D-BRANES IN TOPOLOGICAL MEMBRANES

P. Castelo Ferreira
Dep. de Matemática – Instituto Superior Técnico, Av. Rovisco Pais, 1049-001 Lisboa, Portugal
and
PACT – University of Sussex, Falmer, Brighton BN1 9QJ, U.K.
pcferr@math.ist.utl.pt

I.I. Kogan
Dept. of Physics, Theoretical Physics – University of Oxford, Oxford OX1 3NP, U.K.

R.J. Szabo
Dept. of Mathematics – Heriot-Watt University, Riccarton, Edinburgh EH14 4AS, U.K.
R.J.Szabo@ma.hw.ac.uk

Abstract

It is shown how two-dimensional states corresponding to D-branes arise in orbifolds of topologically massive gauge and gravity theories. Brane vertex operators naturally appear in induced worldsheet actions when the three-dimensional gauge theory is minimally coupled to external charged matter and the orbifold relations are carefully taken into account. Boundary states corresponding to D-branes are given by vacuum wavefunctionals of the gauge theory in the presence of the matter, and their various constraints, such as the Cardy condition, are shown to arise through compatibility relations between the orbifold charges and bulk gauge invariance. We show that with a further conformally-invariant coupling to a dynamical massless scalar field theory in three dimensions, the brane tension is naturally set by the bulk mass scales and arises through dynamical mechanisms. As an auxiliary result, we show how to describe string descendant states in the three-dimensional theory through the construction of gauge-invariant excited wavefunctionals.
Contents

1 INTRODUCTION AND SUMMARY ................................. 1
  1.1 D-Branes ................................................. 2
  1.2 Summary of Main Results ................................. 2
  1.3 Outline .................................................. 4
  1.4 Note Added .............................................. 5

2 BRANE VERTICES ................................................. 6
  2.1 Hamiltonian Formulation ................................. 6
  2.2 Neutral Wavefunctionals ................................ 11
  2.3 Boundary Conditions ................................... 13
  2.4 Charged Wavefunctionals ............................... 15
  2.5 Orbifold Relations ...................................... 18
  2.6 The Continuity Equation ................................. 21
  2.7 The Orbifold Partition Function ......................... 24

3 BRANE STATES ................................................... 29
  3.1 The Fusion Ring .......................................... 30
  3.2 Punctured Wavefunctionals ............................. 31
  3.3 Topological Correlation Functions ....................... 33
  3.4 Surgical Wavefunctionals ................................ 35
  3.5 The Verlinde Formula .................................... 37
  3.6 Ishibashi Wavefunctionals .............................. 40
  3.7 Boundary Couplings and Wilson Lines ................... 42
  3.8 Orbifold Correlation Functions ......................... 46
  3.9 Fundamental Wavefunctionals ........................... 49
  3.10 The Cardy Condition ................................... 52

4 BRANE TENSIONS ............................................... 54
  4.1 Dilaton Coupling ........................................ 54
  4.2 D-Brane Tension ........................................ 57
  4.3 Effective Actions ........................................ 60
  4.4 Excited Wavefunctionals ................................ 61
  4.5 The Regularized Dimension ............................. 66
  4.6 Worldsheet Duality ...................................... 69
1 INTRODUCTION AND SUMMARY

Holography has played an important role over the last few years in many developments centered around string theory and quantum gravity. In many of its incarnations it provides a powerful tool which allows one to extract information about a strongly-coupled theory from the perturbative sector of a holographic, or dual theory. For example, the duality between string theory on anti-de Sitter spacetimes and supersymmetric Yang-Mills theory on the boundaries provides a means of extracting information about the strong curvature limit of supergravity from quantum gauge theory, and it is further hoped that string theory could shed light in this context on the nature of the strong-coupling regime of gauge theories.

In this paper we will study one of the simplest examples of holography. It is based on the representations of spaces of conformal blocks in two-dimensions as the quantum Hilbert spaces of Chern-Simons gauge theories in three-dimensions [1–5]. It extends to a duality between two-dimensional conformal field theories and gravity and three-dimensional topologically massive gauge and gravity theories [6–9]. This has culminated into the topological membrane approach to string theory and has provided various powerful, dynamical descriptions of processes in perturbative string theory which are reachable in the language of three-dimensional gauge and gravity theories. As has been extensively studied in the past [10–28], this formalism starts with a three-dimensional thickened worldsheet on which there lives a topologically massive (or Maxwell-Chern-Simons) gauge theory, together with topologically massive (or Einstein-Chern-Simons) gravity and a propagating three-dimensional scalar field corresponding to the string dilaton field. This entity is known as the “topological membrane”. The usual string worldsheet is split into the left and right moving two-dimensional boundaries of the membrane. As is well-known, even at the level of the topological quantum field theory, pure Chern-Simons theory is not sufficient to describe all of the features of a full conformal field theory (let alone the full string theory). Including a Maxwell kinetic term into the theory lifts the degeneracy of the phase space, leaving four canonical variables (in contrast to the two of pure Chern-Simons theory). This lifting is necessary to enable the proper incorporation of both left and right moving degrees of freedom of the string theory.

The primary motivation behind this approach has been the unification of the various string theories through one fundamental theory, the topological membrane, using the wealth of analytic tools available to field theories in three spacetime dimensions. This framework can thereby be thought of as M-theory at the worldsheet level. In fact, there are many indications already at this level that various 11-dimensional constructs, such as the Hořava-Witten embedding of the $E_8 \times E_8$ heterotic string into M-theory [29,30], can find fundamental realizations as a one dimension extension of the string worldsheet. In other contexts this holographic correspondence has been described using certain spin-foam
models of quantum gravity [31]. In this paper we will concentrate on the incorporation of D-branes and open string Wilson lines into the topological membrane formalism.

### 1.1 D-Branes

To help place our analysis into context, we begin with a brief overview of the nature of D-branes in string theory. Dirichlet boundary conditions were originally introduced for all of the string embedding coordinates in order to probe off-shell string theory [32–35]. Co-existing Dirichlet and Neumann boundary conditions were studied for the first time in [36] as a possible compactification scheme for string theory. Dual string theories [37] incorporate point-like structures [38–40]. This fact strongly suggests that partonic behaviour and hence a temptative realistic description of quantum chromodynamics in string theory is introduced through point-like sources [41–45].

It was noted in [40] that duality interchanges Neumann and Dirichlet boundary conditions. This fact strongly indicates that both types of boundary conditions should coexist in dual theories. Furthermore, it was argued in [46] that the large nonperturbative effects (of order $e^{-1/g_s}$) in string theory have their origin in the same sorts of mechanisms. In [47] it was finally established that D-branes are indeed intrinsic objects in string theory and are required by consistency of the various string dualities.

Recently, the study of D-branes at the level of worldsheet conformal field theory and Wess-Zumino-Novikov-Witten (WZNW) $\sigma$-models have received much attention [48–53] (see [54] for a review). In these approaches, D-branes correspond to vertex operators in boundary conformal field theory. The fruitfulness of these models is that they provide an analytic and tractable means of studying D-branes on curved spaces which are group manifolds. The connection between the target space description of D-branes and the worldsheet framework is best understood within a worldsheet $\sigma$-model path integral (effective action) formalism [55, 56]. These descriptions were brought together in [57] where it was found that the tension of a D-brane is proportional to the suitably regularized dimension of the physical Hilbert space of the worldsheet boundary conformal field theory, and agrees with the description in terms of an effective target space action for the D-brane dynamics. It is these boundary conformal field theory approaches that will naturally emerge from the topological membrane formalism.

### 1.2 Summary of Main Results

In this paper we will expand on the analysis initiated in [28] which is based on constructing Schrödinger wavefunctionals of topologically massive gauge theory that correspond to states of the induced conformal field theory. Partition functions and correlators of the
two-dimensional $\sigma$-models are thereby obtained through inner products between three-dimensional states. The construction of open string states, in particular those describing D-branes, is based on representing the open string sector as a worldsheet orbifold of an associated closed string sector [58–60]. The extension of these constructions to the three-dimensional setting was first described in [61] and further elucidated in [26–28]. Recently, the three-dimensional topological field theory description has been extended to incorporate boundary states and amplitudes appropriate to D-branes in [62–65]. This approach relies on a model-independent axiomatic formalism which gives rise to a modular tensor category describing the Moore-Seiberg data of a rational, chiral conformal field theory. While some of the techniques we introduce in the following are conceptually the same, our goal is to obtain explicit representations of D-brane states directly from the physical wavefunctions of the underlying three-dimensional gauge and gravity theories.

In what follows we will only analyse the simplest instances corresponding to D-branes in flat, compact backgrounds, or equivalently WZNW models based on abelian Lie groups. The advantage of these models over their non-abelian counterparts is that we can make all constructions very explicit without too many technical obstructions, and a lot of what we say can at least in principle be applied also to arbitrary gauge groups. The extensions to non-abelian gauge groups and D-branes in curved backgrounds represents one of the most important generalizations of the ensuing analysis. The topological membrane could thereby prove to be an indispensable tool for discovering the properties of D-branes on group manifolds, which are not yet understood in their entirety. Another important aspect which is not addressed in the following is the emergence of D-brane charge within the three-dimensional context. This requires the proper construction of Ramond-Ramond states in the topological membrane and hence an understanding of how target space supersymmetry arises, which at present is not very well-understood. A better understanding of the latter feature, along with the embedding of the $E_8 \times E_8$ heterotic string in three-dimensions [16,25], may help elucidate the connections between the topological membrane and M-theory.

The main accomplishments of the present paper can be summarized as follows:

- For the first time, D-branes and open string Wilson lines are incorporated into the topological membrane approach. Their vertex operators naturally appear in vacuum membrane wavefunctionals when the topologically massive gauge theory is coupled to external sources, and the orbifold relations are carefully taken into account.

- We find the membrane wavefunctionals corresponding to Ishibashi boundary states. The Cardy condition, which normally selects the appropriate D-brane states, in the three-dimensional case is found to be a compatibility requirement with bulk gauge invariance of the boundary state wavefunctionals.
In addition to the Cardy map, we find natural dynamical interpretations of the various constraints in boundary conformal field theory. For example, locality is a consequence of the cross channel duality in the scattering amplitudes of two external charged particles in the bulk. All of these properties stem from the representation of the Verlinde formula in terms of the Hopf linking of charged particle trajectories.

We show that the mass of a D-brane is naturally set by the bulk mass scales of the three-dimensional theory, and that it also arises as a dynamical property of the topological membrane.

We find the membrane wavefunctionals corresponding to string descendant states. They are gauge-invariant excited eigenstates of the three-dimensional field theoretic Hamiltonian. In particular, the wavefunctionals corresponding to the graviton vertex operator are those which are directly responsible for the appearance of the D-brane tension.

We acquire a new perspective on open/closed string duality.

1.3 Outline

The remainder of this paper is devoted to deriving the results described above, along with many other auxiliary descriptions. It naturally falls into three parts which describe, respectively, in detail how the appropriate brane vertex operators emerge, what the three-dimensional analogs of brane boundary states are, and how the brane tension manifests itself in the topological membrane framework. In the rest of this section we shall outline the structure of each of these three parts.

In section 2 we minimally couple external sources to the quantum field theory living in the three-dimensional membrane. We study their compatibility with the orbifolds of the membrane under its allowed discrete symmetries PT and PCT. The continuity equations for the source currents are found to impose very stringent conditions. Depending on the type of orbifold taken, the currents $J_i$ can be of only two types. For the three-dimensional orbifolds taken with respect to the PT symmetry, we find $J_i \propto \partial_i Y_D$ corresponding to Dirichlet boundary conditions on the induced open string theory, while for the PCT orbifolds we find $J_i \propto \epsilon_{ij} \partial^j Y_N$ corresponding to Neumann boundary conditions. It is shown that the quantity $Y_D$ corresponds to a D-brane collective coordinate, while $Y_N$ parametrizes the photon field of the open string Wilson line. The D-brane vertex naturally emerges in this framework as a boundary term in the induced open string worldsheet action. The analysis of this section motivates the conjecture that one way to include curved backgrounds in this approach, for an abelian gauge group of the three-dimensional theory, is through the proper incorporation of auxiliary gauge fields as external currents.
In section 3 we study, from the point of view of the topological membrane, various aspects of boundary conformal field theory and its description in terms of pure three-dimensional topological field theory, i.e. the gauge theory with just a Chern-Simons term in the action. Following [66,67], we study the correlation functions of Polyakov loop operators in the corresponding finite temperature gauge theory. The wavefunctionals describing the states of conformal field theory are explicitly constructed, and in particular the ones corresponding to Ishibashi boundary states. The Verlinde diagonalization formula is described in a new vein and the constraints defining the fundamental boundary states (corresponding to the D-brane and open string Wilson line vertex operators) are chosen by compatibility with bulk gauge invariance. The Cardy map is obtained by the constraints on the allowed charges of the bulk matter fields.

Finally, in section 4 we derive the D-brane tension directly from the scales of the topological membrane. For this, it is also necessary to introduce on the membrane topologically massive gravity along with a coupled scalar field $D$ which induces the string dilaton field in two-dimensions. The gravitational sector of the membrane induces two-dimensional quantum gravity, i.e. a (deformed) Liouville field theory on the string worldsheet, and the string coupling constant is determined by the vacuum expectation value of the scalar field $D$ as $g_s = \langle D^4 \rangle$. We derive the effective worldsheet $\sigma$-model action including the D-brane collective coordinates and the open string Wilson lines. We obtain explicitly the Born-Infeld action for a wrapped D-brane and relate it to the regularized dimension of the full Hilbert space of physical states of the three-dimensional quantum field theory (including excited states). The latter quantity is also explicitly computed by exploiting a careful bulk renormalization of the topological graviton mass. By doing so, we also solve the long-standing problem of describing excited string states in terms of the topological membrane, and we obtain a novel impetus on the worldsheet modular duality between the open and closed string sectors of the theory.

1.4 Note Added

The work described in this paper and most of the writing of the manuscript were completed by June 4 2003, the tragic day on which Ian Kogan suddenly and unexpectedly passed away. The other two named authors have decided to proceed with submission of the paper as a tribute to his memory, as this was a topic very close and dear to him. He would have taken great pleasure in seeing it finally completed.
2 BRANE VERTICES

In [28] it was remarked for the first time that one could construct collective coordinates for D-branes by minimally coupling external currents to three-dimensional topologically massive gauge fields. The purpose of this section is to make this observation explicit and construct D-brane vertex operators in detail from the vacuum Schrödinger wavefunctionals of topologically massive gauge theory. The basic idea is that the position of the D-brane can be changed by the addition to the two-dimensional \( \sigma \)-model action of the boundary operator \( \partial_{\perp} x^I \) given by the normal derivative to the boundary of the worldsheet of the open string embedding field. The operator \( a_I \partial_{\perp} x^I \) translates the D-brane by the position vector \( a^I \).

Moreover, in this framework the string Wilson line is given by the tangential derivative to the boundary and corresponds to the operator \( A_I \partial_{\parallel} x^I \). The important observation of [28] was that these vertex operators can be induced in the three-dimensional setting by the addition of bulk matter fields. This agrees with the general ideology of topological membrane theory that a change in conformal background, which is described in two-dimensional terms by a deformed conformal field theory, is described in three-dimensions by adding charged matter in the bulk [19]. Thus the D-brane collective coordinate, which controls the background, is now itself controlled by the bulk distribution of charged matter. This unifies the topological membrane pictures for all possible backgrounds. As we will see in the following, the D-brane in this picture simply corresponds to charged matter on an orbifold line in three dimensions.

2.1 Hamiltonian Formulation

We will begin by reviewing and expanding on some of the basic aspects of \( U(1) \) topologically massive gauge theories [6–9] that will be required in the following. The gauge theory is defined on a three-dimensional manifold of the form \( M = \Sigma \times [0, 1] \), where the finite interval \([0, 1]\) parametrizes the time \( t \), and \( \Sigma \) is a compact orientable Riemann surface whose local complex coordinates will be denoted \( z = (z, \bar{z}) \) with integration measure \( d^2 z = |dz \wedge d\bar{z}| \).

We will adopt Gaussian normal coordinates for the Minkowski three-geometry, in which the metric takes the form

\[
ds_{(3)}^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = -dt^2 + h_{ij} \, dx^i \, dx^j .
\] (2.1)

The three-manifold has two boundaries \( \Sigma_0 = \Sigma \times \{0\} \) and \( \Sigma_1 = \Sigma \times \{1\} \) at times \( t = 0 \) and \( t = 1 \), respectively, both of which are copies of \( \Sigma \) with opposite orientation.

The action is a sum of Maxwell and Chern-Simons terms for a \( U(1) \) gauge field \( A \) with
curvature $F$,

$$S_{\text{TMGT}}[A] = \int_0^1 dt \int_S d^2z \left[ -\frac{\sqrt{-g}}{4\gamma} F_{\mu\nu} F^{\mu\nu} + \frac{k}{8\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \sqrt{-g} A_\mu J^\mu \right], \quad (2.2)$$

where we have included the minimal coupling of the gauge fields to a conserved current $J^\mu$,

$$\partial_\mu J^\mu = 0. \quad (2.3)$$

It will be convenient to rescale the spatial components of the current by the coupling constant $\gamma$ such that the continuity equation reads

$$\partial_t \rho = -\gamma \partial_z j^z - \gamma \partial_x j^x \quad (2.4)$$

in terms of the charge density $J^0 = \rho$ and the current densities

$$J^i = 2\gamma j^i. \quad (2.5)$$

The bulk Levi-Civita antisymmetric tensor density $\epsilon^{\mu\nu\lambda}$ induces the tensor density $\epsilon^{ij} = \epsilon^{0ij}$ on the boundaries. As has been extensively studied in the past [1–5, 10–28], the quantum field theory defined by (2.2) induces new degrees of freedom on the boundaries which constitute chiral gauged WZNW models. They are fields belonging to two-dimensional chiral conformal field theories living on $\Sigma_0$ and $\Sigma_1$.

The action (2.2) can be written in a canonical splitting $A_\mu = (A_0, A_i)$ as

$$S_{\text{TMGT}}[A] = \int_0^1 dt \int_S d^2z \left[ -\frac{\sqrt{-g}}{2\gamma} F_{0i} F^{0i} - \frac{\sqrt{-g}}{4\gamma} F_{ij} F^{ij} + \frac{k}{16\pi} \epsilon^{ij} A_0 F_{ij} + \frac{k}{8\pi} \epsilon^{ij} A_i F_{j0} 
+ \sqrt{-g} A_0 J^0 + \sqrt{-g} A_i J^i \right]. \quad (2.6)$$

The canonical momentum conjugate to $A_i$ is

$$\Pi^i = -\frac{\sqrt{-g}}{\gamma} F^{0i} + \frac{k}{8\pi} \epsilon^{ij} A_j, \quad (2.7)$$

while, as usual, the canonical momentum conjugate to $A_0$ is identically zero. The field $A_0$ is therefore non-dynamical and serves as a Lagrange multiplier which imposes the Gauss law constraint

$$0 = \int_S d^2z \left( -\frac{\sqrt{h}}{\gamma} \partial_0 F^{0i} + \frac{k}{8\pi} \epsilon^{ij} F_{ij} + \sqrt{h} \rho \right) - \oint_{\partial\Sigma} \left( -\frac{\sqrt{h}}{\gamma} F^{0i} + \frac{k}{8\pi} \epsilon^{ij} A_j \right) n_i, \quad (2.8)$$

where $n_i$ is a vector normal to the boundary of $\Sigma$. Note that the boundary term in (2.8) is only present when the two-dimensional boundary $\Sigma$ of the underlying three-manifold.
itself has a boundary. Of course for a smooth space this term doesn’t appear because the boundary of a boundary is empty. However, once we quotient the theory by its discrete symmetries new boundaries can emerge at orbifold singularities [26–28, 61]. This extra boundary term is vital for the construction that we will present in the following, because it allows for the imposition of the correct boundary conditions on the induced conformal field theory. Moreover, conformal vertex operators inserted on the boundary are thereby included in the full three-dimensional theory as external fluxes coupled to the gauge fields through the conserved current $J^\mu$, in accordance with the fact that closed string vertex operators correspond to Wilson lines of the three-dimensional gauge theory. We will see later on that the external charges actually allow one to introduce collective coordinates of D-branes. It is also this mechanism that constrains the open string gauge group and therefore the Chan-Paton degrees of freedom [68–70].

The Hamiltonian of the field theory is given by

$$H = \int_\Sigma d^2z \left\{ -A_0 \left[ \partial_i \left( \Pi^i - \frac{k}{8\pi} \epsilon^{ij} A_j \right) + \frac{k}{4\pi} \epsilon^{ij} \partial_i A_j + \sqrt{h} \rho \right] 
+ \partial_i \left[ A_0 \Pi^i \right] + \frac{1}{8\gamma \sqrt{h}} (\epsilon^{ij} F_{ij})^2 
+ \frac{\gamma}{2\sqrt{h}} h_{ij} \left( \Pi^i - \frac{k}{8\pi} \epsilon^{ik} A_k \right) \left( \Pi^j - \frac{k}{8\pi} \epsilon^{jl} A_l \right) - 2 \sqrt{h} \gamma A_i A_j \right\}. \tag{2.9}$$

By defining the electric and magnetic fields as

$$E^i = -\frac{1}{\gamma} F_{0i}, \quad B = \partial_z A_{\tau} - \partial_{\tau} A_z, \tag{2.10}$$

the bulk and boundary Gauss law constraints in (2.8) read

$$\partial_i E^i + \frac{k}{4\pi} B = -\rho \quad \text{in } \Sigma, \quad E^\perp = -i \frac{k}{8\pi} A^\perp \quad \text{on } \partial\Sigma. \tag{2.11}$$

Here we denote components of fields normal and tangential to the worldsheet boundary by the scripts $\perp$ and $\parallel$, respectively, and we use the metric conventions that raised and lowered indices at $\partial\Sigma$ correspond to the interchanges $\perp \leftrightarrow \parallel$. In the quantum field theory, the canonical commutation relations can be written as

$$\left[ E^i(z), E^j(z') \right] = -i \frac{k}{4\pi} \epsilon^{ij} \delta^{(2)}(z - z'), \quad \left[ E^i(z), B(z') \right] = -i \epsilon^{ij} \partial_j \delta^{(2)}(z - z'), \tag{2.12}$$

and the constraints (2.11) lead to an equation that needs to be satisfied by the physical (gauge invariant) states. We will use these equations when we construct the wavefunctions
of the quantum field theory. Note that the $A_0$ dependent terms in (2.9) vanish when the Hamiltonian operator acts on such states.

The generators of time-independent local gauge transformations can be easily defined, for smooth real-valued gauge parameter functions $\Lambda$, as

$$U_{\Lambda} = \exp \left\{ i \int_{\Sigma} d^2 z \, \sqrt{\hbar} \Lambda(z) \left( \partial_i E^i + \frac{k}{4\pi} B + \rho \right) \right\}.$$  \hspace{1cm} (2.13)

For consistency with the boundary Gauss law in (2.11), the normal derivative of the gauge parameter function in (2.13) must obey Neumann boundary conditions at the boundary $\partial \Sigma$, i.e. $\partial_{\perp} \Lambda = 0$. The physical Hilbert space consists of those quantum states which are invariant under the actions of the operators (2.13). In addition, when there are topologically non-trivial gauge field configurations, we must take into account the large gauge transformations of the theory. They are generated by the operators (2.13) obtained by taking $\Lambda = \theta$ to be the multi-valued angle function of the Riemann surface $\Sigma$. Then integrating by parts in (2.13) yields the extra local operator [71–74]

$$V(z_0) = \exp \left\{ -i \int_{\Sigma} d^2 z \, \left[ \left( E^i + \frac{k}{4\pi} \sqrt{\hbar} A_j \right) \epsilon_{ik} \partial^k \ln \frac{E(z, z_0)}{E(z, z')} - \sqrt{\hbar} \theta(z, z_0) \rho \right] \right\},$$  \hspace{1cm} (2.14)

where we have dropped the boundary term using the Gauss law (2.11) on $\partial \Sigma$. Here $E(z, z_0)$ is the prime form of $\Sigma$, $z_0$ is a fixed point on $\Sigma$, and

$$\theta(z, z_0) = \text{Im} \ln \frac{E(z, z_0)}{E(z, z')E(z', z_0)}$$  \hspace{1cm} (2.15)

with $z'$ an arbitrary fixed reference point. Demanding invariance under these operators, i.e. under large gauge transformations, further truncates the physical Hilbert space of the quantum field theory.

From the commutation relations (2.12) we can compute the commutator

$$\left[ B(z), V^n(z_0) \right] = 2\pi n \sqrt{\hbar} V^n(z_0) \delta^{(2)}(z - z_0)$$  \hspace{1cm} (2.16)

for any integer $n$. This means that the operator $V^n(z_0)$ creates a pointlike magnetic vortex at $z_0$ with magnetic flux $\int_{\Sigma} d^2 z \, \sqrt{\hbar} B = 2\pi n$. These objects thereby generate nonperturbative processes which constitute monopoles of the gauge theory. Moreover, from Gauss’ law (2.11) we see that they also carry a bulk electric charge

$$\Delta Q = -\frac{nk}{2}.$$  \hspace{1cm} (2.17)

The electric charge spectrum of the quantum field theory is [18]

$$Q_{m,n} = m + \frac{k}{4} n,$$  \hspace{1cm} (2.18)
where $m$ and $n$ are integers representing, respectively, the contributions from the usual Dirac charge quantization and the monopole flux. Due to the existence of monopole induced processes and linkings between Wilson lines (charge trajectories) it can be shown [25] that, with the correct relative boundary conditions, the insertion of the charge $Q_{m,n}$ at one boundary $\Sigma_0$ (corresponding to a vertex operator insertion in the boundary conformal field theory) necessitates an insertion of the charge

$$
\bar{Q}_{m,n} = m - \frac{k}{4} n
$$

at the other boundary $\Sigma_1$. This fact will be assumed throughout the rest of this paper.

