Various best-choice problems related to the planar homogeneous Poisson process in finite or semi-infinite rectangle are studied. The analysis is largely based on properties of the one-dimensional box-area process associated with the sequence of records. We prove a series of distributional identities involving exponential and uniform random variables, and resolve the Petruccelli-Porosinski-Samuels paradox on coincidence of asymptotic values in certain discrete-time optimal stopping problems.

1 Introduction

On a recent conference on optimal stopping Steve Samuels reported a remarkable coincidence of the asymptotic values in two quite different best-choice problems [29].

Let \((X_j)\) be a sequence of independent uniform \([0, 1]\) random variables. Let \(T_n\) be the family of all stopping times \(\tau \leq n\) adapted to the natural filtration of the sequence, and \(R_n\) be the subclass of stopping times adapted to the sequence \((S_j, T_j)\) where

\[
S_j = \max(X_1, \ldots, X_j) - \min(X_1, \ldots, X_j) \quad \text{and} \quad T_j = 1\{X_j = \max(X_1, \ldots, X_j)\}
\]

are the range and the indicator of an upper record at index \(j\), respectively. For \(N\) uniformly distributed on \([1, \ldots, n]\) and independent of \((X_j)\) define

\[
u_n = \sup_{T_n} P(X_\tau = \max(X_1, \ldots, X_N), \ \tau \leq N)
\]

and define another stopping value

\[
w_n = \sup_{R_n} P(X_\tau = \max(X_1, \ldots, X_n)).
\]

Then, as pointed out by Samuels, the limits are the same

\[
\lim_{n \to \infty} u_n = \lim_{n \to \infty} w_n
\]

and coincide with the value \(v_P := \lim_{n \to \infty} w_n\) established by Petruccelli [21].

The values \(u_n\) and \(w_n\) arise in best-choice models representing very different informational situations of the observer. The value \(u_n\) is the optimum probability of stopping at the maximum of a sequence sampled from known probability distribution, when the observer has incomplete information about the length of the sequence \(N\), see Porosinski [23]. The value \(w_n\) appears as
the minimax probability of stopping at the maximum of a random sequence with definite length 
$n$, but only partial information about the distribution of observations: the observer knows that 
the underlying distribution is uniform on a unit interval but is ignorant of the position of the interval, see Petruccelli [21].

It had been noticed by Porosinski [24] that the two problems have optimal policies with the 
same collection of thresholds and that numerical values of $u_n$ suggest unmistakable convergence 
to $v_P$. However, the coincidence of policies does not imply coincidence of stopping values, as is 
seen from the numerical values tabulated in [23] and in an unpublished Petruccelli’s thesis. In 
[24] Porosinski gave a false argument for (1) which, however, involved a computation with the 
right answer.

Both models are offsprings of the basic problem introduced by Gilbert and Mosteller as 
the ‘full-information game’ [12], where the objective is to maximise $P(X_\tau = \max(X_1,\ldots,X_n))$ 
over $\tau \in T_n$. In this case the observer knows $n$ and the distribution of observations and aims 
to recognise the maximum at the moment it appears. This loose name was attached to the 
problem to stress the contrast with the classical best-choice or secretary problem where no 
information about the distribution is available and the policy is to be based only on relative 
ranks (or, in other version, on record times [11]).

Samuels explained that the correct answer in [24] resulted from yet another coincidence: the 
common optimal policy yields the same best-choice probability in the full-information problem. 
The now threefold coincidence was reinforced by Tamaki and Mazalov [31] who noted that the 
same limit appears in connection with the problem of maximising the inter-record time, as 
studied in [11].

To justify (1) Samuels used Poisson approximation to express the limit values via certain 
multivariate integrals which he evaluated partly analytically, partly using numerical integration. 
He then concluded that this kind of argument does not really explains the phenomenon, because 
the random processes underlying $u_n$ and $w_n$ are of very different nature and do not seem to 
admit a kind of coupling, even asymptotically. As for the coincidence of optimal policies, it 
was derived from the fact that two different mixtures of binomial distributions – one uniform 
in $n$ and another uniform in $p$ – yield the same distribution, see [29].

In this paper we argue that (1) and further coincidences are by no means incidental, rather 
exemplify properties of various Markov chains induced by records from the homogeneous Planar 
Poisson Process (PPP). Essentially the same reason which leads in the discrete-time setting 
to the coincidence of optimal policies unravels in the PPP setting as a characterisation of the 
box-area process which measures the predicted intensity of PPP-records in a given rectangle. 
Our explanation to (1) is that

proper Poisson versions of Porosinski and Petruccelli problems with same size-
parameter $t$ can be reduced to optimal stopping of the same one-dimensional box-
area process for any value of $t$.

We adopt the following well-known framework (also see [7], [9], [10], [12], [16], [22] for similar 
approaches). Consider the PPP restricted to a given rectangle $R$ of area $t$ (with the conventional 
orientation). Suppose the rectangle is scanned from the left to the right by shifting a vertical 
detector and that scanning can be stopped each time an atom of the PPP is detected. Different 
objectives and constraints are considered.
(FI): In the full-information problem $R$ is known and the objective is to stop at the highest PPP-atom in $R$.

(VC): In the vertical cut problem, $R$ is partitioned by a vertical line $V$ drawn through a random uniform point selected on the upper side of the rectangle. The observer, who does not know $V$ aims to stop scanning at the point highest among the Poisson points in $R$ which are to the left from the cut $V$.

(HC): In the horizontal cut problem, $R$ is partitioned by a horizontal line $H$ drawn through a random uniform point on the left side of the rectangle. The observer aims to stop scanning at the point highest among the Poisson points in $R$ below the cut $H$. The observer does not know $H$ but each time an atom is detected she learns if the atom is above or below $H$.

Let $u(t)$ and $w(t)$ be the optimum probabilities in the VC- and HC-problems, respectively. We will show that

$$u(t) \equiv w(t)$$

and give explicit formulas for the value. The common limit $v_P$ will be given interpretations as the optimal probability of the best choice in a $t = \infty$ model.

Generally speaking best-choice problems belong to the province of extremes and records, and there is a well-developed theory of these structures, see [1], [19], [26] and a survey [14]. However, for evaluating stopping policies one needs to consider records satisfying variable constraints, and the theory does not cover this subject yet.

In brief, our plan is as follows. We start in Section 2 with thorough analysis of structures underlying the FI-problem, we present a new complete solution and closed-form formulas, outline connection to an optimal control problem and give various representations of the best-choice probability. A principal novation is the box-area process which we describe as a regenerative process, design a EU-representation (exponential-uniform) for the path and prove a characterisation via the distribution of the number of visits in an interval. In Section 3 we modify the box-area process to adopt it to the VC-problem, derive an analytical expression for $u(t)$ and draw a parallel between the box-area process and the classical Poisson process. In Section 4 we analyse upper and lower record processes and proceed with three different proofs of (2). The relation (2) itself becomes embedded into a series of distributional identities involving rational functions in exponential and uniform random variables. In Section 5 we give a sample of extensions, reduce the duration problem to the VC-problem and finally give a formula for the winning rate, thus fixing a loose end from [12].

2 Records, box areas and the full-information problem.

2.1 Prerequisites. We will consider the homogeneous PPP, which has the Lebesgue measure as intensity. The properties of the PPP which will be used without further reference are:

- The number of PPP-points (referred to here as atoms) in each bounded domain has Poisson distribution with mean equal to the area of the domain.
- The random variables counting the atoms in disjoint domains are independent.
For any rectangle $R$, projections of PPP atoms $a \in R$ on adjacent sides of $R$ yield one-dimensional homogeneous Poisson processes (which are conditionally independent point processes given the number of atoms in $R$).

For any rectangle $R$, conditionally on the number of atoms in $R$, say $n$, the law of PPP in $R$ is the same as that of the point process induced by a sample of $n$ i.i.d. points from the uniform distribution in $R$.

We will use the following notation for exponential integral functions

$$I(t, s) := \int_t^s e^{-\xi} \frac{d\xi}{\xi}, \quad J(t) := \int_0^t \frac{e^{\xi} - 1}{\xi} d\xi, \quad I(s) = I(\infty, s)$$

(see \[8\], \[20\] for detailed study of these and other functions related to the incomplete gamma-function).

We consider only rectangles with sides parallel to coordinate axes. Given a rectangle $R$, an atom $a \in R$ is said to be a record if there are no other atoms in $R$ to the north-west of $a$. The part of $R$ to the north-east of $a$ will be called the box attributed to $a$ and its area $\alpha(a)$ will be called the box area.

If two rectangles $R_1$ and $R_2$ have the same area, there is an affine isomorphism $\phi$ between them which respects both the measure and the natural partial order. It follows that the $\phi$-image of the PPP in $R_1$ is a version of the PPP in $R_2$, with same records and box areas. This kind of self-similarity is crucial for the models to follow, and the only essential parameter of a rectangle will be its area.

Throughout we denote this basic parameter by $t$. Different interpretations are possible: in case $R = [0, t] \times [0, 1]$, the parameter will be implicitly understood as a time horizon for a ‘sequence of marked items arriving in a Poisson manner’, while for $R = [0, 1] \times [0, t]$ one can think of $[0, 1]$ as a time scale and of $[0, t]$ as a scale for ‘qualities of random items’. However, the reader should accept thinking in terms of areas and be prepared for the models like best-choice in a square with side-size $t^{1/2}$. In case $t = \infty$ we consider PPP in the semifinite strip $[0, 1] \times ] - \infty, 0]$.

Denoting $p_j(t)$ the probability of $j$ records in $R$ we have

$$p_j(t) = e^{-t} \sum_{k=j}^{\infty} \frac{t^k}{k!} \frac{\sigma_1(k, j)}{k!}$$

where $\sigma_1(k, j)$ are signless Stirling numbers of the first kind ($= 0$ for $k < j$). This formula follows from the analogous fact about random permutations (see e.g. \[13\]), because if there are $k$ atoms in $R$ all their $k!$ rankings on the vertical scale are equally likely. Two special cases of the formula will be most important:

$$p_0(t) = e^{-t}, \quad p_1(t) = e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{k!} = e^{-t} J(t).$$

Many recursions involving records in $R$ are obtained by conditioning on the area in $R$ to the left from the leftmost atom, say $a$, which is also the first (i.e. leftmost) record. When
\[ R = [0, t] \times [0, 1] \] this area is just the horizontal coordinate of \( a \). In this line, we have for the number of records a recursion

\[
p_j(t) = \int_0^t e^{s-t} \, ds \int_0^1 p_{j-1}(sx) \, dx.
\]

Exchanging the order of integration this becomes

(4)

\[
p_j(t) = e^{-t} \int_0^t p_{j-1}(s) \left( J(t) - J(s) + \log \frac{t}{s} \right) \, ds
\]

and shows that all functions \( p_j(t) \) are obtained by repeated integration of \( p_0(t) = e^{-t} \) with the same kernel. Same recursion in differential form is

(5)

\[
p_j'(t) = -p_j(t) + t^{-1} \int_0^t p_{j-1}(s) \, ds, \quad p_j(0) = 0.
\]

Another recursion can be proved by induction:

(6)

\[
p_j(t) = e^{-t} \int_0^t \frac{(-1)^{j-1}p_{j-1}(-s) - p_{j-1}(s) e^s}{s} \, ds.
\]

Starting from \( p_0(t) = e^{-t} \) this yields already determined \( p_1(t) = e^{-t}J(t) \), then

\[
p_2(t) = e^{-t} \int_0^t \frac{-J(-s)e^s - J(s)}{s} \, ds
\]

and so forth. Note that the power series for \( p_j(t) \)’s define entire functions thus substitution of negative values of \( t \) does make sense.

**Remark.** The sequence of records can be viewed as a north-west Pareto boundary of the Poisson sample. This motivates yet another representations for \( p_j(t) \): as a multidimensional integral over the value of a bivariate sequence of records of length \( j \), or as a one-dimensional integral over the area to the north-west of such a sequence.

### 2.2 Probability of the best choice.

Suppose an observer learns the configuration of PPP atoms by shifting a vertical detector from the left to the right. The objective of the observer is to correctly recognise the highest atom in a rectangle \( R \) at the moment the highest atom is detected. In the full-information problem it is assumed that the observer knows \( R \) exactly.

