The multiplicative unitary as a basis for duality

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Abstract

The classical duality theory associates to an abelian locally compact group a dual companion. Passing to a non-abelian group, a dual object can still be defined, but it is no longer a group. The search for a broader category which should include both the groups and their duals, points towards the concept of quantization. Classically, the regular representation of a group contains the complete information about the structure of this group and its dual. In this article, we follow Baaj and Skandalis and study duality starting from an abstract version of such a representation : the multiplicative unitary. We suggest extra conditions which will replace the regularity and irreducibility of the multiplicative unitary. From the proposed structure of a "quantum group frame", we obtain two objects in duality. We equip these objects with certain group-like properties, which make them into candidate quantum groups.

We consider the concrete example of the quantum $a z + b$-group, and discuss how it fits into this framework. Finally, we construct the crossed product of a quantum group frame with a locally compact group.
1 Introduction

The classical duality theory associates to an abelian locally compact group a dual group. The famous theorem of Pontryagin and Van Kampen then says that this dual contains the complete information about the structure of the group: its dual is again the original group. Several authors (Tannaka, Krein, Tatsuuma) have generalized this result to non-abelian groups, showing that to an arbitrary locally compact group one can still associate a dual object, from which the original group can be recovered (see [6]). But here the beautiful symmetry of the Pontryagin duality is lost: this dual object is no longer a group. And so the search started for a broader category which would include both the groups and their duals.

It appeared that a quantization of the group concept would bring the solution of this problem. The idea behind the process of "quantization" is the following. In a first step, instead of thinking of a space as a set of elements, one considers some set of functions on this space. For instance, let $G$ be a measure space, and consider the set $L_\infty(G)$ of all essentially bounded complex functions on $G$. With pointwise multiplication and supremum norm it becomes a von Neumann algebra. If $G$ is equipped with a (measurable) product, it can be lifted to a $^*$-homomorphism $\Phi : L_\infty(G) \to L_\infty(G \times G)$ given by

$$(\Phi(f))(p, q) = f(pq)$$

for $f \in L_\infty(G)$ and $p, q \in G$. In the process of quantization we now forget about the underlying space $G$ and consider instead the pair $(L_\infty(G), \Phi)$ of a commutative von Neumann algebra with a comultiplication. The next step then is to drop the commutativity condition on the von Neumann algebra.

**Definition 1.1** Let $M$ be a von Neumann algebra. We denote by $M \otimes M$ the von Neumann algebraic tensor product of $M$ with itself. A **comultiplication** on $M$ is a normal, injective $^*$-homomorphism $\Phi : M \to M \otimes M$ such that $\Phi(1) = 1 \otimes 1$ and $\Phi$ satisfies the **coassociativity property** $(\iota \otimes \Phi)\Phi = (\Phi \otimes \iota)\Phi$. ▲

In this definition, and further in this work, we use $\iota$ to denote the identity map.

What extra conditions do we need on $(M, \Phi)$ to have a locally compact quantum group? This question has proved to be a lot more difficult. The first idea that comes to the mind is to generalize the notion of unit and inverse. The first idea that comes to the mind is to generalize the notion of unit and inverse. The first idea that comes to the mind is to generalize the notion of unit and inverse. This brings a lot of problems, since the natural candidates for such generalized unit and inverse are maps which are not well-behaved in general.

A first framework for quantization of locally compact groups, was the theory of Kac algebras (see [4] for an overview). It defines a category which contains the locally compact groups and
which allows a duality theory within the category. Hence it provides a complete answer to the
duality problem formulated above.

However, at this point of time new questions have arisen, to which the Kac algebras cannot
give a full answer. First of all, while studying examples of "quantized groups", sometimes a
structure was found which did not fit into the Kac algebra framework. It looked like the Kac
algebra structure was too restrictive. Moreover, the definition of a Kac algebra claims much
more structure than the definition of a locally compact group. For instance, the existence of a
"Haar measure" is included in the axioms, whereas in the case of a classical group, this is a very
remarkable result of the theory.

For the particular case of compact quantum groups, Woronowicz [23] has developed a theory
which answers all these questions (see [14] for an overview of this theory). For the general case,
the most mature theory at this moment is the one of Kustermans and Vaes [10, 11, 12]. It
realises a large part of the quantization project: the category defined is larger than the Kac
algebra category, and its axioms are more simple. But the existence of the Haar measure is still
included in the axioms, and at this moment a proof for this existence still seems to be far away.

We will follow a different strategy to study duality. Let $G$ be a locally compact group. Fix a left
Haar measure on $G$ and let $\mathcal{H} = L^2(G)$ denote the Hilbert space of (classes of) square integrable
complex functions on $G$. Then the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{H}$ is (isomorphic to) the
Hilbert space $L^2(G \times G)$. Let $W_G \in B(\mathcal{H} \otimes \mathcal{H})$ be the unitary operator defined by

$$(W_G \xi)(p, q) = \xi(p, p^{-1}q)$$

for $\xi \in L^2(G \times G)$ and $p, q \in G$. It was shown by Stinespring [16] for the unimodular, and
by Takesaki [17] for the general case, that $W_G$ contains the complete information about the
algebraic and topological structure of the group $G$. The work [21] of Vanheeswijck where she
studied group duality starting from this operator also is to be situated in this scheme. Kac [7, 8]
defined this operator in an abstract way and proved that the structure of two unimodular Kac
algebras (in his terminology, "ring groups") in duality can equivalently be described in terms of
the associated unitary. Baaj and Skandalis [2] went further in the study of operators "of this
kind" on an abstract level. To a so-called "multiplicative unitary" with certain extra conditions,
they associate two candidate quantum groups in duality. In this work, we continue on this path,
and propose an alternative for the conditions of regularity and irreducibility which Baaj and
Skandalis introduced.

The text is organized as follows. We start in section 2 with a revision of the properties of a
multiplicative unitary, and illustrate these in the case of a classical group. These are not new
results, they are merely included for the convenience of the reader. In the third section, we
introduce our framework and deduce the main results. At the end of this section, we compare our setting with the one of Baaj and Skandalis [2]. In section 4 we study the example of the quantum $az + b$-group. Finally, in section 5, we construct the crossed product with a locally compact group.

This article reflects the work done while the second author was preparing her Ph.D. thesis. A more detailed text can be found in [13].

The story of locally compact quantum groups is told in the language of operator algebras. For a general introduction into the vocabulary and grammar of this language, we refer to [3, 9, 15, 18]. We introduce some notations.

If $H$ is a Hilbert space, we denote by $B(H)$ the von Neumann algebra of all bounded linear operators on $H$. As a general convention, we use the symbol $\odot$ for the algebraic tensor product of vector spaces, and $\otimes$ for completed tensor products. If $H_1$ and $H_2$ are Hilbert spaces, $H_1 \otimes H_2$ denotes the Hilbert space tensor product of $H_1$ and $H_2$. We will denote by $\Sigma$ the flip operator $\Sigma : H_1 \otimes H_2 \to H_2 \otimes H_1$ given by

$$\Sigma(\xi \otimes \eta) = \eta \otimes \xi$$

if $\xi \in H_1$ and $\eta \in H_2$. The corresponding flip acting on operators will be denoted by $\sigma$, so

$$\sigma(x \otimes y) = \Sigma(x \otimes y)\Sigma = y \otimes x$$

for any $x \in B(H_1)$ and $y \in B(H_2)$.

