} |\phi(x) + c|$, where $c \in \mathbb{R}$ is an arbitrary constant.

2. If $C = \mathbb{R} \times 0$, then $\rho(f) = 2\pi \left| \frac{\partial f_0}{\partial x} + f_1 \right|$, where the norm $|\phi|_\infty$ is the sup-norm of the function $\phi$ on the torus $T = \mathbb{R}/2\pi\mathbb{Z}$.

**Proof.** We consider only problem (2) of Theorem 3, because problem (3) is quite similar. We note that, in case (1),

$$|\phi|_\infty = \frac{1}{2} (\sup \phi - \inf \phi).$$

(47)

We note that $\frac{\partial f_0}{\partial x}$ is an odd function, while $f_1$ is even. This implies that the norm $\rho(f)$ is equivalent to (although does not coincide with) $\max \left( \left| \frac{\partial f_0}{\partial x} \right|, |f_1|_\infty \right)$.

To prove the theorem, we define $\rho_0(f)$ by the formula

$$\rho_0(f) = \left| \frac{\partial f_0}{\partial x} + f_1 \right|,$$

(48)

and show that $H(\xi) = \frac{1}{2\pi} \max_{\rho_0(f) \leq 1} \langle f, \xi \rangle$. To this end, let us set

$$\psi_0(t) = \int_0^t \xi_0(x)dx, \quad \psi_1(t) = \xi_1(t), \quad \psi = \psi_0 + \psi_1,$$

(49)

and

$$\phi_0 = \frac{\partial f_0}{\partial x}, \quad \phi_1 = f_1, \quad \phi = \phi_0 + \phi_1.$$
Denote by \( \int_T f \), where \( T = \mathbb{R}/2\pi \mathbb{Z} \), the normalized integral \( \frac{1}{2\pi} \int_0^{2\pi} f(t) dt \). Recall that \( \langle f, \xi \rangle \) stands for \( \int_0^{2\pi} \langle f(t), \xi(t) \rangle dt \).

Then
\[
\int_T \phi \psi = \int_T \phi_0 \psi_0 + \int_T \phi_1 \psi_1 = \frac{1}{2\pi} \langle f, \xi \rangle,
\]
because the integrals \( \int_T \phi_0 \psi_0 dt \) and \( \int_T \phi_1 \psi_0 dt \) vanish (as integrals of odd functions over \( T = \mathbb{R}/2\pi \mathbb{Z} \)). It is clear from Eq. (51) that the maximum of \( \int_T \phi \psi dt \) taken over \( \phi \) such that \( |\phi|_\infty \leq 1 \), coincides with the maximum of \( \langle f, \xi \rangle \) taken over \( f \) such that \( \rho_0(f) \leq 1 \). However, it is trivial that the maximum of \( \int_T \phi \psi \) is \( \int_T |\psi| \). The last value, according to Eq. (38), is equal to \( H(\xi) \).

One can regard our computation of the norm \( \rho \) as an a priori estimate for solutions of the wave equation.

**Theorem 4.** Suppose that \( \dot{f} = (f_0, f_1) \) is a solution of the Cauchy problem
\[
\frac{\partial \dot{f}}{\partial t} = Af + Bu, \quad |u| \leq 1, \quad f(0) = 0,
\]
where \( B = (0, \delta) \) and \( u = u(t) \). If \( T \geq 2\pi \), we then have
\[
\rho(\dot{f}(T)) = 2\pi \left| \frac{\partial f_0}{\partial x} + f_1 \right|_\infty \leq T.
\]

In particular, the following a priori bound holds:

**Corollary 1.** Suppose that \( \dot{f} = (f_0, f_1) \) is a solution of \( \frac{\partial \dot{f}}{\partial t} = Af + Bu, \quad |u| \leq 1 \), while \( \ddot{f} \) is control-free: \( \frac{\partial \ddot{f}}{\partial t} = A\ddot{f} \), and \( \ddot{f}(0) = f(0) \). Then, provided that \( T \geq 2\pi \), we have
\[
\left| (f_1 - \ddot{f}_1)(T) \right|_\infty \leq \frac{1}{2\pi} T.
\]

### 6. COMPUTATION OF THE BASIC CONTROL

Consider Problem (2) in the introduction corresponding to the terminal manifolds \( \mathcal{C} = \mathbb{R} \times 0 \). In order to find the control, we need to solve Eqs. (41) as explicitly as possible. In other words, we have to find a function \( \zeta \) such that
\[
T^{-1} \frac{\partial f_0}{\partial x}(x) \in -\text{sgn} \zeta(x), \quad T^{-1} f_1(x) \in \text{sgn} \zeta(x).
\]

Our discussion of duality, in particular, Eq. (46) shows that the solution is given by
\[
T = \rho(\dot{f}) = 2\pi \left| \frac{\partial f_0}{\partial x} + f_1 \right|_\infty, \quad \zeta = \frac{\partial \rho}{\partial f_1}(\dot{f}).
\]

