An efficient explicit full discrete scheme for strong approximation of stochastic Allen-Cahn equation

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Abstract

A novel explicit full discrete scheme is proposed to numerically solve the stochastic Allen-Cahn equation with cubic nonlinearity, perturbed by additive space-time white noise. The approximation is easily implementable, performing spatial discretization by a spectral Galerkin method and temporal discretization by a kind of nonlinearity-tamed accelerated exponential integrator scheme. Error bounds in a strong sense are analyzed for both the spatial semi-discretization and the spatio-temporal full discretization, with convergence rates in both space and time explicitly identified. It turns out that the obtained convergence rate of the new scheme is, in the temporal direction, twice as high as existing ones in the literature. Numerical results are finally reported to confirm previous theoretical findings.

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Key Words: stochastic Allen-Cahn equation, cubic nonlinearity, strong approximation, spectral Galerkin method, tamed exponential integrator scheme, convergence rates

1 Introduction

As an active area of research, numerical study of evolutionary stochastic partial differential equations (SPDEs) has attracted increasing attention in the past decades (see, e.g., monographs [27, 31, 34] and references therein). Albeit much progress has been made, it is still far from well-understood, especially for numerical analysis of SPDEs with non-globally Lipschitz nonlinearity. The present work attempts to make a contribution in this direction and examine numerical approximations of a typical example of parabolic SPDEs with super-linearly growing nonlinearity, i.e., the stochastic Allen-Cahn equation. The driving noise is a space-time white one, which has a special interest as it can best model the fluctuations generated by microscopic effects in a homogeneous physical system, including, for example, molecular collisions in gases and liquids, electric

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fluctuations in resistors [15]. A lot of researchers carried out numerical analysis of SPDEs subject to the space-time white noise, e.g., [1,3,4,6,7,11,13,16,18,23,24,26,32,33,38,45], to just mention a few.

Numerically solving the continuous problem on a computer forces us to perform both spatial and temporal discretizations. In space, we discretize the underlying SPDE by a spectral Galerkin method, resulting in a system of finite dimensional stochastic differential equations (SDEs). With the spatial discretization, we temporally propose a nonlinearity-tamed accelerated exponential time-stepping scheme (4.1). The approximation errors of both the spatial discretization and the space-time full discrete scheme are carefully analyzed, with strong convergence rates successfully recovered. More accurately, by $X(t_m)$ we denote the unique mild solution of SPDE (2.1) taking values at temporal grid points $t_m = m\tau, m \in \{0, 1, \ldots, M\}$ with uniform time step-size $\tau = \frac{T}{M} > 0$ and by $Y_{t_m}^{M,N}$ the numerical approximations of $X(t_m)$, produced by the proposed fully discrete scheme. The approximation error measured in $L^p(\Omega; H), p \in [2, \infty)$ reads (cf. Theorem 4.11):

$$
\sup_{0 \leq m \leq M} \|X(t_m) - Y_{t_m}^{M,N}\|_{L^p(\Omega; H)} \leq C\left(N^{-\beta} + \tau^\beta\right), \quad \forall \beta \in (0, \frac{1}{2}).
$$

(1.1)

Here $H := L^2((0,1); \mathbb{R})$ and the constant $C$ depends on $p, \beta, T$ and the initial value of SPDE, but does not depend on the discretization parameters $M, N$.

Over the last two years, several research works were reported on numerical analysis of space-time white noise driven SPDEs with cubic (polynomial) nonlinearity [2, 3, 5, 6, 33]. Becker and Jentzen [3] in 2016 introduced two nonlinearity-truncated Euler-type approximations for pure time discretizations of stochastic Ginzburg Landau type equations with slightly more general polynomials. There a strong convergence rate of order almost $\frac{1}{4}$ is identified. More recently when the present manuscript was almost finalized, we were aware of four other preprints [2, 5, 6, 33] submitted online, concerning with numerical approximations of similar SPDEs. Becker, Gess, Jentzen and Kloeden [2] proposed new types of truncated exponential Euler space-time full discrete schemes for the same problem as in [3] and derived strong convergence rates of order almost $\frac{1}{2}$ in space and order almost $\frac{1}{4}$ in time. Later, Bréhier and Goudenège [1] and Bréhier, Cui and Hong [5] analyzed some splitting time discretization schemes and obtained convergence rates of order $\frac{1}{4}$. Liu and Qiao [33] investigated spectral Galerkin backward implicit Euler full discretization, with strong convergence rates of order almost $\frac{1}{2}$ in space and order $\frac{1}{4}$ in time achieved. As clearly implied in (1.1), the spatial convergence rate coincides with those in [2,33], but the convergence rate of our time-stepping scheme can be of order almost $\frac{1}{2}$, twice as high as those in [2,3,5,6,33]. Despite getting involved with linear functionals of the noise process, the newly proposed scheme (1.1) is explicit, easily implementable and does not cost additional computational efforts (see comments in section 5 for the implementation of the linear functionals of the noise process).

It is important to emphasize that, proving the error estimate (1.1) rigorously is rather challenging, confronted with two key difficulties, one being to derive uniform a priori moment bounds for the numerical approximations with super-linearly growing nonlinearity and the other to recover the temporal convergence rate of order almost $\frac{1}{2}$, instead of order (almost) $\frac{1}{4}$ in existing literature. With regard to the former, we first derive certain estimates for deterministic perturbed PDEs (4.7), as elaborated in subsection 4.1. Then the moment bounds are a consequence of a certain bootstrap argument, by showing $\mathbb{E}\left[\mathbb{I}_{\Omega_{R^*,t_m}} \|Y_{t_m}^{M,N}\|_V^p\right] < \infty$ and $\mathbb{E}\left[\mathbb{I}_{\Omega_{R^*,t_m}} \|Y_{t_m}^{M,N}\|_V^2\right] < \infty$,
\[ V := C((0,1), \mathbb{R}), \] for subevents \( \Omega_{R^\tau,t} \) with \( R^\tau \) depending on \( \tau \) carefully chosen (see subsection 4.2). The latter difficulty lies on the estimate of the crucial term \( J_1 \) (cf. (4.61) in subsection 4.4),

\[ J_1 := p \int_0^t \left\| P_N X(s) - Y_{s,M,N}^s \right\|^{p-2} \langle P_N X(s) - Y_{s,M,N}^s, F(Y_{s,M,N}^s) - F(Y_{\lfloor s \rfloor,M,N}^s) \rangle \, ds. \] (1.2)

As usual, such a term is simply treated with the aid of temporal Hölder regularity of \( Y_{s,M,N}^s \) together with the Cauchy-Schwarz inequality and Hölder’s inequality, but to only attain order \( \tau^{\beta/2} \). In our analysis, we decompose \( P_N X(s) - Y_{s,M,N}^s \) in the inner product into three parts, as shown in (4.64). Smoothing property of the analytic semigroup is then fully exploited to handle these three terms, in conjunction with commutativity properties of the nonlinearity and higher temporal Hölder regularity in negative Sobolev space (consult subsection 4.3 and the treatment of \( J_1 \) in the proof of Theorem 4.11 for details). This way we arrive at the desired high convergence rate in time.

Furthermore, we would like to point out that the improvement of convergence rate is essentially credited to fully preserving the stochastic convolution in the time-stepping scheme (1.1). Such a kind of accelerating technique is originally due to Jentzen and Kloeden [26], simulating nearly linear parabolic SPDEs and has been further examined and extended in different settings [25,35,39,45,46], where a globally Lipschitz condition imposed on the nonlinearity is indispensable. When the nonlinearity grows super-linearly and the globally Lipschitz condition is thus violated, one can in general not expect the usual accelerated exponential time-stepping schemes converge in the strong sense, based on the observation that the standard Euler method strongly diverges for ordinary (finite dimensional) SDEs [21]. To address this issue, we introduce a taming technique originally used in [20,22,43,44] for ordinary SDEs, and propose a nonlinearity-tamed version of accelerated exponential Euler scheme for the time discretization. Analyzing the strong convergence rate is, however, much more difficult than that in the finite dimensional SDE setting (see section 4).

Finally, we mention that, just one spatial dimension is considered because the space-time white noise driven SPDE only allows for a mild solution with a positive (but very low) order of regularity in one spatial dimension. It is because of the low order of regularity that the error analysis becomes difficult. Strong and weak convergence analysis of smoother noise (e.g., trace-class noise) driven SPDEs in multiple spatial dimensions, with non-globally Lipschitz nonlinearity, will be our forthcoming works [40] (see also, e.g., [5,17,19,28,30,36,41] for relevant topics).

The rest of this paper is organized as follows. In the next section we collect some basic facts and present the well-posedness of the stochastic problem under given assumptions. Section 3 and Section 4 are, respectively, devoted to the analysis of strong convergence rates for both the spatial semi-discretization and spatio-temporal full discretization of the underlying SPDEs. Numerical results are included in section 5 to test previous theoretical findings.

2 Well-posedness of the stochastic problem

Throughout this article, we are interested in the additive space-time white noise driven stochastic Allen-Cahn equation with cubic nonlinearity, described by

\[ \begin{align*}
\frac{\partial u}{\partial t}(t,x) &= \frac{\partial^2 u}{\partial x^2}(t,x) + f(u(t,x)) + \dot{W}(t,x), \quad x \in D, \quad t \in (0,T], \\
u(0,x) &= u_0(x), \quad x \in D, \\
u(t,0) &= u(t,1) = 0, \quad t \in (0,T].
\end{align*} \] (2.1)
Here \( D := (0, 1) \), \( T > 0 \), \( f: \mathbb{R} \to \mathbb{R} \) is given by \( f(v) = a_3 v^3 + a_2 v^2 + a_1 v + a_0 \), \( a_3 < 0 \), \( a_2, a_1, a_0, v \in \mathbb{R} \), and \( \dot{W}(t, \cdot) \) stands for a formal time derivative of a cylindrical I-Wiener process \([10]\). In order to define a mild solution of \((2.1)\) following the semigroup approach in \([10]\), we attempt to put everything into an abstract framework. Given a real separable Hilbert space \((H, \langle \cdot, \cdot \rangle, \| \cdot \|)\) with \( \| \cdot \| = \langle \cdot, \cdot \rangle^\frac{1}{2} \), by \( \mathcal{L}(H) \) we denote the space of bounded linear operators from \( H \) to \( H \) endowed with the usual operator norm \( \| \cdot \|_{\mathcal{L}(H)} \). Additionally, we denote by \( \mathcal{L}_2(H) \subset \mathcal{L}(H) \) the subspace consisting of all Hilbert-Schmidt operators from \( H \) to \( H \) \([10]\). It is known that \( \mathcal{L}_2(H) \) is a separable Hilbert space, equipped with the scalar product \( \langle \cdot, \cdot \rangle_{\mathcal{L}_2(H)} := \sum_{n \in \mathbb{N}} \langle \Gamma_1 \eta_n, \Gamma_2 \eta_n \rangle \), and norm \( \| \Gamma \|_{\mathcal{L}_2(H)} := (\sum_{n \in \mathbb{N}} \| \Gamma \eta_n \|^2)^\frac{1}{2} \), independent of the particular choice of ON-basis \( \{ \eta_n \}_{n \in \mathbb{N}} \) of \( H \). Below we sometimes write \( \mathcal{L}_2 := \mathcal{L}_2(H) \) for brevity. If \( \Gamma \in \mathcal{L}(H) \) and \( \Gamma_1, \Gamma_2 \in \mathcal{L}_2(H) \), then \( |\langle \Gamma_1, \Gamma_2 \rangle_{\mathcal{L}_2(H)}| \leq \| \Gamma_1 \|_{\mathcal{L}_2(H)} \| \Gamma_2 \|_{\mathcal{L}_2(H)} \), \( \| \Gamma_1 \|_{\mathcal{L}_2(H)} \leq \| \Gamma \|_{\mathcal{L}(H)} \| \Gamma_1 \|_{\mathcal{L}_2(H)} \). By \( \mathcal{L}^\gamma(D; \mathbb{R}), \gamma \geq 1 \) \((\mathcal{L}^\gamma(D) \text{ for short})\) we denote a Banach space consisting of \( \gamma \)-times integrable functions and by \( V := \mathcal{C}(D; \mathbb{R}) \) a Banach space of continuous functions with usual norms. We make the following assumptions.

