IMPACT OF SURFACE DEFECTS ON A CONDENSATE OF ELECTRON PAIRS IN A QUANTUM WIRE

J. Kerner∗

We study the impact of surface defects on a condensate of electron pairs in a quantum wire. Based on previous results, we formulate a simple mathematical model accounting for such surface effects. For a system of noninteracting pairs, we prove the destruction of the condensate in the bulk. Finally, taking repulsive interactions between the pairs into account, we show that the condensate is recovered for pair densities greater than a critical density if the number of the surface defects is not too large.

Keywords: condensation, electron pair, surface defect

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1. Introduction

Our goal here is to create a mathematical model that allows understanding the impact of surface defects on a (Bose–Einstein) condensate of electron pairs in a simple quantum wire defined as the half-line \( \mathbb{R}_+ = (0, \infty) \). We note that this paper is an important extension of the models recently considered in [1], [2]. Indeed, it has long been clear in solid state physics that surface effects play an important role in various situations [3]. For example, it is known that they affect the superconducting behavior of metals [4]–[8]. Hence, starting from [2], where rigorous results regarding condensation of electron pairs in a simple quantum wire (modeled by the half-line) were obtained, we construct a simple mathematical model that allows incorporating surface defects that are small compared with the bulk. After constructing the model, we investigate the condensation of pairs in the bulk. Our first result is that no (eigen)state in the bulk remains macroscopically occupied after the surface defects are taken into account. Hence, from the physical standpoint, it becomes favorable for the pairs to accumulate in the surface defects. In the second step, we introduce repulsive interactions between the pairs and establish the existence of different regimes: one in which condensation in the bulk prevails and one in which it does not. Most importantly, if the number of the surface defects is not too large, then there exists a critical pair density such that pairs condense in the bulk for densities greater than the critical density.

Finally, we refer to [9], where certain aspects of superconductivity of networks were studied.

∗Department of Mathematics and Computer Science, FernUniversität in Hagen, Hagen, Germany, e-mail: Joachim.Kerner@fernuni-hagen.de.
2. Formulation of the model

We consider a quantum wire modeled by the half-line \( \mathbb{R}_+ = (0, \infty) \). On this quantum wire, as in [2], we place a system of two interacting electrons (with the same spin) whose Hamiltonian is formally given by

\[
H_p = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + v_b(|x - y|), \tag{2.1}
\]

where the binding potential \( v_b : \mathbb{R}_+ \to \mathbb{R}_+ \) is defined as

\[
v_b(x) := \begin{cases} 
0 & \text{if } 0 < x < d, \\
\infty & \text{otherwise}.
\end{cases} \tag{2.2}
\]

Because of the binding potential, the two electrons form a pair whose spatial extension is characterized by the parameter \( d > 0 \). We refer to [2] and also to the appendix where a mathematically rigorous realization of (2.1) is given by constructing a suitable quadratic form on

\[
L^2_a(\Omega) := \{ \varphi \in L^2(\Omega): \varphi(x, y) = -\varphi(y, x) \},
\]

where \( \Omega := \{(x, y) \in \mathbb{R}^2_+ : |x - y| < d\} \) is the two-particle configuration space.

Remark 2.1. We note that the fermionic symmetry is not pivotal for our results here. Instead, we could also consider pairs of bosonic particles (see [1] for such a scenario).

To incorporate (localized) surface effects, we now extend our Hilbert space. More explicitly, we work on the direct sum

\[
\mathcal{H} = L^2_a(\Omega) \oplus \ell^2(\mathbb{N}), \tag{2.3}
\]

which means that we couple the (continuous) quantum wire to a discrete graph that should model surface defects. From a physical standpoint, this seems reasonable in a regime where the surface defects are relatively small compared with the bulk. Furthermore, the Hamiltonian of a free pair (meaning without surface–bulk interactions) is given by

\[
H_0 = H_p \oplus \mathcal{L}(\gamma), \tag{2.4}
\]

where \( \mathcal{L}(\gamma) \) is the (weighted) graph Laplacian, i.e., for \( f \in \ell^2(\mathbb{N}) \),

\[
(\mathcal{L}(\gamma)f)(n) := \sum_m \gamma_{nm}(f(m) - f(n)), \tag{2.5}
\]

where \( \gamma = \gamma^T \) with the elements \( \gamma_{n,m} \in \mathbb{R}_+ \) is the associated edge weight matrix [10]. Because we in fact assume that our graph is a path graph (or chain graph), we set \( \gamma_{mn} = \delta_{|n-m|,1} \epsilon_n \), where \( (\epsilon_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+ \).

