THE WARPED PLANE OF THE CLASSICAL KUIPER BELT

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ABSTRACT

By numerically integrating the orbits of the giant planets and of test particles over a period of four billion years, we follow the evolution of the location of the midplane of the Kuiper belt. The Classical Kuiper belt conforms to a warped sheet that precesses with a 1.9 Myr period. The present-day location of the Kuiper belt plane can be computed using linear secular perturbation theory: the local normal to the plane is given by the theory’s forced inclination vector, which is specific to every semimajor axis. The Kuiper belt plane does not coincide with the invariable plane, but deviates from it by up to a few degrees in stable zones. For example, at a semimajor axis of 38 AU, the local Kuiper belt plane has an inclination of 1.9 degrees and a longitude of ascending node of 149.9 degrees when referred to the mean ecliptic and equinox of J2000. At a semimajor axis of 43 AU, the local plane has an inclination of 1.9 degrees and a nodal longitude of 78.3 degrees. Only at infinite semimajor axis does the Kuiper belt plane merge with the invariable plane, whose inclination is 1.6 degrees and nodal longitude is 107.7 degrees. A Classical Kuiper belt object keeps its inclination relative to the Kuiper belt plane nearly constant, even while the plane departs from the trajectory predicted by linear theory. The constancy of relative inclination reflects the undamped amplitude of free oscillation; that is, the homogeneous solution to the forced harmonic oscillator equation retains constant amplitude, even while the inhomogeneous solution cannot be written down accurately because the planetary forcing terms are chaotic. Current observations of Classical Kuiper belt objects are consistent with the plane being warped by the giant planets alone, but the sample size will need to increase by a few times before confirmation exceeds 3σ in confidence. In principle, differences between the theoretically expected plane and the observed plane could be used to infer as yet unseen masses orbiting the Sun, but carrying out such a program would be challenging.

Key words: comets: general – celestial mechanics – Kuiper Belt – solar system: general

1. INTRODUCTION

If we could map, at fixed time, the instantaneous locations in three-dimensional space of all Kuiper belt objects (KBOs), on what two-dimensional surface would the density of KBOs be greatest? We call this surface the plane of the Kuiper belt (KBP), though by “plane” we do not mean to imply that the KBP is flat (we shall find that it is not). The KBP depends on the mass distribution of the solar system—principally, the orbits and masses of the giant planets. There are as many different KBPs as there are dynamical classes of KBO, since each class of object feels a distinct time-averaged force. Here we study the KBP defined by classical KBOs: objects whose fairly circular, low-inclination orbits are not in any strong mean-motion resonance with Neptune (see Elliot et al. 2005 for a classification scheme).

In principle, theoretical determination of the KBP would help observers to discover new KBOs. Conversely, by measuring differences between the theoretical KBP and the actual KBP, we might hope to infer the presence of solar system bodies as yet undetected (“Planet X”; see Gaudi & Bloom 2005 for a summary of current limits).

There is disagreement regarding the location of the KBP. Brown & Pan (2004) analyzed the instantaneous proper motion vectors of hundreds of KBOs irrespective of dynamical class and concluded, with greater than 3σ confidence, that the KBP did not coincide with the invariable plane (IP, the plane perpendicular to the total angular momentum vector of the solar system). They argued that the observed KBP was consistent instead with the forced plane given by linear secular perturbation theory. We will refer to this plane as the BvWP, after Brouwer & van Woerkom (1950), who developed a linear secular theory for the motions of all eight of the major planets. Their theory, in turn, has its origin in the Laplace–Lagrange equations (see, e.g., Murray & Dermott 1999). The BvWP is a warped and time-variable surface whose properties we review in Section 2.

By contrast, Elliot et al. (2005) found that the plane determined by classical KBOs that were observed over multiple epochs was more consistent with the IP (\( \simeq 1\sigma \) difference) than with the BvWP (\( \lesssim 2–3\sigma \) difference). They listed some arguments, none conclusive, for why the IP might be preferred over the BvWP. The low order of the BvW theory, and its inability to account for time variations in semimajor axes, are causes for concern.

We seek to resolve this disagreement using numerical orbit integrations. In Section 2, we review the linear theory and how it equates the KBP with the BvWP. In Section 3, numerical integrations lasting the age of the solar system are used to reveal the theoretical location of the KBP. In Section 4, we compare theory against current observations of the KBP. A summary is given in Section 5, including a brief comment on the prospects for detecting an unseen, outer solar system planet using the KBP.

