SURGERY AND EXCISION FOR FURUTA-OHTA INVARIANTS ON HOMOLOGY $S^1 \times S^3$

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Abstract. We prove a surgery formula and an excision formula for the Furuta-Ohta invariant $\lambda_{FO}$ defined on homology $S^1 \times S^3$, which provides more evidence on its equivalence with the Casson-Seiberg-Witten invariant $\lambda_{SW}$. These formulae are applied to compute $\lambda_{FO}$ of certain families of manifolds obtained as mapping tori under diffeomorphisms of 3-manifolds. In the course of the proof, we give a complete description of the degree-zero moduli space of ASD instantons on 4-manifolds of homology $H_\ast(D^2 \times T^2; \mathbb{Z})$ with a cylindrical end modeled on $[0, \infty) \times T^3$.

1. Introduction

The Furuta-Ohta invariant was introduced by Furuta and Ohta in [6] to study exotic structures on punctured four manifolds. Originally the Furuta-Ohta invariant is defined on manifolds called $\mathbb{Z} \mathbb{Z}$-homology $S^1 \times S^3$, i.e. closed 4-manifolds $X$ with the same homology as $S^1 \times S^3$ whose infinite cyclic cover $\tilde{X}$ has the same homology as $S^3$. The conjecture is that the Furuta-Ohta invariant $\lambda_{FO}(X)$ mod 2 reduces to the Rohlin invariant $\mu(X)$ associated to the 4-manifold. Instead of approaching the conjecture directly, Mrowka-Ruberman-Saveliev [15] considered the Seiberg-Witten correspondence $\lambda_{SW}$ defined over homology $S^1 \times S^3$ where they manage to show that $\lambda_{SW}(X)$ mod 2 reduces to the Rohlin invariant. So the problem has been transformed to prove the equivalence between the Casson-Seiberg-Witten invariant $\lambda_{SW}$ and the Furuta-Ohta invariant $\lambda_{FO}$. The motivation of this article is to study how the Furuta-Ohta invariants change under certain topological operations, which in turn provides more evidence on this equivalence.

In this article we allow the Furuta-Ohta invariants to be defined on a slightly larger class of manifolds which we refer to as admissible homology $S^1 \times S^3$.

Definition 1.1. Let $X$ be a smooth oriented closed 4-manifold with $H_\ast(X; \mathbb{Z}) \cong H_\ast(S^1 \times S^3; \mathbb{Z})$. We call $X$ an admissible homology $S^1 \times S^3$ if it further satisfies the following property: for all non-trivial $U(1)$-representations $\rho : \pi_1(X) \to U(1)$, one has

$$H^1(X; \mathbb{C}_\rho) = 0.$$  

Now let $X$ be an admissible homology $S^1 \times S^3$. After fixing a generator $1_X \in H^1(X; \mathbb{Z})$ as the homology orientation, the Furuta-Ohta invariant is defined to be a quarter of the counting of degree-zero irreducible anti-self-dual $SU(2)$-instantons
on $X$, which is written as
\[ \lambda_{FO}(X) := \frac{1}{4} \# \mathcal{M}_\sigma^*(X). \]

It’s proved in [17] that for a generic small holonomy perturbation $\sigma$, the moduli space $\mathcal{M}_\sigma^*(X)$ is a compact oriented 0-manifold and the counting is well-defined. We note that when $X = S^1 \times Y$ is given by the product of $S^1$ with an integral homology sphere, the Furuta-Ohta invariant coincides with the Casson invariant of $Y$ [17], i.e. $\lambda_{FO}(S^1 \times Y) = \lambda(Y)$.

The first topological operation we consider is the torus surgery. We give a brief description here, and a detailed one in Sectoin [6]. Let $X$ be an embedded integral homology $S^1 \times S^3$ with a fixed generator $1_X \in H^1(X; \mathbb{Z})$, and $\mathcal{T} \hookrightarrow X$ an embedded 2-torus satisfying that the induced map on first homology
\[ H_1(\mathcal{T}; \mathbb{Z}) \to H_1(X; \mathbb{Z}) \]
is surjective. We will refer to such a torus as an essentially embedded torus. We write $\nu(\mathcal{T})$ for a tubular neighborhood of $\mathcal{T}$ in $X$, and fix an identification $\nu(\mathcal{T}) \cong D^2 \times T^2$ as a framing. Let $M = X \setminus \nu(\mathcal{T})$ be the closure of the complement of the neighborhood. It’s straightforward to compute that $H_*(M; \mathbb{Z}) \cong H_*(D^2 \times T^2; \mathbb{Z})$. The framing provides us with a basis $\{\mu, \lambda, \gamma\}$ for $H_1(\partial\nu(\mathcal{T}); \mathbb{Z})$. We require $[\gamma] \in H_1(X; \mathbb{Z})$ to be the dual of the generator $1_X \in H^1(X; \mathbb{Z})$, $\mu$ to be the meridian of $\mathcal{T}$, and $[\lambda]$ to be null-homologous in $M$. Given a relatively prime pair $(p, q)$, performing $(p, q)$-surgery along $\mathcal{T}$ results in the 4-manifold
\[ X_{p,q} = M \cup_{\varphi_{p,q}} D^2 \times T^2, \]
where with respect to the basis $\{\mu, \lambda, \gamma\}$ the gluing map is given by
\[ \varphi_{p,q} = \begin{pmatrix} p & r & 0 \\ q & s & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SL(3, \mathbb{Z}). \tag{1.2} \]

We will see later in Section [6] that the $(1, q)$-surgered manifold $X_{1,q}$ is still an admissible integral homology $S^1 \times S^3$ with a homology orientation induced from that of $X$. We note that $H_*(X_{0,1}; \mathbb{Z}) \cong H_*(S^2 \times T^2; \mathbb{Z})$. We denote by $w_\mathcal{T} \in H^2(X_{0,1}; \mathbb{Z}/2)$ the class that’s dual to the mod 2 class of the core $T^2$ in the gluing $D^2 \times T^2$. We write $D_{w_\mathcal{T}}^0(X_{0,1})$ for the counting of gauge equivalence classes of the irreducible anti-self-dual $SO(3)$-connections on the bundle $SO(3)$-bundle $P$ over $X_{0,1}$ characterized by
\[ p_1(P) = 0 \text{ and } w_2(P) = w_\mathcal{T}. \]

The torus surgery formula relates $\lambda_{FO}(X)$, $\lambda_{FO}(X_{1,q})$, and $D_{w_\mathcal{T}}^0(X_{0,1})$ as follows.

**Theorem 1.2.** After fixing appropriate homology orientations, one has
\[ \lambda_{FO}(X_{1,q}) = \lambda_{FO}(X) + \frac{q}{2} D_{w_\mathcal{T}}^0(X_{0,1}), \quad q \in \mathbb{Z}. \]
In the product case $X = S^1 \times Y$ with $Y$ an integral homology sphere, the Furuta-Ohta invariant of $X$ coincides with the Casson invariant $Y$. We recall that the surgery formula of the Casson invariant is

$$\lambda(Y_1(K)) = \lambda(Y) + \frac{q}{2} \Delta''_K(1),$$

where $K \subset Y$ is a knot, and $\Delta'_K(t)$ is the symmetrized Alexander polynomial of $K$. Comparing with the surgery formula of the Furuta-Ohta invariant we get the following result.

**Corollary 1.3.** Let $K \subset Y$ be a knot in an integral homology sphere. Then

$$D^0_{wp}(S^1 \times Y_0(K)) = \Delta''_K(1).$$

The surgery formula of the Furuta-Ohta invariant should be compared with that of the Casson-Seiberg-Witten invariant proved in [12]:

$$\lambda_{SW}(X_{1,q}) = \lambda_{SW}(X) + q \mathcal{S}W(X_{0,1}),$$

where $\mathcal{S}W(X_{0,1})$ is the Seiberg-Witten invariant of $X_{0,1}$ computed in the chamber specified by small perturbations. The Casson-Seiberg-Witten invariant is defined using Seiberg-Witten theory [15] combining the counting of irreducible monopoles and an index-theoretical correction term. However, the Witten conjecture has not been proved in the case for non-simply connected 4-manifolds with $b^+ = 1$. So one does not get the equivalence of $\lambda_{FO}$ and $\lambda_{SW}$ for families of admissible homology $S^1 \times S^3$ by appealing to the surgery formulae directly.

On the other hand, we apply the surgery formula to give an independent computation of the Furuta-Ohta invariant for manifolds given by the mapping torus of finite order diffeomorphism as in [10]. More precisely, we let $K \subset Y$ be a knot in an integral homology $S^1 \times S^3$. Fix an integer $n > 1$, we denote by $\Sigma_n(Y, K)$ the $n$-fold cyclic cover of $Y$ branched along $K$, and $\tau_n: \Sigma_n(Y, K) \rightarrow \Sigma_n(Y, K)$ the covering translation. It’s shown in [10] Proposition 6.1 that the mapping torus $X_n(Y, K)$ of $\Sigma_n(Y, K)$ under the map $\tau_n$ is an admissible homology $S^1 \times S^3$ whenever $\Sigma_n(Y, K)$ is a rational homology sphere. We can apply the surgery formula to compute $\lambda_{FO}(X_n(Y, K))$ as follows.

**Proposition 1.4.** Assume that $\Sigma_n(Y, K)$ is a rational homology sphere. Then

$$\lambda_{FO}(X_n(Y, K)) = n \lambda(Y) + \frac{1}{8} \sum_{m=1}^{n-1} \text{sign}^{m/n}(Y, K),$$

where $\text{sign}^{m/n}(Y, K)$ is the Tristram-Levine signature of $K$.

This computation is carried out in [10] Theorem 6.4] using a more direct method by relating to the equivariant Casson invariant of $\Sigma_n(Y, K)$. In [12], the author also computed the Casson-Seiberg-Witten invariant for $X_n(Y, K)$ without assuming $\Sigma_n(Y, K)$ is a rational homology sphere, which turns out to be the same formula. However, when $\Sigma_n(Y, K)$ fails to be a rational homology sphere, $X_n(Y, K)$ is not admissible. So the Furuta-Ohta invariant is not defined.
We move to the next topological operation which we call torus excision in this article. The idea is to replace the torus neighborhood $D^2 \times T^2$ with a homology $D^2 \times T^2$ and glue it to the complement by further applying a diffeomorphism. Let $(X_1, T_1)$ and $(X_2, T_2)$ be two pairs of essentially embedded torus in an admissible homology $S^1 \times S^3$ as above. We also fix framings for both $\nu(T_1)$ and $\nu(T_2)$ to get a basis $\{\mu_i, \lambda_i, \gamma_i\}$ of $H_1(\partial \nu(T_i))$ as before. To emphasize the orientation, we write $X_1 = M_1 \cup \nu(T_1)$ and $X_2 = \nu(T_2) \cup M_2$, i.e. the left parts $\partial M_1 = -\nu(T_1), \partial \nu(T_2) = -M_2$ are identified with $T^3$ with a fixed orientation. Let $\varphi : \partial M_2 \to \partial M_1$ be a diffeomorphism so that the glued manifold

$$X_1 \#_{\varphi} X_2 := M_1 \cup_{\varphi} M_2$$

is an admissible homology $S^1 \times S^3$. We let $X_{1,\varphi} = M_1 \cup_{\varphi} D^2 \times T^2$ and $X_{2,\varphi} = D^2 \times T^2 \cup_{\varphi} M_2$. We will give more explanations on what the gluing map means in this context later in Section 7. Roughly $D^2 \times T^2$ is glued to $M_1$ the same way as $M_2$ does, so is to $M_2$. Since the admissible assumption is purely homological, we see that both $X_{1,\varphi}$ and $X_{2,\varphi}$ are admissible. The excision formula of the Furuta-Ohta invariants states as follows.

**Theorem 1.5.** After fixing appropriate homology orientations, one has

$$\lambda_{\text{FO}}(X_1 \#_{\varphi} X_2) = \lambda_{\text{FO}}(X_{1,\varphi}) + \lambda_{\text{FO}}(X_{2,\varphi}).$$

Note that $X_{i,\varphi}$ is obtained from $X_i$ via a torus surgery. When the gluing map $\varphi$ has the form we considered in Theorem 1.2, we further expand the formula as

$$\lambda_{\text{FO}}(X_1 \#_{\varphi_{1,\varphi}} X_2) = \lambda_{\text{FO}}(X_1) + \frac{q}{2} D^0_{\text{wT}}(X_{1,\varphi_{0,1}}) + \lambda_{\text{FO}}(X_2) + \frac{q}{2} D^0_{\text{wT}}(X_{2,\varphi_{0,1}}).$$

We will see later in examples there are certain interesting gluing maps that are not of the form we considered in the surgery formula. Thus we need a generalized surgery formula to compare $\lambda_{\text{FO}}(X_{i,\varphi})$ with $\lambda_{\text{FO}}(X_i)$. It turns out there is an extra term coming out in the formula as we go through the proof of Theorem 1.2 which is caused by the contribution of some ‘bifurcation points’ (c.f. Definition 5.12) in the moduli space of the torus complement. We hope to formulate this extra term in a future article.

We note that the fiber sum operation considered in 1.11 is a special case of the excision. The fiber sum of $(X_1, T_1)$ and $(X_2, T_2)$ is given by gluing the torus complements by a map interchanging the meridian $\mu$ and longitude $\lambda$:

$$X_1 \#_{\tau} X_2 := M_1 \cup_{\varphi_{\tau}} M_2,$$

where with respect to the basis $\{\mu_i, \lambda_i, \gamma_i\}$, $\varphi_{\tau}$ is given by

$$(1.3) \quad \varphi_{\tau} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ 

Note that $M_1 \cup_{\varphi_{\tau}} D^2 \times T^2 = X_1$, $D^2 \times T^2 \cup_{\varphi_{\tau}} M_2 = X_2$. We conclude that the Furuta-Ohta invariant is additive under taking torus fiber sum.
Corollary 1.6. After fixing appropriate homology orientations, one has
\[ \lambda_{FO}(X_1 \# T X_2) = \lambda_{FO}(X_1) + \lambda_{FO}(X_2). \]

In the product case \( X_i = S^1 \times Y_i, T_i = S^1 \times K_i, i=1,2, \) the fiber sum \( X_1 \# T X_2 \) is the product of \( S^1 \) with the knot splicing \( Y_1 \# k Y_2 \) of the pairs \((Y_1, K_1)\) and \((Y_2, K_2)\). Then the fiber sum formula for the Furuta-Ohta invariant recovers the additivity of the Casson invariant under knot splicing. We also note that the same fiber sum formula holds for the Casson-Seiberg-Witten invariant proved in [11]:
\[ \lambda_{SW}(X_1 \# T X_2) = \lambda_{SW}(X_1) + \lambda_{SW}(X_2). \]

Both the proofs of Theorem 1.2 and Theorem 1.5 rely on understanding the anti-self-dual moduli space of the torus complement. The difficulty of analyzing the moduli space of the torus complement arises from two parts. One is \( b^+ = 0 \), which prevents us from using metric perturbations to get rid of the reducible locus. The other is that the gluing boundary is \( T^3 \) whose moduli space has certain non-degeneracy, especially when we consider the trajectories on the torus complement flows to the singular points in the moduli space. To deal with the first issue, we consider holonomy perturbations and analyze the local structure of the moduli space near the reducible locus. To deal with the second issue, we adopt the ‘center manifold’ technique developed in [14].

For the rest of this section, we state the results on the degree zero anti-self-dual moduli space. We let \( Z \) be a 4-manifold with cylindrical end and \( E = Z \times \mathbb{C}^2 \) the trivialized \( \mathbb{C}^2 \)-bundle. The degree-zero perturbed moduli space \( M_{\sigma}(Z) \) of ASD instantons on \( Z \) consists of gauge equivalence classes of \( SU(2) \)-connections \( A \) on \( E \) satisfying the following:

(i) The self-dual part of the curvature equals the perturbation, i.e. \( F_A^+ = \sigma(A) \).
(ii) The curvature of \( A \) is of finite energy, i.e. \( \int_Z |F_A|^2 < \infty \).
(iii) The Chern-Weil integral vanishes, i.e. \( \int_Z \text{tr}(F_A \wedge F_A) = 0 \).

The reason for calling this moduli space degree zero is due to the third requirement on the vanishing of the Chern-Weil integral. The perturbation function \( \sigma \) is gauge-equivariant and satisfy an exponential decay condition along the end:
\[ \| \sigma(A) \|_{L^\infty(\{t\} \times Y)} \leq C e^{-\mu t}, \]
where \( C \) and \( \mu \) are some positive constant independent of \( A \). The space \( \mathcal{P}_\mu \) of perturbations is parametrized by a Banach space denoted by \( (W, \| \cdot \|_W) \). We will write \( \sigma = \sigma_\omega \) for some \( \omega \in W \), and \( \| \sigma \| = \| \omega \|_W \). We write \( Z = M \cup [0, \infty) \times Y \) where \( M \) is a compact 4-manifold with boundary, \( Y \) is a 3-manifold. \([0, \infty) \times Y\) is referred to as the cylindrical part of \( Z \). Let’s write
\[ (1.4) \quad \chi(Y) := \text{Hom}(\pi_1(Y), SU(2))/Ad \]
for the \( SU(2) \)-character variety of the 3-manifold \( Y \) in the cylindrical end. Via the holonomy map, \( \chi(Y) \) is identified with the gauge equivalence classes of the flat connections on \( Y \). The first step to deduce a structure theorem is to establish the existence of the asymptotic map on \( \mathcal{M}_\sigma(Z) \).
Theorem 1.7. Given \([A] \in \mathcal{M}_\sigma(Z)\), the limit \(\lim_{t \to \infty} [A]_{(t) \times Y}\) exists and lies in \(\chi(Y)\). The assignment of \([A]\) to its limit defines a continuous map
\[
\partial_+ : \mathcal{M}_\sigma(Z) \to \chi(Y).
\]

Remark 1.8. To our best knowledge, the existence of the limit of finite energy instantons, or in general the gradient flowlines in banach spaces, has been established in the following cases, none of which fit into our setting. Let’s write \([A]_{(t) \times Y} = B(t)\). The gauge-fixed perturbed ASD equation restricted on the end has the form
\[
\dot{B}(t) = -\operatorname{grad} cs(B(t)) + p(B(t)),
\]
where \(cs\) is the Chern-Simons functional. Consider the following cases

(i) The critical points of \(cs\) are isolated and non-degenerate, and the integral \(\int_{[0, \infty)} \|p(B(t))\|\) is finite.

(ii) There is a perturbed functional \(cs_p\) satisfying
\[
-\operatorname{grad} cs_p(B(t)) = -\operatorname{grad} cs(B(t)) + p(B(t)).
\]
Moreover, either the critical points of \(cs_p\) are Morse-Bott or the perturbed functional \(cs_p\) is analytic.

(iii) \(\|p(B(t))\| \leq \alpha \|\operatorname{grad} cs(B(t))\|\) for some \(\alpha < 1\) after \(t >> 0\).

Perturbations of the form (i) and (ii) are usually considered in establishing the Floer homology. (iii) was considered in [13] where they used metric perturbations. The existence of the limit follows from a modified argument of Simon’s [15]. Sometimes people also consider perturbations of compact support, which falls into case (iii). In the end, generic compact perturbations suffice for our purpose. But to show various transversality results, one has to put perturbations into a Banach space. So we will essentially show the ‘center manifold’ technique in [13] works for perturbations in a completed space. We also note Theorem 1.7 is proved in a slightly more general context in Theorem 3.7 where we shall consider the based moduli space.

We refer to \(\partial_+\) as the asymptotic map. Now we focus on the type of manifolds of our primary interests. Let \(Z\) be a manifold with cylindrical end satisfying the following:

(i) The integral homology of \(Z\) is the same as that of \(D^2 \times T^2\), i.e. \(H_*(Z; \mathbb{Z}) \cong H_*(D^2 \times T^2; \mathbb{Z})\).

(ii) The cylindrical end of \(Z\) is modeled on \([0, \infty) \times T^3\). The torus \(T^3\) is endowed with a flat metric \(h\).

We shall refer to a pillowcase as the quotient of a torus \(T^2\) under the hypoelliptic involution. Thus a pillowcase is an orbifold smoothable to \(S^2\) with 4 singular points having \(\mathbb{Z}/2\) as the isotropy group. One will see later in Section 4 that each central connection in \(\chi(T^3)\) is a singular point, and there are eight of them up to gauge equivalence. We denote by \(\mathcal{C} \subset \chi(T^3)\) the set of these central classes. We now state the structure theorem for the moduli space over \(Z\).
Theorem 1.9. Let \( \sigma \) be a generic perturbation in \( P_\mu \) with \( \| \sigma \| < c \) for some constant \( c > 0 \). We fix an orientation of \( H^1(Z;\mathbb{Z}) \). The degree zero moduli space of perturbed anti-self-dual instantons \( M_\sigma(Z) \) is a compact smooth oriented stratified space with the following structures:

(a) The reducible locus \( M^\text{red}_\sigma(Z) \) is a pillowcase whose singular points consist of the four gauge equivalence classes of central flat \( SU(2) \)-connections on \( Z \).

(b) The irreducible locus \( M^*_\sigma(Z) \) is a smooth oriented 1-manifold of finite components, each of which is diffeomorphic to either the circle \( S^1 \) or the open interval \((0, 1)\).

(c) The ends of the closure of the open arcs in \( M^*_\sigma(Z) \) lie in \( M^\text{red}_\sigma(Z) \) away from the singular points. Near each end \( [A] \in M^\text{red}_\sigma(Z) \), the moduli space \( M_\sigma(Z) \) is modeled on a neighborhood of 0 in the zero set \( o^{-1}(0) \), where

\[
o : \mathbb{R}^2 \oplus \mathbb{R}_+ \longrightarrow \mathbb{C} \\
(x_1, x_2, r) \longmapsto (x_1 + ix_2) \cdot r.
\]

(d) The asymptotic map restricted on the irreducible locus \( \partial_+ : M^*_\sigma(Z) \rightarrow \chi(T^3) \) is \( C^2 \) and transverse to a given submanifold (the choice of the perturbation \( \sigma \) depends on the given submanifold).

