Oriented Lagrangian Orthogonal Matroid Representations

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Several attempts have been made to extend the theory of matroids (here referred to as ordinary or classical matroids) to theories of more general objects, in particular the Coxeter matroids of Borovik, Gelfand and White ([7], first introduced as WP-matroids in [10]), and the \( \Delta \)-matroids and (equivalent but for notation) symmetric matroids of Bouchet (see, for example, [8]). The special cases of Coxeter matroids for the Coxeter groups \( BC_n \) and \( D_n \) and a maximal parabolic subgroup are called symplectic and orthogonal matroids respectively, and may be viewed as collections of \( k \)-element subsets of the \( 2n \)-element set \( \{1,\ldots,n,1^*,\ldots,n^*\} \) with maximality conditions, where \( k \) is between 1 and \( n \). In the case where \( k = n \), these structures are called Lagrangian matroids and are isomorphic in a natural way to Bouchet’s symmetric matroids [6, 11], with orthogonal matroids giving even symmetric matroids. Classical matroids now appear as a special case of even Lagrangian matroids. A concept of representation of even \( \Delta \)- and symmetric matroids by skew-symmetric \( n \times n \) matrices was developed in [9]. In turn, symplectic and orthogonal matroids may be represented by \( k \)-dimensional totally isotropic subspaces of \( 2n \)-dimensional symplectic and orthogonal vector spaces [6, 11]; it is from this that the names of these structures arise.

Attempts have also been made to extend the (classical) theory of oriented matroids to this larger concept. A theory of orientation of Lagrangian symplectic matroids was presented in [4]. However, in the case when the matroid is even (as all orthogonal matroids are), this theory is both uninteresting and trivial; in particular, it is uninteresting for classical matroids. In [12], Wenzel presents an orientation concept for even \( \Delta \)-matroids, and their representations, which includes classical oriented matroids as a special case. In this paper we extend this theory to Lagrangian orthogonal matroids and their representations, and give a completely natural transformation from a representation of a classical oriented matroid to a representation of the same oriented matroid embedded as a Lagrangian orthogonal matroid. We are interested in representations of Lagrangian matroids as isotropic subspaces because such representations arise in the study of maps on surfaces [2, 3], and also because of their natural connections with Schubert cells. Since classical represented matroids correspond to thin Schubert cells in the Grassmannian [5], oriented matroids provide a stratification of the Grassmannian finer than thin Schubert cells but coarser than their connected components. Similarly, these other concepts of orientation provide stratifications of Lagrangian varieties which split thin Schubert cells into unions of connected components.
1 Matroids and Representations

In this section we recall definitions of classical, symmetric and \( \Delta \)-matroids. We then briefly discuss representations of these objects and connections between them, and give an alternative definition of Lagrangian orthogonal matroids.

1.1 Matroids

Let \( I = \{1, \ldots, n\} \). Let \( I_k = \{A \subseteq I \mid |A| = k\} \), the collection of \( k \)-element subsets of \( I \) (we use \emph{collection} for a set of sets to avoid confusion). Set also \( I^* = \{1^*, \ldots, n^*\} \), and \( J = I \cup I^* \). We define the involution \( * \) on \( J \) by setting \( (i^*)^* = i \) for \( i^* \in I^* \) and extend it to sets in the obvious way. A set \( A \subseteq J \) is said to be \emph{admissible} if \( A \cap A^* = \emptyset \), and we set \( J_k \) to be the collection of admissible \( k \)-subsets of \( J \). The \emph{symmetric difference} of two sets \( A \) and \( B \) is written and defined by

\[
A \Delta B = (A \cup B) \setminus (A \cap B).
\]

Then \( M \) is a (classical) matroid if and only if it satisfies Axiom 1 below.

**Axiom 1 (Classical Basis Exchange)** For \( A, B \in \mathcal{B} \) and \( i \in A \setminus B \), there exists \( j \in B \setminus A \) such that \( \langle A \Delta \{i, j\} \rangle \in \mathcal{B} \).

Bouchet, in \cite{8}, defines a \( \Delta \)-matroid as a collection \( \mathcal{B} \) of subsets of \( I \), not necessarily equicardinal, satisfying the following:

**Axiom 2 (Symmetric Exchange Axiom)** For \( A, B \in \mathcal{B} \) and \( i \in A \Delta B \), there exists \( j \in B \Delta A \) such that \( \langle A \Delta \{i, j\} \rangle \in \mathcal{B} \).

It is thus immediately apparent that a classical matroid is also a \( \Delta \)-matroid. Bouchet goes on to define a \emph{symmetric matroid} as essentially a \( \Delta \)-matroid with bases extended to \( n \) elements by adding to \( B \in \mathcal{B} \) all starred elements which do not appear, unstarred, in \( B \). Thus a symmetric matroid is a set \( \mathcal{B} \subseteq J_n \) satisfying:

**Axiom 3** For \( A, B \in \mathcal{B} \) and \( i \in A \Delta B \), there exists \( j \in B \Delta A \) such that \( \langle A \Delta \{i, j, i^*, j^*\} \rangle \in \mathcal{B} \).

