The Likelihood for LSS: Stochasticity of Bias Coefficients at All Orders

Giovanni Cabass, a Fabian Schmidt a

aMax-Planck-Institut für Astrophysik, Karl-Schwarzschild-Str. 1, 85741 Garching, Germany

E-mail: gcabass@mpa-garching.mpg.de, fabians@mpa-garching.mpg.de

Abstract. In the EFT of biased tracers the noise field $\varepsilon_g$ is not exactly uncorrelated with the nonlinear matter field $\delta$. Its correlation with $\delta$ is effectively captured by adding stochasticities to each bias coefficient. We show that if these stochastic fields are Gaussian (the impact of their non-Gaussianity being subleading on quasi-linear scales anyway) it is possible to resum exactly their effect on the conditional likelihood $P[\delta_g|\delta]$ to observe a galaxy field $\delta_g$ given an underlying $\delta$. This resummation allows to take them into account in EFT-based approaches to Bayesian forward modeling. We stress that the resulting corrections to a purely Gaussian conditional likelihood with white-noise covariance are the most relevant on scales where the EFT is under control: they are more important than any non-Gaussianity of the noise $\varepsilon_g$. 
1 Introduction

The effective field theory (EFT) of large-scale structure (LSS) allows for a rigorous, controlled incorporation of the effects of fully nonlinear structure formation on small scales in the framework of cosmological perturbation theory [1, 2]. This is especially important when attempting to infer cosmological information from observed biased tracers such as galaxies, quasars, galaxy clusters, the Lyman-α forest, and others (see [3] for a review; in the following, we will always refer to the tracers as “galaxies” for simplicity): since we currently have no way of simulating the formation of galaxies ab initio to nearly the required accuracy, approaches which rigorously abstract from this imperfect knowledge of the small-scale processes involved in the formation of observed galaxies are highly valuable. The prediction for the galaxy density field
\[ \delta_g(x, \tau) = \frac{n_g(x, \tau)}{\bar{n}_g(\tau)} - 1 \]
can be broken into two parts: a “deterministic” part \( \delta_{g, \text{det}} \), which captures the modulation of the galaxy density by long-wavelength perturbations; and a stochastic residual which fluctuates due to the stochastic small-scale initial conditions. When integrating out small-scale modes, this effectively leads to a noise in the galaxy density.

So far, the calculation of galaxy clustering observables in the EFT has largely been restricted to correlation functions, such as the power spectrum and bispectrum. Recently, Ref. [4] presented a derivation of the likelihood of the entire galaxy density field \( \delta_g(x, \tau) \) given the nonlinear, evolved matter density field, in the context of the EFT. This result offers several advantages over previous approaches restricted to correlation functions:

- It puts the deterministic bias expansion of the galaxy density and the stochasticity of galaxies on the same footing, clarifying the significance of the latter.
- It does not rely on a perturbative expansion of the matter density field. Rather, the likelihood is given in terms of the fully nonlinear density field, which can be predicted for example using N-body simulations, and thus isolates the truly uncertain aspects of the observed galaxy density.
- The likelihood is given by the functional Fourier transform of the generating functional. Since the latter generates correlation functions, the derivation of [4] provides a correspondence between different terms in the likelihood and correlation functions.
- The conditional likelihood of the galaxy density field given the evolved matter density field is precisely the key ingredient required in full Bayesian (“forward-modeling”) inference.
approaches [5,6,7,8,9,10], and can be employed there directly [11,12] (see [13,14,15,16] for related approaches).

The likelihood presented in [4] includes the deterministic bias relation $\delta_{g,\text{det}}$ at all orders in perturbations. At leading order, the noise follows a multivariate Gaussian distribution with scale-independent and spatially-uniform covariance. Ref. [4] identified the most important correction to this noise covariance as being the modulation of the noise amplitude by large-scale density perturbations, and treated this contribution perturbatively. That is, the “field-dependent noise covariance” (or simply “field-dependent covariance”, as we will call it here), was shown to be more relevant than the non-Gaussianity of the noise or its nonlocality (captured by higher-derivative terms in the noise covariance).

In this paper, we show that this correction can be included at all orders in perturbations (while we still stop at leading order in the derivative expansion), and it can also be generalized to take into account the modulation of the noise by other long-wavelength operators. Apart from extending the perturbative reach of the likelihood, this resummation also offers key advantages for its numerical implementation. Together with the deterministic bias relation mentioned above, we thus have resummed the two leading effects in the EFT likelihood of biased tracers.

The outline of the paper is as follows. Our main result is derived in Section 2. Section 3.1 shows how to connect to the perturbative treatment of [4], while in Sections 3.2 and 3.3 we look in more detail at how the numerical implementation would proceed. We conclude in Section 3.4 by discussing how one can (perturbatively) include higher-derivative corrections.