(e) The asymptotic values of irreducible instantons miss the central classes, i.e. \( \partial_+ (M^*_\sigma(Z)) \cap \mathcal{C} = \emptyset \).

The proof of Theorem 1.9 is divided into Proposition 4.3, Proposition 4.7, Corollary 5.7, and Proposition 5.11.

Remark 1.10. The techniques in the proof of Theorem 1.9 can be applied to deduce the structure of moduli spaces of any degree over any end-cylindrical 4-manifold with \( b^+ = 0 \) consisting of instantons asymptotic away from the singular points along the end. The picture will be a space stratified by the type of stabilizers on both the 4-manifold and its asymptotic 3-manifold. The irreducible locus will be a smooth manifold of dimension given by the index of the deformation complex, and the reducible locus will admit a local cone bundle neighborhood. The case we considered in Theorem 1.9 is described completely due to the fact that the dimension of the irreducible locus is low, thus after perturbations, the asymptotic values avoid the singular points.

Outline. Here we give an outline of this article. Section 2 introduces the set-up for the moduli space including the holonomy perturbations. Section 3 establishes the existence of the asymptotic value for the perturbed instantons, i.e. Theorem 1.7. Section 4 deduces the transversality of the irreducible moduli space and the asymptotic map. Section 5 is devoted to describing the reducible locus together with its neighborhood under small generic perturbations. Section 6 and Section 7 prove the surgery formula Theorem 1.2 and the excision formula Theorem 1.5 respectively.
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2. Preliminaries

2.1. The Unperturbed Moduli Space of Manifolds with Cylindrical End.

Definition 2.1. A Riemannian manifold \((Z, g)\) with cylindrical end consists of the following data:

(i) A compact manifold \(M \subset Z\) with boundary \(\partial M = Y\).

(ii) A cylindrical end \([0, \infty) \times Y\) is attached to the boundary \(\partial M\) so that \(Z = M \cup [0, \infty) \times Y\).

(iii) Over the end the metric \(g\) has the form \(g\big|_{[0, \infty) \times Y} = dt^2 + h\), where \(h\) is a metric on \(Y\), \(t\) is the coordinate function on \([0, \infty)\).

(iv) Over a collar neighborhood \((-1, 0] \times Y\) of \(\partial M\) in \(M\) the metric is identified as \(g\big|_{(-1, 0] \times Y} = dt^2 + h\).

Let \((Z, g)\) be a smooth manifold with cylindrical end as above. Consider the trivial \(C^2\)-bundle \(E \to Z\) satisfying

\[ E\big|_{[0, \infty) \times Y} = \pi^*E', \]

where \(E' \to Y\) is the trivial \(C^2\)-bundle on \(Y\), \(\pi: [0, \infty) \times Y \to Y\) is the projection map. Fix \(k \geq 3\), we write \(A_{k, loc}\) for the space of \(L^2_{k, loc} SU(2)\)-connections on \(E\). The gauge group \(G_{k+1, loc}\) consists of \(L^2_{k+1, loc}\) automorphisms of the associated principle bundle \(P\), which is identified with \(L^2_{k+1, loc}(Z, SU(2))\). The gauge action is given by

\[ G_{k+1, loc} \times A_{k, loc} \longrightarrow A_{k, loc} \]

\[ (u, A) \mapsto u \cdot A := A - u^{-1}dAu \]

We say \(A\) is irreducible if \(\text{Stab}(A) = \mathbb{Z}/2\), and reducible if \(\text{Stab}(A) \supset U(1)\). In particular when \(\text{Stab}(A) = SU(2)\), we say \(A\) is central. The configuration space \(A_{k, loc}\) decomposes into the irreducible and reducible parts:

\[ A_{k, loc} = A_{k, loc}^\text{irred} \sqcup A_{k, loc}^\text{red}. \]

Given a connection \(A\) with \(F_A \in L^2(Z, \Lambda^2 T^*Z \otimes \mathfrak{su}(2))\), we denote its Chern-Weil integral by

\[ \kappa(A) := \frac{1}{8\pi^2} \int_Z \text{tr}(F_A \wedge F_A). \]
We denote the energy of a connection $A$ by
\begin{equation}
\mathcal{E}(A) := \int_Z |F_A|^2.
\end{equation}

**Definition 2.2.** The unperturbed moduli space of finite energy instantons is defined to be
\[ \mathcal{M}_k(Z) := \{ A \in \mathcal{A}_{k,\text{loc}} : F_A^+ = 0, \mathcal{E}(A) < \infty, \kappa(A) = 0 \} / \mathcal{G}_{k+1,\text{loc}}. \]

**Remark 2.3.** Unlike the case of closed manifolds, any $SU(2)$-bundle over a manifold with cylindrical end is trivial. But this fact does not imply any ASD connection on $Z$ is flat. So we have imposed the vanishing of the Chern-Weil integral on the definition to ensure the setting is the same when the closed case as we run the neck-stretching argument later. In this way every connection $[A] \in \mathcal{M}_k(Z)$ is actually flat.

To simplify notation, we usually omit $k$ in the notation unless it becomes important. Note that the gauge action $\mathcal{G}$ on $\mathcal{A}$ is not free. To get rid of this issue one can consider the based moduli space defined as follows. Let’s fix a basepoint $z_0 = (0, y_0) \in \{0\} \times Y \subset Z$. Then the gauge group $\mathcal{G}$ acts on the fiber $E_{z_0}$, and acts freely on the product $\mathcal{M}(Z) \times E_{z_0}$. We define the based moduli space to be
\begin{equation}
\tilde{\mathcal{M}}(Z) := \{ (A, v) \in \mathcal{A} \times E_{z_0} : F_A^+ = 0, \mathcal{E}(A) < \infty, \kappa(A) = 0 \} / \mathcal{G}.
\end{equation}

Let $l = k - \frac{1}{2} \geq \frac{5}{2}$. Over the trivial $\mathcal{C}^2$-bundle $E' \rightarrow Y$, one can consider the space of $L^2 \mathcal{C}^2$-connections $\mathcal{A}_l(Y)$ and the gauge group $\mathcal{G}_{l+1}(Y)$. The moduli space of $\tilde{Y}$ consists of equivalence classes of flat connections on $E'$:
\begin{equation}
\tilde{\mathcal{M}}(Y) := \{ B \in \mathcal{A}_l(Y) : F_B = 0 \} / \mathcal{G}_{l+1}(Y).
\end{equation}

Usually we use $B$ to represent a generic connection on $E'$, and $\Gamma$ a generic flat connection. Via the holonomy morphism the moduli space $\tilde{\mathcal{M}}(Y)$ is identified with the character variety
\begin{equation}
\chi(Y) := \text{Hom}(\pi_1(Y), SU(2)) /\text{Ad}.
\end{equation}

Fixing a point $y_0 \in Y$, one can also define the based moduli space to be
\[ \tilde{\mathcal{M}}(Y) := \{ (B, v) \in \mathcal{A}(Y) \times E_{y_0} : F_B = 0 \} / \mathcal{G} \]

Via the holonomy morphism the based moduli space is identified with the representation variety
\begin{equation}
\mathcal{R}(Y) := \text{Hom}(\pi_1(Y), SU(2)).
\end{equation}

Over the cylindrical end $[0, \infty) \times Y$, the ASD equation is related to flat connections on $Y$ in the following way. Let’s fix a smooth flat connection $\Gamma_0$ on $E'$ as a reference connection. The Chern-Simons functional on $\mathcal{A}(Y)$ is
\begin{equation}
\text{cs}(B) = - \int_Y \text{tr} \left( \frac{1}{2} b \wedge d\Gamma_0 b + \frac{1}{3} b \wedge b \wedge b \right),
\end{equation}
where $B = \Gamma_0 + b$ with $b \in L^2_b(T^*Y \otimes \mathfrak{su}(2))$. It’s straightforward to compute that the formal gradient and Hessian are given respectively by
\begin{equation}
\text{grad } cs(B) = *F_B \quad \text{and} \quad \text{Hess } cs|_B(b) = *d_B b.
\end{equation}
Thus the critical points of the Chern-Simons functional consists of flat connections on $Y$. The restricted Hessian $*d_B|_{\ker d_B^*}$ has real, discrete, and unbounded spectrum (c.f. [14, Lemma 2.1.1]).

Given a gauge transformation $u \in G(Y)$ and $B \in \mathcal{A}(Y)$, we have
\begin{equation}
\text{cs}(u \cdot B) - \text{cs}(B) = -4\pi^2 \deg u.
\end{equation}
Thus the Chern-Simons functional descends to an $S^1$-valued function on the quotient space $B(Y)$.

Any connection $A$ on the cylindrical end $[0, \infty) \times Y$ has the form
\begin{equation*}
A = B(t) + \beta(t)dt,
\end{equation*}
where $\beta(t) \in L^2_b(Y, \mathfrak{su}(2))$ is a time-dependent $\mathfrak{su}(2)$-valued 0-form on $Y$. Then
\begin{equation}
F^+_A = \frac{1}{2}(*F_B + \dot{B} - d_B \beta) + dt \wedge (*F_B + \dot{B} - d_B \beta).
\end{equation}
Thus the ASD equation on the cylindrical end reads as
\begin{equation}
\dot{B} = -*F_B + d_B \beta.
\end{equation}
Thus up to gauge transformations the ASD equation on the cylindrical end is the downward gradient flow equation of the Chern-Simons functional. There are two useful methods to choose representatives for a gauge equivalence class. One way is to put $A|[0, \infty) \times Y$ in temporal gauge, i.e. trivialize the bundle $E|[0, \infty) \times Y$ via parallel transport of $A$ so that $A = B(t) + dt$. The other way is to restrict the analysis to a local slice given by a flat connection $\Gamma$ on $Y$. More precisely let’s fix a smooth flat connection $\Gamma \in \mathcal{A}(Y)$. Let
\begin{equation*}
\mathcal{K}_\Gamma := \ker d^*_\Gamma \subset L^2_b(T^*Y \otimes \mathfrak{su}(2)) \quad \text{and} \quad \mathcal{S}_\Gamma := \Gamma + \mathcal{K}_\Gamma \subset \mathcal{A}(Y).
\end{equation*}

**Definition 2.4.** We say a connection $A = B(t) + \beta(t) dt$ on $[0, \infty) \times Y$ in standard form with respect to $\Gamma$ if for all $t \in [0, \infty)$ one has
\begin{equation*}
B(t) \in \mathcal{S}_\Gamma \quad \text{and} \quad \beta(t) \in (\ker \Delta_\Gamma)^\perp,
\end{equation*}
where $\Delta_\Gamma := d^*_\Gamma d_\Gamma : L^2_b(Y, \mathfrak{su}(2)) \rightarrow L^2_{b-2}(Y, \mathfrak{su}(2))$ is the Laplacian twisted by $\Gamma$.

Let’s write $cs|_{\mathcal{S}_\Gamma}$ for the restriction on the Chern-Simons functional on the slice. Then we have the following result:

**Lemma 2.5.** [14] Lemma 2.5.1 Let $\Gamma$ be a smooth flat connection on $E'$. Then there exists a $\text{Stab}(\Gamma)$-invariant neighborhood $U_\Gamma$ of $\Gamma$ in $\mathcal{S}_\Gamma$ and a unique smooth $\text{Stab}(\Gamma)$-equivariant map $\Theta : U_\Gamma \rightarrow L^2_b(Y, \mathfrak{su}(2))$ such that for all $B \in U_\Gamma$ one has
\begin{equation*}
*F_B - d_B(\Theta(B)) \in \mathcal{K}_\Gamma \quad \text{and} \quad \Theta(B) \in (\ker \Delta_\Gamma)^\perp.
\end{equation*}
Furthermore the map $\Theta$ has the following properties. Let $B = \Gamma + b \in U_\Gamma$ be a connection in the slice neighborhood of $\Gamma$. 

(i) The formal gradient of \( cs_\Gamma \) at \( B \) is
\[
\text{grad } cs_\Gamma(B) = - * F_B + d_B(\Theta(B)).
\]

(ii) One has the bounds:
\[
\| \Theta(B) \|_{L^2_j} \leq c(l) \| b \|_{L^2_j} \| \text{grad } cs_\Gamma(B) \|_{L^2_{l-2}},
\]
where \( c(l) > 0 \) is a constant only depending on \( l \).

**Proof.** Only (ii) is different from Lemma 2.5.1 in [14]. However the argument there goes through without any change. We note that
\[
d^*_r d_B(\Theta(B)) = d^*_r(*F_B) = *\[b, F_B\].
\]
Due to the Sobolev multiplication \( L^2_j \times L^2_j \rightarrow L^2_j \) for any \( j \leq l \), we have
\[
\| d^*_r d_B(\Theta(B)) \|_{L^2_{l-2}} \leq \text{const.} \| b \|_{L^2_j} \| F_B \|_{L^2_{l-2}}.
\]
Note that \( d^*_r d_B : L^2_j \cap (\ker \Delta^*_\Gamma)^\perp \rightarrow L^2_{l-2} \) is uniformly invertible for \( B \) close to \( \Gamma \) in \( L^2_j \) norm, we conclude that
\[
\| \Theta(B) \|_{L^2_j} \leq \text{const.} \| b \|_{L^2_j} \| F_B \|_{L^2_{l-2}}.
\]
From (i) it follows that
\[
\| F_B \|_{L^2_{l-2}} \leq \| \text{grad } cs_\Gamma(B) \|_{L^2_{l-2}} + \| d_B(\Theta(B)) \|_{L^2_{l-2}}.
\]
Since \( d_B(\Theta(B)) = d^*_r(\Theta(B)) + [b, \Theta(B)] \), the Sobolev multiplication tells us that
\[
\| d_B(\Theta(B)) \|_{L^2_{l-2}} \leq \text{const.} (1 + \| b \|_{L^2_j}) \| \Theta(B) \|_{L^2_j}.
\]
In summary we have
\[
\| \Theta(B) \|_{L^2_j} \leq \text{const.} \| b \|_{L^2_j} \left( \| \text{grad } cs_\Gamma(B) \|_{L^2_{l-2}} + (1 + \| b \|_{L^2_j}) \| \Theta(B) \|_{L^2_j} \right)
\]
Thus when \( \| b \|_{L^2_j} \) is small, which can be achieved by shrinking \( U_\Gamma \), we get the desired estimate. \( \square \)

Uhlenbeck’s gauge fixing tells us that if the \( L^2 \)-norm of the curvature \( F_B \) is small, one can find a smooth flat connectoin \( \Gamma \) such that \( B \in \mathcal{S}_\Gamma \). Combining [14, Lemma 2.4.3] and the regularity result on ASD connections, we conclude that if the curvature of a connection \( A \) on a cylinder \([t_1, t_2] \times Y\) is small, one can find a gauge tranformation to tranform \( A \) into a standard form \( B(t) + \beta(t) dt \) with respect to a flat connection \( \Gamma \) on \( E' \). From the ASD equation (2.12), we see that
\[
d^*_r * F_B = d^*_r d_B \beta.
\]
Thus by Lemma 2.5 \( \beta(t) = \Theta(B(t)) \) and the ASD equation reads as
\[
\ddot{B}(t) = - \text{grad } cs_\Gamma(B(t)).
\]
2.2. Holonomy Perturbations. To get transversality for moduli spaces, we need to introduce perturbations. Since the manifolds we are interested in have $b_+ = 0$, we cannot use merely metric perturbations as in [13]. We adopt holonomy perturbations instead, following the lines in [4] and [9].

We start with an embedded ball $N \subset \mathbb{Z}$ and a smooth map $q : S^1 \times N \to \mathbb{Z}$ satisfying

(i) $q$ is a submersion;

(ii) $q(1, x) = x$ for any $x \in N$.

Given $x \in N$, we denote by $\text{Hol}_{A_x} \in SU(2)$ the holonomy of a connection $A$ around the loop $q(-, x)$. Denote by $\text{hol}_{A_x} \in \mathfrak{su}(2)$ the traceless part of $\text{Hol}_{A_x}$. We then get a section $\text{hol}_{A_x} \in C^{\infty}(\mathbb{Z}, \mathfrak{su}(2))$. Let $\omega \in \Omega^+(\mathbb{Z})$ be a self-dual 2-form supported on $N$. We form a $\mathfrak{su}(2)$-valued 2-form

$$V_{q,\omega} := \omega \otimes \text{hol}_{A} \in \Omega^{+}(\mathbb{Z}; \mathfrak{su}(2))$$

supported on $N \subset \mathbb{Z}$. The key estimates of $V_{q,\omega}$ are derived in [9].

**Proposition 2.6.** ([9, Proposition 3.1]) Given a submersion $q$, there exist constants $K_n$ such that for any $A \in \mathcal{A}_{k,\text{loc}}$, $\omega \in \Omega^+(\mathbb{Z})$ supported on $N$, one has

$$\|D^n V_{q,\omega}|_{A(a_1, ..., a_n)}\|_{L^2_k(N)} \leq K_n \|\omega\|_{C^k} \prod_{i=1}^n \|a_i\|_{L^2_k(N)},$$

where $D^n V_{q,\omega}$ is the $n$-th differential of $V_{q,\omega}$.

In particular we get a smooth $G_{k+1,\text{loc}}$-equivariant map

$$V_{q,\omega} : \mathcal{A}_{k,\text{loc}} \to L^2_{k,\text{loc}}(\mathbb{Z}, \Lambda^+ \otimes \mathfrak{su}(2)).$$

Now we take a countable family of embedded balls $\{N_\alpha\}_{\alpha \in \mathbb{N}}$ in $\mathbb{Z}$ together with submersions $\{q_\alpha\}_{\alpha \in \mathbb{N}}$ satisfying the condition that for any $x \in \mathbb{Z}$, the countable family of maps

$$\{q_\alpha(-, x) : \alpha \in \mathbb{N}, x \in \text{Int}(N_\alpha)\}$$

is $C^1$-dense in the space of smooth loops in $\mathbb{Z}$ based at $x$.

**Definition 2.7.** Let $\{q_\alpha\}$ be a family of submersions as above. Given $\mu > 0$, the space $\mathcal{P}_\mu$ of holonomy perturbations consists of perturbations of the form

$$\sigma_\omega = \sum_\alpha V_{q_\alpha,\omega_\alpha},$$

where $\omega = \{\omega_\alpha\}$ is a family of self-dual 2-form supported on $N_\alpha$ satisfying the following.

1. Denote by $C_\alpha := \sup\{K_{n,\alpha} : 0 \leq n \leq \alpha\}$, where $K_{n,\alpha}$ is the constant for $q_\alpha$ in Proposition 2.6. Then

$$\sum_\alpha C_\alpha \|\omega_\alpha\|_{C^k}$$

converges.
(2) For $A \in \mathcal{A}_{k,loc}$, one has
\[
\|\nabla^j \sigma(A)\|_{(t) \times Y} \leq C_j \epsilon^{-\mu t} \sum_{\alpha} C_{\alpha} \|\omega_{\alpha}\|_{C^k}, \quad j \leq k,
\]
where $C_j > 0$ only depends on and $\{q_{\alpha}\}$ and $j$.

When there is no confusion, we simply write $\sigma$ for $\sigma_\omega$. We denote by $W_\mu$ the space of sequences $\omega = \{\omega_{\alpha}\}$ satisfying (1) and (2) in Definition 2.7. Note that $W$ forms a Banach space with respect to the norm
\[
\|\omega\|_W := \sum_{\alpha} C_{\alpha} \|\omega_{\alpha}\|_{C^k}.
\]
Since $\sigma$ depends linearly on $\omega$, $P_\mu$ is also a Banach space. Moreover each $\sigma \in P_\mu$ gives rise to a smooth $G_{k+1,loc}$-equivariant map $\sigma : \mathcal{A}_{k,loc} \rightarrow L^2_{k,\mu}(Z, \Lambda^+ \otimes \mathfrak{su}(2))$.

Given $\sigma \in P_\mu$, we define the perturbed moduli space of finite energy instantons to be
\[
\mathcal{M}_\sigma(Z) := \{ A \in \mathcal{A}_{k,loc} : F_\sigma = \sigma(A), \mathcal{E}(A) < \infty, \kappa(A) = 0 \}/G_{k+1,loc}.
\]

3. The Asymptotic Map

In this section we deduce the asymptotic behaviors of the perturbed moduli space $\mathcal{M}_\sigma(Z)$ by modifying the corresponding arguments in [13].

Definition 3.1. Let $\Gamma$ be a smooth flat connection on $E'$. Denote by $\mu_\Gamma$ the smallest nonzero absolute value of eigenvalues of the restricted Hessian $*d_\Gamma|_{\ker d_\Gamma}$.

Since $\chi(Y)$ is compact, and $\mu_\Gamma$ is gauge invariant, we can fix a finite number $\mu > 0$ satisfying
\[
\mu \geq \max\{\mu_\Gamma : \Gamma \text{ is a smooth flat connection}\}.
\]

Given $A \in \mathcal{A}_{k,loc}$, we write
\[
\sigma(A)|_{(t) \times Y} = *\rho_A(t) + dt \land \rho_A(t),
\]
where $\rho_A(t)$ is a $\mathfrak{su}(2)$-valued 1-form on $Y$. Suppose $A|_{[0,\infty) \times Y} = B(t) + \beta(t)dt$ is in standard form with respect to $\Gamma$. Then the perturbed ASD equation restricted on the end $[0, \infty) \times Y$ reads as
\[
\dot{B}(t) = -* F_B(t) + d_B \beta(t) + 2 \rho_A(t).
\]

The decay condition in Definition 2.7 for $\sigma_\omega \in P_\mu$ implies $\rho_A(t)$ decays exponentially on the end as well:
\[
\|\rho_A(t)\|_{L^j_{\infty}(Y)} \leq c_0(j)\|\omega\|_W \cdot e^{-\mu t}, \quad j \leq l,
\]
where the constant $c_0 > 0$ depends on neither $A$ nor $\omega$.

Note that in the non-perturbed case, once $A$ is in standard form we identify $\beta(t) = \Theta(B(t))$, thus the equation has the form of the downward gradient flow
equation. In the perturbed case, this is no longer true. However we still have estimates on how far $\beta(t)$ is from $\Theta(B(t))$.

