A GENERALIZATION OF THE ZERO-DIVISOR GRAPH FOR MODULES

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ABSTRACT. Let $R$ be a commutative ring and $M$ a Noetherian $R$-module. The zero-divisor graph of $M$, denoted by $\Gamma(M)$, is an undirected simple graph whose vertices are the elements of $Z_R(M) \setminus \text{Ann}_R(M)$ and two distinct vertices $a$ and $b$ are adjacent if and only if $abM = 0$. In this paper, we study diameter and girth of $\Gamma(M)$. We show that the zero-divisor graph of $M$ has a universal vertex in $Z_R(M) \setminus \text{Ann}_R(M)$ if and only if $R = \oplus \mathbb{Z}_2 \oplus R'$ and $M = \mathbb{Z}_2 \oplus M'$, where $M'$ is an $R'$-module. Moreover, we show that if $\Gamma(M)$ is a complete graph, then one of the following statements is true:

(i) $\text{Ass}_R(M) = \{m_1, m_2\}$, where $m_1, m_2$ are maximal ideals of $R$.
(ii) $\text{Ass}_R(M) = \{p\}$, where $p^2 \subseteq \text{Ann}_R(M)$.
(iii) $\text{Ass}_R(M) = \{p\}$, where $p^3 \subseteq \text{Ann}_R(M)$.

1. Introduction

Let $R$ be a commutative ring. The concept of the zero-divisor graph of a commutative ring introduced in [6] and studied in [2], they associate a graph, $\Gamma(R)$, to the ring $R$ with vertex set $Z^*(R) := Z(R) \setminus \{0\}$, the set of nonzero zero divisors of $R$, and two distinct vertices $a$ and $b$ are adjacent if and only if $ab = 0$. The zero-divisor graphs of commutative rings have been studied by many authors, see for examples [1, 3, 4]. There are many papers on assigning graphs to rings, see [5, 9, 11]. The concepts of zero-divisor elements and zero divisors graph of a ring, have been generalized to a module in many papers, see for example [7, 10].

Let $R$ be a commutative ring with identity and $M$ an $R$-module. In this paper, we define a zero divisor graph for $M$, denoted by $\Gamma(M)$, which is an undirected simple graph whose vertices are the elements of $Z_R(M) \setminus \text{Ann}_R(M)$, and two distinct vertices $a, b \in Z_R(M) \setminus \text{Ann}_R(M)$ are adjacent if and only if $abM = 0$. It is clear that when $M = R$, $\Gamma(M)$ is exactly the classic zero divisor graph of $R$. Let $M$ be a Noetherian $R$-module. In Section 2, it is shown that $e \in Z_R(M) \setminus \text{Ann}_R(M)$ is a universal vertex of $\Gamma(M)$ if and only if either $Z_R(M) = \text{Ann}_R(cM) \cup \{e\}$ or $Z_R(M) = \text{Ann}_R(cM)$. Furthermore, it is proved that $\Gamma(M)$ is a complete bipartite graph.

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graph if and only if \(|\text{Ass}_R(M)| = 2\), whenever \(r(\text{Ann}_R(M)) = \text{Ann}_R(M)\). In Section 3, we study connectivity, diameter and the girth of \(\Gamma(M)\). Moreover, we show that for a Noetherian \(R\)-module \(M\), the following statements are true:

(i) If \(r(\text{Ann}_R(M)) = \text{Ann}_R(M)\) and there exists \(b \in Z_R(M) \setminus \text{Ann}_R(M)\) such that \(\text{Ann}_M(b) \subseteq \bigcap_{P \in \text{Min}(M)} P\), then \(\Gamma(M)\) is a disconnected graph.

(ii) If \(\Gamma(M)\) is a disconnected graph, then there exists \(b \in Z_R(M) \setminus \text{Ann}_R(M)\) such that \(\text{Ann}_M(b) \subseteq \bigcap_{P \in \text{Min}(M)} P\).

