Isometric Path Partition Number of Butterfly Network and X – Trees

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Abstract. A set S of isometric paths that partition the vertex set of a graph G is called an isometric path partition of G. The isometric path partition number of G is the minimum cardinality of such a set S in G. It is denoted by ip_p(G). The isometric path partition problem is to find a minimum isometric path partition of a given graph G. The isometric path partition problem has wide applications in the designing of an efficient algorithm in many architectures. The problem of determining the isometric path partition of a given graph G is NP-complete and hence it is interesting to compute the exact value of isometric path partition of different networks. In this paper, we study the isometric path partition problem of butterfly networks, complete binary trees, rooted complete binary trees, X – trees and compute the isometric path partition number of these graphs.

1. Introduction
All the graphs G considered in this paper are simple, finite, and undirected. We refer to Bondy & Murthy [1] for the basic definitions. An isometric path is a path that induces the shortest distance between two vertices. An isometric path is also referred to as a geodesic. The problem of isometric path plays an important role in the designing of an efficient algorithm in many networks. An isometric path cover is a set of isometric paths which cover the vertex set V. The isometric path cover number which is denoted by ip_c(G) is the cardinality of a minimum isometric path cover. An isometric path partition is defined as a set of isometric paths that partition the vertex set of G. The isometric path partition number of G is the minimum cardinality of all such isometric path partitions and it is denoted by ip_p(G). An isometric path is called as diametral isometric path if the length of the path is equal to the diameter of the graph.

In the past few decades, the isometric path problems have been researched widely. In designing the VLSI layouts, Aggarwal et al [2] used the concept of the isometric path cover. Later, the isometric path cover number has been computed for trees, cycles, complete bipartite graphs, the Cartesian product of paths, the grid, hypercube Q_r, the block graphs, complete r-partite graphs and Cartesian product of 2 or 3 complete graphs. Fisher and Fitzpatrick [3] have derived a lower bound for the isometric path cover number. However, in the field of isometric path partition, only few results are available [4]. Paul Mauel in [5] showed that the isometric path partition problem is NP-complete. Also he has computed the isometric path partition number of the (r x s)-dimensional grid, the (r x r)-dimensional torus and the r-dimensional benes network.

In this paper, we provide an algorithm to estimate the exact isometric path partition number of butterfly networks, complete binary trees, rooted binary complete trees and X – trees. We use the following result which gives the lower bound for the isometric path partition number.
Theorem 1.1[3]. If \( \text{diam}(G) \) denotes the diameter of a graph \( G \), then
\[
\text{ip}_p(G) \geq \text{ip}_c(G) \geq \left\lceil \frac{|V(G)|}{\text{diam}(G) + 1} \right\rceil.
\]

2. Butterfly Network
Butterfly network \( BF(n) \) is an \( n \)-partite graph. The vertices of an \( n \)-dimensional butterfly network \( BF(n) \) correspond to the set of pairs \([s, l]\) where \( l \) is the level or dimension of the vertex \((0 \leq l \leq n)\) and \( s \) is an \( n \)-bit binary number that denotes the row of the vertex. Two vertices \([s, l]\) and \([s', l']\) are connected by an edge if and only if \( l' = l + 1 \) and either (i) \( s \) and \( s' \) are identical or (ii) \( s \) and \( s' \) differ in precisely the \( l \)-th bit. Edge between \([s, l]\) and \([s', l']\) is referred as a straight edge, whereas the edge between \([s, l]\) and \([s', l']\) is referred as a cross edge. The butterfly network \( BF(n) \) has \((n+1)^2\) vertices since \( BF(n) \) has \((n+1)\) levels and there exit \( 2^n \) vertices at each level. The vertices on level 0 and \( n \) are of degree 2, the remaining vertices are of degree 4 [6,7].

2.1 Diametral isometric paths in \( BF(n) \)
Consider \( BF(n) \). Observe that there are exactly 12 \( n \) disjoint diametral isometric paths namely \( P_1^n, P_2^n, P_3^n, \ldots, P_{2^n-1}^n \) (Figure 1).

![Figure 1. Diametral isometric paths of BF(2) and BF(3).](image)

Also there are \( 2^n-1 \) singleton vertices which are not covered by the above diametral isometric paths and we denote these \( 2^n-1 \) singleton vertices of \( BF(n) \) by \( S_{n}, S_{2n}, S_{3n}, \ldots, S_{2^n-1} \).

2.2. Isometric path partition number of \( BF(n) \)
Input: Butterfly Network \( BF(n), n \geq 1 \).
Algorithm: Label the vertices of \( BF(n) \) as \( b_1, b_2, b_3, \ldots, b_{(n+1)2^n} \) (Figure 1). Let \( S_n \) denote the set of isometric paths that induces the isometric path partition of \( BF(n) \).

