AN APPLICATION OF DESCRIPTIVE SET THEORY TO COMPLEX ANALYSIS

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Abstract. The purpose of this paper is to prove a new general result about rings of complex analytic functions. Let $\Omega$ be an arbitrary nonempty open subset of the complex plane $\mathbb{C}$, $\mathcal{A}(\Omega)$ be the set of holomorphic functions on $\Omega$ viewed as a Polish ring (not a Polish algebra over $\mathbb{C}$) in the usual compact open topology, let $R$ be a Polish ring and let $\varphi : R \to \mathcal{A}(\Omega)$ be an abstract algebraic isomorphism. The main goal of this paper is to prove Theorem 36 that $\varphi$ is a topological isomorphism. A special result of Bers is an easy corollary. Two additional items supplement these results, viz., that $B(\mathbb{D})$, the abstract ring of bounded analytic functions on the unit disk, cannot be made into a Polish ring and that $\mathcal{M}(\Omega)$, the abstract field of meromorphic functions on $\Omega$, cannot be made into a Polish field.

1. Introduction

The purpose of this paper is to prove a new general result about rings of complex analytic functions. Let $\Omega$ be an arbitrary nonempty open subset of the complex plane $\mathbb{C}$, $\mathcal{A}(\Omega)$ be the set of holomorphic functions on $\Omega$ viewed as a Polish ring (not a Polish algebra over $\mathbb{C}$) in the usual compact open topology, let $R$ be a Polish ring and let $\varphi : R \to \mathcal{A}(\Omega)$ be an abstract algebraic isomorphism. The main goal of this paper is to prove Theorem 36, that $\varphi$ is a topological isomorphism. A special result of Bers is an easy corollary. Two additional items help put this theorem into perspective, that $B(\mathbb{D})$, the abstract ring of bounded analytic functions on the unit disk, cannot be made into a Polish ring and that $\mathcal{M}(\Omega)$, the abstract field of meromorphic functions on $\Omega$, cannot be made into a Polish field.

This paper assumes a familiarity with descriptive set theory, especially Polish spaces, analytic sets, sets with the Baire property and

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Borel spaces, as may be found in Parthasarathy ([14]), Kechris ([10]), Becker and Kechris ([2]) and Mackey ([12]).

The by now well known very general approach to the proof of theorems like Theorem 36 is to show that $\varphi$ is measurable with respect to the sets with the Baire property. However, in almost every proof of such theorems the verification of the measurability condition in any particular instance can be intricate and require a fair amount of ingenuity. Such seems to be the case here because of the generality of the result and that a priori there appears so little structure to work with.

A complicating factor in the proof is that $\mathcal{A}(\Omega)$ is viewed as a Polish ring and not as a Polish algebra over $\mathbb{C}$. The proof is largely carried out in a sequence of lemmas and propositions, some of which may be of independent interest. A number of the propositions in the following discussion are given without proofs being presented. In every case either references are given or the proofs are easy and left to the reader.

We always assume unless otherwise stated that if $\Omega \subseteq \mathbb{C}$ is open then $\emptyset \neq \Omega$.

2. Borel Set Preliminaries

The following general theorems and corollaries will prove to be useful.

**Theorem 1** ([2, Theorem 1.2.6]). Let $\varphi : G \to H$ be a Baire Property measurable homomorphism between Polish groups. Then $\varphi$ is continuous. If moreover, $\varphi[G]$ is not meager, then $\varphi$ is also open.

**Theorem 2** ([10, Theorem 12.17]). Let $G$ be a Polish group and $H$ a closed subgroup of $G$. Then there exist a Borel subset $B$ of $G$ so that $B$ meets every $H$-coset at exactly one point.

**Proposition 3** ([1, Proposition 5]). Suppose $G$ is a multiplicative Polish group, $H$ is an analytic subgroup of $G$, and $A \subseteq G$ is an analytic subset of $G$ so that $A$ meets every $H$-coset at exactly one point and $G = AH$. Then $H$ is closed.

**Corollary 4.** Let $G$ be a multiplicative Polish group, $H$ and $K$ be subgroups of $G$ so that $H$ and $K$ are analytic sets, $G = HK$, and $H \cap K = \{e\}$. Then both $H$ and $K$ are closed.

Recall that if $X$ is a set then a family $\mathcal{F}$ of subsets of $X$ separates points if for every pair $x \neq y \in X$, there exists $A \in \mathcal{F}$ so that $x \in A$ and $y \not\in A$.

**Theorem 5** (Mackey [12, Theorem 3.3]). Let $X$ be a Polish space and suppose $\mathcal{F}$ is a countable family of Borel sets which separates points.
Then the family $\mathcal{F}$ generates the Borel structure of $X$. That is, the smallest $\sigma$-algebra containing all members of $\mathcal{F}$ is precisely $\mathcal{B}(X)$.

View $\mathbb{Q}^2 \subset \mathbb{R}^2 = \mathbb{C}$ as complex numbers.

**Corollary 6.** Let $\mathcal{F}$ be a countable family of Borel subsets of $\mathbb{C}$, each with nonempty interior and with the property that, for every $\varepsilon > 0$, there is some $B \in \mathcal{F}$ with $\text{diam}(B) < \varepsilon$. Then the countable collection $\{q + B \mid q \in \mathbb{Q}^2, B \in \mathcal{F}\}$ generates the Borel structure of $\mathbb{C}$, i.e., generates $\mathcal{B}(\mathbb{C})$.

**Proof.** This follows from Theorem 5 since it is easy to check that the countable collection $\{q + B \mid q \in \mathbb{Q}^2, B \in \mathcal{F}\}$ separates points. □

In what follows $D = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$ will denote the open unit disk in the complex plane and $\text{cl}(D) = \{\zeta \in \mathbb{C} \mid |\zeta| \leq 1\}$ is the closed unit disk in the complex plane.

**Corollary 7.** The $\sigma$-algebra generated by the sets $\{\beta + D \mid \beta \in \mathbb{Q}^2\}$ is $\mathcal{B}(\mathbb{C})$. Similarly, the $\sigma$-algebra generated by the sets $\{\beta + \text{cl}(D) \mid \beta \in \mathbb{Q}^2\}$ is $\mathcal{B}(\mathbb{C})$.

**Proof.** Let $\delta > 0$, $\delta \in \mathbb{Q}$ be very small. Then $D \cap (2 - \delta + D)$ is a nonempty open set that fits inside a $\delta \times \sqrt{4\delta - \delta^2}$ box. That is, first translate $D$ to be centered over 2 and then move it slightly left. Now use Corollary 6. The proof for $\text{cl}(D)$ is virtually the same. □

**Lemma 8.** Let $X$ be a Polish space, $\varphi$ a homeomorphism of $X$ and $B \subseteq X$ a subset with the Baire property. Then $\varphi[B]$ has the Baire property.

**Corollary 9.** If $R$ is a Polish ring, $a, b \in R$ with $a$ invertible, $B \subseteq R$ a subset with the Baire property, then $aB + b$ is also a subset with the Baire property.

**Proof.** $\varphi(x) = ax + b$ is a homeomorphism since $a$ is invertible. Now use Lemma 8. □

Though it is consistent with ZFC that the continuous image of a set with the Baire property fails to have the Baire property (see [13] for the existence of $\Delta^1_2$ sets that lack the Baire property), this corollary suggests the following question. If $R$ is a Polish ring, $a \in R$ and $B \subseteq R$ is a set with the Baire property, does $aB$ have the Baire property?

**Lemma 10.** Let $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq 0$ and $f(\zeta) = \alpha \zeta + \beta$. Then, for any $\zeta \in \mathbb{C}$ and $r > 0$, $f[B(\zeta, r)] = B(f(\zeta), |\alpha|r)$. 


Lemma 11. Let $\Omega \subseteq \mathbb{C}$ be a Borel set with $\text{interior}(\Omega) \neq \emptyset$ and $\text{interior}(\Omega^c) \neq \emptyset$. Then

$$\mathcal{U} = \{\alpha \Omega + \beta \mid \alpha, \beta \in \mathbb{Q}^2, \alpha \neq 0\}$$

is a countable family of Borel sets that separates points.

