On the effects of Dvali–Gabadadze–Porrati braneworld gravity on the orbital motion of a test particle

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Abstract

In this paper, we explicitly work out the secular perturbations induced on all the Keplerian orbital elements of a test body to order $O(e^2)$ in the eccentricity $e$ by the weak-field long-range modifications of the usual Newton–Einstein gravity due to the Dvali–Gabadadze–Porrati (DGP) braneworld model. Both the Gauss and the Lagrange perturbative schemes are used. It turns out that the argument of pericentre $\omega$ and the mean anomaly $M$ are affected by secular rates which depend on the orbital eccentricity via $O(e^2)$ terms, but are independent of the semimajor axis $a$ of the orbit of the test particle. For almost circular orbits, the Lue–Starkman (LS) effect on the pericentre is obtained. Some observational consequences are discussed for the Solar System planetary mean longitudes $\lambda$ which would undergo a $1.2 \times 10^{-3}$ arcseconds per century braneworld secular precession. According to recent data analysis over 92 years for the EPM2004 ephemerides, the 1-sigma formal accuracy in determining the Martian mean longitude amounts to $3 \times 10^{-3}$ milliarcseconds, while the braneworld effect over the same time span would be 1.159 milliarcseconds. The major limiting factor is the $2.6 \times 10^{-3}$ arcseconds per century systematic error due to the mismodelling in the Keplerian mean motion of Mars. A suitable linear combination of the mean longitudes of Mars and Venus may overcome this problem. The formal 1-sigma obtainable observational accuracy would be $\sim 7\%$. The systematic error due to the present-day uncertainties in the solar quadrupole mass moment $J_2$, the Keplerian mean motions, the general relativistic Schwarzschild field and the asteroid ring would amount to some tens of per cent.

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1. Introduction

Recently, a braneworld scenario which yields, among other things, long-range modifications of the Newton–Einstein gravity has been put forth by Dvali, Gabadadze and Porrati (DGP) [1, 2]. In such a model, which encompasses an extra flat spatial dimension, there is a free crossover parameter $r_0$, fixed by observations to a value $\lesssim 5$ Gpc, beyond which traditional gravity suffers strong modifications yielding to cosmological consequences which would yield an alternative to the dark energy in order to explain the observed acceleration of the universe. The consistency and stability of the DGP model have recently been discussed in [3].

Interestingly, for $R_g \ll r \ll r_0$, where $R_g = 2GM/c^2$ is the usual Schwarzschild radius for a gravitating body of mass $M$, there are also small modifications to the usual Newton–Einstein gravity which could be detectable in the near future. Indeed, Lue and Starkman (LS) derived in [4] an extra-pericentre advance for the orbital motion of a test particle assumed to be nearly circular. Its magnitude is $\sim 4 \times 10^{-4}$ arcseconds per century ($''\,\text{cy}^{-1}$ in the following). In [4], it has been shown that the sign of such an effect is related to the cosmological expansion phases allowed in this model: the Friedmann–Lemaître–Robertson–Walker (FLRW) phase and the self-accelerating phase. The LS precession is a universal feature of the orbital dynamics of a test particle because, in this approximation, it is independent of its orbital parameters.

Since the present-day accuracy in measuring the non-Newtonian perihelion rate of Mars amounts to $\sim 10^{-3}''\,\text{cy}^{-1}$ (E V Pitjeva 2004 private communication), the possibility of measuring such an effect in the Solar System scenario from the planetary motion data analysis seems to be very appealing. It has been investigated in some detail in [6] where a suitable linear combination of the perihelia of some inner planets has been considered.

In this paper, we work out the secular effects of the DGP model on all the Keplerian orbital elements of a test body to order $O(e^2)$ in the eccentricity $e$ with the Gauss and Lagrange perturbative schemes. Possible observational implications are worked out.

2. The orbital effects

In this section, we will work out the secular effects of the DGP gravity on the Keplerian orbital elements of the orbit of a test body which is depicted in figure 1.

