Dynamics of a self–gravitating neutron source.

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Abstract: We examine the dynamics of a self–gravitating magnetized neutron gas as a source of a Bianchi I spacetime described by the Kasner metric. The set of Einstein-Maxwell field equations can be expressed as a dynamical system in a 4-dimensional phase space. Numerical solutions of this system reveal the emergence of a point–like singularity as the final evolution state for a large class of physically motivated initial conditions. Besides the theoretical interest of studying this source in a fully general relativistic context, the resulting idealized model could be helpful in understanding the collapse of local volume elements of a neutron gas in the critical conditions that would prevail in the center of a compact object.

Keywords: Self-gravitating systems, singularities, magnetic field, degenerate Fermi gases.
1. Introduction

Critical stellar configurations, such as white dwarfs, neutron stars, supermassive stars, relativistic star clusters and black holes are important astrophysical systems in which relativistic effects cannot be ignored. These astrophysical systems, denoted generically by the term “compact objects”, can be thought of as natural laboratories to understand a wealth of phenomena relevant to theoretical and experimental physics under critical conditions. These conditions are ideally suited to propose and test theoretical models of strong magnetic fields associated with self–gravitating sources under critical conditions. In such conditions, gravitation couples with other fundamental interactions (strong, weak and electromagnetic) and from this interplay important clues of their unification could emerge.

The presence and effects of strong magnetic fields in compact objects has been studied in the literature (see [1, 2, 3, 4, 5, 6], and references quoted therein). At a very basic level, the simplest approach is to consider various types of self–gravitating and magnetized plasmas of neutron matter under equations of state that are appropriate for the critical conditions of a compact object [7, 8]. Since astrophysical systems exhibit, in general, angular momentum and pressure anisotropies, it is interesting to examine how these effects could influence their stability.

Following the basic known methodology [1, an equation of state based on a nuclear ferromagnetic model (not related to electric currents) was examined in [2], with the purpose
of studying the interplay of pressure anisotropy and the magnetic field within a Newtonian framework. Hence, in the present work we consider a relativistic generalization that will allows us to examine (under specific restriction of the geometry) the evolution of this type of magnetized neutron gas under strong gravity. In particular, our aim is to provide the minimally basic qualitative and quantitative elements to address the question of whether the magnetic field can “slow down”, or even reverse or stop, gravitational collapse, as well as the issue of the evolution to a stable configuration.

It is evident that a proper and comprehensive study of a magnetized fermion gas, as a source of a compact object, requires a spacetime with axial symmetry (or without symmetries), leading to dynamical equations that must be solved by means of hydrodynamic codes of high complexity [2, 12, 13, 14]. In previous articles [10, 11] we studied a self–gravitating gas of magnetized electrons, described by an appropriate equation of state, as the source of an anisotropic Bianchi-I spacetime described by the Kasner metric. While a Bianchi I model is obviously inadequate as the metric of a compact object, it is among the less complicated geometries compatible with a magnetized source. Thus, the evolution of such a source under a much simplified Bianchi geometry is basically a toy model, but as such it can still be useful to obtain qualitative features of the sources under local critical conditions, specially in conditions near the center of the object where angular momentum plays a minor dynamical role. These qualitative results could provide a better understanding, and/or a useful approximated description, of the behavior of local internal volume elements near the center of a more realistic configuration. In this article we extend previous work on the electron gas to the case of a neutron gas. Hence, we follow a similar methodology based on re–writing the field and conservation equations for the magnetized neutron gas in the Bianchi I geometry as a dynamical system, evolving in a 4–dimensional phase space. This system is then analyzed qualitatively and numerically.

It is worthwhile remarking the basic differences between the magnetized electron gas examined in [1, 10, 11] and the neutron gas that we consider in this article. Electrons interact with a magnetic field through their charge, leading to the Landau diamagnetism characterized by a quantization effect associated with the Landau energy levels. However, neutrons interact with the magnetic field by their anomalous magnetic moment (AMM), in the context of Pauli’s paramagnetism and the equations of Pauli–Dirac. Consequently, one expects the magnetic neutron interaction to be weaker, though conditions of degenerated neutron gases in compact objects are also expected to be more critical (because of the higher densities) than those of degenerated electron gases. Therefore, relativistic effects of gravity are more likely to play a dominant dynamical role in neutron gases. Still, it is important to mention that a self–gravitating and magnetized neutron gas is a simplified model of a source for a compact object, as protons and chemical equilibrium potentials should also be considered to evaluate local effects [1, 4, 8]. Nevertheless, the magnetized neutron gas already exhibits important qualitative and quantitative differences in comparison with electron gases previously examined.

The paper is organized as follows: we derive in section II the equation of state for a magnetized neutron gas. The Einstein–Maxwell field equations for the Kasner metric and this source are derived in section III. In section IV we examine the limit of a weak...
magnetic field. The set of ordinary non–linear differential equations that follows from the field equations is transformed into a set of evolution equations in sections V and VI. This dynamical system is analyzed qualitatively and numerically in section VII. Our conclusions are presented in sections VIII.