2. LINEAR SECULAR THEORY

We study the secular evolution of Classical KBOs by solving the Laplace–Lagrange equations of motion, which neglect all terms higher than second order in orbital eccentricity (\( e \)) and orbital inclination (\( i \)). The solution is detailed in the textbook by Murray & Dermott (1999); we provide a summary here.

We start by describing the motions of the planets. Define an inclination vector \( \mathbf{i} \equiv (q, p) \equiv (i \cos \Omega, i \sin \Omega) \), where \( i \) and \( \Omega \) equal the inclination and longitude of ascending node. Lagrange’s equations governing the inclination vector for the
or $\alpha_{jk}$ of negligible mass with semimajor axis of the $j$th planet are

$$\dot{q}_j = -\frac{1}{n_j a_j^2} \frac{\partial R_j}{\partial q_j}, \quad \dot{p}_j = \frac{1}{n_j a_j^2} \frac{\partial R_j}{\partial p_j},$$

where $n_j$, $a_j$, and $R_j$ are, respectively, the mean motion, semimajor axis, and disturbing function. We consider only the four giant planets so that $j = 1, 2, 3, 4$ represents Jupiter, Saturn, Uranus, and Neptune, respectively. The disturbing function, keeping only the leading terms relevant to the inclination evolution of the $j$th planet, reads

$$R_j = -\frac{n_j^2 a_j^2}{8} \sum_{k=1, k \neq j}^4 \frac{m_k}{M_\odot + m_j} \alpha_{jk} b_{jk}^{(1)}(\alpha_{jk}) \left[ q_j^2 + p_j^2 - 2(q_j q_k + p_j p_k) \right],$$

where $b_{jk}^{(1)}(\alpha_{jk})$ is a Laplace coefficient, $m_j$ is the mass of the $j$th planet,

$$\alpha_{jk} = \begin{cases} a_h/a_j & \text{if } a_j > a_k \\ a_j/a_k & \text{if } a_j < a_k \end{cases},$$

and

$$\tilde{\alpha}_{jk} = \begin{cases} 1 & \text{if } a_j > a_k \\ a_j/a_k & \text{if } a_j < a_k \end{cases}.$$

Equations (1)–(2) yield two coupled systems of first-order differential equations: $(\dot{p}_1, \dot{p}_2, \dot{p}_3, \dot{p}_4)^T = A(q_1, q_2, q_3, q_4)^T$ and $(\dot{q}_1, \dot{q}_2, \dot{q}_3, \dot{q}_4)^T = -A(p_1, p_2, p_3, p_4)^T$, where $A$ is a $4 \times 4$ matrix of constant coefficients depending on the masses and semimajor axes of the planets. These equations describe coupled harmonic oscillators; their solution is

$$q_j = \sum_{k=1}^4 I_{jk} \cos(f_k t + \gamma_k), \quad p_j = \sum_{k=1}^4 I_{jk} \sin(f_k t + \gamma_k),$$

where the frequencies $f_k$ are the eigenvalues of $A$. The elements $I_{jk}$ of the eigenvectors of $A$, and the phases $\gamma_k$, are fitted to the initial inclinations and ascending nodes of the planets. We take initial conditions and planetary data from the NASA JPL Horizons database for JD = 2451544.5 (2000 January 1; $t = 0$ in Equations (3) and (4)). Table 1 lists the resultant values for $f_k$, $I_{jk}$, and $\gamma_k$. We refer to this solution as the BvW solution, even though Brouwer & van Woerkom (1950) included all eight planets.

We now turn to our main concern, the motion of a KBO of negligible mass with semimajor axis $a > a_j$. Its disturbing function is

$$R = -\frac{n^2 a^2}{8} \sum_{j=1}^4 \frac{m_j}{M_\odot} a_j b_{jk}^{(1)}(a_j/a)|q_j^2 + p_j^2 - 2(q_j q_k + p_j p_k)|,$$

Table 1

| $k$ | $f_k$ | $\gamma_k$ |
|-----|-------|------------|
| 1   | 1.5792 | 0.36081 |
| 2   | 1.5792 | -0.89004 |
| 3   | 1.5792 | 0.04067 |
| 4   | 1.5792 | -0.004512 |

Notes.

