Entangled photons from the polariton vacuum in a switchable optical cavity

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We study theoretically the entanglement of two-photon states in the ground state of the intersubband cavity system, the so-called polariton vacuum. The system consists of a sequence of doped quantum wells located inside a microcavity and the photons can interact with intersubband excitations inside the quantum wells. Using an explicit solution for the ground state of the system, operated in the ultrastrong coupling regime, a post-selection is introduced, where only certain two-photon states are considered and analyzed for mode entanglement. We find that a fast quench of the coupling creates entangled photons and that the degree of entanglement depends on the absolute values of the in-plane wave vectors of the photons. Maximally entangled states can be generated by choosing the appropriate modes in the post-selection.

I. INTRODUCTION

With the advent of quantum information theory, the phenomenon of entanglement not only remained a mysterious feature of quantum mechanics but became a resource to perform tasks that are not feasible with classical resources. Examples are quantum communication protocols, which make use of entangled states like quantum key distribution or superdense coding, or the realization of a quantum repeater.

Entangled photon states are often used to implement the protocols mentioned above. Today, there exist several different proposals for the production of bipartite entangled photon states, most prominently type-II parametric down-conversion and biexciton decay in a quantum dot.

The fundamental requirements for such a photon-pair source to be used in quantum information processing are that the states have to possess a sufficient amount of entanglement and that the production of the two photons has to be deterministic and efficient. Determinism means that the release of the photons can be triggered by some external control parameter. Efficiency means that the probability for this event is near unity.

Here, we study the intersubband cavity system, for which the emission of correlated photon pairs was predicted theoretically and can be triggered by modulating the light-matter interaction between microcavity photons and electronic excitations in the quantum wells. Those intersubband transitions are mainly used in quantum well infrared photodetectors and quantum cascade lasers. Embedded in a microcavity, it is possible to reach a regime of ultrastrong light-matter coupling, in which the vacuum-field Rabi frequency can be of the order of the intersubband transition frequency and the ground state of the system, a squeezed vacuum, contains already a non-zero number of photons. Another type of system, which can reach the ultrastrong coupling regime as well, are superconducting circuits, where the emission of quantum vacuum radiation was just recently demonstrated.

In this paper, we analyze the ground state of the intersubband cavity system, the so-called polariton vacuum, related to two-photon entanglement. We use an explicit expression for the polariton vacuum and, after post-selecting certain photonic states, quantify the mode entanglement between the photon pairs via the concurrence.

II. THE INTERSUBBAND CAVITY SYSTEM

The intersubband cavity system was studied theoretically by Ciuti et al. It consists of $n_{\text{QW}}$ identical quantum wells embedded inside a semiconductor optical microcavity. The quantum wells are assumed to be negatively charged with a two-dimensional electron gas (2DEG) with density $N_{\text{2DEG}}$ that populates the first subband. We consider the interaction of intersubband excitations between the two lowest subbands and photons of the fundamental cavity mode.

The ultrastrong coupling regime, in which the vacuum-field Rabi splitting is of the order of the intersubband transition energy, can be reached due to the large dipole moment of intersubband transitions and the collective coupling to all electrons of the 2DEGs. In this regime, the rotating wave approximation is not valid anymore and the full light-matter interaction Hamiltonian including...
FIG. 2. Subband energy structure of a quantum well (QW) formed in a semiconductor heterostructure. (a) In real space, along the growth direction $z$ of the structure, the semiconductors forming the QW are denoted as A and B. The QW contains a two-dimensional electron gas (2DEG). Here, we study intersubband transitions between the first two subbands $n = 1$ and $n = 2$ with transition energy $\hbar \omega_{12}$. (b) The same situation in $k$ space, where the 2DEG populates states up to the Fermi energy $E_F$, with a corresponding wave vector $k_F$.

the antiresonant terms has the form\(^{10}\)

$$H = H_0 + H_{\text{res}} + H_{\text{anti}},$$

which consists of the three terms

$$H_0 = \sum_k \hbar \omega_c(k) \left( a_k^{\dagger} a_k + \frac{1}{2} \right) + \sum_k \hbar \omega_{12} b_k^{\dagger} b_k,$$

$$H_{\text{res}} = \sum_k i\hbar \Omega_{\text{R}}(k) \left( a_k^{\dagger} b_k - a_k b_k^{\dagger} \right) + \sum_k \hbar D(k) \left( a_k^{\dagger} a_k + a_k^{\dagger} a_k + a_k a_k^{\dagger} \right),$$

$$H_{\text{anti}} = \sum_k i\hbar \Omega_{\text{R}}(k) \left( a_k^{\dagger} b_{-k} - a_k b_{-k}^{\dagger} \right) + \sum_k \hbar D(k) \left( a_k^{\dagger} a_{-k} + a_k a_{-k}^{\dagger} \right),$$

where $H_0$ describes the uncoupled photonic and electronic systems, $H_{\text{res}}$ is the resonant part of the light-matter interaction and the antiresonant terms, usually neglected under the rotating wave approximation, are given by $H_{\text{anti}}.$

