Multiplier systems for Hermitian modular groups

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Abstract
Let $\Gamma_{F,n}$ be the Hermitian modular group of degree $n > 1$ in sense of Hel Braun with respect to an imaginary quadratic field $F$. Let $r$ be a natural number. There exists a multiplier system of weight $1/r$ (equivalently a Hermitian modular form of weight $k + 1/r$, $k$ integral) on some congruence group if and only if $r = 1$ or $2$. This follows from a much more general construction of Deligne [De] combining it with results of Hill [Hi], Prasad [P] and Prasad-Rapinchuk [PR]. As far as we know, the systems of weight $1/2$ have not yet been described explicitly. Remarkably Haowu Wang [Wa] gave an example of a modular form of half integral weight. Actually he constructs a Borcherds product of weight $23/2$ for a group of type $O(2,4)$. This group is isogenous to the group $U(2,2)$ that contains the Hermitian modular groups of degree two.

In this paper we want to study such multiplier systems. If one restricts them to the unimodular group

$$\mathcal{U} = \left\{ U; \begin{pmatrix} \hat{U}^{-1} & 0 \\ 0 & U \end{pmatrix} \in \Gamma \right\},$$

one obtains a usual character. Our main result states that the kernel of this character is a non-congruence subgroup. For sufficiently small $\Gamma$ it coincides with the group described by Kubota [Ku] in the case $n = 2$ and by Bass Milnor Serre [BMS] in the case $n > 2$.

Introduction
In the paper [FH] we gave a simple proof of a special case of a theorem of Deligne [De1] that states that the weights of multiplier systems on subgroups of finite index of the Siegel modular group of degree $n > 1$ are integral or half integral. The same result holds for other modular groups as for the Hermitian modular groups, the quaternary modular groups and the orthogonal groups $O(2,n)$, $n \geq 3$. In all these cases multiplier systems of half integral weight do exist. This follows from results of Deligne [De2], Prasad and Rapinchuc [PR] and Prasad [P]. In the case of the Siegel modular forms these multipliers are obtained as theta multiplier systems which can be expressed by means of a
symplectic Gauss sum. In other cases no such explicit description is known as far as I know. In this paper we study the multiplier systems in the Hermitian modular case. Our main result is that the restriction to the unimodular group is a character whose kernel is a non-congruence subgroup as it has been described in [Ku], [BMS].

Our proof rests on techniques from the paper [FH]. Part of this paper can be generalized from the Siegel to the Hermitian modular group. In the first part we generalize results from [FH] to the Hermitian case. Instead of proofs we refer to the corresponding result in [FH] if the generalization is straightforward.

The main difference of the two cases is that the pair \((R, q)\) for the ring \(R\) of integers of an imaginary quadratic field with an distinguished non-zero ideal \(q\) admits a non trivial Mennicke symbol (Definition 5.1) whereas every Mennicke symbol in the case \(R = \mathbb{Z}\) is trivial.

We fix a natural number \(n\) (which later will be 2). We denote by \(E = E^{(n)}\) the \(n \times n\)-unit matrix and by

\[
I = I^{(n)} = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}
\]

the standard alternating matrix. The unitary group \(U(n, n)\) consists of all \(M \in \text{GL}(2n, \mathbb{C})\) with the property \(M'IM = I\). (This is equivalent to \(M'HN = H\), where \(H\) is the Hermitian form \(iI\) of signature \((n, n)\)). The special unitary group \(SU(n, n)\) is the subgroup of elements with determinant one. One has \(SU(1, 1) = \text{SL}(2, \mathbb{R})\).

From now on we fix an imaginary quadratic field \(F = \mathbb{Q}(\sqrt{d})\) of discriminant \(d < 0\) and denote by

\[
\mathfrak{o} = \mathbb{Z} + \mathbb{Z}\omega, \quad \omega = \frac{d + \sqrt{d}}{2},
\]

its ring of integers. The Hermitian modular group \(\Gamma_{F,n}\) is the subgroup of \(U(n, n)\) of matrices with entries in \(\mathfrak{o}\). Let \(q \subset \mathfrak{o}\) be a non zero ideal. Then

\[
\Gamma_{F,n}[q] = \text{kernel}(\Gamma_{F,n} \longrightarrow \text{GL}(2n, \mathfrak{o}/q))
\]

is the (principal) congruence subgroup of level \(q\). Since the field \(F\) is fixed, we can omit the label \(F\) and write

\[
\Gamma_n := \Gamma_{F,n} \quad \text{and} \quad \Gamma_n[q] = \Gamma_{F,n}[q].
\]

For sufficiently small \(q\) the group \(\Gamma_n[q]\) is contained in \(\text{SL}(2n, \mathfrak{o})\). Then the group \(\Gamma_1[q]\) is the usual principal congruence subgroup of the elliptic modular group \(\text{SL}(2, \mathbb{Z})\) of level \(q \cap \mathbb{Z}\).
1. Multiplier systems

We consider the usual action \( MZ = (AZ + B)(CZ + D)^{-1} \) of the unitary group \( U(n, n) \) on the Hermitian upper half plane

\( \mathcal{H}_n = \{ Z \in \mathbb{C}^{n \times n}; \quad Z = X + iY, \quad X = \bar{X}', Y = \bar{Y}', X > 0 \text{ (positive definite)} \} \).

