

A non-grey analytical model for irradiated atmospheres.  
II: Analytical vs. numerical solutions

Vivien Parmentier1,2, Tristan Guillot1,2, Jonathan J. Fortney2, and Mark S. Marley3

1 Laboratoire J.-L. Lagrange, Université de Nice-Sophia Antipolis, CNRS, Observatoire de la Côte d’Azur, BP 4229, 06304 Nice, France e-mail: vivien.parmentier@oca.eu
2 Department of Astronomy and Astrophysics, University of California, Santa Cruz, CA 95064, USA
3 NASA Ames Research Center, MS-245-3, Moffett Field, CA 94035

in preparation

ABSTRACT

Context. The recent discovery and characterization of the diversity of the atmospheres of exoplanets and brown dwarfs calls for the development of fast and accurate analytical models.

Aims. We wish to assess the goodness of the different approximations used to solve the radiative transfer problem in irradiated atmospheres analytically and provide a useful tool for a fast computation of analytical temperature profiles that remains correct over a wide range of atmospheric characteristics.

Methods. We quantify the accuracy of the analytical solution derived in paper I for an irradiated, non-grey atmosphere by comparing it to a state-of-the-art radiative transfer model. Then, using a grid of numerical models, we calibrate the different coefficients of our analytical model for irradiated solar-composition atmospheres of giant exoplanets and brown dwarfs.

Results. We show that the so-called Eddington approximation used to solve the angular dependency of the radiation field leads to relative errors of up to ~ 5% on the temperature profile. For grey or semi-grey atmospheres (i.e. when the visible and thermal opacities respectively can be considered as independent of wavelength), we show that the presence of a convective zone has a limited effect on the radiative atmosphere above it and leads to modifications of the radiative temperature profile of order ~ 2%. However, for realistic non-grey planetary atmospheres, the presence of a convective zone that extends to optical depths smaller than unity can lead to changes in the radiative temperature profile of the order of 20% or more. When the convective zone is located at deeper levels (such as for strongly irradiated “hot Jupiters”), its effect on the radiative atmosphere is again of the same order (~ 2%) as in the semi-grey case. We show that when a strong absorber in the visible, such as TiO/VO, is present in the upper atmosphere, it decreases significantly the deep atmospheric temperature by lowering the “blanketing effect”, an intrinsically non-grey effect. Finally, we provide a functional form for the coefficients of our analytical model for solar-composition giant exoplanets and brown dwarfs. This leads to fully analytical pressure–temperature profiles for the radiative part of the atmospheres with a relative accuracy better than 10% for gravities between 2.5 and 250 m s$^{-2}$ and effective temperatures between 100 and 3000 K. This is a great improvement compared to the commonly used “Eddington boundary condition”.

Key words. extrasolar giant planets – planet formation

1. Introduction

The large diversity of exoplanets in terms of irradiation temperature, gravity, chemical composition discovered around stars with different properties call for the development of fast, accurate and versatile atmospheric models.

In paper I (Parmentier & Guillot 2013), we derived a new analytical model for irradiated atmospheres. Unlike previous models, our model takes into account non-grey opacities both in the visible and in the thermal frequency ranges. Using two different opacity bands in the thermal frequency range, we highlighted the dual role of thermal non-grey opacities in shaping the thermal structure of the atmosphere. Opacities dominated by lines (i.e. opacities where the lowest of the two values is dominant) enable the upper atmosphere to cool down significantly compared to a grey atmosphere whereas opacities dominated by bands (i.e. opacities where the highest of the two values is dominant) lead both to a significant cooling of the upper atmosphere and a significant heating of the deep atmosphere.

The pressure and temperature dependent line-by-line opacities that are used in numerical models to compute accurate temperature profiles are represented in analytical models by only a handful of parameters. Thus, to compute accurate temperature structure from our analytical model for specific planets atmospheres, we need to know how those parameters vary with the physical properties of the planet.

In this study, we apply our model to irradiated, solar-composition, semi-infinite atmospheres e.g., brown dwarfs, giant planets or planets with a surface situated in the optically thick region of the atmosphere. Based on the results from a state-of-the-art numerical model, we assess the goodness of the different approximations inherent in analytical solutions of the radiative transfer equations. Then, using a grid of numerical models, we calibrate the different coefficients of our analytical model and provide a useful tool for a fast computation of analytical temperature profiles for planet atmospheres that remains correct over a wide range of gravity and irradiation temperatures.

As a first step, in Sec. 3 we quantify the accuracy of models derived with the Eddington approximation, a common simplifi-
culation of the radiative transfer equations in analytical model atmospheres. Then in Sec. 4 we build a simple radiative/convective model where the radiative solution of Paper I is replaced by a convective solution whenever the Schwarzschild criterion is verified. We further discuss and quantify the intrinsic error of such a simple model of convective adjustment. Finally, guided by a state-of-the-art numerical integration of the radiative transfer equations, we constrain the parameters of the analytical solution of Paper I to develop a fully analytical solution for the atmospheric temperature/pressure profiles of irradiated giant planets. The solution presented in section 5 reproduces with a 10% accuracy the numerical solutions over a wide range of gravity and irradiances.

2. Models

2.1. Setting

We consider the case of a planet with a thick atmosphere (i.e. a planet with no surface or with a surface at very high optical depth) orbiting at a distance \( a \) from its host star of radius \( R_\star \) and effective temperature \( T_\star \). At the substellar point, the atmosphere receives a flux \( \sigma T_{\text{sub}}^4 \) where the substellar temperature is defined as:

\[
T_{\text{sub}}^4 \equiv T_{\text{eff}}^4 \left( \frac{R_\star}{a} \right)^2
\]

which is the same as the \( T_{\text{int}} \) quantity defined in Guillot (2010). In this paper we define \( T_{\text{int}} \) by:

\[
T_{\text{int}}^4 \equiv (1 - A)T_{\text{eff}}^4 \left( \frac{R_\star}{a} \right)^2
\]

where \( A \) is the Bond albedo of the planet and \( f \) is a parameter modulating the flux received by the planet, useful to compute profiles averaged in latitude and longitude. To first order, planets are spherical and the effective irradiation that a column of atmosphere receives depends on its location on the planet. Thus, the flux that penetrates the modeled slice of atmosphere is given by \( \sigma T_{\mu}^4 \) with:

\[
T_{\mu}^4 \equiv \mu T_{\text{int}}^4 = (1 - A)\mu fT_{\text{eff}}^4 \left( \frac{R_\star}{a} \right)^2
\]

where \( \mu = \cos \theta \), and \( \theta \), the inclination of the stellar irradiation with respect to the local vertical direction. The combination of \( \mu \) and \( f \) can lead to different mean atmospheric profiles. For example, the substellar point profile is obtained by setting \( f = 1 \) and \( \mu = 1 \) whereas the dayside average profile is obtained for \( \mu = 1/\sqrt{3} \), \( f = 0.5 \) and the planet average profile for \( \mu = 1/\sqrt{3} \) and \( f = 0.25 \) (see Guillot 2010). Moreover the planet has an internal flux \( F_{\text{int}} = \sigma T_{\text{int}}^4 \) which leads to the definition the effective temperature

\[
T_{\text{eff},\mu}^4 = T_{\mu}^4 + T_{\text{int}}^4
\]

2.2. Opacities

The interaction between photons and atmospheric gas is described by opacities which are functions of the wavelength of the irradiation considered and the temperature, pressure and composition of the gas. Although the variety of mixtures and cases to be considered is infinite, we choose to limit the present study to one set of opacities because of its very extensive use both in the context of giant exoplanets and brown dwarfs, i.e. the solar-composition opacities provided by Freedman et al. (2008). These opacities have been calculated for a solar-composition mixture in chemical equilibrium. They do not account for the presence of clouds, and any chemical species that condenses at a given temperature and pressure is taken out of the mixture. Although clouds are though to exist in planets atmospheres (see Marley et al. 2013, for a review) and should affect the thermal structure of their atmosphere (e.g., Heng et al. 2012), we do not take into account scattering by cloud particles in this study. However, the first order effect of clouds is to reflect part of the incoming stellar light to the space, which is taken into account in the albedo when calculating the irradiation temperature with eq. (2).

While tens of millions of lines have been used for the calculation of these opacities, we choose to show them in fig. 1 in the same form as they are used by the numerical code described hereafter in Section 2.4. In the so-called correlated-k method, the opacities values are sorted from the lowest to the highest values within a limited number of spectral bins (in our case 196). As long as the spectral bins are small compared to the width of the local Planck function, the error made on the wavelength corresponding to a given opacity is expected to be small and the consequences for the computed temperature profile limited (see Goody & Yung 1999).