The operator $a_k^{\dagger}$ annihilates (creates) a cavity photon with in-plane wave vector $\mathbf{k} = (k_x, k_y)$ and transverse-magnetic (TM) polarization. The reason why only this photon polarization is considered, is the selection rule for intersubband transition.\(^{11}\) The dipole moment points along the growth direction, so the exciting radiation must have a finite electric field component in $z$-direction. As can be seen from Fig. 1, TM-polarized light where the magnetic field component is perpendicular to the plane of incidence, has an electric field component in $z$-direction when it propagates under a finite angle $\theta$. However, if the polarization is not purely transverse-magnetic, it is still sufficient to consider only one polarization direction, since only the TM-polarized part of the radiation is absorbed.

The only bright collective intersubband excitation with in-plane wave vector $\mathbf{k}$ is described by the operators $b_{\mathbf{k}}^{\dagger}$ that fulfill Bose commutation relations $[b_{\mathbf{k}}, b_{\mathbf{k}'}^{\dagger}] \simeq \delta_{\mathbf{k}, \mathbf{k}'}$ in the weak excitation regime, i.e. when the number of intersubband excitations is much less than the number of electrons forming the 2DEG\(^{10,23}\).

The dispersion of the fundamental cavity mode $\omega_c(k)$ is given by

$$\omega_c(k) = \frac{c}{\sqrt{\varepsilon}} \sqrt{k^2 + k_z^2},$$

where $c$ is the speed of light, $\varepsilon$ is the dielectric constant of the material used as cavity spacer and the quantization of $k_z$ can depend in a complicated way on the boundary conditions. In the following, $k = |\mathbf{k}|$ is the length of the in-plane wave vector.

The vacuum Rabi frequency $\Omega_R(k)$ for the intersubband cavity system is given by\(^{10,24}\)

$$\Omega_R(k) = \left( \frac{\alpha^2 N_{\text{2DEG}} a_{\text{eff}}^{\text{QW}} 1/12 \omega_{12}}{2\varepsilon_0 \varepsilon_0 L_{\text{eff}}^{\text{QW}} \sin^2(\theta(k))} \right)^{1/2}.$$ (6)

Here $e$ is the elementary charge, $\varepsilon_0$ is the vacuum permittivity and $m_0$ is the free electron mass. $L_{\text{eff}}^{\text{QW}}$ denotes an effective cavity thickness that depends on the type of cavity mirrors and $n_{\text{QW}}$ is an effective number of embedded quantum wells since not all quantum wells are equally coupled to the photon field. $f_{12}$ is the oscillator strength of the transition between the subbands. For a deep rectangular well, one can use the approximation $f_{12} \simeq n_{\text{QW}}/m^* (m^*$: effective electron mass)\(^{10,11}\).

Finally, $\theta(k)$ is the propagation angle of a cavity photon: $\sin(\theta(k)) = k/\sqrt{k^2 + k_z^2}$. The dispersive coupling parameter $D(k)$ can be approximated by $D(k) \simeq \Omega_R(k) / \omega_{12}$, which is valid for deep rectangular wells and exact for parabolic well potential\(^{20}\).

The Hamiltonian (1) can be diagonalized with an extended Bogoliubov transformation\(^{23}\) also known as Hopfield transformation\(^{23}\) where new bosonic operators

$$p_{j,k} = w_j(k) a_k + x_j(k) b_k + y_j(k) a_k^{\dagger} + z_j(k) b_k^{\dagger}$$

are introduced that describe a quasiparticle called intersubband cavity polaron\(^{29}\) and $j$ indicates weather it belongs to the lower ($j = \text{LP}$) or upper ($j = \text{UP}$) polariton branch. The wave vectors $\mathbf{k}$ are still meant to be in-plane. By an appropriate choice of the Hopfield coefficients $w_j(x_j, y_j) (z_j)$, which are already taken to depend only on $k$, the Hamiltonian becomes diagonal

$$H = E_G + \sum_{j \in \{\text{LP, UP}\}} \sum_k \hbar \omega_j(k) p_{j,k} p_{j,k}^{\dagger}.$$ (8)

