ON PERTURBATIONS OF THE 
ANTI-DE SITTER-SCHWARZSCHILD SPACES OF 
POSITIVE MASS

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Abstract. In this paper we prove the Penrose inequality for metrics 
that are small perturbations of the Anti-de Sitter-Schwarzschild metrics 
of positive mass. We use the existence of a global foliation by weakly 
stable constant mean curvature spheres and the monotonicity of the 
Hawking mass.

1. Introduction

Let $(M^3, g)$ be a complete Riemannian three-manifold, possibly with 
boundary, with exactly one end. Assume that the complement of a compact 
subset of $M$ is diffeomorphic to $\mathbb{R}^3$ minus a ball. If the metric $g$ 
decays sufficiently fast to the hyperbolic metric when expressed in the spherical 
coordinates induced by this chart, $(M, g)$ is called asymptotically hyperbolic. 
Associated to the end of $(M, g)$ there is a geometric quantity $m$ called the 
mass. When $(M, g)$ has scalar curvature $R \geq -6$ and no boundary, this 
invariant must be nonnegative, and equal to zero if and only if $(M, g)$ is 
isometric to the hyperbolic space. This is the Positive Mass Theorem in the 
asymptotically hyperbolic setting, proved by X. Wang \[16\] and also by P. 
Chruściel and M. Herzlich \[4\] under weaker asymptotic conditions.

If $(M, g)$ has scalar curvature $R \geq -6$ and its boundary $\partial M$ is a con-
nected minimal surface which is outermost, i.e., if there are no other closed 
minimal surfaces in $M$, then it is conjectured that the mass $m$ of $(M, g)$ is 
related to the area of $\partial M$ by the following inequality:

\begin{equation}
\left( \frac{|\partial M|}{16\pi} \right) ^{\frac{1}{2}} + 4 \left( \frac{|\partial M|}{16\pi} \right) ^{\frac{3}{2}} \leq m.
\end{equation}

This statement is known as the Penrose Conjecture in the asymptotically 
hyperbolic setting. We refer the reader to the surveys \[3\] and \[9\], where he 
will find a comprehensive discussion on these types of inequalities in various 
settings. In the original asymptotically flat setting, the Penrose Conjecture 
has been proved with different techniques by G. Huisken and T. Ilmanen \[7\] 
and by H. Bray \[2\].

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The Penrose Conjecture contains also a rigidity statement. There are important models, known as the *Anti-de Sitter-Schwarzschild spaces of positive mass*, that satisfy equality in \((1)\). They are obtained as spherically symmetric metrics \(g_m\) on \(M = [0, +\infty) \times S^2\) with constant scalar curvature \(-6\), where the parameter \(m\) is a positive real number that coincides with the mass of \((M, g_m)\) above discussed (see Section 2 for more details). The Penrose Conjecture also asserts that the Anti-de Sitter-Schwarzschild spaces of positive mass are the unique asymptotically hyperbolic manifolds, with scalar curvature \(R \geq -6\) and an outermost minimal boundary, that satisfy the equality in \((1)\).

A special feature of the Anti-de Sitter-Schwarzschild spaces of positive mass is that they are foliated by constant mean curvature spheres which are *weakly stable*. This kind of foliation of an asymptotically hyperbolic manifold is interesting, among other reasons, because of a monotonicity result observed by H. Bray in [1]: if \(R \geq -6\), the so-called Hawking mass functional, defined for closed surfaces \(\Sigma\) in \((M, g)\) by

\[
(2) \quad m_H(\Sigma) := \sqrt{\frac{|\Sigma|}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} (H^2 - 4) d\Sigma\right),
\]

is monotone non-decreasing along these foliations in the direction of increasing area of the leaves.

If \(\partial M\) is a minimal surface in \((M, g)\), its Hawking mass is exactly the left-hand side of \((1)\). This suggests an approach to prove the Penrose inequality \((1)\), at least for the class of asymptotically hyperbolic manifolds \((M, g)\), with \(R \geq -6\) and an outermost minimal boundary, admitting this kind of foliation starting at \(\partial M\) and sweeping out all \(M\): inequality \((1)\) would follow if the Hawking mass of the leaves near the infinity converges to the mass of \((M, g)\).

In [13] and [14], A. Neves and G. Tian proved existence and uniqueness results for foliations by weakly stable CMC spheres near the infinity of certain asymptotically hyperbolic manifolds. See also the works of R. Rigger [15] and R. Mazzeo and F. Pacard [11] for other results of this nature in the asymptotically hyperbolic setting. For the original asymptotically flat setting, see the pioneering work of G. Huisken and S.T. Yau [8].

In the present paper, we consider metrics on \(M = [0, +\infty) \times S^2\) that are small *global* perturbations of the Anti-de Sitter-Schwarzschild metric \(g_m\) in a suitable sense and show that the foliation constructed in [14] outside a compact set can be extended up to \(\partial M\). Following the argument outlined above, we then show the Penrose inequality is true for these asymptotically hyperbolic manifolds.

In order to state our results more precisely, let us introduce some terminology. Given \(m > 0\), the perturbations of \(g_m\) we consider belong to the space \(\mathcal{M}(M, m)\) of Riemannian metrics \(g\) on \(M := \{p = (s, x) \in [0, +\infty) \times S^2\}\).
such that \(\partial M\) is a minimal surface in \((M, g)\) and

\[
d(g, g_m) := \sup_{p \in M} \left( \sum_{i=0}^{3} \exp(-4s) \| (\nabla^m)^i (g - g_m) \|_{g_m} (p) \right) < +\infty.
\]

See Section 2 for the details. All these metrics are asymptotically hyperbolic with mass \(m\) in the sense of [16] and [4].

Our main results can be summarized as follows (see Theorem 7 and Theorem 8):

**Theorem.** Let \(m > 0\). Then there exists \(\epsilon > 0\) such that for every metric \(g \in \mathcal{M}(M, m)\) with \(d(g, g_m) < \epsilon\) the following statements hold:

i) There exists a foliation \(\{\Sigma_t\}_{t \in [0, +\infty)}\) of \((M, g)\) by weakly stable CMC spheres such that \(\Sigma_0 = \partial M\) is an outermost minimal surface.

ii) (The Penrose inequality). If the scalar curvature of \(g\) is greater than or equal to \(-6\), then

\[
\left( \frac{|\partial M|}{16\pi} \right)^{\frac{3}{2}} + 4 \left( \frac{|\partial M|}{16\pi} \right)^{\frac{1}{2}} \leq \lim_{t \to +\infty} m_H(\Sigma_t) = m,
\]

with equality if and only if \((M, g)\) is isometric to the Anti-de Sitter-Schwarzschild space of mass \(m\).

