Orthogonal decompositions for Lévy processes with an application to the gamma, Pascal, and Meixner processes

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Abstract

It is well known that between all processes with independent increments, essentially only the Brownian motion and the Poisson process possess the chaotic representation property (CRP). Thus, a natural question appears: What is an appropriate analog of the CRP in the case of a general Lévy process. At least three approaches are possible here. The first one, due to Itô, uses the CRP of the Brownian motion and the Poisson process, as well as the representation of a Lévy process through those processes. The second approach, due to Nualart and Schoutens, consists in representing any square-integrable random variable as a sum of multiple stochastic integrals constructed with respect to a family of orthogonalized centered power jumps processes. The third approach, never applied before to the Lévy processes, uses the idea of orthogonalization of polynomials with respect to a probability measure defined on the dual of a nuclear space. The main aims of the present paper are to develop the three approaches in the case of a general (R-valued) Lévy process on a Riemannian manifold and (what is more important) to understand a relationship between these approaches. We apply the obtained results to the gamma, Pascal, and Meixner processes, in which case the analysis related to the orthogonalized polynomials becomes essentially simpler and richer than in the general case.

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1 Introduction

It is well known ([13, 16, 35]) that between all processes with independent increments, essentially only the Brownian motion and the Poisson process possess the chaotic representation property (CRP). Thus, in the situation where one has to deal with a general Lévy process, a natural question appears: What is an appropriate analog of the CRP in this case. At least three approaches are possible here.

The first approach was proposed by Itô in [19], see also [13]. Let $X = (X_t)_{t \geq 0}$ be a square integrable Lévy process. One may always choose a version of this process which is right-continuous, has limits from the left, and does not have fixed discontinuities. By the Lévy–Khintchine formula, the process $X$ can be decomposed into a Gaussian process plus a stochastic integral with respect to a compensated Poisson process on $\mathbb{R}_+ \times \mathbb{R}$. By taking the tensor product of the chaos decomposition of the Brownian and Poisson components, one obtains a unitary isomorphism between the canonical $L^2$-space of the process $X$ and a symmetric Fock over $L^2(\mathbb{R}_+ \times \mathbb{R}; \vartheta)$, where the measure $\vartheta$ is derived from the Lévy–Khintchine formula. In what follows, we will call this approach the Fock space decomposition for the Lévy process.

The second approach is due to Nualart and Schoutens [30] (see also the recent book [31]) and consists in the following. Denote $\hat{\nu}(ds) := s^2 \nu(ds)$, where $\nu$ is the Lévy measure of the process $X$, and suppose that the Laplace transform of the measure $\hat{\nu}$ may be extended to an analytic function in a neighborhood of zero, which particularly implies that the measure $\hat{\nu}$ has all moments finite. Instead of considering a single Lévy process $X$, one defines through the original process a family of centered power jump processes $(X_t^m)_{t \geq 0}$, $m \in \mathbb{N}$, which are called by the authors Teugels martingales. (We
Hence, we naturally arrive at a unitary operator between the stochastic integral decomposition. In the general case, the main problem is to identify the scalar measures. One then shows that there exist processes with numbers \( a_{m,n} \in \mathbb{R} \) independent of \( t \), which are orthogonal for different \( m \)'s. A theorem in [27] states that any square-integrable random variable admits a representation as a sum of multiple stochastic integrals with respect to the processes \( (Y_t^{(m)})_{t \geq 0}, m \in \mathbb{N} \). In the case where the CRP holds, either all the processes \( (X_t^{(m)})_{t \geq 0}, m \in \mathbb{N} \), coincide (Poisson case), or \( X_t^{(m)} = 0, t \geq 0, m \geq 2 \) (Gaussian case), and thus one arrives at the classical chaos decomposition. As main examples of application of this approach, the Lévy processes of Meixner's type—the gamma, Pascal, and Meixner processes—were considered in [30, 31].

It should be noticed that the Pascal and Meixner processes were originally introduced in [10] and [32], respectively. In [5], the Meixner process was proposed for a model for risky assets and an analog of the Black–Sholes formula was established. We also refer to the recent paper [36] and the references therein, where many properties of the gamma process are discussed in detail.

Finally, the third approach is based on the idea of orthogonalization of polynomials with respect to a measure defined on the dual of a nuclear space. In the case of a general probability measure, such a procedure was first proposed by Skorohod in [33, Sect. 11]. Suppose that we are given a Lévy noise \( \mu \). In the case where the CRP holds true, \( \mathbb{R} \), is dense in \( L^2(D'; \mu) \). Denoting by \( P^\infty_n(D) \) the closure of all continuous polynomials of power \( \leq n \) in \( L^2(D'; \mu) \), we obtain the orthogonal decomposition

\[
L^2(D'; \mu) = \bigoplus_{n=0}^{\infty} P_n(D'), \quad \tag{1.1}
\]

where \( P_n(D') \) denotes the orthogonal difference \( P_n(D') \ominus P^\infty_{n-1}(D') \). The set of all projections \( :\langle \omega \otimes n, f_n \rangle : \) of continuous monomials \( \langle \omega \otimes n, f_n \rangle \) onto \( P_n(D') \) is dense in \( P_n(D') \). Therefore, we can define, for each \( n \in \mathbb{N} \), a Hilbert space \( \mathcal{F}_n \) as the closure of the set \( D^\otimes n \) in the norm generated by the scalar product

\[
\langle f_n, g_n \rangle_{\mathcal{F}_n} := \frac{1}{n!} \int_{D'} :\langle \omega \otimes n, f_n \rangle : \langle \omega \otimes n, g_n \rangle : \mu(d\omega), \quad f_n, g_n \in D^\otimes n. \quad \tag{1.2}
\]

Hence, we naturally arrive at a unitary operator between \( L^2(D'; \mu) \) and the Hilbert space

\[
\mathcal{F} := \bigoplus_{n=0}^{\infty} \mathcal{F}_n n!, \quad \tag{1.3}
\]

where \( \mathcal{F}_0 := \mathbb{R} \). In the case where the CRP holds true, \( \mathcal{F} \) is the usual symmetric Fock space over \( L^2(\mathbb{R}_+; dx) \), while the obtained unitary operator coincides with the operator given through the multiple stochastic integral decomposition. In the general case, the main problem is to identify the scalar product explicitl.
the Jacobi matrix $J$ defining an (unbounded) self-adjoint operator in $\ell_2$, whose spectral measure is $\mu$, see \cite[Ch. VII, Sect. 1]{1}. More exactly, the Jacobi matrix $J = (J_{i,j})_{i,j=0}^{\infty}$ is given by $J_{i,j} = 0$ if $|i-j| > 1$, $J_{i,i} = J_{i-1,i} = \beta_i$, $i \in \mathbb{N}$, $J_{i,i} = \alpha_i$, $i \in \mathbb{Z}_+$, where $\alpha_i, \beta_i$ are the coefficients of the recurrence relation satisfied by the system $\{Q_i, i \in \mathbb{Z}_+\}$ of the normalized orthogonal polynomials:

$$sQ_i(s) = \beta_{i+1}Q_{i+1}(s) + \alpha_iQ_i(s) + \beta_iQ_i(s), \quad i \in \mathbb{Z}_+, \quad Q_{-1}(s) = 0, \quad Q_1(s) = 1. \quad (1.4)$$

Vice versa, given a Jacobi matrix $J$ which determines a self-adjoint operator in $\ell_2$, there exists a spectral measure $\mu$ of $J$, which is a probability measure on $\mathbb{R}$ whose normalized orthogonal polynomials satisfy the recurrence relation (1.4) with the coefficients $\alpha_i, \beta_i$ defined by $J$ as above.

In the infinite-dimensional case, the role of a Jacobi matrix should be played by a Jacobi field $(A(\varphi))_{\varphi \in \mathcal{D}}$, see the works by Berezansky et al. \cite{2, 3, 4} and \cite{5} for the notion of a Jacobi field. More precisely, each operator $A(\varphi)$ in $\mathfrak{A}$ should correspond in the functional realization to the operator of multiplication by the monomial $(\cdot, \varphi)$, and should have a three-diagonal structure with respect to the orthogonal decomposition (1.3), i.e., each $A(\varphi)$ should be a sum of a creation, neutral, and annihilation operator. However, for a general probability measure on the dual of a nuclear space, the problem of existence of a corresponding Jacobi field is still open. We refer here to the work \cite{6} where a sufficient condition for the existence of the Jacobi field was given in terms of the moments of the measure.

This problem was solved for the gamma process in \cite{21} (see also \cite{22}) and for the processes of Meixner's type—defined even on a general manifold $X$ (instead of $\mathbb{R}_+$)—in the recent paper \cite{25} (see also the paper \cite{4} for the case of the Pascal process on $\mathbb{R}_+$). More precisely, in \cite{21, 23} the Hilbert space $\mathfrak{A}$ was a priori defined as an extended Fock space, a Jacobi field in $\mathfrak{A}$ was constructed, and then it was shown that the spectral measure of the Jacobi field is a corresponding Lévy process.

In \cite{4}, it was shown that the extended Fock space is naturally isomorphic to a direct sum of subspaces of $L^2$-spaces of a special form. In \cite{3, 5}, it was shown that the extended Fock space decomposition of the gamma process can be thought of as an expansion of any $L^2$-random variable in multiple integrals constructed by using the family of the resolutions of the identity of the operators of the corresponding Jacobi field.

The main aims of the present paper are to develop the three approaches in the case of a general (\mathbb{R}-valued) Lévy process without Brownian part on a manifold, and (what is more important) to understand a relationship between these approaches. So, the contents of the present paper is as follows.

In Section 2, we present a definition of a Lévy process on a general Riemannian manifold $X$. In the case where the Lévy measure of the process has the first moment finite, we essentially follow the definition and construction of the process in \cite{3}, using the corresponding Poisson process. In the case of the infinite first moment (which, for a Lévy process on $\mathbb{R}_+$, yields that the trajectories of the process are of unbounded variation on any finite interval of time), we define a Lévy process as a generalized process on a space $\mathcal{D}'$ of distributions on $X$ (which is dual of a nuclear space) through its law—a probability measure $\mu$ on $\mathcal{D}'$ given by its Fourier transform (compare with \cite[Ch. III, Sec. 4]{1}).

In Section 3, using the CRP of the Poisson process on $\mathbb{R} \times X$ with intensity $\nu \otimes \sigma$, we construct a unitary operator between the space $L^2(\mathcal{D}'; \mu)$ and the symmetric Fock space over $L^2(\mathbb{R} \times X; \nu \otimes \sigma)$. Here $\nu$ is the Lévy measure of the process and $\sigma$ is its intensity measure.

In Section 4, using the unitary operator mentioned above, we prove the Nualart–Schoutens chaotic decomposition for the Lévy process on the manifold. We note that even in the standard case where $X = \mathbb{R}$ or $\mathbb{R}_+$, our proof differs from the original one in \cite{30}. Furthermore, we discuss the unitary description of $L^2(\mathcal{D}'; \mu)$ which appears from the obtained chaotic decomposition (the original description of \cite{30} works only in the case of the one-dimensional underlying space). Using our approach, we, in particular, easily derive a formula for multiplication of any multiple stochastic integral by a monomial of first order.

Next, in Section 5, we derive from the Nualart–Schoutens decomposition, the decomposition (1.1) for $L^2(\mathcal{D}'; \mu)$. More exactly, we explicitly identify the scalar product (1.2), and furthermore, we
write down a representation of any function :$(\tilde{\omega}^n, f_n)$: as defined above through multiple stochastic integrals as in Section 4. Thus, we establish a correspondence between the second and third approaches to chaotic decomposition (Corollaries 5.1 and 5.3). Our results also allow one to identify the Jacobi field corresponding to the Lévy process.

Finally, in Section 3, we apply the obtained results to the processes of Meixner’s type—the gamma, Pascal, and Meixner processes. These are characterized by e.g. a special form of their Lévy measure $\nu$, more exactly, the measure $\tilde{\nu}(ds) = s^2 \nu(ds)$ is a probability measure on $\mathbb{R}$ whose orthogonal polynomials with leading coefficient 1 are polynomials of Meixner’s type and satisfy the recurrence relation (6.1) below. In turn, their one-dimensional distributions happen to be again orthogonality measures of polynomials of Meixner’s type. We show that, for these processes, analysis related to the orthogonalized polynomials becomes sufficiently simpler and reacher than in the general case. (Notice that this difference cannot be felt if one restricts himself only to the approach of Itô or that of Nualart and Schoutens!) In particular, we re-derive the Jacobi field of the process as a special case of the general formulas. As a by-product of our considerations, we obtain a more explicit description of the structure of the extended Fock space $\mathfrak{F}$ than that obtained in [6]. Finally, we show that, in the case of the gamma and Pascal processes, the decomposition of each function :$(\tilde{\omega}^n, f_n)$: into a sum of multiple stochastic integrals of the Nualart–Schoutens type may be interpreted as decomposition of a random measure obtained by dividing the whole space into (non-random) disjoint parts (Theorem 6.1).

2 Lévy processes on manifolds

In this section, we construct a Lévy process on a manifold, using ideas of [17, 36].

Let $X$ be a complete, connected, oriented $C^\infty$ (non-compact) Riemannian manifold and let $\mathcal{B}(X)$ be the Borel $\sigma$-algebra on $X$. Let $\sigma$ be a Radon measure on $(X, \mathcal{B}(X))$ that is non-atomic, i.e., $\sigma(\{x\}) = 0$ for every $x \in X$ and non-degenerate, i.e., $\sigma(O) > 0$ for any open set $O \subset X$. (We note that the assumption of the nondegeneracy of $\sigma$ is nonessential and the results below may be generalized to the case of a degenerate $\sigma$.) Note that $\sigma(\Lambda) < \infty$ for each $\Lambda \in \mathcal{B}_c(X)$—the set of all sets from $\mathcal{B}(X)$ with compact closure.

