Panorama of zeta functions

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Introduction

In this essay I will give a strictly subjective selection of different types of zeta functions. Instead of providing a complete list, I will rather try to give the central concepts and ideas underlying the theory.

Talking about zeta functions in general one inevitably is led to start with the Riemann zeta function $\zeta(s)$. It is defined as a Dirichlet series:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s},$$

which converges for each complex number $s$ of real part greater than one. In the same region it possesses a representation as a Mellin integral:

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_{0}^{\infty} \frac{1}{e^t - 1} t^{s} \frac{dt}{t}.$$
This integral representation can be used to show that $\zeta(s)$ extends to a meromorphic function on the complex plane and satisfies the functional equation:

$$\hat{\zeta}(s) = \hat{\zeta}(1 - s),$$

where $\hat{\zeta}$ is the completed zeta function

$$\hat{\zeta}(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

The zeta function has a simple pole at $s = 1$ and is regular otherwise. It has zeros at $-2, -4, \ldots$ which are called the trivial zeros. All other zeros lie in the strip $0 < \text{Re}(s) < 1$ and the famous Riemann Hypothesis states that they should all lie at $\text{Re}(s) = \frac{1}{2}$. This conjecture was stated in the middle of the 19th century and has not been proved to this day.

The Riemann Hypothesis is by no means the only riddle posed by the Riemann zeta function. There is, for example, the question about the spacing distribution of consecutive zeros: Let $(\rho_n)_{n \in \mathbb{N}}$ be the ascending sequence of imaginary parts of the zeros of $\zeta(s)$ in $\{\text{Im}(s) > 0\}$, and let $\tilde{\rho}_n = \frac{\rho_n \log \rho_n}{2\pi}$ be the normalized sequence. Let $\delta_n = \tilde{\rho}_{n+1} - \tilde{\rho}_n$ be the sequence of normalized spacings. Computations of pair correlation functions [18] and extensive numerical calculations [20] then lead one to expect that for any “nice” function $f$ on $(0, \infty)$

$$\frac{1}{N} \sum_{n=1}^{N} f(\delta_n) \to \int_{0}^{\infty} f(s)P(s)ds, \text{ as } N \to \infty.$$ 

Here $P(s)$ is the spacing distribution of a large random Hermitian matrix. The function $P(s)$ vanishes to second order at $s = 0$. So, unlike a Poisson process, the numbers $\tilde{\rho}_n$ “repel” each other.

This expectation is known as the “GUE hypothesis”, where GUE stands for Gaussian Unitary Ensemble and describes the spacing function $P$. The GUE hypothesis is widely accepted, but far from proven.

Using the uniqueness of the prime decomposition of a natural number the Dirichlet series can be turned into an Euler product:

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

which is extended over all prime numbers $p$ and converges for real part of $s$ greater than one. The Euler product indicates that the zeta function may be
viewed as a means to encode the infinite set of data given by the prime numbers into a single object, the meromorphic function $\zeta$. Indeed, the analytic behaviour of the zeta function is exploited in the proof of the prime number theorem [12], which says that the counting function of prime numbers

$$\pi(x) = \#\{p \leq x, \text{ p prime}\}$$

has the asymptotic behaviour

$$\pi(x) \sim \frac{x}{\log x}$$
as $x$ tends to $\infty$.

More generally the term zeta function is used for generating series which encode infinite sets of data such as the numbers of solutions of an algebraic equation over finite fields or the lengths of closed geodesics on a Riemann surface. A zeta function is usually given as a Dirichlet series, an Euler product or a Mellin integral.

Erich Kähler was very interested in zeta functions and he even defined a new one himself. To explain this consider the Euler product of the Riemann zeta function. The set of prime numbers can be identified with the set of all nonzero prime ideals of the ring of integers $\mathbb{Z}$, i.e. the set of all closed points of the scheme $\text{Spec} \mathbb{Z}$. This is the starting point of a generalization to an arbitrary arithmetic scheme $X$, whose so called Hasse-Weil zeta function [24] is defined as a product over the closed points of $X$:

$$\zeta_X(s) = \prod_{x \in |X|} (1 - N(x)^{-s})^{-1},$$

where for $x$ a closed point, $N(x)$ denotes the order of the residue class field $\kappa(x)$ of $x$.

