ABSTRACT THEORY OF DECAY ESTIMATES: PERTURBED HAMILTONIANS

MANUEL LARENAS AND AVY SOFFER

ABSTRACT. For two self-adjoint operators $H, A$ we show that a general commutation relation of type $[H, iA] = Q(H) + K$, in addition to regularity of $H$ and Kato-smoothness of $K$, guarantee pointwise in time decay rates of diverse order. The methodology is based on the construction of a modified conjugate operator $\tilde{A}$ that reduces the problem to previously developed estimates when $K = 0$. Our results apply to energy thresholds and do not rely on resolvent estimates. We discuss applications for the Schrödinger equation (SE) with potential of critical decay, and for the free SE on an asymptotically flat manifold.

CONTENTS

1. Introduction 1
2. Preliminaries 2
3. Assumptions 3
4. The conjugate operator 4
5. Decay estimates 8
6. Higher-order decay estimates 10
7. Applications 12
7.1. Potential of critical decay 12
7.2. Laplacian on manifold 13
References 14

1. INTRODUCTION

In the spectral analysis of self-adjoint operators, methods relying on the positivity of a commutator have very important applications. This approach can be traced back to the work of Putnam in 1967 [P], whose main result relates the condition $[H, iA] \geq 0$ with the absolute continuity of the range of $[H, iA]$. The number of applications of this setting is greatly restricted by the boundedness of the conjugate operator $A$ and the global assumption on the commutator. In the fundamental work of Mourre in 1981 [Mo], the assumptions are more flexible and admit a variety of extensions. In his work, it is required that $[H, iA]$ is dominated by $H$ (for $A$ only self-ajoint) and the positivity assumption is represented by the so-called strict Mourre estimate $E(J)[H, iA]E(J) \geq aE(J)$, where $E$ is the spectral measure of $H$, $a$ is a positive constant and $J$ is a Borel set of $\mathbb{R}$. The main consequences of these conditions are a limiting absorption principle, that is, a control of the resolvent of

The first author was partially supported by NSF DMS-1201394.
Let $H$ be self-adjoint operators on a Hilbert space $L^2(M)$. We are interested in extending the results of [GLS] to the case where $[H, iA]$ is not necessarily equal to a function of $H$. In this work we will assume a more general commutation relation which will yield similar decay estimates by means of a suitable adaptation of the conjugate operator $A$.

We now review some standard definitions in functional analysis. As usual, we write $\langle x \rangle = (1 + x^2)^{1/2}$. Denote $H^1 = D(H)$ the domain of $H$ and consider its adjoint space $H^{-1} = D(H^*)$. The resolvent of $H$ is defined as $R(z) = (H - zI)^{-1}$ for $z \in \rho(z)$ the resolvent set of $H$. For $P$ a bounded operator, we shall say that $P$ commutes with $H$ if for any $t \in \mathbb{R}$ the relation $P e^{itH} = e^{itH}P$ holds in $B(H)$. This is equivalent to $P \varphi(H) = \varphi(H)P$ for any bounded Borel function $\varphi : \mathbb{R} \rightarrow \mathbb{C}$. If $Q$ is unbounded and $D(Q) \supset H^1$ then we say that $Q$ commutes with $H$ if the above identity holds in $B(H^1, H)$ for $Q$.

Acknowledgment: We would like to thank V. Georgescu for helpful remarks.
Consider $Q$ a densely defined operator on $\mathcal{H}$ with $D(Q) \supset \mathcal{H}^1$. We say that $Q$ is $H$-bounded with relative norm $a$ if for some $a, b \in \mathbb{R}$ one has $\|Q\psi\| \leq a\|H\psi\| + b\|\psi\|$, for all $\psi \in \mathcal{H}^1$.

If $S$ is a bounded operator on $\mathcal{H}$ then we denote $[A, S]$ the sesquilinear form on $D(A)$ defined by $[A, S](u, v) = \langle Au Sv \rangle - \langle u | Sv \rangle$. We say that $S$ is of class $C^1 (A)$, and we write $S \in C^1 (A)$, if $[A, S]$ is continuous for the topology induced by $\mathcal{H}$ on $D(A)$ and then we denote $[A, S]$ the unique bounded operator on $\mathcal{H}$ such that $\langle u | [A, S]v \rangle = \langle Au | Sv \rangle - \langle u | Sv \rangle$ for all $u, v \in D(A)$. We consider now the rather subtle case of unbounded operators. Note that we always equip the domain of an operator with its graph topology. If $H$ is a self-adjoint operator on $\mathcal{H}$ then $[A, H]$ is the sesquilinear form on $D(A) \cap D(H)$ defined by $[A, H](u, v) = \langle Au | Hv \rangle - \langle Hu | Av \rangle$. A convenient definition of the $C^1 (A)$ class for any self-adjoint operator is as follows. Let $R(z) = (H - z)^{-1}$ for $z$ in the resolvent set $\rho (H)$ of $H$. We say that $H$ is of class $C^1 (A)$ if $R(z) \in C^1 (A)$ for some (hence for all) $z \in \rho (H)$. In this case, Proposition 6.2.10 in [ABG] shows that $D(A) \cap \mathcal{H}^1$ is dense in $\mathcal{H}^1$ and hence $[A, H]$ extends to a uniquely determined continuous sesquilinear form $[A, H]$ on $\mathcal{H}^1$. For further properties and examples of the $C^1$ regularity we refer to [ABG] and [GG].

Finally, we discuss the notion of Kato-smoothness. We shall say that a closed operator $E$ is $H$-smooth on the range of a bounded operator $P$ if and only if for each $\psi \in \mathcal{H}$ and each $\epsilon \neq 0$, $R(\lambda + i\epsilon) P \psi \in D(E)$ for almost all $\lambda \in \mathbb{R}$ and moreover

$$\|E\|_{H}^{2} = \sup_{\|\psi\| = 1} \frac{1}{4\pi^{2}} \int_{-\infty}^{\infty} (\|ER(\lambda + i\epsilon)P\psi\|^{2} + \|ER(\lambda - i\epsilon)P\psi\|^{2}) \, d\lambda < \infty.$$ 

In particular $E$ is $H$-smooth on the range of $P$ if and only if for all $\psi \in \mathcal{H}$, $e^{itH}P\psi \in D(E)$ for almost every $t \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} \|Ee^{-itH}P\psi\|^{2} \, dt \leq 2\|E\|_{H}^{2} \|P\psi\|^{2}.$$ 

In several applications the smoothness can be controlled more precisely using Sobolev norms as follows. Let $M$ be an $n$-dimensional Riemannian manifold with a smooth Riemannian metric $g_{ij}$. Consider the Hilbert space $\mathcal{H} := L^{2} (M)$ with the inner product defined as $\langle \phi | \psi \rangle = \int_{M} \phi (x) \bar{\psi}(x) \, dg$, where $dg := \sqrt{\det g_{ij}}$. Denote by $\| \cdot \|$ the norm induced by this inner product, that is, $\| \phi \| = \int_{M} |\phi (x)|^{2} \, dg$. The self-adjoint operator $H$ defined on $\mathcal{H}$ enjoys the standard functional calculus and one can define the homogeneous Sobolev norms $\|\psi\|_{H^{s} (M)} := \|H^{s/2} \psi\|$, for $0 \leq s \leq 1$.

