LARGE FRICTION LIMIT OF THE COMPRESSIBLE NAVIER-STOKES EQUATIONS WITH NAVIER BOUNDARY CONDITIONS IN GENERAL THREE-DIMENSIONAL DOMAINS

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ABSTRACT. In this paper, we study the Navier-Stokes equations of compressible, barotropic flow posed in a bounded set in \( \mathbb{R}^3 \) with different boundary conditions. Specifically, we prove that the local-in-time smooth solution of the Navier-Stokes equations with Navier boundary condition converges to the smooth solution of the Navier-Stokes equations with no-slip boundary condition as the Navier friction coefficient tends to infinity.

1. INTRODUCTION

We prove that the local-in-time smooth solution of the Navier-Stokes equations with Navier boundary condition converges to the smooth solution of the Navier-Stokes equations with no-slip boundary condition as the Navier friction coefficient tends to infinity. The present work is intended as the first step in extending the convergence results of large friction limit from the incompressible flows to compressible flows.

The Navier-Stokes equations of a compressible, barotropic flow in a bounded domain \( \Omega \subset \mathbb{R}^3 \) give the conservation of mass and the balance of momentum:

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u^j)_t + \text{div}(\rho u^j u) + (P(\rho))_{x_j} &= \mu \Delta u^j + \lambda (\text{div} u)_{x_j}.
\end{align*}
\]

The above system (1.1) is solved together with initial conditions

\[
(\rho(\cdot, 0), u(\cdot, 0)) = (\rho_0, u_0)
\]

and is equipped with either one of the following boundary conditions, namely

\[
n \cdot u(x) = 0 \text{ and } K \mu (\nabla u) \cdot t(x) = -u \cdot t(x), \quad \text{for } x \in \partial \Omega
\]

or

\[
u(x) = 0, \quad \text{for } x \in \partial \Omega.
\]

The meanings for the functions and symbols are given as follows:

- \( \rho \) and \( u = (u^1, u^2, u^3) \) are functions of \( x \in \Omega \) and \( t \geq 0 \) which represent density and velocity respectively;
- \( P = P(\rho) \) is a given function in \( \rho \) which stands for the pressure;
- \( \mu, \lambda > 0 \) are viscosity constants;

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• $(\cdot)_{x_j}$ and $(\cdot)_t$ stand for the spatial derivative $\frac{\partial}{\partial x_j}$ and time derivative $\frac{\partial}{\partial t}$ respectively;
• $\text{div}(\cdot)$ and $\Delta(\cdot)$ are the usual spatial divergence and Laplace operators;
• $n(x)$ and $t(x)$ are the unit outward normal and tangent vectors respectively on $\partial \Omega$;
• $K > 0$ is a constant and $\alpha := \frac{1}{K}$ is known as the Navier friction coefficient.

To facilitate later discussions, we name the system (1.1)-(1.2) with boundary condition (1.3) the Navier-Stokes equations with Navier boundary condition (NSENC), while we name the system (1.1)-(1.2) with boundary condition (1.4) the Navier-Stokes equations with Dirichlet boundary condition (NSEDC).

The so-called Navier boundary condition (1.3) was first proposed by Navier in [20] which states that the velocity on $\partial \Omega$ is proportional to the tangential component of the stress, while the Dirichlet boundary condition (1.4) (or more precisely the no-slip boundary condition) assumes the fluid will have zero velocity relative to the boundary. Regarding the two different boundary conditions given by (1.3) and (1.4), a very interesting but natural question arises:

(Q): As the constant $K$ vanishes (or equivalently the Navier friction coefficient $\alpha$ tends to infinity), will the solutions to the Navier-Stokes equations with Navier boundary conditions converge to a solution to the Navier-Stokes equations with the usual no-slip boundary conditions?

The present work is devoted to answer the question (Q) by justifying the limit for smooth local-in-time solutions of (1.1)-(1.2) with boundary condition (1.3) as $K$ tends to zero.

We first recall some known existence results regarding the system (1.1)-(1.2). Generally speaking, by imposing different conditions on the initial data, three types of solutions can be shown to exist and they enjoy different properties:

(i) When the initial data is taken to be close to a constant in $H^3(\Omega)$, Matsumura and Nishida [17, 18, 19] obtained global-in-time “small-smooth type” solutions, which were later extended by Danchin [3, 4] to solutions in certain scale-invariant homogeneous Besov spaces. Small-smooth type solutions are constructed via iterative procedure based on asymptotic decay rates for the corresponding linearised system, and they do not exhibit the generic singularities of the system.

(ii) When the initial data is having arbitrarily large energy and with nonnegative density, Lions [16] and Feireisl [6] proved the existence of “large-weak type” solutions (also refer to [13] for improvements in such direction). Large-energy weak solutions by their very nature possess very little regularity, which may even include some non-physical solutions (see [12] for related discussions).

(iii) Different from the two types of solutions as mentioned in (i) and (ii), a third type of “intermediate-weak type” solutions were studied by Hoff [7, 8], Perepelitsa [21], Suen [23, 24, 26] and Suen-Hoff [22] for which initial data are small in fairly weak norms and initial densities are nonnegative and essentially bounded. Such intermediate-weak type solutions have rich physical and mathematical meanings compared to other solution classes: on the one hand, these solutions may exhibit discontinuities in density and velocity gradient across hypersurfaces in $\mathbb{R}^3$, which is not observable from small-smooth
solutions; on the other hand, the solutions would still have enough regularity for the development of a uniqueness and continuous dependence theory \[2, 9, 25\] which seems difficult from the very weak framework for large-weak type solutions. Moreover, such solutions demonstrate fine-structure property near the boundary under no-slip boundary condition \[1.4\] (refer to \[11\] for the instantaneous tangency for density interfaces), which behave drastically different from the case when the Navier boundary condition \[1.3\] is imposed (see \[8\]).

Our goal is to first address the convergence of smooth local-in-time solutions (which can be viewed as the category (i) solutions mentioned above but without the smallness assumption on the initial data), which is the central topic for this paper. The convergences of weak solutions from category (ii) and (iii) will be considered elsewhere.

The convergence issue raised by question (Q) was studied by Kelliher \[14\] and Kim \[15\] for incompressible Navier-Stokes equations which proved the convergence of smooth solutions as \(K\) vanishes. In contrast to the incompressible flows, the compressible system \(1.1\) contains the unknown density function \(\rho\), which contributes to the following difficulties:

- one has to obtain appropriate bounds on \(\rho\) in order to gain control over the velocity \(u\), which is different from the incompressible case for which one can automatically have some control over \(u\) on the boundary (see \[14\] Lemma 9.1);
- one has to ensure some \(K\)-independent bounds on both \(\rho\) and \(u\) before taking the limit \(K \to 0\), which is highly non-trivial compared with the incompressible cases.

In this present work, we try to extend the results from \[14\] and \[15\] to the case for compressible Navier-Stokes equations and obtain convergence of both densities and velocities. The main novelties are:

- We obtain bounds on local-in-time solutions to \(1.1\)-\(1.2\) with boundary condition \(1.3\) which are independent of \(K\);
- We prove the strong convergence of both densities and velocities as \(K\) vanishes, which extends and strengthens the results for incompressible Navier-Stokes equations.