Let $\mathcal{R} := \mathbb{R} \setminus \{0\}$. We endow $\mathcal{R}$ with the relative topology of $\mathbb{R}$ and let $\mathcal{B}(\mathcal{R})$ denote the Borel $\sigma$-algebra on $\mathcal{R}$. Let $\nu$ be a Radon measure on $(\mathcal{R}, \mathcal{B}(\mathcal{R}))$, whose support contains an infinite number of points. Let $\tilde{\nu}(ds) := s^2 \nu(ds)$. We suppose that $\tilde{\nu}$ is a finite measure on $(\mathcal{R}, \mathcal{B}(\mathcal{R}))$, and furthermore, there exists $\varepsilon > 0$ such that

$$\int_{\mathcal{R}} \exp (\varepsilon|s|) \tilde{\nu}(ds) < \infty. \quad (2.1)$$

By (2.1), the Laplace transform of the measure $\tilde{\nu}$ is well defined in a neighborhood of zero and may be extended to an analytic function on $\{z \in \mathbb{C} : |z| < \varepsilon\}$. Therefore, the measure $\tilde{\nu}$ has all moments finite, and moreover, the set of all polynomials is dense in $L^2(\mathcal{R}; \tilde{\nu})$. Next, we evidently have

$$\forall n \geq 2 : \int_{\mathcal{R}} |s|^n \nu(ds) < \infty. \quad (2.2)$$

Notice that (2.2) implies that $\nu(|s| \geq a) < \infty$ for any $a > 0$.

We will first additionally suppose that

$$\int_{\mathcal{R}} |s| \nu(ds) < \infty. \quad (2.3)$$

Let $\Gamma_{\mathcal{R} \times X}$ denote the configuration space over $\mathcal{R} \times X$ defined as follows:

$$\Gamma_{\mathcal{R} \times X} := \{\gamma \subset \mathcal{R} \times X : \sharp(\gamma \cap \{|s| \geq \varepsilon\} \times \Lambda) < \infty \text{ for each } \varepsilon > 0 \text{ and } \Lambda \in \mathcal{B}_c(X)\}.$$
Here, $\sharp(A)$ denotes the cardinality of a set $A$. Each $\gamma \in \Gamma_{\mathcal{R} \times X}$ may be identified with the positive Radon measure
\[ \sum_{(s,x) \in \gamma} \delta_{(s,x)} \in \mathcal{M}_+(\mathcal{R} \times X), \]
where $\delta_{(s,x)}$ denotes the Dirac measure with mass at $(s,x)$, $\sum_{(s,x) \in \emptyset} \delta_{(s,x)} = \text{zero measure}$, and $\mathcal{M}_+(\mathcal{R} \times X)$ denotes the set of all positive Radon measures on the Borel $\sigma$-algebra $\mathcal{B}(\mathcal{R} \times X)$. We endow the space $\Gamma_{\mathcal{R} \times X}$ with the relative topology as a subset of the space $\mathcal{M}_+(\mathcal{R} \times X)$ with the vague topology, i.e., the weakest topology on $\Gamma_{\mathcal{R} \times X}$ with respect to which all maps
\[ \Gamma_{\mathcal{R} \times X} \ni \gamma \mapsto \langle \gamma, f \rangle := \int_{\mathcal{R} \times X} f(s,x) \gamma(ds, dx) = \sum_{(s,x) \in \gamma} f(s,x) \]
are continuous. Here, $f \in C_0(\mathcal{R} \times X)(:=\text{the set of all continuous functions on } \mathcal{R} \times X \text{ with compact support})$.

Let $\pi_{\nu \otimes \sigma}$ denote the Poisson measure on $(\Gamma_{\mathcal{R} \times X}, \mathcal{B}(\Gamma_{\mathcal{R} \times X}))$ with intensity $\nu \otimes \sigma$. This measure can be characterized by its Fourier transform
\[ \int_{\Gamma_{\mathcal{R} \times X}} e^{i \langle \gamma, f \rangle} \pi_{\nu \otimes \sigma}(d\gamma) = \exp \left[ \int_{\mathcal{R} \times X} (e^{if(s,x)} - 1) \nu(ds) \sigma(dx) \right], \quad f \in C_0(\mathcal{R} \times X). \quad (2.4) \]
We refer to e.g. [20] for an explicit construction of the Poisson measure.

Since the measure $\sigma$ is non-atomic, it follows from the construction of the Poisson measure that, for $\pi_{\nu \otimes \sigma}$-a.e. $\gamma \in \Gamma_{\mathcal{R} \times X}$,
\[ \forall (s, x), (s', x') \in \gamma : (s, x) \neq (s', x') \Rightarrow x \neq x'. \quad (2.5) \]
We fix an arbitrary $x_0 \in X$ and let $B(x_0, r)$ denote the closed ball in $X$ centered at $x_0$ and of radius $r$. For any $n, k \in \mathbb{N}$, we then have, by the Mecke identity (e.g. [20]),
\[ \int_{\Gamma_{\mathcal{R} \times X}} \int_{\mathcal{R} \times X} |s|^k \chi_{B(x_0,n)}(x) \gamma(ds, dx) \pi_{\nu \otimes \sigma}(d\gamma) = \int_{\mathcal{R} \times X} |s|^k \chi_{B(x_0,n)}(x) \nu(ds) \sigma(dx) \]
\[ = \int_{\mathcal{R}^+} |s|^k \nu(ds) \sigma(B(x_0,n)) < \infty. \quad (2.6) \]
We denote by $\tilde{\Gamma}_{\mathcal{R} \times X} \subseteq \mathcal{B}(\Gamma_{\mathcal{R} \times X})$ the set of all $\gamma \in \Gamma_{\mathcal{R} \times X}$ for which (2.3) holds and $\int_{\mathcal{R} \times X} |s|^k \chi_{B(x_0,n)}(x) \gamma(ds, dx) < \infty$ for $k, n \in \mathbb{N}$. By (2.6), we get $\pi_{\nu \otimes \sigma}(\tilde{\Gamma}_{\mathcal{R} \times X}) = 1$. Let $\mathcal{B}(\tilde{\Gamma}_{\mathcal{R} \times X})$ denote the trace $\sigma$-algebra of $\mathcal{B}(\Gamma_{\mathcal{R} \times X})$ on $\tilde{\Gamma}_{\mathcal{R} \times X}$.

For each $\gamma \in \tilde{\Gamma}_{\mathcal{R} \times X}$, we define $\omega(\gamma) := \sum_{(s,x) \in \gamma} s \delta_x$, which is a signed Radon measure on $X$. Furthermore, the mapping $\tilde{\Gamma}_{\mathcal{R} \times X} \ni \gamma \mapsto \omega(\gamma) \in \mathcal{M}(X)$ is Borel-measurable. Here, $\mathcal{M}(X)$ denotes the space of all signed Radon measures on $X$ endowed with the vague topology. Let $\Omega(X)$ denote the image of $\Gamma_{\mathcal{R} \times X}$ under the mapping $\gamma \mapsto \omega(\gamma)$ and let $\mathcal{B}(\Omega(X))$ denote the trace $\sigma$-algebra of $\mathcal{B}(\mathcal{M}(X))$ on $\Omega(X)$.

We define a Lévy process on $X$ with intensity measure $\sigma$ and Lévy measure $\nu$ as a generalized process on $\Omega(X)$ whose law is the probability measure $\mu_{\nu, \sigma}$ on $(\Omega(X), \mathcal{B}(\Omega(X)))$ obtained as the image of $\pi_{\nu \otimes \sigma}$ under the measurable mapping
\[ \tilde{\Gamma}_{\mathcal{R} \times X} \ni \gamma \mapsto \omega(\gamma) \in \Omega(X). \quad (2.7) \]
As follows from [20], the Fourier transform of $\mu_{\nu, \sigma}$ is given by
\[ \int_{\Omega(X)} e^{i \langle s, \varphi \rangle} \mu_{\nu, \sigma}(d\omega) = \exp \left[ \int_{\mathcal{R} \times X} (e^{is\varphi(x)} - 1) \nu(ds) \sigma(dx) \right], \quad \varphi \in C_0(X). \]
Here, $C_0(X)$ denotes the set of all continuous functions on $X$ with compact support.

In the case where (2.3) does not hold, such a direct construction of a Lévy process is, of course, impossible, so we proceed as follows.

We denote by $\mathcal{D}$ the space $C^\infty_0(X)$ of all real-valued infinite differentiable functions on $X$ with compact support. This space may be naturally endowed with a topology of a nuclear space, see e.g. [1] for the case $X = \mathbb{R}^d$ and e.g. [2] for the case of a general Riemannian manifold. We recall that

$$
\mathcal{D} = \text{proj lim} \, \mathcal{H}_\tau.
$$

Here, $T$ denotes the set of all pairs $(\tau_1, \tau_2)$ with $\tau_1 \in \mathbb{Z}_+$ and $\tau_2 \in C^\infty(X)$, $\tau_2(x) \geq 1$ for all $x \in X$, and $\mathcal{H}_\tau = \mathcal{H}(\tau_1, \tau_2)$ is the Sobolev space on $X$ of order $\tau_1$ weighted by the function $\tau_2$, i.e., the scalar product in $\mathcal{H}_\tau$, denoted by $(\cdot , \cdot )_\tau$ is given by

$$
(f, g)_\tau = \int_X \left(f(x)g(x) + \sum_{i=1}^{\tau_1} (\nabla_i f(x), \nabla_i g(x))_{L^2(X)}\right)\tau_2(x) \, dx,
$$

where $\nabla_i$ denotes the $i$-th (covariant) gradient, and $dx$ is the volume measure on $X$. For $\tau, \tau' \in T$, we will write $\tau' \geq \tau$ if $\tau'_1 \geq \tau_1$ and $\tau'_2(x) \geq \tau_2(x)$ for all $x \in X$.

The space $\mathcal{D}$ is densely and continuously embedded into the real $L^2$-space $L^2(X; \sigma)$. As easily seen, there always exists $\tau_0 \in T$ such that $\mathcal{H}_{\tau_0}$ is continuously embedded into $L^2(X; \sigma)$. We denote $T' = \{ \tau \in T : \tau \geq \tau_0 \}$ and (2.8) holds with $T$ replaced by $T'$. In what follows, we will just write $T$ instead of $T'$. Let $\mathcal{H}_{-\tau}$ denote the dual space of $\mathcal{H}_\tau$ with respect to the zero space $L^2(X; \sigma)$. Then $\mathcal{D}' = \text{ind lim},_{\tau \in T} \mathcal{H}_{-\tau}$ is the dual of $\mathcal{D}$ with respect to $L^2(X; \sigma)$, and we thus get the standard triple

$$
\mathcal{D}' \supset L^2(X; \sigma) \supset \mathcal{D}.
$$

The dual pairing between any $\omega \in \mathcal{D}'$ and $\xi \in \mathcal{D}$ will be denoted by $\langle \omega, \xi \rangle$. We can evidently consider $\mathcal{M}(X)$ a subset of $\mathcal{D}'$ by identifying any $\omega \in \mathcal{M}(X)$ with $\tilde{\omega} \in \mathcal{D}'$ by setting $\langle \tilde{\omega}, \varphi \rangle := \int_X \varphi(x) \omega(dx)$ for each $\varphi \in \mathcal{D}$. Let $\mathcal{C}(\mathcal{D}')$ denote the cylinder $\sigma$-algebra on $\mathcal{D}'$. Then, the trace $\sigma$-algebra of $\mathcal{C}(\mathcal{D}')$ on $\mathcal{M}(X)$ coincides with $\mathcal{B}(\mathcal{M}(X))$. Thus, any probability measure on $\mathcal{B}(\mathcal{M}(X))$ may also be considered as a probability measure on $\mathcal{C}(\mathcal{D}')$.

We now define a centered Lévy process as a generalized process on $\mathcal{D}'$ whose law is the probability measure on $\langle \mathcal{D}', \mathcal{C}(\mathcal{D}') \rangle$ given by its Fourier transform

$$
\int_{\mathcal{D}'} e^{i\langle \omega, \varphi \rangle} \rho_{\nu, \sigma}(d\omega) = \exp \left[ \int_{\mathbb{R} \times X} \left( e^{is\varphi(x)} - 1 - is\varphi(x) \right) \nu(ds) \sigma(dx) \right], \quad \varphi \in \mathcal{D}.
$$

The existence of $\rho_{\nu, \sigma}$ follows from the Bochner–Minlos theorem. In the case where (2.3) holds, $\rho_{\nu, \sigma}$ coincides with the measure obtained by centering $\mu_{\nu, \sigma}$.

3 The Fock space decomposition for a Lévy process

In this section, we will discuss the Fock space decomposition for a Lévy processes which comes from the multiple stochastic integral decomposition for the corresponding Poisson process.

Let first (2.3) hold. Since the mapping (2.3) is one-to-one and since $\mu_{\nu, \sigma}$ is the image measure of $\pi_{\nu\otimes \sigma}$ under (2.7), we conclude that the operator

$$
L^2(\Gamma_{\mathbb{R} \times X}; \pi_{\nu \otimes \sigma}) \ni F \mapsto UF = (UF)(\omega) := F(\gamma(\omega)) \in L^2(\Omega(X); \mu_{\nu, \sigma})
$$

is unitary. Here, $\omega \mapsto \gamma(\omega)$ is the inverse mapping of (2.7). As well known (see e.g. [3]), the Poisson measure $\pi_{\nu \otimes \sigma}$ possesses the chaotic decomposition property. More exactly, for each $g_n \in \mathcal{D}$,
\( L^2(\mathcal{R} \times X; \nu \otimes \sigma)^{\otimes n} \), \( n \in \mathbb{N} \), one can construct a multiple stochastic integral \( I^{(n)}(g_n) \) with respect to the centered Poisson process on \( \mathcal{R} \times X \) with intensity \( \nu \otimes \sigma \). Here, \( \otimes \) denotes the symmetric tensor product. Furthermore, we have
\[
\| I^{(n)}(g_n) \|_{L^2(\mathcal{R} \times X; \pi_{\nu \otimes \sigma})} = n! \| g_n \|_{L^2(\mathcal{R} \times X; \nu \otimes \sigma)^{\otimes n}},
\]
and any random variable \( G \in L^2(\mathcal{R} \times X; \pi_{\nu \otimes \sigma}) \) can be uniquely represented as a sum of multiple stochastic integrals:
\[
G = \sum_{n=0}^{\infty} I^{(n)}(g_n),
\]
where \( I^{(0)}(g_0) = g_0 \in \mathbb{R} \), and the series converges in \( L^2(\mathcal{R} \times X; \pi_{\nu \otimes \sigma}) \). Thus, we have the unitary operator
\[
\mathcal{F}(L^2(\mathcal{R} \times X; \nu \otimes \sigma)) \ni g = (g_n)_{n=0}^{\infty} \mapsto I g = \sum_{n=0}^{\infty} J^{(n)}(g_n) \in L^2(\mathcal{R} \times X; \nu \otimes \sigma).
\]
Here, for a real Hilbert space \( \mathcal{H} \), \( \mathcal{F}(\mathcal{H}) \) denotes the real Fock space over \( \mathcal{H} \):
\[
\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n} n!, \quad \mathcal{H}^{\otimes 0} := \mathbb{R}
\]
(for a Hilbert space \( \mathcal{H} \) and a constant \( c > 0 \), we denote by \( Hc \) the Hilbert space consisting of the same elements as \( H \) and with scalar product equal to \( c \) times the scalar product in \( H \)). Setting \( J := U1 \), we get the unitary operator
\[
\mathcal{F}(L^2(\mathcal{R} \times X; \nu \otimes \sigma)) \ni g = (g_n)_{n=0}^{\infty} \mapsto J g = \sum_{n=0}^{\infty} J^{(n)}(g_n) \in L^2(\Omega(X); \mu_{\nu, \sigma}),
\]
where \( J^{(n)}(g_n) := U(I^{(n)}(g_n)) \).

