SIMPLE ASSOCIATIVE CONFORMAL ALGEBRAS
OF LINEAR GROWTH

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ABSTRACT. We describe simple finitely generated associative conformal algebras of Gel’fand–Kirillov dimension one.

1. INTRODUCTION

The theory of conformal algebras appeared as a formal language describing the algebraic properties of the operator product expansion (OPE) in two-dimensional conformal field theory \[ \text{[3, 16, 18, 19]} \]. In a few words, the origin of this formalism is as follows: for any two local fields \( a(z), b(z) \) the commutator \( [a(w), b(z)] \) could be written as a (finite) distribution with respect to derivatives of the delta-function:

\[
[a(w), b(z)] = \sum_{n \geq 0} c_n(z) \frac{1}{n!} \partial_z^n \delta(w - z),
\]

where \( \delta(w - z) = \sum_{s \in \mathbb{Z}} w^s z^{-s-1} \). The coefficients \( c_n(z), n \geq 0 \), of this distribution are considered as new "n-products" of \( a(z) \) and \( b(z) \). The algebraic properties of these operations could be formalized by a family of axioms, which gives the notion of a Lie conformal algebra \[ \text{[18]} \]. Associative conformal algebras naturally come from representations of Lie conformal algebras. Moreover, any of those Lie conformal (super-) algebras appeared in physics is embeddable into an associative one (this is not the case in general, see \[ \text{[26]} \]). So the investigation of associative conformal algebras provides some information on the structure and representations of Lie conformal algebras.

From the abstract point of view, a conformal algebra is a vector space \( C \) endowed with a linear map \( D : C \rightarrow C \) and with an infinite family of bilinear operations \( \circ_n: C \times C \rightarrow C \) (\( n \) ranges through non-negative integers), satisfying certain axioms.

In a more general context, a conformal algebra is just an algebra in the pseudotensor category \[ \text{[2]} \] associated with the polynomial Hopf algebra \( H = \mathbb{k}[D] \) (see also \[ \text{[1]} \]). This category consists of left \( H \)-modules endowed with \( H \)-polylinear maps. The notions of associativity, commutativity, etc. are well-defined there. If a conformal algebra \( C \) is a finitely generated \( H \)-module then \( C \) is said to be finite.

The structure theory of finite conformal algebras (and superalgebras) was established in a series of works. In \[ \text{[11, 12, 13, 15]} \], simple and semisimple finite Lie conformal (super)algebras were described. Similar results for associative and Jordan conformal algebras were obtained in \[ \text{[19]} \] and \[ \text{[30]} \], respectively. In \[ \text{[1]} \], the results of \[ \text{[12, 19]} \] were generalized for pseudotensor categories related to arbitrary cocommutative Hopf algebras with finite-dimensional spaces of primitive elements.

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There are two obvious ways how to move beyond the class of finite conformal algebras. The first option is to consider free conformal algebras and study their properties, as it was done in [25, 26, 27], see also [4, 5, 6]. Another way is to introduce a “growth function” of a conformal algebra and move into the next class with respect to the growth rate. The present paper develops the last approach.

In [23], the analogue of Gel’fand–Kirillov dimension (GKdim) for conformal algebras was proposed. As in the case of ordinary algebras, finite conformal algebras have GKdim 0, and there are no conformal algebras with GKdim strictly between 0 and 1. So the natural problem is to explore conformal algebras of linear growth (i.e., those of GKdim 1). Since [23], this class of conformal algebras has been studied in [8, 9, 13, 24, 31].

The difference between ordinary and conformal algebras becomes apparent in the following context. Although there are no (ordinary) finitely generated simple associative algebras with GKdim 1 [28, 29], conformal algebras of this kind do exist. There also exist infinite conformal algebras with finite faithful irreducible representations. It was conjectured in [31] that these families of algebras coincide:

**Conjecture.** An infinite associative conformal algebra $C$ has a finite faithful irreducible representation if and only if $C$ is a finitely generated simple conformal algebra of GKdim 1.

It was shown in [23] that a finitely generated simple associative conformal algebra $C$ of linear growth with a unit (i.e., with an element $e \in C$ such that $e \circ_0 x = x$ for all $x \in C$ and $e \circ_n e = 0$ for $n > 1$) is isomorphic to the conformal algebra $\text{Cend}_N$ of all conformal endomorphisms $[12, 19]$ of the free $N$-generated $k[D]$-module $V_N$. In this case, the conformal algebra $C$ has the finite faithful irreducible representation on $V_N$.

In [8], the converse statement was proved: if a unital associative conformal algebra $C$ has a finite faithful irreducible representation, then $C \simeq \text{Cend}_N$. Therefore, the results of [8, 23] prove the Conjecture for unital conformal algebras, but it is unclear how to join a unit to a conformal algebra.

In [31], the Conjecture (rather its “if” part) was confirmed for conformal algebras with an idempotent (i.e., $e \in C$ such that $e \circ_n e = \delta_{n,0} e$, $n \geq 0$).

On the other hand, the classification of associative conformal algebras with finite faithful irreducible representations was proposed in [19], partially constructed in [8], and completed in [21]. In the present paper, we use the result of [21] in order to prove the Conjecture in general, and get the explicit description of finitely generated simple associative conformal algebras of linear growth.

### 2. Associative Conformal Algebras and Their Representations

From now on, $k$ is an algebraically closed field of zero characteristic, $H$ is the polynomial algebra $k[D]$ with the ordinary derivation $\partial_D = \frac{d}{dD}$, $\mathbb{Z}_+$ denotes the set of non-negative integers. We will also use the notation $x^{(n)}$ for $x^n / n!$, $n \in \mathbb{Z}_+$. It is convenient to set $x^{(n)} := 0$ for $n < 0$.

#### 2.1. Conformal homomorphisms and products

In this section, we introduce the basic definitions from the theory of associative conformal algebras. The exposition follows the one of [11], namely, we adjust the theory of pseudoalgebras by making use of commutativity of $H$. This point of view is also close to [12, 19].
where the notion of a conformal algebra is introduced via λ-brackets, but we avoid using “external” variables.

Denote by $H$-mod the class of all unital left $H$-modules. Let us consider the usual $D$-adic topology on $H$, i.e., suppose the set of ideals $\{(D^n)\mid n \geq 0\}$ to be the system of basic neighborhoods of zero in $H$. For any vector spaces $U$ and $V$ denote by $\text{Hom}(U, V)$ the vector space of $k$-linear maps from $U$ to $V$. Recall the notion of finite topology [17] on $\text{Hom}(U, V)$: the family of basic neighborhoods of a map $\phi \in \text{Hom}(U, V)$ is presented by

$$\{\phi \in \text{Hom}(U, V) \mid \phi(u_i) = \phi_0(u_i), i = 1, \ldots, m\},$$

where $u_1, \ldots, u_m \in U, m \geq 0$. This topology turns $\text{Hom}(U, V)$ into a topological vector space, where a sequence $\{\phi_n\}_{n \geq 0} \subset \text{Hom}(U, V)$ converges to zero if and only if for any $u_1, \ldots, u_m \in U, m \geq 1$, there exists $N \geq 1$ such that $\phi_n(u_i) = 0, i = 1, \ldots, m$, for all $n \geq N$.

**Definition 2.1** ([12]). Let $U, V \in H$-mod. A continuous linear map

$$a : H \to \text{Hom}(U, V) \quad (2.1)$$

is said to be a left conformal homomorphism from $U$ to $V$ if

$$[D, a(h)] = -a(\partial_D h), \quad h \in H. \quad (2.2)$$

A right conformal homomorphism from $U$ to $V$ is a continuous linear map $a$ satisfying

$$a(h)D = -a(\partial_D h), \quad h \in H. \quad (2.3)$$

Denote by $\text{Chom}^l(U, V)$ ($\text{Chom}^r(U, V)$) the vector space of all left (right) conformal homomorphisms from $U$ to $V$. One may consider $\text{Chom}^l(U, V)$ and $\text{Chom}^r(U, V)$ as $H$-modules with respect to

$$(Da)(h) = -a(\partial_D h), \quad a \in \text{Chom}^l(U, V), \quad h \in H, \quad (2.4)$$

$$(Da)(h) = Da(h) + a(\partial_D h), \quad a \in \text{Chom}^r(U, V), \quad h \in H. \quad (2.5)$$

**Proposition 2.2** ([12]). Let $u$ be an element of the $H$-torsion of $U$. Then for any $a \in \text{Chom}^l(U, V)$ or $b \in \text{Chom}^r(U, V)$ we have $a(h)(u) = 0 = b(h)(u)$ for all $h \in H$. \hfill \Box

Given $U, V \in H$-mod, the tensor product $U \otimes V$ could be considered an as $H$-module with respect to three different structures. Let us denote these $H$-modules by $U \otimes^o V$ (outer), $U \otimes^l V$ (left-justified), $U \otimes^r V$ (right-justified):

$$D(u \otimes^o v) = Du \otimes^o v + u \otimes^o Dv,$$

$$D(u \otimes^l v) = Du \otimes^l v,$$

$$D(u \otimes^r v) = u \otimes^r Dv.$$

The following definition is just a slightly modified one from [12, 18, 19] (c.f. [1]).