* The components $I_{jk}$ of the eigenvectors (in degrees), eigenfrequencies $f_k$ (in radians per Myr), and phases $\gamma_k$ (in radians), calculated using data for the giant planets on JD = 2451544.5 from NASA JPL Horizons. All elements are heliocentric and referred to the ecliptic and mean equinox of J2000.

where unsubscripted variables refer to the KBO. Because all masses and semimajor axes are fixed for this secular problem, Lagrange’s equations of motion are of the form

$$\dot{q} = -\frac{1}{n a^2} \frac{\partial R}{\partial p} = -c_0 p + \sum_{j=1}^4 c_j p_j,$$

$$\dot{p} = \frac{1}{n a^2} \frac{\partial R}{\partial q} = +c_0 q - \sum_{j=1}^4 c_j q_j,$$

where the $c$ are constants. Substituting each equation into the time derivative of the other, we find

$$\ddot{q} = -c_0^2 q + \sum_{j=1}^4 c_j [c_0 q_j(t) + p_j(t)],$$

$$\ddot{p} = -c_0^2 p + \sum_{j=1}^4 c_j [c_0 p_j(t) - \dot{q}_j(t)].$$

Equations (5)–(6) describe harmonic oscillators of natural frequency $c_0$, forced by the planetary terms in the sums. The motion is composed of a forced oscillation and a free oscillation:

$$q = q_{\text{forced}} + q_{\text{free}} = q_{\text{forced}} + i_{\text{free}} \cos(ft + \gamma),$$

$$p = p_{\text{forced}} + p_{\text{free}} = p_{\text{forced}} + i_{\text{free}} \sin(ft + \gamma),$$

where $i_{\text{free}}$ and $\gamma$ are constants determined by initial conditions, and the free precession frequency

$$f = f_0 = \sum_{j=1}^4 c_j = -\frac{n}{4} \sum_{j=1}^4 \frac{m_j}{a^2} a_j b_{jk}^{(1)}(a_j/a).$$

The functions $q_{\text{forced}}$ and $p_{\text{forced}}$ depend only on planetary parameters and the KBO semimajor axis $a$:

$$q_{\text{forced}} = -\frac{\mu k}{f - f_k} \cos(f_k t + \gamma_k),$$

$$p_{\text{forced}} = -\frac{\mu k}{f - f_k} \sin(f_k t + \gamma_k).$$

If we declare $a_{jk} = \tilde{\alpha}_{jk} = a_j/a_k$ and allow the Laplace coefficient to take $a_{jk} > 1$ as an argument, then separating the case $a_j < a_k$ from $a_j > a_k$ is not necessary. We stick here with the textbook convention, however.
and either above or below, the separation between the forced poles point normal to the Kuiper belt plane. The free inclination is constant; it is the undamped amplitude of free oscillation, and the forced inclination vector \( \Omega_{\text{forced}} \equiv \arctan(p_{\text{forced}}/q_{\text{forced}}) \), a free inclination \( \Omega_{\text{free}} \equiv \arctan(p_{\text{free}}/q_{\text{free}}) \). Thus free and forced inclinations refer to magnitudes (not vectors). The free inclination is constant; it is the undamped amplitude of the free oscillation.

Figure 1 shows the evolution of \( i_{\text{forced}} \) at various semimajor axes in the Kuiper belt. The open diamonds, located at the endpoints of \( i_{\text{forced}} \), mark what we call the forced poles. At fixed time, the forced poles all lie along a line intersecting the invariable pole, denoted I. To the extent that the \( k = 4 \) mode, driven mainly by Neptune, dominates, the forced poles rotate clockwise (regress) about the invariable pole at a single frequency \( f_4 \), maintaining constant distance to I. Note that the forced poles at \( a > 40.5 \) AU lie diametrically opposite to those at \( a < 40.5 \) AU, and that as \( a \) approaches 40.5 AU from either above or below, the separation between the forced pole and I increases. These latter two properties reflect the \( \nu_2 \) secular resonance at \( a = 40.5 \) AU, where the denominators \( f - f_4 \) of Equations (9)-(10) vanish: the forced response becomes infinite in magnitude at resonance, and changes phase by 180° across resonance (the sign of \( f - f_4 \) switches across \( a = 40.5 \) AU).