This is an open convex domain in \( \mathbb{C}^{n \times n} \). The function

\[ J(M, Z) = \det(CZ + D) \]

has no zeros on the half plane. Since the half plane is convex, there exists a continuous choice \( L(M, Z) = \arg J(M, Z) \) of the argument. We normalize it such that it is the principal value for \( Z = iE \) where \( E \) denotes the unit matrix. Recall that the principal value \( \text{Arg}(a) \) is defined such that it is in the interval \(( -\pi, \pi ] \). So we have

\[ L(M, iE) = \text{Arg}(J(M, i)) \in (-\pi, \pi]. \]

We consider

\[ w(M, N) := \frac{1}{2\pi} \left( (L(MN, Z) - L(M, NZ) - L(N, Z)) \right). \]

Obviously,

\[ e^{2\pi i w(M, N)} = 1. \]

Hence \( w(M, N) \) is independent of \( Z \) and \( w(M, N) \in \mathbb{Z} \). Usually we will compute \( w(M, N) \) by evaluation at \( Z = iE \).

\[ w(M, N) = \frac{1}{2\pi} \left( \text{Arg} J(MN, iE) - \text{arg} J(M, N(iE)) - \text{Arg} J(N, iE) \right), \]

where \( \text{arg} J(M, N(iE)) \) is obtained through continuous continuation of the principal value \( \text{Arg} J(M, iE) \) along a path from \( iE \) to \( N(iE) \). Usually one takes the straight line.

1.1 Remark. The function \( w: U(n, n) \times U(n, n) \rightarrow \mathbb{Z} \) is a cocycle in the following sense:

\[ w(M_1M_2, M_3) + w(M_1, M_2) = w(M_1, M_2M_3) + w(M_2, M_3), \]

\[ w(E, M) = w(M, E) = 0. \]
The computation of \( w(M, N) \) in degree 1 is easy for the following reason. From the definition we have

\[
2\pi w(M, N) = \text{Arg}((c\alpha + d\gamma)i + c\beta + d\gamma) - \text{arg}(cN(i) + d) - \text{Arg}(\gamma i + \delta)
\]

for

\[
M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad N = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}
\]

where \( \text{arg}(cN(i) + d) \) is obtained from the principal value \( \text{Arg}(ci + d) \) through continuous continuation. But \( cz + d \) for \( z \) in the upper half plane never crosses the real axis. Hence the result of the continuation is the principal value too. So all three arguments in the definition of \( w(M, N) \) are the principal values (in degree 1). This makes it easy to compute \( w \). We rely on tables for the values of \( w \) which have been derived by Petersson and reproduced by Maass [Ma], Theorem 16.

**1.2 Lemma.** Let \( M = \begin{pmatrix} \ast & \ast \\ m_1 & m_2 \end{pmatrix} \), \( S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) be two real matrices with determinant 1 and \((m'_1, m'_2)\) the second row of the matrix \( MS \). Then

\[
4w(M, S) = \left\{ \begin{array}{ll}
\text{sgn } c + \text{sgn } m_1 - \text{sgn } m'_1 - \text{sgn } (m_1 m'_1) & \text{if } m_1 m'_1 \neq 0, \\
-(1 - \text{sgn } c)(1 - \text{sgn } m_1) & \text{if } cm_1 \neq 0, m'_1 = 0, \\
(1 + \text{sgn } c)(1 - \text{sgn } m_2) & \text{if } cm'_1 \neq 0, m_1 = 0, \\
(1 - \text{sgn } a)(1 + \text{sgn } m_1) & \text{if } m_1 m'_1 \neq 0, c = 0, \\
(1 - \text{sgn } a)(1 - \text{sgn } m_2) & \text{if } c = m_1 = m'_1 = 0.
\end{array} \right.
\]

**Corollary.** Assume that \( m_1 m'_1 \neq 0 \) and that \( m_1 m'_1 > 0 \) or \( m_1 c < 0 \). Then \( w(M, S) = 0 \).

We give an example.

**1.3 Lemma.** We have

\[
w \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = w \left( \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = 0.
\]

**2. Some special values of the cocycle**

We give some examples for values of \( w \) in degree \( n > 1 \).

**2.1 Lemma.** One has

\[
w \left( \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, M \right) = 0.
\]

The proof is trivial.
2.2 Lemma. Let
\[ P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \]
Set
\[ z := \det(iC + D). \]
Then
\[ w(P, M) = w(M, P) = \text{Arg}(-z) - \text{Arg}(z) - \text{Arg}(-1). \]

Proof. Compare [FH], Lemma 1.2.

\[ \square \]

2.3 Definition. The Siegel parabolic group consists of all elements from SU(n, n) of the form
\[ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}. \]

There is a character on the Siegel parabolic group
\[ \varepsilon \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det(D). \]

For an element \( M \) of the Siegel parabolic group, the expression \( \det(CZ + D) = \det(D) \) is independent of \( Z \). Hence
\[ L(M, Z) = 0 \text{ if } \varepsilon(M) = 1. \]

An immediate consequence is the following lemma.