3. Condensate in the bulk without surface pair interactions

To study the effect of the surface defects on a condensate of electron pairs, we consider the condensation phenomenon similarly to how this was done in [2]. Of course, if we must describe the pair dynamics, then we should add a nondiagonal interaction term to (2.4) describing the coupling between the bulk and the surface. But because we are only interested in quantum statistical properties, we simplify the discussion here by modeling the interaction as a diagonal operator. The coupling between the surface and the bulk
is then realized through the “environment,” also called a “heat bath” in statistical mechanics [11]. More precisely, we consider the one-pair operator
\[ H_\alpha(\gamma) := H_p \oplus (\mathcal{L}(\gamma) - \alpha I), \quad (3.1) \]
where \( \alpha \geq 0 \) is a constant characterizing the “surface tension.” Because we want to investigate Bose–Einstein condensation of pairs, we must now use a suitable thermodynamic limit [12]. For this, we replace the half-line \( \mathbb{R}_+ \) with the interval \( (0, L), \ L > 0 \), and consider the restriction
\[ H_\alpha^L(\gamma) := H_p|_{\mathcal{L}^2(\Omega_L)} \oplus (\mathcal{L}(\gamma) - \alpha I)|_{\mathbb{C}^n(L)} \]
of (3.1) defined on
\[ \mathcal{H}_L = \mathcal{L}^2(\Omega_L) \oplus \mathbb{C}^n(L), \quad (3.2) \]
where \( \Omega_L := \{(x, y) \in \Omega: 0 < x, y < L\} \) and \( n(L) \in \mathbb{N} \) denotes the number of surface defects up to the length \( L \) of the wire.

Because \( H_\alpha^L(\gamma) \) is a direct sum of two operators, we have
\[ \sigma(H_\alpha^L(\gamma)) = \sigma(H_p|_{\mathcal{L}^2(\Omega_L)}) \cup \sigma((\mathcal{L}(\gamma) - \alpha I)|_{\mathbb{C}^n(L)}). \]

Consequently, \( H_\alpha^L(\gamma) \) has a purely discrete spectrum (see [2] and also the appendix for a discussion of \( H_p|_{\mathcal{L}^2(\Omega_L)} \)). In what follows, the eigenvalues of \( H_p^L := H_p|_{\mathcal{L}^2(\Omega_L)} \) are denoted by \( E_n(L) \), and the corresponding eigenfunctions are denoted by \( \varphi_n, \ n \in \mathbb{N}_0 \). Similarly, the eigenvalues of \( (\mathcal{L}(\gamma) - \alpha I)|_{\mathbb{C}^n(L)} \) are denoted by \( \lambda_j(L) \), and the corresponding eigenfunctions are denoted by \( f_j, \ j = 1, \ldots, n(L) \). In both cases, the eigenvalues are counted with multiplicity.

As the first result, we establish the following statement.

**Proposition 3.1.** For all sequences of edge weights \( (e_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+ \), we have
\[ \inf \sigma(H_\alpha^L(\gamma)) = -\alpha. \quad (3.4) \]

**Proof.** With Theorem A.1 in the appendix taken into account, the result follows directly because zero is the lowest eigenvalue of the discrete Laplacian associated with the constant eigenfunction \( (1, \ldots, 1)^T \in \mathbb{C}^n(L) \).

To investigate the condensation of pairs, we work in the grand canonical ensemble, as customary in statistical mechanics [11]–[13]. The associated Gibbs state is
\[ \omega_{\beta, \mu_L}^L(\cdot) := \frac{\text{Tr}_{\mathcal{F}_b}(e^{-\beta(\Gamma(H_\alpha^L(\gamma)) - \mu_L N)}[\cdot])}{Z(\beta, \mu_L)}, \quad (3.5) \]
where \( \beta = 1/T \in (0, \infty) \) is the inverse temperature, \( \mu_L \in (-\infty, \mu_{\text{max}}(L)) \) is the chemical potential (with \( \mu_{\text{max}}(L) \) specified below), and \( Z(\beta, \mu_L) \) is the partition function
\[ Z(\beta, \mu_L) = \text{Tr}_{\mathcal{F}_b}(e^{-\beta(\Gamma(H_\alpha^L(\gamma)) - \mu_L N)}). \]