Let \(G\) be a graph with the vertex set \(V(G)\) and the edge set \(E(G)\). For each pair of vertices \(u, v \in V(G)\), if \(u\) is adjacent to \(v\), then we write \(u - v\). A graph with no edge is called null graph. Recall that \(G\) is connected if there is a path between any two distinct vertices of \(G\). For vertices \(x\) and \(y\) of \(G\), let \(d(x, y)\) be the length of a shortest path from \(x\) to \(y\) (\(d(x, x) = 0\) and \(d(x, y) = \infty\) if there is no such path). The diameter of \(G\) is \(\text{diam}(G) = \sup\{d(x, y) \mid x, y\ \text{are vertices of} \ \Gamma\}\). The girth of \(G\), denoted by \(\text{gr}(G)\), is the length of the shortest cycle in \(G\) (\(\text{gr}(G) = \infty\) if \(G\) contains no cycles). A graph \(G\) is complete if any two distinct vertices are adjacent. The complete graph with \(n\) vertices will be denoted by \(K_n\). A complete bipartite graph is a graph \(G\) which may be partitioned into two disjoint nonempty vertex sets \(A\) and \(B\) such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. If one of the vertex set is a singleton, then we call \(G\) a star graph. A clique of \(G\) is a complete subgraph of \(G\) and the number of vertices in the largest clique of \(G\), denoted by \(\omega(G)\), is called the clique number.

Throughout, \(R\) denotes a commutative ring with nonzero identity and \(M\) is a unitary \(R\)-module, \(Z_R(M)\) its set of zero-divisors. Let \(\text{Ass}_R(M) = \{p \in \text{Spec}(R) : p = \text{Ann}_R(m)\ \text{for some} \ 0 \neq m \in M\}\) denote the set of associated primes of \(M\). Let \(\text{Spec}_R(M)\) denote the set of prime submodules of \(M\) and \(m - \text{Ass}_R(M) = \{P \in \text{Spec}_R(M) : P = \text{Ann}_R(a)\ \text{for some} \ 0 \neq a \in R\}\), where \(\text{Ann}_R(a) = \{m \in M : am = 0\}\), for all \(a \in R\). For notations and terminologies not given in this article, the reader is referred to \([12]\).

2. Properties of zero-divisor graph of a module

In this section we define an undirected graph \(\Gamma(M)\) and study the relations between module theoretic properties of \(M\) and graph theoretic properties of \(\Gamma(M)\).

**Definition 2.1.** Let \(M\) be an \(R\)-module. The zero-divisor graph of \(M\) is the undirected graph \(\Gamma(M)\) with vertices \(Z_R(M) \setminus \text{Ann}_R(M)\) and two distinct vertices \(x, y\) are adjacent if and only if \(xyM = 0\).

**Lemma 2.1.** Let \(M\) be an \(R\)-module. Then \(\Gamma(M)\) is a null graph if and only if \(\text{Ann}_R(M)\) is a prime ideal of \(R\).

**Proof.** Let \(\Gamma(M)\) be a null graph and \(x, y \in R\) such that \(xyM = 0\) and \(xM \neq 0\). So \(y \in Z_R(M)\). If \(y \notin \text{Ann}_R(M)\), then \(x, y \in Z_R(M) \setminus \text{Ann}_R(M)\) and \(xyM = 0\) which is a contradiction. Thus \(y \in \text{Ann}_R(M)\). So \(\text{Ann}_R(M)\) is a prime ideal of \(R\). Now let \(\text{Ann}_R(M)\) be a prime ideal of \(R\). If \(\Gamma(M)\) is not a null graph, then there exist \(x, y \in Z_R(M) \setminus \text{Ann}_R(M)\) such that \(xyM = 0\). So \(x \in \text{Ann}_R(M)\) or \(y \in \text{Ann}_R(M)\) which contracts to definition of vertices. \(\square\)
Lemma 2.2. Let $R$ be a nontrivial commutative ring. Then $R$ is field if and only if $\Gamma(M)$ is a null graph, for each $R$-module $M$.

**Proof.** Suppose that $\Gamma(M)$ is a null graph, for each $R$-module $M$. Let $m$ be a nonzero maximal ideal of $R$ and $0 \neq a \in m$. Set $M := R/m \oplus R$. So we have $(0,a)(1+m,0)M = 0$. Hence, $(0,a)$ and $(1 + m,0)$ are adjacent which is a contradiction. So $m = 0$ and $R$ is a field. The converse is true by Lemma 2.1. □

Example 2.1. Consider $M = Z_p \cong \{a/p^n + Z : a \in Z, n \geq 1\}$ as a $Z$-module. Then $\{p, p^2, p^3, \ldots\} \subseteq Z(M) \setminus \text{Ann}_{Z}(M)$ and $\Gamma(M)$ is a null graph since $\text{Ann}_{Z}(M) = 0$ is a prime ideal of $Z$.

**Theorem 2.1.** Let $c \in Z_R(M) \setminus \text{Ann}_R(M)$. Then $c$ is a universal vertex of $\Gamma(M)$ if and only if either $Z_R(M) = \text{Ann}_R(cM) \cup \{c\}$ or $Z_R(M) = \text{Ann}_R(cM)$.  

