New derivation of soliton solutions to the AKNS$_2$ system via dressing transformation methods

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Abstract
We consider certain boundary conditions supporting soliton solutions in the generalized nonlinear Schrödinger equation (AKNS$_r$) ($r = 1, 2$). Using the dressing transformation (DT) method and the related tau functions, we study the AKNS$_r$ system for the vanishing, (constant) non-vanishing and the mixed boundary conditions, and their associated bright, dark and bright–dark $N$-soliton solutions, respectively. Moreover, we introduce a modified DT related to the dressing group in order to consider the free-field boundary condition and derive generalized $N$ dark–dark solitons. As a reduced submodel of the AKNS$_r$ system, we study the properties of the focusing, defocusing and mixed focusing–defocusing versions of the so-called coupled nonlinear Schrödinger equation ($r$-CNLS), which has recently been considered in many physical applications. We have shown that two-dark–dark-soliton bound states exist in the AKNS$_2$ system, and three- and higher-dark–dark-soliton bound states cannot exist. The AKNS$_r$ ($r \geq 3$) extension is briefly discussed in this approach. The properties and calculations of some matrix elements using level-one vertex operators are outlined.

Dedicated to the memory of S S Costa

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(Some figures may appear in colour only in the online journal)

1. Introduction

Many soliton equations in $1 + 1$ dimensions have integrable multi-component generalizations or more generally integrable matrix generalizations. It is well known that certain coupled multi-field generalizations of the nonlinear Schrödinger equation (CNLS) are integrable and possess soliton-type solutions with rich physical properties (see e.g. [1–4]). The model defined by two coupled NLS systems was earlier studied by Manakov [5]. Another remarkable example
of a multi-field generalization of an integrable model is the so-called generalized sine-Gordon model which is integrable in some regions of its parameter space and has many physical applications [6, 7]. The type of coupled NLS equations finds applications in diverse areas of physics such as nonlinear optics, optical communication, biophysics, multi-component Bose–Einstein condensate, etc (see e.g. [1–3, 8, 9]). The multi-soliton solutions of these systems have recently been considered using a variety of methods depending on the initial-boundary values imposed on the solution. For example, the direct methods, mostly the Hirota method, have been applied in [10, 3], and in [11] a wide class of NLS models have been studied in the framework of the Darboux-dressing transformation. Recently, some properties have been investigated, such as the appearance of stationary bound states formed by two dark–dark solitons in the mixed-nonlinearity case (focusing and defocusing) of the 2-CNLS system [4].

The inverse scattering transform (IST) method for the defocusing CNLS model with non-vanishing boundary conditions (NVBC) has been an open problem for over 30 years. The two-component case was solved in [12] and the multi-component model has very recently been presented in [13]. The inverse scattering [12, 13] and Hirota method [10] results on N dark–dark solitons in the defocusing 2-CNLS model have presented only the degenerate case, i.e. the multi-solitons of both components are proportional to each other and therefore are reducible to the dark solitons of the scalar NLS model.

The $r$-CNLS model is related to the AKNS$_r$ system, which is a model with $2r$ dependent variables. As we will show below, this system reduces to the $r$-CNLS model under a particular reality condition. Moreover, general multi-dark–dark-soliton solutions of generalized AKNS-type systems available in the literature, to our knowledge, are scarce. Recently, there appeared some reports on the general non-degenerate $N$ dark–dark solitons in the $r$-CNLS model [4, 14]. In [4], the model with defocusing and mixed nonlinearity is studied in the context of the KP-hierarchy reduction approach. The mixed focusing and defocusing nonlinearity 2-CNLS model presents a two-dark–dark-soliton stationary bound state, whereas three or more dark–dark solitons cannot form bound states [4]. Here we show that similar phenomena are present in the AKNS$_2$ system. In [14], dark and bright multi-soliton solutions of the $r$-CNLS model are derived from multi-soliton solutions of the AKNS$_r$ system for arbitrary $r$ in the framework of the algebro-geometric approach.

In this paper, we will consider the general constant (vanishing, non-vanishing and mixed vanishing–non-vanishing) boundary value problem for the AKNS$_r$ ($r = 1, 2$) model and show that its particular complexified and reduced version incorporates the focusing and defocusing scalar NLS system for $r = 1$, and the focusing, defocusing and mixed nonlinearity versions of the 2-CNLS system, i.e. the Manakov model, for $r = 2$, respectively. We will consider the dressing transformation (DT) method to solve integrable nonlinear equations, which is based on the Lax pair formulation of the system. In this approach, the so-called integrable highest weight representation (h.w.r.) of the underlying affine Lie algebra is essential to find soliton-type solutions (see [15] and references therein). According to the approach of [15], a common feature of integrable hierarchies presenting soliton solutions is the existence of some special ‘vacuum solutions’ such that the Lax operators evaluated on them lie in some Abelian (up to the central term) subalgebra of the associated Kac–Moody algebra. The boundary conditions imposed to the system of equations must be related to the relevant vacuum connections lying in certain Abelian sub-algebra. The soliton-type solutions are constructed out of those ‘vacuum solutions’ through the so-called soliton specialization in the context of the DT. These developments lead to a quite general definition of tau functions associated with the hierarchies, in terms of the so-called integrable highest weight representations of the relevant Kac–Moody algebra. However, the free-field boundary condition does not allow the vacuum connections to lie in an Abelian sub-algebra, so we will introduce a modified DT in order to
deal with this case. We consider a modified DT relying on the dressing group composition law of two successive DTs [29]. The corresponding tau functions will provide generalized dark–dark-soliton solutions possessing additional parameters as compared to those obtained with constant boundary conditions.

Here we adopt a hybrid of the DT and Hirota methods [7, 18] to obtain soliton solutions of the AKNS system. As pointed out in [17], the Hirota method presents some drawbacks in order to derive a general type of soliton solutions, noticeably it is not appropriate to handle vector NLS-type models in a general form since the process relies on an insightful guess of the functional forms of each component, whereas in the group theoretical approach adopted here, the dependence of each component on the generalized tau functions becomes dictated by the DT and the solutions in the orbit of certain vacuum solutions are constructed in a uniform way. The generalized tau functions for the nonlinear systems are defined as an alternative set of variables corresponding to certain matrix elements evaluated in the integrable h.w.r. of the underlying affine Kac–Moody algebra. In this way, we overcome two difficulties of the methods. The Hirota method needs the tau functions and an expansion for them, but it does not provide a recipe for how to construct them. That is accurately solved through the DT method, which in turn needs the evaluation of certain matrix elements in the vertex operator representation. We may avoid the cumbersome matrix element calculations through the Hirota expansion method. Since this method is recursive, it allows a simple implementation on a computer program for algebraic manipulation like Mathematica.

The paper is organized as follows. In section 2, we present the generalized nonlinear Schrödinger equation (AKNS2) associated with the homogeneous gradation of the Kac–Moody algebra \( \hat{sl}_3 \). The two-component coupled nonlinear Schrödinger (2-CNLS) equation is defined as a particular reduction by imposing certain conditions on the AKNS2 fields. In section 3, we review the theory of the DT, describe the various constant boundary conditions and define the tau functions. In section 3.1, we present the modified DT associated with the dressing group suitable for free-field boundary conditions. In section 4, we apply the DT method to the vanishing boundary conditions and derive the bright solitons. In section 5, we consider the NVBC and derive the AKNS \( r (r = 1, 2) \) dark solitons. Moreover, the general \( N \) dark–dark solitons with free-field NVBC of the AKNS2 are derived through the modified DT method. In section 5.3, we discuss the dark–dark-soliton bound states. In section 6, the dark–bright solitons are derived associated with the mixed boundary conditions. In section 7, we briefly discuss the AKNS \( r (r \geq 3) \) extension in the framework of the DT methods, and in section 8, we discuss the main results of the paper. Finally, we have included the \( \hat{sl}(2) \) and \( \hat{sl}(3) \) affine Kac–Moody algebra properties, in appendices A and B, respectively. In appendix C, the matrix elements in \( \hat{sl}(3) \) have been computed using vertex operator representations.

2. The model

The AKNS2 model can be constructed in the framework of the Zakharov–Shabat formalism using the Hermitian symmetric spaces [19]. In [20], an affine Lie algebraic formulation based on the loop algebra \( g \otimes \mathbb{C}[\lambda, \lambda^{-1}] \) of \( g \in sl(3) \) has been presented. Here instead we construct the model associated with the full affine Kac–Moody algebra \( \mathcal{G} = \hat{sl}(3) \) with homogeneous gradation and a semi-simple element \( E^{(i)} \) (see appendix B). So the connections are given by

\[
A = E^{(1)} + \sum_{i=1}^{2} \Psi^+ E_{\beta_i}^{(0)} + \sum_{i=1}^{2} \Psi^- E_{-\beta_i}^{(0)} + \phi_1 C, \quad \text{(2.1)}
\]
In the basis considered in this paper, one has

$$B = E^{(2)} + \sum_{i=1}^{2} \Psi_{i}^* E_{\beta_i}^{(1)} + \sum_{i=1}^{2} \Psi_{i}^- E_{-\beta_i}^{(1)} + \sum_{i=1}^{r} \partial_i \Psi_{i}^* E_{\beta_i}^{(0)} - \sum_{i=1}^{r} \partial_i \Psi_{i}^- E_{-\beta_i}^{(0)}$$

$$= \left[ \Psi_{i}^* \Psi_{i}^- - \Omega_{1} \right] \left( H_{1}^{(0)} + H_{2}^{(0)} \right) - \left[ \Psi_{2}^* \Psi_{2}^- - \Omega_{2} \right] H_{2}^{(0)}$$

$$- \Psi_{1}^* \Psi_{2}^- E_{\beta_{12}}^{(0)} - \Psi_{2}^* \Psi_{1}^- E_{-\beta_{12}}^{(0)} + \phi \beta C,$$

(2.2)

where $\Psi_{i}^*$, $\Psi_{i}^-$, $\phi_{1}$ and $\phi_{2}$ are the fields of the model. In this construction, we consider the fields as being real; however, some reductions and complexification procedures will be performed below. Note that the auxiliary fields $\phi_{1}$ and $\phi_{2}$ lie in the direction of the affine Lie algebra central term $C$. The existence of this term plays an important role in the theory of the so-called integrable highest weight representations of the Kac–Moody algebra and they will be important below in order to find the soliton-type solutions of the model. Connections (2.1) and (2.2) conveniently incorporate the terms with the constant parameters $\Omega_{1,2}$ in order to take into account the various boundary conditions, so these differ slightly from those in $[20, 21]$. The $\hat{sl}(3)$ Kac–Moody algebra conventions and notations are presented in appendix B. In the basis considered in this paper, one has

$$E^{(l)} = \frac{1}{3} \sum_{a=1}^{2} aH_{a}^{(l)}.$$

(2.3)

The $\beta_i$ in (2.1) and (2.2) are positive roots defined in (B.24) such that

$$\beta_{1} \equiv \alpha_{1} + \alpha_{2}; \quad \beta_{2} \equiv \alpha_{2}, \alpha_{1}, \alpha_{2} \equiv \text{simple roots.}$$

(2.4)

Note that the connections lie in the subspaces defined in (B.12),

$$A \in \mathfrak{g}_{1} + \mathfrak{g}_{2}, \quad B \in \mathfrak{g}_{1} + \mathfrak{g}_{2} \quad [D, \mathfrak{g}_{n}] = n \mathfrak{g}_{n}.$$

(2.5)

The zero-curvature condition $[\partial_{t} - B, \partial_{x} - A] = 0$ supplied with the $\hat{sl}(3)$ commutation relations (B.4)–(B.11) and (B.25)–(B.31) provides the following system of equations:

$$\partial_{t} \Psi_{i}^+ = +\partial_{x}^2 \Psi_{i}^+ - 2 \sum_{j=1}^{2} \Psi_{j}^{+} \Psi_{j}^{-} - \frac{1}{2} (\beta_{i}, \bar{\Omega}) \Psi_{i}^+, \quad i = 1, 2$$

(2.6)

$$\partial_{t} \Psi_{i}^- = -\partial_{x}^2 \Psi_{i}^- + 2 \sum_{j=1}^{2} \Psi_{j}^{+} \Psi_{j}^{-} - \frac{1}{2} (\beta_{i}, \bar{\Omega}) \Psi_{i}^-, \quad i = 1, 2$$

(2.7)

$$\partial_{t} \phi_{1} - \partial_{x} \phi_{2} = 0;$$

(2.8)

$$\bar{\Omega} \equiv \sum_{i=1}^{2} \Omega_{i} \beta_{i}, \quad (\beta_{1}, \bar{\Omega}) = 2 \Omega_{1} + \Omega_{2}, \quad (\beta_{2}, \bar{\Omega}) = 2 \Omega_{2} + \Omega_{1}, \quad \Omega_{1,2} = \text{const.}$$

(2.9)

Note that the auxiliary fields $\phi_{1,2}$ completely decouple from the AKNS2 fields $\Psi_{j}^{\pm}$. The constant parameters $\Omega_{1,2}$ will be related below to certain boundary conditions and some trivial solutions of the system (2.6)–(2.7).

The integrability of the system of equations (2.6) and (2.7), for $\Omega_{1,2} = 0$, and its multi-Hamiltonian structure have been established [21]. A version of (2.6)–(2.7) for arbitrary $r$ has recently been considered in [14]. The system of equations (2.6) and (2.7) supplied with a convenient complexification can be related to some versions of the so-called coupled nonlinear Schrödinger equation (CNLS) [5, 1–3]. For example, by making

$$t \rightarrow -it, \quad [\Psi_{i}^{+}]^{*} = -\mu \partial_{x} \Psi_{i}^{-} \equiv -\mu \partial_{x} \psi_{i},$$

(2.10)
where $\ast$ means complex conjugation, $\mu \in \mathbb{R}_+$, $\delta_i = \pm 1$, we may reduce the system (2.6)–(2.7) to the well-known 2-CNLS,

$$
\begin{align*}
  i \partial_t \psi_k + \delta_k^2 \psi_k + 2\mu \left( \sum_{j=1}^2 \delta_j |\psi_j|^2 - \frac{1}{2}|(\beta_k \cdot \Omega)| \right) \psi_k = 0, \quad k = 1, 2.
\end{align*}
$$

(2.11)

The term $(\beta_k \cdot \Omega)$ is provided in (2.9) and the parameter $\mu > 0$ represents the strength of nonlinearity and the coefficients $\delta_j$ define the sign of the nonlinearity. The system (2.11) can be classified into three classes depending on the signs of the nonlinearity coefficients $\delta_j$. For $\delta_1 = \delta_2 = 1$, this system is the focusing Manakov model which supports bright–bright solitons [5]. For $\delta_1 = \delta_2 = -1$, it is the defocusing Manakov model which supports bright–dark and dark–dark solitons [10, 12, 22]. In the cases $\delta_1 = -\delta_2 = \pm 1$, one has the mixed focusing–defocusing nonlinearities. In this case, these equations support bright–bright solitons [23, 3], bright–dark solitons [24]. The defocusing and mixed nonlinearity cases have recently been considered in [4] through the KP-hierarchy reduction method and dark–dark solitons have been obtained. The system (2.11) has also been considered in the study of oscillations and interactions of dark and dark–bright solitons in Bose–Einstein condensates [25, 26]. The multi-dark–dark solitons in the mixed nonlinearity case are useful for many physical applications such as nonlinear optics, water waves and Bose–Einstein condensates, where the generally coupled NLS equations often appear.

