On the minimal value of global Tjurina numbers for line arrangements

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Dedicated to the memory of my dear friend Ştefan Papadima

1 Introduction and main results

Let $S = \mathbb{C}[x, y, z]$ be the graded polynomial ring in three variables $x, y, z$ and let $C : f = 0$ be a reduced curve of degree $d$ in the complex projective plane $\mathbb{P}^2$. The minimal degree of a Jacobian relation for $f$ is the integer $\text{mdr}(f)$ defined to be the smallest integer $m \geq 0$ such that there is a nontrivial relation

$$af_x + bf_y + cf_z = 0$$
among the partial derivatives $f_x$, $f_y$ and $f_z$ of $f$ with coefficients $a, b, c$ in $S_m$, the vector space of homogeneous polynomials of degree $m$. We say that the plane curve has type $(d, r)$ if $d = \deg(f)$ and $r = \text{mdr}(f)$.

When $\text{mdr}(f) = 0$, then $C$ is a pencil of lines, i.e., a union of lines passing through one point, a situation easy to analyze. We assume from now on that

$$\text{mdr}(f) \geq 1.$$ 

Denote by $\tau(C)$ the global Tjurina number of the curve $C$, which is the sum of the Tjurina numbers of the singular points of $C$. When $C$ is a line arrangement, its global Tjurina number coincides with its global Milnor number $\mu(C)$, and is given by

$$\tau(C) = \sum_p (n(p) - 1)^2,$$  \hspace{2cm} (1.1)

the sum being over all multiple points $p$ of $C$, and $n(p)$ denoting the multiplicity of $C$ at $p$. In this note we consider the minimal values of $\tau(C)$ when $C : f = 0$ is a line arrangement, once we fix its type $(d, r)$. Let $m(C)$ be the maximal multiplicity of a point in $C$, and $n(C)$ the maximal multiplicity of a point in $C \setminus \{p\}$, where $p$ is any point in $C$ of multiplicity $m(C)$. Note that

$$1 \leq n(C) \leq m(C) \leq d.$$ 

Moreover $m(C) = d$ if and only if $\text{mdr}(f) = 0$, and $m(C) = d - 1$ if and only if $\text{mdr}(f) = 1$, see [8, Proposition 4.7]. In addition, the case $2 = n(C) \leq m(C) \leq d - 2$ corresponds to the intersection lattice $L(C)$ being the lattice $L(d, m(C))$ discussed in [8, Proposition 4.7], i.e., the intersection lattice of an arrangement obtained from $m(C)$ concurrent lines by adding $d - m(C)$ lines in general position. The main result of this note is the following improvement of the lower bound of $\tau(C)$ for an arbitrary reduced plane curve $C$, given by du Plessis–Wall in [18], see also Ellia [19]. Their result is stated below in Theorem 2.1.

**Theorem 1.1** Let $C : f = 0$ be an arrangement of $d \geq 4$ lines in $\mathbb{P}^2$ which is not free. If we set $r = \text{mdr}(f) \geq 2$ and $\tau(d, r)_{\text{min}} = (d - 1)(d - r - 1)$, then the following hold:

(i) With the above notation,

$$\tau(C) \geq \tau'(d, r)_{\text{min}} := \tau(d, r)_{\text{min}} + \binom{r}{2} + \binom{n(C)}{2} + 1.$$ 

(ii) If $r \neq d - m(C)$, then the possibly stronger inequality

$$\tau(C) \geq \tau''(d, r)_{\text{min}} := \tau(d, r)_{\text{min}} + \binom{r}{2} + \binom{m(C)}{2} + 1$$

holds.

\[ Springer \]
The line arrangements such that \( r = \text{mdr}(f) \in \{0, 1, 2\} \) are classified, see [24] or [8, Theorem 4.11] for the case \( r = 2 \), which is the only difficult case. For the remaining line arrangements we have the following.

