HYPERGEOMETRIC IDENTITIES IN ELLIPTIC SIGNATURE SIX

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ABSTRACT. Within the Ramanujan theories of elliptic functions, Li-Chien Shen constructed natural elliptic functions in signature three and signature four. When applied in signature six, the same constructions produce non-elliptic functions that nevertheless engender the corresponding hypergeometric identities of Ramanujan.

INTRODUCTION

In his second notebook, Ramanujan laid the foundations for his theory of elliptic functions to alternative bases, in which the hypergeometric function $F\left(\frac{1}{2}, \frac{1}{2}; 1; \bullet\right)$ familiar from classical elliptic function theory is replaced by one of three others, in what are now known as the theories of signature three, signature four, and signature six. See [1] for a substantial account of these developments.

The signature three theory involves $F\left(\frac{1}{3}, \frac{2}{3}; 1; \bullet\right)$. In this case, Shen [9] showed the presence of a certain elliptic function $dn_3$ within the theory; as one of several byproducts, his investigations yielded a fresh derivation of the Borwein cubic identity. Later, the elliptic function $dn_3$ was somewhat differently derived in [4] and considered at greater depth in [7].

The signature four theory involves $F\left(\frac{1}{4}, \frac{3}{4}; 1; \bullet\right)$. In this case, Shen [11] again showed the presence of an elliptic function $dn_2$ within the theory; a new derivation of some hypergeometric identities of Ramanujan was a byproduct of his investigations. Subsequently, the elliptic function $dn_2$ received an alternative treatment, briefly in [5] and more fully in [8].

The signature six theory involves $F\left(\frac{1}{6}, \frac{5}{6}; 1; \bullet\right)$. In this case, Shen [10] brought an elliptic function into the theory, but by an entirely different mechanism, again offering a new perspective on certain hypergeometric identities of Ramanujan. In [6] we showed that the approaches adopted by Shen in signature three and signature four yield functions that are not elliptic when adopted in signature six. Our chief purpose in this brief paper is complementary: to note that, though non-elliptic, these functions still lead to the same hypergeometric identities of Ramanujan; we also connect these non-elliptic functions to the very differently sourced elliptic functions of Shen in this signature.

SIGNATURE SIX

Throughout, our focus will be on real-valued functions of a real variable.

Fix a choice $\kappa \in (0, 1)$ of ‘modulus’. We define a function $f : \mathbb{R} \to \mathbb{R}$ by requiring that if $T \in \mathbb{R}$ then

$$f(T) = \int_0^T F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt.$$ 

As the continuous (indeed, analytic) integrand here is even and strictly positive, the function $f$ is odd and strictly increasing; $f$ is also surjective, as the integrand is periodic. Thus $f$ has an odd inverse function, which we denote by $\phi : \mathbb{R} \to \mathbb{R}$: so if $u \in \mathbb{R}$ then

$$u = \int_0^{\phi(u)} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt.$$
We now define an odd function \( s : \mathbb{R} \to \mathbb{R} \) by following \( \phi \) with the sine function, thus
\[
s = \sin \circ \phi;
\]
we may also similarly define an even function \( c : \mathbb{R} \to \mathbb{R} \) by
\[
c = \cos \circ \phi.
\]
Of course, these two functions satisfy the ‘Pythagorean’ identity
\[
c^2 + s^2 = 1
\]
along with
\[
s(0) = 0 \quad \text{and} \quad c(0) = 1.
\]
The functions \( s \) and \( c \) are counterparts to the Jacobian functions \( \text{sn} \) and \( \text{cn} \) familiar from the classical theory of elliptic functions. However, neither \( s \) nor \( c \) admits an elliptic extension: see [6] for this and more.

