Ultra-chaos in the ABC flow and its relationships to turbulence

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It is well-known that three-dimensional steady-state Arnold-Beltrami-Childress (ABC) flow often has a chaotic Lagrangian structure besides satisfies the Navier-Stokes (NS) equations. Although trajectories of a chaotic system have the sensitive dependence on initial conditions, i.e. the famous “butterfly-effect”, their statistical properties are normally not sensitive to small disturbances. Such kind of chaos (such as those governed by Lorenz equation) is called normal-chaos. However, a kind of new concept, i.e. ultra-chaos, has been reported recently, whose statistics are sensitive to tiny disturbances. Here, we illustrate that ultra-chaos widely exists in trajectories of fluid-particles (in Lagrangian viewpoint) in the unstable ABC flow, which represents a higher disorder than the normal-chaos. Besides, using the ABC flow with small disturbance as initial condition, it is found that trajectories of nearly all fluid-particles become ultra-chaotic after the transition from laminar to turbulence occurs. Our results highly suggest that ultra-chaos should have a close relationship with turbulence.

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I. INTRODUCTION

The sensitivity dependence on initial conditions (SDIC) of trajectory of a chaotic dynamical system was first discovered by Poincaré [1], and then rediscovered by Lorenz [2] with a more popular name “butterfly-effect”. Besides, Lorenz [3, 4] further discovered that trajectories of a chaotic dynamical system have the sensitivity dependence not only on initial conditions (SDIC) but also on numerical algorithms (SDNA), because numerical noises (i.e. truncation error and round-off error) are unavoidable for all numerical algorithms. All of these phenomena are based on the exponential increase of noises (or small disturbances) of chaotic dynamical system. Naturally, this kind of non-replicability/unreliability of chaotic trajectory brought some heated debates on the credence of numerical simulation of chaotic system, and some even made a rather pessimistic conclusion that “for chaotic systems, numerical convergence cannot be guaranteed forever” [5]. Besides, it is currently reported that “shadowing solutions can be almost surely nonphysical”, which “invalidates the argument that small perturbations in a chaotic system can only have a small impact on its statistical behavior” [6].

In order to gain a reproducible/reliable numerical simulation of chaos, Liao [7] suggested a numerical strategy, namely the “Clean Numerical Simulation” (CNS) [7–10], to greatly reduce the background numerical noises (i.e. truncation error and round-off error) in a finite interval of time that is long enough for statistics. In the frame of the CNS [7–10], the spatial and temporal truncation errors are reduced to a required tiny level by means of a fine enough spatial discretization (such as the spatial Fourier expansion) and a high enough order of Taylor expansion in the temporal dimension, respectively. Especially, by means of a large enough number of significant digits to represent all physical and numerical variables/parameters in the multiple precision [11], the round-off error can be reduced to a required tiny level. Furthermore, an additional simulation with the even smaller background numerical noises is performed so as to determine the so-called “critical predictable time” $T_c$ by comparing such two simulations that the numerical noises (caused by truncation and round-off errors) can be negligible, i.e. several orders of magnitude smaller than the physical solution, and thus the computer-generated result of chaos is reproducible/reliable within the whole spatial domain in the time interval $t \in [0, T_c]$. In this way, the CNS can give the reproducible/reliable trajectories of a chaotic dynamical system in an interval of
time $[0, T_c]$ that is long enough for statistics.

The CNS provides us a useful tool to gain reproducible/reliable simulations of chaos in a long enough interval of time. Up to now, the CNS has been successfully applied to solve many chaotic systems, such as the Lorenz equations [10], the two-dimensional Rayleigh-Bénard turbulence [12], the chaotic motion of a free fall desk [13] and some spatiotemporal chaotic systems such as the complex Ginzburg-Landau equation [14], the damped driven sine-Gordon equation [15] and so on. Especially, by means of the CNS, more than 2000 new families of periodic orbits of the three-body system [16–18] have been found, which were reported twice by the popular magazine *New Scientist* [19, 20], because only three families of periodic orbits of the three-body problem were reported in 300 years after Newton mentioned this famous problem in 1687. All of these illustrate the novelty, great potential and validity of the CNS for chaotic dynamic systems.