Let us now turn to the quantization of the topologically massive gauge theory in the functional Schrödinger picture, whereby the physical states are the wavefunctionals $\Psi^{\text{phys}}[A; j]$. By using the canonical quantum commutators (2.12) and the representation

$$
\Pi^i = -i \sqrt{\hbar} \delta \frac{\delta}{\delta A^i},
$$

we impose the Gauss laws (2.11) as constraint equations on the wavefunctionals. By assuming a well-defined decomposition of all fields into independent bulk and boundary degrees of freedom in what follows, we may then factorize the physical states as

$$
\Psi^{\text{phys}}[A; j] = \chi \left[ A^\parallel \right] \Psi \left[ A_z, A_{\Sigma}; j^z, j^\Sigma \right].
$$

For closed surfaces where $\partial \Sigma = \emptyset$, we take $\chi = 1$. But for a generic surface the functional $\chi$ solves the functional boundary Gauss law constraint in (2.11),

$$
-i \frac{\delta}{\delta A^\perp} \chi \left[ A^\parallel \right] = 0.
$$

The bulk gauge constraint in (2.11) reads

$$
\left[ \partial_\Sigma \left( -i \frac{\delta}{\delta A^\Sigma} + \frac{k}{8\pi} \tilde{\epsilon}^\Sigma A_z \right) + \partial_z \left( -i \frac{\delta}{\delta A_z} - \frac{k}{8\pi} \tilde{\epsilon}^z A^\Sigma \right) + \frac{k}{4\pi} \tilde{\epsilon}^{\Sigma z} F^{\Sigma z} + \rho \right] \Psi \left[ A_z, A_{\Sigma}; j^z, j^\Sigma \right] = 0,
$$

where the two-dimensional antisymmetric tensor $\tilde{\epsilon}$ is induced from the bulk by $\tilde{\epsilon}^{ij} = \epsilon^{0ij} / \sqrt{-g}$. By applying the Hamiltonian (2.9) to the physical wavefunctions (2.21) in this polarization, we find that the stationary states satisfy the functional Schrödinger equation

$$
\int d^2 z \sqrt{\hbar} \left\{ -\frac{\gamma}{2} h_{z\Sigma} \left( -i \frac{\delta}{\delta A^\Sigma} - \frac{k}{8\pi} \tilde{\epsilon}^\Sigma A_z \right) \left( -i \frac{\delta}{\delta A_z} + \frac{k}{8\pi} \tilde{\epsilon}^z A^\Sigma \right) + \frac{1}{8\gamma} \left( \tilde{\epsilon}^{\Sigma z} F^{\Sigma z} \right)^2 - \gamma A_z j^z - \gamma A^\Sigma j^\Sigma \right\} \Psi \left[ A_z, A_{\Sigma}; j^z, j^\Sigma \right] = \mathcal{E} \Psi \left[ A_z, A_{\Sigma}; j^z, j^\Sigma \right]
$$

where $\mathcal{E}$ is the energy of the state and we have appropriately normal ordered the Hamiltonian density (This latter point is elucidated in section 4). Since the gauge constraint commutes with the Hamiltonian, these last two equations can be consistently solved.
2.2 Neutral Wavefunctionals

To establish the pertinent properties of the general solutions to (2.23) and (2.24), we will first briefly recall the solution in the absence of external currents [28] and derive its extension to surfaces with boundary. Because of the appearance of the magnetic field in (2.13), in topologically massive gauge theory the physical states are not gauge invariant. Instead, by integrating the infinitesimal gauge constraint (2.23) one finds that the gauge symmetry is represented projectively on wavefunctionals as

\[ U_\Lambda \Psi[A_i; 0] = e^{i\alpha[A_i, \Lambda]} \Psi[A_i + \partial_i \Lambda; 0], \]  

(2.25)

where the projective phase is given by the Polyakov-Wiegmann one-cocycle of the \( U(1) \) Lie algebra as

\[ \alpha[A_i, \Lambda] = \frac{k}{8\pi} \int_\Sigma d^2 z \epsilon^{ij} A_i \partial_j \Lambda. \]  

(2.26)

To separate out the gauge invariant part, one integrates the corresponding cocycle condition

\[ \alpha[A_i, \Lambda + \Lambda'] = \alpha[A_i + \partial_i \Lambda, \Lambda'] + \alpha[A_i, \Lambda] \]  

(2.27)

to get

\[ \Psi[A_z, A_\tau; 0] = \exp \left\{ -\frac{ik}{8\pi} \int_\Sigma d^2 z \sqrt{\hbar} \epsilon^{z\tau} A_z A_\tau \right\} \psi[A_z] \Phi[B], \]  

(2.28)

where \( B \) is the magnetic field. The exponential prefactor in (2.28) is the Kähler potential measure which is required in the holomorphic representation and which compensates the appropriate boundary conditions.

The factor \( \Phi[B] \) is the gauge-invariant solution of Gauss’ law for the pure Maxwell theory \((k = 0)\),

\[ \left[ \partial_\tau \frac{\delta}{\delta A_\tau} + \partial_z \frac{\delta}{\delta A_z} \right] \Phi = 0. \]  

(2.29)

If the fields have non-trivial magnetic charge \( \int_\Sigma d^2 z \sqrt{\hbar} B \neq 0 \), then the wavefunction \( \Phi[B] \) vanishes [75]. When \( \partial \Sigma = \emptyset \), this result is simply the statement that there is overall charge conservation on the closed surface \( \Sigma \). However, locally non-zero magnetic field distributions are still possible. The component \( \psi[A_z] \) solves the gauge constraint of pure Chern-Simons theory \((\gamma \to \infty)\),

\[ \left[ \partial_z \frac{\delta}{\delta A_z} - \frac{ik}{4\pi} \epsilon^{z\tau} \partial_\tau A_z \right] \psi[A_z] = 0. \]  

(2.30)

In this section we will only be interested in the ground state of the field theory (Excited states will be described later on in section 4). This corresponds to a projection onto the lowest Landau level of the quantum spectrum, which is attained in the topological limit.
\( \gamma \to \infty \) (The mass gap between Landau levels is proportional to \( \gamma \)). In this case, \( \Phi = 1 \) and we recover the wavefunctions of pure Chern-Simons gauge theory. These solutions correspond to configurations with weak magnetic field, \( \tilde{\varepsilon}^{z\bar{z}} F_{z\bar{z}} \simeq 0 \). The stationary states are then the eigenfunctions of the first term on the left-hand side of (2.24). In particular, the vacuum state \( \mathcal{E} = 0 \) is determined by the zero mode equation

\[
\left[ \frac{\delta}{\delta A_z} - \frac{k}{8\pi} A_\bar{z} \right] \Psi[A_z, A_\bar{z}; 0] = 0 ,
\]

(2.31)

where we have fixed a complex structure on \( \Sigma \) determined by the worldsheet metric whereby \( \tilde{\varepsilon}^{z\bar{z}} = i \) (In complex Euclidean space the antisymmetric tensor is purely imaginary). Furthermore, by using (2.30) and (2.28) with \( \Phi = 1 \), one finds that it obeys the gauge constraint

\[
\left[ \partial_{\bar{z}} \frac{\delta}{\delta A_\bar{z}} + \frac{k}{8\pi} \partial_{\bar{z}} A_z - \frac{k}{4\pi} F_{z\bar{z}} \right] \Psi[A_z, A_\bar{z}; 0] = 0 .
\]

(2.32)

A solution of (2.31) and (2.32), which is compatible with the gauged WZNW construction for an abelian gauge group, is given by a path integral over an auxiliary, dimensionless worldsheet scalar field \( \varphi \) as

\[
\Psi[A_z, A_\bar{z}; 0] = \int [D\varphi] \exp \left\{ \frac{k}{8\pi} \int_\Sigma d^2 z \left[ A_\bar{z} A_z - 2 A_\bar{z} \partial_z \varphi + \partial_{\bar{z}} \varphi \partial_z \varphi \right] \right\} .
\]

(2.33)

When the worldsheet \( \Sigma \) has a boundary, there is a correction to the projective phase (2.26) given by

\[
\alpha_b[A_i, \Lambda] = -\frac{k}{8\pi} \int_\Sigma d^2 z \varepsilon^{ij} \partial_i (\Lambda A_j) = -\frac{k}{8\pi} \oint_{\partial \Sigma} \Lambda A_\parallel \cdot
\]

(2.34)

Then the functional \( \chi [A^\parallel] \) is also given in terms of the auxiliary scalar field \( \varphi |_{\partial \Sigma} \), the restriction of \( \varphi \) to the boundary of the worldsheet \( \Sigma \), by

\[
\chi [A^\parallel] = \int [D\varphi_\parallel] \exp \left\{ \frac{k}{8\pi} \oint_{\partial \Sigma} \varphi_\parallel A_\parallel \right\} .
\]

(2.35)

The full wavefunction (2.21) is thereby given as

\[
\Psi^{\text{phys}}[A_z, A_\bar{z}; 0] = \int [D\varphi] \exp \left\{ \frac{k}{8\pi} \left[ \int_\Sigma d^2 z \left( A_\bar{z} A_z - 2 A_\bar{z} \partial_z \varphi + \partial_{\bar{z}} \varphi \partial_z \varphi \right) + \oint_{\partial \Sigma} \varphi_\parallel A_\parallel \right] \right\} .
\]

(2.36)

Under a gauge transformation \( A_\mu \to A_\mu + \partial_\mu \Lambda \), this wavefunctional twisted by the \( U(1) \) one-cocycle is indeed invariant after the field redefinition \( \varphi \to \varphi + \Lambda \). The wavefunctional (2.33) together with the cocycle (2.26) is invariant up to a boundary term \( \oint_{\partial \Sigma} \varphi_\parallel \partial^\parallel \Lambda \) [27,28]. By
adding this factor together with the correction (2.34) to the cocycle phase we find that the argument of the wavefunctonal (2.35) transforms to \((\varphi_b - \Lambda) A^\parallel\), and the change can be removed by the shift \(\varphi \to \varphi + \Lambda\).

In these expressions the functional integration measure is assumed to factorize over the bulk and boundary degrees of freedom of \(\varphi\) on \(\Sigma\), and is defined by

\[
[D\varphi] = \sqrt{A_{\Sigma}} \prod_{z \in \Sigma} d\varphi(z) \delta \left( \int_{\Sigma} d^2z \varphi(z) \phi_0 \right) \prod_{x \in \partial \Sigma} d\varphi_b(x) \tag{2.37}
\]

with \(A_{\Sigma} = \int_{\Sigma} d^2z \sqrt{h}\) the area of \(\Sigma\). The factors in the integration measure (2.37) remove the zero modes of the field \(\varphi\) on the Riemann surface \(\Sigma\), which is required for a well-defined functional integral because, by charge conservation on a compact space, the exponential in (2.33) is independent of them. Here \(\phi_0 = 1/\sqrt{A_{\Sigma}}\) is the normalized zero mode eigenfunction of the scalar Laplace operator \(\nabla^2\) on \(\Sigma\), and the delta-function in (2.37), whose argument is the coefficient of \(\phi_0\) in an arbitrary field configuration \(\varphi(z)\), projects out the zero mode integration from the bulk measure \(\prod_{z \in \Sigma} d\varphi(z)\). The worldsheet area factor is included to make the overall combination dimensionless. A more convenient way to use this measure is to change variables from \(\varphi\) to its worldsheet derivatives and compute the Jacobian to get

\[
[D\varphi] = \sqrt{\frac{A_{\Sigma}}{\det \nabla^2}} \prod_{z \in \Sigma} d\left( \partial_z \varphi(z) \right) d\left( \partial_{\bar{z}} \varphi(z) \right) \prod_{x \in \partial \Sigma} d\varphi_b(x) , \tag{2.38}
\]

where the prime on the determinant means that zero-modes are excluded in its evaluation.

### 2.3 Boundary Conditions

In [28] it was demonstrated that the wavefunctions (2.33) are the building blocks of the boundary theories, in that by inserting such states on the boundaries they act as boundary conditions and effectively select the boundary world. The fields \(\varphi\) introduce new propagating degrees of freedom on the boundaries which are absolutely necessary for the consistency of the full three-dimensional theory (on a manifold with boundary) as a well-defined gauge theory. In other words, the requirement of gauge invariance induces new scalar fields \(\varphi\) on the two-dimensional boundaries whose dynamics are governed by chiral, gauged \(U(1)\) WZNW conformal quantum field theories which have partition function given by the path integral in (2.33).

However, once one introduces external sources into the quantum field theory it is not generally possible to obtain a WZNW model on the boundaries. In terms of the boundary wavefunctions this means that we no longer have purely holomorphic or antiholomorphic functionals, and the nice holomorphic factorization property of the quantum field theory
is lost. In the case at hand, this problem may be avoided by bearing in mind that we will eventually orbifold the field theory and identify holomorphic and anti-holomorphic degrees of freedom in boundary operators which are described through insertions of external currents. To this end, we assume that in the chiral sector of the worldsheet field theory only the anti-holomorphic current component \( j^\tau \) is non-vanishing, while in the anti-chiral sector only holomorphic components \( j^z \) remain, i.e. we take the boundary conditions

\[
j^z \bigg|_{\Sigma_0} = 0 , \quad j^\tau \bigg|_{\Sigma_1} = 0 .
\] (2.39)

These conditions are absolutely essential, because the presence of a holomorphic source in the chiral boundary sector would couple extra anti-holomorphic components of the gauge field and hence drastically alter the nature of the degrees of freedom which live there. We will also assume that the electric charge distribution decomposes into holomorphic and anti-holomorphic components as the curl of a vector field \( \tilde{Y}_i \),

\[
\rho = \frac{1}{2} \epsilon^{ij} \partial_i \tilde{Y}_j = i \partial_z \tilde{Y}^z - i \partial_\tau \tilde{Y}^\tau .
\] (2.40)

As for the spatial components of the current, one imposes the conditions

\[
\tilde{Y}^\tau \bigg|_{\Sigma_0} = 0 , \quad \tilde{Y}^z \bigg|_{\Sigma_1} = 0 .
\] (2.41)

Let us stress at this stage that in order to orbifold the theory later on we shall have to extend the wavefunctions to the full, three-dimensional bulk manifold \( \Sigma \times [0,1] \), as in [28]. In the bulk theory the holomorphic and anti-holomorphic degrees of freedom will mix at each time slice. Moreover, in the absence of spatial currents the imposition of the Gauss law is enough to extract the complete ground state wavefunctional. In the present situation one has to be more careful because the presence of spatial currents \( j^i \) does not alter the gauge constraint in (2.23). In order to properly implement our program we must construct a family of distinct Hamiltonian operators \( H_t \) and infinitesimal gauge transformation generators \( G_t \) which vary along the time slices \( t \in [0,1] \) of the quantum field theory. The corresponding vacuum states \( \Psi_t \) at different times obey different Schrödinger equations and Gauss laws given by

\[
H_t \Psi_t = G_t \Psi_t = 0 .
\] (2.42)

This will be carried out explicitly later on in this section.

In particular, for the boundary theories with wavefunctions \( \Psi_0 \) and \( \Psi_1 \) we have the operators

\[
H_0 = \int_{\Sigma_0} d^2z \left\{ -\frac{\gamma}{2} \left( \frac{\delta}{\delta A_\tau} + \frac{k}{8\pi A_z} \right) \left( -\frac{\delta}{\delta A_z} + \frac{k}{8\pi A_\tau} \right) - \gamma A_\tau j^\tau \right\} ,
\] (2.43)
\( G_0 = -i \left[ \partial_z \left( \frac{\delta}{\delta A_z} + \frac{k}{8\pi} A_{\pi} \right) + \partial_{\pi} \left( \frac{\delta}{\delta A_{\pi}} - \frac{k}{8\pi} A_z \right) - \frac{k}{4\pi} F_{z\pi} - \partial_z \bar{Y}^z \right] \bigg|_{\Sigma_0} , \) (2.44)

\( H_1 = \int_{\Sigma_1} d^2z \left\{ \frac{\gamma}{2} \left( \frac{\delta}{\delta A_{\pi}} + \frac{k}{8\pi} A_z \right) - \frac{\delta}{\delta A_{\pi}} \right\} \right] , \) (2.45)

\( G_1 = i \left[ \partial_z \left( \frac{\delta}{\delta A_z} + \frac{k}{8\pi} A_{\pi} \right) + \partial_{\pi} \left( \frac{\delta}{\delta A_{\pi}} - \frac{k}{8\pi} A_z \right) + \frac{k}{4\pi} F_{z\pi} + \partial_z \bar{Y}^z \right] \bigg|_{\Sigma_1} , \) (2.46)

where, as in the previous subsection, we have fixed an appropriate complex structure on the worldsheet \( \Sigma \) and taken the topological limit. We have also substituted in (2.40) and imposed the boundary conditions (2.39) and (2.41). The overall changes in sign between the time \( t = 0 \) and \( t = 1 \) operators are due to the reversal of the orientations between the initial and final worldsheets \( \Sigma_0 \) and \( \Sigma_1 \). The solutions of the four corresponding equations (2.42) will then work as functional boundary conditions for the bulk wavefunctionals \( \Psi_t \) and thereby constrain the dynamics of the bulk quantum field theory.

### 2.4 Charged Wavefunctionals

We will now carry out in detail the computation of the new wavefunctionals for the boundaries \( \Sigma_0 \) and \( \Sigma_1 \) in the presence of the external sources. By using (2.43) the Schrödinger equation in (2.42) at \( t = 0 \) reads explicitly

\[
\int_{\Sigma_0} d^2z \left\{ -\frac{\gamma}{2} \left( \frac{\delta}{\delta A_{\pi}} + \frac{k}{8\pi} A_z \right) \left( -\frac{\delta}{\delta A_{\pi}} + \frac{k}{8\pi} A_z \right) - \gamma A_z j^z \right\} \Psi_0 [A_z, A_{\pi}; j^z] = 0 . \quad (2.47)
\]

Following the analysis of section 2.2, we seek a functional solution of the equation (2.47) similar to (2.33) through the ansatz

\[
\Psi_0 [A_z, A_{\pi}; j^z] = \int [D\varphi] e^{iI_0[\varphi, A_z, A_{\pi}; j^z]} , \quad (2.48)
\]

where

\[
I_0 [\varphi, A_z, A_{\pi}; j^z] = -i \int_{\Sigma_0} d^2z \left[ \frac{k}{8\pi} A_{\pi} A_z + \eta_1 A_z j^z - \left( \frac{k}{4\pi} A_{\pi} + \eta_2 j^z - \bar{Y}^z \right) \partial_z \varphi \right.
\]

\[+ \left. \frac{k}{8\pi} \partial_{\pi} \varphi \partial_z \varphi \right] . \quad (2.49)
\]

The unknown parameters \( \eta_1 \) and \( \eta_2 \) of this ansatz will be found by imposing (2.47) along with gauge invariance.

The terms in (2.49) involving the vector field \( \bar{Y}^i \) arise as follows. In the bulk field theory, the minimal coupling term of the action (2.2) transforms, after an integration by parts over...
time, under bulk gauge transformations as
\[
\int_0^1 dt \int d^2 z \left( A_\mu + \partial_\mu \Lambda \right) J^\mu = \int_0^1 dt \int d^2 z \, A_\mu J^\mu + \left\{ \int_{\Sigma_1} - \int_{\Sigma_0} \right\} d^2 z \, \Lambda \rho .
\] (2.50)

If we add the field \(-i\varphi\rho\) to the Lagrangian of (2.33), then the inhomogeneous term in (2.50) can be absorbed by shifting the scalar field \(\varphi \rightarrow \varphi + \Lambda\), and the full partition function of the topologically massive gauge theory is thereby gauge invariant [28]. Indeed, this modification of the wavefunctional (2.33) solves the Gauss law (2.23) with sources. Upon substituting in the parameterization (2.40) for the charge density and integrating by parts over the worldsheet \(\Sigma_0\), we arrive at the \(\bar{Y}\)-field dependent terms in (2.49).

Let us now compute the action of the Hamiltonian operator (2.43) on the wavefunction (2.48). An easy calculation gives
\[
H_0 \Psi_0 \left[ A_z, A_\pi; j^\pi \right] = \int [D\varphi] \int_{\Sigma_0} d^2 z \left[ \frac{\gamma}{2} \left( \frac{\delta}{\delta A_\pi} + \frac{k}{8\pi} A_z \right) \eta_1 j^\pi - \gamma A_z j^\pi \right] e^{i I_0[\varphi, A_z, A_\pi; j^\pi]}
\]
\[
= \int [D\varphi] e^{i I_0[\varphi, A_z, A_\pi; j^\pi]}
\]
\[
\times \int_{\Sigma_0} d^2 z \left[ \frac{k \eta_1}{8\pi} - 1 \right] A_z j^\pi - \frac{\gamma \eta_1}{8\pi} j^\pi \partial_z \varphi \right] .
\] (2.51)

The second term of the second equality in (2.51) vanishes because we can write the functional integration measure as in (2.38) and shift the new integration variable
\[
\partial_z \varphi \rightarrow \partial_z \varphi + 2A_z + \frac{8\pi}{k} \left( \eta_2 j^\pi - \bar{Y}^z \right)
\] (2.52)
to write (2.49) as a quadratic form in \(\partial_z \varphi \partial_z \varphi\), while leaving both the integration measure and pre-exponential factors in (2.51) unchanged. The resulting functional Gaussian integration then clearly vanishes. If we now set
\[
\eta_1 = \frac{8\pi}{k} ,
\] (2.53)
then the first term also vanishes and we indeed obtain a ground state solution to (2.47).

To fix the remaining constant \(\eta_2\) in (2.49), we will compute the Gauss law constraint using (2.44). Another simple calculation gives
\[
G_0(z) \Psi_0 \left[ A_z, A_\pi; j^\pi \right] = -i \int [D\varphi] \left[ \partial_z \left( \frac{k}{4\pi} A_\pi(z) + \eta_1 j^\pi(z) \right) - \frac{k}{4\pi} \partial_z^2 \varphi(z) \right.
\]
\[
- \frac{k}{4\pi} F_{\pi z}(z) - \partial_z \bar{Y}^z(z) \right] e^{i I_0[\varphi, A_z, A_\pi; j^\pi]}
\]
\[
= -i \int [D\varphi] \left[ \frac{k}{8\pi} \partial_z^2 \varphi(z) - \frac{k}{4\pi} F_{\pi z}(z) + i \frac{\delta I_0}{\delta \varphi(z)} \right.
\]
\[
+ (\eta_1 - \eta_2) j^\pi(z) \right] e^{i I_0[\varphi, A_z, A_\pi; j^\pi]}. \] (2.54)
The third term in the second equality vanishes via an integration by parts in \( \varphi \) space. The last term vanishes if we take 
\[
\eta_1 = \eta_2 .
\]
(2.55)

The remaining terms may be eliminated for smooth field configurations by writing the derivative terms in \( \varphi \) as 
\[
\partial_z \partial_z \varphi + \partial_z \partial_z \varphi
\]
and then shifting \( \partial_z \varphi \) and \( \partial_z \varphi \) by \(-A_z\) and \(A_z\), respectively. This eliminates the local magnetic flux term in (2.54). Then, as was done in (2.51), the remaining functional integrals vanish from shifting derivatives of \( \varphi \) analogously to (2.52) to obtain quadratic forms in (2.49). This is consistent with the integrated form of (2.54), which reads
\[
\int d^2 z \ G_0 \Psi_0 \left[ A_z, A_z; j^z \right] = -i \int [D\varphi] \ e^{i \theta_0 [\varphi, A_z, A_z, j^z]} \times \int d^2 z \left[ \partial_z \left( \frac{k}{4\pi} A_z + \eta_1 j^z - \tilde{Y}^z - \frac{k}{4\pi} \partial_z \varphi \right) - \frac{k}{4\pi} F_{zz} \right].
\]
(2.56)

As discussed in section 2.2, the total magnetic flux on \( \Sigma_0 \) must vanish, and hence so does the last term in (2.56). The remaining terms may then be absorbed into an appropriate shift of \( \partial_z \varphi \) analogous to (2.52), which leaves a quadratic form in (2.49) proportional to \( \partial_z \varphi \partial_z \varphi \) only if (2.55) holds. As in (2.51), the corresponding functional Gaussian integral in the measure (2.38) then vanishes. Therefore, for the parametric values (2.53) and (2.55), the wavefunction (2.48) is a physical ground state of the topologically massive gauge theory.

The equality (2.55) is not at all surprising, since it is required to absorb the gauge transform \( A_i \rightarrow A_i + \partial_i \Lambda \) in (2.49) through the shift \( \varphi \rightarrow \varphi + \Lambda \) [28]. Let us also remark that the assumption that there are no singular \( \varphi \)-field configurations in the bulk is natural because singularities or discontinuities would account for external sources which are responsible for creating additional flux, and would therefore be present in the Gauss law \emph{a priori}. Such fluxes will be present only at the orbifold point on the boundary of the string worldsheet. In other words, they will correspond to insertions of vertex operators in the induced boundary conformal field theory. This aspect will be developed in detail in the next section.

To summarize, the full physical wavefunction (2.21) corresponding to the initial boundary surface \( \Sigma_0 \) is given by
\[
\Psi_{0,\text{phys}}^a [A; j] = \exp \left\{ \int_{\Sigma_0} d^2 z \left( \frac{k}{8\pi} A_z + \frac{8\pi}{k} j^z \right) A_z \right\} \times \int [D\varphi] \exp \left\{ \frac{k}{8\pi} \int_{\Sigma_0} d^2 z \left( \partial_z \varphi - 2A_z - \frac{64\pi^2}{k^2} j^z + \frac{8\pi}{k} \tilde{Y}^z \right) \partial_z \varphi \right\}.
\]
(2.57)
A completely analogous calculation for the final boundary $\Sigma_1$ using (2.45) and (2.46) gives

$$
\Psi_{\text{phys}}^{[A; j]^T} = \exp \left\{ \int_{\Sigma_1} d^2 z \left( - \frac{k}{8\pi} A_z - \frac{8\pi}{k} j^z \right) A_\tau \right\} \times \int [D\varphi] \exp \left\{ \frac{k}{8\pi} \int_{\Sigma_1} d^2 z \left( - \partial_z \varphi + 2A_z + \frac{64\pi^2}{k^2} j^z - \frac{8\pi}{k} \tilde{Y}^\tau \right) \partial_z \varphi \right\} .
$$

(2.58)

This is actually not the end of the story, because $j^i$ and $\tilde{Y}^i$ are in fact related through the continuity equation (2.4) if one demands PT or PCT invariance of the three-dimensional quantum field theory. These symmetries are related to the two possible types of orbifolds of the topological membrane, as has been extensively studied in [26–28] and which will be our focal point for the remainder of this section.

In the presence of a non-empty boundary $\partial \Sigma$ the wavefunctions are given by (2.21) with

$$
\Psi^{\text{phys}} [A; j] = \int [D\varphi] \exp \left\{ i \int_{\Sigma} \varphi_b \left( \tilde{Y}^\parallel - A^\parallel \right) \right\} e^{iH[\varphi, A, A^\parallel, j^\parallel, j^\tau]} .
$$

(2.59)

In the topological membrane picture, the surfaces $\Sigma_0$ and $\Sigma_1$ are already the two-dimensional boundaries of the three-dimensional manifold $M = \Sigma \times [0, 1]$, and thus they cannot have a boundary. Only after an appropriate orbifold operation such as those in [26–28] can we obtain a surface with boundary at the orbifold fixed point $t = 1/2$ in time. Therefore, the extra boundary factor in (2.59) will only be present at that specific point.

### 2.5 Orbifold Relations

We will now describe the orbifolds of the topological membrane obtained by gauging the discrete symmetries of topologically massive gauge theory. In this paper we will only consider worldsheet orbifolds that give rise to oriented surfaces. We are interested in constructing open (oriented) Riemann surfaces $\Sigma^o$ as the $\mathbb{Z}_2$-quotients $\Sigma^o = \Sigma / \mathbb{Z}_2$ of a closed worldsheet $\Sigma$ (the Schottky double of $\Sigma^o$) by an anti-conformal involution $\sigma : \Sigma \rightarrow \Sigma$ whose fixed points correspond to the boundary points of $\Sigma^o$. Given $\Sigma^o$, the Schottky double is constructed as the quotient space $\Sigma = (\Sigma^o \times \{0, 1\}) / \sim$ with respect to the equivalence relation $\sim$ defined by $(x, 0) \sim (x, 1)$ if and only if $x \in \partial \Sigma^o$.