**Corollary 1.2** Let \( C : f = 0 \) be an arrangement of \( d \) lines in \( \mathbb{P}^2 \) which is not free and such that \( r = \text{mdr}(f) \geq 3 \) and \( n(C) \geq 3 \). Then

\[
\tau(C) \geq \tau^N(d, r)_{\min} := \tau(d, r)_{\min} + \binom{r}{2} + 4 \geq \tau(d, r)_{\min} + 7.
\]

**Remark 1.3** Another lower bound \( \tau^O(d, r)_{\min} \) for \( \tau(C) \) was obtained in [16, Theorem 3.5 (2)], where it is shown that

\[
\tau(C) \geq \tau^O(d, r)_{\min} := \tau(d, r)_{\min} + \frac{r^2 - 5r + 10}{2} \geq 2,
\]

for any \( r \geq 3 \), and hence the new result is much stronger.

The above behavior of the lower bounds for the global Tjurina number is in sharp contrast to the behavior of the upper bounds for the global Tjurina number. Indeed, a general upper bound for the global Tjurina number of a reduced complex projective plane curve, given by du Plessis and Wall, is conjecturally realized by line arrangements as soon as \( r = \text{mdr}(f) \leq d - 2 \), see Remark 2.3 for more details.

**Example 1.4** Consider the line arrangements \( C : f = 0 \) of \( d = 7 \) lines such that \( r = \text{mdr}(f) = 3 \). Then \( \tau(7, 3)_{\min} = 6 \cdot 3 = 18 \) and

\[
\tau'(7, 3)_{\min} = 18 + 3 + \binom{n(C)}{2} + 1 = 22 + \binom{n(C)}{2}.
\]

The value \( \tau = 25 \) is obtained for the following three line arrangements. Two of them, say

\[
C_1 : f_1 = xyz(x + y)(x + 3y)(x + 2y + z)(4x + 8y + z) = 0
\]

and

\[
C_2 : f_2 = xyz(x + 2y + z)(y + z)(x - 2y)(x - y) = 0,
\]

satisfy \( m(C_i) = 4 \) and \( n(C_i) = 3 \), since each of them has a quadruple point, a triple point and 12 double points. The two intersection lattices \( L(C_1) \) and \( L(C_2) \) are distinct, since only in \( C_2 \) there is a line in the arrangement containing both points of multiplicity
> 2. Our lower bound for both \( C_1 \) and \( C_2 \) is \( \tau^N(7, 3)_\text{min} = 22 + 3 = 25 \), i.e., a sharp lower bound. The third line arrangement, say 

\[ C_3 : f_3 = xyz(2x - 3y + z)(x - y)(x + z)(y + z) = 0, \]

satisfies \( m(C_3) = n(C_3) = 3 \), since it has four triple points and nine double points. Hence we can use the bounds \( \tau'(d, r)_{\text{min}} = \tau''(d, r)_{\text{min}} = 25, \) since \( 3 = \text{mdr}(C_3) \neq d - m(C_3) = 4 \). Therefore our various bounds are sharp in all these three situations. In all the three cases one has \( \tau^0(d, r)_{\text{min}} = 18 + 5 = 23 \), hence a weaker lower bound.

**Example 1.5** Consider the line arrangements \( C : f = 0 \) of \( d \) lines, with intersection lattice \( L(C) \) of type \( \tilde{L}(m_1, m_2) \) as in [8, Proposition 4.9], where \( 2 \leq m_1 \leq m_2 \). This line arrangement is the union of two pencils of lines, one consisting of \( m_1 \) lines, the other of \( m_2 \) lines, in general position to each other. Then we have \( d' = m_1 + m_2 \), \( r = \text{mdr}(f) = m_1, m(C) = m_2, n(C) = m_1 \), 

\[ \tau(C) = (d - 1)^2 - m_1 m_2 + 1. \]

One has \( \tau(d, r)_{\text{min}} = (d - 1)^2 - m_1 m_2 - (m_1^2 - m_1) \), while the lower bound given by Theorem 1.1 (i) is in this case 

\[ \tau'(d, r)_{\text{min}} = (d - 1)^2 - m_1 m_2 + 1, \]

again a sharp bound.