(i) Let \( S_1 = \{ P_1^n, P_2^n \} \) in \( BF(1) \).

(ii) Select \( BF(2) \). Then \( S_2 = \{ P_1^n, P_2^n, P_3^n, P_4^n \} \).

(iii) Inductively \( S_n = \bigcup_{k=1}^{2^n-1} \{ P_k^n, P_{k+1}^n \} \).

Output: \( \text{ip}_p(BF(n)) = 2^n \).
Proof of correctness: From theorem 1.1, \( \text{ip}_p(BF(n)) \geq 2^{n-1} \) and all the \( 2^{n-1} \) paths are diametral isometric paths. This partition leaves exactly \( 2^{n-1} \) vertices isolated and these vertices cannot be
combined further since all the vertices of degree 4 are already covered by the diametral isometric paths. Hence the proof.

**Theorem 2.2.1.** Let $G$ be a butterfly network $BF(n)$, $n \geq 1$. Then $ip_p(G) = 2^n$.

3. Complete binary tree

An acyclic connected graph is defined as a *tree*. The *binary tree* is one of the most commonly represented trees. In a binary tree every vertex has at most two children. The binary trees are broadly applied in data structures as it can store, manipulate and retrieve the data easily. A complete binary tree is symmetric about the root vertex. Removing the vertices covered by the above path, results in a disconnected graph whose components are the 2 copies of (i) an isolated vertex, (ii) $T_2$, $T_3$, ..., $T_{n-2}$ (Figure 3). Hence by induction, the isometric path partition of $T_n$ is the union of all the isometric path partitions of its components, together with the diametral isometric path of $T_n$.

**Theorem 3.1.1.** Let $G$ be a complete binary tree $T_n$, $n \geq 3$. Then $ip_p(G) = 3 + 2 \sum_{i=2}^{n-2} ip_p(T_i)$.

3.1. Isometric path partition number of Complete Binary tree $T_n$

**Input:** Complete binary tree $T_n$, $n \geq 3$.

**Algorithm:** Label the vertices of $T_n$ from the root vertex and successively at the higher levels as $r_1^0, r_1^1, r_2^0, r_2^1, r_3^0, r_3^1, ..., r_{n-1}^0, r_{n-1}^1, r_n^0, r_n^1$ (Figure 2). Let $S_n$ denote the set of isometric paths that induces an isometric path partition of $T_n$. Let $P_{m,n}$ and $P'_{m,n}$ denote the isometric paths on $n$ vertices contained in $S_m$ of $T_m$.

(i) Assign $S_2 = \{ P_{2,3} \}$.

(ii) Select $T_3$, then $S_3 = \{ P_{3,5}, P_{3,1}, P'_{3,1} \}$ where $P_{3,5}: r_1^2 - r_1^1 - r_1^0 - r_2^1 - r_4^2$, $P_{3,1}: r_2^2$ and $P'_{3,1}: r_3^2$ (Figure 2).

(iii) Select $T_4$, then $S_4 = \{ P_{4,7}, P_{4,3}, P'_{4,3}, P_{4,1}, P'_{4,1} \}$ where $P_{4,7}: r_1^3 - r_1^2 - r_1^1 - r_4^0 - r_2^1 - r_4^2 - r_5^3$, $P_{4,3}: r_3^2 - r_3^1 - r_4^0$, $P_{4,1}: r_3^3 - r_5^3 - r_6^3$, and $P'_{4,1}: r_3^3$ and $P'_{4,1}: r_4^3$ (Figure 2).

(iv) Inductively $S_n = \{ 2 \text{ copies of } (S_2 \cup S_3 \cup ... \cup S_{n-2}) \} \cup \{ P_{n,2n-1}, P_{n,1}, P'_{n,1} \}$.

**Output:** $ip_p(T_n) = 3 + 2 \sum_{i=2}^{n-2} ip_p(T_i)$, $n \geq 3$.

**Proof of correctness:** Consider $T_n$. Identify the diametral isometric path of length $2(n-1)$. A complete binary tree is symmetric about the root vertex. Removing the vertices covered by the above path, results in a disconnected graph whose components are the 2 copies of (i) an isolated vertex ($P_1$) and (ii) $T_2, T_3, ..., T_{n-2}$ (Figure 3). Hence by induction, the isometric path partition of $T_n$ is the union of all the isometric path partitions of its components, together with the diametral isometric path of $T_n$.

