Galerkin-Collocation Manner for The System of Nonlinear Volterra Integral Equations of The Second Type Using Exponential Function

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Summary. In this research, we introduce a new scheme stem on exponential function as test function of Galerkin-Collocation manner (GCm) for disbanding system of nonlinear Volterra integral equations of second type (NLVIEK2) numerically. Our manner involves of diminution the trouble to a collection of nonlinear system of equations. Arithmetic epitome are offered for corroborate the validation of the suggested illation and arithmetic disbanding are contrast with the located manner.

1. Introduction

There is a fundamental relationship between Volterra integral equations and ordinary differential equations. Many initial and boundary value problems can be transformed into integral equations. Problems in the mathematical physics are usually governed by the integral equations [1].

The important of the integral equations in the most competition so that many authors disbanding d integral equations Arithmetically [2 -13]. The reference [14 -21] contain many difference manners for disbanding the integral equations approximately and analytically.

Recently many authors have investigated the Galerkin and Collocation manner for disbanding many type of Integral Equations found in these reference [6, 7, 8, 9, 11, 13, 21, 15, 18, 20, 22, 23].

We use expansion manner to approximate the disbanding of the system of non-linear VIEK2 since one of its uses is to replace complicated functions by some simpler functions so that integral operations can be more easily performed.

Enquire procedure in the approximate process supply for getting resulted is formed in functions, in several time are neared to the correct disbanding for the given model [24].

In this research an extremely easy and effectual Galerkin-Collocation manner is suggested with exponential function as test functions. The drafting used to disbanding system of NLVIEK2 in this form:

\[ u_i (x) = f_i (x) + \int_{a}^{x} k_i (x, t, u_1 (t), \ldots, u_m (t))dt ; x \in I = [a, b], i = 1, 2, \ldots, m \]  

(1)
Where \( f_i, i = 1, 2, \ldots, m \), \( m \in \mathbb{N} \) (\( N \) natural number) be continuous functions on an interval \( I \), \( k_i, i = 1, 2, \ldots, m \) denotes given continuous functions on \( \{(x, t) : a \leq t \leq x \leq b\} \), while \( u_i(x), i = 1, 2, \ldots, m \) are the unknown functions to be determined.

2. Outline of the manner
Galerkin-collocation is stem on approximating the disbanding \( u_i(x), i = 1, 2, \ldots, m \) by partial sum:

\[
S_{iN}(x, t_i) = \sum_{k=0}^{N} a_{ik} \Phi_k(x), i = 1, 2, \ldots, m
\]

(2)

where \( \Phi_k(x) = (e^x)^k \), \( k=0, 1, \ldots, N \) naturally be choosing linearly independent and assuming the function \( S_{iN}(x) \) is a linear combination of \( \Phi_k(x) \), its expansion coefficients \( a_{i0}, a_{i1}, \ldots, a_{IN}, i = 1, 2, \ldots, m \) are to be determined uniquely, see \([4, 25]\).

Consider the functional equation given by

\[
\varphi_i[u_i(x)] = f_i(x), x \in D, \ i = 1, 2, \ldots, m,
\]

(3)

where \( \varphi_i : C[a, b] \rightarrow C[a, b] \) operators of the integral equations defined by

\[
\varphi_i[u_i(x)] = u_i(x) - \int_{a}^{x} k_i(x, t, u_i(t))dt, \ i = 1, 2, \ldots, m
\]

(4)

Substituting the approximate disbanding \( S_{iN}(x) \) given by (2) into equation (3), the result is called "residual function" defined by

\[
E_{iN}(x, a_{i0}, a_{i1}, \ldots, a_{IN}) = \varphi_i[S_{iN}(x)] - f_i(x), \ i = 1, 2, \ldots, m
\]

(5)

This residual function \( E_{iN}(x, a_{i0}, a_{i1}, \ldots, a_{IN}) \) depends on \( x, t [13] \) and the challenge is to choose the coefficients \( a_{i0}, a_{i1}, \ldots, a_{IN} \) so that the residual function is minimized. Throughout this study we shall endeavor to minimize \( E_{iN}(x, a_{i0}, a_{i1}, \ldots, a_{IN}) \) \([18]\).