Let us consider the general case. We still have the unitary operator \( I \) as in (3.2). We now denote by \( \mathcal{P}_{\text{cyl}}(\mathcal{D}') \) the set of all cylindrical polynomials on \( \mathcal{D}' \), i.e., the set of all functions on \( \mathcal{D}' \) that are finite sums of constants and monomials of the form
\[
\langle \cdot, \varphi_1 \rangle \cdots \langle \cdot, \varphi_n \rangle, \quad \varphi_1, \ldots, \varphi_n \in \mathcal{D}, \quad n \in \mathbb{N}.
\]

**Lemma 3.1** \( \mathcal{P}_{\text{cyl}}(\mathcal{D}') \) is a dense subset of \( L^2(\mathcal{D}'; \rho_{\nu, \sigma}) \).

**Proof.** Let \( \mathcal{D}_{\mathbb{C}} \) denote the complexification of the real space \( \mathcal{D} \), and define the Laplace transform of the measure \( \rho_{\nu, \sigma} \) by
\[
L(\theta) := \int_{\mathcal{D}_{\mathbb{C}}} e^{\langle \omega, \theta \rangle} \rho_{\nu, \sigma}(d\omega), \quad \theta \in \mathcal{D}_{\mathbb{C}},
\]
provided the integral on the right hand side of (3.4) exists. Using (2.1) and (2.10), we conclude that the Laplace transform \( L \) is well defined and analytic in a neighborhood of zero in \( \mathcal{D}_{\mathbb{C}} \) (which equivalently means that \( L \) is bounded on this neighborhood and \( \mathcal{G} \)-holomorphic, see e.g. [15]). Then, by (the proof of) [13, Sec. 10, Th. 1], we conclude the statement. \( \blacksquare \)

In what follows, when writing \( I^{(n)}(g_n(s_1, \ldots, s_n); f_n(x_1, \ldots, x_n)) \), we will understand under \( s_1, \ldots, s_n, x_1, \ldots, x_n \) the variables in which the integration in the multiple stochastic integral is carried out.
Lemma 3.2 For any \( \varphi_1, \ldots, \varphi_n \in D \), \( n \in \mathbb{N} \), the distribution of the \( \mathbb{R}^n \)-valued random variable \((\langle \cdot, \varphi_1 \rangle, \ldots, \langle \cdot, \varphi_n \rangle)\) under \( \rho_{\nu, \sigma} \) coincides with the distribution of the \( \mathbb{R}^n \)-valued random variable \((I^{(1)}(s\varphi_1(x)), \ldots, I^{(1)}(s\varphi_n(x)))\) under \( \pi_{\nu \otimes \sigma} \).

Proof. By using (2.4) and (2.10), we see that, for any \( \varphi \in D \), the Fourier transform of the random variable \((\langle \cdot, \varphi \rangle)\) under \( \rho_{\nu, \sigma} \) coincides with the Fourier transform of the random variable \((I^{(1)}(s\varphi(x)))\) under \( \pi_{\nu \otimes \sigma} \). Therefore, by linearity, we conclude that, for any fixed \( \varphi_1, \ldots, \varphi_n \in D \), \( n \in \mathbb{N} \), the Fourier transform of the random variable \((\langle \cdot, \varphi_1 \rangle, \ldots, \langle \cdot, \varphi_n \rangle)\) under \( \rho_{\nu, \sigma} \) coincides with the Fourier transform of the random variable \((I^{(1)}(s\varphi_1(x)), \ldots, I^{(1)}(s\varphi_n(x)))\) under \( \pi_{\nu \otimes \sigma} \). From here the statement follows.

For any \( f \in L^2(\mathcal{R} \times X; \nu \otimes \sigma) \), let \( A(f) \) denote the operator in \( \mathcal{F}(L^2(\mathcal{R} \times X; \nu \otimes \sigma)) \) whose image under the unitary \( I \) is the operator of multiplication by the random variable \( I^{(1)}(f) \). We then have (see e.g. [34])

\[
A(f)g_n = A^+(f)g_n + A^0(f)g_n + A^-(g)g_n, \quad n \in \mathbb{Z}_+
\]

that is, \( A^+(f), A^0(f), A^-(f) \) are creation, neutral, and annihilation operators in the Fock space, respectively.

Lemma 3.3 The linear span of the set

\[
\{1, I^{(1)}(s\varphi_1(x)) \cdots I^{(1)}(s\varphi_n(x)), \varphi_1, \ldots, \varphi_n \in D, n \in \mathbb{N}\}
\]

is dense in \( L^2(\overline{\mathcal{F}}_{\mathcal{R} \times X}; \pi_{\nu \otimes \sigma}) \).

Proof. By (3.5)–(3.8), each function \( I^{(1)}(s\varphi_1(x)) \cdots I^{(1)}(s\varphi_n(x)) \), \( \varphi_1, \ldots, \varphi_n \in D \), \( n \in \mathbb{N} \), indeed belongs to \( L^2(\overline{\mathcal{F}}_{\mathcal{R} \times X}; \pi_{\nu \otimes \sigma}) \). Let \( \mathcal{L} \) denote the closed linear span of the set (3.9), and thus we have to show that \( \mathcal{L} = L^2(\overline{\mathcal{F}}_{\mathcal{R} \times X}; \pi_{\nu \otimes \sigma}) \).

Let us consider the unitary operator

\[
L^2(\mathcal{R}; \nu) \ni f = f(s) \mapsto \frac{f(s)}{s} \in L^2(\mathcal{R}; \nu).
\]

As we already mentioned above, the set of functions \( \{s^n, n \in \mathbb{Z}_+\} \) is total in \( L^2(\mathcal{R}; \nu) \), i.e., its linear span is a dense subset. Therefore, the set of functions \( \{s^n, n \in \mathbb{N}\} \) is total in \( L^2(\mathcal{R}; \nu) \).

Hence, it suffices to show that, for each \( f_n \in L^2(\mathcal{R}; \sigma)^{\otimes n}, n \in \mathbb{N}, k_1, \ldots, k_n \in \mathbb{N}, \)

\[
I^{(n)}((s^{k_1} \cdots s^{k_n} f_n(x_1, \ldots, x_n))^\sim) \in \mathcal{L}.
\]

Let \( O_c(X) \) denote the algebra of sets in \( X \) generated by all open sets in \( X \) with compact closure. For each \( m \in \mathbb{N} \), we introduce a random measure \( X^{(m)} \) on \( X \) by setting, for each \( \Delta \in O_c(X), \)

\[
X^{(m)}(\Delta) := I^{(1)}(s^m \chi_{\Delta}(x)).
\]
It is easy to see (see also the proof of Lemma 4.2 below) that, for any \( f_n \in L^2(X; \sigma)^{\otimes n} \), \( n \in \mathbb{N} \), and any \( k_1, \ldots, k_n \in \mathbb{N} \),

\[
I^n((s_1^{k_1} \cdots s_n^{k_n} f^n(x_1, \ldots, x_n))^{-}) = \int_{X^n} f^n(x_1, \ldots, x_n) d\lambda^{(k_1)}(x_1) \cdots d\lambda^{(k_n)}(x_n),
\]

where the expression on the right hand side denotes the multiple stochastic integral constructed with respect to the random measures \( \lambda^{(k_1)}, \ldots, \lambda^{(k_n)} \). Therefore, it is enough to prove that, for any disjoint \( \Delta_1, \ldots, \Delta_n \in \mathcal{O}_c(X) \),

\[
\lambda^{(k_1)}(\Delta_1) \cdots \lambda^{(k_n)}(\Delta_n) \in \mathcal{L}.
\]

This, in turn, will follow from the next

**Claim.** For \( m \in \mathbb{N} \), let \( \mathcal{L}_m \) denote the closure in \( L^{2m}(\Gamma_{\mathbb{R} \times X}; \pi_v \otimes \sigma) \) of the linear span of the set (3.8). (In particular, we get \( \mathcal{L} = \mathcal{L}_1 \).) Then, we have for any \( n, m \in \mathbb{N} \) and \( \Delta \in \mathcal{O}_c(X) \)

\[
\lambda^{(n)}(\Delta) \in \mathcal{L}_m.
\] (3.12)

**Proof of the Claim.** By (3.3)–(3.8), we see that each element of the set (3.3) indeed belongs to \( L^{2m}(\Gamma_{\mathbb{R} \times X}; \pi_v \otimes \sigma) \). We now prove (3.12) by induction in \( n \in \mathbb{N} \). Let \( \Delta \in \mathcal{O}_c(X) \) and let \( n = 1 \). Approximate \( \chi_{\Delta} \) by a sequence \( \{ \varphi_k, k \in \mathbb{N} \} \subset \mathcal{D} \) such that \( \bigcup_{k \in \mathbb{N}} \text{supp} \varphi_k \text{ is precompact in } X \), \( |\varphi_k(x)| \leq \text{const} < \infty \) for all \( k \in \mathbb{N} \) and \( x \in X \), and \( \varphi_k(x) \to \chi_{\Delta}(x) \) as \( k \to \infty \) for each \( x \in X \). We then get by (3.3)–(3.8)

\[
\int_{\Gamma_{\mathbb{R} \times X}} (\lambda^{(1)}(\Delta) - \lambda^{(1)}(s\varphi_k(x)))^{2m} d\pi_v \otimes \sigma = \int_{\Gamma_{\mathbb{R} \times X}} I^{(1)}(s(\chi_{\Delta}(x) - \varphi_k(x)))^{2m} d\pi_v \otimes \sigma \to 0 \quad \text{as } k \to \infty.
\] (3.13)

Here, \( \Omega := (1, 0, 0, \ldots) \) denotes the vacuum in the Fock space.

Suppose the statement holds for \( \lambda^{(1)}, \ldots, \lambda^{(n)} \) and let us prove it for \( \lambda^{(n+1)} \). By (3.3)–(3.8)

\[
\lambda^{(n+1)}(\Delta) = \lambda^{(1)}(\Delta) \lambda^{(n)}(\Delta) - I^{(2)}((s_1 s_2 \chi_{\Delta}(x_1) \chi_{\Delta}(x_2))^{-}) - \int_R s^{n+1} \nu(ds) \sigma(\Delta).
\]

We evidently have \( \lambda^{(1)}(\Delta) \lambda^{(n)}(\Delta) \in \mathcal{L}_m \) for all \( m \in \mathbb{N} \). Hence, it remains to show that

\[
I^{(2)}((s_1 s_2 \chi_{\Delta}(x_1) \chi_{\Delta}(x_2))^{-}) \in \mathcal{L}_m, \quad m \in \mathbb{N}.
\]

Clearly,

\[
I^{(2)}((s_1 s_2 \chi_{\Delta}(x_1) \chi_{\Delta}(x_2))^{-}) = I^{(2)}((s_1 s_2 \chi_{\Delta \setminus D}(x_1, x_2))^{-}),
\]

where \( D := \{(x_1, x_2) \in X^2 : x_1 = x_2\} \). We approximate the indicator \( \chi_{\Delta \setminus D}(x_1, x_2) \) by a sequence of functions \( \{ f_k, k \in \mathbb{N} \} \) such that each \( f_k \) is a finite sum of functions \( \chi_{\Delta_k}(x_1, x_2) \) with \( \Delta_1, \Delta_2 \in \mathcal{O}_c(X) \), \( \Delta_1 \cap \Delta_2 = \emptyset \), \( \bigcup_{k \in \mathbb{N}} \text{supp} f_k \text{ is precompact in } X^2 \), \( |f_k(x_1, x_2)| \leq 1 \) for all \( k \in \mathbb{N} \) and \( (x_1, x_2) \in X^2 \), and \( f_k(x_1, x_2) \to \chi_{\Delta_1 \setminus D}(x_1, x_2) \) as \( k \to \infty \) for all \( (x_1, x_2) \in X^2 \). For any \( \Delta_1, \Delta_2 \in \mathcal{O}_c(X) \), \( \Delta_1 \cap \Delta_2 = \emptyset \),

\[
I^{(2)}((s_1 s_2 \chi_{\Delta_1}(x_1) \chi_{\Delta_2}(x_2))^{-}) = \lambda^{(1)}(\Delta_1) \lambda^{(n)}(\Delta_2) \in \mathcal{L}_m, \quad m \in \mathbb{N},
\]

which yields

\[
I^{(2)}((s_1 s_2 f_k(x_1, x_2))^{-}) \in \mathcal{L}_m, \quad k, m \in \mathbb{N}.
\] (3.14)

We clearly have

\[
I^{(2)}((s_1 s_2 f_k(x_1, x_2))^{-}) \to I^{(2)}((s_1 s_2 \chi_{\Delta \setminus D}(x_1, x_2))^{-}) \quad \text{in } L^2(\Gamma_{\mathbb{R} \times X}; \pi_v \otimes \sigma)
\] (3.15)
as $k \to \infty$. By (3.14) and (3.15), we will get the inclusion
\[ I^{(2)}((s_1 s_2^m \chi_{\Delta_1 D}(x_1, x_2)) \sim) \in \Sigma_m, \quad m \in \mathbb{N}, \]
provided we show that \( \{ I^{(2)}((s_1 s_2^m f_k(x_1, x_2)) \sim), k \in \mathbb{N} \} \) is a Cauchy sequence in each \( L^{2m}(\Gamma_{\mathbb{R}^k}; \pi_{\nu} \otimes \sigma) \), \( m \in \mathbb{N} \). But this can be easily derived, analogously to (3.13), by using the formula which expresses a product of arbitrary multiple stochastic integrals with respect to the centered Poisson process through a corresponding sum of multiple stochastic integrals, see e.g. [20, Theorem 3] or [34].

\[ \square \]

**Theorem 3.1** We may define a unitary operator
\[ U : L^2(\Gamma_{\mathbb{R}^k}; \pi_{\nu} \otimes \sigma) \to L^2(D'; \rho_{\nu}, \sigma) \]
by setting
\[ U1:=1, \quad U(I^{(1)}(s\varphi_1(x)) \cdots I^{(1)}(s\varphi_n(x))):=\langle \cdot, \varphi_1 \rangle \cdots \langle \cdot, \varphi_n \rangle, \]
\[ \varphi_1, \ldots, \varphi_n \in D, \ n \in \mathbb{N}. \]
Furthermore, by setting \( J:=U1 \), we get a unitary operator
\[ \mathcal{J}(L^2(\mathbb{R} \times X; \nu \otimes \sigma)) \ni g = (g_n)_{n=0}^{\infty} \mapsto \mathcal{J} g = \sum_{n=0}^{\infty} \mathcal{J}^{(n)}(g_n) \in L^2(D'; \rho_{\nu}, \sigma), \]
\( \mathcal{J}^{(n)} \) denoting the restriction of \( \mathcal{J} \) to \( L^2(\mathbb{R} \times X; \nu \otimes \sigma)^{\otimes n} \).

