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Deforming Lie algebras to Frobenius integrable non-autonomous Hamiltonian systems

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Abstract
Motivated by the theory of Painlevé equations and associated hierarchies, we study non-autonomous Hamiltonian systems that are Frobenius integrable. We establish sufficient conditions under which a given finite-dimensional Lie algebra of Hamiltonian vector fields can be deformed into a time-dependent Lie algebra of Frobenius integrable vector fields spanning the same distribution as the original algebra. The results are applied to quasi-Stäckel systems [14].

Keywords: Liouville integrability; Lie algebras; Frobenius integrability; separable systems; quasi-Stäckel systems;

1 Introduction
The importance of the role played by integrable systems is hard to overestimate, given both their manifold applications and their profound connections to a number of areas in pure mathematics, see e.g. [2, 5, 21, 22, 23] and references therein. In particular, finite-dimensional integrable Hamiltonian dynamical systems are well understood, and the key tool in their study is the Liouville theorem [11] relating integrability to existence of sufficiently many independent integrals of motion in involution.

This beautiful and well-studied setup includes a blanket assumption that the systems in question do not involve explicit dependence on the evolution parameter, i.e., time. Allowing for such an explicit dependence is rather far from straightforward and necessitates certain nontrivial modifications of the very notion of integrability, see e.g. [2, 23] and references therein for details. It should be pointed out that the research in this subfield is relatively scarce compared with that centered around the integrable dynamical systems in the setting of the Liouville theorem, cf. e.g. [2, 5, 8, 16] and references therein.
There is, however, an important motivation for the study of explicitly time-dependent dynamical systems and their integrability: the Painlevé equations, which play an important role in many areas of modern mathematics and in applications, cf. e.g. [9, 17] and references therein, as well as certain natural generalizations thereof [1], can be written as time-dependent dynamical systems, see e.g. [1, 9, 18, 25].

In the present paper we take an approach to the study of time-dependent dynamical systems and their integrability that, to the best of our knowledge, was not systematically explored in the earlier literature. Namely, we simultaneously consider several vector fields and the associated dynamical systems, each with its own time, while allowing for an explicit dependence of all vector fields on all times at once and imposing the Frobenius integrability condition guaranteeing local existence of associated multitime solutions, as explained below. In order to construct such vector fields depending on all times at once, we begin with a Lie algebra of time-independent vector fields on the underlying manifold and look for the multiparameter deformations of this algebra having the desired properties.

We expect that this approach, possibly supplemented by certain additional assumptions, will yield new nonautonomous Painlevé-type dynamical systems.

We start by the following definition of Frobenius integrability that is naturally motivated by an important notion of an integrable distribution and by the Frobenius theorem from differential geometry, cf. e.g. [6, 12, 16, 20] and references therein.

**Definition 1** A set of \( n \) non-autonomous vector fields \( Y_i(t_1, \ldots, t_n) \), each depending on \( n \) parameters \( t_i \), on a finite-dimensional manifold \( M \) is called Frobenius integrable if the following zero-curvature condition (Frobenius condition) holds:

\[
\frac{\partial Y_i}{\partial t_j} - \frac{\partial Y_j}{\partial t_i} - [Y_i, Y_j] = 0 \quad \text{for all } i, j = 1, \ldots, n, \tag{1}
\]

where \([-\cdot,\cdot]\) stands for the Lie bracket (commutator) of vector fields.

It is rather straightforward, cf. e.g. [6, 9], to see that if the Frobenius condition (1) is satisfied then the associated set of \( n \) dynamical systems on \( M \)

\[
\frac{dx^\alpha}{dt_i} = Y_i^\alpha(\xi, t_1, \ldots, t_n), \quad \alpha = 1, \ldots, m = \dim M, \quad i = 1, \ldots, n, \tag{2}
\]

possesses a local common multi-time solution \( x^\alpha = x^\alpha(t_1, \ldots, t_n, \xi_0) \) for each point \( \xi_0 \in M \), i.e., for each initial condition \( x^\alpha(0, \ldots, 0, \xi_0) = x_0^\alpha \). Here \( x^\alpha \) are local coordinates on \( M \) on a neighborhood of a point \( \xi_0 \), and \( x_0^\alpha \) are coordinates of \( \xi_0 \) in this coordinate system; \( \xi \in M \) denotes a point on \( M \) and \( Y^\alpha(\xi, t_1, \ldots, t_n) \) is the value of \( \alpha \)-th component of the vector field \( Y \) w.r.t. local coordinate system given by \( x^\alpha \) at the point \( \xi \) at the times \( t_1, \ldots, t_n \). Under certain technical assumptions such local common multi-time solutions from a set of overlapping local coordinate systems can be glued together to define an integral submanifold \( \xi = \xi(\xi_0, t_1, \ldots, t_n) \) passing through \( \xi_0 \). Such a submanifold gives us a natural coordinate-free representation for the solution of the system in question.

Note that equations (1) formally look exactly like the zero-curvature-type equations arising in the study of integrable partial differential dispersionless systems with Lax operators written in terms of vector fields, cf. e.g. [5, 13, 22, 24] and references therein. On
the other hand, if one of the times \( t_i \) is identified with the variable spectral parameter and the vector fields are replaced by matrices, then equations (1) formally look like the isomonodromic representations for the Painlevé and Painlevé-type systems, cf. e.g. [1] and references therein.

