The stable category of preorders in a pretopos II: the universal property

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Abstract

We prove that the stable category associated with the category PreOrd(C) of internal preorders in a pretopos C satisfies a universal property. The canonical functor from PreOrd(C) to the stable category Stab(C) universally transforms a pretorsion theory in PreOrd(C) into a classical torsion theory in the pointed category Stab(C). This also gives a categorical insight into the construction of the stable category first considered by Facchini and Finocchiaro in the special case when C is the category of sets.

Keywords Preorders · Partial orders · Equivalence relations · Pretopos · Stable category · (Pre)torsion theories

Mathematics Subject Classification 06A75 · 18B25 · 18B50 · 18B35 · 18E08 · 18E40

Introduction

This article is meant as the sequel of [2] and it deals with the study of the universal property of the stable category Stab(C) of the category PreOrd(C) of internal preorders in a pretopos C. The canonical functor from PreOrd(C) to Stab(C) universally transforms a pretorsion theory in PreOrd(C) into a classical torsion theory in the pointed category Stab(C). This also gives a categorical insight into the construction of the stable category first considered by Facchini and Finocchiaro in the special case when C is the category of sets.

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C. It reveals the categorical feature of a natural construction due to A. Facchini and C. Finocchiaro in the category PreOrd of preordered sets [5], that we first briefly recall.

The category PreOrd contains the full subcategories Eq and ParOrd whose objects are equivalence relations and partially ordered sets, respectively. The pair of categories (Eq, ParOrd) has two properties making it a pretorsion theory in PreOrd [5, 6], that is, a kind of “non-pointed torsion theory.” More explicitly, any preordered set \((A, \rho)\), where \(\rho\) is a reflexive and transitive relation on the set \(A\), determines an equivalence relation \((A, \sim_{\rho})\), where \(\sim_{\rho} = \rho \cap \rho^o\) and \(\rho^o\) is the opposite relation of \(\rho\), and a partially ordered set \((A/\sim_{\rho}, \pi(\rho))\), where \(\pi : A \to A/\sim_{\rho}\) is the quotient of \(A\) by the equivalence relation \(\sim_{\rho}\), and \(\pi(\rho)\) is the partial order induced by \(\rho\) on the quotient \(A/\sim_{\rho}\). This yields a short \(Z\)-exact sequence

\[
(A, \sim_{\rho}) \xrightarrow{id_A} (A, \rho) \xrightarrow{\pi} (A/\sim_{\rho}, \pi(\rho)) \quad \text{(SES)}
\]

where \(Z\) is the full subcategory of PreOrd whose objects are the “trivial preorders” \((B, =)\), with \(B\) a set and = the equality relation on \(B\). This subcategory \(Z\) determines an ideal of trivial morphisms [4], where a morphism is called trivial if it factors through a trivial object. The fact that the above sequence is \(Z\)-exact means that the identity morphism \(id_A\) above is the \(Z\)-kernel of \(\pi\), and the quotient \(\pi\) is the \(Z\)-cokernel of \(id_A\), where the notions of \(Z\)-kernel and \(Z\)-cokernel are defined by the same universal properties characterizing usual kernels and cokernels, with the only difference that the ideal of 0-morphisms is replaced by the ideal of trivial morphisms [6]. The \(Z\)-exact sequence (SES) has the property that the \(Z\)-kernel belongs to Eq (the “torsion subcategory”) and the \(Z\)-cokernel belongs to ParOrd (the “torsion-free subcategory”). Furthermore, one easily sees that any order preserving morphism from an equivalence relation to a partial order is trivial. These two properties express the fact that (Eq, ParOrd) is a pretorsion theory in PreOrd.

In their study of this pretorsion theory in PreOrd the authors of [5] introduced a new category, the stable category \(\text{Stab}\) of preordered sets: this is a pointed category, arising as a quotient category, with the property that the canonical functor from PreOrd to \(\text{Stab}\) sends the trivial objects in \(Z\) to the zero object in \(\text{Stab}\), and any trivial morphism in PreOrd to a zero morphism in \(\text{Stab}\).

In the first article [2] of this series, we proved that, whenever \(C\) is a coherent category [8], it is possible to give a purely categorical construction of the stable category \(\text{Stab}(C)\) of the category PreOrd\((C)\) of internal preorders in \(C\) (we recall this construction in the first section of this article). Moreover, when \(C\) is a pretopos, the canonical functor \(\Sigma : \text{PreOrd}(C) \to \text{Stab}(C)\) preserves coproducts and sends short \(Z\)-exact sequences in PreOrd\((C)\) to short exact sequences in the pointed category \(\text{Stab}(C)\) (Theorem 7.14 in [2]).

The aim of this article is to prove the universal property of the stable category \(\text{Stab}(C)\), that relies on these two properties of the functor \(\Sigma : \text{PreOrd}(C) \to \text{Stab}(C)\). If we call a functor \(G : \text{PreOrd}(C) \to \mathbb{X}\) a torsion theory functor (Definition 2.1) when it sends the torsion and the torsion-free subcategories of the pretorsion theory \((\text{Eq}(C), \text{ParOrd}(C))\) into the torsion and the torsion-free subcategory, respectively, of a torsion theory \((T, \mathcal{F})\) in the category \(\mathbb{X}\), the universal property can be expressed as follows:

**Theorem 2.3** The canonical functor \(\Sigma : \text{PreOrd}(C) \to \text{Stab}(C)\) is universal among all finite coproduct preserving torsion theory functors \(G : \text{PreOrd}(C) \to \mathbb{X}\), where \(\text{PreOrd}(C)\) is equipped with the pretorsion theory \((\text{Eq}(C), \text{ParOrd}(C))\), and \(\mathbb{X}\) is a pointed category with coproducts equipped with a torsion theory \((T, \mathcal{F})\). This means that any finite coproduct preserving torsion theory functor \(G : \text{PreOrd}(C) \to \mathbb{X}\) factors uniquely through \(\Sigma\):
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\[ \text{PreOrd}(\mathbb{C}) \xrightarrow{\Sigma} \text{Stab}(\mathbb{C}) \]

\[ \forall G \xrightarrow{\exists!} \mathbb{X}, \]

i.e., there is a unique functor \( G \) such that \( G \cdot \Sigma = G \). The induced functor \( G \) preserves finite coproducts, and it is a torsion theory functor.

This theorem reveals the nature of the stable category, namely to transform a pretorsion theory in the sense of [6] into a “classical” torsion theory, universally. Note that some further properties of the stable category \( \text{Stab}(\mathbb{C}) \) can be established when the base category \( \mathbb{C} \) is what we called a \( \tau \)-pretopos in [2], that is a pretopos with the additional property that the transitive closure of any relation on an object exists. Under this assumption, it is possible to show that, for any “suitable” category \( \mathbb{X} \), the induced torsion theory functor \( G \) above preserves kernels and cokernels, hence in particular short exact sequences (Theorem 3.12).

1 Preliminaries

In this work, we shall be mainly interested in the category \( \text{PreOrd}(\mathbb{C}) \) of internal preorders in a pretopos \( \mathbb{C} \). In [7] it was proven that the there is a pretorsion theory \((\text{Eq}(\mathbb{C}), \text{Par}(\mathbb{C}))\) in \( \text{PreOrd}(\mathbb{C}) \). In order to make the present paper as self-contained as possible, we now recall all the definitions needed in the sequel.

Pretorsion theories

We first briefly recall the definition of pretorsion theory for general categories, as defined in [5, 6]. Let \( \mathcal{C} \) be an arbitrary category and consider a pair \((\mathcal{T}, \mathcal{F})\) of two replete full subcategories of \( \mathcal{C} \). Set also \( \mathcal{Z} := \mathcal{T} \cap \mathcal{F} \) and call it the class of “trivial objects”. A morphism \( f : A \to A' \) in \( \mathcal{C} \) is \( \mathcal{Z} \)-trivial if it factors through an object of \( \mathcal{Z} \). Notice that the class of trivial morphisms in \( \mathcal{C} \) is an ideal of morphisms in the sense of Ehresmann [4], and thus it is possible to consider the notions of \( \mathcal{Z} \)-kernel and of \( \mathcal{Z} \)-cokernel, defined by replacing, in the definition of kernel and cokernel, the ideal of zero morphisms with the ideal of trivial morphisms induced by the subcategory \( \mathcal{Z} \). More precisely, we say that a morphism \( \varepsilon : X \to A \) in \( \mathcal{C} \) is a \( \mathcal{Z} \)-kernel of \( f : A \to A' \) if \( f \varepsilon \) is a \( \mathcal{Z} \)-trivial morphism and whenever \( \lambda : Y \to A \) is a morphism in \( \mathcal{C} \) and \( f \lambda \) is \( \mathcal{Z} \)-trivial, then there exists a unique morphism \( \lambda' : Y \to X \) in \( \mathcal{C} \) such that \( \lambda = \varepsilon \lambda' \).