Formally, a policy is a stopping time adapted to the PPP, and the performance index of a policy is the probability of stopping at the highest atom in \( R \). In first turn, we are interested in an optimal policy which maximises the probability of stopping at the highest atom. Since the highest atom is the last (i.e. the rightmost) record in \( R \) it is always optimal to skip non-record observations. On the other hand, when a record \( a \) is observed further records can appear only in the box attributed to \( a \), and because the configuration of atoms in the box is independent on the configuration to the left from \( a \), the box area \( \alpha(a) \) alone determines the conditional probability law for the number of future records and the law of their configuration up to isomorhism. The conditional distribution of the number of records is obtained by substituting \( \alpha(a) \) in place of \( t \) into (3), thus the decision to stop at a record or to skip it should depend only on the box area.
Let $v(t)$ be the optimal probability of stopping at the highest atom. Dynamic programming approach calls for solving the equation (DP-equation)

$$ v(t) = \int_0^t e^{s-t} \, ds \int_0^1 \max(p_0(sx), v(sx)) \, dx $$

which is equivalent to the initial-value problem

$$ v'(t) = -v(t) + t^{-1} \int_0^t \max(p_0(s), v(s)) \, ds, \quad v(0) = 0. $$

It is immediate from (7) that the solution is unique and at least $C^1$-smooth for $t > 0$. However the equation is difficult to deal with directly, unless we learn how to resolve the max operator.

A traditional resolution in the spirit of optimal stopping theory is as follows. Consider equation $p_0(t) = p_1(t)$, which is equivalent to the transcendental equation $J(t) = 1$. There is a single positive root $t_F = 0.804352\ldots$ and we have $p_0(t) > p_1(t) \iff t < t_F$.

Since the box areas can only decrease, this relation implies that we are in the so-called monotone case of optimal stopping and by a well-known argument $v(t) > p_0(t)$ for $t > t_F$ and

$$ v(t) = p_1(t) \quad \text{for} \quad t \leq t_F. $$

We could have come to the same conclusion by a more insightful method we call coupling. Consider a rectangle $R_1 = [0, 1] \times [-t, 0]$ and a smaller rectangle $R_2 = [0, 1] \times [-(t - \delta), 0]$. Obviously, the records in $R_2$ are records in $R_1$ as well, although $R_1$ may contain some more records in the strip $[0, 1] \times [-t, -(t - \delta)]$. If the record sequence in $R_1$ ever enters $R_2$ it stays there forever, in which case the PPP in both rectangles has the same highest atom. Now, any stopping policy $\pi$ in $R_2$ is also a legitimate policy for $R_1$ and if $\pi$ succeeds to pick the highest atom in $R_2$, this is also valid for $R_1$. Since $\pi$ can be arbitrary $R_2$-policy, we have $v(t - \delta) \leq v(t)$, i.e. the value function $v(t)$ is increasing. At the same time, $p_0(t) = e^{-t}$ is decreasing, therefore there is a single match-point under the maximum and a minute thought shows that the match is at $t_F$.

It follows that the optimal policy is to select the first record which has the box area not exceeding $t_F$, if any. For $t \leq t_F$ it is optimal to exploit the greedy policy which selects the very first detected record.

The DP-equation (7) can be easily solved by splitting the integral term at $t_F$. With no extra effort we can do this in a more general framework.

Define a threshold policy $\pi_s$ to be the policy which stops at the first record with box area not exceeding $s$. Clearly, the optimal policy is $\pi_{t_F}$. The definition also covers the greedy policy $\pi_\infty$. (The maximum best-choice probability with $\pi_\infty$ is about 0.51735, attained at $t = 1.50286\ldots$.)

**Warning.** This definition is in terms of box areas, thus incorporates the self-similarity properties of PPP. Stopping rules akin to ‘choose the first atom in $R$ above a given level’ are not threshold policies in our sense.

The probability of the best choice with $\pi_s$ is equal to the probability, which we denote $p_1(t, s)$, that there is a single record in $R$ which has box area not exceeding $s$. In this case the
record is necessarily the last, and it is selected by $\pi_s$ while all preceding records (if any) are skipped. By definition, $p_1(t, s) = p_1(t)$ for $t < s$ and for $t > s$ satisfies

$$\partial_t p_1(t, s) = -p_1(t, s) + t^{-1} \int_s^t p_1(\xi, s) \, d\xi + t^{-1} \int_0^s p_0(\xi) \, d\xi$$

(9)

as it follows by considering the first observed atom in $R$ (which is also the first record). The boundary condition at $s$ is $p_1(s, s) = p_1(s)$. Equation (9) is partial but it is easily reduced to an ordinary differential equation with the help of the next lemma.

**Lemma 1** Given $s > 0$ and a constant $c$ suppose a function $g$ is in $C^1[s, \infty[$ and satisfies equation

$$g'(t) = -g(t) + \frac{1}{t} \int_s^t g(\xi) \, d\xi + \frac{c}{t}, \quad t \in [s, \infty[.$$

Then

$$g(t) = g'(s) s e^s I(t, s) + g(s).$$

(10)

where $g'(s) = -g(s) + cs^{-1}$.

*Proof.* Multiplying by $t$ and differentiating we kill the integral term and reduce the equation to

$$tg''(t) + (t + 1)g'(t) = 0.$$

(11)

Separating variables yields

$$g'(t) = \frac{e^{-t}}{t} s e^s g'(s).$$

Integrating from $s$ to $t$ and matching a boundary condition at $s$ gives the formula. $\square$

**Remark.** Note that $g(t)$ given by (10) is always monotone and for $t \to \infty$ goes to a limit obtained via replacing $I(t, s)$ by $I(s)$.

Applying lemma and writing solution in terms of the exponential integral functions, yields explicit formula for the performance of $\pi_s$

$$p_1(t, s) = I(t, s) e^s s p'_1(s) + p_1(s) = (e^s - 1 - sJ(s)) I(t, s) + e^{-s}J(s), \quad t > s.$$  

(12)

For optimal threshold we have $J(t_F) = 1$, therefore

$$v(t) = (e^{t_F} - t_F - 1)I(t, t_F) + e^{-t_F}, \quad t > t_F.$$  

(13)

We see that for $t > t_F$ the optimal best-choice probability $v(t)$ is a linear transform of the incomplete exponential integral. Passing to limit just amounts to taking the infinite integration bound:

$$v_F := (e^{t_F} - t_F - 1)I(t_F) + e^{-t_F}$$

with the approximate value 0.580164.
History and Remarks. The numerical value of $v_F$ was found in [12] by extrapolation of stopping values from the problem with fixed number of observations $n$. The exact formula for $v_F$ first appeared in [28] and is reproduced (with a sign flop) in [30]. Samuel [29] and Porosinski [24] also derived $p_1(\infty, s)$ (our [12] with $t = \infty$) by computing multidimensional integrals. The Poisson formulation appeared in [27], [3] and a power-series form of $v(t)$ was found in [2], see also [17] and Section 2.3 to follow. The box-area approach, formula (12) and its derivation are new. Partial differential equations for the value function appeared in [3], [27] but they were left unsolved, apparently because the time-space invariance of the problem was not recognised.

The transparent similarity of the finite $t$ and $t = \infty$ formulas, highlighted by (12) and (13), stress a major advantage of the Poisson framework. Also, the convergence rate of $p_1(t, t_F)$ to $v_F$ is better than exponential, determined solely by the convergence of the exponential integral. In the fixed-$n$ framework, the optimal probability decreases to $v_F$, with convergence rate only of the order of $n^{-1}$ (see [16]). Another distinguished feature of the Poisson approach is that solving the stopping problem for arbitrary $t$ essentially amounts to finding the optimum for small $t$, in contrast to discrete-time setting where the solutions differ wildly as $n$ varies.

2.3 Optimising the threshold. The optimal threshold $t_F$ has the property that the function $\partial_t p_1(\xi, s)$ has no break at $\xi = s$ while there is a break for all other thresholds. This property characterises $t_F$ as a root of the equation

$$tp''_1(t) + (t + 1)p'_1(t) = 0$$

which results from equating to 0 the derivative

$$\partial_t p_1(t, s) = e^s I(t, s)(p''(s) + (s + 1)p'(s))$$

and is most closely related to the differential equation (11) of similar form.

A deeper analysis going above the framework of this paper shows connection of the phenomenon with an optimal control problem, which becomes substantial when we consider other objectives and stopping sets more general than $[0, s]$. Here, we only establish the property in the context of a simple variational problem of finding an optimal switch.

Write the objective functional $p_1(t, s)$ as an integral with compound integrand

$$p_1(t, s) = \int_0^s p'_1(\xi) d\xi + \int_s^t \partial_t p_1(\xi, s) ds .$$

Suppose we begin sliding from $\xi = 0$ along the curve $p'_1(\xi)$ and at each time $s < t$ can switch to and keep sliding along another curve $\partial_t p_1(\xi, s)$ to $\xi = t$. Writing the first integrand in the form

$$p'_1(\xi) = \frac{e^{-\xi}}{\xi}(p'_1(\xi) \xi e^{-\xi})$$

we see that switching at $s$ means freezing the bracketed factor and proceeding with the integrand

$$\partial_t p_1(\xi, s) = \frac{e^{-\xi}}{\xi}(p'_1(s) s e^s) ,$$
in accord with Lemma 1.

Direct geometric argument shows that for an optimal switch the integrands must be tangential to each other at the switch location. Indeed, let $\lambda$ be the frozen factor. The quantity $\lambda e^{-\xi} \xi^{-1}$ is increasing in $\lambda$ and goes to 0 or becomes unbounded as $\xi$ goes to $\infty$ or 0, respectively. On the other hand, $p'_1(\xi)$ is positive at 0 and has a unique sign change from + to −, thus only $\lambda > 0$ can correspond to optimal switch. If at some location $s_1$ the integrands meet transversally then there must be a further location $s_2$ where they meet as well. Without loss of generality we can select $s_2$ close enough to $s_1$ to avoid further intersection points between them. In case $s_1 > s_2$ switching at $s_2$ outperforms switching at $s_1$ because in this case $p'_1(\xi)$ crosses $\lambda \xi^{-1}e^{-\xi}$ from above. And in case $s_1 < s_2$ we improve $s_1$ by passing to a tangential point between $s_1$ and $s_2$; thus winning a piece of the area squeezed between the intersection points and ending up with a larger $\lambda$.

A dual argument treats $p_1(t,s)$ as a function of the variable $\lambda$. An optimal value of this parameter is then the largest among those values of $\lambda$ which make $p_1(\xi)$ and $\lambda \xi^{-1} e^{\xi}$ meet at some $s < t$.

Equating derivatives of the integrands in $\xi$ and then substituting $\xi = s$ we get (15). On the other hand, from (3) we find that for any $t$

$$tp''_1(t) - (t+1)p'(t) = p_0(t) - p_1(t)$$

thus (15) is equivalent to $p_0(t) = p_1(t)$ and $t_F$ is the unique optimum switch location. (In case $t < t_F$ it is optimal to keep with the first integrand all the way.)

2.3 Coupling. Coupling allows to consider best-choice problems simultaneously for all values of $t$ and leads eventually to a $t = \infty$ model. The following application of the method leads to a formula for $\partial_t p_1(t,s)$ and, to an extent, unravels (12).

Consider a rectangle $R_1 = [0, 1] \times [-t, 0]$ and a smaller rectangle $R_2 = [0, 1] \times \left[-(t - \delta), 0\right]$. We wish to compare performance of threshold policy $\pi_s$ in $R_1$ and $R_2$ for small $\delta$.

Suppose $t > s$. Clearly, when $\pi_s$ is applied to $R_1$ or $R_2$ the outcomes can be different, but this distinction is limited to the event $A$ that the first atom in $R_1$, say $a$, appears in the small rectangle $[1 - s/t, 1] \times [-t, -(t - \delta)]$, up to a negligible event of probability $o(\delta)$. In the event $A$ there is no stop before the exploration process enters the domain $[1 - s/t, 1] \times [-t, 0]$ and then $\pi_s$ stops at the first available atom. Assuming that $A$ does occur, $\pi_s$ stops at $a$ and this is the correct decision provided there are no further atoms in $[0, 1] \times \left[-(t - \delta), 0\right]$ (which were higher than $a$ with probability complimentary to $o(\delta)$); i.e. when, essentially, $R_2$ contains no PPP-atoms at all. Thus $\pi_s$ performs better in $R_1$ with probability $\delta e^{-t} st^{-1}$. Otherwise, there are some further atoms in $[0, 1] \times \left[-(t - \delta), 0\right]$ and $\pi_s$ picks the first of them, in which case $\pi_s$ fails in $R_1$ but may succeed in $R_2$. Conditioning on the number of atoms in $[1 - st^{-1}, 1] \times \left[-(t - \delta), 0\right]$ yields probability

$$\delta e^{-t} \sum_{k=1}^{\infty} \frac{s^k}{(k+1)!k}$$

in favour of $R_2$. It follows that

$$\partial_t p_1(t,s) = \frac{e^{-t}}{t} \left(s - \sum_{k=1}^{\infty} \frac{s^{k+1}}{(k+1)!k}\right), \quad t > s.$$
Same result in integral form is established by conditioning on the horizontal position of $a$:

\[
\frac{\partial_t p_1(t, s)}{t} = e^{-t} \int_0^s e^\xi (p_0(\xi) - p_1(\xi)) \, d\xi.
\]

For $t < s$, $\pi_s$ coincides with the greedy algorithm and same argument yields an integral formula for the derivative

\[
p_1'(t) = e^{-t} \int_0^t e^\xi (p_0(\xi) - p_1(\xi)) \, d\xi.
\]

Integration yields, once again, the best-choice probability \(12\).

