Almost positively closed models

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Abstract

We introduce the notions of almost positively closed models and positive strong amalgamation property. We study the fundamental properties of these notions and develop some interactions between them.

Introduction

The positive model theory is concerned essentially with the study of h-inductive theories which are built without the use of the negation. Considering positive formulas instead of formulas, and homomorphisms instead of embeddings, the positive logic creates a new situations outside the framework of logic with negation. In this paper, we will explore one of the specific aspects of positive logic which embodies the notions of of algebraic closedness and strong amalgamation, and undertake to study some interactions between these new notions inspired directly or indirectly from the works P.Bacsish [5]. In the first section, after summarising the necessary background of positive model theory, we introduce the general form of symmetric and asymmetric amalgamations. We show that the model-completeness of an h-inductive theory can be characterized by a form of symmetric amalgamation. The second section is devoted to the notions of almost positively closed models and a special class of positive formulas called \((A,T)\)-closed formulas. Note that the terminology "closed formula" here has different meaning of the notion of formulas without free variables. We analyse the class of almost positively closed model and present a characterization through some properties of the class of the \((A,T)\)-closed formulas. In the third section, we introduce the notions of positive strong amalgamation and h-strong amalgamation properties. We show that the class of almost positively closed has the positive strong amalgamation property. Further we give a syntactic characterization of positive strong amalgamation bases.

1 Positive model theory

1.1 Basic definitions and notations

In this subsection we briefly introduce the basic terminology related to the positive logic. For more details, The reader is referred to [1], [3].
Let $L$ be a first order language that contains the symbol of equality and a constant $\bot$ denoting the antilogy. The quantifier-free positive formulas are obtained from atomic formulas using the connectives $\land$ and $\lor$. The positive formulas are build from quantifier-free positive formulas using the logical operators $\land$, $\lor$ and $\exists$ respectively. Eventually, the positive formulas are of the form $\exists \bar{y} \phi(\bar{x}, \bar{y})$, where $\phi(\bar{x}, \bar{y})$ is quantifier-free formula. The variables $\bar{x}$ are said to be free. And a sentence is a formula without free variables.

A sentence is said to be h-inductive, if it is a finite conjunction of sentences of the form:

$$\forall \bar{x} \ (\varphi(\bar{x}) \rightarrow \psi(\bar{x}))$$

where $\varphi(\bar{x})$ and $\psi(\bar{x})$ are positive formulas. The h-universal sentences are the negation of positive sentences.

Let $A$ and $B$ be two $L$-structures. A map $h$ from $A$ to $B$ is a homomorphism if for every tuple $\bar{a} \in A$ and for every atomic formula $\phi$, if $A \models \phi(\bar{a})$ then $B \models \phi(h(\bar{a}))$. So we say that $B$ is a continuation of $A$.

An embedding of $A$ into $B$ is a homomorphism $h : A \rightarrow B$ such that, for every $\bar{a} \in A$ and $\phi$ an atomic formula, if $B \models \phi(h(\bar{a}))$ then $A \models \phi(\bar{a})$. The homomorphism $h : A \rightarrow B$ is said to be an immersion whenever for every $\bar{a} \in A$, and $\phi$ a positive formula, if $B \models \phi(h(\bar{a}))$ then $A \models \phi(\bar{a})$.

A class of $L$-structures is said to be h-inductive if it is closed under the inductive limit of homomorphisms.

parallel to the role of existentially closed structures in the framework of logic with negation, for every h-inductive theory $T$ there exists a class of models of $T$ which represent the theory marvellously, and which enjoy the properties desired by every structures; namely, the h-inductive property of the class, the maximality of types (positive formulas satisfied by an element), amalgamation property, and others. These modules are called positively closed.

**Definition 1** A model $A$ of an h-inductive theory $T$ is said to be positively closed (in short, pc) if every homomorphism from $A$ into $B$ a model of $T$ is an immersion.

The following lemmas announce the main properties of pc models. They will be used without mention.

**Lemma 1 (Lemma14, [3])** A model $A$ of an h-inductive theory is pc if and only if for every positive formula $\varphi$ and $\bar{a} \in A$, if $A \not\models \varphi(\bar{a})$ then there exists a positive formula $\psi$ such that $T \vdash \neg \exists \bar{x} (\varphi(\bar{x}) \land \psi(\bar{x}))$ and $A \models \varphi(\bar{a})$.

