STABLE AUTOMORPHIC FORMS FOR SEMISIMPLE GROUPS

JAE-HYUN YANG

ABSTRACT. In this paper, we introduce the concept of stable automorphic forms for semisimple algebraic groups and use the stability of automorphic forms to study infinite dimensional arithmetic quotients.

Table of Contents

1. Introduction
2. Stable functions on infinite dimensional varieties
3. Stable automorphic forms for a semisimple algebraic group
4. Examples of stable automorphic forms
   4.1. Stable automorphic forms for $Sp(\infty, \mathbb{R})$
   4.2. Stable automorphic forms for $SL(\infty, \mathbb{R})$
5. Applications of the stability to geometry
   5.1. The universal moduli space of abelian varieties
   5.2. The universal moduli space of curves
   5.3. The universal moduli space of polarized real tori

References

1. Introduction

Originally the notion of stable automorphic forms was at first introduced in the symplectic group by E. Freitag [12] in 1977. Those automorphic forms were called stable modular forms by Freitag. He proved that the set of all stable modular forms is a polynomial ring in a countably infinite set of indeterminates over $\mathbb{C}$ (cf. [12, Theorem 2.5, p. 204] or Theorem 5.1). Thereafter R. Weissauer investigated stable modular forms in the sense of Freitag intensively for the study of Eisenstein series [49]. In 2014, Codogni and Shepherd-Barron [9] proved that there do not exist nontrivial stable Schottky-Siegel modular forms for the Jacobian locus (see Theorem 5.5). In 2016, Codogni [8] proved that there exist nontrivial stable Schottky-Siegel modular forms for the hyperelliptic locus (see Theorem 5.6).

In this article, we will deal with the case of stable automorphic forms only for semisimple algebraic groups. The motivation of introducing the notion of stable automorphic forms is...
to investigate the geometric properties of finite or infinite dimensional arithmetic quotients associated with those automorphic forms.

The purpose of this article is to generalize the concept of stable modular forms to that of stable automorphic forms for semisimple algebraic groups and apply the stability of automorphic forms to the study of the universal moduli space of abelian varieties, the universal moduli space of curves and the universal moduli space of polarized real tori. This paper is organized as follows. In section 2, we review the notion of infinite dimensional algebraic varieties due to I. R. Shafarevich [42, 43, 29]. We introduce the notion of stable functions. In section 3, we introduce the notion of stable automorphic forms for semisimple algebraic groups. In the case of a finite dimensional semisimple algebraic group, we follow the definition of automorphic forms given by Harish-Chandra (cf. [22]) and Borel (cf. [2]). In section 4, as examples, we consider stable automorphic forms for both an infinite dimensional symplectic group $Sp(\infty, \mathbb{R})$ and an infinite dimensional special linear group $SL(\infty, \mathbb{R})$. In the final section, using the stability of automorphic forms for $Sp(\infty, \mathbb{R})$ and $SL(\infty, \mathbb{R})$, we characterize the so-called universal (or stable) Satake compactifications and investigate their geometry. We deal with the universal moduli space of abelian varieties, the universal moduli space of curves and the universal moduli space of polarized real tori.

**Notations.** We denote by $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the ring of integers, the field of real numbers and the field of complex numbers respectively. $\mathbb{Z}^+$ and $\mathbb{Z}_+$ denote the set of all positive integers and the set of all nonnegative integers respectively. The symbol “:=” means that the expression on the right is the definition of that on the left. $F^{(k,l)}$ denotes the set of all $k \times l$ matrices with entries in a commutative ring $F$. For a square matrix $A$, $\text{Tr}(A)$ denotes the trace of $A$. For any $M \in F^{(k,l)}$, $^t M$ denotes the transpose of $M$. For $A \in F^{(k,l)}$ and $B \in F^{(k,k)}$, we set $B[A] = ^t ABA$ (Siegel’s notation). $I_n$ denotes the identity matrix of degree $n$. For a complex matrix $A$, $\overline{A}$ denotes the complex conjugate of $A$. $\text{diag}(a_1, a_2, \cdots , a_n)$ denotes the $n \times n$ diagonal matrix with diagonal entries $a_1, \cdots , a_n$. For a smooth manifold, we denote by $C^c_c(X)$ (resp. $C^\infty_c(X)$) the algebra of all continuous (resp. infinitely differentiable) functions on $X$ with compact support.

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

denotes the symplectic matrix of degree $2n$.

$$\mathbb{H}_n = \{ \Omega \in \mathbb{C}^{(n,n)} | \Omega = ^t \Omega, \ \text{Im} \Omega > 0 \}$$

denotes the Siegel upper half plane of degree $n$.

$$Sp(2n, \mathbb{R}) = \{ M \in \mathbb{R}^{(2n,2n)} | ^t M J_n M = J_n \}$$

denotes the symplectic group of degree $g$ and

$$\Gamma_n = \{ \gamma \in \mathbb{Z}^{(2n,2n)} | ^t \gamma J_n \gamma = J_n \} \subset Sp(2n, \mathbb{R})$$

denotes the Siegel modular group of degree $n$. For a positive integer $n$, we denote

$$\mathcal{P}_n = \{ Y \in \mathbb{R}^{(n,n)} | Y = ^t Y > 0 \}$$

and

$$\mathfrak{X}_n = \{ Y \in \mathbb{R}^{(n,n)} | Y = ^t Y > 0, \ \det (Y) = 1 \}.$$ We denote $\mathfrak{X}_n = SL(n, \mathbb{Z}) \backslash \mathcal{X}_n$. 
2. Stable functions on infinite dimensional varieties

First we review the notion of infinite dimensional algebraic groups due to I.R. Shafarevich (cf. [42, 43, 29]).

**Definition 2.1.** By an infinite dimensional algebraic variety over a field $k$ we mean the inductive limit $X$ of a directed system $(X_i, f_{ij})$ of finite dimensional algebraic varieties over the field $k$, where $f_{ij} : X_i \rightarrow X_j \,(i < j)$ are closed embeddings. We write $X := \lim_{i} X_i$.

Throughout this paper, we shall consider only the case where the set of indices is the set $\mathbb{Z}^+$ of all positive integers. Each of the $X_i$ will be considered to be equipped with its Zariski topology and we endow $X$ with the topology of the inductive limit where a set $Z \subset X$ is closed if and only if its preimage in each $X_i$ is closed. In particular, each $X_i$ is closed in $X$.

**Definition 2.2.** A continuous mapping $f : X \rightarrow Y$ of two infinite dimensional algebraic varieties is called a morphism if for any $X_i$ in the system $(X_i)$ defining $X$, there exist at least one $Y_j$ in the system $(Y_j)$ defining $Y$ such that $f(X_i) \subset Y_j$ and the restriction $f : X_i \rightarrow Y_j$ is a morphism of finite dimensional algebraic varieties. Irreducibility and connectedness of an infinite dimensional algebraic variety are defined as irreducibility and connectedness of the corresponding topological space.

**Definition 2.3.** An infinite dimensional algebraic variety $G$ with a group structure is called an infinite dimensional algebraic group if the inverse mapping $x \mapsto x^{-1}$ and the multiplication $(x, y) \mapsto xy$ are morphisms for all $x, y \in G$.

In a similar way, we may define the notions of infinite dimensional smooth manifolds, infinite dimensional complex manifolds, infinite dimensional real or complex Lie groups and so on with a usual topology and suitable morphisms.

Let $X$ be an infinite dimensional space with its directed system $(X_i, f_{ij})$. Let $V$ be a fixed finite dimensional complex vector space. We assume that

(I) to each $X_i$ there is given the vector space $C_i$ of functions on $X_i$ with values in $V$ and that

(II) there is given an inverse system $(C_i, \Phi_{ij})$ of linear maps $\Phi_{ij} : C_j \rightarrow C_i \,(i < j)$ such that $\Phi_{ik} = \Phi_{ij} \circ \Phi_{jk}$ for all $i < j < k$.

Now we let

$$C := \lim_{i} (C_i, \Phi_{ij})$$

$$= \left\{ (f_k) \in \prod_{i \in \mathbb{Z}^+} C_i \mid \Phi_{ij}(f_j) = f_i \text{ for all } i < j \right\}$$

be the inverse limit of the system $(C_i, \Phi_{ij})$. Elements of $C$ are called stable functions.
Example 2.1. Let 
\[ \mathbb{R}^\infty := \lim_{i \to \infty} \mathbb{R}^i \]
be the infinite dimensional Euclidean space, where \( \mathbb{R}^i \) is the real Euclidean space of dimension \( i \). For a positive integer \( i \), we let \( C_c(\mathbb{R}^i) \) be the vector space of all real-valued continuous functions on \( \mathbb{R}^i \) with compact support. For any two positive integers \( i, j \) with \( i < j \), we define 
\[ \psi_{ij} : C_c(\mathbb{R}^j) \to C_c(\mathbb{R}^i) \]
by 
\[ \psi_{ij}(f)(x) := f(x, 0, \cdots, 0) \]
with \( x \in \mathbb{R}^i \), \( (0, \cdots, 0) \) an \( (j-i) \)-dimensional vector. We take the inverse limit 
\[ C = \lim_{\leftarrow i} (C_c(\mathbb{R}^i), \psi_{ij}) \]
of the system \( (C_c(\mathbb{R}^i), \psi_{ij}) \). Elements of \( C \) are stable continuous functions with compact support.

Example 2.2. For any two nonnegative integers \( k, l \in \mathbb{Z}_+ \) with \( k < l \), we define the mapping 
\[ \varphi_{kl} : \mathbb{H}_k \to \mathbb{H}_l \]
by 
\[ \varphi_{kl}(Z) = \begin{pmatrix} Z & 0 \\ 0 & iI_{l-k} \end{pmatrix}, \quad Z \in \mathbb{H}_k. \]
We recall that \( \mathbb{H}_n \) denotes the Siegel upper half plane of degree \( n \) (see Notations). Then the image \( \varphi_{kl}(\mathbb{H}_k) \) is a totally geodesic subspace of \( \mathbb{H}_l \). We let 
\[ \mathbb{H}_\infty = \lim_{k \to \infty} \mathbb{H}_k \]
be the inductive limit of the direct system \( (\mathbb{H}_k, \varphi_{kl}) \). \( \mathbb{H}_\infty \) can be described explicitly as follows:
\[ \left\{ \begin{pmatrix} Z & 0 \\ 0 & iI_{\infty} \end{pmatrix} \mid Z \in \mathbb{H}_k \text{ for some } k \geq 1 \right\}. \]
We can show that \( \mathbb{H}_\infty \) is an infinite dimensional smooth Hermitian symmetric manifold locally closed on \( \mathbb{C}^\infty \), the complex vector space of finite sequences with the finite topology (cf. [15, 19]). \( \mathbb{H}_\infty \) has an invariant Riemannian metric which induces the normalized Riemannian metric on each embedded interior subspace \( \mathbb{H}_k \) in \( \mathbb{H}_\infty \).

The symplectic group \( Sp(2n, \mathbb{R}) \) acts on \( \mathbb{H}_n \) transitively by 
\[ g \cdot \Omega := (A\Omega + B)(C\Omega + D)^{-1}, \]
where \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2n, \mathbb{R}) \) and \( \Omega \in \mathbb{H}_n \). For a fixed nonnegative integer \( k \), a holomorphic function \( f : \mathbb{H}_n \to \mathbb{C} \) is called a Siegel modular form of weight \( k \) if it satisfies the following conditions:

(SM1) \[ f(\gamma \cdot \Omega) = \det(C\Omega + D)^k f(\Omega) \]
for all \( \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \) and \( \Omega \in \mathbb{H}_n \).

(SM2) If \( n = 1 \), \( f \) requires a cuspidal condition, that is, \( f \) is bounded in any domain \( y \geq y_0 > 0 \).
Here $\Gamma_n$ is the Siegel modular group of degree $n$ (See Notations). We denote $[\Gamma_n, k]$ the vector space of all Siegel modular forms of weight $k$. For any two positive integers $i, j$ with $i < j$, we recall the well-known Siegel operator

$$\varphi_{ij} : [\Gamma_j, k] \rightarrow [\Gamma_i, k]$$

declared by

$$(2.4) \quad \varphi_{ij}(f)(\Omega) := \lim_{t \rightarrow \infty} f\left(\Omega 0 \sqrt{-1} t I_{j-i}\right), \quad \Omega \in \mathbb{H}_i.$$ 

It is well known that $\varphi_{ij}$ is a well-defined linear map (cf. [13]). For a fixed nonnegative integer $k$, we take the inverse limit

$$(2.5) \quad [\Gamma_\infty, k] := \lim_{i \leftarrow} ([\Gamma_i, k], \varphi_{ij})$$

of the system $([\Gamma_i, k], \varphi_{ij})$. Elements of $\Gamma_\infty(k)$ are called stable modular forms.

3. Stable automorphic forms for a semisimple algebraic group

Before we introduce the notion of stable automorphic forms for a semisimple algebraic group, we recall automorphic forms on a semisimple algebraic group of finite dimension (cf. [2, 3, 22]).

Let $G$ be a finite dimensional semisimple algebraic group with a maximal compact subgroup $K$. Let $\rho : K \rightarrow GL(V)$ be a given representation of $K$ on a finite dimensional complex vector space $V$. Let $\Gamma$ be an arithmetic subgroup of $G$. A smooth vector-valued function $f : G \rightarrow V$ is called an automorphic form of type $(\rho, \Gamma)$ if it satisfies the following conditions (AF1)–(AF3):

- (AF1) $f(\gamma g k) = \rho(k)^{-1} f(g)$ for all $\gamma \in \Gamma, g \in G$ and $k \in K$.
- (AF2) $f$ is $Z(g)$-finite.
- (AF3) $f$ satisfies a suitable growth condition.

Here $Z(g)$ denotes the center of the universal enveloping algebra $U(g)$ of the Lie algebra $g$ of $G$.

**Theorem 3.1.** The vector space of all automorphic forms of type $(\rho, \Gamma)$ is finite dimensional.

**Proof.** The proof was done by Harish-Chandra [22].

Let $G_\infty$ be an infinite dimensional semisimple algebraic group with its inductive system $(G_i, \phi_{ij})$ of finite dimensional semisimple algebraic groups $G_i$ and the group monomorphisms $\phi_{ij} : G_i \rightarrow G_j$ ($i < j$). We fix a finite dimensional complex vector space $V$.

We now assume that

(I) there is given a sequence $(K_i)$ of compact subgroups such that each $K_i$ is a maximal compact subgroup of $G_i$ and $\phi_{ij}(K_i) \subset K_j$ for all $i < j$. 

(II) there is given a sequence $(\Gamma_i)$ such that each $\Gamma_i$ is an arithmetic subgroup of $G_i$ and $\phi_{ij}(\Gamma_i) \subset \Gamma_j$ for all $i < j$. 

(III) there is given a sequence \((\rho_i)\) such that each \(\rho_i\) is a representation of \(K_i\) on \(V\) compatible with the morphisms \(\phi_{ij}\), that is, if \(i < j\), then \(\rho_j(\phi_{ij}(k)) = \rho_i(k)\) for all \(k \in K_i\).

For each positive integer \(i \in \mathbb{Z}^+\), we let \(A(\Gamma_i, \rho_i)\) be the complex vector space of all automorphic forms of type \((\rho_i, \Gamma_i)\). According to the definition, we see that if \(f \in A(\Gamma_i, \rho_i)\), then \(f\) satisfies the following conditions \((AF1)_i-(AF3)_i:\)

\((AF1)_i\) \(f(\gamma g k) = \rho_i(k)^{-1} f(g)\) for all \(k \in K_i, g \in G_i\) and \(\gamma \in \Gamma_i\).

