YANG–BAXTER OPERATORS FROM ALGEBRA STRUCTURES AND LIE SUPERALGEBRA STRUCTURES

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Abstract

We present solutions for the (constant and spectral-parameter) Yang-Baxter equations and Yang-Baxter systems arising from algebra structures. In the last section, we present enhanced versions of Theorem 1 (from [11]), thus obtaining Yang-Baxter operators from Lie superalgebras.

1 Introduction and preliminaries

The quantum Yang–Baxter equation (QYBE) first appeared in theoretical physics and statistical mechanics. It plays a crucial role in analysis of integrable systems, in quantum and statistical mechanics, in knot theory, and also in the theory of quantum groups. On the other hand, the theory of integrable Hamiltonian systems makes great use of the solutions of the one-parameter form of the Yang-Baxter equation, since coefficients of the power series expansion of such a solution give rise to commuting integrals of motion.

Non-additive solutions of the two-parameter form of the QYBE are referred to as a colored Yang-Baxter operator. They appear in this paper, and, in this case, they are related to the solutions of the one-parameter form of the Yang-Baxter equation.

Yang–Baxter systems emerged from the study of quantum integrable systems, as generalizations of the QYBE related to nonultralocal models.
This paper presents some of the latest results on Yang-Baxter operators from algebra structures and related topics (colored Yang-Baxter operators, Yang-Baxter systems, Yang-Baxter operator from Lie superalgebras). In the last section, we present enhanced versions of Theorem 1 (from [11]). It remains a research project to extend the above theorems for Yang-Baxter systems and spectral-parameter dependent Yang-Baxter equations. We omitted the proofs, but the reader is encouraged to check the results by direct computations.

The following is a short bibliography on QYBE ([8], [7], [3], [11]), and Yang-Baxter systems ([6], [5], [1], [10], [12]).

Throughout this paper $k$ is a field. All tensor products appearing in this paper are defined over $k$. For $V$ a $k$-space, we denote by $\tau : V \otimes V \to V \otimes V$ the twist map defined by $\tau(v \otimes w) = w \otimes v$, and by $I : V \to V$ the identity map of the space $V$.

We use the following notations concerning the Yang-Baxter equation.

If $R : V \otimes V \to V \otimes V$ is a $k$-linear map, then $R^{12} = R \otimes I, R^{23} = I \otimes R, R^{13} = (I \otimes \tau)(R \otimes I)(I \otimes \tau)$.

**Definition 1.1.** An invertible $k$-linear map $R : V \otimes V \to V \otimes V$ is called a Yang-Baxter operator if it satisfies the equation

$$R^{12} \circ R^{23} \circ R^{12} = R^{23} \circ R^{12} \circ R^{23} \tag{1.1}$$

**Remark 1.2.** The equation (1.1) is usually called the braid equation. It is a well-known fact that the operator $R$ satisfies (1.1) if and only if $R \circ \tau$ satisfies the constant QYBE (if and only if $\tau \circ R$ satisfies the constant QYBE):

$$R^{12} \circ R^{13} \circ R^{23} = R^{23} \circ R^{13} \circ R^{12} \tag{1.2}$$

**Remark 1.3.** (i) $\tau : V \otimes V \to V \otimes V$ is an example of a Yang-Baxter operator.

(ii) An exhaustive list of invertible solutions for (1.2) in dimension 2 is given in [4] and in the appendix of [6].

(iii) Finding all Yang-Baxter operators in dimension greater than 2 is an unsolved problem.

Let $A$ be a (unitary) associative $k$-algebra, and $\alpha, \beta, \gamma \in k$. We define the $k$-linear map: $R^A_{\alpha,\beta,\gamma} : A \otimes A \to A \otimes A, \quad R^A_{\alpha,\beta,\gamma}(a \otimes b) = aab \otimes 1 + \beta 1 \otimes ab - \gamma a \otimes b$.

**Theorem 1.4.** (S. Dăscălescu and F. F. Nichita, [2]) Let $A$ be an associative $k$-algebra with $\dim A \geq 2$, and $\alpha, \beta, \gamma \in k$. Then $R^A_{\alpha,\beta,\gamma}$ is a Yang-Baxter operator if and only if one of the following holds:

(i) $\alpha = \gamma \neq 0$, $\beta \neq 0$;
(ii) $\beta = \gamma \neq 0$, $\alpha \neq 0$;
(iii) $\alpha = \beta = 0$, $\gamma \neq 0$. 

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If so, we have $(R^A_{a,\beta,\gamma})^{-1} = R^A_{\frac{\beta}{\gamma}, \frac{\alpha}{\gamma}, \frac{1}{\gamma}}$ in cases (i) and (ii), and $(R^A_{0,0,\gamma})^{-1} = R^A_{0,0,\frac{1}{\gamma}}$ in case (iii).