When $M_1$ and $M_2$ are von Neumann algebras, $M_1 \otimes M_2$ will denote the von Neumann algebraic tensor product of $M_1$ and $M_2$. The predual of a von Neumann algebra $M$ will be denoted by $M^*$. The identity operator on a Hilbert space will be denoted by $1_l$. We will frequently use the leg numbering notation: If $W \in B(H \otimes H)$, we define $W_{12}, W_{23}, W_{13} \in B(H \otimes H \otimes H)$ by $W_{12} = W \otimes 1_l$, $W_{23} = 1_l \otimes W$ and $W_{13} = (1_l \otimes \Sigma)(W \otimes 1_l)(1_l \otimes \Sigma)$. Accordingly, leg numbering will be used for tensor products of more than three copies of $H$, or for tensor products of different Hilbert spaces.

For given vectors $\xi, \eta$ of a Hilbert space $H$, we will denote by $\omega_{\xi, \eta}$ the linear functional on $B(H)$ such that $\omega_{\xi, \eta}(x) = \langle x\xi, \eta \rangle$ for any $x \in B(H)$.

If $G$ is a locally compact group, the set of all complex continuous functions on $G$ with compact support will be denoted by $K(G)$. We will use the left Haar measure, and denote it by $dq$.

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2 The multiplicative unitary

We begin by recollecting some material of Baaj and Skandalis [2]. This section does not contain any new results, but is included for completeness and for convenience of the reader.

**Definition 2.1** Let $\mathcal{H}$ be a Hilbert space. A unitary operator $W \in B(\mathcal{H} \otimes \mathcal{H})$ is said to be **multiplicative** if it satisfies the **Pentagon equation** $W_{12}W_{13}W_{23} = W_{23}W_{12}$.

**Example 2.2** Let $G$ be a locally compact group. Let $W_G \in B(\mathcal{H} \otimes \mathcal{H})$ be the operator defined by $(W_G\xi)(p,q) = \xi(p,p^{-1}q)$ for $\xi \in L^2(G \times G)$ and $p,q \in G$. Then $W_G$ is a multiplicative unitary: the Pentagon equation here amounts to the associativity of the product of $G$.

For the rest of this section, we fix a Hilbert space $\mathcal{H}$ and a multiplicative unitary $W$ on $\mathcal{H}$.

Define the sets

- $S = S(W) = \{(\iota \otimes \omega)(W) \mid \omega \in B(\mathcal{H})_+\}$,
- $\hat{S} = \hat{S}(W) = \{\omega \otimes \iota)(W) \mid \omega \in B(\mathcal{H})_+\}$.

**Proposition 2.3** ([2] proposition 1.4) The vector spaces $S$ and $\hat{S}$ are subalgebras of $B(\mathcal{H})$. They act non-degenerately on $\mathcal{H}$.

Let $M = M(W)$ be the von Neumann algebra generated by $S$, i.e. the smallest von Neumann algebra which contains $S$. Let $\hat{M} = \hat{M}(W)$ be the von Neumann algebra generated by $\hat{S}$.

**Proposition 2.4** $W \in M \otimes \hat{M}$.

**Proof:** Clearly $W$ commutes with $M' \otimes \mathbb{I}$ and with $\mathbb{I} \otimes \hat{M}'$, so $W \in (M' \otimes \hat{M}')'$. By the commutation theorem for tensor products of von Neumann algebras (see e.g. [18] theorem 5.9), this last algebra equals $M \otimes \hat{M}$.

The comultiplication on $M$ will be provided by the multiplicative unitary. For $m \in M$ let $\Phi(m) = W^*(\mathbb{I} \otimes m)W$. Then $\Phi$ maps $M$ into $M \otimes M$: For $\omega \in B(\mathcal{H})$ we have

$$W^*(\mathbb{I} \otimes (\iota \otimes \omega)(W))W = (\iota \otimes \iota \otimes \omega)(W_{12}W_{13}W_{23}) = (\iota \otimes \iota \otimes \omega)(W_{13}W_{23}).$$

Hence by proposition 2.4, it follows that $\Phi(m) \in M \otimes M$ for any $m \in S$. The set $\{m \in M \mid \Phi(m) \in M \otimes M\}$ is a von Neumann algebra. Since it contains $S$, it must be the whole of $M$.

**Proposition 2.5** The map $\Phi : M \to M \otimes M$ is a comultiplication.
The dual multiplicative unitary \( \hat{\mathcal{M}} \) have
\[ M_{\xi,\eta} \in \mathcal{H} \] dimensional Hilbert space. Let
\[ \{ M_{\xi,\eta} \} \]
by \( \hat{\mathcal{W}} \) and similarly \( \hat{\mathcal{W}}_m \). With a similar calculation as the one above, one obtains that the product on \( \mathcal{M} \) establishes a duality between \( \mathcal{W} \). Note that also the flipped map \( \Phi' = \sigma \circ \Phi \) is a comultiplication on \( \mathcal{M} \). With a similar calculation as the one above, one obtains that the product on \( \mathcal{M} \) is dual to this flipped coproduct. For \( m_1, m_2 \in \mathcal{M} \) and \( \hat{m} \in \mathcal{M} \) we have
\[
\langle m_1 \otimes m_2 | \hat{\Phi}'(\hat{m}) \rangle = \langle m_1 m_2 | \hat{m} \rangle.
\]
Example 2.7 Let $G$ be a locally compact group. Let $\lambda : G \to B(L^2(G)) : p \mapsto \lambda_p$ denote the left regular representation of $G$, given by $(\lambda_p \xi)(q) = \xi(p^{-1}q)$ if $\xi \in L^2(G)$ and $p, q \in G$. The von Neumann algebra generated by $\{\lambda_p : p \in G\}$ is the group von Neumann algebra $\mathcal{M}(G)$. It is a well-known result (see [6] theorem 22.11) that $\lambda$ also defines a representation of $L^1(G)$ on $L^2(G)$. We will denote this representation again by $\lambda$. It is given by

$$\langle \lambda(f)\xi, \eta \rangle = \int f(q)(\lambda_q(\xi))(p) dq = \int f(q)\xi(q^{-1}p) dq$$

for $f \in K(G) \subseteq L^1(G)$, for $\xi \in K(G) \subseteq L^2(G)$ and $p \in G$. The von Neumann algebra generated by $\{\lambda(f) : f \in L^1(G)\}$ is again $\mathcal{M}(G)$.

Let $W_G$ be the multiplicative unitary associated to $G$. For $\omega = \omega_{\eta_1, \eta_2}$ with $\eta_1, \eta_2 \in K(G)$, we have that

$$((\iota \otimes \omega)(W_G)\xi)(p) = \xi(p) \int \eta_1(p^{-1}q) \overline{\eta_2(q)} dq$$

if $\xi \in L^2(G)$ and $p \in G$. The von Neumann algebra generated by $\mathcal{M}(W_G)$ is $L^\infty(G)$. The comultiplication induced by $W_G$ is given by $\tilde{\Phi}(f)(p, q) = f(pq)$ if $f \in L^\infty(G)$ and $p, q \in G$. Let $\eta_1, \eta_2 \in K(G)$ and denote the function $\eta_1 \overline{\eta_2}$ in $K(G)$ by $f$. Then we have that

$$((\omega_{\eta_1, \eta_2} \otimes \iota)(W_G)\xi)(p) = \langle \lambda(f)\xi, \eta \rangle$$

(2.1)

if $\xi \in L^2(G)$ and $p \in G$. It follows that $\tilde{\mathcal{M}}(W_G) = \mathcal{M}(G)$. The corresponding comultiplication is given by $\tilde{\Phi}(\lambda_p) = \lambda_p \otimes \lambda_p$ for $p \in G$.

