Bridging the Gap Between $f$-GANs and Wasserstein GANs

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Abstract

Generative adversarial networks (GANs) have enjoyed much success in learning high-dimensional distributions. Learning objectives approximately minimize an $f$-divergence ($f$-GANs) or an integral probability metric (Wasserstein GANs) between the model and the data distribution using a discriminator. Wasserstein GANs enjoy superior empirical performance, but in $f$-GANs the discriminator can be interpreted as a density ratio estimator which is necessary in some GAN applications. In this paper, we bridge the gap between $f$-GANs and Wasserstein GANs (WGANs). First, we list two constraints over variational $f$-divergence estimation objectives that preserves the optimal solution. Next, we minimize over a Lagrangian relaxation of the constrained objective, and show that it generalizes critic objectives of both $f$-GAN and WGAN. Based on this generalization, we propose a novel practical objective, named KL-Wasserstein GAN (KL-WGAN). We demonstrate empirical success of KL-WGAN on synthetic datasets and real-world image generation benchmarks, and achieve state-of-the-art FID scores on CIFAR10 image generation.

1 Introduction

Learning generative models to sample from complex, high-dimensional distributions is an important task in machine learning with many important applications, such as image generation (Kingma and Welling, 2013), imitation learning (Ho and Ermon, 2016) and representation learning (Chen et al., 2016). Generative adversarial networks (GANs, Goodfellow et al. (2014)) introduce a widely popular approach to learning likelihood-free deep generative models (Mohamed and Lakshminarayanan, 2016), where one learns a generative model via finding the equilibrium of a two-player minimax game between a generator and a critic (discriminator). Assuming the optimal critic is obtained, one could typically cast the GAN learning procedure as minimizing some discrepancy measure between the generative model and the data distribution.

Various GAN learning procedures have been proposed for different discrepancy measures. $f$-GANs (Nowozin et al., 2016) minimize a variational approximation of the $f$-divergence between two distributions (Csiszár, 1964; Nguyen et al., 2008), where the critic acts as a density ratio estimator (Uehara et al., 2016; Grover and Ermon, 2017). This includes the original GAN approach (Goodfellow et al., 2014) which can be seen as performing variational Jensen-Shannon divergence minimization. Learning the density ratio between two distributions can be used for importance sampling, and have a range of practical applications such as mutual information estimation (Hjelm et al., 2018), off-policy policy evaluation (Liu et al., 2018), and de-biasing of generative models (Grover et al., 2019).

Another family of GAN approaches are developed based on Integral Probability Metrics (IPMs, Müller (1997)), where the critic is restricted to particular function families. For the family of Lipschitz-1 functions, the IPM reduces to the Wasserstein-1 or earth mover’s distance (Rubner et al., 2000), which motivates the Wasserstein GAN (WGAN, Arjovsky et al. (2017)) approach. Various approaches have been applied to enforce Lipschitzness, including weight clipping (Arjovsky et al., 2017), gradient penalty (Gulrajani et al., 2017) and spectral normalization (Miyato et al., 2018). While Wasserstein GAN approaches have enjoyed strong empirical success in image generation (Karras et al., 2017; Brock et al., 2018), the learned critic cannot be interpreted as a density ratio estimator, which prevents its uses for importance sampling.

In this paper, we propose a generalized view of $f$-GANs and WGANs, from which we derive a new approach named $f$-WGAN. First, we introduce two constraints
that preserves the optimal solution to the variational $f$-divergence estimation problem (Figure 1). We also discuss the connections between these constraints and “change of measure” inequalities.

Next, we relax the problem by considering a minimization problem over a Lagrangian of the constrained problem. By considering the minimum solution over different feasible sets, we are able to generalize the critic objectives of $f$-GAN and WGAN. We then propose the $f$-WGAN critic objective by minimizing over another set, and show $f$-WGAN critic objectives are bounded between $f$-GAN and WGAN ones (Figure 2). Finally, we derive close-form solutions to the minimization problem in $f$-WGAN critic objectives for certain families of $f$, allowing us to bypass iterative minimization inner loops. This results in KL-Wasserstein GAN (KL-WGAN), a practical algorithm that is easy to derive from existing WGAN implementations, has density ratio interpretations, and has similar computational cost in training compared to WGAN. Empirical results demonstrate that KL-WGAN enjoys superior quantitative performance compared to its WGAN counterparts on several benchmarks. Notably, KL-WGAN achieves state-of-the-art FID scores (Heusel et al., 2017) on CIFAR10 image generation with BigGAN architectures (Brock et al., 2018).

2 Preliminaries

Notations Let $X$ denote a random variable with separable sample space $\mathcal{X}$ and let $\mathcal{P}(\mathcal{X})$ denote the set of all probability measures over the Borel $\sigma$-algebra on $\mathcal{X}$. We use $P$, $Q$ to denote probability measures, and $P \ll Q$ to denote $P$ is absolutely continuous with respect to $Q$, i.e. the Radon-Nikodym derivative $dP/dQ$ exists. Under $Q \in \mathcal{P}(\mathcal{X})$, the $p$-norm of a function $r: \mathcal{X} \rightarrow \mathbb{R}$ is defined as

$$\|r\|_p := \left( \int |r|^p dQ \right)^{1/p},$$

with $\|r\|_\infty = \lim_{p \rightarrow \infty} \|r\|_p$. The set of locally $p$-integrable functions is defined as

$$L^p(Q) := \{r: \mathcal{X} \rightarrow \mathbb{R} : \|r\|_p < \infty\},$$

i.e. its norm with respect to $Q$ is finite. We denote $L^p_{\geq 0}(Q) := \{r \in L^p(Q) : \forall x \in \mathcal{X}, r(x) \geq 0\}$ which considers non-negative functions in $L^p(Q)$. The space of probability measures wrt. $Q$ is defined as

$$\{r \in L^1(Q) : \|r\|_1 = 1, \forall x \in \mathcal{X}, r(x) \geq 0\}.$$  

For example, for any $P \ll Q$, $dP/dQ \in \Delta(Q)$ because $\int (dP/dQ) dQ = 1$. We define $\text{im}(\cdot)$ and $\text{dom}(\cdot)$ as image and domain of a function respectively.