The $t = \infty$ model is related to the PPP in the semi-finite ‘rectangle’ $[0, 1] \times -\infty, 0]$. Although the set of records is now infinite with probability one, the number of records above each level $-t$ is finite, and we can therefore speak of a finite best-choice problem embedded in the infinite problem (see \[15\] for details). The value $v_F$ is equal to the optimal probability of the best choice in the infinite problem.

### 2.4 The box-area process.

Fix a rectangle of area $t$ and let $a$ be the leftmost record. The area to the left from $a$ is distributed like $(E - t)_+$ where $E$ is a standard exponential random variable (the distribution has a defect because in the event $E > t$ the PPP puts no atoms in the rectangle). Furthermore, the vertical position of $a$ is uniformly distributed, thus the box area of the first record to observe is distributed like $(E - t)_+ U$ with $U$ being standard uniform.

We find it intuitive to think of detector moving at variable speed adjusted to the configuration of records, so that the area of the current box is explored at unit rate. With this convention, the time between $a$ and the next detected record is distributed like $(E - \alpha(a))_+$.

The random transformation

\[
t \rightarrow (E - t)_+ U
\]

defines a Markov transition function on nonnegative reals. We define the box-area process to be the discrete-time Markov chain with this transition function. Given that the process starts at $t$, its path has the same distribution as the sequence of box areas of consecutive records in a rectangle of area $t$. (Speaking of paths we mean the states visited upon departure from $t$). Each path of the process is decreasing and eventually gets absorbed in 0.

It is seen that the box-area process is a combination of two classical models. With only the first factor present, \[18\] were the homogeneous Poisson process, while setting $E = 0$ we get the stick-breaking transformation $t \rightarrow tU$ (which generates a multiplicative renewal process, i.e. the exponential of the homogeneous Poisson process). An explicit formula for the transition function follows by integrating over the domain \(\{a \in R : \alpha(a) > s\}\) within $R = [0, 1] \times [0, t]$:

\[
P(t, [s, t]) = \int_{s/t}^1 d\xi \int_0^{t-s/x} e^{-\xi} \, d\xi = e^{-t} \left( e^t - e^s - s \int_x^t x^{-1} e^x \, dx \right), \quad s < t,
\]

and the absorption probability is $P(t, \{0\}) = e^{-t}$.

Extending our previous definition define $p_j(t, s)$ to be the probability that the box-area process has $j$ visits in $[0, s]$ conditionally on the initial state $t$ (in case $t < s$ we do not count $t$ as a visit). In terms of the best-choice problem $p_j(t, s)$ can be interpreted as the probability that $\pi_s$ stops at a record followed by $j - 1$ further records, in accord with the former definition of $p_1(t, s)$ in Section 2.2.
Obviously, 
\[ p_j(t, s) = p_j(t) \quad \text{for } t < s, \]
and for \( t > s \) the jump-counts distribution is given by the formula
\[ p_j(t, s) = (se^s p_j'(s))I(t, s) + p_j(s) \]
which extends (12) and appears as a solution to the Cauchy problem
\[
\begin{align*}
\partial_t p_j(t, s) &= -p_j(t, s) + t^{-1} \int_s^t p_j(\xi, t) \, d\xi + t^{-1} \int_0^s p_{j-1}(\xi) \, d\xi \\
p_j(s, s) &= p_j(s),
\end{align*}
\]
in exactly the same way that lead us to (12). Computations with (20) are sometimes facilitated by replacing the derivative using the formula
\[ p'_j(s) = \frac{e^{-s}}{s} \int_0^s e^\xi (p_{j-1}(\xi) - p_j(\xi)) \, d\xi \]
which can be derived from (5) or proved by analogy with (17). Alternative way to treat the derivative is to use recursion (6), to get the solution in the form
\[ p_j(t, s) = (1 - I(t, s)) s e^s p_j(s) + I(t, s)((-1)^{j-1} p_{j-1}(-s) - e^s p_{j-1}(-s)) \]
which also involves a function of negative argument.

Applying (20) we obtain the probability that \( \pi_s \) selects some record
\[ 1 - p_0(t, s) = 1 - e^{-s} + s I(t, s) \quad s < t \]
which is also the probability that the minimum box area (which is attributed to the highest atom) is less than \( s \).

Let \( \phi(t, s, x) \) be the probability that the chain, which starts at \( t \), has the first visit in \([0, s]\) within the subinterval \([x, s]\), \( t \geq s \geq x > 0 \). In extension of (4) we have the relation
\[ p_j(t, s) = \int_0^s p_{j-1}(x) \, dx (1 - \phi(t, s, x)) \]
(differential in \( x \)). Distribution \( \phi(t, s, x) \) satisfies a differential equation of the familiar type
\[
\begin{align*}
\partial_t \phi(t, s, x) &= -\phi(t, s, x) + \frac{1}{t} \int_s^t \phi(\xi, s, x) \, d\xi + \frac{s - x}{t} \\
\phi(s, s, x) &= P(s, [x, s]).
\end{align*}
\]
Solving the equation with the help of Lemma 1 we compute
\[
\partial_x (1 - \phi(t, s, x)) = I(t, s)e^s + (e^{-s} - s I(t, s)) \int_x^s e^y y^{-1} \, dy.
\]
The function \( \partial_x(1 - \phi(t, s, x)) \) is the density of the box area of the record selected by \( \pi_s \). Therefore, the probability of best choice has another integral representation which is a special case of (23)

\[
p_1(t, s) = \int_0^s e^{-x} \, dx \, (1 - \phi(t, s, x)).
\]

One sees that it is the same as (12) by explicit integration based on the identity

\[
\int_0^s e^{-x} \, dx \int_x^s e^y \, dy = J(s).
\]

Setting \( t = \infty \) can be interpreted as a one-point compactification of the state-space of the chain. This corresponds to PPP records in \( R = [0, 1] \times [-\infty, 0] \) and provides natural interpretation to \( t \to \infty \) limits. Thus \( p_j(\infty, s) \) is the distribution of the number of records with box-areas less than \( s \). And \( p_1(\infty, s) \) is the probability of best choice in the infinite problem when \( \pi_s \) is exploited.

Note that the stopping set of \( \pi_s \) is a compact set \([0, s]\), when viewed from the box-area perspective. For \( R \) as above, the corresponding set is an infinite domain between a hyperbola and the right-side of \( R \).

**Remark.** Formula (20) has a touch of mystery. Typically it is hardly possible to directly express the events underlying probabilities (20) and the like in terms of the PPP configuration. For \( R = [0, 1] \times [-t, 0] \) the second term in the RHS of (20) could be interpreted as the probability of \( j \) records above \(-s\), but the first term can be negative, thus it is not at all obvious that the sum is positive.

In case of \( p_0(t, s) \), i.e. for probability of no box areas less than \( s \), a smooth explanation is possible, namely via location of the highest atom. Note that \( e^{-s} \) is the chance for no atoms (thus for no records) above \(-s\). Given this event the ordinate of the highest atom \( a \) has substochastic density 

\[e^{s-\xi} \, 1_{\{\xi \in (s, t)\}} \, d\xi.\]

Given the height \( \xi \), \( a \) must be located to the right from \( 1 - s / (\xi + s) \), to guarantee the box-area not exceeding \( s \), and integrating \( \xi \) out yields \( e^{-s} - s \, I(t, s) \), a probability complimentary to (22).

### 2.5 Path distribution and the EU-representation.

In this section we consider semi-finite rectangle \( R = [0, 1] \times [-\infty, 0] \) and supply random variables associated with the PPP configuration above \(-t\) with subscript \( t \).

Let \((A_j)\) be the sequence of box areas of the records in \( R \) enumerated in time-reverse order. That is to say, \( A_1 \) is the box area of the last record (= the highest atom), \( A_2 \) of the record before the last, etc. Let \((A_{jt})\) be a finite initial subsequence of \((A_j)\), corresponding to the records above \(-t\). Coupling allows to identify \((A_{jt})\) with the collection of states visited by the box-area process started at \( t \) (\( t \) itself excluded). As \( t \to \infty \) the sequence \((A_{jt})\) converges to \((A_j)\) with probability one.

**Warning.** As a point set \((A_{jt})\) is a truncation of \((A_t)\), but it is not \((A_t)\) intersected with \([0, t]\). It it a formidable if at all realistic task to directly derive distribution of \((A_{jt})\) from the distribution of \((A_t)\).

With the convenience \( I(t, s) = 0 \) for \( t < s \) and \( p_{-1}(s) = 0 \) the distribution of \( A_{kt} \) is

\[
1 - P(A_{kt} < s) =
\]
\[
\sum_{j=0}^{k-1} p_j(t,s) =
\]
\[
I(t,s) \sum_{j=0}^{k-1} \int_0^s e^\xi (p_{j-1}(\xi) - p_j(\xi)) d\xi + \sum_{j=1}^{k-1} p_j(s) =
\]
\[
- I(t,s) \int_0^s e^\xi p_k(\xi) d\xi + \sum_{j=1}^{k-1} p_j(s)
\]
as a consequence of (20) and (21).

There is a representation for the time-reversed path \((A_{jt})\) in terms of standard exponential and uniform random variables, which we call the \textit{EU-representation}. Note that the transform \(t \rightarrow (E - t)_+ U\) which defines the box-area chain is related to the following (distributional) construction of the sequence of records in a finite rectangle: skip an exponentially distributed area from the left and then break off a uniform portion of the rectangle from below. The inverse operation amounts to skipping exponentially distributed area from the top and then breaking off a uniform portion from the right.

The inverse operation makes sense also in semi-finite \([0,1] \times [-\infty,0]\), when we identify the uniform breaking with selecting a random point on the upper side of the region \textit{south-west} from record. Calculating the box areas we see that \((A_j)\) become \textit{jointly} represented as

\[
A_k = \left( E_1 + \frac{E_2}{U_1} + \ldots + \frac{E_k}{U_1 \cdots U_{k-1}} \right) (1 - U_1 \cdots U_k)
\]
with \((E_j),(U_j)\) being jointly independent exponential and uniform random variables, respectively. In the event

\[
\left\{ E_1 + \frac{E_2}{U_1} + \ldots + \frac{E_k}{U_1 \cdots U_{k-1}} < t \right\}
\]
same representation is valid for \((A_1,t,\ldots,A_{kt})\). For \(t \rightarrow \infty\) the constraint (26) becomes void and we arrive at an interesting conclusion.

\textbf{Theorem 1} \textit{The distribution of random variable}

\[
A_k = \left( E_1 + \frac{E_2}{U_1} + \ldots + \frac{E_k}{U_1 \cdots U_{k-1}} \right) (1 - U_1 \cdots U_k)
\]
is given by

\[
P(A_k > s) = - I(s) \int_0^s e^\xi p_k(\xi) d\xi + \sum_{j=1}^{k-1} p_j(s).
\]

\textbf{Examples.} We compute

\[
P(A_1 > s) = e^{-s} - s I(s)
\]
\[
P(A_2 > s) = -s I(s) J(s) + e^s I(s) - I(s) - s I(s) + e^{-s} + e^{-s} J(s).
\]
The probability of best choice becomes a difference representation
\[ p_1(\infty, s) = P(A_1 < s < A_2) = P(A_2 > s) - P(A_1 > s) = (e^s - s J(s) - 1)I(s) + e^{-s} J(s). \]

Note that marginal distributions of \( A_1 \) and \( A_2 \) alone do suffice for this computation, because we always have \( A_1 < A_2 \).

**Remark.** Direct computation of the distribution of \( A_k \) from the EU-representation works smoothly only for \( k = 1 \). Already for \( k = 2 \) the computing requires skillful multidimensional integration which was performed in [28], [29] and [24] (the integration could be a bit simplified by expressing the event via marginals and using an explicit formula for the density of sum of exponential variables, as found in Feller’s textbook).

2.6 Characterisation. We will show that the distribution of record-counts (3) uniquely characterises the box-area process as a Markov chain.

**Theorem 2** There exists a unique Markov chain on \([0, \infty)\) which has absorbing state 0, decreasing paths and for any initial state \( t \) the distribution of the number of jumps on \((0, t)\) given by (3).

