SURGERY OF SPLINE-TYPE AND MOLECULAR FRAMES

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Abstract. We prove a result about producing new frames for general spline-type spaces by piecing together portions of known frames. Using spline-type spaces as models for the range of certain integral transforms, we obtain results for time-frequency decompositions and sampling.

1. Introduction and overview

By a spline-type space we mean a normed function space generated by well localized atoms. The most commonly found scenario in the literature is the one of finitely generated shift-invariant spaces. In that case, the atoms are simply lattice shifts of a finite family of functions. Our motivation to consider spline-type spaces in more generality comes from its possible applications to atomic decompositions. Many atomic decompositions of classical function spaces are produced by sampling a continuous integral (wavelet) transform. The form of that integral transform makes it possible to relate its size and spatial localization to its modulus of continuity. This observation plays a central role, for example, in the general theory developed in [17]. In this article we argue that spline-type spaces are a model for the range of certain wavelet transforms and prove a locality statement for them that can then be recast as a result for the corresponding atomic decompositions.

We will prove a locality principle in the form of a surgery scheme for well-localized frames. Our main result asserts that, given a family of frames for a spline-type space, it is possible to construct a new frame for the same space by piecing together arbitrary portions of the original frames, provided that the overlaps between these portions are large enough. Although the result we prove is qualitative, special emphasis is made on how the qualities of the ingredients affect the surgery procedure and what kind of uniformity is to be expected. This is one reason why we work on the Euclidean space and not on a general locally-compact group (although much of the elements involved in our construction have a counterpart in the abstract setting.) The other reason is that we make use of localization theory (see [25], [21] and [3]) and the most interesting examples we could cover by extending the setting to abstract groups, are not covered by it.

For the applications we consider mainly two transforms. The first is the Short Time Fourier Transform (STFT) with a fixed (good) window. This transform maps modulation spaces into spline-type spaces - considered in the general sense - and then yields an application of the surgery scheme to Gabor frames. These results imply a general existence condition for the recently introduced concept of quilted Gabor frame (see [10].) Since the STFT does not exactly map time-frequency shifts into translations - there is an extra phase factor or twist on the STFT side - we see that shift-invariant spaces are not a sufficient model for the range of the transform: we must use general spline-type spaces. As a by-product of this general treatment,
the result we get holds not only for pure time-frequency shifts but also for Gabor molecules concentrated around a general set of nodes.

The second transform we consider is the Kohn-Nirenberg map, which - as shown in [15] - establishes a correspondence between the class of Gabor multipliers (related to different Gabor frames) and the class of (shift-invariant) spline-type spaces (see also [18, Chapter 5]). Gabor multipliers are operators that arise from applying a mask to the coefficients associated with a Gabor frame expansion; hence each of these operators has the form

$$T = \sum_{\lambda \in \Lambda} c_\lambda P_\lambda,$$

where $c_\lambda \in \mathbb{C}$ and $P_\lambda$ is a rank-one operator (essentially a projector onto the subspace generated by a time-frequency atom.) Each operator in a given class of Gabor multipliers can be identified by its associated lower symbol which consists of the Hilbert-Schmidt inner products $\{ \langle T, P_\lambda \rangle \mid \lambda \in \Lambda \}$. Combining the surgery scheme with the KN map and known tools for shift-invariant spaces we get a sufficient condition to identify a class of Gabor multipliers by a mixed lower symbol constructed by using different types of rank-one operators $P_\lambda$ for $\lambda$ in different regions of the time-frequency plane.

Finally, we give an application to irregular sampling. Given a family of sampling sets for which a sampling inequality is known, we can construct new sets for which the sampling inequality still holds. Moreover, given explicit reconstruction formulas for the original sets, we get an approximate reconstruction formula for the new sets.

We now mention two technical aspects of the article. The first one concerns the use of localization theory. In order to develop the surgery scheme, we not only need to know that localized frames have localized dual frames but also what qualities of the original atoms influence the concentration of their dual atoms. To this end, we resort to the constructive proof of Wiener’s lemma for infinite matrices given in [35] (see also [34].)

The second aspect concerns the use of amalgam spaces. When moving to the setting of spaces generated by general atoms, the standard tools for amalgam spaces are not directly applicable and require an extension. In the study of shift-invariant spaces (or more generally, spaces generated by translates) the relevant operators can be expressed as products and convolutions with possibly distributional kernels. Wiener amalgam spaces, as introduced in [12], have proved to be a powerful tool to quantify this formalism. The abstract convolution multiplier theorems allow to deal with smoothness and approximation problems in the context of atoms generated by irregular shifts (see [14].) In the context of general spline-type spaces, the relevant operations are not convolutions but, nonetheless, they are convolution-like. For example, in the proposed applications to (regular) Gabor frames, instead of convolution inequalities for Wiener amalgams we would need twisted convolution inequalities.

Convolution dominated operators (see [4], [29], [26]) and enveloping conditions for irregular atoms ([36], [3], [28]) are now widely used concepts. Here, we will consider an enveloping condition for atoms, not in a pointwise sense, but in the sense of a local - possibly non solid - quantity. We will extend the amalgam norm of a function $f$ to families of functions $F$ in such a way that the condition $\| F \|_{W(B,E)} < \infty$ grants to $F$ the same properties shared by a set of translates of $f$, when $\| f \|_{W(B,E)} < \infty$. When the local norm measures size, the condition
∥F∥_{W(B,E)} < ∞ will amount to certain spatial localization for the family F; when the local norm measures smoothness, it will amount to certain equismoothness property for the family F. Using this extension of the amalgam norm and a simple interpolation argument, we obtain replacements for some of the convolution inequalities in amalgam spaces. These are needed, for example, to extend to the general setting the principle that in a finitely-generated shift-invariant space the smoothness of the generating windows is inherited by the whole space.

The article is organized as follows. Section 2 introduces all the tools required to deal with general sets of atoms and in particular the extension of the amalgam norms to families of functions. In Section 3 we formally introduce spline-type spaces and extend to this setting some of the classic results for shift-invariant spaces. In Section 4 we prove the central result on piecing together various frames to build a new one. The result stated there has the limitation of requiring specific information on the decay of the dual atoms of the frames being pieced together. For clarity, this limitation is addressed separately in Section 5. Section 6 collects the results from the previous sections and gives several applications.

2. Preliminaries

2.1. Frames. Let E be a separable Banach space. A Banach frame for E consists of a countable family \( \{f_k\}_{k \in \Lambda} \subseteq E \) together with a sequence space \( E_d \hookrightarrow \mathbb{C}^\Lambda \) such that the reconstruction operator

\[
R : E_d \to E
\]

\[
(c_k)_k \mapsto \sum_k c_k f_k
\]

is a well-defined bounded retraction. This means that R is bounded and there exists another operator \( C : E \to E_d \), called coefficient operator, such that \( RC = I_E \). We will only consider sequence spaces \( E_d \) for which the set of sequences \( \{\delta^k_j\}_k \) - given by \( \delta^k_j = 1, \text{ if } k = j, \text{ and } 0 \text{ otherwise} \) - is an unconditional basis. In this case, if we only assume that the operator \( R \) maps each sequence \( \delta^k \) to \( f_k \), then it follows that \( R \) has the form prescribed by Equation (1) and that the series in that equation converges unconditionally.

Since \( E_d \hookrightarrow \mathbb{C}^\Lambda \), the coefficient operator \( C \) is implemented by some family of linear functionals \( \{g_k\}_k \subseteq E' \) by means of the formula \( C(f) = \langle f, g_k \rangle \). When a particular choice of a coefficient operator (and hence of coefficient functionals \( \{g_k\}_k \)) is made, we speak of a frame pair \( \{f_k\}_k, \{g_k\}_k \).

In concrete examples, the coefficient functionals may have various representations. For example, if \( E \) is a closed subspace of a Hilbert space \( H \), then each coefficient functional can be represented by various vectors \( g_k \in H \). Each of these choices will be considered to yield a different frame pair \( \{f_k\}_k, \{g_k\}_k \).

It is common in the literature to define Banach frames in terms of the coefficients functionals \( \{g_k\}_k \) rather than the atoms \( \{f_k\}_k \). A family of linear functionals \( \{g_k\}_{k \in \Lambda} \subseteq E' \), together with a sequence space \( E_d \hookrightarrow \mathbb{C}^\Lambda \) is called a Banach frame for E if the coefficient operator

\[
C : E \to E_d
\]

\[
f \mapsto \langle (f, g_k)_k \rangle
\]
is a bounded section; that is, there exists a bounded operator \( R : E_d \to E \) such that \( RC = I_E \). In the abstract setting there is no possible confusion between the two usages since the atoms and the coefficients functionals belong to different spaces. However, in concrete examples where \( E \) is a classical function space and \( E' \) is identified with another classical function space, these two usages can be ambiguous. In the context of Hilbert spaces, since \( C = R^* \), if we use the canonical representation of functionals as elements of the Hilbert space, then the two definitions of frames are equivalent. This equivalence extends to localized frames (see [25].) Frames produced by extension of a localized Hilbert space frame to its associated Banach spaces are Banach frames in both of the senses discussed above (see [21].) In this article, every reference to a Banach frame will be followed by a clarification about its precise meaning.

If \( H \) is a Hilbert space and \( E \subseteq H \) is a closed subspace, a sequence \( F \equiv \{f_k\}_k \subseteq H \) whose orthogonal projection onto \( E \) forms a frame for \( E \) is called an exterior frame for \( E \). Hence, \( F \) is an exterior frame for \( E \) if it satisfies the exterior frame inequality,

\[
A\|g\|_H^2 \leq \sum_k |\langle g, f_k \rangle|^2 \leq B\|g\|_H^2, \quad \text{for all } g \in E,
\]

and some constants \( 0 < A \leq B < +\infty \). Likewise, if the projection of \( F \) onto \( E \), forms a Riesz basis for \( E \), then \( F \) is called a Riesz projection basis for \( E \) (see for example [19].)

### 2.2. Sets.
A subset \( \Lambda \subseteq \mathbb{R}^d \) is called relatively separated if the quantity,

\[
\text{rel}(\Lambda) := \max \left\{ \#(\Lambda \cap ([0,1]^d + x)) \mid x \in \mathbb{R}^d \right\}
\]

is finite. We will call the number \( \text{rel}(\Lambda) \) the relative separation of the set \( \Lambda \). This is somehow an abuse of language since for a very separated set, this quantity is small.

A subset \( \Lambda \subseteq \mathbb{R}^d \) is called L-dense, for some \( L > 0 \), if

\[
\mathbb{R}^d = \bigcup_{k \in \Lambda} B_L(k),
\]

where \( B_L(k) \) denotes the open ball of center \( k \) and radius \( L \). \( \Lambda \) is called relatively dense if it is L-dense for some \( L > 0 \).

### 2.3. Weights.
A weight \( w \) is a function \( w : \mathbb{R}^d \to (0, +\infty) \). For simplicity we will always assume that \( w \) is continuous and symmetric.

A weight \( w \) is said to be submultiplicative if it satisfies,

\[
w(x+y) \leq w(x)w(y), \quad \text{for all } x, y \in \mathbb{R}^d.
\]

As an example, the polynomial weights

\[
w_t(x) := (1 + |x|)^t,
\]

satisfy the submultiplicativity condition if \( t \geq 0 \).

A second weight \( v \) is called \( w \)-moderated if it satisfies,

\[
v(x+y) \leq Cv(x)w(y),
\]

for some constant \( C > 0 \) and every \( x, y \in \mathbb{R}^d \). If the constant in Equation \( 4 \) is 1, we say that \( v \) is strictly moderated by \( w \). The polynomial weight \( w_t \) is strictly \( w_s \)-moderated if \( s \geq 0 \) and \( |t| \leq s \).

For a sequence \( \{c_k\}_{k \in \Lambda} \subseteq \mathbb{C} \), we consider the weighed norm,

\[
\|c\|_{w^p} := \|d\|_{w^p}, \quad \text{where } d_k := |c_k| w(k),
\]
and we denote by $\ell^p_w$ the space of all such sequences having finite norm. Weighed $L^p$ are defined similarly.

For technical reasons, we consider within $\ell^p_w$ the subspace $\ell^p_{w_0}$ defined as the closure of the set of finitely-supported sequences. For $1 \leq p < \infty$ this is just the whole space $\ell^p_w$ and for $p = \infty$ it is $c_0^w$. We make this definition so as not to have to consider the case $p = \infty$ separately.

We now state for future reference some facts about polynomial weights. The first lemma says that polynomial weights are subconvolutive (see [11].)

**Lemma 1.** If $t > d$, then,

$$w_{-t} \ast w_{-t} \leq K w_{-t},$$

for some constant $K \lesssim \max \{1, 1/(t-d)\}$.

There is a corresponding statement for relatively separated index sets. The important point is that the bounds depend only on the relative separation of the sets involved (and this quantity is translation invariant.)

**Lemma 2.** Let $\Gamma \subseteq \mathbb{R}^d$ be a relatively separated set of points and let $t > d$. Then, the following estimates hold for a constant $K \lesssim \max \{1, 1/(t-d)\}$.

(a) $\sum_{\gamma \in \Gamma} w_{-t}(\gamma) \leq K \text{rel}(\Gamma),$

(b) $\sum_{|\gamma| > M} w_{-t}(\gamma) \leq K \text{rel}(\Gamma) M^{-(t-d)},$

(c) $\sum_{\gamma \in \Gamma} w_{-t}(\gamma) w_{-t}(x - \gamma) \leq K \text{rel}(\Gamma) w_{-t}(x)\text{, for all } x \in \mathbb{R}^d.$

For proofs of very similar statements see [24, Lemma 11.1.1], [31, Lemma 1] and [11].

2.4. **Amalgam spaces.** We denote by $\mathcal{D}(\mathbb{R}^d)$ the set of all $C^\infty$, compactly supported, complex-valued functions on $\mathbb{R}^d$, by $C^0(\mathbb{R}^d)$ the set of all continuous functions vanishing at infinity and by $S(\mathbb{R}^d)$ the Schwartz class.

Let $(B, \| \cdot \|_B)$ be a uniformly localizable, isometrically translation invariant Banach space. That is, $B$ satisfies the following axioms.