Suppose now that \( M \) is endowed with a nondegenerate Poisson structure \( \pi \), so we have a Poisson manifold \((M, \pi)\), and the vector fields \( Y_i \) are Hamiltonian, that is, \( Y_i = \pi dH_i \) for some Hamiltonian functions \( H_i \) depending explicitly, in general, on all times \( t_k: H_i = H_i(\xi, t_1, \ldots, t_n) \). We stress that in our setup the Poisson structure \( \pi \) does not depend on any of \( t_k \).

As by definition of the Poisson bracket \( \{·, ·\} \) associated with \( \pi \) we have \([\pi dH_i, \pi dH_j] = -\pi d\{H_i, H_j\}\) (we use the sign convention \( \{H_i, H_j\} = \langle dH_i, \pi dH_j \rangle \), where \( \langle ·, ·\rangle \) is the natural pairing among \( T^*_z M \) and \( T_z M \), although the opposite sign convention also occurs in the literature), we immediately see that for (1) to hold it suffices that the following zero-curvature condition (Frobenius condition) for Hamiltonians \( H_k \) holds:

\[
\frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} + \{H_i, H_j\} = 0, \quad i, j = 1, \ldots, n
\]

(3)

In the case when the vector fields \( Y_i \) do not depend explicitly on the times \( t_i \) the conditions (1) and (3) boil down to \([Y_i, Y_j] = 0 \) for all \( i, j \) and \( \{H_i, H_j\} = 0 \) for all \( i, j \), respectively. In such a case the vector fields \( Y_i \) span an involutive (and thus integrable by the Frobenius theorem [6], [12]) distribution \( \mathcal{D} \) while the Hamiltonian vector fields give rise to a Liouville integrable system under certain additional regularity conditions.

Recall that the members of the hierarchy associated with a given Painlevé equation admit non-autonomous Hamiltonian formulations with evolution parameters \( t_j \) and explicitly time-dependent Hamiltonians that satisfy (3), cf. e.g. [10, 18, 25]. This suggests that, conversely, some of the dynamical systems with the Hamiltonians that satisfy (3) could possess the Painlevé property but we defer the investigation of this idea in more detail to future work.

Motivated by the above, in the present paper we study existence of polynomial-in-times deformations of Lie algebras of autonomous Hamiltonians \( h_i \) (so the associated Hamiltonian vector fields \( X_i = \pi dh_i \) satisfy (6) with \( c_{ij}^k \) being constants) such that the deformed Hamiltonians \( H_i \) satisfy the condition (3). In this way we produce, from the system of non-commuting autonomous vector fields \( X_i = \pi dh_i \), polynomial-in-times vector fields \( Y_i \) that satisfy (1), which guarantees local existence of common multi-time solutions for the set of non-autonomous systems (2). Under certain natural assumptions this deformation is shown to be unique, see Theorem 1 in Section 2 for details. Then, in Section 3, we apply our general theory to the so-called quasi-Stäckel systems [14] and present a way of explicit computation of the deformations in question in this particular setting. As a result, we construct a number of families of non-autonomous Hamiltonian systems with \( n \) degrees of freedom integrable in the Frobenius sense.
Non-autonomous deformations of Lie algebras yielding Frobenius integrability

Consider an \( n \)-dimensional (\( 1 < n < \dim M \)) Lie algebra \( g = \text{span}\{ h_i \in C^\infty(M), i = 1, \ldots, n \} \) of smooth real-valued functions on our Poisson manifold \((M, \pi)\), with the structure constants \( c_{ij}^k \in \mathbb{R} \), so that
\[
\{ h_i, h_j \} = \sum_{k=1}^{n} c_{ij}^k h_k
\]
is the Lie bracket on \( g \), cf. e.g. Definition 6.41 in [19]. We assume that the functions \( h_i : M \to \mathbb{R} \) (the Hamiltonians) are functionally independent.

The functions \( h_i \) define autonomous conservative Hamiltonian systems on \( M \), cf. (2),
\[
\frac{dx^\alpha}{dt_i} = (\pi(\xi)dh_i(\xi, t_1, \ldots, t_n))^\alpha, \quad \alpha = 1, \ldots, m = \dim M, \quad i = 1, \ldots, n.
\]
The Hamiltonian vector fields \( X_i = \pi dh_i \) satisfy
\[
[X_i, X_j] = -\sum_{i=1}^{n} c_{ij}^k X_k
\]
and thus span an involutive, and hence integrable in the sense of Frobenius, distribution \( \mathcal{D} \) on \( M \).

It is well known that if (6) holds then one can choose a basis \( V_1, \ldots, V_n \) of vector fields spanning the distribution \( \mathcal{D} \) such that \( [V_i, V_j] = 0 \) for all \( i, j = 1, \ldots, n \), cf. e.g. [12]. However, a direct (explicit) construction of such a basis is usually not possible and the basis \( V_i \) does not have to consist of Hamiltonian vector fields. We would therefore like to have a method of deforming, in a precise sense defined below, the autonomous vector fields \( X_i \) to (non-autonomous in general) vector fields \( Y_i \) such that

1. The vector fields \( Y_i \) span the same distribution \( \mathcal{D} \) as \( X_i \) do.
2. The vector fields \( Y_i \) are Hamiltonian with respect to \( \pi \) just as \( X_i \), so \( Y_i = \pi dH_i \) for some functions \( H_i \) depending in general on all times \( t_i \).
3. The dynamical systems (2) defined by the vector fields \( Y_i \) possess local common multi-time solutions, so that the condition (1) is satisfied and (3) is valid for \( H_i \).