We shall refer to these two axioms interchangeably as ‘the symmetric exchange axiom’ depending on the structure to which we refer.

We shall now define classical, symplectic and orthogonal matroids in terms of maximality properties. These definitions are drawn from \cite{11}; equivalences with other popular definitions may also be found there. Recall that, given a partial ordering \( \prec \) on a set \( X \), the \emph{Gale ordering} on the set of \( k \)-element subsets \( X_k \) of \( X \) is defined as follows:

for \( A, B \in J_k \), write

\[
A = \{a_1, a_2, \ldots, a_k\}, B = \{b_1, b_2, \ldots, b_k\},
\]

with \( a_i \prec a_{i+1} \) and \( b_j \prec b_{j+1} \) for \( 1 \leq i < k \). Then we write \( A \prec B \) if \( a_i \prec b_j \) for \( 1 \leq i \leq k \).

By a \( B_n \)-admissible ordering, we mean a total ordering on \( J \) satisfying \( i \prec j \) if and only if \( j^* \prec i^* \); that is, an ordering of the form

\[
a_1 \prec a_2 \prec \cdots \prec a_n \prec a_n^* \prec a_{n-1}^* \prec \cdots \prec a_1^*
\]
where \( \{a_1, \ldots, a_n\} \subset J \) is an admissible set. By a \( D_n\)-admissible ordering, we mean a partial ordering on \( J \) of the form

\[
a_1 \prec a_2 \prec \cdots \prec a_n \prec a^*_n \prec \cdots \prec a^*_1
\]

where \( \{a_1, \ldots, a_n\} \subset J \) is an admissible set. Now we have the following (standard) definitions:

1. A collection \( \mathcal{B} \subseteq I_k \) is a (classical) matroid if and only if for every linear ordering \( \prec \) of \( I \) there exists some \( B \in \mathcal{B} \) such that \( A \prec B \) for every \( A \in \mathcal{B} \).

2. A collection \( \mathcal{B} \subseteq J_k \) is a symplectic matroid if and only if for every \( B\)-admissible ordering \( \prec \) of \( J \) there exists some \( B \in \mathcal{B} \) such that \( A \prec B \) for every \( A \in \mathcal{B} \).

3. A collection \( \mathcal{B} \subseteq J_k \) is an orthogonal matroid if and only if for every \( D_n\)-admissible ordering \( \prec \) of \( J \) there exists some \( B \in \mathcal{B} \) such that \( A \prec B \) for every \( A \in \mathcal{B} \).

Clearly, every orthogonal matroid is also a symplectic matroid. A Lagrangian matroid is a symplectic matroid of maximal rank (so that \( k = n \)). Similarly, a Lagrangian orthogonal matroid is an orthogonal matroid of maximal rank \( n \), and Lagrangian orthogonal matroids are Lagrangian matroids.

Finally, we observe that a Lagrangian (symplectic) matroid and a symmetric matroid are the same objects. This follows from the characterisation of symmetric matroids in terms of a greedy algorithm in \([8]\). Furthermore, in \([11]\), it is shown that Lagrangian matroids are orthogonal if and only if they are even; that is, \( B \cap I \) has the same parity for all bases \( B \). Thus, an orthogonal Lagrangian matroid is exactly an even symmetric matroid.

### 1.2 Representations

Concepts of representation of matroids have been introduced in two separate, but closely related, ways. Bouchet introduces a concept of representation by square matrices of ‘symmetric type’ (\([9]\)), whereas in \([6]\) representations are introduced in terms of isotropic subspaces. In this paper we are concerned mainly with representations over the real numbers.

Representable symplectic matroids arise naturally from symplectic and orthogonal geometries, similarly to the way that classical matroids arise from projective geometry.

**Classical representations** We consider a \( k\)-dimensional subspace \( U \) of a vector space \( V \) with basis \( E = \{e_1, \ldots, e_n\} \). Choose a basis \( u_1, \ldots, u_k \) for \( U \) and express it in terms of \( E \) so that \( u_i = \sum_{j=1}^n c_{ij} e_j \). Thus, we have expressed this subspace as the row-space of a \( k \times n \) matrix \( C \) of rank \( k \) with columns indexed by \( I \). Let \( \mathcal{B} \) be the collection of sets of column indices corresponding to non-zero \( k \times k \) minors; then

**Theorem 1** \( \mathcal{B} \) is the collection of bases of a (classical) matroid.

Note that the matroid is independent of the choice of basis \( u_1, \ldots, u_k \). This theorem may be found in any book on matroid theory, for example \([14]\). We now state the corresponding result for symplectic and orthogonal matroids.
Symplectic and orthogonal representations

Let \( V \) be a vector space with basis \( E = \{ e_1, \ldots, e_n, e_1^*, \ldots, e_n^* \} \).