**Definition 2.3.** Let $U, V, W \in H$-mod. A map

$$\mu \in \text{Chom}^l(U \otimes^r V, W), \quad (2.6)$$

is said to be a $W$-valued conformal product of $U$ and $V$ if for any $h \in H$ the map $\mu(h)$ is a homomorphism of $H$-modules $U \otimes^o V$ and $W$. 


It is clear that any conformal product \((2.10)\) could be considered as an element of \(\text{Chom}^1(U \otimes^1 V, W)\), so the notion of a conformal product is “symmetric”. Denote the set of all \(W\)-valued conformal products of \(U\) and \(V\) by \(P(U, V; W)\).

One may interpret \(\mu \in P(U, V; W)\) as an \(H\)-linear map \(\mu_1 : U \rightarrow \text{Chom}^1(V, W)\) or as \(\mu_r : V \rightarrow \text{Chom}^r(U, W)\) via
\[
\mu(h)(u \otimes v) = [\mu_1(u)(h)](v) = [\mu_r(v)(h)](u), \quad h \in H, \ u \in U, \ v \in V.
\]

**Remark 2.4.** Definition 2.3 provides a particular example of an \(H\)-linear map in the corresponding pseudotensor category \([1, 2]\).

### 2.2. Conformal \(n\)-products

Let us fix \(U, V, W \in H\)-mod. For any conformal homomorphisms \(a \in \text{Chom}^1(V, W), b \in \text{Chom}^1(U, W)\) and for any vectors \(u \in U, v \in V\), write
\[
a(D^n)(v) = a \circ_n v, \quad b(D^n)(u) = u \circ_n b, \quad n \in \mathbb{Z}_+.
\]

The sequence of functions \(a(D^n)\) (resp., \(b(D^n)\)), \(n \in \mathbb{Z}_+\), completely describes the corresponding conformal homomorphism.

In particular, a \(W\)-valued conformal product \(\mu\) of \(U\) and \(V\) gives rise to the family of ordinary linear operations \((n\)-products\)
\[
\circ_n : U \otimes V \rightarrow W, \quad n \in \mathbb{Z}_+,
\]
defined as
\[
u \circ_n v = \mu(D^n)(u \otimes v) = [\mu_1(u)(D^n)](v) = [\mu_r(v)(D^n)](u).
\]

It follows from Definition 2.3 that
\[
u \circ_n v = 0, \quad \text{for } n \text{ sufficiently large,} \tag{2.10}
\]
\[
u \circ_n Dv = D(u \circ_n v) + nu \circ_{n-1} v, \tag{2.11}
\]
\[
Dv \circ_n v = -nu \circ_{n-1} v. \tag{2.12}
\]

Moreover, any sequence of \(n\)-products \((2.8)\) satisfying \(2.10–2.12\) uniquely defines a conformal product in the sense of Definition 2.3.

We will use the following notation: for two subspaces \(X \subseteq U, Y \subseteq V\) denote
\[
X \circ_\omega Y = \sum_{n \geq 0} (X \circ_n Y) \subseteq W.
\]

If \(Y\) is an \(H\)-submodule of \(V\), then \(X \circ_\omega Y\) is also an \(H\)-submodule of \(W\) (see \((2.11), (2.12)\)).

Let us consider six modules \(U_1, U_2, U_3, V_1, V_2, W \in H\)-mod with conformal products
\[
\mu_{12} \in P(U_1, U_2; V_1), \quad \nu_1 \in P(V_1, U_3; W), \tag{2.13}
\]
\[
\mu_{23} \in P(U_2, U_3; V_2), \quad \nu_2 \in P(U_1, V_2; W). \tag{2.14}
\]

Each of these products is equivalent to a sequence of \(n\)-products defined by \((2.8)\).

We will denote these operations uniformly by \(\circ_n, n \in \mathbb{Z}_+\).

One may consider the compositions of the conformal products \((2.13), (2.14)\)
\[\nu_1(\mu_{12}, \text{id}_{U_1}), \nu_2(\text{id}_{U_1}, \mu_{23}) : H \otimes H \rightarrow \text{Hom}(U_1 \otimes U_2 \otimes U_3, W)\]
defined as follows:
\[
\nu_1(\mu_{12}, \text{id}_{U_1})(h \otimes g) = \nu_1(g)(\mu_{12}(h), \text{id}_{U_1}), \tag{2.15}
\]
\[
\nu_2(\text{id}_{U_1}, \mu_{23})(h \otimes g) = \nu_2(h)(\text{id}_{U_1}, \mu_{23}(g)), \tag{2.16}
\]

for any $h, g \in H$.

The maps $(2.13), (2.14)$ may satisfy the associativity relation. This relation has the following form:

$$\nu_1(\mu_{12}, \text{id}_{U_3}) = \nu_2(\text{id}_{U_1}, \mu_{23}) \circ F,$$

(2.17)

where $F$ is the formal Fourier transform $[1]$ given by $F = \exp(-\partial D \otimes D)$, i.e.,

$$F(D^n \otimes D^m) = \sum_{s \geq 0} (-1)^s \binom{n}{s} D^{n-s} \otimes D^{m+s}.$$

Note that $F^{-1} = \exp(\partial D \otimes D)$, so

$$F^{-1}(D^n \otimes D^m) = \sum_{s \geq 0} \binom{n}{s} D^{n-s} \otimes D^{m+s}.$$

In terms of $n$-products, the relation (2.17) could be expressed as

$$(u \circ_n v) \circ_m w = \sum_{s=0}^{n} (-1)^s \binom{n}{s} u \circ_{n-s} (v \circ_{m+s} w),$$

(2.18)

or

$$u \circ_n (v \circ_m w) = \sum_{s \geq 0} \binom{n}{s} (u \circ_{n-s} v) \circ_{m+s} w.$$  

(2.19)

for any $n, m \in \mathbb{Z}_+, u \in U_1, v \in U_2, w \in U_3$. It is easy to note that the systems of relations (2.18) and (2.19) are equivalent.

2.3. Conformal algebras and modules.

**Definition 2.5** ([1]). An $H$-module $C \in H$-mod endowed with a conformal product $\mu \in P(C, C; C)$ is called a conformal algebra.

The definition of a conformal algebra could be stated in terms of $n$-products.

**Definition 2.6** ([15]). A vector space $C$ endowed with a linear map $D$ and with a family of linear maps $\circ_n: C \otimes C \to C, n \in \mathbb{Z}_+$, is a conformal algebra if (2.10)–(2.12) hold.

A conformal algebra $C$ is said to be associative if $\mu(\mu, \text{id} C) = \mu(\text{id} C, \mu) \circ F$, i.e., if either of the relations (2.13) or (2.19) holds for any $n, m \in \mathbb{Z}_+$.

Note that for any three subspaces $V_1, V_2, V_3$ of an associative conformal algebra we have

$$V_1 \circ_{\omega} (V_2 \circ_{\omega} V_3) = (V_1 \circ_{\omega} V_2) \circ_{\omega} V_3.$$

A left (right) ideal $I$ of a conformal algebra $C$ is an $H$-submodule $I \subseteq C$ such that $C \circ_{\omega} I \subseteq I$ ($I \circ_{\omega} C \subseteq I$). A conformal algebra $C$ is simple if $C \circ_{\omega} C \neq 0$ and there are no non-zero proper two-sided ideals of $C$. If $C$ is a finitely generated $H$-module, then $C$ is said to be a finite conformal algebra.

