On the Complexity of Mining Itemsets from the Crowd Using Taxonomies*

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ABSTRACT

We study the problem of frequent itemset mining in domains where data is not recorded in a conventional database but only exists in human knowledge. We provide examples of such scenarios, and present a crowdsourcing model for them. The model uses the crowd as an oracle to find out whether an itemset is frequent or not, and relies on a known taxonomy of the item domain to guide the search for frequent itemsets. In the spirit of data mining with oracles, we analyze the complexity of this problem in terms of (i) crowd complexity, that measures the number of crowd questions required to identify the frequent itemsets; and (ii) computational complexity, that measures the computational effort required to choose the questions. We provide lower and upper complexity bounds in terms of the size and structure of the input taxonomy, as well as the size of a concise description of the output itemsets. We also provide constructive algorithms that achieve the upper bounds, and consider more efficient variants for practical situations.

1. INTRODUCTION

The identification of frequent itemsets, namely sets of items that frequently occur together, is a basic ingredient in data mining algorithms and is used to discover interesting patterns in large data sets [1]. A common assumption in such algorithms is that the transactions to be mined (the sets of co-occurring items) have been recorded and are stored in a database. In contrast, there is data which is not recorded in a systematic manner, but only exists in human knowledge. Mining this type of data is the goal of this paper.

As a simple example, consider a social scientist analyzing the life habits of people, in terms of activities (watching TV, jogging, reading, etc.) and their contexts (time, location, weather, etc.). Typically, for large communities, there is no comprehensive database that records all transactions where an individual performs some combination of activities in a certain context. Yet, some trace of the data remains in the memories of the individuals involved. As another example, consider a health researcher who wants to identify new drugs by analyzing the practices of folk medicine (also known as traditional medicine, i.e., medicinal practice that is neither documented in writing nor tested out under a scientific protocol): the researcher may want to determine, for instance, which treatments are often applied together for a given combination of symptoms. For this purpose too, the main source of knowledge are the folk healers and patients themselves.

In a previous work [2, 3] we have proposed to address this challenge using crowdsourcing to mine the relevant information from the crowd. Crowdsourcing platforms (such as, e.g., [3, 13, 27, 29, 32]) are an effective tool for harnessing a crowd of Web users to perform various tasks. In [2, 3] we incorporated crowdsourcing into a crowd mining framework for identifying frequent data patterns in human knowledge, and demonstrated its efficiency experimentally. The goal of the present paper is to develop the theoretical foundations for crowd mining, and, in particular, to formally study the complexity of identifying frequent itemsets using the crowd.

Before presenting our results, let us explain three important principles that guide our solution.

First, in our settings, no comprehensive database can be built. Not only would it be prohibitively expensive to ask all the relevant people to provide all the required information, but it is also impossible for people to recall all the details of their individual transactions such as activity occurrences, illnesses, treatments, etc. [2, 3]. Hence, one cannot simply collect the transactions into a database that could be mined directly. Instead, studies show that people do remember some summary information about their transactions [6], and thus, as demonstrated in [2, 3], itemset frequencies can be learned by asking the crowd directly about them.

Second, as we want to mine the crowd by posing questions about itemset frequencies, we must define a suitable cost model to evaluate mining algorithms. In data mining there are two main approaches for measuring algorithm cost. The first one (see, e.g., [1]) measures running time, including the cost of accessing the database (database scans), which is not suitable for a crowd set-

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ting as there is no database that can be accessed in this manner. The second approach (see, e.g., [25]) assumes the existence of an oracle that can be queried for insights about data patterns (frequency of itemsets, in our case): the cost is then measured by the number of oracle calls. Our setting is closer to this second approach: the crowd serves as an oracle, and we count the number of questions posed to the crowd, namely, crowd complexity. In addition, we study computational complexity, namely, the time required to compute the itemsets about which we want to ask the crowd. There is a clear tradeoff between the costs: investing more computational effort to select questions carefully may reduce the crowd complexity, and vice versa. See Section 7 for a further comparison of our work with existing approaches in data mining.

Finally, for the human-knowledge domains that we consider, one can make mining algorithms more efficient by leveraging semantic knowledge captured by taxonomies. A taxonomy in our context is a partial “is-a” relationship on the domain items, e.g., tennis is a sport, sport is an activity, etc. Many such taxonomies are available, both domain-specific (e.g., for diseases [34]) and general-purpose (e.g., Wordnet [38]). The use of taxonomies in mining is twofold. First, the semantic dependencies between items induce a frequency dependency between itemsets: e.g., because tennis is a sport, the itemset \{sunglasses, sport\} implicitly appears in all transactions where \{sunglasses, tennis\} appears. Hence, if the latter itemset is frequent then so is the former. Second, with taxonomical knowledge we can avoid asking questions about semantically equivalent itemsets, such as \{sport, tennis\} and \{tennis\}. Taxonomies are known to be a useful tool in data mining [35] and we study their use under our complexity measures.

**Results.** For our theoretical results, we harness tools from three areas of computer science: data mining, order theory and Boolean function learning [5, 7, 15, 17, 24, 25, 35]. Order theory is relevant to our discussion, because a taxonomy is in fact a partial order over data items; and Boolean function learning is relevant since the set of frequent itemsets to identify can be represented as a Boolean function indicating whether itemsets are frequent, a connection that was also pointed out in previous works in data mining [25]. Our contribution in this paper is combining and extending these tools to characterize the complexity of crowd mining.

A summary of our main results is presented in Table 1 where we give upper and lower bounds for our two complexity measures. In the first column, we give such bounds as a function of the structure of the input taxonomy \(\Psi\). These bounds are not affected by properties of the output, such as the actual number of frequent itemsets to be identified. In contrast, in the second column, we give complexity bounds as a function of the number of maximal frequent itemsets (MFIs) and minimal infrequent itemsets (MIIs). Intuitively, the MFIs and MIIs (to be defined formally later) are alternative concise descriptions of the frequent itemsets, and thus capture the output of the mining process.

The first row of Table 1 presents crowd complexity results. We show that, given a taxonomy \(\Psi\), the problem of identifying the frequent itemsets has a tight bound logarithmic in \(|S(\Psi)|\) – the number of possible Boolean frequency functions, which depends on \(\Psi\). As reflected in the inequalities at the bottom of Table 1 (and explained in Section 3), \(\log |S(\Psi)|\) is at most exponential in \(|\Psi|\). When the output is considered, our lower complexity bound is the sum of the numbers of MFIs and MIIs, and the upper bound adds the taxonomy size as a multiplicative factor. We provide a constructive algorithm (Algorithm 1) that achieves this bound.

In the second row of the table, we study computational complexity. We focus on “crowd-efficient” algorithms, which achieve the crowd complexity upper bound mentioned above. The crowd complexity lower bound is trivially a lower bound of computational complexity, but w.r.t. the output we obtain a stronger hardness result by showing that the problem is EQ-hard in the taxonomy size and in the numbers of MFIs and MIIs. EQ is a basic problem in Boolean function learning, not known to be solvable in PTIME [5, 10]. As for upper bounds, from Algorithm 1 we obtain an upper computational bound polynomial in \(|I(\Psi)|\) – the number of (relevant) itemsets of \(\Psi\). This size is at most exponential in \(|\Psi|\) (see Section 2). Algorithm 1 is not crowd-efficient w.r.t. the input alone, but for the upper computational complexity bound we relax this requirement in order to achieve a more feasible bound. Omitted from Table 1 are results for the case in which the size of itemsets is bounded by a constant \(k\), which we also study for the problem axes mentioned above.

Finally, given the relatively high lower complexity bounds, we examine two additional approaches. The chain partitioning approach, following a standard technique in data mining and Boolean function learning, suggests an alternative algorithm for crowd mining. We show that while this algorithm is not crowd-efficient in general, it outperforms Algorithm 1 given certain conditions on the frequent itemsets. The greedy approach attempts to maximize, at each question to the crowd, the number of itemsets that are classified as frequent or infrequent. We show that choosing a question that maximizes this number is FP^#P-hard in \(|\Psi|\).

**Paper organization.** We start in Section 2 by formally defining the setting and the problem. Crowd and computational complexity are studied in Sections 3 and 4 respectively. We consider chain partitioning in Section 5 and a greedy approach in Section 6. Related work is discussed in Section 7 and we conclude in Section 8.
2. PRELIMINARIES

We now present the formal model and problem settings for taxonomy-based crowd mining. Table 2 summarizes all the introduced notation. We start by recalling some basic itemset mining definitions from [1] and explain how they apply to our settings.

Let $\mathcal{I} = \{i_1, i_2, i_3, \ldots\}$ be a finite set of distinct item names. Define an itemset (or transaction) $A$ as a subset of $\mathcal{I}$. Define a database $D$ as a bag (multiset) of transactions. $|D|$ denotes the number of transactions in $D$. The frequency or support of an itemset $A \subseteq \mathcal{I}$ in $D$ is $\text{supp}_D(A) := |\{T \in D \mid A \subseteq T\}| / |D|$. A is considered frequent if its support exceeds a predefined threshold $\Theta$: we assume that $0 < \Theta < 1$ as mining is trivial when $\Theta \in \{0, 1\}$. Given a database $D$, we define the predicate $\text{freq}(\cdot)$ which takes an itemset as input and returns true iff this itemset is frequent in $D$ (the dependency on $D$ is omitted from the notation).

For example, in the domain of leisure activities, $\mathcal{I}$ may include different activities, relevant equipment, locations, etc. Each transaction $T$ may represent all the items involved in a particular leisure event (a vacation day, a night out, etc.). If, e.g., the set $\{\text{tennis}, \text{racket}, \text{sunglasses}\}$ is frequent, it means that a racket and sunglasses are commonly used for tennis. Or, if $\{\text{indoor_cycling}, \text{TV}\}$ is frequent, it may imply that indoor cyclists often watch TV while cycling.

In our crowd-based setting, the database of interest $D$ is not materialized and only models the knowledge of people, so we can only access $D$ by asking them questions. As shown in [2], we can ask people for summaries of their personal knowledge, which we can then interpret as data patterns – itemset frequencies in our case. We thus abstractly model a crowd query as follows:

**Definition 2.1 (Crowd Query).** A crowd query takes as input an itemset $A \subseteq \mathcal{I}$ and returns $\text{freq}(A)$.

When using crowdsourcing to answer this type of crowd queries, and when posing questions to the crowd in general, one must deal with imprecise or partial answers. This general problem was studied in previous crowdsourcing works [2, 3, 31]. We can employ one of their methods as a black-box and assume that each crowd query is posed to a sufficient (constant) number of users, so as to gain sufficient confidence in the obtained Boolean answer. Thus, the cost of a crowd mining algorithm can be defined as the number of crowd queries rather than the number of posed questions: the crowd acts as an oracle for itemset frequency. The cost metric does not depend on the size of the hypothetic database $D$, or the number of scans that would be necessary to determine the frequency of the queried itemsets if $D$ were materialized.