2. Magnetized neutron gas: the equation of state.

The main properties of gas of magnetized degenerated neutrons are well known (see [1, 9, 13]). Considering the grand canonical ensemble, a subsystem can be thought of as a local volume of the neutron gas under the influence of a magnetic field $\vec{B}$ associated with the rest of the system (in an astrophysical context this could be good approximation to a suitable volume element inside a compact object). Because of this field, the subsystem becomes polarized, leading to a magnetization vector that satisfies the relationship: $
abla \vec{H} = \vec{B} - 4\pi \vec{M}$. The field $\vec{H}$ can be though of as “external” to the subsystem, while $\vec{B}$ can be “internal” to any particle within the subsystem, which feels (in addition to $\vec{H}$) the contribution $4\pi \vec{M}$ from particles from the rest of the subsystem.

The equation of state of the neutron gas follows from calculating the energy spectrum of the particles that make up the system. We can obtain this energy spectrum from the Dirac equation for neutral particles with anomalous magnetic moment:

$$\left(\gamma^\mu \partial_\mu + m + iq\sigma_\mu F^{\mu\nu}\right)\Psi = 0,$$

where $\sigma_{\mu\lambda} = \frac{1}{2}(\gamma_\mu \gamma_\lambda - \gamma_\lambda \gamma_\mu)$ is the spin tensor, $F^{\mu\nu}$ is the electromagnetic field tensor (we have set $\hbar = c = 1$) and $\Psi$ is the Dirac field. Solving equation (2.1) leads to the following energy spectrum [13], [16], [17]:

$$E_n(p, B, \eta) = \sqrt{p_{\parallel}^2 + \left(\sqrt{p_{\perp}^2 + m_n^2 + \eta q B}^2 \right)}$$

where $p_{\parallel}, p_{\perp}$ are, respectively, the components of momentum in the directions parallel and perpendicular to the magnetic field $B$, $m_n$ is the mass of the neutron, $q = -1.91\mu_N$ is the neutron magnetic moment ($\mu_N = e/2m_p$ is the nuclear magneton), $\eta = \pm 1$ are $\sigma_3$ eigenvalues corresponding to the two possible orientations (parallel or antiparallel) of the neutron magnetic moment with respect to the magnetic field.

The thermodynamical grand potential then takes the form

$$\Omega = -kT \ln Z,$$

where $k$ is the Boltzman constant, $T$ is the temperature, $Z = Tr(\hat{\rho})$ is the partition function of the system, $\rho = e^{-(\hat{H} - \mu\hat{N})/kT}$, $\hat{H}$ is the Hamiltonian, $\mu$ is the chemical potential and $\hat{N}$ is the number of particles operator.

The energy–momentum tensor associated with an external constant magnetic field takes the form:

$$T^\mu_\nu = \left(T \frac{\partial \Omega}{\partial T} + \sum \mu_n \frac{\partial \Omega}{\partial \mu_n}\right)\delta_4^{\mu} \delta_4^{\nu} + 4F^{\mu\gamma} F_{\gamma\nu} \frac{\partial \Omega}{\partial F^2} - \Omega \delta_4^{\mu} \delta_4^{\nu}$$

$$-3-$$
so that in the limit of zero magnetic field limit we obtain the perfect fluid tensor: $T^\mu_\nu = p\delta^\mu_\nu - (p + U)\delta^\mu_4\delta^4_\nu$. The components of the tensor (2.4) are:

$$
\begin{align*}
T^3_3 &= p_\parallel = -\Omega = p, \\
T^1_1 = T^2_2 &= p_\perp = -\Omega - B\mathcal{M} = p - B\mathcal{M}, \\
T^4_4 &= -U = -TS - \mu N - \Omega,
\end{align*}
$$

(2.5) (2.6) (2.7)

where $S$ is the specific entropy, $N = -\partial\Omega/\partial\mu$ is the particle number density, $\mathcal{M} = -\partial\Omega/\partial\mathcal{B}$ is the magnetization, $U$ is the energy density and $p$ is the pressure in the direction of the magnetic field.

The thermodynamical potential can be split in the following two parts:

$$
\Omega = \Omega_{sn} + \Omega_{Vn},
$$

where the first term in the right hand side is the statistical contribution and the second is the vacuum contribution [9]. Explicitly, we have

$$
\Omega_{sn} = -\frac{1}{4\pi^2\xi} \sum_{\eta=1,-1} \int_0^\infty p_\perp dp_\perp dp_3 \ln[f^+(\mu_n, \xi) f^-(\mu_n, \xi)],
$$

(2.8)

where $\xi = 1/k_B T$ and $f^\pm(\mu_n, \xi) = (1 + e^{(E_n + \mu_n)\xi})$ represent, respectively, the contributions from the particles and from the antiparticles. The vacuum term is given by the expression:

$$
\Omega_{Vn} = \frac{1}{4\pi^2\xi} \sum_{\eta=1,-1} \int_0^\infty p_\perp dp_\perp dp_3 E_n,
$$

(2.9)

which is divergent, but can be renormalized, and for fields of intensity $\mathcal{B} < 10^{18}$ G its contribution is not important [9], hence we will neglect this term in the remaining of the present article.