We then generalize the notion of $H$-smoothness taking the supremum on a subspace of $\mathcal{H}$ instead of the whole space. We will say that a closed operator $E$ is $|H|^{-s}$-smooth on the range of a bounded operator $P$ if and only if for all $\psi \in \mathcal{H}^{s} (M)$, $e^{itH}P\psi \in D(E)$ for almost every $t \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} \|Ee^{-itH}P\psi\|^{2} \, dt \leq C_E \|P\psi\|^{2}_{H^{s} (M)}.$$ 

3. Assumptions

Let $H$ and $A$ be two self-adjoint operators on the Hilbert space $\mathcal{H} = L^{2} (M)$, where $M$ is a smooth $n$-dimensional Riemannian manifold equipped with the standard $L^{2}$-inner product $\langle \cdot | \cdot \rangle$ and norm $\| \cdot \|$. Let $P$ be an orthogonal projection and $Q$ be an $H$-bounded operator with relative norm $a \geq 0$. $P$ and $Q$ commute with $H$. Let $E, F$ be linear operators such that $D(E) \cap D(F) \supset P \mathcal{H}^{1}$. The main assumptions of this paper are established as follows.

$$(H) \quad H \text{ is of class } C^{1} (A)$$
- $H$ and $A$ satisfy the commutation relation $P[H, iA]P = P(Q + K)P$ for $K \equiv F^*E$, in the sense that for all $\phi, \psi \in \mathcal{H}^1$

\[ \langle \phi, P[H, iA]P\psi \rangle = (P\phi, QP\psi) + (FP\phi, EP\psi). \]

- $K$ is symmetric on $\mathcal{H}^1$, that is, $\langle \phi | K | \psi \rangle = (K | \phi \rangle | \psi \rangle)$

- $KH^1 \subset \mathcal{H}$ and $(H)^{-s/2}K(H)^{-s/2}$ is a bounded operator on $\mathcal{H}$, for some $s > 0$.

- $E$ and $F$ are $|H|^s$-smooth on the range of $P$.

**Remark 1.** Since $H$ is of class $C^1(A)$, the sesquilinear form $[H, iA]_0$ on $\mathcal{H}^1 \cap D(A)$ extends to a continuous operator in $\mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1})$. Then the commutation relation of $(H)$ implies that the sesquilinear form $(P\phi, QP\psi) + (FP\phi, EP\psi)$ restricted to $\mathcal{H}^1 \cap D(A)$ also extends to a bounded operator in the same space. Therefore, the commutation relation can be written at the level of operators in $\mathcal{B}(\mathcal{H}^1, \mathcal{H}^{-1})$ as $P[H, iA]P = P(Q + K)P$, for a densely defined symmetric operator $K$.

For some of the decay estimates we will require an additional smoothness condition.

\[(Ha)\] The sesquilinear form $P[A, K]_0P$ defined on $D(A) \cap D(K)$ satisfies the identity $P[A, K]_0P(\phi, \psi) = (F'E'P\phi, E'P\psi)$, for all $\phi, \psi \in D(A) \cap D(K)$. Here $E', F'$ hold the same properties of $E, F$ in assumptions $(H)$. Moreover, $P[A, K]_0P$ extends to a densely defined operator $K'$ such that $(H)^{-s/2}K'(H)^{-s/2}$ is bounded on $\mathcal{H}$.

The commutation relation of $(H)$ is the crucial assumption of this paper. It replaces the identity of type $[H, iA] = \varphi(H)$ in [GLS], with a much more general expression. Indeed, it now admits an operator $Q$ (controlled by $H$) in addition to a Kato-smooth perturbation. Observe that the $C^1(A)$ regularity provides a suitable framework in this case as well.

The strategy to derive decay estimates under these generalized conditions will be based on the construction of a new conjugate operator $\tilde{A}$ which will simplify the commutation identity, thus reducing the problem to our preceding estimates. The remaining assumptions of $(H)$ will justify this construction and other algebraic manipulations.

4. THE CONJUGATE OPERATOR

This section aims to construct the conjugate operator $\tilde{A}$. Using the functional calculus define for $s > 0$ the cut-off $h_s(H) := (H)^{-s/2}$. For any operator $X$ denote $X_h := h_s(H)Xh_s(H)$ (omitting the parameter $s$ in $X_h$ for brevity).

We first consider the operator $A_h$. Note that since $H \in C^1(A)$, the operator $h_s(H)$ is of class $C^3(A)$ for $s$ large enough (see the comment before Theorem 3.7 in [GLS]). Moreover, by Lemma 6.2.9 in [ABG], the $C^1(A)$ condition for $h_s(H)$ holds for any $s > 0$. This follows from the continuity of the form $[A, h_s(H)]$ for the topology induced by $\mathcal{H}$, which is a consequence of assumptions $(H)$. Note that $\overline{D(A_h)} \supset D(A)$ and therefore $A_h$ is a densely defined symmetric operator. With some abuse of notation we denote its closure by $\tilde{A}_h$. Lemma 7.2.15 in [ABG] proves that $\tilde{A}_h$ is self-adjoint and $D(A)$ is a core.

The next step is to add a linear perturbation to $A_h$. Define for each $t \in \mathbb{R}$ the operator $h_s(H)U_t h_s(H) := \int_0^t e^{-isH} PK_t P e^{isH} ds$, which is bounded under assumptions $(H)$.

We now show that the limiting operator $B_h := s\lim_{t \to \infty} h_s(H)U_t h_s(H)$ is well-defined.
and bounded as well. Let $\phi, \psi$ in $\mathcal{H}$.
\[
|\langle \phi | h_s(H)(U_t - U_r) h_s(H) | \psi \rangle |
\]
\[
= \left| \int_r^t (h_s(H) P \phi e^{-i s H} K e^{i s H} h_s(H) P \psi) ds \right|
\]
\[
\leq \int_r^t \left| (\langle Fe^{i s H} h_s(H) P \phi | E e^{i s H} h_s(H) P \psi \rangle) \right| ds
\]
\[
\leq \left( \int_0^t \left\| F e^{i s H} h_s(H) P \phi \right\|^2 ds \right)^{1/2} \left( \int_r^t \left\| E e^{i s H} h_s(H) P \psi \right\|^2 ds \right)^{1/2}
\]
\[
\leq C_F \| h_s(H) P \phi \|_{H^r(M)} \left( \int_r^t \left\| E e^{i s H} h_s(H) P \psi \right\|^2 ds \right)^{1/2}
\]
\[
\leq C_F \| \phi \| \left( \int_r^t \left\| E e^{i s H} h_s(H) P \psi \right\|^2 ds \right)^{1/2}.
\] (1)

Since the integrand of the last step is in $L^1(\mathbb{R})$ and $\phi$ is arbitrary, we conclude that the sequence converges strongly to the bounded operator $B_h$.

Finally, define $\tilde{A} := A_h + B_h$. Note that $D(\tilde{A}) = D(A_h) \supset D(A)$. The following result justifies this construction.

**Proposition 2.** Assume (H). Then $\tilde{A}$ defined as above is self-adjoint. Moreover, $H$ is of class $C^1(\tilde{A})$ and the continuous sesquilinear form $[H, i \tilde{A}]_c$ on $\mathcal{H}^1 \cap D(\tilde{A})$ is identified with an operator in $B(\mathcal{H}^1, \mathcal{H})$ satisfying the commutation relation $P[H, i \tilde{A}]_c P = Q_h P$.

**Proof.** Note that $B_h$ is bounded and symmetric by construction, thus the self-adjointness of $A_h$ is guaranteed by the Kato-Rellich theorem. The $C^1(\tilde{A})$ property follows from the identity $[H, A_h] = [H, A]_c$ in $B(\mathcal{H}^1, \mathcal{H}^2)$ of Prop. 7.2.16, in addition to Prop. 6.2.10 in [ABG].