Proof. $\mathcal{U}$ is a countable family of Borel sets since Borel sets are invariant under homeomorphisms. We must see that $\mathcal{U}$ separates points. Let $\zeta_1 \in \mathbb{Q}^2 \cap \text{interior}(\Omega)$, $\zeta_2 \in \mathbb{Q}^2 \cap \text{interior}(\Omega^c)$, and $r > 0$ be so that $B(\zeta_1, r) \subseteq \text{interior}(\Omega)$ and $B(\zeta_2, r) \subseteq \text{interior}(\Omega^c)$.

Now, let $w_1, w_2 \in \mathbb{C}$ be arbitrary so that $w_1 \neq w_2$ and pick $\varepsilon > 0$ so small that

$$2\varepsilon < |w_1 - w_2| \text{ and } |\zeta_1 - \zeta_2| + 2r\varepsilon < |w_1 - w_2|r.$$ 

Pick $\lambda_j \in B(w_j, \varepsilon) \cap \mathbb{Q}^2$ where $1 \leq j \leq 2$. Notice that $B(w_1, \varepsilon) \cap B(w_2, \varepsilon) = \emptyset$ since $2\varepsilon < |w_1 - w_2|$ and therefore $0 < |\lambda_1 - \lambda_2|$. Also easily check that $|w_1 - w_2| < |\lambda_1 - \lambda_2| + 2\varepsilon$ by the triangle inequality.

Let $\varphi(\zeta) = \frac{\lambda_2 - \lambda_1}{\zeta_2 - \zeta_1}(\zeta - \zeta_1) + \lambda_1 = \alpha \zeta + \beta$, where $\alpha = \frac{\lambda_2 - \lambda_1}{\zeta_2 - \zeta_1}$ and $\beta = \frac{\lambda_1 \zeta_2 - \lambda_2 \zeta_1}{\zeta_2 - \zeta_1}$ are both complex rationals. Notice that $\varphi(\zeta_1) = \lambda_1$ and $\varphi(\zeta_2) = \lambda_2$. We will prove that $\alpha \Omega + \beta$ separates $w_1$ and $w_2$.

$$\varphi(\zeta_j) = \lambda_j \text{ and therefore } \varphi[B(\zeta_j, r)] = B\left(\lambda_j, \frac{|\lambda_1 - \lambda_2|}{|\zeta_1 - \zeta_2|}r\right)$$

by Lemma 10.

$$|w_j - \lambda_j| < \varepsilon < \frac{(|w_1 - w_2| - 2\varepsilon)r}{|\zeta_1 - \zeta_2|} < \frac{|\lambda_1 - \lambda_2|}{|\zeta_1 - \zeta_2|}r \quad \text{and therefore } w_j \in \varphi[B(\zeta_j, r)], (1 \leq j \leq 2). \quad \text{Hence, } w_1 \in \alpha \Omega + \beta \text{ and } w_2 \notin \alpha \Omega + \beta.

\[ \square \]

Proposition 12. Let $R$ be a Polish ring and $\varphi : R \to \mathbb{C}$ be an abstract ring isomorphism. If there exists a Borel set $\Omega \subseteq \mathbb{C}$ with $\text{interior}(\Omega) \neq \emptyset$ and $\text{interior}(\Omega^c) \neq \emptyset$ so that $\varphi^{-1}[\Omega]$ has the Baire property, then $\varphi$ is a topological isomorphism.

Proof. First, apply Lemma 11 and then Theorem 5 to see that $\{\alpha \Omega + \beta \mid \alpha, \beta \in \mathbb{Q}^2, \alpha \neq 0\}$ generates the Borel structure of $\mathbb{C}$. Then, since $\varphi^{-1}[\Omega]$ has the Baire property and $R$ is a Polish ring, we have that $\varphi^{-1}(\alpha) \varphi^{-1}[\Omega] + \varphi^{-1}(\beta)$ is a set with the Baire property for every $\alpha, \beta \in \mathbb{C}, \alpha \neq 0$ by Lemma 9. That is, $\varphi$ is a $\mathcal{B}\mathcal{P}$-measurable isomorphism so Theorem 1 applies to conclude that $\varphi$ is a topological isomorphism. \[ \square \]

3. General Polish Ring Preliminaries

This section is devoted to some general facts about Polish rings and Polish $\mathbb{C}$-vector spaces.
Proposition 13. Let $R$ by any Polish ring with unity and let

$$\mathcal{I}_R = \{ x \in R \mid x \text{ has a right inverse } \}.$$  

Then $\mathcal{I}_R$ is an analytic set. $\mathcal{I}_R$ is a Borel set if $R$ is commutative.

Proof. Notice that $\mathcal{I}_R = \{ x \in R \mid \text{there exists } y \in R \text{ with } xy = 1_R \}.$

Let $A = \{ \langle x, y \rangle \in R^2 \mid xy = 1_R \}$ and $\pi : R^2 \to R$ be the projection $\langle x, y \rangle \mapsto x.$ $A$ is closed in $R^2$ since multiplication is continuous. $\mathcal{I}_R$ is an analytic set since it is the range of $\pi.$

Suppose $R$ is commutative. To see that $\mathcal{I}_R$ is Borel, it suffices to check that $\pi \upharpoonright_A$ is injective. To see this, suppose that $\langle x, y_1 \rangle, \langle x, y_2 \rangle \in A.$ Then $xy_1 = 1_R = xy_2 \implies y_1 = y_1xy_2 = xy_1y_2 = y_2.$ \hfill $\square$

Is $\mathcal{I}_R$ always a Borel set?

Proposition 14. Let $R$ be a commutative Polish ring with unity. Then the map $x \mapsto x^{-1}, \mathcal{I}_R \to \mathcal{I}_R,$ is continuous if and only if $\mathcal{I}_R$ is a $G_\delta$ subset of $R.$

Proof. First assume that $x \mapsto x^{-1}, \mathcal{I}_R \to \mathcal{I}_R,$ is continuous. Let $A = \{ \langle x, y \rangle \in R^2 \mid xy = 1_R \}$ and notice that $A$ is closed in $R^2.$ The canonical projection $\pi : R^2 \to R$ defined by $\langle x, y \rangle \mapsto x$ is injective when restricted to $A.$ Since $x \mapsto x^{-1}, \mathcal{I}_R \to \mathcal{I}_R,$ is continuous, we see that $x \mapsto \langle x, x^{-1} \rangle, \mathcal{I}_R \to A,$ is continuous. So $\pi \upharpoonright_A$ is a homeomorphism which implies that $\mathcal{I}_R$ is Polishable and thus is a $G_\delta$ subset of $R.$

Conversely, suppose that $G = \mathcal{I}_R$ is a $G_\delta$ subset of $R.$ Therefore $G$ is Polishable. Notice that the map $x \mapsto x^{-1}, G \to G,$ is a multiplicative group isomorphism. Let $A = \{ \langle x, y \rangle \in G^2 \mid xy = 1_R \}$ and notice that $A$ is closed in $G^2$ since $(x, y) \mapsto xy, G^2 \to G,$ is continuous.

The mappings $A \to G, \langle x, x^{-1} \rangle \mapsto x$ and $\langle x, x^{-1} \rangle \mapsto x^{-1}$ are continuous bijections. Therefore the mapping $x \mapsto \langle x, x^{-1} \rangle$ is a Borel mapping by the Luzin-Suslin Theorem. Hence, the mapping $x \mapsto \langle x, x^{-1} \rangle \mapsto x^{-1}$ is a Borel mapping. Hence, $x \mapsto x^{-1}, \mathcal{I}_R \to \mathcal{I}_R,$ is continuous. \hfill $\square$

From Proposition 14 we see that, for any Polish field $F,$ the multiplicative inversion $x \mapsto x^{-1}, F \setminus \{0\} \to F \setminus \{0\},$ is continuous.