The paper is organized as follows. In Section 2, we describe the geometric properties of the models and define the class of perturbations of \((M, g_m)\) we are going to work with. In Section 3, we construct a foliation of a compact region of \(M\) by weakly stable CMC spheres which begins at the minimal boundary. In Section 4, we recall A. Neves and G. Tian’s result in [14] about existence and uniqueness of foliations near the infinity of certain asymptotically hyperbolic manifolds by weakly stable CMC spheres, stating a version adapted to the space of metrics we consider. In Section 5, we calculate the limit of the Hawking mass of the leaves of that foliation. In Section 6, we argue that the two foliations constructed before glue together if the perturbation is small enough and show that the obtained foliation has the properties described in the above theorem. In Section 7, we prove the Penrose inequality for these small perturbations following the argument outlined above.

We remark that in the asymptotically hyperbolic setting there is also another form of the Penrose Conjecture where the boundary corresponds to some \(H = 2\) surface (see [3] and [9]). In the end of the paper we briefly discuss it and explain the straightforward modifications of the previous results that establish it for small perturbations (see Theorem 9 and Theorem 10).

The approach to the Penrose Conjecture involving the monotonicity of the Hawking mass for a family of surfaces that interpolates the outermost boundary and the infinity was originally suggested in the asymptotically
flat setting by R. Geroch [5], who proposed the inverse mean curvature flow to produce such family. This program was successfully implemented by G. Huisken and T. Ilmanen [7]. In the asymptotically hyperbolic setting, however, there are serious difficulties in using this approach, see the work of A. Neves [12].

In the asymptotically hyperbolic setting, the Penrose Conjecture in its full generality is still an open problem. We remark that F. Girão and L. L. de Lima proved it for another class of asymptotically hyperbolic manifolds, those that are graphical (see [6]).

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2. THE ANTI-DE SITTER-SCHWARZSCHILD SPACES AND ITS PERTURBATIONS

Let \( m \) be a real number. Let \( \rho : (r_0, +\infty) \to \mathbb{R} \) be the function given by \( \rho(r) = \sqrt{1+r^2-2m/r} \), where \( r_0 = r_0(m) \) is the unique positive zero of \( \rho \) (if \( m > 0 \)) or 0 (if \( m \leq 0 \)). Let \((S^2, g_0)\) be the round sphere of constant Gaussian curvature 1 and let \(dS^2\) denote its volume element.

We call \textit{Anti-de Sitter-Schwarzschild space of mass} \( m \) the metric completion of the Riemannian manifold \( ((r_0, +\infty) \times S^2, g_m) \), where using the natural \( r \)-coordinate the metric \( g_m \) is written as

\[
g_m = \frac{dr^2}{1 + r^2 - \frac{2m}{r}} + r^2 g_0.
\]

Although the expression of \( g_m \) in this coordinate system becomes singular at \( r_0 \), it can be proved that when \( m > 0 \) the metric \( g_m \) extends to a smooth Riemannian metric on \( M = [r_0, +\infty) \times S^2 \). On the other hand a smooth extension is not possible if \( m < 0 \). Notice also that if we let the parameter \( m \) to be zero we get the hyperbolic space (this can be easily seen by performing the coordinate change \( r = \sinh s \)).

We will call \textit{coordinate spheres} the surfaces \( S_r = \{ r \} \times S^2 \subset M \). The following proposition describes the geometry of \((M, g_m)\) and of its coordinate spheres.

**Proposition 1.** (Geometry of the Anti-de Sitter-Schwarzschild space of mass \( m \))

i) The Ricci curvature of \( g_m \) is given by

\[
\text{Ric}_m = (\frac{-2m}{r^3}) \frac{1}{\rho^2} dr^2 + (\frac{-2m}{r^3}) r^2 g_0.
\]

ii) The scalar curvature of \( g_m \) is constant and equal to \(-6\).
iii) The coordinate spheres $S_r$ are totally umbilic surfaces with constant mean curvature given by

\[ H_m(r) = \frac{2}{r} \sqrt{1 + r^2 - \frac{2m}{r}}. \]

iv) The Hawking mass of the coordinate spheres $S_r$ is $m$.

v) The Jacobi operator of the coordinate sphere $S_r$ is given by

\[ L_r = \frac{1}{r^2} \left( \Delta_0 + \left( 2 - \frac{6m}{r} \right) \right), \]

where $\Delta_0$ is the Laplacian operator of the round sphere $(S^2, g_0)$.

Proof. A calculation in coordinates. □

Notice that when $m > 0$ the Jacobi operator of $S_r$ is invertible except at $r = 3m$. In any case, it is invertible when restricted to the space of zero mean value functions. However, a degeneration occurs when $r$ goes to infinity: up to normalization, it becomes $\Delta_0 + 2$, which is no more invertible in this restricted space.

From now on we assume $m$ is positive. Let $s$ be the function that gives the distance of a point on $(M, g_m)$ to $\partial M$. Using $s \in [0, +\infty)$ as coordinate, one can write

\[ g_m = \frac{dr^2}{1 + r^2 - \frac{2m}{r}} + r^2 g_0 = ds^2 + \sinh^2(s)v_m(s)g_0, \]

where $v_m$ is a positive function defined on $[0, +\infty)$ that has the following expansion as $s$ goes to infinity:

\[ v_m(s) = 1 + \frac{2m}{3\sinh^3 s} + O(\exp(-5s)). \]

Although we have explicit formulas as in Proposition 1 only when we use the $r$-coordinate, it will be more convenient to use the $s$-coordinate. We will then consider $g_m$ to be defined on $M = [0, +\infty) \times S^2$ by formula (3) above, and as a small abuse of notation we use $s$ both for the first coordinate of a point $p \in M$ and for the function $r \in (r_0, +\infty) \mapsto s(r) \in (0, +\infty)$ that gives the coordinate change described above. It worths noticing that, as a function of $s$, the $r$ coordinate expands as $r = \sinh s(1 + O(\exp(-3s)))$ as $s$ goes to infinity. In particular, for example, the mean curvature of the coordinate spheres $S_s := S_{r(s)}$ behaves as

\[ H_m(s) = \frac{2\cosh s}{\sinh s} - \frac{2m}{\sinh^3 s} + O(\exp(-5s)) \quad \text{as } s \text{ goes to infinity}. \]

Now we define the class of metrics on $M = [0, +\infty) \times S^2$ we are going to work with. Fix some $\alpha \in (0, 1)$.

Definition. Given $m > 0$, let $\mathcal{M}(M, m)$ be the set of Riemannian metrics $g$ on $M = [0, +\infty) \times S^2$ such that
a) There exists a constant $C > 0$ such that for every $p = (s, x) \in M$,
\[
\left( \|g - g_m\| + \|\nabla^m g\| + \|\nabla^m g\|^2 + \|\nabla^m g\|^3 \right)(p) \leq C \exp(-4s).
\]
Here the norm is the $C^{0,\alpha}$-norm calculated with respect to the metric $g_m$ and $\nabla^m$ denotes the Levi-Civita connection of $g_m$.

b) $\partial M$ is minimal in $(M, g)$.

