Completing the spectrum of almost resolvable cycle systems with odd cycle length

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Abstract

In this paper, we construct almost resolvable cycle systems of order $4k + 1$ for odd $k ≥ 11$. This completes the proof of the existence of almost resolvable cycle systems with odd cycle length. As a by-product, some new solutions to the Hamilton-Waterloo problem are also obtained.

Key words: cycle system; almost resolvable cycle system; Hamilton-Waterloo problem

1 Introduction

In this paper, we use $V(H)$ and $E(H)$ to denote the vertex-set and the edge-set of a graph $H$, respectively. We denote the cycle of length $k$ by $C_k$ and the complete graph on $v$ vertices by $K_v$. A factor of a graph $H$ is a spanning subgraph whose vertex-set coincides with $V(H)$. If its connected components are isomorphic to $G$, we call it a $G$-factor. A $G$-factorization of $H$ is a set of edge-disjoint $G$-factors of $H$ whose edge-sets partition $E(H)$. A $C_k$-factorization of $H$ is a partition of $E(H)$ into $C_k$-factors. An $r$-regular factor is called an $r$-factor. Also, a 2-factorization of a graph $H$ is a partition of $E(H)$ into 2-factors.

A $k$-cycle system of order $v$ is a collection of $k$-cycles which partition $E(K_v)$. A $k$-cycle system of order $v$ exists if and only if $3 ≤ k ≤ v$, $v ≡ 1$ (mod 2) and $v(v − 1) ≡ 0$ (mod $2k$) \[2, 20\]. A $k$-cycle system of order $v$ is resolvable if it has a $C_k$-factorization. A resolvable $k$-cycle system of order $v$ exists if and only if $3 ≤ k ≤ v$, $v$ and $k$ are odd, and $v ≡ 0$ (mod $k$), see \[2, 4, 15, 19, 20, 24, 25\]. If $v ≡ 1$ (mod $2k$), then a $k$-cycle system exists, but it is not resolvable. In this case, Vanstone et al. \[27\] started the research of the existence of an almost resolvable $k$-cycle system.

In a $k$-cycle system of order $v$, a collection of $(v − 1)/k$ disjoint $k$-cycles is called an almost parallel class. In a $k$-cycle system of order $v ≡ 1$ (mod $2k$), the maximum possible number

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of almost parallel classes is \((v - 1)/2\), in which case a half-parallel class containing \((v - 1)/2k\) disjoint \(k\)-cycles is left over. A \(k\)-cycle system of order \(v\) whose cycle set can be partitioned into \((v - 1)/2\) almost parallel classes and a half-parallel class is called an \textit{almost resolvable \(k\)-cycle system}, denoted by \(k\text{-ARCS}(v)\).

For recursive constructions of almost resolvable \(k\)-cycle systems, C. C. Lindner, et al. \cite{18} have considered the general existence problem of almost resolvable \(k\)-cycle system from the commutative quasigroup for \(k \equiv 0 \pmod{2}\) and made a hypothesis: if there exists a \(k\text{-ARCS}(2k + 1)\) for \(k \equiv 0 \pmod{2}\) and \(k \geq 8\), then there exists a \(k\text{-ARCS}(2kt + 1)\) except possibly for \(t = 2\). H. Cao et al. \cite{12, 22, 30} continued to consider the recursive constructions of an almost resolvable \(k\)-cycle system for \(k \equiv 1 \pmod{2}\). Many authors contributed to the following known results.

**Theorem 1.1.** \((1, 6, 12, 14, 18, 27)\) Let \(k \geq 3\), \(t \geq 1\) be integers and \(n = 2kt + 1\). There exists a \(k\text{-ARCS}(n)\) for \(k \in \{3, 4, 5, 6, 7, 8, 9, 10, 14\}\), except for \((k, n) \in \{(3, 7), (3, 13), (4, 9)\}\) and except possibly for \((k, n) \in \{(8, 33), (14, 57)\}\).

**Theorem 1.2.** \((22, 30)\) For any odd \(k \geq 11\), there exists a \(k\text{-ARCS}(2kt + 1)\), where \(t \geq 1\) and \(t \neq 2\).

In this paper, we construct almost resolvable cycle systems of order \(4k + 1\) for odd \(k \geq 11\). Combining the known results in Theorems 1.1, 1.2, we will prove the following main result.

**Theorem 1.3.** For any odd \(k \geq 3\), there exists a \(k\text{-ARCS}(2kt + 1)\) for all \(t \geq 1\) except for \((k, t) \in \{(3, 1), (3, 2)\}\).

## 2 Preliminary

In this section we present a basic lemma for the construction of a \(k\text{-ARCS}(4k + 1)\). The main idea is to find some initial cycles with special properties such that all the required almost parallel classes can be obtained from them. We need the following notions for that lemma.

Suppose \(\Gamma\) is an additive group and \(I = \{\infty_1, \infty_2, \ldots, \infty_f\}\) is a set which is disjoint with \(\Gamma\). We will consider an action of \(\Gamma\) on \(\Gamma \cup I\) which coincides with the right regular action on the elements of \(\Gamma\), and the action of \(\Gamma\) on \(I\) will coincide with the identity map. In other words, for any \(\gamma \in \Gamma\), we have that \(x + \gamma\) is the image under \(\gamma\) of any \(x \in \Gamma\), and \(x + \gamma = x\) holds for any \(x \in I\). Given a graph \(H\) with vertices in \(\Gamma \cup I\), the \textit{translate} of \(H\) by an element \(\gamma\) of \(\Gamma\) is the graph \(H + \gamma\) obtained from \(H\) by replacing each vertex \(x \in V(H)\) with the vertex \(x + \gamma\). The \textit{development} of \(H\) under a subgroup \(\Sigma\) of \(\Gamma\) is the collection \(\text{dev}_\Sigma(H) = \{H + x \mid x \in \Sigma\}\) of all translates of \(H\) by an element of \(\Sigma\).

For our constructions, we set \(\Gamma = \mathbb{Z}_k \times \mathbb{Z}_4\). Given a graph \(H\) with vertices in \(\Gamma\) and any pair \((r, s) \in \mathbb{Z}_4 \times \mathbb{Z}_4\), we set \(\Delta_{(r,s)} H = \{x - y \mid \{(x, r), (y, s)\} \in E(H)\}\). Finally, given a list
\( \mathcal{H} = \{H_1, H_2, \ldots, H_t\} \) of graphs, we denote by \( \Delta_{(r,s)} \mathcal{H} = \bigcup_{i=1}^t \Delta_{(r,s)} H_i \) the multiset union of the \( \Delta_{(r,s)} H_i \).

**Lemma 2.1.** Let \( v = 4k + 1 \) and \( \mathcal{C} = \{F_1, F_2\} \) where each \( F_i \) \( (i = 1, 2) \) is a vertex-disjoint union of four cycles of length \( k \) satisfying the following conditions:

(i) \( V(F_i) = (\{(z_k \times Z_4) \cup \{\infty\} \}) \{ (a_i, b_i) \} \) for some \( (a_i, b_i) \in Z_k \times Z_4, \ i = 1, 2; \)

(ii) \( \infty \) has a neighbor in \( Z_k \times \{j\} \) for each \( j \in Z_4; \)

(iii) \( \Delta_{(p,p)} \mathcal{C} = Z_k \setminus \{0\} \) for each \( p \in \{0, 1, 2, 3\}; \)

(iv) \( \Delta_{(q,q)} \mathcal{C} = Z_k \setminus \{0, \pm d_q\} \) for each \( q \in \{2, 3\} \), where \( d_q \) satisfies \( (d_q, k) = 1; \)

(v) \( \Delta_{(r,s)} \mathcal{C} = Z_k \) for each pair \( (r, s) \in Z_4 \times Z_4 \) satisfying \( r \neq s. \)

Then, there exists a k-\text{ARCS}(v).

**Proof:** Let \( V(K_v) = (Z_k \times Z_4) \cup \{\infty\} \). Note that \( 0, d_q, 2d_q, \ldots, (k - 1)d_q \) are \( k \) distinct elements since \( (d_q, k) = 1 \). Then we have the required half parallel class which is formed by the two cycles \( ((0, q), (d_q, q), (2d_q, q), \ldots, ((k - 1)d_q, q)) \), \( q = 2, 3 \). By (i), we know that \( F_i \) is an almost parallel class. All the required \( 2k \) almost parallel classes are \( F_i + (l, 0), \ i = 1, 2, \ l \in Z_k \).

Now we show that the half parallel class and the \( 2k \) almost parallel classes form a k-\text{ARCS}(v). Let \( F' \) be a graph with the edge-set \( \{(a, q), (a + d_q, q)\} \mid a \in Z_k, q = 2, 3 \) and \( \Sigma := Z_k \times \{0\} \). Let \( \mathcal{F} = \text{dev}_2(\mathcal{C}) \cup F' \). The total number of edges – counted with their respective multiplicities – covered by the almost parallel classes and the half parallel class of \( \mathcal{F} \) is \( 2k(4k + 1) \), that is exactly the size of \( E(K_v) \). Therefore, we only need to prove that every pair of vertices lies in a suitable translate of \( \mathcal{C} \) or in \( F' \). By (ii), an edge \( \{(z, j), \infty\} \) of \( K_v \) must appear in a cycle of \( \text{dev}_2(\mathcal{C}) \).

Now consider an edge \( \{(z, j), (z', j')\} \) of \( K_v \) whose vertices both belong to \( Z_k \times Z_4 \). If \( j = j' \in \{2, 3\} \) and \( z - z' \in \{\pm d_q\} \), then this edge belongs to \( F' \). In all other cases there is, by (iii)-(v), an edge of some \( F_i \) of the form \( \{(w, j), (w', j')\} \) such that \( w - w' = z - z' \). It then follows that \( F_i + (w - w' + z', 0) \) is an almost parallel class of \( \text{dev}_2(F_i) \) containing the edge \( \{(z, j), (z', j')\} \) and the conclusion follows.

3 \( \text{ARCS}(4k + 1) \) for \( k \equiv 1 \) (mod 4)

In this section, we will prove the existence of a k-\text{ARCS}(4k + 1) for \( k \equiv 1 \) (mod 4).