Figure 2. Isometric path partition of $T_3$ and $T_4$. 

doi:10.1088/1757-899X/912/6/062052
4. Rooted complete binary tree

A *rooted complete binary tree* RTₙ is attained by attaching a pendant vertex to the root vertex of the complete binary tree. There will be *n* + 1 levels in RTₙ as the newly added pendent vertex will represent the level 0 vertex. The number of vertices of RTₙ is 2ⁿ and the diameter of RTₙ is the same as the diameter of Tₙ [8,9].

4.1. Isometric path partition number of Rooted Complete Binary tree RTₙ

*Input:* Rooted complete binary tree RTₙ, *n* ≥ 3.

*Algorithm:* Label the vertices of RTₙ from the root vertex and proceed successively as *r₁, r₂, r₃, r₄, r₅, r₆,...*, *rₙ, r₁, r₂,...*, *rₙ, r₁, r₂,...*, *rₙ, r₁, r₂,...*, *rₙ, r₁, r₂,...*, *rₙ, r₁, r₂,...*, *rₙ, r₁, r₂,...*. Let *Sₙ* and *Sₙ* denote the isometric path partition of RTₙ and Tₙ respectively. Let *Pₘₙ* and *Pₘₙ* denote the isometric paths on *n* vertices contained in *Sₙ* of Tₙ and *Pₘₙ* denote the isometric path on *n* vertices contained in *Sₙ* of RTₙ.

(i) Select RT₃, then *S₃* = { *P₃₃*, *P₃₃*, *P₃₃* } where *P₃₃* : *r₃² - r₃¹ - r₃⁰*, *P₃₃* : *r₃ - r₂ - r₂*, and *P₃₃* : *r₃ * (Figure 4).
(ii) Select $RT_n$, then $S_4 = \{ P''_{4,5}, P_{4,5}, P_{4,3}, P'_{4,3}, P''_{4,1}\}$ where $P''_{4,5} : r_1^3 - r_1^4 - r_1^2 - r_1^1$, $P_{4,5} : r_3^4 - r_2^3 - r_4^4 - r_5^4$, $P_{4,3} : r_3^4 - r_2^3 - r_4^4$, $P'_{4,3} : r_6^4$, $P'_{4,1} : r_7^4$ and $P''_{4,1} : r_2^4$ (Figure 4).

(iii) Inductively $S_n = \{ P''_{n+1}, P''_{n}\} \cup \{ S'_2 \cup S'_3 \cup \ldots \cup S'_{n-1}\}$.

Output: $ip_p(RT_n) = 2 + \sum_{i=2}^{n-1} ip_p(T_i)$, $n \geq 3$.

**Proof of correctness:** Consider $RT_n$. Choose the isometric path of length $n$ connecting $r_1^0$ to $r_1^n$ through the left child of each level (Figure 5). Removal of the vertices covered by the above path, results in a disconnected graph with the following components (i) an isolated vertex ($P_1$) and (ii) $T_2, T_3, \ldots, T_{n-1}$. Hence the isometric path partition of $RT_n$ is the union of all the isometric path partitions of $T_j$, for $0 \leq j \leq n-1$, together with the initial isometric path of length $n$ and a $P_1$. We ignore the diametral isometric path because, if we choose the diametral isometric path in an even level $RT_n$, in particular then the removal of the vertices covered by this path results in a disconnected graph with more components which contradicts the minimality of the problem.

**Theorem 4.1.1.** Let $G$ be a rooted complete binary tree $RT_n$, $n \geq 3$. Then $ip_p(G) = 2 + \sum_{i=2}^{n-1} ip_p(T_i)$.

5. **X–tree**

An $X$–tree is a tree coined from complete binary tree, by drawing the path $P_i$ through all the vertices at $i$th level of length $2^i - 1$, from left to right for $1 \leq i \leq n$. The $n$-dimensional $X$–tree is denoted by $XT(n)$ (Figure 6).

![Figure 5. Isometric path partition of $RT_6$.](image)

![Figure 6. Isometric path partition of $XT(3)$ and $XT(4)$.](image)
5.1. Isometric path partition number of $X -$ tree $XT(n)$

Input: $X -$ tree $XT(n)$, $n \geq 1$.

Algorithm: Label the vertices of $XT(n)$ from the root vertex and successively as $r_1^n, r'_1, r_2^n, r'_2, ..., r_m^n$. Let $S_n$, $S'_n$, $S_l^n$ and $S'_l^n$ denote the isometric path partition of $XT(n)$, $XT(n) \setminus \{r_1^n\}$, $XT(n) \setminus \{r_1^n, r_2^n, ..., r_m^n\}$ and $XT(n) \setminus \{r_1^n, r_2^n, r_3^n, ..., r_m^n\}$ respectively. Let $P_{m,n}$ and $P'_{m,n}$ denote the isometric paths $P_n$ contained in $S_m$ of $XT(m)$.

