SYNCHROTRON-LOSS SPECTRAL BREAKS IN PULSAR-WIND NEBULAE AND EXTRAGALACTIC JETS

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ABSTRACT

Flows of synchrotron-emitting material can be found in several astrophysical contexts, including extragalactic jets and pulsar-wind nebulae (PWNe). For X-ray synchrotron emission, flow times are often longer than electron radiative lifetimes, so the effective source size at a given X-ray energy is the distance electrons radiating at that energy can convect before they burn off. Since synchrotron losses vary strongly with electron energy, the source size drops with increasing X-ray energy, resulting in a steepening of the synchrotron spectrum. For homogeneous sources, this burnoff produces the well known result of a steepening by 0.5 in the source’s integrated spectral index. However, for inhomogeneous sources, different amounts of steepening are possible. I exhibit a simple phenomenological picture of an outflow of relativistic electrons with bulk nonrelativistic flow speed, with transverse flow-tube radius, magnetic field strength, matter density, and flow velocity all varying as different powers of distance from the injection point. For such a picture, I calculate the value of the spectral index above the break as a function of the power-law indices, and show the possible range of steepenings. I show that these simple calculations are confirmed by full integrations of source luminosity, which also include the spectral “bump” below the break from the accumulation of electrons formerly at higher energies. In many cases, extragalactic jets show X-ray synchrotron emission steeper by more than 0.5 than the radio emission; the same phenomenon is exhibited by many PWNe. It is possible that source inhomogeneities are responsible in at least some cases, so that the amount of spectral steepening becomes a diagnostic for source dynamical or geometrical properties.

Key words: galaxies: jets – radiation mechanisms: non-thermal – supernova remnants – supernovae: individual (B0540-693) – X-rays: ISM

Online-only material: color figures

1. INTRODUCTION

1.1. Spectral Breaks in Jets and Pulsar-wind Nebulae

Broadband spectra from synchrotron radiation characterize a wide variety of astrophysical sources, including active galactic nuclei (AGNs), shell supernova remnants (SNRs), and pulsar-wind nebulae (PWNe). When observed over a sufficiently broad frequency range, such spectra almost invariably show steepening to higher energies. Such steepening can be attributed either to intrinsic spectral structure or to the effects of radiative losses. However, the power-law shape of radio spectra ($S_\nu \propto \nu^{-s}$), with $s \sim 0.5-0.8$ for many sources, suggests an origin of the requisite relativistic electrons in diffusive shock acceleration (DSA), which produces a power-law (or near-power-law, for efficient nonlinear DSA) spectrum of particles $N(E) \propto E^{-s}$ with $s = 2\alpha + 1$ depending on the shock compression ratio. Shocks are clearly present in these astrophysical objects: outer blast waves in SNRs (and perhaps reverse shocks into ejecta as well, for younger objects); relativistic-wind termination shocks in PWNe; and shocks in jets and hot spots in active galaxies. The absence of an obvious mechanism to generate a broken power-law distribution, and the necessity of the operation of radiative losses at some level, has led to the common acceptance of synchrotron losses as the mechanism to bring about spectral steepening.

The standard calculation of the effect of synchrotron losses (reviewed in the next section) for a homogeneous source predicts a steepening of the initial electron spectrum to a second power law one power steeper ($s_2 = s + 1$) than the injection spectrum, implying a radiation spectrum one-half power steeper ($\alpha_2 = \alpha + 0.5$). This is, in fact, rarely observed. In shell supernovae, the maximum energies to which electrons can be accelerated are limited by losses or by finite shock age (or size), but produce an exponential cutoff in $N(E)$, further broadened by inhomogeneities, as observed in a few cases in which synchrotron X-ray emission can be identified (see Reynolds 2008 for a review), so a sharp spectral break to a steeper power law is neither expected nor observed. In PWNe, radio spectra are almost all flatter than $\alpha = 0.5$, a phenomenon not well explained at present, but the steepenings $\Delta \equiv \alpha_{\text{high}} - \alpha_{\text{low}}$ are almost always greater than 0.5 (ranging from 0.7 to 1, in seven of the eight cases tabulated in Chevalier 2005, using updated values for B0540-693 from Williams et al. 2008). Knots in extragalactic jets, when observable in X-rays, show similar too large steepenings (e.g., M87: Perlman & Wilson 2005; Cygnus A: Stawarz et al. 2007). One standard interpretation of knot and hot-spot spectra invokes shock acceleration and subsequent spectral steepening in a uniform post-shock region (Heavens & Meisenheimer 1987), but it cannot explain these larger values of $\Delta$. Coleman & Bicknell (1988) present numerical hydrodynamic simulations and find larger values of $\Delta$, which they apply to observations, but without analytic results allowing the wider application of the results. While there is recognition of the possibility of values of $\Delta$ different from 0.5 (e.g., Kennel & Coroniti 1984a, who find $\Delta = (4 + \alpha)/9$ for the Crab Nebula, or Petre et al. 2007, in a qualitative discussion of the broadband spectrum of the PWN B0540-693), there has, as yet, been no simple characterization of conditions under which values of $\Delta \neq 0.5$ can naturally arise. That characterization is the goal of this paper.

