Gaussian Approximation for Penalized Wasserstein Barycenters

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Abstract—In this work we consider regularized Wasserstein barycenters (average in Wasserstein distance) in Fourier basis. We prove that random Fourier parameters of the barycenter converge to some Gaussian random vector in distribution. The convergence rate has been derived in finite-sample case with explicit dependence on measures count ($n$) and the dimension of parameters ($p$).

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1. INTRODUCTION

Monge–Kantorovich distance or Wasserstein distance is a distance between measures. It represents a transportation cost of measure $\mu_X$ into another measure $\mu_Y$:

$$W_p(\mu_X, \mu_Y) = \min_{\pi \in \Pi[\mu_X, \mu_Y]} \left( \int |x - y|^p d\pi(x, y) \right)^{1/p},$$

(1.1)

where the condition $\pi \in \Pi[\mu_X, \mu_Y]$ means that $\pi(x, y)$ has marginal distributions $\int_y d\pi(x, y) = d\mu_X(x)$ and $\int_x d\pi(x, y) = d\mu_Y(y)$. We focus on regularized $W_1$ distance with probabilistic space $L_1(\mathbb{R}^d) = \{\mathbb{R}^d, \mathcal{B}(\|\cdot\|_2), L_1\}$:

$$\tilde{W}_1(\mu_X, \mu_Y) = \min_{\pi \in \Pi[\mu_X, \mu_Y]} \int |x - y| d\pi(x, y) + R_{\epsilon}(\pi),$$

where $R_{\epsilon}(\pi)$ is a relatively small addition which improves differential properties of the distance. Namely, without $R_{\epsilon}(\pi)$ we can only bound the first derivative, with it we can bound the second derivative as well (we will discuss it in detail further in Theorem 4.3). The choice of the distance with $p = 1$ is motivated by a relative simplicity of the dual formulation of the problem.

Definition 1.1 (W-dual). For two random variables $X$ and $Y \in L_p(\mathbb{R}^d)$ with measures $\mu_X$ and $\mu_Y$ define Wasserstein distance in dual form as

$$W_p^p(\mu_X, \mu_Y) = \max_{\forall x, y : f(x) + g(y) \leq |x - y|^p} \{ \mathbb{E}f(X) + \mathbb{E}g(Y) \}$$

and particularly

$$W_1(\mu_X, \mu_Y) = \max_{\forall x : |\nabla f(x)| \leq 1} \{ \mathbb{E}f(X) - \mathbb{E}f(Y) \},$$

where $\forall x : |\nabla f(x)| \leq 1$ means that function $f$ is 1-Lipschitz.

This definition is equivalent to the original (1.1), which follows from Kantorovich–Rubinstein duality [12, 16]. If $p = 1$ we are dealing with one function instead of two, and also the condition on the gradient is more suitable than the condition on the sum of functions.

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There is a notion of mean in Wasserstein distance, called barycenter. And it is the main object in this research.

**Definition 1.2 (barycenter).** Consider a set of random measures \(\{\mu_i\}_{i=1}^n\), where each \(\mu_i \in L_1(\mathbb{R}^d)\).

In the regularized case *empirical* and *reference* barycenters are

\[
\hat{\mu} = \arg\min_{\mu} \sum_{i=1}^n \tilde{W}_1(\mu, \mu_i)
\]

and

\[
\mu^* = \arg\min_{\mu} \sum_{i=1}^n \mathbb{E}\tilde{W}_1(\mu, \mu_i).
\]

Note, that the use of regularization is very important not only for theoretical research, but also for the computation of barycenters in practical applications. Entropy [11] and quadratic [16] regularizations are the most common in the literature. In addition to simplifying optimization, regularization also allows to calculate a continuous parametric representation of the barycenters [15].

Barycenters are center-of-mass generalization. If we look at the barycenter of a set of uniform measures it extracts a common “shape” of these measures. If the measures are sampled from the same distribution then their barycenter can be treated as a sample approximation of the distribution mean. A simple example is a circles set with means \(\{m_i \in \mathbb{R}^2\}\) and radius’s \(\{r_i\}\)

\[
W_2^2((m_1, r_1), (m_2, r_2)) = \frac{1}{2\pi} \int_0^{2\pi} [(m_2 - m_1) - (r_2 - r_1) \cos(a)]^2 \]

\[+ [(r_2 - r_1) \sin(a)]^2 \, da = (m_2 - m_1)^2 + (r_2 - r_1)^2.
\]

Their \(W_2^2\) barycenter is also a circle with mean \(m = \frac{1}{n} \sum_{i=1}^n m_i\) and radius \(r = \frac{1}{n} \sum_{i=1}^n r_i\). We refer to papers [1, 7, 8] for an overview of the barycenters and related study.

\(W_1\) barycenter in the previous example doesn’t have an explicit formula but we will show below that it has Gaussian approximation. It is well known that the center-of-mass in \(l_2\) norm converges to a Gaussian random vector. As for the barycenter (\(\hat{\mu}\)), it is also expected to have some Gaussian properties. For example, if the measures are Gaussian themselves or one-dimensional or circles set then the Gaussian approximation of the \(W_2^2\) barycenter is proven in papers [1, 14]. In circles set case the mean and radius converge to some Gaussian variables as a sum of independent observations according to Central Limit Theorem. In one-dimensional case, denoting distribution functions by \(F_i(x)\)

\[
W_2^2(\mu_X, \mu_Y) = \int_0^1 |F_X^{-1}(s) - F_Y^{-1}(s)|^2 \, ds,
\]

one gets

\[
\hat{F}^{-1}(s) = \frac{1}{n} \sum_{i=1}^n F_i^{-1}(s).
\]
In the case of Gaussian measures with zero mean and variances \( \{S_i\} \)
\[
W_2^2(\mu_X, \mu_Y) = \text{tr}\{S_X\} + \text{tr}\{S_Y\} - 2\text{tr}\{(S_Y^{1/2}S_XS_Y^{1/2})^{1/2}\}
\]
and for some non-random matrix \( S_x \) (see [18]) the corresponded barycenter’s variance is
\[
\hat{S} = \frac{1}{n} \sum_{i=1}^{n} (S_x^{1/2}S_xS_x^{1/2})^{1/2} + O(1/n).
\]

In both last examples one deals with a mean of independent random variables. Being multiplied by \( \sqrt{n} \) factor, they converge to a Gaussian variable (or to a Gaussian process in case of \( \hat{F}^{-1}(s) \) according to Donsker’s theorem). In general case it appears to be very difficult to reveal such convergence because the barycenter doesn’t have an explicit equation and it is an infinite-dimensional object. In order to handle with this difficulty we propose an approximation of the \( W_1 \) barycenter by a sum of independent variables using projection into Fourier basis and involve some novel results from statistical learning theory. The perspective of Fourier Analysis provides a suitable representation of the Wasserstein distance and it is already studied in [23]. Denote a range of size \( p \) of the barycenter Fourier coefficients by
\[
\hat{\theta}_p = \mathcal{F}_p \left( \frac{d\hat{\mu}(x)}{dx} \right), \quad \theta^*_p = \mathcal{F}_p \left( \frac{d\mu^*(x)}{dx} \right).
\]

The first our result states that for some non-random matrix \( \hat{D} \), independent random vectors \( \{\xi_i\} \), and a depending on \( p \) constant with a high probability
\[
\left\| \hat{D} \left( \hat{\theta}_p - \theta^*_p \right) - \sum_{i=1}^{n} \xi_i \right\| \leq \frac{C(p)}{\sqrt{n}}
\]

Further we show that for some Gaussian vector \( Z \)
\[
W_1 \left( \hat{D}(\hat{\theta}_p - \theta^*_p), Z \right) \leq \frac{C(p)}{\sqrt{n}}
\]
and \( \forall z \in \mathbb{R}_+^n : 
\[
\left| \mathbb{P} \left( ||\hat{D}(\hat{\theta}_p - \theta^*_p)|| > z \right) - \mathbb{P} (||Z|| > z) \right| \leq \frac{C(p)}{\sqrt{n}}.
\]

**Statistical application.** The last statement allows us to obtain the confidence region of parameter \( \hat{\theta}_p \) and describe the distribution inside the region. Besides, the main theorems of this paper may be applied in theoretical bootstrap procedure validity proof [10, 22]. Gaussian approximation is commonly used to justify that the parameter distribution obtained from the bootstrap is close to the distribution of \( \hat{\theta}_p \). Estimation of the parameter distribution is needed to determine the uncertainty of predictions and justification for adding regularization in machine learning. Change point detection methods are often based on bootstrap procedure and confidence intervals for \( \hat{\theta}_p \) (see [2, 19]). When searching for change points with window methods, one has to determine whether there is a statistically significant difference between the models in the left and right parts of the window. More generally, confidence intervals are designed to test hypotheses to determine whether a sample of random measures belongs to a certain parametric family. In [8] the authors demonstrate application of barycentric coordinates that allow to infer missing geometry of an input mesh using a set of 3D models. Recent article [6] shows in experiments that barycenters may be helpful as a loss function in unsupervised face landmarks detection task.

**Contributions of the paper.** We provide a new representation of Wasserstein distance in Fourier space and find upper estimates for the derivatives. Specifying and extending statistical learning theory for our case, we derive some important properties of the barycenter model. Finally, we prove Gaussian approximation in distribution for the Fourier parameters of the barycenter.

The structure of this paper is the following. The main theorems are in Section 2. Section 3 deals with independent parametric models and describes how one can approximate parameter deviations by a sum of independent random vectors \( \{\xi_i\} \). In Section 4 we explore the barycenters model, compute derivatives of the Wasserstein distance using infimal convolution of support functions and check the required assumptions from the Section 3.
2. THE MAIN RESULT

**Definition 2.1 (W-dual-regularised).** For two random variables $X$ and $Y \in \mathcal{L}_1(\mathbb{R}^d)$ with densities $\varphi_X$ and $\varphi_Y$ and some density function $G$ define a regularised Wasserstein distance in dual form as

$$\tilde{W}_1(\varphi_X, \varphi_Y) = \max_{f : ||\nabla f|| \leq 1} \left\{ \mathbb{E}f(X) - \mathbb{E}f(Y) - \varepsilon \int ||\nabla f(x)||^2 G(x)dx \right\}.$$ 

The regulariser term in this definition allows to bound the second derivative of the distance which will be shown below.

Consider a set of random measures (random measure is a measure-valued random element) with densities $\varphi_1, \ldots, \varphi_n$. Let the barycenter measure $\hat{\mu}$ has density $\hat{\varphi}$ and Fourier coefficients $\hat{\theta} = \theta(\hat{\varphi}) \in \mathbb{R}^\infty$. Denote Fourier coefficients of the other measures $\forall i : \theta_i = \theta(\varphi_i) \in \mathbb{R}^\infty$. Define an independent parametric model with dataset $(\theta_1, \ldots, \theta_n)$ and parameter $\theta$:

$$L(\theta) = -\sum_{i=1}^n \tilde{W}_1(\varphi(\theta), \varphi_i(\theta_i)).$$ (2.1)

Note that the model’s Likelihood has minus sign and there is no normalising factor $1/n$. According to notation of the previous section the reference parameter value (coefficients of the reference barycenter) is $\theta^* = \arg\max_{\theta} \mathbb{E}L(\theta)$.