**Assumption 2.1 (Linear operator \( A \))** Let \( D := (0, 1) \) and let \( H = \mathcal{L}^2(D; \mathbb{R}) \) be a real separable Hilbert space, equipped with usual product \( \langle \cdot, \cdot \rangle \) and norm \( \| \cdot \| = \langle \cdot, \cdot \rangle^\frac{1}{2} \). Let \( -A: \text{dom}(A) \subset H \to H \) be the Laplacian with homogeneous Dirichlet boundary conditions, defined by \( -Au = \Delta u, u \in \text{dom}(A) := H^2 \cap H^0_1 \).

The above setting assures that there exists an increasing sequence of real numbers \( \lambda_i = \pi^2 i^2, i \in \mathbb{N} \) and an orthonormal basis \( \{ e_i(x) = \sqrt{2} \sin(i \pi x), x \in (0, 1) \}_{i \in \mathbb{N}} \) such that \( Ae_i = \lambda_i e_i \). In particular, the linear unbounded operator \( A \) is positive, i.e., \( \langle -Av, v \rangle \leq -\lambda_1 \| v \|^2 \), for all \( v \in \text{dom}(A) \). Moreover, \( -A \) generates an analytic semigroup \( E(t) = e^{-tA}, t \geq 0 \) on \( H \) and we can define the fractional powers of \( A \), i.e., \( A^\gamma, \gamma \in \mathbb{R} \) and the Hilbert space \( \dot{H}^\gamma := \text{dom}(A^\gamma) \), equipped with inner product \( \langle \cdot, \cdot \rangle_{\gamma} := \langle A^\frac{\gamma}{2}, A^\frac{\gamma}{2} \rangle \) and norm \( \| \cdot \|_{\gamma} = \langle \cdot, \cdot \rangle_{\gamma}^{\frac{1}{2}} \) \([31]\) Appendix B.2]. Moreover, \( \dot{H}^0 = H \) and \( \dot{H}^\gamma \subset \dot{H}^0, \gamma \geq \delta \). It is well-known that \([37]\)

\[
\| A^\gamma E(t) \|_{\mathcal{L}(H)} \leq Ct^{-\gamma}, \quad t > 0, \gamma \geq 0,
\]

\[
\| A^{-\rho}(I - E(t)) \|_{\mathcal{L}(H)} \leq Ct^\rho, \quad t > 0, \rho \in [0, 1],
\]

and that

\[
\| A^{\frac{\beta - 1}{2}} \|_{\mathcal{L}_2(H)} < \infty, \quad \text{for any } \beta < \frac{1}{2}.
\]

Throughout this paper, by \( C \) and \( \gamma \), we mean various constants, not necessarily the same at each occurrence, that are independent of the discretization parameters.

**Assumption 2.2 (Nonlinearity)** Let \( F: \mathcal{L}^6(D; \mathbb{R}) \to H \) be a deterministic mapping defined by \( F(v)(x) = f(v(x)) := a_3 v^3(x) + a_2 v^2(x) + a_1 v(x) + a_0, x \in (0, 1), a_3 < 0, a_2, a_1, a_0 \in \mathbb{R}, v \in \mathcal{L}^6(D; \mathbb{R}) \).

It is easy to find a constant \( L \in (0, \infty) \) such that

\[
\langle u - v, F(u) - F(v) \rangle \leq L \| u - v \|^2, \quad u, v \in V,
\]

\[
\| F(u) - F(v) \| \leq L(1 + \| u \|^2 + \| v \|^2) \| u - v \|, \quad u, v \in V.
\]
The second property in (2.4) immediately implies
\[ \|F(u)\| \leq C(1 + \|u\|_V^2)\|u\|, \quad u \in V. \] (2.5)

**Assumption 2.3 (Noise process)** Let \( \{W(t)\}_{t \in [0,T]} \) be a cylindrical \( \mathcal{I}\)-Wiener process on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with a normal filtration \( \{\mathcal{F}_t\}_{t \in [0,T]}\), represented by a formal series,
\[ W(t) := \sum_{n=1}^{\infty} \beta_n(t) e_n, \quad t \in [0, T], \] (2.6)
where \( \{\beta_n(t)\}_{n \in \mathbb{N}}, t \in [0, T] \) is a sequence of independent real-valued standard Brownian motions and \( \{e_n = \sqrt{2} \sin(n \pi x), x \in (0, 1)\}_{n \in \mathbb{N}} \) is a complete orthonormal basis of \( H \).

**Assumption 2.4 (Initial value)** Let the initial data \( X_0: \Omega \rightarrow H \), given by \( X_0(\cdot) = u_0(\cdot) \), be an \( \mathcal{F}_0/\mathcal{B}(H) \)-measurable random variable satisfying, for sufficiently large positive number \( p_0 \in \mathbb{N} \),
\[ \mathbb{E}[\|X_0\|_\beta^{p_0}] + \mathbb{E}[\|X_0\|_W^{p_0}] \leq K_0 < \infty, \quad \text{for any } \beta < \frac{1}{2}. \] (2.7)

At the moment, we are prepared to formulate the concrete problem (2.1) as an abstract stochastic evolution equation in the Hilbert space \( H \),
\[
\begin{cases}
  dX(t) + AX(t) \, dt = F(X(t)) \, dt + dW(t), & t \in (0, T], \\
  X(0) = X_0,
\end{cases}
\] (2.8)
where \( X(t, \cdot) = u(t, \cdot) \) and the abstract items \( A, F, X_0 \) are defined as in Assumptions 2.1-2.4. The above assumptions suffice to establish well-posedness and regularity results of SPDE (2.8). Before that, we recall some estimates that can, e.g., be found in [9, Proposition 4.3] and [8, Lemma 6.1.2].

**Lemma 2.5** For any \( p \in [2, \infty) \) and \( \beta \in [0, \frac{1}{2}) \), the stochastic convolution \( \{O_t\}_{t \in [0,T]} \) satisfies
\[ \mathbb{E} \left[ \sup_{t \in [0,T]} \|O_t\|_V^p \right] < \infty, \quad \text{with } O_t := \int_0^t E(t-s) \, dW(s), \] (2.9)
\[ \|O_t - O_s\|_{L^p(\Omega, H)} \leq C(t-s)^{\frac{\beta}{2}}, \quad 0 \leq s < t \leq T. \] (2.10)

**Theorem 2.6** Under Assumptions 2.1-2.4, SPDE (2.8) possesses a unique mild solution \( X: [0, T] \times \Omega \rightarrow V \) with continuous sample path, determined by,
\[ X(t) = E(t)X_0 + \int_0^t E(t-s)F(X(s)) \, ds + O_t \quad \mathbb{P}\text{-a.s.} \] (2.11)
For \( p \in [2, \infty) \) there exists a constant \( C_1 \in [0, \infty) \) depending on \( p, \beta, T \) such that, for any \( \beta < \frac{1}{2} 
\[ \sup_{t \in [0,T]} \|X(t)\|_{L^p(\Omega, V)} \leq C_1 \left( 1 + \|X_0\|_{L^p(\Omega, V)} \right), \] (2.12)
\[ \sup_{t \in [0,T]} \|X(t)\|_{L^p(\Omega, H^\beta)} \leq C_1 \left( 1 + \|X_0\|_{L^p(\Omega, H^\beta)} + \|X_0\|_{L^3(\Omega, V)}^3 \right). \] (2.13)
Moreover, there exists a constant \( C_2 \in [0, \infty) \) depending on \( p, \beta, C_1, T \) and \( X_0 \) such that, for any \( \beta < \frac{1}{2} 
\[ \|X_t - X_s\|_{L^p(\Omega, H)} \leq C_2(t-s)^{\frac{\beta}{2}}, \quad 0 \leq s < t \leq T. \] (2.14)
The uniqueness of the mild solution and the regularity assertion (2.12) are based on [8, Proposition 6.2.2] and (2.9). The rest of estimates in Theorem 2.6 can be verified by standard arguments.
3 Spatial semi-discretization

This section concerns the error analysis for a spectral Galerkin spatial semi-discretization of the underlying problem (2.8). For \( N \in \mathbb{N} \) we define a finite dimensional subspace of \( H \) by

\[
H^N := \text{span}\{e_1, e_2, \ldots, e_N\},
\]

and the projection operator \( P_N : \dot{H}^\alpha \to H^N \) by \( P_N \xi = \sum_{i=1}^N \langle \xi, e_i \rangle e_i \), \( \forall \xi \in \dot{H}^\alpha \), \( \alpha \in \mathbb{R} \). Here \( H^N \) is chosen as the linear space spanned by the \( N \) first eigenvectors of the dominant linear operator \( A \).

It is not difficult to deduce that

\[
\| (P_N - I) \varphi \| \leq \lambda_{N+1}^{-\frac{\alpha}{2}} \| \varphi \|_\alpha \leq N^{-\alpha} \| \varphi \|_\alpha, \quad \forall \varphi \in \dot{H}^\alpha, \alpha \geq 0.
\]

Additionally, define \( A_N : H \to H^N \) as \( A_N = AP_N \), which generates an analytic semigroup \( E_N(t) = e^{-tA_N}, t \in [0, \infty) \) in \( H^N \). Then the spectral Galerkin approximation of (2.8) results in the following finite dimensional SDEs,

\[
\begin{aligned}
\left\{ \begin{array}{l}
dX^N(t) + A_N X^N(t) \, dt = F_N(X^N(t)) \, dt + P_N \, dW(t), \quad t \in (0, T], \\
X^N(0) = P_N X_0,
\end{array} \right.
\end{aligned}
\]

where we write \( F_N := P_N F \) for short. It is clear to see that (3.3) admits a unique solution in \( H^N \).