More precisely this problem can be stated as follows.

**Problem 1** Denote by \( g[t_1, \ldots, t_n] \) the space of \( g \)-valued multivariate polynomials in \( t_1, \ldots, t_n \).

1. Can one find (and, if yes, under which conditions) nonzero polynomials \( H_i \in g[t_1, \ldots, t_n], i = 1, \ldots, n \), such that the non-autonomous Frobenius condition (3) holds and such that \( Y_i = \pi dH_i \) span the same distribution \( \mathcal{D} \) as do \( X_i \)?
2. Is there a unique answer to question 1?
3. Is there an explicit way to calculate \( H_i \)?
Thus, we will look for polynomial-in-times deformations $H_i$ of the Hamiltonians $h_i$ such that $\pi dH_i$ and $\pi dh_i$ span the same distribution $D$ and such that the non-autonomous Hamiltonian systems

$$\frac{dx^\alpha}{dt_i} = (\pi(\xi)dH_i(\xi, t_1, \ldots, t_n))^\alpha, \quad \alpha = 1, \ldots, m = \dim M, \quad i = 1, \ldots, n,$$

(7)

satisfy the Frobenius condition (1) and thus possess local common multi-time solutions.

The first two questions of Problem 1 can be answered in the following general setting.

**Theorem 1** Suppose that in a finite-dimensional Lie algebra $\mathfrak{g}$ there exists a basis $\{h_i\}_{i=1}^n$ such that

i) $\mathfrak{g}_c = \text{span}\{h_i : i = 1, \ldots, d_c\}$, where $d_c \geq 1$, is the center of $\mathfrak{g}$, so that for any $i = 1, \ldots, d_c$ we have $\{h_i, h\} = 0$ for any $h \in \mathfrak{g}$;

ii) $\mathfrak{g}_a = \text{span}\{h_i : i = 1, \ldots, d_a\}$, where $d_a \geq d_c$, is an Abelian subalgebra of $\mathfrak{g}$;

iii) $\{h_i, h_j\} \in \text{span}\{h_1, \ldots, h_{\min(i,j)-1}\}$ for all $i, j \leq n - 1$.

Then there exists a unique multi-time-dependent Lie algebra (multi-time formal deformation of $\mathfrak{g}$) with the generators $H_i \in \mathfrak{g}[t_{d_c+1}, \ldots, t_i-1], i = 1, \ldots, n$ that satisfy Frobenius integrability conditions (3) and the following normalization conditions

a) $H_i = h_i, \quad i = 1, \ldots, d_a,$

b) $H_i|_{t_{d_c+1}=0, \ldots, t_i-1=0} = h_i, \quad i = d_a + 1, \ldots, n.$

The assumptions i)–iii) imply that $\mathfrak{g}_c \subset \mathfrak{g}_a \subset \mathfrak{g}_{n-1} = \text{span}\{h_i : i = 1, \ldots, n - 1\} \subset \mathfrak{g}$. Moreover, iii) implies that $\mathfrak{g}_{n-1}$ is a nilpotent subalgebra of the Lie algebra $\mathfrak{g}$ of codimension one. The theorem of course covers inter alia the case when $\mathfrak{g}$ itself, rather than just $\mathfrak{g}_{n-1}$, is nilpotent. Note also that thanks to the assumption a) we have $H_i = h_i$ for $i = 1, \ldots, d_a$, so the Hamiltonians $h_i$ spanning the Abelian subalgebra $\mathfrak{g}_a$ are not deformed.

We stress that both the statement of Theorem 1 and its proof given below are purely algebraic, so Theorem 1 holds not just for Lie algebras of functions on a Poisson manifold but for an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$ which satisfies the conditions of the theorem, with (3) replaced by

$$\frac{\partial H_i}{\partial t_j} - \frac{\partial H_j}{\partial t_i} + [H_i, H_j] = 0, \quad i, j = 1, \ldots, n,$$

(5’)

where $[\cdot, \cdot]$ denotes the Lie bracket in $\mathfrak{g}$, and $n = \dim \mathfrak{g}$. Of course, then in the proof of the theorem the Poisson bracket $\{\cdot, \cdot\}$ should also be replaced by $[\cdot, \cdot]$.

**Proof.** By virtue of a) we have that $H_i = h_i$ for $i = 1, \ldots, d_a$. Now, as $\partial H_j/\partial t_{d_a+1} = 0$ for $j = 1, \ldots, d_a$ by assumption, the deformed Hamiltonian $H_{d_a+1}$ can be determined from the following part of equations (3):

$$\{H_j, H_{d_a+1}\} - \frac{\partial H_{d_a+1}}{\partial t_j} = 0, \quad j = 1, \ldots, d_a.$$

(8)
This system has a (unique thanks to b)) solution

\[ H_{d_a+1} = \exp \left( - \sum_{i=d_a+1}^{d_a} t_i \text{ad}_{h_i} \right) h_{d_a+1}, \tag{9} \]

as we have \( \text{ad}_{h_i} = 0 \) for \( i = 1, \ldots, d_c \); recall that by definition \( \text{ad}_f(h) = \{f, h\} \) for any \( f, h \in g \). Thus, \( H_{d_a+1} \) depends only on times \( t_{d_a+1}, \ldots, t_{d_a} \). Note that the expression in (9) is a polynomial in \( t_{d_a+1}, \ldots, t_{d_a} \) by virtue of the nilpotency assumption iii). Here and below we tacitly assume that if \( j > k \) then any sum of the form \( \sum_{i=j}^{d_c} \) is identically zero.