The space $\mathcal{M}(M, m)$ has a distance function
\[
d(g_1, g_2) := \sup_{p \in M} \left( \sum_{i=0}^{3} \exp(-4s)\|\nabla^m_i(g_1 - g_2)\|(p) \right).
\]

We remark that each metric in $\mathcal{M}(M, m)$ is asymptotically hyperbolic with mass $m$, according to the definitions of [16] and [4]. Observe also that we do not assume a priori that $\partial M$ is outermost.

Given $g \in \mathcal{M}(M, m)$ one can calculate the expansions of its Ricci tensor, its scalar curvature and the mean curvature of the coordinate spheres in $(M, g)$ as follows: one adds terms of order $O(\exp(-4s))$ to the expansion of the corresponding quantities of $(M, g_m)$ in $s$ coordinate.

We finish this section by discussing the geometry of surfaces in $(M, g)$, $g \in \mathcal{M}(M, m)$.

In this paper, all surfaces considered are closed surfaces $\Sigma \subset M = [0, +\infty) \times S^2$ such that $M \setminus \Sigma$ has two connected components, one of them containing $\partial M$. We define the inner radius and the outer radius of such $\Sigma$ to be
\[
\underline{s} = \inf \{s(x); x \in \Sigma\} \quad \text{and} \quad \overline{s} = \sup \{s(x); x \in \Sigma\},
\]
respectively. We also use the convention that the unit normal vector $N$ points toward the unbounded component of $M \setminus \Sigma$ and that the mean curvature is the trace of the second fundamental form $A$ given by $A(X, Y) := g(\nabla_X N, Y)$ for every pair of vectors $X, Y$ tangent to $\Sigma$.

A constant mean curvature surface $\Sigma$ in $(M, g)$ is called weakly stable when its Jacobi operator,
\[
L_\Sigma = \Delta_\Sigma + \text{Ric}(N, N) + |A|^2,
\]
is such that $-\int_\Sigma L_\Sigma(\phi)\phi d\Sigma \geq 0$ for all functions with $\int_\Sigma \phi d\Sigma = 0$. This analytic definition is equivalent to the geometric one that the second variation of the area of $\Sigma$ under volume preserving variations is nonnegative.

Throughout this paper, we will frequently consider surfaces in $(M, g)$ that are graphical over coordinate spheres $S_s$. Given some function $f$ on $S^2$, we write
\[
S_s(f) := \{(s + f(x), x) \in M; x \in S^2\}.
\]
3. Foliation of compact regions

For metrics $g \in \mathcal{M}(M, m)$ that are close enough to $g_m$, we use the implicit function theorem to construct a family of weakly stable CMC spheres on compact regions of $(M, g)$ containing $\partial M$.

**Theorem 2.** Let $m > 0$. Given $S > 0$, there exists $\epsilon > 0$ and $\eta > 0$ with the following properties:

For every $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ and for every $s \in [0, S]$ there exists a unique function $u(s, g) \in C^{2,\alpha}(S^2)$ with $\|u(s, g)\|_{C^{2,\alpha}} < \eta$ and $\int u(s, g)dS^2 = 0$ such that the surface

$$\Sigma_s(g) := S_s(u(s, g)) = \{(s + u(s, g)(x), x) \in M; x \in S^2\}$$

has constant mean curvature with respect to the metric $g$.

Moreover, for every $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$, $\Sigma_0(g) = \partial M$ and the family $\{\Sigma_s(g)\}_{s \in [0, S]}$ gives a foliation of a compact region of $(M, g)$ by weakly stable CMC spheres, with positive mean curvature if $s \in (0, S]$.

Finally, when $S > s(3m)$, given any constant $\kappa > 0$ and any closed interval $I \subset (s(3m), S]$, it is possible to choose $\epsilon > 0$ above in such way that for every $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ the mean curvature $H_g(s)$ of $\Sigma_s(g)$ is monotone decreasing on $I$ and satisfies $|H_g(s) - H_m(s)| < \kappa$ for every $s \in I$.

**Proof.** For a minor technical reason, we fix some small $a > 0$ and consider $M \subset \tilde{M} = (-a, +\infty) \times S^2$. For each metric $g \in \mathcal{M}(M, m)$ we use the Taylor expansion of $g$ in normal exponential coordinates based on $\partial M$ to define an extension of $g$ to $\tilde{M}$. This space of metrics inherits the distance of $\mathcal{M}(M, m)$. For simplicity we keep using the same notation $\mathcal{M}(M, m)$ for this space of metrics.

Consider the Banach spaces

$$E = \{u \in C^{2,\alpha}(S^2); \int_{S^2} udS^2 = 0\} \text{ and } F = \{u \in C^{0,\alpha}(S^2); \int_{S^2} udS^2 = 0\}.$$  

Given $s \in [0, S]$ and $u \in E$ sufficiently small, we consider the surfaces

$$S_s(u) = \{(s + u(x), x) \in \tilde{M}; x \in S^2\}.$$  

Denote by $H(s, u, g)$ the mean curvature of $S_s(u)$ with respect to a metric $g \in \mathcal{M}(M, m)$. Given a sufficiently small $\eta > 0$, we consider the map $\Phi : [0, S] \times \mathcal{M}(M, m) \times (B(0, \eta) \subset E) \to F$ given by

$$\Phi(s, g, u) = H(s, u, g) - \frac{1}{4\pi} \int_{S^2} H(s, u, g)dS^2.$$  

By definition, $\Phi(s, g, u) = 0$ if and only if $S_s(u)$ is a CMC surface in $(\tilde{M}, g)$. In particular, $\Phi(s, g_m, 0) = 0$ for all $s \in [0, S]$.

We claim that, for every $s \in [0, S]$, $D\Phi(s, g_m, 0)$ is an isomorphism when restricted to $E$. In fact, for every $v \in E$, the family $t \mapsto S_s(tv)$ is a normal
variation of the coordinate sphere $S_s$ in $(M, g_m)$ with speed $v$. Therefore, if $L_s$ is the Jacobi operator of $S_s$ with respect to $g_m$, we have

$$\frac{d}{dt}_{|t=0} \Phi(s, g_m, v) = -L_s(v) + \frac{1}{4\pi} \int_{S^2} L_s(v) dS^2 = -L_s(v).$$

The last equality follows because $L_s(v) = (1/r^2)(\Delta_0 + (2 - 6m/r))(v)$ (see Proposition 1) considering the coordinate change $r = r(s)$ and $v$ has zero mean value. Since $m > 0$, $\Delta_0 + (2 - 6m/r)$ is an invertible operator from $E$ to $F$ for all $s \in [0, S]$ and the claim follows.