**Lemma 2 (theorem 2, [3])** Every model on an h-inductive theory $T$ is continued in a pc model of $T$.

For every positive formula $\varphi$ we denote by $\text{Ctr}_T(\varphi)$ the set of positive formulas $\psi$ such that:

$$T \vdash \neg \exists \bar{x} (\varphi(\bar{x}) \land \psi(\bar{x})).$$

Two h-inductive theories are said to be companion if every model of one of them can be continued into a model of the other, or equivalently if the theories have
Every $h$-inductive theory $T$ has a maximal companion denoted $T_k(T)$, called the Kaiser’s hull of $T$. $T_k(T)$ is the set of $h$-inductive sentences satisfied in each pc models of $T$. Likewise, $T$ has a minimal companion denoted $T_u(T)$, formed by its $h$-universal consequence sentences.

**Remark 1** Let $T_1$ and $T_2$ two $h$-inductive theories. The following propositions are equivalent:

- $T_1$ and $T_2$ are companion.
- $T_k(T_1) \equiv T_k(T_2)$.
- $T_u(T_1) \equiv T_u(T_2)$.

**Definition 2** Let $T$ be an $h$-inductive theory.

- $T$ is said to be model-complete if every model of $T$ is a pc model of $T$.
- We say that $T$ has a model-companion whenever $T_k(T)$ is model-complete theory.

Let $A$ be a model of $T$. We shall use the following notations:

- $\text{Diag}^+(A)$, the set of positive quantifier-free sentences satisfied by $A$ in the language $L(A)$.
- $\text{Diag}(A)$, the set of atomic and negated atomic sentences satisfied by $A$ in the language $L(A)$.
- We denote by $T^+(A)$ the $L(A)$-theory $T \cup \{\text{Diag}^+(A)\}$.
- $T_i(A)$ (resp. $T_u(A)$) denote the set of $h$-inductive (resp. $h$-universal) $L(A)$-sentences satisfied by $A$.
- $T^*_i(A)$ (resp. $T^*_u(A)$) denote the set of $h$-inductive (resp. $h$-universal) $L$-sentences satisfied by $A$.
- $T_k(A)$ (resp. $T^*_k(A)$) denote the Kaiser’s hull of $T_i(A)$ (resp. of $T^*_i(A)$).
- For every subset $B$ of $A$, we denote by $T_i(A,B)$ (resp. $T_u(A,B)$) the set of $h$-inductive (resp. $h$-universal) $L(B)$-sentences satisfied by $A$.

**Definition 3** Let $A$ and $B$ be two $L$-structures and $h$ a homomorphism from $A$ into $B$. $h$ is said to be a strong immersion (in short $s$-immersion) if $h$ is an immersion and $B$ is a model of $T_i(A)$.

**Remark 2** Let $A$ and $B$ two $L$-structures. We have the following properties:

1. If $A$ is immersed in $B$ then $T^*_u(A) = T^*_u(B)$, and $T^*_i(B) \subseteq T^*_i(A)$.
2. $A$ is immersed in $B$ if and only if $T_i(B,A) \subseteq T_i(A)$. 
3. $A$ is strongly immersed in $B$ if and only if $T_i(B,A) = T_i(A)$.

4. If $A$ and $B$ are two pc models of $T$ then every homomorphism from $A$ into $B$ is a strong immersion. Indeed, let $\varphi(\bar{a},\bar{x})$ and $\psi(\bar{a},\bar{x})$ be two positive formulas and let $\chi$ the h-inductive sentence $\forall \bar{x}(\varphi(\bar{a},\bar{x}) \rightarrow \psi(\bar{a},\bar{x}))$. Suppose that $A \models \chi$ and $B \not\models \chi$, then there is $\bar{b} \in B$ such that $B \models \varphi(\bar{a},\bar{b})$ and $B \not\models \psi(\bar{a},\bar{b})$. Given that $B$ is a pc model, there exists $\psi'(\bar{x},\bar{y}) \in \text{Ctr}_T(\psi(\bar{x},\bar{y}))$ such that $B \models \psi'(\bar{a},\bar{b})$. Since $A$ is immersed in $B$, then there is $\bar{a}' \in A$ such that $A \models \varphi(\bar{a},\bar{a}') \land \psi'(\bar{a},\bar{a}')$, which implies $A \models \varphi(\bar{a},\bar{a}')$ and $A \not\models \psi(\bar{a},\bar{a}')$, contradiction.

5. The pc models of the $L(A)$-theory $T^+(A)$ are the pc models of $T$ that are continuation of $A$. Indeed, it is clear that every pc model of $T$ in which $A$ is continued is a model of $T^+(A)$ and then a pc model of $T^+(A)$.

Conversely, let $B$ be a pc model $T^+(A)$ and $C$ a pc model of $T$ in which $B$ is continued by a homomorphism $f$. Then $C$ is a continuation of $A$, so $C$ is a model of $T^+(A)$. Thereby $f$ is an immersion, which implies that $B$ is a pc model of $T$.