Proposition 15. Let $R$ be a Polish ring. If $M \subseteq R$ is a principal one-sided-ideal, then $M$ is an analytic set.

Proof. The left ideal and right ideal cases are similar. For the right-ideal case, let $y \in R$ be the generator for $M.$ That is, $M = \{ yx \mid x \in R \}.$ Since multiplication is continuous, the mapping $x \mapsto yx, R \to R,$ is continuous and $M$ is the image of $R$ under $x \mapsto yx.$ Therefore, $M$ is an analytic set. \hfill $\square$
Proposition 16. Let $X$ be an arbitrary complex topological vector space and $0 \neq x_0 \in X$. The mapping $\lambda \mapsto \lambda x_0$, $C \to X$, is a homeomorphism onto its range and its range is closed in $X$.

Proof. This is a special case of the uniqueness theorem for finite dimensional spaces ([11], 7.3, page 59).

The ring/field of complex numbers $C$ has $2^c$ ring/field automorphisms, only two of which, namely the identity and complex conjugation, are continuous.

Theorem 17 ([9]). Any automorphism of $\mathbb{C}$ which is bounded on an $F_\sigma$ subset of the plane of positive inductive dimension is necessarily continuous.

4. The Ring of Analytic Functions and Special Cases

Let $\Omega \subseteq \mathbb{C}$ be open. $\mathcal{A}(\Omega)$ denotes the collection of complex analytic functions on $\Omega$ endowed with the compact-open topology, i.e., the topology of uniform convergence on compact sets. $\mathcal{A}(\Omega)$ is a commutative Polish $\mathbb{C}$-algebra with unity and therefore a commutative Polish ring with unity with the algebraic operations of point-wise addition and point-wise multiplication. If $\lambda \in \mathbb{C}$, let $c_\lambda \in \mathcal{A}(\Omega)$ be the constant function taking the value $\lambda$. The mapping $\lambda \mapsto c_\lambda$, $\mathbb{C} \to \mathcal{A}(\Omega)$, is continuous and is therefore a homeomorphism onto its range by Proposition 16. In the future, $\lambda$ will be identified algebraically and topologically with $c_\lambda$.

Several of the following propositions are either well known or elementary and their proof is left to the reader.

Proposition 18. Let $\Omega$ be open. Then $\Omega$ is connected if and only if $\mathcal{A}(\Omega)$ is an integral domain.

Proposition 19. Let $\Omega = \bigcup_{n<\kappa} \Omega_n \subseteq \mathbb{C}$ be open, where each $\Omega_n$ is open and connected, where the $\Omega_n$'s are pairwise disjoint and where $\kappa \leq \aleph_0$. The rings $\mathcal{A}(\Omega)$ and $\prod_{n<\kappa} \mathcal{A}(\Omega_n)$ are topologically ring isomorphic.

Ideals in $\mathcal{A}(\Omega)$ will play a key role in what follows, especially the following distinguished class of ideals. For $\alpha \in \Omega$ let

$$M_\alpha = \{ f \in \mathcal{A}(\Omega) \mid f(\alpha) = 0 \}$$

be the associated ideal.

Note that $f \in \mathcal{A}(\Omega)$ is invertible if and only if $f$ is zero-free. Let $\mathcal{I} = \{ f \in \mathcal{A}(\Omega) \mid f \text{ is zero-free} \}$, an abbreviation of $\mathcal{I}_{\mathcal{A}(\Omega)}$.

Recall that if $f \in \mathcal{A}(\Omega)$, $\alpha \in \Omega$ and $f(\alpha) = 0$, then there exists a unique $g \in \mathcal{A}(\Omega)$ so that $f = (z - \alpha)g$. More generally, for any
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If \( f \in A(\Omega) \) and \( \alpha \in \Omega \), there exists a unique \( g \in A(\Omega) \) so that \( f = (z - \alpha)g + f(\alpha) \).

**Lemma 20.** \( I \subseteq A(\Omega) \) is a proper principal maximal ideal if and only if there exists \( \alpha \in \Omega \) so that \( I = M_\alpha \).

**Proof.** \( M_\alpha \) is a proper principal ideal since it is generated by \( z - \alpha \) and is a maximal ideal since \( A(\Omega)/M_\alpha \) is \( \mathbb{C} \).

Conversely, let \( f \) be the generator for an ideal \( I \) in \( A(\Omega) \). If \( f \) is zero-free, then \( f \) is invertible in \( A(\Omega) \), so the constant 1 function is in \( I \) and \( I = A(\Omega) \) contradicting the fact that \( I \) is proper. Hence, there is \( \alpha \in \Omega \) for which \( f(\alpha) = 0 \) and there is \( g \in A(\Omega) \) such that \( f = (z - \alpha)g \). So

\[
I = fA(\Omega) = (z - \alpha)gA(\Omega) \subseteq (z - \alpha)A(\Omega) = M_\alpha.
\]

\( I = M_\alpha \) since \( I \) was assumed to be maximal. \( \square \)

**Lemma 21.** Let \( \Lambda \) be a countable dense subset of \( \mathbb{C} \) and \( \Omega \) be open and connected. Then, for \( f \in A(\Omega) \), \( f \) is constant if and only if

\[
f \in \bigcap_{\lambda \in \Lambda} (\lambda + (\mathcal{I} \cup \{0\})).
\]

**Proof.** The implication is trivial if \( f \) is constant.

Conversely, suppose \( f \in A(\Omega) \) is non-constant. Then, by the Open Mapping Theorem, there is some \( \alpha \in \Omega \) and some \( \lambda \in \Lambda \) so that \( f(\alpha) = \lambda \). It follows that \( f - \lambda \) has a zero which means \( f - \lambda \) is not invertible and, since \( f \) is non-constant, \( f - \lambda \) is not identically zero. That is, \( f - \lambda \not\in \mathcal{I} \cup \{0\} \). \( \square \)

**Lemma 22.** Let \( \Omega \subseteq \mathbb{C} \) be open and connected, let \( R \) be a Polish ring and let \( \varphi : R \to A(\Omega) \) be an abstract isomorphism of rings. Then \( \varphi^{-1}[\mathbb{C}], \varphi^{-1}[\mathbb{C} \setminus \Omega] \) and \( \varphi^{-1}[\Omega] \) are Borel sets.

**Proof.** \( \mathcal{I}_R = \varphi^{-1}[\mathcal{I}] \) is a Borel set by Proposition 13.

\[
\varphi^{-1}[\mathbb{C}] = \bigcap_{\lambda \in \Lambda} (\varphi^{-1}(\lambda) + (\mathcal{I}_R \cup \{0\}))
\]

is a countable intersection of Borel sets and so is a Borel set.

Next,

\[
\mathbb{C} \setminus \Omega = \mathbb{C} \cap \{\alpha \in \mathbb{C} \mid z - \alpha \in \mathcal{I}\} = \mathbb{C} \cap (z + \mathcal{I})
\]

since \( -\mathcal{I} = \mathcal{I} \). Therefore

\[
\varphi^{-1}[\mathbb{C} \setminus \Omega] = \varphi^{-1}[\mathbb{C}] \cap (\varphi^{-1}(z) + \mathcal{I}_R)
\]

is a Borel set.

Finally, \( \varphi^{-1}[\Omega] = \varphi[\mathbb{C}] \setminus \varphi[\mathbb{C} \setminus \Omega] \) is a Borel set. \( \square \)
Lemma 23. Let $\Omega \subseteq \mathbb{C}$ be open and connected, $R$ be a Polish ring and $\varphi : R \to \mathcal{A}(\Omega)$ be an isomorphism of rings. Then

(i) $\varphi^{-1}[\mathbb{C}]$ is closed in $R$ and if $M \subseteq R$ is a proper principal maximal ideal, then $M$ is closed;
(ii) for a proper principal maximal ideal $M$, $\langle x, y \rangle \mapsto x + y, M \oplus \varphi^{-1}[\mathbb{C}] \to R$, is an additive group and topological isomorphism.