2 Main result

In this section we will derive our main result. First, we compute the conditional likelihood including the effect of the stochasticity in the linear bias $b_1$. We then discuss how to include the stochasticities in all bias coefficients.

Let us consider the bias expansion in real space. If we include only the noise in $b_1$, it reads

$$\delta_g(x) = \delta_{g,\text{det}}[\delta](x) + \varepsilon_g(x) + \varepsilon_{g,\delta}(x)\delta(x), \quad (2.1)$$

where the functional $\delta_{g,\text{det}}[\delta]$ contains all the operators constructed from the nonlinear matter field $\delta$. That is, it gives the deterministic bias expansion. Let us write it as

$$\delta_{g,\text{det}}[\delta] = \sum O b_O O[\delta]. \quad (2.2)$$

Then, in real space and up to second order in perturbations (and leading order in derivatives) we have

$$\delta_{g,\text{det}}[\delta](x) = b_1\delta(x) + \frac{b_2}{2}\delta^2(x) + b_{K^2}\delta^2[\delta], \quad (2.3)$$

where $K^2 = K_{ij}K^{ij}$ and the tidal field $K_{ij}[\delta]$ is equal to $(\partial_i\partial_j/\nabla^2 - \delta_{ij}/3)\delta$.

The noise fields $\varepsilon_g$ and $\varepsilon_{g,\delta}$ are uncorrelated with the matter field. If we assume they are Gaussian fields (we will discuss this assumption in more detail in Section 3), their probability distribution is fully characterized by their covariance $C_\varepsilon$ in real or Fourier space. In Fourier space and on large scales this covariance is diagonal and a constant (see Section 2.7 of [3]), i.e.

$$\langle \varepsilon_i(k)\varepsilon_j(k') \rangle = (C_\varepsilon)_{ij} (2\pi)^3 \delta_D^3(k + k'). \quad (2.4)$$
In real space this becomes

\[ \langle \varepsilon(x) \varepsilon(y) \rangle = (C_\varepsilon)_{ij} \delta^{(3)}_\mathcal{D}(x - y). \]  

(2.5)

Hence, we can write the joint PDF of \( \varepsilon_g, \varepsilon_{g,\delta} \) as

\[ P[\varepsilon_g, \varepsilon_{g,\delta}] = \prod_x \frac{1}{2\pi \sqrt{\det C_\varepsilon}} \exp \left( -\frac{1}{2} \int \! d^3x \, \varepsilon(x) \cdot C_\varepsilon^{-1} \cdot \varepsilon(x) \right), \]

(2.6)

where

\[ \varepsilon = (\varepsilon_g, \varepsilon_{g,\delta}), \]

(2.7a)

\[
C_\varepsilon = \begin{pmatrix}
P_{g}^{(0)} & P_{g,\delta}^{(0)} \\
P_{g,\delta}^{(0)} & P_{g,\delta}^{(0)}
\end{pmatrix}.
\]

(2.7b)

In Eq. (2.7b) we denoted with a superscript “\( \{0\} \)” the low-\( \kappa \) limit of the noise auto- and cross-spectra.

Using Eq. (2.1) to rewrite \( \varepsilon_g \) in terms of \( \delta_g, \delta_g, \text{det} [\delta] \) and \( \varepsilon_{g,\delta} \) we can integrate out the field \( \varepsilon_{g,\delta} \) with a procedure analogous to that followed in [11]. In this way we obtain the conditional likelihood \( P[\delta_g|\delta] \). More precisely, since all our expressions are local in real space, the functional integral reduces to a product of ordinary one-dimensional integrals and we find

\[ P[\delta_g|\delta] = \prod_x \frac{1}{\sqrt{2\pi P_\varepsilon[\delta](x)}} \exp \left[ \frac{1}{2} \int \! d^3x \, \frac{\left( \delta_g(x) - \delta_g, \text{det} [\delta](x) \right)^2}{P_\varepsilon[\delta](x)} \right], \]

(2.8)

where we defined the “field-dependent covariance” as

\[ P_\varepsilon[\delta](x) = P_{g}^{(0)} + 2P_{g,\delta}^{(0)} \delta(x) + P_{g,\delta}^{(0)} \delta^2(x). \]

(2.9)

To confirm this result, let us derive it using a different approach. First, we know that the conditional likelihood \( P[\delta_g|\delta] \) is given by the joint likelihood \( P[\delta_g, \delta] \) divided by the likelihood \( P[\delta] \) for \( \delta \). Then, if we know the form of the generating functional \( Z[J_g, J] \), the joint likelihood \( P[\delta_g, \delta] \) can be obtained via its functional Fourier transform as described in [4]. Since the generating functional is obtained by integrating over the initial conditions \( \delta_{in} \), we see that \( P[\delta_g|\delta] \) is given by a functional integral of the following form (we keep track of the overall factor \( N_{\delta}^2 \) coming from two Dirac delta functionals for later convenience)

\[ P[\delta_g|\delta] = N_{\delta}^2 \int \! DX_g DX \, D\delta_{in} \, e^{\int_x \phi_g(x) J_g(x) - S_g[\phi_g]}, \]