Fenchel duality For functions $g: \mathcal{X} \rightarrow \mathbb{R}$ defined over a Banach space $\mathcal{X}$, the Fenchel dual of $g$, $g^*: \mathcal{X}^* \rightarrow \mathbb{R}$ is defined over the dual space $\mathcal{X}^*$ by:

$$g^*(x^*) := \sup_{x \in \mathcal{X}} \langle x^*, x \rangle - g(x),$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing. The dual space of a finite dimensional space $\mathbb{R}^d$ is also $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ is the usual inner product (Rockafellar, 1970).

Generative adversarial networks In generative adversarial networks (GANs, Goodfellow et al. (2014)), the goal is to fit an (empirical) data distribution $P_{\text{data}}$ with an implicit generative model over $X$, denoted as $G_\theta \in \mathcal{P}(X)$. $G_\theta$ is defined implicitly via the process $X = g_\theta(Z)$, where $Z$ is a random variable with a fixed prior distribution. Assuming access to i.i.d. samples from $P_{\text{data}}$ and $G_\theta$, a discriminator $D_\phi: \mathcal{X} \rightarrow [0, 1]$ is used to classify samples from the two distributions, leading to the following objective:

$$\min_{\theta} \max_{\phi} \mathbb{E}_{x \sim P_{\text{data}}} [\log(D_\phi(x))] + \mathbb{E}_{x \sim G_\theta} [\log(1 - D_\phi(x))].$$

If we have infinite samples from $P_{\text{data}}$, and $D_\phi$ and $G_\theta$ are sufficiently expressive, then the above minimax objective will reach an equilibrium where $G_\theta = P_{\text{data}}$ and $D_\phi(x) = 0.5$ for all $x \in \mathcal{X}$.

2.1 Variational Representation of $f$-Divergences

For any convex, lower-semicontinuous function $f: [0, \infty) \rightarrow \mathbb{R}$ satisfying $f(1) = 0$, $f$-divergences (Csiszár, 1964; Ali and Silvey, 1966) between two probabilistic measures $P, Q \in \mathcal{P}(\mathcal{X})$ are defined as:

$$D_f(P||Q) := \mathbb{E}_Q \left[ f \left( \frac{dP}{dQ} \right) \right]$$

$$= \int_{\mathcal{X}} f \left( \frac{dP}{dQ}(x) \right) dQ(x),$$

if $P \ll Q$ and $+\infty$ otherwise. Nguyen et al. (2010) derive a general variational method to estimate $f$-divergences given only samples from $P$ and $Q$.

Lemma 1 (Nguyen et al. (2010)). $\forall P, Q \in \mathcal{P}(\mathcal{X})$ such that $P \ll Q$,

$$D_f(P||Q) = \sup_{T \in L^1(Q)} I_f(T; P, Q),$$

where $I_f(T; P, Q) := \mathbb{E}_P[T] - \mathbb{E}_Q[f^*(T)]$ and the supremum is achieved when $T = f'(dP/dQ)$.

In the context of GANs, Nowozin et al. (2016) proposed variational $f$-divergence minimization where one estimates $D_f(P_{\text{data}}||G_\theta)$ with the variational lower bound
As described earlier in Lemma 1, the optimal solution in Eq.(7) is exactly \( f'(r) \); the special structure of the optimal solution motivates us to perform a reparametrization from \( T \) to \( f'(r) \). Instead of optimizing over \( T \in L_\infty(Q) \) as in Eq.(7), we consider \( r \in L_\infty(Q) \), a subset of \( L_\infty(Q) \) which only includes non-negative functions (recall that \( \text{dom} f = [0, \infty) \)). This still contains the optimal solution as we show in the following proposition.

**Proposition 1.** \( \forall P, Q \in \mathcal{P}(\mathcal{X}) \text{ such that } P \ll Q, \\
D_f(P \| Q) = \sup_{r \in L_\infty(Q)} \mathbb{E}_Q[r] + \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[r \cdot f'(r)] \\
\text{and the supremum is achieved when } r = dP/dQ. \\
\)

This reparametrization allows the optimal solution to be independent of \( f \), since it is the density ratio \( dP/dQ \). From here, we can further restrict our function family by considering the following constraints:

(a) \( r \in \Delta(Q) \), where \( \Delta(Q) \) is defined in Eq.(3);
(b) \( r \in \mathcal{S}_{\mathcal{F}, r}(P, Q) \) where \( \mathcal{S}_{\mathcal{F}, r}(P, Q) \subseteq L_\infty(Q) \),

where \( \mathcal{F} \) is a functional that maps (a function) \( r \) to a subset of \( L_\infty(Q) \) which satisfies \( f'(r) \in \mathcal{F}(r) \).

Constraint (a) allows us to treat the \( r \) in \( \mathbb{E}_Q[r \cdot f'(r)] \) as importance sampling weights:

\[
\mathbb{E}_Q[r \cdot f'(r)] = \mathbb{E}_Q[f'(r)]
\]

where \( dQ_r = r \, dQ \) and \( Q_r \in \mathcal{P}(\mathcal{X}) \). Under constraint (a), constraint (b) becomes \( \text{IPM}_{\mathcal{F}, r}(P, Q_R) = 0 \). We further note that the optimal solution \( r^* \) satisfies both constraints (a) and (b).

Applying constraints (a) and (b) simultaneously leads to the following problem:

\[
\sup_{r \in \Delta(Q)} \mathbb{E}_Q[f(r)] + \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[r \cdot f'(r)] = 0
\]

subject to \( \forall T \in \mathcal{F}_R(r), \mathbb{E}_P[T] - \mathbb{E}_Q[T] = 0 \),

where we can remove the last two terms in Eq.(13) since they sum to zero for feasible \( r \) (as \( f'(r) \in \mathcal{F}(r) \)). The objective in Eq.(13) recovers \( D_f(P \| Q) \) because the optimal solution is still feasible under the constraints (see Figure 1 for a graphical illustration). We state these formally in the following propositions.