2.4 Lemma. For two elements \( P, Q \) of the Siegel parabolic group we have \( w(P, Q) = 0 \) if \( \varepsilon(P) = 1 \).

The proof is trivial.

\[ \square \]

We have to consider two embeddings \( \iota_1, \iota_2 : \text{SL}(2, \mathbb{R}) \to \text{SU}(2, 2) \), namely
\[ \iota_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \iota_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c & d \end{pmatrix}. \]

2.5 Lemma. Let \( M \) be in the image of one of the embeddings \( \iota_\nu \) and \( N \) a Siegel parabolic matrix with \( \varepsilon(N) = 1 \). Then \( w(M, N) = 0 \).

Proof. Compare [FH], Lemma 3.1.
2.6 Lemma. Assume \( n = 2 \). Let
\[
M = \begin{pmatrix} E & S \\ 0 & E \end{pmatrix}, \quad S = \bar{S}'.
\]
Then
\[
w(I, M) = \begin{cases} 
0 & \text{if } \text{tr}(S) \geq 0, \\
-1 & \text{else}.
\end{cases}
\]

Proof. Compare [FH], Lemma 1.6.

2.7 Lemma. Assume \( n = 2 \). Let
\[
M = \begin{pmatrix} E & 0 \\ S & E \end{pmatrix}, \quad S = \bar{S}'.
\]
Then
\[
w(M, I) = \begin{cases} 
-1 & \text{if } \text{tr}(S) > 0, \\
0 & \text{else}.
\end{cases}
\]

Proof. Compare [FH], Lemma 1.7.

3. Multipliers

3.1 Definition. Let \( \Gamma \subset U(n, n) \) be an arbitrary subgroup and let \( r \) be a real number. A system \( v(M), M \in \Gamma \), of complex numbers of absolute value 1 is called a multiplier system of weight \( r \) if
\[
v(MN) \equiv v(M)v(N)\sigma(M, N)
\]
where
\[
\sigma(M, N) = \sigma_r(M, N) := e^{2\pi i r w(M, N)}.
\]

Let now \( \Gamma \) be a normal subgroup of finite index of \( \Gamma_n, n \geq 2 \). Since the congruence subgroup property holds we know that \( \Gamma \) contains a congruence subgroup \( \Gamma_n[q] \). It is easy to show that weights \( r \) of multiplier systems are rational [Ch]. Hence a suitable power of \( v \) is trivial on some congruence subgroup. This shows that there exists a natural number \( l \) such the all values of \( v \) are \( l \)th roots of unity.
For any $L \in \Gamma_n$ we can consider a conjugate multiplier system on $\Gamma$ that is defined by

$$\tilde{v}(M) = v(LML^{-1}) \frac{\sigma(LML^{-1}, L)}{\sigma(L, M)}.$$ 

It is easy to check that this is a multiplier system and that this defines an action of $\Gamma_n$ on the set of all multiplier systems on $\Gamma$. The quotient of two multiplier systems of the same weight is a homomorphism, as we know into a finite group. Since the congruence subgroup problem has been solved for the Hermitian modular group, we obtain $\tilde{v}(M) = v(M)$ on some congruence subgroup. Since the Hermitian modular group is finitely generated, they agree on $\Gamma_n[q]$, $q$ suitable.

3.2 Lemma. Let $v$ be a multiplier system on a subgroup $\Gamma \subset \Gamma_n$ of finite index. In the case $n \geq 2$ there exists an ideal $q \neq 0$ such that $\Gamma[q] \subset \Gamma$ and such that

$$v(M) = v(LML^{-1}) \frac{\sigma(LML^{-1}, L)}{\sigma(L, M)} \quad (M \in \Gamma[q])$$

for all $L \in \Gamma_n$.

Several times we will replace $q$ by a smaller ideal. We then just say “for suitable $q$”. We always assume that suitable $q$ have the property that the only unit of $\mathfrak{o}$ that is congruent to 1 mod $q$ is 1. Then each Siegel parabolic $M \in \Gamma_n[q]$ has the property $\varepsilon(M) = 1$. We also assume that $q \subset 4\mathfrak{o}$.

3.3 Proposition. Let $v$ be a multiplier system on a subgroup $\Gamma \subset \Gamma_2$ of finite index. For suitable $q$ the group $\Gamma_2[q]$ is contained in $\Gamma$ and for each matrix $M$ from $\Gamma_2[q]$ of the form

$$M = \begin{pmatrix} E & 0 \\ * & E \end{pmatrix},$$

we have $v(M) = 1$.