On the other hand, the Dirichlet series of $\zeta$ can be interpreted as a sum over all nonzero ideals of $\mathbb{Z}$, i.e., a sum over all nontrivial closed subsets of $\text{Spec} \mathbb{Z}$. In general, the Hasse-Weil zeta function of an arithmetic scheme $X$ is also expressible as a Dirichlet series, but not one that runs over all nontrivial closed subsets of $X$. So Kähler defined a new zeta function in the following way: For every closed subscheme $Y$ of $X$ he defined $N(Y) = \prod_{y \in Y} N(y)$, and then $Z_X(s) = \sum_Y N(Y)^{-s}$, where $N(Y)^{-s}$ is considered to be zero if $N(Y)$ is infinite. In his thesis [16], Lustig proves that this series converges for real
part of $s$ bigger than 2, provided that the dimension of $X$ does not exceed 2. In his Monadologie [13, 271-331], Kähler determined the zeta function of an arithmetic curve of degree 2. His calculation is reproduced in a modern language in [2]. It is shown that the zeta function in this case is a product of the Riemann zeta functions and a Dirichlet $L$-series with an entire function whose zeros can be given explicitly and lie at $\text{Re}(s) = \frac{1}{2}$. If the dimension of $X$ exceeds 2, Witt ([21 p.369] showed that the series diverges for every complex number $s$.

In the sequel, I will discuss three types of zeta, or $L$-functions. The latter are a slight generalization of zeta functions: they arise, for example by allowing twists by characters in the coefficients of a Dirichlet series. The first section will be on zeta and $L$-functions of arithmetic origin which also might be called “of algebraic geometric origin”, since they are attached to objects of algebraic geometry over the integers.

In the second section we will consider automorphic $L$-functions. They are defined in an analytic setting and look very different from the arithmetic ones of the previous section, however, it is widely believed, and proven in a number of cases, that automorphic $L$-functions and arithmetic $L$-functions are basically the same.

The third section will be devoted to zeta functions of differential geometric origin. These are attached to objects from differential geometry or global analysis, like closed geodesics in Riemannian manifolds.

1 Zeta and $L$-functions of arithmetic origin

This section will be concerned with zeta functions whose defining data come from number theory. Since the Riemann zeta function has an Euler product over the primes, one may view the prime numbers as defining data for $\zeta$, so the Riemann zeta function falls into this category.

Let $K$ be a number field, i.e., a finite extension of the field of rational numbers. The Dedekind zeta function of $K$ is defined as

$$\zeta_K(s) = \sum_{\mathfrak{a}} N(\mathfrak{a})^{-s}, \quad \text{Re}(s) > 1,$$

where the sum runs over all nontrivial ideals $\mathfrak{a}$ of the ring of integers $\mathcal{O}_K$ of $K$, and $N(\mathfrak{a}) = #(\mathcal{O}_K/\mathfrak{a})$ is the norm of the ideal $\mathfrak{a}$. The function $\zeta_K(s)$ can
also be expressed as an Euler product
\[ \zeta_K(s) = \prod_p (1 - N(p)^{-s})^{-1} \]
which runs over all nontrivial prime ideals \( p \) of \( \mathcal{O}_K \). Further, \( \zeta_K(s) \) extends to a meromorphic function on the entire plane satisfying a functional equation \[19\]. There are several generalizations of these zeta functions, called \( L \)-functions, which mostly are obtained by “twisting by characters”. For example
\[ L(\chi, s) = \prod_p (1 - \chi(p)N(p)^{-s})^{-1} \]
is the Hecke \( L \)-function attached to a character \( \chi \) of the group \( I_K \) of all fractional ideals of \( K \) (see \[19\]).