Let $A$ and $B$ two be $L$-structures and $f$ a mapping from $A$ into $B$. We will use the following notations:

- $\text{Hom}(A,B)$ the set homomorphisms from $A$ into $B$.
- $\text{Emb}(A,B)$ the set embeddings from $A$ into $B$.
- $\text{Imm}(A,B)$ the set immersions from $A$ into $B$.
- $\text{Sm}(A,B)$ the set s-immersions from $A$ into $B$.

**Remark 3** Let $A$ and $B$ be two $L$-structures and $f$ a mapping from $A$ into $B$. Consider $B$ as a $L(A)$-structure by interpreting the elements of $A$ in $B$ by $f$. We have the following:

- $f \in \text{Hom}(A,B)$ if and only if $B$ is a model of $\text{Diag}^+(A)$.
- $f \in \text{Emb}(A,B)$ if and only if $B$ is a model of $\text{Diag}(A)$.
- $f \in \text{Imm}(A,B)$ if and only if $B$ is a model of $\text{Diag}^+(A) \cup T_u(A)$.
- $f \in \text{Sm}(A,B)$ if and only if $B$ is a model of $T_i(A)$.

**1.2 Positive amalgamation**

To abbreviate the nominations of homomorphism, embedding, immersion and strong immersion in the definition of the notions of amalgamation, we will use the first letter of each mapping defined above.
Definition 4 Let $\Gamma$ be a class of $L$-structures and $A$ a member of $\Gamma$. We say that $A$ is \([h,e,i,s]\)-amalgamation basis of $\Gamma$, if for every $B,C$ members of $\Gamma$, if $A$ is continued into $B$ by $f$ and embedded into $C$ by $g$, there exist $D \in \Gamma$, $g' \in \text{Imm}(C,D)$ and $f' \in \text{Sm}(B,D)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{g} & C \\
\downarrow{f} & & \downarrow{g'} \\
B & \xrightarrow{f'} & D
\end{array}
$$

By the same way we define the notion of \([\alpha,\beta,\gamma,\delta]\)-amalgamation property for every $(\alpha,\beta,\gamma,\delta) \in \{h,e,i,s\}^4$.

We say that $A$ is an \([\alpha]\)-amalgamation basis of $\Gamma$, if $A$ is an \([\alpha,\alpha,\alpha,\alpha]\)-amalgamation basis of $\Gamma$.

We say that $A$ is \([\alpha,\beta]\)-symmetric amalgamation basis of $\Gamma$, whenever $A$ is an \([\alpha,\beta,\beta,\alpha]\)-amalgamation basis of $\Gamma$.

We say that $A$ is \([\alpha,\gamma]\)-asymmetric amalgamation basis of $\Gamma$, whenever $A$ is an \([\alpha,\beta,\alpha,\beta]\)-amalgamation basis of $\Gamma$.

The following remark list some properties of diver forms of amalgamation with the notations and terminology given in the definition above.

Remark 4 1. Every $L$-structure $A$ is an \([i,h,s,h]\)-amalgamation basis in the class of $L$-structures (lemma 4, [1]). Since every strong immersion is an immersion, it follows that every $L$-structure $A$ is an \([s,h]\)-asymmetric amalgamation basis in the class of $L$-structures.

2. Every $L$-structure $A$ is an \([s,i]\)-asymmetric amalgamation basis in the class of $L$-structures (lemma 5, [1]).

3. Every $L$-structure $A$ is an \([e,s]\)-asymmetric amalgamation basis in the class of $L$-structures (lemma 4, [2]).

4. Every $L$-structure $A$ is an \([i,h]\)-asymmetric amalgamation basis in the class of $L$-structures (lemma 8, [3]).

5. Every pc model of an $h$-inductive theory $T$ is an \([h]\)-amalgamation basis in the class of model of $T$.

Lemma 3 Every $L$-structure is \([s,x]\)-asymmetric amalgamation basis in the class of $L$-structure, where $x$ is a homomorphism, an embedding or an immersion.

Proof. All cases are given in Remark 4.

Lemma 4 A model of $T$ is pc if and only if it has the \([h,i]\)-symmetric amalgamation property in the class of models of $T$.