The notion of \( \mathcal{Z} \)-cokernel is defined dually. A sequence \( A \xrightarrow{f} B \xrightarrow{g} C \) is a short \( \mathcal{Z} \)-exact sequence if \( f \) is the \( \mathcal{Z} \)-kernel of \( g \) and \( g \) is the \( \mathcal{Z} \)-cokernel of \( f \). We say that the pair \((\mathcal{T}, \mathcal{F})\) is a pretorsion theory in \( \mathcal{C} \) if the following two properties are satisfied:

- Any morphism from an object \( T \in \mathcal{T} \) to an object \( F \in \mathcal{F} \) is \( \mathcal{Z} \)-trivial;
- For every object \( X \) of \( \mathcal{C} \) there is a short \( \mathcal{Z} \)-exact sequence

\[ T_X \xrightarrow{f} X \xrightarrow{g} F_X \]

with \( T_X \in \mathcal{T} \) and \( F_X \in \mathcal{F} \).
Pretoposes

In this article, \( \mathcal{C} \) will always be assumed to be a pretopos (see [8] for more details). Let us recall that \( \mathcal{C} \) is a pretopos when

- \( \mathcal{C} \) is exact (in the sense of Barr [1]),
- \( \mathcal{C} \) has finite sums (= coproducts),
- \( \mathcal{C} \) is extensive [3].

The property of extensivity means that \( \mathcal{C} \) has pullbacks along coprojections in a sum and the following condition holds: in any commutative diagram, where the bottom row is the sum of \( A \) and \( B \)

\[
\begin{array}{ccc}
A' & \rightarrow & C \\
\downarrow & & \downarrow \\
A & \xrightarrow{s_1} & A \coprod B
\end{array}
\]

\[
\begin{array}{ccc}
\downarrow & & \downarrow \\
B & \xleftarrow{s_2} & B
\end{array}
\]

the top row is a sum if and only if the two squares are pullbacks. The property saying that the upper row of the diagram is a sum whenever the two squares are pullbacks is usually called the “universality of sums.”

Recall that a sum of two objects \( A \) and \( B \) is disjoint if the coprojections \( s_1 : A \rightarrow A \coprod B \) and \( s_2 : B \rightarrow A \coprod B \) are monomorphisms and their intersection \( A \cap B \) in the pullback

\[
\begin{array}{ccc}
A \cap B & \rightarrow & B \\
\downarrow & & \downarrow \\
A & \xrightarrow{s_1} & A \coprod B
\end{array}
\]

is an initial object in the category \( \text{Sub}(A \coprod B) \) of subobjects of \( A \coprod B \). For a finitely complete category \( \mathcal{C} \) with finite sums, extensivity is equivalent to the property of having disjoint and universal finite sums. In a pretopos the supremum \( A \cup B \) of two disjoint subobjects \( A \rightarrow X \) and \( B \rightarrow X \) is given by the coproduct \( A \coprod B \) of these two objects in \( \mathcal{C} \) (see Corollary 1.4.4 in [8]). Recall also that any pretopos has a strict initial object, namely an initial object 0 with the property that any morphism with codomain 0 is an isomorphism.

Internal preorders

As already said, in this work we shall be mainly interested in the category \( \text{PreOrd}(\mathcal{C}) \) of internal preorders in a pretopos \( \mathcal{C} \), that is defined as follows.

An object \((A, \rho)\) in \( \text{PreOrd}(\mathcal{C}) \) is a relation \( \langle r_1, r_2 \rangle : \rho \rightarrow A \times A \) on \( A \), i.e., a subobject of \( A \times A \), that is reflexive, i.e., it contains the “discrete relation” \( \langle 1_A, 1_A \rangle : A \rightarrow A \times A \) on \( A \) (also denoted by \( \Delta_A \)), and transitive: there is a morphism \( \tau : \rho \times_A \rho \rightarrow \rho \) such that \( r_1 \tau = r_1 p_1 \) and \( r_2 \tau = r_2 p_2 \), where \( (\rho \times_A \rho, p_1, p_2) \) is the pullback

\[
\begin{array}{ccc}
\rho \times_A \rho & \rightarrow & \rho \\
p_1 \downarrow & & \downarrow r_1 \\
\rho & \xrightarrow{r_2} & A
\end{array}
\]
A morphism \((A, \rho) \to (B, \sigma)\) in the category \(\text{PreOrd}(\mathbb{C})\) of preorders in \(\mathbb{C}\) is a pair of morphisms \((f, \hat{f})\) in \(\mathbb{C}\) making the following diagram commute

\[
\begin{array}{ccc}
\rho & \xrightarrow{f} & \sigma \\
\downarrow r_1 & & \downarrow s_1 \\
A & \xrightarrow{f} & B
\end{array}
\]

so that \(fr_1 = s_1 \hat{f}\) and \(fr_2 = s_2 \hat{f}\).

A preorder \((A, \rho)\) is called an equivalence relation if there is a “symmetry,” namely a morphism \(s: \rho \to \rho\) such that \(r_1 s = r_2\) and \(r_2 s = r_1\). Equivalently, the opposite relation \(\rho^\circ\) of \(\rho\) is isomorphic to \(\rho\), hence they determine the same subobject of \(A \times A: \rho^\circ = \rho\).

A preorder \((A, \rho)\) is called a partial order if it “antisymmetric,” i.e., if it has the additional property that \(\rho \cap \rho^\circ\) is equal to the discrete equivalence relation \(\Delta_A\) on \(A\). We write \(\text{Eq}(\mathbb{C})\) and \(\text{Par}(\mathbb{C})\) for the full (replete) subcategories of \(\text{PreOrd}(\mathbb{C})\) whose objects are equivalence relations and partial orders in \(\mathbb{C}\), respectively. The pair \((\text{Eq}(\mathbb{C}), \text{Par}(\mathbb{C}))\) is a pretorsion theory in \(\text{PreOrd}(\mathbb{C})\) [7]. We write \(Z = \text{Eq}(\mathbb{C}) \cap \text{Par}(\mathbb{C})\) for the full (replete) subcategory of trivial objects in \(\text{PreOrd}(\mathbb{C})\) [2], whose objects are “discrete” preorders, i.e., those of the form \((A, \Delta_A)\).

A morphism \((f, \hat{f}): (A, \rho) \to (B, \sigma)\) is called a \(Z\)-trivial morphism if it factors through a trivial object. In the following, we shall often use the terms “trivial morphism” and “trivial object” (dropping the “\(Z\)” of “\(Z\)-trivial”).

Given a morphism \(f: A \to B\), where \((B, \sigma)\) is an object in \(\text{PreOrd}(\mathbb{C})\), we denote by \(f^{-1}(\sigma)\) the inverse image of \(\sigma\) along \(f\), that is the left vertical relation defined by the following pullback:

\[
\begin{array}{ccc}
f^{-1}(\sigma) & \to & \sigma \\
\downarrow & & \downarrow (s_1, s_2) \\
A \times A & \xrightarrow{f \times f} & B \times B
\end{array}
\]

Recall then that, in any category with an initial object 0, a subobject \(\alpha: A \to B\) of an object \(B\) is complemented if there is another subobject \(\alpha^c: A^c \to B\) with the property \(A \cap A^c = 0\) and \(A \cup A^c = B\).

It was observed in [2] (Corollary 5.4) that a subobject \((A, \rho) \hookrightarrow (B, \sigma)\) in \(\text{PreOrd}(\mathbb{C})\) is complemented in \(\text{PreOrd}(\mathbb{C})\) if and only if

1. \(A \hookrightarrow B\) is a complemented subobject in \(\mathbb{C}\), with complement \(A^c \twoheadrightarrow B\); 
2. \(\alpha^{-1}(\sigma) = \rho\), \((\alpha^c)^{-1}(\sigma) = \rho^c\) and all the commutative squares in the diagram

\[
\begin{array}{ccc}
\rho & \xrightarrow{\sigma} & \rho^c \\
\downarrow r_1 & & \downarrow r_1^c \\
A \xrightarrow{\alpha} B = A \bigsqcup A^c & \xleftarrow{\alpha^c} & A^c
\end{array}
\]

(i.e., the ones corresponding to the same index \(i \in \{1, 2\}\)) are pullbacks.