We now prove the commutation relation. Using the functional calculus define $R_\epsilon = (1 + i \epsilon H)^{-1}$ and the bounded operator $H_\epsilon := HH_\epsilon = (i \epsilon)^{-1}(1 - R_\epsilon)$. Note that $P[H_\epsilon, K_h]_c P$ is a continuous sesquilinear form on $\mathcal{H}$ satisfying $P[H_\epsilon, K_h]_c P = i \epsilon^{-1}P[R_\epsilon, K_h]_c P = R_\epsilon P[H, K_h]_c P R_\epsilon$ in form sense. Now we calculate
\[
[H_\epsilon, i h_s(H) U t h_s(H)]_c = \frac{1}{\epsilon} \int_0^t e^{i s H} [R_\epsilon, i P K_h]_c e^{i s H} ds
\]
\[
= R_\epsilon \left( \int_0^t e^{i s H} P[H, i K_h]_c e^{i s H} ds \right) R_\epsilon
\]
\[
= R_\epsilon e^{-i t H} P K_h P e^{i t H} R_\epsilon - R_\epsilon P K_h P R_\epsilon.
\] (2)

Note that for any $\phi, \psi \in \mathcal{H}^1$ one has $(\phi, e^{-i t H} P K_h P e^{i t H} \psi) \rightarrow 0$ for a subsequence $t_k \rightarrow \infty$. This follows from the estimate
\[
\int_0^\infty |(\phi, e^{-i t H} P K_h P e^{i t H} \psi)| dt = \int_0^\infty |(F e^{i t H} h_s(H) P \phi, E e^{i t H} h_s(H) P \psi)| dt
\]
\[
\leq C \int_0^\infty (\| F e^{i t H} h_s P \phi \|^2 + (\| F e^{i t H} h_s P \phi \|^2) dt
\]
\[
\leq C_F \| \phi \|^2 + C_E \| \psi \|^2.
\]

Then let $\epsilon \rightarrow 0$ on the RHS of (2) which converges to $-P K_h P$ in the weak form sense on $\mathcal{H}$. Finally, by making $t_k \rightarrow \infty$ in $[H_\epsilon, i h_s(H) U t h_s(H)]_c$ and then $\epsilon \rightarrow 0$ in weak form sense in $\mathcal{H}^1$, we conclude that the form $[H, i B_h]_c$ with domain $\mathcal{H}^1$ extends to a bounded sesquilinear form on $\mathcal{H}$ satisfying $[H, i B_h]_c = -P K_h P$. 

Now, as sesquilinear forms in \(D(H) \cap D(\tilde{A})\) we have \([H, i\tilde{A}] = [H, iA] + [H, iB]\). By the previous discussion we conclude that \([H, i\tilde{A}]\) is continuous for the topology induced by \(H\) on \(\mathcal{H} \cap D(\tilde{A})\) and moreover
\[
P[H, i\tilde{A}] = P[H, iA] + PK_h = PH_h(1 + iK_h)PH - PK_h = PQ_hP,
\]
which concludes the proof. \(\Box\)

Once we have established a suitable commutation relation for \(H\) and \(\tilde{A}\) in Proposition 2, we recall two important commutator identities used in our previous work. The proofs are analogous and use \(\tilde{A}\) as the conjugate operator.

**Proposition 3.** Let \(H\) be a self-adjoint operator of class \(C^1(\tilde{A})\). Then the restriction of \([\tilde{A}, e^{itH}]\) to \(D(\tilde{A}) \cap \mathcal{H}^1\) extends to a continuous form \([\tilde{A}, e^{itH}]\) on \(\mathcal{H}^1\) and, in the strong topology of space of sesquilinear forms on \(\mathcal{H}^1\), we have
\[
[\tilde{A}, e^{itH}] = \int_0^t e^{i(t-s)H}[H, i\tilde{A}]e^{isH} ds.
\]
Moreover, under the conditions of \((H)\) we have \(P[\tilde{A}, e^{itH}] = tPQ_hPe^{itH}\) as operators in \(B(\mathcal{H}^1, \mathcal{H})\).

**Proof.** Same as Theorem 3.7 in [GLS]. \(\Box\)

**Proposition 4.** Let \(H\) be a self-adjoint operator of class \(C^1(\tilde{A})\). Then \(D(\tilde{A}) \cap \mathcal{H}^1\) is dense in \(\mathcal{H}^1\) and
\[
[\tilde{A}, R(z)] = -R(z)[\tilde{A}, H]R(z) \quad \text{for all } z \in \rho(H).
\]

**Proof.** This is Proposition 6.2.10 in [ABG]. \(\Box\)

In the next section, it will be useful to “commute \(A\) through \(B\)” in the important case \(Q = cH\), for some \(c \neq 0\). To give precise meaning to this, we consider the sesquilinear form \(C_0\) on \(D(\tilde{A})\) defined as \(C_0(\phi, \psi) := [\tilde{A}, e^{itH}]P(\phi, \psi) = (\tilde{A}P\phi, B_hP\psi) - (B_hP\phi, \tilde{A}P\psi)\).

The next result justifies our assertion.

**Proposition 5.** Under conditions \((H)\) and \((Ha)\) in the case \(Q = cH\), the sesquilinear form \(C_0\) defined on \(D(\tilde{A})\) as above can be extended to a bounded operator in \(\mathcal{H}\), which we denote by \(C\).

**Proof.** Step 1: Commutator identity

As before, define \(R_c = (1 + i\epsilon H)^{-1}\) and the bounded operator \(H_c := HH_c\). For \(\phi, \psi\) in \(D(\tilde{A})\) consider the form
\[
D_c(\phi, \psi) := \frac{i((\tilde{A}H, P\phi, B_hR_c^*P\psi) - (B_hH, P\phi, \tilde{A}PR_c^*P\psi))}{D_1} - \frac{i((\tilde{A}PR_c\phi, B_hH^*_cP\psi) - (B_hR_c\phi, \tilde{A}H^*_cP\psi))}{D_2}.
\]

We now use Prop. 3.8 in [GLS] to calculate the commutator identities \(P[H_c, i\tilde{A}]P = cH_hR_c^*P\) and \([H_c, iB_h] = -PR_cK_hR_cP\), which will be used to expand the expression
above.

\[ D_i^2 = \langle [\tilde{A}, H_e] P_\phi, B_h R_e^* P \psi \rangle + i(\langle H \tilde{A} P_\phi, B_h R_e^* P \psi \rangle - \langle P \phi, H_e^* B_h \tilde{A} P R_e^* \psi \rangle) \]

\[ = -\langle c H_e R_e^2 P_\phi, B_h R_e^* P \psi \rangle - \langle \tilde{A} P \phi, H_e^* B_h \tilde{A} P R_e^* \psi \rangle + \langle P \phi, H_e^* B_h \tilde{A} P R_e^* \psi \rangle \]

\[ = -\langle c H_e R_e^2 P_\phi, B_h R_e^* P \psi \rangle - \langle \tilde{A} P \phi, H_e^* B_h \tilde{A} P R_e^* \psi \rangle + \langle P \phi, H_e^* B_h \tilde{A} P R_e^* \psi \rangle \]

\[ = -\langle c H_e R_e^2 P_\phi, B_h R_e^* P \psi \rangle + \langle \tilde{A} P \phi, PR_e K_h (R_e^*)^2 P \psi \rangle \]

\[ = -\langle c H_e R_e^2 P_\phi, B_h R_e^* P \psi \rangle + \langle \tilde{A} P \phi, PR_e K_h (R_e^*)^2 P \psi \rangle \]

\[ = -\langle c H_e R_e^2 P_\phi, B_h R_e^* P \psi \rangle + \langle \tilde{A} P \phi, PR_e K_h (R_e^*)^2 P \psi \rangle \]