So our group $G$ gives rise to two "quantum groups" which are in duality: On the one hand we have the function algebra $L^\infty(G)$ where multiplication is trivial and comultiplication reflects the group product. On the other hand there is the group von Neumann algebra $\mathcal{M}(G)$ where multiplication reflects the group product and comultiplication is trivial. We calculate the duality. Let $\omega \in B(H)$. We saw that $(\iota \otimes \omega)(W_G)$ is (the operator of multiplication by) a function $f \in L^\infty(G)$. Let $\eta_1, \eta_2 \in K(G)$, and let $g = \eta_1 \overline{\eta_2} \in K(G)$. Then equality (2.1) gives us that

$$\lambda(g) = (\omega_{\eta_1, \eta_2} \otimes \iota)(W_G).$$

We have

$$\langle f \mid \lambda(g) \rangle = \langle (\iota \otimes \omega)(W_G) \mid (\omega_{\eta_1, \eta_2} \otimes \iota)(W_G) \rangle$$

$$= \langle \omega_{\eta_1, \eta_2} \otimes \omega(W_G) \rangle = \omega_{\eta_1, \eta_2}(f)$$

$$= \int f(p)g(p) dp.$$

Let us have a look at the case where $G$ is an abelian group. Denote the Pontryagin dual of $G$ by $\hat{G}$, and the duality between $G$ and $\hat{G}$ by $\langle \cdot \mid \cdot \rangle$. By Plancherel’s theorem, for the suitable
choice of the Haar measures on $G$ and $\hat{G}$, the Fourier transform $L^2(G) \to L^2(G) : \xi \to \hat{\xi}$ is an isomorphism of Hilbert spaces. It translates the dual multiplicative unitary $\hat{W}_G$ to the multiplicative unitary $W_{\hat{G}}$ corresponding to the dual group. Indeed, for $\xi \in K(G \times G)$ and $\gamma, \nu \in \hat{G}$ we have

$$(\hat{W}_G \xi)(\gamma, \nu) = \int \int \langle \gamma | p \rangle^{-} \langle \nu | q \rangle^{-} \xi(qp, q) \, dp \, dq$$

$$= \int \int \langle \gamma | q^{-1}p \rangle^{-} \langle \nu | q \rangle^{-} \xi(p, q) \, dp \, dq$$

$$= \int \int \langle \gamma | p \rangle^{-} \langle \gamma^{-1}q^{-1} | q \rangle^{-} \langle \nu | q \rangle^{-} \xi(p, q) \, dp \, dq$$

$$= \hat{\xi}(\gamma, \gamma^{-1} \nu) = (W_{\hat{G}} \hat{\xi})(\gamma, \nu).$$

In this way the theory of multiplicative unitaries generalizes the classical Pontryagin duality.

We would like to remark that this is a very well-known example. Nevertheless we have chosen to include it because of its didactic value. ▲

### 3 Quantum Group Frames

When the von Neumann algebra associated to a multiplicative unitary is commutative, this unitary arises from a classical locally compact group, which can be recovered completely from it ([2], section 2). Unfortunately, the general quantum case is not so straightforward. In order to deduce a number of "group-like" properties, in this section we introduce some extra structure.

At the end of the section, we compare this structure with the conditions of Baaj and Skandalis in [2].

Note that if $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and $W \in B(\mathcal{H} \otimes \mathcal{H})$ is a multiplicative unitary, then $W_{13} = (\iota \otimes \sigma \otimes \iota)(W \otimes 1_{\mathcal{K}} \otimes 1_{\mathcal{K}})$ is a multiplicative unitary on $\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{H} \otimes \mathcal{K}$. Of course this operator will not really carry more information than $W$ itself. To avoid this situation, we introduce the notion "trim". Let $W$ be a multiplicative unitary, and let $S, S^*, \hat{S}, \hat{S}^*, M$ and $\hat{M}$ be as before. We denote by $S\hat{S}^*$ the linear span of the set $\{s\hat{s}^* \mid s \in S \text{ and } \hat{s} \in \hat{S}\}$.

**Proposition 3.1** The $\sigma$-weak closure of $S\hat{S}^*$ is a subalgebra of $B(\mathcal{H})$.

**Proof:** We denote the $\sigma$-weak closure of a set $X$ by $X^-$. For $\omega, \omega' \in B(\mathcal{H})_*$ we have

$$(\omega \otimes \iota)(W^*)(\iota \otimes \omega')(W) = (\omega \otimes \iota \otimes \omega')(W_{12}^*W_{23})$$

$$= (\omega \otimes \iota \otimes \omega')(W_{13}W_{23}W_{12}^*).$$

As $W \in M \otimes \hat{M}$, it can be $\sigma$-weakly approximated by elements of the form $\sum_{i=1}^n m_i \otimes \hat{m}_i$ with $m_i \in M$ and $\hat{m}_i \in \hat{M}$. Using this we obtain that the operator above can be $\sigma$-weakly
approximated by elements
\[ \sum_{i=1}^{n} (\iota \otimes \omega'(\hat{m}_i \cdot))(W) \left( \omega(m_i \cdot) \otimes \iota \right)(W^*) \]
of \( S\hat{S}^* \). So \( \hat{S}^*S \) is contained in \( (S\hat{S}^*)^- \). Hence \((S\hat{S}^*)^- (S\hat{S}^*)^- \) is contained in \((SS\hat{S}^*\hat{S}^*)^- \subseteq (S\hat{S}^*)^- \).

**Definition 3.2** We call the multiplicative unitary \( W \) *trim* if the algebra \( S\hat{S}^* \) is \( \sigma \)-weakly dense in \( \mathcal{B}(\mathcal{H}) \).

We have that the algebra \( S\hat{S}^* \) acts non-degenerately on \( \mathcal{H} \). Hence if the \( \sigma \)-weak closures of \( S \) and \( \hat{S} \) are \( * \)-algebras, the \( \sigma \)-weak closure of \( S\hat{S} \) will be a von Neumann algebra. When this von Neumann algebra is not the whole of \( \mathcal{B}(\mathcal{H}) \), there is a non-trivial projection \( p \in \mathcal{B}(\mathcal{H}) \) which commutes with both \( S \) and \( \hat{S} \). In that case we can “cut down \( W \)” by this projection, and study the multiplicative unitary \( W(p \otimes p) = (p \otimes p)W(p \otimes p) \) on the restricted Hilbert space \( p\mathcal{H} \otimes p\mathcal{H} \). It is clear that this restricted multiplicative unitary still carries essentially all the information of the underlying duality structure.

**Example 3.3** Let \( G \) be a locally compact group, and \( W_G \) be as in example 2.7. Since \( L^\infty(G)' = L^\infty(G) \), we have that \( L^\infty(G)' \cap \mathcal{M}(G)' = \mathbb{C}I \). Hence \( L^\infty(G)\mathcal{M}(G) \) is \( \sigma \)-weakly dense in \( \mathcal{B}(\mathcal{H}) \) and \( W_G \) is trim.