**Remark 2.7** ([12]). It follows from Proposition 2.2 that any simple conformal algebra is a torsion-free $H$-module.

**Lemma 2.8.** If $C$ is a finitely generated associative conformal algebra and $\dim C/DC < \infty$, then $C$ is finite.
Proof. If \( \dim C/DC = n < \infty \), then there exist \( a_1, \ldots, a_n \in C \) such that any element \( x \in C \) could be represented as

\[
x = x_0 + Dz_0, \quad x_0 \in ka_1 + \cdots + ka_n, \quad z_0 \in C.
\]

Further, for any \( s \geq 0 \) there exist \( x_s \in Ha_1 + \cdots + Ha_n, \ z_s \in C \) such that

\[
x = x_s + D^{s+1}z_s, \quad s \geq 0.
\]

Let \( B \) be a finite set of generators of \( C \). Associativity relation implies that

\[
C = \sum_{b \in B} Hb + \sum_{t=0}^{N} H(C \circ_t b)
\]

for sufficiently large \( N \). It follows from (2.20) that the finite set \( B \cup \{ a_i \circ_t b \mid i = 1, \ldots, n, \ b \in B, \ t = 0, \ldots, N \} \) generates \( C \) as an \( H \)-module. \( \square \)

Definitions 2.5 and 2.6 provide the axiomatic description of the following constructions in formal distribution spaces. Let \( A \) be an algebra over \( k \) (non-associative, in general). Then one may define the following operations on the space of formal power series \( A[[z, z^{-1}]] \):

\[
a(z) \circ_n b(z) = \text{Res}_w a(w)b(z)(w-z)^n, \quad n \in \mathbb{Z}_+, \quad (2.21)
\]

where \( \text{Res}_w F(w,z) \) means the coefficient at \( w^{-1} \) of a formal power series \( F(w,z) \) in two variables.

Relations (2.11) and (2.12) hold with \( D = d/dz \), but the locality condition (2.10) does not hold in general. Two series \( a(z), b(z) \in A[[z, z^{-1}]] \) are called local if \( a(w)b(z)(w-z)^N = 0 \) for some \( N \in \mathbb{Z}_+ \). If \( A \)-invariant subspace \( C \subset A[[z, z^{-1}]] \) is closed under all \( n \)-products (2.21) and consists of pairwise mutually local series, then \( C \) is a conformal algebra. If \( A \) is associative, then (2.18) and (2.19) hold, and any conformal algebra \( C \subset A[[z, z^{-1}]] \) is associative. The converse is also true (see, e.g., [19, 25]): an (associative) conformal algebra \( C \) in the sense of Definition 2.6 could be canonically embedded into the space of formal power series \( A[[z, z^{-1}]] \) over some (associative) algebra \( A \).

**Definition 2.9** ([10] [18] [19]). Let \( C \) be an associative conformal algebra with a conformal product \( \mu \in \mathfrak{P}(C, C; C) \). An \( H \)-module \( V \) in \( H \)-mod endowed with a conformal product \( \nu \in \mathfrak{P}(C, V; V) \) is said to be a left conformal \( C \)-module if \( \nu(\mu, \text{id}_V) = \nu(\text{id}_C, \nu) \circ F \). Analogously, \( V \) is said to be a right conformal \( C \)-module if it is endowed with a conformal product \( \nu \in \mathfrak{P}(V, C; V) \) such that \( \nu(\text{id}_V, \mu) \circ F = \nu(\nu, \text{id}_C) \).

It is clear how to express the last definition in terms of \( n \)-products (see, e.g., [10] [12]).

A (left) conformal \( C \)-module \( V \) is called irreducible if \( C \circ V \neq 0 \) and there are no non-trivial conformal \( C \)-submodules of \( V \). If \( V \) is a finitely generated \( H \)-module, then \( V \) is said to be a finite conformal \( C \)-module.

**2.4. Commutativity of conformal algebras.** For any \( V, W \in H \)-mod, the \( H \)-modules \( \text{Chom}^1(V, W) \) and \( \text{Chom}^f(V, W) \) are isomorphic. Namely, for any \( a \in \text{Chom}^1(V, W) \) the map \( \hat{a} : H \to \text{Hom}(V, W) \) defined as

\[
\hat{a}(D^n) = \sum_{s \geq 0} (-1)^{n+s} D^{s+n} a(D^{n+s}), \quad n \in \mathbb{Z}_+, \quad (2.22)
\]
By definition, \( \text{Chom}^r(V, W) \). The same relation defines the inverse map \( \text{Chom}^r(V, W) \rightarrow \text{Chom}^l(V, W) \). It is easy to check that the map \( \text{Chom}^l(V, W) \rightarrow \text{Chom}^r(V, W) \), \( a \mapsto \bar{a} \), is an isomorphism of \( H \)-modules.

Let \( a \in \text{Chom}^l(V, W) \), \( b \in \text{Chom}^l(V, W) \) be conformal homomorphisms, \( V, W \in H \)-mod. For any \( n \in \mathbb{Z}_+ \), denote

\[
\tilde{a}(D^n)(v) = \{a \circ_n v\}, \quad \tilde{b}(D^n)(v) = \{v \circ_n b\}.
\]

By definition,

\[
\{x \circ_n y\} = \sum_{s \geq 0} (-1)^{n+s} D^{(s)}(x \circ_{n+s} y).
\]

Let \( \mu \in \text{P}(U, V; W) \) be a conformal product, \( U, V, W \in H \)-mod. Then the map \( \tilde{\mu} : H \rightarrow \text{Hom}_H(U \otimes V, W) \) constructed by the rule

\[
\tilde{\mu}(h)(u \otimes v) = [\tilde{\mu}_1(h)(u)](v) = [\tilde{\mu}_2(h)(v)](u)
\]

could be considered as a \( W \)-valued conformal product of \( V \) and \( U \): \( \tilde{\mu} \in \text{P}(V, U; W) \), \( \tilde{\mu}_1 = \mu_U \), \( \tilde{\mu}_2 = \mu_V \). Using the notation introduced in \([7]\), denote the corresponding \( n \)-products by \( \{u \circ_n v\} \), \( u \in U \), \( v \in V \), \( n \in \mathbb{Z}_+ \). The relation between operations \( \{\cdot \circ_n \} \) and \( \{\cdot \circ_n \cdot\} \) is given by \((2.23)\).

**Proposition 2.10** \([7, 12, 19]\). For any conformal products \((2.13)\), \((2.14)\) such that the associativity relation \((2.17)\) holds, we also have the following properties:

\[
\begin{align*}
\{u \circ_n \{v \circ_m w\}\} &= \sum_{s \geq 0} (-1)^s \binom{m}{s} \{\{u \circ_{m-s} v\} \circ_{n+s} w\}; \\
\{u \circ_n \{v \circ m \ w\}\} &= \sum_{s \geq 0} (-1)^s \binom{m}{s} \{u \circ_{m+s} \{v \circ_{n-s} w\}\}; \\
\{u \circ_n \{v \circ m \ w\}\} &= \sum_{s \geq 0} (-1)^s \binom{n}{s} u \circ_{m+s} \{v \circ_{n-s} w\},
\end{align*}
\]

\( u \in U_1, v \in U_2, w \in U_3 \).

A conformal algebra \( C \) with a conformal product \( \mu \) is said to be commutative if \( \mu(h)(a \otimes b) = \tilde{\mu}(h)(b \otimes a) \) for all \( h \in H \), \( a, b \in C \), i.e., if \( \alpha \circ_n \beta = \{b \circ_n a\} \). Conformal algebra is commutative if and only if it could be embedded into \( A[[z, z^{-1}]] \) over a commutative algebra \( A \). \([10, 26]\).

