Interior Solutions for Non-singular Gravity and the Dark Star alternative to Black Holes.

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**Abstract**

The general equations describing hydrostatic equilibrium are developed for Non-singular Gravity. A new type of astrophysical structure, a Super Dense Object (SDO) or “Dark Star”, is shown to exist beyond Neutron star field strengths. These structures are intrinsically stable against gravitational collapse and represent the non-singular alternative to General Relativity’s Black Holes.
I. INTRODUCTION

Recently it was discovered that a generalisation of Einstein’s theory of General Relativity (GR) yields an analog of the Schwarzschild solution which is everywhere non-singular [1]. This remarkable result was found for a hitherto neglected sector of Moffat’s Non-symmetric Gravity Theory (NGT) [2]. The neglected degrees of freedom were found to affect drastically the strong field behaviour of the gravitational field. The smallest, non-vanishing contribution from this sector removes event horizons and renders the curvature everywhere finite. This sector of NGT has been dubbed Non-Singular Gravity theory or NSG in order to distinguish this new physics from the singular truncation of NGT studied in the literature to date. The results in Ref. [1] have now been generalised to include the analog of the Reissner-Nordstrom solution. It was found that the electric field is everywhere non-singular and both the electromagnetic energy density and spacetime curvature are everywhere finite [3].

These results immediately raise several questions: (1) What do static structures look like? (2) Can they undergo gravitational collapse? (3) What is the end result of such a collapse? This paper is primarily devoted to answering the first of these questions, although tentative answers are proposed for all three questions. It will be shown that two distinct classes of interior solution exist. The first class is very similar to the usual GR solutions, while the second class describes a new and unusual Super Dense Object (SDO). The GR type structures are unstable against gravitational collapse, although less so than their GR counterparts. The SDOs on the other hand are intrinsically stable since their binding energy tends to zero as they become more compact. Stable SDOs only exist with radii in a small range near their “Schwarzschild radius” $R = 2M$. SDOs with radii much less or much greater than this value are unstable against gravitational expansion. For these reasons, it is conjectured that SDOs represent the non-singular alternative to GR’s Black Holes. Astrophysical phenomena such as Quasi-Stellar Objects (QSOs) and (Active) Galactic Nuclei (AGNs) may well be SDOs.

The outline of this paper is as follows: First the field equations are presented. From these the equations describing hydrostatic equilibrium are derived. The equilibrium equations are then studied in a small region near the origin, where the existence of two geometrically distinct branches of solution is established. Each branch of solution is then studied in turn using a mixture of analytic approximations and numerical surveys. Physical bounds on the “skewness parameter”, which controls the departure from GR, are obtained by considering Neutron stars.

II. FIELD EQUATIONS

Einstein’s General Theory is based on the assumption that the metric of spacetime is a symmetric tensor. By removing this assumption, Moffat was able to construct a non-symmetric theory of gravity where both the metric and connection are non-symmetric.

The NGT Lagrangian with sources is given by [4]

$$\mathcal{L} = \sqrt{-g} g^{\mu \nu} \left( R_{\mu \nu} (\Gamma) + \frac{2}{3} W_{[\mu \nu]} - 8 \pi T_{\mu \nu} + \frac{8}{3} \pi W_{\mu S} \right),$$  \hspace{1cm} (1)
where $R_{\mu\nu}(\Gamma)$ is the generalized Ricci tensor, $W_{\mu}$ is a Lagrange multiplier, $\Gamma$ refers to the torsion-free connection $\Gamma^\lambda_{\mu\nu}$, $T_{\mu\nu}$ is the energy-momentum tensor and $S_{\mu}$ is the conserved current which gives rise to NGT charge. Square brackets denote anti-symmetrisation and units are chosen so that $G = c = 1$ throughout. The field equations that follow from (1) are

$$g_{\mu\nu,\sigma} - g_{\mu\rho} \Lambda^\rho_{\mu\sigma} - g_{\mu\rho} \Lambda^\rho_{\sigma\nu} = 0,$$

(2)

$$\left(\sqrt{-g} g^{[\mu\nu]}\right)_{,\nu} = 4\pi \sqrt{-g} S^\nu,$$

(3)

$$R_{\mu\nu}(\Gamma) = \frac{2}{3} W_{[\nu,\mu]} + 8\pi (T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T).$$

(4)

where $\Lambda^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} + D^\alpha_{\beta\gamma}$, and the tensor $D$ satisfies

$$g_{\rho\nu} D^\rho_{\mu\sigma} + g_{\mu\rho} D^\rho_{\sigma\nu} = -\frac{4}{3} \pi S^\rho \left( g_{\mu\sigma} g_{\rho\nu} - g_{\mu\nu} g_{\rho\sigma} + g_{\mu\nu}[g_{\rho\sigma}] \right).$$

(5)