Proof. Compare [FH], Proposition 2.4. □

3.4 Proposition. Let $v$ be a multiplier system on a subgroup $\Gamma \subset \Gamma_2$ of finite index. For suitable $q$ we have $\Gamma_2[q] \subset \Gamma$ and such the following holds. Let $U$ be an element from the subgroup that is generated by the matrices $(\begin{smallmatrix} 1 & q \\ 0 & 1 \end{smallmatrix})$ and $(\begin{smallmatrix} 1 & 0 \\ q & 1 \end{smallmatrix})$ for $q \in \mathfrak{q}$ and let

$$M = \begin{pmatrix} \bar{U}^{-1} & * \\ 0 & U \end{pmatrix}.$$

Then $v(M) = 1$.

Proof. Compare [FH], Proposition 2.4. □
4. Embedded subgroups

We restrict now to the case \( n = 2 \). The case \( n > 2 \) can be derived from this easily. Besides the embeddings \( \iota_1, \iota_2 \) we have to consider the embedding

\[
\iota : \text{GL}(2, \mathbb{C}) \longrightarrow \text{U}(2, 2), \quad \iota(U) = \begin{pmatrix} \bar{U}^{-1} & 0 \\ 0 & U \end{pmatrix}.
\]

This gives us an embedding \( \text{SL}(2, \mathfrak{o}) \hookrightarrow \Gamma_2 \). We use the notation \( \text{SL}(2, \mathfrak{o})[q] = \text{kernel}(\text{SL}(2, \mathfrak{o}) \longrightarrow \text{SL}(2, \mathfrak{o}/q)) \).

We have \( w(\iota(U), \iota(V)) = 1 \). Hence, for suitable \( q \)

\[
\text{SL}(2, \mathfrak{o})[q] \longrightarrow S^1, \quad U \longrightarrow v(\iota(U)),
\]

is a homomorphism. We mentioned that the values of \( v \) are \( l \)th roots of unity. Hence the kernel is a subgroup of finite index in \( \text{SL}(2, \mathfrak{o})[q] \).

Our method depends on some game between the embeddings \( \iota_1, \iota_2 \) and \( \iota \). We have

\[
P\iota_1(M)P^{-1} = \iota_2(M), \quad P = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

From Lemma 2.2 follows \( w(\iota_2(M), P) = w(P, \iota_1(M)) \). Hence we obtain from Lemma 3.2 the following result.

**4.1 Lemma.** Let \( v \) be a multiplier system on a subgroup \( \Gamma \subset \Gamma_2 \) of finite index. For suitable \( q \) we have \( \Gamma_2[q] \subset \Gamma \) and

\[
v(\iota_1(M)) = v(\iota_2(M))
\]

for \( M \in \Gamma_1[q] \).

For sake of simplicity we write

\[
v(M) = v(\iota_1(M)) = v(\iota_2(M)).
\]

This is a multiplier system in degree 1. We have

\[
w(M, N) = w(\iota_\nu(M), \iota_\nu(N)), \quad \text{for } \nu = 1, 2.
\]

**4.2 Lemma.** Let \( v \) be a multiplier system on a subgroup \( \Gamma \subset \Gamma_2 \) of finite index. For suitable \( q \) the value \( v(M), M \in \Gamma_1[q] \), depends only on the second row of \( M \).

**Proof.** Compare [FH], Lemma 3.2. \( \square \)
4.3 Lemma. Let \( v \) be a multiplier system on a subgroup \( \Gamma \subset \Gamma_2 \) of finite index. For suitable \( q \) we have \( \Gamma_2[q] \subset \Gamma \) and for any 
\[
M_1 = \begin{pmatrix} a & b_1 \\ c_1 & d_1 \end{pmatrix} \in \text{SL}(2, \mathbb{O})[q] \quad \text{and} \quad M_2 = \begin{pmatrix} a & b_2 \\ c_2 & d_2 \end{pmatrix} \in \Gamma_1[q]
\]
(in particular \( a \in \mathbb{Z} \)) the relation
\[
v \begin{pmatrix} \tilde{d}_1 & -\tilde{c}_1 & 0 & 0 \\ -b_1 & a & 0 & 0 \\ 0 & 0 & a & b_1 \\ 0 & 0 & c_1 & d_1 \end{pmatrix} \cdot v \begin{pmatrix} a & 0 & b_2 & 0 \\ 0 & 1 & 0 & 0 \\ c_2 & 0 & d_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = v \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b_1 b_1 b_2 \\ 0 & 0 & 1 & 0 \\ 0 & c_1 c_1 c_2 & 0 & y \end{pmatrix}
\]
holds (where \( y = d_2 - b_2 c_2(d_1 + \tilde{d}_1) + ab_2 c_2 d_1 \tilde{d}_1 \)).