The focusing CNLS system possesses a remarkable type of soliton solution undergoing a shape changing (inelastic) collision property due to the intensity redistribution among its modes. In this context, a novel class of solutions called partially coherent solitons (PCS) which are of substantially variable shape has been found, such that under collisions the profiles remain stationary [1, 27, 28]. Interestingly, the PCSs, namely 2-PCS, 3-PCS, ..., $r$-PCS, are special cases of the well-known 2-, 3-, ..., $r$-soliton solutions of the 2-, 3-, ..., $r$-CNLS equations, respectively [2, 3]. So the understanding of the variable shape collisions and many other properties of these PCS can be studied by providing the higher order soliton solutions of the $r$-CNLS ($r \geq 2$) system considered as the submodel of the relevant AKNS. We believe that the group theoretical point of view of finding the analytical results for the general case of $N$-soliton interactions would facilitate the study of their properties, for example, the asymptotic behavior of trains of $N$-soliton-like pulses with approximately equal amplitudes and velocities, as studied in [16]. Note that the set of solutions of the AKNS model (2.6)–(2.7) is much larger than the solutions of the CNLS system (2.11), since only the solutions of the former which satisfy constraints (2.10) will be solutions of the CNLS model (2.11). This fact will be seen below in many instances, so we believe that the AKNS soliton properties with relevant boundary conditions deserve a further study.

We will show that the three classes mentioned above, i.e. the focusing, defocusing and the mixed focusing–defocusing CNLS model, can be related respectively to the vanishing, non-vanishing and mixed vanishing–non-vanishing boundary conditions of the AKNS, model in the framework of the DT approach adapted conveniently to each case. The (constant) NVBC require the extension of the DT method to incorporate non-zero constant vacuum solutions. Therefore, the vertex operators corresponding to the vanishing boundary case undergo a generalization in such a way that the nilpotency property, which is necessary in obtaining soliton solutions, should be maintained. Using the modified vertex operators, we construct multi-soliton solutions in the cases of (constant) non-vanishing and mixed vanishing–non-vanishing boundary conditions. The free-field NVBC requires a modification of the usual DT of [15] by considering two successive DTs in the context of the dressing group [29]. However,
the same vertex operator generating the dark–dark solitons in the (constant) NVBC will be used in the free-field NVBC case with modified tau functions.

3. DT for AKNS$_2$

In this section, we summarize the so-called DT procedure to find soliton solutions, which works by choosing a vacuum solution and then mapping it onto a non-trivial solution, following the approach of [15]. For simplicity, we concentrate on a version of the DT suitable for vanishing, (constant) non-vanishing and mixed boundary conditions of the AKNS$_2$ model (2.6)–(2.8). The free-field boundary condition requires a modified DT which is developed in subsection 3.1. So let us consider

$$\lim_{x \to -\infty} \Psi_j^{\pm} \to \rho_{j,L}^{\pm}; \quad \lim_{x \to +\infty} \Psi_j^{\pm} \to \rho_{j,R}^{\pm}; \quad \phi_{1,2} \to 0; \quad \rho_{j,L}^{\pm}, \rho_{j,R}^{\pm} = \text{const.} \quad (3.1)$$

We may identify NVBC (3.1) to certain classes of trivial vacuum solutions of the system (2.6)–(2.7).

1) The trivial zero-vacuum solution

$$\Psi_{j,\text{vac}}^{\pm} = \rho_j^{\pm} = 0, \quad j = 1, 2. \quad (3.2)$$

2) The trivial constant vacuum solution

$$\Psi_{j,\text{vac}}^{\pm} = \rho_j^{\pm} \neq 0, \quad j = 1, 2. \quad (3.3)$$

Note that (3.3) is a trivial constant vacuum solution of (2.6)–(2.7) provided that the expression \(\sum_{j=1}^{2} \rho_j^{\pm} \neq \frac{1}{2} (\beta, \Omega)\) vanishes for any \(i = 1, 2\), which is achieved if \(\Omega_1 = \Omega_2 = \Omega\), implying \(\sum_{j=1}^{2} \rho_j^{\pm} = \frac{1}{2} \Omega\).

3) The mixed constant-zero (zero-constant) vacuum solutions

\[
\begin{align*}
(i) \Psi_{1,\text{vac}}^{\pm} &= \rho_1^{\pm} \neq 0, & \Psi_{2,\text{vac}}^{\pm} &= \rho_2^{\pm} = 0, \\
(ii) \Psi_{1,\text{vac}}^{\pm} &= \rho_1^{\pm} = 0, & \Psi_{2,\text{vac}}^{\pm} &= \rho_2^{\pm} \neq 0.
\end{align*}
\]

The first mixed trivial solution (3.4) requires \(2\rho_1^{\pm} \rho_1^{-} = 2\Omega_1 + \Omega_2\), whereas for the second trivial solution (3.5), it must be \(2\rho_2^{\pm} \rho_2^{-} = 2\Omega_2 + \Omega_1\).

Connections (2.1) and (2.2) for the above vacuum solution (3.3) take the form

$$A^{\text{vac}} \equiv E^{(1)} + \rho_1^{\pm} E_{\beta_1}^{(0)} + \rho_1^{-} E_{-\beta_1}^{(0)} + \rho_2^{\pm} E_{\beta_2}^{(0)} + \rho_2^{-} E_{-\beta_2}^{(0)}, \quad (3.6)$$

$$B^{\text{vac}} \equiv E^{(2)} + \rho_1^{\pm} E_{\beta_1}^{(1)} + \rho_1^{-} E_{-\beta_1}^{(1)} + \rho_2^{\pm} E_{\beta_2}^{(1)} + \rho_2^{-} E_{-\beta_2}^{(1)} + \rho_1^{\pm} \rho_1^{-} E_{\beta_1}^{(0)} - \rho_1^{-} \rho_2^{\pm} E_{-\beta_1}^{(0)} - \rho_2^{\pm} \rho_1^{-} E_{\beta_2}^{(0)} - \rho_1^{\pm} \rho_2^{-} E_{-\beta_2}^{(0)} - \rho_1^{-} \rho_2^{-} E_{\beta_1}^{(0)} - \rho_2^{\pm} \rho_1^{-} E_{-\beta_2}^{(0)} - \rho_1^{\pm} \rho_2^{-} E_{\beta_2}^{(0)} - \rho_1^{-} \rho_2^{\pm} E_{-\beta_1}^{(0)}.$$  \( (3.7) \)

Note that \([A^{\text{vac}}, B^{\text{vac}}] = 0\) and in order to define these vacuum connections, it suffices to consider the constant values of \(\rho_{j,L(R)}^{\pm}\) in (3.1) related to one of the limits \(x \to \pm \infty\), say \(\rho_{j,L}^{\pm} \equiv \rho_j^{\pm}\) as above. These connections are related to the group element \(\Psi^{(0)}\) through

$$A^{(\text{vac})} = \partial_\beta \Psi^{(0)} [\Psi^{(0)}]^{-1}, \quad B^{(\text{vac})} = \partial_\beta \Psi^{(0)} [\Psi^{(0)}]^{-1}, \quad (3.8)$$

where

$$\Psi^{(0)} \equiv e^{A^{\text{vac}} + iB^{\text{vac}}}. \quad (3.9)$$
The DT is implemented through two gauge transformations generated by $\Theta_k$ such that the nontrivial gauge connections in the vacuum orbit become [15]

$$
A = \Theta^A_0 A^{(\text{vac})}[\Theta_2^h]^{-1} + \partial_1 \Theta^A_0 [\Theta_2^h]^{-1},
$$

(3.10)

$$
B = \Theta^B_0 B^{(\text{vac})}[\Theta_2^h]^{-1} + \partial_1 \Theta^B_0 [\Theta_2^h]^{-1},
$$

(3.11)

where

$$
\Theta^h_+ = \exp\left(\sum_{n>0} \sigma_n\right), \quad \Theta^h_- \equiv M^{-1}N; \quad M = \exp(\sigma_n)n = \exp\left(\sum_{n>0} \sigma_n\right),
$$

(3.12)

where $[D, \sigma_n] = n \sigma_n$. Therefore, from the relationships

$$
A = \partial_1(\Psi^h)[\Psi^h]^{-1}; \quad B = \partial_1(\Psi^h)[\Psi^h]^{-1}, \quad \Psi^h \equiv \Theta^h_k \Psi^{(0)},
$$

(3.13)

one has

$$
[\Theta^h_+^{-1} \Theta^h_-] = \Psi^{(0)} h[\Psi^{(0)}]^{-1},
$$

(3.14)

where $h$ is some constant group element.

One can relate the fields $\Psi^h$, $\phi_1$ and $\phi_2$ to some of the components in $\sigma_n$. One has

$$
A = A^{\text{vac}} + [\sigma_-, E^{(1)}] + \text{terms of negative grade},
$$

(3.15)

$$
B = B^{\text{vac}} + [\sigma_-, E^{(2)}] + [\sigma_-, E^{(2)}] + \frac{1}{2} [\sigma_-, [\sigma_-, E^{(2)}]] + \left[\sigma_-, \sum_{i=1}^2 \rho_i E_{\beta_i}^{(1)}\right]
$$

$$
+ \left[\sigma_-, \sum_{i=1}^2 \rho_i E_{\beta_i}^{(1)}\right] + \text{terms of negative grade.}
$$

(3.16)

Taking into account the grading structure of the connection $A$ in (2.5), we may write the $\sigma_n$s in terms of the fields of the model. In order to match the zero-grade terms of both the sides of equation (3.15), one must have

$$
\sigma_1 = -2 \sum_{i=1}^2 (\Psi_+ - \rho_i) E_{\beta_i}^{(-1)} + 2 \sum_{i=1}^2 (\Psi^- - \rho_i) E_{\beta_i}^{(-1)} + \sum_{a=1}^2 \sigma_a^1 H_{\nu}^{(-1)}.
$$

(3.17)

In the equation above, the explicit form of $\sigma_1$ in terms of the fields $\Psi^h$ can be obtained by setting the sum of the $(-1)$ grade terms to zero. Nevertheless, the form of the $\sigma_a^1$ will not be necessary for our purposes.

Following the above procedure to match the gradations on both sides of equations (3.15) and (3.16), one notes that the $\sigma_n$s with $n \geq 1$ are used to cancel out the undesired components on the rhs of the equations.

From equations (3.12)–(3.14), one has

$$
\langle \lambda | M^{-1} | \lambda' \rangle = \langle \lambda | [\Psi^{(0)} h \Psi^{(0)-1}] | \lambda' \rangle,
$$

(3.18)

where $\langle \lambda |$ and $| \lambda' \rangle$ are certain states annihilated by $G_<$ and $G_>$, respectively. Defining

$$
\sigma_0 = \sum_{a>0} \sigma_a E_{\alpha}^{(0)} + \sum_{a>0} \sigma_{-a} E_{\alpha}^{(0)} + \sum_{a=1}^2 \sigma_a^1 H_{\nu}^{(0)} + \eta C
$$

(3.19)

and choosing a specific matrix element, one obtains a spacetime dependence for the field $\eta$. 


\[ e^{-\eta} = \langle \lambda_0 | \mathcal{W}(0) | \Psi^{(0)} \rangle | \lambda_0 \rangle \]  

(3.20)

\[ \equiv \tau_0 , \]  

(3.21)

where we have defined the tau function \( \tau_0 \) and \( | \lambda_0 \rangle \) is the highest weight defined in (B.15). Next, we will write the fields \( \Psi^{\pm} \) in terms of certain matrix elements. These functions will be represented as matrix elements in an appropriate representation of the affine Lie algebra \( \hat{sl}(3) \).

We proceed by writing equation (3.14) in the form

\[ \exp \left( - \sum_{n=0} \sigma_{-n} \right) | \lambda_0 \rangle = \left[ \mathcal{W}(0) | \Psi^{(0)} \rangle \right] | \lambda_0 \rangle \tau_0^{-1} , \]  

(3.22)

where equations (3.12), (3.19) and (3.20)–(3.21) have been used.

Then the terms with grade \((-1)\) in both the sides of (3.22) can be written as

\[ - \sigma_{-1} | \lambda_0 \rangle = \frac{\left[ \mathcal{W}(0) | \Psi^{(0)} \rangle \right]_{-1} | \lambda_0 \rangle}{\tau_0 (x, t)} , \]  

(3.23)

or equivalently

\[ \left( \sum_{i=1}^2 (\Psi_+^i - \rho_+^i) E_{\beta_i}^{(-1)} - \sum_{i=1}^2 (\Psi_-^i - \rho_-^i) E_{\beta_i}^{(-1)} - \sum_{a=1}^2 \sigma_{a} \beta_a H_a^{(-1)} \right) | \lambda_0 \rangle \]  

\[ \equiv \left[ \mathcal{W}(0) | \Psi^{(0)} \rangle \right]_{-1} | \lambda_0 \rangle \]  

(3.24)

Acting on the left in equation (3.24) by \( E_{\beta_i}^{(1)} \) and taking the relevant matrix element with the dual highest weight state \( | \lambda_n \rangle \), we may have

\[ \Psi_+^i = \rho_+^i + \frac{\tau_+^i}{\tau_0} \]  

and \( \Psi_-^i = \rho_-^i - \frac{\tau_-^i}{\tau_0} ; \quad i = 1, 2 , \]  

(3.25)

where the tau functions \( \tau_+^i \) and \( \tau_0 \) are defined by

\[ \tau_+^i (x, t) \equiv \langle \lambda_0 | E_{\beta_i}^{(1)} \left[ \mathcal{W}(0) | \Psi^{(0)} \rangle \right]_{-1} | \lambda_0 \rangle , \]  

(3.26)

\[ \tau_-^i (x, t) \equiv \langle \lambda_0 | E_{\beta_i}^{(1)} \left[ \mathcal{W}(0) | \Psi^{(0)} \rangle \right]_{-1} | \lambda_0 \rangle , \]  

(3.27)

\[ \tau_0 (x, t) \equiv \langle \lambda_0 | \left[ \mathcal{W}(0) | \Psi^{(0)} \rangle \right]_{0} | \lambda_0 \rangle . \]  

(3.28)

Note that in order to obtain the above relationships, we have used the commutation rules for the corresponding \( \hat{sl}(3) \) affine Kac–Moody algebra elements (B.4)–(B.11) and (B.25)–(B.31), as well as their properties acting on the highest weight state \( | \lambda_n \rangle \) (B.15).