The generalised Bianchi identities, which result from the diffeomorphism invariance of NGT, give rise to the matter response equations

$$\frac{1}{\sqrt{-g}} \left( g_{\mu\rho} \left( \sqrt{-g} g^{\nu\mu} \right)_{,\nu} + g_{\nu\rho} \left( \sqrt{-g} g^{\mu\nu} \right)_{,\mu} \right) + T^{\mu\nu} (g_{\mu\rho,\nu} + g_{\nu\rho,\mu} - g_{\mu\nu,\rho}) + \frac{2}{3} W_{[\rho,\nu]} S^\nu = 0.$$

(6)

Taking the matter to be described by a perfect fluid, the energy momentum tensor is found to be

$$T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} - pg_{\mu\nu},$$

(7)

where $u^\nu$ is the fluid’s four-velocity and $\rho$ and $p$ are the usual internal energy density and pressure, respectively. The Abelian gauge invariance of the Lagrangian under $W_{\mu} \rightarrow W_{\mu} + \lambda_{,\mu}$ ensures the conservation of NGT current density. This is evidenced in (3) since

$$4\pi (\sqrt{-g} S^\mu)_{,\mu} = (\sqrt{-g} g^{[\mu\nu]})_{,\mu} = 0.$$

(8)

The NGT charge contained in a sphere of radius $r$, in spherical coordinates, is given by

$$l^2(r) = \int S^0 \sqrt{-g} dr d\theta d\phi.$$

(9)

The static spherically symmetric interior case was studied for the truncated version of NGT by Savaria [6]. In this work the pure divergence free parts of $g^{[\mu\nu]}$ were dropped. Even with this truncation, the analysis was more complicated than the usual GR case due to the inclusion of additional source terms arising from the NGT charge. In what follows, matter will be taken to be NGT charge neutral so that $S^\mu = 0$ and $l^2(r) = 0$. This condition explicitly enforces the strong equivalence principle.

The most general, static spherically symmetric metric for NGT was found by Papapetrou [7] to be
\[ g_{\mu\nu} = \begin{pmatrix} \gamma(r) & w(r) & 0 & 0 \\ -w(r) & -\alpha(r) & 0 & 0 \\ 0 & 0 & -\beta(r) & f(r) \sin \theta \\ 0 & 0 & -f(r) \sin \theta & -\beta(r) \sin^2 \theta \end{pmatrix}. \] 

Taking \( l^2(r) \) to be zero demands \( w(r) = 0 \). The field equations are further simplified by making the allowed coordinate choice \( \beta(r) = r^2 \). Both of these simplifications will be made in the derivation of the hydrostatic equilibrium equations.

### III. EQUILIBRIUM EQUATIONS

The field equations can be most compactly expressed in terms of the following quantities:

\[ A = \frac{1}{2} \log(r^4 + f^2), \quad B = \arctan(r^2/f). \] 

The usual GR expressions are recovered when \( A = 2 \log r \) and \( B = \pi/2 \). With \( w(r) = 0 \), the field equation (3) is automatically satisfied. This is a generic feature of NSG, as the divergence-free part of \( g^{[\mu\nu]} \) is the NSG sector of NGT. It was these divergence-free degrees of freedom which were dropped in previous work on NGT. The \((tt), (rr), (\theta\theta)\) and \((\theta\phi)\) components of (4) yield

\[ (\log \gamma)'' - \frac{1}{2} (\log \gamma)' \left( \log \left( \frac{\alpha}{\gamma} \right) \right)' + A' \left( \log \gamma \right)' = 8\pi \alpha (\rho + 3p), \] 

\[ -2A'' + A' (\log \alpha)' - ((A')^2 + (B')^2) - (\log \gamma)'' + \frac{1}{2} (\log \gamma)' \left( \log \left( \frac{\alpha}{\gamma} \right) \right)' = 8\pi \alpha (\rho - p), \] 

\[ 1 + \left( \frac{r B' - r^2 A'}{2\alpha} \right)' + B' \left( \frac{r^2 B' - f A'}{2\alpha} \right) + \left( \frac{f B' - r^2 A'}{4\alpha} \right)' \left( \log \alpha \gamma \right)' = 4\pi r^2 (\rho - p), \] 

\[ q + \left( \frac{r^2 B' + f A'}{2\alpha} \right)' - B' \left( \frac{r B' - r^2 A'}{2\alpha} \right) + \left( \frac{r^2 B' + f A'}{4\alpha} \right)' \left( \log \alpha \gamma \right)' = -4\pi f (\rho - p). \]

The constant \( q \) comes from the Lagrange multiplier field \( W_\phi = 3q \cos \theta/2 \), and prime denotes derivatives with respect to \( r \). One of these equations may be replaced by the \( r \) component of the matter response equation (3), which simplifies to read

\[ p' = -\frac{1}{2} (\rho + p) (\log \gamma)'. \]

This relation is identical to its GR counterpart. The agreement with GR follows from NSG obeying the strong equivalence principle when the NGT charge is zero.

