SIGNING TROPICAL CONVEXITY

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ABSTRACT. We establish a new notion of tropical convexity for signed tropical numbers. We provide several equivalent descriptions involving balance relations and intersections of open halfspaces as well as the image of a union of polytopes over Puiseux series and hyperoperations. Along the way, we deduce a new Farkas lemma and Fourier-Motzkin elimination without the non-negativity restriction on the variables. This leads to a Minkowski-Weyl theorem for polytopes over the signed tropical numbers.

1. INTRODUCTION

Tropical convexity is an important notion with applications in several branches of mathematics. It arises from the usual definition of convexity by replacing + with max and · with +. This notion has been studied for several years involving different approaches from extremal algebra [31], idempotent semirings [15], max-algebra [13], convex analysis [12], discrete geometry [17], matroid theory [19]. So far, it was mainly studied in \( T_{\max} = \mathbb{R} \cup \{ -\infty \} \). Indeed, this is essentially a restriction to the tropical non-negative orthant, as \( r \geq -\infty \) for all \( r \in T_{\max} \), where \(-\infty\) is the tropical zero element. We remedy this restriction by introducing a notion of tropical convexity involving all orthants. We give our main points of motivation for our generalization.

Mean payoff games are equivalent to feasibility of a tropical linear inequalities

\[
\bigoplus_{j \in J} a_j + x_j \geq \bigoplus_{j \in [k] \setminus J} a_j + x_j
\]

where \( a, x \in T_{\max}^k \), see [3]. This problem is in \( \text{NP} \cap \text{co-NP} \) but no polynomial-time algorithm is known [23]. Furthermore, the latter feasibility problem is intimately related to the feasibility problem for classical linear inequality systems [28, 6]. The tropical linear feasibility problem is also a special scheduling problem [26] and it can be considered as a particular disjunctive programming problem [10].

Several polynomial-time algorithms for linear programming are naturally formulated as deciding if the origin is in the convex hull of a set of points, see, e.g., [14]. Our convexity notion provides an analogous formulation for the tropical linear feasibility problem in terms of the signed convex hull of the coefficient vectors, see Corollary 4.2 and Theorem 4.6. This may allow for new algorithmic approaches for mean payoff games.

Furthermore, separation theorems like Farkas’ lemma for linear programming have their easiest formulation in terms of separation from the origin leading to powerful generalizations, see [9]. Our approach allows to formulate an analogous

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theory for tropical linear programming. This gives new possibilities for studying tropical normal fans and tropical hyperplane arrangements.

Additionally, this notion is a natural next step following recent developments in tropical geometry. The concept of signed tropical numbers, a way to model inverse elements for the max-operation, was effectively used in the tropicalization of the simplex method [6]. The study of real tropicalization of semialgebraic sets [24] follows a similar spirit. Another approach to extend from $T_{\text{max}}$ involving signs is to deduce the structure of a variety by ‘unfolding’ it from the positive orthant into the other orthants, which is formalized by the patchworking introduced in [29] that has several applications in algebraic geometry.

1.1. Results. For studying signed tropical convexity over $T_{\pm}$, we need to introduce several relations ‘$\simeq$’, ‘$\triangleright$’, ‘$\geq$’, ‘$>$’. They are essentially just less powerful concepts of equality and inequality, lacking transitivity or compatibility with the operations. Certain properties of the signed tropical convex hull are surprising and, compared to usual convexity, harder to deal with. We provide several different ways to obtain our notion of signed tropical convexity.

While our definition (Definition 3.1) just arises from the usual definition of tropical convexity by replacing equality ‘$=$’ with the balance relation ‘$\simeq$’ (Section 2.3.2), writing convex combinations is a bit more subtle. To deal with sums of positive and negative tropical numbers, which do not just cancel as usual, we resolve the arising balanced numbers by the interval of signed numbers, which have absolute value smaller or equal to the result (Equation (14)). This is essentially the same as using a hyperoperation (Section 3.3). It turns out (Theorem 3.14) that this construction just yields the tropicalization of the union of all possible lifts. Using the machinery developed in Section 4, we also obtain descriptions by tropical halfspaces and the tropical convex hull of the intersection with the orthants (Section 5). As tropical convexity over $T_{\geq 0}$ is well studied, it is useful to have the description of the signed tropical convex hull in terms of certain intersection points with the coordinate hyperplanes (Theorem 5.3). We obtain an equivalent definition as the intersection of all open halfspaces containing a given set of points (Theorem 5.9). The analogous statement with closed tropical halfspaces is not true (Remark 5.10), but we nevertheless derive a Minkowski-Weyl theorem (Theorem 5.12) under some additional assumptions. In particular, the tropical convex hulls of finitely many points are exactly the intersections of closed tropical halfspaces, which yield a tropically convex set.

The duality of signed tropical convex hulls and tropical linear inequality systems is reflected in the dual notions of non-negative kernel (22) and open tropical cones (23). This is formalized in a new version of Farkas’ Lemma (Theorem 4.6) for signed tropically convex sets. We deduce it in a geometric way from new versions of Fourier-Motzkin elimination for signed numbers (Theorem 4.12 and 4.13). Another version (Theorem 4.17) leads to our Minkowski-Weyl theorem. All these elimination schemes profit from omitting the non-negativity constraint of the variables.

1.2. Related work. Our notion of signed tropical convexity heavily relies on the concept of the symmetrized tropical semiring $S$, which goes back to [1], and was further developed in [4, 27], among others. Signed numbers arise in the context of tropical convexity in [6], however only as coefficients for an inequality system. The technically difficult aspects are the necessary properties of equality and order
relations. While [4] also developed different notions replacing orders or equalities, they do not provide all necessary concepts to deal with the new notion of tropical convexity. The relations $\bowtie$, $\vdash$ and $>$ also appear in the context of hyperfields in [24], where images of semi-algebraic sets are studied. The duality of the tropical analog of polar cones in [20] can be considered as a predecessor of our duality in Section 4.1. Infeasibility certificates for linear inequality systems were deduced from the duality of mean payoff games in [22, 8]. A tropical version of Fourier-Motzkin elimination was established in [7]. The latter results rely on the (tropical) non-negativity of the variables and cannot be transferred directly to our setting, as we discuss also in Remark 4.16 and Remark 4.7. The tropicalizations of polytopes [18] or more general semialgebraic sets [24] leads to the image of a single object. However, our construction naturally leads to the tropicalization of a union of polytopes arising as the convex hull of lifts of points. This is in some sense dual to the representation established in [24], where all satisfied equations and inequalities are needed to describe the tropicalization of a single object. Parallel to our work, similar structures for signed numbers are developed in [5, 2].

2. Signed numbers and orderings

We introduce the necessary terminology for our purposes. For a recent comprehensive introduction to signed numbers and the symmetrized semiring, see [4].

2.1. Signed numbers. We define the signed tropical numbers $T_\pm$ by gluing two copies of $(\mathbb{R} \cup -\infty)$ at $-\infty$. One copy is declared the non-negative tropical numbers $T_{\geq 0}$ (this is often denoted by $T_{\max}$ in the literature), the other copy forms the non-positive tropical numbers $T_{\leq 0}$. Most of the time, we denote $-\infty$ by $O$ as it is the tropical zero element. The elements in $T_{\leq 0} \setminus \{O\}$ are marked by the symbol $\ominus$. The signed tropical numbers $T_\pm$ have a natural norm $|.|$ which maps each element of $T_{\geq 0}$ to itself and removes the sign of an element in $T_{\leq 0}$. This gives rise to the order

$$x \leq y \Leftrightarrow \begin{cases} x \in T_{\geq 0} \text{ and } y \in T_{\geq 0} \\ x \leq y \text{ for } x, y \in T_{\geq 0} \\ |x| \geq |y| \text{ for } x, y \in T_{\leq 0} \end{cases}. \quad (2)$$

Furthermore, we obtain the strict order $x < y \Leftrightarrow x \leq y \wedge x \neq y$. The tropical signed space $T_d^\pm$ is the union of $2^d$ orthants which are copies of $T_{\geq 0}^d$ glued along their boundary.

2.2. Balanced numbers. To develop the technical tools for dealing with signed numbers, we use the symmetrized semiring $S$ which forms a semiring containing $T_\pm$, introduced in [1]. This semiring is constructed with a third copy of $\mathbb{R} \cup \{O\}$ by gluing again at $O$. We denote the third copy, the balanced numbers, by $T_\circ$ and mark the elements by the symbol $\circ$. Unfortunately, the symmetrized semiring $S$ cannot be ordered. We extend the norm $|.|$ in such a way that it removes the $\circ$ from an element in $T_\circ$ and leaves the corresponding element in $T_{\geq 0}$. The complementary map $\text{tsgn}$ from $S$ to $\{\ominus, \ominus, \circ, O\}$ remembers only in which of the sets an elements lies: positive tropical numbers $T_{>0} = T_{\geq 0} \setminus \{O\}$, negative tropical numbers $T_{<0} = T_{\leq 0} \setminus \{O\}$, balanced non-zero tropical numbers $T_\circ \setminus \{O\}$ or the tropical zero $\{O\}$. 
Next, we define the binary operations of the semiring. For \( x, y \in \mathbb{S} \), we define the addition by

\[
 x \oplus y = \begin{cases} 
 \arg\max_{x,y}(|x|, |y|) & \text{if } \chi \subseteq \{+, O\} \text{ or } \chi = \{-\} \\
 \bullet \arg\max_{x,y}(|x|, |y|) & \text{else .}
\end{cases}
\]

where \( \chi = \{ \text{sgn}(\xi) \mid \xi \in (\arg\max(|x|, |y|))\} \). Note that we omit the sign for elements in \( T_{\geq 0} \). For the multiplication we set

\[
 x \odot y = (\text{sgn}(x) * \text{sgn}(y)) (|x| + |y|),
\]

where the \(*\)-multiplication table is the usual multiplication of \( \{-1, 1, 0\} \) for \( \{\oplus, \odot, \bullet\} \) with the additional specialty that multiplication with \( O \) yields \( O \).

The operations \( \oplus \) and \( \odot \) extend to vectors and matrices componentwise. Observe that the operations agree with the usual max-tropical operations on \( T_{\geq 0} \).

We can also consider \( \ominus \) as a unary selfmap of the semiring; to this extent, we set

\[
 \ominus x = \begin{cases} 
 \ominus x & \text{if } x \in T_{> 0} \\
 |x| & \text{if } x \in T_{< 0} \\
 x & \text{if } x \in T_ullet.
\end{cases}
\]

The map \( \ominus : \mathbb{S} \to \mathbb{S} \) is a semiring homomorphism. In particular, this justifies to write \( a \ominus b \) for \( a \oplus (\ominus b) \).

Furthermore, the absolute value fulfills \( |a \oplus b| = |a| \odot |b| \) by definition of the addition.

**Example 2.1.** Using the definitions, we see that \(-5\) is positive, \(\ominus 6\) and \(\ominus -6\) are negative, \(\bullet 3\) is balanced. Furthermore, the absolute value of \(-5\) is \(|-5| = -5\), of \(\ominus 6\) is \(|\ominus 6| = 6\), and \(\bullet 3 = 3\). Some simple sums are \(3 \oplus (\ominus 3) = \bullet 3, -3 \oplus 5 = 5, -3 \ominus (\ominus 5) = \ominus 5, \bullet 2 \oplus 4 = 4, \bullet -3 \ominus \ominus 5 = \ominus 3\). Finally some simple products are \(\bullet 3 \ominus 5 = 8, \ominus 4 \odot -6 = \ominus -2, \ominus 1 \ominus \ominus 1 = 2, \bullet 3 \odot O = O, \ominus 4 \odot O = O\).

**2.3. Extending the order.** As already mentioned, the semiring \( \mathbb{S} \) cannot be ordered in a consistent way with respect to its binary operations. However, we will equip it with some binary relations, which partly fulfill the tasks of an order. They occur under a different terminology in [24]; see 3.3.

**2.3.1. Signed order.** Even if \( \mathbb{S} \) cannot be ordered totally, we can extend the ordering from \( T_{\pm} \) partially by setting

\[
 x > y \iff x \ominus y \in T_{> 0}.
\]

This is equivalent to

\[
 x > y \iff \begin{cases} 
 x > y & \text{for } x, y \in T_{\pm}, \text{ see (2)} \\
 x > |y| & \text{for } x \in T_{\pm}, y \in T_ullet \\
 \ominus |x| > y & \text{with } x \in T_ullet, y \in T_{\pm}
\end{cases}
\]

Note that there are pairs in \( T_{\pm} \times T_{\pm} \) and in \( T_{\bullet} \times T_{\pm} \) which are not comparable. In particular, the signed numbers

\[
 \{x \in T_{\pm} \mid x \not\approx a \text{ and } x \not\approx a\}
\]

which are incomparable to \( a \in T_{\bullet} \) via \( '\prec' \), form the interval

\[
 \mathcal{U}(a) := [\ominus |a|, |a|] := \{x \in T_{\pm} \mid \ominus |a| \leq x \leq |a|\}
\]
We also denote the set incomparable to a signed element \( a \in \mathbb{T}_\pm \), which is only the singleton \( \{a\} \), by \( \mathcal{U}(a) \). We extend this to vectors by setting \( \mathcal{U}(v) = \prod_{i \in [d]} \mathcal{U}(v_i) \). Note that also no pair in \( \mathbb{T}_\bullet \times \mathbb{T}_\bullet \) is comparable.