Furthermore, \( \mathcal{F}_b \) is the bosonic Fock space over \( \mathcal{H}_L \),
\[ N = \sum_{j=1}^{n(L)} a_j^* a_j + \sum_{n=0}^{\infty} a_n^* a_n \quad (3.6) \]
is the number operator, and

$$\Gamma(H^L_\alpha(\gamma)) = \sum_{j=1}^{n(L)} (\lambda_j(L) - \alpha) a_j^* a_j + \sum_{n=0}^\infty E_n(L) a_n^* a_n$$  \hspace{1cm} (3.7)$$

is the second quantization of $H^L_\alpha(\gamma)$ (see [14], [15] for more details). We here note that $\{a_n^*, a_j\}$ and $\{a_n, a_j\}$ are the creation and annihilation operators corresponding to the respective states $\{0 \otimes f_j\}_{j=1}^{n(L)}$ and $\{\varphi_n \otimes 0\}_{n=0}^{\infty}$.

Most importantly, in the grand canonical ensemble, there is an explicit formula for the number of pairs occupying a given eigenstate [11]. For every state $\varphi_n \otimes 0$ with the associated number operator $n_{\varphi_n} := a_n^* a_n$, the number of pairs occupying this state is

$$\omega^L_{\beta, \mu_L} (n_{\varphi_n}) = \frac{1}{e^{\beta(E_n(L) - \mu_L)} - 1}. \hspace{1cm} (3.8)$$

An equivalent formula applies to any element of the form $0 \otimes f_j$ with $n_{f_j} := a_j^* a_j$.

For Hamiltonian (3.7), the thermodynamic limit is then realized as the limit $L \to \infty$ such that

$$\rho = \frac{1}{L} \left( \sum_{j=1}^{n(L)} \omega^L_{\beta, \mu_L} (n_{f_j}) + \sum_{n=0}^\infty \omega^L_{\beta, \mu_L} (n_{\varphi_n}) \right)$$  \hspace{1cm} (3.9)$$

holds for all values of $L$, where $\mu_L$ denotes the sequence of chemical potentials and $\rho > 0$ is the pair density. Furthermore, we say that a bulk state $\varphi_n \otimes 0$, $n \in \mathbb{N}_0$, is macroscopically occupied in the thermodynamic limit if

$$\limsup_{L \to \infty} \frac{1}{L} \frac{1}{e^{\beta(E_n(L) - \mu_L)} - 1} > 0. \hspace{1cm} (3.10)$$

It was shown in [2] (see Theorem 4) that if the state of the underlying Hilbert space is simply $L^2(\Omega_L)$, then the state $\varphi_0$ is macroscopically occupied in the thermodynamic limit (also see Theorem A.2 in the appendix). In contrast to this case, when working on $\mathcal{H}_L$, i.e., when including the surface defects, we obtain the following result.

**Theorem 3.1** (Destruction of the condensate in the bulk I). *Let an operator $H^L_\alpha(\gamma)$ and an arbitrary sequence of edge weights $(e_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+$ be given. Then for the associated Gibbs state and all pair densities $\rho > 0$, no bulk state $\varphi_n \otimes 0$, $n \in \mathbb{N}_0$, is macroscopically occupied in the thermodynamic limit. In fact, we have

$$\lim_{L \to \infty} \frac{1}{L} \frac{1}{e^{\beta(E_n(L) - \mu_L)} - 1} = 0, \hspace{1cm} n \in \mathbb{N}_0.$$\hspace{1cm} (3.10)*

**Proof.** Because $\mu_{\max}(L) = \inf \sigma(H^L_\alpha(\gamma))$ in the noninteracting case, Proposition 3.1 implies that $\mu_L \in (-\infty, -\alpha)$. Also, for sufficiently large $L$, Theorem A.1 and Proposition A.1 imply that $E_n(L) - \mu_L > c$ for all $n \in \mathbb{N}_0$ and some positive constant $c > 0$. Consequently, for all $n \in \mathbb{N}_0$ (we here note that $E_0(L)$ is the smallest eigenvalue),

$$\frac{1}{e^{\beta(E_n(L) - \mu_L)} - 1} < M$$

for some $M > 0$ and sufficiently large $L$. The result then readily follows from (3.10) because another factor $1/L$ appears.
4. Condensate in the bulk in the presence of surface pair interactions

In the preceding section, we saw that the condensate of electron pairs in the bulk is destroyed by the presence of surface defects (see Theorem 3.1). But because the volumes of defects in a real wire are relatively small compared with the volume of the bulk, for large pair surface densities, we should take (repulsive) interactions between the pairs into account. To account for those interactions, we use a mean-field approach.