### Itemset Dependency and Taxonomies

The support of different itemsets can be dependent. For example, if $A \subseteq B$ then $B \subseteq T$ implies $A \subseteq T$ so $\text{supp}_D(A) \geq \text{supp}_D(B)$ for every $D$. This fundamental property is used by classic mining algorithms such as Apriori [1].

Moreover, as noted in [35], there may be dependencies between itemsets resulting from semantic relations between items. For instance, in our example from the Introduction, the itemset $\{\text{sunglasses, sport}\}$ is semantically implied by any transaction containing $\{\text{sunglasses, tennis}\}$, since tennis is a sport.

Such semantic dependencies can be naturally captured by a taxonomy [35]. Formally, we define a taxonomy as a partially ordered set (or poset) $\Psi = (\mathcal{I}, \preceq)$ where $\preceq$ is a partial order over the element domain $\mathcal{I}$. $i \preceq i'$ indicates that item $i'$ is more specific than $i$ (any $i'$ is also an $i$). Observe that the antisymmetry of $\preceq$ implies that no two different items in $\mathcal{I}$ are equivalent by $\preceq$. We use $i \preceq i'$ when $i \preceq i'$ and $i \neq i'$, and denote by $\preceq$ the covering relation of $\preceq$: $i \preceq i'$ iff $i < i'$ and there exists no $i''$ s.t. $i < i'' < i'$.

We represent posets as DAGs, whose vertices are the poset elements, and where a directed edge $(i, i')$ indicates that $i \preceq i'$. This is in line with standard representations of posets such as, e.g., Hasse diagrams. We denote by $|\Psi| = O(|\mathcal{I}|^2)$ the size of the taxonomy including the number of elements and pairs in $\preceq$.

**Example 2.2.** Consider the taxonomy $\Psi_1$ shown in Figure 1a. We can label its elements with items, e.g.: $\Psi_1$.

| Crowd | With respect to the input | With respect to the input and output |
|-------|--------------------------|-------------------------------------|
| Lower | $\Omega(|S(\Psi)|)$       | (Prop. 3.6)                        |
| Upper | $O(|S(\Psi)|)$           | (Prop. 3.9)                        |
| Comp. | $\Omega(\log |S(\Psi)|)$ | (Cor. of Prop. 3.6)                |
| Lower | $O(\log |S(\Psi)|)$       | (Cor. 4.4)                        |
| Upper | $O\left(\frac{1}{2}(|\Psi|) \cdot \left(|\Psi|^2 + |I(\Psi)|\right)\right)$ (Prop. 4.3) | |

Table 1: Summary of the main complexity results, where we have $|I(\Psi)| \leq 2^{|S(\Psi)|}$ and $|S(\Psi)| \leq 2^{|I(\Psi)|}$

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1Such equivalence would stand for semantic synonyms such as cycling and biking. We thus assume that every group of synonyms is represented by a single item.
I. cycling, 2. sport, 3. bicycle_touring, 4. indoor_cycling. The interpretation of the taxonomy would then be: both bicycle touring and indoor cycling are types of cycling, and indoor cycling is also a sport.

Let us briefly define some general useful terms in the context of posets. When \( i \leq i' \) we call \( i \) an ancestor and \( i' \) a descendant. Similarly, when \( i < i' \) we call \( i \) a parent and \( i' \) its child. A chain is a sequence of elements \( t_1 < t_2 < \cdots < t_n \). An antichain is a set of elements \( A = \{i_1, \ldots, i_n\} \) that are incomparable with respect to \( \leq \), i.e., there exist no \( i_j \neq i_k \in A \) s.t. \( i_j \leq i_k \). The width of a poset \( P \), denoted by \( w[P] \), is the size of its largest antichain. An order ideal (or lower set) \( A \) of a poset \( P \) is a subset of its elements s.t. if \( i \in A \) then all the ancestors of \( i \) are in \( A \).

**Example 2.3.** The antichains of the example taxonomy \( \Psi \) include the empty antichain \( \{\} \); singleton itemsets such as \( \{3\} \); and antichains of size 2 such as \( \{2,3\} \) (since 2 and 3 are incomparable). There are no larger antichains, and thus \( w[\Psi] = 2 \).

We denote by \( AC[\Psi] \) the domain of antichains of elements from \( \Psi \). Antichains are concise in the sense that they contain no items implied by other items. In this way, e.g., \( \{\text{tennis}\} \) concisely represents \( \{\text{tennis}, \text{sport}\}, \{\text{tennis}, \text{sport, activity}\} \), etc. Thus, unless stated otherwise, whenever we mention itemsets we assume that the items form an antichain. This is also useful for practical purposes: it would be strange, e.g., to ask users whether they simultaneously play tennis and do sport.

Based on \( \leq \), the semantic relationship between items, we can define the corresponding relationship between itemsets. For itemsets \( A, B \) we define \( A \leq B \) iff every item in \( A \) is implied by some item in \( B \). Formally:

**Definition 2.4.** (Itemset taxonomy). Given a taxonomy \( \Psi = (\mathcal{I}, \leq) \) define its itemset taxonomy as the poset \( I(\Psi) = (AC[\Psi], \leq) \). By an abuse of notation, we extend \( \leq \) to itemsets, where for every two itemsets \( A, B \in AC[\Psi] \), \( A \leq B \) iff \( \forall i \in A, \exists i' \in B \) s.t. \( i \leq i' \). Similarly, we extend \( < \) and \( \prec \) to itemsets: \( A < B \) when \( A \leq B \) and \( A \neq B \), and \( A \prec B \) iff \( A < B \) and there exists no \( C \) s.t. \( A < C < B \).

![Figure 1: Example taxonomies](image)

Figure 1b illustrates \( I(\Psi_1) \), the itemset taxonomy of \( \Psi_1 \) from Figure 1a. Observe that, for singleton itemsets, \( \leq \) corresponds to the order on items.

Finally, we redefine support to take \( I(\Psi) \) into account.

**Definition 2.5.** (Itemset support). Let \( A \subseteq I \) be an itemset. We define the support of \( A \) w.r.t. a database \( D \) and a taxonomy \( \Psi \) to be 0 if \( D \) is empty, and as \( \text{supp}_D,A(\Psi) := |\{T \in D \mid A \leq T\}| / |D| \) otherwise.

**Properties of the itemset taxonomy.** By construction, \( I(\Psi) \) is not an arbitrary poset: for instance, it always has a single “root” element, namely the empty itemset, which precedes all other elements by \( \leq \). More generally, the domain of all possible itemset taxonomies can be precisely characterized as the domain of all distributive lattices (see Appendix A for details). We illustrate the structure of \( I(\Psi) \) in two extreme but useful examples.

**Example 2.6.** Figure 1c illustrates a total order or “chain” taxonomy, whose itemset taxonomy is a chain with one more element (Figure 1f). Figure 1g displays a “flat” taxonomy, where all the elements are incomparable. Its itemset taxonomy (Figure 1h) contains all the possible itemsets: it is the Boolean lattice structure explored by classic data mining algorithms such as Apriori [1]. Hence, if a flat \( \Psi \) has \( n \) elements, \( I(\Psi) \) has exactly \( 2^n \) elements and \( \leq \) corresponds exactly to set inclusion.

**Maximal frequent itemsets.** By the definition of support, freq is a (decreasing) monotone predicate over itemsets, i.e., if \( A \leq B \) then freq(\( B \)) implies freq(\( A \)). Consequently, freq can be uniquely and concisely characterized by a set of maximal frequent itemsets (MFIs), namely all the frequent itemsets with no frequent descendants. Equivalently, it can be characterized by a set of minimal infrequent itemsets (MIIs), which are all the infrequent itemsets with no infrequent ancestors. MFIs and MIIs were introduced for knowledge discovery in [27] (where they are called respectively the positive border and negative border), and existing data mining algorithms such as [4] try to identify them as a
to $\Psi$, i.e., it is an arbitrary poset except for the added \{\} element. Compare for example $I^{(1)}(\Psi_1)$ (Figure 1d) with $\Psi_1$ (Figure 1a).

**Problem statement.** Given a known taxonomy $\Psi$ and an unknown database $D$, defining the (also unknown) frequency predicate $\text{freq}$ over the itemsets in $I(\Psi)$, we denote by $\text{MineFreq}$ the problem of identifying, using only crowd queries, all the frequent itemsets in $D$ (or, equivalently, of identifying $\text{freq}$ exactly). We consider interactive algorithms that iteratively compute, based on the knowledge collected so far, which crowd query to pose next, until $\text{MineFreq}$ is solved.

As mentioned in the Introduction, we study the complexity bounds of such algorithms for two metrics. We first consider the number of crowd queries that need to be asked, namely the crowd complexity. Then, we study the feasibility of “crowd-efficient” algorithms, by considering the computational complexity of algorithms that achieve the upper crowd complexity bound. This last restriction is relaxed in the sequel.

### 3. CROWD COMPLEXITY

We now analyze the crowd complexity of $\text{MineFreq}$, first w.r.t. the input taxonomy. Then, we consider the complexity w.r.t. the output, which allows for a finer analysis depending on properties of $\text{freq}$. As a general remark for our analysis, note that we can always avoid querying the same itemset twice, e.g., by caching query answers. Thus, every upper bound $O(X)$ presented in this section is actually $O(\min\{X,|I(\Psi)|\})$.

#### 3.1 With Respect to the Input

To illustrate the problem boundaries, consider the following specific cases of $\Psi$ for which we know the optimal solution strategy. For a chain taxonomy (as in Figure 1c), identifying $\text{freq}$ amounts to a binary search for the single MFI. This can be done in $O(\log |I|)$ steps. For a flat taxonomy (as in Figure 1g), for which the elements of $I(\Psi)$ are the power set of $I$, identifying $\text{freq}$ is equivalent to learning a monotone Boolean function over $n$ variables, where $n = |I|$. For this problem, a tight bound of $\Theta(n^2/\pi)$ is known [21, 22].

We study the solution for a general taxonomy structure. Let us start by defining the following:

**Definition 3.1. (Solution taxonomy).** Given a taxonomy $\Psi$, we define its solution taxonomy $S(\Psi) = I(I(\Psi))$. The domain of elements of $S(\Psi)$ is $\mathcal{A}[I(\Psi)]$, i.e., antichains of itemsets.

We call this construction the solution taxonomy, since its elements correspond, precisely, to the frequency predicates over $I(\Psi)$, i.e., all possible solutions to $\text{MineFreq}$ for a given $\Psi$. More precisely, each element of $S(\Psi)$ is an antichain of itemsets that is exactly the set of MFIs.
If the answer is true can be mapped to an antichain of itemsets, which is an forms an antichain. Thus, every predicate over I(Ψ) its (unique) set of maximal elements, which necessarily

A S(Ψ) are possible in the beginning of the execution, all the elements of ing the execution of mfi.

mfi of some freq predicate. We prove this below but first illustrate the structure via an example.

**Example 3.2.** Figure [2] illustrates parts of the solution taxonomy of the running example, Ψ1 (not provided fully due to its size). Consider, e.g., {{1}, {2}}. This element of S(Ψ1) corresponds to the freq predicate assigning true (only) to {1}, {2} and their ancestor {} in I(Ψ1). Similarly, {} in S(Ψ1) corresponds to a predicate assigning false to every itemset, and {{3, 4}} to the one assigning true to every itemset.

