Fast forward of adiabatic spin dynamics : An application to quantum annealing model in triangle spin systems.

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Abstract. We propose a scheme of the fast forward of adiabatic spin dynamics in a triangular 3 spin cluster of quantum annealing model and reveal the structure and role of driving pair-wise interaction and three-body interactions. We settle the quasi-adiabatic spin dynamics (QASD) by adding the regularization terms to the original Hamiltonian and accelerate it with use of a large time-scaling factor which realizes QASD on shortened time scale. Assuming the candidate regularization Hamiltonian consisting of three-body interactions besides the pair-wise exchange interactions and magnetic field, we solved the regularization terms. These terms multiplied by the velocity function give rise to the state-dependent counter-diabatic terms (CDTs) for each of adiabatic states. Applying this method to quantum annealing model, we find CDTs which include pair-wise interaction and the 3-body interactions. The driving pairwise and three-body interaction in the fast-forward scheme guarantees the complete fidelity of accelerated states.

1. Introduction

If we try to fabricate massive amount of nanoscale structures, we should accelerate the dynamics of each electron, atom or molecules to reach their desired target states in shorter time. In designing quantum computers, the coherence of systems is degraded by their interaction with the environment, and therefore the acceleration of adiabatic quantum dynamics is highly desirable. Masuda and Nakamura [1] proposed a theory of fast forward to accelerate quantum dynamics with use of additional phase and driving potential, and Khujakulov and Nakamura [2] showed that this theory is useful to enhance the quantum tunneling power. The theory of fast-forward guarantees to accelerate any given quantum evolution and to obtain the desired target state on shortened time scale, by fast forwarding the reference quantum dynamics. This theory was further developed to accelerate the quasi-static or adiabatic quantum dynamics [3, 4, 5], and constitutes one of the promising means to the shortcut to adiabaticity (STA) [6, 7, 8, 9, 10, 11, 12]. The relationship between the fast forward and STA is nowadays clear [5, 13, 14]).
While the theory of fast forward has been limited to orbital dynamics of atoms, molecules or Bose-Einstein condensates, we recently proposed a scheme of fast forward of adiabatic spin dynamics of quantum entangled states [15]. Confining to coupled two spin systems, we settled the quasi-adiabatic dynamics by adding the regularization terms to the original (reference) Hamiltonian and then accelerated it with use of a large time-scaling factor. Assuming a candidate regularization Hamiltonian consisting of the pair-wise exchange interactions and magnetic field, we solved the regularization terms. These terms multiplied by the velocity function give rise to the plural number of state-dependent counter-diabatic terms for each of adiabatic states, which is distinct from the issue of the method of transitionless quantum driving by Demirplak-Rice-Berry (to be abbreviated below as DRB) [6, 7, 8].

In the context of fast forward, on the other hand, if we shall stay in two spin dynamics [15], the candidate regularization Hamiltonian is limited to single-spin (e.g., Zeeman energy) and pair-wise interactions (e.g., exchange interactions and their modifications). By treating three or more spin systems, however, we can accommodate three-body interactions in the candidate regularization Hamiltonian. In particular, it is interesting to see (i) how the fast-forward scheme generates three-body interactions in counter-diabatic terms, (ii) what is the nature of such interactions, and (iii) what is the fidelity of the fast-forward state under such interactions.

The triangular 3 spin cluster was extensively studied in the context of quantum chaos [16], topological order parameter [17, 18] and quantum simulation of the frustration [19]. It is now worthwhile to investigate the triangular 3 spin cluster in the context of fast forward, even though the application of the fast forward scheme to $N(\geq 4)$-spin systems will remain as a future work.

2. Fast-forward of adiabatic spin dynamics

Consider the Hamiltonian for spin systems to be characterized by a slowly time-changing parameter $R(t)$ such as the exchange interaction, magnetic field, etc. Then we can study the eigenvalue problem for the time-independent Schrödinger equation:

$$H_0(R)C(R) = E(R)C(R)$$

with

$$C(R) = \begin{pmatrix} C_1(R) \\ \vdots \\ C_N(R) \end{pmatrix},$$

where

$$R \equiv R(t) = R_0 + \epsilon t$$

is the adiabatically-changing parameter with $\epsilon \ll 1$. In Eq.(1), the quantum number $n$ for each eigenvalue and eigenstate is suppressed for simplicity. Let us assume

$$\Psi_0(R(t)) = C(R)e^{-i\int_0^t E(R(t'))dt'} e^{i\xi(R(t))},$$

to be a quasi-adiabatic state, i.e., adiabatically evolving state, where $\xi$ is the adiabatic phase:

$$\xi(R(t)) = i \int_0^t dt' C^{\dagger} \partial_t C = i \epsilon \int_0^t dt' C^{\dagger} \partial_R C.$$  