Proof. Compare [FH], Lemma 3.3. \( \square \)

Now we assume that the multiplier system is of half integral weight. We can restrict it to a subgroup of finite index of the Siegel modular group. Since this group has the congruence subgroup property, \( v \) must agree with the theta multiplier system on a suitable congruence subgroup. Its restriction to the embedded \( \Gamma_1[q] \) is the multiplier system of the classical theta function
\[
1 + 2e^{2\pi i z} + 2e^{2\pi i 4z} + 2e^{2\pi i 9z} + \cdots,
\]
which, for \( q \equiv 0 \mod 4 \), is given by the Kronecker symbol \( \left( \frac{c}{d} \right) \) (see [Di] for details). We will need it only for \( c \neq 0 \) and for odd \( d \). We collect some properties (always assuming this condition)
\[
\left( \frac{c_1 c_2}{d} \right) = \left( \frac{c_1}{d} \right) \left( \frac{c_2}{d} \right), \quad \left( \frac{c}{d_1 d_2} \right) = \left( \frac{c}{d_1} \right) \left( \frac{c}{d_2} \right).
\]

Assume that \( m, n \) are odd coprime numbers such that at least one of them is not negative. Then the usual reciprocity law
\[
\left( \frac{m}{n} \right) \left( \frac{n}{m} \right) = (-1)^{\frac{m-1}{2}} (-1)^{\frac{n-1}{2}}
\]
holds. Assume \( d > 0 \) or \( c_1 c_2 > 0 \). Then
\[
\left( \frac{c_1}{d} \right) = \left( \frac{c_2}{d} \right) \quad \text{if} \quad c_1 \equiv c_2 \mod d.
\]

Also the relation
\[
\left( \frac{c}{d_1} \right) = \left( \frac{c}{d_2} \right) \quad \text{if} \quad \begin{cases} d_1 \equiv d_2 \mod 4c \text{ and } c \equiv 0 \mod 4 \text{ or} \\ d_1 \equiv d_2 \mod c \text{ and } c \equiv 2 \mod 4 \end{cases}
\]
is valid. Finally we mention

\[
\left( \frac{c}{-1} \right) = \begin{cases} 
1 & \text{for } c > 0, \\
-1 & \text{for } c < 0.
\end{cases}
\]

We obtain the existence of a natural number \( q \equiv 0 \mod 4 \) such that \( q \subset (q) \) and such that

\[ v(\iota_1 M) = v(\iota_2 M) = \left( \frac{c}{d} \right) \quad \text{for } M \in \Gamma_1[q]. \]

Since \( ad \equiv 1 \mod bc \) and \( b \equiv 0 \mod 4 \), we get from one of the above rules

\[ \left( \frac{c}{ad} \right) = 1 \quad \text{hence} \quad \left( \frac{c}{a} \right) = \left( \frac{c}{d} \right). \]

Next we assume that \( v \) is trivial on a congruence subgroup inside \( U \). So we can get \( v(H_2) = 1 \). From Lemma 4.3 we get the following result.

There exists a natural number \( q \equiv 0 \mod 4 \) such that for

\[
\begin{align*}
a &\in \mathbb{Z}, \quad c_1 \in \mathfrak{o}, \quad c_2 \in \mathbb{Z}, \\
a &\equiv 1 \mod q, \quad c_1 \equiv 0 \mod q, \quad c_2 \equiv 0 \mod q, \\
a\mathfrak{o} + c_1\mathfrak{o} = \mathfrak{o}, \quad a\mathbb{Z} + c_2\mathbb{Z} = \mathbb{Z}
\end{align*}
\]

the relation

\[ \left( \frac{c_2}{a} \right) = \left( \frac{c_1 \bar{c} c_2}{a} \right) \]

holds. This implies

\[ \left( \frac{cc}{a} \right) = 1 \quad \text{for} \quad a \equiv 1 + q\mathbb{Z}, \ c \equiv q\mathfrak{o}, \ (a, c) = 1. \]

One can apply this relation to \( qc \) for an arbitrary \( c \in \mathfrak{o} \) to obtain

\[ \left( \frac{cc}{a} \right) = 1 \quad \text{for} \quad a \equiv 1 + q\mathbb{Z}, \ c \equiv \mathfrak{o}, \ (a, c) = 1. \]

It is known that there are infinitely many primes of the form \( p = cc [Co] \). We choose one such that \( p \) and \( q \) are coprime. Then we have

\[ \left( \frac{p}{a} \right) = \left( \frac{a}{p} \right) \]

if \( p \) and \( a \) are coprime. We choose \( \alpha \) such that \( \left( \frac{a}{p} \right) = -1 \). We can solve the congruence \( 1 + xq \equiv \alpha \mod p \). Then \( a = 1 + xq \) is coprime to \( p \) and we have

\[ \left( \frac{c\bar{c}}{a} \right) = \left( \frac{p}{a} \right) = \left( \frac{a}{p} \right) = \left( \frac{\alpha}{p} \right) = -1. \]

This is a contradiction. This gives the first part of our main results (for \( n = 2 \) and as a consequence for arbitrary \( n \)).
4.4 Theorem. Let $\Gamma \subset \Gamma_n$ be any subgroup of finite index of a Hermitian modular group of degree $n \geq 2$. Let $v$ be a multiplier system of half integral weight. The restriction of $v$ to the subgroup

$$U = \left\{ U; \begin{pmatrix} U^{-1} & 0 \\ 0 & U \end{pmatrix} \in \Gamma \right\}$$

is a usual character. Its kernel is a non-congruence subgroup of finite index. On a suitable congruence subgroup it agrees with the subgroup constructed in [Ku] in the case $n = 2$ and in [BMS] in the general case.