We shall combine the involution $\sigma$ with time-reversal in the bulk, so that the open worldsheet $\Sigma^o$ may be regarded from the topological membrane perspective as the connecting three-manifold $M/\mathbb{Z}_2 = (\Sigma \times [0, 1]) / \mathbb{Z}_2$ (fig. 1). Due to the presence of both Maxwell and Chern-Simons terms in the action (2.2), the only combinations of discrete spacetime
symmetries in this context which are compatible with the three-dimenional action are the PT and PCT automorphisms of the gauge theory [26]. The roles of the time reversal operation $T : t \mapsto 1 - t$ on the geometry $\Sigma \times [0, 1]$ are to create a single new boundary surface $\Sigma_{1/2}$ at the $T$ orbifold fixed point $t = 1/2$, and also to identify the initial and final surfaces $\Sigma_0 \equiv \Sigma_1$. The open worldsheet $\Sigma^o$ is then obtained by quotienting $\Sigma_{1/2}$ by the discrete $\mathbb{Z}_2$ symmetries generated by $P$ and $PC$ in two dimensions, and its boundary is situated at the branch point locus $x^+ = 0$.

![Figure 1: Orbifold of the topological membrane. The worldsheet $\Sigma_{1/2}$ only feels the discrete symmetries $PC$ and $P$ which each generate the cyclic group $\mathbb{Z}_2$, whose action yields the open string worldsheet $\Sigma^o = \Sigma_{1/2}/\mathbb{Z}_2$.](image)

The topological degrees of freedom of the membrane are reduced by the orbifold operations in the following way. The real homology group $H_1(\Sigma, \mathbb{R}) \cong \mathbb{R}^{2g}$ of the closed genus $g$ Riemann surface $\Sigma$ is generated as a vector space by canonical homology cycles $\alpha_\ell, \beta_\ell$, $\ell = 1, \ldots, g$ which have intersection pairings $\alpha_\ell \cap \alpha_{\ell'} = \beta_\ell \cap \beta_{\ell'} = 0$ and $\alpha_\ell \cap \beta_{\ell'} = -\beta_{\ell'} \cap \alpha_\ell = \delta_{\ell\ell'}$. The intersection form makes $H_1(\Sigma, \mathbb{R})$ into a real symplectic vector space. The orientation-reversing homeomorphism $\sigma : \Sigma \to \Sigma$ induces an involutive isomorphism $\sigma^* : H_1(\Sigma, \mathbb{R}) \to H_1(\Sigma, \mathbb{R})$ which gives the homology group a $\mathbb{Z}_2$-grading as

$$H_1(\Sigma, \mathbb{R}) = \mathcal{L}_+(\Sigma) \oplus \mathcal{L}_-(\Sigma),$$

where $\mathcal{L}_\pm(\Sigma)$ are Lagrangian subspaces of $H_1(\Sigma, \mathbb{R})$, i.e. subspaces of maximal dimension on which the intersection form vanishes, defined as the $\pm 1$ eigenspaces of $\sigma^*$, $\sigma^*\mathcal{L}_\pm(\Sigma) = \pm \mathcal{L}_\pm(\Sigma)$. We will identify $H_1(\Sigma^o, \mathbb{R}) = \mathcal{L}_-(\Sigma)$. This is the canonical choice for the real homology group of the open worldsheet $\Sigma^o$ in the following sense [63]. Let $M_{\Sigma^o}$ be the “solid donut” obtained by filling in the Shottky double of $\Sigma^o$, i.e. $M_{\Sigma^o} = (\Sigma^o \times [0, 1])/\sim$, where $(x, t) \sim (x, 1 - t)$ $\forall t \in [0, 1]$ if and only if $x \in \partial \Sigma^o$. Then $\partial M_{\Sigma^o} = \Sigma$, and the canonical inclusion $i : \partial M_{\Sigma^o} \hookrightarrow M_{\Sigma^o}$ induces a homomorphism $i_* : H_1(\Sigma, \mathbb{R}) \to H_1(M_{\Sigma^o}, \mathbb{R})$ with kernel

$$\ker(i_*) = \mathcal{L}_-(\Sigma).$$
In other words, the Lagrangian subspace $\mathcal{L}_-(\Sigma)$ consists of those homology cycles of $\Sigma$ which are contractible in $M_\Sigma^o$. From the point of view of the membrane geometry $M = \Sigma \times [0, 1]$, the canonical inclusion $j$ of $\partial(M/\mathbb{Z}_2) \cong \Sigma \amalg \Sigma^o$ into $M/\mathbb{Z}_2$ induces a homomorphism into the $\mathbb{Z}_2$-equivariant homology of the topological membrane as

$$H_1(\Sigma, \mathbb{R}) \oplus \mathcal{L}_-(\Sigma) \xrightarrow{j_*} H_1^{\mathbb{Z}_2}(\Sigma \times [0, 1], \mathbb{R}). \quad (2.62)$$

Here and in the following we will choose local complex coordinates at the orbifold line $x^\perp = 0$ which are defined such that $z = x^\parallel + i x^\perp$ and $\bar{z} = x^\parallel - i x^\perp$ (and similarly for the vector fields). This choice is a matter of convention in the definition of the parity operator $\mathcal{P}$. For example, in the case of the sphere $\Sigma = \mathbb{S}^2$ represented stereographically as the complex plane, with these conventions the boundary of the disk $\Sigma^o = \mathbb{D}^2$ obtained by orbifolding the sphere is the real axis [26]. In the remainder of this subsection we will summarize the orbifold transformation rules derived in [26, 28] that we will require in the following.

For the $\text{PT}$ symmetry the field transformation rules are given by

$$\begin{align*}
\text{PT} : \quad & \Lambda \mapsto -\Lambda \\
& \varphi \mapsto -\varphi \\
& A_0 \mapsto A_0 \\
& A_\perp \mapsto A_\perp \\
& A_\parallel \mapsto -A_\parallel \\
& A_z \mapsto A_{\bar{z}} \\
& \partial_i E^i \mapsto -\partial_i E^i \\
& B \mapsto B \\
& Q_{m,n} \mapsto -Q_{m,n} .
\end{align*} \quad (2.63)$$

The orbifold obtained from the quotient under this symmetry corresponds to Dirichlet boundary conditions on the string fields which restricts the charge spectrum to only winding modes

$$Q_{0,n} = \frac{kn}{4} . \quad (2.64)$$

Under the $\text{PCT}$ symmetry the fields transform as

$$\begin{align*}
\text{PCT} : \quad & \Lambda \mapsto \Lambda \\
& \varphi \mapsto \varphi \\
& A_0 \mapsto -A_0 \\
& A_\perp \mapsto -A_\perp \\
& A_\parallel \mapsto A_\parallel \\
& A_z \mapsto -A_{\bar{z}} \\
& \partial_i E^i \mapsto -\partial_i E^i \\
& B \mapsto -B \\
& Q_{m,n} \mapsto Q_{m,n} .
\end{align*} \quad (2.65)$$
The orbifold obtained from the quotient under this symmetry corresponds to Neumann boundary conditions on the string fields which restricts the charge spectrum to only Kaluza-Klein modes

\[ Q_{m,0} = m \, . \] (2.66)

Note that T-duality in this setting is just the interchange of the PT and PCT orbifolds in three-dimensions.

From what we have just described it follows that the orbifold construction is compatible with the constraints on the external current as long as \( \rho|_{\Sigma_0} = -\rho|_{\Sigma_1} \), which is consistent with (2.40) and (2.41). The transformations for the current components are given by

\[
\text{PT} : \quad j^z \mapsto j^\tau, \quad \tilde{Y}^z \mapsto \tilde{Y}^\tau \] (2.67)

and

\[
\text{PCT} : \quad j^z \mapsto -j^\tau, \quad \tilde{Y}^z \mapsto -\tilde{Y}^\tau. \] (2.68)

Of course, these transformations are expected since the current components should transform as vectors under the discrete symmetry operations. We can now use these orbifold relations to systematically construct the Schrödinger wavefunctionals corresponding to the open surface \( \Sigma^o \) from those derived in section 2.4.

Note that at the boundary \( \partial \Sigma \) the scalar field \( \varphi_b \) is not a priori constrained by the orbifold relations, and it has free boundary conditions. It is only after integrating out the gauge field degrees of freedom that we obtain boundary conditions for it [28]. We will derive this fact in the presence of external currents, along with the appropriate extensions of the wavefunctionals to \( \partial \Sigma \), in section 2.7.

### 2.6 The Continuity Equation

Following [27, 28] we will now derive the extension of the wavefunctionals computed above to the entire bulk three-manifold \( \Sigma \times [0, 1] \). For this, we consider two time-dependent functions \( f_0(t) \) and \( f_1(t) \) which obey the boundary conditions

\[
\begin{align*}
    f_0(0) &= -f_1(1) = -1, \\
    f_0(1) &= f_1(0) = 0.
\end{align*}
\] (2.69)

In addition, to ensure compatibility between gauge invariance and the orbifold constructions described above, we must also require that these functions be related by [26, 28]

\[
\begin{align*}
    f_0(1-t) &= -f_1(t), \\
    f_0(1/2) &= -f_1(1/2) = \frac{1}{2}.
\end{align*}
\] (2.70)
Then any integral over the boundaries \( \Sigma_0 \) and \( \Sigma_1 \) of the three-manifold can be extended to the bulk as
\[
\int_{\Sigma_1} d^2 z \, X_1 + \int_{\Sigma_0} d^2 z \, X_0 = \int_0^1 dt \int_{\Sigma} d^2 z \, \partial_t (f_1 X_1 - f_0 X_0) ,
\]
(2.71)
where \( X_0 \) and \( X_1 \) stand for any combinations of the fields at the worldsheet boundaries \( \Sigma_0 \) and \( \Sigma_1 \), respectively.

Within this simple framework one can now evaluate the worldsheet integral at any fixed time slice \( \Sigma_\tau \equiv \Sigma \times \{ \tau \} \), \( \tau \in [0,1] \) by splitting the covering cylinder \( \Sigma \times [0,1] \) over \( \Sigma \) into two pieces, corresponding to the time intervals \( t \in [0,\tau] \) and \( t \in (\tau,1] \), such that one has
\[
\int_{\Sigma_\tau} d^2 z \, X_\tau = \int_{\Sigma} d^2 z \left( f_1(\tau) X_1 - f_0(\tau) X_0 \right) ,
\]
(2.72)
where \( X_\tau, \tau \in [0,1] \) is a one-parameter family of fields defined on the time slices \( \Sigma_\tau \). Of course, for this we need to know the precise dependences of \( f_0(t) \) and \( f_1(t) \) as functions of the time coordinate \( t \). In [28] this problem was not addressed, since for the considerations there it sufficed to know their values solely at the orbifold point \( t = 1/2 \). In this subsection we will use the continuity equation (2.4) for the external currents to derive differential equations for these functions.

For the stationary states of the topologically massive gauge theory, we assume that the components \( j^z, j^\tau \) and \( \rho \) are all time-independent fields. Then, according to the framework just described, the original currents which appear in the action (2.2) are given by the expressions
\[
J^0 = \partial_t (F_0 \rho_0 + F_1 \rho_1) ,
\]
\[
J^z = 2\gamma \partial_t (F_1 j^z) ,
\]
\[
J^\tau = 2\gamma \partial_t (F_0 j^\tau) ,
\]
(2.73)
where the functions \( F_\tau, \tau = 0,1 \) are defined by
\[
F_\tau(t) = \int_{1-\tau}^t dt' \, f_\tau(t') ,
\]
(2.74)
and we have allowed two different but fixed charge distributions \( \rho_\tau \) to live on the two boundaries at \( \tau = 0,1 \). Then the sources indeed do reduce to the desired ones on the boundary surfaces, which we summarize as
\[
\begin{array}{ccc}
\Sigma_0 & \Sigma_1 \\
J^0 & = & -\rho_0 \\
J^z & = & 0 \\
J^\tau & = & -2\gamma j^\tau \\
\end{array}
\]
(2.75)
\[
\begin{array}{ccc}
\Sigma_0 & \Sigma_1 \\
J^0 & = & \rho_1 \\
J^z & = & 2\gamma j^z \\
J^\tau & = & 0 .
\end{array}
\]
The continuity equation (2.4) thereby becomes

\[
\frac{1}{\gamma} \partial_t (f_0 \rho_0 + f_1 \rho_1) = f_0 \partial_x j^x - f_1 \partial_x j^z .
\]  

(2.76)

We can split this equation into two independent differential equations

\[
\partial_t f_0 = -(f_0 + c_0) \lambda_0, \quad \lambda_0 = -\frac{\gamma \partial_x j^x}{\rho_0},
\]

\[
\partial_t f_1 = -(f_1 + c_1) \lambda_1, \quad \lambda_1 = \frac{\gamma \partial_x j^z}{\rho_1},
\]

(2.77)

along with a single constraint which couples them through

\[
c_0 \lambda_0 + c_1 \lambda_1 = 0 .
\]

(2.78)

The source parameters \(c_\tau\) and \(\lambda_\tau\) (\(\tau = 0, 1\)) are constant in time. In this way we obtain the explicit solutions

\[
f_\tau(t) = -c_\tau + c'_\tau \, e^{-\lambda_\tau t} .
\]

(2.79)

The parameters \(\lambda_\tau, c_\tau\) and \(c'_\tau\) (\(\tau = 0, 1\)) of this solution can be unambiguously fixed once we place appropriate orbifold conditions on the field theory.

For this, we note that compatibility with the orbifold identifications requires that the two sets of charge densities be related through \(\rho_0 = -\rho_1\). We further need to consider two sets of functions \(f_\tau\) and \(\tilde{f}_\tau\) defined respectively on the two branches \(t \in [0, 1/2]\) and \(t \in (1/2, 1]\). By setting \(t = 0, 1\) and \(1/2\) in (2.79) and using (2.69) and (2.70), we arrive at the boundary conditions

\[
-c_\tau + c'_\tau = 1 - \tau ,
\]

\[
-c_\tau + c'_\tau \, e^{-\lambda_\tau/2} = \frac{1}{2} - \tau ,
\]

\[
-\tilde{c}_\tau + \tilde{c}'_\tau \, e^{-\tilde{\lambda}_\tau/2} = \frac{1}{2} - \tau ,
\]

\[
-\tilde{c}_\tau + \tilde{c}'_\tau \, e^{-\tilde{\lambda}_\tau} = -\tau .
\]

(2.80)

These equations admit a unique solution in terms of the \(\lambda_\tau\)'s. The constraint (2.78) which couples the two differential equations can be solved by noting the relations (2.75) between the sources on the various boundary surfaces, which imply

\[
\frac{\partial_x j^x}{\rho_0} \bigg|_{\Sigma_t} = -\frac{\partial_x j^z}{\rho_1} \bigg|_{\Sigma_{1-t}} .
\]

(2.81)

This is in fact a global constraint which holds on the entire covering cylinder \(\Sigma \times [0, 1]\), since the variation along the three-manifold is accounted for by the temporal functions which solve the continuity equation above. It follows that \(\lambda_0 = \lambda_1\) and the required constraint (2.78) is satisfied by the solution of (2.80). The quantities (2.81) are dimensionless, and
as such determine an integration constant. We will fix this last arbitrary parameter to the Chern-Simons coupling \( k/8\pi \), as it is the natural expansion coefficient of the model. This choice will conveniently simplify things in the following.

By repeatedly applying the functional relation (2.70) we thereby finally arrive at the full solution

\[
\begin{align*}
  f_0(t) &= -c_0 + (c_0 + 1) e^{-kt/8\pi}, & t \in [0, 1/2], \\
  \tilde{f}_0(t) &= -\frac{1}{2} + f_0(t - 1/2), & t \in (1/2, 1], \\
  f_1(t) &= \frac{1}{2} - f_0(3/2 - t), & t \in [0, 1/2], \\
  \tilde{f}_1(t) &= -f_0(2 - t), & t \in (1/2, 1],
\end{align*}
\]  

(2.82)

where

\[
c_0 = -\frac{1/2}{1 - e^{-k/16\pi}}.
\]  

(2.83)

In addition, from the constraints (2.81) we can deduce an important relation. Namely, under the orbifold involution, the currents \( j^i \) and the charge distribution vector fields \( \tilde{Y}^i \) are related in a very simple way through

\[
\begin{align*}
  j^z &= \frac{k}{8\pi} \tilde{Y}^z, & j^\tau &= \frac{k}{8\pi} \tilde{Y}^\tau.
\end{align*}
\]  

(2.84)

This relationship will drastically simplify the form of the Schrödinger wavefunctionals for the orbifolded gauge theory, whose computation we come to next.

### 2.7 The Orbifold Partition Function

We can now finally demonstrate the explicit appearance of the D-brane vertex operator in the induced two-dimensional conformal field theory. We will assume that the initial (and final) Riemann surface \( \Sigma \) is closed, \( \partial \Sigma = \emptyset \), and denote its genus by \( g \). The partition function for the topological membrane, i.e. for topologically massive gauge theory on the three-geometry \( M = \Sigma \times [0, 1] \), is given by the overlap between the initial and final states (2.57) and (2.58) as [28]

\[
Z_c = \langle \Psi_1 | \Psi_0 \rangle \equiv \int \left[ DA_z \; DA_\tau \right] e^{iS_{TMGT}[A]} \Psi_1^\dagger [A_z, A_\tau; j^z] \Psi_0 [A_z, A_\tau; j^\tau],
\]  

(2.85)

where the functional integral is taken with an appropriately defined gauge-fixed integration measure. To explicitly evaluate (2.85), one decomposes the gauge field \( A_i \) according to a representative of its gauge orbit as

\[
A_i = \bar{A}_i + \partial_i \Lambda,
\]  

(2.86)

where \( \bar{A}_i \) is the gauge-fixed field and \( \Lambda \) is an arbitrary real-valued gauge parameter function. The corresponding change of measure is determined by the relation (2.38). For \( \Sigma \) closed
and \( J^\mu = 0 \), by construction the integrand of (2.85) is independent of the gauge parameter \( \Lambda \), and one finds that the path integral factorizes as

\[
Z^c = Z_{\text{bulk}} Z_\Sigma ,
\]

(2.87)

where \( Z_{\text{bulk}} \) comes from integrating out the bulk parts of the gauge fields \( \vec{A}_i \) and involves only the topologically massive gauge theory action \( S_{\text{TMDGT}}[\vec{A}] \), while \( Z_\Sigma \) contains only boundary degrees of freedoms of the fields and coincides with the partition function of the usual full (non-chiral) gauged \( c = 1 \) WZNW model defined on the closed string worldsheet \( \Sigma \). It is in this way that the closed string sector is reproduced from the bulk dynamics of the topological membrane. In what follows we will normalize the full partition function by setting \( Z_{\text{bulk}} = 1 \).

We will now transform the vacuum amplitude (2.85) into the appropriate partition function for the orbifold \( \sigma \)-model on \( \Sigma^o \). For this, we extend the wavefunctionals (2.57) and (2.58) to the full covering cylinder over \( \Sigma \) by using the prescriptions of the previous subsection to express it in the form

\[
Z^c = \int [DA_z \ DA_\pi] \int [D\varphi] \ e^{iS_{\text{TMDGT}}[A]} \times \exp \left\{ -\frac{k}{8\pi} \int_0^1 dt \int_\Sigma d^2z \ \partial_t \left( A_\pi A_z + \partial_\varphi \partial_z \varphi + \frac{8\pi}{k} \epsilon^{ij} \partial_j \left( \varphi \vec{Y}_i \right) \right) (f_0 + f_1) \right. \\
- \left[ \left( A_\pi - \frac{8\pi}{k} \vec{Y}_z + \frac{64\pi^2}{k^2} j^z \right) \partial_z \varphi - \frac{64\pi^2}{k^2} A_z j^z \right] f_0 \right. \\
- \left[ \left( A_z - \frac{8\pi}{k} \vec{Y}_\pi + \frac{64\pi^2}{k^2} j^\pi \right) \partial_\varphi \varphi - \frac{64\pi^2}{k^2} A_\pi j^\pi \right] f_1 \right\} .
\]

(2.88)

By using the orbifold identifications of the fields described in section 2.5, the explicit time dependences of the continuity equation solutions of the previous subsection, and the relationship (2.84), we find that the currents \( j^i \) completely drop out of the wavefunctionals giving the orbifold amplitude

\[
Z^o = \langle \Psi_{1/2} | \Psi_0 \rangle_\text{orb} \\
\equiv \int [DA_z \ DA_\pi] \ e^{2iS_{\text{TMDGT}}[A]} \Psi_{1/2}^{\text{orb}} \left[ A_z, A_\pi; \vec{Y}_z, \vec{Y}_\pi, \vec{Y}_\parallel \right] ^\dagger \Psi_0^{\text{orb}} \left[ A_z, A_\pi; \vec{Y}_z \right] ,
\]

(2.89)

where the orbifold wavefunction on the initial closed surface \( \Sigma_0 \equiv \Sigma_1 \) is given by

\[
\Psi_0^{\text{orb}} \left[ A_z, A_\pi; \vec{Y}_z \right] = \exp \left\{ \int_\Sigma d^2z \left( \frac{k}{4\pi} A_\pi + \vec{Y}_z \right) A_z \right\} \\
\times \int [D\varphi] \ \exp \left\{ \frac{k}{4\pi} \int_\Sigma d^2z \ (\partial_\varphi - 2A_\pi) \partial_\varphi \right\} ,
\]

(2.90)
while the wavefunctional at the orbifold fixed point $t = 1/2$ corresponding to the open surface $\Sigma_{1/2}/\mathbb{Z}_2 = \Sigma^o$ is

$$
\Psi_{1/2}^{\text{orb}} \left[ A_z, A_\tau; \tilde{Y}^z, \tilde{Y}^\tau \right] = \int [D\varphi] \exp \left\{ \oint_{\partial\Sigma^o} \varphi_b \left( \tilde{Y}^\tau - \frac{k}{8\pi} A^\parallel \right) \right\} 
\times \exp \left\{ -\frac{k}{8\pi} \int_{\Sigma^o} d^2z \left[ A_z \left( \partial_z \varphi - \frac{8\pi}{k} \tilde{Y}^z \right) - A_\tau \left( \partial_\tau \varphi - \frac{8\pi}{k} \tilde{Y}^\tau \right) \right] \right\} .
$$

Here $S_{\text{TMGT}}^{\text{orb}}[A]$ is the topologically massive gauge theory action (2.2) for the orbifold of the three-dimensional theory, with the identification $2S_{\text{TMGT}}^{\text{orb}}[A_{\text{orb}}] = S_{\text{TMGT}}[A]$ and $A$ the extension of the gauge field $A_{\text{orb}}$ from $(\Sigma \times [0,1])/\mathbb{Z}_2$ to the covering cylinder. Analogous relations hold for the other actions, and for brevity we omit the orbifold superscript on all fields. The structure of the wavefunctionals (2.90) and (2.91) shows the physical significance of the specific solutions to the continuity equations that we obtained in the previous subsection. Namely, we can remove the external currents from the bulk by the shifts $\partial_z \varphi \rightarrow \partial_z \varphi + 8\pi \tilde{Y}^z/k$ and $\partial_\tau \varphi \rightarrow \partial_\tau \varphi + 8\pi \tilde{Y}^\tau/k$. In the language of the induced conformal field theory, this means that the change of conformal background is determined solely by boundary deformations, as we hope to find in the end.

We can now use the decomposition (2.86) to compute the conformal field theory partition function $Z_{\Sigma^o}$ for the induced open string WZNW model on the worldsheet $\Sigma^o$. After gauge-fixing, the Hodge decomposition on the cover $\Sigma$ of the gauge field one-form (2.86) is given by [28]

$$
\bar{A}^i = a^i + \epsilon^{ij} \partial_j \xi ,
$$

where $a_i$ are the harmonic degrees of freedom, with quantized periods in $H^1(\Sigma, \mathbb{Z}) \cong \mathbb{Z}^{2g}$, and $\xi$ is a real, compact scalar field on $\Sigma$. The exact term which usually appears in a Hodge decomposition has been set to 0 here by an appropriate gauge choice [28]. We will also use a Hodge decomposition for the external currents of the form

$$
\tilde{Y}^i = \frac{k}{4\pi} \left( \partial^i Y_D + \epsilon^{ij} \partial_j Y_N \right) ,
$$

where again $Y_D$ and $Y_N$ are real compact fields. As the currents are local operators, there are no harmonic degrees of freedom present in this decomposition. While they do change the topology of the worldsheet since a local insertion adds a new boundary to the surface, it is a local deformation only and not a global correction to the topology [76].

For convenience, let us now list the orbifold relations for the various fields appearing in
these decompositions. They are given by

\[
\begin{align*}
\text{PT :} & \quad a_z \mapsto -a_{\tau}, \quad Y_D \mapsto Y_D, \quad \xi \mapsto -\xi, \quad Y_N \mapsto -Y_N, \\
\text{PCT :} & \quad a_z \mapsto -a_{\tau}, \quad Y_D \mapsto -Y_D, \quad \xi \mapsto \xi, \quad Y_N \mapsto Y_N.
\end{align*}
\]

(2.94)

Defining complex fields in terms of their components tangent and transverse to the world-sheet boundary as \(X^z = X^\parallel + i X^\perp\), \(X^\tau = X^\parallel - i X^\perp\), we obtain for the PT orbifold of the topologically massive gauge theory the boundary conditions \(a^\perp = \xi = Y_N = 0\) on \(\partial \Sigma^o\), while \(a^\parallel\) and \(Y_D\) are arbitrary. For the PCT orbifold we have \(a^\perp = a^\parallel = Y_D = 0\) with arbitrary \(\xi\) and \(Y_N\) on \(\partial \Sigma^o\). The reason why \(a^\perp = 0\) at the boundary for both orbifolds is due to the fact that \(a^\perp\) encodes the harmonic degrees of freedom, which depend only on the homology of the Riemann surface \(\Sigma\). When approaching a boundary, \(a^\perp\) cannot have a component normal to it since this would constitute a local singularity, i.e. the failure of a canonical homology cycle of \(\Sigma\) to be closed. This is basically the same argument as to why there is no harmonic component in the Hodge decomposition of the current in (2.93). Note also that for the PCT orbifold \(\partial^\parallel \xi = 0\) at the boundary, because \(A^\perp = a^\parallel - i \partial^\parallel \xi\) from the decomposition (2.92) and for this orbifold \(A^\perp |_{\partial \Sigma^o} = 0\).

Because of the remarks just made, the calculation of the orbifold partition function (2.89) proceeds similarly to that described in [28], except that now the wavefunction (2.91) is completely gauge-invariant, because here we have solved the gauge constraint on \(\partial \Sigma^o\) in (2.8) and so the gauge parameter function \(\Lambda\) gets absorbed into the boundary scalar field degree of freedom \(\varphi_b\). We note again that the boundary field \(\varphi_b\) has for now free boundary conditions, and is arbitrary for both kinds of orbifolds. Only after integration over the gauge degrees of freedom will we encode its boundary conditions in the form of functional Dirac delta-functions in the path integral.