Here, $E_G$ denotes the ground state energy. The resulting lower and upper polaritons dispersion are shown...
and no intersubband excitations present. Without knowing the explicit form of the polariton ground state \( |G\rangle \), we assume GaAs/AlGaAs quantum wells, which have been commonly used experimentally\(^{16,29–31}\). Hence, the material-dependent parameters are \( f_{12} = 14.9 \) (\( m_{GaAs}^* = 0.067 \) and \( \varepsilon = 10 \)). Furthermore, the number of embedded quantum wells \( n_{QW} \), the length of the microcavity \( L_c \), and the intersubband transition frequency \( \omega_{12} \), which is determined by the quantum well depth and thickness, can be adjusted during the manufacturing process. The density of the two-dimensional electron gas can be varied experimentally. To obtain the results of Fig. 3, we chose \( n_{QW} = 50 \), \( L_c = 2 \mu m \), \( \omega_{12} = 150 \) meV and \( N_{2DEG} = 10^{12} \) cm\(^{-2} \) as one particular set of experimentally reasonable values of the parameters mentioned above.

The ground state of the intersubband cavity system, operating in the ultrastrong coupling regime, is not the ordinary vacuum \( |0\rangle \), in which there are no cavity photons and no intersubband excitations present

\[
a_k |0\rangle = b_k |0\rangle = 0,
\]

but a state \( |G\rangle \) that exhibits no intersubband cavity polaritons

\[
p_j,k |G\rangle = 0, \quad j \in \{\text{LP, UP}\}.
\]

Without knowing the explicit form of \( |G\rangle \), one can show that the ground state has some peculiar properties in the ultrastrong coupling regime that were worked out in Ref.\(^{10}\) whereof the essential ones are that it contains a finite number of photons:

\[
\langle G|a_k^+ a_k|G\rangle = |y_{LP}(k)|^2 + |y_{UP}(k)|^2,
\]

and photons with opposite in-plane wave vectors \( k \) and \(-k\) are correlated:

\[
\langle G|a_k^+ a_{-k}|G\rangle = -w_{LP}^*(k)y_{LP}(k) - w_{UP}^*(k)y_{UP}(k).
\]

One can see that only if the light-matter interaction is so strong that the Hopfield coefficients \( y_{LP}(k) \) and \( y_{UP}(k) \) are reasonably large, \( |y_{j}(k)|^2 \sim 0.1 \) (i.e. when the antiresonant terms\(^{3}\) cannot be neglected and therefore the extended Bogoliubov transformation\(^{3}\) is necessary), the ground state \( |G\rangle \) differs significantly from the vacuum state \( |0\rangle \).

The idea is now that the correlations\(^{12}\) can lead to entanglement of two photons propagating in opposite directions. These photons are, however, virtual excitations, but it is conjectured\(^{15,13}\) that they can be released by a non-adiabatic switch-off (quench) of the vacuum Rabi frequency \( \Omega_k \). An experimental approach to this scenario is an ultrafast change of the density \( N_{2DEG} \) of two-dimensional electron gas\(^{16,33,11}\). One mechanism to achieve a modulation of the parameter \( N_{2DEG} \) is a gate voltage, which can lead to the depletion of the QW\(^{39}\). The rapidity is restricted by capacitance of the gates, however. Another implementation uses two asymmetrically coupled QWs, in which one QW can be charged by electron tunneling and this process can happen on the picosecond time scale or faster\(^{11}\). A promising idea to achieve an ultrafast coupling modulation is an all-optical control scheme\(^{10}\), in which electrons from the valence band are resonantly excited to the first subband by a femtosecond laser pulse. In this manner, it could be demonstrated experimentally that the coupling between the cavity photon field and the intersubband transitions in the quantum wells can be switched on in a time shorter than a cycle of light in the microcavity. In Ref.\(^{32}\) and \(^{33}\) the spectrum of the radiation exiting the cavity was derived in more detailed calculations, when a time-dependent coupling \( \Omega_k(k,t) \) is predominant in the system and it is predicted that the vacuum radiation rises above the black body radiation.

III. EXACT GROUND STATE

A pioneering calculation of the polariton ground state of a bulk dielectric was given by Quattropani et al\(^{28}\). The solution is given by independent photon and polarization states. Since the Hamiltonian of the intersubband cavity system is similar to the one in Ref.\(^{28}\) we use their treatment to determine the explicit form of the polariton vacuum \( |G\rangle \) being the ground state of the Hamiltonian\(^{1}\) of the intersubband cavity system. The difference is just that the sums in\(^{1}\) cover all in-plane wave vectors and the intersubband transition frequency \( \omega_{12} \) is taken to be dispersionless.
The ansatz for the polariton vacuum \(|G\rangle\) is:

\[
|G\rangle = \frac{1}{N} e^{\frac{3}{2} \sum_k G(k) (a^+_k a_k + b^+_k b_k) + F(k) (a^+_k b^+_k + b_k a_k)} |0\rangle.
\]