Now we are ready to introduce our framework.

**Definition 3.4** Let \( \mathcal{H} \) be a Hilbert space. A *quantum group frame* on \( \mathcal{H} \) is a triple \( (W, J, \hat{J}) \) such that

1. \( W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \) is a multiplicative unitary;
2. \( W \) is trim (cf. definition 3.2);
3. \( J \) and \( \hat{J} \) are anti-linear operators on \( \mathcal{H} \) such that \( J = J^* \), \( J^2 = I \), \( \hat{J} = \hat{J}^* \), \( \hat{J}^2 = I \) and \( W^* = (\hat{J} \otimes J)W(\hat{J} \otimes J) \);
4. \( JMJ \subseteq M' \) and \( \hat{J}M\hat{J} \subseteq \hat{M}' \).

**Remark 3.5** The definition of a quantum group frame is self-dual. Indeed, \( S(\hat{W})\bar{S}(\hat{W})^* \) is the algebra \( \hat{S}(W)^*S(W) \); by the argument in the proof of proposition 3.1 it has the same closure as the algebra \( S(W)\bar{S}(W)^* \). Therefore \( \hat{W} \) is trim if and only if \( W \) is trim. And if \( (W, J, \hat{J}) \) is a quantum group frame then also \( (\hat{W}, \hat{J}, J) \) is one.
We postpone the discussion about the meaning of the operators $J$ and $\hat{J}$ for a little while, and first show that the $\sigma$-weak closures of the algebras $S$ and $\hat{S}$ associated to a quantum group frame are $^*$-algebras — see theorem 3.10 below. Hence, since $S$ and $\hat{S}$ act non-degenerately on $\mathcal{H}$, these closures will coincide with the von Neumann algebras $M$ and $\hat{M}$. This property is crucial for the further development of the theory. Although we don’t know of any examples where it is not satisfied, it is not a straightforward result. In order to obtain the corresponding result on the $C^*$-algebraic level (i.e. to deduce that the norm-closure of $S$ is a $^*$-algebra), Baaj and Skandalis [2] introduced the notion of regularity. This notion later was weakened by Baaj in his work [1]. We will compare our axioms with their notions in proposition 3.13.

We start with some technical results.

**Lemma 3.6** Let $\mathcal{H}$ be a Hilbert space, and let $B$ be a linear subspace of $\mathcal{B}(\mathcal{H})$ such that the weak closure of $B$ contains $\mathbb{1}$. Then the vector spaces generated by

$$\{\omega(\cdot b) \mid \omega \in \mathcal{B}(\mathcal{H})_*, \ b \in B\}$$

$$\{\omega(b \cdot) \mid \omega \in \mathcal{B}(\mathcal{H})_*, \ b \in B\},$$

are both norm dense subspaces of $\mathcal{B}(\mathcal{H})_*$.  

**Proof:** We will prove the first vector space to be norm dense in $\mathcal{B}(\mathcal{H})_*$. The norm density of the second one follows analogously.

Since the weak closure of $B$ contains $\mathbb{1}$, we have that $B\mathcal{H}$ is weakly dense in $\mathcal{H}$. Hence, because it is a subspace, $B\mathcal{H}$ is also norm dense in $\mathcal{H}$.

From this, it follows that the closed vector space generated by

$$\{\omega_{\xi,\eta} \mid b \in B; \ \xi,\eta \in \mathcal{H}\}$$

in $\mathcal{B}(\mathcal{H})^*$ is the same as the one generated by $\{\omega_{\xi,\eta} \mid \xi,\eta \in \mathcal{H}\}$. The latter subset generates $\mathcal{B}(\mathcal{H})_*$.  

**Lemma 3.7** Let $W$ be a multiplicative unitary on a Hilbert space $H$, and $\hat{J}$, $J$ anti-linear operators on $\mathcal{H}$ such that $J = J^*$, $J^2 = \mathbb{1}$, $\hat{J} = \hat{J}^*$, $\hat{J}^2 = \mathbb{1}$ and $W^* = (\hat{J} \otimes J)W(\hat{J} \otimes J)$. Then $\hat{J}S\hat{J} = S^*$ and $J\hat{S}J = \hat{S}^*$.

**Proof:** For $\omega \in \mathcal{B}(\mathcal{H})_*$ we have that

$$\hat{J}(\iota \otimes \omega)(W)\hat{J} = (\iota \otimes \theta)(W^*),$$

where $\iota$ and $\theta$ are the identity and anti-linear maps, respectively.
where $\theta$ is the linear functional on $\mathcal{B}(\mathcal{H})$ given by $\theta(x) = \omega(JxJ^*)$ for any $x \in \mathcal{B}(\mathcal{H})$. Hence $\hat{J}S\hat{J} = S^*$. The second result follows analogously.

Now let $(W, J, \hat{J})$ be a quantum group frame on a Hilbert space $\mathcal{H}$.

**Lemma 3.8** The linear span of the set

$$\{(x \otimes 1)W(1 \otimes y) \mid x, y \in \mathcal{B}(\mathcal{H})\}$$

is $\sigma$-weakly dense in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})$.

**Proof:** Let $\xi, \eta \in \mathcal{H}$ and let $\omega$ denote the vector functional $\omega_{\xi,\eta}$. The multiplicativity of $W$ gives us

$$1 \otimes (t \otimes \omega)(W) = (t \otimes t \otimes \omega)(W_{13}W_{12}^*W_{23}W_{12}).$$

Let $\{e_i \mid i \in I\}$ be an orthonormal basis for $\mathcal{H}$. Let $p_i$ denote the projection operator on $\mathbb{C}e_i$. Then $\sum_{i \in I} p_i = 1$, where we have convergence in (e.g.) the $\sigma$-weak topology. Now

$$1 \otimes (t \otimes \omega)(W) = \sum_{i \in I} (t \otimes t \otimes \omega)(W_{13}^*(1 \otimes p_i)W_{23}W_{12})$$

$$= \sum_{i \in I} (t \otimes t \otimes \omega_{e_i,\eta})(W_{13}^*)W^*(t \otimes t \otimes \omega_{\xi,e_i})(W_{23})W,$$

with convergence in the $\sigma$-weak topology. We will denote the $\sigma$-weak closure of the linear span of a set $X$ by $\overline{sp}(X)$. The above calculation gives us that

$$1 \otimes S \subseteq \overline{sp}\{(s^* \otimes 1)W^*(1 \otimes s')W \mid s, s' \in S\}.$$ 

Multiplying these sets by the vector space $\mathcal{B}(\mathcal{H}) \otimes \hat{M}'$ on the left hand side (recall that $\otimes$ denotes the algebraic tensor product of vector spaces) gives

$$\mathcal{B}(\mathcal{H}) \otimes \hat{M}'S \subseteq \overline{sp}\{(x \otimes 1)W^*(1 \otimes y)W \mid x, y \in \mathcal{B}(\mathcal{H})\}.$$ 

Axiom (2) of definition 3.4 gives us that the set $SS^*$ is $\sigma$-weakly dense in $\mathcal{B}(\mathcal{H})$, so also $\hat{J}S\hat{J}\hat{J}S^*\hat{J}$ is $\sigma$-weakly dense in $\mathcal{B}(\mathcal{H})$. By lemma 3.7 and axiom (4), the latter set is contained in $S^*\hat{M}'$, which therefore is dense in $\mathcal{B}(\mathcal{H})$ as well. The same then holds for its adjoint $\hat{M}'S$. So we obtain that

$$\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) = \overline{sp}\{(x \otimes 1)W^*(1 \otimes y)W \mid x, y \in \mathcal{B}(\mathcal{H})\}.$$ 

