Perturbed Self-Similar Massless Scalar Field in the Spacetimes with Circular Symmetry in 2 + 1 Gravity

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We present in this work the study of the linear perturbations of the 2 + 1-dimensional circularly symmetric solution, obtained in a previous work, with kinematic self-similarity of the second kind. We have obtained an exact solution for the perturbation equations and the possible perturbation modes. We have shown that the background solution is a stable solution.

I. INTRODUCTION

One of the most outstanding problems in gravitation theory is the study of the relation that exists between the critical phenomena and the process of black hole formation. The studies of non-linearity of the Einstein field equations near the threshold of black hole formation reveal very rich phenomena [1], which are quite similar to critical phenomena in Statistical Mechanics and Quantum Field Theory [2]. In particular, by numerically studying the gravitational collapse of a massless scalar field in 3 + 1-dimensional spherically symmetric spacetimes, Choptuik found that the mass of such formed black holes takes a scaling form,

$$M_{BH} = C(p)(p - p^*)^\gamma,$$

where $C(p)$ is a constant and depends on the initial data, and $p$ parameterises a family of initial data in such a way that when $p > p^*$ black holes are formed, and when $p < p^*$ no black holes are formed. It was shown that, in contrast to $C(p)$, the exponent $\gamma$ is universal to all the families of initial data studied. Numerically it was determined as $\gamma \sim 0.37$. The solution with $p = p^*$, usually called the critical solution, is found also universal. Moreover, for the massless scalar field it is periodic, too. Universality of the critical solution and exponent, as well as the power-law scaling of the black hole mass all have given rise to the name Critical Phenomena in Gravitational Collapse. Choptuik’s studies were soon generalised to other matter fields [3,4], and now the following seems clear: (a) There are two types of critical collapse, depending on whether the black hole mass takes the scaling form (1) or not. When it takes the scaling form, the corresponding collapse is called Type II collapse, and when it does not it is called Type I collapse. In the type II collapse, all the critical solutions found so far have either discrete self-similarity (DSS) or homothetic self-similarity (HSS), depending on the matter fields. In the type I collapse, the critical solutions have neither DSS nor HSS. For certain matter fields, these two types of collapse can co-exist. (b) For Type II collapse, the corresponding exponent is universal only with respect to certain matter fields. Usually, different matter fields have different critical solutions and, in the sequel, different exponents. But for a given matter field the critical solution and the exponent are universal. So far, the studies have been mainly restricted to spherically symmetric case and their non-spherical linear perturbations. Therefore, it is not really clear whether or not the critical solution and exponent are universal with respect to different symmetries of the spacetimes [5,6]. (c) A critical solution for both of the two types has one and only one unstable mode. This now is considered as one of the main criteria for a solution to be critical. (d) The universality of the exponent is closely related to the last property. In fact, using dimensional analysis [7] one can show that

$$\gamma = \frac{1}{|k|},$$

where $k$ denotes the unstable mode.

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From the above, one can see that to study (Type II) critical collapse, one may first find some particular solutions by imposing certain symmetries, such as, DSS or HSS. Usually this considerably simplifies the problem. For example, in the spherically symmetric case, by imposing HSS symmetry the Einstein field equations can be reduced from PDE’s to ODE’s. Once the particular solutions are known, one can study their linear perturbations and find out the spectrum of the corresponding eigenmodes. If a solution has one and only one unstable mode, by definition we may consider it as a critical solution (See also the discussions given in [8]). The studies of critical collapse have been mainly numerical so far, and analytical ones are still highly hindered by the complexity of the problem, even after imposing some symmetries.

Lately, Pretorius and Choptuik (PC) [9] studied gravitational collapse of a massless scalar field in an anti-de Sitter background in 2 + 1-dimensional spacetimes with circular symmetry, and found that the collapse exhibits critical phenomena and the mass of such formed black holes takes the scaling form of Eq.(1) with $\gamma \sim 1.2 \pm 0.02$, which is different from that of the corresponding 3 + 1-dimensional case. In addition, the critical solution is also different, and, instead of having DSS, now has HSS. The above results were confirmed by independent numerical studies [10]. However, the exponent obtained by Husain and Olivier (HO), $\gamma \sim 0.81$, is quite different from the one obtained by PC. It is not clear whether the difference is due to numerical errors or to some unknown physics.

After the above numerical work, analytical studies of the same problem soon followed up [11–14]. In particular, Garfinkle found a class, say, $S[n]$, of exact solutions to the Einstein-massless scalar field equations and showed that in the strong field regime the $n = 4$ solution fits very well with the numerical critical solution found by PC. Lately, Garfinkle and Gundlach (GG) studied their linear perturbations and found that only the solution with $n = 2$ has one unstable mode, while the one with $n = 4$ has three [13]. According to Eq.(2), the corresponding exponent is given by $\gamma = 1/|k| = 4/3$. Independently, Hirschmann, Wu and one of the present authors (HWW) systematically studied the problem, and found that the $n = 4$ solution indeed has only one unstable mode [14]. This difference actually comes from the use of different boundary conditions. As a matter of fact, in addition to the ones imposed by GG [13], HWW further required that no matter field should come out of the already formed black holes. This additional condition seems physically quite reasonable and has been widely used in the studies of black hole perturbations [15]. However, now the corresponding exponent is given by $\gamma = 1/|k| = 4$, which is significantly different from the numerical ones. So far, no explanations about these differences have been worked out, yet.

