THE DULMAGE-MENDELSOHN DECOMPOSITION FOR
b-MATCHINGS

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ABSTRACT. We establish the theory of the Dulmage-Mendelsohn decomposition for b-matchings. The original Dulmage-Mendelsohn decomposition is a classical canonical decomposition of bipartite graphs, which describes the structures of the maximum 1-matchings and the dual optimizers, i.e., the minimum vertex covers. In this paper, we develop analogical properties, and thus obtain the structure of the maximum b-matchings and characterizes the family of b-verifying set.

1. Preliminaries

1.1. Notation.

1.1.1. General Statements. For standard notation of sets, graphs, and algorithms, we mostly refer to Schrijver [8]. In the following, we list exceptions and non-standard definitions. Given a graph or a digraph \( G \), the vertex set is denoted by \( V(G) \); the edge set is denoted by \( E(G) \) if \( G \) is an (undirected) graph; otherwise, \( A(G) \) denotes the arc set. An edge with ends \( u \) and \( v \) is denoted by \( uv \). Similarly, an arc with tail \( u \) and head \( v \) is denoted by \( uv \). As usual, a singleton \( \{x\} \) is often denoted simply by \( x \). We sometimes denote the vertex set of a graph \( G \) simply by \( G \) itself. For \( X \subseteq V(G) \), \( X^c \) denotes \( V(G) \setminus X \).

1.1.2. Operations on Graphs. Given a graph \( G \) and a set of vertices \( X \subseteq V(G) \), the subgraph of \( G \) induced by \( X \) is denoted by \( G[X] \). Given a set of edges \( F \) from a supergraph of \( G \), \( G + F \) denotes the graph obtained by adding \( F \) to \( G \). Given subgraphs \( H_1 \) and \( H_2 \) of \( G \), \( H_1 + H_2 \) denotes the union of \( H_1 \) and \( H_2 \).

1.1.3. Functions on Graphs. Given a set of vertices \( X \) in a graph \( G \), the set of neighbors of \( X \) is denoted by \( N_G(X) \). That is to say, \( N_G(X) \) is the set of vertices that are adjacent to a vertex in \( X \) and themselves are not in \( X \). Given \( X, Y \subseteq V(G) \), the set of edges whose one end is in \( X \) and the other is in \( Y \) is denoted by \( E_G[X,Y] \). The set \( E_G[X,V(G) \setminus X] \) is denoted by \( \delta_G(X) \). The set of edges both of whose ends are in \( X \) is denoted by \( E_G[X] \). With respect to these functions, we often omit the subscript “\( G \)” if it is clear from the contexts.

1.1.4. Paths and Circuits. We treat paths and circuits as graphs. A circuit is a connected graph such that every vertex has the degree two. A path is a connected graph such that every vertex has the degree two or less and it is not a circuit. Given a path \( P \) and vertices \( x, y \in V(P) \), \( xPy \) denotes the subpath between \( x \) and \( y \), namely, the subgraph of \( P \) that is a path with ends \( x \) and \( y \).

1.1.5. Ideals. Let \( \mathcal{P} \) be a poset over a set \( X \). Then, it is easy to observe that, for any lower ideal \( I \subseteq X \), \( X \setminus I \) is an upper ideal of \( \mathcal{P} \); for any upper ideal \( J \subseteq X \), \( X \setminus J \) is a lower ideal of \( \mathcal{P} \). We say a pair of an upper ideal \( I \) and a lower ideal \( J \) is complementary if \( I \cup J = X \).

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1.1.6. *Projective Union.* Let $G$ be a graph, let $I$ be a set of subgraphs of $G$, and let $W \subseteq V(G)$. A *projective union* of $I$ over $W$ is the set $\bigcup_{H \in I} V(H) \cap W$.

1.2. *b-Matchings.* Let $G$ be a graph, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. A set of edges $M \subseteq V(G)$ is a *b-matching* if $|\delta_G(v) \cap M| \leq b(v)$ holds for each $v \in V(G)$. We denote $b$ by 1 if $b(v) = 1$ for each $v \in V(G)$. A 1-matching is often called simply a *matching*. Given a $b$-matching $M$, a vertex $v \in V(G)$ is $M$-loose if $|\delta_G(v) \cap M| < b(v)$ holds; $v$ is $M$-tight if $|\delta_G(v) \cap M| = b(v)$. A maximum $b$-matching is a $b$-matching of $G$ with the greatest number of edges. A $b$-matching $M$ is *perfect* if every vertex is $M$-tight.

An edge is *b-allowed* if it is contained in a maximum $b$-matching; otherwise, it is *b-forbidden*. A $b$-allowed edge $e \in V(G)$ is *b-inevitable* if any maximum $b$-matching contains $e$; otherwise, $e$ is *b-flexible*.

A *b-elementary component* of $G$ is a subgraph $G[V(C)]$, where $C$ is a connected component of the subgraph of $G$ determined by the set of $b$-allowed edges. That is to say, a graph comprises of $b$-elementary components and $b$-forbidden edges that join distinct $b$-elementary components. The *set of $b$-elementary components* of $G$ is denoted by $\mathcal{C}(G, b)$.

A *b-flexible component* is a subgraph $G[V(C)]$, where $C$ is a connected component of the subgraph of $G$ determined by the set of $b$-flexible edges. That is to say, a graph comprises of $b$-flexible components, and $b$-inconsistent and $b$-forbidden edges that join distinct $b$-flexible components. The *set of $b$-flexible components* of $G$ is denoted by $\mathcal{G}(G, b)$.

As such, $b$-elementary components and $b$-flexible components can be viewed, in their respective ways, as fundamental building blocks of a graph in the context of understanding the structure of maximum $b$-matchings. Note that both concepts of $b$-elementary components and $b$-flexible components are canonical by definition. Given a subgraph $C$ of $G$, define $b|_C : V(C) \to \mathbb{Z}_{\geq 0}$ as such that $b|_C(v) := b(v) - k_v$ for each $v \in V(C)$, where $k_v$ is the number of $b$-inconsistent edges in $E_G[v, V(G) \setminus V(C)]$. It is easy to observe that a set of edges is a maximum $b$-matching if and only if it is the disjoint union of maximum $b|_C$-matchings of every $b$-elementary component $C$. Also, a set of edges is a maximum $b$-matching if and only if it is the disjoint union of the set of $b$-inconsistent edges, which join distinct $b$-flexible components, and maximum $b|_C$-matchings of every $b$-flexible component $C$; for, we will later prove the following fact:

**Fact 1.1.** Given a bipartite graph $G$ and a mapping $b : V(G) \to \mathbb{Z}_{\geq 0}$, any edge in a $b$-flexible component is $b$-flexible.

Note that $b$-flexible components give a refinement of $b$-elementary components; that is, for each $C \in \mathcal{C}(G, b)$, there exist $D_1, \ldots, D_k \in \mathcal{G}(G, b)$, where $k \geq 1$, such that $V(C) = V(D_1) \cup \cdots \cup V(D_k)$.

Let $C$ be a subgraph of $G$, which will be typically a $b$-flexible or $b$-elementary component. We say that $C$ is *trivial* if it has only one vertex. A vertex $v \in V(G)$ is *$b$-inactive* if $b(v) = 0$. If $v$ is an inactive vertex, then $G[v]$ forms a trivial $b$-flexible component as well as a trivial $b$-elementary component, which we say that is *$b$-inactive*.

**Definition 1.2.** Given $G$ and $b$, $D(G, b)$ denotes the set of vertices that are $M$-loose for some maximum $b$-matchings.

We say that $C$ of $G$ is *$b$-tight* (resp. *$b$-loose*) if it has no vertices (resp. some vertices) in $D(G, b)$. We say that $C$ is *$b$-inconsistent* if it is $b$-loose or if it is $b$-inactive and its vertex is a neighbor of $D(G, b)$; otherwise, $C$ is *$b$-consistent*.


Observation 1.3. Let $C \in \mathcal{G}(G, b)$ or $C \in \mathcal{C}(G, b)$, and let $M$ be a maximum $b$-matching of $G$. Then, $M \cap E(C)$ is a perfect $b|c$-matching of $C$ if and only if $C$ is $b$-tight.