- $S(\mathbb{R}^d) \hookrightarrow B \hookrightarrow S'(\mathbb{R}^d)$ are continuous embeddings whose composition is the canonical embedding $S(\mathbb{R}^d) \hookrightarrow S'(\mathbb{R}^d)$.
- If $h \in \mathcal{D}(\mathbb{R}^d)$ and $f \in B$, then $hf \in B$ and there is a constant $C = C(h) > 0$ such that $\|hf(\cdot - x)\|_B \leq C \|f\|_B$, for all $x \in \mathbb{R}^d$ and $f \in B$.
- If $f \in B$ and $x \in \mathbb{R}^d$, then $f(\cdot - x) \in B$ and $\|f(\cdot - x)\|_B = \|f\|_B$.
- Complex conjugation defines an isometry on $B$. That is, if $f \in B$, then $f^\ast \in B$ and $\|f\|_B = \|f\|_B$.

We consider the space of distributions that belong to $B$ locally,

$$B_{loc} := \{ f \in S'(\mathbb{R}^d) \mid hf \in B, \text{ for all } h \in \mathcal{D}(\mathbb{R}^d) \}.$$ Given $f \in B_{loc}$ and a non-zero window $\eta \in \mathcal{D}(\mathbb{R}^d)$, we consider the control function

$$F(x) := \|f\eta(\cdot - x)\|, \quad (x \in \mathbb{R}^d.)$$

For a space of functions $E \hookrightarrow L^1_{loc}$, the Wiener amalgam space $W(B, E)$ is defined by

$$W(B, E) := \{ f \in B_{loc} \mid F \in E \},$$

and is given the norm $\|f\|_{W(B, E)} := \|F\|_E$. The space $E$ is normally assumed to be solid and translation invariant. The amalgam $W(B, E)$ is then independent of the
choice of the window $\eta$ in the sense that different windows yield equivalent norms (cf. [12] and [14].) In this article we will always let $E$ be a weighed $L^p$ space.

2.5. Amalgam norm of families. We will consider a relatively separated set of points $\Lambda \subseteq \mathbb{R}^d$, which will be called nodes and a symmetric, submultiplicative, continuous weight $w : \mathbb{R}^d \to (0, +\infty)$. We will also consider a family of measurable functions $f_k : \mathbb{R}^d \to \mathbb{C}$ indexed by the set of nodes $\Lambda$.

For a family $F \equiv \{f_k\}_{k \in \Lambda} \subseteq B_{loc}$ we define its $W(B, L^1_w)$ norm by

$$
\|F\|_{W(B, L^1_w)} := \max \left\{ \left\| \sup_k \|g_k\|_1, \sup_{x} \sum_k |g_k(x)| \right\} \right.,
$$

where $g_k(x) := \|f_k\|_{BQ+x} w(x-k), \quad (x \in \mathbb{R}^d, k \in \Lambda.)$

Here, $\eta \in D(\mathbb{R}^d)$ is any nonzero window function (see Prop. 1 below.)

Observe that if $\|F\|_{W(B, L^1_w)} < +\infty$, then each $F_k$ belongs to $W(B, L^1_w)$. The estimate $\|F\|_{W(B, L^1_w)} < +\infty$ grants, in addition, certain uniformity for the set $\{f_k\}_{k}$, similar to that shared by the translates of an individual atom. Some results to come will give evidence of that. The following proposition shows that, at least, the implicit constant on (c) depends on the relative separation of $\Lambda$.

**Proposition 1.** Let a family $F \equiv \{f_k\}_{k \in \Lambda} \subseteq B_{loc}$ be given.

(a) Let $\|F\|_{W(B, L^1_w)}$ be the norm defined using a nonzero window function $\eta_i \in D(\mathbb{R}^d), (i=1,2)$. Then $\|F\|_{W(B, L^1_w)} \approx \|F\|_{W(B, L^1_w)}$.

(b) For any bounded set $Q \subset \mathbb{R}^d$ with non-empty interior, the norm $\|F\|_{W(B, L^1_w)}$ is also equivalent to the norm $\|F\|_{W(B, L^1_w)}$ defined by

$$
\|F\|_{W(B, L^1_w)} := \max \left\{ \left\| \sup_k \|g_k\|_1, \sup_{x} \sum_k |g_k(x)| \right\} \right.,
$$

where $g_k(x) := \|f_k\|_{BQ+x} w(x-k), \quad (x \in \mathbb{R}^d, k \in \Lambda)$, and

$$
\|f\|_{L^1(Q)} := \inf \{ \|g\|_B : g \equiv f \text{ on } Q \}.
$$

(c) If the family $F$ is given by $f_k = f(-k), k \in \Lambda$ and $\Lambda$ is relatively separated, then $\|F\|_{W(B, L^1_w)} \approx \|f\|_{W(B, L^1_w)}$.

**Remark 1.** The implicit constant on (c) depends on the relative separation of $\Lambda$.

**Proof.** For (a), since $\eta_2$ is compactly supported and not identically 0, it is possible to choose $\alpha > 0$ such that $\sum_{j \in \mathbb{Z}^d} |\eta_2|^2 (\cdot - \alpha j) \approx 1$. This series is locally finite, so the function $m := \eta_1 \left( \sum_{j \in \mathbb{Z}^d} |\eta_2|^2 (\cdot - \alpha j) \right)^{-1}$ is smooth. Choose $\theta \in D(\mathbb{R}^d)$ such that $\theta \equiv 1$ on the support of $\eta_1$. Now,

$$
\eta_1 = \theta \eta_1 = \sum_{j \in \mathbb{Z}^d} \theta m |\eta_2|^2 (\cdot - \alpha j).
$$
Since both \( \theta \) and \( \eta_2 \) are compactly supported, only finitely many terms are not zero and we may write
\[
\eta_1 = \sum_{j=1}^{n} m_j \eta_2(\cdot - x_j),
\]
where \( x_j \in \alpha \mathbb{Z}^d \) and \( m_j := \theta m \eta_2(\cdot - x_j) \in \mathcal{D}(\mathbb{R}^d) \).

Now, for \( x \in \mathbb{R}^d \), and \( k \in \Lambda \),
\[
\|f_k \eta_1(\cdot - x)\|_{B} w(x - k) \leq \sum_{j=1}^{n} \|f_k \eta_2(\cdot - x - x_j)\|_{B} w(x - k)
\]
\[
\leq \sum_{j=1}^{n} \|f_k \eta_2(\cdot - (x + x_j))\|_{B} w((x + x_j) - k) w(x_j)
\]
Consequently,
\[
\|F\|_{W_1(B, \mathcal{L}_1)} \lesssim \|F\|_{W_2(B, \mathcal{L}_1)}.
\]
The other inequality follows by symmetry.

To prove (b), consider first a window \( \eta \in \mathcal{D}(\mathbb{R}^d) \) such that \( \eta \equiv 1 \) on \( Q \). Then for any \( k \in \Lambda \) and \( x \in \mathbb{R}^d \), \( \|f_k\|_{B(Q + x)} \leq \|f_k \eta(\cdot - x)\|_{B} \) and it follows that \( \|F\|_{\tilde{W}(B, \mathcal{L}_1)} \lesssim \|F\|_{W(B, \mathcal{L}_1)} \).

For the other inequality, since \( Q \) has non-empty interior, there exists a non-zero window function \( \eta \in \mathcal{D}(\mathbb{R}^d) \) supported on \( Q \). For any \( k \in \Lambda \), \( x \in \mathbb{R}^d \) and any \( h \in B \) such that \( h \equiv f_k \) on \( Q + x \), we have
\[
\|f_k \eta(\cdot - x)\|_{B} = \|h \eta(\cdot - x)\|_{B} \lesssim \|h\|_{B}.
\]
Therefore, \( \|f_k \eta(\cdot - x)\|_{B} \lesssim \|f_k\|_{B(Q + x)} \), and the desired inequality follows.

Let us now prove (c). For \( x \in \mathbb{R}^d \) and \( k \in \Lambda \), since \( B \) is isometrically translation invariant,
\[
g_k(x) = \|f(\cdot - k) \eta(\cdot - x)\|_{B} w(x - k) = \|f \eta(\cdot - (x - k))\|_{B} w(x - k).
\]
Integrating over \( x \) we get that for any \( k \in \Lambda \),
\[
(7) \quad \|f\|_{W(B, \mathcal{L}_1)} = \|g_k\|_{1}.
\]
This shows that \( \|f\|_{W(B, \mathcal{L}_1)} \leq \|F\|_{W(B, \mathcal{L}_1)} \).

Since by (7) we know that \( \sup_{k} \|g_k\|_{1} \leq \|f\|_{W(B, \mathcal{L}_1)} \), it suffices to show that \( \sup_{x} \sum_{k} g_k(x) \leq \|f\|_{W(B, \mathcal{L}_1)} \).

To this end, let us call \( Q \) the unitary cube centered at 0 and let \( \theta \in \mathcal{D}(\mathbb{R}^d) \) be such that \( \theta \equiv 1 \) on \( \text{supp}(\eta) + Q \). For \( x \in \mathbb{R}^d \), and \( k \in \Lambda \),
\[
g_k(x) = \|f \eta(\cdot - (x - k))\|_{B} w(x - k) = \int_{Q} \|f \eta(\cdot - (x - k))\|_{B} w(x - k) dy
\]
\[
= \int_{Q} \|f \eta(\cdot - (x - k)) \theta(\cdot - (x + y - k))\|_{B} w(x - k) dy
\]
\[
\lesssim \int_{Q} \|f \theta(\cdot - (x + y - k))\|_{B} w(x - k) dy
\]
\[
= \int_{Q+x-k} \|f \theta(-y)\|_{B} w(x - k) dy
\]
Since \( w \) is bounded on \( Q \), for \( y \in Q + x - k \), \( w(x - k) \leq w(y) \sup_Q w \). Therefore, for any \( x \in \mathbb{R}^d \),

\[
\sum_k g_k(x) \lesssim \sum_k \int_{Q + x - k} ||f\theta(\cdot - y)||_B w(y) dy = \int_{\mathbb{R}^d} ||f\theta(\cdot - y)||_B w(y) \sum_k \chi_{Q + x - k}(y) dy
\]

Finally, observe that \( \sum_k \chi_{Q + x - k}(y) \) is bounded by the relative separation of the set of nodes \( \Lambda \). This completes the proof. \( \square \)

**Example 1.** As an easy example of amalgam norm of families, consider a relatively separated set of nodes \( \Lambda \subseteq \mathbb{R}^d \), and a family of measurable functions \( f_k : \mathbb{R}^d \to \mathbb{C} \), \( k \in \Lambda \) satisfying the concentration condition,

\[
|f_k(x)| \leq C w_{-(s+\alpha)}(x - k), \quad x \in \mathbb{R}^d, k \in \Lambda,
\]

for some \( s > d \) and \( \alpha \geq 0 \).

Let \( Q := [0, 1]^d \) be the unit cube. From equation (8) we get that for any \( x \in \mathbb{R}^d \),

\[
||f_k||_{L^\infty(Q+x-z)} \leq C |w_{-(s+\alpha)}||L^\infty(Q+(x-k)) \lesssim C w_{-(s+\alpha)}(x - k),
\]

where the implicit constant depends on \( s + \alpha \). Therefore,

\[
||f_k||_{L^\infty(Q+x-z)w_\alpha(x - k)} \lesssim C w_{-s}(x - k).
\]

Hence by Proposition 7 and Lemma 2 \( ||F||_{W(L^\infty(L^p_{\Lambda}))} \lesssim C_{rel}(\Lambda) \).

However, the concentration condition in Equation (8) is much more precise than the last statement. We will need this stronger condition in Section 4.

### 2.6. Multiplier theorems.

We now introduce a number of multiplier results that will replace in the applications the convolution relations for amalgam spaces. These are easily established for some endpoint spaces and then generalized by interpolation. Throughout this section we will assume the following.

- A relatively separated set of nodes \( \Lambda \subseteq \mathbb{R}^d \) is given.
- \( B \) is a uniformly localizable, isometrically translation invariant, Banach space.
- \( w : \mathbb{R}^d \to (0, +\infty) \) is a symmetric, submultiplicative, continuous weight.
- \( v : \mathbb{R}^d \to (0, +\infty) \) is a symmetric weight moderated by \( w \).

We first show that the synthesis of well-localized atoms is bounded with respect to amalgam space norms.

**Proposition 2.** Let a family \( F \equiv \{f_k\}_{k \in \Lambda} \subseteq B_{loc} \) such that \( ||F||_{W(B,L^p_{\Lambda})} < +\infty \) and \( c \in \ell_p^m \) be given (\( 1 \leq p \leq \infty \)). Then, the series

\[
c \cdot F := \sum_k c_k f_k,
\]

converges in \( W(B,L^p_{\Lambda}) \) and satisfies the following estimate,

\[
||c \cdot F||_{W(B,L^p_{\Lambda})} \lesssim ||c||_{\ell_p^m} ||F||_{W(B,L^p_{\Lambda})}.
\]

**Remark 2.** The implicit constant is the constant in (8).

**Remark 3.** If \( c \in \ell_p^\infty \), then the same conclusion holds but the series is only weak* convergent.
Proof. We will assume that the sequence \( c \) is finitely supported. The general case follows from this one by approximation, using the completeness of \( W(B,L^1_v) \) (see [12].)

Let us set \( f := c \cdot F = \sum_k c_k f_k \). For a window function \( \eta \in D(\mathbb{R}^d) \) and \( x \in \mathbb{R}^d \), we have \( f\eta(\cdot - x) = \sum_k c_k f_k \eta(\cdot - x) \). Therefore

\[
\|f\eta(\cdot - x)\|_{B^v(x)} \leq C \sum_k |c_k| v(k) \|f_k \eta(\cdot - x)\|_{B^v(x-k)},
\]

where the constant \( C \) is the constant in (6).

Now Schur’s lemma (see below) yields the desired inequality. □

In the proof we used part (a) of the following interpolation lemma which we quote for completeness. For a proof see [23, Theorem 1.3.4].

**Lemma 3.** Let \( F \equiv \{f_k\} \) be a family of measurable functions on \( \mathbb{R}^d \) and let \( 1 \leq p \leq \infty \).

