Horizon symmetries of black holes with supertranslation field

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Abstract

Near-horizon symmetries are studied for black hole solutions to Einstein equations containing supertranslation field constructed by Compere and Long. The metric is transformed to variables in which the horizon is located at the surface \( r = 2M \), where \( M \) is the mass of black hole. We consider general diffeomorphisms which preserve the gauge and the near-horizon structure of the metric and find the corresponding transformations of metric components. We review the action of the generators of supertranslations preserving the static gauge of the metric and determine a subgroup of supertranslations preserving the gauge and near-horizon structure of the metric. Variation of the surface charge corresponding to the Killing vectors of asymptotic horizon symmetries is calculated. Sufficient conditions of integrability of the variation of the surface charge to a closed integral form are found and an example of metrics with integrable charge variation is discussed.

1 Introduction

BMS symmetry is a symmetry of the asymptotically flat spaces at the null infinity. The infinite-dimensional group of the BMS transformations extends the Poincare group and contains supertranslations, angular-dependent shifts of retarded time at null infinity [1, 2, 3]. Finite supertranslation diffeomorphisms map field configurations to inequivalent, physically different configurations which differ by structure of a cloud of soft particles [4, 5, 6]. Configurations with different supertranslation fields are physically different in a sense that their corresponding (superrotation) charges are different [7, 8, 9, 10, 11, 12].

The final state of gravitational collapse is diffeomorphic to the Kerr space-time. The set of diffeomorphisms contains supertranslations, and the resulting stationary metric contains supertranslation field. The final state of collapse is parametrised by mass, angular momentum and supertranslation field [12].

The BMS transformations are naturally formulated at the null infinity, and there is a complicated problem of extension of an asymptotically defined metric with supertranslation field in a closed form to the bulk. In paper [11] a family of vacua containing supertranslation field was constructed in the

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bulk. In paper [12] a solution-generation technique was developed and applied to construction of the exact solutions to Einsteins equations which are black hole metrics containing supertranslation field. It is an interesting question, how the infinite-dimensional symmetries of the space-time reveal themselves in the near-horizon physics.

Because many problems of black-hole physics are connected with horizon structure of black holes, and having an explicit example of a metric with supertranslation field, in the present paper we study the near-horizon symmetries of the black hole metric [12] containing supertranslation field.

In Sect.2 the metric of [12] is transformed to a form in which horizon of the black hole is located at the surface \( r = 2M \), where \( M \) is the mass of black hole. Solving the geodesic equations, we show that the surface \( r = 2M \) is the surface of infinite red-shift.

In Sect.3 we study general near-horizon symmetries of the metric preserving the form of the metric components in the leading orders in distance from the horizon \( r - 2M \). Solving the equations for the asymptotic Killing vectors corresponding to near-horizon symmetries of the metric, we find variations of the metric components under the action of generators of the asymptotic near-horizon symmetries.

In Sect.4, first, we review the action of supertranslations preserving the static gauge of the metric. Restricted to the near-horizon region, supertranslations act as near-horizon symmetries. We find a subgroup of the group of supertranslations which preserve both the gauge and the near-horizon form of the metric.

In Sect.5, using the results of Sect.3, we calculate variation of the surface charge corresponding to the asymptotic horizon symmetries. Sufficient conditions on the metric making possible integration of the variation of the charge over the space of metrics to a closed form are found. Example of metrics with supertranslation field depending only on spherical angle \( \theta \) is considered. It is shown that in this case there appear relations between the components of the metric which make possible integration of the variation of the charge to a closed expression.

Sect.6 contains remarks on connection of the results of the present with other papers.

2 Static metric with supertranslation field

In this section, after a short review of black hole with supertranslation field constructed in [12], we transform the metric to a form with the horizon located at the surface \( r = 2M \). Solving the geodesic equations for null geodesics we show that the surface \( r = 2M \) is the surface of infinite red-shift.

Vacuum solution of the Einstein equations containing supertranslation field \( C(z, \bar{z}) \) is

\[
ds^2 = \tilde{g}_{mn}dx^m dx^n = -\frac{(1 - M/2\rho_s)^2}{(1 + M/2\rho_s)^2} dt^2 + (1 + M/2\rho_s)^4 \left[ d\rho^2 + (((\rho - E)^2 + U)\gamma_{ab} + (\rho - E)C_{ab})dz^a dz^b \right],
\]

where \( z^a = z, \bar{z} \). Supertranslation field \( C(z, \bar{z}) \) is a real regular function on the unit sphere. Here

\[
\rho_s(\rho, C) = \sqrt{(\rho - C - C_{00})^2 + D_a D^a C}.
\]

