Some Logically Weak Ramseyan Theorems

Wei Wang

Department of Philosophy, Sun Yat-sen University

May 2013
Reverse Mathematics

- Reverse mathematics: initiated by Harvey Friedman, to formulate theorems in ordinary mathematics in the context of second order arithmetic, and to study proof theoretic strength of theorems.

- The language of second order arithmetic consists of $0, 1, +, \times, \in$, first order variables $x, y, z, \ldots$ and second order variables $X, Y, Z, \ldots$, and quantifiers over first and second order variables.

- A model is a pair $(M, S)$, where $M$ is a model of first order arithmetic and $S$ is a subset of the powerset of $M$.

- Satisfication can be defined from first order satisfication relation by induction. For example, if $\varphi(X)$ is a first order formula with only one second order variable $X$, then $(M, S) \models \forall X \varphi(X)$, iff $(M, U) \models \varphi(U)$ for all $U \in S$. 
Reverse mathematics: initiated by Harvey Friedman, to formulate theorems in ordinary mathematics in the context of second order arithmetic, and to study proof theoretic strength of theorems.

The language of second order arithmetic consists of $0, 1, +, \times, \in$, first order variables $x, y, z, \ldots$ and second order variables $X, Y, Z, \ldots$, and quantifiers over first and second order variables.

A model is a pair $(M, S)$, where $M$ is a model of first order arithmetic and $S$ is a subset of the powerset of $M$.

Satisfaction can be defined from first order satisfaction relation by induction. For example, if $\varphi(X)$ is a first order formula with only one second order variable $X$, then $(M, S) \models \forall X \varphi(X)$, iff $(M, U) \models \varphi(U)$ for all $U \in S$. 
Reverse mathematics: initiated by Harvey Friedman, to formulate theorems in ordinary mathematics in the context of second order arithmetic, and to study proof theoretic strength of theorems.

The language of second order arithmetic consists of 0, 1, +, ×, ∈, first order variables x, y, z, ... and second order variables X, Y, Z, ..., and quantifiers over first and second order variables.

A model is a pair \((M, S)\), where \(M\) is a model of first order arithmetic and \(S\) is a subset of the powerset of \(M\).

Satisfaction can be defined from first order satisfaction relation by induction. For example, if \(\varphi(X)\) is a first order formula with only one second order variable \(X\), then \((M, S) \models \forall X \varphi(X)\), iff \((M, U) \models \varphi(U)\) for all \(U \in S\).
Reverse Mathematics

- Reverse mathematics: initiated by Harvey Friedman, to formulate theorems in ordinary mathematics in the context of second order arithmetic, and to study proof theoretic strength of theorems.

- The language of second order arithmetic consists of 0, 1, +, ×, ∈, first order variables x, y, z, ... and second order variables X, Y, Z, ..., and quantifiers over first and second order variables.

- A model is a pair \((M, S)\), where \(M\) is a model of first order arithmetic and \(S\) is a subset of the powerset of \(M\).

- Satisfication can be defined from first order satisfication relation by induction. For example, if \(\varphi(X)\) is a first order formula with only one second order variable \(X\), then \((M, S) \models \forall X \varphi(X)\), iff \((M, U) \models \varphi(U)\) for all \(U \in S\).
People usually choose $\text{RCA}_0$ as a base system and discuss provability over $\text{RCA}_0$. $\text{RCA}_0$ consists of $\text{PA}^-$, induction for $\Sigma^0_1$ formulas and the Recursive Comprehension Scheme:

$$\forall n(\varphi(n) \leftrightarrow \phi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi$ and $\phi$ are $\Sigma^0_1$ and $\Pi^0_1$ formulas respectively, with or without second order parameters.

$\text{WKL}$ consists of $\text{RCA}_0$ and the statement that every infinite binary tree has an infinite path (Weak König Lemma).

$\text{ACA}_0$ consists of $\text{RCA}_0$ and a comprehension scheme for first order formulas (Arithmetic Comprehension Scheme).

Over $\text{RCA}_0$, $\text{WKL}$ is strictly stronger than $\text{RCA}_0$ and $\text{ACA}_0$ is strictly stronger than $\text{WKL}$. There are systems beyond $\text{ACA}_0$, which are irrelevant in this talk.
People usually choose RCA$_0$ as a base system and discuss provability over RCA$_0$. RCA$_0$ consists of PA$^-$, induction for $\Sigma^0_1$ formulas and the Recursive Comprehension Scheme:

$$\forall n (\varphi(n) \leftrightarrow \phi(n)) \rightarrow \exists X \forall n (n \in X \leftrightarrow \varphi(n)),$$

where $\varphi$ and $\phi$ are $\Sigma^0_1$ and $\Pi^0_1$ formulas respectively, with or without second order parameters.

WKL consists of RCA$_0$ and the statement that every infinite binary tree has an infinite path (Weak König Lemma).

ACA$_0$ consists of RCA$_0$ and a comprehension scheme for first order formulas (Arithmetic Comprehension Scheme).

Over RCA$_0$, WKL is strictly stronger than RCA$_0$ and ACA$_0$ is strictly stronger than WKL. There are systems beyond ACA$_0$, which are irrelevant in this talk.
People usually choose RCA\(_0\) as a base system and discuss provability over RCA\(_0\). RCA\(_0\) consists of PA\(^-\), induction for \(\Sigma^0_1\) formulas and the Recursive Comprehension Scheme:

\[
\forall n (\varphi(n) \iff \phi(n)) \implies \exists X \forall n (n \in X \iff \varphi(n)),
\]

where \(\varphi\) and \(\phi\) are \(\Sigma^0_1\) and \(\Pi^0_1\) formulas respectively, with or without second order parameters.

WKL consists of RCA\(_0\) and the statement that every infinite binary tree has an infinite path (Weak König Lemma).

ACA\(_0\) consists of RCA\(_0\) and a comprehension scheme for first order formulas (Arithmetic Comprehension Scheme).

Over RCA\(_0\), WKL is strictly stronger than RCA\(_0\) and ACA\(_0\) is strictly stronger than WKL. There are systems beyond ACA\(_0\), which are irrelevant in this talk.
People usually choose $\text{RCA}_0$ as a base system and discuss provability over $\text{RCA}_0$. $\text{RCA}_0$ consists of $\text{PA}^-$, induction for $\Sigma^0_1$ formulas and the Recursive Comprehension Scheme:

$$\forall n(\varphi(n) \leftrightarrow \phi(n)) \rightarrow \exists X \forall n(n \in X \leftrightarrow \varphi(n)),$$

where $\varphi$ and $\phi$ are $\Sigma^0_1$ and $\Pi^0_1$ formulas respectively, with or without second order parameters.

$\text{WKL}$ consists of $\text{RCA}_0$ and the statement that every infinite binary tree has an infinite path (Weak König Lemma).

$\text{ACA}_0$ consists of $\text{RCA}_0$ and a comprehension scheme for first order formulas (Arithmetic Comprehension Scheme).