(a) Let \( \{c_k\} \subseteq C \) be a sequence. Then,

\[
\|c \cdot F\|_{L^p} \leq \|c\|_{L^p} \left( \sup_{k} \|f_k\|_{L^1} \right)^{1/p} \left( \sup_{x \in \mathbb{R}^d} \sum_k |f_k(x)| \right)^{1/p'},
\]

(b) Let \( g : \mathbb{R}^d \to [0,\infty] \) be a measurable function. Then,

\[
\|g \cdot F\|_{L^p} \leq \|g\|_{L^p} \left( \sup_{k} \|f_k\|_{L^1} \right)^{1/p'} \left( \sup_{x \in \mathbb{R}^d} \sum_k |f_k(x)| \right)^{1/p},
\]

where

\[
(g \cdot F)_k := \int_{\mathbb{R}^d} g(x)f_k(x)dx.
\]

For both statements, if \( p = 1 \), we interpret \( 1/p' = 0 \).

We now give estimates for transformations operating on families of well-localized atoms. For a matrix of complex numbers \( C \equiv (c_{k,j})_{k,j \in \Lambda} \), we consider the following weighed Schur-type norm,

\[
\|C\|_{S_w} := \max \left\{ \sup_k \sum_j |c_{k,j}| w(k-j), \sup_j \sum_k |c_{k,j}| w(k-j) \right\}.
\]

Furthermore, we denote by \( S_w \) the set of all such matrices having finite norm.

Let us show that these matrices act boundedly on well-concentrated families of atoms.

**Proposition 3.** Let a family \( F \equiv \{f_k\} \subseteq B_{loc} \) such that \( \|F\|_{W(B,L^1_v)} < +\infty \) and a matrix \( C \in S_w \) be given.

Let \( C \cdot F \equiv \{g_k\} \) be the family defined by,

\[
g_k := \sum_j c_{k,j} f_j.
\]

Then, each of the series defining \( g_k \) converges in \( W(B,L^v_w) \) and we have the following estimate,

\[
\|C \cdot F\|_{W(B,L^1_v)} \leq \|C\|_{S_w} \|F\|_{W(B,L^1_v)}.
\]
Proof. Again, by an approximation argument we may assume that $C$ is finitely supported.

First observe that for fixed $k \in \Lambda$, the sequence $\{c_{k,j}\}$ belongs to $\ell^m_n$, where $m$ is the weight given by $m(j) := w(k - j)$. Since $w(j) \leq m(j)w(k)$, it follows from Proposition 3 that the series defining $g_k$ converges in $W(B, L^1_w)$.

Fix a window function $\eta \in D(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. For each $k \in \Lambda$, $g_k \eta(\cdot - x) = \sum_j c_{k,j} f_j \eta(\cdot - x)$. Consequently, if we set $h_k(x) := \|g_k \eta(\cdot - x)\|_{B} w(x - k)$, we get,

\begin{equation}
    h_k(x) \leq \sum_j |c_{k,j}| w(k - j) \|f_j \eta(\cdot - x)\|_B w(x - j).
\end{equation}

Integrating this equation yields,

$$
\|h_k\|_1 \leq \sum_j |c_{k,j}| w(k - j) \|F\|_{W(B, L^1_w)},
$$

so $\sup_k \|h_k\|_1 \leq \|C\|_{S_w} \|F\|_{W(B, L^1_w)}$. From equation (9) we also get,

$$
\sum_k h_k(x) \leq \sum_j \sum_k |c_{k,j}| w(k - j) \|f_j \eta(\cdot - x)\|_B w(x - j)
\leq \|C\|_{S_w} \sum_j \|f_j \eta(\cdot - x)\|_B w(x - j).
$$

Therefore, $\sup_k \sum_k h_k(x) \lesssim \|C\|_{S_w} \|F\|_{W(B, L^1_w)}$. This completes the proof. \qed

We now give a dual estimate.

**Proposition 4.** Let two families $F \equiv \{f_k\}_{k \in \Lambda}, G \equiv \{g_k\}_{k \in \Lambda} \subseteq B_{loc}$ such that $\|F\|_{W(B, L^1_w)}, \|G\|_{W(B, L^1_w)} < +\infty$ be given. Suppose that $B$ is continuously embedded in $L^\infty_{loc}$. Then the cross-correlation matrix $C$, defined by

$$
C_{k,j} := \langle f_k, g_j \rangle,
$$

satisfies $\|C\|_{S_w} \lesssim \|F\|_{W(B, L^1_w)} \|G\|_{W(B, L^1_w)}$.

**Remark 4.** The implicit constant depends on the embedding $B \hookrightarrow L^\infty_{loc}$.

**Proof.** Fix $\eta \in D(\mathbb{R}^d)$ supported on an open ball $B$ around 0 and such that $\eta \equiv 1$ on a smaller concentric ball $B'$. Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$ be a locally integrable function. Given $x \in \mathbb{R}^d$, for almost every $y \in B' + x$,

$$
|f(y)| = |f(y)\eta(y - x)| \leq \|f\eta(\cdot - x)\|_{L^\infty(B')} \lesssim \|f\eta(\cdot - x)\|_B.
$$

Hence

$$
|B_r(x)|^{-1} \int_{B_r(x)} |f(y)| \, dy \lesssim \|f\eta(\cdot - x)\|_B,
$$

for all sufficiently small $r > 0$. This shows that

$$
|f(x)| \lesssim \|f\eta(\cdot - x)\|_B,
$$

at every $x \in \mathbb{R}^d$ that is a Lebesgue point of $f$. Consequently,

$$
|c_{k,j}| w(k - j) \lesssim \int_{\mathbb{R}^d} |f_k \eta(\cdot - x)|_B w(x - k) |g_j \eta(\cdot - x)|_B w(x - j) \, dx.
$$

Taking $\sup_k \sum_j$ and $\sup_j \sum_k$, it follows that $\|C\|_{S_w} \lesssim \|F\|_{W(B, L^1_w)} \|G\|_{W(B, L^1_w)}$. \qed

Finally we show that well localized atoms induce bounded analysis operators.
Proposition 5. Let a family $F \equiv \{ f_k \}_{k \in \Lambda} \subseteq B_{\text{loc}}$ such that $\|F\|_{W(B,L^1_w)} < +\infty$ be given. Suppose that $B$ is continuously embedded in $L^\infty_{\text{loc}}$. For $f \in L^p_v$ ($1 \leq p \leq \infty$) define the analysis sequence,

$$c_k := \langle f, f_k \rangle, \quad k \in \Lambda.$$ 

Then $c$ is well defined, belongs to $\ell^p_v$ and satisfies

$$\|c\|_{\ell^p_v} \lesssim \|f\|_{L^p_v} \|F\|_{W(B,L^1_w)}.$$ 

Remark 5. The implicit constant depends on the embedding $B \hookrightarrow L^\infty_{\text{loc}}$ and the constant in (6).

Proof. Fix $\eta \in D(\mathbb{R}^d)$ supported on an open ball $B$ around 0 and such that $\eta \equiv 1$ on a smaller concentric ball $B'$. As in the proof of Proposition 4 the sequence $c$ satisfies

$$|c_k| \lesssim \int_{\mathbb{R}^d} |f(x)| \|f_k \eta(x - k)\|_B dx,$$

so

$$|c_k| v(k) \lesssim \int_{\mathbb{R}^d} |f(x)| v(x) \|f_k \eta(x - k)\|_B w(x - k) dx.$$ 

The conclusion now follows from part (b) of Lemma 3.

\[ \square \]

2.7. Sets with multiplicity. For technical reasons, sometimes we will need to allow the set of nodes to have repeated elements. A set with multiplicity is simply a map $\Gamma \ni \gamma \mapsto \gamma^* \in \mathbb{R}^d$. Any subset of $\mathbb{R}^d$ can be considered as a set with multiplicity by letting the underlying map be the inclusion. By abuse of notation, we will sometimes refer to a set with multiplicity by the domain of the underlying map. For sets with multiplicity, the relative separation is defined by,

$$\text{rel} (\Gamma) := \max_{x \in \mathbb{R}^d} \# \{ \gamma \in \Gamma / \gamma^* \in [0,1]^d + x \} .$$

Similarly, if a family of functions $F \equiv \{ f_\gamma \}_{\gamma \in \Gamma} \subseteq B_{\text{loc}}$ is indexed by a set with multiplicity, the weighed-amalgam norm is defined by,

$$\|F\|_{W(B,L^1_w)} := \max \left\{ \sup_{\gamma} \|g_\gamma\|_1, \sup_{x} \sum_{\gamma} |g_\gamma(x)| \right\} ,$$

where $g_\gamma(x) := \|f_\gamma \eta(\cdot - x)\|_B w(x - \gamma^*), \quad x \in \mathbb{R}^d, \gamma \in \Gamma.$

Here, $\eta \in D(\mathbb{R}^d)$ is any nonzero window function.

Every proof about “ordinary” relatively separated sets given in this article also works for relatively separated sets with multiplicity. The reader interested in this level of generality should read $\gamma^*$ whenever an element $\gamma$ of a set with multiplicity is used as an element of the Euclidean space (instead of as an index set.)

3. Spline-type spaces

We now formally introduce spline-type spaces. We consider a relatively separated set of points $\Lambda \subseteq \mathbb{R}^d$ which will be called nodes and a family of functions $F \equiv \{ f_k \}_{k \in \Lambda} \subseteq L^\infty_{\text{loc}}$ that will be called atoms.

Let $V^{00}$ be the set of finite linear combination of elements of $F$. For a weight function $v$, and $1 \leq p \leq \infty$, we denote by $V^p_v$ the $L^p_v$-closure of $V^{00}$. If the weight $v$ is the trivial weight 1 we drop it in the notation.
Following the spirit of [17], we do not want to consider each of the spaces \( V^p_w \) individually, but to treat all the range of spaces \( V^p_w \) as a whole. We think of each \( V^p_w \) as variant of a single spline-type space \( V = V(F, \Lambda) \).

It will be assumed that the family \( F \) is a Banach frame for each \( V^p_w \). The general theory of localized frames [25] [21] [3] ensures that this is indeed the case provided that \( F \) is a Hilbert space frame for \( V^2 \) and that \( F \) satisfies a localization property. In our context this property amounts to spatial localization and the adequate technical tool was developed in [27] (see also [35].)

We now formulate precisely the assumptions that we will make on the set of atoms \( F \) and show that under those assumptions, \( F \) is a Banach frame for the whole range of spaces \( V^p_w \).

- We will assume that we have chosen a uniformly localizable and isometrically translation invariant Banach space \( B \), that is continuously embedded into \( L^\infty_{loc} \). An example to keep in mind are fractional Sobolev spaces \( L^s_{loc} \) if either \( q = +\infty \) or if \( s > d/q \) (see [1].)
- We will also assume that \( F \) satisfies the uniform concentration and smoothness condition \( \|F\|_{W(B, L^s_{loc})} < +\infty \), for some weight \( w : \mathbb{R}^d \to (0, \infty) \) that verifies,
  - \( w(x) := \exp \rho(\|x\|) \), for some norm \( \|\cdot\| \) on \( \mathbb{R}^d \) and some continuous, concave function \( \rho : [0, \infty] \to [0, \infty] \) such that \( \rho(0) = 0 \) and \( \lim_{x \to +\infty} \frac{\rho(x)}{x} = 0 \),
  - \( w(x) \geq (1 + \|x\|)^\delta \), for some \( \delta > 0 \).
- Finally, we will assume that \( F \) forms a frame sequence in \( L^2(\mathbb{R}^d) \).

If all the above assumptions are met we say that \( V = V(F, \Lambda) \) is a spline type space.

**Remark 6.** Under the above assumptions, the weight \( w \) satisfies: \( w(0) = 1 \), \( w(x) = w(-x) \) and is submultiplicative (see [27].)

**Remark 7.** The polynomial weights \( w_\alpha \) with \( \alpha > 0 \) and the subexponential weights \( w(x) := e^{\alpha \|x\|^\beta} \) with \( \alpha > 0 \) and \( 0 < \beta < 1 \) satisfy the assumptions above (see [27].)

The first items of the next proposition are a variation of [21] Prop. 2.3), adapted to our context.

**Theorem 1.** Let \( V = V(F, \Lambda) \) be a spline type space, then the following holds.

1. \( G \equiv \{g_k\}_k \), the canonical dual family of \( F \) satisfies \( \|G\|_{W(B, L^s_{loc})} < +\infty \).
2. For any \( 1 \leq p \leq \infty \) and any symmetric, \( w \)-moderated weight \( v \), we have that \( F \) is a Banach frame for \( V^p_v \) with associated sequence space \( z^p_v \).
3. For any \( 1 \leq p \leq \infty \) and any symmetric, \( w \)-moderated weight \( v \), we have the inclusion \( V^p_v \subseteq W(B, L^p_w) \). Moreover, on \( V^p_v \), the \( L^p_v \) and \( W(B, L^p_w) \) norms are equivalent.

**Remark 8.** In this situation, for \( \ell^\infty \), the frame expansion arising from the pair \( (F,G) \) can be extended to the weak* closure of \( V^{00} \) within \( L^\infty_{loc} \), but the series converge only in the weak* topology. For details see [17] and [21].

**Remark 9.** The norm equivalence of item (c) holds uniformly for \( 1 \leq p \leq +\infty \) and any class of \( w \)-moderated weights for which the constant in Equation (6) is bounded.
Proof. Consider the self-correlation matrix $C$, given by $c_{kj} := (f_k, f_j)$. By proposition \[4\] we have that $\|C\|_{S_w} < +\infty$. Since $F$ is a frame sequence in $L^2(\mathbb{R}^d)$, the matrix $C$ has a pseudo-inverse $D \in B(\ell^2)$. It follows from \cite[Theorem 3.4]{1} that $\|D\|_{S_w} < +\infty$. Since $G = D \cdot F$, it follows from Proposition \[3\] that $\|G\|_{W(\mathcal{B}, L^1_{\mathcal{L}})} < +\infty$. This proves (a).