\((AF2)_i\) \(f\) is \(Z(g_i)\)-finite.

\((AF3)_i\) \(f\) satisfies a suitable growth condition.

Here \(Z(g_i)\) denotes the center of the universal enveloping algebra \(U(g_i)\) of the Lie algebra \(g_i\) of \(G_i\).

We also assume that

(IV) there is a sequence \(\{A_i\}\) and \(\{L_{ij}\}\) of linear maps \(L_{ij} : A_j \rightarrow A_i, \quad i < j\)

satisfying the conditions

\[ L_{ik} = L_{ij} \circ L_{jk} \quad \text{for all } i < j < k, \]

where \(A_i\) is a subspace of \(A(\Gamma_i, \rho_i), \quad i = 1, 2, \ldots\). Elements of the inverse limit of the inverse system \((A_i, L_{ij})\)

\[(3.1) \quad A_\infty := \lim_{i \to \infty} A_i\]

are called stable automorphic forms for an infinite dimensional semisimple algebraic group \(G_\infty\). If there is no confusion, we briefly say stable automorphic forms.

We put

\[ K_\infty := \lim_{i \to \infty} K_i, \quad \Gamma_\infty := \lim_{i \to \infty} \Gamma_i \quad \text{and} \quad \rho_\infty := \lim_{i \to \infty} \rho_i. \]

We call \(\rho_\infty\) a stable representation of \(K_\infty\) or simply a stable representation. It is easy to see that a stable automorphic form \(f\) in \(A_\infty\) satisfies the following conditions \((SAF1)-(SAF3):\)

\((SAF1)\) \(f(\gamma g k) = \rho_\infty(k)^{-1} f(g)\) for all \(k \in K_\infty, g \in G_\infty\) and \(\gamma \in \Gamma_\infty\).

\((SAF2)\) \(f\) is \(Z(g_\infty)\)-finite.

\((SAF3)\) \(f\) satisfies a suitable growth condition.

Here

\[ Z(g_\infty) = \lim_{i \to \infty} Z(g_i) \]

denotes the center of the universal enveloping algebra \(U(g_\infty)\) of the Lie algebra \(g_\infty\) of an infinite dimensional semisimple algebraic group \(G_\infty\).

**Theorem 3.2.** The dimension of \(A_\infty\) is finite.

**Proof.** The proof follows from the definition of \(A_\infty\) and the fact that the dimension of \(A(\Gamma_i, \rho_i)\) for each \(i\) is finite due to Harish-Chandra [22] (see also Theorem 3.1). \(\square\)
4. Examples of stable automorphic forms

In this section, we give two examples of stable automorphic forms for both $Sp(\infty, \mathbb{R})$ and $SL(\infty, \mathbb{R})$.

**Example 4.1. Stable automorphic forms for $Sp(\infty, \mathbb{R})$**

First of all, we provide some geometric properties on the Einstein-Kähler Hermitian symmetric manifold $Sp(2n, \mathbb{R})/U(n) \cong \mathbb{H}_n$ that is important geometrically and number theoretically.

We let $G := Sp(2n, \mathbb{R})$ and $K = U(n)$. We recall that $G$ acts on $\mathbb{H}_n$ transitively via the formula (2.3). The stabilizer of the action (2.3) at $iI_n$ is

$$\left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \mid A + iB \in U(n) \right\} \cong U(n).$$

Thus we get the biholomorphic map

$$G/K \rightarrow \mathbb{H}_n, \quad gK \mapsto g \cdot iI_n, \quad g \in G.$$  

It is well known that $\mathbb{H}_n$ is an Einstein-Kähler Hermitian symmetric manifold. Since $\mathbb{H}_n$ is Kähler, it is a symplectic manifold.

For $Z = (z_{ij}) \in \mathbb{H}_n$, we write $Z = X + iY$ with $X = (x_{ij})$, $Y = (y_{ij})$ real. We put $dZ = (dz_{ij})$ and $d\overline{Z} = (d\overline{z}_{ij})$. We also put

$$\frac{\partial}{\partial Z} = \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial z_{ij}} \quad \text{and} \quad \frac{\partial}{\partial \overline{Z}} = \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial \overline{z}_{ij}}.$$  

C. L. Siegel [45] introduced the symplectic metric $ds_{n;A}^2$ on $\mathbb{H}_n$ invariant under the action (2.3) of $Sp(2n, \mathbb{R})$ that is given by

$$(4.1) \quad ds_{n;A}^2 = A \cdot Tr(Y^{-1}dZ Y^{-1}d\overline{Z}), \quad A > 0.$$  

It is known that the metric $ds_{n;A}^2$ is a Kähler-Einstein metric. H. Maass [31] proved that its Laplace operator $\Delta_{n;A}$ is given by

$$(4.2) \quad \Delta_{n;A} = \frac{4}{A} \cdot Tr \left( Y^{-1} \left( Y \frac{\partial}{\partial Z} \right) \frac{\partial}{\partial \overline{Z}} \right).$$  

And

$$(4.3) \quad dv_n(Z) = (\det Y)^{-(n+1)} \prod_{1 \leq i \leq j \leq n} dx_{ij} \prod_{1 \leq i \leq j \leq n} dy_{ij}$$

is a $Sp(2n, \mathbb{R})$-invariant volume element on $\mathbb{H}_n$ (cf. [46, p. 130]).

Siegel [46] proved the following theorem for the Siegel space $(\mathbb{H}_n, ds_{n;1}^2)$.

**Theorem 4.1.** (1) There exists exactly one geodesic joining two arbitrary points $Z_0, Z_1$ in $\mathbb{H}_n$. Let $R(Z_0, Z_1)$ be the cross-ratio defined by

$$R(Z_0, Z_1) = (Z_0 - Z_1)(Z_0 - \overline{Z}_1)^{-1}(\overline{Z}_0 - \overline{Z}_1)(\overline{Z}_0 - Z_1)^{-1}.$$  

$$(Z_0, Z_1) \in \mathbb{H}_n.$$

$$R(Z_0, Z_1) = (Z_0 - Z_1)(Z_0 - \overline{Z}_1)^{-1}(\overline{Z}_0 - \overline{Z}_1)(\overline{Z}_0 - Z_1)^{-1}.$$
For brevity, we put $R_* = R(Z_0, Z_1)$. Then the symplectic length $\rho(Z_0, Z_1)$ of the geodesic joining $Z_0$ and $Z_1$ is given by

$$\rho(Z_0, Z_1)^2 = \text{Tr} \left( \left( \log \frac{1 + R_*^2}{1 - R_*^2} \right)^2 \right),$$

where

$$\left( \log \frac{1 + R_*^2}{1 - R_*^2} \right)^2 = 4 R_* \left( \sum_{k=0}^{\infty} \frac{R_*^k}{2k + 1} \right)^2.$$

(2) For $M \in Sp(2n, \mathbb{R})$, we set

$$\tilde{Z}_0 = M \cdot Z_0 \quad \text{and} \quad \tilde{Z}_1 = M \cdot Z_1.$$

Then $R(Z_1, Z_0)$ and $R(\tilde{Z}_1, \tilde{Z}_0)$ have the same eigenvalues.

(3) All geodesics are symplectic images of the special geodesics

$$\alpha(t) = i \text{diag}(a_1^t, a_2^t, \cdots, a_n^t),$$

where $a_1, a_2, \cdots, a_n$ are arbitrary positive real numbers satisfying the condition

$$\sum_{k=1}^{n} (\log a_k)^2 = 1.$$

Proof. The proof of the above theorem can be found in [45, pp. 289-293]. \qed

Let $\mathbb{D}(\mathbb{H}_n)$ be the algebra of all differential operators on $\mathbb{H}_n$ invariant under the action (2.3). Then according to Harish-Chandra [20, 21],

$$\mathbb{D}(\mathbb{H}_n) = \mathbb{C}[H_1, H_2, \cdots, H_n],$$

where $H_1, H_2, \cdots, H_n$ are algebraically independent invariant differential operators on $\mathbb{H}_n$. We note that $n$ is the rank of $\mathbb{H}_n$, i.e., the rank of $Sp(2n, \mathbb{R})$. That is, $\mathbb{D}(\mathbb{H}_n)$ is a commutative algebra that is finitely generated by $n$ algebraically independent invariant differential operators on $\mathbb{H}_n$. Maass [32, pp. 103–121] found the explicit algebraically independent generators $H_1, H_2, \cdots, H_n$. Let $g_{\mathbb{C}}$ be the complexification of the Lie algebra of $G$. It is known that $\mathbb{D}(\mathbb{H}_n)$ is isomorphic to the center of the universal enveloping algebra of $g_{\mathbb{C}}$ (cf. [23, Chapter II]).

We consider the simplest case $n = 1$ and $A = 1$. Let $\mathbb{H} = \mathbb{H}_1$ be the Poincaré upper half plane. Let $z = x + iy \in \mathbb{H}$ with $x, y \in \mathbb{R}$ and $y > 0$. Then the Poincaré metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2} = \frac{dz \, d\bar{z}}{y^2}$$

is a $SL(2, \mathbb{R})$-invariant Kähler-Einstein metric on $\mathbb{H}$. The geodesics of $(\mathbb{H}, ds^2)$ are either straight vertical lines perpendicular to the $x$-axis or circular arcs perpendicular to the $x$-axis (half-circles whose origin is on the $x$-axis). The Laplace operator $\Delta$ of $(\mathbb{H}, ds^2)$ is given by

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$
and
\[ dv = \frac{dx \wedge dy}{y^2} \]
is a \( SL(2, \mathbb{R}) \)-invariant volume element. The scalar curvature, i.e., the Gaussian curvature is \(-1\). The algebra \( \mathcal{D}(\mathbb{H}) \) of all \( SL(2, \mathbb{R}) \)-invariant differential operators on \( \mathbb{H} \) is given by
\[ \mathcal{D}(\mathbb{H}) = \mathbb{C}[\Delta]. \]

The distance between two points \( z_1 = x_1 + iy_1 \) and \( z_2 = x_2 + iy_2 \) in \( (\mathbb{H}, ds^2) \) is given by
\[
\rho(z_1, z_2) = 2 \ln \left( \frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 + y_1)^2}}{2 \sqrt{y_1 y_2}} \right) \\
= \cosh^{-1} \left( 1 + \frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{2 y_1 y_2} \right) \\
= 2 \sinh^{-1} \frac{1}{2} \sqrt{\frac{(x_2 - x_1)^2 + (y_2 - y_1)^2}{y_1 y_2}}.
\]

For each positive integer \( n \in \mathbb{Z}^+ \), we let
\begin{equation}
G_n := Sp(2n, \mathbb{R}), \quad K_n := U(n), \quad \Gamma_n := Sp(2n, \mathbb{Z})
\end{equation}
be the symplectic group of degree \( n \), the unitary group of degree \( n \) and the Siegel modular group of degree \( n \) respectively. We fix a finite dimensional complex vector space \( V \). And we put \( G_0 = K_0 = \Gamma_0 = \{ \text{identity} \} \). For any two integers \( m, n \in \mathbb{Z}^+ \) with \( m < n \), we define the monomorphism
\begin{equation}
u_{m,n} : K_m \rightarrow K_n
\end{equation}
by
\[ u_{m,n}(A) := \begin{pmatrix} A & 0 \\ 0 & I_{n-m} \end{pmatrix}, \quad A \in K_m := U(m). \]

Let
\[ K_\infty = U(\infty) := \lim_{n \rightarrow \infty} K_n \]
be the inductive limit of the directed system \( (K_n, u_{m,n}) \). Let \( \rho_\infty := (\rho_n) \) be a stable representation of \( K_\infty \), that is,
\[ \rho_\infty := \lim_{n \rightarrow \infty} \rho_n, \]
where \( \rho_n \) is a rational representation of \( K_n \) on \( V \) and for any two positive integers \( m, n \in \mathbb{Z}^+ \) with \( m < n \),
\[ \rho_m(A) := \rho_n \begin{pmatrix} A & 0 \\ 0 & I_{n-m} \end{pmatrix}, \quad A \in K_m. \]

For each positive integer \( n \in \mathbb{Z}^+ \), we let \( A(\rho_n, \Gamma_n) \) be the vector space of automorphic forms of type \( (\rho_n, \Gamma_n) \). See Section 3 for the definition of automorphic forms of type \( (\rho_n, \Gamma_n) \). For each positive integer \( n \in \mathbb{Z}^+ \), we extend \( \rho_n \) to the complexification \( GL(n, \mathbb{C}) \) of \( K_n \) and also denote by \( \rho_n \) the extension of \( \rho_n \) to \( GL(n, \mathbb{C}) \). We note that each coset space \( G_n/K_n \) is an Einstein-Kähler Hermitian symmetric space of noncompact type and is biholomorphic to the Siegel upper half plane
\[ \mathbb{H}_n := \left\{ Z \in \mathbb{C}^{(n,n)} \mid Z = tZ, \quad \text{Im} Z > 0 \right\} \quad \text{(see Notations)}. \]
We recall that $G_n$ acts on $\mathbb{H}_n$ transitively by

$$g \cdot Z := (AZ + B)(CZ + D)^{-1},$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n$ and $Z \in \mathbb{H}_n$. Thus $G_n/K_n$ is identified with $\mathbb{H}_n$ via

$$G_n/K_n \ni gK_n \mapsto g \cdot (iI_n) \in \mathbb{H}_n.$$

Now for each positive integer $n \in \mathbb{Z}^+$, we define the automorphic factor $J_n : G_n \times \mathbb{H}_n \rightarrow GL(V)$ by

$$(4.6) \quad J_n(g, Z) := \rho_n(CZ + D),$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_n$ and $Z \in \mathbb{H}_n$.

For each positive integer $n \in \mathbb{Z}^+$, we denote by $[\Gamma_n, \rho_n]$ the vector space of Siegel modular forms on $\mathbb{H}_n$ of type $\rho_n$. We recall that a Siegel modular form $f$ in $[\Gamma_n, \rho_n]$ is a holomorphic function $f : \mathbb{H}_n \rightarrow V$ satisfying the condition

$$(4.7) \quad f(\gamma \cdot Z) = \rho_n(CZ + D)f(Z) \quad \text{for all } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n \text{ and } Z \in \mathbb{H}_n.$$

For $n = 1$, $f$ requires a cuspidal condition, that is, $f$ is bounded in any domain $Y \supseteq Y_0 > 0$. For all two positive integers $m, n \in \mathbb{Z}^+$ with $m < n$, we have the well-known classical Siegel operator $\Phi_{m,n} : [\Gamma_n, \rho_n] \rightarrow [\Gamma_m, \rho_m]$ defined by

$$(4.8) \quad (\Phi_{m,n} f)(Z) := \lim_{t \to \infty} f \begin{pmatrix} Z & 0 \\ 0 & itI_{n-m} \end{pmatrix}, \quad Z \in \mathbb{H}_m.$$

We observe that (4.8) is well defined and is a linear mapping.

For an element $F \in A(\rho_n, \Gamma_n)$, we define the function $P_n F$ on $\mathbb{H}_n$ by

$$(4.9) \quad (P_n F)(g \cdot iI_n) := J_n(g, iI_n)F(g),$$

where $g \in G_n$.