The relation (6) gives rise to a non-strict relation
\[
x \geq y \iff x > y \text{ or } x = y .
\]
which turns out to be a partial order in Corollary 4.9.

Observe that the ordering is compatible with the reflection map, in the sense that
\[
x \geq y \iff \ominus y \geq \ominus x .
\]
A useful property of strict inequalities is that they can be added together.

**Lemma 2.2.** For \( a, b, c, d \in \mathbb{S} \), we have the implication
\[
a < b \text{ and } c < d \implies a + c < b + d .
\]

**Proof.** By definition, we first get \( b \ominus a > 0 \) and \( d \ominus c > 0 \). As addition is closed in \( \mathbb{T}_{\geq 0} \), this yields \( b \ominus a + d \ominus c > 0 \). The claim follows from (5). \( \square \)

**Remark 2.3.** In general, the strict and non-strict partial order ‘\(<\)’ and ‘\(\leq\)’ on \( \mathbb{S} \) is not compatible with addition. The inequality \( 3 < 4 \) does not imply \( 3 \oplus 5 < 4 \oplus 5 \), and \( 3 \leq 4 \) does not imply \( \ominus 4 = 3 \ominus 4 \leq 4 \ominus 4 = \bullet 4 \). This is the main motivation for introducing the relation ‘\(\vDash\)’ below, which is not an ordering (as it lacks transitivity) but it is compatible with the addition.

An advantage of strict inequalities is the validity of
\[
a \oplus b > c \iff a > c \ominus b .
\]
The analogous reformulation
\[
a \oplus b \geq c \iff a \geq c \ominus b .
\]
is wrong in general. For example, \( 2 \oplus 5 \geq 5 \) but \( 2 \) is incomparable with \( 5 \ominus 5 = \bullet 5 \). However, such reformulations hold for the relation ‘\(\vDash\)’, which we show in Lemma 2.5(a).

2.3.2. **Balanced relations.** The balance relation ‘\(\Delta\)’ was introduced in [1]; we will use the notation \(\bowtie\) in this paper. We define
\[
x \bowtie y \iff x \ominus y \in \mathbb{T}_\bullet .
\]
The following characterizations are immediate from the definitions. For more properties of \(\bowtie\), we refer to [1, §IV].

**Lemma 2.4.** Let \( a, b \in \mathbb{S} \).

(a) \( a \bowtie b \) is equivalent to \( (a \in \mathbb{T}_\bullet, |a| \geq |b|) \lor (b \in \mathbb{T}_\bullet, |b| \geq |a|) \lor (a = b) \).

(b) If \( b \in \mathbb{T}_\pm \), then \( a \bowtie b \) is equivalent to \( b \in \mathcal{U}(a) \).

We introduce the binary relation
\[
x \vDash y \iff x > y \text{ or } x \bowtie y \iff x \ominus y \in \mathbb{T}_{\geq 0} \cup \mathbb{T}_\bullet .
\]
Note that \( a \bowtie b \) is equivalent to \( (a \vDash b) \lor (a \vDash b) \). Recall from Remark 2.3 that bringing terms to the other side of a non-strict inequality with ‘\(\geq\)’ is not valid in general. The next lemma shows, among other simple properties, that ‘\(\vDash\)’ is compatible with the semiring operations.
Lemma 2.5. Let \( a, b, c, d \in S \).

(a) \( a \oplus c \equiv b \iff a \equiv b \ominus c \)

(b) \( a \equiv b \land c \equiv d \implies a \oplus c \equiv b \ominus d \).

(c) If \( c \in T_{\pm} \), then \( b \equiv c \) and \( c \equiv a \) imply \( b \equiv a \).

(d) \( a \equiv b \) implies \( c \ominus a \ominus d \equiv c \ominus b \ominus d \) for \( c \in T_{\geq 0} \) and \( c \oplus b \ominus d \equiv c \ominus a \ominus d \) for \( c \in T_{\leq 0} \).

Proof. (a) The claim follows directly from the definition and the properties of the semiring \( S \).

(b) Using the definition, we obtain \( a \ominus b, c \ominus d \in T_{\geq 0} \cup T_{\bullet} \). This implies already \( a \oplus c \ominus b \ominus d \in T_{\geq 0} \cup T_{\bullet} \).

(c) For a contradiction, assume \( b \not\equiv a \), that is, \( b \ominus a \in T_{< 0} \).

Case I: \( |b| > |a| \). In this case, \( b \in T_{< 0} \). Since \( b \ominus c \in T_{\geq 0} \cup T_{\bullet} \), it follows that \( c \in T_{< 0} \) and \( |c| \geq |b| \), using \( c \in T_{\pm} \). We now get a contradiction to \( c \ominus a \in T_{\geq 0} \cup T_{\bullet} \), since \( |c| \geq |b| > |a| \).

Case II: \( |a| > |b| \). This case follows by an analogous argument. With \( a \in T_{> 0} \), the condition \( c \ominus a \in T_{\geq 0} \cup T_{\bullet} \) implies \( c \in T_{> 0} \) and \( |c| \geq |a| > |b| \). This contradicts \( b \ominus c \in T_{\geq 0} \cup T_{\bullet} \).

Case III: \( |a| = |b| \). In this case, we must have \( b \in T_{\leq 0} \) and \( a \in T_{\geq 0} \). We thus obtain \( c \in T_{< 0} \) as in case I, but also \( c \in T_{> 0} \) as in case II, a contradiction.

(d) The expression \( c \ominus (a \ominus b) = c \ominus a \ominus c \ominus b \) is in \( T_{\geq 0} \) for \( c \in T_{\geq 0} \) and in \( T_{\leq 0} \) for \( c \in T_{\leq 0} \). Now, the statement follows from (b) with \( d \equiv d \).

\( \square \)

Remark 2.6. Note that \( \equiv \) is not a partial order, since transitivity does not hold for all elements, as the example

\[ 1 \equiv \bullet 6, \quad \bullet 6 \not\equiv 3, \quad \text{but } 1 \not\equiv 3 \]

shows.

3. Tropical convexity of signed numbers

3.1 Signed tropical convex combinations. Let us recall the notation that for a matrix \( A \in T_{\geq 0}^{d \times n} \), and a vector \( x \in T_{\geq 0}^{n} \), we denote by \( A \odot x \in T_{\geq 0}^{n} \) the tropical matrix product. The tropical convex hull \( \text{tconv}(A) \) of the columns of a matrix \( A \in T_{\geq 0}^{d \times n} \), studied in [17, 15, 12], is defined as

\[
\text{tconv}(A) = \left\{ A \odot x \mid x \in T_{\geq 0}^{n}, \bigoplus_{j \in [n]} x_j = 0 \right\} \subseteq T_{\geq 0}^{d} .
\]  

(12)

In this definition it is essential that all columns of \( A \) lie in the non-negative orthant \( T_{\geq 0}^{d} \). For general matrices in \( T_{\pm}^{d \times n} \), the product \( A \odot x \) may contain balanced entries. We now extend the notion of the tropical convex hull to \( T_{\pm}^{d} \). Note that we switch freely between a matrix and its set of columns.

Definition 3.1 (Inner hull). The (signed) tropical convex hull of the columns of the matrix \( A \in T_{\pm}^{d \times n} \) is defined as

\[
\text{tconv}(A) = \left\{ z \in T_{\pm}^{d} \mid z \propto A \odot x, x \in T_{\geq 0}^{n}, \bigoplus_{j \in [n]} x_j = 0 \right\} \subseteq T_{\pm}^{d} .
\]  

(13)
Figure 1. The signed tropically convex hull of \{(3,3), (⊖1, ⊖0), (⊖4, ⊖2)\}. We omit labels for the axes as the origin is \((-∞, -∞)\) and therefore infinitely far away.

Such a set is a \((signed)\ tropical\ polytope\). The tropical convex hull of an arbitrary set \(M \subseteq \mathbb{T}_d\) is the union

\[ t\text{conv}(M) = \bigcup_{V \subseteq M, V \text{ finite}} t\text{conv}(V). \]

A subset \(M \subseteq \mathbb{T}_d\) is \(tropical\ convex\) if \(M = t\text{conv}(M)\).

This hull construction generalizes (12) because if \(A \in \mathbb{T}_d^{d \times n}\geq 0\) then \(A \odot x \in \mathbb{T}_d^n\).

In this case, Lemma 2.4(a) implies that \(z \triangleright A \odot x\) holds only for \(z = A \odot x\).

Using Lemma 2.4(b), we can write (13) equivalently as

\[ t\text{conv}(A) = \bigcup \left\{ U(A \odot x) \bigg| x \in \mathbb{T}_n^{\geq 0}, \bigoplus_{j \in [n]} x_j = 0 \right\} \subseteq \mathbb{T}_d^n. \]  

(14)

Example 3.2. The critical points of the tropical convex hull depicted in Figure 1 can be calculated via

\[
\begin{align*}
(-3) \odot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + (\oplus 1, \oplus 0) &= (\oplus 1, \bullet 0), \\
(3) \oplus (-1) \odot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + (\oplus 4, \oplus 2) &= (\bullet 3, \bullet 3),
\end{align*}
\]

A more precise way, how these points can be used to determine the signed tropical convex hull, via the tropical convex hull of the intersection with each orthant is given in Theorem 5.3.

Remark 3.3. There is no unique minimal generating set in the usual sense as the example \(t\text{conv}((0,0), (⊖0, ⊖0)) = t\text{conv}((0, ⊖0), (⊖0, 0))\) shows.

We now derive some elementary properties of this convexity notion. The following are immediate from the definition, as (14) is just a componentwise construction.

Proposition 3.4.

(a) The intersection of tropically convex sets is tropically convex.

(b) The coordinate projection of tropically convex sets is tropically convex.
hull of a set is a tropically convex set (Proposition 3.7). The following technical lemma will be needed for these proofs.

**Lemma 3.5.**
(a) Let \( a \in \mathbb{S} \), \( b \in \mathbb{T}_\pm \), and \( z \in \mathcal{U}(a \oplus b) \). Then there exists an \( a' \in \mathcal{U}(a) \) such that \( z \in \mathcal{U}(a' \oplus b) \).

(b) If \( a \in \mathcal{U}(x) \), \( b \in \mathcal{U}(y) \), and \( c \in \mathbb{T}_\pm \), then \( \mathcal{U}(c \circ a \oplus b) \subseteq \mathcal{U}(c \circ x \oplus y) \).

**Proof.** (a) If \( a \in \mathbb{T}_\pm \), then \( a' = a \) satisfies the requirements. For the rest of the proof, we assume \( a \in \mathbb{T}_\ast \). If \( |b| > |a| \), then we can set \( a' = |a| \). In this case, \( a' \oplus b = a \oplus b = b \in \mathbb{T}_\pm \). Consider now the case \( |a| \geq |b| \), which implies \( a \oplus b = a \). Then \( z \in \mathcal{U}(a \oplus b) \) if and only if \( |z| \leq |a| \). For \( |a| \geq |z| > |b| \), we set \( a' = z \). If \( |a| \geq |b| \geq |z| \), then we set \( a' = \ominus b \). In both cases it is easy to see that \( z \in \mathcal{U}(a' \oplus b) \).

(b) Note that \( |a| \leq |x| \) and \( |b| \leq |y| \), and consequently, \( |c \circ a \oplus b| \leq |c \circ x \oplus y| \). If \( c \circ x \oplus y \) is balanced, then the claim follows: \( \mathcal{U}(c \circ x \oplus y) \) contains all \( r \in \mathbb{T}_\pm \) with \( |r| \leq |c \circ x \oplus y| \); this holds for all \( r \in \mathcal{U}(c \circ a \oplus b) \).

Hence, assume that \( c \circ x \oplus y \) is not balanced. In particular, \( x \) or \( y \) is not balanced. If both \( x, y \in \mathbb{T}_\pm \), then \( a = x \) and \( b = y \) and thus the claim is immediate. The remaining case is when exactly one of \( x \) and \( y \) is balanced. Let us assume \( y \in \mathbb{T}_\pm \); the case \( x \in \mathbb{T}_\pm \) follows similarly. Now we have \( b = y \), and we must also have \( |y| > |c \circ x| \) as otherwise \( c \circ x \oplus y \) would be balanced. Consequently, \( c \circ x \oplus y = y \).

On the other hand, \( |a| \leq |x| \) and \( b = y \) imply \( |c \circ a| < |b| \), and therefore \( c \circ a \oplus b = y \), and the claim follows. \( \square \)

**Proposition 3.6.** An arbitrary subset \( M \subseteq \mathbb{T}_\pm^d \) is tropically convex if and only if \( \text{tconv}(\{p, q\}) \subseteq M \) for all \( p, q \in M \).

**Proof.** For a tropically convex set, the tropical convex hull of all two-element subsets is contained by definition. In the converse direction, we show by induction on \( n \) that if we select any \( n \) vectors from \( M \) as the columns of a matrix \( A \in \mathbb{T}_\pm^{d \times n} \), then \( \mathcal{U}(A \odot x) \subseteq M \) for any \( x \in \mathbb{T}_\pm^d \). \( \mathcal{U}(A \odot x) \) is the \( \ominus \mathcal{T}_\geq 0 \mathcal{U}(A \odot x) \). The case \( n = 2 \) follows by the assumption; consider now \( n \geq 3 \) and assume that the claim holds for \( n - 1 \).