The final expression for the control has the following form:
\[
u(\dot{f}) = -\text{sgn} \zeta(0) = -\text{sgn} f_1(0),
\]
where we take into account Eq. (41) and the vanishing at 0 of the odd part of the function \( x \mapsto \text{sgn} \zeta(x) \). Thus, we obtain indeed a generalization of the dry friction, since it acts with maximal possible amplitude against the velocity, because \( f_1(0) \) is exactly the velocity of the point at which the load is applied. The control (57) leads to a nonlinear wave equation of the following form:
\[
\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2} - \text{sgn} \left( \frac{\partial f}{\partial t}(0) \right) \delta,
\]
governing the damping process. There are no standard existence and uniqueness theorems for the Cauchy problem for Eq. (58). The problem is to extend, to the infinite-dimensional case, the classical results of Filippov [11] concerning the existence and the uniqueness theorem of Boglaevsky [12].
7. RESTATEMENT OF THE MODEL

Previous considerations stress the importance of the function

\[ g = \frac{\partial f_0}{\partial x} + f_1. \]  

(59)

The knowledge of this function is almost equivalent to the knowledge of both the functions \( f_0 \) and \( f_1 \). Indeed, the function \( \frac{\partial f_0}{\partial x} \) is odd, and \( f_1 \) is even. Therefore, the knowledge of these functions is equivalent to the knowledge of the function \( g \). On the other hand, the knowledge of \( \frac{\partial f_0}{\partial x} \) gives a complete information on \( f_0 \) up to an additive constant. This constant is irrelevant if the goal of our damping process is to stop oscillation, or to stop motion of the string at an unspecified point.

The law of the controlled motion given by Eqs. (1), (57), and (58) can be restated as follows:

\[ \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) g(x, t) = \delta(x) u(t), \quad |u| \leq 1. \]  

(60)

This form of governing law has its merits. In particular, it can be made rather explicit. One can rewrite Eq. (60) as follows:

\[ \frac{d}{dt} g(x - t, t) = \delta(x - t) u(t), \]  

(61)

which means that

\[ g(x - t, t) = g(x, 0) + \int_0^t \delta(x - s) u(s) ds = g(x, 0) + \sum_J u(x + 2k\pi), \]  

(62)

where the summation ranges over the set \( I = I_t \) of \( k \in \mathbb{Z} \) such that \( x + 2k\pi \in [0, t] \). By the change of variables \( z = x - t \), we arrive at:

\[ g(z, t) = g(z + t, 0) + \sum_J u(z + t + 2k\pi), \]  

(63)

where the summation is over the set \( J = J_t \) of \( k \in \mathbb{Z} \) such that \( z + 2k\pi \in [-t, 0] \).

Equation (63) should be treated as follows. Here \( g \) is a bounded measurable function of \( x, t \) and \( u \) is a bounded measurable function of \( t \), the curve \( t \mapsto g(\cdot, t) \) is continuous as a mapping from real line to distributions depending on the space variable \( x \). Equation (63) does not hold pointwise, but expresses an equation in the space of curves of distributions with respect to \( x \).

8. EXISTENCE OF THE MOTION UNDER DRY-FRICTION CONTROL

We must obtain an existence theorem for the initial value problem for the nonlinear wave equation (58). By using transformation (59), we reduce the task to the solution of the functional equation

\[ g(z, t) = g(z + t, 0) - \sum_J \text{sgn} g(0, z + t + 2k\pi), \]  

(64)

which in turn can be reduced to the search for the function \( g(0, t), t \geq 0 \), because this defines the control law \( u(t) = -\text{sgn} g(0, t) \).

This is quite nontrivial, because the function \( \phi(t) = g(0, t) \) we are looking for should satisfy a functional equation. The first step in establishing the desired functional equation is to make Eq. (64) hold pointwise. It is explained in the previous section that this equation expresses an equation in the space of curves of distributions of \( x \). In order to make Eq. (64) hold pointwise, we consider the one-sided averaging operators

\[ \text{Av}^{\pm \epsilon} : f(z, t) \mapsto \frac{1}{\epsilon} \int_0^\epsilon f(z + x, t) dx, \quad \text{Av}^\pm : f \mapsto \lim_{\epsilon \to 0} \text{Av}^{\pm \epsilon}(f), \]  

(65)
and a more standard two-sided operator

\[ Av^\varepsilon : f(z,t) \mapsto \frac{1}{2 \varepsilon} \int_{t-\varepsilon}^t f(z + x,t)dx, \quad Av : f \mapsto \lim_{\varepsilon \to 0} Av^\varepsilon(f). \] (66)

We note that, according to the Lebesgue differentiation theorem, the limit averaging operators \( Av^\pm \) and \( Av \) are identities when applied to any L1-function. The reason for application of these operators is that, if operators \( Av^\pm \) are applied to the right-hand and the left-hand sides of Eq. (64), the equation thus obtained holds pointwise. In particular,

\[ Av^\varepsilon(g)(0,t) = Av^\varepsilon(g)(t,0) \]

\[ - \sum \frac{1}{2 \varepsilon} \int_{t-\varepsilon}^t u(z + t + 2k\pi)1_{[-t,0]}(z + t + 2k\pi)dz \]

\[ = Av^\varepsilon(g)(t,0) - \frac{1}{2} Av^-\varepsilon(u)(t) - \sum_{k\neq 0} Av^\varepsilon(u)(t + 2k\pi), \] (67)

where \( u(t) = -\text{sgn} g(0,t) \), the summation ranges over the set of \( k \in \mathbb{Z} \) such that \( 2k\pi \in [-t,0] \), and \( \varepsilon < t \).

