Almost-commuting variety, $\mathcal{D}$-modules, and Cherednik Algebras

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Abstract

We study a scheme $\mathcal{M}$ closely related to the set of pairs of $n \times n$-matrices with rank 1 commutator. We show that $\mathcal{M}$ is a reduced complete intersection with $n + 1$ irreducible components, which we describe.

There is a distinguished Lagrangian subvariety $\mathcal{M}_{\text{nil}} \subset \mathcal{M}$. We introduce a category, $\mathcal{C}$, of $\mathcal{D}$-modules whose characteristic variety is contained in $\mathcal{M}_{\text{nil}}$. Simple objects of that category are analogous to Lusztig's character sheaves. We construct an exact functor of Quantum Hamiltonian reduction from our category $\mathcal{C}$ to the category $\mathcal{O}$ for type $A$ rational Cherednik algebra.

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1 Introduction

1.1 Let $V := \mathbb{C}^n$ and let $\mathfrak{g} := \text{End}(V) = gl_n(\mathbb{C})$ be the Lie algebra of $n \times n$-matrices. We will write elements of $V$ as column vectors, and elements of $V^*$ as row vectors. We consider the following affine closed subscheme in the vector space $\mathfrak{g} \times \mathfrak{g} \times V \times V^*$:

$$\mathcal{M} := \{(X,Y,i,j) \in \mathfrak{g} \times \mathfrak{g} \times V \times V^* \mid [X,Y] + ij = 0\}. \quad (1.1.1)$$

More precisely, let $\mathbb{C}[\mathfrak{g} \times \mathfrak{g} \times V \times V^*] = \mathbb{C}[X,Y,i,j]$ denote the polynomial algebra, and let $J \subset \mathbb{C}[X,Y,i,j]$ be the ideal generated by the $n^2$ entries of the matrix $[X,Y] + ij$. Then, by definition, we have $\mathcal{M} = \text{Spec} \mathbb{C}[X,Y,i,j]/J$, a not necessarily reduced affine scheme.

All the results of this paper are based on Theorem 1.1.2 below that describes the structure of the scheme $\mathcal{M}$. To formulate the Theorem, for each integer $k \in \{0, 1, \ldots, n\}$, let

$$\mathcal{M}'_k := \left\{(X,Y,i,j) \in \mathcal{M} \mid \begin{array}{c} Y \text{ has pairwise distinct eigenvalues,} \\ \dim(\mathbb{C}[X,Y]) = n - k, \\ \dim(j\mathbb{C}[X,Y]) = k \end{array} \right\}$$

and let $\mathcal{M}_k$ be the closure of $\mathcal{M}'_k$ in $\mathcal{M}$.

Theorem 1.1.2. (i) $\mathcal{M}$ is a complete intersection in $\mathfrak{g} \times \mathfrak{g} \times V \times V^*$.

(ii) The irreducible components of $\mathcal{M}$ are $\mathcal{M}_0, \ldots, \mathcal{M}_n$.

(iii) $\mathcal{M}$ is reduced and equidimensional; we have $\dim \mathcal{M} = n^2 + 2n$. 

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As we have learned after completion of the paper, a description of irreducible components of a scheme closely related to \( \mathcal{M} \) was found earlier by Neubauer \([\text{NC}]\).

We may identify \( g \) with its dual, \( g^* \), and write \( g \times g \times V \times V^* \cong T^*(g \times V) \). The cotangent bundle \( T^*(g \times V) \) comes equipped with the standard symplectic structure. A (possibly singular or non-reduced) subscheme \( Z \subset T^*(g \times V) \) is said to be Lagrangian if the generic locus of each irreducible component of \( Z_{\text{red}} \), the scheme \( Z \) taken with reduced structure, is a Lagrangian subvariety in \( T^*(g \times V) \).

We introduce the following closed subset of \( \mathcal{M} \), to be given a scheme structure later, in \([\text{2.4}]\)

\[
\mathcal{M}_{\text{nil}} := \{(X,Y,i,j) \in g \times g \times V \times V^* \mid [X,Y] + ij = 0 \& Y \text{ is nilpotent}\}.
\] (1.1.3)

**Theorem 1.1.4.** The scheme \( \mathcal{M}_{\text{nil}} \) is a Lagrangian (not necessarily reduced) complete intersection in \( T^*(g \times V) \).

**1.2 Applications to the commuting variety.** Write \( Z = \{(X,Y) \in g \times g \mid [X,Y] = 0\} \) for the commuting variety. Again, we regard \( Z \) as (not necessarily reduced) affine subscheme \( Z := \text{Spec} \mathbb{C}[X,Y]/I \), where \( \mathbb{C}[X,Y] = \mathbb{C}[g \times g] \) stands for the polynomial algebra, and \( I \subset \mathbb{C}[X,Y] \) stands for the ideal generated by the \( n^2 \) entries of the matrix \([X,Y]\). The scheme \( Z \) is known to be irreducible (cf. \([\text{E}]\)).

Further, write \( G := \text{GL}(V) \). The group \( G \) acts diagonally on \( g \times g \) via \( g \cdot (X,Y) = (gxg^{-1},gyg^{-1}) \). The induced \( G \)-action on \( \mathbb{C}[X,Y] \) by algebra automorphisms clearly preserves the ideal \( I \).

It is a well known and long standing open question whether or not the scheme \( Z \) is reduced, i.e., whether or not \( \sqrt{I} = I \). We cannot resolve this question. However, during discussions with Pavel Etingof we have realized that Theorem \( 1.1.2 \) combined with some elementary results from \([\text{EG}]\) implies the following

**Theorem 1.2.1.** One has: \( I^G = (\sqrt{I})^G \).

**Remark 1.2.2.** Write \( Z_{\text{red}} \) for the scheme \( Z \) taken with reduced scheme structure. It follows from Theorem \( 1.1.2 \) ii)-(iii) that the map \( (X,Y,i,j) \mapsto (X,Y,i) \) gives an isomorphism of algebraic varieties \( \mathcal{M}_0 \cong Z_{\text{red}} \times V \).

**1.3 Cherednik algebras and quantum Hamiltonian reduction.** Our interest in the geometry discussed in \([\text{E}]\) comes from the theory of rational Cherednik algebras, an important class of associative algebras introduced in \([\text{EG}]\). Below, we will only consider rational Cherednik algebras of type \( A \). Specifically, let \( h = \mathbb{C}^n \) be the tutological \( n \)-dimensional permutation representation of the Symmetric group \( S_n \). The corresponding Cherednik algebra is generated by a copy of the vector space \( h \), a copy of the dual space \( h^* \), and also by the elements \( w \in S_n \). These generators are subject to the defining relations \( 6.1.1 \), in \([\text{E}]\). The resulting algebra \( H_c \) is referred to as the rational Cherednik algebra of \( gl_n \)-type with parameter \( c \in \mathbb{C} \).

Write \( e := \frac{1}{n!} \sum_{\sigma \in S_n} \sigma \in \mathbb{C}[S_n] \subset H_c \) for the symmetrizer idempotent. The subalgebra \( eH_ce \subset H_c \) is called the spherical subalgebra of \( H_c \). It has been argued in \([\text{EG}]\) that the spherical subalgebra may be viewed as a quantization of so-called Calogero-Moser space. Specifically, given \( c \in \mathbb{C}^* \), let \( O_c := \text{Ad} G(\chi_c) \subset g \), be the semisimple conjugacy class of the matrix \( \chi_c := c(\text{Id} - np) \), where \( p = \text{diag}(0,0,\ldots,0,1) \) is the projector on the line \( \ell \) spanned by the last coordinate vector. Then, by an old result due to Kazhdan-Kostant-Sternberg \([\text{KKS}]\), the Calogero-Moser space (with
parameter $c$) may be interpreted as a classical Hamiltonian reduction of the symplectic vector space $\mathfrak{g} \times \mathfrak{g}^*$ over $O_c$, viewed as a coadjoint orbit in $\mathfrak{g}^* \cong \mathfrak{g}$.

Now, the Poisson algebra $\mathbb{C}[\mathfrak{g} \times \mathfrak{g}^*]$ has a natural quantization $\mathcal{D}(\mathfrak{g})$, the algebra of polynomial differential operators on $\mathfrak{g}$. Accordingly, one of the main results of [EG] says that, for generic values of $c$, the algebra $\mathfrak{e}_H e$ may be constructed as a quantum Hamiltonian reduction of $\mathcal{D}(\mathfrak{g})$ with respect to the primitive ideal in the enveloping algebra of $\mathfrak{g}$ that 'quantizes' the coadjoint orbit $O_c$. It has been conjectured that a similar result should hold, in effect, for all values of $c$. This conjecture is proved in the present paper in full generality, see Theorem 6.6.1.

The main novelty of our present approach, as compared to that of [EG], is in replacing the coadjoint orbit $O_c = \text{Ad} G(\chi_c)$ by its standard polarization, an affine Lagrangian subspace of the form $\chi_c + \mathfrak{p}^\perp$, where $\mathfrak{p} \subseteq \mathfrak{g}$ is the maximal parabolic formed by the matrices which preserve the line $\ell \subseteq V$. It is a well known heuristic general principle of representation theory that, usually, performing Hamiltonian reduction (either classical or quantum) with respect to the group $G$ and its coadjoint orbit $O_c$ should be equivalent to performing Hamiltonian reduction with respect to the subgroup $P \subseteq G$, corresponding to the Lie algebra $\mathfrak{p}$, and the one-point coadjoint $P$-orbit $\{\chi_c\} \subseteq \mathfrak{p}^*$. Applying this heuristic principle in our situation, we are thus led to consider quantum Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{g})$ with respect to the group $P$ and its character $\chi_c$. Observe that the group $P$ was defined as the isotropy group of $\ell$, a point in the projective space $\mathbb{P} = \mathbb{P}(V)$. Therefore, a routine argument shows that performing Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{g})$ with respect to $P$ is the same thing as performing Hamiltonian reduction of $\mathcal{D}(\mathfrak{g} \times \mathbb{P})$, a larger algebra, with respect to the group $G$ acting diagonally on $\mathfrak{g} \times \mathbb{P}$. It is this latter reduction that we are doing in the present paper.

Let $\mathcal{D}(\mathfrak{g} \times \mathbb{P}, c)$ be the algebra of $c$-twisted differential operators on $\mathfrak{g} \times \mathbb{P}$, and let $\mathfrak{g}_c$ denote the image of the Lie algebra $\mathfrak{sl}(V) \subseteq \mathfrak{g} = \mathfrak{gl}(V)$ in $\mathcal{D}(\mathfrak{g} \times \mathbb{P}, c)$, see §§5.1, 6.5 for more details and unexplained notation. Our main result stated below gives a construction of the spherical subalgebra $\mathfrak{e}_H e$ in terms of quantum Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{g} \times \mathbb{P}, c)$.

**Theorem 1.3.1.** For any $c \in \mathbb{C}$, let $H_c$ be the rational Cherednik algebra of $\mathfrak{gl}_n$-type with parameter $c$. Then, there is a filtered algebra isomorphism

$$\Phi_c : (\mathcal{D}(\mathfrak{g} \times \mathbb{P}, c)/\mathcal{D}(\mathfrak{g} \times \mathbb{P}, c) \cdot \mathfrak{g}_c)_{\text{ad} \mathfrak{g}_c} \xrightarrow{\sim} \mathfrak{e}_H e$$

such that

$$\Phi_c([\mathbb{C}[\mathfrak{g}]^{\text{Ad} G}] = \mathbb{C}[\mathfrak{h}]^{S_n} \subseteq \mathfrak{e}_H e, \quad \text{and} \quad \Phi_c(\mathfrak{Z}) = (\text{Sym} \mathfrak{h})^{S_n} \subseteq \mathfrak{e}_H e.$$ (1.3.2)

Moreover, the associated graded map gives a graded algebra isomorphism

$$\text{gr} \Phi_c : \text{gr}(\mathcal{D}(\mathfrak{g} \times \mathbb{P}, c)/\mathcal{D}(\mathfrak{g} \times \mathbb{P}, c) \cdot \mathfrak{g}_c)_{\text{ad} \mathfrak{g}_c} \xrightarrow{\sim} \text{gr}(\mathfrak{e}_H e).$$

**Theorem 1.3.1** is a strengthening of [EG] Section 7], esp. Corollary 7.4, (cf. also [BFG], Theorem 7.2.4(i)). In [EG], the homomorphism $\Phi_c$ has been called Harish-Chandra isomorphism for Cherednik algebras.

The Cherednik algebra of $\mathfrak{gl}_n$-type contains, as a subalgebra, the Cherednik algebra of $\mathfrak{sl}_n$-type. The latter is obtained by replacing, in the definition of $H_c$, the $n$-dimensional $S_n$-representation $\mathfrak{h} = \mathbb{C}^n$ by its $(n-1)$-dimensional irreducible subrepresentation. Our proof of Theorem 1.3.1 applies also to the rational Cherednik algebra of $\mathfrak{sl}_n$-type. In that case, the source of the homomorphism $\Phi_c$ should be the algebra $(\mathcal{D}(\mathfrak{sl}_n \times \mathbb{P}, c)/\mathcal{D}(\mathfrak{sl}_n \times \mathbb{P}, c) \cdot \mathfrak{g}_c)_{\text{ad} \mathfrak{g}_c}$, a Hamiltonian reduction of the algebra $\mathcal{D}(\mathfrak{sl}_n \times \mathbb{P}, c)$. 

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1.4 $\Gamma$-analogues. Theorem 1.1.2 has a $\Gamma$-equivariant generalization, where $\Gamma \subset SL_2(\mathbb{C})$ is a finite subgroup.

Fix an integer $n \geq 1$ and let $\mathbb{R} = R^\oplus n$ be the direct sum of $n$ copies of the left regular $\Gamma$-representation. Further, let $x, y$ be the standard basis in $\mathbb{C}^2$, the tautological 2-dimensional $\Gamma$-representation. Using this basis, any linear map $F : R \otimes_c \mathbb{C}^2 \rightarrow \mathbb{R}$ may be identified with a pair of linear maps $X := F(- \otimes x), Y := F(- \otimes y) : \mathbb{R} \rightarrow \mathbb{R}$. Also, given a vector $i \in \mathbb{R}$ and a covector $j \in \mathbb{R}^* := \text{Hom}_C(\mathbb{R}, \mathbb{C})$, we have $i \otimes j \in \mathbb{R} \otimes \mathbb{R}^* = \text{End}_c \mathbb{R}$. As a generalization of (1.1.3), one introduces the following affine scheme, see [EiG, formula (1.11)]:

\[ \mathcal{M}(\Gamma, n) := \{(X, Y, i, j) \in (\text{Hom}_c(\mathbb{R} \otimes_c L, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}^*)^\Gamma \mid [X, Y] + i \otimes j = 0\}. \]

The following theorem reduces, in the special case $\Gamma = \{1\}$, to parts (i) and (iii) of Theorem 1.1.2.

**Theorem 1.4.1.** The scheme $\mathcal{M}(\Gamma, n)$ is reduced and has $n + 1$ irreducible components. Moreover, it is a complete intersection in $(\text{Hom}_c(\mathbb{R} \otimes_c L, \mathbb{R}) \oplus \mathbb{R} \oplus \mathbb{R}^*)^\Gamma$.

We shall describe the irreducible components of $\mathcal{M}(\Gamma, n)$ in Section 3. The proof of Theorem 1.4.1 is similar to the proof of Theorem 1.1.2; the difference is that it makes use of results by Crawley-Boevey [CB] which we were able to avoid in the special case of $\Gamma = \{1\}$. Theorem 1.4.1 plays an important role in the construction of Harish-Chandra homomorphism for wreath-products, see [EiG, formula (1.11)], that reduces in the special case of $\Gamma = \{1\}$ to Theorem 1.3.1 above.

For any nontrivial finite subgroup $\Gamma \subset SL_2(\mathbb{C})$, G. Lusztig has considered in [Lu3, \S 12] a certain closed subset $\mathcal{M}_\text{nil}^L(\Gamma, n) \subset \mathcal{M}(\Gamma, n)$ and proved that this subset is a Lagrangian subscheme. In the special case of the trivial group $\Gamma = \{1\}$, the set $\mathcal{M}_\text{nil}^L(\Gamma, n)$ is still well defined. It turns out that this set is properly contained in, but is not equal to, our set $\mathcal{M}_\text{nil}$, see (1.1.3). Roughly speaking, the difference between the two sets is that, in our definition, only the operator $Y$ in the quadruple $(X, Y, i, j)$ is required to be nilpotent, while in Lusztig’s definition both $X$ and $Y$ are required to be nilpotent.

More generally, a Lagrangian scheme $\mathcal{M}_\text{nil}(\Gamma, n) \subset \mathcal{M}(\Gamma, n)$, that properly contains Lusztig’s Lagrangian scheme $\mathcal{M}_\text{nil}^L(\Gamma, n)$ and which is analogous to our scheme (1.1.3), may be defined in the case of any cyclic (in particular, trivial) group $\Gamma$, that is, in the case of quivers of type $\mathbb{A}$. On the other hand, there seems to be no analogue of such a scheme in the case of finite subgroups $\Gamma \subset SL_2(\mathbb{C})$ of non-cyclic type.

**Remark 1.4.2.** In the quiver language, the case of the trivial group $\Gamma = \{1\}$ corresponds to the quiver with a single vertex and a single edge-loop at that vertex. This case does not fall in the setting of [Lu3], since Lusztig excludes the case of quiver with edge-loops. ♦

1.5 Here are more details about the contents of the paper.

In \S 2, we prove Theorems 1.1.2, 1.1.3 and Theorem 1.2.1. In \S 3, we prove Theorem 1.4.1. In \S 4 we study a certain Lagrangian variety, $\Lambda$, closely related to the variety $\mathcal{M}_\text{nil}$. In \S 5, for each $c \in \mathbb{C}$, we introduce a category $\mathcal{C}_c$ of holonomic $\mathcal{D}(g \times \mathbb{P}, c)$ whose characteristic variety is contained in $\Lambda$. Associated with $\Lambda$, there is a natural stratification of the space $g \times \mathbb{P}(V)$, and $\mathcal{D}$-modules from category $\mathcal{C}_c$ are smooth along the strata of that stratification. Simple
objects of category $C_c$ are analogous to Lusztig’s character sheaves, see [Lu2]. More results about category $C_c$ will be given in [FG].

In Section 6, we remind the definition of rational Cherednik algebra of type $A_{n-1}$, and prove Theorem 1.3.1. In §7, we introduce and study a natural Hamiltonian reduction functor $H : C_c \rightarrow O(eH_c, e)$, an exact functor from category $C_c$ to the category $O$ for the spherical subalgebra $eH_c$, as defined e.g., in [BEG]. In the Appendix (§8), the geometry of the variety $\mathcal{M}$ is applied to deduce a result of M. Haiman on powers of the space of alternating (with respect to the $S_n$-diagonal action) polynomials in $2n$ variables.

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2 The geometry of $\mathcal{M}$

This section is devoted to the proofs of Theorem 1.1.2 and Theorem 1.2.1.

2.1 Linear algebra. Given $X \in \mathfrak{g}$, let $G^X := \{ g \in GL(V) \mid gX = Xg \}$ be the centralizer of $X$ in $G = GL(V)$.

Lemma 2.1.1. For any $X \in \mathfrak{g}$, the group $G^X$ acts on $V$ with finitely many orbits.

Proof. We may assume that $X$ is in Jordan normal form. Suppose first that there is only one Jordan block, and the corresponding eigenvalue of $X$ is $\lambda$. In this case, the non-zero orbits of $G^X$ are the sets

$$\{ v \in V \mid (X - \lambda)^k v \neq 0 \ \& \ (X - \lambda)^{k+1} v = 0 \} \quad \text{where} \ k = 0, 1, \ldots, n - 1.$$  

Let now $X$ have several Jordan blocks. We write a direct sum decomposition $V = \bigoplus_s V_s$ according to the block decomposition of $X$, and let $\overline{G^X} := G^X \cap (\prod_s GL(V_s))$ be the part of the group $G^X$ that respects the direct sum decomposition. We deduce from the above that the group $\overline{G^X}$, hence the larger group $G^X$, has finitely many orbits in $V = \bigoplus_s V_s$.

Let $\mathcal{N} \subset \mathfrak{g}$ be the nil-cone formed by nilpotent matrices, and let $G$ act diagonally on $\mathcal{N} \times V$.

Corollary 2.1.2. The set $\mathcal{N} \times V$ is a finite union of $G$-diagonal orbits.

Proof. The nil-cone $\mathcal{N}$ is a finite union of $G$-orbits, hence we have a finite partition $\mathcal{N} \times V = \bigsqcup O \times V$, where $O$ runs over the $G$-orbits in $\mathcal{N}$. Now, each set $O \times V$ is clearly stable under the $G$-diagonal action. Furthermore, $G$-diagonal orbits in $O \times V$ are in one-to-one correspondence with $G^X$-orbits in $V$, where $X$ is some fixed element of $O$. Thus, we are done by Lemma 2.1.1.
Fix a quadruple \((X,Y,i,j) \in \mathfrak{g} \times \mathfrak{g} \times V \times V^*\). Denote by \(\mathbb{C}[X,Y]i\) the subspace of \(\mathbb{C}^n\) consisting of vectors of the form \(Ai\), where \(A\) is any matrix which can be written as a noncommutative polynomial in \(X\) and \(Y\). The following lemma was due to Nakajima [Na, Lemma 2.9]; we give an alternative shorter proof.

**Lemma 2.1.3.** If \((X,Y,i,j) \in \mathcal{M}\), then \(j\) vanishes on \(\mathbb{C}[X,Y]i\).

**Proof.** Since the rank of the matrix \([X,Y]\) is at most 1, we can simultaneously conjugate \(X\), \(Y\) into upper triangular matrices, cf. [Gu] and also [EG, Lemma 12.7]. Hence, we assume without loss of generality that \(X\), \(Y\) are upper triangular matrices. In this case, for any \(A \in \mathbb{C}[X,Y]\), we have
\[
jAi = \text{Tr}(Ai) = -\text{Tr}(A[X,Y]) = 0
\] since \(A\) is upper triangular and \([X,Y]\) is strictly upper triangular.

### 2.2 The moment map.

We will identify \(\mathfrak{g}^*\) with \(\mathfrak{g}\) via the pairing \(\mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}^*, X \otimes Y \mapsto \text{Tr}(XY)\). Let \(G := \mathfrak{g} \times V\), and view it as a \(G\)-variety with respect to the \(G\)-diagonal action.

The induced \(G\)-action on \(T^*G \cong \mathfrak{g} \times \mathfrak{g} \times V \times V^*\) is given by the formula
\[
g \cdot (X,Y,i,j) = (gXg^{-1}, gYg^{-1}, gi, jg^{-1}) \quad \text{where } g \in G.
\]
This \(G\)-action on \(T^*\mathfrak{g}\) is hamiltonian and the corresponding moment map is
\[
\mu : T^*\mathfrak{g} = \mathfrak{g} \times \mathfrak{g} \times V \times V^* \to \mathfrak{g}^* \cong \mathfrak{g}, \quad (X,Y,i,j) \mapsto [X,Y] + ij. \tag{2.2.1}
\]

We see that the scheme \(\mathcal{M}\), see (1.1.1), may (and will) be identified with \(\mu^{-1}(0) \subset T^*\mathfrak{g}\), the scheme-theoretic zero fiber of the moment map (2.2.1).

Given a conjugacy class \(O\) in \(\mathfrak{g}\), we put \(\mathcal{M}(O) := \{(X,Y,i,j) \in \mathcal{M} | Y \in O\}\), viewed as a (not necessarily closed) reduced scheme.

**Proposition 2.2.2.** \(\mathcal{M}(O)\) is a Lagrangian subscheme in \(T^*\mathfrak{g}\), for any conjugacy class \(O \subset \mathfrak{g}\).