Equation (2.8) can be easily integrated in the case that concerns us ($T = 0$), leading to

$$
\begin{align*}
\Omega_{sn} &= -\lambda \sum_{\eta=1,-1} \left[ \frac{\mu f_\eta^3}{12} + \frac{(1 + \eta\beta)(5\eta\beta - 3)\mu f_\eta}{24} + \frac{(1 + \eta\beta)^3(3 - \eta\beta)}{24} L_\eta - \frac{\eta\beta\mu^3}{6} s_\eta \right],
\end{align*}
$$

(2.10)

where we have introduced the following expressions:

$$
\begin{align*}
f_\eta &= \sqrt{\mu^2 - (1 + \eta\beta)^2}, \\
s_\eta &= \frac{\pi}{2} - \arcsin\left(\frac{1 + \eta\beta}{\mu}\right), \\
\mu &= \frac{\mu_n}{m_n} \\
L_\eta &= \ln\left(\frac{\mu + f_\eta}{1 + \eta\beta}\right), \\
\beta &= \frac{\mathcal{B}}{B_c},
\end{align*}
$$

(2.11) (2.12)

with $B_c = m_n/q \simeq 1.56 \times 10^{20}$ G being the critical field and $\lambda = m_n^4/4\pi^2\hbar^3c^3 = 4.11 \times 10^{36}$ erg cm$^{-3}$.

All thermodynamical variables of the system follow readily from the thermodynamical potential $\Omega$. In particular, by computing neutron density and magnetization in this way
we obtain \( N = N_0 \Gamma_N, M = M_0 \Gamma_M \), where \( N_0 = \lambda/m_n, M_0 = N_0 q \), while the coefficients \( \Gamma_N, \Gamma_M \) take the form:

\[
\Gamma_N = \sum_{\eta=1,-1} \left[ \frac{f_\eta^3}{3} + \frac{\eta \beta(1 + \eta \beta)f_\eta}{2} - \frac{\eta \beta \mu^2}{2}s_\eta \right],
\]

\[
\Gamma_M = -\sum_{\eta=1,-1} \eta \left[ \frac{(1 - 2 \eta \beta) \mu f_\eta}{6} - \frac{(1 + \eta \beta)^2 (1 - \eta / 2)}{3} L_\eta + \mu^3 \frac{s_\eta}{6} \right].
\]

Therefore, given (3) and (2.10), we can write the equation of state for a relativistic degenerated neutron gas in the presence of an external magnetic field as:

\[
U = \mu_n N + \Omega = \lambda \Gamma_U(\beta, \mu), \quad (2.13)
\]

\[
p = -\Omega = \lambda \Gamma_P(\beta, \mu), \quad (2.14)
\]

\[
M = BM = \lambda \beta \Gamma_M(\beta, \mu), \quad (2.15)
\]

where

\[
\Gamma_P = \sum_{\eta=1,-1} \left[ \frac{\mu f_\eta^3}{12} + \frac{(1 + \eta \beta)(5 \eta \beta - 3) \mu f_\eta}{24} + \frac{(1 + \eta \beta)^2 (3 - \eta \beta)}{24} L_\eta - \frac{\eta \beta \mu^3}{6}s_\eta \right],
\]

\[
\Gamma_U = \mu \Gamma_N - \Gamma_P.
\]

Let us remark at this point that in (2.10)-(2.15) we are summing over the magnetic moments parallel (\( \eta = -1 \)) or antiparallel (\( \eta = 1 \)) to the magnetic field (i.e. the well known Pauli Paramagnetism). The choice \( \eta = \pm 1 \), [\ref{3}] is equivalent to consider different phases of the system. The appearance of the threshold of the value of the magnetic field for each one of these cases can be seen if we analyze the expressions of the functions \( f_\eta \) and \( s_\eta \) in (2.11). We have take \( \beta \geq 0 \) therefore starting with \( f_\eta \):

\[
\mu^2 \geq (1 + \eta \beta)^2 = 1 + 2 \eta \beta + \eta^2 \beta^2.
\]

(2.16)

however as \( \eta = -1, 1 \) then always \( \eta^2 = 1 \) and we can rewrite:

\[
\mu^2 \geq (\eta + \beta)^2.
\]

(2.17)

hence;

\[
(\beta + \eta - \mu)(\beta + \eta + \mu) \leq 0.
\]

(2.18)

Now we have two possibilities. But, it is easy to realize that the only acceptable possibility is:

\[
-\mu - \eta \leq \beta \leq \mu - \eta.
\]

(2.19)
This is exactly the restriction that comes from the function $s_\eta$:

$$\left| \frac{1 + \eta \beta}{\mu} \right| \leq 1.$$  \hspace{1cm} (2.20)

Therefore, from either one (2.19) or (2.20) the following constraints are obtained for the magnetic field:

$$\begin{align*}
\text{if} \quad \eta = 1 & \Rightarrow \quad -1 - \mu \leq \beta \leq \mu - 1, \\
\text{if} \quad \eta = -1 & \Rightarrow \quad 1 - \mu \leq \beta \leq 1 + \mu.
\end{align*}$$  \hspace{1cm} (2.21)

These last inequalities restrict the values of the magnetic field. Thus neutrons with AMM aligned to the magnetic field have only magnetic field values in the range between $1 - \mu \leq \beta \leq 1 + \mu$. Similarly, for neutrons with AMM oriented antiparallel to the magnetic field can take magnetic field values in the interval $-1 - \mu \leq \beta \leq \mu - 1$. In the particular case $\mu = 1$ we have.

$$\begin{align*}
\text{if} \quad \eta = -1, & \quad 0 \leq \beta \leq 2, \\
\text{if} \quad \eta = 1, & \quad -2 \leq \beta \leq 0.
\end{align*}$$  \hspace{1cm} (2.23)

This means that for fields smaller (or equal) than two times the critical magnetic field, the neutrons are aligned with $\eta = -1$ (the AMM is parallel to the magnetic field). Hence, neutrons with $\eta = 1$ are forced to invert its sense.