Figure 1 thus provides the opacities for different pressure and temperature points taken along a selected planetary temperature/pressure profile corresponding approximately to a solar composition 1-Jupiter mass and radius planet at 0.05 AU from a Sun-like star. The wavelength range in which the Planck function has 90% and 99% of the total energy is shown by the thick and thin horizontal bars, respectively, for the different temperatures considered. The contribution of the spectral lines to the opacities shapes the cumulative distribution function inside each bin. As one moves progressively from the top to the bottom of the atmosphere, pressure (always) and temperature (generally) increase which broadens the spectral line profiles. This results in a flattening of the cumulative opacity distribution function within each bin. Opacities in the high atmosphere are characterized by very strong variations with wavelength and a comb-like struc-
ture. Deeper-down, the wavelength dependence is mostly due to the presence of molecular bands and takes place on scales significantly larger than our bin size.

An important feature of these opacities is that most of the variations of the opacity with wavelength take place on scales shorter than the characteristic wavelength range of the Planck function. This is certainly the case at low-pressures when the opacity varies extremely quickly with wavelength, but it remains true (to some extent) at high pressures in the band regime. Another feature of irradiated atmospheres is that the temperature variations remain limited so that there is always a significant overlap between the Planck function from the low to the high optical depth levels. These two features justify the use of the picket-fence approximation, and hence of the analytical model of Paper I.

2.3. Analytical model

Although analytical models of irradiated atmospheres can only be obtained for very restrictive approximations on the opacities, they provide nonetheless a useful tool to understand the physics of the radiative transfer and to compute with a low computational cost temperature profiles for a large variety of atmospheric properties. In the particular model derived in Paper I the line by line opacities are modeled by two different homogeneous set of lines, the full opacity function being described by 6 independent parameters.

The first set of lines, described by three parameters, represents the thermal part of the opacities, i.e. the part of the opacity function in the frequency range covered by the local Planck function of the atmospheric thermal emission. The Rosseland mean opacity \( \kappa_R(P, T) \) is the only one of those parameters that can vary with depth in the atmosphere. In particular, it is the relevant opacity to describe accurately the energy transport in the optically thick part of the atmosphere (Mihalas 1984). The other two parameters describe the non-grayness of the opacities, i.e. their variation in frequency. The first one, \( \gamma_p \), is the ratio of the Planck mean opacity to the Rosseland mean opacity, where the Planck mean opacity is dominated by the highest values of the opacities whereas the Rosseland mean is dominated by the lowest values of the opacities. Thus, grey opacities have \( \gamma_p = 1 \) and any departure from the grey model increases \( \gamma_p \). The second parameter, \( \beta_p \), is the relative width of the opacity lines. Values of \( \beta_p \) lower than 0.1 represent opacities dominated by atomic lines whereas values of \( \beta_p \) between 0.1 and 0.9 correspond to opacities dominated by molecular bands. In the following sections, the parameter \( \gamma_p \) will sometimes be replaced by an equivalent parameter \( \nu_1/\nu_2 \), where \( \nu_1 \) is the highest of the two opacities and \( \nu_2 \) the lowest. Value of \( \nu_1/\nu_2 \) between \( 10^4 - 10^6 \) in the upper atmosphere and between \( 10 - 100 \) in the deep atmosphere can be estimated from Fig. 1 for a typical hot-Jupiter. The simple relationship between \( \nu_1/\nu_2 \) and \( \gamma_p \) is described by eq. (87) of Paper I.

The second set of lines, described by three other parameters, represents the visible parts of the opacities, i.e. the part of the opacity function in the frequency range covered by the Planck function of the stellar irradiation. Since the planet’s atmosphere is usually cooler than the stellar photosphere, the two set of opacity lines can be considered as independent of each other. The first two parameters, \( \gamma_{v1} \) and \( \gamma_{v2} \) are the ratio of the the highest (resp. lowest) opacity of the line to the thermal Rosseland mean opacity. These ratios set the strength of the greenhouse effect and the presence of a thermal inversion. The last parameter, \( \beta_v \), describes the relative width of the two visible opacity bands, \( \beta_v = 1 \) being grey visible opacities described by \( \gamma_{v1} \). Whereas the model cannot take into account more than two thermal opacity bands, it can model as many visible bands as needed.

In our analytical model, the Rosseland mean opacity can vary with pressure and temperature. Thus, physical processes producing an overall increase of the opacities, such as the increasing importance of the collision induced absorption with pressure can be accurately taken into account. Our model is the first analytical model to take into account non-gray thermal opacities in irradiated atmosphere. However, the variation of the opacity with frequency cannot change through the atmosphere. Thus, all the other coefficients must remain constant in the whole atmosphere and a physical phenomenon such as the variation of the pressure or thermal broadening of the lines through the atmosphere cannot be taken into account.

2.4. Numerical model

Whereas analytical models are confined to model atmospheres with very simplified opacities, the radiative transfer equations can be solved by numerical integration using the full, line-by-line, frequency, pressure-and temperature-dependent opacities described in section 2.2. Moreover, numerical models can integrate the radiative transfer equations by taking into account an arbitrary high number of angular directions, with no need to invoke the Eddington approximation.

Here, we use the EGP (Extrasolar Giant Planet) code initially developed by McKay et al. (1989) for the study of Titan atmosphere. Since then, it has been extensively modified and adapted for the study of giant planets (Marley & McKay 1999), brown dwarfs (Marley et al. 1996, 2002; Burrows et al. 1997), and hot Jupiters (e.g., Fortney et al. 2005, 2008; Showman et al. 2009). The version of the code we employ solves the radiative transfer equations using the “delta-discrete ordinates” method of Toon et al. (1989) for the incident stellar radiation and the “two-stream source function” method, also of Toon et al. (1989), for the thermal radiative transfer. In some cases incident stellar and emitted thermal radiation bands may overlap, but the radiative transfer is solved separately for each radiation source. Opacities are treated using the correlated-k method (e.g., Goody & Yung 1989). We consider 196 frequency bins ranging from 0.26 to 300 \( \mu \)m; within each bin, the information of typically 10,000 to 100,000 frequency points is compressed inside a single cumulative distribution function that is then interpolated using 8 \( k \)-coefficients. The angular dependency is computed using the Gauss quadrature formula for the fluxes. \( \theta \) being the angular variable, this formula allows to transform an integral over \( \mu = \cos \theta \) into a simple sum over angles:

\[
\int_{-1}^{1} \mu I_\nu(\mu) d\mu = \sum_{n=1}^{N} \omega_n I_\nu(\mu_n)
\]

with the \( \omega_n \) and the \( \mu_n \) being tabulated in Abramowitz & Stegun (1965). Here we use 5 Gauss points. The EGP model calculates a self-consistent radiative/convective solution, deriving the adiabatic gradient using the equation of state of Saumon et al. (1995) but can also look for a fully radiative solution.

Although numerical models were built in order to incorporate the full complexity of the opacity function, it can nonetheless solve the radiative transfer equations with the same simplifications than the ones used in the analytical models. In particular, the \( k \)-coefficient method can be used to easily implement the simplified opacities of Parmentier et al. (2013) by setting a given number of \( k \)-coefficients at \( \nu_1 \) and the other ones at \( \nu_2 \) in
each frequency bin. Moreover, the opacities used to compute the absorption of the stellar flux can be independent from the opacities used to compute the thermal fluxes and we can also use the same visible opacities as in the analytical case.

2.5. Comparison to an asymptotically exact solution

In order to test the validity of the radiative solution found by the numerical model, we compare it to the analytical solution obtained by the method of discrete ordinates in the grey case (Chandrasekhar 1960). This method solves the radiative transfer equations with grey opacities for a non-irradiated atmosphere by replacing the integrals over angle by a gaussian sum. By increasing the number of terms in the sum (i.e. the order of the calculation), it converges towards the exact solution. The first order solution being equivalent to the Eddington approximation.

In fig. 2, we compare the numerical model for the grey, non-irradiated case to these analytical solutions up to the 5th order. The first order analytical solution deviates from the others and from the numerical result by about 2% with the maximum deviation occurring near optical depth unity. We can therefore expect the analytical models based on the Eddington approximation to differ from the exact results by about this value at least – we will come back to that in section 3. The higher order analytical solutions appear to smoothly converge towards the exact solution, but the numerical solution is found to be about ~ 0.5% warmer at low optical depths. This discrepancy arises from a different use of the Gaussian quadrature formula in the two approaches. Whereas the analytical solution uses the gaussian quadrature to compute the integral \( \int_{\mu=1}^{\mu=1} I_{\mu} \, d\mu \), the numerical code uses the quadrature formula to compute the flux integral \( \int_{\mu=1}^{\mu=1} \mu I_{\mu} \, d\mu \).

Therefore, the \( n^{th} \) order analytical solution is formally not the same as the \( n \) Gauss points numerical model and does not converge toward the same solution. We tested that using 8 Gauss points in the numerical model leads to a solution that is correct to 0.1% when compared to the \( n^{th} \) order analytical solution.