Let \( \cdot \) be a bilinear form on \( V \), with the symbol \( \cdot \) often suppressed as usual, with

\[
e_i e_i^* = 1 \quad \text{for all } i \in I
e_i e_j = 0 \quad \text{for all } i, j \in J \text{ with } i \neq j^*.
\]

**Definition 1** The pair \( (V, \cdot) \) is called a *symplectic space* for \( \cdot \) antisymmetric and an *orthogonal space* for \( \cdot \) symmetric. If the vector space is of characteristic 2, it is symplectic. A subspace \( U \) of \( V \) is called *totally isotropic* if \( \cdot \) restricted to \( U \) is identically zero. A *Lagrangian subspace* is a totally isotropic subspace of maximal dimension (easily seen to be \( n \)).

Choose a basis \( u_1, \ldots, u_k \) of such a totally isotropic subspace \( U \) and represent this basis in terms of \( E \), so that

\[
u_i = \sum_{j=1}^n (a_{ij} e_j + b_{ij} e_j^*).
\]

Now we have represented \( U \) as the row space of a \( k \times 2n \) matrix \( C = (A, B) \) with columns indexed by \( J \). Let \( \mathcal{B} \) be the collection of sets of column indices corresponding to non-zero \( k \times k \) minors which are admissible; then

**Theorem 2** If \( U \) is a totally isotropic subspace of a symplectic or orthogonal space, \( \mathcal{B} \) is the collection of bases of a symplectic or orthogonal matroid, respectively. Note that the matroid is independent of the choice of basis \( u_1, \ldots, u_k \) of \( U \).

This is Theorem 5 in [11]; the statement for symplectic matroids only is Theorem 2 in [8].

\( C \) is called a *(symplectic/orthogonal) representation* of \( M = (J, \ast, \mathcal{B}) \), and \( M \) is said to be *(symplecticly/orthogonally) representable*. Note that orthogonal matroids may have symplectic representations. We also note that, when considered in matrix form, the requirement that \( U \) be totally isotropic is equivalent to the requirement that \( AB^t \) be symmetric in the symplectic case and skew-symmetric in the orthogonal case.

In [8], Bouchet considers representations of \( \Delta \)-matroids in terms of matrices of ‘symmetric type’.

**Definition 2** A square matrix \( A = (a_{ij}) \) is said to be *quasi-symmetric* if there exists a function \( \epsilon : I \to \{-1, 1\} \) such that \( \epsilon(i)a_{ij} = \epsilon(j)a_{ji} \) for every \( i, j \in I \). Thus symmetric matrices are quasi-symmetric. \( A \) is said to be of *symmetric type* if it is anti-symmetric or quasi-symmetric.

A *principal minor* of a square matrix is one consisting of those rows and columns indexed by the same set \( H \subseteq I \). Bouchet proves

**Theorem 3** Let the collection of subsets of \( I \) corresponding to non-zero principal minors of a matrix \( A \) of symmetric type be \( S \), and take any \( T \subseteq I \). Then the collection \( \mathcal{B} = \{ A\Delta T \mid A \in S \} \) forms a \( \Delta \)-matroid.
This is part of Theorem 4.1 in [9].

In fact, this result follows as a corollary of Theorem 2, and we can extend it a little in consequence. Take a representation \( C = (A, B) \) of a Lagrangian matroid \( M \), choose a basis \( F \) of it, and set \( T = F \cap I \). Exchange columns \( j, j^* \) for \( j \in T \), and in the symplectic case multiply one of each pair exchanged by \(-1\). We have now moved those columns corresponding to \( F \) into the right-hand side while maintaining (skew-) symmetry of \( AB^t \). Now reduce, by row operations, this non-singular right-hand-side to the identity matrix. The resulting left-hand-side \( A \) is clearly a symmetric matrix in the symplectic case, and skew-symmetric in the orthogonal case. This is now exactly the \( A \) and \( T \) of the above theorem. Other sorts of quasi-symmetric matrices correspond to cases where the right-hand-side has been reduced to a diagonal matrix with entries plus or minus one, and indeed we may alter the definition of ‘symmetric type’ to read simply \( \varepsilon(i)a_{ij} = s\varepsilon(j)a_{ji} \), where \( s = 1 \) or \( s = -1 \). We observe that any such representation is equivalent to one which is strictly symmetric (for \( s = 1 \)) or skew-symmetric (for \( s = -1 \)) and that these produce symplectic and orthogonal Lagrangian matroids respectively.