Figure 2 illustrates how free nodes \( \Omega_{\text{free}} \) phase mix and how such phase mixing helps determine the KBP within the BvW theory. At \( t = 0 \), one hundred KBOs having semimajor axes within 0.5 AU of 43 AU are set down with orbit normals approximately aligned about an arbitrary direction (the orbit poles are actually distributed in a small box in \( p-q \) space). Over tens of Myr, the orbit poles of the particles drift away from one another: the small dispersion in semimajor axis produces a small dispersion in the free precession frequency \( f \) (the rate at which the free inclination vector rotates about the forced pole). While the free nodes distribute themselves over all phases, the free inclinations \( i_{\text{free}} \) remain fixed. Thus, at late times, the collection of free inclination vectors \( i_{\text{free}} \) are distributed axisymmetrically about the mean forced pole at \( a \approx 43 \) AU. The axisymmetry arises practically independently of how the particles’ orbit normals are initially distributed; the only requirements are that many particles share the same \( i_{\text{free}} \) (so that such particles, when phase mixed, trace a full circle) and that there exists a small but non-zero dispersion in semimajor axis (so that there exists a non-zero spread in free precession frequencies, enabling phase mixing). Note in Figure 2 how the invariable pole does not coincide with the center of the circular distribution.

We conclude that according to the BvW theory, the mean orbit normal of a phase-mixed group of particles having approximately the same semimajor axis is given by the forced pole corresponding to that semimajor axis. That local forced pole varies with time (Figure 1), but the particles always encircle it (Figure 2). The KBP is warped because the forced pole changes direction with the KBO semimajor axis (Figure 1).

Note finally that the ability of test particles to keep their free inclinations constant relative to a time-variable forced pole should not be confused with adiabatic invariance. Generally the frequencies \( f \) and \( f_4 \) are not cleanly separated. The constant \( i_{\text{free}} \) simply reflects the undamped amplitude of free oscillation, and

\[
\mu_k = -\sum_{j=1}^{4} c_j I_{jk} = \frac{n}{4} \sum_{j=1}^{4} I_{jk} \frac{a_j}{M_\odot} \frac{a_{j(1)}}{a_{j/2}} (a_j/a).
\]

The inclination vector \( i \) of a KBO is the vector sum of a forced inclination vector \( i_{\text{forced}} \equiv (q_{\text{forced}}, p_{\text{forced}}) \) and a free inclination vector \( i_{\text{free}} \equiv (q_{\text{free}}, p_{\text{free}}) \). Throughout this paper, we refer to a forced inclination \( i_{\text{forced}} \equiv (q_{\text{forced}}^2 + p_{\text{forced}}^2)^{1/2} \), a forced node \( \Omega_{\text{forced}} \equiv \arctan(p_{\text{forced}}/q_{\text{forced}}) \), a free inclination \( i_{\text{free}} \equiv (q_{\text{free}}^2 + p_{\text{free}}^2)^{1/2} \), and a free node \( \Omega_{\text{free}} \equiv \arctan(p_{\text{free}}/q_{\text{free}}) \). Thus free and forced inclinations refer to magnitudes (not vectors). The free inclination is constant; it is the undamped amplitude of the free oscillation.
is set by initial conditions. In the next section, we use numerical simulations to test the constancy of free inclination.

3. NUMERICAL INTEGRATIONS

3.1. Initial Conditions

We calculate the evolution of the KBP at three semimajor axes: \( a = 38, 43, \) and \( 44 \) AU. These are chosen to lie away from strong mean-motion resonances (e.g., the 3:2 resonance resides at \( 39.5 \) AU) and outside the \( a = 40–42 \) AU region of instability carved by the \( \nu_{18}, \nu_{17} \) and \( \nu_{8} \) secular resonances (see, e.g., Chiang et al. 2007). At each \( a \) we lay down \( N_{i} \times N_{i} \) test particles whose initial free inclination vectors are distributed axisymmetrically about the local forced pole. That is, each particle’s initial \( p = p_{\text{forced}} + p_{\text{free}} \) and initial \( q = q_{\text{forced}} + q_{\text{free}}, \) where \( i_{\text{free}} = \left( p_{\text{free}}^{2} + q_{\text{free}}^{2} \right)^{1/2} \) takes 1 of \( N_{i} = 4 \) values (0.01, 0.03, 0.1, 0.3 rad) and \( \Omega_{\text{free}} = \arctan(p_{\text{free}}/q_{\text{free}}) \) takes 1 of \( N_{\Omega} = 20 \) values distributed uniformly between 0 and \( 2\pi. \) This set-up permits us to directly test the BvW theory, which predicts that all \( N_{\Omega} \) particles corresponding to a given \( a \) and given initial \( i_{\text{free}} \) should keep the same \( i_{\text{free}} \) for all time: a circle of points in \( p-q \) space should continue to trace the same-sized circle. Coordinates \( p_{\text{forced}} \) and \( q_{\text{forced}} \) for the initial forced poles are computed using the BvW solution of Section 2, for \( t = 0. \)