Let $W \subseteq V(G)$, which will typically be a color class of a bipartite graph. A $b$-loose subgraph $C$ is hooked up by $W$ if $V(C) \cap D(G, b) \cap W \neq \emptyset$. A $b$-inconsistent subgraph $D$ is hooked up by $W$ if it is $b$-loose and is hooked up by $W$ or if it is $b$-inconsistent and its sole vertex is a neighbor of $D(G, b) \cap W$.

The sets of $b$-consistent and $b$-inconsistent $b$-flexible components are denoted by $\mathcal{G}^+(G, b)$ and $\mathcal{G}^-(G, b)$, respectively. The sets of $b$-inconsistent $b$-flexible components hooked up by $W$ is denoted by $\mathcal{G}_W^-(G, b)$.

Regarding the definitions presented in this section, we will often omit the modifier “$b$-” if no confusion may arise. So will we for the definitions that will appear in later sections, such as $b$-verifying sets.

2. Dulmage-Mendelsohn Decomposition for 1-Matchings

In this section, we present the original Dulmage-Mendelsohn decomposition \cite{DM64}, which is for 1-matchings. Throughout this section, let $G$ be a bipartite graph with color classes $A$ and $B$.

Definition 2.1. A set of vertices $S$ in a graph is a vertex cover if and only if its complement is a stable set. The maximum 1-matching problem forms a min-max theorem as follows:

Theorem 2.2. In a bipartite graph, the number of edges in a maximum 1-matchings is equal to the number of vertices in a minimum vertex cover.

A set of vertices is a vertex cover if and only if its complement is a stable set. Therefore, Theorem 2.2 is equivalent to the following:

Theorem 2.3. In a bipartite graph, the number of edges in a maximum 1-matchings is equal to the value $|Z'|$, where $Z \subseteq V(G)$ is a maximum stable set.

Definition 2.4. Let $W \in \{A, B\}$. Define a binary relation $\leq_W$ over $\mathcal{C}(G, 1)$ as follows: for $C_1, C_2 \in \mathcal{C}(G, 1)$, $C_1 \leq_W C_2$ holds if there exist $D_1, \ldots, D_k \in \mathcal{C}(G, 1)$, where $k \geq 1$, such that $C_1 = D_1$, $C_2 = D_k$, and $E[D_{i+1}\cap W, D_i \cap W^c] \neq \emptyset$ for each $i \in \{1, \ldots, k-1\}$.

Theorem 2.5. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $W \in \{A, B\}$. Then, $\leq_W$ is a partial order over $\mathcal{C}(G, 1)$.

Note that $\leq_A$ and $\leq_B$ are symmetric, that is, for any $C_1, C_2 \in \mathcal{C}(G, b)$, $C_1 \leq_A C_2$ holds if and only if $C_2 \leq_B C_1$ holds.

Definition 2.6. A lower ideal (resp. an upper ideal) $\mathcal{I}$ of the poset $(\mathcal{C}(G, 1), \leq_W)$ is normalized if any inconsistent 1-elementary component hooked up by $W$ (resp. by $W^c$) is in $\mathcal{I}$ and any inconsistent 1-elementary component hooked up by $W^c$ (resp. by $W$) is disjoint from $\mathcal{I}$.

Fact 2.7. If an inconsistent 1-elementary component is hooked up by $A$, then it is not hooked up by $B$, and vice versa.

Fact 2.8. Any inconsistent 1-elementary component hooked up by $A$ (resp. by $B$) is minimal (resp. maximal) in the poset $(\mathcal{C}(G, 1), \leq_A)$.

Accordingly, a lower ideal (resp. an upper ideal) $\mathcal{I}$ of the poset $(\mathcal{C}(G, 1), \leq_A)$ is normalized if and only if there exists a lower (resp. an upper) ideal $\mathcal{I}'$ such that
Using the concept of normalized ideals, the family of maximum stable sets (and accordingly, the family of minimum vertex cover as well) are characterized:

**Theorem 2.9.** Let $G$ be a bipartite graph with color classes $A$ and $B$. A set of vertices $X \subseteq V(G)$ is a maximum stable set if and only if there is a complementary pair of normalized lower and upper ideals $I_A$ and $J_A$ of the poset $(\mathcal{C}(G, 1), \subseteq_W)$ such that $X = X_A \dot{\cup} X_B$, where $X_A$ and $X_B$ are the projective unions of $I_A$ over $A$ and of $I_B$ over $B$, respectively.

Given a maximum 1-matching, the Dulmage-Mendelsohn decomposition can be computed in $O(n + m)$ time, where $n$ and $m$ denote the numbers of vertices and edges, respectively. Therefore, given $G$ and $b$, the Dulmage-Mendelsohn decomposition can be computed in strongly polynomial time.

### 3. Overview of Our New Theory

A min-max relation is known for $b$-matchings in bipartite graphs [7, 8]:

**Theorem 3.1.** Let $G$ be a bipartite graph, and let $b : V(G) \rightarrow \mathbb{Z} \geq 0$. Then, the size of a maximum $b$-matching of $G$ is equal to the minimum value of $b(Z^c) + |E[Z]|$, where $Z$ is taken over all subsets of $V(G)$.

A $b$-verifying set is a set of vertices $Z$ that attains the minimum in Theorem 3.1, namely, whose $b(Z^c) + |E[Z]|$ is equal to the size of a maximum $b$-matchings.

We establish the theory of $b$-matching analogue of the Dulmage-Mendelsohn decomposition, considering $b$-verifying sets and $b$-flexible components. The main theorems that comprise the heart of this new theory are the following:

1. The set of flexible components form a poset with respect to a canonical binary relation, which is analogous to the poset $(\mathcal{C}(G, 1), \subseteq_W)$. (Theorem 4.20)
2. The family of $b$-verifying set is characterized using this poset over flexible components, in totally analogical way to the characterization of the family of maximum stable sets. (Theorem 5.8)

These structures can be computed in strongly polynomial time, which we will see in Theorem 6.7.

If we restrict ourselves to the case $b = 1$, the structure given by our results is generally a refinement of the classical Dulmage-Mendelsohn decomposition for 1-matchings; note that 1-verifying sets and 1-flexible components are more fine-grained concepts than their classical counterparts, namely, than the maximum stable sets and the 1-elementary components. Our results for the case $b = 1$ totally coincide with the classical one if and only if there is no 1-inevitable edge.

### 4. The Dulmage-Mendelsohn Decomposition for $b$-Matchings

#### 4.1. Preparation.

Throughout Section 4 unless otherwise stated, let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \rightarrow \mathbb{Z} \geq 0$. Note that, as the roles of $A$ and $B$ are given arbitrarily, every statement also holds by swapping $A$ and $B$.

In Section 4.1 we present lemmas on the relationship between maximum $b$-matchings and verifying sets, and thus give an observation about verifying sets and flexible components.

The next lemma can be deduced from Theorem 4.1 however we present it with a stand-alone proof so to ensure that the whole new theory of ours is self-contained.
Lemma 4.1. Let $G$ be a bipartite graph, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Then, $|M| \leq b(Z^e) + |E[Z]|$ holds for any $b$-matching $M$ of $G$ and any set of vertices $Z \subseteq V(G)$. The equality holds if and only if

(i) $E[Z] \subseteq M$ holds,
(ii) every vertex in $Z^c$ is $M$-tight, and
(iii) $E[Z^c] \cap M = \emptyset$.

Proof. Obviously, $|M \cap E[Z]| \leq |E[Z]|$ and $|M \setminus E[Z]| \leq b(Z^c)$ hold. Therefore, $|M| \leq b(Z^c) + |E[Z]|$ holds. The necessary and sufficient condition for the equality is easily observed by considering the above two inequality.

Lemma 4.2 implies the following two lemmas:

Lemma 4.2. Let $G$ be a bipartite graph, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. If $Z \subseteq V(G)$ is a verifying set, then, for any maximum $b$-matching $M$, the following hold:

(i) $E[Z] \subseteq M$;
(ii) $E[Z^c] \cap M = \emptyset$;
(iii) Any $M$-loose vertex is contained in $Z$.

