Relation between Dynamic Characteristics and Parameters of a Two-degree-of-freedom Vibro-impact System with Multi-elastic Constraints

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Abstract. A mechanical model of mechanical vibration system with multi-elastic constraints was founded. Based on the multiple parameter and multiple objective cooperative simulation calculation, the three-dimensional parameter space was constructed by the key parameters such as constrained stiffness, exciting frequency and clearance threshold. The large data onto two-dimensional parameter bifurcation diagrams reflecting the correlation between dynamic properties and parameters of the model are obtained. The mode types, existing regions and bifurcation characteristics of periodic impact vibration of the system on the parameter plane of frequency and clearance are described with the change of constrained stiffness. Similarly, the three-dimensional parameter space consisting of mass ratio, or damping ratio and exciting frequency and clearance threshold are analyzed. The results show that the high mass ratio excites rich and wonderful periodic vibration, and different damping ratio determines, the bifurcation characteristics and the number of chattering impacts groups in the small frequency domain. The mode types and transition rules of the fundamental periodic impact vibration in small frequency domain and subharmonic impact vibration in high frequency domain induced by different parameters of the system are revealed.

1. Introduction
Nonlinear vibration of mechanical structure and system is a complex problem in engineering practice. Therefore, to study the mechanism and characteristics of various non-smooth bifurcation phenomena of vibration systems with multi-elastic constraints, so as to describe the dynamic properties of mechanical systems more accurately, it is very important to design and manufacture mechanical systems and structures with excellent safety and performance. By means of qualitative analysis, numerical calculation and experiments, the periodic motion and bifurcation characteristics [1-3], grazing singularity [4-6] and chattering-impact motions systems [7-10] with clearances and constraints have been studied by scholars at home and abroad.

On the basis of Wagg's research, the non-smooth bifurcation problem of a two-degree-of-freedom vibro-impact system with multi-elastic stops is systematically researched by utilizing multiple parameter and multiple objective cooperative simulation calculation. The vibration system with rigid constraints studied by Wagg is only a special case of the vibration system with elastic constraints.
2. Mechanical Model

2.1. Mechanical Model
A two-degree-of-freedom vibration system with multi-elastic stops is expressed in Figure 1. Two masses \( M_1 \) and \( M_2 \) are linked by linear spring damper components \( K_1 \) and \( C_1 \), and mass \( M_2 \) is fastened to the base by a linear spring damper components \( K_2 \) and \( C_2 \). The harmonic exciting force \( P_1 \sin(\omega T + \tau) \) loads on the corresponding mass \( M_i \) (\( i = 1, 2 \)), among them, \( P_1 \), \( \omega \) and \( \tau \) represent the amplitude, frequency and phase angle individually. The elastic stops on the same stiffness \( (K_0) \) were placed on the right stop of mass \( M_1 \) and on the symmetric stops of mass \( M_2 \), the range of constrained stiffness was \( K_0 \in (0, \infty) \). For low amplitude, the system shows free vibration without impact. However, as the vibration amplitude of the system increases sufficiently large, when the displacement of the masses is equal to the clearance threshold of corresponding constraints, the soft impact of the masses occurs to the corresponding constraints [11-13].

![Figure 1. Mechanical model](image)

2.2. Defined Dimensionless Parameters, Variables and Time:

\[
m = \frac{M_2}{M_1 + M_2}, \quad k = \frac{K_2}{K_2 + K_1}, \quad k_0 = \frac{K_0}{K_0 + K_1}, \quad c = \frac{C_2}{C_2 + C_1}, \quad \delta_i = \frac{B_i K_i}{P_2}, \quad f = \frac{P_2}{P_1 + P_2}, \quad x_i = \frac{X_i K_1}{P_1 + P_2}, \quad t = T \sqrt{\frac{K}{M_1}}. \tag{1}
\]

2.3. Dimensionless Differential Equation of Motion

\[
\begin{bmatrix}
1 & 0 \\
0 & \frac{m}{1-m}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+
\begin{bmatrix}
2\zeta & -2\zeta \\
2\zeta & -2\zeta
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+
\begin{bmatrix}
1 & -1 \\
1 & -k
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2
\end{bmatrix}
+
\begin{bmatrix}
f_1(x_1) \\
f_2(x_2)
\end{bmatrix}
= \begin{bmatrix} 1-f \\
f \end{bmatrix}
\sin(\omega t + \tau) \tag{2}
\]