Given the above decompositions, the orbifold wavefunction (2.91) is now rewritten as

\[
\Psi_{1/2}^{orb} [a, \xi; Y_D, Y_N] = \int [D\varphi] \exp \left\{ \frac{k}{8\pi} \oint_{\partial \Sigma^o} \left[ 2 \varphi_b \left( \partial^\perp Y_D + i \partial^\parallel Y_N \right) - i \varphi a^\parallel - i \xi \partial^\perp \varphi \right] \right\} \left. \times \exp \left\{ -\frac{i k}{8\pi} \int_{\Sigma^o} d^2 z \left[ \epsilon^{ij} a_i \left( \partial_j \varphi - 2 \partial_j Y_D - 2 h_{jl} \epsilon^{lk} \partial_k Y_N \right) \right. \right.
\]

\[
\left. \left. + h^{ij} \partial_i \xi \left( \partial_j \varphi - 2 \partial_j Y_D - 2 h_{jl} \epsilon^{lk} \partial_k Y_N \right) \right] \right\}.
\]

(2.95)

By integrating by parts in the bulk of \(\Sigma^o\) it follows that this wavefunction factorizes into
bulk and boundary components as

\[ \Psi_{1/2}^{\text{orb}} = \int [D\varphi] \exp \left\{ -\frac{k}{8\pi} \int_{\Sigma^0} d^2 z \, \xi \nabla^2 (\varphi - 2Y_D) \right\} \Psi_{\partial\Sigma^0}^\dagger, \]  

(2.96)

where the boundary wavefunctional for both orbifold types is given by

PT : \[ \Psi_{\partial\Sigma^0}^\dagger = \int [D\varphi_b] \exp \left\{ \frac{k}{4\pi} \oint_{\partial\Sigma^0} \left[ \varphi_b \partial^\perp Y_D - i a^\parallel (\varphi - Y_D) \right] \right\}, \]  

(2.97)

PCT : \[ \Psi_{\partial\Sigma^0}^\dagger = \int [D\varphi_b] \exp \left\{ \frac{i k}{4\pi} \oint_{\partial\Sigma^0} \left[ \varphi_b \partial^\parallel Y_N + \xi \partial^\perp \varphi_b \right] \right\}. \]  

(2.98)

We thereby find that the integrations over the gauge field degrees of freedom \( a^\parallel \) and \( \xi \) produce, for the two types of orbifold involutions, Dirichlet and Neumann boundary conditions on the scalar field \( \varphi \) as

\[
\begin{align*}
\text{Dirichlet (PT)} : & \quad \delta_{\Sigma^0} \left( \nabla^2 (\varphi - 2Y_D) \right) \delta_{\partial\Sigma^0} (\varphi_b - Y_D), \\
\text{Neumann (PCT)} : & \quad \delta_{\Sigma^0} \left( \nabla^2 (\varphi - 2Y_D) \right) \delta_{\partial\Sigma^0} (\partial^\perp \varphi_b). 
\end{align*}
\]  

(2.99)

It is the \( \Lambda \) independence of (2.95) that leads to boundary conditions in (2.99) which differ from those found in [28]. But just as in [28], the roles played by the wavefunctional (2.95) at the orbifold branch point are to enforce the bulk equation of motion for the free scalar field \( \varphi \) on \( \Sigma^0 \), and also to select the boundary conditions of the open string theory. Otherwise it simply corresponds to inserting the identity character state \( |1\rangle \) into the inner product (2.89).

The main new result here is the appearence of the first exponential factors in the bound-
ary wavefunctionals (2.97) and (2.98) which have emerged from a careful and specific incor-
poration of the external sources and were absent in the analysis of [28]. After an integration
by parts in (2.97), for Dirichlet boundary conditions (PT) it becomes

\[ \mathcal{V}_D = \exp \left\{ -\frac{k}{4\pi} \oint_{\partial\Sigma^0} Y_D \partial^\perp \varphi_b \right\}. \]  

(2.100)

The object (2.100) is just the vertex operator for a D-brane wrapped around a single
compact dimension of radius \( R \) and described by the collective coordinate \( Y_D \). This holds
with the usual identification of the Chern-Simons coupling constant as

\[ k = \frac{2R^2}{\alpha'}, \]  

(2.101)
where $\alpha'$ is the string slope. On the other hand, for Neumann boundary conditions (PCT), the boundary wavefunction (2.98) yields the vertex operator

$$V_N = \exp \left\{ -\frac{ik}{4\pi} \oint_{\partial \Sigma^0} Y_N \partial^\parallel \varphi_b \right\},$$

(2.102)

which is just the Wilson line for the target space gauge field component $R Y_N/2\alpha'$ around the compact direction. As expected by T-duality, the vertex operators (2.100) and (2.102) map into each other under $\partial^\perp \varphi_b \leftrightarrow i \partial^\parallel \varphi_b$.

Thus, as we anticipated at the beginning of this section, the coupling of topologically massive gauge theory to an external current encodes both the Wilson lines and the D-brane collective coordinates of string theory. In the case of a more general topologically massive gauge theory with abelian structure group $U(1)^d \times U(1)^D$, the orbifold of the fields of the $U(1)^d$ sector by the PT involution of the bulk quantum field theory would produce vertex operators of the type $\partial^I \varphi_b$, with $I = 1, \ldots, d$ corresponding to the transverse directions to the brane, while the orbifold of the $U(1)^D$ sector by the PCT involution of the three-dimensional gauge fields $A^I$ would produce target space gauge fields $Y^I_N$, with $I = d + 1, \ldots, d + D$ running along the brane worldvolume directions. Geometrically, both involutions produce the same open surface $\Sigma_1/\mathbb{Z}_2$, but the charge conjugation operation $C$ acts on the charges and the gauge fields $A^I$, and not on the membrane coordinates. In this way we have explicitly demonstrated the emergence of D-branes in the topological membrane approach to string theory through a derivation of the brane vertex operators in the corresponding induced conformal field theory action on an open surface. In section 4 we will analyse D-branes at the level of worldsheet effective actions which are derived from the topological membrane by using the procedure outlined above.

### 3 BRANE STATES

The purpose of this section is to construct the appropriate generalizations of the Schrödinger wavefunctionals of the previous section which correspond to boundary states of D-branes in the induced boundary conformal field theory. We will first describe how these functionals reproduce the well-known three-dimensional descriptions of the fusion algebra and the corresponding Verlinde formula in the closed string sector. Then we shall show how to modify these calculations in the presence of an orbifold of the topological membrane. We shall see that D-branes and string Wilson lines are naturally selected by gauge invariance of the bulk three-dimensional theory as fundamental boundary states. Moreover, in this setting the Cardy map between the set of admissible boundary states and conformal blocks is determined entirely by the spectrum of allowed charges of dynamical matter inside the
topological membrane, i.e. by the number of possible ground states in Chern-Simons theory, which after orbifolding give distinctive boundary conditions.

### 3.1 The Fusion Ring

The $U(1)$ topologically massive gauge theory has a multiply-connected gauge group, and so it corresponds to a $c = 1$ conformal field theory with an extended chiral algebra, the chiral algebra of the rational circle. For this, we take the Chern-Simons coefficient (2.101) to be the rational number

$$k = \frac{2p}{q},$$

where $p$ and $q$ are positive integers. For simplicity we assume that $p$ is even with $p/2$ and $q$ coprime. The spectrum of charges on a closed Riemann surface $\Sigma$ of genus $g$ is given by

$$Q_\ell^\lambda \equiv \frac{\lambda^\ell}{q} = m^\ell + \frac{k n^\ell}{4},$$

$$\bar{Q}^\ell_\lambda \equiv \frac{\bar{\lambda}^\ell}{q} = m^\ell - \frac{k n^\ell}{4}$$

with $m^\ell, n^\ell = 0, 1, \ldots, \frac{pq}{2} - 1, \ell = 1, \ldots, g$ the winding and monopole numbers of a charge as it moves around canonical homology cycles $\beta^\ell$ of $\Sigma$. We assume, for each $\ell$, that the integers $(m^\ell, n^\ell)$ form a Bezout pair with respect to the decompositions (3.2). The resulting conformal field theory is rational as the $U(1)$ charges are now restricted to $\lambda, \bar{\lambda} \in (\mathbb{Z}_{pq})^g$.

For any integer $n$, we shall denote by $[n] \in \mathbb{Z}_{pq}$ its integer part modulo $pq$.

On the punctured (Riemann) sphere $\Sigma = S^2_0$ (equivalently the infinite cylinder $\Sigma = \mathbb{R} \times S^1$), there are $pq$ primary blocks $\phi_{\lambda}(z) = e^{i \lambda \varphi(z)}/q$, $\lambda = 0, 1, \ldots, pq - 1$ of this chiral algebra. Members of the conformal family $[\phi_{\lambda}]$ have charges $Q_\lambda + pl$ with $l \in \mathbb{Z}$. The corresponding fusion rules

$$[\phi_{\lambda}] \times [\phi_{\mu}] = \sum_{\nu=0}^{pq-1} N_{\lambda \mu}^\nu [\phi_{\nu}]$$

may be determined through the representation theory of the group algebra $\mathbb{C}[\mathbb{Z}_{pq}]$ of the cyclic subgroup $\mathbb{Z}_{pq}$ in the $U(1)$ gauge group. In this simple case the fusion coefficients are given by

$$N_{\lambda \mu}^\nu = \delta_{[\lambda + \mu - \nu]},$$

where $\delta_{[\lambda]}$ is the (periodic) delta-function on the finite group $\mathbb{Z}_{pq}$ which is defined by

$$\delta_{[\lambda]}(z) = \frac{1}{pq} \sum_{\mu=0}^{pq-1} e^{2\pi i \mu \lambda/pq}.$$
The natural metric on the fusion algebra is the charge conjugation matrix $C : [\phi_\lambda] \to [\phi_{\lambda^+}]$ and is given here by
\[ C_{\lambda\mu} = \delta_{[\lambda+\mu]} \].
Because of the triviality of (3.6), we shall not distinguish between indices which are raised and lowered with this metric.

### 3.2 Punctured Wavefunctionals

The main goal of this section is to provide a three-dimensional description of the fusion algebra of the previous subsection and its open string counterpart, with an eye to describing those conformal field theoretic states which correspond to D-branes. As we have shown in the previous section, D-brane states arise when we introduce external bulk matter into the topologically massive gauge theory. When these currents correspond to the propagation of charged particles in three-dimensions, they can be equivalently described in terms of the source-free gauge theory in the presence of Wilson lines
\[ W_{Q_i}[A] = \exp \left\{ i Q_i \int_{C_i} A \right\} \] (3.7)

corresponding to the bulk propagation of charges $Q_i = \lambda_i/q$ along the oriented worldlines $C_i \subset M$. In the following we will describe some generic properties of the correlators
\[ \left\langle \prod_{i=1}^s W_{Q_i} \right\rangle_M = \int [DA] \mathcal{W}_M[A] \prod_{i=1}^s W_{Q_i}[A] \ e^{iS_{TMGT}[A]} \] (3.8)
evaluated in the topologically massive gauge theory defined on various three-manifold geometries $M$. As before, all such correlation functions will be evaluated in the topological limit of the quantum field theory. The weight $\mathcal{W}_M$ depends on the particular three-manifold that we are working on.

In this subsection we will be interested in (3.8) for the case of the vertical Wilson lines of the topological membrane (fig. 2(a)), whereby the three-geometry is taken to be $M = \Sigma \times [0, 1]$. Then the operators (3.7) are not invariant under gauge transformations $A_i \to A_i + \partial_i \Lambda$ but instead change by a phase factor which is given by the boundary values of $\Lambda$ on $\Sigma_0$ and $\Sigma_1$. Since such gauge non-invariant terms should be absorbed as usual by shifting the WZNW field as $\varphi \to \varphi + \Lambda$, it follows that insertions of the Wilson line operators (3.8) correspond to insertions of the primary fields $e^{iQ_i \varphi}$ at the boundary points $z_i, \bar{z}_i$ on $\Sigma_0$ and $\Sigma_1$ corresponding to the two endpoints of the worldlines $C_i$. For a collection of unlinked and unknotted particle trajectories in $M$, these Wilson lines thereby correspond to gauge-invariant states of the topological membrane whose wavefunctionals can be built.
by inserting the tachyon fields \( e^{iQ_i \varphi} \) at the points \( z_i \) into the vacuum functional (2.33) to give

\[
\Xi \left[ A_z, A_\Sigma; \left\{ z_1, \ldots, z_s \right\}; 0 \right] = \exp \left\{ \frac{k}{8\pi} \int_\Sigma d^2z \ A_\Sigma A_z \right\} 
\times \int [D\varphi] \prod_{i=1}^s e^{iQ_i (\varphi(z_i) + h(\varphi(z_i)))} \exp \left\{ \frac{k}{8\pi} \int_\Sigma d^2z \ (\partial_\varphi - 2A_\varphi) \partial_\varphi \right\} . \tag{3.9}
\]

The function \( h(\varphi(z)) \) is given by an integral over the harmonic parts \( a_i \) of the gauge field one-forms and it takes into account the large gauge transformations \( \Sigma \rightarrow U(1) \) which wind around the canonical homology cycles of \( \Sigma \). The functionals (3.9) simply coincide with the gauge-invariant charged vacuum states constructed in the previous section in the case that the non-dynamical bulk matter fields correspond to static point charges, as they clearly solve the vacuum wave equation (2.47) and gauge constraint (2.23) with

\[
j^i = 0 , \quad \rho(z) = \sum_{i=1}^s Q_i \delta^2(z - z_i) . \tag{3.10}
\]

They consist of \( s \) gauge-invariant combinations of external particles in the bulk which pierce \( \Sigma \) at the points \( z_i \).

![Figure 2](image_url)

(a) The topological membrane with two linked Wilson lines \( C \) and \( C' \). (b) Identifying the initial and final surfaces of (a) produces the three-geometry \( \Sigma \times S^1 \) with two linked Polyakov loops.

In the present case, invariance under large gauge transformations in the bulk is taken care of by inserting into (3.8) the weight functional [28]

\[
\mathcal{W}_{\Sigma \times [0,1]} \left( \Delta Q_\lambda \right) = \prod_a V^{n_a}(z_a) = \prod_{\ell=1}^g \exp \left\{ i \Delta Q_\lambda^\ell \oint A \right\} , \tag{3.11}
\]
with $\Delta Q^\ell = -n^\ell k/2$ corresponding to a set of primary charges $\vec{\lambda} \in (\mathbb{Z}_{pq})^g$. This operator also incorporates the appropriate non-perturbative linking and monopole-instanton transitions in the bulk. With this definition, the partition function (2.85) of the topological membrane yields, in the absence of sources, the inner product

$$Z_c^e (\Gamma, \bar{\Gamma}) = \langle 1 \rangle_{\Sigma \times [0,1]} = k^{g/2} \sum_{\vec{\lambda} \in (\mathbb{Z}_{pq})^g} \Psi^\dagger_{\vec{\lambda}}(\bar{\Gamma}) \Psi_{\vec{\lambda}}(\Gamma),$$

(3.12)

where $\Psi_{\vec{\lambda}}(\Gamma)$ are the effective topological wavefunctions of topologically massive gauge theory which depend only on the periods $\Gamma$ of the compact Riemann surface $\Sigma$ [28]. They are proportional to the characters of the extended Kac-Moody group in this case. The key feature of (3.12) is that it manifestly admits a holomorphic factorization into components depending only on the left and right moving worldsheets $\Sigma_0$ and $\Sigma_1$. In exactly the same way, the inner product $\langle \Xi_1 | \Xi_0 \rangle$ of the charged wavefunctionals (3.9), defined as in (2.85), reproduces the $c = 1$ conformal field theory partition function on $\Sigma$ with primary state insertions.

### 3.3 Topological Correlation Functions

To describe the chiral algebra of the rational circle, we can exploit the holomorphic factorization of (2.85) (or (3.12)) to restrict it to a single chiral sector of the worldsheet theory. This is achieved by introducing a fixed-point free gluing automorphism along the time direction of the membrane $\Sigma \times [0,1]$ which identifies $\Sigma_0 \equiv \Sigma_1$, i.e. we replace $\Sigma \times [0,1]$ by the quotient space $(\Sigma \times [0,1])/\sim$ with respect to the equivalence relation $\sim$ defined by $(z,0) \sim (z,1) \forall z \in \Sigma$. We then Wick rotate the time direction to Euclidean signature. In this way, the vertical Wilson lines of the topological membrane become Polyakov loops of finite temperature gauge theory on $M = \Sigma \times S^1$ (fig. 2(b)) [66]. Formally, the chiral correlation functions on $\Sigma \times S^1$ are related to those of the topological membrane (on $\Sigma \times [0,1]$) as follows. The crucial point is that in the finite-temperature gauge theory it is no longer possible to fix the temporal gauge $A_0 = 0$, as this is incompatible with invariance under large gauge transformations which wind around the $S^1$. Instead, a natural consistent gauge choice is $\partial_t A_0 = 0$, in which the membrane amplitudes can be evaluated as in the previous section with $A_0$ regarded as an external static source.

For a collection of unlinked and unknotted loops $C_i$, the correlators (3.8) with weight functionals $W_{\Sigma \times S^1} = 1$ depend only on the insertion points $x_i \in \Sigma$. Then, by incorporating the standard Faddeev-Popov gauge-fixing determinant, the trace of the wavefunction correlator (2.85) on $\Sigma \times [0,1]$ in the states (3.9) can be used to generate the Polyakov loop
correlation functions on $\Sigma \times S^1$ through the trace formula

$$\left\langle \prod_{i=1}^{s} W_{Q_i}(x_i) \right\rangle_{\Sigma \times S^1} = \text{Tr} \left\langle \Xi_1 | \Xi_0 \right\rangle$$

$$= \int [DA_0] \int [DA_z DA_{\tau}] \det' \partial_t |_{S^1} e^{iS_{\text{TMTG}}[A]}$$

$$\times \left[ A_z, A_{\tau}; \left\{ \begin{array}{c} x_1 \\ Q_1 \\ \vdots \\ x_s \\ Q_s \end{array} \right\}; A_0 \right] \Xi \left[ A_z, A_{\tau}; \left\{ \begin{array}{c} x_1 \\ Q_1 \\ \vdots \\ x_s \\ Q_s \end{array} \right\}; A_0 \right] ,$$

(3.13)

where we have made the identification $\Xi = \Xi_0 \equiv \Xi_1$ of initial and final membrane states, corresponding to the gluing $\Sigma_0 \equiv \Sigma_1$, which gives a trace. We can now gauge away the field component $A_0 \rightarrow A_0 - \partial_t \Lambda_t = 0$ from the wavefunctionals in (3.13) via the time-dependent gauge transformation $\Lambda_t = tA_0$ with $\Lambda_0 = 0$ and $\Lambda_1 = A_0$. In the membrane picture, this gauge transform only affects the anti-holomorphic spatial components $A_z$ in the wavefunctions $\Xi^\dagger$ on $\Sigma_1$. The Schrödinger amplitudes (2.85) are gauge-invariant, and whence up to an irrelevant normalization the trace (3.13) becomes

$$\left\langle \prod_{i=1}^{s} W_{Q_i}(x_i) \right\rangle_{\Sigma \times S^1} = \int [DA_z DA_{\tau}] e^{iS_{\text{TMTG}}[A]}$$

$$\times \left[ A_z, A_{\tau}; \left\{ \begin{array}{c} x_1 \\ Q_1 \\ \vdots \\ x_s \\ Q_s \end{array} \right\}; 0 \right] \Xi \left[ A_z, A_{\tau}; \left\{ \begin{array}{c} x_1 \\ Q_1 \\ \vdots \\ x_s \\ Q_s \end{array} \right\}; 0 \right] .$$

(3.14)

Since the source-free Hamiltonian $H = H_0$ in (2.43) of this thermodynamical system vanishes in the vacuum sector of the quantum gauge theory, this definition turns the topological inner product into a trace of the identity operator over the finite-dimensional Hilbert space $\mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s)$ of physical ground states of the gauge theory in the presence of Polyakov loop insertions. In two-dimensional language, this space is the same as the vector space of holomorphic conformal blocks of the corresponding WZNW model on a closed surface $\Sigma$ of genus $g$ with $s$ vertex operator insertions [1, 31, 66], or alternatively as the fiber space of the appropriate Friedan-Shenker holomorphic vector bundle over the moduli space of $s$-punctured Riemann surfaces of genus $g$ (with $\Psi_\chi(\Gamma)$ above particular sections of this bundle). It follows that the dimension of the vacuum sector of the matter-coupled topologically massive gauge theory is given by

$$\dim \mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s) = \text{Tr} \mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s)(\Pi) = \left\langle \prod_{i=1}^{s} W_{\lambda_i/q}(x_i) \right\rangle_{\Sigma \times S^1} .$$

(3.15)

In this simple model it is easy to evaluate the topological numbers (3.15) by using the trace formula (3.14). The functional integrations arising after gauge-fixing and insertion of
the explicit forms of the states (3.9) are all Gaussian, as this is a free field theory. By using the decompositions (2.86) and (2.92), the bulk gauge field integrations are readily seen to eliminate the charge independent terms in the scalar fields $\varphi$ after appropriate gauge transformations parametrized by them. Integration over the insertions $\varphi(x_i)$ then produces a delta-function constraint enforcing bulk charge (or flux) conservation on a closed space, $\sum_i Q_i = 0$, and we arrive at

$$\dim \mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s) = \frac{k^{g/2}}{2} \delta_{[\lambda_1 + \ldots + \lambda_s]} \sum_{\vec{\lambda} \in (\mathbb{Z}^{pq})^g} \left( \Psi_{\vec{\lambda}}(\Gamma), \Psi_{\vec{\lambda}}(\Gamma) \right)_a,$$

(3.16)

where $(\ , \ )_a$ is the inner product on the space of harmonic one-forms which is given by an integral over the Jacobian variety of the Riemann surface $\Sigma$. The topological wavefunctions are orthogonal with respect to this inner product [28]

$$\left( \Psi_{\vec{\lambda}}(\Gamma), \Psi_{\vec{\lambda}'}(\Gamma) \right)_a \equiv \frac{1}{2^g \sqrt{\det \Gamma_2}} \int_{\text{Jac}(\Sigma)} \prod_{\ell=1}^g da^\ell \ d\vec{\pi}^\ell \ \Psi_{\vec{\lambda}}^\ell(\vec{a}, \vec{\pi} | \Gamma) \ \Psi_{\vec{\lambda}'}(\vec{a}, \vec{\pi} | \Gamma) = \frac{1}{k^{g/2}} \delta_{\vec{\lambda}, \vec{\lambda}'} ,$$

(3.17)

where $\Gamma_2$ denotes the positive-definite imaginary part of the period matrix $\Gamma$. The dimension (3.16) is therefore given by

$$\dim \mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s) = (pq)^g \delta_{[\lambda_1 + \ldots + \lambda_s]} ,$$

(3.18)

as expected since here there are $(pq)^g$ linearly independent topological wavefunctions $\Psi_{\vec{\lambda}}(\Gamma)$.

3.4 Surgical Wavefunctionals

The final quantities that we shall need to know are the contributions from the linkings of charged particle trajectories, as illustrated in fig. 2. This is a purely bulk effect which can be computed by noting that in the neighbourhood of any such link, the three-manifold may be regarded topologically as the three-sphere $S^3$ and the linking effect as generated by closed contours $C_i$. With $W_{S^3} = 1$ and the bulk normalization of section 2.7 for the partition function, we may then invoke the standard formula [1]

$$\left\langle \prod_{i=1}^s W_{Q_i}[A] \right\rangle_{S^3} = \prod_{i,j=1}^s e^{\frac{2\pi i}{k} Q_i Q_j \#(C_i, C_j)} ,$$

(3.19)

where

$$\#(C_i, C_j) = \oint_{C_i} dx_\mu \oint_{C_j} dy_\nu \ \epsilon_{\mu\nu\lambda} (x - y)_\lambda |x - y|^3$$

(3.20)

is the Gauss linking number of the closed particle trajectories $C_i$ and $C_j$ which counts the number of signed intersections of $C_i$ with the surface spanned by $C_j$. When $i = j$
the self-linking integral (3.20) must be suitably regularized by a framing of the contour \( C_i \) [1]. More precisely, such effects are computed via surgery prescriptions, by gluing three-manifolds with Wilson lines onto the ones of interest to give invariants of links in \( S^3 \) and by using functoriality of the gauge theory amplitudes [1, 63]. In what follows, however, we will only be concerned with deriving relationships between the conformal blocks of open and closed string theories, for which the correspondence described above will suffice.

For the membrane linkings of the sort illustrated in fig. 2(a), the linking effect can be incorporated analytically by introducing the full, non-chiral wave functionals obtained by gluing the left and right moving punctured states (3.9) together using the monopole operators (3.11) and the multi-valued angle function (2.15) of \( \Sigma \). The vacuum functionals associated with the vertical (and possibly linked) propagation of charged particles from the boundary \( \Sigma_0 \) to the boundary \( \Sigma_1 \) are thereby given as

\[
\Phi \left[ A_z, A_{\overline{z}}; \left\{ \frac{z_1}{Q_1}, \ldots, \frac{z_s}{Q_s}, \overline{\frac{z_1}{Q_1}}, \ldots, \overline{\frac{z_s}{Q_s}} \right\} \right] = \prod_{i,j=1}^s e^{\frac{i}{2} \sum_{\ell,\ell' = 1}^g (\Gamma_2^{-1})_{\ell\ell'} \int_{x_i(0)}^{x_i(1)} \int_{x_j(0)}^{x_j(1)} \omega_{\ell'} \overline{\omega_{\ell'}} d\zeta} W_{\Sigma \times [0,1]} \left( \Delta \overline{\theta_{\Sigma}} \right) \Xi_0 \left[ A_z, A_{\overline{z}}; \left\{ \frac{z_1}{Q_1}, \ldots, \frac{z_s}{Q_s} \right\}; 0 \right]
\]

\[
\otimes \Xi_1^\dagger \left[ A_z, A_{\overline{z}}; \left\{ \overline{\frac{z_1}{Q_1}}, \ldots, \overline{\frac{z_s}{Q_s}} \right\}; 0 \right],
\]

(3.21)

where

\[
\theta_{ij}(t) = \theta \left( x_i(t), x_j(t) \right) + 2 \sum_{\ell,\ell' = 1}^g (\Gamma_2^{-1})_{\ell\ell'} \text{Im} \left[ \int_{x_i(0)}^{x_i(1)} (x_j(t) - x_j(0)) \overline{\omega_{\ell'}} + \omega_{\ell'} \right] + \int_{x_i(0)}^{x_i(t)} (\omega_{\ell'} + \overline{\omega_{\ell'}})
\]

(3.22)

is the angle function for particles \( i \) and \( j \) [20] with \( x_i(0) = z_i \) and \( x_i(1) = \overline{z_i} \), and \( \omega_{\ell} = \omega_{\ell}(z) \, dz \), \( \ell = 1, \ldots, g \) form a basis of holomorphic one-differentials on the Riemann surface \( \Sigma \). The function (3.22) changes by \( 2\pi \) whenever the worldline of particle \( i \) wraps exactly once around that of particle \( j \), as this induces an adiabatical rotation of coordinates \((z_i - z_j) \rightarrow e^{2\pi i} (z_i - z_j)\) on \( \Sigma \). For other purely bulk linkings such as those associated with closed Wilson loops in the interior of the membrane, one needs to use the Gauss linking formula (3.19) explicitly. Again we stress that by the usage of (3.19) to compute linking effects we will always be implicitly assuming an appropriate surgery prescription. With this, we have thereby arrived at a complete description of all gauge-invariant vacuum states in the matter-coupled topologically massive gauge theory, and hence at an exact, three-dimensional version of the chiral algebra of the \( c = 1 \) conformal field theory underlying the rational circle.
3.5 The Verlinde Formula

In this subsection we will exploit the simplicity of the present abelian model to describe some physical properties of the Verlinde diagonalization formula for (3.18) [77]. While technically this formula unveils no surprises for the case of the rational circle, we will interpret it here as a non-trivial statement about the requirement of charge conservation in linking processes among charged particles in the bulk. From the perspective of vacuum Schrödinger wavefunctionals, the topological Verlinde numbers express which combinations of the insertions give wavefunctionals (3.9) with non-vanishing inner products, and hence which are the true dimensions of the Hilbert spaces spanned by the states Ξ.