Proof. (i) Let $M \subseteq R$ be a proper principal maximal ideal. It follows that there exists $\alpha \in \Omega$ so that $\varphi[M] = M_\alpha$ and $\mathcal{A}(\Omega) \cong M_\alpha \oplus \mathbb{C}$ as additive groups. Hence, $R \cong M \oplus \varphi^{-1}[\mathbb{C}]$ as additive groups. As $M \cap \varphi^{-1}[\mathbb{C}] = \{0\}$, $M$ is an analytic set by Proposition 15, and $\varphi^{-1}[\mathbb{C}]$ is a Borel set, Corollary 4 implies that both $\varphi^{-1}[\mathbb{C}]$ and $M$ are closed in $R$.

(ii) Since $\varphi^{-1}[\mathbb{C}]$ and $M$ are closed in $R$, they are also additive Polish groups so $M \oplus \varphi^{-1}[\mathbb{C}]$ is an additive Polish group. The mapping $\langle x, y \rangle \mapsto x + y, M \oplus \varphi^{-1}[\mathbb{C}] \to R$, is a homeomorphism by Theorem 1 since it is continuous and an additive group isomorphism. \hfill \Box

Lemma 24. Let $\Omega$ be open and connected and $\varphi : R \to \mathcal{A}(\Omega)$ be a ring isomorphism. For each $\alpha \in \Omega$, the mapping $x \mapsto \varphi^{-1}(\varphi(x)(\alpha)), R \to \varphi^{-1}[\mathbb{C}]$, is continuous.

Proof. For $\alpha \in \Omega$, let $x \mapsto \langle x_\alpha, x_\alpha^* \rangle, R \to \varphi^{-1}[M_\alpha] \oplus \varphi^{-1}[\mathbb{C}]$, be the homeomorphic isomorphism of additive groups guaranteed by Lemma 23. Since projection is continuous, we see that the mapping $x \mapsto x_\alpha^*$, $R \to \varphi^{-1}([\mathbb{C}])$, is continuous.

Since $x = x_\alpha + x_\alpha^*$, $\varphi(x) = \varphi(x_\alpha) + \varphi(x_\alpha^*)$, $\varphi(x)(\alpha) = \varphi(x_\alpha)(\alpha) + \varphi(x_\alpha^*)(\alpha) = 0 + \varphi(x_\alpha^*)(\alpha) = \varphi(x_\alpha^*)(\alpha) = \varphi(x_\alpha^*)$ as $\varphi(x_\alpha) \in M_\alpha$ and $\varphi(x_\alpha^*) \in \mathbb{C}$. Therefore $x \mapsto x_\alpha^* = \varphi^{-1}(\varphi(x_\alpha^*)) = \varphi^{-1}(\varphi(x)(\alpha))$ is continuous. \hfill \Box

Theorem 25. Let $\Omega$ be open and connected. If $R$ is a Polish ring and $\varphi : R \to \mathcal{A}(\Omega)$ is an abstract ring isomorphism so that $\varphi \restriction_{\varphi^{-1}[\mathbb{C}]}$ is $\mathcal{B}\mathcal{P}$-measurable, then $\varphi$ is a homeomorphism.

Proof. $\varphi^{-1}[\mathbb{C}]$ is a closed additive subgroup of $R$ by Lemma 23 and therefore is itself an additive Polish group. $\varphi \restriction_{\varphi^{-1}[\mathbb{C}]}$ is an additive group and topological isomorphism since it is $\mathcal{B}\mathcal{P}$-measurable. Therefore for each $\alpha \in \Omega$ the mapping $x \mapsto x_\alpha^* \mapsto \varphi(x_\alpha^*) = \varphi(x)(\alpha)$ is continuous. Let $\Lambda \subseteq \Omega$ be a countable dense set and let $\psi : \mathcal{A}(\Omega) \to \prod_{\lambda \in \Lambda} \mathbb{C}$ be defined by $\psi(f) = \prod_{\lambda \in \Lambda} f(\lambda)$. $\psi$ is a continuous injection, its range is
a Borel set and $\psi$ is a Borel isomorphism onto its range. Define $\Theta : R \to A(\Omega)$ by $\Theta(x) = \psi^{-1}(\prod_{\lambda \in \Lambda} \varphi(x)(\lambda))$. $\Theta$ is a Borel linear bijection and therefore a topological isomorphism between $R$ and $A(\Omega)$.

**Corollary 26.** Let $A$ be any Polish $\mathbb{C}$-algebra and $\varphi : A \to A(\Omega)$ be an abstract isomorphism of $\mathbb{C}$-algebras, where $\Omega$ is open and connected. Then $\varphi$ is a homeomorphism.

**Proof.** By Proposition 16, we have that $\lambda \mapsto \lambda \varphi^{-1}(1) : \mathbb{C} \to A$ is a homeomorphism onto its range. $\varphi(\lambda \varphi^{-1}(1)) = \lambda$ since $\varphi$ is an isomorphism of $\mathbb{C}$-algebras. Hence $\varphi \restriction_{\varphi^{-1}[\mathbb{C}]}$ is continuous so Theorem 25 applies to ensure that $\varphi$ is a homeomorphism. □

Let $R$ be a Polish ring. $R$ is said to be algebraically determined if, given a Polish ring $S$ and an abstract ring isomorphism, $\varphi : S \to R$, then $\varphi$ is also a topological isomorphism. $\mathbb{C}$ is not an algebraically determined Polish ring since it has many discontinuous automorphisms. On the other hand $\mathbb{R}$ is an algebraically determined Polish ring ([8]), a not totally trivial fact. Similar definitions can obviously be made for algebraically determined Polish groups, Polish algebras, Polish Lie rings, or Polish Lie algebras. The previous corollary states that $A(\Omega)$ is an algebraically determined $\mathbb{C}$-algebra.

**Theorem 27.** Suppose $\Omega$ is open, connected and $\text{interior}(\Omega^c) \neq \emptyset$. Then $A(\Omega)$ is an algebraically determined Polish ring. In particular, this means that $A(\mathbb{D})$ is algebraically determined as a Polish ring.

**Proof.** Let $R$ be a Polish ring and let $\varphi : R \to A(\Omega)$ be an algebraic isomorphism. $\varphi^{-1}[\mathbb{C}]$ is a closed and therefore Polish subring of $R$ by Lemma 23. $\varphi^{-1}[\Omega]$ is a Borel set by Lemma 22. Proposition 12 now guarantees that $\varphi \restriction_{\varphi^{-1}[\mathbb{C}]}$ is a Borel mapping. Conclude the proof by applying Theorem 25. □

**Proposition 28.** Let $\kappa \leq \aleph_0$, suppose $R_n$ is an algebraically determined Polish ring with unity for each $n < \kappa$, and let $R = \prod_{n < \kappa} R_n$. Then $R$ is an algebraically determined Polish ring.

**Proof.** Let $S$ be a Polish ring and let $\varphi : S \to R$ be an algebraic isomorphism. For each $n$, let $\varphi_n(e_n) = 1_R$ if $n = n$ and let $S_n = \varphi^{-1}(e_n)S = S\varphi^{-1}(e_n)$. Each $S_n$ is an analytic set and subring of $S$. If $f_n = \prod_{\ell \neq n} 1_R - e_n$, let $T_n = \varphi^{-1}(f_n)S = S\varphi^{-1}(f_n)$. Then $T_n$ is an analytic set and subring of $S$, $S = S_nT_n = T_nS_n$ and $S_n \cap T_n = \prod_{n < \kappa} \{0\}$. Hence, Corollary 4 implies that each $S_n$ is a closed subring of $S$ and therefore itself is a Polish ring. Let $\varphi_n = \varphi \restriction_{S_n}$. Then $\varphi_n : S_n \to R_n$ is an algebraic isomorphism.
and therefore is a topological isomorphism since $R_n$ is algebraically determined. Therefore $\prod_{n<\kappa} \varphi_n : \prod_{n<\kappa} S_n \rightarrow \prod_{n<\kappa} R_n$ is a topological isomorphism.