Self-similarity is usually divided into two classes, one is the discrete self-similarity mentioned above, and the other is the so-called kinematic self-similarity (KSS) [16], and sometimes it is also called continuous self-similarity (CSS). KSS or CSS is further classified into three different kinds, the zeroth, first and second. The kinematic self-similarity of the first kind is also called homothetic self-similarity, first introduced to General Relativity by Cahill and Taub in 1971 [17]. In Statistical Mechanics, critical solutions with KSS of the second kind seem more generic than those of the first kind [2]. However, critical solutions with KSS of the second kind have not been found so far in gravitational collapse, and it would be very interesting to look for such solutions. We shall present in this work the study of the linear perturbations of the 2 + 1-dimensional circularly symmetric solution, obtained in a previous work [18], with kinematic self-similarity of the second kind and show that the background solution is not critical.

In Section II we present the field equations with kinematic self-similarity of the second kind. In Section III we perturb linearly the field equations. In Section IV we present the solution of the linear perturbation equations. In Section V we apply the boundary conditions in the perturbed solutions and in Section VI we conclude our work.

II. THE FIELD EQUATIONS WITH KINEMATIC SELF-SIMILARITY OF THE SECOND KIND

The general metric for such spacetimes can be written in the form

$$ds^2 = e^{2\Phi(r,t)} dt^2 - e^{2\Psi(r,t)} dr^2 - r^2 S(r,t)^2 d\theta^2. \quad (3)$$

Then, the corresponding non-vanishing components of the Ricci tensor are

$$R_{tt} = e^{2(\Phi - \Psi)} \left[ \Phi_t \left( \Phi_r - \Psi_r + \frac{S_r}{S} + \frac{1}{r} \right) + \Phi_{rr} \right] - \frac{S_t}{S} + \frac{S_t}{S} S_t + \Phi_t \Psi_t - \Psi_t^2 - \Psi_{tt};$$

$$R_{tr} = \frac{\Psi_t}{r} + \Phi_t \frac{S_t}{S} + \Phi_r \frac{S_t}{S} - \frac{S_t}{rS} - \frac{S_t}{S} S_t$$

$$R_{rr} = e^{2(\Psi - \Phi)} \left[ \Psi_{tt} + \Psi_t \left( \Psi_t + \frac{S_t}{S} \Phi_t \right) \right] - \Phi_{rr} + \Phi_r \Psi_r - \Phi_r^2 - \frac{S_t}{S} S_t - 2 \frac{S_t}{rS} + \Phi_t \frac{S_t}{S} + \frac{\Psi_r}{r}$$

$$R_{\theta\theta} = r^2 S_t^2 \left( e^{-2\Phi} \left[ \frac{S_t}{S} (\Psi_t - \Phi_t) + \frac{S_t}{S} S_t \right] - e^{-2\Psi} \frac{S_t}{S} (2 + \frac{S_t}{S}) \right). \quad (4)$$
where the indices $t$ and $r$ denote differentiation to the time coordinate and the radial coordinate, respectively.

Then, introduce two self-similar variables $\tau$ and $x$ via the relations

$$x = \ln \left( \frac{r}{(-t)^{\frac{1}{\alpha}}} \right), \quad \tau = -\ln (-t),$$

or inversely,

$$r = e^{(\alpha x - \tau)/\alpha}, \quad t = -e^{-\tau},$$

where $\alpha$ is a dimensionless constant. For any given function $f(t, r)$, we have

$$f_s = -\frac{1}{\alpha t} (\alpha f_{\tau} + f_x), \quad f_r = \frac{1}{r} f_x,$$

$$f_{ss} = -\frac{1}{\alpha tr} (\alpha f_{\tau \tau} + f_{xx}), \quad f_{rr} = \frac{1}{r^2} (f_{xx} - f_x),$$

$$f_{tt} = \frac{1}{\alpha^2 t^2} (\alpha^2 f_{\tau \tau} + 2\alpha f_{\tau x} + f_{xx} + \alpha^2 f_{\tau} + \alpha f_x),$$

where the comma means differentiation.