**Proposition 2.** \( \forall P, Q \in \mathcal{P}(\mathcal{X}) \text{ such that } P \ll Q, \\
D_f(P \| Q) = \sup_{r \in \Delta(Q)} \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[r \cdot f'(r) - f(r)] \\
= \sup_{r \in \mathcal{S}_{\mathcal{F}, r}(P, Q)} \mathbb{E}_Q[f(r)] \\
\text{and the supremum is achieved when } r = dP/dQ. \\
\)

Proposition 2 is of independent interest, as we will show in Section 5.1 it implies the Donsker-Varadhan inequality (Donsker and Varadhan, 1975) and other known “change of measure” inequalities.
3.1 Generalizing f-GANs and WGANs

We now show that a natural Lagrangian relaxation of the constrained objective leads to a generalization of critic objectives to both f-GANs and WGANs. Let \( \ell_f(T, r; P, Q) \) be a Lagrangian relaxation to the constrained objective in Eq.(13) for some \( T \in \mathcal{F}(r) \), with multipliers \( \lambda = 1 \)

\[
\ell_f(T, r; P, Q) := E_Q[f(r)] + E_P[T] - E_Q, [T] \tag{14}
\]

We consider measuring the discrepancy between \( P \) and \( Q \) with some critic \( T \) using the following objective:

\[
\mathbf{L}^P_T(P, Q) := \inf_{r \in \mathcal{R}} \mathbf{\ell}_f(T, r; P, Q) \tag{15}
\]

where we minimize\(^1\) over \( \mathcal{R} \subseteq \mathcal{L}_\infty(Q) \). We can obtain critic objectives for f-GAN, WGANs, and new objectives via different choices of \( \mathcal{R} \) in Eq.(15).

**f-GAN critic** First, we can recover the critic in the f-GAN objective by setting \( \mathcal{R} = \mathcal{L}^\infty_{>0}(Q) \). Recall from Lemma 1 the f-GAN objective:

\[
D_f(P||Q) = \sup_{T \in \mathcal{L}^\infty(Q)} I_f(T; P, Q) \tag{16}
\]

We show that \( I_f = \mathbf{L}^P_T \) when \( \mathcal{R} = \mathcal{L}^\infty_{\geq 0}(Q) \) as follows.

**Proposition 3.** \( \forall P, Q \in \mathcal{P}(X) \) such that \( P \ll Q \), and \( \forall T \in \mathcal{F} \subseteq \mathcal{L}^\infty(Q) \) such that \( \text{im}(T) \subseteq \text{dom}(f')^{-1} \),

\[
I_f(T; P, Q) = \inf_{r \in \mathcal{L}^\infty_{\geq 0}(Q)} \mathbf{\ell}_f(T, r; P, Q). \tag{17}
\]

**Proof.** Since \( f^{**} = f \) for convex \( f \), we have

\[
E_Q[f(r)] + E_Q[f^*(T)] \geq E_Q[rT] \tag{18}
\]

from Fenchel’s inequality. Therefore:

\[
\mathbf{\ell}_f(T, r; P, Q) = E_Q[f(r)] + E_P[T] - E_Q, [T] \\
\geq E_Q[rT] - E_Q[f^*(T)] + E_P[T] - E_Q, [T] \\
= E_P[T] - E_Q[f^*(T)] = I_f(T; P, Q). \tag{19}
\]

Setting the functional derivative of \( \mathbf{\ell}_f \) with respective to \( r \) to zero, we achieve the equality with \( r = (f')^{-1}(T) \), which exists from assumptions over \( \text{im}(T) \).

**WGAN critic** Next, we recover the WGAN critic objective (IPM) by setting \( \mathcal{R} = \{1\} \), where \( \mathbf{1}(x) = 1 \) is a constant function. First, we can equivalently rewrite the definition of an IPM as follows:

\[
\text{IPM}_F(P, Q) = \sup_{T \in \mathcal{F}} I_W(T; P, Q) \tag{20}
\]

where (recall that \( \mathcal{F} \) is symmetric by assumption)

\[
I_W(T; P, Q) := E_P[T] - E_Q, [T]. \tag{21}
\]

We show that \( I_W = \mathbf{L}^P_T \) when \( \mathcal{R} = \{1\} \) as follows.

\(^{1}\text{In Appendix C, we discuss the rationale behind choosing to minimize over } r \text{ for } \mathbf{\ell}_f(T, r; P, Q).\)}
4 f-Wasserstein GANs

We can then consider the variational divergence minimization objective over $L_f^\Delta(Q)(T; P, Q)$:

$$\min \sup \inf_{Q \in P(X)} \min_{T \in F} \ell_f(T; r; P, Q),$$

(27)

Applying this to WGANs, where $P$ is $P_{data}$, $Q$ is defined by a generator $G_\theta$, and $F$ is the set of 1-Lipschitz functions w.r.t. the Euclidean metric over $X$, we have:

$$\min_{\theta} \max_{\phi} \min \left[ \frac{1}{m} \sum_{r=1}^{m} \mathbb{E}_{G_\theta}[f(r)] + \mathbb{E}_P[D_\phi] - \mathbb{E}_{G_\phi}[D_\phi] \right],$$

where we define $dG_\phi, r = r dG_\phi$.

We name this the “f-Wasserstein GAN” (f-WGAN) objective. As stated in Remark 1, $L_f^\Delta(Q)(T; P, Q)$ could be greater than $I_f(T; P, Q)$ due to restriction over $r \in \Delta(Q)$, so f-WGAN is different from $f$-GAN.

4.1 KL-Wasserstein GANs

While the above objective involves three nested optimizations, we show that for certain choices of $f$, we can obtain close-form solutions for the optimal $r \in \Delta(Q)$; this bypasses the need to perform an inner-loop optimization over $r \in \Delta(Q)$.

**Theorem 1.** Let $f(u) = u \log u$ and $F$ a set of real-valued bounded measurable functions on $\mathcal{X}$. For any fixed choice of $P, Q$, and $T \in F$, we have:

$$\arg \min_{r \in \Delta(Q)} \mathbb{E}_Q[f(r)] + \mathbb{E}_T[T] - \mathbb{E}_Q[T] = \frac{e^T}{\mathbb{E}_Q[e^T]}$$

(28)

**Proof.** In Appendix A.

The above theorem shows that if the $f$-divergence of interest is the KL divergence, then we do not have to solve $r$ via iterative optimization for our $f$-WGAN objective. Therefore, we only need to consider iterative minimax optimization methods on $G_\theta$ and $D_\phi$, which is similar to the optimization procedures used in $f$-GAN and WGANs. In Appendix E, we show a similar argument with $\chi^2$-divergence, and discuss its connections with the $\chi^2$-GAN approach (Tao et al., 2018).