Initial osculating eccentricities are zero and initial mean anomalies are chosen randomly between 0 and \( 2\pi. \) The four giant planets are included in the integration, with initial conditions taken from the JPL Horizons database for JD = 2451544.5 (\( t = 0 \) in the BvW theory). The integration is performed with the swift_rnvs3 code, written by Levison & Duncan (1994) and based on the algorithm developed by Wisdom & Holman (1991). The duration of the integration is 4 Gyr and the timestep is 400 days (about 1/11 the orbital period of Jupiter). We work in a heliocentric coordinate system, the better to compare with the linear secular theory which uses heliocentric elements.

3.2. Results

According to the BvW theory, each set of \( N_{\Omega} = 20 \) particles having the same initial \( a \) and initial \( i_{\text{free}} \) should trace the perimeter of a single circle in \( p-q \) space at any given time, with the center of the circle yielding the local normal to the KBP. Figure 3 tests this prediction; the panels display the \( p-q \) positions of particles initially having \( a = 43 \) AU and \( i_{\text{free}} = 0.1 \) rad, sampled at four different times. Most particles at a given time do lie approximately on one circle, although there are outliers (see crosses in panels (b), (c), and (d)). The outliers represent particles whose inclinations and eccentricities grow to large values. Many of these particles undergo close encounters with Neptune, whereupon they are removed from the integration. We did not identify the cause of the instability, but probably the various high-order mean-motion resonances in the vicinity (Nesvorný & Roig 2001) are to blame. Of the 20 particles shown at \( t = 0 \) in panel (a), only 12 survive to \( t = 4 \) Gyr in panel (d).

Those that survive have semimajor axes that remain constant to within \( \pm 0.5 \) AU.

At each \( t, \) we fit a circle (in a least-squares sense) to those particles known to survive the entire length of the integration. The fit is substantially improved by also discarding particles whose instantaneous eccentricities exceed some value \( e_{\text{cut}} \), as we find that particles on eccentric orbits tend also to be outliers in \( p-q \) space. The value of \( e_{\text{cut}} \) is reduced from one until the fit parameters cease to change significantly. Note that a particle

![Figure 3](https://example.com/figure3.png)

**Figure 3.** Tracking the Kuiper belt plane by numerical integration. In panel (a), we set up \( N_{i} = 20 \) test particles having initial semimajor axes of \( a = 43 \) AU and having orbit poles distributed in a cone of half-width \( i_{\text{free}} = 0.1 \) rad centered on the local forced pole given by the BvW solution. In panels (b) through (d), the orbit poles evolve according to our numerical integration. Solid circles denote test particles that both survive the entire 4 Gyr duration of the integration and have osculating eccentricities less than \( e_{\text{cut}} = 0.08; \) these are fitted to a circle, whose center yields the local normal to the Kuiper belt plane (solid diamond). The “×” symbols denote test particles that do not satisfy these requirements. The Kuiper belt pole so obtained follows closely that predicted by the continuously updated BvW solution (open diamond), and does not point along the invariant pole (“”).

that is discarded from the fit by the \( e_{\text{cut}} \)-criterion at one time can be restored to the fit at a later time (if its eccentricity falls below \( e_{\text{cut}} \) at that later time). Table 2 lists the values of \( e_{\text{cut}} \) chosen for the various combinations of initial \( a \) and initial \( i_{\text{free}}. \)

The circles so fitted are overlaid in Figure 3. A typical fit is excellent and the local normal of the KBP (center of the fitted circle; solid diamond) is confidently identified. Figures 4, 5, and 6 report analogous results for other choices of initial \( a \) and initial \( i_{\text{free}}. \) All 20 particles having \( a = 43 \) AU and \( i_{\text{free}} = 0.3 \) rad are stable for 4 Gyr (Figure 4). By contrast, at \( a = 38 \) AU, where various high-order mean-motion resonances are known to cause instability (Nesvorný & Roig 2001), only two out of 20 particles having \( i_{\text{free}} = 0.3 \) rad remain at the end of 4 Gyr (Figure 6). Fitting a unique circle to two points is impossible. And as is clear from Figure 6, even if we were to try fitting circles at earlier times when more particles are present, such fits would be poor. Only certain combinations of \( a-i_{\text{free}} \) permit determination of the KBP, as documented in our Table 2.