**Remark 2.11** \([19]\). Since a conformal product \( \mu \in \text{P}(U, V; W) \), \( \mu : H \rightarrow \text{Hom}(U \otimes V, W) \) is a continuous map, it is possible to define \( \mu(\exp(\alpha D)) \in \text{Hom}(U \otimes V, W) \), \( \alpha \in k \). This operation is denoted by \( \langle \cdot \rangle_\alpha : U \otimes V \rightarrow W \). \([12, 19]\):

\[
(u_\alpha v) = \sum_{n \geq 0} \alpha^{(n)}(u \circ_n v).
\]

Analogously, we denote

\[
\{u_\alpha v\} = \tilde{\mu}(\exp(\alpha D))(u \otimes v) = \sum_{n \geq 0} \alpha^{(n)}\{u \circ_n v\}.
\]

The following proposition provides a useful description of irreducible modules over associative conformal algebras.
Proposition 2.12. Let \( C \) be an associative conformal algebra, and let \( V \) be an irreducible (left) \( C \)-module. Then there exist \( u \in V \), \( \alpha \in k \) such that \( \{C, u\} = V \).

2.5. Algebra of \( 0 \)-multiplications. Let \( C \) be an associative conformal algebra with a conformal product \( \mu \). Then for every \( a \in C \), the operator \( a(0) = \mu(1)(a) \in \text{End} C \), \( a(0) : x \mapsto a \circ_0 x \), \( x \in C \), is a homomorphism of right \( C \)-modules (see (2.12), (2.19)). The set

\[
A_0 = A_0(C) = \{a(0) \mid a \in C\}
\]

is an associative subalgebra of \( \text{End} C \): \( a(0)b(0) = (a \circ_0 b)(0) \).

Lemma 2.13. If \( C \) is a finitely generated conformal algebra and

\[
a_1 \circ_0 C + \cdots + a_m \circ_0 C = C
\]

for some \( a_1, \ldots, a_m \in C \), then \( A_0 \) is a finitely generated algebra.

Proof. Let \( B \subset C \) be a finite set of generators. Fix an upper bound of locality on \( B \), i.e., a number \( N \) such that \( a \circ_0 b = 0 \) for any \( n > N \), \( a, b, \in B \). Denote by \( I \) the finite set of symbols \( \{1, \ldots, m\} \), and let \( I^* \) be the set of finite words in \( I \) (by \( \varepsilon \) we denote the empty word).

For any \( b \in B \) we may construct a family of elements \( \{b(w) \in C \mid w \in I^*\} \), such that \( b(\varepsilon) = b \) and for any \( u \in I^* \) we have

\[
b(u) = a_1 \circ_0 b(u_1) + \cdots + a_m \circ_0 b(u_m).
\]

Such a family is not unique, but we may fix any of them.

For all \( l, r, n \in \mathbb{Z}_+ \), denote by \( B_{n(l,r)} \) the set of all elements of the form

\[
(a_{i_1} \circ_{s_1} \cdots \circ_{s_{l-1}} a_{s_l} \circ_{t_l} b(u) \circ_{s_{l+1}} a_{n_1} \circ_{r_1} b(u) \circ_{s_{l+2}} a_{n_2} \circ_{r_2} a_{j_1} \circ_{t_1} a_{j_2} \circ_{t_2} \cdots \circ_{t_r} a_{j_r}),
\]

where \( i_k, j_t = 1, \ldots, m \), \( s_k, p_t = 0, \ldots, N \), \( k = 1, \ldots, l' \), \( t = 1, \ldots, r' \), \( b \in B \), \( u \in I^* \), \( \text{length}(u) \leq n \), \( l' \leq l \), \( r' \leq r \), with all possible bracketing schemes. It is clear that \( |B_{n(l,r)}| < \infty \) and \( B = B_{0(0)} \subseteq B_{n(l,r)} \) for all \( l, r, n \in \mathbb{Z}_+ \).

First, let us show that

\[
B_{n_{1}(l_{1},r_{1})} \circ_{n_{n_{2}}(0,r_{2})} B_{n_{1}+n_{2}} \subseteq B^{(n_{1},r_{1}+n_{2})} \circ_{n_{n_{2}}} B^{(n_{2},r_{2})} \circ_{n_{n_{2}}}, \quad n \leq N.
\]

(2.29)

For \( n = 0 \) the relation (2.29) is trivial, so proceed by induction on \( n \). Let \( x \in B^{(0,r_{2})} \), \( y \in B^{(l_{1},r_{1})} \). By definition (see also (2.13)), there exist \( x_1, \ldots, x_m \in B^{(0,r_{2})} \) such that \( x = a_1 \circ_0 x_1 + \cdots + a_m \circ_0 x_m \). Then

\[
y \circ_{n_{1}+1} x = \sum_{j=1}^{m} y \circ_{n_{1}+1} (a_j \circ_0 x_j)
= \sum_{j=1}^{m} \left[ \sum_{s=0}^{n} \binom{n+1}{s} (y \circ_{n_{1}+1-s} a_j) \circ_{s} x_j \right]
+ \sum_{j=1}^{m} (y \circ_0 a_j) \circ_{n_{1}+1} x_j.
\]

(2.30)

Since \( y \circ_{n_{1}+1-s} a_j \in B^{(l_{1},r_{1}+1)} \), the elements \( (y \circ_{n_{1}+1-s} a_j) \circ_{s} x_j \) lie in the space \( B^{(l_{1},r_{1}+1)} \), \( \circ_{s} B^{(l_{1},r_{1}+1)} \), \( a_1 \circ_{0} B^{(l_{1},r_{1}+1)} \) for all \( s = 0, \ldots, n \). The last summand of (2.30) clearly lies in \( B^{(l_{1},r_{1})} \circ_{0} B^{(l_{2},r_{2})} \).
It follows from (2.19) that any element of \( C \) could be presented as an \( H \)-linear combination of left-normed words in \( B \), i.e., conformal monomials of the form
\[
(\ldots (b_1 \circ_{n_1} b_2) \circ_{n_2} \ldots \circ_{n_k} b_{k+1}), \quad b_i \in B = B^{(0,0)}, \quad n_i \leq N,
\]
generate \( C \) as an \( H \)-module. Relation (2.19) implies that such a monomial lies in
\[
(B^{(0,n_1)}_0 \circ_0 B^{(n_1,n_2)}_{n_1} \circ_0 \ldots \circ_0 B^{(n_{k-1},n_k)}_{n_{k-1}} \circ_0 B^{(n_k,0)}_{n_k}),
\]
hence, the finite set \( \{ b(0) \mid b \in B^{(N,N)}_N \} \) generates associative algebra \( A_0 \). \( \square \)

3. Gel’fand–Kirillov dimension of conformal algebras

Let \( C \) be a finitely generated conformal algebra (not necessarily associative), and let \( \{ a_i \mid i \in S \} \subseteq C \) be a finite system of generators (\(| S | < \infty \)). Denote by \( C_n, n \geq 1 \), the \( H \)-linear span of all conformal monomials of the form \( (a_{i_1} \circ_{m_1} \ldots \circ_{m_{k-1}} a_{i_k}) \), \( k = 1, \ldots, n \), with all possible bracketing schemes. For a fixed bracketing scheme, there exist only a finite number of non-zero monomials, so \( C_n \) is finitely generated as an \( H \)-module. In particular, the number \( d_n = \text{rank} C_n \) is finite. It is also clear that \( C = \bigcup_{n \geq 1} C_n \).

From now on, we consider associative conformal algebras only. Throughout the rest of the paper, the term “conformal algebra” means “associative conformal algebra”.

Let \( V \) be a left \( C \)-module over a finitely generated conformal algebra \( C \). Assume that \( V \) is a finitely generated \( C \)-module, and choose a finite system of generators \( \{ v_j \mid j \in T \}, |T| < \infty \), of \( V \) over \( C \). Denote by \( V_1 \) the \( H \)-linear span of \( \{ v_j \mid j \in T \} \).

For \( n > 1 \), define
\[
V_n = V_{n-1} + C_1 \circ_\omega V_{n-1} = V_1 + C_{n-1} \circ_\omega V_1.
\]

It is clear that \( V_n \) is a finitely generated \( H \)-module (\( \text{rank} V_n < \infty \)), and \( V = \bigcup_{n \geq 1} V_n \).

The following definition was originally introduced for the regular module, i.e., for a (non-associative, in general) conformal algebra itself.