Let \( z \in \mathcal{U}(A \odot x) \). Without loss of generality, we can assume that \( x_1 = 0 \). We set \( s = \bigoplus_{\ell=1}^{n-1} x_\ell \odot a^{(\ell)} \in \mathbb{S}^d \), where \( a^{(\ell)} \) is the \( \ell \)-th column of \( A \). We let \( q = a^{(n)} \).

Then, \( A \odot x = s \odot x_n \odot q \).

We can apply Lemma 3.5(a) to each component of \( z, s, \) and \( x_n \odot q \). Thus, we obtain a vector \( p \in \mathcal{U}(s) \) such that \( z \in \mathcal{U}(p \odot x_n \odot q) \). By induction, \( p \in M \), and thus \( z \in \text{tconv}(\{p, q\}) \subseteq M \) by the assumption. This completes the proof. \( \square \)

**Proposition 3.7.** For any matrix \( A \in \mathbb{T}_\pm^{d \times n} \), the set convex hull \( \text{tconv}(A) \) is tropically convex. Consequently, \( \text{tconv}(\text{tconv}(A)) = \text{tconv}(A) \).

**Proof.** Using Proposition 3.6, it suffices to show that if \( p, q \in \text{tconv}(A) \), \( \lambda \in \mathbb{T}_\geq 0 \), \( \lambda \leq 0 \), then \( \mathcal{U}(p \oplus \lambda \odot q) \subseteq \text{tconv}(A) \).

Let \( x, y \in \mathbb{T}_\geq 0, \bigoplus_{\ell \in [n]} x_j = 0 \), \( \bigoplus_{\ell \in [n]} y_j = 0 \) such that \( p \in \mathcal{U}(A \odot x) \) and \( q \in \mathcal{U}(A \odot y) \). We let \( z = x \oplus \lambda \odot q \); clearly, \( z \in \mathbb{T}_\geq 0 \) and \( \bigoplus_{\ell \in [n]} z_j = 0 \). From Lemma 3.5(b), we obtain that \( \mathcal{U}(p \oplus \lambda \odot q) \subseteq \mathcal{U}(A \odot z) \subseteq \text{tconv}(A) \). \( \square \)
Example 3.8. For a subset \( I \subset [d] \) and a point \( a \in \mathbb{T}^d_\pm \), the set
\[
\{ z \in \mathbb{T}^d_\pm \mid z_i = a_i \text{ for all } i \in I \}
\]is tropically convex. Note that all (signed) tropical hyperplanes, which can not be written in the latter form, are not tropically convex. A signed tropical hyperplane of the form
\[
\text{Hyp}(a) = \{ x \in \mathbb{T}^d_\pm \mid a \circ x \in \mathbb{T}_\bullet \}.
\]
Let \( a \in \mathbb{T}^d_\pm \) with \( |\text{supp}(a)| > 1 \), where \( \text{supp}(a) = \{ i \in [d] \mid a_i \neq 0 \} \). We assume that \( \text{supp}(a) \supseteq \{1,2\} \). Then \( p = (\oplus a_2, a_1, 0, \ldots, 0), q = (a_2, \ominus a_1, 0, \ldots, 0) \in \text{Hyp}(a) \). Because of \( p \oplus q = (\bullet a_2, \bullet a_1, 0, \ldots, 0) \), the point \((a_2, a_1, 0, \ldots, 0)\) is contained in \( \text{tconv}(p, q) \). However, it is not an element of \( \text{Hyp}(a) \).

Example 3.9. For a vector \((a_0, a_1, \ldots, a_d) \in \mathbb{T}^{d+1}_+\) we define the open signed tropical halfspace
\[
\mathcal{H}^+(a) = \left\{ x \in \mathbb{T}^d_\pm \mid a \circ \begin{pmatrix} 0 \\ x \end{pmatrix} > 0 \right\}.
\]An open signed tropical halfspace is tropically convex. Let \( c \in \mathbb{T}^d_\pm, c_0 \in \mathbb{T}_\pm, p, q \in \mathbb{T}^d_\pm \) and \( \lambda, \mu \in \mathbb{T}_{\leq 0} \) with \( \lambda \oplus \mu = 0 \). For \( p \) and \( q \) contained in the halfspace, we have \( c \circ (p \oplus c_0) > 0 \) and \( c \circ (q \oplus c_0) > 0 \), and by Lemma 2.2,
\[
\lambda \circ (\lambda \circ (p \oplus \mu \circ q) \oplus c_0) \lambda \circ (c \circ (p \oplus c_0) \oplus \mu \circ (c \circ (q \circ c_0) > 0).
\]
If \( \lambda \circ (\lambda \circ (p \oplus \mu \circ q) \oplus c_0 \) has a balanced component \( b \in \mathbb{T}_\bullet \), then the value of \( \lambda \circ (\lambda \circ (p \oplus \mu \circ q) \oplus c_0 \) cannot depend on this component as it is positive. Hence, we can replace that component by an element in \( \mathcal{U}(b) \) and preserve the inequality (16).

Let \( \mathbb{P} \subset \mathbb{T}^{d \times d}_\pm \) be the set of permutation matrices with 0 as one and \( \mathbb{O} \) as zero, and let \( \mathbb{D} \subset \mathbb{T}^{d \times d}_\pm \) be the set of matrices with diagonal entries from \( \mathbb{T}_\pm \) and \( \mathbb{O} \) else. Their union generates the multiplicative group of signed tropical transformations \( \mathbb{ST} \). This group is the natural group of transformations which leaves the combinatorial structure of a subset of \( \mathbb{T}^d_\pm \) unchanged.

Example 3.10. We want to describe the line segment \( \text{tconv}(p, q) \) for two points \( p, q \in \mathbb{T}^d_\pm \). By suitable scaling with elements from \( \mathbb{ST} \), we can assume that \( p = (0, \ldots, 0) \), and that the entries of \( q \) are ordered by increasing absolute value.

Analogous to the description in [17], one obtains a piecewise-linear structure where the breakpoints are determined by the absolute values of the components of \( q \). As an additional phenomenon, the line segments flip to another orthant at each tropically negative entry of \( q \). If the sign changes in \( \ell \) coordinates at once, the line segment has dimension \( \ell \). We visualize several examples for the two-dimensional case in Figure 2.

Remark 3.11. It is tempting to define a cancellative sum for two numbers \( a, b \in \mathbb{T}_\pm \) by
\[
a \oplus b = \begin{cases} 
    a & |a| > |b| \\
    b & |b| > |a| \\
    a & a = b \\
    0 & a = \ominus b 
\end{cases}.
\]This can be extended componentwise to \( \mathbb{T}^d_\pm \).
A similar construction is used in [11], to define a different version of tropical convexity, see Section 3.4. A conceptional drawback of the cancellative sum is that it is not associative, as the example shows. We use a similar but multi-valued version in Section 4.3 for (34).

**Definition 3.12 (Conic hull).** The signed tropical conic hull of the columns of $A$ is

$$tcone(A) = \bigcup_{\lambda \in \mathbb{T}^d_{\geq O}} U(A \circ \lambda).$$

(17)

The definition together with Proposition 3.7 yields the following.

**Corollary 3.13.** The conic hull of a subset of $\mathbb{T}^d_{\pm}$ is tropically convex.

### 3.2. Image of Puiseux lifts.

The aim of this section is to relate our concept of convexity over $\mathbb{T}^d_{\pm}$ to convexity over $\mathbb{R}$. To achieve this, we move to another ordered field, the field of real Puiseux series $\mathbb{K} = \mathbb{R} \langle t \rangle$. This has proven to be a helpful concept in the study of tropical numbers with signs, see [30, 6, 24]. It is formed by formal Laurent series with exponents in $\mathbb{R}$ and coefficients in $\mathbb{R}$. The exponent sequence is strictly decreasing and it has no accumulation point. This
ordered field is equipped with a non-archimedean valuation val which maps all non-zero elements to their leading exponent and zero to $O = -\infty$. Additionally, the map $sgn: K \rightarrow \{\ominus, 0, \oplus\}$ yields the sign of an element. This gives rise to the signed valuation $sval: K \rightarrow T_\pm$ which maps an element $k \in K$ to $sgn(k) \, \text{val}(k)$. It is enough to think of Puiseux series as polynomials in $t$ with arbitrary exponents and coefficients in $\mathbb{R}$.

The tropicalization of structured sets over $\mathbb{K}$, i.e., the study of the image of a subset of $\mathbb{K}^d$ is a technique which is widely used in tropical geometry. We introduced a concept purely on the tropical side. We will see in Theorem 3.14, that signed tropically convex sets are not the image of the valuation of a single convex hull but of a whole union, ranging over the fibers of tropical points.

In some sense, this is complementary to the main result in [24]. While they consider semialgebraic sets over $\mathbb{K}$ in general, polytopes, i.e., the convex hull of finitely many points in $\mathbb{K}^d$, can be considered as a special case. They show that one has to tropicalize all semialgebraic relations fulfilled by a set to describe its image under the signed valuation map.

Recall that for our concept of tropical convexity over $T_\pm$ the image of a single polytope under the signed valuation may not be tropically convex as the Example 3.18 shows. It is subject to further work to study the special case of polytopes (as semialgebraic sets) from [24] and to see which properties such a notion of signed tropical polytopes could provide.

Note that the next statement is valid for more general fields with a non-trivial non-archimedean valuation val which is surjective onto $T_\geq 0$.

**Theorem 3.14.** The signed hull $tconv(A)$ is the union of the signed valuations for all possible lifts

$$tconv A = \bigcup_{sval(A) = A} sval(\text{conv}(A)).$$

**Proof.** We start with the inclusion ‘$\supseteq$’. Let $A \in T_\pm^{d \times n}$ and fix a lift $A$ of $A$, this means a matrix $A \in \mathbb{K}^{d \times n}$ with $sval(A) = A$. For a vector $\lambda = [\lambda_1, \ldots, \lambda_n]$ with $\sum_{j=1}^n \lambda_j = 1$ the valuation $x = sval(\lambda)$ is in $T_\geq 0$ and fulfills $\bigoplus_{j=1}^n x_i = 0$. We want to show that $b = sval(\lambda_1 \cdot a^{(1)} + \cdots + \lambda_n \cdot a^{(n)}) \in tconv A$. For each $i \in [d]$, let $c_i = \max \left\{ |sval(\lambda_j \cdot a^{(j)})| \mid j \in [n] \right\}$. Furthermore, we define $p = A \odot x = \bigoplus_{j \in [n]} x_j \odot a^{(j)} = \bigoplus_{j \in [n]} sval(\lambda_j \cdot a^{(j)})$. We fix an $i \in [d]$ and we want to show that $b_i \in U(p_i)$. Note that $|p_i| = c_i$. If $p_i$ is not balanced, we already have $b_i = p_i$. Otherwise, we get $|b_i| \leq c_i$ and consequently $b_i \in [\ominus c_i, c_i]$. This finishes the proof of the inclusion ‘$\supseteq$’.

For the other direction, we fix $b \in U(A \odot x)$ for some $x \in T_\geq 0^n, \bigoplus_{j \in [n]} x_j = 0$. We define

$$\lambda_j = t^{x_j} \cdot \left( \sum_{k \in [n]} t^{x_k} \right)^{-1} \quad \text{for each } j \in [n].$$

With this, we get $\lambda \geq 0$ and $\sum_{k \in [n]} \lambda_k = 1$.

For each row $i \in [d]$, we denote by $J_i^+$ the set of indices of the positive elements in $\text{argmax} \left\{ a^{(i)}_j \odot x_j \mid j \in [n] \right\}$, and by $J_i^-$ analogously for negative elements.
Hence, we have \( sval(\cdot) \) and define
\[
a_i^{(j)} = \begin{cases} 
\text{sgn}(a_{ij}) t^{a_{ij}} + \alpha_i & \text{for } j = \ell_i \\
\text{sgn}(a_{ij}) t^{a_{ij}} & \text{else ,}
\end{cases}
\]
where
\[
\alpha_i = \frac{1}{\lambda_\ell} 
\left(-\sum_{k \in [d]} \text{sgn}(a_{ik}) t^{\ell_{ik} + x_\ell} + \text{sgn}(b_i) t^{b_i} \right).
\]

Note that \( |b_i| \leq |a_{\ell_i}| + x_\ell \) and \( |a_j| + x_j - x_\ell \leq |a_{\ell_i}| \) for all \( j \in [n] \). Therefore, \( sval(a_{i}^{(j)}) = a_{ij} \) for all \( i \in [d] \) and \( j \in [n] \). Furthermore, we get
\[
a_i \cdot \lambda = \sum_{k \in [d] \setminus \{\ell_i\}} \lambda_k a_i^{(k)} + \lambda_{\ell_i} a_i^{(\ell_i)}
= \sum_{k \in [d]} \text{sgn}(a_{ik}) t^{a_{ik} + x_\ell} - \sum_{k \in [d]} \text{sgn}(a_{ik}) t^{\ell_{ik} + x_\ell} + \text{sgn}(b_i) t^{b_i}
= \text{sgn}(b_i) t^{b_i}.
\]
Hence, we have \( sval(a_i \cdot \lambda) = b_i \) for all \( i \in [d] \). This concludes the proof.