Because a 0.5% error is significantly smaller than the other sources of uncertainties in the model (the first one being due to the use of the Eddington approximation) and because of convergence problems arising in the 8 Gauss points model, we chose to only use the 5 Gauss points numerical model. We note that this kind of test is unfortunately not possible in the irradiated case (even in the grey approximation) for which no exact analytical solution is known.

3. Consequences of the Eddington approximation

We have seen in section 2.5 that an asymptotically exact solution of the radiative transfer problem can be found in the grey, non-irradiated case. Unfortunately, such a solution does not exist when accounting for external irradiation. The angle dependency of the radiative transfer problem therefore has to be approximated. Analytical models such as that of Paper I use a closure relation between two moments of the intensity field \( I_{\mu} \) (with \( \nu \) the frequency of the radiation):

\[
\int_{\mu=1}^{\mu=1} I_{\mu} \mu^2 \, d\mu \approx \frac{1}{3} \int_{\mu=1}^{\mu=1} I_{\mu} \, d\mu. \tag{6}
\]

This approximation is exact in two specific cases: when the radiation field is isotropic \( (I_{\mu} = cte \gamma \mu) \) and when radiation field is semi-isotropic \( (I_{\mu} = \Gamma'\gamma \mu > 0 \) and \( I_{\mu} = \Gamma'\gamma \mu < 0 \).

In the deep atmosphere, the radiation is quasi-isotropic and this approximation holds. Toward the top of the atmosphere, most of the thermal radiation comes from the deep layers and is therefore close to be semi-isotropic. In-between, the solution is only approximative. In addition, a boundary condition relating two other moments of the intensity field must be adopted:

\[
\left. \int_{\mu=1}^{\mu=1} I_{\mu} \mu \, d\mu \right|_{\mu=0} \approx f_H \left. \int_{\mu=1}^{\mu=1} I_{\mu} \, d\mu \right|_{\mu=0}. \tag{7}
\]

These two conditions form what is called the Eddington approximation.

In the grey, non-irradiated case, those two approximations are linked and \( f_H = 1/2 \). However using equation 6 and imposing \( f_H = 1/ \sqrt{3} \) leads to the exact solution at the top of the atmosphere, even though it lacks of self-consistency. In the irradiated case and in the non-grey case the two approximations are independent and \( f_H \) is usually set to either 1/2 or 1/ \( \sqrt{3} \), following the grey, non-irradiated case (see Paper I, for a complete discussion).

As discussed in section 2.5, the relative uncertainty on the temperature profile resulting from the Eddington approximation is ~ 2% in the grey, non-irradiated case. In order to estimate its magnitude in the grey and non-grey irradiated cases, we must rely on comparison with numerical models. We hereafter adopt the EGP numerical model with 5 Gauss points shown to be a very good approximation to the exact solution.

Now we compare the radiative solutions from our numerical model and different analytical models using the simplified opacities described in Sec. 2.3. Thus the solution can be expressed in function of the Rosseland optical depth \( \tau \) only and is independent of the Rosseland mean opacity or of the gravity. Once normalized by the effective temperature, the temperature in function of the optical depth in each model only depends on the values of \( \gamma_s, \gamma_p \) (or \( \kappa_2/\kappa_1 \)), \( \beta \) and the ratio \( T_{\text{irr}}/T_{\text{int}} \).
3.1. Irradiated semi-grey solutions

In the grey and semi-grey cases, several analytical models have been developed (Chandrasekhar 1960; Hansen 2008; Guillot 2010; Robinson & Catling 2012). As reviewed in Paper I, those models differ mainly by their choice of $f_H$ and their choice of the upper boundary condition. For simplicity, we will compare only three of them: the two different versions of Guillot (2010) (with $f_H = 1/2$ or $f_H = 1/\sqrt{3}$) and the semi-grey limit of the model derived in Paper I with $f_H = 1/2$ which uses as upper boundary condition a mix between the model of Guillot (2010) and the one of Hansen (2008). We compare those models for two different values of the main parameter of semi-grey models: the ratio of the visible to the thermal opacities, $\gamma_v$.

![Figure 3: Comparison solution between the radiative numerical solution (black line), our work (red line) and Guillot (2010) model for two different values of $f_H$ (blue and green lines) for a fully radiative semi-grey atmosphere with $\gamma_v = 0.25$ (plain lines) or $\gamma_v = 10$ (dashed lines). We set $\mu_s = 1/\sqrt{3}$, $T_{\text{int}} = 1288$ K and $T_{\text{int}} = 500$ K.](image)

Figure 3 compares these models for a typical irradiated Jupiter-mass exoplanet close to a solar-type star and shows the magnitude of the error which is due to the Eddington approximation – both the closure relation defined by eq. (6) and the adopted value of $f_H$ – and the chosen upper boundary condition, different between Guillot (2010) and Paper I. The left panel shows the temperature profiles as a function of optical depth which mainly depend on the magnitude of the greenhouse effect: when $\gamma_v$ is small, most of the incoming irradiation is absorbed deep in the atmosphere, the temperature increases monotonously with increasing depth, and the solution behaves like the non-irradiated solution with the same effective temperature (see the 1\textsuperscript{st} order case of Fig. 2). When $\gamma_v$ is large, most of the incoming stellar light is absorbed high up, creating a temperature inversion around visible optical depth unity (and thus thermal optical depth $\tau = 1/\gamma_v$).

The right panel of fig. 3 shows that the magnitude of the difference between the numerical solution and the analytical ones strongly depends on the choice of $f_H$ and of the top boundary condition, but remains of the same order-of-magnitude as for the non-irradiated grey case of section 2.5. Specifically, the Eddington approximation is found to lead to a ~4% uncertainty on the temperature profile and always converges towards zero at large optical depths. Except for the $f_H = 1/\sqrt{3}$ solution, all other analytical solutions, including the one from Paper I, systematically underestimate the temperature at a given depth.

It is obvious from fig. 3 that, unlike the non-irradiated case, no choice of $f_H$ can yield an exact skin temperature $T(\tau = 0)$ (see related discussion in Paper I).

3.2. Irradiated non-grey solutions

We now test the analytical model of Paper I in the non-grey case. In order to do so, we compare the analytical model to the numerical model for different values of the ratio of the thermal opacities $k_2/k_1$ and a single visible channel to the numerical model with the same thermal and visible opacities. We adopt $\beta = 0.86$ and $\gamma_v = 0.25$, typical values needed to reproduce detailed models of hot Jupiters (see section 5 hereafter) and the same irradiation and internal temperature as in the previous section.

Figure 4 shows the resulting temperature-optical depth profiles and the relative difference between the numerical and analytical solutions. As $k_2/k_1$ increases, the temperature profile gets cooler in the upper atmosphere and warmer in the deep atmosphere, an effect described in details in Paper I.

The red curve ($k_2/k_1 = 1$) corresponds to the semi-grey solution already seen in section 3.1 and fig. 3. As shown in the right panel, the discrepancy between the analytical and numerical models increases with the “non-greyness” of the opacities. The maximum error (in absolute terms) increases from about 2% to a little bit less than ~5% when $k_2/k_1$ is increased from 1 to 10\textsuperscript{3}. Moreover, the optical depth range for which the discrepancy is larger than say 1% increases in the same time from [0.1 ~ 10] to [10\textsuperscript{-5} ~ 10\textsuperscript{4}].

This increase in the extent of the region in which the temperature profile departs from the numerical solution is a direct consequence of the Eddington approximation in the two thermal channels with opacities $k_1$ and $k_2$ respectively: At high optical depth, in the diffusion limit, the radiation field is isotropic in each thermal channel and the Eddington approximation is valid. At very low optical depth radiation comes mostly from the levels where the first and the second thermal channels become optically thin, much deeper in the atmosphere. Therefore radiation in the optically thin layers is close to be semi-isotrop which validates the choice of the Eddington approximation. In between, the difference between the analytical and the numerical solutions exhibits two maxima. Those maxima correspond to the levels where the first and the second thermal bands become optically thin. As the ratio $k_2/k_1$ increases, the first channel becomes optically thin at higher Rosseland optical depth and the second channel becomes optically thin at lower Rosseland optical depth, creating the two-peak feature of Fig. 4.

We see however that the error induced by the Eddington approximation remains lower than 5%, with the deep temperatures being colder in the analytical model than in the numerical model. Compared to other sources of uncertainty (in particular our assumptions that $\beta$ and $k_2/k_1$ are uniform in the atmosphere), this is an acceptable level of uncertainty.

4. Consequences of convection on the overlaying radiative solution

At high-enough optical depth, the deep atmospheres of giant planets and brown dwarfs become convective (e.g., Guillot 2006), a consequence of the increase of the opacity with pressure (see Rauscher & Menou 2012). This increase of the opacity in substellar atmospheres is due both to collision-induced ab-
Fig 4: Comparison between the analytical model (plain lines) and the radiative numerical model (dashed lines) for different values of $\kappa_2/\kappa_1$ (left panel). The right panel shows the relative difference between the analytical and the numerical solution for each case. We used $\mu_s = 1/\sqrt{3}$, $T_{\text{irr}} = 1288$ K, $T_{\text{int}} = 500$ K, $\gamma_c = 0.25$ and $\beta = 0.86$.

