Search for Multiple Step Integer Quantum Hall Transitions

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Recent experiments in the integer quantum Hall regime seem to find direct transitions from a quantum Hall state with Hall conductance \( \sigma_{xy} = ne^2/h \) with integer \( n \geq 1 \), to an insulating state in weak magnetic fields. We study this issue using a variation of the tight-binding lattice model for non-interacting electrons. Although quantum Hall transitions with change in Hall conductance \( ne^2/h \) with \( n > 1 \) do exist in our model for special tuning of parameters, they generically split into quantum Hall transitions with the Hall conductance changing by \( e^2/h \) at each transition. This suggests that a generic multiple step quantum Hall transition is incompatible with a non-interacting electron picture.

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For non-interacting two-dimensional electron systems in the presence of a perpendicular applied magnetic field \((B)\), Kivelson, Lee and Zhang [1] proposed a global phase diagram, which predicted that in a spin-polarized system, at a quantum Hall transition, the Hall conductance \( \sigma_{xy} \) can only change by \( e^2/h \) as filling factor \((n)\) changes in the integer quantum Hall effect (IQHE) regime. Stimulated by Kivelson et al.’s proposal, a number of experiments reported the observations of \( \sigma_{xy} = 0 \rightarrow 2e^2/h \rightarrow 0 \) transition [2–4], which is consistent with the global phase diagram, assuming the Zeeman splitting in the lowest Landau level is unresolved. However, more recently, many experiments [5–9], some in systems with spin-split Landau bands, observed direct transitions from an insulating state at low fields to \( \sigma_{xy} = ne^2/h \) IQHE states with \( n \) up to 6. Henceforth in this letter, we dub these transitions from the insulating phase to \( n > 1 \) IQHE states multiple step quantum Hall transitions (MSQHTs).

In a series of papers [10–12], Sheng et al. claimed that the experimental studies of the quantum Hall transitions in weak magnetic fields are consistent with their numerical calculations on a tight-binding lattice model. In particular, the MSQHTs are the consequence of the critical energy in the lowest Landau level floating and merging with critical energies in higher Landau levels [13]. However, it remains unclear how these critical energies can merge, and thus give rise to an MSQHT. An alternative interpretation of the apparent MSQHT is a series of single step \((\Delta \sigma_{xy} = e^2/h)\) plateau transitions that are too close to distinguish numerically.

The two scenarios may be resolved by studying the critical behavior in the vicinity of the transition(s) under investigation. For a single, spin-split Landau level, numerical studies by various groups [13–15] using different microscopic Hamiltonians are consistent with the same critical behavior at the quantum Hall transition (QHT). In all cases, the localization length diverges with an exponent \( \nu \approx 2.3 \pm 0.1 \) at the critical energy, \( E_c \), which is at the band center in the case of electron-hole symmetry. By studying the localization length exponent, for instance, Lee and Chalker [17] demonstrated, in a spin-degenerate Landau level, the existence of a pair of delocalization transitions in the same universality class as the spin-split system.

In this letter, we report our search for MSQHTs by truncating the Hilbert space of the tight-binding lattice model to a single (central) magnetic subband, which contains a multiple of \( 2e^2/h \) Hall conductance, and can therefore support a MSQHT in principle. While this model does not correspond directly to the experimental situation, it allows for a much clearer interpretation regarding the possible existence of a multiple step transition. We also expect the study to lead to a clearer understanding of the tight-binding lattice in the context of weak magnetic fields.

Our results can be summarized as follows: we find that the system can support a MSQHT in the limit of infinite disorder, but in this limit the multiple step comes from a set of noninteracting, single step transitions with accidental degeneracy of the step position. As soon as the disorder is made finite, the multiplet step breaks into single steps, whose critical behavior is the same as that for isolated Landau levels.

We consider a tight-binding model of non-interacting spinless electrons on two-dimensional square lattice, with nearest neighbor hopping \((t)\), a uniform perpendicular magnetic field \(B\), and on-site disorder. The Hamiltonian can be written as:

\[
H = -t \sum_{\langle ij \rangle} (c^\dagger_{i\sigma} c_{j\sigma} + h.c.) + \sum_i \epsilon_i c^\dagger_i c_i, \tag{1}
\]

where \( c^\dagger_i (c_i) \) is fermionic creation (annihilation) operators on site \( i\), \( \langle ij \rangle \) are nearest neighbor sites, \( t \) is the hopping strength, and \( a_{ij} \) are phase factors due to the magnetic field \( B\). \( \epsilon_i \) is a random on-site potential, with a distribution \( P(\epsilon) = (2\pi\sigma^2)^{-1/2} \exp(-\epsilon^2/2\sigma^2) \). The magnetic flux per unit cell is chosen to be \( \phi/\phi_0 = \sum_j a_{ij} = 1/(2N+1) \) with integer \( N\), where \( \phi_0 = hc/e \) is the flux quantum.