By the variation of constant, the corresponding solution can be written as

\[
X^N(t) = E_N(t) P_N X_0 + \int_0^t E_N(t-s) P_N F(X^N(s)) \, ds + \int_0^t E_N(t-s) P_N \, dW(s), \ \mathbb{P}\text{-a.s.}
\]

Before analyzing the spatial discretization error, we need some auxiliary lemmas. The following one is a direct consequence of [4, Lemma 5.4] with \( t_1 = 0 \).

**Lemma 3.1** For any \( p \in [2, \infty) \), the stochastic convolution \( \{O_t\}_{t \in [0, T]} \) satisfies

\[
\sup_{t \in [0, T], N \in \mathbb{N}} \| P_N O_t \|_{L^p(\Omega, V)} < \infty.
\]

Moreover, we can validate the following two lemmas.

**Lemma 3.2** Let \( E(t) = e^{-tA}, t \geq 0 \) be the analytic semigroup defined in section 2. For any \( N \in \mathbb{N} \) and \( \psi \in \dot{H}^\gamma, \gamma \in [0, \frac{1}{2}) \), it holds that

\[
\| P_N E(t) \psi \|_V \leq 2^\gamma \left( \frac{5-2\gamma}{2\pi(1-2\gamma)} \right)^{\frac{1}{2}} t^{\frac{2\gamma-1}{4}} \| \psi \|_\gamma, \quad t > 0, \ \gamma \in [0, \frac{1}{2}).
\]

**Proof of Lemma 3.2** Elementary facts readily yield

\[
\| P_N E(t) \psi \|_V = \sup_{x \in [0,1]} \left| \sum_{i=1}^N e^{-\lambda_i t} \langle \psi, e_i \rangle e_i \right| \leq \sqrt{2} \sum_{i=1}^N e^{-\lambda_i t} \langle \psi, e_i \rangle
\]

\[
\leq \sqrt{2} \left( \sum_{i=1}^N \lambda_i^{-\gamma} e^{-2\lambda_i t} \right)^{1/2} \left( \sum_{i=1}^N \lambda_i^\gamma \| \langle \psi, e_i \rangle \|_2^2 \right)^{1/2}
\]

\[
\leq \sqrt{2} \pi^{-\gamma} \left( \int_0^\infty x^{-2\gamma} e^{-2\pi^2 x^2 t} \, dx \right)^{1/2} \| \psi \|_\gamma
\]

\[
= 2^\gamma \pi^{-\frac{1}{4}} t^{\frac{2\gamma-1}{4}} \left( \int_0^\infty y^{-2\gamma} e^{-y^2/2} \, dy \right)^{1/2} \| \psi \|_\gamma
\]

\[
\leq 2^\gamma \left( \frac{5-2\gamma}{2\pi(1-2\gamma)} \right)^{\frac{1}{2}} t^{\frac{2\gamma-1}{4}} \| \psi \|_\gamma.
\]
as required. □

Lemma 3.3 Let \( \{X(t)\}_{t \in [0,T]} \) be the mild solution to (2.8), defined by (2.11). Then it holds for any \( p \in [2, \infty) \) that

\[
\sup_{N \in \mathbb{N}} \|P_N X(t)\|_{L^p(\Omega,V)} \leq C \gamma \left( 1 + t^{\frac{2\gamma-1}{2}} \right), \quad \gamma \in [0, \frac{1}{2}), \quad t \in (0,T].
\] (3.8)

Proof of Lemma 3.3 Observing that \( E_N(t)P_N = E(t)P_N \) and using Lemmas 3.1, 3.2 show

\[
\|P_N X(t)\|_{L^p(\Omega,V)} \leq \|E(t)P_N X_0\|_{L^p(\Omega,V)} + \int_0^t \|E(t-s)F_N(X(s))\|_{L^p(\Omega,V)} \, ds + \|P_N \mathcal{O}_t\|_{L^p(\Omega,V)}
\]

\[
\leq C_\gamma t^{\frac{2\gamma-1}{4}} \|X_0\|_{L^p(\Omega,H^\gamma)} + C_\gamma \int_0^t (t-s)^{-\frac{1}{4}}\|F(X(s))\|_{L^p(\Omega,H)} \, ds + \|P_N \mathcal{O}_t\|_{L^p(\Omega,V)}
\]

\[
\leq C_\gamma t^{\frac{2\gamma-1}{4}} \|X_0\|_{L^p(\Omega,H^\gamma)} + C_\gamma \sup_{s \in [0,T]} \|F(X(s))\|_{L^p(\Omega,H)} + \|P_N \mathcal{O}_t\|_{L^p(\Omega,V)}
\]

\[
\leq C_\gamma t^{\frac{2\gamma-1}{4}} \|X_0\|_{L^p(\Omega,H^\gamma)} + C_\gamma \left( 1 + \sup_{s \in [0,T]} \|X(s)\|_{L^p(\Omega,V)} \right) + \|P_N \mathcal{O}_t\|_{L^p(\Omega,V)}.
\] (3.9)

Owing to (2.12), (3.3) and Assumption 2.3 one can arrive at the expected estimate. □

Now we are prepared to do convergence analysis for the spectral Galerkin discretization (3.3).

Theorem 3.4 (Spatial error estimate) Let Assumptions 2.1-2.4 hold. Let \( X(t) \) and \( X^N(t) \) be defined through (2.8) and (3.1), respectively. Then it holds, for any \( \beta < \frac{1}{2}, \ p \in [2, \infty) \) and \( N \in \mathbb{N} \),

\[
\sup_{t \in [0,T]} \|X(t) - X^N(t)\|_{L^p(\Omega,H)} \leq C N^{-\beta}.
\] (3.10)

The above convergence rate \( \beta < \frac{1}{2} \) can be arbitrarily close to \( \frac{1}{2} \) but can not reach \( \frac{1}{2} \), since the constant \( C \) explodes when \( \beta \) tends to \( \frac{1}{2} \). This comment also applies to the full approximation error estimates in section 4.

Proof of Theorem 3.4 The triangle inequality along with (3.2) provides us that

\[
\|X(t) - X^N(t)\|_{L^p(\Omega,H)} \leq \|(I - P_N)X(t)\|_{L^p(\Omega,H)} + \|P_N X(t) - X^N(t)\|_{L^p(\Omega,H)}
\]

\[
\leq N^{-\beta} \|X(t)\|_{L^p(\Omega,H^\beta)} + \|e^N_t\|_{L^p(\Omega,H)},
\] (3.11)

where \( e^N_t := P_N X(t) - X^N(t) = \int_0^t E_N(t-s) [F_N(X(s)) - F_N(X^N(s))] \, ds \) satisfies

\[
\frac{d}{dt} e^N_t = -A_N e^N_t + F_N(X(t)) - F_N(X^N(t)) = -A e^N_t + F_N(X(t)) - F_N(X^N(t)).
\] (3.12)

Therefore

\[
\frac{d}{dt} \|e^N_t\|^p = p \|e^N_t\|^{p-2} \langle e^N_t, -A e^N_t + F(X(t)) - F(X^N(t)) \rangle
\]

\[
\leq p \|e^N_t\|^{p-2} \langle e^N_t, F(P_N X(t)) - F(X^N(t)) \rangle + p \|e^N_t\|^{p-2} \langle e^N_t, F(X(t)) - F(P_N X(t)) \rangle
\]

\[
\leq \|l_p\| \|e^N_t\|^{p} + p \|e^N_t\|^{p-1} \|F(X(t)) - F(P_N X(t))\|
\]

\[
\leq (l_p + p - 1) \|e^N_t\|^{p} + \|F(X(t)) - F(P_N X(t))\|^{p}
\]

\[
\leq C \|e^N_t\|^{p} + C (1 + \|X(t)\|^{2p} + \|P_N X(t)\|^{2p}) \|(I - P_N)X(t)\|^p.
\] (3.13)
Choosing $\frac{p-2}{2p} < \gamma < \frac{1}{2}$ in Lemma 3.3 and also considering (2.12), (2.13) and (3.2) assure
\[
\mathbb{E}[\|e_i^N\|^p] \leq C \int_0^t \mathbb{E}[\|e_i^N\|^p] + \mathbb{E}[\| (1 + \|X(s)\|^{2p} + \|P_N X(s)\|^{2p}) (I - P_N) X(s)\|^{p}] \, ds
\]
\[
\leq C \int_0^t \mathbb{E}[\|e_i^N\|^p] + \mathbb{E}[\|X(s)\|^{2p}_{L^{2p}(\Omega,V)} + \|P_N X(s)\|^{2p}_{L^{2p}(\Omega,V)}] \| (I - P_N) X(s)\|^{p}_{L^{2p}(\Omega,H)} \, ds
\]
\[
\leq C \int_0^t \mathbb{E}[\|e_i^N\|^p] \, ds + C N^{-p\beta}. \tag{3.14}
\]
The Gronwall inequality implies the desired error bound. □

4 Spatio-temporal full discretization

This section is devoted to error analysis of a spatio-temporal full discretization, done by a time discretization of the spatially discretized problem (3.3). For $M \in \mathbb{N}$ we construct a uniform mesh on $[0,T]$ with $\tau = \frac{T}{M}$ being the time stepsize, and propose a spatio-temporal full discretization as,
\[
Y_{t_{m+1}}^{M,N} = E_N(\tau) Y_{t_m}^{M,N} \quad + \frac{A_{N}^{-1} (I - E_N(\tau)) F_N(Y_{t_m}^{M,N})}{1 + \tau \| F_N(Y_{t_m}^{M,N}) \|} \quad + \int_{t_m}^{t_{m+1}} E_N(t_{m+1} - s) P_N dW(s) \tag{4.1}
\]
for $m = 0, 1, \ldots, M - 1$ and $Y_0^{M,N} = P_N X_0$. Equivalently, the full discretization (4.1) can be written by $Y_0^{M,N} = P_N X_0$ and for $m = 0, 1, \ldots, M - 1$,
\[
Y_{t_{m+1}}^{M,N} = E_N(\tau) Y_{t_m}^{M,N} \quad + \int_{t_m}^{t_{m+1}} E_N(t_{m+1} - s) F_N(Y_{t_m}^{M,N}) \quad + \int_{t_m}^{t_{m+1}} E_N(t_{m+1} - s) P_N dW(s). \tag{4.2}
\]
Here we invoke a taming technique in [20][22][43][44] for ordinary SDEs, and construct a nonlinearity-tamed accelerated exponential Euler (AEE) scheme as (4.1). The so-called AEE scheme without taming is originally introduced in [20], to strongly approximate nearly linear parabolic SPDEs. Since the stochastic convolution is Gaussian distributed and diagonalizable on $\{e_i\}_{i \in \mathbb{N}}$, the scheme is much easier to simulate than it appears at first sight (see comments in section 5 for the implementation). When the nonlinearity grows super-linearly, one can in general not expect that the usual AEE schemes [25][26][35][39][45][46] converge strongly, based on the observation that the standard Euler method strongly diverges for ordinary (finite dimensional) SDEs [21]. Also, we mention that analyzing the strong convergence rate is much more difficult than that in the finite dimensional SDE setting. We will accomplish it in subsequent subsections.