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Figure 1. (Courtesy of H I M Lichtenegger.) Orbital geometry for a motion around a central mass. Here $L$ denotes the orbital angular momentum of the particle of mass $m$, $J$ is the proper angular momentum of the central mass $M$, $\Pi$ denotes the pericentre position, $f$ is the true anomaly of $m$, which is counted from $\Pi$, $\Omega$, $\omega$ and $i$ are the longitude of the ascending node, the argument of the pericentre and the inclination of the orbit with respect to the inertial frame $\{x, y, z\}$ and the azimuthal angle $\phi$ is the right ascension counted from the $x$ axis.
From the metric for a static, spherical source in a cosmological de Sitter background [4]
\[
(ds)^2 = N^2(r, w)(c\,dr)^2 - A^2(r, w)(dr)^2 - B^2(r, w)[(d\theta)^2 + \sin^2 \theta(d\phi)^2] - (dw)^2,
\]
(1)
where \(w\) is the fourth spatial coordinate and
\[
N \sim 1 - \frac{R_g}{2r} \pm \sqrt{\frac{r R_g}{2r_0^2}},
\]
(2)
the following Lagrangian can be obtained for a spherically symmetric matter source located on the brane around the origin\(^1\) \((r = w = 0)\):
\[
L_{\text{DGP}} = \frac{m}{2} \left[ \left( 1 - \frac{R_g}{2r} \pm \sqrt{\frac{R_g r}{2r_0^2}} \right)^2 c^2 + \dot{x}^2 - \dot{y}^2 - \dot{z}^2 \right].
\]
(3)

2.1. The Gauss perturbative scheme

From
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{v}} \right) - \frac{\partial L}{\partial r} = 0,
\]
(4)
the resulting braneworld acceleration is
\[
A_{\text{DGP}} = \pm \left( \frac{c}{2r_0} \right) \sqrt{\frac{GM}{r}} \hat{r},
\]
(5)
where \(\hat{r}\) is the unit vector in the radial direction. The term (5) can be regarded as a small perturbation whose effects on the Keplerian orbital elements of a test particle can be straightforwardly worked out, e.g., in the Gauss perturbative scheme (e.g. [8]). Note that (5) is purely radial.

The Gauss rate equations for the semimajor axis \(a\), the eccentricity \(e\), the inclination \(i\), the longitude of the ascending node \(\Omega\), the argument of pericentre \(\omega\) and the mean anomaly \(M\) of a test particle are
\[
\frac{da}{dt} = \frac{2}{n\sqrt{1-e^2}} \left[ A_R e \sin f + A_T \left( \frac{p}{r} \right) \right],
\]
(6)
\[
\frac{de}{dt} = \frac{\sqrt{1-e^2}}{na} \left[ A_R \sin f + A_T \left( \cos f + \frac{1}{e} \left( 1 - \frac{r}{a} \right) \right) \right],
\]
(7)
\[
\frac{di}{dr} = \frac{1}{na\sqrt{1-e^2}} A_N \left( \frac{r}{a} \right) \cos(\omega + f),
\]
(8)
\[
\frac{d\Omega}{dr} = \frac{1}{na \sin i \sqrt{1-e^2}} A_N \left( \frac{r}{a} \right) \sin(\omega + f),
\]
(9)
\[
\frac{d\omega}{dr} = -\cos i \frac{d\Omega}{dr} + \frac{\sqrt{1-e^2}}{nae} \left[ -A_R \cos f + A_T \left( 1 + \frac{r}{p} \right) \sin f \right],
\]
(10)