3. Einstein–Maxwell equations.

If a local self–gravitating volume of a magnetized neutron gas evolves in the conditions prevailing inside a compact object, relativistic effects will be important. This implies that local dynamics must be studied in the framework of General Relativity, by means of Einstein’s field equations:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa T_{\mu\nu}. \hspace{1cm} (3.1)$$

together with energy–momentum balance and Maxwell equations:

$$\begin{align*}
\mathcal{T}^{\mu\nu} \delta^\nu_\nu = 0, \\
F^{\mu\nu} \delta^\nu_\nu = 0, \\
F_{[\mu\nu;\alpha]} = 0.
\end{align*}$$  \hspace{1cm} (3.2)

where $\kappa = 8\pi G_N$, with $G_N$ being Newton’s gravitational constant, while square brackets denote anti–symmetrization in $\mu;\nu;\alpha$. The energy–momentum tensor $\mathcal{T}^\mu_\nu$ associated to the magnetized neutron gas is given by (3), with the relevant thermodynamical potentials obtained through Statistical Mechanics and complying with the appropriate equation of state (as discussed in the previous section). This tensor can also be given in terms of the 4-velocity $u^\alpha$ field as:

$$\mathcal{T}^\alpha_\beta = (U + \bar{P}) u^\alpha u_\beta + \bar{P} \delta^\alpha_\beta + \Pi^\alpha_\beta, \quad \bar{P} = p - \frac{2BM}{3}.$$  \hspace{1cm} (3.4)
We will consider the field equations (3.1)–(3.3) with (3.4) as the source tensor of a Bianchi I model described in the coordinate representation known as the Kasner metric:

\[ ds^2 = Q_1(t)^2 \, dx^2 + Q_2(t)^2 \, dy^2 + Q_3(t)^2 \, dz^2 - dt^2. \] (3.5)

which suggests considering a comoving geodesic 4–velocity \( u^\alpha = \delta^\alpha_t \), so that the anisotropic pressure tensor in (3.4) in the coordinates \([x, y, z, t]\) takes the form:

\[ \Pi^\alpha \beta = \text{diag} [\Pi, \Pi, -2\Pi, 0], \quad \Pi = -BM_3, \quad \Pi^\alpha \alpha = 0. \] (3.6)

The field equations (3.1) for (3.4) and (3.6) take the form:

\[ -G^x_x = \dot{Q}_2 \dot{Q}_3 Q_2 - \frac{\dot{Q}_3}{Q_3} = -\kappa(p - BM), \] (3.7)

\[ -G^y_y = \dot{Q}_1 \dot{Q}_3 Q_1 - \frac{\dot{Q}_1}{Q_1} = -\kappa(p - BM), \] (3.8)

\[ -G^z_z = \dot{Q}_1 \dot{Q}_2 Q_1 - \frac{\dot{Q}_2}{Q_2} = -\kappa p, \] (3.9)

\[ -G^t_t = \dot{Q}_1 \dot{Q}_2 Q_1 - \frac{\dot{Q}_2}{Q_2} = \kappa U. \] (3.10)

where \( \dot{A} = A_\alpha u^\alpha = A_t \). Energy–momentum balance (3.2) leads to:

\[ \dot{U} = \dot{Q}_3 (p + U) - \left( \frac{\dot{Q}_1}{Q_1} + \frac{\dot{Q}_2}{Q_2} \right) (BM + p + U). \] (3.11)

while the Maxwell equations equations (3.3) imply:

\[ \frac{\dot{Q}_1}{Q_1} + \frac{\dot{Q}_2}{Q_2} + \frac{\dot{B}}{2B} = 0. \] (3.12)

The Einstein-Maxwell equations (3.7, 3.11 and 3.12) are non–linear second order ordinary differential equations for the metric functions \( Q_1, Q_2, Q_3 \) and \( U \). In order to treat this system numerically, it is necessary to introduce a new set of variables that will transform it into a system of first order evolution equations. However, before undertaking this task, we examine in the following section the weak field limit.

**4. Limit of weak magnetic field.**

The discussion of the limit of weak magnetic field is important in to illustrate the connection between our quantum magnetic field and a classic Maxwellian field in the context of a magnetohydrodynamic treatment. In sections 2 we have shown equations strongly linked with quantum magnetic field.

It is important to emphasize that the term “quantum magnetic field” involves the semi–classical interaction between the magnetic field and the anomalous magnetic moment. This
approach implies a theoretical connection between the equation of state introduced in the previous section and a QED framework. As a consistency condition, this framework must allow for a classical Maxwellian limit that should arise from a series expansion around the zero of the magnetic field $\beta = 0$. The leading term of this expansion should lead to the well known energy-momentum tensor for a Maxwellian magnetic field \[19\]. In general, this type of series expansions can be done as follows:

\[
\begin{align*}
  p_{\perp} &= \sum_{n=0}^{\infty} \frac{\partial^n p_{\perp}}{\partial \beta^n} \bigg|_{\beta=0} \frac{\beta^n}{n!} \simeq p_1 - a_1 \beta^2 + O(\beta^4), \\
  p_{\parallel} &= \sum_{n=0}^{\infty} \frac{\partial^n p_{\parallel}}{\partial \beta^n} \bigg|_{\beta=0} \frac{\beta^n}{n!} \simeq p_1 + a_3 \beta^2 + O(\beta^4), \\
  U &= \sum_{n=0}^{\infty} \frac{\partial^n U}{\partial \beta^n} \bigg|_{\beta=0} \frac{\beta^n}{n!} \simeq U_0 + a_0 \beta^2 + O(\beta^4),
\end{align*}
\]

where it is easy to see that: $p_1 = p_{\parallel}(\beta = 0) \equiv p_{\perp}(\beta = 0)$, $a_0 = (\partial^2 U / \partial \beta^2) \big|_{\beta=0} / 2$, $a_1 = (\partial^2 p_{\perp} / \partial \beta^2) \big|_{\beta=0} / 2$, $a_3 = (\partial^2 p_{\parallel} / \partial \beta^2) \big|_{\beta=0} / 2$, $U_0 = U(\beta = 0)$. Hence, all of the former are functions only of the dimensionless chemical potential $\mu$.

As shown in the examples in the literature of classical magnetic fields in the context of a Bianchi-I geometry \[19, 20\], the leading magnetic field terms are quadratic. This suggests to truncate the series in (4.1)-(4.3) in the quadratic terms, as the higher order terms, like $\beta^4, \beta^6, \ldots$, are (in general) extremely small multipole contributions. If we assume that the velocity fluctuations of the plasma tend to a zero average macroscopically, and that the medium does not undergo any bulk motion, then these contributions can be safely neglected (though typically higher velocities could arise from thermal fluctuations or quantum disorder). Under these assumptions, the energy-momentum tensor of a neutron gas with a minimally coupled magnetic field can always be written in the form \[19\]:

\[
T_{\mu\nu} = (U_0 + U_{mag})u_{\mu}u_{\nu} + (p_0 + p_{mag})h_{\mu\nu} + \Pi_{\mu\nu}^{mag}.
\]

where $p_{mag}$ and $U_{mag}$ are, respectively, the magnetic pressure and the magnetic energy.

It is important to remark that $p_0$ in (4.4) is the isotropic contribution to the pressure of the system, which (in general) can depend on the chemical potential and the magnetic field. On the other hand, $p_1$ is the pressure for the case without magnetic field $\beta = 0$. In the case under consideration we have:

\[
T_{\mu\nu} = \text{diag}[p_1 - a_1 \beta^2, p_1 - a_1 \beta^2, p_1 + a_3 \beta^2, -U_0 - a_0 \beta^2].
\]

Comparison with equations (4.5)-(4.4) leads to:

\[
\begin{align*}
  p_{mag} &= \frac{H^2}{6} = \frac{a_0 \beta^2}{3}, \\
  U_{mag} &= \frac{H^2}{2} = a_0 \beta^2.
\end{align*}
\]
where $H^2 = H^\mu H_\mu = 2a_0\beta^2$ and $H_\mu$ are the components of the magnetic field, which we have assumed to point in the $z$-direction. This assumption is consistent with the fact that a small volume element around the center of a compact object is approximately homogeneous and the rotation axis furnishes a privileged direction, which we can always align with the $z$ axis:

$$H^\alpha = (0, 0, \sqrt{2a_0\beta} Q_3, 0), \quad H_\alpha = (0, 0, \sqrt{2a_0\beta} Q_3, 0).$$

The tensor $\Pi^{mag}_{\mu\nu}$ is then the projected symmetric trace-free tensor representing anisotropic pressures that comes from the magnetic field. It can be written as:

$$(\Pi^{mag})_{\mu\nu} = \text{diag}\left[-\frac{1}{3}(a_1 + a_3)\beta^2, -\frac{1}{3}(a_1 + a_3)\beta^2, \frac{2}{3}(a_1 + a_3)\beta^2, 0\right].$$

Is important to point out that $p_1$ in (4.5) is, in general, different from $p_0$. Thus:

$$p_0 = p_1 - (a_0 + 2a_1 - a_3)\beta^2.$$

where $p_1 = p_{(\beta=0)} = \tilde{P}_{(\beta=0)}$ in (2.14) and (3.4) is the pressure without magnetic field. Only when the magnetic field is zero, $\beta = 0$, the pressures coincide: $p_0 = p_1$ and the energy–momentum tensor becomes that of a perfect fluid with isotropic pressure. In this case $p_0$ and $U_0$ correspond to the pressure and energy density of a classical neutron gas.

5. Local kinematic variables.

Since we are interested on the local evolution of volume elements of the magnetized neutron gas associated with the source (3.4), we need to re–write the dynamical Einstein–Maxwell equations in terms of covariant parameters associated with the local kinematics of volume elements as described by $u^\alpha$. For the Kasner metric in the comoving geodesic 4–velocity frame, the nonzero local kinematic parameters are the expansion scalar, $\Theta$, and shear tensor, $\sigma^{\alpha\beta}$, given by:

$$\Theta = u^\alpha ; \alpha,$$

$$\sigma^{\alpha\beta} = u_{(\alpha;\beta)} - \Theta h_{\alpha\beta},$$

where $h_{\alpha\beta} = u_\alpha u_\beta + g_{\alpha\beta}$ is the projection tensor and round brackets denote symmetrization on the indices $\alpha, \beta$. The geometric interpretation of these parameters is straightforward: $\Theta$ denotes the isotropic proper time change of proper volume of local fluid elements, whereas $\sigma^{\alpha\alpha}$ describes the deformation of local volumes as they expand at different rates along the directions given by its eigenvectors.