In which \( f_1(x_1) \) and \( f_2(x_2) \) were expressed by

\[
f_1(x_1) = \begin{cases}
\frac{k_0}{1-k_0} (x_1 - \delta_1), & x_1 > \delta_1, \\
0, & x_1 \leq \delta_1,
\end{cases}
\]

\[
f_2(x_2) = \begin{cases}
\frac{k_0}{1-k_0} (x_2 - \delta_2), & x_2 > \delta_2, \\
0, & |x_2| \leq \delta_2, \\
\frac{k_0}{1-k_0} (x_2 + \delta_2), & x_2 < -\delta_2.
\end{cases} \tag{3}
\]

Equation (1) allowed us to make a conclusion that the maximum range of partial dimensionless parameters can be identified as \( m \in (0, 1) \), \( k \in (0, 1) \), \( k_0 \in (0, 1) \), \( c \in (0, 1) \). In the above parameter domain, the vibration system with multi-elastic constraints in Figure 1 showed a variety of periodic impact vibration. Symbol \( q/n \) was used to describe the periodic impact vibration of a unilateral constrained mass \( M_1 \), where \( n \) was the ratio of the vibration period \( T_0 = 2\pi n/\omega \) to the periods of the exciting force \( T_0 = 2\pi/\omega \) (\( n = 1, 2, 3\ldots \)), \( q \) means the number of impacts of mass \( M_1 \) at its right elastic constraints during the vibration period (\( q = 0, 1, 2, 3\ldots \)). Meanwhile, the symbol \( n-p-q \) was used to represent the periodic impact vibration characteristics of the mass \( M_2 \) with symmetric constraints, and \( p \) and \( q \) were
used to represent the impact times of the mass $M_2$ with left and right constraints, respectively. More generally, in order to find the values of $n, p$ and $q$, defined four kinds of Poincaré section.

$$\sigma_{II} = \{(x_1, x_1, x_2, x_2, t)R^4 \times T | x_1 = x_{\text{min}}, \text{mod}(t = 2\pi/\omega)\}; \quad \sigma_{II}^2 = \{(x_1, x_1, x_2, x_2, t) \in R^4 \times T | x_1 = \delta_{\omega}, \dot{x}_1 > 0\};$$

$$\sigma_{II}^3 = \{(x_1, x_1, x_2, x_2, t) \in R^4 \times T | x_2 = -\delta_{\omega}, \dot{x}_2 < 0\}; \quad \sigma_{II}^4 = \{(x_1, x_1, x_2, x_2, t) \in R^4 \times T | x_2 = \delta_{\omega}, \dot{x}_2 > 0\}. \quad (4)$$

For periodic impact vibration, the fixed points of Poincaré section $\sigma_{II}$, $\sigma_{II}^2$ and $\sigma_{II}^3$ mean the periodic number $n$ of the mass corresponding to the $k$ constraints and the impact number $p$ and $q$ of the left and right constraints, respectively. The equation of the impact Poincaré mapping $\sigma_{II}^4$ of the system was expressed as

$$X^{(i+1)} = \mathcal{F}(X^{(i)}, \nu), \quad (5)$$

where $X^{(i)} = (x^{(i)}, \dot{x}_1^{(i)}, \dot{x}_2^{(i)}, \dot{x}_3^{(i)})^T$, $X^{(i+1)} = (x^{(i+1)}, \dot{x}_1^{(i+1)}, \dot{x}_2^{(i+1)}, \dot{x}_3^{(i+1)})^T$, $X \in \mathbb{R}^4$, $\nu$ are parameters, $\nu \in \mathbb{R}^4$.