\(^1\) Since we are in the weak-field approximation, it is assumed that \(N(r, w) = 1 + \tilde{n}(r, w), A(r, w) = 1 + \tilde{a}(r, w), B(r, w) = r[1 + \tilde{b}(r, w)]\). The gauge \(\tilde{b}(r, w)|_{w=0} = 0\) has been adopted. The correction only to the Newtonian potential has been considered, i.e. the spatial part of the metric has been considered as Euclidean. Note that the modifications in the cosmological background introduced in [5] do not alter these results.
\[ \frac{d\Omega}{dr} = \frac{1}{na^2 \sqrt{1 - e^2 \sin^2 i}} \frac{\partial R}{\partial \Omega}, \]  
\[ \frac{d\Omega}{dr} = \frac{1}{na^2 \sqrt{1 - e^2 \sin^2 i}} \frac{\partial R}{\partial i}, \]  
\[ \frac{d\omega}{dr} \sim \pm \frac{3c}{8r_0} \left( 1 - \frac{13}{32} e^2 \right), \]  
\[ \frac{dM}{dr} \sim \pm \frac{11c}{8r_0} \left( 1 - \frac{39}{352} e^2 \right). \]  

For \( e \to 0 \), (16) reduces to the result by Lue and Starkman for circular orbits [4]. The upper sign is for the FLRW phase, while the lower sign refers to the self-accelerated phase.

2.2. The Lagrangian perturbative scheme

The Lagrange planetary equations for the rates of change of the Keplerian orbital elements are (e.g. [9])

\[ \frac{da}{dr} = \frac{2}{na} \frac{\partial R}{\partial a}, \]  
\[ \frac{de}{dr} = \frac{(1 - e^2)}{na^2 e} \frac{\partial R}{\partial a} - \frac{\sqrt{1 - e^2}}{na^2 e} \frac{\partial R}{\partial \omega}, \]  
\[ \frac{di}{dr} = \cos i \frac{1}{na^2 \sqrt{1 - e^2 \sin^2 i}} \frac{\partial R}{\partial \omega} - \frac{1}{na^2 \sqrt{1 - e^2 \sin^2 i}} \frac{\partial R}{\partial i}, \]  
\[ \frac{dM}{dr} = \frac{1}{na^2 \sqrt{1 - e^2 \sin^2 i}} \frac{\partial R}{\partial i}. \]
\[
\frac{d\omega}{dt} = -\cos i \frac{1}{n a^2 \sqrt{1 - e^2 \sin^2 i}} \frac{\partial R}{\partial i} + \frac{\sqrt{1 - e^2}}{n a^2 e} \frac{\partial R}{\partial e},
\]
(22)

\[
\frac{dM}{dt} = -\frac{(1 - e^2)}{n a^2 e} \frac{\partial R}{\partial e} - \frac{2}{n a} \frac{\partial R}{\partial a},
\]
(23)

where \( R \) is the perturbing function which accounts for all the departures of the gravitational potential from the Newtonian monopole term. Such a perturbing scheme was applied for the first time to post-Newtonian motions by Rubincam \[10\]. In the case of the DGP braneworld theory from \( \mathcal{H} = p \cdot v - L, p = \partial L/\partial v, \) and (3), it can be obtained that

\[
R_{DGP} = \mp \frac{c}{r_0} \sqrt{\frac{GM}{r}}.
\]
(24)

The DGP perturbing function (24) has to be evaluated on the unperturbed Keplerian ellipse (12) and averaged over one orbital revolution by means of (13) in order to obtain the secular effects. From

\[
(1 + e \cos f)^{-5/2} \sim 1 - \frac{1}{2} e \cos f + \frac{25}{8} e^2 \cos^2 f - \frac{105}{16} e^3 \cos^3 f + \frac{1155}{128} e^4 \cos^4 f,
\]
(25)

one obtains

\[
\langle R \rangle \sim \mp \frac{c}{r_0} \sqrt{\frac{GM}{a}} \left( 1 + \frac{3}{16} e^2 + \frac{9}{1024} e^4 \right).
\]
(26)