**Proof.** Let $c$ be a universal vertex of $\Gamma(M)$. Then $c(Z_R(M) \setminus \{c\}) \subseteq \text{Ann}_R(M)$. So $Z_R(M) \setminus \{c\} \subseteq \text{Ann}_R(M) \setminus c = \text{Ann}_R(cM)$. If $c \notin r(\text{Ann}_R(M))$, then $c = c^n$ for all $n \geq 1$ otherwise $c^{n+1}M = 0$ which contradicts the assumption. Thus $c^2M \neq 0$ and $\text{Ann}_R(cM) \subseteq Z_R(M) \setminus \{c\}$. Hence $\text{Ann}_R(cM) = Z_R(M) \setminus \{c\}$ and so $Z_R(M) = \text{Ann}_R(cM) \cup \{c\}$. Now let $c \in r(\text{Ann}_R(M))$; then $c \neq c^2$, otherwise $c = c^n$ for all $n \geq 1$ and so $c \in \text{Ann}_R(M)$ which is a contradiction. Thus $c^2M = 0$. If $c^2M \neq 0$, then $Z_R(M) = \text{Ann}_R(cM) \cup \{c\}$ otherwise $Z_R(M) = \text{Ann}_R(cM)$. The reverse is obvious. □

**Theorem 2.2.** Let $c \in Z_R(M) \setminus r(\text{Ann}_R(M))$ be a universal vertex of $\Gamma(M)$. Then $R = Z_2 \oplus R'$ and $M = Z_2 \oplus M'$, where $R'$ is a subring of $R$, $M'$ is an $R$-submodule of $M$ and $Z_R(M) = \{(1,0)\} \cup \{(0) \oplus R'\}$.

**Proof.** Suppose that $c \in Z_R(M) \setminus r(\text{Ann}_R(M))$ is a universal vertex of $\Gamma(M)$. By assumption $c^2 = c$. So $R = cR \oplus (1 - c)R$ and $M = cM \oplus (1 - c)M$. Let $R_1 = cR$ and $R' = (1 - c)R$. Then $R_1, R'$ are subrings of $R$. In addition, $M_1 = cM$ is an $R_1$-module and $M' = (1 - c)M$ is an $R'$-module since $M_1, M'$ are submodules of $M$. Moreover, if $r = (r_1, r')$ and $m = (m_1, m')$, then $rm = (cr + (1 - c)r)(cm + (1 - c)m) = c^2rm + (1 - c)^2rm = r_1m_1 + r'm' = (r_1m_1, r'm')$. With regard to above decomposition, we have $c = (1,0)$. So $(1,0)$ is a universal vertex.

Let $0 \neq b \in Z_R(M')$. Then there exists $0 \neq M' \subseteq M'$ such that $bM' = 0$. So $(1,0)(0, m') = (0,0)$ but $(1,0)(M_1 \oplus M') = M_1 \oplus bM' \neq 0$ which means that $(1,0) \in Z_R(M_1 \oplus M') \setminus \text{Ann}_R(M_1 \oplus M')$. Finally by assumption $(1,0)(1,0)M_1 \oplus M' = M_1 \oplus 0 = 0$ which is a contradiction. So $Z_R(M') = 0$. Also, if $0 \neq a \in Z_R(M_1)$, then there exists $0 \neq m_1 \in M_1$ such that $am_1 = 0$. Thus $(a,1)_(0,0) = (0,0)$ and so $(a,1) \in Z_R(M_1 \oplus M') \setminus \text{Ann}_R(M_1 \oplus M')$. Since $(1,0)$ is a universal vertex, $(1,0)(a,1)(M_1 \oplus M') = aM_1 \oplus 0 = 0$. Thus $a \in \text{Ann}_R(M_1)$ which means that $Z_R(M_1) \subseteq \text{Ann}_R(M_1)$. So $Z_R(M_1) = \text{Ann}_R(M_1)$. If $1 \neq a \in R_1 \setminus \text{Ann}_R(M_1)$, then $(a,0) \in Z_R(M_1 \oplus M') \setminus \text{Ann}_R(M_1 \oplus M')$. Thus $(1,0)(a,0)(M_1 \oplus M') = 0$ which is a contradiction. Hence $R_1 \setminus \{1\} \subseteq \text{Ann}_R(M_1)$ and so $R_1 \setminus \{1\} = \text{Ann}_R(M_1)$. In the following, we show that $R_1$ has characteristic 2. Assume that $(1,0) \neq (-1,0)$. Thus $(1,0)(-1,0)(M_1 \oplus M') = -M_1 = 0$ which is a contradiction. Hence $1 = -1 \in R_1$ and so $R_1$ has characteristic 2. If $|R_1| \geq 4$, then $R_1 = \{0,1,a,1+a,\cdots\}$. Thus
\(aM_1 = (1 + a)M_1 = 0\) so \(M_1 = 0\) that is a contradiction. Hence \(R_1 \cong \mathbb{Z}_2\) and \(M_1 \cong \oplus \mathbb{Z}_2\) by [12] Theorem 10.8. Also, \(R_1 \setminus \{1\} = Z_2 \setminus \{1\} = \text{Ann}_{R_1}(M_1) = \{0\}.\) Hence, \(Z_R(M_1 \oplus M') = \{(1, 0)\} \cup \{(0) \oplus \mathbb{R}'\}.\)

