Euclidean Jordan algebras and some conditions over the spectra of a strongly regular graph

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Abstract – Let G be a primitive strongly regular graph G such that the regularity is less than half of the order of G and A its matrix of adjacency, and let $\mathcal{A}$ be the real Euclidean Jordan algebra of real symmetric matrices of order n spanned by the identity matrix of order n and the natural powers of $A$ with the usual Jordan product of two symmetric matrices of order n and with the inner product of two matrices being the trace of their Jordan product. Next the spectra of two Hadamard series of $A$ associated to $A^2$ is analysed to establish some conditions over the spectra and over the parameters of G.

Keywords: Euclidean Jordan algebras, Strongly regular graphs, Admissibility conditions

Introduction

The Euclidean Jordan algebras have many applications in several areas of mathematics. Some authors applied the theory of Euclidean Jordan algebras to interior-point methods [1–10], others applied this theory to combinatorics [11–14], and to statistics [15–17]. More actually, some authors extended the properties of the real symmetric matrices to the elements of simple real Euclidean Jordan algebras, see [18–24].

A good exposition about Jordan algebras can be founded in the beautiful work of K. McCrimmon, A taste of Jordan algebras, see [25], or for a more abstract survey one must cite the work of N. Jacobson, Structure and Representations of Jordan Algebras, see [26] and the PhD thesis of Michael Baes, Spectral Functions and Smoothing Techniques on Jordan Algebras, see [27].

For a well based understanding of the results of Euclidean Jordan algebras we must cite the works of Faraut and Korányi, Analysis on Symmetric Cones, see [28], or for a more abstract survey one must cite the work of N. Jacobson, Structure and Representations of Jordan Algebras, see [26] and the PhD thesis of Michael Baes, Spectral Functions and Smoothing Techniques on Jordan Algebras, see [27].

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For a very readable text on Euclidean Jordan algebras we couldn’t avoid of indicating, the chapter written by F. Alizadeh and S. H. Schmieta, Symmetric Cones, Potential Reduction Methods and Word-By-Word Extensions of the book Handbook of semi-definite programming, Theory, Algorithms and Applications, see [31], and the chapter, written by F. Alizadeh, An Introduction to Formally Real Jordan Algebras and Their Applications in Optimization of the book Handbook on Semidefinite, Conic and Polynomial Optimization, see [32].

In this paper we establish some admissibility conditions in an algebraic asymptotically way over the parameters and over the spectra of a primitive strongly regular graph.

This paper is organized as follows. In the second section we expose the most important concepts and results about Jordan algebras and Euclidean Jordan algebras, without presenting any proof of these results. Nevertheless some bibliography is present on the subject. In the following section we present some concepts about simple graphs and namely strongly regular graphs needed for a clear understanding of this work. In the last section we consider a three dimensional real Euclidean Jordan algebra $\mathcal{A}$ associated to the adjacency matrix of a primitive strongly regular graph and we establish some admissibility conditions over, in an algebraic asymptotic way, the spectra and over the parameters of a strongly regular graph.

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Principal results on Euclidean Jordan algebras

Herein, we describe the principal definitions, results and the more relevant theorems of the theory of Euclidean Jordan algebras without presenting the proof of them.

To make this exposition about Euclidean Jordan algebras we have recurred to the monograph, Analysis on Symmetric Cones of Faraut and Kóranyi [28], and to the book A taste of Jordan algebra of Kevin McCrimmon [25]. But for general Jordan algebras very readable expositions can be found in the book Statistical Applications of Jordan algebras of James D. Malley [17].

Now, we will present only the main results about Euclidean Jordan algebras needed for this paper.

A Jordan algebra $\mathcal{A}$ over a field $\mathbb{K}$ with characteristic $\neq 2$ is a vector space over the field $\mathbb{K}$ with a operation of multiplication $\circ$ such that for any $x$ and $y$ in $\mathcal{A}$, $x \circ y = y \circ x$ and $x \circ (x^{(k)} \circ y) = x^{(k)} \circ (x \circ y)$, where $x^{(k)} = x \circ x \circ \ldots \circ x$ ($k$ times). We will suppose throughout this paper that if $\mathcal{A}$ is a Jordan algebra then $\mathcal{A}$ has a unit element that we will denote it by $e$.

When the field $\mathbb{K}$ is the field of the reals numbers we call the Jordan algebra a real Jordan algebra. Since we are only interested in finite dimensional real Jordan algebras with unit element, we only consider Jordan algebras that are real finite dimensional Jordan algebras and that have an unit element $e$ and that are equipped with an operation of multiplication that we denote by $\circ$.

The real vector space of real symmetric matrices, $\mathcal{A} = \text{Sym}(n, \mathbb{R})$, of order $n$, with the operation $x \circ y = \frac{x + y + xy}{2}$ is a real Jordan algebra.

We must note, that we define the powers of an element $x$ in $\mathcal{A}$ in the usual way $x^{(0)} = e$, $x^{(1)} = x$, $x^{(k)} = x \circ x$ and $x^{(k)} = x \circ x^{(k-1)}$ for any natural number $k$. Hence, we have $x^{2i} = x \circ x = \frac{x + x + xy}{2} = \frac{x + y}{2} = x^2$ and therefore by induction over $\mathbb{N}$ we conclude that $x^{2i} = x^2$ for any natural number $k$, where $x^2$ represents the usual power of order $k$ of a squared symmetric matrix. $\mathcal{A}$ is a Jordan algebra since for $x$ and $y$ in $\mathcal{A}$ we have $x \circ y = \frac{x + y + xy}{2} = y \circ x$ and

$$x \circ (x^{(k)} \circ y) = x \circ (x^2 \circ y) = x \circ \left(\frac{x^2 + xy + yx + y^2}{2}\right) = \frac{x^2 + xy + yx + y^2}{2} = \frac{x^2 + xy + yx + y^2}{2}.$$

Let’s consider another example of Euclidean Jordan algebra. Let consider the real vector space $\mathcal{A}_{n+1} = \mathbb{R}^{n+1}$ equipped with the product $\circ$ such that

$$z \circ w = \begin{bmatrix} z_1 w_1 \\
 z_1 w_1 + w_1 z_1 + z_1 w_1 + w_1 z_1 \end{bmatrix},$$

where $z = \begin{bmatrix} z_1 \\
 z_2 \\
 \vdots \\
 z_n \\
 z_{n+1} \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\
 w_2 \\
 \vdots \\
 w_n \end{bmatrix}$, $z = \begin{bmatrix} z_2 \\
 z_3 \\
 \vdots \\
 z_{n+1} \end{bmatrix}$ and $w = \begin{bmatrix} w_2 \\
 w_3 \\
 \vdots \\
 w_{n+1} \end{bmatrix}$.

Now we will show that $z \circ w = w \circ z$ and that $z \circ (z^{(2)} \circ w) = z^{(2)} (z \circ w)$. We have

$$z \circ w = \begin{bmatrix} z_1 w_1 \\
 z_1 w_1 + w_1 z_1 + z_1 w_1 + w_1 z_1 \end{bmatrix} = w \circ z.$$
Herein, we must say that the element \( e = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) is the unit of the Euclidean Jordan algebra \( \mathcal{A}_{n+1} \). Indeed, we have

\[
\begin{aligned}
e \circ \begin{bmatrix} x_1 \\ y \end{bmatrix} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \circ \begin{bmatrix} x_1 \\ y \end{bmatrix} = \begin{bmatrix} x_1 + y(0) \\ y(0) + 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ y \end{bmatrix}.
\end{aligned}
\]

Now since the operation \( \circ \) is commutative, we showed that

\[
\begin{aligned}
e \circ \begin{bmatrix} x_1 \\ y \end{bmatrix} &= \begin{bmatrix} x_1 \\ y \end{bmatrix} \circ e = \begin{bmatrix} x_1 \\ y \end{bmatrix}.
\end{aligned}
\]

Hence, \( e \) is a unit of the Jordan algebra \( \mathcal{A}_{n+1} \). Now, we have

\[
\begin{aligned}
z \circ (z^2 \circ w) &= z \circ ((z \circ z) \circ w) \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \circ \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \circ \begin{bmatrix} ||z||^2 w_1 + 2z_1 z_2 w_2 \\ ||z||^2 w_2 + 2z_1 z_2 w_1 \end{bmatrix} \\
&= \begin{bmatrix} z_1 w_1 (||z||^2 + 2 ||z||^2) + (||z||^2 + 2z_1 z_2 ||z|| w_2) \\ z_1 w_1 (||z||^2 + 2 ||z||^2) + (||z||^2 + 2z_1 z_2 ||z|| w_2) \end{bmatrix} \\
&= \begin{bmatrix} z_1 |z|^2 w_1 + 2z_1 z_2 w_2 + (2z_1^2 w_1 + ||z||^2 w_1 + 2z_1 z_2 w_2) \end{bmatrix}.
\end{aligned}
\]

Thus, \( \mathcal{A}_{n+1} \) is a Jordan algebra.

Now we define a Jordan subalgebra of the Euclidean Jordan algebra \( \mathcal{A} = \text{Sym}(n, \mathbb{R}) \). Let’s consider the set \( \{B_1, B_2, \ldots, B_l\} \) of symmetric matrices of \( M_n(\mathbb{R}) \) such that

(i) \( B_1 = I_n \),

(ii) \( (B_j)_{jk} \in \{0,1\} \), for \( j, k \in \{1, \ldots, l\} \) and \( B_i = B_i^T \) for \( i = 1, \ldots, l \),

(iii) \( B_1 + B_2 + \cdots + B_l = J_n \),

(iv) \( \forall i, j \in \{1, \ldots, l\} \), \( \forall k \in \{1, \ldots, l\} \exists x^k_{ij} \in \mathbb{R} : B_i B_j = \sum_{k} x^k_{ij} B_k \).