Substituting these equations into Eq. (4), we find that in terms of the self-similar variables, the Ricci tensor is given by

$$R_{tt} = \frac{e^{2(\Phi - \Psi)}}{r^2} \left\{ \Phi_x (\Phi_x - \Psi_x + \frac{S_x}{S}) + \Phi_{xx} \right\} - \frac{1}{\alpha^2 t^2} \left\{ \alpha^2 \left[ \Psi_{\tau \tau} + \psi_r(1 + \Psi_r - \Phi_r) + \frac{S_{\tau \tau}}{S} + \frac{S_{\tau}}{S} (1 - \Phi_r) \right] 
+ \alpha \left[ 2 \Psi_{\tau x} + \psi_r (\Psi_x - \Phi_x) + \psi_r (\Psi_r - \Phi_r + 1) + 2 \frac{S_{\tau x}}{S} + \frac{S_x}{S} (1 - \Phi_r) - \frac{S_{xx}}{S} \Phi_x \right] 
+ \left[ \Psi_{xx} + \psi_x (\Psi_x - \Phi_x) + \frac{S_{xx}}{S} - \frac{S_x}{S} \Phi_x \right] \right\},$$

$$R_{\tau r} = -\frac{1}{\alpha tr} \left\{ \alpha \left[ \Psi_r (1 + \frac{S_x}{S}) + \frac{S_{\tau}}{S} (\Phi_x - 1) - \frac{S_{\tau x}}{S} \right] + \psi_x \left[ 1 + \frac{S_x}{S} \right] + \frac{S_x}{S} (\Phi_x - 1) - \frac{S_{xx}}{S} \right\},$$

$$R_{\tau r} = \frac{e^{2(\Psi - \Phi)}}{\alpha^2 t^2} \left\{ \alpha^2 \left[ \Psi_{\tau \tau} + \psi_r \left( 1 + \psi_r + \frac{S_x}{S} - \Phi_r \right) \right] 
+ \alpha \left[ 2 \Psi_{\tau x} + \psi_r \left( 1 + \psi_r + \frac{S_x}{S} - \Phi_r \right) + \psi_x \left( \psi_x + \frac{S_x}{S} - \Phi_x \right) \right] 
+ \psi_{xx} + \psi_x \left( \psi_x + \frac{S_x}{S} - \Phi_x \right) \right\} + \frac{1}{r^2} \left[ \Phi_x (\Phi_x - \Phi_r + 1) - \Phi_{xx} + \psi_x \left[ 1 + \frac{S_x}{S} \right] - \frac{1}{S} (S_{xx} + S_x) \right],$$

$$R_{\theta \theta} = r^2 S^2 \left\{ \frac{e^{-2\Phi}}{\alpha^2 t^2 S} \left[ \alpha^2 (\psi_r (1 + \psi_r - \Phi_r) + \Psi_{\tau \tau}) + \alpha (\psi_r (\Psi_x - \Phi_x) + S_x (1 + \psi_r + \Phi_r) + 2 S_{\tau x}) + S_x (\Phi_x - \Phi_x + S_{xx}) + \frac{e^{-2\Phi}}{r^2} \left[ \frac{1}{S} (S_x (1 + \Phi_x - \Psi_x) + S_{xx}) + \Phi_x - \Psi_x \right] \right),$$

where the indices $\tau$ and $x$ mean differentiation in respect to these variables.

### III. THE LINEAR PERTURBATION OF THE FIELD EQUATIONS

Once we have the general expressions of $R_{\mu \nu}$ in terms of $\tau$ and $x$, let us consider the following perturbations,

$$\Phi(\tau, x) = \Phi_0(x) + e \Phi_1(x) e^{k \tau},$$

$$\Psi(\tau, x) = \Psi_0(x) + e \Psi_1(x) e^{k \tau},$$

$$S(\tau, x) = S_0(x) + e S_1(x) e^{k \tau},$$

$$\phi(\tau, x) = \phi_0(t) + e \phi_1(x) e^{k \tau},$$

(10)
where $\epsilon$ is a very small real constant, the quantities with subscripts “1” denote perturbations, those with “0” denote the background self-similar solutions.

The background solution is given by

\[
\begin{align*}
\Phi_0(x) &= 0, \\
\Psi_0(x) &= \frac{1}{2} \alpha x, \\
S_0(x) &= \frac{2}{2 - \alpha} e^{-\frac{1}{2} \alpha x}, \\
\phi_0(t) &= 2q \ln(-t),
\end{align*}
\]

and the apparent horizon is given by

\[
r_{AH}(t) = \left[ (2 - \alpha) (-t)^{1/2} \right]^{2/(2 - \alpha)}. \tag{11}
\]

where $\varphi_0$, $\psi_0$ and $s_0$ are integration constants, $q = \pm \frac{1}{\sqrt{2\alpha}}$ and $\alpha < 2$ [18].

It is understood that there may be many perturbation modes for different values (possibly complex) of the constant $k$. The general perturbation will be the sum of these individual modes. Those modes with $Re(k) > 0$ grow as $\tau \to \infty$ and are referred to as unstable modes, while the ones with $Re(k) < 0$ decay and are referred to as stable modes. By definition, critical solutions will have one and only one unstable mode.