**Proof.** The theorem trivially follows from Lemmas 3.1–3.3. \[ \square \]

**Remark 3.1** Evidently, in the case where (2.3) holds, the unitary \( J \) coincides with the operator \( J \) as in (3.3) up to the unitary transformation connected with the centering of the measure.

### 4 The Nualart–Schoutens chaotic decomposition for a Lévy processes

In this section, we will generalize the result of Nualart and Schoutens [30] (see also [31, Section 5.4]) concerning a chaotic decomposition for a usual Lévy process on \( \mathbb{R} \) to the case of a Lévy process on the manifold \( X \).

We introduce polynomials
\[ P_n(s) = s^n + a_{n,n-1} s^{n-1} + \cdots + a_{n,1} s, \quad a_{n,i} \in \mathbb{R}, \ i = 1, \ldots, n-1, \ n \in \mathbb{N}, \]
in such a way that
\[ \int_{\mathbb{R}} P_n(s) P_m(s) \nu(ds) = 0 \quad \text{if } n \neq m. \]

Using unitary (3.10), we see that
\[ P_n(s) = \tilde{P}_{n-1}(s), \quad (\tilde{P}_n(\cdot))_{n=0}^{\infty} \]
where \( (\tilde{P}_n(\cdot))_{n=0}^{\infty} \) is the system of polynomials with leading coefficient 1 that are orthogonal with respect to the measure \( \hat{\nu}(ds) \) on \( \mathbb{R} \). We also evidently have that \( (P_n(\cdot))_{n=1}^{\infty} \) is a total system in
$L^2(\mathcal{R}; \nu)$. Then, by Theorem 3.3, the system of the random variables consisting of the constants $J^{(0)}(f_0)$, $f_0 \in \mathbb{R}$, and

$J^{(n)}((P_{k_1}(s_1)) \cdots P_{k_n}(s_n)f_n(x_1, \ldots, x_n))$, \hspace{1em} $k_1, \ldots, k_n \in \mathbb{N}$, \hspace{1em} $f_n \in L^2(X; \sigma)^n$, \hspace{1em} $n \in \mathbb{N}$, \hspace{1em} is total in $L^2(D'; \rho_{\nu}, \sigma)$.

Denote by $\mathbb{Z}^\infty_{+} \cdot 0$ the set of all sequences $\alpha$ of the form $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n, 0, 0, \ldots)$, $\alpha_i \in \mathbb{Z}_+$, $n \in \mathbb{N}$. Let $|\alpha| := \sum_{i=1}^{\infty} \alpha_i$, evidently $|\alpha| \in \mathbb{Z}_+$. For $\alpha \in \mathbb{Z}^\infty_{+} \cdot 0$, denote

\[ P_{\alpha}(s_1, \ldots, s_{|\alpha|}) := P_{1}(s_1) \cdots P_{1}(s_{\alpha_1}) P_{2}(s_{\alpha_1+1}) \cdots \]

if $|\alpha| \in \mathbb{N}$, and $P_{\alpha}(s_1, \ldots, s_{|\alpha|}) := 1$ if $|\alpha| = 0$. We then see that the system of the random variables \[ T^{\alpha}(f_\alpha) := J^{(|\alpha|)}((P_{\alpha}(s_1, \ldots, s_{|\alpha|})f_\alpha(x_1, \ldots, x_{|\alpha|}))^{\sim}), \] \hspace{1em} $f_\alpha \in L^2(X; \sigma)^n$, \hspace{1em} $\alpha \in \mathbb{Z}^\infty_{+} \cdot 0$, \hspace{1em} is total in $L^2(D'; \rho_{\nu}, \sigma)$. Furthermore, the $T^{\alpha}(\cdot)$'s are pair-wisely orthogonal in $L^2(D'; \rho_{\nu}, \sigma)$ for different $\alpha$'s.

Let

\[ S_n : L^2(\mathcal{R} \times \nu; \nu \otimes \sigma)^n \to L^2_{\text{sym}}((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^n) = L^2(\mathcal{R} \times \nu \otimes \sigma)^n \]

denote the symmetrization projection. For $\alpha \in \mathbb{Z}^\infty_{+} \cdot 0$, $|\alpha| = n \in \mathbb{N}$, denote by

\[ L^2_\alpha((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^n) \]

the subspace of $L^2(\mathcal{R} \times \nu \otimes \sigma)^n$ consisting of those functions $g_n(s_1, x_1, \ldots, s_n, x_n) \in L^2(\mathcal{R} \times \nu \otimes \sigma)^n$ which satisfy

\[ g_n(s_1, x_1, \ldots, s_n, x_n) = g_n(s_{\pi(1)}, x_{\pi(1)}), \ldots, s_{\pi(n)}, x_{\pi(n)}) \]

for $(\nu \otimes \sigma)^n$-a.e. $(s_1, x_1, \ldots, s_n, x_n) \in (\mathcal{R} \times X)^n$ for any permutation $\pi$ of $\{1, \ldots, n\}$ such that

\[ P_{\alpha}(s_1, \ldots, s_n) = P_{\alpha}(s_{\pi(1)}, \ldots, s_{\pi(n)}) \quad \text{for all} \quad (s_1, \ldots, s_n) \in \mathcal{R}^n. \quad (4.3) \]

Evidently,

\[ L^2_\alpha((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^n) = L^2(\mathcal{R} \times \nu \otimes \sigma)^n \otimes L^2_1((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^n) \otimes \cdots. \]

Let

\[ S_\alpha : L^2(\mathcal{R} \times \nu \otimes \sigma)^n \to L^2_\alpha((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^n) \]

denote the orthogonal projection onto $L^2_\alpha((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^n)$. Since

\[ L^2_{\text{sym}}((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^n) \subset L^2((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^n), \]

we get $S_n = S_n S_\alpha$. Next, for each $f_n \in L^2(X; \sigma)^n$, we get by (4.3)

\[ S_\alpha(P_{\alpha}(s_1, \ldots, s_n)f_n(x_1, \ldots, x_n)) = P_{\alpha}(s_1, \ldots, s_n) (S_\alpha f_n)(x_1, \ldots, x_n), \]

where $S_\alpha$ is the orthogonal projection of $L^2(X; \sigma)^n$ onto the subspace $L^2_\alpha((\mathcal{R} \times X)^n; \sigma^{\otimes n})$ consisting of those functions $f_n \in L^2(X; \sigma)^n$ which satisfy

\[ f_n(x_1, \ldots, x_n) = f_n(x_{\pi(1)}, \ldots, x_{\pi(n)}) \]
for $\sigma^{\otimes n}$-a.e. $(x_1, \ldots, x_n) \in X^n$ for any permutation $\pi$ of $\{1, \ldots, n\}$ fulfilling \[4.3\].
Consider the operator
\begin{equation}
L_\alpha^2(X^n; \sigma^{\otimes n}) \ni f_n \mapsto S_n(P_\alpha(s_1, \ldots, s_n)f_n(x_1, \ldots, x_n)) \in L^n_{\text{sym}}((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^{\otimes n}).
\end{equation}
A direct computation shows that
\begin{equation}
\|S_n(P_\alpha(s_1, \ldots, s_n)f_n(x_1, \ldots, x_n))\|^2_{L^n_{\text{sym}}((\mathcal{R} \times X)^n; (\nu \otimes \sigma)^{\otimes n})} = \frac{\alpha_1! \alpha_2! \cdots}{n!} C_\alpha \|f_n\|^2_{L^2(X^n; \sigma^{\otimes n})},
\end{equation}
where $0! := 1$ and
\begin{equation}
C_\alpha := \|P_1\|_{L_2^2(\mathcal{R}, \nu)}^2 \|P_2\|_{L_2^2(\mathcal{R}, \nu)}^2 \cdots.
\end{equation}
Thus, by virtue of \[3.1\] and the definition of $I^\alpha(\cdot)$, we have the following

**Lemma 4.1** For each $\alpha \in \mathbb{Z}_+^n$, $|\alpha| := n \in \mathbb{N}$, and for each $f_n \in L^2(X; \sigma^{\otimes n})$, we have
\begin{equation}
I^\alpha(f_n) = I^\alpha(S_n f_n).
\end{equation}
Furthermore, the mapping
\begin{equation}
L_\alpha^2(X^n; \sigma^{\otimes n}) \ni f_n \mapsto I^\alpha(f_n) \in L^2(D'; \rho_\nu, \sigma)
\end{equation}
is up to a constant factor an isometry; more exactly, for each $f_n \in L_\alpha^2(X^n; \sigma^{\otimes n})$
\begin{equation}
\|I^\alpha(f_n)\|^2_{L^2(D'; \rho_\nu, \sigma)} = \alpha_1! \alpha_2! \cdots C_\alpha \|f_n\|^2_{L^2(X^n; \sigma^{\otimes n})},
\end{equation}
where $C_\alpha$ is given by \[4.4\].

For each $m \in \mathbb{N}$, we define a random measure $X^{(m)}$ on $X$ by setting, for each $\Delta \in \mathcal{O}_c(X)$,
\begin{equation}
X^{(m)}(\Delta) := \mathcal{M}(X^{(m)}(\Delta)) = J^{(1)}(s^m \chi_\Delta(x)).
\end{equation}
Notice that, if \[2.3\] holds, we have for each $\omega = \sum_n s_n \delta_{x_n} \in \Omega(X)$
\begin{equation}
X^{(m)}(\Delta, \omega) = \left\langle \sum_n s_n \delta_{x_n}, \chi_\Delta \right\rangle - \int_X s^m \nu(ds) \sigma(\Delta).
\end{equation}
In \[30, 31\], in the case $X = \mathbb{R}$, the $X^{(m)}$, $m \in \mathbb{N}$, was called a Tengels martingale of order $m$. Let
\begin{equation}
Y^{(m)}(\Delta) := J^{(1)}(P_m(s) \chi_\Delta(x)), \quad \Delta \in \mathcal{O}_c(X).
\end{equation}
By \[1.1\],
\begin{equation}
Y^{(m)}(\Delta) = X^{(m)}(\Delta) + a_{m, m-1} X^{(m-1)}(\Delta) + \cdots + a_{m, 1} X^{(1)}(\Delta).
\end{equation}

**Lemma 4.2** Let $\alpha \in \mathbb{Z}_+^n$, $|\alpha| := n \in \mathbb{N}$. For each $f_n \in L^2(X^n; \sigma^{\otimes n})$, $I^\alpha(f_n)$ coincides with the multiple stochastic integral
\begin{equation}
\int_{X^n} f_n(x_1, \ldots, x_n) \, dY^{(1)}(x_1) \cdots dY^{(1)}(x_{\alpha_1}) \, dY^{(2)}(x_{\alpha_1+1}) \cdots dY^{(2)}(x_{\alpha_1+\alpha_2}) \cdots.
\end{equation}
Proof. For any disjoint $\Delta_1, \ldots, \Delta_n \in \mathcal{O}_c(X)$, we have by the definition of a multiple stochastic integral
\[
\int_{X^n} \chi_{\Delta_1}(x_1) \cdots \chi_{\Delta_n}(x_n) \, d\mathcal{Y}^{(1)}(x_1) \cdots d\mathcal{Y}^{(1)}(x_{\alpha_1}) \, d\mathcal{Y}^{(2)}(x_{\alpha_1+1}) \cdots d\mathcal{Y}^{(2)}(x_{\alpha_1+\alpha_2}) \cdots = \mathcal{J}^{(1)}(\Delta_1) \cdots \mathcal{J}^{(1)}(\Delta_{\alpha_1}) \mathcal{J}^{(2)}(\Delta_{\alpha_1+1}) \cdots \mathcal{J}^{(2)}(\Delta_{\alpha_1+\alpha_2}) \cdots .
\] (4.7)
It follows from Theorem 3.1 and the construction of a multiple stochastic integral with respect to the centered Poisson process that
\[
\mathcal{J}^{(1)}(\Delta_1) \cdots \mathcal{J}^{(1)}(\Delta_{\alpha_1}) \mathcal{J}^{(2)}(\Delta_{\alpha_1+1}) \cdots \mathcal{J}^{(2)}(\Delta_{\alpha_1+\alpha_2}) \cdots = I^{(1)}(P_1(s)\chi_{\Delta_1}(x)) \cdots I^{(1)}(P_1(s)\chi_{\Delta_{\alpha_1}}(x))
\times I^{(1)}(P_2(s)\chi_{\Delta_{\alpha_1+1}}(x)) \cdots I^{(1)}(P_2(s)\chi_{\Delta_{\alpha_1+\alpha_2}}(x)) \cdots
\times I^{(n)}((P_n(s_1, \ldots, s_n)\chi_{\Delta_1}(x_1) \cdots \chi_{\Delta_n}(x_n))').
\] Therefore,
\[
\mathcal{J}^{(1)}(\Delta_1) \cdots \mathcal{J}^{(1)}(\Delta_{\alpha_1}) \mathcal{J}^{(2)}(\Delta_{\alpha_1+1}) \cdots \mathcal{J}^{(2)}(\Delta_{\alpha_1+\alpha_2}) \cdots = I^n(\chi_{\Delta_1} \otimes \cdots \otimes \chi_{\Delta_n}).
\] (4.8)
By (4.7) and (4.8), we have proved the statement for $f_n = \chi_{\Delta_1} \otimes \cdots \otimes \chi_{\Delta_n}$. Since the set of linear combinations of the functions $\chi_{\Delta_1} \otimes \cdots \otimes \chi_{\Delta_n}$ with $\Delta_1, \ldots, \Delta_n \in \mathcal{O}_c(X)$ disjoint is dense in $L^2(X; \sigma)^\otimes n$, we get the statement by the linearity and continuity of the mapping
\[
L^2(X; \sigma)^\otimes n \ni f_n \mapsto I^n(f_n) \in L^2(D'; P_\nu, \sigma)
\]
(see Lemma 4.1). ■

In what follows, we set $L^2_\alpha(X^{[\alpha]}; \sigma^{[\alpha]}) := \mathbb{R}$ for $|\alpha| = 0$.