**Lemma 4.1.** If $F \in A(\rho_n, \Gamma_n)$, then $P_n F$ satisfies the condition (4.7).

**Proof.** For any $Z \in \mathbb{H}_n$, suppose $Z = g \cdot iI_n = \tilde{g} \cdot iI_n$, $g, \tilde{g} \in G_n$ with $\tilde{g} = gk$, $k \in K_n$. We write $k = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \in K_n$. Then for any $F \in A(\rho_n, \Gamma_n)$,

$$(P_n F)(Z) = (P_n F)(\tilde{g} \cdot iI_n) = J_n(\tilde{g}, iI_n)F(gk) = J_n(g, iI_n)J_n(k, iI_n)\rho_n(k)^{-1}F(g) \quad \text{(by the condition (AF1))}$$

$$= J_n(g, iI_n)\rho_n(ib + a)\rho_n(ib + a)^{-1}F(g) = J_n(g, iI_n)F(g) = (P_n F)(g \cdot iI_n).$$

Thus $P_n F$ is well defined.
We put $Z = g \cdot I_n$ for some $g \in G_n$. For any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$ and for any $F \in A(\rho_n, \Gamma_n)$,

\[
(P_n F)(\gamma \cdot Z) = (P_n F)((\gamma g) \cdot i_n) = J_n(\gamma g) F(\gamma g)
\]

Thus the condition (AF1) is satisfied. The condition (AF2) follows from the fact that $f$ holomorphic on domain $\mathbb{H}_n$. Proposition 4.1.

\[
Q_f(g) = (\gamma) \text{ for all } \gamma \in \Gamma_n.
\]

\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \text{ is a linear mapping of } \mathbb{H}_n \text{ into } \mathbb{H}_n.
\]

Therefore $P_n F$ satisfies the condition (4.7). \hfill \square

For an element $f \in [\Gamma_n, \rho_n]$, we define the function $Q_n f$ on $G_n$ by

\[
(Q_n f)(g) = J_n(g, i_n)^{-1} f(g \cdot i_n), \quad g \in G_n.
\]

**Lemma 4.2.** If $f \in [\Gamma_n, \rho_n]$, then $Q_n f$ is contained in $A(\rho_n, \Gamma_n)$.

**Proof.** For any $\gamma \in \Gamma_n$, $g \in G_n$ and $k \in K_n$,

\[
(Q_n f)(\gamma k) = J_n(\gamma g, i_n)^{-1} f((\gamma g) \cdot i_n)
\]

Thus the condition (AF1) is satisfied. The condition (AF2) follows from the fact that $f$ is holomorphic on $\mathbb{H}_n$. The condition (AF3) follows from the fact that $f(Z)$ is bounded in any domain $\{ Z \in \mathbb{H}_n \mid Z = X + i Y, \ Y \geq Y_0 > 0 \}$ for some positive definite matrix $Y_0$ of degree $n$.

From now on, we denote by $A_h(\rho_n, \Gamma_n)$ the image of $[\Gamma_n, \rho_n]$ under $Q_n$.

For all $m, n \in \mathbb{Z}^+$ with $m < n$, we define the **Siegel operator** $L_{m,n}$ on $A_h(\rho_n, \Gamma_n)$ by

\[
(L_{m,n} f)(g) = J_n(g, i_m)^{-1} \lim_{t \to \infty} J_n(g_t, i_n) f(g_t), \quad g \in G_m,
\]

where $f \in A_h(\rho_n, \Gamma_n)$ and $g_t \in G_m$ is defined by

\[
g_t := \begin{pmatrix} A & 0 & B & 0 \\ 0 & t^{1/2} I_{n-m} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & t^{-1/2} I_{n-m} \end{pmatrix}, \quad t > 0
\]

for all $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_m$.

**Proposition 4.1.** The limit in (4.11) exists and $L_{m,n}$ is a linear mapping of $A_h(\rho_n, \Gamma_n)$ into $A_h(\rho_m, \Gamma_m)$.
Proof. Let $F = Q_nf \in \mathbb{A}_h(\rho_n)$ for some $f \in [\Gamma_n, \rho_n]$. Let $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_m$ and let $g_t \in G_n (t > 0)$ be the element in $G_n$ given by the formula (4.12). Then we have

$$(L_{m,n}F)(g) = J_m(g, iI_m)^{-1} \lim_{t \to \infty} J_n(g_t, iI_n)F(g_t)$$

$$= J_m(g, iI_m)^{-1} \lim_{t \to \infty} J_n(g_t, iI_n)(Q_nf)(g_t)$$

$$= J_m(g, iI_m)^{-1} \lim_{t \to \infty} J_n(g_t, iI_n)J_n(g_t, iI_n)^{-1} f(g_t \cdot iI_n) \quad \text{(by (4.10))}$$

$$= J_m(g, iI_m)^{-1} \lim_{t \to \infty} f \begin{pmatrix} (Ai + B)(Ci + D)^{-1} & 0 \\ 0 & itI_{n-m} \end{pmatrix}.$$  

Since $f$ is an element of $[\Gamma_n, \rho_n]$, the limit

$$\lim_{t \to \infty} f \begin{pmatrix} (Ai + B)(Ci + D)^{-1} & 0 \\ 0 & itI_{n-m} \end{pmatrix} = (\Phi_{m,n}f)(g \cdot iI_m)$$

exists and is an element of $[\Gamma_m, \rho_m]$. Here $\Phi_{m,n}$ is the Siegel operator defined by the formula (4.7). Thus the limit in (4.11) exists. On the other hand,

$$(L_{m,n}F)(g) = J_m(g, iI_m)^{-1}(\Phi_{m,n}f)(g \cdot iI_m)$$

$$= (Q_m(\Phi_{m,n}f))(g).$$

Therefore $L_{m,n}F$ is an element of $\mathbb{A}_h(\rho_m)$. Hence $L_{m,n}$ is a linear mapping of $\mathbb{A}_h(\rho_n)$ into $\mathbb{A}_h(\rho_m)$.  

Let $\mathfrak{g}_n$ be the Lie algebra of $G_n$ and $\mathfrak{g}_n^C$ its complexification. Then

$$\mathfrak{g}_n^C = \left\{ \begin{pmatrix} A & B \\ C & -tA \end{pmatrix} \in \mathbb{C}^{(2n, 2n)} \mid B = tB, \ C = tC \right\}.$$  

We let $\widetilde{J}_n := iJ_n$ with $J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. We define an involution $\sigma_n$ of $G_n$ by

$$\sigma_n(g) := \widetilde{J}_n g \widetilde{J}_n^{-1}, \quad g \in G_n.$$  

The differential map $d\sigma_n = \text{Ad}(\widetilde{J}_n)$ of $\sigma_n$ extends complex linearly to the complexification $\mathfrak{g}_n^C$ of $\mathfrak{g}_n$. $\text{Ad}(\widetilde{J}_n)$ has 1 and -1 as eigenvalues. The (+1)-eigenspace of $\text{Ad}(\widetilde{J}_n)$ is given by

$$\mathfrak{k}_n^C := \left\{ \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \in \mathbb{C}^{(2n, 2n)} \mid t^2A = A, \ B = tB \right\}.$$  

We note that $\mathfrak{k}_n^C$ is the complexification of the Lie algebra $\mathfrak{k}_n$ of a maximal compact subgroup $K_n = G_n \cap SO(2n, \mathbb{R}) \cong U(n)$ of $G_n$. The (−1)-eigenspace of $\text{Ad}(\widetilde{J}_n)$ is given by

$$\mathfrak{p}_n^C = \left\{ \begin{pmatrix} A & B \\ B & -A \end{pmatrix} \in \mathbb{C}^{(2n, 2n)} \mid A = t^2A, \ B = tB \right\}.$$  

We observe that $\mathfrak{p}_n^C$ is not a Lie algebra. But $\mathfrak{p}_n^C$ has the following decomposition

$$\mathfrak{p}_n^C = \mathfrak{p}_{n,+} \oplus \mathfrak{p}_{n,-},$$

where

$$\mathfrak{p}_{n,+} = \left\{ \begin{pmatrix} X & iX \\ iX & -X \end{pmatrix} \in \mathbb{C}^{(2n, 2n)} \mid X = tX \right\}$$

and

$$\mathfrak{p}_{n,-} = \left\{ \begin{pmatrix} X & -iX \\ -iX & X \end{pmatrix} \in \mathbb{C}^{(2n, 2n)} \mid X = tX \right\}.$$
and 
\[
p_{n,-} = \left\{ \begin{pmatrix} Y & -iY \\ -iY & -Y \end{pmatrix} \in \mathbb{C}^{(2n,2n)} \left| Y = tY \right. \right\}.
\]

We observe that \( p_{n,+} \) and \( p_{n,-} \) are abelian subalgebras of \( \mathfrak{g}_n^\mathbb{C} \).

**Proposition 4.2.** A function \( F \) in \( A_h(\rho_n, \Gamma_n) \) is characterized by the following properties (Sp1)–(Sp3):

(Sp1) \[ F(\gamma g k) = \rho_n(k)^{-1} F(g) \quad \text{for all } \gamma \in \Gamma_n, \ g \in G_n \text{ and } k \in K_n. \]

(Sp2) \[ X^- F = 0 \quad \text{for all } X^- \in p_{n,-}. \]

(Sp3) For any \( M \in G_n \), the function \( \psi : G_n \to V \) defined by

\[
\psi(g) := \rho_n(Y^{-\frac{1}{2}}) F(M g), \quad g \in G_n, \ g \cdot iI_n := X + iY
\]

is bounded in the domain \( Y \geq Y_0 > 0 \) for some \( Y_0 = tY_0 > 0 \).

**Proof.** (Sp1) follows from (AF1) and (Sp2) follows from the fact that \( f \) is holomorphic on \( \mathbb{H}_n \).

Since \( F \in A_h(\rho_n) := \text{Im} Q_n, \ F = Q_n f \) for some \( f \in [\Gamma_n, \rho_n] \). Let \( Z = X + iY = g \cdot iI_n \in \mathbb{H}_n \) with \( g \in G_n \). Then for any \( M \in G_n \), we have

\[
\psi(g) = \rho_n(Y^{-\frac{1}{2}}) F(M g)
= \rho_n(Y^{-\frac{1}{2}}) J_n(M g, iI_n)^{-1} f((M g) \cdot (iI_n))
= \rho_n(Y^{-\frac{1}{2}}) J_n(g, iI_n)^{-1} J_n(M, Z)^{-1} f(M \cdot Z).
\]

If \( Y_0 \) is sufficiently large, \( \psi(g) \) is bounded. \( \square \)

**Proposition 4.3.** The mapping \( P_n \) and \( Q_n \) are compatible with the Siegel operators \( L_{m,n} \) and \( \Phi_{m,n} \) \((m < n)\). That is, for any \( m, n \in \mathbb{Z}^+ \) with \( m < n \), we have

(4.13) \[ L_{m,n} \circ Q_n = Q_m \circ \Phi_{m,n} \quad \text{on } [\Gamma_n, \rho_n] \]

and

(4.14) \[ P_m \circ L_{m,n} = \Phi_{m,n} \circ P_n \quad \text{on } A_h(\rho_n, \Gamma_n). \]

**Proof.** Let \( f \in [\Gamma_n, \rho_n] \) and \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_m \). Let \( g_t \in G_n \) \((t > 0)\) be the matrix defined by the formula (4.12). Then:

\[
(L_{m,n}(Q_n f))(g) = J_m(g, iI_m)^{-1} \lim_{t \to \infty} J_n(g_t, iI_n)(Q_n f)(g_t)
= J_m(g, iI_m)^{-1} \lim_{t \to \infty} J_n(g_t, iI_n) J_n(g_t, iI_n)^{-1} f(g_t \cdot iI_n)
= J_m(g, iI_m)^{-1} \lim_{t \to \infty} f \begin{pmatrix} (Ai + B)(Ci + D)^{-1} & 0 \\ 0 & it I_{n-m} \end{pmatrix}.
\]

On the other hand,

\[
(Q_m(\Phi_{m,n}(f)))(g) = J_m(g, iI_m)^{-1}(\Phi_{m,n}(f))(g \cdot iI_m)
= J_m(g, iI_m)^{-1} \lim_{t \to \infty} f \begin{pmatrix} g \cdot iI_m & 0 \\ 0 & it I_{n-m} \end{pmatrix}
= J_m(g, iI_m)^{-1} \lim_{t \to \infty} f \begin{pmatrix} (Ai + B)(Ci + D)^{-1} & 0 \\ 0 & it I_{n-m} \end{pmatrix}.
\]
This proves the formula (4.13).

Let $F \in A_h(\rho_m, \Gamma_m)$ and let $Z = g \cdot iI_m \in \mathbb{H}_m$ with $g \in G_m$. Let $g_t \in G_n$ ($t > 0$) be the matrix defined by the formula (4.12). Then

\[
(P_m(L_{m,n}(F))) (Z) = J_m(g, iI_m)(L_{m,n}(F))(g) \\
= J_m(g, iI_m)J_m(g, iI_m)^{-1} \lim_{t \to \infty} J_n(g_t, iI_n)F(g_t) \\
= \lim_{t \to \infty} J_n(g_t, iI_n)F(g_t).
\]

On the other hand,

\[
(\Phi_{m,n}(P_nF)) (Z) = \lim_{t \to \infty} (P_nF) \begin{pmatrix} Z & 0 \\ 0 & itI_{n-i} \end{pmatrix} \\
= \lim_{t \to \infty} (P_nF)(g_t \cdot iI_n) \\
= \lim_{t \to \infty} J_n(g_t, iI_n)F(g_t).
\]

Therefore the formula (4.14) is proved. \hfill \Box

We set

\[
\Gamma_\infty = \lim_{n} \Gamma_n = \lim_{n} Sp(2n, \mathbb{Z}) \quad \text{and} \quad \rho_\infty := \lim_{n} \rho_n.
\]

Using the Siegel operator $\Phi_{m,n}$, we define the inverse limit

\[
(4.15) \quad [\Gamma_\infty, \rho_\infty] := \lim_{n} [\Gamma_n, \rho_n]
\]

For $n \in \mathbb{Z}^+$, we put

\[
\mathbb{M}_n := \bigoplus_{\tau} [\Gamma_n, \tau],
\]

where $\tau$ runs over all isomorphism classes of irreducible rational finite dimensional representations of the general linear group $GL(n, \mathbb{C})$ of degree $n$. For $n = 0$, we set $\mathbb{M}_0 := \mathbb{C}$. For an irreducible finite dimensional representation $\tau = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ of $GL(n, \mathbb{C})$ with $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n$, $\lambda_i \in \mathbb{Z}$ ($1 \leq i \leq n$), the integer $k(\tau) := \lambda_n$ is called the weight of $\tau$. Here $\tau = (\lambda_1, \lambda_2, \cdots, \lambda_n)$ denotes the irreducible finite dimensional representation of $GL(n, \mathbb{C})$ with highest weight $(\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{Z}^n$.

For a positive integer $n \in \mathbb{Z}^+$, we define

\[
\mathbb{M}_n^{*} := \bigoplus_{\tau : \text{even}} [\Gamma_n, \tau],
\]

where $\tau$ runs over all isomorphism classes of irreducible rational finite dimensional \textit{even} representations of $GL(n, \mathbb{C})$ such that the highest weight $\lambda(\tau)$ of $\tau$ is even, i.e., $\lambda(\tau) \in (2\mathbb{Z})^n$. For $n = 0$, we also set $\mathbb{M}_0^{*} := \mathbb{C}$. Clearly for any two positive integers $m, n$ with $m < n$, the Siegel operator $\Phi_{m,n}$ maps $\mathbb{M}_n$ (resp. $\mathbb{M}_n^{*}$) into $\mathbb{M}_m$ (resp. $\mathbb{M}_m^{*}$).