Over $\text{RCA}_0$, $\text{WKL}$ is strictly stronger than $\text{RCA}_0$ and $\text{ACA}_0$ is strictly stronger than $\text{WKL}$. There are systems beyond $\text{ACA}_0$, which are irrelevant in this talk.
Recall that Recursive Comprehension Scheme asserts the existence of sets which are recursive in finitely many second order parameters.

So, if $(M, S) \models \text{RCA}_0$, $X_0, \ldots, X_{n-1} \in S$ and $Y \leq_T X_0 \oplus \cdots \oplus X_{n-1}$, then $Y \in S$.

Such $S$ are called Turing ideal in computability. So, to build models of $\text{RCA}_0$ and some other theorems is to build Turing ideals of special properties.

Computability comes in here.
Recall that Recursive Comprehension Scheme asserts the existence of sets which are recursive in finitely many second order parameters.

So, if \((M, S) \models \text{RCA}_0\), \(X_0, \ldots, X_{n-1} \in S\) and \(Y \leq_T X_0 \oplus \cdots \oplus X_{n-1}\), then \(Y \in S\).

Such \(S\) are called Turing ideal in computability. So, to build models of \(\text{RCA}_0\) and some other theorems is to build Turing ideals of special properties.

Computability comes in here.
Models of RCA$_0$

- Recall that Recursive Comprehension Scheme asserts the existence of sets which are recursive in finitely many second order parameters.

- So, if \((M, S) \models \text{RCA}_0, X_0, \ldots, X_{n-1} \in S\) and \(Y \leq_T X_0 \oplus \cdots \oplus X_{n-1}\), then \(Y \in S\).

- Such \(S\) are called Turing ideal in computability. So, to build models of RCA$_0$ and some other theorems is to build Turing ideals of special properties.

- Computability comes in here.
Recall that Recursive Comprehension Scheme asserts the existence of sets which are recursive in finitely many second order parameters.

So, if $(M, S) \models \text{RCA}_0$, $X_0, \ldots, X_{n-1} \in S$ and $Y \leq_T X_0 \oplus \cdots \oplus X_{n-1}$, then $Y \in S$.

Such $S$ are called \textit{Turing ideal} in computability. So, to build models of $\text{RCA}_0$ and some other theorems is to build Turing ideals of special properties.

Computability comes in here.
Ramsey’s Theorems

- Ramsey theory has been an important subject in reverse mathematics, among which Ramsey’s theorem for pairs (RT$_2^2$) has been the most intensively studied single theorem.

- Recall: for a set $X$ and $n \leq \omega$,

$$[X]^n = \{ \sigma \subset X : |\sigma| = n \},$$

and $[X]^{<n} = \bigcup_{k < n} [X]^k$, $[X]^{\leq n} = \bigcup_{k \leq n} [X]^k$.

Ramsey’s Theorems

(RT$_k^n$) For every $0 < n, k < \mathbb{N}$ and every $f : [N]^n \rightarrow k = \{0, \ldots, k - 1\}$, there exists $H \in [\mathbb{N}]^{\omega}$ s.t. $f$ is constant on $[H]^n$. $H$ is called a homogeneous set for $f$. 
Ramsey’s Theorems

- Ramsey theory has been an important subject in reverse mathematics, among which Ramsey’s theorem for pairs (RT$_2^2$) has been the most intensively studied single theorem.

- Recall: for a set $X$ and $n \leq \omega$,

$$[X]^n = \{\sigma \subset X : |\sigma| = n\},$$

and $[X]^{<n} = \bigcup_{k<n}[X]^k$, $[X]^{\leq n} = \bigcup_{k\leq n}[X]^k$.

Ramsey’s Theorems

(RT$_k^n$) For every $0 < n, k < \mathbb{N}$ and every $f : [N]^n \to k = \{0, \ldots, k - 1\}$, there exists $H \in [\mathbb{N}]^\omega$ s.t. $f$ is constant on $[H]^n$. $H$ is called a homogeneous set for $f$. 
Ramsey’s Theorems

- Ramsey theory has been an important subject in reverse mathematics, among which Ramsey’s theorem for pairs (RT$_2^2$) has been the most intensively studied single theorem.

- Recall: for a set $X$ and $n \leq \omega$,

  $[X]^n = \{\sigma \subset X : |\sigma| = n\}$,

  and $[X]^{<n} = \bigcup_{k < n} [X]^k$, $[X]^{\leq n} = \bigcup_{k \leq n} [X]^k$.

Ramsey’s Theorems

(RT$_k^n$) For every $0 < n, k < \mathbb{N}$ and every $f : [N]^n \rightarrow k = \{0, \ldots, k - 1\}$, there exists $H \in [\mathbb{N}]^\omega$ s.t. $f$ is constant on $[H]^n$. $H$ is called a **homogeneous** set for $f$. 
Around Ramsey’s theorem for pairs
Second order theories

**Theorem**

- *(Jockusch)* $\text{RCA}_0 + \text{WKL} \not\vdash \text{RT}^2_2$.
- *(Seetapun)* $\text{RCA}_0 + \text{WKL} + \text{RT}^2_2 \not\vdash \text{ACA}_0$.
- *(Cholak, Jockusch and Slaman)* $\text{RCA}_0 \vdash \text{RT}^2_2 \iff \text{COH} + \text{SRT}^2_2$, $\text{RCA}_0 + \text{COH} \not\vdash \text{SRT}^2_2$.
- *(Cholak, Jockusch and Slaman)* $\text{RCA}_0 + \text{RT}^2_2$ admits an $\omega$-model consisting only of low$_2$ sets ($X$ is low$_2$ iff $X'' \leq_T \emptyset''$).
- *(Jiayi Liu)* $\text{RCA}_0 + \text{RT}^2_2 \not\vdash \text{WKL}$.
- *(Chong, Slaman and Yang)* $\text{RCA}_0 + \text{SRT}^2_2 \not\vdash \text{COH}$.
(Jockusch) RCA$_0$ + WKL ⊬ RT$_2^2$.

(Seetapun) RCA$_0$ + WKL + RT$_2^2$ ⊬ ACA$_0$.

(Cholak, Jockusch and Slaman) RCA$_0$ ⊢ RT$_2^2$ ↔ COH + SRT$_2^2$, RCA$_0$ + COH ⊬ SRT$_2^2$.

(Cholak, Jockusch and Slaman) RCA$_0$ + RT$_2^2$ admits an ω-model consisting only of low$_2$ sets (X is low$_2$ iff X'' ≤$_T$ ∅").

(Jiayi Liu) RCA$_0$ + RT$_2^2$ ⊬ WKL.