Note that we can ‘embed’ a representation of a classical matroid as a representation of the canonically associated Lagrangian orthogonal matroid. (The classical matroid is a \( \Delta \)-matroid, which is a symmetric matroid upon ‘completing’ all sets in \( B \) with the appropriate starred elements. Since it is even, it is an orthogonal Lagrangian matroid.) We simply make the top \( k \) rows of \( A \) (for a matroid of rank \( k \)) the representation of the classical matroid, and the remaining rows of \( A \) zero; and the top \( k \) rows of \( B \) zero, and the bottom \( n - k \) rows an orthogonal complement of maximal rank of \( A \). This is clearly the required representation, and is both a symplectic and an orthogonal representation simultaneously.

In the case of a general, symplectically represented, symplectic Lagrangian matroid, we assign orientations by considering essentially signs of determinants of principal minors of the above symmetric matrices [4]. Unfortunately, in skew-symmetric matrices that produces uninteresting results, as we shall see; the correct concept is that of the Pfaffian, which we shall define in the next section.

### 2 Orientations

In this section we shall state a definition of classical oriented matroids, give Wenzel’s definition of (even) oriented \( \Delta \)-matroids, and extend it in the obvious way to orthogonal Lagrangian matroids. We remark parenthetically that symplectic Lagrangian matroids (and so \( \Delta \)-matroids, even or otherwise) may be oriented as described in [4]. We go on to discuss representations of these objects, and prove that a representable (classical) oriented matroid is representable as an oriented orthogonal matroid.

#### 2.1 Orientation Axioms

We begin by stating the Grassmann-Plücker relations.

**Theorem 4** For all vectors \( x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{R}^k \) we have that

\[
\det(x_1, x_2, x_3, \ldots, x_k) \cdot \det(y_1, y_2, y_3, \ldots, y_k) = \sum_{j=1}^{k} \det(y_j, x_2, x_3, \ldots, x_k) \cdot \det(y_1, \ldots, y_{j-1}, x_1, y_{j+1}, \ldots, y_k)
\]
The proof of this is simple: observe that the difference of the two sides is an alternating multilinear form in the \( k + 1 \) arguments \( x_1, y_1, y_2, \ldots, y_k \), vectors in a \( k \)-dimensional space. Hence this form is zero.

These relations inspire the chirotope axioms of classical oriented matroid theory:

**Definition 3** A chirotope of rank \( k \) on \( I \) is a mapping \( \chi : I^k \to \{-1, 1, 0\} \) which satisfies:

1. \( \chi \) is not identically zero.
2. \( \chi \) is alternating; that is\( \chi(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = \text{sign}(\sigma)\chi(x_1, \ldots, x_k) \)
   for any \( x_1, \ldots, x_k \in I \), \( \sigma \in \text{Sym}(k) \).
3. For all \( x_1, \ldots, x_k, y_1, \ldots, y_k \in I \) such that
   \( \chi(y_i, x_2, x_3, \ldots, x_k) \cdot \chi(y_1, \ldots, y_{i-1}, x_1, y_{i+1}, \ldots, y_k) \geq 0 \)
   for \( i = 1, \ldots, k \) we have
   \( \chi(x_1, x_2, x_3, \ldots, x_k) \cdot \chi(y_1, y_2, y_3, \ldots, y_k) \geq 0. \)

We then define an oriented matroid as an equivalence class of chirotopes, where two chirotopes are said to be equivalent if \( \chi_1 = \pm \chi_2 \). See [1] for a fuller description of this and other classical oriented matroid definitions. We shall often speak of a chirotope as being an oriented matroid, leaving the equivalence class implicitly understood.

We shall follow Wenzel in [12] by making:

**Definition 4** A map \( p : 2^I \to \mathbb{R} \) is called a twisted Pfaffian map if it satisfies the following:

1. \( p \) is not identically zero.
2. For all \( A, B \subseteq I \) with \( p(A) \neq 0, p(B) \neq 0 \), we have \( \#A = \#B \mod 2 \).
3. If \( A, B \subseteq I \) and \( A \Delta B = \{i_1 < \ldots < i_l\} \) then we have
   \[ \sum_{j=1}^l (-1)^j p(A\Delta\{i_j\}) \cdot p(B\Delta\{i_j\}) = 0. \]

We call two twisted Pfaffian maps equivalent if they differ only by a non-zero constant scalar multiple. In fact, Wenzel makes the definition for a ‘fuzzy ring’ rather than for the real numbers, but we are interested in this paper only in representations over the real numbers. Pfaffian maps may be defined as twisted Pfaffian maps where \( p(\emptyset) = 1 \).