We let

\[
\mathbb{M} := \lim_{n} \mathbb{M}_n \quad \text{and} \quad \mathbb{M}^{*} := \lim_{n} \mathbb{M}_n^{*}.
\]

It is easy to see that both $\mathbb{M}$ and $\mathbb{M}^{*}$ have commutative ring structures compatible with the Siegel operators $\Phi_{*,*}$. Obviously $\mathbb{M}^{*}$ is a subring of $\mathbb{M}$.

Now we obtained the following result.
Proposition 4.4.
\[ M = \bigoplus_{\rho_{\infty}} [\Gamma_{\infty}, \rho_{\infty}], \]
where \( \rho_{\infty} \) runs over all stable representations.

Definition 4.1. Elements of \( M \) are called stable modular forms and elements of \( M^* \) are called even stable modular forms.

Remark 4.1. As mentioned before in the introduction, the concept of stable modular forms was first introduced by E. Freitag [12]. Thereafter the study of stable modular forms was intensively investigated by R. Weissauer [49].

Now we give an example of stable modular forms.

Definition 4.2. A pair \((\Lambda, Q)\) is called a quadratic form if \( \Lambda \) is a lattice and \( Q \) is an integer-valued bilinear symmetric form on \( \Lambda \). The rank of \((\Lambda, Q)\) is defined to be the rank of \( \Lambda \). For \( v \in \Lambda \), the integer \( Q(v, v) \) is called the norm of \( v \). A quadratic form \((\Lambda, Q)\) is said to be even if \( Q(v, v) \) is even for all \( v \in \Lambda \). A quadratic form \((\Lambda, Q)\) is said to be unimodular if \( \det(Q) = 1 \).

Definition 4.3. Let \((\Lambda, Q)\) be an even unimodular positive definite quadratic form of rank \( r \). For a positive integer \( n \), the theta series \( \theta_{Q,n} \) associated to \((\Lambda, Q)\) is defined to be
\[ \theta_{Q,n}(\tau) := \sum_{x_1, \ldots, x_n \in \Lambda} \exp \left( \pi i \sum_{p,q=1}^{n} Q(x_p, x_q) \tau_{pq} \right), \quad \tau = (\tau_{pq}) \in \mathbb{H}_n. \]

Proposition 4.5. Let \((\Lambda, Q)\) be an even unimodular positive definite quadratic form of rank \( r \). Then the collection of all theta series \( \theta_{Q,n} \) associated to \((\Lambda, Q)\) is a stable modular form of weight \( \frac{r}{2} \).

Proof. We note that \( r \equiv 0 \pmod{8} \) (cf. [11]). It is well known that \( \theta_{Q,n}(\tau) \) is a Siegel modular form on \( \mathbb{H}_n \) of weight \( \frac{r}{2} \) (cf. [13]). We easily see that
\[ \Phi_{m,n}(\theta_{Q,n}) = \theta_{Q,m} \quad \text{for all } m, n \text{ with } m < n. \]
Therefore the collection \( \Theta_Q = (\theta_{Q,n})_{n \geq 0} \) is a stable modular form of weight \( \frac{r}{2} \). \qed

Example 4.2. Stable automorphic forms for \( SL(\infty, \mathbb{R}) \)

First of all, we provide some geometric properties on the symmetric manifold \( SL(n, \mathbb{R})/SO(n, \mathbb{R}) \) that is important geometrically and number theoretically.

Definition 4.4. For any positive integer \( n \geq 2 \), we define \( \mathcal{H}_n \) to be the set of all \( n \times n \) real matrices of the form \( z = x \cdot y \), where
\[ x = \begin{pmatrix} 1 & x_{12} & x_{13} & \cdots & x_{1n} \\ 0 & 1 & x_{22} & \cdots & x_{2n} \\ 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]
and
\[ y = \text{diag}(y_1 y_2 \cdots y_{n-1}, y_1 y_2 \cdots y_{n-2}, \cdots, y_1, 1) \]
with \( x_{ij} \in \mathbb{R} \) for \( 1 \leq i < j \leq n \) and \( y_i > 0 \) for \( 1 \leq i \leq n - 1 \).

Let \( \mathcal{X}_n := \{ Y = tY > 0, \det(Y) = 1 \} \).

We can show that \( \mathcal{F}_n \) is diffeomorphic to \( \mathcal{X}_n \). In fact, we have the Iwasawa decomposition
\[ GL(n, \mathbb{R}) = \mathcal{F}_n \cdot O(n) \cdot Z_n, \]
where \( Z_n(\cong \mathbb{R}^\times) \) is the center of \( GL(n, \mathbb{R}) \) (cf. [16, Proposition 1.2.6, pp. 11–12]). Here
\[ O(n) := O(n, \mathbb{R}) = \{ k \in GL(n, \mathbb{R}) | t^k k = k^t k = I_n \} \]
denotes the real orthogonal group of degree \( n \). We see easily that
\[ \mathcal{F}_n \cong GL(n, \mathbb{R})/(O(n) \cdot \mathbb{R}^\times) \cong SL(n, \mathbb{R})/SO(n, \mathbb{R}) \cong \mathcal{X}_n, \]
where \( \cong \) denotes the diffeomorphism.

It is seen that \( GL(n, \mathbb{R}) \) acts on \( \mathcal{F}_n \) by left translation (cf. [16, Proposition 1.2.10, p. 14]). Then we obtain
\[ \mathcal{X}_n := SL(n, \mathbb{Z})/SL(n, \mathbb{R})/SO(n) \cong SL(n, \mathbb{Z})/GL(n, \mathbb{R})/(O(n) \cdot \mathbb{R}^\times), \]
where \( SO(n) := SO(n, \mathbb{R}) = SL(n, \mathbb{R}) \cap O(n) \). Therefore we obtain the following isomorphism
\[ \mathcal{X}_n \cong SL(n, \mathbb{Z})/\mathcal{F}_n. \]

**Proposition 4.6.** Let \( n \geq 2 \). Following the coordinates of Definition 4.4, we put
\[ d^* x = \prod_{1 \leq i < j \leq n} dx_{ij} \quad \text{and} \quad dy^* = \prod_{k=1}^{n-1} y_k^{-n(n-k)-1} dy_k. \]
Then
\[ d^* z = d^* x \cdot d^* y \]
is the left \( SL(n, \mathbb{R}) \)-invariant volume element on \( \mathcal{X}_n \cong \mathcal{F}_n \).

**Proof.** The proof can be found in [16, Proposition 1.5.3, pp. 25–26]. \( \square \)

Following earlier work of Minkowski, Siegel [14] calculated the volume \( \text{Vol}(\Gamma^n \setminus \mathcal{F}_n) \) which is expressed in terms of
\[ \zeta(2) \cdot \zeta(3) \cdots \zeta(n). \]

**Theorem 4.2.** Let \( n \geq 2 \). Then the volume \( \text{Vol}(\Gamma^n \setminus \mathcal{F}_n) \) of \( \Gamma^n \setminus \mathcal{F}_n \) is given by
\[ \text{Vol}(\Gamma^n \setminus \mathcal{F}_n) = \int_{\Gamma^n \setminus \mathcal{F}_n} d^* z = n \cdot 2^{n-1} \cdot \prod_{k=2}^{n} \frac{\zeta(k)}{\text{Vol}(S^{k-1})}, \]
where \( \Gamma^n = SL(n, \mathbb{Z}) \) and
\[ \text{Vol}(S^{k-1}) = \frac{2(\sqrt{\pi})^k}{\Gamma(k/2)} \]
denotes the volume of the \((k-1)\)-dimensional sphere \( S^{k-1} \), \( \zeta(k) = \sum_{n=1}^{\infty} \frac{1}{n^k} \) is the Riemann zeta function and \( \Gamma(p) \) denotes the Gamma function.
Proof. The proof can be found in [16, Theorem 1.6.1, pp. 27–37] or [14]. □

For any positive integer \( n \geq 1 \), we let
\[
P_n := \{ Y \in \mathbb{R}^{(n,n)} \mid Y = Y^t > 0 \}
\]
be the open convex cone in the Euclidean space \( \mathbb{R}^N \) with \( N = \frac{n(n+1)}{2} \). Then \( GL(n, \mathbb{R}) \) acts \( P_n \) transitively by
\[
g \cdot Y = gY^t g \quad \text{for all } g \in GL(n, \mathbb{R}) \text{ and } Y \in P_n.
\]
Since \( O(n) \) is the isotopic subgroup of \( GL(n, \mathbb{R}) \) at \( I_n \), the symmetric space \( GL(n, \mathbb{R})/O(n) \) is diffeomorphic to \( P_n \).

For \( Y = (y_{ij}) \in P_n \), we put
\[
dY = (dy_{ij}) \quad \text{and} \quad \frac{\partial}{\partial Y} = \left( \frac{1 + \delta_{ij}}{2} \frac{\partial}{\partial y_{ij}} \right).
\]

For a fixed element \( A \in GL(n, \mathbb{R}) \), we put
\[
Y^* = A \circ Y = AY^t A, \quad Y \in P_n.
\]
Then
\[
dY^* = A dY^t A \quad \text{and} \quad \frac{\partial}{\partial Y^*} = tA^{-1} \frac{\partial}{\partial Y} A^{-1}.
\]

We can see easily that
\[
ds^2 = Tr((Y^{-1}dY)^2)
\]
is a \( GL(n, \mathbb{R}) \)-invariant Riemannian metric on \( P_n \) and its Laplacian is given by
\[
\Delta = Tr \left( \left( Y \frac{\partial}{\partial Y} \right)^2 \right),
\]
where \( Tr(M) \) denotes the trace of a square matrix \( M \). We also can see that
\[
d\mu_n(Y) = (\det Y)^{-\frac{n+1}{2}} \prod_{i \leq j} dy_{ij}
\]
is a \( GL(n, \mathbb{R}) \)-invariant volume element on \( P_n \).

**Theorem 4.3.** A geodesic \( \alpha(t) \) joining \( I_n \) and \( Y \in P_n \) has the form
\[
\alpha(t) = \exp(tA[V]), \quad t \in [0, 1],
\]
where
\[
Y = (\exp A)[V] = \exp(A[V]) = \exp(t^4 V A V)
\]
is the spectral decomposition of \( Y \), where \( V \in O(n, \mathbb{R}) \), \( A = \text{diag}(a_1, \cdots, a_n) \) with all \( a_j \in \mathbb{R} \).
The distance of \( \alpha(t) \) (\( 0 \leq t \leq 1 \)) between \( I_n \) and \( Y \) is
\[
\left( \sum_{j=1}^{n} a_j^2 \right)^{\frac{1}{2}}.
\]

Proof. The proof can be found in [17] pp. 16-17. □
We consider the following Maass-Selberg (differential) operators \( \delta_1, \delta_2, \cdots, \delta_n \) on \( \mathcal{P}_n \) defined by
\[
(4.20) \quad \delta_k = \text{Tr} \left( \left( Y \frac{\partial}{\partial Y} \right)^k \right), \quad k = 1, 2, \cdots, n,
\]
By Formula (4.18), we get
\[
\left( Y \ast \frac{\partial}{\partial Y} \ast \right)_i = A \left( Y \frac{\partial}{\partial Y} \right)_i A^{-1}
\]
for any \( A \in GL(n, \mathbb{R}) \). So each \( \delta_i \) \( (1 \leq i \leq n) \) is invariant under the action (4.17) of \( GL(n, \mathbb{R}) \).

Selberg \[40\] proved the following.

**Theorem 4.4.** The algebra \( \mathbb{D}(\mathcal{P}_n) \) of all \( GL(n, \mathbb{R}) \)-invariant differential operators on \( \mathcal{P}_n \) is generated by \( \delta_1, \delta_2, \cdots, \delta_n \). Furthermore \( \delta_1, \delta_2, \cdots, \delta_n \) are algebraically independent and \( \mathbb{D}(\mathcal{P}_n) \) is isomorphic to the commutative ring \( \mathbb{C}[x_1, x_2, \cdots, x_n] \) with \( n \) indeterminates \( x_1, x_2, \cdots, x_n \).

**Proof.** The proof can be found in \[32 \] pp.64-66. \( \square \)

Using the Maass-Selberg operators, Brennecken, Ciardo and Hilgert \[6\] found the explicit generators \( E_1, E_2, \cdots, E_{n-1} \) of the algebra of all \( SL(n, \mathbb{R}) \)-invariant differential operators on \( \mathcal{X}_n \). We observe that \( E_1, E_2, \cdots, E_{n-1} \) are algebraically independent.

For any \( \nu = (\nu_1, \nu_2, \cdots, \nu_{n-1}) \), we define the function \( I_\nu : \mathcal{H}_n \longrightarrow \mathbb{C} \) by
\[
(4.21) \quad I_\nu(z) := \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{ij}\nu_j},
\]
where
\[
b_{ij} := \begin{cases} ij, & \text{if } i+j \leq n \\ (n-i)(n-j), & \text{if } i+j \geq n. \end{cases}
\]
We denote by \( \mathbb{D}(\mathfrak{H}_n) \) the algebra of all \( SL(n, \mathbb{R}) \)-invariant differential operators on \( \mathfrak{H}_n \). Then we see that \( I_\nu(z) \) is an eigenfunction of \( \mathbb{D}(\mathfrak{H}_n) \). Let us write
\[
(4.22) \quad DI_\nu(z) = \lambda_D \cdot I_\nu(z) \quad \text{for every } D \in \mathbb{D}(\mathfrak{H}_n).
\]

Since
\[
\lambda_{D_1D_2} = \lambda_{D_1} \lambda_{D_2} \quad \text{for all } D_1, D_2 \in \mathbb{D}(\mathfrak{H}_n).
\]
The function \( \lambda_D \) (viewed as a function of \( D \)) is a character of \( \mathbb{D}(\mathfrak{H}_n) \) which is called the Harish-Chandra character.

Following Goldfeld (cf. \[16 \] Definition 5.1.3, pp.115–116], the notion of a Maass form is defined in the following way.

**Definition 4.5.** Let \( n \geq 2 \) and \( \Gamma^n = SL(n, \mathbb{Z}) \). For any \( \nu = (\nu_1, \nu_2, \cdots, \nu_{n-1}) \in \mathbb{C}^{n-1}, \) a smooth \( f : \Gamma^n \backslash \mathfrak{H}_n \longrightarrow \mathbb{C} \) is said to be a Maass form for \( \Gamma^n \) of type \( \nu \) if satisfies the following conditions \( (M1)-(M3) \):
\[
(M1) \quad F(\gamma z) = f(z) \quad \text{for all } \gamma \in \Gamma^n, \ z \in \mathfrak{H}_n.
\]
We denote by $A_n$ the following properties (SL1)–(SL3):

1. Let $G$ be a symmetric space associated to $\Gamma$ for $\Gamma$. Indeed, $G$ acts on $X_n$ transitively by

$$g \circ Y = gY^t g$$

for all $g \in G$ and $Y \in X_n$.

