Price mediated contagion through capital ratio requirements

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Abstract

We develop a framework for price-mediated contagion in financial systems where banks are forced to liquidate assets to satisfy a risk-weight based capital adequacy requirement. In constructing this modeling framework, we introduce a two-tier pricing structure: the volume weighted average price that is obtained by any bank liquidating assets and the terminal mark-to-market price used to account for all assets held at the end of the clearing process. We consider the case of multiple illiquid assets and develop conditions for the existence and uniqueness of clearing prices. We provide a closed-form representation for the sensitivity of these clearing prices to the system parameters, and use this result to quantify: (1) the cost of regulation faced by the system as a whole and the individual banks, and (2) the value of providing bailouts and bail-ins to consider when such notions are financially advisable. Numerical case studies are provided to study the application of this model to data.

1 Introduction

The modern day financial system is a highly interconnected network. The connections that exist within this network are varied, myriad and complex. These connections might be through direct channels such as interbank debt linkages or through indirect channels such as overlapping portfolios. These connections provide avenues of contagion in the financial networks. Thus negative actions of a bank may cause distress to other firms and eventually affect the entire system. The risk to the financial system, posed by such events, is often called systemic risk.

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It is imperative that we study and understand how this contagion spreads through financial networks in order to prevent and mitigate systemic crises, e.g., the 2007-2009 financial crisis.

In this work, we study contagion in financial systems from fire sale spillovers. These crises originate when a firm is forced to liquidate its assets to meet some obligation or satisfy some regulation. As firms hold overlapping portfolios this causes impacts globally to all other firms from mark-to-market accounting. These firms are now, themselves, forced to liquidate their assets, which exacerbates the crisis by depressing asset prices further. An important factor in the origin of fire sales is the unintended consequence of capital regulations in the form of capital or leverage ratios. Due to these regulatory constraints, banks might be forced to delever, setting off a vicious cycle of contagion due to the pro-cyclical nature of these regulatory environments. Such deleveraging occurred in a large scale in the 2008 financial crisis, resulting in amplification of losses. For further discussion on such mechanisms see [5, 9].

The literature in the study of fire sales may be broadly divided into two different bodies depending on the focus of the study. The first places more emphasis on the development of a general mathematical framework and exploring questions about, e.g., existence and uniqueness of clearing solutions. Among these works, [7] considers the liquidation problem in the context of a capital adequacy ratio. [3, 12] study the fire sale problem when banks are forced to liquidate assets to meet debt obligations. [14] develops an extension to [12] where banks, in addition to meeting their debt obligations, must satisfy a leverage ratio. [5] considers the problem where banks are required to satisfy a risk-weighted capital ratio. [13] extends [5] by considering the price-mediated contagion problem in a continuous-time setting to provide results on existence and uniqueness. The second, broad, body of work on fire sales and price-mediated contagion focuses primarily on the development of an operational modeling framework and the design of stress tests. Typically these results depend on a particular liquidation strategy (e.g., proportional liquidation) and linear price impacts. Some of the notable works in this domain include [15, 11, 9].

The primary innovation of this work that facilitates the mathematical results is the consideration of both mark-to-market prices and volume weighted average prices to characterize the system. In the modeling, assets are liquidated at the volume weighted average price, but unsold assets are marked to the market. This distinction between prices was hinted at by [8], but the implications of that distinction were not fully considered in that work. By considering this pair of prices we are able to determine conditions for existence and uniqueness under capital adequacy and leverage requirements in equilibrium; as far as the authors are aware, this is the
first time that uniqueness results are provided in such a setting. Similar to the dynamic setting of [13], the uniqueness results of this work can be used to calibrate the risk-weights to the liquidity of assets; this is in contrast with the prevailing heuristic methodology used in practice. In addition, we introduce sensitivity results in the fire sale framework which we use to determine two vital statistics: the cost of regulation and the value of a rescue fund (whether an external bailout or a bail-in from other firms in the system). Again, as far as the authors are aware, this is the first time such results are available in the literature. Importantly, all results presented herein hold in markets with multiple assets subject to general price impact functions and with general liquidation strategies satisfying simple, financially meaningful, conditions.

The remainder of this paper is organized as follows: In Section 2, we develop the financial setting for our model and consider liquidation strategies. Notably, we characterize the liquidation strategies as functions of both the mark-to-market and volume weighted average prices. In Section 3, we formulate the clearing liquidations as a fixed point problem and develop conditions for the existence and uniqueness of these equilibrium prices. In Section 4, we formulate the sensitivity analysis of the equilibrium prices with respect to the system parameters as a fixed point problem and provide a closed-form representation for this result. Further, we utilize these results to develop a methodology to evaluate the cost of regulation and study the value of bailouts and bail-ins. Numerical case studies highlighting the applications of this model are presented in Section 5. The proofs of the main results are provided in the Appendix.

2 Financial setting

We begin with some simple notation that will be consistent for the entirety of this paper. Let \( x, y \in \mathbb{R}^n \) for some positive integer \( n \), then

\[
x \land y = (\min(x_1, y_1), \min(x_2, y_2), \ldots, \min(x_n, y_n))^T,
\]

\( x^- = -(x \land 0) \), and \( x^+ = (-x)^- \). Further, to ease notation, we will denote \( [x, y] := [x_1, y_1] \times [x_2, y_2] \times \ldots \times [x_n, y_n] \subseteq \mathbb{R}^n \) to be the \( n \)-dimensional compact interval for \( y - x \in \mathbb{R}^n_+ \). Similarly, we will consider \( x \leq y \) if and only if \( y - x \in \mathbb{R}^n_+ \).

2.1 Price impacts

For our modeling, we note that banks hold both liquid and illiquid assets. In line with [5] [13], we consider two classes of illiquid assets: marketable (stocks or bonds issued by a non-financial
corporation) or non-marketable (loans). The distinction between these two classes of illiquid assets is that non-marketable assets are difficult to sell in the short-run, and hence cannot be liquidated during the crisis we consider herein. For further discussion on non-marketable assets see [5, 10]. Throughout this work we consider a single liquid asset, \( m \geq 1 \) (marketable) illiquid assets, and an arbitrary number of non-marketable assets. Each marketable illiquid asset \( k \) has \( M_k > 0 \) outstanding shares; we will denote the vector of outstanding shares by \( M \in \mathbb{R}^m_+ \). Any marketable illiquid asset, when sold, is subject to price impacts. These price impacts measure the liquidity of an asset; the more liquid an asset the less the prices are affected by market behavior.

With this in mind, we consider the market prices (with the liquid asset acting as the numéraire) modeled by an inverse demand function. That is, the function \( F : [0, M] \to \mathbb{R}^m_+ \) which maps units of illiquid assets being sold into corresponding prices quoted in the market. Thus we are able to provide the mark-to-market prices [MTMP] as a function of the units of assets being sold.

**Assumption 2.1.** The MTMP exhibits no direct cross impacts, i.e., \( F(\Gamma) := (f_1(\Gamma_1), \ldots, f_m(\Gamma_m))^\top \). Additionally, \( f_k : [0, M_k] \to (0, 1] \) is nonincreasing with \( f_k(0) = 1 \) for every asset \( k = 1, \ldots, m \). This implies that \( M_k \) denotes, also, the original market capitalization for each asset \( k \).

However, when liquidating assets, the price attained in the market is a different (higher) price. Remaining with the notion of a static framework common in the literature, e.g., in [7, 3, 12], we consider this price as the average MTMP throughout the liquidations. That is, we define the volume weighted average price [VWAP] as the mapping \( \bar{F} : [0, M] \to \mathbb{R}^m_+ \) such that

\[
\Gamma \in [0, M] \mapsto \bar{F}_k(\Gamma) := \frac{1}{\Gamma_k} \int_0^{\Gamma_k} f_k(\gamma_k) d\gamma_k
\]

(with \( \bar{f}_k(0) = 1 \)) for every asset \( k \). Intuitively, the VWAP encodes the notion that liquidating assets captures the entire price history as prices fall (as modeled by the inverse demand function \( F \)), e.g., the first marginal unit of asset \( k \) is sold at the initial MTMP \( f_k(0) = 1 \) and, after \( \gamma_k \) units of asset \( k \) have been sold, the next marginal unit is liquidated at the MTMP \( f_k(\gamma_k) \). Thus considering these two separate prices makes this model more realistic and enables us to encode a dynamic notion which is absent in the existing literature. Notably, the VWAP naturally satisfies the property that the value obtained from liquidating assets grows as more assets are sold, i.e., \( \Gamma_k \in [0, M_k] \mapsto \Gamma_k \bar{f}_k(\Gamma_k) \) is strictly increasing. This property is the primary one introduced in [3].
Remark 2.2. We wish to note that a similar notion of differentiating between the MTMP and VWAP is introduced in [8]. That work considers a heuristic parameter $\eta \in [0, 1]$, which defines the VWAP by $\bar{f}_k(\Gamma_k) = (1 - \eta) + \eta f_k(\Gamma_k)$. In contrast, we use the MTMP in order to find the actualized VWAP.

We conclude our discussion of the inverse demand function by considering 4 example functions and the resultant VWAPs.

Example 2.3. Throughout these examples, and this work, we use the convention that 0/0 = 1.

(i) First, let us consider a construction of price impacts that follow from a limit order book construction. In the limit order book, there are fixed price levels with limited liquidity at each level. Consider price levels $q_{k,j} > 0$ with liquidity $m_{k,j} > 0$ such that $q_{k,j+1} < q_{k,j}$ for every $j \geq 0$ with $q_{k,0} = 1$ and $m_{k,0} = 0$. Then the MTMP is defined by $x \in [0, M_k] \mapsto \sum_{j \geq 0} q_{k,j} \mathbb{1}_{\{x \in [\sum_{i=j}^{j+1} m_{k,i}, \sum_{i=0}^{j} m_{k,i})\}}$. The resultant VWAP is then given by $\bar{f}_k(x) = \frac{1}{x} \sum_{j \geq 0} q_{k,j} \left[ x \wedge \sum_{i=j}^{j+1} m_{k,i} - x \wedge \sum_{i=0}^{j} m_{k,i} \right]$ for $x \in [0, M_k]$. An example of this inverse demand function setting is provided in Figure 1.