We now give a precise formulation of our results. The parameters \(\Omega, \mu, \lambda, P\) and \(K\) will be assumed to satisfy the following:

\[
\Omega \text{ is a bounded open set in } \mathbb{R}^d \text{ with a } C^4 \text{ boundary;} \quad (1.5)
\]

\[
\mu > 0 \text{ and } \mu + 3\lambda > 0; \quad (1.6)
\]

\[
P \in C([0, \infty)) \cap C^2((0, \infty)); \quad (1.7)
\]

\[
K \text{ is independent of } x \text{ and } t \text{ with } K > 0. \quad (1.8)
\]

Concerning the initial data \((\rho_0, u_0)\), it will be assumed that there is a positive constant \(M_0\) such that

\[
\|\rho_0\|_{L^\infty}, \|\rho_0^{-1}\|_{L^\infty}, \|\rho_0\|_{H^2}, \|u_0\|_{H^3} \leq M_0, \quad (1.9)
\]
and \( u_0 \) further satisfies
\[
    u_0(x) = 0, \quad \text{for } x \in \partial \Omega. \tag{1.10}
\]

We have the following local-in-time existence theorem for smooth solutions to (1.11). It gives the necessary bounds on the smooth solutions and will be useful for proving the convergence results later. The proof will be given in Section 3.

**Theorem 1.1.** Assume that the hypotheses (1.5)-(1.8) hold and let positive numbers \( M_0 \) and \( M_0' > M_0 \) be given. Then there is a positive time \( T > 0 \) and a constant \( N \), both depending on \( \Omega, \mu, \lambda, P, M_0 \) and \( M_0' \) but independent of \( K \) such that if initial data \((\rho_0, u_0)\) is given satisfying (1.9)-(1.10), then there are solutions \((\rho, u)\) and \((\tilde{\rho}, \tilde{u})\) to the initial-boundary value problems (NSENC) and (NSEDC) respectively which are defined on \( \Omega \times [0, T] \) and they satisfy
\[
    \sup_{0 \leq t \leq T} (\|\rho(\cdot, t)\|_{L^\infty}, \|\rho^{-1}(\cdot, t)\|_{L^\infty}, \|\rho(\cdot, t)\|_{H^2}) \\
    + \sup_{0 \leq t \leq T} (\|\tilde{\rho}(\cdot, t)\|_{L^\infty}, \|\tilde{\rho}^{-1}(\cdot, t)\|_{L^\infty}, \|\tilde{\rho}(\cdot, t)\|_{H^2}) \leq M_0' \tag{1.11}
\]
and
\[
    \sup_{0 \leq t \leq T} (\|u(\cdot, t)\|_{H^3}^2, \|u_t(\cdot, t)\|_{H^1}^2) + \int_0^T \|u_t(\cdot, t)\|_{H^2}^2 dt \\
    + \sup_{0 \leq t \leq T} (\|\tilde{u}(\cdot, t)\|_{H^3}^2, \|\tilde{u}_t(\cdot, t)\|_{H^1}^2) + \int_0^T \|\tilde{u}_t(\cdot, t)\|_{H^2}^2 dt \leq N. \tag{1.12}
\]

**Remark 1.2.** Here are some remarks regarding Theorem 1.1.

- In view of Theorem 1.1, for the case of (NSENC), one can replace the compatibility condition (1.10) on \( u_0 \) with a weaker assumption, namely \( n \cdot u_0(x) = 0 \) and \( K \mu (\nabla u_0) \cdot n(x) = -u_0 \cdot t(x), \quad \text{for } x \in \partial \Omega. \)

- The key idea of proving Theorem 1.1 follows closely to the one given in [10], the main difference here is to show the \( K \)-independence of the constants \( M_0' \) and \( N \).

- It is important to have \( M_0' \) and \( N \) both being independent of \( K \), which allow us to apply the bounds (1.11)-(1.12) for proving the convergence of smooth solutions as \( K \) vanishes.

Once we obtain the local-in-time existence of smooth solutions to the systems (NSENC) and (NSEDC), we proceed to study the convergence as \( K \) vanishes. The following theorem is the main result of this paper.

**Theorem 1.3.** Assume that the hypotheses (1.5)-(1.8) hold and let \((\rho_0, u_0)\) be given functions which satisfy (1.9)-(1.10). Suppose \((\rho, u)\) and \((\tilde{\rho}, \tilde{u})\) are smooth local-in-time solutions to the systems (NSENC) and (NSEDC) respectively defined on \( \Omega \times [0, T] \) with the same initial data \((\rho_0, u_0)\), as described by Theorem 1.1. Then there exists \( T^* \in (0, T] \) such that for any \( s_1 \in [0, 2] \) and \( s_2 \in [0, 1] \),
\[
    u \to \tilde{u} \text{ in } L^\infty([0, T^*]; H^{s_1}(\Omega)) \text{ as } K \to 0, \quad \tag{1.13}
\]
\[
    u \to \tilde{u} \text{ in } L^2([0, T^*]; L^2(\partial \Omega)) \text{ as } K \to 0, \quad \tag{1.14}
\]
and
\[
    \rho \to \tilde{\rho} \text{ in } L^\infty([0, T^*]; H^{s_2}(\Omega)) \text{ as } K \to 0. \tag{1.15}
\]
Remark 1.4. As a consequence of Theorem 1.3 if we define the energy functional

$$E(\rho, u, t) = \sup_{0 \leq s \leq t} \left[ \int_{\Omega} (|u|^2 + |\rho|^2)(x) dx + \int_0^t \int_{\Omega} |\nabla u|^2(x, s) dx ds, \right]$$

then for all \( t \in [0, T^*] \), we have \( E(\rho, u, t) \to E(\tilde{\rho}, \tilde{u}, t) \) as \( K \to 0 \).

The rest of the paper is organised as follows. In Section 2, we give some preliminary facts and definitions used in this paper and provide an estimate for the Lamé operator defined in (2.4). In Section 3, we obtain bounds on \( \rho \) and \( u \) which are independent of \( K \) and prove Theorem 1.1. Finally in Section 4, we prove the convergence of smooth solutions to (1.1) as \( K \) vanishes by making use of the bounds obtained in Theorem 1.4, thereby proving Theorem 1.3.

2. Preliminaries and notations

We introduce the following notations and conventions:

- For \( s \geq 0 \) and \( p \geq 1 \), \( W^{s,p}(\Omega) \) is the usual Sobolev space with norm \( \| \cdot \|_{W^{s,p}(\Omega)} \) given by

$$\| f \|_{W^{s,p}(\Omega)} := \sum_{|\beta| \leq s} \left( \int_{\Omega} |D_x^\beta f(x)|^p dx \right)^{\frac{1}{p}}.$$

We write \( H^s(\Omega) := W^{s,2}(\Omega) \). For simplicity, we also write \( H^s = H^s(\Omega) \), \( \| \cdot \|_{L^p} = \| \cdot \|_{L^p(\Omega)} \), \( \| \cdot \|_{W^{s,p}} = \| \cdot \|_{W^{s,p}(\Omega)} \), etc. unless otherwise specified.