1.2. Synchrotron Losses

The first widely known calculation of the behavior of a distribution of electrons subject to synchrotron losses is that of...
Kardashev (1962). While most of these results are well known, it is important to recall the particular conditions under which each is applicable, so I shall beg the reader’s indulgence for a brief review, which can also serve to fix notation. Kardashev solved the kinetic equation for a distribution \( N(E, t) \) of electrons subject to gains by first- and second-order Fermi acceleration, and losses due to radiation or adiabatic expansion, with and without the assumption of new-particle injection and stationarity. He writes the energy-loss rate from a single electron as

\[
\dot{E} = -bB^2 \frac{E^2}{\gamma^3},
\]

where \( B \equiv B \sin \theta, b \equiv (2/3)e^4/(m^2c^3) = 2.37 \times 10^{-3} \text{ cgs} \) (e.g., Pacholczyk 1970), and \( \theta \) is the electron pitch angle between its velocity vector and the magnetic field. Kardashev pointed out that in a pointed magnetic field in the absence of scattering, electrons preserve their value of \( \theta \), since they radiate a beam pattern that is symmetric with respect to their velocity vector. An electron injected into \( B \) at \( t = 0 \) with energy \( E_0 \) has an energy \( E \) after time \( t \) given by the well known result

\[
E(t) = \frac{E_0}{1 + E_0 b B^2 t^2}.
\]

An initially infinitely energetic electron is reduced after time \( t \) to energy \( E_{\text{max}}(t, \theta) = 1/b b B^2 t \). A power-law energy distribution of electrons \( N(E, t) = K E^{-\nu} \), with a single value of pitch angle, will be cut off at \( E_{\text{max}}(t, \theta) \). The electron distribution \( N(E) \) will evolve according to \( N(E) dE = N(E_0) dE_0 \), so that

\[
N(E) = K [E_0(E)]^{-\nu} \frac{dE_0}{dE},
\]

with \( E(E_0) \) given by Equation (2). If \( s < 2 \), the electrons initially above \( E_{\text{max}}(t, \theta) \) are sufficiently numerous to pile up in a “bump” just below \( E_{\text{max}}(t, \theta) \), while if \( s > 2 \), the “bump” disappears. (Numerical solutions to Equation (3) are shown below for inhomogeneous models, illustrating the “bump” effect.)

An initially isotropic distribution of electrons suffers unequal radiation losses, with electrons with large pitch angles being more rapidly depleted. For an initial isotropic power-law distribution, after a time \( t \) one finds no electrons with pitch angles greater than \( \theta_{\text{max}} \) given by \( \sin^2 \theta_{\text{max}} = 1/(b B^2 t) \). Since for synchrotron radiation, an individual electron’s radiation pattern has an angular width of order 1/\( \gamma \) where \( \gamma \) is the individual Lorentz factor, and \( \gamma \gtrsim 10^{10} \) for radio emission and higher frequencies, we can approximate electrons as radiating exactly in their directions of motion. Then \( \theta \) is also the angle between the line of sight and the local magnetic field. An initially isotropic distribution of electrons injected at \( t = 0 \) into a source with a uniform magnetic field making an angle \( \chi \) with the line of sight, observed through its synchrotron emission, would then disappear abruptly at time \( t(\chi) \). More realistically, one might expect that the source has a tangled magnetic field, so that all values of \( \chi \) are achieved in some part of the source. One should then (for an unresolved source) perform an angular integration over the electron distribution. The result is an electron distribution that steepens by one power of \( E \), i.e., \( N(E) \propto E^{-\nu-1} \), above the characteristic energy \( E_b = 1/(b B^2 t) \). (The synchrotron emission from this distribution steepens above \( \nu(E_b) \) by more than the value of 0.5 in spectral index \( \alpha \) because, in this time-dependent case without continuous injection, a correlation exists between \( E \) and \( \theta \) such that more efficiently radiating electrons are depleted most rapidly.)

This situation is still relatively unrealistic, as it ignores any processes by which electrons could change their pitch angles. If electrons scatter in pitch angle on timescales much shorter than the synchrotron-loss timescale, one should simply average the energy-loss rate over angles:

\[
\dot{E} = 1.57 \times 10^{-3} b^2 E^2 \equiv a b^2 E^2,
\]

where \( a \equiv b(\sin^2 \theta) = (2/3)b \). Then the electron distribution will remain isotropic, and will simply cut off at \( E_{\text{max}}(t) \); a source synchrotron spectrum would then cutoff exponentially above \( \nu(E_{\text{max}}(t)) \).

However, the result we all remember from graduate school is neither of these. A source that turns on at \( t = 0 \) with continuous, uniformly spatially distributed injection of a power-law distribution of electrons \( q(E) \equiv J_0 E^{-\nu} \) electrons \( \text{erg}^{-1} \text{ s}^{-1} \text{ cm}^{-3} \), develops a break at energy \( E_b = 1/(a B^2 t) \), where the electron spectrum steepens by one power in \( s \). This corresponds to a steepening of the synchrotron spectrum by one-half power in \( \alpha \) at \( v_b = c_1 B^{-3} \tau^{-2} \) with \( c_1 = 1.12 \times 10^{24} \text{ cgs} \). This relation is frequently used to make a magnetic field strength in a synchrotron source of known age.

The result that synchrotron losses (or inverse-Compton losses, which have the same dependence on electron energy) result in the steepening of the particle spectrum by one power and the steepening of the emitted synchrotron spectrum by a half-power, is a widely held idea. It is this application that will be generalized below. It is important to remember that the standard derivation assumes a distributed injection of electrons in a homogeneous source. For other conditions, it is not correct, as we shall see.