Thus $X_n$ is a smooth manifold diffeomorphic to the symmetric space $G^n / K^n$ through the bijective map

$$G^n / K^n \longrightarrow X_n, \quad gK^n \longmapsto g \circ I_n = g^t g \quad \text{for all } g \in G^n.$$ 

An automorphic form for $\Gamma^n$ is defined to be a real analytic function $f : X_n \longrightarrow \mathbb{C}$ satisfying the following properties (SL1)–(SL3):

1. $f$ is an eigenfunction for all $G^n$-invariant differential operators on $X_n$.
2. $f(\gamma Y^t \gamma) = f(Y)$ for all $\gamma \in G^n$ and $Y \in X_n$.
3. There exist a constant $C > 0$ and $s \in \mathbb{C}^{n-1}$ with $s = (s_1, \cdots, s_{n-1})$ such that $|f(Y)| \leq C |p_{-s}(Y)|$ as the upper left determinants $\det Y_j \longrightarrow \infty \ (1 \leq j \leq n - 1)$, where

$$p_{-s}(Y) := \prod_{j=1}^{n-1} (\det Y_j)^{-s_j}$$

is the Selberg’s power function (cf. [40, 47]).

We denote by $A(\Gamma^n)$ the space of all automorphic forms for $\Gamma^n$. A cusp form $f \in A(\Gamma^n)$ is defined to be an automorphic form for $\Gamma^n$ satisfying the following conditions:

$$\int_{X \in (\mathbb{R}/\mathbb{Z})^{(n,n-j)}} f \left( Y \begin{bmatrix} I_j & X \\ 0 & I_{n-j} \end{bmatrix} \right) dX = 0, \quad 1 \leq j \leq n - 1.$$ 

We denote by $A_0(\Gamma^n)$ the space of all cusp forms for $\Gamma^n$. 

Remark 4.2. In [16], Dorian Goldfeld studied Whittaker functions associated with Maass forms, Hecke operators for $\Gamma^n$, the Godement-Jacquet $L$-function for $\Gamma^n$, Eisenstein series for $\Gamma^n$ and Poincaré series for $\Gamma^n$. 

Let

$$G^n = SL(n, \mathbb{R}), \quad K^n = SO(n) \quad \text{and} \quad \Gamma^n = SL(n, \mathbb{Z}).$$

Let

$$X_n := \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = tY > 0, \quad \det Y = 1 \right\}$$

be a symmetric space associated to $G^n$. Indeed, $G^n$ acts on $X_n$ transitively by

(4.23) $g \circ Y = gY^t g$ for all $g \in G^n$ and $Y \in X_n$.

(M2) $Df(z) = \lambda Df(z)$ for all $D \in \mathbb{D}(\Omega_n)$ with eigenvalue $\lambda_D$ given by (4.22).

(M3) $\int_{\Gamma^n \backslash U} f(uz) du = 0$

for all upper triangular groups $U$ of the form

$$U = \left\{ \begin{pmatrix} I_{r_1} & * & * & * \\
0 & I_{r_2} & * & * \\
0 & 0 & \ddots & * \\
0 & 0 & 0 & I_{r_b} \end{pmatrix} \right\}$$

with $r_1 + r_2 + \cdots + r_b = n$. Here $I_r$ denotes the $r \times r$ identity matrix and $*$ denotes arbitrary real matrices.
Definition 4.6. Let $f \in A(\Gamma^n)$ be an automorphic form for $\Gamma^n$ with eigenvalues determined by $s = (s_1, \ldots, s_{n-1}) \in \mathbb{C}^{(n-1)}$. We set
\[
\xi_1 = \frac{1}{n-1} \sum_{k=2}^{n-1} (n-k)s_k.
\]
We define, for any $f \in A(\Gamma^n)$,
\[
\mathcal{L}_n f(W) := \lim_{v \to \infty} v^{-s_1 - \xi_1} f(Y), \quad v \in \mathbb{R}, \ W \in \mathbb{X}_{n-1}, \ Y \in \mathbb{X}_n,
\]
where $Y$, $v$, $W$ are determined by the unique decomposition of $Y$ given by
\[
Y = \begin{pmatrix} 1 & 0 & 0 \\ x & I_{n-1} \\ 0 & 0 & v^{n-1} W \end{pmatrix}, \quad x \in \mathbb{R}^{(n-1,1)}.
\]

D. Grenier [17] defined the formula (4.24) and proved the following result.

Theorem 4.5. If $f \in A(\Gamma^n)$, then $\mathcal{L}_n f \in A(\Gamma^{n-1})$. Thus $\mathcal{L}_n$ is a linear mapping of $A(\Gamma^n)$ into $A(\Gamma^{n-1})$. Moreover if $f \in A_0(\Gamma^n)$ is a cusp form, then $\mathcal{L}_n f = 0$. In general, $\ker \mathcal{L}_n \neq A_0(\Gamma^n)$.

Proof. See Theorem 2 in [17].

For any $m, n \in \mathbb{Z}^+$ with $m < n$, we define
\[
\xi_{m,n} : \Gamma^m \to \Gamma^n
\]
by
\[
\xi_{m,n}(\gamma) := \begin{pmatrix} \gamma & 0 \\ 0 & I_{n-m} \end{pmatrix}, \quad \gamma \in \Gamma^m.
\]
We let
\[
\Gamma^\infty := \lim_{\rightarrow \infty} \Gamma^n
\]
be the inductive limit of the directed system $(\Gamma^m, \xi_{m,n})$.

Definition 4.7. A collection $(f_n)_{n \geq 1}$ is said to be a stable automorphic form for $\Gamma^\infty$ if it satisfies the following conditions (4.26) and (4.27):
\[
f_n \in A(\Gamma^n), \quad n \geq 1
\]
and
\[
\mathcal{L}_{n+1} f_{n+1} = f_n, \quad n \geq 1.
\]

Let
\[
A^\infty = A(\Gamma^\infty) := \lim_{\rightarrow \infty} A(\Gamma^n)
\]
be the inverse limit of the inverse system $(A(\Gamma^n), \mathcal{L}_n)$, that is, the space of all stable automorphic forms for $\Gamma^\infty$.

We propose the following problems.

Problem 4.1. Discuss the injectivity, the surjectivity and the bijectivity of $\mathcal{L}_n$. 
Problem 4.2. Give examples of stable automorphic forms for $\Gamma^\infty$.

Problem 4.3. Investigate the structure of $A^\infty$.

Remark 4.3. We refer to [55] for more detail on stable automorphic forms for the general linear group $GL(\infty, \mathbb{R})$. We note that the general linear group $GL(n, \mathbb{R}) (n \geq 1)$ is not semisimple.

5. Applications of the stability to geometry

In the final section, we give applications of stable automorphic forms to geometry.

5.1. The universal moduli space of abelian varieties

First of all, for any two nonnegative integers $k, l \in \mathbb{Z}_+$ with $k < l$, we define the mapping $\varphi_{kl}: \mathbb{H}_k \to \mathbb{H}_l$ by

$$\varphi_{kl}(Z) := \begin{pmatrix} Z & 0 \\ 0 & iI_{l-k} \end{pmatrix}, \quad Z \in \mathbb{H}_k.$$  

Then the image $\varphi_{kl}(\mathbb{H}_k)$ is a totally geodesic subspace of $\mathbb{H}_l$. We let

$$\mathbb{H}_\infty = \lim_{k \to} \mathbb{H}_k$$

be the inductive limit of the direct system $(\mathbb{H}_k, \varphi_{kl})$. $\mathbb{H}_\infty$ can be described explicitly as follows:

$$\left\{ \begin{pmatrix} Z & 0 \\ 0 & iI_{\infty} \end{pmatrix} \mid Z \in \mathbb{H}_k \text{ for some } k \geq 1 \right\}.$$ 

We can show that $\mathbb{H}_\infty$ is an infinite dimensional smooth Hermitian symmetric manifold locally closed on $\mathbb{C}^\infty$, the complex vector space of finite sequences with the finite topology (cf. [15, 19]). $\mathbb{H}_\infty$ has an invariant Riemannian metric which induces the normalized Riemannian metric on each embedded interior subspace $\mathbb{H}_k$ in $\mathbb{H}_\infty$.

For each $n \in \mathbb{Z}_+$, we put

$$G_n := Sp(2n, \mathbb{R}), \quad K_n := U(n) \quad \text{and} \quad \Gamma_n := Sp(2n, \mathbb{Z}).$$

For any $k, l \in \mathbb{Z}_+$ with $k < l$, we define the mapping $\pi_{kl}: G_k \to G_l$ by

$$\pi_{kl} \left( \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right) := \begin{pmatrix} A & 0 & B & 0 \\ 0 & I_{l-k} & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & I_{l-k} \end{pmatrix} \quad \text{for all } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_k$$ 

and also define the mapping $\rho_{kl}: \Gamma_k \to \Gamma_l$ by the formula (5.3) with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_k$. Let

$$G_\infty := \lim_{k \to} G_k \quad \text{and} \quad \Gamma_\infty := \lim_{k \to} \Gamma_k.$$
be the inductive limit of the directed systems \((G_k, \pi_{kl})\) and \((\Gamma_k, \rho_{kl})\) respectively. Then \(G_\infty\) and \(\Gamma_\infty\) can be described explicitly as follows:

\[
G_\infty = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & I_\infty & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & I_\infty \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_k \text{ for some } k \geq 1 \right\}
\]

and

\[
\Gamma_\infty = \left\{ \begin{pmatrix} A & 0 & B & 0 \\ 0 & I_\infty & 0 & 0 \\ C & 0 & D & 0 \\ 0 & 0 & 0 & I_\infty \end{pmatrix} \mid \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_k \text{ for some } k \geq 1 \right\}.
\]

We recall that for any two positive integers \(k, l \in \mathbb{Z}^+\) with \(k < l\), the mapping \(u_{k,l} : U(k) \to U(l)\) defined by

\[
u_{k,l}(A + iB) := \begin{pmatrix} A + iB & 0 \\ 0 & I_{l-k} \end{pmatrix} \quad \text{for all } A + iB \in U(k) \text{ with } A, B \in \mathbb{R}^{(n,n)}
\]
yields the inductive limit \(K_\infty := U(\infty)\) of the directed system \((U(k), u_{k,l})\).

**Lemma 5.1.** Let \(k\) and \(l\) be two positive integers with \(k < l\). Then for any \(\gamma \in \Gamma_k\) and \(Z \in \mathbb{H}_k\), we have

\[
\varphi_{kl}(\gamma \cdot Z) = \rho_{kl}(\gamma) \cdot \varphi_{kl}(Z).
\]

**Proof.** (5.5) follows from an easy computation. \(\square\)

For each positive integer \(k \in \mathbb{Z}^+\), we let \(A_k := \Gamma_k \setminus \mathbb{H}_k\) be the Siegel modular variety of degree \(k\). We put \(\mathcal{A}_0 := \{\infty\}\). According to Lemma 5.1, for any \(k, l \in \mathbb{Z}^+\) with \(k < l\), we obtain the canonical embedding \(s_{kl} : A_k \to A_l\) defined by

\[
s_{kl}([Z]) := [\varphi_{kl}(Z)] = \begin{pmatrix} Z & 0 \\ 0 & iI_{l-k} \end{pmatrix},
\]

where \([Z] \in A_k\) with \(Z \in \mathbb{H}_k\) and \([Z]\) denotes the equivalence class of \(Z\). We let

\[
A_\infty := \lim_{k \to \infty} A_k
\]

be the inductive limit of the directed system \((A_k, s_{kl})\).

**Proposition 5.1.** \(G_\infty\) acts on \(\mathbb{H}_\infty\) transitively and \(\Gamma_\infty\) acts on \(\mathbb{H}_\infty\) properly discontinuously. \(\mathbb{H}_\infty\) is isomorphic to \(G_\infty/K_\infty\). And we have

\[
A_\infty = \Gamma_\infty \setminus G_\infty/K_\infty.
\]

\(A_\infty\) is an infinite dimensional Hermitian locally symmetric space. Furthermore \(A_\infty\) has a canonical stratification induced from the canonical stratification of the subspaces \(A_{k+1} \setminus A_k\) \((A_{k+1} \setminus A_k)\) with \(k \geq 1\).

**Proof.** We observe that \(\Gamma_\infty\) is not a finitely generated group. It is countable and an arithmetic discrete subgroup of \(G_\infty\). We see that \(\Gamma_\infty\) acts on \(\mathbb{H}_\infty\) properly discontinuously and holomorphically. The quotient space is Hausdorff. We can show without difficulty that

\[
\Gamma_\infty \setminus G_\infty/K_\infty = \lim_{k \to \infty} A_k = A_\infty.
\]
For each nonnegative integer \( d \in \mathbb{Z}_+ \), we let \([\Gamma_n, d]\) be the vector space of all Siegel modular forms on \( \mathbb{H}_n \) of weight \( d \). We review some properties of the Siegel operator \( \Phi_{n-1,n} : [\Gamma_n, d] \rightarrow [\Gamma_{n-1}, d] \) (c.f. Formula (4.8)). According to the theory of singular modular forms in [11] and [36], \( \Phi_{n-1,n} \) is injective if \( n > 2d \) and \( \Phi_{n-1,n} \) is an isomorphism if \( n > 2d + 1 \).

H. Maass [30] proved that \( \Phi_{n-1,n} \) is an isomorphism if \( d \) is even and \( d > 2n \).

For each nonnegative integer \( n \in \mathbb{Z}_+ \), we put

\[
A_n := \bigoplus_{d=0}^{\infty} [\Gamma_n, d] \quad \text{for } n \geq 1 \quad \text{and} \quad A_0 := \mathbb{C}.
\]

Then \( A_n \) is a \( \mathbb{Z}_+ \)-graded ring which is integrally closed and of finite type over \( \mathbb{C} := [\Gamma_n, 0] \). We observe that for \( m < n \), the Siegel operator \( \Phi_{m,n} \) maps \( M_n \) into \( M_m \) preserving the weights and that \( \Phi_{m,n} \) is a ring homomorphism of \( A_n \) into \( A_m \). Thus \( (A_n, \Phi_{m,n}) \) forms an inverse system of rings over \( \mathbb{Z}_+ \).

We let

\[
A_\infty := \varprojlim_n A_n
\]

be the inverse limit of the system \( (A_n, \Phi_{m,n}) \). That is,

\[
A_\infty = \left\{ (f_k) \in \prod_{l \in \mathbb{Z}_+} A_l \left| \Phi_{k,l}(f_l) = f_k \text{ for any } k < l \right. \right\}.
\]

If \( f = (f_n) \in A_\infty \), then for each \( n \in \mathbb{Z}_+ \), we write

\[
f_n = \sum_{d=0}^{\infty} f_{n,d}, \quad f_{n,d} \in [\Gamma_n, d].
\]

We note that \( \Phi_{m,n}(f_{n,d}) = f_{m,d} \) for all \( m, n \in \mathbb{Z}_+ \) with \( m < n \). For each \( d \in \mathbb{Z}_+ \), the sequence \( \{ (f_{k,d}) | k \in \mathbb{Z}_+ \} \) is called a **stable sequence** of weight \( d \). We denote by \( S_d \) the complex vector space consisting of all stable sequences of weight \( d \). Then it is easy to see that

\[
A_\infty = \bigoplus_{d=0}^{\infty} S_d.
\]

Then \( A_\infty \) is a \( \mathbb{Z}_+ \)-graded ring. It is known that \( \dim_{\mathbb{C}} S_d = \dim_{\mathbb{C}} [\Gamma_n, d] \) if \( k > 2d \) (cf. [12], p. 203).

Let \( S \) be a positive definite even unimodular integral matrix of degree \( m \). Then we define the theta series \( \vartheta_S^{(n)}(Z) \) on \( \mathbb{H}_n \) by

\[
\vartheta_S^{(n)}(Z) := \sum_{U \in \mathbb{Z}^{(m,n)}} e^{\pi i \text{Tr}(S[U]Z)}, \quad Z \in \mathbb{H}_n.
\]

Here \( S[U] := tUSU \) (Siegel’s notation). Then we can show that \( \vartheta_S^{(n)}(Z) \) is a Siegel modular form of weight \( m/2 \) on \( \Gamma_n \).