Note that, in a pretopos \(\mathbb{C}\), this implies that \(\sigma = \rho \bigsqcup \rho^c\).
Partial maps

Before recalling the definition of the stable category of $\text{PreOrd}(\mathbb{C})$, as an intermediate step, we first define the category $\text{PaPreOrd}(\mathbb{C})$ of partial morphisms in $\text{PreOrd}(\mathbb{C})$.

Its objects are the same as the ones of $\text{PreOrd}(\mathbb{C})$, the internal preorders $(A, \rho)$ in $\mathbb{C}$, while a morphism $(A, \rho) \rightarrow (B, \sigma)$ in the category $\text{PaPreOrd}(\mathbb{C})$ is a pair $(\alpha, f)$ depicted as

\[
\begin{array}{c}
(A', \rho') \\
\downarrow \alpha \\
(A, \rho) \quad \leftarrow \quad (\alpha, f) \\
\downarrow \quad \downarrow \\
(B, \sigma)
\end{array}
\]

where $(A', \rho')$ is an internal preorder, $f$ is a morphism in $\text{PreOrd}(\mathbb{C})$, and the monomorphism $\alpha: (A', \rho') \rightarrow (A, \rho)$ is a complemented subobject in $\text{PreOrd}(\mathbb{C})$. Given two composable morphisms $(\alpha, f): (A, \rho) \rightarrow (B, \sigma)$ and $(\beta, g): (B, \sigma) \rightarrow (C, \tau)$ in $\text{PaPreOrd}(\mathbb{C})$, the composite morphism $(\beta, g) \circ (\alpha, f)$ in $\text{PaPreOrd}(\mathbb{C})$ is defined by the external part of the following diagram

\[
\begin{array}{c}
(A'', \rho'') \\
\downarrow \alpha' \\
(A', \rho') \\
\downarrow \alpha \\
(A, \rho) \quad \leftarrow \quad (\alpha, f) \\
\downarrow \quad \downarrow \\
(B', \sigma') \\
\downarrow \beta \\
(B, \sigma) \\
\downarrow \downarrow \\
(C, \tau)
\end{array}
\]

where the upper part is a pullback. In other words,

$(\beta, g) \circ (\alpha, f) = (\alpha \alpha', gf')$.

The well-known properties of pullbacks guarantee that this composition is associative. For any preorder $(A, \rho)$, the identity on it in $\text{PaPreOrd}(\mathbb{C})$ is the arrow

\[
\begin{array}{c}
(A, \rho) \\
\downarrow 1 \\
(A, \rho)
\end{array}
\]

Remark 1.1 Notice that given a partial map $(\alpha, f): (A, \rho) \rightarrow (B, \sigma)$, the subobject $\alpha: (A', \rho') \rightarrow (A, \rho)$ can only be determined up to isomorphism, being the representative of a class of monomorphisms $(A', \rho') \rightarrow (A, \rho)$ in $\text{PreOrd}(\mathbb{C})$. Nevertheless, it is easy to prove that the composition is independent of the choice of representatives. See [10] and the references therein for more details about partial maps.

As explained in [2], there is a functor $I: \text{PreOrd}(\mathbb{C}) \rightarrow \text{PaPreOrd}(\mathbb{C})$ which is the identity on objects and such that, for any $f: (A, \rho) \rightarrow (B, \sigma)$ in $\text{PreOrd}(\mathbb{C})$, its value
\[ I(f) : (A, \rho) \rightarrow (B, \sigma) \] in \( \text{PaPreOrd}(\mathbb{C}) \) is given by the morphism

\[
\begin{array}{c}
(A, \rho) \\
\downarrow f \\
(A, \rho) \rightarrow_{I(f)} (B, \sigma)
\end{array}
\]

To simplify the notation, from now on, we shall write \( A \) instead of \((A, \rho)\) to denote an internal preorder and \( A \rightarrow B \) for a morphism of preorders. The fact that the initial object 0 of \( \text{PreOrd}(\mathbb{C}) \) is strict implies that 0 is a zero object in \( \text{PaPreOrd}(\mathbb{C}) \), and thus the category \( \text{PaPreOrd}(\mathbb{C}) \) is equipped with an ideal \( \mathcal{N} \) of (null) morphisms \([4]\), where \( \mathcal{N} \) is the class of morphisms in \( \text{PaPreOrd}(\mathbb{C}) \) of the form

\[
\begin{array}{c}
\downarrow \\
0 \\
B \rightarrow C.
\end{array}
\]

The stable category \([2]\) is defined as a suitable quotient of the category \( \text{PaPreOrd}(\mathbb{C}) \). In the special case when \( \mathbb{C} \) is the category of sets this construction reduces to the one of the stable category by Facchini and Finocchiaro in \([5]\). In order to define the stable category, the following notion is needed:

**Definition 1.2** A congruence diagram in \( \text{PreOrd}(\mathbb{C}) \) is a diagram of the form

\[
\begin{array}{c}
A_0^1 \\
\alpha_0^1 \downarrow \\
\alpha_0^2 \\
A_0 \\
\downarrow \\
A
\end{array}
\]

\[
\begin{array}{c}
A_1^1 \\
\alpha_1^1 \downarrow \\
\alpha_1^2 \\
A_1 \\
\downarrow \\
B
\end{array}
\]

\[
\begin{array}{c}
A_2^1 \\
\alpha_2^1 \downarrow \\
\alpha_2^2 \\
A_2 \\
\downarrow \\
A_2
\end{array}
\]

where:

- Any arrow of the form \( \xrightarrow{\gamma} \) represents a complemented subobject in \( \text{PreOrd}(\mathbb{C}) \);
- The two triangles commute;
- \( A_0^1 \xrightarrow{\alpha_0^1} A_1 \) is the complement in \( A_1 \) of the subobject \( A_0 \xrightarrow{\alpha_0^2} A_1 \);
- \( f_1 \alpha_0^1 = f_2 \alpha_0^2 \);
- Each \( f_1 \alpha_0^1 \) is a trivial morphism.

Two parallel morphisms \((\alpha_1, f_1)\) and \((\alpha_2, f_2)\) in \( \text{PaPreOrd}(\mathbb{C}) \), depicted as

\[
\begin{array}{c}
\alpha_1 \\
\downarrow \\
A \\
\rightarrow \ B
\end{array}
\]

and

\[
\begin{array}{c}
\alpha_2 \\
\downarrow \\
A \\
\rightarrow \ B
\end{array}
\]
are equivalent if there is a congruence diagram of the form (1.1) between them. In this case one writes \((\alpha_1, f_1) \sim (\alpha_2, f_2)\). As shown in [2], the relation \(\sim\) is an equivalence relation which is also compatible with the composition in \(\text{PaPreOrd}(\mathbb{C})\), and is then a congruence (in the sense of [9]) on the category \(\text{PaPreOrd}(\mathbb{C})\).

**Definition 1.3** [2] The quotient category \(\text{Stab}(\mathbb{C})\) of \(\text{PaPreOrd}(\mathbb{C})\) by the congruence \(\sim\) defined above is called the stable category. If \(\pi : \text{PaPreOrd}(\mathbb{C}) \to \text{Stab}(\mathbb{C})\) is the quotient functor, we also have a functor

\[
\Sigma = \pi \circ I : \text{PreOrd}(\mathbb{C}) \to \text{Stab}(\mathbb{C})
\]

obtained by precomposing \(\pi\) with the functor \(I : \text{PreOrd}(\mathbb{C}) \to \text{PaPreOrd}(\mathbb{C})\).

**Remark 1.4** The definition of the stable category \(\text{Stab}(\mathbb{C})\) of \(\text{PreOrd}(\mathbb{C})\) actually depends on the class \(\mathcal{Z}\) of trivial objects. Thus we should write “the stable category of \(\text{PreOrd}(\mathbb{C})\) with respect to \(\mathcal{Z}\)” and call it the “\(\mathcal{Z}\)-stable category of \(\text{PreOrd}(\mathbb{C})\)”. Nevertheless, we prefer to follow the notation adopted in [5] and refer to \(\text{Stab}(\mathbb{C})\) as the stable category associated with \(\text{PreOrd}(\mathbb{C})\). It is also worth noting that the construction we provide is based on the properties of the class \(\mathcal{Z}\) (such as the fact that it contains both the initial and the terminal objects or that it is closed under coproducts) that may not hold for any pretorsion theory in \(\text{PreOrd}(\mathbb{C})\). Thus, a priori, it is not possible to construct the stable category for any pretorsion theory in \(\text{PreOrd}(\mathbb{C})\).