Since

\[
\frac{\partial \langle R \rangle}{\partial M} = \frac{\partial \langle R \rangle}{\partial \Omega} = \frac{\partial \langle R \rangle}{\partial \omega} = \frac{\partial \langle R \rangle}{\partial i} = 0,
\]
(27)

it is straightforward to obtain

\[
\frac{da}{dt} = \frac{de}{dt} = \frac{di}{dt} = \frac{d\Omega}{dt} = 0.
\]
(28)

The situation is different for the pericentre and the mean anomaly. Indeed,

\[
\frac{\partial \langle R \rangle}{\partial e} = \mp \frac{3c}{8r_0} \sqrt{\frac{GM}{a}} \left( 1 + \frac{3}{32} e^2 \right),
\]
(29)

\[
\frac{\partial \langle R \rangle}{\partial a} = \mp \frac{c}{2r_0} \sqrt{\frac{GM}{a^3}} \left( 1 + \frac{3}{16} e^2 + \frac{9}{1024} e^4 \right).
\]
(30)

As a consequence,

\[
\frac{d\omega}{dt} \sim \mp \frac{3c}{8r_0} \left( 1 - \frac{13}{32} e^2 \right),
\]
(31)

\[
\frac{dM}{dt} \sim \pm \frac{11c}{8r_0} \left( 1 - \frac{39}{352} e^2 \right).
\]
(32)

Note that (31) and (32) are identical to (16) and (17) obtained in section 2.1.

### 3. The possible use of the planetary mean longitudes

In [6], only the use of the planetary perihelia has been examined. First, preliminary observational tests are discussed in [7].

The fact that the mean anomaly is also affected by the DGP gravity with a relatively large effect suggests examining the possibility of using the planetary mean longitudes \( \lambda = \omega + \Omega + M \).
Table 1. Nominal values, in $''$ cy$^{-1}$, of the secular precessions induced on the planetary mean longitudes $\lambda$ by the DGP gravity and by some of the competing Newtonian and Einsteinian gravitational perturbations. For a given planet, the precession labelled ‘Numerical’ includes all the numerically integrated perturbing effects of the dynamical force models used at JPL for the DE200 ephemerides. For example, it also comprises the classical $N$-body interactions, including the Keplerian mean motion $n$. For the numerically integrated planetary precessions, see http://ssd.jpl.nasa.gov/elem_planets.html#rates. The effect labelled ‘GE’ is due to the post-Newtonian general relativistic gravitoelectric Schwarzschild component of the solar gravitational field, that labelled ‘$J_2$’ is due to the classical effect of the Sun’s quadrupole mass moment $J_2$ and that labelled ‘LT’ is due to the post-Newtonian general relativistic gravitomagnetic Lense–Thirring [12] component of the solar gravitational field (not included in the force models adopted by JPL). For $J_2$, the value $1.9 \times 10^{-7}$ has been adopted [11]. For the Sun’s proper angular momentum $\lambda$, which is the source of the gravitomagnetic field, the value $1.9 \times 10^{33}$ kg m$^2$ s$^{-1}$ [13] has been adopted.

| Planet | DGP       | Numerical       | GE        | $J_2$       | LT       |
|--------|-----------|-----------------|-----------|-------------|----------|
| Mercury | $1.2 \times 10^{-3}$ | $5.381\,016\,282 \times 10^8$ | $-8.48 \times 10^3$ | $4.7 \times 10^{-2}$ | $-2 \times 10^{-3}$ |
| Venus   | $1.2 \times 10^{-3}$ | $2.106\,641\,360 \times 10^8$ | $-1.72 \times 10^3$ | $5 \times 10^{-3}$ | $-3 \times 10^{-4}$ |
| Earth   | $1.2 \times 10^{-3}$ | $1.295\,977\,406 \times 10^8$ | $-7.6$ | $1.6 \times 10^{-3}$ | $-1 \times 10^{-4}$ |
| Mars    | $1.2 \times 10^{-3}$ | $6.890\,510\,37 \times 10^7$ | $-2.6$ | $3 \times 10^{-4}$ | $-3 \times 10^{-5}$ |
| Jupiter | $1.2 \times 10^{-3}$ | $1.092\,507\,83 \times 10^7$ | $-1 \times 10^{-4}$ | $5 \times 10^{-6}$ | $-7 \times 10^{-7}$ |
| Saturn  | $1.2 \times 10^{-3}$ | $4.401\,052\,9 \times 10^6$ | $-2 \times 10^{-4}$ | $6 \times 10^{-7}$ | $-1 \times 10^{-7}$ |
| Uranus  | $1.2 \times 10^{-3}$ | $1.542\,547\,7 \times 10^6$ | $-4 \times 10^{-4}$ | $5 \times 10^{-8}$ | $-1 \times 10^{-8}$ |
| Neptune | $1.2 \times 10^{-3}$ | $7.864\,492 \times 10^5$ | $-1 \times 10^{-4}$ | $1 \times 10^{-7}$ | $-5 \times 10^{-9}$ |