(2.10)

where

\[ \phi_g = (X_g, \delta_{in}) \]

(2.11a)

\[ J_g = (i \delta_g, i \delta, 0) \]

(2.11b)

and the “action” \( S_g \) is the sum of a part quadratic in the fields and higher-order interactions, \( S_g, \text{int} \). If we assume to have only a Gaussian noise field \( \varepsilon_g \) with constant power spectrum, that is, we neglect \( \varepsilon_{g,\delta} \), the above equation reduces to

\[ P[\delta_g|\delta] = N_{\delta}^2 \int \! DX_g DX \, D\delta_{in} \, e^{\int_x X_g(x) \delta_g(x) - \int_x X(x) \delta(x)} \]

\[ \times P[\delta_{in}] \, e^{-\frac{1}{2} \int_x P_{g}^{(0)} X_g^2(x)} \]

\[ \times e^{-i \int_x X_g(x) \delta_{g,\text{int}}[\delta_{in}](x)} \, e^{-i \int_x X(x) \delta_{\text{int}}[\delta_{in}](x)}. \]

(2.12)
Using Eq. (2.14), the expression for the joint likelihood then becomes

\[
\mathcal{P}[\delta_g, \delta] = N^{2}\delta(\infty) \int \mathcal{D}X_g \mathcal{D}X \mathcal{D}\delta \int_{x} X_g(x) \left( \delta_g(x) - \delta_{g, \text{fwd}}(\delta_{\in\text{m}}(x)) \right) e^{\int_{x} X(x) \left( \delta(x) - \delta_{\text{fwd}}[\delta_{\text{m}}](x) \right)}
\]

\[
\times P[\delta_{\text{m}}] e^{-\frac{1}{2} \int_{x} P_{\text{fwd}}[\delta_{\text{m}}](x) X_g^2(x)},
\]

The definition of the functionals \( \delta_{\text{fwd}} \) and \( \delta_{g, \text{fwd}} \) is the following. The first is the nonlinear forward model for gravitational evolution of the initial matter field, while the second is the one for galaxies. We can decompose \( \delta_{g, \text{fwd}}[\delta_{\text{in}}] \) as

\[
\delta_{g, \text{fwd}}[\delta_{\text{in}}] = \delta_{g, \text{det}}[\delta_{\text{fwd}}[\delta_{\text{in}}]].
\]

How do we account for the stochasticity of the bias coefficients? They correspond to interactions of the form \( X_g X_g \cdots \delta_{\text{in}} \cdots \) \cite{4}: we recap this in Tab. 1. Most importantly, if we assume that such stochasticities are Gaussian the number of powers of \( X_g \) is equal to 2, i.e. we only have terms of the form \( X_g X_g \delta_{\text{in}} \cdots \). The scaling dimensions of the fields \( X_g \) and \( \delta_{\text{in}} \) can be derived from the quadratic part of the action in Eq. (2.12). We have \( [X_g] = 3/2 \), while for Gaussian initial conditions with a power-law power spectrum \( P_{\text{in}} \propto k^{n_g} \) we obtain \( [\delta_{\text{in}}] = (3 + n_g)/2 \). In our Universe \( n_g \) is close to \(-2\): this tells us that for a given number of external legs the non-Gaussianity of the noise is very suppressed with respect to the interactions we are considering here.

A further simplification arises if we stop at leading order in the derivative expansion. If we assume that only \( P_{\xi_{\delta g, \text{fwd}}}^{(0)} \) is non-vanishing it is possible to write down exactly the form of the interactions \( S_{g, \text{int}} \supset X_g X_g \delta_{\text{in}} \cdots \). Indeed, in \cite{4} (see its Appendix D) we have shown that they are obtained by shifting the bias coefficients \( \delta_{\text{O}} \) of Eq. (2.2) as

\[
\delta_{g, \text{fwd}}[\delta_{\text{in}}] = \delta_{g, \text{det}}[\delta_{\text{fwd}}[\delta_{\text{in}}]].
\]

in real space, where the cross-stochasticity \( P_{\xi_{\delta g, \text{fwd}}}^{(0)} \) is a constant of dimensions of length cubed (as in Eq. (2.7b), for example). Using Eq. (2.14), the expression for the joint likelihood then becomes