(b) is a slight variation of Proposition 2.3 in \cite{21}. We only sketch the argument. Let $v$ be a symmetric, $w$-moderated weight and let $1 \leq p \leq \infty$. The reconstruction operator $R : \ell^p_v \to V^p_v$, $c \mapsto c \cdot F$, is well defined and bounded by Proposition \[2\]. Moreover $\|R\| \lesssim \|F\|_{W(\mathcal{B}, L^1_{\mathcal{L}})}$. Proposition \[5\] implies that the coefficients mapping $C : L^p_v \to \ell^p_v$, given by $f \mapsto \{(f, g_k)\}_k$ is well defined and satisfies $\|C\| \lesssim \|G\|_{W(\mathcal{B}, L^1_{\mathcal{L}})}$. Moreover, if $f$ is a finite linear combination of functions of $F$, we have that $RC(f) = f$. It follows that $(F, G)$ determines a Banach frame pair.

Now (c) follows easily from Proposition \[4\]. Since $\mathcal{B} \hookrightarrow L^\infty, loc \hookrightarrow L^p, loc$, we have the inclusion $W(\mathcal{B}, L^p_v) \hookrightarrow W(L^p, L^p_v)$. Therefore, for $f \in V^p_v$, $f = RC(f)$ and,

\[
\|f\|_{L^p_v} \lesssim \|f\|_{W(\mathcal{B}, L^p_v)} \lesssim \|F\|_{W(\mathcal{B}, L^1_{\mathcal{L}})}\|C(f)\|_{\ell^p_v} \lesssim \|F\|_{W(\mathcal{B}, L^1_{\mathcal{L}})}\|G\|_{W(\mathcal{B}, L^1_{\mathcal{L}})}\|f\|_{L^p_v}.
\]

We now observe that, in order to bound an operator on a spline-type space, we just need to control its behavior on the atoms.

**Proposition 6.** Let $V = V(F, \Lambda)$ be a spline-type space. Let $v$ be a symmetric, $w$-moderated weight, $1 \leq p \leq \infty$ and let $T : V^p_v \to L^p_v$ be a linear operator. Then,

\[
\|T\|_{V^p_v \to L^p_v} \lesssim \|T(F)\|_{W(\mathcal{B}, L^1_{\mathcal{L}})}.
\]

**Proof.** If $f = c \cdot F$ for a finitely supported sequence $c \in \ell^p(\Lambda)$, then $T(f) = c \cdot T(F)$. Theorem \[1\] implies that,

\[
\|T(f)\|_{L^p_v} \lesssim \|T(f)\|_{W(L^\infty, L^p_v)} \lesssim \|T(f)\|_{W(\mathcal{B}, L^p_v)} \lesssim \|c\|_{\ell^p} \|T(F)\|_{W(\mathcal{B}, L^1_{\mathcal{L}})} \lesssim \|f\|_{L^p_v} \|T(F)\|_{W(\mathcal{B}, L^1_{\mathcal{L}})}.
\]

The conclusion extends to general $f$ by an approximation argument.

Finally we observe that, as a consequence of Theorem \[1\] there is a universal projector $P : L^p_v \to V^p_v$, for all $1 \leq p \leq \infty$ and $w$-moderated weights $v$. More precisely, we have the following statement.

**Theorem 2.** Let $V = V(F, \Lambda)$ be a spline-type space and let $P : L^2 \to V^2$ be the orthogonal projector onto $V^2$. Then, for all $1 \leq p < \infty$ and $w$-moderated weights $v$, the restriction of $P$ to $S(\mathbb{R}^d)$ extends by density to a bounded projector $P : L^p_v \to V^p_v$. For $p = \infty$ the same statement is true replacing $L^\infty_v$ for $C^0$.

Moreover, the norm of $P$ is uniformly bounded for $1 \leq p \leq +\infty$ and any class of $w$-moderated weights for which the constant in Equation \[6\] is bounded.

**Proof.** We only need to check that the restriction of $P$ to $S(\mathbb{R}^d)$ is bounded in the norm of $L^p_v$. The projector $P$ is given by

\[
P(f) = \sum_{k \in \Lambda} \langle f, g_k \rangle f_k,
\]
where $G$ is the family of dual atoms given by Theorem 1 (a). Using Propositions 2 and 3 we get,

\[ \|P(f)\|_{L^p_E} \lesssim \|P(f)\|_{W(B, L^p_E)} \lesssim \|F\|_{W(B, L^p_E)} \|G\|_{W(B, L^p_E)} \|f\|_{L^p_E}. \]

\[\Box\]

4. Frame surgery

In this section we consider the following locality problem. We are given a spline type space $V$ and several exterior frames \( \{\varphi_k^i\}_{k \in \Lambda_i}, i \in I \), for it. For each of these frames, we arbitrarily select a region of the euclidean space $E_i$ where we want to use it. The family \( \{E_i\}_{i} \) must form a covering of $\mathbb{R}^d$. We argue that, if for each $i \in I$ we pick from the frame \( \{\varphi_k^i\}_{k \in \Lambda_i} \) those elements that are concentrated near $E_i$, then the resulting family \( \{\varphi_k^i\}_{k \in \Delta_i, i \in I} \) forms an exterior frame for $V$. Moreover, given (possibly non-canonical) dual frames for each of the original exterior frames, we provide an approximate reconstruction operator for the new exterior frame.

Since we are not dealing with frames for the whole space $L^2(\mathbb{R}^d)$, we cannot take a function $f$, break it into pieces $f_i$ supported on $E_i$, expand each $f_i$ using the exterior frame \( \{\varphi_k^i\}_{k \in \Lambda_i} \), and then add all those expansions. This approach does not work in our context because for $f \in V$, the localized pieces $f_i$ do not belong to $V$ and consequently cannot be expanded using the frame \( \{\varphi_k^i\}_{k} \).

Instead, we argue that for each member of the covering $E_i$, the norm of a function $f \in V$ restricted to $E_i$ should depend mainly on the atoms concentrated around $E_i$. Then we glue these local estimates by means of an almost-orthogonality principle, which is implicit in the computations below.

To be able to quantify the approximation scheme we will work with frames polynomially localized in space.

4.1. The approximation scheme. We now give the precise assumptions for this section.

- We assume that $V = V(F, \Lambda)$ is a spline-type space where the atoms $F \equiv \{f_k\}_{k \in \Lambda}$ and a given system of dual atoms $G \equiv \{g_k\}_{k \in \Lambda}$ satisfy,

\[ |f_k(x)|, |g_k(x)| \leq C (1 + |x - k|)^{-(s+\alpha)} \quad (x \in \mathbb{R}^d, k \in \Lambda), \]

for some constants $C > 0$, $s > d$ and $\alpha \geq 0$. It is well-known \cite{25} that if this condition holds for the atoms $F$, then it is automatically satisfied by some system of dual atoms $G$.

- We are given a family of frame pairs for $V^2$, \( \{\psi_k^i\}_{k \in \Lambda_i}, \{\varphi_k^i\}_{k \in \Lambda_i} \) \quad (i \in I),

that satisfy the following uniform concentration condition around their nodes $\Lambda_i$,

\[ |\psi_k^i(x)|, |\psi_k^i(x)| \leq C (1 + |x - k|)^{-(s+\alpha)} \quad (x \in \mathbb{R}^d, k \in \Lambda_i, i \in I), \]

for some constant $C > 0$, that, for simplicity, is assumed to be equal to the constant in (10).

\[\text{1Remember that the analyzing atoms } \{\varphi_k^i\}_{k} \text{ need not to belong to } V^2.\]
Observe that we are requiring all the frames and the dual frames to be uniformly localized. Given a concrete family of uniformly localized (exterior) frames, it can be difficult to decide if they possess a corresponding family of dual frames sharing a common spatial localization. This problem is addressed in Section 5.

- We have a (measurable) covering of \( \mathbb{R}^d \), \( \mathcal{E} \equiv \{ E_i \}_{i \in I} \) that is uniformly locally finite. This means that for some (or any) cube \( Q \),

\[
\#_{\mathcal{E},Q} := \max_{x \in \mathbb{R}^d} \# \{ i \in I \mid (Q + x) \cap E_i \neq \emptyset \} < \infty.
\]

Observe that this assumption in particular implies that number of overlaps of \( \mathcal{E} \) is finite. That is,

\[
\#_{\mathcal{E}} := \max_{x \in \mathbb{R}^d} \# \{ i \in I \mid x \in E_i \} < \infty.
\]

- We suppose that the set of nodes \( \Lambda_i \) are uniformly relatively separated. That is,

\[
\sup_{i \in I} \text{rel}(\Lambda_i) < \infty.
\]

Observe that this assumption, together with the uniform localization of the dual frames, implies that the original frames have a uniform common lower bound.

We now prove the central approximation result. In Section 4.2 we apply this result to the construction of new frames.

**Theorem 3.** Let \( \{ \eta_i \}_{i \in I} \) be a (measurable) partition of the unity subordinated to \( \mathcal{E} \). That is, each \( \eta_i \) is nonnegative, \( \sum_i \eta_i \equiv 1 \) and \( \text{supp}(\eta_i) \subseteq E_i \).

For each \( r > 0 \) consider the sets

\[
\Lambda_i^r := \{ k \in \Lambda_i/d(k,E_i) \leq r \}
\]

and choose any set \( \Delta_i^r \) such that \( \Lambda_i^r \subseteq \Delta_i^r \subseteq \Lambda_i \). Consider also, the following approximate reconstruction operator,

\[
A^r(f) := \sum_{i \in I} \sum_{k \in \Delta_i^r} \langle f, \varphi_k \rangle \psi_k^i \eta_i.
\]

Then, for any \( 1 \leq p \leq \infty \), any \( w_\alpha \)-moderated weight \( v \) and any \( f \in V_v^p \),

\[
\| A^r(f) - f \|_{L^p_v} \leq K \#_{\mathcal{E}} \| f \|_{L^p_v} r^{-(s-d)},
\]

where \( K > 0 \) is a constant that only depends on \( d, C, s, \alpha \), the set of nodes \( \Lambda_i \) and the common relative separation of all the sets of nodes \( \Lambda_i \) and the constant in Equation (6) using the weight \( w_\alpha \) as moderator.

**Remark 10.** The fact that the covering is uniformly locally finite is not used in the proof of the theorem; only the weaker condition in Equation (13) is needed. However, the stronger assumption of Equation (12) is required for the applications of Section 7.

**Proof.** Observe first that we can always add more nodes to the set \( \Lambda \) and extend the set of atoms \( F \) and dual atoms \( G \) by associating 0 to the new nodes. All the assumptions on the atoms are preserved by this extension, but the relative separation of the set of nodes changes. By adding to \( \Lambda \) any fixed relatively separated
Consequently, $F, G$.

Since (Lemma 3 implies that), we can assume that $\Lambda$ is $L$-dense (c.f. Equation (3)).

The relative separation of the resulting set can be bounded by $\text{rel}(\Lambda) + \text{rel}(\Gamma)$.

For all $i \in I$, every $f \in V^0$ admits the expansion,

$$f = \sum_{j \in \Lambda_i} \langle f, \varphi^i_j \rangle \psi^i_j.$$  

Averaging all these expansions yields,

$$f = \sum_{i \in I} \sum_{j \in \Lambda_i} \langle f, \varphi^i_j \rangle \psi^i_j \eta_i.$$  

Since $f$ also admits the expansion $f = \sum_k \langle f, g_k \rangle f_k$, it follows that

$$f = \sum_{k \in \Lambda} \langle f, g_k \rangle \sum_{i \in I} \sum_{j \in \Delta^i_0} \langle f_k, \varphi^i_j \rangle \psi^i_j \eta_i.$$  

Similarly,

$$A^r(f) = \sum_{i \in I} \sum_{j \in \Delta^i_0} \langle f, \varphi^i_j \rangle \psi^i_j \eta_i$$

$$= \sum_{k \in \Lambda} \langle f, g_k \rangle \sum_{i \in I} \sum_{j \in \Delta^i_0} \langle f_k, \varphi^i_j \rangle \psi^i_j \eta_i.$$  

Therefore, for $f \in V^0$,

$$f - A^r(f) = \sum_{k \in \Lambda} \langle f, g_k \rangle \sum_{i \in I} \sum_{j \in \Delta^i_0} \langle f_k, \varphi^i_j \rangle \psi^i_j \eta_i.$$  

Consequently, if we set $c_k := \langle f, g_k \rangle$, by Lemma [1]

$$|f - A^r(f)| \lesssim \sum_{k \in \Lambda} |c_k| \sum_{i \in I} \sum_{j \notin \Delta^i} \langle w_{-(s+\alpha)}(\cdot - k), w_{-(s+\alpha)}(\cdot - j) \rangle \chi_E,$$

$$\lesssim \sum_{k \in \Lambda} |c_k| \sum_{i \in I} \sum_{j \notin \Delta^i} w_{-(s+\alpha)}(k - j) w_{-(s+\alpha)}(\cdot - j) \chi_E.$$  

If we define,

$$E^r_k := \sum_{i \in I} \sum_{j \notin \Delta^i} w_{-(s+\alpha)}(k - j) w_{-(s+\alpha)}(\cdot - j) \chi_E,$$

Lemma [3] implies that,

$$||f - A^r(f)||_{L^p_\alpha} \lesssim ||c||_{L^p_\alpha} \left( \sup_k ||E^r_k w_\alpha(\cdot - k)||_1 \right)^{1/p} \left( \sup_{x \in \mathbb{R}^d} \sum_k |E^r_k(x) w_\alpha(x - k)| \right)^{1/p'}.$$  

Since $(F, G)$ is a frame pair for $V^p_\alpha$, $||c||_{L^p_\alpha} \leq K ||f||_{L^p}$. for some constant $K$ that only depends on $d, C, s, \alpha$, the constant in Equation (13) (taking $w = w_\alpha$) and $\Lambda$.

Consequently,

$$||f - A^r(f)||_{L^p_\alpha} \leq K ||f||_{L^p} \left( \sup_k ||E^r_k w_\alpha(\cdot - k)||_1 \right)^{1/p} \left( \sup_{x \in \mathbb{R}^d} \sum_k |E^r_k(x) w_\alpha(x - k)| \right)^{1/p'}.$$  

Now observe that, since $w_\alpha(x - k) \leq w_\alpha(x - j) w_\alpha(k - j),

$$|E^r_k(x) w_\alpha(x - k)| \leq \sum_{i \in I} \sum_{j \notin \Delta^i} w_{-(s+\alpha)}(k - j) w_{-(s+\alpha)}(\cdot - j) \chi_E(x).$$
For every $i \in I$, since $\Lambda$ is now assumed to be $L$-dense, there exists a map $\mu_i : \Lambda_i \to \Lambda$ such that $|k - \mu_i(k)| \leq L$, for all $k \in \Lambda_i$. This map will be used to reduce the proof to the case where all the index sets are equal. This same argument was used in [3], where irregularly distributed phase-space points are assigned a near point in a regular reference system by means of a ‘round-up’ map.