(ii) Second, consider a generalization of the linear price impact function common in the literature (e.g., [5, 11, 9, 15]). Let the MTMP be defined by $x \in [0, M_k] \mapsto f_k(x) = 1 - bx^a$ for $a \geq 0$ and $b \in [0, M^{-a})$. The resultant VWAP is then given by $\bar{f}_k(x) = 1 - \frac{b}{1+a} x^a$ for any $x \in [0, M_k]$. In particular, this satisfies the construction of the VWAP from [8] with $\eta = \frac{1}{1+a}$ as described in Remark 2.2. We wish to note that many commonly used inverse demand functions exist in this setting: if $a = 1$ then this is the linear inverse demand function.

Figure 1: Illustration of the mark-to-market price and associated volume weighted average price from a limit order book construction as in Example 2.3(i).
(iii) Third, consider a different generalization of the linear price impact function. Let the MTMP be defined by \( x \in [0, M_k] \mapsto f_k(x) = (1 - bx)^a \) for \( b \leq M^{-1} \) with \( ab \geq 0 \). The resultant VWAP is then given by

\[
\bar{f}_k(x) = \begin{cases} 
\frac{1}{1+b} \left[ 1 - (1 - bx)^{1+a} \right] & \text{if } a \neq -1 \\
-\frac{\log(1-bx)}{b} & \text{if } a = -1
\end{cases}
\]

for any \( x \in [0, M_k] \).

(iv) Finally, consider the exponential inverse demand function for the MTMP, i.e., \( x \in [0, M_k] \mapsto f_k(x) = \exp(-bx) \) for \( b \geq 0 \). The resultant VWAP is then given by \( \bar{f}_k(x) = \frac{x}{b} [1 - \exp(-bx)] \) for any \( x \in [0, M_k] \).

2.2 The stylized balance sheet

We will consider two time points \( t \in \{0, 1\} \). At \( t = 0 \), a bank or firm holds \( x \geq 0 \) liquid assets (e.g., cash). As mentioned in the prior section, we will assume, without loss of generality, that the price of this asset stays constant at 1 at all times. In addition to this liquid asset, the bank portfolio is comprised of illiquid assets. We assume that each bank holds \( s \in [0, M] \subseteq \mathbb{R}^m \) shares of the marketable illiquid assets and \( \ell \geq 0 \) of non-marketable assets. Without loss of generality, we assume that the price of all the (marketable) illiquid assets are \( F(\vec{0}) = \bar{F}(\vec{0}) = \vec{1} \) at time 0.

On the other side of the balance-sheet each bank or firm has \( \bar{p} \geq 0 \) in liabilities. For simplicity in this work, we will assume that all liabilities are not held by any other firms in this system; additionally, we will assume that no liabilities come due during the (short time horizon) of the fire sale cascade under study, but are liquid enough that they can frictionless be paid off early with liquid assets. Thus, at time 0, this bank has a capital of \( x + \ell + \vec{1}^T s - \bar{p} \). This is depicted in Figure 2.

At time \( t = 1 \), a firm or bank may liquidate some of its marketable illiquid assets. In particular, the bank sells \( \gamma \in [0, s] \) at the VWAP \( \bar{q} \in (0, 1]^m \) to obtain \( \vec{q}^T \gamma \geq 0 \) liquid assets. The remaining \( s - \gamma \) marketable illiquid assets in the banking book are valued at the MTMP \( q \in (0, \bar{q}] \). We will denote this space of joint MTMP and VWAP prices by the lattice

\[
\mathbb{D} := \{(q, \bar{q}) \in [F(M), F(\vec{0})] \times [\bar{F}(M), \bar{F}(\vec{0})] \mid q \leq \bar{q}\}.
\]

The capital at time \( t = 1 \) is thus given by \( C := x + \ell + \vec{q}^T \gamma + q^T [s - \gamma] - \bar{p} \). This is depicted in
### Remark 2.4

While computing the capital of a bank, the proceeds from liquidation is given by $q^\top \gamma$. This is in contrast to much of the existing literature on fire sales (e.g., [5, 14]) where no distinction is made between liquidation price and current price; in prior literature the proceeds would be computed as $q^\top s$. By introducing the MTMP and VWAP prices, this is a more realistic scenario and it offers a better interpretation of the capital of a firm. If a single price had been considered, the capital of a firm $(x + \ell + q^\top s - \bar{p})$ would have been independent of liquidations $\gamma$ and would depend only on the price $q$.

For the remainder of this work we consider $n \geq 1$ banks. We will denote the set of all banks in the system by $\mathcal{N} := \{1, 2, ..., n\}$. In terms of vector notation, at time $t = 0$, the banks are holding an amount $x \in \mathbb{R}^n_{+}$ of liquid assets, $\ell \in \mathbb{R}^n_{+}$ shares of non-marketable illiquid assets, $S = (s_{ik}) \in \mathbb{R}^{n \times m}_{+}$ shares of marketable illiquid assets, and an amount $\bar{p} \in \mathbb{R}^n_{+}$ in liabilities. By construction, we will always set $\sum_{i \in \mathcal{N}} s_{ik} \leq M_k$ for all assets $k$. At time $t = 1$, the banks are liquidating $\Gamma = (\gamma_{ik}) \in [0, S]$ of the marketable illiquid assets.

#### 2.3 Capital adequacy ratio and liquidation strategies

The Basel regulation mandates the use of a capital adequacy ratio (also called the risk-weighted capital ratio) to assess the solvency of banks. The risk-weighted capital ratio is defined as

$$\text{Risk-weighted capital ratio} = \frac{\text{Total Capital}}{\text{Risk-Weighted Assets}}$$
The determination of the risk-weights of different assets requires the consideration of a number of complex factors. In this case, we make the assumption that these risk-weights are known to us and given by $0$ for the liquid asset and $\alpha_k \geq 0$ for marketable illiquid asset $k = 1, \ldots, m$. For the non-marketable asset we let the risk-weight be dependent on each bank and let $\alpha_{\ell,i} \geq 0$ be the risk-weight for the non-marketable assets of bank $i \in \mathcal{N}$. To simplify notation, we define $A := \text{diag}(\alpha_1, \ldots, \alpha_m)$ to be the diagonal matrix of the risk-weights of the marketable illiquid assets.

Thus at time $t = 0$, the risk-weighted capital ratio $\theta_i$ of bank $i$ is given by

$$\theta_i(0) = \frac{x_i + \ell_i + \mathbf{1}^T s_i - p_i}{\mathbf{1}^T A s_i + \alpha_{\ell,i} \ell_i}$$

According to banking regulations, banks are required to maintain a minimum capital ratio $\theta_{\min} > 0$, e.g., 8% in Basel III regulations. We wish to note that this capital adequacy ratio is related to a leverage requirement if all risk-weights are set to 1.

At time $t = 0$, banks may or may not be in compliance with this regulatory constraint. In [5], it is assumed that an external shock occurs at time $t = 0^+$ to which banks must react at time $t = 1$; herein, we do not explicitly differentiate between times $t = 0$ and $t = 0^+$ to simplify notation. If we were to consider the shock setting, this could be a shock to the value of liquid or illiquid assets (marketable or non-marketable) as is standard in the literature (see, e.g., [5, 15, 11]) or a shock in the risk-weights, i.e., where the risk-weight of an asset jumps due to, e.g., a credit downgrade.

Depending on the state of the balance sheets of the financial institutions at $t = 0$, the capital ratio of some banks may fall below the regulatory minimum $\theta_{\min}$. In this situation, banks typically have two options: issue new equity or liquidate portions of the banking book. At the time of a crisis, issuing new equity might not be feasible. As such, banks will be forced to liquidate assets to meet the regulatory constraint. However, these liquidations can set off a fire sale causing additional losses to the system.

When a bank is forced to liquidate assets, they do so in a minimal way. That is, they liquidate the minimal value necessary to recover

$$\theta_i(1) := \frac{x_i + \ell_i + \tilde{q}^T \gamma_i + q^T [s_i - \gamma_i] - \tilde{p}_i}{q^T A [s_i - \gamma_i] + \alpha_{\ell,i} \ell_i} \geq \theta_{\min}$$

and will do nothing if they already satisfy this regulatory requirement. This practice is in line with the existing literature [5, 15, 11] as well as empirical evidence [1]. It might be entirely
possible that even when a bank liquidates all its assets it cannot restore its capital ratio to $\theta_{\text{min}}$ at time 1. In this situation we will assume that such a bank is insolvent and costlessly liquidated at $t = 1$ (while liquidating all of its marketable assets in the market) along the lines of [5].

To this end, bank $i \in \mathcal{N}$ can belong to any of the following three mutually exclusive and exhaustive sets given the prices $(q, \bar{q}) \in \mathbb{D}$. To shorten notation throughout this work we will define the shortfall of bank $i$ as $h_i := \bar{p}_i - x_i - (1 - \alpha_{\ell,i,\theta_{\text{min}}}) \ell_i$.

- **Solvent and liquid**: Let us denote this set by $S(q, \bar{q})$. For any bank $i \in S(q, \bar{q})$, $\gamma_i(q, \bar{q}) = 0$. In fact, this set is characterized by $h_i \leq q^\top [I - A\theta_{\text{min}}] s_i$.

- **Solvent but illiquid**: Let us denote this set by $L(q, \bar{q})$. For any bank $i \in L(q, \bar{q})$, $q^\top [I - A\theta_{\text{min}}] s_i < h_i < q^\top s_i$.

- **Insolvent**: Let us denote this set by $D(q, \bar{q})$. For any bank $i \in D(q, \bar{q})$, $\gamma_i(q, \bar{q}) = s_i$. In fact, this set is characterized by $h_i \geq \bar{q}^\top s_i$.

These conditions are encoded mathematically in Assumption 2.5 below.