Then the real vector space \( \mathcal{B} \) spanned by the set \( \{B_1, B_2, \ldots, B_l\} \) with the Jordan product \( \circ \) is a Jordan algebra. Firstly, we must say that \( \mathcal{B} \) is closed for the Jordan product. Indeed, for \( C \) and \( D \) in \( \mathcal{B} \) we have \( C = x_1 B_1 + \cdots + x_l B_l \) and \( D = \beta_1 B_1 + \cdots + \beta_l B_l \). Therefore

\[
\begin{aligned}
C^T &= (x_1 B_1 + \cdots + x_l B_l)^T = x_1 B_1^T + \cdots + x_l B_l^T \\
&= x_1 B_1 + \cdots + x_l B_l = C
\end{aligned}
\]

and

\[
\begin{aligned}
D^T &= (\beta_1 B_1 + \cdots + \beta_l B_l)^T = \beta_1 B_1^T + \cdots + \beta_l B_l^T \\
&= \beta_1 B_1 + \cdots + \beta_l B_l = D.
\end{aligned}
\]

Since \( C \circ D = \frac{C^T D + D^T C}{2} \) then
\((C \odot D)^T = \frac{(CD + DC)^T}{2} = \frac{(CD) + (DC)^T}{2} = \frac{D^T C^T + C^T D^T}{2} = \frac{DC + CD}{2} = \frac{CD + DC}{2} = C \odot D.\)

So \(C \odot D\) is a symmetric matrix of \(M_n(\mathbb{R})\), but from property iv) we conclude that \(C \odot D = \sum_{k=1}^{r-1} g_k B_k\), therefore \(C \odot D \in B\). Now, we must say, that for any matrices \(X\) and \(Y \in B\) we have \(X \odot Y = Y \odot X\) and \(X \odot (X^2 \odot Y) = X^2 \odot (X \odot Y)\) since \(B\) is a subalgebra of \(A = \text{Sym}(n, \mathbb{R})\).

Let \(A\) be a \(n\)-dimensional Jordan algebra. Then \(A\) is power associative, this is, is an algebra such that for any \(x\) in \(A\) the algebra spanned by \(x\) and \(e\) is associative. For \(x\) in \(A\) we define rank \((x)\) as being the least natural number \(k\) such that \(\{e, x^1, \ldots, x^k\}\) is a linearly dependent set and we write \(\text{rank}(x) = k\). Now since \(\dim(A) = n\) then \(\text{rank}(x) \leq n\). The rank of \(A\) is defined as being the natural number \(r = \text{rank}(A) = \max \{ \text{rank}(x) : x \in A\}\). An element \(x\) in \(A\) is regular if \(\text{rank}(x) = r\). Now, we must observe that the set of regular elements of \(A\) is a dense set in \(A\). Let’s consider a regular element \(x\) of \(A\) and \(r = \text{rank}(x)\).

Then, there exist real numbers \(x_1(x), x_2(x), \ldots, x_{r-1}(x)\) and \(x_r(x)\) such that
\[
x^r - x_1(x)x^{(r-1)} + \cdots + (-1)^{r-1}x_r(x)e = 0, \tag{1}
\]
where \(0\) is the zero vector of \(A\). Taking into account (1) we conclude that the polynomial \(p(x, -)\) define by the equality (2).
\[
p(x, \lambda) = x^r - x_1(x)x^{(r-1)} + \cdots + (-1)^r x_r(x), \tag{2}
\]
is the minimal polynomial of \(x\). When \(x\) is a non regular element of \(A\) the minimal polynomial of \(x\) has a degree less than \(r\). The polynomial \(p(x, -)\) is called the characteristic polynomial of \(x\). Now, we must say that the coefficients \(x_i(x)\) are homogeneous polynomials of degree \(i\) on the coordinates of \(x\) on a fixed basis of \(A\). Since the set of regular elements of \(A\) is a dense set in \(A\) then we extend the definition of characteristic polynomial to non regular elements of \(A\) by continuity.

The roots of the characteristic polynomial \(p(x, -)\) of \(x\) are called the eigenvalues of \(x\). The coefficient \(x_i(x)\) of the characteristic polynomial \(p(x, -)\) is called the trace of \(x\) and we denote it by \(\text{Trace}(x)\) and we call the coefficient \(x_r(x)\) the determinant of \(x\) and we denote it by \(\text{Det}(x)\).

Let \(A\) be a real finite dimensional associative algebra with the bilinear map \((x, y) \mapsto xy\). We introduce on \(A\) a structure of Jordan algebra by considering a new product \(\circ\) defined by \(x \circ y = (x \odot y + y \odot x)/2\) for all \(x\) and \(y\) in \(A\). The product \(\circ\) is called the Jordan product of \(x\) by \(y\). Let \(A\) be a real Jordan algebra and \(x\) be a regular element of \(A\). Then we have \(\text{rank}(x) = r = \text{rank}(A)\). We define the linear operator \(L_x\) such that \(L_x(z) = x \circ z\), \(\forall z \in A\). We define the real vector space \(\mathbb{R}[x]\) by \(\mathbb{R}[x] = \{z \in A : \exists \gamma_0, \gamma_1, \ldots, \gamma_{r-1} \in \mathbb{R} : z = \gamma_0 e + \gamma_1 x^1 + \cdots + \gamma_{r-1}x^{(r-1)}\}\). The restriction of the linear operator \(L_x\) to \(\mathbb{R}[x]\) we call \(L^x(x)\). We must note now that \(\text{trace}(L^x(x)) = a_1(x) = \text{Trace}(x)\) and \(\text{det}(L^x(x)) = a_r(x) = \text{Det}(x)\).

A Jordan algebra is simple if and only if it does not contain any nontrivial ideal. An Euclidean Jordan algebra \(A\) is a Jordan algebra with an inner product \(\cdot\) such that \(L_x(y)z = y L_x(x)z\), for all \(x, y\) and \(z\) in \(A\). Herein, we must say that an Euclidean Jordan algebra is simple if and only if it can’t be written as a direct sum of two Euclidean Jordan algebras. But it is already proved that every Euclidean Jordan algebra is a direct orthogonal sum of simple Euclidean Jordan algebras.

The Jordan algebra \(A = \text{Sym}(n, \mathbb{R})\) equipped with the Jordan product \(x \circ y = \frac{xy + yx}{2}\) with \(xy\) and \(yx\) the usual products of matrices of order \(n\) by \(y\) and of \(y\) by \(x\), and with the inner product \(x|y = \text{trace}(L_x(x)y)\) for \(x\) and \(y\) in \(A\) is an Euclidean Jordan algebra. Indeed, we have
\[
L_x(y)|z = x \circ y|z
\]
\[
= \text{trace}\left((x \circ y) \circ z\right)
\]
\[
= \text{trace}\left(\frac{(xy + yx)}{2} \circ z\right)
\]
\[
= \text{trace}\left(\frac{(xy)z + (yx)z}{2}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4} + \frac{(yx)z}{4} + \frac{(yx)z}{4} + \frac{(yx)z}{4}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right)
\]
\[
= \text{trace}\left(\frac{(xy)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right) + \text{trace}\left(\frac{(yx)z}{4}\right)
\]
Therefore, the set \( \mathcal{X}_{n-1} = \{ \mathbf{e}, x^{10}, x^{20}, \ldots, x^{n-10} \} \) is a linearly independent set of \( \mathcal{A} \) if and only if the set

\[
S_{n-1} = \{ (1,1,\ldots,1), (\lambda_1, \lambda_2, \ldots, \lambda_n), (\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2), \ldots, (\lambda_1^{n-1}, \lambda_2^{n-1}, \ldots, \lambda_n^{n-1}) \}
\]

is a linearly independent set of \( \mathbb{R}^n \). But, since

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\lambda_1 & \lambda_2 & \cdots & \lambda_n \\
\lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}
\end{vmatrix} = \prod_{1 \leq i < j \leq n} (\lambda_j - \lambda_i) \neq 0
\]

is a linearly independent set of \( \mathbb{R}^n \), we conclude that \( \mathcal{A}_{n+1} = \mathbb{R}^{n+1} \) is an Euclidean Jordan algebra relatively to the inner product.
then the set \( S_{n-1} \) is a linearly independent set of \( \mathbb{R}^n \) and therefore the set
\[
\mathcal{S}_{n-1} = \{ e, x^1, x^{20}, \ldots, x^{n-10} \}
\]
is a linearly independent set of \( \mathcal{A} \). The set \( \mathcal{S}_n = \{ e, x^1, x^{20}, \ldots, x^n \} \) is a linearly dependent set of \( \mathcal{A} \) since the set
\[
S_n = \{ (1, 1, \ldots, 1), (\lambda_1, \lambda_2, \ldots, \lambda_n), (\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}), (\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2) \}
\]
is a linearly dependent set of \( \mathbb{R}^n \) because the dimension of \( \mathbb{R}^n \) is \( n \). Therefore, we conclude that \( \text{rank}(x) = n \).

Let \( x \) be an element of \( \mathcal{A} \) with \( k \) distinct non null eigenvalues \( \lambda_j \) and let \( v_0, v_1, \ldots, v_{n-1} \) and \( v_i \) be an orthonormal basis of eigenvectors of \( x \) associated with \( \lambda_j \), this \( x_1 = v_0 \lambda_1, v_1, \ldots, v_{n-1} \). Now, we consider the elements
\[
u_i = \sum_{j=1}^n v_j
\]
and we have
\[
x = \lambda_i u_i u_i^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_k u_k u_k^T.
\]
Therefore, we have
\[
\mathcal{S}_k = \{ e, x^1, x^{20}, \ldots, x^{n-10} \}
\]
is a linearly independent set of \( \mathcal{A} \) and therefore \( \text{rank}(x) = k \). If \( x \) has \( k \) distinct eigenvalues where \( k-1 \) eigenvalues are non null and one is null then one proves one a similar way that \( \text{rank}(x) = k \).