We state the result obtained by Freitag [12].
**Theorem 5.1.** \( \mathbb{A}_\infty \) is a polynomial ring in a countably infinite set of indeterminates over \( \mathbb{C} \) given by

\[
\mathbb{A}_\infty = \mathbb{C}[\vartheta_S^{(n)}(Z) \mid n \in \mathbb{Z}_+],
\]

where \( S \) runs over the set of all equivalence classes of irreducible positive definite symmetric, unimodular even integral matrices.

*Proof.* See Theorem 2.5 in [12]. \( \square \)

**Remark 5.1.** The homogeneous quotient field \( Q(\mathbb{A}_\infty) \) of \( \mathbb{A}_\infty \) is a rational function field with countably infinitely many variables. But in general, \( \mathbb{A}_n \) is not a polynomial ring. It is well known that the homogeneous function field \( Q(\mathbb{A}_n) \) of \( \mathbb{A}_n \) is an algebraic function field with the transcendence degree \( \frac{1}{2}n(n+1) \).

For any two nonnegative integers \( m, n \in \mathbb{Z}_+ \) with \( m < n \), the Siegel operator \( \Phi_{m,n} : \mathbb{A}_n \rightarrow \mathbb{A}_m \) induces the morphism \( \Phi_{m,n}^* : \text{Proj} \mathbb{A}_m \rightarrow \text{Proj} \mathbb{A}_n \) of projective schemes. The Satake compactification \( A_n^* = \text{Proj} \mathbb{A}_n \) of \( A_n \) contains \( A_n \) as a Zariski open dense subset. As a set, \( A_n^* \) is the disjoint union of \( A_n \) and its rational boundary components, i.e.,

\[
A_n^* = A_n \cup A_{n-1} \cup \cdots \cup A_1 \cup A_0, \quad A_0 = \{ \infty \}.
\]

We refer to Satake’s paper [38]. W. Baily [1] proved that \( A_n^* \) is a normal projective variety. Obviously \( (A_n^*, \Phi_{m,n}^*) \) forms an inductive system of schemes over \( \mathbb{Z}_+ \). We let

\[
A_\infty^* := \lim_{\longrightarrow} A_n^* \tag{5.13}
\]

be the inductive limit of \( (A_n^*, \Phi_{m,n}^*) \). We call the infinite dimensional variety \( A_\infty^* \) the universal (or stable) Satake compactification.

**Theorem 5.2.** The universal Satake compactification \( A_\infty^* \) has the following properties:

(1) \( A_\infty^* = \text{Proj} \mathbb{A}_\infty \).

(2) \( A_\infty^* \) is an infinite dimensional projective variety which contains \( A_\infty \) as a Zariski open dense subset. So \( A_\infty^* \) is also called the Satake compactification of \( A_\infty \).

*Proof.* The proofs of (1) and (2) follows from Theorem 5.1 and the following facts (a)–(c):

(a) \( A_n^* = \text{Proj} \mathbb{A}_n \) as schemes (see (5.8)).

(b) For sufficiently large \( m, n \in \mathbb{Z}_+ \) with \( n > m > 2d + 1 > 0 \), the Siegel operator

\[
\Phi_{m,n} : [\Gamma_n, d] \rightarrow [\Gamma_m, d]
\]

is an isomorphism.

(c) \( A_n \) is a Zariski open dense subset of \( A_n^* \). \( \square \)

Now we shall describe the analytic local ring of the image of the boundary point in \( A_n^* \) under \( f_n^* : A_n^* \rightarrow A_\infty^* \) \((n \in \mathbb{Z}_+^+)\) is the canonical morphism. Let \([Z_k] \in A_k \) \((0 \leq k \leq n - 1, Z_k \in \mathbb{H}_k)\) be a boundary point in \( A_n^* \setminus A_n \) (\( A_n^* \) setminus \( A_n \)). We set

\[
Z^*_{k,\infty} := f_k^*([Z_k]).
\]

**Theorem 5.3.** The analytic local ring at \( Z^*_{k,\infty} \) in \( A_\infty^* \) consists of all sequences \( (f_m)_{m=0}^\infty \) with \( \Phi_{m,m+1}f_{m+1} = f_m \) such that each \( f_{k+m} \) \((m \geq 1)\) is a convergent series of type

\[
f_{k+m}(Z, W_m, \Omega_m) = \sum_{T_m} \phi_{T_m}(Z, W_m) e^{2\pi i T_m Tr(T_m \Omega_m)},
\]
where $Z$ is an element in a sufficiently small open neighborhood $V$ of $z_k$ in $\mathbb{H}_k$ invariant under the action of $\Gamma_k$, $W_m \in \mathbb{C}^{(k,m)}$, $\Omega_m \in \mathbb{H}_m$ and $T_m$ runs over the set of all semi-positive symmetric half-integral matrices of degree $m$. In addition, each $\phi_{T_m}(Z, W_m)$ $(m \geq 1)$ is a Jacobi form of weight 0 and index $T_m$ defined on $V \times \mathbb{C}^{(k,m)}$.

Proof. The proof can be found in [24].

We refer to [50, 51, 52, 53, 54, 55, 59] for the notion of Jacobi forms and more results on Jacobi forms.

Ji and Jost [28] describe $A^*_k$ in a somewhat different way. Since $\mathbb{H}_k$ is a Hermitian symmetric space of noncompact type, it can be embedded into its compact dual $\mathcal{Y}_k$ which is a complex projective variety via the Borel embedding. The description of the compact dual $\mathcal{Y}_k$ is given as follows. We suppose that $\Lambda = (\mathbb{Z}^{2k}, \langle , \rangle)$ is a symplectic lattice with a symplectic form $( , )$. We extend scalars of the lattice $\Lambda$ to $\mathbb{C}$.

Let

$$\mathcal{V}_k := \left\{ L \subset \mathbb{C}^{2k} \mid \dim \mathbb{C} L = k, \langle x, y \rangle = 0 \quad \text{for all } x, y \in L \right\}$$

be the complex Lagrangian Grassmannian variety parameterizing totally isotropic subspaces of complex dimension $k$. For the present time being, for brevity, we put $G = Sp(2k, \mathbb{R})$ and $K = U(k)$. The complexification $G_\mathbb{C} = Sp(2k, \mathbb{C})$ of $G$ acts on $\mathcal{V}_k$ transitively. If $H$ is the isotropy subgroup of $G_\mathbb{C}$ fixing the first summand $\mathbb{C}^k$, we can identify $\mathcal{V}_k$ with the compact homogeneous space $G_\mathbb{C}/H$. We let

$$\mathcal{V}_k^+ := \left\{ L \subset \mathcal{V}_k \mid -i\langle x, \bar{x} \rangle > 0 \quad \text{for all } x(\neq 0) \in L \right\}$$

be an open subset of $\mathcal{V}_k$. We see that $G$ acts on $\mathcal{V}_k^+$ transitively. It can be shown that $\mathcal{V}_k^+$ is biholomorphic to $G/K \cong \mathbb{H}_k$. A basis of a lattice $L \in \mathcal{V}_k^+$ is given by a unique $2k \times k$ matrix $t(-I_k, \tau) \in \mathbb{C}^{(2k,k)}$ with $\tau \in \mathbb{H}_k$. Therefore we can identify $L$ with $\tau$ in $\mathbb{H}_k$. In this way, we embed $\mathbb{H}_k$ into $\mathcal{V}_k$ as an open subset of $\mathcal{V}_k$.

The closure $\overline{\mathcal{V}}_k$ of $\mathcal{V}_k$ in $\mathcal{V}_k$ is compact. The standard embedding $\varphi_{kl}$ (see Formula (5.1)) of $\mathbb{H}_k$ into the boundary of $\overline{\mathcal{V}}_l$ with $k < l$ and the translates by $\Gamma_l := Sp(2l, \mathbb{Z})$ of these standard boundary components give all the rational boundary components (briefly rbc) of $\mathbb{H}_k$. We denote by $\overline{\mathbb{H}}_{k,Q}$ the union of $\mathbb{H}_k$ with these rbc. Then there exists the so-called Satake topology on $\overline{\mathbb{H}}_{k,Q}$ such that $\Gamma_k$ acts continuously on $\overline{\mathbb{H}}_{k,Q}$. Then we obtain the Satake compactification of $\mathbb{H}_k$:

$$A^*_k = \Gamma_k \backslash \overline{\mathbb{H}}_{k,Q}.$$

From the increasing sequence

$$\overline{\mathbb{H}}_{1,Q} \leftrightarrow \overline{\mathbb{H}}_{2,Q} \leftrightarrow \overline{\mathbb{H}}_{3,Q} \leftrightarrow \cdots \leftrightarrow \overline{\mathbb{H}}_{k,Q} \cdots ,$$

we get the inductive limit

$$\overline{\mathbb{H}}_{\infty,Q} = \lim_{k \to \infty} \overline{\mathbb{H}}_{k,Q}.$$

$\overline{\mathbb{H}}_{\infty,Q}$ can be realized as

$$\overline{\mathbb{H}}_{\infty,Q} = \left\{ \begin{pmatrix} \Omega & 0 \\ 0 & i\tau_{\infty} \end{pmatrix} \mid \Omega \in \overline{\mathbb{H}}_{k,Q} \text{ for some } k \right\}.$$
Taking the quotient of $\mathbb{H}_\infty, \mathbb{Q}$ by $\Gamma_\infty$, we obtain the completion of $A_\infty$,

$$A^\text{Sat}_\infty = \Gamma_\infty \backslash \mathbb{H}_\infty, \mathbb{Q}.$$ 

Since

$$A^\text{Sat}_\infty = \lim_{\rightarrow} A^*_k = \bigcup_{k \geq 0} A^*_k$$

under the inclusion $A^*_k \hookrightarrow A^*_{k+1}$, we see that $A^\text{Sat}_\infty = A^*$ (cf. Theorem 5.2).

Ji and Jost [28] obtain the following result.

**Proposition 5.2.** The universal Satake compactification $A^*_\infty$ admits the following decomposition

$$A^*_\infty = A_\infty \bigcup (A_0 \sqcup A_1 \sqcup A_2 \sqcup \cdots),$$

where

$$\bigcup_{k \geq 0} A_k$$

is the boundary, and $A_\infty$ is the interior in some sense, which can also be decomposed into a non-disjoint union of $A_k, k \geq 0$. Every $A_k$ can appear in $A^*_\infty$ in two ways: either in the interior $A_\infty$ or in the boundary $\bigcup_{k \geq 0} A_k$.

**Proof.** The proof can be found in section 3 of the paper [28] of Ji and Jost. $\square$

### 5.2. The universal moduli space of curves

For each positive integer $g \in \mathbb{Z}^+$, we let $M_g$ be the moduli space of projective curves of genus $g$ and $A_g = \text{Sp}(2g, \mathbb{Z}) \backslash \mathbb{H}_g$ the Siegel modular variety of degree $g$. According to Torelli’s theorem, the Jacobi mapping

$$(5.14) \quad T_g : M_g \rightarrow A_g$$

defined by

$$C \mapsto J(C) := \text{the Jacobian of } C$$

is injective, and in fact it is an embedding. $T_g$ induces an embedding

$$T^*_g : M^*_g \rightarrow A^*_g,$$

where $M^*_g$ (resp. $A^*_g$) is the Satake compactification of $M_g$ (resp. $A_g$). The Jacobian locus $J_g := T_g(M_g)$ is a $(3g - 3)$-dimensional subvariety of $A_g$ if $g \geq 2$. Let $J^*_g := T^*_g(M^*_g)$ be the Satake compactification of $J_g$, which is equal to the closure of $J_g$ in $A^*_g$ for the Satake topology.

For convenience, we set

$$J_0 = J^*_0 = A_0 = A^*_0 = \{\infty\}, \text{ one point}.$$ 

We define

$$J_\infty := \bigcup_{g \geq 0} J_g$$

and

$$A_\infty = \bigcup_{g \geq 0} A_g, \quad A^*_\infty = \bigcup_{g \geq 0} A^*_g = \lim_{\rightarrow} A^*_g \quad (\text{see (5.7) and (5.13)}).$$
Proposition 5.3. (1) The boundary of $J_g^*$ is the union of $J_{g_1} \times \cdots \times J_{g_k}$, where $g_1 + \cdots + g_k \leq g$ with $k \geq 1$.

(2) For any two positive integers $k, g$ with $k < g$, if $J_k$ appears in the boundary of $J_g^*$, then the closure of $J_k$ is equal to the Satake compactification $J_k^*$ of $J_k$.

(3) The subspace $J_g^*$ of $A_\infty^*$ has a canonical stratification such that the closure of each stratum is a projective variety over $\mathbb{C}$, and $J_\infty^*$ is the Satake compactification of $J_\infty^*$. $J_g^*$ can appear in many different ways in $J_\infty^*$.

(4) $J_g^*$ is connected in $A_\infty^*$.

(5) For any $g \in \mathbb{Z}^+$, there is a unique way to embed $J_g$ into $J_{g+1}^*$ which is the closure of $J_{g+1}$ inside $J_g^*$. Under this inclusion, we get an increasing sequence of spaces

$$J_0^* \hookrightarrow J_1^* \hookrightarrow J_2^* \hookrightarrow J_3^* \hookrightarrow \cdots$$

and

$$J_\infty^* := \bigcup_{g \geq 0} J_g^* = \lim_{g \to \infty} J_g^*.$$

Proof. The proof can be found in section 4 of the paper [28] of Ji and Jost.

\[\square\]

Theorem 5.4. For any $g \in \mathbb{Z}^+$, there exists a Riemannian metric on $J_g^*$ that induces a Riemannian metric on each stratum. As a result, there exists a measure on $J_g^*$ that induces a finite volume measure on each stratum.

Proof. The proof can be found in [28, Theorem 5.2 and Corollary 5.3].

\[\square\]

Definition 5.1. A modular form $f \in [\Gamma_g, k]$ is called a Schottky-Siegel form of weight $k$ for $J_g$ (resp. $Hyp_g$) if it vanishes along $J_g$ (resp. $Hyp_g$). A collection $(f_g)_{g \geq 0}$ is called a stable Schottky-Siegel form of weight $k$ for the Jacobian locus (resp. the hyperelliptic locus) if $(f_g)_{g \geq 0}$ is a stable modular form of weight $k$ and $f_g$ vanishes along $J_g$ (resp. $Hyp_g$) for every $g \geq 0$.

G. Codogni and N. I. Shepherd-Barron [9] proved the following.

Theorem 5.5. There do not exist stable Schottky-Siegel form for the Jacobian locus.

Proof. See [9, Theorem 1.3 and Corollary 1.4].

\[\square\]

Remark 5.2. Let

$$\varphi_g(\tau) := \theta_{E_8 \oplus E_8, g}(\tau) - \theta_{D_{16}^+, g}(\tau), \quad \tau \in \mathbb{H}_g$$

be the Igusa modular form, that is, the difference of the theta series in genus $g$ associated to the two distinct positive even unimodular quadratic forms $E_8 \oplus E_8$ and $D_{16}^+$ of rank 16. We see that $\varphi_g(\tau)$ is a Siegel modular form on $\mathbb{H}_g$ of weight 8. Since $\Phi_{g-1, g} \varphi_g = \varphi_{g-1}$ for all $g \geq 1$, a collection $(\varphi_g)_{g \geq 0}$ is a stable modular form of weight 8. Igusa [23, 26] showed that the Schottky-Siegel form discovered by Schottky [39] is an explicit rational multiple of $\varphi_4$. In [23], he also showed that the Jacobian locus $J_4$ is reduced and irreducible, and so cuts out exactly $J_4$ in $A_4$. Indeed, $\varphi_4(\tau)$ is a degree 16 polynomial in the Thetanullwerte of genus 4. On the other hand, Grushkov and Salvati Manni [18] showed that the Igusa modular form $\varphi_5$ of genus 5 cuts out exactly the trigonal locus in $J_5$ and so does not vanish along $J_5$. Thus $(\varphi_g)_{g \geq 0}$ is not a stable Schottky-Siegel form.
G. Codogni proved the following.

