On cosmological constant of Generalized Robertson-Walker space-times

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Abstract We study Einstein’s equation in \((m+n)D\) and \((1+n)D\) warped spaces \((\bar{M}, \bar{g})\) and classify all such spaces satisfying Einstein equations \(\bar{G} = -\bar{\Lambda}\bar{g}\). We show that the warping function not only can determine the cosmological constant \(\bar{\Lambda}\) but also it can determine the cosmological constant \(\Lambda\) appearing in the induced Einstein equations \(\hat{G} = -Ah\) on \((M, h)\). Moreover, we discuss on the origin of the 4D cosmological constant as an emergent effect of higher dimensional warped spaces.

Keywords Einstein equation · cosmological constant · generalized Robertson-Walker spacetime · Lorentzian warped product

Mathematics Subject Classification (2010) Primary 83F05 · 35Q76 · Secondary 53C25

1 Introduction

One of the most fruitful generalizations of the direct product of two pseudo-Riemannian manifolds is the warped product defined by Bishop and O’Neill in Ref. [4]. The notion of warped products plays very important roles in differential geometry as well as in mathematical physics, especially in general relativity.
Many basic solutions of the Einstein field equations are warped products. For instance, both Schwarzschild and Robertson-Walker models in general relativity are warped products. Schwarzschild spacetime is the best relativistic model which describes the outer spacetime around a massive star or a black hole and the Robertson-Walker model describes a simply connected homogeneous isotropic expanding or contracting universe.

In Lorentzian geometry, it was first noticed that some well known solutions to Einstein’s field equations can be expressed in terms of warped products [1] and afterwards Lorentzian warped products have been used to obtain more solutions to Einstein’s field equations. On the other hand, first attempts for finding a static solution of the field equations of general relativity, applied for the cosmology, was done by Einstein who introduced the cosmological constant [8]. Since then the cosmological constant has played a very important role in the context of cosmology. However, the problem of origin and the large disagreement between the theoretical prediction, based on the quantum field theory calculations, and the observational bound on the value of this cosmological term has remained as the well known “cosmological constant problem” [7,12]. Here, we do not propose a solution for this problem, but we aim to present a new origin for the cosmological constant, based on the warped product of two pseudo-Riemannian manifolds. In this work, we study the generalized Robertson-Walker warped spacetimes with a cosmological constant.

Generalized Robertson-Walker spacetime models and standard static spacetime models are two well known solutions to Einstein’s field equations which can be expressed as Lorentzian warped products. Bejancu et al in Ref. [3], by using the extrinsic curvature of the horizontal distribution, obtained the classification of all spaces $(\bar{M}, \bar{g})$ satisfying Einstein equations $\bar{G} = -\bar{\Lambda}\bar{g}$, where $(\bar{M}, \bar{g})$ is a 5D warped space defined by 4D spacetime $(M, g)$ and the warped function. They described all the exact solutions for the warped metric by means of 4D exact solutions. We generalize the formalism of Bejancu et al in 5D warped space to $(m + n)D$ and $(1 + n)D$ warped spaces. The main motivation of this work is to describe the cosmological constant in Einstein’s equation as a resultant of the geometry of warped spaces.

In section 2, we recall some necessary details on warped product of pseudo-Riemannian manifolds, Ricci tensor, scalar curvature and Einstein gravitational equation on warped space. In section 3, we find necessary and sufficient conditions for the validity of Einstein equations on warped product space. Finally, in section 4, we construct five classes of exact solutions of Einstein equation for generalized Robertson-Walker spacetime.

2 Preliminaries

Let $\psi$ be a smooth function on a pseudo-Riemannian $n$-manifold $(M, g)$. Then the Hessian tensor field of $\psi$ is given by $H^\psi(X, Y) = XY\psi - (\nabla_X Y)\psi$, and the Laplacian of $\psi$ is given by $\Delta \psi = \text{trace}(H^\psi)$, or equivalently, $\Delta = \text{div} (\text{grad})$, where $\nabla$, div and grad are Levi-Civita connection of $M$, the divergence and
the gradient operators, respectively. (see p. 85 of [9]). Furthermore, we will frequently use the notation \( \| \text{grad} \ f \|_2 = g(\text{grad} \ f, \text{grad} \ f) \). In terms of a coordinate system \((x^1, \ldots, x^n)\), we have

\[
\Delta \psi = g^{ij} \left( \frac{\partial^2 \psi}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial \psi}{\partial x^k} \right),
\]

where \( \Gamma^k_{ij} \) are the Christoffel symbols on \((M, g)\) given by

\[
\Gamma^k_{ij} = \frac{1}{2} g^{kl} \left( \frac{\partial g_{il}}{\partial x^j} + \frac{\partial g_{jl}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^l} \right).
\]

We state the following Lemmas for later uses.