Our starting point is with the topological wavefunctions at genus 1 which are given by [28]

\[ \Psi_\lambda(\tau) = \frac{1}{(k\tau^2)^{1/4}\eta(\tau)} \sum_{r=-\infty}^{\infty} \exp \left\{ \frac{2\pi i r}{k} \left( rp + \frac{\lambda}{q} \right) \right\}, \quad \lambda \in \mathbb{Z}_{pq}, \]  

(3.23)

where the modular parameter \( \tau \) is a complex number of imaginary part \( \tau_2 > 0 \) and

\[ \eta(\tau) = e^{\pi i \tau/12} \prod_{r=1}^{\infty} \left( 1 - e^{2\pi i r \tau} \right) \]  

(3.24)

is the Dedekind function. Using the completeness of these states, we define the S-matrix through the modular transform

\[ \Psi_\lambda \left( -\frac{1}{\tau} \right) = \sum_{\lambda' = 0}^{pq-1} S_{\lambda'\lambda} \Psi_{\lambda'}(\tau). \]  

(3.25)

It can be computed explicitly by using the definition (3.1), the modular transformation properties

\[ \left( -\frac{1}{\tau} \right)^2 = \frac{\tau_2}{|\tau|^2}, \quad \eta \left( -\frac{1}{\tau} \right) = \sqrt{-i\tau} \eta(\tau), \]  

(3.26)

and the Poisson resummation formula

\[ \sum_{r'=-\infty}^{\infty} e^{-\pi hr'^2-2\pi i br'} = \frac{1}{\sqrt{h}} \sum_{r=-\infty}^{\infty} e^{-\pi (r-h)^2/h} \]  

(3.27)

to get

\[ \Psi_\lambda \left( -\frac{1}{\tau} \right) = \frac{1}{\sqrt{pq} (k\tau^2)^{1/4}\eta(\tau)} \sum_{r'=-\infty}^{\infty} \exp \left\{ \frac{2\pi i r'}{pq} \left( \frac{\tau r'}{2} + \lambda \right) \right\}. \]  

(3.28)

By writing \( r' = \lambda' + pq r \) and summing (3.28) over \( \lambda' = 0, \ldots, pq-1 \) and \( r \in \mathbb{Z} \), we arrive at (3.25) with

\[ S_{\lambda\lambda'} = \frac{1}{\sqrt{pq}} e^{2\pi i \lambda \lambda'/pq}. \]  

(3.29)
Thus the modular transformation (3.25) is just a finite Fourier transform on the cyclic group $\mathbb{Z}_{pq}$. The $S$-matrix (3.29) is symmetric and unitary,

$$\sum_{\lambda'}^{pq-1} S_{\lambda}^{\lambda'} S_{\lambda'}^{1} = \delta_{[\lambda-\mu]}.$$  \hspace{1cm} (3.30)

The key property of (3.29) within the present context is that it can be defined in terms of a completely non-perturbative, three-dimensional bulk process. Namely, it corresponds to the statistical exchange phase between two charged particles whose bulk trajectories $C$ and $C'$ are linked according to $\#(C,C') = +1$ and $\#(C,C') = \#(C',C') = 0$, as illustrated in fig. 3. This equality follows from the Gauss linking formula (3.19) along with (3.1), (3.2) and (3.29) which determine the loop correlator in fig. 3 as $e^{4\pi i Q Q'/k} = S_{\lambda\lambda}/S_{0\lambda}$. This expression for the topological invariant of the Hopf link in $S^3$ can be derived more generally by using a surgery prescription [1], in which a solid torus surrounding the link is cut out of $S^3$ and then glued back after performing a modular transformation on its boundary.

\[ \langle Q \quad Q' \rangle_{S^3} = \frac{S_{\lambda\lambda'}}{S_{0\lambda}} \]

Figure 3: The Hopf linking of two charges $Q = \lambda/q$ and $Q' = \lambda'/q$ in the bulk can be used to define the modular $S$-matrix.

Let us now examine the fusion coefficients (3.4) for the rational circle. By using (3.5) they may be expressed as a sum of statistical exchange phases from linked particle trajectories in the bulk, which from (3.19) and (3.29) may each be written as

$$e^{4\pi i Q(Q_1+Q_2+Q_3)/k} = \frac{S_{\lambda_1\lambda} S_{\lambda_2\lambda} S_{\lambda_3\lambda}}{(S_{0\lambda})^3}. \hspace{1cm} (3.31)$$

The fusion coefficients can thereby be written as

$$N_{\lambda_1\lambda_2\lambda_3} = \sum_{\lambda=0}^{pq-1} \frac{S_{\lambda_1\lambda} S_{\lambda_2\lambda} S_{\lambda_3\lambda}}{S_{0\lambda}}, \hspace{1cm} (3.32)$$

which is the celebrated Verlinde diagonalization for the three-punctured Riemann sphere [77]. It implies, in particular, that the eigenvalues $n^{(\lambda)}_{\lambda'}$ of the fusion matrix $(N_{\lambda})_{\lambda'\mu} = N_{\lambda\lambda'\mu}$ are given by

$$n^{(\lambda)}_{\lambda'} = \frac{S_{\lambda\lambda'}}{S_{0\lambda}}. \hspace{1cm} (3.33)$$
The statistical exchange phases (3.33) are discrete characters of the \( \mathbb{Z}_{pq} \) gauge subgroup of the topologically massive gauge theory which themselves obey the fusion algebra

\[
N^{(\lambda)}_{\lambda'} n^{(\lambda)}_{\mu} = \sum_{\nu=0}^{pq-1} N^{(\lambda)}_{\lambda'} n^{(\lambda)}_{\mu} n^{(\lambda)}_{\nu}
\]

for all \( \lambda \in \mathbb{Z}_{pq} \).

On the other hand, according to (3.15) and (3.18), the fusion algebra of the rational circle is determined by the three-point function of Polyakov loops on the three-geometry \( \mathbb{S}^2 \times \mathbb{S}^1 \) [66,67]. A heuristic derivation of (3.32) is depicted schematically in fig. 4. Again it can be derived more generally via surgery on \( \mathbb{S}^2 \times \mathbb{S}^1 \) [1].

\[
N_{\lambda_1 \lambda_2 \lambda_3} = \left< Q_1 Q_2 Q_3 \right>_{\mathbb{S}^3} = \frac{1}{pq} \sum_{\lambda=0}^{pq-1} \left< Q_1 Q_2 Q_3 \right>_{\mathbb{S}^3}
\]

Figure 4: Schematic derivation of the Verlinde formula for the three-punctured two-sphere. In the first equality we note that the puncturing of \( \Sigma = \mathbb{S}^2 \) by Polyakov loops, as represented in fig. 2(b), can be interpreted as the single linking of each of the three loops with a loop in the trivial \( Q = 0 \) representation of the cyclic group \( \mathbb{Z}_{pq} \). In the second equality we use (3.5) to replace this trivial loop by the sum over a complete set of Polyakov loops. By (3.19), each term in this sum coincides with the linking phases (3.31).

The general case (3.18) is treated similarly, giving the general Verlinde formula

\[
\dim \mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s) = \sum_{\lambda=0}^{pq-1} (S_{0\lambda})^{2-2g-s} \prod_{i=1}^{s} S_{\lambda_i \lambda}
\]

for the dimension of the corresponding vacuum Hilbert space of the three-dimensional gauge theory. Physically, we consider \( s \) static charges \( Q_1, \ldots, Q_s \) moving in the bulk of \( \Sigma \times \mathbb{S}^1 \) which puncture the genus \( g \) Riemann surface \( \Sigma \) at fixed, distinct points \( x_1, \ldots, x_s \). Such a punctured Riemann surface can be represented by \( 2g + s - 2 \) “pants” (three-holed or three-punctured Riemann spheres) glued together along the holes. One then takes a sum over charge labellings of the internal hole boundaries of the product of \( N_{\lambda_1 \lambda_2 \lambda_3} \), one for each pair of pants. For example, in the source-free case we have

\[
\dim \mathcal{H}_{g,0,k} = \sum_{\lambda=0}^{pq-1} \sum_{\lambda_1, \ldots, \lambda_s} \sum_{\mu_1, \ldots, \mu_g} \sum_{\nu_1, \ldots, \nu_g} \sum_{\lambda'_{g-1}} N_{\lambda \lambda'} \mu_1 N_{\mu_1 \lambda_1} \nu_1 N_{\nu_1 \lambda_2} \mu_2 N_{\mu_2 \lambda_2} \nu_2 \cdots N_{\nu_{g-1} \lambda_g} \mu_g N_{\mu_g \lambda_g}.
\]
By using (3.32) and unitarity (3.30) of the S-matrix, we then arrive at (3.35). It is in this way that we may interpret the Verlinde formula dynamically as the statement of overall charge conservation for arbitrary, non-perturbative linking processes involving charged matter in the bulk quantum field theory. In other words, the formula (3.35) expresses the construction of the three-dimensional wavefunctions (3.9), which carry information about the physical spectrum of the induced string theory, from different gluings of three-manifolds with boundaries by pants.

### 3.6 Ishibashi Wavefunctionals

In this subsection we will construct a basis of wavefunctionals which generate the three-dimensional counterparts of closed string, brane boundary states of the rational circle. The basic idea is that the diagonal, modular invariant torus partition function \[ Z_c(\tau, \bar{\tau}) = \sqrt{k} \sum_{\lambda=0}^{pq-1} \Psi_\lambda(\tau) \otimes \Psi_\lambda^*(\bar{\tau}) \] (3.37)
encodes information about the closed string Hilbert space \( \mathcal{H}_c \) in radial quantization of the two-dimensional \( \sigma \)-model on the infinite cylinder \( \Sigma = \mathbb{R} \times S^1 \). It can be regarded as an element of the infinite-dimensional vector space
\[ \mathcal{H}_c = \bigoplus_{\lambda=0}^{pq-1} [\phi_\lambda] \otimes [\bar{\phi}_\lambda], \] (3.38)
where the conformal block \([\phi_\lambda]\) is regarded now (under the operator-state correspondence) as the irreducible Virasoro module built on the Fock state \( |Q\rangle \) corresponding to the primary field \( \phi_\lambda = e^{iQ \varphi} \). This identification makes use of the inner products on \([\phi_\lambda]\) and \([\bar{\phi}_\lambda]\) in the usual way.

With this in mind, let us consider the wavefunctionals (3.9) of the topological membrane with \( \Sigma = S^2_0 \) the punctured sphere. Their charges \( Q_i \) label the irreducible representations of the cyclic group \( \mathbb{Z}_{pq} \), and so they act on (one-dimensional) vector spaces spanned by the Fock vacuum states \( |Q_i\rangle \). Via application of the corresponding Virasoro descendent fields, these vector spaces can be extended to the modules \([\phi_\lambda]\). In other words, the wavefunctional (3.9) is an operator acting on the product of the corresponding representation spaces, so that
\[ \Xi_{S^2_0, \Sigma; \{z_1, \ldots, z_s; \}} \in \bigotimes_{i=1}^{s} [\phi_\lambda]^* \otimes [\phi_\lambda]. \] (3.39)
In the next section we will see how the membrane amplitudes also incorporate the string descendent fields into the actions of the operators in (3.39).
Given this interpretation, we will now study the one-point operators (3.39) (with $s = 1$) in some detail. The structure of a vertical Wilson line operator $W_Q[A]$ corresponding to the bulk propagation of a non-self-linking charge $Q$ from the right-moving sector $\Sigma_0$ to the left-moving sector $\Sigma_1$ of the string worldsheet depends crucially on what discrete symmetries we require of the topologically massive gauge theory. Let us first consider the simplest case whereby the bulk quantum field theory possesses the full PCT invariance. Then from (2.63), and the fact that the time inversion operation $T$ reverses the orientation of the contour $C$ defining the external particle worldline, it follows that the PCT involution of the quantum field theory acts on vertical Wilson lines as

$$\text{PCT} : W_Q[A] \mapsto W_Q[A].$$

(3.40)

This implies that charge is conserved for external particles which propagate along a Wilson line from $\Sigma_0$ to $\Sigma_1$, i.e. $Q = Q = m$. In this case, the charge non-conserving monopole induced processes are suppressed, $\mathcal{W}_{S^2 \times [0,1]} = 1$, and from (3.39) it follows that the one-punctured state (3.9) may be regarded as an operator

$$\Xi^{S^2_0} \left[ A_z, A_{\bar{z}}; \left\{ \frac{z}{Q} \right\}; 0 \right] : [\phi_{\lambda}] \longrightarrow [\phi_{\lambda}].$$

(3.41)

By gluing together the left and right moving sectors as we did in (3.21), we can then form the vacuum wavefunctionals corresponding to PCT-invariant states of the membrane with the appropriate bulk propagation of external charges as

$$\Phi^D_m \left[ A_z, A_{\bar{z}}; z, \bar{z} \right] = \Xi^{S^2_0} \left[ A_z, A_{\bar{z}}; \left\{ \frac{z}{Q} \right\}; 0 \right] \otimes \Xi^{S^2_0} \left[ A_z, A_{\bar{z}}; \left\{ \frac{\bar{z}}{Q} = Q \right\}; 0 \right].$$

(3.42)

Here we have made the identification $\Xi = \Xi_0 \equiv \Xi_1$ as is dictated by the orbifold symmetry (with respect to $P$) which identifies left and right movers $\Sigma_0 \equiv \Sigma_1$. The operator (3.42) acts only in the diagonal, left-right symmetric product $[\phi_{\lambda}] \otimes [\phi_{\bar{\lambda} = \lambda}] \subset \mathcal{H}$, and as such is proportional to the orthogonal projection $P_{\lambda} = \sum_l |\lambda, l\rangle \langle \lambda = \lambda, l|$ onto this subspace, with $|\lambda, l\rangle$, $l \in \mathbb{Z}_+$ the elements of the corresponding number basis. By choosing the proportionality constant to be one and using the inner product on $[\phi_{\lambda}]$, it follows that the homomorphism (3.42) on $[\phi_{\lambda}] \rightarrow [\phi_{\lambda}]$ is in a one-to-one correspondence with the coherent state

$$|m\rangle^D = \sum_{l=0}^{\infty} |\lambda, l\rangle \otimes U_P U_C |\lambda, l\rangle,$$

(3.43)

where the anti-unitary operators $U_P$ and $U_C$ respectively implement the parity and charge conjugation automorphisms on the right-moving Hilbert space with $U_P U_C |\lambda, l\rangle \in [\phi_{\lambda}]$. This vector is recognized as the Dirichlet Ishibashi state for the $c = 1$ Gaussian model [78], of Kaluza-Klein momentum $m$ about the compactified direction. The set of states $|m\rangle^D$,
\( m \in \mathbb{Z}_{pq/2} \) span the space of boundary states of the rational circle corresponding to Dirichlet boundary conditions on the open string embedding fields. We shall therefore refer to (3.42) as a Dirichlet Ishibashi wavefunctional. It is the unique vacuum state of the topologically massive gauge theory which carries definite \( U(1) \) charge \( Q = m \) and which respects the PCT symmetry.

The analysis is similar in the case where only the PT sub-invariance of the quantum field theory is assumed. From (2.65) it follows that the action on vertical Wilson lines is now

\[
\text{PT} : W_Q[A] \mapsto W_{-Q}[A],
\]

and there is a vortex operator of charge (2.17) which ruins charge conservation in propagation along a Wilson line from \( \Sigma_0 \) to \( \Sigma_1 \), i.e. \( \bar{Q} = -Q = -kn/4 \). Again the one-punctured state is the operator (3.41) but, because of the non-trivial monopole process, instead of (3.42) one must now form the vacuum functional

\[
\Phi^N_n[A_z, A_{\bar{z}}; z, \bar{z}] = W_{S_0^2 \times [0,1]}(\Delta Q) \Xi_{S_0^2}^{S_n^0} \left[ A_z, A_{\bar{z}}; \left\{ \frac{z}{Q} \right\}; 0 \right] \otimes \Xi_{S_0^2}^{S_n^0} \left[ A_z, A_{\bar{z}}; \left\{ \frac{\bar{z}}{-Q} \right\}; 0 \right]^\dagger.
\]

It coincides with the orthogonal projection onto the subspace \([\phi_\lambda] \otimes [\phi_{\lambda,-\lambda}] \subset \mathcal{H}^c\), and therefore with the closed string coherent states

\[
|kn/4\rangle^N = \sum_{l=0}^{\infty} |\lambda, l\rangle \otimes U_p|\lambda, l\rangle,
\]

where \( U_p|\lambda, l\rangle \in [\phi_{-\lambda}] \). These are just the Neumann Ishibashi boundary states of the rational circle, of vanishing total momentum along the compact dimension, and hence we will call (3.45) the Neumann Ishibashi wavefunctionals of the topological membrane. They are the unique vacuum states of definite charge which are invariant under three-dimensional PT transformations. The anticipated properties of closed string Ishibashi states are thereby a consequence of boundary gauge invariance of the bulk wavefunctions in three-dimensions.

### 3.7 Boundary Couplings and Wilson Lines

For the remainder of this section we will analyse generic boundary wavefunctionals which are expansions in the Ishibashi wavefunctionals (3.42) and (3.45) of the form

\[
\psi_B = \sum_m B^m_D \Phi^D_m + \sum_n B^n_N \Phi^N_n.
\]

We wish to determine what sorts of combinations (3.47) correspond to three-dimensional states of D-branes. In boundary conformal field theory, such states would be constrained.
by various sewing and locality conditions, the most important of which is the celebrated Cardy condition \cite{Cardy}. The constants $B$ in (3.47) characterize the boundary condition of the corresponding two-dimensional state. We will only deal with those boundary conditions that preserve the original $U(1)$ gauge symmetry of the topological membrane. Then the sums in (3.47) run through all $m, n \in \mathbb{Z}_{pq/2}$ for the case of the rational circle. As we will see in the following, the three-dimensional formalism naturally selects the “fundamental” branes that satisfy the Cardy relations.

The key to this analysis will be a study of the behaviour of Wilson line correlators of the bulk three-dimensional gauge theory after application of the pertinent orbifold involutions. In this subsection we shall make some general remarks about the explicit construction of conformal boundary conditions from the membrane perspective. For brevity we consider only the PT orbifold of the topologically massive gauge theory and work to lowest order in string perturbation theory. Consider a two-dimensional boundary conformal field theory with a set of boundary conditions labelled by $\alpha, \beta, \ldots$. The basic observation here is that a D-brane vertex operator insertion corresponds to a change in boundary conditions from $\alpha$ to $\beta$ and that in terms of the full membrane picture it is interpreted as a Wilson line insertion on the boundary when the theory is subjected to Dirichlet boundary conditions, as depicted in fig. 5. The simplest counterpart of this insertion in the full three-dimensional membrane corresponds to a configuration with two Wilson lines and is pictured in fig. 6.

\[
q_{\alpha \beta} = \frac{k}{4} n_{\alpha \beta}
\]

Figure 5: A Wilson line insertion playing the role of a boundary vertex operator.

In the absence of boundary terms in the Schrödinger wavefunctionals, the PT orbifold does not allow non-trivial Wilson line insertions to live on the boundary of the string worldsheet \cite{PT}. However, they are now possible due to the boundary terms that we have derived in the orbifold wavefunctions which induce the vertex operators (2.1). The field
Figure 6: The PT orbifold of the topological membrane in the presence of two Wilson lines with one boundary vertex insertion. The charges propagating along the Wilson lines are constrained by bulk charge conservation.

$Y_D$ must in this way have a discontinuity at the insertion point of the Wilson line which generates some charge $q_{\alpha\beta}$. The full three-dimensional theory must generate a mirror charge at that insertion point to screen it and its pair somewhere in the bulk of $\Sigma^o$. Then the new pair “propagates” through the bulk to the other boundary $\Sigma_0 \equiv \Sigma_1$, where the charges emerge with opposite signs due to the monopole and linking processes described earlier (recall that here the orbifold charge spectrum is of the form $kn/4$).

A natural assumption is that the discontinuity is generated by a magnetic flux which is responsible for the appearence of charge at the insertion point. As a toy model we can thereby take

$$Y_D(x) = 2\pi n_\alpha + 2\pi n_{\alpha\beta} \Theta(x - x_0) ,$$

(3.48)

where $x$ is the coordinate of the boundary $\partial \Sigma^o$ and $x_0$ is the point of insertion on the boundary. Here $\Theta$ denotes the usual Heaviside function, and $n_{\alpha\beta} = n_\beta - n_\alpha$ with $n_\alpha$ and $n_\beta$ integers. The components of the corresponding current on $\partial \Sigma^o$ are then given by

$$j^\parallel(x) = \frac{k^2}{4\pi} n_{\alpha\beta} \delta(x - x_0) ,$$

$$\rho(x) = i k n_{\alpha\beta} \partial_\parallel \delta(x - x_0) ,$$

(3.49)

and integrating the bulk Gauss law (2.11) locally in a neighbourhood $\sigma_{x_0} \subset \Sigma^o$ of the insertion point yields the total magnetic flux

$$\int_{\sigma_{x_0}} d^2z \ B = \frac{4\pi}{k} \int_{\sigma_{x_0}} d^2z \ \rho = 2\pi n_{\alpha\beta} .$$

(3.50)
As required, the total flux is an integer multiple of $2\pi$, and the integers $n_\alpha$ and $n_\beta$ are related to the decomposition of the boundary states. Note again that the total charge in the bulk theory vanishes, and only at the orbifold fixed point does a non-trivial charge emerge. This same argument holds for the infinite strip with a Wilson line insertion on one of the two boundaries.

However, the above picture is not quite complete, because here we are working with compact manifolds, meaning that after orbifolding our boundaries are always closed. In other words, instead of the complex plane we should consider the parametrization of a disk, and instead of an infinite strip we have an annulus. Thus we must consider at least two vertex insertions, as depicted in fig. 7, such that

$$Y_D(x) = 2\pi n_\alpha + 2\pi n_\alpha \beta \Theta(x - x_1) - 2\pi n_\alpha \beta \Theta(x - x_2)$$  \hspace{1cm} (3.51)

in order to induce the appropriate changes of boundary conditions. The simplest counterpart in the full three-dimensional membrane corresponds to a configuration with two Wilson lines ending at the boundaries as illustrated in fig. 8. Note that in the absence of Wilson line insertions at the boundary, the boundary condition is nonetheless fixed by a constant value for the field $Y_D$ on $\partial \Sigma^o$. This same picture is valid for the annulus diagram with each of the two boundaries treated independently.

![Figure 7: Boundary conditions on a disk: Two-dimensional perspective.](image)

To quantify these arguments somewhat, let us consider the charge insertions at the boundary of $\Sigma^o$ as they appear in the punctured wavefunctionals (3.9) and rewrite them as boundary integrals over an effective charge distribution $Q(x - x_0)$ which represents a discontinuity at the charge insertion point in terms of the Heaviside function, as explained above. Then in the presence of a D-brane vertex (2.100) we get the total exponential factor

$$V_D \exp \left\{ i Q \varphi(x_0) \right\} = \exp \left\{ \int_{\partial \Sigma^o} dx \left[ Q(x - x_0) - \frac{k}{4\pi} Y_D(x) \right] \partial^\perp \varphi(x) \right\}.$$ \hspace{1cm} (3.52)
By shifting the brane collective coordinate as $Y_D(x) \to Y_D(x) + (4\pi/k) Q(x-x_0)$, we recover the standard D-brane vertex operator (2.100) but now $Y_D(x)$ has a discontinuity at $x = x_0$ as in (3.48). The same constructions can be carried out for multiple Wilson line insertions.

### 3.8 Orbifold Correlation Functions

From the analysis of the previous subsection we may deduce an important property of the computation of correlators of the orbifold gauge theory, which may be regarded as a three-dimensional version of the bulk-boundary correspondence of two-dimensional conformal field theory [80]. This correspondence is based on the fact that each bulk point of an open surface $\Sigma^o = \Sigma/\mathbb{Z}_2$ has two (mirror) pre-images on its Schottky double $\Sigma$, and it states that the $s$-point correlators on $\Sigma^o$ are in a one-to-one correspondence with the chiral $2s$-point correlation functions on $\Sigma$. The interaction of local fields with the boundary of $\Sigma^o$ (in the form of boundary conditions on $\partial \Sigma^o$) is then simulated by the interaction between mirror images of the same holomorphic field on $\Sigma$, carrying conjugate primary charges $\lambda, \bar{\lambda} = \pm \lambda$.

The basic example of this correspondence in the membrane picture is provided by the orbifold partition function (2.89) at genus $g$ obtained from (3.12), which reads

$$Z^o(\Gamma) = k^{g/4} \sum_{\tilde{\lambda} \in (\mathbb{Z}_{pq}/2)^g} \Psi_{\tilde{\lambda}}^{\text{orb}}(\Gamma),$$

where $\Psi_{\tilde{\lambda}}^{\text{orb}}(\Gamma)$ are the reduced topological wavefunctions whose quantum numbers $\tilde{\lambda}$ are either purely winding or momentum integers according to which of the PT or PCT orbifolds of the quantum field theory has been taken.
More generally, let us assume that the boundary of $\Sigma^o$ consists of $b$ connected components $C'_\alpha \cong S^1$, $\alpha = 1, \ldots, b$, so that

$$\partial \Sigma^o = \prod_{\alpha=1}^{b} C'_\alpha. \quad (3.54)$$

The pre-image of each $C'_\alpha$ on the double cover $\Sigma$ is a $\mathbb{Z}_2$-invariant, equatorial loop. It therefore corresponds to a Wilson loop on a chosen time slice in the covering cylinder $\Sigma \times [0, 1]$, which becomes a circle of singular points in the corresponding three-dimensional orbifold [61–63]. This fact follows immediately from comparison of the orbifold wavefunctionals (2.90) and (2.91), and from the remarks made in the last subsection. In particular, as each component $C'_\alpha$ corresponds to the bulk propagation of an external $U(1)$ particle, it carries an integral charge $Q'_\alpha$, again dependent on the type of orbifold taken.