\[
\mathcal{P}[\delta_g, \delta] = N^{2}\delta(\infty) \int \mathcal{D}X_g \mathcal{D}X \mathcal{D}\delta \int_{x} X_g(x) \left( \delta_g(x) - \delta_{g, \text{fwd}}[\delta_{\text{in}}](x) \right) e^{\int_{x} X(x) \left( \delta(x) - \delta_{\text{fwd}}[\delta_{\text{in}}](x) \right)}
\]

\[
\times P[\delta_{\text{in}}] e^{-\frac{1}{2} \int_{x} P_{\text{fwd}}[\delta_{\text{m}}](x) X_g^2(x)},
\]
where the field-dependent covariance $P_\varepsilon[\delta_{\text{fwd}}[\delta_{\text{in}}]](x)$ is defined similarly to Eq. (2.9), i.e.

$$P_\varepsilon[\delta_{\text{fwd}}[\delta_{\text{in}}]](x) = P_{\varepsilon g}^{(0)} + 2 \sum_{O} P_{\varepsilon g, O}^{(0)} O[\delta_{\text{fwd}}[\delta_{\text{in}}]](x) .$$

Thanks to locality (the field $X_g$ is always evaluated at the same position $x$ at the order in derivatives we are working at) it is now straightforward to carry out the functional integral in $X_g$, since it is a Gaussian integral. We obtain

$$\mathcal{P}[\delta_g, \delta] = \mathcal{N}_{\delta(\infty)} \int DX D\delta_{\text{in}} \prod_x \frac{1}{\sqrt{2\pi P_\varepsilon[\delta_{\text{fwd}}[\delta_{\text{in}}]](x)}}$$

$$\times \mathcal{P}[\delta_{\text{in}}] e^{\int_x X(x) \left( \delta(x) - \delta_{\text{fwd}}[\delta_{\text{in}}](x) \right)}$$

$$\times \exp \left[ -\frac{1}{2} \int d^3x \frac{\left( \delta_g(x) - \delta_{g, \text{fwd}}[\delta_{\text{in}}](x) \right)^2}{P_\varepsilon[\delta_{\text{fwd}}[\delta_{\text{in}}]](x)} \right] .$$

Finally, we can carry out the integrals over $X$ and $\delta_{\text{in}}$. The integral over $X$ gives a Dirac delta functional

$$\mathcal{N}_{\delta(\infty)}^{-1} \delta(\delta - \delta_{\text{fwd}}[\delta_{\text{in}}]) ,$$

and integrating over $\delta_{\text{in}}$ sets $\delta_{\text{in}} = \delta_{\text{fwd}}[\delta]$. Following the same steps for the matter likelihood, it is then possible to recognize in Eq. (2.17) the conditional likelihood $\mathcal{P}[\delta_g|\delta]$

$$\mathcal{P}[\delta_g|\delta] = \frac{\mathcal{P}[\delta_g, \delta]}{\mathcal{P}[\delta]} = \prod_x \frac{1}{\sqrt{2\pi P_\varepsilon[\delta](x)}} \exp \left[ -\frac{1}{2} \int d^3x \frac{\left( \delta_g(x) - \delta_{g, \text{fwd}}[\delta](x) \right)^2}{P_\varepsilon[\delta](x)} \right] .$$

Using Eq. (2.16) with all the stochasticities of $b_O$ set to zero except for $O = \delta$ we recognize the result of Eqs. (2.8), (2.9).

So far we have discussed the case of only $\varepsilon_{g, \delta}$ being different from zero, which led us to Eqs. (2.8), (2.9), and the case of all the noises $\varepsilon_{g, O}$ being non-vanishing but considering only the impact of $P_{\varepsilon g, O}^{(0)}$, which resulted in Eqs. (2.16), (2.19). Before proceeding let us then briefly discuss what happens if we turn on all $\varepsilon_{g, O}$ but do not put $P_{\varepsilon g, O}^{(0)}$ to zero. For example, let us consider the stochasticity in $O = \delta^2$. The calculation leading to Eqs. (2.8), (2.9) can be straightforwardly extended to accommodate the corresponding stochastic field $\varepsilon_{g, \delta^2}$. Thanks to locality we now have to solve a two-dimensional integral at each point $x$. The resulting conditional likelihood has the same form as before, only with a different field-dependent covariance. Indeed, $P_\varepsilon[\delta]$ is now given by

$$P_\varepsilon[\delta](x) = P_{\varepsilon g}^{(0)} + 2 \sum_{O} P_{\varepsilon g, O}^{(0)} O[\delta](x) + \sum_{O, O'} P_{\varepsilon g, O}^{(0)} O[\delta](x) O'[\delta](x) ,$$

with

$$O, O' \in \{ \delta, \delta^2 \} .$$

Combined with the result of Eq. (2.16) this equation strongly suggests that once we include the stochasticities of all the bias coefficients the field-dependent noise keeps the same form as in Eq. (2.20), but with $O, O'$ running over all the operators of the deterministic bias expansion.
3 Discussion and conclusions