4.1 Ingredients in the deterministic setting

At first, we show some estimates involved with the semigroup.

Lemma 4.1 Let $t > 0$ and let $P_N, E(t)$ be defined as in the above sections. Then
\[
\| P_N E(t) \psi \|_{L^1(D)} \leq t^{-\frac{1}{2}} \| \psi \|, \quad \| P_N E(t) \psi \|_{V} \leq (\frac{t}{2})^{-\frac{1}{2}} \| \psi \|_{L^1(D)}, \tag{4.3}
\]
\[
\| P_N E(t) \psi \|_{L^p(D)} \leq t^{-\frac{1}{p}} \| \psi \|, \quad \| P_N E(t) \psi \|_{L^\infty(D)} \leq (\frac{t}{2})^{-\frac{1}{2}} \| \psi \|_{L^\infty(D)}.
\]
Proof of Lemma 4.1. The first assertion can be proved directly with the aid of (3.6). One can, for example, see [4, Lemma 5.6] for its proof. To arrive at the second one, we use (3.6) with \( \gamma = 0 \) to get
\[
\| P_N E(t) \psi \|_V = \| P_N E(\frac{t}{2}) P_N E(\frac{t}{2}) \psi \|_V \leq (\frac{t}{2})^{-\frac{1}{4}} \| P_N E(\frac{t}{2}) \psi \| = (\frac{t}{2})^{-\frac{1}{4}} \sup_{\| \phi \| \leq 1} \| \phi \| \| P_N E(\frac{t}{2}) \psi \| \phi \| \leq 1
\]
\[
= (\frac{t}{2})^{-\frac{1}{4}} \sup_{\| \phi \| \leq 1} \| \phi \| \| P_N E(\frac{t}{2}) \phi \| \leq (\frac{t}{2})^{-\frac{1}{4}} \sup_{\| \phi \| \leq 1} \| \phi \| \| P_N E(\frac{t}{2}) \phi \| \| P_N E(\frac{t}{2}) \psi \|_V \leq \| \psi \| \| P_N E(\frac{t}{2}) \psi \|_V \leq \| \psi \| \| P_N E(\frac{t}{2}) \psi \|_V \leq \| \psi \| \| P_N E(\frac{t}{2}) \psi \|_V \leq \| \psi \|^3.
\] (4.4)

Concerning the last inequality, one similarly acquires
\[
\| P_N E(t) \psi \|_{L^2(D)} \leq \| P_N E(t) \psi \|_V \leq \| \psi \|^2 \cdot t^{-\frac{1}{4}} \| \psi \| = t^{-\frac{1}{4}} \| \psi \|^3.
\] (4.5)

The proof is now completed. \( \square \)

In the sequel, we restrict ourselves to the following problem in \( H^N, N \in \mathbb{N} \),
\[
\begin{cases}
\frac{\partial u^N}{\partial t} = -A_N v^N + P_N F(v^N + z^N), & t \in (0, T], \\
u^N(0) = 0,
\end{cases}
\] (4.7)

where \( F \) comes from Assumption 2.2 and \( z^N, v^N : H^N \times [0, T] \to H^N \). It is easy to see, (4.7) has a unique solution in \( H^N \), which can be expressed by
\[
v^N(t) = \int_0^t E_N(t-s) P_N F(v^N(s) + z^N(s)) \, ds.
\] (4.8)

Define norms \( \| u \|_{L^q(D \times [0, t])} := \left( \int_0^t \| u(s) \|_{L^q(D)}^q \, ds \right)^{\frac{1}{q}}, q \geq 1, t \in [0, T] \). For the particular case \( q = 2 \), \( L^2(D \times [0, t]) \) (\( L^q \) for brevity) becomes a Hilbert space with \( \langle u, v \rangle_{L^2(D \times [0, t])} := \int_0^t \langle u(s), v(s) \rangle \, ds \). The forthcoming estimate plays an essential role in proving moment bounds of the approximations.

**Lemma 4.2** Let \( v^N, N \in \mathbb{N} \) be the solution to (4.7). For any \( t \in [0, T] \), there exists a constant \( C \), independent of \( N \), such that
\[
\| v^N(t) \|_V \leq C(1 + \| z^N \|_{L^2(D \times [0, t])}), \quad \forall t \in [0, T].
\] (4.9)

**Proof of Lemma 4.2** The proof is divided into two steps.
Step 1. For any fixed $t \in [0, T]$, we claim first that, by setting $\varrho_t := 5t^{\frac{1}{4}} \max\{\frac{|a_2|}{a_3^2}, \frac{|a_1|}{a_3^2}, \frac{|a_0|}{a_3^2}\}$,
\[
\|v^N\|_{L^4(D \times [0,t])} \leq 5\|z^N\|_{L^4(D \times [0,t])} \quad \text{or} \quad \|v^N\|_{L^4(D \times [0,t])} \leq \varrho_t. \tag{4.10}
\]
By deterministic calculus and noting $A_N v^N = A v^N$ for any $v^N \in H^N$, we infer
\[
0 \leq \frac{1}{2}\|v^N(t)\|^2 = \int_0^t \langle v^N(s), -A_N v^N(s) + P_N F(v^N(s) + z^N(s)) \rangle \, ds \leq \int_0^t \langle v^N(s), F(v^N(s) + z^N(s)) \rangle \, ds = \langle v^N, F(v^N + z^N) \rangle_{L^2}. \tag{4.11}
\]
Noticing that $a_3 \leq 0$, for any $v, z \in \mathbb{R}$,
\[
v f (v + z) = a_3 v (v + z)^3 + a_2 v (v + z)^2 + a_1 v (v + z) + a_0 v \\
\leq a_3 v^4 + 3a_3 v^3 z + a_3 v z^3 + a_2 v^3 z + a_2 v^2 z^2 + a_1 v^2 z + a_1 v z + a_0 v. \tag{4.12}
\]
After using the fact $a_3 \leq 0$ and the Hőlder inequality, one derives
\[
\langle v^N, F(v^N + z^N) \rangle_{L^2} \leq a_3 \|v^N\|_{L^4}^4 + 3a_3 \|v^N\|_{L^2}^2 \|z^N\|_{L^4} + |a_3| \|v^N\|_{L^4} \|z^N\|_{L^4}^2 + |a_2| \|v^N\|_{L^4} \|v\|_{L^4} \|z^N\|_{L^4}^2 + |a_1| \|v\|_{L^4} \|z^N\|_{L^4}^2 + |a_0| \|v\|_{L^4} \|z^N\|_{L^4}^2 \tag{4.13}
\]
Assume the claim (4.10) is false, namely, $\|z^N\|_{L^4(D \times [0,t])} < \frac{1}{5} \|v^N\|_{L^4(D \times [0,t])}$ and $\|v^N\|_{L^4(D \times [0,t])} > \varrho_t$. This enables us to derive
\[
\langle v^N, F(v^N + z^N) \rangle_{L^2} \leq \left( a_3 + \frac{3|a_3|}{5} + \frac{|a_1|}{125} \right) \|v^N\|_{L^4}^4 + \left( |a_2| \frac{1}{5} \frac{4}{25} + \frac{|a_2|}{25} \right) \|v^N\|_{L^4}^3 \tag{4.14}
\]
which contradicts (4.11).
Step 2. Apparently, (4.10) implies
\[
\|v^N\|_{L^4(D \times [0,t])} \leq 5 \|z^N\|_{L^4(D \times [0,t])} + \varrho_t, \quad \forall t \in [0, T]. \tag{4.15}
\]
This together with the last inequality in (4.13), the property of the cubic nonlinearity and the Hőlder inequality yields, for any $t \in [0, T]$,
\[
\|v^N(t)\|_{L^4(D)} \leq \int_0^t \|E(t - s) P_N F(v^N(s) + z^N(s))\|_{L^4(D)} \, ds \leq \int_0^t \left( \frac{1}{2} \right)^{\frac{9}{32}} \|F(v^N(s) + z^N(s))\|_{L^4(D)} \, ds \leq C \int_0^t \left( \frac{1}{2} \right)^{\frac{9}{32}} \left( 1 + \|v^N(s)\|_{L^4(D)}^3 + \|z^N(s)\|_{L^4(D)}^3 \right) \, ds \tag{4.16}
\]
\[
\leq C \left( \int_0^t \left( \frac{1}{2} \right)^{\frac{8}{32}} \, ds \right)^{\frac{1}{3}} \left( \frac{1}{2} \right)^{\frac{9}{32}} \left( \int_0^t 1 + \|v^N(s)\|_{L^4(D)}^4 + \|z^N(s)\|_{L^4(D)}^4 \, ds \right)^{\frac{3}{4}} \leq C \left( 1 + \|z^N\|_{L^4(D \times [0,t])}^3 \right).
\]
Likewise, by virtue of the second inequality in (4.13) instead, one obtains, for any \( t \in [0, T] \),

\[
\|v^N(t)\|_V \leq \int_0^t \|P_N E(t-s)F(v^N(s) + z^N(s))\|_V \, ds
\]
\[
\leq C \int_0^t (t-s)^{-\frac{1}{2}} \|F(v^N(s) + z^N(s))\|_{L^1(D)} \, ds
\]
\[
\leq C \int_0^t (t-s)^{-\frac{1}{2}} (1 + \|v^N(s)\|_{L^3(D)}^3 + \|z^N(s)\|_{L^3(D)}^3) \, ds
\]
\[
\leq C \left(1 + \|z^N\|_{L^2(D \times [0,t])}^9 + \int_0^t (t-s)^{-\frac{2}{3}} \|z^N(s)\|_{L^3(D)}^3 \, ds \right)
\]
\[
\leq C(1 + \|z^N\|_{L^2(D \times [0,t])}^9).
\]

The proof of Lemma 4.2 is thus finished. \( \square \)

### 4.2 A priori moment bounds of the approximations

This subsection aims to obtain a priori estimates of the full discrete approximation, which require estimates in the previous subsection as well as a certain bootstrap argument. First, we define

\[
[t] := t_i, \quad \text{for} \ t \in [t_i, t_{i+1}), \ i \in \{0, 1, ..., M-1\}
\]

and introduce a continuous version of the full discrete scheme (4.2) as,

\[
Y_{i}^{M,N} = E_N(t)Y_{0}^{M,N} + \int_0^t \frac{E_N(t-s)F_N(Y_{i}^{M,N})}{1 + \tau \|E_N(Y_{i}^{M,N})\|} \, ds + \mathcal{O}_t^N, \quad \text{with} \quad \mathcal{O}_t^N := P_N \mathcal{O}_t, \ t \in [0, T].
\]

By \( B^c \) and \( \mathbb{1}_B \), we denote the complement and indicator function of a set \( B \), respectively. Additionally, we introduce a sequence of decreasing subevents

\[
\Omega_{R,t_i} := \left\{ \omega \in \Omega : \sup_{j \in \{0,1,...,i\}} \|Y_{t_j}^{M,N}(\omega)\|_V \leq R \right\}, \quad R \in (0, \infty), \ i \in \{0, 1, ..., M\}.
\]

It is clear that \( \mathbb{1}_{\Omega_{R,t_i}} \in \mathcal{F}_{t_i} \) for \( i \in \{0,1,...,M\} \) and \( \mathbb{1}_{\Omega_{R,t_i}} \leq \mathbb{1}_{\Omega_{R,t_j}} \) for \( t_i \geq t_j \) since \( \Omega_{R,t_i} \subseteq \Omega_{R,t_j}, t_i \geq t_j \). Besides, we put additional assumptions on the initial data.