(13)

\(N\) is a normalization constant and the expansion coefficients \(G(k)\) and \(F(k)\) have to be determined in order to satisfy the definition of the polariton vacuum \(|10\rangle\):

\[
p_{j,k} |G\rangle = 0, \quad j \in \{LP, UP\}.
\]

We anticipate that the functions \(G(k)\) and \(F(k)\) will only depend on the absolute value of the in-plane wave vector \(k\). After some algebra using commutation relations, which is explicitly given in Ref. [28], the action of \(a_k\) and \(b_k\) on \(|G\rangle\) is, e.g.:

\[
a_k |G\rangle = (G(k)a^+_k + F(k)b^+_k) |G\rangle, \quad (14)
\]

\[
b_k |G\rangle = (G(k)b^+_k + F(k)a^+_k) |G\rangle. \quad (15)
\]

Inserting (14) and (15) into the definitions (7) and (10), one obtains a system of equations for the coefficients \(G(k)\) and \(F(k)\):

\[
w_j(k)G(k) + x_j(k)F(k) + y_j(k) = 0, \quad (16)
\]

\[
w_j(k)F(k) + x_j(k)G(k) + z_j(k) = 0, \quad (17)
\]

which has the solutions:

\[
G(k) = \frac{x_j(k)z_j(k) - w_j(k)y_j(k)}{w_j^2(k) - x_j^2(k)}, \quad (18)
\]

\[
F(k) = \frac{x_j(k)y_j(k) - w_j(k)z_j(k)}{w_j^2(k) - x_j^2(k)}. \quad (19)
\]

That this can be fulfilled simultaneously by the Hopfield coefficients of the lower and upper polariton, can be seen from the following relations [27, 28]:

\[
w_{LP}(k) = x_{UP}(k), \quad x_{LP}(k) = w_{UP}(k),
\]

\[
y_{LP}(k) = z_{UP}(k), \quad z_{LP}(k) = y_{UP}(k). \quad (20)
\]

By inserting the explicit expressions of the Hopfield coefficients, the expansion coefficients can be rewritten as [28]:

\[
G(k) = \frac{\omega_{12} + \omega_k - \omega_{LP}(k) - \omega_{UP}(k)}{\omega_{12} - \omega_k - \omega_{LP}(k) - \omega_{UP}(k)}, \quad (21)
\]

\[
F(k) = -\frac{\omega_{12}}{\Omega_R(k)}. \quad (22)
\]

Finally, the polariton vacuum \(|G\rangle\) is calculated to be

\[
|G\rangle = \frac{1}{N} e^{\frac{3}{2} \sum_k G(k) (a^+_k a_k + b^+_k b_k - 2\frac{\omega_{12}}{\Omega_R(k)} a^+_k b^+_k)} |0\rangle,
\]

(23)

because the last two terms in the exponential can be combined, and with the normalization \(N\) given by

\[
N = \prod_k \left( |w_{LP}(k)|^2 + |x_{LP}(k)|^2 \right)^{\frac{1}{2}}.
\]

IV. PHOTON ENTANGLEMENT

As seen in the previous section, the intersubband cavity system contains a finite number of photons if it is in the ultrastrong coupling regime. One possibility of a photon pair in the ground state \(|G\rangle\) to be entangled will be studied below. After specifying the type of entanglement, it is quantified by the concurrence [24, 25].

A. Mode entanglement

We first define the type of photon entanglement. Since the transverse-magnetic polarization of the interacting photons is fixed by the selection rule for intersubband transitions [11], polarization entanglement as achieved with parametric down-conversion or the biexciton decay is out of the question. But there exist anomalous correlations between photons with opposite in-plane momentum [12].

\[
\langle G|a_{k\rightarrow k} G\rangle = -w^*_{LP}(k)y_{UP}(k) - w^*_{UP}(k)y_{LP}(k).
\]

Our idea is to test the photonic states in \(|G\rangle\) for mode or frequency entanglement [29].
will be described in the following and could be realized if we choose \( k \) the same absolute value of the in-plane wave vector. I.e. each subsystem \( L \) and \( R \), respectively, that have, however allowed modes. We only consider two different modes in this section, the post-selection is even more restrictive in terms of requirements, in linear order are

\[
|G(2)\rangle = 1 \text{ and the operator } \mathcal{T}_k \text{ is}
\]

\[
\mathcal{T}_k = a_k^\dagger a_{-k} + b_k^\dagger b_{-k} - 2i \frac{\omega_{12}}{\Omega_R(k)} a_k^\dagger b_{-k}.
\]