Therefore, we conclude that \( \text{rank}(\mathcal{A}) = n \) and the regular elements of \( \mathcal{A} \) are the elements \( x \) of \( \mathcal{A} \) with \( n \) distinct eigenvalues.

The characteristic polynomial of a regular element of \( \mathcal{A} \) is a monic polynomial of minimal degree \( n = \text{rank}(A) \). Now, let \( x \) be an element of \( \mathcal{A} \) with \( n \) distinct eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_{n-1} \) and \( \lambda_n \) then by the Theorem of Cayley-Hamilton we conclude that the polynomial \( p(x) \) such that
\[
p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_n)
\]
is the monic polynomial of minimal degree of \( x \). Therefore since the monic polynomial of minimal degree of element \( x \) is unique we conclude that the characteristic polynomial of \( x \), \( p(x, \lambda) \) is such that \( p(x, \lambda) = p(x) \). This is \( p(x, \lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) \). So, we have \( p(x, \lambda) = x^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n) x^{n-1} + \cdots + (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n \).

Therefore, we have Trace \( (x) = \lambda_1 + \lambda_2 + \cdots + \lambda_n \) and Det \( (x) = \lambda_1 \lambda_2 \cdots \lambda_n \).

Now, we will show that \( \text{rank}(\mathcal{A}_{n+1}) = 2 \). To came to this conclusion, we will firstly show that for \( x \neq 0 \), \( \text{rank} \left( \begin{bmatrix} x_1 \\ x \end{bmatrix} \right) = 2 \) and that for \( x_1 \neq 0 \), \( \text{rank} \left( \begin{bmatrix} 1 \\ x_1 \end{bmatrix} \right) = 1 \). So, let suppose \( x \neq 0 \) then, we have
\[
x \begin{bmatrix} x_1 \\ x \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff (x = 0) \land (\beta = 0).
\]
Therefore, the set \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x \end{bmatrix} \right\} \) is a linearly independent set of \( \mathcal{A}_{n+1} \). Now, we have \( \begin{bmatrix} x_1 \\ x \end{bmatrix}^{20} = \begin{bmatrix} x_1^2 + ||x||^2 \\ 2x_1x \end{bmatrix} \) and since
\[
\frac{x_1^2 + ||x||^2}{2x_1x} = \alpha \begin{bmatrix} x_1 \\ x \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \end{bmatrix} \iff (x = 2x_1) \land (\beta = ||x|| - x_1^2)
\]
we conclude that \( \text{rank} \left( \begin{bmatrix} x_1 \\ x \end{bmatrix} \right) = 2 \). If \( x_1 \neq 0 \), we have \( \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \). Then the set \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x_1 \\ x \end{bmatrix} \right\} \) is a linearly dependent set of \( \mathcal{A}_{n+1} \) then \( \text{rank} \left( \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \right) = 1 \). And, therefore \( \text{rank}(\mathcal{A}_{n+1}) = 2 \). And the regular elements of \( \mathcal{A}_{n+1} \) are the elements of \( \mathcal{A}_{n+1} \) such that \( x \neq 0 \).
Since, when \( x \neq \overline{0} \) we have

\[
\begin{bmatrix} x_1 \\ x \end{bmatrix}^{20} - 2x_1 \begin{bmatrix} x_1 \\ x \end{bmatrix} + (x_1^2 - ||x||^2) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\]

Then, supposing \( x \neq \overline{0} \) and considering the notation \( x = \begin{bmatrix} x_1 \\ x \end{bmatrix} \), we conclude that the characteristic polynomial of \( x \) is \( p(x, \lambda) = \lambda^2 - 2x_1 \lambda + (x_1^2 - ||x||^2) \). Therefore, the eigenvalues of \( x \) are \( \lambda_1(x) = x_1 - ||x|| \) and \( \lambda_2(x) = x_1 + ||x|| \). Trace \( (x) = 2x_1 \) and \( \det(x) = x_1^2 - ||x||^2 \).

Let \( A \) be a real Euclidean Jordan algebra with unit element \( e \). An element \( f \) in \( A \) is an idempotent of \( A \) if \( f^{20} = f \). Two idempotents \( f \) and \( g \) of \( A \) are orthogonal if \( f \circ g = 0 \). A set \( \{g_1, g_2, \ldots, g_k\} \) of nonzero idempotents is a complete system of orthogonal idempotents of \( A \) if \( g_i^{20} = g_i \), for \( i = 1, \ldots, k \), \( g_i \circ g_j = 0 \), for \( i = j \), and \( \sum_i g_i = e \). An element \( g \) of \( A \) is a primitive idempotent if it is a non null idempotent of \( A \) and if cannot be written as a sum of two orthogonal nonzero idempotents of \( A \). We say that \( \{g_1, g_2, \ldots, g_k\} \) is a Jordan frame of \( A \) if \( \{g_1, g_2, \ldots, g_k\} \) is a complete system of orthogonal idempotents such that each idempotent is primitive.

Let consider the Euclidean Jordan algebra \( A = \text{Sym}(n, \mathbb{R}) \) with \( n = 1, \ldots, n \) where \( E_{ij} \) is the square matrix of order \( n \) such that \( (E_{ij})_{ij} = 1 \) and \( (E_{ij})_{ik} = 0 \) if \( i \neq j \) or \( k \neq j \).

Let \( k \) be a natural number such that \( 1 < k < n \). Then \( S = \{E_{11} + E_{22} + \cdots + E_{kk}, E_{kk+1}, \ldots, E_{nm}\} \) is a complete system of orthogonal idempotents of \( A \) and \( S' = \{E_{11}, E_{22}, \ldots, E_{nm}\} \) is a Jordan frame of \( A \).

Let consider the Euclidean Jordan algebra \( A_{n+1} \) and \( x \) non zero element of \( A_{n+1} \). Then the set \( S = \{g_1, g_2\} = \{ \frac{1}{2} \begin{bmatrix} 1 \\ -\overline{x} \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ \overline{x} \end{bmatrix} \} \) is a Jordan frame of the Euclidean Jordan algebra \( A_{n+1} \). Indeed, we have:

(i) \( g_1^{20} = g_1 \) and \( g_2^{20} = g_2 \)

\[
g_1^{20} = g_1 \circ g_1 = \frac{1}{2} \begin{bmatrix} 1 \\ -\overline{x} \end{bmatrix} \circ \frac{1}{2} \begin{bmatrix} 1 \\ \overline{x} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = g_1
\]

and, we have

\[
g_2^{20} = g_2 \circ g_2 = \frac{1}{2} \begin{bmatrix} 1 \\ \overline{x} \end{bmatrix} \circ \frac{1}{2} \begin{bmatrix} 1 \\ \overline{x} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = g_2
\]

(ii)

\[
g_1 + g_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -\overline{x} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ \overline{x} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = e
\]

(iii)

\[
g_1 \circ g_2 = \begin{bmatrix} 1 \\ -\overline{x} \end{bmatrix} \circ \begin{bmatrix} 1 \\ \overline{x} \end{bmatrix} = \begin{bmatrix} 1 - ||\overline{x}||^2 \\ \overline{x} - \frac{\overline{x} \overline{x}}{||\overline{x}||^2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Therefore we conclude that \( \{g_1, g_2\} \) is a Jordan frame of \( A_{n+1} \) since rank \( (A_{n+1}) = 2 \).
Theorem 1. ([28], p. 43). Let $\mathcal{A}$ be a real Euclidean Jordan algebra. Then for $x$ in $\mathcal{A}$ there exist unique real numbers $\beta_1(x), \beta_2(x), \ldots, \beta_k(x)$, all distinct, and a unique complete system of orthogonal idempotents $\{g_1, g_2, \ldots, g_k\}$ such that

$$x = \beta_1(x)g_1 + \beta_2(x)g_2 + \cdots + \beta_k(x)g_k.$$  (3)

The numbers $\beta_i(x)$'s of (3) are the eigenvalues of $x$ and the decomposition (3) is the first spectral decomposition of $x$.

Theorem 2. ([28], p. 44). Let $\mathcal{A}$ be a real Euclidean Jordan algebra with rank $(\mathcal{V}) = r$. Then for each $x$ in $\mathcal{A}$ there exists a Jordan frame $\{g_1, g_2, \ldots, g_r\}$ and real numbers $\beta_1(x), \cdots, \beta_{r-1}(x)$ and $\beta_r(x)$ such that

$$x = \beta_1(x)g_1 + \beta_2(x)g_2 + \cdots + \beta_r(x)g_r.$$ (4)

Remark 1. The decomposition (4) is called the second spectral decomposition of $x$. And we have

$$\text{Det}(x) = \prod_{j=1}^r \beta_j(x), \quad \text{Tr}(x) = \sum_1^r \beta_j(x) \quad \text{and} \quad \alpha_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq r} \beta_{i_1}(x) \cdots \beta_{i_k}(x).$$

Example 1. For $x \neq 0$, the second spectral decomposition of $x = \begin{bmatrix} x_1 \\ x \end{bmatrix}$ relatively to the Jordan frame $S = \{g_1, g_2\} = \left\{ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right\}$ of $\mathcal{A}_{n+1}$ is

$$x = (x_1 - \|x\|)g_1 + (x_1 + \|x\|)g_2.$$

An Euclidean Jordan algebra is called simple if and only if have only trivial ideals.