As an analogue of the classical complete elliptic integral, we introduce
\[
K = f(\frac{1}{2}\pi) = \int_0^{\frac{1}{2}\pi} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt,
\]
this being a function of the modulus \( \kappa \) (rather, of its square). Thus
\[
\phi(K) = \frac{1}{2}\pi
\]
so that
\[
s(K) = 1 \quad \text{and} \quad c(K) = 0.
\]

Not surprisingly, the functions \( s \) and \( c \) are periodic, with \( 4K \) as period. As a step towards the establishment of this fact, note first the identity
\[
\int_0^{\pi} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt = f(\pi) = 2K.
\]
In fact, the substitution \( t = \tau + \pi \) gives
\[
\int_0^{\pi} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt = \int_{-\pi}^{0} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \kappa^2 \sin^2 \tau\right) d\tau = K
\]
since \( \sin(\tau + \pi) = -\sin \tau \) and the integrand is even.

**Theorem 1.** If \( u \in \mathbb{R} \) then \( s(u + 2K) = -s(u) \) and \( c(u + 2K) = -c(u) \).

**Proof.** Let \( T = \phi(u) \). Splitting the integral,
\[
f(\pi + T) = \int_0^{\pi} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt + \int_{\pi}^{\pi + T} F\left(\frac{1}{6}, \frac{5}{6}; \frac{1}{2}; \kappa^2 \sin^2 t\right) dt.
\]
On the right, the first integral is \( 2K \) (as seen above) while the second reduces to \( f(T) \) after a \( \pi \) shift in the variable. Thus
\[
f(\pi + T) = 2K + f(T) = 2K + u
\]
and so
\[
\phi(2K + u) = \pi + T = \pi + \phi(u).
\]
The first identity of the Theorem follows upon applying the sine:
\[
s(2K + u) = \sin \circ \phi(2K + u) = \sin(\pi + \phi(u)) = -\sin(\phi(u)) = -s(u);
\]
the second follows upon application of the cosine. \( \square \)

**Theorem 2.** The functions \( s \) and \( c \) have period \( 4K \).

**Proof.** Let \( g \) stand for \( s \) or \( c \). If \( u \in \mathbb{R} \) then
\[
g(u + 4K) = g(u + 2K + 2K) = -g(u + 2K) = -(g(u)) = g(u).
\]
\( \square \)
The ‘complete integral’ \( K = f(\frac{1}{2}\pi) \) itself has a familiar value in hypergeometric terms.

**Theorem 3.** \( K = \frac{1}{2}\pi F(\frac{1}{6}, \frac{5}{6}, 1; \kappa^2) \).

**Proof.** Expand the hypergeometric integrand and integrate the resulting series term-by-term, using the standard integral formula

\[
\int_0^{\frac{\pi}{2}} \sin^{2n} t \, dt = \frac{1}{2^n} \pi \frac{(2n)!}{(2^n n!)}. 
\]

□

Up until now, the hypergeometric parameters need not have been \( \frac{1}{6} \) and \( \frac{5}{6} \); henceforth, we shall require the parameters to have these precise values.

An alternative explicit integral formula for \( K \) stems from the hypergeometric evaluation

\[
F(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; \sin^2 \varphi) = \frac{\cos \frac{2}{3} \psi}{\cos \psi} 
\]

for which we refer to page 101 in Volume 1 of the Bateman Manuscript Project [2].

For the following, let \( \alpha \in (0, \frac{1}{2}\pi) \) be the ‘modular’ angle defined by

\[
\kappa = \sin \alpha. 
\]

**Theorem 4.**

\[
K = \sqrt{2} \int_0^{\alpha} \frac{\cos \frac{2}{3} \psi}{\sqrt{\cos 2\psi - \cos 2\alpha}} \, d\psi. 
\]

**Proof.** In the definition

\[
K = \int_0^{\frac{\pi}{2}} F(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; \kappa^2 \sin^2 \varphi) \, d\varphi 
\]
change the integration variable from \( \varphi \in [0, \frac{1}{2}\pi] \) to \( \psi \in [0, \alpha] \) where

\[
\sin \psi = \kappa \sin \varphi. 
\]