2. BASIC CALCULATION

Energy-loss breaks from a synchrotron source in which relativistic electrons are advected systematically away from an injection region (such as a wind-termination shock) can be described as being due to shrinking of the effective source size with frequency. At a high enough observing frequency, electrons drop below the energy required to emit at that frequency before they reach the edge of the object, hence limiting its effective size at that frequency. Thus, all results depend on the critical energy an electron may have after suffering synchrotron losses and convecting at speed \( v \). Equation (4) gives the synchrotron-loss rate from a single electron. In a nonrelativistic, constant-density flow (i.e., neglecting adiabatic losses), but allowing the possibility of spatially varying \( B \), we generalize the homogeneous results slightly to obtain

\[
E_c = \left[ \int a B^2 dr/v \right]^{-1}
\]

for the energy an initially infinitely energetic electron would have after moving at \( v \) through a field \( B \) for a distance \( r \). So any injected electron distribution must cut off at this energy. For a one-dimensional flow of electrons streaming at constant speed \( v \) in a constant magnetic field \( B \), the effective length \( r(E) \) of the source is found from

\[
E(r) = \left( \frac{v}{a B^2} \right) r^{-1}.
\]

An electron of this energy radiates chiefly at frequency \( v = c_m E^2 B = c_m (v^2/a^2 B^2) r^{-2} \) where \( c_m = 1.82 \times 10^{18} \text{ cgs} \) (e.g., Pacholczyk 1970). Then at any frequency \( v \), the source has an
effective length
\[ r(v) = e^{1/2} \left( \frac{v}{a B^{3/2}} \right) v^{-1/2}. \]  

For synchrotron emission with a spectral index \( \alpha \), the observed flux density would then vary as
\[ S_\nu \propto v^{-\alpha} r(v) \propto v^{-\alpha - 1/2} \]

—the famous steepening by one-half power in the spectral index. (Note that at no position \( r \) in the source there is a break in the electron energy distribution \( N(r, E) \) to a new power law, although such a distribution does result after integrating over the entire source volume to obtain \( N(E) \).)

This kind of argument can be generalized to inhomogeneous sources, for which the steepening may be larger or smaller than 0.5. For instance, if the synchrotron emissivity increases with distance from the center, then as the effective source size shrinks, more emission will be lost than if the source were homogeneous, and the steepening can be greater than 0.5.

Here, we consider a simple model in which electrons are injected at some initial radius \( r_0 \) in a “jet” whose full width \( w \) rises with dimensionless distance \( l \equiv r/r_0 \) as \( w = w_0 l^\epsilon \). A conical jet (or piece of spherical outflow) then has \( \epsilon = 1 \); a confined jet has \( \epsilon < 1 \). Then, the cross-sectional area increases as \( A_\perp \propto l^{2\epsilon} \) (see Figure 1). We shall assume ad hoc power-law dependences of quantities on dimensionless length \( l \), ignoring any transverse variations, making the problem one-dimensional. Let the \( l \)-dependence of basic quantities be given by
\[ B = B_0 l^{m_B}, \quad v = v_0 l^{m_v}, \quad \rho = \rho_0 l^{m_\rho}. \]

While this is completely general, physical constraints may couple the \( m \)'s. For instance, in the absence of some mechanism such as mass loading (Lyutikov 2003) or entrainment to alter the effective density \( \rho \), conservation of mass gives
\[ \text{Mass conservation:} \quad \rho A_\perp = \text{const} \Rightarrow m_\rho + m_v = -2\epsilon. \]  

In the absence of turbulent amplification or reconnection, magnetic flux will be conserved, giving different relations for components of \( \mathbf{B} \) parallel and perpendicular to the jet axis:

Longitudinal (radial) field: \( B_\parallel : B A_\perp = \text{const} \Rightarrow m_b = -2\epsilon \).

Transverse (toroidal) field: \( B_\perp : B w v = \text{const} \Rightarrow m_b = -m_v - \epsilon = m_\rho + \epsilon \),

where the last form for transverse \( B \) involves invoking mass conservation. Of course, unless the magnetic field is exactly radial, any toroidal component will eventually dominate, barring extremely peculiar and probably unphysical behaviors (e.g., \( m_\rho < -3 \)). At any rate, here our emphasis is on the greatest generality; any particular set of dynamical assumptions will then confine a model to a subregion of the parameter space of the \( m \)'s.

In this formalism, Equation (5) implies \( E_c \propto l^{-(1+2m_\rho-m_\nu)} \) in the absence of other energy-loss mechanisms such as adiabatic expansion losses (nonconstant density). In the presence of adiabatic losses, it can be shown (e.g., Kennel & Coroniti 1984a; Reynolds 1998) that the critical energy is given by
\[
E_c = \rho^{1/3} \left[ \int_{l_0}^{l} \frac{a B^2}{v} \rho^{1/3} r_0 dl \right]^{-1} = \rho_0^{m_v/3} \left[ \int_{l_0}^{l} av_0^{-1} r_0 B_0^2 l^2 m_\rho - m_\nu + m_\rho/3 \right]^{1-(1+2m_\rho-m_\nu)} = A m^{m_\rho}, \]

where we have assumed the source is long enough such that \( l(E) \gg 1 \). Note that the effects of adiabatic losses have canceled out (except for a small change in the integration constant), leaving \( E_c \) with the same \( l \)-dependence as in the constant-density case. We have also made the assumption that the integral in Equation (13) is dominated by the upper limit at \( l \), demanding that \( m_\rho = 2m_\nu - m_\nu + m_\rho/3 < -1 \). Otherwise, \( E_c \propto l^{m_\rho/3} \) and radiative losses play no role in the spectral behavior, so that the calculation is not self-consistent. Even if this condition is met, we still wish to exclude situations in which the gradients conspire to arrange adiabatic gains of electrons as they convect, i.e., \( m_E > 0 \). This situation appears both unphysical and unlikely. Note that \( m_\rho = m_\nu/3 - m_E - 1 \). These two conditions rule out some volume in the parameter space of \( (\epsilon, m) \) and must be checked for any particular choices of those parameters. The constraints are related; \( m_\rho > -1 \) requires \( m_E < m_\rho/3 \), so that we ultimately require \( m_\rho < 0 \).