Define a local region around $\theta^*$

$$\Omega(\mathbf{r}) = \{\theta : ||D(\theta - \theta^*)|| \leq \mathbf{r}\},$$

where $D$ is Fisher matrix of the model

$$D^2 = -\nabla^2 \mathbb{E}L(\theta^*).$$

Here and below, operator $\nabla$ denotes Frechêt derivative. We also assume below that matrix $D$ is invertible. This assumption is not strict, since instead of it we can use pseudo-inverse matrix $D^+$ for multiplication. It operates identically and preserves property $D^+D = I$. And $D^+$ may be obtained through singular value decomposition $USU^T$, where $S^+$ is formed from $S$ by taking reciprocal of non-zero elements on the diagonal and leaving the zeros in place. Since we do not divide the model by factor $n$, the matrix $D^2$ has an order $O(n)$ (see [21], Section 2.1).

**Theorem 2.2.** Assume that exists a differentiable Fourier basis $\{\hat{\psi}_k(x)\}_{k=1}^\infty$ with Gram matrix $G(x)$ in which for some constant $C_D$ and minimal eigenvalue it holds

$$\lambda_{\min}(DK \circ GD) \geq C_D n,$$

where

$$K \circ G = \int \nabla^T \hat{\psi}(x) \nabla \hat{\psi}^T(x) G(x) dx.$$ 

Let the random Fourier parameters of the dataset have a common density: $\theta_1 \ldots \theta_n \sim q(\theta)$, and it fulfills condition

$$\int_{\mathbb{R}^\infty} ||D^{-1} \nabla q(\theta)|| d\theta = \frac{C_q}{\sqrt{n}}.$$

Let $\hat{\theta}, \theta^* \in \mathbb{R}^\infty$ be the Fourier coefficients of the empirical and reference barycenter defined above and $\varepsilon$ be the regulariser constant of $\tilde{W}_1$. Then with probability $1 - e^{-t}$

$$\left\| D(\hat{\theta} - \theta^*) - D^{-1} \nabla L(\theta^*) \right\| \leq \diamond (\mathbf{r}, t),$$

$$\diamond (\mathbf{r}, t) \leq \frac{O(H_1 + t)}{\varepsilon C_D^3 \sqrt{n}} + \frac{O(t C_q)}{\varepsilon C_D^2 \sqrt{n}} + O\left(\frac{H_2}{n}\right),$$

where $H_1, H_2, t \in \mathbb{R}$.
where $H_1, H_2$ are components of ellipsoid entropy with matrix $D^2$ and eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$, such that for some absolute constant $C$ and $\alpha > 1$

$$H_1 \leq C(\alpha - 1)^{-1/2} \sqrt{\sum_{i} \frac{\log^\alpha(\lambda_i^2/\lambda_{\min}^2)}{\lambda_i^2/\lambda_{\min}^2}}$$

and

$$H_2 \leq C \sum_{i} \frac{\lambda_{\min}}{\lambda_i}.$$ 

We have shown that deviations of the parameter $\hat{\theta}$ may be approximated by expression $D^{-1}\nabla L(\theta^*)$ which is a sum of independent random vectors. Next one may derive Gaussian approximation for the last term and find the correspondent normal vector $Z$. Define additional Fisher matrix corresponded to the projection into the first $p$ elements of the parameter $\theta$ (we will describe it in more detail in Section 3).

$$\tilde{D}^2 = D_{p\times p}^2 - D_{p\times \infty}^2 D_{\infty \times \infty}^{-2} D_{\infty \times p}^2$$

such that

$$D^2 = \begin{pmatrix} D_{p\times p}^2 & D_{p\times \infty}^2 \\ D_{\infty \times p}^2 & D_{\infty \times \infty}^2 \end{pmatrix}$$

and define the gradient of the projection into the first $p$ elements of the parameter $\theta$. 

$$\tilde{\nabla} = \nabla_{1...p} - D_{p\times \infty}^2 D_{\infty \times \infty}^{-2} \nabla_{p...\infty}.$$ 

The next result shows that vector $\hat{\theta}$ is close in distribution to Gaussian vector 

$$Z \sim \mathcal{N}(0, \text{Var}[\tilde{D}^{-1}\tilde{\nabla} L(\theta^*)]).$$ 

**Theorem 2.3.** Let $\hat{\theta}_p, \theta^*_p \in \mathbb{R}^p$ be the first $p$ Fourier coefficients of the empirical and reference barycenters. Then under assumptions from Theorem 2.2 with probability $(1 - e^{-t})$

$$W_1(\tilde{D}(\hat{\theta}_p - \theta^*_p), Z) \leq \diamondsuit(r, t) + O\left(\frac{p \log n}{\sqrt{CDn}}\right)$$

and $\forall z \in \mathbb{R}_+$

$$\left| \mathbb{P}(\|\tilde{D}(\hat{\theta}_p - \theta^*_p)\| > z) - \mathbb{P}(\|Z\| > z) \right| \leq O\left(\frac{\diamondsuit(r, t)}{\sqrt{p}} + \frac{\sqrt{p} \log^2 n}{\sqrt{CDn}}\right),$$

and for $\diamondsuit$ we provide an upper bound in Theorem 2.2.

### 3. STATISTICAL LEARNING THEORY

In this section we consider an infinite dimensional statistical model $L(\theta)$. Let parameter $\theta$ consists of two parts $(u, v)$, such that $u = \theta_{1...p} \in \mathbb{R}^p$. Working with a finite dataset of size $n$ we are going to find the deviations of argmax$L(\theta)$ basing on three model assumptions listed below. Further we will make some specifications for independent models and check that the barycenter model (2.1) satisfies to these assumptions.

**General Approach**

The Likelihood function $L(\theta) = L(\theta, \mathbb{Y})$ depends on the parameters vector $\theta = (u, v)$ and a random dataset $\mathbb{Y}$ of size $n$. Denote parameter’s MLE and reference values:

$$\hat{\theta} = \arg\max_{\theta} L(\theta),$$

$$\bar{\theta} = \arg\max_{\theta} L_{\bar{\theta}}(\theta).$$
\[ \theta^* = \arg\max_{\theta} \mathbb{E}L(\theta). \]

We are going to study deviations of \( \hat{\theta} \) and \( u \) in the following sense. For Fisher matrix \( D^2 \) and projection matrix defined below \( \hat{D}^2 \) the deviation \( D(\hat{\theta} - \theta^*) \) may be approximated by \( D^{-1}L(\theta^*) \) and analogically \( \hat{D}(\hat{u} - u^*) \) by \( \hat{D}^{-1}\nabla L(\theta^*) \). This approach bases on [21]. Remind that by definition Fisher matrix is

\[ D^2 = -\nabla^2\mathbb{E}L(\theta^*) = \begin{pmatrix} D^2_u & D^2_{uv} \\ D^2_{vu} & D^2_v \end{pmatrix}. \]  

(3.1)

Here and below operator \( \nabla \) means Fréchet derivative. We also assume below that matrix \( D \) is invertible. Denote the stochastic part of the Likelihood

\[ \zeta(\theta) = L(\theta) - \mathbb{E}L(\theta). \]

It would be easier to deal with the model if the matrix \( D^2 \) has block-diagonal view \( (D^2_{uv} = D^2_{vu} = 0) \). One can make parameter replacement in order to satisfy to this condition. Define a new parameter \( \vartheta = \vartheta(u, v) \) such that

\[ \nabla_u \nabla_{\vartheta}^T \mathbb{E}L(\theta^*) = -\nabla_{\vartheta} \nabla_u^T \mathbb{E}L(\theta^*) = 0 \]

and

\[ \vartheta = v + D_v^{-2}D^2_{vu}u. \]

Or in other words the parameter’s transformation matrix is

\[ S = \begin{pmatrix} I & 0 \\ D_v^{-2}D^2_{vu} & I \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} I & 0 \\ -D_v^{-2}D^2_{vu} & I \end{pmatrix}. \]

The gradient in the new coordinates \((u, \vartheta)\) may be obtained by rule \( \nabla(u, \vartheta) = (S^{-1})^T \nabla(u, v) \). Use notation \( \nabla\) for its first part

\[ \nabla = \nabla_u(u, \vartheta) = \nabla_u - D^2_{uv}D_v^{-2}\nabla_v. \]

Fisher matrix after parameters replacement changes according to rule \( D^2(u, \vartheta) = (S^{-1})^T D^2 S^{-1} \), so in the new coordinates it has view

\[ D^2(u, \vartheta) = -\nabla^2\mathbb{E}L(u^*, \vartheta^*) = \begin{pmatrix} \hat{D}^2 & 0 \\ 0 & D^2_{\vartheta} \end{pmatrix}, \]

\[ \hat{D}^2 = D^2_u - D^2_{uv}D_v^{-2}D^2_{vu}. \]

The last equation clarifies the origins of the projection matrix structure. It makes possible to study the deviations \( u - u^* \) independently on the other part of \( \theta \). We derive it below. Define a local region around point \( \theta^* \)

\[ \Omega(r) = \{ \theta : ||D(\theta - \theta^*)|| \leq r \}. \]  

(3.2)

Now we will write down three conditions on the Likelihood derivatives which are essential for the deviations of \( \hat{\theta} \). The first and second conditions should be satisfied in the local region \( \Omega(r) \). The third condition is required to make expansion of the previous two conditions to the whole parameter space \( \mathbb{R}^\infty \). Further we will show that these conditions are also sufficient for deviation bounds of the parameter \( u \), namely from deviations bound of \( \hat{\theta} \) follows analogical bound for \( u \).

**Assumption 1.** In the region \( \Omega(r) \)

\[ ||-D^{-1}\{\nabla\mathbb{E}L(\theta) - \nabla\mathbb{E}L(\theta^*)\} - D(\theta - \theta^*)|| \leq \delta(r)r. \]

**Assumption 2.** In the region \( \Omega(r) \) with probability \( 1 - e^{-t} \)

\[ \sup_{\theta \in \Omega(r)} ||D^{-1}\{\nabla\zeta(\theta) - \nabla\zeta(\theta^*)\}|| \leq z(t)r. \]
Assumption 3. The Likelihood function is convex \((-\nabla^2 L(\theta) \geq 0)\) or the expectation of Likelihood function is upper-bounded by a strongly convex function \((\exists b > 0 : \mathbb{E}L(\theta^*) - \mathbb{E}L(\theta) \geq b\|D(\theta - \theta^*)\|^2)\).

The first two assumptions guarantee approximation of the parameter difference \(D(\theta - \theta^*)\) by the difference of the gradients of the Likelihood function inside the region \(\Omega(r)\) with normalization matrix \(D^{-1}\). Below we will obtain that in case \(\theta = \hat{\theta}\) the expression \(D(\hat{\theta} - \theta^*)\) has corresponded approximation \(D^{-1}\nabla L(\theta^*)\). Random vector \(\nabla L(\theta^*)\) is convenient for analyzing statistical properties because it does not depend on parameter \(\hat{\theta}\) and in many cases is a sum of independent random vectors. The last property is important for Gaussian approximation. In the following sections we will prove that for the barycenter model \(\nabla L(\theta^*)\) is close in distribution to some Gaussian vector.