**Proof.** In general, assume first that \(\mathfrak{g}\) is an arbitrary smooth \(G\)-variety. Write \(\mathcal{S}\) for the set (possibly infinite) of all \(G\)-orbits in \(\mathfrak{g}\) and, given a \(G\)-orbit \(S \in \mathcal{S}\), let \(T^*_S \mathfrak{g} \subset T^*\mathfrak{g}\) denote the conormal bundle to \(S\). Then, the natural \(G\)-action on \(T^*\mathfrak{g}\) is Hamiltonian with moment map \(\mu : T^*\mathfrak{g} \to \mathfrak{g}^*\), and it is well known that
\[
\mu^{-1}(0) = \bigcup_{S \in \mathcal{S}} T^*_S \mathfrak{g}. \tag{2.2.3}
\]

Now, return to our case \(\mathfrak{g} = \mathfrak{g} \times V\) and write \(\text{pr} : \mathfrak{g} \times V^* \to \mathfrak{g}\) for the first projection. We see from (2.2.3) that proving the Proposition amounts to showing that, for any conjugacy class \(O\) in \(\mathfrak{g}\), the set \(\text{pr}^{-1}(O)\) is a finite union of \(G\)-orbits. It is clear that this last statement is equivalent to Lemma 2.1.1. □
2.3 A flat morphism. Write $\mathbb{C}^{(n)}$ for the set of unordered $n$-tuples of complex numbers. Let
\[ \pi : T^*\mathfrak{G} = g \times g \times V \times V^* \to \mathbb{C}^{(n)}, \quad (X,Y,i,j) \mapsto \text{Spec} Y \] (2.3.1)
be the map that sends $(X,Y,i,j)$ to the unordered $n$-tuple $\text{Spec} Y$ of eigenvalues of $Y$, counted with multiplicities.

**Proposition 2.3.2.** The morphism $\mu \times \pi : T^*\mathfrak{G} \to g \times \mathbb{C}^{(n)}$ is flat. In particular, all nonempty (scheme-theoretic) fibers of this morphism are equidimensional, of dimension $n^2 + n$.

**Proof.** For any unordered $n$-tuple $y = (y_1, \ldots, y_n) \in \mathbb{C}^{(n)}$, the set of all the matrices $Y \in g$ such that $\text{Spec} Y = y$ is clearly a finite union of conjugacy classes. Therefore, the zero fiber of the map $\mu \times \pi$ is equal, as a set, to a finite union of Lagrangian subschemes of the form $\mathcal{M}(O)$, as in Proposition 2.2.2. In particular, the dimension of the zero fiber is $\leq \frac{1}{2} \dim T^*\mathfrak{G} = n^2 + n = \dim T^*\mathfrak{G} - \dim(g \times \mathbb{C}^{(n)})$.

Next, we define a $\mathbb{C}^\times$-action on each of the varieties $T^*\mathfrak{G}$, $g$, and $\mathbb{C}^{(n)}$, as follows. Let $z \in \mathbb{C}^\times$ act on $g \times g \times V \times V^*$, resp., on $\mathbb{C}^n$, by scalar multiplication by $z$. This gives a $\mathbb{C}^\times$-action on $T^*\mathfrak{G}$, resp., on $\mathbb{C}^{(n)} = \mathbb{C}^n/S_n$. Further, we let $z \in \mathbb{C}^\times$ act on $g$ by scalar multiplication by $z^2$. This gives a $\mathbb{C}^\times$-action on $g$ such that the map $\mu \times \pi$ becomes a $\mathbb{C}^\times$-equivariant morphism. Thus, the standard argument based on the asymptotic cone construction, cf. eg. [CG] §2.3.9 or [Kr] ch.I, §6, shows that the dimension of any fiber of the map $\mu \times \pi$ cannot be greater than the dimension of the zero fiber.

Hence, since $g \times \mathbb{C}^{(n)}$ is smooth, we conclude that the morphism $\mu \times \pi$ is flat, cf. [Gr] Proposition 6.1.5].

Composing the flat morphism $\mu \times \pi$ with the first projection $g \times \mathbb{C}^{(n)} \to g$, we deduce the following special case of [Gr] Theorem 4.4].

**Corollary 2.3.3.** The morphism $\mu$ is flat.

Let $\pi = \pi|_{\mathcal{M}}$ be the restriction of $\pi$ to the closed subscheme $\mathcal{M} \subset T^*\mathfrak{G}$.

**Corollary 2.3.4.** The scheme $\mathcal{M}$ is a complete intersection in $T^*\mathfrak{G}$ and $\dim \mathcal{M} = n^2 + 2n$. Furthermore, $\pi : \mathcal{M} \to \mathbb{C}^{(n)}$, $(X,Y,i,j) \mapsto \text{Spec} Y$ is a flat morphism. All fibers of this morphism are $(n^2 + n)$-dimensional, Lagrangian subschemes in $T^*\mathfrak{G}$.

**Proof.** First of all, any quadruple of the form $(0,Y,0,0)$ belongs to $\mu^{-1}(0)$. It follows that all fibers of the restriction of the map $\pi$ to $\mathcal{M}$ are nonempty.

Now, flat base change with respect to the imbedding $(0) \times \mathbb{C}^{(n)} \to g \times \mathbb{C}^{(n)}$ implies that the scheme $\mathcal{M} = \mu^{-1}(0) = (\mu \times \pi)^{-1}((0) \times \mathbb{C}^{(n)})$ is a complete intersection in $T^*\mathfrak{G}$ and, moreover, that the morphism $\pi : \mathcal{M} \to \mathbb{C}^{(n)}$ is flat. In particular, the dimension of any irreducible component of any fiber of this morphism equals $\dim T^*\mathfrak{G} - \dim(g \times \mathbb{C}^{(n)}) = n^2 + n$.

Further, it is clear that each fiber of the map $\pi$ is equal, as a set, to a finite union of Lagrangian subschemes of the form $\mathcal{M}(O)$, see Proposition 2.2.2. Furthermore, we have proved that each irreducible component of the corresponding scheme-theoretic fiber has the same dimension as the dimension of $\mathcal{M}(O)$. Thus, any such irreducible component must be the closure of an irreducible component of the set of the form $\mathcal{M}(O)$, hence, it is a Lagrangian subscheme. 


2.4 The scheme $\mathcal{M}_{nil}$. First of all, recall that the nil-cone $\mathcal{N} \subset \mathfrak{g}$ is equal, as a set, to the zero fiber of the map $\mathfrak{g} \to \mathbb{C}^{(n)}$, $Y \mapsto \text{Spec } Y$, cf. (2.3.1). Thus, we make $\mathcal{N}$ a scheme by giving it the scheme structure of the scheme-theoretic zero fiber of the map $Y \mapsto \text{Spec } Y$. It is known that this scheme is an irreducible reduced scheme of dimension $n^2 - n$; moreover, it is a complete intersection in $\mathfrak{g}$.

We consider the projection $\text{pr}_Y : \mathcal{M} \to \mathfrak{g}, (X, Y, i, j) \mapsto Y$. It is clear that set-theoretically we have $\mathcal{M}_{nil} = (\text{pr}_Y)^{-1}(\mathcal{N})$, see (2.3.3). We define a scheme structure on $\mathcal{M}_{nil}$ to be the natural one on the scheme-theoretic inverse image of the scheme $\mathcal{N}$ under the morphism $\text{pr}_Y$. We do not know whether or not the scheme $\mathcal{M}_{nil}$ is reduced.

The above discussion shows that $\mathcal{M}_{nil}$ may be identified, as a scheme, with the scheme-theoretic zero fiber of the morphism $\pi : \mathcal{M} \to \mathbb{C}^{(n)}$. Thus, by Corollary 2.3.4, $\mathcal{M}_{nil}$ is a Lagrangian complete intersection. This proves Theorem 1.1.4.

2.5 Generic locus of $\mathcal{M}$. Our proof of Theorem 1.1.2 follows the strategy of [Wi] §1, in which Wilson considered the equation $[X, Y] + \mathbb{I}d = ij$ instead of our equation $[X, Y] + ij = 0$. Wilson’s situation was somewhat simpler since in his case the corresponding variety $\mathcal{M}$ was smooth and irreducible.

Lemma 2.5.1. Let $(X, Y, i, j) \in \mathcal{M}$. Suppose that the eigenvalues of $Y$ are pairwise distinct, $\dim \mathbb{C}[X, Y]i \leq n - k$, and $\dim j\mathbb{C}[X, Y] \leq k$. Then the $G$-orbit of $(X, Y, i, j)$ contains a representative such that:

(i) $Y$ is diagonal, say $Y = \text{diag}(y_1, \ldots, y_n)$;
(ii) $i = (0, \ldots, 0, 1, \ldots, 1)$ and $j = (1, \ldots, 1, 0, \ldots, 0)$ for some $k' \leq k \leq k''$;
(iii) $X = (X_{rs})_{1 \leq r, s \leq n}$ has the entries

$$X_{rs} = \begin{cases} x_r & \text{if } r = s, \\ \frac{1}{y_r - y_s} & \text{if } r > k'' \text{ and } s \leq k', \\ 0 & \text{else,} \end{cases}$$

for some $x_1, \ldots, x_n$.

Conversely, the data $(X, Y, i, j)$ defined by (i), (ii) and (iii) for any choices of $x_1, \ldots, x_n$, $y_1, \ldots, y_n$, $k'$, $k''$ (with $y_1, \ldots, y_n$ pairwise distinct, $k' \leq k \leq k''$) belongs to $\mathcal{M}$, moreover, we have $\dim(\mathbb{C}[X, Y]i) = n - k''$, $\dim(j\mathbb{C}[X, Y]) = k'$.

Proof. (cf. [Wi] Proposition 1.10.) We may assume that $Y = \text{diag}(y_1, \ldots, y_n)$. We may assume furthermore that $j = (1, \ldots, 1, 0, \ldots, 0)$ for some $k'$. By the Vandermonde determinant, we see that $j\mathbb{C}[Y]$ is the $k'$ dimensional subspace of $\mathbb{C}^n$ whose last $n - k'$ coordinates are 0. Hence, $k' \leq k$, and by Lemma 2.4.3 we may assume that $i = (0, \ldots, 0, 1, \ldots, 1)$ for some $k'' \geq k'$. By the Vandermonde determinant again, we see that $\mathbb{C}[Y]i$ is the $n - k''$ dimensional subspace of $\mathbb{C}^n$ whose first $k''$ coordinates are 0, hence $k'' \geq k$. Now, solving the equation $[X, Y] + ij = 0$ for $X$ gives (iii). Note that since $X$ is lower-triangular, we have $j\mathbb{C}[X, Y] = j\mathbb{C}[Y]$ is $k'$ dimensional,
and \( \mathbb{C}[X,Y]i = \mathbb{C}[Y]i \) is \( n - k'' \) dimensional. Thus, we have proved that the \( G \)-orbit has a representative satisfying (i), (ii) and (iii), and we have also proved the last statement of the lemma.

2.6 Irreducible components of \( \mathcal{M} \). For each \( k = 0, \ldots, n \), we introduce the following subset of \( \mathcal{M} \):

\[
\mathcal{M}_k'' := \left\{ (X,Y,i,j) \in \mathcal{M} \mid \begin{array}{l}
Y \text{ has pairwise distinct eigenvalues,} \\
\dim(\mathbb{C}[X,Y]i) \leq n - k, \quad \dim(j\mathbb{C}[X,Y]) \leq k, \\
\text{and } \dim(\mathbb{C}[X,Y]i) + \dim(j\mathbb{C}[X,Y]) < n
\end{array} \right\}.
\]

Lemma 2.6.1. (i) The dimension of \( \mathcal{M}_k' \) is equal to \( n^2 + 2n \), and \( \mathcal{M}_k' \) is connected. Moreover, both the actions of \( G \) and \( g \) on \( \mathcal{M}_k' \) are free.

(ii) The dimension of \( \mathcal{M}_k'' \) is strictly less than \( n^2 + 2n \).

Proof. Consider any \( (X,Y,i,j) \) of the form given in Lemma 2.5.1. Suppose first that \( (X,Y,i,j) \in \mathcal{M}_k' \), i.e. \( k' = k'' = k \). In this case, it is easy to see that the isotropy in \( G \) and \( g \) of the quadruple \( (X,Y,i,j) \), hence of the quadruple \( (X,Y,i,j) \), are trivial. Moreover, the choice of a representative of the form in Lemma 2.5.1, in the same orbit of \( (X,Y,i,j) \), is unique up to the action of \( S_k \times S_{n-k} \), where \( S_k \) permutes the first \( k \) coordinates of \( \mathbb{C}^n \) and \( S_{n-k} \) permutes the last \( n - k \) coordinates of \( \mathbb{C}^n \). Denote by \( \Delta \) the big diagonal in \( \mathbb{C}^n \). By Lemma 2.5.1 we have

\[
\mathcal{M}_k' \simeq G \times S_k \times S_{n-k} \left( \mathbb{C}^n \times (\mathbb{C}^n \setminus \Delta) \right),
\]

where the action of \( S_k \), resp. \( S_{n-k} \), on \( \mathbb{C}^n \times (\mathbb{C}^n \setminus \Delta) \) is by permutation of the pairs \( (x_1,y_1), \ldots, (x_k,y_k) \), resp. \( (x_{k+1},y_{k+1}), \ldots, (x_n,y_n) \). This proves (i).

Suppose now that \( (X,Y,i,j) \in \mathcal{M}_k'' \), so that \( k'' > k' \). In this case, the subgroup of \( G \) consisting of matrices of the form

\[
\text{diag}(1,\ldots,1,a,b,\ldots,c,1,\ldots,1) \quad \text{where } a,b,\ldots,c \in \mathbb{C}\setminus\{0\},
\]

acts trivially on \( (X,Y,i,j) \). This is a subgroup of strictly positive dimension. Hence, Lemma 2.5.1 implies (ii).

Proof of Theorem 1.1.2. Let \( \Delta \) be the big diagonal in \( \mathbb{C}^n \). By Lemma 2.1.3, we have a set-theoretic equality

\[
\mathcal{M} = \bigcup_{k=0}^n \mathcal{M}_k' \sqcup \bigcup_{k=0}^n \mathcal{M}_k'' \sqcup \pi^{-1}(\Delta).
\]

We claim that the decompositions above imply that \( \mathcal{M}_0, \ldots, \mathcal{M}_n \) are precisely all the irreducible components of \( \mathcal{M} \). To this end, observe that, by Lemma 2.6.1, \( \mathcal{M}_k' \) is connected and has dimension \( n^2 + 2n \), and \( \mathcal{M}_k'' \) has dimension strictly less than \( n^2 + 2n \). Further, since \( \dim \Delta < n \), Corollary 2.5.3 yields \( \dim \pi^{-1}(\Delta) < n + (n^2 + n) = n^2 + 2n \).

Now, by Corollary 2.3.1, the scheme \( \mathcal{M} \) is a complete intersection. Hence, each irreducible component of \( \mathcal{M} \) must have dimension \( n^2 + 2n \). Thus, parts (i) and (ii) of the Theorem both follow from 2.6.2.
Finally, since the $g$-action on $\mathcal{M}_k^G$ is free, the moment map $\mu$ is a submersion at generic points of $\mathcal{M}$. Hence $\mathcal{M}$ is generically reduced. But $\mathcal{M}$ is a complete intersection, hence it is Cohen-Macaulay. It follows that $\mathcal{M}$ is reduced, cf. \cite[Theorem 2.2.11]{CG} or \cite[Exercise 18.9]{E}. This proves (iii). 

\[\square\]

### 2.7 Around commuting variety.

Let $I_1 \subset \mathbb{C}[g \times g] = \mathbb{C}[X, Y]$ be the ideal generated by all the $2 \times 2$ minors of the matrix $[X, Y]$. Recall also the ideals $J \subset \mathbb{C}[X, Y, i, j]$ and $I \subset \mathbb{C}[X, Y]$ defined in \S 2.8. Further, let $p^*: \mathbb{C}[X, Y] \rightarrow \mathbb{C}[X, Y, i, j]$ be the pullback morphism induced by the projection map $p: (X, Y, i, j) \mapsto (X, Y)$. It is clear from definitions that one has

$$I \supset I_1 \quad \text{and} \quad p^*(I_1) \subset J.$$ 

Thus, there are natural algebra morphisms

$$\mathbb{C}[Z] := \mathbb{C}[X, Y]/I \leftarrow \mathbb{C}[X, Y]/I_1 \xrightarrow{p^*} \mathbb{C}[X, Y, i, j]/J.$$ 

By \cite[Theorem 12.1]{EG}, we know that $I^G = I_1^G$. Hence, taking $G$-invariants, we obtain the following diagram

$$\mathbb{C}[Z]^G = (\mathbb{C}[X, Y]/I)^G \xleftarrow{p^*} (\mathbb{C}[X, Y]/I_1)^G \xrightarrow{p^*} (\mathbb{C}[X, Y, i, j]/J)^G.$$ 

#### Proposition 2.7.2.

The morphism $p^*$ in (2.7.1) is an isomorphism.

**Proof.** To prove surjectivity, we note by Weyl’s fundamental theorem of invariant theory that $\mathbb{C}[X, Y, i, j]^G$ is generated as an algebra by polynomials of the form $\text{Tr}(P(X, Y, i j))$, where $P$ is any noncommutative polynomial. But this is equal to $p^*(\text{Tr}(P(X, Y, -[X, Y])))$ modulo $J$, and $\text{Tr}(P(X, Y, -[X, Y]))$ is contained in $\mathbb{C}[X, Y]^G$.

For injectivity, since we know that $I^G = I_1^G$, it suffices to show that any element $f \in \mathbb{C}[X, Y]^G \cap J$ is contained in $I$. From the definition of $J$, we can write $f = \sum f_a d_a$ where $f_a \in \mathbb{C}[X, Y, i, j]$ and $d_a = [X, Y]_{rs} + i_j r_s$ for some $r, s$. Let $\phi : g \times g \hookrightarrow g \times V \times V^*$, $(X, Y) \mapsto (X, Y, 0)$ be the natural imbedding, and $\phi^*: \mathbb{C}[X, Y, i, j] \rightarrow \mathbb{C}[X, Y]$ the restriction morphism, so $\phi^*(i) = \phi^*(j) = 0$. Since $f$ is contained in $\mathbb{C}[X, Y]$, we have $f = \phi^*(f) = \sum a \phi^*(a) \phi^*(d_a)$. But $\phi^*(d_a) = [X, Y]_{rs}$. Hence, $f$ is contained in $I$. \[\square\]

**Proof of Theorem 1.2.2.** By Theorem 1.2.2, $\mathcal{M}$ is reduced, that is, $J = \sqrt{J}$, and so $J^G = (\sqrt{J})^G$. But $(\sqrt{J})^G$ is the radical of $J^G$ in $\mathbb{C}[X, Y, i, j]^G$. By Proposition 2.7.2 it follows that $I^G$ is equal to its radical in $\mathbb{C}[X, Y]^G$, hence $I^G = (\sqrt{I})^G$. \[\square\]

### 2.8 We are going to state a corollary of Theorem 1.2.2 that will be used later in applications to Cherednik algebras.

To this end, let $\mathfrak{h} := \mathbb{C}^n$ be the permutation representation of the Symmetric group $S_n$, and let $S_n$ act diagonally on $\mathfrak{h} \times \mathfrak{h}$. The quotient $(\mathfrak{h} \times \mathfrak{h})/S_n$ has a natural structure of algebraic variety, with coordinate ring $\mathbb{C}[(\mathfrak{h} \times \mathfrak{h})/S_n] = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^{S_n}$.

We may identify $\mathfrak{h}$ with the Cartan subalgebra in $g$ formed by diagonal matrices, so we have a tautological imbedding $\mathfrak{h} \times \mathfrak{h} \hookrightarrow g \times g$. Write $i_o$ for the vector $i_o := (1, 1, \ldots, 1) \in V$. We define the following closed imbedding

$$\varepsilon : \mathfrak{h} \times \mathfrak{h} \hookrightarrow g \times g \times V \times V^*, \quad (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto (\text{diag}(x_1, \ldots, x_n), \text{diag}(y_1, \ldots, y_n), i_o, 0).$$

(2.8.1)
Note that $S_n$, viewed as the subgroup of permutation matrices in $G$, fixes $i_o$. Thus, the image of $ε$ is an $S_n$-stable subset in $g \times g \times V \times V^*$. Let $\mathcal{M}_{\text{red}}$ denote the scheme $\mathcal{M}$ with reduced scheme structure (of course, we already know by Theorem 1.1.2 (iii) that $\mathcal{M}$ is reduced, but we prefer not to use the Theorem at this point). It is clear that the above defined map $ε$ gives an $S_n$-equivariant closed imbedding $ε : h \times h \hookrightarrow \mathcal{M}_{\text{red}}$.

Further, it is a well known and easy consequence of Weyl’s fundamental theorem on $GL_n^*$-invariants that restriction of polynomial functions from $g \times g$ to $h \times h$ induces an algebra isomorphism $q : (C[X,Y]/\sqrt{T})^G \to C[h \times h]^S_n$. (Here and below we use the notation of (2.7.1)). Thus, using (2.7.1), we obtain the following chain of algebra isomorphisms

$$C[\mathcal{M}]^G = (C[X,Y,i,j]/I)^G \cong (C[X,Y]/I_1)^G \cong (C[X,Y]/I)^G$$

$$= C[X,Y]^G/I^G = C[X,Y]^G/(\sqrt{T})^G \cong C[h \times h]^{S_n}.$$  

Let $f : \mathcal{M}_{\text{red}} \to (h \times h)/S_n$ be the morphism of schemes induced by the composite algebra isomorphism in (2.8.2). Set theoretically, the morphism $f$ can be described in more geometric terms as follows.

Given an upper triangular matrix $X$, let $X_{\text{diag}} \in h$ denote the diagonal part of $X$. Recall further that, for any quadruple $(X,Y,i,j) \in \mathcal{M}_{\text{red}}$, the matrices $X,Y$ can be simultaneously put into upper triangular form, by [23, Lemma 12.7]. We assign to such a quadruple $(X,Y,i,j) \in \mathcal{M}_{\text{red}}$, where $X,Y$ are upper triangular matrices, the pair $(X_{\text{diag}},Y_{\text{diag}}) \in h \times h$, taken up to $S_n$-diagonal action on $h \times h$. The resulting map $\mathcal{M}_{\text{red}} \to (h \times h)/S_n$ is clearly constant on $G$-diagonal orbits in $\mathcal{M}_{\text{red}}$. Furthermore, it is easy to verify that this map of sets corresponds to the scheme morphism $f$ defined earlier using the chain of algebra isomorphisms in (2.8.2).

**Lemma 2.8.3.** Restriction of functions via $ε$, resp. pull-back of functions via $f$, induce mutually inverse graded algebra isomorphisms $ε^*$ and $f^*$ in the diagram below:

$$C[h \times h]^{S_n} = C[(h \times h)/S_n] \xrightarrow{~f^*~} C[\mathcal{M}_{\text{red}}]^G \xrightarrow{\sim} C[\mathcal{M}]^G.$$  

**Proof.** It is straightforward to verify that $ε^* \circ f^* = \text{Id}_{C[h \times h]^{S_n}}$. Further, the map $f^*$ is an isomorphism by construction, see (2.8.2). It follows that $ε^*$ is the inverse of $f^*$.

### 3 Generalization to quiver moment maps

#### 3.1 Quiver setting

Throughout this section, we let $Q$ be a quiver with vertex set $I$. The double $Q$ of $Q$ is the quiver obtained from $Q$ by adding a reverse edge $a^* : j \to i$ for each edge $a : i \to j$ in $Q$. If $a : i \to j$ is an edge in $Q$, we call $t(a) := i$ its tail, and $h(a) := j$ its head. The opposite quiver $Q^{\text{op}}$ is the quiver with the same underlying graph as $Q$ but with all the edges oriented in the opposite direction to the ones in $Q$.