According to the solitonic specialization in the context of the DT method, the soliton solutions are determined by choosing the suitable constant group elements \( h \) in equations (3.26)–(3.28). In order to obtain \( N \)-soliton solutions, the general prescription is to parameterize the orbit of the vacuum as a product of exponentials of eigenvectors of the operators \( e_i \) \( (e_1 = A^\alpha \) and \( e_2 = B^\alpha \) defined in (3.6)–(3.7), i.e \( h = \prod_{i=1}^N e^{E_i} \), where \( [e_j, F_i] = \lambda_i^j F_i \) \), such that \( (F_i)^m \neq 0 \) only for \( m < m_i \), with \( m_i \) being some positive integer. The relationships between DT, solitonic specialization and the Hirota method have been presented in [15] for any hierarchy of integrable models possessing a zero-curvature representation in terms of an affine Kac–Moody algebra. The DT method provides a relationship between the fields of the model and the relevant tau functions, and it explains the truncation of the Hirota expansion. The Hirota method is a recursive method which can be implemented through a computer program for algebraic manipulation like Mathematica. On the other hand, the DT method requires the computation of matrix elements as in equations (3.26)–(3.28) in the vertex operator representations of the affine Kac–Moody algebra. Actually, these matrix element calculations are very tedious in the case of higher soliton solutions.
3.1. Free-field boundary conditions and dressing group

As a generalization of the constant NVBC (3.3), consider the free-field NVBC

$$\lim_{x \to \pm \infty} \Psi(x,t) \equiv \rho^\pm e^{a^+ x + b^+ t},$$

(3.29)

where \(\rho_0^\pm, a_0^\pm, b_0^\pm\) are some constants. Note that (3.29) is a free-field solution of (2.6)–(2.7) such that \(b^\pm_j = \pm (a^\pm_j)^2 \mp 2 \Lambda_j, \) for \(\Lambda_j \equiv \left| \sum_{i=1}^j \rho_i^+ - \frac{1}{2} (\beta_j, \Omega) \right| \) const. However, a direct application of the DT approach of [15] is not possible since the relevant connections \(A^\text{vac}\) and \(B^\text{vac}\) do not belong to an Abelian subalgebra (up to the central term). So in order to consider NVBC (3.29), we resort to the dressing group [29] composition law of two successive DTs. In fact, the relevant connections can be related to \(A^\text{vac}\), \(B^\text{vac}\) in (3.6) and (3.7) through certain gauge transformations generated by \(\Theta^\pm\) such that

$$\hat{V}^\text{vac} = \Theta^\pm(x,t) \hat{V}^\text{vac}[\Theta^\pm(x,t)]^{-1} + \partial_x \Theta^\pm(x,t) \left[ \Theta^\pm(x,t) \right]^{-1}, \quad \hat{V}_1 = \hat{A}^\text{vac}, \quad \hat{V}_2 = \hat{B}^\text{vac},$$

(3.30)

where \(i = 1, 2, x_1 = x, x_2 = t, V_1 = A^\text{vac}, V_2 = B^\text{vac}\) and \(\Theta^\pm \in SL(3).\) One must have

$$\Psi^{(0)} \rightarrow \hat{\Psi}^{(0)} = \Theta^\pm(x,t) \Psi^{(0)} \Theta^\pm(x,t)^{-1}, \quad \hat{V}_1 = \partial_x \hat{\Psi}^{(0)} [\hat{\Psi}^{(0)}]^{-1},$$

(3.31)

where \(g\) is an arbitrary constant group element and

$$\Theta^\pm = e^{\beta_0} e^{\sum_{\alpha=1}^n \beta_\alpha \chi}, \quad [D, \chi_\alpha] = n \chi_\alpha.$$  

(3.32)

From (3.30), one has \(e^{-\rho E^{(l)}} e^{\rho E^{(o)}} = E^{(o)}\), so

$$\chi_0 = \chi_0^o + \frac{2}{\rho_{1\beta}} \chi_{\beta}^o + \sum_{a=1}^2 \chi_{\alpha}^o H_{\alpha}^{(o)} + \chi C$$

(3.33)

Therefore, a modified DT procedure can be implemented with [29]

$$[\Theta^{-\frac{a}{\alpha}}]^{-1} \Theta^{\frac{b}{\beta}} \equiv \hat{\Psi}^{(0)} h \left[ \Psi^{(0)} \right]^{-1} = \Theta^{\frac{a}{\alpha}}(x,t) \Psi^{(0)} h \left[ \Psi^{(0)} \right]^{-1} \left[ \Theta^{\frac{b}{\beta}}(x,t) \right]^{-1}; \quad h = gh',$$

(3.35)

through

$$A' = \Theta^{\frac{a}{\alpha}} A^{\text{vac}} \left[ \Theta^{\frac{b}{\beta}} \right]^{-1} + \partial_x \Theta^{\frac{a}{\alpha}} \left[ \Theta^{\frac{b}{\beta}} \right]^{-1}, \quad B' = 0 \Theta^{\frac{a}{\alpha}} B^{\text{vac}} \left[ \Theta^{\frac{b}{\beta}} \right]^{-1} + \partial_x \Theta^{\frac{a}{\alpha}} \left[ \Theta^{\frac{b}{\beta}} \right]^{-1}. \quad (3.36)$$

So equation (3.35) can be used instead of (3.14) in order to derive general dark–dark solitons with free-field NVBC.

4. Bright solitons and vanishing boundary conditions

For simplicity, we first apply the DT method to VBC (3.2) and show the existence of bright–soliton solutions [30]. The connections \(A^{\text{vac}} = E^{(1)}, \ B^{\text{vac}} = E^{(2)}\) are related to the group element \(\Psi_0\) as

$$A^{\text{vac}} = \partial_x \Psi_0 [\Psi_0]^{-1}, \quad B^{\text{vac}} = \partial_x \Psi_0 [\Psi_0]^{-1}, \quad \Psi_0 \equiv e^{E^{(1)}+tE^{(2)}}.$$  

(4.1)

In order to obtain soliton solutions, the simultaneous adjoint eigenstates of the elements \(E^{(1)}\) and \(E^{(2)}\) play a central role. In the case at hand, one has the eigenstates \(F_j\) and \(G_j\) in (C.1) such that

$$[\chi E^{(1)} + t E^{(2)}, F_j] = - \phi_j(x,t) F_j; \quad \phi_j(x,t) = v_j(x + \rho_j t)$$

(4.2)

$$[\chi E^{(1)} + t E^{(2)}, G_j] = \eta_j(x,t) G_j; \quad \eta_j(x,t) = \rho_j(x + \rho_j t).$$

(4.3)
4.1. jth-component one-bright-soliton solution

Consider the product
\[ h = e^{\epsilon_1 F_1} e^{\epsilon_2 G_2}, \]
where \( j_1 \) and \( j_2 \) are some indexes chosen from \{1, 2\}. Using the nilpotency properties of \( F_{j_1} \) and \( G_{j_2} \) (see appendix C), one obtains
\[ [\psi_0 h \psi_0^{-1}] = (1 + e^{-\phi_j} a_j F_{j_1}) (1 + e^{\eta_j} b_j G_{j_2}) \]
\[ = 1 + e^{-\phi_j} a_j F_{j_1} + e^{\eta_j} b_j G_{j_2} + a_j b_j e^{\phi_j} F_{j_1} G_{j_2}, \]
with \( \phi_j \) and \( \eta_j \) given in (4.2) and (4.3), respectively. The corresponding tau functions become
\[ \tau_0 = 1 + a_j b_j C_{j_1,j_2} e^{-\phi_j} e^{\eta_j}, \quad C_{j_1,j_2} = \frac{v_j G_j}{(v_j - \rho_j)^2}, \]
\[ \tau_i^+ = \delta_i b_j \rho_i e^{\phi_j}, \quad \tau_i^- = \delta_i a_j v_i e^{-\phi_j}, \]
where the matrix element \( C_{j_1,j_2} \) has been presented in (C.2). In order to construct one-soliton solutions, we must have \( j_1 = j_2 = 1 \) in (4.7). Therefore, one obtains\(^3\)
\[ \psi_i^+ = \frac{b_i \rho_i e^{\phi_i}}{1 + a_i b_i C_{i,j} e^{-\phi_i} e^{\eta_i}}, \quad \psi_i^- = -\frac{a_i v_i e^{-\phi_i}}{1 + a_i b_i C_{i,j} e^{-\phi_i} e^{\eta_i}}, \quad i = j; \]
\[ \psi_i^\pm = 0, \quad i \neq j. \]
Imposing the relationships \( \rho_j^* = -v_j, \quad b_j^* = -\mu \delta a_j v_j, \quad a_j \in \mathbb{C} \), so from (4.7) one has \( C_{jj} = -(\frac{v_j}{2})^2 \). Equations (4.9) and (4.10) with complexification (2.10) provide a solution of the CNLS system (2.11),
\[ \psi_i(x, t) = \begin{cases} -\frac{(a_i v_i) v_i}{v_j} e^{\phi_j} \text{sech} \left( \frac{\psi_j + X_0}{2} \right), & i = j \\ 0, & i \neq j \end{cases}, \]
where \( e^{X_0} = \frac{\mu (a_i v_i) v_i}{v_j + i v_j}, \quad \psi_j = v_j (x - i v_j t) \equiv \psi_j R + i \psi_j I \). It must be \( \delta = +1 \), and solution (4.11) possesses two complex \( (a_j, v_j) \) parameters plus the real coupling \( \mu > 0 \).

Solution (4.11) for the \( j \)th component is known as a ‘bright soliton’ in the context of the scalar nonlinear Schrödinger equation (NLS).

4.2. One-bright–bright-soliton solution

The main observation in the last construction of the \( j \)th-component one-soliton is that it has been excited by the group element \( h \) in (4.4) such that \( j_1 = j_2 = j \). So in order to excite the two components of \( \Psi^\pm \) \( (i = 1, 2) \) and reproduce a bright soliton for each component, let us consider the group element
\[ h = e^{\epsilon_i F_i} e^{\delta_i G_i} e^{\phi_i F_i} e^{\eta_i G_i}, \]
where the exponential factors contain \( F_s \) and \( G_s \) of type (C.1). The tau functions become
\[ \tau_j^+ = b_j \rho_j e^{\phi_j}, \quad \eta_j = \rho_j (x + \rho_j t) + \rho_{0j}, \quad j = 1, 2. \]
\(^3\) If \( j_1 \neq j_2 \) in (4.7) and (4.8), one still has certain trivial solutions, since in this case \( C_{j_1,j_2} = 0 \) implying \( \tau_0 = 1 \).
\[
\tau_j = a_j \psi_j e^{-\psi_j}, \quad \psi_j = v_j (x + v_j t) + v_{0j}, \quad j = 1, 2. \tag{4.14}
\]

where \( C_{jj} = \frac{\partial^2}{\partial \psi_j^2} \). Let us set \( \delta' = -\mu \delta \), \( \rho' = -v_j \equiv -v_1 \); then \( \eta' = -\psi_j \equiv -\psi_1 = -(\psi_{1R} + i \psi_{1I}) \). \( (\psi_j, a_j \in \mathbb{C}) \) in relations (4.13)–(4.15). Therefore, the vector soliton solution of the 2-CNLS equation (2.11) arises:

\[
\begin{pmatrix}
\psi_1(x, t) \\
\psi_2(x, t)
\end{pmatrix} = \begin{pmatrix}
A_1 \\
A_2
\end{pmatrix} v_1 b \text{sech} \left( \frac{\psi_{1R} + \frac{X_0}{2}}{\varepsilon} \right) e^{i\phi},
\]

where \( A_i = -\frac{\delta_i}{\mu^2} \sum_{j=1}^4 a_{ij} \psi_j \).

This solution is valid for \( \sum_j \delta_j |a_j \psi_j|^2 > 0 \). Note that this one-bright–bright-soliton possesses, apart from the real \( \mu > 0 \), four arbitrary complex parameters, namely \( a_1, a_2, v_1, v_2 \). This solution in the case of mixed nonlinearity \( \delta_1 = -\delta_2 = 1 \) may have a singular behavior if the sum \( \sqrt{\sum_{j=1}^4 \delta_j |a_j \psi_j|^2} \) in the denominator of the expression of \( A_i \) vanishes, such that the soliton amplitude in (4.16) diverges. The \( N \)-bright–bright-soliton requires the generalization of the group element in (4.12) as \( h = e^{iF_1 \phi_1 G_1 \cdots e^{iF_N \phi_N G_N} \phi_0 G_n} \).

5. AKNS \((r = 1, 2)\), NVBC and dark solitons

In this section, we tackle the problem of finding dark soliton-type solutions of the system (2.6)–(2.7). The associated 2-CNLS model (2.11) with NVBC has been considered in the framework of direct methods, such as the Hirota tau function approach (see e.g. [10, 31, 22, 32]), and recently in the IST approach [12, 13]. In the last approach, the relevant Lax operators have remarkable differences and rather involved spectral properties as compared to their counterparts with vanishing boundary conditions (see [34] and references therein), e.g. in the NVBC case the spectral parameter requires the construction of certain Riemann sheets [34–37]. So it would be interesting to give the full Lie algebraic construction of the tau functions and soliton solutions for the system (2.6)–(2.7) with NVBC.

5.1. AKNS1: N dark solitons

For simplicity, first we describe the entire process for the system (2.6)–(2.7) with just two fields \( \Psi^\pm \). So let us consider the \( \hat{s}l(2) \) affine Kac–Moody algebra in the Weyl–Cartan (WC) basis

\[
[H^{(m)}, E^{(n)}] = \frac{m}{2} \delta_{m+n,0} C, \quad [H^{(m)}, E^{(n)}] = \pm E^{(m+n)},
\]

\[
[H^{(m)}, E^{(n)}] = 2H^{(m+n)} + m \delta_{m+n,0} C. \tag{5.1}
\]

In order to study a NVBC for the system \( \hat{s}l(2) \)-AKNS, consider the Lax pair

\[
A = E^{(1)} + \Psi^+ E^{(0)} + \Psi^- E^{(0)} + \phi_1 C, \tag{5.2}
\]

\[
B = E^{(2)} + \Psi^+ E^{(1)} + \Psi^- E^{(1)} + \partial_\psi E^{(0)} - \partial_\psi \Psi^+ E^{(0)} - 2(\Psi^+ \Psi^- - \rho^+ \rho^-) H^{(0)} + \phi_2 C. \tag{5.3}
\]

This basis differs from the Chevalley (Ch) basis in (A.1)–(A.4) by the rescaling of the generator \( H^{(0)}_{WC} \) to \( \frac{1}{2} H^{(0)}_{Ch} \).
where $\Psi^+$ and $\Psi^-$ are the fields of the model ($\rho^\pm = $ constant). In this case, one considers the generator $E^{(1)} = H^{(1)}$, $\phi_1$ and $\phi_2$ are introduced as auxiliary fields. Therefore, the equations of motion suitable to treat NVBC become

$$
\partial_t \Psi^+ = \partial_x^2 \Psi^+ - 2(\Psi^+ \Psi^- - \rho^+ \rho^-) \Psi^+,
$$

$$
\partial_t \Psi^- = - \partial_x^2 \Psi^- + 2(\Psi^+ \Psi^- - \rho^+ \rho^-) \Psi^-,
$$

$$
\partial_t \phi_1 - \partial_t \phi_2 = 0.
$$

The AKNS$_1$ model (5.4)–(5.5) is recovered from the AKNS$_2$ extension (2.6)–(2.7) simply by setting to zero the additional fields. In [33], a complexified version of the system (5.4)–(5.5) (for $\rho^\pm = 0$) has been introduced and its reduction to the focusing and defocusing NLS system has been presented, as well as its soliton solutions. In fact, the $\mathfrak{sl}(2)$-AKNS model (5.4)–(5.5) as well, through a particular reduction, contains as a sub-model the scalar defocusing NLS system

$$
i \partial_t \psi + \partial_x^2 \psi - 2(|\psi|^2 - \rho^2)\psi = 0.
$$

This equation is suitable for treating NVBC [37, 34, 35],

$$
\psi(x,t) = \begin{cases} 
\rho, & x \to -\infty \\
\rho \epsilon^z, & x \to +\infty ;
\end{cases} \quad \epsilon = e^{\omega/2}, \quad \rho \in \mathbb{R}.
$$

In fact, this boundary condition is manifestly compatible with the equation of motion (5.7). The defocusing NLS equation (5.7) with the boundary condition (5.8) is exactly integrable by the inverse scattering technique [38]. This model has soliton solutions in the form of localized ‘dark’ pulses created on a constant or stable continuous wave background solution.