**Definition 5** Let \( S'_{2m} = \{ \sigma \in S_{2m} | \sigma(2k - 1) = \min_{2k-1 \leq j \leq 2m} \sigma(j) \text{ for } 1 \leq k \leq m \} \), and let \( A \) be a skew-symmetric matrix. Then the Pfaffian of \( A \) is defined by

\[ \text{Pf}((a_{ij})_{1 \leq i, j \leq 2m}) = \sum_{\sigma \in S'_{2m}} \text{sign} \sigma \prod_{k=1}^m a_{\sigma(2k-1)\sigma(2k)}. \]

The Pfaffian of the empty set is 1, by definition.
It can be shown that the square of the Pfaffian of a (skew-symmetric) matrix is the determinant of that matrix.

**Theorem 5** If $A$ is a skew-symmetric $n \times n$ matrix, $I_1, I_2 \subseteq I$ and $I_1 \Delta I_2 = \{i_1, \ldots, i_l\}$ with $i_j < i_{j+1}$ for $1 \leq j \leq l - 1$ then

\[
\sum_{j=1}^{l} (-1)^j p(I_1 \Delta \{i_j\}) p(I_2 \Delta \{i_j\}) = 0
\]

where $p(S) = \text{Pf}((a_{ij})_{i,j \in S})$ for any $S \subseteq I$.

This is Proposition 2.3 in [13].

Thus a skew-symmetric matrix with real coefficients yields a Pfaffian map, and in fact Pfaffian maps to a given ring (here, to the reals) are in $1-1$ correspondence with skew-symmetric matrices over the same ring (this is Theorem 2.2 in [13]). It is thus clear from Theorem 3 that the subsets of $I$ corresponding to non-zero values of the twisted Pfaffian map form a $\Delta$-matroid.

We now follow [12, Definition 2.10] in making

**Definition 6** An oriented even $\Delta$-matroid is an equivalence class of maps $p : 2^I \to \{+1, -1, 0\}$ satisfying

1. $p$ is not identically zero.
2. For all $A, B \subseteq I$ with $p(A) \neq 0, p(B) \neq 0$, we have $\#A = \#B \mod 2$.
3. If $A, B \subseteq I$ and $A\Delta B = \{i_1 < \ldots < i_l\}$ and for some $w \in \{+1, -1\}$ we have

\[
\kappa_j = w(-1)^j p(A \Delta \{i_j\}) \cdot p(B \Delta \{i_j\}) \geq 0
\]

for $1 \leq j \leq l$, then $\kappa_j = 0$ for all $1 \leq j \leq l$.

We shall often speak of a map as an oriented even $\Delta$-matroid, with the equivalence class implicitly understood.

The bases of the oriented even $\Delta$-matroid are those subsets of $F \subseteq I$ for which $p(F) \neq 0$. We observe that every Pfaffian map yields an oriented $\Delta$-matroid by simply ignoring magnitudes.

**Lemma 6** The collection of bases of an oriented $\Delta$-matroid is a $\Delta$-matroid.

**Proof** Recall that a collection of sets is a $\Delta$-matroid if and only if it satisfies the symmetric exchange axiom, Axiom 2:

\[
\text{for } E, F \in \mathcal{B}, e \in E \Delta F \text{ there exists } f \in E \Delta F \text{ such that } E \Delta \{e, f\} \in \mathcal{B}.
\]

Set, without loss of generality, $i_1 = e$ in Condition 3 above (there is no loss of generality since we are not concerned with signs or orderings). Set $A = E \Delta \{e\}, B = F \Delta \{e\},$ and $w$ such that $\kappa_1$ is 1. Thus some other $\kappa_j$ must be $-1$; let $f = i_j$. Now, from the defining equation for $\kappa_j$, we have

\[
0 \neq p(A \Delta \{f\}) p(B \Delta \{f\}) = p(E \Delta \{e, f\}) p(F \Delta \{e, f\})
\]
and so we obtain $E \Delta \{e, f\}, F \Delta \{e, f\} \in \mathcal{B}$, which is more than we need.

We now make the obvious definition: Take a Lagrangian orthogonal matroid $\mathcal{B}$, with an equivalence class of signs assigned to its bases. Two sets of signs are said to be equivalent when they are either identical on all bases or opposite on all bases. We express this as an equivalence class of maps

$$p : J_n \to \{+,-,0\}$$

with

$$\mathcal{B} = \{A \in J_n \mid p(A) \neq 0\}$$

and equivalence given by $p \sim -p$. Consider the corresponding even $\Delta$-matroid and equivalence class of signs $p'$ obtained by ignoring starred elements; that is, $p'(A) = p(B)$, where $B \in J_n$ is the unique element with $B \cap I = A$. Now we say that $p$ is an oriented orthogonal matroid exactly when $p'$ is an oriented even $\Delta$-matroid.

### 2.2 Oriented Representations

We first state two now-obvious theorems.