To write out the desired functional equation, consider the function \( \phi(t) = Av^\varepsilon(g)(0,t) \). It follows from Eq. (67), by passing to the limit \( \varepsilon \to 0 \), that this function does exist in \( L_\infty(0,\infty) \). It also follows from Eq. (67) that

\[ \phi(t) = G(t) - \frac{1}{2} \text{sgn} \phi(t) - \sum_{k\neq 0,2k\pi \in [-t,0]} \text{sgn} \phi(t + 2k\pi), \] (68)

where \( G(t) = g(t,0) \) is the given initial 2\( \pi \)-periodic function. The solution of this equation gives at the same time a rigorously defined solution to the nonlinear wave equation (58).

Thus, we have to solve the equation

\[ \phi(t) + \frac{1}{2} \text{sgn} \phi(t) + \sum_{k\neq 0,2k\pi \in [-t,0]} \text{sgn} \phi(t + 2k\pi) = G(t), \] (69)

where \( G \) is a given function, and \( \phi \) is an unknown. Note that the function \( \phi \) need not be periodic. It should be defined for nonnegative \( t \). Note also that, if \( t < 2\pi \), the last equation reduces to a very simple one:

\[ \phi(t) + \frac{1}{2} \text{sgn} \phi(t) = G(t), \] (70)

which, obviously, has a unique solution, since the mapping \( x \mapsto x + \text{sgn} x \) is a strictly monotone increasing (multivalued) function. More explicitly, the solution is \( \phi(t) = G(t) - \frac{1}{2} \) if \( G(t) > \frac{1}{2} \) and \( \phi(t) = G(t) + \frac{1}{2} \) if \( G(t) < -\frac{1}{2} \). Otherwise, \( \phi(t) = 0 \). Note that \( |\phi(t)| \leq |G(t)| \) in the interval \([0,2\pi]\) under consideration of values of the argument \( t \).

It is better to rewrite the above equation (70) in the form

\[ \phi(t) + \frac{1}{2} v(t) = G(t), \quad v(t) = \text{sgn} \phi(t), \] (71)

where the sgn-function is regarded as a multivalued one: \( \text{sgn}(0) = [-1,1] \). Then the a priori multivalued function \( \text{sgn} \phi(t) \) is defined by (71) uniquely. If \( t < 2\pi \), we obtain from Eq. (69) and from the periodicity of \( G(t + 2\pi) = G(t) \) that

\[ \phi(t + 2\pi) + \frac{1}{2} \text{sgn} \phi(t + 2\pi) = G(t) - \text{sgn} \phi(t), \] (72)

which allows us to extend, by the preceding arguments, the function \( \phi(t) \) from \( t \in [0,2\pi) \) to any positive value of \( t \). By the arguments used above, we obtain \( |\phi(t)| \leq |G(t)| \) for all \( t \geq 0 \). We then have the following precise statement.
Theorem 5. The Cauchy problem for the nonlinear wave equation
\begin{equation}
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) g(x, t) = -\delta(x) \operatorname{sgn} g(0, t),
\end{equation}
where \(g(x, 0)\) is a given bounded (Borel-measurable) function, possesses a unique bounded solution for \(t \geq 0\). The functions \(\phi(t) = g(0, t)\) and \(u(t) = -\operatorname{sgn} g(0, t)\) form the unique solution of the functional equation (69).

The flow \(g = g(\cdot, 0) \mapsto \Phi_t(g) = g(\cdot, t)\) in the space of measurable bounded functions, where \(t \geq 0\), will be called the dry-friction flow.

9. ASYMPTOTIC OPTIMALITY: PROOF

9.1. Asymptotic Optimality of Control: Polar-Like Coordinate System

Here we present reasons, at an intuitive level, for the asymptotic optimality of the control law (5). The rigorous treatment of asymptotic optimality is performed below. We define a polar-like coordinate system, well suited for the representation of the motion under the control \(u\). Every state \(0 \neq f\) of the string can be represented uniquely as
\begin{equation}
f = \rho \phi, \text{ where } \rho = \rho(x) \text{ is a positive factor, and } \phi \in \partial \Omega. \tag{74}
\end{equation}
The pair \(\rho, \phi\) is the coordinate representation for \(x\), and \(\rho(\phi) = 1\) is the equation of the “sphere” \(\omega = \partial \Omega\). It is important that the set \(\omega\) is invariant under free (uncontrolled) motion of our system (6). This follows from the similar invariance of the support function \(H_{\Omega}(\rho)\) under the evolution governed by \(\dot{\rho} = -A^*p\). The latter invariance is clear, because the support function is an ergodic mean of the function \(\xi_1(0, t)\) under the free motion. This implies the invariance of the dual function \(\rho = \rho(f)\), and thus \(\langle \partial \rho/\partial f, A f \rangle = 0\). Therefore, under the control \(u\) in (6.3), the total (Lie) derivative of \(\rho\) takes the following form:
\begin{equation}
\dot{\rho} = \left\langle \frac{\partial \rho}{\partial f}, A f + Bu \right\rangle = \left\langle \frac{\partial \rho}{\partial f}, Bu \right\rangle = -\left| \left\langle \frac{\partial \rho}{\partial f}, B \right\rangle \right|, \tag{75}
\end{equation}
where the last identity holds because \(\partial \rho/\partial f\) is the outer normal to the set \(\rho \Omega\). In particular, the “radius” \(\rho\) is a monotone nonincreasing function of time. Moreover, the right-hand side of Eq. (75) is necessarily equal to -1 if \(f_1(0) \neq 0\). For any other admissible control, we have
\begin{equation}
\dot{\rho} \geq -\left| \left\langle \frac{\partial \rho}{\partial f}, B \right\rangle \right|. \tag{76}
\end{equation}
The evolution of \(\phi\), by virtue of system (6), is described by
\begin{equation}
\dot{\phi} = A \phi + \frac{1}{\rho} (Bu - \phi \dot{\rho}) = A \phi + \frac{1}{\rho} \left( Bu + \phi \left| \left\langle \frac{\partial \rho}{\partial f}, B \right\rangle \right| \right). \tag{77}
\end{equation}
We note that the right-hand side \(\left| \left\langle \frac{\partial \rho}{\partial f}(\phi), B \right\rangle \right| \) of Eq. (75) equals \(\left| \left\langle \frac{\partial \rho}{\partial f}(\phi), B \right\rangle \right|\). Thus, the evolution of the right-hand side of Eq. (75) is determined by the evolution of \(\phi\) by Eq. (77). It is clear that, if \(\rho\) is large, then the second term on the right-hand side of (77) is \(O(1/\rho)\) and affects the motion of \(\phi\) over the “sphere” \(\omega\) only slightly. Our next task is to compute approximately the “ergodic mean”
\begin{equation}
E_T = \frac{1}{T} \int_0^T \left| \left\langle \frac{\partial \rho}{\partial f}, B \right\rangle \right| \, dt. \tag{78}
\end{equation}
of the right-hand side of Eq. (75) provided that \( \rho \) is large. Here \( B \) is a constant vector, while, according to the preceding arguments, the vector function