Therefore we can apply the implicit function theorem: there exists a small ball $B$ around $(s, g_m)$ in $[0, S] \times \mathcal{M}(M, m)$, some possibly smaller $\eta > 0$ and a function $(\tilde{s}, g) \in B \mapsto u(\tilde{s}, g) \in B(0, \eta) \subset E$ such that $u(s, g_m) = 0$ and $u(\tilde{s}, g)$ is uniquely defined in $B(0, \eta)$ by the equation $\Phi(\tilde{s}, g, u(\tilde{s}, g)) = 0$ for all $(\tilde{s}, g) \in B$.

By compactness, we can choose sufficiently small $\eta > 0$ and $\epsilon > 0$ such that $u$ is uniquely defined on $[0, S] \times \{g \in \mathcal{M}(M, m) ; d(g, g_m) < \epsilon\}$ and takes values on $B(0, \eta) \subset E$. Therefore for each metric $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ we have constructed a family

$$\{\Sigma_s(g)\}_{s \in [0, S]} := \{S_s(u(s, g))\}_{s \in [0, S]}$$

of CMC spheres in $(M, g)$, where $u(s, g) \in E$ has norm $\|u(s, g)\|_{C^{2,0}} < \eta$. Notice that $\{\Sigma_s(g_m)\}$ must be precisely the foliation of $M$ by the coordinate spheres.

Since $\partial M = S_0(0)$ is minimal for all metrics $g \in \mathcal{M}(M, m)$, the uniqueness of the function $u$ above constructed implies that $u(0, g) = 0$, i.e., $\Sigma_0(g) = \partial M$ for every $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$.

In order to prove that $\{\Sigma_s(g)\}$ is a foliation of some region of $M$, we have to analyze the sign of its lapse function, that is, its normal speed. Since for $g_m$ the constructed family $\{\Sigma_s(g_m)\}$ is a foliation, its lapse function is positive on $[0, S]$, hence the lapse function of $\{\Sigma_s(g)\}_{s \in [0, S]}$ with respect to all $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ is also positive, at least when we choose a possibly smaller $\epsilon$. Since each of these families starts at $\partial M = \{0\} \times S^2 \subset \tilde{M}$, the families $\{\Sigma_s(g)\}_{s \in [0, S]}$ foliate a compact region of $M$.

To see that $\Sigma_g(s)$ has positive mean curvature in $(M, g)$ for all $s \in (0, S]$, observe that is true for $\{\Sigma_s(g_m)\}$, and also that, by Proposition 1, $H_m'(0) = 2/r_0^2(2 - 6m/r_0) > 0$. Hence, by continuity we can arrange $\epsilon$ in such way that for every $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ the surface $\Sigma_g(s)$ has positive mean curvature for all $s \in (0, S]$.

Now we argue that the leaves are weakly stable. In fact, since $m > 0$, for every $s \in [0, S]$ the Jacobi operator $L_s$ of $\Sigma_s(g_m)$ in $(M, g)$ satisfies

$$-\int L_s(\phi) \phi d\Sigma_s(g_m) = \int |\nabla_0 \phi|^2 - \left(2 - \frac{6m}{r(s)}\right) \phi^2 dS^2 \geq \frac{6m}{r(S)} \int \phi^2 dS^2$$

for every $\phi \in C^\infty(\Sigma_s(g_m))$ with $\int \phi d\Sigma_s(g_m) = 0$. Choosing some possibly smaller $\epsilon$, there exists a constant $c > 0$ such that for every $g \in \mathcal{M}(M, m)$
with $d(g,g_m) < \epsilon$ and every $s \in [0,S]$ the Jacobi operator $L_{(s,g)}$ of the surface $\Sigma_s(g)$ in $(M,g)$ is such that

$$- \int L_{(s,g)}(\phi) \phi d\Sigma_s(g) \geq c \int \phi^2 d\Sigma_s(g) \geq 0$$

for every $\phi \in C^\infty(\Sigma_s(g))$ with $\int \phi d\Sigma_s(g) = 0$, i.e., $\Sigma_s(g)$ is weakly stable.

The last statement of the theorem also follows by continuity, since $H'_m(s) < 0$ on $(s(3m), +\infty)$. □

Having in mind the gluing argument (see Section 6), we finish this section showing the existence and uniqueness of small graphs over coordinate spheres $S_s$ with prescribed mean curvature $H_m(s)$ in $(M,g)$, $g \in \mathcal{M}(M,m)$, at least when $d(g,g_m)$ is small enough and $s$ is large enough. More precisely, we have:

**Theorem 3.** Let $m > 0$. Given a closed interval $I \subset (s(3m), +\infty)$, there exists $\epsilon > 0$ and $\eta > 0$ with the following properties:

For every $g \in \mathcal{M}(M,m)$ with $d(g,g_m) < \epsilon$ and for every $s \in I$ there exists a unique function $h(s,g) \in C^{2,\alpha}(S^2)$ with $\|h(s,g)\|_{C^{2,\alpha}} < \eta$ such that the surface

$$S_s(h(s,g)) = \{(s + h(s,g))(x), x \in M; x \in S^2\}$$

has constant mean curvature $H_m(s)$ in $(M,g)$.

Moreover, the family $\{S_s(h(s,g))\}_{s \in I}$ gives a foliation of a compact region of $(M,g)$ by weakly stable CMC spheres.

**Proof.** Following the notations of Theorem 2 given $\eta > 0$ sufficiently small we consider the map $\Phi : I \times \mathcal{M}(M,m) \times (B(0,\eta) \subset C^{2,\alpha}(S^2)) \to C^{0,\alpha}(S^2)$ given by $\Phi(s,g,u) = H(s,u,g)$. Notice that $\Phi(s,g_m,0) = H_m(s)$ for all $s \in I$.

Given $s \in I$ and $g \in \mathcal{M}(M,m)$, we want to solve the equation $\Phi(s,g,u) = H_m(s)$ for some $u \in C^{2,\alpha}(S^2)$. The linearization of $\Phi$ at $(s,g_m,0)$ is such that, for every $v \in C^{2,\alpha}(S^2)$,

$$D\Phi_{(s,g_m,0)}(0,0,v) = -L_s(v) = -\frac{1}{r^2}(\Delta_0 + (2 - \frac{6m}{r})), $$

where we use the coordinate $r = r(s)$, see Proposition 1. Since $I \subset (s(3m), +\infty)$, the map $v \in C^{2,\alpha}(S^2) \mapsto L_s(v) \in C^{0,\alpha}(S^2)$ is an isomorphism for all $s \in I$. Hence, we can apply the implicit function theorem. The last statement follows by the same arguments of Theorem 2. □