- We adopt the usual notation for Hölder seminorms, namely for \( v : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( \beta \in (0, 1] \),

$$\langle v \rangle^\beta = \sup_{x_1, x_2 \in \mathbb{R}^3 \atop x_1 \neq x_2} \frac{|v(x_2) - v(x_1)|}{|x_2 - x_1|^\beta};$$

and for \( v : Q \subseteq \mathbb{R}^3 \times [0, \infty) \to \mathbb{R}^3 \) and \( \beta_1, \beta_2 \in (0, 1] \),

$$\langle v \rangle^{\beta_1, \beta_2}_{Q} = \sup_{(x_1, t_1), (x_2, t_2) \in Q \atop (x_1, t_1) \neq (x_2, t_2)} \frac{|v(x_2, t_2) - v(x_1, t_1)|}{|x_2 - x_1|^\beta_1 + |t_2 - t_1|^\beta_2}.$$

- Regarding the constants used in this work, \( C \) shall denote a positive and sufficiently large constant, whose value may change from line to line.

We recall the following standard facts (see [3] and [29] for example) which will be useful for later analysis:

- There is a constant \( C = C(\Omega) \) such that for all \( \varphi \in H^2 \),

$$\| \varphi \|_{L^\infty} \leq C \| \varphi \|_{H^2}. \quad (2.1)$$

- There is a constant \( C = C(\Omega) \) such that for \( p \in [2, 6] \) and \( \varphi \in H^1 \),

$$\| \varphi \|_{L^p} \leq C \left( \| \varphi \|_{L^2} + \| \varphi \|_{L^{6,p}}^{\frac{6-p}{2p}} \| \nabla \varphi \|_{L^2} \right). \quad (2.2)$$

- Assume that \( \partial \Omega \) is \( C^1 \), then there exists a constant \( C = C(\Omega) \) such that for all \( \varphi \in W^{1,1} \),

$$\int_{\partial \Omega} |\varphi(x)|^2 dS_x \leq C \int_{\Omega} \left\{ |\varphi(x)|^2 + |\varphi(x)||\nabla \varphi(x)| \right\} dx. \quad (2.3)$$
For $\mu > 0$ and $\mu + 3\lambda > 0$, we let $L$ denote the Lamé operator given by

$$ (Lu)^j = \mu \Delta u^j + \lambda \text{div}(u_x^j). \quad (2.4) $$

With respect to the Lamé operator $L$, we define the system

$$
\begin{cases}
Lu = -g \text{ in } \Omega, \\
\mathbf{n} \cdot u = 0 \text{ and } K\mu(\nabla u) \cdot t = -u \cdot t \text{ on } \partial\Omega,
\end{cases}
\quad (2.5)
$$

where $u : \bar{\Omega} \to \mathbb{R}^3$ is the unknown function and $g : \Omega \to \mathbb{R}^3$ is given.

The following lemma gives an estimate on $u$ in terms of $g$ which is crucial for obtaining bounds on solutions to the system (NSENC) later.

**Lemma 2.1.** Assume that $\mu > 0$ and $\mu + 3\lambda > 0$ and let $m = 0$ or 1. Assume in addition that $\Omega$ is a bounded open set in $\mathbb{R}^3$ with a $C^{m+3}$ boundary. Then there is a constant $C = C(\Omega, \mu, \lambda)$ independent of $K$ such that if $u$ is a solution of (2.5) with $g \in H^m(\Omega)$, then $u \in H^{m+2}(\Omega)$ and

$$ \|u\|_{H^{m+2}} \leq C(\|u\|_{L^2} + \|g\|_{H^m}) \quad (2.6) $$

**Proof.** The proof is almost identical to the one given in [10, Lemma 2.2] (also refer to [1] and [28] for more details). To see why the constant $C$ is independent of $K$, we give the estimates on $\|\nabla u\|_{L^2}$ as an example. We multiply (2.5) by $u$ and integrate to get

$$ \int_\Omega Lu \cdot u \, dx = -\int_\Omega g \cdot u \, dx. \quad (2.7) $$

Upon integrating by parts and applying the boundary condition (2.5) 2, the integral on the left of (2.7) can be rewritten as follows

$$ \int_\Omega Lu \cdot u \, dx = -\alpha \int_{\partial\Omega} |u|^2 \, dS_x - \int_\Omega [\mu|\nabla u|^2 + \lambda(\text{div}(u))^2] \, dx, $$

where $\alpha = \frac{1}{K}$. Hence (2.7) becomes

$$ \int_\Omega [\mu|\nabla u|^2 + \lambda(\text{div}(u))^2] \, dx \leq \int_\Omega g \cdot u \, dx - \alpha \int_{\partial\Omega} |u|^2 \, dS_x $$

$$ \leq \int_\Omega g \cdot u \, dx, $$

where the last inequity follows since $\alpha > 0$. Therefore, we can see that $\|\nabla u\|_{L^2}$ can be bounded in terms of $\|g\|_{L^2}$ and is independent of $K$, which gives the required interior regularity estimates for $u$. The case when $u$ is supported in the intersection of $\Omega$ with a small neighborhood of a point on the boundary $\partial\Omega$ follows by the same argument given in [10] (which is somewhat simpler in our case here since $K$ is now just a positive constant) and we omit the details for the sake of brevity. \qed

**Remark 2.2.** We observe that $K$ (or equivalently $\alpha$) has the correct sign, which allows us to discard those corresponding boundary terms appeared in the above analysis. This is also important for proving the $K$-independence for the constant $C$ as in (2.6).
3. Existence of local-in-time smooth solutions: Proof of
Theorem 1.1

In this section, we give the proof of Theorem 1.1. We only focus on the system (NSENC) since the local-in-time existence of (NSEDC) can be proved in the same way as in [27]. Throughout this section $M_0$ will be fixed as in the statement of Theorem 1.1.

We first give the following definitions of function spaces which will be useful for later analysis. More details can be found in [10].

**Definition 3.1.** For $M \geq M_0$ and $T > 0$, $\sum_{N,T}^{M}$ is the set of maps $\rho : [0, T] \rightarrow H^2(\Omega)$ such that $\rho(\cdot, 0) = \rho_0$, $\rho \in C([0, T]; H^1(\Omega))$, $\rho_t \in C([0, T]; L_\infty(\Omega))$, and
\[
\sup_{0 \leq t \leq T} (\|\rho(\cdot, t)\|_{L_\infty}, \|\rho^{-1}(\cdot, t)\|_{L_\infty, 1}, \|\rho(\cdot, t)\|_{H^2}) \leq M.
\]
(3.1)

And for $N > 0$, $\sum_{N,T}^{M}$ is the set of maps $u : [0, T] \rightarrow H^3(\Omega)$ such that $u \cdot u(x) = 0$, $u(\cdot, 0) = u_0$, $u \in C([0, T]; H^2(\Omega))$, $u_t \in C([0, T]; L^2(\Omega))$ and
\[
\sup_{0 \leq t \leq T} (\|u(\cdot, t)\|_{H^2}^2, \|u_t(\cdot, t)\|_{H^1}^2) + \int_0^T \|u_t(\cdot, t)\|_{H^2}^2 \leq N.
\]
(3.2)

We recall the following theorem from [10] Theorem 3.2 which shows that given $u \in \sum_{N,T}^{M}$, there is a corresponding solution $\rho$ to the mass equation (1.1) with initial data $\rho_0$.