Accordingly, any edge in $E[Z]$ is inevitable, whereas any edge in $E[Z^c]$ is forbidden.

Lemma 4.3. Let $G$ be a bipartite graph, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $M$ be a $b$-matching. If a set of vertices $Z \subseteq V(G)$ satisfies

(i) $E[Z] \subseteq M$,
(ii) $E[Z^c] \cap M = \emptyset$,
(iii) any $M$-loose vertex is contained in $Z$,

then $M$ is a maximum $b$-matching and $Z$ is a verifying set.

A set of vertices is separating if it is empty or is the union of vertex sets of some flexible components. We can now present a fundamental observation about relationship between verifying sets and flexible components.

Lemma 4.4. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $Z$ be a verifying set, and let $Z_1 := (Z \cap A) \cup (B \setminus Z)$ and $Z_2 := (A \setminus Z) \cup (Z \cap B)$. Then,

(i) $Z_1$ and $Z_2$ are separating,
(ii) any $C \in G^+_A(G, b)$ (resp. $C \in G^-_B(G, b)$) satisfies $V(C) \subseteq Z_1$ (resp. $V(C) \subseteq Z_2$), and
(iii) $G^+_A(G, b) \cap G^-_B(G, b) = \emptyset$.

Proof. From the last statement of Lemma 4.2, it is obvious that (i) follows. Additionally, Lemma 4.2 (i) implies $D(G, b) \subseteq Z$. Hence, we have $V(C) \subseteq Z_1$ for any $C \in G^+_A(G, b)$. Therefore, from Lemma 4.2 (i), if $v \in B$ is an inactive vertex with $N_G(v) \cap D(G, b) \neq \emptyset$, then $v \in Z_1$ holds. Therefore, we obtain (ii). As $Z_1$ and $Z_2$ are disjoint, this immediately proves (iii).

4.2. Structure of Inconsistent Flexible Components.

4.2.1. Canonical Verifying Set. The goal of this section is to obtain Theorem 4.9 which claims the existence of two special verifying sets.

Let $M$ be a $b$-matching. We say a path or a circuit $M$-alternating if edges in $M$ and not in $M$ appear alternately. More precisely, a circuit $C$ is $M$-alternating if $|\delta_C(v) \cap M| = 1$ for every $v \in V(C)$. We define three types of $M$-alternating paths: A path $P$ with ends $x$ and $y$ is $M$-wedge from $x$ to $y$ if $|\delta_P(v) \cap M| = 1$ for each $v \in V(P) \setminus \{y\}$ whereas $\delta_P(y) \cap M = \emptyset$; A path $P$ with ends $x$ and $y$ is $M$-saturated (resp. $M$-exposed) between $x$ and $y$ if $|\delta_P(v) \cap M| = 1$ (resp. $|\delta_P(v) \cap (E(G) \setminus M)| = 1$) for each $v \in V(P)$.

The next one is easy to confirm:
Lemma 4.5. Let $G$ be a bipartite graph, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $M$ be a maximum $b$-matching, and let $v \in V(G)$ be an $M$-loose vertex. Let $P$ be an $M$-wedge path from a vertex $u \in V(G)$ to the vertex $v$. Then, $M \triangle E(P)$ is also a maximum $b$-matching of $G$. Accordingly, all edges of $P$ are contained in the same flexible component.

The first statement and its proof of the next lemma have been known; see Pap [7], which claims that, given a maximum $b$-matching, we can construct a verifying set. The other statements prove that this verifying set is, in fact, canonical, according to its relationship with inconsistent flexible components:

Lemma 4.6. Let $G$ be a bipartite graph with color classes $A$ and $B$, let $b : V(G) \to \mathbb{Z}_{\geq 0}$, and let $M$ be a maximum $b$-matching of $G$. Let $U_A$ be the set of $M$-loose vertices in $A$. Let $S_A \subseteq A$ be the set of vertices from which to vertices in $A$ there exist $M$-wedge paths, and let $T_A \subseteq B$ be the set of vertices between which and vertices in $U_A$ there exit $M$-exposed paths. Then,

(i) $S_A \cup (B \setminus T_A)$ is a verifying set, and

(ii) $S_A = D(G, b) \cap A$.

Accordingly, if $Z_1 := S_A \cup T_A$, then $Z_1$ is the disjoint union of vertex sets of all inconsistent flexible components hooked up by $A$.

Proof. First we prove $E[S_A, B \setminus T_A] \subseteq M$ and $E[A \setminus S_A, T_A] \cap M = \emptyset$. Suppose there is an edge $xy \in E(G) \setminus M$ with $x \in S_A$ and $y \in B \setminus T_A$. By definition, there is an $M$-wedge path $P$ from $x$ to a vertex $z \in U_A$. Then, $P + xy$ is an $M$-exposed path between $z$ and $y$, which is a contradiction. Next suppose there is an edge $xy \in M$ with $x \in A \setminus S_A$ and $y \in T_A$. By definition, there is an $M$-exposed path $P$ between a vertex $z \in U_A$ and $y$. Then, $P + xy$ is an $M$-wedge path from $x$ to $z$, which is again a contradiction.

Next we prove that all vertices in $T_A$ are $M$-tight. Suppose a vertex $y \in T_A$ is $M$-loose, and let $P$ be an $M$-exposed path between some vertex in $U_A$ and $y$. Then, $M \triangle E(P)$ is a $b$-matching of $G$ that is larger than $M$, which is a contradiction. Therefore, $Z_1$ contains all $M$-loose vertices, and hence, from Lemma 4.3, the statement (i) is proved.

For the statement (ii) we first prove $S_A \subseteq D(G, b) \cap A$. By definition, for any $x \in S_A$, there is an $M$-wedge path from $x$ to a vertex $z \in U_A$. Then, $M \triangle E(P)$ is a maximum $b$-matching in which $x$ is $M \triangle E(P)$-loose. Therefore, $x \in D(G, b)$ holds, and we have $S_A \subseteq D(G, b) \cap A$. On the other hand, according to (i) and Lemma 4.2(iii) any vertex in $A \setminus S_A$ is $M'$-tight with respect to any maximum $b$-matching $M'$. Therefore, we have $S_A \supseteq D(G, b) \cap A$. Thus, (ii) is proved.

From (i) and Lemma 4.3(iii) it follows that $S_A \cup T_A$ is a separating set. As $S_A = D(G, b) \cap A$, any loose flexible component $C$ hooked up by $A$ satisfies $V(C) \subseteq S_A \cup T_A$; conversely, if $C \subseteq G \setminus (G, b) \cap A \neq \emptyset$ satisfies $V(C) \subseteq S_A \cup T_A$, then $C$ is a loose flexible component hooked up by $A$. Therefore, $S_A \cup T_A$ consists of the vertex sets of all loose flexible components hooked up by $A$ and some trivial flexible components with their sole vertices in $T_A$; we prove in the following that those trivial are exactly the inactive flexible components hooked up by $A$.

Let $v \in T_A$ be such that $G[v]$ is a trivial flexible component. By the definition of $T_A$, there is an $M$-exposed path $P$ between $v$ and an $M$-loose vertex $w \in A$. If $b(v) > 0$ holds, then from Lemma 4.2(iii) there exists a vertex $u \in A$ with $uw \in M$. Then, $P + uw$ is an $M$-wedge path from $u$ to $w$. From Lemma 4.5, this implies that $v$ and $w$ are in the same flexible component. This is a contradiction. Hence, $v$ is an inactive vertex. Moreover, if $z \in V(P)$ is such that $zv \in E(P)$, then of course $zv \notin M$ holds, and, as $wPz$ is an $M$-wedge path from $z$ to $w$, we have $z \in S_A$; namely, $N_G(v) \cap S_A \neq \emptyset$. 
Conversely, if \( v \in B \) is an inactive vertex with \( N_G(v) \cap S_A \neq \emptyset \), then, from Lemma 4.3(ii), \( v \in T_A \) holds. Therefore, \( S_A \cup T_A \) is the union of vertex sets of all inconsistent flexible components hooked by \( A \).

\[ \square \]

**Remark 4.7.** Note that Lemmas 4.1 and 4.6 provide the proof of Theorem 3.1.

**Definition 4.8.** Let \( W \in \{ A, B \} \). The inconsistent unit of \( G \) hooked up by \( W \) is the disjoint union of vertex sets of inconsistent flexible component hooked up by \( W \), and is denoted by \( \Sigma_W(G) \).