The goal of G-Cm is to choose the coefficients \( a_{i0}, a_{i1}, \ldots, a_{IN} \) so that \( E_{iN}(x, a_{i0}, a_{i1}, \ldots, a_{IN}) \) becomes small (in fact zero) over a chosen domain. In integral form this can be achieved with the condition

\[
\int_{D} w_i(x)E_{iN}(x, a_{i0}, a_{i1}, \ldots, a_{IN})dx = 0, \ i = 1, 2, \ldots, N
\]

(6)

where

\[
w_i(x) = \frac{\partial S_{iN}(x)}{\partial a_k}; \ k = 0, 1, 2, \ldots, N, \ i = 1, 2, \ldots, m
\]

where \( S_{iN}(x) \) is the approximated disbanding of the problem, therefore by

\[
\int_{D} \frac{\partial S_{iN}(x)}{\partial a_k}E_{iN}(x, a_{i0}, a_{i1}, \ldots, a_{IN})dx = 0, \ k = 0, 1, 2, \ldots, N.
\]

(7)

This will provide \( m \times (N + 1) \)'s non-linear simultaneous equations to determine the coefficients \( a_{i0}, a_{i1}, \ldots, a_{IN}, i = 1, 2, \ldots, m \).
3. Disbanding of a System of Non-linear VIEK2 by Galerkin-Collocation Manner

In this section we approximate the disbanding of a system of non-linear VIEK2 by GCm, described in the previous section.

Using operator’s form (3), system (8) can be written as follows:

$$\phi_i(u_i(x)) = u_i(x) - \sum_{k=0}^{N} a_k \exp(k \beta x) - \int_a^x K_i(x,t,U(t)) \, dt, \quad i = 1, 2, \ldots, m$$

(8)

Where the unknown functions $u_i(x)$ are approximated by $S_{iN}(x)$ which is given by equation (2).

Now the approximate disbanding (2) substituting in the system (9) to obtain:

$$\phi_i[S_{iN}(x)] = \sum_{k=0}^{N} a_k \exp(k \beta x) - \int_a^x K_i(x,t,\sum_{k=0}^{N} a_{1k} \exp(k \beta x), \ldots, \sum_{k=0}^{N} a_{mk} \exp(k \beta x)) \, dt$$

From equation (5) we obtain the following residual equations

$$E_{iN}(x,a_o, a_{11}, a_{12}, \ldots, a_{1N}) = \sum_{k=0}^{N} a_k \exp(k \beta x) - \int_a^x K_i(x,t,\sum_{k=0}^{N} a_{1k} \exp(k \beta x), \ldots, \sum_{k=0}^{N} a_{mk} \exp(k \beta x)) \, dt - f_i(x)$$

(9)

The V non-linear equations in $a_{i0}, a_{i1}, \ldots, a_{iN}, i = 1, 2, \ldots, m$ are gained by assuming the integration of the residual $E_{iN}(x,a_o, a_{i1}, \ldots, a_{iN})$ which defined by the equation (7) over the intervals $[a,x_p], \beta = 0, 1, \ldots, V$ are vanishing, i.e.

$$\int_a^{x_p} \frac{\partial S_{iN}(x)}{\partial a_k} E_{iN}(x,a_o, a_{i1}, \ldots, a_{iN}) \, dx = 0.$$  

(10)

Substituting $E_{iN}(x,a_o, a_{i1}, \ldots, a_{iN})$ in the equation (10) into the equation (10) yields:

$$\int_a^{x_p} \beta \exp(\beta x) \left[ \sum_{k=0}^{N} a_k \exp(k \beta x) - \int_a^x K_i(x,t,\sum_{k=0}^{N} a_{1k} \exp(k \beta x), \ldots, \sum_{k=0}^{N} a_{mk} \exp(k \beta x)) \, dt \right] \, dx - \int_a^{x_p} \exp(\beta x) f_i(x) \, dx = 0$$  