Theorem 4.1 Each $F \in L^2(D'; P_\nu, \sigma)$ may be uniquely represented as a sum of multiple stochastic integrals
\[
F = \sum_{\alpha \in \mathbb{Z}^n_+} I^n(f_\alpha), \quad f_\alpha \in L^2_\alpha(X^{[\alpha]}; \sigma^{[\alpha]}),
\]
and
\[
I^n(f_\alpha) = \int_{X^{[\alpha]}} f_\alpha(x_1, \ldots, x_{|\alpha|}) \, d\mathcal{Y}^{(1)}(x_1) \cdots d\mathcal{Y}^{(1)}(x_{\alpha_1}) \, d\mathcal{Y}^{(2)}(x_{\alpha_1+1}) \cdots d\mathcal{Y}^{(2)}(x_{\alpha_1+\alpha_2}) \cdots
\]
for $|\alpha| \in \mathbb{N}$ and $I^n(f_\alpha) = f_\alpha$ for $|\alpha| = 0$. Furthermore,
\[
\|F\|_{L^2(D'; P_\nu, \sigma)}^2 = \sum_{\alpha \in \mathbb{Z}^n_+} \alpha_1! \alpha_2! \cdots C_\alpha \|f_\alpha\|_{L^2(X^{[\alpha]}; \sigma^{[\alpha]})}^2,
\]
where $C_\alpha$ is given by (4.3).

Proof. The statement follows by Theorem 3.1 and Lemmas 4.1, 4.2. ■

We define a Hilbert space
\[
H := \bigoplus_{\alpha \in \mathbb{Z}^n_+} H_\alpha, \quad H_\alpha := L^2_\alpha(X^{[\alpha]}; \sigma^{[\alpha]}) \alpha_1! \alpha_2! \cdots C_\alpha.
\] (4.9)
As a trivial consequence of Theorem 4.1, we get
Corollary 4.1 We have the unitary operator

\[ H \ni f = (f_\alpha)_{\alpha \in \mathbb{Z}_+^\infty} \rightarrow \mathcal{I}f := \sum_{\alpha \in \mathbb{Z}_+^\infty} \mathcal{I}^\alpha(f_\alpha) \in L^2(D'; \rho_{\nu, \sigma}). \]

For each \( \varphi \in D \), we have \( \langle \cdot, \varphi \rangle = \mathcal{I}^{(1,0,\ldots)}(\varphi) \) \( \rho_{\nu, \sigma} \)-a.e., and hence, we define \( \langle \cdot, \varphi \rangle \in L^2(D'; \rho_{\nu, \sigma}) \) for any \( \varphi \in L^2(X; \sigma) \) as \( \mathcal{I}^{(1,0,\ldots)}(\varphi) \). We will now obtain a formula for the multiplication of a multiple stochastic integral \( \mathcal{I}^\alpha(f_\alpha), f_\alpha \in L^2_2(X^{[\alpha]}; \sigma^{\otimes[\alpha]}) \), by a random variable \( \langle \cdot, \varphi \rangle \) (compare with formulas (4.3)–(3.8) in the Poisson case).

By the Favard theorem, the system of orthogonal polynomials \( (\tilde{P}_n)_{n=0}^{\infty} \) fulfills the recurrence formula

\[ s\tilde{P}_n(s) = \tilde{P}_{n+1}(s) + a_n \tilde{P}_n(s) + b_n \tilde{P}_{n-1}(s), \quad n \in \mathbb{Z}_+, \quad \tilde{P}_0(s) := 0, \]

with real numbers \( a_n \) and positive numbers \( b_n \). Using unitary (3.10), we then get

\[ sP_n(s) = P_{n+1}(s) + a_{n-1}P_n(s) + b_{n-1}P_{n-1}(s), \quad n \in \mathbb{N}, \quad P_0(s) := 0. \]  

(4.10)

For \( \alpha \in \mathbb{Z}_+^\infty \) and \( n \in \mathbb{N} \), we denote

\[ \alpha \pm 1_n := (\alpha_1, \ldots, \alpha_{n-1}, \alpha_n \pm 1, \alpha_{n+1}, \ldots), \]

and let \( \mathcal{I}(f_\alpha) := 0 \) if \( \alpha_n < 0 \) for some \( n \in \mathbb{N} \).

Corollary 4.2 Let \( \varphi \in L^1(X; \sigma) \cap L^\infty(X; \sigma) \). Then, for any \( \alpha \in \mathbb{Z}_+^\infty \), \( |\alpha| = n \), and any \( f_\alpha \in L^2_2(X; \sigma)^{\otimes n} \), we have

\[
\langle \cdot, \varphi \rangle \mathcal{I}^\alpha(f_\alpha) = \mathcal{I}^{\alpha+1}(S_{\alpha+1}(\varphi(x_1)f_\alpha(x_2, \ldots, x_{n+1})) \\
+ \frac{\alpha!}{(n-1)!} \int_\mathbb{R} s^2 \nu(ds) \mathcal{I}^{\alpha-1}(S_{\alpha-1}(\varphi(x)f_\alpha(x, x_1, \ldots, x_{n-1}) \sigma(dx))) \\
+ \sum_{n \geq 1} \alpha_n \left[ \mathcal{I}^{\alpha-1}(S_{\alpha-1}(\varphi(x_{\alpha_1} + \ldots + \alpha_n)f_\alpha(x_1, \ldots, x_n)) \\
+ a_{n-1} \mathcal{I}^\alpha(S_{\alpha}(\varphi(x_{\alpha_1} + \ldots + \alpha_n)f_\alpha(x_1, \ldots, x_n))) \\
+ b_{n-1} \mathcal{I}^{\alpha-1}(S_{\alpha+1-1_n}(\varphi(x_{\alpha_1} + \ldots + \alpha_n)f_\alpha(x_1, \ldots, x_n))) \right].
\]

(4.11)

Proof. The corollary easily follows from the definition of \( \mathcal{I}^\alpha(\cdot) \), Lemma 4.1, (3.3)–(3.8), and (4.10).

\[ \blacksquare \]

5 Orthogonalization of continuous polynomials

We denote by \( \mathcal{P}(D') \) the set of continuous polynomials on \( D' \), i.e., functions on \( D' \) of the form

\[ F(\omega) = \sum_{i=0}^n \langle \omega^{\otimes i}, f_i \rangle, \quad \omega^{\otimes 0} := 1, \quad f_i \in D^{\otimes i}, \quad i = 0, \ldots, n, \quad n \in \mathbb{Z}_+. \]

The greatest number \( i \) for which \( f^{(i)} \neq 0 \) is called the power of a polynomial. We evidently have \( \mathcal{P}_{cyl}(D') \subset \mathcal{P}(D') \). We denote by \( \mathcal{P}_n(D') \) the set of continuous polynomials of power \( \leq n \).

By [33] Sect. 11, \( \mathcal{P}(D') \) is a dense subset of \( L^2(D'; \rho_{\nu, \sigma}) \). Let \( \mathcal{P}_n^-(D') \) denote the closure of \( \mathcal{P}_n(D') \) in \( L^2(D'; \rho_{\nu, \sigma}) \), let \( \mathcal{P}_n(D') \), \( n \in \mathbb{N} \), denote the orthogonal difference \( \mathcal{P}_n^-(D') \subset \mathcal{P}_{n-1}^-(D') \), and let \( \mathcal{P}_0(D') := \mathcal{P}_0^-(D') \).
Theorem 5.1 We have the orthogonal decomposition

\[ L^2(D'; \rho_{\nu, \sigma}) = \bigoplus_{n=0}^{\infty} P_n(D'), \]  

and furthermore,

\[ P_n(D') = I_H_n, \]

where

\[ H_n := \bigoplus_{\alpha \in \mathbb{Z}_{+}^{\infty} : 1\alpha_1 + 2\alpha_2 + \cdots \leq n} H_{\alpha}, \quad n \in \mathbb{Z}_+. \]

Proof. The orthogonal decomposition (5.1) is clear, so we have to prove (5.2), (5.3), or equivalently

\[ P_{\sim n}(D') = \bigoplus_{i=0}^{n} I_{H_i}, \quad n \in \mathbb{Z}_+. \]

Fix any \( \varphi \in D \) and consider the continuous polynomial \( \langle \cdot, \varphi \rangle_n \). Since \( J^{-1}(\langle \cdot, \varphi \rangle_n) = A(s\varphi(x))^n\Omega \), we conclude, using formulas (3.5)–(3.8), that

\[ \langle \cdot, \varphi \rangle_n \in \bigoplus_{i=0}^{n} I_{H_i}. \]

Since the Laplace transform of the measure \( \rho_{\nu, \sigma} \) is analytic in a neighborhood of zero in \( D_C \), by [23, Lemma 3.9], there exist \( \tau \in T \) and \( \varepsilon_\tau > 0 \) such that

\[ \int_{D'} \exp \left( \varepsilon_\tau \|\omega\|_\tau \right) \rho_{\nu, \sigma}(d\omega) < \infty, \]

where \( \| \cdot \|_\tau \) denotes the norm in the Hilbert space \( H_\tau \). Therefore,

\[ \int_{D'} \|\omega\|_\tau^n \rho_{\nu, \sigma}(d\omega) < \infty, \quad n \in \mathbb{N}, \]

which yields the continuity of the mapping

\[ H_\tau^{\otimes n} \ni f_n \mapsto \langle \cdot \otimes_n, f_n \rangle \in L^2(D'; \rho_{\nu, \sigma}) \]

for each \( n \in \mathbb{N} \). Therefore, any polynomial \( \langle \cdot \otimes_n, f_n \rangle, f_n \in D^{\otimes n} \), can be approximated in the \( L^2(D'; \rho_{\nu, \sigma}) \) norm by linear combinations of polynomials of the form \( \langle \cdot, \varphi \rangle_n, \varphi \in D \). Since \( \bigoplus_{i=0}^{n} I_{H_i} \) is a linear closed subspace of \( L^2(D'; \rho_{\nu, \sigma}) \), (5.5) implies the inclusion \( P_{\sim n}(D') \subset \bigoplus_{i=0}^{n} I_{H_i} \).

Let us prove the inverse inclusion \( \bigoplus_{i=0}^{n} I_{H_i} \subset P_{\sim n}(D') \). It suffices to show that, for each \( f_m \in L^2(X; \sigma)^{\otimes m}, m \in \{1, \ldots, n\} \),

\[ J^{(m)}((s_1^{k_1} \cdots s_m^{k_m} f_m(x_1, \ldots, x_m))^-) \in P_{\sim n}(D'), \quad k_1, \ldots, k_m \in \mathbb{N}, \quad k_1 + \cdots + k_m = n. \]

Analogously to the proof of Lemma 4.2, we get

\[ J^{(m)}((s_1^{k_1} \cdots s_m^{k_m} f_m(x_1, \ldots, x_m))^-) = \int_{X^m} f_m(x_1, \ldots, x_m) dX^{(k_1)}(x_1) \cdots dX^{(k_m)}(x_m), \]
By using [26, Theorem 2] or [34], we express the right hand side of (5.12) as a sum of multiple

\[ \sum_{\Delta_1, \ldots, \Delta_m \in \mathcal{O}_c(X)} X^{(k_1)}(\Delta_1) \cdots X^{(k_m)}(\Delta_m) \in \mathcal{D}_n^{-m}(\mathcal{D}'), \quad k_1 + \cdots + k_m = n. \]  \hspace{1cm} (5.7)

But it follows from Theorem 3.1 and the Claim from the proof of Lemma 3.3 that, for any \( m, n \in \mathbb{N} \) and any \( \Delta \in \mathcal{O}_c(X) \),

\[ X^{(n)}(\Delta) \in \mathcal{D}_n^{-m}(\mathcal{D}'), \]  \hspace{1cm} (5.8)

where \( \mathcal{D}_n^{-m}(\mathcal{D}') \) denotes the closure of \( \mathcal{P}_n(\mathcal{D}') \) in \( L^{2m}(\mathcal{D}'; \rho_{\nu, \sigma}) \). Finally, (5.8) implies (5.7). \( \blacksquare \)

For a monomial \( \langle \cdot \otimes n, f_n \rangle, f_n \in \mathcal{D}^{\otimes n} \), we denote by \( \langle \cdot \otimes n, f_n \rangle \) the orthogonal projection of \( \langle \cdot \otimes n, f_n \rangle \) onto \( \mathcal{P}_n(\mathcal{D}') \). Since for each monomial \( \langle \cdot \otimes k, f_k \rangle, f_k \in \mathcal{D}^{\otimes k} \), with \( k < n \), the projection of \( \langle \cdot \otimes k, f_k \rangle \) onto \( \mathcal{P}_n(\mathcal{D}') \) equals zero, the set \( \{ \langle \cdot \otimes n, f_n \rangle : f_n \in \mathcal{D}^{\otimes n} \} \) is dense in \( \mathcal{P}_n(\mathcal{D}') \). Our next aim is to find an explicit representation of \( \langle \cdot \otimes n, f_n \rangle \) through multiple stochastic integrals \( I(\cdot) \)’s.