To make $\Psi_0(R(t))$ in Eq.(4) to satisfy the TDSE, we must regularize the Hamiltonian as

$$H^{\text{reg}}_0(R(t)) = H_0(R(t)) + \epsilon \tilde{H}_n(R(t)).$$
Then TDSE becomes
\[ i\hbar \frac{\partial}{\partial t} \Psi_0(R(t)) = (H_0 + \epsilon \hat{H}_n)\Psi_0(R(t)). \] (7)

Here \( \hat{H}_n \) is the \( n \)-th state-dependent regularization term. Substituting \( \Psi_0(R(t)) \) in Eq.(4) into the above TDSE, we see Eq.\((4)\) in order of \( O(\epsilon^0) \), and in order of \( O(\epsilon^1) \)
\[ \hat{H}_n C(R) = i\hbar \partial_R C(R) - i\hbar (C^\dagger \partial_R C)C(R), \] (8)

which is the core equation of the present work.

The fast-forward state is defined by
\[ \Psi_{FF}(t) \equiv C(R(\Lambda(t))) e^{-\frac{i}{\hbar} \int_0^t E(R(\Lambda(t'))) dt'} e^{i\xi(R(\Lambda(t)))}, \] (9)

where \( \Lambda(t) \) is an advanced time defined by
\[ \Lambda(t) = \int_0^t \alpha(t') dt', \] (10)

with the standard time \( t \). \( \alpha(t) \) is a magnification time-scale factor given by \( \alpha(0) = 1 \), \( \alpha(t) > 1(0 < t < T_{FF}) \) and \( \alpha(t) = 1(t \geq T_{FF}) \). We consider the fast forward dynamics with a new time variable which reproduces the target state \( \Psi_0(R(T)) \) in a shorter final time \( T_{FF} \) defined by
\[ T = \int_0^{T_{FF}} \alpha(t) dt. \] (11)

The explicit expression for \( \alpha(t) \) in the fast-forward range \( (0 \leq t \leq T_{FF}) \) is typically given by [3]
as :
\[ \alpha(t) = \bar{\alpha} - (\bar{\alpha} - 1) \cos \left( \frac{2\pi}{T_{FF}} t \right), \] (12)

where \( \bar{\alpha} \) is the mean value of \( \alpha(t) \) and is given by \( \bar{\alpha} = T/T_{FF} \). Then by taking the time derivative of \( \Psi_{FF} \) in Eq.(9) and using the equalities \( \partial_t C(R(\Lambda(t))) = \alpha \partial_R C \) and \( \partial_t \xi(R(\Lambda(t))) = iC^\dagger \partial_t C = i\alpha C^\dagger \partial_R C \), we have
\[ i\hbar \frac{\partial \Psi_{FF}}{\partial t} = \left[ i\hbar \alpha \epsilon (\partial_R C - (C^\dagger \partial_R C)C) + EC \right] \times e^{-\frac{i}{\hbar} \int_0^t E(R(\Lambda(t'))) dt'} e^{i\xi(R(\Lambda(t)))}. \] (13)

The first and second terms in the angular bracket on the r.h.s are replaced by \( \alpha \epsilon \hat{H}_n C(R(\Lambda(t))) \) and \( H_0 C(R(\Lambda(t))) \), respectively, by using Eqs.(8) and (1). Using the definition of \( \Psi_{FF}(t) \) and taking the asymptotic limit \( \bar{\alpha} \to \infty \) and \( \epsilon \to 0 \) under the constraint \( \bar{\alpha} \cdot \epsilon \equiv \bar{\nu} = finite \), we obtain
\[ i\hbar \frac{\partial \Psi_{FF}}{\partial t} = \left( v(t) \hat{H}_n(R(\Lambda(t))) + H_0(R(\Lambda(t))) \right) \Psi_{FF} \equiv H_{FF} \Psi_{FF}. \] (14)

Here \( v(t) \) is a velocity function available from \( \alpha(t) \) in the asymptotic limit:
\[ v(t) = \lim_{\epsilon \to 0, \bar{\alpha} \to \infty} \epsilon \alpha(t) = \bar{\nu} \left( 1 - \cos \frac{2\pi}{T_{FF}} t \right). \] (15)

Consequently, for \( 0 \leq t \leq T_{FF} \),
\[ R(\Lambda(t)) = R_0 + \lim_{\epsilon \to 0, \bar{\alpha} \to \infty} \epsilon \Lambda(t) = R_0 + \int_0^t v(t') dt' \]
\[ = R_0 + \bar{\nu} \left[ t - \frac{T_{FF}}{2\pi} \sin \left( \frac{2\pi}{T_{FF}} t \right) \right]. \] (16)

