A unified representation of the q-oscillator and the q-plane

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Using deformations inspired by relativistic considerations and phase space symmetry, we deform the position and momentum operators in one dimension. The resulting algebra is shown to yield the q-oscillator algebra in one limiting case and the q-plane commutation relation in another.

In 1992 Arik and Mungan anticipated a connection between the q-oscillator and relativity, as was previously done by R.M.Mir-Kasimov in a different context. Quoting from Arik and Mungan, the momentum position commutator of a n-dimensional q-oscillator is given by

\[ [P_j, Q_j] = -i \left( 1 + \frac{q^2 - 1}{q^2 + 1} \sum_{k=1}^{j} (P_k^2 + Q_k^2) \right). \]  \tag{1}

Here \( P \)'s and \( Q \)'s are dimensionless quantities. To express them in terms of the physical position and momentum operators we perform the usual replacement

\[ Q_k \rightarrow \sqrt{\frac{\hbar \omega}{m} Q_k}, \quad P_k \rightarrow \sqrt{\frac{\hbar \omega}{m} P_k}. \]  \tag{2}

If we also choose the dimensionless variable \( q \) as

\[ q = 1 + \frac{\hbar \omega}{mc^2} \ldots, \]  \tag{3}

where the omitted terms denote higher powers of \( \hbar \omega/mc^2 \), it is evident that the contraction can be accomplished in different ways. Letting \( q \rightarrow 1 \) as \( c \rightarrow \infty \) in (1), we obtain the Heisenberg algebra suggesting that the algebra (1) is relativistic. Another contraction is the limit \( q \rightarrow 1 \) as \( \omega \rightarrow 0 \), which corresponds to turning off the oscillator interaction. In view of the former we expect that this will yield a relativistic free particle algebra. In the latter limit the commutator (1) becomes

\[ [P_j, Q_j] = -i \hbar \left( 1 + \sum_{k=1}^{j} \frac{P_k^2}{mc^2} \right). \]  \tag{4}

In the remaining part of their paper Arik and Mungan have proved that this relation is indeed a free relativistic algebra.

The one dimensional case of (1) is

\[ [P, X] = -i\hbar \left( 1 + \frac{P^2}{m^2 c^2} \right). \]  \tag{5}

This commutator can be viewed as a definition for \( P \) if we are given \( X = x \). The conclusion is that deforming the momentum operator and leaving the position operator undeformed yields an algebra which describes a relativistic free motion.

We now deform the position operator in the same way as we have deformed the momentum operator

\[ X = \frac{\hbar}{\tau} \operatorname{Sinh}(\frac{\tau}{\hbar} x), \]  \tag{6}

\[ P = \frac{\hbar}{\delta} \operatorname{Sinh}(-i\delta D). \]  \tag{7}

where the parameter \( \tau \) is a free parameter having the dimensions of momentum. We point out that the deformation of the position operator in is based on the same footing as the deformation of the momentum operator in , that is both can be considered as a difference operator in the imaginary direction. Thus the symmetry between \( x \)-space and \( k \)-space is maintained. Now we introduce two dimensionless constants \( \mu \) and \( \nu \) by

\[ \delta = \frac{\mu h}{mc}, \]  
\[ \tau = \nu mc. \]  \tag{8}

It can be shown that the commutation relation of the deformed position and the deformed momentum operator in (1) is

\[ [P, X] = -i\hbar \frac{\sin(\mu \nu)}{\mu \nu (1 + \cos \mu \nu)} \times \]  
\[ \left\{ \frac{\mu}{\sqrt{1 + \mu^2 \frac{P}{mc}^2}} \sqrt{1 + \nu^2 \frac{mcX}{h}^2} \right\}, \]  \tag{9}

where \{a,b\} is the anti-commutator. We will now consider two limiting cases of the algebra defined by (1). Although we have used the definitions (1) to calculate the commutation relation (1), we can disregard them when considering the limiting cases we have claimed.

An expansion to the first non trivial order of (1) around \( \mu^2 = 0, \nu^2 = 0 \) yields,

\[ [P, X] = -i\hbar \left( 1 + \frac{1}{2} \left( \frac{\mu}{mc} \right)^2 P^2 + \frac{1}{2} \left( \frac{\nu mc}{h} \right)^2 X^2 \right). \]  \tag{10}

The commutator (10), to lowest order in \( \mu \) and \( \nu \), is the same as the q-oscillator commutation...
\[
[P, Q] = -i \left( 1 + \frac{q^2 - 1}{q^2 + 1} \right) (P^2 + Q^2). \tag{11}
\]

Here \( P \) and \( Q \) are the dimensionless momentum and position operators. We can embed the oscillator interaction by letting \( Q \to (m\omega/\hbar)^{1/2}Q \) and \( P \to (m\omega/\hbar)^{-1/2}P \).

Comparing the two commutators (10) and (11) after this replacement we find the correspondence relations

\[
\frac{\nu}{\mu} = \hbar \frac{\omega}{mc^2},
\]

\[
q = 1 + \frac{\nu}{2} + \cdots. \tag{12}
\]

The \( \nu/\mu \) in (12) is the usual dimensionless quantity encountered in the parameterization of \( q \). As we have noted Arik and Mungan showed that by identifying \( q = 1 + \hbar \omega/mc^2 \) and considering the limit \( \omega \to 0 \) one gets a relativistic free particle algebra. This limit corresponds to turning off the oscillator interaction. Furthermore we see that \( \omega \) is proportional to \( \nu \), that is the \( \omega \to 0 \) limit can be achieved by letting \( \nu \to 0 \). Therefore we can argue that the deformation of the momentum operator alone yields a relativistic free particle algebra, in addition deforming the position operator yields an interacting system with interaction parameter \( \omega \) being proportional to the deformation parameter \( \nu \).

Another interesting limit of (11) can be calculated by taking

\[
\nu = \beta \sqrt{\alpha + 2\pi n}, \quad \mu = \beta^{-1} \sqrt{\alpha + 2\pi n}
\]

and letting \( n \to \infty \). Here \( \alpha \) and \( \beta \) are arbitrary. This results in

\[
\text{classical} \quad q \to 1
\]

\[
\hbar \quad \hbar \to 0
\]

\[
\text{quantum} \quad q
\]

\[
\mu, \nu \sim 0
\]

\[
\delta\text{-deformed}
\]

\[
\mu, \nu \to 0 \equiv q \to 1
\]

FIG. 1. The diagram of \( \hbar, q \) and \( \delta \) deformations

\[
P X = q X P, \quad q = (1 + e^{-i\alpha})/(1 + e^{+i\alpha}). \tag{13}
\]

The relations (13) describes the generators that form the classically deformed phase space \( R^2_q \). Interestingly we obtain relation (13) after "quantization" whereas in [3] it is postulated as a q-deformation of the classical \( R^2 \) without quantization. This is not a contradiction at all since in [3] \( \hbar \) has disappeared. This deformation limit can also be obtained by taking \( \hbar \to 0 \) which turns off the quantization. Therefore we can argue that relation (13) is an example of the "diagram" (see FIG.1) of \( \hbar \) and q deformations presented in [3], since we have obtained the q-deformed classical relation (13) from a \( \delta \)-deformed quantum relation (13) by taking \( \hbar \to 0 \). Both relations (13) and (13) allow us to state that the difference operator and the deformed position operator are closely related to q-deformations and to relativity.

Thus we have, at least in one dimension shown that a close connection exists between relativity, q-oscillators and difference operators. This work is based on a previous unpublished preprint by the author and M.Arik [4].

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