**Theorem 3.2.** Given $N > 0$ and $M'_0 > M_0$, there is $T_1 = T_1(M_0, M'_0, N) > 0$ such that if $u \in \sum_{N,T}^{M}$ for some $T > 0$, then there is a unique $\rho \in \sum_{N,T}^{M}$ such that the pair $(\rho, u)$ satisfies the equation (1.1) such that
\[
\int_0^{\min\{T, T_1\}} |\nabla \rho|^2 dxdt \leq 1
\]
(3.3)
and
\[
\sup_{0 \leq t \leq \min\{T, T_1\}} \|\rho_t(\cdot, t)\|_{L^2} \leq C_1 M_0^2
\]
(3.4)
for a constant $C_1 > 0$ which depends only on $\Omega$. Also, there is a constant $C = C(M_0, M'_0, N)$ such that
\[
\langle \rho \rangle_{\Omega \times [0, T]} \leq C
\]
(3.5)
and
\[
\|\rho(\cdot, t_1) - \rho(\cdot, t_2)\|_{H^1}, \|\rho_t(\cdot, t_1) - \rho_t(\cdot, t_2)\|_{L^2} \leq C|t_2 - t_1|
\]
(3.6)
for all $t_1, t_2 \in [0, \min\{T, T_1\}]$.

Next, we reverse the role of $\rho$ and $u$ and obtain estimates on $u$ determined by a given density $\rho \in \sum_{N,T}^{M}$. The results are summarised in the following lemma.

**Lemma 3.3.** Given $M \geq M_0$, there is a positive time $T_2 = T_2(M_0, M)$ and a constant $N = N(M_0, M)$ independent of $K$ such that if $\rho \in \sum_{N,T}^{M}$ and satisfies
Step 1. Preliminary $L^2$ bound: We multiply (1.1) by $u'$, apply the boundary condition (1.3) and sum over $j$ to obtain
\[
\frac{d}{dt} \int_{\Omega} \frac{1}{2} \rho |u|^2 dx + \mu \int_{\Omega} |\nabla u|^2 dx + K^{-1} \int_{\partial \Omega} |u|^2 dS_x \\
\leq N \int_{\Omega} (|\nabla P| |u| + |\rho_t + \text{div}(\rho u)||u|^2) dx.
\]
Since $K > 0$, the boundary integral can be discarded from the left side, and hence we apply the bound (3.1) on $\rho$ to conclude
\[
\frac{d}{dt} \int_{\Omega} |u|^2 dx + \int_{\Omega} |\nabla u|^2 dx \leq N(1 + \|u(\cdot, t)\|_{H^1}^3). \tag{3.9}
\]

Step 2. $H^1$ bound: Next we multiply (1.1) by $u'_t$, apply the boundary condition (1.3) and sum over $j$ to obtain
\[
\frac{d}{dt} \left[ \int_{\Omega} \frac{1}{2} (\mu |\nabla u|^2 + \lambda (\text{div}(u))^2) dx + K^{-1} \int_{\partial \Omega} \frac{1}{2} |u|^2 dS_x \right] + N^{-1} \int_{\Omega} |u_t|^2 dx \\
\leq N \left[ 1 + \int_{\Omega} |u|^2 |\nabla u|^2 dx \right].
\]
Using Lemma 2.1 and the embedding (2.2), we have
\[
\int_{\Omega} |\Delta u|^2 dx \leq N \left( \int_{\Omega} (|u_t|^2 + |\rho|^2) + \int_{\Omega} |u|^2 |\nabla u|^2 \right), \tag{3.10}
\]
and the last integral on the right of the above can be bounded by
\[
\int_{\Omega} |u|^2 |\nabla u|^2 dx \leq N\|u\|_{H^1}^4 + \|u\|_{H^1}^3 \|\Delta u\|_{L^2} \leq N \left( \|u\|_{H^1}^4 + \|u\|_{H^1}^3 \left( \int_{\Omega} (|u_t|^2 + |\rho|^2) + \int_{\Omega} |u|^2 |\nabla u|^2 \right) \right),
\]
which gives
\[
\int_{\Omega} |u|^2 |\nabla u|^2 dx \leq N(1 + \|u\|_{H^1}^6 + \|u\|_{H^1}^3 \|u\|_{L^2}).
\]
Hence there is a positive time $T_2 = T_2(M_0, M)$ which is independent of $K$ such that $u$ satisfies
\[ \sup_{0 \leq t \leq \min\{T_2, T\}} \|u(\cdot, t)\|^2_{H^1} + \int_0^{\min\{T_2, T\}} \int_{\Omega} |u_t|^2 dx dt \leq N. \tag{3.11} \]

**Step 3. $H^2$ bound:** We then multiply (3.1) by $u_t$, apply the boundary condition (1.3) and sum over $j$ to obtain
\[ \frac{d}{dt} \int \frac{1}{2} \rho |u| dx + N^{-1} \int_{\Omega} |\nabla u|^2 dx + K^{-1} \int_{\partial\Omega} |u|^2 dS_x \leq N(1 + \|u\|^4_{L^2}). \]
Again since $K > 0$, the boundary integral as appeared above can be discarded. It follows that for a new time $T_2 = T_2(M_0, M)$,
\[ \sup_{0 \leq t \leq \min\{T_2, T\}} \int \Omega |u_t|^2 dx + \int_0^{\min\{T_2, T\}} \int_{\Omega} |\nabla u|^2 dx dt \leq N. \tag{3.12} \]
Furthermore, by applying the bounds (3.11) and (3.12) on (3.10), we can see that $\|\mathcal{L}u\|_{L^2}$ can be bounded by $N$, which implies that
\[ \sup_{0 \leq t \leq \min\{T_2, T\}} \|u(\cdot, t)\|_{H^2} \leq N. \tag{3.13} \]