Electrons with energy \( E_c \) (at position \( l \) where the magnetic field strength is \( B(l) \)) radiate chiefly at a frequency
\[ v(E_c) = c \frac{v_0^2}{a^2 v_0^2 + B_0^2 l^2} \left( 1 + 2m_\nu - m_\nu + m_\rho/3 \right)^2 \times \int_{m_\rho-2(1+2m_\rho-m_\nu)}^{m_\rho} A_\nu v^{1/m_\nu}, \]

defining \( A_\nu \) and \( m_\nu \). Note \( m_\nu = m_\rho + 2m_E = -2 - 3m_b + 2m_v \). Then,
\[ l(v) = A_\nu^{-1/m_\nu} v^{1/m_\nu} \equiv A_\nu v^{1/m_\nu}. \]

We are focusing on conditions such that the source size shrinks with increasing frequency, i.e., \( m_\nu < 0 \). It can be easily shown that the condition on \( m \) above is equivalent to \( m_\nu < (m_\rho + m_\nu) \). If mass conservation is assumed, then \( m_\rho + m_\nu = -2\epsilon \) and \( m_\nu < 0 \) always. Otherwise, this condition must be checked; but for reasonable values of the parameters (such as those in all examples described below), it is always fulfilled.

Now to discuss the synchrotron flux, we assume a power-law electron distribution \( N(E) = K E^{-(s-\gamma)} \) between energies \( E_i \) and \( E_h \gg E_i \) (we take \( s > 1 \)). As the flow expands, conservation of electron number gives
\[ n_e = \frac{K}{s-1} \rho E_i^{1-s} \propto \rho \Rightarrow \rho = K \approx K \rho E_i^{1-s-1} \propto \rho^{(1+2s-1)/3} \]
\[ = \rho^{(s+3)/3} \equiv \rho^{(2\epsilon+3)/3}, \]

where the observed flux density grows as a power \( l^{1/3} \).
We have assumed that relativistic, but too low for radiative losses to be important. We can write the synchrotron emissivity (following Pacholczyk 1970) as

\[ j_v = c_j K B^{1+\alpha} v^{-\alpha} = c_j K_0 B_0^{1+\alpha} \left( 2a + 3m_\mu / 3 + (1 + \alpha) m_b \right) v^{-\alpha} \equiv A_j l^{m_j - 1} v^{-\alpha} \]  

(18)

Here \( c_j(s) \equiv c(s)/(2c_1) \) in the notation of Pacholczyk; for \( s = 1.5 \), \( c_j = 1.34 \times 10^{-10} \) ergs, and we have defined another important index, \( m_j \equiv (2a + 3)m_\mu / 3 + (1 + \alpha) m_b \). Assume for the time being that we view the jet directly perpendicular to the axis. Then the line-of-sight path length through the jet at any position \( l \) is just \( w(l) = w_0 l^\nu \) (through the center; averaged over lines of sight intersecting a circular cross section, we obtain a mean line-of-sight path of \( (\pi/4) w \)). We recall that we are assuming all jet quantities to be constant in cross section, that is, along these lines of sight. So if at a distance \( D \), the integrated flux density \( S_v \) is given by

\[ S_v = \int I_v d\Omega = \frac{1}{D^2} \int \int_{l_0}^{l_v} w r_0 dl (w l^{m_j}) v^{-\alpha} \]  

(19)

\[ = \frac{\pi A_j}{4D^2} \int_{l_0}^{l_v} w r_0 dl (w l^{m_j}) v^{-\alpha} \]  

(20)

\[ = \frac{\pi A_j r_0 w_0^2}{4D^2 (1 + 2\epsilon + m_j)} l_v^{1 + 2\epsilon + m_j - 1 - \alpha} \]  

(21)

\[ = \frac{\pi c_j K_0 B_0^{1+\alpha} r_0 w_0^2}{4D^2 [1 + 2\epsilon + (2a + 3)m_\mu / 3 + (1 + \alpha) m_b]} \times A_j l_v^{1 + 2\epsilon + (2a + 3)m_\mu / 3 + (1 + \alpha) m_b} \times l^{m_j - 1} v^{-\alpha - \Delta}, \]

where the last equation defines \( \Delta \), the amount of spectral steepening:

\[ \Delta = 1 + 2\epsilon + m_j - \frac{1 + 2\epsilon + m_j}{m_b + 2m_E} m_b = \frac{1 + 2\epsilon + (2a + 3)m_\mu / 3 + (1 + \alpha) m_b}{2 + 3m_b - 2m_\mu}. \]  

(22)

We have assumed that \( m_\mu < 0 \), and that the flux integral depends on the outer, not the inner, limit of integration, that is,

\[ 1 + 2\epsilon + m_j > 0. \]  

(23)

This latter condition can be restrictive.

While we have assumed a jet seen from the side, since the emission is optically thin the observed flux density should be independent of viewing angle. This can be shown explicitly for the case of jets seen exactly end-on, for which the flux integral Equation (19) becomes

\[ S_v = \frac{1}{D^2} \int_{l_0}^{l_v} w_0^2 \frac{\pi}{2} w d\omega \int_{l_0}^{l_v} c_j K_0 B_0^{1+\alpha} r_0 v^{-\alpha} l^{m_j} dl. \]  

(24)

Here, the line of sight is parallel to the jet axis, beginning at a value of \( l \equiv l_i \) dependent on \( w \) (if \( w < w_0 \), \( l_i = 1 \)). The upper limit \( w_\text{max} \) is just the value of \( w \) at which \( l_i = l_v \), i.e., \( w_\text{max} = w(l_v) \equiv w_0 [l_v]^{\nu} \). Now we need a slightly more restrictive condition for the emission to be dominated by the outer limit of integration \( l(v) \): \( 1 + m_j > 0 \). If this is the case, the two integrals in Equation (24) decouple, giving

\[ S_v = \frac{\pi}{2D^2} c_j K_0 B_0^{1+\alpha} r_0 w_0^2 \frac{1}{2} \int_{l_0}^{l_v} [l(v)]^{1 + m_j}. \]  

(25)

But \( w_\text{max} = w_0 [l(v)]^\nu \), so

\[ S_v = \frac{\pi}{2D^2} c_j K_0 B_0^{1+\alpha} r_0 w_0^2 \frac{1}{2} \int_{l_0}^{l_v} A_j l_v^{1 + 2\epsilon + m_j} [l(v)]^{1 + m_j - \alpha}, \]

the same as Equation (21) except for a factor \((1 + m_j)/(1 + 2\epsilon + m_j)\).