Next, we prove the claim about the correspondence between S(Ψ) elements and frequency predicates by first showing a bijective correspondence between elements of S(Ψ) and (decreasing) monotone predicates over I(Ψ).

Then, we show that each such predicate can indeed serve as a frequency predicate for some database.

**Proposition 3.3.** There exists a bijective mapping from AC[I(Ψ)] to the monotone predicates over I(Ψ).

The proof maps every predicate in a general poset to its (unique) set of maximal elements, which necessarily forms an antichain. Thus, every predicate over I(Ψ) can be mapped to an antichain of itemsets, which is an element of S(Ψ) (see formal proof in Appendix B.1).

**Proposition 3.4.** For every threshold 0 < Θ < 1, every monotone predicate F over I(Ψ) is the frequency predicate freq of some database D.

Proof. Given F, let M be its set of maximal elements. If M is empty, i.e., no itemset should be frequent, F is realized by the empty database. Otherwise, construct D to consist of the following d transactions: n “full” transactions with all the items of I, one transaction per A ∈ M containing exactly A, and d − n − |M| empty transactions. We choose d and n s.t. every (non-empty) itemset B is frequent in D if it is supported by > n transactions, or, equivalently, if it is supported by at least one of the |M| non-trivial transactions. To do that, pick an integer d that is large enough such that there exists an integer n such that 0 ≤ n/d < Θ < (n + 1)/d < (n + |M|)/d ≤ 1. When this holds, the frequent itemsets of D are exactly the ancestors of itemsets in M, so freq = F as desired.

We have now shown that the MineFreq problem (identifying freq) is equivalent to finding the element of S(Ψ) corresponding to freq, namely mfi. Before we study the complexity of this last task, let us first describe abstractly how the solutions space is narrowed down during the execution of any algorithm that solves MineFreq. In the beginning of the execution, all the elements of S(Ψ) are possible. The algorithm uses some decision method to pick an itemset A ∈ AC[Ψ] and queries it. If the answer is true (A is frequent), this means that mfi contains A or one of its descendants in I(Ψ), so we can eliminate all MFI sets of S(Ψ) that do not have this property, which we can show are exactly the non-descendants of {A} in S(Ψ). Conversely, if the answer is false (A is infrequent), we can eliminate all the descendants of {A} in S(Ψ) (including {A}). The last remaining element in S(Ψ) at the end corresponds to the correct freq predicate, because it is the only one consistent with the observations.

**Example 3.5.** Consider again S(Ψ1) in Example 3.2. By discovering that, e.g., freq({1}), we know e.g.: that {2} cannot be the only MFI so we can eliminate the solution element {{2}}, that its ancestor {} is not an MFI so we can eliminate {{}}, and so on. In total, all non-descendants of {{1}} in S(Ψ1) can be eliminated.

**Lower Bound.** We now give a lower crowd complexity bound for solving MineFreq in terms of the input, which is proved to be tight in the sequel. The proof relies on the fact that solving MineFreq amounts to searching for an element in S(Ψ); it can be given as a simple information-theoretic argument (see Appendix B.1), but we present it in connection with [24] as we will reuse this link to obtain our upper bound.

**Proposition 3.6.** The worst-case crowd complexity of identifying freq is Ω( log(|S(Ψ)|)).

Proof. This is implied by the analogous claim of [24] about order ideals in general posets. By characterizing an order ideal by its maximal elements (whose descendants are not in the ideal) we obtain an antichain, which defines a bijective correspondence between order ideals and antichains. Thus, we can map antichains to order ideals, and use the observation in [24] directly to obtain the same lower bound.

In the worst case, log |S(Ψ)| can be linear in |I(Ψ)|, which itself may be exponential in |Ψ| (e.g., for a flat taxonomy). When this is the case, a trivial algorithm achieves the complexity bound by querying every element in I(Ψ). However, for some taxonomy structures (e.g., chain taxonomies), the size of S(Ψ) is much smaller. We now use w[I(Ψ)] to deduce a more explicit lower bound for Prop. 3.6.

**Proposition 3.7.** w[I(Ψ)] ≥ (w[Ψ]/2).

Proof. By definition, there exists at least one itemset in I(Ψ) of size w[Ψ]. This itemset has (w[Ψ]/2) subsets of size [w[Ψ]/2]. These itemsets are also in I(Ψ), since they only contain incomparable items. Moreover, they are pairwise incomparable in I(Ψ), and thus form an antichain whose size yields the lower bound.

By replacing Ψ with I(Ψ) we get a lower bound for w[S(Ψ)]. We can thus prove the following bound which,
though weaker, is more explicit than Prop. 3.6 as it is expressed in terms of the original ontology width rather than |S(Ψ)|.

**Corollary 3.8.** The worst-case crowd complexity of identifying freq is \( \Omega(2^{w[Ψ]}/\sqrt{|I(Ψ)|}) \).

**Proof.** \(|S(Ψ)| > w[S(Ψ)] \geq \left(\frac{w[I(Ψ)]}{2}\right)^{\frac{1}{|I(Ψ)|}}\). We obtain \( \log |S(Ψ)| \geq \Omega(\log (2^{w[I(Ψ)]}/\sqrt{|I(Ψ)|})) = \Omega(w[I(Ψ)]) \) using Stirling’s approximation; and finally, using the lower bound of \( w[I(Ψ)] \) and applying the approximation again, we express the bound in terms of \( w[Ψ] \). \( \square \)

**Upper Bound.** We now state a tight upper bound (i.e., that matches the lower bound up to a multiplicative constant). The proof relies on Theorem 1.1 of [24], which shows that, in any poset, there exists an element such that the proportion of order ideals (or, in our case, elements of \( S(Ψ) \)) that contain the element is within a constant range of \( 1/2 \). Hence, a greedy strategy that queries such elements will eliminate a constant fraction of the possible solutions at each step and completes in a time that is logarithmic in the size of the search space. The proof details are deferred to Appendix B.1.

**Proposition 3.9.** The worst-case crowd complexity of identifying freq is \( O(\log |S(Ψ)|) \).

**3.2 With Respect to the Input and Output**

So far our results only relied on the structure and size of the input taxonomy \( Ψ \). However, as noted in Section 2, the characteristics of the output freq predicate may have a crucial effect on the problem complexity, because, in practical scenarios, the number of MFIs and MIIs is usually small. For instance, when dealing with leisure habits, the number of activities that are common among people is typically very small w.r.t. all the combinations that the taxonomy allows. Hence, we next study the effect of the output on the crowd complexity boundaries of MineFreq.

**Lower Bound.** Since each of the sets of MFIs and MIIs uniquely represents the freq predicate, one could hope that it would be sufficient to identify only one of them to solve MineFreq. However, it turns out that one must query at least all the MIIs to verify that the MFIs are maximal, and vice versa. This result is well-known for Boolean lattices [15]; in our setting it follows from the more general Thm. 2 of [25] (which concerns any partial order rather than just distributive lattices).

**Proposition 3.10.** The worst-case crowd complexity of identifying freq is \( \Omega(|mfi| + |mii|) \).

Hence, though we can describe the output by its set of MFIs (or MIIs), we need to query both the MFIs and MIIs. This implies that the crowd complexity may be exponential even in the minimal output size, since the difference between \(|mfi|\) and \(|mii|\) may be large though only one suffices to describe the output. This is derived from a known result in Boolean function learning [7].

**Corollary 3.11.** (See [7]). The worst-case crowd complexity of identifying freq is \( \Omega(2^{\min(|mfi|, |mii|)}) \).

We note that the current lower bound is not tight: for instance, over a chain taxonomy, \(|mfi| + |mii| \leq 2\) for any freq predicate, but we already noted in Section 3.1 that the worst-case crowd complexity in this case is \( \Omega(\log |I|) \).

**Upper bound.** We next show an upper bound that is within a factor \(|I|\) of the lower bound of Prop. 3.10. It generalizes known MFI and MII identification algorithms for the case where there is no underlying taxonomy, such as the monotone Boolean function learning algorithm of [15] and the Dualize and Advance algorithm of [17] [18]; see Section 7 for an in-depth comparison. Intuitively, our algorithm traverses the elements of \( I(Ψ) \) in an efficient way to identify an MFI or an MII, and repeats this process as long as there are unclassified elements in \( I(Ψ) \), i.e., elements that are not known to be frequent or infrequent. Using this method we can find each MFI or MII in time \( O(|I|) \), and the bound follows.

**Theorem 3.12.** Algorithm 1 identifies freq in crowd complexity \( O(|I| \cdot (|mfi| + |mii|)) \).

```
Data: Ψ: a taxonomy
Result: M = mfi and N = mii, for the correct freq predicate over I(Ψ)
M, N ← ∅;
while there is an unclassified element A ∈ I(Ψ) do
    if freq(A) then
        /* A is an ancestor of an MFI, search for it by traversing A’s frequent descendants. */
        for i ∈ I do
            B ← get-AC(A ∪ anc(i))
            if A < B and freq(B) then A ← B
            /* A’s descendants are infrequent */
            mark-freq(A); add A to M;
        else /* A is a descendant of an MII, search for it by traversing A’s infrequent ancestors. */
            for i ∈ I do
                B ← get-AC(A \ desc(i))
                if B < A and ¬freq(B) then A ← B
                /* A’s ancestors are frequent */
                mark-infreq(A); add A to N;
        return M, N;
```

Algorithm 1: Identify mfi and mii
**Proof.** We explain the course of Algorithm 1 show that it is correct (i.e., identifies freq correctly), and analyze its computational complexity.

Algorithm 1 uses a few sub-routines: mark-freq($A$) (resp., mark-infreq($A$)) classifies the itemset $A$ and its ancestors (resp., descendants) as frequent (resp., infrequent). get-AC($A$) removes from $A$ all the items that are implied by other items (i.e., all $i \in A$ such that $i < i'$ for some $i' \in A$) so that get-AC($A$) returns an antichain representing $A$. anc($i$) and desc($i$) return, respectively, the ancestors and descendants of $i$ in $\Psi$ (including $i$).

We argue that each iteration of the main while loop of Algorithm 1 identifies exactly one new MFI or MII. First, an unclassified node $A \in I(\Psi)$ is chosen. If $A$ is frequent (first if statement), it is either an MFI or an ancestor of an MFI. Since it used to be unclassified, at this point each of its descendants is unclassified or infrequent: in particular, $A$ is not an ancestor of an already discovered MFI. We thus start traversing descendants of $A$ by adding items from $\mathcal{Z}$ to $A$ and using get-AC to turn the result into an antichain. Either the current $A$ is an MFI so all of its children are infrequent, the inner for loop ends, and we identify $A$ as an MFI. Otherwise, as $A$ is frequent but not maximal, there exists some frequent $B \in I(\Psi)$ s.t. $B = get-AC(A \cup anc(i'))$ for some item $i'$. If $i'$ had already been considered by the for loop but was dismissed, it would mean that we dismissed an ancestor of $B$ as infrequent, contradicting the assumption that $B$ is frequent. Thus, $i'$ cannot have been considered by the for loop yet, so we will replace $A$ by $B$ before the for loop terminates. Hence, at the end of the for loop, we identify a new MFI. In the same manner, the code within the else part identifies an MII by traversing infrequent ancestors until reaching an infrequent element that has only frequent parents.