**Step 4. $H^3$ bound:** Finally, we multiply (3.1) by $\mathcal{L}u_t$ and sum over $j$ to obtain
\[ \int_{\Omega} \rho(u_t + \nabla u u) + \nabla P(\rho) - \mathcal{L}u_t dx = 0. \tag{3.14} \]
We compute the term $\int_{\Omega} \rho u_t \cdot \Delta u_t$ as appeared in (3.14). Using the boundary condition (1.3),
\[ \int_{\Omega} \rho u_t \cdot \Delta u_t = -\frac{d}{dt} \int \frac{1}{2} \rho |\nabla u_t|^2 dx + \int \frac{1}{2} \rho |\nabla u_t|^2 dx \]
\[ - K^{-1} \int_{\partial\Omega} \rho |u_t|^2 dS_x - \frac{1}{2} \int_{\partial\Omega} \rho_t u_t \cdot \nabla u_t dS_x, \]
and by applying (2.3), the term $- \int_{\partial\Omega} \frac{1}{2} \rho_t u_t \cdot \nabla u_t dS_x$ can be bounded by
\[ \left| \int_{\partial\Omega} \frac{1}{2} \rho_t u_t \cdot \nabla u_t dS_x \right| \leq \frac{1}{2} \left( \int_{\partial\Omega} |\rho_t u_t|^2 \right)^{\frac{1}{2}} \left( \int_{\partial\Omega} |\nabla u_t|^2 \right)^{\frac{1}{2}} \]
\[ \leq N \|u_t\|_{H^2} \left\{ \left( \int_{\Omega} |\rho_t u_t|^2 \right)^{\frac{1}{2}} + \left( \int_{\Omega} |\rho_t u_t|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\nabla \rho_t|^2 |u_t|^2 \right)^{\frac{1}{2}} \right\} \]
\[ + N \|u_t\|_{H^2} \left\{ \left( \int_{\Omega} |\rho_t u_t|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\rho_t^2 |u_t|^2 \right)^{\frac{1}{2}} \right\}. \]
Hence we obtain
\[ \int_{\Omega} \rho u_t \cdot \Delta u_t \leq -\frac{d}{dt} \left( \int \frac{1}{2} \rho |\nabla u_t|^2 dx + K^{-1} \int_{\partial\Omega} \rho |u_t|^2 dS_x \right) \]
\[ + N \|u_t\|_{H^2} \left\{ \left( \int_{\Omega} |\rho_t u_t|^2 \right)^{\frac{1}{2}} \right\} \left( \int_{\Omega} |\nabla \rho_t|^2 |u_t|^2 \right)^{\frac{1}{2}} \]
\[ + N \|u_t\|_{H^2} \left\{ \left( \int_{\Omega} |\rho_t u_t|^2 \right)^{\frac{1}{2}} \left( \int_{\Omega} |\rho_t^2 |u_t|^2 \right)^{\frac{1}{2}} \right\}. \tag{3.15} \]
Notice that since $K^{-1} \int_{\partial \Omega} \rho |u| |\nu| dS_x \geq 0$, it can be dropped off from the analysis after we integrate over time. The other terms in (3.14) can be treated in a similar way as we did before so that by applying (3.15) on (3.14), integrating over time, using the bounds available for $\rho \in \tilde{\Sigma}^{M,T}$ and performing a long but straightforward sequence of estimates, we conclude that

$$\sup_{0 \leq t \leq \min(T_2,T)} \int_{\Omega} |\nabla u| |u_t| |\nu| dS_x + \int_{0}^{\min \{ T_2, T \}} \int_{\Omega} |L u| |u_t| |\nu| dS_x dt \leq N. \quad (3.16)$$

Together with (3.13) and the result (2.6) obtained in Lemma 2.1, we have

$$\sup_{0 \leq t \leq T} \left( \| u(t) \|_{H^3}, \| u_t(t) \|_{H^1} \right) + \int_{0}^{\min \{ T_2, T \}} \left\| u_t(s) \right\|_{H^2} |\nu| ds \leq N, \quad (3.17)$$

which implies $u \in \tilde{\Sigma}^{N, \min \{ T_2, T \}}$ as claimed. □

4. Convergence of smooth solutions: Proof of Theorem 1.3

In this section, we give the proof of Theorem 1.3 which will be carried out in a sequence of lemmas. We make use of the $K$-independent bounds obtained in Theorem 1.1 in controlling both the densities and velocities. To begin with, suppose $(\rho, \tilde{u})$ and $(\rho, u)$ are smooth classical solutions to the system (1.1) which are defined on $\Omega \times [0, T]$ with boundary conditions (1.3) and (1.4) respectively satisfying the bounds (1.11)-(1.12), and assume that $(\rho, \tilde{u})$ and $(\rho, u)$ are having the same initial data $(\rho_0, u_0)$ which satisfy (1.9)-(1.10). Define

$$w := u - \tilde{u}, \quad \phi := \rho - \tilde{\rho}.$$

Then we have $w|_{t=0} = 0$ and $\phi|_{t=0} = 0$. Furthermore, for all $t \in [0, T]$, $w$ and $\phi$ satisfy the following integral equations respectively:

$$\frac{1}{2} \int_{\Omega} |\phi|^2 dx + \int_{0}^{t} \int_{\Omega} \phi \text{div}(\rho u - \tilde{\rho} \tilde{u}) dx ds = 0. \quad (4.1)$$
and
\[
\int_0^t \int_\Omega (\rho u - \tilde{\rho} \tilde{u}) \cdot w dxds + \int_0^t \int_\Omega \text{div}(\rho u \otimes u - \tilde{\rho} \tilde{u} \otimes \tilde{u}) \cdot w dxds \\
+ \int_0^t \int_\Omega (\nabla P(\rho) - \nabla P(\tilde{\rho})) \cdot w dxds + \mu \int_\Omega |\nabla w|^2 + \lambda \int_\Omega (\nabla w)^2 dxds \\
= - \int_0^t \int_{\partial \Omega} (K^{-1} u - n \cdot \nabla \tilde{u}) \cdot \omega dxds.
\]

With the help of those \( K \)-independent bounds obtained in Theorem 1.1, we can bound \( \phi \) and \( w \) which will be given in subsequent lemmas.

We first prove the following lemma which gives an estimate on \( \phi \):

Lemma 4.1. For all \( t \in [0, T] \), we have
\[
\int_\Omega \phi^2(x, t) dx \leq C \left( \int_0^t \int_{\partial \Omega} |u|^2 ds dx \right)^{\frac{1}{2}} + C \int_0^t \int_\Omega |\phi|^2 dxds \\
+ C \left( \int_0^t \int_\Omega |\phi|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\nabla \tilde{u}|^2 dxds \right)^{\frac{1}{2}} \\
+ C \left( \int_0^t \int_\Omega |\phi|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\phi|^2 dxds \right)^{\frac{1}{2}},
\]

where \( C \) is a positive constant which only depends on \( T \) and on \( M_0, M'_0 \) and \( N \) as described in Theorem 1.1 and is independent of \( K \).

Proof. In view of (4.4), since \( \rho u - \tilde{\rho} \tilde{u} = \phi u + \tilde{\phi} w \), we can decompose the integral \( \int_0^t \int_\Omega \phi \text{div}(\rho u - \tilde{\rho} \tilde{u}) dxds \) as follows:
\[
\int_0^t \int_\Omega \phi \text{div}(\rho u - \tilde{\rho} \tilde{u}) dxds = \int_0^t \int_\Omega \phi \text{div}(\phi u + \tilde{\phi} w) dxds := I_1 + I_2,
\]
where
\[
I_1 := \int_0^t \int_\Omega \phi \text{div}(\phi u) dxds, \quad I_2 := \int_0^t \int_\Omega \phi \text{div}(\tilde{\phi} w) dxds.
\]