Since the sets $\Lambda_i$ are assumed to be uniformly relatively separated, there exists a number $N \in \mathbb{N}$, that depends only on $L$ and the relative separation of all the sets of nodes, such that

$$\#\mu_i^{-1}(\{j\}) \leq N, \text{ for every } j \in \Lambda.$$ 

Suppose initially that $r > 2L$, define $R := r - L$ and estimate,

$$|E^r_k(x)w_\alpha(x - k)| \leq \sum_{i \in I} \sum_{j \in \Lambda, l \in \mu_i^{-1}(j), \ i \not\in \Lambda^R_i} w_-(k - l)w_-(x - l)\chi_{E_i}(x).$$

If $l \in \mu_i^{-1}(j)$, then $|j - l| \leq L$, so $w_-(k - l) \lesssim w_-(k - j)$ and $w_-(x - l) \lesssim w_-(x - j)$. (Here the implicit constants depend on $L$ and $s$.) If in addition $l \not\in \Lambda^R_i$, then $j \not\in \Omega^R_i$, where

$$\Omega^R_i := \{ k \in \Lambda \mid d(k, E_i) \leq R \}.$$ 

Consequently,

$$|E^r_k(x)w_\alpha(x - k)| \lesssim N \sum_{i \in I} \sum_{j \not\in \Omega^R_i} w_-(k - j)w_-(x - j)\chi_{E_i}(x).$$

Using this estimate, we bound the weighed Schur norm of the kernel $E^r_k$.

For every $k \in \Lambda$,

$$\|E^r_k w_\alpha(\cdot - k)\|_1 \lesssim \sum_{i \in I} \sum_{j \not\in \Omega^R_i} w_-(j - k) \int_{E_i} w_-(x - j) dx$$

$$= \sum_{j \in \Lambda} \sum_{d(j, E_i) > R} w_-(j - k) \int_{E_i} w_-(x - j) dx$$

$$\leq \#\mathcal{E} \sum_{j \in \Lambda} w_-(j - k) \int_{\bigcup E_i} w_-(x - j) dx,$$

where the union in the last integral ranges over all $i \in I$ such that $d(j, E_i) > R$.

Since the complement of the cube $Q_R(j)$ contains that union, we get,

$$\|E^r_k w_\alpha(\cdot - k)\|_1 \leq \#\mathcal{E} \sum_{j \in \Lambda} w_-(j - k) \int_{\mathbb{R}^d \setminus Q_R(j)} w_-(x - j) dx$$

$$= \#\mathcal{E} \sum_{j \in \Lambda} w_-(j - k) \int_{\mathbb{R}^d \setminus Q_R(0)} w_-(x) dx.$$ 

The set $\Lambda - k$ has the same relative separation that $\Lambda$, so Lemma [2] implies that

$$\|E^r_k w_\alpha(\cdot - k)\|_1 \lesssim \#\mathcal{E} \int_{\mathbb{R}^d \setminus Q_R(0)} w_-(x) dx$$

$$\lesssim \#\mathcal{E} R^{-(s-d)}.$$
Since $r > 2L$, it follows that $R > r/2$ and

\[ \sup_k \| E_k^x w_\alpha(x - k) \|_1 \lesssim \# \varepsilon r^{-(s-d)}. \]  

Using again the estimate in Equation (16), we bound now sup \( x \sum_k |E_k^x(x)w_\alpha(x - k)|. \)

Fix \( x \in \mathbb{R}^d \) and let

\[ I_x := \{ i \in I \mid x \in E_i \}. \]

From Equation (13) we know that \( \# I_x \leq \# \varepsilon. \) We now estimate,

\[ \sum_{k \in \Lambda} |E_k^x(x)w_\alpha(x - k)| \lesssim \sum_{i \in I_x} \sum_{j \notin \Omega_i^R} \sum_{k \in \Lambda} w_{-s}(k - j) w_{-s}(x - j). \]

Since \( \Lambda \) and \( \Lambda - \{ j \} \) have the same relative separation, Lemma 2 implies that,

\[ \sum_{k \in \Lambda} w_{-s}(k - j) \lesssim 1, \]

so,

\[ \sum_{k \in \Lambda} |E_k^x(x)w_\alpha(x - k)| \lesssim \sum_{i \in I_x} \sum_{j \notin \Omega_i^R} w_{-s}(x - j). \]

For \( i \in I_x \) and \( j \notin \Omega_i^R \), we have that \( |x - j| \geq d(j, E_i) > R \). It follows that,

\[ \sum_{k \in \Lambda} |E_k^x(x)w_\alpha(x - k)| \leq \sum_{i \in I_x} \sum_{|x - j| > R} w_{-s}(x - j). \]

Since the sets \( \Gamma \) and \( x - \Gamma \) have the same relative separation, Lemma 2 implies that,

\[ \sum_{k \in \Lambda} |E_k^x(x)w_\alpha(x - k)| \lesssim \# \varepsilon R^{-(s-d)}. \]

Using again the fact that \( r > 2L \), it follows that,

\[ \sup_{x \in \mathbb{R}^d} \sum_{k \in \Lambda} |E_k^x(x)w_\alpha(x - k)| \lesssim \# \varepsilon r^{-(s-d)}. \]  

Combining the estimates in Equations (17), (18) and (19), it follows that

\[ \| A^r(f) - f \|_{L^p} \lesssim \| f \|_{L^p} \# \varepsilon r^{-(s-d)}, \]

for \( r > 2L \).

It remains to show that a similar estimate holds for \( 0 < r \leq 2L \). In this case, \( r^{-(s-d)} \gtrsim 1 \). So, it suffices to observe that \( \| A^r \|_{V^p_{-s} \to L^p} \lesssim \# \varepsilon \), uniformly on \( r \). Reexamining the estimates given for the error kernel \( E^r \), the desired conclusion follows.

\[ \Box \]

**Remark 11.** The technique in the proof of the theorem of using the frame expansion twice is somehow analogous to the use of reproducing formulas in the classical decomposition results for function spaces (see for example [22] and [17].)

The formula defining the operator \( A^r \) makes sense in \( L^p_r \), but the bound given in the theorem is only valid in the smaller subspace \( V_{-s}^p \), where the “reproducing formula” (the frame expansion) is valid. By means of it, the task of bounding the operator is reduced in the proof to the one of controlling its behavior on atoms, much in the spirit of the classical atomic decompositions (see [22] and also [28].)
4.2. Constructing new frames. We now interpret the approximation result of Section 4.1 as a method to produce new frames. Observe that, however, for some applications, the estimate provided by Theorem 3 is all that is needed. If concrete atoms and dual atoms are known, then the estimate in the theorem provides an approximate reconstruction operator for the new system of atoms.

Consider again the ingredients of Section 4.1 and let \( \{\eta_i\}_{i \in I} \) be a (measurable) partition of the unity subordinated to \( \mathcal{E} \) (e.g., \( \eta_i = (\sum_j \chi_{E_j})^{-1} \chi_{E_i} \)).

Let \( v \) be a \( w_\alpha \)-moderated weight and let \( P : L^p_v \to \mathcal{V}_v^p \) be the universal projector onto \( \mathcal{V}_v^p \) (cf. Theorem 2).

Fix a value of \( r > 0 \) and consider the operator \( B_r : \mathcal{V}_v^p \to \mathcal{V}_v^p \) given by \( B_r := P \circ A^r \), where

\[
A^r(f) := \sum_{i \in I} \sum_{k \in \Lambda'_r} \langle f, \varphi^i_k \rangle \psi^i_k \eta_i,
\]

and, as before, \( \Lambda'_r := \{k \in \Lambda_i / d(k, E_i) \leq r\} \).

For each \( i \in I \), let \( (\mathcal{V}_v^p)_i \) be the \( L^p_v \)-closed linear space generated by the atoms \( \{\psi^i_k \eta_i\}_{k \in \Lambda'_r} \). These spaces, of course, depend on \( r \).

Consider the direct sum \( \oplus_i (\mathcal{V}_v^p)_i \) as a subspace of \( \ell^p_{L^p_v} \), the space of \( \ell^p_v \)-valued \( \ell^p \) families; more precisely, \( \oplus_i (\mathcal{V}_v^p)_i \) is the closure of the algebraic direct sum within \( \ell^p_{L^p_v} \). Let \( \iota : \oplus_i (\mathcal{V}_v^p)_i \to \mathcal{V}_v^p \) be the operator given by

\[
\iota((f^i)_i) := \sum_i f^i.
\]

Since \( \mathcal{E} \) is locally finite, \( \iota \) is well-defined and bounded uniformly on \( p \) and \( v \). Indeed, for \( 1 \leq p < \infty \),

\[
\|\sum_i f^i\|^p_{\ell^p_{L^p_v}} = \int_{\mathbb{R}^d} \left| \sum_i f_i(x) \right|^p v(x) \, dx 
\leq \int_{\mathbb{R}^d} \left( \sum_i |f_i(x)|^p \right)^p v(x) \, dx,
\]

where \( I_x := \{i \in I \mid x \in E_i\} \). Since \( \#I_x \leq \#\mathcal{E} \),

\[
\|\iota((f^i)_i)\|^p_{\ell^p_{L^p_v}} \leq \#\mathcal{E} \int_{\mathbb{R}^d} \sum_i |f_i(x)|^p v(x) \, dx 
= \#\mathcal{E} \sum_i \|f_i\|^p_{L^p_v} 
= \#\mathcal{E} \|((f^i)_i)\|^p_{\ell^p_{L^p_v}}.
\]

So, \( \|\sum_i f^i\|_{L^p_v} \leq \#\mathcal{E} \|((f^i)_i)\|_{\ell^p_{L^p_v}} \). For \( p = \infty \), a similar computation establishes the same estimate. Composing \( \iota \) with the projector \( P \), we get a synthesis operator \( S\gamma : \oplus_i (\mathcal{V}_v^p)_i \to \mathcal{V}_v^p \).

For each \( i \in I \), let \( Q_i : \mathcal{V}_v^p \to (\mathcal{V}_v^p)_i \) be given by

\[
Q_i(f) := \sum_{k \in \Lambda'_r} \langle f, \varphi^i_k \rangle \psi^i_k \eta_i.
\]

The concentration conditions on Equation (10) imply that all these operators are uniformly bounded. Moreover, they determine a map \( Q : \mathcal{V}_v^p \to \oplus_i (\mathcal{V}_v^p)_i \), given
by $Q(f) := (Q_i(f))$. We will prove below that $Q$ is well defined and bounded. Assuming this for the moment, we have a commutative diagram,

$$
\begin{array}{ccc}
V^p_v & \xrightarrow{Q} & \bigoplus_i (V^p_v)_i \\
\downarrow B^r & & \downarrow S_N \\
V^p_v & \xrightarrow{} & V^p_v
\end{array}
$$

It follows from Theorem 3 that for a sufficiently large value of $r > 0$, $B^r$ is invertible and consequently $Q$ is left-invertible and $S_N$ is right-invertible. This provides two ways of viewing $V^p_v$ as a retract of $\bigoplus_i (V^p_v)_i$. One is $Q$ (with retraction $(B^r)^{-1} S_N$) and the other is $Q(B^r)$ (with retraction $S_N$.) In the spirit of [30] and [6], this can be called an exterior Banach fusion frame or an exterior stable splitting (see also [16] and [13].)

Now observe that each of the maps $Q_i$ can be factored through $z^p_v$, $V^p_v \xrightarrow{Q} (\bigoplus_i (V^p_v)_i) \xrightarrow{C_i} z^p_v(\Lambda^*_i) \xrightarrow{R_i} z^p_v(\Lambda^*_i)$

where $C_i(f) := (\langle f, \varphi^i_k \rangle)_k$ and $R_i(c) := \sum_k c_k \psi^i_k \eta_i$.

This induces a commutative diagram,

$$
\begin{array}{ccc}
V^p_v & \xrightarrow{Q} & \bigoplus_i (V^p_v)_i \\
\downarrow C & & \downarrow R \\
\bigoplus_i z^p_v(\Lambda^*_i) & \xrightarrow{} & \bigoplus_i z^p_v(\Lambda^*_i)
\end{array}
$$

where in $\bigoplus_i z^p_v(\Lambda^*_i)$ we use the $p$-norm; that is $\|((c^i)_i)_i\| := \|((\|c^i\|_p)_i)_i\|_p$. This is just a weighed $\ell^p$ norm; this way of presenting it is due to the structure of the index sets. The boundedness of the operators $C$ and $R$ is proved in Theorem 4 below. Assuming this fact for the moment, observe that if $B^r$ is invertible, then $Q$ is left-invertible and so is $C$. We formalize this in the following Theorem.

**Theorem 4.** Suppose that the assumptions of Section 4.1 are satisfied. Let $v$ be a $w_\alpha$-moderated weight. Then for all sufficiently large values of $r > 0$,

$\{ \varphi^i_k : i \in I, k \in \Lambda^*_i \}$

is a Banach frame for $V^p_v$.

More precisely, if we define the index set $\Gamma := \bigcup_{i \in I} \Lambda^*_i \times \{ i \}$ and the weight $V(k,i) := v(k)$, then the analysis map

$V^p_v \to z^p_v(\Gamma)$

$f \mapsto (\langle f, \varphi^i_k \rangle)_{(k,i)}$

is bounded and left-invertible, for all sufficiently large values of $r > 0$.

Moreover, the value of $r$ may be chosen uniformly for all $1 \leq p \leq \infty$ and every class of $w_\alpha$-moderated weights for which the respective constant (cf. Equation 6) is uniformly bounded.
Remark 12. Observe that although we are constructing a new frame \( \{ \varphi_k^i \}_{i \in I, k \in \Lambda_i^r} \) out of the pieces \( \{ \varphi_k^i \}_{k \in \Lambda_i^r} \), we do not claim that each of these pieces forms a frame sequence. This construction should be compared to the methods in [2, 20] and [6] where a global frame is built from local (possibly exterior) frames for certain subspaces.