The proof will therefore be complete if we prove that $S$ and $\prod_{n<\kappa} S_n$ are topologically isomorphic. Next, note that $S$ and $S_n \times T_n$ are homeomorphic since the natural mapping $S_n \times T_n \rightarrow S$ is a continuous bijection of additive Polish groups. Thus there is a natural continuous ring bijection $p : S \rightarrow \prod_{n<\kappa} S_n$ which therefore is a topological isomorphism.

\[ \square \]

**Corollary 29.** Suppose $\Omega$ is open and disconnected. Then $A(\Omega)$ is an algebraically determined Polish ring.

**Proof.** Choose $\kappa \leq \aleph_0$ so that $\Omega = \bigcup \{ \Omega_n \mid n < \kappa \}$ where the $\Omega_n$'s are open, connected for $n < \kappa$ and pairwise disjoint. Therefore Proposition 19 implies that $A(\Omega)$ and $\prod_{n<\kappa} A(\Omega_n)$ are algebraically and topologically isomorphic. Since $\text{interior}(\Omega_n^c) \neq \emptyset$ for each $n$, Theorem 27 implies that each $A(\Omega_n)$ is algebraically determined for each $n < \kappa$. Now apply Proposition 28 to conclude the proof. \[ \square \]

The results of this section, in particular Corollary 26, Theorem 27, and Corollary 29 may be considered as test cases and exercises in the techniques developed in this section and are preliminary to the general result in the next section.

### 5. The General Case for the Ring of Analytic Functions

The purpose of this section is to prove that if $\Omega \subseteq \mathbb{C}$ is open, then $A(\Omega)$ is an algebraically determined Polish ring. First recall some basic, easily proved facts about conformal mappings.

$z$, as usual denotes the identity mapping. Recall that a conformal map between two open subsets $\Omega_1$, $\Omega_2$ of $\mathbb{C}$ is a bijection $\gamma : \Omega_1 \rightarrow \Omega_2$ so that both $\gamma$ and $\gamma^{-1}$ are analytic. A bijective map $\gamma : \Omega_1 \rightarrow \Omega_2$ is said to be anti-conformal if $\overline{\gamma} : \Omega_1 \rightarrow \overline{\mathbb{C}[\Omega_2]}$ is a conformal mapping. Here, of course, the overline denotes complex conjugation. A simple example of a conformal mapping is $f(\zeta) = \alpha \zeta + \beta$, where $\alpha, \beta \in \mathbb{C}$ and $\alpha \neq 0$.

**Lemma 30.** Let $\Omega_1$ be connected open and define $\Omega_2 = \overline{\mathbb{C}[\Omega_1]}$. Then the map $\varphi : A(\Omega_1) \rightarrow A(\Omega_2)$ defined by

$$\varphi(f) = \overline{f} \circ \overline{\varphi(\Omega_2)}$$

is a topological isomorphism of Polish rings along with the property that $\varphi(f)' = \varphi(f')$ for each $f \in A(\Omega_1)$. 
Proposition 31. Suppose $\gamma : \Omega_1 \to \Omega_2$ is a conformal bijection where $\Omega_1, \Omega_2$ are both open. Then $\varphi : A(\Omega_1) \to A(\Omega_2)$ defined by $\varphi(f) = f \circ \gamma^{-1}$ is a topological isomorphism of Polish rings. More generally, $A(\Omega_1)$ and $A(\Omega_2)$ are topologically isomorphic as Polish rings if $\Omega_1$ and $\Omega_2$ are open subsets of $\mathbb{C}$ along with the property that they are conformally or anti-conformally equivalent.

From this we see that deciding whether or not $A(\Omega)$ is algebraically determined for open and connected $\Omega \subseteq \mathbb{C}$ reduces to deciding it for some conformal or anti-conformal representative.

Definition 32. Let $R$ be a Polish ring and let $S \subseteq R$. Let

$$ls(S) = \{x \in R | x^n \in S \text{ for some } n \geq 1 \text{ and } x^m \neq x^n \text{ for all } 1 \leq m < n\}$$

and

$$li(S) = \{x \in R | x^n \in S \text{ for all } n \geq 1 \text{ and } x^m \neq x^n \text{ for all } 1 \leq m < n\}.$$

Comment: the notation is supposed to bring to mind limsup and liminf.

Lemma 33. If $R$ is a Polish ring and $S \subseteq R$ is a Borel set (respectively, an analytic set), then $ls(S)$ and $li(S)$ are both Borel sets (respectively, analytic sets).

Proof.

$$ls(S) = \left( \bigcup_{n \geq 1} \{x \in R | x^n \in S\} \right) \cap \left( \bigcup_{1 \leq m < n} \{x \in R | x^m = x^n\} \right)^c$$

and

$$li(S) = \left( \bigcap_{n \geq 1} \{x \in R | x^n \in S\} \right) \cap \left( \bigcup_{1 \leq m < n} \{x \in R | x^m = x^n\} \right)^c.$$ 

\qed

Theorem 34 (Weierstrass, [4, Theorem 7.32]). Let $\Omega \subseteq \mathbb{C}$ be open, $A \subseteq \Omega$ be relatively discrete, and $n : A \to \mathbb{Z}$ be a function with $n(\alpha) \geq 1$ for each $\alpha \in A$. Then there exists a holomorphic function $f : \Omega \to \mathbb{C}$ so that $f$ has a zero of order $n(\alpha)$ at every $\alpha \in A$ and no other zeros.

Lemma 35. Let $cl(D) \subseteq \Omega \subseteq \mathbb{C}$ be open, $\lambda \in li(\Omega)$ and $A^*(\Omega) = A(\Omega) \setminus \{0\}$. Then the following are equivalent:

(i) $|\lambda| > 1$;
(ii) $\{\lambda^n | n \geq 1\}$ is relatively discrete in $\Omega$;
(iii) there exists $f \in A^*(\Omega)$ such that $f(\lambda^n) = 0$ for all $n \geq 1$.
Proof. (i) \(\implies\) (ii). Suppose that \(\lambda \in li(\Omega)\) with \(|\lambda| > 1\). Then an easy induction shows that \(|\lambda|^n \geq 1 + n(|\lambda| - 1)\) (Bernoulli’s Inequality). Therefore \(\{(|\lambda|^n) | n \in \mathbb{N}\}\) is a sequence of strictly increasing numbers which tends towards infinity. This guarantees that \(\{\lambda^n | n \in \mathbb{N}\}\) has no accumulation points in \(\mathbb{C}\) let alone in \(\Omega\).

(ii) \(\implies\) (iii). This follows immediately from Theorem 34.

(iii) \(\implies\) (i). Suppose \(\lambda \in li(\Omega)\) and \(|\lambda| \leq 1\). Then the distinct sequence \(\{\lambda^n\}_{n \geq 1}\) has a limit point in \(cl(\mathbb{D}) \subseteq \Omega\), which in turn implies that \(f\) is identically zero, a contradiction. \(\square\)

**Theorem 36.** If \(\Omega \subseteq \mathbb{C}\) is open, then \(A(\Omega)\) is an algebraically determined Polish ring.

Proof. Let \(R\) be a Polish ring and let \(\varphi : R \rightarrow A(\Omega)\) be an algebraic isomorphism. Our goal is to prove that \(\varphi\) is a topological isomorphism. We can and do assume that \(\Omega\) is connected by Corollary 29.

We can assume that \(cl(\mathbb{D}) \subseteq \Omega \subseteq \mathbb{C}\). If not, apply a simple conformal mapping of the form \(\varsigma \mapsto a\varsigma + b\) and apply Proposition 31.