To estimate \( I_1 \), we notice that
\[
\int_0^t \int_\Omega \phi \text{div}(\phi u) = \int_0^t \int_\Omega \frac{1}{2} \text{div}(\phi^2 u) + \int_0^t \int_\Omega \frac{1}{2} \phi^2 \text{div}(u),
\]
and therefore by the bounds (1.12) and (2.1) and the boundary condition (1.3),
\[
I_1 = \frac{1}{2} \int_0^t \int_{\partial \Omega} \phi^2 n \cdot u ds dx + \frac{1}{2} \int_0^t \int_\Omega \phi^2 \text{div}(u) dxds \\
\leq \frac{1}{2} \|\text{div}(u)\|_{L^\infty} \int_0^t \int_\Omega |\phi|^2 dxds \leq C \int_0^t \int_\Omega |\phi|^2 dxds.
\]

For the term \( I_2 \), using the bounds (1.11) - (1.12) and (2.1), we readily have
\[
I_2 \leq C \int_0^t \int_\Omega |\tilde{\rho}| \nabla \omega \|\phi\| dxds + C \int_0^t \int_\Omega |\nabla \tilde{\rho}| \|\phi\| dxds \\
\leq C \|\tilde{\rho}\|_{L^\infty} \int_0^t \int_\Omega |\nabla \omega| \|\phi\| dxds + C \int_0^t \|\nabla \tilde{\rho}\|_{L^4} \|\omega\|_{L^4} \|\phi\|_{L^2} dxds.
\]
Using the bound (1.11) on $\hat{\rho}$, the term $C\|\hat{\rho}\|_{L^4} \int_0^t \int_\Omega |\nabla w| |\phi| dx ds$ can be bounded by
\[ C\left( \int_0^t \int_\Omega |\phi|^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\nabla w|^2 dx ds \right)^{\frac{1}{2}}, \]
and for $C \int_0^t \|\nabla \hat{\rho}\|_{L^2} \|w\|_{L^4} \|\phi\|_{L^2} dx ds$, we apply the embedding (2.2) and the bound (1.11) to obtain
\[ C \int_0^t \|\nabla \hat{\rho}\|_{L^4} \|w\|_{L^4} \|\phi\|_{L^2} dx ds \]
\[ \leq C \int_0^t \|\nabla \hat{\rho}\|_{L^2} + \|\nabla \hat{\rho}\|_{L^2} \|\nabla \hat{\rho}\|_{L^2} \|w\|_{L^2} + \|w\|_{L^2} \|\nabla w\|_{L^2} \|\phi\|_{L^2} \]
\[ \leq C\left( \int_0^t \int_\Omega |\phi|^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\nabla w|^2 dx ds \right)^{\frac{1}{2}} + C\left( \int_0^t \int_\Omega |\phi|^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |w|^2 dx ds \right)^{\frac{1}{2}}. \]
Hence we have
\[ I_2 \leq C\left( \int_0^t \int_\Omega |\phi|^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\nabla w|^2 dx ds \right)^{\frac{1}{2}} + C\left( \int_0^t \int_\Omega |\phi|^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |w|^2 dx ds \right)^{\frac{1}{2}}. \]
By (4.5)
We apply the bounds (4.4) and (4.5) on (4.1), and the assertion (4.3) follows.

Next we prove the following lemma which consists of the estimate on $w$:

**Lemma 4.2.** For all $t \in [0, T]$, we have
\[ \int_\Omega \rho|w(x, t)|^2 dx + \int_0^t \int_\Omega |\nabla w|^2 dx ds \]
\[ \leq C \int_0^t \int_\Omega |w|^2 dx ds + C\left( \int_0^t \int_\Omega |w|^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\nabla w|^2 dx ds \right)^{\frac{1}{2}} + C\left( \int_0^t \int_\Omega |\nabla w|^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\phi|^2 dx ds \right)^{\frac{1}{2}} \]
\[ + C\left( \int_0^t \int_\Omega |u|^2 dx ds \right)^{\frac{1}{2}} + C\left( \int_0^t \int_\Omega |\nabla w|^2 dx ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\partial \Omega |u|^2 dS_x ds \right)^{\frac{1}{2}}, \]
where $C$ is a positive constant which only depends on $T$ and on $M_0$, $M'_0$ and $N$ as described in Theorem 1.11 and is independent of $K$.

**Proof.** In view of (4.2), we first estimate the left side of (4.2). Define
\[ I_3 := \int_0^t \int_\Omega (\rho u - \hat{\rho} u) \cdot w dx ds, \quad I_4 := \int_0^t \int_\Omega \div(\rho u \otimes u - \hat{\rho} u \otimes \hat{u}) \cdot w dx ds \]
\[ I_5 := \int_0^t \int_\Omega (\nabla P(\rho) - \nabla P(\hat{\rho})) \cdot w dx ds, \quad I_6 := \mu \int_\Omega |\nabla w|^2 + \lambda \int_\Omega (\div w)^2 dx ds. \]
To estimate $I_3$, we note that

$$I_3 = \frac{1}{2} \int_0^t \rho |\mathbf{w}|^2 \, dx + I_{3,1} + I_{3,2} + I_{3,3}$$  \hspace{1cm} (4.7)

where

$$I_{3,1} := \int_0^t \int_\Omega \frac{1}{2} \rho_i |\mathbf{w}|^2 \, dx \, ds, \quad I_{3,2} := \int_0^t \int_\Omega \phi_i \tilde{\mathbf{u}} \cdot \mathbf{w} \, dx \, ds, \quad I_{3,3} := \int_0^t \int_\Omega \phi_i \tilde{\mathbf{u}} \cdot \mathbf{w} \, dx \, ds.$$

The term $I_{3,3}$ is readily bounded by

$$|I_{3,3}| \leq C \|\tilde{\mathbf{u}}_i\|_{H^2} \left( \int_0^t \int_\Omega |\mathbf{w}|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\phi|^2 \, dx \, ds \right)^{\frac{1}{2}}.$$

For $I_{3,1}$, we use the mass equation (1.1) for $(\rho, \mathbf{u})$ and notice that $\mathbf{w} = \mathbf{u}$ and $\mathbf{n} \cdot \mathbf{u} = 0$ on $\partial \Omega$, we have, by the bounds (1.11)-(1.12) and the estimate (2.1) on $\|\rho \mathbf{u}\|_{L^\infty}$ that

$$|I_{3,1}| \leq C \int_0^t \int_\Omega |\rho \mathbf{u}| \|\nabla \mathbf{w}\| |\mathbf{w}| \, dx \, ds$$

$$\leq C \|\rho \mathbf{u}\|_{L^\infty} \left( \int_0^t \int_\Omega |\mathbf{w}|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\nabla \mathbf{w}|^2 \, dx \, ds \right)^{\frac{1}{2}}$$

$$\leq C \left( \int_0^t \int_\Omega |\mathbf{w}|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\nabla \mathbf{w}|^2 \, dx \, ds \right)^{\frac{1}{2}},$$

while for the term $I_{3,2}$, we notice that

$$\int_0^t \int_\Omega \phi_i \tilde{\mathbf{u}} \cdot \mathbf{w} \, dx \, ds$$

$$= - \int_0^t \int_\Omega \text{div}(\rho \mathbf{u} - \tilde{\rho} \tilde{\mathbf{u}}) \tilde{\mathbf{u}} \cdot \mathbf{w} \, dx \, ds = - \int_0^t \int_\Omega \text{div}(\phi \mathbf{u} + \tilde{\rho} \tilde{\mathbf{u}}) \mathbf{w} \cdot \mathbf{w} \, dx \, ds$$