**Theorem 7** Given a $k \times n$ real matrix $C$, let

$$\chi(S \in I^k) = \text{sign det}(c_{ij} \mid j \in S).$$

Then $\chi$ is an oriented matroid; further, the underlying (unoriented) matroid is the matroid represented by $C$. The oriented matroid represented is not altered when standard row operations are performed on $C$.

**Theorem 8** Given an $n \times n$ square skew-symmetric real matrix $A$ and $T \subseteq I$, define $p : 2^I \to \{+1, -1, 0\}$ by setting $p(B)$ to be the sign of the Pfaffian of the principal minor indexed by $B \Delta T$. Then $p$ is an oriented even $\Delta$-matroid, and the underlying $\Delta$-matroid is that represented by $A$ and $T$.

The first theorem is classical, and the second from [12]; both should now be obvious from the definitions and earlier theorems.

An oriented classical matroid is described by a map

$$\chi : I^k \to \{+,-,0\}, \quad \chi \sim -\chi$$

and an oriented even $\Delta$-matroid by a map

$$p : 2^I \to \{+,-,0\}, \quad p \sim -p.$$

Given $\chi$, we widen the domain by setting $\chi(A) = 0$ whenever $\#A \neq k$, and obtain a map which is a candidate to be an even $\Delta$-matroid. Given $p$ satisfying $p(A) = 0$ whenever $\#A \neq k$, some fixed $k$, we can restrict to a candidate to be an oriented matroid. It is natural to ask when these candidates succeed.
Theorem 9 Every oriented matroid is an oriented even Δ-matroid, and every oriented even Δ-matroid whose bases are all of rank $k$ is an oriented matroid.

Furthermore, a representation $C$ of an oriented matroid $M$ yields a representation $A$ of it as an oriented even Δ-matroid as follows. Choose a basis $T$ of $M$. Now set $a_{ij} = \det(T \Delta \{i, j\}) / \det(T)$, where by the determinant of a set we mean the determinant of the appropriate $k$ columns of $C$, or 0 if the set is not of cardinality $k$. Now $A$ is the required orientation.

This follows from [13, Theorem 4.1].

We now move on to define a representation of an oriented orthogonal matroid.

Definition 7 Given $C$, an orthogonal representation of an orthogonal matroid $M$ over $\mathbb{R}$, we construct the oriented orthogonal matroid represented by $C$ as follows. Choose a basis $F$ of $M$, and swap columns $j$ and $j^*$ for $j \in T = F \cap I$ so that all columns of $F$ are in the right-hand $n$ places. Now perform row operations so that the right-hand $n$ columns become the identity matrix. Now the left-hand side, $A'$, is a skew-symmetric matrix (this is exactly the procedure discussed after Theorem 3). Since we have $A'$ and $T$, we have a representation of an oriented even Δ-matroid. Unfortunately, this oriented even Δ-matroid is dependent on the initial choice of $F$, although the underlying non-oriented Δ-matroid is not, so we modify $A'$ as follows.

Set $\varepsilon_0 = 1$ and $\varepsilon_i = \left\{ \begin{array}{ll} \varepsilon_{i-1} & i \notin T \\ -\varepsilon_{i-1} & i \in T \end{array} \right.$ for $i > 0$. Then set $a_{ij} = \varepsilon_i \varepsilon_j a'_{ij}$. $A = (a_{ij})$ is again skew-symmetric, with rows and columns indexed by $I$, and we assign to the basis $B$ the sign of the Pfaffian of the principal minor of $A$ indexed by $(B \Delta F) \cap I$. If we consider instead that we have permuted column labels with columns, then the indices giving rise to this Pfaffian are those of the columns of $A$ labelled by elements of $B$. Note that this corresponds to the oriented even Δ-matroid represented by $A, T$.

Theorem 10 The above procedure obtains an oriented orthogonal matroid, which is independent of choice of $F$.

The fact that this is an oriented orthogonal matroid is obvious from considering the oriented even Δ-matroid represented by $A, T$; we need only show independence of choice of $F$. It is enough to show that a representation $(A, I_n)$ yields the same orientation using $A$ directly and going through the above procedure with $\#T = 2$. The symmetric exchange axioms of the first section and the evenness tell us that any two bases are connected by a path where adjacent bases differ in this way.