\( C_{00} \) is the lowest spherical harmonic mode of \( C(z, \bar{z}) \). In the following we do not write \( C_{00} \) explicitly understanding \( C \rightarrow C - C_{00} \). Covariant derivatives \( D_a, a = z, \bar{z} \) are defined with respect to the metric on the sphere \( ds^2 = \gamma_{zz}dz d\bar{z}, \gamma_{zz} = \cot^2 \theta e^{2\psi}, \gamma_{z\bar{z}} = 2e^{-2\psi}, \psi = \ln(1 + |z|^2) \). Also we consider the metric (1) written in spherical coordinates with the metric on the sphere \( ds^2 = d\theta^2 + \sin^2 \theta d\phi^2 \).
The tensor $C_{ab}$ and the functions $U$ and $E$ are defined as

$$C_{ab} = -(2D_a D_b - \gamma_{ab} D^2) C,$$

$$U = \frac{1}{8} C_{ab} C^{ba},$$

$$E = \frac{1}{2} D^2 C + C,$$

and in $z, \bar{z}$ representation are equal to

$$C_{zz} = -2D_z D_z C, \quad C_{\bar{z}\bar{z}} = -2D_{\bar{z}} D_{\bar{z}} C, \quad C_{z\bar{z}} = 0,$$

$$U = \frac{1}{4} (\gamma_{z\bar{z}})^2 D_z^2 D_{\bar{z}}^2 C,$$

$$E = \gamma_{z\bar{z}} D_z D_{\bar{z}} C + C.$$

We introduce a new variable $r = r(\rho, z^a)$ chosen so that the $\tilde{g}_{tt}$ component of the metric is equal to $1 - 2M/r$ [13]. Variable $r$ is defined the by the relation

$$r = \rho s(\rho, C) \left(1 + \frac{M}{2\rho s(\rho, C)}\right)^2.$$

Inversely, $\rho$ is expressed through $r$ as

$$\rho = C + \sqrt{\frac{K^2}{4} - D_a C D^a C},$$

where we introduced the functions

$$K = r - M + r V^{1/2}, \quad V = 1 - \frac{2M}{r}.$$

Introducing

$$b_a = \frac{2\partial_a C}{K}, \quad b^2 = b_a b^a, \quad a = z, \bar{z},$$

we have the expression for $d\rho(r, z^a)$ in a form

$$d\rho = \frac{K}{2\sqrt{1 - b^2}} \left[f_a dz^a + \frac{dr}{r V^{1/2}}\right],$$

$$f_a = b_a \sqrt{1 - b^2} - \frac{\partial_a b^2}{2}.$$

Writing

$$((\rho - E)^2 + U) \gamma_{z\bar{z}} = \left(\frac{K}{2}\right)^2 \hat{g}_{z\bar{z}},$$

$$(\rho - E) C_{zz} = \left(\frac{K}{2}\right)^2 \hat{g}_{zz},$$

$$(\rho - E) C_{\bar{z}\bar{z}} = \left(\frac{K}{2}\right)^2 \hat{g}_{\bar{z}\bar{z}},$$

$$C_{zz} = -2D_z D_z C,$$

$$C_{\bar{z}\bar{z}} = -2D_{\bar{z}} D_{\bar{z}} C,$$

$$C_{z\bar{z}} = 0,$$

$$U = \frac{1}{4} (\gamma_{z\bar{z}})^2 D_z^2 D_{\bar{z}}^2 C,$$

$$E = \gamma_{z\bar{z}} D_z D_{\bar{z}} C + C.$$
we express the metric (1) in a form
\[ ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{rz}drdz + g_{zz}dz^2 + 2g_{z\bar{z}}dzd\bar{z} = -Vdt^2 + \frac{dr^2}{V(1 - b^2)} + \frac{r(f_{,z}dz + f_{,\bar{z}}d\bar{z})dr}{V^{1/2}(1 - b^2)} + r^2\left[\left(\frac{f_z^2}{1 - b^2} + \hat{g}_{zz}\right)dz^2 + \left(\frac{f_{\bar{z}}^2}{1 - b^2} + \hat{g}_{\bar{z}\bar{z}}\right)d\bar{z}^2 + 2\left(\frac{f_zf_{\bar{z}}}{1 - b^2} + \hat{g}_{z\bar{z}}\right)dzd\bar{z}\right]. \] (15)

To obtain the metric in a form (15), we have used the relations
\[ \frac{(1 - M/2\rho_s)^2}{(1 + M/2\rho_s)^3} = V, \quad (1 + M/2\rho_s)^4 = \frac{4r^2}{K^2}. \] (16)

For the above expressions to be well-defined, we require that \(1 - b^2 > 0\). Because \(K\) is the increasing function of \(r\) having its minimum at \(r = 2M\) the sufficient condition is \(1 - |2\partial_s C/M|^2 > 0\). In (15) we separated the factors \(V\) which are most singular in the near-horizon limit \(r \to 2M\).

To show that the surface \(r = 2M\) is the infinite red-shift surface, we solve the null geodesic equations in the metric (15) in the limit \(V \to 0\). Geodesic equations can be written either in a form using the Christoffel symbols, or starting from the Lagrangian corresponding to the metric [14]. Separating the leading in \(V \to 0\) parts of the metric components, we obtain the Lagrangian in a form
\[ \mathcal{L} = -\frac{V\dot{t}^2}{2} + \frac{\dot{r}^2g_{rr}}{2V} + \frac{\ddot{r}g_{rr}}{V} + \frac{\dot{r}g_{rr,r}}{2V^2} + \frac{\dot{r}^2g_{rrr}}{2V^2} - \frac{\ddot{r}g_{rr}}{V^{1/2}} - \frac{\dot{r}^2g_{rr}}{V^{1/2}} + \frac{\dot{r}^2V_r}{2} + \frac{\ddot{r}g_{rr}}{2V^{1/2}} + \frac{\ddot{r}g_{rr}}{2V^{1/2}} + \ddot{g}_{ab}\dot{z}_a\dot{z}_b + \ddot{g}_{ab}\dot{z}_b + \ddot{g}_{ab}\dot{z}_b + \frac{1}{2}\ddot{g}_{bc,\bar{z}}\dot{z}_b\dot{z}_c = 0. \] (17)