Substituting Eq.(10) into Eq.(8) and then we will have

\[
R_{\mu\nu} = R_{\mu\nu}(\tau, x, \epsilon). \tag{13}
\]

Now considering $R_{\mu\nu}$ is function of $\epsilon$ only, then we expand it in terms of $\epsilon$,

\[
R_{\mu\nu}(\tau, x, \epsilon) = \frac{1}{(-t)^{1/2}} \left\{ R_{\mu\nu}^{(0)}(x) + \epsilon R_{\mu\nu}^{(1)}(x)e^{k\tau} + O(\epsilon^2) \right\}, \tag{14}
\]

where $R_{\mu\nu}^{(0)}(x)$ is the part of the Ricci tensor corresponding background, and $R_{\mu\nu}^{(1)}(x)$ the perturbation part, which is function of the background, $\Phi_0(x), \Psi_0, S_0(x)$ and the linear perturbations, $\Phi_1(x), \Psi_1, S_1(x)$. In the paper of Hirschmann, Wang & Wu [14] it was calculated them for the self-similar solutions of the first kind but in double null coordinates [cf. Eq.(65) given there].

To first order in $\epsilon$, it can be shown that the non-vanishing components of the Ricci tensor are given by

\[
R_{\mu\nu}^{(1)}(x) = e^{2(\Phi_0 - \Psi_0 + x/2)} \left\{ \Phi_0' \left( 2\Phi_1' - \Psi_1' \right) - \Phi_1' \Psi_0' + \Phi_1'' + 2 (\Phi_1 - \Psi_1) \left[ \Phi_0' \left( \Psi_0' - \Psi_0' \right) + \Phi_0'' \right] \\
+ \frac{1}{S_0} \left[ - \frac{S_0^{'}/S_0'}{S_0} \Phi_0' + 2 (\Phi_1 - \Psi_1) \Phi_0'S_0' + \Phi_0'S_1' + \Phi_1'S_0' \right] \right\} + \\
+ \frac{e^{2\tau}}{\alpha^2} \left\{ \Phi_0' \left( ak\Phi_1 + \Psi_1' \right) + \Psi_0' \left( ak\Phi_1 + \Phi_1' \right) - 2\Psi_0' \left( ak\Psi_1 + \Psi_1' \right) - \alpha^2 k^2 \Psi_1 - 2\alpha k \Psi_1' - \Psi_1'' \\
- \alpha^2 k \Psi_1 - \alpha \Psi_1' + \\
+ \frac{1}{S_0} \left[ \Phi_0' \left( akS_1 + S_1' \right) + S_0' \left( ak\Phi_1 + \Phi_1' \right) - \alpha^2 k^2 S_1 - 2\alpha k S_1' - S_1'' - \alpha^2 k S_1 - \alpha S_1' \right. \right. \\
\left. \left. - \frac{1}{S_0} \left( S_0'S_1\Phi_0' - S_0''S_1 - \alpha S_0'S_1 \right) \right] \right\} \tag{15}
\]

\[
R_{\tau\tau}^{(1)}(x) = -\frac{e^{2(\Psi_0 - \Phi_0 + x/2)} + x}{\alpha S_0} \left[ - \frac{S_1}{S_0} \left( S_0' - \Phi_0' S_0' - \Psi_0' S_0' + S_0'' - S_0 \Psi_0' \right) - \Phi_0' \left( akS_1 + S_1' \right) - \Psi_0' S_1' + \\
- S_0' \left( ak\Psi_1 + \Psi_1' + \Phi_1' \right) + akS_1 + S_1' - \Psi_0' S_1 - S_0 \left( ak\Psi_1 + \Psi_1' \right) + akS_1 + S_1' \right] \tag{16}
\]

\[
R_{\tau\tau}^{(1)}(x) = \frac{e^{2(\Psi_0 - \Phi_0 + x/2)}}{\alpha^2} \left[ 2 (\Psi_1 - \Phi_1) \left( \Psi_0' - \Phi_0' S_0' + \phi_0'S_0' \right) + 2ak\Psi_1 S_0' + 2\Psi_0' \Psi_1' + \alpha^2 k^2 \Psi_1 + \\
+ 2\alpha k \Psi_1' + \Psi_1'' - \alpha^2 k \Psi_1 + \alpha \Psi_1' - ak\Psi_1 \phi_0' - \Psi_0' \phi_0' + \phi_0' \Psi_0' + \phi_0' \phi_0' \right].
\]