**Correctness.** The above implies that the algorithm terminates, that each identified MFI and MII is correct, and that all elements are correctly marked as frequent and infrequent. To prove completeness, consider an MFI $A$. By the end of the algorithm, $A$ is known to be frequent; since it has no frequent descendants, mark-freq($A$) was necessarily called, which implies that $A$ was added to $M$. The proof for MIs is similar.

**Complexity.** Since Algorithm 1 identifies an MFI or MII in each while iteration, there can be at most $|mfi| + |mii|$ iterations. The inner loop performs $O(|\mathcal{Z}|)$ queries, and thus the total complexity is as stated above. ☑

Following an idea of [17], we observe that the bound can be improved to $O(|mii| + |\mathcal{Z}| \cdot |mfi|)$ if we always choose the unclassified element $A$ to be minimal, because this ensures that no queries need to be performed whenever we are in the else branch. Moreover, if we run two instances of Algorithm 1 in parallel, one choosing maximal unclassified elements for $A$ and the other one choosing minimal unclassified elements for $A$, we improve the bound to $O(|mfi| + |mii| + |\mathcal{Z}| \cdot \min\{|mfi|, |mii|\})$.

### 3.3 Restricted Itemset Size

We next consider the $k$-itemset taxonomy, $I^{(k)}(\Psi)$. Beyond the practical motivations for using $I^{(k)}(\Psi)$ (see Section 2), restricting the number of MFIs and MIs may naturally improve the complexity bounds.

As explained in Section 2, $I^{(k)}(\Psi)$ is not necessarily a distributive lattice; and the size of $I^{(k)}(\Psi)$ is always polynomial while that of $I(\Psi)$ may be exponential (w.r.t. $|\mathcal{I}|$). However, for every $I(\Psi)$ such that $k \geq w[I(\Psi)]$, it holds that $I^{(k)}(\Psi) = I(\Psi)$.

Note that in Section 3.1 we did not make any assumptions on the itemset taxonomy structure, so our results apply to any poset and in particular to $I^{(k)}(\Psi)$. We obtain the following, where $S^{(k)}(\Psi) := I(I^{(k)}(\Psi))$.

**Corollary 3.13.** The worst-case crowd complexity of identifying freq over $I^{(k)}(\Psi)$ is $\Omega(\log(|S^{(k)}(\Psi)|))$; and there exists an algorithm to identify freq over $I^{(k)}(\Psi)$ in crowd complexity $O(\log(|S^{(k)}(\Psi)|)) \leq O(|\mathcal{I}|^k)$.

For the complexity w.r.t. the output over restricted itemsets, the lower bound of Thm. 3.10 holds as well, using the same proof. For the upper bound, however, we cannot use Algorithm 1 for a $k$-itemset taxonomy, adding (or removing) a single item to a $k$-itemset does not necessarily yield a $k$-itemset. Improving the trivial upper bound remains an open problem.

### 4. COMPUTATIONAL COMPLEXITY

We next study the feasibility of “crowd-efficient” algorithms, by considering the computational complexity of algorithms that achieve the upper crowd complexity bound. We follow the same axes as in the previous section. In all problem variants, we have the crowd complexity lower bound as a simple (and possibly not tight) lower bound. For some variants, we show that, even when the crowd complexity is feasible, the underlying computational complexity may still be infeasible.

#### 4.1 With Respect to the Input

As a simple lower bound, we know that the computational complexity of MineFreq is higher than the crowd complexity, and is thus $\Omega(\log(|S(\Psi)|))$.

The problem of finding tighter bounds for computational complexity w.r.t. the input remains open. Many works [10][12] provide efficient algorithms for computing a good split element in particular types of posets, but no efficient algorithm is known for the more general case of distributive lattices (or for arbitrary posets). We now give evidence suggesting that no such algorithm exists.

At any point of a MineFreq-solving algorithm, we define the best-split element as the element of $I(\Psi)$ which is
guaranteed to eliminate the largest number of solutions of $S(\Psi)$ when queried. Following the proof of Prop. 4.2, if we could efficiently compute the best-split element, we would obtain a computationally efficient greedy algorithm that is also crowd-efficient. We show below that this is not possible for the case of bounded-size itemsets (and their corresponding restricted itemset taxonomies). This, of course, does not prove that there exists no computationally efficient non-greedy algorithm; however, it suggests that it is unlikely that such an algorithm exists, because of the close relationship between finding best-split elements and counting the antichains of $I(\Psi)$. This is similar to a result of [12], which proves that identifying a good-split element (which guarantees eliminating a constant fraction of the solutions) is computationally equivalent to a relative approximation of the number of order ideals (though this is not known to be $\#P$-complete).

**Theorem 4.1.** The problem of identifying, given $\Psi$ and $k$, the best-split element in $I^{(k)}(\Psi)$ is $\text{FP}^{\#P}$-complete in $|\Psi|$.  

**Proof.** (Sketch). To prove membership, we show a reduction from our problem to counting antichains in a general poset, which is known to be in #P [33]. Using an oracle for antichain counting, we can count the number of eliminated antichains in $S^{(k)}(\Psi)$ for every element of $I(\Psi)$, and thus find the best-split element.

The more challenging part is proving hardness. For that, we show a reduction from the problem of counting antichains (which is $\text{FP}^{\#P}$-hard) to our problem. Let us call ancestor and descendant solutions of $A$ the solutions (elements of $S^{(k)}(\Psi)$) that are eliminated if an itemset $A$ is discovered to be frequent or infrequent respectively. For any poset $P$ and natural number $n$, we show that we can construct a $k$-itemset taxonomy $I^{(k)}(\Psi)$ with an itemset $A_0$ such that, for some increasing affine function $F$, $A_0$ has $F(|AC[P]|)$ descendant solutions and $F(n)$ ancestor solutions. As the best-split element $A^*$ in $I^{(k)}(\Psi)$ has a roughly equal number of ancestor and descendant solutions, comparing the position of $A_0$ and $A^*$ allows us to compare $|AC[P]|$ and $n$: if $A^*$ is an ancestor of $A_0$, it has more descendant solutions than $A_0$, and hence $|AC[P]| < n$. Similarly, if $A^*$ is a descendant of $A_0$, $|AC[P]| > n$. Using this decision method, it is possible to perform a binary search on values of $n$ between 0 and $2^{|P|}$ and find the exact value of $|AC[P]|$. 

As for upper bounds, our results for complexity w.r.t. the input, namely Cor. 4.1 will follow from the results w.r.t. the input and output that we present in the next section.

\[\#P\] is the class of counting problems that return the number of solutions of NP problems. $\text{FP}^{\#P}$ is the class of function problems that can be computed in polynomial time using a $\#P$ oracle.

### 4.2 With Respect to the Input and Output

**Lower Bound.** As shown by Algorithm 1 finding an MFI or MII requires a number of queries linear in $|I|$. However, note that the algorithm assumes that at any point we are able to determine if the set of unclassified elements of the itemset taxonomy is empty. We next show that this is a non-trivial problem. We recall the definition of problem EQ [5]. Let $B_n = \{0, 1\}^n$ be the set of Boolean vectors of length $n$. Define the order $\leq$ on $B_n$ by $x \leq y$ iff $x_i \leq y_i$ for all $i$. For $X \subseteq B_n$, write $T(X) = \{y \in B_n \mid \exists z \in X, z \leq y\}$ and $F(X) = \{y \in B_n \mid \exists z \in X, y \leq z\}$. Problem EQ is the following: given $X, Y \subseteq B_n$ such that $T(X) \cap F(Y) = \emptyset$, decide whether $T(X) \cup F(Y) = B_n$.

**Proposition 4.2.** If MineFreq can be solved in computational time $O(|I| \cdot (|I|^2 + |mfi| + |mii|))$ then there exists a PTIME solution for problem EQ from [5].

It is unknown whether EQ is solvable in polynomial time (see [11, 16] for a survey); the connection between frequent itemset mining and EQ (and its other equivalent formulations, such as monotone dualization or hypergraph transversals) was already noted in [25]. Note that the proof above uses the fact that the itemset size is not restricted. For $k$-itemset taxonomies, finding a tighter lower bound than the trivial $|mfi| + |mii|$ remains an open problem.

**Upper Bound.** We consider again Algorithm 1 whose complexity we analyzed in Section 3.2. By completing some implementation details, we can now analyze its computational complexity as well, and obtain an upper bound. For simplicity, this bound is presented in the Introduction with $|\Psi|$ which is $\leq |I|$.

**Proposition 4.3.** There exists an algorithm to solve MineFreq in computational time $O(|I| \cdot (|I|^2 + |mfi| + |mii|))$.

**Proof.** Algorithm 1 uses a computation of an unclassified element of $I(\Psi)$. Since by Prop. 4.2 this is probably non-polynomial, we can use the brute-force method of materializing the itemset taxonomy $I(\Psi)$. We use a hash table to find any element in the $I(\Psi)$ structure in time linear in the element size. The implementation of mark-freq and mark-infreq locates $A$ in $I(\Psi)$ using the hash table, traverses its ancestors or descendants respectively, and marks them as (in)freq.

To compare itemsets efficiently, we represent each itemset $A$ by an ordered list of the items in its order ideal, i.e., $\downarrow A = \{i \in I \mid \exists i' \in A, i' \leq i\}$. In this case, $A \leq B$ if $\downarrow A \subseteq \downarrow B$, which can be verified in time $O(|A| + |B|) \leq O(|I|)$. Using this representation, we do not need the sub-routine get-AC. We generate once, for every $i \in I$, two ordered lists: desc($i$) and
ancl(i), holding its descendants and ancestors respectively. These lists can be computed in time $O(|I(Ψ)|)$ by building the transitive closure of $I(Ψ)$, and can be used to compute $\downarrow A \cup \text{ancl}(i)$ and $\downarrow A \text{desc}(i)$ in time $O(|I(Ψ)|)$.

Let us analyze the overall complexity of the suggested implementation. We construct $I(Ψ)$ (where each element has both its antichain and order ideal representations) in $O(|AC[Ψ]| + |I(Ψ)|^2)$ according to Prop. A.4 (devised to the Appendix), and construct $\text{ancl}(i)$ and $\text{desc}(i)$ in time $O(|I(Ψ)|^2)$. Now, we run $|mfi| + |mii|$ times the body of the outer while loop, which 1. finds an unclassified element by a brute-force search taking time $|AC[Ψ]|$, 2. runs $O(|I(Ψ)|)$ times the body of one of the for loops that computes $\downarrow A \cup \text{ancl}(i)$ or $\downarrow A \text{desc}(i)$ and verifies $\leq$ in time $O(|I(Ψ)|)$, and 3. calls mark-freq or mark-infreq which takes time $O(|I(Ψ)| + |I(Ψ)|)$ to locate the itemset in $I(Ψ)$ and traverse its ancestors or descendants. Summing these numbers and simplifying the expression yields the claimed complexity bound.