**Lemma 3.1.** For any \( k \geq 13 \) and \( k \equiv 1 \) (mod 4), there exists a k-\text{ARCS}(4k + 1).

**Proof:** Let \( v = 4k + 1 \) and \( k = 4n + 1, \ n \geq 3 \). We use Lemma 2.1 to construct a k-\text{ARCS}(v) with \( V(K_v) = (Z_k \times Z_4) \cup \{\infty\} \). The required parameters in (i) and (iv) of Lemma 2.1 are \( (a_1, b_1) = (0, 3), \ (a_2, b_2) = (0, 2), \ d_2 = 2, \) and \( d_3 = \frac{k-1}{2} \). The required 8 cycles in \( F_1 = \{C_1, C_2, C_3, C_4\} \) and \( F_2 = \{C_5, C_6, C_7, C_8\} \) are listed as below.
The cycle \( C_1 \) is the concatenation of the sequences \( T_1, (0, 0), \) and \( T_2, \) where
\[
T_1 = ((n, 0), (−n, 0), \ldots, (n−i, 0), (−(n−i), 1), \ldots, (1, 0), (−1, 1)), 0 ≤ i ≤ n − 1;
T_2 = ((1, 1), (−1, 0), \ldots, (1 + i, 1), (−(1 + i), 0), \ldots, (n, 1), (−n, 0)), 0 ≤ i ≤ n − 1.
\]

**Note:** Actually \( T_1 \) can be viewed as the concatenation of the sequences \( T_1^0, T_1^1, \ldots, T_1^{n−1}, \)
where the general formula is \( T_1^i = ((n − i, 0), (−(n − i), 1)), 0 ≤ i ≤ n − 1. \) Thus, for brevity,
we just list the first sequence at the beginning of \( T_1 \) and the last sequence at the end of \( T_1, \)
and use the underlined sections to give the general formula in the middle of \( T_1. \) We give the
range of \( i \) after the sequence \( T_1 \) so that the reader can easily calculate the number of vertices
in \( T_1. \) Similarly, this partial underlying happens ahead in some places as well.

The cycle \( C_2 \) is the concatenation of the sequences \( T_1, T_2, \) and \( (0, 2), \) where
\[
T_1 = ((1, 2), (−1, 3), \ldots, (1 + i, 2), (−(1 + i), 3), \ldots, (n, 2), (−n, 3)), 0 ≤ i ≤ n − 1;
T_2 = ((−n, 2), (n, 3), \ldots, (−(n − i), 2), (n − i, 3), \ldots, (−1, 2), (1, 3)), 0 ≤ i ≤ n − 1.
\]

The cycle \( C_3 \) is the concatenation of the sequences \( T_1, T_2, \) and \( T_3, \) where
\[
T_1 = ((n + 1, 1), (−(n + 1), 2), \ldots, (n + i, 1), (−(n + i), 2), \ldots, (2n − 1, 1), (−(2n − 1), 2)), 0 ≤ i ≤ n − 2;
T_2 = ((−2n, 1), (2n, 2), \ldots, (−(2n − i), 1), (2n − i, 2), \ldots, (−(n + 1), 1), (n + 1, 2)), 0 ≤ i ≤ n − 1;
T_3 = ((−2n, 0), (−2n, 2), (0, 1)).
\]

The cycle \( C_4 \) is the concatenation of the sequences \( ∞, T_1, T_2, \) and \( T_3, \) (By a slight abuse
of notation, here \( ∞ \) is regarded as a sequence.), where
\[
T_1 = ((−2n, 3), (2n, 3), \ldots, (−(2n − i), 3), (2n − i, 3), \ldots, (−(n + 1), 3), (n + 1, 3)), 0 ≤ i ≤ n − 1;
T_2 = ((n + 1, 0), (−(n + 1), 0), \ldots, (n + i, 0), (−(n + i), 0), \ldots, (2n − 1, 0), (−(2n − 1), 0)), 0 ≤ i ≤ n − 2;
T_3 = ((2n, 0), (2n, 1)).
\]

The cycle \( C_5 \) is the concatenation of the sequences \( T_1, (0, 1), \) and \( T_2, \) where
\[
T_1 = ((n, 1), (−n, 3), \ldots, (n − i, 1), (−(n − i), 3), \ldots, (1, 1), (−1, 3)), 0 ≤ i ≤ n − 1;
T_2 = ((1, 3), (−1, 1), \ldots, (1 + i, 3), (−(1 + i), 1), \ldots, (n, 3), (−n, 1)), 0 ≤ i ≤ n − 1.
\]

The cycle \( C_6 \) is the concatenation of the sequences \( T_1, T_2, T_3, \) and \( (−n, 0), \) where
\[
T_1 = ((−(n + 1), 0), (n + 1, 3), \ldots, (−(n + 1 + i), 0), (n + 1 + i, 3), \ldots, (−(2n − 1), 0), (2n − 1, 3)), 0 ≤ i ≤ n − 2;
T_2 = ((−2n, 0), (0, 3));
T_3 = ((2n, 0), (−2n, 3), \ldots, (2n − i, 0), (−(2n − i), 3), \ldots, (n + 1, 0), (−(n + 1), 3)), 0 ≤ i ≤ n − 1.
\]

Next, we consider the cycles \( C_7 \) and \( C_8. \)

For \( k = 13, 25, \) the two cycles are listed as follows.

\( k = 13: \)
\[
C_7 = ((1, 2), (−5, 1), (−5, 2), (0, 0), (3, 2), (−2, 0), (4, 2), (3, 0), (5, 2), (1, 0), (−3, 2), (−1, 0), (−2, 2));
C_8 = (∞, (2, 0), (−4, 2), (−6, 1), (5, 1), (4, 1), (−4, 1), (6, 1), (6, 3), (−6, 2), (2, 2), (6, 2), (−1, 2)).
\]

\( k = 25: \)
\[
C_7 = ((1, 2), (−11, 1), (−11, 2), (0, 0), (2, 2), (1, 0), (4, 2), (5, 0), (3, 2), (6, 0), (10, 2), (2, 0), (7, 2), (−5, 0), (5, 2),
(−4, 0), (11, 2), (4, 0), (−8, 2), (−1, 0), (−10, 2), (−2, 0), (−7, 2), (−3, 0), (8, 2));
C_8 = (∞, (3, 0), (9, 2), (7, 1), (−7, 1), (8, 1), (−8, 1), (9, 1), (11, 1), (−10, 1), (10, 1), (−9, 1), (12, 1), (12, 1),

(12, 3), (−12, 2), (−6, 2), (6, 2), (−3, 2), (12, 2), (−2, 2), (−5, 2), (−1, 2), (−9, 2), (−4, 2)).

For \(k \geq 17\) and \(k \neq 25\), we distinguish the following two cases.

**Case 1:** \(k \equiv 5 \pmod{8}\) and \(k \geq 21\).

In this case, \(C_7\) is the concatenation of the sequences \(S_1, S_2, \ldots, S_6\) as follows.

\[
\begin{align*}
S_1 &= ((−2n−3, 2), (−2n−1, 1)); \\
S_2 &= ((1, 2), (−1, 0), \ldots, (1 + i, 2), (−(1 + i), 0), \ldots, (\frac{n+1}{2}, 2), (−\frac{n+1}{2}, 0)), 0 \leq i \leq \frac{n−3}{2}; \\
S_3 &= ((−\frac{n+1}{2}, 2), (\frac{n+1}{2}, 0), \ldots, (\frac{n+1}{2} + i, 2), (\frac{n+1}{2} + i, 0), \ldots, (−(n − 1), 2), (n − 1, 0)), 0 \leq i \leq \frac{n−3}{2}; \\
S_4 &= ((2n−1, 2), (0, 0), (−(2n−1), 2)); \\
S_5 &= ((1, 0), (−1, 2), \ldots, (1 + i, 0), (−(1 + i), 2), \ldots, (\frac{n+1}{2}, 0), (−\frac{n+1}{2}, 2)), 0 \leq i \leq \frac{n−3}{2}; \\
S_6 &= ((−\frac{n+1}{2}, 0), (\frac{n+1}{2}, 2), \ldots, (\frac{n+1}{2} + i, 0), (\frac{n+1}{2} + i, 2), \ldots, (−(n − 1), 0), (n − 1, 2)), 0 \leq i \leq \frac{n−3}{2}.
\end{align*}
\]

For the last cycle \(C_8\), when \(k = 21, 29\), it is listed as below respectively.

\(k = 21:\)

\[
C_8 = (\infty, (5, 0), (−6, 2), (−6, 1), (6, 1), (−7, 1), (7, 1), (9, 1), (−8, 1), (8, 1), (−10, 1), (10, 1), (10, 3), (−10, 2), (8, 2), (−5, 2), (7, 2), (−8, 2), (6, 2), (10, 2), (5, 2)).
\]

\(k = 29:\)

\[
C_8 = (\infty, (7, 0), (−8, 2), (−8, 1), (8, 1), (−9, 1), (9, 1), (−10, 1), (10, 1), (−14, 1), (12, 1), (−12, 1), (13, 1), (12, 1), (−11, 1), (11, 1), (14, 1), (14, 3), (−14, 2), (7, 2), (11, 2), (14, 2), (8, 2), (−7, 2), (9, 2), (−9, 2), (10, 2), (−10, 2), (12, 2), (−12, 2)).
\]

When \(k \geq 37\), \(C_8\) is the concatenation of the sequences \(\infty, T_1, T_2, \ldots, T_8\), where

\[
T_1 = ((n, 0), (−(n + 1), 2));
\]

\[
T_2 = ((−(n + 1), 1), (n + 1, 1), \ldots, (−(n + 1 + i), 1), (n + 1 + i, 1), \ldots, (−\frac{n+1}{2}, 1), (\frac{n+1}{2}, 1)), 0 \leq i \leq \frac{n−3}{2}.
\]

For the other 6 sequences \(T_3, T_4, \ldots, T_8\), we distinguish the following 3 subcases.