The idea is to show that linear combinations of functions \( p_j(\cdot) \) span \( C[0, t] \) for any \( t \). With a Stone-Weierstrass argument in mind, we see that the functions separate points and linearly independent, thus span an infinite-dimensional space. However it is not clear whether the set of finite linear combinations of functions \( p_j(t) \) (or of power-series \( e^t p_j(t) \)) is closed under multiplication (apparently not). We will resolve the complication by proving an inversion formula, expressing quasi-monomials as infinite series in the \( p_j \)'s. This will imply that infinite series in functions \( e^t p_j(t) \) do form a ring.

**Lemma 2** (inversion formula) For any \( j \)
\[
e^{-t} \frac{t^j}{j!} = \sum_{k=j}^{\infty} \sigma_2(k, j) \frac{j!}{k!} (-1)^{k-j} p_k(t),
\]
where \( \sigma_2(k, j) \) are Stirling numbers of the second kind.

**Proof.** For any \( m \) the matrix \( (\sigma_2(k, j) j! (-1)^{k-j})_{i,j=1}^{m} \) is inverse to \( (\sigma_1(k, j)/k!)_{i,j=1}^{m} \), and both matrices are lower-triangular. We need to show that formal inversion of the analogous infinite matrices makes sense, i.e. that the involved series converge.

Splitting the sum in (3) at \( m \), swapping summations and using the finite inversion we obtain
\[
e^{-t} \sum_{k=j}^{m} \sigma_2(k, j) j! (-1)^{k-j} p_k(t) =
\]
\[
e^{-t} \frac{t^j}{j!} + e^{-t} \sum_{k=j}^{m} \sigma_2(k, j) j! (-1)^{k-j} \sum_{i=m+1}^{\infty} \frac{\sigma_1(i, k) t^i}{i!} \frac{i!}{i!}.
\]
Denote the rest term by $\rho_m$. Pulling out the homogeneous factor, we get

$$\sum_{i=m+1}^{\infty} \frac{\sigma_1(i,k)}{i!} \frac{t^i}{i!} = \frac{t^{m+1}}{(m+1)!} \left( \frac{\sigma_1(m+1,k)}{(m+1)!} + \frac{\sigma_1(m+2,k)}{(m+2)!} \frac{t}{m+2} + \ldots \right) <$$

$$< \frac{t^{m+1}}{(m+1)!} \left( 1 + \frac{t}{m+2} + \frac{t^2}{(m+2)(m+3)} + \ldots \right) < \frac{t^{m+1}}{(m+1)!} \cdot \text{const}$$

where the constant does not depend on $m$. By definition, $\sigma_2(k,j)$ is the number of partitions of a set with $k$ elements in $j$ parts hence it does not exceed the number of labelled partitions in at most $j$ parts, which is $j^k$. Using the bound, we estimate

$$\rho_m < \text{const} \cdot \frac{t^{m+1}}{(m+1)!} \sum_{k=j}^{m} j^k < \text{const} \cdot \frac{(tj)^{m+1}}{(m+1)!}$$

where the constant depends on $t$ and $j$ but not on $m$. Obviously, $\rho_m \to 0$ as $m \to \infty$, hence the series in the inversion formula converges to the conjectured $e^{-t^j}/j!$.

**Example.** The simplest instance of the inversion formula is

$$e^{-t} t = p_1(t) - p_2(t) + p_3(t) - \ldots$$

For higher order monomials the coefficients in the series are unbounded.

**Proof of the Theorem.** By the Stone-Weierstrass theorem, linear combinations of monomials $e^{-s}s^k$ (quasi-polynomials) are dense in $C[0,t]$. By the inversion formula, each quasi-polynomial in $s$ is representable as a converging series in $p_k$’s, hence these functions span $C[0,t]$ as well. It follows that any finite measure $\mu$ on $[0,t]$ is uniquely determined by the ‘moments’

$$\int_0^t p_k(s) \mu(ds).$$

Thus, if a Markov chain has transition measure $\mu(t,ds)$, decreasing paths, and the distribution of jump counts as given by (3), we must have

$$p_k(t) = \int_0^t p_{k-1}(s) \mu(t,ds)$$

which determines $\mu(t,\cdot)$ inambiguously. Because this holds for arbitrary $t$, the transition function must coincide with that for the box-area process (19).

**Remark.** With more work, the path monotonicity condition in the theorem can be omitted.

### 3 Random horizon problem – vertical cut.

**3.1 The vertical cut problem.** Fix a rectangle $R$ of area $t$ and suppose it is partitioned by a vertical line $V$ which splits off $Ut$ units of the area from the right, where $U$ is a standard
uniform random variable independent of the PPP. Suppose the rectangle is scanned from the left to the right, and the objective of the observer is to maximise the probability of stopping at the atom which is highest among atoms in $R$ to the left from $V$.

In the VC-problem the observer knows $R$ and the distribution of $V$, but the exact position of random horizon is unknown. A selection policy should be adapted to the PPP but not to $U$.

We make distinction between two versions of the problem. According to version I, the observer always knows whether the vertical cut has been approached or not; and, of course, stops scanning when $V$ is reached. In version II the observer never learns the position of $V$.

The additional information in version I is worthless because there is no essential updating of the position of $V$, and thus the optimal policies are the same. However, formulas for the conditional distribution of the predicted number of records are different and only version I has a smooth formulation in terms of box areas. We will consider here version I but will return to version II in Section 5 on different occasion.

As in the full-information problem, the shape of $R$ does not matter because the affine isomorphism of rectangles with same area also respects a uniform random cut. Each time a record to the left from $V$ is detected the conditional distribution of $U$ becomes scaled uniform, and this implies readily that an optimal policy must be adapted to the box-area process in $R$ which is to be truncated properly to take into account approaching $V$.

The exposition to follow is based on same ideas as in the full-information problem, therefore we omit many details. Loosely speaking, it is all much the same, but $I(s, t)$ must be replaced by the exponential integral of degree 2

\[ I_2(t, s) := \int_s^t \frac{e^{-\xi}}{\xi^2} d\xi = \frac{e^{-s}}{s} - \frac{e^{-t}}{t} - I(t, s), \quad I_2(s) := I(\infty, s). \]

Let $q_j(t)$ be the probability of $j$ records to the left from $V$. Conditionally on $k$ atoms in $R$, the distribution of the number of atoms to the left from $V$ is uniform on $\{0, \ldots, k\}$, hence

\[ q_j(t) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!} \sum_{i=0}^{k} \frac{\sigma_1(i, j)}{i!}. \]

In particular,

\[ q_0(t) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{(k+1)!}, \quad q_1(t) = e^{-t} \sum_{k=1}^{\infty} \frac{t^k}{(k+1)!} h(k) \]

where $h(k) := 1 + 2^{-1} + \ldots + k^{-1}$ is the harmonic number. Two most important cases are

\[ q_0(t) = \frac{1 - e^{-t}}{t}, \quad q_1(t) = \frac{-J(-t) - e^{-t} J(t)}{t}. \]

A basic relation with functions (3) is

\[ q_j(t) = t^{-1} \int_0^t p_j(s) \, ds. \]

as one sees by averaging over the random horizon. Yet another relation appears when we write (3) in the form $p_j'(t) = -p_j(t) + q_{j-1}(t)$ and compare it with what is obtained by differentiating $e^t p_j(t)$ using recursion (3):

\[ q_j(t) = \frac{e^{-t}}{t} \left( (-1)^{j-1} p_{j-1}(-t) - e^t p_{j-1}(t) \right). \]
The counterpart of (5) becomes
\[ q'_j(t) = -(1 + t^{-1})q_j(t) + t^{-1} \int_0^t q_{j-1}(s) \, ds. \]
with the newly appearing factor \((1 + t^{-1})\) reflecting the risk of approaching \(V\) at probability rate \(dt/t\). But we also have another differential equation which follows from (28)
\[ tq'_j(t) = p_j(t) - q_j(t). \]

Let \(u(t)\) be the optimal probability of stopping at the highest atom to the left from \(V\). The DP-equation (dynamic programming) for \(u\) becomes
\[ u'(t) = -u(t)(1 + t^{-1}) + t^{-1} \int_0^t \max (q_0(s), u(s)) \, ds, \quad u(0) = 0, \]
and is resolved by the same method we applied to (4). Define \(t_P = 2.11982\ldots\) to be the unique positive root of any of four equivalent equations:
\[
\begin{align*}
q_0(t) &= q_1(t) \\
-J(-t) - e^{-t}J(t) &= 1 - e^{-t} \\
p_1(t) - p_0(t) &= 1 - J(-t) \\
\sum_{j=2}^{\infty} \frac{1}{j} \sum_{k=j+1}^{\infty} \frac{t^{k-1}}{k!} &= 1.
\end{align*}
\]
The uniqueness follows by monotonicity and for the same reason
\[ q_0(t) > q_1(t) \iff t < t_P. \]

It follows that we are again in the monotone case of optimal stopping, hence an optimal policy is the threshold policy \(\pi_{t_P}\), prescribing to choose the first record to the left from \(V\) with the box area less than \(t_P\) (if any).

Let \(q_j(t, s)\) be the probability of \(j\) records to the left from \(V\) with box areas less than \(s\), in a rectangle of area \(t\). Then \(q_1(t, s)\) is the probability of best choice with \(\pi_s\) and the optimal probability equals \(u(t) = q_1(t, t_P)\).

The relevant Cauchy problem becomes
\[ \partial_t q_j(t, s) = -q_j(t, s)(1 + t^{-1}) + t^{-1} \int_s^t q_j(\xi, t) \, d\xi + t^{-1} \int_0^s q_j(\xi, t) \, d\xi, \quad t > s \]
with the initial condition \(q_j(s, s) = q_j(s)\). The analogue of Lemma 1 carries over in the form of

**Lemma 3** Given \(s > 0\) and a constant \(c\) suppose a function \(g\) is in \(C^1[s, \infty)\) and satisfies equation
\[ g'(t) = -g(t) \left(1 + \frac{1}{t} \right) + \frac{1}{t} \int_s^t g(\xi) \, d\xi + \frac{c}{t}, \quad t \in [s, \infty[. \]

Then
\[ g(t) = g'(s) s^2 e^s I_2(t, s) + g(s) \]
where \(g'(s) = (cs^{-1} - (1 + s^{-1}) g(s))\).
and leads to the solution

\[(32) \quad q_j(t, s) = I_2(t, s)s^2 e^s q'_j(s) + q_j(s) \quad t > s.\]

Using (30) the formula in case \(j = 1\) takes form

\[q_1(t, s) = I_2(t, s) e^s s(p_1(s) - q_1(s)) + q_1(s) = I_2(t, s)(sJ(s) + e^s J(-s) + J(s)) - \frac{1}{s}(J(-s) + e^{-s} J(s))\]

which is simplified in the limit, when we express \(I_2(s)\) via \(I_1(s)\), as

\[q_1(\infty, s) = -I(s)(J(-s)e^s + J(s) + sJ(s)) + e^{-s} J(s)\]

(probability of best choice with \(\pi_s\) in the infinite problem). The optimal best-choice probability is obtained by substituting \(s = t_P\):

\[u(t) = q_1(t, t_P) = I_2(t, s) e^s s(sJ(s) - e^s + 1) + \frac{1 - e^{-s}}{s}\]

and for \(t = \infty\) this further simplifies to

\[(33) \quad q_1(\infty, t_P) = I(t_P) \left( e^{t_P} - t_P J(t_P) - 1 \right) + e^{-t_P} J(t_P).\]

which is also the optimal probability of best choice in the infinite VC-problem. The right-hand side of the last formula is the Petruccelli’s value \(v_P\).

The function \(\partial_t q_1(t, s)\) has a break at \(t = s\) for any \(s \neq t_P\). Similarly to \(t_F\) in Section 2.3 threshold \(t_P\) can be interpreted as an optimal switching location where \(q'_1(s)\) becomes tangential to a curve \(\lambda e^{-s}s^2\). The ‘no-corner’ condition at \(t_P\) characterises this threshold as a unique root of \(tq''_1(t) + (t + 2)q'_1(t) = q_0(t) - q_1(t)\), and this equation is equivalent to \(q_1(t) = q_0(t)\) because \(q_1\) satisfies the differential equation

\[tq''_1(t) + (t + 2)q'_1(t) = q_0(t) - q_1(t)\]

which in turn is a consequence of \((29)\).

Remark. The right-hand side of (33) appeared first in [21] as the limit best-choice probability in fixed-\(n\) partial information problem, as described in the Introduction.