**Assumption 2.5.** Given a coupled MTMP and VWAP $(q, \bar{q}) \in \mathbb{D}$, the system of banks will liquidate $\Gamma(q, \bar{q}) \in [0, S]$ marketable illiquid assets satisfying the minimal liquidation condition for each bank $i$:  

$$ (\bar{q} - [I - A\theta_{\text{min}}] q)^\top \gamma_i(q, \bar{q}) = (h_i - q^\top [I - A\theta_{\text{min}}] s_i)^+ \wedge (\bar{q} - [I - A\theta_{\text{min}}] q)^\top s_i $$  

for shortfall $h_i := \bar{p}_i - x_i - (1 - \alpha_{\ell,i,\theta_{\text{min}}}) \ell_i$.

We refer to (1) as the minimal liquidation condition as it provides the minimal amount needed to be liquidated in order to satisfy the capital adequacy requirement $\theta_i(1) \geq \theta_{\text{min}}$, or liquidate all assets if insolvent. This is similar to the liquidity constraint used in [12, 14].

Before considering some sample liquidation functions, we wish to give one more consideration of the risk-weights.

**Assumption 2.6.** Throughout this work we will assume that $\alpha_k \theta_{\text{min}} < 1$ for every asset $k = 1, 2, ..., m$.

If $\alpha_k \theta_{\text{min}} \geq 1$ for any $k$, then the setting of this paper implies that as price drops in that asset, the bank will always satisfy the capital regulation which is opposite to the scenario that we are modeling. For further discussion see Remark 2.2 of [13].
We complete this section by discussing a few sample liquidation functions satisfying Assumption 2.5.

**Example 2.7.**

(i) Consider a setting with only $m = 1$ marketable illiquid asset. In this setting, $\gamma(q, \bar{q}) = \left(\frac{h - q(1 - \alpha\theta_{\min})s}{q - (1 - \alpha\theta_{\min})q}\right)^+ \wedge s$ for any $(q, \bar{q}) \in \mathbb{D}$. We wish to note that this liquidation function $\gamma$ is continuous and nonincreasing in $(q, \bar{q}) \in \mathbb{D}$.

(ii) Consider a setting in which a bank wishes to maintain its current portfolio ratio, i.e., when it liquidates it sells shares of its full portfolio. Proportional liquidation has been widely explored in the existing literature (e.g., [11, 15, 8]) for the analysis of fire sales. Such a strategy is defined by:

$$\gamma(q, \bar{q}) = \left[\left(\frac{h - q^\top[I - A\theta_{\min}]s}{\bar{q} - [I - A\theta_{\min}]q}\right)^+ \wedge 1\right] s$$

for any $(q, \bar{q}) \in \mathbb{D}$. We wish to note that this liquidation function $\gamma$ is continuous and nonincreasing in $(q, \bar{q}) \in \mathbb{D}$.

(iii) Consider now the first of two strategies based on utility maximization. In this first setting, a bank decides on its liquidation strategy $\gamma$ so as to maximize some strictly concave utility function $u$. As such we consider firms to be utility maximizers of the following problem rather than following a mechanical property:

$$\gamma(q, \bar{q}) = \arg\max_{\gamma \in G(q, \bar{q})} u(\gamma)$$

with constraint set provided by a relaxation of the minimal liquidation condition:

$$G(q, \bar{q}) = \{\gamma \in [0, s] \mid \bar{q} - [I - A\theta_{\min}]q)^\top \gamma \geq (h - q^\top[I - A\theta_{\min}]s)^+ \wedge [(\bar{q} - [I - A\theta_{\min}]q)^\top s]\}$$

for any $(q, \bar{q}) \in \mathbb{D}$. By strict concavity and using Berge’s maximum theorem, we can show that this utility maximizing liquidation function is continuous. If $u$ is also decreasing then this will satisfy the minimal liquidation condition (I).

(iv) Consider now an extension of the utility maximizer, but one that also depends on the
actions of other firms. Consider bank $i$ to follow this strategy and the aggregate liquidations of all other banks is given by $\gamma_{-i} \in \mathbb{R}_+^m$:

$$
\gamma_i(q, \bar{q}; \gamma_{-i}) = \arg\max_{\gamma_i \in \mathcal{G}(q, \bar{q})} u_i(\gamma_i, \gamma_{-i})
$$

for prices $(q, \bar{q}) \in \mathcal{D}$. We call this an equilibrium strategy since, if multiple firms follow this strategy, the actualized liquidations would be the solution of a fixed point problem. Notably, so long as the utility functions are strictly concave then this liquidation strategy is continuous w.r.t. the prices and liquidations of the other firms. As with the utility maximizer liquidation function, if $u_i$ is also decreasing then this strategy will satisfy the minimal liquidation condition (1). Finally, if all firms utilizing this liquidation strategy construct a diagonally strictly concave game (see, e.g., [17]) then there exists a unique fixed point liquidation strategy, and that strategy is continuous in the prices $(q, \bar{q}) \in \mathcal{D}$.

3 Clearing formulation and solutions

With this thorough discussion of the financial setting, we now wish to consider the problem of finding the clearing prices. Though the minimal liquidation condition (1) gives information about the liquidations that bank $i$ performs under the MTMP and VWAP $(q, \bar{q}) \in \mathcal{D}$, when actualizing these sales the prices will adjust according to the inverse demand functions $F$ and $\bar{F}$. Therefore, this problem can accurately be modeled using a fixed point equation.

As discussed above, consider the matrix of liquidations $\Gamma(q, \bar{q}) \in [0, S]$ at MTMP and VWAP of $(q, \bar{q}) \in \mathcal{D}$. We seek an equilibrium price so that the resultant price from liquidations is equal to the prices that generate those liquidations. That is, we seek the fixed point to the function $\Phi : \mathcal{D} \to \mathcal{D}$ defined by:

$$
(q, \bar{q}) \in \mathcal{D} \mapsto \Phi(q, \bar{q}) := \left(F(\Gamma(q, \bar{q})^\top \mathbb{I}), \bar{F}(\Gamma(q, \bar{q})^\top \mathbb{I})\right).
$$

We call $(q, \bar{q}) \in \mathcal{D}$ a clearing solution or clearing price if

$$(q, \bar{q}) = \Phi(q, \bar{q}).$$

In this section, we develop conditions for existence and uniqueness of the clearing prices. In the existing literature on price-mediated contagion due to leverage and capital adequacy ratio requirements, existence of (static) clearing solutions has been explored for the one asset case
in [5] and for the multi-asset case in [14]. However in these works, uniqueness of the clearing prices has not been previously solved. This is tackled directly in Theorem 3.1.

**Theorem 3.1.** Consider the regulatory and balance sheet assumptions from Section 2.

(i) Let the inverse demand function \( F : [0, M] \to (0, 1]^m \) be continuous. If the liquidation function \( \Gamma : D \to [0, S] \) is continuous, then there exists a clearing price \((q^*, \bar{q}^*)\).

(ii) If the liquidation function \( \Gamma : D \to [0, S] \) is nonincreasing, then there exists a greatest and least clearing price \((q^\uparrow, \bar{q}^\uparrow) \geq (q^\downarrow, \bar{q}^\downarrow)\).

(iii) Let the inverse demand function be such that \( \Gamma^* \in [0, M] \mapsto \bar{F}(\Gamma^*)^\top \Gamma^* + F(\Gamma^*)^\top [I - A\theta_{\min}](M - \Gamma^*) \)

is strictly increasing. If the liquidation function \( \Gamma : D \to [0, S] \) is nonincreasing and satisfies Assumption 2.5, then there exists a unique clearing price \((q^*, \bar{q}^*)\).

**Remark 3.2.** The adoption of the VWAP in this framework, besides providing a more realistic financial framework, offers significant mathematical advantages, particularly in the analysis of uniqueness as is evident from the preceding theorem as this was unable to be proven in, e.g., [5, 14].

**Remark 3.3.** We want to point out the similarity in the uniqueness condition presented in this work to one of the very few uniqueness result in the fire sale literature as presented in [3, 12]. In those works, the liquidation occurs based on a leverage ratio \((\alpha = 1)\) with no leverage allowed \((\theta_{\min} = 1)\). Our new condition on the inverse demand function in Theorem 3.1 reduces exactly to the condition from those papers. In fact, given our construction of the VWAP from the MTMP, the property \( \Gamma^* \in [0, M] \mapsto \bar{F}(\Gamma^*)^\top \Gamma^* \) strictly increasing is automatically satisfied.

Theorem 3.1 provides a condition for the uniqueness of solution for an equilibrium price \((q^*, \bar{q}^*)\) in terms of the inverse demand function \( F \). However this condition also depends on the risk-weights \( \alpha \). We can make this dependence explicit by stating the uniqueness condition in terms of the inverse demand function \( F \) and the risk-weights \( \alpha \) along the lines of [13]. This is described in the following corollary. This is important as it allows us to calibrate the risk-weights to the liquidity of the assets.

**Corollary 3.4.** Let the inverse demand function \( F \) be differentiable and such that \( \Gamma^*_k \in [0, M_k] \mapsto \frac{(M_k - \Gamma^*_k)^f(\Gamma^*_k)}{f_k(\Gamma^*_k)} \) is nondecreasing for every asset \( k = 1, 2, .., m \). Additionally, let \( \alpha_k \in \frac{1}{\theta_{\min}} \times \)
\begin{equation*}
\left(-\frac{M_k f_k'(0)}{1-M_k f_k'(0)}, 1\right)
\end{equation*}
for every asset \(k\). If the liquidation function \(\Gamma : \mathbb{D} \to [0, S]\) is nonincreasing and satisfies Assumption 2.5, then there exists a unique clearing price \((q^*, \bar{q}^*)\).

**Remark 3.5.** Corollary 3.4 provides sufficient conditions for the uniqueness property given in Theorem 3.1(iii). As such, it is a stronger set of conditions than presented in Theorem 3.1(iii). However, we feel that the conditions of Corollary 3.4 are easier to test and evaluate as it separates the conditions on the inverse demand function and the risk-weights, while also providing a nice interpretation in terms of calibrating risk-weights. We wish to note that the conditions of Corollary 3.4 exactly coincide with those provided in Lemma 3.11 of [13] in a continuous-time model of the proportional liquidation setting. The condition on the inverse demand function \((\Gamma_k^* \in [0, M_k])\mapsto (\frac{M_k - \Gamma_k^* f_k'(\Gamma_k^*)}{f_k'(\Gamma_k^*)}, 1)\) is nondecreasing) is discussed in detail in Remark 3.5 of that work. In short, this condition implies that a financial institution does not need to increase the speed it is selling the illiquid assets solely to counteract its own impacts.