**Case 1.1:** \(k \equiv 5 \pmod{24}\) and \(k \geq 53\).

\[
\begin{align*}
T_3 &= ((\frac{3n+5}{2}, 1), (−\frac{3n+5}{2}, 1), (\frac{3n+3}{2}, 1), (−\frac{3n+3}{2}, 1), (\frac{3n+1}{2}, 1), (−\frac{3n+1}{2}, 1), \ldots, (\frac{3n+5}{2} + 3i, 1), (−\frac{3n+5}{2} + 3i, 1), \\
&\quad (\frac{3n+4}{2} + 3i, 1), (−\frac{3n+4}{2} + 3i, 1), (\frac{3n+3}{2} + 3i, 1), (−\frac{3n+3}{2} + 3i, 1), \ldots, (2n − 7, 1), (−(2n − 7), 1), \\
&\quad (2n − 8, 1), (−(2n − 8), 1), (2n − 9, 1), (−(2n − 9), 1)), 0 \leq i \leq \frac{n+19}{2}; \\
T_4 &= ((2n − 4, 1), (−2n − 3, 1), (2n − 3, 1), (−2n − 2, 1), (2n − 2, 1), (−2n − 6, 1), (2n − 6, 1), (−(2n − 5), 1), \\
&\quad (2n − 5, 1), (−(2n − 4), 1), (−2n, 1), (2n − 1, 1), (2n, 1), (2n, 2), (−2n, 2), (n, 2); \\
T_5 &= ((n+2, 2), (−(n+2), 2), (n+1, 2), (−(n+4), 2), (n+3, 2), (−(n+3), 2), \ldots, (n + 2 + 3i, 2), (−(n + 2 + 3i), 2), \\
&\quad (n + 1 + 3i, 2), (−n + 4 + 3i), 2), (n + 3 + 3i, 2), (−(n + 3 + 3i), 2), \ldots, (\frac{3n−2}{2}, 2), (−\frac{3n−2}{2}, 2), (\frac{3n−11}{2}, 2), \\
&\quad (−\frac{3n−5}{2}, 2), (\frac{3n−5}{2}, 2), (−\frac{3n−3}{2}, 2)), 0 \leq i \leq \frac{n+11}{2}; \\
T_6 &= ((\frac{3n−3}{2}, 2), (−\frac{3n−3}{2}, 2), (\frac{3n−1}{2}, 2), (−\frac{3n−1}{2}, 2), (\frac{3n−5}{2}, 2), (−\frac{3n−5}{2}, 2)), \\
&\quad (\frac{3n−4}{2}, 2), (−\frac{3n−4}{2}, 2), (\frac{3n−2}{2}, 2), (−\frac{3n−2}{2}, 2), (\frac{3n−7}{2} + 3i, 2), (−\frac{3n−7}{2} + 3i, 2), \\
&\quad (\frac{3n−5}{2} + 3i, 2), (−\frac{3n−5}{2} + 3i, 2), (\frac{3n−9}{2} + 3i, 2), (−\frac{3n−9}{2} + 3i, 2), \ldots, (2n − 7, 2), (−(2n − 8), 2), \\
&\quad (2n − 9, 2), (−(2n − 7), 2), (2n − 5, 2), (−(2n − 9), 2)), 0 \leq i \leq \frac{n+19}{2}; \\
T_7 &= ((2n − 4, 2), (−(2n − 4), 2), (2n, 2), (−(2n − 5), 2), (2n − 6, 2), (−(2n − 4), 2), (2n − 3, 2), (−(2n − 6), 2), \\
&\quad (2n − 2, 2)).
\end{align*}
\]
**Case 1.2:** $k \equiv 13 \pmod{24}$ and $k \geq 37$.

$$T_3 = \left( \left( \frac{n+5}{2}, 1 \right), \left( -\frac{n}{2}, 1 \right), \left( \frac{n}{2}, 1 \right), \left( -\frac{n}{2}, 1 \right), \left( \frac{n}{2}, 1 \right), \left( \frac{n}{2}, 1 \right), \left( -\frac{n}{2}, 3i, 1 \right), \left( -\frac{n}{2}, 3i, 1 \right), \ldots, \left( \frac{n}{2}, 3i, 1 \right), \left( -\frac{n}{2}, 3i, 1 \right), \left( -\frac{n}{2}, 3i, 1 \right), \ldots, \left( 2n - 2, 1 \right), \left( -2n - 2, 1 \right), \left( 2n - 3, 1 \right), \left( -2n - 3, 1 \right), \left( 2n - 4, 1 \right), \left( -2n - 4, 1 \right) \right), 0 \leq i \leq \frac{n-4}{2};$$

$$T_4 = \left( \left( 2n - 1, 1 \right), \left( 2n - 1, 1 \right), \left( 2n - 3, 1 \right), \left( -2n - 2, 1 \right), \left( -2n - 2, 1 \right) \right);$$

$$T_5 = \left( \left( n+2, 2 \right), \left( -n+2, 2 \right), \left( n+1, 2 \right), \left( -n+4, 2 \right), \left( n+3, 2 \right), \left( -n+3, 2 \right), \ldots, \left( n+2+3i, 2 \right), \left( -n+2+3i, 2 \right), \left( n+3+3i, 2 \right), \left( -n+3+3i, 2 \right), \ldots, \left( 2n+1, 2 \right), \left( -2n+1, 2 \right), \left( 3n-2, 2 \right), \left( -3n-2, 2 \right), 0 \leq i \leq \frac{n-15}{6};$$

$$T_6 = \left( \left( \frac{n-2}{2}, 2 \right), \left( -\frac{n+2}{2}, 2 \right), \left( \frac{n-2}{2}, 2 \right), \left( -\frac{n+2}{2}, 2 \right), \left( \frac{n-2}{2}, 2 \right), \left( -\frac{n+2}{2}, 2 \right) \right);$$

**Case 1.3:** $k \equiv 21 \pmod{24}$ and $k \geq 45$.

$$T_3 = \left( \left( \frac{n}{2}, 1 \right), \left( -\frac{n}{2}, 1 \right), \left( \frac{n}{2}, 1 \right), \left( -\frac{n}{2}, 1 \right), \ldots, \left( \frac{n}{2}, 3i, 1 \right), \left( -\frac{n}{2}, 3i, 1 \right), \left( \frac{n}{2}, 3i, 1 \right), \left( -\frac{n}{2}, 3i, 1 \right), \left( \frac{n}{2}, 3i, 1 \right), \left( -\frac{n}{2}, 3i, 1 \right), \ldots, \left( 2n - 6, 1 \right), \left( -2n - 6, 1 \right), \left( 2n - 7, 1 \right), \left( -2n - 7, 1 \right), \left( 2n - 8, 1 \right), \left( -2n - 8, 1 \right) \right), 0 \leq i \leq \frac{n-15}{6};$$

$$T_4 = \left( \left( 2n - 3, 1 \right), \left( -2n - 3, 1 \right), \left( 2n - 2, 1 \right), \left( -2n - 2, 1 \right), \left( 2n - 5, 1 \right), \left( -2n - 5, 1 \right), \left( 2n - 4, 1 \right), \left( -2n - 4, 1 \right), \left( 2n - 1, 1 \right), \left( -2n - 1, 1 \right), \left( 2n - 2, 1 \right), \left( -2n - 2, 2 \right), \left( n, 2 \right), \left( -n, 2 \right) \right);$$

$$T_5 = \left( \left( n+2, 2 \right), \left( -n+2, 2 \right), \left( n+1, 2 \right), \left( -n+4, 2 \right), \left( n+3, 2 \right), \left( -n+3, 2 \right), \ldots, \left( n+2+3i, 2 \right), \left( -n+2+3i, 2 \right), \left( n+3+3i, 2 \right), \left( -n+3+3i, 2 \right), \ldots, \left( \frac{n+13}{2}, 2 \right), \left( -\frac{n+13}{2}, 2 \right), \left( \frac{n-10}{2}, 2 \right), \left( -\frac{n-10}{2}, 2 \right), \left( \frac{n-11}{2}, 2 \right), \left( -\frac{n-11}{2}, 2 \right) \right), 0 \leq i \leq \frac{n-17}{6};$$

**Case 2:** $k \equiv 1 \pmod{8}$, $k \geq 17$ and $k \neq 25$.

In this case, $C_7$ is the concatenation of the sequences $S_1$, $S_2$, $(-n, 2)$, $S_3$, $S_4$, $S_5$, and $S_6$.

$$S_1 = \left( \left( 2n - 3, 2 \right), \left( -2n - 1, 1 \right) \right);$$

$$S_2 = \left( \left( 1, 2 \right), \left( -1, 0 \right), \ldots, \left( 1 + i, 0 \right), \left( -1 + i, 0 \right), \ldots, \left( n - 1, 2 \right), \left( -n - 1, 0 \right), 0 \leq i \leq n - 2 \right);$$

$$S_3 = \left( \left( 1, 0 \right), \ldots, \left( 1 + i, 0 \right), \left( -1 + i, 0 \right), 0 \leq i \leq \frac{n-4}{2} \right);$$

$$S_4 = \left( \left( 2, 0 \right), \left( -2, 2 \right) \right);$$

$$S_5 = \left( \left( \frac{n+2}{2}, 0 \right), \left( -\frac{n+2}{2}, 2 \right), \ldots, \left( \frac{n+2}{2} + i, 0 \right), \left( -\frac{n+2}{2} + i, 2 \right), \ldots, \left( n - 1, 0 \right), \left( -n - 1, 2 \right), 0 \leq i \leq \frac{n-4}{2} \right);$$

$$S_6 = \left( \left( n, 0 \right), \left( n + 1, 2 \right) \right).$$

For the cycle $C_8$, when $k = 17, 33, 41$, it is listed as below respectively.
$k = 17$:

$C_k = (\infty, (0, 0), (7, 2), (7, 1), (6, 1), (8, 1), (8, 3), (8, 2), (6, 2), (2, 2), (7, 2), (4, 2), (8, 2))$.

$k = 33$:

$C_k = (\infty, (0, 0), (15, 2), (15, 1), (9, 1), (9, 1), (10, 1), (10, 1), (11, 1), (11, 1), (12, 1), (13, 1), (12, 1), (14, 1), (14, 1), (16, 1), (16, 1), (16, 3), (16, 2), (8, 2), (12, 2), (14, 2), (16, 2), (11, 2), (12, 2), (14, 2), (9, 2), (10, 2), (10, 2), (11, 2), (4, 2), (13, 2), (15, 2))$.

$k = 41$:

$C_k = (\infty, (0, 0), (19, 2), (19, 1), (12, 1), (13, 1), (11, 1), (11, 1), (12, 1), (15, 1), (14, 1), (14, 1), (18, 1), (15, 1), (16, 1), (17, 1), (18, 1), (20, 1), (16, 1), (20, 1), (20, 3), (20, 2), (10, 2), (13, 2), (13, 2), (14, 2), (11, 2), (12, 2), (12, 2), (17, 2), (5, 2), (16, 2), (19, 2), (18, 2), (14, 2), (20, 2), (16, 2), (15, 2), (18, 2))$.

When $k \geq 49$, the cycle $C_k$ is the concatenation of the sequences $\infty, T_1, T_2, \ldots, T_9$, where

$T_1 = (0, 0), (2n - 1, 2), (2n - 1, 1)$.

For the other 8 sequences $T_2, T_3, \ldots, T_9$, we distinguish 3 subcases.