**Corollary 2.1.** Let \(M\) be an \(R\)-module. Then the graph \(\Gamma(M)\) has a universal vertex in \(Z_R(M) \setminus r(\text{Ann}_{R}(M))\) if and only if \(R = \mathbb{Z}_2 \oplus \mathbb{R}'\) and \(M = \oplus \mathbb{Z}_2 \oplus M',\) where \(M'\) is an \(\mathbb{R}'\)-module and \(Z_R(M) = \{(1, 0)\} \cup \{(0) \oplus \mathbb{R}'\}.\)

**Corollary 2.2.** Let \(r(\text{Ann}_{R}(M)) = 0.\) Then \(\Gamma(M)\) is a complete graph if and only if \(\Gamma(M) = K_2.\)

**Proof.** By Corollary 2.1, \((1, 0)\) and \((0, 1)\) are universal vertices. So \(R = \mathbb{Z}_2 \oplus \mathbb{Z}_2.\)

**Theorem 2.3.** Let \(M\) be a Noetherian \(R\)-module and \(\Gamma(M)\) a complete graph. Then one of the following statements is true:

(i) \(\text{Ass}_R(M) = \{m_1, m_2\}\) where \(m_1, m_2\) are maximal ideals of \(R.\)

(ii) \(\text{Ass}_R(M) = \{p\}\) where \(p^2 \subseteq \text{Ann}_{R}(M).\)

(iii) \(\text{Ass}_R(M) = \{p\}\) where \(p^3 \subseteq \text{Ann}_{R}(M).\)

**Proof.** (i) If there exists \(c \in Z_R(M) \setminus \text{Ann}_{R}(M)\) such that \(c = c^2,\) then by the proof of Theorem 2.2 it follows that \(R = \mathbb{Z}_2 \oplus \mathbb{Z}_2\) and \(M = M_1 \oplus M_2\) where \(M_1 = \oplus \mathbb{Z}_2\) and \(M_2 = \oplus \mathbb{Z}_2\) are \(\mathbb{Z}_2\)-modules. In this case \(\text{Ass}_R(M) = \{m_1 = Z_2 \oplus 0, m_2 = 0 \oplus \mathbb{Z}_2\}.\)

(ii) Suppose that \(c \neq c^2\) for all \(c \in Z_R(M) \setminus \text{Ann}_{R}(M)\) and \(a, b\) are two distinct elements of \(Z_R(M) \setminus \text{Ann}_{R}(M).\) Since \(\Gamma(M)\) is a complete graph, \(abM = 0.\) So \(ab \in \text{Ann}_{R}(M)\) and \(\{ab : a, b\ \text{are distinct elements of} \ Z_R(M) \setminus \text{Ann}_{R}(M)\} \subseteq \text{Ann}_{R}(M).\) If for all \(c \in Z_R(M) \setminus \text{Ann}_{R}(M),\) \(c^2M = 0,\) then \(\{ab : a, b \in Z_R(M)\} \subseteq \text{Ann}_{R}(M).\) Let \(p, p_1 \in \text{Ass}_R(M).\) Thus \(p^2 \subseteq p_1\) and \(p_1^2 \subseteq p.\) So \(p = p_1.\) Hence, \(\text{Ass}_R(M) = \{p\}\) and \(Z_R(M) = r(\text{Ann}_R(M)) = p\) which implies that \(p^2 \subseteq \text{Ann}_{R}(M).\)