For the random horizon problem, our argument seems to be the first complete proof that Petruccelli’s formula also yields \(q_1(\infty, t_P)\). Porosinski [24] attempted to show that \(v_P\) is the limit in the discrete-time problem with uniform random number of observations, but his argument has a gap. On bottom of p. 325 he confused conditional and unconditional best-choice probabilities and left without proof an equality on bottom line 2 (which was nevertheless correct by coincidence with the FI-problem, as discovered by Samuels [29]). Samuels [29] expressed \(q_1(\infty, t_P)\) as a multidimensional integral and partly using numerical integration justified the value with the precision Mathematica can give.

3.2 Box-area process. The box-area Markov chain related to the the VC-problem is the sequence of box areas associated with the records to the left from \(V\). To make clear distinction
with the process introduced in Section 2.4 let us call the new chain $Q$-process, and the basic box-area process the $P$-process.

One-step transition of the $Q$-process is given by the scheme

$$t \rightarrow (t - E)U_1 1_{\{E < tU_2\}}$$

where $E, U_1, U_2$ are independent exponential and uniform random variables, respectively. This can be given a continuous time interpretation, as follows. Starting with area $t$, during a period of length $E$ the area is explored at unit rate unless the process gets absorbed in the meantime, with absorption probability rate being $ds/s$. If the absorption does not occur, at time $t - E$ the new box-area is obtained by stick-breaking $(t - E) \rightarrow (t - E)U_1$.

We will denote $(B_{jt})$ the time-reverse sequence of states visited by the $Q$-process conditioned on start at $t$ and $(B_j)$ the sequence associated with records in the semi-finite rectangle. To unify exposition, let us consider the semi-finite compactified rectangle $[0, 1] \times [-\infty, 0]$, with obvious interpretation of the random vertical cut. The sequence $(B_j)$ is associated with records to the left from $V$ and the sequence $(B_{jt})$ with records which are also above $-t$. This is just the coupling approach for the VC-problem.

Since $Q$-process is obtained by truncating the set of records, the sequence $(B_j)$ (or $(B_{jt})$) is a random shift of $(A_j)$ (respectively $(A_{jt})$) by a few positions. However, there is no transparent distributional connection between the processes.

Digression. Given $(A_j)$ the shift-size depends on the full sequence. This claim is based on the following fact about the shape of record sequence (see [14] and [9]. In the (unlikely) event that a fixed rectangle contains a large number of records they tend to concentrate near diagonal, thus a random cut splits away a large part of $(A_j)$, which is certainly not typical. Whatever the values of, say $A_1, \ldots, A_k$, the number of records in the rectangle is likely to be moderate, and the cut isolates a few of the records. Thus looking at a finite piece of $(A_j)$ does not allow to definitely decide how many of the entries should be removed to get $(B_j)$.

Multiplying the integrand in (19) by $1 - \xi/t$ we compute the transition function for the $Q$-process as

$$Q(t, [s, t]) = \frac{e^{s-t} + t - 1 - s}{t}$$

and the absorption probability is $Q(t, \{0\}) = q_0(t) = (1 - e^{-t})/t$. Another piece of transition function is

$$Q(t, [0, s]) = \frac{e^{-t} - e^{s-t} + s}{t}, \quad t > s$$

and they are related through $Q(t, [0, s]) + Q(t, [s, t]) = 1 - q_0(t)$ for $t > s > 0$.

The transition function satisfies a differential equation

$$\partial_t Q(t, [0, s]) = -Q(t, [0, s]) \left(1 + \frac{1}{t}\right) + \frac{\min(s, t)}{t}$$

obtained by conditioning on the first observation. The equation is valid for arbitrary $t$ and $s$ and can be solved directly by separating variables and variation of constant. For future reference we note that $Q(t) = Q(t, [0, s])$ also satisfies

$$tQ'' + (t + 2)Q' + Q - 1_{\{t < s\}}(t) = 0$$
as obtained by differentiation.

A $Q$-analogue of $\phi(t, s, x)$, the probability that the process has its first visit on $]0, s]$ within subinterval $[x, s]$ is

$$
\psi(t, s, x) = I_2(t, s) \partial_s Q(s, [x, s]) + Q(s, [x, s]).
$$

and an integral representation of best-choice probability $q_1(t, s)$ follows as in Section 2.4.

The distribution of counts $q_j(t)$ uniquely characterises the $Q$-process. One way to show this is to use an explicit inversion formula which represents monomials $e^{-t}t^j/(j + 1)!$ as series in $q_j(t)$’s, namely with coefficients $\sigma_2(j, k)k! - \sigma_2(j, k + 1)(k + 1)!$. But once we have established a similar result for the $P$-process a reduction is possible.

**Theorem 3** There exists a unique Markov chain on $[0, \infty)$ which has absorbing state $0$, decreasing paths and for any initial state $t$ the distribution of the number of jumps on $[0, t]$ given by (27).

**Proof.** Let us show that the functions $q_j(s), s \in [0, t]$ span a dense subspace in $C[0, t]$. Integrating the inversion formula in Lemma 2 we obtain

$$
i_k(t) = \sum_{j=k}^{\infty} \sigma_2(j, k)k!(-1)^{k-j}q_j(t)
$$

where

$$
i_k(t) = \frac{1}{t} \int_0^t e^{-s} \frac{s^k}{k!} ds.
$$

But since $i_k$’s are representable via $q_j$’s, same applies to quasipolynomials which can be recovered by recursion

$$
i_k(t) = -e^{-t}t^{k-1}/k! + i_{k-1}(t).
$$

The density claim follows, and the rest is as in the proof of Theorem 2. □

### 3.3 Coupling.

We will derive now a formula for $\partial_t q_1(t, s)$ to demonstrate some combinatorics behind (32). Consider $R = [0, 1] \times [-\infty, 0]$ sectioned by a random vertical cut $V$, identified with uniform r.v. $U$.

Suppose $\pi_s$ is applied to finite rectangles $R_1$ and $R_2$ as in Section 2.3. The outcomes in $R_1$ or $R_2$ can be different only in the event $B$ that the leftmost atom in $R_1$, say $a$, appears in a random rectangle $[1 - s/t, U] \times [-t, -(t - \delta)]$ (which is an empty set in case $U < 1 - s/t$) in which case $\pi_s$ restricted to $R_1$ selects $a$.

Assuming that $B$ occurs, $\pi_s$ does right if there are no further atoms in $[0, U] \times [-(t - \delta), 0]$, as it happens when $U$ separates $a$ from these atoms. Conditioning on the total number $k$ of atoms in $R_1$ we find that the best-choice probability in favour of the larger rectangle $R_1$ is

$$
\delta \frac{s}{t} e^{-t} t^k \sum_{k=1}^{\infty} \frac{t^k}{(k + 1)!},
$$

where the factor $s/t$ stays for the probability of $U > 1 - s/t$. 

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On the other hand, the advantage for \( R_2 \) appears when \( B \) occurs, some further atoms are located to the right from \( a \) and to the left from \( U \), and the leftmost of these atoms is the highest in \([0, U] \times [-t, 0] \). Conditioning on the total number \( k \) of atoms in \( R_1 \) yields probability
\[
\frac{\delta}{t} e^{-t} s \sum_{k=2}^{\infty} \frac{t^k}{(k+1)!} h(k - 1)
\]
to the advantage of \( \pi_s \) in \( R_2 \). Putting two parts together yields the derivative
\[
\partial_t q(t, s) = e^{-t} \sum_{k=2}^{\infty} \frac{s^k}{(k+1)!} h(k - 1) - e^{-t} \sum_{k=1}^{\infty} \frac{s^k}{(k+1)!}
\]
which is a (quasi-) power-series form of the formula
\[
\partial_t q_j(t, s) = e^{-t} \int_{0}^{\min\{s,t\}} e^{\xi} (q_{j-1}(\xi) - q_j(\xi)) d\xi
\]
(36)

analogous to (16). Integrating we re-derive (32).

3.4 EU-representation. The EU-representation of the path for the \( Q \)-process differs from that for the \( P \)-process only in the first step of the algorithm: obtaining \( B_1 \) involves uniform breaking then exponential skip and repeated breaking, with the first break corresponding to the vertical cut. It follows that \( (B_j) \) can be jointly represented as
\[
B_k = \left( \frac{E_1}{U} + \frac{E_2}{U U_1} + \ldots + \frac{E_k}{U U_1 \ldots U_{k-1}} \right) (1 - U U_1 \ldots U_k)
\]
(37)

with the same notation as in (25). The first members of this sequence coincide with \( B_{kt} \) as long as the first bracketed factor does not exceed \( t \), and the finite sequence \( (B_{kt}) \) converges to \( (B_k) \) almost surely.

**Theorem 4** The distribution of random variable (37) is given by
\[
P(B_k > s) = -I_2(s) \int_{0}^{s} e^{\xi} q_k(\xi) d\xi + \sum_{j=1}^{k-1} q_j(s)
\]

**Example.** Expressing \( I_2(s) \) via \( I(s) \) we have
\[
P(B_1 > s) = (1 - e^s + s) I(s) + e^{-s}
\]
\[
P(B_2 > s) = e^{-s}(1 + J(s)) - I(s)(1 - e^s + e^s J(-s) + J(s) + s + s J(s))
\]
and a difference representation of the best-choice probability follows via
\[
q_1(t, s) = P(B_2 > s) - P(B_1 > s).
\]

3.5 Duality. There is a wonderful duality between \( P \) - and \( Q \)-processes which reveals as coincidence of probabilities of record counts in some finite rectangles and the semi-finite
rectangle $R = [0, 1] \times [-\infty, 0]$. A consequence is a series of coincidences in related stopping problems.

Recall that when $(A_j)$ and $(B_j)$ are considered as functions of the same record sequence in $R$ we have $A_j \leq B_j$. On the other hand, from the EU-representations of the sequences follows that if we construct $(A_j)$ through $E_j, U_j$ then a new sequence $(B'_j)$ defined by

$$B'_j = A_{j+1} - E_1(1 - U_1 \ldots U_{j+1})$$

has the same distribution as $(B_j)$. It follows that for any $s > 0$

$$P(A_1 > s) < P(B_1 > s) < P(A_2 > s) < P(B_2 > s) < \ldots$$

which means that sequences $(A_j)$ and $(B_j)$ are stochastically interlacing.

Because

$$p_j(\infty, s) = P(A_{j+1} > s) - P(A_j > s),$$

$$q_j(\infty, s) = P(B_{j+1} > s) - P(B_j > s)$$

we can expect that for certain values of $s$ we have

(38) $q_j(\infty, s) = p_j(\infty, s)$

and for some other $s$ we have

(39) $q_j(\infty, s) = p_{j+1}(\infty, s)$.

We stress that the quantities involved are related to record counts in the infinite $R$. The miracle is that the values of $s$ which solve the equations can be identified as the roots of analogous equations involving record counts in a finite rectangle.

**Theorem 5** For positive $s$ equation (38) is equivalent to $q_{j-1}(s) = q_j(s)$. Similarly, equation (39) is equivalent to $p_j(s) = p_{j+1}(s)$.

**Proof.** The equations relating two kinds of functions are

(40) $p'_j(t) = -p_j(t) + q_{j-1}(t)$

(41) $tq'_j(t) = p_j(t) - q_j(t)$

(the second follows from the definition of $q_j$). Expressing $I_2(s)$ via $I(s)$ and using (41) we find from (32)

$$q_j(\infty, s) = p_j(s) - I(s) e^s s(p_j(s) - q_j(s)).$$

Now if $q_{j-1}(s) = q_j(s)$ holds then by (41) and (20) also (38) is valid, and vice versa. Same argument works for $p_j(s) = p_{j+1}(s)$.

**Example.** First of all, $q_1(\infty, t_P) = p_1(\infty, t_P)$. That is to say, the optimal policy in the VC-problem has the same best choice probability $v_P$ in both VC- and FI-problems. Another coincidence is $q_0(\infty, t_P) = p_1(\infty, t_P)$, saying that the probability that no record is selected by $\pi_{t_P}$ in the VC-problem equals the optimal best-choice probability in the FI-problem.
Remark. Unwillingly, Porosinski proved that $p_1(\infty, t_P)$ coincides with Petruccelli’s $v_P$. It is this coincidence which vualised a gap in his argument for $q_1(\infty, t_P) = v_P$, see [24] and [29].

It is not hard to show that equation $q_j(t, s) = p_j(t, s)$ always has a solution $s$ for all $t$ sufficiently large. Explicitly, for $j = 1$ the equation becomes

$$(q_1(s) - q_0(s))I(t, s) = \frac{e^{-t}}{t} s q_1'(s)$$

and has a solution at least for $t > 3$. Analogous fact is also valid for the finite $-t$ counterpart of (39). These solutions depend on $t$ but they converge to the solutions characterised by the theorem exponentially fast.