Let \( I_f \) denote the saturation of the ideal \( J_f \) with respect to the maximal ideal \( m = (x, y, z) \) in \( S \) and consider the local cohomology group 

\[ N(f) = I_f / J_f = H^0_{\text{m}}(M(f)). \]

The graded \( S \)-module \( N(f) \) satisfies a Lefschetz type property with respect to multiplication by generic linear forms, see [10]. This implies, in particular, the inequalities 

\[ 0 \leq n(f)_0 \leq n(f)_1 \leq \cdots \leq n(f)_{[T/2]} \geq n(f)_{[T/2]+1} \geq \cdots \geq n(f)_T \geq 0, \]

where \( T = 3d - 6 \) and \( n(f)_k = \dim N(f)_k \) for any integer \( k \). We set 

\[ v(C) = \max_j \{ n(f)_j \}, \]

and call \( v(C) \) the *freeness defect* of the curve \( C \). It is known that a curve \( C \) is free (resp. nearly free) if and only if \( v(C) = 0 \) (resp. \( v(C) = 1 \)), see [6]. Theorem 1.1 yields the following.

**Corollary 1.6** Let \( C : f = 0 \) be an arrangement of \( d \) lines in \( \mathbb{P}^2 \) which is not free. If we have \( 2 \leq r = \text{mdr}(f) < d/2 \), then 

\[ v(C) \leq v^N(C) := \frac{r(r + 1)}{2} - \left( \frac{n(C)}{2} \right) - 1 \leq \frac{r(r + 1)}{2} - 2. \]
It is known that a line arrangement with \( r = 2 \) is either free or nearly free, see [8, Theorem 4.11]. Hence in this case \( \nu(C) \in \{0, 1\} \), and the upper bound given by Corollary 1.6 is 1, hence this bound is sharp in this case.

**Remark 1.7** Another upper bound \( \nu^O(C) \) for the freeness defect \( \nu(C) \) of a line arrangement is discussed in [14]. For instance, for the case when \( d = |C| \) is odd, it is shown that

\[
\nu(C) \leq \nu^O(C) := \frac{1}{4} \left( (d - 1)(d - 3) - \tau(C) + \sum_{j \text{ even}} \nu_j \right),
\]

where \( \nu_j \) denotes the number of points of multiplicity \( j \) in the line arrangement \( C \). For the arrangements \( C_1 \) and \( C_2 \) from Example 1.4 we have \( \nu^N(C) = 6 - 3 - 1 = 2 \) and

\[
\nu^O(C) = \frac{1}{4} (24 - 25 + 1 + 12) = 3.
\]

Hence in this case the new upper bound \( \nu^N(C) \) from Corollary 1.6 is sharper than the old bound \( \nu^O(C) \) from [14]. On the other hand, consider the line arrangement

\[
\mathcal{B}: f = y(y + x)(y - x)(y + x - 2z)(y - x - 2z)(3y + x - 2z)(3y - x - 2z) = 0.
\]

Here \( d = 7, \tau(C) = 26, r = \text{mdr}(f) = 3, \) and there are five triple points and six double points. Hence in this case one has \( \nu^N(C) = 6 - 3 - 1 = 2 \) as above, and

\[
\nu^O(C) = \frac{1}{4} (24 - 26 + 6) = 1.
\]

Therefore in this case the new upper bound \( \nu^N(C) \) from Corollary 1.6 is weaker than the old bound \( \nu^O(C) \) from [14].

For any curve reduced \( C : f = 0 \) in \( \mathbb{P}^2 \), we consider the gradient mapping

\[
\psi_f = \text{grad}(f) : \mathbb{P}^2 \rightarrow \mathbb{P}^2.
\]

It is a rational map, defined by

\[
(x : y : z) \mapsto (f_x(x, y, z) : f_y(x, y, z) : f_z(x, y, z)).
\]

It is known that

\[
\deg(\psi_f) = (d - 1)^2 - \sum_{\mathcal{P} \in C} \mu(C, \mathcal{P}),
\]

see [9, p.487]. When \( C \) is a line arrangement, the above sum of local Milnor numbers is exactly the global Tjurina number \( \tau(C) \), just recall formula (1.1). In particular, when \( C \) is a free line arrangement of type \((d, r)\), then \( \deg(\psi_f) = r(d - 1) - r^2 \). Moreover, Theorems 1.1 and 2.1 imply the following.
Corollary 1.8 Let $C : f = 0$ be an arrangement of $d$ lines in $\mathbb{P}^2$ which is not free. If we have $2 \leq r = mdr(f) < d/2$, then
\[
r(d-1) - r^2 < \deg(\psi_f) \leq r(d-1) - {r \choose 2} - \left( \frac{n(C)}{2} \right) - 1.
\]

2 Global Tjurina numbers and related results

The following result is due to du Plessis and Wall, see [18, Theorem 3.2], and was the motivation of a number of papers in this area, see for instance [6,19].