For the remaining \( H_i \) we proceed by induction. Suppose that \( H_j = H_j(t_{d_c+1}, \ldots, t_j) \), \( j = 1, \ldots, k \) are already known. Then \( H_{k+1} \) can be uniquely determined from equations (3). Since \( \partial H_j / \partial t_{k+1} = 0 \) for \( j = 1, \ldots, k \), the said equations read

\[ \{ H_j, H_{k+1} \} - \frac{\partial H_{k+1}}{\partial t_j} = 0, \quad j = 1, \ldots, k. \tag{10} \]

The first \( d_c \) of these equations yield

\[ \frac{\partial H_{k+1}}{\partial t_j} = 0, \quad j = 1, \ldots, d_c, \]

as for \( j = 1, \ldots, d_c \) we have \( \{ H_j, H_{k+1} \} = 0 \) by virtue of the assumption i). This means that \( H_{k+1} \) does not depend on \( t_1, \ldots, t_{d_c} \). The remaining equations in (10) therefore have a (unique thanks to b)) solution of the form (cf. e.g. [7] and references therein)

\[ H_{k+1} = \mathcal{P} \exp \left( - \int \sum_{i=d_c+1}^{k} \text{ad}_{H_i} dt_i \right) h_{k+1} \tag{11} \]

where \( \gamma \) is any (smooth) curve in (an open domain of) \( \mathbb{R}^{k-q_0} \) connecting the points 0 and \( (t_{d_c+1}, \ldots, t_k) \) and where \( \mathcal{P} \exp \) denotes the path-ordered exponential, see e.g. [20] and references therein. This integral does not depend on a particular choice of \( \gamma \) because of the zero-curvature equations (10). Parameterizing the curve \( \gamma \) by a parameter \( \tau' \in [0, \tau] \) so that \( \gamma(0) = 0 \) and \( \gamma(\tau) = (t_{d_c+1}, \ldots, t_k) \) yields

\[ \int \gamma \sum_{i=d_c+1}^{k} \text{ad}_{H_i} dt_i = \int_0^\tau \sum_{i=d_c+1}^{k} \text{ad}_{H_i}|_{t_j=t_j(\tau')} dt_i(\tau') \equiv - \int_0^\tau F_k(\tau') d\tau' \tag{12} \]

where \( F_k \) is an \( \text{End}(g) \)-valued function of the parameter \( \tau' \). The path-ordered exponential can be computed using the following formal Magnus expansion, see e.g. [20] and references therein:

\[ \mathcal{P} \exp \left( \int_0^\tau F_k(\tau') d\tau' \right) = \sum_{s=0}^{\infty} \Omega_{s}^k = \sum_{s=0}^{\infty} \frac{1}{s!} \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{s-2}} d\tau_{s-1} \int_0^{\tau_{s-1}} F_k(\tau'_1) \cdots F_k(\tau'_s) d\tau'_s. \tag{13} \]
To complete the proof it remains to establish the polynomiality of $H_{k+1}$ in $t_{d_c+1}, \ldots, t_k$. This is achieved by observing that $\Omega^k_s$ for all $k = d_a + 1, \ldots, n$ involve only $\text{ad}_{H_j}$ with $j = d_c + 1, \ldots, n - 1$ but do not involve $\text{ad}_{H_n}$; therefore $\Omega^k_s$ vanish for sufficiently large $s$ as the expressions like
$$\text{ad}_{H_{r_1}} \text{ad}_{H_{r_2}} \cdots \text{ad}_{H_{r_j}}$$
will all vanish for sufficiently large $j$ if all $r_i$ belong to $d_c + 1, \ldots, n - 1$ as $g_{n-1}$ is nilpotent by virtue of assumption iii).

Notice that the non-autonomous Hamiltonian systems (7) are conservative by construction, as the $i$-th Hamiltonian $H_i$ does not depend on its own evolution parameter $t_i$. Moreover, for $k > j$ the Frobenius conditions (3) read
$$\frac{\partial H_k}{\partial t_j} - \{H_j, H_k\} = \frac{\partial H_k}{\partial t_j} + \{H_k, H_j\} = \left( \frac{\partial}{\partial t_j} + L_{Y_j} \right) H_k = 0, \quad k > j$$
where $L_{Y_j}$ is the Lie derivative along the vector field $Y_j$, so all $H_k$ with $k > j$ are time-dependent integrals of motion for the $j$-th flow.

**Remark 1** Note that by the very construction of the Hamiltonians $H_i$ the vector fields $Y_i = \pi dH_i$ span the same distribution $\mathcal{D}$ as $X_i = \pi dh_i$, as required in part 1) of Problem 1.