Lemma 4.10 proved that \( \Sigma_A(G) \) and \( \Sigma_B(G) \) determine verifying sets that are canonical. We can further observe, from Lemma 4.4, that what they give are the “nucleus” of every verifying set:

**Theorem 4.9.** Let \( G \) be a bipartite graph with color classes \( A \) and \( B \), and let \( b : V(G) \rightarrow \mathbb{Z}_{\geq 0} \).

(i) Then, \((\Sigma_A(G) \cap A) \cup (B - \Sigma_B(G))\) and \((\Sigma_B(G) \cap B) \cup (A - \Sigma_A(G))\) are verifying sets.

(ii) Any verifying set \( Z \) contains \((\Sigma_A(G) \cap A) \cup (\Sigma_B(G) \cap B)\) and is disjoint from \((\Sigma_A(G) \cap B) \cup (\Sigma_B(G) \cap A)\).

4.2.2. Inner Structure of Inconsistent Unit. In this section, we investigate relationships between inconsistent flexible components. The results here will later utilized in Section 4.3 to prove that inconsistent flexible components are minimal or maximal in the poset formed by the set of flexible components, or in Section 6 to construct an algorithm.

**Lemma 4.10.** Let \( G \) be a bipartite graph with color classes \( A \) and \( B \), and let \( b : V(G) \rightarrow \mathbb{Z}_{\geq 0} \). Let \( C \) be a loose flexible component hooked up by \( A \). Denote \( A_C := V(C) \cap A \) and \( B_C := V(C) \cap B \). Let \( M \) be a maximum \( b \)-matching of \( G \), and let \( U_A \subseteq A \) be the set of \( M \)-loose vertices in \( A \). Then,

(i) \( A_C \cap U_A \neq \emptyset \), and \( A_C \) is the set of vertices from which to \( M \)-loose vertices in \( A_C \) there exist \( M \)-wedge paths of \( C \); and,

(ii) \( B_C \) is the set of vertices in \( B \) that are contained in \( M \)-wedge paths from vertices in \( A_C \) to vertices in \( U_A \).

Accordingly, for any vertex \( v \in B_C \), \( C \) has an \( M \)-exposed path between \( v \) and a vertex in \( U_A \), and an \( M \)-saturated path between \( v \) and a vertex in \( A_C \).

**Proof.** From Lemma 4.9, for any \( v \in A_C \), there an \( M \)-wedge path \( P \) from \( v \) to an \( M \)-loose vertex \( u \in A \). From Lemma 4.9, \( P \) is a path of \( C \) and thus \( u \in V(C) \) follows. Therefore, \( A_C \) has some vertices in \( U_A \), and for any \( v \in A_C \), \( C \) has an \( M \)-wedge path from \( v \) to a vertex in \( U_A \). The converse direction of (i) is obvious. Hence, (i) is proved.

As \( B_C \subseteq \Sigma_A(G) \cap B \), Lemma 4.9 implies that, for any vertex \( v \in B_C \), there is an \( M \)-exposed path \( Q \) from \( v \) and a vertex in \( U_A \). As \( A_C \neq \emptyset \), we have \( b(v) > 0 \) for any \( v \in B_C \). Hence, there is a vertex \( w \in A \) with \( wv \in M \), and \( Q + vw \) is an \( M \)-wedge path from \( w \) to \( v \). From Lemma 4.9, this path \( Q + vw \) is contained in \( C \), and accordingly so is \( Q \). As the converse direction of (i) is obvious, this proves (ii). From (ii), the remaining statement follows. \[ \square \]

Lemma 4.10 derives the next lemma. Lemma 4.11 will imply in Section 4.3 that any two distinct loose flexible components hooked up by the same color class are not compatible in the poset.
Lemma 4.11. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $C_1$ and $C_2$ be two distinct loose flexible components hooked up by $A$. Then, $E[C_1, C_2] = \emptyset$.

Proof. Suppose to the contrary, namely, that there exist $u \in V(C_1)$ and $v \in V(C_2)$ with $uv \in E(G)$. Without loss of generality, assume $u \in A$ and $v \in B$. Let $M$ be an arbitrary maximum $b$-matching of $G$. First consider the case with $uv \notin M$.

From Lemma 4.10 $C_1$ has an $M$-wedge path $P_1$ from $u$ to an $M$-loose vertex $w$ in $A \cap V(C_1)$. According to the last statement of Lemma 4.10 $C_2$ has an $M$-saturated path $P_2$ between $v$ and a vertex $z \in A \cap V(C_2)$. Then, $P_1 + wP_2$ is an $M$-wedge path from $z$ to $w$, which implies, from Lemma 4.5, that $z$ and $w$ are in the same flexible component. Hence, we reach a contradiction for this case.

Second, consider the case with $uv \in M$. According to the last statement of Lemma 4.10 $C_2$ has an $M$-exposed path $Q$ between $v$ and an $M$-loose vertex $x \in A \cap V(C_2)$. Then, $uv + Q$ is an $M$-wedge path from $u$ to $x$, which implies, from Lemma 4.5, that $u$ and $x$ are in the same flexible component. Thus, we again reach a contradiction, and this completes the proof. $\square$

From Lemma 4.12 and Theorem 4.9 the next lemma is obtained.

Lemma 4.12. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $C$ be an inconsistent flexible component hooked up by $A$. Then, for any $D \in \mathcal{G}(G,b) \setminus \mathcal{g}_A(G,b)$, the edges in $E[V(C) \cap A, V(D) \cap B]$ are inevitable, whereas the edges in $E[V(D) \cap A, V(C) \cap B]$ are forbidden.

From Lemma 4.11, the next lemma, which will be used in Section 6, is also derived easily:

Lemma 4.13. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $S$ be the set of inactive vertices in $\Sigma_A(G)$. Then, the set of loose flexible components hooked up by $A$ is equal to the set of connected components of $G[S]$.

4.3. Structure of Consistent Flexible Components. In this section, we obtain Theorem 4.20 which states that the set of consistent flexible components forms a poset with respect to a certain canonical partially order we define here.

A graph is $b$-flexible connected if it has only one flexible component.

Lemma 4.14. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. If $G$ is a flexible connected graph with a perfect $b$-matching $M$,

(i) then, for any $x \in A$ and any $y \in B$, there is an $M$-wedge path from $x$ to $y$;

(ii) for any $x \in A$ and any $y \in B$, there is an $M$-saturated path between $x$ and $y$; and,

(iii) for any $x \in A$ and any $y \in B$, there is an $M$-exposed path between $x$ and $y$.

Proof. Let $x \in A$ be an arbitrary vertex. Let $S$ be the set of vertices in $A$ from which $x$ is not flexible connected, which $x$ is an $M$-saturated path. Then, by this definition, the edges in $E[S, B \setminus T]$ are disjoint from $M$, whereas the edges in $E[A \setminus S, T]$ are in $M$. Let $Z := (A \setminus S) \cup T$. As $G$ has no $M$-loose vertices, $Z$ is a verifying set, according to Lemma 4.9. Hence, if neither $S \cup T = \emptyset$ nor $S \cup T = V(G)$ holds, then Lemma 4.3 implies that is not flexible connected, which is a contradiction. Obviously, $S \cup T \neq \emptyset$. Therefore, we have $S \cup T = V(G)$, that is to say, $T = T$. Thus, (i) and (ii) are proved.

To prove (iii), let $S'$ be the set of vertices from which to $x$ there is an $M$-wedge path, and let $T'$ be the set of vertices between which and $x$ there is an $M$-exposed path. From the symmetric argument similar to the above, we can prove (iii). $\square$
The next lemma is easy to confirm:

**Lemma 4.15.** Let $G$ be a bipartite graph, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. If $C \in \mathcal{G}^+(G, b)$ holds, then $C$ is a $b|_C$-flexible connected graph with a perfect $b|_C$-matching.