**Example 3.6.** We now wish to consider our example inverse demand functions to determine under which scenarios they satisfy the differentiability and monotonicity condition of Corollary 3.4 with nonincreasing liquidation functions (e.g., proportional liquidation). Though the utility maximizing and equilibrium liquidation strategies have clearing solutions, the results of this work are not strong enough to guarantee uniqueness of those clearing prices.

(i) The limit order book setting is not differentiable, and thus cannot be used with the results of Corollary 3.4. In fact, due to the jumps in the MTMP, the uniqueness condition of Theorem 3.1(iii) cannot hold globally, though we can still guarantee a maximal (and minimal) clearing solution.

(ii) If \(f(x) = 1 - bx^a\) then Corollary 3.4 is satisfied so long as \(a \leq 1\) and \(b \in [0, M^{-a})\). Though, if \(a < 1\) then the condition for \(\alpha\) results in an empty interval. This is confirmed by observing the results of Theorem 3.1(iii) to determine that uniqueness holds so long as \(a \geq 1\) with \(\alpha \in \frac{1}{\theta_{\min}} \times (\frac{a(M-x_f)-M(1-bMx_f^{a-1})}{(1+a)(M-x_f)-M(1-bMx_f^{a-1})}, 1)\) for \(x_f \in [0, M]\) solving \(f(x) = a(1 - \frac{x}{M})\). Note that \(x_f\) is unique if \(a \geq 1\) but nonexistent if \(a < 1\).

(iii) If \(f(x) = (1 - bx)^a\) then Corollary 3.4 is satisfied for any \(b < M^{-1}\) with \(ab \geq 0\). The risk-weight \(\alpha\) is then constrained by \(\alpha \in \frac{1}{\theta_{\min}} \times (\frac{abM}{1+abM}, 1)\).

(iv) If \(f(x) = \exp(-bx)\) then Corollary 3.4 is satisfied for any \(b \geq 0\). The risk-weight \(\alpha\) is then constrained by \(\alpha \in \frac{1}{\theta_{\min}} \times (\frac{M}{1+abM}, 1)\).

Intriguingly, the two generalizations of the linear inverse demand function provide very different results related to uniqueness of the clearing solution. Further consideration of this discrepancy...
and deduction of the appropriate shape of the inverse demand function would be an important follow-up.

4 Stability and sensitivity analysis

In this section, we perform sensitivity analysis of the equilibrium prices with respect to the system parameters. This is a critical exercise as the exact system parameters are often unknown and the results depend on how these parameters are calibrated. As far as the authors are aware, a systematic study on the dependence of clearing prices to system parameters has not been undertaken previously. We characterize the sensitivity analysis as a fixed point problem and prove the existence and uniqueness of the solution to this problem. In particular, we are interested in studying the sensitivity of the clearing solution to changes in the risk-weights $\alpha_k$, the regulatory threshold $\theta_{\text{min}}$, and the firm shortfall $h_i$. Sensitivity to risk-weights provides a first-order approximation for the impacts of a credit downgrade on the health of the financial system. We study the sensitivity with respect to the regulatory threshold, not because it is potentially unknown, but because it allows us to quantify the cost of regulation to the different banks. We elaborate on this point in Section 4.2. Finally, in considering sensitivity with respect to the firm shortfall, we are able to quantify a measure of how uncertainty in the shortfall might impact the health of the financial system. In addition, we propose a methodology to determine when there are incentives for a bailout or bail-ins from other institutions. We elaborate on this point in Section 4.3.

4.1 Sensitivity analysis

Throughout this section let $\#$ denote an arbitrary parameter of the fire sale model considered in this work. For example, $\#$ can denote $\alpha_k$ for some asset $k$. We utilize this general notation as we will see that the sensitivity analysis can be constructed in this general setting with only minor adjustments.

**Theorem 4.1.** Consider the setting of Theorem 3.1(iii) in which the liquidation function $\Gamma : \mathbb{D} \to [0, S]$ is such that $\gamma_i$ is strictly decreasing on $\{(q, \bar{q}) \in \mathbb{D} \mid i \in \mathbb{L}(q, \bar{q})\}$ for any bank $i$. Define $\Gamma^* : \mathbb{D} \to [0, M]$ as $(q, \bar{q}) \in \mathbb{D} \mapsto \Gamma(q, \bar{q})^\top \mathbf{1}$. The partial derivatives describing the sensitivity of
clearing prices \((q^*, \bar{q}^*)\) to the parameter \# are defined by:

\[
\begin{pmatrix}
\frac{\partial q^*}{\partial \#} \\
\frac{\partial \bar{q}^*}{\partial \#}
\end{pmatrix} = \left[ I - \begin{pmatrix}
\text{diag } [F'(\Gamma^*(q^*, \bar{q}^*))] & 0 \\
0 & \text{diag } [\bar{F}'(\Gamma^*(q^*, \bar{q}^*))]
\end{pmatrix} \right]^{-1} \left[ \begin{pmatrix}
J_{\Gamma^*}(q^*, \bar{q}^*) \\
J_{\Gamma^*}(q^*, \bar{q}^*)
\end{pmatrix} \right]^{-1}
\times \begin{pmatrix}
\text{diag } [F'(\Gamma^*(q^*, \bar{q}^*))] \\
\text{diag } [\bar{F}'(\Gamma^*(q^*, \bar{q}^*))]
\end{pmatrix} \frac{\partial \Gamma^*(q^*, \bar{q}^*; \#)}{\partial \#}
\]

where \(J_{\Gamma^*}(q^*, \bar{q}^*)\) denotes the Jacobian of \(\Gamma^*\) at \((q^*, \bar{q}^*)\) and, to simplify notation, \(F'(\Gamma^*) := (f'_1(\Gamma^*_1), ..., f'_m(\Gamma^*_m))^{\top}\) and similarly with \(\bar{F}'(\Gamma^*)\).

The representation of \(\frac{\partial q^*}{\partial \#}\) provided in Theorem 4.1 follows from implicit differentiation on the clearing mechanism (2). To verify that this representation is well-defined, in Appendix B.1 we prove the invertibility of the matrix on the right-hand side as a Leontief inverse. This matrix inverse provides the additional gains or losses caused solely by the clearing mechanism due to contagion effects.

One simple application of the sensitivity analysis provided by Theorem 4.1 is in studying the impacts of (marginal) changes in the risk-weights of the various assets under study in this work. For instance, if an asset is subject to a credit downgrade then its risk-weight will increase. Though in reality these take discrete values, we are able to analytically describe the impact to the equilibrium prices \((q^*, \bar{q}^*)\) of a marginal change in the risk-weight \(\alpha_k\) of asset \(k\). In particular, a change in the risk-weight for a single asset can cause cross-impacts to all other assets through the fire-sale mechanism as well. The total impact to market capitalization from a parallel jump in risk-weights (e.g., from a market downturn causing a system-wide reevaluation of credit ratings) can be quantified as

\[
\sum_{k=1}^{m} \frac{\partial M^{\top} q^*}{\partial \alpha_k} = M^{\top} \sum_{k=1}^{m} \frac{\partial q^*}{\partial \alpha_k}.
\]

As a market downturn increases the risk of credit default events, all risk-weights are subject to shifts during exactly the financial crises studied in this work. As such, this drop in market capitalization can be considered as a stability check on the clearing prices in order to account for these feedback effects, which are otherwise not considered.

### 4.2 Cost of regulation

A particularly interesting application of the sensitivity analysis provided by Theorem 4.1 is in the development of a scheme for computing the cost of regulation incurred by each bank. This is
based on the idea that a tightened regulatory threshold \( \theta_{\text{min}} \) will result in an increased loss for a bank. Therefore computing the loss incurred for a marginal increase in the regulatory threshold gives a measure of the regulatory cost for a bank. The loss incurred, and hence the cost of regulation, may be quantified in three different ways which we will consider. In the following computations, we wish to provide a reminder that the shortfall \( h_i \) of bank \( i \) also depends on \( \theta_{\text{min}} \), which must also be taken into account.

- **Cost of regulation on markets:** Consider the post-fire sale total market capitalization \( M^\top q^* \) where \((q^*, \bar{q}^*)\) denotes the clearing prices. As the regulatory threshold \( \theta_{\text{min}} \) increases, the stressed banks have to liquidate additional assets to satisfy the tighter regulatory environment. This causes prices to drop and feedback effects to the system. Ultimately, these effects are also felt by the market as a whole by causing a drop in the market capitalizations. Mathematically, the losses to the market capitalization caused by this marginal tightening of the regulatory threshold, denoted by \( CR \), is provided by

\[
CR := \frac{\partial M^\top (\bar{1} - q^*)}{\partial \theta_{\text{min}}} = -M^\top \frac{\partial q^*}{\partial \theta_{\text{min}}}.
\]

- **Cost of regulation from realized losses:** Let us consider the situation where, under the current regulatory regime, a bank has to liquidate a part of its assets. As the threshold \( \theta_{\text{min}} \) increases, that bank has to liquidate even more of its assets to satisfy the capital adequacy requirements. We can use the marginal change in losses from implementing the sales from a marginal change in \( \theta_{\text{min}} \) to quantify the cost of regulation. Let \((q^*, \bar{q}^*)\) be the clearing prices and let \( \gamma_i(q^*, \bar{q}^*) \) be the liquidation strategy of bank \( i \) under the current regulatory environment \( \theta_{\text{min}} \). Mathematically, for bank \( i \), we represent cost of regulation from realized loss, denoted \( CRL_i \), as

\[
CRL_i := \frac{\partial (\bar{1} - \bar{q}^*)^\top \gamma_i(q^*, \bar{q}^*; \theta_{\text{min}})}{\partial \theta_{\text{min}}} = -\left( \frac{\partial \bar{q}^*}{\partial \theta_{\text{min}}} \right)^\top \gamma_i(q^*, \bar{q}^*) + \left( \bar{1} - \bar{q}^* \right)^\top \frac{\partial \gamma_i(q^*, \bar{q}^*; \theta_{\text{min}})}{\partial \theta_{\text{min}}}.
\]

We note that \( \frac{\partial \bar{q}^*}{\partial \theta_{\text{min}}} \) can be computed using the results of Theorem 4.1 whereas \( \frac{\partial \gamma_i(q^*, \bar{q}^*; \theta_{\text{min}})}{\partial \theta_{\text{min}}} \) depends on liquidation strategy and can be computed explicitly from that construction.