\( H_{FF} \) is the driving Hamiltonian and \( \hat{H}_n \) is the regularization term obtained from Eq.(8) to generate the fast-forward scheme in spin system.
3. Triangular 3 spin cluster

As a model of three-spin systems, we consider the triangular 3 spin cluster shown in Fig.1. On the basis of this configuration, we shall investigate quantum annealing model.

\[ H = J_1(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) + J_2(\sigma_i^x \sigma_k^x + \sigma_i^y \sigma_k^y + \sigma_i^z \sigma_k^z) + J_3(\sigma_j^x \sigma_k^x + \sigma_j^y \sigma_k^y + \sigma_j^z \sigma_k^z) \]

(17)

Here, the number of independent equations in Eq.(8) is less than that of the unknown \{\tilde{H}_{ij}\} \((1 \leq i, j \leq 8)\), noting the dimension of Hilbert space= 2^3 for the 3-spin system. Therefore some extra strategy should be introduced. We assume a form for the regularization term (\tilde{H}) in Eq.(8) so as to include the diagonal-exchange interaction \(\tilde{J}_1 = \tilde{J}_1(\epsilon_t)\), \(\tilde{J}_2 = \tilde{J}_2(\epsilon_t)\), \(\tilde{J}_3 = \tilde{J}_3(\epsilon_t)\), off-diagonal-exchange interaction \(W_1 = W_1(\epsilon_t)\), \(W_2 = W_2(\epsilon_t)\), \(W_3 = W_3(\epsilon_t)\), three-body interaction \(Q = Q(\epsilon_t)\), and 3-component magnetic field \(\tilde{B} = \tilde{B}(\epsilon_t)\). The candidate for regularization Hamiltonian \(\tilde{H}\) takes the following form:

\[ \tilde{H} = \tilde{J}_1(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) + \tilde{J}_2(\sigma_i^x \sigma_k^x + \sigma_i^y \sigma_k^y + \sigma_i^z \sigma_k^z) + \tilde{J}_3(\sigma_j^x \sigma_k^x + \sigma_j^y \sigma_k^y + \sigma_j^z \sigma_k^z) \]

+ \(\tilde{W}_1[(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) + (\sigma_j^x \sigma_k^x + \sigma_j^y \sigma_k^y + \sigma_j^z \sigma_k^z)]\)

+ \(\tilde{W}_2[(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) + (\sigma_j^x \sigma_k^x + \sigma_j^y \sigma_k^y + \sigma_j^z \sigma_k^z)]\)

+ \(\tilde{W}_3[(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z) + (\sigma_j^x \sigma_k^x + \sigma_j^y \sigma_k^y + \sigma_j^z \sigma_k^z)]\)

+ \(\tilde{Q}[\sigma_i^x(\sigma_j^y \sigma_k^z + \sigma_j^z \sigma_k^y) + \sigma_i^y(\sigma_j^z \sigma_k^x + \sigma_j^x \sigma_k^z) + \sigma_i^z(\sigma_j^x \sigma_k^y + \sigma_j^y \sigma_k^x)]\)

+ \(\tilde{B}\).

The single-body (\(\tilde{B}\)) and two-body (\(\tilde{J}, \tilde{W}\)) terms are just the extension of the corresponding terms in the two-spin system in [15]. The three-body interaction (\(\tilde{Q}\)) is new and is introduced by being inspired by the scalar chirality [17, 18, 19]:

\[ \sigma_i \cdot (\sigma_j \times \sigma_k) = \sigma_i^x(\sigma_j^y \sigma_k^z - \sigma_j^z \sigma_k^y) + \sigma_i^y(\sigma_j^z \sigma_k^x - \sigma_j^x \sigma_k^z) + \sigma_i^z(\sigma_j^x \sigma_k^y - \sigma_j^y \sigma_k^x), \]

(18)

which is invariant against the cyclic change of site indices. As we shall investigate in each of the following Subsections, however, the term in Eq.(18) turns out not to contribute to the counter-diabatic term and therefore we have included in Eq.(17) a slightly-different variant which is symmetric with respect to exchange of the sites of spin variables.
Arranging the bases as $|↑↑↑⟩$, $|↑↓↓⟩$, $|↓↑↓⟩$, $|↓↓↑⟩$, $|↓↓↓⟩$, $|↑↑↓⟩$, $|↑↓↑⟩$, $|↓↑↑⟩$, $|↓↓↑⟩$ and $|↓↓↓⟩$, we obtain