In the presence of \( B\), the tight-binding energy spectrum splits into \((2N+1)\) magnetic subbands. Each of the
2N side subbands carries a Hall conductance \( \sigma_{xy} = e^2/h \), while the center subband carries \( \sigma_{xy} = -2Ne^2/h \). In contrast to other approaches \([9,12,18]\), we first truncate the Hamiltonian to the Hilbert space spanned by the eigenstates in the center subband in the absence of disorder, before adding the disorder. This results in a single subband with \( \sigma_{xy} = -2Ne^2/h \) for any disorder strength. The negative sign of \( \sigma_{xy} \) is not essential to the discussion here, since we are primarily interested in the scaling behavior near the critical point(s) of the center subband.

We calculate, within the center subband, the Thouless number \([19]\), as a measure of the diagonal conductance \( \sigma_{xx} \) in units of \( e^2/h \), for different lattice sizes \( L \times L \) (\( L \) is the number of sites in each dimension):

\[
g_L(E) = \frac{\langle |\delta E| \rangle}{\Delta E} \sim \frac{h}{e^2} \sigma_{xx},
\]

where \( \langle |\delta E| \rangle \) is the average shift in the energy level due to a change of boundary condition from periodic to anti-periodic, and \( \Delta E = 1/L^2 \rho(E) \) the mean energy level separation, \( \rho(E) \) being the density of states. \( g_L(E) \) is dimensionless, and by finite-size scaling, is expected to be a function of \( L \) only through the ratio \( L/\xi \), where \( \xi \) is the localization length characterizing the wavefunctions at the energy \( E \), which diverges at critical energies \( E_c \), as \( \xi \sim |E - E_c|^{-\nu} \). As a consequence, the square root of the second moment of \( g_L(E) \),

\[
W(L) = \left[ \frac{\int_{-\infty}^{\infty} (E - E_c)^2 g_L(E) dE}{\int_{-\infty}^{\infty} g_L(E) dE} \right]^{1/2} \sim L^{-1/\nu} \] (3)

is expected to scale as \( W(L) \sim L^{-1/\nu} \) in the thermodynamic limit for a single critical energy.

We first consider the case \( \phi/\phi_0 = 1/3 \), where the central subband carries \( \sigma_{xy} = -2e^2/h \), and to further simplify the situation, we begin with the infinite disorder limit \( \sigma/t \to \infty \). Equivalently, we set \( t = 0 \) after truncating the Hilbert space to the center subband only, so that the energy scale is finite. Figure 1 shows \( g_L(E) \) for different \( L \) and \( L \) averaged over 250 to 2500 samples, depending on \( L \). The inset shows the width \( W(L) \) of the curves vs. \( L \) for even \( L \) on a double logarithmic plot. The straight line fit implies a single critical energy \( E_c = 0 \), and a critical exponent \( \nu = 2.2 \pm 0.1 \), consistent with the value obtained for a single step quantum Hall transition in a sideband \([20]\) or an isolated Landau level \([13]\). The data are well fit by a gaussian form, \( g_L(E) = g_0 e^{-(E^2/2W(L)^2)} \), with \( g_0 \approx 0.38 \), approximately twice the value \( g_0 \approx 0.2 \) in a side subband \([20,21]\).

To understand this result, we examined the structure of the projected Hamiltonian. Upon application of the projection operator, the on-site random potential becomes non-local, giving rise to (short-range) hopping. In the \( t = 0 \) limit, there is no direct hopping, and the hopping generated by the random on-site potential due to projection, leads to a bipartite Hamiltonian. Thus, for even-sized lattices, the system splits into two disconnected sublattices interpenetrating each other, with identical statistical properties, since hopping from one sublattice to the other is not generated. This explains why we see a single QHT at \( E = 0 \), with the same value of \( \nu \) as a single-step QHT, but twice the Thouless conductance \( g_0 \), we simply have two independent transitions at the same energy.
pect curves for \(W(L)\) versus \(L\) to be straight lines on a log-log plot for large sizes, with a universal slope, related to the localization length critical exponent for a double step QHT. Instead, we find (see Fig. 2) a series of straight lines with a slope that depends on the disorder \((\sigma/t)\); furthermore, the peak value, \(g_L(0)\) is dependent on \(\sigma/t\) as well. (Note that the odd-even effect gets smaller as \(\sigma/t\) decreases). Non-universal \(\nu\) for other disorder distributions \(P(e)\) have been observed as well [27].

![Graph](image)

**FIG. 2.** Width of Thouless number curves, \(W(L)\), as a function of lattice size \(L\) for \(\sigma/t = 1, 2, 4, \text{ and } 8\).

Figure 3(a) shows the full Thouless number curves for \(\sigma/t = 5\) for \(L = 15, 21, 30\), averaged over 120-600 samples; the curves show a distinct departure from the gaussian form of the scaling curve for the single-step QHT, with a pronounced flat-top shape.