Here dot is derivative with respect to an affine parameter along the geodesic \(\tau\), bar over metric component means that the leading in \(V \to 0\) factor is written explicitly. The Lagrange equations are
\[ \frac{d(iV)}{d\tau} = 0, \] (18)
\[ \frac{\ddot{r}g_{rr}}{V} + \frac{\dot{r}^2g_{rr}}{V} + \frac{\ddot{r}g_{rr}}{2V^2} + \frac{\dot{r}g_{rr,r}}{2V^2} + \frac{\dot{r}^2g_{rrr}}{2V^2} - \frac{\ddot{r}g_{rr}}{V^{1/2}} - \frac{\dot{r}^2g_{rr}}{V^{1/2}} + \frac{\dot{r}^2V_r}{2} + \frac{\ddot{r}g_{rr}}{2V^{1/2}} + \frac{\ddot{r}g_{rr}}{2V^{1/2}} + \ddot{g}_{ab}\dot{z}_a\dot{z}_b = 0, \] (19)
\[ -\frac{\ddot{g}_{ar}V_r}{2V^{3/2}} + \frac{\ddot{g}_{ar}}{V^{1/2}} + \frac{\ddot{g}_{ar}}{V^{1/2}} + \ddot{g}_{ab}\dot{z}_b + \ddot{g}_{ab}\dot{z}_b + \frac{1}{2}\ddot{g}_{bc,\bar{z}}\dot{z}_b\dot{z}_c = 0. \] (20)

We look for a solution in the asymptotic region \(V \to 0\) in a form
\[ i = \frac{E}{V}, \] (21)
\[ \dot{r} = C + C_1V^{1/2} + \cdots, \] (22)
\[ \dot{z}_a = A_aV^{-1/2} + A_1a + \cdots. \] (23)

Substituting the Ansatz in Eqs. (18)-(19) and separating the leading in \(V \to 0\) terms, we have
\[ V^{-2}[E^2 - \ddot{g}_{rr}C^2 - \ddot{g}_{ra}A_rC] = 0, \] (24)
\[ V^{-3/2}[(\ddot{g}_{ra}C + \ddot{g}_{ab}A_b) = 0. \] (25)

Solving the system (24)-(25), we obtain
\[ A_b = -(g^{-1})_{ba}\ddot{g}_{ar}, \] (26)
\[ E^2 = [\ddot{g}_{rr} - \ddot{g}_{rb}(g^{-1})_{ba}\ddot{g}_{ar}]C^2. \] (27)
Because of translation invariance of the metric in $t$, the system of equations (18)-(20) has the first integral $\mathcal{L} = \mathcal{H} = 0$. Substituting the Ansatz with the solution (26)-(27), we find that in the main order in $V \to 0$ the relation $\mathcal{H} = 0$ is satisfied identically. From (21) and (22) in the main order in $V \to 0$ we obtain

$$\frac{dr}{dt} = \pm |E|V(r).$$

(28)

From this relation it follows that the surface $r = 2M$ is the surface of infinite redshift [15].

### 3 Diffeomorphisms preserving the near-horizon form of the metric

In this section we study general diffeomorphisms which preserve the form of the metric in the near-horizon. To discuss the near-horizon geometry, we consider the leading in $x = r - 2$ terms in the metric. Requiring that the functional form of the leading terms is preserved under the action of diffeomorphisms, we obtain the restrictions on the vector fields generating diffeomorphisms. Transformations preserving the gauge conditions $g_{rt} = g_{ta} = 0$ result in time independence of generators of transformations, from transformations of other metric components follow restrictions on the $x$-dependence of generators.

Separating the leading in $x$ behavior in the metric components, we present the metric (15) as

$$ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{ra}drdz^a + g_{ab}dz^a dz^b =$$

$$= (-g_{tt,1}x + O(x^2)) dt^2 + \left(\frac{g_{rr,-1}}{x} + O(x^{-1/2})\right) dx^2 + 2 \left(\frac{g_{ra,-1/2}}{x^{1/2}} + O(x^0)\right) dx dz^a +$$

$$+ (g_{ab,0} + O(x^{1/2})) dz^a dz^b.$$

(29)

Under the diffeomorphisms generated by vector fields $\chi^k$ the metric components are transformed as

$$L_{\chi} g_{mn} = \chi^k \partial_k g_{mn} + \partial_m \chi^k g_{kn} + \partial_n \chi^k g_{mk}$$

(30)

We look for the components of the vector fields $\chi^k$ in a form of expansion in powers in $x^{1/2}$

$$\chi^k = \chi^k_0 + x^{1/2} \chi^k_{1/2} + x \chi^k_1 + \cdots$$

The metric (1) is written in the gauge

$$g_{rt} = g_{ta} = 0.$$

Transformations preserving the gauge conditions are

$$L_{\chi} g_{rt} = \partial_r \chi^r g_{tt} + \partial_t \chi^r g_{rr} + \partial_k \chi^a g_{ar} = 0$$

(31)

$$L_{\chi} g_{at} = \partial_a \chi^r g_{tt} + \partial_t \chi^a g_{ra} + \partial_t \chi^b g_{ba} = 0.$$  

(32)

From the equations (31)- (32) it follows that

$$\dot{\chi}^r_0 = \dot{\chi}^r_{1/2} = \dot{\chi}^r_1 = 0,$$

$$\dot{\chi}^a_0 = \dot{\chi}^a_{1/2} = 0.$$

(33)

(34)

Here "dot" denotes differentiation over $t$. 