Obviously, a chaotic numerical simulation given by the CNS can be used as a benchmark solution to study the influence of numerical noise to chaos. Using the CNS as a tool, it is found that, for some chaotic systems, such as the Lorenz equations [2], which has one positive Lyapunov exponent, and the so-called hyper-chaotic Rossler system [21], which has two positive Lyapunov exponents, their statistics always keep the same, although their trajectories are rather sensitive to small disturbances. We call them the normal-chaos [22]. However, it was found that the statistical properties (such as the probability density function) of some chaos are extremely sensitive to the tiny noise/disturbance [13–15], which is called ultra-chaos [22]. In this letter, we use the Arnold-Beltrami-Childress (ABC) flow as an example to illustrate that ultra-chaos indeed widely exists and is in a higher disorder than a normal-chaos. Besides, our results highly suggest that turbulence should have a close relationship with ultra-chaos.

II. ULTRA-CHAOS IN THE ABC FLOWS

The Arnold-Beltrami-Childress (ABC) flow

\[
\mathbf{u}_{ABC}(x, y, z) = [A \sin(z) + C \cos(y)] \mathbf{e}_x + [B \sin(x) + A \cos(z)] \mathbf{e}_y + [C \sin(y) + B \cos(x)] \mathbf{e}_z \tag{1}
\]
describes a kind of stationary flow of the incompressible fluid with periodic boundary conditions, where $u_{ABC}$ is the velocity vector field, $A$, $B$ and $C$ are arbitrary constants, $x$, $y$ and $z$ are Cartesian coordinates, $e_x$, $e_y$ and $e_z$ are the direction vectors of Cartesian coordinate system, respectively. It was first discovered as a class of analytical solutions of the Euler/Navier-Stokes equations by Arnold [23], and since then the Lagrangian chaotic property [23–27] as well as the so-called Beltrami property, i.e. substantial helicity \( u_{ABC} \times (\nabla \times u_{ABC}) = 0 \), of this kind of flow have aroused wide concern in nonlinear dynamics, hydrodynamics and magnetohydrodynamics. The property of exponential deviation of a fluid particle (i.e. Lagrangian chaos) in the ABC flow is typical for chaotic dynamical systems [23–25, 28–32] and essential for the development of turbulent flows [25, 27, 33], and this feature as well as the above-mentioned substantial helicity is also essential for the fast dynamo action (i.e. fast generation of magnetic field in conducting fluids) and for the origin of magnetic field of large astrophysical objects [26, 29, 34–41].

Let $x(t)$, $y(t)$ and $z(t)$ represent the location coordinates of a fluid particle, $\dot{x}(t)$, $\dot{y}(t)$ and $\dot{z}(t)$ denote their temporal derivatives, respectively. Thus, in the Lagrangian sense, the motion of a fluid particle of the ABC flow, which is reported to be a typical chaotic dynamical system in many cases, is governed by

\[
\begin{align*}
\dot{x}(t) &= A \sin[z(t)] + C \cos[y(t)], \\
\dot{y}(t) &= B \sin[x(t)] + A \cos[z(t)], \\
\dot{z}(t) &= C \sin[y(t)] + B \cos[x(t)],
\end{align*}
\]

(2)

with the initial condition

\[
(x(0), y(0), z(0)) = r_0,
\]

(3)

where $r_0$ denotes a starting point of the fluid particle. Without loss of generality, let us consider the case of $A = 1$ and different values of $B$ and $C$. It should be emphasized here that, by means of the CNS, we can always gain a reproducible/reliable trajectory of the chaotic motion of a fluid particle of the ABC flow in a long enough interval of time. To investigate the influence of small disturbance on trajectory of the fluid-particle in the ABC flow (1) starting from $r_0 = (x(0), y(0), z(0))$, we compare the trajectories of two close fluid-particles of the ABC flow, starting from the initial positions $r_0$ and $r_0' = r_0 + (0, 0, 1) \times \delta$, respectively, where $\delta = |r_0 - r_0'|$ is a tiny constant. Note that $\delta = 0$ when $r_0 = r_0'$.