**Lemma 3.2.** Suppose $A|_{[0,\infty)} = B(t) + \beta(t)dt$ is the restriction of a perturbed ASD connection $A$ on $Z$ such that $A$ is in standard form with respect to $\Gamma$ and $B(t) \in U_\Gamma$ as in Lemma 2.5. Then

$$\|\beta(t) - \Theta(B(t))\|_{L^2_j} \leq 2\alpha(j - 2)\|\omega\|_{W^\mu t}, \ j \leq l.$$  

**Proof.** We note that the perturbed ASD equation gives us

$$\dot{B} = -\operatorname{grad} cs_\Gamma(B) + d_B(\beta(t) - \Theta(B(t))) + 2\rho_A(t).$$

By construction $\beta(t) - \Theta(B(t)) \in (\ker \Delta_\Gamma)^\perp$ and $\dot{B} + \operatorname{grad} cs_\Gamma(B) \in \ker d^*_\Gamma$. Since $d^*_\Gamma d_B : L^2_t \cap (\ker \Delta_\Gamma)^\perp \to L^2_{t-2}$ is uniformly invertible for $B$ close to $\Gamma$ in $L^2_t$ norm, we have

$$\|\beta(t) - \Theta(B(t))\|_{L^2_j} \leq \text{const.} \|d^*_\Gamma d_B \beta(t) - \Theta(B(t))\|_{L^2_{j-2}}$$

$$\leq \text{const.} \|p_A(t)\|_{L^2_{j-2}}$$

$$\leq 2\alpha_0(j - 2)\|\omega\|_{W^\mu t}.$$  

To simplify notations, we write

$$(3.4) \quad p_A(t) := d_B(\beta(t) - \Theta(B(t))) + 2\rho_A(t).$$

Then the perturbed ASD equation over the end in standard form reads as

$$(3.5) \quad \dot{B}(t) = -\operatorname{grad} cs_\Gamma(B(t)) + p_A(t),$$

where the extra perturbation term satisfies

$$(3.6) \quad \|p_A(t)\|_{L^2(Y)} \leq \alpha_0\|\omega\|_{W^\mu t}$$

for some constant $\alpha_0 > 0$ independent of $A$ and $\omega$.

To derive the convergence of a perturbed ASD connection $[A] \in \mathcal{M}_\rho(Z)$, we start with a sequence of lemmas establishing the estimate for the length of the flowline $B(t)$ corresponding to $A$.

**Lemma 3.3.** [13, Proposition 4.2.1] Let $\Gamma$ be a smooth flat connection on $E'$. Then there exists a neighborhood $U_\Gamma \subset S_\Gamma$ of $\Gamma$, and constant $\theta \in (0, \frac{1}{2}]$ so that for any connection $B \in U_\Gamma$ one has

$$(3.7) \quad |cs(B) - cs(\Gamma)|^{1-\theta} \leq \|\operatorname{grad} cs_\Gamma(B)\|_{L^2(Y)}.$$  

This is a Łojasiewicz type inequality in the infinite dimensional setting originally proved by Simon [18].

**Lemma 3.4.** Let $\Gamma, U_\Gamma$, and $\theta$ be as in Lemma 3.3. Let $A$ be a perturbed instanton on $E$ of the form in (3.3) such that $B(t) \in U_\Gamma$ for $t \in [t_1, t_2]$. Suppose

$$(3.8) \quad \|p_A(t)\|_{L^2} \leq \alpha \|\operatorname{grad} cs_\Gamma(B(t))\|_{L^2},$$
for some constant $\alpha \in (0, 1)$. Then

$$\int_{t_1}^{t_2} \| \dot{B}(t) \|_{L^2} dt \leq \frac{1}{\theta^2 \sqrt{1 - \alpha^2}} | \mathrm{cs}(B(t_1)) - \mathrm{cs}(B(t_2)) |^\theta .$$

**Proof.** From (3.5) we get

$$\| \dot{B}(t) \|^2_{L^2} + \| \text{grad} \mathrm{cs}_\Gamma(B(t)) \|^2_{L^2} = -2 \langle \dot{B}(t), \text{grad} \mathrm{cs}_\Gamma(B(t)) \rangle + \| p_A(t) \|^2_{L^2} \leq -2 \langle \dot{B}(t), \text{grad} \mathrm{cs}_\Gamma(B(t)) \rangle + \alpha^2 \| \text{grad} \mathrm{cs}_\Gamma(B(t)) \|^2_{L^2}.$$ 

Thus

$$2 \sqrt{1 - \alpha^2} \| \dot{B}(t) \| \cdot \| \text{grad} \mathrm{cs}_\Gamma(B(t)) \| \leq \| \dot{B}(t) \|^2 + (1 - \alpha^2) \| \text{grad} \mathrm{cs}_\Gamma(B(t)) \|^2 \leq -2 \langle \dot{B}(t), \text{grad} \mathrm{cs}_\Gamma(B(t)) \rangle = -2 \frac{d}{dt} \mathrm{cs}(B(t)).$$

In particular we conclude that Chern-Simons functional $\mathrm{cs}$ is non-increasing along a path solving (3.5). Let’s assume that $\mathrm{cs}(B(t_1)) > \mathrm{cs}(B(t_2)) > \mathrm{cs}(\Gamma)$. The other cases can be proved similarly. Then applying Lemma 3.3 we get

$$-\frac{d}{dt} \left( \mathrm{cs}(B(t)) - \mathrm{cs}(\Gamma) \right) \geq \theta \sqrt{1 - \alpha^2} \| \dot{B}(t) \| \| \text{grad} \mathrm{cs}_\Gamma(B(t)) \| \geq \theta \sqrt{1 - \alpha^2} \| \dot{B}(t) \|.$$ 

Thus integrating both sides we get

$$\int_{t_1}^{t_2} \| \dot{B}(t) \| dt \leq \frac{1}{\theta^2 \sqrt{1 - \alpha^2}} \left( (\mathrm{cs}(B(t_1)) - \mathrm{cs}(\Gamma))^\theta - (\mathrm{cs}(B(t_2)) - \mathrm{cs}(\Gamma))^\theta \right),$$

$$\leq \frac{1}{\theta^2 \sqrt{1 - \alpha^2}} | \mathrm{cs}(B(t_1)) - \mathrm{cs}(B(t_2)) |^\theta .$$

$\square$

Although the Chern-Simons functional $\mathrm{cs}$ do not necessarily decay along the path $B(t)$ corresponding to a perturbed ASD connection $A$, one can show after adding a term of exponential decay it’s decreasing.

**Lemma 3.5.** Let $\Gamma, U_\Gamma$, and $\theta$ be as in Lemma 3.3. Let $A$ be a perturbed instanton on $E$ of the form in (3.5) such that $B(t) \in U_\Gamma$ for $t \in [t_1, t_2]$. Then the function

$$\mathrm{cs}_A(t) := \mathrm{cs}(B(t)) + \frac{c_0^2 \| \omega \|^2_{W^2}}{2\mu} e^{-2\mu t}$$

is non-increasing.
Proof. We compute that
\[
- \frac{d}{dt} cs_A(t) = - \langle \dot{B}(t), \text{grad } cs(B(t)) \rangle + c_0^2 \| \omega \|_{W}^2 e^{-2\mu t}
\]
\[
\geq - \langle \dot{B}(t), \text{grad } cs(B(t)) \rangle + \| p_A(t) \|_{L^2}^2
\]
\[
= \frac{1}{2} (\| \dot{B}(t) \|_{L^2}^2 + \| \text{grad } cs(B(t)) \|_{L^2}^2)
\]
\[
\geq \| \dot{B}(t) \|_{L^2} \cdot \| \text{grad } cs(B(t)) \|_{L^2}
\]
\[
\geq 0.
\]

□

Now we give an estimate of the length of a perturbed flowline \( B(t) \), which is based on discussion with Cliff Taubes.

**Proposition 3.6.** Let \( \Gamma, U_\Gamma, \) and \( \theta \) be as in Lemma 3.3. Let \( A \) be a perturbed instanton on \( E \) of the form in (3.5) such that \( B(t) \in U_\Gamma \) for \( t \in [0, \infty) \). Then one can find \( T_1 > 0 \) such that

1. either
\[
\int_{T_1+1}^{\infty} \| \dot{B}(t) \| dt \leq c'_1 (\| F_A \|_{L^2([T_1+1, \infty) \times Y)}^\theta + e^{-2\mu \theta T_1}),
\]
where \( c'_1 \) only depends on \( \theta \) and \( \mu \);
2. or
\[
\int_{T_1+1}^{\infty} \| \dot{B}(t) \|_{L^2} dt \leq c''_1 e^{-\mu \theta T_1},
\]
where \( c''_1 \) only depends on \( c_0, \| \omega \|_{W}, \mu, \theta \) and \( \| F_A \|_{L^2([T_1+1, \infty) \times Y)} \).

Proof. We first explain how the time \( T_1 \) comes into the picture. By the standard elliptic theory for ASD connections, see for our case explicitly [14, Lemma 3.5.1], there is a positive number \( c_0 > 0 \) such that whenever a perturbed ASD connection \( A \) on \( E \) satisfies \( \| F_A \|_{L^2([T_1, \infty) \times Y)} < c_0 \), one has
\[
\| F_B(s) \|_{L^2}^2 \leq \text{const.} (\| F_A \|_{L^2([T_1, \infty) \times Y)}^2 + e^{-2\mu s}), \text{ when } s \geq T_1 + 1.
\]
The finiteness of \( \| F_A \|_{L^2(Z)} \) guarantees us the existence of such a \( T_1 \).

Now consider the following two cases:

1. \( \| p_A(t) \|_{L^2} \leq \frac{1}{2} \| \text{grad } cs(B(t)) \|_{L^2}. \)
2. \( \| p_A(t) \|_{L^2} \geq \frac{1}{2} \| \text{grad } cs(B(t)) \|_{L^2}. \)
If (1) holds for all $t \in [0, \infty)$, then Lemma 3.4 tells us that for any $T' > T_1 + 1$ we have
\[
\int_{T_1+1}^{T'} \|\dot{B}(t)\| \, dt \leq \text{const.} (\text{cs}(B(T_1)) - \text{cs}(B(T')))^\theta
\]
\[
= \text{const.} \left( \frac{1}{2} \int_{T_1+1}^{T'} \int_Y \text{tr}(F_A \wedge F_A) \right)^\theta
\]
\[
= \text{const.} \left( \frac{1}{2} \int_{T_1+1}^{T'} \int_Y |F_A|^2 + 2 \text{tr}(F_A^+ \wedge F_A^+) \right)^\theta
\]
\[
\leq \text{const.} (\|F_A\|^\theta_{L^2([T_1+1, T'] \times Y)} + e^{-2\mu T_1}).
\]
We conclude that
\[
\int_{T_1+1}^{\infty} \|\dot{B}(t)\| \, dt \leq \epsilon_1 (\|F_A\|^\theta_{L^2([T_1+1, \infty) \times Y)} + e^{-2\mu T_1}).
\]

If (2) holds for all $t \in [0, \infty)$, then the exponential decay on both $p_A(t)$ and $\text{grad \, cs}_\Gamma(B(t))$ implies the result.

Now we may assume (2) holds at $t = T_1$. Let $[a_0, b_0], \ldots, [a_n, b_n], \ldots$ be a sequence of intervals with integer end points such that

(i) $a_0 > T_1$, and $a_i > b_{i-1}$ for $i \geq 1$.

(ii) (1) holds for all $t \in [a_i, b_i]$, $i \geq 0$.

(iii) For any $k$ with $b_i \leq k < a_{i+1}$, there exists $t_k \in [k, k+1]$ such that (2) holds at $t = t_k$.

We need to estimate the length of $B(t)$ over $[a_i, b_i]$ and $[b_i, a_{i+1}]$ respectively.

Step 1. Let’s first consider the case over $[a_i, b_i]$ where (1) always holds. We choose
\[
t_{a_i} = \max \{ t \in [a_i - 1, a_i] : (2) \text{ holds at } t \}
\]
and
\[
t_{b_i} = \min \{ t \in [b_i, b_i + 1] : (2) \text{ holds at } t \}.
\]

From Lemma 3.3 and Lemma 3.3 we know that
\[
\int_{t_{a_i}}^{t_{b_i}} \|\dot{B}(t)\| \, dt \leq \frac{2}{\sqrt{3\theta}} (|\text{cs}(B(a_i)) - \text{cs}(\Gamma)|^\theta + |\text{cs}(B(b_i)) - \text{cs}(\Gamma)|^\theta)
\]
\[
\leq \frac{2}{\sqrt{3\theta}} (\|\text{grad \, cs}_\Gamma(B(t_{a_i}))\|^{\theta/\theta} + \|\text{grad \, cs}_\Gamma(B(t_{b_i}))\|^{\theta/\theta})
\]
\[
\leq \text{const.} (e^{-\mu a_i \cdot \theta/\theta} + e^{-\mu b_i \cdot \theta/\theta})
\]
\[
\leq \text{const.} e^{-\frac{\mu}{\theta/\theta}} (a_i - 1)
\]

Step 2. Next we consider the case over $[b_i, a_{i+1}]$. Let $k$ be an integer in $[b_i, a_{i+1}]$, and $t_k \in [k, k+1]$ such that (2) holds at $t = t_k$. Applying the Cauchy-Schwarz
inequality we get
\[
\int_{t_k}^{k+2} \|\dot{B}(t)\| dt \leq \sqrt{2} \int_{t_k}^{k+2} \langle \dot{B}(t), -\text{grad} cs_F(B(t)) + p_A(t) \rangle dt \\
\leq \sqrt{2} \left( |cs(B(t_k)) - cs(B(k + 2))|^{\frac{3}{2}} + |\text{grad} cs(B(k + 2)) + c_s(G)|^{\frac{3}{2}} + e^{-\mu t_k} \right) \\
+ \int_{t_k}^{k+2} \langle -\text{grad} cs_F(B(t)) + p_A(t), p_A(t) \rangle dt^{\frac{1}{2}}.
\]
(3.9)

Corollary 2.5.2 in [14] tells us that when \( B \in U_T \) (one may shrink \( U_T \) in the first place to apply the result). We continue with (3.9) to get
\[
\int_{t_k}^{k+2} \|\dot{B}(t)\| dt \leq \text{const.} \left( |cs(B(t_k)) - cs(G)|^{\frac{3}{2}} + |cs(B(k + 2)) - cs(G)|^{\frac{3}{2}} + e^{-\mu t_k} \right) \\
\leq \text{const.} \left( \|\text{grad} cs_F(B(t_k))\| \|\mu\|^{\frac{1}{2}} + e^{-\mu t_k} \right) \\
\leq \text{const.} (e^{-\mu t_k} + e^{-\mu T} + e^{-\mu t_k}),
\]
where the second inequality uses Lemma [3.5] to bound \( cs(B(k + 2)) - cs(G) \) via \( cs(B(t_k)) - cs(G) \).

Combining Step 1 and Step 2 we conclude that if neither of 1 nor 2 holds for all \( t \in [0, \infty) \), one has
\[
\int_{T+1}^{\infty} \|\dot{B}(t)\| dt \leq \text{const.} \sum_{n=0}^{\infty} e^{-\mu t_k} + e^{-\mu t_n} + e^{-\mu t_k} + e^{-\mu n} \\
\leq c_1'' e^{-\mu T},
\]
where the constant \( c_1'' \) only depends on \( c_0, \|\omega\|_{W}, \mu, \) and \( \|F_A\|_{L^2([T, T+1] \times Y)} \).

With the estimate of the length of the perturbed gradient flowlines, we are able to obtain the existence of the asymptotic map from \( \mathcal{M}_\sigma(Z) \) to the character variety \( \chi(Y) \).

**Theorem 3.7.** Let \( A \) be a finite energy perturbed ASD connection on the bundle \( E \to Z \) of a manifold with cylindrical end \( [0, \infty) \times Y \). Given \( v \in E_{y_0} \) we get a path \( v(t) \in E'_{y_0} \) by parallel transporting \( v \) via \( A \) along the path \( [0, \infty) \times \{y_0\} \). Let \( \tilde{B}(t) = A(t)_{|_{Y \times X}} \). Then the path of equivalence classes \([B(t), v(t)]\) has a limit \([B_0, v_0] \in \mathcal{R}(Y)\) . This defines a continuous \( SU(2) \)-equivariant map
\[
\tilde{\partial}_+: \tilde{\mathcal{M}}_\sigma(Z) \to \mathcal{R}(Y).
\]

**Proof.** This is the main result of Chapter 4 in [14] where they have given a complete proof in the case when one adopts metric perturbations. We sketch the proof here and point out the modification we need in our case. For each flat connection \( \Gamma \) on \( E' \), one choose a neighborhood \( U_T \subset S_T \) such that Proposition [3.6] holds. Since
$\mathcal{R}(Y)$ is compact, it follows from Uhlenbeck’s gauge fixing that one can choose $\epsilon_0 > 0$ such that whenever $\|F_B\|_{L^2(Y)} < \epsilon_0$ for a connection $B$ on $E'$, one can find a gauge transformation $u$ such that $u \cdot B \in U_{T_0}$.

Given a perturbed ASD connection $A$ on $E$, we write $B(t) = A|_{Y_t}$. Since $A$ has finite energy, one can find $T_1$ and $\epsilon_1$ such that $\|F_A\|_{L^2([T_1, \infty) \times Y)} < \epsilon_1$ and $\|F_{B(t)}\|_{L^2(Y)} < \epsilon_0$ due to the elliptic theory of ASD connections as mentioned in the proof of Proposition 3.6. Thus $B(T_1) \in U_{T_0}$ for some flat connection $\Gamma$. Following the proof of Proposition 4.3.1 in [14], the length estimate in Proposition 3.6 implies that after gauge transformations $A|_{[T_1, \infty) \times Y}$ is in standard form with respect to $\Gamma$ and $B(t) \in U_{T_0}$ for all $t > T_1$ by possibly choosing larger $T_1$ and smaller $\epsilon_1$. Due to the non-increasingness of the modified function $cs_A(t)$, we know $\lim_{t \to \infty} cs(B(t))$ exists. Again we can shrink $U_{T_0}$ if necessary so that $U_{T_0} \cap U_{T_0'} = \emptyset$ if $cs(\Gamma) = cs(\Gamma')$ and $[\Gamma]$, $[\Gamma']$ are in different component. Thus the limit set of $[B(t)]$ is contained in a single component of $\chi(Y)$. After gauge transformations, $B(t)$ now has finite length, which implies that the limit set has to be a point. In this way we get a map

$$\partial_+ : \mathcal{M}_\sigma(Z) \longrightarrow \chi(Y).$$

The continuity of this map follows from the argument in [14] Page 71] verbatimly. The proof of the based version $\tilde{\partial}_+$ is the same as that of the original version in [14] Theorem 4.6.1] once the existenc of $\partial_+$ is established. 

After establishing the existence of the asymptotic map, we can apply the Uhlenbeck’s compactness argument to the perturbed ASD connections as in [9]. Since the bundle $E$ in our case is trivial, there is no bubbling nor energy escape. Thus we obtain the compactness of $\mathcal{M}_\sigma(Z)$.

**Corollary 3.8.** The perturbed moduli space $\mathcal{M}_\sigma(Z)$ is compact.

To study the behavior of a perturbed flowline $[B(\cdot)]$ given by an instanton $[A]$ asymptotic to a limit $[\Gamma_A] \in \chi(Y)$, we recall the notion of center manifolds in [14] Definition 5.1.2].

**Definition 3.9.** Let $H = H_0 \oplus H_0^\perp$ be an orthogonal decomposition of a Hilbert space $H$ with $H_0$ a finite dimensional subspace. Let $U \subset H$ be a neighborhood of $0$ in $H$, and $\nu : U \to H$ a vector field over $U$. A $C^k$-center manifold for the pair $(U, \nu)$ is a submanifold $\mathcal{H} \subset H$ given by the graph of a $C^k$-map $f : U_0 \to H_0^\perp$, where $U_0 \subset H_0$ is a neighborhood of $0$ in $H_0$ satisfying the following conditions:

1. $\mathcal{H} \subset U$, and $T_0\mathcal{H} = H_0$.
2. $\nu(x, f(x)) \in T_{(x, f(x))}\mathcal{H}$ for any $x \in U_0$.
3. $\text{Crit}(\nu) \cap U' \subset \mathcal{H}$, where $U' \subset U$ is a smaller open neighborhood of $0$ in $H$. Here $\text{Crit}(\nu)$ is a set of critical points of the vector field $\nu$.

Roughly speaking a center manifold is a finite-dimensional submanifold that is locally preserved by the flow of $\nu$ and contains all nearby critical points. Now we take $\Gamma$ to be a smooth flat connection on $Y$ with a neighborhood $U_{\Gamma}$ in $\mathcal{S}_\Gamma$.
satisfying Lemma 2.5. We take $U \subset K_\Gamma$ so that $U_\Gamma = \{ \Gamma \} + U$. The deformation complex at $\Gamma$ is given by

$$L^2_{t+1}(su(2)) \xrightarrow{dt} L^2_t(T^*Y \otimes su(2)) \xrightarrow{dt} L^2_{t-1}(\Lambda^2 T^*Y \otimes su(2)),$$

We identify $H^1(Y; ad \Gamma) = \text{ker} \, d_t \cap \text{ker} \, d^*_t$ as a subspace in $K_\Gamma$. Let $H^1_\Gamma \subset K_\Gamma$ be the $L^2$-orthogonal complement of $H^1(Y; ad \Gamma)$. Corollary 5.1.4 in [14] ensures the existence of a $\text{Stab}(\Gamma)$-invariant $C^2$-center manifold $\mathcal{H}_\Gamma$ for the pair $(U, - \text{grad} \, cs_\Gamma)$. We write $W^s_\Gamma \subset \mathcal{H}_\Gamma$ for the stable set of $- \text{grad} \, cs_\Gamma$ on the center manifold, i.e. for any $B \in W^s_\Gamma$ the flowline of $- \text{grad} \, cs_\Gamma$ starting at $B$ converges to some point in $\mathcal{H}_\Gamma$. The most important property of the center manifold is that any perturbed ASD connection on the end can be approximated exponentially closely by a gradient flowline on the center manifold after a sufficiently long time.