**Remark 4.16.** Under Lemma 4.15, the claim of Lemma 4.14 can be applied for each consistent flexible component by treating it as a flexible connected graph.

We can now prove Fact 1.1.

**Proof of Fact 1.1.** Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $C \in \mathcal{G}(G, b)$, and let $uv \in E(C)$, where $u \in A$ and $v \in B$.

First consider the case with $C \in \mathcal{G}^+(G, b)$. Let $M$ be an arbitrary maximum $b$-matching of $G$. Under Lemma 4.14 if $uv \in M$ holds, then let $P$ be an $M$-exposed path between $u$ and $v$; otherwise, let $P$ be an $M$-saturated path between $u$ and $v$. Then, $P + uv$ is an $M$-alternating circuit, and hence, $M \Delta E(P + uv)$ is also a maximum $b$-matching. The edge $uv$ is exclusively contained in either $M$ or $M \Delta E(P + uv)$, and therefore, $uv$ is a flexible edge.

Next consider the case with $C \in \mathcal{G}^-(G, b)$. It suffices to consider the case where $C$ is a loose flexible component hooked up by $A$. According to Lemma 4.10 there is a maximum $b$-matching $M$ such that $u$ is $M$-loose. Under Lemma 4.10 if $uv \in M$ holds, then let $P$ be an $M$-exposed path between $u$ and $v$ and an $M$-loose vertex in $V(C) \cap A$; otherwise, let $P$ be an $M$-saturated path between $u$ and $v$ in $V(C) \cap A$. Then, $P + uv$ is an $M$-alternating circuit or an $M$-wedge path from an $M$-loose vertex to a vertex in $V(C) \cap A$. Hence, $M$ and $M \Delta E(P + uv)$ are both maximum $b$-matchings, exactly one of which contains $uv$. Therefore, again $uv$ is a flexible edge. \qed

In the following, we first define a binary relation $\preceq_W$ over $\mathcal{G}(G, b)$, and then, using this, further define a binary relation $\preceq_W$ over $\mathcal{G}^+(G, b)$, where $W \in \{A, B\}$; this $\preceq_W$ will turn out, in Theorem 4.20, to be a partial order.

**Definition 4.17.** Let $W \in \{A, B\}$. We define a binary relation $\preceq_W$ over $\mathcal{G}(G, b)$ as follows: For $C_1, C_2 \in \mathcal{G}(G, b)$, $C_1 \preceq_W C_2$ holds if $C_1 = C_2$ or if there is an inevitable edge between $V(C_1) \cap W$ and $V(C_2) \cap W^c$, or is an forbidden edge between $V(C_2) \cap W$ and $V(C_1) \cap W^c$. Furthermore, we define a binary relation $\preceq_W$ over $\mathcal{G}^+(G, b)$ as follows: For $C_1, C_2 \in \mathcal{G}^+(G, b)$, $C_1 \preceq_W C_2$ holds if there exist $D_1, \ldots, D_k \in \mathcal{G}^+(G, b)$ with $k \geq 1$ such that, $D_1 = C_1$, $D_2 = C_2$, and $D_1 \preceq_W \cdots \preceq_W D_k$ hold.

**Remark 4.18.** The two binary relations $\preceq_A$ and $\preceq_B$ are symmetric. That is, $C_1 \preceq_A C_2$ holds if and only if $C_2 \preceq_B C_1$ holds.

From Lemmas 4.14 and 4.15 we derive the following lemma:

**Lemma 4.19.** Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $C_1$ and $C_2$ be consistent flexible components with $C_1 \preceq_A C_2$. Let $D_1, \ldots, D_k \in \mathcal{G}^+(G, b)$, where $k \geq 1$, be such that $C_1 = D_1$, $C_2 = D_k$, and $D_1 \preceq_A \cdots \preceq_A D_k$. Let $A_i := V(C_i) \cap A$ and $B_i := V(C_i) \cap B$ for each $i \in \{1, 2\}$. Then, for any maximum $b$-matching $M$ of $G$, the following hold:

(i) For any $x \in A_1$ and any $y \in A_2$, there is an $M$-wedge path from $x$ to $y$;

(ii) For any $x \in A_1$ and any $y \in B_2$, there is an $M$-saturated path between $x$ and $y$;

(iii) For any $x \in A_2$ and any $y \in B_1$, there is an $M$-exposed path between $x$ and $y$; and,

(iv) For any $x \in B_1$ and any $y \in B_2$, there is an $M$-wedge path from $y$ to $x$.

Additionally, these paths can be taken so that their vertices are contained in $V(D_1) \cup \cdots \cup V(D_k)$. 


Proof. We proceed by induction on $k$. If $k = 1$, then, from Lemmas 4.14 and 4.15, the statements hold. Let $k > 1$, and assume the statements hold for the cases where the parameter is less. Note that under this hypothesis, the statements hold for $D_1$ and $D_{k-1}$, which satisfy $D_1 \preceq_{A} \cdots \preceq_{A} D_{k-1}$. If $D_k$ is equal to one of $D_1, \ldots, D_{k-1}$, then, of course, we are done. Hence, in the following, assume $D_k \neq D_i$ for any $i \in \{1, \ldots, k-1\}$. Let $x$ be an arbitrary vertex from $V(D_1)$, and let $y$ be an arbitrary vertex from $V(D_k)$.

First consider the case where there exists $uv \in M$ with $u \in V(D_{k-1}) \cap A$ and $v \in V(D_k) \cap B$. If $x \in A$ holds, then let $P$ be an $M$-wedge path $P$ from $x$ to $u$; otherwise, let $P$ be an $M$-exposed path between $u$ and $x$. From the hypothesis, we can take $P$, so that $V(P) \subseteq V(D_1) \cup \cdots \cup V(D_{k-1})$ holds. On the other hand, under Lemmas 4.14 and 4.15 if $y \in A$ holds, then let $Q$ be an $M$-exposed path of $D_k$ between $y$ and $v$; otherwise, let $Q$ be an $M$-wedge path of $D_k$ from $y$ to $v$. Then, $P + uv + Q$ is a path with $V(P + uv + Q) \subseteq V(D_1) \cup \cdots \cup V(D_k)$, which is $M$-wedge from $x$ to $y$, $M$-saturated between $x$ and $y$, $M$-exposed between $y$ and $x$, or $M$-wedge from $y$ to $x$, according to the cases with $x \in A$ and $y \in A$, with $x \in A$ and $y \in B$, with $x \in B$ and $y \in A$, or with $x \in Y$ and $y \in Y$, respectively. Thus, the statements hold for $D_1$ and $D_k$ for this case.

Next consider the case where there exists $uv \in E(G) \setminus M$ with $u \in V(D_{k-1}) \cap B$ and $v \in V(D_k) \cap B$. In this case, the statements are also proved to hold for $D_1$ and $D_k$, in the similar way as the above. This completes the proof.

From Lemma 4.19, we can now prove that $(\mathcal{G}^+(G, b), \preceq_W)$ forms a poset for each $W \subseteq \{A, B\}$.

Theorem 4.20. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Then, the binary relation $\preceq_A$ is a partial order over $\mathcal{G}^+(G, b)$.

Proof. As reflexivity and transitivity are obvious from the definition, we prove antisymmetry in the following. Let $C_1, C_2 \in \mathcal{G}^+(G, b)$ be such that $C_1 \preceq_A C_2$ and $C_2 \preceq_A C_1$. Let $D_1, \ldots, D_k \in \mathcal{G}^+(G, b)$, where $k \geq 1$, be such that $C_1 = D_1$, $C_2 = D_k$, and $D_1 \preceq_{A} \cdots \preceq_{A} D_k$. Let $D_{l}, \ldots, D_1 \in \mathcal{G}^+(G, b)$, where $l \geq k$ be such that $D_{l} = D_1$ and $D_k \preceq_{A} \cdots \preceq_{A} D_{l}$. Suppose antisymmetry fails, that is, suppose $C_1 \neq C_2$. Then, we can suppose $l \geq 3$. Without loss of generality, we can assume $D_i \neq D_{i+1}$ for each $i \in \{1, \ldots, l-1\}$. Let $p \in \{3, \ldots, l\}$ be the smallest number with $D_p \in \{D_1, \ldots, D_{p-1}\}$. Let $q \in \{1, \ldots, p-1\}$ be such that $D_p = D_q$. Note that $D_q \neq D_{p-1}$ and $D_q \preceq_{A} \cdots \preceq_{A} D_{p-1} \preceq_{A} D_p$ hold.