The set of equations (12)-(15) are best studied as three useful combinations. The first combination is \( f \) times (14) plus \( r^2 \) times (15), which yields

\[ f + q r^2 + \left( \frac{(r^4 + f^2) B'}{2\alpha} \right)' - \left( \frac{(r^4 + f^2)}{2\alpha} \right) A' B' + \left( \frac{(r^4 + f^2) B'}{4\alpha} \right)' (\log \alpha \gamma)' = 0. \]
The second combination is $f$ times (15) minus $r^2$ times (14), which yields

$$fq - r^2 + \left(\frac{r^4 + f^2}{2\alpha}A'\right)' - \left(\frac{r^4 + f^2}{2\alpha}\right)(A')^2 + \left(\frac{r^4 + f^2}{4\alpha}\right)(\log \alpha)' = 4\pi (r^4 + f^2)(p - \rho).$$

(18)

The third useful combination is (13) plus (12) minus $4\alpha/(r^4 + f^2)$ times (18) which gives

$$\frac{2q^2 - fq}{r^4 + f^2} - \frac{1}{r^4 + f^2} \left(\frac{r^4 + f^2}{\alpha}A'\right)' + \frac{(A')^2 - (B')^2 - 2A''}{2\alpha} = 16\pi \rho.$$

(19)

These three equations, along with the conservation equation (16), provide the most useful, complete set of field equations describing hydrostatic equilibrium in NS G.

Starting with the first of these combinations, it is straightforward to verify that $f = -qr^2$ solves equation (17). With this form for $f$, all the field equations collapse to be identical to their GR counterparts. Unfortunately, this simple solution is incompatible with the external Wyman solution [8,1] (the NSG analog of the Schwarzschild metric) since having $q$ non-zero leads to a jump discontinuity in the curvature invariant $R_{\mu\nu}R^{\mu\nu}$ at the edge of the structure. This follows from the fact that $W_\phi = 0$ for the Wyman solution, while $W_\phi = 3q \cos \theta/2$ in the interior. Consequently, the matching of these solutions at the edge of the structure demands $q = 0$.

The complexity of the field equations makes it difficult to combine them into a simple, fundamental hydrostatic equilibrium equation. However, considerable insight into the nature of the interior solutions can be gained by considering power-series solutions about the origin. Taking the following expansions:

$$p = p_0 + p_1 r + p_2 r^2 + \ldots$$

(20)

$$\rho = \rho_0 + \rho_1 r + \rho_2 r^2 + \ldots$$

(21)

$$\gamma = \gamma_0 + \gamma_1 r + \gamma_2 r^2 + \ldots$$

(22)

$$\alpha = \alpha_0 + \alpha_1 r + \alpha_2 r^2 + \ldots$$

(23)

$$f = f_0 + f_1 r + f_2 r^2 \ldots$$

(24)

and substituting them into (14,19) leads to two distinct branches of solution. The first is characterised by having $\alpha_0 = 1$ and $f_0 = 0$, while the second branch has $\alpha_0 = 0$ and $f_0 \neq 0$. The first branch is very similar to the usual interior solutions found in GR, while the second branch is unlike anything seen before.

Physically, the first branch can be thought of as a matter fields perturbing Minkowski space, while the second branch looks like matter fields perturbing the Wyman metric. The existence of this second branch will be seen to be intimately related to the non-singular nature of the theory. The fact that there are two branches of solution does not lead to any lack of uniqueness in the description of a structure since the two branches describe objects of very different field strengths. For example, the Wymanian branch of solutions cannot describe the sun, or a white dwarf or even a neutron star. Instead, this branch describes super dense objects (SDOs), which typically have radii of $R \leq 2M$. Each branch will now be considered in turn.
IV. MINKOWSKIAN SOLUTIONS

The Minkowkian-type solutions have expansions about the origin which are very similar to their GR counterparts:

\[ f = \sigma r^3 \left( 1 + \frac{2}{3} \pi \rho_0 - \frac{2}{3} \pi p_0 + \frac{5}{8} \sigma^2 x^2 + \ldots \right) , \tag{25} \]

\[ p = p_0 - \frac{2}{3} \pi (\rho_0 + p_0)(\rho_0 + 3p_0) r^2 + \ldots , \tag{26} \]

\[ \gamma = \gamma_0 \left( 1 + \frac{4}{3} \pi (\rho_0 + 3p_0) r^2 + \ldots \right) , \tag{27} \]

\[ \alpha = \left( 1 - \frac{2 \mathcal{M}(r)}{r} \right)^{-1} , \tag{28} \]

where

\[ \mathcal{M}(r) = 4\pi \int_0^r \left[ \rho + \frac{21}{8\pi} \sigma^2 + \sigma^2 \left( \frac{13}{12} \rho_0 - \frac{11}{4} p_0 + \frac{5}{16\pi} \sigma^2 \right) x^2 + \ldots \right] x^2 \, dx . \tag{29} \]