**Assumption 4.3** For sufficiently large positive number \( p_0 \in \mathbb{N} \), the initial data \( X_0 \) obeys

\[
\sup_{N \in \mathbb{N}} \|P_N X_0\|_{L^{p_0}(\Omega, V)} < \infty.
\]

Due to the Sobolev embedding inequality, (4.21) is fulfilled provided \( \|P_N X_0\|_{L^{p_0}(\Omega, V)} < \infty \) for any \( \gamma > \frac{3}{2} \). Next we start the bootstrap argument, by showing \( \mathbb{E}[\mathbb{1}_{\Omega_{R^\tau,t_m}} \|Y_{t_m}^{M,N}\|_V^p] < \infty \) and \( \mathbb{E}[\mathbb{1}_{\Omega_{R^\tau,t_m}} \|Y_{t_m}^{M,N}\|_V^p] < \infty \) for subevents \( \Omega_{R^\tau,t} \) with \( R^\tau \) depending on \( \tau \) carefully chosen.
Lemma 4.4 Let \( p \in [2, \infty) \) and \( R^\tau := \tau^{-\beta} \) for any \( \beta \in (0, \frac{1}{2}) \). Under Assumptions 2.1, 2.3, 4.3, the approximation process \( Y_{t_i}^{M,N}, i \in \{0, 1, ..., M\} \) produced by (1.1) obeys

\[
\sup_{M,N \in \mathbb{N}} \sup_{i \in \{0, 1, ..., M\}} \mathbb{E} \left[ \mathbbm{1}_{\Omega_{R^\tau, t_i-1}} \| Y_{t_i}^{M,N} \|_V^p \right] < \infty, \tag{4.22}
\]

where we set \( \mathbbm{1}_{\Omega_{R^\tau, t_i-1}} = 1 \).

Proof of Lemma 4.4. The proof heavily relies on the use of Lemma 4.2. In order to apply it, we introduce a process \( Z_t^{M,N} \) given by,

\[
Z_t^{M,N} := E_N(t)Y_0^{M,N} + \int_0^t E_N(t-s) \left[ \frac{P_NF(Y_s^{M,N})}{1+\|P_NF(Y_s^{M,N})\|} - P_NF(Y_s^{M,N}) \right] ds + \mathcal{O}_t^{N} \\
= E_N(t)Y_0^{M,N} + \int_0^t E(t-s)P_N \left[ F(Y_s^{M,N}) - F(Y_s^{M,N}) \right] ds \\
+ \int_0^t E(t-s) \left[ \frac{P_NF(Y_s^{M,N})}{1+\|P_NF(Y_s^{M,N})\|} - P_NF(Y_s^{M,N}) \right] ds + P_N\mathcal{O}_t, \quad t \in [0, T]. \tag{4.23}
\]

With this, one can rewrite (4.19) as

\[
Y_t^{M,N} = \int_0^t E_N(t-s)P_NF(Y_s^{M,N}) ds + Z_t^{M,N}. \tag{4.24}
\]

Further, we define \( \bar{Y}_t^{M,N} \) as

\[
\bar{Y}_t^{M,N} := Y_t^{M,N} - Z_t^{M,N}, \quad \text{with} \quad \bar{Y}_0^{M,N} = 0. \tag{4.25}
\]

Once again, we recast (4.24) as

\[
\bar{Y}_t^{M,N} = \int_0^t E_N(t-s)P_NF(\bar{Y}_s^{M,N} + Z_s^{M,N}) ds, \quad t \in [0, T], \tag{4.26}
\]

which satisfies

\[
\frac{d}{dt} \bar{Y}_t^{M,N} = -A_N\bar{Y}_t^{M,N} + P_NF(\bar{Y}_t^{M,N} + Z_t^{M,N}), \quad t \in (0, T], \quad \bar{Y}_0^{M,N} = 0. \tag{4.27}
\]

Now one can employ Lemma 4.2 to deduce,

\[
\| \bar{Y}_t^{M,N} \|_V \leq C(1 + \| Z_t^{M,N} \|_{L^9(D \times [0, t])}^9), \quad t \in [0, T], \tag{4.28}
\]

where \( Z_t^{M,N} \) is defined by (4.23). Thus, for any \( i \in \{0, 1, ..., M\} \),

\[
\mathbb{E} \left[ \mathbbm{1}_{\Omega_{R^\tau, t_i-1}} \| \bar{Y}_{t_i}^{M,N} \|_V^p \right] \leq C \left( 1 + \mathbb{E} \left[ \mathbbm{1}_{\Omega_{R^\tau, t_i-1}} \| Z_{t_i}^{M,N} \|_{L^9(D \times [0, t_i])}^9 \right] \right) \leq C \left( 1 + \mathbb{E} \left[ \mathbbm{1}_{\Omega_{R^\tau, t_i-1}} \int_0^{t_i} \| Z_s^{M,N} \|_V^p ds \right] \right), \tag{4.29}
\]
where, for \( s \in [0, t_i] \), \( i \in \{0, 1, \ldots, M\} \), it stands that

\[
\mathbb{1}_{\Omega_{R,t_i-1}} \|Z_{s}^{M,N}\|_{V} \leq \|E_{N}(s)Y_{0}^{M,N}\|_{V} + \mathbb{1}_{\Omega_{R,t_i-1}} \left\| \int_{0}^{s} E(s-r)P_{N} \left[ F(Y_{r}^{M,N}) - F(Y_{[r]}^{M,N}) \right] dr \right\|_{V} \\
+ \mathbb{1}_{\Omega_{R,t_i-1}} \left\| \int_{0}^{s} E(s-r)P_{N}F(Y_{[r]}^{M,N}) \tau \|P_{N}F(Y_{[r]}^{M,N})\|_{1+\tau\|P_{N}F(Y_{[r]}^{M,N})\|} dr \right\|_{V} + P_{N}\mathcal{O}_{s}\|_{V}
:= \|E_{N}(s)Y_{0}^{M,N}\|_{V} + I_{1} + I_{2} + P_{N}\mathcal{O}_{s}\|_{V}.
\]

Before proceeding further, we claim

\[
\mathbb{1}_{\Omega_{R,t_i-1}} \|Y_{r}^{M,N}\|_{V} \leq C(1 + R + \tau^{\frac{3}{2}}R^{3}), \quad \forall r \in [0, t_i].
\]  

(4.31)

For the case \( r \in (t_{i-1}, t_{i}] \), the definition of \( Y_{r}^{M,N} \), boundedness of the semigroup \( E(t) \) in \( V \) and (3.6) with \( \gamma = 0 \) promise

\[
\mathbb{1}_{\Omega_{R,t_i-1}} \|Y_{r}^{M,N}\|_{V} \leq \mathbb{1}_{\Omega_{R,t_i-1}} \left( \|E(r-t_{i-1})Y_{t_{i-1}}^{M,N}\|_{V} + \int_{t_{i-1}}^{r} \|E(r-u)F_{N}(Y_{[u]}^{M,N})\|_{V} du \right. \\
+ \left. \left\| \int_{t_{i-1}}^{r} E(r-u)P_{N} dW_{u}\right\|_{V} \right) \leq C(R + \tau^{3/4}R^{3} + 1).
\]

(4.32)

For the case \( r \in [0, t_{i-1}] \), we recall \( \mathbb{1}_{\Omega_{R,t_i-1}} \leq \mathbb{1}_{\Omega_{R,[r]}} \), which allows us to get \( \mathbb{1}_{\Omega_{R,t_i-1}} \|Y_{r}^{M,N}\|_{V} \leq \mathbb{1}_{\Omega_{R,[r]}} \|Y_{r}^{M,N}\|_{V} \). Then repeating the same arguments as used in (4.32) shows (4.31). With the aid of (2.4) and (4.31), the first term \( I_{1} \) can be treated as follows,

\[
I_{1} \leq \mathbb{1}_{\Omega_{R,t_i-1}} \int_{0}^{s} (s-r)^{-\frac{1}{2}} \|F(Y_{r}^{M,N}) - F(Y_{[r]}^{M,N})\| dr \\
\leq \mathbb{1}_{\Omega_{R,t_i-1}} C(1 + R^{2} + \tau^{\frac{3}{2}}R^{6}) \int_{0}^{s} (s-r)^{-\frac{1}{2}} \|Y_{r}^{M,N} - Y_{[r]}^{M,N}\| dr,
\]

(4.33)

where \( r \in [0, s], \ s \in [0, t_{i}] \),

\[
Y_{r}^{M,N} - Y_{[r]}^{M,N} = [E(r) - E([r])]Y_{0}^{M,N} + \int_{0}^{r} E(r-u)P_{N}F(Y_{[u]}^{M,N}) \frac{P_{N}F(Y_{[u]}^{M,N})}{1+\tau\|P_{N}F(Y_{[u]}^{M,N})\|} du \\
- \int_{0}^{[r]} E([r]-u)P_{N}F(Y_{[u]}^{M,N}) \frac{P_{N}F(Y_{[u]}^{M,N})}{1+\tau\|P_{N}F(Y_{[u]}^{M,N})\|} du + P_{N}\mathcal{O}_{r} - P_{N}\mathcal{O}_{[r]}.
\]

(4.34)

This suggests that

\[
\mathbb{1}_{\Omega_{R,t_i-1}} \|Y_{r}^{M,N} - Y_{[r]}^{M,N}\| \\
\leq \tau^{\frac{3}{2}}\|Y_{0}^{M,N}\|_{\beta} + \mathbb{1}_{\Omega_{R,t_i-1}} \left\| \int_{0}^{[r]} E([r]-u)(E(r-[r]) - I) \frac{P_{N}F(Y_{[u]}^{M,N})}{1+\tau\|P_{N}F(Y_{[u]}^{M,N})\|} du \right\| \\
+ \mathbb{1}_{\Omega_{R,t_i-1}} \left\| \int_{[r]}^{r} E(r-u) \frac{P_{N}F(Y_{[u]}^{M,N})}{1+\tau\|P_{N}F(Y_{[u]}^{M,N})\|} du \right\|_{H} + \mathbb{1}_{\Omega_{R,t_i-1}} \|P_{N}(\mathcal{O}_{r} - \mathcal{O}_{[r]})\| \\
\leq \tau^{\frac{3}{2}}\|X_{0}\|_{\beta} + C(1 + R^{3})(\tau^{\frac{3}{2}} + \tau) + \|\mathcal{O}_{r} - \mathcal{O}_{[r]}\|.
\]