3.1 Connection to perturbative treatment

Let us first study the structure of the result of Eqs. (2.8), (2.19). The noise auto- and cross-spectra all have dimensions of length cubed. Factoring out the power spectrum of $\varepsilon_g$ we have that

$$\frac{1}{P_{\varepsilon}[\delta](x)} = \frac{1}{P_{\varepsilon}^{[0]}} \sum_{n=0}^{+\infty} c_n \delta^n(x), \quad (3.1)$$

where $c_n$ are tracer-dependent dimensionless constants which are expected to be of order unity, and we have restricted the set of bias operators to powers of the matter density for simplicity.

Therefore the logarithm $\psi[\delta_g|\delta] = -2 \ln P[\delta_g|\delta]$ of the conditional likelihood contains only terms of the form (forgetting for a moment about the determinant of the inverse covariance)

$$\psi[\delta_g|\delta] = \sum_{n=0}^{+\infty} c_n \int d^3 x \delta^n(x) \left( \frac{\delta_g(x) - \delta_g,\text{det}[\delta](x)}{P_{\varepsilon}^{[0]}} \right)^2. \quad (3.2)$$

This had to be expected given the structure of the interaction terms in $S_{g,\text{int}}$ that describe the Gaussian noise of the bias coefficients, and matches with the tree-level calculation carried out in [4].

We can further connect with the perturbative calculation of [4] by studying the size of these corrections with respect to the Gaussian conditional likelihood with field-independent covariance. We see immediately that on quasi-linear scales, where the EFT of biased tracers is under control, the additional terms that we obtain in Eq. (3.2) are subleading since we include only modes below some cutoff $\ll k_{\text{NL}}$, and consequently the typical size of a fluctuation $\delta(x)$ is smaller than unity.

We can also discuss the relative importance of the terms in Eq. (3.2) with respect to corrections coming from the non-Gaussianities of the noise. Noise non-Gaussianities are captured by interactions with more than two powers of $X_g$ in $S_{g,\text{int}}$ (see Tab. 1). They correspond to terms of higher order in the difference $\delta_g - \delta_g,\text{det}[\delta]$ in $\psi[\delta_g|\delta]$. Therefore, in an expansion

$$\psi[\delta_g|\delta] = \sum_{m=2}^{+\infty} \sum_{n=0}^{+\infty} d_{m,n} \int d^3 x \delta^n(x) \left( \frac{\delta_g(x) - \delta_g,\text{det}[\delta](x)}{P_{\varepsilon}^{[0]}} \right)^m, \quad (3.3)$$

again involving dimensionless coefficients $d_{m,n}$ assumed to be of order unity, the terms coming from the stochasticity in the bias coefficients are always more relevant than non-Gaussianities at a fixed $m + n$ (i.e. at a fixed number of external legs in $S_{g,\text{int}}$). Indeed, they are always enhanced by powers of the ratio

$$\sqrt{\frac{P_L(k)}{P_{\varepsilon}^{[0]}}}, \quad (3.4)$$

where $P_L$ is the linear matter power spectrum. Note that if we compare contributions at different $m + n$ it is very much possible for terms coming from noise non-Gaussianities to be more important than the ones we are keeping non-perturbatively in the field-dependent covariance.

On the other hand, if we expand also the square of the difference between $\delta_g$ and $\delta_g,\text{det}[\delta]$ in Eq. (3.2) around a linear bias relation we see that including an operator $O[\delta]$ in $\delta_g,\text{det}[\delta]$ is always more important on large scales than including the stochasticity in its bias coefficient.
the former comes with an enhancement by one power of the ratio in Eq. (3.4) with respect to the latter.

Finally, let us discuss how to treat the determinant of the field-dependent noise in Eqs. (2.8), (2.19). More precisely, we want to make the connection with one of the results of [4]. There we have shown that once the stochasticity of bias coefficients is included, loops of the field $X_g$ generate counterterms in the action that carry only powers of the initial field $\delta_{in}$. These new interactions give rise to terms in the log-likelihood $\mathcal{L}[\delta]$ that do not depend on the “data” $\delta_g$, but only on the matter field $\delta$: i.e. to terms with $m = 0$ in Eq. (3.3) (which we haven’t included there). We can straightforwardly see that they correspond exactly to the determinant in Eqs. (2.8), (2.19) by using the relation

$$\prod_x \frac{1}{\sqrt{2\pi P_\xi[\delta](x)}} = e^{-\frac{1}{2} \int_x \ln 2\pi P_\xi[\delta](x)}.$$  

(3.5)

3.2 About the numerical implementation and the field-dependent covariance

The result of this paper allows for an incorporation of the stochasticities of bias coefficients in EFT-based approaches to Bayesian forward modeling [11,12]. Without the resummation of these corrections at all orders in the matter field $\delta$ this would not have been possible. Indeed, a perturbative calculation would only include them via the Edgeworth-like expansion\(^1\) of Eq. (3.2) that is not normalizable and hence not amenable to numerical sampling techniques. This is in contrast to the likelihood of Eqs.(2.8), (2.19), which is a properly normalized Gaussian with $\delta$-dependent covariance (that can be sampled straightforwardly) and reduces to a Dirac delta functional in the limit of vanishing noise amplitudes.