The above quantity is related to the gravitational mass beneath radius \( r \). The relation is not exact, however, as the matching at the edge of the structure leads to a complicated relation between \( \mathcal{M}(R) \) and the mass of the structure \( M \). The skew field, \( f \), increases the mass of a structure above that of a structure with a similar matter distribution, \( \rho(r) \), in GR. From (29) it is clear the mass increase results from the energy density \( \sigma^2 \) of the skew field. Since \( \sigma^2 \) has the dimensions of energy density, it is convenient to parameterise this quantity as some fraction of the central density:

\[ \sigma^2 = \epsilon \rho_0 . \tag{30} \]

Since the contribution to the effective energy density from \( f \) increases from \( 21\sigma^2/4\pi \) with increasing \( r \), it is no surprise that even a small value of \( \epsilon \) causes significant changes. Numerical studies reveal that large departures from the usual GR results occur once \( \epsilon \) reaches \( \epsilon_{\text{crit}} \sim 0.1(2M/R) \). The fact that the critical value for \( \epsilon \) scales with the dimensionless “gravitational potential” \( 2M/R \) is in keeping with the general physical picture of NSG. Large departures from GR occur when the gravitational potential is large. This relationship is made more precise by considering the matching of the interior solutions to the exterior Wyman metric.

A. Matching Conditions

The boundary between the interior and exterior solutions occurs at a radius \( r = R \) defined by \( p(R) = 0 \). The exterior Wyman metric is described by

\[ \gamma = e^{\nu} , \tag{31} \]

\[ \alpha = \frac{M^2(\nu')^2e^{-\nu}(1 + s^2)}{(\cosh(\alpha \nu) - \cos(\beta \nu))^2} , \tag{32} \]
\[ f = \frac{2M^2e^{-\nu}[\sinh(av) \sin(b\nu) + s(1 - \cosh(av) \cos(b\nu))]}{(\cosh(av) - \cos(b\nu))^2}, \]  
(33)

where
\[ a = \sqrt{\frac{\sqrt{1 + s^2} + 1}{2}}, \quad b = \sqrt{\frac{\sqrt{1 + s^2} - 1}{2}}, \]  
(34)

and \( \nu \) is given implicitly by the relation:
\[ e^{\nu}(\cosh(av) - \cos(b\nu))^2 \frac{r^2}{2M^2} = \cosh(av) \cos(b\nu) - 1 + s \sinh(av) \sin(b\nu). \]  
(35)

The dimensionless quantity \( s \) arises as a constant when solving the vacuum field equations \[8\]. Unlike the constant of integration \( M \), which is related to a Gaussian integral of a conserved charge, the constant \( s \) is not connected with any conserved quantity in the theory. Unlike \( M \), there is nothing in the theory which could explain why \( s \) might vary from one body to another. For this reason, it is reasonable to expect that \( s \) is a fundamental constant of nature. This conjecture is supported in the weak field regime where \( s \) is seen to be a coupling constant setting the relative strength of the skew and symmetric fields. In what follows, \( s \) will be taken to be a universal constant of nature.

Formally matching the solutions at \( r = R \) is not very enlightening, since \( \gamma \) is only known as an implicit function of \( r \). More can be learnt by matching onto a small \( M/r \) expansion of the Wyman metric, where the metric functions take the near-Schwarzschild form:
\[ \gamma = 1 - \frac{2M}{r} + \frac{s^2M^5}{15r^5} + \frac{4s^2M^6}{15r^6} + \ldots, \]  
(36)
\[ \alpha = \left(1 - \frac{2M}{r} + \frac{2s^2M^4}{9r^4} + \frac{7s^2M^5}{9r^5} + \frac{87s^2M^6}{45r^6} + \ldots\right)^{-1}, \]  
(37)
\[ f = \frac{sM^2}{3} + \frac{2sM^3}{3r} + \frac{6sM^4}{5r^2} + \ldots. \]  
(38)

These expansions are only valid for \( 2M/R \leq 1 \), which is not a serious restriction seeing as most structures have \( 2M/R \ll 1 \).

Matching \( \alpha \) and \( f \) at \( r = R \) reveals:
\[ M = M_0 + \frac{s^2M_0^4}{18R^4} \left(11M_0 + \frac{15}{4}R - \frac{7\pi}{15}(\rho_0 + 3p_0)R^3 + \ldots\right), \]  
(39)
\[ \sigma^2 = \frac{s^2M^4}{9R^6} \left(1 + \frac{4M}{R} + \frac{4\pi}{15}(3p_0 - 5\rho_0)R^2 + \ldots\right), \]  
(40)

where \( M_0 \) is the usual GR expression for the mass
\[ M_0 = M^{\text{GR}} = 4\pi \int_0^R \rho \, r^2 \, dr. \]  
(41)
Since the corrections to the GR expression for the mass of the structure are always positive, the gravitational mass of a body in NSG will exceed that of a body with a similar density distribution in GR. This increase in mass is primarily due to a decrease in the binding energy, which occurs as a result of the repulsive contributions to the gravitational force coming from the skew sector. If $s$ is too large, the repulsive contributions from the skew sector can cause a structure to become unbound. More moderate values of $s$ serve to stabilise a structure relative to its GR counterpart. The leading term in (40) can be used to express $\epsilon$ in terms of $s$:

$$\epsilon = \frac{4\pi}{27} \left( \frac{M}{R} \right)^3 \left( \frac{\bar{\rho}}{\rho_0} \right) s^2,$$