Remark 1.5. The Yang–Baxter equation plays an important role in knot theory. Turaev has described a general scheme to derive an invariant of oriented links from a Yang–Baxter operator, provided this one can be "enhanced". In [9], we considered the problem of applying Turaev’s method to the Yang–Baxter operators derived from algebra structures presented in the above theorem. We concluded that Turaev’s procedure invariably produces from any of those enhancements the Alexander polynomial of knots.

We now present the matrix form of the operator obtained in the case (i) of the previous theorem, $R = R^A_{a,\beta,\alpha} : A \otimes A \to A \otimes A$, $R(a \otimes b) = a\alpha \otimes 1 + \beta b \otimes a - a\alpha \otimes b$. We consider the algebra $A = k[X]/(X^2 - mX - n)$, where $m, n$ are scalars. Then $A$ has the basis $\{1, x\}$, where $x$ is the image of $X$ in the factor ring. We consider the basis $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$ of $A \otimes A$ and represent the operator $R$ in this basis:

$R(1 \otimes 1) = \beta 1 \otimes 1$

$R(1 \otimes x) = (\beta - \alpha) 1 \otimes x + \alpha x \otimes 1$

$R(x \otimes 1) = \beta 1 \otimes x$

$R(x \otimes x) = (\alpha + \beta)n 1 \otimes 1 + \beta m 1 \otimes x + \alpha m x \otimes 1 - \alpha x \otimes x$

In matrix form, this operator reads:

$$
\begin{pmatrix}
\beta & 0 & 0 & 0 \\
0 & \beta - \alpha & \alpha & 0 \\
0 & 0 & \beta & 0 \\
(\alpha + \beta)n & \beta m & \alpha m & -\alpha
\end{pmatrix}
$$

(1.3)

Let us observe that $R' = R \circ \tau$ is a solution for the equation (1.2). It is convenient to get rid of the auxiliary parameters and to consider the simplest form of $R'$:

$$
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 - q & q & 0 \\
\eta & 0 & 0 & -q
\end{pmatrix}
$$

(1.4)

where $\eta \in \{0, 1\}$, and $q \in k - \{0\}$. The matrix form (1.4) was obtained as a consequence of the fact that isomorphic algebras produce isomorphic Yang–Baxter operators.
2 The two-parameter form of the QYBE

Formally, a colored Yang-Baxter operator is defined as a function $R : X \times X \to \text{End}_k(V \otimes V)$, where $X$ is a set and $V$ is a finite dimensional vector space over a field $k$. Thus, for any $u, v \in X$, $R(u, v) : V \otimes V \to V \otimes V$ is a linear operator. We consider three operators acting on a triple tensor product $V \otimes V \otimes V$, $R^{12}(u, v) = R(u, v) \otimes I$, $R^{23}(v, w) = I \otimes R(v, w)$, and similarly $R^{13}(u, w)$ as an operator that acts non-trivially on the first and third factor in $V \otimes V \otimes V$.

If $R$ satisfies the two-parameter form of the QYBE:

$$R^{12}(u, v)R^{13}(u, w)R^{23}(v, w) = R^{23}(v, w)R^{13}(u, w)R^{12}(u, v) \quad (2.5)$$

$\forall u, v, w \in X$, then it is called a colored Yang-Baxter operator.

Theorem 2.1. (F. F. Nichita and D. Parashar, [10]) Let $A$ be an associative $k$-algebra with $\text{dim } A \geq 2$, and $X \subset k$. Then, for any two parameters $p, q \in k$, the function $R : X \times X \to \text{End}_k(A \otimes A)$ defined by

$$R(u, v)(a \otimes b) = p(u - v)1 \otimes ab + q(u - v)ab \otimes 1 - (pu - qv)b \otimes a, \quad (2.6)$$

satisfies the colored QYBE $(2.5)$.

Remark 2.2. If $pu \neq qv$ and $qu \neq pv$ then the operator $(2.6)$ is invertible. Moreover, the following formula holds:

$$R^{-1}(u, v)(a \otimes b) = \frac{p(u-v)}{(pu-v)(pv-q)} ba \otimes 1 + \frac{q(u-v)}{(qu-p)(pv-q)} b \otimes a.$$

Remark 2.3. Let us consider the Theorem 2.1.

If we let $v=0$ and $u=1$, we obtain the operator $R(a \otimes b) = p1 \otimes ab + qab \otimes 1 - pb \otimes a$, which satisfies the constant QYBE $(1.2)$. Notice that $\tau \circ R$ is the Yang-Baxter operator from the Theorem 1.4, case (i).