### 3 Non-autonomous deformations of quasi-Stäckel Hamiltonians

In this section we apply Theorem 1 to quasi-Stäckel systems constructed in [14]; cf. also e.g. [3, 4, 14] for general background on Stäckel and quasi-Stäckel systems. In this particular setting we will be able to compute the expressions in (9) and (11), thus answering the question 3 of Problem 1.

Fix an $n \in \mathbb{N}$, $n \geq 2$. Consider a $2n$-dimensional Poisson manifold $M$ and a particular set $(\lambda_i, \mu_i)$ of local Darboux (canonical) coordinates on $M$, so that $\{\mu_i, \lambda_j\} = \delta_{ij}$, $i, j = 1, \ldots, n$ while all $\{\lambda_i, \lambda_j\}$ and $\{\mu_i, \mu_j\}$ are zero. Fix also $m \in \{0, \ldots, n + 1\}$ and consider the following system of linear quasi-Stäckel separation relations [14] (cf. also [15])
$$\sum_{j=1}^{n} \lambda_i^{n-j} h_j = \frac{1}{2} \lambda_i^m \mu_i^2 + \sum_{k=1}^{n} v_{ik}(\lambda) \mu_k, \quad i = 1, \ldots, n,$$
where $\lambda = (\lambda_1, \ldots, \lambda_n)$,
$$\sum_{k=1}^{n} v_{ik}(\lambda) \mu_k = \begin{cases} -\sum_{k \neq i} \frac{\mu_i - \mu_k}{\lambda_i - \lambda_k}, & \text{for } m = 0, \\ -\lambda_i^{m-1} \sum_{k \neq i} \frac{\lambda_i \mu_i - \lambda_k \mu_k}{\lambda_i - \lambda_k} + (m - 1) \lambda_i^{m-1} \mu_i, & \text{for } m = 1, \ldots, n, \\ -\lambda_i^{n-1} \sum_{k \neq i} \frac{\lambda_i^2 \mu_i - \lambda_k^2 \mu_k}{\lambda_i - \lambda_k} + (n - 1) \lambda_i^n \mu_i, & \text{for } m = n + 1. \end{cases}$$
Solving (15) with respect to $h_j$ yields, for each choice of $m \in \{0, \ldots, n+1\}$, $n$ Hamiltonians on $M$:

$$h_1 = E_1 = \frac{1}{2} \mu^T G \mu, \quad h_i = E_i + W_i, \quad i = 2, \ldots, n$$

where $\mu = (\mu_1, \ldots, \mu_n)^T$, and the quantities

$$E_i = \frac{1}{2} \mu^T A_i \mu, \quad W_i = \mu^T Z_i, \quad i = 2, \ldots, n,$$

are generated by the first, respectively second, term on the right-hand side of (15). We chose to omit the index $m$ in the above notation to simplify writing.

Here

$$G = \text{diag} \left( \frac{\lambda_1^m}{\Delta_1}, \ldots, \frac{\lambda_n^m}{\Delta_n} \right), \quad \text{where} \quad \Delta_i = \prod_{j \neq i} (\lambda_i - \lambda_j),$$

can be interpreted as a contravariant metric tensor on an $n$-dimensional manifold $Q$, $E_1$ can then be interpreted as the geodesic Hamiltonian of a free particle in the pseudo-Riemannian configuration space $(Q, g = G^{-1})$ so that $M = T^*Q$ [3, 4]. Next, $A_r = K_r G$, where $K_r$ are $(1, 1)$-Killing tensors for metric $g$ with any chosen $m \in \{0, \ldots, n+1\}$, and are given by

$$K_i = (-1)^i \text{diag} \left( \frac{\partial \sigma_i}{\partial \lambda_1}, \ldots, \frac{\partial \sigma_i}{\partial \lambda_n} \right), \quad i = 1, \ldots, n$$

where $\sigma_r(\lambda)$ are elementary symmetric polynomials in $\lambda$. Moreover, $E_i$ are integrals of motion for $E_1$ as in fact they all pairwise commute: $\{E_i, E_j\} = 0$, $i, j = 1, \ldots, n$.

The vector fields $Z_i$ are in this setting the Killing vectors of the metric $g$ for any $m \in \{0, \ldots, n+1\}$ as $L_{Z_i} g = 0$, and they take the form [14]

$$(Z_i)^{\alpha} = \sum_{k=1}^{i-1} (-1)^{i-k} k \sigma_{i-k-1} \frac{\lambda_{\alpha}^{m+k-1}}{\Delta_\alpha}, \quad i \in I_1^m$$

and

$$(Z_i)^{\alpha} = \sum_{k=1}^{n-i+1} (-1)^{i+k} k \sigma_{i+k-1} \frac{\lambda_{\alpha}^{m-k-1}}{\Delta_\alpha}, \quad i \in I_2^m$$

where

$$I_1^m = \{2, \ldots, n - m + 1\}, \quad I_2^m = \{n - m + 2, \ldots, n\}, \quad m = 0, \ldots, n + 1.$$ 

Note that the above notation implies that

$$I_1^0 = \{2, \ldots, n\}, \quad I_1^n = I_1^{n+1} = \emptyset, \quad I_2^0 = I_2^1 = \emptyset.$$ 

It was shown in [14] that the Hamiltonians $h_i$ constitute a Lie algebra $\mathfrak{g} = \text{span}\{h_i \in C^\infty(M): i = 1, \ldots, n\}$ with the following commutation relations:

$$\{h_1, h_i\} = 0, \quad i = 2, \ldots, n,$$
and
\[ \{h_i, h_j\} = \begin{cases} 0, & \text{for } i \in I_1^m \text{ and } j \in I_2^m, \\ (j-i)h_{i+j-(n-m+2)}, & \text{for } i, j \in I_1^m, \\ -(j-i)h_{i+j-(n-m+2)}, & \text{for } i, j \in I_2^m, \end{cases} \] (16)
where \( i, j = 2, \ldots, n \). We use here the convention that \( h_i = 0 \) for \( i \leq 0 \) or \( k > n \).