Table 2. Formal standard deviations, in milliarcseconds (mas), of the planetary mean longitudes as from table 4 of [11] for the EPM2004 ephemerides. Note that realistic errors may be an order of magnitude larger. About 300,000 position observations (1911–2003) of different types (optical, radar, spacecraft, LLR) have been used. The braneworld shift for $\lambda$ amounts to 1.159 mas over a 92-year time span.

| Planet | Mercury | Venus | Mars | Jupiter | Saturn | Uranus | Neptune |
|--------|---------|-------|------|--------|--------|--------|---------|
|        | $3.75 \times 10^{-1}$ | $1.87 \times 10^{-1}$ | $3 \times 10^{-3}$ | $1.109$ | $3.474$ | $8.818$ | $3.5163 \times 10^{1}$ |

3.1. The observational sensitivity

The sizes of the DGP secular precessions and of many Newtonian and Einsteinian competing secular precessions of the mean longitudes of the Solar System planets are reported in table 1. It can be noted that the DGP secular precessions amount to $1.2 \times 10^{-3}''$ cy$^{-1}$ for all the planets: the corrections due to the eccentricities are very small amounting to almost $10^{-5}''$ cy$^{-1}$. With regard to the observational sensitivity, in table 2, retrieved from table 4 of [11], the most recent results for the EPM2004 ephemerides are presented. They are based on the processing of a vast number of data of different kinds (optical, radar, spacecraft, LLR) ranging from 1911 to 2003. It can be noted that Mars is the best candidate for extracting the DGP effect because the formal standard deviation in $\lambda_{\text{Mars}}$ amounts to $3 \times 10^{-3}$ milliarcseconds (mas) only, while the braneworld shift for the same time span is 1.159 mas.

4. Some systematic errors

In this section, we will examine various competing classical and general relativistic effects which would act as sources of systematic errors in order to see if it is possible to use only the Martian mean longitude for the proposed test.
4.1. The impact of the solar quadrupole mass moment and of the Einsteinian gravitoelectric force

With regard to the systematic errors which would be induced by the other Newtonian and Einsteinian competing effects, the solar quadrupole mass moment $J_2$ is presently known with a 15% accuracy (table 7 of [11]), so that the mismodelled part of its secular precession would be two orders of magnitude smaller than the effect of interest.

More important would be the impact of the Einsteinian gravitoelectric precession. Indeed, the general relativistic perihelion precession has been measured to a $10^{-4}$ relative accuracy via the PPN parameters $\beta$ and $\gamma$ (table 8 of [11]): assuming that it would also hold for $\lambda$, this would yield a $2 \times 10^{-4}$ $\omega$ mismodelled effect. However, it should be noted that these are merely the formal 1-sigma errors: realistic bounds might be one order of magnitude larger.