Thus, \( D_\epsilon(\phi, \psi) = [B_h, c H_h R_h^2 P]_{\omega}(\phi, \bar{\psi}_\epsilon) + [\tilde{A} P, PR, K_h R_h]_{\omega}(\phi, \bar{\psi}_\epsilon), \) where \( \bar{\psi}_\epsilon = R_e^* \psi. \) Note that the expression on the RHS converges weakly in formsense as \( \epsilon \to 0. \)

The first term of the limit can be extended to the bounded operator \(-icPK_hP\) (see proof of Prop. 2). On the other hand, by conditions (H) and (Ha) the second term can be expanded into Kato-smooth operators

\[ [\tilde{A} P, PK_h P]_{\omega}(\phi, \psi) = [A_h P, PK_h P]_{\omega}(\phi, \psi) + [B_h P, PK_h P]_{\omega}(\phi, \psi) \]

\[ = (F' P \phi, E' P \psi) + (F P \phi, E P \psi) - (E P \phi, F P B_h P \psi), \]

and thus the sesquilinear form can be extended to a bounded operator in \( \mathcal{H} \) satisfying \( ||[\tilde{A} P, PK_h P]|| \leq C(E, F, E', F'). \)

We conclude that \( D_\epsilon \) converges to a sesquilinear form that can be extended to a bounded operator in \( \mathcal{H}. \)

Step 2: Convergence as \( t \to \infty \)

Fix \( \epsilon > 0 \) and for \( \phi, \psi \in D(\tilde{A}) \) denote \( \phi_\epsilon = R_e \phi, \psi_\epsilon = R_e \psi. \) Define the sesquilinear form \( C_t(\phi, \psi) := \langle \tilde{A} e^{itH} \phi, B_h e^{itH} \psi \rangle - \langle B_h e^{itH} P \phi, \tilde{A} e^{itH} P \psi \rangle. \) Note that \( \frac{\partial}{\partial \epsilon} C_t(\phi_\epsilon, \psi_\epsilon) = D_\epsilon(e^{itH} \phi, e^{itH} \psi). \) Now we use an estimate similar to (1) to prove convergence in \( t. \)

\[ \| (C_t - C_r)(\phi_\epsilon, \psi_\epsilon) \| \]

\[ \leq \int_r^t \| D_\epsilon(e^{itH} \phi, e^{itH} \psi) \| \, dt \]

\[ \leq \left[ \int_r^t \| [P \phi, e^{itH} c K_h e^{itH} P \psi] \| \, dt \right] + \left[ \int_r^t \| [\tilde{A} P, PR_e K_h R_e] \psi \| \, dt \right] \]

\[ \leq C(F) ||\phi|| \left( \int_r^t \| E e^{itH} h_x P \psi_\epsilon \|^2 \, ds \right)^{1/2} + C(F') ||\phi|| \left( \int_r^t \| E' e^{itH} h_x P \psi_\epsilon \|^2 \, ds \right)^{1/2}. \]

By Kato-smoothness of the RHS we conclude that the sequence is Cauchy in \( t \) (for fixed \( \epsilon \)) and moreover, \( C_t(\phi_\epsilon, \psi_\epsilon) \to 0 \) as \( t \to \infty. \)

We proceed analogously to prove the result of the theorem.

\[ |C_0(\phi_\epsilon, \psi_\epsilon)| \leq \limsup_t \left( |C_t(\phi, \psi)| + \int_0^t |D_\epsilon(e^{itH} \phi, e^{itH} \psi)| \, dt \right) \]

\[ \leq C(E, F, E', F') ||\phi_\epsilon|| ||\psi_\epsilon||. \]
Since \( R, D(A) \) is dense in \( \mathcal{H} \), we conclude by letting \( \epsilon \to 0 \) that the sesquilinear form defined above \( C_\varphi (\phi, \psi) = (AP\phi, B_hP\psi) - (B_hP\phi, AP\psi) \) restricted to \( D(\tilde{A}) \) extends to a bounded operator on \( \mathcal{H} \).

**Corollary 6.** \( B_h \) leaves invariant the domain of \( \tilde{A} \), that is, \( B_hD(\tilde{A}) \subset D(\tilde{A}) \).

**Proof.** Let \( \epsilon > 0 \) and \( \psi \in D(\tilde{A}), \phi \in \mathcal{H} \). Denote \( R_\epsilon (\tilde{A}) = (1 + i\epsilon\tilde{A})^{-1} \) and \( \tilde{A}_\epsilon = \tilde{A}R_\epsilon (\tilde{A}) \). Recall also that \( PB_hP = B_h \) and \([P, \tilde{A}]\) extends to a bounded operator since \( H \in C^1(\tilde{A}) \).

\[
|\langle \phi | \tilde{A}, B_h \psi \rangle| = |\langle P \tilde{A}R_\epsilon (\tilde{A})^* \phi | B_hP \psi \rangle|
\]

\[
= |\langle [P, \tilde{A}] (1 - i\epsilon \tilde{A})^{-1} \phi | B_hP \psi \rangle + \langle \tilde{A}P(1 - i\epsilon \tilde{A})^{-1} \phi | B_hP \psi \rangle|
\]

\[
\leq C||\phi|| ||\psi|| + |\langle (1 - i\epsilon) \tilde{A}^{-1} \phi | P[\tilde{A}, B_h]P \psi \rangle| + |\langle B_hP(1 - i\epsilon \tilde{A})^{-1} \phi | P \tilde{A}P \psi \rangle|
\]

\[
\leq C||\phi|| ( ||\psi|| + ||\tilde{A}\psi|| ) ,
\]

where the constant \( C \) is independent of \( \epsilon \). Thus, \( ||\tilde{A}, B_h \psi|| \leq C(||\psi|| + ||\tilde{A}\psi||) \) and we conclude by Fatou’s lemma.

### 5. Decay Estimates

**Definition 7.** For \( u \in \mathcal{H} \), define the function \( \psi_u(t) := \langle u, e^{itH}u \rangle \), \( t \in \mathbb{R} \) and the set

\[
\mathcal{E} = \{ u \in \mathcal{H} : \psi_u \in L^2(\mathbb{R}) \}.
\]

For \( u \in \mathcal{E} \) denote \( [u]_H = ||\psi_u||^{1/2}_{L^2} \).

In [ABG] it was shown that \( \mathcal{E} \) is a dense linear subspace of the absolutely continuity subspace of \( H \) and \([.]_H \) is a complete norm on it. In this work we will consider the following additional assumption on the space \( \mathcal{E} \), which will be relevant for Propositions 9 and 12 where no assumption of positivity of \( Q \) is made.

\((Hb)\) For any \( u \in \mathcal{H} \) one has \( (A_h+i)^{-1}u \in \mathcal{E} \).

We now proceed to prove the main results of this work.

**Proposition 8.** Assume \((H)\) with \( Q \geq 0 \). Then for \( u \in \mathcal{H}^s \) such that \( Pu \in D(\tilde{A}) \cap D(Q^{1/2}) \) we have the estimate \( ||\psi_{Q^{1/2}Pu}(t)|| \leq C_u(t)^{-1} \).