In the system of equations (5.4) and (5.5), consider the transformation

$$
t \to -it, \quad x \to -ix,
$$

$$
\Psi^\pm \to i \Psi^\pm \epsilon^z,
$$

where the factor $\epsilon^z$ is introduced for later convenience. Furthermore, the identification

$$
\psi \equiv \Psi^+ = (\Psi^-)^*,
$$

where the star stands for complex conjugation, such that $\rho^+ \rho^- \to -\rho^2$ allows one to reproduce the defocusing NLS equation (5.7).

Let us take as the vacuum solution of (5.4)–(5.5) the constant background configuration

$$
\Psi^\pm = \rho^\pm \epsilon^z, \quad \phi_{1,2} = 0, \quad \rho, \epsilon = \text{const}.
$$

Therefore, the gauge connections (5.2) and (5.3) for the vacuum solution (5.12) become

$$
A^{\text{vac}} \equiv \epsilon_1 = E^{(1)} + \rho^+ E^{(0)} + \rho^- E^{(0)}
$$

$$
B^{\text{vac}} = \epsilon_2 = E^{(2)} + \rho^+ E^{(1)} + \rho^- E^{(1)}.
$$

We follow equation (3.8) in order to write the connections in the form

$$
A^{(\text{vac})} = \partial_t \Psi_{\text{mbc}}^{(0)} \left[ \Psi_{\text{mbc}}^{(0)} \right]^{-1}, \quad B^{(\text{vac})} = \partial_t \Psi_{\text{mbc}}^{(0)} \left[ \Psi_{\text{mbc}}^{(0)} \right]^{-1},
$$

with the group element

$$
\Psi_{\text{mbc}}^{(0)} \equiv e^{\epsilon x + \epsilon z},
$$

$$
= \exp \left[ \epsilon (E^{(1)} + \rho^+ E^{(0)} + \rho^- E^{(0)}) + t (E^{(2)} + \rho^+ E^{(1)} + \rho^- E^{(1)}) \right].
$$
Let us emphasize that this group element differs fundamentally from that associated with the case with VBC. First, the difference originates from the constant boundary condition terms added to the relevant vacuum gauge connections. However, they must be related since \( \lim_{\rho \to 0} \Psi^{(0)}_{\text{nvbc}} \to \Psi_0 \), where \( \Psi_0 \) is the group element in the VBC case (4.1). Second, the vacuum connections \( \varepsilon_{1,2} \) in the DT procedure will require another eigenvectors under the adjoint actions in analogy to equations (4.2) and (4.3), as we will see below.

Another proposal for this group element \( \Psi^{(0)}_{\text{nvbc}} \) has been introduced in [39]. There it has been considered a group element inspired in the inverse scattering method, possessing double-valued functions of the spectral parameter \( \lambda \). This fact motivates the introduction of an affine parameter to avoid constructing Riemann sheets. This construction involves some complications when used in the context of the DT method. The main difficulty arises in the computation of the matrix elements associated with the highest states in order to find the relevant tau functions. However, the group element given in (5.16) will turn out to be more suitable in the DT procedure. In fact, a similar group element has been proposed in [40] to tackle NVBC soliton solutions in the so-called negative even-grade mKdV hierarchy.

The DT procedure in this case follows all the way verbatim as in section 3 adapted to the \( sl(2) \) case. It follows by relating the relevant connections (5.2) and (5.3) to the connections in equations (5.13) and (5.14) corresponding to the vacuum solution (5.12).

Next, we will write the fields \( \Psi^{\pm} \) in terms of the relevant tau functions. According to the development in section 3, these functions will be written as certain matrix elements in an integrable h.w.r. of the affine Lie algebra \( \hat{sl}(2) \). So it is a straightforward task to obtain the following relationships:

\[
\Psi^{+} = \rho^{+} \frac{\tau^{+}}{\tau_0} \quad \text{and} \quad \Psi^{-} = \rho^{-} \frac{\tau^{-}}{\tau_0},
\]

where the \( \tau \)-functions \( \tau^{\pm} \) and \( \tau_0 \) are given by

\[
\tau^{+} = \langle \lambda_{\nu} | \hat{E}^{\uparrow(1)} [\Psi^{(0)}_{\text{nvbc}} h \Psi^{(0)\dagger}_{\text{nvbc}}]_{-1} | \lambda_{\nu} \rangle, 
\]

\[
\tau^{-} = \langle \lambda_{\nu} | \hat{E}^{\downarrow(1)} [\Psi^{(0)}_{\text{nvbc}} h \Psi^{(0)\dagger}_{\text{nvbc}}]_{-1} | \lambda_{\nu} \rangle, 
\]

\[
\tau_0 = \langle \lambda_{\nu} | \hat{h} \Psi^{(0)}_{\text{nvbc}} \Psi^{(0)\dagger}_{\text{nvbc}} | 1_{0} \rangle | \lambda_{\nu} \rangle,
\]

with the group element \( \Psi^{(0)}_{\text{nvbc}} \) given by (5.16).

### 5.1.1. AKNS\(_1\) and one-dark-soliton of defocusing scalar NLS

Let us consider the group element

\[
h^q = e^{ad^q V^q (\gamma, \rho^{\pm})} \quad (q = 1, 2),
\]

where \( a \) is a constant. The vertex operator \( V^q \) satisfies \( \hat{V}^q = 2V^q \) (see below) where \( \hat{V}^q \) is defined in (A.10).

Next, one must look for the eigenvalues of \( \varepsilon_1 \) and \( \varepsilon_2 \), respectively, in their adjoint actions on the vertex operator \( V^q \). In order to achieve this, let us note that these Weyl–Cartan basis elements \( \varepsilon_{1,2} \) are related to the corresponding Chevalley basis elements \( \hat{\varepsilon}_1 \) and \( \hat{\varepsilon}_2 \) in (A.12) through \( \hat{\varepsilon}_n = 2 \varepsilon_n \) \( (n = 1, 2) \), provided that \( \hat{\rho}^{\pm} = 2 \rho^{\pm} \) and the relationship between the Weyl–Cartan and Chevalley bases \( \hat{H}^{(n)}_{\text{WC}} = \frac{1}{2} H^{(n)}_{\text{Ch}} \) is considered for the \( sl(2) \) element \( H^{(n)} \). So the vertex operators are related by \( \hat{V}^q = 2V^q \). Then from (A.11), one has

\[
[\varepsilon_1, V^q] = \gamma V^q, \quad [\varepsilon_2, V^q] = (-1)^{q-1} \gamma (\gamma^2 - 4 \rho^+ \rho^-)^{1/2} V^q, \quad q = 1, 2.
\]
These solutions are the vector one-soliton of the AKNS1 model. This solution possesses four propagation of the soliton. Figure 1 displays the component solitons of this vector one-soliton.

Note that these eigenstates \( V^q \) exhibit a more complex structure in comparison to their counterparts \( F_j \) and \( G_j \) corresponding to the VBC as in (4.2) and (4.3). Therefore, one has

\[
[x \varepsilon_1 + t \varepsilon_2, V^q(\gamma, \rho^{\pm})] = [\gamma (x + (-1)^{q-1}vt) V^q(\gamma, \rho^{\pm})],
\]

where \( v = \sqrt{\gamma^2 - 4 \rho^+ \rho^-} \). Denoting \( \varphi \gamma = \gamma (x + (-1)^{q-1}vt) \), one can write

\[
[\Psi^{(0)}_{\text{nbc}}, \Psi^{(0)\dagger}_{\text{nbc}}] = \exp(e^{e\gamma t} V^q) = 1 + e^{e\gamma t} a_q V^q,
\]

where we have used \( (V^q)^n = 0 \) for \( n \geq 2 \), that is, \( V^q \) is nilpotent, (A.17)–(A.18).

So in order to find the explicit tau functions in (5.18)–(5.20), it remains to compute the relevant matrix elements. Using the properties presented in appendix A and equations (3.26)–(3.28), one obtains the tau functions \( \tau^{(q)}_1 = 1 + c^{(q)} e^{e\gamma t} ; \tau^{(q)}_2 = s^{(q)} a q^{(q)} \rho^\pm e^{e\gamma t} \), where the matrix elements \( c^{(q)} = d^{(q)} (\lambda_o | V^q | \lambda_o) \), \( s^{(q)} = d^{(q)}(\rho^\pm)^{-1} (\lambda_o | E^{(q)}_{a q} V^q | \lambda_o) \) have been considered. They are given in appendix A, equations (A.13) and (A.14), respectively. So one has \( s^{(q)} = \frac{\rho^e}{\rho^{\pm}} (\gamma \pm (-1)^{q-1} \sqrt{\gamma^2 - 4 \rho^+ \rho^-} \). Without loss of generality, we can assume \( \gamma > 0 \), so the matrix element \( (\lambda_o | V^q | \lambda_o) \) from (A.13) allows us to determine the sign of \( c^{(q)} \), which is completely fixed by \( \text{sign}[c^{(q)}] = e_q \text{sign}[d^{(q)}] \). We will see below that the signs of the free real parameter \( c^{(q)} \) determine two different types of solutions.

Consider \( c^{(q)} > 0 \). This case follows if \( a^{(q)} \) and \( e_q \) have equal sign. Then, relations (5.17) supplied with the above tau functions provide

\[
\Psi^{(q)}(x, t) = \rho^+ \left[ 1 - \frac{s^{(q)}}{2 e^{(q)}} \right] - \frac{\rho^e s^{(q)}}{2 e^{(q)}} \tanh \left\{ \frac{\gamma [x + (-1)^{q-1}vt] + \frac{\log c^{(q)}}{2}}{2} \right\}.
\]

These solutions are the vector one-soliton of the AKNS1 model. This solution possesses four real parameters, i.e. \( \rho^\pm, \gamma, c^{(q)} \). Note that the value of the index \( q \) determines the direction of propagation of the soliton. Figure 1 displays the component solitons of this vector one-soliton.

The true one-dark-soliton of the AKNS1 model is obtained as the product

\[
[\Psi^{-(q)} \Psi^{+(q)}](x, t) = \rho^+ \rho^- \left[ 1 - \frac{s^{(q)}}{2 e^{(q)}} \right] - \frac{\rho^e s^{(q)}}{2 e^{(q)}} \tanh \left\{ \frac{\gamma [x + (-1)^{q-1}vt] + \frac{\log c^{(q)}}{2}}{2} \right\}.
\]

Figure 1. The real vector \( (\Psi^+, \Psi^-) \) one-soliton profile of the AKNS1 model for \( q = 1 \) in (5.26). They were plotted for \( \gamma_1 = 8, \rho^+ = \rho^- = 3.9, \log c^{(1)} = 8 \). The solid and dashed lines correspond to \( \Psi^+ \) and \( \Psi^- \), respectively.
Figure 2. The one-dark-soliton profile of the AKNS$_1$ model emerges as $\Psi^+ - \Psi^-$. This is the soliton traveling to the left, the case $q = 1$ in (5.27). It was plotted for $\gamma_1 = 8$, $\rho^+ = \rho^- = 3.9$, $\log |c^{(1)}| = 8$.

Figure 3. The singular solitons of the AKNS$_1$ model. The solid lines travel to the left ($q = 1$) and the dashed lines to the right ($q = 2$). They are plotted for $t = -0.07$, $\gamma = 10$, $\rho^+ = \rho^- = 2$, $\log |c^{(2)}| = 10$.

It has been plotted in figure 2 for the case $q = 1$.

Next, consider $c(q) < 0$. This case follows if $a(q)$ and $e_q$ have the opposite sign. Similarly, relations (5.17) and the above tau functions provide

$$
\Psi^{\pm}(x, t) = \rho^\pm \left[ 1 - \frac{\chi^{\pm}(q)}{2e^{(q)}} \right] - \frac{\rho^\pm \chi^{\pm}(q)}{2e^{(q)}} \coth \left\{ \gamma [x + (-1)^{q-1}vt]/2 + \frac{\log |c^{(q)}|}{2} \right\}.
$$

These solutions are the singular solitons of the AKNS$_1$ model and, to our knowledge, they have never yet been reported. As above, the value of the index $q$ determines the direction of propagation of the soliton. They possess four real parameters, namely $\rho^\pm$, $\gamma$, $e^{(q)}$. Similarly, let us consider the product of the components,

$$
[\Psi^{-(q)} \Psi^{+(q)}](x, t) = \rho^+ \rho^- - (-1)^{q-1} \frac{\gamma^2}{4} \csc h^2 \left\{ \gamma [x + (-1)^{q-1}vt]/2 + \frac{\log |c^{(q)}|}{2} \right\}.
$$

These functions are displayed in figure 3. Note that these solitons are singular for $[\gamma [x + (-1)^{q-1}vt]/2 + \frac{\log |c^{(q)}|}{2}] = 0$. 
Some comments are in order here.

First, the one-dark-soliton solutions (5.27) and the singular solitons (5.29), respectively, have the same boundary values in \( x \to \pm \infty \), i.e. \( \lim_{x \to \pm \infty} \Psi^{-}(q)(x, t)\Psi^{+}(q)(x, t) = \rho^{+}\rho^{-} \). In both types of solitons, this behavior is in contrast with the different boundary values assumed by the corresponding components in both the limits. For the one-dark-soliton components, one has \( \lim_{x \to -\infty} \Psi^{\pm} \to \rho^\pm \), whereas \( \lim_{x \to +\infty} \Psi^{\pm} \to \rho^\pm [1 - \frac{\gamma}{2\rho^\pm}(\gamma \pm \sqrt{\gamma^2 - 4\rho^\pm \rho^-}] \). This fact is shown in figure 1.

Second, the center of the one-dark-solitons is given for \( \gamma[x + (-1)^{q-1}v\ell t]/2 + \log\epsilon^\gamma(q) = 0 \). In fact, in this point the functions \( \Psi^{-}(q)\Psi^{+}(q)|_{\text{center}} = \rho^+\rho^- - \gamma^2/4 \) (for \( q = 1, 2 \)) take the smallest intensity. So their intensity dips are controlled by the value of the parameter \( \gamma \).