A generic vertical Wilson line correlator in the orbifold theory with the boundary values $Q'_1, \ldots, Q'_b$ of the field $Y = Y_D$ or $Y_N$ on $\partial \Sigma^o$ may then be computed by using those of section 3.2 through the prescription

$$\left\langle \prod_{i=1}^{s} W_{Q_i} \right\rangle_{\text{orb}}^{Q'_1 \cdots Q'_b} = \prod_{i,j=1}^{s} e^{\mp Q_i Q_j (\theta_i (1/2) - \theta_i (0))} \left[ D A_z \ D A_\tau \right] \times \Xi_{1/2}^{\text{orb}} \left[ A_z, A_\tau; \left\{ \tilde{Q}_1 = \pm Q_1, \ldots, \tilde{Q}_s = \pm Q_s \right\} ; Y \mid C'_\alpha = 4\pi \lambda'_\alpha \right] \hat{1} \times \mathcal{W}_{\Sigma \times [0,1]}(\Delta \tilde{Q}_X) \Xi_0^{\text{orb}} \left[ A_z, A_\tau; \left\{ \frac{z_1}{Q_1}, \ldots, \frac{z_s}{Q_s} \right\} ; 0 \right] . \quad (3.55)$$

This is of course just the orbifold inner product $\langle \Xi_{1/2} | \Xi_0 \rangle_{\text{orb}}$ defined in (2.89) on the wavefunctional (3.21) with the prescribed boundary values of the external currents. By using a finite temperature correspondence completely analogous to that of (3.13,3.14) in the presence of Polyakov loops, we may evaluate (3.55) as the ordinary Wilson line correlator

$$\left\langle \prod_{i=1}^{s} W_{Q_i} \right\rangle_{\text{orb}}^{Q'_1 \cdots Q'_b} = \left\langle \prod_{\alpha=1}^{b} W_{Q_\alpha} \prod_{i=1}^{s} \mathcal{W}_{\Sigma \times [0,1]}(\Delta Q_i) \right\rangle_{\Sigma \times S^1} W_{Q_i, Q_i = \pm Q_i}, \quad (3.56)$$

where it is understood that the weight $\mathcal{W}_{\Sigma \times [0,1]}(\Delta Q_i)$ is unity whenever the bulk monopole processes associated with particle $i$ are suppressed, i.e. $\tilde{Q}_i = Q_i$. While this correspondence doesn’t completely determine the functional dependence of the correlators on $\Sigma^o$, it suffices to determine the relationships between the conformal blocks corresponding to the left and right hand sides of (3.56). We have also assumed that the Wilson line insertions $W_{Q_i}$ only pierce the interior of the open string worldsheet $\Sigma^o$ as illustrated in fig. 9, so that the left-hand side of (3.56) corresponds to a bulk insertion of primary fields in the boundary conformal field theory. In order to properly account for the topological dependences of
the correlation functions, one would need to further specify an appropriate homological truncation to the Lagrangian subspace (2.61). This will not be required in the following.

As is seen from (2.91) and discussed above, the equality (3.56) can be understood from the fact that the branch point locus of the orbifold is equivalent to point-like insertions of curvature singularities for the gauge field in the quantum theory [61]. In addition, the charges of these closed Wilson loops have the very natural interpretation as boundary conditions on the open string worldsheet. This provides a very simple and elegant derivation of the well-known result that the allowed conformal boundary conditions correspond to specific integral reductions of the primary charges of the rational circle, i.e. that the boundary conditions are in a one-to-one correspondence with the primary fields of the chiral conformal field theory. The equality (3.56) now incorporates the worldsheet boundary effects through the linkings of the vertical Wilson lines with the closed boundary loops, as depicted in fig. 9. The case of the sphere $\Sigma = \mathbb{S}^2$ with orbifold the disk $\Sigma^o = D^2$ is depicted in fig. 10. In the presence of Wilson lines in the full three-dimensional membrane, each sphere living at each fixed time slice will be pierced and each of these piercings constitutes a new boundary. Consider two Wilson lines of charges $q$ and $q^c$ piercing $\Sigma_{1/2}$ at points $z$ and $z^c$ which are related by the action of the orbifold group generator [26]. The piercing is considered to live in the bulk of the disk as pictured in fig. 10.

As in section 3.5, the linking of external particle $i$ with the boundary component $C'_\alpha$ of (3.54) produces the statistical phase factor $e^{4\pi i Q_i Q'_\alpha/k} = S_{\lambda_i \lambda'_\alpha}/S_{0 \lambda'_\alpha}$, so that in this simple
In this case the boundary effects may be unravelled to write the orbifold correlator (3.56) as

\[
\left\langle \prod_{i=1}^{s} W_{Q_i} \right\rangle_{\text{orb}} = \prod_{\alpha=1}^{b} \frac{1}{S_{0\lambda_\alpha}} \left\langle \prod_{i=1}^{s} S_{\lambda_i,\lambda_\alpha} W_{\Sigma \times [0,1]}(\Delta Q_i) W_{Q_i} W_{\bar{Q}_i} = \pm Q_i \right\rangle_{\Sigma \times S^1}. \tag{3.57}
\]

Let us stress again that in (3.57) the allowed particle charges $Q_i, \bar{Q}_i = \pm Q_i$ and $Q'_\alpha$ are prescribed by the discrete PT or PCT symmetries in three-dimensions. This formula is the key identity which will allow us to completely specify the membrane states of D-branes.

### 3.9 Fundamental Wavefunctionals

We are now ready to analyse the expansion (3.47) into Ishibashi wavefunctionals. Branes in the $S^1$ background are completely characterized by their couplings to closed string modes, and these couplings are encoded in the expansion coefficients $B$ of (3.47). Not all linear combinations of Ishibashi states can couple to the bulk conformal field theory. To determine the appropriate couplings, we will compute the one-point functions of bulk fields on the interior of the disk $D^2$.

Consider the propagation of a charge $Q$ from the Schottky double $S^2$ to the orbifold $D^2$ whose boundary carries a charge $Q'$. According to the general relation (3.57), the orbifold Wilson line correlator can be represented in terms of a bulk Polyakov loop correlator in the finite temperature gauge theory as

\[
\left\langle W_{Q} \right\rangle_{\text{orb}}^{Q'} = \frac{S_{\lambda \lambda'}}{S_{0\lambda'}} \left\langle W_{S^2 \times [0,1]}(\Delta Q) W_{Q} W_{\bar{Q}} = \pm Q \right\rangle_{S^2 \times S^1}. \tag{3.58}
\]

According to (3.55), the correlator on the right-hand side of (3.58) coincides with the membrane inner product (2.85) on Ishibashi wavefunctionals as

\[
\left\langle W_{Q} \right\rangle_{\text{orb}}^{Q'} = \frac{S_{\lambda \lambda'}}{S_{0\lambda'}} \left\langle 1 \mid \Phi_{\lambda}^{B} \right\rangle, \tag{3.59}
\]
where $B = D, N$ labels the particular type of Ishibashi wavefunctional (3.43) or (3.46) obtained from the given PCT or PT symmetry of the three-dimensional gauge theory. These latter states are defined on the punctured sphere $S^2_0$, whereby a fictitious screening charge is present at the puncture in order to render the right-hand side of (3.59) non-vanishing, as explained in section 3.7. Such screenings will always be implicitly assumed in the following.

Consider now a generic membrane state $\psi^B_\lambda$ of the geometry $S^2_0 \times [0, 1]$ which is labelled by the boundary charge $Q'$, and which is invariant under the given orbifold involution, i.e. an expansion in the appropriate Ishibashi wavefunctionals as

$$\psi^B_\lambda = \sum_{\mu \in \mathbb{Z}, \bar{\mu} = \pm \mu} B^B_{\lambda \mu} \Phi^B_{\mu}.$$  

We wish to consider in particular those wavefunctionals $\psi^B_\lambda$ which are “fundamental”, in the sense that their membrane inner products are determined by the trivial $Q = 0$ Wilson line in the orbifold theory with boundary condition $Q'$. This is achieved via the insertion of a complete set of orbifold Wilson lines, and by using (3.59) we find

$$\langle 1 | \psi^B_\lambda \rangle = \left\langle \sum_{Q:Q=\pm Q} W_Q \right\rangle_{Q'}^{Q} = \sum_{\lambda \in \mathbb{Z}, \lambda = \pm \lambda'} S^\lambda_{\lambda'} \langle 1 | \Phi^B_\lambda \rangle.$$  

From (3.61) it follows that the coupling coefficients of (3.60) are then given by

$$B^B_{\lambda \mu} = \frac{S^\lambda_{\lambda'}}{S^0_{0\lambda'}},$$  

and we have thereby arrived at a remarkably simple derivation of the celebrated Cardy solution (up to an irrelevant normalization by $\sqrt{S^0_{0\lambda'}}$) of the sewing constraints in boundary conformal field theory [79]. As before, a more general argument uses an appropriate surgery prescription [63].

The couplings (3.62) (trivially) obey the factorization constraints

$$B^B_{\mu} \lambda B^B_{\nu} \lambda = \sum_{\lambda' \in \mathbb{Z}} N_{\mu \nu} \lambda' F_{\lambda 1} \left[ \begin{array}{cc} \mu \\ \rho \\ \nu \\ \sigma \end{array} \right] B^B_{\lambda' \lambda},$$  

where $F$ are the fusing matrices of the rational circle which are given by

$$F_{\lambda 1} \left[ \begin{array}{cc} \mu \\ \rho \\ \nu \\ \sigma \end{array} \right] = \delta_{[\mu+\nu+\sigma-\rho]} \delta_{[\nu+\sigma-\lambda]} \delta_{[\mu+\nu-\lambda']} \times \exp \left\{ \pi i (\mu + \sigma + 1) \frac{pq}{\nu} \left[ \nu \left( \mu + \nu + \sigma - [\mu + \nu + \sigma] \right) + (\nu + \sigma) \left( \mu + 2\nu + \sigma - [\mu + \nu] - [\nu + \sigma] \right) \right] \right\}.$$  

50
They may be defined in terms of the interactions between Wilson lines in the bulk three-dimensional gauge theory, as depicted schematically in fig. 11. The classifying algebra (3.63) thereby completely determines the brane moduli in the form (3.62), whose physical interpretation now depends on the type of orbifold taken.

\[ F_{\lambda\lambda'} = \sum_{\lambda'} F_{\lambda\lambda'} \left[ \begin{array}{cc} \mu & \nu \\ \rho & \sigma \end{array} \right] \]

Figure 11: The fusing matrices \( F \) are defined through the worldline junctions associated with the charge conserving interactions of four charged particles in the bulk labelled by \( \mu, \nu, \rho, \sigma \). They represent a cross channel duality in the intersections of the corresponding Wilson lines and are related to the statistical exchange phases that arise from the linkings of the charged particle trajectories.

For \( B = D \), by using (2.64), (2.101), (3.1) and (3.60) we thereby find that the most general fundamental, \( U(1) \) gauge symmetry preserving and PCT-invariant membrane wavefunctionals are given by

\[ \psi_a[A_z, A_{\bar{z}}; z, \bar{z}] = \sum_{m=0}^{pq/2-1} e^{ima/R} \Phi^D_m[A_z, A_{\bar{z}}; z, \bar{z}], \quad (3.65) \]

where

\[ a = \frac{2 \pi l q R}{p}, \quad l \in \mathbb{Z}_{pq/2}. \quad (3.66) \]

The state (3.65) can be given the following physical interpretation. Let us compute the inner product of (3.65) with the action of an unknotted vertical Wilson line \( W_{m'}[A] \) corresponding to the propagation of a charge \( m' \in \mathbb{Z}_{pq/2} \) from \( \Sigma_0 \) to \( \Sigma_1 \). By using the explicit forms (3.9) of the Ishibashi wavefunctionals (3.42), and the fact that the effect of the Wilson line \( W_{m'}[A] \) is to induce the \( S^1 \)-valued operators \( \phi_{m'} = e^{i m' \varphi} \) on the string worldsheet, we find

\[ \langle 1 | W_{m'} \psi_a \rangle = \sum_{m=0}^{pq/2-1} e^{ima/R} \langle 1 | W_{m'} \Phi^D_m \rangle = \sum_{m=0}^{pq/2-1} e^{ima/R} \langle 1 | \Phi^D_{m+m'} \rangle = e^{-i m'a/R} \langle 1 | \psi_a \rangle. \quad (3.67) \]

We interpret (3.67) to mean that as a particle propagates through the bulk of the membrane in the presence of the particle state described by (3.65), it undergoes a non-local interaction with this particle defect and acquires a statistical phase factor \( e^{-4 \pi i m'/m/k} \) from linking. In the worldsheet picture, this defect clearly corresponds to a D-brane situated at the point
Note that in this case the brane moduli (3.62) are $p^{th}$ roots of unity, and so the positions of the D-branes are restricted to lie at the vertices of a regular $p$-gon.

For the case $B = N$, from (2.66) it follows that the most general fundamental, gauge symmetry preserving and PT-invariant membrane states are

$$
\psi_{w}[A_z, A_{\bar{z}}; z, \bar{z}] = \sum_{n=0}^{pq/2-1} e^{i w n R} \phi_n^N[A_z, A_{\bar{z}}; z, \bar{z}].
$$

(3.68)

They describe Neumann branes which carry Wilson lines

$$
w = \frac{2\pi l q}{p R}, \quad l \in \mathbb{Z}_{pq/2}.
$$

(3.69)

In this case the target $S^1$ Wilson line is induced by those of charged particles in the bulk of the topological membrane.

Let us stress again that the particular coupling coefficients (3.62), which correspond to the statistical exchange phase between two Hopf linked charged particles, arise from the assumption that the brane states (3.60) are compatible with the full $U(1)$ gauge symmetry of the topological membrane. This property was captured by the first equality of (3.61), which took into account the sum over all $U(1)$ charges. It is possible in this simple case to find other expansions (3.60) which preserve a smaller symmetry group and still obey the necessary conformal factorization constraints (3.63). These rational boundary symmetries would then restrict the D-brane moduli to a smaller subset of those found above. However, these branes do not constitute very natural objects from the point of view of the underlying topologically massive gauge theory, and within the present framework we may use gauge invariance as a guiding principle to conclude that the special branes found above account for all the relevant branes of the topological membrane.

### 3.10 The Cardy Condition

In the previous subsection we saw that the closed string coupling coefficients of membrane wavefunctionals were determined unambiguously by the requirement of gauge invariance. We recall that a similar principle was used in the derivation of the Verlinde formula (see fig. 4). This suggests that there could be a natural relationship between them. In boundary conformal field theory this connection is known as the Cardy condition [79]. Here we shall investigate how it arises as a natural feature of the membrane formulation of the open string sector, and thereby provide a nice consistency check of the present formalism.

For this, we will study the one-loop, open string annulus amplitude. The Schottky double of the annulus is the torus of purely imaginary modulus $\tau = i \tau_2$, and the Neumann
The orbifold partition function (3.70) is a vector in the open string Hilbert space

\[ \mathcal{H}^\alpha = \bigoplus_{n=0}^{pq/2-1} \big[ \phi_n^{\text{orb}} \big] . \] (3.72)

Now let us consider the annulus with prescribed charges \( n_1 \) and \( n_2 \) on its two boundary circles. By using bulk charge conservation and the fusion coefficients (3.4), we may write the annulus partition function restricted to this sector as

\[ Z^\alpha_{\lambda_1,\lambda_2}(\tau_2) = k^{1/4} \sum_{\lambda=0}^{pq/2-1} N_{\lambda_1,\lambda_2}^\lambda \Psi^{\text{orb}}_{\lambda}(i \tau_2) . \] (3.73)

On the other hand, the Dirichlet cylinder amplitude for closed string propagation in a time span \( 2\pi \tilde{\tau}_2 \) is obtained from (3.37) via a PT orbifold in three-dimensions and may be written as [28]

\[ \tilde{Z}^\alpha(\tilde{\tau}_2) = k^{1/4} \sum_{m=0}^{pq/2-1} \tilde{\Psi}^{\text{orb}}_{m}(i \tilde{\tau}_2) , \] (3.74)

where

\[ \tilde{\Psi}^{\text{orb}}_{m}(i \tilde{\tau}_2) = \frac{1}{2(k \tilde{\tau}_2)^{1/4} \eta(i \tilde{\tau}_2)} \sum_{r'=\infty}^{\infty} \exp \left\{ -\frac{2\pi \tilde{\tau}_2}{k} \left( r'p + m \right)^2 \right\} . \] (3.75)

Worldsheet duality is the statement that the partition functions (3.70) and (3.74) coincide after a modular transformation,

\[ \tilde{Z}^\alpha(\tilde{\tau}_2 = \frac{2}{\tau_2}) = Z^\alpha(\tau_2) , \] (3.76)

which may be derived explicitly by using the Poisson resummation formula (3.27) [28]. In the next section we will give a dynamical interpretation of this duality in the finite temperature gauge theory. When restricted to a fixed boundary charge sector as in (3.73), application of the definition (3.25) to (3.74) reproduces the Verlinde diagonalization formula (3.32). This is thereby a three dimensional version of the open string derivation of the Verlinde formula through the Cardy relations [79].

53
Formally, the Cardy condition arises because the fixed boundary charge restriction of the partition function (3.74) is, by definition, the membrane inner product

\[ \tilde{Z}_{\lambda,\lambda'}^{\circ}(\tilde{\tau}_2) = \langle \psi^D_{\lambda'} | \psi^D_{\lambda} \rangle \]  

(3.77)

of fundamental wavefunctionals (3.60). By using the expansion coefficients (3.62) and comparing (3.77) with (3.73) after a modular transformation, we arrive at the Verlinde formula. Alternatively, the three dimensional dynamical description of this condition follows from surgery [63], analogously to the derivation depicted in fig. 4. In the present case, we use the fact, discussed earlier, that the annulus graph can be interpreted in three dimensions as the correlator of two Wilson lines carrying the prescribed boundary charges \( Q_1 \) and \( Q_2 \). We may thereby represent (3.73) as the \( S^3 \) correlator of three Wilson loops of charges \( Q_1, Q_2 \) and \( Q_3 = 0 \) linked with a trivial \( Q = 0 \) loop, as in the first equality of fig. 4. By rewriting the trivial loops as sums over complete sets of Polyakov loops, we again arrive at the anticipated result above.

4 BRANE TENSIONS

In this final section we shall describe what is perhaps the most important aspect of the three-dimensional representation of D-branes, that the topological membrane approach yields the correct tension of a D-brane. This will require two additional steps beyond what we have developed thus far. First, we shall have to properly introduce and study the mass scales of the bulk three-dimensional theory and discuss how they generate those of the induced worldsheet theory. Second, we will introduce the excited eigenstates of the gauge theory Hamiltonian and study the Hilbert space of physical states beyond the topological vacuum sector. With these developments we will show that the appropriate D-brane tension arises both as a dynamical property of the topological membrane and as an intrinsic quantity set by the mass scales of the bulk theory. As we will see, the careful incorporation of the bulk mass scales is absolutely crucial for the appropriate appearence of the D-brane mass, as they are required to remove the genuine quantum field theoretic divergences which arise above the vacuum sector in the three-dimensional formulation. As additional bonuses, we shall discover the proper formulation of descendent string states in three-dimensions and also acquire a new perspective on worldsheet modular duality.

4.1 Dilaton Coupling

In our derivation of the D-brane tension in three dimensions, we shall first describe how the appropriate factors of the string coupling constant \( g_s \) appear. For this, we need to describe how to incorporate the string dilaton field into the topological membrane formalism.
This problem was addressed in [24] and consists of examining the conformal coupling of topologically massive gauge theory to topologically massive gravity through the action

\[ S_{\text{CTMGT}}[A, \omega; D] = \int_M d^3x \sqrt{-g} \left[ \kappa D^2 R(\omega) + 8\kappa \partial_\mu D \partial^\mu D - \frac{1}{4\gamma} D^2 F_{\mu\nu} F^{\mu\nu} \right] \]

\[ - 8\kappa \oint_{\partial M} D \partial_\perp D + S_{\text{CS}}[A, \omega] \]  

(4.1)

which is defined on a three-manifold \( M \). Here \( \kappa \) is the three-dimensional Planck mass, \( R(\omega) \) is the curvature of the torsion-free, \( SO(2,1) \) Lie algebra valued spin-connection \( \omega_a^\mu \) of the frame bundle of \( M \), and \( D \) is a dimensionless scalar field in three spacetime dimensions. The first term in (4.1) is a modification of the Einstein-Hilbert action, while

\[ S_{\text{CS}}[A, \omega] = \int_M d^3x \left[ \frac{k}{8\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \frac{k'}{8\pi} \epsilon^{\mu\nu\lambda} \left( \omega_a^\mu \partial_\nu \omega_a^\lambda + \frac{2}{3} \epsilon^{abc} \omega_a^\mu \omega_b^\nu \omega_c^\lambda \right) \right] \]  

(4.2)

is the sum of the gauge and gravitational Chern-Simons actions.

The action (4.1) is invariant under three-dimensional conformal transformations [24]. In the naive vacuum \( \langle D^2 \rangle = 0 \), the spectrum of the corresponding quantum field theory contains a massless scalar particle but no graviton or gauge degrees of freedom, so that in this phase the model is equivalent to the pure topological field theory defined by the Chern-Simons actions (4.2). On the other hand, in the phase where the scalar field \( D^2 \) has a non-vanishing vacuum expectation value, there is a propagating graviton mode with mass \( \frac{8\pi k \langle D^2 \rangle}{k'} \) and a photon with topological mass

\[ \mu = \frac{\gamma k \langle D^2 \rangle}{4\pi} . \]  

(4.3)

The zero-point quantum fluctuations of the field \( D \) thereby set the mass scales of the bulk theory.

For the membrane geometry \( M = \Sigma \times [0,1] \), in addition to the usual conformal \( \sigma \)-model of central charge \( c = 1 \) living at the boundary \( \Sigma \), the gravitational sector of the three-dimensional theory (4.1) induces Liouville theory coupled to a worldsheet dilaton field on \( \Sigma \) [24]. After the usual boundary identifications and an appropriate rescaling of the fields, the effective two-dimensional gravity action is given by

\[ S_{\text{L}}[D, \phi] = \int_{\Sigma} d^2z \left[ -\frac{1}{4\pi} \left( \ln D^4 + \phi \right) R^{(2)} - 2\kappa D \partial_\perp D + \frac{1}{16\pi} \partial_\perp \phi \partial_\perp \phi + \Lambda_\Sigma e^{-\phi} \right] , \]  

(4.4)

where \( \phi \) is the Liouville field, \( R^{(2)} \) is the usual two-dimensional curvature of the worldsheet \( \Sigma \), and \( \Lambda_\Sigma \) is the worldsheet cosmological constant which is induced by the dreibein
condensate and topological graviton mass of the gravitational sector of the original three-
dimensional theory (4.1), and by the vacuum expectation value of the scalar field $D^2$. From
(4.4) it is clear that $D$ is the three-dimensional version of the string dilaton field, and
that the string coupling constant, like the three dimensional mass scales, is dynamically
generated by the vacuum expectation value
$$g_s = \langle D^4 \rangle. \tag{4.5}$$

Then, from (4.4) it follows that the contribution to the string statistical sum from a Riemann
surface $\Sigma$ of genus $g$ with $b$ boundaries and $c$ crosscaps will contain the factors
$$\left\langle \exp \left\{ -\frac{1}{4\pi} \int_{\Sigma} d^2z \ (\ln D^4) \ R^{(2)} \right\} \right\rangle = (g_s)^{-\chi(\Sigma)}, \tag{4.6}$$
where
$$\chi(\Sigma) = 2 - 2g - b - c = \frac{1}{4\pi} \int_{\Sigma} d^2z \ R^{(2)} \tag{4.7}$$
is the Euler number of $\Sigma$.

If $\Sigma$ is a closed surface, then (4.6) gives the contribution to the closed string sector of
the worldsheet theory. The crucial point now is that the open string sector can be obtained
as a $\mathbb{Z}_2$-orbifold of the closed string sector. In the usual way, the original action $S_\Sigma$ defined
on the closed surface $\Sigma$ is twice that of its orbifold, $S_\Sigma = 2S_{\Sigma/\mathbb{Z}_2}$ [60]. Thus the open string
contribution is given by $(g_s)^{-\chi(\Sigma)/2}$. Heuristically, this feature manifests itself within the
three dimensional picture in the membrane scattering amplitudes $\langle \Psi_1 | \Psi_0 \rangle$ between states
$\Psi_0$ and $\Psi_1$ inserted at the initial and final surfaces $\Sigma_0$ and $\Sigma_1$ in the closed string picture
(see fig. 1). Inserting a complete set of wavefunctionals $\Psi_{1/2}$ at the orbifold fixed point
t = 1/2 in the bulk of the topological membrane determines this amplitude as [28]
$$\langle \Psi_1 | \Psi_0 \rangle = \sum_{\Psi_{1/2}} \langle \Psi_1 | \Psi_{1/2} \rangle \langle \Psi_{1/2} | \Psi_0 \rangle. \tag{4.8}$$

Under the orbifold operation, the initial and final surfaces are identified, $\Sigma_0 \equiv \Sigma_1$, and the
only wavefunction that can live at $\Sigma_{1/2}$ is the identity character state $|\Psi_{\lambda=0}\rangle = |1\rangle$, so that the right-hand side of (4.8) reduces to $\langle |1| \Psi_0 \rangle^2$. Thus the decomposition (4.8) heuristically
means that closed strings come in a double volume of open strings, as usual. It follows that
the open string membrane amplitudes should be identified with the square roots of closed
string ones,
$$\langle \Psi_1 | \Psi_0 \rangle_{\text{orb}} = \sqrt{\langle \Psi_1 | \Psi_0 \rangle}. \tag{4.9}$$
For example, while the leading order closed string diagram $S^2$ varies with the string coupling constant as $1/(g_s)^2$, the disk amplitude $D^2$ behaves like $1/g_s$, which is the standard
coupling dependence of the D-brane tension at tree-level in string perturbation theory. These arguments can also be used to account for the missing square root dependences of the Wilson line correlators (3.57).

4.2 D-Brane Tension

To derive a formula for the brane tension one can proceed in one of two ways. The first one, which we shall begin developing in section 4.4, computes the three-dimensional version of the one-point tadpole insertion on a disk, i.e. the Wilson line correlator (3.58) in the presence of a monopole. This uses the fact that, in the worldsheet theory, the mass is measured by the one-point function with the graviton vertex operator and it requires a development of excited string states in the topological membrane. The second approach, which we will pursue in this subsection and the next, is to use our bulk construction of the brane coordinates $Y_N$ and $Y_D$ to derive the Dirac-Born-Infeld action from the membrane partition function. This exploits the fact that the tension appears as the boundary entropy factor in the overall normalization of the open string partition function.