(42)

where $\bar{\rho}$ is the “average density” of the structure,

$$\bar{\rho} = \frac{3M}{4\pi R^3}.$$

(43)

The above expression relating $\epsilon$ and $s$ indicates that the critical value of $s$ for which large departures from GR begin to be seen will scale as $s_{\text{crit}} \approx R/M$. This indicates that the most stringent bounds on $s$ will come from the study of Neutron stars, which is no surprise considering Neutron stars are the only observationally verified astrophysical objects which require a relativistic description. The numerical results will show that if $s$ exceeds $s_{\infty} \sim 20$, Neutron stars become unbound.

One unusual feature of the matching at $r = R$ is that the matching of the metric functions does not ensure the matching of the gradients of the metric functions. In GR, any pressure and density profile which smoothly approaches zero at the edge of the structure leads, via the field equations, to a smooth matching of $\alpha'$ and $\gamma'$ once $\alpha$ and $\gamma$ have been matched. This in turn ensures that all curvature invariants smoothly match at $r = R$, which is the physically important condition. In NSG the curvature invariants can smoothly match at the edge of the structure even though there are jump discontinuities in the gradients of the metric functions, so long as these jumps obey certain relations which can be derived from the field equations. For example, the jump in $\gamma'$ is given by the relation

$$4R^3 \left[ \frac{\gamma'}{\gamma} \right] = \left( (f')^2 \right) \left( \frac{R^4 - f^2}{R^4 + f^2} \right) - \left[ f' \right] \frac{8R^3}{R^4 + f^2} - \left[ \frac{\gamma' f'}{\gamma} \right] 2f,$$

(44)

where square brackets denote the jump in the enclosed quantity at $r = R$. In the GR limit, $f = 0$ and the above equation yields $[\gamma'] = 0$. In NSG there may or may not be a jump in $\gamma'$, depending on whether there is a jump in $f'$. For the Minkowskian branch of solutions $f'$ is always positive in the interior while $f'$ is always negative in the exterior (if $r > M$), leading to a jump in $f'$. Equation (44) serves as a useful check on the accuracy of the numerical solutions, and was found to be satisfied within error tolerances.

**B. Binding Energy**

The internal energy of a body is defined in the usual way to be $E = M - m_N N$, where $m_N$ is the rest mass of the $N^{th}$ constituent and $N$ is the conserved particle number for this constituent:
\[ N = \int \sqrt{-g} J^0_N \, dr \, d\theta \, d\phi = \int_0^R 4\pi \sqrt{\alpha(r)(r^4 + f(r)^2)} \, n(r) \, dr , \quad (45) \]

and \( n(r) \) is the proper number density for this constituent. The total internal energy can then be broken up into its thermal and gravitational components as \( E = T + V \). The thermal energy is given in terms of the proper internal material energy density, \( e(r) = \rho(r) - m_N n(r) \), by

\[ T = \int_0^R 4\pi \sqrt{\alpha(r)(r^4 + f(r)^2)} \, e(r) \, dr , \quad (46) \]

while the gravitational potential energy \( V \) is given by

\[ V = M - \int_0^R 4\pi \sqrt{\alpha(r)(r^4 + f(r)^2)} \, \rho(r) \, dr . \quad (47) \]

The gravitational binding energy, \( \Omega = -V \), is always smaller in NSG than it would be for the same mass distribution in GR. To leading order, the binding energy is given by

\[ \Omega = \Omega^{GR} - \frac{15}{8} \sigma^2 R^3 - \ldots \]

\[ = \Omega^{GR} - \epsilon M \frac{3}{4\pi} \left( \frac{\rho_0}{\bar{\rho}} \right) - \ldots \]

\[ = \Omega^{GR} - \frac{15Ms^2}{72} \left( \frac{M}{R} \right)^3 - \ldots , \quad (48) \]

which explains why a large value of \( \epsilon \) (or \( s \)) causes structures to become unbound. For polytrope equations of state:

\[ p = \kappa \rho^{(1+1/n)} , \quad (49) \]

the Newtonian expression for the binding energy becomes

\[ \Omega^{\text{Newt.}} = \frac{3}{5 - n} \frac{M^2}{R} , \quad (50) \]

which can be combined with (48) to give an expression for the value of \( s \) for which the structure becomes unbound:

\[ s_{\infty} = \left( \frac{72}{5(5 - n)} \right)^\frac{1}{2} \left( \frac{R}{M} \right) . \quad (51) \]