Absorption by hydrogen molecules increasing with density (above roughly $10^{-2}$ g cm$^{-3}$) and eventually to new opacity sources linked to a larger abundance of electrons at temperatures ~ 2000 K and above. Generally, exoplanets and brown dwarfs with low-irradiation levels (i.e. such that $T_{\text{irr}} \lesssim T_{\text{int}}$) have a convective zone extending all the way from the deep interior to the $\tau \sim 1$ optical depths. This is for example the case of Jupiter, whose atmosphere becomes convective at pressures of order $P \sim 0.3$ bar – but with considerable heterogeneity depending on the latitude and longitude on the planet (e.g. Magalhaes et al. 2002; West et al. 2004). However, in very close-in exoplanets and brown dwarfs, the high stellar irradiation maintains the atmosphere in a very hot state and pushes the radiative/convective transition down to very high optical depths (see Guillot et al. 1996; Guillot 2006).

Numerical models naturally account for these convective zones by imposing a temperature gradient set by convection when a condition such as the Schwarzschild or Ledoux criterion is met. The temperature profile in the radiative part(s) of the atmosphere is recalculated iteratively to fulfill the radiative transfer equations. While it is easy to implement the first condition in analytical atmospheric models, it is generally not possible to implement the second one and modify the radiative solution due to the presence of a convective region. In the specific case of the grey and semi-grey model, Robinson & Catling (2012) recently derived a radiative-convective model that satisfies these two conditions, although it necessitates a small numerical integration. For non-grey thermal opacities, no analytical model solves self-consistently for the convective and the radiative parts of the atmosphere. In the specific case of the model of Paper I, the boundary condition of the radiative atmosphere lays in the optically thick layers and a the solution cannot be modified to account for a change in the temperature gradient at deep levels.

We want to estimate the error made when using the Schwarzschild criterion to include a convective zone at depth without recalculating the temperature of the radiative zone above it. We build our analytical radiative/convective model by switching from our radiative solution to the adiabatic solution whenever the convective gradient becomes lower than the radiative one. We compare the resulting analytical solution to the numerical solution in which both the depth of the radiative/convective boundary and the atmospheric temperature profile are converged iteratively. As the presence and depth of a convective zone depends on the exact value of the opacities, we need to specify the Rosseland mean opacity in our model. In this section, in order to facilitate the comparison, we fix the Rosseland mean opacity in function of pressure to its value in our fiducial model, described in Fig. 1. However, all our results will be relative to the depth of the convective zone and thus independent from the exact Rosseland mean opacity function used.

We only consider the case for which the atmosphere transitions from being radiative at high altitudes to being convective at depth (i.e. we do not include the possibility of alternating radiative and convective zones). In the convective zone, we assume that the temperature gradient is exactly adiabatic (i.e. we do not account for the superadiabatic gradient required to transport the heat flux – see e.g., Guillot (2006)).

4.1. Non-irradiated grey case

We first compare the solutions obtained in the non-irradiated grey case. In order to see how the location of the radiative/convective zone influences the solutions, we artificially modify the adiabatic gradient by a factor that varies from $1/4^\beta$ to 4.

Fig 5: Comparison of our numerical and analytical radiative-convective models for different adiabatic gradients in the non-irradiated, grey case. The thin line is the radiative zone and the thick one represents the convective zone. We used $T_{\text{int}} = 500$ K and $g = 25$ m/s$^2$. Note that the cases $\nabla_{\text{ad}} \times 2$ (green) and $\nabla_{\text{ad}} \times 4$ (red) are superimposed.

When the radiative/convective transition occurs below optical depth unity (red, green, blue and purple curves in Fig. 5), the difference between the analytical and numerical solutions is unchanged (the corresponding curves are indistinguishable on the right panel) and entirely due to the Eddington approximation as discussed in the previous section. This error is frozen at the radiative/convective boundary and propagates in the convective zone leading to an estimate of the deep temperature profile that is at most 2% percent off. For a convective zone that crosses the $\tau = 1$ limit (orange and black curves of Fig. 5), the lower boundary condition used in the analytical radiative model...
– that the deep atmosphere reaches the diffusion limit – is no more valid. The error becomes dependent on the location of the radiative/convective transition (and value of the adiabatic gradient). It however remains of the same order as the one due to the Eddington approximation. This validates models calculating the radiative/convective boundary of the deep convective zone without re-calculating the upper radiative profile. However, the presence of detached convective zones cannot be modeled correctly with this method, and an approach similar to Robinson & Catling (2012) is needed.

4.2. Non-irradiated non-grey case

We now turn, with Fig. 6, to the non-irradiated non-grey case, using the fiducial values $\kappa_2/\kappa_1 = 10^2$ and $\beta = 0.83$. As in the grey case, the errors are dominated by the Eddington approximation as long as the radiative/convective boundary occurs at optical depths larger than unity in the 2 thermal channels that are considered. (The error at low optical depths is larger but this is because the error due the Eddington approximation is increased in the non-grey case). However, as soon as the convective zone extends to levels of optical depth unity or smaller, the discrepancy between the analytical and numerical solutions increases significantly: the upper atmosphere warms up and our analytical solution is no more a good representation of the radiative atmosphere. This is clearly due to a non-grey effect. In a given spectral interval, the thermal flux present a given pressure is set by the integrated thermal emission of all the atmospheric layers below it in this specific spectral interval. At large optical depth, the emission is thermalized and the thermal flux per wavelength emitted in both spectral channels is the same regardless of the temperature gradient. At optical depth close to unity, the thermal flux in each channel depends on the actual temperature gradient. The analytical solution assumes that the temperature gradient is set by radiation transport everywhere and thus calculates inaccurately the flux emitted in the two spectral channels if convection extends to optical depths smaller than unity. The resulting temperature profile can differ by tens of percents from the numerical one. In addition, because the relative error is “frozen” at the one obtained at the radiative/convective transition, it does not tend towards zero at large optical depths as was the case with the purely radiative solutions.

Considerable caution should therefore be exerted when switching from radiative to convective gradient without recalculating the radiative solution in the general (non-grey) case. Specifically, when the atmosphere becomes convective at optical depths smaller than unity, the resulting temperature profile may be inaccurate by several tens of percents.

4.3. Irradiated non-grey case

We now consider the effect of irradiation with our fiducial “hot Jupiter” atmosphere. As already discussed, the strong irradiation tends to push the radiative/convective zone towards deep levels (see Guillot 2006). This is seen in the profiles of Fig. 6 which all occur at optical depths ~ 100 or deeper, with only a small dependence on the value of the chosen adiabatic gradient. As expected, this suppresses the changes of the temperature profile in the purely radiative atmosphere. The errors are almost independent of the assumed adiabatic gradient and mostly due to the Eddington approximation. For hot Jupiters, and generally for strongly irradiated atmospheres, the presence of a deep convective zone may be accounted for by adopting a purely radiative solution and switching to the convective one when the Schwarzschild criterion is verified.

Of course, for a smaller irradiation level and/or larger values of the $k_2/k_1$ ratio, the presence of a convective zone reaching optical depths closer to unity (in one of the thermal channels at least) will lead to an increase on the error of the calculated temperature profile. We expect this error to be approximately bounded by that of the non-grey, non-irradiated case.

5. Modeling the non-grey effects

Analytical model atmospheres are useful to understand the key physical processes of the radiative transfer in planetary atmo-
spheres. Unfortunately they cannot take into account the complex variation of the opacities with frequency, temperature and pressure. However, when modeling a specific planet atmosphere with a given chemical composition, the knowledge of the line-by-line opacities should drive the scientist in his choice of parameters when using the analytical models. In this section we wish to understand what characteristics of the opacities shape the temperature/pressure profile of a planet atmosphere and find a method to derive the simplified opacities of our analytical model from the line-by-line opacities. Ideally, the resulting analytical temperature/pressure profile should be a good approximation of the numerical solution computed with the full frequency, temperature and pressure dependent opacities.

A first approach to determine our coefficients is an a-posteriori determination i.e. to choose the coefficients such that the analytical and the numerical profiles match correctly. Although this should give the best results in terms of goodness of the fit, the retrieved coefficients might not be physically realistic and it could be difficult to relate them to the real atmospheric opacities. Another approach is to find a-priori values, directly from the opacities. This requires a deep understanding of the opacities and how they shape the temperature profile. A last possibility is to combine the two approaches: using an a-priori determination when possible and adjusting the remaining coefficients a-posteriori to fit the numerical profile.

5.1. A priori determination of the coefficients

5.1.1. Visible coefficients

The visible coefficients control at which depth the stellar flux is absorbed in the atmosphere. When the visible absorption is strong, the stellar flux is absorbed in the upper part of the atmosphere and radiated back to space. At the opposite, when the visible absorption is weak, the incoming irradiation is deposited at depth where the thermal optical depth is large and the deep atmosphere warms up. This is the well-known greenhouse effect.