The stable category is pointed, and the zero object 0 of \(\text{Stab}(\mathbb{C})\) is the image by the functor \(\Sigma\) of the initial object in \(\text{PreOrd}(\mathbb{C})\). As shown in [2], an object \(A\) in \(\text{PreOrd}(\mathbb{C})\) is such that \(\Sigma(A) = 0\) if and only if \(A\) is a “discrete” object, that is an object \(A\) equipped with the preorder given by the discrete equivalence relation \(\Delta_A\) on \(A\). Moreover, \(f : A \to B\) in \(\text{PreOrd}(\mathbb{C})\) is a trivial morphism if and only if \(\Sigma(f) = 0\) in \(\text{Stab}(\mathbb{C})\). More generally, one has the following result, where we write \(<\alpha,f>\) for the image of the morphism

\[
\begin{array}{ccc}
(A, \rho) & \xrightarrow{f} & (B, \sigma) \\
\downarrow \alpha & & \downarrow \quad \\
(A', \rho') & & \\
\end{array}
\]

by the functor \(\pi\):

**Lemma 1.5** For a morphism \(A \xrightarrow{<\alpha,f>} B\) in \(\text{Stab}(\mathbb{C})\) the following conditions are equivalent:

1. \(<\alpha,f>=0>\)
2. \(f\) is a trivial morphism in \(\text{PreOrd}(\mathbb{C})\).
**Proof** The assumption $<\alpha, f> = 0$ implies that there is a congruence diagram of the form

![Diagram](image)

hence $A_0 = 0 = A_0^{2c}$ (since 0 is a strict initial object). This implies that $A_0^{1c} = A'$, $\alpha_0^{1c} = 1_{A'}$ and $f$ is trivial on $A'$.

Conversely, when $f$ is a trivial morphism, it suffices to build the following congruence diagram

![Diagram](image)

showing that $<\alpha, f> = 0$.

Note also that the “intuition” here should be that a diagram

![Diagram](image)

“represents” a morphism $<\alpha, f>$ whose restriction on the (complemented) subobject $(A', \rho')$ of $(A, \rho)$ is $f$, and that is “trivial” on the complement of $(A', \rho')$ in $(A, \rho)$, as explained in the following:
Proposition 1.6 If \( A \xrightarrow{\langle \alpha, f \rangle} B \) is a morphism in \( \text{Stab}(\mathbb{C}) \), then the following diagram is commutative in \( \text{Stab}(\mathbb{C}) \), where \( A^\complement \xrightarrow{\alpha^\complement} A \) is the complement of \( A' \) in \( A \) in \( \text{PreOrd}(\mathbb{C}) \):

\[
\begin{array}{c}
\Sigma(A') \\
\downarrow \Sigma(f) \\
\Sigma(A) \\
\downarrow \Sigma(\alpha') \\
\Sigma(A^\complement) \\
\end{array}
\]

\[
\begin{array}{c}
\Sigma(B) \\
\downarrow 0 \\
\Sigma(A) \\
\downarrow \Sigma(\alpha) \\
\Sigma(A^\complement) \\
\end{array}
\]

Proof In order to see that \( \langle \alpha, f \rangle \Sigma(\alpha) = \Sigma(f) \) it suffices to consider the diagram

\[
\begin{array}{c}
A' \\
\downarrow \alpha \\
A' \\
\downarrow \alpha \\
A' \\
\end{array}
\]

\[
\begin{array}{c}
A' \\
\downarrow \langle \alpha, f \rangle \\
B \\
\end{array}
\]

where the upper quadrangle is a pullback. On the other hand, the assumption that \( A' \cap A^\complement = 0 \) implies that \( \langle \alpha, f \rangle \Sigma(\alpha^\complement) = 0 \)

\[
\begin{array}{c}
0 \\
\downarrow \alpha^\complement \\
A^\complement \\
\downarrow \alpha \\
A \\
\end{array}
\]

\[
\begin{array}{c}
\downarrow \langle \alpha, f \rangle \\
B \\
\end{array}
\]

since the composite \( 0 \to A' \to B \) is obviously trivial. One concludes by Lemma 1.5. \( \square \)

The following result (Lemma 7.11 in [2]) will also be useful:

Lemma 1.7 [2] Let us consider a morphism \( \langle \alpha, f \rangle \) in \( \text{Stab}(\mathbb{C}) \) represented by

\[
\begin{array}{c}
\alpha \\
\downarrow f \\
A' \\
\downarrow A \\
B \\
\end{array}
\]

and assume that for any complemented subobject \( B' \xrightarrow{f^{-1}} B \) the induced morphism \( f^{-1}(B') \to B' \) has a \( \mathbb{Z} \)-cokernel in \( \text{PreOrd}(\mathbb{C}) \). Then the cokernel of \( \langle \alpha, f \rangle \) exists in \( \text{Stab}(\mathbb{C}) \), and

\[
coker(\langle \alpha, f \rangle) = \Sigma(\mathbb{Z}\text{-coker}(f)).
\]
2 The universal property of \text{Stab}(\mathbb{C})

**Definition 2.1** Let \((\mathcal{A}, \mathcal{T}, \mathcal{F})\) be a category \(\mathcal{A}\) with a given pretorsion theory \((\mathcal{T}, \mathcal{F})\) in \(\mathcal{A}\). If \((\mathcal{B}, \mathcal{T}', \mathcal{F}')\) is a pointed category \(\mathcal{B}\) with a given torsion theory \((\mathcal{T}', \mathcal{F}')\) in it, we say that a functor \(G : \mathcal{A} \rightarrow \mathcal{B}\) is a *torsion theory functor* if the following two properties are satisfied:

1. \(G(A) \in \mathcal{T}'\) for any \(A \in \mathcal{T}\), \(G(B) \in \mathcal{F}'\) for any \(B \in \mathcal{F}\);
2. If \(T(A) \rightarrow A \rightarrow F(A)\) is the canonical short \(\mathcal{Z}\)-exact sequence associated with \(A\) in the pretorsion theory \((\mathcal{T}, \mathcal{F})\), then

\[
0 \rightarrow G(T(A)) \rightarrow G(A) \rightarrow G(F(A)) \rightarrow 0
\]

is a short exact sequence in \(\mathcal{B}\).

When \(\mathbb{C}\) is a pretopos, we write \((\text{Eq}(\mathbb{C}), \text{ParOrd}(\mathbb{C}))\) for the pretorsion theory in \(\text{PreOrd}(\mathbb{C})\) where \(\text{Eq}(\mathbb{C})\) is the category of equivalence relations and \(\text{ParOrd}(\mathbb{C})\) the category of partial orders in \(\mathbb{C}\).

**Proposition 2.2** The functor \(\Sigma : \text{PreOrd}(\mathbb{C}) \rightarrow \text{Stab}(\mathbb{C})\) is a torsion theory functor that preserves finite coproducts and monomorphisms.

**Proof** The fact that \((\text{Eq}(\mathbb{C}), \text{ParOrd}(\mathbb{C}))\) is a pretorsion theory was observed in [7] (for any exact category \(\mathbb{C}\)), while the preservation of finite coproducts and monomorphisms by the functor \(\Sigma\) was established in Proposition 6.2 and Proposition 6.1 in [2], respectively. It remains to prove that \((\text{Eq}(\mathbb{C}), \text{ParOrd}(\mathbb{C}))\) is a torsion theory in the pointed category \(\text{Stab}(\mathbb{C})\). Consider any morphism \(<\alpha, f \succ : (A, \rho) \rightarrow (B, \sigma)\), where \(\rho\) is an equivalence relation on \(A\) and \(\sigma\) a partial order on \(B\), depicted as

\[
\begin{array}{ccc}
(A', \rho') & \xrightarrow{\alpha} & (A, \rho) \\
\downarrow{f} & & \downarrow{f} \\
(B, \sigma) & \xleftarrow{f'} & (A', \rho')
\end{array}
\]

The fact that \(\rho\) is an equivalence relation and \(\alpha\) a complemented subobject in \(\text{PreOrd}(\mathbb{C})\) implies that also \(\rho' = \alpha^{-1}(\rho)\) is an equivalence relation (on \(A'\)). It follows that the morphism \(f : (A', \rho') \rightarrow (B, \sigma)\) is trivial in \(\text{PreOrd}(\mathbb{C})\) (since \((\text{Eq}(\mathbb{C}), \text{ParOrd}(\mathbb{C}))\) is a pretorsion theory in \(\text{PreOrd}(\mathbb{C})\)), hence a zero morphism in \(\text{Stab}(\mathbb{C})\).

Next, let us prove that the canonical short \(\mathcal{Z}\)-exact sequence in \(\text{PreOrd}(\mathbb{C})\)

\[
(A, \sim_{\rho}) \xrightarrow{i} (A, \rho) \xrightarrow{\pi} (A / \sim_{\rho}, \pi(\rho))
\]

associated with any internal preorder \((A, \rho)\), where \(\sim_{\rho} = \rho \cap \rho^{\circ}\) and \(i\) is the canonical inclusion, becomes a short exact sequence in \(\text{Stab}(\mathbb{C})\).