Since we know that $|mfi| + |mii| \leq |AC[Ψ]|$, we can plug $|AC[Ψ]|$ in the complexity formula and obtain an upper bound that does not depend on the numbers of MFIs and MIs. In this manner we achieve a bound polynomial in $|I(Ψ)|$ and improve the upper bound described in Section 4.1. However, note that this is in fact a relaxation of our requirement for crowd-efficient algorithms, since Algorithm 1 is not crowd-efficient w.r.t. the upper bound of Prop. 3.9 in terms of the input.

This result is also simplified in the Introduction, replacing $|I(Ψ)|$ by $|Ψ|$ which is $\geq |I(Ψ)|$, and $|AC[Ψ]|$ by $|I(Ψ)|$ which is $\geq |AC[Ψ]|$.

COROLLARY 4.4. There exists an algorithm to solve MineFreq in computational complexity $O(|I(Ψ)| \cdot |I(Ψ)|^2 + |AC[Ψ]|)$.

5. CHAIN PARTITIONING

Recall that in the beginning of Section 3.1 we mentioned the special case of chain taxonomies, for which a binary search achieves a tight complexity bound, both crowd and computational, of $Θ(|AC[Ψ]|)$. We generalize this insight to solve MineFreq for taxonomies partitioned in disjoint chain taxonomies. Chain partitioning is a standard technique in Boolean function learning [21, 23], that splits the Boolean lattice elements into disjoint chains, and then performs a binary search for the maximal frequent element on each chain. The following easy proposition holds (we justify how the partition $P$ is obtained at the end of the section):

PROPOSITION 5.1. Given a partition $P$ of $I(Ψ)$ into $w[I(Ψ)]$ chains, freq can be identified in both crowd and computational complexity $O(w[I(Ψ)] \cdot \log |I(Ψ)|)$.

The $\log |I(Ψ)|$ factor comes from the binary search in the chains. To understand intuitively why their length is at most $|I|$, notice that the worst case is achieved by the full Boolean lattice, and that, in this case, for every chain of the form $A_0 \leq \ldots \leq A_n$, it holds that $|A_i| + 1 \leq |A_{i+1}|$, so at most $|I| + n$ items can be added to $A_0$ in total (see Figure 1h).

Let us compare the result of Prop. 5.1 with previous results. In terms of crowd complexity, if $|S(Ψ)|$ is close to its lower bound, $2^{w[I(Ψ)]}$, then the partition binary search performs more queries by a multiplicative factor of $\log |I|$ than the upper bound of Prop. 3.9. On the other hand, since we know that the bound of Prop. 3.9 is tight, we get an upper bound for $|S(Ψ)|$ that depends on $w[I(Ψ)]$ (in addition to the trivial upper bound $2^{\Omega(|Ψ|)}$).

COROLLARY 5.2. $|S(Ψ)| \leq 2^{w[I(Ψ)]} \log |I(Ψ)|$.

When $|mfi| + |mii| = \Omega(w[I(Ψ)])$, the crowd complexity of the partition binary search is asymptotically smaller than that of Algorithm 1 $O(|I(Ψ)| \cdot (|mfi| + |mii|))$. Intuitively, this is because Algorithm 1 in the worst case, can traverse a full chain for every MFI and MII, taking linear time whereas the partition binary search takes logarithmic time. However, when $|mfi| + |mii|$ is small w.r.t. $w[I(Ψ)]$, Algorithm 1 considers significantly less chains and is thus more efficient.

It remains to explain how to obtain the partition $P$. By Dilworth’s theorem, it is possible to partition the poset $I(Ψ)$ into exactly $w[I(Ψ)]$ chains [9]. Computing the partition can be done in $O(poly(|I(Ψ)|))$, by a reduction to maximum matching (or maximal join) in a bipartite graph [14]. See Appendix B.3 for a discussion on the complexity of taxonomy chain partitioning.

6. GREEDY ALGORITHMS

In the previous sections, we have attempted to fully identify freq. The solutions that we presented try to do so by maximizing the number of eliminated solutions, or identifying MFIs or MIs. However, we may not be able to pose enough questions to identify freq exactly. In a dynamic crowd setting we could assume, e.g., that the cost of obtaining answers from the crowd (both in terms of money and latency) is not controlled, and that the identification of freq may be interrupted at any time. In such cases, our algorithms would perform badly:

EXAMPLE 6.1. Assume that the unclassified part of the itemset taxonomy $I(Ψ)$ contains a chain $C$ of even length $2n$, for some $n > 1$, and one incomparable itemset $A$. There are exactly $2 + 4n$ antichains in this poset (one empty, $1 + 2n$ of size 1 and $2n$ of size 2), which is also the number of possible solutions. Asking about $A$ eliminates exactly half of the possible solutions for freq and finds an MII or MFI. However, if we have to interrupt the computation after only one query, we have only obtained information about $A$. It would have been better to query a middle element of $C$: though this eliminates less solutions and does not identify an MII of MFI, it classifies $\geq n$ itemsets.
Motivated by this example, in this section, we assume that the computation can be halted at any time, and look at the intuitive strategy that tries to maximize the number of classified itemsets at halting time using the following greedy approach: compute, for every itemset, what is the worst-case (minimal) number of itemsets that could be classified if we query it; then query the greedy best-split itemset, namely the itemset which maximizes this number. To perform the greedy best-split computation, we need to count the number of ancestors and descendants of each element; this may be done in time linear in \(|I(Ψ)|\) per itemset. In terms of \(|Ψ|\), we can show that this computation is hard.

**Proposition 6.2.** Finding the greedy best-split itemset is \(\text{FP}^\#\text{P}\)-hard w.r.t. \(|Ψ|\). There exists an algorithm which finds it in time \(O(|\mathcal{AC}(Ψ)| \cdot |I|^2 + |I(Ψ)|))\).

To prove this, we first observe that the structure of \(S^{(1)}(Ψ)\) is almost identical to that of \(I(Ψ)\). Then, for the structure used in the proof of Thm. 4.1, we show a reduction from finding the best-split element in \(S^{(1)}(Ψ)\) to finding the greedy best-split element in \(I^{(1)}(Ψ)\). See Appendix B for the details. The second part follows from the brute-force method described above, in combination with the complexity of materializing \(I(Ψ)\) (see the upper bound in Appendix A).

### 7. RELATED WORK

Throughout the paper, we have combined and extended results from order theory, Boolean function learning and data mining \([5, 7, 15, 17, 18, 24]\), to obtain our characterization of the complexity of the crowd mining problem. We now discuss further related work.

Several recent works consider the use of crowdsourcing platforms as a powerful means of data procurement (e.g., \([13, 27, 32]\)). As the crowd is an expensive resource, many works focus on minimizing the number of questions posed to the crowd to perform a certain task: for instance, computing common query operators such as filter, join and max \([8, 20, 25, 31, 26]\), performing entity resolution \([37]\), etc. The present work considers the mining of data patterns from the crowd, and thus is closely related to this line of work.

The most relevant work, by some of the present authors, is \([2]\), which proposes a general crowd mining framework. That work focused on a technique to estimate the confidence in a mined data pattern and how much it increases if more answers are gathered: we could use this technique to implement the crowd query black-box mechanism in our context. However, \([2]\) did not address the issue of the dependencies between rules, or study the implied complexity boundaries, which is the objective of the present paper. Another particularly relevant work is \([30]\), which considers a crowd-assisted search problem in a graph. While it is possible to reformulate some of our problems as graph searches in the itemset and solution taxonomies, there are two important differences between our setting and theirs. First, our itemset and solution taxonomies may be exponential in the size of the original taxonomy but have a specific structure, which allows, in some cases, to perform the search without materializing them. Second, we allow algorithms for \texttt{MineFreq} to choose crowd queries interactively based on the answers to previous queries, whereas \([30]\) studies “offline” algorithms where all questions are selected in advance. Consequently, our algorithms and complexity results are inherently different.

Frequent itemset discovery is a fundamental building block in data mining algorithms (see, e.g., \([1]\)). The idea of using taxonomies in data mining was suggested in \([35]\), which we use as a basis for our definitions.

Another line of works in data mining models the discovery of interesting data patterns through oracle calls \([25]\). This work is closely connected to ours by (i) the use of oracles, which may be seen as an abstraction of the crowd (compared to our setting), and (ii) the separation between the complexity analysis of the number of oracle calls (crowd complexity in our case) and of the computational process. However, because our motivation is to query the crowd, we focus on the specific problem of mining under a taxonomy over the itemsets (and related variants such as limiting the itemset size) which is not studied in itself in this line of work. On the one hand, \([25]\) studies a generalization of our setting, namely the problem of finding all interesting sentences given a specialization relation on sentences. They introduce the notion of border (corresponding to MFIs and MIIs) as a way to bound the number of oracle calls. However, in this general setting, they are not able to give complexity bounds on the performance of applicable algorithms (e.g., Algorithm AllMSS from \([19]\)) to match the bounds that we obtain for the more specific setting of mining frequent itemsets under a taxonomy. On the other hand, the aforementioned papers also study the restricted case of Boolean lattices and give complexity bounds in this case (e.g., for the Dualize and Advance algorithm \([17, 18]\)); however, those algorithms exploit the connection with hypergraph traversals which is very specific to the Boolean lattice. Hence, these algorithms cannot be used to mine frequent itemsets under a taxonomy, which is very natural when working with the crowd, and their complexity bounds are not applicable to our problem.

Finally, among the many works that discuss the connection of data mining and hypergraph traversals, we note the recent work \([16]\) which is relevant to our EQ-hardness result (Prop. 4.2) as it sheds more light on the (still open) complexity of EQ.

### 8. CONCLUSION

In this paper, we have considered the identification of frequent itemsets in human knowledge domains by pos-
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**APPENDIX**

**A. ITEMSET TAXONOMY STRUCTURE**

*Itemset taxonomies and distributive lattices.* We noted in Section 4 that itemset taxonomies are not arbitrary posets. In fact, by the observation that antichains correspond to order ideals (or lower sets), from the proof of Prop. 3.6 their structure is characterized by Birkhoff’s representation theorem\(^3\). A *distributive lattice* is a standard mathematical structure where join \(\wedge\) and meet \(\vee\) operations are defined (which roughly correspond to AND and OR) and where these operations distribute over each other \((x \wedge y) \vee z = (x \vee z) \wedge (y \vee z)\) and likewise if we exchange \(\vee\) and \(\wedge\).