In his thesis \[25\], J. Tate used harmonic analysis on the adele ring \( \mathbb{A} = \mathbb{A}_K \) and the idele group \( \mathbb{A}^\times = \mathbb{A}_K^\times \) to give quite far reaching generalizations of the theory of \( L \)-functions which we will now explain. A rational differential form \( \omega \) will give a Haar measure on \( \mathbb{A}^\times \), which, by the product formula, being independent of the choice of \( \omega \), is called the Tamagawa measure on \( \mathbb{A}^\times \). The continuous homomorphisms of \( \mathbb{A}^\times/K^\times \) to \( \mathbb{C}^\times \), also called quasi-characters, are all of the form \( z \mapsto c(z) = \chi(z)|z|^s \) for a character \( \chi \) and a complex number \( s \). For such a quasi-character and a sufficiently regular function \( f \) on \( \mathbb{A} \) the general zeta function of Tate is defined by
\[ \zeta(f, c) = \int_{\mathbb{A}^\times} f(z)c(z)d^\times z, \]
the integration being with respect to the Tamagawa measure and convergent if \( \text{Re}(s) > 1 \). Let \( \hat{f} \) be the additive Fourier transform of \( f \). Considered as an analytic function in \( c \) the zeta function is regular except for two simple poles at \( c_0 \) and \( c_1 \), where \( c_0(z) = 1 \) and \( c_1(z) = |z| \), with residues \( -\kappa f(0) \) and \( \kappa \hat{f}(0) \) respectively, where
\[ \kappa = \frac{2^{r_1}(2\pi)^{r_2}hR}{w\sqrt{d}}. \]
Here \( r_1 \) and \( r_2 \) are the numbers of real embeddings, resp. pairs of complex embeddings of \( K \), \( h \) is the class number, \( R \) the regulator, \( w \) the order of the
group of roots of unity in \( K \), and \( d \) is the absolute value of the discriminant. Furthermore, there is a functional equation
\[
\zeta(f, c) = \zeta(\hat{f}, \hat{c}),
\]
where \( \hat{c}(z) = |z|c(z)^{-1} \).

One recovers classical zeta and \( L \)-functions by restricting \( c \) and \( f \) to special cases. For example \( c(z) = |z|^s \) gives, for suitable \( f \), the (completed) Dedekind zeta function of \( K \).

For \( f \) of simple type the integral over \( \mathbb{A}^\times \) defining \( \zeta(f, c) \) can be written as a product of integrals over the completions of \( K \), so \( \zeta(f, c) \) becomes a product of local zeta functions which themselves satisfy local functional equations and bear other interesting information.

J. Igusa \([11]\) replaced the characters in Tate’s local zeta functions by characters composed with polynomials and so defined a new rich class of local zeta functions with applications to other areas of number theory.

Perhaps the most important generalization of the Riemann zeta function is the class of motivic \( L \)-functions \([5]\). To explain this let us go back to the Hasse-Weil zeta function \( \zeta_X(s) = \prod_{x \in |X|} (1 - N(x))^{-1} \) of an arithmetic scheme \( X \). If \( X \) is proper and flat over \( \text{Spec } \mathbb{Z} \) with smooth generic fibre \( X \otimes \mathbb{Q} \), then it will have good reduction at almost all primes \( p \), we call such primes “good”. The Lefschetz trace formula for \( l \)-adic cohomology implies that the zeta function equals a finite number of Euler factors multiplied by
\[
\prod_{\nu=0}^{2\dim X} \prod_{p \text{ good}} \det_{\mathbb{Q}_l}(1 - p^{-s} Fr_p^* | H^\nu(X \otimes \overline{\mathbb{Q}_p}, \mathbb{Q}_l))(-1)^{\nu+1}.
\]