More precisely, we replace the first term on the right-hand side of (3.7) (i.e., the free Hamiltonian model, and we have

\[
\sum_{j=1}^{n(L)} (\lambda_j(L) - \alpha + \lambda \rho_s(\mu, L))a_j^*a_j := h_\mu(\alpha, \lambda),
\]

where \( \rho_s(\mu, L) \geq 0 \) is the pair density in the surface defects, i.e., on \( C^n(L) \) (see Eq. (4.4) below). Furthermore, \( \lambda > 0 \) is the interaction strength associated with the repulsive interactions between the pairs.

**Remark 4.1.** We note that the interaction term in the standard mean-field approach is \( \lambda N^2/V \), where \( V \) is the associated volume [16], [12].

Because we can write \( h_\mu(\alpha, \lambda) \) as in (4.1), we conclude that the eigenvalues \( \lambda_j(L) \) are effectively only shifted by \( \lambda \rho_s(\mu, L) - \alpha \). Accordingly, the problem thus reduces to an effective noninteracting “particle” model, and we have

\[
\mu_k < \min\{\lambda \rho_s(\mu, L) - \alpha, E_0(L)\}
\]

for the sequence of chemical potentials \( \mu_k \) [17], taking into account that the lowest eigenvalue of the Laplacian is zero. In particular,

\[
\mu_{\max}(L) = \min\{\lambda \rho_s(\mu, L) - \alpha, E_0(L)\}.
\]

Furthermore, we choose \( \mu_k \) and the surface pair density \( \rho_s(\mu, L) \) such that

\[
\rho = \lim_{k \to \infty} \frac{1}{L} \left( \sum_{j=1}^{n(L)} e^{\beta(\lambda_j(L) - \alpha + \lambda \rho_s(\mu, L)) - \mu_k} - 1 \right) + \sum_{n=0}^{\infty} e^{\beta(E_n(L) - \mu_k)} - 1
\]

for a subsequence \( \mu_{k_k} \) together with

\[
\rho_s(\mu, L) = \frac{1}{n(L)} \sum_{j=1}^{n(L)} e^{\beta(\lambda_j(L) - \alpha + \lambda \rho_s(\mu, L)) - \mu_k} - 1.
\]

Hereafter in this section, we assume that \( L/n(L) \) is bounded from above and \( \mu_{k_k} \) converges to a (possibly negative infinite) limit value \( \mu \leq E_0 \) (from Eqs. (4.3) and (4.4), we indeed conclude that there are values and, in particular, arbitrarily large or small values of \( \rho > 0 \) for which such sequences exist). We note that for notational simplicity in what follows, we restrict ourself to subsequences without further notice.

**Theorem 4.1.** Let \( \mu_k \in (-\infty, E_0(L)) \) be a corresponding sequence of chemical potentials with a limit value \( \mu \leq E_0 \). Then

\[
\lim_{L \to \infty} \left( \frac{n(L)}{L} \rho_s(\mu, L) + \rho_0(\mu, L) \right) = \rho - \frac{\sqrt{\pi}}{\pi} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{1}{e^{3\beta x^2 - \mu} e^{\beta(x^2) - \mu}} \, dx,
\]

where

\[
\rho_0(\mu, L) := \frac{\omega_{\mu, L}(n_{\phi_0})}{L}.
\]
Proof. The theorem follows directly from Eq. (A.3) with Eq. (4.3) as the starting point.

We set
\[ \lim_{L \to \infty} \frac{L}{n(L)} =: \delta, \quad 0 \leq \delta < \infty. \]
For \( \rho_0 := \lim_{L \to \infty} \rho_0(\mu, L) \), we then have
\[ \lim_{L \to \infty} \rho_s(\mu, L) =: \rho(\mu, \delta) - \delta \rho_0 \]
if the limit value is \( \mu \leq E_0 \). From the physical standpoint, it is also interesting to write
\[ \rho(\mu, \delta) = \delta \rho - \rho_{\text{exc}}, \]
where \( \rho_{\text{exc}} = \rho_{\text{exc}}(\beta, \mu) \) equals the second term in the parentheses in Eq. (4.6). We note that \( \rho_{\text{exc}} \) is the density of pairs occupying all excited (eigen)states (i.e., \( \varphi_n \oplus 0 \) with \( n \geq 1 \)) in the bulk in the thermodynamic limit.