**Case 2.1:** $k \equiv 1 \pmod{24}$ and $k \geq 49$.

$T_2 = ((-n+2), 1), (n+3, 1), (-n+3, 1), (n+2, 1), (n+1, 1), (-n+2+3i), 1, (n+3+3i), 1, (-n+3+3i), 1, (-n+2+3i), 1, (n+3+3i), 1, 1, \ldots, (-n+2+3i), 1, (n+3+3i), 1, (n+2+3i), 1, (n+2+3i), 1, \ldots, (\frac{3n-10}{2}, 1), (\frac{3n-10}{2}, 1), (\frac{3n-10}{2}, 1), \ldots, 0 \leq i \leq \frac{n-12}{6}$;

$T_3 = ((\frac{3n+8}{2}, 1), (\frac{3n+8}{2}, 1), (\frac{3n+8}{2}, 1), (\frac{3n+8}{2}, 1), (\frac{3n+8}{2}, 1), (\frac{3n+8}{2}, 1), (\frac{3n+8}{2}, 1), (\frac{3n+8}{2}, 1)$;

$T_4 = ((\frac{3n+8}{2}, 1), (\frac{3n+10}{2}, 1), (\frac{3n+10}{2}, 1), (\frac{3n+10}{2}, 1), (\frac{3n+10}{2}, 1), (\frac{3n+10}{2}, 1), (\frac{3n+10}{2}, 1), (\frac{3n+10}{2}, 1)$;

$T_5 = ((-2n-7), 1), (2n-4, 1), (2n-3, 1), (2n-6, 1), 0 \leq i \leq \frac{n-16}{6}$;

$T_6 = ((-2n-1), 1), (2n-1), (2n-2), (2n-1), (2n-1), (2n-1), (2n-3), 1, (2n-2), 1, (2n-2), (2n-2), 1, \ldots, 0 \leq i \leq \frac{n-12}{6}$;

$T_7 = ((\frac{3n+2}{2}, 2), (\frac{3n+2}{2}, 2), (\frac{3n+2}{2}, 2), (\frac{3n+2}{2}, 2), (\frac{3n+2}{2}, 2), (\frac{3n+2}{2}, 2)$;

$T_8 = ((\frac{3n+2}{2}, 2), (\frac{3n+4}{2}, 2), (\frac{3n+4}{2}, 2), (\frac{3n+4}{2}, 2), (\frac{3n+4}{2}, 2), (\frac{3n+4}{2}, 2)$;

$T_9 = ((-2n-7), 2), (2n-7, 2), (2n-6, 2), (2n-6, 2), 0 \leq i \leq \frac{n-18}{6}$;

$T_9 = ((-2n-5), 2), (2n-2), (2n-2), (2n-4), 1, (2n-4), 1, (2n-3), 2, (2n-1), 2, (2n-1), 2, (2n-1), 2, (2n-1), 2, (2n-1), 2, (2n-1), 2$.

**Case 2.2:** $k \equiv 9 \pmod{24}$ and $k \geq 57$.

$T_2 = ((-n+2), 1), (n+3, 1), (-n+3, 1), (n+2, 1), (n+1, 1), (n+2, 1), (n+1, 1), (n+2, 1), (n+3, 1), (n+3, 1), (n+3, 1), (n+2, 1), (n+1, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), (n+3, 1), (n+2, 1), \ldots, (\frac{3n-4}{2}, 1), (\frac{3n-4}{2}, 1), 0 \leq i \leq \frac{n-8}{6}$;

$T_3 = ((-\frac{3n+2}{2}, 1), (\frac{3n-6}{2}, 1), (\frac{3n-6}{2}, 1))$;
\[ T_4 = (\left( \frac{3n}{2}, 1 \right), -\left( \frac{3n+8}{2}, 1 \right), -\left( \frac{3n+2}{2}, 1 \right), -\left( \frac{3n+10}{2}, 1 \right), -\left( \frac{3n+6}{2}, 1 \right), -\left( \frac{3n+4}{2}, 1 \right), -\left( \frac{3n+8}{2}, 1 \right), -\left( \frac{3n+3}{2}, 1 \right), (\frac{3n+10}{2}, 1), (\frac{3n+6}{2}, 1), (\frac{3n+4}{2}, 1), (\frac{3n+8}{2}, 1), (\frac{3n}{2} + 3i, 1), (\frac{3n+10}{2} + 3i, 1), (\frac{3n+6}{2} + 3i, 1), (\frac{3n+4}{2} + 3i, 1), (2n - 7, 1), (2n - 3, 1), (2n - 6, 1), (2n - 5, 1), (2n - 2, 1), (2n - 1, 1), 0 \leq i \leq \frac{n-14}{6} ;
\]
\[ T_5 = ((2n - 4, 1), (2n - 3, 1), (2n - 2, 1), (2n, 1), (n + 3, 2), (n + 2, 2), (n + 1, 2) ;
\]
\[ T_6 = ((-n+1, 2), (n+5, 2), (n+6, 2), (n+7, 2), (n+4, 2), (n+3, 2), (n + 1 + 3i, 2), (n + 5 + 3i, 2), (n + 6 + 3i, 2), (n + 3 + 3i, 2), (n + 4 + 3i, 2), (2n - 9, 2), (2n - 6, 2), (2n - 7, 2), (2n - 5, 2), (2n - 2, 2), (2n - 1, 2), 0 \leq i \leq \frac{n-16}{6} ;
\]
\[ T_7 = ((-\frac{3n}{2}, 1), (\frac{3n}{2} - 1, 1), (\frac{3n}{2} - 2, 1), (\frac{3n}{2} - 3, 1), (\frac{3n}{2} - 4, 1) ;
\]
\[ T_8 = ((\frac{3n+1}{2}, 1), (\frac{3n+2}{2}, 1), (\frac{3n+3}{2}, 1), (\frac{3n+4}{2}, 1), (\frac{3n+5}{2}, 1), (\frac{3n+6}{2}, 1), (\frac{3n+7}{2}, 1), (\frac{3n+8}{2}, 1), (\frac{3n+9}{2}, 1), (\frac{3n+10}{2}, 1), (\frac{3n+11}{2}, 1), (\frac{3n+12}{2}, 1), (\frac{3n+13}{2}, 1), (\frac{3n+14}{2}, 1), (\frac{3n+15}{2}, 1), (\frac{3n+16}{2}, 1), (\frac{3n+17}{2}, 1), (\frac{3n+18}{2}, 1), (\frac{3n+19}{2}, 1), (\frac{3n+20}{2}, 1), (\frac{3n+21}{2}, 1), (\frac{3n+22}{2}, 1), 0 \leq i \leq \frac{n-16}{6} ;
\]
\[ T_9 = ((-\frac{3n}{2}, 2), (\frac{3n}{2} - 1, 2), (\frac{3n}{2} - 2, 2), (\frac{3n}{2} - 3, 2), (\frac{3n}{2} - 4, 2), (\frac{3n}{2} - 5, 2), (\frac{3n}{2} - 6, 2), (\frac{3n}{2} - 7, 2), (\frac{3n}{2} - 8, 2), (\frac{3n}{2} - 9, 2), (\frac{3n}{2} - 10, 2), (\frac{3n}{2} - 11, 2), (\frac{3n}{2} - 12, 2), (\frac{3n}{2} - 13, 2), (\frac{3n}{2} - 14, 2), 0 \leq i \leq \frac{n-18}{6} ;
\]
\[ T_{10} = ((2n - 5, 2), (2n - 6, 2), (2n - 7, 2), (2n - 8, 2), (2n - 9, 2), (2n - 10, 2), (2n - 11, 2), (2n - 12, 2), (2n - 13, 2), (2n - 14, 2), (2n - 15, 2), (2n - 16, 2), (2n - 17, 2), (2n - 18, 2), (2n - 19, 2), (2n - 20, 2) ;
\]

\section*{4 \(k\)-ARCS\((4k + 1)\) for \(k \equiv 3 \pmod{4}\)}

In this section, we will prove the existence of a \(k\)-ARCS\((4k + 1)\) for \(k \equiv 3 \pmod{4}\).
Lemma 4.1. For any $k \geq 11$ and $k \equiv 3 \pmod{4}$, there exists a $k$-ARCS$(4k+1)$.

Proof: Let $v = 4k + 1$ and $k = 4n + 3$, $n \geq 2$. We use Lemma 2.1 to construct a $k$-ARCS$(v)$ with $V(K_v) = (\mathbb{Z}_k \times \mathbb{Z}_4) \cup \{\infty\}$. Three of the required parameters are $(a_1, b_1) = (0, 3)$ and $d_3 = \frac{4k+1}{2}$. The other required parameters $a_2, b_2, d_2$ and 8 cycles in $F_1 = \{C_1, C_2, C_3, C_4\}$ and $F_2 = \{C_5, C_6, C_7, C_8\}$ are listed as below.

The cycle $C_1$ is the concatenation of the sequences $T_1$, $(0, 0)$, and $T_2$, where

$T_1 = ((n, 0), (n, 1), \ldots, (n-i, 0), (-(n-i), 1), \ldots, (1, 0), (-1, 1)), 0 \leq i \leq n-1$;

$T_2 = ((1, 1), (-1, 0), \ldots, (1+i, 1), (-1+i, 0), \ldots, (n+1, 1), (-(n+1), 0)), 0 \leq i \leq n$.

The cycle $C_2$ is the concatenation of the sequences $T_1$, $T_2$, $T_3$, and $(0, 2)$, where

$T_1 = ((1, 2), (0, 2), \ldots, (1+i, 2), (-1+i, 3), \ldots, (n, 2), (-n, 3)), 0 \leq i \leq n-1$;

$T_2 = ((n+1, 2), (n+1, 3));$

$T_3 = ((n, 3), \ldots, (n-i, 2), (n-i, 3), \ldots, (1, 3), (0, 1)), 0 \leq i \leq n-1$.

The cycle $C_3$ is the concatenation of the sequences $T_1$, $T_2$, and $T_3$, where

$T_1 = ((n+2, 1), (n+2, 2), \ldots, (n+2+i, 1), (-(n+2+i), 2), \ldots, (2n, 1), (-2n, 2)), 0 \leq i \leq n-2$;

$T_2 = ((-2n+1, 1), (2n+1, 2), \ldots, (-2n+1-i, 1), (2n+1-i, 2), \ldots, (-n+2, 1), (n+2, 2)), 0 \leq i \leq n-1$;

$T_3 = ((-n+1, 1), (-n+1, 2)), (-(2n+1), 0), (-2n+1), 2), (0, 1))$.