**Remark 2** The Lie algebra \( \mathfrak{g} \) splits into a direct sum of Lie subalgebras \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \) where
\[
\mathfrak{g}_1 = \text{span}\{h_1\} \oplus \text{span}\{h_r : r \in I_1^m\} \quad \text{and} \quad \mathfrak{g}_2 = \text{span}\{h_r : r \in I_2^m\}.
\]

In order to successfully apply Theorem 1 and formulas (9) and (11) we will now focus on the cases \( m = 0,1 \), when \( \mathfrak{g} = \mathfrak{g}_1 \), since \( I_2^m \) is then empty. Note also that for these cases the Lie algebra \( \mathfrak{g} \) is nilpotent. Then (16) reads
\[
\text{ad}_{h_s}h_i = \{h_i, h_s\} = (i, s_1)h_{i+s_1-(n-m+2)} \quad \text{with} \quad (i, s_1) = s_1 - i
\]
from which it immediately follows that for any \( k \in \mathbb{N} \)
\[
\text{ad}_{h_{s_k}} \cdots \text{ad}_{h_{s_1}} h_i = (i, s_1, \ldots, s_k)h_{i+s_1+\cdots+s_k-k(n-m+2)} \tag{17}
\]
where
\[
(i, s_1 \ldots, s_k) = (i, s_1, \ldots, s_{k-1})[s_k - s_{k-1} - \cdots - s_1 - i + s(n-m+2)]. \tag{18}
\]

Note that in (17) we put \( h_s = 0 \) for \( s < 1 \).

**Theorem 2** Suppose that \( m = 0 \) or \( m = 1 \) (then \( I_2^m \) is empty while \( d_+ = 2 \) for \( m = 0 \) and \( d_+ = 1 \) for \( m = 1 \)). Then the conditions of Theorem 1 are satisfied and the polynomial-in-times deformation of \( \mathfrak{g} \) given by formulas (9) and (11) can be written in the form
\[
H_i = h_i - \sum_{r_1=d_+}^{i-1} (\text{ad}_{h_{r_1}} h_i) t_{r_1} + \sum_{r_1=d_+}^{i-1} \sum_{r_2=r_1}^{i-1} \alpha_{i r_1 r_2} (\text{ad}_{h_{r_2}} \text{ad}_{h_{r_1}} h_i) t_{r_1} t_{r_2} - \sum_{r_1=d_+}^{i-1} \sum_{r_2=r_1}^{i-1} \sum_{r_3=r_2}^{i-1} \alpha_{i r_1 r_2 r_3} (\text{ad}_{h_{r_3}} \text{ad}_{h_{r_2}} \text{ad}_{h_{r_1}} h_i) t_{r_1} t_{r_2} t_{r_3} + \cdots, \tag{19}
\]
and the real constants \( \alpha_{i r_1 \cdots r_k} \) can be uniquely determined from the Frobenius integrability condition (3).

**Proof.** The formula (17) implies that the center \( \mathfrak{g}_c \) for \( m = 0 \) is two-dimensional and given by \( \mathfrak{g}_c = \text{span}\{h_1, h_2\} \) while for \( m = 1 \) the center \( \mathfrak{g}_0 \) is one-dimensional and spanned by \( h_1 \) only. The same formula implies also that in both cases \( \{h_i, h_j\} \in \text{span}(h_1, \ldots, h_{\min(i,j)-1}) \) for all \( i, j = 1, \ldots, n \) so the conditions i)–iii) of Theorem 1 are satisfied. The explicit form (19) of deformations (9) and (11) is obtained by a direct computation using the formulas given in the proof of Theorem 1 and taking a straight line for \( \gamma \). ■
Notice that from (16) it follows that the dimension $d_a$ of the Abelian subalgebra $g_a$ of $g$ is given by
\[ d_a = \left\lfloor \frac{n + 3 - m}{2} \right\rfloor, \quad m = 0, 1 \] (20)
so by Theorem 1 the first $d_a$ Hamiltonians $h_i$ will not be deformed, and that in (19) $i = d_a + 1, \ldots, n$.

Theorem 2 gives us an effective way of calculating the sought-for deformations, as it will be demonstrated in the following examples. Of course, the highest degree of polynomials in $t_j$ arising in this way depends on $n$.