Suppose $a_{ij} \neq 0$, and set $T = \{i, j\}$ with $i < j$. Let $B$ be the skew-symmetric matrix obtained as follows. Take the compound matrix $(A, I_n)$, swap the $i$-th and $j$-th columns of $A$ with those of $I_n$, and reduce using row operations to the form $(B, I_n)$. It is helpful to know about the form of $B$. When we write $A_S$, we mean the Pfaffian of the minor of $A$ indexed by $S$. By $[a_{ij}a_{kl}]$, with $k \neq l$, $\{i, j\} \cap \{k, l\} = \emptyset$, we mean $\pm A_{\{i,j,k,l\}}$ with the sign chosen such that the term $a_{ij}a_{kl}$ has positive sign.
Lemma 11  The skew-symmetric matrix $B$ satisfies:

$$b_{kl} = \begin{cases} 
-1/a_{ij} & k = i, l = j, k \neq l \\
-1/a_{ij} & k = i, l \neq j, k \neq l \\
-1/a_{ij} & k = j, l \neq i, k \neq l \\
0 & k = l \\
[a_{ij}a_{kl}]/a_{ij} & |\{i, j, k, l\}| = 4 
\end{cases}$$

Proof  The first six statements are immediately clear from the construction of $B$. From consideration of determinants, which can be more readily seen, the final part is correct up to sign. But the term $a_{ij}a_{kl}$ appears in some sense ‘early’ in the construction of $B$ from $A$ and cannot then change sign, so this is the correct sign also. \(\diamond\)

Now, without loss of generality, $k < l$, since $b_{kl} = -b_{lk}$. Define, for $|i, j, k, l| = 4$, 

$$\epsilon_{ijkl} = \begin{cases} 
-1 & i < k < j < l \\
-1 & k < i < l < j \\
+1 & \text{otherwise} 
\end{cases}$$

(we leave $\epsilon_{ijkl}$ undefined when its subscripts are not all distinct). Clearly, from our formula for Pfaffians, $[a_{ij}a_{kl}] = \epsilon_{ijkl} A_{\{i, j, k, l\}}$.

Let us define a matrix $C$ from $B$ by multiplying rows $k$ for $i \leq k < j$ and the corresponding columns by $-1$. Then we have

Lemma 12  The Pfaffian minor $C_S$ satisfies $C_S = A_{\Lambda_{\{i,j\}}}/a_{ij}$.

Proof  Throughout, $i < j$ and $k < l$. Define $\rho_{ijk} = -1$ if $i < k < j$ and $+1$ otherwise. Thus $\epsilon_{ijkl} = \rho_{ijk}\rho_{ijl}$ wherever $\epsilon_{ijkl}$ is defined, and $c_{kl} = \rho_{ijkl}b_{kl}$. Thus, the elements of the skew-symmetric matrix $C$ satisfy:

$$c_{ij} = 1/a_{ij} \quad c_{ik} = \rho_{ijk}a_{jk}/a_{ij} \quad c_{ik} = \rho_{ijk}a_{ik}/a_{ij} \quad c_{kl} = A_{\{i,j,k,l\}}/a_{ij}$$

(for $|\{i, j, k, l\}| = 4$). It is easy to see that the lemma holds for determinants rather than Pfaffians of minors, so $C_S = \pm A_S$. Each term of $C_S$, rewritten in terms of the $a_{kl}$, corresponds to several terms of $A_S/a_{ij}$; thus we need check only that one of these has the same sign in $C_S$ as in $A_S$.

Let 

$$f_1 < \cdots < f_p < i < f_{p+1} < \cdots < f_q < j < f_{q+1} < \cdots < f_m$$

and write $F = \{f_1, \ldots, f_m\}$. We divide the proof into the four cases $S = F$, $S = F \cup \{i\}$, $S = F \cup \{j\}$, $S = F \cup \{i, j\}$. We shall divide these each into sub-cases depending on whether $p$ and $q$ are odd or even.

First we take $S = F = \{f_1, \ldots, f_m\}$; we may assume $m$ is even (as otherwise $A_S = C_S = 0$). Now, take

$$\tau = c_{f_1f_2} \cdots c_{f_{m-1}f_m},$$

which has positive sign in $C_S$; this contains the signed term

$$\left(\prod_{l=1}^{n/2} \epsilon_{f_{2l-1}f_{2l}}/a_{f_{2l-1}f_{2l}}\right) \tau/a_{ij},$$

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where

$$\bar{a} = a_{f_1 f_2} \ldots a_{f_{m-1} f_m} a_{ij}.$$  

Now we consider our sub-cases. If both $p$, $q$ are even, then $\bar{\mathbf{a}}$ has positive sign in $\mathbf{A}_{S \setminus i, j}$, and in fact all the $\epsilon_{ij f_2} / f_2$ are positive, so the term has positive sign in $\mathbf{C}_S$ as well. If $p$ is odd but $q$ is even then $\bar{\mathbf{a}}$ has negative sign in $\mathbf{A}_{S \setminus i, j}$, and all the $\epsilon$ are positive except for $\epsilon_{ij f_p f_{p+1}}$, so again $\bar{\mathbf{a}}$ has the correct sign. Similarly, if $q$ is odd but $p$ is even then $\bar{\mathbf{a}}$ has negative sign, and all the $\epsilon$ are positive except for $\epsilon_{ij f_q f_{q+1}}$. Finally, if both $p, q$ are odd, then $\bar{\mathbf{a}}$ has positive sign in $\mathbf{A}_{S \setminus i, j}$, and all the $\epsilon$ are positive except for $\epsilon_{ij f_p f_{p+1}}$ and $\epsilon_{ij f_q f_{q+1}}$. This disposes of the first case.