**Lemma 1** (Ref. [5]) Every harmonic function on a compact Riemannian manifold is constant.

**Lemma 2** (Hopf’s lemma, in Ref. [5]) Let \( M \) be a compact Riemannian manifold. If \( \psi \) is a differentiable function on \( M \) such that \( \Delta \psi \geq 0 \) everywhere on \( M \) (or \( \Delta \psi \leq 0 \) everywhere on \( M \)), then \( \psi \) is a constant function.

Let \((M_1, g)\) and \((M_2, h)\) are two pseudo-Riemannian manifolds and \( f \) be a positive smooth function on \( M_1 \). Then the warped product \( \bar{M} = M_1 \times_f M_2 \) is the product manifold \( M_1 \times M_2 \) endowed with the pseudo-Riemannian metric \( \bar{g} = \pi^* g + (f \circ \pi)^2 \sigma^* h \), where \( \pi \) and \( \sigma \) are the projections of \( M_1 \times M_2 \) onto \( M_1 \) and \( M_2 \) respectively. The function \( f \) is called a warping function and also \((M_1, g)\) and \((M_2, h)\) is called a base manifold and a fiber manifold, respectively [9].

The warped product \((\bar{M}, \bar{g})\) is a Lorentzian warped product if \((M_2, h)\) are Riemannian and either \((M_1, g)\) is Lorentzian or else \((M_1, g)\) is a one-dimensional manifold with a negative definite metric \(-dt^2\), Ref. [2]. We recall the de sitter space with cosmological constant \( \Lambda > 0 \), and

\[
d s^2 = -dt^2 + e^2(\frac{t}{2})^2 (dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)),
\]

where \((r, \theta, \phi)\) are spherical coordinates [2]. The de sitter space is an example of the Lorentzian warped product.

One of the main properties of \( \bar{M} \) is that the two factors \((M_1, g)\) and \((M_2, h)\) are orthogonal with respect to \( \bar{g} \). For a vector field \( X \) on \( M_1 \), the lift of \( X \) to \( M_1 \times_f M_2 \) is the vector field \( \hat{X} \) whose value at each \((p, q)\) is the lift \( X_p \) to \((p, q)\). Thus the lift of \( X \) is the unique vector field on \( M_1 \times_f M_2 \) that is \( \pi_1 \)-related to \( X \) and \( \pi_2 \)-related to the zero vector field on \( M_2 \). For a warped product \( M_1 \times_f M_2 \), let \( \mathcal{D} \) denotes the distribution obtained from the vectors tangent to the horizontal lifts of \( M_1 \).

Let \( \bar{M} = (M_1 \times_f M_2, \bar{g}) \) be a warped product of pseudo-Riemannian manifolds \((M_1, g)\) and \((M_2, h)\) with metric \( \bar{g} = g \times_f h \). If \( X, Y, Z \in \mathcal{D}_1 \) and \( V, W, U \in \mathcal{D}_2 \), then

\[
\nabla_X Y = \nabla^1_X Y,
\]

\[
\nabla_X V = \nabla^V X = \frac{X(f)}{f} V,
\]

\[
\nabla_V W = \nabla^V W - \frac{g(V, W)}{f} \text{grad} f,
\]
where $\nabla, \nabla^1$ and $\nabla^2$ are the Levi-Civita connection of metrics $\bar{g}, g$ and $h$, respectively.

\[
\bar{R}(X, Y)Z = R^1(X, Y)Z. \tag{6}
\]

\[
\bar{R}(V, X)Y = -\frac{H^1_f(X, Y)V}{f}, \text{ where } H^1_f \text{ is the Hessian of } f. \tag{7}
\]

\[
\bar{R}(X, Y)W = \bar{R}(V, W)X = 0. \tag{8}
\]

\[
\bar{R}(V, W)U = R^2(V, W)U - \|\text{grad } f\|^2/f^2(g(V, U)W - g(W, U)V), \tag{9}
\]

where $\bar{R}, R^1$ and $R^2$ are the curvature tensor of metrics $\bar{g}, g$ and $h$, respectively.