\section{Almost positively closed structures}

In this section, we introduce the notions of almost and $\Gamma$-almost positively closed models, we give a syntactic characterisation and a characterization via the closed formulas which turns out to be an essential tool in the study of the notion of $\Delta$-almost positively closedness.

\begin{definition}
Let $T$ be an $h$-inductive theory, and let $\Delta$ be a subset of $L$-quantifier-free positive formulas. A model $A$ of $T$ is said to be:

\begin{itemize}
  \item \textbf{Almost positively closed (apc in short)}, if for every model $B \models T$, $f \in \text{Hom}(A, B)$ and $\varphi(\vec{x}, \vec{y})$ a quantifier-free positive formula, if $B \models \exists \vec{y} \varphi(\vec{a}, \vec{y})$ and $\vec{a} \in A$ then there is $\vec{a}' \in A$ such that $B \models \varphi(\vec{a}, \vec{a}')$.
  
  \item \textbf{$\Delta$-almost positively closed ($\Delta$-apc in short)}, if for every model $B \models T$, $f \in \text{Hom}(A, B)$ and $\varphi(\vec{x}, \vec{y})$ a quantifier-free positive formula if $B \models \exists \vec{y} \varphi(\vec{a}, \vec{y})$ and $\vec{a} \in A$ then there is $\vec{a}' \in A$ such that $B \models \varphi(\vec{a}, \vec{a}')$.
  
  \item \textbf{Weakly almost positively closed (wpc in short)}, if for every pc model $B \models T$, $f \in \text{Hom}(A, B)$ and $\varphi(\vec{x}, \vec{y})$ a quantifier-free positive formula if $B \models \exists \vec{y} \varphi(\vec{a}, \vec{y})$ and $\vec{a} \in A$ then there is $\vec{a}' \in A$ such that $B \models \varphi(\vec{a}, \vec{a}')$.
  
  \item \textbf{$\Delta$-weakly almost positively closed ($\Delta$-wpc in short)}, if for every pc model $B \models T$, $f \in \text{Hom}(A, B)$ and $\varphi(\vec{x}, \vec{y}) \in \Gamma$, if $B \models \exists \vec{y} \varphi(\vec{a}, \vec{y})$ and $\vec{a} \in A$ then there is $\vec{a}' \in A$ such that $B \models \varphi(\vec{a}, \vec{a}')$.
\end{itemize}

\end{definition}

\begin{theorem}
Let $A$ be a model of an $h$-inductive $L$-theory, let $\Delta$ be a set of $L(A)$-quantifier-free positive formula. $A$ is $\Delta$-apc of $T$ if and only if for every $\varphi(\vec{a}, \vec{x}) \in \Gamma$, there exists a quantifier free positive formula $\psi(\vec{a}, \vec{a}') \in \text{Diag}^+(A)$ such that

\[ T \vdash \forall \vec{x} \vec{y} ((\psi(\vec{x}, \vec{y}) \land \exists \vec{z} \varphi(\vec{x}, \vec{z})) \rightarrow \varphi(\vec{x}, \vec{y})). \]

\end{theorem}
Proof. Assume that $A$ is a $\Delta$-apc model of $T$ and let $\varphi(\bar{a}, \bar{x}) \in \Gamma$. Considering $T^* = T \cup \text{Diag}^+(A) \cup \{\exists \bar{x} \varphi(\bar{a}, \bar{x})\}$ is consistent and $A$ is $\Delta$-apc then $T^* \cup \{\neg \varphi(\bar{a}, \bar{a}') | \bar{a}' \in A\}$ is inconsistent. Thereby there are $\bar{a}' \in A$ and $\psi(\bar{a}, \bar{a}') \in \text{Diag}^+(A)$ such that $T \cup \{\psi(\bar{a}, \bar{a}'), \neg \varphi(\bar{a}, \bar{a}', \exists \bar{x} \varphi(\bar{a}, \bar{x})\}$ is inconsistent, which implies $T \vdash \forall \bar{x}, \bar{y}((\psi(\bar{x}, \bar{y}) \land \exists \bar{z} \varphi(\bar{x}, \bar{z})) \rightarrow \varphi(\bar{x}, \bar{y}))$.

The other direction is clear.

Corollary 1 Let $A$ be a model of $T$ and $\Delta$ a set of quantifier-free positive $L(A)$-formulas. If $A$ is immersed in an $\Delta'$-apc model $B$ of $T$ and $\Delta \subseteq \Delta'$, then $A$ is an $\Delta$-apc model of $T$.

Proof. Let $\varphi(\bar{a}, \bar{x}) \in \Delta$. Given that $\varphi(\bar{a}, \bar{x}) \in \Delta'$ and $B$ is $\Delta'$-apc model of $T$, by theorem \[ there exists $\psi(\bar{a}, \bar{b}) \in \text{Diag}^+(B)$ such that

$$T \vdash \forall \bar{x}, \bar{y}((\psi(\bar{x}, \bar{y}) \land \exists \bar{z} \varphi(\bar{x}, \bar{z})) \rightarrow \varphi(\bar{x}, \bar{y})).$$

On the other hand, since $A$ is immersed in $B$ then there is $\bar{a}' \in A$ such that $\psi(\bar{a}, \bar{a}') \in \text{Diag}^+(B)$, hence $A$ is an $\Delta$-apc of $T$ by theorem \[.