Any simple Euclidean Jordan algebra is isomorphic to one of the five Euclidean Jordan algebras that we describe below:

(i) The spin Euclidean Jordan algebra $\mathcal{A}_{n+1}$.

(ii) The Euclidean Jordan algebra $\mathcal{A} = \text{Sym}(n, \mathbb{R})$ with the Jordan product of matrices and with an inner product of two symmetric matrices as being the trace of their Jordan product.

(iii) The Euclidean Jordan algebra $\mathcal{A} = H_n(\mathbb{C})$ of hermitian matrices of complexes of order $n$ equipped with the Jordan product of two hermitian matrices of complexes and with the scalar product of two Hermitian matrices of complexes as being the real part of the trace of their Jordan product.

(iv) The Euclidean Jordan algebra $\mathcal{A} = H_n(\mathbb{H})$, of hermitian matrices of quaternions of order $n$ equipped with the Jordan product of hermitian matrices of quaternions and with the scalar product of two hermitian matrices of quaternions as being the real part of the trace of their the Jordan product.

(v) The Euclidean Jordan algebra $\mathcal{A} = H_3(\mathbb{O})$ of hermitian matrices of octonions of order $n$ equipped with the Jordan product of two hermitian matrices of octonions and with the inner product of two hermitian matrices of octonions as being the real part of the trace of their Jordan product.

We now describe the Pierce decomposition of an Euclidean Jordan algebra relatively to one of its idempotents. But, first we must say that for any nonzero idempotent $q$ of an Euclidean Jordan algebra $\mathcal{A}$ the eigenvalues of the linear operator $L_q(g)$ are $0, \frac{1}{2}$ and 1 and this fact permits us to say that, considering the eigenspaces $\mathcal{A}(g, 0) = \{x \in A : L_q(g)(x) = 0x\}, \mathcal{A}(g, \frac{1}{2}) = \{x \in A : L_q(g)(x) = \frac{1}{2}x\}$ and $\mathcal{A}(g, 1) = \{x \in A : L_q(g)(x) = 1x\}$ of $L_q(g)$ associated to these eigenvalues we can decompose $\mathcal{A}$ as an orthogonal direct sum $\mathcal{A} = \mathcal{A}(g, 0) + \mathcal{A}(g, \frac{1}{2}) + \mathcal{A}(g, 1)$. Now, we will describe the Pierce decomposition of the Euclidean Jordan algebra $\mathcal{A} = \text{Sym}(n, \mathbb{R})$ relatively to an idempotent of the form

$$C = \begin{bmatrix} I_k & O_{k \times n-k} \\ O_{k \times n-k} & O_{n-k \times n-k} \end{bmatrix}.$$ 

Let $\mathcal{A} = \text{Sym}(n, \mathbb{R})$ and $C = \begin{bmatrix} I_k & O_{k \times n-k} \\ O_{k \times n-k} & O_{n-k \times n-k} \end{bmatrix}$. Now, we will show that $\mathcal{A}(C, 1) = \begin{bmatrix} X_{k \times k} & O_{k \times n-k} \\ O_{k \times n-k} & O_{n-k \times n-k} \end{bmatrix}$.

$X_{k \times k} \in \text{Sym}(k, \mathbb{R})$. 

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We have

$$
\begin{bmatrix}
I_k & O_{k \times n-k} \\
O^T_{n-k \times k} & O_{k \times k}
\end{bmatrix}
\circ
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
= \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\downarrow
\begin{bmatrix}
\frac{1}{2}
\begin{bmatrix}
I_k & O_{k \times n-k} \\
O^T_{n-k \times k} & O_{k \times k}
\end{bmatrix}
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
+ \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\begin{bmatrix}
I_k & O_{k \times n-k} \\
O^T_{n-k \times k} & O_{k \times k}
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\downarrow
\begin{bmatrix}
A_{k \times k} \\
A^T_{k \times n-k}
\end{bmatrix}
\frac{1}{2}A_{k \times n-k}
O_{n-k \times n-k}
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
= \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\downarrow
\begin{bmatrix}
A_{k \times k} \\
A^T_{k \times n-k}
\end{bmatrix}
\frac{1}{2}A_{k \times n-k}
O_{n-k \times n-k}
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
= \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\downarrow
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\frac{1}{2}A_{k \times n-k}
O_{n-k \times n-k}
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
= \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\downarrow
\begin{bmatrix}
A_{k \times k} = O_{k \times k} \land (A_{n-k \times n-k} = O_{n-k \times n-k})
\end{bmatrix}
$$

Therefore, we have

$$\mathcal{A}(C, 1) = \left\{ \begin{bmatrix}
A_{k \times k} & O_{k \times n-k} \\
O^T_{k \times n-k} & O_{n-k \times n-k}
\end{bmatrix} : A_{k \times k} \in M_k(\mathbb{R}) \land (A_{k \times k} = A^T_{k \times k}) \right\}$$

Now, let calculate $\mathcal{A}(C, \frac{1}{2})$. So, let consider the following equivalences.

$$
\begin{bmatrix}
I_k & O_{k \times n-k} \\
O^T_{n-k \times k} & O_{k \times k}
\end{bmatrix}
\circ
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
= \frac{1}{2}
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\downarrow
\begin{bmatrix}
\frac{1}{2}
\begin{bmatrix}
I_k & O_{k \times n-k} \\
O^T_{n-k \times k} & O_{k \times k}
\end{bmatrix}
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
+ \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\begin{bmatrix}
I_k & O_{k \times n-k} \\
O^T_{n-k \times k} & O_{k \times k}
\end{bmatrix}
\end{bmatrix}
= \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\downarrow
\begin{bmatrix}
A_{k \times k} \\
A^T_{k \times n-k}
\end{bmatrix}
\frac{1}{2}A_{k \times n-k}
O_{n-k \times n-k}
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
= \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\downarrow
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\frac{1}{2}A_{k \times n-k}
O_{n-k \times n-k}
\begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
= \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & A_{n-k \times n-k}
\end{bmatrix}
\downarrow
\begin{bmatrix}
A_{k \times k} = O_{k \times k} \land (A_{n-k \times n-k} = O_{n-k \times n-k})
\end{bmatrix}
$$

Hence, we have $\mathcal{A}(C, \frac{1}{2}) = \left\{ \begin{bmatrix}
O_{k \times k} & A_{k \times n-k} \\
A^T_{k \times n-k} & O_{n-k \times n-k}
\end{bmatrix} : A_{k \times n-k} \in M_{k \times n-k}(\mathbb{R}) \right\}$. Now, let’s calculate $\mathcal{A}(C, 0)$. So let consider the following calculations:
\[
\begin{bmatrix}
I_k & O_{k \times n-k} \\
O_{n-k \times k}^T & O_{n-k \times n-k}
\end{bmatrix} \circ \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A_{k \times n-k}^T & A_{n-k \times n-k}
\end{bmatrix} = 0 \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A_{k \times n-k}^T & A_{n-k \times n-k}
\end{bmatrix}
\]
\[
\frac{1}{2} \left( \begin{bmatrix}
I_k & O_{k \times n-k} \\
O_{n-k \times k}^T & O_{n-k \times n-k}
\end{bmatrix} \circ \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A_{k \times n-k}^T & A_{n-k \times n-k}
\end{bmatrix} \right) = \begin{bmatrix}
O_{k \times k} & O_{k \times n-k} \\
O_{k \times n-k}^T & O_{n-k \times n-k}
\end{bmatrix}
\]
\[
\frac{1}{2} \left( \begin{bmatrix}
A_{k \times k} & A_{k \times n-k} \\
A_{k \times n-k}^T & A_{n-k \times n-k}
\end{bmatrix} \right) + \frac{1}{2} \begin{bmatrix}
A_{k \times k} & O_{k \times n-k} \\
O_{k \times n-k}^T & O_{n-k \times n-k}
\end{bmatrix} = \begin{bmatrix}
O_{k \times k} & O_{k \times n-k} \\
O_{k \times n-k}^T & O_{n-k \times n-k}
\end{bmatrix}
\]
\[
\begin{bmatrix}
\frac{1}{4} A_{k \times k} & A_{k \times n-k} \\
A_{k \times n-k}^T & O_{n-k \times n-k}
\end{bmatrix} = \begin{bmatrix}
O_{k \times k} & O_{k \times n-k} \\
O_{k \times n-k}^T & O_{n-k \times n-k}
\end{bmatrix}
\]
\[
(A_{k \times k} = O_{k \times k} \land (A_{k \times n-k} = O_{k \times n-k}).
\]

Hence we have \( A(C, 0) = \left\{ \begin{bmatrix}
O_{k \times k} & O_{k \times n-k} \\
O_{k \times n-k}^T & O_{n-k \times n-k}
\end{bmatrix} : A_{n-k \times n-k} \in \text{Sym} (n-k, \mathbb{R}) \right\} \).

For, other hand if we consider a Jordan frame \( S = \{g_1, g_2, \cdots, g_r\} \) of an Euclidean Jordan algebra \( \mathcal{A} \) and considering the spaces \( A_i = \{x \in \mathcal{A} : L_i(g_i)x = x\} \) and the spaces \( A_{ij} = \{x \in \mathcal{A} : L_i(g_i)x = \frac{1}{2} x \land L_j(g_j)x = \frac{1}{2} x\} \) then we obtain the decomposition of \( \mathcal{A} \) as an orthogonal direct sum of the vector spaces \( A_i \) and \( A_{ij} \) in the following way: \( \mathcal{A} = \sum_{i=1}^r A_i + \sum_{1 \leq i < j \leq r} A_{ij} \).