(i) Let $S_2 = \{P_{2,4}, P_{2,2}, P_{2,1}\}$ for $XT(2)$, where $P_{2,4} = r_3^n - r_1^n - r_2^n - r_4$, $P_{2,2} = r_2^n - r_3^n$ and $P_{2,1}$ = $r_1^n$.

(ii) Select $XT(3)$. Then $S_3 = \{P_{3,6}, P'_3, P_{3,2}, P_{3,1}\}$ where $P_{3,6} = r_3^n - r_1^n - r_2^n - r_3 - r_4$, $P'_3 = r_3^n - r_2^n - r_3 - r_4 - r_5$, $P_{3,2} = r_2^n - r_3^n$ and $P_{3,1}$ = $r_1^n$ (Figure 6).

(iii) Choose $XT(4)$. Then $S_4 = \{P_{4,8}, P'_{4,8}, P_{4,6}, P_{4,3}, P'_{4,3}, P_{4,2}, P_{4,1}\}$ where $P_{4,8} = r_4^n - r_3^n - r_1^n - r_2^n - r_4 - r_5 - r_6$, $P'_{4,8} = r_4^n - r_3^n - r_2^n - r_3^n - r_4 - r_5 - r_6 - r_7$, $P_{4,6} = r_4^n - r_3^n - r_2^n - r_3^n - r_4^n - r_5^n$, $P_{4,3} = r_4^n - r_3^n - r_2^n - r_3^n$, $P'_{4,3} = r_4^n - r_3^n - r_2^n - r_3^n$, $P_{4,2} = r_4^n - r_3^n$ and $P_{4,1}$ = $r_1^n$ (Figure 6).

(iv) Select $XT(3) \setminus \{r_1^n, r_2^n, r_3^n\}$. Then $S'_4 = \{P'_{3,6}, P_{3,2}, P_{3,1}\}$.

(v) Select $XT(4) \setminus \{r_1^n\}$. Then $S'_4 = \{P_{4,8}, P'_{4,8}, P_{4,6}, P_{4,3}, P'_{4,3}, P_{4,2}\}$.

(vi) Select $XT(3) \setminus \{r_1^n, r_2^n, r_3^n\}$. Then $S'_3 = \{P'_{3,6}, P_{3,2}, P_{3,1}\}$.

(vii) Select $XT(5)$. Then $S_5 = \{P_{5,10}, P_{5,1}\} \cup \{S'_3\} \cup \{S'_4\} \cup \{S'_5\}$ (Figure 7).

(viii) Inductively $S_n = \{P_{n,2n}, P_{n,1}\} \cup \{S'_{n-2}\} \cup \{S'_n\} \cup \{S'_{n-2}\}$.

Output: $ip_p(XT(n)) = ip_p(XT(n-1)) + 2$ $ip_p(XT(n-2)) - 1$, $n \geq 5$.

Proof of correctness: In $XT(n)$ for $n \geq 5$, select the diametral isometric path from $r_1^n$ to $r_2^n$ passing through the left and right boundary. On removal of this diametral isometric path, the graph $XT(n)$ gets disconnected into two components namely an isolated vertex ($P_i$) and a connected subgraph $H$. One can observe that, the connected subgraph $H$ can be partitioned into three subgraphs namely (i) $XT(n-1) \setminus \{r_1^n\}$, (ii) $XT(n-2) \setminus \{r_1^n, r_2^n, r_3^n, ..., r_m^n\}$ and (iii) $XT(n-2) \setminus \{r_1^n, r_2^n, r_3^n, ..., r_m^{n-2}\}$ (Figure 7). By induction, the isometric path partition of $H$ is obtained by the union of the isometric path partitions of the above three subgraphs denoted by $S'_n$, $S'_n$, and $S'_n$ respectively. Hence the proof.

Theorem 5.1.1. Let $G$ be a $X -$ tree $XT(n)$, $n \geq 5$. Then $ip_p(G) = ip_p(XT(n-1)) + 2$ $ip_p(XT(n-2)) - 1$.

6. Conclusion

In this paper, we have investigated the isometric path partition problem and computed the exact isometric path partition number of butterfly networks, complete binary trees, rooted complete binary
trees and $X$–trees. The isometric path partition number of hypercube, circular networks and cactus networks are under further study.

Figure 7. Isometric path partition of XT(5).

7. References

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