Third, solution (5.26) in the case \( q = 1 \) allows the imposition of the conditions (5.11) in order to satisfy the defocusing NLS equation (5.7). So let us make transformation (5.9) and the complexification condition (5.11) and \( \Psi = i e^{i/2} [\rho + (\frac{\sqrt{\gamma^2 - 2\rho^2}}{2\rho}) e^{i\epsilon(q)/2}] \). \( \psi(x, t) = \gamma(x - \tilde{v}_p t) + \gamma_0 \), obtained in this way is a solution of the defocusing NLS equation (5.7). In fact, the function \( \psi \) can be written as

\[
\psi(x, t) = \frac{1}{2} \tan[i\gamma(x - \tilde{v}_p t)/2 + \gamma_0/2] + \gamma/2 - i\rho e^{-i\epsilon(q/2)}.
\]

which is the dark soliton solution of the defocusing NLS equation (5.7) (see e.g. [31]). Note that this solution satisfies the NVBC \( \lim_{x \to \infty} |\psi(x, t)|^2 \to \rho^2 \).

Fourth, similarly, one can consider the complexification condition (5.11) in the case \( q = 2 \). In this case, one obtains \( \Psi = i e^{i/2} \frac{1}{2} \frac{e^{2\epsilon(q)/2}}{\epsilon(q)} \tan[i\gamma(x + \tilde{v}_p t)/2 + \gamma_0/2] + \gamma_0/2] + \gamma/2 - i\rho e^{-i\epsilon(q/2)} \). \( \delta = \tan^{-1}(\tilde{v}_p/\gamma) \). However, they do not satisfy condition (5.11); hence, these functions would not be solutions of (5.7).

From the previous constructions, it is clear that the \( N \) solitons of the types dark, singular or mixed dark-singular can be constructed by choosing convenient signs of the parameters \( a_j^{(q)} \) and \( e_j \) associated with the group element

\[
h = e^{i\pi \epsilon(q)/2} e^{i\pi \epsilon(q)/2} e^{i\pi \epsilon(q)/2} \cdots e^{i\pi \epsilon(q)/2}, \quad q = 1, 2.
\]

5.2. AKNS: \( N \) dark–dark solitons

Consider the NVBC for the system (2.6)–(2.7) in the form (3.1) and its associated constant vacuum solution (3.3). The vacuum connections (3.6) and (3.7) associated with this trivial solution commute \( [A^{avc}, B^{avc}] = 0 \) if one takes \( \Omega_1 = \Omega_2 = \frac{3}{2}(\rho^+_1 \rho^-_1 + \rho^+_2 \rho^-_2) \). Consider the notation \( A^{avc} = \epsilon_1 \), \( B^{avc} = \epsilon_2 \).

One-dark–dark-soliton. Let us consider equations (3.26)–(3.28) and choose the group element \( h^q = e^{i\theta W^{(k)}} \), with \( \theta \) and \( k \) being some parameters, where the vertex operator \( W^{(k)}(k) \) is defined in (C.7), such that

\[
[e_1, W^{(q)}] = k^q W^{(q)}; \quad q = 1, 2;
\]

\[
[e_2, W^{(q)}] = w^q W^{(q)},
\]

where

\[
w^q = c_{q} k^q \sqrt{(k^q)^2 - 4\rho^+_1 \rho^-_1 - 4\rho^+_2 \rho^-_2}, \quad c_q \equiv (-1)^{q-1}.
\]
So one has
\[
[x e_1 + t e_2, W^q] = (k^q x + w^q t) W^q.
\] (5.35)

The vertex operator \( W^q \) (see (C.7)) is associated with the one-dark–dark-soliton and it can be shown to be nilpotent (C.8). So using (3.26)–(3.28), the following tau functions correspond to the element \( h^q \) given above:
\[
\tau_0^{(q)} = 1 + c^{(q)} e^{k t + w^q t},
\] (5.36)
\[
\tau_i^{±(q)} = \rho_i^{±(q)} e^{k t + w^q t}, \quad i = 1, 2,
\] (5.37)
where \( c^{(q)} = d^q (\lambda_o |W^q|\lambda_o), \ s^{±(q)} = (\rho_i^{±(q)})^{-1} a^q (\lambda_o |E^q| W^q |\lambda_o) \). The above tau functions provide a 1-dark–dark solution for the fields \( \Psi^q \) of the system (2.6)–(2.7), such that the following relationships hold between the parameters:
\[
c^{(q)} = s^{+(q)} s^{-(q)}; \quad (k^q)^2 = \frac{(s^{+(q)} - s^{-(q)})^2 (\sum_k \rho_k^+ \rho_k^-)}{s^{+(q)} s^{-(q)}}; \quad w^q = s^{+(q)} + s^{-(q)} (k^q)^2.
\] (5.38)

We will concentrate on the regular solutions of the AKNS2 model since they will give rise to the dark solitons, as was clear from the AKNS1 construction. So in what follows, we will require \( c^{(q)} > 0 \). Then, relations (3.25) provide
\[
\Psi_j^{±(q)} = \frac{\rho_j^{±}}{2} (1 + (y^q)^{±1}) + (-1 + (y^q)^{±1}) \tanh[(k^q x + w^q t + \log c^{(q)})/2]),
\] (5.39)
where \( j = 1, 2; \ y^q \equiv \frac{s^{+(q)}}{s^{-(q)}} \). The condition \( c^{(q)} > 0 \) requires that \( y^q > 1 \) and \( y^q < 0 \). This vector soliton possesses six real parameters, i.e. \( \rho_j^{±}, k^q, c^{(q)} \).

Taking into account (C.7), one can compute the relevant matrix elements, which define \( s^{±(q)} \), in order to obtain
\[
y^q = \frac{k^q + e_q \sqrt{ (k^q)^2 - 4 \sum_j \rho_j^+ \rho_j^-}}{k^q - e_q \sqrt{ (k^q)^2 - 4 \sum_j \rho_j^+ \rho_j^-}}.
\] (5.40)

Note that for any value of the index \( q \), if \( \sum_j \rho_j^+ \rho_j^- > 0 \), the last relationship implies \( y^q < 0 \) and if \( \sum_j \rho_j^+ \rho_j^- < 0 \), then \( y^q > 0 \).

Some remarks are in order here.

First, the solitons associated with the pair of components \( (\Psi_j^+, \Psi_j^-) \) and \( (\Psi_j^+, \Psi_j^-) \), respectively, are proportional, so they are degenerate and reducible to the single dark soliton of the \( \mathrm{N}(2) \) AKNS1 model (5.26), whereas the solitons associated with the pair of components \( (\Psi_j^+, \Psi_j^-) \), respectively, are not proportional, so they are non-degenerate presenting in general different degrees of ‘darkness’ in each component. In this way, the solution in (5.39) presents a non-degenerate one-dark–dark-soliton in the components, say \( (\Psi_j^+, \Psi_j^-) \), of the AKNS2 model. A non-degenerate one-dark–dark-soliton in the defocusing 2-CNLS has been given in [22, 10] through Hirota’s method. The inverse scattering [12, 13] and Hirota method [10] results on \( N \) dark–dark solitons in the defocusing 2-CNLS model have presented only the degenerate case.

Second, as in the AKNS1 solution (5.26), one can show that the one-dark–dark-solitons (5.39) allow the imposition of conditions (2.10) for the pair of fields \( (\Psi_j^+, \Psi_j^-) \), \( i = 1, 2 \), in order to satisfy the 2-CNLS equation (2.11). In fact, consider the parameters \( y^q = \frac{e_q}{y^q} \) and a complexification procedure as in (2.10) provided with a convenient set of complex parameters.
in (5.39), \(k^q \rightarrow i k^q\), \(L^\pm \rightarrow i L^\pm\) such that \(y_0 = e^{i2\phi_0}\), \(z_0^q = \frac{\alpha_0}{i\pi E(4j\rho_j)\rho_j^2} - (k^q)^2\), with \(\phi_0\) being real, so equation (2.10) can be satisfied if
\[
\left(\rho_j^+\right)^* = \mu \delta_j \rho_j^+.
\] (5.41)

Note that, as was shown in the AKNS1 case and its one-dark-soliton (5.30) particular reduction, in order to obtain 2-CNLS one-dark-solitons, one must consider \(q = 1\) in both components (\(\Psi_1^+ \, \Psi_1^-\)). Therefore, the relationship between the parameters \(\rho_j^\pm\) determines clearly if these solutions will correspond to the defocusing (\(\delta_j = -1\)) or to the mixed nonlinearity (\(\delta_1 = -\delta_2 = \pm 1\)) 2-CNLS system, respectively.

Third, let us discuss the extension of our results to the case of free-field NVBC (3.29). In order to compute the relevant tau functions analogous to those in (3.26)–(3.28), one must consider expression (3.35) instead of (3.14). So the tau functions become

\[
\hat{\tau}_j^+(x, t) \equiv \langle \lambda_0 | E^{(1)}_{\alpha, \beta}(x, t) \Psi(0) h[\Psi(0)]^{-1} | \lambda_0 \rangle e^{-\chi},
\] (5.42)

\[
= (\rho_j^+ + \rho_j^+)^2 \hat{\tau}_0^+(x, t) + \langle \lambda_0 | e^{-\chi} E^{(1)}_{\alpha, \beta} e^{\chi} \Psi(0) h[\Psi(0)]^{-1} | \lambda_0 \rangle \, e^{-\chi},
\] (5.43)

\[
\hat{\tau}_j^-(x, t) \equiv \langle \lambda_0 | E^{(1)}_{\alpha, \beta}(x, t) \Psi(0) h[\Psi(0)]^{-1} | \lambda_0 \rangle \, e^{-\chi},
\] (5.44)

\[
= (-\rho_j^- + \rho_j^-)^2 \hat{\tau}_0^-(x, t) + \langle \lambda_0 | e^{-\chi} E^{(1)}_{\alpha, \beta} e^{\chi} \Psi(0) h[\Psi(0)]^{-1} | \lambda_0 \rangle \, e^{-\chi},
\] (5.45)

\[
\hat{\tau}_0^\pm(x, t) \equiv \langle \lambda_0 | \Psi(0) h[\Psi(0)]^{-1} | \lambda_0 \rangle,
\] (5.46)

where the group element \(\Theta^\pm(x, t)\) is defined in (3.31) and (3.32) and \(\chi(x, t)\) is an ordinary function. We have used \(\Theta^\pm\) from equation (3.32) and decomposition (3.33) for \(\chi_0\) as well as properties (B.15) and (B.16) of the h.w.r. The tau functions (5.42)–(5.46) exhibit the modifications to be done on the previous equations (3.26)–(3.28) in order to satisfy NVBC (3.29). These amount to introduce the group element \(\Theta^\pm\) and the factor \(e^{-\chi}\). Then from (5.42)–(5.46) and the analog to (3.25) \(\Psi_j^\pm \equiv \rho_j^+ + \frac{\hat{\tau}_j^+}{\tau_0} \) and \(\Psi_j^- \equiv \rho_j^- - \frac{\hat{\tau}_j^-}{\tau_0}\), one can obtain

\[
\hat{\tau}_j^\pm \equiv \langle \lambda_0 | e^{-\chi} E^{(1)}_{\alpha, \beta} e^{\chi} \Psi(0) h[\Psi(0)]^{-1} | \lambda_0 \rangle, \quad \hat{\tau}_0^\pm \equiv \frac{\hat{\tau}_j^\pm}{\tau_0} \equiv \langle \lambda_0 | \Psi(0) h[\Psi(0)]^{-1} | \lambda_0 \rangle.
\] (5.47)

In fact, relations (5.47) can be formally obtained from those in (3.25) by making the changes \(\rho_j^\pm \rightarrow \hat{\rho}_j^\pm (x, t)\), \(\tau_j^\pm \rightarrow \hat{\tau}_j^\pm\) and \(\tau_0 \rightarrow \hat{\tau}_0\). The soliton-type solutions can be obtained following similar steps as the previous ones, i.e. for one-soliton, choose \(h^q = e^{\alpha W_0^q}\) and the adjoint eigenvectors of \(\epsilon_{1,2}\) become precisely the vertex operator \(W_0^q\) as in (5.32)–(3.33); then the expressions \(k^q x + w^q t\) of the tau functions (5.36)–(5.37) do not change.

Next, let us describe the general properties of solutions (5.47) and (5.48). (i) Note that the form of \(\hat{\tau}_0\) remains the same as that in (5.36). (ii) The modified DT and the related tau functions \(\hat{\tau}_0\) (5.48) will introduce some new parameters into the tau functions \(\hat{\tau}_j^\pm\) in (5.37). In (5.37), the factor \(\rho_j^\pm\) must be replaced with \(\hat{\rho}_j^\pm (x, t)\) and \(s^q(\phi)\) in general will depend on the index ‘\(i\)‘ and the spacetime, i.e. \(s^q(\phi) \rightarrow s^q(\phi|q)(x, t)\). One obtains the explicit form from \(s^q(\phi|q)(x, t) \equiv \langle \hat{\rho}_j^\pm (x, t) | c_{ij}^\pm \langle \lambda_0 | e^{-\chi} E^{(1)}_{\alpha, \beta} e^{\chi} \Psi(0) h[\Psi(0)]^{-1} | \lambda_0 \rangle\). In the case of one-dark–dark-soliton, an element \(\chi_0\) is required such that \(s^q(\phi|q)(x, t) \equiv s^q(\phi) \chi_0^q(x, t)\), with \(s^q(\phi)\) and \(\chi_0^q(x, t)\).
being constant parameters. So under the above modified DT, one has the general form of the one-dark–dark-soliton,  

\[
\Psi_j^{\pm(q)} = \frac{\rho_j^+ e^{\Omega_j^{q} + \beta_j t}}{2} \left[ \left( 1 + (y_j^{q})^{\pm 1} \right) + \left( 1 - (y_j^{q})^{\pm 1} \right) \tanh \left( (k^q x + w^q t + \text{log} \Omega_j^{q}) / 2 \right) \right].
\]  

(5.49)

where \( j = 1, 2, y_j^{q} \equiv \frac{e^{\pm \Omega_j^{q}}}{\rho_j}, \) and

\[
b_i^\pm = \pm (a_i^+)^2 \pm 2 \left[ \sum_{j=1}^{2} \rho_j^+ \rho_j^- \left( \frac{1}{2} \beta_j \beta_i \right) \right] ; \quad \beta_1, \beta_2 \neq \beta_2, \beta_2. \]

(5.50)

Solutions (5.49) are the general one-dark–dark-soliton components of the AKNS2 model. The index \( q = 1, 2 \) refers to the solitons traveling to the left and right, respectively, and this solution has 13 real parameters, namely \( \rho_1^+, \rho_2^+, k^1, k^2, a_1^+, a_2^+, \Omega_1, \Omega_2, \) i.e. additional seven real parameters as compared with the solution in (5.39). A complexified version of this soliton for \( q = 1 \), which is a solution of the 2-CNLS model, has recently been reported in [4]. Moreover, the general form has also been reported in [14] for AKNS, using another approach. As in the AKNS1 case (5.27), it is clear that the one-dark-soliton profiles can be recovered by plotting the functions \( [\Psi_j^{\pm(q)}(x, t)] (j = 1, 2; q = 1, 2) \).

Moreover, similar steps can be followed in order to obtain the singular solutions of the AKNS2 model related to \( c_j^{(q)} < 0 \). They will give rise to singular solitons as in the case of the AKNS1 construction (5.28)–(5.29). It is also possible to construct mixed dark-singular solitons by conveniently choosing the positive and negative values of the parameters \( c_j^{(q)} \) of each component of the vector \( (\Psi_j^+) \).