The expansion scalar and components of the shear tensor for the Kasner metric are:

$$\Theta = \frac{\dot{Q}_1}{Q_1} + \frac{\dot{Q}_2}{Q_2} + \frac{\dot{Q}_3}{Q_3},$$

$$\sigma^{\mu\nu} = \text{diag}\left[\sigma^x_1, \sigma^y, \sigma^z, 0\right] = \text{diag}\left[\Sigma_1, \Sigma_2, \Sigma_3, 0\right],$$

where $\dot{Q}_1, \dot{Q}_2, \dot{Q}_3$ are the time derivatives of the Kasner exponents.
where
\[ \Sigma_a = \frac{2\dot{Q}_a}{3Q_a} - \frac{\dot{Q}_b}{3Q_b} - \frac{\dot{Q}_c}{3Q_c}, \quad a \neq b \neq c, \quad (a, b, c = 1, 2, 3). \] (5.5)

Since the shear tensor is trace–free: \( \sigma^{\alpha}_\alpha = 0 \), we can eliminate one of the quantities \((\Sigma_1, \Sigma_2, \Sigma_3)\) in terms of the other two. In fact, for the Bianchi I model in the Kasner metric, one of these quantities is enough to fully represent \( \sigma^{\alpha}_\alpha \), though for mathematical convenience we will keep two of these variables by eliminating \( \Sigma_1 \) in terms of \((\Sigma_2, \Sigma_3)\).

By means of (5.3) and (5.5), all second order derivatives of the metric functions in (3.7), (3.11) and (3.12) can be re–written as first order derivatives of \( \Theta \), \( \Sigma_2 \) and \( \Sigma_3 \). After some algebraic manipulation we can re–write (3.7), (3.11) and (3.12) as the following autonomous first order system of evolution equations

\[ \dot{U} = -(U + p - \frac{2}{3}BM)\Theta - BM\Sigma_3, \] (5.6)
\[ \dot{\Sigma}_2 = -\frac{\kappa BM}{3} - \Theta\Sigma_2, \] (5.7)
\[ \dot{\Sigma}_3 = \frac{2}{3}\kappa BM - \Theta\Sigma_3, \] (5.8)
\[ \dot{\Theta} = \kappa(BM + \frac{3}{2}(U - p)) - \Theta^2, \] (5.9)
\[ \dot{\beta} = \frac{2}{3}\beta(3\Sigma_3 - 2\Theta). \] (5.10)

Together with these equations, we have the following algebraic constraint
\[ -\Sigma_2^2 - \Sigma_2\Sigma_3 + \frac{\Theta^2}{3} - \Sigma_3^2 = \kappa U. \] (5.11)

that follows from (3.10). This non–linear first order system in the variables \( U, \beta, \Theta, \Sigma_2, \Sigma_3 \) and the constraint (5.11) become fully determined once we use the thermodynamical equations of the previous section to express \( M \) and \( B \) in terms of \( \beta = B/B_c \). The solution of this system describes the dynamical evolution of local volumes of a magnetized neutron gas that could be taken as a rough approximation to a grand canonical subsystem of this source near the center of a compact object.

6. Dynamical equations.

The system of evolution equations (5.6)–(5.10) can be transformed into a proper dynamical system by introducing the following dimension–less evolution parameter
\[ H = \frac{\Theta}{3}, \quad \frac{d}{d\tau} = \frac{1}{H_0} \frac{d}{dt}, \] (6.1)

together with the dimension–less functions:
\[ \mathcal{H} = \frac{H}{H_0}, \quad S_2 = \frac{\Sigma_2}{H_0}, \quad S_3 = \frac{\Sigma_3}{H_0}, \quad \beta = \frac{B}{B_c}, \] (6.2)

where \( H_0 \) is a constant inverse length scale, which for convenience we choose as \( 3H_0^2 = \kappa \lambda \Rightarrow |H_0| = 1.66 \times 10^{-4} \text{ cm}^{-1} \). Note that \( H_0 \) is not the cosmological Hubble constant, given
by $H_0^{\text{cosm}} = 0.59 \times 10^{-28} \, \text{cm}^{-1}$, but a constant that provides a length scale $1/H_0 \sim 6 \, \text{km}$ that is adequate for the characteristic length scale of the system under consideration. The functions $S_2$ and $S_3$ are the shear tensor component normalized to this scale, while the dimensionless time $\tau$ can be positive or negative, depending on the sign of $H_0 = \pm \sqrt{\kappa \lambda/3}$.