The inner product runs over all but finitely many primes. Here \( l = l_p \) is a prime different from \( p \) and \( Fr_p \) is the geometric Frobenius at \( p \). The characteristic polynomial is known to have coefficients in \( \mathbb{Q} \) which are independent of \( l \) (for \( p \) good). It has turned out in a number of cases that the individual factors
\[
\prod_{p \text{ good}} \det_{\mathbb{Q}_l}(1 - p^{-s} Fr_p^* | H^\nu(X \otimes \overline{\mathbb{Q}_p}, \mathbb{Q}_l))(-1)^{\nu+1}
\]
themselves have a meromorphic continuation to \( \mathbb{C} \), and indeed satisfy a functional equation if suitably completed at the bad primes. This then is considered to be the \( L \)-function attached to the motive \( H^\nu(X) \). Motives form
a conjectural category in which a scheme \( X \) decomposes into \( H^\nu(X) \) for \( \nu = 0, \ldots, 2 \dim X \) and all usual cohomology theories factor over this category. This category is supposed to be large enough to contain twists and all \( L \)-functions mentioned so far can be realized as motivic \( L \)-functions.

There are various conjectures which relate vanishing orders or special values of motivic \( L \)-functions to other arithmetic quantities. We will here only give one example, the conjecture of Birch and Swinnerton-Dyer. To explain this let \( E \) be an elliptic curve over a number field, i.e., a projective curve of genus one with a fixed rational point. Then there is a natural structure of an abelian algebraic group on \( E \). The group \( E(K) \) of rational points is known to be finitely generated, so its rank, which is defined by \( r = \dim_Q E(K) \otimes \mathbb{Q} \), is finite. The Birch and Swinnerton-Dyer conjecture states that \( r \) should be equal to the vanishing order of the Hasse-Weil zeta function \( \zeta_E(s) \) at \( s = 1 \). There is also a more refined version 26 of this conjecture giving an arithmetical interpretation of the first nontrivial Taylor coefficient of \( \zeta_E(s) \) at \( s = 1 \).

## 2 Automorphic \( L \)-functions

Let \( f \) be a cusp form of weight \( 2k \) for some natural number \( k \) as in 24, i.e., the function \( f \) is holomorphic on the upper half plane \( \mathbb{H} \) in \( \mathbb{C} \), and has a certain invariance property under the action of the modular group \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{H} \). Then \( f \) admits a Fourier expansion

\[
 f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi i n z}.
\]

Define its \( L \)-function for \( \text{Re}(s) > 1 \) by

\[
 L(f, s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.
\]

The easily established integral representation

\[
 \hat{L}(f, s) = (2\pi)^{-s} \Gamma(s) L(f, s) = \int_0^{\infty} f(it) t^{s} \frac{dt}{t},
\]
implies that $L(f, s)$ extends to an entire function satisfying the functional equation

$$\hat{L}(f, s) = (-1)^k \hat{L}(f, 2k - s).$$

With $\Lambda(f, s) = \hat{L}(f, 2ks)$ this becomes

$$\Lambda(f, s) = (-1)^k \Lambda(f, 1 - s).$$

This construction can be extended to cusp forms for suitable subgroups of the modular group. These $L$-functions look like purely analytical objects with no connection to the $L$-functions of arithmetic origin mentioned earlier. Thus it was particularly daring of A. Weil, G. Shimura, and Y. Taniyama in 1955 to propose the conjecture that the zeta function of any elliptic curve over $\mathbb{Q}$ coincides with a $\Lambda(f, s)$ for a suitable cusp form $f$. This conjecture was proved in part by A. Wiles and R. Taylor [27, 28] providing a proof of Fermat’s Last Theorem as a consequence. Subsequently, the conjecture has been proved in full by Breuil, Conrad, Diamond and Taylor [3].