\[
\frac{\partial \rho}{\partial f}(t) := \frac{\partial \rho}{\partial f}(f(t))
\]  

(79)

behaves approximately as \( e^{A^*t} \frac{\partial \rho}{\partial f}(0) \). Therefore, the ergodic mean \( E_T \) is well approximated by

\[
E_T = \frac{1}{T} \int_0^T \langle e^{A^*t} \xi, B \rangle \, dt,
\]  

(80)

where \( \xi = \frac{\partial \rho}{\partial f}(0) \). We know from Theorem 2 that, as \( T \to \infty \), the ergodic mean \( E_T \) tends to \( H(\xi) = H(\frac{\partial \rho}{\partial f}) \). However, according to one of the basic “duality relations” (46), we know that \( H(\frac{\partial \rho}{\partial f}) = 1 \).

Therefore, we conclude, by using the abbreviation \( \rho(t) = \rho(\frac{\partial \rho}{\partial f}(f(t))) \), that

\[
\frac{(\rho(0) - \rho(T))}{T} = 1 + o(1) \quad \text{as} \quad T \to \infty,
\]  

(81)

whenever we use the dry-friction control (57).

Under any other admissible control, according to Theorem 4,

\[
\frac{(\rho(0) - \rho(T))}{T} \leq 1 + o(1).
\]  

(82)

The relations (81) and (82) express the asymptotic optimality we sought for.

9.2. Formal Proof

Here we prove the asymptotic optimality of the control (57) using the function \( g(x, t) \) of Eq. (59). The law of motion (63) is

\[
g(z, t) = g(z + t, 0) - \sum_{J} \text{sgn} g(0, z + t + 2k\pi),
\]  

(83)

where the set \( J = J_t \) consists of \( k \in \mathbb{Z} \) such that \( z + 2k\pi \in [-t, 0] \). The functional \( \rho \) has the form \( \rho(g) = 2\pi \sup_{x \in \mathbb{R}/2\pi\mathbb{Z}} |g(x, t)| \). The control \( \text{sgn} g(0, z + t) \) is not affected by the scaling transformation

\[
g \mapsto \Phi = g/\rho.
\]  

(84)

However, if \( \rho \) is large, then our previous considerations reveal that the function \( \Phi = g/\rho \) moves in an almost uncontrollable way. This means that, approximately,

\[
\Phi(x, t) \approx \Phi(x + t, 0),
\]  

(85)

so that we come to the approximate equation

\[
\text{sgn} g(0, z) \approx \text{sgn} g(z, 0).
\]

More precisely, suppose that, on the time-interval \([0, T]\), we have \( \rho(g_t) \geq 2\pi M \), where \( M \) is a (large) constant. In view of Eq. (69), we have

\[
g(0, t) = g(t, 0) - \frac{1}{2} \text{sgn} g(0, t) - \sum_{k \neq 0, 2k\pi \in [-t, 0]} \text{sgn} g(0, t + 2k\pi),
\]  

(86)
and therefore,
\[ |g(0, t) - g(t, 0)| \leq \frac{t}{2\pi}. \]

(87)

Since \( \rho(g) \geq M \), there exist points \( x \in \mathbb{R}/2\pi\mathbb{Z} \) at which either \( g(x, 0) \geq M-1 \) or \( g(x, 0) \leq -(M-1) \). Assume for definiteness that \( g(x, 0) \geq M-1 \). Then, in view of Eq. (87), \( \text{sgn} g(0, t + 2k\pi) = +1 \) for \( t \in [0, T] \) provided that \( \frac{T}{2\pi} \leq M-1 \). For instance, this is the case if \( M \) is large and \( T = O(\sqrt{M}) \).