For example, without loss of generality, let us consider the motion of a fluid-particle of
the ABC flow (in the Lagrangian sense) starting from the point \( r_0 = (0, 0, 0) \) in the case of \( A = 1 \) and different values of \( B \) and \( C \). To investigate its chaotic property, we compare its trajectory with that starting from a very close one \( r'_0 = r_0 + (0, 0, 1) \times \delta \), where we choose either \( \delta = 10^{-5} \) or \( 10^{-10} \), respectively. It is found that in each case we can always gain a reproducible chaotic simulation in a quite long interval \( t \in [0, 10000] \) by means of a parallel algorithm of the CNS using the 200th-order Taylor expansion with the time-step \( \Delta t = 0.01 \) and representing all data in 500-digit multiple-precision (MP), whose replicability/reliability is guaranteed via another CNS result with even smaller background numerical noises, given by the 205th-order Taylor expansion (with the same time-step) and the 520-digit multiple-precision.

For example, in the case of \( A = 1, B = 0.7 \) and \( C = 0.42 \), the fluid-particle starting from \( r_0 = (0, 0, 0) \) has a chaotic motion (with the maximum Lyapunov exponent \( \lambda_{\text{max}} = 0.01 \)) in a restricted spatial domain, as shown in Fig. 1(a) for its phase plot \((x, z)\). In the case of \( \delta = 10^{-5} \) and \( \delta = 10^{-10} \), although the chaotic trajectories of the two fluid-particles, starting from the points very close to \( r_0 = (0, 0, 0) \), are rather sensitive to the starting point, their attractors and statistical properties such as the probability density function (PDF) are almost the same as those given by the chaotic trajectory starting from \( r_0 = (0, 0, 0) \) that corresponds to \( \delta = 0 \), as shown in Fig. 1 (b), (c) and (d), respectively. Thus, in the case of \( A = 1, B = 0.7 \) and \( C = 0.42 \), the motion of the fluid-particle starting from \( r_0 = (0, 0, 0) \) is a normal-chaos, since its statistical properties such as the PDF of \( z(t) \) are not sensitive to the small disturbances of the starting point.

However, in the case of \( A = 1, B = 0.7 \) and \( C = 0.43 \), i.e. with a small change of \( C \), the chaotic motion (with the maximum Lyapunov exponent \( \lambda_{\text{max}} = 0.06 \)) of the fluid particle of the ABC flow starting from \( r_0 = (0, 0, 0) \) becomes quite different from that in the case of \( A = 1, B = 0.7 \) and \( C = 0.42 \) mentioned above: the fluid-particle moves far and far away from \( r_0 \) and besides its phase plot \((x, z)\) becomes very sensitive to the small disturbance of the starting position, as shown in Fig. 2(a). These are quite different from the results in the case of \( A = 1, B = 0.7 \) and \( C = 0.42 \). Since the ABC flow is periodic, we normalize the values of \( z(t) \) to \([-\pi, \pi] \), i.e.

\[
z'(t) = z(t) + 2\pi n_z,
\]

where the values of \( n_z \) are integers, and \(-\pi \leq z' < +\pi\). Note that, as illustrated in Fig. 2(b),
FIG. 1. Influence of the tiny disturbances on the phase plot $x - z$ and the probability density function (PDF) of a normal-chaotic motion of the fluid-particle in the ABC flow. The curves are based on the CNS results in $t \in [0, 10000]$ of the normal-chaotic motion of the fluid-particle, governed by the ABC flow (2) with (3) in the case of $A = 1$, $B = 0.7$, and $C = 0.42$ (with the maximum Lyapunov exponent $\lambda_{max} = 0.01$), from the starting point $r'_0 = (0, 0, 0) + (0, 0, 1) \times \delta$ when $\delta = 0$ (red), or $\delta = 10^{-5}$ (black), or $\delta = 10^{-10}$ (blue), respectively. (a) The phase plot $(x, z)$ when $\delta = 0$; (b) The phase plot $(x, z)$ when $\delta = 10^{-5}$; (c) The phase plot $(x, z)$ when $\delta = 10^{-10}$; (d) The PDFs of $z(t)$.