The third assumption in combination the 1st and 2nd allows us to bound the radius of \(\Omega(r)\) which includes \(\hat{\theta}\). Use notation
\[
\Diamond = \{\delta(r) + z(t)\} r.
\]
Consider a convex Likelihood function. From \((-\nabla^2 L(\theta) \geq 0)\) and \((L(\hat{\theta}) > L(\theta^*))\) follows that the local region \(\Omega(r)\) that includes \(\hat{\theta}\) should cover the next region
\[
\Omega(r) \supset \{\theta : L(\theta) \geq L(\theta^*)\}.
\]
Use notation \(D^2(\theta) = -\nabla^2 \mathbb{E}L(\theta)\). Estimate the minimal possible radius \(r\) that satisfy to the previous condition. Let \(\theta_0\) be some point between \(\theta\) and \(\theta^*\) that is used in Taylor expansion with central point \(\theta^*\).
\[
0 \geq L(\theta^*) - L(\theta)
= -(\theta - \theta^*)^T \nabla L(\theta^*) - \frac{1}{2}(\theta - \theta^*)^T \nabla^2 \zeta(\theta_0)(\theta - \theta^*) + \frac{1}{2}\|D(\theta_0)(\theta - \theta^*)\|^2.
\]
Assumption 1 provides
\[
\|D(\theta_0)(\theta - \theta^*)\|^2 = \|D(\theta - \theta^*)\|^2 + \|(D(\theta_0) - D)(\theta - \theta^*)\|^2
\geq r^2 - \delta(r)r^2.
\]
Assumption 2 provides with probability \((1 - e^{-t})\)
\[
(\theta - \theta^*)^T \nabla^2 \zeta(\theta_0)(\theta - \theta^*) \leq z(t)r^2.
\]
Put these two expressions into the initial inequality and obtain bound for radius \(r\)
\[
0 \geq -\|D^{-1}\nabla L(\theta^*)\|r - \frac{z(t)}{2}r^2 + \frac{1 - \delta(r)}{2}r^2,
\]
\[
r(1 - \delta(r) - z(t)) \leq 2\|D^{-1}\nabla L(\theta^*)\|.
\]

Theorem 3.1 (proof in Appendix A). Let the Likelihood function \(L\) be convex \((-\nabla^2 L(\theta) \geq 0)\) and for \(r \leq r_0\) it holds \(\delta(r) + z(t) \leq 1/2\). Then under Assumptions 1,2 with probability \(1 - e^{-t}\)
\[
\|D(\hat{\theta} - \theta^*) - D^{-1}\nabla L(\theta^*)\| \leq \Diamond(r, t),
\]
\[
\|\hat{D}u - u^* - D^{-1}\nabla L(\theta^*)\| \leq \Diamond(r, t),
\]
and
\[
r \leq r_0 = 4\|D^{-1}\nabla L(\theta^*)\|.
\]

Remark 3.2. Assumption 3 includes two options: convex model or upper-bounded model by a strongly convex function. In the previous theorem we have considered only the first option because the barycenter model (2.1) is convex (we will prove it below). The second option with a non-convex function \(L\) is more general and considered in [21] (Theorem 2.1). We have described convex models separately to get a more accurate bound for the region \(\Omega(r)\).
Independent Models

Consider independent models (models with independent observations) and obtain a simpler variant of Assumption 2. Involve three basic lemmas for that.

**Lemma 3.3** (Bernstein’s inequality Theorem 2.10[9]). Let real-valued random variables $X_1 \ldots X_n$ be independent. Assume that there exist positive numbers $v$ and $R$ such that

$$\mathbf{v}^2 = \sum_{i=1}^{n} \mathbb{E}X_i^2$$

and for all integers $q \geq 3$

$$\sum_{i=1}^{n} \mathbb{E}[X_i]^q \leq \frac{q!}{2}v^2 R^{q-2}.$$

Then for all $\lambda \in (1, 1/R)$

$$\log \mathbb{E}e^{\lambda \sum_{i}(X_i - \mathbb{E}X_i)} \leq \frac{\mathbf{v}^2 \lambda^2}{2(1 - R\lambda)}.$$

**Lemma 3.4** (Dudley’s entropy integral Lemma 13.1[9]). Let $\Omega$ be a finite pseudometric space and let $f(\theta)$ ($\theta \in \Omega$) be a collection of random variables such that for some constants $a, v, R > 0$, for all $\theta_1, \theta_2 \in \Omega$ and all $0 < \lambda < (Rd(\theta_1, \theta_2))^{-1}$

$$\log \mathbb{E}\exp\{\lambda(f(\theta_1) - f(\theta_2))\} \leq a\lambda d(\theta_1, \theta_2) + \frac{\mathbf{v}^2 \lambda^2 d(\theta_1, \theta_2)}{2(1 - R\lambda d(\theta_1, \theta_2))}.$$

Then for any $\theta_0 \in \Omega$,

$$\mathbb{E}[\sup_{\theta} f(\theta) - f(\theta_0)] \leq 3ar + 12v \int_{0}^{r/2} \sqrt{\log N(\varepsilon, \Omega)} d\varepsilon + 12R \int_{0}^{r/2} \log N(\varepsilon, \Omega) d\varepsilon,$$

where $r = \sup_{\theta \in \Omega} d(\theta, \theta_0)$ and $N(\varepsilon, \Omega)$ is covering number.

**Lemma 3.5** (Bousquet’s inequality Theorem 12.5[9]). Consider independent random variables $X_1 \ldots X_n$ and let $\mathcal{F} : X \to \mathbb{R}$ be countable set of functions that satisfy conditions $\mathbb{E}f(X_i) = 0$ and $\|f\|_{\infty} \leq R$. Define

$$Z = \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} f(X_i).$$

Let $v^2 \geq \sup_{f \in \mathcal{F}} \sum_{i=1}^{n} \mathbb{E}f^2(X_i)$ then with probability $1 - e^{-t}$

$$Z < \mathbb{E}Z + \sqrt{2t(v^2 + 2R\mathbb{E}Z)} + \frac{tR}{3}.$$

Apply these lemmas in order to simplify Assumption 2 for independent models. The likelihood of independent model is a sum of independent functions:

$$(L - \mathbb{E}L)(\theta) = \zeta(\theta) = \sum_{i=1}^{n} \zeta_i(\theta).$$

Note that $\zeta_i$ depends from the implicit $i$th element from the dataset, such that $\zeta_i(\theta) = \zeta_i(\theta, Y_i)$.

**Definition 3.6** (H). Entropy of the model’s parameter space $\Omega(r)$ with metric $\|D(\theta_1 - \theta_2)\|$

$$H_1 = \int_{0}^{1/2} \sqrt{\log N(\varepsilon r, \Omega)} d\varepsilon, \quad H_2 = \int_{0}^{1/2} \log N(\varepsilon r, \Omega) d\varepsilon,$$

where $N(\varepsilon, \Omega)$ is covering number.
Theorem 3.7. Let $D$ be the matrix from Assumption 2 and $\forall \theta \in \Omega(\gamma)$ (ref. Definition 3.2)

$$\sup_{||u||=1} \sum_{i=1}^n \mathbb{E}(u^T D^{-1} \nabla^2 \zeta_i(\theta) D^{-1} u)^2 \leq \mathbf{v}^2$$

and

$$||D^{-1} \nabla^2 \zeta_i(\theta) D^{-1}|| \leq R.$$ 

Then Assumption 2 fulfills inside $\Omega(\gamma)$ with probability $1 - e^{-t}$ and

$$\mathbb{I}(t) \leq \mathbf{v}(12\sqrt{2}H_1 + \sqrt{2t}) + R(24H_2 + 24\sqrt{H_2t} + t/3),$$

where $H_1$ and $H_2$ are the components of ellipsoid entropy (3.4).

Remark 3.8. We have obtained a new form of Assumption 2 for independent models with bounded components $\nabla^2 \zeta_i(\theta)$. It is much easier to check it for a specific model (such as a barycenter model), because instead of a stochastic process $\nabla^2 \zeta_i(\theta)$ we are dealing with its second moment.

Proof. Set a random process for each $i$:

$$X_i(\gamma, \theta) = \frac{1}{r} \gamma^T \{\nabla \zeta_i(\theta) - \nabla \zeta_i(\theta^*)\}.$$

Such that

$$\sup_{||D\gamma|| \leq r} \sum_{i} X_i(\gamma, \theta) = ||D^{-1} \{\nabla \zeta(\theta) - \nabla \zeta(\theta^*)\}||$$

$\forall$ fixed $(\gamma, \theta) \in \Omega(\gamma, 0) \times \Omega(\gamma, \theta^*)$ and $||u|| = 1$:

$$\sup_{u} \mathbb{E} \sum_{i=1}^n (\nabla_{\theta} X_i(\gamma, \theta)^T D^{-1} u)^2 = \sup_{u} \mathbb{E} \sum_{i=1}^n \frac{1}{r} (\gamma \nabla^2 \zeta_i(\theta)^T D^{-1} u)^2$$

$$\leq \sup_{u} \mathbb{E} \sum_{i=1}^n (u^T D^{-1} \nabla^2 \zeta_i(\theta)^T D^{-1} u)^2 \leq \mathbf{v}^2.$$ 

Analogically

$$\sup_{u} \mathbb{E} \sum_{i=1}^n (\nabla_{\gamma} X_i(\gamma, \theta)^T D^{-1} u)^2 \leq \mathbf{v}^2$$

$\forall i \in 1, \ldots, n$:

$$||D^{-1} \nabla X_i(\gamma, \theta)|| \leq R.$$ 

Apply Lemma 3.3 for the sum of random variables $X(\gamma, \theta) = \sum_i X_i(\gamma, \theta)$ when $(\gamma, \theta)$ are fixed

$$\log \mathbb{E}\exp \lambda(X(\gamma, \theta_1) - X(\gamma, \theta_2))+ \log \mathbb{E}\exp \lambda((\theta_1 - \theta_2)^T \nabla_{\theta} X(\gamma, \theta))$$

$$\leq \sup_u \log \mathbb{E}\exp \lambda(||D(\gamma_1 - \gamma_2)||^2 u^T D^{-1} \nabla_{\gamma} X(\gamma, \theta))$$

$$+ \sup_u \log \mathbb{E}\exp \lambda(||D(\theta_1 - \theta_2)||^2 u^T D^{-1} \nabla_{\theta} X(\gamma, \theta))$$

$$\leq \frac{\mathbf{v}^2 \lambda^2 ||D(\gamma_1 - \gamma_2)||^2}{2(1 - R\lambda||D(\gamma_1 - \gamma_2)||)} + \frac{\mathbf{v}^2 \lambda^2 ||D(\theta_1 - \theta_2)||^2}{2(1 - R\lambda||D(\theta_1 - \theta_2)||)} \leq \frac{\mathbf{v}^2 \lambda^2 d_{12}^2}{2(1 - R\lambda d_{12})},$$

$$d_{12}^2 = ||D(\theta_2 - \theta_1)||^2 + ||D(\gamma_2 - \gamma_1)||^2.$$ 

Denote

$$\mathbb{Y} = \Omega(\gamma) \times \Omega(\gamma)$$
such that $\log N(\varepsilon, \Upsilon) = 2 \log N(\varepsilon, \Omega(r))$. Then with Lemma 3.4 we obtain
\[
\mathbb{E} \sup_{\gamma, \theta} X(\gamma, \theta) \leq 12v \int_0^{r/2} \sqrt{\log N(\varepsilon, \Upsilon)} d\varepsilon + 12R \int_0^{r/2} \log N(\varepsilon, \Upsilon) d\varepsilon.
\]
Application of Lemma 3.5 to the random variable $Z = \sup_{\gamma, \theta} X(\gamma, \theta)$ leads to the final result. Making variable replacement $\varepsilon = r\varepsilon'$ and dividing the last expression by $r$ we obtain
\[
\mathcal{Z}(t) \leq \mathbb{E} + \sqrt{2t(v^2 + 2RE)} + \frac{tR}{3},
\]
where
\[
\mathbb{E} = 12\sqrt{2v}H_1 + 24RH_2.
\]
In order to simplify this inequality we have also used property $H_1^2 \leq H_2$.