$$
\tilde{\mathcal{H}} = \begin{pmatrix}
E & A & A & B + D & B + D & B + D & 0 \\
A^* & E_1 & E_2 & E_2 & C & C & 0 & B + D^* \\
A^* & E_1^* & E_2 & C & 0 & C & B + D^* \\
A^* & E_2^* & E_2^* & E_1 & C & C & B + D^* \\
B^* + D^* & C^* & C^* & 0 & E_1 & E_2^* & E_2^* & F \\
B^* + D^* & C^* & 0 & C^* & E_2 & E_1 & E_2^* & F \\
B^* + D^* & 0 & C^* & C^* & E_2 & E_2 & E_1 & F \\
0 & B^* + D & B^* + D & B^* + D & F^* & F^* & F^* & E_3
\end{pmatrix}, \quad (19)
$$

where $A = -i2\tilde{W}_2 + 2\tilde{W}_3 + C$, $B = \tilde{J}_1 - \tilde{J}_2 - 2i\tilde{W}_1$, $C = \frac{1}{2}(\tilde{B}_x - i\tilde{B}_y)$, $D = -2i\tilde{Q}$, $E = 3\tilde{J}_3 + \frac{3\tilde{B}_z}{2}$, $E_1 = -\tilde{J}_3 + \frac{2\tilde{B}_z}{2}$, $E_2 = \tilde{J}_1 + \tilde{J}_2$, $E_3 = 3\tilde{J}_3 - \frac{3\tilde{B}_z}{2}$, and $F = i2\tilde{W}_2 - 2\tilde{W}_3 + C$. The explicit expression for $\tilde{\mathcal{H}}$ in Eq.(19) will help us to solve Eq.(8).

4. Quantum annealing model

Our interest here lies in showing a variety of driving fields or counter-diabatic terms for three-spin systems. Therefore, more practical subjects of quantum annealing, such as finding the ground state of many-spin systems described by a very complicated Hamiltonian and applying the fast-forward protocol to accelerate the quantum adiabatic computation when the final ground state is unknown, are outside of the scope of the present work. The triangular spin analogue of the quantum annealing model [20] is described by the following Hamiltonian:

$$
H_0 = -J(\sigma_1^x\sigma_2^x + \sigma_2^x\sigma_3^x + \sigma_3^x\sigma_1^x) - \frac{1}{2}(\sigma_1^y + \sigma_2^y + \sigma_3^y)B_x \\
- \frac{1}{2}(\sigma_1^z + \sigma_2^z + \sigma_3^z)B_z,
$$

where $J$ and $B_z$ are positive constants and $B_x = B_x(R(t))$ plays the role of tunneling among spin up and down states. Here $J > 0$ and $J < 0$ correspond to the ferromagnetic and antiferromagnetic interactions, respectively. By decreasing $B_x$ from a large value towards 0, the entangled state tends to the ground state of the Ising model. By using the bases : $|↑↑↑⟩$, $|↑↓↓⟩$, $|↓↑↓⟩$, $|↓↓↑⟩$, $|↑↑↓⟩$, $|↑↓↑⟩$, $|↓↑↑⟩$ and $|↓↓↓⟩$, we obtain

$$
H_0 = \begin{pmatrix}
-3J - \frac{3B_z}{2} & -B_x & -B_x & 0 & 0 & 0 & 0 \\
-B_x & -B_x & 0 & 0 & 0 & -B_x & -\frac{B_x}{2} \\
B_x & 0 & J - \frac{B_x}{2} & 0 & -\frac{B_x}{2} & 0 & -\frac{B_x}{2} \\
B_x & 0 & 0 & -B_x & 0 & -\frac{B_x}{2} & 0 \\
0 & B_x & 0 & 0 & J + \frac{B_x}{2} & 0 & -\frac{B_x}{2} \\
0 & 0 & B_x & 0 & 0 & J + \frac{B_x}{2} & -\frac{B_x}{2} \\
0 & 0 & 0 & -B_x & 0 & 0 & J + \frac{B_x}{2} \\
0 & 0 & 0 & 0 & -B_x & 0 & J + \frac{B_x}{2} \\
\end{pmatrix}.
$$