To make the derivation as explicit and general as possible, we will consider, in this subsection and the next only, a more general topologically massive gauge theory with structure group $U(1)^d$, abelian gauge fields $A^I$, and corresponding field strengths $F^I$, where $I = 1, \ldots, d$. The source-free action is given by

$$S^{(d)}_{TMGT}[A; D] = \int_0^t dt \int_{\Sigma} d^2 z \left[ -\frac{\sqrt{-g}}{4\gamma D^2} F^I_{\mu\nu} F^I_{\mu\nu} - \frac{2}{\pi} K_{IJ} \epsilon^{\mu\nu\lambda} A^I_\mu \partial_\nu A^J_\lambda \right].$$ (4.10)

The inclusion of external sources, as in (2.2), will be described in the next subsection. The Chern-Simons coefficient (2.101) is now modified to the rank 2 constant tensor

$$K_{IJ} = \frac{1}{\alpha'} \left( G_{IJ} + i B_{IJ} \right),$$ (4.11)

where $G_{IJ}$ is a symmetric real matrix which is interpreted as the target space graviton condensate, while $B_{IJ}$ is an antisymmetric real matrix which is interpreted as a constant background NS–NS two-form field. Note that in complex worldsheet coordinates, the antisymmetric part of the Chern-Simons coefficient matrix is purely imaginary, owing to the Euclidean signature of the target space. The previous results derived in section 2 generalize straightforwardly with the inclusion of the symmetric part $G_{IJ}$. On the other hand, the terms containing $B_{IJ}$ are total derivatives and thereby contribute only at worldsheet boundaries. The bulk gauge theory (4.10) induces the two-dimensional $c = d$ conformal $\sigma$-model with target space the $d$-dimensional torus $T^d$ of metric $G_{IJ}$. The closed string conformal field theory is characterized by a Narain lattice $\Gamma(K) \subset \mathbb{R}^{d,d}$ with rational-valued moduli.
We shall now present a careful evaluation of the partition function $s$ associated with the quantum field theory defined by (4.10), keeping track in particular of the normalization factors that were neglected in the analysis of section 2.7. For closed strings, after identifying both boundaries $\Sigma_0$ and $\Sigma_1$ with opposite orientations [16, 24–28], we find that the generalization of (2.85, 2.87) is given by the path integral

$$Z^c = (g_s)^{-\chi(\Sigma)}\mathcal{N} \times \prod_{I=1}^{d} \int [D\bar{A}_z^I D\bar{A}_{\bar{z}}^I] \int [D\varphi^I] \exp \left\{-\frac{K_{IJ}}{8\pi} \int_{\Sigma} d^2z \left[ 2 \left( \bar{A}_z^I - \partial_z \varphi_0^I \right) \left( \bar{A}_{\bar{z}}^I - \partial_{\bar{z}} \varphi_1^I \right) + \partial_z (\varphi_0^I - \varphi_1^I) \partial_{\bar{z}} (\varphi_0^I - \varphi_1^I) \right] \right\}.$$  

(4.12)

Here we have identified the gauge field degrees of freedom $\bar{A}^I$ on both boundaries, and assumed that the path integral over the Liouville field is performed with its contribution absorbed in an appropriate overall normalization constant included in $\mathcal{N}$. We have also frozen the dilaton field $D$ at its classical value, as will be discussed further in section 4.4.

The chiral scalar fields $\varphi_0^I$ and $\varphi_1^I$ are induced by $U(1)^d$ gauge invariance of the corresponding Schrödinger wavefunctionals on the surfaces $\Sigma_0$ and $\Sigma_1$, respectively, which combine into the non-chiral fields $\varphi^I = \varphi_0^I - \varphi_1^I$ on $\Sigma$.

In keeping with the discussion of the previous subsection, we will fix $\mathcal{N}$ such that the closed string partition function (4.12) contributes the usual perturbative weight $(g_s)^{-\chi(\Sigma)}$, along with the usual topological contributions as in (3.12) which to avoid clutter we will not display explicitly. The functional Gaussian integrations over $\bar{A}^I$ and $\varphi^I$ in (4.12) may be done explicitly as described in section 2.7, and by exploiting the fact that the fields are defined on a compact surface to expand them in a countable basis of functions on $\Sigma$. In genus $g = 0$ this is the Laurent basis on the complex plane which is labelled by $\mathbb{Z}$, for $g = 1$ it is the Fourier basis on $\mathbb{S}^1 \times \mathbb{S}^1$ which is indexed by $\mathbb{Z}^2$, while for $g \geq 2$ it is given by the Krichever-Novikov basis which is labelled by $\mathbb{Z}$ for $g$ even and by $\mathbb{Z} + 1/2$ for $g$ odd. Infinite constants are then regulated by using the standard zeta-function regularizations

$$\prod_{n=1}^{\infty} c = c^{-1/2}, \quad \prod_{n+1/2=1}^{\infty} c = 1.$$  

(4.13)

The $\bar{A}^I$ integrals are performed after an appropriate shift of the fields $(\bar{A}_z^I, \bar{A}_{\bar{z}}^I) \rightarrow (\bar{A}_z^I - \partial_z \varphi_0^I, \bar{A}_{\bar{z}}^I - \partial_{\bar{z}} \varphi_1^I)$. In this way the contribution to the normalization constant comes from two complex functional Gaussian integrations and results in

$$\mathcal{N} = \left( \frac{\sqrt{2}}{8\pi^2} \left| \det(K_{IJ}) \right| \right)^{\nu_g},$$  

(4.14)
where the index $\nu_g$ depends on the genus of the surface and is given by $\nu_0 = 1$, $\nu_1 = 2$, while $\nu_g = 1$ for $g \geq 2$ even and $\nu_g = 0$ for $g > 2$ odd.

In the case of open strings, the worldsheet $\Sigma_1$ is, under the given orbifold involution, identified with $\Sigma_0$ and the wavefunction at $\Sigma_{1/2}$ consists only of terms on the boundary $\partial \Sigma^o$. This point was already used in (3.56) and will be further elucidated in the next subsection. Then we have only the integral over one of the boundaries, say $\Sigma = \Sigma_0$, and the orbifold of (4.12) yields the mass formula

$$M^2 = \alpha' (g_s)^{-\chi(\Sigma)} N \left( \frac{\det(G_{IJ}/\alpha')}{\sqrt{32\pi^2}} \right)^{\nu_g/2} \times \prod_{I=1}^d \int [D\bar{A}_I] \int [D\phi^I] \exp \left\{ -\frac{1}{4\pi\alpha'} \int_{\Sigma} d^2z \left[ G_{IJ} \bar{A}_I^I \partial_{z} \phi^J \right. \right. \right.$$

$$\left. \left. - G_{IJ} \partial_{\bar{z}} \phi^J \bar{A}_I^I + \alpha' K_{IJ} \left( \bar{A}_I^I - \partial_{\bar{z}} \phi^I \right) \left( \bar{A}_J^J - \partial_z \phi^J \right) \right] \right\} ,$$

(4.15)

where the overall metric determinant factor in (4.15) comes from the regularized integrations enforcing the functional Dirac delta-functions which impose the boundary conditions (2.97,2.98). Remembering the orbifold correspondence rule (4.9), we have identified (4.15) as the square of the orbifold contribution to the normalization of (2.89) determining the brane tension. As before, to determine the precise topological dependence here requires an appropriate truncation to the Lagrangian subspace (2.61). This is not required to determine the dependence of the mass formula on the couplings. Note that on the boundary $\partial \Sigma^o$ only the antisymmetric part $B_{IJ}$ of $K_{IJ}$ is present, and generally its symmetric part $G_{IJ}$ acts as a metric for the raising and lowering of the $U(1)^d$ group indices (target space directions in the induced string theory). As with the topological contributions, we suppress these boundary terms in all formulas of this subsection.

Let us first consider the $\text{PT}$ orbifold of the three-dimensional gauge theory (4.10), corresponding to a $d$-brane wrapping $T^d$. Integration over the gauge field components in (4.15) as described in section 2.7 then yields

$$\mathcal{M}^2 = \alpha' (g_s)^{-\chi(\Sigma)} N \left( \frac{\pi}{\sqrt{2}} \left| \frac{\det(G_{IJ}/\alpha')}{\det(K_{IJ})} \right| \right)^{\nu_g/2} \times \prod_{I=1}^d \int [D\phi^I] \exp \left\{ -\frac{\tilde{K}_{IJ}}{4\pi} \int_{\Sigma} d^2z \left[ \partial_{\bar{z}} \phi^I \partial_z \phi^J \right] \right\} ,$$

(4.16)

where

$$\tilde{K}_{IJ} = \frac{1}{(\alpha')^2} G_{IP} \left( K^{-1} \right)^{I'J'} G_{J'J} \equiv \frac{1}{\alpha'} \left( \tilde{G}_{IJ} + i \tilde{B}_{IJ} \right)$$

(4.17)

is the T-dual background to the Chern-Simons coupling matrix (4.11), with symmetric and antisymmetric parts $\tilde{G}_{IJ}$ and $\tilde{B}_{IJ}$, respectively. Carrying out the remaining path integration
in (4.16) and inserting (4.14) then yields finally
\[ M^2 = \alpha' (g_s)^{-\chi(\Sigma)} \left( \frac{\det(K_{IJ})}{\sqrt{4\pi \det(G_{IJ}/\alpha')}} \right)^{\nu_g}. \] (4.18)

The effective coupling (4.18) is the familiar Born-Infeld action for a flat background \( K_{IJ} \) times the right power of the string coupling constant \( g_s \), giving the familiar contribution to the D-brane tension on \( T^d \) [57, 76]. Note that it is determined in part by the effective \( \sigma \)-model action of (4.16) in the T-dual background fields, as expected.

For the PCT orbifold of the gauge theory (4.10), the terms involving solely the metric \( G_{IJ} \) disappear. Working through as above one finds that the final result of the open string path integral is \( M^2 \propto \det(G_{IJ})^{-\nu_g/2} \), yielding the correct dependence of the tension appropriate for Neumann boundary conditions along each of the directions of \( T^d \) [57, 76]. Thus from the topological membrane approach we naturally obtain the anticipated spacetime supergravity field and string coupling dependences of the tensions for both Neumann and Dirichlet branes. Note that this derivation is completely independent of the induced boundary vertex operators that we obtained in section 2.7, and is simply a consequence of the orbifold properties of the topological membrane.

### 4.3 Effective Actions

We will now carry the analysis of the previous subsection one step further and incorporate sources into the orbifold partition functions, as we did in section 2. For this, it suffices to consider the following construction. At time \( t = 1/2 \) the wavefunctional (2.96) simply sets boundary conditions on the fields \( \varphi \) and produces the appropriate boundary vertex operator insertions of section 2.7, while at \( t = 0 \) (identified by the orbifold involution with \( t = 1 \)) the wavefunctional (2.90) produces the actual effective \( \sigma \)-model with a constant background \( \tilde{K}_{IJ} \), as is apparent from (4.16)–(4.18). So let us identify each point on \( \Sigma^o = \Sigma_{1/2}/\mathbb{Z}_2 \) with two points on \( \Sigma_0 = \Sigma \), so that the closed Riemann surface \( \Sigma \) constitutes a double covering of \( \Sigma^o \). In this way, as described at the end of section 2.7, from a \( U(1)^d \times U(1)^D \) topologically massive gauge theory one obtains the open bosonic string partition function in constant backgrounds with D-branes and Wilson lines,

\[ Z^o = (g_s)^{-\chi(\Sigma)/2} \prod_{l=1}^{d+D} \int [D\varphi^l] \prod_{a=1}^{d} \delta_{\partial\Sigma^o} (\varphi_a^o) \prod_{m=d+1}^{d+D} \delta_{\partial\Sigma^o} (\partial \varphi^m) e^{-S_{\text{eff}}[\varphi]}, \] (4.19)

where the fields \( \varphi^a, a = 1, \ldots, d \) obey Dirichlet boundary conditions and \( \varphi^m, m = d + 1, \ldots, d+D \) obey Neumann boundary conditions on the boundary of the open string world-sheet \( \Sigma^o \). As before, the boundary of \( \Sigma^o \) in the three-dimensional membrane is situated
at the branch point locus $x^\perp = 0$ of the orbifold, and we have already used the prescription (4.9). By using the shifts $\bar{A}_I^a \rightarrow \bar{A}_I^a - 4\pi (K^{-1})^{IJ} G_{J,J'} \bar{Y}_{z'}$ and $\varphi^a \rightarrow \varphi^a - Y_{D}^a$ in the original orbifold path integral, the effective action in (4.19) is given modulo topological contributions by

$$S_{\text{eff}}[\varphi] = \frac{1}{8\pi\alpha'} \int d^2 z \left[ \sqrt{h} \epsilon^{ij} \tilde{G}_{IJ} + i \epsilon^{ij} \tilde{B}_{IJ} \right] \partial_i \varphi^I \partial_j \varphi^J + \frac{1}{4\pi} \oint_{x^\perp = 0} \left[ Y_{D,a} \partial^\perp \varphi^a_b + i Y_{N,m} \partial^\parallel \varphi^m_b \right],$$

(4.20)

where we have used a Hodge decomposition of the sources analogous to (2.93) and a suitable source rescaling.

This is the usual string theoretic, deformed worldsheet $\sigma$-model action in the presence of D-branes and Wilson lines. The effective target space action is obtained in the standard way by integrating out the fields $\varphi^a$ with Dirichlet boundary conditions $[55]$. In particular, by using the boundary conditions $\varphi^a_b = Y_{D,b} \left|_{\partial \Sigma} \right.$ in the static gauge for constant background $\varphi^m$, the orbifold partition function (4.19) integrates (modulo topological terms) to the Dirac-Born-Infeld action $[56]$

$$S_{\text{DBI}} = (g_s)^{-\chi(\Sigma)/2} \int d^D \varphi \left| \det(\mathcal{K}_{mn} + \mathcal{F}_{mn}) \right|^{\nu_g/2},$$

(4.21)

where

$$\mathcal{K}_{mn}(\varphi^m) = K_{IJ} \frac{\partial \varphi^I}{\partial \varphi^m} \frac{\partial \varphi^J}{\partial \varphi^n},$$

$$\mathcal{F}_{mn}(\varphi^m) = \frac{\partial Y_{N,m}}{\partial \varphi^n} - \frac{\partial Y_{N,n}}{\partial \varphi^m},$$

$$\varphi^I = (\varphi^m, Y_{D}^a).$$

(4.22)

Since we are working with constant background $K_{IJ}$, the D-brane action (4.21) is manifestly invariant under T-duality.

### 4.4 Excited Wavefunctionals

In this subsection we will construct excited states of the topological membrane which will enable an alternative, intrinsic derivation of the D-brane tension within the three-dimensional formalism. At the same time, this solves a long-standing problem in the topological membrane approach to string theory, the proper construction of states in the three-dimensional gauge theory which correspond to excited, descendent string states in two-dimensions. For this, we rewrite the non-vanishing electric field commutators in (2.12) in complex coordinates as

$$\left[ E_z(z), E_{z'}(z') \right] = -\frac{k}{4\pi} \delta^{(2)}(z-z').$$

(4.23)
This is a functional Heisenberg oscillator algebra which in the Schrödinger picture can be represented as in section 2.1 by the electric field operators

\[
E_z = -i \frac{\delta}{\delta A_z} + \frac{i k}{8\pi} A_z, \\
E_T = -i \frac{\delta}{\delta A_T} - \frac{i k}{8\pi} A_T. 
\]  
(4.24)

The Hamiltonian of the dilaton-coupled topologically massive gauge theory part of the action (4.1) is straightforward to obtain in the classical background of the dilaton field with \( \langle D^2 \rangle \neq 0 \). As usual, the vacuum energy density arising from normal ordering the Hamiltonian operator using the commutation relations (4.23) is a divergent term \( \delta^{(2)}(0) \). The normal ordered energy density is then defined such that the divergent term drops out,

\[ \circ h_{ij} E_i E_j \circ \equiv E_T E_z, \]

and the dilaton-coupled gauge theory Hamiltonian in the temporal gauge \( A_0 = 0 \) is

\[
H_D = \frac{1}{4} \int_\Sigma \, d^2 z \, \frac{1}{D^2} \left[ \gamma E_T E_z + \frac{1}{\gamma} B^2 \right]. 
\]  
(4.25)

The equations of motion obtained by varying the classical action (4.1) with respect to the dilaton field \( D \) read

\[
\frac{1}{2\gamma D^3} F_{\mu\nu} F^{\mu\nu} - 16\kappa \nabla^2 D + 2\kappa D R(\omega) = 0 , 
\]  
(4.26)

which we note carefully uses the boundary term over \( \partial M \) in (4.1) that ensures the dilaton field theory has a classical extremum. By using the Minkowski signature of the three-dimensional metric (2.1) to compute \( F_{\mu\nu} F^{\mu\nu} = -\gamma^2 E_T E_z + B^2 \), we can use the classical value of the dilaton field determined from (4.26) to express the magnetic field term \( B^2 \) in (4.25) in terms of the field theoretic oscillator number operator \( E_T E_z \). We find

\[
H_D = \frac{1}{2} \int_\Sigma \, d^2 z \, \left[ \frac{\gamma}{D^2} E_T E_z + 2\kappa V(D,\omega) \right], 
\]  
(4.27)

where the dilaton potential is given by

\[
V(D,\omega) = 8 \, D \nabla^2 D - D^2 \, R(\omega) 
\]  
(4.28)

and is essentially the three-dimensional energy density of the Liouville field theory (4.4). Note that substituting \( D^2 \) by its vacuum expectation value given from (4.26) breaks the quantum S-duality symmetry \( D^2 \mapsto (\text{const.})/D^2 \).

From (2.31) it follows that the vacuum states (3.9) are zero-modes of the field theoretic harmonic oscillator annihilation operator \( E_z \) in (4.24),

\[
E_z \equiv \left[ \begin{array}{c} A_z \\ A_T \\ \{ z_1 \ldots z_s \\ Q_1 \ldots Q_s \} \end{array} \right] ; 0 = 0 . 
\]  
(4.29)
Physically, from (4.27) it then follows that the dilaton coupling to the gravitational sector induces a shift in the ground state energy of the membrane from \( E_0 = 0 \) to
\[
E_0 = \kappa \left< V(D, \omega) \right>.
\]
This zero-point energy vanishes in the pure Chern-Simons limit \( \kappa \to 0 \) of topologically massive gravity (equivalently in the phase of the topological field theory with \( \langle D^2 \rangle = 0 \)). Geometrically, the electric field operators (4.24) define a connection on a principal \( U(q) \)-bundle \( P \) over the space of gauge fields of the given (non-trivial) complex line bundle over the Riemann surface \( \Sigma \) [5]. This connection, according to (4.23), has constant curvature \( k \equiv \frac{\mathcal{R}}{q} \). By using in addition the constraint (2.32), the condition (4.29) simply means that the vacuum wavefunctionals (3.9) of the topological membrane are gauge-invariant holomorphic sections of the bundle \( P \). This property is of course tied to the fact that the vacuum sector is a topological field theory, whose quantization yields the holomorphic Friedan-Shenker vector bundles of the induced conformal field theory on \( \Sigma \).

We will now construct the eigenstates of the Hamiltonian (4.27) by using these observations. The natural gauge-invariant excited states of the topological membrane are the Landau levels created by the field theoretic harmonic oscillator creation operator \( E_{z_i} \) of (4.24). They are obtained by acting on the wavefunctionals (3.9) with non-negative powers \( l_i \) of the operators \( E_{z_i}(z_i) \) at the primary field insertion points \( z_i \in \Sigma \) to give
\[
\Psi^s \left[ A_z, A_{\bar{z}}; \left\{ \begin{array}{c}
\frac{z_1}{l_1} \\
\vdots \\
\frac{z_s}{l_s}
\end{array} \right\}; 0 \right] \\
= \prod_{i=1}^s \frac{1}{\sqrt{l_i!}} \left( \frac{4\pi i}{k} E_{z_i}(z_i) \right)^{l_i} \Xi \left[ A_z, A_{\bar{z}}; \left\{ \begin{array}{c}
\frac{z_1}{l_1} \\
\vdots \\
\frac{z_s}{l_s}
\end{array} \right\}; 0 \right] \\
= \exp \left\{ \frac{k}{8\pi} \int_{\Sigma} d^2 z A_z A_{\bar{z}} \right\} \int [D\varphi] \exp \left\{ \frac{k}{8\pi} \int_{\Sigma} d^2 \varphi \left( \frac{\partial_{z\varphi}}{2A_{\bar{z}}} \right) \partial_{z\varphi} \right\} \\
\times \prod_{i=1}^s \frac{1}{\sqrt{l_i!}} \left( A_z(z_i) - \partial_{z\varphi}(z_i) \right)^{l_i} e^{iQ_i(\varphi(z_i) + h_{\varphi}(z_i))}. \tag{4.31}
\]
As before, the excited states \( \Psi^s \) are gauge invariant up to a projective phase, and as in (3.39) in the cases that the worldsheet is the punctured sphere, \( \Sigma = S^2_0 \), they act on the product of the corresponding Virasoro modules,
\[
\Psi^s \left[ A_z, A_{\bar{z}}; \left\{ \begin{array}{c}
\frac{z_1}{l_1} \\
\vdots \\
\frac{z_s}{l_s}
\end{array} \right\}; 0 \right] \in \bigotimes_{i=1}^s [\phi_{\lambda_i}]^* \otimes [\phi_{\lambda_i}]. \tag{4.32}
\]
They are the proper gauge invariant wavefunctionals that correspond to string descendent states of higher level numbers

\[ N = \sum_{i=1}^{s} l_i \geq 0. \tag{4.33} \]

By construction, the wavefunctionals \((4.31)\) are eigenstates of the Hamiltonian operator \((4.27)\), \(H_D \Psi_s = \mathcal{E}_N \Psi_s\), where the excited state energies are those of Landau levels,

\[ \mathcal{E}_N = \mathcal{E}_0 + \frac{\mu}{\langle D^4 \rangle} N, \tag{4.34} \]

with \(\mu\) the topological photon mass \((4.3)\). There is therefore an infinite, continuous degeneracy labelled by the locations \(z_i \in \Sigma\) of the insertions. From the three-dimensional perspective, higher string states thus correspond to Landau levels which consist of \(N = \sum l_i\) gauge invariant combinations of external charged particles and photons, situated at the points \(z_i\). As expected, in the naive vacuum \(\langle D^2 \rangle = 0\), the excited states become infinitely massive and decouple from the topological sector of the quantum field theory with \(\mathcal{E} = \mathcal{E}_0 = 0\). For a generic dilaton coupling to topologically massive gravity, the scale of the theory is just the photon mass \(\mu\).

In what follows we shall need the inner product of two states of the form \((4.31)\) in the finite temperature gauge theory on \(\Sigma \times S^1\). The calculation proceeds along the lines of that in section 3.3, and in keeping with the conventions of the rest of this section we absorb all functional determinant factors into an appropriate bulk normalization. The inner product of two chiral excited states is given explicitly by the path integral

\[
\begin{align*}
\text{Tr} \left\langle \Psi^s_1 \left\{ x_i, Q_i, l_i \right\} | \Psi^s_0 \left\{ x_{i'}, Q_{i'}, l_{i'} \right\} \right\rangle &= \int [DA_z] [DA_\tau] \exp \left\{ \frac{k}{4\pi} \int_{\Sigma} d^2z A_\tau A_z \right\} \exp \left\{ \frac{k}{8\pi} \int_{\Sigma} d^2z \left( \partial_z \varphi - 2A_\tau \right) \partial_z \varphi \right\} \\
& \times \int [D\varphi] \exp \left\{ \frac{k}{8\pi} \int_{\Sigma} d^2z \left( \partial_z \varphi - 2A_\tau \right) \partial_z \varphi \right\} \\
& \times \prod_{i=1}^{s} \frac{1}{\sqrt{l_i}} \left( A_z(x_i) - \partial_z \varphi(x_i) \right)^{l_i} e^{-iQ_i \left( \varphi(x_i) + h_\varphi(x_i) \right)} \\
& \times \prod_{i'=1}^{s'} \frac{1}{\sqrt{l_{i'}}} \left( A_\tau(x_{i'}) - \partial_\tau \varphi(x_{i'}) \right)^{l_{i'}} e^{iQ_{i'} \left( \varphi(x_{i'}) + h_\varphi(x_{i'}) \right)}. \tag{4.35}
\end{align*}
\]

We shift \(A_z \rightarrow A_z + \partial_z \varphi\), \(A_\tau \rightarrow A_\tau + \partial_\tau \varphi\) in \((4.35)\) and use gauge invariance of the topologically massive gauge theory action. By substituting in the gauge orbit decomposition

64
(2.86) and using the bulk normalization described in section 2.7, we arrive at

\[
\text{Tr} \left\langle \Psi_1^s \left\{ \frac{x_i}{Q_i} \right\} | \Psi_0^{s'} \left\{ \frac{x_{i'}}{Q_{i'}} \right\} \right\rangle \\
= \int [DA_z DA_{\pi}] \exp \left\{ \frac{k}{4\pi} \int d^2z \ \tilde{A}_z \tilde{A}_z \right\} \prod_{i=1}^s \frac{1}{\sqrt{l_i!}} \tilde{A}_z(x_i)^{l_i} \prod_{i'=1}^{s'} \frac{1}{\sqrt{l'_{i'}!}} \tilde{A}_{\pi}(x'_{i'})^{l'_{i'}} \\
\times \int [D\varphi] [D\varphi'] \exp \left\{ \frac{k}{8\pi} \int d^2z \ \partial_z(\varphi + \varphi') \partial_z(\varphi + \varphi') \right\} \\
\times \prod_{i=1}^s e^{-iQ_i \langle (\varphi(x_i) + h_\varphi(x_i)) \rangle} \prod_{i'=1}^{s'} e^{iQ_{i'} \langle (\varphi(x'_{i'}) + h_{\varphi'}(x'_{i'})) \rangle}.
\] (4.36)

We evaluate the functional integrals in (4.36) exactly as we did in section 3.3. The new feature here is the insertions of the gauge fields in the holomorphic and antiholomorphic sectors. The Gaussian integrals over these insertion points require \( s = s' \), and that the collections \( \{x_i; l_i\} \) and \( \{x'_i; l'_i\} \) be equal as unordered sets, i.e. up to a permutation \( \pi \in S_s \) on \( s \) letters. This is of course just the usual Wick expansion, which is weighted by the combinatorical factor \( 1/s! \). Each of these integrals also introduces a factor \( l_i! \left(4\pi/k\right)^{l_i+1} \), and additional normalization terms from the harmonic degrees of freedom of the gauge fields due to these extra insertions are taken care of as described in section 4.2. The integrals over the insertions \( \varphi(x_i) \) and \( \varphi'(x'_i) \) then impose charge conservation \( \sum_i(Q_i + Q'_i) = 0 \). The rest of the calculation proceeds exactly as before.