Thus, the two-photon states with opposite in-plane wave vector, i.e. the states that fulfill the post-selection requirements, in linear order are

\[
a_k^\dagger a_{-k}^\dagger |0\rangle_a = |k\rangle_L |k\rangle_R,
\]

\[
a_q^\dagger a_{-q}^\dagger |0\rangle_a = |q\rangle_L |q\rangle_R,
\]

and in second order

\[
a_k^\dagger a_{-k}^\dagger b_k^\dagger b_{-k} |0\rangle_b = |k\rangle_L |k\rangle_R \otimes b_k^\dagger b_{-k} |0\rangle_b,
\]

\[
a_q^\dagger a_{-q}^\dagger b_q^\dagger b_{-q} |0\rangle_b = |q\rangle_L |q\rangle_R \otimes b_q^\dagger b_{-q} |0\rangle_b,
\]

\[
a_k^\dagger a_{-k}^\dagger b_q^\dagger b_{-q} |0\rangle_b = |k\rangle_L |q\rangle_R \otimes b_q^\dagger b_{-q} |0\rangle_b,
\]

\[
a_q^\dagger a_{-q}^\dagger b_k^\dagger b_{-k} |0\rangle_b = |q\rangle_L |k\rangle_R \otimes b_k^\dagger b_{-k} |0\rangle_b.
\]

Here, the explicit expression of the tensor product \( |0\rangle_a \otimes |0\rangle_b \) of the individual vacuum states for photons and intersubband excitations is used and \( k' \) can be an arbitrary in-plane wave vector.

We carry out the post-selection by projecting onto these states with a projection operator \( \mathcal{P}_{LR} \),

\[
|\psi_{LR}\rangle = \frac{1}{N_{LR}} \mathcal{P}_{LR} |G(2)\rangle,
\]

with \( N_{LR} \) being a necessary normalization constant, since the operation is a projection. As an intermediate result, we obtain the pure state \( |\psi_{LR}\rangle \) in which all the two-photon states fulfilling the conditions of the post-selection are extracted. We give an explicit expression for \( |\psi_{LR}\rangle \) in Appendix A.

Hence, one possible product basis of \( \mathcal{H}_{LR} \) is

\[
|k\rangle_L \otimes |q\rangle_R,
\]

\[
|k\rangle_L \otimes |q\rangle_R,
\]

\[
|q\rangle_L \otimes |k\rangle_R,
\]

\[
|q\rangle_L \otimes |k\rangle_R.
\]

For all further calculations, the polariton vacuum \( |G\rangle \) is expanded in the second order in the small expansion coefficient \( G(k) \),

\[
|G\rangle = \frac{1}{N} e^{\frac{i}{2} \sum_k G(k) (a_k^\dagger a_{-k} + b_k^\dagger b_{-k} - 2i \frac{\omega_{12}}{\Omega_R(k)} a_k^\dagger b_{-k})} |0\rangle
\]

\[
\approx \frac{1}{N} \left[ 1 + \frac{1}{2} \sum_k G(k) \mathcal{T}_k^\dagger + \frac{1}{8} \left( \sum_k G(k) \mathcal{T}_k^\dagger \right)^2 \right] |0\rangle
\]

\[
\equiv |G(2)\rangle,
\]

where \( \tilde{N} \) is a new normalization constant to preserve \( \langle G(2)|G(2)\rangle = 1 \) and the operator \( \mathcal{T}_k \) is

\[
\mathcal{T}_k = a_k^\dagger a_{-k} + b_k^\dagger b_{-k} - 2i \frac{\omega_{12}}{\Omega_R(k)} a_k^\dagger b_{-k}.
\]

The post-selection needs to fulfill the following requirements: first, we only allow for states in which two photons with opposite in-plane wave vectors appear. In addition, the post-selection is even more restrictive in terms of allowed modes. We only consider two different modes in each subsystem \( L \) and \( R \), respectively, that have, however the same absolute value of the in-plane wave vector. I.e. we choose \( k \) and \( q \), with \( k \neq q \), and consider the modes \( k = (k, 0) \) and \( -k = (-k, 0) \) and accordingly \( q = (q, 0) \) and \( -q = (-q, 0) \), where all the wave vectors point along the \( x \)-direction. So the basis states of \( \mathcal{H}_L \) are

\[
|k\rangle_L = a_{-k}^\dagger |0\rangle_a,
\]

\[
|q\rangle_L = a_{-q}^\dagger |0\rangle_a,
\]

where \( |0\rangle_a \) is the photon vacuum. The basis of \( \mathcal{H}_R \) is

\[
|k\rangle_R = a_k^\dagger |0\rangle_a,
\]

\[
|q\rangle_R = a_q^\dagger |0\rangle_a.
\]