**Definition 3.1** (23). Let \( C \) be a finitely generated conformal algebra, and let \( V \) be a finitely generated left \( C \)-module. Then the value
\[
\text{GKdim}_C V = \limsup_{n \to \infty} \log_n (\text{rank} V_n) \in \mathbb{R}_+ \cup \{ \infty \}
\]
is called the Gel’fand–Kirillov dimension of \( C \)-module \( V \).

If \( V = C \) is the regular module, then \( \text{GKdim} C = \text{GKdim}_C C \) is called the Gel’fand–Kirillov dimension of \( C \).

The similar construction is valid for right \( C \)-modules.

The definition of Gel’fand–Kirillov dimension could be expanded to infinitely generated conformal algebras and modules in the usual way:
\[
\text{GKdim}_C V = \sup_{C' \subseteq_{f.g} C, V' \subseteq_{f.g} V} \text{GKdim}_{C'} V'.
\]

As in the case of usual algebras, Gel’fand–Kirillov dimension does not depend on the system of generators of \( C \) (and of \( V \)). If \( V' \) is a conformal submodule or homomorphic image of \( V \), then \( \text{GKdim}_C V' \leq \text{GKdim}_C V \) (see 23). It is also easy to obtain the following properties (well-known for usual algebras).
Lemma 3.2. Let $C$ be a finitely generated conformal algebra, and let $V$ be a finitely generated $C$-module. Then

(i) $\text{GKdim}_C V = 0$ if and only if $V$ is a finite conformal module;

(ii) $\text{GKdim}_C V > 0$ implies $\text{GKdim}_C V \geq 1$.

Proof. Although the statements are clear, let us state the proof in the case of right modules. First, consider the case when $V$ is a torsion-free $H$-module, i.e., $\text{tor} \ V = 0$. Assume that $\text{rank} V_n = \text{rank} V_{n+1}$ for some $n \geq 1$. Then there exists $f \in H$ such that $fV_{n+1} \subseteq V_n$, so the $H$-module $V_{n+1}/V_n$ coincides with its torsion. Since every generator $a \in C_1$ defines a conformal homomorphism in $\text{Chom}^/(V_{n+1}/V_n, V_{n+2}/V_{n+1})$ by the rule

$$(v + V_n) \circ_m a = v \circ_m a + V_{n+1}, \quad v \in V_{n+1}, \ m \in \mathbb{Z}_+,$$

Proposition 2.2 implies $V_{n+1} \circ_1 C_1 \subseteq V_{n+1}$. Therefore, the “right analog” of (3.1) implies $V_{n+1} = V_{n+2} = \cdots = V_n$, so rank $V < \infty$.

We have proved that either rank $V < \infty$ or the sequence rank $V_n$, $n \geq 1$, is strictly increasing. In the last case, rank $V_n \geq n$ for every $n \geq 1$, so $\text{GKdim}_C V \geq 1$.

If $\text{tor} V \neq 0$, then we may apply the arguments above to the torsion-free module $\overline{V} = V/\text{tor} V$. Since $\text{GKdim}_C \overline{V} \leq \text{GKdim}_C V$, it is sufficient to show that rank $(\overline{V}) < \infty$ implies rank $V < \infty$. Indeed, if $\overline{V}$ is finitely generated as an $H$-module, then there exist $v_1, \ldots, v_m \in V$ such that an arbitrary $v \in \overline{V}$ could be presented as

$$v = f_1 v_1 + \cdots + f_m v_m + u, \quad f_i \in H, \ u \in \text{tor} V. \quad (3.2)$$

Since $V = \bigcup_{n \geq 1} V_n$, there exists $n \geq 1$ such that $v_1, \ldots, v_m \in V_n$. In particular, every element $v \in V_{n+1}$ can be presented as (3.2), so by Proposition 2.2 we have $V_{n+1} \circ_1 C_1 \subseteq V_{n+1}$. Hence, rank $V < \infty$ as above. \hfill \Box

Corollary 3.3. (i) Let $V$ be an arbitrary conformal module over a conformal algebra $C$. Then $\text{GKdim}_C V < 1$ implies $\text{GKdim}_C V = 0$.

(ii) There are no conformal algebras of Gel’fand–Kirillov dimension strictly between one and zero. \hfill \Box

It is well-known for usual algebras, that if an ideal $I$ of an associative algebra $A$ contains a regular element, then $\text{GKdim} A/I \leq \text{GKdim} A - 1$. We will use an analogous fact for conformal modules.

Let us consider a finitely generated conformal algebra $C$, and fix a system of generators. A homomorphism of (right) $C$-modules $L : C \to C$ is called bounded if there exists $m = m(L) \geq 0$ such that $L(C_n) \subseteq C_{n+m}$ for any $n \geq 0$. The set of all bounded homomorphisms is an associative subalgebra $\mathcal{L}(C)$ of $\text{End} C$. For example, $H$-linear maps of the form $x \mapsto \alpha x + a \circ_0 x$, $\alpha \in k$, $a \in C$, lie in $\mathcal{L}(C)$.

For every $L \in \mathcal{L}(C)$ consider

$$I_L = L(C) + \bigcup_{n \geq 1} \text{Ker}_C L^n \subseteq C. \quad (3.3)$$

It is clear that $I_L$ is a right ideal of $C$.

Proposition 3.4. Let $C$ be a finitely generated conformal algebra such that $C \circ_1 C = C$, and let $L \in \mathcal{L}(C)$. If $I_L \neq C$, then there exists an irreducible finitely generated right $C$-module $V$ such that

$$\text{GKdim}_C V \leq \text{GKdim} C - 1.$$
Proof. Let \( J_1 \subseteq J_2 \subseteq J_3 \subseteq \ldots \) be an increasing chain of proper right ideals of \( C \). If \( J_m = \bigcup_{k \geq 1} J_k \) is equal to \( C \), then there exists \( m \geq 1 \) such that \( J_m \) contains all generators of \( C \). In this case \( J_m = C \) is not a proper ideal. Therefore, \( J_m \) is a proper right ideal of \( C \). By the Zorn lemma, we conclude that every proper right ideal of \( C \) can be embedded into a maximal right ideal of \( C \). Let \( I \) be a maximal right ideal which contains \( I_L \neq C \).

Note that \( V = C/I \) is an irreducible finitely generated right \( C \)-module (\( V \circ_\omega C \neq 0 \) since \( I \not\supseteq C \circ_\omega C = C \), this is the only place where we use this condition).

In particular, \( V \) is a torsion-free \( H \)-module. Let us fix a finite system of generators \( \{ a_i \mid i \in S \} \) for \( C \), and consider \( V \) to be generated by the set \( \{ \bar{a}_i = a_i + I \mid i \in S \} \).

Let \( d(n) = \text{rank} V_n, n \geq 1 \), and let \( \bar{a}_1, \ldots, \bar{a}_{d(n)} \) be an \( H \)-basis of \( V_n \). Then the set of elements

\[
\{ u_j, L(u_j), \ldots, L^n(u_j) \mid j = 1, \ldots, d(n) \} \subset C_{n+mn}, \quad m = m(L),
\]

is \( H \)-linearly independent. Indeed, if there exist some polynomials \( f_{k,j}, k = 0, \ldots, n, j = 1, \ldots, d(n) \), such that

\[
\sum_j f_{0,j} u_j + \sum_j f_{1,j} L(u_j) + \cdots + \sum_j f_{n,j} L^n(u_j) = 0,
\]

then \( \sum_j f_{0,j} u_j \in L(C) \subseteq I \), so \( f_{0,j} = 0 \) for all \( j = 1, \ldots, d(n) \). The relation implies that

\[
\sum_j f_{1,j} u_j + \cdots + \sum_j f_{n,j} L^{n-1}(u_j) \in \ker C \}
\]

so \( \sum_j f_{1,j} u_j \in I + L(C) = I \), and all \( f_{1,j} \) are zero. In the same way, \( f_{k,j} = 0 \) for all \( k = 0, \ldots, n, j = 1, \ldots, d(n) \).

Since \( C_{n+mn} \) contains at least \( (n+1)d(n) \) \( H \)-linearly independent elements, we may conclude that \( \text{rank} C_{n+mn} \geq nd(n) \). Now it is left to apply the usual arguments to deduce \( \text{GKdim} C \geq 1 + \text{GKdim}_C V \).

\[ \square \]

Remark 3.5. The same statement could be proved for “right” bounded operators, i.e., for homomorphisms \( R : C \to C \) of left \( C \)-modules, such that \( R(C_n) \subseteq C_{n+m} \) for some \( m \geq 0 \) and for all \( n \geq 1 \). For example, the maps \( x \mapsto \alpha x + \{ x \circ_0 a \}, \alpha \in k, a \in C \), satisfy these conditions.