\[
R_{\tau\tau}^{(1)}(x) = \frac{e^{2(\Psi_0 - \Phi_0 + x/2)}}{\alpha^2} \left[ 2 (\Psi_1 - \Phi_1) \left( \Psi_0' - \Phi_0' S_0' + \phi_0'S_0' \right) + 2ak\Psi_1 S_0' + 2\Psi_0' \Psi_1' + \alpha^2 k^2 \Psi_1 + \\
+ 2\alpha k \Psi_1' + \Psi_1'' - \alpha^2 k \Psi_1 + \alpha \Psi_1' - ak\Psi_1 \phi_0' - \Psi_0' \phi_0' + \phi_0' \Psi_0' + \phi_0' \phi_0' \right].
\]
\[
R_{\mu\nu}^{(1)}(x) = -e^{-2\phi_0} \left[ (S_1 - 2S_0 \Phi_1) (\Phi_0' S_0' + \Phi_0' S_0 - \Psi_0' S_0' - \Psi_0' S_0 + S_0' + S_0') + + S_0 (\Phi_0' S_1' + \Phi_0' S_1 + \Phi'_1 S_0 - \Psi_0' S_1' - \Psi_0' S_1 - \Psi'_1 S_0' - \Psi'_1 S_0 + S_1'' + S_1') \right] - e^{\alpha(x - r - x - \phi_0)} \left[ (S_1 - 2S_0 \Phi_1) (\Phi_0' S_0' - \Psi_0' S_0' - S_0'' - \alpha S_0') + + S_0 (\alpha k \Phi_1 S_1' + \Phi_0' S_1' + \alpha k \Psi_0' - \Psi_0' S_1' - \alpha k \Psi_0' - \Psi_0' S_1' - \alpha k \Psi_0' + - \Psi'_1 S_0' - \alpha k^2 S_1 - 2akS_1'' + 2\alpha k S_1' - S_1'' - \alpha^2 k S_1' - \alpha S_1') \right],
\]

where the prime denotes differentiation in respect to \( x \).

Once we have \( R_{\mu\nu}^{(1)}(x) \), we have to calculate the quantities
\[
A_{\mu\nu} = \phi_{\mu,\nu}.
\]

Substituting Eqs.(10) into the above equations, we have
\[
A_{\mu\nu}(x, t, \epsilon) = \frac{1}{(-t)^2} \left\{ A_{\mu\nu}^{(0)}(x) + \epsilon A_{\mu\nu}^{(1)}(x)e^{\epsilon \tau} + O(\epsilon^2) \right\},
\]

where \( A_{\mu\nu}^{(0)}(x) \) is the part of the background, and \( A_{\mu\nu}^{(1)}(x) \) the perturbation part, which is function of the background, \( \phi_0(t) \), and the linear perturbation, \( \phi_1(x) \), and given by
\[
A_{\mu\nu}^{(1)} = \frac{e^{2\tau}}{\alpha} \left\{ 4q (\alpha k \phi_1 + \phi'_1) \right\}
\]

\[
A_{\mu\nu}^{(1)}(x) = \frac{e^{\alpha(x - r - \phi_0)}}{\alpha} \left[ 2q \phi'_1 \right]
\]

\[
A_{\mu\nu}^{(1)}(x) = 0
\]

\[
A_{\mu\nu}^{(1)}(x) = 0
\]

where the dot means time differentiation.

Once we have \( A_{\mu\nu}^{(1)}(x) \) and \( R_{\mu\nu}^{(1)}(x) \), the linear perturbation equations are given by
\[
R_{\mu\nu}^{(1)}(x) = A_{\mu\nu}^{(1)}(x),
\]

which in general are complicated.

After we have the general linear perturbation equations (21), then we turn to consider the background solutions given by Eqs.(11). By virtue of the simple form of the solutions and the fact \( \phi_0(x) = 0 \), Eqs.(21) can be solved in our case.

\[
\alpha^2 k \Phi_1 + \alpha \Phi'_1 + \alpha^2 k^2 \Psi_1 + 2\alpha k \Psi_1' + \Psi'' + \left( \frac{2 - \alpha}{4} \right) e^{\phi_0} \left( \frac{2}{\alpha} k^2 + 2k + \frac{1}{2} \right) \alpha^2 S_1 + 2\alpha (2k + 1) S_1' + 2S_1'' = 0,
\]

\[
\Phi_1'' = 0,
\]

\[
(\alpha - 2) [\alpha k \Psi_1 + \Psi_1'' + \alpha \Phi_1' + \frac{2 - \alpha}{2} e^{\phi_0} \left( \alpha (2k + 1) S_1 + (2\alpha k + \alpha + 2) S_1' + 2S_1'' \right) = -2q \phi'_1,
\]

\[
\Phi_1'' = 0,
\]

\[
(\alpha - 2) \left[ \alpha k \Psi_1 + \Psi_1' + 2\Psi_1'' + \alpha (\alpha k \Phi_1 + \Phi_1') - \alpha e^{\phi_0} \left( \frac{2 - \alpha}{4} \right) \left( 2\alpha k + \alpha \right) S_1 + 2S_1'' \right] = 0,
\]
\[(2 - \alpha) \Psi' + (2 - \alpha) \Phi' - 2\Phi'' - \frac{(2 - \alpha)e^{\Psi x}}{2} \left[\alpha S_1 + (2 + \alpha) S'_1 + 2 S''_1\right] = 0, \quad (26)\]

\[\alpha (\alpha k \Psi_1 + \Psi'_1) - \alpha (\alpha k \Phi_1 + \Phi'_1) - \frac{(2 - \alpha)e^{\Psi x}}{2} \left[\alpha^2 k (1 + 2k) S_1 + \alpha (1 + 4k) S'_1 + 2 S''_1\right] = 0, \quad (27)\]

\[(2 - \alpha) \Psi' - (2 - \alpha) \Phi' - \frac{(2 - \alpha)e^{\Psi x}}{2} \left[\alpha S_1 + (2 + \alpha) S'_1 + 2 S''_1\right] = 0. \quad (28)\]