Remark 5 Let $T$ be an $h$-inductive $L$-theory and $\Delta$ a set of quantifier-free positive $L$-formulas. We have the following properties:

1. If $A$ is apc then $A$ is wpc of $T$.
2. Every pc model of $T$ is an apc (resp. $\Delta$-apc) model of $T$.
3. The classes of apc and wpc (resp. $\Delta$-apc and $\Delta$-wpc) models of $T$ are $h$-inductive.
4. If $A$ is an apc model of $T$ and $B$ a model of $T$, then $\text{Emb}(A, B) = \text{Imm}(A, B)$.
5. Let $\Delta \subseteq \Delta'$ be two sets of free quantifier positive formulas. If $A$ is $\Delta$-apc (resp. $\Delta$-wpc) then $A$ is $\Delta$-apc (resp. $\Delta$-wpc).
6. A model $A$ of $T$ is apc if and only if for every positive quantifier-free formulas $\varphi(\bar{x}, \bar{y})$, there exists $\psi(\bar{a}, \bar{a}') \in \text{Diag}^+(A)$ such that

$$T \vdash \forall \bar{x}, \bar{y}((\psi(\bar{x}, \bar{y}) \land \exists \bar{z} \varphi(\bar{x}, \bar{z})) \rightarrow \varphi(\bar{x}, \bar{y})).$$

7. Every apc model of $T$ has the property of $[e, h]$-asymmetric amalgamation (property 4 of Remark \[ and the property 4 of Remark \[).

Example 1 Let $L = \{f\}$ functional language. Let $T$ be the $h$-inductive theory $\forall x, y (f(x) = f(y) \rightarrow x = y)$. The theory $T$ has a model companion axiomatized by $T_k(T) = T \cup \{\forall x y (x = y)\}$. The class of apc model of $T$ is elementary and axiomatized by the $h$-inductive theory $T \cup \{\exists x, f(x) = x\} \cup \{\forall x \exists y (f(y) = x)\}$. 

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2. Let $L$ and $T$ respectively the functional language and the theory defined above. Let $T''$ the $h$-inductive theory $T \cup \{\neg \exists x \ (f(x) = x)\}$. The class of apc model of $T''$ is axiomatized by the $h$-inductive theory

$$T \cup \{\forall x \exists y (f(y) = x)\} \cup \{\exists x f^p(x) = x \mid p \text{ prime number}\}.$$ 

3. Let $T_f$ the theory of fields. Since the negation of equality $x = y$ is defined by the positive formula $\exists z \ ((x - y) \cdot z = 1)$ and every homomorphism is an embedding then the classes of apc fields, pc fields and existentially closed fields are equals.

**Definition 6** Let $T$ be an $h$-inductive $L$-theory and $A$ a model of $T$.

- A positive formula $\varphi(x)$ is said to be $T$-algebraic if there exists a positive formula $\psi(y_1, \ldots, y_n)$ such that $\psi \not\equiv \bot$ modulo $T$ (ie, $\psi(y_1, \ldots, y_n)$ has a realisation in some model of $T$) and:

$$T \vdash \forall x, \bar{y}((\varphi(x) \land \psi(\bar{y})) \rightarrow \bigvee_i x = y_i).$$

We denote by $\mathbb{A}l_T$ the set of $T$-algebraic quantifier-free positive $L$-formulas.

- For every formula $\varphi(\bar{x})$, we denote by $E(\varphi, T)$ the set of positive formulas $\psi(\bar{y})$ such that $\psi \not\equiv \bot$ modulo $T$ that satisfy the property:

$$T \vdash \forall \bar{x}, \bar{y}((\varphi(\bar{x}) \land \psi(\bar{y})) \rightarrow \bigvee_{i,j} x_i = y_j).$$

- A positive formula $\varphi(\bar{x}, \bar{y})$ is said to be $(A,T)$-closed if for every pc model continuation $B$ of $A$, if $B \models \varphi(\bar{a}, \bar{b})$ for some $\bar{a} \in A$ then $\bar{b} \in A$.

**Remark 6** 1. A quantifier-free positive formula is $T$-algebraic if and only if its is algebraic in the sense of Robinson [4].

2. Given that the class of pc models of $T^+(A)$ coincides with the class of pc models of $T$ that are continuation of $A$ (bullet 5 of remark [3], then a formula is $(A,T)$-closed if and only if it is $(A,T^+(A))$-closed.