**Theorem 5.6.** There exist non-trivial stable Schottky-Siegel form for the hyperelliptic locus. Precisely the ideal of stable Schottky-Siegel forms for the hyperelliptic locus is generated by differences of theta series

\[ \Theta_P - \Theta_Q, \]

where \( P \) and \( Q \) are positive definite even unimodular quadratic forms of the same rank. See Definition 4.3 for the definition of \( \Theta_P \).

**Proof.** The proof can be found in [8, Theorem 1.2]. □

**Remark 5.3.** Let \( \varphi_g(\tau) \) be the Igusa modular form defined by the formula (5.15). We denote by \( [\Gamma_g, k]_0 \) be the space of all Siegel cuspidal Hecke eigenforms on \( \mathbb{H}_g \) of weight \( k \). It is known that \( [\Gamma_4, 8]_0 = \mathbb{C} \cdot \varphi_4 \) (for a nice proof of this, we refer to [10]). Poor showed that \( \varphi_g(\tau) \) vanishes on the hyperelliptic locus \( \text{Hyp}_g \) for all \( g \geq 1 \), and the divisor of \( \varphi_g(\tau) \) in \( \mathbb{A}_g \) is proper and irreducible for all \( g \geq 4 \). And Ikeda [27] proved that if \( g \equiv k (\text{mod} 2) \), there exists a canonical lifting

\[ I_{g,k} : [\Gamma_1, 2k]_0 \longrightarrow [\Gamma_{2g}, g + k]_0. \]

Considering the special cases of the Ikeda lift maps \( I_{2,6} \) and \( I_{6,6} \), Breulman and Kuss showed that

\[ I_{2,6}(\Delta) = c \varphi_4, \quad c(\neq 0) \in \mathbb{C}, \]

and constructed a nonzero Siegel cusp form of degree 12 and weight 12 which is the image of \( \Delta(\tau) \) under the lifting \( I_{6,6} \), where

\[ \Delta(\tau) = (2\pi i)^{12} \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q := e^{2\pi i \tau}, \quad \tau \in \mathbb{H}_1 \]

is a cusp form of weight 12.

5.3. The universal moduli space of polarized real tori

Let

\[ \mathcal{P}_n := \left\{ Y \in \mathbb{R}^{(n,n)} \mid Y = {}^t Y > 0 \right\} \]

be the cone of positive definite symmetric real matrices of degree \( n \). Then \( GL(n, \mathbb{R}) \) acts on \( \mathcal{P}_n \) transitively by

\[ g \circ Y := g Y {}^t g, \quad g \in GL(n, \mathbb{R}), \ Y \in \mathcal{P}_n. \]

First we recall the concept of polarized real tori (cf. [57, p. 295]).

**Definition 5.2.** A real torus \( T = \mathbb{R}^n / \Lambda \) with a lattice \( \Lambda \) in \( \mathbb{R}^n \) is said to be polarized if the associated complex torus \( \mathfrak{A} = \mathbb{C}^n / L \) is a polarized real abelian variety, where \( L = \mathbb{Z}^n + i \Lambda \) is a lattice in \( \mathbb{C}^n \). Moreover if \( \mathfrak{A} \) is a principally polarized real abelian variety, \( T \) is said to be principally polarized. Let \( \Phi : T \longrightarrow \mathfrak{A} \) be the smooth embedding of \( T \) into \( \mathfrak{A} \) defined by

\[ \Phi(v + \Lambda) := i v + L, \quad v \in \mathbb{R}^n. \]
Let $\mathcal{L}$ be a polarization of $\mathfrak{A}$, that is, an ample line bundle over $\mathfrak{A}$. The pullback $\Phi^* \mathcal{L}$ is called a polarization of $T$. We say that a pair $(T, \Phi^* \mathcal{L})$ is a polarized real torus.

**Example 5.1.** Let $Y \in \mathcal{P}_n$ be a $n \times n$ positive definite symmetric real matrix. Then $\Lambda_Y = Y \mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$. Then the $n$-dimensional torus $T_Y = \mathbb{R}^n/\Lambda_Y$ is a principally polarized real torus. Indeed,

$$\mathfrak{A}_Y = \mathbb{C}^n/\Lambda_Y, \quad L_Y = \mathbb{Z}^n + i \Lambda_Y$$

is a principally polarized real abelian variety. Its corresponding Hermitian form $H_Y$ is given by

$$H_Y(x, y) = E_Y(i x, y) + i E_Y(x, y) = t^x Y^{-1} \overline{y}, \quad x, y \in \mathbb{C}^n,$$

where $E_Y$ denotes the imaginary part of $H_Y$. It is easily checked that $H_Y$ is positive definite and $E_Y(L_Y \times L_Y) \subset \mathbb{Z}$ (cf. [34, pp. 29–30]). The real structure $\sigma_Y$ on $\mathfrak{A}_Y$ is a complex conjugation. In addition, if $\det Y = 1$, the real torus $T_Y$ is said to be special. We refer to [57] pp. 275–279 for more details about real structure.

**Example 5.2.** Let $Q = \begin{pmatrix} \sqrt{2} & \sqrt{3} \\ \sqrt{3} & -\sqrt{5} \end{pmatrix}$ be a $2 \times 2$ symmetric real matrix of signature $(1, 1)$. Then $\Lambda_Q = Q \mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$. Then the real torus $T_Q = \mathbb{R}^2/\Lambda_Q$ is not polarized because the associated complex torus $\mathfrak{A}_Q = \mathbb{C}^2/\Lambda_Q$ is not an abelian variety, where $\Lambda_Q$ is a lattice in $\mathbb{C}^2$.

**Definition 5.3.** Two polarized tori $T_1 = \mathbb{R}^n/\Lambda_1$ and $T_2 = \mathbb{R}^n/\Lambda_2$ are said to be isomorphic if the associated polarized real abelian varieties $\mathfrak{A}_1 = \mathbb{C}^n/\Lambda_1$ and $\mathfrak{A}_2 = \mathbb{C}^n/\Lambda_2$ are isomorphic, where $L_i = \mathbb{Z}^n + i \Lambda_i$ ($i = 1, 2$), more precisely, if there exists a linear isomorphism $\varphi : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that

$$\varphi(L_1) = L_2,$$

$$\varphi_*(E_1) = E_2,$$

$$\varphi_*(\sigma_1) = \varphi \circ \sigma_1 \circ \varphi^{-1} = \sigma_2,$$

where $E_1$ and $E_2$ are polarizations of $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively, and $\sigma_1$ and $\sigma_2$ denotes the real structures (in fact complex conjugations) on $\mathfrak{A}_1$ and $\mathfrak{A}_2$ respectively.

**Example 5.3.** Let $Y_1$ and $Y_2$ be two $n \times n$ positive definite symmetric real matrices. Then $\Lambda_i := Y_i \mathbb{Z}^n$ is a lattice in $\mathbb{R}^n$ ($i = 1, 2$). We let

$$T_i := \mathbb{R}^n/\Lambda_i, \quad i = 1, 2$$

be real tori of dimension $n$. Then according to Example 5.1, $T_1$ and $T_2$ are principally polarized real tori. We see that $T_1$ is isomorphic to $T_2$ as polarized real tori if and only if there is an element $A \in GL(n, \mathbb{Z})$ such that $Y_2 = A Y_1^t A$.

**Example 5.4.** Let $Y = \begin{pmatrix} \sqrt{2} & \sqrt{3} \\ \sqrt{3} & \sqrt{5} \end{pmatrix}$. Let $T_Y = \mathbb{R}^2/\Lambda_Y$ be a two dimensional principally polarized torus, where $\Lambda_Y = Y \mathbb{Z}^2$ is a lattice in $\mathbb{R}^2$. Let $T_Q$ be the torus in Example 5.2. Then $T_Y$ is diffeomorphic to $T_Q$. But $T_Q$ is not polarized. $T_Y$ admits a differentiable embedding into a complex projective space but $T_Q$ does not.

Let

$$G^n = SL(n, \mathbb{R}), \quad K^n = SO(n) \quad \text{and} \quad \Gamma^n = GL(n, \mathbb{Z})/\{\pm I_n\}.$$
We observe that $\Gamma^n = SL(n, \mathbb{Z})/\{ \pm I_n \}$ if $n$ is even, and $\Gamma^n = SL(n, \mathbb{Z})$ if $n$ is odd. 

Let 
\[ X_n := \{ Y \in \mathcal{P}_n \mid \det Y = 1 \} \]
be the subspace of $\mathcal{P}_n$. We see that $G^n$ acts on $X_n$ transitively via (5.16), and $K^n$ is the stabilizer at $I_n$. Thus $X_n$ is a symmetric space which is diffeomorphic to the homogeneous space $G^n/K^n$ through the following correspondence 
\[ G^n/K^n \rightarrow X_n, \quad gK^n \mapsto g \circ I_n = g^t g, \quad g \in G_n. \]

The arithmetic variety 
\[ \mathfrak{R}_n = GL(n, \mathbb{Z}) \backslash \mathcal{P}_n = GL(n, \mathbb{Z}) \backslash GL(n, \mathbb{R})/O(n) \]
is the moduli space of principally polarized real tori of dimension $n$. Let 
\[ X_n := \Gamma^n \backslash X_n = \Gamma^n \backslash G^n/K^n \]
be the moduli space of special principally polarized real tori of dimension $n$. 

For any two positive integers $m, n \in \mathbb{Z}^+$ with $m < n$, we define 
\[ \eta_{m,n} : G^m \rightarrow G^n \]
by 
\[ \eta_{m,n}(A) := \begin{pmatrix} A & 0 \\ 0 & I_{n-m} \end{pmatrix} \quad \text{for all } A \in G^m. \]

We let 
\[ G^\infty := \lim_{n \rightarrow \infty} G^n, \quad K^\infty := \lim_{n \rightarrow \infty} K^n \quad \text{and} \quad \Gamma^\infty := \lim_{n \rightarrow \infty} \Gamma^n \]
be the inductive limits of the directed systems $(G^n, \eta_{m,n})$, $(K^n, \eta_{m,n})$ and $(\Gamma^n, \eta_{m,n})$ respectively. 

Let $J_n$ (resp. Hyp$_n$) be the Jacobian locus (resp. the hyperelliptic locus) in the Siegel modular variety $A_n$. We define 
\[ X_{n,J} := \{ Y \in X_n \mid \mathfrak{A}_Y \text{ is the Jacobian of a curve of genus } n, \ \text{i.e., } [\mathfrak{A}_Y] \in J_n \} \]
and 
\[ X_{n,H} := \{ Y \in X_n \mid \mathfrak{H}_Y \text{ is the Jacobian of a hyperelliptic curve of genus } n \} \]. 
See Example 5.1 for the definition of $\mathfrak{A}_Y$. We see that $\Gamma^n$ acts on both $X_{n,J}$ and $X_{n,H}$ properly discontinuously. So we may define 
\[ X_{n,J} := \Gamma^n \backslash X_{n,J} \quad \text{and} \quad X_{n,H} := \Gamma^n \backslash X_{n,H}. \]

$X_{n,J}$ and $X_{n,H}$ are called the Jacobian real locus and the hyperelliptic real locus respectively. 

**Problem.** Characterize the Jacobian real locus.

Let $X^n_S$ be the standard or maximal Satake compactification of $X_n$. For the details of the standard or maximal Satake compactification of a locally symmetric space, we refer to [4], pp. 286–291], [5] and [48], pp. 7–9. We denote by $X^n_{S,J}$ (resp. $X^n_{S,H}$) the standard or
maximal Satake compactification of $\mathcal{X}_{n,J}$ (resp. $\mathcal{X}_{n,H}$). We can show that $\mathcal{X}_{n,J}$ (resp. $\mathcal{X}_{n,H}$) is the closure of $\mathcal{X}_{n,J}$ (resp. $\mathcal{X}_{n,H}$) inside $\mathcal{X}_{n}^{S}$. We have the increasing sequences

$$\mathcal{X}_{1}^{S} \hookrightarrow \mathcal{X}_{2}^{S} \hookrightarrow \mathcal{X}_{3}^{S} \hookrightarrow \cdots,$$

$$\mathcal{X}_{1,J}^{S} \hookrightarrow \mathcal{X}_{2,J}^{S} \hookrightarrow \mathcal{X}_{3,J}^{S} \hookrightarrow \cdots$$

and

$$\mathcal{X}_{1,H}^{S} \hookrightarrow \mathcal{X}_{2,H}^{S} \hookrightarrow \mathcal{X}_{3,H}^{S} \hookrightarrow \cdots.$$

We put

$$\mathcal{X}_{\infty}^{S} := \lim_{n \to \infty} \mathcal{X}_{n}^{S}, \quad \mathcal{X}_{\infty,J}^{S} := \lim_{n \to \infty} \mathcal{X}_{n,J}^{S} \quad \text{and} \quad \mathcal{X}_{\infty,H}^{S} := \lim_{n \to \infty} \mathcal{X}_{n,H}^{S}.$$

For any two positive integers $m, n \in \mathbb{Z}^+$ with $m < n$, we embed $\mathcal{X}_{m}$ into $\mathcal{X}_{n}$ as follows:

$$\psi_{m,n} : \mathcal{X}_{m} \to \mathcal{X}_{n}, \quad Y \mapsto \begin{pmatrix} Y & 0 \\ 0 & I_{n-m} \end{pmatrix}$$

for all $Y \in \mathcal{X}_{m}$.

We let

$$\mathcal{X}_{\infty} = \lim_{n \to \infty} \mathcal{X}_{n}$$

be the inductive limit of the directed system $(\mathcal{X}_{n}, \psi_{m,n})$. We can show that

$$\mathcal{X}_{\infty} = G_{\infty}/K_{\infty}.$$

Now we have the Grenier operator

$$\mathcal{L}_{n} : A(\Gamma^{n}) \to A(\Gamma^{n-1})$$

defined by the formula (4.24).

**Definition 5.4.** An automorphic form $f \in A(\Gamma^{n})$ is said to be a Grenier–Schottky automorphic form for the Jacobian real locus (resp. the hyperelliptic real locus) if it vanishes along $\mathcal{X}_{n,J}$ (resp. $\mathcal{X}_{n,H}$). A collection $(f_{n})_{n \geq 1}$ is called a stable Grenier–Schottky automorphic form for the Jacobian real locus (resp. the hyperelliptic real locus) if it satisfies the following conditions (SGS1) and (SGS2):

1. (SGS1) $f_{n}$ is a Grenier-Schottky automorphic form for the Jacobian real locus (resp. the hyperelliptic real locus) for each $n \geq 1$.

2. (SGS2) $\mathcal{L}_{n} f_{n} = f_{n-1}$ for all $n > 1$.

The following natural question arises:

**Question 5.1.** Are there stable Grenier-Schottky automorphic forms for the Jacobian real locus (resp. the hyperelliptic real locus) ?