The apparent non-universal exponent is based on the assumption that the critical energy remains at \(E = 0\); if instead, there were two single step QHT at \(\pm E_c\) in the thermodynamic limit, one would expect the \(g_L(E)\) curves to be made up of two contributions. Indeed, a fit of the form

\[
g_L(E) = g(L^{1/\nu}|E - E_c|) + g(L^{1/\nu}|E + E_c|),
\]

is found to work extremely well for all \(|E| \geq E_c\) for all values of the disorder studied with a disorder dependent critical energy \(E_c\) with a single value of \(\nu\). The universal exponent obtained from such a fit is \(\nu = 2.3 \pm 0.2\), which is just the result for a single step QHT. Furthermore, the shape of the scaling curve \(g(L^{1/\nu}|E - E_c|)\) (see Fig. 3(b)), is very well fit the gaussian form obtained for a single step QHT. Finally, the dependence of \(E_c\) on \(\sigma\), shown in the inset of Fig. 3(b), is found to be \(E_c/\sigma = 0.78(t/\sigma)^{0.67}\) for \(0.03 < t/\sigma < 1.0\), consistent with \(E_c/\sigma \to 0\) in the limit of \(t/\sigma \to 0\). The fit of Eq. 4 does not work as well between the two critical energies presumably because of the extremely large localization lengths, much beyond the sizes of systems studied, for this region. Thus the data implies that the accidental double step for infinite disorder splits into two single steps for the generic, finite disorder case, leading to a behavior similar to that obtained in double-layer systems [22] or for electrons with spin-orbit coupling [17].

![Graph](image)

**FIG. 3.** (a) Thouless number \(g_L(E)\) of the center subband \((N = 1)\) of lattices of linear dimensions \(L = 15, 21\) and \(30\) for \(\sigma/t = 5\). (b) Scaling curve of the Thouless number, \(g(L^{1/\nu}|E - E_c|)\), with \(E > E_c\) and \(\nu = 2.5\). Dashed line shows the gaussian curve found for single step quantum Hall transitions. Inset shows that the \(E_c\), in units of \(\sigma\), can be fit to a power law (dashed line) as a function of \(t/\sigma\).

We next consider the case with \(\phi/\phi_0 = 1/5 (N = 2)\), in which the center subband has total Hall conductance \(-4e^2/h\). Since this case allows for many more possibilities, we restrict our attention to the infinite disorder \((t = 0)\) case. In this limit we again obtain a Hamiltonian that is bipartite for even-sized lattices. This implies that the transition in the center subband can be either (a) two double step quantum Hall transitions at \(E = 0\), or (b) a pair of split, single step quantum Hall transitions, each pair occurring at \(E = \pm E_c\). Figure 4 shows \(g_L(E)\) of even-size lattices for infinite disorder, averaged over 80 to 1800 samples, depending on system size.

Forcing the width \(W(L)\) to scale as a power law in \(L\) over the sizes studied yields an exponent \(\nu = 4.2\). However, the shape of the \(g_L(E)\) curves is rather flat-topped compared to a gaussian, and the almost doubled value of \(\nu\) also suggests that a more natural scenario is (b)
above. Indeed, fitting to the form given by Eq. (1) for $|E| > E_c$ yields $E_c = 0.3$, $g_0 = 0.4$, $\nu = 2.4$, and furthermore a gaussian shape for the scaling curve (see inset of Fig 4). Thus, with a single adjustable parameter, $E_c$, the data are consistent with a pair of degenerate single step transitions, with exactly the same Thouless number peak, critical exponent, and scaling curve. This splitting of the critical energy in each sublattice argues strongly for single step quantum Hall transitions being the generic situation for non-interacting electrons.

![Graph showing Thouless number $g_c(E)$](image)

**FIG. 4.** Thouless number $g_c(E)$ of the center subband ($N = 2$) with infinite disorder ($\sigma = 1$, $t = 0$) for lattices with $L = 20$, 30, 40, and 50. The inset shows the scaling collapse of the Thouless number $g(L^{1/\nu}|E - E_c|)$, using the four $g(E)$ curves for $E > E_c = 0.3$ and $\nu = 2.4$.

Our study has raised questions about the interpretation of the results found in the full tight-binding lattice model [9-12]. In particular, a genuine MSQHT must be associated with degenerate critical energies, which is less likely to appear in the full model with finite amount of disorder and critical energies in all side subbands that can interact with those in the center subband. Very recently, Koschny et al. [23] have claimed to resolve two separate transitions with filling factor $n = 2$ in the full lattice model, and thus contradicted the direct transition from insulator to high Hall plateaus suggested by Sheng et al. Nevertheless, the difficulty of unambiguously resolving this issue, as well as the obstacle that forbids us to obtain the scaling behavior between adjacent critical energies, is consistent with the existence of the large localization length well beyond the sizes accessible to present numerical studies. The length scale may, indeed, agree with the large crossover length argued by Huckestein [24] in the framework of the standard scaling theory of the IQHE.

To summarize, we have found that for non-interacting electrons, even in a single band with total Hall conductance a multiple of $2e^2/h$, in the presence of disorder, a sequence of single step quantum Hall transitions appears to be the generic situation. The only multiple step transition we have been able to obtain is by special tuning of parameters, for which there exist accidental degeneracies of critical energies along with zero coupling of these degenerate critical energies. In such cases, the multiple step transition is a trivial superposition of single step transitions, with the same exponents for the localization length ($\nu = 2.3$) and scaling curves for the Thouless conductance. This calls into question the interpretation of the apparent multiple step transitions seen in experiment [9] using a non-interacting model [12].

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