3. Let $A$ be a model of $T$. Denote by $C_A$ the set of quantifier-free formulas that are $(A,T)$-closed. By definition of $C_A$ we observe that $A$ is $C_A$-wpc.

4. If every formula in $\mathbb{A}l_{T^+(A)}$ is $(A,T)$-closed, by the property 2 above and the definition of formulas $(A,T)$-closed, we conclude that $A$ is $\mathbb{A}l_{T^+(A)}$-wpc. Considering that $\mathbb{A}l_T \subset \mathbb{A}l_{T^+(A)}$, $A$ is also $\mathbb{A}l_T$-wpc.

**Definition 7** For every quantifier-free positive formulas $\varphi(\bar{x})$ such that $\varphi(\bar{x}) \not\equiv \bot$ modulo $T$, we denote by $E_T$ the set of quantifier-free positive formula $\varphi(\bar{x})$ such that $E(\varphi, T) \neq \emptyset$. 
Lemma 5  Let $A$ be an $h$-amalgamation basis of $T$. If $A$ is $E_{T^+(A)}$-wpc (resp. $E_{T^+(A)}$-apc) then every formula in $Al_{T^+(A)}$ is $(A,T)$-closed.

Proof. Let $A$ be an $E_{T^+(A)}$-wpc and an $h$-amalgamation basis of $T$. Assume the existence of a formula $\varphi(\bar{a},y) \in Al_{T^+(A)}$ such that $\varphi(\bar{a},y)$ is not $(A,T)$-closed. So, there exist $B$ a pc models of $T$ and $b \in B - A$ such that $B \models \varphi(\bar{a},b)$. Let $\psi(\bar{a},\bar{x}) \in E(\varphi, T^+(A))$. Let $C$ be a pc model of $T^+(A)$ and $\bar{c} \in C$ such that $C \models \psi(\bar{a},\bar{c})$. Given that $\psi(\bar{a},\bar{y}) \in E_{T^+(A)}$ and $A$ is an $E_{T^+(A)}$-wpc model of $T$, then there is $\bar{a}'$ in $A$ such that $C \models \psi(\bar{a},\bar{a}')$. Let $D$ be a model of $T$ that amalgamate commutatively the diagram $C \leftarrow A \rightarrow B$, so $D$ is immersed in $D$ and $B \models \varphi(a,b) \wedge \psi(a,a')$. Which implies $\bigvee_i b = a_i$, contradiction.

The proof of the case where $A$ is $Al_T$-apc is an immediate consequence of this Lemma.

3 Strong amalgamation

In this section we introduce the notions of positive strong amalgamation and $h$-strong amalgamation. We investigate their properties and interactions with the notion of almost positively closedness.

3.1 positive strong amalgamation

Definition 8 Let $T$ be an $h$-inductive theory. A model $A$ of $T$ is said to be a positive strong amalgamation basis (in short PSA) (resp. $h$-strong amalgamation basis (in short $h$-SA)) of $T$, if for every pc models (resp. models) $B$ and $C$ of $T$, if $A$ is continued into $B$ and $C$ by two homomorphisms $f$ and $g$ respectively, then there exist $D$ a model of $T$, and $f', g'$ two homomorphisms such that the following diagram commutes:

$$
\begin{array}{c}
A \\
g \\
\downarrow \\
C \\
g' \\
\downarrow \\
D
\end{array}
\begin{array}{c}
\rightarrow \\
f \\
\rightarrow \\
f'
\end{array}

$$

and satisfies the following property:

$\forall (b,c) \in B \times C$, if $g'(c) = f'(b)$ then there is $a \in A$ such that $c = g(a)$ and $b = f(a)$.

Example 2 1. Let $T$ be an $h$-inductive theory such that a model $A$ of $T$ is pc if and only if $A \models \forall x, y x = y$. Then $T$ has the positive strong amalgamation property. As examples of these theories we have the theory of groups and the theory of partially ordered sets.
Lemma 6 Let $A, B, C$ be three $L$-structures. Let $i \in \text{Imm}(A, B)$ and $h \in \text{Hom}(A, C)$. Then there exist $D$ a $L$-structure, $h'$ a homomorphism and $s$ an $s$-immersion, such that the following diagram commutes:

\[
\begin{array}{ccc}
A & \xrightarrow{i} & B \\
\downarrow h & & \downarrow h' \\
C & \xrightarrow{s} & D
\end{array}
\]

and satisfies the following property:

$\forall (b, c) \in B \times C$, if $h'(b) = s(c)$, then there exists $a \in A$ such that $c = h(a)$ and $b = i(a)$.