Theorem 2.1 For a positive integer $r$, define two integers by
\[
\tau(d,r)_{\text{min}} = (d-1)(d-r-1) \quad \text{and} \quad \tau(d,r)_{\text{max}} = (d-1)(d-r-1) + r^2.
\]

For any reduced plane curve $C : f = 0$ of type $(d,r)$, one has the inequalities
\[
\tau(d,r)_{\text{min}} \leq \tau(C) \leq \tau(d,r)_{\text{max}}.
\]

Moreover, suppose that $r = mdr(f) > (d-1)/2$. Then one has the stronger inequality
\[
\tau(C) \leq \tau(d,r)_{\text{max}} - \left( \frac{2r + 2 - d}{2} \right).
\]

In particular, if $d$ is even and $r = d/2$, then $\tau(C) \leq \tau(d,r)_{\text{max}} - 1$.

It is easy to see that the lower bound is sharp, i.e., there are curves $C$ of type $(d,r)$ such that $\tau(C) = \tau(d,r)_{\text{min}}$ for any degree $d$ and $1 \leq r \leq d - 1$, see [16, Example 4.5] in general and [19, Lemma 21] for $(d,r) = (d,d-2)$. Such curves are called minimal Tjurina curves and they are completely described in [16, Theorem 3.5 (1)]. The curves $C$ for which the maximal values for $\tau(C)$ are attained have special properties. Indeed, we have the following result, see [6].

Theorem 2.2 Let $C : f = 0$ be a reduced plane curve of type $(d,r)$. Then the following hold:

(i) One has $\tau(C) = \tau(d,r)_{\text{max}}$ if and only if $C : f = 0$ is a free curve with exponents $(r,d-1-r)$, and then $r < d/2$.

(ii) One has $\tau(C) = \tau(d,r)_{\text{max}} - 1$ if and only if $C : f = 0$ is a nearly free curve with exponents $(r,d-r)$, and then $r \leq d/2$.

The free and nearly free curves, as well as related objects, are actively investigated since some time, see [1,3,4,6,7,12,13,19,23–25].

Remark 2.3 Note that there are free (resp. nearly free line arrangements) in $\mathbb{P}^2$ for any fixed type $(d,r)$ with $r < d/2$ (resp. with $r \leq d/2$), i.e., the maximal values for $\tau(C)$ can be attained with $C$ a line arrangement, at least when $r \leq d/2$. It is conjectured that, for any $d/2 \leq r \leq d - 2$, there are Tjurina maximal line arrangements, i.e.,
arrangements $C : f = 0$ of $d$ lines such that $r = \text{mdr}(f)$ and $\tau(C) = \tau(d, r)_{\text{max}}$, see [2,17] for many supporting examples.

**Remark 2.4** One can easily check that the function $\tau : \{1, 2, \ldots, d-1\} \rightarrow \mathbb{Z}$ given by $\tau(r) = \tau(d, r)_{\text{max}}$ for $r < d/2$ and

$$\tau(r) = \tau(d, r)_{\text{max}} - \left(\frac{2r + 2 - d}{2}\right)$$

for $d/2 \leq r \leq d - 1$ is strictly decreasing.

The relation between the invariants $\nu(C)$ and $\text{mdr}(f)$ is given by the following result, see [7, Theorem 1.2].

**Theorem 2.5** Let $C : f = 0$ be a reduced plane curve of degree $d$ and let $r = \text{mdr}(f)$.

Then the following hold:

(i) If $r < d/2$, then

$$\nu(C) = (d - 1)^2 - r(d - 1 - r) - \tau(C) = \tau(d, r)_{\text{max}} - \tau(C).$$

(ii) If $r \geq (d - 2)/2$, then

$$\nu(C) = \left\lceil \frac{3}{4} (d - 1)^2 \right\rceil - \tau(C).$$

Here, for any real number $u$, $\lceil u \rceil$ denotes the round up of $u$, namely the smallest integer $U$ such that $U \geq u$.