5
Transformations preserving the leading in $x$ behavior of other metric components are

$$L_{\chi}g_{tt} = \chi^r \partial_r g_{tt} + 2 \partial_r \chi^r g_{tt} = O(x),$$
$$L_{\chi}g_{rr} = \chi^r \partial_r g_{rr} + \partial_a \chi^a g_{rr} + 2 \partial_r \chi^r g_{rr} = O(x^{-1}),$$
$$L_{\chi}g_{ar} = \chi^r \partial_r g_{ar} + \chi^b \partial_b g_{ar} + \partial_a \chi^r g_{ar} + \partial_r \chi^r g_{ar} = O(x^{-1/2}),$$
$$L_{\chi}g_{ab} = (\chi^r \partial_r + \chi^c \partial_c) g_{ab} + \partial_a \chi^r g_{rb} + \partial_a \chi^r g_{cb} + \partial_b \chi^r g_{ra} + \partial_b \chi^c g_{ca} = O(x^0).$$

From the transformation of the $g_{rr}$-component it follows that

$$\chi_0^r = \chi_{1/2}^r = 0. \quad (36)$$

From transformations (35) we extract transformations of the leading-order terms of metric components

$$\delta_{\chi}g_{tt} = \chi_1^r g_{tt} + \chi_0^a \partial_a g_{tt} + \chi_0^r g_{tt},$$
$$\delta_{\chi}g_{rr,-1} = \chi_1^r g_{rr,-1} + \chi_{1/2}^a \partial_a g_{rr,-1} + \chi_0^r \partial_0 g_{rr,-1},$$
$$\delta_{\chi}g_{ar,-1/2} = \frac{1}{2} \chi_1^r g_{ar,-1/2} + \chi_0^b \partial_b g_{ar,-1/2} + \partial_a \chi_0^b g_{br,-1/2} + \partial_0 \chi_0^b g_{br,-1/2} + \frac{1}{2} \chi_{1/2}^r g_{ab,0},$$
$$\delta_{\chi}g_{ab,0} = \chi_0^c \partial_c g_{ab,0} + \partial_a \chi_0^c g_{cb,0} + \partial_b \chi_0^c g_{ca,0},$$
$$\delta_{\chi}g_{ab,1/2} = \frac{1}{2} \chi_1^r g_{ab,1/2} + \partial_a \chi_1^r g_{rb,1/2} + \partial_b \chi_1^r g_{ra,1/2} + \chi_0^c \partial_c g_{ab,1/2} + \chi_{1/2}^r \partial_c g_{ab,0} + \frac{1}{2} \chi_0^c g_{cb,1/2} + \partial_a \chi_1^r g_{cb,1/2} + \partial_b \chi_0^c g_{ca,1/2} + \partial_b \chi_{1/2}^r g_{ca,0}. \quad (37)$$

The leading-order parts of the metric components are transformed through the functions

$$x_{\chi_1}(z, \bar{z}), \quad x_{\bar{\chi}_0}(z, \bar{z}), \quad x_{\bar{\chi}_0}(z, \bar{z}), \quad x^{1/2}_{\chi_1/2}(z, \bar{z}), \quad x^{1/2}_{\chi_1/2}(z, \bar{z}), \quad x^0_{\theta}(t, z, \bar{z}). \quad (38)$$

The variations of the metric components are used in Sect.5 for calculation of the surface charge corresponding to asymptotic horizon symmetries. The Lie brackets of vector fields generating the near-horizon transformations are

$$[\chi_{(1)}, \chi_{(2)}]^k = \chi_{(12)}^k, \quad (39)$$

where

$$\chi_{(12),0}^t = \chi_{(1),0}^t \partial_t \chi_{(2),0}^t + \left( \chi_{(1),0}^b \partial_b \chi_{(2),0}^t - (1 \leftrightarrow 2) \right),$$
$$\chi_{(12),1}^t = \chi_{(1),0}^t \partial_0 \chi_{(2),1}^t - (1 \leftrightarrow 2),$$
$$\chi_{(12),0}^a = \chi_{(1),0}^b \partial_b \chi_{(2),0}^a,$$
$$\chi_{(12),1/2}^a = \left( \chi_{(1),0}^b \partial_b \chi_{(2),1/2}^a - (1 \leftrightarrow 2) \right) + 1/2 \left( \chi_{(1),1} \chi_{(2),1/2}^a - (1 \rightarrow 2) \right). \quad (40)$$

4 Supertranslutions preserving the static gauge of the metric

The metric (1) was obtained in [12] in the static gauge $\tilde{g}_{\rho a} = \tilde{g}_{ta} = 0$. The aim of this section is to find which supertranslutions preserve both the gauge and the form of the metric (15) at the horizon. We shortly review the supertranslutions preserving the gauge of (1). Next, we write the generator of supertranslutions in the case of the metric (15). We find conditions on the generator of
supertranslations under which the functional form of the component $g_{tt}$ at the horizon is preserved. It is shown that these conditions are sufficient to preserve the near-horizon form of all the components of the metric.