The eigenvalues are : $\lambda_1 = \lambda_2 = Z + J$, $\lambda_3 = \lambda_4 = J - Z$, $\lambda_5 = \sqrt{\lambda_3 + \lambda_4 + \lambda_6}$, $\lambda_6 = -\sqrt{\lambda_3 + \lambda_4 + \lambda_6}$, $\lambda_7 = \sqrt{\lambda_3 + \lambda_4 + \lambda_6}$, $\lambda_8 = \sqrt{\lambda_3 + \lambda_4 + \lambda_6}$, where $Z = \frac{1}{2}\sqrt{B_x^2 + B_z^2}$, $S_\pm = \pm 2^{-7/6}3^{1/2}\alpha \eta^{-1/2}3^{1/6}$, $G = -\frac{2}{11}(12 \times 21^3/\beta - 1/3/\mu + 21^3/\beta - 1/3/\rho^2 + 2 - 1/3/\beta^3/3 - 4)$, and $K_\pm = \pm 2^{-5/6}3^{-1/2}\eta^{-1/3}/4 - J$, with $\alpha = 4\nu J$, $\beta = 27\alpha^2 + 3\sqrt{3}\kappa + 72\mu \rho - 2\rho^2$, $\nu = B_x^2 - 2B_z^2$, and $\kappa = \sqrt{27\alpha^4 + 144\alpha^2\mu \rho - 4\alpha^2\rho^2 - 256\mu^2\rho^2 - 128\mu^2\rho^2 - 16\mu^2\rho^2}$.
Meanwhile, let’s consider the ferromagnetic case \((J > 0)\). \(B_x\) dependence of eigenvalues are shown in Fig.2(a).

The eigenvector for the ground state \((\lambda_8)\) are \(C_1 = B_1 \zeta, C_2 = B_2 \zeta, C_3 = B_3 \zeta, C_4 = B_4 \zeta, C_5 = B_5 \zeta, C_6 = B_6 \zeta, C_7 = B_7 \zeta,\) and \(C_8 = B_8 \zeta,\) where

\[
B_1 = \frac{\lambda_1^2 (2B_z - \frac{4J}{3})}{B_x^3} + 4\Gamma^3 \frac{B_x^3}{3B_x^2}
- \frac{\lambda_1 \Gamma \left(7B_x^2 + B_z^2 - 4B_z J + \frac{20J^2}{3}\right)}{B_x^3},
- \frac{3B_x^6 - 3B_x^2 J + B_z^2 J - 2B_z J^2 + 4J^3}{B_x^3},
\]

\[
B_2 = B_3 = B_4 = \frac{B_z^2 - B_x^2 + 2J^2}{B_x^3}
+ \frac{\lambda_2 \Gamma \left(\frac{4B_z}{3} - \frac{4J}{3}\right)}{B_x^2} + \frac{2\Gamma^2}{3B_x^2},
\]

\[
B_5 = B_6 = B_7 = \frac{B_z - 2J}{B_x} + \frac{2\Gamma}{3B_x},
B_8 = 1,
\]

and

\[
\zeta = \frac{1}{\sqrt{B_1^2 + 3B_2^2 + 3B_5^2 + B_8}}
\]

\[
\Gamma = \sqrt{S_- + G} - K_-. \tag{23}
\]

Here we see the symmetry \(C_1, C_2 = C_3 = C_4, C_5 = C_6 = C_7\) and \(C_8\) and the adiabatic phase \(\xi = 0\). From the normalization \((C_1^2 + 3C_2^2 + 3C_5^2 + C_8^2 = 1),\) we obtain

\[
C_1 \frac{\partial C_1}{\partial R} + 3C_2 \frac{\partial C_2}{\partial R} + 3C_5 \frac{\partial C_5}{\partial R} + C_8 \frac{\partial C_8}{\partial R} = 0. \tag{24}
\]

Due to the symmetry of \(\{C_j\}\) and using the candidate regularization Hamiltonian in Eq.(19), the equation for regularization terms is written as

\[
\begin{align*}
\text{i}h \frac{\partial C_1}{\partial R} &= \hat{\mathcal{H}}_{11} C_1 + (\hat{\mathcal{H}}_{12} + \hat{\mathcal{H}}_{13} + \hat{\mathcal{H}}_{14}) C_2 + (\hat{\mathcal{H}}_{15} + \hat{\mathcal{H}}_{16} + \hat{\mathcal{H}}_{17}) C_5 + \hat{\mathcal{H}}_{18} C_8,
\text{i}h \frac{\partial C_2}{\partial R} &= \hat{\mathcal{H}}_{21} C_1 + (\hat{\mathcal{H}}_{22} + \hat{\mathcal{H}}_{23} + \hat{\mathcal{H}}_{24}) C_2 + (\hat{\mathcal{H}}_{25} + \hat{\mathcal{H}}_{26} + \hat{\mathcal{H}}_{27}) C_5 + \hat{\mathcal{H}}_{28} C_8,
\text{i}h \frac{\partial C_5}{\partial R} &= \hat{\mathcal{H}}_{51} C_1 + (\hat{\mathcal{H}}_{52} + \hat{\mathcal{H}}_{53} + \hat{\mathcal{H}}_{54}) C_2 + (\hat{\mathcal{H}}_{55} + \hat{\mathcal{H}}_{56} + \hat{\mathcal{H}}_{57}) C_5 + \hat{\mathcal{H}}_{58} C_8,
\text{i}h \frac{\partial C_8}{\partial R} &= \hat{\mathcal{H}}_{81} C_1 + (\hat{\mathcal{H}}_{82} + \hat{\mathcal{H}}_{83} + \hat{\mathcal{H}}_{84}) C_2 + (\hat{\mathcal{H}}_{85} + \hat{\mathcal{H}}_{86} + \hat{\mathcal{H}}_{87}) C_5 + \hat{\mathcal{H}}_{88} C_8.
\end{align*}
\]

\[
\tag{25}
\]