(Chong, Slaman and Yang) RCA$_0$ + SRT$_2^2$ ⊬ COH.
Around Ramsey’s theorem for pairs
Second order theories

Theorem

- (Jockusch) $\text{RCA}_0 + \text{WKL} \not\vdash \text{RT}_2^2$.
- (Seetapun) $\text{RCA}_0 + \text{WKL} + \text{RT}_2^2 \not\vdash \text{ACA}_0$.
- (Cholak, Jockusch and Slaman) $\text{RCA}_0 \vdash \text{RT}_2^2 \iff \text{COH} + \text{SRT}_2^2$, $\text{RCA}_0 + \text{COH} \not\vdash \text{SRT}_2^2$.
- (Cholak, Jockusch and Slaman) $\text{RCA}_0 + \text{RT}_2^2$ admits an $\omega$-model consisting only of low$_2$ sets ($X$ is low$_2$ iff $X'' \leq_T \emptyset''$).
- (Jiayi Liu) $\text{RCA}_0 + \text{RT}_2^2 \not\vdash \text{WKL}$.
- (Chong, Slaman and Yang) $\text{RCA}_0 + \text{SRT}_2^2 \not\vdash \text{COH}$.
Around Ramsey’s theorem for pairs
Second order theories

Theorem

- (Jockusch) $\text{RCA}_0 + \text{WKL} \not\vdash \text{RT}_2^2$.
- (Seetapun) $\text{RCA}_0 + \text{WKL} + \text{RT}_2^2 \not\vdash \text{ACA}_0$.
- (Cholak, Jockusch and Slaman) $\text{RCA}_0 \vdash \text{RT}_2^2 \iff \text{COH} + \text{SRT}_2^2$, $\text{RCA}_0 + \text{COH} \not\vdash \text{SRT}_2^2$.
- (Cholak, Jockusch and Slaman) $\text{RCA}_0 + \text{RT}_2^2$ admits an $\omega$-model consisting only of $\text{low}_2$ sets ($X$ is $\text{low}_2$ iff $X'' \leq_T \emptyset''$).
- (Jiayi Liu) $\text{RCA}_0 + \text{RT}_2^2 \not\vdash \text{WKL}$.
- (Chong, Slaman and Yang) $\text{RCA}_0 + \text{SRT}_2^2 \not\vdash \text{COH}$.
Around Ramsey’s theorem for pairs
Second order theories

### Theorem

- (Jockusch) $\text{RCA}_0 + \text{WKL} \not\vdash \text{RT}^2_2$.
- (Seetapun) $\text{RCA}_0 + \text{WKL} + \text{RT}^2_2 \not\vdash \text{ACA}_0$.
- (Cholak, Jockusch and Slaman) $\text{RCA}_0 \vdash \text{RT}^2_2 \leftrightarrow \text{COH} + \text{SRT}^2_2$, $\text{RCA}_0 + \text{COH} \not\vdash \text{SRT}^2_2$.
- (Cholak, Jockusch and Slaman) $\text{RCA}_0 + \text{RT}^2_2$ admits an $\omega$-model consisting only of $\text{low}_2$ sets ($X$ is $\text{low}_2$ iff $X'' \leq_T \emptyset''$).
- (Jiayi Liu) $\text{RCA}_0 + \text{RT}^2_2 \not\vdash \text{WKL}$.
- (Chong, Slaman and Yang) $\text{RCA}_0 + \text{SRT}^2_2 \not\vdash \text{COH}$. 
Around Ramsey’s theorem for pairs
Second order theories

Theorem

- (Jockusch) $\text{RCA}_0 + \text{WKL} \not\vdash \text{RT}_2^2$.
- (Seetapun) $\text{RCA}_0 + \text{WKL} + \text{RT}_2^2 \not\vdash \text{ACA}_0$.
- (Cholak, Jockusch and Slaman) $\text{RCA}_0 \vdash \text{RT}_2^2 \iff \text{COH} + \text{SRT}_2^2$, $\text{RCA}_0 + \text{COH} \not\vdash \text{SRT}_2^2$.
- (Cholak, Jockusch and Slaman) $\text{RCA}_0 + \text{RT}_2^2$ admits an $\omega$-model consisting only of $\text{low}_2$ sets ($X$ is $\text{low}_2$ iff $X'' \leq_T \emptyset''$).
- (Jiayi Liu) $\text{RCA}_0 + \text{RT}_2^2 \not\vdash \text{WKL}$.
- (Chong, Slaman and Yang) $\text{RCA}_0 + \text{SRT}_2^2 \not\vdash \text{COH}$.
$I\Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \bar{p}(\varphi(0, \bar{p}) \land \forall x(\varphi(x, \bar{p}) \rightarrow \varphi(x + 1, \bar{p}) \rightarrow \forall x \varphi(x, \bar{p})).$$

$B\Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \bar{p}, a(\forall x < a \exists y \varphi(x, y, \bar{p}) \rightarrow \exists b \forall x < a \exists y < b \varphi(x, y, \bar{p})).$$

**Theorem**

- (Kirby and Paris) $I\Sigma_1 \leftrightarrow B\Sigma_2 \leftrightarrow I\Sigma_2 \leftrightarrow \cdots$ is a proper hierarchy.
- (Hirst; Cholak, Jockusch and Slaman) $\text{RCA}_0 + \text{SRT}^2_2 \vdash B\Sigma_2$.
- (Chong, Slaman and Yang) $\text{RCA}_0 + \text{RT}^2_2 \nvdash I\Sigma_2$.
$I \Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \vec{p} (\varphi(0, \vec{p}) \land \forall x (\varphi(x, \vec{p}) \rightarrow \varphi(x + 1, \vec{p}) \rightarrow \forall x \varphi(x, \vec{p}))$$.

$B \Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \vec{p}, a (\forall x < a \exists y \varphi(x, y, \vec{p}) \rightarrow \exists b \forall x < a \exists y < b \varphi(x, y, \vec{p}))$$.

**Theorem**

- (Kirby and Paris) $I \Sigma_1 \leftarrow B \Sigma_2 \leftarrow I \Sigma_2 \leftarrow \cdots$ is a proper hierarchy.
- (Hirst; Cholak, Jockusch and Slaman) $\text{RCA}_0 + \text{SRT}^2_2 \vdash B \Sigma_2$.
- (Chong, Slaman and Yang) $\text{RCA}_0 + \text{RT}^2_2 \not\vdash I \Sigma_2$
$I\Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \vec{p}(\varphi(0, \vec{p}) \land \forall x(\varphi(x, \vec{p}) \rightarrow \varphi(x + 1, \vec{p}) \rightarrow \forall x \varphi(x, \vec{p}))).$$

$B\Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \vec{p}, a(\forall x < a \exists y \varphi(x, y, \vec{p}) \rightarrow \exists b \forall x < a \exists y < b \varphi(x, y, \vec{p}))).$$

**Theorem**

- *(Kirby and Paris)* $I\Sigma_1 \leftarrow B\Sigma_2 \leftarrow I\Sigma_2 \leftarrow \cdots$ is a proper hierarchy.
- *(Hirst; Cholak, Jockusch and Slaman)* $\text{RCA}_0 + \text{SRT}_2^2 \models B\Sigma_2$.
- *(Chong, Slaman and Yang)* $\text{RCA}_0 + \text{RT}_2^2 \not\models I\Sigma_2$.
$I \Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \bar{p}(\varphi(0, \bar{p}) \land \forall x(\varphi(x, \bar{p}) \rightarrow \varphi(x + 1, \bar{p}) \rightarrow \forall x \varphi(x, \bar{p}))).$$

$B \Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \bar{p}, a(\forall x < a \exists y \varphi(x, y, \bar{p}) \rightarrow \exists b \forall x < a \exists y < b \varphi(x, y, \bar{p})).$$

**Theorem**

- *(Kirby and Paris)* $I \Sigma_1 \leftarrow B \Sigma_2 \leftarrow I \Sigma_2 \leftarrow \cdots$ *is a proper hierarchy.*
- *(Hirst; Cholak, Jockusch and Slaman)* $\text{RCA}_0 + \text{SRT}_2^2 \vdash B \Sigma_2$.
- *(Chong, Slaman and Yang)* $\text{RCA}_0 + \text{RT}_2^2 \not\vdash I \Sigma_2$.
$I\Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \bar{p}(\varphi(0, \bar{p}) \land \forall x (\varphi(x, \bar{p}) \rightarrow \varphi(x + 1, \bar{p}) \rightarrow \forall x \varphi(x, \bar{p}))$$.