\[
\bar{Ric}(X, Y) = M_1^{\bar{Ric}}(X, Y) - \frac{n}{f}H^{\bar{Ric}}_f(X, Y). \tag{11}
\]

\[
\bar{Ric}(X, V) = 0. \tag{12}
\]

\[
\bar{Ric}(V, W) = M_2^{\bar{Ric}}(V, W) - g(V, W)\left(\frac{\Delta f}{f} + (n - 1)\frac{\|\text{grad } f\|^2}{f^2}\right), \tag{13}
\]

where $\bar{Ric}, M_1^{\bar{Ric}}$ and $M_2^{\bar{Ric}}$ are the Ricci tensor of metrics $\bar{g}, g$ and $h$, respectively.

\[
\bar{S} = S^{M_1} + \frac{S^{M_2}}{f^2} - 2\frac{n}{f}\frac{\Delta f}{f} - n(n - 1)\frac{\|\text{grad } f\|^2}{f^2}, \tag{14}
\]

where $S^{M_1}$ is scalar curvature of $(M_1, g)$ and $S^{M_2}$ is scalar curvature of $(M_2, h)$ [9].

We use the Einstein convention, that is, repeated indices with one upper index and one lower index denote summation over their range. If not stated otherwise, throughout the paper we use the following ranges for indices: $i, j, k, \ldots \in \{1, \ldots, m\}$; $\alpha, \beta, \ldots \in \{m + 1, \ldots, m + n\}$; $a, b, c, \ldots \in \{1, \ldots, n + m\}$.

In what follows we take $(x^i, x^\alpha)$ as a coordinate system on $M_1 \times f M_2$, where $(x^i)$ and $(x^\alpha)$ are the local coordinates on $M_1$ and $M_2$, respectively.

Suppose that $\bar{g}$ is a pseudo-Riemannian metric on $\bar{M}$ defined by $\bar{g} = g \times f h$ given by its local components:

\[
\bar{g}_{ij}(x^a) = g_{ij}(x^k), \tag{15}
\]

\[
\bar{g}_{i\alpha}(x^a) = 0, \tag{16}
\]

\[
\bar{g}_{\alpha\beta}(x^a) = f^2(x^k)h_{\alpha\beta}(x^\mu), \tag{17}
\]

where $g_{ij}(x^k)$ are the local components of $g$ and $h_{\alpha\beta}(x^\mu)$ are the local components of $h$. Let $\bar{M} = M_1 \times f M_2$ be a warped product manifold. Then

\[
\bar{R}_{ij} = R_{ij} - \frac{n}{f}H^1_f, \tag{18}
\]

\[
\bar{R}_{i\alpha} = 0, \tag{19}
\]

\[
\bar{R}_{\alpha\beta} = R_{\alpha\beta} - \left(\frac{\Delta f}{f} + (n - 1)\frac{\|\text{grad } f\|^2}{f^2}\right)\bar{g}_{\alpha\beta}, \tag{20}
\]

where $\bar{R}, R^1$ and $R^2$ are the curvature tensor of metrics $\bar{g}, g$ and $h$, respectively.
where $R_{ij} = \text{Ric}(\partial_i, \partial_j)$ are the local components of Ricci tensor of $(M_1, g)$ and $R_{\alpha\beta} = \text{Ric}(\partial_\alpha, \partial_\beta)$ are the local components of Ricci tensor of $(M_2, h)$.

We denote by $\overline{G}$ the Einstein gravitational tensor field of $(\overline{M}, \overline{g})$, that is, we have,

$$\overline{G} = \text{Ric} - \frac{1}{2}\overline{S}\overline{g}, \quad (21)$$

where $\text{Ric}$ and $\overline{S}$ are Ricci tensor and scaler curvature of $\overline{M}$, respectively.

**Proposition 1.** Let $\overline{G}$ be the Einstein gravitational tensor field of $(\overline{M}, \overline{g})$, then we have following equations:

$$\overline{G}_{ij} = G_{ij} - \frac{n}{\overline{f}} H_{ij} - \frac{1}{2} \left( \frac{S_{M_2}}{\overline{f}^2} - 2n\frac{\Delta f}{\overline{f}} - n(n-1)\frac{\|\text{grad} f\|^2}{\overline{f}^2} \right) g_{ij} , \quad (22)$$

$$\overline{G}_{\alpha\beta} = G_{\alpha\beta} - \overline{f}^2 \left( \frac{\Delta f}{\overline{f}} \left( 1 - n \right) + \frac{1}{2} \frac{S_{M_1}}{m} + \left( n-1 \right) \frac{2-n}{2} \frac{\|\text{grad} f\|^2}{\overline{f}^2} \right) h_{\alpha\beta} , \quad (23)$$

$$\overline{G}_{i\alpha} = 0 , \quad (24)$$

where $G_{ij}$ and $G_{\alpha\beta}$ are the local components of the Einstein gravitational tensor field of $(M_1, g)$ and $(M_2, h)$, respectively.