Further, we wish to note that for the situation where banks are not liquidating any assets (i.e., \( i \in S(q^*, \bar{q}^*) \)), increasing \( \theta_{\text{min}} \) by a marginal amount will not result in increased liquidation losses; indeed in such instances \( CRL_i \) is equal to 0.
• **Cost of regulation from marked-to-market losses:** As we noted in the discussion of the cost of regulation from realized losses, increasing $\theta_{\text{min}}$ will not result in increased realized liquidation losses for solvent and liquid banks $i \in S(q^*, \bar{q}^*)$. However, increasing $\theta_{\text{min}}$ might cause some other bank to sell additional illiquid assets which would depreciate the price of these assets. As banks hold overlapping portfolios this causes impacts globally to all other banks due to mark-to-market accounting even if that particular bank was not liquidating assets itself; for bank $i$, we represent cost of regulation from marked-to-market losses, denoted $CMI_i$, as

$$CMI_i := -\frac{\partial}{\partial \theta_{\text{min}}} \left[ x_i + \ell_i + (\bar{q}^*)^T \gamma_i(q^*, \bar{q}^*; \theta_{\text{min}}) + (q^*)^T \left[ s_i - \gamma_i(q^*, \bar{q}^*; \theta_{\text{min}}) \right] - \bar{p}_i \right]$$

$$= - \left[ \left( \frac{\partial \bar{q}^*}{\partial \theta_{\text{min}}} \right)^T \gamma_i(q^*, \bar{q}^*) + \left( \frac{\partial q^*}{\partial \theta_{\text{min}}} \right)^T \left[ s_i - \gamma_i(q^*, \bar{q}^*) \right] + (\bar{q}^* - q^*)^T \frac{\partial \gamma_i(q^*, \bar{q}^*; \theta_{\text{min}})}{\partial \theta_{\text{min}}} \right].$$

Thus for bank $i$, $CMI_i$ is the (negative of the) sensitivity of its equity with respect to $\theta_{\text{min}}$. Hence $CMI_i$ captures the losses that are not reflected in $\text{CRL}_i$. We note that in the situation, where none of the banks need to liquidate a fraction of their assets (i.e., if $S(q^*, \bar{q}^*) \cup D(q^*, \bar{q}^*) = \mathcal{N}$), increasing $\theta_{\text{min}}$ will not result in any price depreciation or mark-to-market losses. Thus only in that case do we find that $CMI_i = 0$ for every bank $i$.

### 4.3 Value of rescue funds

Another interesting application of the sensitivity analysis provided by Theorem 4.1 is in quantifying the value of providing a marginal bailout or bail-in to the financial system. We define a bailout to be an external rescue fund used to prop up the health of the financial system, whereas a bail-in has funds made available by other banks in the financial system. We consider two possible structures for the use of these rescue funds: direct rescue by providing extra liquidity to a distressed bank or indirect rescue by purchasing troubled assets. We will consider both bailouts and bail-ins along with the appropriate determinations for when these strategies would be appropriate.

#### 4.3.1 Direct bailouts and bail-ins

By providing a small amount of additional (liquid) assets to a bank, the capital adequacy requirement becomes easier to attain and fewer assets need to be liquidated. In doing so, the clearing MTMPs will be improved. Of note, it is possible that providing a marginal amount of additional liquidity to a solvent but illiquid bank $i \in \mathcal{L}(q^*, \bar{q}^*)$ can have outsized effects on the
health of the system via the feedback effects inherent in the clearing mechanism.

- **Value of a direct bailout of bank $i$:** By providing additional liquidity to a bank, that bank improves its balance sheet and therefore need not sell as many assets. This causes the clearing MTMPs to grow, thus also raising the (time $t = 1$) market capitalization of the various assets. Of note, it is possible that providing a marginal bailout (to a solvent but illiquid firm $i \in \mathcal{L}(q^*, \bar{q}^*)$) can provide outsized effects on the market capitalization of the system, thus providing incentives to undertake this action. In particular, if the impacts of the bailout are larger than the initial cost of the bailout, then such incentives exist. Herein we will measure the value to society of a bailout by the impact on the total market capitalization. Specifically, as a bailout to bank $i$ increases the liquid holdings $x_i$, the value of a (marginal) direct bailout is the difference between the gains to market capitalization due to reducing shortfall $h_i$ and the costs of the bailout itself. Mathematically, we define this bailout decision structure as

$$DBO_i := - \left( \frac{\partial M^\top q^*}{\partial h_i} + 1 \right) = - \left( M^\top \frac{\partial q^*}{\partial h_i} + 1 \right).$$

In particular, the sign of $DBO_i$ is important. If $DBO_i > 0$ then a (marginal) bailout should be undertaken to prop up bank $i$, otherwise the bailout ultimately costs more than it benefits the system. Notably, this would only be given to firms that are solvent but illiquid $\mathcal{L}(q^*, \bar{q}^*)$ (but possibly not all of those such banks).

- **Value of a direct bail-in of bank $i$ from bank $j$:** In contrast to the bailout considered above, a bail-in from bank $j \notin \mathcal{D}(q^*, \bar{q}^*)$ to bank $i$ is a cash payment from $j$ to $i$. This type of payment simultaneously decreases the assets available to bank $j$ but improves the health of bank $i$. In particular, to consider such a bail-in, the capital of firm $j$ is studied given these positive and negative changes to the shortfall of banks $j$ and $i$ respectively. Recall that $h_j = \bar{p}_j - x_j - (1 - \alpha_{t,j} \theta_{\text{min}}) \ell_j$, therefore we can rewrite the capital of bank $j$ in terms of the shortfall $h_j$ by $C_j = -h_j + (q^*)^\top \gamma_j(q^*, \bar{q}^*; h_j) + (q^*)^\top [s_j - \gamma_j(q^*, \bar{q}^*; h_j)] + \alpha_{t,j} \theta_{\text{min}} \ell_j$. As such, similar to the study of bailouts, we can define the bail-in decisions via

$$DBI_{ji} := \frac{\partial C_j(h_i)}{\partial h_j} - \frac{\partial C_j(h_i)}{\partial h_i}$$

$$= \left( \frac{\partial q^*}{\partial h_j} - \frac{\partial \bar{q}^*}{\partial h_i} \right)^\top \gamma_j(q^*, \bar{q}^*) + \left( \frac{\partial q^*}{\partial h_j} - \frac{\partial \bar{q}^*}{\partial h_i} \right)^\top [s_j - \gamma_j(q^*, \bar{q}^*)]$$

$$+ (\bar{q}^* - q^*)^\top \left[ \frac{\partial \gamma_j(q^*, \bar{q}^*; h_j)}{\partial h_j} - \frac{\partial \gamma_j(q^*, \bar{q}^*; h_i)}{\partial h_i} \right] - 1.$$
We wish to note that, typically, \( \frac{\partial \gamma_j(q^*, \bar{q}^*)}{\partial h_i} = 0 \) as the behavior of bank \( j \) does not depend directly on the shortfall of bank \( i \); we refer back to the, e.g., the proportional liquidation strategy of Example 2.7(ii). As with the bailout above, if \( DBI_{ji} > 0 \) then bank \( j \) has the incentive to provide a marginal rescue fund to bank \( i \). Notably this would only be given to firms that are solvent but illiquid \( \mathcal{L}(q^*, \bar{q}^*) \) (but possibly not all of those such banks).

### 4.3.2 Indirect bailouts and bail-ins

In contrast to the direct bailouts and bail-ins mentioned above, it might be more politically palatable to directly purchase distressed assets. This was undertaken in the United States following the 2008 financial crisis through the Troubled Asset Relief Program. In particular, by providing additional liquidity to an asset, the health of each individual firm will improve through feedbacks inherent in the clearing mechanism. We wish to note that, in this setting, we are considering a modified version of the sensitivity analysis presented in Theorem 4.1.

To consider this setting we need to introduce the modified inverse demand function with the troubled asset purchasing. Let \( \beta \geq 0 \) be the cash invested in this indirect rescue of asset \( k \), then the modified clearing problem can be written as:

\[
(q, \bar{q}) = \left( F \left( \left[ \sum_{i \in \mathcal{N}} \gamma_i(q, \bar{q}) - \text{diag}[\bar{q}]^{-1}\beta \right]^+ \right), \bar{F} \left( \left[ \sum_{i \in \mathcal{N}} \gamma_i(q, \bar{q}) - \text{diag}[\bar{q}]^{-1}\beta \right]^+ \right) \right);
\]

as the rescue fund is also monotonic, the uniqueness argument presented in Theorem 3.1(iii) still holds. In considering this bailout or bail-in setting, we are interested in taking the derivative w.r.t. the rescue fund \( \beta_k \) at \( \beta = 0 \). Implicit differentiation and rearranging terms provides us with the form for \( \frac{\partial q^*}{\partial \beta_k} \) and \( \frac{\partial \bar{q}^*}{\partial \beta_k} \) as the same as that given in Theorem 4.1, but with new right-hand side, i.e.,

\[
\left( \frac{\partial q^*}{\partial \beta_k}, \frac{\partial \bar{q}^*}{\partial \beta_k} \right) = - \left[ I - \begin{pmatrix} \text{diag}[F'(\Gamma^*(q^*, \bar{q}^*))] & 0 \\ 0 & \text{diag}[\bar{F}'(\Gamma^*(q^*, \bar{q}^*))] \end{pmatrix} \right]^{-1} \begin{pmatrix} J\Gamma^*(q^*, \bar{q}^*) \\ J\Gamma^*(q^*, \bar{q}^*) \end{pmatrix} \times \left( \frac{f_k'(\Gamma^*(q^*, \bar{q}^*))}{q_i^*} \right) e_k \left( \frac{\bar{f}_k'(\Gamma^*(q^*, \bar{q}^*))}{\bar{q}_i^*} \right) e_k
\]

where \( e_k \in \mathbb{R}^m \) is the unit vector with a single 1 in its \( k^{th} \) component.