We recall now the construction of the Bourbaki ideal $B(C, \rho_1)$ associated to a degree $d$ reduced curve $C : f = 0$ and to a minimal degree nonzero syzygy $\rho_1 \in \text{AR}(f)$, see [15]. For any choice of the syzygy $\rho_1 = (a_1, b_1, c_1) \in \text{AR}(f)$ with minimal degree $r = d_1$, we have a morphism of graded $S$-modules

$$S(-r) \xrightarrow{u} \text{AR}(f), \quad u(h) = h \cdot \rho_1.$$  

For any homogeneous syzygy $\rho = (a, b, c) \in \text{AR}(f)$, consider the determinant $\Delta(\rho) = \det M(\rho)$ of the $3 \times 3$ matrix $M(\rho)$ which has as first row $x, y, z$, as second row $a_1, b_1, c_1$ and as third row $a, b, c$. Then it turns out that $\Delta(\rho)$ is divisible by $f$, see [6], and we define thus a new morphism of graded $S$-modules

$$\text{AR}(f) \xrightarrow{v} S(r - d + 1), \quad v(\rho) = \frac{\Delta(\rho)}{f}. \quad (2.1)$$

and a homogeneous ideal $B(C, \rho_1) \subset S$ such that $\text{im} \ v = B(C, \rho_1)(r - d + 1)$. It is known that the ideal $B(C, \rho_1)$, when $C$ is not a free curve, defines a 0-dimensional subscheme $Z(C, \rho_1)$ in $\mathbb{P}^2$, see [15, Proposition 2.1]. This construction yields the following exact sequence:

$$0 \rightarrow S(m - d) \xrightarrow{u} \text{AR}(f) \xrightarrow{v} B(C, \rho_1)(-m + 1) \rightarrow 0. \quad (2.2)$$
3 Proof of Theorem 1.1 and of Corollary 1.6

Consider the graded $S$-submodule $\text{AR}(f) \subset S^3$ of all relations involving the derivatives of $f$, namely

$$\rho = (a, b, c) \in \text{AR}(f)_{q}$$

if and only if $af_x + bf_y + cf_z = 0$ and $a, b, c$ are in $S_q$, the space of homogeneous polynomials of degree $q$. We have the following.

**Lemma 3.1** Let $C : f = 0$ be a reduced plane curve of degree $d$ such that any irreducible component of $C$ is rational. Then

$$\tau(C) = \dim \text{AR}(f)_{d-2} + \frac{(d - 1)(d - 2)}{2}.$$ 

**Proof** Use [11, Formula (3.4)] and the vanishing of $N(f)_{2d-3}$, provided by [14, Theorem 2.7]. \hfill $\Box$

Let $\rho_1 \in \text{AR}(f)_r$ be a nontrivial relation of minimal degree. The inclusion $S \cdot \rho_1 \subset \text{AR}(f)$ implies that

$$\dim \text{AR}(f)_{d-2} = \dim S_{d-r-2} + \dim \overline{\text{AR}(f)}_{d-2},$$

where $\overline{\text{AR}(f)} = \text{AR}(f)/(S\rho_1)$. This implies

$$\tau(C) = \dim \text{AR}(f)_{d-2} + \frac{(d - 1)(d - 2)}{2}$$

$$= (d - 1)^2 - r\left(d - \frac{r + 1}{2}\right) + \dim \overline{\text{AR}(f)}_{d-2}.$$ 

The exact sequence (2.2) shows that the graded $S$-module $\overline{\text{AR}(f)}$ is torsion free. Let $p \in C$ (resp. $q \in C$) be two distinct points such that $p$ (resp. $q$) has multiplicity $m = m(C)$ (resp. $n = n(C)$). Proceeding as in [4, Section (2.2)], we get two syzygies, namely $\rho_p \in \text{AR}(f)_{d-m}$ and $\rho_q \in \text{AR}(f)_{d-n}$. We have the following.