$B\Sigma_n$ is the following scheme for all $\Sigma_n$ formulas $\varphi$:

$$\forall \bar{p}, a(\forall x < a \exists y \varphi(x, y, \bar{p}) \rightarrow \exists b \forall x < a \exists y < b \varphi(x, y, \bar{p}))$$.

**Theorem**

- *(Kirby and Paris)* $I\Sigma_1 \leftarrow B\Sigma_2 \leftarrow I\Sigma_2 \leftarrow \cdots$ is a proper hierarchy.
- *(Hirst; Cholak, Jockusch and Slaman)* $\text{RCA}_0 + \text{SRT}_2^2 \vdash B\Sigma_2$.
- *(Chong, Slaman and Yang)* $\text{RCA}_0 + \text{RT}_2^2 \nvdash I\Sigma_2$.
A $\Pi_1^1$ sentence is a sentence of the form $\forall X \psi$, where $\psi$ is of first order.

**Theorem**

- (Cholak, Jockusch and Slaman) COH is $\Pi_1^1$-conservative over $\text{RCA}_0$, id est, if $\varphi$ is a $\Pi_1^1$ sentence provable in $\text{RCA}_0 + \text{COH}$ then it is provable in $\text{RCA}_0$.

- (Cholak, Jockusch and Slaman) $\text{RT}_2^2$ is $\Pi_1^1$-conservative over $\text{RCA}_0 + I\Sigma_2$.

- (Chong, Slaman and Yang) COH is $\Pi_1^1$-conservative over $\text{RCA}_0 + B\Sigma_2$.

**Question**

Is $\text{SRT}_2^2$ or $\text{RT}_2^2$ $\Pi_1^1$-conservative over $\text{RCA}_0 + B\Sigma_2$.
A $\Pi^1_1$ sentence is a sentence of the form $\forall X \psi$, where $\psi$ is of first order.

**Theorem**

- (Cholak, Jockusch and Slaman) \( \text{COH} \) is $\Pi^1_1$-conservative over $\text{RCA}_0$, id est, if $\varphi$ is a $\Pi^1_1$ sentence provable in $\text{RCA}_0 + \text{COH}$ then it is provable in $\text{RCA}_0$.

- (Cholak, Jockusch and Slaman) $\text{RT}_2^2$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + I\Sigma_2$.

- (Chong, Slaman and Yang) \( \text{COH} \) is $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma_2$.

**Question**

Is $\text{SRT}_2^2$ or $\text{RT}_2^2$ $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma_2$.?
A $\Pi^1_1$ sentence is a sentence of the form $\forall X \psi$, where $\psi$ is of first order.

**Theorem**

- (Cholak, Jockusch and Slaman) $\text{COH}$ is $\Pi^1_1$-conservative over $\text{RCA}_0$, id est, if $\varphi$ is a $\Pi^1_1$ sentence provable in $\text{RCA}_0 + \text{COH}$ then it is provable in $\text{RCA}_0$.

- (Cholak, Jockusch and Slaman) $\text{RT}^2_2$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + I\Sigma_2$.

- (Chong, Slaman and Yang) $\text{COH}$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma_2$.

**Question**

Is $\text{SRT}^2_2$ or $\text{RT}^2_2$ $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma_2$.
A $\Pi^1_1$ sentence is a sentence of the form $\forall X \psi$, where $\psi$ is of first order.

| Theorem |
|---------|
| (Cholak, Jockusch and Slaman) COH is $\Pi^1_1$-conservative over RCA$_0$, *id est*, if $\varphi$ is a $\Pi^1_1$ sentence provable in RCA$_0 + \text{COH}$ then it is provable in RCA$_0$. |
| (Cholak, Jockusch and Slaman) RT$_2^2$ is $\Pi^1_1$-conservative over RCA$_0 + I\Sigma_2$. |
| (Chong, Slaman and Yang) COH is $\Pi^1_1$-conservative over RCA$_0 + B\Sigma_2$. |

| Question |
|----------|
| Is SRT$_2^2$ or RT$_2^2$ $\Pi^1_1$-conservative over RCA$_0 + B\Sigma_2$. |
A $\Pi^1_1$ sentence is a sentence of the form $\forall X \psi$, where $\psi$ is of first order.

**Theorem**

- *(Cholak, Jockusch and Slaman)* $\text{COH}$ is $\Pi^1_1$-conservative over $\text{RCA}_0$, id est, if $\varphi$ is a $\Pi^1_1$ sentence provable in $\text{RCA}_0 + \text{COH}$ then it is provable in $\text{RCA}_0$.

- *(Cholak, Jockusch and Slaman)* $\text{RT}^2_2$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + I\Sigma_2$.

- *(Chong, Slaman and Yang)* $\text{COH}$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma_2$.

**Question**

Is $\text{SRT}^2_2$ or $\text{RT}^2_2$ $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma_2$.
Ramsey’s theorems for longer tuples

Theorem (Jockusch)

Fix $n, k \in \mathbb{N}$.

- Every computable $f : [\mathbb{N}]^n \to k$ admits an infinite homogeneous set which is $\Pi^0_n$.

- There exists a computable $g : [\mathbb{N}]^n \to k$, which admits no infinite $\Sigma^0_n$ homogeneous set.

- If $n > 2$ then $\text{RCA}_0 \vdash \text{RT}_k^n \iff \text{ACA}_0$. 
Ramsey’s theorems for longer tuples

Theorem (Jockusch)

Fix $n, k \in \mathbb{N}$.

- Every computable $f : [\mathbb{N}]^n \to k$ admits an infinite homogeneous set which is $\Pi^0_n$.
- There exists a computable $g : [\mathbb{N}]^n \to k$, which admits no infinite $\Sigma^0_n$ homogeneous set.
- If $n > 2$ then $\text{RCA}_0 \vdash \text{RT}^n_k \leftrightarrow \text{ACA}_0$. 

Ramsey’s theorems for longer tuples

**Theorem (Jockusch)**

Fix $n, k \in \mathbb{N}$.

- Every computable $f : [\mathbb{N}]^n \to k$ admits an infinite homogeneous set which is $\Pi_0^n$.