B. Post-selection

FIG. 5. Two photons with different frequencies (colors) leaving the cavity in opposite directions. The two subsystems left (L) and right (R) are defined via the sign of \( k_x \). Since there is a difference in frequency, the photons have different in-plane wave vectors.
Here, the matrix representation is in the basis and the abbreviations

\[
\begin{align*}
X(k, q) &= |F(k)|^2 |F(q)|^2, \\
Y(k, q) &= G(k)G(q) \times \left[ 1 + \frac{1}{2} S \right] - |F(k)|^2 - |F(q)|^2, \\
Z(k) &= X(k, k) + Y(k, k),
\end{align*}
\]  

(41)

(42)

(43)

where introduced and \( S \) is the sum over all expansion coefficients squared

\[
S = \sum_{k'} G^2(k').
\]  

(44)

The value of \( S \) (see Appendix B) depends on the sample area \( A \). In the limit of \( A \gg \frac{2\pi}{\omega_{12}} \), one can take the limit \( S \to \infty \) and obtains

\[
\varrho_{S}^{\infty} \to \frac{1}{G^2(k) + G^2(q)} \left( \begin{array}{ccc} G^2(k) & 0 & 0 \\ 0 & 0 & 0 \\ G(k)G(q) & 0 & 0 \end{array} \right).
\]  

(45)

This corresponds to the pure photon state

\[
|\psi^{\infty}\rangle = \frac{1}{\sqrt{G^2(k) + G^2(q)}} \left( \begin{array}{c} (G(k) |k\rangle_L |k\rangle_R + G(q) |q\rangle_L |q\rangle_R \end{array} \right).
\]  

(46)

\section{Measure of entanglement}

The state \( \varrho_{S}^{\infty} \), which we derived from \( |G^{(2)}\rangle \), describes two photons that propagate with opposite in-plane wave vectors in the microcavity and that can potentially be released by an appropriate time-modulation (quench) of the Rabi frequency \( \Omega_{R}(k) \). Since one chooses the modes \( k \) and \( -k \) and accordingly \( q \) and \( -q \) via the post-selection, the photons effectively form a two-qubit system. For such a system, the entanglement for mixed states can be calculated analytically without evaluating a convex roof explicitly from the density matrix by way of the concurrence \( C(\varrho_{S}^{\infty}) \). With this function, the so-called entanglement of formation \( E_{F}(\varrho_{S}^{\infty}) \) of two qubits can be easily calculated via

\[
E_{F}(\varrho_{S}^{\infty}) = h \left( 1 + \sqrt{1 - C^2(\varrho_{S}^{\infty})} \right),
\]  

(47)

with the binary entropy

\[
h(x) = -x \log_2(x) - (1-x) \log_2(1-x).
\]  

(48)

The concurrence itself is given by

\[
C(\varrho_{S}^{\infty}) = \max \{ 0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 \}.
\]  

(49)

Here, \( \lambda_1 \) to \( \lambda_4 \) are, in decreasing order, the square roots of the eigenvalues of the matrix \( \rho \tilde{\sigma} \) and \( \tilde{\sigma} \) is a transformation of the density matrix given by

\[
\tilde{\rho} = (\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y),
\]  

(50)

where \( \sigma_y \) is the Pauli \( y \) matrix and the \( * \) denotes complex conjugation.

\subsection{Analytical results}

For the photonic state \( \varrho_{S}^{\infty} \), the parameters \( \lambda_1 \) to \( \lambda_4 \) are found to be

\[
\begin{align*}
\lambda_1 &= \frac{1}{N_{LR}^2} \left( \sqrt{Z(k)Z(q) + Y(k, q)} \right), \\
\lambda_2 &= \frac{1}{N_{LR}^2} \left( \sqrt{Z(k)Z(q) - Y(k, q)} \right), \\
\lambda_3,4 &= \frac{1}{N_{LR}^2} X(k, q).
\end{align*}
\]  

(51)

(52)

(53)

Hence from (49), we obtain in connection with (41)-(43) for the concurrence

\[
C(\varrho_{S}^{\infty}) = C(k, q) = \frac{2}{N_{LR}^2} \left[ G(k)G(q) \right] \left[ 1 + \frac{1}{2} S \right] - |F(k)|^2 - |F(q)|^2 \right] - |F(k)|^2 |F(q)|^2.
\]  

(54)

The concurrence thus only depends on the absolute values \( k \) and \( q \) via the expansion coefficients, which were given in (21) and (22).

\[
G(k) = \frac{\omega_{12} + \omega_{c}(k) - \omega_{LP}(k) - \omega_{UP}(k)}{\omega_{12} - \omega_{c}(k) - \omega_{LP}(k) - \omega_{UP}(k)},
\]  

\[
|F(k)| = \frac{\omega_{12}}{\Omega_{R}(k)} G(k).
\]  