Two dark–dark solitons. In order to obtain the two-dark–dark-soliton (\( N = 2 \)) solution, one must choose the group element \( h_2^\pm = e^{a_1 W_1(k_1)} e^{a_2 W_2(k_2)} \). Then following similar steps, one has

\[
\tau_0 = 1 + c_1 e^{k_1 x + w_1 t} + c_2 e^{k_2 x + w_2 t} + c_{12} d_0 e^{k_1 x + w_1 t} e^{k_2 x + w_2 t},
\]

(5.51)

\[
\tau_j^\pm = \rho_j^+ \left[ s_j^+ e^{k_1 x + w_1 t} + s_j^- e^{k_2 x + w_2 t} + s_j^+ s_j^- d_0 e^{k_1 x + w_1 t} e^{k_2 x + w_2 t} \right], \quad i = 1, 2,
\]

(5.52)

where

\[
d_0 = \left( \frac{s_1^+ s_2^- - s_1^- s_2^+}{\sqrt{s_1^+ s_2^-} - \sqrt{s_1^- s_2^+}} \right)^2 ; \quad d^\pm = -\frac{(s_1^+ s_2^- - s_1^- s_2^+)}{(s_1^+ - s_1^-)(s_2^+ - s_2^-)} d_0;
\]

(5.53)

\[
c_n = \frac{s_n^+ s_n^-}{s_n^+ - s_n^-} ; \quad (k_n)^2 = -\frac{(s_n^+ - s_n^-)^2}{s_n^+ s_n^-} \sum_{i=1}^{2} \rho_i^+ \rho_i^- ; \quad w_n = \frac{s_n^+ + s_n^-}{s_n^+ - s_n^-} \left( n = 1, 2 \right).
\]

(5.54)

Note that the above tau functions related to two-dark–dark-soliton must require \( c_j > 0 \). This two-soliton has eight real parameters, i.e. \( \rho_j^+, k_1, k_2, c_1, c_2 \). The above process can be extended for \( N \)-dark–dark-soliton solutions; in that case the group element in (3.26)–(3.28) must take the form \( h_N^\pm = e^{a_1 W_1(k_1)} e^{a_2 W_2(k_2)} \ldots e^{a_N W_N(k_N)} \). The relevant tau functions follow

\[
\tau_0 = \left[ \delta_{mn} + \frac{s_m^+ s_n^-}{s_m^+ s_n^-} e^{k_m x + w_m t} \right]_{N \times N} ; \quad m, n = 1, 2, 3, \ldots, N
\]

(5.55)

\[
\tau_i^\pm = \rho_i^+ \left[ \delta_{mn} + s_n^+ \left( 1 + c_m e^{k_m x + w_m t} \right) \right]_{N \times N} \right]^{-1} ; \quad i = 1, 2,
\]

(5.56)
where \(|N\times N|\) stands for the determinant. \(c_{mn} = \frac{c_{n}c_{l} - c_{l}c_{n}}{(\alpha_{n} - \alpha_{l})(\beta_{n} - \beta_{l})} \left(\frac{\sqrt{\alpha_{n}^2 + \beta_{n}^2}}{\sqrt{\alpha_{l}^2 + \beta_{l}^2}}\right)^2\) and \(w_{mn}, k_{n}\)
are the same as in (5.54). Note that \(c_{mn} = c_{nm}, c_{nn} = 0\). We have omitted the index \((q)\) in all the parameters above. The \(N\)-dark–dark-soliton (5.55) possesses \(2N + 4\) real parameters, i.e. \(\rho_{j}^{\pm}, k_{m}, c_{m}\). Let us mention that we have verified these solutions up to \(N = 3\) using the Mathematica program. The above solutions associated with the two- (5.51)–(5.52) or higher-order (5.55)–(5.56) dark–dark solitons are in general non-degenerate. Note that in the CNLS case, the two and higher dark–dark solitons derived in [10] are actually degenerate and reducible to scalar NLS dark solitons. So regarding this property, our solutions resemble those recently obtained in [4] for the \(r\)-CNLS system and in [14] for the AKNS model.

Let us remark that the free-field NVBC (3.29) requires the introduction of more parameters into the \(N\)-soliton tau functions (5.55)–(5.56) through the modified DT. So following similar steps to obtain (5.49), one has that (i) the parameters \(s_{n}^{\pm}\) entering the tau function \(\tau_{0}\) in (5.55) remain the same; (ii) in (5.56), the factor \(\rho_{j}^{\pm}\) must be changed to \(\rho_{j}^{\pm} e^{i\rho_{j}^{\pm}\tau_{j}^{\pm}}\); and (iii) the parameters \(s_{n}^{\pm}\) entering the tau functions \(\tau_{j}^{\pm}\) in (5.56) must be changed as \(s_{n}^{\pm} \rightarrow s_{n}^{\pm} x_{j}^{\pm}\). This general \(N\)-soliton will have, in addition to the \(\Omega_{1,2}\) parameters, \(3N + 8\) real parameters, i.e. \(\rho_{j}^{\pm}, k_{m}, c_{m, j}, a_{j}^{\pm}\).

Similar constructions can be performed to obtain the singular and the mixed dark-singular \(N\) solitons. In [4], the general non-degenerate \(N\) dark–dark solitons in the \(r\)-CNLS model with defocusing and mixed nonlinearity have been reported in the context of the KP-hierarchy reduction approach, and in [14], the bright and dark multi-soliton solutions of the AKNS system have been addressed in the algebro-geometric approach.

5.3. Dark–dark-soliton bound states

So far, reports on multi-dark–dark-soliton bound states in integrable systems, to our knowledge, are very limited. Recently, it has been shown that in the mixed-nonlinearity case of the 2-CNLS system, two dark–dark solitons can form a stationary bound state [4]. Then, in order to have multi-dark–dark-soliton bound states in the \(N\)-soliton solution (5.55)–(5.56), the constituent solitons should have the same velocity, i.e. denoting \(y_{n} = \frac{x_{n}^{+}}{x_{n}^{-}}\) (\(y_{n} \notin [0, 1]\)), then \(\frac{w_{n}}{k_{n}} = \frac{y_{n} + 1}{y_{n} - 1} k_{n} = v\) (assume \(v > 0\) for certain soliton parameters) where \(n = 1, 2, \ldots\). We show that the signs of the sum \(\sum_{j} \rho_{j}^{+}\rho_{j}^{-}\) determine the existence of these bound states; for the positive sign, bound states can be formed, whereas for the negative sign, they do not exist.

First, consider \(\sum_{j} \rho_{j}^{+}\rho_{j}^{-} > 0\); then from (5.54), it follows that \(\epsilon_{n} = \frac{\sqrt{|\rho_{n}^{+}\rho_{n}^{-}|}}{\sqrt{|\rho_{n}^{+}\rho_{n}^{-}|}} (\sum_{j} \rho_{j}^{+}\rho_{j}^{-})^{1/2}\) for \(y_{n} < 0\). Therefore, in order for multi-dark–dark-soliton bound states to exist, the equation

\[
\frac{|y_{n}| - 1}{|y_{n}| + 1} \sqrt{|y_{n}|} = \frac{v}{\sqrt{\sum_{j} \rho_{j}^{+}\rho_{j}^{-}}} \equiv c > 0
\]

(5.57)

must give at least two distinct positive solutions for \(|y_{n}|\). In fact, one has the following two solutions:

\[
|y_{1,2}| = 1 + \frac{c^{2}}{2} \pm \frac{\sqrt{16 + c^{2}}}{2} \mp \frac{\sqrt{12c^{2} + c^{4} + \Delta}}{2\sqrt{2}},
\]

(5.58)

where \(\Delta \equiv \frac{6c^{2} + 20c^{4} + c^{6}}{16 + 2c^{2}}\). These exhaust all the possible solutions with condition (5.57). The right-hand side of (5.58) with either + or − signs provides \(|y_{1,2}| > 0\) for any \(c > 0\). The other possible solutions \(y_{1,3,4}\) do not satisfy the above requirements, since they possess a term of the form \(\sqrt{12c^{2} + c^{4} - \Delta}\), which is imaginary for any positive value \(c\). Therefore, two-dark–dark-soliton bound states exist in the \(sl(3)\)-AKNS system, and three- and higher-dark–dark-soliton bound states cannot exist. These results hold for any value of the index \(q\). Note that in order
to reduce to the 2-CNLS system, the parameter relationship (5.41) must be satisfied. So, from (5.41), one has \( \sum_{i} \rho_i^+ \rho_i^- = \frac{1}{\mu} \left[ |\rho_i^+|^2 + |\rho_i^-|^2 \right] < 0 \) (remember that under \( \rho_i^+ \to i\rho_i^+ \), the sum \( \sum_{i} \rho_i^+ \rho_i^- \) reverses sign), so it is possible if \( \delta_1 = -\delta_2 \) (e.g. \( \delta_1 = -\delta_2 = -1 \) for \( |\rho_i^+| > |\rho_i^-| \)). This case corresponds to the 2-CNLS with mixed focusing and defocusing nonlinearities. Thus, the two-dark–dark-soliton bound state solution we have obtained here corresponds to the Manakov model with mixed nonlinearity. The case \( \delta_i = -1 \) \((i = 1, 2)\) defines the defocusing Manakov model which does not support multi-dark–dark-soliton bound states [4].

Second, the condition \( \sum_{i} \rho_i^+ \rho_i^- < 0 \) will provide only a single positive solution for \( y^d > 0 \) in the equation \( \frac{d}{dx} = v > 0 \). So one cannot obtain two or more solitons with the same velocity and therefore bound states in this case are not possible.

6. Mixed boundary conditions and dark–bright solitons

One may ask about the mixed boundary conditions for the system (2.6)–(2.7), i.e. NVBC for one of the field components of the system, say \( \Psi_1^+ \), and VBC for the other field component, \( \Psi_2^\pm \). So we will deal with the mixed boundary condition (3.4). Moreover, taking into account the sequence of conditions as in (2.10)

\[
t \to -it, \quad [\Psi_i^+] = -\mu \delta_i \Psi_i^- \equiv -\mu \delta_i \psi_i,
\]

the system (2.6)–(2.7), for \((\beta, \bar{\rho})\) being real, can be written as

\[
i\hbar \psi_k + \partial_x^2 \psi_k + 2\mu \left[ \sum_{j=1}^{2} \delta_j |\psi_j|^2 - \frac{1}{2} (\beta_k, \bar{\rho}) \right] \psi_k = 0, \quad k = 1, 2.
\]

Precisely, the system (6.2) in the defocusing case \((\delta_1 = \delta_2 = -1)\) and possessing the first trivial solution (3.4), i.e. \( \rho_1 \neq 0; \rho_2^\mp = 0 \), has been considered in [26] in order to investigate dark–bright solitons which describe an inhomogeneous two-species Bose–Einstein condensate. The system (2.6)–(2.7) with the mixed trivial solution (3.4) can be written as

\[
\partial_t \Psi_1^\pm = \pm \partial_x^2 \Psi_1^\pm + 2 \left[ \sum_{j=1}^{2} \Psi_j^+ \Psi_j^- - \rho_1^+ \rho_1^- \right] \Psi_i^\pm,
\]

\[
\partial_t \Psi_2^\pm = \pm \partial_x^2 \Psi_2^\pm + 2 \left[ \sum_{j=1}^{2} \Psi_j^+ \Psi_j^- - \frac{1}{2} \rho_1^+ \rho_1^- - \frac{3}{2} \Omega_2 \right] \Psi_i^\pm.
\]

From the previous sections, we can make the following observation: the vacuum connections relevant to each type of solutions must exhibit the fact that dark solitons are closely related to NVBC, whereas bright solitons to the relevant VBC one. Since \( \Omega_2 \) and \( \Omega_1 \) are some parameters satisfying \( 2\Omega_1 + \Omega_2 = 2\rho_1^+ \rho_1^- \), for simplicity we will assume \( \Omega_2 = 0, \Omega_1 = \rho_1^+ \rho_1^- \) in (3.6)–(3.7). Therefore, connections (3.6)–(3.7) for the mixed (constant-zero) boundary condition (3.4) take the form

\[
\hat{A}_\text{vac} \equiv e^{\tilde{\xi}_{\beta_1}} = E^{(1)} + \rho_1^+ E_{\beta_1}^{(0)} + \rho_1^- E_{-\beta_1}^{(0)},
\]

\[
\hat{B}_\text{vac} \equiv e^{\tilde{\xi}_{\bar{\rho}_1}} = E^{(2)} + \rho_1^+ E_{\beta_1}^{(1)} + \rho_1^- E_{-\beta_1}^{(1)}.
\]

The relevant group element \( \Psi_{\text{mbc}}^{(0)} \) is given by

\[
\Psi_{\text{mbc}}^{(0)} \equiv e^{i\rho_1^+ \tilde{\xi}_{\beta_1} + i\rho_1^- \tilde{\xi}_{\bar{\rho}_1}}.
\]
In order to construct the soliton solutions, we must look for the common eigenstates of the adjoint action of the vacuum connections (6.5) and (6.6). So one has

$$[\hat{F}^\pm_{\rho_1}, \Gamma_{\pm\rho_1}] = \pm k^\pm \Gamma_{\pm\rho_1},$$

$$[\hat{E}^2_{\rho_1}, \Gamma_{\pm\rho_1}] = \pm w^\pm \Gamma_{\pm\rho_1}, \quad w^\pm = (k^\pm)^2 - \rho_1^+ \rho_1^-,$$

where the vertex operators \(\Gamma_{\pm\rho_1}(k^\pm, \rho_1^\pm)\) are defined in (C.3). We expect that these vertex operators will be associated with the bright–dark-soliton solutions of the model.