For the second case, take $S = F \cup \{i, j\}$. Once again $m$ is even in the non-trivial case. Now

$$\tilde{a} = a_{f_1 f_2} \ldots a_{f_{m-1} f_m}$$

has positive sign in $\mathbf{A}_{S \setminus \{i, j\}}$, and

$$\tilde{a} = c_{f_1 f_2} \ldots c_{f_{m-1} f_m} c_{ij}$$

yields the term

$$\left( \prod_{t=1}^{m/2} \epsilon_{ij f_2 \ldots f_2} \right) \bar{a}_{ij} / a_{ij}.$$  

Similarly to the first case, this is $\bar{\mathbf{a}} / a_{ij}$ when $p, q$ are both even or both odd, and $-\bar{\mathbf{a}} / a_{ij}$ when exactly one of $p, q$ is even. However, $\bar{\mathbf{a}}$ has positive sign in $\mathbf{C}_S$ exactly when $p, q$ are both even or both odd. This disposes of the second case.

Now take $S = F \cup \{i\}$. Here the non-trivial case has $m$ odd. Suppose first that $q$ is odd. Take

$$\bar{a} = a_{f_1 f_2} \ldots a_{f_{q-1} f_q} a_{f_q} a_{f_{q+1} f_{q+2}} \ldots a_{f_{m-1} f_m},$$

which has positive sign in $\mathbf{A}_{F \cup \{j\}}$. Now take

$$\bar{a} = c_{f_1 f_2} \ldots c_{f_{q-1} f_q} c_{f_q} c_{f_{q+1} f_{q+2}} \ldots c_{f_{m-1} f_m}.$$  

This contains the term

$$\left( \prod_{t=1}^{(q-1)/2} \epsilon_{ij f_2 \ldots f_2} \right) \left( \prod_{t=1}^{(m-1)/2} \epsilon_{ij f_2 f_{q+1}} \right) \rho_{ij q} a_{ij}.$$  

Now, $\bar{\mathbf{a}}$ has positive sign in $\mathbf{C}_S$ exactly when $p$ is odd also. Since $i < f_q < j$, $\rho_{ij q}$ is negative, and all the $\epsilon$ are positive except for $\epsilon_{ij f_p f_{p+1}}$, which appears exactly when $p$ is odd. This disposes of the sub-cases where $q$ is odd. The remaining cases, for $q$ even and for $S = F \cup \{j\}$, are similar.

Since $(\mathbf{C}, \mathbf{I}_n)$ is the form that would be obtained by following Definition $\bar{\mathbf{F}}$, we have proven Theorem $\bar{\mathbf{F}}$, as the signs differ only by a constant scalar multiple. Finally, we state the following:

**Theorem 13** Let $\mathbf{B}$ be a representation of the oriented matroid $M$, with columns indexed by $I$. Then

$$\left( \begin{array}{cc} \mathbf{B} & 0 \\ 0 & \mathbf{D} \end{array} \right)$$

is an orthogonal representation of the corresponding oriented orthogonal Lagrangian matroid, where $\mathbf{D}$ is an orthogonal complement to $\mathbf{B}$, with columns indexed by $I^\circ$. 

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Proof Let $M$ be of rank $k$, and suppose without loss of generality that the leftmost $k$ columns of $B$ form a basis of $M$. Since performing row operations on representations of classical oriented matroids does not alter the oriented matroid represented, we may assume that these $k$ columns form an identity matrix in the first $k$ rows, and that the rightmost $n-k$ columns of the orthogonal complement form an identity matrix in the last $n-k$ rows also. We swap these first $k$ columns into the right-hand-side, and make the appropriate multiplications, obtaining a matrix $(A I)$, where

$$
A = \begin{pmatrix}
0 & \cdots & 0 & (-1)^k b_{1,k+1} & \cdots & (-1)^k b_{1,n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 0 & (-1)^k b_{k,k+1} & \cdots & (-1)^k b_{k,n} \\
(-1)^k d_{1,1} & \cdots & (-1)^k d_{1,k} & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
(-1)^k d_{n-k,1} & \cdots & (-1)^k d_{n-k,k} & 0 & \cdots & 0
\end{pmatrix}
$$

Now we see that $a_{ij} = \det(\{1, \ldots, k\} \Delta i, j)$ where $\det$ is the determinant of the appropriate $k$ columns of $B$, and $0$ if its argument has more or less than $k$ elements. Now the result follows at once from Theorem [\[].

\[ \diamond \]

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