- There exists a computable $g : [\mathbb{N}]^n \to k$, which admits no infinite $\Sigma_0^n$ homogeneous set.

- If $n > 2$ then $\text{RCA}_0 \vdash \text{RT}_k^n \leftrightarrow \text{ACA}_0$. 
Erdős and Rado introduced a partition relation $\kappa \rightarrow [\lambda]^n_k, < d$: for every $f : [\kappa]^n \rightarrow k$, there exists a set $H \subset \kappa$ of cardinality $\lambda$ with $|f([H]^n)| < d$. We call these Achromatic Ramsey Theorems, and denote $\omega \rightarrow [\omega]^n_k, < d+1$ by $\text{ART}^n_{k,d}$.

Harvey Friedman introduced a similar family. For each $f : [\omega]^n \rightarrow \omega$, a set $H$ is $f$-thin if $f([H]^n) \neq \omega$. $\text{TS}^n$ is the statement that every $f$ on $[\omega]^n$ admits an infinite thin set. Clearly, $\text{RCA}_0 \vdash \text{ART}^n_{d+1,d} \rightarrow \text{TS}^n$.

**Theorem (WW)**

For each $n > 0$ there exists $d$ such that $\text{RCA}_0 + \text{ART}^n_{k,d} \nvdash \text{ACA}_0$ for all $k$.

$d$ can be taken to be the $(n-1)$-th Schröder number $S_{n-1}$:

$$S_0 = 1, S_n = S_{n-1} + \sum_{k<n} S_k S_{n-k-1}.$$
Consequences of Ramsey’s theorems
Achromatic Ramsey theorems and Thin Set theorems

Erdős and Rado introduced a partition relation $\kappa \to [\lambda]^n_k, <d$: for every $f : [\kappa]^n \to k$, there exists a set $H \subset \kappa$ of cardinality $\lambda$ with $|f([H]^n)| < d$. We call these Achromatic Ramsey Theorems, and denote $\omega \to [\omega]^n_k, <d+1$ by $\text{ART}_k^d$.

Harvey Friedman introduced a similar family. For each $f : [\omega]^n \to \omega$, a set $H$ is $f$-thin if $f([H]^n) \neq \omega$. $\text{TS}_n$ is the statement that every $f$ on $[\omega]^n$ admits an infinite thin set. Clearly, $\text{RCA}_0 \vdash \text{ART}_{d+1, d}^n \rightarrow \text{TS}_n$.

**Theorem (WW)**

For each $n > 0$ there exists $d$ such that $\text{RCA}_0 + \text{ART}_k^n \nvdash \text{ACA}_0$ for all $k$.

$d$ can be taken to be the $(n-1)$-th Schröder number $S_{n-1}$:

$$S_0 = 1, S_n = S_{n-1} + \sum_{k<n} S_k S_{n-k-1}.$$
Erdős and Rado introduced a partition relation $\kappa \rightarrow [\lambda]^n_k, < d$: for every $f : [\kappa]^n \rightarrow k$, there exists a set $H \subset \kappa$ of cardinality $\lambda$ with $|f([H]^n)| < d$. We call these Achromatic Ramsey Theorems, and denote $\omega \rightarrow [\omega]^n_k, < d + 1$ by $\text{ART}^n_{k,d}$.

Harvey Friedman introduced a similar family. For each $f : [\omega]^n \rightarrow \omega$, a set $H$ is $f$-thin if $f([H]^n) \neq \omega$. $\text{TS}^n$ is the statement that every $f$ on $[\omega]^n$ admits an infinite thin set. Clearly, $\text{RCA}_0 \vdash \text{ART}^n_{d + 1,d} \rightarrow \text{TS}^n$.

**Theorem (WW)**

For each $n > 0$ there exists $d$ such that $\text{RCA}_0 + \text{ART}^n_{k,d} \nvdash \text{ACA}_0$ for all $k$.

$d$ can be taken to be the $(n - 1)$-th Schröder number $S_{n-1}$:

$$S_0 = 1, S_n = S_{n-1} + \sum_{k<n} S_k S_{n-k-1}.$$
Erdős and Rado introduced a partition relation $\kappa \rightarrow [\lambda]_k^n, <_d$: for every $f : [\kappa]^n \rightarrow k$, there exists a set $H \subset \kappa$ of cardinality $\lambda$ with $|f([H]^n)| < d$. We call these Achromatic Ramsey Theorems, and denote $\omega \rightarrow [\omega]_k^n, <_{d+1}$ by $\text{ART}^n_{k,d}$.

Harvey Friedman introduced a similar family. For each $f : [\omega]^n \rightarrow \omega$, a set $H$ is $f$-thin if $f([H]^n) \neq \omega$. $\text{TS}^n$ is the statement that every $f$ on $[\omega]^n$ admits an infinite thin set. Clearly, $\text{RCA}_0 \vdash \text{ART}^n_{d+1,d} \rightarrow \text{TS}^n$.

**Theorem (WW)**

For each $n > 0$ there exists $d$ such that $\text{RCA}_0 + \text{ART}^n_{k,d} \nvdash \text{ACA}_0$ for all $k$.

$d$ can be taken to be the $(n-1)$-th Schröder number $S_{n-1}$:

$$S_0 = 1, \quad S_n = S_{n-1} + \sum_{k<n} S_k S_{n-k-1}.$$
H. Friedman introduced the so-called Free Set theorems. For \( f : [\omega]^n \rightarrow \omega \), a set \( H \) is \( f \)-free, if \( f(\sigma) \notin H - \sigma \) for all \( \sigma \in [H]^n \). \( FS^n \) asserts that for each \( f \) on \([\omega]^n \) there exists an infinite free set.

**Theorem**

Fix a finite positive \( n \).

- (Cholak et al.) \( RCA_0 \vdash RT_2^n \rightarrow FS^n \rightarrow TS^n \).

- (Cholak et al.) Each computable \( f : [\omega]^n \rightarrow \omega \) admits an infinite \( \Pi^0_n \) free set. But there exists a computable \( g : [\omega]^n \rightarrow \omega \) admitting no infinite \( \Sigma^0_n \) free set.

- (WW) \( RCA_0 + FS^n \nvdash ACA_0 \).
H. Friedman introduced the so-called Free Set theorems. For \( f : [\omega]^n \to \omega \), a set \( H \) is \( f \)-free, if \( f(\sigma) \notin H - \sigma \) for all \( \sigma \in [H]^n \). \( FS^n \) asserts that for each \( f \) on \( [\omega]^n \) there exists an infinite free set.

**Theorem**

*Fix a finite positive \( n \).*

- *(Cholak et al.)* \( RCA_0 \vdash RT_2^n \to FS^n \to TS^n \).
- *(Cholak et al.)* Each computable \( f : [\omega]^n \to \omega \) admits an infinite \( \Pi^0_n \) free set. But there exists a computable \( g : [\omega]^n \to \omega \) admitting no infinite \( \Sigma^0_n \) free set.
- *(WW)* \( RCA_0 + FS^n \not\vdash ACA_0 \).
H. Friedman introduced the so-called Free Set theorems. For 
\( f : [\omega]^n \to \omega \), a set \( H \) is \( f \)-free, if 
\( f(\sigma) \notin H - \sigma \) for all \( \sigma \in [H]^n \). \( FS^n \) 
asserts that for each \( f \) on \([\omega]^n\) there exists an infinite free set.

