Sparse Distributed Learning via Heterogeneous Diffusion Adaptive Networks

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Abstract—In-network distributed estimation of sparse parameter vectors via diffusion LMS strategies has been studied and investigated in recent years. In all the existing works, some convex regularization approach has been used at each node of the network in order to achieve an overall network performance superior to that of the simple diffusion LMS, albeit at the cost of increased computational overhead. In this paper, we provide analytical as well as experimental results which show that the convex regularization can be selectively applied only to some chosen nodes keeping rest of the nodes sparsity agnostic, while still enjoying the same optimum behavior as can be realized by incorporating unregularized learning at a subset of nodes, chosen nodes keeping rest of the nodes sparsity agnostic, while convex regularization can be selectively applied only to some analytical as well as experimental results which show that the incorporation of unregularized learning at a subset of nodes, less computational cost is needed in the proposed approach. We also provide a guideline for selection of the sparsity aware nodes and a closed form expression for the optimum regularization parameter.

Index terms—Adaptive network, diffusion LMS, Sparse systems, excess mean square error, adaptive filter, $l_1$ norm.

I. INTRODUCTION

Diffusion strategies [1]-[3] were first invented to solve distributed estimation problems in real-time environments where data are continuously streamed. Here, all nodes employ adaptive filter algorithms to process the streaming data, and simultaneously share their instantaneous estimates with their neighbors. These approaches are also very useful to model many self-organizing systems [4].

Recently, in [5]-[6], diffusion LMS schemes have been used to estimate sparse vectors, or equivalently, to identify FIR systems that have most of the impulse response coefficients either zero or negligibly small. In these papers, certain sparsity promoting norms of the filter coefficient vectors have been used to regularize the standard LMS cost function, prominent amongst them being the $l_1$ norm of the coefficient vector that leads to the sparsity aware, zero attracting LMS (ZA-LMS) [7]-[11] form of weight adaptation. These diffusion sparse LMS algorithms manifest superior performance in terms of lesser steady state network mean square deviation (NMSD) compared with the simple diffusion LMS.

In this paper, we show that the minimum level of the steady state NMSD achieved using ZA-LMS based update at all the nodes of the network can also be obtained by a heterogeneous network with only a fraction of the nodes using the ZA-LMS update rule (referred as sparsity aware nodes in this paper) while the rest employing the standard LMS update (referred as sparsity agnostic nodes in this paper), provided the nodes using the ZA-LMS are distributed over the network maintaining some “uniformity”. Note that reduction in the number of sparsity aware nodes reduces the overall computational burden of the network, especially when more complicated sparsity aware algorithms involving significant amount of computation are deployed to exploit sparsity. As shown in this paper, the only adjustment to be made to achieve the above reduction in the number of sparsity aware nodes is a proportional increase in the value of the optimum zero attracting coefficient. Analytical expressions explaining the above behavior are provided and the claims made are validated via detailed simulation studies. Finally, the proposed analysis, though restricted to the $l_1$-norm regularized algorithm (i.e., ZA-LMS) only, can be trivially extended to the case of more general norms and thus similar behavior can also be expected from the corresponding heterogeneous networks.

II. BRIEF REVIEW OF DIFFUSION SPARSE LMS ALGORITHMS

We consider a connected network consisting of $N$ nodes that are spatially distributed. At every time index $n$, each $k^{th}$ node collects some scalar measurement $d_k(n)$ and some $M \times 1$ vector $u_k(n)$ which are related by the following model:

$$d_k(n) = u_k^T(n)v_0 + v_k(n),$$

where $v_k(n)$ is the measurement noise at the $k^{th}$ node and $w_0$ is the unknown $M \times 1$ vector, known a priori to be sparse, which is required to be estimated. Both $u_k(n)$ and $v_k(n)$ are variates generated from some Gaussian distributions, with $u_k(n)$ and $v_k(m)$ being mutually independent for all $n, m$.

In the diffusion scheme, every $k$-th node, $k = 1, 2, \ldots, N$ deploys a $M \times 1$ adaptive filter $w_k(n)$ to estimate $w_0$, which takes $d_k(n)$ and $u_k(n)$ respectively as the local desired response and input vectors. The estimates of $w_0$, i.e., $w_k(n)$ for each $k$ are exchanged with the neighbors of the $k$-th node, i.e., nodes directly connected to it, and are used to refine the estimates in one of the two following manners: (A) Adapt-then-Combine (ATC) where $w_k(n)$ is first updated
to an intermediate estimate \(v_k(n+1)\), which is then linearly combined with similar estimates received from the neighbors, and (B) Combine-then-Adapt (CTA) where \(w_k(n)\) is first linearly combined with similar estimates received from the neighbors and then updated. Originally, the diffusion schemes were proposed assuming LMS form of weight adaptation at each node [2]-[3]. In the context of sparse estimation, certain sparsity exploiting norms of \(w_k(n)\) were added to the corresponding LMS cost function [5]-[6], the most popular of them being the \(l_1\) norm penalty \(||w_k(n)||_1\) which results in the introduction of the zero attracting terms \(sgn[w_k(n)]\) in the LMS update equations [7]-[9]. The resulting diffusion ZA-LMS algorithm for the ATC scheme, popularly termed as ZA-ATC diffusion algorithm [6], is shown in Table I and is considered by us in this paper.