For each \( \alpha \in \mathbb{Z}_+^\infty, 1\alpha_1 + 2\alpha_2 + \cdots = n, n \in \mathbb{N} \), and for any function \( f_n : X^n \to \mathbb{R} \) we define a function \( D_\alpha f_n : X^{[\alpha]} \to \mathbb{R} \) by setting

\[ (D_\alpha f_n)(x_1, \ldots, x_\alpha) := f(x_1, \ldots, x_{\alpha_1}, x_{\alpha_1+1}, x_{\alpha_1+1}, x_{\alpha_1+2}, x_{\alpha_1+2}, \ldots, x_{\alpha_1+\alpha_2}, x_{\alpha_1+\alpha_2}, \ldots). \]  \hspace{1cm} (5.9)

**Corollary 5.1** For each \( f_n \in \mathcal{D}^{\otimes n}, n \in \mathbb{N} \), we have

\[ \langle \cdot \otimes n, f_n \rangle := \sum_{\alpha \in \mathbb{Z}_+^\infty : 1\alpha_1 + 2\alpha_2 + \cdots = n} R_\alpha I(\alpha) D_\alpha f_n. \]  \hspace{1cm} (5.10)

where

\[ R_\alpha = \frac{n!}{\alpha_1! (1!)^{\alpha_1} \alpha_2! (2!)^{\alpha_2} \alpha_3! (3!)^{\alpha_3} \cdots}. \]  \hspace{1cm} (5.11)

**Remark 5.1** The \( R_\alpha \) given by (5.11) describes the number of all possible partitions of a set consisting of \( n \) elements into \( \alpha_1 \) sets containing one element, \( \alpha_2 \) sets containing two elements, and so forth.

**Proof.** Suppose that \( f_n = \varphi^{\otimes n}, \varphi \in \mathcal{D} \). We have

\[ \mathcal{U}^{-1}(\langle \cdot, \varphi \rangle^n) = I^{(1)}(s\varphi(x))^n. \]  \hspace{1cm} (5.12)

By using [26, Theorem 2] or [34], we express the right hand side of (5.12) as a sum of multiple stochastic integrals with respect to the centered Poisson process:

\[ I^{(1)}(s\varphi(x))^n = \sum_{\alpha \in \mathbb{Z}_+^\infty : 1\alpha_1 + 2\alpha_2 + \cdots = n} R_\alpha I^{[\alpha]}(s_1 \cdots s_\alpha) {\varphi}^{2 \alpha_1 + 1} \cdots {\varphi}^{2 \alpha_2} \cdots (D_\alpha \varphi^{\otimes n})(x_1, \ldots, x_\alpha) + G_\alpha, \]  \hspace{1cm} (5.13)
where
\[ G_{\alpha} \in \bigoplus_{\alpha \in \mathbb{Z}_+^\infty : 1\alpha_1 + 2\alpha_2 + \cdots \leq n - 1} U^{-1}T_{\alpha}. \] (5.14)

By (4.12), \( P_k(s) - s^k \) is a polynomial of order \( k - 1 \), which yields by (5.13),
\[ I^{(1)}(s\varphi(x))^n = \sum_{\alpha \in \mathbb{Z}_+^\infty : 1\alpha_1 + 2\alpha_2 + \cdots = n} R_{\alpha} I^{(|\alpha|)} ((P_\alpha(s_1, \ldots, s_{|\alpha|})(D_\alpha \varphi^{\otimes n})(x_1, \ldots, x_{|\alpha|}))^\sim) + G'_{\alpha}, \] (5.15)

where \( G'_{\alpha} \) also belongs to the space from (5.14). By (5.12), (5.15), and Theorem 5.1, we get (5.10) for \( f_n = \varphi^{\otimes n} \).

Next, by the continuity of the mapping (5.6), we conclude that each mapping
\[ \mathcal{H}_n \ni f_n \mapsto \langle \cdot^{\otimes n}, f_n \rangle_{L^2(\mathcal{D}')} \] (5.16)
is also continuous. Let us fix \( n \in \mathbb{N} \). Without loss of generality, we can suppose that, for all \( \alpha \in \mathbb{Z}_+^\infty : 1\alpha_1 + 2\alpha_2 + \cdots = n \), the mapping
\[ \mathcal{H}_n \ni f_n \mapsto D_\alpha f_n \in L^2(X; \sigma^{\otimes |\alpha|}) \] is continuous. Now, the formula (5.10) in the general case follows by an approximation of \( f_n \in \mathcal{D}^{\otimes n} \) by linear combinations of functions of the form \( \varphi^{\otimes n}, \varphi \in \mathcal{D} \), in the \( \mathcal{H}_n^{\otimes n} \) norm. \qed

**Corollary 5.2** For any \( f_n, g_n \in \mathcal{D}^{\otimes n}, n \in \mathbb{N}, \) we have
\[ \int_{\mathcal{D}'} \langle \omega^{\otimes n}, f_n \rangle : \langle \omega^{\otimes n}, g_n \rangle : L^2(\rho\nu, \sigma)(d\omega) = n! \sum_{\alpha \in \mathbb{Z}_+^\infty : 1\alpha_1 + 2\alpha_2 + \cdots = n} K_{\alpha} \int_{X'||\alpha|} (D_\alpha f_n(x_1, \ldots, x_{|\alpha|})) \times (D_\alpha g_n(x_1, \ldots, x_{|\alpha|}) \sigma^{\otimes |\alpha|}(dx_1, \ldots, dx_{|\alpha|}), (5.17) \]
where
\[ K_{\alpha} = \frac{n!}{\alpha_1! \alpha_2! \cdots \prod_{k \geq 1} \left( \frac{\| P_k \|_{L^2(\mathcal{R}_\nu)}}{k!} \right)^{2\alpha_k}}. \] (5.18)

**Proof.** The corollary follows directly from Theorem 4.1 and Corollary 5.1. \qed

Analogously to (5.3), we define, for each \( n \in \mathbb{Z}_+ \), a Hilbert space
\[ \mathcal{F}_n := \bigoplus_{\alpha \in \mathbb{Z}_+^\infty : 1\alpha_1 + 2\alpha_2 + \cdots = n} \mathcal{F}_{n, \alpha}, \quad \mathcal{F}_{n, \alpha} := L^2(X'||\alpha|; \sigma^{\otimes |\alpha|}) K_{\alpha}. \] (5.19)
In this section, we consider an application of the above results to the processes of Meixner’s type.

### 6 Processes of Meixner’s type

For each \( \lambda \geq 0 \), let \( \tilde{P}_\lambda \) denote the probability measure on \( (\mathbb{R}, \mathcal{B}(\mathbb{R})) \) whose orthogonal polynomials \( (\tilde{P}_\lambda)_n \) with leading coefficient 1 satisfy the recurrence relation

\[
s\tilde{P}_\lambda, n(s) = \tilde{P}_\lambda, n+1(s) + \lambda(n+1)\tilde{P}_\lambda, n(s) + n(n+1)\tilde{P}_\lambda, n-1(s),
\]

where \( \cdot_n \) denotes the \( \mathfrak{F}_n, \alpha \)-component of an element of \( \mathfrak{F}_n \). Since \( \{\langle \cdot_n, f_n \rangle; f_n \in D_n \} \), is a dense subset of \( P_n(D') \), we get by Theorem 5.1 and Corollaries 5.1 and 5.2 that \( \mathcal{E}_n D_n \) is a dense subset of \( \mathfrak{F}_n \). In what follows, for simplicity of notations, we will just write \( f_n = g_n \) in \( D_n \) if and only if \( f_n = g_n \) in \( \mathfrak{F}_n \).

Let

\[
\mathfrak{F} := \bigoplus_{n=0}^{\infty} \mathfrak{F}_n n!, \quad \mathfrak{F}_0 := \mathbb{R}.
\]

There evidently exists a unitary operator \( \mathcal{A} : H \to \mathfrak{F} \) which acts between any \( \mathbf{H}_\alpha \)- and \( \mathfrak{F}_\alpha \)-component of the space \( H \), respectively \( \mathfrak{F} \) as a constant (depending on \( \alpha \)) times the identity operator.

We evidently have the following

**Corollary 5.3** We have the unitary operator

\[
\mathfrak{U} : \mathfrak{F} \to L^2(D'; \rho_{\nu, \sigma})
\]

that is defined through

\[
\mathfrak{U} f_n := \langle \cdot_n, f_n \rangle; \quad f_n \in D_n, \quad n \in \mathbb{Z}_+,
\]

and then extended by linearity and continuity to the whole space \( \mathfrak{F} \). Furthermore,

\[
\mathfrak{U} = IA^{-1}.
\]

**Remark 5.2** Using Theorem 6.1 and Corollaries 1.2, 3.1 one can explicitly identify the Jacobi filed of the measure \( \rho_{\nu, \sigma} \), that is, the family of commuting selfadjoint operators \( a(\varphi), \varphi \in D \), acting in the Hilbert space \( \mathfrak{F} \) and satisfying

\[
\mathfrak{U} a(\varphi) \mathfrak{U}^{-1} = \langle \cdot, \varphi \rangle, \quad \varphi \in D,
\]

where \( \langle \cdot, \varphi \rangle \cdot \) denotes the operator of multiplication by \( \langle \cdot, \varphi \rangle \) in \( L^2(D'; \rho_{\nu, \sigma}) \). As easily seen, for each \( f_n \in D_n \), we have

\[
a(\varphi) f_n = a^+ (\varphi) f_n + a^0 (\varphi) f_n + a^- (\varphi) f_n,
\]

where \( a^+(\varphi) f_n \in \mathfrak{F}_{n+1}, a^0(\varphi) f_n \in \mathfrak{F}_n \), and \( a^-(\varphi) f_n \in \mathfrak{F}_{n-1} \). Though we always have \( a^+(\varphi) f_n = a^-(\varphi) f_n \), \( a^+ \) is a usual creation operator, the structure of the neutral operator \( a^0(\varphi) \) and the annihilation operator \( a^-(\varphi) \) is, in general, quite complicated. In the next section, we will consider a family of Lévy processes for which these operators have a much simpler form.

**Remark 5.3** Let us suppose that (2.3) holds. For each \( n \in \mathbb{N} \), we now define the random measure \( Y^{(n)} \) on \( X \) by setting, for each \( \Delta \in \mathcal{O}_c(X) \),

\[
Y^{(n)} (\Delta) := J^{(1)} (P_n(\mu_\Delta)(x)).
\]

Then, as easily seen, all the results of Sections 4 and 5 remain true if we change \( L^2(D'; \rho_{\nu, \sigma}) \) for the space \( L^2(\Omega(X); \mu_{\nu, \sigma}) \) and, in the formulation of Corollary 4.2 and Remark 5.2, \( \langle \cdot, \varphi \rangle \) for \( \langle \cdot, \varphi \rangle \cdot \).
By [28] (see also [12] Ch. VI, sect. 3), \((\tilde{P}_\lambda)_{n=0}^\infty\) is a system of polynomials of Meixner’s type, the measure \(\tilde{\nu}_\lambda\) is uniquely determined by the above condition and is given as follows: For \(\lambda \in [0, 2)\),

\[
\tilde{\nu}_\lambda(ds) = \frac{\sqrt{4-\lambda^2}}{2\pi} \times |\Gamma(1+i(4-\lambda^2)^{-1/2}s)|^2 \exp \left[ -s2(4-\lambda^2)^{-1/2} \arctan \left( \frac{\lambda - \lambda^2}{4} \right) \right] ds
\]

(\(\tilde{\nu}_\lambda\) is a Meixner distribution), for \(\lambda = 2\)

\[
\tilde{\nu}_2(ds) = \chi_{(0,\infty)}(s)e^{-s}ds
\]

(\(\tilde{\nu}_2\) is a gamma distribution), and for \(\lambda > 2\)

\[
\tilde{\nu}_\lambda(ds) = (\lambda^2 - 4) \sum_{k=1}^\infty P^k_{\lambda}k\delta_{\sqrt{s^2-4}k}, \quad p_\lambda = \frac{\lambda - \sqrt{\lambda^2 - 4}}{\lambda + \sqrt{\lambda^2 - 4}}
\]

(\(\tilde{\nu}_\lambda\) is now a Pascal distribution).

Since \(\tilde{\nu}_\lambda(\{0\}) = 0\) for each \(\lambda \geq 0\), we can define a Lévy measure \(\nu_\lambda\) by setting

\[
\nu_\lambda(ds) = \frac{1}{s^2} \tilde{\nu}_\lambda(ds), \quad \lambda \geq 0.
\]

The integral \(\int_\mathbb{R} |s| \nu_\lambda(ds)\) is finite for \(\lambda \geq 2\) and infinite for \(\lambda \in [0, 2)\). Therefore, we will consider, for each \(\lambda \geq 2\), the Lévy process with law \(\tilde{\varrho}_\lambda:=\mu_{\nu_\lambda,\sigma}\), and for \(\lambda \in [0, 2)\) the centered Lévy process with law \(\varrho_\lambda:=\rho_{\nu_\lambda,\sigma}\).