To show the dependence of the concurrence on \( k \) and \( q \), we have to evaluate the sum \( S \) explicitly. Using a sample area \( A = (200 \, \mu m)^2 \), we find \( S = 0.716 \). The result is presented in Fig. 7 for the same parameters as used before, where \( C(k, q) \) is shown in a density plot as a function of the modes \( k \) and \( q \). Below it, we show cuts for different values of \( q \) to illustrate the depandency of the concurrence better. One can observe two branches of high entanglement that appear for large values of \( k \) and/or \( q \). Their appearance can be explained by the characteristics of the expansion coefficient \( G(k) \) for large \( k \). \( G(k) \) tends to zero as \( 1/k^2 \), hence \( |F(k)|^2 \) scales as \( 1/k^3 \). Hence for
modes are far from each other. We give a more precise analysis of expression (55) in the next subsection, where the limit of large sample areas is worked out.

2. Large-cavity limit

The case of a large cavity, i.e. a large sample area $A$, is described by the limit $S \to \infty$, see Appendix B. The concurrence in this case is calculated to be

$$C(k, q) = \frac{2G(k)G(q)}{G^2(k) + G^2(q)}. \quad (56)$$

We show the result in Fig. 7. As one can see, the concurrence always has two maxima if $q$ is held constant. One maximum appears for $G(k) = G(q)$ and $k \neq q$. There, photons are maximally entangled since we have $C = 1$. However, this maximum is relatively sharply peaked and if one realizes the post-selection experimentally by choosing a certain finite $k$-range, the entanglement will be reduced. The other maximum appears when $k = q$, which seems to be an artefact of the calculation, since this case was excluded in the calculations above. The reason for the exclusion is that the two-photon states would be separable and hence not entangled. In this case, however, the maximum is quite broad. Therefore, for a given $q$ there exists a wide range of corresponding modes $k$, for which the two photons are almost maximally entangled. In the second plot of Fig. 7 we show a magnification for wave vectors up to $0.5 \cdot 10^7 \text{ m}^{-1}$ to show the dependence of $C(k, q)$ on $k$ clearer around the first maximum, which is not visible in the previous plot. In particular at the intersubband resonance, which is around $0.2 \cdot 10^7 \text{ m}^{-1}$, the two maxima approach each other so that by selecting different modes around the resonance, the entanglement of the photons can be made almost maximal. The corresponding photon energies are in the mid infrared, about 150 meV.

V. CONCLUSION

An efficient and deterministic source of entangled photons is needed in quantum information processing. In this work, we examined a new scheme of photon production, based on the emission of quantum vacuum radiation from the intersubband cavity system. Because the triggered photon emission is based on a non-adiabatic modulation of the system’s ground state, an exact expression for this state could be used. Since the ground state consists of an infinite number of photonic and electronic states, we propose a post-selective measurement to reduce the photonic system to an effective two-qubit system, in which the qubit state was defined as two different in-plane wave vectors. The so-called mode entanglement of the photons is quantified by the concurrence. We found an analytical expression for the concurrence, which depends on the absolute values of the chosen wave vectors. We find that the concurrence, and therefore the entanglement of the

![Graphs showing concurrence as a function of $k$ and $q$ for GaAs/AlGaAs quantum wells.](image)

FIG. 6. (a) The concurrence $C(k, q)$ as a function of $k$ and $q$ for GaAs/AlGaAs quantum wells. (Parameter values: $\varepsilon = 10$, $f_{12} = 14.9$, $n_{\text{eff}}^0 = 50$, $L_c = 2 \mu\text{m}$, $\hbar \omega_{12} = 150 \text{ meV}$, $N_{2\text{DEC}} = 10^{12} \text{ cm}^{-2}$.) In (b), we plot $C(k, q)$ for fixed values of $q$ (given in the respective legend) and for a large range of $k$ from 0 to $3 \cdot 10^7 \text{ m}^{-1}$, which corresponds to a photon energy of 1.9 eV (450 THz, red) and in (c) a magnification of the range up to $k = 0.5 \cdot 10^7 \text{ m}^{-1}$ (320 meV, 80 THz, mid infrared).

the diagonal branch, i.e. $k \approx q$, the $|F|$ terms in (54) and (A2) can be neglected and we obtain

$$C(k, q) \approx \frac{2G(k)G(q)}{G^2(k) + G^2(q)} \approx 1. \quad (55)$$

Accordingly, photons in the visible regime are almost maximally entangled if their wave numbers are of the same size.