4.2. The $N$-body perturbations

The mean longitude can be written as $\lambda = nt + \epsilon$, so that [15]

$$\frac{d\lambda}{dt} = n + \frac{d\epsilon}{dt} = \frac{dn}{dt}t + \frac{d\epsilon}{dt}, \quad (33)$$

where

$$\frac{d\epsilon}{dt} = -\frac{2}{na} \frac{\partial R}{\partial a} + \frac{\sqrt{1-e^2}(1-\sqrt{1-e^2})}{na^2e} \frac{\partial R}{\partial e} + \frac{\tan(\epsilon/2)}{na^2\sqrt{1-e^2}} \frac{\partial R}{\partial \epsilon}. \quad (34)$$

4.2.1. The direct and indirect effects on the Keplerian mean motion due to the other planets and the asteroids.

With regard to the Keplerian mean motion, it turns out to be the major limiting factor. Indeed, the uncertainty in the solar $GM \sigma_{GM} = 8 \times 10^9$ m$^3$ s$^{-2}$ (table 1 of [14]) yields $\sigma_{nGM} = 2.6 \times 10^{-5}$ $\omega$. However, if we adopt the position of keeping fixed the solar $GM$, the formal 1-sigma error due to the semimajor axis $\sigma_{nax} = 6.57 \times 10^{-1}$ m (table 4 of [11]) only reduces to $\sigma_{nax} = 3 \times 10^{-4}$ $\omega$.

Another source of potential bias is represented by the indirect perturbations on the Keplerian mean motion induced by the variations in the semimajor axis

$$\Delta n = \frac{dn}{dt} t = -\frac{3}{2} \frac{d\epsilon}{dt} t = -\frac{3}{2} \frac{\partial R}{\partial \epsilon} t. \quad (35)$$

According to [15], there are no secular effects on the semimajor axis: instead, the so-called resonant perturbations affect this Keplerian orbital element. They are induced by those terms in the expansion of the $N$-body disturbing function which retain the mean longitudes of the perturbed and the perturbing bodies. Such kinds of harmonic perturbations, due to the asteroids for Mars, are potentially very insidious because they may have large amplitudes and extremely long periods. This topic has been treated in [16]. Table 4 of [16] lists the most important of such perturbations. The nominal amplitudes of the perturbations induced, e.g., by (1) Ceres, (2) Pallas, (4) Vesta and (7) Iris are $4.7 \times 10^{-3}$ $\omega$, $1.2 \times 10^{-2}$ $\omega$, $5.7 \times 10^{-3}$ $\omega$ and $5 \times 10^{-3}$ $\omega$, respectively. According to the results of table 6 of [11], their mismodelled parts amount to $7 \times 10^{-5}$ $\omega$, $3 \times 10^{-5}$ $\omega$, $4 \times 10^{-5}$ $\omega$ and $8 \times 10^{-5}$ $\omega$, respectively. It must also be noted that the integrated shift of $\Delta n$ grows quadratically in time.

4.2.2. The perturbations on the mean longitude due to the other planets and the asteroid ring.

Here we will deal with $d\epsilon/dt$ which is responsible for the secular perturbations on $\lambda$.

The Newtonian secular perturbations induced on the Mars mean longitude by the other planets of the Solar System are of the order of $10^3$ $\omega$. The major source of uncertainty
Table 3. Sources of systematic errors, in $''$cy$^{-1}$, affecting the mean longitude of Mars. The reported figures are at the 1-sigma level. The total effect, obtained by summing up the various errors, is more than 90% of the braneworld signature.