**Proof** By using (21), (18), (17) and (14), we obtain (22), (23) and (24). \(\square\)

### 3 Einstein field equations with cosmological constant

Let $(\overline{M}, \overline{g})$ be the warped space and $f$ be the warped function. Suppose that the Einstein gravitational tensor field $\overline{G}$ of $(\overline{M}, \overline{g})$ satisfies the Einstein equations with cosmological constant $\overline{\Lambda}$ as

$$\overline{G} = -\overline{\Lambda}\overline{g} . \quad (25)$$

First, we prove the following theorem.

**Theorem 1** The Einstein equations on $(\overline{M}, \overline{g})$ with cosmological constant $\overline{\Lambda}$ are equivalent with the following equations

$$- \overline{\Lambda} = \left( \frac{m+n-2}{2m} \right) \left( \frac{n}{\overline{f}} \frac{\Delta f}{\overline{f}} - S_{M_1} \right), \quad (26)$$

$$\overline{G}_{\alpha\beta} = \overline{f}^2 \left[ \frac{n}{2} \frac{\Delta f}{\overline{f}} \left( \frac{1-n}{m} \right) + \frac{S_{M_1}}{m} + (n-1) \frac{\|\text{grad} f\|^2}{\overline{f}^2} \right] h_{\alpha\beta} , \quad (27)$$

Moreover, we have

$$R_{\alpha\beta} = \overline{f}^2 \left[ \frac{n}{2} \frac{\Delta f}{\overline{f}} \left( \frac{1-n}{m} \right) + \frac{S_{M_1}}{m} + (n-1) \frac{\|\text{grad} f\|^2}{\overline{f}^2} \right] h_{\alpha\beta} . \quad (28)$$
Proof Using (22) and Einstein equation $\tilde{G} = -\tilde{A}\tilde{g}$, we have

$$G_{ij} - \frac{n}{f} H_{ij} - \frac{1}{2} g_{ij} \left( \frac{S_{M^2}}{f^2} - 2n \frac{\Delta f}{f} - n(n-1) \frac{\|\text{grad} f\|^2}{f^2} \right) = -\tilde{A}g_{ij}. \quad (29)$$

Contracting (29) with $g^{ij}$, and consider $G_{ij} g^{ij} = S_{M^1} - \frac{m}{f} S_{M^2}$ and $H_{ij} g^{ij} = \Delta f$, we obtain

$$\tilde{A} = \left( S_{M^1} \left( \frac{1}{m} - \frac{1}{2} \right) + n(1 - \frac{1}{m}) \frac{\Delta f}{f} - \frac{1}{2} \frac{S_{M^2}}{f^2} + \frac{n}{2} (n-1) \frac{\|\text{grad} f\|^2}{f^2} \right). \quad (30)$$

Then using (23), we obtain

$$G_{\alpha\beta} = f^2 h_{\alpha\beta} \left( 1 - \frac{n}{m} \frac{\Delta f}{f} + (n-1)(1 - \frac{n}{2}) \frac{\|\text{grad} f\|^2}{f^2} + \frac{1}{2} S_{M^1} - \tilde{A} \right). \quad (31)$$

By putting $\tilde{A}$ from (30) in (31), we have

$$G_{\alpha\beta} = f^2 h_{\alpha\beta} \left( 1 - \frac{n}{m} \frac{\Delta f}{f} + (n-1) \left( 1 - \frac{n}{m} \frac{\Delta f}{f} + \frac{1}{2} \frac{S_{M^2}}{f^2} + \frac{n}{2} \frac{S_{M^1}}{m} \right) \right). \quad (32)$$

Now, contracting (32) with $h^{\alpha\beta}$ we obtain

$$\frac{S_{M^2}}{f^2} = n \left( 1 - \frac{n}{m} \frac{\Delta f}{f} + (n-1) \frac{\|\text{grad} f\|^2}{f^2} + \frac{n}{m} S_{M^1} \right). \quad (33)$$

By using (30) and (33), we obtain

$$-\tilde{A} = \left( \frac{m+n-2}{2m} \right) \left( n \frac{\Delta f}{f} - S_{M^1} \right). \quad (34)$$

We now use (33) in (32) and obtain

$$G_{\alpha\beta} = f^2 h_{\alpha\beta} \left( 1 - \frac{n}{2} \left( 1 - \frac{n}{m} \frac{\Delta f}{f} + \frac{S_{M^1}}{m} + (n-1) \frac{\|\text{grad} f\|^2}{f^2} \right) \right). \quad (35)$$

Now, from the above and by using (33) and (21), we obtained (28).