Generator of supertranslations preserving the static gauge of (1) was found in [12] and has the form

$$
\xi_T = T_{00}\partial_t - (T - T_{00})\partial_\rho + F^{ab}D_aT D_b,
$$

where

$$
F^{ab} = \frac{C^{ab} - 2\gamma^{ab}(\rho - E)}{2((\rho - E)^2 - U)^2}.
$$

Horizon of the metric (1) is located at the surface

$$
\rho_H(z, \bar{z}) = C + \sqrt{1/4 - D_a C D^a C}.
$$

Horizon of the metric (15) is located at the surface $r = 2$.

Generator of supertranslations preserving the static gauge of the metric (15) is obtained from the generator (41) by the coordinate transformation

$$
\chi^r_T = \xi^r\frac{\partial r}{\partial \rho} + \xi^a\frac{\partial r}{\partial z^a} = \xi^r\frac{\partial r}{\partial \rho} + \xi^a\frac{\partial r}{\partial z^a},
$$

$$
\chi^a_T = \xi^\rho\frac{\partial z^a}{\partial \rho} + \xi^\rho\frac{\partial z^a}{\partial \bar{z}^a} = \xi^a_T.
$$

From (2) and (7), we have

$$
\frac{\partial r}{\partial \rho} = \frac{K^2 - 1}{K^2}, \quad \frac{\partial \rho}{\partial \rho} = \sqrt{1 - b^2}, \quad \frac{\partial \rho}{\partial z^a} = \frac{K}{4}[-2b_a\sqrt{1 - b^2} + D_a b^2]
$$

In variables $t, r$ of the metric (15) generator of supertranslations takes the form

$$
\chi_T = \chi_T^r \partial_t + \chi_T^r \partial_r + \chi_T^a \partial_a,
$$

where

$$
\chi_T^r = T_{00},
$$

$$
\chi_T^r = \frac{K^2 - 1}{K^2} \left( -(T - T_{00})\sqrt{1 - b^2} + \frac{K}{4}F^{ab}D_b T(-2b_a\sqrt{1 - b^2} + D_a b^2) \right),
$$

$$
\chi_T^a = F^{ab}D_b T.
$$

Here $F^{ab}(r, z^a) = F^{ab}(\rho = \rho(r, z^a), z^a)$. In the near-horizon region $r = 2 + x, |x| \ll 1$, we have

$$
K \simeq 1 + \sqrt{2}x, \quad b_a = b_{a0}(1 - \sqrt{2}x), \quad b_{a0} = 2\partial_a C.
$$

For $x \ll 1$ the component $g_{tt}$ is

$$
g_{tt} = \frac{x}{2} + O(x^2).
$$

For $x \ll 1$ acting by the generator of supertranslations on the component $g_{tt}$, we obtain

$$
L_{\chi_T} g_{tt} = \frac{2}{r^2} \frac{K^2 - 1}{K^2} \left( -(T - T_{00})\sqrt{1 - b^2} + \frac{K}{4}(-2b_a\sqrt{1 - b^2} + D_a b^2) \right).
$$
In the near-horizon region \((K^2 - 1)/K^2 = O(x^{1/2})\). To have the same form of the transformed component \(g_{tt}\) as in (47), the second factor in brackets in (48) should be of order \(O(x^{1/2})\) or less, i.e.

\[-(T - T_{00})\sqrt{1 - \bar{b}^2} + \frac{K}{4} F^{ab} D_b T (-2b_a\sqrt{1 - \bar{b}^2} + D_a b^2) = O(x^{1/2}). \tag{49}\]

This imposes condition on \(T(z, \bar{z})\)

\[-(T - T_{00})\sqrt{1 - \bar{b}^2} + \frac{1}{4} F^{ab} D_b T (-2b_a\sqrt{1 - \bar{b}^2} + D_a b^2)]_{r=0} = 0. \tag{50}\]

Eq. (50) for \(T(z, \bar{z})\) is solved in Appendix A. From condition (49) it follows that the generator of supertranslations in the near-horizon region has the following structure

\[\chi_T = O(x^0)\partial_t + O(x)\partial_z + O(x^0)\partial_a. \tag{51}\]

It is seen that the generator is of the form found in Sect. 3, and its action preserves the near-horizon behavior of all the metric components.

Transformations (41) and (44) form a commutative algebra under the modified bracket [12]

\[[\xi_1, \xi_2]_{\text{mod}} = [\xi_1, \xi_2] - \delta_{T_1}\xi_2 + \delta_{T_2}\xi_1. \tag{52}\]

5 Surface charge of asymptotic horizon symmetries

In this section, first, we calculate the variation of the surface charge corresponding to the asymptotic horizon symmetries, and, next, discuss conditions on the form of the metric at the horizon which allow for integrating the variation of the charge over the space of metrics to a closed form. We consider a particular example of such metrics.