Lemma 3.6. Let \( C \) be a conformal algebra, and let \( A_0 = A_0(C) \subseteq L(C) \) be the algebra of 0-multiplications defined by \( L(C) \). If \( C \) is a torsion-free \( H \)-module, then \( \text{GKdim} A_0 \leq \text{GKdim} C \).

Proof. Consider a subalgebra \( A'_0 \subseteq A_0 \) generated by a finite family of elements \( a_1(0), \ldots, a_n(0) \in A_0, n \geq 1 \), and let \( C' \) be the conformal subalgebra of \( C \) generated by \( \{ a_1, \ldots, a_n \} \). It is clear that \( A'_0 \subseteq A_0(C') \).

It is sufficient to show that if \( u_1(0), \ldots, u_d(0) \in A_0(C') \), \( d \geq 1 \), are \( k \)-linearly independent, then \( u_1, \ldots, u_d \in C' \) are \( H \)-linearly independent. Assume

\[
f_1(D) u_1 + \cdots + f_d(D) u_d = 0
\]

for some \( f_1, \ldots, f_d \in H \). Then

\[
f_1(0) u_1(0) + \cdots + f_d(0) u_d(0) = 0.
\]

Since \( C' \) is a torsion-free \( H \)-module, there exists \( i \in \{ 1, \ldots, d \} \) such that \( f_i(0) \neq 0 \), so \( u_1(0), \ldots, u_d(0) \) are linearly dependent over \( k \).
Therefore, $\text{GKdim} A'_0 \leq \text{GKdim} C' \leq \text{GKdim} C$ for any finitely generated subalgebra $A'_0$ of $A_0$. Thus, $\text{GKdim} A_0 \leq \text{GKdim} C$. □

4. The Conformal Algebra $\text{Cend}_N$ and Its Irreducible Subalgebras

Let $V$ be a unital left $H$-module. Denote the $H$-modules $\text{Chom}(V, V)$ and $\text{Chom}'(V, V)$ by $\text{Cend}^1V$ and $\text{Cend}'V$, respectively. The vector spaces $\text{Cend}^1V$, $\text{Cend}'V$ (of left and right \textit{conformal endomorphisms}) are considered as $H$-modules with respect to (4.1), (4.2). There exist natural conformal products

$$\nu_1 \in \text{P}(\text{Cend}^1V, V; V), \quad \nu_2 \in \text{P}(V, \text{Cend}'V; V)$$

defined as follows:

$$\nu_1(h)(a \otimes v) = a(h)(v), \quad \nu_2(h)(v \otimes b) = b(h)(v),$$

$h \in H$, $v \in V$, $a \in \text{Cend}^1V$, $b \in \text{Cend}'V$.

If $V$ is a finitely generated $H$-module, then $\text{Cend}^1V$ and $\text{Cend}'V$ can be endowed with conformal products $\mu_1$ and $\mu_2$, respectively, such that

$$\nu_1(\mu_1, \text{id}_V) = \nu_1(\text{id}_{\text{Cend}^1V}, \nu_1) \circ \mathcal{F}, \quad \nu_2(\mu_2, \text{id}_V) = \nu_2(\nu_2, \text{id}_{\text{Cend}^1V}) \circ \mathcal{F}^{-1}.$$

Therefore, $\text{Cend}^1V$ and $\text{Cend}'V$ are (associative) conformal algebras (see, e.g., [12, 19, 19]).

Let $V_N$ be a free $N$-generated $H$-module, $N \geq 1$. The conformal algebra $\text{Cend}^1V_N = \text{Cend}_N^1$ can be presented as follows. For a fixed $H$-basis $\{e_1, \ldots, e_N\}$ of $V_N$, one may define the operation $\partial_D$ on $V_N$ as on $H \otimes \mathbb{k}[x]$ ($\partial_D e_i = 0, i = 1, \ldots, N$). For any matrix $A = A(x) \in M_N(\mathbb{k}[x])$ with polynomial entries define the conformal endomorphism

$$A : H \rightarrow \text{End} V_N, \quad A(x)(D^n) : u \mapsto A(D)\partial_D^n(u), \quad (4.1)$$

where $n \in \mathbb{Z}_+$. Proposition 4.1 (12, 18, 19, 23). The $H$-module generated by the conformal endomorphisms (4.2), $A \in M_N(\mathbb{k}[x])$, is isomorphic to $\text{Cend}_N^1$. Therefore, the conformal algebra $\text{Cend}_N^1$ could be identified with $M_N(\mathbb{k}[D, x]) \cong H \otimes M_N(\mathbb{k}[x])$, where the family of $n$-products is defined by

$$A(x) \circ_n B(x) = A\partial^n_D(B(x)), \quad n \in \mathbb{Z}_+. \quad (4.2)$$

The map $a \mapsto \tilde{a}$ given by (4.2) provides an anti-isomorphism of conformal algebras $\text{Cend}_N^1$ and $\text{Cend}_N^1$. Namely, we have

$$\tilde{a} \circ_n b = \{b \circ_n a\}, \quad a, b \in \text{Cend}_N^1, \quad n \in \mathbb{Z}_+. \quad (4.3)$$

Combining this map with an arbitrary anti-automorphism of $\text{Cend}_N^1$ (see [8] or [21]), we get an isomorphism of conformal algebras $\text{Cend}_N^1$ and $\text{Cend}_N^1$. From now on, we will denote this conformal algebra by $\text{Cend}_N^1$.

It is clear from (4.1) that

$$\{a(h) \mid a \in \text{Cend}_N, \quad h \in H\} = M_N(W), \quad (4.4)$$

where $W$ is the first Weyl algebra (see [21] for details):

$$W = \mathbb{k}(D, \partial_D \mid \partial_DD - D\partial_D = 1) \subset \text{End} V_N.$$
Example 4.2. Consider $S_0 = M_N(k[\partial_D]) \subset M_N(W)$. The set
\[ \{a \in \text{Cend}_N \mid a(H) \subset S_0 \} = \text{Cur}_N \] (4.5)
is a conformal subalgebra of Cend$_N$ called the current subalgebra. The image of Cur$_N$ in $M_N(k[D,x]) \simeq$ Cend$_N$ (see Proposition 4.1) is given by $M_N(k[D])$.

Example 4.3. For any matrix $Q = Q(D) \in M_N(k[D])$ the set
\[ \{a \in \text{Cend}_N \mid a(H) \subset M_N(W)Q \} = \text{Cend}_{N,Q} \] (4.6)
is a conformal subalgebra of Cend$_N$. This conformal subalgebra could be identified with $M_N(k[D,x])Q(-D + x)$.

Remark 4.4 ([8, 19, 30]). Conformal algebras Cend$_{N,Q}$, $N \geq 1$, $\det Q \neq 0$, are simple, finitely generated, and GKdim Cend$_{N,Q} = 1$. Moreover, these are infinite conformal algebras with finite faithful irreducible modules.

The following statement was conjectured in [19]. In [21], it was proved for left modules, but it also holds in the case of right modules since Cend$_N \simeq$ Cend$_N$.

Theorem 4.5 ([21]). Let $C$ be a conformal algebra with a finite faithful irreducible module. Then either $C \simeq$ Cur$_N$ or $C \simeq$ Cend$_{N,Q}$ for some $N \geq 1$, where $Q$ has the canonical diagonal form, i.e., $Q = \text{diag}(f_1, \ldots, f_N)$, $f_i$ are non-zero monic polynomials, and $f_i[f_2| \ldots |f_N] \neq 0$.

Theorem 4.6 ([21]). Let $C$ be a simple associative conformal algebra with a non-trivial finite module. Then either $C$ is finite, or $C \simeq$ Cend$_{N,Q}$ as in Theorem 4.5.

Corollary 4.7. Let $C$ be a finitely generated simple conformal algebra such that GKdim $C = 1$. Then either $C \simeq$ Cend$_{N,Q}$ ($N \geq 1$, $\det Q \neq 0$) or for any $L \in \mathcal{L}(C)$ there exists $n \geq 1$ such that $L(C) + \text{Ker}_C L^n = C$.