These power-law spectra can hold only over a frequency range related to the size range of the source by \( l(v) \). For instance, for conical, constant-velocity, mass-conserving flow with tangential magnetic field, we have \( \epsilon = 1 \), \( m_\mu = -2 \), and \( m_b = -1 \), giving \( l(v) \propto v^{-1} \). Thus, a source showing the corresponding value of \( \Delta \) (in this case, \( \Delta = 7\alpha/3 \)) between frequencies \( v_1 \) and \( v_2 \) must shrink over a range of sizes given by \( r_1 / r_2 = v_2 / v_1 \). In general, sources whose spectra are set by effective variations of size with frequency are predicted to have sizes varying as

\[ l(v) \propto v^{1/m_\mu} \equiv v^{1/(1 - 2 - 3m_\mu + 2m_b)}. \]  

(27)

Equation (22) relates the size exponent \( 1/m_\mu \) to \( \Delta / 1 + m_j \). (Thus, a source with observed \( \Delta \) will have a smaller rate of shrinkage with frequency for a larger value of \( 1 + m_j \).) The source subtends a solid angle on the sky of

\[ \Delta \Omega = \frac{1}{D^2} \int_{l_1}^{l(v)} (w_0 l^\nu) (r_0 dl) = \frac{1}{D^2} \int_{l_1}^{l(v)} [l(v)]^{1 + \epsilon} dl. \]  

(28)

If the source is only marginally resolved, one may consider an “average” angular size \( \langle \theta \rangle \) defined by

\[ \langle \theta \rangle \equiv \langle \Delta \Omega \rangle^{1/2} = \frac{\sqrt{r_0 w_0}}{D} \frac{1}{\sqrt{1 + \epsilon}} [l(v)]^{(1 + \epsilon)/2}, \]  

(29)

so that \( \langle \theta \rangle \propto v^{1/(1 + \epsilon) + 2m_b} \). For spherical or conical flows, i.e., \( \epsilon = 1 \), \( \langle \theta \rangle \propto v^{1/m_\mu} \) as before, but for confined flows \( \epsilon < 1 \), the average angular size decreases much more slowly with frequency. If a confined jet is seen end-on, the size variation with frequency is reduced even further, as the apparent diameter is now proportional to \( w_\text{max} \propto [l(v)]^{\nu} \) instead of \( l(v) \):

\[ \theta \propto v^{1/m_\mu} \equiv v^{1/(m_\mu + 2m_b)} \equiv v^{1/(1 - 2 - 3m_\mu + 2m_b)}. \]  

(30)

This may result in a significantly reduced size effect, since the primary change in emitting volume is shrinkage along the line of sight.

### 3. SPECIAL CASES

We can examine a few special cases. First, in the case of a plane constant-velocity flow, we have \( \epsilon = 0 \), \( m_\mu = 0 \), \( m_b = 0 \), and we recover \( \Delta = 1/2 \). Next, consider tangential magnetic field and conical outflow \((\epsilon = 1)\) with constant density and assuming mass and flux conservation. This corresponds
to the inner parts of a Kennel & Coroniti (1984a, 1984b, hereafter KC) MHD spherical flow. Then \(m_ρ = 0, m_v = -2, m_b = m_ρ + \epsilon = 1\), and we have

\[
\Delta = \frac{4 + \alpha}{9},
\]

which was derived in Kennel & Coroniti (1984b, Equation (4.11b)). This already indicates that values of \(\Delta \neq 0.5\) can be obtained from reasonable situations. It also suggests that obtaining values very different from 0.5 may be difficult. The size effect predicted above is quite weak: \(m_ν \neq 0\) still gives the correct scaling for spheres. Let the dimensionless radius be \(R \equiv r/r_0\), with injection at \(R = 1\) and burnoff at

\[
R(v) = A_R v^{1/m_ν} \equiv A_R (1/(1 - 2m_ρ + 2m_b) )
\]

—that is, the same expression as for \(l(v)\). (Sphericity should not change the expression, only, perhaps, the values of the \(m_ν\)’s.) Similarly, we should have the same dependence of \(j_ν(R)\) in the spherical case as we had for \(j_ν(l)\) in the jet case:

\[
j_ν(R) = c_j k_0 R^{1+α}_0 v^{-α} R^{m_ν}
\]

with

\[
m_ν = \frac{2α + 3}{3} m_ρ + (1 + α) m_b.
\]

Then the total flux from this spherically symmetric, optically thin source is just

\[
S_ν = \frac{4π}{D^2} \int_1^{R(v)} 4π j_ν(R) R^2 dR
= \frac{4π}{D^2} (4π c_j k_0 R^{1+α}_0 v^{-α}) \frac{[R(v)]^{m_ν+3}}{m_ν + 3},
\]

which gives

\[
\Delta \equiv -\frac{m_ν + 3}{m_ν} = \frac{(2α + 3) m_ρ/3 + (1 + α) m_b + 3}{2 + 3m_b - 2m_ν}.
\]

This is the same result as can be obtained by setting \(\epsilon = 1\) in the previous expression for \(\Delta\), Equation (22).