4. Foliation near the infinity

The next theorem is the version of the existence and uniqueness theorem of A. Neves and G. Tian adapted to the asymptotically hyperbolic manifolds $(M,g)$ where $g$ belongs to the space of metrics $\mathcal{M}(M,m)$ (we refer the reader to Theorem 2.2 and the proof of Theorem 8.2 in [14]).
Theorem 4 (see [14]). Let $m > 0$. Given $\epsilon_0 > 0$, there exists $\delta > 0$, $C > 0$ and $\tilde{s}_0 > s(3m)$ with the following properties:

1) Given $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon_0$, for all $l \in (2, 2 + \delta)$, there exists a unique sphere $\Sigma = \Sigma_l(g) \subset M$ such that
   a) $M \setminus \Sigma$ has two connected components, one of them containing $\partial M$;
   b) $\Sigma$ is a weakly stable constant mean curvature sphere in $(M, g)$ with mean curvature $H = l$; and
   c) The inner radius $\underline{s}_l$ and the outer radius $\bar{s}_l$ of $\Sigma$ satisfy
      \[ \underline{s}_l \geq \tilde{s}_0 \quad \text{and} \quad \bar{s}_l - \underline{s}_l \leq 1. \]

2) The family $\{\Sigma_l\}_{l \in (2, 2 + \delta)}$ gives a smooth foliation of the complement of a compact set of $M$ and $\lim_{l \to 2} \Sigma_l = +\infty$.

3) Given $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon_0$ and $l \in (2, 2 + \delta)$, if for the above surface $\Sigma_l$ in $(M, g)$ we define $\tilde{s}_l$ by the equality
      \[ |\Sigma_l| = 4\pi \sinh^2 \tilde{s}_l, \]

then:
   a) If we set $w_l(p) = s(p) - \tilde{s}_l$ for $p \in \Sigma_l$, then
      \[ \sup_{\Sigma_l} |w_l| \leq C \exp(-\underline{s}_l) \quad \text{and} \quad \int_{\Sigma_l} |\partial_\Sigma|^2 d\Sigma \leq C \exp(-2\underline{s}_l). \]
   b) \[ \int_{\Sigma_l} |\hat{A}_l|^2 d\Sigma \leq C \exp(-4\underline{s}_l). \]
   c) There exists a function $f \in C^2(S^2)$ with $\|f\|_{C^2} \leq C$ such that
      \[ \Sigma_l = S_{\tilde{s}_l}(f) = \{(\tilde{s}_l + f(x), x) \in M; x \in S^2\}. \]

Remark 1. Using the terminology of [14], the metrics in $\mathcal{M}(M, m)$ satisfy condition (H) with same constants $r_1, C_2$ and $C_3$, and the constant $C_1$ depends only on $d(g, g_m)$. Therefore, if we fix the constant $C_4 = 1$, all constants appearing in their theorem depend only on the distance between the metric $g \in \mathcal{M}(M, m)$ and $g_m$. This uniform dependence on $d(g, g_m)$ is crucial to the gluing argument.

Remark 2. Theorem 4 is proven by the continuity method, see Theorem 8.2 in [14]. In particular, for a fixed closed interval $I \subset (\tilde{s}_0, +\infty)$, there exists $\epsilon \in (0, \epsilon_0)$ and $\eta > 0$ with the following property: for every $g \in \mathcal{M}(M, m)$ with $d(g, g_m) < \epsilon$ and for every $s \in I$, there exists a unique function $h \in C^{2, \alpha}(S^2)$ with $\|h\|_{C^{2, \alpha}} \leq \eta$ such that the surface $\Sigma_l$ in $(M, g)$ with $l = H_m(s)$ given by Theorem 4 can be written as $\Sigma_l = S_s(h) = \{(s + h(x), x) \in M; x \in S^2\}$.

In other words, on compact sets of the form $I \times S^2 \subset M$, where $I$ is an interval contained in $(s(3m), +\infty)$, all the surfaces of the foliation $\{\Sigma_l\}$ of $(M, g)$ with mean curvature $l = H_m(s)$ for $s \in I$ are obtained by using the implicit function theorem as in Theorem 3 at least for metrics $g \in \mathcal{M}(M, m)$. 

sufficiently close to \( g_m \). This remark contains all the information needed for the gluing argument (see Section 6).

5. LIMIT OF THE HAWKING MASS

Let \( g \) be a metric in \( \mathcal{M}(M, m) \) with scalar curvature \( R \geq -6 \). Recall that the Hawking mass of a closed surface \( \Sigma \) in \( (M, g) \) is

\[
m_H(\Sigma) = \sqrt{\frac{\int_{\Sigma} H^2}{16\pi}} = \left(1 - \frac{1}{16\pi} \int_{\Sigma} (H^2 - 4)d\Sigma\right).
\]

We want to calculate the limit of the Hawking mass of the weakly stable CMC spheres \( \Sigma \) in \( (M, g) \) given by Theorem 4 as they approach the infinity. In order to do this, we need the following consequence of Gauss equation (see \[13\]).

**Lemma 5.** Given \( g \in \mathcal{M}(M, m) \), let \( \{\Sigma_t\}_{t>0} \) be a family of constant mean curvature spheres in \( (M, g) \) such that \( s_t \to +\infty \) as \( t \) goes to infinity. Then

\[
(H_t^2 - 4)|\Sigma_t| = 16\pi - \frac{8m - 12m|\partial_s|_g^2}{\sinh s}d\Sigma_t + 2\int_{\Sigma_t} |A_t|d\Sigma_t + |\Sigma_t|O(\exp(-4s_t)).
\]

**Proof.** Let \( K_t \) be the Gaussian curvature of \( \Sigma_t \). The Gauss equation for \( \Sigma_t \) in \( (M, g) \) can be written as

\[
2K_t = R - 2\text{Ric}(N_t, N_t) + H_t^2 - |A_t|^2
\]

\[
=(R + 6) - 2(\text{Ric}(N_t, N_t) + 2) + \frac{H_t^2 - 4}{2} - |A_t|^2.
\]

By Proposition 1 for metrics \( g \in \mathcal{M}(M, m) \), if \( \{\partial_s, e_1, e_2\} \) is a \( g_m \)-orthonormal referential, then

\[
\text{Ric}(\partial_s, \partial_s) = -2 - \frac{2m}{\sinh s} + O(\exp(-4s)),
\]

\[
\text{Ric}(e_i, e_j) = (-2 + \frac{m}{\sinh s})\delta_{ij} + O(\exp(-4s)),
\]

\[
\text{Ric}(\partial_s, e_i) = O(\exp(-4s)), \quad \text{and}
\]

\[
R + 6 = O(\exp(-4s)).
\]

Considering the \( g_m \)-orthogonal decomposition \( N_t = a\partial_s + X \), it follows that

\[
4K_t = (H_t^2 - 4) + \left(\frac{8m - 12m|X|_{g_m}^2}{\sinh s}\right) - 2|A_t|^2 + O(\exp(-4s)).
\]