(11)

From (12) we get the following non-linear system of equations

$$\int_a^{x_p} \beta \exp(\beta x) \left[ \sum_{k=0}^{N} a_k \exp(k \beta x) \right] \, dx - \int_a^{x_p} \beta \exp(\beta x) K_i(x,t,\sum_{k=0}^{N} a_{1k} \exp(k \beta x), \ldots, \sum_{k=0}^{N} a_{mk} \exp(k \beta x)) \, dt \, dx$$  

$$\beta = 0, 1, \ldots, V.$$  

The unknown coefficients $a_{i0}, a_{i1}, \ldots, a_{iN}, i = 1, 2, \ldots, m$ are to be specified by disband the above non-linear system of equation (12), and substitute the magnitude of coefficients into the equation (2), we obtain the approximate disbanding of equation (1).

4. Illustrative Epitomes

In order to assess both the applicability and accuracy of the theoretical results in section 3, we have applied them to a variety of non-linear system of VIEK2’s in the following epitomes:

Epitome 1 [26]

Disbanding a system of non-linear VIEK2’s:

$$u_i(x) = x - x^2 + \int_0^x (u_1(t) + u_2(t)) \, dt$$
\[ u_2(x) = x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \int_0^x (u_1'(t) + u_2(t))dt \]

We have the correct disbanding of the system are \( u_1(x) = x \) and \( u_2(x) = x \). We view that the approximate disbanding gained by this manner has perfect pact with the correct disbanding.

**Table 1.** comparison of the errors of Epitome 1.

|   | \( u_1(x) \) | \( u_2(x) \) |
|---|---|---|
| N=1 | \( 2.6854 \times 10^{-2} \) | \( 2.5521 \times 10^{-2} \) |
| N=2 | \( 2.0255 \times 10^{-3} \) | \( 1.9737 \times 10^{-3} \) |
| N=3 | \( 3.6170 \times 10^{-4} \) | \( 2.3227 \times 10^{-4} \) |

In table 2, taking \( N=2, 3 \), we contrast the error of the offered manner (Manner 1) with [17], [20], [19] and [16] respectively.

**Table 2.** The result of contrast of Epitome 1.

| Manners | N=2 | N=3 |
|---|---|---|
|   | \( u_1(x) \) | \( u_2(x) \) | \( u_1(x) \) | \( u_2(x) \) |
| Manner 1 | \( 2.0255 \times 10^{-3} \) | \( 1.9737 \times 10^{-3} \) | \( 3.6170 \times 10^{-4} \) | \( 2.3227 \times 10^{-4} \) |
| Manner 2 | 0 | 0 | 0 | 0 |
| Manner 3 | 0 | 0 | 0 | 0 |
| Manner 4 | MBLM2 | \( 7.6115 \times 10^{-2} \) | \( 7.3900 \times 10^{-2} \) | \( 2.5163 \times 10^{-3} \) | \( 1.2612 \times 10^{-3} \) |
| MBLM3 | \( 2.2870 \times 10^{-3} \) | \( 1.0959 \times 10^{-3} \) | \( 1.2124 \times 10^{-4} \) | \( 4.1972 \times 10^{-5} \) |

**Epitome 2 [27]**

Disbanding a system of non-linear VIEK2’s:

\[ u_1(x) = \sec(x) - x + \int_0^x ((u_1')^2 - (u_2')^2)dt \]
\[ u_2(x) = 3\tan(x) - x - \int_0^x ((u_1')^2 + (u_2')^2)dt \]

We have the correct disbanding of the system are \( u_1(x) = \sec(x) \) and \( u_2(x) = \tan(x) \). We view the approximate disbanding gained by this manner has perfect pact with the correct disbanding. In table 3 taking \( N=2, 3 \), we contrast the error of the offered manner (Manner 1) with [17], [20], [19] and [16] respectively.
Table 3. The result of contrast of Epitome 2.