The parameter \(\rho\) in Table I is the zero-attracting coefficient which is a very very small, positive constant taken same for all the nodes and \(\mathcal{N}_k\) denotes the set of nodes in the neighborhood of the node \(k\) (including itself).

### TABLE I
THE ZA-ATC DIFFUSION ALGORITHM [6]

| \(e_k(n)\) | \(d_k(n) - w_k^T(n)u_k(n)\) |
| \(v_k(n+1)\) | \(w_k(n) + \mu_k u_k(n)e_k(n)\) |
| \(w_k(n+1)\) | \(\sum_{j \in \mathcal{N}_k} c_{i,j}w_j(n+1)\) |

The combining coefficients \(c_{i,j}\) are non-negative constants which are usually chosen satisfying the following [1]:

\[
c_{i,j} > 0 \quad \text{if} \quad i \in \mathcal{N}_k
\]

\[
\text{and} \quad \sum_{i \in \mathcal{N}_k} c_{i,j} = 1.
\]

There exist several standard schemes in the literature to choose the coefficients \(c_{i,j}\), e.g., the uniform combination rule, the metropolis rule, the Laplacian rule and the nearest neighbor rule to name a few. Using these coefficients, a combination matrix \(C'\) is defined for the network, where \([C']_{i,j} = c_{i,j}\).

### III. PROPOSED HETEROGENEOUS NETWORK AND ITS NMSD BEHAVIOR

Before presenting the proposed heterogeneous network and its par behavior with the ZA-ATC based diffusion network of [6], it will be useful to consider some of the major results of [6] here. For this, we first define the average network mean-square deviation at the \(n^{th}\) time index as,

\[
MSD_{\text{net}}(n) = \frac{1}{N} \sum_{k=1}^{N} MSD_k(n),
\]

where \(MSD_k(n)\) is the individual mean-square deviation of the \(k^{th}\) node at the \(n^{th}\) time index, i.e.,

\[
MSD_k(n) = E[||w_0 - w_k(n)||^2] = E[||\tilde{w}_k(n)||^2]
\]

where \(\tilde{w}_k(n) = w_0 - w_k(n)\) is the weight deviation vector for the \(k^{th}\) node at \(n^{th}\) index.

The expression for steady-state \(MSD_{\text{net}}(\infty)\) (i.e., \(MSD_{\text{net}}(\infty)\)) of the ZA-ATC algorithm was derived analytically in [6]. However, [6] considered a more general form of diffusion, in which both \(d_k(n)\) and \(u_k(n)\) are also exchanged with the neighbors along with the local estimates \(w_l(n)\). In contrast, in this paper, we consider exchange of only \(w_l(n)\) which is also the most common form of diffusion. Additionally, we introduce a few more simplifications in [6]. Firstly, we assume same step-size \(\mu\) for all nodes. Next, both the input signal and noise at each node are assumed to be spatially and temporally i.i.d. Under these, it is easy to check that the \(MSD_{\text{net}}(\infty)\) expression for the ZA-ATC algorithm [6] simplifies to the following:

\[
MSD_{\text{net}}(\infty) = \frac{\mu^2\sigma_2^2\sigma_2^2}{N} [vec(C^T C)]^T(I - F)^{-1} q + \frac{1}{N} (\beta(\infty) - \alpha(\infty)),
\]

where

\[
\alpha(\infty) = -2\mu E[sgn[\tilde{w}(\infty)]^T \Omega C C^T(I - \mu D)\tilde{w}(\infty)],
\]

and

\[
\beta(\infty) = \mu^2 E[||sgn[\tilde{w}(\infty)||^2_{\Omega C C^T}] > 0.
\]

The function \(\phi(\rho)\) has two zero-crossing points, one at \(\rho = 0\) and the other at \(\rho = \frac{\alpha'(\infty)}{2\beta'(\infty)}\), and between them, \(\phi(\rho)\) takes only negative values with the minima occurring at \(\rho = \frac{\alpha'(\infty)}{2\beta'(\infty)}\), which, from (6), also minimizes \(MSD_{\text{net}}(\infty)\). For systems