The largest mass White Dwarfs are well approximated by a \( n = 3 \) polytrope, and have \( M/R \approx 4 \times 10^{-4} \), which bounds \( s \) to be below \( s_{\infty} = 6.7 \times 10^3 \). The major source of error in (51) comes from the approximate expression for the binding energy, which can be off by a factor of two for White Dwarfs. The second term in (48) is essentially exact for White Dwarfs and agrees with the numerical results within 0.1%. A plot of the dependence of \( \Omega/M \) on \( s \) is displayed in Fig. 1. for a White Dwarf with \( M/R = 2 \times 10^{-5} \).
Fig. 1. \((\Omega/M \times 10^5)\) as a function of the skewness constant \(s\) for a White Dwarf with central density \(\rho_0 = 1.45 \times 10^{-13} \text{ km}^{-2}\). The lower line, (a), is the exact numerical result while the upper line, (b), is the analytic result employing the Newtonian approximation for the binding energy.

The largest mass Neutron stars can be approximated by a \(n = 3/2\) polytrope, and have \(M/R \approx 0.1\), which gives the far more stringent bound on \(s\) of \(s_\infty = 20\). This value of \(s_\infty\) will be roughly 30\% too large due to the approximate value used for the binding energy. Since Neutron stars are known to exist, and since they are well described by GR, a tighter bound on \(s\) can be obtained by requiring that the repulsive force causes at most a 10\% reduction in the binding energy. This then gives a maximum allowed value of \(s_{\text{crit}} = 6\). Limiting NSG to be a 1\% correction over GR for Neutron stars gives \(s_{\text{crit}} = 2\), and so on.

Fig. 2. \(f(r)\) in the interior of a Neutron star of radius \(R = 0.8322 \rho_c^{-1/2} = 12.5 \text{ km}\) when \(s = 8.9\). The upper line, (a), is an analytic approximation, while the lower line, (b), is the exact numerical result.

Even if \(s\) is large enough to cause a 20\% reduction in the binding energy, numerical
evaluation shows that the density profile is almost identical to that in GR, with the difference in $\rho(r)$ never exceeding 0.1% anywhere in the structure. These studies were done using the equation of state

$$p = \frac{\rho c}{5} \left( \frac{\rho}{\rho_c} \right)^{5/3},$$

(52)

where $\rho_c = 1/(72\pi)$ km$^{-2}$ is a critical density. Above this density the equation of state goes over to a $n = \infty$ polytrope. Choosing a central density of $\rho_0 = 0.2\rho_c$ and taking $s = 0$ yields a Neutron star with mass $M = 0.0782$, radius $R = 0.8325$ and binding energy $\Omega^{GR} = 0.0074$ in units of $\rho_c^{-1/2}$. From (51), these values lead to the prediction that $s = 9.7$ will cause a 20% reduction in the binding energy. The numerical results showed a 20% change was caused by having $s = 8.9$, and gave rise to a Neutron star with mass $M = 0.0798$ and radius $R = 0.8322$. Since the graphs of $\alpha$, $\gamma$, $\rho$ and $p$ are essentially the same for both GR and NSG, the only function worthy of display is the skew field $f$. A plot of $f$ is displayed in Fig. 2. for the above mentioned Neutron star. The exact numerical result is shown, along with the power series approximation given in (25). Even for relativistic situations, such as Neutron stars, the approximate solution for $f$ gives remarkably good results.

It is interesting to note that the strongest bounds on the coupling of matter to NGT charge came from Neutron star calculations, where the NGT charge caused an increase in the binding energy [9]. The tightest bound corresponded to a 25% increase in the binding energy. If both sectors of NGT are taken into account, this bound is removed since a moderate value of $s$ can cause a compensatory 25% decrease in the binding energy. This possibility is worthy of further investigation.

If $s$ is a fundamental, dimensionless constant of nature, like the fine structure constant, it is not unreasonable to expect $s$ to lie in the range $0.001 < s < 1$. For this range of values, NSG has essentially no effect on the description of any known astrophysical object, save Black Holes. Since both the Chandrasekar and Oppenheimer-Volkoff limits for the mass of White Dwarfs and Neutron stars rely on quasi-equilibrium reasoning, and since the equilibrium description of these structures is unchanged, the usual sequence of gravitational collapse will occur. The departure of NSG from GR comes as the collapse continues, not to a Black Hole, but to the SDOs to be described in the next section.