Substituting the variables (6.2) into the system (5.6) and the constraint (5.11) we get

$$\mu, \tau = \frac{1}{\Gamma U, \mu} \left[ (2H - S_3)(\Gamma_M - 2\Gamma_{U,\beta})\beta - 3\mathcal{H}(\Gamma_P + \Gamma_U) \right],$$

$$S_{2,\tau} = -\beta \Gamma_M - 3S_2 \mathcal{H},$$

$$S_{3,\tau} = 2\beta \Gamma_M - 3S_3 \mathcal{H},$$

$$\mathcal{H}, \tau = \beta \Gamma_M - \frac{3\Gamma_P}{2} - \frac{1}{2} S_2 S_3 - \frac{3}{2} \mathcal{H}^2 - \frac{1}{2}(S_2^2 + S_3^2),$$

$$\beta, \tau = 2\beta (S_3 - 2\mathcal{H}),$$

$$- S_2^2 - S_3^2 - S_2 S_3 + 3\mathcal{H}^2 = 3\Gamma_U.$$  \hspace{1cm} (6.8)

where $\tau$ indicates derivative with respect to $\tau$. We have also replaced the variable $U$ for $\mu$, because from (2) we have: $U = U(\beta, \mu) \Rightarrow U, \tau = \lambda(\Gamma_U, \mu, \tau + \Gamma_{U,\beta,\tau})$, which allows us to obtain $\mu, \tau$ from $U, \tau$. We will solve numerically the system (6.3) in the following section.

7. Numerical solutions and physical discussion

Since we are interested in studying a collapsing magnetized neutron gas configuration, we need to consider the local collapse of volume elements. Hence, we solve the constraint (6.8) to obtain the two roots of $H$, so that the condition for a collapsing evolution follows by selecting the negative root. To ensure the local collapse we shall demand in the solution of equations (6.3) that the initial expansion, $\Theta$, is negative, which implies for an initial time hypersurface $\tau = 0$ that $H(0) < 0$. This follows from (5.3) and (6.2) expressed in terms of the local proper volume $V = \sqrt{\text{det} g_{\alpha\beta}} = Q_1 Q_2 Q_3$ as:

$$V = V(0) \exp \left( 3 \int_{\tau=0}^\tau \mathcal{H} d\tau \right).$$  \hspace{1cm} (7.1)

To investigate the direction dependence of the collapse, in terms of the directions given by the space coordinates $(x, y, z)$, we can relate by means of (5.3), (5.5), (6.1) and (6.2) the spatial metric components to the combination $\mathcal{H} + S_j$, leading to:

$$Q_j(\tau) = Q_j(0) \exp \left[ \int (\mathcal{H} + S_j) d\tau \right], \quad j = 1, 2, 3.$$  \hspace{1cm} (7.2)

where $Q_j(0)$ are constants that can be identified with initial values for $Q_j(\tau)$. To solve the system (5.3), we shall use a wide range of initial conditions associated with typical conditions prevailing in a neutron star, for example: $\mu = 2 \Rightarrow \rho \sim 10^{15} \, \text{g/cm}^3$, $\beta_0 = 10^{-2} - 10^{-5}$, for magnetic fields between $10^{18}$ gauss and $10^{15}$ gauss. We shall impose in all numerical trials the condition of volume collapse: $\mathcal{H}(0) < 0$, together
with: \( S_2(0) = 0, \pm 1, S_3(0) = 0, \pm 1 \), which correspond to cases with zero initial deformation and initial deformation in the direction of the axes \( y \) or \( z \). The numeric solution for the function \( H \), displayed in Figure 1 for different initial conditions, shows that \( H \rightarrow -\infty \), regardless of the selected initial conditions. The magnetic field tends to increase, but always remains below the critical field \( B = B_c \). This behavior is shown in Figure 1 for the full range of initial conditions. The plots displayed in Figure

**Figure 1**: The left panel shows the behavior of \( H \) vs \( \tau \) for different initial conditions. The right panel shows the magnetic field intensity \( (\beta = B/B_c) \), it has tendency to rise, but remains below the value of the critical field.

and in the left panel of Figure 2 describe the collapse of the fluid elements from the solutions of (1.3). It is evident from these figures that the quantities \( S_1 + H \rightarrow -\infty \), so that the spatial metric coefficients tend to zero \( (Q_1, Q_2, Q_3 \rightarrow 0) \), which clearly shows that volume elements collapse to a point like isotropic singularity.

### 8. Phase Space.

As it was done in [10], we can use the constraint (6.8) to transform the evolution system

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**Figure 2**: Behavior of \( (S_1 + H) \) and \( (S_2 + H) \) versus \( \tau \). We can see this quantities tending to \(-\infty\), besides different collapse times for different initial conditions.
Figure 3: The left panel shows behavior of $\left(S_3 + H\right)$ versus $\tau$. The behavior is similar to the trajectories given in Fig. 2, therefore the quantity $\left(S_3 + H\right) \to -\infty$. The right panel shows paths in the small section of the space phase $(S_3, \beta, \mu)$. The points represent the unbranded initial conditions, the point “a” represents the attractor point which coordinates are $(S_3 = 0, \beta = 0, \mu = 1, H = 0)$.

(6.3) to a reduced system of equations in the variables $S_3, \beta, \mu, H$

\[
\mu, \tau = \frac{1}{\Gamma_{U,\mu}} \left[ \left( 2H - S_3 \right) (\Gamma_M - 2\Gamma_{U,\beta}) \beta - 3H (\Gamma_P + \Gamma_U) \right],
\]

(8.1)

\[
S_3, \tau = 2\beta \Gamma_M - 3S_3\mathcal{H},
\]

(8.2)

\[
\mathcal{H}, \tau = \beta \Gamma_M + \frac{3}{2} (\Gamma_U - \Gamma_P) - 3\mathcal{H}^2,
\]

(8.3)

\[
\beta, \tau = 2\beta (S_3 - 2\mathcal{H}),
\]

(8.4)

where we note that the only equation that was modified is the equation for $\mathcal{H}$, hence (8.1) is equivalent to (6.3).