Let us write the following expressions:

$$[x\hat{E}^2_{\rho_1} + \hat{F}^2_{\rho_1}, \Gamma_{\pm\rho_1}] = \psi^+(x, t)\Gamma_{\rho_2}; \quad \psi^+(x, t) = k^+ x + w^+ t;$$

$$(x\hat{E}^2_{\rho_1} + \hat{F}^2_{\rho_1}, \Gamma_{-\rho_2}) = -\psi^-(x, t)\Gamma_{-\rho_2}; \quad \psi^-(x, t) = k^- x + w^- t.$$  (6.10)

The one-dark–bright-soliton is constructed taking \(h\) in (3.26)–(3.28) as

$$h = e^{\gamma^+ \Gamma_{+\rho_1}} e^{\gamma^- \Gamma_{-\rho_1}}, \quad \gamma^\pm = \text{constants},$$

where the vertex operators in (C.3) have been considered. Note that due to the nilpotency property of the vertex operators, as presented in appendix C, the exponential series must truncate. So replacing the group element (6.12) in (3.26)–(3.28), one has the following tau functions:

$$\tilde{\tau}^\pm_1 = e^{\gamma^+ - w^-} \gamma^+ \gamma^- (\lambda_o) |{E}^{(1)}_{\pm\rho_1} \Gamma_{+\rho_1} \Gamma_{-\rho_1} |\lambda_o)$$

$$\tilde{\tau}^\pm_2 = e^{\gamma^+ \gamma^-} (\lambda_o) |{E}^{(1)}_{\pm\rho_1} \Gamma_{\pm\rho_1} |\lambda_o)$$

$$\tilde{\tau}_0 = 1 + e^{\gamma^+ - w^-} \gamma^+ \gamma^- (\lambda_o) |\lambda_o) \Gamma_{+\rho_1} \Gamma_{-\rho_1} |\lambda_o).$$  (6.13–15)

These tau functions resemble those in (5.36) and (5.37) for the one-dark-soliton (5.39) and equations (4.7) and (4.8) for the one-bright-soliton (4.9) components, respectively. The matrix elements of the type (6.13)–(6.15) can be computed, and will depend only on the parameters \(k^\pm\) and \(\rho_1^\pm\). So one has six independent parameters \(\gamma^\pm, k^\pm, \rho_1^\pm\) associated with these tau functions. Let us write the tau functions in the form \(\tilde{\tau}_0 = 1 + c_0 e^{k^+ x + w^+ t} e^{-k^- x - w^- t}, \quad \tilde{\tau}_1^\pm = a^\pm e^{k^+ x + w^+ t} e^{-k^- x - w^- t}, \quad \tilde{\tau}_2^\pm = b^\pm e^{k^+ x + w^+ t},\) where \(w^\pm = (k^\pm)^2 - \rho_1^+ \rho_1^-; \quad c_0 = a^+ a^-; \quad a^\pm = k^\pm \rho_1^-; \quad b^+ b^- = (\frac{a^+}{a^-})(k^+ - 1)(k^- + \rho_1^+ \rho_1^-).\) So using these tau functions in (3.25), one has

$$\Psi_1^\pm = \rho_1^\pm \pm \left(\frac{a^\pm}{2c_0}\right) \left[1 + \tanh \left(\frac{(k^- - k^+) x + (w^+ - w^-) t}{2}\right)\right]$$

$$\Psi_2^\pm = \frac{b^\pm}{2\sqrt{c_0}} e^{\frac{1}{2}(k^+ + k^-) x + (w^+ - w^-) t} \text{sech} \left(\frac{(k^- - k^+) x + (w^+ - w^-) t}{2}\right).$$

This is the one-dark–bright-soliton of the \(\hat{sl}(3)\) AKNS model (6.3)–(6.4) (for \(\Omega_2 = 0\)). Note that this solution has six independent real parameters, say \(a^-, b^+, k^\pm, \rho_1^\pm\).

The construction of the two dark–bright solitons follows similar steps. The group element

$$h = e^{\gamma^+ \Gamma_{+\rho_1}^{(1)}} e^{\gamma^- \Gamma_{-\rho_1}^{(2)}} e^{\gamma^+ \Gamma_{+\rho_1}^{(3)}} e^{\gamma^- \Gamma_{-\rho_1}^{(4)}}$$

$$5 \quad \text{There exist other eigenstates} \ |\pm\rangle \text{ defined in (C.5). However, these eigenstates are related to purely dark solitons for the first component} \Psi_1^\pm, \text{e.g. if one takes the group element} h = e^{\gamma}, \text{it will not excite the second component} \Psi_2^\pm \text{since the matrix elements of the type} (\lambda_o) |{E}^{(1)}_{\pm\rho_1} \lambda_o) \text{vanish.}$$
The 2-CNLS model. The generalization to ±τ does the job. We record the relevant tau functions in (C.3) will be associated with the roots β0 in (3.24). This tau function resembles those in [24] provided for the 2-CNLS model. The generalization to N dark–bright solitons requires the group element h to be

\[ h = e^{i \phi^{j} \Gamma_{+j}^{(k_{j})}} e^{i \phi^{j} \Gamma_{-j}^{(k_{j})}} e^{i \phi^{j} \Gamma_{+j}^{(k_{j})}} e^{i \phi^{j} \Gamma_{-j}^{(k_{j})}} \ldots e^{i \phi^{j} \Gamma_{+j}^{(k_{j})}} e^{i \phi^{j} \Gamma_{-j}^{(k_{j})}}. \]

7. Generalization to the AKNSr (r \geq 3) model

The procedures presented so far can directly be extended to the AKNSr (r \geq 3) model for the affine Kac–Moody algebra sl(n) furnished with the homogeneous gradation. According to the construction in [20], in this case the equations of motion will describe the dynamics of the fields \( \Psi^{j}_{+} \) (j = 1, 2, \ldots, r) associated with the generators \( E_{k_{j}}^{(j)} \), where the \( \beta_{j} \) are the positive roots defined by \( \beta_{j} = \alpha_{j} + \alpha_{j+1} + \ldots + \alpha_{r} \) (\( \alpha_{j} \) = simple roots). The outcome will be the equations in (2.6)–(2.7) with 2r real fields.

The DT methods would be applied following similar steps as in section 3 and subsection 3.1, for constant and free-field NVBCs, respectively. In particular, in the constant NVBC, the form of relationships (3.25) and (3.26)–(3.28) will remain the same, except that \( i = 1, 2, \ldots, r \). In the case of free-field NVBC, the relationships (5.47) and (5.48) would be satisfied with \( i = 1, 2, \ldots, r \).

The VBC and the bright solitons will be associated with the vertex operators \( F_{j}, G_{j} \) (see (C.1)) as in section 4. In this case, one requires \( \rho_{0}^{j} = 0 \) (i = 1, 2, \ldots, r) in (3.25). The dark solitons, as in section 5, will require the vertex operator of type \( W^{n}(k, \rho_{0}^{j}) \) (j = 1, 2, \ldots, r), the analog of the operator in (C.7) incorporating additional terms. Finally, the mixed boundary conditions and the dark–bright solitons will emerge by extending the discussion in section 6.

In the case of the vector one-soliton solution, it is possible to form the combination (m, r – m), where m = the number of dark components and r – m = the number of bright components. So the vertex operators analogous to \( \Gamma_{\pm \beta_{j}}^{(k_{j}),} \rho_{0}^{j} \) in (C.3) will be associated with the roots \( \pm \beta_{j} \) (j = 1, 2, \ldots, m) such that \( \Gamma_{\pm \beta_{j}}^{(k_{j}),} \rho_{0}^{j} \) (j = m + 1, m + 2, \ldots, r).
8. Discussion

We have considered soliton-type solutions of the AKNS model supported by the various boundary conditions (3.2)–(3.5): vanishing, (constant) non-vanishing and mixed vanishing–non-vanishing boundary conditions related to bright, dark and bright–dark-soliton solutions, respectively, by applying the DT approach as presented in [15]. The set of solutions of the AKNSr system (2.6)–(2.7) is much larger than the solutions of the r-CNLS system (2.11). A subset of solutions of the AKNSr system (2.6)–(2.7) for r = 2 and (5.4)–(5.5) for r = 1, respectively, solve the scalar NLS (5.7) and 2-CNLS system (2.11), under relevant complexifications.

Moreover, the free-field boundary condition (3.29) for dark solitons is considered in the context of a modified DT approach associated with the dressing group [29], and the general N-dark–dark-soliton solutions of the AKNS2 system have been derived. These soliton components are not proportional to each other and thus they do not reduce to the AKNS1 solitons; in this sense they are not degenerate. We showed that these solitons under convenient complexifications reduce to the general N-dark–dark solitons derived previously in the literature for the CNLS model [4, 14]. In addition, we have shown that two-dark–dark-soliton bound states exist in the \( \hat{sl}(3) \)-AKNS system, and three- and higher-dark–dark-soliton bound states cannot exist. These results hold for any value of the index \( q \). In the case of reduced 2-CNLS when focusing and defocusing nonlinearities are mixed, this result corresponds to two-dark–dark-soliton stationary bound state [4].

In the mixed constant boundary conditions, we derived the dark–bright solitons of the \( \hat{sl}(3) \) AKNS model. These solitons under complexification (6.1) reduce to the solitons of the 2-CNLS model (6.2) which will be useful in order to investigate dark–bright solitons appearing in an inhomogeneous two-species Bose–Einstein condensate [26].

The relevant steps toward the AKNSr (r \( \geq 3 \)) extension were briefly discussed in the framework of the DT methods. In particular, the vertex operator calculations can be extended in a direct way following the same steps as in appendices B and C and the \( \hat{sl}(n) \) h.w.r. [42].

Another point we should highlight relies upon the possible relevance of the CNLS tau functions to its higher order generalizations. We expect that the tau functions of the higher order CNLS generalization are related somehow to the basic tau functions of the usual CNLS equations. This fact is observed for example in the case of the coupled scalar NLS+ derivative-NLS system in which the coupled system possesses a composed tau function depending on the basic scalar NLS tau functions [41].

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Appendix A. \( \hat{sl}(2) \) matrix elements

The commutation relations for the \( \hat{sl}(2) \) affine Kac–Moody algebra elements are

\[
[H^{(m)} , H^{(n)}] = 2m\delta_{m+n,0}C ,
\]

\[
[H^{(m)} , E^{(n)}_{\pm}] = \pm 2E^{(m+n)}_{\pm} ,
\]

\[
[E^{(m)}_{\pm} , E^{(n)}_{\mp}] = H^{(m+n)} + m\delta_{m+n,0}C ;
\]
\[
[D, T^{(m)}_a] = m T^{(m)}_a; \quad T^{(m)}_a = \{ H^{(m)}, E^{(m)}_+ \}.
\]

The central extension ensures h.w.r. of the affine algebra (see e.g. [15]). So in the h.w.r. \(|\lambda_a\rangle, |\lambda_1\rangle\), one has the following relationships:

\[
E^{(0)}_+ |\lambda_a\rangle = 0 \quad (A.5)
\]

\[
E^{(m)}_\pm |\lambda_a\rangle = 0, \quad m > 0 \quad (A.6)
\]

\[
H^{(m)} |\lambda_a\rangle = 0, \quad m > 0; \quad (A.7)
\]

\[
C |\lambda_a\rangle = \delta_{a,1} |\lambda_a\rangle, \quad (A.8)
\]

\[
\langle \lambda| a \rangle = |\lambda_a\rangle, \quad (A.9)
\]

where \(a = 0, 1\). The adjoint relations \((E^{(m)}_\pm)^\dagger = E^{(-m)}_\pm, (H^{(m)})^\dagger = H^{(-m)}\) allow one to know their actions on the \(|\lambda_a\rangle\). Next, consider the vertex operators

\[
\hat{V}^q(\gamma, \hat{\rho}) = \sum_{n=-\infty}^{\infty} (\gamma^2 - \hat{\rho}^2)^{-n/2} e_q^n \left[ H^{(n)} - \frac{\hat{\rho}^+}{\gamma - e_q(\gamma^2 - \hat{\rho}^2)^{1/2}} E^{(0)}_+ \\
+ \frac{\hat{\rho}^-}{\gamma + e_q(\gamma^2 - \hat{\rho}^2)^{1/2}} E^{(0)}_- \right] + e_q \left( \frac{\gamma^2 - \hat{\rho}^2}{\gamma^2} \right)^{1/2} \delta_{n,0} C; \quad q = 1, 2, \quad (A.10)
\]

where \(e_q \equiv (-1)^{q-1}\) and \(\hat{\rho}^2 \equiv \hat{\rho}^+ \hat{\rho}^-\). The vertex operator \(\hat{V}^q(\gamma, \hat{\rho})\) satisfies

\[
[\hat{\epsilon}_1, \hat{V}^q] = 2\gamma \hat{V}^q, \quad [\hat{\epsilon}_2, \hat{V}^q] = (-1)^{q-1} 2\gamma (\gamma^2 - \hat{\rho}^2)^{1/2} \hat{V}^q, \quad q = 1, 2, \quad (A.11)
\]

where

\[
\hat{\epsilon}_1 = H^{(1)} + \hat{\rho}^+ E^{(0)}_+ + \hat{\rho}^- E^{(0)}_-, \quad \hat{\epsilon}_2 = H^{(2)} + \hat{\rho}^+ E^{(1)}_+ + \hat{\rho}^- E^{(1)}_- \quad (A.12)
\]

The following matrix elements can be computed using properties (A.5)–(A.9):

\[
\langle \lambda_a | \hat{V}^q | \lambda_a \rangle = e_q \frac{(\gamma^2 - \hat{\rho}^2)^{1/2}}{\gamma} \quad (A.13)
\]

\[
\langle \lambda_a | E^{(1)}_\pm \hat{V}^q | \lambda_a \rangle = e_q \frac{2\hat{\rho}^\mp}{\gamma \mp e_q(\gamma^2 - \hat{\rho}^2)^{1/2}} (\gamma^2 - \hat{\rho}^2)^{1/2} \quad (A.14)
\]

The matrix element \(\langle \lambda_a | \hat{V}^q(\gamma_1, \hat{\rho}) \hat{V}^q(\gamma_2, \hat{\rho}) | \lambda_a \rangle\) can be computed by developing the products and keeping only non-trivial terms; then one makes use of the commutation rules to change the order, and eventually to obtain some central terms \(C\). The double sum can be simplified to a single sum and each term can be substituted by power series like \(\sum_{n=0}^{\infty} x^n = \frac{1}{1-\gamma}\), \(\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}\). So one can obtain

\[
\langle \lambda_a | \hat{V}^q(\gamma_1, \hat{\rho}) \hat{V}^q(\gamma_2, \hat{\rho}) | \lambda_a \rangle = [2 + 2K(\gamma_1, \gamma_2)] - \frac{K_0(\gamma_1, \gamma_2)}{[1 - K_0(\gamma_1, \gamma_2)]^2} \quad (A.15)
\]

where \(K_0(\gamma_1, \gamma_2) \equiv \frac{\sqrt{\gamma_1^2 - \hat{\rho}^2}}{\sqrt{\gamma_2^2 - \hat{\rho}^2}}\) and \(K(\gamma_1, \gamma_2) \equiv \frac{\sqrt{\gamma_1^2 - \hat{\rho}^2} \sqrt{\gamma_2^2 - \hat{\rho}^2} - \gamma_1 \gamma_2}{\gamma_1 \gamma_2}\). \quad (A.16)
In order to prove the nilpotency property of the vertex operator $V^q(\gamma_1, \hat{\rho})$, when evaluated within the state $|\lambda_0\rangle$, it is convenient to write (A.15) in the following Laurent series expansion:

$$
\langle \lambda_0 | \hat{V}^q(\gamma_1, \hat{\rho}) \hat{V}^q(\gamma_2, \hat{\rho}) | \lambda_0 \rangle = -6\hat{\rho}^2 \sqrt{y_1^2 - \hat{\rho}^2} \sqrt{y_2^2 - \hat{\rho}^2} \left( \frac{\sqrt{y_1^2 - \hat{\rho}^2} + \sqrt{y_2^2 - \hat{\rho}^2}}{y_2^2 - \hat{\rho}^2} \right)^2 \\
\times \left[ \frac{1}{4!} - \frac{10y_2}{(y_2^2 - \hat{\rho}^2)^{3/2}} \frac{(y_1 - y_2)^2}{5!} + \frac{15(y_2^2 + \hat{\rho}^2)}{(y_2^2 - \hat{\rho}^2)^2} \frac{(y_1 - y_2)^2}{6!} \\
- \frac{420y_2(2y_2^2 + \hat{\rho}^2)}{(y_2^2 - \hat{\rho}^2)^3} \frac{(y_1 - y_2)^3}{7!} + \ldots \right].
$$
(A.17)