For the $f$-WGAN objective in Eq.(27), the trivial algorithm would have to perform iterative updates to $G_\theta$, $D_\phi$, and $r$. However, from Theorem 1, we can directly obtain the optimal $r \in \Delta(Q)$ using Eq.(28) for any fixed $D_\phi$, which is $e^{D_\phi} / \mathbb{E}_Q[e^{D_\phi}]$. Then, we can apply this $r$ to the f-WGAN objective, and perform gradient descent updates on $G_\theta$ and $D_\phi$ only, as $r$ is now a function of $D_\phi$. Avoiding the optimization procedure over $r$ allows us to propose practical algorithms that are similar to existing GAN procedures.

Algorithm 1 Pseudo-code for KL-Wasserstein GAN

1: **Input:** the (empirical) data distribution $P_{data}$;
2: **Output:** implicit generative model $G_\theta$.
3: Initialize generator $G_\theta$ and discriminator $D_\phi$.
4: **repeat**
5: Draw $P_m := m$ i.i.d. samples from $P_{data}$;
6: Draw $Q_m := m$ i.i.d. samples from $G_\theta(x)$.
7: Compute $D_1 := \mathbb{E}_{P_m}[D_\phi(x)]$ (real samples)
8: for all $x \in \mathcal{Q}_m$ do (fake samples)
9: Compute $r_0(x) := e^{D_\phi(x)} / \mathbb{E}_{Q_m}[e^{D_\phi(x)}]$.
10: end for
11: Compute $D_0 := \mathbb{E}_{Q_m}[r_0(x)D_\phi(x)]$.
12: Perform SGD over $\theta$ with $-\nabla_\theta D_0$;
13: Perform SGD over $\phi$ with $\nabla_\phi (D_0 - D_1)$.
14: Regularize $D_\phi$ to satisfy $k$-Lipschitzness.
15: **until** stopping criterion
16: **return** learned implicit generative model $G_\theta$.

In Algorithm 1, we describe KL-Wasserstein GAN (KL-WGAN), a practical algorithm motivated by the $f$-Wasserstein GAN objectives based on the observations in Theorem 1. We note that $\nabla_\phi (D_0 - D_1)$ corresponds to maximizing $L_f^\Delta(T; P, Q)$ with the critic and $\nabla_\theta D_1$ corresponds to minimizing $L_f^\Delta(T; P, Q)$ with the generator. Under the ideal assumption that our $D_\phi$ and $G_\theta$ are infinitely powerful and the batch size $m \to \infty$, Algorithm 1 corresponds to solving Eq.(27), in which $G_\theta$ will recover $P_{data}$.

In terms of implementation, the only differences between KL-WGAN and WGAN are between lines 8 and 11, where WGAN will assign $r_0(x) = 1$ for all $x \sim Q_m$. In contrast, KL-WGAN “importance weights” the samples using the critic, in the sense that it will assign higher weights to samples that have large $D_0(x)$ and lower weights to samples that have low $D_\phi(x)$. This will encourage $G_\theta(x)$ to emphasize on samples that have high critic scores. We can perform backpropagation directly through $r_0(x)$; therefore, it is relatively easy to implement the KL-WGAN algorithm from an existing WGAN implementation, as we only need modify how the loss function is implemented. We present an implementation of KL-WGAN losses (in PyTorch) in Appendix B.

While the mini-batch estimation for $r_0(x)$ provides a biased estimate to the optimal $r \in \Delta(Q)$ (which according to Theorem 1 is $e^{D_\phi(x)} / \mathbb{E}_Q[e^{D_\phi(x)}]$), i.e., normalized with respect to $Q$ instead of over a minibatch of $m$ samples as done in line 8), we found that this does not affect performance significantly, possibly due to the Lipschitz constraints over $D_\phi$. We further note that computing $r_0(x)$ does not require additional network evaluations, so the computational cost for each iteration is nearly identical between WGAN and KL-WGAN.
5 Related Work

5.1 Restricted $f$-Divergence Estimation and “Change of Measure” Inequalities

We show that adding constraints (a) and (b) in Section 3 allows us to directly prove “change of measure” inequalities of KL and Rényi’s α-divergences. First, we show that Proposition 2 directly implies the Donsker-Varadhan inequality (Donsker and Varadhan, 1975).

**Corollary 1.** $\forall P, Q \in \mathcal{P}(X)$ such that $P \ll Q$,

$$D_{KL}(P||Q) = \sup_{T \in L^\infty(Q)} E_P[T] - \log E_Q[e^T].$$

**Proof.** $\forall T \in L^\infty(Q)$, we consider $r(T) = e^T / E_Q[e^T]$, where $r(T) \in \Delta(Q)$ by definition. If $T = \log(dP/dQ)$, we have $r(T) = dP/dQ$. From Proposition 2:

$$D_{KL}(P||Q) = \sup_{r \in \Delta(Q)} E_P[\log r + 1] - E_Q[r]$$

$$= \sup_{T \in L^\infty(Q)} E_P[T] - \log E_Q[e^T]$$

which completes the proof. $\square$

Proposition 2 also directly implies the “change of measure” inequality of Rényi α-divergences (Atar et al., 2015; Bégin et al., 2016).

**Theorem 2.** $\forall P, Q \in \mathcal{P}(X)$ such that $P \ll Q$, and for all $\alpha > 1$, we have

$$D_\alpha(P||Q) = \sup_{T \in L^\infty(Q)} \alpha \log E_P[T^{\alpha-1}] / (\alpha - 1) - \log E_Q[T^\alpha]$$

where the optimal $r = dP/dQ$, and

$$D_\alpha(P||Q) = \frac{1}{(\alpha - 1)} \log \left( E_Q \left[ \left( \frac{dP}{dQ} \right)^\alpha - 1 \right] \right).$$

is the Rényi α-divergence.

We sketch the proof as follows. Consider

$$r(T) = (T \cdot E_P[T^{\alpha-1}]) / (E_Q[T^\alpha])$$

which is an element of $S_{\mathcal{F}_\theta}(P,Q)$. This allows us to apply Proposition 2 to complete the proof. We present the detailed proof in Appendix D.

5.2 $f$-divergences, IPMs and GANs

Variational $f$-divergence minimization and IPM minimization paradigms are widely adopted in GANs. A non-exhaustive list includes $f$-GAN (Nowozin et al., 2016), Wasserstein GAN (Arjovsky et al., 2017), MMD-GAN (Li et al., 2017), WGAN-GP (Gulrajani et al., 2017), SNGAN (Miyato et al., 2018), LSGAN (Mao et al., 2017), etc. The $f$-divergence paradigms enjoy better interpretations over the role of learned discriminator (in terms of density ratio estimation), whereas IPM-based paradigms enjoy better training stability and empirical performance. Prior work have connected IPMs with $\chi^2$ divergences between mixtures of data and model distributions (Mao et al., 2017; Tao et al., 2018; Mroueh and Sercu, 2017); our approach can be applied to $\chi^2$ divergences as well, and we discuss its connections with $\chi^2$-GAN in Appendix E.