**Example 1** Consider the case $n = 6, m = 0$. Then the formulas (16) yield the following matrix of commutators $\{h_i, h_j\}$:
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3h_1 & 0 \\
0 & 0 & 0 & h_1 & 2h_2 & 0 \\
0 & 0 & -h_1 & 0 & h_3 & 0 \\
0 & -3h_1 & -2h_2 & -h_3 & 0 & 0
\end{pmatrix},
\]
and clearly $d_c = 2$ while $d_a = 4$. The explicit values of the expansion coefficients $\alpha_{ir_1 \ldots r_k}$ can be obtained by plugging (19) into (3). Having done this we obtain
\[ H_i = h_i, \quad i = 1, \ldots, 4, \quad H_5 = h_5 + h_1 t_4, \quad H_6 = h_6 + 3h_1 t_3 + 2h_2 t_4 + h_3 t_5. \]

**Example 2** For the case $n = 6, m = 1$ the formulas (16) yield the following matrix of commutators $\{h_i, h_j\}$:
\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 4h_1 \\
0 & 0 & 0 & 0 & 0 & 2h_1 \\
0 & 0 & 0 & 0 & h_2 & 3h_2 \\
0 & 0 & -2h_1 & -h_2 & 0 & h_4 \\
0 & -4h_1 & -3h_2 & -2h_3 & -h_4 & 0
\end{pmatrix},
\]
so now $d_c = 1$ while $d_a = 4$ as in the previous example. Inserting (19) into (3) yields
\[
H_i = h_i, \quad i = 1, \ldots, 4, \quad H_5 = h_5 + 2h_1 t_3 + h_2 t_4,
\]
\[
H_6 = h_6 + 4h_1 t_2 + 3h_2 t_3 + 2h_3 t_4 + h_4 t_5 - \frac{1}{2} h_2 t_5^2.
\]

Now turn to the general study of the cases $m = n, n + 1$, where $g = \text{span} \{h_1\} \oplus g_{I_2}$ (since $I_1^m$ is empty). The constants $(i_1, \ldots, i_{s+1})$ in (18) are the same as in the previously considered cases up to the sign, i.e., $(i_1, \ldots, i_{s+1}) \rightarrow (-1)^s(i_1, \ldots, i_{s+1})$. From (16) it follows that
\[
\text{for} \quad m = n : \quad g_c = \text{span} \{h_1\}, \quad g_a = \text{span} \{h_1, h_{n-k+1}, \ldots, h_n\},
\]
\[
\text{for} \quad m = n + 1 : \quad g_c = \text{span} \{h_1, h_n\}, \quad g_a = \text{span} \{h_1, h_{n-k+1}, \ldots, h_n\},
\]
where \( k = \left[ \frac{m}{2} \right] \). Thus, \( d_c = \dim g_c = 1 \) for \( m = n \), \( d_c = \dim g_c = 2 \) for \( m = n + 1 \) while \( d_a = \dim g_a = k + 1 \) in both cases. If we now rearrange the Hamiltonians \( h_i \) so that \( h_i' \equiv h_i \) for \( i = 2, \ldots, n \) we observe that in this ordering the assumptions of Theorem 1 are all satisfied. Actually, for \( m = n + 1 \) the algebra \( g \) is nilpotent, while for \( m = n \) we have \( \{ h_i', h_j' \} \in \text{span} \{ h_1', \ldots, h_n' \} \) which means that \( g \) is a codimension one extension by derivation of the nilpotent Lie algebra \( g_{n-1} \). Thus, we obtain the following

**Corollary 1** Suppose that \( m = n \) or \( m = n + 1 \) (so \( I^m \) is empty). Then the conditions of Theorem 1 are satisfied and the polynomial-in-times deformation of \( g \) given by formulas (9) and (11), for the original ordering of the Hamiltonians \( h_i \), can be written in the form

\[
H_i = h_i - \sum_{r_1=i+1}^{n} (\text{ad}_{h_{r_1}} h_i) t_{r_1} + \sum_{r_1=i+1}^{n} \sum_{r_2=r_1}^{n} \alpha_{ir_1 r_2} \left( \text{ad}_{h_{r_2}} \text{ad}_{h_{r_1}} h_i \right) t_{r_1} t_{r_2} - \sum_{r_1=i+1}^{n} \sum_{r_2=r_1}^{n} \sum_{r_3=r_2}^{n} \alpha_{ir_1 r_2 r_3} \left( \text{ad}_{h_{r_3}} \text{ad}_{h_{r_2}} \text{ad}_{h_{r_1}} h_i \right) t_{r_1} t_{r_2} t_{r_3} + \cdots
\]

(21)

where the real constants \( \alpha_{ir_1...r_k} \) can be uniquely determined from the Frobenius integrability condition (3).

Note that \( d_c \) does not enter the above formula; in the case of \( m = n + 1 \) when \( d_c = 2 \) the sums in (21) end already at \( n - 1 \) since then \( \text{ad}_{h_n} = 0 \) as \( h_n \) is part of the center of the algebra. As before, the highest degree of \( t \)-polynomials depends on \( n \). By analogy with the previous case, the \( \left[ \frac{m}{2} \right] + 1 \) Hamiltonians spanning the Abelian subalgebra \( g_a \), that is, \( h_1 \) and \( h_{n-k+1}, \ldots, h_n \) with \( k = \left[ \frac{m}{2} \right] \), are not deformed.