4. BREAKS GREATER THAN 0.5

Obtaining values of \(\Delta\) considerably greater than 0.5, as seems to be required by observations of most PWNe, requires relaxing some of the assumptions. The relation from mass conservation of \(m_ρ = -2\epsilon - m_v, m_b\) might not hold if some form of mass loading occurs, for instance, by evaporation of material from thermal filaments, or entrainment of material from a confining medium. Flux freezing for the magnetic field will not hold in the presence of either turbulent amplification of magnetic field or of reconnection. As an example, consider (for algebraic simplicity) the case \(α = 0\), but conical, constant-density flux (so \(m_v = -2\)), still conserving mass. Now in this case, \(m_j = m_b\) and \(m_ρ = -(3 + 2m_b)\), so

\[
\Delta = -\frac{3 + m_b}{m_ρ - 2(3 + 2m_b)} = \frac{3 + m_b}{6 + 3m_b} \Rightarrow m_ρ = \frac{-6Δ - 3}{-3Δ - 1}.
\]

Now we can obtain \(Δ = 2/3\) with the value \(m_b = -1\) in this case: the magnetic field drops as the first power of distance (faster than if frozen-in and tangential, slower than if longitudinal). This does not seem like an unreasonable possibility. Note that the conditions are met: \(1 + 2ε + m_j = 3 + m_b = 2 > 0, m_ρ = -1 < 0\), and \(m_ν \equiv 2m_ρ - m_v + m_ρ/3 = 0 \geq -1\). The source size would obey

\[
l(v) \propto v^{1/(3m_b-2m_ν)} = v^{1/(1 - 2(1))} = v^{-1/3}.
\]

To obtain weaker size effects, one is driven to larger values of \(1 + 2ε + m_j\), for the same observed \(Δ\).

For practical use, it is convenient to consider \(m_b\) a dependent variable, in terms of the other quantities. First, solve the equation for \(Δ\) for \(m_b\), in terms of \(Δ, m_ρ, \epsilon, \) and \(α\), with no presumed relationship between \(m_b\) and \(m_ν\). The result is

\[
m_b = \frac{3(2m_v - 2) + (2α + 3) m_ν/3 + 1 + 2ε}{3α - 1 - α}.
\]

Two of the conditions are satisfied if \(m_ν < \min(0, m_ρ/3)\), or

\[
1 + 2m_b - m_ν > \max(0, -m_ρ/3).
\]

Assuming mass conservation, this becomes

\[
1 + 2ε + 2m_b + m_ρ > \max(0, -m_ρ/3).
\]

In addition, we still require \(1 + 2ε + m_j > 0\).

An application to a particular source (i.e., an object of known \(α\) and \(Δ\), presumably) can then be made by inserting those values. Various possibilities for \(α\) and \(Δ\) can be tried: for each value of \(m_b\) obtained in this way, the condition on \(m_ρ\) must be checked by hand. For example, consider B0540-693, with a radio spectrum of \(α = 0.25\) (Manchester et al. 1993), and a break with \(Δ \sim 1\) at around 20 \(μm\) (Williams et al. 2008). A constant-density spherical (or conical) outflow cannot produce this \(Δ\). Inserting the values of \(α\) and \(Δ\) into Equation (41), and assuming for simplicity \(\epsilon = 1\), we find

\[
m_b = \frac{1 + 2m_ν + 1.2m_ρ}{1.7}.
\]
Then at the price of abandoning mass conservation, we can choose $m_E = 1$ and $m_r = -2$, giving $m_b = -1.06$, or about $-1$. The consistency conditions are all met: $m_c = 1/3 > -1$, $m_f = -(1 + 2m_b - m_e) = -1 < 0$, and $1 + 2\epsilon + m_j = 2.9 > 0$. The source effective radius decreases as

$$r \propto \nu^{1/(m_0 + 2m_E)} \propto \nu^{-0.33},$$

which may be a serious problem, since we need the slope of $-\alpha - \Delta \approx -1.2$ to hold from about 20 $\mu$m to somewhere in the blue or near UV—say 0.2 $\mu$m, requiring that the source shrink between these two wavelengths by a factor of 4.6—perhaps unlikely. (Though what is shrinking is really the region containing the dominant flux; there could be a faint halo contributing a small amount of flux in which a brighter, shrinking core is embedded). A numerical calculation of this model is shown in Figure 2, along with observations. The physics which could cause these values of $m_r$ and $m_b$, is, of course, completely unknown.

Figures 3 and 4 plot $m_b$ versus $\Delta$ for Equation (39) and its generalizations to the pairs $(\alpha, \epsilon) = (0.3, 1), (0, 0.5)$, and $(0.3, 0.5)$, respectively. Mass conservation is still invoked. The consistency condition on $m_c$ places upper limits on $\Delta$ (lower limits on $m_b$) shown as the squares on curves on each plot. (The condition $1 + 2\epsilon + m_j > 0$ is met for all curves shown.) It is difficult to obtain values of $\Delta > 0.7$; flows with rapidly dropping density (such as $m_r = -2\epsilon$, for constant-velocity, mass-conserving flows) seem unable to do so. Substantial deceleration seems to be required, as well as rapid decreases in the magnetic field strength. While there is some parameter space available for accomplishing this, especially for sources with very flat radio spectra, the most physically reasonable way to bring about the required deceleration seems to be mass loading, which also considerably expands the available parameter space of source gradients.