Remark 13. As a related result, we mention Lemma 4.7 in [33] where it is shown that if \( \{ 2^{k/2} \psi(2^k \cdot - j) \mid k,j \in \mathbb{Z} \} \) is a wavelet frame for \( L^2(\mathbb{R}) \) and the wavelet \( \psi \) satisfies a mild smoothness condition, then for all sufficiently large values of \( r > 0 \), the system of fine scales \( \{ 2^k \psi(2^k \cdot + j) \mid k \in \mathbb{Z}, j \geq 0 \} \) forms an exterior frame for the subspace \( H_r := \{ f \in L^2(\mathbb{R}) \mid \hat{f} \equiv 0 \text{ on } [-r, r] \} \).

Proof. Using Theorem [3] choose a value of \( r > 0 \) such that the operator \( B^r \) is invertible. By the discussion above, it only remains to bound the operators \( C \) and \( R \). Consider the index set \( \Gamma \) as a set with multiplicity (cf. Section 2.7), where we map each of the sets \( \Lambda_i^r \times \{ i \} \) into \( \mathbb{R}^d \) by discarding the second coordinate.

The fact that the sets \( \Lambda_i \) are uniformly relatively separated (cf. Equation (13)) and that the covering \( E \) is uniformly locally finite (cf. Equation (12)) implies that \( \Gamma \) is relatively separated. Indeed, let \( Q \) be the unit cube and \( Q' := Q + [-r, r]^d \).

For any \( x \in \mathbb{R}^d \),

\[
\{(k,i) \in \Gamma \mid k \in Q + \{ x \} \} = \bigcup_{i \in I} \{ k \in \Lambda_i \cap (Q + \{ x \}) \mid d(k,E_i) \leq r \} \times \{ i \} \\
\subseteq \bigcup_{i \in I_x} (\Lambda_i \cap (Q + \{ x \})) \times \{ i \},
\]

where \( I_x := \{ i \in I \mid E_i \cap (Q' + \{ x \}) \neq \emptyset \} \). Hence \( \text{rel}(\Gamma) \leq \#E \cdot Q' \sup_i \text{rel}(\Lambda_i) < \infty \) (cf. Equation (12)).

The family of atoms \( \{ \varphi_k^i \}_{(k,i) \in \Gamma} \) satisfies a polynomial concentration condition. By Example [1] the family has finite \( W(L^\infty, L^1_w) \) norm. The boundedness of the operator \( C \) now follows from Propositions [4].

For the boundedness of \( R \), observe that the families \( \{ \psi_k^i \eta_i \}_{k \in \Lambda_i^r} \) satisfy a uniform polynomial concentration condition and their nodes are uniformly relatively separated. Hence, by Example [1]

\[
M := \sup_i \| \{ \psi_k^i \eta_i \} \|_{W(L^\infty, L^1_w)} < \infty.
\]

Consequently, by Proposition [2] for \( 1 \leq p < \infty \),

\[
\| R(c) \|_{L^p_{c\cdot}}^p \leq \| \sum_i \| \sum_{k \in \Lambda_i^r} c_k^i \psi_k^i \eta_i \|_{L^p_{c\cdot}} \|
\leq MP \sum_i \| c^i \|_{L^p_{c\cdot}}^p
= MP \| c \|_{L^p_{c\cdot}}^p.
\]

So, \( \| R(c) \|_{L^p_{c\cdot}} \lesssim \| c \|_{L^p_{c\cdot}} \). A similar computation shows that the same estimate is valid for \( p = \infty \). \( \square \)
5. QUALITY STATEMENTS ABOUT DUAL ATOMS

The theory of localized frames asserts that, if the atoms in a frame are sufficiently localized (in an abstract sense), then the dual atoms are also localized. Theorem 5 below shows that if a family of frames is sufficiently and uniformly localized, then the respective dual families are also uniformly localized. This is relevant to the construction in Section 4.

To obtain this qualitative statement we need to know not only that the polynomial off-diagonal decay of a matrix \( M \) is inherited by its pseudo-inverse \( M^\dagger \), but also what qualities of the original matrix \( M \) determine the constants governing the off-diagonal decay of \( M^\dagger \). This extra information is essentially provided by the main result in [35]. Lemma 4 below states the required qualitative statement and Theorem 5 is just an application of it to our setting.

Lemma 4. Let \( \Gamma \subseteq \mathbb{R}^d \) be a relatively separated set and let \( M \in B(\ell^2(\Gamma)) \) be a positive operator. Assume the following.

- \( M \) satisfies,
  \[
  |M_{k,j}| \leq C(1 + |k - j|)^{-s} \quad (k, j \in \Gamma),
  \]
  for some constants \( C > 0 \) and \( s > d \).
- The spectrum of \( M \), satisfies,
  \[
  (\sigma(M) \setminus \{0\}) \cap B_A(0) = \emptyset,
  \]
  for some \( A > 0 \) (here \( B_A(0) \subseteq \mathbb{C} \) is the ball of radius \( A \) centered at 0.)
- \( \text{rel}(\Gamma) \leq R \), for some \( 0 \leq R < \infty \).

Then \( M^\dagger \), the Moore-Penrose pseudo-inverse of \( M \), satisfies,

\[
|M^\dagger_{k,j}| \leq C'(1 + |k - j|)^{-s} \quad (k, j \in \Gamma),
\]

for some constant \( C' \) that only depends on \( C, s, d, A \) and \( R \).

Proof. If we modify the hypothesis of the theorem so that \( M \) is assumed to be an invertible normal operator instead of a positive pseudo-invertible one (and hence \( M^\dagger = M^{-1} \)), then the conclusion follows from [35, Theorem 4.1].

The case of the pseudo-inverse is treated in [35], but no explicit reference to the qualities involved in the off-diagonal decay of the pseudo-inverse is made. However, the proof given in [35, Theorem 5.1] (see also [21]) can be slightly adapted to obtain a quantitative conclusion. We only sketch the modifications.

Under the assumptions of the theorem,

\[
(20) \quad M^\dagger = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z} (zI - M)^{-1} \, dz,
\]

where the curve \( \gamma \) is the rectangle with vertices \( A/2 \pm i, \|M\| + A/2 \pm i \) oriented anti-clockwise; here \( \|M\| \) denotes the norm of \( M \) in \( B(\ell^2(\Gamma)) \). Consequently, for \( k, j \in \Gamma \),

\[
(21) \quad M^\dagger_{k,j} = \frac{1}{2\pi i} \int_{\gamma} \frac{1}{z} (zI - M)_{k,j}^{-1} \, dz.
\]

Observe that \( \|M\| \) can be bounded in terms of \( d, s, C \) and \( R \) (by interpolating its \( \ell^1 \to \ell^1 \) and \( \ell^\infty \to \ell^\infty \) norm) and that the length of \( \gamma \) is \( 2\|M\| + 2 \). For \( z \) in the curve \( \gamma, |z| \lesssim \|M\| + 1 \) and \( |1/z| \leq 2/A \). Hence, it suffices to bound the off-diagonal decay of the resolvent \( (zI - M)^{-1} \) in terms of the allowed parameters.
Let \( z \) lie in the curve \( \gamma \). The distance from \( z \) to \( \sigma(M) \) is at least \( m := \min \{1, A/2\} \), so \( (\sigma(zI - M) \setminus \{0\}) \cap B_m(0) = \emptyset \). Moreover, for \( k, j \in \Gamma \),
\[
|z| \delta_{kj} + C(1 + |k - j|)^{-s} \lesssim (C + \|M\| + 1)(1 + |k - j|)^{-s}.
\]
By [33] Theorem 4.1, the off-diagonal decay of \( (zI - M)^{-1} \) is bounded by a constant depending only on allowed parameters. \( \square \)

Now we apply this estimate to spline-type spaces.

**Theorem 5.** Let \( V \equiv V^2(F, \Lambda) \) be a spline-type space, where the atoms \( F \) satisfy,
\[
|f_k(x)| \leq C(1 + |x - k|)^{-s} \quad (x \in \mathbb{R}^d, k \in \Lambda),
\]
for some constant \( C > 0 \) and \( s > 0 \). Assume the following.

- For each \( i \in I \), we have a family of measurable functions \( \{\varphi^i_k\}_{k \in \Lambda_i} \) that satisfy the following uniform concentration condition around their nodes \( \Lambda_i \):
  \[
  |\varphi^i_k(x)| \leq C'(1 + |x - k|)^{-s} \quad (x \in \mathbb{R}^d, k \in \Lambda_i),
  \]
  for some constant \( C' > 0 \) (independent of \( i \)).
- The set of nodes \( \Lambda_i \) are uniformly relatively separated. That is,
  \[
  \sup_{i \in I} \text{rel}(\Lambda_i) < \infty.
  \]
- Each family \( \{\varphi^i_k\}_{k} \) satisfies the (exterior) frame inequality\(^2\),
  \[
  A\|f\|_2^2 \leq \sum_{k} |\langle f, \varphi^i_k \rangle|^2 \leq B\|f\|_2^2,
  \]
  for \( f \in V^2 \) and constants \( 0 < A \leq B < \infty \) that are independent of \( i \).

Then, the respective families of canonical dual frame sequences \( \{\psi^i_k\}_{k} \subseteq V^2 \) satisfy,
\[
|\psi^i_k(x)| \leq D(1 + |x - k|)^{-s} \quad (x \in \mathbb{R}^d, k \in \Lambda_i),
\]
for some constant \( D \), independent of \( i \).

**Proof.** Let \( G \equiv \{g_k\}_{k} \) be the canonical dual frame of \( F \). By the results in [21], there exists a constant \( C'' > 0 \) such that
\[
|g_k(x)| \leq C''(1 + |x - k|)^{-s} \quad (x \in \mathbb{R}^d, k \in \Lambda).
\]
For each \( i \in I \) and \( k \in \Lambda_i \), let \( \varphi^i_k \) be the orthogonal projection of \( \varphi^i_k \) on \( V^2 \). Each of the functions has the expansion,
\[
\varphi^i_k = \sum_{j \in \Lambda} \langle \varphi^i_k, g_j \rangle f_j.
\]
Consequently using Lemmas 1 and 2,
\[
|\varphi^i_k| \lesssim CC'C'' \sum_{j \in \Lambda} w_{-s}(k - j)w_{-s}(\cdot - j)
\]
\[
\lesssim CC'C'' \text{rel}(\Lambda)w_{-s}(\cdot - k).
\]

\(^2\)Note that the functions \( \varphi^i_k \) need not to belong to \( V^2 \), cf. Section 2.
Since the exterior frame condition in the hypothesis is also satisfied by the functions \( \{\varphi_k^i\}_{k_i} \), we can replace each \( \varphi_k^i \) by \( \varphi_k^i \) and assume without loss of generality that \( \varphi_k^i \in V^2 \).

For each \( i \in I \), consider the Gram matrix \( M^i \) given by,

\[
M^i_{kj} := \langle \varphi_k^i, \varphi_j^i \rangle \quad (k, j \in \Lambda_i).
\]

By Lemma 1 it follows that

\[
\left| M^i_{kj} \right| \leq K (1 + |k - j|)^{-s} \quad (k, j \in \Lambda_i),
\]

for some constant \( K \) that depends on \( s \) and \( C' \). Moreover, since each \( \{\varphi_k^i\}_k \) is a frame with bounds \( A \) and \( B \), the spectrum of \( M^i \) satisfies,

\[
\sigma(M^i) \subseteq \{0\} \cup [A, B].
\]

By Lemma 4 the pseudo-inverse of \( M^i \) satisfies

\[
\left| (M^i)^{\dagger}_{kj} \right| \leq K'(1 + |k - j|)^{-s} \quad (k, j \in \Lambda_i),
\]

for some constant \( K' \) independent of \( i \).

Each of the dual elements \( \psi_k^i \) is given by,

\[
\psi_k^i = \sum_{j \in \Lambda_i} (M^i)^{\dagger}_{kj} \varphi_j^i.
\]

Therefore,

\[
\left| \psi_k^i(x) \right| \leq CK' \sum_{j \in \Lambda_i} w_{-s}(k - j)w_{-s}(j - x)
\]

Using Lemma 2 (c) with \( \Gamma := \Lambda_i - \{x\} \) and \( k' := k - x \), it follows that

\[
\left| \psi_k^i(x) \right| \leq K'' \text{rel}(\Gamma)w_{-s}(x - k) = K'' \text{rel}(\Lambda_i)w_{-s}(x - k).
\]

For some constant that \( K'' \) independent of \( i \). Since the sets of nodes are uniformly relatively separated, the conclusion follows. \( \square \)

6. Applications

6.1. Spline-type spaces. We now combine the results of Sections 4.2 and 5 in a concrete statement.

**Theorem 6.** Let \( V = V(F, \Lambda) \) be a spline-type space. Assume the following.

- The atoms \( F \) satisfy the polynomial concentration condition around their nodes,

\[
|f_k(x)| \leq C (1 + |x - k|)^{-s-\alpha} \quad (x \in \mathbb{R}^d, k \in \Lambda),
\]

for some constants \( C > 0 \), \( s > d \) and \( \alpha \geq 0 \).

- We are given a family of exterior frames for \( V^2 \), \( \{\varphi_k^i\}_{k \in \Lambda_i}, i \in I \), that satisfy the following uniform polynomial concentration condition around their nodes \( \Lambda_i \),

\[
|\varphi_k^i(x)| \leq C' (1 + |x - k|)^{-s-\alpha} \quad (x \in \mathbb{R}^d, k \in \Lambda_i, i \in I),
\]

for some constant \( C' > 0 \).
The exterior frames \( \{ \{ \varphi^i_k \} \}_{k \in I} \) share a uniform lower (and upper) bound. That is,
\[
A \|f\|_2^2 \leq \sum_k |\langle f, \varphi^i_k \rangle|^2 \leq B \|f\|_2^2 \quad (f \in V^2),
\]
holds for some constants \( 0 < A \leq B < \infty \). \[\□\]

The sets of nodes \( \Lambda_i \) are uniformly relatively separated (cf. Equation (14).)

We have a measurable covering of \( \mathbb{R}^d \), \( \mathcal{E} \equiv \{ E_i \}_{i \in I} \) that is uniformly locally finite (cf. Equation (12).)

Then, for all sufficiently large values of \( r > 0 \),
\[
\{ \varphi^i_k : i \in I, d(k, E_i) \leq r \}
\]
is an exterior Banach frame for \( V^p \).