The trajectories in the 3–dimensional subsection of the phase space, parametrized by $(S_3, \beta, \mu)$, are shown in Figure 3. The evolution of the system is determined by the sign of $H_0$. For $\tau < 0 \Rightarrow H_0 = -\sqrt{\kappa\lambda/3}$, the system evolves towards the stable attractor (point marked by a), while for $\tau > 0 \Rightarrow H_0 = \sqrt{\kappa\lambda/3}$, the trajectories evolve towards a singularity. A similar study was conducted for the remaining 3–dimensional subsections of the space phase, obtaining qualitatively similar results. which was obtained coordinates of attractor are: $(S_3 = 0, \beta = 0, \mu = 1, \mathcal{H} = 0)$.

9. Conclusions.

We have used a Bianchi I model to study the evolution of a magnetized neutron gas characterized by a physically motivated and fully relativistic equation of state. As far as we are aware, this equation of state has not been considered previously in a general relativistic context. Besides the general theoretical interest in undertaking such a study, we argue that the simplified Bianchi geometry roughly approximates a grand canonical subsystem of a magnetized neutron source in the conditions prevailing near the center of a compact object, hence our treatment can be conceived as a toy model that can be
useful in understanding the local evolution of volume elements of this source under these conditions. However, a proper examination of the specific limitations of the dynamics of this toy model and/or its connection with concrete astrophysical studies of actual compact objects lies beyond the scope of the present article. As we comment further ahead, we will consider these important tasks in forthcoming articles by resorting to perturbation techniques, more elaborate numerical methods and less idealized sources.

The Einstein-Maxwell field equations for a magnetized neutron gas in the Bianchi I geometry were transformed into a set of non-linear evolution equations, which were solved numerically for generic collapsing initial conditions and analyzed qualitatively as a proper dynamical system. The results that we found are:

- The final state in the collapsing evolution of local volume elements is an isotropic point-like singularity. This final state occurred for a wide range of initial conditions associated with parameter values that would be typical in a compact object.

- The magnetic field increases, but its values are always below the critical field. This result is consistent with numerical values of maximal field intensities compatible with stability in numerical studies of magnetized rotating configurations (see \cite{2}).

- The study of the phase space associated with the dynamical equations shows that the system evolves, for $H_0 < 0$, to an equilibrium point, (i.e. into a stable configuration). It is possible to introduce a temperature dependence in the equation of state. By doing so, the evolution of the neutron gas could be associated with high temperature neutron sources in the context of early universe conditions in cosmological models dealing with primordial magnetic field \cite{22}.

It is important to remark that, unlike the dynamical study of a magnetized electron gas \cite{10}, anisotropic “cigar-like” singularities did not occur for a wide range of initial conditions. Since a dynamical effect of the magnetic field under critical conditions is a final state anisotropic singularity aligned in the direction of the field, the exclusive occurrence of point-like isotropic in a magnetized neutron gas suggests that the final stage of the evolution of this gas is more intensely dominated by the focusing effect associated with strong gravity than the electron gas. This is consistent with the fact that electrons are strongly coupled to the magnetic field through their electric charge, whereas neutrons have a weaker coupling associated with their anomalous magnetic moment.

It is important to stress that the geometry of Bianchi I models in a comoving frame has stringent limitations in dealing adequately with the dynamical effects associated with a magnetic field. This is important when considering neutrons as a source in which electric charge vanishes but not the magnetic moment. By being spatially homogeneous with a 4-velocity orthogonal to flat 3-dimensional hypersurfaces of maximal symmetry, the Lorentz force is necessarily zero: $f^b = qu_aF^{ab} = 0$. Also, the fact that the Bianchi I model is spatially flat makes it inadequate to examine (even as a toy model) the interplay of local collapse and the magnetic field in localized objects, as such interplay is necessarily associated with strong positive spatial curvature. However, in our case these inadequacy can be overcome (at least partially) by considering a general perturbation scheme on a
Bianchi I model, in which spatial curvature and 4-acceleration are perturbative but not zero. In this case it is possible to examine the effects of the magneto–curvature coupling associated with a non–trivial Lorentz force and a nonzero deceleration parameter in the Raychaudhuri equation (see [18] for general detail). The use of such a perturbed Bianchi I model for the description of the neutron gas source considered in this article is presently under consideration in a separate article.

Besides the introduction of a perturbation scheme in a Bianchi I model, another possible improvement on the dynamical description of the source under consideration would be to consider Bianchi models I, V, VII or IX with a tilted 4–velocity, which are endowed with more degrees of dynamical freedom, including even the possibility of nonzero rotation (see [23] and references quoted therein). These models would allow for a less restrictive study of the dynamical effects, reported in [18], in which magnetic tension and gravitational collapse may present non–trivial coupling with a nonzero and non–perturbative 4–acceleration and vorticity with the magnetic field.

Finally, as we mentioned in the introduction, a magnetized gas consisting only of neutrons can be theoretically interesting but it is too idealized as a potential source for a compact object. Thus, we will consider as an extension of this work, besides the extra degrees of freedom in the dynamics (mentioned above), a gas mixture of neutrons, electrons and protons, complying with suitable balance conditions and adequate chemical potentials, in comparison with other types of equations of state [2, 4, 8]. These extension of the present work are also under consideration for future articles.

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