In the case when the Euclidean symmetric Jordan algebra is \( \mathcal{A} = \text{Sym} (n, \mathbb{R}) \) and we consider the Jordan frame of \( \mathcal{A} \), \( S = \{E_{11}, E_{22}, \cdots, E_{nn}\} \) we obtain the following spaces \( A_i = \{A \in M_n(\mathbb{R}) : \exists x \in \mathbb{R} : A = xE_i\} \) and the spaces \( A_{ij} = \{A \in M_n(\mathbb{R}) : \exists x_{ij} \in \mathbb{R} : A = x_{ij}(E_{ij} + E_{ji})\} \), where the matrices \( E_{ij} \) are the matrices with 1 in the entry \( i \) and with the others entries zero and the matrix \( E_{ij} \) is the matrix with 1 in the entry \( ij \) and zero on the others entries. Therefore the Pierce decomposition of \( \mathcal{A} \) relatively to the Jordan frame of \( \mathcal{A} \) is \( \mathcal{A} = \sum_{i=1}^n A_i + \sum_{1 \leq i < j \leq n} A_{ij} \). Therefore, we can write, any matrix of the \( \text{Sym} (n, \mathbb{R}) \) in the form \( A = \sum_{i=1}^n a_i E_{ii} + \sum_{1 \leq i < j \leq n} a_{ij}(E_{ij} + E_{ji}) \).

Let consider the spin Euclidean Jordan algebra \( A_{n+1} \) and let consider the idempotent \( c = \frac{1}{2} \begin{bmatrix} 1 \\ x \end{bmatrix} \) with \( \|x\| = 1 \). We, will obtain the spaces \( A_{n+1}(c, 1), A_{n+1}(c, \frac{2}{3}) \) and \( A_{n+1}(c, 0) \). We have, the following equivalences

\[
c \circ \begin{bmatrix} a_i \\ a_j \end{bmatrix} = \begin{bmatrix} a_i \\ a_j \end{bmatrix} \iff \frac{1}{2} \begin{bmatrix} 1 \\ x \end{bmatrix} \circ \begin{bmatrix} a_i \\ a_j \end{bmatrix} = \begin{bmatrix} a_i \\ a_j \end{bmatrix}
\]
\[
\iff \frac{1}{2} a_i + \frac{1}{2} x a_j = \begin{bmatrix} a_i \\ a_j \end{bmatrix}
\]
\[
\iff a_i = x a_j \land a_j = (x a_i) x
\]
\[
\iff \begin{bmatrix} a_i \\ a_j \end{bmatrix} = (2x a_i) \begin{bmatrix} 1 \\ x \end{bmatrix} \iff \begin{bmatrix} a_i \\ a_j \end{bmatrix} = (2x a_i) c.
\]

So, we conclude that \( A_{n+1}(c, 1) = \{xc, x \in \mathbb{R}\} \). Now, we will calculate \( A_{n+1}(c, 0) \).

\[
c \circ \begin{bmatrix} a_i \\ a_j \end{bmatrix} = 0 \begin{bmatrix} a_i \\ a_j \end{bmatrix} \iff \frac{1}{2} \begin{bmatrix} 1 \\ x \end{bmatrix} \circ \begin{bmatrix} a_i \\ a_j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]
\[
\iff \frac{1}{2} a_i + \frac{1}{2} x a_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff a_i = -x a_j \land a_j = (x a_i) x
\]
\[
\iff \begin{bmatrix} a_i \\ a_j \end{bmatrix} = (-2x a_i) \begin{bmatrix} 1 \\ x \end{bmatrix} = \frac{1}{-x} \begin{bmatrix} 1 \\ x \end{bmatrix}.
\]
Therefore, we obtain, \( \mathcal{A}_{n+1}(c, 0) = \left\{ x^T \begin{bmatrix} 1 \\ -1 \end{bmatrix}, x \in \mathbb{R} \right\} \). We can rewrite \( \mathcal{A}_{n+1}(c, 0) = \left\{ x \begin{bmatrix} 1 \\ O_{1 \times n} \\ -I_n \end{bmatrix}, c, x \in \mathbb{R} \right\} \). Now, we will obtain \( \mathcal{A}_{n+1}(c, \frac{1}{2}) \).

So, we have

\[
c \circ \begin{bmatrix} a_1 \\ \frac{1}{2}a_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} a_1 \\ \frac{1}{2}a_1 \end{bmatrix} \Leftrightarrow \frac{1}{2} \begin{bmatrix} a_1 \\ \frac{1}{2}a_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} a_1 \\ \frac{1}{2}a_1 \end{bmatrix}
\]

\[
\Leftrightarrow \begin{bmatrix} \frac{1}{2}a_1 + \frac{1}{2}x|a| \\ \frac{1}{2}a_1 + \frac{1}{2}a_1x \end{bmatrix} = \begin{bmatrix} \frac{1}{2}a_1 \\ \frac{1}{2}a_1 \end{bmatrix} \Leftrightarrow x|a| = 0 \land a_1x = 0
\]

\[
\Leftrightarrow (a_1 = 0) \land (x|a| = 0).
\]

Therefore, we obtain,

\[
\mathcal{A}_{n+1}(c, \frac{1}{2}) = \left\{ x \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} : x|a| = 0, x \in \mathbb{R} \right\}.
\]

Now, we will analyse the Pierce decomposition of \( \mathcal{A}_{n+1} \) relatively to the Jordan frame

\[
\{c_1, c_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\}.
\]

As in the previous example, we have \((\mathcal{A}_{n+1})_{11} = \mathcal{A}_{n+1}(c_1, 1) = \{xc_1 : x \in \mathbb{R}\}\), and \((\mathcal{A}_{n+1})_{22} = \mathcal{A}_{n+1}(c_2, 1) = \{xc_2 : x \in \mathbb{R}\}\). Now, we will calculate the vector space

\[
(\mathcal{A}_{n+1})_{12} = \left\{ x \in \mathcal{A}_{n+1} : (L_{c_1}(c_1)x = \frac{1}{2}x) \land (L_{c_2}(c_2)x = \frac{1}{2}x) \right\}
\]

So, we have the following calculations

\[
\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \circ \begin{bmatrix} x_1 \\ x_2 \\ \frac{1}{2}x \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \\ \frac{1}{2}x \end{bmatrix} \Leftrightarrow \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \\ \frac{1}{2}x \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \frac{1}{2}x \end{bmatrix}
\]

\[
\Leftrightarrow \begin{bmatrix} \frac{3}{2}x_1 + \frac{1}{2}x_2 \\ \frac{3}{2}x_2 + \frac{1}{2}x_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_1 \\ \frac{1}{2}x_2 \end{bmatrix}
\]

\[
(x_1 = 0) \land (x_2 = 0).
\]

Therefore \( \mathcal{A}_{n+1}(c_1, \frac{1}{2}) = \left\{ \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}x \end{bmatrix} : x \in \mathbb{R}^{n-1} \right\} \). Now, we will calculate \( \mathcal{A}_{n+1}(c_2, \frac{1}{2}) \).

\[
\frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \circ \begin{bmatrix} x_1 \\ x_2 \\ \frac{1}{2}x \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \\ \frac{1}{2}x \end{bmatrix} \Leftrightarrow \frac{1}{2} \begin{bmatrix} x_1 \\ x_2 \\ \frac{1}{2}x \end{bmatrix} + x_1 \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \frac{1}{2}x \end{bmatrix}
\]

\[
\Leftrightarrow \begin{bmatrix} \frac{3}{2}x_1 - \frac{1}{2}x_2 \\ \frac{3}{2}x_2 - \frac{1}{2}x_1 \\ x \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_1 \\ \frac{1}{2}x_2 \\ \frac{1}{2}x \end{bmatrix} \Leftrightarrow (x_1 = 0) \land (x_2 = 0).
\]

Hence, we have also that \( \mathcal{A}_{n+1}(c_2, \frac{1}{2}) = \mathcal{A}_{n+1}(c_1, \frac{1}{2}) \). Therefore we have

\[
(\mathcal{A}_{n+1})_{12} = \left\{ \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2}x \end{bmatrix} : x \in \mathbb{R}^{n-1} \right\}.
\]

Therefore for any element \( x = \begin{bmatrix} x_1 \\ x_2 \\ \frac{1}{2}x \end{bmatrix} \) of \( \mathcal{A}_{n+1} \) we conclude that
and we must observe that the diagonal entries of this matrix are null. The number of edges incident to a vertice say that a graph is a simple graph if it has no multiple edges (more than one edge between two vertices) and if it has no loops. Sometimes we make a sketch to represent a graph like the one presented on the Figure 1.

An edge is incident on a vertice of a graph if it has one of its extreme points. Two vertices of a graph are adjacent if they are connected by an edge. The adjacency matrix of a simple graph is a symmetric matrix and we call the dimension of this matrix are null. The number of edges incident to a vertice of a simple graph is a regular matrix and we must observe that the diagonal entries of this matrix are null. The number of edges incident to a vertex of a simple graph is the number of vertices of order n, a square matrix of order n, A such that $A = [a_{ij}]$ where $a_{ij} = 1$ if $v_i,v_j \in E(X)$ and 0 otherwise. The adjacency matrix of a simple graph is a symmetric matrix and we must observe that the diagonal entries of this matrix are null. The number of edges incident to a vertex v of a simple graph is called the degree of v. And, we call a simple graph a regular graph if each of its vertices have the same degree and we say that a graph $G$ is a $k$-regular graph if each of its vertices have degree k.