The Gaussian form of the conditional likelihood, however, is only sensible if we are sure that the covariance is positive-definite. Is $P_\xi[\delta](x)$ a positive number? The perturbative analysis of the previous section ensures that the answer is yes if we restrict ourselves to scales where the EFT is under control, since the corrections proportional to $P^{(0)}_{\xi_g[\delta]}$ and $P^{(0)}_{\xi_g[\delta,O]}$ carry additional powers of $\delta$. Notice that the same perturbative arguments apply even in the simple case of only Gaussian, scale-dependent noise $\xi_g$ discussed in [4,11]. In that case, if we implement the scale dependence of $P_{\xi_g}(k)$ via its local expansion in powers of $k^2$, we must restrict to scales such that these higher-derivative corrections are subleading with respect to the constant part $P^{(0)}_{\xi_g}$.

It would nevertheless be nice to show the positivity of the covariance non-perturbatively. In order to do this let us consider the manifestly nonnegative combination

$$\left(\xi_g(x) + \sum_O \xi_{g,O}(x) O[\delta](x)\right)^2.$$  

(3.6)

If we average this combination\(^2\) over the noise fields $\xi_g$ and $\xi_{g,O}$, using the fact that they are uncorrelated with the matter field $\delta$, we obtain exactly Eq. (2.20) times an irrelevant factor proportional to a real-space $\delta^{(3)}_\delta(0)$. We then conclude that the field-dependent covariance that appears in our likelihood is positive-definite at all orders in perturbations.

Once we know that the covariance is positive-definite, Eq. (2.8) can be straightforwardly built into the framework described in [12]. The main difference with the implementation

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\(^1\)More precisely, the loop expansion employed in [4] leads to a functional Taylor series of the logarithm of the likelihood.

\(^2\)Technically, we multiply it by the joint likelihood $\mathcal{P}[\xi_g,\xi_g,\delta,\ldots]$ for the noise fields (given by Eq. (2.6) in the case of $O = \delta$ only) and functionally integrate over all noise fields.
presented there is that the likelihood is now evaluated by summing over the grid on which \( \delta_g \), \( \delta_{g,\text{det}}[\delta] \) are discretized in real space, after applying a sharp-\( k \) filter on the scale \( k_{\text{max}} \) to both fields.

3.3 Marginalizing over bias parameters

Ref. [12] showed how the deterministic bias parameters can be marginalized over analytically in case of the Fourier-space likelihood with no stochasticities of the \( b_O \). This becomes very useful once one wants to numerically sample the likelihood via Bayesian methods. Here we show the same for the real-space likelihood in Eq. (2.8). In the following, we assume that, in preparation, all fields \( y \in \{\delta_g, O\} \) are transformed to Fourier space, where we set

\[
y(k) = 0 \quad \forall \{k = 0, |k| \geq k_{\text{max}}\}.
\]

Then, the fields are transformed back to real space.

Let us rewrite Eq. (2.2) as

\[
\delta_{g,\text{det}}(x) = \mu(x) + \sum_{O \in O_{\text{marg}}} b_O O(x) \quad , \quad \mu(x) = \sum_{O \in O_{\text{all}} \setminus O_{\text{marg}}} b_O O(x) ,
\]

where \( O_{\text{marg}} \) denotes the subset of operators whose bias parameters we wish to marginalize over (we denote the cardinality of this set as \( n_{\text{marg}} \)), and we will replace the arguments \( [\delta](x) \) with \( x \) throughout this and the next section for notational clarity. We can then write the likelihood Eq. (2.8) as (we still keep the continuous integral \( \int d^3x \); in practical applications this turns into a sum over the grid on which \( \delta_g \), \( \delta_{g,\text{det}} \) are discretized in real space)

\[
P[\delta_g | \delta, \{b_O\}] = \exp \left( -\frac{1}{2} \int d^3x \ln 2\pi P_\varepsilon(x) \right) \times \exp \left[ -\frac{1}{2} \int d^3x \frac{(\delta_g(x) - \mu(x))^2}{P_\varepsilon(x)} \right.
\]