**Lemma 3.2** Assume that $C$ is a line arrangement of $d$ lines. Then both syzygies $\rho_p$ and $\rho_q$ are primitive, and moreover

$$\rho_q \not\in S_{m-n} : \rho_p.$$ 

**Proof** Following [4, Section (2.2)], we recall the construction of the two syzygies $\rho_p$ and $\rho_q$. First choose the coordinates on $\mathbb{P}^2$ such that $p = (1 : 0 : 0)$ and $q = (0 : 1 : 0)$. To construct $\rho_p$, we write $f = g(y, z)h(x, y, z)$, where $g = 0$ is the equation of the $m = m(C)$ lines passing through $p$, and $h = 0$ is the equation of the remaining $d - m$ lines in $C$. With this notation, one has

$$\rho_p = (xh_x - d \cdot h, yh_x, zh_x).$$
where \( h_x \) denotes the partial derivative of \( h \) with respect to \( x \). Moreover, one has \( f_x = gh_x \). Similarly, one can write \( f = \tilde{g}(x, z)\tilde{h}(x, y, z) \), where \( \tilde{g} = 0 \) is the equation of the \( n = n(C) \) lines passing through \( q \), and \( \tilde{h} = 0 \) is the equation of the remaining \( d - n \) lines in \( C \). With this notation, one has

\[
\rho_q = (x\tilde{h}_y, y\tilde{h}_y - d\cdot\tilde{h}, z\tilde{h}_y),
\]

where \( \tilde{h}_y \) denotes the partial derivative of \( \tilde{h} \) with respect to \( y \). Moreover, one has \( f_y = \tilde{g}\tilde{h}_y \).

The fact that the syzygy \( \rho_p \) is primitive, that is it cannot be written as \( \rho_p = h\rho \) for a homogeneous polynomial \( h \in S \) with \( \text{deg}(h) > 0 \) and some syzygy \( \rho \in \text{AR}(f) \) is proven in [4, Section (2.2)]. The same argument applies to \( \rho_q \).

Assume there is a polynomial \( A \in S_{m-n} \) such that \( \rho_q = A\rho_p \). Looking at the third coordinate, we get \( \tilde{h}_y = Ah_x \). Then the equality on the first coordinate implies \( Ah = 0 \), which is a contradiction. \( \square \)

Now we complete the proof of Theorem 1.1, and in view of Example 1.5 we may assume that \( C \) is not of type \( \tilde{L}(m_1, m_2) \). We apply [4, Theorem 1.2] and discuss the following cases, in order to prove claim (i).

**Case 1:** \( r = \text{mdr}(f) = d - m \). Then we can take \( \rho_1 = \rho_p \) and the class of \( \rho_q \) in \( \text{AR}(f) \) is nonzero, because of Lemma 3.2 and of the fact that the syzygies constructed in [4, Section (2.2)] are primitive, i.e., they are not multiple of strictly lower degree syzygies. It follows that the vector space \( S_{n-2}\rho_q \) is naturally embedded in \( \text{AR}(f)_{d-2} \).

Let \( h_q = v(\rho_q) \in B_{d-n-m+1} \), where \( v \) is the morphism defined in (2.1). We get

\[
S_{n-2}h_q \subset \text{AR}(f)_{d-2} = B_{d-m-1}.
\]

On the other hand we know, see [22, Corollary 3.5] or [8, Corollary 3.6], that the Castelnuovo–Mumford regularity \( \text{reg}(\text{AR}(f)) \) of the graded \( S \)-module \( \text{AR}(f) \) satisfies \( \text{reg}(\text{AR}(f)) \leq d - 2 \) as soon as \( d \geq 4 \). In particular, all the generators in a minimal set of generators for \( \text{AR}(f) \) have degrees \( \leq d - 2 \). It follows that our ideal \( B \) is generated in degrees \( \leq d - m - 1 \). Moreover we know that the subscheme in \( \mathbb{P}^2 \) defined by the ideal \( B \) is 0-dimensional, [15, Proposition 2.1], and hence \( B \) cannot be generated by a single polynomial \( h_q \). It follows that there is at least one polynomial \( \hat{h} \in B_{d-m-1} \setminus S_{n-2}h_q \) and hence

\[
\dim \text{AR}(f)_{d-2} \geq \dim S_{n-2} + 1.
\]

This completes the proof of claim (i) in this case.

**Case 2:** \( r = \text{mdr}(f) = m - 1 \) and \( C \) is free, the case which is discarded by our hypotheses.

**Case 3:** \( m \leq r = \text{mdr}(f) \leq d - m - 1 \). Then let \( \rho_1 \in \text{AR}(f)_r \) be a minimal degree nonzero syzygy. Exactly as in Case 1 above, it follows that the vector space \( S_{m-2}\rho_p \) is naturally embedded in \( \text{AR}(f)_{d-2} = B_{r-m-1} \), and this proves claims (i) and (ii) in
this case as well, exactly as above. Indeed, by Lemma 3.2, we know that $\rho_p \notin S \cdot \rho_1$, since $\rho_p$ is a primitive syzygy.