Panels (d) of Figures 7 and 8 plot the radii of the fitted circles versus time, for two choices of initial \( a-i_{\text{free}}. \) The BvW linear theory predicts that these radii should be constant. In fact they are nearly so, varying by at most one part out of six.

How can we further test the BvW theory when we know that it fails to predict the orbits of the planets on Gyr timescales? We compute instead a semi-analytic, BvW-based solution as follows. At each time in the numerical integration, we output the inclinations and ascending nodes of the giant planets and use these to recompose the eigenvectors—and thus the forced poles—of the linear theory. Thus we obtain a prediction of where the KBP should reside according to the linear theory at a given instant, using the simulation results for the planets at that instant to supply the integration constants. This “continuously
Table 2

| $a$ (AU) | $i_{\text{free}}$ (rad) | Number of survivors (out of 20) | $e_{\text{max}}$ of survivors | $e_{\text{cut}}$ | Permits KBP determination? |
|---------|-------------------------|---------------------------------|-------------------------------|------------------|-----------------------------|
| 38      | 0.01                    | 17                              | 0.05                          | None applied     | Yes                         |
|         | 0.03                    | 16                              | 0.05                          | None applied     | Yes                         |
|         | 0.10                    | 8                               | 0.07                          | None applied     | No                          |
|         | 0.30                    | 2                               | 0.02                          | None applied     | No                          |
| 43      | 0.01                    | 19                              | 0.12                          | 0.04             | Yes                         |
|         | 0.03                    | 20                              | 0.04                          | None applied     | Yes                         |
|         | 0.10                    | 12                              | 0.12                          | 0.08             | Yes                         |
|         | 0.30                    | 20                              | 0.02                          | None applied     | Yes                         |
| 44      | 0.01                    | 20                              | 0.11                          | 0.03             | Yes                         |
|         | 0.03                    | 20                              | 0.11                          | 0.04             | Yes                         |
|         | 0.10                    | 20                              | 0.15                          | 0.06             | Yes                         |
|         | 0.30                    | 20                              | 0.29                          | 0.08             | Yes                         |

Figure 4. Same as Figure 3, except that initial $a_K = 43$ AU, initial $i_{\text{free}} = 0.3$ rad, and no $e_{\text{cut}}$ criterion is applied. The numerically fitted Kuiper belt pole (solid diamond) tracks the continuously updated BvW pole (open diamond) well; both precess about the invariable pole ("+").

Figure 5. Same as Figure 3, except that initial $a_K = 38$ AU, initial $i_{\text{free}} = 0.1$ rad, and no $e_{\text{cut}}$ criterion is applied. Here again the numerically determined Kuiper belt pole (solid diamond) is well described by the continuously updated BvW pole (open diamond), not the invariable pole ("+").

Figure 6. Same as Figure 3, except that initial $a_K = 38$ AU, initial $i_{\text{free}} = 0.3$ rad, and no $e_{\text{cut}}$ criterion is applied. Only two test particles survive the 4 Gyr duration of the integration, rendering determination of the Kuiper belt pole impossible.

4. THEORY VERSUS OBSERVATION

We have shown by numerical simulations in Section 3 that the classical KBP is given by the BvWP, i.e., by linear secular theory. Here we assess whether observations of KBOs bear out this result, by locating the actual poles of the KBP near the updated BvW solution" for the local normal (forced inclination vector) is shown as an open diamond in Figures 3–6. Its location tracks that of the numerically fitted pole (solid diamond) well—much better than does the invariable pole (upright cross); see especially Figures 3 and 5. Figures 7 and 8 also demonstrate that the continuously updated BvWP hews closely to the numerically fitted KBP.

To summarize the results of our numerical simulations: linear secular theories like BvW correctly predict how the warped KBP evolves with time, provided the parameters of those analytic theories are continuously updated using either observations or numerical simulations of the giant planets' orbits. The KBP today is accurately predicted by the updated BvW solution. The KBP does not, in general, coincide with the invariable plane, except at infinite distance from the planets.
Figure 7. Results for test particles having initial $a_K = 43$ AU and initial $i_{\text{free}} = 0.1$ rad, demonstrating that the numerically obtained KBP follows the continuously updated BvWP, not the IP. (a) Inclination of the KBP (solid circles), BvWP (open circles), and IP (line), all relative to the ecliptic. (b) Mutual inclination between the KBP and BvWP (solid diamonds), the BvWP and the IP (open diamonds), and the KBP and the IP (solid line). (c) Longitude of ascending node of the KBP (solid circles), BvWP (open circles), and IP (line), all relative to the ecliptic and mean equinox of J2000. (d) Numerically obtained free inclination (solid circles; these are the radii of the fitted circles in Figure 3), compared against the initial $i_{\text{free}}$ (line). The BvW theory predicts that the free inclination should be constant; while it is nearly so in our numerical integration, its mean value is offset by 0.6 deg (10%) relative to its initial value. Data are sampled every $10^8$ yr and do not resolve the precession of the KBP occurring with a $\sim 10^6$ yr period (see Figure 10).