Observe that if \( \nu \) is the unit normal of a coordinate sphere in \( (M, g) \), then \( \partial_s = \nu + W \), where \( g(\nu, W) = O(\exp(-4s)) \) and \( |W|_g = O(\exp(-4s)) \). Hence, the \( g \)-orthogonal decomposition \( N_t = b\nu + Y \) is such that \( |X|_{g_m}^2 = |Y|_g^2 + O(\exp(-4s)) \). On the other hand, if \( \nu = bN_t + \nu^\top \) is the \( g \)-orthonormal decomposition of \( \nu \) corresponding to the tangent space of \( \Sigma_t \) and its \( g \)-normal \( N_t \), we have \( |Y|_g^2 = |\nu^\top|_g^2 \). Therefore one can change \( |X|_{g_m}^2 \) by \( |\partial_s|_g^2 \) in (5).
Since $\Sigma_t$ is a sphere of constant mean curvature, the lemma follows after integration of (5). □

**Proposition 6.** Let $m > 0$. Given $g \in \mathcal{M}(M, m)$ a metric with scalar curvature $R \geq -6$, the family $\{\Sigma_t\}_{t \in (2,2+\delta)}$ in $(M, g)$ given by Theorem 4 is such that

$$
\lim_{t \to +\infty} m_H(\Sigma_t) = \lim_{t \to +\infty} \frac{\sqrt{|\Sigma_t|} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_t} (H_t^2 - 4)d\Sigma_t \right)}{16\pi} = m.
$$

**Proof.** Let $\hat{s}_t$ be defined by $|\Sigma_t| = 4\pi \sinh^2 \hat{s}_t$. We use the informations given by Theorem 4, item 3), and calculate all expansions as $\partial M$ goes to infinity.

Lemma 5 and the estimates of Theorem 4, item 3) on the behavior of $|\partial^3 \Sigma_t|$ and $|\hat{A}_t|$ implies

(6)

$$
m_H(\Sigma_t) = \frac{|\Sigma_t|^{1/2}}{8\pi^{3/2}} \left( \int_{\Sigma_t} \frac{m}{\Sigma_t \sinh^3 s} d\Sigma_t + |\Sigma_t|O(\exp(-4\hat{s}_t)) + O(\exp(-4\hat{s}_t)) \right).
$$

In order to analyze (6), observe first that $|\Sigma_t - \hat{s}_t| = |\min_{x \in S^2} \{f(x)\}| \leq C$

for all $l$. Hence

(7)

$$
|\Sigma_t| = 4\pi \sinh^2 \hat{s}_t = O(\exp(2\hat{s}_t)).
$$

On the other hand, for every $p = (s, x) \in \Sigma_t$, since $|w_t(p)| \leq C \exp(-\hat{s}_t)$,

$$
\frac{\sinh \hat{s}_t}{\sinh s} = \frac{\sinh(s - w_t(p))}{\sinh s} = \frac{\cosh w_t(p) - \cosh s}{\sinh s} \sinh w_t(p) = 1 + O(\exp(-\hat{s}_t)).
$$

Therefore

(8)

$$
\int_{\Sigma_t} \frac{|\Sigma_t|^{1/2}}{\sinh^3 s} d\Sigma_t = \frac{(4\pi)^{3/2}}{|\Sigma_t|} \int_{\Sigma_t} \left( \frac{\sinh \hat{s}_t}{\sinh s} \right)^3 d\Sigma_t = 8\pi^{3/2} (1 + O(\exp(-\hat{s}_t))).
$$

Combining (6), (7) and (8) we conclude that $\lim_{t \to +\infty} m_H(\Sigma_t) = m$. □

6. Gluing argument and properties of the global foliation

We now argue that when a metric $g \in \mathcal{M}(M, m)$ is sufficiently close to $g_m$ it is possible to glue together the foliations of $(M, g)$ obtained in Theorems 2 and 4.

**Theorem 7.** Let $m > 0$. There exists $\epsilon > 0$ with the following property:

If $g \in \mathcal{M}(M, m)$ is such that $d(g, g_m) < \epsilon$, then there exists a foliation $\{\Sigma_t\}_{t \in [0, +\infty)}$ of $M$ such that:

i) Each $\Sigma_t$ is a weakly stable CMC sphere in $(M, g)$, with positive mean curvature when $t > 0$; and

ii) $\Sigma_0 = \partial M$ is an outermost minimal surface in $(M, g)$.

Moreover, if $g$ has scalar curvature greater than or equal to $-6$, then

iii) $\lim_{t \to +\infty} m_H(\Sigma_t) = \lim_{t \to +\infty} \sqrt{\frac{|\Sigma_t|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_t} (H_t^2 - 4)d\Sigma_t \right) = m.$
Proof. Given an arbitrary \( \epsilon_0 > 0 \), let \( \delta > 0 \), \( C > 0 \) and \( s_0 > s(3m) \) be given by Theorem 4. Recall that the function \( H_m(s) \) is monotone decreasing on the interval \((s(3m), +\infty)\) and converges to 2 as \( s \) goes to infinity. Let \( s_0 > s_0 \) be such that \( H_m(s_0) < 2 + \delta \).

Let \( S > s_0 + 1 \). Given this choice of \( S \), we can choose \( \epsilon \in (0, \epsilon_0) \) and \( \eta < 1 \) sufficiently small in such way that Theorem 2 holds for \( S \) and Theorem 3 holds for the interval \([S - 1, S]\) for every metric \( g \in \mathcal{M}(M, g) \) with \( d(g, g_m) < \epsilon \).

In particular, we can assume that for every metric \( g \in \mathcal{M}(M, g) \) with \( d(g, g_m) < \epsilon \) and for every \( s \in [S - 1, S] \) the surface \( \Sigma_l = \Sigma_l(g) \) in \((M, g)\) described in Theorem 4 with \( l = H_m(s) \) is given by

\[
\Sigma_l = S_s(h),
\]

where \( h \) is the unique function in \( C^{2,\alpha}(S^2) \) with norm \( < \eta \) such that its graph over \( S_s \) has constant mean curvature \( H_m(s) \), see the remarks after Theorem 4.