More precisely, if we define the index set \( \Gamma_r := \bigcup_{i \in I} \Lambda^i_r \times \{ i \} \) and the weight \( V(\cdot) := v(\cdot) \), then the analysis map
\[
V^p \rightarrow z^p(\Gamma_r) \quad f \mapsto (\langle f, \varphi^i_k \rangle)_{(k,i)}
\]
is bounded and left-invertible.

Moreover, the value of \( r \) may be chosen uniformly for all \( 1 \leq p \leq \infty \) and every class of \( w_\alpha \)-moderated weights for which the respective constants (cf. Equation (6)) are uniformly bounded.

**Proof.** Combine Theorems 4 and 5. \[\□\]

### 6.2. Shift invariant spaces.

As a corollary of Theorem \[\□\] we describe a method to piece together bases of lattice translates. First recall some notation and facts for shift-invariant spaces (see for example \[32\], \[9\] and \[5\].) Given a lattice \( \Lambda \subseteq \mathbb{R}^d \) and \( f, g \in L^2(\mathbb{R}^d) \), the bracket product is defined by,
\[
[f, g]_\Lambda(x) := \sum_{\lambda \in \Lambda^\perp} \hat{f}(x + \lambda^\perp)\hat{g}(x + \lambda^\perp) \quad (x \in \mathbb{R}^d).
\]

Here, \( \hat{f}(w) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x w} dx \) is the Fourier transform of \( f \) and \( \Lambda^\perp := \{ \lambda^\perp \in \mathbb{R}^d \mid \langle \lambda, \lambda^\perp \rangle \in \mathbb{Z}, \text{ for all } \lambda \in \Lambda \} \) is the orthogonal lattice of \( \Lambda \). Since the bracket \([f, g] \) is \( \Lambda^\perp \) periodic, it can be considered as a function on the torus \( \mathbb{R}^d/\Lambda^\perp \).

The \( \Lambda \) translates of a finite set of functions \( \{ f_1, \ldots, f_N \} \) form a Riesz sequence in \( L^2(\mathbb{R}^d) \) if an only if the matrix of functions \( \hat{G} \equiv \left( \hat{G}_{n,m} \right)_{1 \leq k,j \leq N} \) given by
\[
\hat{G}(x)_{n,m} := [f_n, f_m]_\Lambda(x) \quad (x \in \mathbb{R}^d),
\]
is uniformly invertible in the sense that all its eigenvalues are bounded away from 0 and \( \infty \), uniformly on \( x \) (up to sets of null measure.)

Combining the theory of shift-invariant spaces with Theorem \[\□\] we get the following.

---

\[3\]Observe that Equation (26) already implies the existence of a uniform upper bound \( B \).

\[4\]By a lattice \( \Lambda \) we mean a set of the form \( \Lambda = AZ^d \), where \( A \in \mathbb{R}^{n \times n} \) is an invertible matrix. This is sometimes called a full-rank lattice.
Theorem 7. Let \( \Lambda \subseteq \mathbb{R}^d \) be a lattice and let \( V^2 = V^2(F, \Lambda \times \{1, \ldots, N\}) \) be a finitely-generated shift invariant space where the atoms are given by,

\[
F \equiv \{ f_n(\cdot - \lambda) : 1 \leq n \leq N, \lambda \in \Lambda \}.
\]

Assume the following.

- The atoms form a Riesz basis of \( V^2 \) and satisfy the following decay condition,

\[
|f_n(x)| \leq C(1 + |x|)^{-s} \quad (1 \leq n \leq N),
\]

for some constants \( C > 0, \alpha \geq 0 \) and \( s > d \).
- We have a measurable covering of \( \mathbb{R}^d \), \( E \equiv \{ E_i \}_{i \in I} \) that is uniformly locally finite (cf. Equation (12)).
- We are given a family of measurable functions \( \{ g_n^i : \mathbb{R}^d \to \mathbb{C} \} \) for each \( i \in \Lambda \) satisfying the decay condition,

\[
|g_n^i(x)| \leq C'(1 + |x|)^{-s} \quad (1 \leq n \leq N),
\]

for some constant \( C' > 0 \) (independent of \( i \) and \( n \)).
- The matrices of functions \( \{ \hat{G}^i_{n,m} \}_{1 \leq n,m \leq N} \) given by

\[
\hat{G}^i_{n,m} := [f_n, g_m^i]_\Lambda(x) \quad (x \in \mathbb{R}^d/\Lambda^\perp),
\]

are uniformly bounded and invertible in the sense that each \( \hat{G}^i(x) \) is invertible and

\[
\sup_{x,i} \left\| \hat{G}^i(x) \right\|, \sup_{x,i} \left\| \hat{G}^i(x)^{-1} \right\| < \infty.
\]

Then, for all sufficiently large values of \( r > 0 \), the set

\[
\{ g_n^i(\cdot - \lambda) : i \in I, 1 \leq n \leq N, \lambda \in \Lambda, d(\lambda, E_i) \leq r \},
\]

is a Banach frame for \( V^p_v \), for all \( 1 \leq p \leq \infty \) and all strictly \( w_\alpha \)-moderated weights \( v \). More precisely, if we define the index set

\[
\Gamma^r := \{ (i,n,\lambda) \in I \times \{1, \ldots, N\} \times \Lambda \mid d(\lambda, E_i) \leq r \}
\]

and the weight \( V(i,n,\lambda) := v(\lambda) \) on it, then the analysis map

\[
V^p_v \to z^p_v(\Gamma^r)
\]

\[
f \mapsto (\langle f, g_n^i \rangle)_{(i,n,\lambda)}
\]

is bounded and left-invertible.

Remark 14. The theorem is stated for bases just for simplicity. Using the tools from [32], [9] and [5] it can be reformulated for frames.

Proof. Let \( A \) and \( B \) be the Riesz basis bounds of \( F \). Also let \( A' := \sup_{x,i} \| \hat{G}^i(x)^{-1} \| \) and \( B' := \sup_{x,i} \| \hat{G}^i(x) \| \). Using the fiberization theory in [32], [9] and [5], for each \( x \in \mathbb{R}^d/\Lambda^\perp \), the system \( \{ \hat{f}_1(x+k)_k, \ldots, (\hat{f}_N(x+l))_k \} \) is a Riesz basis with bounds \( A \) and \( B \) for some subspace \( V^2_x \subseteq \ell^2(\Lambda^\perp) \). Since its cross-gramian matrix with the system \( \{ \hat{g}_1(x+k)_k, \ldots, (\hat{g}_N(x+k))_k \} \) is invertible, it follows that this latter system is a Riesz projection basis for the subspace \( V^2_x \) with bounds \( B^{-1}A'^{-2} \).
and $(B')^2 A^{-1}$. Invoking again the fiberization theory, it follows that the $\Lambda$ translates of $\{g_1, \ldots, g_N\}$ are a projection basis for $V^2$ with bounds $\approx B^{-1} A^{-2}$ and $(B')^2 A^{-1}$ (the implicit constant depends on the volume of the lattice $\Lambda$.) We can now apply Theorem 6. □

Remark 15. In [5] no results for projection bases nor exterior frames are explicitly given. However, it is proved there (and also in [9]) that the orthogonal projector onto a shift-invariant space operates fiberwise, so the desired extension follows. For further results on exterior frames for shift-invariant spaces see [7] and [8].

6.3. Sampling. Applying Theorem 6 to the reproducing kernels of a (smooth) spline-type space we get the following.

Theorem 8. Let $V = V(F, \Lambda)$ be a spline-type space generated by a family of continuous atoms $F \subseteq C^0(\mathbb{R}^d)$ that satisfy,

$$|f_k(x)| \leq C (1 + |x - k|)^{- (s + \alpha)}, \quad (x \in \mathbb{R}^d, k \in \Lambda),$$

for some $s > d$, $C > 0$ and $\alpha \geq 0$.

Assume the following.

- $\mathcal{E} \equiv \{E_i\}_{i \in I}$ is a uniformly locally finite measurable covering of $\mathbb{R}^d$ (cf. Equation (12)).
- For each $i \in I$, we have a set $X_i \subseteq \mathbb{R}^d$ and this collection of sets is uniformly relatively separated (i.e. $\sup_i \text{rel}(X_i) < \infty$.)
- For each of the sets $X_i$, the following sampling inequality (30)

$$A \|f\|_2 \leq \sum_{x \in X_i} |f(x)|^2 \leq B \|f\|_2^2,$$

holds for all $f \in V^2$ and some constants $0 < A \leq B < \infty$ independent of $i$.

For each $r > 0$, let $X^r := \{(i, x) : i \in I, x \in X_i, d(x, E_i) \leq r\}$.

Then, for all sufficiently large $r > 0$, there exists constants $0 < A^r \leq B^r < \infty$ such that the sampling inequality, (31)

$$A^r \|f\|_{L^p_v} \leq \left( \sum_{(i, x) \in X^r} |f(x)|^p v(x)^p \right)^{1/p} \leq B^r \|f\|_{L^p_v},$$

holds for all $1 \leq p \leq \infty$ (with the usual adjustment for $p = \infty$), all strictly $w_\alpha$-moderated weights $v$, and all $f \in V^p_v$.

Remark 16. For any class of $w_\alpha$-moderated weights for which the respective constants (cf. Equation (6)) are uniformly bounded, the conclusion of the theorem still holds.

Proof. First observe that since $F \subseteq C^0(\mathbb{R}^d)$, Theorem [1] applies with $B = C^0$ and consequently $V^p_v \subseteq C^0$. The norm equivalence of Theorem [1] also implies that $V^2$ is a reproducing-kernel Hilbert space. We already know that $F$ has a dual frame $G \equiv \{g_k\}_k$ satisfying a polynomial decay condition,

$$|g_k(x)| \leq C' (1 + |x - k|)^{- (s + \alpha)}.$$
for some constant $C' > 0$. The functional $f \mapsto f(x_0)$ is represented by the function $K_{x_0} \in V^2$ given by
\begin{equation}
K_{x_0} = \sum_{k \in \Lambda} g_{k}(x_0) f_k.
\end{equation}
We will apply Theorem 6 to the family of frames,
\[ \{K_x\}_{x \in X_i}, \quad (i \in I). \]
To this end, observe that Equation (30) implies that this family satisfies the condition on Equation (27) of Theorem 6. We only need to check the condition on Equation (26) for the family of reproducing kernels.

For $x \in X_i$, using Equation (32), we estimate,
\[ |K_x| \leq CC' \sum_{k \in \Lambda} w_{-(s+\alpha)}(x-k)w_{-(s+\alpha)}(\cdot - k). \]
Using Lemma 2 (c) with $\Gamma := \Lambda - \{x\}$, it follows that
\[ |K_x| \leq CC' \text{rel}(\Gamma) w_{-(s+\alpha)}(\cdot - x) = K'' \text{rel}(\Lambda) w_{-(s+\alpha)}(\cdot - x). \]
Now we can apply Theorem 6 to obtain the desired conclusion. □

6.4. **Gabor molecules.** Let $\phi : \mathbb{R}^d \to \mathbb{R}$, $\phi(x) := \pi^{-d/4}e^{-\frac{|x|^2}{2}}$ be the Gaussian normalized in $L^2$. The Short-Time Fourier Transform with respect to $\phi$ of a test function $f \in S(\mathbb{R}^d)$ is defined by
\begin{equation}
\mathcal{V}_\phi f(x, w) := \langle f, M_w T_x \phi \rangle.
\end{equation}
Here, $T_x$ is the translation operator given by
\[ T_x(f)(y) := f(y - x), \]
and $M_w$ is the modulation operator given by
\[ M_w(f)(y) := e^{2\pi i w y} f(y). \]
The definition in Equation (33) extends to tempered distributions. The time-frequency shift $\pi(x, w)$ is defined by $\pi(x, w) := M_w T_x$.

For $1 \leq p \leq \infty$ and a weight $v$, the modulation space $M^p_v$ is defined as
\[ M^p_v := \{ f \in S'(\mathbb{R}^d) \mid \mathcal{V}_\phi f \in L^p_v(\mathbb{R}^{2d}) \}, \]
and given the norm $\|f\|_{M^p_v} := \|\mathcal{V}_\phi f\|_{L^p_v}$, which makes it a Banach space (see [24 Chapter 11]). $M^p_v$ is similarly defined using $C^0_v$ instead of $L^\infty_v$.

For an adequate lattice, $\Lambda \subseteq \mathbb{R}^d \times \mathbb{R}^d$, the Gabor system $\{M_w T_x \phi : (x, w) \in \Lambda\}$ is a frame for $L^2(\mathbb{R}^d)$. Consider the family of functions $F \equiv \{ f_k \}_{k \in \Lambda} \subseteq L^2(\mathbb{R}^d \times \mathbb{R}^d)$ defined by $f_k := \mathcal{V}_\phi(M_w T_x \phi)$, where $k = (x, w)$. Since $\mathcal{V}_\phi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R}^d)$ is an isometry, it follows that $F$ forms a frame sequence in $L^2(\mathbb{R}^d \times \mathbb{R}^d)$.

Since $\mathcal{V}_\phi \phi \in S(\mathbb{R}^d)$ (see [24 Theorem 11.2.5]), for any $s > 0$ there exists a constant $C_s > 0$ such that
\[ |\mathcal{V}_\phi \phi(z)| \leq C_s (1 + |z|)^{-s}. \]
Since $|f_k| = |\mathcal{V}_\phi \phi(-k)|$ (see [24 Equation 3.14]) it follows that,
\begin{equation}
|f_k(z)| \leq C_s (1 + |z-k|)^{-s} \quad (z \in \mathbb{R}^{2d}, k \in \Lambda).
\end{equation}
Consequently, by Example 1 we know that $V = V(F, \Lambda)$ is a spline-type space.
Observe that for polynomially moderated weights $v$ and $1 \leq p < \infty$, $V_\varphi$ maps, by definition, the modulation space $M^p_v$ isometrically onto $V^p_v$. For $p = \infty$, the same statement is true replacing $M^\infty_v$ for $M^0_v$.

In view of this, Theorem 4 can be reformulated for Gabor molecules.