The complement graph of a graph $X$ denoted by $\overline{X}$ is a graph with the same set of vertices of $X$ and such that two distinct vertices are adjacent vertices of $\overline{X}$ if and only if they are non adjacent vertices of $X$.

Along this paper we consider only non-empty, simple and non complete graphs.

Strongly regular graphs were firstly introduced by R. C. Bose in the paper [33].

A graph $X$ is called a $(n, k; \lambda, \mu)$-strongly regular graph if is $k$-regular and any pair of adjacent vertices have $\lambda$ common neighbors and any pair of non-adjacent vertices have $\mu$ common adjacent vertices.

The adjacency matrix $A$ of a $(n, k; \lambda, \mu)$-strongly regular $X$ satisfies the equation $A^2 = kI_n + \lambda A + \mu(J_n - A - I_n)$, where $J_n$ is the all ones real matrix of order n.

The eigenvalues $\theta$, $\tau$, $k$ and the multiplicities $m_\theta$ and $m_\tau$ respectively of a $(n, k; \lambda, \mu)$-strongly regular graph $X$, see, for instance [34, 35], are defined by the equalities (5):

$$\begin{align*}
\theta &= (\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})/2, \\
\tau &= (\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})/2, \\
m_\theta &= \frac{|\tau|n + \tau - k}{n(\theta - \tau)}, \\
m_\tau &= \frac{\theta n + k - \theta}{n(\theta - \tau)}.
\end{align*}$$

(5)

Therefore necessary conditions for the parameters of a $(n, k; \lambda, \mu)$-strongly regular graph are that $\frac{|\tau|n + \tau - k}{n(\theta - \tau)}$ and $\frac{\theta n + k - \theta}{n(\theta - \tau)}$ must be natural numbers, they are known as the integrability conditions of a strongly regular graph, see [34].

**Figure 1.** A simple graph $X$. 

$$x = (x_1 + x_2)c_1 + (x_1 - x_2)c_2 + \begin{bmatrix} 0 \\ 0 \\ x \end{bmatrix}.$$
We note that a graph $X$ is a $(n, k, \lambda, \mu)$-strongly regular graph if and only if its complement graph $\overline{X}$ is a $(n, n - k - 1; n - 2 - 2k + \mu, n - 2k + \lambda)$. Now we will present some admissibility conditions on the parameters of a $(n, k, \lambda, \mu)$-strongly regular graph. The parameters of a $(n, k, \lambda, \mu)$-strongly regular graph $X$ verifies the admissibility condition (6).

$$k(k - 1 - \lambda) = \mu(n - k - 1).$$

(6)

The inequalities (7) are known as the Krein conditions of $X$, see [36].

$$(k + \theta)(\tau + 1) \geq (\theta + 1)(k + \theta + 2\tau),$$

$$ (k + \tau)(\theta + 1) \geq (\tau + 1)(k + \tau + 2\theta).$$

(7)

Given a graph $X$, we call a path in $X$ between two vertices $v_i$ and $v_{k+1}$ to a non null sequence of distinct vertices, exceptionally the first vertex and the last vertex can be equal, and distinct edges, $W = v_1v_2v_3\cdots v_{k}v_{k+1}$ whose terms are vertices and edges alternated and such that for $1 \leq i \leq k$ the vertices $v_i$ and $v_{i+1}$ define the edge $e_i$. The path is a closed path if and only if the only repeated vertices are the initial vertex and the final vertex.

A graph $Y$ is a subgraph of a graph $Y$ and we write $Y' \subseteq Y$ if $V(Y') \subseteq V(Y)$ and $E(Y') \subseteq E(Y)$. If $Y' \subseteq Y$ and $Y' \neq Y$, we say that $Y'$ is a proper subgraph of $Y$. We must observe that for any non empty subset $V$ of $V(Y)$ we can construct a subgraph of $Y$ whose set of vertices is $V'$ and whose set of edges is formed by the edges of $E(Y)$ whose extreme points are vertices in $V$ which we call the induced subgraph of $Y$ and which we denote by $Y(V')$. Two vertices $v_i$ and $v_j$ of a graph $X$ are connected if there is a path between $v_i$ and $v_j$ in $X$. This relation between vertices is a relation of equivalence in the set of vertices of the graph $X$, $V(X)$, whereby there exists a partition of $V(X)$ in non empty subsets $V_1, V_2, \cdots, V_1$ of $V(X)$ such that two vertices are connected if and only if they belong to the same set $V_i$ for a given $i \in \{1, 2, \cdots, l\}$. The subgraphs $X(V_1), X(V_2), \cdots, X(V_l)$ are called the connected components of $X$. If $X$ has only one component then we say that the graph $X$ is connected.

Finally, since from now on, we only consider primitive strongly regular graphs, we note that a $(n, k, \lambda, \mu)$-strongly regular graph is primitive if and only $X$ and $\overline{X}$ are connected graphs.

$$P' = [C_{k \times k} O_{k \times n-k} D_{n-k \times k} E_{n-k \times n-k}]$$

(8)

where $k$ is such that $1 \leq k \leq n-1$, if doesn’t exist such matrix $P$ we say that the matrix $A$ is irreducible. If $A$ is a reducible symmetric of order $n$ then $D_{n-k \times k} = O_{n-k \times k}$. From the Theorem of Frobenius we know that if $A$ is a real square irreducible matrix with non negative entries then $A$ has an eigenvector $\mathbf{u}$ such that $A \mathbf{u} = \mathbf{r} \mathbf{u}$ with all entries positive and such that $|\lambda| \leq r$ for any eigenvalue of $A$ and $r$ is a simple positive eigenvalue of $A$. Now since a graph is connected if and only if its matrix of adjacency is irreducible then, if a graph $X$ is a connected strongly regular graph then the graph $A$ has a connected strongly regular graph then the greatest eigenvalue of its adjacency matrix $A$ is a simple eigenvalue of $A$ with an eigenvector with all components positive. Hence the regularity of a connected strongly regular graph $X$ is a simple eigenvalue of the adjacency matrix of $X$.

Some relations on the parameters of a strongly regular graph

Let $m$ be a natural number. We denote the set of real matrices of order $m$ by $M_m(\mathbb{R})$ and the set of symmetric of $M_m(\mathbb{R})$ by $\text{Sym}(m, \mathbb{R})$. For any two matrices $H = [h_{ij}]$ and $L = [l_{ij}]$ of $M_m(\mathbb{R})$, we define the Hadamard product of $H$ and $L$ as being the matrix $H \odot L = [h_{ij}l_{ij}]$ and the Kronecker product of matrices $H$ and $L$ as being the matrix $H \otimes L = [h_{ij}L]$. For any matrix $P$ of $M_m(\mathbb{R})$ and for any nonnegative integer number $j$ we define the Schur (Hadamard) power of order $j$ of $P$, as being the matrix $P^j$ in the following way: $P^0 = I_m, P^1 = P$ for any natural number $j \geq 2$ we define $P^{j+1} = P \odot P^j$.

In this section we will establish some inequalities over the parameters and over the spectra of a primitive strongly regular graph.
Let’s consider a primitive \((n, k, \lambda, \mu)\)-strongly regular graph \(G\) such that \(\frac{\mu}{k} > k > \mu > 0\) and with \(\mu < \lambda\), and let \(A\) be its adjacency matrix with the distinct eigenvalues \(\tau, \theta, k\). Now, we consider the Euclidean Jordan algebra \(\mathcal{A} = \text{Sym}(n, \mathbb{R})\) with the Jordan product \(x \circ y = \frac{xy + yx}{2}\) and with the inner product \(|x|y = \text{trace}(x \circ y)\), where \(xy\) and \(yx\) are the usual products of \(x\) by \(y\) and the usual product of \(y\) by \(x\). Now we consider the Euclidean Jordan subalgebra \(\mathcal{A}\) of \(\text{Sym}(n, \mathbb{R})\) spanned by \(I_n\) and the natural powers of \(A\). We have that rank \((A) = 3\) since has three distinct eigenvalues and is a three dimensional real Euclidean Jordan algebra. Let \(\mathcal{B} = \{E_1, E_2, E_3\}\) be the unique Jordan frame of \(\mathcal{A}\) associated to \(A\), where

\[
E_1 = \frac{1}{n} I_n + \frac{1}{n} A + \frac{1}{n} (J_n - A - I_n),
\]

\[
E_2 = \left[\frac{1}{n} \tau n + \tau - k \right] \frac{1}{n(\theta - \tau)} I_n + \frac{1}{n(\theta - \tau)} A + \frac{\tau - k}{n(\theta - \tau)} (J_n - A - I_n),
\]

\[
E_3 = \frac{\theta n + k - \theta}{n(\theta - \tau)} I_n + \frac{-n + k - \theta}{n(\theta - \tau)} A + \frac{k - \theta}{n(\theta - \tau)} (J_n - A - I_n).
\]

Let’s consider the right positive number \(x\) such that \(x \leq 1\), and let’s consider the binomial Hadamard series

\[
S_x = \sum_{i=0}^{\infty} (-1)^i \binom{-x}{i} \left(\frac{(\frac{\theta}{\theta - \tau})^i}{i!}\right) (J_n - A - I_n),
\]

The second spectral decomposition of \(S_x\) relatively to the Jordan frame \(\mathcal{B}\) is

\[
S_x = \sum_{i=1}^{\infty} q_{ix} E_i,
\]

Now, we show that the eigenvalues \(q_{ix}\) of \(S_x\) are positive.