**Proposition 3.10.** Let $\mathcal{H}_\Gamma$ be a center manifold of $\Gamma$ with respect to $(U, \text{grad} \, cs_\Gamma)$. Let $[A] \in \mathcal{M}_{\sigma_\omega}(Z)$ with $|\omega|_W \leq 1$ such that $A = B(t) + \beta(t)dt$ is in standard form with respect to $\Gamma$ over the end $[0, \infty) \times Y$. Then there exists a neighborhood $V_\Gamma \subset U_\Gamma$ of $\Gamma$, and positive constants $T_\Gamma, \epsilon_\Gamma, \kappa_\Gamma > 0$ so that the following are satisfied.

(i) $B(t) \in V_\Gamma$ for all $t \geq T_\Gamma - 1$.

(ii) $\|F_A\|_{[T_\Gamma - 1, \infty) \times Y} \|L^2 < \epsilon_\Gamma$.

(iii) There is a unique downward gradient flowline, $B_\Gamma : [T_\Gamma - 1, \infty) \rightarrow \mathcal{H}_\Gamma$, of $cs_\Gamma$ on the center manifold of the following property. The ASD connection $A_\Gamma = B_\Gamma(t) + \Theta(B_\Gamma(t))dt$ induced from $B_\Gamma$ satisfies

$$\|A - A_\Gamma\|_{L^2_1([t - \frac{1}{2}, t + \frac{1}{2}] \times Y)} < \kappa_\Gamma e^{-\frac{\mu}{2}(t - T_\Gamma)}, \forall t \geq T_\Gamma,$$

where $\mu_\Gamma$ is the smallest nonzero absolute value of eigenvalues of the Hessian $*d\Gamma|_{\text{ker} \, d^*_t}$.

**Proof.** This is Theorem 5.2.2 in [14] when one adopts metric perturbations. Its proof carries through our case without any change. From the finite length of $B(t)$ and finite energy of $A$, (1) and (2) follow immediately. Over $[T_\Gamma - 1, \infty) \times Y$, we decompose $A = \Gamma + b(t) + c(t) + \beta(t)dt$, where $b(t) \in \mathcal{H}_\Gamma$, $c(t) \in H^1_\Gamma$. Lemma 5.4.1] tells us that

$$\|c(t)\|_{L^2} \leq \text{const.} e^{-\frac{\mu}{2}(t - T_\Gamma)}.$$  

From (3.5) and $\mu > \frac{\mu_\Gamma}{2}$, we conclude that

$$\|b(t) + \text{grad} \, cs_\Gamma(B(t))\|_{L^2} \leq \text{const.} e^{-\frac{\mu_\Gamma}{2}(t - T_\Gamma)}$$

Then [14] Lemma 5.3.1] gives us the unique gradient flowline $B_\Gamma = \Gamma + b_\Gamma : [T_\Gamma - 1, \infty) \rightarrow \mathcal{H}_\Gamma$ such that

$$\|b_\Gamma(t) - b(t)\|_{L^2} \leq \text{const.} e^{-\frac{\mu_\Gamma}{2}(t - T_\Gamma)}.$$ 

From Lemma 2.5] and Lemma 3.2 we get

$$\|\beta(t) - \Theta(B_\Gamma(t))\|_{L^2} \leq \|\beta(t) - \Theta(B(t))\|_{L^2} + \|\Theta(B) - \Theta(B_\Gamma)\|_{L^2} \leq \text{const.} e^{-\frac{\mu_\Gamma}{2}(t - T_\Gamma)}.$$
Now the result follows from the standard bootstrapping argument, see [14] Lemma 3.3.2 for our particular case.

\[ \square \]

**Remark 3.11.** Note that \( T_\Gamma, \epsilon_\Gamma, \kappa_\Gamma \) depend continuously on \([\Gamma]\) and \([A]\). The compactness of \( \chi(Y) \) and \( \mathcal{M}_\sigma(Z) \) implies that those parameters can be chosen uniformly, which we denote by \( T_0, \epsilon_0, \) and \( \kappa_0 \) respectively. We also choose \( V_\Gamma \) for each \( \Gamma \) uniform to all \([A] \in \mathcal{M}_\sigma(Z)\) with \( ||\omega||_W \leq 1 \). However there is certain dependence among \( V_\Gamma \), \( T_\Gamma \), and \( \epsilon_\Gamma \). Choosing \( \epsilon_0 \) smaller forces \( T_0 \) larger, which in turn enables us to shrink \( V_\Gamma \) to ensure the estimates hold. In this way, we can choose three neighborhoods \( V_\Gamma \subset V'_\Gamma \subset U_\Gamma \) so that all perturbed ASD connections \([A]\) would enter \( V_\Gamma \) at \( T_0 \) and stay within \( V'_\Gamma \) ever since for some \( \Gamma \). Meanwhile the center manifold \( \mathcal{H}_\Gamma \) is defined inside \( U_\Gamma \).

We write \( \mathcal{M}_\sigma(Z, V_\Gamma) \) for the set of equivalence classes \([A]\) of ASD connections on \( Z \) for which one can pick up a gauge class representative \( A \) whose restriction on the end \([T_0, \infty) \times Y\) satisfies Proposition 3.10. Then the assignment

\[ Q_\Gamma : \mathcal{M}_\sigma(Z, V_\Gamma) \longrightarrow W^s_\Gamma \]

\[ [A] \mapsto B_\Gamma(T_0) \tag{3.13} \]

defines a continuous map following the same argument as in [14] Proposition 5.2.2. Note that any two gauge transformations transforming the restriction of \( A \) on the end into a connection of standard form with respect to \( \Gamma \) differ by a constant gauge transformation in Stab(\( \Gamma \)), thus the map \( Q_\Gamma \) is well-defined. The map \( Q_\Gamma \) refines the asymptotic map \( \partial^+ \) in the sense that \( \partial^+ \) is the composition of \( Q_\Gamma \) with the map sending \( B_\Gamma(T_0) \) to the limit point in \( V_\Gamma \) following the downward gradient flowline of \( cs_\Gamma \). We also have the based version

\[ \tilde{Q}_\Gamma : \tilde{\mathcal{M}}_\sigma(Z, V_\Gamma) \to W^s_\Gamma \times_{\text{Stab}\Gamma} E'_{y_0}. \tag{3.14} \]

4. **Transversality on the Irreducible Moduli Space**

As considered by Morgan-Mrowka-Ruberman in [14], to improve the regularity of the map \( Q_\Gamma \) one can first embed the moduli space \( \mathcal{M}_\sigma(Z, V_\Gamma) \) into a larger one which they refer to as a thickened moduli space, then prove transversality results there.

The thickening moduli space is defined with the help of thickening data about smooth flat connections on \( E' \), which we now recall from [14]. The motivation for introducing the thickening data is to resolve the issue that the gradient vector field \( \text{grad} \ cs_\Gamma \) is incomplete. So one artificially truncates this vector field via a cut-off function. In this way all the local properties near \( \Gamma \) are preserved, and one can apply analysis tools without worrying about the incompleteness.

**Definition 4.1.** Let \( \Gamma \) be a smooth flat connection on \( E' \). We choose

(a) a center manifold \( \mathcal{H}_\Gamma \) of \( \Gamma \),
(b) neighborhoods \( V_\Gamma \subset V'_\Gamma \subset U_\Gamma \) as in Remark 3.11.
a cut-off function \( \varphi : H_\Gamma \rightarrow [0, 1] \) such that \( \varphi \equiv 1 \) on \( V'_\Gamma \cap H_\Gamma \) and 
\( \text{supp} \varphi \subset U'_\Gamma \cap H_\Gamma \) for some \( U'_\Gamma \subset U_\Gamma \).

We refer to the triple \( \mathcal{T}_\Gamma = (H_\Gamma, V_\Gamma, \varphi_\Gamma) \) as a set of thickening data about \( \Gamma \).

Given a thickening triple \( \mathcal{T}_\Gamma \), we write
\[
H_\Gamma^{out} := \varphi_\Gamma^{-1}(0, 1) \quad \text{and} \quad H_\Gamma^{in} := \varphi_\Gamma^{-1}(1).
\]
We denote by
\[
\Xi_\Gamma^{tr} := - \varphi_\Gamma \cdot \text{grad} \; cs_\Gamma |_{H_\Gamma}
\]
the truncated downward gradient vector field over the center manifold \( H_\Gamma \). Then \( \Xi_\Gamma^{tr} \) is a complete vector field over \( H_\Gamma \) despite that \( H_\Gamma \) is only defined near \( \Gamma \). For each \( h \in H_\Gamma^{out} \), we let \( B_h : [T_0, \infty) \rightarrow H_\Gamma \) be the unique flowline of \( \Xi_\Gamma^{tr} \) such that \( B_h(T_0) = h \). Now we extend the connection \( B_h(t) + \Theta(B_h(t))dt \) smoothly to a connection \( A_h \) on the entire manifold \( Z \) so that over the compact part \( A_h|_{Z\backslash [T_0, \infty) \times Y} \) depends on \( h \) smoothly. To put the weighted Sobolev space into the package, we choose a weight \( \delta_\Gamma \in (0, \frac{1}{2}) \) for each \( [\Gamma] \in \chi(Y) \).

**Definition 4.2.** Given \( h \in H_\Gamma^{out} \), we write
\[
A_{k, \delta_\Gamma}(Z, \mathcal{T}_\Gamma, h) := \{ A \in A_{k, loc}(Z) : A - A_h \in L_{k, \delta_\Gamma}^2(Z, T^*Z \otimes \mathfrak{su}(2)) \}.
\]
We denote the union by
\[
A_{k, \delta_\Gamma}(Z, \mathcal{T}_\Gamma) := \bigcup_{h \in H_\Gamma^{out}} A_{k, \delta_\Gamma}(Z, \mathcal{T}_\Gamma, h).
\]
The gauge group \( G_{k+1, \delta_\Gamma}(Z, \mathcal{T}_\Gamma) \) that preserves \( A_{k, \delta_\Gamma}(Z, \mathcal{T}_\Gamma) \) consists of all \( L_{k+1, loc} \) gauge transformations \( u \) such that there exists \( \tau \in \text{Stab}(\Gamma) \) satisfying
\[
u |_{[T_0, \infty) \times Y} \circ \tau - \text{id} \in L_{k+1, \delta_\Gamma}^2([T_0, \infty) \times Y, SU(2))
\]Finally we pick a cut-off function \( \varphi : Z \rightarrow [0, 1] \) such that \( \varphi |_{[0, T_0] \times Y} \equiv 1 \) and \( \varphi |_{[T_0, \infty) \times Y} \equiv 0 \). The thickened moduli space with respect to the thickening data \( \mathcal{T}_\Gamma \) perturbed by \( \sigma \in P_\mu \) is defined to be
\[
M_\sigma(Z, \mathcal{T}_\Gamma) := \{ A \in A_{k, \delta_\Gamma}(Z, \mathcal{T}_\Gamma) : F_A^+ - \varphi F_A^+ = \sigma(A), \kappa(A) = 0 \}/G_{k+1, \delta_\Gamma}(Z, \mathcal{T}_\Gamma).
\]
The thickened based moduli space \( \check{M}_\sigma(Z, \mathcal{T}_\Gamma) \) is defined similarly.

For the rest of this section, we are concerned with irreducible connections. The construction of the thickened moduli space gives us a map
\[
P_\Gamma : M_\sigma^*(Z, \mathcal{T}_\Gamma) \rightarrow H_\Gamma
\]
(4.1)
\[
[A] \mapsto h,
\]
where \( h \) is the element specified in Definition 4.2. We also have the based version
\[
\check{P}_\Gamma : \check{M}_\sigma^*(Z, \mathcal{T}_\Gamma) \rightarrow H_\Gamma \times_{\text{Stab}(\Gamma)} E_{y_0}'.
\]
Note that the perturbations $\sigma : A_{k,loc} \to L^2_{k,\mu}$ are smooth maps. Following from [14] Section 7.3] the based thickened moduli space $\tilde{M}^*_\sigma(Z, T')$ is a $C^2$-manifold and $\tilde{P}_\Gamma$ is a $C^2$-map. From Proposition 3.10 for $|\omega|_W \leq 1$ we have an identification

$$\tilde{M}^*_\sigma(Z, V_{T'}) \simeq \tilde{P}_\Gamma^{-1}(W^{\sigma}_{T'} \cap V_{T'} \times \text{Stab}_\Gamma E'_{y_0}).$$

Denote the embedding by $j : \tilde{M}^*_\sigma(Z, V_{T'}) \hookrightarrow \tilde{M}^*_\sigma(Z, T')$. We obtain the following commutative diagram:

$$\begin{array}{ccc}
\tilde{M}^*_\sigma(Z, V_{T'}) & \xrightarrow{j} & \tilde{M}^*_\sigma(Z, T') \\
\downarrow \tilde{Q}_T & & \downarrow \tilde{P}_T \\
W^{\sigma}_{T'} \times \text{Stab}_\Gamma E'_{y_0} & \hookrightarrow & \mathcal{H}_{T'} \times \text{Stab}_\Gamma E'_{y_0}
\end{array}$$

As mentioned above, we introduce the thickened moduli space mainly to establish the transversality result. The following result is a variance of [14] Theorem 9.0.1 in our case.

**Proposition 4.3.** The map $\tilde{P}_\Gamma$ is transverse to any finite set of smooth submanifold in $\mathcal{H}_{T'} \times \text{Stab}_\Gamma E'_{y_0}$ with respect to a generic perturbation $\sigma$. Moreover the dimension of the based thickened irreducible moduli space $\tilde{M}^*_\sigma(Z, T')$ is given by

$$\dim \tilde{M}^*_\sigma(Z, T') - \dim \mathcal{M}^*_\sigma(Z, T') = 2(\chi(Z) + \sigma(Z)) + \frac{h^1_\Gamma - h^0_\Gamma}{2} + \frac{\rho(\Gamma)}{2} + 3,$$

where $h^i_\Gamma = \dim H^i(Y; \text{ad } \Gamma)$, $i = 0, 1$, $\rho(\Gamma)$ is the Atiyah-Potadi-Singer $\rho$-invariant of the odd signature operator twisted by $\Gamma$ in [2], $\chi(Z)$ is the Euler characteristic, and $\sigma(Z)$ is the signature.

**Proof.** It only remains to show the transversality of $\tilde{P}_\Gamma$. The computation of the formal dimension is the same as that in [14] Chapter 8. Consider the map

$$\mathcal{F} : \mathcal{P}_\mu \times A_{k,\delta_\Gamma}(Z, T') \hookrightarrow L^2_{k-1,\delta_\Gamma}(\Lambda^+ T^* Z \otimes \mathfrak{su}(2))$$

$$(\sigma_\omega, A) \mapsto F^+_{\omega} - \varphi F^+_{\omega} - \sigma_\omega(A).$$

Denote by $\mathcal{F}_h := \mathcal{F}|_{\mathcal{P}_\mu \times A_{k,\mu}(Z, T', h)}$ the restricted map. Then the differential of $\mathcal{F}_h$ is

$$DF_h|_{(\sigma_\omega, A)}(\nu, a) = d^+_A a - D\sigma_\omega|_A a - \sigma_\nu(A),$$

where $a \in L^2_{k,\mu}(T^* Z \otimes \mathfrak{su}(2))$, $\nu \in W_{\mu}$. Recall that the perturbation has the form $\sigma_\nu(A) = \sum_{a} V_{q_\alpha} (A)$, where $q_\alpha$ gives us a dense family of loops in the loop space of $Z$. As in [9] Lemma 13, the irreducibility of $A$ implies the image of $\sigma_\nu(A)$ is dense as we vary $\nu$. On the other hand, the operator

$$d^+_A \circ e^{-\tau\delta_\Gamma} d^+_A e^{\tau\delta_\Gamma} : L^2_{k,\delta_\Gamma}(T^* Z \otimes \mathfrak{su}(2)) \to L^2_{k-1,\delta_\Gamma}(\Lambda^+ T^* Z \otimes \mathfrak{su}(2)) \oplus L^2_{k-1,\delta_\Gamma}(su(2)),$$

where $\tau : Z \to \mathbb{R}$ is a smooth function such that $\tau|_{\{t\} \times Y} = t$, is Fredholm except at a discrete set of $\mathbb{R}$ from the theory of Atiyah-Patodi-Singer [1]. We may choose $\delta_\Gamma$ in the first place to make this operator Fredholm. In particular we see that the
image of $d^+_A$ is closed and has finite dimensional cokernel. Thus we conclude that $\mathcal{F}_h$ is a submersion.

Consider the map
\[
\tilde{P}^t_\Gamma : \mathcal{P}_\mu \times (A^*_k, (Z, T_\Gamma) \times G E_{20}) \rightarrow \mathcal{H}_\Gamma \times \text{Stab}(\Gamma) E'_{y_0}
\]
sending $(\sigma, [A, v])$ to the pair $[h, v']$, where $v'$ is the limit of $v$ under the parallel transport by $A$. By the construction of $A^*_k, (Z, T_\Gamma)$, the map $\tilde{P}^t_\Gamma$ is a submersion. Since $\mathcal{F}$ is gauge equivariant and $\mathcal{F}_h$ is a submersion, we conclude that the restricted map
\[
\tilde{P}^t_\Gamma : \mathcal{F}^{-1}(0) \times G E_{20} \rightarrow \mathcal{H}_\Gamma \times \text{Stab}(\Gamma) E'_{y_0}
\]
is also a submersion. Denote by $\Pi : \mathcal{P}_\mu \times (A^*_k, (Z, T_\Gamma) \times G E_{20}) \rightarrow \mathcal{P}_\mu$ the projection onto the first factor. Then the Sard-Smale theorem tells us that $\tilde{P}^t_\Gamma = \tilde{P}^t_\Gamma \mid_{\Pi^{-1}(\sigma)}$ is transverse to a given smooth submanifold for a generic perturbation $\sigma$. □

Before extracting more information of the asymptotic map from Proposition 4.3, let’s first recall the Kuranishi obstruction map at a flat connection $\Gamma$.

**Theorem 4.4.** [14, Theorem 12.1.1] Let $\Gamma$ be a $C^\infty$ flat connection on $Y$. Then there exists a $\text{Stab}(\Gamma)$-invariant neighborhood $V$ of 0 in $H^1(Y; \text{ad } \Gamma)$, a $\text{Stab}(\Gamma)$-invariant neighborhood $U$ of $\Gamma$ in $S_\Gamma$, and $C^\infty$ $\text{Stab}(\Gamma)$-equivariant maps
\[
(p_\Gamma, o_\Gamma) : V \rightarrow U \text{ and } \sigma_\Gamma : V \rightarrow H^2(Y; \text{ad } \Gamma)
\]
satisfying
\begin{enumerate}
  \item $p_\Gamma$ is an embedding whose differential at 0 is the inclusion $H^1(Y; \text{ad } \Gamma) \hookrightarrow \mathcal{K}_\Gamma$.
  \item The restriction of $p_\Gamma \mid_{\sigma_\Gamma^{-1}(0)}$ is a homeomorphism onto the space of flat connections in $U$.
\end{enumerate}

**Remark 4.5.** Roughly speaking, the proof of Theorem 4.4 makes use of the implicit function theorem to obtain a map
\[
q_\Gamma : V \rightarrow d^+_\Gamma \subset L^2_1(T^*Y \otimes \mathfrak{su}(2)),
\]
which is characterized by the fact that
\[
\Pi_\Gamma^t F_{\Gamma+b+q_\Gamma(b)} = 0,
\]
where $\Pi_\Gamma^t : L^2_{l-1}(\Lambda^2 T^*Y \otimes \mathfrak{su}(2)) \rightarrow \text{im } d_\Gamma$ is the $L^2$-orthogonal projection. Then
\[
p_\Gamma(b) = \Gamma + b + q_\Gamma(b) \text{ and } \sigma_\Gamma(b) = \Pi_\Gamma^t F_{p_\Gamma(b)},
\]
where $\Pi_\Gamma : L^2_{l-1}(\Lambda^2 T^*Y \otimes \mathfrak{su}(2)) \rightarrow \ker d_\Gamma$ is the $L^2$-orthogonal projection. The zero set of map $\sigma_\Gamma$ provides a local structure of the character variety $\chi(Y)$ near $[\Gamma]$.

**Definition 4.6.** For a flat connection $\Gamma$ on $Y$, the map $\sigma_\Gamma$ in Theorem 4.4 is called the Kuranishi obstruction map. $[\Gamma] \in \chi(Y)$ is said to be a smooth point if the Kuranishi map of $\Gamma$ vanishes on $V_\Gamma$, i.e. $\sigma_\Gamma \equiv 0$, otherwise a singular point.
Proposition 4.7. \[\text{Corollary 9.3.1}\] Let $[\Gamma] \in \chi(Y)$ be a smooth point. Then there exists a center manifold $\mathcal{H}_\Gamma$ consisting of flat connections. Moreover for a generic perturbation $\sigma$ the asymptotic map
\begin{equation}
\bar{\partial}_+ : \tilde{M}_\sigma(Z, V_\Gamma) \to R(Y)
\end{equation}
is $C^2$ and transverse to a given submanifold in its range.

Proof. When $[\Gamma] \in \chi(Y)$ is a smooth point, one can take a center manifold $\mathcal{H}_\Gamma$ to be the graph of the map $q_\Gamma$. Indeed in this case $\mathcal{H}_\Gamma$ consists of flat connections which are the critical points of $cs_\Gamma$ near $\Gamma$, and are preserved by the gradient flow of $cs_\Gamma$. Since every point on the center manifold $\mathcal{H}_\Gamma$ is a critical point, all gradient flowlines on $\mathcal{H}_\Gamma$ is constant. Thus the map $\tilde{Q}_\Gamma$ coincides with the asymptotic map $\bar{\partial}_+$. Then result now follows from Proposition 4.3. \qed

Now we restrict our attention to the case when $Y = T^3$. For each singular point in $\chi(T^3)$, Gompf and Mrowka have constructed a center manifold. We recall their construction below and use it to prove that there are no irreducible ASD connections asymptotic to those singular points.