V. SUPER DENSE OBJECTS

The second branch of interior solution is characterised by having a rich geometrical structure near the origin. The metric near $r = 0$ is of the form

$$\gamma = \gamma_0 + \gamma_2 r^2 + \mathcal{O}(r^4),$$

(53)

$$\alpha = \alpha_2 r^2 + \mathcal{O}(r^4),$$

(54)

$$f = f_0 + f_2 r^2 + \mathcal{O}(r^4),$$

(55)

so that the proper volume scales as $V_p \sim r^2$, while two-surfaces of constant $r$ have surface area $S = 4\pi r^2$. The curvature invariants are all constant near the origin, and are proportional to $(S/V_p)^2$. For the vacuum Wyman metric the leading coefficients are given by
\[
\gamma_0 = \exp\left[ -\frac{\pi}{s} - 2 - \frac{\pi s}{8} + \mathcal{O}(s^2) \right], 
\]

\[
\alpha_2 = \frac{4\gamma_0}{M^2 s^2} \left( 1 + \mathcal{O}(s^2) \right), 
\]

\[
f_0 = M^2 \left( 4 - \frac{s\pi}{2} + s^2 + \mathcal{O}(s^3) \right), 
\]

where a small \( s \) expansion has been employed. It should be noted that the limit \( f \to 0 \) is achieved by taking \( M \to 0 \), not \( s \to 0 \). The Wymanian interior solutions possess the same geometry near the origin, although the coefficients in the expansion are affected by the presence of matter. Interestingly, the ratio \( 4\gamma_0 / \alpha_2 = s^2 M^2 \) remains almost completely unchanged in the presence of matter. This serves to highlight a major technical difficulty in obtaining numerical solutions to the field equations for this branch of solution. The standard method for obtaining numerical solutions relies on power series solutions near \( r = 0 \) to initiate the integration out from the centre of the structure. This is usually no problem since the region near \( r = 0 \) is essentially Minkowski space, and no \textit{a priori} knowledge of global parameters, such as the structure’s mass, is required. For the Wymanian branch of solutions this is not the case since the mass of the structure must be known before it can be determined!

FIG. 3. The density profile \( \rho(r) \) for a typical “neutron SDO”.

An obvious solution to this quandary is to break with convention and integrate from the edge of a structure in toward the centre. This can be successfully accomplished in GR by specifying a mass and radius and integrating to find the corresponding central density (although unphysical choices for the mass and radius will give rise to negative central densities). The success of this operation in GR is predicated on the fact that smooth matching of \( \alpha \) and \( \gamma \) automatically results in the smooth matching of all gradients. In NSG the jump in the gradients must also be specified. For the Minkowskian branch of solutions non-zero jumps could not be avoided, making this method difficult to implement. Thankfully, the Minkowskian branch can be treated using the conventional method. For the Wymanian branch of solutions \( f' \) has the same sign in the interior and exterior solutions, so it is not
imperative that there be a jump in $f'$. In the absence of a proof that the magnitudes of $f'$ should also match, it is difficult to proceed. While there is no guarantee that the derivatives match, there is no doubt that choosing all gradients to match is one acceptable possibility. This assumption will now be made in order to proceed.

A wide numerical survey was made in order to cover a range of equations of state and values of $s$. While the details differ, all SDO solutions studied did fit a standard picture. The generic features include a decrease in binding energy for radii above or below a central value, $R_\Omega$, which typically lay in the range $1.1M < R_\Omega < 2.1M$. The central density exhibited similar behaviour, but always peaked at a slightly smaller value of radius, $R_\rho < R_\Omega$, and dropped more rapidly for larger radii. SDOs with radii slightly larger than $R_\rho$ had density profiles which grew and then dropped, while SDOs with radii below $R_\rho$ had monotonically increasing density profiles. A comparison of binding energies at fixed $\rho_0$ for SDOs with radii on either side of $R_\rho$, revealed that the SDOs with radii less than $R_\rho$ were more tightly bound and thus energetically favoured. This is in agreement with the usual expectation that stable structures must have monotonically increasing density profiles. Of course, usual expectations should not be relied upon for these unusual objects. Varying the value of $s$ did not change this overall picture, although it did increase $R_\Omega$ and $R_\rho$ somewhat, and tended to reduce peak binding energies (which could reach huge values for small values of $s$).

FIG. 4. The skew function $f(r)$ inside a typical “neutron SDO”.

The most important of the above properties is the intrinsic stability of all SDOs against gravitational collapse. If one tries to crush a SDO into a smaller radius its binding energy decreases, and like a cosmic rubber ball, the SDO will spring back to its equilibrium radius. This intrinsic stability allows stable SDOs to exist for any mass. SDOs represent a non-singular endpoint to gravitational collapse, and provide nature with a sane alternative to GR’s Black Holes. While a SDO resides deep inside a gravitational well, and can have a surface redshift so large that it is essentially a “black star”, the gravitational gradients are mollified to the extent that matter pressure is able to support the structure.