First consider the case where $D_{p-1} \preceq_{A} D_2$ is given by an edge $uv \in M$. If $u \in V(D_{p-1}) \cap A$ and $v \in V(D_p) \cap B$. From Lemma 4.19 as $D_q \preceq_{A} \cdots \preceq_{A} D_{p-1}$ holds, there is an $M$-exposed path $P$ between $u$ and $v$. Then, $P + uv$ is an $M$-alternating circuit that shares some vertices with more than one flexible component. Therefore, $M \Delta E(P + uv)$ is a maximum $b$-matching of $G$, which excludes some inevitable edges or contains some forbidden edges. This is a contradiction.

Next consider the case where there is an edge $uv \in E(G) \setminus M$ with $u \in V(D_{p-1}) \cap B$ and $v \in V(D_p) \cap A$. In this case, take $P$ as an $M$-saturated path between $u$ and $v$. Then, $P + uv$ is again an $M$-alternating circuit, and in the same way, we reach a contradiction. Therefore, we obtain $C_1 = C_2$, and this completes the proof.

4.4. Extension over All Flexible Components. In this section, we prove, in Theorem 4.23 the canonical partially ordered structure over the set of all flexible components. From Lemma 4.12 if $D_1, \ldots, D_k \in \mathcal{G}(G, b)$, where $k \geq 1$, with $D_1 \preceq_{A} \cdots \preceq_{A} D_k$ satisfy $D_1, D_2 \in \mathcal{G}^+(G, b)$, then $D_k \in \mathcal{G}^+(G, b)$ holds for each $i \in \{1, \ldots, k\}$. Therefore, the definition of $\preceq_W$ can be compatibly extended over $\mathcal{G}^+(G, b)$ as follows.
Definition 4.21. Let \( W \in \{A, B\} \). We define a binary relation \( \preceq_W \) over \( G(G, b) \) as follows: For \( C_1, C_2 \in G(G, b) \), \( C_1 \preceq_W C_2 \) holds if there exist \( D_1, \ldots, D_k \in G(G, b) \), where \( k \geq 1 \), with \( D_1 \preceq_W \cdots \preceq_W D_k \).

The next lemma follows from Lemmas 4.11 and 4.12.

Lemma 4.22. Let \( G \) be a bipartite graph with color classes \( A \) and \( B \), and let \( b : V(G) \to \mathbb{Z}_{\geq 0} \). If \( C \in G^+(G, b) \) satisfies \( D \preceq_A C \) for \( D \in G(G, b) \setminus \{C\} \), then \( D \) is an inactive flexible component hooked up by \( C \) and \( A \) is a loose flexible component hooked up by \( A \) with \( N_G(D) \cap V(C) \neq \emptyset \). If \( C \in G^+(G, b) \) satisfies \( C \preceq_B D \) for \( D \in G(G, b) \setminus \{C\} \), then \( D \) is an inactive flexible component hooked up by \( B \) and \( C \) is a loose flexible component hooked up by \( B \) with \( N_G(D) \cap V(C) \neq \emptyset \).

From Lemma 4.22 and Theorem 4.20 we derive Theorem 4.23.

Theorem 4.23. Let \( G \) be a bipartite graph with color classes \( A \) and \( B \), and let \( b : V(G) \to \mathbb{Z}_{\geq 0} \). Then, the binary relation \( \preceq_A \) is a partial order over \( G(G, b) \).

Proof. Reflexivity and transitivity are obvious from the definition, hence we prove antisymmetry in the following. Let \( C_1, C_2 \in G^+(G, b) \) be such that \( C_1 \preceq_A C_2 \) and \( C_2 \preceq_A C_1 \). Let \( D_1, \ldots, D_k \in G^+(G, b) \), where \( k \geq 1 \), be such that \( C_1 = D_1, C_2 = D_k \), and \( D_1 \preceq_A \cdots \preceq_A D_k \). Let \( D_1, \ldots, D_k \in G^+(G, b) \), where \( l \geq k \) be such that \( D_l = C_1 \) and \( D_k \preceq_A \cdots \preceq_A D_l \). If all \( D_1, \ldots, D_l \) are consistent flexible components, then, from Theorem 4.20 we obtain \( C_1 = C_2 \). Hence, in the following, consider the case where \( \{D_1, \ldots, D_l\} \) has some inconsistent flexible components. Assume \( D_i \in G^+_A(G, b) \) for some \( i \in \{1, \ldots, l\} \). Then, Lemma 4.22 implies \( D_1, \ldots, D_l \in G^+_A(G, b) \). As \( D_1 = D_i \), this implies \( D_i \in G^+_A(G, b) \), which further implies \( D_1, \ldots, D_l \in G^+_A(G, b) \). If \( |D_1, \ldots, D_l| > 1 \), then, from Lemma 4.22 \( D_1 \) is an inactive flexible component and \( D_i \) is a loose flexible component. That is, \( D_1 \neq D_i \), which is a contradiction. Hence, we obtain \( |D_1, \ldots, D_l| = 1 \), namely, \( C_1 = C_2 \). We can also obtain \( C_1 = C_2 \) for the counterpart case, where \( D_i \in G^+_B(G, b) \), by a similar argument. This completes the proof.

5. Characterization of Verifying Sets

This section is devoted to obtain Theorem 5.3 which characterizes the family of verifying sets under Theorem 4.23 using the concept of normalized ideals. Throughout this section, unless otherwise stated, let \( G \) be a bipartite graph with color classes \( A \) and \( B \), and let \( b : V(G) \to \mathbb{Z}_{\geq 0} \). Note that, as the roles of \( A \) and \( B \) are given arbitrarily, every statement also holds by swapping \( A \) and \( B \).

From Lemma 4.11 it is easy to observe the following lemma:

Lemma 5.1. Let \( G \) be a bipartite graph with color classes \( A \) and \( B \), and let \( b : V(G) \to \mathbb{Z}_{\geq 0} \). Let \( C \in G(G, b) \). For any verifying set \( Z \subseteq V(G) \) of \( G \), either one of the following holds:

(i) \( V(C) \cap A \subseteq Z \) and \( V(C) \cap B \subseteq Z^c \); or,

(ii) \( V(C) \cap A \subseteq Z^c \) and \( V(C) \cap B \subseteq Z \).

The next lemma is easy to confirm from Lemma 4.22.

Lemma 5.2. Let \( G \) be a bipartite graph, and let \( b : V(G) \to \mathbb{Z}_{\geq 0} \). Let \( M \) be a maximum \( b \)-matching of \( G \). Let \( x, y \in V(G) \). If \( x \in Z \) and \( xy \in E(G) \setminus M \) hold, then \( y \in Z^c \) holds. If \( x \in Z^c \) and \( xy \in M \) hold, then \( y \in Z \) holds.

We define the normalized upper and lower ideals in the poset \( (G(G, b), \preceq_W) \), where \( W \in \{A, B\} \), in a similar way to those defined in Section 2.
Definition 5.3. A lower ideal $\mathcal{I}$ of the poset $(G(b), \preceq_A)$ is normalized if $G^-(A, b) \subseteq \mathcal{I}$ and $G^-(B, b) \cap \mathcal{I} = \emptyset$. An upper ideal $\mathcal{J}$ of $(G(b), \preceq_A)$ is normalized if $G^+(A, b) \cap \mathcal{I} = \emptyset$ and $G^+(B, b) \subseteq \mathcal{I}$.