Lemma 4.1 (Destruction of the condensate in the bulk II). Let \( \lambda > 0, \mu \) be the limit point of \( \mu_L \), and
\[ \rho(\mu, \delta) < \frac{E_0 + \alpha}{\nu \lambda} \]
for some \( \nu > 1 \). Then
\[ \lim_{L \to \infty} \frac{1}{L} e^{\beta(\frac{E_0}{L} - \mu_L) - 1} = 0, \quad n \in \mathbb{N}_0. \]

Proof. Based on relation (4.2) and the assumptions in the lemma, we have \( \mu < E_0 - \varepsilon \) with some constant \( \varepsilon > 0 \) for the limit point of \( \mu_L \). Hence, using Proposition 3.1, we obtain
\[ \lim_{L \to \infty} \frac{1}{L} e^{\beta(\frac{E_0}{L} - \mu_L) - 1} = 0, \]
which yields the statement because the bulk ground state is the most occupied.

We immediately obtain the following corollary, which is particularly interesting from the physical standpoint.

Corollary 4.1. Let \( \rho = 0 \). Then for all \( \lambda > 0 \), Eq (4.9) holds for the states \( \varphi_n \oplus 0, n \in \mathbb{N}_0 \). More generally, if
\[ \delta \rho < \frac{E_0 + \alpha}{\nu \lambda} \]
for some \( \nu > 1 \) and \( \lambda > 0 \), then Eq. (4.9) holds for the states \( \varphi_n \oplus 0, n \in \mathbb{N}_0 \).

Remark 4.2. Corollary 4.1 implies that the condensate of electron pairs in the bulk (which by Theorem A.2 exists if no surface defects are present) is destroyed for any repulsive mean-field interaction strength \( \lambda > 0 \) if
\[ \lim_{L \to \infty} \frac{L}{n(L)} = 0, \]
i.e., if the number of defects per interval \((0, L)\) increases faster than \( L \) in the limit \( L \to \infty \). In addition, Lemma 4.1 implies that the condensate in the bulk is destroyed for arbitrarily large pair densities \( \rho > 0 \) if
the interaction strength \( \lambda > 0 \) is sufficiently small or the surface tension \( \alpha \geq 0 \) is sufficiently large.
Finally, we obtain the following result.

**Theorem 4.2** (Reconstruction of the condensate). If \( \delta, \lambda > 0 \), then there is a critical pair density \( \rho_{\text{crit}} = \rho_{\text{crit}}(\beta, \delta, \alpha, \lambda) > 0 \) such that for all pair densities \( \rho > \rho_{\text{crit}} \), we have

\[
\lim_{L \to \infty} \frac{1}{L} \frac{1}{e^{\beta E_0(L)} - 1} > 0.
\]

(4.12)

**Proof.** We suppose the contrary: such a critical pair density does not exist. Then there exists an arbitrarily large \( \rho > 0 \) for which \( \rho_0 = 0 \).

For given values \( \beta, \delta, \alpha, \lambda > 0 \), we first choose a large \( \rho \) such that

\[
\tilde{\rho}(E_0, \delta) > 2 E_0 + \alpha \lambda.
\]

(4.13)

Using the same reasoning as in the proof of Lemma 4.1 and taking Proposition 3.1 into account, we then conclude that

\[
\mu \leq E_0
\]

(4.14)

for the limit point of the associated sequence of chemical potentials.

We then use (4.13) and (4.14) in (4.4) to conclude that for such \( \rho \), there exists a constant \( C > 0 \) such that \( |\rho_\lambda(\mu_L, L)| < C \) for all \( L \geq L_0 \), where \( L_0 \) is sufficiently large. Finally, a further increase of \( \rho \) yields a contradiction with (4.6), and we consequently obtain the statement in the theorem.

**Remark 4.3.** We note that the results in this section hold for an arbitrary sequence \((e_n)_{n \in \mathbb{N}} \subset \mathbb{R}_+\) of edge weights.

**Appendix**

In this appendix, we collect some results from [2] that are used in this paper. The spectral properties of Hamiltonian (2.1) defined on the Hilbert space \( L^2_a(\Omega) \) or \( L^2_a(\Omega_L) \) were studied in [2]. The spectral information was then used to investigate the Bose–Einstein condensation of electron pairs in the spirit of Theorems 3.1 and 4.2.