\[
+ \sum_{O \in O_{\text{marg}}} b_O \int d^3x \frac{(\delta_g(x) - \mu(x))O(x)}{P_\varepsilon(x)}
\]

\[
- \frac{1}{2} \sum_{O,O' \in O_{\text{marg}}} b_O b_{O'} \int d^3x \frac{O(x)O'(x)}{P_\varepsilon(x)} \right],
\]

where we have added the argument “\( \{b_O\} \)” to make more clear that the likelihood for the data \( \delta_g \) is conditioned also on the set of bias parameters to be marginalized over is Gaussian, so that it can be written as

\[
P_{\text{prior}}(b_O : O \in O_{\text{marg}}) = \frac{(2\pi)^{-n_{\text{marg}}}}{\sqrt{\det C_{\text{prior}}}} \exp \left[ -\frac{1}{2} \sum_{O,O' \in O_{\text{marg}}} (b_O - \bar{b}_O)(C_{\text{prior}}^{-1})_{OO'}(b_{O'} - \bar{b}_{O'}) \right],
\]

where \( \bar{b}_O \) denotes the central value of the prior on the parameter \( b_O \) and \((C_{\text{prior}})^{-1}_{OO'}\) denotes the (inverse) covariance.
Including the prior, Eq. (3.9) can be more compactly written as

\[
\mathcal{P}[\delta_g|\delta, \{b_O\}] = \frac{(2\pi)^{n_{\text{marg}}}}{\sqrt{\det C_{\text{prior}}}} \exp \left[ -\frac{1}{2} \sum_{O,O' \in \mathcal{O}_{\text{marg}}} \tilde{b}_O (C_{\text{prior}}^{-1})_{OO'} \tilde{b}_{O'} - \frac{1}{2} \int d^3x \ln 2\pi \mathcal{P}_\epsilon(x) \right] 
\times \exp \left[ -\frac{1}{2} C + \sum_{O \in \mathcal{O}_{\text{marg}}} b_O B_O - \frac{1}{2} \sum_{O,O' \in \mathcal{O}_{\text{marg}}} b_O b_{O'} A_{OO'} \right],
\]

(3.11)

where

\[
C = \int d^3x \frac{1}{\mathcal{P}_\epsilon(x)} (\delta_g(x) - \mu(x))^2,
\]

(3.12a)

\[
B_O = \int d^3x \frac{(\delta_g(x) - \mu(x))O(x)}{\mathcal{P}_\epsilon(x)} + \sum_{O' \in \mathcal{O}_{\text{marg}}} (C_{\text{prior}}^{-1})_{OO'} \tilde{b}_{O'},
\]

(3.12b)

\[
A_{OO'} = \int d^3x \frac{O(x)O'(x)}{\mathcal{P}_\epsilon(x)} + (C_{\text{prior}}^{-1})_{OO'}.
\]

(3.12c)

Note that \(A_{OO'}\) is a Hermitian and positive-definite matrix. The former is obvious from its definition. The latter follows from the fact that the field-dependent covariance is strictly positive, so that the integral \(\int d^3x 1/\mathcal{P}_\epsilon(x)\) defines a scalar product, and the fact that the operators \(O\) are linearly independent. Eq. (3.11) then allows us to perform the Gaussian integral over the \(b_O\). The result is

\[
\mathcal{P}[\delta_g|\delta, \{b_O\}_{\text{unmarg}}] = \left( \prod_{O \in \mathcal{O}_{\text{marg}}} \int db_O \right) \mathcal{P}[\delta_g|\delta, \{b_O\}]
\]

\[
= \frac{1}{\sqrt{\det C_{\text{prior}} \det A}} \exp \left[ -\frac{1}{2} \int d^3x \ln 2\pi \mathcal{P}_\epsilon(x) \right] 
\times \exp \left[ -\frac{1}{2} C + \frac{1}{2} \sum_{O,O' \in \mathcal{O}_{\text{marg}}} B_O (A^{-1})_{OO'} B_{O'} \right],
\]

(3.13)

where, as it is clear from Eqs. (3.8), (3.12a), (3.12b), \(C\) and \(B_O\) depend only on \(\delta\) and on the bias parameters that we have not marginalized over.

We have thus reduced the parameter space from \(\{b_O\}\) to \(\{b_O\}_{\text{unmarg}}\). This marginalization applies whatever the number \(n_{\text{marg}}\) of bias coefficients to be marginalized over. Notice that \(A_{OO'}\) depends on the parameters entering the variance \(\mathcal{P}_\epsilon(x)\), Eq. (2.9), and thus has to be recomputed when those change.

### 3.4 Including higher-derivative stochasticity

In this paper we have shown how a field-dependent stochasticity can be incorporated into the EFT likelihood at all orders in perturbations if we stop at the lowest (zeroth) order in derivatives.