The tiny disturbances of the starting position can lead to huge deviations of the PDFs of the normalized chaotic simulations $z'(t)$ in $t \in [0, 10000]$. In other words, in the case of $A = 1$, $B = 0.7$ and $C = 0.43$ of the ABC flow (2), even statistical properties of the chaotic motion of the fluid particle starting from $r_0 = (0, 0, 0)$ are very sensitive to the initial position, and thus the corresponding motion of the particle is a kind of ultra-chaos. Obviously, this kind of ultra-chaos is at a higher-level of disorder than that normal-chaos, as shown in Fig. 1 and Fig. 2. This example illustrates that ultra-chaos indeed exists in the famous ABC flow.
FIG. 2. Influences of tiny disturbances on the phase plot \( x - z \) and the probability density function (PDF) of an ultra-chaotic motion of the fluid-particle in the ABC flow. The curves are based on the CNS results in \( t \in [0, 10000] \) of the ultra-chaotic motion of the fluid-particle, governed by the ABC flow \( (2) \) with \( (3) \) in the case of \( A = 1.0, B = 0.7, \) and \( C = 0.43 \) (with the maximum Lyapunov exponent \( \lambda_{\text{max}} = 0.06 \)) from the starting point \( r'_0 = (0, 0, 0) + (0, 0, 1) \times \delta \) when \( \delta = 0 \) (red), or \( \delta = 10^{-5} \) (black), or \( \delta = 10^{-10} \) (blue), respectively. (a) The phase plots \( (x, z) \); (b) The PDFs of the normalized results \( z'(t) \).

Besides the PDF, let us further investigate other statistics such as the ensemble average to demonstrate the higher-level of disorder given by the ultra-chaos than the normal-chaos mentioned above. Here we consider the ensemble average of the chaotic trajectories of a fluid-particle starting from the point \( r'_0 = r_0 + (0, 0, 1) \times \delta_i \) with 1000 different tiny initial disturbances \( \delta_i \) \( (i = 1, 2, 3, \ldots, 1000) \), which are given by the Gaussian random number generator in the case of the standard deviation \( \sigma_d = \sqrt{\langle \delta_i^2 \rangle} \) as well as zero mean, i.e. \( \mu_d = \langle \delta_i \rangle = 0 \), where \( \langle \rangle \) denotes the average operator. It is found that, in the case of \( A = 1, B = 0.7, C = 0.42 \) and \( r_0 = (0, 0, 0) \), corresponding to the normal-chaotic motion of the fluid particle, the ensemble averages of the phase plots \( x - z \), which are given respectively either by \( \sigma_d = 10^{-5} \) or \( \sigma_d = 10^{-10} \), are almost the same, as shown in Fig. 3(a) and (b). On the contrary, in the case of \( A = 1, B = 0.7 \) and \( C = 0.43 \), the ensemble averages of the phase plots \( x - z \) (of the ultra-chaotic motions of the fluid particle), which are given respectively either by \( \sigma_d = 10^{-5} \) or \( \sigma_d = 10^{-10} \), are totally different, as shown in Fig. 3(c). Furthermore, the PDFs of the ensemble-averaged trajectories of the ultra-chaotic fluid-particles starting from \( r_0 = (0, 0, 0) \) is also very sensitive to the starting position, which are completely different from those given by the normal-chaotic fluid-particles, as illustrated in Fig. 4. All of these
FIG. 3. **Influences of the tiny disturbances on the $x-z$ phase plots of the ensemble-averaged trajectories of the normal-chaotic or ultra-chaotic fluid-particle in the ABC flow.** The $x-z$ phase plots of the ensemble-averaged trajectories are based on the CNS results in $t \in [0,10000]$ of the normal-chaotic or the ultra-chaotic fluid-particle of the ABC flow in the case of $A = 1$, $B = 0.7$ and either $C = 0.42$ or $C = 0.43$ from the starting point $r_0 = (0,0,0) + (0,0,1) \times \delta_i$, $1 \leq i \leq 1000$, with either $\sigma_d = \sqrt{\langle \delta_i^2 \rangle} = 10^{-5}$ (black) or $\sigma_d = 10^{-10}$ (blue), respectively. (a) The $x-z$ phase-plot of the normal-chaotic fluid-particle when $C = 0.42$ with $\sigma_d = 10^{-5}$; (b) The $x-z$ phase-plot of the normal-chaotic fluid-particle when $C = 0.42$ with $\sigma_d = 10^{-10}$; (c) The $x-z$ phase-plot of the ultra-chaotic fluid-particle when $C = 0.43$ with either $\sigma_d = 10^{-5}$ or $10^{-10}$.

results illustrate that, unlike the normal-chaos, even the ensemble-averaged quantities as well as their corresponding PDFs of the ultra-chaos in the ABC flow are rather sensitive to the tiny disturbances. It further illustrates that this kind of ultra-chaotic motion in the ABC flow is indeed at a higher-level of disorder than the normal-chaos.