3.6 A digression. In reply to Samuels’ challenge to explain the coincidence $v_P = p_1(\infty, t_P)$ we feel that there are indeed good reasons to further seek for an explanation but we will not dwell thereon. Instead we will show that the phenomenon is not isolated and even a stronger coincidence holds for ... an ordinary Poisson process. To stress the similarity we will use in this subsection notation confronting with the rest of the paper.

Consider the homogeneous PP on the positive half-axis, scanned from finite $t$ or $\infty$ to 0. Let $V$ be a standard exponential r.v. independent of the configuration of atoms. The number of atoms within $[0, s]$ has distribution $p_k(s) = e^{-s}s^k/k!$, and because occurrence of atoms to the right from $s$ does not affect the configuration to the left from $s$ we have trivially $p_k(t, s) = p_k(s)$.

Given $V < t$ the conditional distribution of the number of atoms on $(V, t)$ is

$$\frac{1}{1 - e^{-t}} q_k(t) = \frac{1}{1 - e^{-t}} \int_0^t p_k(t - \xi) e^{-\xi} d\xi = \frac{e^{-t}}{1 - e^{-t}} \frac{t^{k+1}}{(k+1)!}$$

and the distribution of the number of atoms on $[V, s]$ is

$$q_k(t, s) = \frac{1 - e^{-s}}{1 - e^{-t}} q_k(s).$$

Observe the identity $p_k(\infty, s) \equiv q_{k-1}(\infty, s)$. There is no need to write formulas: this follows from independence and the fact that the leftmost atom $A_1$ has the same distribution as $V$. Furthermore $s = k$ is the unique positive root of $p_k(s) = p_{k-1}(s)$ and $s = k + 1$ is the unique positive root of $q_k(s) = q_{k-1}(s)$.

We have therefore 4-fold coincidence

$$p_j(\infty, j) = p_{j-1}(\infty, j) = q_{j-1}(\infty, j) = q_{j-2}(\infty, j) = e^{-j} \frac{j^j}{j!}.$$

In optimal stopping terms this reads as follows. Denoting $A_j$ the $j$th smallest atom and $B_j$ the $j$th smallest atom among the atoms to the right from $V$ the rule ‘stop at the first atom to the left from $j$’ is optimal for recognising $A_j$, optimal for recognising $B_{j-1}$, and suboptimal but has the very same performance for recognising $A_{j-1}$ and same for $B_{j-2}$ (when the value of index $j$ makes sense).

Optimality of threshold $j$ for stopping on $A_j$ was derived by Bruss and Paindaveine [1] in a related context of optimal stopping at the $j$th last success in a sequence of independent trials.
Example. For \( j = 1 \) and \( j = 2 \) there is a relation to the Poisson versions of the ‘classical, no-information secretary problem’ and the ‘no-information secretary problem with uniform random horizon’ (see [25] and [30] for discrete time formulations). Suppose the observer of PPP in \([0, 1] \times [-\infty, 0]\) exploits a policy ‘stop at the leftmost record in \([s, 1] \times [-\infty, 0]\)’. This kind of policy is of ‘no-information’ type in the sense that it is adapted to the one-dimensional process of record times, making a decision independent on ‘actual value of item but solely on its relative rank’. If the objective is to pick the last record, the optimal \( s \) is \( e^{-1} \), and if the objective is to pick the last record before random vertical cut the optimum is at \( e^{-2} \), as everybody knows (and can derive either directly or from results for the discrete-time setting).

To put the problem into framework of this subsection, recall that the projection of the set of records onto horizontal axis is a PP (of record times) to the intensity \( dt/t \); thus applying the \( -\log \) transform we obtain a homogeneous PP on the positive half-axis, and the cut becomes an exponential r.v.

So we have \( p_1(\infty, 1) = p_0(\infty, 1) = q_0(\infty, 1) = e^{-1} \) which means that the optimum best-choice probability in the classical problem equals no-stop probability in this problem and also equals no-stop probability with same policy in the random horizon problem. And \( q_1(\infty, 2) = p_2(\infty, 2) = p_1(\infty, 2) = q_0(\infty, 2) = 2e^{-2} \) reads as boring as: the optimum probability in the random horizon problem equals the optimum probability with same policy in the poissonised classical problem, equals the no-stop probability with same policy in the random horizon problem.

A good occasion to celebrate the 40th anniversary of secretary problems.

4 Partial information - horizontal cut.

4.1 Motivation and setup. We start with a Poisson version of Petruccelli’s ‘partial information’ problem. Suppose the observer aims to select the highest PPP atom in a finite rectangle \( R = [0, t] \times [\theta, \theta - 1] \) of known shape but with unknown vertical position \( \theta \). Suppose the online information of the observer consists of the PPP configuration in \( R \) (but not outside the rectangle), to the left from the detector. Evaluating a policy by its worst-case performance, the question of interest is about the maximin policy and maximin probability of best choice.

A minimal sufficient statistics for \( \theta \) is a pair \((X, Y)\) where \( X \) is the vertical position of the lowest atom, and \( Y \) is the vertical position of the highest atom to the left from the current position of detector. From spatial independence of the PPP and the nature of the performance index follows that we can restrict consideration to policies adapting decisions to these variables. A newly appearing feature is that we need to take into account not only the records we considered before, which are the upper records, but also lower records (such that there are no other atoms to the south-west), because these are exactly the observations necessary to update the information about \( R \).

The problem has obvious shift-invariance in the sense that performance of a policy \( \pi \) when \( \theta = \theta_0 \) is the same as performance of a (properly defined) \( x \)-shift of \( \pi \) when \( \theta = \theta_0 + x \), for any \( x \). Invoking the ‘Hunt-Stein invariance principle’ of statistics one sees that we can further restrict to invariant policies, whose performance does not depend on the unknown parameter. Since a shift-invariant function of \((X, Y)\) depends in effect only on the range \( Y - X \), the range and the horizontal position of upper record are the sole parameters of interest when such a
record is detected.

Analysis of invariant policies and related structures is the subject of this section. Because performance of invariant policy is independent of $\theta$, we lose no generality when assuming that the rectangle is standardised to $R = [0, t] \times [0, 1]$.

When an upper record with horizontal position $s$ is detected, the conditional distribution of $Y$ given the range $r = Y - X$ is uniform on $[1 - r, 1]$. It follows easily that the distribution of the number of forthcoming upper records is $q_j((1 - r)(t - s))$ with $q_j(\cdot)$ as in Section 3 (by symmetry same applies to lower records). Repeating the familiar argument, the optimal decision whether to stop on upper record or not should be based on the criterion $(1 - r)(t - s) < t_P$.

This suggests that $(1 - r)(t - s)$ is a proper analogue of the box area from the VC-problem, and motivates the following definition. For $R = [0, t] \times [0, 1]$ we call $1 - r$ the corange and the quantity $(1 - r)(t - s)$ the corange-box area. The definition extends obviously to arbitrary rectangles. We stress that the corange-box area attributed to an upper record $a$ is determined via $a$ and the adjoint record, i.e. the rightmost lower record to the left from $a$.

So does the coincidence of stopping policies imply coincidence of best-choice probabilities? It is a ‘yes’ we wish to show, but the correct answer in the problem as we formulated it is ‘no’ for a very simple reason: the initial state in the partial information problem is not $t$. To be precise, speaking of the ‘initial state’ is inappropriate because the range is not defined before the leftmost atom in $R$ is detected. In fact, the first observed atom plays a special role: while being a unique upper and lower record, it serves as a cut which splits $R$ in two subrectangles supporting independent streams of upper and lower records. For conformity with the VC-model we shall assume that the range is 0 and the corange-box area is $t$ when the observation starts, this is equivalent to assuming that we start with unknown random reference value – the vertical position of an observation which is not counted as a record, but must be taken into account when establishing if a PPP atom is a record.

The final step in formulation of our model is swapping the subrectangles resulting from the random cut, without changing the orientation. The cutting line becomes the bottom of new rectangle while the bottom and the top sides merge into a new random cut.

The reason for this surgery is threefold. Firstly, we avoid considering two disjoint rectangles. Secondly, when $R$ is fixed requiring that a policy should be range-adapted is somewhat artificial and it is much more intuitive to think of the problem where the actual coordinates of atoms are observed, despite a bit nebulous reward function – probability of best-choice under unknown reference value. Finally, we make upper and lower records converge rather than diverge, and this point is crucial for a $t = \infty$ extension of the model.

To summarise, our final formulation of the HC-problem is this. A fixed rectangle $R$ of area $t$ is sectioned by a random uniform horizontal cut $H$. An observer knows $R$ and the distribution of $H$ but not position of the cut. An upper record is defined to be an atom $a$ which is below $H$ and is higher than all atoms below $H$ to the left from $a$; and a lower record is defined to be an atom $a$ which is above $H$ and is lower than all atoms above $H$ to the left from $a$. Each time an atom $a$ is detected, the observer learns the coordinates of $a$ and also learns whether $a$ is above or below $H$. The objective is to recognise the last upper record at the moment it is detected.

The corange attributed to upper record $a$ is the vertical distance between $a$ and adjoint lower record $b$, or the vertical coordinate of $a$ if $b$ is not defined. All rectangles with same area are affinely isomorphic, and the isomorphism respects the PPP, a uniform horizontal cut, and the structure of upper and lower record processes. At each stage the conditional distribution of
$H$ is uniform within the corange interval spanned on the current upper record and its adjoint. An optimal policy, say $\hat{\pi}_{t_P}$, stops at first atom which has corange-box area less than $t_P$.

4.2 VC=HC: quick proof. Apparently, the most complex and confusing feature in the HC-problem is that both upper and lower records affect the state. Let us look at the evolution of the corange in details. Start with $R = [0, t] \times [0, 1]$, thus the initial range is 0 and corange 1. The waiting time for the first change is a truncated exponential r.v. which is related with the leftmost atom $a$ to detect. The vertical position of $a$, say $Z$, is uniform, independent of $H$, thus the new range has the same distribution as a spacing, i.e. the size of interval between $H$ and $Z$, and the new corange has the same distribution as $\max(U_1, U_2)$ for two uniform r.v.'s. It follows that one-step decrement of the corange-box area is described by scheme $t \to (t - E)_+ \max(U_1, U_2)$. Independently of the decrement, $Z < H$ or $Z > H$ with same probability $1/2$, by exchangeability. In the event $Z < H$ we have an upper record, and a lower record otherwise (the upper records occur below $H$). It is seen that an upper record occurs after a geometric number of lower-record observations, provided the corange-process is not absorbed at 0 in the meantime.

This description allows to write a DP-equation for the best-choice probability $w(t)$. Using the form of $\hat{\pi}_{t_P}$,

$$w'(t) = -w(t) + \frac{1}{2} \int_0^{\min(1, t_P/t)} q_0(tx) \, dx^2 + \frac{1}{2} \int_{\min(1, t_P/t)}^1 w(tx) \, dx^2 + \frac{1}{2} \int_0^1 w(tx) \, dx^2.$$ 

Integration is over corange decrement $x$ having the $\max(U_1, U_2)$-distribution $x^2$; the third integral term stands for the event that the first atom to observe is a lower record, the first and second integral terms stand for the events that the first observation is an upper record and it is selected or skipped, respectively. It is instructive to put DP-equation for the VC-problem in similar form

$$u'(t) = -u(t)(1 + t^{-1}) + \int_0^{\min(1, t_P/t)} q_0(tx) \, dx + \int_{\min(1, t_P/t)}^1 u(tx) \, dx.$$ 

To see that the equations are equivalent assume $t > t_P$, substitute $x = t\xi$ and differentiate. This yields same $tu'' = -(t + 2)u'$ (recall that it was $tv'' = -(t + 1)v'$ in the FI-problem). From optimality of $t_P$ follows that solutions coincide for $t < t_P$ (as can also be seen from the equations directly) and both $u', w'$ are equal at $t_P$ and continuous, thus passing to higher order differential equation does not alter solution.

Although this argument offers a little of an explanation, the promised coincidence follows.

4.3 Corange-box area process. We define corange-box area process only for upper-record observations. Thus between two upper records, arbitrarily many lower records can contribute to the change of state.

**Theorem 6** The corange-box area Markov chain associated with the HC-problem has the same distribution as the $Q$-process in the VC-problem.