The character variety $\chi(T^3)$ is identified as a copy of the quotient $T^3/\sim$, where $\sim$ is given by the hypoelliptic involution consisting of 8 fixed points corresponding to central connections. When $[\Gamma] \in \chi(Y)$ is a noncentral connection, the first homology $H^1(T^3; \text{ad} \Gamma)$ is computed as
\begin{equation}
H^1(T^3; \text{ad} \Gamma) \cong H^1(T^3) \otimes H^0(T^3; \text{ad} \Gamma),
\end{equation}
where $H^1(T^3)$ is the space of harmonic 1-forms on $T^3$, $H^0(T^3; \text{ad} \Gamma) \cong i\mathbb{R}$ is the Lie algebra of the stabilizer of $\Gamma$. Thus each $b \in H^1(T^3; \text{ad} \Gamma)$ gives a flat connection $\Gamma + b$ due to $b \wedge b = 0$. This implies that $q_\Gamma(b) = 0$, and then the Kuranishi map $\sigma_\Gamma(b) = 0$ with $b$ in a small neighborhood $V_\Gamma$ of 0. We conclude that $\Gamma$ is a smooth point in the sense of Definition 4.6. When $[\Gamma] \in \chi(Y)$ is a central connection, we have
\begin{equation}
H^1(T^3; \text{ad} \Gamma) \cong H^1(T^3) \otimes \mathfrak{su}(2).
\end{equation}
Now we fix an orthonormal frame $\{e^1, e^2, e^3\}$ of $H^1(T^3)$ with respect to the product metric. Then each $b \in H^1(T^3; \text{ad} \Gamma)$ has the form
\[b = \sum_i e^i \otimes X_i, \quad X_i \in \mathfrak{su}(2).\]
Thus the curvature of $\Gamma + b$ has the form
\begin{equation}
F_{\Gamma+b} = \frac{1}{2} \sum_{i,j} e^i \wedge e^j \otimes [X_i, X_j].
\end{equation}
In particular $F_{\Gamma+b} \in \mathcal{H}^2(T^3)$. From Remark 4.5 we conclude that $q_\Gamma(b) = 0$. Thus the Kuranishi map is
\[\sigma_\Gamma(b) = \frac{1}{2} \sum_{i,j} e^i \wedge e^j \otimes [X_i, X_j].\]
Proposition 4.8. [17 Proposition 15.2] Let $\Gamma$ be a smooth central flat connection on $E' \to T^3$. Then $H^1(T^3; \text{ad}\, \Gamma)$ is a center manifold of $\Gamma$. Moreover the stable manifold of the origin is given by

$$W^0_\Gamma = \{ b = \sum_i e^i \otimes X_i \in H^1(T^3) \otimes \mathfrak{su}(2) : \Vert X_i \Vert = \Vert X_j \Vert, \langle X_i, X_j \rangle = 0, \langle X_1, [X_2, X_3] \rangle \leq 0 \}.$$  

(4.11)

Corollary 4.9. Let $\Gamma$ be a central flat connection on the trivial $SU(2)$-bundle $E'$ over $T^3$. Then for a generic perturbation $\sigma$, one has

$$\partial_+^{-1}([\Gamma]) \cap \mathcal{M}_\sigma^* (Z) = \emptyset,$$

where $Z = M \cup [0, \infty) \times T^3$ is a homology $D^2 \times T^2$.

Proof. From the commutative diagram [4.3] and Proposition 4.3 we know that the map

$$\tilde{P}_\Gamma : \tilde{\mathcal{M}}_\sigma^* (Z, T\Gamma) \to H^1_T \times \text{Stab}\, \Gamma E'_y$$

is transverse to the stable set $W^0_\Gamma \times \text{Stab}\, \Gamma E'_y$ for a generic perturbation $\sigma$. Moreover dim $\mathcal{M}_\sigma^* (Z, T\Gamma) = 6$ from (4.4), and the stratified space $W^0_\Gamma$ has codimension 4 in $H^1_T$ from (4.11). Thus $\partial_+^{-1}([\Gamma]) \cap \mathcal{M}_\sigma^* (Z)$ lies in a 2-dimensional $C^2$-manifold $\tilde{P}_\Gamma^{-1}(W^0_\Gamma \times \text{Stab}\, \Gamma E'_y)$. Since $\partial_+^{-1}([\Gamma]) \cap \mathcal{M}_\sigma^* (Z)$ is the quotient of the free smooth $SO(3)$-action on $\partial_+^{-1}([\Gamma]) \cap \mathcal{M}_\sigma^* (Z)$, we conclude that it has to be empty due to dimension counting.

So far we have a complete description of the irreducible moduli space $\mathcal{M}_\sigma^* (Z)$ for a generic perturbation $\sigma$:

(i) $\mathcal{M}_\sigma^* (Z)$ is a smooth oriented 1-manifold.

(ii) $\mathcal{M}_\sigma^* (Z)$ misses all central connections in $\chi(T^3)$ under the asymptotic map.

(iii) $\partial_+: \mathcal{M}_\sigma^* (Z) \to \chi(T^3)$ is transverse to any prefixed subcomplex in $\chi(T^3)$.

5. THE REDUCIBLE LOCUS

In this section, we study the structure of the reducible locus $\mathcal{M}_\sigma^{\text{red}}(Z)$ with respect to a generic perturbation. We continue to assume that $Y = T^3$ and $H_*(Z; \mathbb{Z}) \cong H_*(D^2 \times T^2; \mathbb{Z})$. The first part is to give a global description of $\mathcal{M}_\sigma^{\text{red}}(Z)$. The second part is to give a local description of the moduli space near points that are approached by a sequence of irreducible instantons.

Before diving into the details, we would like to comment on the choice of the weight $\delta_Y$. There are two things we need to take care of. The first is to ensure the moduli space lies in the thickened moduli space. This is achieved by choosing $\delta_Y \in (0, \frac{\mu_Y}{2})$, where $\mu_Y$ is the smallest nonzero absolute value of eigenvalues of the restricted Hessian $\ast d\Gamma_{\mid \ker d\Gamma}$. The second is to ensure the deformation complex at an instanton $[A] \in M_\sigma(Z)$ is Fredholm. Later we will see it’s equivalent to the Fredholmness of the following complex:

$$(F_{\delta_Y, \sigma}) L^2_{k+1, \delta_Y}(Z, \mathfrak{su}(2)) \xrightarrow{-d_{\lambda}} L^2_{k, \delta_Y}(T^*Z \otimes \mathfrak{su}(2)) \xrightarrow{d_{\lambda}} L^2_{k-1, \delta_Y}(\Lambda^+ T^*Z \otimes \mathfrak{su}(2)).$$
where $d^+_{A,\sigma} = d^+_A - D\sigma|_A$. We may ignore the perturbation part, since the Fredholmness is preserved under small or compact perturbations. According to [14, Lemma 8.3.1] the complex $(F_{\delta\sigma})$ is Fredholm if and only if $\frac{d^+_A}{\delta}$ is not an eigenvalue of $-\ast d_{[\operatorname{im}d^+_A]^2}$. Since $\operatorname{im}d^+_A|_{\Omega^2} \subset \ker d^+_A|_{\Omega^2}$, we may simply consider the smallest absolute value of eigenvalues of $\ast d^+_A|_{\ker d^+_A}$. Note that when $\Gamma$ is not a central connection, we have $H^1(T^3, \operatorname{ad}\Gamma) = 1$. Thus only when approaching the central connections can the smallest absolute value of eigenvalues approach 0. Let’s fix a neighborhood $\mathcal{O}_c$ of the central connections in $\chi(T^3)$. Then we may choose a uniform weight $\delta > 0$ for all instantons $[A]$ with $\partial_+[\delta] A \in \chi(T^3) \setminus \mathcal{O}_c$.

5.1. The Global Picture. Recall that an $SU(2)$-connection $A$ on $E$ is reducible if $A$ preserves a splitting $E = L \oplus L^*$ for some line bundle $L$. We write $A = A_L \oplus A_L^*$ corresponding to the splitting of $E$. The holonomy perturbation decomposes correspondingly as

$$\sigma(A) := \begin{pmatrix} \sigma_L(A_L) & 0 \\ 0 & \sigma_L(A_L^*) \end{pmatrix} \in \Omega^+(Z, \mathfrak{su}(2)),$$

where $\sigma_L$ is defined by

$$\sigma_L(A_L) = \frac{1}{2} \sum_\alpha \left( \operatorname{Hol}_{q_\alpha} A_L - \operatorname{Hol}_{q_\alpha} A_L^* \right) \otimes \omega_\alpha.$$

Note that $\sigma_L(A_L) = -\sigma_L(A_L^*)$. Thus the perturbed ASD equation $F_A^+ = \sigma(A)$ is equivalent to

$$F_{A_L}^+ = \sigma_L(A_L),$$

where $\sigma_L(A_L) \in \Omega^+(Z, i\mathbb{R})$.

**Lemma 5.1.** Let $[A] \in \mathcal{M}_\mathcal{O}^\text{red}(Z)$ be a reducible class of perturbed ASD connections. Then one can choose a representative $A$ which preserves the splitting $E = \mathbb{C} \oplus \mathbb{C}$.

**Proof.** As in the proof of Proposition 3.7, one can choose a representative $A$ such that $A |_{\{T, t_0\} \times T^3}$ is in standard form with respect to a flat connection $\Gamma$ on $E'$. Let $E = L \oplus L^*$ be the decomposition that $A$ preserves. Let’s write $A = B + \beta dt$ and $A_L = B_L + \beta_L dt$. Then we have

$$F_{A_L} = F_{B_L} + dt \wedge (\dot{B}_L - d_{B_L} \beta_L).$$

We denote by $T_s \subset \{s\} \times T^3$ the 2-torus representing a generator $H_2(Z; \mathbb{Z}) \cong \mathbb{Z}$. Then the Chern-Weil formula gives us that

$$c_1(L) \cdot 1_Z = \frac{1}{2\pi i} \int_{T_s} F_{A_L} = \frac{1}{2\pi i} \int_{T_s} F_{B_L(s)}.$$

Due to the finite energy of $A$, elliptic theory tells us that for $T >> 0$ one has

$$\|F_{B(s)}\|_{L^2_T}^2 \leq \text{const.}(\|F_A\|_{L^2((T, \infty) \times Y)}^2 + e^{-2\mu s}), \text{ when } s \geq T + 1.$$
We may take \( l = 2 \). Then the Sobolev embedding \( L^2_{\text{tr}} \hookrightarrow C_0 \) for 3-manifolds implies that \( \|F_{B(s)}\|_{C^0} \to 0 \) as \( s \to \infty \). Since \( \|F_{BL(s)}\|_{C^0} \leq \|F_{B(s)}\|_{C^0} \), we conclude that

\[
c_1(L) \cdot 1_Z = \lim_{s \to \infty} \frac{1}{2\pi i} \int_{T_s} F_{BL(s)} = 0.
\]

Thus \( c_1(L) = 0 \in H^2(Z; \mathbb{Z}) \) meaning \( L \) is equivalent to the trivial bundle \( \mathbb{C} \). \( \square \)

**Lemma 5.1** leads us to consider the moduli space of ASD \( U(1) \)-connections on the trivial bundle \( \mathbb{C} \). We write \( \mathcal{A}^{U(1)}_{k,\text{loc}}(Z) \) for the space of \( L^2_{k,\text{loc}} U(1) \)-connections on the trivial line bundle \( \mathbb{C} \) of \( Z \). Fixing the product connection as the reference connection, \( \mathcal{A}^{U(1)}_{k,\text{loc}} \) is identified with \( L^2_{k,\text{loc}}(T^*Z \otimes i\mathbb{R}) \). We write \( \mathcal{G}^{U(1)}_{k+1,\text{loc}} \) for the \( L^2_{k+1,\text{loc}} \) gauge transformations of the \( U(1) \)-bundle \( \mathbb{C} \). Then \( \mathcal{G}^{U(1)}_{k+1,\text{loc}} \) is given by \( L^2_{k+1,\text{loc}}(Z, U(1)) \).

**Definition 5.2.** The moduli space of perturbed anti-self-dual \( U(1) \)-connections is defined to be

\[
\mathcal{M}^{U(1)}_{\gamma}(Z) := \{ A_L \in \mathcal{A}^{U(1)}_{k,\text{loc}} : F_{A_L}^+ = \sigma_L(A_L), \int_Z |F_{A_L}|^2 < \infty \}/\mathcal{G}^{U(1)}_{k+1,\text{loc}}.
\]

If we write \( A_L = d + a_L \) with \( a_L \in L^2_{k,\text{loc}}(T \ast Z \otimes i\mathbb{R}) \), the induced dual connection has the form \( A_L^* = d - a_L \). Thus \( F_{A_L} = -F_{A_L^*} \). Combining with the fact that \( \sigma_L(A_L) = -\sigma_L(A_L^*) \), we get an involution \( \tau \) on \( \mathcal{M}^{U(1)}_{\gamma}(Z) \) given by \( \tau([A]) = [A^*] \).

**Lemma 5.3.** The quotient of \( \mathcal{M}^{U(1)}_{\gamma}(Z) \) under \( \tau \) is the reducible locus \( \mathcal{M}^{\text{red}}_{\gamma}(Z) \). Moreover the set of fixed points of \( \tau \) consists of classes of flat connections whose holonomy groups lie in \( \{-1, 1\} \subset U(1) \).

**Proof.** Note that the map \( A_L \mapsto A_L \oplus A_L^* \) descends to a surjective map \( \mathcal{M}^{U(1)}_{\gamma}(Z) \to \mathcal{M}^{\text{red}}_{\gamma}(Z) \) by Lemma 5.1. We also note the Weyl group of \( SU(2) \) is \( \mathbb{Z}/2 \) generated by the matrix representative

\[
\eta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Thus any \( SU(2) \) gauge transformation of \( E \) preserves the splitting \( E = \mathbb{C} \oplus \mathbb{C} \) is either a \( U(1) \) gauge transformation or a \( U(1) \) gauge transformation multiplied by \( \eta \). The effect of multiplying \( \eta \) is applying the involution \( \tau \). This identifies \( \mathcal{M}^{U(1)}_{\gamma}(Z)/\tau = \mathcal{M}^{\text{red}}_{\gamma}(Z) \).

Let \( [A_L] \in \mathcal{M}^{U(1)}_{\gamma}(Z) \) be a fixed point of \( \tau \). Then there exists \( u \in \mathcal{G}^{U(1)} \) such that \( A_L^* = u \cdot A_L \). Let’s write \( A_L = d + a_L \). Then this is equivalent to

\[
-a_L = a_L - u^{-1} du.
\]

Since \( u^{-1} du \) is a closed 1-form, we conclude that \( da_L = 0 \), which implies that \( A_L \) is a flat connection. In particular we have \( F_{A_L}^+ = 0 \). From (5.1) we see that this requires \( \text{Hol}_L A_L = \text{Hol}_L A_L^* \) for any loop \( \gamma \) in \( Z \). We conclude that the holonomy of \( A_L \) has to be real in \( U(1) \), which lies in \( \{-1, 1\} \). \( \square \)
Consider the non-central stratum are nonzero, the argument of Proposition 4.3 implies that reversing diffeomorphism, thus \( \rho \) is transverse to any given submanifold with respect to generic perturbations. The center manifold is \( \sigma \)-invariant of the deformation complex at a flat \( U(1) \)-connection \( \Gamma_L \) over \( T^3 \):

\[
(5.3) \quad L^2_i(T^3, i\mathbb{R}) \xrightarrow{d} L^2_{i-1}(T^*T^3 \otimes i\mathbb{R}) \xrightarrow{sd} L^2_{i-2}(T^*T^3 \otimes i\mathbb{R}).
\]

The \( U(1) \)-version Chern-Simons functional is

\[
cs^{U(1)}(B_L) = -\frac{1}{2} \int_{T^3} b_L \wedge db_L,
\]

where \( B_L = \nabla_L + b_L \), \( \nabla_L \) is the product connection, and \( b_L \in L^2(T^*T^3 \otimes i\mathbb{R}) \). The gradient of \( cs^{U(1)} \) is given by

\[
\text{grad} cs^{U(1)} |_{B_L} = \star db_L.
\]

We denote by \( H^1_L := \ker d \cap \ker d^* \in L^2(T^*T^3 \otimes i\mathbb{R}) \). We claim that \( \Gamma_L + H^1_L \) is the center manifold for the pair \( (H^1_L, - \text{grad} cs^{U(1)}) \), i.e. the center manifold is the graph of the zero map. Indeed, \( \Gamma_L + H^1_L \) consists of all critical points of \( \text{grad} cs^{U(1)} \) in \( \ker d^* \) and is preserved by the gradient flowlines. Moreover, the gradient vector field \( \text{grad} cs^{U(1)} \) is already complete over \( H^1_L \). So there is no need to consider the thickened moduli space. The argument in Proposition 4.3 implies that all connections in \( M^{U(1)}_\sigma(Z) \) have exponential decay to their asymptotic value. We denote by \( R^{U(1)}(T^3) := \text{Hom}(T^3, U(1)) \) the space of \( U(1) \)-representations of \( T^3 \), and write the asymptotic map as

\[
\partial_+ : M^{U(1)}_\sigma(Z) \longrightarrow R^{U(1)}(T^3).
\]

Since the center manifold is \( C^\infty \), the asymptotic map is \( C^\infty \) as well. If we only consider the non-central stratum \( M^{U(1),*}_\sigma(Z) \) where the holonomy perturbations are nonzero, the argument of Proposition 4.3 implies that

\[
\partial_+ : M^{U(1),*}_\sigma(Z) \longrightarrow R^{U(1)}(T^3)
\]

is transverse to any given submanifold with respect to generic perturbations. The computation of the dimension is given by considering the deformation complex at \( [A_L] \in M^{U(1),*}_\sigma(Z) \) as in [13] Chapter 8:

\[
d = -\left( \chi(Z) + \sigma(Z) \right) + \frac{h^1 + h^0}{2} + \frac{\rho(\Gamma_L)}{2},
\]

where \( \Gamma_L \) is the asymptotic value of \( A_L \), \( h^i \) is the dimension of the \( i \)-the homology of the deformation complex (5.3) at \( \Gamma_L \) as above, and \( \rho(\Gamma_L) \) is \( \rho \)-invariant of the odd signature operator twisted by \( \Gamma_L \). Note that \( T^3 \) admits an orientation-reversing diffeomorphism, thus \( \rho(\Gamma_L) = 0 \). Since \( h^1 = 3, h^0 = 1 \), we conclude...
that $d = 2$. Combining with the transversality result, we see that the image 
$\partial_+(\mathcal{M}_\sigma^U(1)\ast(Z))$ misses the points in $\mathcal{R}^U(1)(T^3)$ whose holonomy groups lie in $\{\pm 1\}$. We summarize the above discussion as follows.

**Proposition 5.5.** Let $Z$ be a Riemannian smooth manifold with a cylindrical end modeled on $[0, \infty) \times T^3$ satisfying $H_\ast(Z; \mathbb{Z}) \cong H_\ast(D^2 \times T^2; \mathbb{Z})$. Then given a generic perturbation $\sigma \in \mathcal{P}_\mu$ we have the following description for the non-central moduli space $\mathcal{M}_\sigma^U(1)\ast(Z)$.

(i) $\mathcal{M}_\sigma^U(1)\ast(Z)$ is an oriented smooth 2-manifold.

(ii) The asymptotic map $\partial_+ : \mathcal{M}_\sigma^U(1)\ast(Z) \to \mathcal{R}^U(1)(T^3)$ is smooth and transverse to any given submanifold of $\mathcal{R}^U(1)(T^3)$.

(iii) The image $\partial_+(\mathcal{M}_\sigma^U(1)\ast(Z))$ misses all connections in $\mathcal{R}^U(1)(T^3)$ whose holonomy groups lie in $\{\pm 1\}$.

Note that when there are no perturbations, the $U(1)$-moduli space $\mathcal{M}_\sigma^U(1)(Z)$ is the space of gauge equivalence classes of flat $U(1)$-connections on $Z$. The holonomy map identifies $\mathcal{M}_\sigma^U(1)(Z)$ with the space $\mathcal{R}^U(1)(Z)$ consisting of $U(1)$-representations of $\pi_1(Z)$, thus is identified as a 2-torus. Our next step is to show that this feature is preserved under small generic perturbations.

Note that all connections in $\mathcal{M}_\sigma^U(1)(Z)$ has exponential decay after a fixed time $T_0$ given by Remark 3.11. Moreover the decay rate is given by $\frac{\mu}{2}$, where $\mu$ is the smallest absolute value of eigenvalues of $\ast d |_{\text{ker } d^\ast}$. We choose $\delta < \frac{\mu}{2}$ as the weight. Now we can narrow down the ambient connection space to be

$$A^U_{k,\delta}(Z) := \{ A_L \in A^U_{k,\delta}(Z) : \exists b_L \in \mathcal{H}^1(T^*T^3; i\mathbb{R}) \text{ such that } A_L - \varphi b_L \in L^2_{2k,\delta}(T^*Z \otimes i\mathbb{R}) \},$$

where $\varphi : Z \to \mathbb{R}$ is the cut-off function in Definition 4.2. The gauge group that preserves $A^U_{k,\delta}(Z)$ is

$$G_{k,\delta+1} := \{ u \in G_{k,\delta+1}(Z) : u|_{[T_0, \infty) \times T^3} = u_0 \cdot e^\xi, \text{ where } u_0 \in S^1 \},$$

where $\xi \in L^2_{k+1,\delta}([T_0, \infty) \times T^3, i\mathbb{R})$. Let $[A_L] \in \mathcal{M}_\sigma^U(1)(Z)$. The Lie algebra of the gauge group $G^{U(1)}_{k+1,\delta}(Z)$ is

$$L^2_{k+1,\delta}(T^*Z \otimes i\mathbb{R}) := \{ \xi \in L^2_{k+1,\delta}(Z, i\mathbb{R}) : d\xi \in L^2_{k,\delta}(Z, i\mathbb{R}) \}.$$
where \( d^+_\sigma := d^+ - D\sigma_L|_{A_L} \) is the linearization of the perturbed ASD equation at \( A_L \). Sitting inside of the complex \( (E_{\delta,\sigma})^{U(1)} \) is a subcomplex:

\[
(F_{\delta,\sigma}^{U(1)}) \quad L^2_{k+1,\delta}(Z, i\mathbb{R}) \xrightarrow{-d} L^2_{k,\delta}(T^*Z \otimes i\mathbb{R}) \xrightarrow{d^+_\sigma} L^2_{k-1,\delta}(\Lambda^+ T^* Z \otimes i\mathbb{R}).
\]

When \( \sigma = 0 \), the quotient of \( (E_{\delta}^{U(1)}) \) by \( (F_{\delta}^{U(1)}) \) is identified as

\[
i\mathbb{R}^0 \xrightarrow{0} H^1(T^3; i\mathbb{R}) \to 0.
\]

As mentioned in the beginning of this section, the choice of \( \delta \) also ensures that the complex \( (E_{\delta,\sigma})^{U(1)} \) is Fredholm with respect to small or compact perturbations.