In view of (83), this means that
\[
g(z - t, t) = g(z, 0) - \frac{t}{2\pi} \text{sgn} g(z, 0) + O(1),
\]

(88)

where \(|O(1)| \leq 1\), and \( g(z, 0) \geq M-1 \). This implies
\[
\sup_z g(z, t) = \sup_z g(z, 0) - \frac{t}{2\pi} + O(1),
\]

(89)

since \( \text{sgn} g(z, 0) = +1 \) if \( g(z, 0) \geq M-1 \). Since \( \rho(t) = 2\pi \sup_x g(z, t) \), we obtain the approximate equation
\[
(\rho(0) - \rho(t))/t = 1 + O(1/t),
\]

(90)

provided that the length \( T \) of the time interval is less than \( 2\pi(M-1) \).

Partitioning any sufficiently long interval of time \([0, T]\) into many equal intervals of length \( \leq 2\pi(M-1) \), we come to the following precise result.

**Theorem 6.** Consider the evolution \( \rho(t) = \rho(g_t) \) of \( \rho \) under the control (83). Let
\[
M = \min\{\rho(0), \rho(T)\}.
\]

(91)

Suppose that \( M \to +\infty \) as \( T \to +\infty \). Then
\[
(\rho(0) - \rho(T))/T = 1 + O(1/T + 1/M).
\]

(92)

Under any other admissible control,
\[
(\rho(0) - \rho(T))/T \leq 1 + O(1/T + 1/M).
\]

(93)

The preceding arguments of this section prove (92) and the assertion (93) following from Theorem 4.

10. FEATURES OF THE DRY-FRICTION FLOW

Methods used in Section 8 allow us to reveal the basic properties of the dry-friction flow. In particular, it is possible to derive the asymptotic optimality of the dry-friction flow directly from Eqs. (70)–(72).

10.1. Far from the Target

Indeed, it follows from equations (70), (71) that, if \( \sup_x g(x, 0) > 1/2 \), then
\[
\sup_{t \in [0, 2\pi]} \phi(t) = \sup_x g(x, 0) - 1/2.
\]

The same estimates hold with essential supremum vraisup instead of the plain sup. In particular,
\[
\|\phi_0\| = \|g\| - \frac{1}{2},
\]

(94)
where \( \phi_0 \) is the restriction of \( \phi \) to the interval \([0, 2\pi]\), \( g(x) = g(x, 0) \) is the initial data, and \( \|f\| \) stands for the \( L_\infty \)-norm of \( f \) over the same interval \([0, 2\pi]\). It follows from equation (72) that
\[
\|\phi_1\| = \|\phi_0\| - 1, \tag{95}
\]
where \( \phi_1 \) is the restriction of \( \phi(t + 2\pi) \) to the interval \([0, 2\pi]\). Denote by \( F_\tau \), where \( \tau \geq 0 \), the shift of the argument, \((F_\tau \phi)(t) := \phi(t + \tau)\). It is clear from the definition of the dry-friction flow \( \Phi_\tau \) and Eq. (94) that
\[
\|F_\tau \phi\| = \|\Phi_\tau g\| - \frac{1}{2}, \tag{96}
\]
provided that \( \|\Phi_\tau g\| \geq \frac{1}{2} \). Equation (95) means that
\[
\|F_{2\pi} \phi\| = \|\phi\| - 1. \tag{97}
\]
These identities imply that, for positive integers \( k \),
\[
\|\Phi_{2k\pi} g\| = \|g\| - k, \tag{98}
\]
provided that \( \|g\| \geq k + \frac{1}{2} \). This can be restated in the notation of the preceding section, Section 9, as follows:
\[
\frac{\rho(0) - \rho(2k\pi)}{2k\pi} = 1, \tag{99}
\]
and gives a very precise form of the asymptotic optimality (see Theorem 6).

10.2. Near the Target

On the contrary, if \( \|g\| \leq \frac{1}{2} \), then the dry-friction flow does not help to damp the string. Under this condition, we obtain from Eq. (70) and Eq. (71) that
\[
\phi_0 = 0, \quad \text{sgn} \, \phi_0 = -2g, \tag{100}
\]
and from equation (72) that
\[
\phi(t + 2\pi) = 0, \quad \text{sgn} \, \phi(t + 2\pi) = 2g(t) = 2g(t, 0) = -\text{sgn} \, \phi(t), \tag{101}
\]
whenever \( t \in [0, 2\pi] \). This means that the dry-friction flow is given by solution of the Cauchy problem for the linear equation
\[
\left( \frac{\partial}{\partial t} - \frac{\partial}{\partial x} \right) g(x, t) = -2(-1)^k \delta(x) g(t, 0) \quad \text{if} \quad t \in [2k\pi, 2(k + 1)\pi). \tag{102}
\]
The norm \( \|\Phi_{2k\pi} g\| = \|\phi(t + 2k\pi)\| \) does not depend on the positive integer \( k \).

It is easy to solve the Cauchy problem (102) explicitly. The solution \( g(x, t) \) is determined via the initial data \( G(x) = g(x, 0) \) as \( g(x, t) = (-1)^k G(x + t) \) provided that \( x \in [0, 2\pi] \) and \( t \in [2k\pi, 2(k + 1)\pi] \).

11. CONCLUSION

The subject of the present paper arose as a natural extension of our preceding study of finite systems of oscillators [2, 3]. There we stressed the case of nonresonant systems. Here we study the string, which is an infinite system of highly resonant oscillators. In both cases, the basic new results are the existence and uniqueness of the motion under the dry-friction control and the asymptotic optimality of the control.