| Source of systematic error | Mismodelled amplitude ($''$cy$^{-1}$) |
|---------------------------|-------------------------------------|
| Keplerian mean motion     | $3 \times 10^{-4}$                  |
| Asteroid ring             | $3 \times 10^{-4}$                  |
| Schwarzschild GE field    | $2 \times 10^{-4}$                  |
| (7) Iris                  | $8 \times 10^{-5}$                  |
| (1) Ceres                 | $7 \times 10^{-5}$                  |
| Solar $J_2$               | $4.5 \times 10^{-5}$                |
| (4) Vesta                 | $4 \times 10^{-5}$                  |
| (2) Pallas                | $3 \times 10^{-5}$                  |
| Lense–Thirring GM field (assumed unmodelled) | $3 \times 10^{-5}$ |
| Total                     | $1.1 \times 10^{-3}$                |

is represented by the $GM$ of the perturbing bodies among which Jupiter plays a dominant role, especially for Mars. According to [17], the Jovian $GM$ is known with a relative accuracy of $10^{-8}$; this would imply for the red planet a mismodelled precession induced by Jupiter of the order of $10^{-5}''$cy$^{-1}$. The $GM$ of Saturn is known with a $1 \times 10^{-6}$ relative accuracy [18]. However, the ratio of the secular precession induced on the Martian Keplerian elements by Saturn to that induced on the Mars perihelion by Jupiter is proportional to $(M_{Sat}/M_{Jup})(a_{Jup}/a_{Sat})^{3} \sim 5 \times 10^{-2}$. This would assure that also the effect of Saturn is of the order of $10^{-5}''$cy$^{-1}$. The situation with the precessions induced by Uranus and Neptune is even more favourable. Indeed, for Uranus $(M_{Ura}/M_{Jup})(a_{Jup}/a_{Ura})^{3} \sim 1 \times 10^{-3}$ and the relative uncertainty in the Uranian $GM$ is $2 \times 10^{-6}$ [19]. For Neptune, $(M_{Nep}/M_{Jup})(a_{Jup}/a_{Nep}) \sim 3 \times 10^{-4}$ and $\sigma_{GM}/GM = 2 \times 10^{-6}$ [20].

A source of potentially non-negligible perturbations on the Martian mean longitude is the asteroid ring, i.e. the ensemble of the minor asteroids whose impact can be modelled as due to a solid ring in the ecliptic plane [21]. The perturbations due to it can be worked out, e.g., with the Lagrangian approach and the disturbing function of the appendix of [21]. By using the values of [11] for the ring’s radius and mass, it turns out that the secular perturbation on $\dot{\lambda}_{Mars}$ amounts to $-3.4 \times 10^{-3}''$cy$^{-1}$, with an uncertainty of $3 \times 10^{-4}''$cy$^{-1}$.

4.3. The total systematic error on the mean longitude of Mars

In table 3, we summarize the various systematic errors affecting the mean longitude of Mars. It turns out that if only the mean longitude of Mars was analysed, the 1-sigma total error would amount to $\sim 1.1 \times 10^{-3}''$cy$^{-1}$, i.e. more than 90% of the braneworld effect.

In the next section, we will outline a possible strategy to suitably combine the data of Mars with those of other inner planets in order to cancel out, by construction, many systematic errors. A numerical example is explicitly worked out.

5. The linear combination approach

In order to cancel out the impact of the various sources of systematic bias, it is possible to suitably linearly combine the mean longitudes of Mars and Venus following an approach adopted in, e.g., [22, 23]. Let us assume that we have at our disposal $N$ Keplerian orbital
elements $K$ whose time evolution is supposed to be affected by a certain number of Newtonian and post-Newtonian effects, say
\[ \dot{K} = \dot{K}_{DGP} + \dot{K}_{J2} + \dot{K}_{GE} + \dot{K}_{N-body} + \dot{K}_{LT} + \cdots. \] (36)
If we are interested in isolating one particular feature, say $\dot{\lambda}_{DGP}$, and we know it is smaller than other larger effects which affect the same Keplerian element, we can explicitly write down the expressions of the observational `residuals' $\delta \dot{K}_{\text{obs}}$ in terms of the feature of interest—which will be assumed to be entirely (or partly) present in the residuals—and of the main larger aliasing effects—which will affect the residuals with their mismodelled part only—so that the number of terms in the sum on the right-hand side of (36) which represent the effect of interest and the other most relevant larger bias is equal to the number $N$ of Keplerian orbital elements we have at our disposal. Now we have a system of $N$ equations in $N$ unknowns which we can solve for the effect we are interested in. The resulting expression will be, by construction, independent of the other larger aliasing effects.