Since \((-1)^i \left(\frac{-x}{i}\right) = (-1)^i \left(\frac{-1}{i}\right) (-z)(-z - 1)(-z - 2) \cdots (-z - l + 1)\) then

\[
(-1)^i \left(\frac{-x}{i}\right) = (-1)^i \left(\frac{-1}{i}\right) (z + 1)(z + 2) \cdots (z + l - 1) \geq 0.
\]

Now, we have \(S_{x^2} = \sum_{i=0}^{n} (-1)^i \left(\frac{-x^2}{i}\right) \left(\frac{(\frac{\theta}{\theta - \tau})^i}{i!}\right) (J_n - A - I_n)\). Since \(A^2 = kI_n + \lambda A + \mu (J_n - A - I_n)\) then we conclude that

\[
\frac{(A^2 - \theta^2 I_n)^{2\theta}}{k^4} = \frac{(kI_n + \lambda A + \mu (J_n - A - I_n) - \theta^2 I_n)^{2\theta}}{k^4}
\]

\[
= \frac{(k - \theta^2)^2 I_n + \lambda A + \mu (J_n - A - I_n)}{k^4}
\]

\[
= \frac{(k - \theta^2)^2 I_n + \lambda^2 A + \mu^2 (J_n - A - I_n)}{k^4}
\]

\[
= \frac{(k - \theta^2)^2 I_n + k^2 A + \mu^2 (J_n - A - I_n)}{k^4}.
\]

Let’s consider the second spectral decomposition \(S_{x^2} = q_{1ix} E_1 + q_{2ix} E_2 + q_{3ix} E_3\). Since \(\lambda > \mu\) then we have that \(|\tau| < \theta\) and therefore \(|\tau|^2 < \theta^2\) and since \(\theta^2 \leq \kappa^2\) then the eigenvalues of \(\frac{e^{|\tau|^2} I_n}{\theta^2}\) are positive. Since for any two matrices \(C\) and \(D\) of \(M_n(\mathbb{R})\) we have \(\lambda_{\text{min}}(C) \leq \lambda_{\text{min}}(C \circ D)\) and since \(B\) is a Jordan frame of \(\mathcal{A}\) that is a basis of \(\mathcal{A}\) and \(\mathcal{A}\) is closed for the Schur product of matrices we deduce that the eigenvalues of \(\left(\frac{e^{|\tau|^2} I_n}{\theta^2}\right)\) are positive. So we conclude that the eigenvalues \(q_{ix}\) for \(i = 1, \ldots, 3\) of \(S_{x^2}\) are all positive.

Since \(q_{1x} = \lim_{x \to +\infty} q_{1ix}\), \(q_{2x} = \lim_{x \to +\infty} q_{2ix}\), \(q_{3x} = \lim_{x \to +\infty} q_{3ix}\) then we have that \(q_{1x} \geq 0, q_{2x} \geq 0\) and \(q_{3x} \geq 0\). We must say that \(S_1 E_1 = q_{1ix} E_1, S_2 E_2 = q_{2ix} E_2\) and \(S_3 E_3 = q_{3ix} E_3\), hence we have: \(q_{1x} = \frac{1}{\left(\frac{1}{\kappa} + \frac{1}{\kappa^2} \theta\right) k^2} + \frac{1}{\left(\frac{1}{\kappa} + \frac{1}{\kappa^2} \tau\right) k} + \frac{1}{\left(\frac{1}{\kappa} + \frac{1}{\kappa^2} \tau\right) k} (n - k - 1),\)

\(q_{2x} = \frac{1}{\left(\frac{1}{\kappa} + \frac{1}{\kappa^2} \theta\right) k^2} + \frac{1}{\left(\frac{1}{\kappa} + \frac{1}{\kappa^2} \tau\right) k} (\theta - 1)\) and \(q_{3x} = \frac{1}{\left(\frac{1}{\kappa} + \frac{1}{\kappa^2} \theta\right) k^2} + \frac{1}{\left(\frac{1}{\kappa} + \frac{1}{\kappa^2} \tau\right) k} (\tau - 1)\).

Let’s consider the element \(S_{x^2} = S_{x^2} \circ S_x\) of \(\mathcal{A}\). Since the eigenvalues of \(E_3\) and of \(S_x\) are positive and since \(\lambda_{\text{min}}(E_3) \lambda_{\text{min}}(S_x) \leq \lambda_{\text{min}}(E_3 \circ S_x)\) then the eigenvalues of \(E_3 \circ S_x\) are positive. Now since \(k < \frac{\mu}{k}\) and \(\lambda > \mu\) and by an asymptotical analysis of the spectrum of \(E_3 \circ S_x\) we will deduce the inequalities (16) and (21) of the Theorems 3 and 4 respectively are verified.

Now, we consider the second spectral decomposition \(E_3 \circ S_x = q_{1ix} E_1 + q_{2ix} E_2 + q_{3ix} E_3\). Then, we have

\[
q_{1ix} = \frac{\theta n + k - \theta}{n(\theta - \tau)} \left(\frac{\theta^2}{\theta - \tau}\right) + \frac{-n + k - \theta}{n(\theta - \tau)} \left(\frac{\theta^2}{\theta - \tau}\right) k - \frac{k - \theta}{n(\theta - \tau)} \left(\frac{\theta^2}{\theta - \tau}\right) (n - k - 1),
\]

(11)
Using the fact that \( k < \frac{\mu}{2} \) then we conclude that \( \frac{\mu-k+\theta}{\theta+k+\theta} > \frac{1}{2\theta+1} \).

**Theorem 3.** Let \( n, k, \mu \) and \( \lambda \) be natural numbers with \( n-1 > k > \mu \) and \( X \) be a \((n, k; \lambda, \mu)\)-primitive strongly regular graph with distinct eigenvalues \( k, \theta \) and \( \tau \). If \( k < \frac{\mu}{2} \) and \( \lambda > \mu \) then

\[
\frac{k^4 - \mu \lambda}{k^4 - (k-\theta^2)^2} > \frac{k^4 - \mu \lambda}{k^4 - \lambda^2}.
\]

**Proof.** Since \( q_{iz}^1 \geq 0 \) and recurring to the equality (14) we deduce the equality (17).

\[
\frac{\theta n + k - \theta}{n(\theta - \tau)} \left( \frac{k^4 - \mu^2}{k^4 - (k-\theta^2)^2} \right)^2 - \frac{n + k - \theta}{n(\theta - \tau)} \left( \frac{k^4 - \mu^2}{k^4 - (k-\theta^2)^2} \right)^2 \geq 0.
\]

Making, some algebraic manipulation of equality (17) we obtain the inequality (18).

\[
\frac{k^4 - \mu^2}{k^4 - (k-\theta^2)^2} \geq \frac{n - k + \theta}{\theta n + k - \theta} \left( \frac{k^4 - \mu^2}{k^4 - (k-\theta^2)^2} \right)^2 + \frac{n + k - \theta}{n(\theta - \tau)} \left( \frac{k^4 - \mu^2}{k^4 - (k-\theta^2)^2} \right)^2 \geq 0.
\]

Calculating the limits of the expressions of both hand sides of (18) when \( x \) approaches zero we obtain the equality (19).

\[
1 \geq \frac{n - k + \theta}{\theta n + k - \theta} \left( \frac{k^4 - \mu^2}{k^4 - (k-\theta^2)^2} \right)^2 + \frac{n + k - \theta}{n(\theta - \tau)} \left( \frac{k^4 - \mu^2}{k^4 - (k-\theta^2)^2} \right)^2.
\]

Now since when \( k < \frac{\mu}{2} \) we have \( \frac{\mu-k+\theta}{\theta+k-\theta} > \frac{1}{2\theta+1} \) and after some algebraic manipulation of the inequality (19) we obtain the inequality (20).

\[
0 > \frac{k}{2\theta + 1} \left( \frac{\ln \left( \frac{k^4 - \mu^2}{k^4 - (k-\theta^2)^2} \right)}{\ln \left( \frac{k^4 - \mu^2}{k^4 - (k-\theta^2)^2} \right)} \right).
\]

**Theorem 4.** Let \( n, k, \mu \) and \( \lambda \) be natural numbers and let \( X \) be a \((n, k; \lambda, \mu)\)-primitive strongly regular graph with \( n-1 > k > \mu \), \( \frac{\mu}{2} > k, \mu < \lambda \) and with the distinct eigenvalues \( \theta, \tau \) and \( k \). Then
\[
\left( \frac{k^4 - \mu^2}{k^4 - (k - \theta)^4} \right)^{2n+1} > \left( \frac{k^4 - \mu^2}{k^4 - k^2} \right) \theta.
\]

**Proof.** Now since \(q_k^3 \geq 0\) and recurring to the equality (15) we obtain that
\[
\frac{\theta n + k - \theta}{n(\theta - \tau)} \cdot \left( \frac{(k^4 - \mu^2)^2 - (k^4 - (k - \theta)^4)^2}{(k^4 - (k - \theta)^4)^2 (k^4 - k^2)^2} \right)^{\theta} + \frac{-n + k - \theta}{n(\theta - \tau)} \left( \frac{(k^4 - \mu^2)^2 - (k^4 - (k - \theta)^4)^2}{(k^4 - k^2)^2 (k^4 - (k - \theta)^4)^2} \right) \theta \geq 0.
\]

From (22) we deduce the inequality (23).
\[
\left( \frac{k^4 - \mu^2}{k^4 - (k - \theta)^4} \right)^{\theta} \geq \frac{n - k + \theta}{\theta n + k - \theta} \left( \frac{(k^4 - \mu^2)^2 - (k^4 - (k - \theta)^4)^2}{(k^4 - k^2)^2 (k^4 - (k - \theta)^4)^2} \right) \theta.
\]