(iii) Now assume that there exists \(c \in Z_R(M) \setminus \text{Ann}_{R}(M)\) such that \(c^2M \neq 0.\) In this case, we have \(c^2M = 0.\) Thus \(\{abc : a, b, c \in Z_R(M)\} \subseteq \text{Ann}_{R}(M).\) Let \(p, p_1 \in \text{Ass}_R(M).\) Then \(p^3 \subseteq p_1\) and \(p_1^3 \subseteq p.\) So \(p = p_1.\) Hence, \(\text{Ass}_R(M) = \{p\}\) and \(Z_R(M) = r(\text{Ann}_R(M)) = p\) which implies that \(p^3 \subseteq \text{Ann}_{R}(M).\)

**Corollary 2.3.** Let \(M\) be a Noetherian \(R\)-module, \(\text{Ass}_R(M) = \{p\}\) and \(p^2 \subseteq \text{Ann}_{R}(M).\) Then \(\Gamma(M)\) is a complete graph. In particular, if \(R\) is a Noetherian ring, then \(\Gamma(R/p^2)\) is a complete graph, where \(p\) is a prime ideal of \(R.\)

**Proof.** Let \(a \in Z_R(M) \setminus \text{Ann}_{R}(M).\) Then for each \(b \in Z_R(M) \setminus \text{Ann}_{R}(M),\) we have \(ab \in p^2 \subseteq \text{Ann}_{R}(M)\) since \(\text{Ass}_R(M) = \{p\}\) and \(Z_R(M) = p.\) So \(abM = 0.\) For the second part it is obvious that \(\text{Ass}_R(R/p^2) = \{p\}\) and \(p^2 \subseteq \text{Ann}_{R}(R/p^2).\) According to the above, the proof is completed.

**Example 2.2.** Let \(p\) be a prime number and \(M = \mathbb{Z}/p^3\mathbb{Z}\). Then \(\text{Ass}_{\mathbb{Z}}(M) = \{p\mathbb{Z}\}\) and \(p^3\mathbb{Z} \subseteq \text{Ann}_{\mathbb{Z}}(M)\) but \(\Gamma(M)\) is not a complete graph.

**Corollary 2.4.** Let \(R\) be a local ring and \(\Gamma(M)\) be a complete graph. Then \(\text{Ass}_R(M) = \{p\}\) where either \(p^2 \subseteq \text{Ann}_{R}(M)\) or \(p^3 \subseteq \text{Ann}_{R}(M).\)
Proof. In this case for all \( a \in Z_{R}(M) \setminus \text{Ann}_{R}(M) \), \( a \neq a^{2} \). So the result follows by the proof of Theorem 2.3.

\[ \square \]

**Theorem 2.4.** Let \( M \) be a Noetherian \( R \)-module and \( r(\text{Ann}_{R}(M)) = \text{Ann}_{R}(M) \). Then \( \Gamma(M) \) is a complete bipartite graph if and only if \( |\text{Ass}_{R}(M)| = 2 \).

**Proof.** Suppose that \( I = \text{Ann}_{R}(M) \). Then \( r(\text{Ann}_{R}(M))/I = r(\text{Ann}_{R/I}(M)) \) since \( M \) is an \( R/I \)-module. Thus by hypothesis \( r(\text{Ann}_{R/I}(M)) = 0 \). Moreover, for each \( a \in R \), we have \( a \in Z_{R}(M) \setminus \text{Ann}_{R}(M) \) if and only if \( 0 \neq a + I \in Z_{R/I}(M) \) and \( p \in \text{Ass}_{R}(M) \) if and only if \( p/I \in \text{Ass}_{R/I}(M) \) since \( p = \text{Ann}_{R}(m) \) if and only if \( p/I = \text{Ann}_{R}(m)/I = \text{Ann}_{R/I}(m) \) for all \( 0 \neq m \in M \). So we can and do assume \( r(\text{Ann}_{R}(M)) = 0 \) and we have to show that \( \Gamma(M) \) is a complete bipartite graph if and only if \( |\text{Ass}_{R}(M)| = 2 \).