Figure 8. Same as Figure 7, except for initial $a_K = 38$ AU and initial $i_{\text{free}} = 0.1$ rad.

38 AU and 43 AU. These semimajor axes lie to either side of the $v_{18}$ resonance; theory predicts that the corresponding poles should lie to either side of the invariable pole, with all three orbit normals lying in one plane (see Figure 1). Our dataset consists of KBOs listed on the Minor Planet Center Web site on 2008 January 22 whose (a) astrometric arcs extend longer than 50 days (many objects in our sample have much longer arcs), (b) eccentricities are less than 0.1, (c) inclinations are less than 10°, and (d) are classified by the Deep Ecliptic Survey (DES) as “Classical” in at least two out of their three orbital integrations (see Elliot et al. 2005 for a description of the classification scheme). In the vast majority of cases, all three DES integrations yield a classification of “Classical.” Moreover, the typical $3\sigma$ uncertainties in semimajor axes are
Table 3

| Sample | Objects |
|--------|---------|
| 38 AU  | 1998 WV24, 1999 OJ4, 2000 YB2, 82157, 2003 FD128, 2003 QA92, 2003 YL179, 2003 QO91, 144897, 119951 |
| 43 AU  | 19255, 1994 EV5, 1996 TK66, 33001, 1998 WY24, 1998 WX24, 1999 CN153, 1999 RT214, 1999 ON4, 1999 XY143, 1999 RW214, 1999 CH154, 1999 RU215, 1999 HV11, 1999 DA, 1999 HJ12, 1999 CW131, 2000 PU29, 2000 PX29, 134860, 2000 CU105, 2000 ON67, 2000 PC30, 2000 FS53, 2000 WV12, 2000 WL183, 2000 OU69, 2000 YU1, 88268, 2001 QB298, 2001 QD298, 2001 XR254, 2001 OQ297, 2001 HZ58, 2001 RW131, 2001 OK108, 2001 DB106, 88267, 2001 XU254, 2001 FK185, 2001 OZ108, 2002 CD251, 2002 PX170, 2002 PV170, 2002 FW26, 2002 WL21, 160256, 2002 VB131, 2002 PY170, 2002 CS154, 2002 PD155, 2003 SN137, 2003 UT291, 2003 FK127, 2003 QG91, 2003 FA130, 2003 HY56, 2003 QY90, 2003 TK58, 2003 QF91, 2003 QL01, 2003 QL12, 2003 YR179, 2003 QY111, 2003 QD03, 2003 TL58, 2003 QU90, 2003 YT179, 2003 XY179, 2003 YJ179, 2004 UD10, 2004 DM71, 2005 JZ174, 2005 GD87, 2005 JP179, 2005 XU100, 2006 HA123 |

Figure 9. Pole positions of observed classical KBOs having semimajor axes near 38 AU (open circles) and 43 AU (filled circles), in \( q = i \cos \Omega, p = i \sin \Omega \) space, referenced to the J2000 ecliptic. According to theory (Sections 2–3), the average pole position at a given semimajor axis (triangles with error bars) should match the forced pole positions (diamonds) calculated from BvW. They do to within \( 2\sigma \) (error bars are \( \pm 3\sigma \), where \( \sigma \) is the standard deviation of the mean). Unfortunately, the alternative hypothesis that the average pole positions are given by the invariable pole (bold cross) cannot be ruled out with greater confidence.

We assemble two samples, one for which \( 38.09 \text{ AU} < a < 39.10 \text{ AU} \) (the “38 AU” sample, containing 10 objects) and another for which \( 42.49 \text{ AU} < a < 43.50 \text{ AU} \) (the “43 AU” sample, containing 80 objects). Object designations are given in Table 3.