Calculating the limits of the expressions of both hand sides of (23) when \(x\) approaches zero we obtain the equality (24).
\[
1 \geq \frac{n - k + \theta}{\theta n + k - \theta} \left( \frac{\ln \left( \frac{k^4 - \mu^2}{k^4 - (k - \theta)^4} \right)}{\ln \left( \frac{k^4 - \mu^2}{k^4 - k^2} \right)} \right).
\]

Now, since when \(k < \frac{\pi}{2}\) we have \(\frac{\mu - k}{\ln + k - \theta} > \frac{1}{2n+1}\) and after some algebraic manipulation of the inequality (24) we obtain the inequality (25).
\[
1 > \frac{\theta}{2n+1} \left( \frac{\ln \left( \frac{k^4 - \mu^2}{k^4 - (k - \theta)^4} \right)}{\ln \left( \frac{k^4 - \mu^2}{k^4 - k^2} \right)} \right).
\]

And, finally we conclude that
\[
\left( \frac{k^4 - \mu^2}{k^4 - (k - \theta)^4} \right)^{2n+1} > \left( \frac{k^4 - \mu^2}{k^4 - k^2} \right) \theta.
\]

**Preliminares on quaternions and octonions**

This section is a brief introduction on quaternions and octonions. Good readable texts on this algebraic structures can be found on the works [37, 38]. Now, we consider the real linear space \(\mathcal{A}\) of quaternions spanned by the basis \(\mathcal{B} = \{1, i, j, k\}\), where the elements of \(\mathcal{B}\) verify the following rules of multiplication \(\bar{p}^2 = \bar{i}^2 = k^2 = -1\) and

1) \(ij = -ji = k;\)
2) \(jk = -kj = i;\)
3) \(ki = -ik = j.\)

So, we can write \(\mathcal{A} = \{x_01 + x_1i + x_2j + x_3k, x_0, x_1, x_2, x_3 \in \mathbb{R}\}.\) Given a quaternion \(x = x_01 + x_1i + x_2j + x_3k\) we call to \(x_0\) the real part of \(x\) we denote it by \(\text{Re} (x)\) and we call to \(x_1i + x_2j + x_3k\) the imaginary part of \(x\) and we write \(\text{Im} (x) = x_1i + x_2j + x_3k.\)

One says that a quaternion \(x\) is a pure quaternion if \(\text{Re} (x) = 0\) and if \(x = \text{Re} (x)\) one says that the quaternion \(x\) is a real number.

For discovering the quaternions on multiplication described above we must use the diagram of Fano, see Figure 2.

When we multiply to elements of the set \(\{i, j, k\}\) we use the rule: when we multiply two elements in clockwise sense we get the next element, so for instance we have \(jk = i\), but if we multiply them in the counterclockwise sense we obtain the next but with minus sign \(jk = -i.\)

And, therefore we obtain Table 1 of multiplication of two elements of \(\mathcal{A}.\)

If we consider \(x = x_01 + x_1i + x_2j + x_3k\) and \(y = y_01 + y_1i + y_2j + y_3k\) we obtain the following expression for the product \(x \cdot y\),
\[
x \cdot y = x_0y_0 - (x_1y_1 + x_2y_2 + x_3y_3) + x_0(y_1i + y_2j + y_3k) + y_0(x_1i + x_2j + x_3k) + (x_2y_3 - x_3y_2)i + (x_3y_1 - x_1y_3)j + (x_1y_2 - x_2y_1)k.
\]

Using the notation \(\text{Re} (x) = x_0\) and \(\text{Im} (x) = x_1i + x_2j + x_3k\) we conclude that
\[
x \cdot y = \text{Re} (x) \text{Re} (y) - \text{Im} (x) \text{Im} (y) + \text{Re} (x) \text{Im} (y) + \text{Re} (y) \text{Im} (x) + \text{Im} (x) \times \text{Im} (y).
\]
Table 1. Table of multiplication of quaternions.

| x * y | 1   | i   | j   | k   |
|-------|-----|-----|-----|-----|
| 1     | 1   | i   | j   | k   |
| i     | i   | -1  | k   | -j  |
| j     | j   | -k  | -1  | i   |
| k     | k   | j   | -i  | -1  |

For a quaternion \( x = x_0 + x_1 i + x_2 j + x_3 k \) we define we define \(|x|\) by the the equality \(|x| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}\). Now, we will show that \( \overline{x * y} = y * \overline{x} \). Hence, we have the following calculations:

\[
\overline{x * y} = (\Re(x) + \Im(x)) * (\Re(y) + \Im(y))
\]

\[
= \Re(x) \Re(y) - \Im(x) \Im(y) + \Re(x) \Im(y) + \Re(y) \Im(x) + \Im(x) \times \Im(y)
\]

\[
= (\Re(y) - \Im(y)) * (\Re(x) - \Im(x))
\]

\[
= \overline{\Re(y) + \Im(y)} * \overline{\Re(x) + \Im(x)}
\]

\[
= y * \overline{x}.
\]

In a similar way, we deduce that \(|x| = \sqrt{x^* x}\). The inverse of a nonzero quaternion \( x \) is defined as an element \( x^{-1} \) such that \( x * x^{-1} = x^{-1} * x = 1 \). But since \( x * \overline{x} = x * x = |x|^2 = x^2 \), where \( x^2 = x * x \). Then, \( x^{-1} = \frac{\overline{x}}{|x|^2} \).

Now, we will introduce the real linear space of octonions. \( \mathcal{A} = \{x_0 + x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 + x_5 f_5 + x_6 f_6 + x_7 f_7, x_i \in \mathbb{R}, x_0 \in \mathbb{R}, x_i \in \mathbb{R}, i = 1, \ldots, 7 \} \) where the elements of the basis \( B = \{1, f_1, f_2, f_3, f_4, f_5, f_6, f_7\} \) of \( \mathcal{A} \) satisfy the rules of multiplication presented in Table 2 below and deduced using the diagram presented in Figure 3.

Now, we fulfill the table recurring to the diagram of Fano, for instance we have \( f_0 f_5 = f_6 \) but we have \( f_5 f_2 = -f_6 \) since the sense from \( f_1 \) to \( f_2 \) is contrary to the sense of line that contains \( f_1 \) and \( f_4 \).

The conjugate of the octonion \( x = x_0 + x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 + x_5 f_5 + x_6 f_6 + x_7 f_7 \) is \( \overline{x} = x_0 - x_1 f_1 - x_2 f_2 - x_3 f_3 - x_4 f_4 - x_5 f_5 - x_6 f_6 - x_7 f_7 \) and the real part of the octonion \( x \) is \( \Re(x) = x_0 \) and the imaginary part of the octonion is \( \Im(x) = x_1 f_1 + x_2 f_2 + \cdots + x_7 f_7 \). We define

\[
|\overline{x}| = \sqrt{x^* x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + \cdots + x_7^2}.
\]

Table 2. Table of multiplication of octonions.

| x * y | 1   | f_1 | f_2 | f_3 | f_4 | f_5 | f_6 | f_7 |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|
| 1     | 1   | f_1 | f_2 | f_3 | f_4 | f_5 | f_6 | f_7 |
| f_1   | f_1 | -1  | f_2 | f_3 | f_4 | f_5 | f_6 | f_7 |
| f_2   | f_2 | -f_3 | -1  | f_1 | f_6 | f_7 | f_4 | f_5 |
| f_3   | f_3 | -f_2 | -f_1 | -1  | f_7 | f_6 | f_5 | f_4 |
| f_4   | f_4 | -f_5 | -f_6 | -f_7 | -1  | f_1 | f_2 | f_3 |
| f_5   | f_5 | f_4 | -f_7 | f_6 | -f_1 | -1  | f_3 | f_2 |
| f_6   | f_6 | f_5 | f_4 | -f_7 | f_1 | -1  | f_2 | f_3 |
| f_7   | f_7 | f_6 | f_5 | f_4 | f_1 | -1  | f_3 | f_2 |
Using the table of multiplication 2 we obtain for $x = x_0 + x_1f_1 + x_2f_2 + x_3f_3 + x_4f_4 + x_5f_5 + x_6f_6 + x_7f_7$ and for $y = y_0 + y_1f_1 + y_2f_2 + y_3f_3 + y_4f_4 + y_5f_5 + y_6f_6 + y_7f_7$ we obtain

$$x*y = x_0y_0 + x_0(y_1f_1 + y_2f_2 + y_3f_3 + y_4f_4 + y_5f_5 + y_6f_6 + y_7f_7) +$$
$$+ y_0(x_1f_1 + x_2f_2 + x_3f_3 + x_4f_4 + x_5f_5 + x_6f_6 + x_7f_7) +$$
$$- x_1y_1 + x_2y_2 - x_3y_3 - x_4y_4 - x_5y_5 - x_6y_6 - x_7y_7 +$$
$$+ (x_1y_3 - x_3y_1)f_1 + (x_1y_2 - x_2y_1)f_2 + \cdots + (x_1y_2 - x_2y_1)f_7.$$  

We must note that: $x*y = \text{Re}(x)\text{Re}(y) + \text{Re}(x)\text{Im}(y) + \text{Re}(y)\text{Im}(x) - \text{Im}(x)\text{Im}(y) + \text{Im}(x)\times\text{Im}(y)$ Proceeding like we have done for the quaternions we deduce that $\|x\| = \sqrt{x_0^2 + x_1^2 + x_2^2 + \cdots + x_7^2}$. We, must say, again that $\|x\| = \sqrt{x^*x}$. Let $x$ be an octonion such that $\|x\| \neq 0$ then the inverse of $x$ is $x^{-1} = \frac{x}{\|x\|^2}$, since $x \cdot x^{-1} = x^{-1} \cdot x = 1$.

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