1The networks presented in [6] and also in [5] are “homogeneous” in the sense that these networks deploy only sparsity aware nodes.
that are highly sparse, it follows from [7] that \( \alpha'(\infty) > 0 \)
and conversely, for non-sparse systems, \( \alpha'(\infty) < 0 \). Since for
proper zero attraction, \( \rho \) must be positive, the optimum value
of \( \rho \) is then given by

\[
\rho_{opt} = \max\left[0, \frac{\alpha'(\infty)}{2\beta'(\infty)}\right].
\]

(9)

The corresponding minimum value of \( \phi \) (when \( \rho_{opt} > 0 \)) is
then given as

\[
\phi_{min} = -\frac{\alpha'^2(\infty)}{4N\beta'(\infty)}.
\]

(10)

**The Proposed Heterogeneous Diffusion Network** :

In this section, we show that the same level of
\( \phi_{min} \) can be reached by a heterogeneous network as well, where
only a fraction of the nodes are sparsity aware and rest are
sparsity agnostic, provided the network is designed satisfying
the assumptions I.A and I.B as given in the box below where
\( S \) denotes the set of indices of the sparsity aware nodes and :

**Assumption I**

I.A

We assume that the matrix \( C' \) is doubly stochastic, i.e., \( \forall i, j, \sum_{i=1}^{N} c'_{i,j} = 1 \) and \( \sum_{j=1}^{N} c'_{i,j} = 1 \). This is
valid for many practical rules used to select combiner coefficients.[1]

I.B

We also assume that the sparsity-aware nodes are distributed over the network in such a way that \( \forall j, \sum_{i \in S} c'_{i,j} = \frac{N}{N} \).

The physical interpretation of this assumption is
that it ensures a uniform influence of the sparsity
aware nodes on each node of the network. This can
be employed as a design criterion.

In order to show the above, we replace the
matrix \( \Omega \) by a new one defined as \( \Omega_s = \text{diag}[\rho_1 I_{M \times M}, \rho_2 I_{M \times M}, \ldots, \rho_N I_{M \times M}] \), where
\( \rho_k = \rho, \text{ if } k \in S, \text{ else } \rho_k = 0 \).

Using this and the fact that \( I - \mu D = (1 - \mu \sigma_n^2)I, \alpha(\infty) \)
and \( \beta(\infty) \) modify to \( \alpha_1(\infty) \) and \( \beta_1(\infty) \), given as follows :

\[
\alpha_1(\infty) = -2\mu(1 - \mu \sigma_n^2)E[\text{sgn}(\mathbf{w}(\infty))]^T \Omega_s \mathbf{C}^T \mathbf{w}(\infty)]
\]

(11)

and

\[
\beta_1(\infty) = \mu^2 E[\text{sgn}(\mathbf{w}(\infty))]^T \Omega_s \mathbf{C}^T \Omega_s \text{sgn}(\mathbf{w}(\infty)]
\]

(12)

Note that unlike \( \alpha(\infty) \) and \( \beta(\infty) \), it is lot more difficult to
express \( \alpha_1(\infty) \) and \( \beta_1(\infty) \) as a function of \( \rho \), since unlike
\( \Omega, \Omega_s \) can not be written simply as \( \rho I \). Instead, one needs
to analyze the RHS of (11) and (12) to express \( \alpha_1(\infty) \) and
\( \beta_1(\infty) \) in terms of \( \rho \). Towards this, we make the following assumptions :

**Assumption II**

II.A

\[
E[\text{sgn}(\mathbf{w}_i(\infty) | \mathbf{w}_m^T(\infty))] \approx \theta, (\forall i, m) \text{ is a matrix independent of } i \text{ and } m, \text{ when } m \in \mathbb{N}_j, \forall j \in \mathbb{N}_i
\]

II.B

\[
E[\text{sgn}(\mathbf{w}_i(\infty) | \text{sgn}(\mathbf{w}_m^T(\infty)))] \approx \psi, (\forall i, m) \text{ is a matrix independent of } i \text{ and } m, \text{ when } m \in \mathbb{N}_j, \forall j \in \mathbb{N}_i.
\]

In words, the above assumptions tell that, at the
steady-state, any pair of nodes having overlapping
neighborhood (including directly connected nodes, and
the same node) show approximately same
cross-covariance and similar cross-moments. This
is motivated by the fact that all nodes have same
step-size, same input and noise statistics, and the
above mentioned pairs continuously exchange their
intermediate estimates using diffusion strategy.