(4.35)
Inserting this into (4.33) results in
\[
I_1 \leq C(1 + R^2 + \tau^2 R^6) \int_0^s (s - r)^{-\frac{1}{2}} \left[ \tau^2 |X_0| + C(1 + R^3) \tau \frac{3}{4} + \|O_r - O_{[r]}\| \right] \, dr \\
\leq C(1 + R^2 + \tau^2 R^6)\tau^2 |X_0| s^\frac{3}{4} + C(1 + R^2 + \tau^2 R^6)(1 + R^3)\tau^\frac{3}{4} s^\frac{3}{4} + C\int_0^s (s - r)^{-\frac{1}{2}} (1 + R^2 + \tau^2 R^6)\|O_r - O_{[r]}\| \, dr.
\]

Therefore, letting \( R := \tau^{-\frac{3}{4}} \) and considering (2.10) one can further infer that
\[
\|I_1\|_{L^{9p}(|\Omega, R|)} \leq C(1 + \|X_0\|_{L^{9p}(|\Omega, H^0|)}).
\]

In a similar manner, choosing \( R := \tau^{-\frac{3}{4}} \) enables us to arrive at
\[
I_2 \leq \mathbb{I}_{\Omega_{R,t_i-1}} \int_0^s \| E(s - r) P_N F(Y_{[r]}^{M,N}) \| \cdot \| P_N F(Y_{[r]}^{M,N}) \| \, dr \\
\leq \mathbb{I}_{\Omega_{R,t_i-1}} \tau \int_0^s (s - r)^{-\frac{1}{4}} \| F(Y_{[r]}^{M,N}) \|^2 \, ds \leq C\tau (R^6 + 1)
\]

Bearing (4.37), (4.38) and (3.5) in mind, one can deduce from (4.30) that, for any \( s \in [0, t_i] \),
\[
\mathbb{E}[\mathbb{I}_{\Omega_{R,t_i-1}} \| Z_s^{M,N} \|^p_{V}] \leq C < \infty.
\]

This together with (4.29) and (4.39) immediately implies
\[
\mathbb{E}[\mathbb{I}_{\Omega_{R^\tau,t_i-1}} \| \tilde{Y}_{t_i}^{M,N} \|_{V}] \leq C < \infty.
\]

Combining this with (4.25) verifies the desired assertion (4.22). □

**Theorem 4.5** Let Assumptions 2.1-2.4, 4.3 be fulfilled. Then for any \( p \in [2, \infty) \),
\[
\sup_{M,N \in \mathbb{N}} \sup_{m \in \{0,1,...,M\}} \mathbb{E}[\|Y_{t_m}^{M,N} \|_{V}] < \infty.
\]

**Proof of Theorem 4.5** Since (4.22) and the fact that \( \Omega_{R,t_i} \subset \Omega_{R,t_i-1} \) ensure
\[
\sup_{M,N \in \mathbb{N}} \sup_{i \in \{0,1,...,M\}} \mathbb{E}[\mathbb{I}_{\Omega_{R^\tau,t_i-1}} \| Y_{t_i}^{M,N} \|_{V}] \leq \sup_{M,N \in \mathbb{N}} \sup_{i \in \{0,1,...,M\}} \mathbb{E}[\mathbb{I}_{\Omega_{R^\tau,t_i-1}} \| Y_{t_i}^{M,N} \|_{V}] < \infty,
\]

it remains to estimate \( \sup_{M,N \in \mathbb{N}} \sup_{m \in \{0,1,...,M\}} \mathbb{E}[\mathbb{I}_{\Omega_{R^\tau,t_m-1}} \| Y_{t_m}^{M,N} \|_{V}] \). It is evident to check that
\[
\|Y_{t_m}^{M,N}\|_V \leq \|E(t_m)P_NX_0\|_V + \|P_NO_{t_m}\|_V + \int_0^{t_m} \| E(t_m - s) \frac{P_N F(Y_{[s]}^{M,N})}{1 + \tau \|P_N F(Y_{[s]}^{M,N})\|} \|_V \, ds \\
\leq C t_m^{2 \beta - 1} \|X_0\|_\beta + \int_0^{t_m} \left( \|A_{t_m}^2 E(t_m - s) \frac{P_N F(Y_{[s]}^{M,N})}{1 + \tau \|P_N F(Y_{[s]}^{M,N})\|} \|_H \right) \, ds + \|P_N O_{t_m}\|_V \\
\leq C t_m^{2 \beta - 1} \|X_0\|_\beta + C\tau^{-1} + C,
\]

\( m \in \{1,2,...,M\} \).
Meanwhile, one can learn that
\begin{equation}
\Omega_{R^*,t_m} = \Omega_{R^*,t_{m-1}} + \Omega_{R^*,t_{m-1}} \cdot \{ \omega \in \Omega : \| Y^{M,N}_{t_m} \|_V > R^* \},
\end{equation}
and as a result
\begin{equation}
1_{\Omega_{R^*,t_m}} = 1_{\Omega_{R^*,t_{m-1}}} + 1_{\Omega_{R^*,t_{m-1}}} \cdot 1_{\{ \| Y^{M,N}_{t_m} \|_V > R^* \}} = \sum_{i=0}^{m} 1_{\Omega_{R^*,t_{i-1}}} \cdot 1_{\{ \| Y^{M,N}_{t_i} \|_V > R^* \}},
\end{equation}
where we recall $1_{\Omega_{R^*,t_{-1}}} = 0$. By Assumption 4.3 and the Chebyshev inequality, one can show, for any $M \in \mathbb{N}$ and $m \in \{0, 1, \ldots, M\}$,
\begin{align}
\mathbb{E}[1_{\Omega_{R^*,t_m}} \| Y^{M,N}_{t_m} \|_V^{p}] &= \sum_{i=0}^{m} \mathbb{E}[\| Y^{M,N}_{t_i} \|_V^{p} \cdot 1_{\Omega_{R^*,t_{i-1}}} \cdot 1_{\{ \| Y^{M,N}_{t_{i}} \|_V > R^* \}}] \\
&\leq \sum_{i=0}^{m} \left( \mathbb{E}[\| Y^{M,N}_{t_i} \|_V^{2p}] \right)^{\frac{1}{2}} \cdot \left( \mathbb{E}[1_{\Omega_{R^*,t_{i-1}}} \cdot 1_{\{ \| Y^{M,N}_{t_{i}} \|_V > R^* \}}] \right)^{\frac{1}{2}} \\
&\leq \sum_{i=0}^{m} C(1 + \tau^{-p}) \cdot \left( \mathbb{P}(1_{\Omega_{R^*,t_{i-1}}} \| Y^{M,N}_{t_i} \|_V > R^*) \right)^{\frac{1}{2}} \\
&\leq C(1 + \tau^{-p}) \sum_{i=0}^{m} \tau^{p+1} \left( \mathbb{E}[1_{\Omega_{R^*,t_{i-1}}} \| Y^{M,N}_{t_i} \|_V^{\frac{8(p+1)}{\eta}}] \right)^{\frac{1}{2}} \leq C(1 + \tau^{-p}) \sum_{i=0}^{m} \tau^{p+1} \left( \mathbb{E}[1_{\Omega_{R^*,t_{i-1}}} \| Y^{M,N}_{t_i} \|_V^{\frac{8(p+1)}{\eta}}] \right)^{\frac{1}{2}} < \infty.
\end{align}
This estimate together with (4.42) yields the required estimate (4.41). 

With Theorem 4.5 at hand, one can use standard arguments to validate the coming corollaries.

**Corollary 4.6** Under conditions in Theorem 4.3, for any $p \in [2, \infty)$ and $\beta < \frac{1}{2}$ we obtain,
\begin{equation}
\sup_{M,N \in \mathbb{N}, t \in [0,T]} \| Y^{M,N}_{t} \|_{L^p(\Omega, \mathcal{H})} + \sup_{M,N \in \mathbb{N}, t \in [0,T]} \| Y^{M,N}_{t} \|_{L^p(\Omega, V)} < \infty.
\end{equation}

**Corollary 4.7** Under conditions in Theorem 4.3, for any $p \in [2, \infty)$ and $\beta < \frac{1}{2}$ we get,
\begin{equation}
\| Y^{M,N}_{t} - Y^{M,N}_{s} \|_{L^p(\Omega, H)} \leq C(t-s)^{\frac{\beta}{2}}, \quad 0 \leq s < t \leq T.
\end{equation}

### 4.3 Further technical lemmas

In addition to the above preparations, we still rely on the following results, which are essential to identify the expected temporal convergence rates of order almost $\frac{1}{2}$.

**Lemma 4.8** Let $F : L^6(D; \mathbb{R}) \to H$ be a mapping determined by Assumption 2.2. Then it holds for any $\beta \in (0, \frac{1}{2})$ and $\eta > \frac{1}{2}$ that
\begin{equation}
\| F'(\nu) \nu \|_{-\eta} \leq C(1 + \max \{ \| \chi \|_V, \| \chi \|_{-\beta} \}) \| \nu \|_{-\beta}, \quad \chi \in V \cap \mathcal{H}^{\beta}, \nu \in V.
\end{equation}
Proof of Lemma 4.8. As $\beta \in (0, \frac{1}{2})$, standard arguments with the Sobolev-Slobodeckij norm (see, e.g., [12] (19.14)) and properties of the nonlinearity guarantee

$$
\|F'(\chi)\psi\|_{\beta}^2 \leq C\|F'(\chi)\psi\|^2 + C\int_0^1 \int_0^1 \frac{|f'(\chi(x))\psi(x) - f'(\chi(y))\psi(y)|^2}{|x - y|^{2\beta + 1}} \, dy \, dx
$$

$$
\leq C\|F'(\chi)\psi\|^2 + C\int_0^1 \int_0^1 \frac{|f'(\chi(x))(\psi(x) - \psi(y))|^2}{|x - y|^{2\beta + 1}} \, dy \, dx
$$

$$
+ C\int_0^1 \int_0^1 \frac{|[f'(\chi(x)) - f'(\chi(y))]\psi(y)|^2}{|x - y|^{2\beta + 1}} \, dy \, dx
$$

$$
\leq C\|F'(\chi)\psi\|^2 + C\int_0^1 \int_0^1 \frac{|f'(\chi(x))(\psi(x) - \psi(y))|^2}{|x - y|^{2\beta + 1}} \, dy \, dx
$$

(4.50)