\(\mathbb{K} = \varphi^{-1}[\mathbb{C}]\) is a closed (and therefore Polish) subring of \(R\) by Lemma 23. To prove the theorem it suffices to show that \(\varphi : \mathbb{K} \rightarrow \mathbb{C}\) is \(B\mathcal{P}\)-measurable by Theorem 25. This will be accomplished by Proposition 12 if we prove that \(\varphi^{-1}[cl(\mathbb{D})] \subseteq \mathbb{K}\) has the Baire property. We know already that \(\varphi^{-1}[\mathbb{C} \setminus \Omega] \subseteq \mathbb{K}\) and \(\varphi^{-1}[\Omega] \subseteq \mathbb{K}\) are Borel sets by Lemma 22. Furthermore \(ls(\mathbb{C} \setminus \Omega)\) is a Borel set by Lemma 33. \((\mathbb{C} \setminus \Omega) \cap cl(\mathbb{D}) = \emptyset\), so if \(\lambda^n \in \mathbb{C} \setminus \Omega\), then \(|\lambda| > 1\) and therefore \(ls(\mathbb{C} \setminus \Omega) \cap cl(\mathbb{D}) = \emptyset\). Notice that \(cl(\mathbb{D}) = ls(\mathbb{C} \setminus \Omega) \cup \{\lambda \in li(\Omega) | |\lambda| > 1\}\). We will be done if we prove that \(\varphi^{-1}[\{\lambda \in li(\Omega) | |\lambda| > 1\}]\) has the Baire property.

Define

\[A = \{(x, a, y, b) \in (R \times \varphi^{-1}[\Omega] \times R \times \mathbb{K}) | x = (\varphi^{-1}(z) - a)y + b\}\]

and consider the continuous map \(\pi : R \times \varphi^{-1}[\Omega] \times R \times \mathbb{K} \rightarrow R \times \varphi^{-1}[\Omega]\) defined by \(\pi(x, a, y, b) = (x, a)\). \(A\) is relatively closed in \(R \times \varphi^{-1}[\Omega] \times R \times \mathbb{K}\) and therefore is a Borel subset of \(R \times \varphi^{-1}[\Omega] \times R \times \mathbb{K}\).

We will show that \(\pi \upharpoonright A\) is an injective mapping onto \(R \times \varphi^{-1}[\Omega]\).

Suppose

\[
\langle x, a, y_1, b_1 \rangle, \langle x, a, y_2, b_2 \rangle \in A.
\]

It follows that \((\varphi^{-1}(z) - a)y_1 + b_1 = (\varphi^{-1}(z) - a)y_2 + b_2\) necessitating

\[
(z - \varphi(a))\varphi(y_1) + \varphi(b_1) = (z - \varphi(a))\varphi(y_2) + \varphi(b_2).
\]

Since \(\varphi(b_1), \varphi(b_2) \in \mathbb{C}\), if we evaluate at \(\varphi(a)\), we see that \(\varphi(b_1) = \varphi(b_2)\). Consequently,

\[
(\varphi^{-1}(z) - a)\varphi(y_1) = (\varphi^{-1}(z) - a)\varphi(y_2) \implies \varphi(y_1) = \varphi(y_2).
\]
Hence, \( y_1 = y_2 \) and \( b_1 = b_2 \) which establishes that \( \pi \upharpoonright_A \) is an injection.

To see that \( \pi \upharpoonright_A \) is a surjection onto \( R \times \varphi^{-1}([\Omega]) \), let \( \langle x, a \rangle \in R \times \varphi^{-1}([\Omega]) \). Let \( f = \varphi(x) \) and \( \varphi(a) = \alpha \), then there exists \( g \in A(\Omega) \) so that \( f = (z - \alpha)g + f(\alpha) \). If \( y = \varphi^{-1}(g) \) and \( b = \varphi^{-1}(f(\alpha)) \), then \( x = (\varphi^{-1}(z) - a)y + b \), \( \langle x, a, y, b \rangle \in A \) and \( \pi(x, a, y, b) = \langle x, a \rangle \).

Now, as \( \pi \upharpoonright_A \) is a continuous bijection onto \( R \times \varphi^{-1}([\Omega]) \), let \( \pi^* : R \times \varphi^{-1}([\Omega]) \to R \times \varphi^{-1}([\Omega]) \times R \times \mathbb{K} \) be the Borel mapping \( \pi^* = (\pi \upharpoonright_A)^{-1} \).

Let \( p : R \times \varphi^{-1}([\Omega]) \times R \times \mathbb{K} \to \mathbb{K} \) be the projection onto the fourth coordinate, \( p(x, a, y, b) = b \).

For each \( n \geq 1 \), notice that \( \langle x, a \rangle \mapsto \langle x, a^n \rangle, R \times \mathbb{K} \to R \times \mathbb{K} \), is continuous. Lemma 33 implies that \( li(\varphi^{-1}([\Omega])) \) is a Borel set since \( \varphi^{-1}([\Omega]) \) is a Borel set. Note that \( a^n \in li(\varphi^{-1}([\Omega])) \) for all \( n \geq 1 \) if \( a \in li(\varphi^{-1}([\Omega])) \).

With all this in hand, we now define

\[
B = \{ \langle x, a \rangle \in R^* \times li(\varphi^{-1}([\Omega])) \mid (p \circ \pi^*)(x, a^n) = 0 \text{ for all } n \geq 1 \}
\]

where \( R^* = R \setminus \{0\} \). Notice that \( B \) is a Borel subset of \( R \times li(\varphi^{-1}([\Omega])) \) as \( R^* \times \varphi^{-1}([\Omega]) \) is a Borel subset of \( R \times \mathbb{K} \) and \( \langle x, a \rangle \mapsto (p \circ \pi^*)(x, a^n) \), \( R \times \varphi^{-1}([\Omega]) \to \mathbb{K} \), is a Borel mapping for each \( n \geq 1 \). Hence, letting \( P : R \times \varphi^{-1}([\Omega]) \to \varphi^{-1}([\Omega]) \) be the projection onto the second coordinate, we see that \( P[B] \) is an analytic set.

The next step is to see that

\[
P[B] = \varphi^{-1}[\{ \lambda \in li(\Omega) \mid |\lambda| > 1 \}].
\]

Toward this end, we show that \( (p \circ \pi^*)(x, a) = \varphi^{-1}(\varphi(x)(\varphi(a))) \). Let \( \langle x, a, y, b \rangle = \pi^*(x, a) \) and notice that

\[
\varphi(x) = (z - \varphi(a))\varphi(y) + \varphi(b) \implies \varphi(x)(\varphi(a)) = \varphi(b) \implies b = \varphi^{-1}(\varphi(x)(\varphi(a))).
\]

This establishes that \( p \circ \pi^*(x, a) = \varphi^{-1}(\varphi(x)(\varphi(a))) \).

Notice that \( \varphi^{-1}[li(\Omega)] = li(\varphi^{-1}([\Omega])) \). Now suppose \( a \in P[B] \) and let \( x \in R^* \) be so that \( \langle x, a \rangle \in B \). By our definition of \( B \), \( a \in li(\varphi^{-1}([\Omega])) \) which, by our initial observation, implies that \( \varphi(a) \in li(\Omega) \). By virtue of \( \langle x, a \rangle \in B \), we have that

\[
0 = (p \circ \pi^*)(x, a^n) = \varphi^{-1}(\varphi(x)(\varphi(a^n))) = \varphi^{-1}(\varphi(x)(\varphi(a))^n)
\]

\[
\implies \varphi(x)(\varphi(a)^n) = 0
\]

for each \( n \geq 1 \). Hence, since \( x \neq 0 \) necessitates \( \varphi(x) \neq 0 \), we apply Lemma 35 to conclude that \( |\varphi(a)| > 1 \). Therefore \( a \in P[B] \) implies that \( a \in \varphi^{-1}[\{ \lambda \in li(\Omega) \mid |\lambda| > 1 \}] \) and \( P[B] \subseteq \varphi^{-1}[\{ \lambda \in li(\Omega) \mid |\lambda| > 1 \}] \).