When taking into account only one visible band (i.e. $\beta_v = 1$, as in Guillot (2010)), a natural choice for the parameter $\gamma_1$ is the ratio of the mean Rosseland visible opacity (using the stellar Planck function to weight the line by line opacities) to the mean Rosseland thermal opacity (using the local Planck function to weight the line by line opacities). Unfortunately, this ratio can vary significantly with height. We find that choosing the ratio at $\tau_v = 2/3$ (where $\tau_v$ is the Rosseland visible optical depth) leads to a correct representation of the absorbed stellar flux and could be used, together with a correct modeling of thermal non-grey effects, to get a first guess of the deep temperature. The part of the stellar flux that heats up the deep atmosphere is the one that propagates down to the $\tau > 1$ level. Thus, the opacities that determine the relevant strength of the visible absorption are the lowest visible opacities. The Rosseland mean is a good estimate of the weakest opacities over a given frequency range and is thus a suitable estimate.

However, when a significant portion of the stellar radiation is absorbed in the upper atmosphere of the planet, in particular when strong visible absorbers such as titanium oxide or sodium are present in the atmosphere, the stellar flux that reaches the $\tau > 1$ level depends strongly on the amount of absorption in each spectral channels in the upper atmosphere. The knowledge of $\gamma_v$ at a given level is not sufficient anymore for a correct estimate of the deep temperature. A more sophisticated model of the visible absorption is then needed.

The radiative transfer equations are linear with respect to the absorbed stellar flux. As shown in Paper I, our model can take into account as many spectral bands in the visible as needed with the condition that the different values $\gamma_i = k_{v_i}/k_{R}$ in each visible bands are constant through the atmosphere. If well chosen, constant non-grey visible opacities can relatively well approximate the absorbed stellar flux at all atmospheric levels. We therefore adopt the following method: using the line by line opacities from Freedman et al. (2008) and the actual numerical PT profile, we calculate the total absorbed flux at each layer of the atmosphere. We then adjust the relative contributions of the different visible opacity bands in order to correctly match the absorbed visible flux from the numerical simulation. The stellar flux absorbed by $n$ spectral bands of width $\beta_{v_i}$ is:

$$ F(\tau) = F_0 \sum_{i=1}^{n} \beta_{v_i} e^{-\gamma_{v_i} \tau_{v_i}}, \quad (8) $$

where the visible bands are homogeneously distributed in frequency (similar to the thermal bands), $F_0$ is the total incident stellar flux and the $\beta_{v_i}$ must verify: $\sum \beta_{v_i} = 1$. We apply this method using one to four opacity bands. As seen in Fig. 8, the absorbed flux can be described with a 2% accuracy with the two bands model and with a 0.5% accuracy for the four bands model. Our analytical model is limited to two spectral bands in the thermal channels. We consider that using two bands in the visible is a good compromise between complexity and accuracy. Fig. 9 compares the numerical model in black, taking into account all the line-by-line opacities and the semi-grey model (blue line) where the visible opacities are adjusted in order to have the same absorbed flux as in the numerical model but where the thermal opacities remain grey. The semi-grey model, even though it models correctly the absorbed flux in function of depth, lays far from the numerical solution. Clearly, non-grey thermal opacities are needed.

5.1.2. Thermal coefficients

The thermal coefficients describe how well the atmosphere is able to retain its energy. As explained qualitatively by Pierrehumbert (2010) and quantitatively in Paper I, the pres-
enence of non-grey thermal opacities can strongly affect the temperature profile of the planet. Because it is tied to the emission and absorption of the thermal flux, only the opacity variations that have an extent smaller or comparable to the local Planck function can contribute to the non-grey effects. The cumulative distribution function of the opacities in the frequency range covered by the local Planck function should thus contain enough information to constrain the non-grey effects. As a grey atmosphere cools down principally by emission from the τ = 2/3 level, the non-greyness of the opacities at this level should determine the strength of the non-grey effects.

We plot in Fig. 10 the cumulative distribution function of the opacities at this level. It represents the relative spectral width over which the opacities are lower than a given opacity k0 as a function of k0. The opacities cover a wide range of value (6 orders of magnitude in the specific example shown in Fig. 10). Our analytical model can describe the non-grey thermal opacities with only two parameters: the ratio of the Planck mean opacity to the Rosseland mean opacity, γp, and the relative size of the two bands, β. Unlike in the visible case, the thermal effects are local effects that do not depend of the behavior of the rest of the atmosphere. The value of γp can hence be calculated as a function of pressure and temperature from tables available in the community (e.g., Freedman et al. 2008).

The parameter β describes the relative amount of the opacities which are in the first band compared to the second band. The Rosseland mean opacity is determined by the smallest values of the opacities, which is the second band opacity in our model. We decide to use as β the fraction of the opacities in the spectral range covered by the local Planck function that are higher than the Rosseland mean opacity. This can be derived directly from the cumulative distribution function of the opacities plotted in Fig. 10. In the specific example of Fig. 10, 25% of the opacities lay below the Rosseland mean opacity hence β = 0.75.

5.2. Application/Different models

In order to test the goodness of our analytical model and derive reasonable estimates of the coefficients, we use the EGP numerical code to build a grid of atmospheric radiative/convective models for giant planets with a solar composition atmosphere, three different gravity (2.5, 25, and 250 m/s²), and an internal temperature of Tin = 100 K. We consider the case of a planet orbiting a sun-like star at various distances corresponding to irradiation temperatures from 100 K to 3000 K. All the profiles where calculated using μ* = 1/√3. Figure 11 shows different models obtained for different estimates of our coefficients (top panel) and a comparison between the numerical profiles and the resulting analytical profiles (bottom panel). In all models but model D, we use as Rosseland mean opacity the one calculated by the numerical model directly from the line-by-line opacities. In model D, we use the functional fit of the Rosseland mean opacities of Freedman et al. (2008) provided by Valencia et al. (2013). Model D is therefore a fully analytical model that can be downloaded and implemented by the community. We now describe the different models.

Model A: In this model, we adjust all our coefficients a-posteriori in order to have the best match to the numerical profiles. It leads to temperature profiles in agreement within 5% with the numerical ones. Therefore, it shows that our analytical model can represent a large variety of atmospheric temperature profiles and goes beyond the limitation of previous semi-grey models (Parmentier et al. 2013). However, the spread of the retrieved value of the coefficients makes it difficult to derive a trustable functional form and a better approach is needed in order to get a fully analytical model.

Model B: Here we use the methods of Sec. 5.1 to determine a-priori the various coefficients. The visible coefficients have four different behaviors in function of Tμ. Those behaviors reflect changes in chemical composition with the irradiation tem-

![Fig. 9: Pressure-temperature profiles calculated using the numerical model and the full set of opacities (black), the semi-grey (blue) and the non-grey (red) analytical. As un Fig. 1, g = 25 m/s², μ* = 1/√3, Tin = 100 K and Tm = 1253 K. The coefficients used for the analytical models are taken from table 1. The non-grey model is a much better match to the numerical profile than the semi-grey one.](image1)

![Fig. 10: Cumulative distribution function of the opacities at P = 0.85 bar and T = 1464 K, corresponding to the τ = 2/3 level of an atmosphere with Tm = 1253 K and a gravity g = 25 m/s². The Y axis represents the fraction of frequency where the monochromatic opacities are lower than the corresponding k0 of the X-axis. The red line shows the value of the Rosseland mean opacity and the black line the Planck mean opacity. We can see that 25% of the frequency range have monochromatic opacities smaller than the Rosseland mean opacity whereas 90% have monochromatic opacities smaller than the Planck mean opacity.](image2)
Fig. 11: Top panel: coefficients $\gamma_p$, $\beta$, $\beta_v$, $\gamma_{v1}$, and $\gamma_{v2}$ obtained for the six different models described in Sec. 5.2 in function of the irradiation temperature for planets of solar composition with different gravity and an internal temperature of 100K.