Accordingly, for any $\beta \in (0, \frac{1}{2})$ and $\eta > \frac{1}{2}$, one uses the Sobolev embedding inequality to derive

$$
\|F'(\chi)\nu\|_{-\eta} = \sup_{\|\varphi\| \leq 1} \left| \left\langle A^{-\frac{\eta}{2}} F'(\chi)\nu, \varphi \right\rangle \right| = \sup_{\|\varphi\| \leq 1} \left| \left\langle \nu, (F'(\chi))^* \cdot A^{-\frac{\eta}{2}} \varphi \right\rangle \right|
$$

$$
= \sup_{\|\varphi\| \leq 1} \left| \left\langle A^{-\frac{\eta}{2}} \nu, A^{\frac{\eta}{2}} F'(\chi)A^{-\frac{\eta}{2}} \varphi \right\rangle \right|
$$

$$
\leq \sup_{\|\varphi\| \leq 1} \|\nu\|_{-\beta} \cdot \|F'(\chi)A^{-\frac{\eta}{2}} \varphi\|_{\beta}
$$

(4.51)

This completes the proof. □

Lemma 4.9. Letting Assumptions 2.1, 2.4, 4.3 be fulfilled, for any $p \in [2, \infty)$ and $\beta < \frac{1}{2}$ we have

$$
\|Y^{M,N}_t - Y^{M,N}_s\|_{L^p(\Omega, H^{-\beta})} \leq C(t - s)^\beta, \quad 0 \leq s < t \leq T.
$$

(4.52)

Proof of Lemma 4.9. The definition (1.19) implies, for $0 \leq s < t \leq T$,

$$
Y^{M,N}_t - Y^{M,N}_s = (E_N(t - s) - I)Y^{M,N}_s + \int_s^t E_N(t - r)\frac{F_N(Y^{M,N}_r)}{1 + r \|F_N(Y^{M,N}_r)\|} \, dr + \int_s^t E_N(t - r)P_N \, dW(r).
$$

(4.53)

Making use of (2.2), (2.3) and the inequality $\|\Gamma_1\|_{L^2} \leq \|\Gamma\|_{L\Gamma_1 \|L^2} \leq \|\Gamma\|_{L^2} \leq \|\Gamma_1\|_{L^2(H)}$, $\Gamma \in L(H), \Gamma_1 \in L_2(H)$ gives

$$
\left\| \int_s^t E_N(t - r)P_N \, dW(r) \right\|_{L^p(\Omega, H^{-\beta})} \leq C \left( \int_s^t \|A^{-\frac{\beta}{2}} E_N(t - r)P_N\|^2_{L^2(H)} \, dr \right)^{\frac{1}{2}} \leq C(t - s)^\beta.
$$

(4.54)
A combination of (2.2) and Corollary 4.6 shows
\[ \| (E_N(t-s) - I)Y_{s,N} \|_{L^p(\Omega,H^{-\beta})} \leq \| A^{-\beta}(E(t-s) - I) \|_{L^p(\Omega,H)} \cdot \| Y_{s,N} \|_{L^p(\Omega,H^\beta)} \]
\[ \leq C(t-s)^{\beta}. \]  
(4.55)

Now we proceed to estimate the left term in (4.53), with the help of (4.41) and (2.4),
\[ \| \int_s^t E(t-r) \frac{F_N(Y_{s,N})}{1 + \tau \| F_N(Y_{s,N}) \|} \, dr \|_{L^p(\Omega,H^{-\beta})} \leq \int_s^t \| F(Y_{s,N}) \|_{L^p(\Omega,H)} \, dr \leq C(t-s). \]  
(4.56)

Gathering (4.54), (4.55) and (4.56) we deduce from (4.53) that (4.52) is true. □

In view of (4.41), (4.49), (4.52) and Corollary 4.6, one can see the following corollary.

**Corollary 4.10** Under conditions in Lemma 4.9, for any \( 0 < \beta < \frac{1}{2} \) and \( \eta > \frac{1}{2} \) it holds
\[ \| F(Y_{t,N}) - F(Y_{s,N}) \|_{L^p(\Omega,H^{-\eta})} \leq C(t-s)^{\beta}, \quad 0 < s < t < T. \]  
(4.57)

### 4.4 Main results: error bounds for the full discretization

Equipped with these results in previous subsections, we are now ready to prove the main result.

**Theorem 4.11 (Error bounds for the full discretization)** Let Assumptions 2.1-2.4, 4.3 hold. There is a generic constant \( C \) independent of \( N \) and \( M \) such that, for any \( p \in [2, \infty) \),
\[ \sup_{0 \leq m \leq M} \| X(t_m) - Y_{t_m} \|_{L^p(\Omega,H)} \leq C \left( N^{-\beta} + \tau^\beta \right). \]  
(4.58)

**Proof of Theorem 4.11** Denoting \( e_{t,N} := P_N X(t) - Y_{t,N} \), we note that
\[ \| X(t_m) - Y_{t_m} \|_{L^p(\Omega,H)} \leq \| X(t_m) - P_N X(t_m) \|_{L^p(\Omega,H)} + \| e_{t,N} \|_{L^p(\Omega,H)}, \]  
(4.59)

where
\[ \frac{d}{dt} e_{t,N} = -A_N e_{t,N} + F_N(X(t)) - \frac{F_N(Y_{s,N})}{1 + \tau \| F_N(Y_{s,N}) \|}. \]  
(4.60)

This in conjunction with (2.4) tells us that
\[ \| e_{t,N} \|^p = p \int_0^t \| e_{s,N} \|^{p-2} \langle e_{s,N}, -A_N e_{s,N} + F_N(X(s)) - \frac{F_N(Y_{s,N})}{1 + \tau \| F_N(Y_{s,N}) \|} \rangle \, ds \]
\[ \leq p \int_0^t \| e_{s,N} \|^{p-2} \langle e_{s,N}, F_N(X(s)) - F_N(P_N X(s)) \rangle + F_N(Y_{s,N}) - \frac{F_N(Y_{s,N})}{1 + \tau \| F_N(Y_{s,N}) \|} \rangle \, ds + pL \int_0^t \| e_{s,N} \|^p \, ds \]
\[ = pL \int_0^t \| e_{s,N} \|^p \, ds + p \int_0^t \| e_{s,N} \|^{p-2} \langle e_{s,N}, F_N(X(s)) - F(P_N X(s)) \rangle \, ds \]
\[ + p \int_0^t \| e_{s,N} \|^{p-2} \langle e_{s,N}, F_N(Y_{s,N}) - F(Y_{s,N}) \rangle \, ds \]
\[ + p \int_0^t \| e_{s,N} \|^{p-2} \langle e_{s,N}, \tau \| F_N(Y_{s,N}) \| \| F(Y_{s,N}) \| \rangle \, ds \]
\[ := pL \int_0^t \| e_{s,N} \|^p \, ds + J_0 + J_1 + J_2. \]  
(4.61)
Following the same lines as in estimates of (3.13) and (3.14) can bound the item $J_0$ as,

$$
\mathbb{E}[J_0] \leq p \mathbb{E} \int_0^t \|e_{s,N}^{M,N}\|^{p-1} \|F(X(s)) - F(P_N X(s))\| \, ds \\
\leq (p - 1) \int_0^t \mathbb{E} [\|e_{s,N}^{M,N}\|^p] \int_0^t \mathbb{E} [\|F(X(s)) - F(P_N X(s))\|^p] \\
\leq (p - 1) \int_0^t \mathbb{E} [\|e_{s,N}^{M,N}\|^p] + C(\frac{1}{N})^{p\beta}.
$$

(4.62)

The term $J_2$ is also easy to be treated, after taking the H"{o}lder inequality and (4.41) into account:

$$
\mathbb{E}[J_2] \leq p \mathbb{E} \int_0^t \|e_{s,N}^{M,N}\|^{p-1} \cdot \tau [\mathbb{E} [F(Y_{[s]}^{M,N})]^2] \, ds \\
\leq (p - 1) \int_0^t \mathbb{E} [\|e_{s,N}^{M,N}\|^p] \int_0^t \mathbb{E} [\|F(Y_{[s]}^{M,N})\|^{2p}] \, ds \\
\leq (p - 1) \int_0^t \mathbb{E} [\|e_{s,N}^{M,N}\|^p] \int_0^t \mathbb{E} [\|X_0\|^{6p}] \tau^p.
$$

(4.63)

The remaining term $J_1$ must be handled more carefully. To do so we recall that $e_{s,N}^{M,N} = P_N X(s) - Y_{s,N}^{M,N} = \int_0^s E(s - r)(F_N(X(r)) - \frac{F_N(Y_{[r]}^{M,N})}{1 + \tau F_N(Y_{[r]}^{M,N})}) \, dr$ and split $J_1$ into three terms:

$$
J_1 = p \int_0^t \|e_{s,N}^{M,N}\|^{p-2} \left( \int_0^s E(s - r) \left( F_N(X(r)) - \frac{F_N(Y_{[r]}^{M,N})}{1 + \tau F_N(Y_{[r]}^{M,N})} \right) \, dr, F(Y_{[s]}^{M,N}) - F(Y_{[s]}^{M,N}) \right) \, ds \\
= p \int_0^t \|e_{s,N}^{M,N}\|^{p-2} \left( \int_0^s E(s - r) \left( F_N(X(r)) - F_N(Y_{[r]}^{M,N}) \right) \, dr, F(Y_{[s]}^{M,N}) - F(Y_{[s]}^{M,N}) \right) \, ds \\
+ p \int_0^t \|e_{s,N}^{M,N}\|^{p-2} \left( \int_0^s E(s - r) \left( F_N(Y_{[r]}^{M,N}) - F_N(Y_{[r]}^{M,N}) \right) \, dr, F(Y_{[s]}^{M,N}) - F(Y_{[s]}^{M,N}) \right) \, ds \\
+ p \int_0^t \|e_{s,N}^{M,N}\|^{p-2} \left( \int_0^s E(s - r) \left( \frac{\tau F_N(Y_{[r]}^{M,N})}{1 + \tau F_N(Y_{[r]}^{M,N})} \right) \, dr, F(Y_{[s]}^{M,N}) - F(Y_{[s]}^{M,N}) \right) \, ds \\
:= J_{11} + J_{12} + J_{13}.
$$

(4.64)