In addition to this contribution (and the non-Gaussianity of the stochasticity discussed in Section 3.1), however, we also have higher-derivative stochastic terms. These correspond
to a series in $k^2$ in the Fourier-space covariance of Eq. (2.4). Refs. [11,12] argued that these contributions can be resummed when writing the likelihood in Fourier space.

In terms of scaling dimensions, in [4] we have shown that the field-dependent stochasticity is more relevant than the higher-derivative stochasticity; specifically, relative to the leading (constant) Gaussian stochasticity the former is suppressed by $(3 + n_\delta)/2 \sim 0.8$, while the latter is suppressed by 2. Since one cannot resum both of these contributions at the same time in closed form, it thus makes sense to resum the more relevant one, as done here.

Nevertheless it is possible to incorporate the higher-derivative stochasticity in the result of this work in a perturbative way. Let us now show how. First, we extend Eq. (2.4) to

$$\langle \varepsilon_i(k) \varepsilon_j(k') \rangle = \left( C_\varepsilon + C_\varepsilon^{(2)} k^2 \right)_{ij} (2\pi)^3 \delta_D^{(3)}(k + k') .$$

In real space this becomes

$$\langle \varepsilon_i(x) \varepsilon_j(y) \rangle = \left( C_\varepsilon - C_\varepsilon^{(2)} \nabla^2 \right)_{ij} \delta_D^{(3)}(x - y) .$$

Then, expanding to leading order in $C_\varepsilon^{(2)}/C_\varepsilon$, we can write the joint PDF of $\varepsilon_g, \varepsilon_{g,\delta}$ as (we neglect the expansion of the determinant in the following, since this section is meant to give only a qualitative discussion)

$$P[\varepsilon_g, \varepsilon_{g,\delta}] = \prod_x \frac{1}{2\pi \sqrt{\det C_\varepsilon}} \exp \left( -\frac{1}{2} \int d^3x \varepsilon(x) \cdot C_\varepsilon^{−1} \cdot \left( \mathbb{1} + C_\varepsilon^{−1} \cdot C_\varepsilon^{(2)} \nabla^2 \right) \cdot \varepsilon(x) \right) .$$

Let us now only keep the $i = j = 1$ entry of the matrix $(C_\varepsilon^{−1} \cdot C_\varepsilon^{(2)})_{ij}$, defining

$$R_\varepsilon^2 \propto \left( C_\varepsilon^{−1} \cdot C_\varepsilon^{(2)} \right)_{11} ,$$

where $R_\varepsilon^2$ can have either sign. We then obtain

$$P[\varepsilon_g, \varepsilon_{g,\delta}] = \prod_x \frac{1}{2\pi \sqrt{\det C_\varepsilon}} \exp \left( -\frac{1}{2} \int d^3x \varepsilon(x) \cdot C_\varepsilon^{−1} \cdot \varepsilon(x) \right) - \frac{1}{2} \frac{R_\varepsilon^2}{\tilde{P}_{\varepsilon g}^{(0)}} \int d^3x \varepsilon_g(x) \nabla^2 \varepsilon_g(x) \right) .$$

We can now integrate out $\varepsilon_g$ and $\varepsilon_{g,\delta}$ as before. Rewriting $\varepsilon_g$ in terms of the other fields via Eq. (2.1) gives, again at leading order in $R_\varepsilon^2$, a contribution of the form

$$P[\delta_g|\delta] = \prod_x \frac{1}{\sqrt{2\pi \tilde{P}_\delta(x)}} \exp \left[ -\frac{1}{2} \int d^3x \left( \frac{\delta_g(x) - \delta_{g,\text{det}}(x)}{\tilde{P}_\delta(x)} \right)^2 \right] - \frac{1}{2} \frac{R_\varepsilon^2}{\tilde{P}_{\varepsilon g}^{(0)}} \int d^3x \left( \delta_g(x) - \delta_{g,\text{det}}(x) \right) \nabla^2 \left( \delta_g(x) - \delta_{g,\text{det}}(x) \right) \right) .$$

The term in the second line can be evaluated straightforwardly in Fourier space, and it corresponds to the leading higher-derivative stochastic contribution when expanding the result of [11] at first order in $k^2$. There will be other contributions in addition to this one, e.g. of the form (dropping an overall dimensionless coefficient)

$$\varphi[\delta_g|\delta] \supset \frac{R_\varepsilon^2}{\tilde{P}_{\varepsilon g}^{(0)}} \int d^3x \, \delta''(\delta_g(x) - \delta_{g,\text{det}}(x)) \nabla^2 \left( \delta_g(x) - \delta_{g,\text{det}}(x) \right) .$$

These are less relevant on large scales than the one in Eq. (3.19).
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