- **Value of an indirect bailout of asset \( k \):** By providing additional liquidity to an asset, the MTMP and VWAP of that asset would improve. Immediately this causes the clearing
prices to grow and, due to the pro-cyclical nature of the capital adequacy ratio, results in banks liquidating fewer assets as well. Therefore such an indirect bailout can easily have outsized benefits to the health of the financial system, even to assets other than the one getting the additional liquidity; this provides an incentive for this bailout to take place. As before, if the impacts of the bailout are larger than its initial cost, then the incentive structure is in place for a bailout to occur. Also as before, herein we will measure the value to society of a bailout by the impact on the total market capitalization. Mathematically, we define this bailout decision structure via

\[
IBO_k := \frac{\partial M^\top q^*}{\partial \beta_k} - 1 = M^\top \frac{\partial q^*}{\partial \beta_k} - 1.
\]

In particular, the sign of \(IBO_i\) is important. If \(IBO_i > 0\) then a (marginal) bailout should be undertaken to provide extra liquidity to asset \(k\), otherwise the bailout ultimately costs more than it benefits the system. Notably, this would only be given to assets in distress \(q^*_k < 1\) (but possibly not all of those such assets).

- **Value of an indirect bail-in of asset \(k\) from bank \(j\):** As with the indirect bailout, an indirect bail-in provides additional liquidity to a specific asset. This additional market liquidity can cause large feedback gains on the clearing prices. In order to participate in such a bail-in, we assume that the bank is solvent, i.e., \(j \not\in \mathcal{D}(q^*, q^*)\). Such a bank would decide to participate in an indirect bail-in if the effects from providing the liquidity (on firm capital) and holding marginal more units of asset \(k\) outweigh the costs from also increasing shortfall in tandem. As such, the value of an indirect bail-in from firm \(j\) on asset \(k\) can be computed mathematically via

\[
IBI_{jk} := \left. \frac{\partial C_j(h_j)}{\partial \beta_k} + \frac{1}{q_k} \frac{\partial C_j(s_{jk})}{\partial s_{jk}} + \frac{\partial C_j(h_j)}{\partial \beta_k} \right|_{\beta=0}
\]

\[
= \left( \frac{\partial q^*}{\partial h_j} + \frac{1}{q_k} \frac{\partial q^*}{\partial s_{jk}} + \frac{\partial q^*}{\partial \beta_k} \right)^\top \gamma_j(q^*, q^*) + \left( \frac{\partial q^*}{\partial h_j} + \frac{1}{q_k} \frac{\partial q^*}{\partial s_{jk}} + \frac{\partial q^*}{\partial \beta_k} \right)^\top \left[ s_j - \gamma_j(q^*, q^*) \right]
\]

\[
+ (q^* - q^*)^\top \left[ \left( \frac{\partial q^*}{\partial h_j} + \frac{1}{q_k} \frac{\partial q^*}{\partial s_{jk}} \right) \frac{\partial C_j}{\partial s_{jk}} \right] - \left( 1 - \frac{q_k^*}{q_k} \right).
\]

As with the indirect bailout above, if \(IBI_{jk} > 0\) then bank \(j\) has the incentive to provide a marginal rescue fund to prop up the value of asset \(k\). Notably, this would only be given to assets in distress \(q^*_k < 1\) (but possibly not all of those such assets).

**Remark 4.2.** We conjecture that a direct bailout (of a solvent but illiquid institution) will typically outperform an indirect bailout of the assets being held. This is due to leveraging
effects, i.e., if a bank is given additional capital, through leverage, they can *avoid* (in first order effects) liquidating a greater value of assets than they obtained in capital in the first place. An indirect bailout will always (in first order effects) compensate for exactly the value of the bailout. Additionally, leverage ratios are typically larger than 1, hence our conjecture.

5 Case studies

In this section we consider two case studies to discuss the implications of our model. For simplicity, each of the case studies is undertaken with a linear inverse demand function. We restrict the risk-weights $\alpha$ to the bound discussed in Corollary 3.4 and Example 3.6. Briefly, the two case studies are as follows:

(i) First, we consider a two asset, two bank system in order to explore the implications of diversification under different liquidation functions. We compare this case study with that of, e.g., [6].

(ii) Second, we consider a system of six large banks participating in the 2015 CCAR stress test as considered in [5]. We use this data to study the both the cost of regulation and the value of rescue funds in a real financial system.

5.1 Effects of liquidation strategies and diversification

In this case study, we consider a two bank ($n = 2$) and two asset ($m = 2$) system. We assume that the banks do not hold any liquid or non-marketable asset, i.e., $x_i = \ell_i = 0$ for $i = 1, 2$. We assume that both banks have liabilities $\bar{p}_i = 1$ and the total (pre-fire sale) market capitalization of each asset is 2, i.e., $M_k = s_{1k} + s_{2k} = 2$ for $k = 1, 2$.

We study the impact of diversification in this system by varying the composition of the illiquid asset holdings of each bank. To do this, we use a similar setting to Example 5.4 of [13]. That is, consider a parameter $\lambda \in [0, 2]$ and set $s_{11} = \lambda$, $s_{12} = M_2 - \lambda$, $s_{21} = M_1 - \lambda$, and $s_{22} = \lambda$. When $\lambda \in \{0, 2\}$, the banks are holding non-overlapping portfolios, corresponding to a *fully diverse system*. When $\lambda = 1$, the portfolios of the banks are identical, corresponding to a *fully diversified system*. Due to symmetry between the banks, we will only consider $\lambda \in [0, 1]$. Thus as $\lambda$ increases, the system moves from fully diverse to fully diversified.

For the purpose of this example, we will consider the linear inverse demand functions $f_k(\Gamma_k) = 1 - \frac{\Gamma_k}{\delta}$ for both assets $k = 1, 2$. Further, we will consider two liquidation functions in this case study in order to determine the impacts such choices have on the clearing
prices and market capitalizations; these liquidation functions are:

- **proportional liquidation** as discussed in Example 2.7(ii);

- **equilibrium liquidation** as discussed in Example 2.7(iv) in which both banks are trying to minimize their realized loss, i.e., $u_i(\gamma_i, \gamma^*_i) = -\gamma_i^T(1 - F(\gamma_i + \gamma^*_i))$. We note that this is a strictly decreasing and concave utility function on $\gamma_i \in [0, s_i]$, therefore the minimal liquidation assumption holds. Further this is a continuous, strictly concave, and diagonally strictly concave function (see, e.g., [4] for proof of such), therefore this strategy exists, is unique, and is continuous in the prices $(q, \bar{q})$ as discussed when introduced in Example 2.7.

**Remark 5.1.** Though the equilibrium liquidation strategy has a clearing solution (Theorem 3.1(i)), it need not be unique. We compute the clearing prices by Picard iterations beginning at $(q^0, \bar{q}^0) = (\vec{1}, \vec{1})$, i.e., no impacts. We wish to note that this method converged to a clearing solution for every choice of parameters tested hinting at a stronger property than proven thus far.

Consider the regulatory environment with $\theta = 0.2$ and with risk-weights $\alpha_1 = \alpha_2 = 2$ at time $t = 0$. With these parameters, both banks satisfy the capital adequacy requirements at $t = 0$ without any fire sales occurring. However, consider at time $t = 0^+$ the first asset has a credit downgrade causing its risk-weight to double, i.e., $\alpha_1 = 4$. This stress precipitates a fire sale of one or both banks depending on their investments at time $t = 1$ in order to satisfy the capital adequacy requirements. Notably, we consider the stress to the system to be a credit downgrade rather than a shock to the balance sheet of a bank as is typically assumed (see, e.g., [5, 13]). The results of this fire sale are displayed in Figure 3, which demonstrate the significant impacts that the choice of liquidation function has on the clearing prices.

First, in Figure 3a, we see that cross-asset contagion is a significant factor in the proportional liquidation scenario. That is, even though asset 2 is not shocked, under proportional liquidation its price monotonically decreases in $\lambda \in [0, 1]$. The equilibrium liquidation strategy appears to have cross-asset contagion, but only up to a point. This form of contagion is stronger than the proportional strategy for highly diverse portfolios ($\lambda \leq 0.3$), but reaches a limiting amount of contagion for more diversified portfolios. Second, in Figure 3b, we see that the total market capitalization of the system reaches its maximum, in the proportional liquidation setting, at $\lambda \approx 0.4$. In fact, the worst case in that setting is for the fully diversified portfolio. In contrast, the equilibrium liquidation strategy has nondecreasing total market capitalization as diversification increases. That is, diversification has the stabilizing effects typically assigned to it. In aggregate, the proportional liquidation strategy outperforms the equilibrium strategy for
(a) Clearing MTMP under proportional and equilibrium liquidation strategies with varying levels of diversification.

Figure 3: Section 5.1: The impacts of diversification of portfolio holdings and liquidation strategies on system health.

(b) Clearing total market capitalization under proportional and equilibrium liquidation strategies with varying levels of diversification.

diverse systems ($\lambda \leq 0.55$) but underperforms for diversified investments ($\lambda \geq 0.55$). Previous research, which generally consider full diversification harmful from a contagion perspective, may have suffered from the bias of focusing on the proportional liquidation strategy. In actuality, policy should be designed considering the strategic aspect in liquidation functions.