**IV. THE SOLUTIONS OF THE LINEAR PERTURBATION EQUATIONS**

We will solve the system of the perturbed Eqs.(22)-(28). From Eq.(23) we have

\[\Phi_1 = ax + b. \quad (29)\]

From Eqs.(26) and (28) we have

\[(2 - \alpha) \Phi'_1 = 0, \quad (30)\]

which solutions are \(\alpha = 2\) (which is out of range of our solution) or \(\Phi'_1 = 0\). Thus, from Eq.(29) we have that

\[\Phi_1 = b = \text{constant}. \quad (31)\]

Using Eq.(31) and summing the Eq.(26) and Eq.(28) we get

\[\Psi'_1 = \frac{\alpha e^{\Psi x}}{2} \left[\alpha S_1 + (2 + \alpha) S'_1 + 2 S''_1\right] \quad (32)\]

Using Eq.(31) and substituting Eq.(32) into Eq.(27) we get

\[\Psi_1 = b - \frac{\Psi'_1}{\alpha k} + \frac{(2 - \alpha)e^{\Psi x}}{2\alpha^2 k} \left[\alpha^2 k (1 + 2k) S_1 + \alpha (1 + 4k) S'_1 + 2 S''_1\right] \quad (33)\]

Substituting Eq.(32) into Eq.(33) and differentiating it, we have

\[A S_1 + BS'_1 + CS''_1 + 4S''_1 + E e^{\Psi x} = 0, \quad (34)\]

where

\[A = \frac{\alpha^2}{2}(-8\alpha k^3 + 4\alpha k + \alpha + 16k^3 - 4k), \quad (35)\]

\[B = \alpha(-8\alpha k^2 + 4\alpha k + 3\alpha + 16k^2), \quad (36)\]

\[C = 2(8\alpha k - \alpha - 4k + 4), \quad (37)\]

\[E = 4\alpha^2 k^2. \quad (38)\]

Since our background solution with second kind self-similarity is identical to the solution with self-similarity of the first kind [18], we will study hereinafter only the case \(\alpha = 1\). Thus,

\[A = 4k^3 + \frac{1}{2}, \quad (39)\]

\[B = 8k^2 + 4k + 3, \quad (40)\]

\[\]
\[ C = 8k + 6, \]

\[ E = 4b^2, \]  

and the solution of equation (34) is given by

\[ S_1(x) = -\frac{be^{-\frac{x}{k}}}{k - 1} + c_1e^{-\frac{x}{k}(1+2k)} + c_2e^{-\frac{x}{k}(k+1+\sqrt{\Delta})} + c_3e^{-\frac{x}{k}(k+1-\sqrt{\Delta})}, \]

where

\[ \Delta = -k(3k - 4). \]  

In the next section we will apply the boundary conditions for two special cases: \( \Delta > 0 \) and \( \Delta < 0 \).

\section*{V. THE BOUNDARY CONDITIONS FOR THE PERTURBED SOLUTIONS}

We will apply the boundary conditions only at two regions of the spacetime: at the centre of the spacetime \( r = 0 \) and at the event horizon \( r_{AH} \) given by equation (12), that furnishes

\[ r_{AH} = -t. \]

Thus, the metric at the apparent horizon is given by

\[ ds^2_{AH} = dt^2 - dr^2 - 4(-t)^2 d\theta^2. \]

It can be easily seen from this metric that the apparent horizon is singular, in this case, only at \( t = 0 \). Then the final state of the collapse is a marginally naked singularity.

We would like to note that for the perturbed part of the metric (3) to represent circular symmetry, some physical and geometrical conditions needed to be imposed [19]. For gravitational collapse, we impose the following conditions at \( r = 0 \):

(i) There must exist a symmetry axis, which can be expressed as

\[ X \equiv \left| \xi^\mu_{(\theta)} \xi^\nu_{(\theta)} g_{\mu\nu} \right| \to 0, \]

as \( r \to 0 \), we have chosen the radial coordinate such that the axis is located at \( r = 0 \), and \( \xi^\mu_{(\theta)} \) is the Killing vector with a close orbit, and given by \( \xi^\alpha_{(\theta)} \partial_\alpha = \partial_\theta \).