3. Results and Discussion

3.1. Effect of Constraint Stiffness Ratio on System Dynamic Properties

Considering a two-degree-of-freedom forced vibration system with multi-elastic constraints, the interaction between the dynamic properties of the system and its parameter constrained stiffness ratio $k_0$ is revealed. Firstly, the effects of constraints (constrained stiffness $k_0$, clearance threshold $\delta$) and exciting frequency $\omega$ of the dynamic characteristics of the system are studied. Selection of dimensionless parameters: $m = 0.5$, $k = 0.5$, $c = 0.5$, $\zeta = 0.1$. The range of constrained stiffness ratio $k_0 \in (0,1)$, the values are discretized at equal intervals. Figure 2 representatively shows the mode types and occurrence regions of the periodic vibration of the system in the $(\omega, \delta)$-parameter plane in Figure 1 corresponding to three sets of $k_0$. When the constrained stiffness is taken as a value $k_0 \in (0,0.5)$, the system masses $M_1$ and $M_2$ exhibit simple and conventional periodic impact vibration in the parameter domain of figure 2 (a). With the increase in $k_0$, the dynamic characteristics of the system show complexity and diversity, see figure 2 (b). As the frequency $\omega$ decreases in the small clearance $\delta$ region, a series of grazing contacts occur to between each mass and its corresponding constraints, resulting in the transition of $q/1$ vibration to $(q+1)/1$ vibration via Grazing bifurcation. As a result, the number of impacts $p$ (or $q$) of each mass in the system increases step by step during the fundamental periodic. In the small frequency $\omega$ and clearance $\delta$ region of the lower left corner of the parametric plane, the mass $M_2$ shows the different types of fundamental periodic motion groups, while the mass $M_1$ clearly shows tongue-shaped non-hysteretic transition region along the boundary of adjacent fundamental periodic impact vibration. The fundamental periodic motion groups ($q/1$ or $1-p-q$) of the system exhibit a chattering-impact motions characteristic when the number of impacts of the system masses to become large enough under the condition of minimal frequency $\omega$. When $k_0 > 0.95$, with the increase of frequency $\omega$, the sequence of periodic doubling bifurcation may be interrupted by grazing contact between a periodic vibration trajectory and elastic constraints, and the system shows complex dynamic characteristics. Based on the above analysis, the constrained stiffness $k_0 = 0.95$ is taken as the reference parameter, as shown in Figure 2 (c), and The effects of other parameters of the model on the mode types and bifurcation characteristics of periodic vibration are further analyzed. In the biparametric bifurcation diagram, 60% of the unlabeled black regions are chaotic.
3.2. Effect of Mass Ratio on System Dynamic Properties

The range of mass distribution ratio is $m \in (0, 1)$, and the values are discretized at equal intervals. The damping distribution ratio $c$ of the system parameters is calculated by the same method. In Figure 3 are plotted the dimensionless reference parameters: $k_0 = 0.95$, $k = 0.5$, $c = 0.5$, $\zeta = 0.1$, the typical three groups of $m$ values correspond to the mode types and bifurcation characteristics of periodic vibration on the $(\omega, \delta)$-parameter plane. When the mass ratio $m$ is small, the equivalent mass mainly concentrates on the mass $M_1$. In the low clearance $\delta$ and frequency $\omega$ region of the lower left corner of the parametric plane, the mass $M_1$ presents the fundamental periodic vibration groups, and tongue-shape non-hysteretic transition regions are formed along the boundary of two adjacent fundamental periodic vibrations $(q/1$ and $(q+1)/1)$. There are different types of fundamental periodic vibration groups $(1-p-q)$ in the mass $M_2$, and the number of impacts on the left stop $p$ is more than that on the right stop $q$, namely $p > q$. In the high frequency $\omega$ and small clearance $\delta$ region, As the frequency $\omega$ decreases, mass $M_1$ exhibits subharmonic periodic vibration of $1/2$, $2/2$ and $2/3$, or doubling bifurcation or grazing bifurcation, resulting in subharmonic periodic vibration of $2/4$, $3/4$, $3/3$ and $6/6$. Mass $M_2$ induces periodic doubling bifurcation or grazing bifurcation, subharmonic periodic vibrations of $3-1-0$, $2-1-1$, $2-2-1$, $2-3-1$, $2-3-0$, $2-2-0$ and $2-1-0$ are shown. When the mass ratio $m$ is large, the equivalent mass focuses on the mass $M_2$, and the system exhibits abundant dynamic characteristics. The parameter domain of $(q/1$ $(q \geq 2))$ the fundamental motion groups of mass $M_1$ rapidly extends to the large clearance threshold $\delta$. However, $(1-p-q)$ the different types of fundamental motion groups of mass $M_2$ are significantly reduced and the existing regions are obviously contracted with the direction of frequency $\omega$ reduction, see Figures 3 (b) and 3 (c). In the clearance $\delta$ and frequency $\omega$ region of the upper right corner of the parametric plane, the system displays $(0/1$ or $1-0-0$) non-impact vibration. The results show that with the increase $m$ of system parameters, the fundamental motion groups of