On the other hand suppose that \( a \in \varphi^{-1}[\{ \lambda \in li(\Omega) \mid |\lambda| > 1 \}] \) then \( \varphi(a) \in li(\Omega) \). Pick a non-zero \( f \in A(\Omega) \) so that \( f(\varphi(a)^n) = 0 \) for each \( n \geq 1 \) by Theorem 34. Hence, \( \langle \varphi^{-1}(f), a \rangle \in B \), so \( a \in P[B] \),
Lemma 39. Let $\gamma: \Omega_1 \to \Omega_2$ be a conformal mapping where $\Omega_1$ and $\Omega_2$ are open and $\Omega_1$ is connected. Then $\varphi: \mathcal{A}(\Omega_1) \to \mathcal{A}(\Omega_2)$ defined by $\varphi(f) = f \circ \gamma^{-1}$ is a topological isomorphism of rings. The same is true if $\gamma$ is anti-conformal.

Proof. $\varphi(1) = 1$, therefore $\varphi(m) = m$ for $m \in \mathbb{Z}$ and furthermore $\varphi(r) = r$ for all $r \in \mathbb{Q}$. $\varphi$ is continuous by Theorem 36 and therefore $\varphi(x) = x$ for all $x \in \mathbb{R}$. $\varphi(i)^2 = \varphi(i^2) = \varphi(-1) = -1$ so $\varphi(i) = \pm i$. Hence $\varphi[\mathbb{C}] = \mathbb{C}$ and $\varphi$ behaves either like the identity or complex conjugation on $\mathbb{C}$. \qed

Lemma 38. Let $\varphi: \mathcal{A}(\Omega_1) \to \mathcal{A}(\Omega_2)$ be a ring isomorphism where $\Omega_1$ and $\Omega_2$ are connected. Then $\varphi[\mathbb{C}] = \mathbb{C}$ and $\varphi$ is either the identity or complex conjugation on $\mathbb{C}$.

Proof. For $\alpha \in \Omega_1$ and $\beta \in \Omega_2$, let $M_{\alpha} = \{f \in \mathcal{A}(\Omega_1) \mid f(\alpha) = 0\}$ and $M_{\beta}^* = \{f \in \mathcal{A}(\Omega_2) \mid f(\beta) = 0\}$. Now, let $\alpha \in \Omega_1$ be arbitrary and choose $\gamma(\alpha) \in \Omega_2$, using Lemma 20, so that

$$M_{\gamma(\alpha)}^* = \varphi[M_{\alpha}].$$

$\gamma: \Omega_1 \to \Omega_2$ is a bijection since $\varphi$ is an isomorphism of rings.

Notice that $f - f(\alpha) \in M_{\alpha}$. Then $\varphi(f) - \varphi(f(\alpha)) \in M_{\gamma(\alpha)}$.

Since $\varphi(f(\alpha)) \in \mathbb{C}$ by Lemma 38, we see that $\varphi(f)(\gamma(\alpha)) = \varphi(f(\alpha)) = 0$. That is, $\varphi(f(\alpha)) = \varphi(f)(\gamma(\alpha))$. \qed

Theorem 40 (Bers [3]). Let $\varphi: \mathcal{A}(\Omega_1) \to \mathcal{A}(\Omega_2)$ be an abstract ring isomorphism where $\Omega_1$ is open and connected. Then there exists a conformal or anti-conformal mapping $\gamma: \Omega_1 \to \Omega_2$ so that

$$\varphi(f) = f \circ \gamma^{-1} \text{ or } \varphi(f) = \overline{f} \circ \gamma^{-1}.$$

Proof. $\varphi$ is a homeomorphism by Theorem 36. Let $\gamma: \Omega_1 \to \Omega_2$ be the bijection provided by Lemma 39. We proceed now by cases.
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Case I. Suppose $\varphi \restriction C = z$. Then, note that, for $\alpha \in \Omega_1$, 
$$\alpha = z(\alpha) = \varphi(z(\alpha)) = \varphi(z)(\gamma(\alpha)).$$
Hence, $\varphi(z) \circ \gamma = z \restriction_{\Omega_1}$. Since $\gamma$ is a bijection, we see that $\gamma^{-1} = \varphi(z)$ so $\gamma$ is a conformal mapping.

Now, for any $f \in \mathcal{A}(\Omega_1)$ and $\alpha \in \Omega_1$, notice that 
$$f(\alpha) = \varphi(f(\alpha)) = \varphi(f)(\gamma(\alpha)) \implies f = \varphi(f) \circ \gamma \implies \varphi(f) = f \circ \gamma^{-1}.$$

Case II. Suppose $\varphi \restriction C = \overline{z}$. Then, note that, for $\alpha \in \Omega_1$, 
$$\overline{\alpha} = z(\alpha) = \varphi(z(\alpha)) = \varphi(z)(\gamma(\alpha)).$$
Hence, $\varphi(z) \circ \gamma = \overline{z} \restriction_{\Omega_1}$. It follows that $\overline{\gamma^{-1}} = \varphi(z)$ so $\gamma$ is an anti-conformal mapping.

Now, for any $f \in \mathcal{A}(\Omega_1)$ and $\alpha \in \Omega_1$, notice that 
$$\overline{f(\alpha)} = \varphi(f(\alpha)) = \varphi(f)(\gamma(\alpha)) \implies \overline{z} \circ f = \varphi(f) \circ \gamma \implies \overline{z} \circ f \circ \gamma^{-1} = \varphi(f) \implies \varphi(f) = \overline{f \circ \gamma^{-1}}.$$

\[\square\]

7. The Bounded Analytic Functions on the Disk

Recall that Liouville’s Theorem informs us that the only bounded entire functions are the constants. So in this trivial case the bounded entire functions form a Polish ring. But what about the bounded analytic functions on other domains? The results of this section should be compared with the classic results of Kakutani ([6] and [7]).

Let $B(\mathbb{D})$ be the abstract ring of bounded analytic functions on $\mathbb{D}$. Let $H^\infty$ be the ring $B(\mathbb{D})$ endowed with the topology of uniform convergence (the topology compatible with the sup norm metric) and identify each scalar $\lambda \in \mathbb{C}$ with the constant function taking value $\lambda$. In this section we assume that the abstract ring $B(\mathbb{D})$ is given a fixed Polish ring topology and we will show that this leads to a contradiction.

The following proposition must be well known. A proof is hinted at in https://math.stackexchange.com/questions/1689215/h-infty-is-not-separable.

**Proposition 41.** The space $H^\infty$ is complete metrizable but not separable.

For a hint of the proof, for each $\lambda \in S^1$, let 
$$f_\lambda = \exp \left( \frac{z + \lambda}{z - \lambda} \right).$$
Then show that \( \|f_{\lambda}\| \leq 1 \) and \( \|f_{\lambda_1} - f_{\lambda_2}\| \geq 1 \) for all \( \lambda_1, \lambda_2 \in S^1 \) with \( \lambda_1 \neq \lambda_2 \).

**Lemma 42.** For any \( f \in B(\mathbb{D}) \) and \( \alpha \in \mathbb{D} \), there exists \( g \in B(\mathbb{D}) \) so that \( f = (z - \alpha)g + f(\alpha) \). Consequently, \( M_{\alpha} = \{ f \in B(\mathbb{D}) \mid f(\alpha) = 0 \} \) is a principal maximal ideal.

**Proof.** We need only check the statement for non-constant functions as the statement obviously holds for constant functions (\( g \) is taken to be zero when \( f \) is constant). Define \( g : \mathbb{D} \to \mathbb{C} \) by the rule

\[
g(\zeta) = \begin{cases} 
\frac{f(\zeta) - f(\alpha)}{\zeta - \alpha}, & \zeta \neq \alpha; \\
\frac{f'(\alpha)}{\zeta - \alpha}, & \zeta = \alpha.
\end{cases}
\]

\( g \) is analytic in \( \mathbb{D} \) so we need only check it is bounded. Let \( U \) be a connected open set so that \( \alpha \in U \subseteq \text{cl}(U) \subseteq \mathbb{D} \) and, for each \( \zeta \in U \),

\[
\left| \frac{f(\zeta) - f(\alpha)}{\zeta - \alpha} \right| < |f'(\alpha)| + 1.
\]

Let \( D = \min\{ |\zeta - \alpha| \mid \zeta \in \mathbb{D} \setminus U \} \) and notice that \( D > 0 \) since \( U \) is an open set containing \( \alpha \). Now, for any \( \zeta \in \mathbb{C} \),

\[
|g(\zeta)| \leq \max \left\{ |f'(\alpha)| + 1, \frac{2\|f\|_\infty}{D} \right\}.
\]

and therefore \( g \) is bounded.