The cycle $C_4$ is the concatenation of the sequences $\infty$, $T_1$, $T_2$, $T_3$, and $T_4$, where

$T_1 = ((-2n+1), 3), (2n+1, 3), \ldots, (-2n+1-i, 3), (2n+1-i, 3), \ldots, (-n+2, 3), (n+2, 3)), 0 \leq i \leq n-1$;

$T_2 = ((-n+1), 3), (n+1, 0));$

$T_3 = ((n+2, 0), (-n+2, 0), \ldots, (n+2+i, 0), (-(n+2+i), 0), \ldots, (2n, 0), (-2n, 0)), 0 \leq i \leq n-2$;

$T_4 = ((2n+1+0), (2n+1, 1))$.

For $k = 11$, the other four cycles are listed as below.

$k = 11$: $(a_2, b_2) = (\frac{4k+1}{2}, -2), d_2 = 2$.

$C_5 = ((0, 1), (0, 3), (1, 1), (2, 3), (4, 1), (1, 3), (5, 1), (-4, 3), (2, 1), (5, 3), (-1, 1));$

$C_6 = ((0, 0), (3, 3), (1, 0), (-2, 3), (3, 0), (4, 3), (4, 0), (-3, 3), (1, 0), (-5, 3), (4, 0));$

$C_7 = ((2, 2), (-3, 1), (1, 2), (-5, 0), (4, 2), (-3, 0), (5, 2), (2, 0), (3, 2), (-2, 0), (-3, 2));$

$C_8 = ((\infty, 0), (-2, 2), (-4, 1), (-2, 1), (3, 1), (-5, 1), (-1, 3), (0, 2), (-4, 2), (-1, 2))$.

For $k \geq 15$, the two cycles $C_5$ and $C_6$ are defined as below.

The cycle $C_5$ is the concatenation of the sequences $T_1$, $T_2$, $(0, 1)$, $T_3$, and $T_4$, where

$T_1 = ((n+2, 1), (-(n+1), 3));$

$T_2 = ((n, 1), (-n, 3), \ldots, (n-i, 1), (-(n-i), 3), \ldots, (1, 1), (-1, 3)), 0 \leq i \leq n-1$;

$T_3 = ((1, i), (1, 1), \ldots, (1+i, 3), (-(1+i), 1), \ldots, (n-1, 3), (-(n-1), 1)), 0 \leq i \leq n-2$;

$T_4 = ((n-3, 1), (-n, 1))$.

The cycle $C_6$ is the concatenation of the sequences $T_1$, $T_2$, $T_3$, and $(n+1, 0)$, where

$T_1 = ((-n+2, 0), (n+2, 3), \ldots, (-(n+2+i), 0), (n+2+i, 3), \ldots, (-2n+1, 0), (2n+1, 3)), 0 \leq i \leq n-1$;

$T_2 = ((2n+1), 0), (0, 3), (-1, 0), (-2n+1, 3));$

$T_3 = ((2n, 0), (-2n, 3), \ldots, (2n-i, 0), (-2n-i, 3), \ldots, (n+2, 0), (-n+2, 3)), 0 \leq i \leq n-2$.  

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For the last two cycles $C_7$ and $C_8$, when $k = 15, 19, 23, 27, 35$, they are listed as below.

$k = 15$: $(a_2, b_2) = \left( \frac{k+1}{2}, 2 \right)$, $d_2 = 4$.

$C_7 = ((0, 2), (−7, 1), (−1, 2), (0, 0), (1, 2), (3, 0), (7, 2), (1, 0), (−2, 2), (−4, 0), (3, 2), (−2, 0), (6, 2), (−3, 0), (−7, 2));$

$C_8 = ((\infty, 2, 0), (−3, 2), (−5, 1), (6, 1), (7, 1), (−6, 1), (−3, 1), (4, 1), (4, 3), (5, 2), (2, 2), (−4, 2), (−6, 2), (4, 2)).$

$k = 19$: $(a_2, b_2) = \left( \frac{k+1}{2}, 2 \right)$, $d_2 = 2$.

$C_7 = ((0, 2), (−9, 1), (−7, 2), (0, 0), (1, 2), (2, 0), (4, 2), (1, 0), (7, 2), (−4, 0), (−8, 2), (3, 0), (−2, 2), (4, 0), (−6, 2), (−3, 0), (2, 2), (−5, 0), (5, 2));$

$C_8 = ((\infty, −2, 0), (−4, 2), (7, 1), (−8, 1), (9, 1), (−7, 1), (−6, 1), (8, 1), (−4, 1), (5, 1), (5, 3), (6, 2), (3, 2), (9, 2), (−3, 2), (8, 2), (−1, 2), (−5, 2)).$

$k = 23$: $(a_2, b_2) = \left( \frac{k+1}{2}, 2 \right)$, $d_2 = 2$.

$C_7 = ((1, 2), (−9, 1), (2, 2), (−2, 0), (4, 2), (−3, 0), (5, 2), (−4, 0), (6, 2), (−5, 0), (8, 2), (5, 0), (−4, 2), (4, 0), (3, 2), (1, 0), (−1, 2), (3, 0), (−3, 2), (0, 0), (−5, 2), (−6, 0), (10, 2));$

$C_8 = ((\infty, 2, 0), (−9, 2), (−11, 1), (8, 1), (−7, 1), (11, 1), (10, 1), (−10, 1), (−8, 1), (9, 1), (−5, 1), (6, 1), (6, 3), (7, 2), (0, 2), (−11, 2), (−8, 2), (9, 2), (−10, 2), (−2, 2), (11, 2), (−7, 2)).$

$k = 27$: $(a_2, b_2) = \left( \frac{k+1}{2}, 2 \right)$, $d_2 = 2$.

$C_7 = ((0, 2), (−13, 1), (−11, 2), (0, 0), (1, 2), (2, 0), (4, 2), (1, 0), (5, 2), (−7, 0), (2, 2), (4, 0), (9, 2), (−6, 0), (7, 2), (−4, 0), (3, 2), (−5, 0), (13, 2), (3, 0), (−5, 2), (−2, 0), (−8, 2), (−3, 0), (−7, 2), (6, 0), (−4, 2));$

$C_8 = ((\infty, 5, 0), (−2, 2), (13, 1), (11, 1), (−12, 1), (−11, 1), (−8, 1), (9, 1), (−9, 1), (12, 1), (−10, 1), (10, 1), (−6, 1), (7, 1), (7, 3), (8, 2), (11, 2), (−6, 2), (6, 2), (12, 2), (−10, 2), (10, 2), (−3, 2), (−12, 2), (−1, 2), (−9, 2)).$

$k = 35$: $(a_2, b_2) = \left( \frac{k+1}{2}, 2 \right)$, $d_2 = 2$.

$C_7 = ((1, 2), (−15, 1), (2, 2), (−2, 0), (3, 2), (−3, 0), (4, 2), (−5, 0), (5, 2), (−6, 0), (6, 2), (−7, 0), (7, 2), (−8, 0), (8, 2), (−9, 0), (11, 2), (8, 0), (9, 2), (7, 0), (−9, 2), (5, 0), (−8, 2), (4, 0), (−7, 2), (3, 0), (−6, 2), (−4, 0), (−5, 2), (0, 0), (−4, 2), (2, 0), (−1, 2), (6, 0), (−2, 2));$

$C_8 = ((\infty, 1, 0), (−16, 2), (17, 1), (−16, 1), (16, 1), (−12, 1), (−17, 1), (13, 1), (13, 1), (−14, 1), (15, 1), (14, 1), (−10, 1), (11, 1), (−11, 1), (12, 1), (−8, 1), (9, 1), (9, 3), (10, 2), (−12, 2), (12, 2), (−11, 2), (14, 2), (−13, 2), (16, 2), (−10, 2), (−15, 2), (13, 2), (−3, 2), (15, 2), (0, 2), (−14, 2), (17, 2));$

For the last two cycles $C_7$ and $C_8$ of other values, we distinguish the following 2 cases.

Case 1: $k \equiv 3 \pmod{8}$ and $k \geq 43$. Here, $(a_2, b_2) = \left( \frac{k+1}{2}, 2 \right)$, $d_2 = 2$.

The cycle $C_7$ is the concatenation of the sequences $S_1$, $S_2$, ..., $S_9$, $S_5$, $S_6$, and $S_7$, where

$S_1 = ((1, 2), (−2n − 1, 1));$

$S_2 = ((2, 2), (−2, 0), ..., (2 + i, 2), (−2 + i, 0), ..., \left( \frac{n−2}{2}, 2 \right), (−\frac{n−2}{2}, 0)), 0 \leq i \leq \frac{n−6}{2};$

$S_3 = ((\frac{n−4}{2}, 0), ..., (\frac{n−4}{2} + 2, (−\frac{n−4}{2} + i), 0), ..., (n, 2), (−(n + 1), 0)), 0 \leq i \leq \frac{n−4}{2};$

$S_4 = ((n + 3, 2), (n, 0), (n + 1, 2), (n − 1, 0), (−(n + 1), 2));$

$S_5 = ((n − 3, 0), (−n, 2), ..., (n − 3 − i, 0), (−(n − i), 2), ..., (\frac{n−4}{2}, 0), (−\frac{n−4}{2}, 2)), 0 \leq i \leq \frac{n−4}{2}.$

For the sequences $S_6$, $S_7$, and the cycle $C_8$, we distinguish 3 subcases.

Case 1.1: $k \equiv 3 \pmod{24}$ and $k \geq 51$. 

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$S_0 = ((-\frac{n+3}{2}, 2), (\frac{n-4}{2}, 0), (-\frac{n}{2}, 2), (\frac{n-4}{2}, 0), \ldots, ((-\frac{n+3}{2} - 3i), 2), (\frac{n-8}{2} - 3i), 0), (-\frac{n}{2} - 3i), 2), (\frac{n-8}{2} - 3i), 0), (-\frac{n}{2} - 3i), 2), (\frac{n-8}{2} - 3i), 0), \ldots, (-10, 2), (5, 0), (-9, 2), (7, 0), (-5, 2), (6, 0)), 0 \leq i \leq \frac{n-15}{6};$

$S_7 = ((-7, 2), (3, 0), (-6, 2), (2, 0), (-4, 2), (0, 0), (-3, 2), (4, 0), (-1, 2), (n - 2, 0), (-2, 2))$.