The hypergeometric integrand evaluates as

\[
F(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; \kappa^2 \sin^2 \varphi) = F(\frac{1}{6}, \frac{5}{6}, \frac{1}{2}; \sin^2 \psi) = \frac{\cos \frac{2}{3} \psi}{\cos \psi} 
\]

alongside which the change of variable introduces the factor

\[
\frac{d\varphi}{d\psi} = \frac{\cos \varphi}{\kappa \cos \varphi} = \frac{\cos \psi}{\sqrt{\kappa^2 - \sin^2 \psi}} 
\]

in which trigonometric duplication gives

\[
\kappa^2 - \sin^2 \psi = \sin^2 \alpha - \sin^2 \psi = \frac{1}{2} (\cos 2\psi - \cos 2\alpha). 
\]

□

We shall now transform this integral expression for \( K \) until it assumes a decidedly elliptic appearance.

**Theorem 5.**

\[
K = \sqrt{\frac{3}{2}} \int_{\cos \frac{3}{2}(\pi - \alpha)}^{\cos \frac{3}{2}(\pi + \alpha)} \frac{dx}{\sqrt{4x^3 - 3x - (1 + 2\kappa^2)}}. 
\]
Proof. First of all, as the cosine function is even,
\[
\sqrt{\frac{2}{3}} K = \frac{1}{\sqrt{3}} \int_{-\alpha}^{\alpha} \frac{\cos \frac{2}{3} \psi}{\sqrt{\cos 2 \psi - \cos 2\alpha}} \, d\psi = \frac{2}{3} \sin \frac{2}{3} \pi \int_{-\alpha}^{\alpha} \frac{\cos \frac{2}{3} \psi}{\sqrt{\cos 2 \psi - \cos 2\alpha}} \, d\psi.
\]
Next, the addition formula for the (odd) sine function yields
\[
\sqrt{\frac{2}{3}} K = \frac{2}{3} \int_{-\alpha}^{\alpha} \frac{\sin \frac{2}{3} (\pi + \psi)}{\sqrt{\cos 2 \psi - \cos 2\alpha}} \, d\psi
\]
whence the substitution \( \theta = \pi + \psi \) leads to
\[
\sqrt{\frac{2}{3}} K = \frac{2}{3} \int_{\pi-\alpha}^{\pi+\alpha} \frac{\sin \frac{2}{3} \theta}{\sqrt{\cos 2 \theta - \cos 2\alpha}} \, d\theta.
\]
Finally, the substitution \( x = \cos \frac{2}{3} \theta \) produces the integral formula
\[
\sqrt{\frac{2}{3}} K = \int_{\cos \frac{2}{3} (\pi-\alpha)}^{\cos \frac{2}{3} (\pi+\alpha)} \frac{dx}{\sqrt{4x^3 - 3x - \cos 2\alpha}}
\]
wherein \( \cos 2\alpha = 1 - 2 \sin^2 \alpha = 1 - 2\kappa^2 \).

The idea of this proof comes straight from Theorem 3.2 in [10], though we have varied its place in the scheme of things.

As announced, this integral formula is elliptic in nature. Indeed, let \( \wp \) be the Weierstrass function with invariants
\[
g_2 = 3 \quad \text{and} \quad g_3 = 1 - 2\kappa^2
\]
so that
\[
(\wp')^2 = 4\wp^3 - 3\wp - (1 - 2\kappa^2) = 4\wp^3 - 3\wp - \cos 2\alpha.
\]
The invariants being real and the discriminant
\[
\Delta = g_2^3 - 27g_3^2 = 108\kappa^2 (1 - \kappa^2)
\]
being positive, \( \wp \) has a rectangular period lattice with a positive fundamental half-period \( \omega \).