In the rest of the paper we will give the proof of the second part. (The case $n = 2$ is enough.)

5. Mennicke symbol

We recall the notion of a Mennicke symbol. Let $R$ be a Dedekind domain and $q \subset R$ a non-zero ideal. We introduce the set

$$C(R, q) := \{ (a, b) \in R \times R; \quad Ra + Rb = R, \quad a \equiv 1 \mod q, \quad b \equiv 0 \mod q \}.$$

Every pair $(a, b)$ is the first column of a matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in \text{SL}(2, R)$. Multiplying $M$ with a matrix of the type $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \in \text{SL}(2, R)$ one can achieve $c \equiv 0 \mod q$ and $d \equiv 1 \mod q$.

5.1 Definition. A Mennicke symbol mod $q$ is a map

$$C(R, q) \rightarrow G, \quad (a, b) \mapsto \begin{bmatrix} b \\ a \end{bmatrix},$$

into some group $G$ such that the following properties hold.

MS1 It is invariant under the transformations $(a, b) \mapsto (a + xb, b)$ and $(a, b) \mapsto (a, b + qay)$ for integral $x, y$.

MS2 It satisfies the rule

$$\begin{bmatrix} b_1 b_2 \\ a \end{bmatrix} = \begin{bmatrix} b_1 \\ a \end{bmatrix} \begin{bmatrix} b_2 \\ a \end{bmatrix}.$$

In our context, the group $G$ will be the group of complex numbers of absolute value one. Mennicke symbols have been classified in [BSM] for Dedekind domains of arithmetic type. If $R$ is the ring of algebraic integers in a number field that is not totally imaginary, then the Mennicke symbols are trivial. In the case of a totally imaginary field they can be described explicitly by means of power residue symbols.

The main result of this section is the following theorem.
5.2 Theorem. Let $v$ be a multiplier system of half integral weight on a subgroup of finite index of a Hermitian modular group of degree two. Then there exists a non-zero ideal $q \subset \mathfrak{o}$ with the following properties.

1) $\Gamma_2[q] \subset \Gamma$.

2) There exists a Mennicke symbol $[\cdot]$ for $(\mathfrak{o}, q)$ such that for all $U \in \text{SL}(2, \mathfrak{o})[q]$ one has

$$[c] = v\left(\bar{U}'^{-1} \begin{pmatrix} 0 & 0 \\ 0 & U \end{pmatrix}\right), \quad U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Proof. We mention that the kernel of $v$ on $\text{SL}(2, \mathfrak{o})[q]$ agrees for suitable $q$ with a non congruence subgroup constructed by Kubota [Ku]. The proof of the theorem is given during the rest of this section.

We have to consider also the embeddings $\iota_1, \iota_2 : \Gamma_1[q] \rightarrow \Gamma_2[q]$. As in the Hermitian case we have $v(\iota_1(M)) = v(\iota_2(M))$ an this depends only on the second row of $M \in \Gamma_1[q]$. Hence we can define

$$\{c\} = v\left(\begin{pmatrix} a & 0 & b & 0 \\ 0 & 1 & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right)^{-1} = v\left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & c & 0 & d \end{pmatrix}\right).$$

The elements of $C(\mathfrak{o}, q)$ are the second columns of the matrices in $\text{GL}(2, \mathfrak{o})[q]$. Hence

$$[b] = \left(\iota\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\right)$$

is well-defined on $C(\mathfrak{o}, q)$. We claim that this symbol satisfies MS1. We notice that $w$ is trivial on the image of $\iota$. Hence $v$ is a character on this group. The invariance under $(a, b) \mapsto (a, b + qay)$ follows from the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & qy \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b + qay \\ * & * \end{pmatrix}.$$

To prove the invariance under $(a, b) \mapsto (a + xb, b)$, we consider

$$\begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} = \begin{pmatrix} a + xb & b \\ * & * \end{pmatrix}.$$

Due to Lemma 3.2 we can assume that $v(\iota(M))$ is invariant under conjugation with $\iota\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$. This proves MS1.

We would like to have also MS2. To get a result in this direction, we make use of

$$v(\iota_\nu(M^{-1})) = v(\iota_\nu(M))^{-1}, \quad \nu = 1, 2.$$
This is true since in degree 1 one has $w(M, M^{-1}) = 0$. (This is a general rule for $c \neq 0$ and also for $c = 0$ and $a > 0$. But in our case $c = 0$ implies $a = 1$ since we assume $q > 2$.) From the analogue of Lemma 4.3 we get the general rule (compare Lemma 13.3 in [BMS].)

$$\left[ \begin{array}{c} c_1 \\ a \end{array} \right] \left[ \begin{array}{c} c_2 \\ a \end{array} \right] = \left\{ \frac{c_1 c_2}{a} \right\}.$$  

We insert $c_2 = 1 - a$.