The upper half plane is a homogeneous space of the group $SL_2(\mathbb{R})$, and so cusp forms may be viewed as functions on this group, in particular, they are vectors in the natural unitary representation of $SL_2(\mathbb{R})$ on the space

$$L^2(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})).$$

Going even further one can extend this quotient space to the quotient of the adele group $GL_2(\mathbb{A})$ modulo its discrete subgroup $GL_2(\mathbb{Q})$, so cusp forms become vectors in

$$L^2(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}^1),$$

where $GL_2(\mathbb{A}^1)$ denotes the set of all matrices in $GL_2(\mathbb{A})$ whose determinant has absolute value one. Now $GL_2$ can be replaced by $GL_n$ for $n \in \mathbb{N}$ and one can imitate the methods of Tate’s thesis (the case $n = 1$) to arrive at a much more general definition of an automorphic $L$-function: this is an Euler product $L(\pi, s)$ attached to an automorphic representation $\pi$ of $GL_n(\mathbb{A})^1$, i.e. an irreducible subrepresentation $\pi$ of $L^2(GL_n(\mathbb{Q}) \backslash GL_n(\mathbb{A}^1))$. As in the $GL_1$-case it has an integral representation as a Mellin transform and it extends to a meromorphic representation, which is entire if $\pi$ is cuspidal and $n > 1$. Furthermore it satisfies a functional equation

$$L(\pi, s) = \epsilon(\pi, s)L(\bar{\pi}, 1 - s),$$

where $\bar{\pi}$ is the contragredient representation and $\epsilon(\pi, s)$ is a constant multiplied by an exponential [7].
Extending the Weil-Shimura-Taniyama conjecture, R.P. Langlands conjectured in the 1960s that any motivic $L$-function coincides with $L(\pi, s)$ for some cuspidal $\pi$. An affirmative answer to this question would prove many other, older conjectures in number theory. It has local and characteristic $p$ analogues, which have been proved [10, 14].

3 Zeta functions of differential-geometric origin

In this section we will be concerned with zeta functions attached to objects arising in differential geometry. These will serve as a measure of complexity of the geometric objects. We start with an oversimplified example: a finite graph $X$. If it has no closed paths, it is topologically trivial, so the set of primitive closed paths gives a measure of complexity. Primitivity here means that a path (or a walk, as some people say) $c_0$ is not the power of a shorter one, so you walk each closed path only once. The number of closed paths will in general be infinite, so one considers the following zeta function as a formal power series at first:

$$Z_X(T) = \prod_{c_0} (1 - T^{l(c_0)}),$$

where the product runs over all primitive closed paths and $l(c_0)$ denotes the length of a given path $c_0$. Then it turns out [11] that $Z_X(T)$ is in fact a polynomial, for it can be written as $\det(1 - TA)$ for some generalized adjacency operator $A$ for the graph $X$.

If one replaces the graph $X$ by a compact Riemannian surface $Y$ of genus $\geq 2$, then one can attach a natural hyperbolic metric to $Y$ and replace the paths by closed geodesics. One ends up with the Ruelle Zeta Function

$$R_Y(s) = \prod_c (1 - e^{-sl(c)}),$$

where the product runs over all primitive closed geodesics in $Y$ and $l(c) > 0$ is the length of the geodesic $c$. The Ruelle Zeta Function $R_Y(s)$ equals $Z_Y(s)/Z_Y(s + 1)$, where $Z_Y(s)$ is the Selberg Zeta Function attached to $Y$. 
defined by
\[ Z_Y(s) = \prod_c \prod_{n \geq 0} (1 - e^{(s+n)l(c)}). \]

The Selberg Zeta Function can be studied using harmonic analysis and one can prove that it extends to an entire function having all its zeros in the set \( \mathbb{R} \cup (\frac{1}{2} + i\mathbb{R}) \), i.e., \( Z_Y(s) \) satisfies a generalized Riemann Hypothesis \([9]\).

Note that in this section, Euler products often occur without the \((-1)\) in the exponent. This is more than a matter of taste, since, for example, the Selberg zeta function is entire this way round. The reason why one has to take different exponents becomes transparent when one generalizes the Selberg zeta function to spaces of higher rank \([4]\): the natural exponents are certain Euler characteristics which can take positive or negative values.