**Theorem 6.** \( K = \sqrt{\frac{2}{3}} \omega \).

*Proof.* The roots \( e_1 > e_2 > e_3 \) of the cubic \( 4x^3 - 3x - (1 - 2\kappa^2) \) are precisely
\[
e_1 = \cos \frac{2}{3} \alpha, \quad e_2 = \cos \frac{2}{3} (\alpha - \pi), \quad e_3 = \cos \frac{2}{3} (\alpha + \pi).
\]
Simply substitute these values into the formula
\[
\omega = \int_{e_3}^{e_2} \frac{dx}{\sqrt{4x^3 - 3x - (1 - 2\kappa^2)}}
\]
for which we refer to Section 51 of the classic text [3] by Greenhill. \( \square \)

Having identified \( K \) in classical elliptic terms, we may of course also identify \( K \) in terms of the ‘classical’ hypergeometric function \( F\left(\frac{1}{2}, \frac{1}{2}; 1; \bullet\right) \).

For this, it is convenient to rescale the acute modular angle \( \alpha \) and define \( \beta \in (0, \frac{1}{3}\pi) \) by
\[
3\beta = 2\alpha.
\]

**Theorem 7.**
\[
K = (1 - k^2 + k^4)^{1/4} \frac{1}{\pi} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)
\]
where the (classical) modulus \( k \) is given by
\[
k^2 = \frac{2 \sin \beta}{\sin \beta + \sqrt{3} \cos \beta}.
\]
Proof. Further reference to Section 51 of [3] turns up the following formula for the positive fundamental half-period \( \omega \) of \( \nu \):

\[
\omega = \sqrt{\frac{1}{e_1 - e_3}} K\left( \frac{e_2 - e_3}{e_1 - e_3} \right)
\]

where \( K(k^2) \) is the complete elliptic integral with \( k \) as classical modulus, which evaluates hypergeometrically as

\[
K(k^2) = \frac{4}{\pi} F\left( \frac{1}{4}, \frac{1}{2}; 1; k^2 \right).
\]

The formulae

\[
g_2 = 2 (e_1^2 + e_2^2 + e_3^2) = -4 (e_2 e_3 + e_3 e_1 + e_1 e_2)
\]

relating the invariant \( g_2 \) to the midpoint values are standard: see Section 53 of [3] for example. With

\[
k^2 = \frac{e_2 - e_3}{e_1 - e_3}
\]

it follows that

\[
(e_1 - e_3)^2 (1 - k^2 + k^4) = (e_1^2 + e_2^2 + e_3^2) - (e_2 e_3 + e_3 e_1 + e_1 e_2) = \frac{4}{9} g_2 = \frac{9}{4}
\]

here, whence

\[
\sqrt{\frac{3}{2}} \sqrt{\frac{1}{e_1 - e_3}} = (1 - k^2 + k^4)^{1/4}.
\]

In the proof of Theorem 6 we recorded the midpoint values

\[
e_1 = \cos \beta, \quad e_2 = \cos \left( \beta - \frac{2}{3} \pi \right), \quad e_3 = \cos \left( \beta + \frac{2}{3} \pi \right).
\]

From these formulae, it follows that \( e_1 - e_3 = \sqrt{3} \sin (\beta + \frac{1}{4} \pi) \) and \( e_2 - e_3 = \sqrt{3} \sin \beta \), whence

\[
k^2 = \frac{\sin \beta}{\sin \left( \beta + \frac{1}{4} \pi \right)} = 2 \frac{\sin \beta}{\sin \beta + \sqrt{3} \cos \beta}.
\]

Assemble the pieces to conclude the proof. \( \square \)

We are now in possession of a hypergeometric identity that relates signature six to the classical signature. Direct comparison of Theorem 4 and Theorem 7 reveals that

\[
F\left( \frac{1}{6}, \frac{5}{6}; 1; \kappa^2 \right) = (1 - k^2 + k^4)^{1/4} F\left( \frac{1}{2}, \frac{1}{2}; 1; k^2 \right)
\]

where

\[
\kappa = \sin \alpha \text{ and } k^2 = 2 \frac{\sin \beta}{\sin \beta + \sqrt{3} \cos \beta}
\]

with

\[
2 \alpha = 3 \beta.
\]

We can recast this relationship into more familiar form, as the following hypergeometric identity.