**Theorem**

*Fix a finite positive \( n \).*

- *(Cholak et al.)* \( \text{RCA}_0 \vdash \text{RT}_2^n \to \text{FS}^n \to \text{TS}^n \).

- *(Cholak et al.)* Each computable \( f : [\omega]^n \to \omega \) admits an infinite \( \Pi^0_n \) free set. But there exists a computable \( g : [\omega]^n \to \omega \) admitting no infinite \( \Sigma^0_n \) free set.

- *(WW)* \( \text{RCA}_0 + \text{FS}^n \not\vdash \text{ACA}_0 \).
H. Friedman introduced the so-called Free Set theorems. For $f : [\omega]^n \to \omega$, a set $H$ is $f$-free, if $f(\sigma) \not\in H - \sigma$ for all $\sigma \in [H]^n$. $FS^n$ asserts that for each $f$ on $[\omega]^n$ there exists an infinite free set.

**Theorem**

Fix a finite positive $n$.

- (Cholak et al.) $RCA_0 \vdash RT^2_n \rightarrow FS^n \rightarrow TS^n$.

- (Cholak et al.) Each computable $f : [\omega]^n \rightarrow \omega$ admits an infinite $\Pi^0_n$ free set. But there exists a computable $g : [\omega]^n \rightarrow \omega$ admitting no infinite $\Sigma^0_n$ free set.

- (WW) $RCA_0 + FS^n \nvdash ACA_0$. 
Consequences of Ramsey’s theorems
Rainbow Ramsey theorems ...

\[ f : [\omega]^n \rightarrow \omega \text{ is } k\text{-bounded, if } |f^{-1}(c)| \leq k \text{ for all } c. \text{ A rainbow for } f \text{ is a set } H \text{ such that } f \text{ is injective on } [H]^n. \text{ RRT}_k^n \text{ states that every } k\text{-bounded } f \text{ admits an infinite rainbow.} \]

**Theorem**

- (Galvin) RCA₀ ⊢ RT₂^n → RRT₂^n.
- (Csima and Mileti) Each computable 2-bounded \( f : [\omega]^n \rightarrow \omega \) admits an infinite \( \Pi^0_n \) rainbow. But there exists a computable 2-bounded \( g : [\omega]^n \rightarrow \omega \) admitting no infinite \( \Sigma^0_n \) rainbow.
- (Csima and Mileti) For each 2-random \( R \), there exists an \( \omega \)-model of RCA₀ + RRT₂, in which every set is \( R \)-computable. Thus, RCA₀ + RRT₂ does not imply WKL, RT₂, RRT₂ etc.
Consequences of Ramsey’s theorems

Rainbow Ramsey theorems ...

\[ f : [\omega]^n \to \omega \text{ is } k\text{-bounded, if } |f^{-1}(c)| \leq k \text{ for all } c. \] A rainbow for \( f \) is a set \( H \) such that \( f \) is injective on \([H]^n\). \( RRT^n_k \) states that every \( k \)-bounded \( f \) admits an infinite rainbow.

**Theorem**

- (Galvin) \( \text{RCA}_0 \vdash RT^2_n \to RRT^2_n \).
- (Csima and Mileti) Each computable 2-bounded \( f : [\omega]^n \to \omega \) admits an infinite \( \Pi^0_n \) rainbow. But there exists a computable 2-bounded \( g : [\omega]^n \to \omega \) admitting no infinite \( \Sigma^0_n \) rainbow.
- (Csima and Mileti) For each 2-random \( R \), there exists an \( \omega \)-model of \( \text{RCA}_0 + RRT^2_2 \), in which every set is \( R \)-computable. Thus, \( \text{RCA}_0 + RRT^2_2 \) does not imply \( \text{WKL}, RT^2_2, RRT^3_2 \) etc.
Consequences of Ramsey’s theorems
Rainbow Ramsey theorems ...

\[ f : [\omega]^n \to \omega \] is \( k \)-bounded, if \( |f^{-1}(c)| \leq k \) for all \( c \). A rainbow for \( f \) is a set \( H \) such that \( f \) is injective on \([H]^n\). \( \text{RRT}_k^n \) states that every \( k \)-bounded \( f \) admits an infinite rainbow.

**Theorem**

- (Galvin) \( \text{RCA}_0 \vdash \text{RT}_2^n \to \text{RRT}_2^n \).
- (Csima and Mileti) Each computable 2-bounded \( f : [\omega]^n \to \omega \) admits an infinite \( \Pi^0_n \) rainbow. But there exists a computable 2-bounded \( g : [\omega]^n \to \omega \) admitting no infinite \( \Sigma^0_n \) rainbow.
- (Csima and Mileti) For each 2-random \( R \), there exists an \( \omega \)-model of \( \text{RCA}_0 + \text{RRT}_2^2 \), in which every set is \( R \)-computable. Thus, \( \text{RCA}_0 + \text{RRT}_2^2 \) does not imply \( \text{WKL} \), \( \text{RT}_2^2 \), \( \text{RRT}_2^3 \) etc.
Consequences of Ramsey’s theorems
Rainbow Ramsey theorems ...

\[ f : [\omega]^n \rightarrow \omega \text{ is } k\text{-bounded, if } |f^{-1}(c)| \leq k \text{ for all } c. \] A rainbow for \( f \) is a set \( H \) such that \( f \) is injective on \([H]^n\). \( \text{RRT}_k^n \) states that every \( k \)-bounded \( f \) admits an infinite rainbow.

Theorem

- **(Galvin)** \( \text{RCA}_0 \vdash \text{RT}_2^n \rightarrow \text{RRT}_2^n \).
- **(Csima and Mileti)** Each computable 2-bounded \( f : [\omega]^n \rightarrow \omega \) admits an infinite \( \Pi^0_n \) rainbow. But there exists a computable 2-bounded \( g : [\omega]^n \rightarrow \omega \) admitting no infinite \( \Sigma^0_n \) rainbow.
- **(Csima and Mileti)** For each 2-random \( R \), there exists an \( \omega \)-model of \( \text{RCA}_0 + \text{RRT}_2^2 \), in which every set is \( R \)-computable. Thus, \( \text{RCA}_0 + \text{RRT}_2^2 \) does not imply \( \text{WKL} \), \( \text{RT}_2^2 \), \( \text{RRT}_2^3 \) etc.
Consequences of Ramsey’s theorems