**Theorem A.1** (Birkhoff ’37). Any itemset taxonomy is isomorphic to a distributive lattice, and vice versa.

*The covering relation.* When representing \(I(\Psi)\) as a DAG (like in Figure 12), with edges representing the covering relation \(\preceq\), it is not easy to see which relationship holds between the itemsets at each end of an edge. However, looking more closely, we can characterize those edges as two different types:

- **Addition edge:** An edge between two itemsets \(A, B\) where there exists \(i \in I\) s.t. \(B = A \cup \{i\}\), i.e., one item \(i\) is added to \(A\) to obtain \(B\). (The item \(i\) is necessarily one of the maximal items that are not implied by those of \(A\), i.e., necessarily every \(j\) such that \(j \leq i\) is implied by \(A\).)

- **Specialization edge:** An edge between two itemsets \(A, B\) where there exists \(i, i' \in I\) s.t. \(i < i'\) and \(B = A - \text{parents}[i'] \cup \{i'\}\), i.e., at least one item in \(A\) is made more specific to obtain \(B\).

The reason for these two edge types becomes clear if we represent itemsets by their order ideals. Denote by \(\downarrow A\) and \(\downarrow B\) the order ideals of itemsets \(A\) and \(B\) respectively. The following easy proposition holds.

**Proposition A.2.** \(A \prec B\) in \(I(\Psi)\) iff there exists \(i \notin A\) s.t. \(\downarrow B = \downarrow A \cup \{i\}\) and parents\([i]\) \(\subseteq \downarrow A\)

**Proof.** First observe that \(A \leq B\) iff \(\downarrow A \subseteq \downarrow B\) (this follows from the correspondence between taxonomies and distributive lattices).

For one direction, assume \(A \leq B\). Then \(B \not\leq A\), so \(\downarrow B \not\subseteq \downarrow A\). Let \(i\) be an element of \(\downarrow B \setminus \downarrow A\) that is minimal for the \(\leq\) order on \(I\). Clearly \(A \leq A \cup \{i\}\), and \(\downarrow A \cup \{i\} \subseteq \downarrow B\) so \(A \cup \{i\} \leq B\). Because \(i \not\leq \downarrow A\), \(A \cup \{i\} \neq B\), so \(A \leq A \cup \{i\} \leq B\) and \(A \prec B\) implies that \(B = A \cup \{i\}\). To show that parents\([i]\) \(\subseteq \downarrow A\), observe that for any \(j \in \text{parents}[i] \setminus \downarrow A\) we would have \(j \in \downarrow B \setminus \downarrow A\), contradicting the minimality of \(i\).

Conversely, let \(A\) and \(B\) be itemsets such that there exists \(i \notin A\) s.t. \(\downarrow B = \downarrow A \cup \{i\}\) and parents\([i]\) \(\subseteq \downarrow A\). Because \(\downarrow A \subseteq \downarrow B\), \(A \leq B\). Consider \(C\) such that \(A \leq C \leq B\). Because \(\downarrow A \subseteq \downarrow C \subseteq \downarrow B\), and because the condition on \(i\) imposes that \(|\downarrow B| = |\downarrow A| + 1\), we must have \(\downarrow C = \downarrow A\) or \(\downarrow C = \downarrow B\), so \(C = A = C = B\), thus \(A \leq B\). \(\square\)

By the proposition above, there are two cases in \(I(\Psi)\) in which \(A \leq B\): let \(i\) be the item s.t. \(\downarrow B = \downarrow A \cup \{i\}\). If all the parents of \(i\) are in \(\downarrow A\) but not in \(A\) (including if \(i\) has no parents), then necessarily \(i\) has no ancestors in \(A\) (by maximality of the elements of \(A\) in \(\downarrow A\)) and an *addition edge* will connect \(A\) and \(B\). Otherwise, the parents of \(i\) in \(A\) must be removed, or *specialized*, in order to obtain \(B\).

**Example A.3.** In Figure 2 the solid arrows stand for addition edges, and the double arrows stand for specialization edges. (We “overload” this representation and use double arrows in the item taxonomies \(P\) and \(\Psi\), in order to denote the semantic relationship between items.) For instance, in \(S(1)(\Psi)\) there is an addition edge between \(\{5\}\) and \(\{\{5\}, \{6\}\}\) since the parent of \(\{6\}\) in \(I(1)(\Psi)\), \(\{\}\), is implied by \(\{5\}\) and thus \(\{6\}\) can be added; and there is a specialization edge from \(\{\{5\}, \{6\}\}\) to \(\{7\}\) since both \(\{5\}\) and \(\{6\}\) are specialized into their child in \(I(1)(\Psi)\), \(\{7\}\).

**Materializing the itemset taxonomy.** We next describe an explicit process to materialize \(I(\Psi)\) and \(I(k)(\Psi)\), which is used in Section 4 of this paper. There are naturally different possible representations for both itemsets (e.g., the itemset and the corresponding order ideal) and orders (Hasse diagram, full transitive closure of the relation...). We choose a representation that allows for an efficient itemset taxonomy construction, and which can be later translated into other representations.

We assume some total order on all items of \(\Psi\), respecting \(\leq\), which may be, e.g., a topological ordering of \(\Psi\). For each element \(A\) of the itemset taxonomy we keep 2 ordered lists: \(E_A\) which contains the items in \(A\) and \(O_A\) which contains all the elements in the order ideal \(\downarrow A\) corresponding to \(A\), or in other words, all the ancestors of the items in \(E_A\) including the items of \(E_A\) themselves. We also assume that given an element in \(\Psi\), accessing one of its children or parents can be done in O(1).

**Algorithm 2** constructs, given \(\Psi\), the itemset taxonomy \(I(\Psi)\). The following proposition proves its correctness and complexity.

**Proposition A.4.** Given a taxonomy \(\Psi\) as input, Algorithm 2 constructs \(I(\Psi)\) in time \(O(|\text{AC}(\Psi)| \cdot |\mathcal{I}|^2)\).

**Proof.** Completeness. We show the following induction: whenever \(A\) is constructed properly, \(A \prec B\), and
Data: $\Psi$: a taxonomy
Result: $I(\Psi)$: the itemset taxonomy of $\Psi$

\[
E_0, O_0 \leftarrow \emptyset; \\
/\* Q contains the itemsets to handle. */ \\
Q \leftarrow \text{FIFO queue with only } \emptyset; \\
/\* D contains the handled itemsets. */ \\
D \leftarrow \text{an empty hash table}; \\
O \leftarrow \text{an empty set of itemset pairs}; \\
\text{while } Q \text{ is not empty do} \\
\quad A \leftarrow \text{pop}[Q]; \\
\quad \text{Add } A \text{ to } D; \\
\quad \text{for } i \in I \text{ do} \\
\quad \quad \text{if } i \notin O_A \text{ and parents}[i] \subseteq O_A \text{ then} \\
\quad \quad \quad \text{Create a new itemset } B; \\
\quad \quad \quad E_B \leftarrow (E_A - \text{parents}[i]) \cup \{i\}; \\
\quad \quad \quad O_B \leftarrow O_A \cup \{i\}; \\
\quad \quad \quad \text{Add } (A, B) \text{ to } O; \\
\quad \quad \quad \text{if } B \notin D \text{ then} \\
\quad \quad \quad \quad \text{Add } B \text{ to } Q; \\
\text{return } I(\Psi) = (D, O); \\
\]

Algorithm 2: Construct $I(\Psi)$

If $A \in D$, then $B$ is constructed properly, $B \in D$ and $(A, B) \in O$. The base case of $A = \emptyset$ is straightforward.

Consider the iteration at which $A$ is handled in the while loop. By Prop. A.2 there exists $i \in B$ s.t. $\downarrow B = \downarrow A \cup \{i\}$ and parents$[i] \subseteq \downarrow A$. Because $A$ was properly constructed, we have $i \notin O_A$ and parents$[i] \subseteq O_A$ and therefore we enter the then block. The construction of $O_B$ is clearly correct. As $E_A$ was correctly constructed, the only ancestors of $i$ in $E_A$ can be direct parents of $i$, which are removed. No descendants of $i$ can be in $E_A$ by definition. Thus, by removing the parent of $i$ we obtain the antichain representing the order ideal $\downarrow B$ i.e., the correct $E_B$. Finally, we add $(A, B)$ to $O$, and if $B$ is not yet in $D$ it is added to $Q$ so will be added to $D$ later.

We therefore conclude by induction that $\mathcal{AC}[\Psi] \subseteq D$ and $\prec \subseteq O$.

Correctness. From the above it is immediate that $D \subseteq \mathcal{AC}[\Psi]$. It remains to prove that we do not add incorrect pairs to $O$. Assume some pair of antichains $(A, B)$ was added by the algorithm to $O$. This means that their order ideals differ by a single item satisfying the conditions of Prop. A.2 but then $\prec$ must hold for the pair.

Complexity. The while loop traverses all the elements in $I(\Psi)$ - there are $|\mathcal{AC}[\Psi]|$ of them. For each element the inner for loop traverses $|I|$ items. Other actions within the loop – verifying that $i \notin O_A$ and all of $i$’s parents are in $O_A$, constructing $B$, etc. – take $O(|I|)$ actions. Thus, the total complexity is as stated.

\[\text{Algorithm 2 is fairly simple and we do not claim it is optimal. That said, for some taxonomies, e.g., Boolean lattices, } |I(\Psi)| = \Theta(|I| \cdot |\mathcal{AC}[\Psi]|). \text{ Hence, for such taxonomies our algorithm is at most within a multiplicative factor of } |I| \text{ from the construction lower complexity bound.} \]

Now, we turn to construct $I^{(k)}(\Psi)$. We cannot use a straightforward adaptation of Algorithm 2 for this purpose, since it relies on some assumptions that are not valid in for $k$-itemset taxonomies. For instance, if $A \preceq B$ in $I^{(k)}(\Psi)$, then $A$ and $B$ are not necessarily separated by a single item. However, since $I^{(k)}(\Psi)$ is small in size, we can propose the following straightforward algorithm:

**Proposition A.5.** Given a taxonomy $\Psi$, $I^{(k)}(\Psi)$ can be materialized in time $O(|I|^{2k+1})$.

**Proof.** If $k = 0$, construct a single itemset, $\emptyset$. Otherwise, find for every $i \in I$ its set of ancestors, lexicographically ordered. This can be done in $O(|I|^2)$. Generate all the subsets of $I$ of size up to $k$ in time $O(|I|^k)$. For each such itemset $A$, check whether it is an antichain by checking for every $i \in A$, whether one of the ancestors of $i$ are in $A$. This takes $O(|I|^2)$ operations per itemset. Keep only the antichains – these are the elements of $I^{(k)}(\Psi)$. For each antichain also compute its order ideal, by computing the union of ancestors of $i \in A$, in time $O(|I|^2)$. Finally, for each pair of antichains $A, B$ check in $O(|I|)$ whether $A \preceq B$ – if the order ideal of $B$ contains the one of $A$. The total complexity is $O(|I|^2 + |I|^k \cdot |I|^2 + |I|^{2k+1} \cdot |I|)$, which, for $k \geq 1$, is $O(|I|^{2k+1})$. □

Remark. The above algorithm for $k \geq 1$ computes the full relation $\leq$. If we are interested in $\prec$, we can perform the transitive reduction of the DAG represented by the pairs of $\leq$. In particular, transitive reduction can be computed by performing, for every node, a linear-time longest path search from this node, and keeping only the paths of length 1. The total complexity is the number of nodes (in our case $O(|I|^k)$) times the number of edges, and thus the total complexity is $O(|I|^{3k})$.