Now let \([A_L] \in \mathcal{M}^{U(1)}_\sigma(Z)\) be a central connection. The differential of \( \sigma_L \) at \( A_L \) is

\[
(D\sigma_L|_{A_L}(a))_x = \frac{1}{2} \sum_\alpha \left( \int_{q_{\alpha,x}} a \right) \cdot (\text{Hol}_{q_{\alpha,x}} A^*_L - \text{Hol}_{q_{\alpha,x}} A_L) \otimes \omega_\alpha |_x,
\]

where \( a \in L^2_{k,\delta}(T^*Z \otimes i\mathbb{R}) \), \( x \in Z \). Since \( \text{Hol}_x A_L = \text{Hol}_x A^*_L \), we conclude that \( D\sigma_L|_{A_L} = 0 \) whenever \( A_L \) is central. [12, Proposition 3.12] identifies \( H^2(E_{\delta}^{U(1)}) \cong \hat{H}_c^+(Z; i\mathbb{R}) \), where \( \hat{H}_c^+(Z; i\mathbb{R}) \) is the image of \( H_c^+(Z; i\mathbb{R}) \) in \( H^2(Z; i\mathbb{R}) \) under the inclusion map. Due to the fact that \( b^+(Z) = 0 \), we conclude that \( H^2(E_{\delta}^{U(1)}) = 0 \).

From this fact we learn two things:

(a) Each central class \([A_L] \) is a smooth point in \( \mathcal{M}^{U(1)}_\sigma(Z) \).

(b) The non-perturbed \( U(1) \)-moduli space \( \mathcal{M}^{U(1)}(Z) \) is regular, i.e. it’s smoothly cut-out by the defining equation.

**Proposition 5.6.** \( \mathcal{M}^{U(1)}_\sigma(Z) \) is diffeomorphic to a 2-torus with respect to a generic perturbation \( \sigma_\omega \) with \( \|\omega\|_W \leq c_2 \) for some constant \( c_2 > 0 \).

**Proof.** The proof is similar to the transversality result as in Proposition 4.3. Let \( \sigma_\omega \in \mathcal{P}_\mu \) be a generic perturbation so that \( \mathcal{M}^{U(1)}_\sigma(Z) \) is regular. Pick a path \( \sigma_t \) from 0 to \( \sigma_\omega \) in \( \mathcal{P}_\mu \). Now we consider the map

\[
\mathcal{F}^{U(1)} : \mathcal{P}_\mu \times \ker d^+_\delta \longrightarrow L^2_{k-1,\delta}(\Lambda^+ T^* Z \otimes i\mathbb{R})
\]

\[
(\sigma, a_L) \longmapsto d^+_\sigma a_L,
\]

where \( d^+_\sigma \equiv e^{-\delta \tau} d^* e^{\delta \tau} : L^2_{k,\delta}(T^*Z \otimes i\mathbb{R}) \to L^2_{k-1,\delta}(Z, i\mathbb{R}) \) is the formal \( L^2 \)-adjoint of \( d \). Our discussion above implies that \( \mathcal{F}^{U(1)} \) is a submersion. We denote by \( \mathcal{Z}^{U(1)} = (\mathcal{F}^{U(1)})^{-1}(0) \). By construction, \( \sigma_0 \) and \( \sigma_1 \) are two regular values of the projection map \( \pi : \mathcal{Z}^{U(1)} \to \mathcal{P}_\mu \). We approximate the path \( \sigma_t \) relative to boundary by a generic path \( \sigma'_t \) transverse to the map \( \pi : \mathcal{Z}^{U(1)} \to \mathcal{P}_\mu \). Then the union

\[
\mathcal{Z}_t^{U(1)} := \bigcup_{t \in [0,1]} \pi^{-1}(\sigma'_t) \cap \mathcal{Z}^{U(1)}
\]

is a cobordism from \( \mathcal{M}^{U(1)}(Z) \) to \( \mathcal{M}^{U(1)}_{\sigma_1}(Z) \). Since \( \sigma_0 = 0 \) is a regular value of \( \pi|_{\mathcal{Z}^{U(1)}} \), we conclude that whenever \( \|D\sigma_L\| \) is small, \( \sigma \) is a regular value as
well. Since \( \|D\sigma_{\omega,L}\| \leq \text{const.} \|\omega\|_W \), we can choose \( \omega \) to have small norm, say less than \( \epsilon_2 \) so that each point \( \sigma'_i \) in the path is a regular value of \( \pi|_{Z^U(1)} \). Thus the cobordism \( Z^U(1) \) is a product. This shows that \( M^{U(1)}_{\sigma}(Z) \) is diffeomorphic to \( M^U(Z) \) which is a 2-torus.

**Corollary 5.7.** Given a generic perturbation \( \sigma_\omega \) satisfying \( \|\omega\|_W < \epsilon_2 \), the reducible locus \( M^\text{red}_{\sigma}(Z) \) is identified as a pillowcase, i.e. the quotient of \( T^2 \) by the hypoelliptic involution.

**Proof.** Lemma 5.3 tells us that \( M^\text{red}_{\sigma}(Z) \) is the quotient of \( M^U_{\sigma}(Z) \) under an involution whose fixed point set consists of flat connection on \( Z \) with holonomy group inside \( \{ \pm 1 \} \). Since \( |b^1(Z;Z/2)| = 4 \), there are four of them. Moreover each of them are smooth in \( M^U_{\sigma}(Z) \). Now we know \( M^U_{\sigma}(Z) \) is a 2-torus. The result follows.

\[ \square \]

5.2. The Kuranishi Picture. Now we analyze the Kuranishi picture about a reducible instanton \( [A] \) inside the entire moduli space \( M_{\sigma}(Z) \). Let \( [A] \in M^\text{red}_{\sigma}(Z) \) with the form \( A = A_L \oplus A^*_L \) with respect to a reduction \( E = \mathbb{C} \oplus \mathbb{C} \).

**Lemma 5.8.** Any central instanton \( [A] \in M_{\sigma}(Z) \) is isolated from the irreducible locus \( M^\text{red}_{\sigma}(Z) \) for a small perturbation \( \sigma \in P_\mu \).

**Proof.** Since \( \sigma(A) = 0 \) for any perturbation \( \sigma \in P_\mu \), we know that \( A \) is actually flat. Then it suffices to prove the result in the non-perturbed case.

Note that the non-perturbed moduli space \( M(Z) \) is identified with the space of gauge equivalent classes of flat connections. Following the same argument as in the paragraph above Proposition 4.8, we see the Kuranishi obstruction map at \( [A] \) in the \( M(Z) \) is given by

\[
\sigma_A : H^1(Z;\mathfrak{su}(2)) \rightarrow H^2(Z;\mathfrak{su}(2))
\]

(5.4)

\[ e^1 \otimes X_1 + e^2 \otimes X_2 \mapsto e^1 \wedge e^2 \otimes [X_1,X_2], \]

where \( \{ e^1,e^2 \} \) is an orthonormal frame of \( H^1(Z) \). Since the stabilizer of \( A \) is \( SU(2) \), a neighborhood of \( [A] \) in \( M(Z) \) is identified with a neighborhood in the \( SU(2) \)-quotient \( \sigma_A^{-1}(0)/SU(2) \simeq \mathbb{R}^2/(\mathbb{Z}/2) \). This proves that a neighborhood of \( [A] \) in \( M(Z) \) is the same as a neighborhood of \( [A] \) in the reducible locus \( M^\text{red}_{\sigma}(Z) \).

Thus \( [A] \) is isolated from the irreducible locus.

\[ \square \]

Given a generic small perturbation \( \sigma \), any instanton \( [A] \in M_{\sigma}(Z) \) asymptotic to a central connection in \( \chi(T^3) \) has to be central itself. Moreover Lemma 5.8 tells us that the central instantons \( [A] \in M_{\sigma}(Z) \) is isolated from the reducible locus. Thus we can choose the neighborhood \( O_c \) of central connections in \( \chi(T^3) \) such that all irreducible instantons \( [A] \in M^\text{red}_{\sigma}(Z) \) has their asymptotic values outside \( O_c \). This in turn enables us to pick a weight \( \delta > 0 \) uniformly for all irreducible instantons in the Fredholm package.

Now let \( [A] \in M^\text{red}_{\sigma}(Z) \) be a non-central reducible instanton satisfying \( \partial_+( [A] ) \notin O_c \). We may write \( A = d + a \) with \( a \in L^2_{k,\delta}(Z,\mathfrak{su}(2)) \) and \( A_L = d + a_L \) with
\(a_L \in \hat{L}_{k, \delta}^2(Z, i\mathbb{R})\). The perturbed deformation complex at \([A]\) is

\[
(E_{\delta, \sigma}) \quad \hat{L}_{k+1, \delta}^2(Z, \mathfrak{su}(2)) \xrightarrow{d_A} \hat{L}_{k, \delta}^2(T^*Z \otimes \mathfrak{su}(2)) \xrightarrow{d_{A, \sigma}^+} \tilde{L}_{k-1, \delta}^2(\Lambda^+ T^*Z \otimes \mathfrak{su}(2)),
\]

where \(d_{A, \sigma}^+ = d_A^+ - D\sigma|_A\). With respect to the isomorphism

\[
i\mathbb{R} \oplus \mathbb{C} \longrightarrow \mathfrak{su}(2)
\]

\[
(v, z) \longmapsto \begin{pmatrix} v & z \\ - \bar{z} & -v \end{pmatrix}
\]

the induced connection on \(\mathfrak{su}(2)\)-forms \(\Omega^i(Z, \mathfrak{su}(2))\) splits as \(d \oplus A_C\) with \(A_C = A_{C, \sigma}^\mathbb{R} \oplus \mathbb{C}\), the holonomy perturbation splits as \(\sigma_L \oplus \sigma_C\), and the deformation complex \((E_{\delta, \sigma})\) splits as the direct sum of the following two complexes:

\[
(E_{U(1)}^{(1)}) \quad \hat{L}_{k+1, \delta}^2(Z, i\mathbb{R}) \xrightarrow{d} \hat{L}_{k, \delta}^2(T^*Z \otimes i\mathbb{R}) \xrightarrow{d_A^+} \tilde{L}_{k-1, \delta}^2(\Lambda^+ T^*Z \otimes i\mathbb{R})
\]

and

\[
(E_{C}^\mathbb{C}) \quad L_{k+1, \delta}^2(Z, \mathbb{C}) \xrightarrow{d_{A, \sigma}^+} L_{k, \delta}^2(T^*Z \otimes \mathbb{C}) \xrightarrow{d_{A, \sigma}^+} \tilde{L}_{k-1, \delta}^2(\Lambda^+ T^*Z \otimes \mathbb{C}),
\]

where \(d_{A, \sigma}^+ = d_{A, \sigma}^+ \oplus - D\sigma|_C\). More precisely, let \(a \in L_{k, \delta}^2(T^*Z \otimes \mathbb{C})\). The differential \(D\sigma|_C\) evaluating at \(a\) is given by

\[
(D\sigma|_C(a))_x = - \sum_\alpha \left(\int_{\theta_e} a \cdot (\text{Hol}_{\theta_e, x}(A_L))^2 \otimes \omega\right)_x.
\]

Since we are working with weighted Sobolev spaces, the homology of the complex is defined to be

\[
H^0_A(E_{\delta, \sigma}) := \ker d_A, \quad H^1_A(E_{\delta, \sigma}) := \ker d_{A, \delta}^+ \cap \ker d_{A, \delta}^*, \quad H^2_A(E_{\delta, \sigma}) := \ker d_{A, \sigma, \delta}^+,
\]

where

\[
d_{A, \delta}^* = e^{-\delta\tau} d_A^* e^{\delta\tau}, \quad d_{A, \sigma, \delta}^* = e^{-\delta\tau} d_{A, \sigma}^* e^{\delta\tau}
\]

are the \(L^2\)-adjoints. We further note that the complex \((E_{\delta, \sigma})\) is \(U(1)\)-equivariant, where the \(U(1)\)-action on \(\mathfrak{su}(2)\)-valued forms are induced by the action on the Lie algebra \(\mathfrak{su}(2)\):

\[
e^{i\theta} \cdot (v, z) = (v, e^{i2\theta} z).
\]

Earlier in Theorem 1.4 we have considered the Kuranishi obstruction map at a flat connection on a 3-manifold to study the local structure. The same strategy can be applied to the four dimensional case as well.

Following [14, Theorem 12.1.1] there exists a \(U(1)\)-invariant neighborhood \(V_A\) of 0 in \(H^1_A(E_{\delta, \sigma})\) together with a \(U(1)\)-equivariant map

\[
\sigma_A : V_A \rightarrow H^2_A(E_{\delta, \sigma})
\]

such that the \(U(1)\)-quotient of \(\sigma_A^{-1}(0)\) is isomorphic to a neighborhood of \([A] \in \mathcal{M}_\sigma(Z)\) as a stratified space. In particular when \(H^1_A(E_{\delta, \sigma}^\mathbb{C}) = H^2_A(E_{\delta, \sigma}^\mathbb{C}) = 0\) at \(A\), we have \(H^2_A(E_{\delta, \sigma}) = 0\). We see that

\[
\sigma_A^{-1}(0) = H^1_A(E_{\delta, \sigma}^{U(1)}) \simeq H^2(Z, i\mathbb{R}) = i\mathbb{R} \oplus i\mathbb{R}.
\]
Thus $[A]$ is isolated from the irreducible part $\mathcal{M}_0^r(Z)$. The next proposition shows this situation fits with all but finitely many reducibles $[A] \in \mathcal{M}_{\sigma}^{\text{red}}(Z)$.

**Proposition 5.9.** With respect to a small generic perturbation $\sigma \in \mathcal{P}_\mu$, for all but finitely many noncentral reducible instantons $[A] \in \mathcal{M}_{\sigma}^{\text{red}}(Z)$ satisfying $\partial_+([A]) \notin \mathcal{O}_c$ one has

$$H^1_A(E^C_{\delta,\sigma}) = 0.$$  
Moreover $H^1_A(E^C_{\delta,\sigma}) \cong \mathbb{C}$ for the finitely many exceptional reducibles.

**Proof.** Given $A_L \in \mathcal{A}_{k,\delta}^{U(1)}(Z)$, we write $A_C := A_L^{\otimes 2}$ for the connection on the trivial line bundle $\mathcal{L} \to Z$. Let $\tilde{A} = A_L \oplus \tilde{A}_L^*$ be a noncentral flat connection satisfying $\partial_+([\tilde{A}]) \notin \mathcal{O}_c$ and $H^1_A(E^C_{\delta}) \neq 0$. We write $\mathcal{H}_1 := H^1_A(E^C_{\delta})$, $\mathcal{H}_2 := H^2_A(E^C_{\delta})$. Note that $i\mathcal{R} = H^0(Z; \text{ad} \tilde{A}) = H^0(Z; i\mathcal{R}) \oplus H^0_A(E^C_{\delta})$. Thus $H^0_A(E^C_{\delta}) = 0$. Denote by $\Pi : L^2_{k-1}(\Lambda^+ T^* Z \otimes \mathbb{C}) \to \text{im} d^+_{A_C}$ the orthogonal projection to the image of $d^+_{A_C}$. Note that

$$\text{ind } E^C_{\delta} = \text{ind } E_\delta - \text{ind } E_\delta^{U(1)} = 1 - 1 = 0$$
for all $A \in \mathcal{M}^{\text{red}}(Z)$ with $\partial_+([A]) \notin \mathcal{O}_c$. We conclude $\text{dim}_C \mathcal{H}_1 = \text{dim}_C \mathcal{H}_2$. Let’s consider the map

$$\eta : \mathcal{P}_\mu \times \mathcal{A}_{k,\delta}^{U(1)}(Z) \times \ker d^+_{A,\delta} \to \text{im} d^+_{A_C}$$

$$(\sigma, A_L, b) \mapsto \Pi(d^+_{A_C,\sigma} b).$$

The differential of $\eta$ at $(0, \tilde{A}_L, b)$ on the third component is given by

$$D\eta|_{(0, \tilde{A}_L, b)}(0, 0, \beta) = \Pi(d^+_{A_C} b),$$

which is surjective. By the implicit function theorem we can find a neighborhood $U \times V \subset \mathcal{P}_\mu \times \mathcal{A}_{k,\delta}^{U(1)}(Z)$ of $(0, \tilde{A}_L)$ and a map $h : U \times V \times \mathcal{H}^1 \to \ker d^+_{A,\delta}$ such that for all $(\sigma, A_L, b) \in U \times V \times \mathcal{H}^1$ one has

$$\eta(\sigma, A_L, b + h(\sigma, A_L, b)) = 0.$$  
In particular $d^+_{A_C,\sigma}(b + h(\sigma, A_L, b)) \in \mathcal{H}^2$. This leads us to a map

$$\xi : U \times V \to L^2_{k-1,\delta}(\Lambda^+ T^* Z \otimes i\mathcal{R}) \times \text{Hom}_C(\mathcal{H}_1, \mathcal{H}_2)$$

$$(\sigma, A_L) \mapsto (F^+_{A_L} - \sigma_A(L), b \mapsto d^+_{A_C,\sigma}(b + h(\sigma, A_L, b))).$$

We write $\xi = \xi_1 \times \xi_2$ for its decomposition into the two factors in its range. The argument in Proposition 1.3 implies that $\xi_1$ is a submersion. Since the loops $g_\alpha$ constructed in the holonomy perturbation are dense at each point, Proposition 65 in [8] implies that we only need to vary finitely many components in $\omega = \{\omega_\alpha\}$ to ensure that $\xi_2$ is a submersion. Thus we conclude the map $\xi$ is a submersion. Let $S_i \subset \text{Hom}_C(\mathcal{H}_1, \mathcal{H}_2)$ be the stratum consisting of linear maps of complex codimension-$i$ kernel. Then the projection map $\pi : \xi^{-1}(\{0\} \times S_i) \to U$ is Fredholm of real index $2 - 2i^2$. By the Sard-Smale theorem, for a generic perturbation $\sigma \in U$ only the top two strata $S_0$ and $S_1$ survive in the image of $\xi|_{\{\sigma\} \times V}$, which
The irreducible part of a neighborhood $\exists$ there are only finitely many such instantons, all of which are noncentral.

Let $\exists$ Proof.

for some noncentral, and has a neighborhood $\exists$ Proposition 5.11.

Given a small generic perturbation $\exists$ to define the cohomology.

$\exists$ with $\exists$ to the irreducible part $\exists$.

$\exists$ is also isolated from $\exists$. Due to the compactness of $\exists$, we only need to run the argument above for finitely many reducible instantons $\exists A$ satisfying $\exists H^1_A(E^C_{2,\sigma}) = \mathbb{C}$. □

Remark 5.10. The only reason we impose the condition that $\exists_+[A] \notin \mathcal{O}_c$ is to ensure the complex $\exists (E_{2,\sigma})$ is Fredholm. Since we know all central instantons are isolated from the irreducible part $\exists^*(Z)$, any reducible instanton $\exists [A] \in \mathcal{M}^\text{red}(Z)$ with $\exists_+[A] \in \mathcal{O}_c$ is also isolated from $\exists^*(Z)$ once we choose $\mathcal{O}_c$ small enough.

From the perspective of representation variety, this is equivalent to the vanishing of the twisted cohomology $\exists H^1(Z; \mathbb{C}_{\mathcal{A}_c}) = 0$. This property is also preserved under small perturbations. However in the perturbed case we need to vary the weight $\delta$ to define the cohomology.

With the help of Proposition 5.9 we have the following description of a neighborhood of the reducible locus $\exists M^\text{red}_\sigma(Z)$ in the total moduli space $\exists M_\sigma(Z)$.

**Proposition 5.11.** Given a small generic perturbation $\exists \sigma \in \mathcal{P}_\mu$, all but finitely many reducible instantons $\exists [A] \in \mathcal{M}^\text{red}_\sigma(Z)$ are isolated from the irreducible moduli space $\exists M^*_\sigma(Z)$. Moreover any reducible instanton $\exists [A]$ not isolated from $\exists M^*_\sigma(Z)$ is noncentral, and has a neighborhood $\exists U_{[A]}$ in $\exists M_\sigma(Z)$ such that $\exists U_{[A]} \cap M^*_\sigma(Z) \simeq [0, \epsilon)$ for some $\epsilon > 0$.

Proof. Let $\exists [A] \in \mathcal{M}_\sigma(Z)$ have $\exists H^1_A(E^C_{2,\sigma}) = \mathbb{C}$. From Proposition 5.9 and Remark 5.10 there are only finitely many such instantons, all of which are noncentral.