\section{Barycenters Model}

We are going to show that Assumptions 1–3 are fulfilled for the barycenters model (2.1) defined in Section 2. Also we need to estimate $\Delta(r, t)$ from expression (3.3). It will allow us to apply Theorem 3.1. Remind that we deal with the Likelihood function $L(\theta) = L(\theta, \{\theta_i\}_{i=1}^n)$, where the implicit random vectors $\{\theta_i\}_{i=1}^n$ is the dataset of Fourier coefficients corresponded to the random measures $\{\mu_i\}_{i=1}^n$.

We start with derivatives study of the model and show that Wasserstein distance in some differentiable Fourier basis $\{\psi_i(x)\}_{i=1}^\infty$ is a support function. Remind that by definition support function for a convex body $E$ is
\[
s_E(\theta) = \sup_{\eta \in E} \theta^T \eta.
\]
Consider Wasserstein distance in the dual form (ref. definition 2.1 in Section 2). Let $G(x)$ be the Gram function of this Fourier basis. Decompose an arbitrary function $f(x)$ in this basis
\[
f(x) = \sum_k \eta_k(f) \psi_k(x),
\]
where
\[
\eta_k(f) = \langle f, \psi_k \rangle_G = \int f(x) \psi_k(x) G(x) dx.
\]
Now we can rewrite the expectation difference from the dual Wasserstein definition as
\[
\mathbb{E} f(X) - \mathbb{E} f(Y) = \left\langle f, \frac{\varphi_X}{G} \right\rangle_G - \left\langle f, \frac{\varphi_Y}{G} \right\rangle_G = \langle \eta(f), \theta(\varphi_X) \rangle - \langle \eta(f), \theta(\varphi_Y) \rangle,
\]
where
\[
\theta_k(\varphi) = \int \varphi(x) \psi_k(x) dx.
\]
Define positive symmetric matrices
\[
K_x = \begin{pmatrix}
\nabla^T \psi_1(x) \\
\vdots \\
\nabla^T \psi_k(x) \\
\vdots \\
\nabla^T \psi_\infty(x)
\end{pmatrix}
\begin{pmatrix}
\nabla \psi_1(x) & \ldots & \nabla \psi_k(x) & \ldots
\end{pmatrix} = (\nabla^T \psi(x))(\nabla \psi^T(x))
\]
and
\[
K \circ G = \int K_x G(x) dx.
\]
Each $K_x$ is positive, since $\eta^T K_x \eta = ||\nabla f(x)||^2$. Condition $\forall x : ||\nabla f(x)|| \leq 1$ is equivalent in Fourier basis to

$$\eta \in \bigcap \mathcal{E}_x = \left\{ \eta : \forall x : \left( \sum_k \eta_k \psi_k(x) \right)^2 = \eta^T K_x \eta \leq 1 \right\}.$$  
(4.1)

An important remark is that

$$\bigcap \mathcal{E}_x \subset \{ \eta : \eta^T (K \circ G) \eta \leq 1 \}.$$  
(4.2)

We have got a useful intermediate result.

**Lemma 4.1.** Let random vectors $X$ and $Y$ have densities $\varphi_X$ and $\varphi_Y$ with Fourier coefficients $\theta_X$ and $\theta_Y$, then the Wasserstein distance is the support function of the convex set $\bigcap \mathcal{E}_x$ (4.1), i.e.,

$$W_1(\varphi_X, \varphi_Y) = \max_{\eta \in \bigcap \mathcal{E}_x} \langle \eta, \theta_X - \theta_Y \rangle.$$  

Moreover, for the regularised case defined in Section 2 it holds

$$\tilde{W}_1(\varphi_X, \varphi_Y) = \max_{\eta \in \bigcap \mathcal{E}_x} \langle \eta, \theta_X - \theta_Y \rangle - \varepsilon \eta^T K \circ G \eta.$$  

Remind that barycenter’s Likelihood consists of independent components $l_i(\theta - \theta_i)$ with random vectors $\theta_i \in \mathbb{R}^\infty$ and parameter $\theta \in \mathbb{R}^\infty$, such that

$$l(\theta - \theta_i) = \max_{\eta \in \bigcap \mathcal{E}_x} \langle \eta, \theta - \theta_i \rangle - \varepsilon \eta^T K \circ G \eta$$

$$= \max_{\eta} \langle \eta, \theta - \theta_i \rangle - \varepsilon \eta^T K \circ G \eta - \delta \bigcap \mathcal{E}_x(\eta).$$

Note that by definition the dual function of $l$ is

$$l^*(\eta) = \delta \bigcap \mathcal{E}_x(\eta) + \varepsilon \eta^T K \circ G \eta.$$  

Consequently from Lemma 4.10 follows that

$$l(\theta - \theta_i) = \delta \bigcap \mathcal{E}_x(\theta - \theta_i) \oplus (\varepsilon \eta^T K \circ G \eta)^*(\theta - \theta_i)$$

$$= \left( \max_{\eta \in \bigcap \mathcal{E}_x} \langle \eta, \theta - \theta_i \rangle \right) \oplus \frac{1}{\varepsilon} (\theta - \theta_i)^T (K \circ G)^{-1} (\theta - \theta_i).$$  
(4.3)

Symbol $\oplus$ denotes infimal convolution (ref. definition 4.9 in Appendix C). Application of Theorem 4.17, taking into account property (4.2), provides the following bounds on the derivatives of function $l$.

**Theorem 4.2.** The gradient upper bound of the function $l$ is

$$||D^{-1} \nabla l|| \leq \frac{1}{\lambda_{\min}^{1/2} (DK \circ GD)}.$$  

**Proof.** Denote

$$\eta^*(\theta) = \arg\max_{\eta \in \bigcap \mathcal{E}_x} \eta^T \theta.$$  

Use Eq. (4.3). By the consequence of Lemmas 4.12 and 4.13 $\exists \theta_0$:

$$\nabla l(\theta - \theta_i) = \eta^*(\theta_0).$$

Since $||(K \circ G)^{1/2} \eta^*|| \leq 1$, which follows from expression (4.2),

$$||D^{-1} \eta^*|| = ||D^{-1} (K \circ G)^{-1/2} (K \circ G)^{1/2} \eta^*|| \leq ||D^{-1} (K \circ G)^{-1/2}||.$$  

**Theorem 4.3.** Upper bounds of the second derivative of function $l$ are

$$||D^{-1} \partial \nabla l(\theta - \theta_i) D^{-1}|| \leq \frac{1}{\min_x \lambda_{\min} (DK_x D)|| (K \circ G)^{-1/2} (\theta - \theta_i)||}.$$  

and
\[ \| D^{-1} \partial \nabla^T l(\theta - \theta_1) D^{-1} \| \leq \frac{1}{\varepsilon \lambda_{\min}(D K \circ G D)}. \]

**Remark 4.4.** Matrix \( K_x \) may be singular which makes the first bound non-informative. The second bound comes from the regulariser \( \varepsilon \eta^T K \circ G \eta \) and has big coefficient \( (1/\varepsilon) \). It is a weak part of the current theory and requires an improvement or probably an example which shows that this bound it tight.

**Proof.** Consider support function with one ellipsoid
\[ s_x(\theta) = \max_{\eta^T K_x \eta \leq 1} \langle \eta, \theta \rangle = \| K_x^{-1/2} \theta \|. \]

Denote \( \eta^*(\theta) = \arg\max \langle \eta, \theta \rangle \), and account that \( \eta^T K_x \eta \leq 1 \)
\[ \eta^*(\theta) = \frac{K_x^{-1} \theta}{\| K_x^{-1/2} \theta \|}. \]

\[ \frac{\partial \eta^*(\theta)}{\partial \theta} = \frac{K_x^{-1} \theta K_x^{-1} \theta - K_x^{-1} \theta^T K_x^{-1} (\theta^T K_x^{-1})^{3/2}}{\| (\theta^T K_x^{-1})^{1/2} \|}. \]

For some vector \( \| \gamma \| = 1 \) by means of property \( \| a \|^2 \| b \|^2 \geq (a^T b)^2 \)
\[ \gamma^T K_x^{-1} \gamma \theta^T K_x^{-1} \theta - \gamma^T K_x^{-1} \theta^T K_x^{-1} \gamma \leq \| K_x^{-1} \| \theta^T K_x^{-1} \theta, \]
\[ \left\| \frac{\partial \eta^*(\theta)}{\partial \theta} \right\| \leq \frac{\| K_x^{-1} \|}{(\theta^T K_x^{-1})^{1/2}}. \]

Apply Theorem 4.17 that gives the first bound
\[ \| D^{-1} \partial \nabla^T l(\theta - \theta_1) D^{-1} \| \leq \max_x \left[ D^{-1} \frac{\partial \eta^*(\theta^*_{\theta})}{\partial \theta} D^{-1} \right] \frac{s_x(\theta^*_{\theta})}{s(\theta - \theta_1)} \leq \frac{\max_x \| D^{-1} K_x^{-1} D^{-1} \|}{\| (K \circ G)^{-1/2} (\theta - \theta_1) \|}. \]

The second bound for this norm follows directly from Lemma 4.15 and Eq. (4.3). \( \square \)

**Remark 4.5.** Wasserstein distance also may be differentiated directly. Paper [20] contains corresponded lemma about directional derivative. For directions \( h_1, h_2 \) it holds
\[ W_1^t(\mu_X, \mu_Y)(h_X, h_Y) = \max_{(u, v) \in \Phi(\mu_X, \mu_Y)} -\langle u, h_X \rangle - \langle v, h_Y \rangle, \]
where
\[ \Phi = \{(u, v) : \langle u, \mu_X \rangle + \langle v, \mu_Y \rangle = W_1(\mu_X, \mu_Y), \forall (x, y) : u(x) + v(y) \leq \| x - y \| \}. \]

Now we prove three essential properties of the barycenters model that corresponds to Assumptions 1–3 from Section 3.