To solve the 4-component simultaneous linear equations for \(\{\hat{\mathcal{H}}_{ij}\}\) in Eq.(25), we should note the nature of \(\hat{\mathcal{H}}\) in the candidate for regularization terms in Eq.(19), which is Hermitian and traceless and has other symmetries. This setup implies that we could choose four independent real variables instead of ten real variables in Eq.(19). Among \(10C_4 = \frac{10 \cdot 9}{4 \cdot 3!}\) choices, however, we
should pick up only the cases where $4 \times 4$ coefficient matrix for the unknown $\{\hat{J}, \hat{W}, \hat{Q}, \hat{B}\}$ is regular and each of four solutions is real. For example there is a choice where $\hat{J}_3$, $\hat{Q}$, $\hat{W}_1$, and $\hat{W}_2$, are independent variable with others zero, such that Eq.\((25)\) can be rewritten as

$$
\begin{align*}
\frac{i \hbar \partial C_1}{\partial R} &= 3 \hat{J}_3 C_1 + (-i 6 \hat{W}_2) C_2 + (-6i \hat{W}_1 - 6i \hat{Q}) C_3, \\
\frac{i \hbar \partial C_2}{\partial R} &= (i 2 \hat{W}_2) C_1 - \hat{J}_3 C_2 + (-2i \hat{W}_1 + 2i \hat{Q}) C_3, \\
\frac{i \hbar \partial C_3}{\partial R} &= (2i \hat{W}_1 + 2i \hat{Q}) C_1 - \hat{J}_3 C_3 + (2i \hat{W}_2) C_3, \\
\frac{i \hbar \partial C_4}{\partial R} &= (6i \hat{W}_1 - 6i \hat{Q}) C_2 + (-6i \hat{W}_2) C_3 + 3 \hat{J}_3 C_4.
\end{align*}
$$

By solving Eq.\((26)\), we obtain the solution

$$
\begin{align*}
\hat{J}_3 &= \frac{a C_1 + 3b C_2 + 3c C_3 + C_8 d}{3 (C_4^2 - C_5^2 - C_6^2 + C_8^2) } = 0, \\
\hat{Q} &= \frac{i ( -a C_8 + 3b C_5 - 3c C_2 + C_1 d)}{12 (C_1 C_2 - C_5 C_3)}, \\
\hat{W}_1 &= \frac{i}{12 (C_1 C_2 - C_5 C_3)} \left[ a \{ C_1^2 C_8 \\ - 2C_1 C_2 C_5 + C_2^2 C_8 + C_5^2 C_8 - C_3^2 C_8 \} \\ - 3b \{ C_5 (C_2^2 + C_5^2 - C_6^2) - 2C_1 C_2 C_5 + C_5 C_8^2 \} \\ - 3c \{ C_1^2 C_2 - 2C_1 C_3 C_5 + C_2 (-C_4^2 + C_5^2 + C_8^2) \} \\ + d \{ -C_1^3 + C_1 C_2^2 + C_1 C_5^2 + C_1 C_8^2 - 2C_2 C_3 C_5 \} \right], \\
\hat{W}_2 &= \frac{-i}{6 (C_1 C_2 - C_5 C_8) \left[ a C_1 + 3c C_5 (C_2^2 - C_3^2) + (3b C_2 + C_8 d) (C_1^2 - C_3^2) \right]}
\end{align*}
$$

where $a = i \hbar \frac{\partial C_1}{\partial R}$, $b = i \hbar \frac{\partial C_2}{\partial R}$, $c = i \hbar \frac{\partial C_3}{\partial R}$, $d = i \hbar \frac{\partial C_4}{\partial R}$. The regularization terms is given by