On $C^I$, we have the standard inner product $\alpha \cdot \beta := \sum_{i \in I} \alpha_i \beta_i$, and we write $|\alpha|^2 := \sum_{i \in I} \alpha_i^2$. We will also use the Ringel form of $Q$, a (not necessarily symmetric) bilinear form on $Z^I$ defined by

$$\langle \alpha, \beta \rangle := \sum_{i \in I} \alpha_i \beta_i - \sum_{a \in Q} \alpha_{t(a)} \beta_{h(a)}, \quad \text{where } \alpha = (\alpha_i)_{i \in I}, \ \beta = (\beta_i)_{i \in I}.$$
Let \((\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle\) be its symmetrization. The corresponding quadratic form \(q(\alpha) := \langle \alpha, \alpha \rangle = \frac{1}{2}(\alpha, \alpha)\) is the Tits form; we set

\[ p(\alpha) := 1 - q(\alpha) = 1 + \sum_{a \in Q} \alpha_t(a) \alpha_h(a) - |\alpha|^2. \]

Let \(e_i \in \mathbb{Z}^I\) denote the coordinate vector corresponding to the vertex \(i \in I\).

The representations of \(Q\) of dimension vector \(\alpha \in \mathbb{N}^I\) are the elements of the vector space

\[ \text{Rep}(Q, \alpha) := \bigoplus_{a \in Q} \text{Mat}(\alpha_{h(a)} \times \alpha_t(a), \mathbb{C}). \]

Let \(G(\alpha) := \left( \prod_{i \in I} \text{GL}(\alpha_i, \mathbb{C}) \right)/\mathbb{C}^*. \) The group \(G(\alpha)\) acts on \(\text{Rep}(Q, \alpha)\) by conjugation and the orbits are the isomorphism classes of representations of \(Q\) of dimension vector \(\alpha\). Let

\[ \text{End}(\alpha)_0 := \left\{ (g_i)_{i \in I} \mid \sum_{i \in I} \text{Tr}(g_i) = 0 \right\} \subseteq \text{End}(\alpha) := \bigoplus_{i \in I} \text{Mat}(\alpha_i, \mathbb{C}). \]

We identify \(\text{Rep}(Q, \alpha)\) with the cotangent bundle \(T^*\text{Rep}(Q^{op}, \alpha)\), and \(\text{End}(\alpha)_0\) with the dual of the Lie algebra of \(G(\alpha)\). Then, we have a moment map

\[ \mu_\alpha : \text{Rep}(Q, \alpha) \longrightarrow \text{End}(\alpha)_0, \quad \mu_\alpha(x) := \sum_{a \in Q} [x_a, x_a^*]. \]

Let \(\lambda \in \mathbb{C}^I\). The fiber \(\mu_\alpha^{-1}(\lambda)\) is a (not necessarily reduced) scheme. In particular, the zero fiber, \(\mu_\alpha^{-1}(0)\), is the union (possibly infinite) of the conormal bundles to \(G(\alpha)\)-orbits in \(\text{Rep}(Q^{op}, \alpha)\).

### 3.2 Irreducible components of \(\mu_\alpha^{-1}(0)\).

Fix \(\lambda \in \mathbb{C}^I\), and let \(R^+_\lambda\) be the set of positive roots \(\alpha\) with \(\alpha \cdot \lambda = 0\). Let \(\Sigma'_\alpha\), resp., \(\Sigma_\lambda\), be the set of \(\alpha \in R^+_\lambda\) with the property that

\[ p(\alpha) \geq p(\beta^{(1)}) + \cdots + p(\beta^{(r)}), \quad \text{resp.}, \quad p(\alpha) > p(\beta^{(1)}) + \cdots + p(\beta^{(r)}), \quad \text{(3.2.1)} \]

for any decomposition \(\alpha = \beta^{(1)} + \cdots + \beta^{(r)}\) where \(\beta^{(t)} \in R^+_\lambda, \forall t = 1, \ldots, r\), and \(r \geq 2\).

Given \(\alpha \in \Sigma'_\lambda\), let \(\Sigma'_\lambda(\alpha)\) be the set of decompositions \(\alpha = \beta^{(1)} + \cdots + \beta^{(r)}\) such that the inequality in (3.2.1) is an equality, i.e., such that we have \(p(\alpha) = p(\beta^{(1)}) + \cdots + p(\beta^{(r)})\).

**Theorem 3.2.2.** Assume \(\alpha \in \Sigma'_\lambda\). Then

(i) The scheme \(\mu_\alpha^{-1}(\lambda)\) is equidimensional of dimension \(|\alpha|^2 - 1 + 2p(\alpha)\); furthermore, it is a complete intersection in \(\text{Rep}(Q, \alpha)\).

(ii) The irreducible components of \(\mu_\alpha^{-1}(\lambda)\) are in 1-1 correspondence with elements of the set \(\Sigma'_\lambda(\alpha)\).

Note that \(\Sigma_\lambda \subset \Sigma'_\lambda\). Also, if \(\alpha \in \Sigma_\lambda\), then \(\Sigma'_\lambda(\alpha) = \{ \alpha \}\). Crawley-Boevey proved in [CB, Theorem 1.2] that, for any \(\alpha \in \Sigma_\lambda\), the scheme \(\mu_\alpha^{-1}(\lambda)\) is reduced and irreducible.

**Conjecture 3.2.3.** For any \(\alpha \in \Sigma'_\lambda\), the scheme \(\mu_\alpha^{-1}(\lambda)\) is reduced.
Proof of theorem 3.2.2. Part (i) of the Theorem is [CB, Theorem 4.4].

To prove (ii) we need to introduce some notation. Given an algebraic group G acting on an algebraic variety $Z$, let $\dim_G Z$ denote the maximal number of parameters for $G$-orbits in $Z$, see [CB, §3].

For any integer $d \geq 1$ and dimension vector $\beta$, let $I^d(\beta)$ be the set of indecomposable representations $\rho \in \Rep(Q^{op}, \beta)$, such that $\dim G(\beta) \cdot \rho = d$. It is known, see [KR, pp.142-143], that each set $I^d(\beta)$ has the structure of a locally closed reduced subscheme in $\Rep(Q^{op}, \beta)$. Thus, $I(\beta) = \sqcup_{d \geq 0} I^d(\beta)$ is the constructible set of indecomposable representations of $Q^{op}$ of dimension $\beta$.

Given an $r$-tuple of dimension vectors $\beta^{(1)}, \ldots, \beta^{(r)} \in \mathbb{Z}^I$, we consider the action of the group $H := G(\beta^{(1)}) \times \ldots \times G(\beta^{(r)})$ on the set $I(\beta^{(1)}) \times \ldots \times I(\beta^{(r)})$. It is known that $\dim_H I(\beta^{(1)}) \times \ldots \times I(\beta^{(r)}) \leq \sum_{t=1}^r p(\beta^{(t)})$, by Kac’s theorem, [K]. Furthermore, it was pointed out to us by Crawley-Boevey that, according to the last remark in [KR, p.144], one has

Claim 3.2.4. For any $r$-tuple $\beta^{(1)}, \ldots, \beta^{(r)}$ of positive roots, there exists exactly one $r$-tuple $(d_1, \ldots, d_r)$ and exactly one irreducible component, $Z(\beta^{(1)}, \ldots, \beta^{(r)})$, of the set $I^{d_1}(\beta^{(1)}) \times \ldots \times I^{d_r}(\beta^{(r)})$ such that $\dim_H Z(\beta^{(1)}, \ldots, \beta^{(r)}) = \sum_{t=1}^r p(\beta^{(t)})$.

Moreover, for any $H$-stable constructible set $\mathcal{Y} \subset I(\beta^{(1)}) \times \ldots \times I(\beta^{(r)})$, we have

$$\dim_H \mathcal{Y} < \dim_H Z(\beta^{(1)}, \ldots, \beta^{(r)}) \quad \text{whenever} \quad \dim(\mathcal{Y} \cap Z(\beta^{(1)}, \ldots, \beta^{(r)})) < \dim Z(\beta^{(1)}, \ldots, \beta^{(r)}).$$

(3.2.5)

Now, fix a dimension vector $\alpha$ and a decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$. We have a natural (block diagonal) imbedding $H := G(\beta^{(1)}) \times \ldots \times G(\beta^{(r)}) \hookrightarrow G(\alpha)$, and a similar imbedding $I(\beta^{(1)}) \times \ldots \times I(\beta^{(r)}) \subset \Rep(Q^{op}, \beta^{(1)}) \times \ldots \times \Rep(Q^{op}, \beta^{(r)}) \hookrightarrow \Rep(Q^{op}, \alpha)$.

Write $I(\beta^{(1)}, \ldots, \beta^{(r)}) \subset \Rep(Q^{op}, \alpha)$ for the set of representations whose indecomposable summands have dimensions $\beta^{(t)}$, $t = 1, \ldots, r$. Equivalently, we have

$$I(\beta^{(1)}, \ldots, \beta^{(r)}) := G(\alpha) \cdot (I(\beta^{(1)}) \times \ldots \times I(\beta^{(r)})),$$

is the $G(\alpha)$-saturation of the set $I(\beta^{(1)}) \times \ldots \times I(\beta^{(r)})$. It is clear that the space $\Rep(Q^{op}, \alpha)$ is a disjoint union of the sets $I(\beta^{(1)}, \ldots, \beta^{(r)})$ for various decompositions of $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ such that each $\beta^{(t)} \in R^+_\mathbb{N}$.

Let $\pi : \mu_\alpha^{-1}(\Lambda) \hookrightarrow \Rep(Q^{op}, \alpha) \rightarrow \Rep(Q^{op}, \alpha)$ be the restriction of the vector bundle projection $T^*\Rep(Q^{op}, \alpha) \rightarrow \Rep(Q^{op}, \alpha)$. Thus, the set $\mu_\alpha^{-1}(\Lambda)$ breaks up into a disjoint union of various pieces $\pi^{-1}(I(\beta^{(1)}, \ldots, \beta^{(r)}))$. Fix one such piece corresponding to a decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$, where each $\beta^{(t)} \in R^+_\mathbb{N}$. It follows from [CB, Lemma 3.4] and [CB, Lemma 4.3] that $\dim \pi^{-1}(I(\beta^{(1)}, \ldots, \beta^{(r)})) \leq |\alpha|^2 - 1 + 2p(\alpha)$ with equality if and only if the decomposition $\beta^{(1)}, \ldots, \beta^{(r)}$ is in $\Sigma'_\alpha(\alpha)$. Therefore, by part (i) of the theorem, we see that any irreducible component of $\mu_\alpha^{-1}(\Lambda)$ has to be the closure of an irreducible component of the set $\pi^{-1}(I(\beta^{(1)}, \ldots, \beta^{(r)}))$ such that the decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ belongs to $\Sigma'_\alpha(\alpha)$; moreover, this irreducible component must have dimension equal to $|\alpha|^2 - 1 + 2p(\alpha)$.

To complete the proof of part (ii) of the Theorem, it remains to show that, for each $r$-tuple $\beta^{(1)}, \ldots, \beta^{(r)} \in \Sigma'_\alpha(\alpha)$, the set $\pi^{-1}(I(\beta^{(1)}, \ldots, \beta^{(r)}))$ contains only one irreducible component of required dimension. To this end, recall the set $Z(\beta^{(1)}, \ldots, \beta^{(r)})$ introduced earlier. This is an $H$-stable irreducible subvariety in $\Rep(Q^{op}, \alpha)$. Let $d$ denote the dimension of the
The set $Z_{\text{reg}}(\beta^{(1)}, \ldots, \beta^{(r)})$ of all points $\rho \in Z(\beta^{(1)}, \ldots, \beta^{(r)})$ such that $\dim G(\alpha) \cdot \rho = d$ is a Zariski open dense subset. Using [CB] Lemma 3.4 [and [CB] Lemma 4.1] we find that $\dim \pi^{-1}(G(\alpha) \cdot Z_{\text{reg}}(\beta^{(1)}, \ldots, \beta^{(r)})) = |\alpha|^2 - 1 + 2p(\alpha)$. Furthermore, since all $G(\alpha)$-orbits in $G(\alpha) \cdot Z_{\text{reg}}(\beta^{(1)}, \ldots, \beta^{(r)})$ have the same dimension, we deduce that all fibers of the projection $\pi^{-1}(G(\alpha) \cdot Z_{\text{reg}}(\beta^{(1)}, \ldots, \beta^{(r)})) \rightarrow G(\alpha) \cdot Z_{\text{reg}}(\beta^{(1)}, \ldots, \beta^{(r)})$ are affine-linear spaces of the same dimension, hence, the set $\pi^{-1}(G(\alpha) \cdot Z_{\text{reg}}(\beta^{(1)}, \ldots, \beta^{(r)}))$ is irreducible.

On the other hand, put $\mathcal{Y} := (I(\beta^{(1)}) \times \ldots \times I(\beta^{(r)})) \setminus Z_{\text{reg}}(\beta^{(1)}, \ldots, \beta^{(r)})$. Then, from [CB] Lemma 3.4 and [CB] Lemma 4.1, we obtain

$$\dim_{G(\alpha)} G(\alpha) \cdot \mathcal{Y} = \dim H \mathcal{Y} < \dim H Z(\beta^{(1)}, \ldots, \beta^{(r)}) = \sum_{t=1}^{r} p(\beta^{(t)}).$$

Thus, using [CB] Lemma 3.4 we deduce that

$$\dim \pi^{-1}(I(\beta^{(1)}, \ldots, \beta^{(r)})) \setminus G(\alpha) \cdot Z_{\text{reg}}(\beta^{(1)}, \ldots, \beta^{(r)}) = \dim \pi^{-1}(G(\alpha) \cdot \mathcal{Y}) < |\alpha|^2 - 1 + 2p(\alpha),$$

and the theorem follows. \(\Box\)

**Remark 3.2.6.** We do not know if $\pi^{-1}(I(\beta^{(1)}, \ldots, \beta^{(r)}))$ is irreducible, for any $(\beta^{(1)}, \ldots, \beta^{(r)}) \in \Sigma(\alpha)$.

### 3.3 Extended Dynkin case.

One of the goals of this section is to prove Theorem 1.4.1. Using the McKay correspondence, one may reformulate the theorem in the language of quivers, see eg. [EG] §11. This will allow us to use the results from [CB].

From now on, for the rest of this section, we let $Q$ be an affine Dynkin quiver, let $\alpha$ be an extending vertex of $Q$, and let $S$ be the quiver obtained from $Q$ by adjoining one vertex $s$ and one arrow $s \rightarrow \alpha$.

Fix a positive integer $n$. Let $I$ be the vertex set of $S$, let $\delta \in \mathbb{N}I$ be the minimal positive imaginary root of $Q$, and let $\alpha := n\delta + \epsilon_s$. We have $q(\delta) = 0$ and $q(\alpha) = 1 - n$, so $p(\delta) = 1$ and $p(\alpha) = n$.

We will use some of the notation introduced in the proof of Theorem 3.2.2. In particular, for any decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$, where $\beta^{(t)} \in \mathbb{N}I \setminus \{0\}$, let $I(\beta^{(1)}, \ldots, \beta^{(r)})$ be the subset of $\text{Rep}(S^{\text{op}}, \alpha)$ consisting of the representations of $S^{\text{op}}$ whose indecomposable summands have dimension $\beta^{(t)}$, $t = 1, \ldots, r$.

Fix $\lambda \in \mathbb{C}^I$ such that $\lambda_s = 0$ and $\lambda \cdot \delta = 0$. We have the moment map

$$\mu_\alpha : \text{Rep}(S, \alpha) \rightarrow \text{End}(\alpha)_0.$$

Denote by $\pi : \mu_\alpha^{-1}(\lambda) \rightarrow \text{Rep}(S^{\text{op}}, \alpha)$ the projection map.

**Lemma 3.3.1.** For any decomposition $\alpha = \beta^{(1)} + \cdots + \beta^{(r)}$ with $\beta^{(t)} \in \mathbb{N}I \setminus \{0\}$, we have

$$\dim \pi^{-1}(I(\beta^{(1)}, \ldots, \beta^{(r)})) \leq n^2|\delta|^2 + 2n$$

with equality if and only if all but one of the $\beta^{(t)}$ are equal to $\delta$.

**Proof.** The lemma follows from [CB] Lemma 9.2, [CB] Lemma 4.3, and [CB] Lemma 3.4. \(\Box\)
Let $\sigma : \text{Rep}(S^{\text{op}},\alpha) \rightarrow \text{Rep}(Q^{\text{op}},n\delta)$ be the projection map. Let $U \subset \text{Rep}(Q^{\text{op}},n\delta)$ be the subset consisting of all representations $Y$ which have a decomposition $Y \cong Y_1 \oplus \cdots \oplus Y_n$, where each $Y_i$ is indecomposable with dimension vector $\delta$, and $\dim \text{End}(Y) = n$ (this implies in particular that $\text{End}(Y_i) = \mathbb{C}$, for all $t = 1,\ldots,n$). It is well known that the canonical decomposition of $n\delta$ is $\delta + \cdots + \delta$, and moreover, the subset $\{Y \mid \dim \text{End}(Y) \leq n\}$ is open in $\text{Rep}(Q^{\text{op}},n\delta)$, for any $n$; see [Ka] and [Sc]. Hence, $U$ is a dense open subset of $\text{Rep}(Q^{\text{op}},n\delta)$.

Now, we have the following composition of maps:

$$\text{Rep}(\mathbb{S},\alpha) \xrightarrow{\pi} \text{Rep}(S^{\text{op}},\alpha) \xrightarrow{\sigma} \text{Rep}(Q^{\text{op}},n\delta).$$

**Lemma 3.3.2.** We have $\dim \pi^{-1}(\sigma^{-1}(\text{Rep}(Q^{\text{op}},n\delta) \setminus U)) < n^2|\delta|^2 + 2n$.

**Proof.** This is immediate from [CB, Lemma 9.3] and [CB, Lemma 3.4].

A representation of a quiver is called a brick if its endomorphism algebra is of dimension one. Let $\mathcal{M}_k$ (resp., $\mathcal{M}_k''$) be the subset of $\pi^{-1}(I(k\delta + \epsilon_s,\delta,\ldots,\delta))$ consisting of all $(X,Y,i,j) \in \text{Rep}(Q,n\delta) \times \text{Rep}(Q^{\text{op}},n\delta) \times X \times V^*$ such that $Y \in U$, and $(X,Y,i,j)$ is a brick (resp., $(X,Y,i,j)$ is not a brick). Define $\mathcal{M}_k$ to be the closure of $\mathcal{M}_k'$ in $\mu^{-1}_k(\lambda)$.

The main result of this section is Theorem 3.3.3 below, which is a more precise version of Theorem 3.2.3. Note that, for the quiver $S$ and $\alpha$, $\lambda$ defined above, in the notation of Theorem 3.2.3, we have $\alpha \in \Sigma_\lambda$, by [CB, Lemma 9.2]. Thus, part (i) in the theorem below is a special case of Theorem 3.2.3(i) while part (ii) in the theorem below provides an explicit description of the $1$-$1$ correspondence from Theorem 3.2.2(ii).

Our proof of Theorem 3.3.3 will be independent of the proof of Theorem 3.2.2. We remark also that, most important in the theorem below, is its part (iii), which is a special case of Conjecture 3.2.3.

**Theorem 3.3.3.** (i) $\mu^{-1}_k(\lambda)$ is equidimensional, we have $\dim \mu^{-1}_k(\lambda) = n^2|\delta|^2 + 2n$; furthermore, the scheme $\mu^{-1}_k(\lambda)$ is a complete intersection in $\text{Rep}(\mathbb{S},\alpha)$.

(ii) The irreducible components of $\mu^{-1}_k(\lambda)$ are $\mathcal{M}_0,\ldots,\mathcal{M}_n$.

(iii) The scheme $\mu^{-1}_k(\lambda)$ is reduced.

First, recall the following standard result, cf. [CB, Lemma 3.1].

**Lemma 3.3.4.** If $y = (y_a)_{a \in Q} \in \text{Rep}(Q^{\text{op}},\alpha)$, then there is an exact sequence

$$0 \rightarrow \text{Ext}^1(y,y)^* \xrightarrow{c} \text{Rep}(Q,\alpha) \xrightarrow{\epsilon} \text{End}(\alpha) \xrightarrow{\epsilon^*} \text{End}(y)^* \rightarrow 0,$$

where the map $c$ sends $(x_a) \in \text{Rep}(Q,\alpha)$ to $\sum_{a \in Q}[x_a,y_a^*]$, the map $\epsilon$ sends $(g_i)$ to the linear map $\text{End}(y) \rightarrow \mathbb{C}: (\phi_i) \mapsto \sum_i \text{Tr}(g_i\phi_i)$, and the map $\epsilon^*$ was defined in [CB, §3].

Next, we prove the following lemma.

**Lemma 3.3.5.** (i) We have $\dim \mathcal{M}_k' = n^2|\delta|^2 + 2n$, and $\mathcal{M}_k'$ is irreducible.

(ii) We have $\dim \mathcal{M}_k'' < n^2|\delta|^2 + 2n$. 

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Proof. Let $R = \operatorname{Rep}(Q^{\text{op}}, \delta) \times \cdots \times \operatorname{Rep}(Q^{\text{op}}, \delta)$, and consider it as a subset of $\operatorname{Rep}(Q^{\text{op}}, n\delta)$ using block-diagonal matrices. Let $I'_k$ be the subset of $\operatorname{Rep}(S^{\text{op}}, \alpha)$ consisting of the elements $(Y, j)$ such that $Y \in R \cap U$ and $j = (1, \ldots, 1, 0, \ldots, 0)$. We have $I'_k \subset I(k\delta + \epsilon_s, \delta, \ldots, \delta)$. It is clear that $\pi^{-1}(I'_k)$ is contained in the disjoint union $\mathcal{M}''_k \cup \mathcal{M}''_k$. Moreover, any element in $\mathcal{M}''_k \cup \mathcal{M}''_k$ is $G(\alpha)$-conjugate to an element in $\pi^{-1}(I'_k)$. Therefore, the $G(\alpha)$-saturation of $\pi^{-1}(I'_k)$ is $\mathcal{M}''_k \cup \mathcal{M}''_k$.

Now let $y = (Y, j) \in I'_k$, and write $Y = Y_1 \oplus \cdots \oplus Y_n$, where $Y_i \in \operatorname{Rep}(Q^{\text{op}}, \delta)$. Observe that $\dim \operatorname{End}(Y) = n - k + 1$, hence by Lemma 3.3.4 the fiber $\pi^{-1}(y)$ is an affine space of dimension $2n - k$. Since $I'_k$ is an irreducible, it follows that the scheme $\pi^{-1}(I'_k)$ is irreducible and has dimension $n|\delta|^2 + 2n - k$. Hence, $\mathcal{M}''_k \cup \mathcal{M}''_k$ is irreducible.

We begin the proof of the inequality of part (ii) of the lemma. Let $p: \pi^{-1}(y) \rightarrow V$ be the projection map $p(X, Y, i, j) = i$.

Claim: The vector $i$ is in the image of $p$ if and only if it is of the form $i = (0, \ldots, 0, i_{k+1}, \ldots, i_n)$. If $i$ is in the image of $p$, then $p^{-1}(i)$ is an affine space of dimension $n$.

Proof of Claim: Let $N = \sum \delta_i$. We shall consider $X$ and $Y$ as endomorphisms of a vector space of dimension $nN$. Recall that we write $Y$ as a block-diagonal matrix $Y_1 \oplus \cdots \oplus Y_n$. Similarly, we write $X \in \operatorname{Rep}(Q, n\delta)$ as a $n \times n$ block matrix, where each block is a $N \times N$ matrix.