**Proposition 2** The Einstein equations $\tilde{G} = -\tilde{A}\tilde{g}$ on $(\tilde{M}, \tilde{g})$ with cosmological constant $\tilde{\Lambda}$ induces the Einstein equations $G_{\alpha\beta} = -\Lambda h_{\alpha\beta}$ on $(M_2, h_{\alpha\beta})$, where the cosmological constant $\Lambda$ is given by

$$\Lambda = f^2 \left( 1 - \frac{n}{2} \left( \frac{\Delta f}{f} - \frac{S_{M^1}}{m} \right) + (n-1) \frac{\|\text{grad} f\|^2}{f^2} \right). \quad (36)$$

Proof By using (27) and Einstein equations $G_{\alpha\beta} = -\Lambda h_{\alpha\beta}$, we can obtain (36). Also, by using (28) and Theorem 3.3 of [13, page 38],

$$f^2 \left( \frac{\Delta f}{f} - \frac{S_{M^1}}{m} + (n-1) \frac{\|\text{grad} f\|^2}{f^2} \right) \quad (37)$$

must be a constant, and therefore $(M_2, h)$ is an Einstein manifold.

Now using (26) and Lemma 1 and 2, we deduce the following corollaries.
Corollary 1 The warping function $f$ is an eigenfunction of the Laplacian operator $\Delta$ with eigenvalue $\frac{2m\Lambda + (m + n - 2)S^{M_1}}{n(m + n - 2)}$.

Corollary 2 Let $\bar{M} = M_1 \times_f M_2$ and satisfy an Einstein equation with a cosmological constant $\bar{\Lambda}$. If $M_1$ is a compact Riemannian manifold and Ricci flat then $f$ is constant.

Corollary 3 Let $\bar{M} = M_1 \times_f M_2$ which satisfy an Einstein equation with a cosmological constant $\bar{\Lambda} = 0$. If $M_1$ is Ricci flat then $f$ is harmonic.

4 Generalized Robertson-Walker spacetimes

An $(n+1)$-dimensional generalized Robertson-Walker (GRW) spacetime with $n > 1$ is a Lorentzian manifold which is a warped product manifold $\bar{M} = I \times_f M$ of an open interval $I$ of the real line $\mathbb{R}$ and a Riemannian $n$-manifold $(M, g)$ endowed with the Lorentzian metric

$$\bar{g} = -\pi^*(dt^2) + f(t)^2 \sigma^*(g),$$

where $\pi$ and $\sigma$ denote the projections onto $I$ and $M$, respectively, and $f$ is a positive smooth function on $I$. In a classical Robertson-Walker (RW) spacetime, the fiber is three dimensional and of constant sectional curvature, and the warping function $f$ is arbitrary [6]. Such spaces include the Einstein-de Sitter space, the Friedman cosmological models, and the de Sitter space.

The following formula can be directly obtained from the previous result and noting that on a generalized Robertson-Walker spacetime $\text{grad}_I f = -f'$, $\|\text{grad}_I f\|^2 = -f'^2$, $g(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}) = -1$, $H(f(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})) = f''$ and $\Delta f = -f''$. We denote the usual derivative on the real interval $I$ by the prime notation (i.e., $'$) from now on.

Putting $m = 1$ in Theorem 1 and by using (26) and (27), we have

$$\bar{\Lambda} = \frac{1}{8} n(n - 1) (B^2 + 2B'),$$

$$G_{\alpha\beta} = -\frac{1}{4} (n - 1)(n - 2)f^2 B' h_{\alpha\beta},$$

where $B = 2f'$. Moreover, Einstein Equation (40) on $(M, \bar{g})$ with cosmological constant $\bar{\Lambda}$ induce the Einstein Equation $G = -\Lambda g$ on $(M, g)$ where the cosmological constant $\Lambda$ is given by

$$\Lambda = \frac{1}{4} n(n - 1)(n - 2)f'^2 B',$$

and is constant. Next, we assume that $\bar{\Lambda}$ given by (39) is a constant. Thus, we must solve the differential equation $-\frac{1}{8} n(n - 1)(B^2 + 2B') = k$, where $k$ is a constant. We can state the following classification theorem.