Since the estimates of $\mathbb{E}[J_{11}]$ and $\mathbb{E}[J_{12}]$ are demanding, we handle the item $\mathbb{E}[J_{13}]$ first. Utilizing (2.4), the Hölder inequality, (4.41) and (4.48) results in

$$
\mathbb{E}[J_{13}] \leq p \mathbb{E} \int_0^t \int_0^s \|e_{s,N}^{M,N}\|^{p-2} \cdot \tau [\mathbb{E} [F(Y_{[r]}^{M,N})]^2 \cdot \|F(Y_{[s]}^{M,N}) - F(Y_{[s]}^{M,N})\|] \, drds \\
\leq C \tau \mathbb{E} \int_0^t \int_0^s \|e_{s,N}^{M,N}\|^{p-2} \|F(Y_{[r]}^{M,N})\|^2 (1 + \|Y_{[s]}^{M,N}\|^2 + \|Y_{[s]}^{M,N}\|^2 \|Y_{s,N}^{M,N} - Y_{[s]}^{M,N}\|) \, drds \\
\leq C \int_0^t \mathbb{E} [\|e_{s,N}^{M,N}\|^p] \, ds + C \tau^p \int_0^t \mathbb{E} \left[ \left( \int_0^s \|F(Y_{[r]}^{M,N})\|^2 (1 + \|Y_{[s]}^{M,N}\|^2 \|Y_{s,N}^{M,N} - Y_{[s]}^{M,N}\|) \, dr \right)^{\frac{p}{2}} \right] \, ds \\
\leq C \int_0^t \mathbb{E} [\|e_{s,N}^{M,N}\|^p] \, ds + C \tau^p (1 + \frac{2}{7}).
$$

(4.65)
At the moment we come to the estimate of $\mathbb{E}[J_{11}]$ and use the Taylor formula, the self-adjointness of operators $F'(u)$ and $P_N$ to infer that

$$
\mathbb{E}[J_{11}] = p \mathbb{E} \int_0^t \left\| e_s^{M,N} \right\|^{p-2} \left( \int_0^s E(s-r) (F_N(X(r)) - F_N(Y_r^{M,N})) \, dr \right) \, F(Y_r^{M,N}) - F(Y_{[s]}^{M,N}) \, ds
$$

$$
= p \mathbb{E} \int_0^t \left\| e_s^{M,N} \right\|^{p-2} \left( \int_0^s E(s-r) P_N \int_0^1 \left[ \sigma \right] \right) \, X(r) - Y_r^{M,N} \, dr \, F(Y_r^{M,N}) - F(Y_{[s]}^{M,N}) \, ds
$$

$$
= p \mathbb{E} \int_0^t \int_0^s \int_0^1 \left\| e_s^{M,N} \right\|^{p-2} \left( X(r) - Y_r^{M,N} \right) \, F'(Y_r^{M,N} + \sigma(X(r) - Y_r^{M,N})) \, d\sigma \, dr \, ds
$$

Further, employing Young’s inequality, Hölder’s inequality and the transformation of integral domain and taking $\frac{1}{p} < \eta < 1$ give

$$
\mathbb{E}[J_{11}] \leq C \mathbb{E} \int_0^t \int_0^s (s-r)^{-\frac{\eta}{2}} \left\| e_s^{M,N} \right\|^{p} \, dr \, ds + C \mathbb{E} \int_0^t \int_0^s (s-r)^{-\frac{\eta}{2}} \left\| X(r) - Y_r^{M,N} \right\|^{\frac{2}{p}} \, dr \, ds
$$

$$
\times \left( 1 + \left\| X(r) \right\|^{2p} + \left\| Y_r^{M,N} \right\|^{2p} \right) \left( 1 + \left\| X(r) \right\|^{2p} + \left\| Y_r^{M,N} \right\|^{2p} \right) \, ds
$$

$$
\leq C \int_0^t \mathbb{E} \left[ \left\| e_s^{M,N} \right\|^{p} \right] \, ds + C \int_0^t \mathbb{E} \left[ \left\| X(s) - Y_s^{M,N} \right\|^{p} \right] \, ds
$$

$$
+ C \int_0^t \mathbb{E} \left[ \left\| X(s) - Y_s^{M,N} \right\|^{p} \right] \, ds
$$

$$
\leq C \int_0^t \mathbb{E} \left[ \left\| e_s^{M,N} \right\|^{p} \right] \, ds + C \int_0^t \mathbb{E} \left[ \left\| X(s) - Y_s^{M,N} \right\|^{p} \right] \, ds
$$

$$
+ C \int_0^t \mathbb{E} \left[ \left\| X(s) - Y_s^{M,N} \right\|^{p} \right] \, ds
$$

To proceed further, we resort to Corollary 4.11 as well as (2.12), (4.41) and achieve

$$
\mathbb{E}[J_{11}] \leq C \int_0^t \mathbb{E} \left[ \left\| e_s^{M,N} \right\|^{p} \right] \, ds + C \lambda_{N+1}^{-\eta/2} + C \int_0^t \left( \mathbb{E} \left[ \left\| A^{-\frac{\eta}{2}} \left( F(Y_s^{M,N}) - F(Y_{[s]}^{M,N}) \right) \right\|^{2p} \right] \right)^{\frac{1}{2}} \, ds
$$

$$
\leq C \int_0^t \mathbb{E} \left[ \left\| e_s^{M,N} \right\|^{p} \right] \, ds + C \left( \frac{1}{N} \right)^{p\beta} + C \tau^{p\beta}. \tag{4.68}
$$
Finally, it remains to deal with the estimate of $E[J_{12}]$. By the Hölder inequality and putting $\frac{1}{2} < \eta < 1$ one can derive that

$$E[J_{12}] = pE \int_0^t \|e^M_N\|^{p-2} \left( \int_0^s A^\frac{\eta}{2} \left( F_N(Y^M_s) - F_N(Y^M_{[s]}) \right) \right) ds$$

$$\leq pE \int_0^t \int_0^s \|e^M_N\|^{p-2} \cdot C(s-r)^{-\eta} \|A^{-\frac{\eta}{2}}(F(Y^M_s) - F(Y^M_{[s]}))\|$$

$$\times \|A^{-\frac{\eta}{2}}(F(Y^M_s) - F(Y^M_{[s]}))\| ds$$

$$\leq C \int_0^t \int_0^s (s-r)^{-\eta} E[\|e^M_N\|^p] ds + C E \int_0^t \int_0^s (s-r)^{-\eta} \|A^{-\frac{\eta}{2}}(F(Y^M_s) - F(Y^M_{[s]}))\|^\frac{p}{2}$$

$$\times \|A^{-\frac{\eta}{2}}(F(Y^M_s) - F(Y^M_{[s]}))\|^\frac{p}{2} dr ds$$

$$\leq C \int_0^t E[\|e^M_N\|^p] ds + C \int_0^t \int_0^s (s-r)^{-\eta} E[\|A^{-\frac{\eta}{2}}(F(Y^M_s) - F(Y^M_{[s]}))\|^p] dr ds$$

$$+ C \int_0^t \int_0^s (s-r)^{-\eta} E[\|A^{-\frac{\eta}{2}}(F(Y^M_s) - F(Y^M_{[s]}))\|^p] dr ds. \quad (4.69)$$

Again, the use of Corollary 4.10 leads us to

$$E[J_{12}] \leq C \int_0^t E[\|e^M_N\|^p] ds + C \tau^{\beta p}, \quad (4.70)$$

which together with (4.63), (4.68) forces us to recognize from (4.64) that

$$E[J_1] \leq C \int_0^t E[\|e^M_N\|^p] ds + C \left( \frac{1}{\tau^2} \right)^{p \beta} + C \tau^{\beta p}. \quad (4.71)$$

Plugging this and (4.62), (4.63), into (4.61) and applying the discrete version of the Gronwall inequality gives the desired error bound. $\square$

## 5 Numerical experiments

Some numerical experiments are performed in this section to test previous theoretical findings. Consider the stochastic Allen-Cahn equation with additive space-time white noise, described by

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u - u^3 + \dot{W}(t), & t \in (0, 1), \ x \in (0, 1), \\
u(0, x) = \sin(\pi x), & x \in (0, 1), \\
u(t, 0) = \nu(t, 1) = 0, & t \in (0, 1). \end{cases} \quad (5.1)$$

Here $\{W(t)\}_{t \in [0, T]}$ is a cylindrical $I$-Wiener process represented by (2.6). In what follows, we will use the new full discrete scheme (4.1) to approximate the continuous problem (5.1). Error bounds are always measured in terms of mean-square approximation errors at the endpoint $T = 1$, caused
by spatial and temporal discretizations and the expectations are approximated by computing averages over 1000 samples.

Before proceeding further with numerical simulations, it is helpful to mention that the stochastic convolution in the scheme (5.1) is easily implementable once one realize that \[
\int_{t_m}^{t_{m+1}} E_N(t_{m+1} - s) P_N dW(s) = \sum_{i=1}^{N} \Lambda_i e_i,
\]
where \(\Lambda_i = \int_{t_m}^{t_{m+1}} e^{-(t_{m+1} - s)} \lambda_i d\beta(s), 1 \leq i \leq N\) are independent, zero-mean normally distributed random variables with explicit variances \(E[|\Lambda_i|^2] = \frac{1-e^{-2\lambda_i \tau}}{2\lambda_i}\). For more details on the implementation of so-called AEE schemes, one can consult [26, section 3] and [45, section 4.1].

To visually inspect the convergence rates in space, we identify the “exact” solution by using the full discretization with \(M_{\text{exact}} = N_{\text{exact}} = 2^{11} = 2048\). The spatial approximation errors \(\|X(1) - X^N(1)\|_{L^2(\Omega;H)}\) with \(N = 2^i, i = 2, 3, ..., 7\) are depicted in Fig.1 against \(\frac{1}{N}\) on a log-log scale, where one can observe that the resulting spatial errors decrease at a slope close to \(1/2\). This is consistent with the previous theoretical result (3.10).

![Mean-square errors of the spectral Galerkin method](image)

Figure 1: The convergence rate of the spectral Galerkin spatial discretization.

Moreover, we attempt to illustrate the error bound (4.58) for the full discrete scheme (5.1). As implied by (4.58), the convergence rate in space is identical to that in time. Consequently, we take \(M = N, p = 2, \beta = \frac{1}{2} - \epsilon\) with arbitrarily small \(\epsilon > 0\) in (4.58) to arrive at

\[
\|X(1) - Y_{t,N}^{N,N}\|_{L^2(\Omega;H)} \leq C \epsilon N^{-\frac{1}{2} + \epsilon}.
\]

(5.2)

To see (5.2), we, similarly as above, do a full discretization on a very fine mesh with \(M_{\text{exact}} = N_{\text{exact}} = 2^{11} = 2048\) to compute the “exact” solution. Six different mesh parameters \(N = 2^i, i = 2, 3, ..., 7\) are then used to get six full discretizations. The resulting errors are listed in Table I and plotted in Fig.2 on a log-log scale. From Fig.2, one can observe the expected convergence rate of order almost \(\frac{1}{2}\), which agrees with that indicated in (5.2).
Table 1: Computational errors of the full discrete scheme with $M = N$

| $N = 2^k$ | $N = 2^k$ | $N = 2^k$ | $N = 2^k$ | $N = 2^k$ | $N = 2^k$ |
|-----------|-----------|-----------|-----------|-----------|-----------|
| 0.106381  | 0.077172  | 0.055174  | 0.039209  | 0.027624  | 0.019225  |

Figure 2: The convergence rate of the space-time full discretization.

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