Given \( \kappa \in (0, \eta) \), let \([c, d]\) be the image of the interval \([S - 3\kappa/4, S - \kappa/4]\) under the map \( H_m(s) \). We can moreover assume that \( \epsilon \in (0, \epsilon_0) \) is sufficiently small in such way that for every metric \( g \in \mathcal{M}(M, m) \) with \( d(g, g_m) < \epsilon \) the foliation \( \{\Sigma^1_s(g)\}_{s \in [0, S]} \) constructed in Theorem 2 has in particular the following properties:

a) For every \( s \in [S - 1, S] \), there exists a function \( u(s, g) \in C^{2,\alpha}(S^2) \) with \( ||u(s, g)||_{C^{2,\alpha}} < \eta/2 \) such that

\[
\Sigma^1_s(g) = S_s(u(s, g)).
\]

b) The mean curvature \( H_g(s) \) of \( \Sigma^1_s(g) \) in \((M, g)\) is a decreasing function on the interval \([S - 1, S]\) with

\[
|H_g(s) - H_m(s)| < (d - c)/4
\]

for all \( s \in [S - 1, S] \). Hence, there exists an interval \((a, b) \subset [S - 3\kappa/4, S - \kappa/4]\) such that for every \( s \in (a, b) \) there exists a unique \( \tilde{s} \in (S - 3\kappa/4, S - \kappa/4) \) with \( H_g(s) = H_m(\tilde{s}) \). This defines \( \tilde{s} \) as a smooth function of \( s \).

We now prove that the theorem is true for this choice of \( \epsilon \).

In fact, fix some \( g \in \mathcal{M}(M, m) \) with \( d(g, g_m) < \epsilon \). By item a) and b) above, given \( s \in (a, b) \), if we define the function \( \tilde{h} = s - \tilde{s}(s) + u(s, g) \in C^{2,\alpha}(S^2) \), then

\[
||\tilde{h}||_{C^{2,\alpha}} \leq |s - \tilde{s}(s)| + ||u||_{C^{2,\alpha}} < \kappa/2 + \eta/2 < \eta
\]

and the graph

\[
S_{\tilde{s}(s)}(\tilde{h}) = S_s(u(s, g)) = \Sigma^1_s(g)
\]

has constant mean curvature \( H_g(s) = H_m(\tilde{s}(s)) \). These are the conditions that uniquely characterize the function that gives \( \Sigma_l \) in \((M, g)\) with \( l = H_m(\tilde{s}(s)) \) as a graph over \( S_{\tilde{s}(s)} \). Therefore \( \Sigma^1_s(g) = \Sigma_l(g) \) where \( l = H_m(\tilde{s}(s)) \) for all \( s \in (a, b) \).

This proves that the foliations \( \{\Sigma^1_s(g)\} \) and \( \{\Sigma_l(g)\} \) given by Theorems 2
and, respectively, glue together. The foliation obtained, \( \{ \Sigma_t \}_{t \in [0, +\infty)} \), is a foliation of \((M, g)\) by weakly stable CMC spheres, starting at the minimal \( \Sigma_0 = \partial M \), such that each \( \Sigma_t \) has positive mean curvature for \( t > 0 \), and such that \( \lim_{t \to +\infty} m_H(\Sigma_t) = m \) when \( g \) has scalar curvature \( R \geq -6 \) (see Theorem 2, Theorem 4 and Proposition 6). It remains only to prove that \( \partial M \) is an outermost minimal surface in \((M, g)\). This is a consequence of the Maximum Principle, since we proved \( M \setminus \partial M \) is foliated by surfaces with positive mean curvature. \( \square \)

7. The Penrose inequality for perturbations of the Anti-de Sitter-Schwarzschild spaces of positive mass

Using the foliation by weakly stable CMC spheres constructed above on \((M, g)\), \( g \in \mathcal{M}(M, m) \) sufficiently close to \( g_m \), and the remark of H. Bray that the Hawking mass is monotone non-decreasing in such families (see [7]), we prove the Penrose inequality for this class of asymptotically hyperbolic manifolds.

\textbf{Theorem 8.} Given \( m > 0 \), let \( \epsilon > 0 \) be given by Theorem 7. If \( g \in \mathcal{M}(M, m) \) with \( d(g, g_m) < \epsilon \) has scalar curvature \( R \geq -6 \), then

\[
\left( \frac{|\partial M|}{16\pi} \right)^\frac{1}{2} + 4 \left( \frac{|\partial M|}{16\pi} \right)^\frac{3}{2} \leq m.
\]

Moreover, equality holds if and only if \((M, g)\) is isometric to \((M, g_m)\).

\textbf{Proof.} Let \( \{ \Sigma_t \}_{t \geq 0} \) be the foliation of \((M, g)\) by weakly stable CMC spheres constructed in Theorem 7. We assume \( g \) has scalar curvature \( R \geq -6 \), so that \( \{ \Sigma_t \} \) has all the properties i), ii) and iii) described there.

We claim that the Hawking mass of \( \Sigma_t \),

\[
m_H(\Sigma_t) = \sqrt{\frac{|\Sigma_t|}{16\pi}} \left( 1 - \frac{1}{16\pi} \int_{\Sigma_t} (H_t^2 - 4) d\Sigma_t \right),
\]

is monotone non-decreasing in \( t \). In fact, we can be more precise:

\textbf{Claim:} \( m'_H(\Sigma_t) \geq 0 \). Moreover, \( m'_H(\Sigma_t) \) is zero at \( t = t_0 > 0 \) if and only if \( \Sigma_{t_0} \) satisfies the following properties:

a) \( R \) is constant and equal to \(-6\) along \( \Sigma_{t_0} \);

b) \( \Sigma_{t_0} \) is totally umbilic; and

c) \( \Sigma_{t_0} \) has constant Gaussian curvature.

The proof of the claim goes as follows. Choose a parametrization of this foliation by some function \( F : [0, +\infty) \times S^2 \to M \) such that for each \( t \in [0, +\infty) \), \( F_t : S^2 \to M \) is a parametrization of \( \Sigma_t \) and \( \partial_t F \) does not vanish. Let \( \rho_t \) be the lapse function, i.e., \( \rho_t = g(N_t, \partial_t F) \) where \( N_t \) is the unit normal of the surface \( \Sigma_t \) pointing to infinity. Since we have a foliation, \( \rho_t > 0 \) on \( \Sigma_t \) for all \( t \). We also define the mean value \( \overline{\rho}_t = \int \rho_t d\Sigma_t / |\Sigma_t| \).
By the first variation formula of the Hawking mass (see, for example, [10]), we get
\[(16\pi)^{3/2} m'_H(\Sigma_t) = 2|\Sigma_t|^{1/2} \int_{\Sigma_t} (\Delta_t H_t + Q_t H_t) \rho_t d\Sigma_t,\]
where
\[Q_t = \frac{1}{2}(R + 6) + \left(\frac{4\pi}{|\Sigma_t|} - K_t\right) + \frac{1}{2} \left(|A_t|^2 - \frac{1}{2|\Sigma_t|} \int_{\Sigma_t} H_t^2 d\Sigma_t\right).\]

Since \(H_t\) is constant for each \(t\),
\[(16\pi)^{3/2} m'_H(\Sigma_t) = 2|\Sigma_t|^{1/2} H_t \int Q_t \rho_t d\Sigma_t\]
\[= 2|\Sigma_t|^{1/2} H_t \int (\Delta_t + Q_t)(\rho_t - \overline{\rho}_t) d\Sigma_t + 2|\Sigma_t|^{1/2} H_t \overline{\rho}_t \int Q_t d\Sigma_t\]
\[= 2|\Sigma_t|^{1/2} H_t \int L_t(\rho_t - \overline{\rho}_t) d\Sigma_t + 2|\Sigma_t|^{1/2} H_t \overline{\rho}_t \int Q_t d\Sigma_t.\]

In the last line, we used the Gauss equation (11) to see that the operator \(\Delta_t + Q_t\) and the Jacobi operator \(L_t = \Delta_t + \text{Ric}(N_t, N_t) + |A_t|^2\) of \(\Sigma_t\) differ by a constant.