In this way we find that the dimension formula (3.18) is modified by these excited states to

\[
\text{Tr} \left\langle \Psi_1^s \left\{ \frac{x_i}{Q_i} \right\} | \Psi_0^{s'} \left\{ \frac{x_{i'}}{Q_{i'}} \right\} \right\rangle = \left( \frac{4\pi}{k} \right)^{N-\nu_g} (pq)^g \delta_{ss'} \delta_{[\lambda_1 + \lambda'_1 + \ldots + \lambda_s + \lambda'_s]} \\
\times \frac{1}{s!} \sum_{\pi \in S_s} \prod_{i=1}^s \delta_{l_i, l'_{\pi(i)}} \delta^{(2)}(x_i - x'_{\pi(i)}),
\] (4.37)

with the level number \( N \) defined by (4.33) and the index \( \nu_g \) defined in section 4.2. Thus the states (4.31) are orthogonal, and Dirac delta-function normalizable with norm given by \( (pq)^g/2 \left(4\pi/k\right)^{(N-\nu_g)/2} \). Note that (4.37) depends explicitly on the actual insertion points on \( \Sigma \), in contrast to what occurs in the vacuum sector where the gauge theory is topological. The formula (4.37) can be loosely thought of as describing the modification of the dimension of the vacuum Hilbert space from the excited membrane states. In the next subsection we show how to make this statement more precise, and use it to provide another derivation of the D-brane tension formula from the topological membrane approach.
4.5 The Regularized Dimension

In [57] it was shown that there is a universal formula for the tension of a D-brane in terms of the suitably regulated dimension of the space of physical open string states of the associated boundary conformal field theory. In this subsection we will show that, quite remarkably, there is an analog of this dimension formula in the topological membrane approach. The key feature is that once we consider the entire Hilbert space $\mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s) = \bigoplus_{\{l_i\}} \mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s; l_1, \ldots, l_s)$ of all states of the form (4.31) with definite $U(1)$ charges, the Hamiltonian no longer vanishes and has energy spectrum given by (4.30,4.34). This implies that the statistical mechanical partition function $\text{Tr} \mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s) (e^{-\beta H_D})$ is no longer independent of the inverse temperature $\beta$ which is the circumference of the thermal circle $S^1$. However, in the high-temperature limit $\beta \to 0$ whereby the circle shrinks to a point, the partition function is the trace of the identity operator and yields the dimension of the physical state space $\mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s)$. Of course this dimension is formally infinite above the vacuum sector (containing infinitely many Landau levels) as it includes propagating states of the three-dimensional photon which has mass (4.3). However, by carefully taking the limit we will see that the temperature compactification provides a well-defined regularization and yields a dimension formula which reproduces the D-brane mass (4.18). This is completely analogous to the way the dimension formula gives the brane tension in [57]. In the context of the topological membrane approach, this derivation is far more desirable than our previous one, because it provides a more intrinsic three-dimensional definition of the tension in terms of the mass scales of the bulk theory, with no explicit reference to its string theoretic origin.

Having to sum over all states (4.31) required to compute the trace over $\mathcal{H}_{g,s,k}(\lambda_1, \ldots, \lambda_s)$ would be an arduous task. We can, however, simply the calculation by the following observation. Since the electric field operators obey $[E_z(z), E_z'(z')] = 0 = [E_{-z}(z), E_{-z}(z')]$, we can take a generic wavefunctional (4.31) and adiabatically transport every one of its insertion points to a single point $z \in \Sigma$. The only effect this operation will have is to induce phase factors due to the linkings of the charged particle trajectories in the bulk (see section 3.4). Although these phase factors constrain the charge spectrum of the quantum gauge theory [25] (and hence the momentum lattice of string theory), for the calculation that follows they play no role and will simply drop out of all formulas with the appropriate normalizations. Thus without loss of generality we may lift some of the infinite degeneracy at each level, such that the contribution to the trace over physical states depends on only a single representative excited wavefunctional for each Landau level $N$. In other words, we compute the trace only over the single insertion states (4.31) which have $s = 1$ and $N = l$. We denote the Hilbert space in which these states live as $\mathcal{H}_{g,k}(\lambda) = \bigoplus_{N \geq 0} \mathcal{H}_{g,1,k}(\lambda; N)$. In the string theory picture, this reduction simply means that the mass is measured by
the coupling of a brane boundary state with the graviton vertex operator. Its relation to Ishibashi states and the Cardy formula is obtained through the analysis of section 3.9.

The regulated dimension of the Hilbert space is defined as

\[
\text{reg dim } \mathcal{H}_{g,k}(\lambda) = \lim_{\beta \to 0} \text{Tr} \mathcal{H}_{g,k}(\lambda) \left( e^{-\beta H_D} \right).
\]

(4.38)

The first thing we need to do is define precisely the trace over \( s = 1 \) states (4.31) contributing to the statistical mechanical partition function in (4.38). We sum over all Landau levels \( N \geq 0 \) and positions \( x \in \Sigma \) of external particles. As there are \( N \) identical ways to insert the particles via the electric field creation operator in (4.31), we need to divide by the total number \( N! \) of permutations in each Landau level. Furthermore, we normalize the trace by dividing it by \((pq)^g\), so that (4.38) coincides with the number of linearly independent topological wavefunctions when restricted to the vacuum sector of the topologically massive gauge theory. Thus the formal definition of (4.38) is given by

\[
\text{reg dim } \mathcal{H}_{g,k}(\lambda) = \lim_{\beta \to 0} \sum_{N=0}^{\infty} \frac{1}{N!} \int_{\Sigma} d^2 x \frac{\langle \Psi_1 \{ x \} \{ Q \} \rangle}{(pq)^g} e^{-\beta H_D} \langle \Psi_0 \{ x \} \{ Q \} \rangle \). 
\]

(4.39)

Substituting the energy eigenvalues (4.34) and the wavefunctional normalizations (4.37) into (4.39), and summing over all Landau levels thereby yields

\[
\text{reg dim } \mathcal{H}_{g,k}(\lambda) = \left( \frac{k}{4\pi} \right)^{\nu_g} \lim_{\beta \to 0} A_\Sigma \delta^{(2)}(0) \exp \left\{ -\beta \mathcal{E}_0 + \frac{4\pi}{k} e^{-\beta \mu/(D^4)} \right\}, 
\]

(4.40)

where \( A_\Sigma \) is the area of the Riemann surface \( \Sigma \) and the ground state energy \( \mathcal{E}_0 \) is given by (4.30). As expected from bulk charge conservation, the expression (4.40) is independent of \( Q \). However, it is singular. Due to the Dirac delta-function normalizability of the excited physical states, the contribution from each Landau level diverges as \( \delta^{(2)}(0) \). To get a finite result, we remove these quantum field theoretic infinities by an appropriate renormalization of the gravitational sector of the bulk theory at finite temperature. For this, we regulate the delta-function at the origin as \( A_\Sigma \delta^{(2)}(0) = \Lambda_{uv}/\mu \), where \( \Lambda_{uv} \to \infty \) is a fundamental ultraviolet cutoff on the string worldsheet \( \Sigma \). In the given non-trivial dilaton background, it follows from (4.40) that the three-dimensional Planck mass \( \kappa \) should then scale logarithmically with the cutoff and linearly with the temperature as

\[
\kappa \left\langle V(D,\omega) \right\rangle = \frac{1}{\beta} \ln \left( \frac{\Lambda_{uv}}{\mu} \right)
\]

(4.41)

in the limits \( \beta \to 0 \) and \( \Lambda_{uv} \to \infty \). Then our derivation of the dimension formula is non-singular provided that the mass scale of the gravitational sector is logarithmically close to
the ultraviolet cutoff (and the topological photon mass) and linearly close to the (infinite) temperature.

However, this is not quite what we want, as one easily checks that even after this renormalization, the formula (4.40) does not reproduce the correct target space radius dependence of the D-brane tension that we found in section 4.2, except in the topological limit \( \mu \to \infty \) or \( \langle D^2 \rangle = 0 \). The reason for this is that from the point of view of the bulk three-dimensional dynamics, we should be decompactifying the temperature circle rather than shrinking it to a point, in order to map the excited state contributions onto those appropriate for the topological membrane dynamics. Thus we instead compactify the Euclidean time direction on the dual circle of radius \( \tilde{\beta} = 1/\mu^2 \beta \), and take the equivalent limit \( \tilde{\beta} \to \infty \). The expression (4.38) is thereby computed as

\[
\text{reg dim } \mathcal{H}_{g,k}(\lambda) = \lim_{\tilde{\beta} \to \infty} \text{Tr}_{\mathcal{H}_{g,k}(\lambda)} \left( e^{-\tilde{\beta} H_D} \right).
\]

(4.42)

Everything proceeds in precisely the same way as above but with the temperature replaced by its dual. In particular, by taking the limits \( \tilde{\beta} \to \infty \) and \( \Lambda_{uv} \to \infty \) with \( \ln(\Lambda_{uv})/\tilde{\beta} \) held fixed, the topological graviton mass now undergoes a finite renormalization. With this, the above calculations show that only the lowest Landau level \( N = 0 \) contributes in the limit \( \tilde{\beta} \to \infty \), and we arrive at the final expression

\[
\text{reg dim } \mathcal{H}_{g,k}(\lambda) = \left\langle \Psi_1 \left\{ \begin{array}{c} x \\ Q \\ 0 \end{array} \right\} \middle| \Psi_0 \left\{ \begin{array}{c} x \\ Q \\ 0 \end{array} \right\} \right\rangle_{\text{ren}} = \left( \frac{k}{4\pi} \right)^{\nu_g} \quad \text{(4.43)}
\]

for the regulated dimension of the physical Hilbert space. Note that this formula could also have been obtained by taking the instead the limit \( \beta \to 1/\mu \) in (4.40), appropriate to the membrane geometry \( \Sigma \times [0, 1] \), in the topological sector where \( \langle D^2 \rangle = 0 \).

Comparing with (4.18), we thereby arrive at the dimension formula

\[
\mathcal{M}^2 = \left( \frac{\pi}{64} \right)^{\nu_g/2} \alpha'(g_s)^{-\chi(\Sigma)} \sqrt{\text{reg dim } \mathcal{H}_{g,k}(\lambda)}.
\]

(4.44)

The appearance of the square root here is not surprising when we apply the orbifold correspondence rule (4.9) to the closed string amplitude which appears in (4.43), i.e. it produces the appropriate regularized dimension for the open string sector. This applies to the PT orbifold, corresponding to a D-brane wrapping the target space circle \( S^1 \) of radius \( R \) given by the formula (2.101), in which case the physical state normalization formula (4.37) may be applied directly. In the case of the PCT orbifold, corresponding to a Neumann brane along the target space circle, one needs to modify the contribution of harmonic modes to the normalization functional integral (4.36) along the lines described in section 4.2. It is straightforward to then show in a completely analogous way that the regulated dimension of the space of physical membrane states is given by \( \text{reg dim } \mathcal{H}_{g,k}(\lambda) = (4\pi/k)^{\nu_g} \), and hence
that once again the formula (4.44) holds. Thus (4.44) is a universal result for the physical
states of the topological membrane.

4.6 Worldsheet Duality

The truncation to the lowest Landau level $N = 0$ used to arrive at (4.43) is the analog of the
fact that the dominant contribution to the regularized dimension of the open string Hilbert
space in the associated boundary conformal field theory comes from the identity represent-
ation in the closed string picture [57]. This is completely consistent with the membrane
interpretation, as these are the only contributions relevant to the orbifold of the amplitude
(4.8). In fact, the passing to the dual temperature compactification $\beta \to \tilde{\beta}$ is the analog
of the worldsheet modular transformation that is used to compute the boundary conformal
field theoretic dimension of states and which induces the usual worldsheet duality between
the open and closed string sectors. The Cardy condition in this context simply states the
equality of the closed string cylinder amplitude between two boundary states and the one-
loop annulus amplitude with corresponding prescribed open string boundary conditions.
This is essentially what was computed in section 3.10. It follows from the calculation of
the previous subsection that this worldsheet duality is simply a duality between Wilson
and Polyakov loop correlators in the three-dimensional gauge theory. In fact, as we now
explain, the topological membrane approach gives an interesting new perspective on this
duality.

Consider the one-loop open string vacuum diagram with boundary conditions labelled
by $\alpha$ and $\beta$. In the membrane picture, it corresponds to an annulus $\Sigma^o = C^2$ which sits at
three-dimensional time $t = 1/2$ (see fig. 1). Its Schottky double is the torus $\Sigma = T^2$ which
sits at the times $t = 0$ and $t = 1$. In the open string channel, as the string propagates around
the cycle of the annulus, each point on it is described by two pre-images on the torus. In the
closed string channel, it thereby indeed corresponds to closed string propagation, but now
between the two corresponding Ishibashi states. With left and right moving worldsheets
 glued onto each other in the orbifold picture, i.e. $\Sigma_0 \equiv \Sigma_1$, the propagation in worldsheet
time $\tau \in [0, 1]$ is interlaced with the membrane propagation time $t \in [0, 1]$ in such a way
that the equivalence between the open and closed string channels is clear. These processes
are all depicted in fig. 12.

Acknowledgments

The work of P.C.F. was supported by Grant SFRH/BPD/5638/2001 from FCT-MCT (Port-
tugal). The work of I.I.K. was supported by PPARC Grant PPA/G/0/1998/00567 and EU
Grant FMRXCT960090. The work of R.J.S. was supported by a PPARC Advanced Fel-
Figure 12: Membrane representation of worldsheet duality. (a) An open string has two closed string pre-images. (b) The interlacing between worldsheet and membrane time evolutions. Propagation of the string between its initial and final Ishibashi states $|\alpha\rangle$ and $|\beta\rangle$ requires a superposition of the initial and final membrane wavefunctionals $\Psi_0$ and $\Psi_1$. Here the horizontal direction is propagation along the membrane time $t$. (c) The propagation of the closed string between the Ishibashi states naturally induces a worldsheet annulus diagram. Here the horizontal direction is propagation along the worldsheet time $\tau$. 

lowship.
References

[1] E. Witten, Quantum Field Theory and the Jones Polynomial, Commun. Math. Phys. 121 (1989) 351–399.

[2] J.M.F. Labastida and A.V. Ramallo, Operator Formalism for Chern-Simons Theories, Phys. Lett. B227 (1989) 92; Chern-Simons Theory and Conformal Blocks, Phys. Lett. B228 (1989) 214.

[3] W. Ogura, Path Integral Quantization of Chern-Simons Gauge Theory, Phys. Lett B229 (1989) 61.

[4] M. Bos and V.P. Nair, U(1) Chern-Simons Theory and c = 1 Conformal Blocks, Phys. Lett. B223 (1989) 61; Coherent State Quantization of Chern-Simons Theory, Int. J. Mod. Phys. A5 (1990) 959.

[5] E. Witten, On Holomorphic Factorization of WZW and Coset Models, Commun. Math. Phys. 144 (1992) 189–212.

[6] J.F. Schonfeld, A Mass Term for Three-Dimensional Gauge Fields, Nucl. Phys. B185 (1981) 157.

[7] S. Deser, R. Jackiw and S. Templeton, Three-Dimensional Massive Gauge Theories, Phys. Rev. Lett. 48 (1982) 975–978; Topologically Massive Gauge Theories, Ann. Phys. 140 (1982) 372–411.

[8] G. Moore and N. Seiberg, Taming the Conformal Zoo, Phys. Lett. B220 (1989) 422.

[9] S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, Remarks on the Canonical Quantization of the Chern-Simons-Witten Theory, Nucl. Phys. B326 (1989) 108.

[10] I.I. Kogan, The Off-Shell Closed String as Topological Open Membranes: Dynamical Transmutation of World Sheet Dimension, Phys. Lett. B231 (1989) 377.

[11] S. Carlip and I.I. Kogan, Quantum Geometrodynamics of the Open Topological Membrane and String Moduli Space, Phys. Rev. Lett. 64 (1990) 1487; Three-Dimensional Gravity and String Ghosts, Phys. Rev. Lett. 67 (1991) 3647–3649, hep-th/9110005.

[12] S. Carlip and I.I. Kogan, Three-Dimensional Topological Field Theories and Strings, Mod. Phys. Lett. A6 (1991) 171–181.

[13] I.I. Kogan, Quantum Mechanics on the Moduli Space from the Quantum Geometrodynamics of the Open Topological Membrane, Phys. Lett. B256 (1991) 369; Quantum Liouville Theory from Topologically Massive Gravity: 1 + 1 Cosmological Constant as Square of 2 + 1 Graviton Mass, Nucl. Phys. B375 (1992) 362.
[14] S. Carlip, *Inducing Liouville Theory from Topologically Massive Gravity*, Nucl. Phys. **B362** (1991) 111–124.

[15] S. Carlip, *(2 + 1)-Dimensional Chern-Simons Gravity as a Dirac Square Root*, Phys. Rev. **D45** (1992) 3584–3590, hep-th/9109006.

[16] L. Cooper and I.I. Kogan, *Boundary Conditions and Heterotic Construction in Topological Membrane Theory*, Phys. Lett. **B383** (1996) 271–280, hep-th/9602062.

[17] G. Amelino-Camelia, I.I. Kogan and R.J. Szabo, *Conformal Dimensions from Topologically Massive Quantum Field Theory*, Nucl. Phys. **B480** (1996) 413–456, hep-th/9607037; *Gravitational Dressing of Aharonov-Bohm Amplitudes*, Int. J. Mod. Phys. **A12** (1997) 1043–1052, hep-th/9610057.

[18] L. Cooper, I.I. Kogan and K.-M. Lee, *String Winding Modes from Charge Non-Conservation in Compact Chern-Simons Theory*, Phys. Lett. **B394** (1997) 67–74, hep-th/9611107.

[19] I.I. Kogan, *Three-Dimensional Description of the $\Phi_{1,3}$ Deformation of Minimal Models*, Phys. Lett. **B390** (1997) 189–196, hep-th/9608031.

[20] L. Cooper, I.I. Kogan and R.J. Szabo, *Mirror Maps in Chern-Simons Gauge Theory*, Ann. Phys. **268** (1998) 61–104, hep-th/9710179.

[21] L. Cooper, I.I. Kogan and R.J. Szabo, *Dynamical Description of Spectral Flow in $N = 2$ Superconformal Field Theories*, Nucl. Phys. **B498** (1997) 492–510, hep-th/9702088.

[22] I.I. Kogan and R.J. Szabo, *Liouville Dressed Weights and Renormalization of Spin in Topologically Massive Gravity*, Nucl. Phys. **B502** (1997) 383–418, hep-th/9703071.

[23] I.I. Kogan and A. Lewis, *Vacuum Instability in Chern-Simons Theory, Null Vectors and Two-Dimensional Logarithmic Operators*, Phys. Lett. **B431** (1998) 77–84, hep-th/9802102.

[24] I.I. Kogan, A. Momen and R.J. Szabo, *Induced Dilaton in Topologically Massive Quantum Field Theory*, J. High Energy Phys. **9812** (1998) 013, hep-th/9811006.

[25] P. Castelo Ferreira, I.I. Kogan and B. Tekin, *Toroidal Compactification in String Theory from Chern-Simons Theory*, Nucl. Phys. **B589** (2000) 167–195, hep-th/0004078.

[26] P. Castelo Ferreira and I.I. Kogan, *Open and Unoriented Strings from Topological Membrane: I. Prolegomena*, J. High Energy Phys. **0106** (2001) 056, hep-th/0012188.

[27] P. Castelo Ferreira, *Heterotic, Open and Unoriented Strings from Topological Membrane*, D.Phil. Thesis – Oxford University (unpublished), hep-th/0110067.
[28] P. Castelo Ferreira, I.I. Kogan and R.J. Szabo, Conformal Orbifold Partition Functions from Topologically Massive Gauge Theory, J. High Energy Phys. 0204 (2002) 035, hep-th/0112104.

[29] P. Hořava and E. Witten, Heterotic and Type I String Dynamics from Eleven-Dimensions, Nucl. Phys. B460 (1996) 506–524, hep-th/9510209.

[30] P. Hořava and E. Witten, Eleven-Dimensional Supergravity on a Manifold with Boundary, Nucl. Phys. B475 (1996) 94–114, hep-th/9603142.

[31] L. Freidel and K. Krasnov, 2D Conformal Field Theories and Holography, hep-th/0205091.

[32] J.H. Schwarz, Off Mass-Shell Dual Amplitudes without Ghosts, Nucl. Phys. B65 (1973) 131–140.

[33] J.H. Schwarz and C.C. Wu, Off Mass-Shell Dual Amplitudes. 2., Nucl. Phys. B72 (1974) 397.

[34] E. Corrigan and D.B. Fairlie, Off-Shell States in Dual Resonance Theory, Nucl. Phys. B91 (1975) 527.

[35] M.B. Green, Locality and Currents for the Dual String, Nucl. Phys. B103 (1976) 333.

[36] W. Siegel, Strings with Dimension-Dependent Intercept, Nucl. Phys. B109 (1976) 244.

[37] J. Scherk, An Introduction to the Theory of Dual Models and Strings, Rev. Mod. Phys. 47 (1975) 123–164.

[38] M.B. Green, Dynamical Point-Like Structure and Dual Strings, Phys. Lett. B69 (1977) 89.

[39] M.B. Green, Modifying the Bosonic String Vacuum, Phys. Lett. B201 (1988) 42–48.

[40] M.B. Green, Spacetime Duality and Dirichlet String Theory, Phys. Lett. B266 (1991) 325.

[41] M.B. Green, Temperature Dependence of String Theory in the Presence of Worldsheet Boundaries, Phys. Lett. B282 (1992) 380–386, hep-th/9201054.

[42] Z. Yang, Asymptotic Freedom and Dirichlet String Theory, hep-th/9211092.

[43] M. Li, Dirichlet Strings, Nucl. Phys. B420 (1994) 339–362, hep-th/9307122.

[44] M. Li, Dirichlet String Theory and Singular Random Surfaces, hep-th/9309108.
[45] E. Witten, *Branes and the Dynamics of QCD*, Nucl. Phys. **B507** (1997) 658–690, hep-th/9706109.

[46] J. Polchinski, *Combinatorics of Boundaries in String Theory*, Phys. Rev. **D50** (1994) 6041–6045, hep-th/9407031.

[47] J. Polchinski, *Dirichlet Branes and Ramond-Ramond Charges*, Phys. Rev. Lett. **75** (1995) 4724–4727, hep-th/9510017.

[48] A. Recknagel and V. Schomerus, *D-Branes in Gepner Models*, Nucl. Phys. **B531** (1998) 185–225, hep-th/9712186.

[49] A.Yu. Alekseev and V. Schomerus, *D-Branes in the WZW Model*, Phys. Rev. **D60** (1999) 061901, hep-th/9812193.

[50] L. Birke, J. Fuchs and C. Schweigert, *Symmetry Breaking Boundary Conditions and WZW Orbifolds*, Adv. Theor. Math. Phys. **3** (1999) 671–726, hep-th/9905038.

[51] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *The Geometry of WZW branes*, J. Geom. Phys. **34** (2000) 162–190, hep-th/9909030.

[52] J. Fuchs, L.R. Huiszoon, A.N. Schellekens, C. Schweigert and J. Walcher, *Boundaries, Crosscaps and Simple Currents*, Phys. Lett. **B495** (2000) 427–434, hep-th/0007174.

[53] L.R. Huiszoon, K. Schalm and A.N. Schellekens, *Geometry of WZW Orientifolds*, Nucl. Phys. **B624** (2002) 219–252, hep-th/0110267.

[54] M.R. Gaberdiel, *D-Branes from Conformal Field Theory*, in: “Quantum Structure of Spacetime and the Geometric Nature of Fundamental Interactions”, eds. C. Kounnas, D. Lüst and S. Theisen (Wiley, 2002), hep-th/0201113.

[55] E.S. Fradkin and A.A. Tseytlin, *Effective Field Theory from Quantized Strings*, Phys. Lett. **B158** (1985) 316–322.

[56] A.A. Tseytlin, *Self-Duality of Born-Infeld Action and Dirichlet 3-Brane of Type IIB Superstring Theory*, Nucl. Phys. **B469** (1996) 51–67, hep-th/9602064.

[57] J.A. Harvey, S. Kachru, G. Moore and E. Silverstein, *Tension is Dimension*, J. High Energy Phys. **0003** (2000) 001, hep-th/9909072.

[58] J.A. Harvey and J.A. Minahan, *Open Strings in Orbifolds*, Phys. Lett. **B188** (1987) 44–50.

[59] G. Pradisi and A. Sagnotti, *Open String Orbifolds*, Phys. Lett. **B216** (1989) 59–67.
[60] P. Hořava, *Strings on Worldsheet Orbifolds*, Nucl. Phys. **B327** (1989) 461–484.

[61] P. Hořava, *Chern-Simons Gauge Theory on Orbifolds: Open Strings from Three Dimensions*, J. Geom. Phys. **21** (1996) 1–33, hep-th/9404101.

[62] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *Conformal Boundary Conditions and Three-Dimensional Topological Field Theory*, Phys. Rev. Lett. **84** (2000) 1659–1662, hep-th/9909140.

[63] G. Felder, J. Fröhlich, J. Fuchs and C. Schweigert, *Correlation Functions and Boundary Conditions in Rational Conformal Field Theory and Three-Dimensional Topology*, Compos. Math. **131** (2002) 189, hep-th/9912239.

[64] J. Fuchs, I. Runkel and C. Schweigert, *Conformal Correlation Functions, Frobenius Algebras and Triangulations*, Nucl. Phys. **B624** (2002) 452–468, hep-th/0110133.

[65] J. Fuchs, I. Runkel and C. Schweigert, *TFT Construction of RCFT Correlators I: Partition Functions*, Nucl. Phys. **B646** (2002) 353–497, hep-th/0204148.

[66] M. Blau and G. Thompson, *Derivation of the Verlinde Formula from Chern-Simons Theory and the G/G Model*, Nucl. Phys. **B408** (1993) 345–390, hep-th/9305010.

[67] G. Grignani, P. Sodano, G.W. Semenoff and O. Tirkkonen, *G/G Models as the Strong Coupling Limit of Topologically Massive Gauge Theory*, Nucl. Phys. **B489** (1997) 360–386, hep-th/9609228.

[68] N. Marcus and A. Sagnotti, *Tree Level Constraints on Gauge Groups for Type I Superstrings*, Phys. Lett. **B119** (1982) 97.

[69] N. Marcus and A. Sagnotti, *Group Theory from ‘Quarks’ at the Ends of Strings*, Phys. Lett. **B188** (1987) 58.

[70] A. Sagnotti, *Anomaly Cancellations and Open String Theories*, in: “From Superstrings to Supergravity”, Erice 1992 Proceedings, pp. 116–125, hep-th/9302099.

[71] E.C. Marino, *Quantum Theory of Nonlocal Vortex Fields*, Phys. Rev. **D38** (1988) 3194.

[72] A. Kovner, B. Rosenstein and D. Eliezer, *Photon as a Goldstone Boson in (2 + 1)-Dimensional Abelian Gauge Theories*, Nucl. Phys. **B350** (1991) 325.

[73] A. Kovner and B. Rosenstein, *Topological Interpretation of Electric Charge, Duality and Confinement in (2 + 1)-Dimensions*, Int. J. Mod. Phys. **A7** (1992) 7419.

[74] I.I. Kogan and A. Kovner, *Compact QED in Three Dimensions: A Simple Example of a Variational Calculation in a Gauge Theory*, Phys. Rev. **D51** (1995) 1948.
[75] M. Asorey, F. Falceto and S. Carlip, *Chern-Simons States and Topologically Massive Gauge Theories*, Phys. Lett. B312 (1993) 477–485, hep-th/9304081.

[76] J. Polchinski, *String Theory* (Cambridge University Press, 1998).

[77] E. Verlinde, *Fusion Rules and Modular Transformations in 2D Conformal Field Theory*, Nucl. Phys. B300 (1988) 360.

[78] N. Ishibashi, *The Boundary and Crosscap States in Conformal Field Theories*, Mod. Phys. Lett. A4 (1989) 251.

[79] J.L. Cardy, *Boundary Conditions, Fusion Rules and the Verlinde Formula*, Nucl. Phys. B324 (1989) 581–596.

[80] J.L. Cardy and D.C. Lewellen, *Bulk and Boundary Operators in Conformal Field Theory*, Phys. Lett. B259 (1991) 274–278.