**Property 1.** Let \( \theta \in \Theta(\mathbf{r}) \) and \( \theta_0, \theta_0' \) be some vectors on the line between \( \theta \) and \( \theta^* \)
\[ \delta(\mathbf{r}) = \frac{1}{\mathbf{r}} \left\| -D^{-1} \{ \nabla^3 L(\theta) - \nabla^3 L(\theta^*) \} \right\| \]
\[ \leq \| D^{-1} \{ \nabla^2 L(\theta_0) - \nabla^2 L(\theta_0') \} D^{-1} \| \leq \| D^{-1} \{ \nabla^3 L(\theta_0) D^{-1} \} D^{-1} \| \mathbf{r}. \]

Let \( q(\theta_i) \) be distribution of each \( \theta_i \) then
\[ \nabla^3 E_i L(\theta - \theta_i) = \sum_{i=1}^n \int \nabla^3 L(\theta - \theta_i) q(\theta_i) d\theta_i = -\sum_{i=1}^n \int \nabla^2 L(\theta - \theta_i) \times \nabla q(\theta_i) d\theta_i \]
\[
\|D^{-1}\{\nabla^2 L(\theta)D^{-1}\}D^{-1}\| \leq \int \|D^{-1}\nabla^2 L(\theta - \theta_x)D^{-1}\|\|D^{-1}\nabla q(\theta_x)\|d\theta_x.
\]

And from Theorem 4.3 one gets
\[
\|D^{-1}\nabla^2 l(\theta - \theta_i)D^{-1}\| \leq \frac{1}{\varepsilon \lambda_{\min}(DK \circ GD)}.
\]

Subsequently
\[
\delta(r) = \frac{r}{\varepsilon \lambda_{\min}(DK \circ GD)} \int \|D^{-1}\nabla q(\theta)\|d\theta.
\]

**Property 2.** From Theorem 3.7 and Theorem 4.3 follows that one may set
\[
v^2 = \frac{n}{\varepsilon^2 \lambda_{\min}^2(DK \circ GD)}
\]
and
\[
R = \frac{1}{\varepsilon \lambda_{\min}(DK \circ GD)}.
\]

Then
\[
\hat{z}(t) \leq v(12\sqrt{2}H_1 + \sqrt{2t}) + R(24H_2 + 24\sqrt{H_2}t + t/3)
\]
\[
\leq \frac{\sqrt{n}(12\sqrt{2}H_1 + \sqrt{2t}) + 24H_2 + 24\sqrt{H_2}t + t/3}{\varepsilon \lambda_{\min}(DK \circ GD)}.
\]

**Property 3.** Each model component \(l(\theta - \theta_i)\) without regularisation is convex since
\[
l(\lambda\theta_1 + (1 - \lambda)\theta_2 - \theta_i) = l(\lambda(\theta_1 - \theta_i)) + (1 - \lambda)(\theta_2 - \theta_i))
\]
\[
= \max_{\eta \in \mathcal{E}_x} \langle \eta, \lambda(\theta_1 - \theta_i) + (1 - \lambda)(\theta_2 - \theta_i) \rangle
\]
\[
\leq \max_{\eta \in \mathcal{E}_x} \langle \eta, \lambda(\theta_1 - \theta_i) \rangle + \max_{\eta \in \mathcal{E}_x} \langle \eta, (1 - \lambda)(\theta_1 - \theta_i) \rangle = \lambda l(\theta_1 - \theta_i) + (1 - \lambda)l(\theta_2 - \theta_i).
\]

Note that the regularised \(l\) is also convex as a composition of convex functions and the complete model \(L\) is convex \((-\nabla^2 L > 0)\) as a positive aggregation of convex functions.

Combination of these properties is used in the next proof which gives the deviation bound of the parameter \(\hat{\theta}\).

**Proof of Theorem 2.2.** Basing on Theorem 3.1 and Properties 1–3 which correspond to Assumptions 1–3 from Section 3 we have
\[
\left\| D(\hat{\theta} - \theta^*) - D^{-1}\nabla L(\theta^*) \right\| \leq \{\delta(r) + \hat{z}(t)\}r = \circ (r, t)
\]
with probability \(1 - e^{-t}\). The values of \(\delta(r)\) and \(\hat{z}(t)\) are estimated in Properties 1, 2, where
\[
\delta(r) = \frac{r}{\varepsilon \lambda_{\min}(DK \circ GD)} \int \|D^{-1}\nabla q(\theta)\|d\theta,
\]
\[
\hat{z}(t) = \frac{\sqrt{n}(12\sqrt{2}H_1 + \sqrt{2t}) + 24H_2 + 24\sqrt{H_2}t + t/3}{\varepsilon \lambda_{\min}(DK \circ GD)}.
\]

Simplify the previous expressions. Account that according to the theorem conditions
\[
\lambda_{\min}(DK \circ GD) \geq C_D n \quad \text{and} \quad \int \|D^{-1}\nabla q(\theta)\|d\theta = \frac{C_q}{\sqrt{n}},
\]
\[
\delta(r) = \frac{r}{\varepsilon C_D \sqrt{n}}.
\]
\[ z(t) = \frac{12\sqrt{2}H_1 + \sqrt{2t}}{\varepsilon C_D n^{3/2}} + O\left(\frac{H_2}{n}\right). \]

Lemma 4.2 provides bound \( \forall \theta \)

\[ ||D^{-1}\nabla l(\theta)|| \leq \frac{1}{\lambda_{min}^{1/2}(DK \circ GD)}. \]

From this bound and Hoeffding’s inequality (Theorem 2.8 [9]) follows the bound for \( ||D^{-1}\nabla L(\theta^*)|| \). With probability \( 1 - e^{-t} \)

\[ r \leq 4||D^{-1}\nabla L(\theta^*)|| \leq \frac{8\sqrt{n}(1 + \sqrt{2t})}{\lambda_{min}^{1/2}(DK \circ GD)} \leq \frac{8 + 8\sqrt{2t}}{\sqrt{C_D}}. \]

Finally taking into account definition (3.3)

\[ \diamond (r, t) \leq \frac{O(H_1 + t)}{\varepsilon C_D n^{3/2}} + \frac{O(tC_D)}{\varepsilon C_D n^{1/2}} + O\left(\frac{H_2}{n}\right). \]

**Proof of Theorem 2.3.** Bind Theorems 2.2 and 4.23. Form Theorem 3.1 follows that the bound of Theorem 2.2 also holds for projection of the parameter \( \theta \):

\[ ||\tilde{D}(\hat{\theta}_p - \theta^*_p) - \tilde{D}^{-1}\nabla L(\theta^*)|| \leq \diamond (r, t). \]

So with probability \( 1 - e^{-t} \)

\[ W_1(\tilde{D}(\hat{\theta}_p - \theta^*_p), Z) = \min_{\pi(\theta, Z)} E||\tilde{D}(\hat{\theta}_p - \theta^*_p) - Z|| \leq W_1(\tilde{D}^{-1}\nabla L(\theta^*), Z) + \diamond (r, t). \]

Furthermore from Theorem 4.23 follows

\[ W_1(\tilde{D}^{-1}\nabla L(\theta^*), Z) \leq \sqrt{2}\mu_3 \left( \log(6p \sqrt{\text{tr}\{\Sigma\}}) - \log(\mu_3) \right), \]

where \( \Sigma = \text{Var}[\tilde{D}^{-1}\nabla L(\theta^*)] \) and setting \( X_i = \tilde{D}^{-1}\nabla l(\theta^* - \theta_i) \) with independent copy \( X'_i \)

\[ \mu_3 = \sum_{i=1}^{n} \mathbb{E}X_i^T \Sigma^{-1} X_i ||X_i - X'_i|| \leq 2 \max ||X_i|| \sum_{i=1}^{n} \mathbb{E}X_i^T \Sigma^{-1} X_i. \]

Using the next two properties

\[ \sum_{i=1}^{n} \mathbb{E}X_i^T \Sigma X_i = \text{tr} \left\{ \Sigma^{-1} \sum_{i=1}^{n} \mathbb{E}X_i X_i^T \right\} = p, \]

\[ \max ||X_i|| = ||\tilde{D}^{-1}\nabla l(\theta^* - \theta_i)|| \leq ||D^{-1}\nabla l(\theta^* - \theta_i)|| \leq \frac{1}{\lambda_{min}^{1/2}(DK \circ GD)}, \]

we get

\[ \mu_3 \leq \frac{2p}{\sqrt{C_D n}} \]

and

\[ W_1(\tilde{D}^{-1}\nabla L(\theta^*), Z) \leq O\left(\frac{p\log(np)}{\sqrt{C_D n}}\right). \]

Analogically one can derive a consequence from Theorems 2.2 and 4.26. Let \( C_A \) be the anti-concentration constant of the distribution \( \mathbb{P}(||Z|| > z) \), defined in Theorem 4.26. Then \( \forall z \in \mathbb{R} \)

\[ \mathbb{P}(||\tilde{D}(\hat{\theta}_p - \theta^*_p) > z) - \mathbb{P}(||Z|| > z). \]
Assumption 1 provides
\[ \Pr(\|\hat{D}(\theta) - \theta\| > z) \leq \Pr(\|Z\| > z) + C_A \delta(r,t) \]
and
\[ \Pr(\|\hat{D}(\theta) - \theta\| > z) - \Pr(\|Z\| > z) \leq C_A \mu_3 O(\log^2 n). \]
As for the anti-concentration constant it can be estimated as proposed in ([13], Theorem 2.7), where was derived \( C_A = O(1/\sqrt{p}) \) for Euclidean norm. Grouping all together we get
\[ \Pr(\|\hat{D}(\theta) - \theta\| > z) - \Pr(\|Z\| > z) \leq O\left(\frac{\sqrt{p} \log^2 n}{\sqrt{C_D n}} + \delta(r,t)\right). \]

Appendix A

PROOF OF THEOREM 3.1

From \((-\nabla^2 L(\theta) \geq 0)\) and \((L(\hat{\theta}) > L(\theta^*))\) follows that the local region \( \Omega(r) \) that includes \( \hat{\theta} \) should cover the next region
\[ \Omega(r) \supset \{ \theta : L(\theta) \geq L(\theta^*) \}. \]
Use notation
\[ D^2(\theta) = -\nabla^2 L(\theta). \]
Estimate the minimal possible radius \( r_0 \) that satisfy to the previous condition. Let \( \theta_0 \) be some point between \( \theta \) and \( \theta^* \) that is used in Taylor expansion with central point \( \theta^* \)
\[ 0 \geq L(\theta^*) - L(\theta) = - (\theta - \theta^*)^T \nabla L(\theta^*) - \frac{1}{2} (\theta - \theta^*)^T \nabla^2 L(\theta) (\theta - \theta^*) + \frac{1}{2} ||D(\theta_0)(\theta - \theta^*)||^2. \]
Assumption 1 provides
\[ ||D(\theta_0)(\theta - \theta^*)||^2 = ||D(\theta - \theta^*)||^2 + ||D(\theta_0) - D(\theta - \theta^*)||^2 \geq r^2 \delta(r) r^2. \]
Assumption 2 provides with probability \((1 - e^{-t})\)
\[ (\theta - \theta^*)^T \nabla^2 \zeta(\theta_0)(\theta - \theta^*) \leq \zeta(t) r^2. \]
Put these two properties into the initial inequality
\[ 0 \geq -||D^{-1} \nabla L(\theta^*)|| \cdot r - \frac{\zeta(t)}{2} r^2 + \frac{1 - \delta(r)}{2} r^2 \]
\[ r (1 - \delta(r) - \zeta(t)) \leq 2 ||D^{-1} \nabla L(\theta^*)||. \]
So one may set under the assumption \( \delta(r) + \zeta(t) \leq 1/2 \)
\[ r_0 = 4 ||D^{-1} \nabla L(\theta^*)||. \]
From Assumptions 1, 2 also follows that
\[ ||D(\hat{\theta} - \theta^*) + D^{-1} \{ \nabla L(\hat{\theta}) - \nabla L(\theta^*) \} || \leq \delta(r,t). \]
Since \( \nabla L(\hat{\theta}) = 0 \) we have
\[ ||D(\hat{\theta} - \theta^*) - D^{-1} \nabla L(\theta^*)|| \leq \delta(r,t). \]
Note that for the coordinates transform \( S \) there exists the following invariant:
\[
\begin{pmatrix}
\hat{D} & 0 \\
0 & D_{\theta}
\end{pmatrix}
\begin{pmatrix}
u - u^* \\
\theta - \theta^*
\end{pmatrix}
+ \begin{pmatrix}
\hat{D}^{-1} & 0 \\
0 & D_{\theta}
\end{pmatrix}
\begin{pmatrix}
\nabla L(u, \theta) - \hat{\nabla} L(u^*, \theta^*) \\
\nabla_{\theta} L(u, \theta) + \nabla_{\theta} L(u^*, \theta^*)
\end{pmatrix}
\leq \delta(r,t). \]
\[ = \|D(\theta - \theta^*) + D^{-1} \{ \nabla L(\theta) - \nabla L(\theta^*) \} \|. \]