5. INFERRING PHYSICAL PARAMETERS

Energy-loss spectral breaks are commonly used to infer source magnetic field strengths in PWNe. One requires a source age $t$; for PWNe in SNR shells, one can use modeling of the shell emission to estimate an age, while in some cases, a source size divided by a mean flow speed (estimated one way or another) can give an age estimate. Then one simply assumes a homogeneous source for which $E_c = (aB^2t)^{-1} = 637/B^2t$ and from Equation (5),

$$B_b = \left(\frac{c_m}{a^2}\right)^{1/3} v_b^{-1/3} t^{-2/3} = 0.90 \nu_{\text{GHz}}^{-1/3} t_{\text{yr}}^{-2/3} \text{ G} \quad (46)$$

(where we have averaged over pitch angles). It is of interest to compare this to the magnetic field that would be inferred.
for a flow model assuming (incorrectly) that the source is homogeneous. Let the source have a size $L$ and break frequency $v_b$ (so that $L = r_0 L/v_b$). For a flow model, we can deduce the initial magnetic field $B_0$ from Equation (15):

$$\left(\frac{L}{r_0}\right)^{m_v} = \frac{a^2}{\epsilon_m} \left(\frac{r_0}{v_b}\right)^{1/3} \frac{1}{f^2} B_0^3 v_b,$$

(47)

where $f \equiv 1 + 2 m_b - m_\nu + m_\rho/3$. This implies

$$B_0 = \left(\frac{L}{r_0}\right)^{m_v/3} \left(\frac{C_m}{\alpha^2}\right)^{1/3} v_b^{-1/3} \frac{r_0^{2/3}}{v_b^{1/3}} f^{-2/3}.$$  

(48)

Then

$$B_h/B_0 = \left(\frac{r_0}{v_0 L}\right)^{2/3} f^{-2/3} \left(\frac{L}{r_0}\right)^{-m_v/3}.$$  

(49)

Of course, with substantial magnetic field gradients, $B_h/B_0$ can range widely either below or above 1. Some source properties, such as the initial ratio of energy input in magnetic field to that in particles (KC’s $\sigma$ parameter), require knowledge of $B_0$. For those properties, estimation of source magnetic field from the homogeneous assumption can lead to significant error. However, it is possible to show that $B_h$ does give a good approximation to the mean magnetic field averaged over the lifetime of a particle moving with the flow, as of course it must. The total magnetic field energy in a flow model is

$$U_B = \frac{1}{8\pi} \int_1^{L/r_0} r_0 dl \pi \left(\frac{w(t)}{2}\right)^2 B_0^{2m_v}.$$  

(50)

$$= \frac{1}{32} \left[ \frac{w_0^2 B_0^2 r_0}{1 + 2 \epsilon + 2 m_b} \right] \left(\frac{L}{r_0}\right)^{1+2\epsilon+2m_b}.$$  

(51)

where we have assumed $1 + 2 \epsilon + 2 m_b > 0$ and $L \gg r_0$. The total volume in the flow is

$$V = \int_1^{L/r_0} \pi \left(\frac{w(t)}{2}\right)^2 r_0 dl = \frac{\pi r_0 w_0^2}{4(1 + 2 \epsilon)} \left(\frac{L}{r_0}\right)^{1+2\epsilon}.$$  

(52)

Then the mean magnetic energy density $u_B$ is

$$\langle u_B \rangle \equiv \frac{U_B}{V} = \left(\frac{B_0^2}{8\pi}\right) \left(\frac{1 + 2 \epsilon}{1 + 2 \epsilon + 2 m_b}\right) \left(\frac{L}{r_0}\right)^{2m_v}.$$  

(53)

and the homogeneous energy density $u_B(hom) \equiv B_0^2/8\pi$ satisfies

$$\langle u_B \rangle = \frac{f^{-4/3}}{v_0} \left(\frac{r_0}{v_0 L}\right)^{4/3} \left(\frac{1 + 2 \epsilon + 2 m_b}{1 + 2 \epsilon}\right) \left(\frac{L}{r_0}\right)^{4(1-m_v)/3},$$  

(54)

where the exponent of $L/r_0$ has been rewritten using $m_v = 2 m_b - 2 - 3 m_\rho$.

If the value of $t$ used in the homogeneous relation Equation (46) is the actual transit time $t_{trans}$ of an electron from $r_0$ to $L$, one obtains a similar result. That $t_{trans}$ is given by

$$t_{trans} \equiv \int_0^{r_0} dl = \frac{r_0}{v_0} \int_1^{L/r_0} l^{-m_v} dl = \frac{r_0}{v_0} \frac{1}{1 - m_v} \left(\frac{L}{r_0}\right)^{1-m_v}.$$  

(55)

Then

$$\left(\frac{r_0}{v_0 t_{trans}}\right)^{4/3} \left(\frac{L}{r_0}\right)^{4(1-m_v)/3} = (1 - m_v)^{4/3}.$$  

(56)

independent of physical parameters. (We are assuming $m_\nu \leq 0$, i.e., we exclude accelerating flows.) This means that all factors in Equation (54) are of order unity, so that there is no large discrepancy between the true mean magnetic field energy density and that inferred under the assumption that the source is homogeneous. However, if the source lifetime is used for $t$ (which may differ substantially from $t_{trans}$), or an estimate of $t_{trans}$ is made from the measured expansion velocity of the outer boundary of the PWN, serious errors may be made in inferring $B$.

6. NUMERICAL CALCULATIONS AND APPLICATIONS TO OBSERVED SOURCES

These results can easily be confirmed by numerical integration of the appropriate equations. In particular, Equation (3) can be used to find the detailed particle distribution, accounting for the pileup of particles at energies just below $E_{max}(t)$ as they migrate down in energy (for $s < 2$). This pileup can produce a detectable “bump” in the spectrum just below $v_b$. The numerical calculations can also show how sharp a break can be achieved in practice. (Similar results were presented in Reynolds 2003.)