First, Proposition 7.1 in [2] implies that \(\Sigma(i) : (A, \sim_{\rho}) \rightarrow (A, \rho)\) is the kernel of \(\Sigma(\pi) : (A, \rho) \rightarrow (A / \sim_{\rho}, \pi(\rho))\). To see that \(\Sigma(\pi)\) is the cokernel of \(\Sigma(i)\) we shall use Lemma 1.7. To apply this result, observe that, for any complemented subobject \((A', \rho')\) of \((A, \rho)\), the upper horizontal morphism in the pullback

\[
\begin{array}{ccc}
(A', \rho'') & \xrightarrow{i} & (A, \rho \cap \rho^{\circ}) \\
\downarrow{i'} & & \downarrow{i} \\
(A', \rho') & \rightarrow & (A, \rho)
\end{array}
\]
is again a complemented subobject. This implies that $\rho''$ is the restriction to $A'$ of the equivalence relation $\rho \cap \rho^o$ on $A$, i.e., the following square is a pullback in $\mathbb{C}$:

$$
\begin{array}{ccc}
\rho'' & \rightarrow & \rho \cap \rho^o \\
\downarrow & & \downarrow \\
A' \times A' & \rightarrow & A \times A
\end{array}
$$

This implies that $\rho''$ is an equivalence relation, and then the $\mathbb{Z}$-cokernel of $i'$ exists (by Proposition 7.3 in [2]). The result then follows from Lemma 1.7.

**Theorem 2.3** Let $\mathbb{C}$ be a pretopos. The functor $\Sigma : \text{PreOrd}(\mathbb{C}) \rightarrow \text{Stab}(\mathbb{C})$ has the following property: it is universal among all finite coproduct preserving torsion theory functors $G : \text{PreOrd}(\mathbb{C}) \rightarrow \mathbb{X}$, where $\mathbb{X}$ has a torsion theory $(T, F)$ and finite coproducts. This means that any finite coproduct preserving torsion theory functor $G : \text{PreOrd}(\mathbb{C}) \rightarrow \mathbb{X}$ factors uniquely through $\Sigma$:

$$
\begin{array}{ccc}
\text{PreOrd}(\mathbb{C}) & \xrightarrow{\Sigma} & \text{Stab}(\mathbb{C}) \\
\forall G & \downarrow & \exists ! G \\
\mathbb{X} & \xrightarrow{k} & \mathbb{Y}
\end{array}
$$

Moreover, the induced functor $\overline{G}$ preserves finite coproducts, and is a torsion theory functor.

**Proof** Since $\text{PreOrd}(\mathbb{C})$ and $\text{Stab}(\mathbb{C})$ have the same objects it is clear that the definition of the functor $\overline{G}$ on the objects is “forced” by $G : \overline{G}(A) = G(A)$, for any object $A$ in $\text{Stab}(\mathbb{C})$. Let then $<\alpha, f> : A \rightarrow B$ be a morphism in $\text{Stab}(\mathbb{C})$ [as in (1.4)], and recall that it is then $[f, 0]$, the morphism induced by the universal property of the coproduct $A' \coprod A'^c = A$, since the diagram (1.3) in Proposition 1.6 commutes. Again, the condition $\overline{G} \circ \Sigma = G$ and the fact that $\overline{G}$ has to preserve binary coproducts force the definition of the functor $\overline{G}$ on morphisms:

$$
\overline{G}(<\alpha, f>) = [G(f), 0].
$$

The above arguments already prove the uniqueness of the functor $\overline{G}$ with the above properties. We still need to check that $\overline{G}$ is well-defined on morphisms, i.e., if we have $<\alpha, f> = <\alpha, \overline{f}>$ in $\text{Stab}(\mathbb{C})$, then $\overline{G}(<\alpha, f>) = \overline{G}(<\alpha, \overline{f}>)$ or, equivalently,

$$
[G(f), 0] = [G(\overline{f}), 0].
$$

Now, the assumption $<\alpha, f> = <\alpha, \overline{f}>$ gives a congruence diagram

\[\text{Diagram here}\]
In \( \text{PreOrd}(\mathbb{C}) \), if we write \( A'' \) and \( \overline{A''} \) for the complements of \( A' \) and \( \overline{A'} \) in \( A \), respectively, we have the decompositions
\[
A = A' \bigsqcup A'' = A_0 \bigsqcup A_1 \bigsqcup A''
\]
and
\[
A = \overline{A'} \bigsqcup \overline{A''} = \overline{A_0} \bigsqcup \overline{A_1} \bigsqcup \overline{A''}.
\]
Accordingly, by taking into account the distributivity law for subobjects (see Lemma 1.4.2 in [8] and recall that coproducts in \( \text{PreOrd}(\mathbb{C}) \) are computed “componentwise” [2, Proposition 5.3]) we get the following equalities:
\[
A = (A_0 \bigsqcup A_1 \bigsqcup A'') \cap (\overline{A_0} \bigsqcup \overline{A_1} \bigsqcup \overline{A''})
\]
\[
= (A_0 \cap A_0) \bigsqcup (A_0 \cap \overline{A_1}) \bigsqcup (A_0 \cap \overline{A''}) \bigsqcup (A_1 \cap A_0) \bigsqcup (A_1 \cap \overline{A_1}) \bigsqcup (A_1 \cap \overline{A''}) \bigsqcup (A'' \cap A_0) \bigsqcup (A'' \cap \overline{A_1}) \bigsqcup (A'' \cap \overline{A''}).
\]
By taking into account the equalities \( A_0 \cap A_0 = A_0 \) and
\[
A_0 \cap \overline{A_1} = A_0 \cap \overline{A''} = A_1 \cap A_0 = A'' \cap A_0 = 0,
\]
we see that
\[
A = A_0 \bigsqcup (A_1 \cap \overline{A_1}) \bigsqcup (A_1 \cap \overline{A''}) \bigsqcup (A'' \cap \overline{A_1}) \bigsqcup (A'' \cap \overline{A''}).
\]
We then observe that:
- \( f = \overline{f} \) on \( A_0 \);
- \( f \) is trivial on \( A_1 \) and \( \overline{f} \) is trivial on \( \overline{A_1} \), hence \( f \) and \( \overline{f} \) are trivial on \( A_1 \cap \overline{A_1} \);
- \( f \) is trivial on \( A_1 \) and \( \langle \alpha, \overline{f} \rangle \) is zero in \( \text{Stab}(\mathbb{C}) \) on \( \overline{A''} \), hence \( \langle \alpha, f \rangle \) and \( \langle \alpha, \overline{f} \rangle \) are zero morphisms on \( A_1 \cap \overline{A''} \) in \( \text{Stab}(\mathbb{C}) \);
- similarly, \( \langle \alpha, f \rangle \) and \( \langle \alpha, \overline{f} \rangle \) are zero morphisms on \( A'' \cap \overline{A_1} \);
- \( \langle \alpha, f \rangle \) is zero on \( A'' \) and \( \langle \alpha, \overline{f} \rangle \) is zero on \( \overline{A''} \), and this implies that \( \langle \alpha, f \rangle \) and \( \langle \alpha, \overline{f} \rangle \) are both zero on \( A'' \cap \overline{A''} \) in \( \text{Stab}(\mathbb{C}) \).

By assumption \( G \) is a torsion theory functor, hence it sends the trivial morphisms in \( \text{PreOrd}(\mathbb{C}) \) to zero morphisms in \( \mathbb{X} \). A zero morphism in \( \text{Stab}(\mathbb{C}) \) is a morphism of the form
\[
\begin{array}{c}
A' \quad g \\
\downarrow \beta \\
A
\end{array}
\quad \quad \begin{array}{c}
B \\
\downarrow \beta \\
A
\end{array}
\]
where \( g \) is trivial in \( \text{PreOrd}(\mathbb{C}) \). Accordingly, in this case, \( \overline{G}(\langle \beta, g \rangle) = [0, 0] \) is the zero morphism from \( A \) to \( B \) in \( \mathbb{X} \). By assumption \( G \) preserves finite coproducts, hence
\[
G(A) = G(A_0) \bigsqcup G(A_1 \cap \overline{A_1}) \bigsqcup G(A_1 \cap \overline{A''}) \bigsqcup G(A'' \cap \overline{A_1}) \bigsqcup G(A'' \cap \overline{A''}),
\]
and from the observations above we know that \( [f, 0] \) and \( [\overline{f}, 0] \) coincide on \( G(A_0) \) and are zero morphisms on all the other components. It follows that \( [f, 0] = [\overline{f}, 0] \), and the definition of \( \overline{G} \) is compatible with the congruence defining the morphisms in \( \text{Stab}(\mathbb{C}) \).