**Example 3** Consider the case \( n = 6, m = n \). The matrix of commutators \( \{ h_i, h_j \} \) (16) reads

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -h_3 & -2h_4 & -3h_5 & -4h_6 \\
0 & h_3 & 0 & -h_5 & -2h_6 & 0 \\
0 & 2h_4 & h_5 & 0 & 0 & 0 \\
0 & 3h_5 & 2h_6 & 0 & 0 & 0 \\
0 & 4h_6 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and since \( k = \left[ \frac{m}{2} \right] = 3 \) the Hamiltonians \( h_i \) with \( i = 1, 4, 5, 6 \) span an Abelian subalgebra \( g_a \) and are thus not deformed while \( H_2 \) and \( H_3 \) are found by inserting (21) into (3) in order to determine the constants \( \alpha_{ir_1...r_k} \). The result is

\[
H_i = h_i, \quad i = 1, 4, 5, 6, \\
H_2 = h_2 + h_3 t_3 + 2h_4 t_4 + 3h_5 t_5 + 4h_6 t_6 + h_5 t_3 t_4 + 2h_6 t_3 t_5, \\
H_3 = h_3 + h_5 t_4 + 2h_6 t_5.
\]

**Example 4** Consider now the case \( n = 6, m = n + 1 \). The matrix of commutators
\(\{h_i, h_j\}\) reads now

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -h_4 & -2h_5 & -3h_6 & 0 \\
0 & h_4 & 0 & -h_6 & 0 & 0 \\
0 & 2h_5 & h_6 & 0 & 0 & 0 \\
0 & 3h_6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Again, \(h_i\) are not deformed for \(i = 1, 4, 5, 6\) while \(H_2\) and \(H_3\) are found just by substituting (21) into (3), which yields

\[H_i = h_i, \ i = 1, 4, 5, 6, \ \ H_2 = h_2 + h_4t_3 + 2h_5t_4 + 3h_6t_5 - \frac{1}{2}h_6t_3^2, \ \ H_3 = h_3 + h_6t_4.\]

Finally, let us again return to the general theory and find integrable deformations of our algebra \(\mathfrak{g}\) in the case when \(1 < m < n\). In this case both components \(\mathfrak{g}_{I_1}\) and \(\mathfrak{g}_{I_2}\) in the splitting \(\mathfrak{g} = \mathfrak{g}_{I_1} \oplus \mathfrak{g}_{I_2}\) are nontrivial, with \(\dim \mathfrak{g}_{I_1} = n - m + 1\) and \(\dim \mathfrak{g}_{I_2} = m - 1\). Each of the components has an Abelian subalgebra of its own. We denote the maximal Abelian subalgebras of \(\mathfrak{g}_{I_1}\) and \(\mathfrak{g}_{I_2}\) by \(\mathfrak{g}_{a_1}\) and \(\mathfrak{g}_{a_2}\), respectively, with (cf. (20))

\[
\dim \mathfrak{g}_{a_1} = \left\lceil \frac{n + 3 - m}{2} \right\rceil \equiv d_{a_1},
\]

\[
\dim \mathfrak{g}_{a_2} = \left\lfloor \frac{m}{2} \right\rfloor \equiv d_{a_2},
\]

so

\[
\dim \mathfrak{g}_a = \dim \mathfrak{g}_{a_1} + \dim \mathfrak{g}_{a_2},
\]

\(\mathfrak{g}_{a_1}\) and \(\mathfrak{g}_{a_2}\) are given by

\[
\mathfrak{g}_{a_1} = \text{span} \{h_1, h_2, \ldots, h_{d_{a_1}}\}
\]

\[
\mathfrak{g}_{a_2} = \text{span} \{h_{n-d_{a_2}+1}, \ldots, h_n\}
\]

so

\[
\mathfrak{g}_a = \text{span} \{h_1, h_2, \ldots, h_{d_{a_1}}, h_{n-d_{a_2}+1}, \ldots, h_n\}.
\]

Therefore, the Hamiltonians \(h_{d_{a_1}+1}, \ldots, h_{n-m+1}\) belonging to \(\mathfrak{g}_{I_1}\) should then be deformed by formulas (19) with \(d_c = 1\) (for \(m = n + 1\) the center is two-dimensional, spanned by \(h_1\) and \(h_n\), but \(h_n\) does not belong to \(\mathfrak{g}_{I_1}\)) while the Hamiltonians \(h_1, \ldots, h_{d_{a_1}}\) remain unchanged. Likewise, the Hamiltonians \(h_{n-m+2}, \ldots, h_{n-d_{a_2}}\) should be deformed according to (21) while the last \(d_{a_2}\) Hamiltonians remain unchanged.
Example 5 Consider the case \( n = 11, m = 6 \). The matrix of commutators \( \{h_i, h_j\} \) is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

If we perform the deformation on each subalgebra separately, as described above, we obtain

\[
H_i = h_i, \quad i = 1, \ldots, 4, 9, \ldots, 11, \quad H_5 = h_5 + h_2 t_4 + 2h_1 t_3,
\]

\[
H_6 = h_6 + 4h_1 t_2 + 3h_2 t_3 + 2h_3 t_4 + h_4 t_5 - \frac{1}{2} h_2 t_5^2,
\]

\[
H_7 = h_7 + h_8 t_8 + 2h_9 t_9 + 3h_{10} t_{10} + 4h_{11} t_{11} + h_{10} t_8 t_9 + 2h_{11} t_8 t_{10},
\]

\[
H_8 = h_8 + h_{10} t_9 + 2h_{11} t_{10}.
\]

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