and recall the fact that $\tilde{\mathbf{u}} = 0$ on $\partial \Omega$ to obtain

$$|I_{3,2}| = \left| \int_0^t \int_\Omega \text{div}(\rho \mathbf{u} - \tilde{\rho} \tilde{\mathbf{u}}) \tilde{\mathbf{u}} \cdot \mathbf{w} \, dx \, ds \right| = \left| \int_0^t \int_\Omega (\phi \mathbf{u} + \tilde{\rho} \tilde{\mathbf{u}}) \cdot \nabla (\tilde{\mathbf{u}} \cdot \mathbf{w}) \, dx \, ds \right|,$$

which gives

$$|I_{3,2}| \leq C \int_0^t \int_\Omega |\mathbf{w}|^2 \, dx \, ds + C \left( \int_0^t \int_\Omega |\mathbf{w}|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\nabla \mathbf{w}|^2 \, dx \, ds \right)^{\frac{1}{2}}$$

$$+ C \left( \int_0^t \int_\Omega |\mathbf{w}|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\phi|^2 \, dx \, ds \right)^{\frac{1}{2}}$$

$$+ C \left( \int_0^t \int_\Omega |\nabla \mathbf{w}|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\phi|^2 \, dx \, ds \right)^{\frac{1}{2}}.$$
Hence we can bound $I_{3,1} + I_{3,2} + I_{3,3}$ by

$$\begin{align*}
|I_{3,1} + I_{3,2} + I_{3,3}| &\leq C \int_0^t \int_{\Omega} |w|^2 dxds + C \left( \int_0^t \int_{\Omega} |w|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\Omega} |\nabla w|^2 dxds \right)^{\frac{1}{2}} \\
& \quad + C \left( \int_0^t \int_{\Omega} |\nabla w|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\Omega} |\phi|^2 dxds \right)^{\frac{1}{2}} \\
& \quad + C(\|\tilde{u}\|_{H^2} + 1) \left( \int_0^t \int_{\Omega} |w|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\Omega} |\phi|^2 dxds \right)^{\frac{1}{2}}.
\end{align*}$$

(4.8)

Next we consider $I_4$. Upon integrating by parts, using the boundary condition that $\mathbf{n} \cdot \mathbf{u} = 0$ on $\partial \Omega$ and applying bounds (1.11)-(1.12), it can be estimated as follows (summation over repeated indexes is understood):

$$\begin{align*}
|I_4| &\leq \left| \int_0^t \int_{\Omega} \nabla w^j \cdot (\rho u^j - \tilde{\rho} \tilde{u}^j) dxds \right| \\
& \leq C \left( \int_0^t \int_{\Omega} |\nabla w|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\Omega} |\phi|^2 dxds \right)^{\frac{1}{2}} \\
& \quad + C \left( \int_0^t \int_{\Omega} |\nabla w|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\Omega} |\phi|^2 dxds \right)^{\frac{1}{2}}.
\end{align*}$$

(4.9)

For the term $I_5$, we again integrate by parts and use the boundary condition to obtain

$$\begin{align*}
|I_5| &\leq \left| \int_0^t \int_{\partial \Omega} \mathbf{n} \cdot \mathbf{u} (P(\rho) - P(\tilde{\rho})) dS_z ds \right| + \left| \int_0^t \int_{\Omega} (P(\rho) - P(\tilde{\rho})) \cdot \nabla w dxds \right| \\
& \leq C \left( \int_0^t \int_{\Omega} |\nabla w|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\Omega} |\rho - \tilde{\rho}|^2 dxds \right)^{\frac{1}{2}}.
\end{align*}$$

(4.10)

Now we estimate the right side of (4.12). Recalling that $w = u$ on $\partial \Omega$ and applying the estimate (2.3) on $\nabla \tilde{u}$ to get

$$\begin{align*}
-\mu \int_0^t \int_{\partial \Omega} (K^{-1} \mathbf{u} - \mathbf{n} \cdot \nabla \tilde{u}) \cdot \mathbf{w} ds_z ds \\
= -\mu \int_0^t \int_{\partial \Omega} K^{-1} |\mathbf{u}|^2 ds_z ds + \mu \int_0^t \int_{\partial \Omega} (\mathbf{n} \cdot \nabla \tilde{u}) \cdot \mathbf{u} ds_z ds \\
& \leq C \left( \int_0^t \int_{\partial \Omega} |\nabla \tilde{u}|^2 ds_z ds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\partial \Omega} |\mathbf{u}|^2 ds_z ds \right)^{\frac{1}{2}} \\
& \leq C \left( \int_0^t \int_{\partial \Omega} \{ |\nabla \tilde{u}|^2 + |\nabla \tilde{u}|^2 \} ds_z ds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\partial \Omega} |\mathbf{u}|^2 ds_z ds \right)^{\frac{1}{2}} \\
& \leq C \left( \int_0^t \int_{\partial \Omega} |\mathbf{u}|^2 ds_z ds \right)^{\frac{1}{2}}.
\end{align*}$$

(4.11)
Combining (4.7), (4.8), (4.9), (4.10), (4.11) with (4.2), we conclude that
\[ \|u\|_{L^2(0,T)} \leq C \int_0^T \int_\Omega |\nabla u|^2 dxds \]
and hence we obtain from (4.13) that
\[ \frac{1}{2} \int_0^t \int_\Omega \rho |\omega|^2 dx + \frac{1}{2} \int_0^t \int_\Omega |\nabla \omega|^2 dxds \]
\[ \leq C \left( \int_0^t \int_\Omega |\nabla w|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\nabla w|^2 dxds \right)^{\frac{1}{2}} + C \left( \int_0^t \int_\Omega |\nabla \omega|^2 dxds \right)^{\frac{1}{2}} \]
\[ + C \left( \int_0^t \int_\Omega |\nabla \omega|^2 dxds \right)^{\frac{1}{2}} + C \left( \int_0^t \int_\Omega |\nabla \omega|^2 dxds \right)^{\frac{1}{2}} + \int_0^t \int_\Omega \mu |\nabla \omega|^2 dxds \]
and the estimate (4.12) follows.

The following lemma contains the crucial bound on $\|u\|_{L^2(\partial \Omega)}$ in terms of $K$ which will be used for proving Theorem 1.3.