(\( \Rightarrow \)) Let \( \Gamma(M) \) be a complete bipartite graph and \( V_{1}, V_{2} \) be two distinct sets of vertices. We prove that \( V_{i} \cup \{0\} \), for \( i = 1, 2 \), is a prime ideal of \( R \). To show this let \( a, b \in V_{i} \) and \( a + b \neq 0 \). Then \( a + b \in V_{i} \). Now, suppose that \( a, b \in V_{1} \). Thus there exist \( x, y \in V_{2} \) such that \( xaM = 0 = ybM \). Hence, \( (a + b)xyM = 0 \) and \( xyM = 0 \) implies \( a + b \in Z_{R}(M) \). If \( a + b \in V_{1} \), then \( a + b \in V_{2} \) where \( a(a + b)M = 0 \) and \( b(a + b)M = 0 \). So \( (a + b)^{2}M = 0 \) which means that \( a + b \in \text{rad}(\text{Ann}_{R}(M)) = 0 \) and this is a contradiction. Thus \( a + b \in V_{1} \). Let \( a, b \in R \) and \( ab \in V_{1} \). We show that \( a \in V_{1} \) or \( b \in V_{1} \). If \( a = 0 \) or \( b = 0 \), then the proof is completed. So let \( a \neq 0, b \neq 0 \). If \( ab = 0 \), then by hypothesis \( a \in V_{1} \) or \( b \in V_{1} \). Now, suppose that \( ab \neq 0 \) and \( a, b \in V_{2} \). Thus \( a^{2}b^{2}M = 0 \) and \( (ab)^{2}M = 0 \) which implies that \( ab \in \text{rad}(\text{Ann}_{R}(M)) = 0 \) and this is a contradiction.

(\( \Leftarrow \)) Let \( \text{Ass}(M) = \{p_{1}, p_{2}\} \). Then \( r(\text{Ann}_{R}(M)) = p_{1} \cap p_{2} = 0 \). Assume that \( a, b \in p_{1} \setminus \{0\} \) and \( abM = 0 \). Thus \( ab \in p_{2} \) and so either \( a \in p_{2} \) or \( b \in p_{2} \), which implies that \( a = 0 \) or \( b = 0 \) and this is a contradiction. Hence, two elements of \( p_{2} \setminus \{0\} \) are not adjacent. By a similar argument, we can show that the elements of \( p_{2} \setminus \{0\} \) are not adjacent. Let \( a \in p_{1}^{*} \setminus \{0\} \) and \( b \in p_{2}^{*} \setminus \{0\} \). Then \( ab \in p_{1}^{*}p_{2}^{*} \subseteq p_{1} \cap p_{2} = 0 \). So \( abM = 0 \) which means that two elements of \( p_{1}^{*} \) and \( p_{2}^{*} \) are adjacent. Hence, \( \Gamma(M) \) is a complete bipartite graph.

\[ \square \]

**Theorem 2.5.** Let \( M \) be a Noetherian \( R \)-module and \( r(\text{Ann}_{R}(M)) = \text{Ann}_{R}(M) \). If \( \Gamma(M) \) is a complete \( r \)-partite graph, \( r \geq 3 \), then at most one of the parts has more than one vertex.

**Proof.** Suppose that \( V_{1}, \ldots, V_{r} \) are distinct parts of \( \Gamma(M) \). Let \( V_{i} \) and \( V_{s} \) have more than one vertex. Choose \( x \in V_{i} \) and \( y \in V_{s} \). Let \( V_{t} \) be a part of \( \Gamma(M) \) such that \( V_{t} \neq V_{i} \) and \( V_{t} \neq V_{s} \). Let \( z \in V_{t} \). Since \( \Gamma(M) \) is a complete \( r \)-partite graph, \( \text{Ann}_{R}(xM) = \bigcup_{1 \leq i \leq r} V_{i} \cup \{0\} \), \( \text{Ann}_{R}(yM) = \bigcup_{1 \neq i \leq r} V_{i} \cup \{0\} \) and \( \text{Ann}_{R}(zM) = \bigcup_{1 \leq i \leq r} V_{i} \cup \{0\} \). Hence, \( \text{Ann}_{R}(zM) \subseteq \bigcup_{1 \leq i \leq r} \text{Ann}_{R}(xM) \cup \text{Ann}_{R}(yM) \). So \( \text{Ann}_{R}(zM) \subseteq \text{Ann}_{R}(xM) \) or \( \text{Ann}_{R}(zM) \subseteq \text{Ann}_{R}(yM) \). Let \( \text{Ann}_{R}(zM) \subseteq \text{Ann}_{R}(xM) \) and let \( x' \in V_{i} \) be such that \( x' \neq x \). Then \( x' \in \text{Ann}_{R}(zM) \cup \text{Ann}_{R}(xM) \) this is a contradiction.

\[ \square \]
3. Study of the zero-divisor graph of a module by annihilator submodules

In this section we study the relations between the set of annihilator prime submodules of a module and zero-divisor graph of a module. Let $M$ be an $R$-module. A proper submodule $P$ of $M$ is said to be prime submodule whenever for $r \in R$ and $m \in M$, $rm \in P$ implies $m \in P$ or $r \in \text{Ann}_R(M/P)$. Let $\text{Spec}_R(M)$ denote the set of prime submodules of $M$ and $m - \text{Ass}_R(M) = \{ P \in \text{Spec}_R(M) : P = \text{Ann}_M(a) \text{ for some } 0 \neq a \in R \}$, where $\text{Ann}_M(a) = \{ m \in M : am = 0 \}$, for $a \in R$.