Finally we give the following remark.
Remark 5.4. We consider the non-reductive group
\[ G_{\ast, n} := SL(n, \mathbb{R}) \ltimes \mathbb{R}^n \]
which is the semidirect product of \( SL(n, \mathbb{R}) \) and \( \mathbb{R}^n \) with multiplication law
\[ (A, a) \circ (B, b) := (AB, a'B^{-1} + b), \quad A, B \in SL(n, \mathbb{R}), \quad a, b \in \mathbb{R}^n. \]
Then we have the natural action of \( G_{\ast, n} \) on the Minkowski-Euclid space \( \mathbb{X}_{\ast, n} := \mathbb{X}_n \times \mathbb{R}^n \)
defined by
\[ (A, a) \cdot (Y, \zeta) := (AY^tA, (\zeta + a)^tA), \quad (A, a) \in G_{\ast, n}, \quad Y \in \mathbb{X}_n, \quad \zeta \in \mathbb{R}^n. \]
It is easily seen that \( K_{\ast, n} := \{ (\lambda, 0) \in G_{\ast, n} \mid \lambda \in SO(n) \} \)
is the stabilizer of the action (5.23) at \((I_n, 0)\). Thus \( \mathbb{X}_{\ast, n} \) is a smooth manifold diffeomorphic
to the non-symmetric homogeneous space \( G_{\ast, n} / K_{\ast, n} \) via the following correspondence
\[ G_{\ast, n} / K_{\ast, n} \rightarrow \mathbb{X}_{\ast, n}, \quad (A, a) \cdot K_{\ast, n} \mapsto (A^tA, a^tA) \quad \text{for all} \quad (A, a) \in G_{\ast, n}. \]
We let
\[ \Gamma_{\ast, n} = \Gamma_n \ltimes \mathbb{Z}^n \]
be the discrete subgroup of \( G_{\ast, n} \). Then \( \Gamma_{\ast, n} \) acts on \( \mathbb{X}_{\ast, n} \) properly discontinuously. We show that by associating a special principally polarized real torus of dimension \( n \) to each equivalence class in \( \mathbb{X}_n \), the quotient space
\[ \mathbb{X}_{\ast, n} := \Gamma_{\ast, n} \backslash \mathbb{X}_{\ast, n} = \Gamma_{\ast, n} G_{\ast, n} / K_n \]
may be regarded as a family of special principally polarized real tori of dimension \( n \). We refer to [56, 57] for related topics about \( \mathbb{X}_{\ast, n} \). In a similar way, we may investigate the infinite dimensional arithmetic variety
\[ \mathbb{X}_{\infty, \ast} := \lim_{n \rightarrow} \mathbb{X}_{\ast, n}. \]
For any two positive integers \( m, n \in \mathbb{Z}^+ \) with \( m < n \), we define the injective mapping
\[ \chi_{m, n} : G_{\ast, n} \rightarrow G_{\ast, m} \]
by
\[ \chi_{m, n} ((A, a)) := \left( \begin{array}{c} A \\ 0 \\ I_{n-m} \end{array} \right), \quad (a, 0), \]
where \( A \in SL(m, \mathbb{R}), \quad a \in \mathbb{R}^m \) and \( (a, 0) \in \mathbb{R}^n \). We let
\[ G_{\infty, \ast} := \lim_{n \rightarrow} G_{\ast, n}, \quad K_{\infty, \ast} := \lim_{n \rightarrow} K_{n, \ast} \quad \text{and} \quad \Gamma_{\infty, \ast} := \lim_{n \rightarrow} \Gamma_{n, \ast} \]
be the inductive limits of the directed systems \((G_{n, \ast, \chi_{m, n}}), (K_{n, \ast, \chi_{m, n}}) \) and \((\Gamma_{n, \ast, \chi_{m, n}}) \)
respectively. For any two positive integers \( m, n \in \mathbb{Z}^+ \) with \( m < n \), we define the injective mapping
\[ \mu_{m, n} : \mathbb{X}_{\ast, m} \rightarrow \mathbb{X}_{\ast, n} \]
by
\[ \mu_{m, n} ((Y, \zeta)) := \left( \begin{array}{c} Y \\ 0 \\ I_{n-m} \end{array} \right), \quad (\zeta, 0) \]
where \( Y \in X_m \) and \( \zeta \in \mathbb{R}^m \). We let
\[
X_{\infty,0} := \lim_{n \to} X_{n,0}
\]
be the inductive limit of the directed system \((X_{n,0}, \mu_{m,n})\). Then \( G_{\infty,0} \) acts on \( X_{\infty,0} \) transitively and \( K_{\infty,0} \) is the stabilizer at the origin. Moreover \( \Gamma_{\infty,0} \) acts on \( X_{\infty,0} \) properly discontinuously. Thus we obtain
\[
X_{\infty,0} = G_{\infty,0}/K_{\infty,0}
\]
and
\[
X_{\infty,0} = \Gamma_{\infty,0}/G_{\infty,0}/K_{\infty,0} = \Gamma_{\infty,0}/X_{\infty,0}.
\]
We can define the notion of automorphic forms on \( X_{n,0} \) (cf. Definition 8.1 in [56]) and define the generalized Grenier operator \( \mathcal{L}_{n,0} \). So we can study the stability of these automorphic forms.

**References**

[1] W.L. Baily, *Satake’s compactification of \( V^n_+ \)*, Amer. J. Math. 80 (1958), 348–364.
[2] A. Borel, *Introduction to automorphic forms*, Proc. Sympos. Pure Math., Vol. 9, Amer. Math. Soc., Providence, R.I. (1966), 199-210.
[3] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, Proc. Symposia in Pure Math., Vol. XXXIII, Part 1 (1979), 189-202.
[4] A. Borel and L. Ji, *Compactifications of symmetric and locally symmetric spaces*, Math. Theory Appl. Birkhäuser Boston, Inc., Boston, MA, 2006. xvi+479 pp. ISBN:978-0-8176-3247-2 ISBN:0-8176-3247-6
[5] A. Borel and L. Ji, *Compactifications of symmetric spaces. (English summary)*, J. Differential Geom. 75 (2007), no. 1, 1–56.
[6] D. Brennecken, L. Ciardo and J. Hilgert, *Algebraically Independent Generators for the Algebra of Invariant Differential Operators on \( SL_n(\mathbb{R})/SO_n(\mathbb{R}) \)*, arXiv:2008.07479v2 [math.RT] 28 Oct 2020.
[7] S. Breulmann and M. Kuss, *On a conjecture of Duke-Imamoğlu*, Proc. Amer. Math. Soc. 128 (2000), 1595–1604.
[8] G. Codogni, *Hyperelliptic Schottky problem and stable modular forms*, Doc. Math. 21 (2016), 445–466.
[9] G. Codogni and N. I. Shepherd-Barron, *The non-existence of stable Schottky forms*, Compos. Math. 150 (2014), no. 4, 679–690.
[10] W. Duke and Ö. Imamoğlu, *Siegel modular forms of small weight*, Math. Ann. 310 (1998), 73–82.
[11] E. Freitag, *Holomorphe Differentialformen zu Kongruenzgruppen der Siegelschen Modulgruppe*, Invent. Math. 30 (1975), 181–196.
[12] E. Freitag, *Stabile Modulformen*, Math. Ann. 230 (1977), 197–211.
[13] E. Freitag, *Siegsche Modulfunktionen*, Springer (1983).
[14] P. Garret, *Volume of \( SL_n(\mathbb{Z})/SL_n(\mathbb{R}) \) and \( Sp_n(\mathbb{Z})/Sp_n(\mathbb{R}) \)*, [http://www.math.umn.edu/~garrett/m/v/volumes.pdf](http://www.math.umn.edu/~garrett/m/v/volumes.pdf) (April 20, 2014).
[15] H. Glöckner, *Direct limit Lie groups and manifolds*, J. Math. Kyoto Univ. 43 (2003), no. 1, 2–26.
[16] D. Goldfeld, *Automorphic forms and L-functions for the group \( GL(n,R) \)* With an appendix by Kevin A. Brougham Cambridge Stud. Adv. Math., 99. Cambridge University Press, Cambridge (2006). xiv+493 pp. ISBN:978-0-521-83771-2 ISBN:0-521-83771-5
[17] D. Grenier, *An analogue of Siegel’s \( \phi \)-operator for automorphic forms for \( GL(n,\mathbb{Z}) \)*, Trans. Amer. Math. Soc. 331, No. 1 (1992), 463-477.
[18] S. Grushevsky and R. Salvati Manni, *The superstring cosmological constant and the Schottky form in genus 5*, Amer. J. Math. 133 (4) (2011), 1007–1027.
[19] V. Hansen, Some theorems on direct limits of expanding sequences of manifolds, Math. Scand. 29 (1971), 5–36.

[20] Harish-Chandra, Representations of a semisimple Lie group on a Banach space. I., Trans. Amer. Math. Soc. 75 (1953), 185-243.

[21] Harish-Chandra, The characters of semisimple Lie groups, Trans. Amer. Math. Soc. 83 (1956), 98-163.

[22] Harish-Chandra, Automorphic forms on semi-simple Lie groups, Notes by J.G.M. Mars, Lecture Notes in Math., vol. 62, Springer-Verlag, Berlin-Heidelberg-New York (1968).

[23] S. Helgason, Groups and geometric analysis, Academic Press, New York (1984).

[24] J. Igusa, A desingularization problem in the theory of Siegel modular functions, Math. Ann. 168 (1967), 228–260.

[25] J. Igusa, On the irreducibility of Schottky divisor, J. Fac. Sci. Tokyo 28 (1981), 531–545.

[26] J. Igusa, Schottky’s invariant and quadratic forms, E. B. Christoffel: the influence of his work on mathematics and the physical sciences, Birkhäuser, Basel (1981), 352–362.

[27] T. Ikeda, On the lifting of elliptic cusp forms to Siegel cusp forms of degree 2n, Ann. Math. 154 (2001), 641–681.

[28] L. Ji and J. Jost, Universal moduli spaces of Riemann surfaces, arXiv:1611.08732v1 [math.AG] 26 Nov 2016.

[29] T. Kambayashi, Reviews of two papers (1966, 1981) of Igor R. Shafarevich, AMS Math. Reviews. MR0485898 (58 5697) and MR0607583 (84a:14021).

[30] H. Maass, Über die Darstellung der Modulformen n-ten Grades durch Poincaré’sche Reihen, Math. Ann. 123 (1951), 125–151.

[31] H. Maass, Die Differentialgleichungen in der Theorie der Siegelschen Modulfunktionen, Math. Ann. 126 (1953), 44–68.

[32] H. Maass, Siegel modular forms and Dirichlet series, Lecture Notes in Math., vol. 216, Springer-Verlag, Berlin-Heidelberg-New York (1971).

[33] M. Matone and R. Volpato, Vector-valued modular forms from the Mumford forms, Schottky-Igusa form, product of Thetanullwerte and the amazing Klein formula, Proc. Amer. Math. Soc. 141, no. 8 (2013).

[34] D. Mumford, Abelian Varieties. Oxford University Press (1970): Reprinted (1985).

[35] C. Poor, Schottky’s form and the hyperelliptic locus, Proc. Amer. Math. Soc. 124 (1996), 1987–1991.

[36] H. Resnikoff, Automorphic forms of singular weight are singular forms, Math. Ann. 215 (1975), 173-193.

[37] P. Ryan, Representatives of infinite Dimensional Reductive Lie Groups, Lecture Notes (December 5, 2014).

[38] I. Satake, On the compactification of the Siegel space, J. Indian Math. Soc., 20 (1956), 259–281.

[39] F. Schottky, Zur Theorie der Abelschen Funktionen von vier Variablen, J. Reine Angew. Math. 102 (1888), 304-352.

[40] A. Selberg, Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series, J. Indian Math. Soc. B. 20 (1956), 47-87.

[41] J-P. Serre, A Course in Arithmetic, Graduate Texts in Math., Vol. 7, Springer-Verlag (1973).

[42] I. R. Shafarevich, On some infinite dimensional groups, Rend. Mat. e. Appl.(5), vol. 25 (1966), no. 1–2, 208–212.

[43] I. R. Shafarevich, On some infinite dimensional groups. II (Russian), Izv. Akad. Nauk SSSR Ser. Mat.45(1981), no.1, 214–226, 240. [English translation: Math. USSR. Izvestija, vol. 18, No. 1 (1982), 185–194.]

[44] C. L. Siegel, The volume of the fundamental domain for some infinite groups, Trans. Amer. Math. Soc. vol. 30 (1936), 209–218.

[45] C. L. Siegel, Symplectic Geometry, Amer. J. Math. 65 (1943), 1-86; Academic Press, New York and London (1964); Gesammelte Abhandlungen, no. 41, vol. II, Springer-Verlag (1966), 274-359.

[46] C. L. Siegel, Topics in Complex Function Theory: Abelian Functions and Modular Functions of Several Variables, vol. III, Wiley-Interscience, 1973.

[47] A. Terras, Harmonic analysis on symmetric spaces and applications II, Springer-Verlag (1988).

[48] T. Weissauer, Absence of principal eigenvalues for higher rank locally symmetric spaces, arXiv:2205.03167v2 [math.SP] 28 Apr 2023.

[49] R. Weissauer, Stabile Modulformen und Eisensteinreihen, Lecture Notes in Math. 1219, Springer-Verlag, Berlin-Heidelberg-New York (1986).
[50] J.-H. Yang, *Remarks on Jacobi forms of higher degree*, Proc. of the 1993 Workshop on Automorphic Forms and Related Topics, edited by Jin-Woo Son and Jae-Hyun Yang, the Pyungsan Institute for Mathematical Sciences (1993), 33–58.

[51] J.-H. Yang, *The Siegel-Jacobi Operator*, Abh. Math. Sem. Univ. Hamburg 63 (1993), 135–146.

[52] J.-H. Yang, *Singular Jacobi Forms*, Trans. Amer. Math. Soc. 347 (6) (1995), 2041–2049.

[53] J.-H. Yang, *Construction of vector valued modular forms from Jacobi forms*, Canadian J. of Math. 47 (6) (1995), 1329–1339.

[54] J.-H. Yang, *A geometrical theory of Jacobi forms of higher degree*, Kyungpook Math. J. 40 (2) (2000), 209–237 or arXiv:math.NT/0602267.

[55] J.-H. Yang, *Geometry and Arithmetic on the Siegel-Jacobi Space*, Geometry and Analysis on Manifolds, In Memory of Professor Shoshichi Kobayashi (edited by T. Ochiai, A. Weinstein et al), Progress in Mathematics, Volume 308, Birkhäuser, Springer International Publishing AG Switzerland (2015), 275–325.

[56] J.-H. Yang, *Invariant differential operators on the Minkowski-Euclid space*, J. Korean Math. Soc. 50 (2013), No. 2, pp. 275–306.

[57] J.-H. Yang, *Polarized real tori*, J. Korean Math. Soc. 52 (2015), No. 2, pp. 269–331.

[58] J.-H. Yang, *Stable automorphic forms for the general linear group*, to appear in J. Korean Math. Soc.

[59] C. Ziegler, *Jacobi Forms of Higher Degree*, Abh. Math. Sem. Hamburg 59 (1989), 191–224.

YANG INSTITUTE FOR ADVANCED STUDY  
HYUNDAI 41 TOWER, NO. 1905  
293 MOKDONDONG-RO, YANGCHEON-GU  
SEOUL 07997, KOREA

DEPARTMENT OF MATHEMATICS  
INHA UNIVERSITY  
INCHEON 22212, KOREA

Email address: jhyang@inha.ac.kr or jhyang8357@gmail.com