Several works (Liu et al., 2017; Farnia and Tse, 2018) considered restricting function classes directly over the $f$-GAN objective. This differs from our approach as discussed in Remark 1. Husain et al. (2019) show that restricted $f$-GAN objectives are lower bounds to Wasserstein autoencoder (Tolstikhin et al., 2017) objectives, aligning with our argument for $f$-GAN and WGAN (Figure 2).

Our approach is most related to regularized variational $f$-divergence estimators (Nguyen et al., 2010; Ruderman et al., 2012) and linear $f$-GANs (Liu et al., 2017; Liu and Chaudhuri, 2018) where the function family F is a RKHS with fixed “feature maps”. Different from these approaches, we directly motivate our approach via variational $f$-divergence estimation on restricted function families, specifically, $\Delta(Q)$ and $S_{\mathcal{F}_\theta}(P,Q)$. Our approach naturally allows the “feature maps” to be learned. Moreover, considering both restrictions allows us to bypass inner-loop optimization via closed-form solutions in certain cases (such as KL or $\chi^2$ divergences); this leads to our KL-WGAN approach which is easy to implement from existing WGAN implementations.

5.3 Reweighting of Generated Samples

The learned discriminators in GANs can further be used to perform reweighting over the generated samples (Tao et al., 2018); these include rejection sampling (Azadi et al., 2018), importance sampling (Grover et al., 2019), and Markov chain monte carlo (Turner et al., 2018). These approaches can only be performed after training has finished, unlike our KL-WGAN case where discriminator-based reweighting are performed during training. Moreover, prior reweighting approaches assume that the discriminator learns to approximate some (fixed) function of the density ratio $dP_{\text{data}}/dG_\theta$, which does not apply directly to general IPM-based GAN objectives (such as WGAN); in KL-WGAN, we interpret the discriminator outputs as (un-normalized, regularized) log density ratios, introducing the density ratio interpretation to the IPM paradigm. We note that post-training discriminator-based reweighting can also be applied to our approach, and is orthogonal to our contributions; we leave this as future work.
Table 1: Negative Log-likelihood (NLL) and Maximum mean discrepancy (MMD, multiplied by $10^3$) results on six 2-d synthetic datasets. Lower is better for both evaluation metrics. $W$ denotes the original WGAN objective, and $KL-W$ denotes the proposed KL-WGAN objective.

| Metric | GAN | MoG  | Banana | Rings | Square | Cosine | Funnel |
|--------|-----|------|--------|-------|--------|--------|--------|
|        |     | NLL  |        |       |        |        |        |
| W      | 2.65 ± 0.00 | 3.61 ± 0.02 | 4.25 ± 0.01 | 3.73 ± 0.01 | 3.98 ± 0.00 | 3.60 ± 0.01 |
| KL-W   | 2.54 ± 0.00 | 3.57 ± 0.00 | 4.25 ± 0.00 | 3.72 ± 0.00 | 4.00 ± 0.01 | 3.57 ± 0.00 |
|        |     | MMD  |        |       |        |        |        |
| W      | 25.45 ± 7.78 | 3.33 ± 0.59 | 2.05 ± 0.47 | 2.42 ± 0.24 | 1.24 ± 0.40 | 1.71 ± 0.65 |
| KL-W   | 6.51 ± 3.16 | 1.45 ± 0.12 | 1.20 ± 0.10 | 1.10 ± 0.23 | 1.33 ± 0.23 | 1.08 ± 0.23 |

Figure 3: Histograms of samples from the data distribution, (spectral normalized) WGAN and our KL-WGAN.

6 Experiments

6.1 Synthetic and UCI Benchmark Datasets

We first demonstrate the effectiveness of KL-WGAN on synthetic and UCI benchmark datasets (Asuncion and Newman, 2007) considered in (Wenliang et al., 2018). The 2-d synthetic datasets include Mixture of Gaussians (MoG), Banana, Ring, Square, Cosine and Funnel; these datasets cover different modalities and geometries. We use RedWine, WhiteWine and Parkinsons from the UCI datasets. We use the same SNGAN (Miyato et al., 2018) architectures for WGAN and KL-WGANs (detailed in Appendix F).

After training, we draw 5,000 samples from the generator and then evaluate two metrics over a fixed validation set. One is the negative log-likelihood (NLL) of the validation samples on a kernel density estimator fitted over the generated samples (we use identical kernel bandwidths for all cases); the other is the maximum mean discrepancy (MMD, Borgwardt et al. (2006)) between the generated samples and validation samples.

We report the mean and standard error for the NLL and MMD results in Tables 1 and 2 (with 5 random seeds in each case) for the synthetic datasets and UCI datasets respectively. The results demonstrate that our KL-WGAN approach (which uses spectral normalization to enforce Lipschitzness) outperforms its WGAN counterpart on all but the Cosine dataset. From the histograms of samples in Figure 3, we can visually observe where our KL-WGAN performs significantly better than WGAN. For example, WGAN fails to place modes in the center of Gaussians in MoG and fails to learn a proper square in Square, whereas in our KL-WGAN approaches we place modes correctly in MoG and learns the square boundaries in Square.

For the MoG, Square and Cosine datasets, we further show the estimated divergences over a batch of 256 samples in Figure 4, where WGAN uses $I_W$ and KL-WGAN uses the proposed $\mathcal{L}_\Delta^W(Q)$. While both estimated divergences decrease over the course of training, our KL-WGAN divergence is more stable on all three cases. In addition, we evaluate the number of occurrences when a negative estimate of the divergences was produced for an epoch (which contradicts the fact that divergences should be non-negative); over 500 batches, WGAN has 46, 181 and 55 occurrences on MoG, Square and Cosine respectively, while KL-WGAN has 29, 100 and 7 occurrences, notably lower than WGAN.