Proof. Suppose there exists $L \in \mathcal{L}(C)$ such that for any $n \geq 1$ the ideal $L(C) + \text{Ker}_C L^n \subset C$ is proper (see the proof of Proposition 5.1). Since $C$ is a finitely generated conformal algebra, the union of these ideals (equal to $I_L$) is also proper. Then by Proposition 5.1 we conclude that there exists an irreducible finitely generated $C$-module $V$ such that GKdim$_C V = 0$. By Lemma 5.2 $V$ is finite, so Theorem 4.5 implies that $C$ is isomorphic to Cend$_{N,Q}$, $N \geq 1$, $\det Q \neq 0$.

We are going to show that the second case is impossible, so Cend$_{N,Q}$, $N \geq 1$, $\det Q \neq 0$, exhaust all simple finitely generated associative conformal algebras of linear growth.

5. Classification theorem

The purpose of this work is to prove the following statement.

Theorem 5.1. Let $C$ be a finitely generated simple associative conformal algebra such that GKdim $C = 1$. Then $C \simeq$ Cend$_{N,Q}$, $N \geq 1$, $\det Q \neq 0$.

Throughout this section, $C$ is assumed to satisfy the conditions of Theorem 5.1. This theorem proves the conjecture from [21] (see Section 1) and generalizes the following results.

Theorem 5.2 ([23]). If there exists an element $e \in C$ such that $e \circ_0 a = a$ for any $a \in C$, and $e \circ_n e = 0$ for any $n \geq 1$, then $C \simeq$ Cend$_N$, $N \geq 1$. 
Such an element $e \in C$ is called a unit of a conformal algebra. Note that a conformal unit is not unique.

**Theorem 5.3 (31).** If $C$ contains an element $e$ such that $e \circ_n e = \delta_{n,0}e$ for any $n \geq 0$, then $C \simeq \text{Cend}_{N,Q}$, $N \geq 1$, $Q = \text{diag}(1, f_2, \ldots, f_N)$.

An element $e \in C$ satisfying the condition described in the last statement is called an idempotent of a conformal algebra.

We will deduce a statement which is more general than Theorem 5.2. Moreover, we will prove Theorem 5.3 using arguments different from [31].

Let us first sketch the idea of the proof of Theorem 5.1. Recall the associative algebra $A_0 = A_0(C) = \{ a(0) : x \mapsto a \circ_0 x \mid a \in C \} \subseteq L(C)$ which was defined in (2.28). By Lemma 3.6 $\text{GKdim} A_0 \leq 1$.

If there exists $L \in L(C)$ such that $L(C) + \text{Ker} C L^n \neq C$ for all $n \geq 1$, then we may use Corollary 4.7.

Therefore, it is sufficient to consider the case when for any $L \in L(C)$ there exists $n \geq 1$ such that $L(C) + \text{Ker} C L^n = C$. In particular, for any $a \in C$ we may assume

$$a \circ_0 C + \text{Ker} C a(0)^n = C$$

for an appropriate $n \geq 1$. Relation (5.1) implies

$$a(0)A_0 + \text{Ann}_A a(0)^n = A_0.$$  

(5.2)

It remains to show that either of (5.1) or (5.2) leads to a contradiction.

**Proposition 5.4.** If $A$ is a finitely generated algebra of at most linear growth (i.e., $\text{GKdim} A \leq 1$), and for any $a \in A$ there exists $n \geq 1$ such that $aA + \text{Ann}_A (a^n) = A$, then $A$ is finite-dimensional.

**Proof.** It was shown in [29] that $A$ is a PI algebra and its prime radical $R = R(A)$ is nilpotent (moreover, it coincides with the Jacobson radical $J = J(A)$, see, e.g., [20]). So $\bar{A} = A/R$ is a finitely generated semiprime algebra which is left and right Goldie [28, 29].

Let $\bar{a} \in \bar{A}$ be a right regular element, i.e., $\bar{a}\bar{x} = 0$ implies $\bar{x} = 0$ in $\bar{A}$ (such an element necessarily exists in any semiprime Goldie algebra). Then $\text{Ann}_A(a^n) \subseteq R$ for any $n \geq 1$, and

$$aA + R = A.$$  

It is easy to derive that $\bar{A}$ contains a unit and any (right) regular element is invertible. Hence, $\bar{A}$ coincides with its classical quotient algebra $Q(A)$. By the Goldie theorem, $\bar{A} = Q(\bar{A})$ is semisimple Artinian, so it is equal to a finite direct sum of simple (and finitely generated) algebras of at most linear growth. The main result of [28] implies that any algebra of this kind is finite-dimensional, so $\text{dim} \bar{A} < \infty$. Finally, one may use the Kaplansky theorem and nilpotency of $R$ to deduce that $A$ is finite-dimensional itself. \hfill $\square$

The following statement generalizes the result of [23] (c.f. Theorem 5.2).

**Proposition 5.5.** If there exists an element $a \in C$ such that $a \circ_0 C = C$, then $C \simeq \text{Cend}_{N'}$. 
Proof. First, consider the homomorphism of left $C$-modules $R : C \to C$, $x \mapsto \{ x \circ_0 a \}$. If $x \in \text{Ker}_C R$, then by $\text{(2.2d)}$ we have
\[ 0 = \{ x \circ_0 a \} \circ_\omega C = x \circ_\omega (a \circ_0 C) = x \circ_\omega C, \]
so $x = 0$. Proposition 3.4 and Remark 3.5 imply that either $C = \{ C \circ_0 a \}$, or $C$ has a finite faithful irreducible right module. In the last case, $C \simeq \text{Cend}_{N,Q}$ by Theorem 4.6.

By Lemma 5.6, $A_0 = A_0(C)$ is finitely generated. If the condition (5.2) fails for some $b(0) \in A_0$, then by Corollary 5.7 we have $C \simeq \text{Cend}_{N,Q}$. If (5.2) holds for any $b(0) \in A_0$, then $\dim A_0 < \infty$ by Proposition 5.3.

Therefore, either $C$ is isomorphic to $\text{Cend}_{N,Q}$ as in Theorem 4.6 or $\{ C \circ_0 a \} = C$ and $\dim A_0 < \infty$. In the last case, $\dim(C \circ_0 a) < \infty$, but $C = \{ C \circ_0 a \} = C \circ_0 (a + DC)$, so $\dim C/DC < \infty$. Then Lemma 2.8 implies $C$ to be of finite type, but $C$ is assumed to be of linear growth.

Hence, $C$ is necessarily isomorphic to $\text{Cend}_{N,Q}$, $\det Q \neq 0$. Since there exists $a \in C$ such that $a \circ_0 C = C$, the matrix $Q$ has to be invertible, so $\text{Cend}_{N,Q} = \text{Cend}_{N}$.

**Lemma 5.6 (c.f. [11]).** If there exists an element $a \in C$ such that $a \circ_0 (a \circ_0 C) = a \circ_0 C \neq 0$, then $C_0 = a \circ_0 \{ C \circ_0 a \}$ is a finitely generated conformal algebra such that every proper ideal $I$ of $C_0$ satisfies $a \circ_0 I = 0$.

Proof. Since $C$ is simple, we have
\[ \{ C \circ_0 a \} \circ_\omega (a \circ_0 C) = C \]
(15.2)

(15.3)

(15.4)

(15.5)

In particular, every element $b \in B$ (where $B$ is a finite set of generators) could be represented as $\{ X \circ_0 a \} \circ_\omega a \circ_0 \{ (X \circ_\omega X) \circ_0 a \}$

generates $C_0$ as a conformal algebra.