Proof. The proof consists in the verification that the set

\[T_i(C) \cup \text{Diag}^+(B) \cup \text{Diag}^+(C) \cup \{b \neq c | b \in B - A, c \in C - h(A)\}\]

is $L(B \cup C)$-consistent.

Assume, to the contrary that the set above is $L(B \cup C)$-inconsistent. Then there exist $\varphi(h(\bar{a}), \bar{c}) \in \text{Diag}^+(C)$, $\psi(\bar{a}, \bar{b}) \in \text{Diag}^+(B)$ where $\bar{c} \in C - h(A)$ and $\bar{b} \in B - A$, such that:

\[T_i(C) \vdash \forall \bar{y} \ ((\varphi(h(\bar{a}), \bar{c}) \land \psi(\bar{a}, \bar{y})) \rightarrow \bigvee_{i,j} y_i = c_j).
\]

Given that $B \models \psi(\bar{a}, \bar{b})$ and $A$ is immersed in $B$, there is $\bar{a}' \in A$ such that $A \models \psi(\bar{a}, \bar{a}')$. Consequently, $C \models \varphi(h(\bar{a}), \bar{c}) \land \psi(h(\bar{a}), h(\bar{a}'))$. Thereby $C \models \bigvee_{i,j} h(\bar{a}')_i = c_j$, which is a contradiction.

Corollary 2 Every pc model $A$ of $T$ is a $h$-strong amalgamation basis of $T$.

Proof. Immediate from Lemma [6]

Proposition 2 Let $A$ and $B$ be two models of an $h$-inductive theory $T$, and let $i \in \text{Imm}(A, B)$. If $B$ is a $h$-SA basis of $T$ then $A$ is a PSA basis of $T$.

Proof. Let $A_1$ and $A_2$ be two pc mode of $T$, $f \in \text{Hom}(A, A_1)$ and $g \in \text{Hom}(A, A_2)$. Considering the diagrams $A_1 \leftarrow A \rightarrow B$ and $A_1 \leftarrow A \rightarrow B$, by Lemma [6] we get the commutative diagrams (1) and (2) below, where $f', g'$ are homomorphisms and $i_1, i_2$ are strong immersions.

Now, given that $B$ has the $h$-strong amalgamation property, we get the commutative diagram (3) below:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{i_1} & B_1 \\
\downarrow f & & \downarrow f' \\
A & \xrightarrow{i} & B \\
\downarrow g & & \downarrow g' \\
A_2 & \xrightarrow{i_2} & B_2
\end{array}
\]
where $f'', g''$ are homomorphisms.

We claim that $C$ makes the diagram $A_1 \leftarrow A \to A_2$ strongly amalgamable. Indeed, let $a_1 \in A_1$ and $a_2 \in A_2$ such that $f'' \circ i_1(a_1) = g'' \circ i_2(a_2)$, by the $h$-strong amalgamation property of the diagram (3), there is $b \in B$ such that $f'(b) = i_1(a_1)$ and $g'(b) = i_2(a_2)$. Considering the properties of the diagrams (1) and (2), we get two elements $a$ and $a'$ from $A$ such that:

\[
\begin{cases}
  f(a) = a_1, & i(a) = b \\
  g(a') = a_2, & i(a') = b.
\end{cases}
\]

Given that $i$ is an immersion, then $a = a'$ and $f(a) = a_1, g(a) = a_2$. So $A$ is a PSA basis of $T$.

**Lemma 7** Let $A$ be an $h$-amalgamation basis of $T$, $B$ a pc model of $T$ and $f \in \text{Hom}(A, B)$. $A$ is PSA basis of $T$ if and only if for every formula $\varphi(\bar{a}, \bar{x}) \in E_{T^+(A)}$ and for every $b_1, \cdots, b_n \in B - f(A)$, we have $B \models \varphi(\bar{a}, b_1 \cdots b_n)$.

**Proof.** Let $A$ be a PSA basis of $T$. Suppose that there are $\varphi(\bar{a}, \bar{x}) \in E_{T^+(A)}$, $B$ a pc model of $T$ and $f \in \text{Hom}(A, B)$ such that $B \models \varphi(\bar{a}, b_1 \cdots b_n)$, where $b_1, \cdots, b_n \in B - f(A)$. Let $\psi(\bar{a}, \bar{y}) \in E(\varphi, T^+(A))$, $C$ a pc model of $T$ and $g \in \text{Hom}(A, C)$ such that $C \models \psi(\bar{a}, \bar{c})$.