Dataset code is contained in https://github.com/kevin-w-li/deep-kexpfam.
Table 2: Negative Log-likelihood (NLL, top two rows) and Maximum mean discrepancy (MMD, multiplied by $10^3$, bottom two rows) results on real-world datasets. Lower is better for both evaluation metrics. W denotes the original WGAN objective, and KL denotes the proposed KL-WGAN objective.

|       | RedWine | WhiteWine | Parkinsons |
|-------|---------|-----------|------------|
| W     | 14.55 ± 0.04 | 14.12 ± 0.02 | 20.24 ± 0.08 |
| KL    | 14.41 ± 0.03 | 14.08 ± 0.02 | 20.16 ± 0.05 |

Table 3: Inception and FID scores for CIFAR10. We list comparisons with results reported by WGAN-GP (Gulrajani et al., 2017), Fisher GAN (Mroueh and Sercu, 2017), $\chi^2$ GAN (Tao et al., 2018), MoLM (Ravuri et al., 2018), SNGAN (Miyato et al., 2018), NCSN (Song and Ermon, 2019), BigGAN (Brock et al., 2018) and Sphere GAN (Park and Kwon, 2019). (*) denotes our experiments with the PyTorch BigGAN implementation.

| Method   | Inception | FID      |
|----------|-----------|----------|
| **CIFAR10 Unconditional** |             |          |
| WGAN-GP  | 7.86 ± .07 |          |
| Fisher GAN | 7.90 ± .05 |          |
| MoLM     | 7.90 ± .10 | 18.9     |
| SNGAN    | 8.22 ± .05 | 21.7     |
| Sphere GAN | 8.39 ± .08 | 17.1     |
| NCSN     | **8.91**  | 25.32    |
| BigGAN*  | 8.60 ± .10 | 15.71    |
| KL-BigGAN* | **8.66 ± .09** | **13.87** |

| Method   | Inception | FID      |
|----------|-----------|----------|
| **CIFAR10 Conditional** |             |          |
| Fisher GAN | 8.16 ± .12 |          |
| WGAN-GP*  | 8.42 ± .10 |          |
| $\chi^2$-GAN | 8.44 ± .10 |          |
| SNGAN     | 8.60 ± .08 | 17.5     |
| BigGAN    | **9.22**  | **14.73** |
| BigGAN*   | 9.08 ± .11 | 8.21     |
| KL-BigGAN* | **9.20 ± .09** | **7.94**  |

6.2 Image Generation

We demonstrate our KL-WGAN’s practical usefulness on image generation tasks with the CIFAR10 dataset. Our experiments are based on the BigGAN (Brock et al., 2018) PyTorch implementation$.^3$ We use a smaller network than the one reported in Brock et al. (2018) (implemented on TensorFlow), using the default architecture in the PyTorch implementation. We compare training a BigGAN network with its original objective and training same network with our proposed KL-WGAN algorithm. To prevent division by zero in step 9 at early stages of training, we add an additional $10^{-8}$ to both the numerator and denominator.

We report two common benchmarks for image generation: Inception scores (Salimans et al., 2016) and Fréchet Inception Distance (FID) (Heusel et al., 2017) in Table 3 and compare with existing approaches. Despite the strong performance of BigGAN, our method (KL-BigGAN) is able to achieve superior inception scores and FID scores. This demonstrates that the KL-WGAN algorithm is practically useful, and can serve as a viable drop-in replacement for the existing WGAN objective even on state-of-the-art GAN models, such as BigGAN. We show model samples in Appendix G.

7 Conclusions

In this paper, we discuss a generalized perspective over $f$-GANs and WGANs. We show that adding constraints (a) and (b) preserves the optimal solution and can be used to prove “change of measure” inequalities. Relaxing the constraints leads to a Lagrangian that generalizes both $f$-GAN and WGAN critic objectives though minimization over specific sets. Considering alternative sets leads to the general $f$-WGAN objective, and a practical KL-WGAN algorithm. We demonstrate the effectiveness of KL-WGAN on several tasks. In future work, we are interested in considering other constraints that could lead to alternative objectives and/or inequalities. It would also be interesting to investigate the KL-WGAN approach on high-dimensional density ratio estimation tasks.

$^3$https://github.com/ajbrock/BigGAN-PyTorch
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A Proofs

Proposition 1. \(\forall P, Q \in \mathcal{P}(X)\) such that \(P \ll Q\),
\[
D_f(P \| Q) = \sup_{r \in \mathbb{L}^\infty_{\geq 0}(Q)} \mathbb{E}_P[f(r)] + \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[r \cdot f'(r)]
\]
and the supremum is achieved when \(r = dP/dQ\).

Proof. Let us consider \(h(u, x) := f'(r(x)) \cdot u - f(u)\), where \(f^*(f'(r(x))) = \sup_{u \in R} h(u, x)\). We have \(\partial h/\partial u = f'(r(x)) - f'(u)\), since \(f^*\) is convex, \(f'\) is non-decreasing, so \(\partial h/\partial u\) is zero when \(u = r(x)\), non-negative when \(u < r(x)\) and non-positive when \(u > r(x)\). Therefore, \(f^*(f'(r(x))) = f'(r(x)) \cdot r(x) - f(r(x))\) for all \(x \in X\), and from Lemma 1 we have:

\[
D_f(P \| Q) = \sup_{T \in \mathbb{L}^\infty(Q)} \mathbb{E}_P[T] - \mathbb{E}_Q[f^*(T)]
\]
\[
= \sup_{r \in \mathbb{L}^\infty_{\geq 0}(Q)} \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[f^*(f'(r))]
\]
\[
= \sup_{r \in \mathbb{L}^\infty_{\geq 0}(Q)} \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[r \cdot f'(r) - f(r)],
\]
where the second equality holds because the optimal \(T = f'(dP/dQ)\) (from Lemma 1) and \(dP/dQ \in \mathbb{L}^\infty_{\geq 0}(Q)\) (from the fact that \(P \ll Q\), \(\forall x \in X\), \(x \in \text{supp}(P) \Rightarrow x \in \text{supp}(Q)\), so \(dP(x)/dQ(x)\) is bounded) so the supremum in Eq.(35) can be achieved.

\[\square\]

Proposition 2. \(\forall P, Q \in \mathcal{P}(X)\) such that \(P \ll Q\),
\[
D_f(P \| Q) = \sup_{r \in \Delta(Q)} \mathbb{E}_Q[f(r)]
\]
and the supremum is achieved when \(r = dP/dQ\).