(ii) The spacetime near the symmetry axis is locally flat, which can be written as [20]

\[ \frac{X_{,\alpha} X_{,\beta} g^{\alpha\beta}}{4X} \to -1, \]

as \( r \to 0 \). Note that solutions failing to satisfy this condition sometimes are also acceptable. For example, when the left-hand side of the above equation approaches a finite constant, the singularity at \( r = 0 \) may be related to a point-like particle [21].

(iii) No closed timelike curves (CTC’s). In spacetimes with circular symmetry, CTC’s can be easily introduced. To ensure their absence, we assume that the condition

\[ \xi^\mu_{(\theta)} \xi^\nu_{(\theta)} g_{\mu\nu} < 0, \]

holds in the whole spacetime.
A. Case $\Delta > 0$

In this case we have from equation (33) that

$$
\Psi_1(x) = \frac{1}{k-1} \left[ k^2 c_3 e^{\frac{1}{2}x(1+\sqrt{k(3k-4)})} + k^2 c_2 e^{\frac{1}{2}x(1-\sqrt{k(3k-4)})} + 3e^{xk} k be^{\frac{1}{2}x(k-\sqrt{k(3k-4)})} 
+ c_3 \sqrt{k(3k-4) k^2 e^{\frac{1}{2}x(k-\sqrt{k(3k-4)})} - 2c_3 e^{\frac{1}{2}x(k+\sqrt{k(3k-4)})} - 2c_2 e^{\frac{1}{2}x(k-\sqrt{k(3k-4)})} c_3 e^{\frac{1}{2}x(k+\sqrt{k(3k-4)})} \right] e^{-xk} + c_4 e^{-xk}. \tag{50}
$$

In order to apply the first boundary condition (47), we have to calculate the quantity $\sqrt{X} = rS_1$, which can be written as

$$
rS_1 = \frac{b}{k-1} (-rt)^{\frac{1}{2}} + c_1 r^{\frac{1}{2}k} (-t)^{\frac{1}{2}k} + c_2 r^{\frac{1}{2}(k+1+\sqrt{k})} (-t)^{\frac{1}{2}(k+1+\sqrt{k})} + c_3 r^{\frac{1}{2}(k+1-\sqrt{k})} (-t)^{\frac{1}{2}(k+1-\sqrt{k})}. \tag{51}
$$

Since the limit of $rS_1$ must vanishes when $r \to 0$, all the exponents of $r$ must be greater than zero. It is easily shown that some the exponents cannot satisfy this condition, then the first condition is not fulfilled. Thus, these perturbations are limited by the boundary conditions.

B. Case $\Delta < 0$

In this case we have from equation (33) that

$$
\Psi_1(x) = -e^{\frac{1}{2}xk} \frac{1}{k-1} \left[ -2c_2 k^2 \cos \left( \frac{1}{2} \sqrt{k(3k-4)x} \right) + 4c_2 k \cos \left( \frac{1}{2} \sqrt{k(3k-4)x} \right) - 3k be^{\frac{1}{2}xk} 
+ 2c_2 k \sqrt{k(3k-4)} \cos \left( \frac{1}{2} \sqrt{k(3k-4)x} \right) + 2c_2 \sqrt{k(3k-4)} \cos \left( \frac{1}{2} \sqrt{k(3k-4)x} \right) - 2c_2 \cos \left( \frac{1}{2} \sqrt{k(3k-4)x} \right) \right] e^{-xk} + c_4 e^{-xk}. \tag{52}
$$

Then we have now two possibilities for the arbitrary constants $c_2$ and $c_3$ in order to get a real function $S_1(x)$:

$c_2 = c_3$ and $c_2 = -c_3$.

For $c_2 = c_3$ we get

$$
S_1(x) = \frac{b}{k-1} \left[ -e^{\frac{1}{2}x} + c_1 e^{\frac{1}{2}(1+2k)x} + 2c_2 e^{\frac{1}{2}(k+1)x} \cos \left( \frac{1}{2} \sqrt{-\Delta x} \right) \right], \tag{53}
$$

and $rS_1$ is given by

$$
rS_1 = \frac{b}{k-1} \left[ (-rt)^{\frac{1}{2}} + c_1 r^{\frac{1}{2}k} (-t)^{\frac{1}{2}k} + c_2 r^{\frac{1}{2}(k+1)} \cos \left( \frac{1}{2} \sqrt{-\Delta} \ln \left( \frac{r}{t} \right) \right) (-t)^{\frac{1}{2}(k+1)}. \tag{54}
$$