Bottom panel: Mean relative difference between the numerical and the analytical model for the six different models described in Sec. 5.2. The first line is the mean difference for $10^{-4}$ bar $< P < 10^{-2}$ bar, the second one for $10^{-2}$ bar $< P < 10^0$ bar and the third one for $10^0$ bar $< P < 10^2$ bar.
Parmentier: A non-grey analytical model for irradiated atmospheres II

Table 1: Functional form of the coefficients of the analytical model of Paper I valid for solar composition atmospheres. We use $X = \log_{10}(T_{\text{eff}, \mu})$.

| Coefficient | Expression | $T_{\text{eff}, \mu} < 300K$ | $300K < T_{\text{eff}, \mu} < 900K$ | $900K < T_{\text{eff}, \mu} < 1600K$ | $T_{\text{eff}, \mu} > 1600K$ |
|-------------|------------|-----------------------------|---------------------------------|---------------------------------|-----------------------------|
| $\log_{10}(\gamma_{c2})$ | $a + bX$ | $a = -0.076$ | $a = -11.8$ | $a = -8.14$ | $a = 0.95$ |
| | | $b = -0.94$ | $b = 3.75$ | $b = 2.46$ | $b = -0.32$ |
| $\log_{10}(\gamma_{c1})$ | $a + bX$ | $a = -0.064$ | $a = -9.65$ | $a = -16.1$ | $a = 11.9$ |
| | | $b = -0.043$ | $b = 3.98$ | $b = 5.40$ | $b = -3.15$ |
| $\beta_\alpha$ | $a + bX$ | $a = -0.039$ | $a = 2.23$ | $a = 1.37$ | $a = 0.90$ |
| | | $b = 0.19$ | $b = -0.75$ | $b = -0.32$ | $b = -0.12$ |
| $\beta$ | $a + bX + cX^2$ | $a = -3.43, b = 2.60, c = -0.39$ | |
| $\log_{10}(\gamma_p)$ | $a + bX + cX^2$ | $a = -22.2, b = 16.3, c = -2.83$ | |

Temperature (a plot of the line-by-line opacities for the four different regimes is shown in appendix):

- For $T_{\text{eff}, \mu} < 300K$, the visible coefficients are mostly constant with $T_{\text{eff}, \mu}$. At those low temperatures, the visible opacities are dominated by Rayleigh scattering and therefore exhibit a slight dependance with the gravity.

- For $300K < T_{\text{eff}, \mu} < 900K$, the visible opacities are dominated by the sodium lines at great depth, where the profile is warm enough to have sodium in gaseous state, whereas it is dominated by much smaller lines in the upper atmosphere. As $T_{\text{eff}, \mu}$ increases, the atmospheric Planck function shifts toward smaller wavelength, where the Rosseland mean opacity is smaller. Because the visible opacities stay roughly constant on this temperature range, their ratio to the Rosseland mean opacity, $\gamma_{c1}$ and $\gamma_{c2}$ increases with $T_{\text{eff}, \mu}$.

- For $900K < T_{\text{eff}, \mu} < 1600K$ the sodium and potassium become the main gaseous absorbers in the upper atmosphere, leading to a strong visible absorption and thus a sudden increase in the parameter $\beta_\alpha$.

- For $T_{\text{eff}, \mu} > 1600K$ titanium and vanadium oxides become the main gaseous absorbers in the upper atmosphere, creating again a sudden increase in the parameter $\beta$.

The thermal coefficients do not exhibit such discontinuities with $T_{\text{eff}, \mu}$. $\beta$ is rather constant and equal to $\approx 0.8$. This high value of $\beta$ can be interpreted as a predominance of the molecular bands (i.e. the water and methane bands) to the atomic lines in the non-grey opacities. Around $T_{\text{eff}, \mu} \approx 200K$, $\beta$ reaches values even closer to 1. At these temperatures, the Planck function at the atmospheric levels of $\tau \approx 2/3$ overlaps with the 5$\mu$m window in the opacities which is consistent with large values of $\beta$. In the other hand, $\gamma_p$ varies significantly with $T_{\text{eff}, \mu}$ with a "saddle-like" shape with two maxima at 200K and 1000K. At low temperatures, the Planck function of the atmosphere shifts towards large wavelengths ($> 10\mu$m) for which the opacities are almost constant, leading to small values of $\gamma_p$. At very high $T_{\text{eff}, \mu}$, the Planck function of the atmosphere shifts towards smaller wavelengths ($< 1\mu$m) for which the TiO broad-band absorption significantly flattens the opacities, leading to small values of $\gamma_p$.

In between, when the atmospheric Planck function is between 1 and 10$\mu$m, the opacities are dominated by the water and methane bands, which raises the value of $\gamma_p$ to $\approx 100$.

Although this model gives a correct estimate of the profile at high pressure, it leads to errors of $\approx 40\%$ at medium and low pressure. Given that the coefficients were all guessed $a$-priori,
reaching a 40% accuracy can be a fair, first guess of the temperature profile. This method could be extended to planets with very different opacities without going through the whole numerical integration of the radiative transfer equations. However, as proven by model A, a much better accuracy can be obtained by the analytical profile and a mixed method with some coefficients derived \textit{a-priori} and others \textit{a-posteriori} can be a good compromise.

Model C: In this model we use a mixed method to derive the coefficients of the analytical model, with some of them being derived \textit{a-priori} and some of them \textit{a-posteriori}. The method to determine the visible coefficients seems robust, as it can give the correct absorbed flux as a function of optical depth in the atmosphere with a 2% accuracy. The method to determine the thermal coefficients is more subject to caution as it is unclear whether the value of $\gamma_p$ in our analytical model should correspond to the value of $\gamma_p$ derived from the real opacities. Moreover, our criteria to choose $\beta$ (the fraction of the opacities that are higher than the Rosseland mean opacity) is ad-hoc and does not rely on strong physical arguments. At last, there is no strong argument to choose the depth at which those coefficients are calculated. We thus decided to obtain the visible coefficients from the a-priori solution and to fit the thermal ones by adjusting the analytical profile to the numerical profile. The resulting analytical solutions lead to an estimate of the temperature profile that always differs by less than 10% from the numerical solution.

Compared to model B, only the thermal coefficients are changed in model C. $\gamma_p$ keeps the same dependency with $T_{\text{eff},\mu}$ but is one order of magnitude smaller. This tends to suggest that the relevant opacities to calculate $\gamma_p$ are at deeper levels than the $\tau = 2/3$ level, as was done in model B. The parameter $\beta$ has a much higher dependency with $T_{\text{eff},\mu}$ than in model B. In particular, it decreases significantly at small $T_{\text{eff},\mu}$, enhancing the fact that, for small $T_{\text{eff},\mu}$, non-grey thermal effects become less important.

Model D: In order to have a fully analytical model we fit a functional form to the coefficients derived in model C as a function of $T_{\text{eff},\mu}$. Following the different regimes that we just described, we fit four different affine functions to the visible coefficients. The thermal coefficients having a much smoother variation with the irradiation temperature, we use only a 2nd order polynomial to fit the coefficients over the whole temperature range. The functional form of the coefficients are presented in Table 1. The resulting model matches the numerical profiles over a wide range of irradiation temperatures and planet gravity with an accuracy always better than 10% at all pressures (see Fig. 12 and column D of Fig. 11). Whereas for previous models the Rosseland mean opacities used in the analytical solution where calculated by the numerical model directly from the line-by-line opacities, Model D uses the fit of the Freedman et al. (2008) Rosseland mean opacities provided by Valencia et al. (2013). This makes model D a self-consistently fully analytical model.

Model E: Here, the importance of non-grey effects are tested. This model has grey thermal opacities (i.e. $\gamma_p = 1$) but uses the functional form derived in model D for the visible coefficients. Therefore, model E is a good representation of the absorption of the stellar irradiation by the atmosphere but lacks the non-grey effects. As expected, at low $T_{\text{eff},\mu}$, where non-grey effects were proven to be negligible, model E gives a reasonable estimate of the temperature profile. Conversely, when $T_{\text{eff},\mu}$ is higher than 300 K, the analytical profile lays 20 to 50% away from the numerical solution, compared to $\approx 10\%$ when including non-grey effects. The non-grey absorption of the stellar irradiation therefore cannot, by itself, explain the temperature structure in planetary atmospheres. Non-grey thermal effects, such as the ones considered in model D, are necessary.

Model F: In this model, a comparison with the previous estimate of Guillot (2010) is done. We use grey thermal opacities and the visible coefficients provided by Guillot (2010). Those coefficients were derived in order to match the deep temperature of highly irradiated planets, which it does well. However, for smaller irradiation model F fails to represent the numerical temperature profiles and the relative error between the two models can reach 40% at all atmospheric pressures.

### 5.3. The role of TiO and VO

The most irradiated planets have dayside atmospheric temperatures high enough such that, for a solar composition atmosphere some metal oxides such as titanium and vanadium oxides (TiO and VO respectively), are chemically stable in gas phase (Lodders 2002). Several studies (e.g., Hubeny et al. 2003; Fortney et al. 2008) show that, if present in solar abundance in the upper atmosphere of irradiated planets, titanium and vanadium oxides could change significantly the temperature structure of those atmospheres, accounting for the strong thermal inversion at high altitude inferred from secondary eclipse measurements (e.g., Knutson et al. 2008; Fortney et al. 2008). Moreover, the presence of TiO/VO changes the deep temperature profiles, the planets with TiO having cooler deep temperatures than planets without TiO, what can affects the cooling rate and thus the long term evolution of gas giant exoplanets (Budaj et al. 2012; Parmentier & Guillot 2011).