We use the short notations of Sect. 3:

\[
g_{mn} = \begin{pmatrix}
-\bar{g}_{tt} x & 0 & 0 & 0 \\
0 & \bar{g}_{rr}/x & \bar{g}_{rz}/\sqrt{x} & \bar{g}_{zz}/\sqrt{x} \\
0 & \bar{g}_{rz}/\sqrt{x} & \bar{g}_{zz} & \bar{g}_{zz} \\
0 & \bar{g}_{zz}/\sqrt{x} & \bar{g}_{zz} & \bar{g}_{zz}
\end{pmatrix}, \quad g^{mn} = \begin{pmatrix}
-\bar{g}^{tt}/x & 0 & 0 & 0 \\
0 & \bar{g}^{rr}/x & \bar{g}^{rz}/\sqrt{x} & \bar{g}^{zz}/\sqrt{x} \\
0 & \bar{g}^{rz}/\sqrt{x} & \bar{g}^{zz} & \bar{g}^{zz} \\
0 & \bar{g}^{zz}/\sqrt{x} & \bar{g}^{zz} & \bar{g}^{zz}
\end{pmatrix}, \tag{53}\]

where \(\bar{g}_{mn} = g_{mn,k}\) and \(g_{mn,k}\) are the leading-order in \(x\) parts of metric components. The elements of the corresponding matrix of metric variations calculated in Sect. 3 are denoted as \(\delta g_{mn} = h_{mn}, \delta \bar{g}_{mn} = \bar{h}_{mn}\). The inverse variations are defined as \(h^{mn} = g^{mk}h_{kl}g^{ln}\), and the trace of variations is \(h = h_{mn}g^{mn} = \bar{h}\).

Variation of the surface charge of the asymptotic symmetries is calculated following [16]

\[
\delta \chi \hat{Q}(g, \bar{h}) = \frac{1}{4\pi} \int (d^2x)_{ab} \sqrt{|g|} \chi^a \nabla^b h - \chi^a \nabla^a h_{\sigma}^b + \chi_\sigma \nabla^a h_{\sigma}^b + \frac{1}{2} h \nabla^a \chi^b + \frac{1}{2} h^a_\sigma (\nabla^b \chi_\sigma - \nabla_\sigma \chi^b) - (r \leftrightarrow t). \tag{54}\]

In present case \((d^2x)_{ab} = (1/4)\varepsilon_{abmn}dx^m dx^n\) where \(m, n = z, \bar{z}\) and \(a, b = r, t\). Integration is performed over the sphere of the radius \(r = 2 + x\), determinant of the metric is \(g^{(2)} = g_{zz}g_{\bar{z}z} - g_{\bar{z}z}^2\), where \(g^{(2)} = \bar{g}^{(2)}/(2 + x)^2\). The limit \(x = 0\) is taken at the end of the calculation.
Calculating contributions of the leading-order in \( x \to 0 \) parts of the five terms in the integrand of (54), we obtain

1. \[ \chi' \nabla^t h - \chi' \nabla^r h = \chi' g^{tt} \partial_t h - \chi' g^{rr} \partial_r h - \chi' g^{ra} \partial_a h = O(x^{1/2}). \]
2. \[ -\chi' \nabla^s h^{ts} + \chi' \nabla^s h^{rs} = -\chi' \nabla_t h^{tt} + \chi' (\nabla_r h^{rr} + \nabla_a h^{ra} + \nabla_t h^{tt}) = \left(-\frac{g^{rr} \bar{h}_{tt}}{\bar{g}_{tt}} + \bar{h}^{rr}\right) + O(x^{1/2}). \]
3. \[ \chi_s \nabla^r h^{ts} - \chi_s \nabla^t h^{rs} = \chi_t \nabla^r h^{tt} - \chi_r \nabla^t h^{rr} - \chi_a \nabla^t h^{ra} = \frac{\lambda_0}{2} \left(\frac{g^{rr} \bar{h}_{tt}}{\bar{g}_{tt}} - \bar{h}^{rr}\right) + O(x^{1/2}). \]
4. \[ \frac{h}{2} (\nabla^r \chi^t - \nabla^t \chi^r) = \frac{h}{2} \left[ (\bar{g}^{rr} \nabla_r + \bar{g}^{ra} \nabla_a) \chi^t - \bar{g}^{tt} \nabla_t \chi^t \right] = \frac{\lambda_0}{2} \bar{g}^{rr} h + O(x^{1/2}). \]
5. \[ \frac{1}{2} (h^{rs} \nabla^t \chi_s - h^{ts} \nabla^r \chi_s) = \frac{1}{2} (h^{rr} \nabla^t \chi_r + h^{ra} \nabla^t \chi_a - h^{tt} \nabla^t \chi_t) = \frac{\lambda_0}{4} g^{rr} \left[ \frac{\bar{h}_{tt}}{\bar{g}_{tt}} - (\bar{h}^{rr} \bar{g}_{rr} \bar{g}_{rr} + \bar{h}^{ra} \bar{g}_{ar} \bar{g}_{rr} + \bar{h}^{rr} \bar{g}_{ra} \bar{g}_{ar} + \bar{h}^{ra} \bar{g}_{rb} \bar{g}_{ar}) \right] + O(x^{1/2}) = -\frac{\lambda_0}{4} \left( \frac{g^{rr} \bar{h}_{tt}}{\bar{g}_{tt}} + \bar{h}^{rr} \right) + O(x^{1/2}). \]
6. \[ -\frac{1}{2} (h^{rs} \nabla^t \chi^t - h^{ts} \nabla^r \chi^r) = -\frac{1}{2} (h^{rr} \nabla_r \chi^t + h^{ra} \nabla_a \chi^t - h^{tt} \nabla_t \chi^t) = -\frac{\lambda_0}{4} \left( \frac{g^{rr} \bar{h}_{tt}}{\bar{g}_{tt}} + \bar{h}^{rr} \right) + O(x^{1/2}). \]