Multiplying these spaces on the right hand side with the unitary $W^*$, we have

$$\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) = \overline{sp}\{(x \otimes 1)W^*(1 \otimes y) \mid x, y \in \mathcal{B}(\mathcal{H})\}.$$ 

Using $W^* = (\hat{J} \otimes J)W(\hat{J} \otimes J)$ now gives the result. ■
Following Baaj and Skandalis [2], we introduce the vector space

$$C = \{(\iota \otimes \omega)(\Sigma W) \mid \omega \in \mathcal{B}(\mathcal{H})_*\}. \quad (3.2)$$

Similarly, let $D$ be the vector space

$$D = \{((\omega \otimes \iota)\Sigma W) \mid \omega \in \mathcal{B}(\mathcal{H})_*\} = \{((\iota \otimes \omega)(W\Sigma) \mid \omega \in \mathcal{B}(\mathcal{H})_*\}. \quad (3.3)$$

**Lemma 3.9** The vector spaces $C$ and $D$ are $\sigma$-weakly dense in $\mathcal{B}(\mathcal{H})$.

**Proof:** We give the proof for $C$. The result for $D$ is proved analogously, or follows from the previous by using that $C^* = JD\bar{J}$. Let $\xi, \xi', \eta, \eta' \in \mathcal{H}$ and let $x, y$ be the rank one operators $x = (\cdot, \xi)\xi'$ and $y = (\cdot, \eta)\eta'$. We have that

$$\Sigma(x \otimes 1\mathbb{I})W(1\mathbb{I} \otimes y) = (1\mathbb{I} \otimes x)\Sigma W(1\mathbb{I} \otimes y) = (\iota \otimes \omega_{\eta', \xi})(\Sigma W) \otimes (\cdot, \eta)\xi'. \quad (3.2)$$

By lemma 3.8, the closure of the vector space spanned by

$$\{(\iota \otimes \omega)(\Sigma(x \otimes 1\mathbb{I})W(1\mathbb{I} \otimes y)) \mid \omega \in \mathcal{B}(\mathcal{H})_*; x, y \text{ are rank one operators on } \mathcal{H}\}$$

is $\mathcal{B}(\mathcal{H})$. Hence we obtain that the closure of $C$ is $\mathcal{B}(\mathcal{H})$. \hfill \blacksquare

We then can use the technique of Baaj and Skandalis ([2] proposition 3.5) to obtain the following result.

**Theorem 3.10** Let $(W, J, \tilde{J})$ be a quantum group frame. Then the $\sigma$-weak closure of $S$ is a $\ast$-algebra. Hence it coincides with the von Neumann algebra $\mathcal{M}$. Similarly, the $\sigma$-weak closure of $\hat{S}$ is the von Neumann algebra $\hat{\mathcal{M}}$.

**Proof:** As before, we denote the $\sigma$-weak closure of the linear span of a set $X$ by $\mathcal{SP}(X)$. Since by lemma 3.9, the set $D$ is $\sigma$-weakly dense in $\mathcal{B}(\mathcal{H})$, it follows from lemma 3.6 that the set $\{\omega'(z \cdot) \mid z \in D, \omega \in \mathcal{B}(\mathcal{H})_*\}$ is dense in $\mathcal{B}(\mathcal{H})_*$. We have

$$(S^*)^- = \mathcal{SP}\{(\iota \otimes \omega(\cdot z))(W^*) \mid z = (\iota \otimes \omega')(W\Sigma); \omega, \omega' \in \mathcal{B}(\mathcal{H})_*\} = \mathcal{SP}\{(\iota \otimes \omega \otimes \omega')(W_{12}W_{23}\Sigma_{23}) \mid \omega, \omega' \in \mathcal{B}(\mathcal{H})_*\} = \mathcal{SP}\{(\iota \otimes \omega \otimes \omega')(W_{13}W_{23}W_{12}^\ast\Sigma_{23}) \mid \omega, \omega' \in \mathcal{B}(\mathcal{H})_*\} = \mathcal{SP}\{(\iota \otimes \omega')(W(1\mathbb{I} \otimes y)W^*) \mid y = (\omega \otimes \iota)(W\Sigma); \omega, \omega' \in \mathcal{B}(\mathcal{H})_*\} = \mathcal{SP}\{(\iota \otimes \omega')(W(1\mathbb{I} \otimes y)W^*) \mid \omega' \in \mathcal{B}(\mathcal{H})_*; y \in C\}. \quad (3.3)$$

Since the $\sigma$-weak closure of $C$ is $\mathcal{B}(\mathcal{H})$, it follows that $(S^*)^-$ is self-adjoint, so also $S^-$ is self-adjoint. By duality we obtain that also the $\sigma$-weak closure of $\hat{S}$ is self-adjoint. \hfill \blacksquare
Let us now have a closer look at the operators \( J \) and \( \hat{J} \). We first return to the example 3.3 of a classical group \( G \). Let \( J_G \) and \( \hat{J}_G \) be the anti-linear operators on \( L^2(G) \) defined by

\[
(J_G \xi)(p) = \xi(p), \quad (\hat{J}_G \xi)(p) = \Delta(p)^{-\frac{1}{2}} \xi(p^{-1}),
\]

where \( \Delta \) denotes the modular function of \( G \). It is straightforward to check that axioms (3) and (4) are satisfied. We argued in example 3.3 that also axiom (2) is satisfied. Hence \((W_G, J_G, \hat{J}_G)\) is a quantum group frame.

So in the group example, we see that \( \hat{J} \) is linked with the inverse operation. In the quantum world, the role of the inverse is played by the antipode. In general this antipode will not be a bounded operator, but \( \hat{J} \) will implement its unitary part (and so will \( J \) for the dual picture) — see [10]. Note that by lemma 3.7, we have that \( \hat{J}M\hat{J} = M \) and \( J\hat{J}M = \hat{M} \). So there are linear maps

\[
R : M \to M : m \mapsto \hat{J}m\hat{J}, \\
\hat{R} : \hat{M} \to \hat{M} : \hat{m} \mapsto J\hat{m}J.
\]

In the group case, \( R : L^\infty(G) \to L^\infty(G) \) is given by \((Rf)(p) = f(p^{-1})\) whenever \( f \in L^\infty(G) \), and \( \hat{R} : \mathcal{M}(G) \to \mathcal{M}(G) \) is given by \( \hat{R}(\lambda_p) = \lambda_{p^{-1}} \) for \( p \in G \).