$$
\tilde{H} = \begin{pmatrix}
0 & -i 2 \tilde{W}_2 & -i 2 \tilde{W}_2 & -i 2 \tilde{W}_2 & P + L & P + L & P + L & 0 \\
-i 2 \tilde{W}_2 & 0 & 0 & 0 & 0 & 0 & 0 & P + L^* \\
i 2 \tilde{W}_2 & 0 & 0 & 0 & 0 & 0 & 0 & P + L^* \\
i 2 \tilde{W}_2 & 0 & 0 & 0 & 0 & 0 & 0 & P + L^* \\
0 & P^* + L & P^* + L & P^* + L & -i 2 \tilde{W}_2 & -i 2 \tilde{W}_2 & -i 2 \tilde{W}_2 & 0
\end{pmatrix}
$$

with $P = -i 2 \tilde{W}_1$, $L = -i 2 \tilde{Q}$, and $h = 1$. The fast-forward Hamiltonian is written as

$$
\begin{align*}
H_{FF} &= -J (\sigma_1^x \sigma_2^x + \sigma_2^x \sigma_3^x + \sigma_3^x \sigma_1^x) \\
&\quad - \frac{1}{2} (\sigma_1^x + \sigma_2^x + \sigma_3^x) B_2 (R(L(t))) \\
&\quad - \frac{1}{2} (\sigma_3^x + \sigma_2^x + \sigma_3^x) B_2 + v \tilde{H}_n
\end{align*}
$$
with \( \tilde{\mathcal{H}}_n = v(t)\tilde{Q}(R(\Lambda(t))) \left[ \sigma_3^n (\sigma_3^n \sigma_3^x + \sigma_3^y \sigma_3^y) + \sigma_3^y (\sigma_3^y \sigma_3^x + \sigma_3^x \sigma_3^y) + \sigma_3^x (\sigma_3^x \sigma_3^x + \sigma_3^y \sigma_3^y) \right] + v(t)\tilde{W}_0(R(\Lambda(t))) \left[ (\sigma_2^x \sigma_3^y + \sigma_2^y \sigma_3^y) + (\sigma_3^x \sigma_3^x + \sigma_3^y \sigma_3^y) \right] + v(t)\tilde{W}_2(R(\Lambda(t))) \left[ (\sigma_2^x \sigma_3^x + \sigma_3^x \sigma_3^y) + (\sigma_2^y \sigma_3^y + \sigma_3^y \sigma_3^y) \right] \). Again one might conceive that, by including in Eq. (17) antisymmetric terms like \( \sigma_2^x \sigma_3^y - \sigma_3^x \sigma_3^y \) and \( \sigma_3^y \sigma_3^x - \sigma_3^x \sigma_3^y \), etc, more solutions would be available. However, we find that there is no solution that includes such terms. Therefore the ansatz in Eq. (17) is justified as in the previous Subsection.

Choosing the above solutions, we show in Fig. 2 the time dependence of counter-diabatic terms, i.e., dynamics of \( \tilde{Q}, \tilde{W}_0, \tilde{W}_2 \) multiplied by \( v(t) \), respectively. Here \( v(t) \) in Eq. (15) and \( R(\Lambda(t)) \) in Eq. (16) are used, and parameter values are: \( J = 1, B_z = 0.1, B_x = B_0 - R(\Lambda(t)) \) with \( \tilde{v} = 100, T_{FF} = 0.1, R_0 = 0, \) and \( B_0 = 10 \).

![Figure 2](image1.png)

Figure 2. Ferromagnetic case \((J > 0)\). (a) \( B_z \) dependence of eigenvalues (\( \lambda_j \)). The time dependence of counter-diabatic terms : (b) \( Q = v(t)\tilde{Q} \) (dashed line), \( W_1 = v(t)\tilde{W}_1 \) (dotted line), \( W_2 = v(t)\tilde{W}_2 \) (solid line). \( J = 1 \). Other parameter values are given in the text.

With use of solution for the counter-diabatic term in Fig. 2, we numerically solve TDSE in Eq. (14). The result is shown in Fig. 3, where parameter values are the same as used in Fig. 2. We see in Fig. 3, as \( B_x \) is decreased, the dynamics shows a rapidly change from the initial states \((C_1 = 0.4177, C_2 = C_3 = C_4 = 0.3346, C_5 = C_6 = C_7 = 0.3304, \) and \( C_8 = 0.4024 \)) as a linear combination of \( |\uparrow\uparrow\uparrow\rangle, |\uparrow\uparrow\downarrow\rangle, |\uparrow\downarrow\uparrow\rangle, |\uparrow\downarrow\downarrow\rangle, |\downarrow\uparrow\uparrow\rangle, |\downarrow\downarrow\uparrow\rangle, \) and \( |\downarrow\downarrow\downarrow\rangle \) states at \( t = 0 \) to the ferromagnetic \( |\uparrow\uparrow\uparrow\rangle \) state at \( t = T_{FF} \). We numerically evaluated the time-dependent fidelity of the wavefunction solution \( \Psi_{FF} \) to the eigenfunction \( \Psi_0 \) and confirmed the fidelity = 1 during the fast-forward time range \( 0 \leq t \leq T_{FF} \).