Now consider Lemma 3.3.4 for the quiver $Q$ and dimension vector $n\delta$. Since $Y \in U$, we have $\dim \operatorname{End}(Y) = n$, so the cokernel of the map $c$ in Lemma 3.3.4 has dimension $n$. For any $X \in \operatorname{Rep}(Q, n\delta)$, the element $c(X) = [X, Y]$ is a $n \times n$ block matrix such that each of the $n$ blocks on the diagonal have trace 0. Hence, the image of $c$ is the subspace of all $n \times n$ block matrix such that each of the $n$ blocks on the diagonal have trace 0. Since $\lambda \cdot \delta = 0$ and $j = (1, \ldots, 1, 0, \ldots, 0)$, the element $\lambda - ij$ is in the image of $c$ if and only if $i$ is of the form $i = (0, \ldots, 0, i_{k+1}, \ldots, i_n)$, and in this case, by Lemma 3.3.4, $p^{-1}(i)$ is an affine space of dimension $n$.

This completes the proof of the Claim.

We can now prove that $\dim \mathcal{M}''_k < n^2|\delta|^2 + 2n$. Suppose $(X, Y, i, j) \in \pi^{-1}(I'_k)$. By the above Claim, we have $i = (0, \ldots, 0, i_{k+1}, \ldots, i_n)$. If $i_{k+1}, \ldots, i_n$ are all nonzero, then it is clear that $(X, Y, i, j)$ is a brick. Define $Z_k$ to be the closed subset of $\pi^{-1}(I'_k)$ defined by the equation

\[i_{k+1}i_{k+2} \cdots i_n = 0\]

(that is, one of the last $n - k$ entries of $i$ is equal to 0). Then $\mathcal{M}''_k$ is contained in the $G(\alpha)$-saturation of $Z_k$. By the above Claim, $Z_k$ is properly contained in $\pi^{-1}(I'_k)$, so we have $\dim Z_k < n|\delta|^2 + 2n - k$.

Let $GL(n\delta) := \prod_k GL(n\delta_k)$, and let $H_k \subset GL(\delta)^n$ be the subgroup of $GL(n\delta)$ consisting of diagonal block matrices whose component at the extending vertex is of the form $\det(1, \ldots, 1, g_{k+1}, \ldots, g_n)$. We have $\dim H_k = n|\delta|^2 - k$. Moreover, $Z_k$ is stable under the action $H_k$.

Hence,

\[\dim \mathcal{M}''_k \leq \dim Z_k + \dim GL(n\delta) - \dim H_k\]

\[<(n|\delta|^2 + 2n - k) + n^2|\delta|^2 - (n|\delta|^2 - k) = n^2|\delta|^2 + 2n\]

Therefore, by Lemma 3.3.4 and Lemma 3.3.2 it follows that $\mathcal{M}''_k$ is of dimension $n^2|\delta|^2 + 2n$. Moreover, $\mathcal{M}''_k$ is open dense in the $G(\alpha)$-saturation of $\pi^{-1}(I'_k)$, hence it is irreducible. \hfill $\square$

Proof of Theorem 3.3.3 Observe that $\mu^{-1}_\alpha(\lambda)$ is defined by $n^2|\delta|^2$ equations in the $2n^2|\delta|^2 + 2n$ dimensional space $\operatorname{Rep}(S, \alpha)$. Thus, the irreducible components of $\mu^{-1}_\alpha(\lambda)$ must
have dimension of at least $n^2|\delta|^2 + 2n$. By Lemma 3.3.4, it follows that $\mu_{\alpha}^{-1}(\lambda)$ is a complete intersection and equidimensional of dimension $n^2|\delta|^2 + 2n$. Further, set theoretically, we have

$$\mu_{\alpha}^{-1}(\lambda) = \left( \bigcup_{k=0}^{n} \mathcal{L}'_{k} \right) \sqcup \left( \bigcup_{k=0}^{n} \mathcal{L}''_{k} \right) \sqcup \pi^{-1}(\sigma^{-1}(Q^p, n\delta) \setminus U).$$

It follows, by Lemma 3.3.2 and Lemma 3.3.5, that the irreducible components of $\mu_{\alpha}^{-1}(\lambda)$ are $\mathcal{L}_0, \ldots, \mathcal{L}_n$.

Since the representations in $\mathcal{L}'_k$ are bricks, the moment map $\mu_{\alpha}$ is a submersion at generic points of $\mu_{\alpha}^{-1}(\lambda)$, and so $\mu_{\alpha}^{-1}(\lambda)$ is generically reduced. Moreover, since $\mu_{\alpha}^{-1}(\lambda)$ is a complete intersection, it is Cohen-Macaulay. Hence, $\mu_{\alpha}^{-1}(\lambda)$ is reduced (cf. [CG Theorem 2.2.11] or [E], Exercise 18.9)].

4 A Lagrangian variety

4.1 Let $\mathbb{P} := \mathbb{P}(V)$ be the projective space (of dimension $n - 1$). We identify the total space of the cotangent bundle to $\mathbb{P}$ with

$$T^{*}\mathbb{P} = \{(i, j) \in (V \setminus \{0\}) \times V^* \mid \langle j, i \rangle = 0\}/\mathbb{C}^*,$$

where the multiplicative group $\mathbb{C}^*$ acts naturally on $V \times V^*$ by $t(i, j) = (t \cdot i, t^{-1} \cdot j)$. The group $G = GL(V)$ acts naturally on $\mathbb{P}$ and the induced $G$-action on $T^{*}\mathbb{P}$ is Hamiltonian with moment map $(i, j) \mapsto ij \in \mathfrak{g} = \mathfrak{g}^*.$

Next, we set $\mathfrak{X} := \mathfrak{g} \times \mathbb{P}$ and view it as a $G$-variety with respect to $G$-diagonal action. We remark that this action clearly factors through the quotient $PGL(V) = GL(V)/\mathbb{C}^*$, in particular, any $G$-orbit in $\mathfrak{X}$ may also be regarded as either $SL(V)$- or $PGL(V)$-orbit. Further, we have

$$T^{*}\mathfrak{X} = T^{*}\mathfrak{g} \times T^{*}\mathbb{P} = \{(X, Y, i, j) \in \mathfrak{g} \times (V \setminus \{0\}) \times V^* \mid \langle j, i \rangle = 0\}/\mathbb{C}^*,$$

where $\mathbb{C}^*$ acts on $(i, j)$ as above and does not act on $X, Y$. Again, the induced $G$-action on $T^{*}\mathfrak{X}$ is Hamiltonian with moment map given, essentially, by formula (4.2).

4.2 A stratification of $\mathfrak{X}$. Given a direct sum decomposition $V = V_1 \oplus \ldots \oplus V_l$, write $L = GL(V_1) \times \ldots \times GL(V_l)$ for the corresponding Levi subgroup in $G = GL(V)$, formed by block-diagonal matrices, Let $\mathfrak{l} = \text{Lie}(L) = \mathfrak{gl}(V_1) \oplus \ldots \oplus \mathfrak{gl}(V_l)$ be the corresponding Levi subalgebra in $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathcal{K}_l = \mathcal{N} \cap \mathfrak{l}$ the nilpotent variety of the reductive Lie algebra $\mathfrak{l}$. The group $L$ acts on $\mathcal{K}_l$ with finitely many orbits.

We have a direct sum decomposition $\mathfrak{l} = \mathfrak{z}_l \oplus [\mathfrak{l}, \mathfrak{l}]$, where $\mathfrak{z}_l$, the center of the Lie algebra $\mathfrak{l}$, may be identified naturally with $\mathbb{C}^l$. Let $\mathfrak{z}_l^0 \subset \mathfrak{z}_l = \mathbb{C}^l$ denote the Zariski-open dense subset formed by the elements of $\mathbb{C}^l$ with pairwise distinct coordinates.

G. Lusztig introduced, see [L1], a certain stratification of the Lie algebra of an arbitrary reductive group $G$. The strata of Lusztig’s stratification are smooth locally-closed $\text{Ad}G$-stable subvarieties labelled by the set of $G$-conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{o})$, where $\mathfrak{l} \subset \text{Lie}G$ is a Levi subalgebra and $\mathfrak{o} \subset \mathfrak{l}$ is a nilpotent $\text{Ad}L$-conjugacy class.

We are going to introduce a similar stratification of the variety $\mathfrak{X}$. The strata of our stratification will be labelled by the set $\mathcal{S}$ formed $G$-conjugacy classes of pairs $(\mathfrak{l}, \mathfrak{o})$, where $\mathfrak{l} \subset \mathfrak{g}$ is a
Proof. To prove (i), let an element of \( X \) be the corresponding Levi subalgebras with Lie algebras \( l = \oplus_r V_r \) and \( l' = \oplus_s V'_s \), respectively. Let \( z \in \mathfrak{g}_r, z' \in \mathfrak{g}_r' \), and let \((x, v) \in \mathcal{M}_r \times V, (x', v') \in \mathcal{M}_r' \times V\). Write \( \Omega = L(x, v) \), \( \Omega' = L'(x', v') \), for the corresponding orbits.

Suppose there exists \( g \in G \) such that \((z' + x', \mathbb{C}v') = g(z + x, \mathbb{C}v)\). By uniqueness of Jordan decomposition, we must have \( gzg^{-1} \subset \mathfrak{g}_r \) and \( gzxg^{-1} \subset \text{Ad} L'(x') \). Thus, the decompositions \( V = \oplus_r V_r \) and \( V = \oplus_s V'_s \) have the same number of direct summands, moreover, we have \( g = wa \), where \( w \) is a permutation matrix such that each block of \( l \) is mapped to a block of \( l' \), and \( a \in L \). Hence, the action of \( g \) must take the set \( \{ (z_1 + x_1) \mid z_1 \in \mathfrak{g}_r, (x_1, \mathbb{C}v_1) \in \Omega \} \) into the set \( \{ (z_2 + x_2) \mid z_2 \in \mathfrak{g}_r', (x_2, \mathbb{C}v_2) \in \Omega' \} \). Part (i) follows.

To prove (ii), we observe that there are only finitely many \( \text{Ad} L \)-orbits \( O \subset \mathcal{M}_r \). But for such an orbit \( O \), the number of \( L \)-diagonal orbits in \( O \times \mathbb{P} \) is finite, due to Lemma \[2.1.1\]. Further, the centralizer in \( GL(V) \) of any \( z \in \mathfrak{g}^\circ \) is \( L \). Part (ii) follows, since it is clear by Jordan normal form that any element of \( \mathfrak{X} \) belongs to some \( \mathfrak{X}(l, \Omega) \).

Thus, \( \bigsqcup_{(l, \Omega) \in \mathcal{S}} T^* \mathfrak{X}(l, \Omega) \) is a (reducible) singular Lagrangian subvariety in \( T^* \mathfrak{X} \).

4.3 Relevant strata. We remind the reader that the following properties of a linear map \( X : V \to V \) are equivalent:

- The map \( X \) has a cyclic vector, i.e., there is a line \( \ell \in \mathbb{P} \) such that \( \mathbb{C}[X] \ell = V \);
- \( X \) is a regular element of \( \mathfrak{g} \), i.e., such that \( \dim \mathfrak{g}^X = n \);
- In Jordan normal form for \( X \), different Jordan blocks have pairwise distinct diagonal entries.

Definition 4.3.1. A pair \((X, \ell) \in \mathfrak{g} \times \mathbb{P}\) is said to be relevant if \( X \) is a regular element in \( \mathfrak{g} \) and the subspace \( \mathbb{C}[X] \ell \subset V \) has an \( X \)-stable complement.

Let \( \mathfrak{X}_{\text{relevant}} \subset \mathfrak{X} \) be the set of all relevant pairs \((X, \ell) \in \mathfrak{g} \times \mathbb{P}\). It is clear that \( \mathfrak{X}_{\text{relevant}} \) is a \( G \)-stable dense subset in \( \mathfrak{X} \) containing, in particular, all pairs \((X, \ell) \) such that \( \mathbb{C}[X] \ell = V \).

Now, fix a pair \((l, \Omega) \in \mathcal{S} \). We observe that the pairs \((X, \ell) \) in the corresponding stratum \( \mathfrak{X}(l, \Omega) \) are either all relevant or not, i.e., either \( \mathfrak{X}(l, \Omega) \subset \mathfrak{X}_{\text{relevant}} \), in which case we call the stratum \( \mathfrak{X}(l, \Omega) \) relevant, or else \( \mathfrak{X}(l, \Omega) \subset \mathfrak{X} \setminus \mathfrak{X}_{\text{relevant}} \).

\[1\]This has been done by Rupert Yu by generalizing Broer results [Bro] on the classical Jordan classes.

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Explicitly, let \((X, \ell) \in \mathcal{X}(l, \Omega)\) be such that \(X = z + x\) and \(\ell = Ci\), where \(z \in \mathfrak{g}_l^0\) and \((x, \ell) \in \Omega\). Write \(l = \mathfrak{gl}(V_1) \oplus \ldots \oplus \mathfrak{gl}(V_l)\), \(X = \oplus_{k=1}^l X_k\), and \(i = i_1 + \ldots + i_l\), where \(X_k \in \mathfrak{gl}(V_k)\), \(i_k \in V_k\). Then, the pair \((X, \ell)\) is relevant if and only if the following two conditions hold:

- For each \(k = 1, \ldots, l\), in Jordan normal form, the element \(X_k\) has a single Jordan block.
- For any \(k = 1, \ldots, l\), the vector \(i_k\) is either a cyclic vector for \(X_k\) or else \(i_k = 0\) (different alternatives for different values of \(k = 1, \ldots, l\) are allowed).

4.4 The scheme \(\Lambda\). Recall the subscheme \(\mathcal{M}_\text{nil} \subset T^*(\mathfrak{g} \times V)\) defined by formula (4.4.1). We use the identification in (4.4.4) and define a subscheme in \(T^*\mathcal{X}\) as follows:

\[
\Lambda := \left( \mathcal{M}_\text{nil} \cap \left[ T^*\mathfrak{g} \times T^*(V \setminus \{0\}) \right] \right) / \mathbb{C}^\times
\]

\[
= \left( \{(X, Y, i, j) \in \mathfrak{g} \times \mathfrak{g} \times (V \setminus \{0\}) \times V^* \mid [X, Y] + ij = 0 \& Y \text{ is nilpotent} \} \right) / \mathbb{C}^\times.
\]

Note that the equation \([X, Y] + ij = 0\) implies that \((j, i) = \text{Tr}(ij) = -\text{Tr}[X, Y] = 0\). Thus, the second line in (4.4.1) does define a subscheme of \(T^*\mathcal{X}\). It is clear that \(\Lambda\) is a \(G\)-stable closed subscheme of \(T^*\mathcal{X}\).

The following result is a much more precise version of Theorem 1.1.4.

Theorem 4.4.2. \(\Lambda\) is a Lagrangian subscheme in \(T^*\mathcal{X}\). More precisely, with the notation of (4.4.3) we have

\[
\Lambda = \bigcup_{\{l, \Omega \in \mathcal{S} \mid \mathcal{X}(l, \Omega) \subset \mathcal{X}_{\text{relevant}}\}} \bar{T}_{\mathcal{X}(l, \Omega)}^*\mathcal{X},
\]

where bar denotes the closure and the union on the right is taken over all relevant strata.

Proof. Fix a pair \((l, \Omega) \in \mathcal{S}\), so \(l = \mathfrak{g}_l^0 \oplus [l, l]\). Let \(p = (X, \ell) \in \mathcal{X}(l, \Omega)\). By \(G\)-equivariance, it suffices to analyze the situation in the case where \(X = z + x\) and \(\ell = Ci\), for some \(z \in \mathfrak{g}_l^0\) and \((x, i) \in \Omega\). We will use the notation of (4.4.3). So,

\[
X = z + x, \quad \text{where} \quad x = \oplus_{k=1}^l x_k, \quad x_k \in \mathfrak{gl}(V_k) \text{ is nilpotent, and} \quad z = \oplus_{k=1}^l z_k \cdot \text{Id}_{V_k}, \tag{4.4.3}
\]

for some pairwise distinct complex numbers \(z_1, \ldots, z_l \in \mathbb{C}\).

Step 1. We need an explicit description of various tangent and cotangent bundles.

The tangent space to \(\mathcal{X}\) at \(p = (X, \ell)\) has a direct sum decomposition \(T_p\mathcal{X} = \mathfrak{g} \oplus T_l\mathbb{P}\), where \(T_l\mathbb{P}\) denotes the tangent space to \(\mathbb{P}\) at the point \(\ell \in \mathbb{P}\). Similarly, for cotangent spaces, we have \(T_p^*\mathcal{X} = \mathfrak{g}^* \oplus T_l^*\mathbb{P}\). We write \(T_p \subset T_p\mathcal{X}\) for the tangent, resp. \(N_p \subset T_p^*\mathcal{X}\) for the conormal, space to the stratum \(\mathcal{X}(l, \Omega)\) at \(p\).

Let \(S \subset \mathfrak{g}\) be any subset. It will be convenient to use shorthand notation and, given a subset \(U\) in either \(\mathcal{X} = \mathfrak{g} \times \mathbb{P}\) or in \(T_p\mathcal{X} = \mathfrak{g} \oplus T_l\mathbb{P}\), resp. in \(T_p^*\mathcal{X} = \mathfrak{g}^* \oplus T_l^*\mathbb{P}\), write \(S + U := \{(s + x', \ell') \mid s \in S, (x', \ell') \in U\}\).

With this notation, we have \(s + L(x', \ell') = L(s + x', \ell')\), for any \(s \in \mathfrak{g}_l^0\), in particular, \(\mathfrak{g}_l^0 + \Omega = L(\mathfrak{g}_l^0 + x, \ell)\). Therefore, we get \(\mathcal{X}(l, \Omega) = G(\mathfrak{g}_l^0 + \Omega) = G(L(\mathfrak{g}_l^0 + x, \ell))\). Hence, \(\mathcal{X}(l, \Omega) = G(\mathfrak{g}_l^0 + x, \ell)\) and, for the tangent spaces, we deduce \(T_p = \mathfrak{g}_l^0 + T_p G(p)\), where \(T_p G(p)\) stands for the tangent space at \(p\) to the \(G\)-orbit through \(p\). Dually, we obtain

\[
N_p = (\mathfrak{g}_l^0 + 0) \cap T_{G(p)}^*\mathcal{X} = (\mathfrak{g}_l^0 \oplus T_l^*\mathbb{P}) \cap \mu^{-1}(0), \tag{4.4.4}
\]
where in the rightmost equality we have used that the preimage of 0 under the moment map is the union of the conormal bundles to the \(G\)-orbits in \(\mathfrak{X}\). From now on, we will identify \(\mathfrak{g}^*\) with \(\mathfrak{g}\) via the trace form, and thus view \(\mathfrak{z}_t^+\) as a subspace in \(\mathfrak{g}\).

**Step 2.** We begin the proof of the Theorem by showing that

\[
\Lambda \subset \bigcup_{(l,\Omega) \in \mathcal{S}} T_{\mathfrak{X}(l,\Omega)}^* \mathfrak{X}. \quad (4.4.5)
\]

To this end, let \((X, Y, i, j) \in \Lambda\). Thus, \(Y\) is nilpotent and \([X, Y] + ij = 0\). Hence, there exists a complete flag in \(V\) that is stable under the action of both \(X\) and \(Y\). Therefore, it is also stable under the action of \(z\). Thus, all three matrices \(X, z, Y\) can be simultaneously made upper-triangular. Furthermore, since \(Y\) is nilpotent, in the upper-triangular form, \(Y\) has vanishing diagonal entries. We conclude that \(\text{Tr}(z^m \cdot Y) = 0\) for all \(m = 1, 2, \ldots\). Since \(z = \oplus_{k=1}^r z_k \cdot \text{Id}_{V_k}\), cf. (4.4.3), we see by the Vandermonde determinant, that \(\text{Tr}(s \cdot Y) = 0\) for any \(s \in \mathfrak{z}_t\). Thus, \(Y \in \mathfrak{z}_t^+\) and (4.4.4) shows that \((Y, j) \in \mathbb{N}_p\). This yields (4.4.5).

**Step 3.** By Theorem 1.1.4, we know that \(\Lambda\) is a closed Lagrangian scheme. This Lagrangian scheme is (set-theoretically) contained in the RHS of (4.4.3), which is also a Lagrangian scheme. It follows that each irreducible component of \(\Lambda\) must be at the same time an irreducible component of the scheme in the RHS of (4.4.3), hence has the form \(T_{\mathfrak{X}(l,\Omega)}^* \mathfrak{X}\), for some pair \((l, \Omega)\) such that \(T_{\mathfrak{X}(l,\Omega)}^* \mathfrak{X} \subset \Lambda\). Thus, we have proved that

\[
\Lambda = \bigcup_{\{(l,\Omega) \in \mathcal{S} \mid T_{\mathfrak{X}(l,\Omega)}^* \mathfrak{X} \subset \Lambda\}} T_{\mathfrak{X}(l,\Omega)}^* \mathfrak{X}. \quad (4.4.6)
\]

We see that completing the proof of the theorem amounts to showing that the stratum \(\mathfrak{X}(l,\Omega)\) is relevant if and only if \(T_{\mathfrak{X}(l,\Omega)}^* \mathfrak{X} \subset \Lambda\), that is, if and only if for some (hence any) point \(p = (X, \ell) \in \mathfrak{X}(l,\Omega)\), the following holds

\[
(Y, j) \in \mathbb{N}_p \implies Y \text{ is nilpotent}. \quad (4.4.7)
\]

**Step 4.** We claim first that if \(X\) is not regular in \(\mathfrak{g}\) then (4.4.5) does not hold.

We argue by contradiction. If \(X\) is not regular in \(\mathfrak{g}\) then there exists \(k \in \{1, \ldots, r\}\) such that Jordan normal form of the matrix \(X_k\) consists of more than one block. Hence, there exists a nonzero semisimple element \(Y_k \in \mathfrak{s}(V_k)\) that commutes with \(X_k\). We let \(Y\) be the matrix whose \(k\)-th component equals \(Y_k\) and all other components vanish. Clearly, for any \(s \in \mathfrak{z}_t\), we have \(\text{Tr}(s \cdot Y) = 0\), so \(Y \in \mathfrak{z}_t^+\). Further, the quadruple \((X, Y, i, j)\), where \(i \in \ell\) and \(j = 0\) satisfies the equation \([X, Y] + ij = 0\). Thus, \((Y, j) \in \mathbb{N}_p\). Moreover, it is clear that \((X, Y, i, j) \notin \Lambda\) since \(Y\) is a nonzero semisimple, hence not nilpotent, element. Our claim is proved.

Next, for \(p = (X, \ell)\) as in (4.4.3), we claim that

\[
p = (X, \ell) \text{ is relevant iff one has: } (Y, j) \in \mathbb{N}_p \& Y \in \mathcal{I} \implies Y \in \mathcal{N}. \quad (4.4.7)
\]

We write \(X = \oplus_{k=1}^r X_k\), where \(X_k \in \mathfrak{gl}(V_k)\). Clearly, it suffices to prove our claim for each \(X_k\) separately. Therefore, we may assume that \(\mathcal{I} = \mathfrak{g}\), so \(X = z \cdot \text{Id}_V + x\), for some \(z \in \mathbb{C}\) and \(x \in \mathcal{N}\). By the claim at the beginning of Step 4, we may further restrict our attention to the case where \(X\) is regular. Thus, we have reduced the proof of (4.4.7) to the following
Step 5. Proof of (4.4.7) assuming that \( X \) consists of a single Jordan \( n \times n \)-block.