The cycle $C_8$ is the concatenation of the sequences $\infty$, $T_1, T_2, \ldots, T_9$, where

$T_1 = ((1, 0), (-2n, 2), (2n + 1, 1), (-2n, 1), (-2n + 1, 1), (-2n - 3, 1), (2n - 1, 1), (-2n - 2, 1), (2n, 1));$

$T_2 = ((-2n - 5, 1), (2n - 2, 1), (-2n - 4, 1), (2n - 6, 1), (2n - 3, 1), \ldots, ((-2n - 5 - 3i), 1), (2n - 2 - 3i), 1), (-2n - 4 - 3i), 1), (2n - 4 - 3i), 1), \ldots, (-\frac{3n+2}{2}, 1), (\frac{3n+1}{2}, 1), (-\frac{3n-5}{2}, 1), (\frac{3n-5}{2}, 1), 0 \leq i \leq \frac{n-15}{6};$

$T_3 = ((\frac{3n-6}{2}, 1), (-\frac{3n-6}{2}, 1), (\frac{3n-4}{2}, 1));$

$T_4 = ((\frac{3n-6}{2}, 1), (\frac{3n-6}{2}, 1), (-\frac{3n-8}{2}, 1), (\frac{3n-8}{2}, 1), \ldots, ((-\frac{3n-6}{2} - 3i), 1), (\frac{3n-2}{2} - 3i, 1), (-\frac{3n-6}{2} - 3i), 1), (\frac{3n-8}{2} - 3i, 1), \ldots, (-n + 3, 1), (n + 5, 1), (-n + 4, 1), (n + 3, 1), (-n + 2, 1), (n + 4, 1)), 0 \leq i \leq \frac{n-15}{6};$

$T_5 = ((-n + 1, 1), (n + 1, 1), (n + 3, 1));$

$T_6 = ((n - 2, 2), (-n - 4, 2), (n - 4, 2), (n - 6, 2), (-n - 5, 2), \ldots, (n + 2 + 3i, 2), (n + 4 + 3i, 2), (n + 6 + 3i, 2), (n + 5 + 3i, 2), \ldots, (\frac{3n-6}{2} - 2), (\frac{3n-4}{2} - 2), (\frac{3n-4}{2}, 2), (-\frac{3n-4}{2}, 2), (-\frac{3n-2}{2}, 2), (\frac{3n-2}{2}, 2), \ldots, (-n - 10, 2), (-n - 8, 2), \ldots, (-n - 10, 2), (-n - 8, 2), \ldots, (-n - 10, 2), (-n - 8, 2), \ldots, (-n - 15, 2), \ldots, (2n - 6, 2), (2n - 7, 2), (2n - 2, 2), (-2n - 6, 2), 0 \leq i \leq \frac{n-15}{6};$

$T_7 = ((\frac{3n-6}{2} - 2), (-\frac{3n-6}{2} - 2), (\frac{3n-4}{2} - 2), (-\frac{3n-4}{2} - 2), (\frac{3n-2}{2}, 2), \ldots, (-\frac{3n-6}{2} - 3i, 2), (\frac{3n-8}{2} - 3i, 0), \ldots, (-8, 0), (-7, 2), (5, 0), (-3, 2), (4, 0)), 0 \leq i \leq \frac{n-15}{6};$

$S_7 = ((-5, 2), (0, 0), (-4, 2), (2, 0), (-1, 2), (n - 2, 0), (-2, 2))$.

The cycle $C_8$ is the concatenation of the sequences $\infty$, $T_1, T_2, \ldots, T_9$, where

$T_1 = ((1, 0), (-2n, 2), (2n + 1, 1), (-2n, 1), (2n, 1), (-2n - 4, 1), (-2n + 1, 1), (-2n - 3, 1), (2n - 1, 1), (-2n - 2, 1), (2n, 1));$

$T_2 = ((-2n - 7, 1), (2n - 2, 1), (-2n - 5, 1), (2n - 6, 1), (-2n - 4, 1), \ldots, ((-2n - 7 - 3i), 1), (2n - 2 - 3i), 1), (-2n - 4 - 3i), 1), (2n - 6 - 3i), 1), \ldots, (-\frac{3n+1}{2}, 1), (\frac{3n+1}{2}, 1), (-\frac{3n-5}{2}, 1), (\frac{3n-5}{2}, 1), 0 \leq i \leq \frac{n-15}{6};$

$T_3 = ((\frac{3n+4}{2}, 1), (\frac{3n-4}{2} - 1), 1);$

$T_4 = ((\frac{3n+4}{2}, 1), (\frac{3n-4}{2} - 1), 1), (-\frac{3n-6}{2}, 1), (\frac{3n-2}{2} - 3i, 1), (-\frac{3n-10}{2}, 1), \ldots, (-\frac{3n-4}{2} - 3i, 1), (\frac{3n-4}{2} - 3i, 1), (\frac{3n-6}{2} - 3i, 1), (\frac{3n-2}{2} - 3i, 1), (\frac{3n-4}{2} - 3i, 1), (\frac{3n-8}{2} - 3i, 1), (\frac{3n-14}{2} - 3i, 1), (\frac{3n-18}{2} - 3i, 1), \ldots, (n + 7, 1), (n + 6, 1), \ldots$
\[(n+5,1),(-(n+4),1), (n+6,1),(-(n+2),1)), 0 \leq i \leq \frac{n-14}{6};\]

\[T_5 = ((n+3,1), -(n+3,1), (n+4,1), -(n,1), (n+1,1), (n+1,3));\]

\[T_6 = ((n+2,2), -(n+4),2), (n+4,2), -(n+3),2), (n+6,2), -(n+5),2), \ldots, (n+2+3i,2), -(n+4+3i,2),\]

\[(n+4+3i,2), (n-3+3i,2), (n+6+3i,2), (n-5+3i,2), \ldots, (\frac{n+14}{3},2), (\frac{n-7}{3},2), (\frac{n+5}{3},2),\]

\[(\frac{n+14}{3},2), (\frac{n+12}{3},2), (\frac{n+10}{3},2), 0 \leq i \leq \frac{n+2}{3};\]

\[T_7 = ((\frac{n+2}{2}),\frac{n+2}{2}, (\frac{n+6}{2}), (\frac{n+4}{2}), (\frac{3n+10}{2}), (\frac{n+2}{2}));\]

\[T_8 = ((\frac{n+14}{3},2), (\frac{n+14}{3},2), (\frac{n+16}{3},2), (\frac{n+18}{3},2), (\frac{n+18}{3}+3i,2), (\frac{n+18}{3}+3i,2),\]

\[(\frac{n+18}{3}+3i,2), (\frac{n+18}{3}+3i,2), (\frac{n+18}{3}+3i,2), (\frac{n+18}{3}+3i,2), \ldots, (2n-4,2), (2n-5,2),\]

\[(2n-6,2), (2n-4,2), (2n-2,2), (2n-2,2), 0 \leq i \leq \frac{n+20}{6};\]

\[T_9 = ((-n-1,2), (2n-2), 2), (2n-3,2), (2n-1,2), (2n,2), (2n+1,2), (2n-3,2), (2n+2,2)).\]

**Case 1.3:** \(k \equiv 19 \pmod{24} \) and \( k \geq 43.\)

\[S_6 = ((\frac{n-2}{3},2), (\frac{n-2}{3},0), (\frac{n-2}{2}), (\frac{n-2}{2}), 0, (\frac{n-2}{2}), (\frac{n-2}{2}), \ldots, (\frac{n-2}{3} - 3i), 2), (\frac{n-2}{3} - 3i), 0), (\frac{n-2}{3} - 3i), 0);\]

\[0 \leq i \leq \frac{n-16}{6};\]

\[S_7 = ((-6,2), (0,0), (-3,2), (2,0), (-5,2), (3,0), (-1,2), (n-2,0), (-2,2)).\]

The cycle \(C_8\) is the concatenation of the sequences \(\infty, T_1, T_2, \ldots, T_9,\) where

\[T_1 = ((1,0), -(2n-2), (2n+1,1), (2n-1,1), (2n+1,1), (2n-1,1));\]

\[T_2 = ((2n-4,1), -(2n-3,1), (2n-2,1), -(2n-4-1), (2n,1), -(2n-2), 1), (2n-4-3i,1),\]

\[(-(2n-3-3i,1), (2n-2 - 3i,1), (2n-4 - 3i,1), (2n-3,1), (2n-4 - 3i,1),\]

\[((\frac{n+14}{3},1), (\frac{n+14}{3},1), (\frac{n+16}{3},1), (\frac{n+18}{3},1), (\frac{n+18}{3}+3i,1), 0 \leq i \leq \frac{n+20}{6};\]

\[T_3 = ((\frac{n+2}{2},1), (\frac{n+2}{2},1));\]

\[T_4 = ((\frac{n+2}{2},1), (\frac{n+2}{2},1), (\frac{n+2}{2},1), (\frac{n+2}{2},1), (\frac{n+2}{2},1), \ldots, (\frac{n+2}{2} - 3i,1), (\frac{n+2}{2} - 3i,1),\]

\[(-n+4,1), (n+3,1), (n+2,1), (n+1,1), 0 \leq i \leq \frac{n+10}{6};\]

\[T_5 = ((-n,1), (n+1,1), (n+1,3));\]

\[T_6 = ((-n+2,2), (n+4), 2), (n+4,2), (n-3), 2), (n+6,2), (n+5,2), \ldots, (n + 2 + 3i, 2), (n + 4 + 3i, 2),\]

\[(n + 4 + 3i, 2), (n + 6 + 3i, 2), (n + 5 + 3i, 2), \ldots, (\frac{n-4}{3}, 2), (\frac{n-2}{3}, 2), (\frac{n-2}{3}, 2),\]

\[0 \leq i \leq \frac{n+10}{6};\]

\[T_7 = ((\frac{n+2}{2},2), (\frac{n+2}{2},2));\]

\[T_8 = ((\frac{n+14}{3},2), (\frac{n+14}{3},2), (\frac{n+16}{3},2), (\frac{n+18}{3},2), (\frac{n+18}{3}+3i,2), (\frac{n+18}{3}+3i,2),\]

\[(-\frac{n+18}{3}+3i,2), (\frac{n+18}{3}+3i,2), (\frac{n+18}{3}+3i,2), (\frac{n+18}{3}+3i,2), \ldots, (2n-4,2), (2n-5,2),\]

\[(2n-6,2), (2n-4,2), (2n-2,2), (2n-2,2), (2n-4,2), (2n-3,2), (2n-1,2), (2n,2), (2n+2,2)).\]

**Case 2:** \(k \equiv 7 \pmod{8}\) and \( k \geq 31.\)

In this case the cycle \(C_7\) is the concatenation of the sequences \(S_1, S_2, S_3, (n+3,2), S_4, S_5, S_6,\) and \(S_7,\) where
$S_1 = ((1, 2), (-2n - 1, 1));$
$S_2 = ((2, 2), (-2, 0), . . . , (2 + i, 2), (-2 + i, 0), . . . , (\frac{n + 5}{2}, 2), (-\frac{n + 3}{2}, 0)), 0 \leq i \leq \frac{n - 3}{2};$
$S_3 = ((\frac{n + 5}{2}, 2), (-\frac{n + 3}{2}, 0), . . . , (\frac{n + 5}{2}, i + 2), (-\frac{n + 3}{2}, i + 2), . . . , (n + 1, 2), (-n, 0)), 0 \leq i \leq \frac{n - 3}{2};$
$S_4 = ((n, 0), (-n - 1, 2), . . . , (n - i, 0), (-n - 1 - i, 2), . . . , (\frac{n + 5}{2}, 0), (-\frac{n + 3}{2}, 2)), 0 \leq i \leq \frac{n - 3}{2};$
$S_5 = ((\frac{n + 5}{2}, 0), (\frac{n + 3}{2}, 2)).$

For the sequences $S_6$, $S_7$, and the cycle $C_8$, we distinguish 3 subcases.