Applying again the condition (47), we can see that all the exponents of $r$ must be greater than zero only when $k < 0$, which admits only stable modes for the perturbation. For the second boundary condition (48), we have surveyed several sets of values of $b$, $c_1$, $c_2$, $c_4$ and $k$, and we have found at least one set that satisfies this condition: $b = 0$, $c_1 = 0$, $c_2 = 1$, $c_4 = 1$, $k = -1$. In this case we get

$$
lim_{r \to 0} rS_1 = -7 - 4\sqrt{7}, \tag{55}
$$

or

$$
lim_{r \to 0} rS_1 = -4 + 4\sqrt{7}. \tag{56}
$$

For $c_2 = -c_3$ we get

$$
S_1(x) = -\frac{be^{\frac{1}{2}x}}{k-1} + c_1 e^{-\frac{1}{2}(1+2k)x} - 2ic_2 e^{-\frac{1}{2}(k+1)x} \sin \left( \frac{1}{2} \sqrt{-\Delta x} \right). \tag{57}
$$

Since this case is analogous to the case where $c_2 = c_3$, we will not present it.
VI. CONCLUSIONS

We have presented in this work the study of the linear perturbations of the 2+1-dimensional circularly symmetric solution, obtained by Chan, da Silva, Villas da Rocha & Wang [18], with kinematic self-similarity of the second kind. We have shown there that the solution is Kantowski-Sachs like [20] and it may be considered as representing a Friedmann-like cosmological model in a 2+1 dimensional spacetime. We have obtained in this paper an exact solution for the perturbed equations, which admits only stable modes, showing that our Friedmann-like background solution in 2+1 dimension is stable. This result is in agreement with the conclusions of the work of Hirschmann, Wang & Wu [14].

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[1] M. W. Choptuik, Phys. Rev. Lett. 70, 9 (1993); “Critical Behavior in Massless Scalar Field Collapse,” in Approaches to Numerical Relativity, Proceedings of the International Workshop on Numerical Relativity, Southampton, December, 1991, Edited by Ray d’Inverno; “Critical Behavior in Scalar Field Collapse,” in Deterministic Chaos in General Relativity, Edited by D. Hobill et al. (Plenum Press, New York, 1994), pp. 155-175.
[2] G.I. Barenblatt, Similarity, Self-Similarity, and Intermediate Asymptotics (Consultants Bureau, New York, 1979); N. Goldenfeld, Lectures on Phase Transitions and the Renormalization Group (Addison Wesley Publishing Company, New York, 1992).
[3] C. Gundlach, “Critical phenomena in gravitational collapse: Living Reviews,” gr-qc/0001046 (2000), and references therein.
[4] A.Z. Wang, “Critical Phenomena in Gravitational Collapse: The Studies So Far,” gr-qc/0104073, Braz. J. Phys. 31, 188 (2001), and references therein.
[5] M.W. Choptuik, E.W. Hirschmann, S.L. Liebling, and F. Pretorius, Phys. Rev. D68, 044007 (2003).
[6] A.Z. Wang, Phys. Rev. D68, 064006 (2003).
[7] C. R. Evans and J. S. Coleman, Phys. Rev. Lett. 72, 1782 (1994); T. Koike, T. Hara, and S. Adachi, ibid., 74, 5170 (1995); C. Gundlach, ibid., 75, 3214 (1995); E. W. Hirschmann and D. M. Eardley, Phys. Rev. D52, 5850 (1995).
[8] P. R. Brady, M. W. Choptuik, C. Gundlach, and D. W. Neilson, Class. Quantum Grav. 19, 6359 (2002).
[9] F. Pretorius and M. W. Choptuik, Phys. Rev. D62, 124012 (2000).
[10] V. Hussain and M. Olivier, Class. Quantum Grav. 18, L1 (2001).
[11] D. Garfinkle, Phys. Rev. D63, 044007 (2001).
[12] G. Clément and A. Fabbri, Class. Quantum Grav. 18, 3665 (2001); Nucl. Phys. B630, 269 (2002).
[13] D. Garfinkle and C Gundlach, Phys. Rev. D66, 044015 (2002).
[14] E.W. Hirschmann, A.Z. Wang, and Y. Wu, Class. Quant. Grav. 21, 1791 (2004).
[15] S. Chandrasekhar, The Mathematical Theory of Black Holes (Clarendon Press, Oxford University Press, Oxford, 1983).
[16] B. Carter and R.N. Henriksen, Ann. Physique Suppl. 14, 47 (1989). See also, A.A. Coley, Class. Quantum Grav. 14, 87 (1997).
[17] M.E. Cahill and A.H. Taub, Commun. Math. Phys. 21, 1 (1971).
[18] R. Chan, M.F.A. da Silva, J.F. Villas da Rocha, A. Wang, Int. J. Mod. Phys. D, in press, gr-qc/0406026 (2005).
[19] A. Y. Miguelote, N. A. Tomimura, and A.Z. Wang, Gen. Rel. Grav. 36, 1883 (2004).
[20] D. Kramer, H. Stephani, E. Herlt, and M. MacCallum, Exact Solutions of Einstein’s Field Equations (Cambridge University Press, Cambridge, England, 1980).
[21] A. Vilenkin and E. P. S. Shellard, Cosmic String and Other Topological Defects (Cambridge University Press, Cambridge, 1994).