Recall that \( \ell = C \cdot i \), and let \( m \in \{0, 1, \ldots, n\} \) be the unique integer such that \( i \in \text{Ker}(x^m) \setminus \text{Ker}(x^{m-1}) \), where we put \( \text{Ker}(x^0) := \{0\} \) and \( \text{Ker}(x^{-1}) := \emptyset \). Let \( (Y, j) \in N_p \). We write \( i = (i_1, \ldots, i_n) \) and \( j = (j_1, \ldots, j_n) \), as usual. Lemma 2.1.13 shows that, conjugating the triple \( (Y, i, j) \) by an element of the group \( G^X \) if necessary, we may assume that

\[
\begin{align*}
i &= (1, \ldots, 1, 0, \ldots, 0), \quad \text{and} \quad j &= (0, \ldots, 0, j_{m+1}, \ldots, j_n),
\end{align*}
\]

for some (not necessarily nonzero) \( j_{m+1}, \ldots, j_n \in \mathbb{C} \).

Now, writing out the equation \([X, Y] + ij = 0\) for \( i, j \) as above and \( X \) an \( n \times n \) Jordan block, we find that \( Y \) must be upper triangular and, moreover, that

\[
Y_{rs} = Y_{r-1,s-1} - i r^{-1} j_s, \quad \text{for all } 1 \leq r \leq s \leq n.
\]

We see that one can choose, arbitrarily, the first row of \( Y \) and then all other entries of \( Y \) are uniquely determined from the equations above. Solving these equations recursively yields

\[
Y_{rs} = Y_{1,s-r+1} - i_1 j_s - i_{r-2} j_{s-r+1} - \cdots - i_1 j_{s-r+2}, \quad \text{for all } 1 \leq r \leq s \leq n.
\]

In particular, using (4.4.8), for \( r = s \) we find

\[
Y_{rr} = \begin{cases} 
Y_{11} & \text{if } 0 < r \leq m \leq n \\
Y_{11} - j_{m+1} & \text{if } 0 < m < r \leq n \\
Y_{11} & \text{if } 0 = m < r \leq n.
\end{cases}
\]

Assume first that \( 0 < m < n \). We choose some complex number \( j_{m+1} \neq 0 \), and find \( Y_{11} \) from the equation \( n \cdot Y_{11} - (n-m) \cdot j_{m+1} = 0 \). Then, formula (4.4.10) shows that if we let \( j \) be any row vector with the chosen \( m+1 \)-th entry \( j_{m+1} \), then there exists a trace zero upper triangular matrix \( Y \) that satisfies \([X, Y] + ij = 0\) and such that \( Y_{11} \neq 0 \). This matrix \( Y \) is clearly not nilpotent, hence, we have found a pair \( (Y, j) \in N_p \) with non-nilpotent \( Y \). Thus, condition 4.4.6 does not hold for \( X \). Hence, the conormal bundle \( T^*_X(\Omega(X)) \) is not contained in \( \Lambda \).

Assume now that either \( m = 0 \) or \( m = n \). Then, formula (4.4.10) combined with the requirement that \( \text{Tr} Y = 0 \) forces \( Y_{rr} = 0 \), for all \( r = 1, \ldots, n \). Hence, we have shown that, for any \( (Y, j) \in N_p \), the matrix \( Y \) is necessarily nilpotent. Thus, condition 4.4.6 holds for \( X \).

Finally, we observe that in the case \( m = 0 \) we have \( i = 0 \), while in the case \( m = n \) the vector \( i = (1, \ldots, 1) \) is a cyclic vector for \( X \). Thus, \( (X, Ci) \) is a relevant pair.

This completes the proof of (4.4.7).

Step 6 Claim (4.4.7) implies, in particular, that if (4.4.6) holds, then the pair \((X, \ell)\) is relevant. Now, let \( p = (X, \ell) \) be any relevant pair. To complete the proof of the Theorem, we must show that (4.4.6) holds for \( p \).

To this end, we use Definition 1.3.1 and write \( V = V^1 \oplus V^2 \), where \( V^1 = C[X] \ell \) and \( V^2 \) is an \( X \)-stable complement. Accordingly, we write an arbitrary element \( Y \in \mathfrak{gl}(V) \) in block form, \( Y = \|Y^r\| \), where \( Y^r \in \text{Hom}(V^r, V^s) \), \( r, s \in \{1, 2\} \). The matrix \( X \) is block-diagonal, so we have \( X = X^1 \oplus X^2 \), where \( X^r = z^r + x^r \in \mathfrak{gl}(V^r) \), \( r = 1, 2 \), are both regular.

For each \( r, s \in \{1, 2\} \) we have the map

\[
(ad X)^r s : \text{Hom}(V^r, V^s) \to \text{Hom}(V^r, V^s), \quad Y^r s \mapsto X^s Y^r s - Y^r s X^r.
\]
This map has Jordan decomposition \((\text{ad} \, X)^{rs} = (\text{ad} \, z)^{rs} + (\text{ad} \, x)^{rs}\). Since the eigenvalues \(z_1, \ldots, z_l\) of \(z\) are pairwise distinct, see (4.4.3), the maps \((\text{ad} \, X)^{12}\) and \((\text{ad} \, X)^{21}\) are both invertible.

Now, fix a pair \((Y, j) \in N_p\), as in (4.4.21). Write \(j = j^1 \oplus j^2\). Also, we have \(\ell = Ci, \) and \(i \in V^1\) is a cyclic vector for \(X^1\). Writing out the equation \([X, Y] + ij = 0\) block-by-block, we find

\[
(\text{ad} \, X)^{rs}(Y^{rs}) + i^r j^s = 0, \quad \forall r, s \in \{1, 2\}.
\]

Since \(i \in V^1\) and \(X^{12}\) is an invertible map, equation (4.4.11) forces \(Y^{12} = 0\). Therefore, to show that \(Y\) is nilpotent, it suffices to show that both \(Y^{11}\) and \(Y^{22}\) are nilpotent. Hence, we may ignore the block \(Y^{21}\). Replacing \(Y^{21}\) by zero does not affect equation (4.4.11) for diagonal blocks, so from now on we assume that \(Y = Y^{11} \oplus Y^{22}\).

From equation (4.4.11) for the block corresponding to \(\text{Hom}(V^2, V^2)\), using that \(i \in V^1\), we derive [\(X^2, Y^{22}\)] = 0. For the block corresponding to \(\text{Hom}(V^1, V^1)\), we use that \(i\) is a cyclic vector of the operator \(X^1\) and apply Lemma 2.1.3. We deduce that \(j^1 = 0\), hence equation (4.4.11) yields \([X^1, Y^{11}] = 0\). We see that \([X, Y] = 0\). This implies that \(Y\) commutes with \(z\), hence we get \(Y \in I\).

Thus, we are in the situation as at the end of Step 4. Specifically, since \((X, \ell)\) is relevant, from (4.4.7) we obtain that \((Y, j) \in \Lambda\). This completes the proof of the Theorem. \(\blacksquare\)

### 4.5 ‘Fourier dual’ description of irreducible components.

We recall the standard canonical isomorphism of symplectic manifolds:

\[
T^*(g \times \mathbb{P}) = g \times g^* \times T^* \mathbb{P} \xrightarrow{\text{symplectic\ conjugate}} g^* \times g \times T^* \mathbb{P} = T^*(g^* \times \mathbb{P}).
\]

Explicitly, using the identification \(g^* = g\) and formula (4.4.2), one rewrites isomorphism (4.5.1) in down-to-earth terms:

\[
T^*(g \times \mathbb{P}) = \{(X, Y, i, j) \in g \times g \times (V \setminus \{0\}) \times V^* \mid (j, i) = 0\}/\mathbb{C}^\times = T^*(g^* \times \mathbb{P}),
\]

where the matrix \(X\) is viewed as an element of \(g\) while the matrix \(Y\) is viewed as an element of \(g^*\).

We have vector bundle projections:

\[
p : T^*(g \times \mathbb{P}) \to g \times \mathbb{P}, \quad (X, Y, i, j) \mapsto (X, i); \quad p^\vee : T^*(g^* \times \mathbb{P}) \to g^* \times \mathbb{P}, \quad (X, Y, i, j) \mapsto (Y, j).
\]

Let \(\Lambda^\vee \subset T^*(g^* \times \mathbb{P})\) be the image of \(\Lambda \subset T^*(g \times \mathbb{P})\) under the canonical isomorphism in (4.5.1). It is clear that \(\Lambda^\vee\) is a closed \(G\)-stable Lagrangian subscheme of \(T^*(g^* \times \mathbb{P})\). Furthermore, if we view \(\mathcal{N} \times \mathbb{P}\) as a closed subset in \(g^* \times \mathbb{P}\) (rather than in \(g \times \mathbb{P}\) then, by definition of \(\Lambda\), we have \(p^\vee(\Lambda^\vee) \subset \mathcal{N} \times \mathbb{P}\).

**Lemma 4.5.3.** The irreducible components of \(\Lambda^\vee\) are the sets \(\overline{T^S(g^* \times \mathbb{P})}\), where \(S\) runs through the (finite) set of all \(G\)-diagonal orbits in \(\mathcal{N} \times \mathbb{P}\).

**Proof.** It is clear that, for any smooth \(G\)-stable locally closed subvariety \(S\) contained in \(\mathcal{N} \times \mathbb{P}\), we have \(\overline{T^S(g^* \times \mathbb{P})} \subset \Lambda^\vee\). Thus, since \(\Lambda^\vee\) is closed, we deduce that \(\overline{T^S(g^* \times \mathbb{P})} \subset \Lambda^\vee\), for any \(G\)-diagonal orbit \(S \subset \mathcal{N} \times \mathbb{P}\).

To prove the converse, observe that \(\Lambda^\vee\) is stable under dilations along the fibers of the projection \(p^\vee : T^*(g^* \times \mathbb{P}) \to g^* \times \mathbb{P}\), i.e., it is a Lagrangian cone-subvariety. But any irreducible
sections of the sheaf well-defined for any $A$ finitely generated.

Definition 5.2.1. For each $c \in \mathbb{C}$, let $\mathcal{O}_X(c)$ be the corresponding standard invertible sheaf on $\mathbb{P}$, and $\mathcal{O}_X(c)$ its pull-back via the second projection $X = g \times P \to P$. Write $\mathcal{D}_X(c)$ for the sheaf of algebraic twisted differential operators on $X$ acting on the sections of $\mathcal{O}_X(c)$. Although the sheaf $\mathcal{O}_X(c)$ exists for integral values of $c$ only, the corresponding sheaf $\mathcal{D}_X(c)$ is well-defined for any $c \in \mathbb{C}$, cf. [BB2]. We write $\mathcal{D}(X,c) := \Gamma(X, \mathcal{D}_X(c))$ for the algebra of global sections of the sheaf $\mathcal{D}_X(c)$.

The action of any element of the Lie algebra $g$ gives rise to a vector field on $g \times X$. Thus, we have a morphism of the Lie algebra $g$ into the Lie algebra of first order differential operators on $g \times X$. The latter morphism extends uniquely to a filtration preserving algebra homomorphism $\tau : \mathcal{U}g \to \mathcal{D}(g \times X)$. For each $c \in \mathbb{C}$, there is also a similar algebra homomorphism $\tau_c : \mathcal{U}g \to \mathcal{D}(X,c)$.

Let $1$ denote the identity matrix viewed as a base element in the center of the Lie algebra $g$. It is clear that the $\text{ad} 1$-action on $g$ is trivial and the action of $1$ on $V$ generates the $C^\infty$-action on $V$ by dilations. It follows from definitions that $\tau_c(1) = c$ for any $c \in \mathbb{C}$. Furthermore, the following canonical algebra isomorphisms are, in effect, both isomorphisms

$$\mathcal{D}(g \times X) / \mathcal{D}(g \times X) \cdot \tau(1) \cong (\mathcal{D}(g \times V) / \mathcal{D}(g \times V) \cdot (\tau(1) - c)) \cong \mathcal{D}(X,c). \quad (5.1.1)$$

Here, the first isomorphism is a special case of [K11] and the second isomorphism follows from the known description of the algebra $\Gamma(P, \mathcal{D}_P(c))$.

5.2 Let $Z$ be the algebra of Ad $G$-invariant constant coefficient differential operators on $g$. Thus, we have a natural algebra isomorphism $Z \cong (\text{Sym } g)^{\text{Ad } G}$. Write $Z_+$ for the augmentation ideal in $Z$ formed by differential operators without constant term. Now, any differential operator on $g$ may be identified with a differential operator on $g \times P$ that acts trivially along the $P$-factor. This way, we obtain an algebra map $i : Z \hookrightarrow \mathcal{D}(X,c)^G$.

Let $g = \mathbb{C} \cdot 1 \oplus \mathfrak{sl}(V)$ be an obvious Lie algebra direct sum decomposition. We put $g_c := \tau_c(\mathfrak{sl}(V)) \subset \mathcal{D}(X,c)$. Further, let $eu \in \mathcal{D}(X,c)$ denote a first order differential operator corresponding to the Euler vector field along the factor $g$ in the cartesian product $X = g \times P$. The differential operator $eu$ is clearly $G$-invariant, i.e., $eu$ commutes with $g_c$. We write $\mathcal{U}$ for the associative subalgebra in $\mathcal{D}(X,c)$ generated by $g_c$ and $eu$.

Motivated by [H1] and [La1, La2, La3], we are going to introduce a subcategory of the category of finitely generated left $\mathcal{D}(X,c)$-modules.

Definition 5.2.1. A finitely generated $\mathcal{D}(X,c)$-module $M$ is called admissible if the following holds:

- The action on $M$ of the subalgebra $i(Z_+)$ is locally-nilpotent, i.e., for any $u \in M$ there exists an integer $k = k(u) \gg 0$ such that $i(Z_+)^k u = 0$;
• The action on $M$ of the subalgebra $\mathcal{U}$ is locally-finite.

Let $\mathcal{C}_c$ be the full subcategory of $\mathcal{D}(\mathfrak{X}, c)$-$\text{mod}$ whose objects are admissible $\mathcal{D}(\mathfrak{X}, c)$-modules.

**Remark 5.2.2.** In [FG], we study the group analog of the notion of admissible $\mathcal{D}$-module. The space $\mathfrak{X} = \mathfrak{g} \times \mathbb{P}$ is replaced there by the space $\mathfrak{X}^G := G \times \mathbb{P}$. The group $G$ acts on itself by conjugation and this makes $\mathfrak{X}^G$ a $G$-variety with respect to diagonal action. This way, one gets an algebra map $\tau^G : \mathcal{U}(\mathfrak{g}_c) \rightarrow \mathcal{D}(\mathfrak{X}^G, c)$.

Further, let $Z^G \cong (\mathcal{U}\mathfrak{g})^{\text{Ad}G}$ be the algebra of bi-invariant differential operators on $G$ and $Z^G_+ := Z^G \cap (\mathfrak{g} \cdot \mathcal{U}\mathfrak{g})$ its augmentation ideal. One constructs similarly an algebra imbedding $\tau^G : Z^G \rightarrow \mathcal{D}(\mathfrak{X}^G, c)$. Thus, the two conditions of Definition [5.1.1] may be replaced by their $G$-counterparts, with $Z_+$ being replaced by $Z^G_+$, resp. $\mathcal{U}$ being replaced by $\mathcal{U}(\mathfrak{g}_c)$. This leads to the notion of admissible $\mathcal{D}(\mathfrak{X}^G, c)$-module (the extra condition of local finiteness of $\mathfrak{g}$-action has no group analogue and may be dropped from definition).

**Remark 5.2.3.** Recall that a (possibly infinite dimensional) $\mathfrak{g}_c$-module $M$ is said to be locally-finite if $\dim(\mathcal{U}(\mathfrak{g}_c) \cdot m) < \infty$, for any $m \in M$.

It follows from complete reducibility of finite dimensional $\mathfrak{sl}(V)$-modules that any locally finite $\mathfrak{g}_c$-module splits into (possibly infinite) direct sum of finite dimensional simple $\mathfrak{sl}(V)$-modules. Furthermore, the group $SL(V)$ being simply-connected, any finite dimensional $\mathfrak{sl}(V)$-module can be exponentiated to a rational representation of the algebraic group $SL(V)$. Thus, a locally finite $\mathfrak{g}_c$-module is the same thing as a (possibly infinite) direct sum of finite dimensional simple rational representations of the algebraic group $SL(V)$ (on which the central element $1$ acts as $c \cdot \text{Id}$).

### 5.3

The projective space $\mathbb{P}$ is a partial flag manifold for $G$, hence, Beilinson-Bernstein theorem [BB1] holds for $\mathbb{P}$. The space $\mathfrak{g}$ being affine, from [BB1] we deduce

**Proposition 5.3.1.** For any $c \geq 0$, the functor of global sections provides an equivalence between the abelian categories of $\mathcal{D}_X(c)$-coherent sheaves and of finitely-generated $\mathcal{D}(\mathfrak{X}, c)$-modules, respectively.

Given a $\mathcal{D}_X(c)$-coherent sheaf $\mathcal{L}$, we write $\text{Supp} \mathcal{L}$ for its support as an $\mathcal{O}_X$-module, and $\text{Ch} \mathcal{L}$ for its characteristic cycle, a $C^\infty$-stable algebraic cycle in $T^*X$. Abusing the notation, we will also write $\text{Ch} \mathcal{L}$ for the set-theoretic support of the algebraic cycle $\text{Ch} \mathcal{L}$. Thus, we have $p(\text{Ch} \mathcal{L}) = \text{Supp} \mathcal{L}$, where $p : T^*X \rightarrow X$ is the projection.

Using Proposition 5.3.1 we will often identify an object of the category $\mathcal{C}_c$ with a $\mathcal{D}_X(c)$-coherent sheaf. Thus, we write $\text{Ch} \mathcal{L}$ for the characteristic cycle, resp., $\text{Supp} \mathcal{L}$ for the support, of the $\mathcal{D}_X(c)$-coherent sheaf corresponding to an object $\mathcal{L} \in \mathcal{C}_c$.

**Proposition 5.3.2.** Let $\mathcal{M}$ be a finitely generated $\mathcal{D}(\mathfrak{X}, c)$-module which is locally finite as a $\mathcal{U}$-module. Then, we have $\mathcal{M} \in \mathcal{C}_c$, i.e. if and only if $\text{Ch} \mathcal{M} \subset \Lambda$.

Furthermore, any object of $\mathcal{C}_c$ is a regular holonomic (twisted) $\mathcal{D}$-module which is smooth along the strata $\mathfrak{X}(I, \Omega)$, $(I, \Omega) \in \mathcal{I}$.

**Proof.** We will exploit the theory of Fourier transform of $\mathcal{D}$-modules, see [Br].

Recall that the Fourier transform gives an equivalence

$$F_\mathcal{D} : \mathcal{D}(\mathfrak{g} \times \mathbb{P}, c)-\text{mod} \rightarrow \mathcal{D}(\mathfrak{g}^* \times \mathbb{P}, c)-\text{mod}, \quad \mathcal{M} \mapsto F_\mathcal{D} \mathcal{M}, \quad (5.3.3)$$
from the category of finitely generated (twisted) \( \mathcal{D} \)-modules on the total space of the (trivial) vector bundle \( \mathcal{F} = g \times \mathbb{P} \to \mathbb{P} \) to a similar category of finitely generated twisted \( \mathcal{D} \)-modules on the total space of the dual vector bundle \( g^* \times \mathbb{P} \to \mathbb{P} \).

We will generalize earlier notation slightly, and write \( \mathfrak{eu} \) for the Euler vector field along the fibers of any vector bundle, e.g., along the fibers of the vector bundle \( g \times \mathbb{P} \to \mathbb{P} \) or of \( g^* \times \mathbb{P} \to \mathbb{P} \). Recall that a (twisted) \( \mathcal{D} \)-module on a vector bundle is said to be \textit{monodromic} if it is locally finite as an \( \mathfrak{eu} \)-module.

It is known that the Fourier transform functor acts especially nicely on monodromic modules. In particular, it takes monodromic (twisted) \( \mathcal{D} \)-modules into monodromic (twisted) \( \mathcal{D} \)-modules. Furthermore, it is known (see [Br]) that given a monodromic (twisted) \( \mathcal{D} \)-module \( M \), one has:

- \( \text{Ch}(M) = \text{Ch}(\mathcal{F}_g M) \) under the canonical isomorphism in \( \mathcal{D} \).
- \( M \) has regular singularities \( \iff \mathcal{F}_g M \) has regular singularities.

From now on, we fix an \( \mathcal{U} \)-locally finite, finitely generated \( \mathcal{D}(\mathcal{X}, c) \)-module \( M \) such that \( \text{Ch}(M) \subset \Lambda \), and let \((\text{Sym} g)^\mathfrak{ad} G \) be the set of \( G \)-invariant polynomials on \( g^* \cong g \) without constant term.

It is immediate from the equation \( \text{Ch}(M) = \text{Ch}(\mathcal{F}_g M) \) and definition of \( \Lambda \) that we have \( \text{Supp}(\mathcal{F}_g M) \subset \mathcal{N} \times \mathbb{P} \). Any element of \((\text{Sym} g)^\mathfrak{ad} G\) vanishes on \( \mathcal{N} \). Hence, the action of \((\text{Sym} g)^\mathfrak{ad} G\) on \( \mathcal{F}_g M \) is locally nilpotent. It follows that the action of the subalgebra \( \mathfrak{i}(\mathbb{Z}_+^*) \) on \( M \) is also locally nilpotent. This proves the implication \( \text{Ch} M \subset \Lambda \implies M \in \mathcal{C}_c \).

Next, we have

\textbf{Claim 5.3.4.} Any \( \mathcal{U}g \)-locally finite, finitely generated \( \mathcal{D}(\mathcal{X}, c) \)-module \( M \) such that \( \text{Supp} M \subset \mathcal{N} \times \mathbb{P} \) is automatically a holonomic module with regular singularities.

\textit{Proof of Claim.} As we have explained in Remark 5.2.3, any \( \mathcal{U}(g_\mathbb{C}) \)-locally finite \( \mathcal{D}(\mathcal{X}, c) \)-module is automatically \( SL(V) \)-equivariant. It is clear that any \( G \)-diagonal orbit \( S \subset \mathcal{N} \times \mathbb{P} \) is also a single \( SL(V) \)-orbit. It follows that the algebraic variety \( \mathcal{N} \times \mathbb{P} \) is partitioned into finitely many \( SL(V) \)-orbits, by Corollary 2.1.2. But it is a well-known result that if \( H \) is a linear algebraic group and \( \mathcal{X} \) is an arbitrary \( H \)-variety with finitely many \( H \)-orbits then, any \( H \)-equivariant \( \mathcal{D} \)-module on \( \mathcal{X} \) is a holonomic module with regular singularities. This proves that, for any \( \mathcal{U} \)-locally finite, finitely generated \( \mathcal{D}(\mathcal{X}, c) \)-module \( M \) such that \( \text{Ch}(M) \subset \Lambda \), the \( \mathcal{D} \)-module \( \mathcal{F}_g M \) has regular singularities. Therefore, by the properties of Fourier transform mentioned above, \( M \) also has regular singularities. This proves Claim 5.3.4. \( \square \)

To complete the proof of the Proposition, we must show that \( M \in \mathcal{C}_c \implies \text{Ch} M \subset \Lambda \). This is entirely analogous to the proof of implication (i) \( \implies \) (ii) in [Gi1], Theorem 1.4.2. We leave details to the reader. \( \square \)

\textbf{Corollary 5.3.5. \( \mathcal{C}_c \) is an abelian, artinian category with finitely many simple objects; in particular, every object of \( \mathcal{C}_c \) has finite length.} \( \square \)

\textbf{Remark 5.3.6.} There is a natural group analog of the Lagrangian subscheme \( \Lambda \subset T^*(g \times \mathbb{P}) \). It is a Lagrangian subscheme \( \Lambda^G \subset T^*(G \times \mathbb{P}) \) defined by

\[ \Lambda^G = \{(g,Y,i,j) \in G \times g \times (V \setminus \{0\}) \times V^* \mid \text{Ad}_g(Y) - Y + ij = 0 \& \ Y \text{ is nilpotent}\}/\mathbb{C}^\times. \]

One can then formulate and prove a group version of Proposition 5.3.2 for \( \mathcal{D}(\mathcal{X}^G, c) \)-modules, cf. Remark 5.3.2. \( \square \)
6 Cherednik algebra and Hamiltonian reduction.