Case 2.1: $k \equiv 7 \pmod{24}$ and $k \geq 31$. Here, $(a_2, b_2) = (\frac{3k - 5}{4}, 2)$, $d_2 = 4.$

$S_6 = ((\frac{n + 5}{2}, 0), (-\frac{n + 3}{2}, 2), (\frac{n + 5}{2}, 0), (-\frac{n + 3}{2}, 2), (-\frac{n + 3}{2}, 0), (-\frac{n + 3}{2}, 2), . . . , (\frac{n + 3}{2}, 0), (-\frac{n + 5}{2} - 3i, 2), (-\frac{n + 5}{2} - 3i, 2), . . . , (5, 0), (-4, 2), (6, 0), (-5, 2), (7, 0), (-6, 2)), 0 \leq i \leq \frac{n - 3}{2};$

$S_7 = ((2, 0), (0, 2), (3, 0), (-2, 2), (4, 0), (-3, 2), (1, 0), (-n, 2), (-n + 1, 0), (2n, 2)).$

For $k = 31, 55$, the cycle $C_8$ is listed as below.

$k = 31: C_8 = (\infty, (0, 0), (-15, 2), (14, 1), (12, 1), (-14, 1), (-15, 1), (-12, 1), (15, 1), (-10, 1), (13, 1), (-11, 1), (10, 1), (-9, 1), (11, 1), (-7, 1), (8, 1), (8, 3), (9, 2), (11, 2), (-11, 2), (15, 2), (-1, 2), (-13, 2), (-10, 2), (13, 2), (-4, 2), (-14, 2), (-8, 2), (12, 2), (-12, 2)).$

$k = 55:$

$C_8 = (\infty, (0, 0), (-27, 2), (26, 1), (-26, 1), (23, 1), (-27, 1), (27, 1), (-24, 1), (22, 1), (-23, 1), (25, 1), (-22, 1), (20, 1), (24, 1), (-19, 1), (19, 1), (-18, 1), (21, 1), (-21, 1), (20, 1), (-16, 1), (16, 1), (-15, 1), (18, 1), (-17, 1), (17, 1), (-13, 1), (14, 1), (14, 3), (15, 2), (-18, 2), (17, 2), (-17, 2), (19, 2), (-19, 2), (18, 2), (21, 2), (20, 2), (-20, 2), (22, 2), (-22, 2), (21, 2), (-25, 2), (27, 2), (-1, 2), (25, 2), (-7, 2), (24, 2), (-26, 2), (23, 2), (-24, 2), (14, 2), (16, 2), (-23, 2)).$

For any $k \geq 79$, the cycle $C_8$ is the concatenation of the sequences $\infty, T_1, T_2, . . . , T_9$, where

$T_1 = ((0, 0), (-2n + 1, 2), (2n, 1), (-2n, 1), (2n - 3, 1)(-2n + 1, 1)(2n + 1, 1)(-2n - 2, 1));$

$T_2 = ((2n - 6, 1), (-2n - 3, 1), (2n - 1, 1), (-2n - 4, 1), (2n - 2, 1), (-2n - 5, 1), . . . , (2n - 6 - 3i, 1), (-2n - 3 - 3i, 1), (-2n - 4 - 3i, 1), (2n - 2 - 3i, 1), (-2n - 5 - 3i, 1), . . . , (\frac{3n - 3}{2}, 1), (-\frac{3n + 3}{2}, 1), (\frac{3n + 5}{2}, 1), (-\frac{3n + 7}{2}, 1), (\frac{3n + 9}{2}, 1), (-\frac{3n + 11}{2}, 1), 0 \leq i \leq \frac{n - 1}{2};$

$T_3 = ((\frac{3n + 5}{2}, 1), (-\frac{3n + 7}{2}, 1), (\frac{3n + 9}{2}, 1), (-\frac{3n + 11}{2}, 1), (\frac{3n - 3}{2}, 1), (-\frac{3n - 1}{2}, 1), (\frac{3n - 3}{2}, 1), (-\frac{3n - 1}{2}, 1)) ;$

$T_4 = ((\frac{3n - 7}{2}, 1), (-\frac{3n - 9}{2}, 1), (\frac{3n - 3}{2}, 1), (-\frac{3n - 5}{2}, 1), (\frac{3n - 7}{2}, 1), (-\frac{3n - 9}{2}, 1), . . . , (\frac{3n - 7}{2} - 3i, 1), (-\frac{3n - 9}{2} - 3i, 1), (\frac{3n - 7}{2} - 3i, 1), (-\frac{3n - 9}{2} - 3i, 1), . . . , (n + 3, 1), (-n + 2, 1), (n + 5, 1), (-n + 4, 1), (n + 4, 1), (-n + 1, 1), (-n + 1, 1), (-n + 1, 3));$

$T_5 = ((n + 1, 1), (n + 1, 3));$

$T_6 = ((n + 2, 2), (-n + 5, 2), (n + 4, 2), (-n + 4, 2), (n + 6, 2), (-n + 6, 2), . . . , (n + 2 + 3i, 2), (-n + 5 + 3i, 2), (n + 4 + 3i, 2), (-n + 4 + 3i, 2), (n + 6 + 3i, 2), (-n + 6 + 3i, 2), . . . , (\frac{3n - 3}{2} - 3i, 1), (-\frac{3n - 1}{2} - 3i, 1), (\frac{3n - 3}{2} - 3i, 1), (-\frac{3n - 1}{2} - 3i, 1), 0 \leq i \leq \frac{n - 3}{2};)$
\[T_7 = (\frac{3n+3}{2}, 2, \frac{3n+13}{2}, 2, -\frac{3n+9}{2}, 2, \frac{3n+7}{2}, 2, -\frac{3n+11}{2}, 2, \frac{3n+9}{2}, 2, -\frac{n+1}{2}, 2)\];
\[T_8 = (\frac{3n+11}{2}, 2, -\frac{3n+11}{2}, 2, -\frac{3n+13}{2}, 2, -\frac{3n+15}{2}, 2, -\frac{3n+17}{2}, 2), \ldots, \frac{3n+11}{2} + 3i, 2, \frac{3n+19}{2} + 3i, 2, \ldots, (2n - 7, 2), (2n - 4, 2), (2n - 5, 2), (2n - 3, 2), (2n - 2, 2), 0 \leq i \leq \frac{n-35}{6};\]
\[T_9 = ((2n - 4, 2), (2n - 3, 2), (2n - 2, 2), (2n - 1, 2), (n + 1, 2), (n - 1, 2), (n - 2, 2), (n - 6, 2), (n - 7, 2), (n - 1, 2));\]

**Case 2.2:** \(k \equiv 15 \pmod{24}\) and \(k \geq 39\). Here, \((a_2, b_2) = (\frac{3k-5}{4}, 2), d_2 = 4\).

\[S_6 = ((\frac{n-3}{2}, 0), (\frac{n-3}{2}, 2), (\frac{n-3}{2}, 0), (\frac{n-1}{2}, 2), \ldots, (\frac{n-3}{2} - 3i, 0), (\frac{n-5}{2} - 3i, 2), (\frac{n-3}{2} - 3i, 0), (\frac{n-5}{2} - 3i, 2), \ldots, (6, 0), (\frac{5}{2}, 0), (\frac{6}{2}, 0), (8, 0), (\frac{7}{2}, 0), 0 \leq i \leq \frac{n-35}{6};\]
\[S_7 = ((3, 0), (3, 2), (4, 0), (4, 2), (5, 0), (5, 2), (2, 0), (2, 2), (1, 0), (1, 2), (0, 2), (0, 1), (0, 1), (0, 2), (2n, 2)).\]

For \(k = 39\), the cycle \(C_8\) is listed as below.
\[C_8 = (\infty, (0, 0), (19, 2), (18, 1), (18, 1), (19, 1), (19, 1), (16, 1), (14, 1), (16, 1), (15, 1), (17, 1), (12, 1), (12, 1), (11, 1), (14, 1), (13, 1), (13, 1), (9, 1), (10, 1), (10, 3), (11, 2), (14, 2), (13, 2), (13, 2), (15, 2), (14, 2), (15, 2), (14, 2), (15, 2), (16, 2), (16, 2), (17, 2), (12, 1), (17, 2), (12, 2), (10, 2), (18, 2).\]