As $G^-(A, b) \cap G^+(B, b) = \emptyset$, we have the following lemma:

Lemma 5.4. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. If $\mathcal{I}$ is a normalized lower ideal of the poset $(G(b), \preceq_A)$, then $G(G(b), \mathcal{I})$ is normalized lower ideal of $(G(b), \preceq_A)$. If $\mathcal{J}$ is a normalized upper ideal of the poset $(G(b), \preceq_A)$, then $G(G(b), \mathcal{J})$ is a normalized lower ideal of $(G(b), \preceq_A)$.

Remark 5.5. From Lemma 4.22, a lower ideal (resp. an upper ideal) $\mathcal{I}$ of the poset $(G(b), \preceq_A)$ is normalized if and only if there exists a lower (resp. an upper) ideal $\mathcal{J}$ such that $\mathcal{I} = \mathcal{J} \cup G^-(A, b)$ (resp. $\mathcal{I} = \mathcal{J} \cup G^+(B, b)$).

The next lemma provides the sufficiency part of Theorem 5.8.

Lemma 5.6. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $Z \subseteq V(G)$ be a verifying set. Then, there exist a complementary pair of normalized lower and upper ideals $\mathcal{I}_A$ and $\mathcal{I}_B$ of the poset $(G(b), \preceq_A)$ such that $Z = Z_A \cup Z_B$, and $Z_A$ and $Z_B$ are the projective unions of $\mathcal{I}_A$ and $\mathcal{I}_B$ over $A$ and $B$, respectively.

Proof. Let $\mathcal{I}_A$ and $\mathcal{I}_B$ be the sets of flexible components that have some vertices in $Z \cap A$ and $Z \cap B$, respectively. By this definition, if we let $Z_A$ and $Z_B$ be the projective unions of $\mathcal{I}_A$ and $\mathcal{I}_B$ over $A$ and $B$, then $Z = Z_A \cup Z_B$. In the following, we prove that $\mathcal{I}_A$ and $\mathcal{I}_B$ forms a complementary pair of normalized lower and upper ideals. First we prove that $\mathcal{I}_A$ is a lower ideal. Let $C \in \mathcal{I}_A$, and let $D \in G(G(b), \mathcal{I}_A)$ be such that $D \preceq_A C$. We prove $D \in \mathcal{I}_A$. By the definition of $\preceq_A$, to prove this lemma, it suffices to consider the case where there is an edge $uv \in M$ with $u \in V(D) \cap A$ and $v \in V(C) \cap B$ and the case where there is an edge $uv \in E(G) \setminus M$ with $u \in V(D) \cap B$ and $v \in V(C) \cap A$. As for the first case, Lemma 5.1 implies $V(C) \cap B \subseteq Z_v$, and accordingly $v \in Z_v$. This further implies, from Lemma 5.2, $u \in Z$. Therefore, from Lemma 5.1 again, we obtain $V(D) \cap A \subseteq Z$. Thus, $D \in \mathcal{I}_A$ is obtained. The other case is also proved by a similar argument. Hence, $\mathcal{I}_A$ is a lower ideal in $(G(b), \preceq_A)$; moreover, from Theorem 5.8, $\mathcal{I}_A$ is normalized.

From Lemma 5.1 $\mathcal{I}_A \cup \mathcal{I}_B = G(G(b), \mathcal{I})$. Therefore, from Lemma 5.4, $\mathcal{I}_B$ is a normalized upper ideal of $(G(b), \preceq_A)$. This completes the proof.

The next lemma is the necessity part of Theorem 5.8.

Lemma 5.7. Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \to \mathbb{Z}_{\geq 0}$. Let $\mathcal{I}_A$ and $\mathcal{I}_B$ be a complementary pair of normalized lower- and upper-ideals of the poset $(G(b), \preceq_A)$. Let $Z_A$ and $Z_B$ be the projective unions of $\mathcal{I}_A$ and $\mathcal{I}_B$ over $A$ and $B$, respectively. Then, $Z_A \cup Z_B$ is a verifying set of $G$.

Proof. Let $Z := Z_A \cup Z_B$. Let $M$ be an arbitrary maximum $b$-matching of $G$.

First, note that all $M$-loose vertices are contained in $Z$, because $\mathcal{I}_A$ and $\mathcal{I}_B$ are normalized. Second, we prove $E[Z] \subseteq M$. Suppose there is an edge $uv \in E[Z] \setminus M$, with $u \in Z_A$ and $v \in Z_B$. Then, there exist $C \in \mathcal{I}_A$ and $D \in \mathcal{I}_B$ such that $u \in V(C)$ and $v \in V(D)$. This implies $D \preceq_A C$, which contradicts $\mathcal{I}_A$ being a lower ideal. Hence, we obtain $E[Z] \subseteq M$.

Thirdly, we prove $E[Z] \cap M = \emptyset$. Suppose there is an edge $uv \in E[Z] \cap M$. This case can be also proved in the same way as the above. Finally, from Lemma 4.3 we obtain that $Z$ is a verifying set.
Combining Lemmas 5.3 and 5.4, we now obtain the characterization of the verifying sets as follows:

**Theorem 5.8.** Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \rightarrow \mathbb{Z}_{\geq 0}$. A set of vertices $Z \subseteq V(G)$ is a verifying set if and only if there is a complementary pair of normalized lower ideal $\mathcal{I}_A$ and upper ideal $\mathcal{I}_B$ of the poset $(G(G, b), \preceq_A)$ such that $Z = Z_A \cup Z_B$, where $Z_A$ and $Z_B$ are the projective unions of $\mathcal{I}_A$ and $\mathcal{I}_B$ over $A$ and $B$, respectively.

6. **Algorithm for Computing the $b$-Matching Dulmage-Mendelsohn Decomposition**

6.1. **General Statements.** In Section 6 we provide an algorithm to compute the $b$-matchings Dulmage-Mendelsohn decomposition. That is to say, we show that, given a bipartite graph, the set of flexible components and the poset can be computed in strongly polynomial time. This algorithm first obtains an arbitrary maximum $b$-matching $M$ and then constructs a certain kind of auxiliary digraphs using $M$. In the remainder of Section 6, let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \rightarrow \mathbb{Z}_{\geq 0}$.

**Definition 6.1.** Given a set of edges $M \subseteq E(G)$, the digraph $\text{Aux}(G; A, B; M)$ is defined as follows:

(i) $V(\text{Aux}(G; A, B; M)) := V(G)$;
(ii) $uv$ is an arc of $\text{Aux}(G; A, B; M)$ if $uv \in M$ holds for $u \in A$ and $v \in B$;
(iii) $uv$ is an arc of $\text{Aux}(G; A, B; M)$ if $uv \in E(G) \setminus M$ holds for $u \in A$ and $v \in B$.

We will construct $\text{Aux}(G; A, B; M)$ to determine the inconsistent unit $\Sigma_A(G)$, and then $\text{Aux}(G; B, A; M)$ to compute $\Sigma_B(G)$, and the consistent flexible components and the poset.

6.2. **Computing Inconsistent Flexible Components.** The next lemma immediately follows from Lemma 1.6

**Lemma 6.2.** Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \rightarrow \mathbb{Z}_{\geq 0}$. Let $M$ be a maximum $b$-matching of $G$. Let $U_A$ be the set of $M$-loose vertices in $A$. Then, $\Sigma_A(G)$ is equal to the set of vertices that can be reached by directed paths from $U_A$ in $\text{Aux}(G; A, B; M)$.

6.3. **Computing Consistent Flexible Components.** In Section 6.3 we show how to compute the consistent flexible components and the poset, by revealing their relationship with the strongly connected components decomposition of the auxiliary digraph.

**Definition 6.3.** Let $M$ be a maximum $b$-matching of $G$. A path $P$ with ends $u \in V(G)$ and $v \in V(G)$ is $(M; A, B)$-ascending from $u$ to $v$ if $P$ satisfies the following:

(i) If $u \in A$ and $v \in A$ hold, then $P$ is an $M$-wedge path from $u$ to $v$;
(ii) If $u \in A$ and $v \in B$ hold, then $P$ is an $M$-saturated path between $u$ and $v$;
(iii) If $u \in B$ and $v \in A$ hold, then $P$ is an $M$-exposed path between $u$ and $v$;
(iv) If $u \in B$ and $v \in B$ hold, then $P$ is an $M$-wedge path from $u$ to $v$.