The irreducible part of a neighborhood $\exists U_{[A]}$ is identified with the $\exists U(1)$-quotient of $\exists \bar{\sigma}_A^{-1}(0) \cap V_A$, where $\exists V_A \subset H^1_A(E_{2,\sigma})$ is a neighborhood of the origin. We identify $\exists H^1_A(E_{2,\sigma}) \cong \mathbb{C} \oplus \mathbb{R} \oplus \mathbb{C}$, and $\exists H^2(E_{2,\sigma}) \cong \mathbb{C}$ so that the $\exists U(1)$-action are given respectively by $\exists e^{i\theta} \cdot (x_1, x_2, z) = (a, b, e^{2i\theta} z), \quad e^{i\theta} \cdot w = e^{2i\theta} w.$

To get a better understanding of how $\exists \sigma_A$ looks like, we recall its construction as follows. One first considers the map

$$
\ker d^\sigma_{A,\delta} \longrightarrow \im d^\sigma_{A,\delta} \\
a \mapsto \Pi(F^\sigma_{A+a} - \sigma(A + a)),
$$

where $\Pi : L_{k-1,\delta}^2(A^+T^*Z \otimes \mathfrak{su}(2)) \rightarrow \im d^\sigma_{A,\delta}$ is the $L_2^\delta$ orthogonal projection onto the image of $d^\sigma_{A,\delta}$. Since this map is a submersion, the implicit function theorem gives us a function $\exists \Phi_A : V_A \rightarrow \im d^\sigma_{A,\delta}^+$ so that $\exists A + a + \Phi_A(a) \in H^2(E_{2,\sigma})$. Then we let

$$
\sigma_A(a) := F^\sigma_{A+a+\Phi_A(a)} - \sigma(A + a + \Phi_A(a)).
$$

Note that $\sigma_A$ is analytic and vanishes at least up to second order by the virtue of its construction. Thus the $\exists U(1)$-equivariance forces the Kuranishi map to take
the following form
\[ \varrho_{A}(x_1, x_2, z) = f(x_1, x_2, |z|) \cdot z. \]
where \( f : i\mathbb{R} \oplus i\mathbb{R} \oplus \mathbb{C} \to \mathbb{C} \) vanishes at least up to first order. We further write
\[ f(x_1, x_2, |z|) : = \sum_{i \geq 0} f_i(x_1, x_2)|z|^i. \]

We note that \( \varrho_{A} \) vanishes at least to second order. Thus the second order term of \( \varrho_{A} \) at 0 is given by \( D\varrho_{A,\sigma}^{-1}|_{0} \), which is nonvanishing due to the transversality in Proposition 5.9. So up to an orientation-preserving change of coordinates, we may take \( f_0(x_1, x_2) = x_1 \pm ix_2 \). Since the complex \( (E^{U}_{\delta,\sigma}) \) is complex linear, we know that \( \varrho_{A}(0, 0, z) = f(0, 0, |z|) \cdot z \) is complex linear. Thus \( f_i(0, 0) = 0 \) for all \( i \geq 1 \).

It now follows that the zero set of \( \varrho_{A} \) is given by
\[ \varrho_{A}^{-1}(0) = \{x_1 = x_2 = 0\} \cup \{z = 0\}. \]

We then conclude that the normal part is identified with \( \mathbb{C}/U(1) \simeq [0, \infty) \).

\[ \square \]

**Definition 5.12.** Any reducible instanton \([A] \in \mathcal{M}^{red}_{\sigma}(Z)\) in Proposition 5.9 satisfying \( H_{\lambda}^1(E^{C}_{\delta,\sigma}) = \mathbb{C} \) is called a bifurcation point of \( \mathcal{M}^{red}_{\sigma}(Z) \).

### 5.3. The Orientation

At the end of this section, we discuss how we orient the perturbed moduli spaces. Formally an orientation of the moduli space \( \mathcal{M}_{\sigma}(Z) \) is a trivialization of the determinant line of the index bundle associated with the deformation complex parametrized by connections in \( \mathcal{M}_{\sigma}(Z) \). As we noted above, one cannot choose a uniform weight \( \delta \) so that the deformation complex \( (E_{\delta,\sigma}) \) is Fredholm for all instantons \([A] \in \mathcal{M}_{\sigma}(Z)\). So we only orient the portion of the moduli space that makes \( (E_{\delta,\sigma}) \) Fredholm.

Choose a weight \( \delta \) and a neighborhood \( \mathcal{O}_c \subset \chi(T^3) \) of the central connections as before. We write
\[ \mathcal{M}_{\sigma}(Z, \mathcal{O}_c^\prime) := \{[A] \in \mathcal{M}_{\sigma}(Z) : \partial_+ [A] \notin \mathcal{O}_c\} \]
for the portion of the moduli space consisting of instantons \([A]\) that are not asymptotic to an element in \( \mathcal{O}_c \). We first orient the unperturbed reducible locus \( \mathcal{M}^{red}_{\sigma}(Z) \) as follows. Note the unperturbed deformation complex \( (E^{U}_{\delta}(1)) \) is independent of \([A] \in \mathcal{M}^{red}_{\sigma}(Z)\), thus the corresponding index bundle is trivialized automatically once we fix a trivialization at a single point. According to [12], Proposition 3.12, its determinant is identified with
\[ \det \text{Ind}(E^{U}_{\delta,(1)}) = \lambda^{\max}H^0(Z; i\mathbb{R})^* \otimes \lambda^{\max}H^1(Z; i\mathbb{R}). \]

Following the path \( (E^{U}_{\delta,(t)}(1)), t \in [0, 1], \) we use the orientation on \( \text{Ind}(E^{U}_{\delta,(t)}(1)) \) to orient the perturbed index bundle \( \text{Ind}(E^{U}_{\delta,\sigma}(1)) \) for all small perturbations \( \sigma \). The \( SU(2) \) deformation complex \( (E_{\delta,\sigma}) \) splits into the direct sum of two complexes \( (E^{U}_{\delta,\sigma}(1)) \) and \( (E^{C}_{\delta,\sigma}) \) at a reducible instanton \([A] \in \mathcal{M}^{red}_{\sigma}(Z, \mathcal{O}_c^\prime)\). The complex structure on \( (E^{C}_{\delta,\sigma}) \) provides us with a canonical orientation. Combining with an orientation on \( (E^{U}_{\delta,\sigma}(1)) \), we get a trivialization of \( \det \text{Ind}(E_{\delta,\sigma}) \) on the reducible locus \( \mathcal{M}^{red}_{\sigma}(Z, \mathcal{O}_c^\prime)\).
Now we discuss how we orient the irreducible moduli space $\mathcal{M}_\sigma^*(Z)$. Let $[A] \in M_\sigma^*(Z)$. Recall that we only allow small perturbations $\sigma$ so that $\partial_+ [A] \notin O_c$. We choose a path $[A_i]$ in the space of connections asymptotic to flat connections not in $O_c$ with exponential decay rate $-\delta$ so that $[A_i] = [A]$ and $[A_0] \in \mathcal{M}_{\text{red}}(Z, O_c)$. Then the orientation at $[A_0]$ will induce one at $[A]$. The orientation does not depend on the choice of $[A_0]$ since the index bundle $\text{Ind}(E)$ is trivialized over $\mathcal{M}_{\text{red}}(Z, O_c^c)$.

To conclude, an orientation of $H^0(Z; i\mathbb{R})^* \oplus H^1(Z; i\mathbb{R})$ induces an orientation on $\mathcal{M}_\sigma(Z, O_c^c)$ for all small perturbations $\sigma$. The orientation of $Z$ induces an orientation on $H^0(Z; i\mathbb{R})$. We fix an orientation on $H^1(Z; i\mathbb{R}) \cong i\mathbb{R} \oplus i\mathbb{R}$ which is referred to as a homology orientation. Moreover to each bifurcation point $[A] \in \mathcal{M}_\sigma^*(Z)$ we assign a sign as follows.

**Definition 5.13.** Let $[A] \in \mathcal{M}_\sigma^*(Z)$ be a bifurcation point as in Proposition 5.9. We assign $+1$ (resp. $-1$) to $[A]$ if $f_0 : i\mathbb{R} \oplus i\mathbb{R} \to \mathbb{C}$ is orientation-preserving (resp. orientation-reversing), where $f_0$ is given by the Kuranishi obstruction map $\sigma_A$ in the proof of Proposition 5.9.

**Remark 5.14.** Since the local structure the moduli space near a bifurcation $[A]$ is modeled on $\sigma_A^{-1}(0)$, the $'+1'$ assignment describes the case when the path of irreducible instantons is pointing away from $[A]$, and the $'\!-1'$ assignment corresponds to the case when the path is pointing into $[A]$.

### 6. The Surgery Formula

#### 6.1. The Set-Up

We first give a more explicit description of the surgery operation. Let $X$ be an admissible integral homology $S^1 \times S^3$, and $T \hookrightarrow X$ an embedded torus inducing a surjective map on first homology, i.e. the map $H_1(T; \mathbb{Z}) \to H_1(X; \mathbb{Z})$ given by the inclusion is surjective. We fix a generator $1_X \in H^1(X; \mathbb{Z})$ serving as a homology orientation. We fix a framing of $T$ by choosing an identification $\nu(T) \cong D^2 \times T^2$. We write

$$
\mu = \partial D^2 \times \{pt.\} \times \{pt.\}, \lambda = \{pt.\} \times S^1 \times \{pt.\}, \gamma = \{pt.\} \times \{pt.\} \times S^1.
$$

Let’s denote by $M := X \setminus \nu(T)$ the closure of the complement of the tubular neighborhood. Then we have $H_*(M; \mathbb{Z}) \cong H_*(D^2 \times T^2; \mathbb{Z})$. We require the framing is chosen so that $[\lambda]$ generates $\ker (H_1(\partial M; \mathbb{Z}) \to H_1(M; \mathbb{Z}))$ and $1_X : [\gamma] = 1$.

Under this choice, the isotopy class of $\mu$ and $\lambda$ are fixed, but there is still ambiguity in choosing $\gamma$ which we will allow. Since the diffeomorphism type of the surgered manifold $X_{p,q} = M \cup_{\varphi_{p,q}} D^2 \times T^2$ is determined by the isotopy class of $[\varphi_{p,q}(\mu)] = p[\mu] + q[\lambda]$, the surgery operation is well-defined despite the framing ambiguity.

Note that only when $p = 1$ can the $(p, q)$-surgered manifold have the same homology as that of $S^1 \times S^3$. For simplicity, we write

$$
(6.1)\quad X_q = X_{1,q}, \quad X_0 = X_{0,1} \quad \text{and} \quad \varphi_q = \varphi_{1,q}, \quad \varphi_0 = \varphi_{0,1}.
$$

We also write $N = D^2 \times T^2$ and identify $T^3 = \partial M = -\partial N$. In this way, $X_q = M \cup_{\varphi_q} N$. Since the gluing map $\varphi_q$ preserves $[\gamma]$ for all $q$, we abuse the notation
Theorem 6.1. \([\gamma] \in H_1(X_q; \mathbb{Z})\) as a chosen generator. To define the Furuta-Ohta invariant one needs \(X_q\) to be admissible for \(q \neq 0\). This can be seen as follows. Let \(q \neq 0\). Any representation \(\rho : \pi_1(X_q) \to U(1)\) is determined by the image \(\rho([\gamma]) \in U(1)\) of the generator \([\gamma]\). So every representation on \(X_q\) comes from one on \(X\). Consider the following portion of the Mayer-Vietoris sequence:

\[
0 \to H^1(X_q; \mathbb{C}_\rho) \to H^1(M; \mathbb{C}_\rho) \oplus H^1(N; \mathbb{C}_\rho) \xrightarrow{j_q} H^1(T^3; \mathbb{C}_\rho),
\]

where \(j_q(\alpha, \beta) = \alpha|_{\partial M} - \varphi^*_q(\beta|_{\partial N})\). Since \(H^1(X, \mathbb{C}_\rho) = 0\), we conclude that \(\forall \alpha \in H^1(M, \mathbb{C}_\rho), \beta \in H^1(N, \mathbb{C}_\rho)\)

\[
\alpha|_{\partial M} - \beta|_{\partial N} = 0 \iff \alpha = 0, \beta = 0.
\]

Denote by \(r_M : H^1(M; \mathbb{C}_\rho) \to H^1(T^3; \mathbb{C}_\rho)\) and \(r_N : H^1(N; \mathbb{C}_\rho) \to H^1(T^3; \mathbb{C}_\rho)\) the restriction map. Note that \(\text{im } r_M \cap \text{im } r_N = \text{im } r_M \cap \text{im } \varphi^*_q \circ r_N\). Thus \(j_q(\alpha, \beta) = 0\) implies that \(\exists \beta' \in H^1(N; \mathbb{C}_\rho)\) such that \(\beta'|_{\partial N} = \varphi^*_q(\beta|_{\partial N})\), which further implies that \(\alpha = 0, \beta' = 0\), thus \(\beta = 0\). This shows that \(H^1(X_q; \mathbb{C}_\rho) = 0\).

Let \(E = \mathbb{C}^2 \times X\) be a trivialized \(\mathbb{C}^2\)-bundle over an admissible integral homology \(S^1 \times S^3\). We denote by \(\mathcal{M}_\sigma(X)\) the moduli space of perturbed ASD \(SU(2)\)-connections on \(E\). The vanishing of the twisted first homology ensures that the reducible locus \(\mathcal{M}^\text{red}_\sigma(X)\) is isolated from the irreducible locus \(\mathcal{M}^*_\sigma(X)\) for all small perturbations. Moreover the irreducible locus \(\mathcal{M}^*_\sigma(X)\) is an oriented compact 0-manifold. The Furuta-Ohta invariant \([\text{17}]\) is defined to be the signed count of irreducible instantons under a generic small perturbation:

\[
\lambda_{\text{FO}}(X) := \frac{1}{4} \# \mathcal{M}^*_\sigma(X).
\]

The proof of the surgery formula is based on a neck-stretching argument which we set up as follows. Recall we have the decomposition \(X = M \cup N\) with \(N = \nu(T)\) the tubular neighborhood of the embedded torus, and \(T^3 = \partial M = -\partial N\). Identify a neighborhood of \(T^3\) in \(X\) by \((-1, 1) \times T^3\). Let \(h\) be a flat metric on \(T^3\). We pick a metric \(g\) on \(X\) so that

\[
g|_{(-1, 1) \times T^3} = dt^2 + h.
\]

Given \(L > 0\), we stretch the neck \((-1, 1) \times T^3\) of \(X\) to obtain \((X_L, g_L)\):

\[
X_L = M \cup [-L, L] \times T^3 \cup N.
\]

The geometric limit is denoted by \(M_\circ := M \cup [0, \infty) \times T^3, N_\circ := (-\infty, 0] \times T^3 \cup N\), and \(X_\circ = M_\circ \cup N_\circ\).

Instead of proving Theorem \([\text{12}]\) directly, we prove the following special case when \(q = 1\).

**Theorem 6.1.** After fixing appropriate homology orientations, one has

\[
\lambda_{\text{FO}}(X_1) = \lambda_{\text{FO}}(X) + \frac{1}{2} D^0_{\text{wT}}(X_0).
\]

We explain why the special case is sufficient. Let’s denote by \(\mathcal{T}_q \subset X_q\) the image of the core \(\{0\} \times T^2 \subset D^2 \times T^2\) in \(X_q\) after the surgery performed. Then with respect to the framing of \(\mathcal{T}_q\) given by the gluing copy \(D^2 \times T^2\), \((1, 1)\)-surgery along
\( T_q \) results in \( X_{q+1} \), and \((0,1)\)-surgery along \( T_q \) results in \( X_0 \). Thus Theorem 6.1 is derived by applying Theorem 6.1 repetitively.

6.2. The Proof of Theorem 6.1. Recall \((X,g)\) is an admissible integral homology \( S^1 \times S^3 \) decomposed as \( X = M \cup_{T^3} N \), where \( N = \nu(T) \) is a tubular neighborhood of an embedded torus \( T \subset X \). We have framed \( N \cong D^2 \times T^2 \) with a basis \( \{\mu, \lambda, \gamma\} \) on \( \partial N = -T^3 \). The surgered manifolds \( X_1 \) and \( X_0 \) are given respectively by

\[
X_1 = M \cup_{\varphi_1} N \quad \text{and} \quad X_0 = M \cup_{\varphi_0} N,\]

where under the basis \( \{\mu, \lambda, \gamma\} \) on \(-T^3\) we have

\[
\varphi_1 = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix} \quad \text{and} \quad \varphi_0 = \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

We put a metric \( g_1 \) and \( g_0 \) on \( N \) such that \((\varphi^1)^*(g_1|_{\partial N}) = (\varphi^0)^*(g_0|_{\partial N}) = h\). Then we get the neck-stretched manifolds \( X_{1,L} \) and \( X_{0,L} \). Theorem 1.9 tells us that \( \partial_+ : M^*_\mu(M_0) \to \chi(T^3) \) is transverse to any given submanifold in \( \chi(T^3) \) and misses the singular points in \( \chi(T^3) \) for small perturbations. Note that \( \pi_1(N) \) is abelian. Thus the unperturbed moduli space consists of only reducible instantons.

Let's write

\[
\mathcal{M}_1 := M^*_\mu(M_0) \quad \text{and} \quad \mathcal{M}_2 := M^\text{red}(N_0).
\]

It follows from the standard gluing theorem (see for instance [5], [13], [16]) that

\[
\#M^*_\mu(X_L) = \#(\partial_+(\mathcal{M}_1) \cap \partial_-(\mathcal{M}_2))
\]

\[
\#M^*_\mu(X_{1,L}) = \#(\partial_+(\mathcal{M}_1) \cap \varphi_1^* \circ \partial_-(\mathcal{M}_2)),
\]

where \( \varphi_1^* : \chi(T^3) \to \chi(T^3) \) is the map induced by \( \varphi_1 \). To compare the difference we now put coordinates on the character variety \( \chi(T^3) \).

Let's first consider the \( U(1) \)-character variety \( \mathcal{R}^{U(1)}(T^3) \) which is a double cover of \( \chi(T^3) \). Recall that we have fixed a basis \( \{\mu, \lambda, \gamma\} \) for \( \partial M = T^3 \). To any \( U(1) \)-connection \( A_L \) we assign a coordinate \((x(A_L), y(A_L), z(A_L))\) given by

\[
x(A_L) = \frac{1}{2\pi i} \int_{\mu} a_L, \quad y(A_L) = \frac{1}{2\pi i} \int_{\lambda} a_L, \quad z(A_L) = \frac{1}{2\pi i} \int_{\gamma} a_L,
\]

where \( a_L = A - d \in \Omega^1(T^3, i\mathbb{R}) \) is the difference between \( A_L \) and the product connection. Modulo \( U(1) \)-gauge transformations, the coordinates \( x, y, z \) take values in \( \mathbb{R}/\mathbb{Z} \). Then the holonomies of \( A_L \) around \( \mu, \lambda, \gamma \) are given respectively by

\[
\text{Hol}_\mu A_L = e^{-2\pi ix}, \quad \text{Hol}_\lambda A_L = e^{-2\pi iy}, \quad \text{Hol}_\gamma A_L = e^{-2\pi iz}.
\]

If we restricts to the fundamental cube

\[
\mathcal{C}_{T^3} := \{(x, y, z) : x, y, z \in [-\frac{1}{2}, \frac{1}{2}]\},
\]

the \( SU(2) \)-character vareity \( \chi(T^3) \) is identified with the quotient of \( \mathcal{C}_{T^3} \) under the equivalence relations \((x, y, z) \sim (-x, -y, -z), (-\frac{1}{2}, y, z) \sim (\frac{1}{2}, y, z), (x, -\frac{1}{2}, z) \sim \)
(x, \frac{1}{2}, z), and (x, y, -\frac{1}{2}) \sim (x, y, \frac{1}{2}). We shall restrict further to the following portion of the fundamental cube involved in the proof:
\[ C_{T^3}^0 := \{(x, y, z) : x \in [-\frac{1}{2}, 0], y \in [0, \frac{1}{2}], z \in [-\frac{1}{2}, \frac{1}{2}]\}. \]

Then the equivalence relations above only identify points on the lower-strata of \( C_{T^3}^0 \), i.e. strata of dimension less than 3 consisting of the faces, edges, and vertices of the cube. Then the image \( \partial_-(M_2) \) is given by the quotient of the plane
\[ P_N := \{(0, y, z) : y \in [0, \frac{1}{2}], z \in [-\frac{1}{2}, \frac{1}{2}]\} \]
whose quotient \([P_M] \subset \chi(T^3)\) is a pillowcase. The image \( \partial_+(M_{o}^{\text{red}}) \) of the unperturbed reducible locus on the manifold \( M_o \) is given by the quotient of the plane
\[ P_M := \{(x, 0, z) : x \in [-\frac{1}{2}, 0], z \in [-\frac{1}{2}, \frac{1}{2}]\} \]
whose quotient \([P_M]\) is also a pillowcase. The image \( \varphi^*_+ \circ \partial_-(M_2) \) is given by the quotient of the plane
\[ P_1 := \{(x, -x, z) : x \in [-\frac{1}{2}, 0], z \in [-\frac{1}{2}, \frac{1}{2}]\} \]
whose quotient \([P_1] \subset \chi(T^3)\) is a cylinder \([0, 1] \times S^1\). Finally we consider a parallel copy of \( P_M \) given by
\[ P_0 := \{(x, \frac{1}{2}, z) : x \in [-\frac{1}{2}, 0], z \in [-\frac{1}{2}, \frac{1}{2}]\} \]

Figure 1. The Cube Portion \( C_{T^3}^0 \)
whose quotient \([P_0] \subset \chi(T^3)\) is again a pillowcase. We then orient the portion of the fundamental cube \(C_{T^3}^0\) by \(dx \wedge dy \wedge dz\). It’s straightforward to see that the equivalence relations defined on the faces of \(C_{T^3}^0\) is orientation-preserving. Thus all the top strata of the quotient of the planes \(P_N, P_M, P_1,\) and \(P_0\) are oriented by the orientation induced from that of \(C_{T^3}^0\). Let’s consider a solid

\[
V := \{(x, y, z) : x + y \geq 0, x \in [-1/2, 0], y \in [0, 1/2], z \in [-1/2, 1/2]\}.
\]

Then the quotient \([V]\) is enclosed by \([-[P_1], [P_N], \) and \([P_0]\) in \(\chi(T^3)\). Note that

\[
V \cap P_M = \{(0, 0, z) : z \in [-1/2, 1/2]\}.
\]

Let \([A] \in \mathcal{M}^{red}(M_o)\) be a non-central instanton such that \(\partial_+([A]) \in [V] \subset \chi(T^3)\). The admissibility of \(X\) implies that \(H^1(M_o; \text{ad} A_C) = 0\) since \([A]\) comes from a reducible instanton on \(X\). Thus \([V]\) avoids the asymptotic values of the bifurcation points in \(\mathcal{M}_\sigma(M_o)\) with respect to small perturbations. By choosing generic perturbations making \(\partial_+\) transverse to \([V]\), we conclude that

\[
\#\mathcal{M}_\sigma^+(X_1) - \#\mathcal{M}_\sigma^+(X) = \#(\partial_+(M_1) \cap [P_1]) - \#(\partial_+(M_1) \cap [P_N]) = \#\partial_+^{-1}([P_0]).
\]

Now the proof has been reduced to the following result.