Lastly, as the ring homomorphism \( f \mapsto f(\alpha), B(\mathbb{D}) \to \mathbb{C} \), has kernel \( M_{\alpha} \), we see that \( M_{\alpha} \) is a principal maximal ideal in \( B(\mathbb{D}) \). \( \square \)

The next lemma finds inspiration from a comment in the proof of [15, Proposition 3] where H. L. Royden suggests an algebraic characterization of the constant functions in any ring consisting of meromorphic functions.

**Lemma 43.** Let \( \varphi : B(\mathbb{D}) \to H^\infty \) be the identity map. Then \( \varphi^{-1}[\mathbb{C}] \) is an analytic set.

**Proof.** Let \( \Lambda \) be a countable dense subset of \( \mathbb{C} \). First, we will see that \( f \) is constant if and only if, for all \( \lambda \in \Lambda \), there exists \( g \in B(\mathbb{D}) \) such that \( g^2 = f - \lambda \).

If \( f \) is a constant function, then so is \( f - \lambda \) and \( f - \lambda \) has a square root.

Now suppose \( f \) is non-constant and pick \( \alpha \in \mathbb{D} \) so that \( f'(\alpha) \neq 0 \). Pick \( r > 0 \) so that for all \( \zeta \in \text{cl}(B(\alpha, r)) \), \( f'(\zeta) \neq 0 \). Since \( f \) is analytic,
f[B(\alpha, r)] is open so pick \lambda \in f[B(\alpha, r)] \cap \Lambda and \zeta \in B(\alpha, r) so that 
f(\zeta) = \lambda. Suppose we have \gamma \in B(\mathbb{D}) so that \gamma^2 = f - \lambda. Notice that 
g^2(\zeta) = f(\zeta) - \lambda = 0 and g(\zeta) = 0. Next 2g\gamma = f' which gives 
2g(\zeta)g'(\zeta) = f'(\zeta) \neq 0,
contradicting the fact that g(\zeta) = 0. So f - \lambda has no square root.

Now, notice that 
\[ A_\lambda = \{ \langle f, g \rangle \in B(\mathbb{D})^2 \mid g^2 = f - \lambda \} \]
is closed so the projection \pi : B(\mathbb{D})^2 \to B(\mathbb{D}) onto the first coordinate guarantees that \pi[A_\lambda] is an analytic set. Finally, analytic sets are closed under countable intersections so 
\[ A = \bigcap \{ \pi[A_\lambda] \mid \lambda \in \Lambda \} \]
is an analytic set. We are done since \[ A = \varphi^{-1}[\mathbb{C}] \]. \hfill \Box

**Lemma 44.** Let \varphi : B(\mathbb{D}) \to H^\infty be the identity map and let \alpha \in \mathbb{D}. Then

(i) \( M_\alpha \) is closed in \( B(\mathbb{D}) \);
(ii) \( \varphi^{-1}[\mathbb{C}] \) is closed in \( B(\mathbb{D}) \);
(iii) \( f \mapsto f(\alpha), B(\mathbb{D}) \to \varphi^{-1}[\mathbb{C}], \) is continuous.

**Proof.** (i)-(ii) Lemma 42 implies that \( M_\alpha \) is a principal ideal in \( B(\mathbb{D}) \), so \( M_\alpha \) is an analytic set. Since Lemma 43 establishes that \( \varphi^{-1}[\mathbb{C}] \) is an analytic additive subgroup of \( B(\mathbb{D}) \), both \( M_\alpha \) and \( \varphi^{-1}[\mathbb{C}] \) are closed subsets of \( B(\mathbb{D}) \) by an application of Lemma 4.

(iii) The natural decomposition \( f \mapsto \langle f - f(\alpha), f(\alpha) \rangle, B(\mathbb{D}) \to M_\alpha \oplus \varphi^{-1}[\mathbb{C}] \), is a homeomorphism qua a continuous group isomorphism between additive Polish groups. Lastly, \( f \mapsto f(\alpha), B(\mathbb{D}) \to \varphi^{-1}[\mathbb{C}], \) is continuous. \hfill \Box

**Theorem 45.** The abstract ring of bounded analytic functions on the disk cannot be made into a Polish ring.

**Proof.** Let \( F_n = \{ f \in B(\mathbb{D}) \mid \| f \|_\infty \leq n \} \). Since the mapping \( B(\mathbb{D}) \to \mathbb{C}, f \mapsto f(\alpha) \), is continuous for each \( \alpha \in \mathbb{D} \), each \( F_n \) is closed in \( B(\mathbb{D}) \). Since \( B(\mathbb{D}) = \bigcup_{n \geq 1} F_n \), the Baire Category Theorem implies that some \( F_n \), say \( F_m \), contains a nonempty open subset \( U \subseteq F_m \subseteq B(\mathbb{D}) \). But then \( 0 \in V = U - U \subseteq F_m - F_m \subseteq F_{2m} \), where \( V \) is now an open neighborhood of \( 0 \in B(\mathbb{D}) \). Furthermore, if \( \mu > 0 \), then \( 0 \in \mu \cdot V \subseteq \mu \cdot F_{2m} = F_{2m} \), where \( \mu \cdot V \) is an open neighborhood of \( 0 \in B(\mathbb{D}) \) since multiplication by \( \mu \) is a homeomorphism in \( B(\mathbb{D}) \). From this it is elementary to check that every open subset of \( H^\infty \) is open in \( B(\mathbb{D}) \), i.e., the identity mapping \( B(\mathbb{D}) \to H^\infty \) is continuous. But
the continuous image of a separable space is separable, contradicting Proposition 41.

\[ \square \]

8. THE FIELD OF MEROMORPHIC FUNCTIONS

In what follows \( \mathcal{M}(\Omega) \) denotes the abstract field of meromorphic functions on \( \Omega \). The result of this section is presented for its own interest and as a complement to Theorem 36.

**Theorem 46.** The abstract field \( \mathcal{M}(\Omega) \) cannot be made into a Polish field.

**Proof.** Suppose that \( \mathcal{M}(\Omega) \) can be given a Polish field topology. Then \( G = \mathcal{M}(\Omega) \setminus \{0\} \) may be viewed as a multiplicative Polish group. For each \( \alpha \in \Omega \) let \( o(f, \alpha) \in \mathbb{Z} \) be the order (positive for zeros, negative for poles, and 0 otherwise) of \( f \) at \( \alpha \). Each of the mappings \( f \mapsto o(f, \alpha) \), \( G \to (\mathbb{Z}, +) \), is a homomorphism of groups which is continuous by a theorem of Dudley [5]. Since arbitrary subsets of the discrete space \( \mathbb{Z} \) are both open and closed, then \( \mathcal{A}(\Omega) \setminus \{0\} = \bigcap_{\alpha \in \Omega} \{f \in G \mid o(f, \alpha) \geq 0\} \) is closed in \( G \). Therefore \( \mathcal{A}(\Omega) \) is a closed subset and hence a Polish subring of \( \mathcal{M}(\Omega) \). Theorem 36 implies that \( \mathcal{A}(\Omega) \subseteq \mathcal{M}(\Omega) \) has its usual topology. Then \( z^2 + z/n \to z^2 \) in the usual topology, but \( 1 = o(z^2 + z/n, 0) \not\to o(z^2, 0) = 2 \), a contradiction. \[ \square \]

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