The results of both studies are similar, but the methods are rather different. The similarity of studies is especially far reaching in the first parts of both of them, where we investigate the reachable sets and limiting capabilities of admissible controls. In some aspects, the present case of a string is simpler than that of finitely many nonresonant oscillators. For example, we do not use any special function, like the hypergeometric function in the sense of Gel’fand or the DiPerna–Lions theory [14], which play a decisive part in control of finite system of oscillators (see [3]). On the other hand, the infinite-dimensional case is related to well-known and quite real-analytic difficulties which are present (for details, see Appendix III).
Therefore, the control is piecewise differentiable with jumps at the form
\( g(x, t) = \delta(x)u(t), \quad u = -\sgn(g(0, t)), \) (A1)
the control is not uniquely defined by the current state of the string, i.e., \( g(0, t) \equiv 0. \)

In order to construct a motion of this kind, we use the spectral decompositions
\[ g(x, t) = \sum g_\mu(x) e^{i\mu t}, \quad \text{and} \quad u(t) = \sum u_\mu e^{i\mu t}. \]

This almost periodic function should be bounded: \(|u| \leq 1\) Then the functions \( g_\mu \) should satisfy
\[ i\mu g_\mu - \frac{\partial}{\partial x} g_\mu = -\delta u_\mu \quad \text{and} \quad g_\mu(0) = 0. \] (A2)

The first equation (A2) guarantees that 0 is a point of discontinuity of \( g_\mu \), so that the second equation (A2) should be treated cautiously. In fact, the discussion of Sec. 8 shows that we have to take \( \frac{1}{2}(g_\mu(0^+) + g_\mu(0^-)) \) for \( g_\mu(0) \). Indeed, according to the first equation (A2), the function \( g_\mu \) is piecewise differentiable with jumps at \( x = 0 \). Therefore, the \( 2\pi \)-periodic function \( g_\mu \) should have the form
\[ g_\mu(x) = C_\mu e^{i\mu x} \quad \text{for} \quad x \in [0, 2\pi), \] (A3)
where the constant is \( C_\mu = (1 - e^{2\pi i\mu})^{-1} u_\mu \). The condition \( g_\mu(0) = 0 \) gives
\[ 1 + e^{2\pi i\mu} = 0, \] (A4)
which implies that \( \mu \) should have the form \( \mu = \frac{1}{2} \nu \), where \( \nu \) is an odd integer, and \( C_\mu = u_\mu/2. \)

Therefore, the control \( u(t) = \sum u_\mu e^{i\mu t} \) is not simply almost periodic but \( 4\pi \)-periodic. Moreover, we have
\[ g(x, t) = \sum g_\mu(x) e^{i\mu t} = \frac{1}{2} \sum u_\mu e^{i\mu(t+x)} = \frac{1}{2} u(t) + x \quad \text{for} \quad x \in [0, 2\pi). \] (A5)

**APPENDIX II. SINGULAR ARCS, II**

It is possible to analyze, in a more general fashion, the singular arcs of the motion governed by the second-order nonlinear wave equation (58). We shall do this by the direct finite-dimensional approximation of the string by finitely many harmonics. These arcs are the time-intervals of the motion, where the semi-flow \( \tilde{t} \rightarrow \phi_t(\tilde{t}) = \tilde{t} \) leaves the “hyperplane” \( f_1(0) = 0 \) invariant (on which the control \( u = -\sgn(f_1(0)) \) is not uniquely defined). We put the word “hyperplane” into quotation marks because the value \( f_1(0) \) is badly defined within the natural state space of the string. In order to be correct, we introduce a cut-off. Namely, we start with an approximation of the string with first \( N \) harmonics:
\[ f_0(x) = \sum_{k=0}^{N} a_k \cos kx \quad \text{modulo constants} \quad a_0, \quad f_1(x) = \sum_{k=0}^{N} b_k \cos kx, \] (A6)
and consider, in this \( 2N + 1 \)-dimensional space, the correctly defined ODE
\[ \dot{a}_0 = b_0, \quad \dot{a}_k = b_k, \quad \dot{b}_0 = u, \quad \dot{b}_k = -k^2 a_k + 2u, \quad k = 1, \ldots, N. \] (A7)
which is the natural finite-dimensional approximation of the wave equation.

We require that \( \sum_{k=0}^{N} b_k = 0 \), which is a restatement of the condition \( f_1(0) = 0 \). This immediately implies that \( u = \frac{1}{2N+1} \sum_{k=0}^{N} k^2 a_k \). Thus, we obtain a linear differential equation

\[
\dot{a}_k = b_k, \quad \dot{b}_k = -k^2 a_k, \quad k = 1, \ldots, N,
\]

in the space \( \mathbb{R}^{2N} \) of sequences \( a_k, b_k, k = 1, \ldots, N \). We are going to solve system (A8) and then remove the cut-off; in other words, we pass to the limit \( N \to \infty \) in the solution thus obtained. To obtain the solution, we first find the “spectral decomposition” of the linear operator given by the right-hand side of Eq. (A8). The corresponding eigenvalue problem is as follows:

\[
\lambda a_l = b_l, \quad \lambda b_l = -l^2 a_l + \frac{2}{2N+1} \sum_{k=1}^{N} k^2 a_k, \quad l = 1, \ldots, N,
\]

which is equivalent to

\[
(l^2 + \lambda^2) a_l = \frac{2}{2N+1} \sum_{k=1}^{N} k^2 a_k, \quad l = 1, \ldots, N.
\]