**Proposition 6.2.** Continuing with notations above, one has

\[
\#\partial_+^{-1}([P_0]) = 2D_{w_T}^0(X_0),
\]

where \(D_{w_T}^0(X_0)\) counts the irreducible anti-self-dual \(SO(3)\)-instantons on the \(\mathbb{R}^3\)-bundle \(E_0 \rightarrow X_0\) characterized by

\[
p_1(E_0) = 0, \ w_2(E_0) = \text{PD}[T_0] \in H^2(X_0; \mathbb{Z}/2),
\]

and \(T_0 \subset X_0\) is the core torus of the gluing \(D^2 \times T^2\).

**Proof.** From its construction, \(\#\partial_+^{-1}([P_0])\) counts the irreducible \(SU(2)\)-instantons on \(M_o\) whose asymptotic holonomy around \(\lambda\) is \(\text{Diag}(-1) \subset SU(2)\). Through the isomorphism \(\text{ad} : \mathfrak{su}(2) \rightarrow \mathfrak{so}(3)\), every \(SU(2)\)-connection gives rise to an \(SO(3)\)-connection. Let \([A] \in \partial_+^{-1}([P_0])\). Then \(A\) corresponds to a perturbed anti-self-dual \(SO(3)\)-connection \(A'\) whose asymptotic holonomy around \(\lambda\) is the identity. Since performing 0-surgery amounts to gluing \(D^2 \times T^2\) by sending the meridian \(\partial D^2 \times \{\text{pt}\} \times \{\text{pt}\}\) to \(\lambda\), the gluing theorem for \(SO(3)\)-instantons glues \([A']\) to an \(SO(3)\)-instanton \([A'_0]\) on \(E_0\). The fact that \(A'_0\) fails to lift to an \(SU(2)\)-connection forces \(w_2(E_0) = \text{PD}[T_0]\).

To see why there is a factor of ‘2’ in the equation, we note the \(SU(2)\)-gauge group and \(SO(3)\)-gauge group fit into the exact sequence

\[
\text{Map}(M_o, \pm 1) \rightarrow G^{SU(2)}(M_o) \rightarrow G^{SO(3)}(M_o).
\]

Thus each \(SU(2)\)-gauge equivalence class corresponds to two \(SO(3)\)-gauge equivalence classes. □
6.3. An Application to Finite Order Diffeomorphisms. This subsection is devoted to the proof of Proposition 4.4 using the surgery formula. We briefly recall the set-up. Let $\mathcal{K} \subset Y$ be a knot in an integral homology $S^1 \times S^3$. We write $\Sigma_n$ for the $n$-fold cyclic cover of $Y$ branched along $\mathcal{K}$, and $\tau_n : \Sigma_n \to \Sigma_n$ for the covering translation. We assume $\Sigma_n$ is a rational homology sphere, and denote by $X_n$ the mapping torus of $\Sigma_n$ under the covering translation $\tau_n$.

Proof of Proposition 4.4. The argument is exactly the same as in the case of the Casson-Seiberg-Witten invariant [12, Proposition 1.2]. We denote by $\mathcal{T} \subset X_n$ the mapping torus of the branching set $\tilde{\mathcal{K}} \subset \Sigma_n$, and $X'_n$ the manifold resulted from performing $(1,1)$-surgery of $X_n$ along $\mathcal{T}$. Lemma 7.1 in [12] tells us that the restriction of the covering translation $\tau_n$ on the knot complement extends to a free self-diffeomorphism $\tau'_n : \Sigma'_n \to \Sigma'_n$ with $\Sigma'_n$ the manifold given by performing 1-surgery of $\Sigma_n$ along $\tilde{\mathcal{K}}$. Moreover $X'_n$ is the mapping torus of $\Sigma'_n$ under $\tau'_n$. From Corollary 7.7 in [17], we have

$$\lambda_{FO}(X'_n) = n\lambda(Y) + \frac{1}{8} \sum_{m=0}^{n-1} \text{sign}^{m/n}(Y, \mathcal{K}) + \frac{1}{2} \Delta_{\mathcal{K}}'(1).$$

Let's denote by $X_0^n$ the manifold obtained by performing $(0,1)$-surgery of $X_n$ along $\mathcal{T}$. In the proof of [12, Proposition 1.2], it has been shown that $X_0^n = S^1 \times Y_0(\mathcal{K})$. Combining the surgery formula and Corollary 1.3 we get

$$\lambda_{FO}(X_n) = n\lambda(Y) + \frac{1}{8} \sum_{m=0}^{n-1} \text{sign}^{m/n}(Y, \mathcal{K}). \quad \square$$

7. The Excision Formula

7.1. The Set-Up. We start with a more explicit description of the excision operation. Let $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ be two pairs of admissible homology $S^1 \times S^3$ with an essentially embedded torus. We choose an identification $\nu(\mathcal{T}_i) \cong D^2 \times T^2$ for a tubular neighborhood of $\mathcal{T}_i$ as in Section 6 so that we get a basis $\{\mu_i, \lambda_i, \gamma_i\}$ of $\partial \nu(\mathcal{T}_i) = \partial \mathcal{T}_i$ for each $i$. Let $\varphi : \partial M_2 \to \partial M_1$ be a diffeomorphism so that the manifold

$$X_1 \#_{\varphi} X_2 := M_1 \cup_{\varphi} M_2$$

is an admissible homology $S^1 \times S^3$. Let $A_{\varphi}$ be the matrix representing the induced map $\varphi_* : H_1(\partial M_2; \mathbb{Z}) \to H_1(\partial M_1; \mathbb{Z})$ under the basis $\{[\mu_i], [\lambda_i], [\gamma_i]\}$. Over $D^2 \times T^2$, we write

$$\mu' = \{pt.\} \times S^1 \times \{pt.\}, \lambda' = \partial D^2 \times \{pt.\} \times \{pt.\}, \gamma' = \{pt.\} \times \{pt.\} \times S^1.$$  

We let $X_1, \varphi := M_1 \cup_{\varphi_1} D^2 \times T^2$ and $X_2, \varphi := D^2 \times T^2 \cup_{\varphi_2} M_2$ with the gluing map $\varphi_i$ inducing the matrix $A_{\varphi}$ on first homology groups with respect to the bases $\{[\mu_i], [\lambda_i], [\gamma_i]\}$ and $\{[\mu'], [\lambda'], [\gamma']\}$.

Since we require $X_1 \#_{\varphi} X_2$ to be an admissible homology $S^1 \times S^3$, the form of the matrix $A_{\varphi}$ can be described more explicitly. Due to the ambiguity of the
choice of $\gamma_i$, one can find a framing of $\nu(\mathcal{T}_i)$ by adding to $\gamma_1$ certain multiples of $\mu_1$ and $\lambda_1$ so that $A_{\varphi}$ has the form

$$A_{\varphi} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ p & q & 1 \end{pmatrix}. $$

Since $A_{\varphi}$ is orientation-reversing, we have $\det A_{\varphi} = -1$. With the help of the Mayer-Vietoris sequence, one can show that $X_1 \#_{\varphi} X_2$ is an integral homology $S^1 \times S^3$ if and only if $\gcd(aq,b) = 1$ and $(aq)^2 + b^2 \neq 0$. If we wish to repeat the argument in Section 6 to derive the admissibility of $X_1 \#_{\varphi} X_2$ from that of $X_1$ and $X_2$ purely on the homology level, i.e. $\text{im} r_{M_1} \cap \text{im} r_{M_2} = \text{im} r_{M_1} \cap \text{im} \varphi^* \circ r_{M_2}$, we get

$$b = \pm 1, q = 0. $$

We note that the diffeomorphism type of $X_{1,\varphi}$ is determined by the image $\varphi_1,*(|\mathcal{X}|) \in H_1(\partial M_1; \mathbb{Z})$. Thus in this case, $X_{1,\varphi}$ is obtained from $X_1$ via the $(1,d)$-surgery. The same can be derived for $X_{2,\varphi}$. In general we need to know more about the topology of $X_1$ and $X_2$ to determine whether $X_1 \#_{\varphi} X_2$ is admissible with a given matrix $A_{\varphi}$.

When we take the fiber sum of $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$, the gluing map $\varphi_\mathcal{T}$ corresponds to the matrix:

$$A_{\varphi_{\mathcal{T}}} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

We see that $X_{i,\varphi_{\mathcal{T}}} = X_i$, $i = 1, 2$, and the fiber sum $X_1 \#_{\mathcal{T}} X_2$ is an admissible integral homology $S^1 \times S^3$ as $b = 1, q = 0$ in this case.

To apply the neck-stretching argument, we put metrics $g_1$ on $X_1$ and $g_2$ on $X_2$ so that when restricting to collar neighborhoods of $\partial M_1$ and $\partial M_2$ they are respectively of the form

$$dt^2 + \varphi^* h \text{ and } dt^2 + h, $$

where $h$ is a flat metric on $T^3$. We also let $L$ be the length-parameter of the neck, and write various neck-stretched manifolds as in Section 6.

We say one more word about the choice of perturbations. The purpose of perturbing the anti-self-dual equation is to achieve various transversality properties, i.e. Proposition 4.3, Proposition 4.7, Proposition 5.6, and Proposition 5.9 which correspond to the surjectivity of certain differential operators. Thus one can use perturbations of compact support since the transversality is an open condition. So given perturbations of compact support $\sigma_i$ on the end-cylindrical manifold $M_{i,o}$, we get a perturbation $\sigma_1 \#_{\varphi} \sigma_2$ on $X_1 \#_{\varphi} X_2$ when the neck is stretched long enough.

7.2. The Proof of Theorem 1.5. We shall omit the neck-length parameter $L$ in the notations, which is chosen to be large enough to apply the gluing argument. To simplify the notation, we write $N = D^2 \times T^2$. Depending on the context, $N_o$
could mean either \(N \cup [0, \infty) \times T^3\) or \((\infty, 0] \times T^3 \cup N\). The gluing theorem tells us that
\[
\#M^*_{\sigma_1}(X_{1, \varphi}) = \#(\partial_+ (M^*_{\sigma_1}(M_{1, o})) \cap \varphi^* \circ \partial_- (M^\text{red}(N_o)))
\]
\[
\#M^*_{\sigma_2}(X_{2, \varphi}) = \#(\partial_+ (M^\text{red}(N_o)) \cap \varphi^* \circ \partial_- (M^*_{\sigma_2}(M_{2, o})))
\]
and moreover
\[
\#M^*_{\sigma_1 \# \sigma_2}(X_{1 \# \varphi} X_2) = \#(\partial_+ (M^\text{red}(M_{1, o})) \cap \varphi^* \circ \partial_- (M^*_{\sigma_2}(M_{2, o})))
\]
\[+ \#(\partial_+ (M^*_{\sigma_1}(M_{1, o})) \cap \varphi^* \circ \partial_- (M^*_{\sigma_2}(M_{2, o})))
\]
\[+ \#(\partial_+ (M^*_{\sigma_1}(M_{1, o})) \cap \varphi^* \circ \partial_- (M^\text{red}(M_{2, o})))\].

The counting on right hand side of the third equation above makes sense because we can first fix a generic \(\sigma_2\), then choose a generic \(\sigma_1\) so that the asymptotic map \(\partial_+ : M^*_{\sigma_1}(M_{1, o}) \to \chi(T^3)\) is transverse to \(\varphi^* \circ \partial_- (M^*_{\sigma_2}(M_{2, o}))\).

As in the proof of Theorem 6.1, we regard the character variety \(\chi(T^3)\) as the quotient the fundamental cuber \(C_T\) under appropriate relations. We identify the copy \(T^3 = \partial M_1\) with a basis given by \(\{\mu_1, \lambda_1, \gamma_1\}\). Then
\[
\partial_+ (M^\text{red}(M_{1, o})) = [P_M] \text{ and } \varphi^* \circ \partial_- (M^\text{red}(M_{2, o})) = \varphi^*[P_M],
\]
where \(P_M\) is defined in (6.4). The admissibility of \(X_{1 \# \varphi} X_2\) ensures there is no bifurcation points on neither \(M^\text{red}(M_{1, o})\) nor \(M^\text{red}(M_{2, o})\) asymptotic to \([P_M] \cap \varphi^*[P_M]\). Since \(\dim M^*_{\sigma_1}(M_{1, o}) = \dim M^*_{\sigma_2}(M_{2, o}) = 1\), the transversality of the asymptotic maps implies that
\[
(7.1) \quad \partial_+ (M^*_{\sigma_1}(M_{1, o})) \cap \varphi^* \circ \partial_- (M^*_{\sigma_2}(M_{2, o})) = \emptyset.
\]

The proof of Proposition 4.3 gives us an isotopy from the unperturbed reducible locus \(M^\text{red}(M_{2, o})\) to the perturbed one \(M^\text{red}(M_{2, o})\). Thus we conclude the counting
\[
\#(\partial_+ (M^*_{\sigma_1}(M_{1, o})) \cap \varphi^* \circ \partial_- (M^*_{\sigma_2}(M_{2, o})))
\]
is equal to
\[
\#(\partial_+ (M^*_{\sigma_1}(M_{1, o})) \cap \varphi^* \circ \partial_- (M^\text{red}(M_{2, o}))).
\]
Note that \(\varphi^* \circ \partial_- (M^\text{red}(M_{2, o})) = \varphi^* \circ \partial_- M^\text{red}(N_o)\). Thus
\[
(7.2) \quad \#(\partial_+ (M^*_{\sigma_1}(M_{1, o})) \cap \varphi^* \circ \partial_- (M^\text{red}(M_{2, o}))) = \#M^*_{\sigma_1}(X_{1, \varphi}).
\]
Similarly we have
\[
(7.3) \quad \#(\partial_+ (M^\text{red}(M_{1, o})) \cap \varphi^* \circ \partial_- (M^*_{\sigma_2}(M_{2, o}))) = \#M^*_{\sigma_2}(X_{2, \varphi}).
\]
Combining (7.1), (7.2), and (7.3), we conclude that
\[
\#M^*_{\sigma_1 \# \sigma_2}(X_{1 \# \varphi} X_2) = \#M^*_{\sigma_1}(X_{1, \varphi}) + \#M^*_{\sigma_2}(X_{2, \varphi})
\]
which finishes the proof.
7.3. Examples. In this subsection we compute the Furuta-Ohta invariants for two families of admissible integral homology $S^1 \times S^3$ arisen from mapping tori under diffeomorphisms of infinite order.

Example 7.1. Let $(Y_1, \mathcal{K}_1)$ and $(Y_2, \mathcal{K}_2)$ be two pairs of integral homology sphere with an embedded knot. Fix two integers $n_1, n_2 > 1$. In what follows, $j = 1$ or $j = 2$. We denote by $\Sigma_j$ the cyclic $n_j$-fold cover of $Y_1$ branched along $\mathcal{K}_j$, and $\tilde{\mathcal{K}}_j$ the preimage of $\mathcal{K}_j$ in the cover $\Sigma_j$.

Now we take $X_j$ to be the mapping torus of $\Sigma_j$ under the covering translation, and $T_j$ the mapping torus of $\tilde{\mathcal{K}}_j$. Then the fiber sum formula tells us that the Furuta-Ohta invariant of $X_1 \#_T X_2$ is given by

$$\lambda_{FO}(X_1 \#_T X_2) = n_1 \lambda(Y_1) + \frac{1}{8} \sum_{m_1=1}^{n_1-1} \text{sign}^{m_1/n_1}(Y_1, \mathcal{K}_1)$$

$$+ n_2 \lambda(Y_2) + \frac{1}{8} \sum_{m_2=1}^{n_2-1} \text{sign}^{m_2/n_2}(Y_2, \mathcal{K}_2).$$

We claim that the fiber $X_1 \#_T X_2$ is the mapping torus of the knot splicing, denoted by $\Sigma_1 \#_{T_1} \Sigma_2$, under certain self-diffeomorphism. We denote by $T_j$ the covering translation on $\Sigma_j$. A $\tau_j$-invariant neighborhood of $\tilde{\mathcal{K}}_j$ is identified with $S^1 \times D^2$ where $\tau_j$ acts as

$$\tau_j(e^{i\eta_j}, re^{i\theta_j}) = (e^{i\eta_j}, re^{i(\theta_j + \frac{2\pi}{n_j})}).$$

A neighborhood of $T_j$ is now identified with $[0, 1] \times S^1 \times D^2 / \sim$, for which we identify with $D^2 \times D^2$ as follows:

$$[t, e^{i\eta_j}, e^{i\theta_j}] \mapsto ([t, e^{i\eta_j}, re^{i(\theta_j + \frac{2\pi}{n_j})}]).$$

Under the identifications above, along the mapping circle the knot complements $V_1 := \Sigma_1 \setminus \nu\tilde{\mathcal{K}}_1$ and $V_2 := \Sigma_2 \setminus \nu\tilde{\mathcal{K}}_2$ are glued at time $t$ via the map

$$\phi_t : \partial V_2 \longrightarrow \partial V_1$$

$$(e^{i\eta_2}, e^{i\theta_2}) \longmapsto (e^{i(\theta_2 + \frac{2\pi}{n_2})}, e^{i(\eta_2 - \frac{2\pi}{n_1})}).$$

We write $F_t := V_1 \cup_{\phi_t} [0, 1] s \times D^2 \cup_{\partial V_2} V_2$ for the fiber at time $t$. We identify $F_t$ with $F_0$ by inserting the isotopy $\phi_0^{-1} \circ \phi_{(1-s)}$, from $\phi_0^{-1} \circ \phi_t$ to $\text{id}$ along $[0, 1] \times D^2$, and denote by $f_t : F_t \rightarrow F_0$ this identification. From its construction, $F_0$ is the knot splicing $\Sigma_1 \#_{T_1} \Sigma_2$. To see how the monodromy map looks like, we note that for $x \in \{0\} \times D^2 \subset F_1$, one has

$$\phi_0 \circ \tau_2(x) = \tau_1 \circ \phi_1(x).$$

Thus $\tau_1$ and $\tau_2$ combine to a map $\tau_1 \# \tau_2 : F_1 \rightarrow F_0$. So the monodromy map is given by $\tau_1 \# \tau_2 \circ f^{-1}_1 : F_0 \rightarrow F_0$ whose restriction to the neck $[0, 1] \times D^2 \subset F_0$ has the form

$$(s, e^{i\eta}, e^{i\theta}) \longmapsto (s, e^{i(\eta + (1-s) \frac{2\pi}{n_1})}, e^{i(\theta + s \frac{2\pi}{n_2})}).$$
In particular the monodromy map is of infinite order.

**Example 7.2.** We consider the ‘Dehn twist’ along a torus in this example. Let $Y = Y_1 \#_k Y_2$ be a splicing of two integral homology spheres along embedded knots. We denote by $V_i$ the knot complement in $Y_i$, and write $Y = V_1 \cup [0, 1]_s \times T^2 \cup V_2$. We use $(e^{i\eta}, e^{i\theta}) \in S^1 \times S^1$ to parametrize $T^2$ so that $S^1 \times \{pt.\}$ is null-homologous in $V_1$ and $\{pt.\} \times S^1$ is null-homologous in $V_2$. Let $p, q$ be a relatively prime pair and $c : [0, 1] \to T^2$ be a curve

$$c(t) := (e^{i(\eta t + 2\pi p)}, e^{i(\theta t + 2\pi q)}).$$

The Dehn twist along $c$ is a diffeomorphism $\tau_c : Y \to Y$ whose restriction on $V_1$ and $V_2$ is identity, and on the neck $[0, 1] \times T^2$ is given by

$$\tau_c(s, e^{i\eta}, e^{i\theta}) = (s, e^{i(\eta + s2\pi p)}, e^{i(\theta + s2\pi q)}).$$

Then we see that $\tau_c$ has infinite order. Let $X_c$ be the mapping torus of $Y$ under $\tau_c$. Since $Y$ is an integral homology sphere, $X_c$ is an admissible homology $S^1 \times S^3$.

We claim that $X_c$ is given by torus excision. Let $M_1 = S^1 \times V_1$ and $M_2 = S^1 \times V_2$. We regard $X_c = [0, 1]_t \times Y / \sim$, where $(0, \tau_c(y)) \sim (1, y)$. Since $\tau_c|_{[0, 1] \times T^2}$ is isotopic to identity, we can identify $[0, 1]_t \times ([0, 1]_s \times T^2 \cup V_2)/ \sim$ with $[0, 1]_t \times T^2 \cup M_2$ by

$$(t, s, e^{i\eta}, e^{i\theta}) \mapsto (s, e^{2\pi t}, e^{i(\eta + (1-t)s2\pi p)}, e^{i(\theta + (1-t)s2\pi q)}).$$

Then $[0, 1]_t \times T^2 \cup M_2$ is glued to $M_1$ by

$$(e^{i\xi}, e^{i\eta}, e^{i\theta}) \mapsto (e^{i\xi}, e^{i(\eta+p\xi)}, e^{i(\theta-p\xi)}).$$

In terms of the gluing matrix, the gluing map $\phi$ is given by

$$A_\phi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -p & -q & 1 \end{pmatrix}.$$  

If we write $X_{1,\phi} = M_1 \cup_\phi D^2 \times T^2$, $X_{2,\phi} = D^2 \times T^2 \cup_\phi M_2$, then

$$\lambda_{\text{FO}}(X_c) = \lambda_{\text{FO}}(X_{1,\phi}) + \lambda_{\text{FO}}(X_{2,\phi}).$$

Finally, we note that $X_{i,\phi}$ is obtained from $S^1 \times Y_i$ by a torus surgery. However, the gluing map $\phi$ is not the type we considered in Theorem 1.2 and we don’t know how to compare $\lambda_{\text{FO}}(X_{i,\phi})$ with $\lambda_{\text{FO}}(S^1 \times Y_i)$ in this case.

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