Using $A = 1$ but various values of $B$ and $C$, it is found that there exist the non-chaos,
FIG. 4. Influences of the tiny disturbances on the PDFs of the ensemble-averaged trajectories of the normal-chaotic or ultra-chaotic fluid-particles in the ABC flow. The PDFs of the ensemble-averaged trajectories are based on the CNS results in \( t \in [0, 10000] \) of the normal-chaotic or the ultra-chaotic fluid-particle of the ABC flow \(^2\) with \(^3\) in the case of \( A = 1, B = 0.7 \) and either \( C = 0.42 \) or \( C = 0.43 \) from the starting point \( r_0 = (0, 0, 0) + (0, 0, 1) \times \delta_i, 1 \leq i \leq 1000 \), with either \( \sigma_d = \sqrt{\langle \delta_i^2 \rangle} = 10^{-5} \) (black) or \( \sigma_d = 10^{-10} \) (blue), respectively. (a) The PDFs of \( z(t) \) of the normal-chaotic fluid-particle when \( C = 0.42 \); (b) The PDFs of the normalized results \( z'(t) \) of the ultra-chaotic fluid-particle when \( C = 0.43 \).

normal-chaos, and the ultra-chaos for the motion of the fluid particle starting from \( r_0 = (0, 0, 0) \), as shown in Fig. 5. For a non-chaotic motion, its trajectory is not sensitive to the tiny disturbances of starting position. For a normal-chaotic motion, although its trajectory is rather sensitive to the tiny disturbances of starting position, its attractor and especially its statistical properties are not sensitive to the tiny disturbances. However, for an ultra-chaotic motion, even its statistical properties are sensitive to the tiny disturbances of starting position. Note that, for a normal-chaotic motion, the fluid particle starting from \( r_0 = (0, 0, 0) \) always moves in a restricted spatial domain (i.e. its position is in a restricted domain of the phase plot \( x - z \)). However, for an ultra-chaotic motion, the fluid-particle starting from \( r_0 = (0, 0, 0) \) departs from its starting point far and far away. This further illustrates that an ultra-chaotic motion of the fluid-particle in the ABC flow has a higher disorder than a normal-chaotic motion, although the velocity field of the ABC flow itself as a whole is periodic and steady-state.

On the other hand, keeping \( A = 1, B = 0.7, C = 0.43 \) and using various positions of the starting point \( r_0 = (x(0), y(0), z(0)) \) in the ABC flow, where \(-\pi \leq x(0), y(0), z(0) \leq +\pi,\)
FIG. 5. Chaotic states of the fluid-particle in the ABC flow (2) starting from $r_0 = (0, 0, 0)$ for different values of $B$ and $C$ when $A = 1$. Red domain: ultra-chaos; Blue domain: normal-chaos; Gray domain: non-chaos.

FIG. 6. States of chaos for the motions of the fluid-particles starting from different points $r_0 = (x(0), y(0), z(0))$ in the ABC flow (2) in the case of $A = 1$, $B = 0.7$ and $C = 0.43$. (a) on $z(0) = 0$; (b) on $z(0) = \pi/8$; (c) on $z(0) = \pi/4$; (d) on $z(0) = 3\pi/8$; (e) on $z(0) = 7\pi/16$; (f) on $z(0) = \pi/2$. Red points: ultra-chaos; Blue points: normal-chaos.
TABLE I. Statistical values of the maximum Lyapunov exponents $\lambda_{max}$ given by the normal-chaos and ultra-chaos for the motions of the fluid particles in the ABC flow. These results are obtained via solving the ABC flow (2) in the case of $A = 1$, $B = 0.7$ and $C = 0.43$, using various starting points $r_0 = (x(0), y(0), z(0))$ of the fluid particles, where $-\pi \leq x(0), y(0), z(0) \leq +\pi$.