In order to illustrate these general features, the density profile and skew field are displayed for a “neutron SDO”. The equation of state is designed to roughly model a relativistic core
coated with a crust of non-relativistic neutrons:

\[ p = \begin{cases} \frac{1}{7}\rho, & \rho \geq 1 \\ \frac{1}{5}\rho^{\frac{4}{3}}, & \rho < 1 \end{cases} \] (59)

Here units have been chosen where \( \rho_c = 1 \). The density profile shown in Fig. 3. is for an SDO with \( s = 0.1, M = 1, R = 1.298 \). This SDO had a central density of \( \rho_0 = 9.63 \times 10^6 \), and a binding energy of \( \Omega = 26.02 \). The skew field \( f \) is displayed in Fig. 4., and illustrates the primary difference between the Wymanian and Minkowskian branch of solutions. The Wymanian branch is characterised by having a large central value of \( f \), with \( f_0 \approx 4M^2 \). The field undergoes a small initial increase before dropping sharply toward the edge of the structure. The Minkowskian branch, on the other hand, has \( f \) starting from zero and sharply increasing. Interestingly, the energetically disfavoured SDOs with radii above \( R_\rho \) have skew functions which start near zero and increase to a peak value before decreasing. This suggest a series of quasi-equilibrium configurations may exist between the Minkowskian and Wymanian branch of solutions, although time dependent collapse calculations will have to be performed to verify this conjecture. Work is currently in progress [10] to study the evolution of SDOs through gravitational collapse in NSG.

The aforementioned neutron SDO has a surface redshift of \( z = 2.09 \times 10^5 \), which suggests the name Dark Star is appropriate for these objects. As \( s \) tends to zero, the surface redshifts associated with SDOs tend to infinity, making Dark Stars as black as Black Holes.

The neutron SDOs with \( s = 0.1 \) become unbound for radii less than \( R \approx 1.264 \). The dependence of binding energy and central density on radii was studied in detail for the stiff equation of state:

\[ p = \rho^{\frac{4}{3}} \] (60)

This involved finding the density profiles for over fifty different radii in order to produce plots of \( \Omega(R) \) and \( \rho_0(R) \).

FIG. 5. The binding energy \( \Omega \) as a function of radius \( R \) for a SDO with a stiff equation of state.
For most choices of $s$ and equation of state fewer radii were studied, since it was sufficient to check that the general picture was the same. In Fig. 5, the binding energy is plotted against radius for this stiff equation of state with $s = 0.1$. A similar plot of the central density versus $R$ for these SDOs is displayed in Fig. 6. For radii beyond $R = 2.04$ a new complication enters the picture since the central density is zero and there is a region of empty space near the origin. This would require an inner and outer matching to the vacuum Wyman solution, although this possibility was not pursued since such annular distributions become unbound.

Fig. 6. The central density $\rho_0$ as a function of radius $R$ for a SDO with a stiff equation of state.

The above description of SDOs is at best a rough sketch, based on a wide, but by no means exhaustive, numerical survey. Since little is known about the behaviour of matter for the kinds of densities being considered for SDOs, it is difficult to choose a sensible equation of state. On the positive side, the generic features described for SDOs do not seem to be greatly affected by the choice of equation of state. The possibility that the gradients of $\alpha$, $\gamma$ and $f$ exhibit jumps at the edge of some SDOs remains to be investigated. It is possible that a proof can be found to show the Wymanian branch of solutions never exhibit jumps in metric gradients. No work has yet been done to study the feasibility of describing QSOs or AGNs in terms of SDOs, although the prospects are very good. In short, much work remains to be done to understand these fascinating objects.

VI. DISCUSSION

The non-singular gravitational theory described in this paper reproduces every prediction of GR for weak to moderate gravitational fields to any desired accuracy (taking $s$ small enough). Any experimental prediction based on systems with $2M/R < 1$ will fail to tell NSG and GR apart, unless $s$ is larger than $\sim 1$. If $s$ is larger than this value there is some hope that detailed studies of Neutron stars may tell the two theories apart.

The only unique signal for any value of $s$ would come for the super dense objects believed
to exist at the centre of most galaxies. If these galactic nuclei have masses above $10^8 M_\odot$, GR predicts that a star passing by will be torn apart by tidal forces once it is within the horizon of the massive Black Hole, so the star would vanish without a trace. If the galactic nuclei was a SDO on the other hand, the star’s death would be visible [11]. Unfortunately, if $s$ is too small, the radiation given off as the star is torn apart would be so heavily redshifted that no signal could be seen above background.

In the absence of experimental tests to tell NSG and GR apart, there is only the following aesthetic distinction: Any non-zero value of $s$ leads to a non-singular endpoint to gravitational collapse - a Super Dense Object or Dark Star. In contrast, GR is always fated to have a singular endpoint to collapse - a Black Hole. Why cling to the concept of a Black Hole when the smallest imaginable change to GR banishes these objects from our picture of the universe?

Infinities belong in mathematics, not nature.

ACKNOWLEDGMENTS

I am grateful for the support provided by a Canadian Commonwealth Scholarship. I thank Norm Frankel, Janna Levin, John Moffat and Pierre Savaria for their interest in this work and for carefully reading the manuscript. I would also like to thank Dick Bond and Glenn Starkmann for several useful discussions concerning NSG.
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