**Lemma 4.3.** There exists $T^* \in (0,T]$ such that for all $t \in [0,T^*)$, we have
\[ \int_0^t \int_{\partial \Omega} |u|^2 \, dS \, ds \leq M_0 K. \] (4.12)

**Proof.** We multiply the momentum equation (1.1) by $u^j$, sum over $j$ and integrate to obtain
\[ \int_\Omega \rho \frac{|u|^2}{2} dx + \int_0^t \int_\Omega u \cdot \nabla P dxds + \int_0^t \int_\Omega \mu \frac{|\nabla u|^2}{2} + \lambda (\text{div}(u))^2 \} dxds = \int_\Omega \rho \frac{|u_0|^2}{2} dx + \int_0^t \int_{\partial \Omega} \mu (\nabla u) \cdot u \, ds \, dxds + \int_0^t \int_{\partial \Omega} \lambda \text{div}(u) \, ndS \, ds \]
\[ = \int_\Omega \rho \frac{|u_0|^2}{2} dx - \int_0^t \int_{\partial \Omega} K^{-1} |u|^2 \, ds \, ds, \] (4.13)
where the last equality of (4.13) follows from the boundary condition (1.3). Upon integrating by parts and using the boundary condition that $n \cdot u = 0$ on $\partial \Omega$, we have
\[ \left| \int_0^t \int_\Omega u \cdot \nabla P dxds \right| \leq \left( \int_0^t \int_\Omega |\nabla u|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |P|^2 dxds \right)^{\frac{1}{2}} \]
\[ \leq C \left( \int_0^t \int_\Omega |\nabla u|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |P|^2 dxds \right)^{\frac{1}{2}}, \]
and hence we obtain from (4.13) that
\[ \int_\Omega \rho \frac{|u|^2}{2} dx + \int_0^t \int_\Omega \mu \frac{|\nabla u|^2}{2} + \lambda (\text{div}(u))^2 \} dxds \]
\[ \leq C \left( \int_0^t \int_\Omega |\nabla u|^2 dxds \right)^{\frac{1}{2}} \left( \int_0^t \int_\Omega |\rho|^2 dxds \right)^{\frac{1}{2}} + \int_\Omega \rho \frac{|u_0|^2}{2} dx \]
\[ - \int_0^t \int_{\partial \Omega} K^{-1} |u|^2 \, ds \, ds. \] (4.14)
On the other hand, we make use of the boundary condition (1.3) and the bound (1.11) on $\rho$ to obtain
\[
\int_{\Omega} \frac{\rho^2}{2} \, dx = \int_{\Omega} \frac{\rho_0^2}{2} \, dx - \int_0^t \int_{\Omega} \rho \text{div}(\rho u) \, dx \, ds
\leq \int_{\Omega} \frac{\rho_0^2}{2} \, dx + \int_0^t \int_{\Omega} |\nabla \rho| |u| \, dx \, ds
\leq \int_{\Omega} \frac{\rho_0^2}{2} \, dx + C \left( \int_0^t \int_{\Omega} |\rho|^2 \, dx \, ds \right)^{\frac{1}{2}} \left( \int_0^t \int_{\Omega} |u|^2 \, dx \, ds \right)^{\frac{1}{2}}.
\] (4.15)

We sum up (4.14) and (4.15), recall the positive lower bound (1.11) on $\rho$ and choose $T^* \in (0, T]$ small enough, then for $t \in [0, T^*)$, we have
\[
\sup_{s \in [0, t]} \left( \int_{\Omega} |\rho|^2 \, dx + \int_{\Omega} |u|^2 \, dx \right) \leq \int_{\Omega} \rho_0 |u_0|^2 \, dx + \int_{\Omega} |\rho_0|^2 \, dx + \int_0^t \int_{\partial \Omega} K^{-1} |u|^2 \, dS \, ds.
\] (4.16)

We deduce from (4.16) that
\[
\int_0^t \int_{\partial \Omega} |u|^2 \, dS \, ds \leq M_0 K, \quad \text{for all } t \in [0, T^*],
\] which implies (4.12).

We are now ready to give the proof of Theorem 1.3:

**Proof of Theorem 1.3.** We sum up (4.3) and (4.6), apply the positive lower bound (1.11) on $\rho$ and apply Young’s inequality to obtain, for $t \in [0, T]$,
\[
\int_{\Omega} \left\{ \phi^2(x, t) + |w|^2(x, t) \right\} \, dx + \int_0^t \int_{\Omega} |\nabla w|^2 \, dx \, ds
\leq C \left( \|\tilde{u}_1\|_{H^2} + 1 \right) \int_0^t \int_{\Omega} \left\{ \phi^2 + |w|^2 \right\} \, dx \, ds
+ C \int_0^t \int_{\partial \Omega} |u|^2 \, dS \, ds + \left( \int_0^t \int_{\partial \Omega} |u|^2 \, dS \, ds \right)^{\frac{1}{2}}.
\] (4.17)

Applying Grönwall’s inequality on (4.17) and using the bound (1.12) on the time integral $\int_0^T \|\tilde{u}_1(\cdot, t)\|_{H^2}^2 \, dt$, we further get
\[
\int_{\Omega} \left\{ \phi^2(x, t) + |w|^2(x, t) \right\} \, dx
\leq C \left\{ \int_0^t \int_{\partial \Omega} |u|^2 \, dS \, ds + \left( \int_0^t \int_{\partial \Omega} |u|^2 \, dS \, ds \right)^{\frac{1}{2}} \right\} e^{Ct}, \quad \text{for all } t \in [0, T].
\] (4.18)

Let $T^* > 0$ be chosen as in Lemma 4.3. We apply (4.12) on (4.18) to obtain, for all $t \in [0, T^*)$,
\[
\int_{\Omega} \left\{ \phi^2(x, t) + |w|^2(x, t) \right\} \, dx \leq C \left\{ M_0 K + M_0^\frac{1}{2} K^\frac{1}{2} \right\} e^{Ct}.
\] (4.19)

Hence by taking $K \to 0$ in (4.19), we have that
\[
\begin{cases}
\rho \to \tilde{\rho} \text{ in } L^\infty([0, T^*]; L^2(\Omega)) \text{ as } K \to 0, \\
u \to \tilde{u} \text{ in } L^\infty([0, T^*]; L^2(\Omega)) \text{ as } K \to 0.
\end{cases}
\] (4.20)
Moreover, by applying the convergences given in (4.20) on (4.19), we have
\[ u \to \tilde{u} \text{ in } L^2([0, T^*]; H^1(\Omega)) \text{ as } K \to 0. \]

Since \( \tilde{u} = 0 \text{ on } \partial\Omega \), using (4.12) from Lemma 4.3, we conclude that \( u \to \tilde{u} \text{ in } L^2([0, T^*]; L^2(\partial\Omega)) \text{ as } K \to 0 \) and (1.14) follows. Finally, by Sobolev inequality, for \( s_1 \in (0, 2] \), there exists \( \sigma = \sigma(s_1) \in (0, 1) \) such that for \( t \in [0, T^*] \),
\[ \|(u - \tilde{u})(\cdot, t)\|_{H^{s_1}} \leq \|(u - \tilde{u})(\cdot, t)\|_{L^2} \|(u - \tilde{u})(\cdot, t)\|^{1-\sigma}_{H^{-\sigma}}, \tag{4.21} \]
so that by applying the bound (4.12) on \( \|(u - \tilde{u})(\cdot, t)\|_{H^{s_1}}^{1-\sigma} \), we have
\[ \sup_{0 \leq t \leq T^*} \|(u - \tilde{u})(\cdot, t)\|_{H^{s_1}} \to 0 \text{ as } K \to 0 \]
and (1.13) follows. The proof of (1.16) is just similar and we finish the proof of Theorem 1.3. \( \square \)

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