... Rainbow Ramsey theorems

Theorem

- (WW) $\text{RCA}_0 \vdash \text{FS}^n \rightarrow \text{RRT}^n_2$, so $\text{RCA}_0 + \text{RRT}^n_2 \nvdash \text{ACA}_0$.
- (Xiaojun Kang) $\text{RCA}_0 + \text{RRT}^2_2$ does not imply either $\text{TS}^2_2$ or $\text{FS}^2_2$.
- (WW) Every computable 2-bounded coloring of triples admits an infinite $\text{low}_3$ rainbow $X$ (i.e., $X''' \leq_T \emptyset'''$). Thus, $\text{RCA}_0 + \text{RRT}^3_2 \nvdash \text{RRT}^4_2$.
- (WW) Every computable 2-bounded coloring of triples admits an infinite rainbow, which does not compute a completion of the first order arithmetic. Thus, $\text{RCA}_0 + \text{RRT}^3_2 \nvdash \text{WKL}$.
- (Conidis and Slaman; WW) $\text{RRT}^2_2$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma_2$. 
| Theorem |
|--------|
| (WW) $\text{RCA}_0 \vdash \text{FS}^n \to \text{RRT}_2^n$, so $\text{RCA}_0 + \text{RRT}_2^n \not\vdash \text{ACA}_0$. |
| (Xiaojun Kang) $\text{RCA}_0 + \text{RRT}_2^2$ does not imply either $\text{TS}^2$ or $\text{FS}^2$. |
| (WW) Every computable 2-bounded coloring of triples admits an infinite $\text{low}_3$ rainbow $X$ (i.e., $X''' \leq_T \emptyset'''$). Thus, $\text{RCA}_0 + \text{RRT}_2^3 \not\vdash \text{RRT}_2^4$. |
| (WW) Every computable 2-bounded coloring of triples admits an infinite rainbow, which does not compute a completion of the first order arithmetic. Thus, $\text{RCA}_0 + \text{RRT}_2^3 \not\vdash \text{WKL}$. |
| (Conidis and Slaman; WW) $\text{RRT}_2^2$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma_2$. |
Consequences of Ramsey’s theorems

... Rainbow Ramsey theorems

| Theorem |
|---------|

- 

(WW) \( \text{RCA}_0 \vdash \text{FS}^n \rightarrow \text{RRT}^n_2 \), so \( \text{RCA}_0 + \text{RRT}^n_2 \nvdash \text{ACA}_0 \).

- (Xiaojun Kang) \( \text{RCA}_0 + \text{RRT}^2_2 \) does not imply either \( \text{TS}^2 \) or \( \text{FS}^2 \).

- (WW) Every computable 2-bounded coloring of triples admits an infinite \( \text{low}_3 \) rainbow \( X \) (i.e., \( X'''' \leq_T \emptyset'''' \)). Thus, \( \text{RCA}_0 + \text{RRT}^3_2 \nvdash \text{RRT}^4_2 \).

- (WW) Every computable 2-bounded coloring of triples admits an infinite rainbow, which does not compute a completion of the first order arithmetic. Thus, \( \text{RCA}_0 + \text{RRT}^3_2 \nvdash \text{WKL} \).

- (Conidis and Slaman; WW) \( \text{RRT}^2_2 \) is \( \Pi^1_1 \)-conservative over \( \text{RCA}_0 + B\Sigma_2 \).
Consequences of Ramsey’s theorems

... Rainbow Ramsey theorems

Theorem

- \((WW)\) \(\text{RCA}_0 \vdash \text{FS}^n \rightarrow \text{RRT}_2^n\), so \(\text{RCA}_0 + \text{RRT}_2^n \not\vdash \text{ACA}_0\).
- \(\text{(Xiaojun Kang)}\) \(\text{RCA}_0 + \text{RRT}_2^2\) does not imply either \(\text{TS}_2\) or \(\text{FS}_2\).
- \((WW)\) Every computable 2-bounded coloring of triples admits an infinite \(\text{low}_3\) rainbow \(X\) (i.e., \(X^{'''} \leq_T \emptyset^{'''}\)). Thus, \(\text{RCA}_0 + \text{RRT}_2^3 \not\vdash \text{RRT}_2^4\).
- \((WW)\) Every computable 2-bounded coloring of triples admits an infinite rainbow, which does not compute a completion of the first order arithmetic. Thus, \(\text{RCA}_0 + \text{RRT}_2^3 \not\vdash \text{WKL}\).
- \((\text{Conidis and Slaman}; \text{WW})\) \(\text{RRT}_2^2\) is \(\Pi^1_1\)-conservative over \(\text{RCA}_0 + B\Sigma_2\).
Consequences of Ramsey’s theorems

... Rainbow Ramsey theorems

Theorem

- (WW) $\text{RCA}_0 \vdash \text{FS}^n \rightarrow \text{RRT}^n_2$, so $\text{RCA}_0 + \text{RRT}^n_2 \not\vdash \text{ACA}_0$.
- (Xiaojun Kang) $\text{RCA}_0 + \text{RRT}^2_2$ does not imply either $\text{TS}^2$ or $\text{FS}^2$.
- (WW) Every computable $2$-bounded coloring of triples admits an infinite $\text{low}_3$ rainbow $X$ (i.e., $X''' \leq_T \emptyset'''$). Thus, $\text{RCA}_0 + \text{RRT}^3_2 \not\vdash \text{RRT}^4_2$.
- (WW) Every computable $2$-bounded coloring of triples admits an infinite rainbow, which does not compute a completion of the first order arithmetic. Thus, $\text{RCA}_0 + \text{RRT}^3_2 \not\vdash \text{WKL}$.
- (Conidis and Slaman; WW) $\text{RRT}^2_2$ is $\Pi^1_1$-conservative over $\text{RCA}_0 + B\Sigma_2$. 
A partial picture below ACA₀
Questions ...

- Let \( \varphi \) be a sentence of the form \( \forall X \exists Y \psi \), where \( \psi \) is of first order. We say that \( \varphi \) admits low\(_n\) solutions, if for every computable \( X \) there exists a low\(_n\) \( Y \) such that \( \psi(X, Y) \) holds.

- \( \mathsf{RT}_2^2 \) admits low\(_2\) solution by Cholak, Jockusch and Slaman, and \( \mathsf{RRT}_2^3 \) admits low\(_3\) solutions by WW.

- For \( \varphi \) being one of \( \mathsf{ART}_{k,d}^n \) (for reasonable \( d \)), \( \mathsf{TS}^n \), \( \mathsf{FS}^n \), \( \mathsf{RRT}_2^n \), does \( \varphi \) admits low\(_n\) solutions?

- If the above question has an affirmative answer then we may have proper hierarchies of combinatorial principles.
Let $\varphi$ be a sentence of the form $\forall X \exists Y \psi$, where $\psi$ is of first order. We say that $\varphi$ admits $\text{low}_n$ solutions, if for every computable $X$ there exists a $\text{low}_n$ $Y$ such that $\psi(X, Y)$ holds.

$\text{RT}_2^2$ admits $\text{low}_2$ solution by Cholak, Jockusch and Slaman, and $\text{RRT}_2^3$ admits $\text{low}_3$ solutions by WW.

For $\varphi$ being one of $\text{ART}_{k,d}^n$ (for reasonable $d$), $\text{TS}^n$, $\text{FS}^n$, $\text{RRT}_2^n$, does $\varphi$ admits $\text{low}_n$ solutions?