To this date, there has been no firm direct detection of TiO in exoplanets atmospheres (see Désert et al. 2008; Huisen et al. 2013; Sing et al. 2013). Several studies show that condensation in a vertical cold trap (Spiegel et al. 2009), in an horizontal cold trap (Parmentier et al. 2013) or dissociation by stellar radiation (Knutson et al. 2010) could significantly deplete the upper atmosphere of irradiated planets in TiO and VO.
We calculated a grid of pressure/temperature profiles and derived the same analytical models as in the previous section in the case where TiO and VO has been removed from the whole atmosphere by any of the aforementioned processes. The resulting coefficients are presented in Fig. A.2 and Table A.1 in appendix. As expected, the absence of TiO/VO changes significantly how the atmosphere absorbs the stellar irradiation. Whereas γ1 and γ2 remain almost unaffected, the parameter β, vanishes at high temperatures in the case without TiO/VO whereas it converges toward 0.5 in the case with TiO/VO: in the first case (β, → 0), all the stellar irradiation is absorbed deep in the atmosphere by the second visible band whereas in the second case (β, → 0.5) half of the stellar irradiation is absorbed at much lower pressures by the large TiO absorption band. This enhanced stellar absorption in the upper atmosphere creates the inversion observed in the profiles at high $T_{\text{eff, } \mu}$, in the case with TiO/VO (e.g., Fig. 12) and not in the case without (see Fig. 13). TiO and VO also affect the thermal coefficients. In particular, $\gamma_3$ decreases from 20 to 2 when $T_{\text{eff, } \mu}$, goes from 1000K to 3000K in the case with TiO/VO whereas it remains around ≈ 20 in the case without TiO/VO (see Fig. A.2). TiO has a rather broad band opacity that fills in the gaps due to the water opacities in the 0.4 – 1 µm wavelength range (see the difference between the two columns of Fig. A.1 at high effective temperatures) whereas absorption by VO is less significant. For large atmospheric temperatures, the local Planck function extends below 1 µm in wavelength (see Fig. A.1). When TiO is present the opacities are flatter below 1 µm, therefore non-grey thermal effects are lower in the case with TiO, explaining the trend observed in $\gamma_3$.

As a conclusion, the concomitant increase in temperature in the upper atmosphere and decrease in the deep atmosphere created by the presence of TiO/VO is caused not only by the absorption of part of the stellar flux in the upper atmosphere but also by the weakening of the non-grey blanketing effects due to the presence of broad-band TiO opacities in the 0.4 – 1 µm wavelength range. Both effects contribute equally to the relatively low effective temperatures in the deep atmosphere of gas giant planets when TiO and VO are present. Conversely, when TiO/VO are absent from the dayside atmosphere the stellar irradiation penetrates deeper and the blanketing is stronger, heating the deep atmosphere.

5.4. Low irradiation planets and brown dwarfs

Gravitational contraction and deuterium burning can be a significant source of internal luminosity in young giant planets and brown dwarfs respectively. This luminosity can overtake the stellar irradiation as the dominant heating source in the atmosphere. We calculated temperature/pressure profiles for planets with an internal temperature of 300K and 1000K and derived the same analytical models as in the case with $T_{\text{int}} = 100K$ presented in Sec. 5.2. Figs. 14 and 15 show that, as long as $T_{\text{int}} \approx T_{\mu}$ (i.e. $T_{\text{eff, } \mu,} \approx T_{\mu}$), model D of of Sec. 5.2 – derived considering an internal temperature of 100K – correctly matches the numerical temperature/pressure profile for higher internal temperatures. However, when $T_{\text{int}} \gg T_{\mu}$ (i.e. $T_{\text{eff, } \mu,} \approx T_{\mu}$), our analytical model cannot reproduce the temperature/pressure profiles predicted by the numerical model with the same accuracy than in the low internal temperature case (see Fig. A.3 in appendix). In particular, it can be seen from Figs. 14 and 15 that whenever $T_{\text{int}} \gg T_{\mu}$, our model is systematically hotter than the numerical model at low pressures with a discrepancy up to 40% between the two models. A possible interpretation is that whenever the internal temperature becomes the dominant heating source in the atmosphere, the stellar irradiation cannot balance anymore the non-grey thermal cooling of the upper atmosphere. As a consequence, the temperature gradient in the optically thin part of the atmosphere is larger than in highly irradiated atmospheres with the same $T_{\text{eff, } \mu}$. Therefore, the local atmospheric Planck function shifts by a frequency range of the order of its own extent along the atmosphere, an effect not taken into account in the model of Paper I where no opacity structures larger than the Planck function are considered. Moreover, when the internal luminosity dominates over the stellar irradiation, the convective zone can reach low optical depths which is another limit of our analytical model (see Sec. 4).

In summary, model D of Sec. 5.2 can be used to model irradiated planets atmospheres as long as the internal temperature is smaller than the irradiation temperature.

5.5. Recommended model

When modeling gas giant planets of solar composition, we recommend the use of model D of Sec. 5.2. This model uses the solution of the radiative transfer equations given by Paper I where the first five parameters describing the opacities are expressed as a function of the effective temperature (see table 1) whereas the analytical Rosseland mean opacities are given by Valencia et al. (2013). Model D is fully analytical, yet achieves an overall accuracy of 10% in temperature (at a given pressure) for irradiated giant planet atmospheres of solar composition with gravities in the range 2.5 – 250 m/s² and effective temperatures from 100 to 3000K assuming a smaller internal temperature. When the internal flux dominates over the external flux, model D becomes less accurate with an error that can reach ≈ 30% in the deep atmosphere and more than 40% at low optical depths.

The model has a proper behavior, but tends to predict temperature inversion absent from the numerical solutions. This is attributed to the simplification of the two thermal opacities when in reality many characteristic values should be used.

This accuracy is to be compared to that of simpler models. For example, models where the temperature is set to the effective temperature at $r = 2/3$ and the profile is assumed to follow the diffusion approximation below (i.e. the so-called Eddington boundary condition). We calculated that this commonly used prescription (e.g., Bodenheimer et al. 2003; Batygin et al. 2011, among many others ) lead to an error in the temperature profile below the $r = 2/3$ level of the order of ≈ 30% except fortuitously for 800K < $T_{\text{eff, } \mu} < 1200$K where the error is lower than 10%. Such an error on the boundary condition of interior models can strongly affects internal structure and planetary evolution calculations. Even semi-grey model (e.g., Hansen 2008; Guillot 2010) cannot reach an accuracy better than 20%, even with adjusted variable opacity coefficients.

A FORTRAN implementation of model D, including both cases with and without TiO/VO is available for download on the internet.

6. Conclusion

Analytical solutions of the radiative transfer equations, although derived using very restrictive (but necessary) approximations, offer a deep insight in the physical processes shaping the temperature profile of planetary atmospheres and can provide fast and roughly accurate solutions to be incorporated in more complex planetary models.

1 http://www.oca.eu/parmentier/nongrey
In this study we used hand-in-hand the analytical model derived in Paper I, that includes non-grey visible and thermal effects, and a state-of-the-art numerical model that solves the radiative transfer equations considering their full frequency and angular dependency.

We first quantified the validity of the Eddington approximation. We showed that this approximation leads to errors in the temperature profile of at most 2% in the grey case and 4% in the non-grey case.

Planets with a thick atmosphere usually become convective below a certain depth. Thus, a common way to produce a radiative/convective temperature profile is to switch from a radiative solution to a convective solution whenever the Schwarzschild criterion is met, considering that the radiative solution remains unaffected by the presence of a convective zone below it. We showed that this approach is always valid in the grey case – the error due to the Eddington approximation being frozen at the radiative/convective boundary and propagated along the convective zone. However, for non-grey atmospheric opacities, we showed that this method is valid only as long as the radiative/convective boundary remains in the optically thick layer of the atmosphere. When the radiative/convective boundary is in the optically thin region of the atmosphere, the radiative solution is very sensitive to the precise location of the radiative/convective boundary and this common approach can lead to relative errors of tens of percents when estimating the upper, radiative, atmospheric temperatures.

We showed that non-grey visible effects are not sufficient to explain the atmospheric temperature profiles that consider the full frequency dependent opacities and that non-grey thermal effects need to be taken into account. We provided a reliable method to obtain the visible coefficients of our analytical model directly from the opacities and explored how the thermal coefficients could also be directly derived from the knowledge of the line-by-line atmospheric opacities.

In particular, we showed that the presence of TiO can warm up the upper atmosphere and cool down the deep atmosphere not only because it absorbs a significant amount of stellar irradiation in the upper atmosphere, but also because its broad band opacity reduces the non-grey thermal “blanketing effect”.

Finally, using an a-priori determination of the visible coefficients and an a-posteriori determination of the thermal coefficients, we provide a fully analytical model for solar composition optically thick atmospheres. This model agrees with the numerical calculations within 10% over a wide range of gravities and effective temperatures. Our model leads to a much better estimate of the deep temperature profile than the previous analytical estimates. Therefore, when modeling the atmospheric structure of giant planets, we recommend the use of Model D described in Sec. 5.5 that uses the analytical expressions derived in Paper I with the first five parameters given in table 1 in the case with TiO and by table A.1 in the case without TiO and the Rosseland mean opacities given by Valencia et al. (2013). For convenience, we provide an implementation in FORTRAN of our model at the adress http://www.oca.eu/parmentier/ongrey.