**Lemma 3.1.** Let $M$ be an $R$-module, $a, b, c \in R$ and $\text{Ann}_M(a)$ be a prime submodule of $M$. Then the following statements are true:

(i) If $\text{Ann}_M(b)$ is a prime submodule of $M$, then $abM = 0$.
(ii) If $\text{Ann}_M(b) \not\subseteq \text{Ann}_M(a)$, then $abM = 0$.
(iii) If $b \in r(\text{Ann}_R(M)) \setminus \text{Ann}_R(M)$, then $abM = 0$.
(iv) If $abM \neq 0$, then $\text{Ann}_M(b)$ is not a prime submodule of $M$, $\text{Ann}_M(b) \subseteq \text{Ann}_M(a)$ and $b \not\in r(\text{Ann}_R(M))$.
(v) If $\text{Ann}_M(b) \subseteq \text{Ann}_M(a)$ and $a \not\in r(\text{Ann}_R(M))$, then $abM = 0$.
(vi) If $bcM = 0$, then either $abM = 0$ or $acM = 0$.

**Proof.** (i) Let $P_1 = \text{Ann}_M(a)$, $P_2 = \text{Ann}_M(b)$ be two distinct prime submodules of $M$. Assume that $m \in P_1 \setminus P_2$. Thus $ma = 0 \in P_2$ which implies that $aM \subseteq P_2 = \text{Ann}_M(b)$ since $m \not\in P_2$. Hence, $abM = 0$ and so $a, b$ are adjacent in $\Gamma(M)$.

(ii) It follows by a similar argument to that of (i).

(iii) Let $b \in r(\text{Ann}_R(M)) \setminus \text{Ann}_R(M)$. Then there exists $t \in \mathbb{N}$ such that $b^tM = 0$. So $b^tM = 0 \subseteq \text{Ann}_M(a)$ which implies that $bM \subseteq \text{Ann}_M(a)$ and $abM = 0$.

(iv) It is contrapositive of (i), (ii) and (iii).

(v) If $abM = 0$, then $aM \subseteq \text{Ann}_M(b) \subseteq \text{Ann}_M(a)$. So $a \in r(\text{Ann}_R(M))$ which contradicts the assumption. Thus $abM \neq 0$.

(vi) Suppose that $bcM = 0$ and $m \in M \setminus P_1 = \text{Ann}_M(a)$. Then $bcm \in P_1$ which implies that $bc \in \text{Ann}_R(M/P_1)$ such that $\text{Ann}_R(M/P_1)$ is a prime ideal of $R$. Hence, either $b \in \text{Ann}_R(M/P_1)$ or $c \in \text{Ann}_R(M/P_1)$. So either $abM = 0$ or $acM = 0$. 

**Theorem 3.1.** Let $M$ be a Noetherian $R$-module. Then the following statements are true:

(i) If $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$ and there exists $b \in Z_R(M) \setminus \text{Ann}_R(M)$ such that $\text{Ann}_M(b) \subseteq \bigcap_{P \in m - \text{Ass}_M(M)} P$, then $\Gamma(M)$ is a disconnected graph.

(ii) If $\Gamma(M)$ is a disconnected graph, then there exists $b \in Z_R(M) \setminus \text{Ann}_R(M)$ such that $\text{Ann}_M(b) \subseteq \bigcap_{P \in m - \text{Ass}_M(M)} P$.

**Proof.** (i) We show that $b$ is an isolated vertex of $\Gamma(M)$. Let $a \in Z_R(M) \setminus \text{Ann}_R(M)$ and let $a$ be adjacent to $b$ in $\Gamma(M)$. We know that $X = \{ \text{Ann}_M(c) : c \notin \text{Ann}_R(M) \}$ has a maximal element and it is easy to see that the maximal element
of $X$ is a prime submodule. So there exists $c \in Z_R(M) \setminus \text{Ann}_R(M)$ such that $\text{Ann}_M(a) \subseteq \text{Ann}_M(c)$ and $\text{Ann}_M(c)$ is a prime submodule of $M$. Now, $abM = 0$ implies $bcM = 0$ which contradicts the Lemma 3.1(iv). Hence, $b$ is an isolated vertex of $\Gamma(M)$.