|                      | Normal-chaos | Ultra-chaos |
|----------------------|--------------|-------------|
| Maximum value of $\lambda_{max}$ | $1.3 \times 10^{-2}$ | $8.7 \times 10^{-2}$ |
| Minimum value of $\lambda_{max}$ | $8.5 \times 10^{-5}$ | $4.3 \times 10^{-2}$ |
| Mean                 | $9.7 \times 10^{-4}$ | $6.9 \times 10^{-2}$ |
| Standard deviation   | $7.5 \times 10^{-4}$ | $1.0 \times 10^{-2}$ |

It is found that both the normal-chaos and ultra-chaos (for the motions of the fluid particle starting from $r_0$) widely exist, and these two states of chaos co-exist at the same time in the ABC flow, as shown in Fig. 6. The statistical values of their maximum Lyapunov exponents $\lambda_{max}$ are given in Table I. Statistically speaking, the maximum Lyapunov exponents $\lambda_{max}$ of an ultra-chaotic motions of the fluid particle in the ABC flow is about two-order magnitude larger than that of a normal-chaos.

Note that, when $z(0)$ increases from 0 to $\pi/2$, there exists a kind of structure constituted by the starting positions $(x(0), y(0))$ of the fluid particles with the normal-chaotic motions (corresponding to blue points) as well as the ultra-chaotic motions (corresponding to red points), and this kind of structure has continuous deformations, as shown in Fig. 6. Let $\alpha(x(0), y(0), z(0)) = 0$ or 1 denote the normal-chaotic motion or the ultra-chaotic motion of fluid particle starting from the point $r_0 = (x(0), y(0), z(0))$, respectively. It is found that, for $-\pi \leq x(0), y(0) \leq +\pi$, there exist the following relationships (symmetries):

$$\alpha(x(0), y(0), z(0)) = \alpha(-x(0), y(0), \pi - z(0)), \quad (5)$$

where $z(0) \in [\pi/2, \pi]$,

$$\alpha(x(0), y(0), z(0)) = \alpha(x(0), \pi - y(0), -z(0)), \quad (6)$$
TABLE II. Values of the parameter $C$ versus $N_{\text{ultra}}/N_{\text{all}}$, where $N_{\text{ultra}}$ denotes the number of the starting points corresponding to the ultra-chaos (for the motion of a particle in the ABC flow) and $N_{\text{all}} = 20^3$ denotes the total number of the equidistant starting points, respectively. These results are obtained via solving the chaotic dynamical system (2) in $t \in [0,10000]$ by means of the CNS, in the case of $A = 1.0, B = 0.7, 0 \leq C \leq 0.43$, using various starting points $r_0 = (x(0), y(0), z(0))$ of the fluid particles, where $-\pi \leq x(0), y(0), z(0) \leq +\pi$.

| $C$   | $N_{\text{ultra}}/N_{\text{all}}$ |
|-------|----------------------------------|
| 0.43  | 49%                             |
| 0.2   | 47%                             |
| 0.1   | 43%                             |
| 0.01  | 20%                             |
| 0.001 | 6%                              |
| 0.0001| 2%                              |
| 0.0   | 0%                              |

where $y(0) \in [0, \pi]$, $z(0) \in [-\pi, 0]$, and

$$
\alpha(x(0), y(0), z(0)) = \alpha(x(0), -\pi - y(0), -z(0)),
$$

(7)

where $y(0) \in [-\pi, 0]$, $z(0) \in [-\pi, 0]$.

Let $\beta$ denote the ratio of the numbers of the starting fluid-particles with ultra-chaotic trajectories to the whole particle numbers in $-\pi \leq x, y, z \leq +\pi$. In theory it holds that

$$
\beta = \frac{1}{(2\pi)^3} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} \int_{-\pi}^{+\pi} \alpha(x, y, z) \, dx \, dy \, dz,
$$