References

Abramowitz, M. & Stegun, I. A. 1965. Handbook of mathematical functions with formulas, graphs, and mathematical tables, ed. Abramowitz, M. & Stegun, I. A.
Batygin, K., Stevenson, D. J., & Bodenheimer, P. H. 2011, ApJ, 738, 1
Bodenheimer, P., Laughlin, G., & Lin, D. N. C. 2003, ApJ, 592, 555
Budaj, J., Hubeny, I., & Burrows, A. 2012, A&A, 537, A115
Burrows, A., Marley, M., Hubbard, W. B., et al. 1997, ApJ, 491, 856
Chandrasekhar, S. 1960, Radiative transfer
Désert, J.-M., Vidal-Madjar, A., Lecavelier Des Etangs, A., et al. 2008, A&A, 492, 585
Fortney, J. J., Lodders, K., Marley, M. S., & Freedman, R. S. 2008, ApJ, 678, 1419
Fortney, J. J., Marley, M. S., Lodders, K., Saumon, D., & Freedman, R. 2005, ApJ, 627, L69
Freedman, R. S., Marley, M. S., & Lodders, K. 2008, ApJS, 174, 504
Goody, R. M. & Yung, Y. L. 1989, Atmospheric radiation : theoretical basis
Guillot, T. 2006, in Saas-Fee Advanced Course 31: Extrasolar planets, ed. D. Queloz, S. Udry, M. Mayor, W. Benz, P. Cassen, T. Guillot, & A. Quirrenbach, 243–368
Guillot, T. 2010, A&A, 520, A27+
Guillot, T., Burrows, A., Hubbard, W. B., Lunine, J. I., & Saumon, D. 1996, ApJ, 459, L35
Hansen, B. M. S. 2008, ApJS, 179, 484
Heng, K., Hayek, W., Pont, F., & Sing, D. K. 2012, MNRAS, 420, 20

Parmentier: A non-grey analytical model for irradiated atmospheres II
Hubeny, I., Burrows, A., & Sudarsky, D. 2003, ApJ, 594, 1011
Hubeny, C. M., Sing, D. K., Pont, F., et al. 2013, ArXiv e-prints
Knutson, H. A., Charbonneau, D., Allen, L. E., Burrows, A., & Megeath, S. T. 2008, ApJ, 673, 526
Knutson, H. A., Howard, A. W., & Isaacson, H. 2010, ApJ, 720, 1569
Lodders, K. 2002, ApJ, 577, 974
Magalhaes, J. A., Seifff, A., & Young, R. E. 2002, Icarus, 158, 410
Marley, M. S., Ackerman, A. S., Cuzzi, J. N., & Kitzmann, D. 2013, ArXiv e-prints
Marley, M. S. & McKay, C. P. 1999, Icarus, 138, 268
Marley, M. S., Saumon, D., Guillot, T., et al. 1996, Science, 272, 1919
Marley, M. S., Seager, S., Saumon, D., et al. 2002, ApJ, 568, 335
McKay, C. P., Pollack, J. B., & Courtin, R. 1989, Icarus, 80, 23
Mihalas. 1984, Foundation of Radiation Hydrodynamics (Oxford university press)
Parmentier, V. & Guillot, T. 2011, in EPSC-DPS Joint Meeting 2011, 1367
Parmentier, V. & Guillot, T. 2013, submitted to A&A
Parmentier, V., Showman, A. P., & Lian, Y. 2013, A&A, 558, A91
Pierrehumbert, R. T. 2010, Principles of Planetary Climate
Rauscher, E. & Menou, K. 2012, ApJ, 750, 96
Robinson, T. D. & Catling, D. C. 2012, ApJ, 757, 104
Saumon, D., Chabrier, G., & van Horn, H. M. 1995, ApJS, 99, 713
Showman, A. P., Fortney, J. J., Lian, Y., et al. 2009, ApJ, 699, 564
Sing, D. K., Lecavelier des Etangs, A., Fortney, J. J., et al. 2013, MNRAS
Spiegel, D. S., Silverio, K., & Burrows, A. 2009, ApJ, 699, 1487
Toon, O. B., McKay, C. P., Ackerman, T. P., & Santhanam, K. 1989, J. Geophys. Res., 94, 16287
Valencia, D., Guillot, T., Parmentier, V., & Freedman, R. S. 2013, ApJ, 775, 10
West, R. A., Baines, K. H., Friedson, A. J., et al. 2004, Jovian clouds and haze, ed. F. Bagenal, T. E. Dowling, & W. B. McKinno, 79–104
Appendix A: Additional material

The opacities in the form of k-coefficients used in the numerical model and discussed in Sec. 5.2 are presented in Fig. A.1 for a solar composition atmosphere and for an atmosphere without TiO/VO.

The analytical model adjusted to match the temperature/pressure profile of an atmosphere without TiO, discussed in Sec. 5.3 is presented in Fig. A.2 and Table A.1.

The effect of a strong internal luminosity on the analytical model, discussed in Sec. 5.4 is presented if Fig. A.3.

Table A.1: Functional form of the coefficients with $X = \log_{10}(T_{\text{eff},\mu})$ for atmospheres where TiO has been artificially removed.

| Coefficient | Expression | $T_{\text{eff},\mu} < 300K$ | $300K < T_{\text{eff},\mu} < 900K$ | $900K < T_{\text{eff},\mu} < 1600K$ | $T_{\text{eff},\mu} > 1600K$ |
|-------------|------------|-----------------|-----------------|-----------------|-----------------|
| $\log_{10}(y_{c2})$ | $a + bX$ | $a = -0.076$ | $a = -11.8$ | $a = -4.75$ | $a = -7.48$ | $b = -0.94$ | $b = 3.75$ | $b = 1.34$ | $b = 2.13$ |
| $\log_{10}(y_{c1})$ | $a + bX$ | $a = -0.064$ | $a = -9.60$ | $a = -10.8$ | $a = -18.5$ | $b = -0.043$ | $b = 3.97$ | $b = 3.66$ | $b = 5.88$ |
| $\beta_v$ | $a + bX$ | $a = -0.039$ | $a = 2.23$ | $a = 1.68$ | $a = 4.93$ | $b = 0.19$ | $b = -0.75$ | $b = -0.41$ | $b = -1.41$ |
| $\beta$ | $a + bX + cX^2$ | | | | | | | | $a = -7.25$, $b = 5.49$, $c = -0.93$ |
| $\log_{10}(y_p)$ | $a + bX + cX^2$ | | | | | | | | $a = -11.9$, $b = 8.46$, $c = -1.34$ |
Fig. A.1: Opacities from Freedman et al. (2008) organized as k-coefficient inside each bin of wavelength for a solar composition atmosphere (left column) and an atmosphere without TiO/VO (right column). The different colors are for different temperature and pressure taken along the corresponding numerical P-T profile. The thick bars on top represents the wavelength range where 90% of the thermal flux is emitted, the thin bars where 99% of the thermal flux is emitted. The four lines are for different effective temperature. We used $T_{int} = 100 \, \text{K}$, $\mu_\ast = 1/\sqrt{3}$ and $g = 25 \, \text{m/s}^2$. 
Fig. A.2: Top panel: coefficients obtained for the six different models described in Sec. 5.2 in function of the irradiation temperature for planets with different gravities and an internal temperature of 100K, 300K and 1000K. Here, TiO and VO have been artificially removed from the atmosphere. Bottom panel: Mean relative difference between the numerical and the analytical model for the six different models described in Sec. 5.2. The first line is the mean difference for $10^{-4}$ bar $< P <$ $10^{-2}$ bar, the second line for $10^{-2}$ bar $< P <$ $10^0$ bar and the third line for $10^0$ bar $< P <$ $10^2$ bar.
Fig. A.3: Top panel: coefficients obtained for the six different models described in Sec. 5.2 in function of the irradiation temperature for planets with a solar composition atmosphere with different gravities and an internal temperature of 100K, 300K and 1000K. The model of column D use the functional form of the coefficients derived in the case $T_{\text{int}} = 100$K only. The outliers at $T_{\text{eff}, \mu} = 300$K and $T_{\text{eff}, \mu} = 1000$K are due to the models with $T_{\text{int}} = 300$K and $T_{\text{int}} = 1000$K respectively. Bottom panel: Mean relative difference between the numerical and the analytical model for the six different models described in Sec. 5.2. The first line is the mean difference for $10^{-4}$ bar $< P < 10^{-2}$ bar, the second line for $10^{-2}$ bar $< P < 10^{0}$ bar and the third line for $10^{0}$ bar $< P < 10^{2}$ bar.