5.2 CCAR 2015 case study

In this case study, we use our model to study a stress test of a real financial system. For this, we use the Comprehensive Capital Analysis and Review (CCAR) 2015 data and consider the six global systemically important banks with large trading operations, i.e., Bank of America, Citigroup, Goldman Sachs, JP Morgan Chase, Morgan Stanley, and Wells Fargo, as was done in [5]. The data for these organizations is shown in Table 1 which has been replicated from [5]. This data allows us to directly characterize the (pre-fire sale) value of the banks’ balance sheets (with liabilities equal to the difference between total assets and capital). For a detailed discussion of the CCAR dataset we refer to [5].

| Asset | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| $\alpha$ | 0.07 | 0.08 | 0.12 | 0.15 | 0.18 | 0.2 | 0.25 | 0.35 | 0.5 | 0.6 | 0.75 | 1  | 2.5 | 4.25 | 6.5 |

Table 2: Section 5.2: Risk-weights $\alpha$ for the marketable illiquid assets.
From the CCAR data set we can immediately find the risk-weight $\alpha_{\ell,i}$ of each firm’s non-marketable portfolio by dividing the non-marketable risk-weighted assets by the value of the non-marketable assets $(\alpha_{\ell,i} \ell_i / \ell_i)$. In order to calibrate each bank’s marketable portfolio $s$, we make use of the risk-weights for commonly traded assets. We assume that there are $m = 16$ illiquid marketable asset and choose $\alpha$ for these 16 assets to cover a wide range of risk-weights as depicted in Table 2. For each bank $i$, the individual portfolio $s_i$ is chosen as the minimizer of the following minimum norm problem

$$\min_{\tilde{s}_i \in \mathbb{R}_m^+} \left\{ \| \tilde{s}_i \|_2 \mid \tilde{1}^T \tilde{s}_i = \tilde{1}^T s_i, \tilde{1}^T A \tilde{s}_i = \tilde{1}^T A s_i \right\}$$

where the value of the assets $\tilde{1}^T s_i$ and the risk-weighted assets $\tilde{1}^T A \tilde{s}_i$ are provided in Table 1. Additionally, we use these risk-weights to calibrate the linear inverse demand functions $f_k(\Gamma_k) = 1 - b_k \Gamma_k$ by setting $b_k = \frac{4 \theta_{\min}}{\frac{3(1 - \alpha_k \theta_{\min}) M_k}}$; this choice of liquidity parameter guarantees the uniqueness properties of Corollary 3.4. Finally, in accordance with Basel III regulations, we set $\theta_{\min} = 0.08$. We wish to emphasize that the purpose of this calibration is to provide a demonstrative data set for the case study. An accurate calibration of the financial system is an interesting problem in itself and beyond the scope of the current work.

For simplicity and comparison to prior literature, we assume all banks follow a proportional liquidation strategy. Under the setting considered, no fire sale occurs. This validates our modeling assumptions as no large market event occurred during the period this data covers. Consider now a 5% shock to the non-marketable assets $\ell_i$ for each of the 6 banks. Under this stress regime, four banks (Citigroup, Goldman Sachs, Morgan Stanley and Wells Fargo) do not need to liquidate any assets in the fire sale. However, JP Morgan Chase is solvent but illiquid, and Bank of America is insolvent in this stress scenario.

With this stress scenario, we first wish to provide the costs of regulation (as discussed in
Section 4.2). This is displayed in Figure 4. Consistent with the theory, only Bank of America and JP Morgan Chase show a non-zero CRL, all 6 banks have a non-zero CMI. This highlights that even though four banks are not liquidating under the current stress regime, they will incur mark-to-market losses if \( \theta_{\text{min}} \) is increased. We also wish to highlight the costs incurred to the market as a whole due to increased regulatory oversight \( CR = 3276.8 \) far exceeds the costs incurred by any individual bank.

![Figure 4: Section 5.2: Cost of Regulation CRL and CMI.](image)

Finally, we wish to consider the value of direct and indirect rescues of this system. For direct rescues, we look only at the values of supporting only JP Morgan Chase, as all other banks are either liquid or insolvent and, as such, would be unaffected by a marginal influx of liquidity. Notably, we find that there is a benefit to society for providing such a bailout to JP Morgan Chase with \( DBO_{JPM} = 0.3507 \). However, no solvent and liquid institution benefits enough from joining a bail-in in this scenario \( DBI_{JPM} \approx -0.8 \). As conjectured in Remark 4.2, the value of the indirect bailout of every asset is below that of the direct bailout of JP Morgan Chase. In fact, the value of the indirect bailout is negative (with the high risk-weighted assets having \( IBO \) closer to 0 but still negative). We did not compute the indirect bail-in values due to the combination of results for the direct bail-in and indirect bailouts.
A Proofs from Section 3

A.1 Proof of Theorem 3.1

Proof. (i) This is a trivial application of Brouwer’s fixed point theorem.

(ii) This is a trivial application of Tarski’s fixed point theorem.

(iii) As discussed in the preceding section, a bank $i$ can belong to any of the following three mutually exclusive and exhaustive sets:

- solvent and liquid: $S(q, \bar{q}) = \{i \in \mathcal{N} \mid h_i \leq q^\top [I - A\theta_{\min}] s_i\};$
- solvent but illiquid: $L(q, \bar{q}) = \{i \in \mathcal{N} \mid q^\top [I - A\theta_{\min}] s_i < h_i < \bar{q}^\top s_i\};$ or
- insolvent: $D(q, \bar{q}) = \{i \in \mathcal{N} \mid h_i \geq \bar{q}^\top s_i\}.$

Using the minimal liquidation condition (i), under any MTMP and VWAP prices $(q, \bar{q}) \in \mathbb{D}$ for any bank $i$:

$$(\bar{q} - [I - A\theta_{\min}] q)^\top \gamma_i(q, \bar{q}) = \begin{cases} 0 & \text{if } i \in S(q, \bar{q}) \\ h_i - q^\top [I - A\theta_{\min}] s_i & \text{if } i \in L(q, \bar{q}) \\ (\bar{q} - [I - A\theta_{\min}] q)^\top s_i & \text{if } i \in D(q, \bar{q}). \end{cases}$$

Using (iii), there exists a greatest and least clearing price $(q^\uparrow, \bar{q}^\downarrow) \geq (q^\downarrow, \bar{q}^\uparrow).$ Further from $\Gamma$ nonincreasing, $\Gamma^\uparrow := \Gamma(q^\uparrow, \bar{q}^\downarrow)^\top \Gamma \leq (\Gamma(q^\downarrow, \bar{q}^\uparrow))^\top \Gamma =: \Gamma^\downarrow.$

Assume that there does not exist a unique clearing price, i.e., there exists some asset $k$ such that either $q^\uparrow_k > q^\downarrow_k$ or $\bar{q}^\uparrow_k > \bar{q}^\downarrow_k$ (and thus $\Gamma^\uparrow_k < \Gamma^\downarrow_k$). Thus considering that both $(q^\uparrow, \bar{q}^\downarrow)$ and $(q^\downarrow, \bar{q}^\uparrow)$ are clearing solutions, we utilize the additional property on the inverse demand functions to find a contradiction:

$$0 > [(\bar{q}^\top)^\top \Gamma^\top + (q^\top)^\top [I - A\theta_{\min}] (M - \Gamma^\top)] - [(\bar{q}^\top)^\top \Gamma^\downarrow + (q^\top)^\top [I - A\theta_{\min}] (M - \Gamma^\downarrow)]$$
$$\geq [(\bar{q}^\top)^\top \Gamma^\top + (q^\top)^\top [I - A\theta_{\min}] (\sum_{i=1}^n s_i - \Gamma^\top)] - [(\bar{q}^\top)^\top \Gamma^\downarrow + (q^\top)^\top [I - A\theta_{\min}] (\sum_{i=1}^n s_i - \Gamma^\downarrow)]$$
$$= \sum_{i \in D^\top \cap D^\downarrow} (\bar{q}^\top - q^\top)^\top s_i + \sum_{i \in S^\top \cap D^\downarrow} (h_i - (\bar{q}^\top)^\top s_i) + \sum_{i \in S^\top \cap S^\downarrow} ([I - A\theta_{\min}] q^\top - \bar{q}^\downarrow)^\top s_i$$
$$+ \sum_{i \in L^\top \cap L^\downarrow} (h_i - h_i) + \sum_{i \in S^\top \cap L^\downarrow} ((\bar{q}^\top)^\top [I - A\theta_{\min}] s_i - h_i) + \sum_{i \in S^\top \cap S^\downarrow} (q^\top - q^\downarrow)^\top [I - A\theta_{\min}] s_i$$
$$> 0$$

where $S^\top := S(q^\top, \bar{q}^\top)$ and $S^\downarrow, L^\top, L^\downarrow, D^\top, D^\downarrow$ are defined likewise.
A.2 Proof of Corollary 3.4

Proof. By Theorem 3.1, we need only prove that \( \Gamma^* \in [0, M] \mapsto \tilde{F}(\Gamma^*)^\top \Gamma^* + F(\Gamma^*)^\top [I - A\theta_{\text{min}}] (M - \Gamma^*) \) is strictly increasing. By taking derivatives and rearranging terms, this is true if for every asset \( k \)

\[
\alpha_k > -\frac{1}{\theta_{\text{min}} f_k(\Gamma_k^*)} \frac{(M_k - \Gamma_k^*) f'_k(\Gamma_k^*)}{(M_k - \Gamma_k^*) f''_k(\Gamma_k^*)} \quad \forall \Gamma_k^* \in [0, M_k].
\]

The additional condition on the inverse demand function \( F \) imposed in this corollary is sufficient to ensure that the right-hand side of (3), i.e., \(-\frac{1}{\theta_{\text{min}} f_k(\Gamma_k^*)} \frac{(M_k - \Gamma_k^*) f'_k(\Gamma_k^*)}{(M_k - \Gamma_k^*) f''_k(\Gamma_k^*)}\) is nonincreasing in \( \Gamma_k^* \in [0, M_k] \)

Thus to ensure (3), we require \( \alpha_k \) to satisfy the inequality at \( \Gamma_k^* = 0 \). Using this fact and Assumption 2.6, uniqueness is ensured by the conditions of this corollary.

\[\square\]