For any \(k \geq 63\), the cycle \(C_8\) is the concatenation of the sequences \(\infty, T_1, T_2, \ldots, T_9\), where
\[T_1 = ((0, 0), (2n + 1, 2), (2n + 1, 2), (2n + 1, 2), (2n + 1, 2), (2n + 1, 1), (2n - 2, 1));\]
\[T_2 = ((2n - 6, 1), (2n - 3, 1), (2n - 1, 1), (2n - 4, 1), (2n - 2, 1), (2n - 5, 1), \ldots, (2n - 6 - 3i, 1), (2n - 3 - 3i, 1), (2n - 4 - 3i, 1), (2n - 2 - 3i, 1), (2n - 5 - 3i, 1), \ldots);\]
\[T_3 = ((\frac{n-3}{2}, 1), (\frac{n-3}{2}, 1), (\frac{3n}{2} - 3i, 1), (\frac{3n}{2} - 3i, 1), (\frac{3n}{2} - 3i, 1));\]
\[T_4 = ((\frac{3n-1}{2}, 1), (\frac{3n-1}{2}, 1), (\frac{3n-1}{2}, 1), (\frac{3n-1}{2}, 1), (\frac{3n-9}{2} - 3i, 1), (\frac{3n-9}{2} - 3i, 1), (\frac{3n-9}{2} - 3i, 1), \ldots, (n+3, 1), (n+2, 1), (n+5, 1), (n+4, 1), (n+1, 1), (n-1, 1), 0 \leq i \leq \frac{n-29}{6});\]
\[T_5 = ((n+1, 1), (n+1, 3));\]
\[T_6 = ((n+2, 2), (n+5, 2), (n+4, 2), (n+4, 2), (n+6, 2), (n+6, 2), (n+2 + 3i, 2), (n+5 + 3i, 2), (n+4 + 3i, 2), (n+4 + 3i, 2), (n+6 + 3i, 2), (n+6 + 3i, 2), (\frac{3n-1}{2}, 2), (\frac{3n-1}{2}, 2), \ldots, (\frac{3n-5}{2}, 2), \ldots, (\frac{3n-5}{2}, 2), (\frac{3n-7}{2}, 2), (\frac{3n-7}{2}, 2), 0 \leq i \leq \frac{n-29}{6});\]
\[T_7 = ((\frac{3n+1}{2}, 2), (\frac{3n+1}{2}, 2), (\frac{3n+1}{2}, 2), (\frac{3n+1}{2}, 2));\]
\[T_8 = ((\frac{3n+13}{2}, 2), (\frac{3n+13}{2}, 2), (\frac{3n+13}{2}, 2), (\frac{3n+13}{2}, 2), \ldots, (\frac{3n+13}{2} + 3i, 2), (\frac{3n+17}{2} + 3i, 2), (\frac{3n+17}{2} + 3i, 2), \ldots, (2n - 7, 2), (2n - 2, 2), (2n - 8, 2), (2n - 6, 2), (2n - 6, 2), (2n - 7, 2), 0 \leq i \leq \frac{n-35}{6});\]
\[T_9 = ((2n - 4, 2), (2n - 4, 2), (2n - 3, 2), (2n - 2, 2), (2n - 1, 2), (1, 2), (2n - 1, 2), (2n - 2, 2), (1, 2) \ldots);\]

**Case 2.3:** \(k \equiv 23 \pmod{24}\) and \(k \geq 47\). Here, \((a_2, b_2) = (\frac{3k-1}{4}, 2), d_2 = 2\).
where completes the proof of the existence of almost resolvable cycle systems with odd cycle length.

\[ S_6 = ((\frac{n-3}{2}, 0), (-\frac{n-5}{2}, 2), (\frac{n+1}{2}, 0), (-\frac{n+1}{2}, 2), (\frac{n+1}{2}, 0), (-\frac{n+1}{2}, 2), \ldots, (\frac{n-3}{2} - 3i, 0), (-\frac{n-5}{2} - 3i, 2), (-\frac{n}{2} - 3i, 0), (-\frac{n-1}{2} - 3i, 2), (\frac{n+1}{2} - 3i, 0), (-\frac{n+1}{2} - 3i, 2), \ldots, (7, 0), (-6, 2), (8, 0), (-7, 2), (9, 0), (-8, 2)), 0 \leq i \leq \frac{n-17}{6}; S_7 = ((4, 0), (-3, 2), (3, 0), (-3, 2), (6, 0), (-4, 2), (5, 0), (0, 2), (2, 0), (-2, 2), (1, 0), (n, 2), (-n+1, 0), (2n, 2)). \]

For \( k = 47 \), the cycle \( C_k \) is listed as below.

\[ C_k = \langle (0, 0), (0, -23, 2), (22, 1), (-22, 1), (19, 1), (-23, 1), (23, 1), (-20, 1), (20, 1), (-19, 1), (17, 1), (21, 1), (-16, 1), (17, 1), (-18, 1), (18, 1), (-14, 1), (14, 1), (-13, 1), (16, 1), (-15, 1), (15, 1), (-11, 1), (12, 1), (12, 3), (13, 2), (-16, 2), (18, 2), (-15, 2), (17, 2), (-14, 2), (16, 2), (19, 2), (-6, 2), (21, 2), (-21, 2), (15, 2), (-20, 2), (1, 2), (23, 2), (-17, 2), (20, 2), (-19, 2), (13, 2), (-22, 2), (-18, 2) \rangle. \]

For any \( k \geq 71 \), the cycle \( C_k \) is the concatenation of the sequences \( \infty, T_1, T_2, \ldots, T_9 \), where

\[ T_1 = \langle (0, 0), (0, -2n + 1), (2n, 1), (-2n, 1), (2n - 3, 1), (-2n + 1, 1), (2n + 1, 1), (-2n - 2, 1) \rangle; \]

\[ T_2 = \langle (2n - 6, 1), (-2n - 3, 1), (2n - 1, 1), (-2n - 4, 1), (2n - 2, 1), (-2n - 5, 1), \ldots, (2n - 6 - 3i, 1), (2n - 1 - 3i, 1), (-2n - 4 - 3i, 1), (2n - 2 - 3i, 1), (-2n - 5 - 3i, 1), \ldots, (\frac{3n+5}{2}, 1), (\frac{3n+11}{2}, 1), (\frac{3n+15}{2}, 1), (\frac{3n+18}{2}, 1), (-\frac{3n+7}{2}, 1) \rangle, 0 \leq i \leq \frac{n-17}{6}; \]

\[ T_3 = \langle (\frac{3n+7}{2}, 1), (\frac{3n+11}{2}, 1), (-\frac{3n+13}{2}, 1), (\frac{3n+15}{2}, 1), (-\frac{3n+15}{2}, 1), (\frac{3n+13}{2}, 1), (\frac{3n+5}{2}, 1) \rangle; \]

\[ T_4 = \langle (\frac{3n+7}{2}, 1), (\frac{3n+11}{2}, 1), (\frac{3n+13}{2}, 1), (\frac{3n+15}{2}, 1), (-\frac{3n+15}{2}, 1), (\frac{3n+13}{2}, 1), (\frac{3n+5}{2}, 1) \rangle; \]

\[ T_5 = \langle (n+1, 1), (n+1, 3) \rangle; \]

\[ T_6 = \langle (n+2, 2), (n+3, 2), (n+5, 2), (n+7, 2), (n+4, 2), (n+6, 2), (-n+3, 2), \ldots, (n+2+3i, 2), (-n+5+3i, 2), (n+7+3i, 2), (-n+4+3i, 2), (n+6+3i, 2), (-n+3+3i, 2), \ldots, (\frac{3n+3}{2}, 2), (\frac{3n+5}{2}, 2), (\frac{3n+7}{2}, 2), (\frac{3n+9}{2}, 2), (\frac{3n+11}{2}, 2), 0 \leq i \leq \frac{n-11}{6}; \]

\[ T_7 = \langle (\frac{3n+11}{2}, 2), (\frac{3n+13}{2}, 2), (\frac{3n+15}{2}, 2), (\frac{3n+17}{2}, 2), (\frac{3n+19}{2}, 2), (\frac{3n+21}{2}, 2) \rangle; \]

\[ T_8 = \langle (\frac{3n+11}{2}, 2), (\frac{3n+13}{2}, 2), (\frac{3n+15}{2}, 2), (\frac{3n+17}{2}, 2), (\frac{3n+19}{2}, 2), (\frac{3n+21}{2}, 2), (\frac{3n+23}{2}, 2), (-2n - 8, 2), (2n - 5, 2), (-2n - 7, 2), (2n - 7, 2), 0 \leq i \leq \frac{n-23}{6}; \]

\[ T_9 = \langle (2n - 3, 2), (2n - 2, 2), (2n - 5, 2), (2n - 1, 2), (-2n - 1, 2), (-1, 2), (2n + 1, 2), (2n - 4, 2), (2n - 4, 2), (-2n - 2, 2) \rangle. \]

5 Concluding remarks

Combining Theorems 1.1, 1.2 and Lemmas 3.1, 3.1, we have proved Theorem 1.3. This completes the proof of the existence of almost resolvable cycle systems with odd cycle length. For the even case, Lemma 2.1 is still useful for certain subcases.

We are working on the case \( k = 4m + 2 \). But it should be mentioned that Lemma 2.1 can
not be applied to solve the existence of a $k$-almost resolvable cycle system when $k = 2^m$. So there is still a long way to go before the whole problem can be solved completely.

As an application, almost resolvable cycle systems can be used to construct some solutions to the Hamilton-Waterloo problem \([30]\). The Hamilton-Waterloo problem $HWP(v; m, n; \alpha, \beta)$ is the problem of determining whether the complete graph $K_v$ (for $v$ odd) or $K_v$ minus a 1-factor $I$ (for $v$ even) has a 2-factorization in which there are exactly $\alpha C_m$-factors and $\beta C_n$-factors. We denote by $HWP(v; m, n; \alpha, \beta)$ the set of $(\alpha, \beta)$ for which a solution to $HWP(v; m, n; \alpha, \beta)$ exists. For recent results on the Hamilton-Waterloo problem, we refer the reader to \([5, 7–11, 16, 17, 21, 23, 28, 29]\). As a by-product, the following theorem can be obtained by combining Theorem 1.3 of this paper, and Theorems 1.4 and 3.5, Constructions 3.11 and 3.13 in \([30]\).

**Theorem 5.1.** If $k \geq 3$ is odd and $t \geq 1$, then $(\alpha, \beta) \in HWP(k(2kt+1); k, 2kt+1)$ if and only if $\alpha, \beta \geq 0$ and $\alpha + \beta = \frac{(2kt+1)^2 - 1}{2}$, except possibly when:

1. $t = 1$.
   
   \[ k = 5 : \beta \in \{1, 2, 3\}; \]
   \[ k = 7 : \beta \in \{1, 2, 3, 5\}; \]
   \[ k \geq 9 : \beta \in \{1, 2, 3, 5, 7\}. \]

2. $t = 2$.
   
   \[ k = 5, 7, 9 : \beta \in \{1, 2, 3, 5, 7\}. \]

3. $t \geq 3$.
   
   \[ k = 3 : \]
   
   - $t$ is odd: $\beta \in \{1, 3, 5, \ldots, 3t - 4, 3t - 2\} \cup \{2, 9t - 3, 9t - 1\}$
   - $t$ is even: $\beta \in \{1, 3, 5, \ldots, 3t + 3\} \cup \{2, 9t - 3, 9t - 1\}$
   \[ k \geq 5 : \beta \in \{1, 2, 3, 5, 7\}. \]

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