It follows from (6.1) that, for each \(\lambda \geq 0\),

\[
\|\tilde{P}_{\lambda,n}\|^2_{L^2(\mathbb{R};\tilde{\nu}_\lambda)} = n!(n+1)!, \quad n \in \mathbb{Z}_+,
\]

and hence

\[
\|P_{\lambda,n}\|^2_{L^2(\mathbb{R};\nu_\lambda)} = (n-1)!n!, \quad n \in \mathbb{N},
\]

where \(P_{\lambda,n}(s) := \tilde{P}_{\lambda,n-1}(s)s\). By (5.18) and (6.2), we conclude that, for each \(\alpha \in \mathbb{Z}_{\geq 0}^\infty, \alpha_1+2\alpha_2+\cdots = n\),

\[
K_\alpha = \frac{n!}{\alpha_1!\alpha_2!\alpha_2!\cdots}.
\]

**Remark 6.1** Let us give a combinatoric interpretation of the number \(K_\alpha\) in (6.3). Under a loop \(\kappa\) connecting points \(x_1, \ldots, x_m, m \geq 2\), we understand a class of ordered sets \((x_{\pi(1)}, \ldots, x_{\pi(m)})\), where \(\pi\) is a permutation of \(\{1, \ldots, m\}\), which coincide up to a cyclic permutation. Let us also interpret a set \(\{x\}\) as a “one-point” loop \(\kappa\), i.e., a loop that comes out of \(x\). Let \(\vartheta_n = \{\kappa_1, \ldots, \kappa_{|\vartheta_n|}\}\) be a collection of \(|\vartheta_n|\) loops \(\kappa_j\) that connect points from the set \(\{x_1, \ldots, x_n\}\) so that every point \(x_i \in \{x_1, \ldots, x_n\}\) goes into one loop \(\kappa_j = \kappa_{j(i)}\) from \(\vartheta_n\). Then, for \(\alpha \in \mathbb{Z}_{\geq 0}^\infty, \alpha_1+2\alpha_2+\cdots = n\), \(K_\alpha\) is the number of all different collections of loops connecting points from the set \(\{x_1, \ldots, x_n\}\) and containing \(\alpha_1\) one-point loops, \(\alpha_2\) two-point loops, etc.

In the following proposition, we will explicitly identify the Jacobi field \(a_\lambda(\varphi), \varphi \in \mathcal{D}\), of the measure \(\varrho_\lambda\) (see Remarks 5.2, 5.3).
Proposition 6.1 For each \( \lambda \geq 0 \), we have for all \( \varphi \in \mathcal{D} \) and all \( f_n \in \mathcal{D}^{\otimes n} \), \( n \in \mathbb{Z}_+ \),
\[
a_\lambda(\varphi)f_n = a^+(\varphi)f_n + \lambda a^0(\varphi)f_n + a^-(\varphi)f_n.
\]
Here, \( a^+(\xi) \) is the standard creation operator:
\[
a^+(\varphi)f_n := \varphi \hat{\otimes} f_n,
\]
a\(^0(\varphi) \) is the standard neutral operator:
\[
a^0(\varphi) f_n(x_1, \ldots, x_n) = (\varphi(x_1) + \cdots + \varphi(x_n)) f_n(x_1, \ldots, x_n), \quad n \in \mathbb{N}, \ a^0(\varphi)f_0 = 0,
\]
and \( a^-(\varphi) = a_1^-(\varphi) + a_2^{-}(\varphi) \), where \( a_1^-(\varphi) \) is the standard annihilation operator:
\[
(a_1^-(\varphi)f_n)(x_1, \ldots, x_{n-1}) = n \int_X \varphi(x) f_n(x_1, \ldots, x_{n-1}) \sigma(dx), \quad n \in \mathbb{N}, \ a^0(\varphi)f_0 = 0,
\]
and
\[
(a_2^{-}(\varphi)f_n)(x_1, \ldots, x_{n-1}) = n(n-1)(\varphi(x_1)f_n(x_1, x_2, x_3, \ldots, x_{n-1}))^+, \quad n \geq 2,\ a_2^{-}(\varphi)f_0 = 0, \quad i = 0, 1.
\]

Proof. The proposition easily follows in the way described in Remark 5.2 (see also Remark 5.3) through formula (6.1). \( \blacksquare \)

Corollary 6.1 For each \( \lambda \geq 0 \) and \( \omega \in \mathcal{D}' \), define \( \omega^{\otimes n; \lambda} \in \mathcal{D}'^{\otimes n} \) by the recurrence formula
\[
\omega^{\otimes (n+1); \lambda} = \omega^{\otimes n; \lambda}; (x_1, \ldots, x_{n+1}) = \left(\omega^{\otimes n; \lambda}(x_1, \ldots, x_n) \omega(x_{n+1})\right)^+, \quad n \in \mathbb{N},
\]
\[
= - n \left(\omega^{\otimes (n-1); \lambda}(x_1, \ldots, x_{n-1}) \delta(x_{n+1} - x_n)\right)^+ \quad - n(n-1) \left(\omega^{\otimes (n-1); \lambda}(x_1, \ldots, x_n) \delta(x_{n+1} - x_n) \delta(x_{n-1} - x_n)\right)^+ \quad - \lambda n \left(\omega^{\otimes n; \lambda}(x_1, \ldots, x_n) \delta(x_{n+1} - x_n)\right)^+ - c_\lambda \left(\omega^{\otimes n; \lambda}(x_1, \ldots, x_n) 1(x_{n+1})\right)^+ \quad - \omega^{\otimes 0; \lambda} = 1, \quad \omega^{\otimes 1; \lambda} = \omega - c_\lambda, \quad (6.4)
\]
where \( c_\lambda := 0 \) for \( \lambda \in [0, 2) \) and \( c_\lambda := 2/(\lambda + \sqrt{\lambda^2 - 4}) \) for \( \lambda \geq 2 \). Then, for each \( f_n \in \mathcal{D}^{\otimes n} \), \( n \in \mathbb{N} \),
\[
\langle \omega^{\otimes n}, f_n \rangle = \langle \omega^{\otimes n; \lambda}, f_n \rangle, \quad \forall \lambda \text{-a.e. } \omega \in \mathcal{D}'.
\]

Proof. Since \( \int_\mathbb{R} s \nu_\lambda(ds) = c_\lambda \) for each \( \lambda \geq 2 \), the statement trivially holds for \( n = 1 \). Suppose the statement holds for all \( n \leq m \) and let us prove it for \( n = m + 1 \). By Proposition 6.1, we then get for any sequence \( \{ \varphi_i, i \in \mathbb{N} \} \subset \mathcal{D} \)
\[
\langle \omega^{\otimes (m+1)}, \varphi_i^{\otimes (m+1)} \rangle = \langle \omega^{\otimes (m+1); \lambda}, \varphi_i^{\otimes (m+1)} \rangle, \quad i \in \mathbb{N}, \quad \forall \lambda \text{-a.e. } \omega \in \mathcal{D}'.
\]
There exist \( \tau_1, \tau_2 \in T, \tau_2 > \tau_1 \), such that \( \theta_\lambda(H_{\tau_1}) = 1 \), for each \( n \in \{2, \ldots, m+1\} \) the mapping \( (5.16) \) with \( \tau = \tau_2 \) is continuous, and for each \( \omega \in H_{\tau_1} \omega^{\otimes (m+1); \lambda} \in H_{\tau_2}^{\otimes (m+1)} \). Choose \( \{ \varphi_i, i \in \mathbb{N} \} \subset \mathcal{D} \) which is a total set in \( H_{\tau_2} \). Approximate an arbitrary \( f_{m+1}^{(k)} \in \mathcal{D}^{\otimes (m+1)} \) in the \( H_{\tau_2}^{\otimes (m+1)} \) topology by a sequence \( \{ f_{m+1}^{(k)} \}, k \in \mathbb{N} \) such that each \( f_{m+1}^{(k)} \) is a linear combination of functions \( \varphi_i^{\otimes (m+1)} \). Then, (6.5) implies the statement. \( \blacksquare \)

By using (6.4) and (25), the following proposition was proved in (23) (see also (2) and (31, Chs. 4, 5)).
Proposition 6.2  The Fourier transform of the measure \( \varrho_\lambda \) is given, in a neighborhood of zero, by the following formula: for \( \lambda = 2 \)

\[
\int_{\mathcal{D}'} e^{i \langle \omega, \varphi \rangle} d\mu_2(\omega) = \exp \left[ - \int_X \log(1 - i\varphi(x)) \sigma(dx) \right], \quad \varphi \in \mathcal{D}, \quad \|\varphi\|_\infty := \sup_{x \in X} |\varphi(x)| < 1,
\]

and for \( \lambda \neq 2 \)

\[
\int_{\mathcal{D}'} e^{i \langle \omega, \varphi \rangle} d\mu_\lambda(\omega) = \exp \left[ \frac{-1}{\alpha \beta} \int_X \log \left( \frac{\alpha e^{-i\beta\varphi(x)} - \beta e^{-i\alpha\varphi(x)}}{\alpha - \beta} \right) \sigma(dx) + ic_\lambda \int_X \varphi(x) \sigma(dx) \right]
\]

for all \( \varphi \in \mathcal{D} \) satisfying

\[
\left\| \frac{\alpha(e^{-i\beta\varphi} - 1) - \beta(e^{-i\alpha\varphi} - 1)}{\alpha - \beta} \right\|_\infty < 1.
\]

Here, \( \alpha, \beta \in \mathbb{C} \) are defined through the equation \( 1 + \lambda z + z^2 = (1 - \alpha z)(1 - \beta z) \), \( z \in \mathbb{R} \). Furthermore, we have, for \( \lambda = 2 \),

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \langle \omega^{\otimes n}, \varphi^{\otimes n} \rangle = \exp \left[ - (\log(1 + \varphi)) + \left\langle \omega, \frac{\varphi}{\varphi + 1} \right\rangle \right],
\]

and for \( \lambda \neq 2 \)

\[
\sum_{n=0}^{\infty} \frac{1}{n!} \langle \omega^{\otimes n}, \varphi^{\otimes n} \rangle = \exp \left[ \frac{-1}{\alpha - \beta} \left( \log \left( \frac{(1 - \beta\varphi)^{1/\beta}}{(1 - \alpha\varphi)^{1/\alpha}} \right) \right) + \frac{1}{\alpha - \beta} \left\langle \omega - c_\lambda \sigma(\Delta) \log \left( \frac{1 - \beta\varphi}{1 - \alpha\varphi} \right) \right\rangle \right].
\]

Formulas (6.4), (6.5) hold for each \( \omega \in \mathcal{M}(X) \) and for each \( \varphi \in \mathcal{D} \) satisfying \( \|\varphi\|_\infty < 1 \) for (6.4) and \( \|\varphi\|_\infty < (\max(|\alpha|, |\beta|))^{-1} \) for (6.7). More generally, for each fixed \( \tau \in T \), there exists a neighborhood of zero in \( \mathcal{D} \) (depending on \( \lambda \)), denoted by \( \mathcal{O}_\tau \), such that (6.7), respectively (6.4), holds for all \( \omega \in \mathcal{H}_{-\tau} \) and all \( \varphi \in \mathcal{O}_\tau \).

As a direct corollary of this proposition, we get, for each \( \Delta \in \mathcal{O}_\tau(X) \), an explicit formula for the distribution of the random variable \( \langle \cdot, \chi_\Delta \rangle \) under \( \varrho_\lambda \).

Corollary 6.2 (25) For each \( \Delta \in \mathcal{O}_\tau(X) \), the distribution \( \varrho_{\lambda, \Delta} \) of the random variable \( \langle \cdot, \chi_\Delta \rangle \) under \( \varrho_\lambda \) is given as follows: For \( \lambda > 2 \), \( \varrho_{\lambda, \Delta} \) is the negative binomial (Pascal) distribution

\[
\varrho_{\lambda, \Delta}(ks) = (1 - p_\lambda)^{\sigma(\Delta)} \sum_{k=0}^{\infty} \frac{(\sigma(\Delta))_k}{k!} p_\lambda^k \delta_{\sqrt{\lambda^2 - 4}k},
\]

where for \( r > 0 \) \( (r)_0:=1 \), \( (r)_k:=r(r+1)\cdots(r+k-1) \), \( k \in \mathbb{N} \). For \( \lambda = 2 \), \( \varrho_{2, \Delta} \) is the Gamma distribution

\[
\varrho_{2, \Delta}(ds) = \frac{s^{(\sigma(\Delta)-1)} e^{-s}}{\Gamma(\sigma(\Delta))} \chi_{(0, \infty)}(s) ds.
\]

Finally, for \( \lambda \in [0, 2) \)

\[
\varrho_{\lambda, \Delta}(ds) = \frac{(4 - \lambda^2)^{(\sigma(\Delta)-1)/2}}{2\pi \Gamma(\sigma(\Delta))} \left| \Gamma(\sigma(\Delta)/2 + i(4 - \lambda^2)^{-1/2}(s + \lambda\sigma(\Delta)/2)) \right|^2
\[
\times \exp \left[ - (2s + \lambda\sigma(\Delta))(4 - \lambda^2)^{-1/2} \arctan \left( \lambda(4 - \lambda^2)^{-1/2} \right) \right] ds.
\]

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Let us dwell upon a representation of the random measure $Y^{(n)}$, $n \in \mathbb{N}$, through $\omega^{\otimes n}_{\lambda}$. We fix $\Delta \in \mathcal{O}_{c}(X)$. We take a sequence $\{\varphi_{k}, k \in \mathbb{N}\} \subset \mathcal{D}$ such that the $\varphi_{k}$’s are uniformly bounded, $\bigcup_{k \in \mathbb{N}}\text{supp} \varphi_{k}$ is a precompact set in $X$, $\varphi_{k}(x) \to \chi_{\Delta}(x)$ as $k \to \infty$ for each $x \in X$, and a sequence $\{\psi_{k}, k \in \mathbb{N}\} \subset \mathcal{D}^{\otimes n}$ such that $\psi_{k}$’s are uniformly bounded, $\bigcup_{k \in \mathbb{N}}\text{supp} \psi_{k}$ is a precompact set in $X^{n}$, and the $\psi_{k}$’s converge point-wisely to the indicator of the set $\{(x_{1}, \ldots, x_{n}) \in B(x_{0}, r)^{n} : x_{1} = x_{2} = \cdots = x_{n}\}$ as $k \to \infty$. Here, $B(x_{0}, r)$ is a ball in $X$ such that $\Delta \subset B(x_{0}, r)$. We set $G_{k}^{\omega}(x_{1}, \ldots, x_{n}) = (\varphi_{k}(x_{1})\psi_{k}(x_{1}, \ldots, x_{n}))$. Evidently,

$$G_{k}^{\omega}(x_{1}, \ldots, x_{n}) \to \chi_{\Delta}(x_{1})\chi_{\{x_{1}=x_{2}=\cdots=x_{n}\}}(x_{1}, x_{2}, \ldots, x_{n})$$

for each $(x_{1}, \ldots, x_{n}) \in X^{n}$.