If the above question has an affirmative answer then we may have proper hierarchies of combinatorial principles.
Questions ...

Let \( \varphi \) be a sentence of the form \( \forall X \exists Y \psi \), where \( \psi \) is of first order. We say that \( \varphi \) admits low\(_n\) solutions, if for every computable \( X \) there exists a low\(_n\) \( Y \) such that \( \psi(X, Y) \) holds.

RT\(^2\) admits low\(_2\) solution by Cholak, Jockusch and Slaman, and RRT\(^3\) admits low\(_3\) solutions by WW.

For \( \varphi \) being one of ART\(_{k,d}^n\) (for reasonable \( d \)), TS\(^n\), FS\(^n\), RRT\(_2^n\), does \( \varphi \) admits low\(_n\) solutions?

If the above question has an affirmative answer then we may have proper hierarchies of combinatorial principles.
Let $\varphi$ be a sentence of the form $\forall X \exists Y \psi$, where $\psi$ is of first order. We say that $\varphi$ admits low$_n$ solutions, if for every computable $X$ there exists a low$_n$ $Y$ such that $\psi(X, Y)$ holds.

RT$^2_2$ admits low$_2$ solution by Cholak, Jockusch and Slaman, and RRT$^3_2$ admits low$_3$ solutions by WW.

For $\varphi$ being one of ART$^n_{k,d}$ (for reasonable $d$), TS$^n$, FS$^n$, RRT$^n_2$, does $\varphi$ admits low$_n$ solutions?

If the above question has an affirmative answer then we may have proper hierarchies of combinatorial principles.
A weakened but perhaps related question concerns theorems like Infinite Pigeonhole Principle. Recall that an infinite $C$ is cohesive for $\vec{R} = (R_n : n < \omega)$, if for each $n$ either $C \cap R_n$ or $C - R_n$ is finite.

Every computable $\vec{R}$ admits a low$_2$ cohesive set, by Stephan and Jockusch; every $\emptyset'$-computable $\vec{R}$ admits a low$_3$ cohesive set, by WW.

Does every $\emptyset^{(n)}$-computable $\vec{R}$ admit a low$_{n+2}$ cohesive set?

Even weaker: does every $\emptyset^{(n)}$-computable $f : [\omega]^1 \to 2$ admit a low$_{n+1}$ infinite homogeneous set?

Questions can also be raised about relative strength of theorems in different families (e.g., is TS$^n$ or RRT$^n_2$ strictly weaker than FS$^n$?), and about first order theories of these theorems (e.g., which $\varphi$'s considered above imply $B \Sigma_2$ over RCA$_0$?).
Questions

A weakened but perhaps related question concerns theorems like Infinite Pigeonhole Principle. Recall that an infinite \( C \) is cohesive for \( \vec{R} = (R_n : n < \omega) \), if for each \( n \) either \( C \cap R_n \) or \( C - R_n \) is finite.

Every computable \( \vec{R} \) admits a low\(_2\) cohesive set, by Stephan and Jockusch; every \( \emptyset' \)-computable \( \vec{R} \) admits a low\(_3\) cohesive set, by WW.

Does every \( \emptyset^{(n)} \)-computable \( \vec{R} \) admit a low\(_{n+2}\) cohesive set?

Even weaker: does every \( \emptyset^{(n)} \)-computable \( f : [\omega]^1 \to 2 \) admit a low\(_{n+1}\) infinite homogeneous set?

Questions can also be raised about relative strength of theorems in different families (e.g., is TS\(_n\) or RRT\(_2^n\) strictly weaker than FS\(_n\)?), and about first order theories of these theorems (e.g., which \( \varphi \)'s considered above imply \( B\Sigma_2 \) over RCA\(_0\)?).
... Questions

- A weakened but perhaps related question concerns theorems like Infinite Pigeonhole Principle. Recall that an infinite $C$ is cohesive for $\vec{R} = (R_n : n < \omega)$, if for each $n$ either $C \cap R_n$ or $C - R_n$ is finite.

- Every computable $\vec{R}$ admits a low$_2$ cohesive set, by Stephan and Jockusch; every $\emptyset'$-computable $\vec{R}$ admits a low$_3$ cohesive set, by WW.

- Does every $\emptyset^{(n)}$-computable $\vec{R}$ admit a low$_{n+2}$ cohesive set?

- Even weaker: does every $\emptyset^{(n)}$-computable $f : [\omega]^1 \rightarrow 2$ admit a low$_{n+1}$ infinite homogeneous set?

- Questions can also be raised about relative strength of theorems in different families (e.g., is $TS^n$ or $RRT_2^n$ strictly weaker than $FS^n$?), and about first order theories of these theorems (e.g., which $\phi$'s considered above imply $B\Sigma_2$ over $RCA_0$?).
Questions

- A weakened but perhaps related question concerns theorems like Infinite Pigeonhole Principle. Recall that an infinite $C$ is cohesive for $\vec{R} = (R_n : n < \omega)$, if for each $n$ either $C \cap R_n$ or $C - R_n$ is finite.

- Every computable $\vec{R}$ admits a $\text{low}_2$ cohesive set, by Stephan and Jockusch; every $\emptyset'$-computable $\vec{R}$ admits a $\text{low}_3$ cohesive set, by WW.

- Does every $\emptyset^{(n)}$-computable $\vec{R}$ admit a $\text{low}_{n+2}$ cohesive set?

- Even weaker: does every $\emptyset^{(n)}$-computable $f : [\omega]^1 \to 2$ admit a $\text{low}_{n+1}$ infinite homogeneous set?

- Questions can also be raised about relative strength of theorems in different families (e.g., is $\text{TS}^n$ or $\text{RRT}_2^n$ strictly weaker than $\text{FS}^n$?), and about first order theories of these theorems (e.g., which $\varphi$'s considered above imply $B\Sigma_2$ over RCA$_0$?).
A weakened but perhaps related question concerns theorems like Infinite Pigeonhole Principle. Recall that an infinite \( C \) is cohesive for \( \vec{R} = (R_n : n < \omega) \), if for each \( n \) either \( C \cap R_n \) or \( C - R_n \) is finite.

Every computable \( \vec{R} \) admits a \text{low}_2 cohesive set, by Stephan and Jockusch; every \( \emptyset' \)-computable \( \vec{R} \) admits a \text{low}_3 cohesive set, by WW.

Does every \( \emptyset^{(n)} \)-computable \( \vec{R} \) admit a \text{low}_{n+2} cohesive set?

Even weaker: does every \( \emptyset^{(n)} \)-computable \( f : [\omega]^1 \rightarrow 2 \) admit a \text{low}_{n+1} infinite homogeneous set?

Questions can also be raised about relative strength of theorems in different families (e.g., is TS\(^n\) or RRT\(^n_2\) strictly weaker than FS\(^n\)?) and about first order theories of these theorems (e.g., which \( \varphi \)'s considered above imply B\(\Sigma_2\) over \( \text{RCA}_0 \)?).
Thanks!