This, in turn, is equivalent to the system

\[
a_k = \frac{R}{k^2 + \lambda^2}, \quad \frac{2}{2N+1} \sum_{k=1}^{N} \frac{k^2}{k^2 + \lambda^2} = 1.
\]

Here \( R = R_\lambda \) is an arbitrary \( k \)-independent constant. Thus, the eigenvalue problem reduces to the solution of

\[
\frac{2}{2N+1} \sum_{k=1}^{N} \frac{k^2}{k^2 + \lambda^2} = 1.
\]

It is easy to show that problem (A12) has \( 2N \) purely imaginary eigenvalues \( \lambda = i\mu \). Indeed, this statement is equivalent to the fact that the polynomial equation of degree \( N \)

\[
\frac{2}{2N+1} \sum_{k=1}^{N} \frac{k^2}{k^2 - t} = 1
\]

has \( N \) real roots. It is clear, at least when \( N \) is large, that there is a positive root \( t \) in the close vicinity of zero, and that the function \( t \mapsto \sum_{k=1}^{N} \frac{1}{k^2 - t} \) tends to \( +\infty \) as \( t \to k^2 - 0 \) and to \( -\infty \) as \( t \to k^2 + 0 \) for \( k = 1, \ldots, N \). This implies that each interval \([k, k+1]\) for \( k = 0, \ldots, N - 1\) contains a root \( t_k \).

Now we pass to the limit as \( N \to \infty \) in Eq. (A12) and Eq. (A13). To do this, we rewrite Eq. (A12) in the following form:

\[
\sum_{k=1}^{N} \frac{k^2 - \mu^2}{k^2 - \mu^2} + \mu^2 \sum_{k=1}^{N} \frac{1}{k^2 - \mu^2} = N + \frac{1}{2}.
\]

which is equivalent to

\[
\sum_{k=1}^{N} \frac{1}{k^2 - \mu^2} = \frac{1}{2\mu^2}.
\]
The function
\[ g_N(\mu) = \sum_{k=1}^{N} \frac{1}{k^2 - \mu^2} \]
has then a well-defined limit as \( N \to \infty \). Namely,
\[ \lim_{N \to \infty} g_N(\mu) = \sum_{k=1}^{\infty} \frac{1}{k^2 - \mu^2} = \frac{\pi}{2\mu} \cotg(\pi \mu) = g(\mu), \quad (A16) \]
and the limit eigenvalues \( i\mu \) are given by the roots \( \mu_k = k + \frac{1}{2} \) of \( \cotg(\pi \mu) \). Here \( \mu_k \) is the unique solution of \( \cotg(\pi \mu) \) on the interval \([k, k+1]\). The spectral decomposition is defined by the correspondence
\[ a_k = \sum_{\mu} \frac{R_\mu}{k^2 - \mu^2} \quad (A17) \]
between the infinite sequences \( a_k \) and \( R_\mu \).

**Remark.** We have the following result.

**Theorem A.1.** (cf. the “Eisenstein”-part of [13])
\[ \sum_{k=1}^{\infty} \frac{1}{k^2 - \mu^2} \cos kx - \frac{1}{2\mu^2} = -\frac{1}{|\mu|} \sin \|\mu x\|_\mu, \quad \text{where} \quad \|y\|_\mu = \inf_{n \in \mathbb{Z}} |y + 2\pi mn|. \quad (A18) \]

**Proof.** Indeed, the operator \( L = \frac{\partial^2}{\partial x^2} + \mu^2 \), when applied to the left-hand side and right-hand side of (A18), gives \( -\delta(x) \), and the kernel of \( L \) in the space of \( 2\pi \)-periodic functions is \( \{0\} \).

This fact allows us to rewrite the transform (A17) in its functional form as follows:
\[ f_0(x) = \sum a_k \cos kx = -\sum_{\mu} \frac{\Re R_\mu}{|\mu|} \sin \|\mu x\|_\mu, \]
\[ f_1(x) = \sum b_k \cos kx = -\sum_{\mu} (\text{sgn} \mu) \Im R_\mu \sin \|\mu x\|_\mu. \quad (A19) \]

The sequence of complex numbers \( R_\mu \) is self-adjoint, meaning that \( R_{-\mu} = \overline{R_\mu} \). Thus, the solution of Eq. (52) along a singular arc has the form \( f(x,t) = \sum a_k(t) \cos kx \), where
\[ a_k(t) = \sum_{\mu} \frac{R_\mu e^{i\mu t}}{k^2 - \mu^2} = 2\sum_{\mu > 0} \frac{\Re (R_\mu e^{i\mu t})}{k^2 - \mu^2}, \quad (A20) \]
and \( \mu \) ranges over the roots \( \mu_k = k + \frac{1}{2} \) of the function \( \tan(\pi \mu) \).