By using, e.g., the figures of table 1 for the numerically integrated total precessions, which encompasses all the Newtonian and general relativistic effects, it is possible to obtain
\[ \delta \dot{\lambda}_{\text{Mars}} + c_1 \delta \dot{\lambda}_{\text{Venus}} = \mp 8 \times 10^{-4} \text{''cy}^{-1}, \] (37)
where
\[ c_1 = -\frac{\dot{\lambda}^{\text{(Num)}}_{\text{Venus}}}{\dot{\lambda}^{\text{(Num)}}_{\text{Mars}}} = -3.270 \times 10^{-1} \] (38)
and $\delta \dot{\lambda}$ are the time series residuals of the mean longitude built up in order to entirely absorb all the non-Newtonian and non-Einsteinian gravity.

The combination (37) is not affected, by construction, by all the competing Newtonian and general relativistic perturbations acting upon $\lambda$, at least to the level of accuracy of the dynamical force models used in calculating $\lambda^{\text{(Num)}}$. Of course, the coefficient $c_1$ can, in principle, be more precisely recalculated by using future, more accurate ephemerides. Indeed, from the previous discussion, the importance of also including the asteroids should be clear, to the best of our knowledge. While in the mathematical model of DE200 the perturbations of only three of the major asteroids are present, in the more advanced DE410 and EPM2004 ephemerides the perturbations of 300 asteroids and also of the asteroid ring have been included.

From table 2, it is possible to obtain for the formal 1-sigma observational accuracy in (37) a 7% value over 92 years. The systematic error due to the solar quadrupole mass moment, the Keplerian mean motions, the general relativistic Schwarzschild field and the asteroid ring amounts to some tens of per cent by assuming the formal 1-sigma level of uncertainty of [11] for $J_2$, $\omega_{\text{Mars}}$, $\omega_{\text{Venus}}$, $\beta$, $\gamma$ and $M_{\text{ring}}$.

6. Conclusions

The Dvali–Gabadadze–Porrati braneworld model, in the Lue–Starkman extension related to a spherically symmetric central mass, is very interesting because it predicts, among other things, small modifications of the Newton–Einstein gravity in the weak-field approximation which have testable phenomenological implications over the Solar System length scale.

In this paper, we have explicitly worked out its effects on the Keplerian orbital elements of the orbit of a test particle without restricting to circular orbits. It turns out that the pericentre $\omega$

\[2\] Here we speak about residuals of Keplerian orbital elements in a, strictly speaking, improper sense. The Keplerian orbital elements and their rates are not directly observable: they can only be computed and determined as fitted parameters in the usual least-squares procedure. The basic observable quantities are ranges, range-rates and angles.
and the mean anomaly $M$ undergo secular precessions which are independent of the size and the shape of the orbit. The sizes of these rates are $\sim \mp 4 \times 10^{-4}$''cy$^{-1}$ and $\pm 1.4 \times 10^{-3}$''cy$^{-1}$, respectively. The first nonvanishing corrections due to the eccentricity are of order $O(e^2)$. They are of the order of $10^{-5}$''cy$^{-1}$.

The possibility of observing such effects in the orbital motions of the inner planets of the Solar System has been examined, with particular emphasis on Mars. The mean longitude $\lambda$ has been considered along with various competing Newtonian and Einsteinian effects. For $\lambda$, the braneworld shift for all planets amounts to $\sim 1$ mas over almost one century while the present-day observational accuracy, based on the processing of almost 300,000 data of various kinds for the EPM2004 ephemerides spanning a 92-year temporal interval, is $3 \times 10^{-3}$ mas for Mars. A suitable linear combination with Venus would allow reduction of the impact of the systematic errors. The observational error would be $\sim 7\%$.

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