\( \square \)

**Remark 3.15.** Theorem 3.14 generalizes [18, Proposition 2.1], since \( val \) is a semiring homomorphism from \( K_{\geq 0} \) to \( T_{\max} = T_{\geq 0} \).

**Corollary 3.16.** The tropical convex hull is the union of the convex hulls of the lifts, i.e.,
\[
tconv(A) = sval(conv(sval^{-1}(A))) .
\]

### 3.3. Convex hull from hyperoperation

We introduce the necessary notions for hyperfields to define a signed convex hull and compare our binary operations with hyperfield operations (20). Let us briefly introduce the real plus-tropical hyperfield \( \mathbb{H} \), see [30]. It has the multiplicative group \( (T_{\pm}, \cdot) \) and its additive hyperoperation on \( T_{\pm} \) is given by
\[
x \boxplus y = \begin{cases} 
\arg\max_{x,y} (|x|, |y|) & \text{if } \chi \subseteq \{+, 0\} \text{ or } \chi = \{-\}
\{\ominus |x|, |x|\} & \text{else .}
\end{cases}
\]

We see that the latter addition for non-balanced numbers \( x, y \in T_{\pm} \) differs from the Definition in (3) in that it has a multi-valued result in the powerset of \( T_{\pm} \). A balanced outcome \( z \in T_{\mp} \) is replaced with the interval \( \mathcal{U}(z) = [-|z|, |z|] \). One can extend the operations again componentwise and use the symbol \( \boxplus \) for the product of two matrices or vectors. In particular, the operation \( \ominus \) agrees with \( \ominus \) on \( T_{\pm} \). The addition is set-valued in \( \mathbb{H} \) if and only if it would be balanced in \( \mathbb{S} \). It agrees with \( \ominus \) on \( T_{\geq 0} \).

We recall the order relations used in [24] for the multiplicative real tropical hyperfield. Note that they use the multiplication \( \ominus = \cdot \) instead of our approach with \( \cdot \). A polynomial over the real tropical hyperfield is a formal expression
\[
F(x) = \ominus d_1, \ldots, d_n \in \mathbb{Z} \cdot d_1 \cdot \ldots \cdot x_n^{d_n}
\]
which can be evaluated at an element \( \zeta \in \mathbb{H}^n \). This yields a subset
\[
F(\zeta) = \ominus d_1, \ldots, d_n \in \mathbb{Z} \cdot d_1 \cdot \ldots \cdot \zeta_n^{d_n} \subseteq \mathbb{H}^n .
\]
Note that we mainly deal with linear polynomials, where the exponent vector \((d_1, \ldots, d_n) \in \mathbb{Z}^d\) is just a unit vector.

One can define the sets
\[
\begin{align*}
\{ F = 0 \} & := \{(x_1, \ldots, x_n) \in T^d_{=0} : 0 \in F(x_1, \ldots, x_n)\} \\
\{ F \geq 0 \} & := \{(x_1, \ldots, x_n) \in T^d_{=0} : F(x_1, \ldots, x_n) \cap T_{\geq 0} \neq \emptyset\} \\
\{ F > 0 \} & := \{(x_1, \ldots, x_n) \in T^d_{=0} : F(x_1, \ldots, x_n) \in T_{>0}\} .
\end{align*}
\]

Observe that \(F = 0\) is indeed equivalent to \(F \geq 0 \lor -F \leq 0\) due to the structure of the set (19). Translating (20) to the symmetrized semiring \(S\) yields the relations ‘\(\gg\)' and ‘\(\ll\)’.

To motivate the next construction, we consider a tropical polytope generated by \(V \in T_{d \times k}^\geq\) as the image of the tropical standard simplex in the sense that
\[
tconv(V) = \{ V \odot \lambda \mid \bigoplus_{\ell \in [k]} \lambda_\ell = 0 \} .
\]

For a matrix \(A \in T_{d \times n}^\geq\), we define the balanced image of the tropical standard simplex \(\Delta_d = \{ \lambda \mid \bigoplus_{\ell \in [k]} \lambda_\ell = 0 \}\) by
\[
A \odot \Delta_d := \left\{ A \odot x \mid \bigoplus_{j \in [n]} x_j = 0, x \geq 0 \right\} \subset S^d . \tag{21}
\]

With this notion, one can write \(tconv(A) = \bigcup_{z \in A \odot \Delta_d} \mathcal{U}(z)\).

By using the hyperoperations in \(H\), we can naturally consider the image of the tropical standard simplex \(\Delta_d = \{ \lambda \mid \bigoplus_{\ell \in [k]} \lambda_\ell = 0 \}\) with respect to matrix multiplication by \(V \in T_{d \times k}^\geq\) as a subset of \(T_{d \times k}^\geq\).

\textbf{Proposition 3.17.} \(V \square \Delta_d = tconv(V)\).

\textit{Proof.} This follows directly by the definition of the set-valued addition in (18) from (13) with \(\mathcal{U}(z) = [\odot|z|, |z|]\) for \(z \in T_{=}^\bullet\). \hfill \Box

3.4. \textbf{Connection with \(B\)-convexity.} Parallel to the development of tropical convexity, the more general notion of \(B\)-convexity was developed starting with [12]. The notion of \(B\)-convexity boils down to convexity defined over the semiring \(\mathbb{R}_{\geq 0}\) with operations ‘\(\oplus\)’ = ‘max’ and ‘\(\odot\)’ = ‘\(+\)’, see [12, Theorem 2.1.1]. Taking logarithms transforms these operations to ‘\(\oplus\)’ = ‘\(\cdot\)’ and ‘\(\odot\)’ = ‘\(+\)’ on \(\mathbb{R} \cup \{-\infty\}\). This gives rise to a transferred version of \(B\)-convexity on \(T_{=\pm}\) by considering the images of \(B\)-convex sets in \(\mathbb{R}^d\) under the map \(x \mapsto \text{sgn}(x) \log(|x|)\).

The following example shows that our notion of signed tropical convexity is an even more restrictive notion than \(B\)-convexity and \(B^\leq\)-convexity [11].

\textbf{Example 3.18.} The tropical convex hull of \(A = \{(\odot 2, \odot 1), (2, 1)\}\) is the set
\[
[\odot 2, 2] \times [\odot 1, 1] .
\]

However, the set \(\text{Co}^r(A)\) is
\[
L = \{ (2 \odot \lambda, \lambda) \mid \lambda \in [\odot 1, 1] \} .
\]

for all \(r \in \mathbb{N}\). In particular, also \(\text{Co}^\infty(A)\) equals \(L\). This implies that \(B(L) = L\). We depict both in Figure 3. Hence, \(tconv(A)\) strictly contains \(B(A)\). Furthermore, [11, Corollary 4.2.4] shows that \(L\) is also \(B^\leq\)-convex.
Interestingly, the set $L$ is also the image under the signed valuation of the set
\[
\text{conv}\left(\left(-t^2, \frac{t^2}{t}\right)\right).
\]
Here, we mean the convex hull over the Puiseux series $R\{t\}$. So $L$ is the tropicalization of a single line segment while our hull construction yields the union of line segments whose spanning points tropicalize to $A$, as we saw in Section 3.2. For example, we get the set \(\{\ominus2\} \times [\ominus1, 1] \cup [\ominus2, 2] \times \{1\}\) as the tropicalization of \(\text{conv}\left(\left(-2t^2, \frac{t^2}{2t}\right)\right)\).

![Figure 3. Distinction between B-convex line and tropical line segment through the origin.](image)

### 4. Farkas’ Lemma and Fourier-Motzkin Elimination

#### 4.1. Convexity and tropical linear feasibility.

For a matrix $A = (a_{ij}) \in T_{d \times n}^\pm$, we define the non-negative kernel
\[
\ker_+(A) = \{ x \in T_{n \geq O} \setminus \{O\} \mid A \odot x \bowtie O \}.
\]  
This corresponds to the classical definition of a polyhedral cone in the form $Ax = 0, x \geq 0, x \neq 0$. We replace ‘=’ by ‘$\bowtie$’ and ‘$\geq$’ by ‘$\bowtie$’. In terms of the non-negative kernel, we can express containment in the convex hull as follows.

**Proposition 4.1.** For $A \in T_{d \times n}^\pm$ and $b \in T_d^\pm$ we have
\[
b \in \text{tconv}(A) \Leftrightarrow \ker_+\left(\begin{pmatrix} A & \ominus b \\
0 & \ominus O \end{pmatrix}\right) \neq \emptyset.
\]

**Proof.** The condition $b \in \text{tconv}(A)$ is equivalent to the existence of an element $x \in T_{n \geq O}$ with $\bigoplus_{j \in [n]} x_j = 0$ and $A \odot x \bowtie b$.

Let $(x, t)$ be a vector in the non-negative kernel, where $x \in T_{n \geq O}$ and $t \in T_{\geq 0}$ denotes the last component. First, we claim that $t \neq O$. Indeed, $t = O$ would yield $\bigoplus_{j \in [n]} x_j \bowtie O$, which implies $x_j = O$ for all $j \in [n]$. Thus, we obtain $(x, t) = O$, a contradiction. Since $t \neq O$, we can scale $(x, t)$ such that $t = 0$. In this case, the definition of the kernel gives $A \odot x \bowtie O$ and $\bigoplus_{j \in [n]} x_j \bowtie O \bowtie O$. The latter inequality yields $\bigoplus_{j \in [n]} x_j = 0$. This is the same as the combination witnessing $b \in \text{tconv}(A)$ as above. \qed
Corollary 4.2. The origin $O$ is in the convex hull $\text{tconv}(A)$ if and only if the non-negative kernel $\text{ker}_+(A)$ is not empty.

Proof. Setting $b = O$ in Proposition 4.1 implies the equivalence with the definition from (22). \hfill \Box

We now define the open tropical cone as the dual to the non-negative kernel (22).

$$\text{sep}_+(A) = \{ y \in T^d_{\pm} \mid y^T \odot A > O \}.$$ (23)

The name is motivated by the use of the elements of $\text{sep}_+(A)$ as separators of the columns of $A$ from the origin. Note that the condition ‘$>$ $O$’, in particular, means that the product ‘$y^T \odot A$’ is comparable with $O$ and, equivalently, in $T_{>0}$.

We can also define $\text{ker}_+(A)$ and $\text{sep}_+(A)$ for $A \in \mathbb{S}^{d \times n}$. However, this does not provide a wider class of objects. This follows by replacing a balanced number by $O$ in $\text{ker}_+(A)$ and applying Lemma 4.11 for $\text{sep}_+(A)$. We still extend the definition to these more general matrices, as it will lead to simplified arguments.

Example 4.3. The dotted black shape in Figure 4 depicts the $(\max, \cdot)$-convex hull of the columns of the matrix $A = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 4 & 3 \end{pmatrix}$. The critical points are

$$\frac{1}{2} \cdot \begin{pmatrix} -2 \\ 3 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \begin{pmatrix} \bullet 1 \\ 4 \end{pmatrix} \quad \text{and} \quad \frac{2}{3} \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} \bullet 2 \\ 3 \end{pmatrix}.$$

The blue shaded area is $\text{sep}_+(A)$. The configuration visualizes Theorem 4.4, as $\text{sep}_+(A)$ is not empty and the $(\max, \cdot)$-convex hull of $A$ does not contain the origin.

Let us now show that weak duality holds between the non-negative kernel and the open tropical cone:

Theorem 4.4 (Weak duality). For a matrix $A \in T^d_{\pm \times n}$ at least one of the sets $\text{ker}_+(A)$ or $\text{sep}_+(A)$ is empty. Equivalently, for an arbitrary vector $y \in T^d_{\pm}$ and a non-negative vector $x \in T^n_{\geq O} \setminus \{O\}$ at most one of $A \odot x \in T^d_+$ and $y^T \odot A > O$ can hold.
Proof. The claim follows from the fact that $T_\bullet$ is a left- and right-ideal of $S$, see [4, Definition 2.6]. This means that for $x \in \ker_+(A)$, the product $y^\top \odot A \odot x$ is in $T_\bullet$, while $y^\top \odot A > O \Rightarrow y^\top \odot A \odot x > O$. □

Remark 4.5. We give a more explicit construction for the former statement. We consider the product $y^\top \odot A \odot x$. Scaling the rows of $A$ by arbitrary numbers in $T_\pm$ does not change whether $x \in \ker_+(A)$, just as scaling the columns of $A$ by a non-negative number in $T_\geq O$ does not change whether $y \in \text{sep}_+(A)$. Hence, we can assume that $x$ and $y$ only have the entries 0 or $O$. Let $a_{pq}$ be the entry of $A$ with maximal absolute value. For $x \in \ker_+(A)$, there is an index $r \in [n]$ such that $a_{pr} = \ominus a_{pq}$. We can assume that $a_{pq} > O > a_{pr}$. From this we conclude that $y \notin \text{sep}_+(A)$ since the $r$th column of $y^\top \odot A$ cannot be positive.

The key result of this section will be showing the appropriate version of Farkas’ lemma. The proof will follow via Fourier-Motzkin elimination.