To prove that \( \overline{G} \) is a functor consider two composable morphisms in \( \text{Stab}(\mathbb{C}) \)
\[
\begin{array}{c}
A \quad \alpha \quad B \\
\downarrow \beta \quad \downarrow \\
\end{array}
\quad \quad \begin{array}{c}
C
\end{array}
\]

\( \overline{G} \) sends them to the corresponding morphisms in \( \mathbb{X} \).
and the following composition diagram where the upper square is a pullback

\[
\begin{array}{ccc}
A^c & \xrightarrow{\alpha^c} & A \\
\downarrow & & \downarrow \gamma \\
A' & \xrightarrow{f} & B' \\
\downarrow & & \downarrow \beta \\
A & \xrightarrow{(\alpha, f)} & B & \xrightarrow{(\beta, g)} & C
\end{array}
\]

and \(A'^c, B'^c\) are the complements of \(A'\) and \(B'\) in \(A\) and in \(B\), respectively. We also consider the pullback

\[
\begin{array}{ccc}
A^c & \xrightarrow{f} & B' \\
\downarrow & & \downarrow \beta \\
A' & \xrightarrow{f} & B
\end{array}
\]

expressing the fact that \(A^c = f^{-1}(B'^c)\), and we observe that

\[
A = A' \bigsqcup A'^c = A^c \bigsqcup A \bigsqcup A'^c.
\]

In \(\mathbb{K}\) we have to check that \([G(g), 0][G(f), 0]\) and \([G(g \bar{f}), 0]\) coincide on

\[
G(A) = G(A^c) \bigsqcup G(A) \bigsqcup G(A'^c).
\]

On \(G(A)\) we have \(G(g)G(f)\) in both cases, hence

\[
\bar{G}(< \beta, g >) \cdot \bar{G}(< \alpha, f >) = \bar{G}(< \beta, g > < \alpha, f >).
\]

Now, on \(A^c\) the morphism \(f\) factors through \(B'^c\) (see diagram (2.1)), hence

\[
\bar{G}(< \beta, g >) \cdot \bar{G}(< \alpha, f >)
\]

is the zero morphism on \(G(A^c)\). But \(\bar{G}(< \beta, g > < \alpha, f >)\) is also the zero morphism on \(G(A^c)\), hence these two morphisms are equal on \(G(A^c)\). Finally, on \(A'^c\) we have \(< \alpha, f > = 0\), hence again

\[
\bar{G}(< \beta, g >) \cdot \bar{G}(< \alpha, f >) = 0 = \bar{G}(< \beta, g > < \alpha, f >)
\]
on \(G(A'^c)\), completing this part of the proof.

One clearly has that \(\bar{G}(1_A) = G(1_A) = 1_{G(A)}\), since \(G\) is a functor. To see that \(\bar{G}\) preserves finite coproducts one has to observe that \(G\) preserves finite coproducts and these are calculated in \(\text{Stab}(\mathbb{C})\) as in \(\text{PreOrd}(\mathbb{C})\) (see Corollary 6 in [2]). In order to check that \(\bar{G}\) is a torsion theory functor, since \(\bar{G}\) and \(G\) coincide on objects, it will suffice to prove that \(\bar{G}\) preserves the canonical short exact sequences in the torsion theory. This follows from Proposition 2.2, since the canonical short exact sequence in the torsion theory in \(\text{Stab}(\mathbb{C})\) is the image by \(\Sigma\) of the canonical short \(\mathcal{Z}\)-exact sequence in the pretorsion theory in \(\text{PreOrd}(\mathbb{C})\) and, by assumption, \(G\) preserves this kind of sequences. \(\square\)
3 The case of $\tau$-pretoposes

The aim of this section is to prove that, when $C$ is a $\tau$-pretopos (in the sense of Definition 3.7), all the short exact sequences in $\text{Stab}(C)$ are images (up to isomorphism) by the functor $\Sigma : \text{PreOrd}(C) \to \text{Stab}(C)$ of a short $\mathbb{Z}$-exact sequence in $\text{PreOrd}(C)$.

**Proposition 3.1** The stable category $\text{Stab}(C)$ has disjoint binary coproducts.

**Proof** Consider any commutative diagram in $\text{Stab}(C)$

$$
\begin{array}{ccc}
C & \xrightarrow{<\alpha, f>} & A \\
\downarrow{<\beta, g>} & & \downarrow{\Sigma(s_A)} \\
B & \xrightarrow{\Sigma(s_B)} & A \bigsqcup B,
\end{array}
$$

where $s_A$ and $s_B$ denote the coprojections of the coproduct in $\text{PreOrd}(C)$. This means that there is a congruence diagram

$$
\begin{array}{ccc}
A_0 & \xrightarrow{\alpha_0} & A_1 \\
\downarrow{\alpha_1} & & \downarrow{\alpha} \\
A_0 & \xrightarrow{\beta} & A \\
\downarrow{\alpha_2} & & \downarrow{\beta} \\
A_2 & \xrightarrow{g} & B \\
\end{array}
$$

In the category $\text{PreOrd}(C)$ the equality $s_A f \alpha_0 = s_B g \alpha_1$ induces a unique morphism $A_0 \to A \times_A A \bigsqcup B$ to the pullback $A \times_A A \bigsqcup B$ of $s_A$ and $s_B$. Since $A \times_A A \bigsqcup B$ is the initial object $0$ in $\text{PreOrd}(C)$ and $0$ is strict, it follows that $A_0 = 0$. This implies that $A_{0}^{c} = A_1$ and $A_{0}^{2c} = A_2$, and the morphisms $s_A f$ and $s_B g$ are both trivial. Since $\Sigma(s_A)$ and $\Sigma(s_B)$ are monomorphisms in $\text{Stab}(C)$ (by Proposition 2.2), it follows that $<\alpha, f> = 0$ and $<\beta, g> = 0$. From the fact that $0$ is a zero object in $\text{Stab}(C)$ it follows that the square

$$
\begin{array}{ccc}
0 & \xrightarrow{} & A \\
\downarrow & & \downarrow{\Sigma(s_A)} \\
B & \xrightarrow{\Sigma(s_B)} & A \bigsqcup B,
\end{array}
$$

is a pullback in $\text{Stab}(C)$, as desired. \qed
Definition 3.2 Let $\mathbb{X}$ be a category with binary coproducts. One says that binary coproducts in $\mathbb{X}$ are pre-universal if, given any morphism $f : C \to A \sqcup B$, there exists a commutative diagram of the form

$$
\begin{array}{c}
A' \xrightarrow{s_{A'}} C \xleftarrow{s_{B'}} B' \\
\downarrow f \quad \downarrow f \\
A \xrightarrow{s_A} A \sqcup B \xleftarrow{s_B} B
\end{array}
$$

where the top row of the diagram is a sum (i.e., $C = A' \sqcup B'$ and $s_A'$ and $s_B'$ are the coprojections).

Notice that the (non-extensive) category of pointed sets has pre-universal binary coproducts. We now want to prove that also the stable category $\text{Stab}(\mathbb{C})$ has this property. For ease of notation, since in $\text{Stab}(\mathbb{C})$ binary coproducts (exist and) are computed as in $\text{PreOrd}(\mathbb{C})$ [2, Corollary 6.3], in the sequel we shall often write $A \xrightarrow{s_A} A \sqcup B \xleftarrow{s_B} B$ for the coprojections of the coproduct of $A$ and $B$ both in $\text{Stab}(\mathbb{C})$ and in $\text{PreOrd}(\mathbb{C})$.

Proposition 3.3 The stable category $\text{Stab}(\mathbb{C})$ has pre-universal binary coproducts.