### 3.1 Proof of Corollary 1.6

One has

$$
\nu(C) = \tau(d, r)_{\max} - \tau(C) \leq \tau(d, r)_{\max} - \tau'(d, r)_{\min} = \frac{r(r + 1)}{2} - \left(\frac{n(C)}{2}\right) - 1 \leq \frac{r(r + 1)}{2} - 2.
$$

### 4 An application to Terao’s conjecture

Terao conjectured that if $\mathcal{A}$ and $\mathcal{A}'$ are hyperplane arrangements in $\mathbb{P}^n$ with isomorphic intersection lattices $L(\mathcal{A}) = L(\mathcal{A}')$, and if $\mathcal{A}$ is free, then $\mathcal{A}'$ is also free, see for details [5, 21, 25]. Using Theorem 1.1, we get the following partial positive answer in the case of line arrangements, which is an improvement of [6, Corollary 2.3].

**Theorem 4.1** Let $C : f = 0$ and $C' : f' = 0$ be two arrangements of $d$ lines in $\mathbb{P}^2$ with isomorphic intersection lattices $L(C) = L(C')$. If $C$ is free, then there is a unique integer $r \geq 0$ such that $r < d/2$ and

$$
\tau(C) = (d - 1)(d - r - 1) + r^2.
$$

If this integer $r$ satisfies

$$
r < \frac{-3 + \sqrt{8d + 41}}{2},
$$

then the line arrangement $C'$ is also free.

**Proof** We can assume that $n(C) \geq 3$, since in the other cases Terao’s conjecture is obvious. Indeed, for $n(C) \leq 2$, the intersection lattice $L(C)$ has the type $L(d, m(C))$, and the freeness of $C$ depends only on $m(C)$, see [8, Proposition 4.7].

It is clear that $r$ should be equal to $\text{mdr}(f)$ by Theorem 2.2 (i). Let $s = \text{mdr}(f')$. Then Theorem 2.1 and Remark 2.4 imply that $s \leq r$. Indeed, if we suppose $s > r$, we get a contradiction as follows. First one has $\tau(C) = \tau(C')$, since this number depends only on the intersection lattice $L(C) = L(C')$. Then Remark 2.4 implies that

$$
\tau(C) = \tau(d, r)_{\max} = \tau(r) > \tau(s),
$$

since $C$ is free. Hence we get

$$
\tau(C') = \tau(C) > \tau(s) = \tau(d, s)_{\max},
$$

because $s < r$.
in contradiction with Theorem 2.1.

When \( n(C) \geq 3 \), a direct computation shows that for \( s < (-3 + \sqrt{8d + 41})/2 \), the intervals \([\tau^N(d, s)_{\min}, \tau(d, s)_{\max}]\) and \([\tau^N(d, s - 1)_{\min}, \tau(d, s - 1)_{\max}]\) are disjoint, since

\[
\tau(d, s)_{\max} < \tau^N(d, s - 1)_{\min}.
\]

It follows that each value \( \tau(d, s)_{\max} \) uniquely determines the corresponding \( s \) when \( s < (-3 + \sqrt{8d + 41})/2 \). It follows that \( \text{mdr}(f') = \text{mdr}(f) = r \) and hence \( C' \) is free by Theorem 2.2 (i).

\[\square\]

**Remark 4.2** Note that the assumption \( L(C) = L(C') \) in Theorem 4.1 can be replaced by the much weaker conditions \( \tau(C) = \tau(C'), n(C) \geq 3 \) and \( n(C') \geq 3 \).

**Example 4.3** When \( d = 100 \), then we have

\[
\frac{-3 + \sqrt{8d + 41}}{2} = \frac{-3 + 29}{2} = 13.
\]

Hence Terao’s conjecture holds for any free arrangement \( C : f = 0 \) of 100 lines with \( r \leq 13 \). The old result [6, Corollary 2.3] when \( d = 100 \) gives the bound \( r \leq \sqrt{d - 2} \), hence \( r \leq 9 \).

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