Given that $A$ is a PSA basis of $T$, we obtain the following commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow{g} & & \downarrow{i} \\
C & \xrightarrow{i'} & D
\end{array}
\]

where $i$ and $i'$ are immersion, and $D$ a model of $T$ that satisfies the property:

\[
\forall (b, c) \in B \times C; \ i(b) = i'(c) \Rightarrow \exists a \in A, (f(a) = b \land g(a) = c).
\]

Now, since $D \models \varphi(\bar{a}, i(\bar{b})) \land \psi(\bar{a}, i'(\bar{c}))$ then $D \models \bigvee_{i,j} i(b_i) = i'(c_j)$, which implies the existence of an element $a' \in A$ such that $i(b_i) = i'(c_j) = i \circ f(a')$.

Contradiction.

We shall prove the other direction by contrapositive. Let $B$ and $C$ be two pc models of $T$, $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(A, C)$ such that the diagram

\[
\begin{array}{ccc}
C & \xrightarrow{g} & A & \xrightarrow{f} & B \\
& & & &
\end{array}
\]

is not $h$-strongly amalgamable. Thereby there exist $\varphi(f(\bar{a}), b_1 \cdots b_n) \in \text{Diag}^+(B)$, $\psi(g(\bar{a}), c_1 \cdots c_m)) \in \text{Diag}^+(C)$ where $b_1, \cdots, b_n \in B - A$ and $c_1, \cdots, c_m \in C - A$, such that

\[
T^+(A) \vdash \forall \bar{y} \ ((\varphi(\bar{a}, \bar{x}) \land \psi(\bar{a}, \bar{y})) \to \bigvee_{i,j} x_i = y_j).
\]

Thus $\varphi(\bar{a}, \bar{x}) \in E_{T^+(A)}$ and $B \models \varphi(\bar{a}, b_1 \cdots b_n)$. 

Theorem 2 Let $A$ be an $h$-amalgamation basis of $T$, we have the following properties:

1. If $A$ is a $E_{T+(A)}$-wpc model of $T$ then $A$ is a PSA basis of $T$.

2. If $A$ is a PSA basis of $T$ then $A$ is $Al_{T+(A)}$-wpc.

Proof.

1. Let $A$ be an $h$-amalgamation basis and $E_{T+(A)}$-wpc model of $T$. Let $B$ and $C$ be two pc models of $T$, $f \in \text{Hom}(A, B)$ and $g \in \text{Hom}(A, C)$. Let $D$ a model of $T$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{f} & B \\
g \downarrow & & \downarrow i_1 \\
C & \xrightarrow{i_2} & D
\end{array}
$$

where $i_1$ and $i_2$ are immersions.

We claim that the set $T \cup \text{Diag}^+(B) \cup \text{Diag}^+(C) \cup \{b \neq c \mid b \in B - A, c \in C - A\}$ is $L(B \cup C)$-consistent (note that the element of $A$ are interpreted by the same symbols of constants in $B$ and $C$). Suppose to the contrary that the set above is inconsistent. Thus there are $\bar{a} \in A$, $\bar{b} \in B - A$, $\bar{c} \in C - A$, $\varphi(\bar{a}, \bar{b}) \in \text{Diag}^+(B)$ and $\psi(\bar{a}, \bar{c}) \in \text{Diag}^+(C)$ such that

$$T \cup \{\varphi(\bar{a}, \bar{b}), \psi(\bar{a}, \bar{c}), \bigwedge_{i,j} b_i \neq c_j\}$$

is $L(B \cup C)$-inconsistent, thereby

$$T^+(A) \vdash \forall \bar{y}, \bar{z} \ ((\varphi(\bar{a}, \bar{y}) \land \psi(\bar{a}, \bar{z})) \rightarrow \bigvee_{i,j} y_i = z_j).$$

Now, since $C \models \psi(\bar{a}, \bar{c})$, $\psi \in E_{T+(A)}$, and $A$ is an $E_{T+(A)}$-wpc model, then there is $\bar{a}' \in A$ such that $C \models \psi(\bar{a}, \bar{a}')$. Thereby $D \models \psi(\bar{a}, \bar{a}')$, so $B \models \psi(\bar{a}, \bar{a}') \land \varphi(\bar{a}, \bar{b})$. Which implies $B \models \bigvee_{i,j} b_i = a'_j$, contradiction. Thus $A$ is a PSA basis of $T$.

2. Suppose that $A$ is PSA of $T$. Since $Al_{T+(A)} \subseteq E_{T+(A)}$, by Lemma 7 every formula in $Al_{T+(A)}$ is $(A, T)$-closed. which implies that $A$ is a $Al_{T+(A)}$-wpc model of $T$ by Remark 6 (4).

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