In item 2 we have used that

\[ \nabla_r h^{rr} = \partial_r h^{rr} + 2 \Gamma^r_{rs} h^{rr} + 2 \Gamma^r_{ra} h^{ra} = \bar{h}^{rr} - \bar{h}^{rr} (\bar{g}^{rr} \bar{g}_{rr} + \bar{g}^{ra} \bar{g}_{ar}) + O(x^{1/2}) = O(x^{1/2}), \]

and in item 5 we used the identities

\[ \bar{g}^{rr} \bar{g}_{rr} + \bar{g}^{ra} \bar{g}_{ar} = 1, \]
\[ \bar{g}^{rr} \bar{g}_{ra} + \bar{g}^{rb} \bar{g}_{ba} = 0. \]

Substituting the expressions (55) in (54), we obtain

\[ \rho_\chi Q(g, h) = \frac{1}{16 \pi} \int dz d\bar{z} \sqrt{g(2)} \frac{\chi^t}{2} \left( \bar{g}^{rr} h - \bar{h}^{rr} - \frac{\bar{g}^{rr} \bar{h}_{tt}}{\bar{g}_{tt}} \right). \]

The combination in the integrand can be presented as

\[ \bar{g}^{rr} h - \bar{h}^{rr} - \frac{\bar{g}^{rr} \bar{h}_{tt}}{\bar{g}_{tt}} = (\bar{g}^{rr} \bar{g}^{ab} - \bar{g}^{ra} \bar{g}^{rb}) \bar{h}_{ab}. \]

Denoting determinant of the 3D part of the metric as \( \bar{g}^{(3)} \), and using the identity

\[ \bar{g}^{rr} \bar{g}^{ab} - \bar{g}^{ra} \bar{g}^{rb} = \bar{g}_{ab} \bar{g}^{(3)}, \]

we can write (57) as

\[ (\bar{g}^{rr} \bar{g}^{ab} - \bar{g}^{ra} \bar{g}^{rb}) \bar{h}_{ab} = (\bar{g}_{zz} \delta \bar{g}_{zz} + \bar{g}_{zz} \delta \bar{g}_{zz} - 2 \bar{g}_{zz} \delta \bar{g}_{zz}) / \bar{g}^{(3)} = \delta \bar{g}^{(2)} / \bar{g}^{(3)} \]
A sufficient condition of integrability of (56) is $\bar{g}^{(3)} = f(\bar{g}^{(2)})$, where $f$ is an integrable function. This case is realized, if supertranslation field depends on $|z^2|$, or, in spherical coordinates $\theta, \varphi$, only on $\theta$, i.e. $C(z, \bar{z}) = C(|z^2|) = \bar{C}(\theta)$. In this case, because of the identity $\bar{g}_{rr} \bar{g}_{\theta\theta} - \bar{g}_{r\theta}^2 = \bar{g}_{\theta\theta}$ (see Appendix B), we have $\bar{g}^{(3)} = (\bar{g}_{rr} \bar{g}_{\theta\theta} - \bar{g}_{r\theta}^2) \bar{g}_{\varphi\varphi} = \bar{g}^{(2)}$. The charge is

$$\hat{Q} = \frac{1}{16\pi} \int \chi_0^f (\bar{g}_{\theta\theta} \bar{g}_{\varphi\varphi})^{1/2} d\theta d\varphi \sin \theta + \hat{Q}_0.$$ \hfill (59)

With $\chi_0^f$ independent of $\theta$ the charge is proportional to the surface of the horizon, i.e. to the entropy of the black hole.

### 6 Conclusions

In this paper we studied the near-horizon symmetries of the metric of a black hole containing supertranslation field. To study general transformations preserving the near-horizon form of the metric, we transformed the metric to a coordinate system in which the horizon of the metric is located at the surface $r = 2M$. Solving the geodesic equations for null geodesics, it was shown that the surface $r = 2M$ is the surface of infinite redshift. We reviewed the action of the generators of supertranslations preserving the gauge of the metric and found a class of supertranslations which also preserve the near-horizon form of the metric.

Next, we determined the form of generators of asymptotic horizon symmetries preserving the near-horizon form of the metric with supertranslation field. Variations of the metric components under the action of generators of near-horizon symmetries were obtained. Using the variations of the metric components, we obtained variation of the surface charge corresponding to the asymptotic horizon symmetries.