![Figure 3](image2.png)

Figure 3. Ferromagnetic case \((J > 0)\). The time dependence of \( |C_1^{FF}|^2 \) (solid line), \( |C_2^{FF}|^2 = |C_3^{FF}|^2 = |C_4^{FF}|^2 \) (dotdashed line), \( |C_5^{FF}|^2 = |C_6^{FF}|^2 = |C_7^{FF}|^2 \) (dotted line), \( |C_8^{FF}|^2 \) (dashed line). \( J = 1 \).
5. Conclusion
We presented a scheme of the fast forward of adiabatic spin dynamics in a triangular spin cluster. We settled the quasi-adiabatic dynamics by adding the regularization terms to the original Hamiltonian and then accelerated it with use of a large time-scaling factor. The candidate regularization Hamiltonian can include three-body interactions which are absent in the original Hamiltonian consisting of the Zeeman energy and the pair-wise exchange interactions. The driving 3-body interactions are symmetric with respect to exchange of the sites of spin variables like $\sigma_1^z(\sigma_2^x\sigma_3^y + \sigma_2^y\sigma_3^x)$. The present scheme guarantees the complete fidelity of accelerated adiabatic states. In quantum annealing model, we find CDTs which include 3-body interactions. In case of quantum annealing we obtain three kinds of CDTs; two of them are pair-wise interaction i.e $\tilde{W}_1[(\sigma_1^x\sigma_2^y + \sigma_1^y\sigma_2^x) + (\sigma_2^x\sigma_3^y + \sigma_2^y\sigma_3^x) + (\sigma_3^x\sigma_1^y + \sigma_3^y\sigma_1^x)]$, $\tilde{W}_2[(\sigma_1^x\sigma_2^z + \sigma_1^z\sigma_2^x) + (\sigma_2^x\sigma_3^z + \sigma_2^z\sigma_3^x) + (\sigma_3^x\sigma_1^z + \sigma_3^z\sigma_1^x)]$, and another one is 3-body interactions $\tilde{Q}(\sigma_1^x\sigma_2^z + \sigma_2^x\sigma_3^z) + \sigma_1^y(\sigma_2^z\sigma_3^x + \sigma_3^z\sigma_2^x)$. The driving pairwise interaction and three-body interaction in the present scheme guarantees the complete fidelity of accelerated adiabatic states.

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References
[1] Masuda S and Nakamura K 2008 Phys. Rev. A 78 062108
[2] Khujakulov A and Nakamura K 2016 Phys. Rev. A 93, 022101
[3] Masuda S and Nakamura K 2010 Proc. R. Soc A 466 1135
[4] Masuda S and Nakamura K 2011 Phys. Rev. A 84 043434
[5] Nakamura K, Khujakulov A, Avazbaev S and Masuda S 2017 Phys. Rev. A 95 062108
[6] Demirplak M and Rice S A 2003 J. Phys. Chem. A 89 9937
[7] Demirplak M and Rice S A 2005 J. Phys. Chem. B 109 6838
[8] Berry M V 2009 J. Phys. A 42 365303
[9] Lewis H R and Riesenfeld W B 1969 J. Math. Phys. A 10 1458
[10] Chen X, Ruschhaupt A, del Campo A, Guery-Odelin A and Muga J G 2010 Phys. Rev. Lett. 104 063002
[11] Torrontegui E, Ibanez M, Martinez-Garaot M, Modugno A, del Campo A, Guery-Odelin D, Ruschhaupt A, Chen X and Muga J G 2013 Adv. At. Mol. Opt. Phys. 62, 117
[12] Masuda S, Nakamura K and del Campo A 2014 Phys. Rev. Lett. 113 063003
[13] Torrontegui E, Martinez-Garaot M, Ruschhaupt A and Muga J G, 2012 Phys. Rev. A 86 013601
[14] Takahashi K 2014 Phys. Rev. A 89 042113
[15] Setiawan I, Gunara B E, Masuda S and Nakamura K 2017 Phys. Rev. A 96 052106
[16] Nakamura K 1993 Quantum Chaos-A New Paradigm of Nonlinear Dynamics Cambridge Univ. Press, Cambridge
[17] Kawamura H 1998 J. Phys. C 10 4707
[18] Georgeot G and Milla F 2010 Phys. Rev. Lett. 104, 200502
[19] Kim K et al 2010 Nature 465 590
[20] Kadowaki K and Nishimori H 1998 Phys. Rev. E 58 5355