Proof. From Proposition 1, we have:
\[
D_f(P \| Q) = \sup_{r \in \mathbb{L}^\infty_{\geq 0}(Q)} \mathbb{E}_Q[f(r)] + \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[r \cdot f'(r)]
\]
Since \(\Delta(Q) \subset \mathbb{L}^\infty_{\geq 0}(Q)\), we have:
\[
D_f(P \| Q) \geq \sup_{r \in \Delta(Q)} \mathbb{E}_Q[f(r)] + \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[r \cdot f'(r)]
\]
Moreover, since \(dP/dQ \in \Delta(Q)\) we can achieve \(D_f(P \| Q)\) with the supremum, so the inequality becomes an equality.

Since \(S_{F_R}(P, Q) \subset \mathbb{L}^\infty_{\geq 0}(Q)\), we have:
\[
D_f(P \| Q) \geq \sup_{r \in S_{F_R}(P, Q)} \mathbb{E}_Q[f(r)] + \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[r \cdot f'(r)]
\]
(36)
\[
= \sup_{r \in S_{F_R}(P, Q)} \mathbb{E}_Q[f(r)]
\]
where the equality holds from the definition of \(S_{F_R}(P, Q)\) and the fact that \(f'(r) \in F_R(r)\). \[\square\]

Theorem 1. Let \(f(u) = u \log u\) and \(F\) a set of real-valued bounded measurable functions on \(X\). For any fixed choice of \(P, Q,\) and \(T \in F\), we have
\[
\arg\min_{r \in \Delta(Q)} \mathbb{E}_Q[f(r)] + \mathbb{E}_P[T] - \mathbb{E}_Q[T] = \frac{e^T}{\mathbb{E}_Q[e^T]} \tag{28}
\]
Proof. Consider the following Lagrangian:

\[ h(r, \lambda) := \mathbb{E}_Q[f(r)] - \mathbb{E}_Q[r \cdot T] + \lambda(\mathbb{E}_Q[r] - 1) \tag{37} \]

where \( \lambda \in \mathbb{R} \) and we formalize the constraint \( r \in \Delta(r) \) with \( \mathbb{E}_Q[r] - 1 = 0 \). Taking the functional derivative \( \partial h/\partial r \) and setting it to zero, we have:

\[ f'(r) \, dQ - T \, dQ + \lambda \]

\[ = (\log r + 1) \, dQ - T \, dQ + \lambda = 0, \tag{38} \]

so \( r = \exp(T - (\lambda + 1)) \). We can then apply the constraint \( \mathbb{E}_Q[r] = 1 \), where we solve \( \lambda + 1 = \mathbb{E}_Q[e^T] \), and consequently the optimal \( r = e^T/\mathbb{E}_Q[e^T] \).

\[ \square \]

B KL-WGAN Implementation in PyTorch

```python
def loss_kl_dis(dis_fake, dis_real):
    
    ***
    Critic loss for KL-WGAN.
    dis_fake, dis_real are the critic outputs for generated samples and real samples.
    We use the hinge loss from BigGAN PyTorch implementation.
    ***
    loss_real = torch.mean(F.relu(1. - dis_real))
    dis_fake_norm = torch.exp(dis_fake).mean() + 1e-8
    dis_fake_ratio = (torch.exp(dis_fake) + 1e-8) / dis_fake_norm
    dis_fake = dis_fake * dis_fake_ratio
    loss_fake = torch.mean(F.relu(1. + dis_fake))
    return loss_real, loss_fake

def loss_kl_gen(dis_fake):
    
    ***
    Generator loss for KL-WGAN.
    dis_fake is the critic outputs for generated samples.
    We use the hinge loss from BigGAN PyTorch implementation.
    ***
    dis_fake_norm = torch.exp(dis_fake).mean() + 1e-8
    dis_fake_ratio = (torch.exp(dis_fake) + 1e-8) / dis_fake_norm
    dis_fake = dis_fake * dis_fake_ratio
    loss = -torch.mean(dis_fake)
    return loss
```

C Argument about Minimization over \( r \in \mathcal{R} \) for Lagrangian

It may seem counter-intuitive that the objective in Eq.(15) consider minimization over \( r \in \mathcal{R} \) instead of maximization (as are described for variational \( f \)-divergence estimation). However, we note that the \( f \)-divergence estimation objective:

\[ D_f(P||Q) = \sup_{r \in L_{\infty}^0(Q)} \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[f^*(f'(r))] \tag{39} \]

\[ = \sup_{r \in L_{\infty}^0(Q)} \mathbb{E}_P[f'(r)] - \mathbb{E}_Q[r \cdot f'(r) - f(r)] \tag{40} \]

is concave with respect to \( r \), since \( f'(r) \) is non-decreasing and \( f^* \) is convex. In contrast, the relaxed objective in the Lagrangian:

\[ \ell_f(T; r; P, Q) := \mathbb{E}_Q[f(r)] + \mathbb{E}_P[T] - \mathbb{E}_Q[T] \tag{41} \]
is no longer concave with respect to \( r \), but convex! Here, \( T \) does not depend on \( r \), whereas in the \( f \)-divergence case \( T = f'(r) \) depends on \( r \). Therefore, it seems more reasonable to minimize this unconstrained convex objective for \( \ell_f(T; r; P, Q) \) (instead of maximizing it, which could be unbounded); as we have demonstrated, minimization over different choices of \( R \) leads to different objectives, such as \( f \)-GANs and WGANs.

\section{Restricted \( f \)-Divergence Estimation and “Change of Measure” Inequalities}

\textbf{Theorem 2.} \( \forall P, Q \in \mathcal{P}(\mathcal{X}) \) such that \( P \ll Q \), and for all \( \alpha > 1 \), we have

\[
D_\alpha(P\|Q) = \sup_{T \in L^\infty(Q)} \frac{\alpha \log E_P[T^{\alpha - 1}]}{\alpha - 1} - \log E_Q[T^{\alpha}]
\]

where the optimal \( r = dP/dQ \), and

\[
D_\alpha(P\|Q) = \frac{1}{(\alpha - 1)} \log \left( E_Q \left[ \left( \frac{dP}{dQ} \right)^\alpha - 1 \right] \right)
\]

is the Rényi \( \alpha \)-divergence.