**Proof.** Assume first that \( u \in \mathcal{H}^1 \) and define \( v = \langle H \rangle^{s/2}u \).

\[
t\psi_{Q^{1/2}Pu}(t) = (Q^{1/2}Pu|t e^{itH}Q^{1/2}Pu) = \langle v|PQhe^{itH}Pv \rangle = \langle v|Pe^{itH}, \tilde{A}Pv \rangle = \langle e^{-itH}Pv|\tilde{A}Pv \rangle - \langle \tilde{A}Pv|e^{itH}Pv \rangle .
\]

(4)

We now expand the first term of the last expression, the second one is analogous.

\[
|\langle e^{-itH}Pv|\tilde{A}Pv \rangle| = |\langle e^{-itH}Pv|A_hPv \rangle + \langle e^{-itH}Pv|B_hPv \rangle| \leq ||u|| ||PAPu|| + C||\langle H \rangle^{s/2}Pu|| .
\]

Hence \( ||t\psi_{Q^{1/2}Pu}(t)|| \leq 2(||u|| ||PAPu|| + C||\langle H \rangle^{s/2}Pu||) \).
For general $u$ we define $u_\varepsilon := R_\varepsilon u \in \mathcal{H}$. Note that $P_{u_\varepsilon} \in D(Q^{1/2})$ since $Q$ commutes with $H$ and thus from (4) we obtain the estimate
\[
|t\psi_{Q^{1/2}P_{u_\varepsilon}}(t)| \leq 2\|v\|\|P\tilde{A}Pv\|. \tag{5}
\]
Since $H \in C^1(\tilde{A})$, from Proposition 12 in [GLS] we obtain that $PR_\varepsilon v \in D(\tilde{A})$. Moreover, Proposition 4 implies that $[\tilde{A}, R_\varepsilon] = cR_\varepsilon[H, i\tilde{A}]R_\varepsilon$, hence $P[\tilde{A}, R_\varepsilon]P = cPR_\varepsilon QPR_\varepsilon P$ as operators in $\mathcal{H}$. Now we use that $Q$ is $H$-bounded with relative norm $a$, which yields
\[
\begin{aligned}
\|P\tilde{A}Pv\| &= \|P[\tilde{A}, R_\varepsilon]Pv\| + \|PR_\varepsilon \tilde{A}Pv\| \\
&\leq \varepsilon (a\|HR_\varepsilon Pv\| + b\|R_\varepsilon Pv\|) + \|\tilde{A}Pv\| \\
&\leq (a + b\|Pv\| + \|PAPu\| + C\|Pv\| \\
&\leq C\|(H)^{s/2}Pu\| + \|PAPu\|.
\end{aligned}
\]
Finally, let $\varepsilon \to 0$ and use Fatou’s lemma on the lhs side of (5) to conclude
\[
|t\psi_{Q^{1/2}P_{u_\varepsilon}}(t)| \leq 2\|(H)^{s/2}u\|\left(C\|(H)^{s/2}Pu\| + \|PAPu\|\right).
\]

**Proposition 9.** Assume $(H)$, $(Ha)$ and $(Hb)$ in the special case $Q = cH$ and $s = 1/2$. Then for $u \in D(A)$ such that $P_{u}$ and $PAP_{u}$ are in $\mathcal{E}$, one has $|\psi_{P_{u}}(t)| \leq C_{u}(t)^{-1/2}$.

**Proof.** Define $\chi = \chi([-M,M])$ the characteristic function of the interval in $\mathbb{R}$. We decompose $u := u_1 + u_2$, where $u_1 = \chi(H)u$ and $u_2 = (1 - \chi(H))u$. Note that $\psi_{P_{u}} = \psi_{P_{u_1}} + \psi_{P_{u_2}}$. We will show that $\psi_{u_1} = O(t^{-1/2})$ and $\psi_{u_2} = O(t^{-1})$.

By Corollary 8.2 in [GLS] it suffices to prove that $\delta\psi_{P_{u_1}}(t) := t\psi_{P_{u_1}}(t)$ is in $L^2(\mathbb{R})$. Define $v = (H)^{s/2}u_1$.
\[
\begin{aligned}
ict\psi_{P_{u_1}}(t) &= \langle Pu_1 | tHe^{iH}Pu_1 \rangle \\
&= \langle Pv | t(H)^{s/2}e^{iH}(H)^{-s/2}e^{iH}Pv \rangle \\
&= \langle Pv | tHb e^{iH}Pv \rangle \\
&= \langle v | P[e^{iH}, A]Pv \rangle \\
&= \langle e^{-iH}v | P\tilde{A}Pv \rangle - \langle P\tilde{A}Pv | e^{iH}v \rangle.
\end{aligned}
\]

We now expand the first term of the last expression, the second one is analogous.
\[
\begin{aligned}
\langle e^{-iH}v | P\tilde{A}Pv \rangle &= \langle e^{-iH}v | PApv \rangle + \langle e^{-iH}v | PBhv \rangle \\
&= \langle e^{-iH}u_1 | PAPu_1 \rangle + \langle e^{-iH}Pv | (A_1 + i)^{-1}(A_1 + i)Bhv \rangle.
\end{aligned}
\]

Note that $(A_1 + i)Bhv = [A_1, Bh]Pv + B_h(A_1 + i)Pv$, which is in $\mathcal{H}$ by Proposition 5 and the fact that $v \in D(A_1)$. Thus
\[
c\|\delta\psi_{P_{u_1}}\| \leq 2\|P_{u_1}H[PAP_{u_1}H + ([A_1 + i]^{-1}(CP(H)^{s/2}u_1 + B_h(H)^{-s/2}AP_{u_1})_H),
\]

and we conclude $|\psi_{u_1}(t)| \leq C_{u_1}(t)^{-1/2}$.

To estimate the decay of $u_2$ we now consider $v \in \mathcal{H}$ such that $u_2 = |H|^{1/2}(H)^{-1/2}v$. Then
\[
\begin{aligned}
ict\psi_{P_{u_2}}(t) &= \langle Pu_2 | tHe^{iH}Pu_2 \rangle \\
&= \langle Pv | tHe^{iH}Hb \text{sgn}(H)Pv \rangle \\
&= \langle v | P[e^{iH}, A] \text{sgn}(H)Pv \rangle. \tag{6}
\end{aligned}
\]
Note that \( g(H) := |H|^{-1/2}(H)^{1/2}(1 - \chi(H)) \) is a bounded smooth function, so \( \|v\| \leq \|\psi\| \) and \( P \tilde{g}(H)P = g(H)H_{0}P + P g(H)\tilde{A}P \). Hence the first term of the commutator in (6) is bounded \( |\langle e^{-itH}v|PAPv\rangle| \leq \|\psi\| (\|u\| + \|\tilde{A}Pu\|) \) and we conclude that \( |\psi_{P\psi}(t)| \leq C_{u}(t)^{-1} \) as desired. \( \square \)

**Remark 10.** In case assumptions (H), (Ha) and (Hb) are met with \( s = 0 \), the construction of the conjugate operator is simpler because there is no need to introduce the cut-off \( h_{s} \) (and then \( \tilde{A} = A + B \)). Hence Propositions 8 and 9 remain valid. We state them here for completeness, the proofs are analogous.

**Proposition 11.** Assume (H) with \( Q \geq 0 \) and \( s = 0 \). Then for \( u \in \mathcal{H} \) such that \( Pu \in D(A) \cap D(Q^{1/2}) \) we have the estimate \( |\psi_{Q^{1/2}Pu}(t)| \leq C_{u}(t)^{-1} \).

**Proposition 12.** Assume (H), (Ha) and (Hb) in the special case \( Q = cH \) and \( s = 0 \). Then for \( u \in D(A) \) such that \( Pu \) and \( PAPu \) are in \( \mathcal{E} \), one has \( |\psi_{Pu}(t)| \leq C_{u}(t)^{-1/2} \).

We now study the particular case \( H = H_{0} + V(x) \) on \( \mathcal{H} = L^{2}(\mathbb{R}^{n}) \), where \( H_{0} \) is a self-adjoint operator and \( V(x) \) is smooth real-valued function. Let \( P \) be a projection that commutes with \( H \). Let us consider the following assumptions.