B Proofs from Section 4

B.1 Proof of Theorem 4.1

Proof. Consider the clearing procedure 2. Implicit differentiation w.r.t. \( \# \) at the equilibrium prices \((q^*, \bar{q}^*)\) provides the pair of linear equations for every asset \( k \):

\[
\frac{\partial q_k^*}{\partial \#} = \frac{\partial f_k(\Gamma_k^*(q^*, \bar{q}^*; \#))}{\partial \#}
= f_k(\Gamma_k^*(q^*, \bar{q}^*; \#)) \sum_{l=1}^{m} \left( \frac{\partial \Gamma_k^*(q^*, \bar{q}^*; \#)}{\partial q_l^*} \frac{\partial q_l^*}{\partial \#} + \frac{\partial \Gamma_k^*(q^*, \bar{q}^*; \#)}{\partial \bar{q}_l^*} \frac{\partial \bar{q}_l^*}{\partial \#} \right)
+ \frac{\partial \Gamma_k^*(q^*, \bar{q}^*; \#)}{\partial \#}
\]

\[
\frac{\partial \bar{q}_k^*}{\partial \#} = \frac{\partial f_k(\Gamma_k^*(q^*, \bar{q}^*; \#))}{\partial \#}
= f_k(\Gamma_k^*(q^*, \bar{q}^*; \#)) \sum_{l=1}^{m} \left( \frac{\partial \Gamma_k^*(q^*, \bar{q}^*; \#)}{\partial q_l^*} \frac{\partial q_l^*}{\partial \#} + \frac{\partial \Gamma_k^*(q^*, \bar{q}^*; \#)}{\partial \bar{q}_l^*} \frac{\partial \bar{q}_l^*}{\partial \#} \right)
+ \frac{\partial \Gamma_k^*(q^*, \bar{q}^*; \#)}{\partial \#}
\]

In matrix notation, this problem reduces to solving the linear system:

\[
[I - W] \begin{pmatrix} \frac{\partial q}{\partial \#} \\ \frac{\partial \bar{q}}{\partial \#} \end{pmatrix} = \begin{pmatrix} \text{diag} [F'(\Gamma^*(q^*, \bar{q}^*))] \\ \text{diag} [\tilde{F}'(\Gamma^*(q^*, \bar{q}^*))] \end{pmatrix} \frac{\partial \Gamma^*(q^*, \bar{q}^*)}{\partial \#}
\]

\[
W = \begin{pmatrix} \text{diag} [F'(\Gamma^*(q^*, \bar{q}^*))] & 0 \\ 0 & \text{diag} [\tilde{F}'(\Gamma^*(q^*, \bar{q}^*))] \end{pmatrix} \begin{pmatrix} J \Gamma^*(q^*, \bar{q}^*) \\ J \Gamma^*(q^*, \bar{q}^*) \end{pmatrix}
\]

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Thus the result is proven so long as $I - W$ is invertible. Note that $W$ is independent of the choice of parameter $\#$.

We will now prove that $W$ is an invertible matrix by considering standard input-output analysis. To do so, we first wish to point out that $(q, \bar{q})$ and $I_q$ for any $q$, $(\hat{\Gamma}(q, \bar{q}))$ and $\hat{\Gamma}(q, \bar{q})$ are both invertible matrices by considering standard input-output analysis. We will now prove that $W$ is invertible. Note that $(q, \bar{q})$ is independent of the

For simplicity of notation, define $L^* := L(q^*, q^*)$. Let $(q, \bar{q}) \in \mathbb{D} \mapsto \hat{\Gamma}(q, \bar{q}) := \sum_{i \in L(q, \bar{q})} \gamma_i(q, \bar{q})$ and $(q, \bar{q}) \in \mathbb{D} \mapsto \hat{H}(q, \bar{q}) := \sum_{i \in L(q, \bar{q})} h_i$. We wish to note that \(\frac{\partial \gamma_i(q, \bar{q})}{\partial q_k} = \frac{\partial \gamma_i(q, \bar{q})}{\partial \bar{q}_k} = 0\) for any asset $k$ with prices $(q, \bar{q}) \in \mathbb{D}$ such that $i \in S(q, \bar{q}) \cup D(q, \bar{q})$.

The minimal liquidation condition (II) implies that:

$$(\bar{q} - [I - A\theta_{\min}]q)^\top \hat{\Gamma}(q, \bar{q}) = \hat{H}(q, \bar{q}) - q^\top [I - A\theta_{\min}] \sum_{i \in L(q, \bar{q})} s_i$$

for any $(q, \bar{q}) \in \mathbb{D}$. Assuming no bank is at the boundary between $L(q, \bar{q})$ and either $S(q, \bar{q})$ or $D(q, \bar{q})$ (if so then one-sided derivatives would be required), we are able to determine by implicit differentiation that for any prices $(q, \bar{q}) \in \mathbb{D}$ and asset $k$

$$[1 - \alpha_k\theta_{\min}] \left( \sum_{i \in L(q, \bar{q})} s_{ik} - \hat{\Gamma}_k(q, \bar{q}) \right) = -(\bar{q} - [I - A\theta_{\min}]q)^\top \frac{\partial \hat{\Gamma}^*(q, \bar{q})}{\partial q_k}$$

$$\hat{\Gamma}_k(q, \bar{q}) = -(\bar{q} - [I - A\theta_{\min}]q)^\top \frac{\partial \hat{\Gamma}^*(q, \bar{q})}{\partial \bar{q}_k}. \quad (4)$$

By assumption and previous discussions of firm behaviors, $(q, \bar{q}) \in \mathbb{D} \mapsto F(\Gamma^*(q, \bar{q})) \hat{\Gamma}(q, \bar{q}) + F(\Gamma^*(q, \bar{q}))^\top [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q, \bar{q}) \right)$ is strictly decreasing on $(q, \bar{q}) \in \{(q, \bar{q}) \in \mathbb{D} \mid L(q, \bar{q}) = L^*\}$. By differentiation w.r.t. the MTMP $q_k$, this implies at the clearing prices $(q^*, \bar{q}^*)$

$$0 > F(\Gamma^*(q^*, \bar{q}^*))^\top \frac{\partial \Gamma^*(q^*, \bar{q}^*)}{\partial q_k} + \left( \frac{\partial F(\Gamma^*(q^*, \bar{q}^*))}{\partial q_k} \right)^\top \hat{\Gamma}(q^*, \bar{q}^*)$$

$$- F(\Gamma^*(q^*, \bar{q}^*))^\top [I - A\theta_{\min}] \frac{\partial \Gamma^*(q^*, \bar{q}^*)}{\partial q_k} + \left( \frac{\partial F(\Gamma^*(q^*, \bar{q}^*))}{\partial \bar{q}_k} \right)^\top [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right)$$

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for any asset $k$. That is, for any asset $k$,

$$
\left( \frac{\partial F^*(q^*, \bar{q}^*)}{\partial q_k} \right) ^\top \hat{\Gamma}(q^*, \bar{q}^*) + \left( \frac{\partial F^*(q^*, \bar{q}^*)}{\partial q_k} \right) ^\top [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right)
$$

$$
< - (\bar{F}(\Gamma^*(q^*, \bar{q}^*))) - [I - A\theta_{\min}]F(\Gamma^*(q^*, \bar{q}^*)) ^\top \frac{\partial \Gamma^*(q^*, \bar{q}^*)}{\partial q_k}.
$$

Through comparison with (4), we are able to conclude that

$$
\left( \frac{\partial \bar{F}(\Gamma^*(q^*, \bar{q}^*))}{\partial q_k} \right) ^\top \hat{\Gamma}(q^*, \bar{q}^*) + \left( \frac{\partial F^*(q^*, \bar{q}^*)}{\partial q_k} \right) ^\top [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right)
$$

$$
< [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right).
$$

Through the same analysis but applying (5) to the derivatives w.r.t. $\bar{q}_k$, we can also conclude

$$
\left( \frac{\partial \bar{F}(\Gamma^*(q^*, \bar{q}^*))}{\partial \bar{q}_k} \right) ^\top \hat{\Gamma}(q^*, \bar{q}^*) + \left( \frac{\partial F^*(q^*, \bar{q}^*)}{\partial \bar{q}_k} \right) ^\top [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right)
$$

$$
< \hat{\Gamma}(q^*, \bar{q}^*).
$$

Finally, we consider the vector

$$
v := \begin{pmatrix} [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right) \\ \hat{\Gamma}(q^*, \bar{q}^*) \end{pmatrix} \geq 0.
$$

By construction of the matrix $W$ and an application of (6) and (7), we find that

$$
W^\top v = \begin{pmatrix} \left( \frac{\partial \bar{F}(\Gamma^*(q^*, \bar{q}^*))}{\partial q_{l_1}} \right) ^\top \hat{\Gamma}(q^*, \bar{q}^*) + \left( \frac{\partial F^*(q^*, \bar{q}^*)}{\partial q_{l_1}} \right) ^\top [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right) \\ \vdots \\ \left( \frac{\partial \bar{F}(\Gamma^*(q^*, \bar{q}^*))}{\partial q_{l_m}} \right) ^\top \hat{\Gamma}(q^*, \bar{q}^*) + \left( \frac{\partial F^*(q^*, \bar{q}^*)}{\partial q_{l_m}} \right) ^\top [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right) \\ \left( \frac{\partial \bar{F}(\Gamma^*(q^*, \bar{q}^*))}{\partial \bar{q}_{l_1}} \right) ^\top \hat{\Gamma}(q^*, \bar{q}^*) + \left( \frac{\partial F^*(q^*, \bar{q}^*)}{\partial \bar{q}_{l_1}} \right) ^\top [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right) \\ \vdots \\ \left( \frac{\partial \bar{F}(\Gamma^*(q^*, \bar{q}^*))}{\partial \bar{q}_{l_m}} \right) ^\top \hat{\Gamma}(q^*, \bar{q}^*) + \left( \frac{\partial F^*(q^*, \bar{q}^*)}{\partial \bar{q}_{l_m}} \right) ^\top [I - A\theta_{\min}] \left( \sum_{i \in L^*} s_i - \hat{\Gamma}(q^*, \bar{q}^*) \right) \end{pmatrix}
$$

$$
< v.
$$

Thus an application of, e.g., Theorem 2.1 of [13], $(I - W)^{-1}$ exists and is given by the Leontief inverse. \[\square\]
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