(ii) Let $b \in Z_R(M) \setminus \text{Ann}_R(M)$ be an isolated vertex of $\Gamma(M)$. If $|m - \text{Ass}(M)| = 1$, then by the proof of (i) we have $\text{Ann}_M(b) \subseteq \bigcap_{P \in m - \text{Ass}(M)} P$. If $|m - \text{Ass}(M)| \geq 2$, then $\text{Ann}_M(b)$ is not a prime submodule, see Lemma 3.1(i). So $\text{Ann}_M(b) \subseteq \bigcap_{P \in m - \text{Ass}(M)} P$ by Lemma 3.1(ii).

**Theorem 3.2.** If $\Gamma(M)$ is a connected graph, then $\text{diam}(\Gamma(M)) \leq 3$.

**Proof.** Let $x, y \in Z_R(M) \setminus \text{Ann}_R(M)$ be distinct vertices of $\Gamma(M)$. If $xyM = 0$, then $d(x, y) = 1$. Suppose that $xyM$ is nonzero. Then there exist $a, b \in Z_R(M) \setminus \text{Ann}_R(M) \cup \{x, y\}$ with $axM = byM = 0$. If $a = b$, then $x - a - y$ is a path of length 2. Thus we may assume that $a \neq b$. If $abM = 0$, then $x - a - b - y$ is a path of length 3 so $d(x, y) \leq 3$. If $abM \neq 0$, then $x - ab - y$ is a path of length 2. Hence $d(x, y) = 2$. Thus $d(x, y) \leq 3$ and so $\text{diam}(\Gamma(M)) \leq 3$. □

**Theorem 3.3.** Let $M$ be a Noetherian $R$-module and let $\Gamma(M)$ have a cycle. Then $\text{gr}(\Gamma(M)) \leq 4$.

**Proof.** Let $c_1 - c_2 - \cdots - c_7$ be a cycle such that $c_i = \text{Ann}_M(b_i)$ for each $1 \leq i \leq 7$. If $|m - \text{Ass}_R(M)| \geq 3$, then $\text{gr}(\Gamma(M)) \leq 3$ by Lemma 3.1(i). Let $|m - \text{Ass}_R(M)| \leq 2$ and $P = \text{Ann}_M(a) \in m - \text{Ass}_R(M)$. From $b_1b_2M = 0$ it follows that either $b_1aM = 0$ or $b_2aM = 0$, by Lemma 3.1(vi). Let $b_1aM = 0$. If $b_2aM = 0$, then $\text{gr}(\Gamma(M)) \leq 3$. Otherwise $b_2b_3M = 0$ implies $ab_3M = 0$. So $\text{gr}(\Gamma(M)) \leq 4$. If $a$ be one of the $b_i$’s, then $\text{gr}(\Gamma(M)) \leq 4$. □

**Theorem 3.4.** Let $M$ be a Noetherian $R$-module and $r(\text{Ann}_R(M)) = \text{Ann}_R(M)$. If $\omega(\Gamma(M))$ is finite, then $\omega(\Gamma(M)) = |m - \text{Ass}(M)|$.

**Proof.** Let $I = \text{Ann}_R(M)$. Then it is easy to see that $M$ is an $R/I$-module and $r(\text{Ann}_R(M)/I) = r(\text{Ann}_{R/I}(M))$. Thus by hypothesis $r(\text{Ann}_{R/I}(M)) = 0$. Moreover, for each $a \in R$, we have $\text{Ann}_M(a) = \text{Ann}_M(a + I)$. So we can and do assume that $r(\text{Ann}_R(M)) = 0$. Let $|m - \text{Ass}(M)| = n$. By Lemma 3.1(i), all elements of $m - \text{Ass}(M)$ are adjacent. Now, let $\omega(\Gamma(M)) = k > n$. Then there exist $a, b \in Z_R(M) \setminus \text{Ann}_R(M)$ such that $abM = 0$ and there is $P = \text{Ann}_M(c) \in m - \text{Ass}(M)$ where $\text{Ann}_M(a), \text{Ann}_M(b) \subseteq \text{Ann}_M(c)$. If $abM = 0$ then $bM \subseteq \text{Ann}_M(a) \subseteq \text{Ann}_M(c)$. So $bcM = 0$. Similarly, $acM = 0$. Also, $bcM = 0$ implies $c^2M = 0$ which is a contradiction. So $a, b$ are not adjacent. Therefore $\omega(\Gamma(M)) = |m - \text{Ass}(M)| = n$. □

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