Figure 9 plots the observed \((q, p)\) positions for the two samples, and compares with theory. Theory predicts that the average \((\bar{q}, \bar{p})\) measured for each sample should equal \((q_{\text{forced}}, p_{\text{forced}})\) calculated for the average semimajor axis of the sample. The good news is that the data are consistent with this prediction; the differences between the observed poles and the forced poles are less than \( 2\sigma \) for both samples (the error bars in Figure 9 are \( \pm 3\sigma \), where \( \sigma \) is the standard deviation of the mean). The bad news is that the observed poles are also consistent with the invariable pole, at a similar confidence level. While it is encouraging for the theory that the observed pole at 38 AU indeed lies to the left (toward smaller \( q \)) of the invariable pole, and that the observed pole at 43 AU lies to the right, there are not enough observations to make more precise statements and to rule out the hypothesis that the KBP equals the IP with greater than \( 3\sigma \) confidence.

Note that by selecting our sample to have orbital inclinations less than \( 10^\circ \) with respect to the ecliptic plane, we bias our measurement of the average pole position towards the ecliptic pole. This systematic error is probably still smaller, however, than our random error. For example, the observed pole at 38 AU actually lies further from the ecliptic pole than does the theoretically expected forced pole. See Elliot et al. (2005) for ways of reducing this systematic bias.

We also performed a two-dimensional Kolmogorov–Smirnov test to see whether the \((q, p)\) distributions for the two samples differ (they should). The probability that they do not is 4.9%—
small enough to be suggestive of a real difference, but in our judgement too large to be conclusive.

5. SUMMARY AND DISCUSSION

Classical Kuiper belt objects trace a sheet that warps and precesses in response to the planets. The current location and shape of this sheet—the “plane” of the classical Kuiper belt—can be computed using linear secular theory, with the observed masses and current orbits of the planets as input parameters (Brown & Pan 2004). The local normal to the plane is given by the theory’s forced inclination vector, which is specific to every semimajor axis. At infinite distance from the planets, the plane coincides with the invariable plane. The deviations of the Kuiper belt plane away from the invariable plane, while generally non-zero, are typically small: less than 3° outside the secularly unstable gap at \( a \approx 40.5 \pm 1 \) AU (inside the gap no KBOs have been observed, as expected). As the semimajor axis varies from \(<40.5 \) AU to \(>40.5 \) AU, the ascending node of the Kuiper belt plane on the invariable plane rotates by very nearly 180°, a result of the sign change in the forced response across the \( \nu_{18} \) resonance.

These conclusions are supported by our numerical integrations of giant planet and test particle orbits lasting 4 Gyr. These integrations show that a Kuiper belt object maintains a nearly fixed orbital inclination with respect to the time-variable Kuiper belt plane. This accords with linear theory, in which the free inclination represents the undamped, constant amplitude of a test particle’s free oscillation. It may seem surprising that the linear theory is vindicated in this regard while it cannot accurately predict planetary motions and hence the future location of the Kuiper belt plane. But the inaccuracies accrue only slowly—Figure 10 shows that it takes several precession periods for the linear theory to diverge from numerical simulation in predicting the location of the plane, and the analytic and numerical solutions are always qualitatively similar. Referring back to the equations of motion (5) and (6), we see that if we take the planetary forcing terms \( q_j(t) \) and \( p_j(t) \) to be given by the more realistic numerical integration—instead of the inaccurate but qualitatively similar analytic solution (3) and (4)—then the free component of the motion (the homogeneous solution of the differential equation) would still be given by (7) and (8), irrespective of the forced component (the inhomogenous solution). Thus an object can keep its free inclination with respect to the forced plane fixed, even though the location of the forced plane itself cannot be forecast analytically.

Currently the data on actual KBOs are consistent with, but do not conclusively verify, our theoretical finding that the Kuiper belt plane warps by a few degrees to either side of the secularly unstable gap. Our analysis of the observations, like that of Elliot et al. (2005), suffers from large random errors. The study by Brown & Pan (2004) does not, but at the expense of including objects of all dynamical classes and not following variations in the pole position with semimajor axis. Quadrupling the sample size of low-\(i\), low-\(e\) objects at 38 AU, where there are currently ten usable objects, can increase our confidence in the reality of the warp to greater than 3\( \sigma \).

Detecting “Planet X” via its influence on the Kuiper belt plane will be substantially more challenging. We calculate that a 100-\(M_\oplus\) planet with a semimajor axis of 300 AU and an orbital inclination with respect to the ecliptic of 10° would shift the forced poles in the 43–50 AU region by 0.1°.

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