**5.3 Lemma.** We have 

$$\left\{ \begin{array}{c} 1 - a \\ a \end{array} \right\} = 1$$

for $a \equiv 1 \mod q$.

**Proof.** We use

$$\left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ a - 1 & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -1 \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} 2 - a & a - 1 \\ 1 - a & a \end{array} \right).$$

Now we obtain 

$$\left[ \begin{array}{c} c \\ a \end{array} \right] = \left\{ \frac{c(1 - a)}{a} \right\}.$$  

Before we continue, we mention that $\{\}$ is not a Mennicke symbol. It does not satisfy MS1. Nevertheless it is closely related to $[\cdot]$.  

**5.4 Lemma.** We have 

$$\left\{ \begin{array}{c} c \\ d \end{array} \right\} = \left\{ \begin{array}{c} c \\ d + yc \end{array} \right\}$$

and

$$\left\{ \begin{array}{c} c + xqd \\ d \end{array} \right\} = \left\{ \begin{array}{c} c \\ d \end{array} \right\} e^{2 \pi i rs} \text{ where } s = w\left( \left( \begin{array}{cc} * & * \\ c & d \end{array} \right), \left( \begin{array}{cc} 1 & 0 \\ qx & 1 \end{array} \right) \right).$$

**Proof.** The first relation can be derived from

$$\left( \begin{array}{cc} 1 & -y \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} * & * \\ c & d \end{array} \right) \left( \begin{array}{cc} 1 & y \\ 0 & 1 \end{array} \right) = \left( \begin{array}{cc} * & * \\ c & d + cy \end{array} \right).$$

To derive the second one we consider the relation

$$\left( \begin{array}{cc} * & * \\ c & d \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ qx & 1 \end{array} \right) = \left( \begin{array}{cc} * & * \\ c + dxq & d \end{array} \right).$$

It shows

$$\left\{ \begin{array}{c} c + dxq \\ b \end{array} \right\} = \left\{ \begin{array}{c} c \\ d \end{array} \right\} e^{2 \pi i rs}.$$
The \( w \)-value \( s \) is usually not zero. \( \Box \)

But from the corollary of the table of Maass in the introduction we get

\[
\begin{pmatrix}
1 & 0 \\
-qc & 1
\end{pmatrix} = 0.
\]

Using this, we get

\[
\begin{pmatrix}
c \bar{c}(1-a) \\
a
\end{pmatrix} = \begin{pmatrix}c \bar{c} \end{pmatrix}.
\]

So we obtain

\[
\begin{pmatrix}c \bar{c} \end{pmatrix} = \begin{pmatrix}a \bar{a} \end{pmatrix}
\]

and moreover

\[
\begin{pmatrix}c_1c_2 \bar{a} \\
\bar{a}
\end{pmatrix} = \begin{pmatrix}c_1 \bar{c}_1c_2 \bar{c}_2 \\
a
\end{pmatrix} = \begin{pmatrix}c_1 \bar{c}_1 \\
\bar{c}_2 \bar{c}_2
\end{pmatrix} = \begin{pmatrix}c_1 \bar{c}_1 \\
\bar{c}_2 \bar{c}_2
\end{pmatrix}.
\]

This is part of the condition MS2. (We assume up to now \( a \in \mathbb{Z} \)).

5.5 Lemma. Let \((a, b)\) be two elements of \( \mathfrak{o} \) such that \((a, b) = \mathfrak{o}\). Then there exists \( x \in \mathfrak{o} \) such that \( a + xb \) is not divisible by any natural number \( > 1 \).

Proof. In the case \( b = 0 \) we can take \( x = 0 \). Hence we can assume that \( b \neq 0 \).

We write an element \( a \in \mathfrak{o} \) in the form

\[
a = \hat{a} + \bar{a}\omega, \quad \omega = \frac{d + \sqrt{d}}{2}.
\]

We will use

\[
\omega^2 = -N(\omega) + d\omega.
\]

From a solution \( ax + by = 1 \) we derive that the 4 integers

\[
\hat{a}, \; \hat{b}, \; \hat{a}N(\omega), \; \hat{b}N(\omega)
\]

are coprime. We have to find \( x \in \mathfrak{o} \) such that

\[
\hat{a} + \hat{x}\bar{b} = \bar{x}bN(\omega), \quad \hat{a} + \hat{x}\bar{b} = \bar{x}(\hat{b} + \bar{b}d)
\]

are coprime. We consider the greatest common divisors

\[
g = \gcd(\hat{a}, \bar{b}, \bar{b})
\]