Since
\[
\left\| \begin{pmatrix} \tilde{D} & 0 \\ 0 & D_\theta \end{pmatrix} \begin{pmatrix} u \\ \vartheta \end{pmatrix} \right\|^2 = \theta^T S^T [(S^{-1})^T D^2 (S^{-1})] S \theta = \|D\theta\|^2,
\]
\[
\left\| \begin{pmatrix} \tilde{D}^{-1} & 0 \\ 0 & D_\theta^{-1} \end{pmatrix} \begin{pmatrix} \nabla \\ \nabla_\vartheta \end{pmatrix} \right\|^2 = \nabla^T S^{-1} [(S^{-1})^T D^2 (S^{-1})]^{-1} (S^{-1})^T \nabla = \nabla^T D^{-2} \nabla,
\]
\[
\begin{pmatrix} \nabla \\ \nabla_\vartheta \end{pmatrix} = \nabla^T S^{-1} S \theta = \nabla^T \theta.
\]

Subsequently, basing on this invariant we obtain the bound for projection
\[
\|\tilde{D}(\tilde{u} - u^*) - \tilde{D}^{-1} \tilde{\nabla} L(\theta^*)\|
\]
\[
\leq \|D(\hat{\theta} - \theta^*) - D^{-1} \nabla L(\theta^*)\| \leq \odot (r, t).
\]

**Appendix B**

**ELLIPSOID ENTROPY**

The upper bound \( \xi(t) \) of the random process in Assumption 2 with parameter \( \theta \in \Omega(r) \) requires entropy computation of the ellipsoid \( \Omega(r) \). It will be useful to us in the next section and below we provide a short excerpt on this topic. The general formula for the covering number \( N(\varepsilon, \Omega) \) of a convex set \( \Omega \) in \( \mathbb{R}^p \) with Euclidean metric is
\[
N(\varepsilon, \Omega) \leq \frac{\text{volume}(\Omega + (\varepsilon/2) B_1)}{\text{volume}(B_1)} \left( \frac{2 \varepsilon}{\varepsilon} \right)^p,
\]
where \( B_1 \) is the unit ball. Remind that \( N(\varepsilon, \Omega) \) equals to the minimal count of balls with radius \( \varepsilon \) that is sufficient to cover \( \Omega \). We will need two components of the ellipsoid entropy:
\[
H_1 = \int_0^{1/2} \sqrt{\log N(\varepsilon r, \Omega(r))} d\varepsilon, \quad H_2 = \int_0^{1/2} \log N(\varepsilon r, \Omega(r)) d\varepsilon.
\]

**Lemma 4.6.** Let \( \|D^{-2}\| = 1 \). Then for the entropy components (3.4) of the ellipsoid \( \Omega(r) \) with matrix \( D^2 \) defined in expression (3.2) it holds
\[
H_1 \leq C(\alpha - 1)^{-1/2} \sqrt{\sum_i \frac{\log \lambda_i^2(D)}{\lambda_i^2(D)}}
\]
and
\[
H_2 \leq C \sum_i \frac{1}{\lambda_i(D)}.
\]
where \( C \) is some absolute constant and \( \alpha > 1 \).

**Proof.** Function \( \log N(\varepsilon r, \Omega(r)) \) is monotone-decreasing. One may split the integration interval of (3.4) into the following parts:
\[
\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 4 & 8 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 8 & 16 \end{pmatrix}, \quad \cdots
\]
Take corresponded values $N(r/4, \Omega(r))$, $N(r/8, \Omega(r))$, $N(r/16, \Omega(r))$, \ldots and obtain integral approximation by the histogram

$$H_1 \leq \sum_{k=2}^{\infty} \frac{1}{2^k} \sqrt{\log N \left( \frac{r}{2^k}, \Omega(r) \right)}$$

and

$$H_2 \leq \sum_{k=2}^{\infty} \frac{1}{2^k} \log N \left( \frac{r}{2^k}, \Omega(r) \right).$$

Theorem H.7.1 in [21] provides upper bounds for the right parts of the previous expressions and completes the proof. □

Appendix C

SUPPORT FUNCTIONS

Bounds for the first and second derivatives of the Likelihood of barycenters model (2.1) involves additional theory from convex analysis.

**Definition 4.7 (s).** Legendre–Fenchel transform of a function $f : X \to \mathbb{R}$ or the convex conjugate function calls

$$f^*(y) = \sup_{x \in X} (\langle x, y \rangle - f(x)).$$

**Definition 4.8 (s).** Support function for a convex body $E$ is

$$s_E(\theta) = \sup_{\eta \in E} \theta^T \eta.$$  

Note that for the indicator function $\delta_E(\eta)$ of a convex set $E$ the conjugate function is support function of $E$, i.e.,

$$\delta^*_E(\theta) = s_E(\theta).$$

**Definition 4.9 (s).** Let $f_1, f_2 : E \to \overline{\mathbb{R}}$ be convex functions. The infimal convolution of them is

$$(f_1 \oplus f_2)(x) = \inf_{x_1 + x_2 = x} (f_1(x_1) + f_2(x_2)).$$

**Lemma 4.10 (Proposition 13.21 [3]).** Let $f_1, f_2 : E \to \overline{\mathbb{R}}$ be convex lower-semi-continuous functions. Then

$$(f_1 \oplus f_2)^* = f_1^* + f_2^*,$$

$$(f_1 + f_2)^* = f_1^* \oplus f_2^*.$$
Lemma 4.11. The support function of intersection $E = E_1 \cap E_2$ is infimal convolution of support functions for $E_1$ and $E_2$

$$s_E(\theta) = \inf_{\theta_1 + \theta_2 = \theta} (s_{E_1}(\theta_1) + s_{E_2}(\theta_2)).$$

Proof. According to the previous lemma

$$\delta_{E_1 \cap E_2}(\eta) = \delta_{E_1}(\eta) + \delta_{E_2}(\eta),$$

$$(\delta_{E_1} + \delta_{E_2})^* = \delta_{E_1}^* \oplus \delta_{E_2}^*.$$

With additional property

$$\text{intdom } \delta_{E_1} \cap \text{dom } \delta_{E_2} = \text{int } E_1 \cap E_2 \neq \emptyset$$

one have

$$(\delta_{E_1} + \delta_{E_2})^* = \delta_{E_1}^* \oplus \delta_{E_2}^*.$$ 

Lemma 4.12. Let a support function $s_E(\theta)$ be differentiable, then its gradient belongs to the border of corresponded convex set $E$

$$\nabla s_E(\theta) = \eta^*(\theta) \in \partial E,$$

where

$$\eta^*(\theta) = \arg\max_{\eta \in E} \eta^T \theta.$$

Proof. It follows from the convexity of $E$ and linearity of the optimization functional.

$$\frac{\partial \eta^*(\theta)}{\partial ||\theta||} = 0 \Rightarrow \frac{\partial \eta^*(\theta)^T}{\partial \theta} \theta = 0,$$

$$\nabla s_E(\theta) = \frac{\partial \eta^*(\theta)^T}{\partial \theta} \theta + \eta^*(\theta) = \eta^*(\theta).$$

Lemma 4.13 (Proposition 16.48 [3]). Let $f_1, f_2 : E \to \mathbb{R}$ be convex continuous functions. Then the subdifferential of their infimal convolution can be computed by formula

$$\partial(f_1 \oplus f_2)(x) = \bigcup_{x = x_1 + x_2} \partial f(x_1) \cap \partial f(x_2).$$

Corollary 4.14. If in addition $f_1, f_2$ are differentiable, then their infimal convolution is differentiable and $\exists x_1, x_2 : x = x_1 + x_2$ and

$$\nabla (f_1 \oplus f_2)(x) = \nabla f_1(x_1) = \nabla f_2(x_2).$$

Lemma 4.15. Let $f_1, \ldots, f_m : E \to \mathbb{R}$ be convex and two times differentiable functions. There is an upper bound for the second derivative of the infimal convolution $\forall t : \sum_{i=1}^{m} t_i = 1$

$$\partial^2 \nabla T(f_1 \oplus \ldots \oplus f_m)(x) \preceq \sum_{i=1}^{m} t_i^2 \nabla^2 f(x_i),$$

where $\sum_{i=1}^{m} x_i = x$.

Proof. Use notation $f = f_1 \oplus \ldots \oplus f_m$. Let

$$f(x) = \sum_i f_i(x_i).$$

According to Lemma 4.13 if all the functions are differentiable then

$$\nabla f(x) = \sum_i t_i \nabla f_i(x_i).$$
From the definition $\oplus$ also follows that
\[ f(x + z) \leq \sum_i f_i(x_i + t_i z). \]

Make Taylor expansion for the left and right parts and account equality of the first derivatives
\[ z^T \partial \nabla^T f(x + \theta z) z \leq \sum_i t_i^2 z^T \nabla^2 f_i(x_i + \theta z) z. \]

Since the direction $z$ was chosen arbitrarily, dividing both parts of the previous equation by $||z||^2 \to 0$, we come to inequality
\[ \partial \nabla^T f(x) \leq \sum_i t_i^2 \nabla^2 f_i(x_i). \]

\[ \square \]

**Remark 4.16.** One can find another proof of the similar theorem in ([3], Theorem 18.15).

**Theorem 4.17.** Let $f_1, \ldots, f_m : E \to \mathbb{R}$ be convex and two times differentiable functions. There is an upper bounds for infinitesimal convolution $f = f_1 \oplus \ldots \oplus f_m$ derivatives $\forall \gamma \exists x_1, \ldots, x_m$:
\[ \gamma^T \partial \nabla^T f(x) \gamma \leq \max_i \gamma^T \nabla^2 f_i(x_i) \gamma \frac{f_i(x_i)}{f(x)} \]
and
\[ \gamma^T \partial \nabla^T f^2(x) \gamma \leq 2(\gamma^T \nabla f(x))^2 + 2 \max_i \gamma^T \nabla^2 f_i(x_i) \gamma f_i(x_i). \]

**Proof.** Choosing appropriate $\{t_i\}$ in Lemma 4.15 one get the required upper bounds. Set
\[ t_i = \frac{f_i(x_i)}{\sum_{j=1}^m f_j(x_j)} \]
and since
\[ \sum_{j=1}^m f_j(x_j) = f(x), \]
\[ \sum_i t_i^2 \gamma^T \nabla^2 f_i(x_i) \gamma \leq \max_i t_i \gamma^T \nabla^2 f_i(x_i) \gamma = \max_i \gamma^T \nabla^2 f_i(x_i) \gamma f_i(x_i). \]

In order to prove the second formula apply this inequality in
\[ \partial \nabla^T f^2 = 2 \nabla f \nabla^T f + 2 f \partial \nabla f. \]

\[ \square \]

**Corollary 4.18.** Let $s_1, \ldots, s_m : E^* \to \mathbb{R}$ be support functions of the bounded convex smooth sets $E_1, \ldots, E_m$. There are upper bounds for the derivatives of support function $s$ of intersection $E_1 \cap \ldots \cap E_m$, such that $\forall i$
\[ \gamma^T \partial \nabla^T s(\theta) \gamma \leq \max_i \gamma^T \eta_i^s / \partial \theta_i \gamma s_i(\theta_i) \]
and
\[ \gamma^T \partial \nabla^T s^2(\theta) \gamma \leq 2(\gamma^T \eta_i^s)^2 + 2 \max_i \gamma^T \eta_i^s / \partial \theta_i \gamma s_i(\theta_i). \]

**Proof.** It follows from Theorem 4.17 and Lemma 4.12.  

\[ \square \]
GAUSSIAN APPROXIMATION

Multivariate analogues of Berry–Esseen Theorem have many modifications depending on space dimension of random vectors and functions set used for measures comparison. Bentkus in [4, 5] has presented excellent results related to this topic. Namely for a sequence of i.i.d random vectors with identity covariance matrix \( \{X_i\}_{i=1}^{\infty}, X_i \in P(\mathbb{R}^p) \), any convex set \( A \) and Gaussian vector \( Z \in \mathcal{N}(0, I) \) it holds

\[
\left| \mathbb{P} \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \in A \right) - \mathbb{P}(Z \in A) \right| \leq \frac{400p^{1/4}E||X_1||^3}{\sqrt{n}}
\]

and

\[
W_1 \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i, Z \right) \leq O \left( \frac{E||X_1||^3}{\sqrt{n}} \right).
\]

We extend these two statements for independent random vectors with non-identity covariance \( \Sigma \). Additionally we remove factor \( p^{1/4} \) replacing it with anti-concentration constant defined below.