The maps \( R \) and \( \hat{R} \) are *-anti-automorphisms. Moreover, they flip the comultiplication :

**Proposition 3.11** We have that

\[
\Phi \circ R = \sigma \circ (R \otimes R) \circ \Phi, \\
\hat{\Phi} \circ \hat{R} = \sigma \circ (\hat{R} \otimes \hat{R}) \circ \hat{\Phi}.
\]

**Proof:** We have

\[
W_{12}^*W_{23}W_{12} = W_{13}W_{23} = (\hat{J} \otimes \hat{J} \otimes J)W_{13}^*W_{23}^*(\hat{J} \otimes \hat{J} \otimes J) \\
= (\sigma \otimes \iota)((\hat{J} \otimes \hat{J} \otimes J)W_{13}^*W_{23}^*(\hat{J} \otimes \hat{J} \otimes J)) \\
= (\sigma \otimes \iota)((\hat{J} \otimes \hat{J} \otimes J)W_{12}^*W_{23}^*W_{12}(\hat{J} \otimes \hat{J} \otimes J)).
\]

Let \( \omega \in \mathcal{B}(\mathcal{H})_* \) and let \( m = (\iota \otimes \omega)(W) \). Define \( \theta \in \mathcal{B}(\mathcal{H})_* \) by \( \theta(x) = \omega(Jx^*J) \). So \( (\iota \otimes \theta)(W) = R(m) \). Applying \( (\iota \otimes \iota \otimes \theta) \) to the above equality gives \( \Phi(R(m)) = \sigma((R \otimes R)\Phi(m)) \). By theorem 3.10, such elements \( m \) form a \( \sigma \)-weakly dense part of the von Neumann algebra \( M \). So the first statement follows. The other equality follows by duality. 

\[ \blacksquare \]
In the following, we will discuss how our extra conditions on $W$ relate to the conditions of regularity and irreducibility which Baaj and Skandalis impose in [2]. Let us fix a Hilbert space $H$ and a multiplicative unitary $W$ on $H$.

As before, let $C = \{(\iota \otimes \omega)(\Sigma W) \mid \omega \in \mathcal{B}(H)_+\}$. Without extra conditions lemma 3.9 is not valid anymore. But we have the following property:

\textbf{Lemma 3.12} ([2] proposition 3.2)

$C$ is an algebra which acts non-degenerately on $H$.

Baaj and Skandalis [2] define $W$ to be \textit{regular} if the norm closure of $C$ equals $\mathcal{K}(H)$, the $C^*$-algebra of compact operators on $H$. Baaj [1] defines $W$ to be \textit{semi-regular} if the norm closure of $C$ contains $\mathcal{K}(H)$. The other extra condition which is imposed on $W$ in [2] is the irreducibility.

The multiplicative unitary $W$ is called \textit{irreducible} if there exists a unitary $U \in \mathcal{B}(H)$ such that

\begin{align*}
(1) &\quad U^2 = 1 \text{ and } (\Sigma (\mathbb{1} \otimes U)W)^3 = 1, \\
(2) &\quad \text{the unitaries } \overline{W} = \Sigma (U \otimes \mathbb{1})W(U \otimes \mathbb{1})\Sigma \text{ and } \widetilde{W} = (U \otimes U)\overline{W}(U \otimes U) \text{ are multiplicative.}
\end{align*}

In our setting, the product $\hat{J}J$ will play the role of $U$. Putting $U = \hat{J}J$, it is not immediately clear whether $U^2 = 1$, so we will define $\overline{W}$ and $\widetilde{W}$ as

\begin{align*}
\overline{W} &= \Sigma (U^* \otimes \mathbb{1})W(U \otimes \mathbb{1})\Sigma = (J \otimes J)\Sigma W^* \Sigma (J \otimes J) \\
\widetilde{W} &= (U \otimes U)\overline{W}(U^* \otimes U^*) = (\hat{J} \otimes \hat{J})\Sigma W^* \Sigma (\hat{J} \otimes \hat{J}).
\end{align*}

\textbf{Proposition 3.13} Let $\mathcal{H}$ be a Hilbert space and $W \in \mathcal{B}(H \otimes H)$ a multiplicative unitary. Suppose that $J$ and $\hat{J}$ are anti-linear operators on $\mathcal{H}$ and that $J = J^*$, $J^2 = \mathbb{1}$, $\hat{J} = \hat{J}^*$, $\hat{J}^2 = \mathbb{1}$ and $W^* = (\hat{J} \otimes J)W(\hat{J} \otimes J)$. Then the following two sets of conditions are equivalent:

\begin{enumerate}
\item $W$ is \textit{trim},
\item $JMJ \subseteq M'$ and $\hat{J}M\hat{J} \subseteq \hat{M}'$;
\end{enumerate}

and

\begin{enumerate}
\item[(1')] $C$ is $\sigma$-weakly dense in $\mathcal{B}(\mathcal{H})$,
\item[(2')] $(\mathbb{1} \otimes \hat{J}J)\Sigma \overline{W}W\widetilde{W}$ is a scalar multiple of $\mathbb{1}$.
\end{enumerate}

Condition (1') is weaker than regularity and semi-regularity. Of course, the existence of the two anti-linear operators $J$ and $\hat{J}$ is a stronger claim than the existence of only their product. But from the examples it will become clear that these operators play a very natural role, so that it is reasonable to work with $J$ and $\hat{J}$ instead of their product. In this setting, the unitaries $\overline{W}$ and $\widetilde{W}$ are automatically multiplicative. This is another reason why these are natural conditions to
impose. Condition (2') slightly weakens the further irreducibility condition \( (\Sigma(1 \otimes U)W)^3 = 1 \), which is equivalent to \((1 \otimes U)\Sigma W W \bar{W} = 1\). In the example of the quantum \( ab + c\)-group (see section 4) we will indeed obtain a scalar which is not 1. Nevertheless, this turns out not to be really essential, as we will see in the discussion after proposition 3.14.

Altogether, we think our conditions are more natural than conditions like (1') and (2'), and therefore easier to verify in examples. However, a drawback is that we have not been able to deduce that they imply the norm closure of \( S \) to be a C*-algebra. But we did obtain that its \( \sigma \)-weak closure is a von Neumann algebra.

**Proof:** Suppose that the conditions (1) and (2) are satisfied.

- Lemma 3.9 gives us that condition (1') then holds.

- We will show that \((1 \otimes \hat{J}J)\Sigma W W \bar{W}\) commutes with \(M \otimes 1\). Similar calculations show that it also commutes with \(\hat{M} \otimes 1\), \(1 \otimes M\) and \(1 \otimes \hat{M}\). Hence by (1), it must be a scalar multiple of 1.

First note that using the multiplicativity of \( W \) we have

\[
(\hat{J} \otimes \hat{J} \otimes J)W_{12}^*W_{23}^* (\hat{J} \otimes \hat{J} \otimes J) = (\hat{J} \otimes \hat{J} \otimes J)W_{13}^*W_{12}^* (\hat{J} \otimes \hat{J} \otimes J) = \Sigma_{12} (\hat{J} \otimes \hat{J} \otimes J)W_{23}^*W_{23}^* \Sigma_{12} (\hat{J} \otimes \hat{J} \otimes J).
\]

Replacing \( W^* \) by \((\hat{J} \otimes J)W(\hat{J} \otimes J)\) we obtain that

\[
(1 \otimes \hat{J}J \otimes 1)W_{12}(1 \otimes J\hat{J} \otimes 1)W_{23} = \Sigma_{12} W_{13} W_{23} (\hat{J} J \otimes 1 \otimes 1) \Sigma_{12} W_{12} (1 \otimes J\hat{J} \otimes 1)
\]

and using the multiplicativity once again, this gives

\[
(1 \otimes \hat{J}J \otimes 1)W_{12}(1 \otimes J\hat{J} \otimes 1)W_{23} = \Sigma_{12} W_{12}^* W_{23} W_{12} (\hat{J} J \otimes 1 \otimes 1) \Sigma_{12} W_{12} (1 \otimes J\hat{J} \otimes 1).
\]