Proof Let us consider any morphism $<\alpha, f> : C \to A \sqcup B$, and then the diagram

$$
\begin{array}{c}
C' \xrightarrow{\Sigma(f)} C \xleftarrow{\Sigma(y)} C'' \\
\downarrow \Sigma(\alpha) \quad \downarrow 0 \\
C \xrightarrow{<\alpha, f>} A \sqcup B
\end{array}
$$

in $\text{Stab}(\mathbb{C})$ and

$$
\begin{array}{c}
A'' \xrightarrow{s_{A''}} C' \xleftarrow{s_{B''}} B'' \\
\downarrow f \quad \downarrow f \\
A \xrightarrow{s_A} A \sqcup B \xleftarrow{s_B} B
\end{array}
$$

in $\text{PreOrd}(\mathbb{C})$, respectively, where in the second one $A''$ and $B''$ are the inverse images along $f$, and then $C' = A'' \sqcup B''$. Note that, by Proposition 1.6, in the stable category we have the
equality $<\alpha, f> \Sigma(\gamma) = 0$. There is then the following factorization $\Sigma(f_A)$ in $\text{Stab}(C)$:

\[ A'' \xrightarrow{\sigma_A''} A'' \bigsqcup C'^c \xrightarrow{\Sigma(f_A)} A \]

where $\sigma_A''$ and $\sigma_{C'^c}$ are the coproduct coprojections. One then sets $A' = A'' \bigsqcup C'^c$ and gets the diagram

\[ A' \xrightarrow{s_A'} C \xleftarrow{\Sigma(\alpha)s_{B''}} B'' \]

\[ A \xrightarrow{s_A} A \bigsqcup B \xleftarrow{s_B} B, \]

whose commutativity can be checked as follows. We have

\[ C = C' \bigsqcup C'^c = A'' \bigsqcup B'' \bigsqcup C'^c = A' \bigsqcup B''. \]

By Proposition 1.6, we know that

\[ <\alpha, f> s_A'' \sigma_{A''} = <\alpha, f> \Sigma(\alpha)s_{A''} = \Sigma(f)s_{A''} \]

and

\[ <\alpha, f> s_A'' \sigma_{C'^c} = <\alpha, f> \Sigma(\gamma) = 0. \]

Since we also have that

\[ s_A \Sigma(f_A) \sigma_{A''} = s_A \Sigma(f''_A) = \Sigma(f)s_{A''}. \]

and

\[ s_A \Sigma(f_A) \sigma_{C'^c} = s_A 0 = 0, \]

we conclude that $<\alpha, f> s_A' = s_A \Sigma(f_A)$, as desired. On the other hand, the following equalities show that the right-hand side of diagram (3.2) commutes:

\[ <\alpha, f> \Sigma(\alpha)s_{B''} = \Sigma(f)s_{B''} = s_B \Sigma(f''_B). \]

\[ \square \]

**Remark 3.4** Let us observe that the choice of the objects $A'$ and $B'$ in Definition 3.2 is by no means unique. Indeed, in the proof of Proposition 3.3, we could as well have chosen $A' = A''$ and $B' = B'' \bigsqcup C'^c$ (with reference to diagram (3.2)). So the pre-universality of binary coproducts could be rephrased as the existence of three objects $A'', B'', C''$, respectively, mapped by $f$ in $A, B, 0$, and such that $C = A'' \bigsqcup B'' \bigsqcup C''$. 

\[ \Box \]
Lemma 3.5 Let \( \mathcal{X} \) be a category with a zero object and binary coproducts which are disjoint and pre-universal. Assume that

\[
\begin{array}{ccc}
K & \xrightarrow{k} & A \\
& \xrightarrow{f} & B
\end{array}
\]

and

\[
\begin{array}{ccc}
N & \xrightarrow{n} & C \\
& \xrightarrow{g} & D
\end{array}
\]

are composable morphisms such that \( k = \ker(f) \) and \( n = \ker(g) \). Then the morphism

\[
\begin{array}{ccc}
k \coprod n : K \coprod N & \rightarrow & A \coprod C
\end{array}
\]

is the kernel of \( f \coprod g : A \coprod C \rightarrow B \coprod D \).

**Proof** First of all the composite \((f \coprod g)(k \coprod n)\) is clearly the zero morphism:

\[
(f \coprod g)(k \coprod n) = fk \coprod gn = 0 \coprod 0 = 0.
\]

Next consider any arrow \( h : E \rightarrow A \coprod C \) such that \((f \coprod g)h = 0\). By the pre-universality of binary coproducts we can form the commutative diagram

\[
\begin{array}{ccc}
E_1 & \xrightarrow{s_1} & E & \leftarrow & E_2 \\
\downarrow{h_1} & & \downarrow{h} & & \downarrow{h_2} \\
A & \xrightarrow{s_A} & A \coprod C & \leftarrow & C
\end{array}
\]

where \( E = E_1 \coprod E_2 \). We have the equality

\[
s_B fh_1 = (f \coprod g)s_A h_1 = (f \coprod g)hs_1 = 0,
\]

where \( s_B : B \rightarrow B \coprod D \) is a monomorphism, hence there is a unique morphism \( m_1 \) such that \( km_1 = h_1 \). Similarly, one can prove that there is a unique \( m_2 \) such that \( nm_2 = h_2 \). The universal property of the coproduct \( E = E_1 \coprod E_2 \) gives a unique morphism \( m = m_1 \coprod m_2 : E \rightarrow K \coprod N \) such that

\[
(k \coprod n)ms_1 = (k \coprod n)s_K m_1 = s_A km_1 = s_A h_1 = hs_1.
\]

Symmetrically, one has that \((k \coprod n)ms_2 = hs_2\), yielding the equality \((k \coprod n)m = h\). To check the uniqueness of the factorization consider then another morphism \( r : E \rightarrow K \coprod N \) such that \((k \coprod n)r = h\). Again by pre-universality we have a commutative diagram
where $E = \hat{E}_1 \coprod \hat{E}_2$. Consider the following commutative diagram, where $E_1 = \hat{E}_{11} \coprod \hat{E}_{12}$ and $E_2 = \hat{E}_{21} \coprod \hat{E}_{22},$

![Diagram](image)

that exists by the pre-universality of binary coproducts. By assumption the restrictions to $\hat{E}_{11}$ of $m$ and $r$ are equal when composed with $k \coprod n$

$$(k \coprod n)rs_1 \hat{s}_{11} = (k \coprod n)ms_1 \hat{s}_{11}$$

and these composites both factor through $s Ak$

$$(k \coprod n)ms_1 \hat{s}_{11} = hs_1 \hat{s}_{11} = s_A h_1 \hat{s}_{11} = s_A km_1 \hat{s}_{11}$$

and $s_A k$ is a monomorphism (as a composite of monomorphisms). This means that $m$ and $r$ induce a unique morphism $m_2 \hat{s}_{22} = r_2 \hat{s}_{22}: \hat{E}_{22} \to N$. On $\hat{E}_{12}$ we get the following equalities

$$s_A h_1 \hat{s}_{12} = h s_1 \hat{s}_{12} = (k \coprod n)ms_1 \hat{s}_{12} = (k \coprod n)rs_1 \hat{s}_{12} = (k \coprod n)r s_2 \hat{s}_{12} = (k \coprod n)N s_2 r s_{12} = s_C n r s_{12}$$

showing that the induced morphisms $\hat{E}_{12} \to A$ and $\hat{E}_{12} \to C$ are the zero morphisms, since the coproduct $A \coprod C$ is disjoint. But $k$ and $n$ are both monomorphisms, hence both the morphisms $\hat{E}_{12} \to K$ and $\hat{E}_{12} \to N$ are zero as well. Similarly, the morphisms $\hat{E}_{21} \to K$ and $\hat{E}_{21} \to N$ are also zero. By composing with $s_K: K \to K \coprod N$ and $s_N: N \to K \coprod N$ we obtain that the restrictions $\hat{E}_{11} \to K \coprod N$ of $m$ and $r$ are equal. Similarly, the restrictions $\hat{E}_{12} \to K \coprod N$ of $m$ and $r$ are both zero, hence the restriction of $m$ and $r$ to $E_1$ are equal. In a similar way one checks that the restrictions of $m$ and $r$ to $E_2$ are equal, and, finally, $m = r$.

\[\square\]

**Proposition 3.6** The category $\text{Stab}(\mathbb{C})$ has kernels. If $<\alpha, f>: A \to B$ is a morphism in $\text{Stab}(\mathbb{C})$ as in diagram (1.2), its kernel is given by

$$\Sigma(k \coprod 1_{A'}) : K \coprod A' \rightarrow A' \coprod A'^c$$

where $k$ is the $\mathbb{Z}$-kernel of $f$ in $\text{PreOrd}(\mathbb{C})$ and $A'^c$ is the complement of $A'$ in $A$.