The next lemma states the converse of Lemma 1.19

**Lemma 6.4.** Let $G$ be a bipartite graph with color classes $A$ and $B$, and let $b : V(G) \rightarrow \mathbb{Z}_{\geq 0}$. Let $M$ be a maximum $b$-matching of $G$. Let $C, D \in G^+(G, b)$, and let $u \in V(C)$ and $v \in V(D)$. If there is an $(M; A, B)$-ascending path from $u$ to $v$, then $C \preceq_A D$ holds.
Proof. We proceed by induction on \(|E(P)|\). If \(|E(P)| = 0\), then the statement trivially holds. Next assume \(|E(P)| > 1\), and the lemma holds for any case where \(|E(P)|\) is less. Let \(w \in V(P)\) be such that \(vw \in E(P)\). Let \(D'\) be the flexible component with \(w \in V(D')\). In the cases (i) and (iii), \(v \in A \cap V(D)\) and \(w \in B \cap V(D')\) hold, whereas in the cases (ii) and (iv) \(v \in B \cap V(D)\) and \(w \in A \cap V(D')\) hold. Therefore, in every case, \(D' \sqsupseteq_A D\) holds. On the other hand, \(P - vw\) is a path shorter than \(P\) that is \(M\)-saturated between \(u\) and \(w\), or \(M\)-wedge from \(u\) to \(w\), \(M\)-saturated between \(u\) and \(w\), or \(M\)-exposed between \(u\) and \(w\), according to the cases (i) (ii) (iii) (iv) respectively. Therefore, by the induction hypothesis, \(C \sqsubseteq_A D\) holds. Thus, we have \(C \sqsubseteq_A D\).

Combining Lemmas 4.19 and 6.4, the next lemma follows:

**Lemma 6.5.** Let \(G\) be a bipartite graph with color classes \(A\) and \(B\), and let \(b : V(G) \to \mathbb{Z}_{\geq 0}\). Let \(M\) be a maximum \(b\)-matching of \(G\). Let \(C, D \in G^+(G, b)\). Then, the following three properties are equivalent:

(i) \(C \sqsubseteq_A D\) holds;

(ii) for any \(u \in V(C)\) and any \(v \in V(D)\), there is an \((M; A, B)\)-ascending path from \(u\) to \(v\);

(iii) there exist \(u \in V(C)\) and \(v \in V(D)\) such that there is an \((M; A, B)\)-ascending path from \(u\) to \(v\).

Given a digraph \(D\), denote \(u \to v\) for \(u, v \in V(D)\) if there is a directed path from \(u\) to \(v\). Then, \(\to\) is a pseudo order over \(V(D)\) and is naturally reduced to a partial order over the set of strongly connected components of \(D\). We also denote this reduced partial order by \(\to\).

Under Lemma 6.5, the consistent \(b\)-flexible components of \(G\) and the strongly connected components of \(\text{Aux}(G; B, A; M)\) are associated as follows:

**Lemma 6.6.** Let \(G\) be a bipartite graph with color classes \(A\) and \(B\), and let \(b : V(G) \to \mathbb{Z}_{\geq 0}\). Let \(M\) be a maximum \(b\)-matching of \(G\). Let \(V_0 := V(G) \setminus \Sigma_A(G) \setminus \Sigma_B(G)\), and let \(V_0 \neq 0\). Let \(C_1, \ldots, C_k\), where \(k \geq 1\), be the strongly connected components of the digraph \(\text{Aux}(G[V_0]; B \cap V_0, A \cap V_0, M \cap E[V_0])\). Then, the family \(\{V(C_i) : i = 1, \ldots, k\}\) is equal to the family \(\{V(H) : H \in G^+(G, b)\}\). Additionally, \(C_i \to C_j\) holds for \(i, j \in \{1, \ldots, k\}\) if and only if \(H \sqsubseteq_A I\) holds for \(H, I \in G^+(G, b)\), where \(V(C_i) = V(H)\) and \(V(C_j) = V(I)\).

### 6.4. Concluding Algorithms

Combining results in preceding sections, we now obtain the following:

**Theorem 6.7.** Let \(G\) be a bipartite graph, and let \(b : V(G) \to \mathbb{Z}_{\geq 0}\). Given a maximum \(b\)-matching of \(G\), the \(b\)-matching Dulmage-Mendelsohn decomposition can be computed in \(O(n + m)\) time, where \(n = |V(G)|\) and \(m = |E(G)|\).

*Proof.* See Algorithm 1 in the table. The correctness follows from Lemmas 6.2 and 6.6. Each of Lines 1 to 5 and Lines 7 to 20 in total can obviously done in \(O(n + m)\) time; as the strongly connected component decomposition of a digraph can be computed in linear time (see, e.g., Cormen et al. [1]), Line 6 can be also computed in \(O(n + m)\) time. □

As a maximum \(b\)-matching of a graph can be computed in strongly polynomial time (see Schrijver [8], which lists various kinds of such algorithms), Theorem 6.7 implies the following:

**Theorem 6.8.** Give a bipartite graph and a mapping \(b : V(G) \to \mathbb{Z}_{\geq 0}\), the \(b\)-matching Dulmage-Mendelsohn decomposition can be computed in strongly polynomial time.
**Algorithm 1** The $b$-Matching Dulmage-Mendelsohn Decomposition

**Require:** a bipartite graph $G$ with color classes $A$ and $B$, a mapping $b : V(G) \rightarrow \mathbb{Z}_{\geq 0}$, a maximum $b$-matching $M$

**Ensure:** the $b$-matching Dulmage-Mendelsohn decomposition of $G$

1: compute $\text{Aux}(G; A, B; M)$; compute $\Sigma_A(G)$;
2: compute $\text{Aux}(G; B, A; M)$; compute $\Sigma_B(G)$;
3: let $I_A$ and $I_B$ be the sets of inactive vertices in $\Sigma_A(G)$ and $\Sigma_B(G)$, respectively;
4: recognize $G[v]$ as an inactive flexible component in $\mathcal{G}_A^-(G, b)$ or $\mathcal{G}_B^-(G, b)$, for each $v \in I_A \cup I_B$; recognize $G[V(C)]$ as a member of $\mathcal{G}_A^+(G, b)$ or $\mathcal{G}_B^+(G, b)$ for each connected component $C$ of $G[\Sigma_A(G) \setminus I_A]$ and $G[\Sigma_B(G) \setminus I_B]$, respectively;
5: compute $D := \text{Aux}(G[V_0]; B \cap V_0, A \cap V_0; M \cap E[V_0])$, where $V_0 = V(G) \setminus \Sigma_A(G) \setminus \Sigma_B(G)$;
6: compute the strongly connected component decomposition of $D$; recognize $G[V(C)]$ as a member of $\mathcal{G}_A^+(G, b)$ for each strongly connected component $C$; let $C_1 \preceq_A C_2$ for each pair of strongly connected components $C_1$ and $C_2$ with $C_1 \rightarrow C_2$;
7: for all $C \in \mathcal{G}_A^+(G, b)$ do
8: for all $D \in \mathcal{G}_A^+(G, b)$ with $N_C(C) \cap V(D) \neq \emptyset$ do
9: let $C \preceq_A D$;
10: end for
11: end for
12: for all $C \in \mathcal{G}_B^+(G, b)$ do
13: for all $D \in \mathcal{G}(G, b) \setminus \mathcal{G}_B^+(G, b)$ with $N_C(C) \cap V(D) \neq \emptyset$ do
14: let $D \preceq_A C$;
15: end for
16: end for
17: for all inactive flexible component $C$ hooked up by $A$ do
18: for all $D \in \mathcal{G}_A^+(G, b)$ with $N_C(C) \cap V(D) \neq \emptyset$ do
19: let $C \preceq_A D$;
20: end for
21: end for
22: for all inactive flexible component $C$ hooked up by $B$ do
23: for all $D \in \mathcal{G}_B^+(G, b)$ with $N_C(C) \cap V(D) \neq \emptyset$ do
24: let $D \preceq_A C$;
25: end for
26: end for

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