| Manners  | N=2          | N=3          |
|----------|--------------|--------------|
|          | $u_1(x)$     | $u_2(x)$     | $u_1(x)$     | $u_2(x)$     |
| Manner 1 | $1.8826 \times 10^{-3}$ | $4.2660 \times 10^{-2}$ | $4.8405 \times 10^{-3}$ | $2.1359 \times 10^{-4}$ |
| Manner 2 | $1.8302 \times 10^{-1}$ | $5.2323 \times 10^{-1}$ | $1.8302 \times 10^{-1}$ | $6.8365 \times 10^{-2}$ |
| Manner 3 | $1.6240 \times 10^{-1}$ | $3.7186 \times 10^{-1}$ | $1.1991 \times 10^{-1}$ | $5.0132 \times 10^{-2}$ |
| Manner 4 | MBLM2       | $3.4937 \times 10^{-1}$ | $1.0005 \times 10^{-1}$ | $4.6744 \times 10^{-3}$ |
|          | MBLM3       | $5.1058 \times 10^{-1}$ | $3.2951 \times 10^{-1}$ | $1.8302 \times 10^{-3}$ |
|          | MBLM5       | $3.3693 \times 10^{-3}$ | $4.7832 \times 10^{-3}$ | $8.9688 \times 10^{-4}$ |

Epitome 3 [14]

Disbanding a system of non-linear VIEK2’s:

$$ u_1(x) = \frac{1}{4} - \frac{1}{4} e^{2x} + \int_{0}^{x} (x-t)u_2^2(t) dt $$

$$ u_2(x) = -xe^{x} + 2e^{x} - 1 + \int_{0}^{x} \frac{1}{t} e^{-2\gamma(t)} dt $$

We have the correct disbanding of the system are $u_1(x) = -0.5x$ and $u_2(x) = e^{x}$. We view that the approximate disbanding gained by this manner has perfect pact with the correct disbanding. In table 4 taking N=2, 3, we contrast the error of the offered manner (Manner 1) with [17], [20], [19] and [16] respectively.

Table 4. The result of contrast of Epitome 3.

| Manners  | N=2          | N=3          |
|----------|--------------|--------------|
|          | $u_1(x)$     | $u_2(x)$     | $u_1(x)$     | $u_2(x)$     |
| Manner 1 | $3.7628 \times 10^{-4}$ | $3.5221 \times 10^{-2}$ | $3.8931 \times 10^{-5}$ | $2.8554 \times 10^{-5}$ |
| Manner 2 | $9.5006 \times 10^{-6}$ | $2.0700 \times 10^{-1}$ | $1.5006 \times 10^{-6}$ | $1.0700 \times 10^{-1}$ |
| Manner 3 | $5.9824 \times 10^{-2}$ | $5.4301 \times 10^{-2}$ | $1.5734 \times 10^{-3}$ | $2.0101 \times 10^{-4}$ |
| Manner 4 | MBLM2       | $7.7723 \times 10^{-3}$ | $2.2726 \times 10^{-2}$ | $6.5937 \times 10^{-5}$ |
|          | MBLM3       | $1.7864 \times 10^{-3}$ | $9.7120 \times 10^{-4}$ | $6.7330 \times 10^{-5}$ |
|          | MBLM5       | $1.7864 \times 10^{-3}$ | $9.7120 \times 10^{-4}$ | $6.7330 \times 10^{-5}$ |
5. Conclusion
In this research an extremely easy and effectual Galerkin-Collocation manner stem on exponential function as test function has been expanded to disbanding second kind nonlinear system of Volterra integral equations. The Arithmetic results a gained by the offered manner are in perfect pact with the correct disbanding. An accuracy of the arithmetic disbanding increase by increasing the number of exponential function in the approximations also the efficiency and the accuracy depended on a suitable choice of collocation points. Also by comparing the result of our way with [17], [20], [19] and [16] we gained a perfect approximation disbanding for equation (2).

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