**First proof.** The number of visits in each interval $[0, t]$ has the same distribution $q_j(t)$ as for the $Q$-process. But by Theorem 3 such a process is unique, thus the processes have the same distribution. □
**Second proof** is based on computing the transition function for the corange-box area chain. Denoting temporarily the transition probability \( \hat{Q} \) we can write

\begin{equation}
\partial_t \hat{Q}(t, [0, s]) = -\hat{Q}(t, [0, s]) + \frac{1}{2} \int_0^1 \hat{Q}(tx, [0, s]) dx^2 + \frac{1}{2} \int_0^{\min(s/t, 1)} dx^2.
\end{equation}

The first integral term stands for the event that the first atom to observe is a lower record, in which case there is no transition from \( t \) to \([0, s]\) and the new corange is \( tx \). The second term stands for the event that the first observation is an upper record, and the decrement is larger than \( t - s \) in case \( t > s \) or arbitrary in case \( t < s \).

To transform (42) change the variable of integration to \( \xi = tx \) – this yields factor \( t^{-2} \) at the integral – then multiply equation by \( t^2 \), differentiate and divide by \( t \). The integral goes and we see that \( \hat{Q}(t, [0, s]) \) satisfies (35), same equation as for \( Q(t, [0, s]) \). Both functions coincide with \( 1 - q_0(t) \) for \( t < s \) and there is no break at \( t = s \), thus by uniqueness

\( \hat{Q}(t, [0, s]) = Q(t, [0, s]). \)

It follows that the corange-box area process in the HC-problem is identical, stochastically, with the \( Q \)-process of genuine box areas from VC-problem.

### 4.4 Hor-Ver choice

A randomised model enables to couple VC- and HC-problems and to introduce some symmetry. Suppose a square is partitioned by uniform random horizontal and vertical cuts \( H \) and \( V \) which meet at point \( O \). Two observers Ver and Hor learn the PPP configuration in the square as the same vertical detector moves from the left to the right. Each of the observers can drop out each time an atom is detected and the stop is a win if the last detected atom is the highest among the PPP atoms in the square south-west from \( O \). Hor knows the position of \( V \) but not \( H \); each time an atom is detected she is told if the atom is above \( H \) or below. Ver knows the position of \( H \) but not \( V \).

Call an atom \( a \) ‘upper record’ if \( a \) is the highest among all the PPP atoms below \( H \) seen so far. Both Hor and Ver hunt for the last upper record in the rectangle with vertex \( O \). Call an atom \( a \) ‘lower record’ if \( a \) is the lowest among all PPP atoms above \( H \) seen so far.

The appeal of this model is that the observers learn the same configuration and have the same objective. The surprise is that they perform equally well by using very different policies, optimal for different kinds of information flows. Clearly, the PPP configuration to the right from \( V \) is of no interest for Hor, who will stop at the first upper record \( a \) which has the corange area less than \( t_P \). Similarly, the configuration above \( H \) will be ignored by Ver, who will stop at the first upper record \( a \) with the area of 2-dim interval \( (a, O) \) less than \( t_P \).

Generically, they stop at different atoms, but both succeed with same probability

\[ \frac{1}{t} \int_0^t q_1(s, t_P) ds \]

which is close to \( v_P \) when the side of the square \( t^{1/2} \) is sufficiently large.

### 4.5 EU-representation

A model of infinite record processes leads to a EU-representation of the corange-box area chain, and offers a framework for asymptotic considerations in the HC-problem. The role of these considerations is somewhat limited by the fact that there is no obvious infinite analogue of the stopping problem nor embedding of finite-\( t \) record processes.
Consider PPP in the infinite strip \( R = R_+ \cup R_- \) with \( R_- = [0, 1] \times [-\infty, 0] \) and \( R_+ = [0, 1] \times [0, \infty] \). Define an atom to be a lower record if \( a \in R_+ \) and is lower than all atoms in \( R_+ \) to the left from \( a \). Define an atom \( a \in R_- \) to be an upper record if \( a \in R_- \) and is higher than all atoms in \( R_- \) to the left from \( a \). The definition agrees with that of Section 4.1 when the horizontal axis is understood as a fixed cut.

Enumerate the upper records \( a_j \), from the right to the left (in reverse observation order). A lower record \( b_j \) is called adjoint to \( a_j \) if \( b_j \) is the rightmost lower record to the left from \( a_j \).

Theorem 7 \( \) Sequence \( (C_k) \) can be jointly represented as

\[
C_k = \left( E_1 + \frac{E_2}{U_1} + \ldots + \frac{E_{k+1}}{U_1 \cdots U_k} \right) \left( 1 - U_1 \cdots U_k \right)
\]

Proving marginal representation is easy. Indeed, for \( j \) fixed a EU-representation for coordinates of \( a_j \) is

\[
U_1 \cdots U_j \quad \text{and} \quad - \left( E_1 + \ldots + \frac{E_j}{U_1 \cdots U_{j-1}} \right)
\]

On the other hand, given \( U_i = u_i, \ i \leq j \) the ordinate of the adjoint lower record \( b_j \) is conditionally independent of the ordinates of \( a_i, i \leq j - 1 \), and is distributed like \( E_{j+1}(u_1 \cdots u_j)^{-1} \). This yields the representation of corange at \( a_j \). Justifying the joint distribution is more involved, requiring some preparation.

Lemma 4 \( \) Let \( E_1, E_2 \) be i.i.d. exponential r.v.'s, independent of uniform \( V \). Then

\[
\frac{E_1}{u_1} + \frac{E_2}{u_1 u_2} 1\{V > u_2\} \overset{d}{=} \frac{E}{u_1 u_2}
\]

where \( u_1, u_2 \in [0, 1] \) and \( E \) is a standard exponential r.v.

Proof. Expanding the \( n \)th power of the LHS yields an expression

\[
\frac{E_1^n}{u_1^n} + \sum_{k=0}^{n-1} \binom{n}{k} \frac{E_1^k E_2^{n-k}}{u_1^n u_2^{n-k}} 1\{V > u_2\}
\]

which has expectation

\[
\frac{n!}{u_1^n} + \sum_{k=0}^{n-1} \binom{n}{k} \frac{k!(n-k)!}{u_1^n u_2^{n-k}} (1 - u_2) = \frac{n!}{u_1^n u_2^n}
\]
equal to the $n$th moment of the RHS. Since the moments characterise the exponential distribution uniquely we are done. □

Proof of the theorem. Consider

$$a_j = \left( U_1 \cdots U_j, - \left( F_1 + \ldots + \frac{F_j}{U_1 \cdots U_{j-1}} \right) \right)$$

a coordinate-wise EU-representation for upper records. Given $(U_j) = (u_j)$ we will construct a distributional copy of the corange sequence. To this end, we need a further supply of independent exponential and uniform r.v.’s $(G_j)$ and $(V_j)$, also independent of $(F_j)$.

We have $a_1 = (u_1, -F_1)$ and the adjoint lower record can be written as $b_1 = (u_1 V_1, G_1 u_1^{-1})$ so that

$$C_1 = \left( F_1 + \frac{G_1}{u_1} \right) (1 - u_1)$$

is the smallest corange-box area. Note that the first component of $b_1$ is (conditionally) independent of $a_1$ and $C_1$. If $V_1 u_1 < u_1 u_2$ then $b_1 = b_2$ and if $V_1 u_1 > u_1 u_2$ there is an increment and $b_2 = G_1 u_1^{-1} + G_2 (u_1 u_2)^{-1}$. Continuing so forth, given $a_1, b_1, \ldots, a_j, b_j$ and given $C_1, \ldots, C_j$ the horizontal position of $b_{j+1}$ is distributed like $V_j u_1 \cdots u_j$ and we have a repetition $b_{j+1} = b_j$ exactly when $V_j u_1 \cdots u_j < u_1 \cdots u_{j+1}$.

Each time $V_j > u_{j+1}$ both $a_{j+1}$ and $b_{j+1}$ contribute to the corange increment (or decrement when viewed in the right observation order). Thus we arrive at a representation which should be clear from the $j = 3$ case:

$$C_3 = \left( F_1 + \frac{F_2}{u_1 u_2} + \frac{F_3}{u_1 u_2} + \frac{G_1}{u_1} \right) 1_{\{V_1 > u_2\}} + \frac{G_2}{u_1 u_2 u_3} 1_{\{V_2 > u_3\}} \right) (1 - u_1 u_2 u_3).$$

The terms

$$\frac{F_3}{u_1 u_2} + \frac{G_3}{u_1 u_2 u_3} 1_{\{V_2 > u_3\}}$$

are present neither in $C_1$ nor in $C_2$ thus we can painlessly replace them by $E_3 (u_1 u_2 u_3)^{-1}$, without destroying the joint distribution of $(C_1, C_2, C_3)$. The next substitution

$$\frac{F_2}{u_1} + \frac{G_2}{u_1 u_2} 1_{\{V_1 > u_2\}} \overset{d}{=} \frac{E_2}{u_1 u_2}$$

should be performed simultaneously in $C_2$ and $C_3$. A complete proof follows by induction in $j$. □

Note that the representation does not show the cumulative contribution of upper records versus cumulative contribution of lower records. The theorem implies a distributional identity.

Corollary. $(B_j) \overset{d}{=} (C_j)$.

Example. The simplest instance of the distributional identity is $B_1 = C_1$, which is

$$\frac{E_1}{U_1} (1 - U_1 U_2) \overset{d}{=} \left( E_1 + \frac{E_2}{U_1} \right) (1 - U_1).$$
The reader is advised to visually compare the EU-representations for \((A_j), (B_j)\) and \((C_j)\) and to attempt deducing \(B_j \overset{d}{=} C_j\) for \(j = 1, 2\) by integration (see [29]).

5 Extensions and compliments.

5.1 Duration problem. Consider the PPP in \(R = [0, t] \times [0, 1]\), with horizontal axis interpreted as time scale. Suppose that stopping at a record at time \(s\) yields a reward equal to the horizontal distance between the record selected and the next record to observe, or equal to \(t - s\) if no record follows. This is the ‘full-information case of the duration problem’ introduced in by Ferguson et al [11], p. 55. It was shown in [11] that the optimal rule is \(\pi_{t_P}\), and recently, in fixed-\(n\) context, the value is asymptotic to \(t v_P\), see [31].

It is the aim of this section to show that the duration problem is nothing else but a minor variation of the VC-problem, namely its vualised version II.

Suppose the first atom to observe is in the origin \(a = (0, 0)\). The expected reward from stopping is then
\[
\int_0^t e^{-x} x \, dx + te^{-t} = 1 - e^{-t} = t q_0(t)
\]
where the second term in the LHS stands for the event that no further records occur. Similarly, stopping at atom \(a\) at time \(s\) yields a reward \((t - s)q_0(\alpha(a))\), where the box area is given by \(\alpha(a) = (1 - x)(t - s)\) for \(a = (s, x)\).

Now recall that in version II of the VC-problem the observer does not know if the horizon has been approached. Thus when an atom \(a\) is detected the conditional probability of best choice is equal to
\[
\frac{t - s}{t} q_0(\alpha(a))
\]
where the first factor is the chance that \(a\) is to the left from \(V\) and the second factor is the conditional probability of best choice given that \(a\) is indeed to the left from \(V\). Thus the payoff in version II differs by constant factor \(t^{-1}\) from that in the duration problem. But version II is equivalent to version I, therefore in the duration problem the expected reward with \(\pi_s\) is simply \(t q_1(t, s)\), the optimal policy is \(\pi_{t_P}\) and the ‘maximum expected duration of holding a record’ is \(t u(t) = q_1(t, t_P)\). For any \(t\).

5.2 Bin-packing. Suppose there is a bin of unit capacity. To-be-packed items of random uniform-\([0, 1]\) size arrive at the epochs of a homogeneous Poisson process. An item is irrevocably packed immediately at the time of arrival provided there is enough room in the bin left (greedy policy). The problem is to recognise the last packing at the time it occurs.

A minute thought shows that the state variable in the problem is the product of the remaining capacity and the expected number of Poisson epochs to come. The probability law of this process is stochastically equivalent to the box-area process. And this implies that the problem is equivalent to the FI best-choice problem.

5.3 Additive representations of best-choice probability. Of some interest are representations in the form of a sum of probabilities of events expressed explicitly via PPP configuration. Decompositions of this kind are tractable logically, but not analytically because they cannot be expressed in invariant terms, i.e. using box areas.
Samuels developed such decompositions for FI-, VC- and HC-problems [29]. In the FI case his representation of

\[ p_1(\infty, s) = P(A_1 < s < A_2) = P(E_1(1 - U_1) < s < (E_1 + E_2/U_1)(1 - U_1U_2)) \]

is based on testing the inequality \( E_2/U_1 > s/(1 - U_1) - E_1 \) and has two parts

\[
P(E_1(1 - U_1) < s < (E_1 + E_2/U_1)(1 - U_1)) = (e^s - 1)I(s)
\]

\[
P((E_1 + E_2/U_1)(1 - U_1) < s < (E_1 + E_2/U_1)(1 - U_1U_2)) = (e^{-s} - sI(s))J(s).
\]

Loosely speaking, Samuels’ decomposition makes distinction between the cases when the vertical distance between the last and second last records is large or small.