Theorem 4.6 (Farkas’ lemma). For a matrix $A \in T_{d \times n}$ exactly one of the sets $\ker_+(A)$ or $\text{sep}_+(A)$ is nonempty.

Remark 4.7. Theorem 4.6 is similar to [22, Corollary 3.12]. Through a suitable replacement of the balanced coefficients and a careful analysis of the occurring signs, Theorem 4.6 may be deduced from [22]. Note however, that we allow for unconstrained variables in the definition of $\text{sep}_+(A)$ which is not directly covered by [22, Corollary 3.12].

4.2. Technical properties of the order relations. The next lemma is a version of transitivity and it is a preparation for the elimination of a variable in a system of inequalities in Section 4.3.

Proposition 4.8. Let $A, B \subset S$ be two finite sets. There is an element $c \in S$ with $c \ominus a > O$ and $b \ominus c > O$ for all $a \in A, b \in B$.(24)

if and only if $b \ominus a > O$ for all $(a, b) \in A \times B$. (25)

Furthermore, the element $c$ can chosen to be signed.

Proof. For each pair $(a, b) \in A \times B$, we add the inequalities in (24) using Lemma 2.2 and obtain $b \ominus a \ominus c \ominus c > O$. This implies $b \ominus a > |c| \geq O$ and, hence, (25).

For the other direction, note that $A \subset T_{\pm}$ or $B \subset T_{\pm}$, as two balanced elements are not comparable by ‘>’. Because of the symmetry (9), we can assume that $B \subset T_{\pm}$. Let $\beta$ denote the minimum of $B$. Furthermore, we define $\alpha$ as the maximum of $A \cap T_\pm$ and $\{|a| \mid a \in A \cap T_\bullet\}$, where either of these two sets could also be empty. We obtain from (25) that $\beta > \alpha$, where we use that $b > a \iff b > |a|$ for $a \in T_\bullet$. An arbitrary element $c$ in the interval $\beta > \alpha$ fulfills (24). As the elements in $B$ are totally ordered, the claim for the inequalities involving $B$ follows immediately. Distinguishing the balanced and signed elements yields the claim for the inequalities involving $A$.

Corollary 4.9. The relation defined in (8) is a partial order.

Proof. Reflexivity and antisymmetry follow directly from (8) and (6). Proposition 4.8 implies transitivity.
Proposition 4.10. Let $A, B \subset S$ be two finite sets. There is an element $c \in \mathbb{T}_\pm$ with
\[ c \ominus a \Vdash O \text{ and } b \ominus c \Vdash O \text{ for all } a \in A, b \in B \] (26)
if and only if
\[ b \ominus a \Vdash O \text{ for all } (a, b) \in A \times B . \] (27)

Proof. The first direction from (26) to (27) follows from Lemma 2.5.a and 2.5.c because of $c \in \mathbb{T}_\pm$.

For the other direction, let
\[ \alpha_0 = \argmin \{|a| \mid a \in A \cap \mathbb{T}_* \} , \]
\[ \alpha_1 = \max \{ a \mid a \in A \cap \mathbb{T}_\pm \} , \]
\[ \beta_0 = \argmin \{|b| \mid b \in B \cap \mathbb{T}_* \} , \]
\[ \beta_1 = \min \{ b \mid b \in B \cap \mathbb{T}_\pm \} , \] (28)

with respect to the ordering ‘$\geq$’. By construction, we get from (27) the relation $\beta_1 \ominus \alpha_1 \Vdash O$, which yields $\beta_1 \geq \alpha_1$. Furthermore, we obtain $\beta_1 \ominus \alpha_0 \Vdash O$ implying $\beta_1 \geq \ominus |\alpha_0|$ and $\beta_0 \ominus \alpha_1 \Vdash O$ implying $|\beta_0| \geq \alpha_1$. We conclude that
\[ \underline{\beta} := \min(|\beta_0|, \beta_1) \geq \max(\ominus |\alpha_0|, \alpha_1) =: \overline{\alpha} , \]
using also the trivial inequality $|\beta_0| \geq \ominus |\alpha_0|$. Let $\gamma$ be an arbitrary element in the interval
\[ \{ x \in \mathbb{T}_\pm \mid \underline{\beta} \geq x \geq \overline{\alpha} \} \neq \emptyset . \]

By checking all possibilities arising from the list in (28), we see that the element $\gamma$ fulfills $b \ominus \gamma \Vdash O$ and $\gamma \ominus a \Vdash O$ for all $a \in A, b \in B$. □

To deal with geometric objects in $\mathbb{T}_d^\pm$, we will use balanced numbers because this allows for explicit calculations in the semiring $S$. However, as we are only interested in the signed part of the sets. We provide a first tool to resolve balanced numbers in inequalities. While this is for strict inequalities, Proposition 5.15 provides a tool for the relation $\vdash$.

Lemma 4.11. For $a, b \in S$, we have an equivalence of
\begin{enumerate}
\item $a \ominus b \ominus a > \emptyset$
\item $a \ominus b > \emptyset$ and $\ominus b \ominus a > \emptyset$
\item For all $c \in [\ominus |a|, |a|] = U(a \ominus a)$, it holds $c \ominus b > \emptyset$.
\end{enumerate}

Proof. The condition (3) clearly implies (2), as the latter is just a special case. The implication from (2) to (1) follows by adding up the positive values in (2). For the direction from (1) to (3), we use that $a \ominus a$ is balanced and, hence, incomparable. Therefore, we have $b > |a|$. Because of $|a| = |\ominus a| = |a \ominus a|$, the claim follows. □

We are ready to define two important geometric objects in $\mathbb{T}_d^\pm$ associated with a matrix.
4.3. **Fourier-Motzkin.** We derive three versions of Fourier-Motzkin elimination, which will be useful for deriving further description of signed tropical convex sets in Section 5. As the elimination process produces balanced coefficients for the inequalities, it is convenient to have an elimination procedure which can directly deal with those (Theorem 4.12). We also need to derive explicit inequalities with signed coefficients (Theorem 4.12) to describe the dual convex hull in Section 5.1. The version with non-strict inequalities (Theorem 38) will be used in constructing an exterior description by closed tropical halfspaces (Theorem 5.12).

For a subset $M$ of $\mathbb{T}_d^d$, we define its coordinate projection $\rho_i(M)$ with $i \in [d]$ by

$$
\rho_i(M) = \{(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d) \mid (x_1, \ldots, x_d) \in M\}.
$$

To simplify notation, we set $i = d$ in the following. For a matrix $A = (a_{ij}) \in \mathbb{S}^{d \times n}$ let $J^+ = \{j \in [n] \mid a_{dj} > \mathbf{O}\}$, $J^- = \{j \in [n] \mid a_{dj} < \mathbf{O}\}$, $J^* = \{j \in [n] \mid a_{dj} \in \mathbb{T}_* \setminus \{\mathbf{O}\}\}$, $J^0 = \{j \in [n] \mid a_{dj} = \mathbf{O}\}$.

Furthermore, we define $T = (t_{j,p}) \in \{0, \mathbf{O}\}^{n \times (J^+ \cup J^*) \times (J^* \cup J^\prime)}$ as the incidence matrix

$$
t_{j,p} = \begin{cases} 
0 & j = k \text{ or } j = \ell \text{ for } p = (k, \ell) \in (J^+ \cup J^*) \times (J^* \cup J^\prime) \\
0 & j = p \text{ for } p \in J^0 \\
0 & \text{else}
\end{cases}.
$$

We denote by $A_{-d}$ the matrix obtained from $A$ by removing the last row.

4.3.1. *Strict inequalities.*

**Theorem 4.12** (Fourier–Motzkin for strict inequalities with balanced numbers). The $d$th coordinate projection of the open tropical cone $\rho_d(\text{sep}_+(A))$ for the matrix $A \in \mathbb{S}^{d \times n}$ is the open tropical cone $\text{sep}_+(A_{-d} \odot T)$.

**Proof.** We can assume that the absolute value of each entry in the $d$th row of $A$ is either $0$ or $\mathbf{O}$. This can be achieved by multiplying each column of $A$ indexed by $J^+ \cup J^\prime$ with the inverse of the entry in its $d$th row. The inequality $y^\top \odot A \succ \mathbf{O}$ remains valid in this transformation.

Using Lemma 4.11 and ordering the inequalities according to the partition from (30), we get the system

$$
y_d \oplus (y_1, \ldots, y_{d-1}) \odot (a_{1, j}, \ldots, a_{d-1, j})^\top \succ \mathbf{O} \text{ for } j \in J^+ \cup J^* \tag{32}
y \odot (y_1, \ldots, y_{d-1}) \odot (a_{1, j}, \ldots, a_{d-1, j})^\top \succ \mathbf{O} \text{ for } j \in J^- \cup J^\prime.
$$

Proposition 4.8 implies that (32) has a solution $(y_1, \ldots, y_{d-1}, y_d) \in \mathbb{T}_d^d$ if and only if $(y_1, \ldots, y_{d-1}) \odot A_{-d} \odot T \succ \mathbf{O}$.

We make more explicit what this version of tropical Fourier–Motzkin elimination means for a matrix $A = (a_{ij}) \in \mathbb{T}_d^{d \times n}$ of signed numbers.

Let $u, v \in \mathbb{T}_d^d$ be two points with $u_d > \mathbf{O}$ and $v_d < \mathbf{O}$. We define their **multi-valued cancellation** with respect to the $d$th entry. Setting

$$
\lambda_u = \ominus u_d^{-1} \ominus (u_d^{-1} \ominus v_d^{-1})^{-1} \text{ and } \lambda_v = u_d^{-1} \ominus (u_d^{-1} \ominus v_d^{-1})^{-1},
$$

(34)
we have $\lambda_u \oplus \lambda_v = 0$ and $\lambda_u, \lambda_v \geq \mathbf{0}$. We define $\tilde{u} = \lambda_u \odot u$, $\tilde{v} = \lambda_v \odot v$ and $I = \{i \in [d-1] \mid \tilde{u}_i = \ominus \tilde{v}_i\}$. By construction, the $d$th entries fulfill $\tilde{u}_d = \ominus \tilde{v}_d$. We set

$$z^1_i = \begin{cases} \tilde{u}_i + \tilde{v}_i & \text{for } i \notin I \\ \tilde{u}_i & \text{for } i \in I \end{cases} \quad \text{and} \quad z^2_i = \begin{cases} \tilde{u}_i + \tilde{v}_i & \text{for } i \notin I \\ \tilde{v}_i & \text{for } i \in I \end{cases}$$

and define $\zeta_d(u, v) = \{z^1, z^2\}$. Note that $z^1$ and $z^2$ only differ by the signs of coordinates indexed by $I$, and the set $\zeta_d(u, v)$ has only one element exactly if $I = \emptyset$.

From this, we form a new matrix $\zeta_d(A)$ using the definition of the index sets from (30), where $J^\perp$ is empty in this case. For each pair $(k, \ell) \in J^\perp \times J^\perp$, we introduce one or two columns, namely the element(s) in $\zeta_d(a_{*, k}, a_{*, \ell})$. The columns of $A$ indexed by $J^\perp$ are taken over into $\zeta_d A$.

**Theorem 4.13 (Fourier–Motzkin for strict inequalities with signed numbers).** The $d$th coordinate projection of the open tropical cone $\rho_d(\text{sep}_+(\zeta_d(A)))$ is the open tropical cone $\text{sep}_+(\zeta_d(A)_{-d})$.

**Proof.** We get the claim from Theorem 4.12 with Lemma 4.11. □

We can define the matrix $\zeta_d(A)$ for an arbitrary row $i \in [d]$, not only for the $d$th row.

**Definition 4.14.** The matrix $\zeta_i(A)$ is the $i$th elimination matrix of $A$.

**Remark 4.15.** The crucial difference to classical Fourier–Motzkin elimination happens in the treatment of balanced numbers occurring in the calculation. While classically other variables could also just be eliminated, here we have to deal with their balanced left-overs. For strict inequalities, Lemma 4.11 provides a tool to resolve them by introducing two inequalities instead of one. We will see how to resolve them for non-strict inequalities in Proposition 5.15.

**Remark 4.16.** The classical technique of Fourier-Motzkin for polytopes, see e.g. [16] has already been successfully adapted to tropical linear inequality systems in [7]. In the latter work, an algorithmic scheme to determine a projection of a tropical inequality system is described. In our Theorem 4.13, we do not have the non-negatively constrained variables but allow arbitrary elements of $\mathbb{T}_\pm$. Classically, one can represent an unconstrained variable as the difference of a pair of non-negative variables. Applying this technique to a system of the form $y^\top \odot A > \mathbf{0}$ with unconstrained $y \in \mathbb{T}_\pm^d$ for a matrix $A \in \mathbb{T}_\pm^{m \times n}$ yields the system

$$(u^\top, v^\top) \odot \left( \begin{array}{c} A \\ \ominus A \end{array} \right) > \mathbf{0} \text{ with } u, v \in \mathbb{T}_\pm^d.$$

Reordering terms with coefficients in $\mathbb{T}_{<0}$ to the other side of the inequality yields a system which allows to apply [7, Theorem 11]. However, the differences of non-negative variables are harder to resolve as there is no cancellation but it results in balanced entries. This makes our direct approach more tractable for unconstrained variables. Furthermore, we are also interested in the structure of the resulting inequalities in Section 5, whence our approach is more suitable for this setting.