(8)

since $\alpha = 1$ and $\alpha = 0$ correspond to an ultra-chaos and a normal-chaos, respectively. In practice, we use the Monte Carlo method to approximately calculate the ratio

$$
\beta \approx N_{\text{ultra}}/N_{\text{all}},
$$

(9)

where $N_{\text{all}}$ denotes the number of whole randomly chosen starting fluid-particles $r_0 \in \Omega = \{(x, y, z) : -\pi \leq x, y, z \leq +\pi\}$, and $N_{\text{ultra}}$ is the number of the starting fluid-particles with
an ultra-chaotic trajectory. Obviously, the larger $N_{all}$, the more accurate the result $\beta$ given by the Monte Carlo method. In the case of $A = 1$, $B = 0.7$ and $0 \leq C \leq 0.43$, it is found using $N_{all} = 8000$ that the ratio $\beta \approx N_{ultra}/N_{all}$ is dependent upon the value of $C$, as shown in Table II. Especially, when $C \leq 0.1$, there is a power-law relationship between $\beta \approx N_{ultra}/N_{all}$ and $C$, i.e.
\[
\beta \approx N_{ultra}/N_{all} \approx C^{0.4},
\] as illustrated in Fig. 7. Thus, when the parameter $C$ decreases, the value of $N_{ultra}$, i.e. the number of the starting fluid-particles with an ultra-chaotic trajectory, decreases until $N_{ultra} = 0$ when $C = 0$. This is reasonable since it is well-known that the ABC flow in the case of $C = 0$ is stable and thus chaos does not exist in $C = 0$.

III. RELATIONSHIP BETWEEN ULTRA-CHAOS AND TURBULENCE

The velocity $\mathbf{u}_{ABC}$ of the famous Arnold-Beltrami-Childress (ABC) flow (1) was first discovered by Arnold [23] as a steady-state solution of the dimensionless Navier-Stokes equations
\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\nabla p + \frac{1}{Re} \Delta \mathbf{u} + \mathbf{f},
\] (11)
\[
\nabla \cdot \mathbf{u} = 0,
\] (12)
where $t \geq 0$ denotes the time, $\nabla$ is the Hamilton operator, $\Delta$ is the Laplace operator, $Re$ is the Reynolds number, $p$ is the pressure and $\mathbf{f} = \mathbf{u}_{ABC}(x, y, z)/Re$ is the given external force.
FIG. 8. (a) Total kinetic energy; (b)-(f) Module values $|\omega|$ of instantaneous vorticity fields at $t = 30$, $t = 50$, $t = 60$, $t = 100$, and $t = 300$, respectively, in the case of $A = 1.0$, $B = 0.7$ and $C = 0.43$.

per unit mass, respectively, with the periodic boundary conditions at $x = \pm \pi, y = \pm \pi, z = \pm \pi$.

Without loss of generality, let us consider the case of $A = 1.0$, $B = 0.7$ and $0 \leq C \leq 0.43$. As reported by Podvigina and Pouquet [33], the Reynolds number $Re = 50$ corresponds to a turbulent flow if the initial velocity field $u_{ABC}$ is under a small disturbance at order of magnitude $10^{-3}$. Such kind of turbulent flow is solved numerically in $t \in [0, 500]$: the spatial domain $[-\pi, +\pi]^3$ is discretized by a uniform mesh with $128^3$ points for the spatial Fourier expansion, where the maximum grid spacing is less than the minimum Kolmogorov scale [42], and the 3/2 rule for dealiasing [42] is used, with the time-step $\Delta t = 10^{-3}$.

Let us first use the unstable ABC flow in the case of $A = 1.0$, $B = 0.7$ and $C = 0.43$ as the initial condition of the NS equations (11) and (12). It is found that, at the beginning, since the time is not long enough for the tiny velocity disturbances to transfer into the
TABLE III. $T_{tran}$ versus $C$ by means of using the ABC flow in the case of $A = 1$, $B = 0.7$ and $C \leq 0.1$ as the initial condition under a small disturbance at order of magnitude $10^{-3}$ for the NS equations (11) and (12), where $T_{tran}$ denotes the time of the transition occurrence.

| $C$  | $T_{tran}$ |
|------|------------|
| 0.1  | 50.5       |
| 0.01 | 59.5       |
| 0.001| 70.0       |
| 0.0001| 80.0    |
| 0.0  | −          |

FIG. 9. Relationship between $C$ and $T_{tran}$ by means of using the ABC flow in the case of $A = 1$, $B = 0.7$ and $C \leq 0.1$ as the initial condition under a small disturbance at order of magnitude $10^{-3}$ for the NS equations (11) and (12), where $T_{trans}$ denotes the time of transition occurrence.