We now consider the algebra $A = \frac{k[X]}{(X^2 - \sigma)}$, where $\sigma$ is a scalar. Then $A$ has the basis $\{1, x\}$, where $x$ is the image of $X$ in the factor ring. We consider the basis $\{1 \otimes 1, 1 \otimes x, x \otimes 1, x \otimes x\}$ of $A \otimes A$ and represent the operator $(2.6)$ in this basis:

$$R(u, v)(1 \otimes 1) = (qu - pv)1 \otimes 1$$
$$R(u, v)(1 \otimes x) = p(u - v)1 \otimes x + (q - p)ux \otimes 1$$
$$R(u, v)(x \otimes 1) = (q - p)v1 \otimes x + (u - v)x \otimes 1$$
$$R(u, v)(x \otimes x) = \sigma(p + q)(u - v)1 \otimes 1 - (pu - qv)x \otimes x$$

In matrix form, this operator reads

$$R(u, v) = \begin{pmatrix}
qu - pv & 0 & 0 & \sigma(p + q)(u - v) \\
0 & p(u - v) & (q - p)v & 0 \\
0 & (q - p)u & q(u - v) & 0 \\
0 & 0 & 0 & qv - pu
\end{pmatrix} \quad (2.7)$$
3 Yang-Baxter systems

It is convenient to describe the Yang-Baxter systems in terms of the Yang-Baxter commutators.

Let $V, V', V''$ be finite dimensional vector spaces over the field $k$, and let $R : V \otimes V' \rightarrow V \otimes V'$, $S : V \otimes V'' \rightarrow V \otimes V''$ and $T : V' \otimes V'' \rightarrow V' \otimes V''$ be three linear maps. The Yang–Baxter commutator is a map $[R, S, T] : V \otimes V \otimes V' \otimes V'' \rightarrow V \otimes V \otimes V' \otimes V''$ defined by

$$ [R, S, T] := R_{12}S_{13}T_{23} - T_{23}S_{13}R_{12}. \quad (3.8) $$

Note that $[R, R, R] = 0$ is just a short-hand notation for writing the constant QYBE (1.2).

A system of linear maps $W : V \otimes V \rightarrow V \otimes V, Z : V' \otimes V' \rightarrow V' \otimes V', X : V \otimes V' \rightarrow V \otimes V'$ is called a $WXZ$–system if the following conditions hold:

$$ [W, W, W] = 0 \quad [Z, Z, Z] = 0 \quad [W, X, X] = 0 \quad [X, X, Z] = 0 \quad (3.9) $$

It was observed that $WXZ$–systems with invertible $W, X$ and $Z$ can be used to construct dually paired bialgebras of the FRT type leading to quantum doubles. The above is one type of a constant Yang–Baxter system that has recently been studied in [10] and also shown to be closely related to entwining structures [1].

**Theorem 3.1.** (F. F. Nichita and D. Parashar, [10]) Let $A$ be a $k$-algebra, and $\lambda, \mu \in k$. The following is a $WXZ$–system:

- $W : A \otimes A \rightarrow A \otimes A, \quad W(a \otimes b) = ab \otimes 1 + \lambda 1 \otimes ab - b \otimes a,$
- $Z : A \otimes A \rightarrow A \otimes A, \quad Z(a \otimes b) = \mu ab \otimes 1 + 1 \otimes ab - b \otimes a,$
- $X : A \otimes A \rightarrow A \otimes A, \quad X(a \otimes b) = ab \otimes 1 + 1 \otimes ab - b \otimes a.$

4 Applications and conclusions

Using the techniques from above we now present enhanced versions of Theorem 1 (from [11]).

**Theorem 4.1.** Let $V = W \otimes kc$ be a $k$-space, and $f, g : V \otimes V \rightarrow V \otimes V$ $k$-linear maps such that $f, g = 0$ on $V \otimes c + c \otimes V$. Then, $R : V \otimes V \rightarrow V \otimes V, \quad R(v \otimes w) = f(v \otimes w) \otimes c + c \otimes g(v \otimes w)$ is a solution for QYBE (1.2).

Let $(L, [\cdot, \cdot])$ be a Lie superalgebra over $k$, and $Z(L) = \{ z \in L : [z, x] = 0 \text{ } \forall \text{ } x \in L \}$.

For $z \in Z(L)$, $|z| = 0$ and $\alpha \in k$ we define:
\[ \phi_{\alpha}^L : L \otimes L \longrightarrow L \otimes L \]

\[ x \otimes y \mapsto \alpha [x, y] \otimes z + (-1)^{|x||y|} y \otimes x. \]

Its inverse is:

\[ \phi_{\alpha}^{-1} : L \otimes L \longrightarrow L \otimes L \]

\[ x \otimes y \mapsto \alpha z \otimes [x, y] + (-1)^{|x||y|} y \otimes x. \]

**Theorem 4.2.** Let \((L, [,])\) be a Lie superalgebra over \(k\), \(z \in Z(L), |z| = 0\), and \(\alpha \in k\). Then: \(\phi_{\alpha}^L\) is a YB operator.

**Remark 4.3.** It remains a research project to extend the above theorems for Yang-Baxter systems and spectral-parameter dependent Yang-Baxter equations.

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