**Definition 4.19 (\( H_k \)).** The multivariate Hermite polynomial is

\[
H_k(x) = (-1)^{|k|} e^{x^T \Sigma^{-1} x/2} \frac{\partial^{|k|}}{\partial k_1 \ldots \partial k_p} e^{-x^T \Sigma^{-1} x/2},
\]

where \( x \in \mathbb{R}^p \) and \( |k| = k_1 + \ldots + k_p \).

**Lemma 4.20 [17].** Consider a Gaussian vector \( Z \sim \mathcal{N}(0, \Sigma) \) and two functions \( h \in C^1 \) and \( f_h \) such that

\[
f_h(x) = -\int_0^1 \frac{1}{2t} \mathbb{E} \tilde{h}(Z(x, t)) dt,
\]

where for \( t \in [0, 1] \)

\[
\tilde{h}(Z(x, t)) = h(\sqrt{t}x + \sqrt{1-t}Z) - \mathbb{E}h(Z).
\]

Then \( f_h \) is a solution of Stein’s equation

\[
\tilde{h}(x) = (\text{tr}\{\nabla^2 \Sigma\} - x^T \nabla) f_h(x)
\]

and

\[
\frac{\partial^{|k|}}{\partial k_1 \ldots \partial k_p} f_h(x) = -\int_0^1 \frac{1}{2} \frac{t^{k_1-1}}{(1-t)^{k_p}} \mathbb{E} H_k(Z) \tilde{h}(Z(x, t)) dt.
\]

**Proof.** One may verify this statement through substituting the solution \( f_h \) into Stein’s equation. It has been done in Lemma 1 of [17]. \( \square \)

In the following discussion we will need difference between the second derivatives of function \( f_h \).

**Corollary 4.21.** Let \( f_h \) be the solution of Stein’s equation, then

\[
\nabla^2 f_h(x) - \nabla^2 f_h(y) = -\int_0^1 \frac{1}{2(1-t)} \mathbb{E} H_2(Z) \{ h(Z(x, t)) - h(Z(y, t)) \} dt,
\]

where

\[
H_2(Z) = \Sigma^{-1/2} \{(\Sigma^{-1/2} Z)(\Sigma^{-1/2} Z)^T - I\} \Sigma^{-1/2}.
\]
Use notation \( \forall i \) \( X_{-i} \) for sum of \( \{X_j\}_{j=1}^n \) without the \( i \)-th element and \( X'_i \) for an \textit{independent copy} of \( X_i \).

Use the following notation for conditional expectation

\[
\mathbb{E}_{-i} = \mathbb{E}(\cdot | X_i, X'_i),
\]

\[
X_{-i} = \sum_{j=1, j \neq i}^n X_j.
\]

**Lemma 4.22.** Consider a Gaussian vector \( Z \in \mathcal{N}(0, \Sigma) \) and a sequence of independent zero-mean random vectors \( X = \sum_{i=1}^n X_i \) in \( \mathbb{R}^p \) with the same non-singular variance matrix

\[
\mathbb{E}XX^T = \Sigma.
\]

Then for any function with bounded the first derivative \( h \in C^1(\mathbb{R}^p) \)

\[
\mathbb{E}h(X) - \mathbb{E}h(Z) \leq A \log \left( \frac{6B}{A} \right),
\]

where

\[
A = \sum_{i=1}^n \mathbb{E}X_i^T \Sigma^{-1} X_i A_i, \quad B = \sum_{i=1}^n \mathbb{E}X_i^T \Sigma^{-1} X_i B_i,
\]

and \( \forall \alpha > 0 \) on interval \( t \in [0, 1 - \alpha] \)

\[
A_i \geq \frac{1}{\sqrt{t}} \sup_{|\gamma|=1, \theta \in [0,1]} \mathbb{E}_{-i} J_t(\gamma, \theta, X_i, X'_i),
\]

and on interval \( t \in [1 - \alpha, 1] \) for the same \( \alpha \)

\[
B_i \geq \frac{1}{2\sqrt{1 - t}} \sup_{|\gamma|=1, \theta \in [0,1]} \mathbb{E}_{-i} J_t(\gamma, \theta, X_i, X'_i),
\]

and

\[
J_t(\gamma, \theta, X_i, X'_i) = \{ (\gamma^T \Sigma^{-1/2} Z)^2 - 1 \} \{ h(Z(X_{-i} + \theta X_i, t)) - h(Z(X_{-i} + X'_i, t)) \},
\]

where

\[
Z(x, t) = \sqrt{t} x + \sqrt{1 - t} Z.
\]

**Proof.** From Lemma 4.20 follows that for any function \( h \) with the first bounded derivative

\[
\mathbb{E}h(X) - \mathbb{E}h(Z) = \mathbb{E}\text{tr}\{\nabla^2 \Sigma\} f_h(X) - \mathbb{E} \sum_{i=1}^n X_i^T \nabla f_h(X).
\]

Let \( \theta \) be some value in \( [0, 1] \). Decompose \( \nabla f_h(X) \) by Taylor formula

\[
\nabla f_h(X) = \nabla f_h(X_{-i}) + \nabla^2 f_h(X_{-i} + \theta X_i) X_i.
\]

Note that

\[
\mathbb{E}\text{tr}\{\nabla^2 \Sigma\} f_h(X) = \mathbb{E} \sum_{i=1}^n X_i^T \nabla^2 f_h(X_{-i} + X'_i) X_i.
\]

Substitute them into the first expression

\[
\mathbb{E}h(X) - \mathbb{E}h(Z) = \sum_{i=1}^n \mathbb{E} X_i^T \left\{ \nabla^2 f_h(X_{-i} + X'_i) - \nabla^2 f_h(X_{-i} + \theta X_i) \right\} X_i
\]

\[
= \sum_{i=1}^n \mathbb{E}(\Sigma^{-1/2} X_i)^T \Sigma^{1/2} \{ \nabla^2 f_h(X_{-i} + X'_i) - \nabla^2 f_h(X_{-i} + \theta X_i) \} \Sigma^{1/2} \Sigma^{-1/2} X_i.
\]
From the consequence of Lemma 4.20 use equality for the second derivative difference. For a unit vector $||\gamma|| = 1$ and conditional expectation $E_{-i} = E(\cdot|X_i, X'_i)$

$$
\gamma^T E_{-i} \Sigma^{1/2} \{\nabla^2 f_h(X_{-i} + X'_i) - \nabla^2 f_h(X_{-i} + \theta X_i)\} \Sigma^{1/2} \gamma
$$

$$
= \int_0^1 \frac{1}{2(1-t)} E_{-i} \{(\Sigma^{-1/2} Z)^T \gamma \}^2 - 1 \{ h(Z(X_{-i} + \theta X_i, t)) - h(Z(X_{-i} + X'_i, t)) \} dt
$$

$$
\leq \int_0^{1-\alpha} \frac{\alpha t^{1/2}}{2(1-t)} A_i dt + \int_{1-\alpha}^1 \frac{1}{(1-t)^{1/2}} B_i dt \leq -\frac{A_i}{2} \log(\alpha) + 2B_i \sqrt{\alpha}.
$$

Sum it with $X^T \Sigma^{-1} X_i$ and finally we obtain

$$
Eh(X) - Eh(Z) \leq -\frac{A}{2} \log(\alpha) + 2B \sqrt{\alpha}
$$

$$
\leq A \left(1 + \log \left(\frac{2B}{A}\right)\right) \leq A \log \left(\frac{6B}{A}\right).
$$

\[\square\]

**Theorem 4.23** (Multivariate Berry–Esseen with Wasserstein distance). Consider a sequence of independent zero-mean random vectors $X = \sum_{i=1}^n X_i$ in $\mathbb{R}^p$ with a variance matrix $EXX^T = \Sigma$.

Then 1-Wasserstein distance between $X$ and Gaussian vector $Z \in \mathcal{N}(0, \Sigma)$ has the following upper bound

$$
W_1(X, Z) \leq \sqrt{2} \mu_3 \log \left(\frac{6p \sqrt{\text{tr}(\Sigma)}}{\mu_3}\right),
$$

where

$$
\mu_3 = \sum_{i=1}^n E X_i^T \Sigma^{-1} X_i ||X_i - X'_i||,
$$

and each $X'_i$ is an independent copy of $X_i$.

**Remark 4.24.** In i.i.d case with $\Sigma = I_p$

$$
W_1(X, Z) = O \left(\frac{p^{3/2} \log(n)}{\sqrt{n}}\right).
$$

These is the same theorem with a different proof in paper [4].

**Proof.** Basing on Lemma 4.22 we consider $h$ with property $||\nabla h(\cdot)|| \leq 1$ and involve definition of $A_i$ and $B_i$. This property comes from the dual definition of $W_1$, Section 2. We decompose $A_i$ extracting $\sqrt{t}(X_i - X'_i)$ and decompose $B_i$ extracting $\sqrt{1-t}Z$

$$
A_i = ||X_i - X'_i|| E_{-i} ||(\gamma^T \Sigma^{-1/2} Z)^2 - 1||,
$$

$$
B_i = E_{-i} ||(\gamma^T \Sigma^{-1/2} Z)^2 - 1|| ||Z||.
$$

Note that $(\gamma^T \Sigma^{-1/2} Z)^2$ has chi-square distribution $\chi_1$. And its variance equals 2. So we obtain by means of Cauchy–Bunyakovsky inequality

$$
E_{-i} ||(\gamma^T \Sigma^{-1/2} Z)^2 - 1|| \leq \sqrt{2}
$$

and

$$
E_{-i} ||(\gamma^T \Sigma^{-1/2} Z)^2 - 1|| ||Z|| \leq \sqrt{E ||(\gamma^T \Sigma^{-1/2} Z)^2 - 1||^2 \sqrt{E ||Z||^2}} = \sqrt{2 \text{tr}(\Sigma)}.
$$
Subsequently
\[ A \leq \sqrt{2} \sum_{i=1}^{n} \mathbb{E}X_i^T \Sigma^{-1} X_i \|X_i - X_i'\| = \sqrt{2}\mu_3 \]
and
\[ B \leq \sqrt{2 \text{tr}(\Sigma)} \sum_{i=1}^{n} \mathbb{E}X_i^T \Sigma^{-1} X_i = p\sqrt{2 \text{tr}(\Sigma)}. \]

For the next result we will need the following technical lemma.