Another decomposition appears when we concentrate on both the highest atom \( a \) and an atom \( b \) which is the highest among PPP atoms below \(-s\), within the \( R = [0, 1] \times [-\infty, 0] \). Indeed, suppose the event \( A_1 < s < A_2 \) occurs. There are three cases: \( \alpha(b) > s \), or \( \alpha(b) < s \) and \( b \) is a record, or \( \alpha(b) < s \) and \( b \) is no record. Let \( \xi < -s \) be the vertical position of \( b \).

In the first case the horizontal position of \( b \) must be within \([1 - s/\xi, 1]\), and we must have \( A_2 \geq \alpha(b) \) and \( a \) as the unique record above \(-s\). Integrating yields

\[
\int_{s}^{\infty} e^{s - \xi}(1 - s/\xi) p_1(s) d\xi = (e^{-s} - sI(s))J(s)
\]

In the second case \( b \) coincides with \( a \). In the third case there must be exactly one record above \(-s\) to the left from \( b \) and no atoms above \(-s\) to the right from \( b \). We failed to evaluate probabilities in the two last cases directly, as it involves the not-so-easy integration of \( x^{-1} \exp(-x + c/x) \) (an instance of generalised incomplete gamma function studied in [8]). Thus we could deduce the total probability of these cases, \( (e^s - 1)I(s) \), only from the formula for \( p_1(\infty, s) \) and the first case.

The moral of this is that the second decomposition yields the same two terms as that of Samuels, although it is based on a completely different principle. This offers a new puzzle because distribution of \( b \) does not fit in the EU-representation for records since \( b \) need not be a record at all.

5.4 Beyond the box areas. The box-areas approach is good for study ‘time-space invariant’ functionals of the PPP records, but is of limited value when we need to explicitly separate the coordinates. Nevertheless, the invariance helps to study more general functionals as well. Next examples illustrate the matters in the context of FI problem.

Example: distribution of stopping time. Consider threshold policy \( \pi_s \) in \( R = [0, 1] \times [-\infty, 0] \). Being a stopping time, \( \pi_s \) accepts some value within \([0, 1]\) – coordinate of the selected atom – or is indefinite if no atom is selected. Let \( f(t, \xi, s) \) be the probability that the selected atom is above \(-t\) and to the left from \( \xi \) for \( t \in [0, \infty] \), \( \xi \in [0, 1] \). For \( t < s \) we have \( f(t, \xi, s) = 1 - e^{-\xi t} \) because \( \pi_s \) stops if there is such an atom. For \( t > s/(1 - \xi) \) we have \( \partial_t f(t, \xi, s) = 0 \), as is easily seen by drawing a hyperbolic stopping boundary for \( \pi_s \). And for \( t \in [s, s/(1 - \xi)] \)

\[
\partial_t f(t, \xi, s) = (\xi + st^{-1} - 1)e^{-\xi t}
\]
because the choices in two close rectangles of heights \( t - \delta \) and \( t \) are only different when the atom highest for the configuration on \([0, \xi] \times [-\infty, 0]\) is in the \( \delta \)-strip. Integrating we find that for all \( t \geq s/(1 - \xi) \)

\[
f(t, \xi, s) = \frac{\xi - 1}{\xi} \left( e^{-\xi s} - e^{-\xi s/(1-\xi)} \right) + s I \left( \frac{s \xi}{1 - \xi}, s \xi \right) + 1 - e^{-s \xi},
\]

independently of \( t \), it is therefore the distribution for semi-finite \( R \). When \( \xi \to 1 \), we have \( f(\infty, \xi, s) \to 1 - e^{-s} + s I(s) \) which is \( 1 - p_0(\infty, s) \), probability that \( \pi_s \) ever selects an atom.

In Section 2.4 we derived an integral representation (24) of the best-choice probability in terms of the box-area process. Next example gives similar ‘real-time’ rate, a Poisson analogue of the ‘probability of win at a given draw’ introduced in [12], p. 57.

**Example:** the best-choice probability rate. In the framework of the previous example, let \( g(t, \xi, s) \) be the probability that the last record appears before \( \xi \) and is selected by \( \pi_s \), \( \xi \in [0, 1] \). Think of \( \partial_\xi g(\infty, \xi, s) \) as a winning probability rate at time \( \xi \in [0, 1] \), so that the total best-choice probability \( p_1(\infty, s) \) is obtained by integration over \( \xi \in [0, 1] \).

It is not hard to see that \( \partial_\xi g(t, \xi, s) = 0 \) for \( t > s/(1 - \xi) \), because the atoms south-west from the point \((\xi, -t)\) are outside the stopping region \( \{ (x, -t) : (1 - x)t < s \} \). It follows that \( g(\infty, \xi, s) = g(s/(1 - \xi), \xi, s) \).

For \( t < s/(1 - \xi) \) we will find the derivative \( \partial_\xi g(\xi, s) \) by the coupling technique. Consider two rectangles \( R_1 = [0, \xi] \times [-t, 0] \) and \( R_2 = [0, \xi] \times [-\xi, 0] \). Policy \( \pi_s \) stops at distinct atoms in \( R_1 \) and \( R_2 \) if the first record, say \( a \), with box area less than \( s \) appears in the strip \([0, s/t \xi \times [-\xi, 0] \). Let \( x \) be the horizontal coordinate of \( a \). When \( a \) is the overall last record \( \pi_s \) wins in \( R_1 \) but not in \( R_2 \). The counterpart is more complex: \( \pi_s \) wins in \( R_2 \) but not in \( R_1 \) if after \( a \) there are \( k > 0 \) atoms in \([x, \xi] \times [-\xi, 0] \), the leftmost of these atoms appears within \([x, \xi] \times [-\xi, 0] \) and it is the last record; the probability of this event is computed via distribution of the minimum in a sample of size \( k \) and using the fact that vertical ranking is independent of the arrival time. Integrating over \( x \) yields

\[
\partial_\xi g(t, \xi, s) = e^{-t} \int_{(1-s)/t}^{\xi} \left( 1 - \sum_{k=1}^{\infty} \left( \frac{t^k (1-x)^k}{k!} - \frac{t^k (1-\xi)^k}{k!} \right) \right) \, dx.
\]

Differentiating in \( \xi \) and then integrating over \( t \) from 0 to \( s/(1 - \xi) \) and finally converting the series into exponential integral functions we obtain a formula missed in the fundamental 1966 paper:

\[
\partial_\xi g(\infty, \xi, s) = -e^{-s} + \frac{-e^{-s \xi} - \xi e^{-s}}{1 - \xi} + e^{-s \xi} - e^{-s \xi/(1-\xi)} \frac{s}{1 - \xi} \left( I \left( \frac{s \xi}{1 - \xi}, s \xi \right) - I \left( \frac{s}{1 - \xi}, s \right) \right)
\]

– complicated but correct!

The grouping of terms was selected to show that the rate is an entire function in \( \xi \). For \( \xi = 0 \) and 1 the values are \( 1 - e^s \) and \( e^{-s} \), respectively, in accord with Figure 3 from [12], corresponding to the optimal threshold \( s = t_F \).

**Remark.** At the end of Section 3d, Gilbert and Mosteller write: “Theory we do not give shows that, for large \( n \), the probability of winning on any draw with the optimum strategy ...
is roughly \((1 - e^{-c})/n \ldots\) (with \(c = t_F = 0.804\)) Now we know that this roughly means, in spirit of their one-paragraph Section 3e, precisely that up to higher order terms, probability of win at draw \(i\) is \(n^{-1} \partial_\xi g(\infty, i/n, t_F)\) where the function is close to 0.6 for most of the range \(\xi \in [0, 1]\). A Mathematica-drawn graph of \(\partial_\xi g(\infty, \cdot, t_F)\) demonstrates perfect agreement with Figure 3 in [12], p. 58.

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References

[1] Arnold, B.C., Balakrishnan, N. and Nagaraja, H.N. (1998) Records, Wiley, NY.

[2] Berezovsky, B.A. and Gneden, A.V. (1984) The Best Choice Problem, Nauka, Moscow.

[3] Bojdecki, T. (1978) On optimal stopping of a sequence of independent random variables – probability maximizing approach, Stoch. Proc. Appl. 6, 153-163.

[4] Bojdecki, T. (1977) On optimal stopping of independent random variables appearing to a renewal process with random time horizon, Bol. Soc. Math. Mexicana 22, 35-40.

[5] Brown, S. (1993) Records, mixed Poisson processes and optimal selection: an intensity approach, Preprint.

[6] Bruss, F.T. and Paindaveine, D. (2000) Selecting a sequence of last successes in independent trials, J. Appl. Prob. 37, 389-399.

[7] Bruss, F.T. and Rogers L.C.G. (1991) Embedding optimal selection problems in a Poisson process, Stoch. Proc. Appl. 38 267-278.

[8] Chaudhry, M.A. and Zubair, S.M. (2002) On a Class of Incomplete Gamma Functions with Applications, Boca Raton: Chapman and Hall, London.

[9] Deuschel, J.-D., and Zeitouni, O. (1995) Limiting curves for i.i.d. records, Ann. Prob. 23, 852-878.

[10] Dynkin, E.B. and Yushkevitch, A.A. (1969) Markov Processes: Theorems and problems, Plenum Press, NY.

[11] Ferguson, T.S, Hardwick, J.P. and Tamaki, M. (1992) Duration of owning a relatively best object, Contemporary Math. 125, 37-57.

[12] Gilbert, J. and Mosteller, F. (1966) Recognizing the maximum of a sequence, J. Amer. Stat. Assoc. 61, 35-73.
[13] Goldie, C.M. (1989) Records, permutations and greatest convex minorants, *Math. Proc. Camb. Phil. Soc* **106**, 169-177.

[14] (1999) Record sequences and their applications, Preprint.

[15] Goldie, C.M. and Resnick, S.I. (1995) Many multivariate records, *Stoch. Proc. Appl.*, **59**, 185-216.

[16] Gnedin, A.V. (1996) On the full-information best-choice problem, *J. Appl. Prob.* **33**, 678-687.

[17] Gnedin, A.V. and Sakaguchi, M. (1992) On a best-choice problem related to the Poisson process, *Contemporary Math.* **125**, 59-64.

[18] Kühne, R. and Rüschendorf, L. (2000) Approximation of optimal stopping problems, *Stoch. Proc. Appl.* **90**, 301-325.

[19] Nevzorov, V.B. (2001) *Records*, Transl. Math. Monographs, AMS, Providence.

[20] Nielsen, N. (1906) *Theorie des Integrallogarithmus und verwandter Transzendenten*, Teubner, Leipzig (reprinted by Chelsea in 1965, available via the electronic library of Cornell University [http://cdl.library.cornell.edu/math_N.html](http://cdl.library.cornell.edu/math_N.html)).

[21] Petruccelli, J.D. (1980) On a best choice problem with partial information, *Ann. Stat.* **8**, 1171-1174.

[22] Pfeifer, D. (1989) Extremal processes, secretary problems and the 1/e law, *J. Appl. Prob.* **27**, 722-733.

[23] Porosinski, Z. (1987) The full-information best choice problem with a random number of observations, *Stoch. Proc. Appl.* **bf 24**, 293-307.

[24] Porosinski, Z. (2002) On best choice problems having similar solutions, *Stat. Prob. Letters* **56**, 321-327.

[25] Presman, E.L. and Sonin, I.M. (1972) The best choice problem for a random number of objects, *Theor. Probab. Appl.* **20**, 770-781.

[26] Resnick, S. (1987) *Extreme Values Regular Variation and Point Processes*, Springer, NY.

[27] Sakaguchi, M. (1976) Optimal stopping problems for randomly arriving offers, *Math. Japonica* **21**, 201-217.

[28] Samuels, S.M. (1982) Exact solutions for the full information best choice problem, *Purdue Univ. Stat. Dept. Mimeo Series* **82-17**.

[29] Samuels, S.M. (2002) Two (or maybe three) quite different best-choice problems are extraordinary similar, Workshop on Optimal Stopping and Stochastic Games, 1-7 July, Bedlewo, Poland (unpublished notes).
[30] Samuels, S.M. (1991) Secretary problems. Chapter 16 of *Handbook of Sequential Analysis* (B.K. Ghosh and P.K. Sen eds), Marcel Dekker, NY.

[31] Tamaki, M. and Mazalov, V.V. (2002) An explicit formula for the limiting gain in the full information duration problem, report on the Workshop on Optimal Stopping and Stochastic Games, Bedlewo, Poland.

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