From the mathematical standpoint, Hamiltonian (2.1) acts on (suitable) functions \( \varphi \in L^2_a(\Omega) \) or \( \varphi \in L^2_a(\Omega_L) \) as the standard two-dimensional Laplacian \( -\Delta \). More precisely, \( H_p \) on \( L^2_a(\Omega) \) (which is also denoted by \( H_p|_{L^2_a(\Omega)} \) here) is rigorously defined in terms of its associated quadratic form

\[
q[\varphi] = \int_{\Omega} |\nabla \varphi|^2 \, dx
\]

(A.1)

with the domain

\[
D_q := \{ \varphi \in H^1(\Omega) \cap L^2_a(\Omega) : \varphi|_{\partial \Omega_D} = 0 \},
\]

where \( \partial \Omega_D := \{(x, y) \in \overline{\Omega} : |x - y| = d\} \). Similarly, to realize \( H_p \) on \( L^2_a(\Omega_L) \) (here denoted by \( H_p|_{L^2_a(\Omega_L)} \)), we introduce

\[
q_L[\varphi] = \int_{\Omega_L} |\nabla \varphi|^2 \, dx
\]

(A.2)

with the domain

\[
D^L_q := \{ \varphi \in H^1(\Omega) \cap L^2_a(\Omega) : \varphi|_{\partial \Omega_L} = 0 \},
\]
where \( \partial \Omega^L_\perp := \{(x, y) \in \Omega^L : |x - y| = d \text{ or } x = d \text{ or } y = d\} \). Both forms are densely defined, positive, and closed, which based on the representation theorem for forms [15] implies that there exists a unique self-adjoint operator in each case. Most importantly, by the definition of the forms \( q[\cdot] \) or \( q_{\perp}[\cdot] \), this operator is just the respective two-dimensional Laplacian \(-\Delta\) with some domain \( D(-\Delta) \subset D_q \) or \( D_{\perp}(-\Delta) \subset D_{\perp}^L \).

Because \( H_p|_{L_2(\Omega)} = (-\Delta, D(-\Delta)) \) is defined on an unbounded Lipschitz domain \( \Omega \), the spectrum consists of an essential part possibly together with a discrete part. Let \( E \) consists of an essential part possibly together with a discrete part. Let \( E \) be an eigenvalue of multiplicity one. In addition, \( E \) satisfies the inequality

\[
\frac{\pi^2}{2d^2} \leq E \leq 0.93 \cdot \frac{2\pi^2}{d^2}.
\]

Also, the following proposition was shown in Lemma 1 in [2]. We here note that \( H_p|_{L_2(\Omega_L)} \) has only a purely discrete spectrum because \( \Omega_L \) is a bounded Lipschitz domain.

**Proposition A.1.** Let \( E_0(L) \) denote the ground state eigenvalue of \( H_p|_{L_2(\Omega_L)} \). Then we have

\[
\lim_{L \to \infty} E_0(L) = E_0.
\]

In Theorem 4.1, we use formula (18) in [2], which is an explicit expression for the density of pairs occupying all states except the ground state at a given inverse temperature \( \beta \in (0, \infty) \) and a given chemical potential \( \mu \in (-\infty, 2\pi^2/d^2) \):

\[
\lim_{L \to \infty} \frac{1}{L} \sum_{n=1}^{\infty} e^{\beta(E_n(L) - \mu)} - 1 = \frac{\sqrt{2}}{\pi} \sum_{n=1}^{\infty} \int_{0}^{\infty} \frac{1}{e^{\beta n^2/d^2} e^{\beta(x^2 - \mu)}} - 1 dx.
\]

(A.3)

We then use this formula to prove the Bose–Einstein condensation of pairs of electrons where each pair is described by \( H_p|_{L_2(\Omega_L)} \). In the first step, we introduce a sequence of chemical potentials \( \mu_L \) such that

\[
\frac{1}{L} \sum_{n=0}^{\infty} e^{\beta(E_n(L) - \mu_L)} - 1 = \rho
\]

for all values of \( L > 0 \), where \( \rho > 0 \) denotes the pair density. Moreover, with a slight abuse of notation, we let \( \varphi_0 \in L_2^2(\Omega_L) \) denote the ground state eigenfunction associated with \( E_0(L) \). We say that \( \varphi_0 \) is macroscopically occupied in the thermodynamic limit if

\[
\limsup_{L \to \infty} \frac{1}{L} e^{\beta(E_0(L) - \mu_L)} - 1 > 0.
\]

We can then formulate Theorem 4 in [2] as follows.

**Theorem A.2.** There exists a critical density \( \rho_{\text{crit}} = \rho_{\text{crit}}(\beta) > 0 \) such that \( \varphi_0 \) is macroscopically occupied in the thermodynamic limit for all pair densities \( \rho > \rho_{\text{crit}} \).
Conflicts of interest. The author declares no conflicts of interest.

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