Theorem 2 Let $(\bar{M}, \bar{g})$ be an $(n+1)$-dimensional generalized Robertson-Walker (GRW) spacetime with $n > 1$. We have
1. The cosmological constant of \((\bar{M}, \bar{g})\) is given by
\[
\bar{\Lambda} = -\frac{n(n-1)}{2L^2}, \quad L \neq 0,
\] (42)
and \(\bar{g}\) is given by one of the expressions
\[
ds^2 = \frac{1}{k} e^{2t/L} g_{\alpha\beta}(x^{\mu}) dx^\alpha dx^\beta - dt^2,
\] (43)
\[
ds^2 = \frac{1}{k} \left( \cosh \frac{b+t}{L} \right)^2 g_{\alpha\beta}(x^{\mu}) dx^\alpha dx^\beta - dt^2,
\] (44)
\[
ds^2 = \frac{1}{k} \left( \sinh \frac{b+t}{L} \right)^2 g_{\alpha\beta}(x^{\mu}) dx^\alpha dx^\beta - dt^2, \quad t \neq -b,
\] (45)
where \(k > 0\), and \(b \in \mathbb{R}\). Moreover, the spacetime \((M, g)\) must be Ricci flat in case of the metric (43), and an Einstein space in cases of both metrics (44) and (45), with cosmological constants
\[
\Lambda = -\frac{(n-1)(n-2)}{2kL^2},
\] (46)
and
\[
\Lambda = \frac{(n-1)(n-2)}{2kL^2},
\] (47)
respectively.

2. The cosmological constant of \((\bar{M}, \bar{g})\) and \(\bar{g}\) are given by
\[
\bar{\Lambda} = \frac{n(n-1)}{2L^2}, \quad L \neq 0,
\] (48)
and
\[
ds^2 = \frac{1}{k} \left( \cos \frac{b+t}{L} \right)^2 g_{\alpha\beta}(x^{\mu}) dx^\alpha dx^\beta - dt^2,
\] (49)
\[\text{where } k > 0\text{, and } b \in \mathbb{R}. \text{ In this case, } (M, g) \text{ is an Einstein space with cosmological constant}
\]
\[
\Lambda = \frac{(n-1)(n-2)}{2kL^2},
\] (50)

3. The cosmological constant of \((\bar{M}, \bar{g})\) is \(\bar{\Lambda} = 0\), that is, \((\bar{M}, \bar{g})\) is Ricci flat, and \(\bar{g}\) is given by
\[
ds^2 = \frac{1}{L^2} (b-t)^2 g_{\alpha\beta}(x^{\mu}) dx^\alpha dx^\beta - dt^2, \quad t \neq -b,
\] (51)
where \(L \neq 0\), and \(b \in \mathbb{R}\). Moreover, \((M, g)\) must be an Einstein space with cosmological constant
\[
\Lambda = \frac{(n-1)(n-2)}{2L^2}.
\] (52)
Putting $n = 4$, indicating the five dimensional spacetime, we obtain

$$\Lambda = \frac{3}{L^2}. \quad (53)$$

This is in agreement with the current observed value of $4D$ cosmological constant provided that $L$ be the current size of the universe, namely $10^{28}$ meters. Such a good agreement between the theoretical value of cosmological constant obtained in this higher dimensional model of warped spaces and the experimental value of cosmological constant may account for the possible higher dimensional origin of the cosmological constant.

5 Conclusion

In this article, using the formalism of Bejancu et al, we have studied Einstein’s equation in $(m+n)D$ and $(1+n)D$ warped spaces $(\bar{M}, \bar{g})$ where $\bar{M} = M_1 \times f M_2$ is the product manifold $M_1 \times M_2$ endowed with the pseudo-Riemannian metric $\bar{g} = \pi^* g + (f \circ \pi)^2 \sigma^* h$. We have classified all such spaces satisfying Einstein equations $\bar{G} = -\Lambda \bar{g}$. We have shown that the warping function $f$ can determine both the cosmological constants $\Lambda$, and $\Lambda$ appearing in the induced Einstein equations $G = -A h$ on $(M_2,h)$. Moreover, we have discussed on the origin of the $4D$ cosmological constant as an emergent effect of higher dimensional warped spaces and confronted its numerical value with its observed value.

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