The elimination procedure derived in the last section allows to prove the desired separation in $\mathbb{T}_\pm^d$. 


Proof of Theorem 4.6. At first, we show the claim for \( d = 1 \). The set \( \text{sep}_+(A) \) is non-empty if and only if either all entries are positive or all entries are negative. Otherwise, we can select a balanced entry or a pair of entries with opposite sign by multiplication from the right.

If \( \text{sep}_+(A) \) is not empty, then Theorem 4.4 tells us that \( \text{ker}_+(A) \) is empty. So, we assume that \( \text{sep}_+(A) \) is empty. As the scaling of the columns of \( A \) does not change the sets \( \text{sep}_+(A) \) or \( \text{ker}_+(A) \), we can assume that the absolute value of the entries in the last row of \( A \) is 0 or \( O \). Let \( T \) be the matrix from (31). Then Theorem 4.12 shows that \( \text{sep}_+(A_{-d} \odot T) \) is empty. By induction, there is an element \( z \) in \( \text{ker}_+(A_{-d} \odot T) \). We show that \( T \odot z \in \text{ker}_+(A) \). By definition of \( T \), the elements in the \( d \)th row of \( A \odot T \) are all in \( T \). This implies that the \( d \)th row of \( A \odot T \odot z \) is in \( T \). Additionally, the choice of \( z \) yields \( A_{-d} \odot T \odot z \in T^{d-1}_p \). This finishes the proof. \( \square \)

4.3.2. Non-strict inequalities. Next, we derive the analogous statement to Theorem 4.13 for the relation ‘\( = \)’ instead of ‘\( > \)’.

**Theorem 4.17** (Fourier–Motzkin for non-strict inequalities with signed numbers). The \( d \)th coordinate projection of the set

\[
\{ y \in T^d_\pm \mid (0, y^\top) \odot A \models O \}
\]

for \( A \in T^{(d+1) \times n}_\pm \) is the set

\[
\{ z \in T^{d-1}_\pm \mid (0, z^\top) \odot A_{-d} \odot T \models O \}.
\]

Proof. We can assume that the absolute value of each entry in the \( d \)th row of \( A \) is either 0 or \( O \). This can be achieved by multiplying each column of \( A \) indexed by \( J^+ \cup J^- \) with the inverse of the entry in its \( d \)th row. The inequality \( y^\top \odot A \models O \) remains valid in this transformation.

Ordering the inequalities according to the distinction from (30), we get the system

\[
y_d \oplus (0, y_1, \ldots, y_{d-1}) \odot (a_{0,j}, a_{1,j}, \ldots, a_{d-1,j})^\top \models O \text{ for } j \in J^+
\]

\[
y_d \ominus (0, y_1, \ldots, y_{d-1}) \odot (a_{0,j}, a_{1,j}, \ldots, a_{d-1,j})^\top \models O \text{ for } j \in J^-
\]

Proposition 4.10 implies that (39) has a solution \((0, y_1, \ldots, y_{d-1}, y_d) \in T^d_\pm \) if and only if

\[
(0, y_1, \ldots, y_{d-1}) \odot A_{-d} \odot T \models O
\]

has a solution \((0, y_1, \ldots, y_{d-1}) \in T^{d-1}_\pm \), where \( T \) is the matrix defined in (31) with \( d \) replaced by \( d + 1 \) and \( J^\bullet = \emptyset \). \( \square \)

**Example 4.18.** We will see how to obtain an exterior description by closed halfspaces in Theorem 5.12. To determine the exterior description of the one dimensional line segment from \( \ominus 0 \) to 1 in \( T_\pm \), one can eliminate \( x_1 \) and \( x_2 \) from the system

\[
\ominus 0 \odot x_1 \ominus 1 \odot x_2 \ominus z \models O
\]

\[
0 \odot x_1 \ominus 1 \odot x_2 \ominus z \models O
\]

\[
0 \odot x_1 \ominus 0 \odot x_2 \ominus 0 \models O
\]

\[
\ominus 0 \odot x_1 \ominus 0 \odot x_2 \ominus 0 \models O
\]

\[
0 \odot x_1 \models O
\]

\[
0 \odot x_2 \models O
\]
Eliminating $x_1$ yields

\[
\begin{align*}
\bullet 1 \odot x_2 \bullet z \models O & \quad \text{from (41a)\&(41b)} \quad (42a) \\
1 \odot x_2 \odot z \odot 0 \models O & \quad \text{from (41a)\&(41c)} \quad (42b) \\
1 \odot x_2 \odot z \models O & \quad \text{from (41a)\&(41e)} \quad (42c) \\
\ominus 1 \odot x_2 \odot z \odot 0 \models O & \quad \text{from (41d)\&(41b)} \quad (42d) \\
\bullet 0 \odot x_2 \bullet 0 \models O & \quad \text{from (41d)\&(41c)} \quad (42e) \\
\ominus 0 \odot x_2 \odot 0 \models O & \quad \text{from (41d)\&(41e)} \quad (42f) \\
0 \odot x_2 \models O & \quad \text{from (41f)} \quad (42g)
\end{align*}
\]

From further elimination of $x_2$ we get by ignoring redundant inequalities of (42)

\[
\begin{align*}
\bullet 0 \odot z \odot 0 \models O & \quad \text{from (42b)\&(42d)} \quad (43a) \\
\ominus (1) \odot z \odot 0 \models O & \quad \text{from (42b)\&(42f)} \quad (43b) \\
(1) \odot z \odot (1) \models O & \quad \text{from (42g)\&(42d)} \quad (43c) \\
0 \models O & \quad \text{from (42g)\&(42f)} \quad (43d)
\end{align*}
\]

This yields the exterior description $z \models \ominus 0$ and $z \models \ominus 1$.

5. Orthants and halfspaces

As the intersection of a signed tropically convex set with an orthant is just a tropically convex set over $T_{\max}$, this allows to study signed tropical convex sets through the existing theory of unsigned tropically convex sets. The proofs are based on the duality between the non-negative kernels and open tropical cones. We then use the separation results and Fourier-Motzkin elimination to describe signed tropically convex sets as intersections of tropical halfspaces.

5.1. Connection with tropical convexity in $T_{\max}$. The next construction connects the signed tropical convex hull with the hull (12) in each orthant of $T_{d,\pm}$.

5.1.1. Generators in each orthant. Fix a matrix $A \in T_{d,\pm}^{d \times n}$ and a coordinate hyperplane $H_i := \{ x \in T_{d,\pm}^{d} \mid x_i = O \}$. Recall the definition of the $i$th elimination matrix $\zeta_i(A)$ of $A$ from Definition 4.14.

**Proposition 5.1.** The intersection $tconv(A) \cap H_i$ is generated by $\zeta_i(A)$.

**Proof.** Using the definition (35) and the fact that $\ominus |z|, |z| \in U(z)$ for $z \in T_\circ$, the inclusion $\zeta_i(A) \subseteq tconv(A) \cap H_i$ follows from Definition 3.1. Example 3.8 and Proposition 3.4.a imply that $tconv(\zeta_i(A)) \subseteq tconv(A) \cap H_i$.

For the other inclusion, assume that there is a point $z \in tconv(A)$ with $z_i = O$ which is not contained in $tconv(\zeta_i(A))$. By Proposition 4.1, this implies that

$$\ker_+ \left( \begin{array}{cc} 0 & \ominus 0 \\ \zeta_i(A) & \ominus z \end{array} \right) = \emptyset.$$

The Farkas Lemma 4.6 implies that

$$\text{sep}_+ \left( \begin{array}{cc} 0 & \ominus 0 \\ \zeta_i(A) & \ominus z \end{array} \right) \neq \emptyset.$$
Furthermore, we get
\[
\text{sep}_+ \left( \zeta, \begin{pmatrix} 0 & 0 & \vdots & 0 \\ A & \odot z \end{pmatrix} \right) \neq \emptyset
\]
as, because of \( z_i = \mathcal{O} \), the last column is unchanged by \( \zeta \) and the first row remains the same due to the definition of \( \lambda_u \) and \( \lambda_v \) for (35).

However, by Theorem 4.13, then also
\[
\text{sep}_+ \left( \begin{pmatrix} 0 & 0 & \vdots & 0 \\ A & \odot z \end{pmatrix} \right) \neq \emptyset.
\]

Using again the Farkas Lemma 4.6, this implies \( z \notin \text{tconv}(A) \), a contradiction. \( \square \)

There is a natural bijection between \( \Delta_2 = \{ (\nu, \mu) \in \mathbb{T}^2_{\geq 0} \mid \max(\nu, \mu) = 0 \} \) and \( \overline{\mathbb{T}} = \mathbb{R} \cup \{-\infty, \infty\} \) given by
\[
(\nu, \mu) \mapsto \mu - \nu.
\]

We denote the inverse image of an element \( \eta \in \overline{\mathbb{T}} \) with respect to this map by \( \Psi(\eta) \). This leads to a parametrization of a tropical line segment for \( a, b \in \mathbb{T}^d_{\geq 0} \) via \( \text{tconv}(a, b) = \{ L_\eta(a, b) \mid \eta \in \overline{\mathbb{T}} \} \) where \( L_\eta(a, b) = \Psi(\eta) \odot a \oplus \Psi(\eta) \odot b \). Note that \( L_{-\infty}(a, b) = a \) and \( L_\infty(a, b) = b \).

**Proposition 5.2.** The intersection \( \text{tconv}(A) \cap \mathbb{T}^d_{\geq 0} \) is generated by
\[
\left( A \cup \bigcup_{i \in [d]} (\text{tconv}(A) \cap H_i) \right) \cap \mathbb{T}^d_{\geq 0}.
\]

**Proof.** Let \( z \in \text{tconv}(A) \cap \mathbb{T}^d_{\geq 0} \) be an element of \( \mathcal{U}(A \odot \lambda) \) with \( \lambda \in \Delta_d \). We consider the tropical line segments from \( z \) to the columns of \( A \). For a fixed column \( a^{(j)} \) of \( A \) with \( a^{(j)} \notin \mathbb{T}^d_{\geq 0} \), there is a minimal \( \eta \in \overline{\mathbb{T}} \) such that a component of \( L_\eta(z, a^{(j)}) \) is balanced.

**Intermediate claim I:** All entries of \( L_\eta(z, a^{(j)}) \) are either balanced or in \( \mathbb{T}_{\geq 0} \).

For an arbitrary row \( i \in [d] \), the expression \( \Psi(\eta) \odot z_i \odot \Psi(\eta) \odot a^{(j)}_i \) is in \( \mathbb{T}_{\geq 0} \) for \( \eta = -\infty \). The claim now follows from the piecewise definition of the addition in terms of the absolute values.

Using Proposition 3.6, we see that the point \( b^{(j)} \) obtained from \( L_\eta(z, a^{(j)}) \) by replacing all balanced entries with \( \mathcal{O} \) is in \( \text{tconv}(A) \cap \mathbb{T}^d_{\geq 0} \). For \( a^{(j)} \in \mathbb{T}^d_{\geq 0} \) we set \( b^{(j)} = a^{(j)} \).

**Intermediate claim II:** The point \( z \) is in the convex hull of \( \{ b^{(j)} \mid j \in [n] \} \). It is enough to show that
\[
\ker_+ \left( \begin{pmatrix} 0 & \vdots & 0 \\ a^{(1)} & \vdots & a^{(n-1)} & b^{(n)} \end{pmatrix} \oplus \begin{pmatrix} 0 & \vdots & 0 \odot z \end{pmatrix} \right) \neq \emptyset
\]
because than we can iteratively replace the columns \( a^{(i)} \) by \( b^{(i)} \). Let \( b^{(1)} \in \nu \ominus z \oplus \mu \ominus a^{(1)} \). Pick an element \( x \) scaled such that \( x_1 = \mu \). Then
\[
\left( \begin{pmatrix} \nu & \mathcal{O} & \vdots & \mathcal{O} & \mu \ominus \nu \\ \nu \ominus z & \mathcal{O} & \vdots & \mathcal{O} \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \vdots & 0 \odot \mathcal{O} \end{pmatrix} \right) \odot x
is in $T^d_*$. Therefore $(0, \ldots, 0)$ is in the non-negative kernel of
\[
\begin{pmatrix}
\nu \odot x_1 & x_2 & \cdots & x_n & \ominus(x_n+1) \\
\nu \odot z \odot x_1 \odot a(1) & x_2 \odot a(2) & \cdots & x_n \odot a(n) & \ominus(x_n+1) \\
0 & x_2 \odot a(2) & \cdots & x_n \odot a(n) & \ominus(x_n+1)
\end{pmatrix}
\]

(44)

For fixed $i \in [d]$, if $\nu \odot z_i \odot \mu \odot a_i^{(1)}$ is not balanced or the maximum absolute value is attained somewhere else in the row, we can replace it by $b_i^{(1)}$ and $(0, \ldots, 0)$ is still in the non-negative kernel.