Let $I$ be an ideal of $C_0$. Then
\[ J = \{ C \circ_0 a \} \circ_\omega I \circ_\omega (a \circ_0 C) \]
is an ideal of $C$, so either $J = 0$ or $J = C$. Note that
\[ a \circ_0 \{ J \circ_0 a \} = (a \circ_0 \{ C \circ_0 a \}) \circ_\omega I \circ_\omega (a \circ_0 \{ C \circ_0 a \}) \subseteq I. \]

Hence, $J = C$ implies $I = C_0$. If $J = 0$, then $\{ C \circ_0 a \} \circ_\omega I \circ_\omega (a \circ_0 C) = C \circ_\omega (a \circ_0 I) \circ_\omega (a \circ_0 C)$ and
\[ (a \circ_0 I) \circ_\omega (a \circ_0 C) = 0. \]

Now, consider an arbitrary element $x \in I$, $x = a \circ_0 \{ y \circ_0 a \}$. For any $b \in C$ there exists an element $c \in C$ such that $a \circ_0 b = a \circ_0 a \circ_0 c$. Thus, $a \circ_0 x \circ_\omega b = a \circ_0 a \circ_0 y \circ_\omega (a \circ_0 b) = a \circ_0 a \circ_0 (y \circ_\omega (a \circ_0 a \circ_0 c)) = a \circ_0 a \circ_0 (\{ y \circ_0 a \} \circ_\omega (a \circ_0 c)) = (a \circ_0 x) \circ_\omega (a \circ_0 c) = 0$ by (3.6). Hence, $a \circ_0 I$ annihilates the whole $C$, so $a \circ_0 I = 0$. \[ \square \]
We would like to state a simple proof of Theorem 5.3 which is important for further considerations.

Proof of Theorem 5.3. It is sufficient to show that $C \simeq \text{Cend}_{N,Q}$, $N \geq 1$, $\det Q \neq 0$. It is easy to note that $\text{Cend}_{N,Q}$ contains an idempotent if and only if the canonical diagonal form of $Q$ is of the form $\text{diag}(1, f_2, \ldots, f_N)$.

Fix a finite set of generators $B \subset C$. Let $e \in C$ be an idempotent. Since $e \circ_0 e = \{e \circ_0 e\} = e$, the subalgebra $C_0 = e \circ_0 \{C \circ_0 e\}$ is simple and finitely generated by Lemma 5.6. Moreover, $C_0 = H(e \circ_0 C \circ_0 e)$, and $C_0$ contains a unit, e.g., $e = e \circ_0 \{e \circ_0 e\} \in C_0$. Hence, either $C_0$ is finite, or $C_0 \simeq \text{Cend}_N$ for some $N \geq 1$ (see Theorem 5.2 or Proposition 5.3).

If $C_0$ is finite, then $C$ is finite itself. Indeed, one may present

$$C = HB + H(B \circ_0 B) + H(B \circ_0 C \circ_0 B)$$

and note that there exists a finite-dimensional subspace $X \subset C$ such that $B \subset X \circ_0 e \circ_0 X$ (since $C \circ_0 e \circ_0 C$ is a non-zero ideal of $C$). Thus,

$$C = HB + H(B \circ_0 B) + H(X \circ_0 C_0 \circ_0 X)$$

is a finite conformal algebra.

If $C_0 \simeq \text{Cend}_N$, then there exists an element $v \in C_0$ corresponding to $xI_N \in M_N(\mathbb{k}[D, x]) \simeq \text{Cend}_N$. We may assume that there exists $n \geq 1$ such that (5.1) holds for $v$, i.e.,

$$v \circ_0 C + \text{Ker}_C v(0)^n = C$$

(otherwise, Corollary 4.7 implies the claim). In particular, for the element $a = v^{\circ_0 n} = v \circ_0 \ldots \circ_0 v \in C_0$ (corresponding to $x^n I_N$) there exists $y \in C$ such that

$$a \circ_0 a \circ_0 y = a \circ_0 e_1 a = a.$$  \hspace{1cm} (5.5)

Here $e_1$ is the element of $C_0$ corresponding to $I_N \in \text{Cend}_N$ (in fact, the isomorphism between $C_0$ and $\text{Cend}_N$ could be chosen in such a way that $e = e_1$, see [23]). Consider $z = e_1 \circ_0 \{y \circ_0 e_1\} \in C_0$, and note that

$$a \circ_0 a \circ_0 z = a \circ_0 a \circ_0 e_1 \circ_0 \{y \circ_0 e_1\} = \{a \circ_0 e_1\} = a.$$  \hspace{1cm} (5.6)

This relation is clearly impossible in $\text{Cend}_N$. \hspace{1cm} $\square$

The following lemma is a particular case of the general statement from 5.1.

Lemma 5.7. Let $C$ be a conformal algebra, and let $I$ be an ideal of $C$ such that $C \circ_0 I = 0$. If $\overline{C} = C/I$ contains an idempotent $\overline{e}$, then there exists a preimage $e \in C$ which is also an idempotent.

Proof. Let $e \in C$ be an arbitrary preimage of the idempotent $\overline{e} \in \overline{C}$. Then $e - e \circ_0 e \in I$, and $e \circ_n e \in I$ for any $n \geq 1$. In particular, $e_1 = e \circ_0 e$ is also a preimage of $\overline{e}$. Since $C \circ_0 I = 0$, we have $e_1 \circ_0 e_1 = e_1$, and $e_1 \circ_n e_1 = 0$ for any $n \geq 1$. Therefore, $e_1$ is an idempotent of $C$. \hspace{1cm} $\square$

Proof of Theorem 5.7. Let us suppose that $C \neq \text{Cend}_{N,Q}$. Then by Corollary 4.7 for every $L \in \mathcal{L}(C)$ there exists $n \geq 1$ such that $L(C) + \text{Ker}_C L^n = C$. In particular, every element of $C$ satisfies the condition (5.1).

Assume that there exists an element $e \in C$ such that $e$ is not nilpotent with respect to the 0-product. Then by (5.1) there exists an integer $n \geq 1$ such that

$$e \circ_0 C + \text{Ker}_C e(0)^n = C,$$
For $a = e(0)^{n-1}e \in C$ we have $a \circ_0 (a \circ_0 C) = a \circ_0 C \neq 0$. By Lemma 5.6, the conformal algebra $C_0 = a \circ_0 \{C \circ_0 a\}$ is finitely generated, and for every proper ideal $I$ of $C_0$ we have $C_0 \circ_\omega I = 0$. Moreover, there exists $x = a \circ_0 \{a \circ_0 a\} \in C_0$ such that $x \circ_0 C_0 = C_0$.

Let us choose a maximal ideal $I$ of $C_0$, and consider the simple finitely generated associative algebra $C_1 = C_0/I$. Since $\text{GKdim} C_1 \leq \text{GKdim} C = 1$, we may use Proposition 4.7 and Theorem 4.8 to conclude that $C_1$ is isomorphic either to $\text{Cend}_N$ or to $\text{Cur} M_N(\mathbb{k})$. In any case, $C_1$ contains a unit, so $C_0$ has an idempotent by Lemma 5.7. Theorem 5.8 implies that $C \simeq \text{Cend}_N,Q$, $N \geq 1$, $Q = \text{diag}(1,f_2,\ldots,f_N)$. But this algebra clearly does not satisfy the condition $\delta_i^i$.

It remains to consider the case when every element of $C$ is nilpotent with respect to the 0-product. Let us choose any irreducible module $V$ of $C$. As it was shown in [8] (see Proposition 2.12), there exist $0 \neq u \in V$ and $\gamma \in \mathbb{k}$ such that $\{C,u\} = V$. In particular, there exists an element $b \in C$ such that $\{b,u\} = u$. Note that

$$u = \{b_1\{b_2,\ldots,\{b_\gamma \ldots u\} \} = \{\{b_\gamma \ldots u\} \} = 0$$

(see (2.10) and (2.20)), so $u = 0$ in contradiction with $V \neq 0$. 

**Theorem 5.8 ([8]).** Two conformal algebras $\text{Cend}_{N_1},P$ and $\text{Cend}_{N_2},Q$ ($\det P(x), \det Q(x) \neq 0$) are isomorphic if and only if $N_1 = N_2 = N$ and there exists $\alpha \in \mathbb{k}$ such that $P(x)$ and $Q(x + \alpha)$ have the same canonical diagonal form, i.e., there exist matrices $A,B \in M_N(\mathbb{k}[x])$, $\det A(x), \det B(x) \in \mathbb{k}\{\{0\}$, such that $P(x)A(x) = B(x)Q(x + \alpha)$. 

Theorems 5.1 and 5.8 provide the complete description of simple finitely generated associative conformal algebras of linear growth up to isomorphism.

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