I illustrate these effects with several models. The example parameters for B0540-693 mentioned above ($\alpha = 0.25$, $\epsilon = 1$, $m_\rho = 1$, $m_b = -1$, and $m_\nu = -2$) produce distribution functions $N(E, l)$ at various points in the flow shown in Figure 2,
at positions $l \equiv r/r_0 = 10, 30, 50, 70,$ and $90$. (The energies are in units of a fiducial energy $E_f \equiv a B_0^2 r_0/v_0$, the energy an initially infinitely energetic electron would have after radiating for a time $r_0/v_0$ in a magnetic field $B_0$.) The sharp cutoff energy $E_c$, decreasing down the flow, is apparent, as is the spike just below it of electrons formerly above $E_c$. The rising density produces adiabatic gains in the density of electrons of too low energy to be subject to radiative losses. Spatial integration over these electron distributions produces the model spectral-energy distribution also shown in Figure 2, reproduced from Williams et al. (2008), which fits the data surprisingly well, apart from the anomalous X-ray flux. (A technical problem, “pileup” in the Chandra detectors due to the bright X-ray pulsar in B0540-693, makes the absolute determination of the X-ray flux of the nebula difficult; see Petre et al. 2007.) For the relatively nonrelativistic (not an egregious approximation), we can obtain some relatively coarse quantity such as the 50% enclosed power radius or FWHM, as cited for the Crab by Kennel & Coroniti (1984b).

For an AGN jet example, Stawarz et al. (2007) quote hot spots A and D in Cygnus A as having break frequencies around 3 GHz, with low-frequency spectral indices $\alpha_i = 0.28$ and 0.21, respectively, steepening with $\Delta \equiv 0.9$ in both cases. If we assume these hot spots result from post-shock emission in a shocked relativistic flow, and approximating the flow as nonrelativistic (not an egregious approximation), we can obtain $\Delta = 0.9$ with $s = 1.6$ for hot spot A, and $m_\rho = 1$ (entraining material; again, without such assumptions, large values of $\Delta$ are problematic), $m_\rho = -1$ (decelerating flow), and $\epsilon = 0.5$ (confined flow, as suggested by mass entrainment). Inserting these values in Equation (41) gives $m_v = -0.29$ or about $-0.3$. These values satisfy all consistency conditions, and give a size effect of $\theta \propto \nu^{-0.32}$. The available data have a range of angular resolutions, but at least qualitatively, hot-spot sizes do decrease between 5 and 230 GHz (Wright & Birkinshaw 2004).

7. SUMMARY OF RESULTS

Here, I collect the principal results and consistency requirements. The basic result is the expression for $\alpha_2 - \alpha_1 \equiv \Delta$, the amount of spectral steepening, Equation (22):

$$\Delta = \frac{1 + 2 \epsilon + m_j}{m_b + 2m_E} = \frac{1 + 2 \epsilon + (2\alpha + 3)m_\rho/3 + (1 + \alpha)m_b}{2 + 3m_b - 2m_v}.$$  

This expression holds if several consistency requirements are met: $m_E < \min(0, m_\rho/3)$, where $m_E \equiv -1 - 2m_\rho + m_\nu$, so that the burnoff energy at position $l$ depends on $l$, and $E_h$ drops with $l$; and $1 + 2 \epsilon + (2\alpha + 3)m_\rho/3 + (1 + \alpha)m_b > 0$, so that the integrated
flux density $S_\nu$ depends on the outer limit of integration. Finally, the effective source size should shrink with frequency: $m_\nu < 0$, a condition always met in the presence of mass conservation, and almost always met for reasonable parameters otherwise. If the conditions are met, the source size (some measure of the region from which the bulk of the emission originates) decreases with frequency as

$$I(\nu) \propto \nu^{1/m_\nu} \equiv \nu^{-1/(2+3m_\nu-2m_\nu)}$$  (58)

if seen more or less from the side; if the flow is nearly along the line of sight, the expression becomes

$$\theta \propto \nu^{\epsilon/m_\nu} = \nu^{-\epsilon/(2+3m_\nu-2m_\nu)}$$  (59)

(which is the same for spherical or conical flows where $\epsilon = 1$).

8. CONCLUSIONS

My basic conclusion is just that synchrotron-loss spectral breaks differing from 0.5 can be produced naturally in inhomogeneous sources. I have treated the inhomogeneities resulting from flows, which seem most natural, using simple power-law parameterizations, but more complex functional dependencies can be treated the same way. Other types of inhomogeneities may be possible as well. These results are most straightforwardly applied to PWNe or knots in extragalactic jets, but may have applications wherever bulk flows of relativistic material are involved. In particular, energy-loss breaks seen in gamma-ray burst afterglows (e.g., Sari et al. 1998; Galama et al. 1998; and much later work) may provide opportunities for the application of these results. For nearby sources, the simplest test of the models is the detection of the size effect; every model predicts both a particular $\Delta$ and some rate of decrease of size with frequency (really of volume, since the size decrease may take place along the line of sight). For the same $\Delta$, some variation in the strength of the size effect is possible.

Fortunately, assuming that an inhomogeneous source is actually homogeneous does not drastically alter the inferred mean magnetic field (averaged over the history of a fluid element); but since there are by assumption large gradients of most quantities, local values of the magnetic field strength, such as those at the injection radius, may depart substantially from the mean values. This may affect inferences of the KC magnetization parameter $\sigma$. Furthermore, the inference requires knowledge of the actual flow time across the source—knowledge that may be hard to come by in the presence of large velocity gradients.

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