(H1)  
(i) There is a self-adjoint first-order operator \( A \) so that \( H_{0} \) is of class \( C^{1}(A) \) and \( [H_{0}, iA] = cH_{0} \) for some \( c \neq 0 \)
(ii) The functions \( V(x) \) and \( w(x) := [V(x), iA] \) are bounded

Conditions (H1) ensure that \( H \) is of class \( C^{1}(A) \) and it follows that the sesquilinear form \( \langle H, iA \rangle = cH - cV(x) + w(x) \) on \( D(H) \cap D(A) \) can be identified with an operator \( [H, iA] \) in \( B(\mathcal{H}, \mathcal{H}) \). In order to satisfy assumptions (H) with \( Q = cH \) and \( K = -cV(x) + w(x) \) it remains to show the Kato-smoothness condition. We can derive this from decay estimates as follows.

Let \( \sigma > 0 \) and \( P_{\omega}(H) \) be the projection onto the space of absolute continuity of \( H \). Consider the local decay estimate

\[
\int_{\mathbb{R}} \|(x)^{-\sigma/2}e^{-itH}P_{\omega}u\|^{2}dt \leq C\|u\|^{2}, \tag{7}
\]

for all \( u \in \mathcal{H} \) and some \( C > 0 \). Then from Proposition 9 we obtain the following result.

**Proposition 13.** Let \( H = H_{0} + V(x) \) and \( A \) as in (H1), with \( V \) such that

\[
\sup_{x \in \mathbb{R}^{n}} \left( (x)^{\sigma}[V(x)] + (x)^{\sigma+1}|\nabla V(x)| \right) < \infty.
\]

Assume also the local decay estimate (7). Then for \( u \in D(A) \) such that \( P_{\omega}AP_{\omega}u \) is in \( \mathcal{E} \), then \( |\psi_{P_{\omega}u}(t)| \leq C_{u}(t)^{-1/2} \).

6. Higher-Order Decay Estimates

We now improve our results by iteration of the previous method. In order to obtain higher-order decay estimates, the main difficulty lies in extending Proposition 5 for higher powers of \( A_{h} \) and \( B_{0} \). This task is extremely laborious with the current methods so it will not be pursued here. Proposition 5 allows to construct the operator \( \tilde{A}^{2} = (A_{h} + B_{0})^{2} \) on \( D(\tilde{A}^{2}) \supset D(A^{2}) \). This will be enough to increase the time decay rate by a power of one as shown in the propositions below.

In this section, we will restrict ourselves to the case \( s = 0 \) and \( P = 1 \) in (H), (Ha) and (Hb). So here \( \tilde{A} = A + B \) defined on \( D(\tilde{A}) = D(A) \), \( B \) is bounded on \( \mathcal{H} \) and the commutation relation reads \( [H, \tilde{A}] = Q \).
We now recall the necessary formalism of higher-order regularity of operators (see [ABG] and [GLS] for further discussion). Let $A$ be a self-adjoint operator on a Hilbert space $H$ and $k \in \mathbb{N}$. We say that a bounded operator $S$ is of class $C^k(A)$, and we write $S \in C^k(A)$, if the map $\mathbb{R} \ni t \mapsto e^{-itA}Se^{itA}S \in B(H)$ is of class $C^k$ in the strong operator topology. It is clear that $S \in C^k(A)$ if and only if $S \in C^k(A)$ and $[S, A] \in C^k(A)$. Clearly $C^k(A)$ is a $*$-subalgebra of $B(H)$ and if $S \in B(H)$ is bijective and $S \in C^k(A)$ then $S^{-1} \in C^k(A)$.

For any $S \in B(H)$ let $\tilde{A}(S) = [S, iA]$ considered as a sesquilinear form on $D(A)$. We may iterate this and define a sesquilinear form on $D(\tilde{A}^k)$ by:

$$S^{(k)} \equiv \tilde{A}^k(S) = i^k \sum_{i+j=k} \frac{k!}{i!j!} (-\tilde{A})^i S \tilde{A}^j.$$

Then $S \in C^k(A)$ if and only if this form is continuous for the topology induced by $H$ on $D(\tilde{A}^k)$. We keep the notation $\tilde{A}^k(S)$ for the bounded operator associated to its continuous extension to $H$.

Now let $H$ be a self-adjoint operator on $H$ and $R(z) = (H - z)^{-1}$ for $z$ in the resolvent set $\rho(H)$ of $H$. We say that $H$ is of class $C^k(A)$ if $R(z) \in C^k(A)$ for some $z_0 \in \rho(H)$; then we shall have $R(z) \in C^k(A)$ for all $z \in \rho(H)$ and more generally $\varphi(H) \in C^k(A)$ for a large class of functions $\varphi$ (e.g. rational and bounded on the spectrum of $H$).

We shall say that a densely defined operator $S$ on $H$ is boundedly invertible if $S$ is injective, its range is dense, and its inverse extends to a continuous operator on $H$. If $S$ is symmetric this means that $S$ is essentially self-adjoint and 0 is in the resolvent set of its closure.

**Proposition 14.** Let $H$ be of class $C^2(A)$. Assume $(H)$ and $(Ha)$ with $s = 0$ and $Q = Q(H)$ such that $Q'$ is a bounded function. Then for $u \in H$ such that $u \in D(A^2) \cap D(Q)$ we have the estimate $|\psi_{Qu}(t)| \leq C_u(t)^{-2}$.

**Proof.** Note that as sesquilinear forms in $D(Q^2) \cap D(\tilde{A})$ one has the identity

$$t^2Q^2e^{itH} = \tilde{A}^2(e^{itH}) - Q'(H)\tilde{A}(e^{itH}).$$

Note also that $\tilde{A}^2(H) = Q'(H)Q(H)$, hence $H \in C^2(\tilde{A})$ by Propositions 7.2.16 and 6.2.10 in [ABG].

Assume first that $u \in H^1$, then

$$|t^2 \psi_{Qu}(t)| = |\langle u | t^2Q^2e^{itH}u \rangle| \leq |\langle u | \tilde{A}^2(e^{itH})u \rangle| + |\langle u | Q'(H)\tilde{A}(e^{itH})u \rangle| \leq C(\|u\| \|\tilde{A}^2u\| + \|\tilde{A}u\|^2) + \|u\| \|\tilde{A}Q'(H)u\| \leq C\|u\| (\|\tilde{A}u\| + \|\tilde{A}^2u\|) + \|u\|^2.$$

By Proposition 5, $(A + B)^2u$ is well defined and $\|\tilde{A}^2u\| \leq \|\tilde{A}u\|^2 + C\|Au\| + \|u\|$, thus $|\psi_{Qu}(t)| \leq C_u(t)^{-2}$, with $C_u = C\|u\| (\|u\| + \|Au\| + \|A^2u\|) + \|Au\|^2$.

For general $u \in D(A^2) \cap D(Q)$ we use $u_c = R_cu \in H^4$. Note that $u_c \in D(\tilde{A}^2)$ because $R_c \in C^2(\tilde{A})$. Proceeding like in the proof of Proposition 8 and letting $\epsilon \to 0$ we obtain the desired result. □

**Remark 15.** Note that if the operator $Q(H)$ is assumed boundedly invertible, the estimate of Proposition 14 holds for $\psi_u(t)$, with $u \in D(A^2) \cap D(Q)$.