Otherwise, $\ominus\nu \odot z_i = \mu \odot a_i^{(1)}$ and $\nu \oplus x_{n+1} = \nu$. But $(0, \ldots, 0)$ is also in the non-negative kernel of
\[
\begin{pmatrix}
\mu \odot a^{(1)} & x_2 & \cdots & x_n & \ominus x_{n+1} \\
\mu \odot a^{(1)} & x_2 \odot a^{(2)} & \cdots & x_n \odot a^{(n)} & x_{n+1} \ominus (\ominus z) \\
0 & a^{(2)} & \cdots & a^{(n)} & (\ominus z)
\end{pmatrix}
\]

Since $\mu \odot a_i^{(1)} = \ominus\nu \odot z_i$ has the same sign as $\ominus x_{n+1} \odot z_i$ and we have $\nu \geq x_{n+1}$, there has to be an $\ell \in [n]$ such that $x_\ell \odot a_i^{(\ell)} = \ominus\mu \odot a_i^{(1)} = \nu \odot z_i$. Therefore, we can replace $\nu \odot z_i \oplus x_1 \odot a(1)$ by $O$ and $(0, \ldots, 0)$ remains in the non-negative kernel.

Therefore, $(0, x_2, \ldots, x_n, (\nu \oplus x_{n+1}))$ is in the non-negative kernel of
\[
\begin{pmatrix}
0 & 0 & \cdots & 0 & \ominus 0 \\
(\nu \oplus x_{n+1}) & 0 & \cdots & 0 & \ominus z
\end{pmatrix}
\]

This finishes the proof of claim II.

\[\square\]

We fix a (finite) set $M \subseteq T^d_*$ and interpret $M$ as a matrix whose columns list the points. Each permutation $\sigma$ in the symmetric group on $d$ elements $S_d$ gives rise to a sequence of matrices $\zeta_{\sigma(1)}(M), \zeta_{\sigma(2)}(\zeta_{\sigma(1)}(M))$ until $\zeta_{\sigma(d)}(\zeta_{\sigma(1)}(\zeta_{\sigma(2)}(\cdots(M)\cdots))$. We denote the concatenation of these $d$ matrices by $\zeta_{\sigma}(M)$. The concatenation of the matrices for all $\sigma \in S_d$ forms a matrix $\zeta(M)$.

**Theorem 5.3.** The convex hull $t\text{conv}(M)$ of $M$ is the union
\[
\bigcup_{O \text{ closed orthant of } T^d_*} t\text{conv}(O \cap \zeta(M)).
\]

**Proof.** By Proposition 5.2, we know that $t\text{conv}(M)$ is generated by the projections on the boundary of the orthants and the generators in the interior. Iteratively applying Proposition 5.1 yields the claim. \[\square\]

**Remark 5.4.** For sufficiently generic matrices, one can use the cancellative sum from Remark 3.11 to determine the tropical convex hull in each orthant (and hence, by Theorem 5.3, the whole tropical convex hull. If their are no antipodal points, then the multi-valued cancellation, see (34), which was used for the definition of $\zeta(M)$, is just single-valued. Hence, the iterative construction of a single intersection point with a coordinate hyperplane suffices.

**Example 5.5.** Looking at the points from Example 3.2, we see how we can determine the tropical convex hull of $\{(3,3), (\ominus1,0), (\ominus4,2)\}$. It is the union of the tropical convex hulls $t\text{conv}(\{(3,3), (O,3), (O,1)\})$, $t\text{conv}(\{(\ominus1,0), (O,1), (O,3), (\ominus4,0)\})$, $t\text{conv}(\{(\ominus1,0), (\ominus4,2), (\ominus1,0), (\ominus4,0)\})$.
On the other hand, to get the tropical convex hull \( \text{tconv}((0,0), (\ominus 2, \ominus 2)) \) in Figure 2a, one needs actual multi-valued cancellation.

We get \( \zeta_1 \left( \begin{pmatrix} 0 \\ 0 \\ \ominus 2 \\ \ominus 2 \end{pmatrix} \right) = \left\{ \begin{pmatrix} 0 \\ 0 \\ \ominus 2 \\ \ominus 2 \end{pmatrix} \right\} \) and \( \zeta_2 \left( \begin{pmatrix} 0 \\ 0 \\ \ominus 2 \\ \ominus 2 \end{pmatrix} \right) = \left\{ \begin{pmatrix} \ominus 2 \\ 0 \\ \ominus 2 \\ 0 \end{pmatrix} \right\} \). From applying \( \zeta_1(\zeta_2(.)) \), we additionally obtain \( \{(0,0)\} \). For the positive orthant, this yields the generators \( \{(0,0), (0,-2), (-2,0), (0,0)\} \). The other orthants are derived analogously.

**Corollary 5.6.** Tropically convex sets are contractible.

**Proof.** The space \( \mathbb{T}_d^+ \) inherits the topology of \( \mathbb{R}^d \) via the map \( x \mapsto \text{sgn}(x) \log(|x|) \), where the origin is mapped to the all-\( \mathbb{O} \)-point. As tropically convex sets in all orthants are contractible [17, Theorem 2], we can contract to the boundary of the orthants. The claim follows by induction on the dimension \( d \). \( \square \)

For the definition of the covector decomposition in \((\mathbb{R} \cup \{-\infty\})^d \) and its connection with regular subdivisions of \( \Delta_d \times \Delta_n \) we refer the reader to [25].

**Corollary 5.7** (Covector decomposition). The combinatorics of the tropically convex hull of a matrix \( A \in \mathbb{T}^{d \times n}_\pm \) can be described by \( 2^d \) regular subdivisions of \( \Delta_d \times \Delta_n \).

5.2. **Description by halfspaces.** An important property of classical polytopes is the duality between the representation as convex hull and as intersection of finitely many halfspaces. This is more subtle for tropical polytopes over \( \mathbb{T}_\pm \). While we establish a description as intersection of open tropical halfspaces (Theorem 5.9) containing a set of points, we need additional properties to formulate a Minkowski-Weyl theorem (Theorem 5.12).

**Definition 5.8** (Outer hull). The outer hull of a set \( M \subseteq \mathbb{T}_d^+ \) is the intersection of its containing open halfspaces

\[
\bigcap_{M \subseteq H^+(a)} H^+(a) .
\]  

(46)

**Theorem 5.9.** The outer and the inner hull of a matrix \( A \in \mathbb{T}_\pm^{d \times n} \) coincide, i.e., \( \text{tconv}(A) = \bigcap_{A \subseteq H^+(v)} H^+(v) \), where we identify \( A \) with the set of its columns.

**Proof.** The inclusion \( \text{tconv}(A) \subseteq \bigcap_{A \subseteq H^+(v)} H^+(v) \) follows by combining Proposition 3.4.a and Example 3.9.

For the other inclusion, assume that there is a point

\[
z \in \bigcap_{A \subseteq H^+(v)} H^+(v) \setminus \text{tconv}(A) .
\]

By Proposition 4.1, we get that \( \ker_+(B) = \emptyset \), where

\[
B = \begin{pmatrix} 0 & \ominus 0 \\ A & \ominus z \end{pmatrix} .
\]

Theorem 4.6 implies that \( \text{sep}_+(B) \neq \emptyset \). Hence, there is a separator \((u_0, \bar{u}) = (u_0, u_1, \ldots, u_d) \in \mathbb{T}_\pm^{d+1} \) with \( u_0 \ominus \bar{u}^\top \ominus a^{(j)} > 0 \) for all columns \( a^{(j)} \) of \( A \), and \( \ominus u_0 \ominus \ominus \bar{u}^\top \ominus z > \mathbb{O} \Leftrightarrow u_0 \ominus \bar{u}^\top \ominus z < \mathbb{O} \). This means that the columns of \( A \) lie in the halfspace \( H^+(u) \) but \( z \) does not. This contradicts the choice of \( z \) in \( \bigcap_{A \subseteq H^+(v)} H^+(v) \). \( \square \)
For a vector \((a_0, a_1, \ldots, a_d) \in T^{d+1}_\pm\), we define a *closed (signed) tropical halfspace* by
\[
\mathcal{H}^+(a) = \left\{ x \in T^d_\pm \mid a \odot \begin{pmatrix} 0 \\ x \end{pmatrix} \in T_{\geq 0} \cup T_\bullet \right\}.
\] (47)

**Remark 5.10.** Closed tropical halfspaces are not as suitable for the hull construction in Definition 5.8 as open tropical halfspaces. The inner hull of \(M = \{(\odot 1, 1), (1, \odot 1)\}\) should contain the origin \(O\). However, the closed halfspace \(x_1 \oplus x_2 \geq \odot 0\) contains those two points but not the origin. Taking the analogous intersection as in (46) with closed tropical halfspaces for \(M\) yields again \(M\). Note that this is the same as the intersection of the balanced image \(M \odot \Delta_2\) with \(T^2_\pm\).

Example 3.8 shows that such a closed signed tropical halfspace is in general not tropically convex. However, a finite intersection of such halfspaces can be tropically convex, as Figure 6 shows.

**Lemma 5.11.** *The closed signed tropical halfspace \(\mathcal{H}^+(a)\) is the topological closure of the open signed halfspace \(\mathcal{H}^+(a)\).*

**Proof.** Let \(z \in T^d_{\geq 0}\) be an element of \(\mathcal{H}^+(a) \setminus \mathcal{H}^+(a)\). Set
\[
J = \text{argmax} \left\{ |c_j \odot z_j| \mid j \in [d]_0 \right\},
\]
where \(z_0 = 0\), and let \(\ell \in J\) with \(c_\ell \odot z_\ell > O\). Denoting the \(k\)th unit vector \((0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{R}^d\) by \(e_k\) for \(k \in [d]\) and \(e_0 = (-1, \ldots, -1)\), we define a sequence
\[
y^{(m)} = z + \frac{1}{m} \cdot e_\ell.
\]
The sequence converges to \(z\) but each element of the sequence is an element of \(\mathcal{H}^+(a)\). \(\square\)

Observe that the former statement is wrong for inequalities with balanced numbers as coefficients.

In the remaining part of the section, we will prove a Minkowski-Weyl theorem for tropical polytopes over \(T^d_\pm\). From Example 3.8, we saw that closed tropical halfspaces are not tropically convex. Hence, one has to adapt the condition in the characterization of finite intersections of closed tropical halfspaces.
Theorem 5.12. For each finite set $V \subset \mathbb{T}_d^+$, there are finitely many closed tropical halfspaces $H$ such that \( \text{tconv}(V) \) is the intersection of the halfspaces.

For each finite set $H$ of closed halfspaces, whose intersection $M$ is tropically convex, there is a finite set of points $V \in \mathbb{T}_d^+$ such that $M = \text{tconv}(V)$.

Corollary 5.13. A tropically convex set is the intersection of its containing closed halfspaces.

Remark 5.14. The crucial difference with Theorem 5.9 is that for open tropical halfspaces the generators are enough, while for closed tropical halfspaces, we have to take the whole set into account.

We will use the Fourier-Motzkin version for the relation ‘‘≡’’ (Theorem 4.17) to deduce an exterior description of a tropical polytope. However, the system describing the projection in (38) may contain balanced coefficients. We address this issue in the next statement. It is an existence argument statement that a balanced coefficient can be replaced by a signed coefficient, see also Figure 7.

Proposition 5.15 (Resolving balanced coefficients). Let $M$ be a tropically convex set and $c = (c_0, c_1, \ldots, c_d) \in \mathbb{S}^d$ with $c_i \in \mathbb{T}_\bullet$ for some $i \in [d]_0$ such that

$$M \subseteq H(c) = \left\{ x \in \mathbb{T}_d^+ \mid c \odot \begin{pmatrix} 0 \\ x \end{pmatrix} \models \mathbf{O} \right\}.$$  

Then there is an element $b \in U(c_i)$ such that $M$ is contained in

$$\left\{ x \in \mathbb{T}_d^+ \mid (c_0, \ldots, c_{i-1}, b, c_{i+1}, \ldots, c_d) \odot \begin{pmatrix} 0 \\ x \end{pmatrix} \models \mathbf{O} \right\}.$$  

Proof. If $i = 0$, then we can set $b = |c_i|$.

Assuming without loss of generality that $i = d$, we set $c_{-d} = (c_0, c_1, \ldots, c_{d-1})$.

Fix $u \in \mathbb{T}_d^{d-1}$, $v \in \mathbb{T}_d^+$ such that $(u, v) \in M$. We define $w = w(u) = c_{-d} \odot \begin{pmatrix} 0 \\ u \end{pmatrix}$.

If $v \in \mathbb{T}_< \mathbf{O}$, then

$$\lambda(u, v) = \arg \max \{ \lambda \in U(c_d) \mid w \odot \lambda \odot v \models \mathbf{O} \}.$$  

Let

$$\lambda = \min \{ \lambda(u, v) \mid (u, v) \in M, v \leq \mathbf{O} \}.$$  

If $v \in \mathbb{T}_> \mathbf{O}$, then

$$\lambda(u, v) = \arg \min \{ \lambda \in U(c_d) \mid w \odot \lambda \odot v \models \mathbf{O} \}.$$  

Let

$$\lambda = \max \{ \lambda(u, v) \mid (u, v) \in M, v > \mathbf{O} \}.$$  

Figure 6. Exterior description of $\text{tconv}((\ominus 1, \ominus 1), (2, 2))$ by closed halfspaces.
We derive a contradiction to the convexity of $M$, if $\lambda > \bar{\lambda}$.