Therefore, moving some of the factors of this equality of operators to the other side, we have

\[
W_{23}^* W_{12} (\hat{J} J \otimes 1 \otimes 1) \Sigma_{12} W_{12} (1 \otimes J\hat{J} \otimes 1)
= W_{12} (\hat{J} J \otimes 1 \otimes 1) \Sigma_{12} W_{12} (1 \otimes J\hat{J} \otimes 1) W_{23}^*.
\]

It follows that \((\Sigma W(\hat{J} J \otimes 1) \Sigma W(1 \otimes J\hat{J})) \Sigma_{12}\) commutes with \( W_{13} \), and hence the algebra \(M \otimes 1\) commutes with \(\Sigma W(\hat{J} J \otimes 1) \Sigma W(1 \otimes J\hat{J}) \Sigma\). Using the fact that \(\hat{J} M \hat{J} = M\) and assumption (2) \(JMJ \subseteq M'\), we may conclude that \(M \otimes 1\) commutes with

\[
(\hat{J} J \otimes \hat{J} J)W(\hat{J} J \otimes 1) \Sigma W(\hat{J} J \otimes 1) \Sigma W(1 \otimes J\hat{J}) \Sigma = (1 \otimes \hat{J}J) \Sigma W W \bar{W}.
\]
Conversely, suppose that \((1 \otimes \hat{J}J)\Sigma W \hat{W} = k1 \otimes 1\) for some \(k \in \mathbb{C}\).

- Following the proof of proposition 3.11 we have

\[
W_{12}^* W_{23} W_{12} = \Sigma_{12}(\hat{J} \otimes \hat{J} \otimes J)W_{12}^* W_{23} W_{12}(\hat{J} \otimes \hat{J} \otimes J)\Sigma_{12}
\]
\[
= \Sigma_{12}(\hat{J} \otimes \hat{J} \otimes J)W_{12}(J \otimes \hat{J} \otimes J)W_{23}(J \otimes \hat{J} \otimes J)W_{12}(\hat{J} \otimes \hat{J} \otimes J)\Sigma_{12}.
\]

Commuting \(\Sigma_{12}\) with \((\hat{J} \otimes \hat{J} \otimes J)\) and moving the factors \(W_{12}\) of the left hand side and \((\hat{J} \otimes \hat{J} \otimes J)\Sigma_{12} W_{12}^* (J \otimes \hat{J} \otimes J)\) of the right hand side to the other side, we obtain

\[
(J \otimes \hat{J} \otimes J)W_{12}\Sigma_{12}(\hat{J} \otimes \hat{J} \otimes J)W_{12}^* W_{23}
\]
\[
= W_{23}(J \otimes \hat{J} \otimes J)W_{12}\Sigma_{12}(\hat{J} \otimes \hat{J} \otimes J)W_{12}^*.
\]

By assumption, we have that

\[
W = k\Sigma(1 \otimes J\hat{J})\Sigma \hat{W}^* = k(J \otimes J)\Sigma \hat{W}^* = k(1 \otimes \hat{J}J)\Sigma \hat{W}^*.
\]

So equality (3.4) gives us that \(W_{23}\) and \(\hat{W}_{12}\) commute. Using the fact that

\[
JS^* J = \{(\omega \otimes \iota)((J \otimes J)\Sigma \hat{W}^* \Sigma (J \otimes J)) \mid \omega \in \mathcal{B}(\mathcal{H})_*\}
\]
\[
= \{(\omega \otimes \iota)(\hat{W}) \mid \omega \in \mathcal{B}(\mathcal{H})_*\},
\]

we conclude that \(JMJ \subseteq M'\).

- We show that also the triple \((\hat{W}, \hat{J}, J)\) satisfies property \((2')\). Then it will follow from the above that \(\hat{J}M\hat{J} \subseteq \hat{M}'\). We have

\[
\left( (1 \otimes J\hat{J})\Sigma \hat{W} \right)^* \hat{W} = \Sigma(J \otimes J)\Sigma \hat{W}^* \Sigma \hat{W}(\hat{J} \otimes \hat{J})\Sigma \hat{W}^* \Sigma \hat{W}(1 \otimes J\hat{J})
\]
\[
= (\Sigma \hat{W}^* \Sigma \hat{W})(1 \otimes J\hat{J}) = k(1 \otimes J\hat{J}) = k(1 \otimes 1).
\]

Hence \((1 \otimes J\hat{J})\Sigma \hat{W} \hat{W} = k1 \otimes 1\).

- Finally we will show that \(W\) is trim. For \(\omega \in \mathcal{B}(\mathcal{H})_*\), we have that

\[
(\iota \otimes \omega)(\hat{W}) = (\iota \otimes \omega)((\hat{J} \otimes \hat{J})W^* \Sigma (\hat{J} \otimes \hat{J}));
\]

the set of these elements is \(\hat{C}^* \hat{J}\). From the assumption it then follows that

\[
\hat{C}^* \hat{J} = \{(\iota \otimes \omega)(\Sigma W^* \Sigma (J \otimes J)W(J \otimes \hat{J})) \mid \omega \in \mathcal{B}(\mathcal{H})_*\}. \tag{3.5}
\]
Let $\xi, \eta \in H$ and let $\omega$ denote the vector functional $\omega_{\xi, \eta}$. Let $\theta \in \mathcal{B}(H)_*$ be given by $\theta(x) = \omega(JxJ^-)$, so $\theta = \omega_{\hat{J}x, \hat{J}x}$. Like in the proof of lemma 3.8, let $\{e_i \mid i \in I\}$ be an orthonormal basis for $H$. Let $p_i$ denote the projection operator on $\mathbb{C}e_i$. We have

$$\begin{align*}
(t \otimes \omega)(\Sigma W^* \Sigma (J \otimes J)W(J \otimes \hat{J})) & = J ((t \otimes \theta)(\Sigma W \Sigma (\mathbb{1} \otimes \hat{J} J)W)) J \\
 & = J \left( \sum_{i \in I}(t \otimes \omega_{\hat{J}x_i, \hat{J}x_i})(\Sigma W \Sigma (\mathbb{1} \otimes p_i) (\mathbb{1} \otimes \hat{J} J)W) \right) J \\
 & = J \left( \sum_{i \in I}(t \otimes \omega_{e_i, \hat{J}x_i})(\Sigma W \Sigma (t \otimes \omega_{\hat{J}x_i, e_i})((\mathbb{1} \otimes \hat{J} J)W)) \right) J \\
 & = J \left( \sum_{i \in I}(\omega_{e_i, \hat{J}x_i} \otimes t)(W)(t \otimes \omega_{\hat{J}x_i, e_i})((\mathbb{1} \otimes \hat{J} J)W) \right) J,
\end{align*}$$

with convergence in (e.g.) the $\sigma$-weak topology. This element belongs to the closure of $J\hat{M}MJ$. Together with equality (3.5), this gives us that $\hat{J}C^*\hat{J}$ is contained in the closure of $J\hat{M}MJ$. Therefore if $C$ is $\sigma$-weakly dense in $\mathcal{B}(H)$, then $W$ is trim.

We can now show that $J$ and $\hat{J}$ commute up to a scalar:

**Proposition 3.14** Let $(W, J, \hat{J})$ be a quantum group frame. There exists a complex number $\lambda$ with modulus 1 such that $\hat{J}C^*\hat{J} = \lambda J$.