**Proposition 16.** Let $H$ be of class $C^2(A)$. Assume $(H)$, $(Ha)$ and $(Hb)$ in the special case $Q = cH$ and $s = 0$. Let $u \in H$ be of the form $u = |H|^{1/2}v$, for some $v \in D(A^2)$ such that $A^jv \in \mathcal{E}$, $j = 0, 1, 2$ and $ABv \in \mathcal{E}$. Then $|\psi_u(t)| \leq C_u(t)^{-3/2}$. 
Proof. Define \( \chi = \chi[-M,M] \) the characteristic function of the interval in \( \mathbb{R} \). We decompose

\[ u := u_1 + u_2, \]

where \( u_1 = \chi(H)u \) and \( u_2 = (1 - \chi(H))u \). Note that \( \psi_{pu} = \psi_{pu_1} + \psi_{pu_2} \).

We will consider the following additional assumptions. We will show that \( \psi_{pu_1} = O(t^{-3/2}) \) and \( \psi_{pu_2} = O(t^{-2}) \). To study \( \psi_{pu_1} \) we rely on Corollary 8.3 in [GLS], that is, we need to prove that both \( \psi_{pu_1}(t) \) and \( \psi_{pu_1}(t) \) are functions in \( L^2(\mathbb{R}) \). We only show the latter, the former is an analogous calculation. Set \( v_1 = \chi(H)v \).

Note that

\[ H = \Delta + \chi \quad \text{in place of} \quad L, \]

we only show the latter, the former is an analogous calculation. Set

\[ \psi_{pu_2}(t) = \|c(t^2H\psi_{pu_2}(t))\| \leq C\|v_1\|\|A(H\psi_{pu_2}(t))\| \leq C\|v_1\|\|A\psi_{pu_2}(t)\|^2, \]

which concludes the proof.

\[ \square \]

7. APPLICATIONS

7.1. Potential of critical decay. Here we consider the equation \( i\partial_t u + \Delta u - V(x)u = 0 \) in \( \mathbb{R}^n (n \geq 3) \), with initial condition \( u(0, x) = f(x) \). In [BPSS], a resolvent estimate is used to obtain weighted \( L^2 \) estimates for time-independent potentials \( V(x) \in C^1(\mathbb{R}^n \setminus \{0\}) \) satisfying the following assumptions.

\[ (A1) \ \sup_{x \in \mathbb{R}^n} |x|^2 |V| < \infty \]

\[ (A2) \ \text{The operator} \ \Delta + |x|^2V + \lambda^2 \ \text{is positive on every sphere, i.e., there is a} \ \delta > 0 \ \text{such that for every} \ r > 0, \]

\[ \int_{|x|=r} |\nabla u(x)|^2 + (\lambda^2 + |x|^2V(x))|u(x)|^2 d\sigma(x) \geq \delta^2 \int_{|x|=r} |u(x)|^2 d\sigma(x) \]

\[ (A3) \ \text{The operator} \ \Delta + |x|^2\tilde{V} + \lambda^2 \ \text{is positive on every sphere, i.e., (A2) holds with} \ \tilde{V} \ \text{in place of} \ V. \]

Here \( \Delta \) represents the spherical Laplacian, \( \tilde{V} := \partial_r (rV(x)) \) and \( \lambda = (n - 2)/2 \).

In Section 3 of that paper the following Morawetz estimate is obtained

\[ \|\| |x|^{-1} e^{-itH} f \|_{L^2} \leq C \|f\|_{L^2}. \quad (8) \]

We will use this result to obtain pointwise estimates using the generator of dilations \( A = -\frac{1}{2} (x \cdot \nabla + \nabla \cdot x) \) as the conjugate operator. In order to verify the \( C^1(A) \) regularity for \( H \) and the other conditions of \( (H) \) we will consider the following additional assumptions.
(A4) $V$ is nonnegative and locally integrable in $\mathbb{R}^n$
(A5) $\sup_{x \in \mathbb{R}^n} |x|^2 |\nabla V| < \infty$ and $\|(x \cdot \nabla V)/(V)^{-1}\|_{L^\infty} < \infty$

In [Ka95] Theorem 4.6a the domain of the Friedrich extension of $H = -\Delta + V(x)$ is characterized as the set of all $i) u \in \mathcal{H}$ such that $\nabla u$ belongs to $(L^2(\mathbb{R}^n))^n$, ii) $\int (x|u|^2 \, dx < \infty$, and iii) $\Delta u$ exists and $-\Delta u + V(x)u$ belongs to $\mathcal{H}$. It is easy to check that if the dilation group leaves $D(H)$ invariant, in fact ii) and iii) are preserved under condition (A5). In this scenario, the $C^1(A)$ regularity follows from the commutation relation $[H, iA] = -2\Delta - x \cdot \nabla V = 2H - (2V + x \cdot \nabla V)$, which clearly satisfies $\|(H u, Au)\| \leq C(\|H u\| + \|u\|)$ by (A4) and (A5). Note that here $Q = 2H$ and $K = 2V + x \cdot \nabla V$.

Now write $K = \text{sign}(K)[K^{1/2}]K^{1/2}$. Assumptions (A1) and (A4) yield the bound $\|K^{1/2}e^{itH}f\|_{L^2} \leq C\|x\|^{-1}e^{itH}f\|_{L^2}$, and then estimate (8) implies that $K$ is the product of two Kato-smooth operators. Therefore, the assumptions of (H) hold with $s = 0$ and the estimate of Proposition 12 reads as follows.

**Proposition 17.** Let $H$ and $A$ be as above and $P$ a smooth projection commuting with $H$. Assume conditions (A1)-(A5) and (Hb). Then for $u \in D(A)$ such that $Pu$ and $PAPu$ are in $\mathcal{E}$, one has the estimate $\|\psi_{Pu}(t)\| \leq C_u(t)^{-1/2}$.

**Remark 18.** We can use the results of [BPSS] to characterize the space $\mathcal{E}$. For instance, the endpoint Strichartz estimate $\|e^{-itH}u\|_{L^4(\mathbb{R})} \leq C\|u\|_{L^4}$ for $n = 3$ yields that for $u \in L^{10/3}(\mathbb{R}^3)$, we have $\|\psi_u\|_{L^4} \leq \|u\|_{L^{10/3}} \leq \|u\|_{L^{5/3}} \leq C\|u\|_{L^4}$. Thus $\|\psi_u\|_{L^2} \leq \|\psi_u\|_{L^4} \leq \|\psi_u\|_{L^2}$. We can use the estimate (8) in dimension $n$, to show that if $xu \in L^2(\mathbb{R}^n)$, one has $\|\psi_u\|_{L^2} \leq \|\psi_u\|_{L^2}$.

### 7.2. Laplacian on manifold

Let $(M, g) = (\mathbb{R}^3, g)$ be a compact perturbation of $\mathbb{R}^3$, i.e. $M$ is $\mathbb{R}^3$ endowed with a smooth metric $g$ which equals the Euclidean metric outside of a ball $B(0, R_0) = \{x \in \mathbb{R}^3 : |x| \leq R_0\}$ for some fixed $R_0$. In [RT] global-in-time decay estimates were obtained for solutions to the Schrödinger equation $iu_t = Hu$, where $H = -\frac{1}{2}\Delta_M$ is the Laplace-Beltrami operator on $M$. It has been shown that $H$ defined on $S := C^\infty(M)$ (smooth functions of compact support) is essentially self-adjoint and its domain is the Sobolev space $H^2(M)$. The main result of their paper is the following.