The other branch appears if $G(k) \approx G(q)$ and the modes are far from each other. We give a more precise
post-selected photons, is non-zero. In the limiting case of large sample areas, there exists a continuous set of mode pairs for which the concurrence is 1, i.e. the photons are maximally entangled. Also, in this case it turns out that for photon energies around the intersubband resonance, which is in the mid infrared regime of the electromagnetic spectrum, the photons are almost maximally entangled, the concurrence being close to 1. This is fundamentally important for the possible use in quantum information processing. Furthermore, a high degree of entanglement can be achieved if the modes chosen in the post-selection are close to each other. Therefore, one could extract entangled photon pairs in technologically relevant frequency domains like a telecom wavelength.

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Appendix A: Post-selected state

The pure state \( |\psi_{LR}\rangle \), which contains all two-photon states fulfilling the conditions of the post-selection, is given as

\[
|\psi_{LR}\rangle = \frac{1}{N_{LR}} \mathcal{P}_{LR} |G^{(2)}\rangle \\
= \frac{1}{N_{LR}} \left[ (G(k)|k\rangle_L|k\rangle_R + G(q)|q\rangle_L|q\rangle_R) \otimes \left( |0\rangle_b + \frac{1}{2} \sum_{k'} G(k')b^\dagger_{k'}b_{-k'}|0\rangle_b \right) + F^2(k)|k\rangle_L|k\rangle_R \otimes b^\dagger_{-k}b_{k}|0\rangle_b \\
+ F^2(q)|q\rangle_L|q\rangle_R \otimes b^\dagger_{-q}b_{q}|0\rangle_b + F(k)F(q)|q\rangle_L|k\rangle_R \otimes b^\dagger_{-k}b_{q}|0\rangle_b + F(k)F(q)|k\rangle_L|q\rangle_R \otimes b^\dagger_{-q}b_{k}|0\rangle_b \right] \]

(A1)

with \( N_{LR} \) being a normalization constant,

\[
N_{LR}^2 = \langle G^{(2)}|\mathcal{P}_{LR}|G^{(2)}\rangle = (G^2(k) + G^2(q)) \left( 1 + \frac{1}{2} S \right) + (|F(k)|^2 + |F(q)|^2)^2 - 2G^2(k)|F(k)|^2 - 2G^2(q)|F(q)|^2. \quad (A2)
\]

We already introduced the sum \( S \) as being

\[
S = \sum_{k'} G^2(k'). \quad (A3)
\]

Appendix B: Continuum limit

When taking the sum over all two-dimensional in-plane wave vectors \( \mathbf{k} \) in Eqn. \( 44 \), the appearing vectors depend on the boundary conditions. We choose periodic boundary conditions, and hence

\[
k_x = \frac{2\pi}{L_x} n_x, \quad n_x = 0, \pm 1, \ldots, \quad (B1)
\]

\[
k_y = \frac{2\pi}{L_y} n_y, \quad n_y = 0, \pm 1, \ldots, \quad (B2)
\]

where \( L_{x(y)} \) is the cavity length in \( x(y) \)-direction and \( n_{x(y)} \) is an integer. Every discrete wave vector \( \mathbf{k} \) has a volume \( \Delta \) in \( \mathbf{k} \)-space:

\[
\Delta = \Delta k_x \Delta k_y = \frac{(2\pi)^2}{L_x L_y} = \frac{(2\pi)^2}{A} \quad (B3)
\]
and $\Delta k(x,y)$ is the difference between two adjacent wave vectors in $x(y)$-direction, $A$ the sample area. In the continuum limit, the $k$ vectors lie close in the reciprocal space and the sum can be replaced by an integral

$$S = \sum_k G^2(k) = \frac{1}{\Delta} \sum_k \Delta G^2(k) \rightarrow \frac{1}{\Delta} \int \, d^2k \, G^2(k)$$

$$= \frac{A}{2\pi} \int_{k=0}^\infty \, dk \, k \, G^2(k)$$

$$= \frac{A}{2\pi} \left( \frac{c}{\omega_{12}} \right)^2 I,$$  \hspace{1cm} (B4)

where we use polar coordinates to evaluate the integral and carried out the polar-angle integration. In the last step, we make a substitution and introduce the dimensionless variable $\tilde{k} := \frac{c}{\omega_{12}} k$ to get the dimensionless integral $I$,

$$I = \int_{k=0}^\infty \, dk \, k \, G^2\left(\frac{c}{\omega_{12}} \tilde{k}\right).$$  \hspace{1cm} (B5)

One can show that the expansion coefficient $G(k)$ decreases like $1/k^2$ for large $k$. Consequently, the integrand has the asymptotics

$$k \, G^2(k) \rightarrow \infty \frac{1}{k^3}$$  \hspace{1cm} (B6)

and hence, the integral converges. We evaluate $I$ numerically using the same parameters as above and find $I = 1.9 \cdot 10^{-4}$. 

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