By Dirichlet’s prime number theorem we can find \( y \in \mathfrak{o} \) such that

\[
\hat{a} + \hat{y}\bar{b} + y(\hat{b} + \bar{b}d) = gp
\]
where \( p \) is a prime number. There are infinitely many choices for \( p \). Hence we can get that \( p \) is coprime to \( \tilde{b}^2(d^2 - d)/4 + \tilde{b}^2 + \tilde{b} \tilde{d} \). (This expression equals \( N(\tilde{b} + \tilde{b}\omega) \) which is positive.) Now we set

\[
\hat{x} = \hat{y} + t(\tilde{b} + \tilde{b}d), \quad \check{x} = \hat{y} - \tilde{t}\tilde{b} \quad (t \in \mathbb{Z}).
\]

Then we have still

\[
\check{a} + \check{x} \tilde{b} - \check{x} \tilde{b} N(\omega) = \check{a} + \hat{y} \tilde{b} - \tilde{y} \tilde{b} N(\omega) + t(\tilde{b}^2 + \tilde{b} \tilde{d} + \tilde{b}^2 N(\omega)).
\]

Now we consider the greatest common divisor

\[
g' = (\check{a} + \hat{y} \tilde{b} - \tilde{y} \tilde{b} N(\omega), \tilde{b}^2 + \tilde{b} \tilde{d} + \tilde{b}^2 N(\omega)).
\]

We can choose \( t \) such that

\[
\check{a} + \check{x} \tilde{b} - \check{x} \tilde{b} N(\omega) = g'p'
\]

where \( p' \) is a prime. We can choose \( p' \) coprime to \( gp \). Our goal was to get \( gp \) and \( g'p' \) coprime. This means that \( g, g' \) are coprime. But

\[
\text{ggT}(g, g') = \text{ggT}(\check{a}, \tilde{b}, \check{a} + \hat{y} \tilde{b} - \tilde{y} \tilde{b} N(\omega), \tilde{b}^2 + \tilde{b} \tilde{d} + \tilde{b}^2 N(\omega)) = \text{ggT}(\check{a}, \tilde{b}, \check{a}, \tilde{b}) = 1.
\]

This proves Lemma 5.5. \( \square \)

Now we come to the proof of MS2. We can write it in the form

\[
\begin{bmatrix} q^2b_1b_2 \\ a \end{bmatrix} = \begin{bmatrix} qb_1 \\ a \end{bmatrix} \begin{bmatrix} qb_2 \\ a \end{bmatrix}
\]

where \( a, b_1, b_2 \) are in \( \mathfrak{o} \) such that \( a \equiv 1 \mod q \) and such that \( (a, b_1) = (a, b_2) = \mathfrak{o} \). This formula is invariant under the replacement \( b_1 \mapsto b_1 + xa \). By Lemma 5.5 we can assume that \( b_1 \) is not divisible by any natural number. We also want to make an replacement for \( b_2 \). For this we consider the ray class of the principal ideal \( (b_2) \mod \) the ideal \( (a) \). (Recall that two ideals \( b_1, b_2 \) are in the same ray class mod an ideal \( a \) if there exist \( \beta_1 \equiv \beta_2 \equiv 1 \mod a \) such that \( \beta_1 b_1 = \beta_2 b_2 \).) Our product formula does not change if one replaces \( b_2 \) by \( \beta b_2 \) for \( \beta \equiv 1 \mod (a) \). Hence we may replace \( (b_2) \) by any other \( (b'_2) \) in the same ray class. In each ray class there are infinitely many primes. Hence we can assume that \( b_2 \) is coprime to \( N(b_1) \). Now we make the replacement \( b_2 + xaN(b_1) \). Again we make use of Lemma 5.5 to reduce to the case where \( b_2 \) is not divisible by any natural number, and, in addition, is coprime to \( Nb_1 \). We claim that then also \( b_1b_2 \) is not divisible by any natural number. We argue indirectly and assume that there is prime number \( p \) that divides \( b_1b_2 \). Then \( p \) splits in \( \mathfrak{o} \) into two prime
ideals, \((p) = \mathfrak{p}\mathfrak{p}\). Since \(p\) does not divide \(b_1\) and \(b_2\), we can assume \(\mathfrak{p}|b_1\) and \(\mathfrak{p}|b_2\). But then \(\mathfrak{p}|\bar{b}_1\) which contradicts to the fact that \(\bar{b}_1\) and \(b_2\) are coprime.

Now we make the stronger assumption \(a \equiv 1 \mod q^2\). Then \(\bar{a} \equiv 0 \mod q^2\). We can make the replacement \(a \mapsto a + xq^2b_1b_2\) without changing the product formula. This means that we replace \(\bar{a} \mapsto q^2(\bar{a}/q^2 + \bar{y})\) where \(y = xb_1b_2\). Since \(b = b_1b_2\) is not divisible by any natural number, \(\bar{y}\) runs through all rational integers if \(x\) runs through all integers in \(\mathfrak{o}\). This shows that can transform \(\bar{a}\) to 0. So we can assume \(a \in \mathbb{Z}\). But in this case the product formula has been proved. Now we replace \(q\) by \(q^2\) to obtain Theorem 5.2. \(\square\)


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