### B. Supplementary Proofs

#### B.1 Crowd Complexity

**Proof (Prop. 3.3)**. We prove the following more general result: for any poset $P$, there exists a bijective mapping $\varphi$ between $\mathcal{AC}[P]$ and the set of monotone predicates over $P$. Our original claim will follow from $P = I(\Psi)$. Consider the mapping $\varphi$ associating, to a monotone predicate $F$ over $P$, the set of its maximal elements. First, observe that $\varphi$ is clearly injective. Next, observe that the set of maximal elements is an antichain of $P$, because if two such elements are comparable, it contradicts the maximality of one of them. Thus, the range of $\varphi$ is actually $\mathcal{AC}[P]$. Conversely, from any antichain $A \in \mathcal{AC}[P]$ we can define a predicate $F$ which returns true for $e$ iff $\exists e' \in A$ $e \leq e'$. By...
this definition, \( F \) is monotone and has \( A \) as its set of maximal elements. Hence, \( \varphi \) is also surjective.

Proof (Prop. \ref{prop:3.6} direct). With every query of an itemset, either some set of solutions in \( S(\Psi) \) is marked as impossible, or its complement is. Thus, in the worst-case, a query eliminates at most half of the possible solutions. Hence, by induction, in order to eliminate all solutions except the correct one, every algorithm must pose a number of queries that is at least logarithmic in the solution space, i.e., \( \Omega(\log |\mathcal{AC}|[I(\Psi)]) \) or, equivalently, \( \Omega(\log |S(\Psi)|) \).

Proof (Prop. \ref{prop:3.9}). While we search for the right element in \( S(\Psi) \), we can mark the itemsets of \( I(\Psi) \) as frequent, infrequent and unclassified by using the result of our queries and the monotonicity property: an itemset is marked as frequent if it is an ancestor of an itemset that was queried and identified as frequent (remember that an itemset is an ancestor of itself), as infrequent if it is the descendant of an itemset identified as infrequent, and as unclassified in other cases. At any point of the algorithm, define \( U \subseteq I(\Psi) \) as the set of itemsets that are marked unclassified, and \( R \subseteq I(\Psi) \) as the set of itemsets marked frequent that have no descendants marked frequent. We claim that there is an order-preserving bijection from the antichains of \( U \) to the possible elements of \( S(\Psi) \). Consider the function \( \varphi \) associating, to an antichain \( A \) of \( U \), the subset \( \varphi(A) \) of \( I(\Psi) \) formed by \( A \cup B \), where \( B \) is the set of elements in \( R \) that have no descendant in \( A \). \( \varphi(A) \) is an antichain of \( I(\Psi) \) because \( A \) is an antichain, \( B \) is a subset of \( R \) which is an antichain, and any element of \( U \) is incomparable to any element of \( R \). \( \varphi(A) \) is possible in \( S(\Psi) \), because it is consistent with the observations. \( \varphi \) is injective because \( A \mapsto \varphi(A) \cap R \) is the identity. The range of \( \varphi \) is the possible elements of \( S(\Psi) \), because any possible element of \( S(\Psi) \) must be an antichain of \( R \cup U \) including a descendant of every itemset of \( R \). \( \varphi \) is order-preserving. Thus, we can characterize the possible elements of \( S(\Psi) \) using only \( U \), and in fact, they form a poset isomorphic to the itemset taxonomy \( I(U) \).

By \cite{24}, in any poset there exists an element \( e \) such that the fraction of order ideals that contain \( e \) is between \( \delta_0 \approx 0.17 \) and \( 1 - \delta_0 \). Thus, there exists an itemset \( A \in U \) such that the fraction of possible solutions in \( S(\Psi) \) that contain \( A \) or one of its descendants is between \( \delta_0 \) and \( 1 - \delta_0 \). Consequently, when querying \( A \) we are guaranteed to eliminate at least \( \delta_0 \) of the possible elements in \( S(\Psi) \).

We can define an algorithm which achieves this upper bound as follows: at each iteration, choose a "good split" element, e.g., by counting for each element in \( I(\Psi) \) the fraction of possible solutions in \( S(\Psi) \) which contain this element or its descendants. This algorithm terminates after \( O(\log_{1/(1-\delta_0)} |S(\Psi)|) = O(\log |S(\Psi)|) \) queries.

Proof (Prop. \ref{prop:3.10}). Assume w.l.o.g. that we have not queried some MII \( A \). (The argument is similar if \( A \) is an MII.) Because \( A \) is frequent, we have \( \text{freq}(A) = \text{true} \). Denote by \( \text{freq}' \) the predicate s.t. \( \text{freq}'(A) = \text{false} \) and \( \text{freq}'(B) = \text{freq}(B) \) for any \( B \neq A \). As we have not queried \( A \), we cannot distinguish between \( \text{freq} \) and \( \text{freq}' \); since all of \( A \)'s children are infrequent, and all of its ancestors are frequent, determining \( \text{freq}(A) \) can only be done by querying \( A \) directly. Since \( mfi \cap mii = \emptyset \), \( \text{freq} \) cannot be identified by less than \( |mfi| + |mii| \) queries.

B.2 Finding the Best-split Element

We give here the full details of the proof for Thm. \ref{thm:4.1}. We start by a few auxiliary results about the relationship between antichain counting and best-split identification, which will be needed to prove hardness.

Define the concatenation operator \( \circ \) on two posets \( P \) and \( Q \) as follows: \( P \circ Q \) is a poset, whose elements consist of a copy of \( P \) and a copy of \( Q \) that are disjoint (for simplicity, we abuse notation and call these copies \( P \) and \( Q \) ), plus a new element \( e \) that is neither in \( P \) nor in \( Q \). The order relation over \( P \circ Q \) is such that its restriction to \( P \times P \) and \( Q \times Q \) matches the original order on \( P \) and \( Q \), and such that \( p \leq e \) for every element \( p \) in \( P \) and \( e \leq q \) for every element \( q \) in \( Q \). (Note that this implies that the order is total on \( P \times Q \): for all \( p \in P \) and \( q \in Q \), we have \( p \leq q \) by transitivity). Equivalently, \( \circ \) can be defined as a series composition of \( P, e \) and \( Q \), which is a standard operator in order theory.\footnote{See, e.g., Möhring, R. H. Computationally tractable classes of ordered sets. Springer Netherlands, 1988.} The \( \circ \) operation is clearly associative.

We first define a few useful Lemmas, and then prove the main claim.

Lemma B.1. Given two posets \( P \) and \( Q \) with distinct elements, \(|\mathcal{AC}[P \circ Q]| = |\mathcal{AC}[P]| + |\mathcal{AC}[Q]|\).

Proof. Consider the antichains in \( P \circ Q \). First, it is easy to see that \( \mathcal{AC}[P] \cup \mathcal{AC}[Q] \cup \{\{e\}\} \subseteq \mathcal{AC}[P \circ Q] \). Now, every antichain that contains an element of \( P \) must contain only elements from \( P \), since elements of \( Q \cup \{e\} \) are comparable to every element in \( P \); the same applies to antichains containing an element of \( Q \). As \( e \) is comparable to any other element, the only antichain containing \( e \) is the singleton \( \{e\} \). Thus \( \mathcal{AC}[P] \cup \mathcal{AC}[Q] \cup \{\{e\}\} = \mathcal{AC}[P \circ Q] \). The union on the left hand side is a disjoint union except for the empty antichain that is common to \( \mathcal{AC}[P] \) and \( \mathcal{AC}[Q] \); the empty itemset; thus \( |\mathcal{AC}[P] \cup \mathcal{AC}[Q] \cup \{\{e\}\}| = |\mathcal{AC}[P]| + |\mathcal{AC}[Q]| - 1 + 1 = |\mathcal{AC}[P \circ Q]| \).

Lemma B.2. There exists a family \( (\Gamma_n) \) of posets such that for every natural number \( n \), \(|\Gamma_n| = O(\log^2 n) \) and \(|\mathcal{AC}[\Gamma_n]| = n \).
(a) Original input $P$

(b) $Ψ = Γ_{12} ⊕ P_9 ⊕ P_3 ⊕ P ⊕ P$

(c) $I^{(1)}(Ψ)$

(d) $S^{(1)}(Ψ)$

Figure 2: Example posets for the proof of Lemma B.3
Proof. For \( n \) s.t. \( n = 2^m \) for some natural \( m \), take \( \Gamma_n \) to be the flat poset with \( m \) elements: the antichains of \( \Gamma_n \) are exactly its subsets, so \( |\mathcal{AC}[\Gamma_n]| = 2^m = n \), and we have \( |\Gamma_n| = m = \log_2 n \). Now, any natural number can be expressed in the binary form \( n = n_1 + n_2 + \cdots + n_p \) where \( n_i = 2^m_i \), \( 0 \leq n_1 < n_2 < \cdots < n_p \) and \( p = O(\log n) \). Then define \( \Gamma_n = \Gamma_{n_1} \circ \cdots \circ \Gamma_{n_p} \).

By Lemma B.1, \( \Gamma_n \) has \( |\mathcal{AC}[\Gamma_{n_1}]| + \cdots + |\mathcal{AC}[\Gamma_{n_p}]| = n_1 + \cdots + n_p = n \) antichains. The number of elements in \( \Gamma_n \) is \( \log n_1 + \cdots + \log n_p = m_1 + \cdots + m_p = O(m^2) = O(\log^2 n) \), and there are at most two edges per each \( \Gamma_n \) element. Thus, the total size of \( \Gamma_n \) is \( O(\log^2 n) \).

Using the construction of \( \Gamma_n \), we can show that identifying the best split gives us some information about the number of antichains in a given poset.

Lemma B.3. Assume there exists an algorithm \( A \) to identify the best-split element in \( I^{(k)}(\Psi) \) in \( O(\text{poly}(|\Psi|)) \). Then, for any given poset \( P \) and \( n \leq 2^{|P|} \), \( A \) can decide in \( O(\text{poly}(|P|)) \) whether \( |\mathcal{AC}[P]| \) is less than, greater than or equal to \( n \).

Proof. We prove the lemma by constructing a poset with two parts: one corresponding to \( \Gamma_n \), and one corresponding to \( P \circ P \). Intuitively, finding the best split element allows us to compare the number of antichains in \( \Gamma_n \) (namely, \( 2n \)) and the number of antichains of \( P \circ P \), thus comparing \( n \) and \( |\mathcal{AC}[P]| \). We use \( 2n \) and \( P \circ P \) instead of \( n \) and \( P \) to ensure that the number of antichains is even, so there is only one best-split element, which simplifies the analysis.

Formally, define \( \Psi \) to be \( \Gamma_{2n} \circ P_0 \circ P_0 \circ P_0 \circ P \circ P \), where \( P_0 \) is an empty poset. Let \( e_{-1}, e_0, e_1, e_2 \), and \( e_3 \) be the “\( e \)" elements created by the successive concatenations. Since the size of \( \Gamma_{2n} \) is \( O(\log^2(n2n)) \) which is itself \( O(|P|^2) \), \( \Psi \) can be computed in polynomial time.