6.1 Reminder. Recall the permutation representation of functions from $\mathfrak{h} = \mathbb{C}^n$ of $W = S_n$, the Symmetric group. We write $y_1, \ldots, y_n$ for the standard basis of $\mathfrak{h}$, and $x_1, \ldots, x_n \in \mathfrak{h}^*$ for the corresponding dual coordinate functions. Let $s_{ij} \in S_n$ denote the transposition $i \leftrightarrow j$.

Let $c \in \mathbb{C}$. The rational Cherednik algebra of type $A_{n-1}$, as defined in [EG], is an associative $\mathbb{C}$-algebra $H_c$ with generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ and the group $S_n$, and the following defining relations:

$$s_{ij} \cdot x_i = x_j \cdot s_{ij}, \quad s_{ij} \cdot y_i = y_j \cdot s_{ij}, \quad \forall i, j \in \{1, 2, \ldots, n\}, \ i \neq j$$

$$[y_i, x_j] = c \cdot s_{ij}, \quad [x_i, x_j] = 0 = [y_i, y_j], \quad \forall i, j \in \{1, 2, \ldots, n\}, \ i \neq j$$

$$[y_k, x_k] = 1 - c \cdot \sum_{i \neq k} s_{ik}.$$  \hspace{1cm} (6.1.1)

In [EG] we have mentioned the spherical subalgebra $\mathfrak{e}H_c \subset H_c$. The assignment $a \mapsto a \cdot e = e \cdot a$ gives algebra imbeddings $(\text{Sym } \mathfrak{h})^W = \mathbb{C}[y_1, \ldots, y_n]^W \hookrightarrow \mathfrak{e}H_c \subset H_c$ and $\mathbb{C}[\mathfrak{h}]^W = \mathbb{C}[x_1, \ldots, x_n]^W \hookrightarrow \mathfrak{e}H_c$. We will identify the algebras $(\text{Sym } \mathfrak{h})^W$ and $\mathbb{C}[\mathfrak{h}]^W$ with their images in $\mathfrak{e}H_c$. It is known that these two algebras generate $\mathfrak{e}H_c \subset H_c$ as an algebra.

It is immediate to see from the defining relations (6.1.1) that the following assignment

$$w \mapsto w \ (\forall w \in S_n), \quad x_i \mapsto y_i, \quad y_i \mapsto -x_i, \quad \forall i = 1, \ldots, n,$$  \hspace{1cm} (6.1.2)

extends to an algebra automorphism $H_c \to H_c$, called Fourier automorphism, cf. [EG] §7. By restriction, we also get the automorphism $\mathfrak{e}H_c \to \mathfrak{e}H_c$ and, by transport of structure, an auto-equivalence $\mathbb{F}_H : \mathfrak{e}H_c - \text{mod} \to \mathfrak{e}H_c - \text{mod}$, called Fourier transform functor.

The Cherednik algebra has an increasing filtration such that all elements of $S_n$ as well as the generators $x_1, \ldots, x_n \in H_c$ have filtration degree zero, and the generators $y_1, \ldots, y_n \in H_c$ have filtration degree 1. We equip $\mathfrak{e}H_c$ with the induced filtration. Then, the Poincaré-Birkhoff-Witt theorem for Cherednik algebras, see [EG], yields a graded algebra isomorphism

$$\text{gr}(\mathfrak{e}H_c) \cong \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^W.$$  \hspace{1cm} (6.1.3)

The rest of this section is devoted to the proof of our main result, Theorem 1.1.3.1.

6.2 We have the moment map $\mu_x : T^* X \to \mathfrak{g}^*$ and its scheme-theoretic zero fiber:

$$\mu_x^{-1}(0) = \{(X, Y, i, j) \in \mathfrak{g} \times \mathfrak{g} \times (V \setminus \{0\}) \times V^* \mid [X, Y] + ij = 0\}/\mathbb{C}^*.$$  

(as has been explained in [EG] this is indeed a subscheme in $T^* X$ since the equation $[X, Y] + ij = 0$ implies $(j, i) = 0$). It is clear that $\mu_x^{-1}(0)$ is a closed $G$-stable subscheme of $T^* X$, furthermore, Theorem 1.1.2(iii) implies that $\mu_x^{-1}(0)$ is a reduced complete intersection. By definition, restriction of functions from $T^* X$ to $\mu_x^{-1}(0)$ gives an algebra isomorphism:

$$\mathbb{C}[T^* X]/\mathbb{C}[T^* X] \cdot \mu_x^*(\mathfrak{g}_c) = \mathbb{C}[\mu_x^{-1}(0)].$$  \hspace{1cm} (6.2.1)

where the Lie algebra $\mathfrak{g}_c$ is identified with the vector space of linear functions on $\mathfrak{g}_c$.

Lemma 6.2.2. There is a graded algebra isomorphism

$$\mathbb{C}[\mu_x^{-1}(0)]^G = (\mathbb{C}[T^* X]/\mathbb{C}[T^* X] \cdot \mu_x^*(\mathfrak{g}_c))^G \cong \mathbb{C}[\mathfrak{h} \times \mathfrak{h}]^W.$$
Proof. We write $O_1 \subset \mathfrak{g}$ for the conjugacy class of rank one nilpotent matrices, and let $\overline{O}_1 = O_1 \cup \{0\}$ be the closure of $O_1$ in $\mathfrak{g}$. The moment map $\pi : T^*P \to \mathfrak{g}^* = \mathfrak{g}$, $(i, j) \mapsto ij$, see formula (6.2.1), gives a birational isomorphism $T^*P \longrightarrow \overline{O}_1$.

Let $I_1 \subset \mathbb{C}[\mathfrak{g}] = \mathbb{C}[Z]$ be the ideal generated by all the $2 \times 2$ minors of the matrix $Z$ and also by the linear function $Z \mapsto \text{Tr} Z$. This is known to be a prime ideal whose zero scheme equals $\overline{O}_1$. Furthermore, the pull-back morphism $\pi^* : \mathbb{C}[\mathfrak{g}] / I_1 = \mathbb{C}[\overline{O}_1] \to \mathbb{C}[T^*P]$ is known to be a graded algebra isomorphism.

Next, we write $T^*X = T^*\mathfrak{g} \times T^*P$. It is clear that the moment map $\mu_x : T^*X \to \mathfrak{g}^*$ may be factored as the following composite map:

$$T^*X = T^*\mathfrak{g} \times T^*P \overset{\pi_x}{\longrightarrow} \mathfrak{g} \times \mathfrak{g}^* \times \overline{O}_1 = \mathfrak{g} \times \mathfrak{g} \times \overline{O}_1 \overset{\theta}{\longrightarrow} \mathfrak{g}^* = \mathfrak{g}^*,$$  

(6.2.3)

where $\pi_x := \text{Id}_{T^*\mathfrak{g}} \times \pi$, and where the map $\theta$ is given by $\mathfrak{g} \times \mathfrak{g} \times \overline{O}_1 \ni (X, Y, Z) \mapsto [X, Y] + Z$. Observe that, by the last sentence of the preceding paragraph, the map $\pi_x$ induces a graded algebra isomorphism

$$\pi^*_x : \mathbb{C} [\mathfrak{g} \times \mathfrak{g}] \otimes (\mathbb{C}[\mathfrak{g}] / I_1) \cong \mathbb{C} [\mathfrak{g} \times \mathfrak{g} \times \overline{O}_1] \longrightarrow \mathbb{C}[T^*X].$$  

(6.2.4)

Write $\mathbb{C}[X, Y, Z] := \mathbb{C}[\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}]$, and let $\mathbb{C}[X, Y, Z][([X, Y] + Z)$ denote the ideal in $\mathbb{C}[X, Y, Z]$ generated by all matrix entries of the matrix $[X, Y] + Z$. We also consider a larger ideal $I := \mathbb{C}[X, Y] \otimes I_1 + \mathbb{C}[X, Y, Z][([X, Y] + Z) \subset \mathbb{C}[X, Y, Z]$. Thus, from (6.2.3) and (6.2.4) we find

$$\mathbb{C}[T^*X] / \mathbb{C}[T^*X] \cdot \mu^*_x(g) \cong \mathbb{C}[\mathfrak{g} \times \mathfrak{g} \times \overline{O}_1] / \mathbb{C}[\mathfrak{g} \times \mathfrak{g} \times \overline{O}_1] \cdot \theta^*(g) = \mathbb{C}[X, Y, Z] / I.$$

We define an algebra homomorphism $r : \mathbb{C}[X, Y, Z] \to \mathbb{C}[X, Y]$ to be the map sending a polynomial $P \in \mathbb{C}[X, Y, Z]$ to the function $(X, Y) \mapsto P(X, Y, -[X, Y])$. The homomorphism $r$ clearly induces an isomorphism $\mathbb{C}[X, Y, Z] / \mathbb{C}[X, Y, Z][([X, Y] + Z) \cong \mathbb{C}[X, Y]$. Observe that the linear function $P : (X, Y, Z) \mapsto \text{Tr} Z = \text{Tr}([X, Y] + Z)$ belongs to the ideal $\mathbb{C}[X, Y, Z][([X, Y] + Z)$. We deduce that the map $r$ sends the subspace $\mathbb{C}[X, Y] \otimes I_1 \subset \mathbb{C}[X, Y, Z]$ to $I_1 \subset \mathbb{C}[X, Y]$, the ideal $I_1$ considered in Section 2.7. Thus, we obtain algebra isomorphisms

$$\mathbb{C}[T^*X] / \mathbb{C}[T^*X] \cdot \mu^*_x(g) \cong \mathbb{C}[X, Y, Z] / I \cong \mathbb{C}[X, Y] / I_1.$$

The statement of the Lemma is now immediate from isomorphisms (6.2.3). \hfill \square

6.3 The Dunkl homomorphism. Let $\mathfrak{h}^{ve} \subset \mathfrak{h} = \mathbb{C}^n$ be the Zariski open dense subset formed by $n$-tuples with pairwise distinct coordinates. The Dunkl homomorphism is an algebra imbedding $\Theta : \mathfrak{e}_H, e \hookrightarrow \mathfrak{g}(\mathfrak{h}^{ve})^W$ defined as follows, see e.g. [EG].

Write $\mathbb{C}[\mathfrak{h}^*] / \mathfrak{w}$ for the smash product algebra, and let $\text{triv} : \mathbb{C}[\mathfrak{h}^*] / \mathfrak{w} \to \mathbb{C}$ be the homomorphism that sends every element $w \in \mathfrak{w}$ to $1$, and acts on the polynomial algebra $\mathbb{C}[\mathfrak{h}^*]$ by $f \mapsto f(0)$. We view $\mathbb{C}[\mathfrak{h}^*] / \mathfrak{w}$ as a subalgebra of $H_c$ and let $H_c \otimes \mathbb{C}[\mathfrak{h}^*] / \mathfrak{w} \text{triv}$ be the induced left $H_c$-module.

The natural imbedding $\mathbb{C}[\mathfrak{h}] \to H_c$ yields, by the Poincaré-Birkhoff-Witt theorem for the Chevènement algebra $H_c$, cf. [EG], a vector space isomorphism $\mathbb{C}[\mathfrak{h}] \cong H_c \otimes \mathbb{C}[\mathfrak{h}^*] / \mathfrak{w} \text{triv}$. The left $H_c$-action on $H_c \otimes \mathbb{C}[\mathfrak{h}^*] / \mathfrak{w} \text{triv}$ gets transported, via the isomorphism, to an $H_c$-action on $\mathbb{C}[\mathfrak{h}]$. The resulting action of the spherical subalgebra $\mathfrak{e}_H, e$ preserves the subspace $\mathbb{C}[\mathfrak{h}] / \mathfrak{w} = e \cdot \mathbb{C}[\mathfrak{h}]$, of symmetric polynomials. Moreover, a direct calculation shows that, for any $u \in \mathfrak{e}_H, e$, the
corresponding action-map \( u : \mathbb{C}[h]^W \to \mathbb{C}[h]^W \) is given by a \( W \)-invariant differential operator, \( \Theta(u) \), with rational coefficients. More precisely, all coefficients of the differential operator \( \Theta(u) \) turn out to be regular functions on \( \mathfrak{h}^\text{reg} \).

The assignment \( u \mapsto \Theta(u) \) gives the desired Dunkl homomorphism \( \Theta : \mathfrak{h}.e \to \mathcal{D}(\mathfrak{h}^\text{reg})^W \).

This is a filtration preserving injective algebra homomorphism, and we let \( \text{gr} \Theta \) denote the corresponding associated graded map. The composite map

\[
\mathbb{C}[h \times h^*]^W = \text{gr}(\mathcal{E}_c) \xrightarrow{\text{gr} \Theta} \text{gr}(\mathcal{D}(\mathfrak{h}^\text{reg})^W) = \mathbb{C}[h^\text{reg} \times h^*]^W
\]

is known to be equal to the natural restriction map \((j \times \text{Id})^* : \mathbb{C}[h \times h^*]^W \to \mathbb{C}[h^\text{reg} \times h^*]^W\), induced by the open embedding \( j : \mathfrak{h}^\text{reg} \to \mathfrak{h} = \mathbb{C}^n \).

### 6.4 The radial part map.

Recall that \( g_c \) denotes the image in \( \mathcal{D}(\mathfrak{X}, c) \) of the Lie subalgebra \( \mathfrak{s}(V) \subset \mathfrak{g} \), and let \( \mathcal{D}(\mathfrak{X}, c) \cdot g_c \subset \mathcal{D}(\mathfrak{X}, c) \) be the left ideal generated by \( g_c \).

Using (6.1.1), for any \( c \in \mathbb{C} \), one obtains

\[
(\mathcal{D}(\mathfrak{X}, c)/\mathcal{D}(\mathfrak{X}, c) \cdot g_c)^{\text{ad} g} \cong \left( \mathcal{D}(g \times V)/\mathcal{D}(g \times V) \cdot (\tau(1) - c) + \mathcal{D}(g \times V) \cdot \tau(\mathfrak{sl}(V)) \right)^{\text{ad} g}.
\]

Here, \( \text{ad} g \)-invariants are taken with respect to the ‘adjoint’ action defined by the formula \( \text{ad} g : u \mapsto \tau(g) \cdot u - u \cdot \tau(g) \), \( \forall g \in \mathfrak{g} \) (or a similar formula with \( \tau_c \) instead of \( \tau \)). The object on each side of (6.4.1) is the result of a certain Hamiltonian reduction, see [7.1] below and \[BFG\] §3.4. Thus, each side in (6.4.1) acquires a natural algebra structure and the isomorphism in (6.4.1) is an algebra isomorphism. In this paper, we will only work with the algebra on the left hand side of (6.4.1). Thus, we will neither use nor prove the isomorphism in (6.4.1).

We now recall the construction, due to \[BFG\] Proposition 5.3.6], of the following filtered algebra homomorphism, called ‘radial part’ map

\[
\Psi_c : (\mathcal{D}(\mathfrak{X}, c)/\mathcal{D}(\mathfrak{X}, c) \cdot g_c)^{\text{ad} g_c} \longrightarrow \mathcal{D}(\mathfrak{h}^\text{reg})^W, \quad \text{such that}
\]

\[
\Theta(\mathbb{C}[h]^W) = \Psi_c(\mathbb{C}[g]^G), \quad \text{and} \quad \Theta((\text{Sym}\mathfrak{h})^W) = \Psi_c(\mathbb{Z}).
\]

Let \( \mathfrak{g}^\text{reg} \subset \mathfrak{G} = \mathfrak{g} \times V \) be the subset formed by the pairs \((x, v)\) such that \( v \) is a cyclic vector for the operator \( x : V \to V \). It is clear that \( \mathfrak{G}^\text{reg} \) a \( G \)-stable, Zariski open dense subset of \( \mathfrak{G} \). We compose the first projection \( \mathfrak{G} = \mathfrak{g} \times V \to \mathfrak{g} \) with the adjoint quotient map \( \mathfrak{g} \to \mathfrak{g}/\text{Ad} G = \mathfrak{h}/W \), and restrict the resulting morphism to the subset \( \mathfrak{G}^\text{reg} \subset \mathfrak{G} \). This way we get a morphism \( p : \mathfrak{G}^\text{reg} \to \mathfrak{h}/W \). It turns out that the group \( G \) acts freely along the fibers of \( p \), and this makes the map \( p : \mathfrak{G}^\text{reg} \to \mathfrak{h}/W \) a principal \( G \)-bundle over \( \mathfrak{h}/W \), cf. \[BFG\] Lemma 5.3.3).

Let \( \mathfrak{X}^\text{reg} \) be the image of \( \mathfrak{G}^\text{reg} \) under the projection \( \mathfrak{g} \times (V \setminus \{0\}) \to \mathfrak{g} \times \mathbb{P} \). The map \( p \) descends to \( \mathfrak{X}^\text{reg} \) and makes it a principal \( PGL(V) \)-bundle on \( \mathfrak{h}/W \). Now, the standard description of differential operators on the base of a principal bundle in terms of those on the total space of the bundle yields an algebra isomorphism

\[
(\mathcal{D}(\mathfrak{X}^\text{reg}, c)/\mathcal{D}(\mathfrak{X}^\text{reg}, c) \cdot g_c)^G \xrightarrow{\text{res}} \mathcal{D}(\mathfrak{h}/W), \quad \forall c \in \mathbb{C}.
\]

We have (strict) inclusions \( \mathcal{D}(\mathfrak{h})^W \subset \mathcal{D}(\mathfrak{h}/W) \subset \mathcal{D}(\mathfrak{h}^\text{reg})^W \). The map \( \Psi_c \) in (6.4.2) is defined as the following composite homomorphism

\[
\xrightarrow{\text{restriction}} (\mathcal{D}(\mathfrak{X}^\text{reg}, c)/\mathcal{D}(\mathfrak{X}^\text{reg}, c) \cdot g_c)^G \xrightarrow{\text{res}} \mathcal{D}(\mathfrak{h}/W) \hookrightarrow \mathcal{D}(\mathfrak{h}^\text{reg})^W.
\]

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More explicitly, fix a nonzero volume element $v^* \in \wedge^n V^*$ and let
$$(x, v) \mapsto s(x, v) := \langle v^*, v \wedge x(v) \wedge \ldots \wedge x^{n-1}(v) \rangle,$$
a polynomial function on $\mathfrak{g}$. Further, for any integer $c \in \mathbb{Z}$, we put
$$\mathcal{O}(\mathfrak{g}^{reg}, c) := \{ f \in \mathbb{C}[\mathfrak{g}^{reg}] \mid g^*(f) = (\det g)^c \cdot f, \ \forall g \in G \}.$$ 
Observe that $s \in \mathcal{O}(\mathfrak{g}^{reg}, 1)$ and $\mathfrak{g}^{reg} = \mathfrak{g} \setminus s^{-1}(0)$, in particular, $\mathfrak{g}^{reg}$ is an affine variety.

It is clear that pull-back via the bundle projection $p: \mathfrak{g}^{reg} \to \mathfrak{h}/W$ makes the vector space $\mathcal{O}(\mathfrak{g}^{reg}, c)$ a $\mathbb{C}[\mathfrak{h}/W]$-module. Furthermore, one shows that this is in effect a rank one free $\mathbb{C}[\mathfrak{h}/W]$-module with generator $s^c$, so one has a bijection $\mathbb{C}[\mathfrak{h}/W] \xrightarrow{\sim} \mathcal{O}(\mathfrak{g}^{reg}, c), f \mapsto s^c \cdot p^*(f)$. The isomorphism in (6.4.3) is obtained by transporting the action of differential operators on $\mathfrak{g}^{reg}$ via the bijection.

6.5 Proof of Theorem 1.3.1 Our argument follows the strategy of [BRC, §5] (that, in its turn, is based on an argument from [EC]), and we will freely use the notation from [BRC].

STEP 1. We claim that the image of $\Theta$ is contained in the image of $\Psi_c$, i.e., we have
$$\Theta(e_{\mathbb{H},c}) \subseteq \Psi_c((\mathcal{D}(\mathfrak{X}, c)/\mathcal{D}(\mathfrak{X}, c) \cdot g_c)^{ad \cdot g_c}). \quad (6.5.1)$$
To see this, recall that the two subalgebras $\mathbb{C}[\mathfrak{h}]^W, (\text{Sym } \mathfrak{h})^W \subset e_{\mathbb{H},c}$ generate $e_{\mathbb{H},c}$ as an algebra. It follows that $\Theta(\mathbb{C}[\mathfrak{h}]^W)$ and $\Theta(\text{Sym } \mathfrak{h})^W$ generate $\Theta(e_{\mathbb{H},c})$ as an algebra. Thus, equations (6.5.2) yield the inclusion in (6.5.1).

From (6.5.1) we deduce an imbedding of the corresponding associated graded algebras:
$$\text{gr } \Theta(e_{\mathbb{H},c}) \hookrightarrow \text{gr } \Psi_c((\mathcal{D}(\mathfrak{X}, c)/\mathcal{D}(\mathfrak{X}, c) \cdot g_c)^{ad \cdot g_c}), \quad (6.5.2)$$

STEP 2. For any smooth manifold $\mathcal{Y}$, the sheaf $\mathcal{D}_Y$, of differential (or twisted differential) operators on $\mathcal{Y}$, comes equipped with the standard increasing filtration by the order of differential operator. Let $\mathcal{D}(\mathcal{Y}) = \Gamma(\mathcal{Y}, \mathcal{D}_Y)$ and write gr $\mathcal{D}(\mathcal{Y})$ for the associated graded algebra. The principal symbol map provides a canonical graded algebra imbedding gr $\mathcal{D}(\mathcal{Y}) \hookrightarrow \mathbb{C}[T^* \mathcal{Y}]$.

In the special case $\mathcal{Y} = \mathfrak{g}$, the imbedding gr $\mathcal{D}(\mathfrak{g}) \hookrightarrow \mathbb{C}[\mathfrak{g} \times \mathfrak{g}^*]$ is clearly an isomorphism. Also, for $\mathcal{Y} := \mathbb{P} = \mathbb{P}(V)$, the principal symbol map gr $\mathcal{D}(\mathbb{P}, c) \hookrightarrow \mathbb{C}[T^* \mathbb{P}]$ is well known to be an isomorphism. We deduce that for $\mathfrak{X} = \mathfrak{g} \times \mathbb{P}$ the principal symbol map yields an isomorphism gr $\mathcal{D}(\mathfrak{X}, c) \hookrightarrow \mathbb{C}[T^* \mathfrak{X}]$.

The imbedding $g_c \hookrightarrow \mathcal{D}(\mathfrak{X}, c)$ extends, by multiplicativity, to a filtered algebra map $\mathcal{U}(g_c) \to \mathcal{D}(\mathfrak{X}, c)$. This algebra map induces an associated graded homomorphism
$$\text{Sym } g_c = \text{gr } \mathcal{U}(g_c) \to \text{gr } \mathcal{D}(\mathfrak{X}, c) = \mathbb{C}[T^* \mathfrak{X}].$$
The latter homomorphism is well known to be the pull-back via the moment map $\mu_\chi : T^* \mathfrak{X} \to \mathfrak{g}_c^*$. Hence, in $\mathbb{C}[T^* \mathfrak{X}] = \text{gr } \mathcal{D}(\mathfrak{X}, c)$, we have $\mathbb{C}[T^* \mathfrak{X}] \cdot \mu_\chi^*(g_c) = (\text{gr } \mathcal{D}(\mathfrak{X}, c)) \cdot g_c \subseteq \text{gr } (\mathcal{D}(\mathfrak{X}, c) \cdot g_c)$.