\textit{Proof.} Consider the \( f \)-divergence with \( f_\alpha(u) = u^\alpha - 1 \). For any \( T \in L^\infty(Q) \), we consider a corresponding function \( r(T) \in L^\infty(Q) \) such that

\[
r(T) = (T \cdot E_P[T^{\alpha - 1}])/E_Q[T^\alpha].
\]

Let \( \mathcal{F}_R(r) = \{ f'_\alpha(r) \} \); we show that \( r(T) \) is an element of \( S_{\mathcal{F}_R}(P, Q) \) for all \( T \in L^\infty(Q) \). Since

\[
\frac{E_P[f'_\alpha(r(T))]}{E_Q[r(T) \cdot f'_\alpha(r(T))]} = \frac{E_P[r(T)^\alpha - 1]}{E_Q[r(T)^\alpha]}
\]

\[
= \frac{E_P[T^{\alpha - 1}]}{E_Q[T^\alpha]} \cdot \frac{E_Q[T^\alpha]}{E_P[T^{\alpha - 1}]} = 1,
\]

we have \( r(T) \in S_{\mathcal{F}_R}(P, Q) \) by definition. Moreover, \( r(dP/dQ) = dP/dQ \), so we obtain the supremum with \( T = dP/dQ \). Therefore, from Proposition 2:

\[
\exp((\alpha - 1)D_\alpha(P\|Q)) = \sup_{r \in S_{\mathcal{F}_R}(P, Q)} E_Q[r^\alpha - 1] + 1
\]

\[
= \sup_{T \in L^\infty(Q)} \frac{(E_P[T^{\alpha - 1}])^\alpha}{(E_Q[T^\alpha])^{\alpha - 1}}
\]

Taking the logarithms and divide \( (\alpha - 1) \) on both sides completes the proof. \hfill \Box

Theorem 2 has been presented in (Bégin et al., 2016) with a different proof based on Hölder’s inequality. However, our proof uncovers the connection between Theorem 2 and the more general variational \( f \)-divergence lower bounds (Nguyen et al., 2008), where the former is a direct consequence of the latter.

\section{Argument about \( \chi^2 \)-Divergences}

We present a similar argument to Theorem 1 to \( \chi^2 \)-divergences, where \( f(u) = (u - 1)^2 \).

\textbf{Theorem 3.} Let \( f(u) = (u - 1)^2 \) and \( \mathcal{F} \) is a set of real-valued bounded measurable functions on \( \mathcal{X} \). For any fixed choice of \( P, Q \), and \( T \in \mathcal{F} \), we have

\[
\arg \min_{r \in \Delta(Q)} E_Q[f(r)] + E_P[T] - E_{Q_r}[T] = \frac{T - E_Q[T] + 2}{2}
\]
Proof. Consider the following Lagrangian:

\[ h(r, \lambda) := \mathbb{E}_Q[f(r)] - \mathbb{E}_Q[r \cdot T] + \lambda(\mathbb{E}_Q[r] - 1) \]  

(46)

where \( \lambda \in \mathbb{R} \) and we formalize the constraint \( r \in \Delta(r) \) with \( \mathbb{E}_Q[r] = 1 \). Taking the functional derivative \( \partial h/\partial r \) and setting it to zero, we have:

\[
\frac{f'(r)}{dQ} - T \frac{dQ}{dQ} + \lambda (47) = 2r \frac{dQ}{dQ} - T \frac{dQ}{dQ} + \lambda = 0,
\]

so \( r = (T - \lambda)/2 \). We can then apply the constraint \( \mathbb{E}_Q[r] = 1 \), where we solve \( \lambda = \mathbb{E}_Q[T] - 2 \), and consequently the optimal \( r = (T - \mathbb{E}_Q[T])/2 \).

If we plug in this optimal \( r \), we obtain the following objective:

\[
\mathbb{E}_P[T] - \mathbb{E}_Q[T] + \frac{1}{4} \mathbb{E}_Q[T^2] + \frac{1}{4} (\mathbb{E}_Q[T])^2 = \mathbb{E}_P[T] - \mathbb{E}_Q[T] - \frac{\text{Var}_Q[T]}{4}. \]

(48)

Let us now consider \( P = P_{\text{data}}, Q = \frac{P_{\text{data}} + G_\theta}{2} \), then the \( f \)-divergence corresponding to \( f(u) = (u - 1)^2 \):

\[
D_f(P || Q) = \int_X \frac{(P(x) - Q(x))^2}{P(x)+Q(x)} \, dx,
\]

(49)

is the squared \( \chi^2 \)-distance between \( P \) and \( Q \). So the objective becomes:

\[
\min_{\theta} \max_{\phi} \mathbb{E}_{P_{\text{data}}}[D_\theta] - \mathbb{E}_{G_\theta}[D_\phi] - \text{Var}_{M_\phi}[D_\phi],
\]

(50)

where \( M_\theta = \frac{P_{\text{data}} + G_\theta}{2} \) and we replace \( T/2 \) with \( D_\phi \). In comparison, the \( \chi^2 \)-GAN objective (Tao et al., 2018) for \( \theta \) is:

\[
\frac{(\mathbb{E}_{P_{\text{data}}}[D_\theta] - \mathbb{E}_{G_\theta}[D_\phi])^2}{\text{Var}_{M_\phi}[D_\phi]}.
\]

(51)

They do not exactly minimize \( \chi^2 \)-divergence, or a squared \( \chi^2 \)-divergence, but a normalized version of the 4-th power of it, hence the square term over \( \mathbb{E}_{P_{\text{data}}}[D_\theta] - \mathbb{E}_{G_\theta}[D_\phi] \).

F Additional Experimental Details

For 2d experiments, we consider the WGAN and KL-WGAN objectives with the same architecture and training procedure. Specifically, our generator is a 2 layer MLP with 100 neurons and LeakyReLU activations on each hidden layer, with a latent code dimension of 2; our discriminator is a 2 layer MLP with 100 neurons and LeakyReLU activations on each hidden layer. We use spectral normalization (Miyato et al., 2018) over the weights for the generators and consider the hinge loss in (Miyato et al., 2018). Each dataset contains 5,000 samples from the distribution, over which we train both models for 500 epochs with RMSProp (learning rate 0.2). The procedure for tabular experiments is identical except that we consider networks with 300 neurons in each hidden layer with a latent code dimension of 10.
G  Samples

We show uncurated samples from BigGAN trained with WGAN and KL-WGAN loss in Figures 5a and 5b.

(a) Samples from BigGAN trained with WGAN.

(b) Samples from BigGAN trained with KL-WGAN.