Recall that the only difference between the structures of \( \Psi \) and \( I^{(1)}(\Psi) \) is the additional root element, representing the empty itemset. By Lemma B.1, the total number of antichains of \( \Psi \) is \( 2n + 3 + 2 |\mathcal{AC}[P]| \), and the number of antichains of \( I^{(1)}(\Psi) \) is \( 1 + 2n + 3 + 2 |\mathcal{AC}[P]| \) (due to the addition of the root). Out of them, exactly \( 2 \cdot |\mathcal{AC}[P]| + 2 \) are supersets of \( \{e_0\} \): \( \{e_0\}, \{e_1\}, \{e_2\} \) and the antichains of \( P \circ P \), excluding the empty antichain. There are exactly \( 2n + 2 \) other antichains — the empty antichain, the antichain of the empty itemset \( \{\emptyset\} \), the antichains of itemsets from \( \Gamma_{2n} \), and \( \{e_{-1}\} \).

Now, apply \( A \) on \( \Psi \) (and \( k = 1 \)) to obtain the best-split element \( A \) in \( I^{(1)}(\Psi) \). If \( |\mathcal{AC}[P]| = n \), then the best-split element is \( \{e_0\} \), because the number of antichains containing its descendants \( 2 |\mathcal{AC}[P]| + 2 \) is exactly the number of the rest of the antichains \( 2n + 2 \), and there are no other best-split elements, since every ancestor of \( \{e_0\} \) has at least one more antichain containing it or its descendants \( \{\{e_{-1}\}\} \), every descendant of \( \{e_0\} \) has one less antichain containing it or its descendants \( \{\{e_0\}\} \), and every element is comparable to \( e_0 \) so it is either an ancestor or a descendant. This is where we use the fact that the number of antichains above and below \( \{e_0\} \) is even, since otherwise we might have gotten more than one best-split element.

In a similar manner, we can show that when \( |\mathcal{AC}[P]| = n + 1 \), \( \{e_1\} \) is the only best-split element, and when \( |\mathcal{AC}[P]| = n - 1 \), \( \{e_{-1}\} \) is the only best-split element. In general, when \( |\mathcal{AC}[P]| \) is larger than \( n \) (resp., smaller than \( n \)), the best-split will be an descendant of \( \{e_1\} \) (resp., an ancestor of \( \{e_{-1}\} \)). We thus decide as follows: if \( A = \{e_0\}, |\mathcal{AC}[P]| = n \); if \( A = \{e_1\} \) or one of its descendants, \( |\mathcal{AC}[P]| > n \), and otherwise \( |\mathcal{AC}[P]| < n \).

In order to visualize the structures used in the proof, consider the following example.

Example B.4. Assume that we are interested in finding the number of antichains in the poset \( P \) depicted in Figure 2a. Since \( P \) has 4 elements, the number of antichains is at most \( 2^4 = 16 \). Assume that we are currently trying to compare it to \( n = 6 \). Based on the proof above, we define \( \Psi = \Gamma_4 \circ P_0 \circ P_0 \circ P_0 \circ P \circ P \). The resulting poset is depicted in Figure 2b. \( I^{(1)}(\Psi) \), illustrated in Figure 2c is similar to \( \Psi \), but has an additional root. Now, consider \( S(I^{(1)}(\Psi)) \), illustrated in Figure 2d. In this small example, we can count the number of descendants and non-descendants of \( \{e_0\} \). Both turn out to be \( 14 = 2n + 2 \). In this case the best-split element would be \( \{e_0\} \), and we can determine that the number of antichains in \( P \) is 6. This is true: the antichains are \( \{1\} \{2\}, \{3\}, \{2, 3\} \) and \( \{4\} \).

Finally, we prove Thm. 4.1, namely that the problem of identifying the best-split element in \( I^{(k)}(\Psi) \) is \( \#P \)-complete w.r.t. \( \Psi \).

Proof (Thm. 4.1). Membership. Given a taxonomy \( \Psi \) and a number \( k \), we can construct \( I^{(k)}(\Psi) \) in polynomial time using Algorithm 2. Assume that we have an oracle that given a poset returns the number of antichains in it. This problem is known to be \( \#P \)-complete [33]. First, use the oracle to count the number of antichains in \( I^{(k)}(\Psi) \), which is equal to the number of possible solutions (sets of MFIs) for this instance of MineFreq. Then, for each element \( A \) in \( I^{(k)}(\Psi) \), generate a copy \( \Phi \) of \( I^{(k)}(\Psi) \) which excludes \( A \) and its descendants. The construction of \( \Phi \) is naturally in PTIME.

Use the oracle to count the antichains in the resulting poset. These are exactly the solutions that are possible if \( A \) is infrequent. \( \min_{\mathcal{AC}[\Phi]} |\mathcal{AC}[\Phi] - \mathcal{AC}[I^{(k)}(\Psi)]| \) gives the number of solutions that are guaranteed to be eliminated if \( A \) is queried. The element \( A \) that maximizes this formula is the best-split element, and for a constant \( k \), there is a polynomial number of elements in \( I^{(k)}(\Psi) \), so the best-split element is identified after a polynomial number of invocations of the oracle.

Hardness. Because antichain counting is an \( FP_{\#P} \)-complete problem, to show \( FP_{\#P} \)-hardness of finding
the best split it suffices to show a PTIME reduction from best-split computation to antichain counting. Let \( \mathcal{A} \) be an oracle taking as input \( \Psi \) and returning the best-split element in \( I(\Psi) \). Given a poset \( P \), we show how to determine its number of antichains in PTIME using oracle \( \mathcal{A} \). We perform a binary search for the number of antichains of \( P \). Our initial range of values for \(|\mathcal{A}[P]|\) is \( 1 \ldots 2^{|P|} \) because \( \emptyset \) is an antichain and the number of antichains in clearly bounded by the number of subsets of \( P \). We can use \( \mathcal{A} \) and the method in Lemma B.3 to search within this range. In the worst case, the binary search tests \(|P|\) possible values for \(|\mathcal{A}[P]|\), and performs each test in polynomial time with one invocation of \( \mathcal{A} \), so we have the desired PTIME reduction. \( \Box \)

B.3 Chain Partitioning

We next describe a simple algorithm partition, which computes a partition of an itemset taxonomy \( I(\Psi) \) into disjoint chains.

Start by materializing \( I(\Psi) \), which can be done in time \( O(|\mathcal{A}[\Psi]| \cdot |\mathcal{I}|^2) \) by Prop. A.4. Based on Dilworth’s theorem, chain partitioning can be computed by solving a maximum matching (or maximal join) in a bipartite graph \([14]\). The construction involves creating one edge for each pair of itemsets \( A \leq B \); this requires computing the full (transitive) \( \leq \) relation in time \( O(|\mathcal{A}[\Psi]|^2) \). Finding the matching can be done, e.g., in \( O(|I(\Psi)| \sqrt{|\mathcal{A}[\Psi]|}) \), using the Hopcroft-Karp algorithm\([5]\). By summing the complexity of all the steps and simplifying the result, we obtain the algorithm complexity \( O(|\mathcal{A}[\Psi]| \cdot |\mathcal{I}|^2 + |\mathcal{A}[\Psi]|^2 + |I(\Psi)| \sqrt{|\mathcal{A}[\Psi]|}) \). It may be possible to further simplify this expression, depending on the structure of \( \Psi \).

B.4 Greedy Algorithms

Proof (Thm. 6.2). We want to prove that finding the greedy best-split itemset is \( \text{FP}^\#P \)-hard w.r.t. \(|\Psi|\).

Observation 1. The structure of \( S^{(1)}(\Psi) = I(\mathcal{I}^{(1)}(\Psi)) \) is almost identical to that of \( I(\Psi) \), up to an additional root element. This follows from the fact that the structures of \( \Psi \) and \( I(\mathcal{I}^{(1)}(\Psi)) \) are almost identical.

Observation 2. If the itemset \( A \) is frequent, then all the solutions that do not contain any \( B \in \text{desc}(A) \) are impossible. Similarly, if \( A \) is infrequent, all the solutions that contain some \( B \in \text{desc}(A) \) are impossible. Consequently, querying \( A \) eliminates half of the solution space iff \( \{A\} \) has the same number of descendants and non-descendants in the solution taxonomy.

Using these observations, we show a PTIME reduction from finding the greedy best-split element to counting antichains. Let \( \mathcal{A} \) be an oracle taking a taxonomy \( \Psi \) as input and returning the greedy best-split of \( I(\Psi) \). Given a poset \( P \), we show how to determine its number of antichains in PTIME using oracle \( \mathcal{A} \). As in the proof of Thm. \([4]\), define \( \Psi = \Gamma_{2n} \circ P_0 \circ P_0 \circ P \circ P_0 \circ P \), where \( P \) is the input to the antichain counting problem. Define \( \Psi' = I^{(1)}(\Psi) \). It is enough to show that by finding the greedy best-split element in \( I(\Psi') = S^{(1)}(\Psi) \), we can decide whether the best-split element in \( I^{(1)}(\Psi) \) is \( \{e_0\} \), one of its ancestors or one of its descendants. We have three cases:

1. The number of descendants and non-descendants of \( \{e_0\} \) in \( S^{(1)}(\Psi) \) is identical. This allows us to deduce two things. First, by observation 2, \( \{e_0\} \) is the best-split element of \( \Psi' = I^{(1)}(\Psi) \). Second, \( \{e_0\} \) is the greedy best-split element of \( I(\Psi') \), because the non-descendants of \( \{e_0\} \) are exactly its ancestors since \( \{e_0\} \) is comparable to all other elements of \( S^{(1)}(\Psi) \).

2. \( \{e_0\} \) has more descendants than non-descendants in \( I(\Psi') \). In this case, by a similar reasoning we can show that the greedy best-split element of \( I(\Psi') \) must be \( \{e_1\} \) or one of its descendants, and using observation 2 we can show that the best-split element of \( I^{(1)}(\Psi) \) is one of \( \{e_1\} \)’s descendants.

3. \( \{e_0\} \) has less descendants than non-descendants. Similarly to the previous case, we can show that the greedy best-split element of \( I(\Psi') \) must be \( \{e_{-1}\} \) or one of its ancestors and that the best-split element of \( I^{(1)}(\Psi) \) is \( \{e_{-1}\} \) or one of its ancestors. So, by applying \( \mathcal{A} \) to \( \Psi' \) we can determine if \( \{e_0\} \) in \( I^{(1)}(\Psi) \) is the best-split element. We can therefore count the number of antichains in \( P \) in a similar manner to the proof of Thm. \([4]\) \( \Box \)

\[\text{Hopcroft, John E., and Richard M. Karp. (1973). “An} \ n^{5/2} \text{algorithm for maximum matchings in bipartite graphs.” SIAM Journal on computing 2(4): 225–231.}\]