**Theorem 9.**

- Let $E = \{E_i\}_{i \in I}$ be a uniformly locally finite measurable covering of $\mathbb{R}^d \times \mathbb{R}^d$ (cf. Equation 12.)
- For each $i \in I$, let $G^i = \{g^i_k\}_{k \in \Lambda_i}$ be a frame for $L^2(\mathbb{R}^d)$ with lower bound $A_i$ and suppose that $A := \inf_i A_i > 0$.
- Suppose that the sets of time-frequency nodes $\Lambda_i \subseteq \mathbb{R}^d \times \mathbb{R}^d$ are uniformly relatively separated (i.e., $\sup_{i \in I} \text{rel}(\Lambda_i) < \infty$.)
- Assume that the molecules $G^i$ satisfy the following uniform time-frequency concentration condition,

$$|V_\varphi g^i_k(z)| \leq C (1 + |z - k|)^{-(s+\alpha)} \quad (z \in \mathbb{R}^d \times \mathbb{R}^d, k \in \Lambda_i),$$

for some constants $C > 0$, $s > 2d$ and $\alpha \geq 0$ (independent of $i$.)

Then, for all sufficiently large $r > 0$, the system

$$\{g^i_k : i \in I, k \in \Lambda_i, d(k, E_i) \leq r\}$$

is a Banach frame simultaneously for all the modulation spaces $M^p_v$, for all strictly $w_\alpha$-moderated weights $v$ and $1 \leq p < \infty$. The same is true for $p = \infty$, replacing $M^\infty_v$ for $M^0_v$.

More precisely, if we set,

$$\Gamma^r := \{(i, k) : i \in I, k \in \Lambda_i, d(k, E_i) \leq r\},$$

and define a weight $V$ on $\Gamma^r$ by

$$V(k, i) := v(k),$$

then, the coefficients map given by

$$M^p_v \to z^p_v(\Gamma^r)$$

$$f \mapsto (\langle f, g^i_k \rangle)_{(i, k)}$$

is bounded and left-invertible, for all sufficiently large values of $r$.

**Remark 17.** For any class of $w_\alpha$-moderated weights for which the respective constants (cf. Equation 6) are uniformly bounded, it is also possible to choose a value of $r > 0$ for which the conclusion of the theorem holds.

**Proof.** Consider the spline-type space $V^2 = V_\varphi(L^2(\mathbb{R}^d))$ from the discussion above.

Define the functions,

$$\varphi^i_k := V_\varphi(g^i_k) \quad (i \in I, k \in \Lambda_i).$$

Since $V_\varphi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d \times \mathbb{R}^d)$ is an isometry, each of the families $\{\varphi^i_k\}_{k}$ is a frame for $V^2$ with lower bound $A$. Moreover, Equation 35 implies that these families share a uniform polynomial concentration condition. This condition is also shared by the atoms $\{f_k\}_k$ because Equation 44 holds for any value of $s > 0$.

The theorem now follows from Theorem 6 and the fact that $V_\varphi : M^p_v \to V^p_v$ is a surjective isometry (with the discussed modification for $p = \infty$.)$\square$
Remark 18. Observe that since we have identified the range of the STFT (with a fixed window) with a spline-type space, we get from Theorem 7 that, on the range of the STFT, the $L^p_v$ and $W(L^\infty,L^p_v)$ norms are equivalent (the class of weights $v$ for which this is true depends on the time-frequency localization of the window function; in the case of the Gaussian window, any polynomial weight $w_\alpha$ with $\alpha \geq 0$ will work.) Results of this kind can be found in Chapter 12 of [24], see for example Proposition 12.1.11 there.

Remark 19. Finally observe that the argument given can be used to combine not only time-frequency concentrated frames for $L^2(\mathbb{R}^d)$ but also frames for proper subspaces $S \subseteq L^2(\mathbb{R}^d)$. Simply let $V^2 = \mathcal{V}_\phi(S)$ and apply the same argument as above.

For completeness, we give a version of Theorem 11 for pure time-frequency atoms. This gives general sufficient conditions for the existence of the so called quilted Gabor frames, recently introduced in [10].

Corollary 1.

- Let $E \equiv \{E_i\}_{i \in I}$ be a uniformly locally finite measurable covering of $\mathbb{R}^d \times \mathbb{R}^d$ (cf. Equation (12)).
- For each $i \in I$, let $G^i \equiv \{T_jM_k g^i : (k,j) \in \Lambda_i\}$ be a Gabor frame for $L^2(\mathbb{R}^d)$ with lower bound $A_i$ and suppose that $A := \inf_i A_i > 0$.
- Suppose that the sets of time-frequency nodes $\Lambda_i \subseteq \mathbb{R}^d \times \mathbb{R}^d$ are uniformly relatively separated.
- Assume that the windows $\{g^i\}_i$ satisfy the following uniform time-frequency concentration condition,

$$C := \sup_i \|g^i\|_{M^\infty_{w_{s+\alpha}}} < +\infty,$$

for some constants $s > 2d$ and $\alpha \geq 0$ (independent of $i$).

Then, for all sufficiently large $r > 0$, the system

$$\{T_jM_k g^i : i \in I, (k,j) \in \Lambda_i, d((k,j), E_i) \leq r\}$$

is a Banach frame simultaneously for all the modulation spaces $M_p^\infty$, for all strictly $w_\alpha$-moderated weights $v$ and $1 \leq p < \infty$. The same is true for $p = \infty$, replacing $M^\infty_{w_\alpha}$ for $M^0_\alpha$.

Proof. Observe that,

$$|\mathcal{V}_\phi(T_jM_k g^i)| = |\mathcal{V}_\phi(g^i)(\cdot - \lambda)| \leq C w_{-(s+\alpha)}(z - \lambda),$$

where $\lambda := (j,k) \in \Lambda_i$. Therefore, we can apply Theorem 11.

6.5. Gabor multipliers. Now we give an application of the frame surgery scheme to Gabor multipliers. We follow largely the approach in [15]. For a general background on Gabor multipliers see [18] Chapter 5.

Given a lattice in the time-frequency plane $\Lambda \subseteq \mathbb{R}^{2d}$ and two families of functions $F \equiv \{f_1, \ldots, f_N\}$, $G \equiv \{g_1, \ldots, g_N\} \subseteq L^2(\mathbb{R}^d)$ we consider the class of operators,

$$G_{F,G} := \{ \sum_{n=1}^N \sum_{\lambda \in \Lambda} m_n(\lambda)(-\pi(\lambda)g_n) \pi(\lambda)f_n \mid m_n \in \ell^2(\Lambda) \},$$

where $\pi(\lambda)$ is the time-frequency shift $\pi(\lambda) := M_\lambda T_x$, if $\lambda = (x,w)$. The convergence of the series defining the class $G$ requires additional assumptions (see
The operators in this class are called the *Gabor multipliers* associated with the time-frequency atoms \((F,G)\) and the lattice \(\Lambda\).

For \(f, g \in L^2(\mathbb{R}^d)\) we use the notation \(P_{f,g} := \langle -, g \rangle f\) for the corresponding rank-one operator. Furthermore, for a point \((x, w) \in \mathbb{R}^d \times \mathbb{R}^d\) we let the time-frequency shifts act on an operator \(T\) by

\[
\rho(x, w)(T) := M_w T x M_w = \pi(x, w) T \pi(x, w)^* .
\]

Every linear operator \(T\) mapping continuously \(S(\mathbb{R}^d)\) into \(S'(\mathbb{R}^d)\) admits a distributional kernel \(K(T) \in S'(\mathbb{R}^{2d})\). The *Kohn-Nirenberg symbol* of \(T\) is defined in terms of \(K\) by

\[
\sigma(T)(x, w) := \int_{\mathbb{R}^d} K(T)(x, x - s) e^{-2\pi i s w} ds .
\]

From this definition it follows that the Kohn-Nirenberg map defines and isometry between the class of Hilbert-Schmidt operators and \(L^2(\mathbb{R}^d)\). The important property for us is that the Kohn-Nirenberg map intertwines the action \(\rho\) with the regular action of \(\mathbb{R}^d \times \mathbb{R}^d\) (by translations.) That is,

\[
\sigma(\rho(z)T) = \sigma(T)(\cdot - z) \quad (z \in \mathbb{R}^d \times \mathbb{R}^d) .
\]

We see then that the Kohn Nirenberg map \(KN : T \mapsto \sigma(T)\) relates the class \(G\) to a shift-invariant space \(V^2(F,G) := KN(G_{F,G})\) given by,

\[
V^2 = \left\{ \sum_{n=1}^{N} \sum_{\lambda \in \Lambda} m_n(\lambda) \sigma(P_{f_n,g_n})(\cdot - \lambda) \mid (m_n) \in \ell^2(\Lambda) \right\} .
\]

The Kohn-Nirenberg symbol of the projector \(P_{f,g}\) is explicitly given by,

\[
\sigma(P_{f,g})(x, w) = f(x)\overline{g(w)} e^{-2\pi i x w},
\]

so its 2d Fourier transform is

\[
\overline{\sigma(P_{f,g})(x, w)} = V_g f(-w, x).
\]

Consequently, the inner product between the building blocks of \(V^2\) is given by

\[
\langle \sigma(P_{f_n,g_n}), \sigma(P_{f_m,g_m}) \rangle = \langle V_{g_n} f_n, V_{g_m} f_m \rangle ,
\]

whereas, with the notation \(z^* = (-w, x)\) for \(z = (x, w)\), their bracket product (see Section 6.2) is given by

\[
|\sigma(P_{f_n,g_n}), \sigma(P_{f_m,g_m})|_\Lambda(z) = \sum_{\lambda^+ \in \Lambda^+} V_{g_n} f_n(z^* - \lambda^+) \overline{V_{g_m} f_m(z^* - \lambda^+)}. \tag{38}
\]

Hence, the theory of shift-invariant spaces (see [32], [31] and [5]) implies the following.

**Proposition 7.** The set \(\{ \langle - , \pi(\lambda) g_n \rangle \pi(\lambda) f_n \mid \lambda \in \Lambda \}\) is a Riesz sequence in the space of Hilbert-Schmidt operators if and only if the matrix of functions \(\tilde{G} = \tilde{G}(F,G) \equiv (\tilde{G}_{n,m})_{1 \leq n, m \leq N}\), given by,

\[
\tilde{G}_{n,m}(z) = \sum_{\lambda^+ \in \Lambda^+} V_{g_n} f_n(z - \lambda^+) \overline{V_{g_m} f_m(z - \lambda^+)}, \tag{39}
\]

is uniformly bounded and invertible (that is, its eigenvalues are bounded away from \(0\) and \(\infty\), uniformly on \(z\).)
Remark 20. Observe that, for time-frequency concentrated windows, since by Remark 18 the STFT of an $L^2$ function belongs to the amalgam space $W(C_0, L^2)$, it follows that the entries of the matrix in Equation (39) are continuous periodic functions. Therefore, that matrix will be uniformly invertible if it is invertible at every point.

Proof. The only observation to complete the proof is that, since the condition in Equation (39) is required for every $z \in \mathbb{R}^d$, we can drop the change of coordinates $z \mapsto z^*$ in the bracket product. 

Consequently, in the situation of Proposition 7, any operator $T \in G(F,G)$ can be stably recovered from its lower symbol

$$\langle T, P_{\pi(\lambda)f_n, \pi(\lambda)g_n} \rangle_{HS} : \lambda \in \Lambda,$$

where $\langle \cdot, \cdot \rangle_{HS}$ denotes the Hilbert-Schmidt inner product. We can now reformulate Theorem 7 in this context.

Theorem 10. Let a lattice $\Lambda \subseteq \mathbb{R}^d$ and a uniformly locally finite measurable covering of the time-frequency plane $E \equiv \{E_i\}_{i \in I}$ be given.

Let $f_1, \ldots, f_N, g_1, \ldots, g_N \in L^2(\mathbb{R}^d)$ be such that the matrix $\hat{G}(F,G)$ on Equation (39) is uniformly invertible and suppose that these atoms satisfy,

\begin{align}
|f_n(x)| & \leq C (1 + |x|)^{-s}, \\
|\hat{g}_n(w)| & \leq C (1 + |w|)^{-s},
\end{align}

for some constants $C > 0$ and $s > d$.

Let families $\{f_1, \ldots, f_N\}, \{g_1, \ldots, g_N\} \subseteq L^2(\mathbb{R}^d), i \in I$ be given. Assume the following.

- The given families satisfy,

\begin{align}
|f_n^i(x)| & \leq C'(1 + |x|)^{-s}, \\
|\hat{g}_n^i(w)| & \leq C'(1 + |w|)^{-s},
\end{align}

for some constant $C' > 0$ (independent of $i$ and $n$.)

- The matrices of functions $\left(\hat{G}^i_{n,m}\right)_{1 \leq n, m \leq N}$ given by

$$\hat{G}^i(z)_{n,m} := \sum_{\lambda^+ \in \Lambda^+} \mathcal{V}_{g_n} f_n(z - \lambda^+) \mathcal{V}_{\hat{g}_m} \hat{f}_m(z - \lambda^+) \quad (z \in \mathbb{R}^d \times \mathbb{R}^d).$$

are uniformly bounded and invertible in the sense that each $\hat{G}^i(z)$ is invertible and

$$\sup_{z, i} \|\hat{G}^i(z)\|, \sup_{z, i} \|\hat{G}^i(z)^{-1}\| < \infty.$$

Then, for all sufficiently large values of $r > 0$, any Gabor multiplier $T \in G(F,G)$ can be stably recovered in Hilbert-Schmidt norm from its mixed lower symbol

$$\langle T, P_{\pi(\lambda)f_n, \pi(\lambda)g_n} \rangle_{HS} : i \in I, 1 \leq n \leq N, \lambda \in \Lambda, d(\lambda, E_i) \leq r \rangle.$$

Remark 21. As we have seen, the theorem also establishes an uniform equivalence between the $\ell_2^n$ norm of the coefficients in Equation (44) and the $L^p_n$ norm of the Kohn-Nirenberg symbol of $T$, for $1 \leq p \leq \infty$ and a certain class of weights.
Proof. By the discussion above, in order to apply Theorem 7 we need to observe that the Kohn-Nirenberg symbols of all the atoms are adequately localized. This follows from Equation 37 and the fact that $w_{-s}(x)w_{-s}(u) \leq w_{-2s}(x,w)$, for $x, w \in \mathbb{R}^d$. □

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