The point with coordinates \( a_k, b_k \) belongs to the singular arc at the cut-off level \( N \) if the control \( u = \frac{1}{N+1} \sum_{k=0}^{N} k^2a_k \) satisfies the bound \( |u| \leq 1 \). The problem is to pass to the limit \( N \to \infty \) in this condition. In terms of the variables \( R_\mu \), the condition means that
\[ \left| \frac{1}{N+1} \sum_{k \in [1,N], \mu \in (0,N)} \frac{k^2\Re (R_\mu)}{k^2 - \mu^2} \right| \leq \frac{1}{2}. \quad (A21) \]
The sum

\[
\frac{1}{N+1} \sum_{k=1}^{N} \frac{k^2}{k^2 - \mu^2} = \frac{1}{N+1} (N + \mu^2 g_N(\mu)) = \frac{1}{N+1} (N + 1 + \mu^2 g_N(\mu) - 1)
\]  

(A22)
equals to 1 because of equation (A15): \( \mu^2 g_N(\mu) = 1 \). Therefore, the condition (A21) at the cut-off level \( N \) can be restated as follows:

\[
\left| \sum_{\mu \in (0,N)} \Re(R_\mu) \right| \leq \frac{1}{2}.
\]  

(A23)

The formal passage to the limit \( N \to \infty \) transforms (A23) into

\[
\left| \sum_{\mu > 0} \Re(R_\mu e^{i \mu t}) \right| \leq \frac{1}{2}.
\]  

(A24)

We note that the summation in Eq. (A23) and Eq. (A24) goes over different sets of roots \( \mu \): in the first case, over the roots of \( \mu^2 g_N(\mu) = 1 \), and in the other case, over the roots of \( \mu^2 g(\mu) = 1 \).

The time-limits of the arc are determined by the inequality

\[
\left| \sum_{\mu > 0} \Re(R_\mu e^{i \mu t}) \right| \leq \frac{1}{2},
\]  

(A25)

which says that the control \( u \) satisfies the inequality \( |u| \leq 1 \). Equations (A21) and (A20) together determine, at a formal level, the singular motion of the system. In order to make these considerations applicable to the “real” string, we need to know the continuity properties of the function

\[
t \mapsto \sum_{\mu > 0} \Re(R_\mu e^{i \mu t}),
\]  

(A26)

when the function \( \frac{\partial f_0}{\partial x} + f_1 \) is bounded and related to the sequence \( R_\mu \) via Eq. (A20).

APPENDIX III. CONTRACTING PROPERTIES OF THE DRY-FRICTION CONTROL

We consider the semi-flow \( \phi_t(f) = f_t \) defined by the Cauchy problem for the nonlinear string equation (6) of second order. Here we shall show that, at formal level, the semi-flow \( \phi_t(f) = f_t \) is continuous and even contracting with respect to the “spacial” argument \( f \). Indeed, consider the energy functional

\[
E(f) = \frac{1}{2} \| f_1 \|^2 + \frac{1}{2} \left\| \frac{\partial f_0}{\partial x} \right\|^2,
\]  

(A27)

where \( \| \cdot \| \) is the \( L_2(0,2\pi) \)-norm. We have the following basic a priori estimate:

\[
\frac{d}{dt} E(\phi_t(f) - \phi_t(g)) \leq 0.
\]  

(A28)

Indeed, by formal computation, we have for \( E(t) = E(\phi_t(f) - \phi_t(g)) \) that

\[
\frac{d}{dt} E = A + B + C, \quad \text{where} \quad A = \left\langle \frac{\partial}{\partial x} (f_{10} - g_{10}), \frac{\partial}{\partial x} (f_{t1} - g_{t1}) \right\rangle,
\]

\[
B = \langle f_{t1} - g_{t1}, \Delta (f_{10} - g_{10}) \rangle, \quad C = - (f_{10}(0) - g_{10}(0)) (\text{sgn} f_{t1}(0) - \text{sgn} g_{t1}(0)).
\]  

(A29)
We have $A + B = 0$, because
\[
\langle v, \Delta u \rangle = -\left\langle \frac{\partial v}{\partial x}, \frac{\partial u}{\partial x} \right\rangle
\]  
(A30)
for any pair $u, v$ of periodic functions. Therefore,
\[
\dot{E} = A + B + C = C.
\]  
(A31)
Denote $x = f_t(0)$, $y = g_t(0)$. We then have
\[
C = -(x - y)\text{sgn} x - \text{sgn} y \leq 0,
\]  
(A32)
because the sgn-function is monotone. This implies inequality (A28) which, in turn, implies that the mapping $f \mapsto \Phi_t(f)$ is contracting with respect to the energy-norm for $t \geq 0$. These considerations are formal, because are based on the formal differentiation of a product of distributions.

This becomes even more clear if we rewrite the above formal computation in the simpler case of the nonlinear string equation (60) of first order. In this case, the energy is as follows:
\[
E(g) = \frac{1}{2} \|g\|^2 = \frac{1}{2} \int g(x)^2 \, dx,
\]  
(A33)
where the integration is over the torus $\mathbb{R}/2\pi \mathbb{Z}$. Formally, if
\[
E(t) := E(\Phi_t(g) - \Phi_t(f)),
\]  
(A34)
then
\[
\frac{d}{dt} E = A + B, \quad \text{where} \quad A = \left\langle \frac{\partial}{\partial x} (f - g), (f - g) \right\rangle,
\]  
(A35)
\[
B = -(f(0) - g(0)) (\text{sgn} f(0) - \text{sgn} g(0)).
\]
The term $A$ vanishes for any pair of periodic functions $f, g$, and the term $B$ is nonpositive since the sgn-function is monotone. Thus, the dry-friction semi-flow $\Phi_t$ is contracting with respect to the energy-norm.

It is not clear to us whether or not the dry-friction flow rigorously constructed in Section 8 is contracting indeed with respect to the energy norm.

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