Studying the form of the variation of the surface charge, we found a sufficient condition of integrability of the variation of the surface charge over the space of metrics to a closed expression. In the case of metrics with the supertranslation field depending on $|z^2|$, or, in spherical variables $\theta, \varphi$, with dependence only on $\theta$, it was shown that due to specific relations between the metric components, it is possible to integrate the variation of the surface charge and obtain the charge of horizon symmetries in a closed form. The charge is proportional to the horizon surface and can be interpreted as entropy of the black hole.

Horizon symmetries of a class of metrics with the near-horizon form

$$ds^2 = -2\kappa pdv^2 + 2dv dp + 2\rho h_a(x) dv dx^a + (\omega_{ab}(x) + \rho \lambda_{ab}(x)) dx^a dx^b, \quad x^a = z, \bar{z}$$ \hfill (60)

were previously studied in many papers, as examples, in [17, 18, 19, 20, 21, 22, 23]. Horizon of the metric is located at $\rho = 0$, and $\rho$ is a distance of the surface $\rho = \text{const}$ from the horizon. The metric is written in the gauge $g_{\rho\rho} = g_{\rho a} = 0$.

In the near-horizon region the metric components are expanded in power series in $\rho$ [24]. In contrast to this case, the metrics considered in the present paper are expanded in series of $(r - 2M)^{1/2}$. In the limit of vanishing supertranslation field the terms with fractional powers of $r - 2M$ vanish.

The charge of the asymptotic horizon symmetries for metrics (60), in the case $\kappa = \text{const}$ was obtained in [20, 21] in a form

$$Q = \frac{1}{16\pi G} \int dz d\bar{z} \sqrt{\gamma} (2T \kappa \Omega - y^a \theta_a \Omega).$$
Here \( T(z, \bar{z}) \) is a part of the asymptotic Killing vector \( \chi^v(z, \bar{z}, v) \), and the metric component \( \omega_{ab} \) has diagonal form \( \omega_{ab} = \gamma_{ab} \Omega \). The volume \( \sqrt{\gamma} \Omega \, dz \, d\bar{z} \) is an analog of the volume \( \sqrt{\bar{g}_{\theta\theta}} \sin \theta d\theta d\phi \) in (59), and \( T \) is an analog of \( \chi^t \). It is seen that the corresponding structures of the charges in both cases are similar.

The near-horizon transformations for the pure Schwarzschild metric were considered in [18] and refs. therein. In the limit of vanishing supertranslation field the results of these papers and the present paper are identical.

7 Appendix A

In this Appendix we find general solution of Eq. (50)

\[
[ -T\sqrt{1-b^2} + \frac{1}{4} F_{ab} D_b T(-2b_a \sqrt{1-b^2} + D_a b^2) ]_{r=2} = 0 \tag{A1}
\]

which can be presented in a form

\[
T + F^a D_a T = 0, \tag{A2}
\]

where

\[
F^a = \frac{1}{2} F^{ac} (b_c + \partial_c \sqrt{1-b^2})|_{r=2}
\]

Following the general rules of solving the differential equations with partial derivatives [25], we consider a function \( W(T, z, \bar{z}) \) satisfying the equation

\[
T \frac{\partial W}{\partial T} + F_z \frac{\partial W}{\partial z} + F_{\bar{z}} \frac{\partial W}{\partial \bar{z}} = 0. \tag{A3}
\]

Eq. (A3) is solved by writing the system of ordinary differential equations

\[
\frac{dT}{T} = \frac{dz}{F_z} = \frac{d\bar{z}}{F_{\bar{z}}}. \tag{A4}
\]

Let the independent first integrals of the Eq. (A4) be

\[
\psi_1(T, z, \bar{z}) = C_1, \quad \psi_2(T, z, \bar{z}) = C_2. \tag{A5}
\]

The general solution of the Eq. (A3) for \( W(T, z, \bar{z}) \) is

\[
W = f(\psi_1, \psi_2), \tag{A6}
\]

where \( f \) is an arbitrary smooth function. The function \( T(z, \bar{z}) \) is determined from the equation

\[
f(\psi_1, \psi_2) = 0. \tag{A7}
\]

8 Appendix B

If supertranslation field depends \( C(z, \bar{z}) \) depends only on \( |z|^2 \), it is convenient to parametrise the unit sphere by \( \theta, \varphi \) coordinates. In these coordinates the metric (15) takes a form

\[
ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -V \, dt^2 + \frac{dr^2 g_{rr}}{V} + \frac{2dr d\theta g_{r\theta}}{V^{1/2}} + d\theta^2 g_{\theta\theta} + d\varphi^2 \sin^2 \theta g_{\varphi\varphi}
= -V \, dt^2 + \frac{dr^2}{V(1-b^2)} + 2dr d\theta \frac{br(\sqrt{1-b^2} - b')}{(1-b^2)V^{1/2}} + \\
+ d\theta^2 \frac{(\sqrt{1-b^2} - b')^2}{(1-b^2)} + d\varphi^2 r^2 \sin^2 \theta (b \cot \theta - \sqrt{1-b^2})^2, \tag{B1}
\]
where $b = 2\partial_\theta C(\theta)/K$. It is explicitly verified that the metric components are connected by a relation

$$g_{rr} g_{\theta\theta} - g_{r\theta}^2 = \frac{g_{\theta\theta}}{V}.$$  

(B2)

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