**Theorem.** Let $M$ be a smooth compact perturbation of $\mathbb{R}^3$ which is nontrapping and smoothly diffeomorphic to $\mathbb{R}^3$. Then for any Schwartz solution $u(x, t)$ and any $\sigma > 0$ we have

$$\int_{\mathbb{R}} \|(x)^{-1/2 + \sigma} \nabla e^{-itH}u_0\|_{L^2(M)}^2 + \|(x)^{-3/2 - \sigma} e^{-itH}u_0\|_{L^2(M)}^2 \, dt \leq C_{\sigma, M} \|u_0\|_{H^{1/2}(M)}^2. \tag{9}$$

We choose the conjugate operator $A = -i/2(x \cdot \nabla + \nabla \cdot x)$ defined on $S$. Note that the dilation group leaves $H^2(M)$ invariant. Assuming that metric is smooth and bounded, the $C^1(A)$ condition follows from the commutation relation $[H, iA] = 2H + K$, where $K$ is a second order operator supported in $B(0, R_0)$. More explicitly, a straightforward calculation on $S$ shows that $K$ is an operator of the form $K = k_1(x)H + ik_2(x)\nabla + ik_3(x)$, where the $k_i$’s are smooth and bounded real functions supported in $B(0, R_0)$. Observe also that $(H)^{-1/2}K(H)^{-1/2}$ is bounded. Taking $E = F = \sqrt{|K|}$, the Kato-smoothness is a direct consequence of the estimate (9) and therefore the assumptions of (H) are met with $s = 1/2$. Condition (Ha) holds since $[A, K]$ is a second order differential operator as well. Proposition 9 yields the following result.

**Proposition 19.** Let $H$ and $A$ be above and $P$ a smooth projection commuting with $H$. Assume that condition (Hb) holds for the space $\mathcal{E}$. Then for $u \in D(A)$ such that $Pu$ and $PAPu$ are in $\mathcal{E}$, one has $\|\psi_{Pu}(t)\| \leq C_u(t)^{-1/2}$. 
REFERENCES

[ABG] W. Amrein, A. Boutet de Monvel, V. Georgescu: $C_0$-groups, commutator methods and spectral theory of N-body Hamiltonians, Birkhäuser, Basel-Boston-Berlin, 1996.

[BCD] V. Banica, R. Carles, T. Duyckaerts: On scattering for NLS: from Euclidean to hyperbolic space. Discrete Contin. Dyn. Syst. 24(4):1113-1127, 2009.

[BS] M. Beals, W. Strauss: $L^p$ estimates for the wave equation with a potential. Comm. Partial Differential Equations 18(7-8):1365-1397, 1993.

[BGM] A. Boutet de Monvel, V. Georgescu, M. Măntoiu: Locally smooth operators and the limiting absorption principle for N-body Hamiltonians. Rev. Math. Phys. 5: 105189, 1993.

[BSM] A. Boutet de Monvel, G. Kazantseva, M. Măntoiu: Some anisotropic Schrödinger operators without singular spectrum. Helv. Phys. 69: 13-25, 1996.

[BPSS] N. Burq, F. Planchon, J. Stalker, A. S. Tahvildar-Zadeh: Strichartz estimates for the Wave and Schrödinger Equations with Potentials of Critical Decay. Indiana Univ. Math. J., 53:1665–1680, 2004.

[BSo1] P. Blue and A. Soffer: Semilinear wave equations on the Schwarzschild manifold I: Local decay estimates. Advances in Diff. Eqs. 8(5):595-614, 2003.

[BSo2] P. Blue and A. Soffer: Phase space analysis on some black hole manifolds. J. Funct. Anal. 256(1):1-90, 2009.

[DR] M. Dafermos and I. Rodnianski: The black hole stability problem for linear scalar perturbations, in Proceedings of the Twelfth Marcel Grossmann Meeting on General Relativity, Singapore, 2011, 132-189. arXiv: 1010.5137

[DSR] R. Donninger, W. Schlag and A. Soffer: A proof of Price’s Law on Schwarzschild blackhole manifolds for all angular momenta. Advances in Mathematics 225:484-540, 2011.

[EGG] B. Erdogan, M. Goldberg and W. Green: Dispersive estimates for four dimensional Schrödinger and wave equations with obstructions at zero energy. Comm. PDE. 39(10):1936-1964, 2014.

[Ger] C. Gérard: A proof of the abstract limiting absorption principle by energy estimates. J. Funct. Anal. 254:2070-2074, 2008.

[GG] V. Georgescu, C. Gérard: On the virial theorem in quantum mechanics. Communications in Mathematical Physics 208(2):275-281, 1999.

[GLS] V. Georgescu, M. Larenas, A. Soffer: Abstract theory of pointwise decay with applications to wave and Schrödinger equations.

[Gold] M. Goldberg: Dispersive Estimates for the Three-Dimensional Schrödinger Equation with Rough Potentials. American Journal of Mathematics 128(3):731-750, 2006.

[GS] M. Goldberg and W. Schlag: Dispersive estimates for Schrödinger operators in dimensions one and three. Comm. Math. Phys. 251(1):157-178, 2004.

[HSS] W. Hunziker, I. M. Sigal and A. Soffer: Minimal escape velocities. Comm. Partial Differential Equations 24:2279–2295, 1999.

[JK] A. Jensen and T. Kato: Spectral properties of Schrödinger operators and time-decay of the wave functions. Duke Math. J. 46(3):583-611, 1979.

[JN1] A. Jensen and G. Nenciu: A unified approach to resolvent expansions at thresholds. Rev. Math. Phys. 13(6):717-754, 2001.

[JN2] A. Jensen and G. Nenciu: Erratum: “A unified approach to resolvent expansions at thresholds”. Rev. Math. Phys. 16(5):675-677, 2004. [Rev. Math. Phys. 13(6):717-754, 2001]

[JSR] J.-L. Journé, A. Soffer and C. D. Sogge: Decay estimates for Schrödinger operators. Comm. Pure Appl. Math. 44:573-604, 1991.

[Ka95] T. Kato: Perturbation Theory for Linear Operators. Springer-Verlag Berlin Heidelberg, 1995.

[MRT] M. Măntoiu, S. Richard, R. Tiedra de Aldecoa: Spectral analysis for adjacency operators on graphs. Ann. Henri Poincaré 8(7): 1401-1423, 2007.

[MT] M. Măntoiu, R. Tiedra de Aldecoa: Spectral analysis for convolution operators on locally compact groups. J. Funct. Anal. 253(2): 675-691, 2007.

[Mo] E. Mourre: Absence of singular continuous spectrum for certain self-adjoint operators. Communications in Mathematical Physics 78(3):519-567, 1981.

[P] C.R. Putnam. Commutation properties of Hilbert space operators and related topics. Springer, Berling, Heidelberg 1967.

[RS] S. Richard: Some Improvements in the Method of the Weakly Conjugate Operator. Letters in Mathematical Physics 76: 27-36, 2006.

[RT] M. Reed, B. Simon: Methods of modern mathematical physics, 4 volumes, Academic Press.

[RT] I. Rodnianski, T. Tao: Long time decay estimates for the Schrödinger equation on manifolds. Mathematical Aspects of Nonlinear Dispersive Equations, Ann. of Math. Stud., vol. 163, Princeton Univ. Press, Princeton, NJ (2007), pp. 223-253.

[Sch] W. Schlag: Dispersive estimates for Schrödinger operators: a survey. Mathematica aspects of nonlinear dispersive equations, Ann. of Math. Stud. 1:255-285, 2007.
I. M. Sigal and A. Soffer: Local decay and velocity bounds for time-independent and time-dependent Hamiltonians. Preprint, Princeton, 1987.

D. Tataru: Local decay of waves on asymptotically flat stationary space-times. *American Journal of Mathematics* 135(2):361-401, 2013.

R. Weder: $L^p$-$L^p$ estimates for the Schrödinger equation on the line and inverse scattering for the nonlinear Schrödinger equation with a potential. *Journal of Functional Analysis* 170(1):37-68, 2000.

K. Yajima: Dispersive estimates for Schrödinger equations with threshold resonance and eigenvalue. *Commun. Math. Phys.* 259:475-509, 2005.