It is then possible to prove the following :

**Theorem 1:** For a network satisfying the
Assumptions I.A and II.A as given above, we have,

\[
\alpha_1(\infty) = -2\mu(1 - \mu \sigma_n^2)T \mathbf{r} | \mathbf{\theta}_s N_s.
\]

(13)

Proof: Skipped due to page limitation.

**Theorem 2:** For a network satisfying the
Assumptions I.A, I.B and II.B as given above, we have,

\[
\beta_1(\infty) = \frac{\mu^2 \rho^2 T \mathbf{r} | \mathbf{\psi} N_s^2}{N}.
\]

(14)

Proof: Skipped due to page limitation.

Substituting \( \alpha_1(\infty) \) and \( \beta_1(\infty) \) in \( \phi(\rho) = \frac{1}{\rho}(\beta_1(\infty) - \alpha_1(\infty)) \), then differentiating w.r.t. \( \rho \) and equating the derivative
to zero, we obtain,

\[
\rho_{opt} = \max\left[0, \frac{(1 - \mu \sigma_n^2)T \mathbf{r} | \mathbf{\theta}_s N_s}{\mu T \mathbf{r} | \mathbf{\psi} N_s}\right].
\]

(15)

The corresponding minimum value of \( \phi(\rho) \) [when \( \rho_{opt} > 0 \),
\text{i.e.}, the system is sparse], say, \( \phi'_{min} \) is given as

\[
\phi'_{min} = -\frac{(1 - \mu \sigma_n^2)^2 T \mathbf{r} | \mathbf{\theta}_s^2}{T \mathbf{r} | \mathbf{\psi}^2}.
\]

(16)

Note that \( \phi'_{min} \) as given in (16) is independent of \( N_s \).
Therefore, its value remains same when \( N_s = N \), \text{i.e.}, when
the network becomes homogeneous with all nodes being sparsity aware. This also implies that if $\phi_{\text{min}}$ as given by (10) is analyzed using the assumptions I and II, it would give rise to the same expression as that of $\phi'_{\text{min}}$ (i.e., (16)). From this and (15), we then make the following two conclusions:

- The $\min[MSD_{\text{net}}(\infty)]$ does not change when the network changes from being homogeneous to heterogeneous, with only $N_s$ of the total $N$ (0 $\leq N_s \leq N$) nodes employing sparsity-aware adaptation.
- For sparse systems, the $\rho_{\text{opt}}$ minimizing $\phi(\rho)$ and thus $[MSD_{\text{net}}(\infty)]$ (i.e., $-(1-\mu^2)\psi(\theta|N_s)$ as given in (15)) is inversely proportional to $N_s$, meaning that while maintaining the same $\min[MSD_{\text{net}}(\infty)]$, one can reduce the number of sparsity-aware nodes by introducing proportional increase in the value of $\rho$.

IV. SIMULATION STUDIES

To test the performance of the heterogeneous networks, we use a strongly connected network of $N = 30$ nodes placed randomly in a geographic region. The weights of the edges are determined by the uniform combination rule [1]. The goal of the network is to estimate a 128×1 vector $\mathbf{w}_0$ which is highly sparse (only one coefficient being non-zero). We choose the same step-size $\mu = 6 \times 10^{-3}$ for all the nodes. Among these 30 nodes, $N_s$ number of nodes use the ZA-LMS and rest of the nodes use simple LMS update, with the former spaced 'uniformly' (i.e., satisfying assumptions I.A and I.B) over the network. The input signals and noise variables are drawn from Gaussian distributions, and they are temporally and spatially independent. Also, the input and noise statistics are same for all the nodes, with $\sigma^2_u = 1$, and $\sigma^2_v = 1 \times 10^{-4}$. To start with, the value of $\rho$ is kept fixed at $2 \times 10^{-6}$ for all the $N_s$ sparsity-aware nodes. The simulation is then carried out for 3000 iterations and the network steady state MSD is evaluated by taking ensemble average over 1000 independent runs. This is done for different values of $N_s$ (ranging from 0 to 30) and based on this, the network steady state MSD is plotted as a function of $N_s$. The value of $\rho$ is then increased progressively to take the following five values: $4 \times 10^{-6}, 6 \times 10^{-6}, 1 \times 10^{-5}, 2 \times 10^{-5}, 4 \times 10^{-5}$, one at a time for all the ZA-LMS based nodes. Fig. 1 displays the network steady state MSD vs. $N_s$ plots with $\rho$ as a parameter. It is easily seen from Fig. 1 that (i) the minima reached by each MSD-vs-$N_s$ plot is same for all the plots, and (ii) as $\rho$ increases, the value of $N_s$ where the minima occurs reduces and vice versa. In other words, Fig. 1 validates the theoretical conjectures made in the previous section.

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