Thus, using (6.2.1), we obtain the following chain of graded algebra morphisms
$$\mathbb{C}[\mu_\chi^{-1}(0)] = \mathbb{C}[T^* \mathfrak{X}] / \mathbb{C}[T^* \mathfrak{X}] \cdot \mu_\chi^*(g_c) = \text{gr } \mathcal{D}(\mathfrak{X}, c) / (\text{gr } \mathcal{D}(\mathfrak{X}, c) \cdot g_c) \xrightarrow{\text{proj}} \text{gr } (\mathcal{D}(\mathfrak{X}, c) / \mathcal{D}(\mathfrak{X}, c) \cdot g_c).$$
Restricting these morphisms to \( \text{ad} \mathfrak{g}_c \)-invariants and using \( (6.4) \), we obtain the following graded algebra morphisms

\[
\mathbb{C}[\mu^{-1}_x(0)]^{\text{ad} \mathfrak{g}_c} = \left( \text{gr} \mathcal{D}(X, c) / (\text{gr} \mathcal{D}(X, c) \cdot \mathfrak{g}_c) \right)^{\text{ad} \mathfrak{g}_c} \xrightarrow{\text{proj}} \text{gr} \left( (\mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c))^{\text{ad} \mathfrak{g}_c} \right). \tag{6.5.3}
\]

**Step 3.** Let \( \text{Id}_h \times \kappa : \mathbb{C}[h \times h^*]^W \rightarrow \mathbb{C}[h \times h]^W \) be the algebra isomorphism arising from the bijection \( \kappa : h^* \rightarrow h \) induced by the trace pairing on \( \mathfrak{g} \). Thus, we obtain the following diagram:

\[
\begin{array}{ccc}
\mathbb{C}[h \times h^*]^W & \xrightarrow{\text{Id}_h \times \kappa} & \mathbb{C}[h \times h]^W \\
\text{gr}(\mathfrak{e}H, \mathfrak{e}) & \xrightarrow{\Theta} & \text{gr} \Psi_c \left( (\mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c))^{\text{ad} \mathfrak{g}_c} \right) \\
\text{gr} \Theta(\mathfrak{e}H, \mathfrak{e}) & \xrightarrow{\Theta} & \text{gr} \Psi_c \left( (\mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c))^{\text{ad} \mathfrak{g}_c} \right)
\end{array}
\]

Now, it is straightforward to verify that this diagram of graded algebra maps *commutes*. This forces both surjections in the diagram, as well as the injective map \( (6.5.2) \) in the bottom row of the diagram, all to be *bijective*. Therefore, we deduce the following isomorphisms:

\[
\text{gr} \left( \mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c) \right)^{\text{ad} \mathfrak{g}_c} \cong \text{gr} \Psi_c \left( (\mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c))^{\text{ad} \mathfrak{g}_c} \right) \cong \text{gr} \Theta(\mathfrak{e}H, \mathfrak{e}) \cong \text{gr}(\mathfrak{e}H, \mathfrak{e}), \tag{6.5.4}
\] where the equality indicated as \( \psi \) holds inside the bigger algebra \( \text{gr} \mathcal{D}(h^*_{\text{reg}})^W \).

As has been proved earlier, the algebra \( \Theta(\mathfrak{e}H, \mathfrak{e}) \) is contained in \( \Psi_c \left( (\mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c))^{\text{ad} \mathfrak{g}_c} \right) \). The equality \( \psi \) may be obtained as the associated graded map corresponding to the imbedding \( \Theta(\mathfrak{e}H, \mathfrak{e}) \rightarrow \Psi_c \left( (\mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c))^{\text{ad} \mathfrak{g}_c} \right) \). It follows that the imbedding itself is, in effect, an equality (that holds in the larger algebra \( \mathcal{D}(h^*_{\text{reg}})^W \)). Hence, we may invert the (injective) map \( \Theta \) and define a graded algebra morphism \( \Phi_c \) as the following composite map, very similar to the one used in \( [BG] \):

\[
\Phi_c : (\mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c))^{\text{ad} \mathfrak{g}_c} \xrightarrow{\Psi_c} \Psi_c \left( (\mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c))^{\text{ad} \mathfrak{g}_c} \right) \xrightarrow{\Theta^{-1}} \mathfrak{e}H, \mathfrak{e}.
\]

It is immediate from \( (6.5.4) \) that the corresponding associated graded map gives a bijection \( \text{gr} \Phi_c : \text{gr} \left( \mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c) \right) \cong \text{gr}(\mathfrak{e}H, \mathfrak{e}) \). Thus, the map \( \Phi_c \) is itself a bijection. The theorem is proved.

**Corollary 6.5.5.** The projection \( \text{proj} \) in \( (6.5.2) \) is a bijection; in particular, one has a graded algebra isomorphism \( \text{gr} \left( (\mathcal{D}(X, c) / (\mathcal{D}(X, c) \cdot \mathfrak{g}_c))^{\text{ad} \mathfrak{g}_c} \right) \cong \mathbb{C}[\mu^{-1}_x(0)]^G \).

**6.6 An application to [BG].** We fix \( c \in \mathbb{C} \) and use other notation of previous sections. The natural \( GL(V) \)-action on \( P = P(V) \) gives an algebra homomorphism \( \mathcal{U}\mathfrak{g} \rightarrow \mathcal{D}(P, c) \). The kernel of this homomorphism is known to be a *primitive* ideal \( \mathcal{I} \subset \mathcal{U}\mathfrak{g} \), moreover, it is exactly the primitive ideal considered in \([BG]\).
The group \( G = GL(V) \) acts on \( \mathfrak{g} \) via the adjoint action. Differentiating this action gives rise to an associative algebra homomorphism \( \text{ad} : \mathcal{U}\mathfrak{g} \to \mathcal{D}(\mathfrak{g}) \). Let \( \mathcal{D}(\mathfrak{g}) \cdot \text{ad} \mathfrak{g} \subset \mathcal{D}(\mathfrak{g}) \) denote the left ideal in \( \mathcal{D}(\mathfrak{g}) \) generated by the image of \( \mathfrak{g} \subset \mathcal{U}\mathfrak{g} \) under this homomorphism. The space \( (\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad} \mathfrak{g})^{\text{ad} \mathfrak{g}} \) inherits from \( \mathcal{D}(\mathfrak{g}) \) a natural filtered algebra structure.

One of the main results of [EG] is a construction, for any \( c \in \mathbb{C} \), of an algebra homomorphism

\[
\Phi'_c : (\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad} \mathfrak{g})^{\text{ad} \mathfrak{g}} \longrightarrow eH_c e.
\]

This homomorphism is compatible with the filtrations on \( (\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad} \mathfrak{g})^{\text{ad} \mathfrak{g}} \) and \( eH_c e \) introduced above. So, there is a well-defined associated graded algebra homomorphism

\[
gr(\Phi'_c) : \text{gr}(\mathcal{D}(\mathfrak{g})/\mathcal{D}(\mathfrak{g}) \cdot \text{ad} \mathfrak{g})^{\text{ad} \mathfrak{g}} \longrightarrow \text{gr}(eH_c e).
\]

**Theorem 6.6.1.** For any \( c \in \mathbb{C} \), the maps \( \Phi'_c \) and \( \text{gr}(\Phi'_c) \) are both isomorphisms.

**Proof.** It has been explained in Remark (ii) at the end of [EG] §10 that both statements are immediate consequences of Theorem 12.4 (the latter hasn’t been known at the time the paper [EG] was written).

**Remark 6.6.2.** Surjectivity of the maps \( \Phi'_c \) and \( \text{gr}(\Phi'_c) \) has been already established in [EG]. The injectivity part was proved in [EG] for all values of \( c \in \mathbb{C} \) except possibly an (unknown) finite set, see [EG], Theorem 7.3.

7 The functor of Hamiltonian reduction.

7.1 Generalities on Quantum Hamiltonian reduction. Let \( a \) be an arbitrary finite dimensional Lie algebra. Given a \( a \)-module \( M \), we write \( M^a := \{m \in M \mid \rho(x)m = 0, \forall x \in a\} \), for the vector space of \( a \)-invariants, and \( M_a := aM \setminus M \), for the vector space of \( a \)-coinvariants.

Let \( A \) be an associative algebra, viewed as a Lie algebra with respect to the commutator Lie bracket. Given a Lie algebra homomorphism \( \rho : a \to A \), one has an adjoint \( a \)-action on \( A \) given by \( \text{ad} x : a \mapsto \rho(x) \cdot a - a \cdot \rho(x), x \in a, a \in A \). The left ideal \( A \cdot \rho(a) \) is stable under the adjoint action. Furthermore, one shows that multiplication in \( A \) induces a well defined associative algebra structure on

\[
\mathfrak{A}(A, a, \rho) := (A/A \cdot \rho(a))^{\text{ad} a},
\]

the space of \( \text{ad} a \)-invariants in \( A/A \cdot \rho(a) \). The resulting algebra \( \mathfrak{A}(A, a, \rho) \) is called quantum Hamiltonian reduction of \( A \) at \( \rho \).

If \( A \), viewed as an \( \text{ad} a \)-module, is semisimple, i.e., splits into a (possibly infinite) direct sum of irreducible finite dimensional \( a \)-representations, then the operations of taking \( a \)-invariants and taking the quotient commute, and we may write

\[
\mathfrak{A}(A, a, \rho) = (A/A \cdot \rho(a))^{\text{ad} a} = A^{\text{ad} a}/(A \cdot \rho(a))^{\text{ad} a}.
\]

Observe that, in this formula, \( (A \cdot \rho(a))^{\text{ad} a} \) is a two-sided ideal of the algebra \( A^{\text{ad} a} \).
Let \( Q := A/A \cdot \rho(a) \), a left \( A \)-module. If \( a \in A \) is such that the element \( a \mod A \cdot \rho(a) \in A/A \cdot \rho(a) \) is ad \( a \)-invariant, then the operator of right multiplication by \( a \) descends to a well-defined left \( A \)-linear map \( R_a : A/A \cdot \rho(a) \to A/A \cdot \rho(a) \). This gives the space \( Q \) a right \( \mathfrak{A}(A, a, \rho) \)-module structure, hence makes it an \( A \)-\( \mathfrak{A}(A, a, \rho) \)-bimodule. Moreover, the right \( \mathfrak{A}(A, a, \rho) \)-action on \( Q \) induces an algebra isomorphism \( \mathfrak{A}(A, a, \rho) = (\text{End}_A Q)^{op} \).

Let \( A \)-mod, resp., \( \mathfrak{A}(A, a, \rho) \)-mod, be the abelian category of all left \( A \)-modules, resp., left \( \mathfrak{A}(A, a, \rho) \)-modules. Any \( A \)-module \( M \) may be viewed also as a \( a \)-module, via the homomorphism \( \rho \). Clearly, each of the spaces \( M^a \) and \( M_a \) has a natural \( A^{ad^a} \)-module structure.

We let \( (A, a) \)-mof be the full subcategory of \( A \text{-mod} \) whose objects are finitely generated \( A \)-modules which are, in addition, completely reducible as \( a \)-modules. We have a canonical \( A^{ad^a} \)-module isomorphism

\[
M^a \rightsquigarrow M_a, \quad \text{for any } M \in (A, a)\text{-mof}. \tag{7.1.2}
\]

Further, let \( \mathfrak{A}(A, a, \rho) \)-mof be the full subcategory of \( \mathfrak{A}(A, a, \rho) \)-mod whose objects are finitely generated \( \mathfrak{A}(A, a, \rho) \)-modules.

### 7.2 The functor \( \mathbb{H} \)

We define the following functor, called Hamiltonian reduction functor

\[
\mathbb{H} : (A, a)\text{-mod} \longrightarrow \mathfrak{A}(A, a, \rho)\text{-mod},
\]

\[M \mapsto \mathbb{H}(M) = \text{Hom}_A(Q, M) = \text{Hom}_A(A/A \cdot \rho(a), M) = M^a.
\]

Here, the action of \( \mathfrak{A}(A, a, \rho) \) on \( \mathbb{H}(M) \) comes from the tautological right action of \( \text{End}_A(Q) \) on \( Q \), via the above mentioned isomorphism \( \mathfrak{A}(A, a, \rho) = (\text{End}_A(Q))^{op} \).

**Proposition 7.2.2.** Assume that \( A \) is a left Noetherian algebra and, moreover, that \( A \) is a semisimple ad \( a \)-module. Then

(i) The algebra \( \mathfrak{A}(A, a, \rho) \) is left Noetherian and \( Q \) is an object of \((A, a)\text{-mof}\).

(ii) The functor \( \mathbb{H} \) induces an exact functor \( \mathbb{H} : (A, a)\text{-mof} \to \mathfrak{A}(A, a, \rho)\text{-mof} \).

(iii) The functor \( \mathbb{H} \) in \( (ii) \) has a left adjoint functor

\[
\mathbb{H}^\top : \mathfrak{A}(A, a, \rho)\text{-mof} \longrightarrow (A, a)\text{-mof}, \quad E \mapsto Q \otimes_{\mathfrak{A}(A, a, \rho)} E.
\]

Furthermore, the canonical adjunction morphism \( E \longrightarrow \mathbb{H}(\mathbb{H}^\top(E)) \) is an isomorphism for any \( E \in \mathfrak{A}(A, a, \rho)\text{-mof} \).

**Proof.** First of all, we observe that the left \( a \)-action on \( Q = A/A \cdot \rho(a) \) coincides with the ad \( a \)-action on \( Q \). The adjoint action of \( a \) on \( A \), hence on \( Q \), is completely reducible. We deduce that \( Q \) is completely reducible as a left \( a \)-module. This proves the second claim of part (i). Also, we deduce the following natural isomorphisms of left \( A^{ad^a} \)-modules (the rightmost isomorphism below is due to (7.1.2)):

\[
\mathfrak{A}(A, a, \rho) = (A/A \rho(a))^{ad^a} = Q^{ad^a} = Q^a = \mathbb{H}(Q) \cong Q_a. \tag{7.2.3}
\]

Below, we use the notation \( \mathfrak{A} := \mathfrak{A}(A, a, \rho) \), and identify this algebra with a quotient of the algebra \( A^{ad^a} \), via (7.1.1), since the ad \( a \)-action on \( A \) is completely reducible. Observe that the action of \( A^{ad^a} \) on each of the objects in \( \mathfrak{A} \text{-mof} \) descends to an action of the quotient algebra \( \mathfrak{A} \). Thus, we may view \( \mathfrak{A} \text{-mof} \) as a chain of isomorphisms of left \( \mathfrak{A} \)-modules.
The classical argument due to Hilbert shows that if $A$ is a noetherian algebra, then so is $A^{\text{ad}a}$. A similar argument shows that, for any $M \in A\text{-mod}$, the space $M^a$ is a finitely generated $A^{\text{ad}a}$-module. Furthermore, the functor $M \mapsto M^a$ is exact on the category of completely reducible $a$-modules. Therefore, similar statements hold for the algebra $\mathcal{A}$, a quotient of $A^{\text{ad}a}$. Parts (i)-(ii) of the Proposition follow.

Next, let $E$ be a finitely generated left $\mathcal{A}$-module. Then $Q \otimes_{\mathcal{A}} E$ is clearly finitely generated over $A$. Moreover, $Q \otimes_{\mathcal{A}} E$ is isomorphic, as a left $a$-module, to a quotient of a direct sum of finitely many copies of $Q$. The latter $a$-module being completely reducible by (i), we conclude that $Q \otimes_{\mathcal{A}} E$ is a completely reducible left $a$-module.

Further, by general ‘abstract nonsense’, there is a categorical isomorphism:

\[ \text{Hom}_A(Q \otimes_{\mathcal{A}} E, L) = \text{Hom}_A((A/A \cdot \rho(a)) \otimes_{\mathcal{A}} E, L) \cong \text{Hom}_A(E, L^a), \quad \forall E \in \mathcal{A}\text{-mod}, \; L \in (A, a)\text{-mod}. \]

The isomorphism shows that the functor $\mathcal{H}$ is indeed a left adjoint of $\mathcal{H}$.

Using the isomorphisms of $\mathcal{A}$-modules from (7.2.3), we obtain $\text{Tor}^A_1(aQ, -) = \text{Tor}^\mathcal{A}_1(Q, -) = \text{Tor}^\mathcal{A}_1(\mathcal{A}, -) = 0$. Therefore, for any $\mathcal{A}$-module $E$, one has a short exact sequence

\[ 0 \rightarrow aQ \otimes_{\mathcal{A}} E \rightarrow Q \otimes_{\mathcal{A}} E \rightarrow (aQ \otimes Q) \otimes_{\mathcal{A}} E \rightarrow 0. \]

Thus, by exactness of $\mathcal{H}$ and the rightmost isomorphism in (7.2.3), we compute

\[ \mathcal{H}(\mathcal{T}(E)) = (Q \otimes_{\mathcal{A}} E)_a = a(Q \otimes_{\mathcal{A}} E) \setminus (Q \otimes_{\mathcal{A}} E) = (aQ \otimes Q) \otimes_{\mathcal{A}} E = \mathcal{A} \otimes_{\mathcal{A}} E = E. \]

This completes the proof of part (iii) of the Proposition. \(\square\)

Write $\text{Ker} \mathcal{H}$ for the full subcategory of $(A, a)\text{-mod}$ formed by the objects $L$ such that $\mathcal{H}(L) = 0$. Since $\mathcal{H}$ is exact, the category $\text{Ker} \mathcal{H}$ is a Serre subcategory in $(A, a)\text{-mod}$. Let $(A, a)\text{-mod}/\text{Ker} \mathcal{H}$ be the corresponding quotient category.

**Corollary 7.2.4.** The functor $\mathcal{H}$ induces an equivalence $(A, a)\text{-mod}/\text{Ker} \mathcal{H} \rightarrow \mathcal{A}(A, a, \rho)\text{-mod}$.

**Proof.** The equivalence of categories stated in the proposition is known, by ‘abstract nonsense’, to be a formal consequence of the existence of a left adjoint functor, $\mathcal{T}$, such that the canonical adjunction gives an isomorphism of functors $\text{Id}_{(A,a)\text{-mod}} \rightarrow \mathcal{H} \circ \mathcal{T} \mathcal{H}$. The latter isomorphism is nothing but Proposition 7.2.2(iii). \(\square\)

### 7.3 We return to the setting of Cherednik algebras.

Fix $c \in \mathbb{C}$. We put $a := g_c, A := \mathcal{D}(\mathfrak{X}, c)$, and let $\rho : g_c \rightarrow \mathcal{D}(\mathfrak{X}, c)$ be the tautological imbedding. The algebra $\mathcal{D}(\mathfrak{X}, c)$ is clearly both left and right noetherian. The Hamiltonian reduction algebra $\mathcal{A}(\mathcal{D}(\mathfrak{X}, c), g_c, \rho) = (\mathcal{D}(\mathfrak{X}, c)/\mathcal{D}(\mathfrak{X}, c) \cdot g_c)_{\text{ad}g_c}$, c.f. (6.4.1), is isomorphic to the spherical Cherednik algebra $\mathcal{E}H_c, e$, by our main Theorem 7.3.1.

Thus, applying Proposition 7.2.2 and Corollary 7.2.4 in our present setting, we obtain the following result

**Proposition 7.3.1.** (i) The left $\mathcal{D}(\mathfrak{X}, c)\text{-module } Q = \mathcal{D}(\mathfrak{X}, c)/\mathcal{D}(\mathfrak{X}, c) \cdot g_c$ is an object of the abelian category $(\mathcal{D}(\mathfrak{X}, c), g_c)\text{-mod}$. 

\[ \text{33} \]
(ii) The Hamiltonian reduction functor gives an exact functor

\[ \mathbb{H} : (\mathcal{D}(X, c), g_c)\text{-mof} \longrightarrow eH_e\text{-mof}, \quad M \mapsto \text{Hom}_{\mathcal{D}(X, c)}(Q, M) = M^\# . \]

This functor induces an equivalence \((\mathcal{D}(X, c), g_c)\text{-mof}/\text{Ker} \mathbb{H} \simeq eH_e\text{-mof} \). □

Further, from definitions, one easily derives the following (cf. [EG, Proposition 7.6]).

**Proposition 7.3.2.** The functor \( \mathbb{H} \) intertwines Fourier transforms of \( \mathcal{D} \)-modules and \( eH_e \)-modules, respectively, i.e., there is a natural isomorphism of functors

\[ F_H \circ \mathbb{H} \cong \mathbb{H} \circ F_\mathcal{D} . \] □

### 7.4 The Harish-Chandra \( \mathcal{D} \)-module and category \( \mathcal{O}(eH_e) \)

Recall that \( Z_+ \) denotes the augmentation ideal in \( Z \cong (\text{Sym } g)^{ad} \# \), the algebra of ad \( g \)-invariant constant coefficients differential operators on \( g \). We set

\[ F := Q/Q \cdot Z_+ = \mathcal{D}(X, c)/(\mathcal{D}(X, c) \cdot g_c + \mathcal{D}(X, c) \cdot Z_+) . \]

It is clear that \( F \) is an admissible left \( \mathcal{D}(X, c) \)-module; it may be called Harish-Chandra \( \mathcal{D}(X, c) \)-module. This name is motivated by the works [HK] and [G2], where the authors considered a similar \( \mathcal{D} \)-module for \( \text{Ad } G \)-invariant eigen-distributions on an arbitrary semisimple Lie algebra. The analogy with loc. cit. will be studied further in [FG].

Next, let \( I := (\text{Sym } h)^W_+ \) be the augmentation ideal in the algebra \((\text{Sym } h)^W \). Given an algebra \( A \) and an algebra imbedding \((\text{Sym } h)^W \hookrightarrow A\), we will use the notation \( \langle I \rangle := A \cdot I \) for the left ideal in \( A \) generated by \( I \).

Recall that the algebras \((\text{Sym } h)^W \) and \( \mathbb{C}[h]^W \) may be viewed as subalgebras in \( eH_e \). In particular, \( I \) is a subalgebra of \( eH_e \).

The space \( eH_e/\text{eH}_e \cdot (\text{Sym } h)^W_+ = eH_e/\langle I \rangle \) has an obvious left \( eH_e \)-module structure. We also consider the following left \( H_c \)-module \( P := H_c \otimes (\text{Sym } h)^W_+ \) \((\text{Sym } h/\langle I \rangle) \).

**Lemma 7.4.1.** We have a natural \( eH_e \)-module isomorphism \( eH_e/\langle I \rangle \cong eP \).

**Proof.** Consider the cross product algebra \((\text{Sym } h)^W_+ \) and the natural algebra imbeddings \((\text{Sym } h)^W \hookrightarrow (\text{Sym } h)^W_+ \hookrightarrow (\text{Sym } h)^W \). We get a map \( f : (\text{Sym } h)^W \hookrightarrow (\text{Sym } h)^W_+ \hookrightarrow \text{Sym } h \cdot I = \text{Sym } h/\langle I \rangle \).

Tensoring this map with the obvious inclusion \( r : eH_e \hookrightarrow eH_e \) we obtain a chain of maps

\[
\text{eH}_e/(\text{eH}_e \cdot I) = eH_e/\text{eH}_e \otimes (\text{Sym } h)^W/(\text{Sym } h/\langle I \rangle) \xrightarrow{r \otimes f} eH_e \otimes (\text{Sym } h)^W_+ \otimes (\text{Sym } h/\langle I \rangle) = eH_e \otimes (\text{Sym } h/\langle I \rangle) = eP .
\]

All the maps in this chain are filtration preserving morphisms of left \( eH_e \)-modules. We claim that the composite morphism is, in effect, a bijection.