**Lemma 4.25.** Let a random variable \( \varepsilon \) has a tail bound \( \forall x \geq x_0 \)
\[ P(\varepsilon > h(x)) \leq e^{-x}. \]
Then for a function \( g : \mathbb{R}_+ \to \mathbb{R}_+ \) with derivative \( g' : \mathbb{R}_+ \to \mathbb{R}_+ \)
\[ \mathbb{E}[\mathbb{1}[\varepsilon > h(x_0)] g(\varepsilon) \leq g(h(x_0))e^{-x_0} + \int_{x_0}^{\infty} e^{-x} g'(h(x))h'(x)dx. \]

In particular
\[ \mathbb{E}[\mathbb{1}[\varepsilon > h(x_0)] \varepsilon \leq h(x_0) e^{-x_0} + \int_{x_0}^{\infty} e^{-x} h'(x)dx, \]
\[ \mathbb{E}[\mathbb{1}[\varepsilon > h(x_0)] \varepsilon^r \leq h(x_0)^r e^{-x_0} + r \int_{x_0}^{\infty} e^{-x} h(x)^{r-1} h'(x)dx. \]

**Theorem 4.26** (Multivariate Berry–Esseen). Consider a sequence of independent zero-mean random vectors \( X = \sum_{i=1}^{n} X_i \) in \( \mathbb{R}^p \) with a variance matrix
\[ \mathbb{E}XX^T = \Sigma. \]
Let \( \varphi : \mathbb{R}^p \to \mathbb{R}_+ \) be some norm function (sub-additive and homogeneous) and with Gaussian vector \( Z \in \mathcal{N}(0, \Sigma) \) fulfills anti-concentration property, such that \( \forall x \in \mathbb{R}_+ \)
\[ P(\varphi(Z) > x) - P(\varphi(Z) > x + \Delta) \leq C_A \Delta. \]
Then the measure difference between \( X \) and Gaussian vector \( Z \) has the following upper bound \( \forall x \)
\[ |P(\varphi(X) > x) - P(\varphi(Z) > x)| \leq 22C_A \mu_3 \log \left( \frac{3p}{C_A \mu_3} \right) \log \left( \frac{\sqrt{2 \mathbb{E}\varphi^2(Z)p}}{10C_A \mu_3^2} \right) \leq C_A \mu_3 O(\log^2 n), \]
where
\[ \mu_3 = \sum_{i=1}^{n} \mathbb{E}X_i^T \Sigma^{-1} X_i 2\varphi(X_i). \]

**Proof.** Make some preliminary computations. Define a smooth indicator function
\[ g_{x, \Delta}(t) = \begin{cases} 0, & t < x - \Delta \\ (x - t)/\Delta, & t \in [x - \Delta, x] \\ 1, & t > x. \end{cases} \]
Set $h = g_{x, \Delta} \circ \varphi$. Denote the required bound by $\delta$:
$$
\sup_{x \in \mathbb{R}^+} \left| \mathbb{P}(\varphi(X) > x) - \mathbb{P}(\varphi(Z) > x) \right| \leq \delta.
$$

Note that from sub-additive property of the function $\varphi$ follows
$$
g_{x, \Delta}(\varphi(X + dX)) \leq g_{x, \Delta}(\varphi(X) + \varphi(dX))
$$
and
$$
g'_{x, \Delta}(t) = \frac{1}{\Delta} \mathbb{1}[x - \Delta \leq t \leq x].
$$
By the anti-concentration property
$$
\mathbb{E}g'_{x, \Delta}(\varphi(Z)) = \frac{1}{\Delta} \left( \mathbb{P}(\varphi(Z) > x - \Delta) - \mathbb{P}(\varphi(Z) > x) \right) \leq C_A.
$$
And using definition of $\delta$
$$
\mathbb{E}g'_{x, \Delta}(\varphi(Z, t)) \leq \frac{1}{\Delta} \left( \mathbb{P}(\varphi(Z) > x - \Delta) - \mathbb{P}(\varphi(Z) > x) \right) + \frac{2\delta}{\Delta}
$$
(4.6)
$$
\leq C_A + \frac{2\delta}{\Delta}. \tag{4.7}
$$

Now we will bound $J_t(\gamma, \theta, X_i, X'_i)$ required in Lemma 4.22. Remind that by definition
$$
J_t(\gamma, \theta, X_i, X'_i) = \{ (\gamma^T \Sigma^{-1/2} Z)^2 - 1 \} \{ h(Z_{X_i} + \theta X_i, t) - h(Z_{X_i} + X'_i, t) \},
$$
where
$$
Z(x, t) = \sqrt{t} x + \sqrt{1-t} Z.
$$
For some $\theta' \in [0, 1]$ using sub-additivity of $\varphi$ and Taylor formula we get
$$
h(Z_{X_i} + \theta X_i, t) - h(Z_{X_i} + X'_i, t)
\leq g_{x, \Delta}(\varphi(Z_{X_i} + X'_i, t) + \varphi(\sqrt{t}(\theta X_i - X'_i))) - g_{x, \Delta}(\varphi(Z_{X_i} + X'_i, t))
\leq g'_{x, \Delta}(\varphi(Z_{X_i} + X'_i, t)) + \theta' \varphi(\sqrt{t}(\theta X_i - X'_i)) \varphi(\sqrt{t}(\theta X_i - X'_i)).
$$
Together with (4.7)
$$
\mathbb{E}_{-i} h(Z_{X_i} + \theta X_i, t) - \mathbb{E}_{-i} h(Z_{X_i} + X'_i, t) \leq \sqrt{t} \left( C_A + \frac{2\delta}{\Delta} \right) \varphi(\theta X_i - X'_i).
$$
Analogically one can obtain the same inequality for the opposite sign and subsequently we get inequality with module
$$
| \mathbb{E}_{-i} h(Z_{X_i} + X'_i, t) - \mathbb{E}_{-i} h(Z' + \theta X_i, t) | \leq \sqrt{t} \left( C_A + \frac{2\delta}{\Delta} \right) \varphi(\theta X_i - X'_i).
$$
Using the previous expression and notation
$$
\varepsilon^2 = (\gamma^T \Sigma^{-1/2} Z)^2 \sim \mathcal{N}^2(0, 1),
$$
estimate the upper bound of $J_t$
$$
\frac{1}{\sqrt{t}} \mathbb{E}_{-i} J_t(\gamma, \theta, X_i, X'_i) \leq | \tau - 1 | \left( C_A + \frac{2\delta}{\Delta} \right) \varphi(\theta X_i - X'_i) + \mathbb{E}[\varepsilon^2 > \tau] |\varepsilon^2 - 1|.
$$
For $\varepsilon \sim \mathcal{N}(0, 1)$ we have
$$
\mathbb{P}(\varepsilon > \sqrt{2} x) \leq e^{-x} \quad \text{and} \quad \mathbb{P}(\varepsilon^2 > 2 x + 2 \log 2) \leq e^{-x},
$$
and by means of Lemma 4.25 we get for all $\tau \geq 1$
$$
\mathbb{E}[\varepsilon^2 > \tau] |\varepsilon^2 - 1| \leq 2(\tau + 1) e^{-\tau/2}.$$
and

\[ A_i = \sup_{||\gamma||=1, \theta \in [0, 1]} \frac{1}{\sqrt{t}} \mathbb{E}_{-i} J_i (\gamma, \theta, X_i, X_i^\prime) \]

\[ \leq |\tau - 1| \left( C_A + \frac{2\delta}{\Delta} \right) \varphi(\theta X_i - X_i^\prime) + 2(\tau + 1)e^{-\tau/2}. \]

One should find optimal values for the arbitrary parameters \( \Delta > 0 \) and \( \tau \geq 1 \). Setting \( \Delta = \delta/(2C_A) \) and \( \tau = 2 \log(3p/(CA\mu_3)) \) we obtain that

\[ A = \sum_{i=1}^{n} \mathbb{E}X_i^T \Sigma^{-1} X_i A_i \leq 5|\tau - 1|C_A\mu_3 + 2(\tau + 1)e^{-\tau/2} \]

\[ \leq 11C_A\mu_3 \log \left( \frac{3p}{C_A\mu_3} \right). \]

We need also the other upper bound for \( B_i \) when \( t \) is close to 1.

\[ \mathbb{E}_{-i} \{ e^2 - 1 \} h(Z(X_{-i} + X_i^\prime, t)) \]

\[ = \mathbb{E}_{-i} \{ e^2 - 1 \} \{ h(\sqrt{t}(X_{-i} + X_i^\prime) + \sqrt{1-t}Z) - h(\sqrt{t}(X_{-i} + X_i^\prime)) \} \]

\[ \leq \sqrt{1-t} \mathbb{E}_{-i} e^2 - 1 |g_{x, \Delta}^t| \varphi(Z) \leq \sqrt{1-t} \frac{\Delta}{\Delta} \sqrt{2\mathbb{E}\varphi^2(Z)}. \]

In the last expression we have applied Cauchy–Bunyakovsky inequality and the upper bound for \( |g_{x, \Delta}^t| \leq 1/\Delta \). Accounting condition \( \Delta = \delta/(2C_A) \) one may derive that

\[ B_i = \frac{2C_A}{\delta} \sqrt{2\mathbb{E}\varphi^2(Z)} \]

and furthermore

\[ B = \sum_{i=1}^{n} \mathbb{E}X_i^T \Sigma^{-1} X_i B_i \leq \frac{2C_A}{\delta} \sqrt{2\mathbb{E}\varphi^2(Z)} p. \]

In order to make step from \( h \) expectation difference to the probabilities difference we will use the next inequality:

\[ \mathbb{P}(\varphi(X) > x) \leq \mathbb{E}h(X) = \mathbb{E}h(Z) + \mathbb{E}h(X) - \mathbb{E}h(Z) \]

\[ \leq \mathbb{P}(\varphi(Z) > x - \Delta) + \mathbb{E}h(X) - \mathbb{E}h(Z) \leq \mathbb{P}(\varphi(Z) > x) + \mathbb{E}h(X) - \mathbb{E}h(Z) + C_A\Delta, \]

which gives

\[ \delta \leq |\mathbb{E}h(X) - \mathbb{E}h(Z)| + C_A\Delta \leq |\mathbb{E}h(X) - \mathbb{E}h(Z)| + \frac{\delta}{2}, \]

\[ \delta \leq 2|\mathbb{E}h(X) - \mathbb{E}h(Z)|. \]

Basing on Lemma 4.22 we consider \( h = g_{x, \Delta} \circ \varphi \) and we have already estimated the main values of \( A \) and \( B \). Assuming \( \delta > A \) we obtain

\[ \delta \leq 2A (\log(6B) - \log(A)) \leq 2A (\log(6B\delta) - \log(\delta) - \log(A)) \]

\[ \leq 2A (\log(6B\delta) - 2\log(A)) \leq 22C_A\mu_3 \log \left( \frac{3p}{C_A\mu_3} \right) \log \left( \frac{p\sqrt{2\mathbb{E}\varphi^2(Z)}}{10C_A\mu_3^2} \right). \]

\[ \square \]

**Remark 4.27.** In i.i.d case with \( \Sigma = I_p \) and \( \varphi(x) = O(||x||) \)

\[ |\mathbb{P}(\varphi(X) > x) - \mathbb{P}(\varphi(Z) > x)| = O \left( \frac{p\log^2(n)}{\sqrt{n}} \right). \]

Note that Lemma 4.26 improves the classical Multivariate Berry–Esseen Theorem [5] for the case of norm functions \( \varphi(x) = O(||x||) \). Namely, instead of the dependence on the dimension \( p^{7/4} \) we got a linear dependence.
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