The set of closed geodesics on \( Y \) is in bijection with the set of closed orbits of the geodesic flow \( \phi \) on the sphere bundle \( SY \). So \( R_Y(s) = R_\phi(s) \) is a special case of a dynamical zeta function, since it counts closed orbits of a dynamical system. Actually D. Ruelle proved the meromorphicity of \( R_\phi(s) \) in a more general setting: he needed \( \phi \) only to be a smooth flow satisfying a certain hyperbolicity condition. He used the theory of Markov families to express \( R_\phi(s) \) as a quotient of certain transfer operators. For the geodesic flow of the modular curve \( \text{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \) the eigenvectors of these transfer operators can be identified with modular functions \([15]\).

Also for discrete dynamical systems there is a theory of zeta functions. Let most generally \( f \) be an invertible self map of a set \( X \) and define its zeta function as the formal power series
\[ Z_f(T) = \prod_p (1 - T^{l(p)}), \]

where the product runs over the set of periodic orbits and \( l(p) \in \mathbb{N} \) is the period of \( p \). One has to put restrictions on \( f \) in order for \( Z_f \) to be well defined. For \( f \) being a diffeomorphism of a compact manifold satisfying certain natural hyperbolicity conditions A. Manning has shown in \([17]\), that \( Z_f(T) \) indeed is a rational function. This in particular means that all fixed point data of \( f \) can be exhibited from the finite set of poles and zeros of \( Z_f(T) \).

The Hasse-Weil zeta function of a smooth projective variety \( V \) over a finite
field $\mathbb{F}_q$ with $q$ elements can be viewed as a dynamical zeta function too. It coincides with $Z_{Fr}(q^{-s})$, where $Fr$ is the Frobenius acting on $V(\mathbb{F}_q)$.

One feature that makes zeta functions of geometric origin so attractive is that they tend to satisfy Lefschetz formulae. For example, as Artin, Grothendieck and Verdier have shown, the Hasse-Weil zeta function of a smooth projective variety $V$ over $\mathbb{F}_q$ satisfies

$$\zeta_V(s) = \prod_{\nu=0}^{2\dim V} \det(1 - q^{-s}Fr|H^\nu(V, \mathbb{Q}_l))^{(-1)^{\nu+1}}.$$

The Selberg zeta function satisfies a similar Lefschetz formula involving the Frobenius vector field and the tangential cohomology of the contracting foliation [8, 21, 4]. This fact has given rise to some far reaching conjectures whether these formulae are valid for more general systems and if they even could be part of a cohomological framework which would explain most of the conjectural properties of zeta and $L$-functions of arithmetic schemes [6].

4 Closing remarks

In this brief survey we have missed out many other important classes of zeta functions. It was not our aim to give an exhaustive list but to show some general lines of development. Summarizing we find that zeta functions, given by a Dirichlet series, an Euler product or a Mellin integral, encode infinite sets of data. The first thing one usually asks for, is convergence, next meromorphicity and functional equation. If one is lucky the function turns out to be rational, this then means that the infinite set of data can be recovered from the finitely many poles and zeros. One finally starts to ask in which way the analytic behaviour of the zeta function reflects properties of the encoded objects. Prominent examples of this are the Prime Number Theorem, in which the position of the pole and the zeros of the zeta function give information on the growth of the data, or results or conjectures on special values like the Birch and Swinnerton-Dyer conjecture.

Seeking harmony and simplicity the human mind is always tempted to believe that objects of similar behaviour, although of very different origin, should be of the same nature. Remarkably enough this has turned out true in the case
of the Taniyama-Shimura-Weil conjecture and has shown strong evidence in the case of the Langlands conjecture.

Whenever some entities are counted with some mathematical structure on them, it is likely that a zeta function can be set up and often enough it will extend to a meromorphic function. Zeta functions show up in all areas of mathematics and they encode properties of the entities counted which are well hidden and hard to come by otherwise. They easily give fuel for bold new conjectures and thus drive on the progress of mathematics. It is a fairly safe assertion to say that zeta functions of various kinds will stay in the focus of mathematical attention for times to come.

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