**Theorem 8.** For the increasing bijection \((0,1) \rightarrow (0,1) : x \mapsto \xi \) given by

\[
4\xi (1 - \xi) = \frac{27}{4} \frac{x^2 (1 - x)^2}{(1 - x + x^2)^3}
\]

there holds the identity

\[
F\left( \frac{1}{6}, \frac{5}{6}; 1; \xi \right) = (1 - x + x^2)^{1/4} F\left( \frac{1}{2}, \frac{1}{2}; 1; x \right).
\]

**Proof.** Two increasing bijections are defined by

\[
(0, \frac{1}{2} \pi) \rightarrow (0,1) : \alpha \mapsto \sin^2 \alpha = \kappa^2 =: \xi
\]

and

\[
(0, \frac{1}{3} \pi) \rightarrow (0,1) : \beta \mapsto 2 \frac{\sin \beta}{\sin \beta + \sqrt{3} \cos \beta} = k^2 =: x.
\]
In light of the identity revealed just prior to this Theorem, we need only check that when the rescaling \(2\alpha = 3\beta\) is used to coordinate these bijections, there results the bijection of the Theorem. Thus, let \(0 < \beta < \frac{1}{3}\pi\): from

\[
\frac{1}{x} = \frac{1}{2} (1 + \sqrt{3} \cot \beta)
\]

it follows that

\[
\frac{1}{x^2} - \frac{1}{x} + 1 = \frac{1}{4} (3 + 3 \cot^2 \beta) = \frac{3}{4} \csc^2 \beta
\]

and therefore that

\[
\frac{3}{4} \frac{x^2}{1 - x + x^2} = \sin^2 \beta =: S
\]

say; now

\[
3 - 4S = 3 \frac{1 - x}{1 - x + x^2}
\]

so that by trigonometric triplication

\[
\frac{27}{4} \frac{x^2 (1 - x)^2}{(1 - x + x^2)^3} = S(3 - 4S)^2 = \sin^2 3\beta
\]

and the coordinating rescale gives

\[
\sin^2 3\beta = \sin^2 2\alpha
\]

where by trigonometric duplication

\[
\sin^2 2\alpha = 4 \sin^2 \alpha \cos^2 \alpha = 4\xi (1 - \xi).
\]

\(\square\)

We spoke of this hypergeometric identity as being familiar. In fact, it appears as equation (1.5) in [10]; the proof in Section 4 of that paper rests in part on the theory of theta functions.

Additional hypergeometric identities follow from this one. Note first that the bijection in Theorem 8 is invariant under the simultaneous replacements \(x \mapsto 1 - x\) and \(\xi \mapsto 1 - \xi\). Now, let \(0 < p < 1\): on the one hand, if we put

\[
x = \frac{p(2 + p)}{1 + 2p}
\]

and

\[
\xi = \frac{27}{4} \frac{p^2 (1 + p)^2}{(1 + p + p^2)^3}
\]

then

\[
1 - x + x^2 = \left( \frac{1 + p + p^2}{1 + 2p} \right)^2
\]

and Theorem 8 exactly reproduces Theorem 11.1 of [1]; on the other hand, if we make the aforementioned simultaneous replacements

\[
x = \frac{1 - p^2}{1 + 2p}
\]

and

\[
\xi = \frac{1}{4} \frac{(1 - p)^2 (1 + 2p)^2 (2 + p)^2}{(1 + p + p^2)^3}
\]

then \(1 - x + x^2\) remains the same and Theorem 8 exactly reproduces Corollary 11.2 of [1]. Of course, the proofs of these identities in [1] are rather different, being based instead on judicious manipulations of other hypergeometric identities recorded in the Bateman Manuscript Project [2] (specifically: (42) on page 114 with \(a = 1/12\); (32) on page 113 with \(a = b = 1/2\); (2) on page 111 with \(a = 1/12\) and \(b = 5/12\).
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