To prove the claim, we consider the corresponding associated graded map. We have \( \text{gr}(eH_e) = \mathbb{C}[h \times h^*] \) is a projective, hence flat, \( \mathbb{C}[h^*] \#W \)-module. We deduce, using the identification \( \mathbb{C}[h^*] = \text{Sym } h \), that \( eH_e \) is a flat right \((\text{Sym } h)^W \)-module, moreover, we have

\[
\text{gr}(eH_e \otimes (\text{Sym } h)^W/(\text{Sym } h/\langle I \rangle)) = \text{gr}(eH_e) \otimes (\mathbb{C}[h^*]/\langle I \rangle) = (\mathbb{C}[h \times h^*] \otimes (\mathbb{C}[h^*]/\langle I \rangle)) = (\mathbb{C}[h \times h^*] \otimes (\mathbb{C}[h^*]/\langle I \rangle)).
\]
Similarly, we obtain $\text{gr}(\mathcal{E}_{\mathfrak{c}}/\langle I \rangle) = \mathbb{C}[\hbar \times \mathfrak{h}^{*}]^{W}/\langle I \rangle$. Thus, we compute

$$\text{gr}(\mathcal{E}_{\mathfrak{c}}/\langle I \rangle) = \mathbb{C}[\hbar \times \mathfrak{h}^{*}]^{W}/\langle I \rangle = (\mathbb{C}[\hbar \times \mathfrak{h}^{*}] \otimes_{\text{triv}} \mathbb{C}[\hbar \times \mathfrak{h}^{*}]^{W}/\langle I \rangle)$$

$$= (\mathbb{C}[\hbar \times \mathfrak{h}^{*}] \otimes_{(\text{Sym} \mathfrak{h})^{W}} \text{Sym} \mathfrak{h})/\langle I \rangle$$

$$= \mathbb{C}[\hbar \times \mathfrak{h}^{*}] \otimes_{(\text{Sym} \mathfrak{h})^{W}} (\text{Sym} \mathfrak{h})/\langle I \rangle$$

$$= \text{gr}(\mathcal{E}_{\mathfrak{c}} \otimes_{(\text{Sym} \mathfrak{h})^{W}} \text{Sym} \mathfrak{h})/\langle I \rangle) = \text{gr}(\mathcal{E}P).$$

We leave to the reader to check that the composite isomorphism above is nothing but the associated graded map corresponding to the $\mathcal{E}_{\mathfrak{c}}$-module morphism $\mathcal{E}P$.

Let $\mathcal{O}(\mathcal{E}_{\mathfrak{c}})$ denote category $\mathcal{O}$ for the spherical subalgebra $\mathcal{E}_{\mathfrak{c}}$, see [BEC]. This is a full subcategory of $\mathcal{E}_{\mathfrak{c}}$-mod whose objects are locally nilpotent as $I$-modules. It is clear that $\mathcal{O}(\mathcal{E}_{\mathfrak{c}})$ is an object of $\mathcal{O}(\mathcal{E}_{\mathfrak{c}})$.

Next, recall the Hamiltonian reduction functor, see Proposition 7.4.3. Write $\text{Ker} \mathbb{H}$ for the full subcategory of $\mathcal{E}_{\mathfrak{c}}$ formed by the objects $L$ such that $\mathbb{H}(L) = 0$.

**Proposition 7.4.3.** The Hamiltonian reduction functor restricts to an exact functor $\mathbb{H} : \mathcal{E}_{\mathfrak{c}} \to \mathcal{O}(\mathcal{E}_{\mathfrak{c}})$. The latter functor induces an equivalence $\mathcal{E}_{\mathfrak{c}}/\text{Ker} \mathbb{H} \to \mathcal{O}(\mathcal{E}_{\mathfrak{c}})$.

Furthermore, we have $\mathbb{H}(F) = \mathcal{E}_{\mathfrak{c}}/\langle I \rangle$.

**Proof.** We know that $\Phi_{\mathfrak{c}}(Z_{+}) = (\text{Sym} \mathfrak{h})^{W}$, see (5.2.2). This immediately implies that $\mathbb{H}(L)$ is a $(\text{Sym} \mathfrak{h})^{W}$-locally nilpotent $\mathcal{E}_{\mathfrak{c}}$-module, for any $Z_{+}$-locally nilpotent $\mathcal{P}(X, c)$-module $L$. We deduce that the functor $\mathbb{H}$ takes category $\mathcal{E}_{\mathfrak{c}}$ into $\mathcal{O}(\mathcal{E}_{\mathfrak{c}})$.

To prove the last statement of the Proposition, we observe that the canonical $g_{\mathfrak{c}}$-equivariant projection $Q \to Q^{g_{\mathfrak{c}}}$ is a morphism of right $Z$-modules. Hence, we have $(Q : Z_{+})^{g_{\mathfrak{c}}} = Q^{g_{\mathfrak{c}}} : Z_{+}$. We compute

$$\mathbb{H}(F) = \mathbb{H}(Q/Q : Z_{+}) = \mathbb{H}(Q)/\mathbb{H}(Q : Z_{+}) = Q^{g_{\mathfrak{c}}}/(Q : Z_{+})^{g_{\mathfrak{c}}} = Q^{g_{\mathfrak{c}}}/Q^{g_{\mathfrak{c}}} : Z_{+} = \mathcal{E}_{\mathfrak{c}}/\mathcal{E}_{\mathfrak{c}} : (\text{Sym} \mathfrak{h})^{W} = \mathcal{E}_{\mathfrak{c}}/\langle I \rangle,$$

where we have used the exactness of the functor $\mathbb{H}$, and the isomorphisms in (5.2.3).

### 7.5 Relation to the Hilbert scheme.

Let $U \subset T^{*}X$ be the set formed by the quadruples $(X, Y, i, j) \in T^{*}X$ such that $V = \mathbb{C}[X, Y, i, j]$ is such that $i$ is a cyclic vector for the pair $(X, Y)$. It is clear that $U$ is a $G$-stable Zariski open subset in $T^{*}X$. Furthermore, the group $G$ acts freely on $U$ and there is a universal geometric quotient morphism $\Upsilon : U \to \text{Hilb}^n \mathbb{C}^2$, where $\text{Hilb}^n \mathbb{C}^2$ denotes the Hilbert scheme of $n$ points in the plane, see [Na].

An irreducible component $Z$ of the Lagrangian scheme $\Lambda$ is said to be stable if the set $Z \cap U$ is dense in $Z$. In such a case, $Z \cap U$ is a $G$-stable closed subset in $U$, hence, we have $Z \cap U = \Upsilon^{-1}(Z^{\text{Hilb}})$, where $Z^{\text{Hilb}} := \Upsilon(Z)$, a closed Lagrangian subscheme in $\text{Hilb}^n \mathbb{C}^2$.

Now, let $L \in \mathcal{E}_{\mathfrak{c}}$ and let $\text{Ch}(L) = \sum m_k \cdot Z_k$, be the characteristic cycle of $L$, a formal integral combination of closed irreducible subvarieties $Z_k \subset T^{*}X$. We define $\text{Ch}^{\text{Hilb}}(L) := \sum (Z_k \text{ is stable}) \cdot m_k \cdot Z_k^{\text{Hilb}}$, the formal integral combination of Lagrangian subschemes in $\text{Hilb}^n \mathbb{C}^2$ corresponding, as explained above, to the stable irreducible components $Z_k$. Thus, $\text{Ch}^{\text{Hilb}}(L)$ is a Lagrangian cycle in $\text{Hilb}^n \mathbb{C}^2$. 
On the other hand, in the recent paper \([GS]\), Gordon and Stafford have attached to any object \(E \in \mathcal{O}(\mathfrak{eH},e)\) a Lagrangian cycle \(\text{Ch}^{\mathcal{G}\mathcal{S}}(E)\) in \(\text{Hilb}^n \mathbb{C}^2\). The construction used in \([GS]\) is totally different from the approach of the present paper. Nevertheless, it is likely (cf. also \([FG]\)) that one has:

\[
\text{Ch}^{\mathcal{H} \text{H}}(\mathcal{T} \mathcal{H}(E)) = \text{Ch}^{\mathcal{G}\mathcal{S}}(E), \quad \forall E \in \mathcal{O}(\mathfrak{eH},e).
\]

## 8 Appendix: A remark on a theorem of M. Haiman

**VICTOR GINZBURG**

### 8.1 Main result.
Write \(\mathbb{C}[x, y] := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]\) for a polynomial ring in two sets of variables \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\). The Symmetric group \(S_n\) acts naturally on the \(n\)-tuples \(x\) and \(y\), and this gives rise to an \(S_n\)-diagonal action on the algebra \(\mathbb{C}[x, y]\). We write \(\mathbb{C}[x, y]^{S_n} \subset \mathbb{C}[x, y]\) for the subalgebra of \(S_n\)-invariant polynomials and \(\mathcal{A} := \mathbb{C}[x, y]^e \subset \mathbb{C}[x, y]\) for the subspace of \(S_n\)-alternating polynomials. The space \(A\) is stable under multiplication by elements of the algebra \(\mathbb{C}[x, y]^{S_n}\), in particular, it may be viewed as a module over \(\mathbb{C}[y]^{S_n} \subset \mathbb{C}[x, y]^{S_n}\), the subalgebra of symmetric polynomials in the last \(n\) variables \(y_1, \ldots, y_n\).

For each \(k = 1, 2, \ldots\), let \(A^k\) be the \(\mathbb{C}\)-linear subspace in \(\mathbb{C}[x, y]\) spanned by the products of \(k\) elements of \(A\). The action of \(\mathbb{C}[y]^{S_n}\) on \(A\) induces one on \(A^k\), hence each space \(A^k\), \(k = 1, 2, \ldots\), acquires a natural \(\mathbb{C}[y]^{S_n}\)-module structure.

The goal of this Appendix is to give a direct proof of the following special case of a much stronger result due to M. Haiman \([Ha2]\) Proposition 3.8.1).

**Theorem 8.1.1.** For each \(k = 1, 2, \ldots\), the space \(A^k\) is a free \(\mathbb{C}[y]^{S_n}\)-module.

In an earlier paper, Haiman showed, cf. \([Ha1]\) Proposition 2.13, that the above theorem holds for all \(k \gg 0\). The corresponding statement for all \(k\) follows from Haiman’s proof of his his Polygraph theorem, the main technical result in \([Ha2]\).

### 8.2 Geometric interpretation of \(A^k\).
The group \(G = GL(V)\) acts naturally on \(\mathcal{M}\), see \([2]\). Thus, we get a \(G\)-action \(g : f \mapsto g(f)\), on the coordinate ring \(\mathbb{C}[[\mathcal{M}]]\) by algebra automorphisms.

For each \(k = 1, 2, \ldots\), we set

\[
\mathbb{C}[[\mathcal{M}]]^{(k)} := \{ f \in \mathbb{C}[[\mathcal{M}]] \mid g(f) = (\det g)^k \cdot f, \quad \forall g \in G\}.
\]

We will use the notation introduced in \([2,8]\). There is an obvious identification \(\mathbb{C}[h \times h] = \mathbb{C}[x, y]\). In particular, we may view the vector space \(A^k\), see \([8,1]\) as a subspace in \(\mathbb{C}[h \times h]\).

A key ingredient in our approach to Theorem 8.1.1 is the following result

**Proposition 8.2.1.** For each \(k = 1, 2, \ldots\), restriction of functions via the imbedding \(\varepsilon\), see \([2,8,1]\), induces a vector space isomorphism \(\varepsilon^* : \mathbb{C}[[\mathcal{M}]]^{(k)} \rightarrow A^k\).

**Remark 8.2.2.** Lemma \([2,8,8]\) may be viewed as a version of Proposition 8.2.1 for \(k = 0\).

**Remark 8.2.3.** For each \(k = 1, 2, \ldots\), M. Haiman constructed in \([Ha1]\) a natural map \(A^k \rightarrow \Gamma(\text{Hilb}^n \mathbb{C}^2, \mathcal{O}(k))\), where \(\mathcal{O}(1)\) is a natural ample line bundle on \(\text{Hilb}^n \mathbb{C}^2\), cf. \([Ha1]\). Moreover, it follows from the results of \([Ha2]\) that this map is, in effect, an isomorphism.
8.3 Proof of Proposition 8.2.1 Fix nonzero volume elements $v \in \wedge^n V$ and $v^* \in \wedge^n V^*$, respectively. Given an $n$-tuple $f = (f_1, \ldots, f_n)$, $f_r \in \mathbb{C}(x, y)$, of noncommutative polynomials in two variables, we consider polynomial functions $\psi, \phi \in \mathbb{C}[g \times g \times V \times V^*]$ of the form

$$\psi_f(X, Y, i, j) = \langle v^*, f_1(X, Y)i \wedge \ldots \wedge f_n(X, Y)i \rangle,$$

$$\phi_f(X, Y, i, j) = (jf_1(X, Y) \wedge \ldots \wedge jf_n(X, Y), v),$$

where $f_r(X, Y)$ denotes the matrix obtained by plugging the two matrices $X, Y \in g$ in the noncommutative polynomial $f(x, y)$. We will keep the notation $\psi_f, \phi_f$ for the restriction of the corresponding function to the closed subvariety $\mathcal{M} \subset g \times g \times V \times V^*$. It is clear that the imbedding $\varepsilon : h \times h \hookrightarrow \mathcal{M}$ that, restricting these functions further to the subset $h \times h$, one has $\varepsilon^* \psi_f \in A$ and $\varepsilon^* \phi_f = 0$.

By Theorem 11.1.2 we know that $\mathcal{M} = \mathcal{M}_0 \cup \ldots \cup \mathcal{M}_n$, is a union of $n + 1$ irreducible components. It is immediate from the definition of the set $\mathcal{M}_r$, cf. 11.1.1 that, for any choice of $n$-tuple $f = (f_1, \ldots, f_n)$, the function $\psi_f$ vanishes on $\mathcal{M}_r$ whenever $r \neq 0$, while $\phi_f$ vanishes on $\mathcal{M}_r$ whenever $r \neq n$. Since each irreducible component is reduced, by Theorem 11.1.2 the above vanishings hold scheme-theoretically:

$$\psi_f|_{\mathcal{M}_r} = 0 \quad \forall r \neq 0, \quad \text{and} \quad \phi_f|_{\mathcal{M}_r} = 0 \quad \forall r \neq n. \quad (8.3.2)$$

Next, similarly to $\mathbb{C}[\mathcal{M}]^{(k)}$, for each $k \in \mathbb{Z}$, we introduce the space $\mathbb{C}[g \times g \times V \times V^*]^{(k)}$ of polynomial functions on $g \times g \times V \times V^*$ that satisfy the equation $g(f) = (\det g)^k \cdot f, \quad \forall g \in G$. It is clear that

$$\psi_f \in \mathbb{C}[g \times g \times V \times V^*]^{(1)} \quad \text{resp.}, \quad \phi_f \in \mathbb{C}[g \times g \times V \times V^*]^{(-1)}, \quad \forall f = (f_1, \ldots, f_n).$$

Fix $k \in \mathbb{Z}$ and observe that $\mathbb{C}[g \times g \times V \times V^*]^{(k)}$ is naturally a $\mathbb{C}[g \times g \times V \times V^*]^G$-module. Applying Weyl's fundamental theorem on $GL_n$-invariants we deduce that this $\mathbb{C}[g \times g \times V \times V^*]^G$-module is generated by products of the form $\psi_1 \cdot \ldots \cdot \psi_p \cdot \phi_1 \cdot \ldots \cdot \phi_q$, where $p - q = k$ and where each factor $\psi_i, \phi_i$, resp. each factor $\phi_i, \psi_i$, is of the form $\psi_f, \phi_f$, resp.

The action of $G$ on $\mathbb{C}[g \times g \times V \times V^*]$ being completely reducible, we deduce that restricting functions from $g \times g \times V \times V^*$ to $\mathcal{M}$ yields a surjection $\mathbb{C}[g \times g \times V \times V^*]^{(k)} \rightarrow \mathbb{C}[\mathcal{M}]^{(k)}$. It follows that $\mathbb{C}[\mathcal{M}]^{(k)}$, viewed as a $\mathbb{C}[\mathcal{M}]^G$-module, is again generated by the products $\psi_1 \cdot \ldots \cdot \psi_p \cdot \phi_1 \cdot \ldots \cdot \phi_q$, with $p - q = k$. Furthermore, from (8.3.2), we see that for $k \geq 0$ we must have $p = 0 \& q = 0$. On the other hand, for $k \leq 0$ we must have $p = 0 \& q = k$.

From now on, we assume that $k \geq 1$. Thus, the imbedding $\mathcal{M}_0 \rightarrow \mathcal{M}$ induces a bijection $\mathbb{C}[\mathcal{M}]^{(k)} \rightarrow \mathbb{C}[\mathcal{M}_0]^{(k)}$. It follows that $\mathbb{C}[\mathcal{M}]^{(k)}$ is generated, as a $\mathbb{C}[\mathcal{M}]^G$-module, by the products $\psi_1 \cdot \ldots \cdot \psi_k$. Since $\varepsilon^* \psi_f \in A$ for any $f$, we find that $\varepsilon^*(\psi_1 \cdot \ldots \cdot \psi_k) \in A^k$, hence $\varepsilon^*(\mathbb{C}[\mathcal{M}]^{(k)}) = \varepsilon^*(\mathbb{C}[\mathcal{M}_0]^{(k)}) \subset A^k$.

To prove injectivity of the restriction map $\varepsilon^* : \mathbb{C}[\mathcal{M}]^{(k)} \rightarrow A^k$, we observe that $G \cdot \varepsilon(h \times h)$, the $G$-saturation of the image of the imbedding $\varepsilon$, is an irreducible variety of dimension $n^2 + n = \dim \mathcal{M}$. Furthermore, for any diagonal matrix $Y \in h$ with pairwise distinct eigenvalues, we have $\mathbb{C}[Y]i_n = V$. Hence, a Zariski open subset of $G \cdot \varepsilon(h \times h)$ is contained $\mathcal{M}_0$, cf. 11.1.1. Since $\mathcal{M}_0 = \mathcal{M}_0$ and $G \cdot \varepsilon(h \times h)$ is irreducible, we conclude that $G \cdot \varepsilon(h \times h) \subset \mathcal{M}_0$ and, moreover, the set $G \cdot \varepsilon(h \times h)$ is Zariski dense in $\mathcal{M}_0$. Thus, for any $f \in \mathbb{C}[\mathcal{M}_0]^{(k)}$ such that $\varepsilon^*(f) = 0$ we must have $f = 0$. This proves injectivity of the map $\varepsilon^*$.

We have seen above that $\mathbb{C}[\mathcal{M}]^{(k)}$ is generated, as a $\mathbb{C}[\mathcal{M}]^G$-module, by the $k$-fold products $\psi_1 \cdot \ldots \cdot \psi_k$. Therefore, it suffices to prove surjectivity of the map $\varepsilon^*$ for $k = 1$. To prove the
latter, we identify $A = \mathbb{C}[x, y]^*$ with $\wedge^n \mathbb{C}[x, y]$, the $n$-th exterior power of the vector space $\mathbb{C}[x, y]$ of polynomials in 2 variables. With this identification, the space $A$ is spanned by wedge products of the form $f_1 \wedge \ldots \wedge f_n, f_1, \ldots, f_n \in \mathbb{C}[x, y]$.

Now, by definition of the irreducible component $\mathcal{M}_0$, for any $(X, Y, i, j) \in \mathcal{M}_0$, we have $[X, Y] = [X, Y] + ij = 0$. Therefore, for any $f \in \mathbb{C}[x, y]$, the expression $f(X, Y)$ is a well-defined matrix. In other words, for any lift of $f$ to a noncommutative polynomial $\hat{f} \in \mathbb{C}(x, y)$, i.e., for any $f$ in the preimage of $\hat{f}$ under the natural projection $\mathbb{C}(x, y) \to \mathbb{C}[x, y]$, we have $\hat{f}(X, Y) = f(X, Y)$. Thus, given an $n$-tuple $f_1, \ldots, f_n \in \mathbb{C}[x, y]$, we have a well-defined element

$$\psi_f = \langle \psi^*, f_1(X, Y)i \wedge \ldots \wedge f_n(X, Y)i \rangle \in \mathbb{C}[\mathcal{M}]^{\langle 1 \rangle}.$$ 

It is straightforward to verify that for such an element one has $\varepsilon^* \psi_f = f_1 \wedge \ldots \wedge f_n$. This proves surjectivity of the map $\varepsilon^*$ and completes the proof of the Proposition.

### 8.4 Proof of Theorem 8.1.1

We have the standard grading $\mathbb{C}[y]^{S_n} = \oplus_{d \geq 0} \mathbb{C}^d[y]^{S_n}$, by degree of the polynomial. Write $\mathbb{C}[y]^{S_n}_+ := \oplus_{d > 0} \mathbb{C}^d[y]^{S_n}$ for the augmentation ideal.

In general, let $E$ be any flat nonnegatively graded $\mathbb{C}[y]^{S_n}$-module. Then, choosing representatives in $E$ of a $\mathbb{C}$-basis of the vector space $E/\mathbb{C}[y]^{S_n}_+ E$ yields a free $\mathbb{C}[y]^{S_n}$-basis in $E$. Hence, any flat nonnegatively graded $\mathbb{C}[y]^{S_n}$-module is free.

Next, we have a $\mathcal{C}^*$-action on $\mathcal{M}$ by dilations, given for any $z \in \mathbb{C}^n$ by the assignment $(X, Y, i, j) \mapsto (z \cdot X, z \cdot Y, z \cdot i, z \cdot j)$. This $\mathcal{C}^*$-action gives rise to a natural grading $\mathbb{C}[\mathcal{M}] = \oplus_{d \geq 0} \mathbb{C}^d[\mathcal{M}]$, on the algebra $\mathbb{C}[\mathcal{M}]$. With this grading, the pull-back morphism $\pi^* : \mathbb{C}[y]^{S_n} \to \mathbb{C}[\mathcal{M}]$ induced by the map $\pi : (X, Y, i, j) \mapsto \text{Spec} Y$, is a graded algebra morphism, so $\mathbb{C}[\mathcal{M}]$ may be viewed as a graded $\mathbb{C}[y]^{S_n}$-module.

A key point of the proof is that the map $\pi : \mathcal{M} \to \mathbb{C}^{(n)}$, $(X, Y, i, j) \mapsto \text{Spec} Y$ is a flat morphism, by Corollary 25.3. In algebraic terms, this means that $\mathbb{C}[\mathcal{M}]$ is a flat (nonnegatively graded) $\mathbb{C}[y]^{S_n}$-module. As has been explained at the beginning of the proof, this implies that $\mathbb{C}[\mathcal{M}]$ is free over $\mathbb{C}[y]^{S_n}$. Further, by complete reducibility of the $G$-action on $\mathbb{C}[\mathcal{M}]$, we deduce that $\mathbb{C}[\mathcal{M}]^{(k)}$, viewed as a graded $\mathbb{C}[y]^{S_n}$-submodule in $\mathbb{C}[\mathcal{M}]$, splits off as a direct summand, for any $k = 1, 2, \ldots$. Hence, $\mathbb{C}[\mathcal{M}]^{(k)}$ is projective, in particular, flat over $\mathbb{C}[y]^{S_n}$, as a direct summand of a free $\mathbb{C}[y]^{S_n}$-module.

To complete the proof of Theorem 8.1.1, we observe that $\mathbb{C}[\mathcal{M}]^{(k)}$ is again a positively graded $\mathbb{C}[y]^{S_n}$-module. Thus, we conclude as above that this graded module must be free over $\mathbb{C}[y]^{S_n}$.

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