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Figure 2. Under different constraint stiffness ratios, the system presents dynamic characteristics in the $(\omega, \delta)$-parameter plane: (a1) $M_1$: $k_0 = 0.25$; (b1) $M_1$: $k_0 = 0.75$; (c1) $M_1$: $k_0 = 0.95$; (a2) $M_2$: $k_0 = 0.25$; (b2) $M_2$: $k_0 = 0.75$; (c2) $M_2$: $k_0 = 0.95$
mass $M_1$ exhibit diversity, and the different types of fundamental motion groups of mass $M_2$ exhibit fierce competition in the parameter domain. The larger $m$ of system parameters, the more complex the dynamic characteristics of the system.

**Figure 3.** Under different mass ratios, the system presents dynamic characteristics in the $(\omega, \delta)$-parameter plane: (a1) $M_1$: $m = 0.25$; (b1) $M_1$: $m = 0.75$; (c1) $M_1$: $m = 0.95$; (a2) $M_2$: $m = 0.25$; (b2) $M_2$: $m = 0.75$; (c2) $M_2$: $m = 0.95$

3.3. Effect of Damping Ratio on System Dynamic Properties

The typical three groups of damping distribution ratios $c$ correspond to the mode types and occurrence areas of periodic impact vibration on the $(\omega, \delta)$-parameter plane, as seen in figure 4. The results are compared with those under reference parameters (1), see figure 2 (c). When $c < 0.5$, the mode types and bifurcation characteristics of mass $M_1$ and $M_2$ show little change, and they all show complex bifurcation characteristics in the low frequency $\omega$ and small clearance threshold $\delta$ region. When $c$ increased, the total area of $(q/1 \ (q \geq 2))$ the fundamental motion groups of the low frequency region of the mass $M_1$ have almost no change, but the number of impacts $(q)$ decreases gradually. At the same time, the tongue-shaped non-hysteretic transition regions of the adjacent fundamental periodic impact vibration are obviously reduced, and even disappeared, and the existence area of subharmonic periodic vibrations in the high frequency $\omega$ region of mass $M_1$ is also reduced. The mode types and existing areas of the different types of fundamental motion groups of mass $M_2$ are obviously reduced. When the value of $c$ is larger, the peak values of the corresponding fundamental periodic vibration regions and the subharmonic periodic vibration regions of the two masses decrease with the increase of parameter $c$, see figures 4 (b) and 4 (c). The results show that the change of damping ratio $c$ in the system parameters has a great influence on the mode types and bifurcation characteristics of the fundamental motion groups corresponding to the masses $M_1$ and $M_2$, and the dynamic characteristics of masses $M_1$ and $M_2$ show similar evolution regularity.
Figure 4. Under different damping ratios, the system presents dynamic characteristics in the \((\omega, \delta)\) - parameter plane: (a1) \(M_1\): \(c = 0.25\); (b1) \(M_1\): \(c = 0.75\); (c1) \(M_1\): \(c = 0.95\); (a2) \(M_2\): \(c = 0.25\); (b2) \(M_2\): \(c = 0.75\); (c2) \(M_2\): \(c = 0.95\)

4. Conclusion
In summary, the constrained stiffness ratio \(k_0\) determines the number of fundamental motion groups of the system. Under the condition of large mass ratio \(m\), the dynamic characteristics of masses \(M_1\) and \(M_2\) show complexity and diversity. The lower the damping ratio \(c\) is, the more significant the non-smooth characteristic of the system is.

The numerical results can reveal the essential relationship between the dynamic characteristics of mechanical vibration system with multiple-elastic constraints and its parameters and the matching law. Mechanical systems with different functions can determine the reasonable matching range of their parameters, and realize the collaborative optimization of dynamic characteristics and functional objectives of the system.

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