macro-level, it is very close to the ABC flow, i.e. about 49% starting fluid-particles are ultra-chaotic, according to Table II. The transition from laminar flow to turbulence occurs approximately at $t \approx 50.0 = T_{trans}$, as shown in Fig. 8, where $T_{trans}$ denotes the time of the transition occurrence. Knowing the velocity field $\mathbf{u}$ of the NS equations (11) and (12), we can investigate the chaotic property of trajectory of the fluid-particle starting from $\mathbf{r}_0$ in a
similar way as mentioned in § 2. When $t = 50$, we use the Monte Carlo method to randomly choose 10000 starting fluid-particles in $-\pi \leq x, y, z < +\pi$ and found that all trajectories of fluid-particles starting from them are ultra-chaotic. This strongly suggest that ultra-chaotic trajectories of all fluid-particles should be a necessary condition of turbulence.

Similarly, let us consider the stable ABC flow in the case of $A = 1$, $B = 0.7$ and $C = 0$. Using the ABC flow $\mathbf{u}_{ABC}$ under a small perturbation at order of magnitude $10^{-3}$ as the initial condition, we numerically solve the NS equations (11) and (12) in $t \in [0,2000]$ and investigate the chaotic property of trajectory of the fluid-particles. It is found that in the case of $C = 0$ the transition from laminar to turbulence never occurs and besides there is no ultra-chaotic motion of fluid-particles in the whole $t \in [0,2000]$. This suggests from another side that ultra-chaotic trajectories of fluid-particle should have a close relationship with turbulence.

Let us further consider the ABC flows in the case of $A = 1$, $B = 0.7$ and $0 \leq C < 0.43$. It is found that the time $T_{\text{trans}}$ of transition from laminar flow to turbulence increases as $C$ decreases from 0.1 to 0.0001, as shown in Table III. When $0 \leq C \leq 0.1$, there is a linear relationship:

$$T_{\text{tran}} \approx -10 \log_{10}(C) + 40,$$

as illustrated in Fig. 9 indicating that $T_{\text{tran}} \to +\infty$ as $C \to 0$, say, the transition from laminar flow to turbulence should never occur when $C = 0$, which agrees with our numerical simulation in the case of $C = 0$ mentioned above. In all cases of $A = 1$, $B = 0.7$ and $0 \leq C \leq 0.43$ under consideration, we use the Monte Carlo method to randomly choose 10000 starting points in $-\pi \leq x, y, z < +\pi$ and found that all trajectories of fluid-particles starting from these randomly chosen 10000 fluid-particles are ultra-chaotic. Thus, when $A = 1$, $B = 0.7$ and $0 < C \leq 0.43$, the less the number of ultra-chaotic fluid-particles at beginning, corresponding to an unstable ABC flow with smaller $C$, the larger the transition time $T_{\text{trans}}$, say, more time is needed for all fluid-particles to become ultra-chaotic at $t = T_{\text{trans}}$. All of these highly suggest that ultra-chaotic trajectories of nearly all fluid-particles should be a necessary condition of transition occurrence from laminar to turbulence for the viscous flow governed by the NS equations (11) and (12) considered here.
IV. CONCLUDING REMARKS

In this paper, we illustrate that trajectories of many fluid-particles in the unstable steady-state ABC flow \[1\] are ultra-chaotic, say, their statistical properties are sensitive to tiny disturbances of their starting position. Obviously, such kind of ultra-chaotic motions of fluid-particles represent a higher disorder than the normal-chaotic ones. Besides, using the ABC flow as an initial condition of the NS equations \[11\] and \[12\] with a small disturbance at the order $10^{-3}$ of magnitude, it is found that trajectories of all fluid-particles become ultra-chaotic after the transition from laminar to turbulent flow occurs. Thus, ultra-chaos should have a close relationship with turbulence. Our results highly suggest that trajectories of nearly all fluid-particles become ultra-chaotic should be a necessary condition of transition occurrence from laminar to turbulence, at least for the viscous flows considered in this paper.

Note that the chaotic property of the ABC flow is essential for the development of turbulence \[25, 27, 33\]. Hopefully, the ultra-chaos as a new concept \[22\] might open a brand-new door to study the chaos theory, turbulence and especially their relationships.

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