Since \(\partial M = \Sigma_0\) is minimal, the derivative of \(m_H(\Sigma_t)\) is zero at \(t = 0\). When \(t > 0\), \(\Sigma_t\) has positive mean curvature. Observe also that \(\int Q_t d\Sigma_t \geq 0\), since \(R \geq -6\). Therefore
\[(16\pi)^{3/2} m'_H(\Sigma_t) \geq 2|\Sigma_t|^{1/2} H_t \int L_t(\rho_t - \overline{\rho}_t) d\Sigma_t.\]

Now we use the weak stability of \(\Sigma_t\). Since \(H_t\) is constant for each \(t\), \(L_t(\rho_t) = H'_t\) is also constant on \(\Sigma_t\). Hence the stability inequality gives
\[0 \leq -\int L_t(\rho_t - \overline{\rho}_t)(\rho_t - \overline{\rho}_t) d\Sigma_t = \int L_t(\rho_t - \overline{\rho}_t)(\rho_t - \overline{\rho}_t) d\Sigma_t = \overline{\rho}_t \int L_t(\rho_t - \overline{\rho}_t) d\Sigma_t.\]

This implies that \(m'_H(\Sigma_t)\) is nonnegative. If \(m'_H(\Sigma_t) = 0\) at \(t > 0\), then
\[\int Q_t d\Sigma_t = 0 \quad \text{and} \quad \int L_t(\rho_t - \overline{\rho}_t)(\rho_t - \overline{\rho}_t) d\Sigma_t = 0.\]

The first equality implies that \(\Sigma_t\) satisfies a) and b). By the weak stability, the second equality implies that \(L_t(\rho_t - \overline{\rho}_t)\) is constant, for it must be orthogonal to every function on \(\Sigma_t\) with zero mean value. Since \(L_t(\rho_t)\) is constant, this implies that \(L_t(\overline{\rho}_t)\) is also constant. Then c) follows from a), b) and the Gauss equation (11).

Once we proved the claim, inequality (9) follows immediately:
\[\left(\frac{\partial M}{16\pi}\right)^{1/2} + 4 \left(\frac{\partial M}{16\pi}\right)^{3/2} = m_H(\Sigma_0) \leq \lim_{t \to +\infty} m_H(\Sigma_t) = m.\]
Now we analyze the equality. In this case, \( m_H(\Sigma_t) \) must be constant and equal to \( m \). By the second part of the claim, each \( \Sigma_t \) satisfies a), b) and c) for all \( t > 0 \). Then, possibly after a change of the parametrization \( F : [0, +\infty) \times S^2 \to M^+ \), \( F^* g \) is a metric on \( M = [0, +\infty) \times S^2 \) that can be written in the form \( ds^2 + V^2(s)g_0 \), has constant scalar curvature \( R = -6 \), and is such that all slices \( \{s\} \times S^2 \) have Hawking mass \( m \). These conditions uniquely characterize the metric \( g_m \). This finishes the proof.

Another Penrose inequality. In Proposition 4, we saw that the mean curvature of the coordinate spheres in the Anti-de Sitter-Schwarzschild spaces of mass \( m > 0 \) is given by the function \( H_m(r) = \frac{2}{r}\sqrt{1 + r^2 - 2m/r} \). Observe that \( H_m(2m) = 2 \) and that \( H_m(r) > 2 \) for all \( r > 2m \). In particular, the Maximum Principle implies that there are no other closed surfaces with constant mean curvature 2 in \((2m, +\infty) \times S^2, g_m)\).

Let \((M, g)\) be an asymptotically hyperbolic three-manifold with connected boundary \( \partial M \). Assume that \((M, g)\) has scalar curvature \( R \geq -6 \) and that \( \partial M \) is an outermost \( H = 2 \) surface, meaning that there are no closed surfaces in \( M \) with constant mean curvature \( H = 2 \) other than \( \partial M \). In this setting, the Penrose Conjecture is that the area of \( \partial M \) and the mass \( m \) of \((M, g)\) are related by the inequality

\[
\left( \frac{|\partial M|}{16\pi} \right)^{\frac{1}{2}} \leq m,
\]

and equality holds if and only if \((M, g)\) is isometric to the piece of the Anti-de Sitter-Schwarzschild space of mass \( m \) outside the region bounded by the coordinate sphere of mean curvature 2.

Given \( m > 0 \), we analogously define the space \( \mathcal{M}(M_2, m) \) of metrics \( g \) on \( M_2 := [s(2m), +\infty) \times S^2 \) such that \( \partial M_2 \) has constant mean curvature 2 in \((M_2, g)\) and \( d(g, g_m) < +\infty \). The analogous versions of Theorem 7 and Theorem 8 follows immediately by the same arguments.

**Theorem 9.** Let \( m > 0 \). There exists \( \epsilon > 0 \) with the following property:

If \( g \in \mathcal{M}(M_2, m) \) is such that \( d(g, g_m) < \epsilon \), then there exists a foliation \( \{\Sigma_t\}_{t \in [0, +\infty)} \) of \( M_2 \) such that:

i) Each \( \Sigma_t \) is a weakly stable CMC sphere in \((M, g)\), with mean curvature \( H_t > 2 \) when \( t > 0 \); and

ii) \( \Sigma_0 = \partial M_2 \) is an outermost \( H = 2 \) surface in \((M, g)\).

Moreover, if \( g \) has scalar curvature \( R \geq -6 \), then

iii) \( \lim_{t \to +\infty} m_H(\Sigma_t) = m \).

**Theorem 10.** Given \( m > 0 \), let \( \epsilon > 0 \) be given by Theorem 9. If \( g \in \mathcal{M}(M_2, m) \) with \( d(g, g_m) < \epsilon \) has scalar curvature \( R \geq -6 \), then

\[
\left( \frac{|\partial M|}{16\pi} \right)^{\frac{1}{2}} \leq m.
\]

Moreover, equality holds if and only if \((M_2, g)\) is isometric to \((M_2, g_m)\).
ON PERTURBATIONS OF THE ANTI-DE SITTER-SCHWARZSCHILD SPACES

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