Then, by Corollary 6.2 and the majorized convergence theorem, we easily get

$$Y^{(n)}(\Delta) = \lim_{k \to \infty} \langle \cdot, \omega^{\otimes n}_{\lambda} \rangle_{\mathcal{M}(\mathcal{D}^{\otimes n}); G_{k}^{\omega}}$$

and we informally write

$$Y^{(n)}(\Delta, \omega) = \langle \omega^{\otimes n}_{\lambda}(x_{1}, \ldots, x_{n}), \chi_{\Delta}(x_{1})\chi_{\{x_{1}=x_{2}=\cdots=x_{n}\}}(x_{1}, x_{2}, \ldots, x_{n}) \rangle_{\mathcal{D}^{\otimes n}; \varrho_{\lambda}}.$$ 

Furthermore, by (6.4), $\omega^{\otimes n}_{\lambda} \in \mathcal{M}(X^{n})$ for each $\omega \in \Omega(X)$. Therefore, in the case where $\varrho_{\lambda}$ is concentrated on $\Omega(X)$, i.e., $\lambda \geq 2$, using the majorized convergence theorem, we conclude from the above the following

**Proposition 6.3** For each $\varrho_{n} \in \mathcal{M}(X^{n})$, $n \in \mathbb{N}$, define $D_{n}\varrho_{n} \in \mathcal{M}(X)$ by setting, for each $\Delta \in \mathcal{B}_{c}(X)$

$$D_{n}\varrho_{n}(\Delta) := \varrho_{n}\{(x_{1}, \ldots, x_{n}) \in X^{n} : x_{1} \in \Delta, x_{1} = x_{2} = \cdots = x_{n}\}.$$ 

Let $\lambda \geq 2$. Then, for each $\Delta \in \mathcal{O}_{c}(X)$ and $n \in \mathbb{N},$

$$Y^{(n)}(\Delta, \omega) = \omega^{\otimes n}_{\lambda}(\Delta)$$

for $\varrho_{\lambda}$-a.e. $\omega \in \Omega(X),$

where

$$\omega^{\otimes n}_{\lambda} := D_{n}\omega^{\otimes n}_{\lambda}.$$ 

**Remark 6.2** Let $\lambda \geq 2$ and $\omega \in \Omega(X)$. As easily seen from (6.4), the $\omega^{\otimes n}_{\lambda}$’s satisfy the recurrence relation

$$\omega^{\otimes n+1}_{\lambda} = D_{2}(\omega^{\otimes n}_{\lambda} \otimes \omega) - \lambda n \omega^{\otimes n}_{\lambda} - n^{2} \omega^{\otimes n-1}_{\lambda},$$

$$\omega^{\otimes 0}_{\lambda} = \sigma, \quad \omega^{\otimes 1}_{\lambda} = \omega - c_{\lambda}\sigma,$$

which is, of course, equivalent to the recurrence relation satisfied by the polynomials $(P_{\lambda, n})_{n=0}^{\infty}$.

Finally, we will give a representation of the kernel of a multiple stochastic integral $\mathcal{I}^{\alpha}(f_{\alpha})$ through $\omega^{\otimes n}_{\lambda}$, where $n = \alpha_{1} + 2\alpha_{2} + \cdots$. Just as in Proposition 6.3 we suppose that $\lambda \geq 2$.

So, let us fix any $\alpha \in \mathbb{Z}_{>0}^{\infty}$, $|\alpha| \in \mathbb{N}$, and let $n := 1\alpha_{1} + 2\alpha_{2} + \cdots$. Let

$$X_{\alpha} := \{(x_{1}, \ldots, x_{n}) \in X^{n} : x_{\alpha_{1}+1} = x_{\alpha_{1}+2}, \ldots, x_{\alpha_{1}+2\alpha_{2}-1} = x_{\alpha_{1}+2\alpha_{2}}, \ldots, x_{1} \neq x_{2} \neq \cdots \neq x_{n} \neq x_{\alpha_{1}+2} \neq \cdots \neq x_{\alpha_{1}+2\alpha_{2}} \neq \cdots \neq x_{\alpha_{1}+2\alpha_{2}+3\alpha_{3}} \neq \cdots \neq x_{1} \}.$$ 

Here, the writing $y \neq y \neq \cdots \neq y$ means that $y \neq y$ if $i \neq j$. For any permutation $\pi$ of $\{1, \ldots, n\}$, denote

$$X_{\alpha}^{\pi} := \{(x_{\pi(1)}, \ldots, x_{\pi(n)}) : (x_{1}, \ldots, x_{n}) \in X_{\alpha}\}.$$
Evidently, all the sets $X^{\alpha}$ either coincide or disjoint, and there are exactly $R_{\alpha}$ disjoint sets between them (see formula (5.11) and Remark 5.1). We denote these disjoint sets by $X^{(1)}_{\alpha}, \ldots, X^{(R_{\alpha})}_{\alpha}$, where $X^{(1)}_{\alpha} = X_{\alpha}$.

Now, we fix $n \in \mathbb{N}$ and take all $\alpha \in \mathbb{Z}_{\geq 0}^{\infty}$ such that $1\alpha_1 + 2\alpha_2 + \cdots = n$. Then, we get the following representation of $X^n$ as a union of pair-wisely disjoint sets $X^{(i)}_{\alpha}$:

$$X^n = \bigcup_{\alpha \in \mathbb{Z}_{\geq 0}^{\infty}} \bigcup_{1\alpha_1 + 2\alpha_2 + \cdots = n} R_{\alpha} X^{(i)}_{\alpha}.$$  

(6.8)

Thus, each measure $\vartheta \in M(X^n)$ can be represented as

$$\vartheta = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{\infty}} \sum_{1\alpha_1 + 2\alpha_2 + \cdots = n} R_{\alpha} \vartheta^{(i)}_{\alpha},$$

where

$$\vartheta^{(i)}_{\alpha}(\Delta) := \vartheta(\Delta \cap X^{(i)}_{\alpha}), \quad \Delta \in \mathcal{B}(X^n).$$

Let us suppose that the measure $\vartheta$ is invariant under the action of the group of permutations of $\{1, \ldots, n\}$ on $X^n$, i.e., for each permutation $\pi$, $\vartheta(\Delta) = \vartheta(\pi \Delta)$ for all $\Delta \in \mathcal{B}(X^n)$, where

$$\pi \Delta := \{(x_{\pi(1)}, \ldots, x_{\pi(n)}) : (x_1, \ldots, x_n) \in \Delta\}.$$

For any $f_n \in D^{\otimes n}$, because of symmetricity, we then obtain:

$$\langle \vartheta, f_n \rangle = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{\infty}} \sum_{1\alpha_1 + 2\alpha_2 + \cdots = n} R_{\alpha} \langle \vartheta^{(i)}_{\alpha}, f_n \rangle,$$

(6.9)

where $\vartheta^{(1)}_{\alpha} := \vartheta^{(i)}_{\alpha}$.

Denote

$$\tilde{X}^{[\alpha]} := \{(x_1, \ldots, x_{|\alpha|}) \in X^{[\alpha]} : x_i \neq x_j \text{ if } i \neq j\}.$$

There exits a natural bijection between $\tilde{X}^{[\alpha]}$ and $X_{\alpha}$ given by

$$\tilde{X}^{[\alpha]} \ni (x_1, \ldots, x_n) \mapsto T_{\alpha}(x_1, \ldots, x_n) := (x_1, \ldots, x_{\alpha_1}, x_{\alpha_1+1}, x_{\alpha_1+1}, \ldots, x_{\alpha_1+1}, \underbrace{x_{\alpha_1+1}, x_{\alpha_1+1}, x_{\alpha_1+1}}_{\text{2 times}}, x_{\alpha_1+2}, \underbrace{x_{\alpha_1+2}, x_{\alpha_1+2}}_{\text{2 times}}, x_{\alpha_1+2}, \underbrace{x_{\alpha_1+2}, x_{\alpha_1+2}}_{\text{3 times}}, \ldots) \in X_{\alpha}.$$  

Evidently $T_{\alpha}$ and its inverse, $T_{\alpha}^{-1}$, are measurable mappings with respect to the corresponding trace $\sigma$-algebras.

Let us consider $\vartheta_{\alpha}$ as a measure on $X_{\alpha}$, and let $\hat{\vartheta}_{\alpha}$ denote the image measure of $\vartheta_{\alpha}$ under $T_{\alpha}^{-1}$. As easily seen, for $f_n \in D^{\otimes n}$,

$$\langle \vartheta_{\alpha}, f_n \rangle = \langle \hat{\vartheta}_{\alpha}, D_{\alpha} f_n \rangle.$$

(6.10)

where $D_{\alpha}$ was defined in (5.9). By (6.9) and (6.10), taking $\vartheta = \omega^{\otimes n}_{\lambda}$ for any $\omega \in \Omega(X)$, we get

$$\langle \omega^{\otimes n}_{\lambda}, f_n \rangle = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^{\infty}} \sum_{1\alpha_1 + 2\alpha_2 + \cdots = n} R_{\alpha} \langle \omega^{\otimes n}_{\lambda, \alpha}, D_{\alpha} f_n \rangle,$$

(6.11)

where $\omega^{\otimes n}_{\lambda, \alpha}$ is the corresponding $\hat{\vartheta}_{\alpha}$ measure for $\vartheta = \omega^{\otimes n}_{\lambda}$. 

Theorem 6.1 Let $\lambda \geq 2$, let $\alpha \in \mathbb{Z}^\infty_{+,0}$, $m:=|\alpha| \in \mathbb{N}$, and let $n:=1\alpha_1+2\alpha_2+\cdots$. Let $f_\alpha \in D^{\hat{\omega}_{\alpha_1}} \otimes D^{\hat{\omega}_{\alpha_2}} \otimes \cdots \subset D^{\hat{\omega}_m}$. Then,

$$I^\alpha(f_\alpha) = \langle \omega^{\hat{\omega}_m}, \hat{\omega}_\alpha, f_\alpha \rangle \quad \theta_\lambda \text{-a.s.,}$$

where for $\omega \in \Omega(X)$ : $\omega^{\hat{\omega}_m, \hat{\omega}_\alpha}$ was extended to a measure on $X^m$ by setting it zero on $X^m \setminus \widetilde{X}^m$.

Remark 6.3 It follows from Theorem 6.1 that the representation of $\langle \omega^{\hat{\omega}_n}, f_n \rangle$: obtained in Corollary 5.1 in the case of $\varrho_\lambda$, $\lambda \geq 2$, may be understood as taking the partition (6.8) of the space $X^n$, constructing the corresponding decomposition of each measure $\omega^{\hat{\omega}_n, \lambda}$ for $\omega \in \Omega(X)$, and then applying this decomposition to $\langle \omega^{\hat{\omega}_n, \lambda}, f_n \rangle$.

Proof. We fix $f_\alpha$ as in the formulation of the theorem. For any $x_0 \in X$, we choose $r > 0$ such that $\text{supp } f_\alpha \subset B(x_0, r)^n$. Let $\{\varphi_i, i \in \mathbb{N}\} \subset D^{\hat{\omega}_m}$ be such that $|\varphi_i(x_1, \ldots, x_m)| \leq 1$ for all $i \in \mathbb{N}$ and $(x_1, \ldots, x_m) \in X^m$, $\text{supp } \varphi_i \subset B(x_0, r+1)^m$, $i \in \mathbb{N}$, and $\varphi_i$’s converge point-wisely to the indicator of the set $\{(x_1, \ldots, x_m) \in B(x_0, r)^m : x_1 \neq x_2 \neq \ldots \neq x_m\}$. For $k \in \{2, \ldots, n\}$, let $\{\psi_i^{(k)}, i \in \mathbb{N}\} \subset D^{\hat{\omega}_k}$ be such that $|\psi_i^{(k)}(x_1, \ldots, x_k)| \leq 1$ for all $i \in \mathbb{N}$ and $(x_1, \ldots, x_k) \in X^k$, $\text{supp } \psi_i^{(k)} \subset B(x_0, r+1)^k$, $i \in \mathbb{N}$, and $\psi_i^{(k)}$’s converge point-wisely to the indicator of the set $\{(x_1, \ldots, x_k) \in B(x_0, r)^k : x_1 = x_2 = \cdots = x_k\}$. For $i \in \mathbb{N}$, we define $G_i \in D^{\hat{\omega}_n}$ setting

$$G_i(x_1, \ldots, x_n):=\left(f_\alpha \varphi_i\right)(x_1, x_2, \ldots, x_\alpha_1, x_{\alpha_1+2}, x_{\alpha_1+4}, \ldots, x_{\alpha_1+2\alpha_2}, \ldots) \times \psi_i^{(2)}(x_{\alpha_1+1}, x_{\alpha_1+2}) \cdots \psi_i^{(2)}(x_{\alpha_1+2\alpha_2-1}, x_{\alpha_1+2\alpha_2}) \times \psi_i^{(3)}(x_{1+2\alpha_2+1}, x_{1+2\alpha_2+2}, x_{1+2\alpha_2+3}) \cdots \psi_i^{(3)}(x_{1+2\alpha_2+3\alpha_3-2}, x_{1+2\alpha_2+3\alpha_3-1}, x_{1+2\alpha_2+3\alpha_3}) \cdots .$$

Let $G_i^\sim \in D^{\hat{\omega}_n}$ denotes the symmetrization of $G_i$, $i \in \mathbb{N}$. As easily seen, we get for each $\omega \in \Omega(X)$:

$$\langle \omega^{\hat{\omega}_n, \lambda}, G_i^\sim \rangle \rightarrow \langle \omega^{\hat{\omega}_n, \lambda}, f_\alpha \rangle$$

(6.12)

as $i \rightarrow \infty$. On the other hand, by Theorem 5.1 and Corollary 5.1,

$$\langle \cdot^{\hat{\omega}_n}, \hat{\omega}_\alpha, G_i^\sim \rangle \rightarrow I^\alpha(D_\alpha f_\alpha)$$

(6.13)

as $i \rightarrow \infty$ in $L^2(\Omega(X); \rho_\lambda)$. Comparing (6.13) and (6.13), we conclude the statement. ■

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