**Proof** In $\text{PreOrd}(\mathbb{C})$ we have the $\mathbb{Z}$-kernels

$$K \xrightarrow{k} A' \xrightarrow{f} B$$

and

$$A'^c \xrightarrow{\tau} A'^c \xrightarrow{\tau} 1.$$
By Proposition 7.1 in [2] we know that $\Sigma : \text{PreOrd}(C) \to \text{Stab}(C)$ sends these $\mathcal{Z}$-kernels to the kernels

$$K \xrightarrow{k} A' \xrightarrow{f} B$$

and

$$A^{\text{rc}} \longrightarrow A'^{\text{rc}} \longrightarrow 0$$

in $\text{Stab}(C)$, respectively. By Lemma 3.5 we know that $k \bigsqcup 1_{A^{\text{rc}}}$ is then the kernel of $f \bigsqcup 0 : A' \bigsqcup A^{\text{rc}} \to B \bigsqcup 0 = B$. From Proposition 1.6 it follows that in the following diagram in $\text{Stab}(C)$

$$K \bigsqcup A^{\text{rc}} \xrightarrow{k \bigsqcup 1_{A^{\text{rc}}}} A \xrightarrow{<\alpha, f>} B$$

the morphism $k \bigsqcup 1_{A^{\text{rc}}}$ is the kernel of $<\alpha, f>$, as desired. \(\Box\)

We now recall the following definition that had a role in proving some of the results in [2]:

**Definition 3.7** A $\tau$-pretopos is a pretopos $C$ with the property that the transitive closure of any relation on an object in $C$ exists in $C$.

Any $\sigma$-pretopos (i.e., a pretopos admitting denumerable unions of subobjects, that are preserved by pullbacks) is in particular a $\tau$-pretopos [8], as well as any elementary topos [2, Proposition 7.7].

**Proposition 3.8** When $C$ is a $\tau$-pretopos, two composable morphisms in $\text{Stab}(C)$

$$A \xrightarrow{<\alpha, f>} B \xrightarrow{<\beta, g>} C$$

form a short exact sequence if and only if, up to isomorphism, they are the image by $\Sigma$ of a short exact sequence in $\text{PreOrd}(C)$.

**Proof** One implication follows from Theorem 7.14 in [2] (for which the assumption that $C$ is a $\tau$-pretopos is needed). Conversely, consider a short exact sequence

$$A \xrightarrow{<\alpha, f>} B \xrightarrow{<\beta, g>} C$$

in $\text{Stab}(C)$. By Proposition 3.6 we know that the kernel of $<\beta, g>$ is a morphism of type $\Sigma(n)$:

$$<\alpha, f> = \ker <\beta, g> = \Sigma(n).$$

More precisely, going back to the construction of the kernel $\text{Ker}(<\beta, g>)$ as in Proposition 3.6 we have the diagram

```
K \bigsqcup B^{\text{rc}} \xrightarrow{k \bigsqcup 1_{B^{\text{rc}}}} B \xrightarrow{\beta^{\text{rc}}} B^{\text{rc}} \xrightarrow{\beta} B' \xrightarrow{\beta^{\text{rc}}} B^{\text{rc}} \xrightarrow{g} C
```

and we set \( n = k \coprod B^\vee \). When \( \mathbb{C} \) is a \( \tau \)-pretopos the functor \( \Sigma : \text{PrOrd}(\mathbb{C}) \to \text{Stab}(\mathbb{C}) \) sends \( \mathbb{Z} \)-cokernels to cokernels (Corollary 7.13 in [2]): this implies that

\[
< \beta , g > = \text{coker}(< \alpha , f >) = \text{coker}(\Sigma(n)) = \Sigma(\mathbb{Z} - \text{coker}(n)).
\]

The construction of the cokernel \( \text{coker}(< \alpha , k >) \) as in Lemma 7.11 of [2] shows that the \( \mathbb{Z} \)-cokernel \( q : B \to Q \) of \( n \) is such that

\[
\Sigma(q) = \text{coker}(< 1_k \coprod B^\vee , k \coprod 1_{B^\vee} >) = \text{coker}(\Sigma(n)).
\]

Since the sequence (3.3) is exact, \( < \beta , g > \) is isomorphic to \( \Sigma(q) \), and the sequence (3.3) is isomorphic to the exact sequence

\[
\Sigma(K \coprod B^\vee) \to \Sigma(B) \to \Sigma(Q),
\]

as desired.

**Corollary 3.9** When \( \mathbb{C} \) is a \( \tau \)-pretopos, under the assumptions of Theorem 2.3, the functor \( \overline{G} \) preserves the short exact sequences whenever the functor \( G \) sends short \( \mathbb{Z} \)-exact sequences to short exact sequences.

**Proof** This follows immediately from Proposition 3.8.

**Corollary 3.10** When \( \mathbb{C} \) is a \( \tau \)-pretopos, under the assumptions of Theorem 2.3, the functor \( \overline{G} \) preserves cokernels whenever the functor \( G \) sends \( \mathbb{Z} \)-cokernels to cokernels.

**Proof** Let \( < \alpha , f > : A \to B \) be a morphism in \( \text{Stab}(\mathbb{C}) \) and \( q : B \to C \) its cokernel. As we have noticed in the Proof of Proposition 3.8, this cokernel is the image by \( \Sigma \) of the \( \mathbb{Z} \)-cokernel of \( f \), i.e., \( q = \Sigma(\mathbb{Z} - \text{coker}(f)) \). In the category \( \mathbb{X} \) we have that \( \overline{G}(q) = G(q) = \text{coker}(G(f)) = \text{coker}(\overline{G}(f)) \). Consider then the following diagram

\[
\begin{array}{ccc}
G(A) = G(A') \coprod G(A^\vee) & \xrightarrow{[G(f), 0]} & [G(A), 0] = \mathbb{Z} - \text{coker}(f) \\
\downarrow G(\alpha) & & \downarrow \text{coker} \\
G(A') & \xrightarrow{G(f)} & G(B) \xrightarrow{G(q)} G(Q) \\
\downarrow G(\alpha^\vee) & & \downarrow \text{coker} \\
G(A^\vee) & \xrightarrow{0} & G(B) \\
\end{array}
\]

One sees that \( G(q)G(f) = 0 \), since \( G(q) = \text{coker}(G(f)) \) and, obviously, \( G(0) = 0 \), hence \( G(q)[G(f), 0] = 0 \). Then, if \( x[G(f), 0] = 0 \) we get

\[
xG(f) = x[G(f), 0]G(\alpha) = 0G(\alpha) = 0,
\]

yielding a unique factorization \( y \) of \( x \) through \( G(q) = \text{coker}(G(f)) \). It follows that \( G(q) = \text{coker}([G(f), 0]) \), and this implies that

\[
\overline{G}(q) = G(q) = \text{coker}([G(f), 0]) = \text{coker}(\overline{G}(< \alpha , f >)).
\]

\( \square \)
Proposition 3.11 If in Theorem 2.3 we also assume that

- $C$ is a $\tau$-pretopos,
- $X$ has finite coproducts that are disjoint and pre-universal,
- $G : \text{PreOrd}(C) \to X$ sends $Z$-kernels to kernels,

then the functor $\overline{G} : \text{Stab}(C) \to X$ preserves kernels.

Proof Consider a morphism $<\alpha, f> : A \to B$ in $\text{Stab}(C)$ and its kernel $k \bigsqcup 1_{A^{c}}$ in $\text{Stab}(C)$ as in Proposition 3.6

where $k$ is the $Z$-kernel of $f$. From the assumptions, it follows that

\[\overline{G}(k) = G(k) = \ker(G(f)) = \ker(\overline{G}(f)),\]

and one also has

\[1_{\overline{G}(A^{c})} = \ker(0), \quad \text{where } 0 : \overline{G}(A^{c}) \to 0.\]

From Lemma 3.5 and Theorem 2.3, we get the equalities

\[\overline{G}(k \bigsqcup 1_{A^{c}}) = \overline{G}(k) \bigsqcup 1_{\overline{G}(A^{c})} = \ker(\overline{G}(f) \bigsqcup 0) = \ker(\overline{G}(<\alpha, f>)),\]

hence $\overline{G}$ preserves the kernel of $<\alpha, f>$. \qed

Thus, in case of a $\tau$-pretopos, we have the following result.

Theorem 3.12 Let $C$ be a $\tau$-pretopos. The functor $\Sigma : \text{PreOrd}(C) \to \text{Stab}(C)$ is universal among all finite coproduct preserving torsion theory functors $G : \text{PreOrd}(C) \to X$, where $X$ has a torsion theory $(T,F)$, and it has binary coproducts that are disjoint and pre-universal. Consider any finite coproduct preserving torsion theory functor $G : \text{PreOrd}(C) \to X$ that sends $Z$-kernels and $Z$-cokernels (then in particular short $Z$-exact sequences) to kernels, cokernels (and short exact sequences). The functor $G$ then factors uniquely through $\Sigma$

\[
\begin{array}{ccc}
\text{PreOrd}(C) & \xrightarrow{\Sigma} & \text{Stab}(C) \\
\forall G & \downarrow & \exists \overline{G} \\
\downarrow X, & & \end{array}
\]

and the induced functor $\overline{G}$ preserves finite coproducts, is a torsion theory functor that preserves kernels and cokernels (then in particular short exact sequences).

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