From this, it is clear that

$$
\lim_{\gamma_2 \to y} \langle \lambda_0 | \hat{V}^q(\gamma_1, \hat{\rho}) \hat{V}^q(\gamma_2, \hat{\rho}) | \lambda_0 \rangle \to 0, \quad q = 1, 2.
$$
(A.18)

### Appendix B. The affine Kac–Moody algebra $\hat{sl}_3(C)$

In the following, we provide some results about the affine Kac–Moody algebra $G = \hat{sl}_3(C)$ relevant to our discussions above. We follow closely [42, 18]. The elements of the $sl_3(C)$ Lie algebra are all $3 \times 3$ complex matrices with zero trace. Consider the corresponding root system $\Delta = \{ \pm\alpha_1, \pm\alpha_2, \pm\alpha_3 \}$, such that the three positive roots are $\alpha_i, i = 1, 2, 3$, with $\alpha_i, a = 1, 2$, being the simple roots and $\alpha_3 = \alpha_1 + \alpha_2$. We choose a standard basis for the Cartan subalgebra $H$ such that

$$
H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},
$$
(B.1)

and the generators of the root subspaces corresponding to the positive roots are chosen as

$$
E_{+\alpha_1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{+\alpha_2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{+\alpha_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$
(B.2)

For negative roots, one has $E_{-\alpha} = (E_{+\alpha})^T$. The invariant bilinear form on $sl_3(C)$, $(x | y) = \text{tr}(xy)$; $x, y \in sl_3(C)$ induces a non-degenerate bilinear form on $H^*$ which we also denote by $(\cdot | \cdot)$. This definition allows one to write

$$
(\alpha_1 | \alpha_1) = 2, \quad (\alpha_2 | \alpha_2) = 2, \quad (\alpha_1 | \alpha_2) = -1.
$$
(B.3)

On the other hand, the generators $T^{(m)} = [H^{(m)}, H^{(m)}_2, E^{(m)}_a]$, where $m \in \mathbb{Z}$ and $\alpha \in \Delta$, together with the central $C$ and the ‘derivation’ operator $D ([D, T^{(m)}] = mT^{(m)})$ form a basis for $sl_3(C)$. These generators satisfy the commutation relations

$$
[H^{(m)}_a, H^{(n)}_b] = m(\alpha_a | \alpha_b)C\delta_{m+n,0},
$$
(B.4)

$$
[H^{(m)}_a, E^{(n)}_{\pm\alpha_b}] = \pm(\alpha_a | \alpha_b)E^{(m+n)}_{\pm\alpha_b},
$$
(B.5)

$$
[E^{(m)}_{\alpha_a}, E^{(n)}_{-\alpha_b}] = H^{(m+n)}_a + mC\delta_{m+n,0},
$$
(B.6)

$$
[E^{(m)}_{\alpha_a}, E^{(n)}_{\alpha_b}] = H^{(m+n)}_1 + H^{(m+n)}_2 + mC\delta_{m+n,0}.
$$
(B.7)
and for the subspaces $G$ and therefore it follows from (B.13) and (B.14) that

$$ [E_{a_1}^{(m)}, E_{a_2}^{(n)}] = E_{a_1}^{(m+n)} , $$ (B.8)

$$ [E_{a_1}^{(m)}, E_{\bar{a}_1}^{(n)}] = -E_{a_2}^{(m+n)} , $$ (B.9)

$$ [E_{a_1}^{(m)}, E_{\bar{a}_2}^{(n)}] = E_{a_1}^{(m+n)} , $$ (B.10)

$$ [D, C] = 0 , $$ (B.11)

where $a, b = 1, 2, i = 1, 2, 3$ and $m, n \in \mathbb{Z}$. The remaining non-vanishing commutation relations are obtained by using the relation $[E_{a}^{(m)}, E_{\bar{a}}^{(n)}] = -[E_{a}^{(-m)}, E_{\bar{a}}^{(-n)}]$. In this paper, we use the homogeneous $\mathbb{Z}$-gradation of $\hat{sl}_3(C)$ which is defined by the grading operator $D$, such that

$$ \hat{sl}_3(C) = \bigoplus_{m \in \mathbb{Z}} \hat{g}_m , \quad [\hat{g}_m, \hat{g}_n] \subset \hat{g}_{m+n} , $$ (B.12)

where $\hat{g}_m = \{ x \in \hat{sl}_3(C) | [D, x] = m x ; m \in \mathbb{Z} \}$.

The subspace $\hat{g}_0$ is a subalgebra of $\hat{sl}_3(C)$ given by $\hat{g}_0 = CH_1 \oplus CH_2 \oplus CE_{\alpha}^{(0)} \oplus CC \oplus CD$ (B.13)

and for the subspaces $\hat{g}_m (m \neq 0)$ we have

$$ \hat{g}_m = C H_1^{(m)} \oplus C H_2^{(m)} \oplus C E_{\alpha_1}^{(m)} \oplus C E_{\bar{a}_2}^{(m)} \oplus C E_{\bar{a}_1}^{(m)} \oplus C E_{\bar{a}_2}^{(m)} \oplus C E_{\bar{a}_1}^{(m)} , $$ (B.14)

We use in the paper the fundamental h.w.r. $|\lambda_0 \rangle$, satisfying

$$ H_a^{(0)} |\lambda_0 \rangle = 0 , \quad E_a^{(0)} |\lambda_0 \rangle = 0 , \quad C |\lambda_0 \rangle = |\lambda_0 \rangle $$ (B.15)

for $a, b = 1, 2$, and $\alpha \in \Delta$. Such a state is annihilated by all positive-grade subspaces,

$$ \hat{g}_m |\lambda_0 \rangle = 0 , \quad m > 0 , $$ (B.16)

and all the representation space is spanned by the states obtained by acting on $|\lambda_0 \rangle$ with negative-grade generators. This representation space can be supplied with a scalar product such that one has

$$ (H_a^{(m)})^\dagger = H_{a}^{-m} , \quad (E_a^{(m)})^\dagger = E_{-a}^{-m} \quad (B.17)$$

$$ C^\dagger = C , \quad D^\dagger = D . $$ (B.18)

It follows from (B.13) and (B.14) that

$$ (\hat{g}_m)^\dagger = \hat{g}_{-m} , $$ (B.19)

and therefore

$$ |\lambda_0 \rangle \hat{g}_{-m} = 0 , \quad m > 0 . $$ (B.20)

In addition to the subalgebra $g_0$, it is also convenient to consider two additional subalgebras:

$$ \hat{g}_{<0} = \bigoplus_{m>0} \hat{g}_m , \quad \hat{g}_{>0} = \bigoplus_{m<0} \hat{g}_{-m} . $$ (B.21)

These subalgebras and the corresponding Lie groups play an important role in the DT method.

The next relationships are useful in the AKNS$_r (r = 2)$ model construction. The special element $E^{(i)}$ in the basis presented above can be written as

$$ E^{(i)} = \frac{1}{2} (H_1^{(i)} + 2H_2^{(i)}), \quad [D, E^{(i)}] = i E^{(i)} . $$ (B.22)
Then the matrix $E^{(0)}$ becomes

$$E^{(0)} = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$  \hfill (B.23)

The roots entering in the AKNS$_2$ construction are

$\beta_1 = \alpha_3 = \alpha_1 + \alpha_2; \quad \beta_2 = \alpha_2.$ \hfill (B.24)

Moreover, the following commutation relations hold:

$$[E^{(i)}, H_a^{(m)}] = i\delta_{am}C\delta_{i+m,0}; \quad a = 1, 2,$$

$$[E^{(j)}(n), E^{(m)}_{\pm\beta_1}] = \pm E^{(l+m)}_{\pm\beta_1}, \quad j, l = 1, 2. \hfill (B.25)$$

$$[E^{(j)}(n), E^{(m)}_{\pm\beta_1}, \pm\beta_2] = 0, \hfill (B.26)$$

$$[H_2^{(m)}, E^{(n)}_{\pm\beta_1}] = \pm E^{(m+n)}_{\pm\beta_1}, \quad [H_1^{(m)}, E^{(l)}_{\pm\beta_1}, \pm\beta_2] = \mp E^{(m+n)}_{\pm\beta_2} \hfill (B.27)$$

$$[H_2^{(m)}, E^{(l)}_{\pm\beta_1}, \pm\beta_2] = \mp E^{(m+n)}_{\pm\beta_2}. \hfill (B.28)$$

$$[H_1^{(m)}, E^{(n)}_{\pm\beta_1}, \pm\beta_2] = \pm 2E^{(m+n)}_{\pm\beta_2} \hfill (B.29)$$

$$[H_2^{(m)}, E^{(n)}_{\pm\beta_1}, \pm\beta_2] = \mp E^{(m+n)}_{\pm\beta_2}. \hfill (B.30)$$

**Appendix C. $\hat{s}(3)$ matrix elements**

Consider the vertex operators associated with bright-soliton solutions,

$$F_j = \sum_{n=-\infty}^{+\infty} v_j^n E^{-n}_{\beta_1}, \quad G_j = \sum_{n=-\infty}^{+\infty} \rho_j^n E^{-n}_{\beta_1}; \quad j = 1, 2; \quad v_j, \rho_j \in \mathbb{C}. \hfill (C.1)$$

It can be shown that they are nilpotent, i.e. $F_j^2 = 0, G_j^2 = 0$. The matrix element $\langle \lambda_o | F_j G_k | \lambda_o \rangle$ can be computed by developing the products and keeping only non-trivial terms; then one makes use of the commutation rules to obtain the central term $C$. The double sum can be simplified to a single sum, which provides the power series $\sum_{n=1}^{\infty} nx^n = \frac{1}{(1-x)^2}$. So one has

$$\langle \lambda_o | F_j G_k | \lambda_o \rangle = \frac{v_j \rho_k}{(v_j - \rho_k)^2} \delta_{j,k}. \hfill (C.2)$$

Let us consider the deformation of the vertex operators $F_2, G_2$ as

$$\Gamma_{\pm\beta_2}(k^\pm, \rho_1^\pm) = \sum_{n=-\infty}^{+\infty} \left( \frac{w^\pm}{k^\pm} \right) \left[ k^\pm E^{(n)}_{\pm\beta_2} - \rho_1^\pm E^{(n)}_{\pm\beta_1, \pm\beta_2} \right], \quad w^\pm = (k^\pm)^2 - \rho_1^\pm \rho_1^- \hfill (C.3)$$

It is a direct computation to show the nilpotency of these operators, i.e. $\Gamma_{\pm\beta_2}^2 = 0$. Similar computations to that in (C.2) provide the following matrix element:

$$\langle \lambda_o | \Gamma_{\pm\beta_2}(k^+) \Gamma_{-\beta_2}(k^-) | \lambda_o \rangle = \frac{w^+ w^- k^+ k^-}{(k^+ k^- + \rho_1^\pm \rho_1^-)(k^+ - k^-)^2}. \hfill (C.4)$$
Consider the vertex operator analogous to that in (A.10),

\[ V^q_{\beta_i}(\lambda, \rho_0) = \sum_{n=-\infty}^{\infty} \left\{ (\lambda^2 - \rho_0^2)^{-n/2} [\epsilon_q]^n \left[ \left( \frac{1}{2} (H_1^{(n)} + H_2^{(n)}) - \frac{\rho_1^+}{\lambda} - e_q(\lambda^2 - \rho_0^2)^{1/2} E^{(n)}_{\beta_i} \right) + \frac{\rho_1^-}{\lambda + e_q(\lambda^2 - \rho_0^2)^{1/2} E^{(n)}_{\beta_i}} \right] + e_q(\lambda^2 - \rho_0^2)^{1/2} \delta_{n,0} C \right\}; \quad q = 1, 2, \]  

(C.5)

where \( e_q \equiv (-1)^{q-1} \) and \( \rho_0^2 = 4\rho_1^+ \rho_1^- \). The next matrix element computation follows similar steps to that performed to arrive at (A.15), except that one must take into account the \( s\bar{l}(3) \) commutation rules. So one has

\[ \langle \lambda, \alpha \mid V^q_{\beta_i}(\lambda_1, \rho_0) V^q_{\beta_i}(\lambda_2, \rho_0) \mid \lambda, \alpha \rangle = \frac{1}{4} \left\{ 2 + 2K(\lambda_1, \lambda_2) \right\} \frac{K_0(\lambda_1, \lambda_2)}{[1 - K_0(\lambda_1, \lambda_2)]^2} \]  

\[ + \left[ K(\lambda_1, \lambda_2) \rho_0^2 + 1 \right] \}; \quad q = 1, 2, \]  

(C.6)

where \( K_0 \) and \( K \) are given in (A.16). Since this two-point function, except for an overall constant factor, is similar to that in (A.15), one can use relationships (A.17)–(A.18) to show that the operator \( V^q_{\beta_i}(\lambda_1, \rho_0) \) is nilpotent.

The vertex operator generating the dark–dark-soliton solution becomes

\[ W^q(k, \rho_{1,2}^\pm) = \sum_{n=-\infty}^{\infty} \left\{ \left( k^2 - 4 \sum_{i=1}^2 \rho_i^+ \rho_i^- \right)^{-n/2} \left[ \frac{\epsilon_q^n}{\lambda} \left[ s_1 H_1^{(n)} + s_2 H_2^{(n)} + \sum_{i=1}^2 \epsilon_{i(q)} E^{(n)}_{\beta_i} \right] \right. \right. \]  

\[ + \left. \left. 2 \sum_{i=1}^2 \epsilon_{i(q)} E^{(n)}_{\beta_i} + \epsilon_{12} E^{(n)}_{\beta_{12}} + \epsilon_{12} E^{(n)}_{\beta_{12}} \right] + e_q(\lambda - 4 \sum_{i=1}^2 \rho_i^+ \rho_i^-)^{1/2} \delta_{n,0} C \]  

\[ \epsilon_{i(q)}^{\pm} = k + \frac{2 \rho_i^+ \rho_i^-}{k - \epsilon_q(k^2 - 4 \sum \rho_i^+ \rho_j^-)^{1/2}}; \quad s_2 = \frac{1}{2}; \quad s_1 = \frac{1}{2} \sum \rho_i^+ \rho_i^-; \quad \epsilon_{12} = \frac{1}{2} \sum \rho_i^+ \rho_j^-; \quad e_q \equiv (-1)^{q-1}. \]  

(C.7)

Note that the vertex operator (C.7) reduces to that in (C.5) in the limit \( \rho_2^\pm \to 0 \). The nilpotent property of this vertex operator can be verified as follows:

\[ \langle \lambda, \alpha \mid W^1(k_1, \rho_{1,2}^\pm) W^1(k_2, \rho_{1,2}^\pm) \mid \lambda, \alpha \rangle = \left( \frac{x_1 x_2}{4} \right) \frac{x_2 - x_1}{x_1^2 + S} \left( x_1 + \frac{1}{4} S - 4x_1 \right) (x_2 - x_1 + \ldots), \]  

(C.8)

where \( x_1 = \sqrt{k_1^2 - 4S}, \quad x_2 = \sqrt{k_2^2 - 4S}, \quad S = \sum \rho_i^+ \rho_j^- \). In the limit \( x_2 \to x_1 \) (or \( k_2 \to k_1 \)), the rhs of equation (C.8) vanishes.

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