Let $(p,q)$ and $(r,s)$ attain $\lambda$ and $\bar{\lambda}$, respectively. In particular, we have $q < 0$ and $s > 0$. The inequality $\bar{\lambda} > \lambda$ implies that $\bar{\lambda} > \circ|c_d|$ and that $\lambda < |c_d|$. We can assume that $w(p)$ and $w(r)$ are signed numbers, as we otherwise can replace them by their absolute value without changing the admissible values of $\lambda$ in (48) and (50).

Now, the construction of $\lambda$ in (48) implies that $w(p) = \ominus \lambda \odot q$ and (50) yields $w(r) = \ominus \lambda \odot s$. This implies $w(r) \odot s^{-1} = \ominus \lambda < \ominus \lambda = w(p) \odot q^{-1}$. \hfill (52)

We consider the point in the convex combination of $(p,q)$ and $(r,s)$ given by $z = (\ominus q^{-1} \odot s^{-1})^{-1} \odot (\ominus q^{-1} \odot p \odot s^{-1} \odot r, 0)$. Then

$$c \odot \begin{pmatrix} 0 \\ z \end{pmatrix} = c \odot \left( (\ominus q^{-1} \odot s^{-1})^{-1} \odot (\ominus q^{-1} \odot p \odot s^{-1} \odot r) \right) =$$

$$= (\ominus q^{-1} \odot s^{-1})^{-1} \odot \left( \ominus q^{-1} \odot c_d \odot \begin{pmatrix} 0 \\ 0 \\ 0 \\ O \end{pmatrix} \odot s^{-1} \odot c_d \odot \begin{pmatrix} 0 \\ r \end{pmatrix} \right) =$$

$$= (\ominus q^{-1} \odot s^{-1})^{-1} \odot \left( \ominus q^{-1} \odot w(p) \odot s^{-1} \odot w(r) \right)$$

By (52), this implies $c \odot \begin{pmatrix} 0 \\ z \end{pmatrix} < 0$. As $M$ is convex, this yields $z \in M \subseteq H(c)$, a contradiction.

So, we can conclude that $\lambda \leq \bar{\lambda}$. Fix any $b$ in the interval $[\lambda, \bar{\lambda}] \neq \emptyset$. Let $(u,v) \in M$ with $v < 0$. Then

$$(c_d, b) \odot \begin{pmatrix} 0 \\ u \\ v \end{pmatrix} = w(u) \odot b \odot v \equiv w(u) \odot \lambda \odot v \equiv w(u) \odot \lambda(u,v) \odot v \equiv O,$$

where we use $b \leq \lambda$. (49) and Lemma 2.5.d. The proof for $v > 0$ goes analogously. \hfill $\Box$

*Example 5.16.* A pathological example for the last statement arises from resolving the tautological relation $\bullet (-1) \odot x \oplus \bullet (-1) \odot y \oplus \bullet 0 \equiv O$. One obtains the chain of relations

$\bullet (-1) \odot x \oplus \bullet (\ominus 1) \odot y \oplus \bullet 0 \equiv O \Leftrightarrow$

$\bullet (-1) \odot x \oplus \bullet (-1) \odot y \oplus \bullet 0 \equiv O \Leftrightarrow$

$\bullet (\ominus 1) \odot x \odot O \odot y \oplus \bullet 0 \equiv O \Leftrightarrow$

$O \odot x \odot O \odot y \oplus \bullet 0 \equiv O$.

*Remark 5.17.* Indeed, any value of $b \in U(c_d)$ in the former proof could occur as Figure 7 shows. Any part of the dashed line without opposite (with respect to the origin) points is a tropical line segment.
Figure 7. Finding a containing halfspace without balanced coefficients

Example 5.18. The shaded area in Figure 7 shows the feasible region
\[
\{(x, y) \in T^2_\pm \mid \ominus (-1) \odot x_1 \oplus (\bullet 2) \odot x_2 \oplus 0 \not\triangleright O\}.
\]
The red line marks the inequality \(\ominus (-1) \odot x_1 \oplus 2 \odot x_2 \oplus 0 \not\triangleleft O\), while the green line marks the inequality \(\ominus (-1) \odot x_1 \oplus (\ominus 2) \odot x_2 \oplus 0 \not\triangleleft O\). The yellow line corresponds to the inequality \(\ominus (-1) \odot x_1 \oplus 0 \not\triangleleft O\). The blue dashed line interpolates between these three possible extreme closed halfspaces contained in the feasible region.

Our proof of Theorem 5.12 is based on eliminating variables from the canonical exterior description (13). For using those halfspaces, we need to show the additional requirement of tropical convexity for their intersection.

Lemma 5.19. The set
\[
\{(x, z) \in T^{n+d}_\pm \mid A \odot x \preceq z, x \geq O\}
\]
is tropically convex.

Proof. It is enough to show that, for fixed \(a \in T^n_\pm\), the set
\[
H = \{(x, z) \in T^{n+1}_\pm \mid a \odot x \preceq z, x \geq O\}
\]
is tropically convex, then the claim follows from Proposition 3.4(a). Let \((p, q), (r, s) \in H\), and \(\lambda \in T_{\geq 0}, \lambda \leq 0\). We need to show that \(\mathcal{U}(p \oplus \lambda \odot r, q \oplus \lambda \odot s) \subseteq H\).

Note that since \(p, r \geq O\), we have \(p \oplus \lambda \odot r \in T_\pm\) and therefore \(\mathcal{U}(p \oplus \lambda \odot r) = \{p \oplus \lambda \odot r\}\). By Lemma 2.4(b), we have that \(q \in \mathcal{U}(a \odot p)\) and \(s \in \mathcal{U}(a \odot r)\). Using Lemma 3.5(b), we see that
\[
\mathcal{U}(q \oplus \lambda \odot s) \subseteq \mathcal{U}((a \odot p) \oplus \lambda \odot (a \odot q)) = \mathcal{U}(a \odot (p \oplus \lambda \odot r)),
\]
completing the proof. \(\square\)

Proof of Theorem 5.12. Equation (13) provides a description by halfspaces involving additional variables. The convex hull of \(V\) is the set of those \(z \in T^d_\pm\) for which
there is an \( x \in \mathbb{T}_n \) with

\[
\begin{pmatrix}
A & \ominus I_d & 0 \\
\ominus A & I_d & 0 \\
0 & O_d & \ominus 0 \\
\ominus 0 & O_d & 0 \\
I_n & O_d & 0
\end{pmatrix} \odot \begin{pmatrix}
x \\
z \\
0
\end{pmatrix} \equiv 0 ,
\]

(54)

where \( I \) is a tropical identity matrix with 0 on the diagonal and \( O \) elsewhere.

By Lemma 5.19 and the tropical convexity of \( \{ x \in \mathbb{T}_n \mid \bigoplus_{j \in [n]} x_j = 0, x \geq 0 \} \), the set of \( (x, z) \in \mathbb{T}_n^{n+d} \) fulfilling (54) is the intersection of tropically convex sets and, by Proposition 3.4.a, tropically convex as well.

Hence, we can use Theorem 4.17 to successively project out the \( x \)-variables. As the inequalities arising from a projection may contain balanced coefficients, we use Proposition 5.15 to replace them by signed coefficients. This yields a description of \( \text{tconv}(A) \) by non-strict inequalities.

If the intersection of closed tropical halfspaces \( M \) is tropically convex, then its intersection with any orthant is tropically convex. By the tropical Minkowski-Weyl theorem in the non-negative orthant [21], each of the parts in the orthants are finitely generated. Taking the tropical convex hull of the union of all these generators yields \( M \), as \( M \) is tropically convex.

\[
\square
\]

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References

1. M. Akian, G. Cohen, S. Gaubert, R. Nikoukhah, and J. P. Quadrat, Linear systems in (max, +) algebra, 29th IEEE Conference on Decision and Control, Dec 1990, pp. 151–156 vol.1.
2. Marianne Akian, Xavier Allamigeon, Stéphane Gaubert, and Sergei Sergeev, Tropical optimization with signs, in preparation.
3. Marianne Akian, Stéphane Gaubert, and Alexander Guterman, Tropical polyhedra are equivalent to mean payoff games, Internat. J. Algebra Comput. 22 (2012), no. 1, 1250001, 43. MR 2900854
4. Marianne Akian, Stéphane Gaubert, and Alexander Guterman, Tropical Cramer determinants revisited, Tropical and idempotent mathematics and applications, Contemp. Math., vol. 616, Amer. Math. Soc., Providence, RI, 2014, pp. 1–45. MR 3221324
5. Marianne Akian, Stéphane Gaubert, and Louis Rowen, Linear algebra over systems, in preparation.
6. Xavier Allamigeon, Pascal Benchimol, Stéphane Gaubert, and Michael Joswig, Tropicalizing the simplex algorithm, SIAM J. Discrete Math. 29 (2015), no. 2, 751–795. MR 3336300
7. Xavier Allamigeon, Uli Fahrenberg, Stéphane Gaubert, Ricardo D. Katz, and Axel Legay, Tropical Fourier-Motzkin elimination, with an application to real-time verification, International Journal of Algebra and Computation 24 (2014), no. 05, 569–607.
8. Xavier Allamigeon, Stéphane Gaubert, and Ricardo D. Katz, Tropical polar cones, hypergraph transversals, and mean payoff games, Linear Algebra Appl. 435 (2011), no. 7, 1549–1574. MR 2810655
9. Achim Bachem and Walter Kern, Linear programming duality: an introduction to oriented matroids., Berlin: Springer-Verlag, 1992 (English).
10. Egon Balas, *Disjunctive programming*, Ann. Discrete Math. 5 (1979), 3–51, Discrete optimization (Proc. Adv. Res. Inst. Discrete Optimization and Systems Appl., Banff, Alta., 1977), II. MR 558566
11. Walter Briec, *Some remarks on an idempotent and non-associative convex structure*, J. Convex Anal. 22 (2015), no. 1, 259–289. MR 3346189
12. Walter Briec and Charles Horvath, *E-convexity*, Optimization 53 (2004), no. 2, 103–127. MR 2058292
13. Peter Butkovič, Hans Schneider, and Sergei Sergeev, *Generators, extremals and bases of max cones*, Linear Algebra Appl. 421 (2007), no. 2-3, 394–406. MR 2294351
14. Sergei Chubanov, *A polynomial projection algorithm for linear feasibility problems*, Math. Program. 153 (2015), no. 2, Ser. A, 687–713. MR 3397077
15. Guy Cohen, Stéphane Gaubert, Jean-Pierre Quadrat, and Ivan Singer, *Max-plus convex sets and functions*, Idempotent mathematics and mathematical physics, Contemp. Math., vol. 377, Amer. Math. Soc., Providence, RI, 2005, pp. 105–129. MR 2149000
16. Michele Conforti, Gérard Cornuéjols, and Giacomo Zambelli, *Integer programming*, vol. 271, Cham: Springer, 2014 (English).
17. Mike Develin and Bernd Sturmfels, *Tropical convexity*, Doc. Math. 9 (2004), 1–27. MR 2054977
18. Mike Develin and Josephine Yu, *Tropical polytopes and cellular resolutions*, Experiment. Math. 16 (2007), no. 3, 277–291. MR 2367318 (2009j:52009)
19. Alex Fink and Felipe Rincón, *Stiefel tropical linear spaces*, J. Combin. Theory Ser. A 135 (2015), 291–331. MR 3366480
20. Stéphane Gaubert and Ricardo D. Katz, *The tropical analogue of polar cones*, Linear Algebra Appl. 431 (2009), no. 5-7, 608–625. MR 2535537
21. Stéphane Gaubert and Ricardo D. Katz, *Minimal half-spaces and external representation of tropical polyhedra*, J. Algebraic Combin. 33 (2011), no. 3, 325–348. MR 2772536
22. Dima Grigoriev and Vladimir V. Podolskii, *Tropical effective primary and dual Nullstellensätze*, Discrete Comput. Geom. 59 (2018), no. 3, 507–552. MR 3770201
23. V. A. Gurvich, A. V. Karzanov, and L. G. Khachiyan, *Cyclic games and finding minmax mean cycles in digraphs*, Zh. Vychisl. Mat. i Mat. Fiz. 28 (1988), no. 9, 1407–1417, 1439. MR 967535
24. Philipp Jell, Claus Scheiderer, and Josephine Yu, *Real tropicalization and analytification of semialgebraic sets*, 2018.
25. Michael Joswig and Georg Loho, *Weighted digraphs and tropical cones*, Linear Algebra Appl. 501 (2016), 304–343. MR 3485070
26. Rolf H. Möhring, Martin Skutella, and Frederik Stork, *Scheduling with AND/OR precedence constraints*, SIAM J. Comput. 33 (2004), no. 2, 393–415. MR 2048448
27. Louis Rowen, *Algebras with a negation map*, 2016.
28. Sven Scheure, *From parity and payoff games to linear programming*, Mathematical foundations of computer science 2009, Lecture Notes in Comput. Sci., vol. 5734, Springer, Berlin, 2009, pp. 675–686. MR 2539531
29. Oleg Viro, *Patchworking real algebraic varieties*, 2006.
30. , *Hyperfields for tropical geometry i. hyperfields and dequantization*, 2010.
31. Karel Zimmermann, *A general separation theorem in extremal algebras*, Ekonom.-Mat. Obzor 13 (1977), no. 2, 179–201. MR 0453607

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