**Proof:** Proposition 3.13 tells us that there exists $k \in \mathbb{C}$ such that

$$k(1 \otimes J\hat{J}) = \Sigma W^* W \hat{W}$$

$$= (J \otimes J)W^* (J \otimes J)\Sigma W \Sigma (J \otimes \hat{J})W^* (\hat{J} \otimes \hat{J})\Sigma \quad (3.6)$$

$$= (J\hat{J} \otimes \mathbb{1}) W(\hat{J} J \otimes \mathbb{1}) \Sigma W \Sigma (\mathbb{1} \otimes \hat{J} J)W(\mathbb{1} \otimes J\hat{J})\Sigma \quad (3.7)$$

Taking adjoints of equality (3.6) we get

$$\overline{k}(1 \otimes J\hat{J}) = \Sigma (J \otimes \hat{J})W(\hat{J} \otimes \hat{J})\Sigma W^* (J \otimes J)W(J \otimes J)$$

$$= \Sigma (J \otimes \hat{J})(J\hat{J} \otimes \mathbb{1}) W(\hat{J} J \otimes \mathbb{1}) \Sigma W (\mathbb{1} \otimes \hat{J} J) W(\mathbb{1} \otimes J\hat{J}) \Sigma (J \otimes \hat{J}),$$

and using equality (3.7) we obtain

$$\overline{k}(1 \otimes J\hat{J}) = \Sigma (J \otimes \hat{J}) k(1 \otimes J\hat{J}) \Sigma (J \otimes \hat{J}) = \overline{k}(J\hat{J} J \otimes \hat{J} J).$$

Hence $J\hat{J} J \otimes J\hat{J} J \hat{J} = \mathbb{1}$. So there is $\lambda \in \mathbb{C}$ such that $J\hat{J} J \hat{J} = \lambda \mathbb{1}$, and hence $J\hat{J} = \lambda \hat{J} J$. Since both $J\hat{J}$ and $\hat{J} J$ are unitary operators, $\lambda$ must have modulus 1.

\[ \square \]
Note that when $J$ is multiplied by a complex number $\gamma$ of modulus 1, the properties $J^2 = \mathbb{1}$ and $J^* = J$ remain unchanged. Also the anti-automorphism $R$ is not altered. Hence if $(W, J, \hat{J})$ is a quantum group frame, then so is $(W, \gamma J, \hat{J})$. Choosing $\gamma$ such that $\gamma^{-2} = \lambda$ gives us

$$(\gamma J)\hat{J} = \gamma \lambda \hat{J} J = \overline{\gamma} \hat{J} = \hat{J}(\gamma J).$$

This means that we have some choice when we define $J$, and we can determine $J$ in such a way that $J$ and $\hat{J}$ commute. However, often there are natural candidates for $J$ and $\hat{J}$, which do not necessarily commute — see the example of the $az + b$-group in section 4. We will say something more about this matter in remark 3.17.

As we mentioned in the introduction, the most mature theory of locally compact quantum groups is the one introduced by Kustermans and Vaes in [10, 11, 12]. In the following we argue that the quantum groups which satisfy their definition, give rise to a quantum group frame. Since we work in the von Neumann algebra setting, we will use the definition on the von Neumann algebra level. Let us first recall this definition.

**Definition 3.15** ([12] Definition 1.1) Consider a von Neumann algebra $M$ together with a unital normal $^*$-homomorphism $\Phi : M \to M \otimes M$ such that $(\Phi \otimes \iota)\Phi = (\iota \otimes \Phi)\Phi$. Assume moreover the existence of

1. a normal semi-finite faithful (n.s.f.) weight $\phi$ on $M$ that is left invariant, i.e. such that $\phi((\omega \otimes \iota)\Phi(x)) = \phi(x)\omega(\mathbb{1})$ for all $\omega \in M^+_*$ and $x \in M^+$ such that $\phi(x) < \infty$;

2. a normal semi-finite faithful weight $\psi$ on $M$ that is right invariant, i.e. such that $\psi((\iota \otimes \omega)\Phi(x)) = \psi(x)\omega(\mathbb{1})$ for all $\omega \in M^+_*$ and $x \in M^+$ such that $\psi(x) < \infty$.

Then the pair $(M, \Phi)$ is called a locally compact quantum group. ▲

Let $(M, \Phi)$ be a locally compact quantum group. Let $\varphi$ be a n.s.f. left invariant weight on $(M, \Phi)$, and let $(\mathcal{H}, \iota, \Lambda)$ be a GNS-construction for $\varphi$. Denote the modular conjugation of $\varphi$ with respect to this GNS-construction by $J$. Theorem 2.1 of [12] and the comments thereafter state that there exists a unique unitary element $W \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H})$ such that

$$W^*(\Lambda(x) \otimes \Lambda(y)) = (\Lambda \otimes \Lambda)(\Phi(y)(x \otimes \mathbb{1}))$$

for all $x, y \in M$ such that $\varphi(x^*x) < \infty$ and $\varphi(y^*y) < \infty$. This unitary satisfies the Pentagon equation. The $\sigma$-weak closure of the set $S = \{(\iota \otimes \omega)(W) \mid \omega \in \mathcal{B}(\mathcal{H})_*\}$ is the von Neumann algebra $M$, and for $x \in M$ one has that $\Phi(x) = W^*(\mathbb{1} \otimes x)W$. (Note that in [12] the $\sigma$-strong $^*$-closure is considered; but since the norm closure of $S$ is a $^*$-algebra, its $\sigma$-strong $^*$-closure coincides with the $\sigma$-weak closure.)

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Define $\widehat{M}$ to be the $\sigma$-weak closure of the set \((\omega \otimes 1)(W) \mid \omega \in B(H)^*\). Then $\widehat{M}$ is a von Neumann algebra. The correspondence $\widehat{\Phi}(x) = \Sigma W(x \otimes 1)W^*\Sigma$ defines a unique normal $\ast$-homomorphism $\widehat{\Phi} : \widehat{M} \to \widehat{M} \otimes \widehat{M}$. The pair $(\widehat{M}, \widehat{\Phi})$ is again a locally compact quantum group, called the dual of $(M, \Phi)$. The dual left invariant weight $\widehat{\varphi}$ is constructed together with its GNS-construction, having $\mathcal{H}$ as the underlying Hilbert space. The modular conjugation of $\widehat{\varphi}$ with respect to this GNS-construction is denoted as $\widehat{J}$.

**Proposition 3.16** The triple $(W, J, \widehat{J})$ is a quantum group frame on $\mathcal{H}$.

**Proof:** It follows from the theory of weights that $J$ and $\widehat{J}$ are isometric involutions of $\mathcal{H}$ such that $M' = JMJ$ and $\widehat{M}' = \widehat{J}\widehat{M}\widehat{J}$. The equality $(\widehat{J} \otimes J)W(\widehat{J} \otimes J) = W^*$ is proved in corollary 2.2 of [12]. From this equality we obtain that $\widehat{J}JM\widehat{J} = M'$ and $J\widehat{M}J = \widehat{M}'$. One of the comments after definition 1.6 of [12] says that $M \cap \widehat{M} = \mathbb{C}1$. Hence also $\mathbb{C}1 = \widehat{J}JM \widehat{J}J \cap J\widehat{M}\widehat{J}J = M' \cap \widehat{M}'$. So $M\widehat{M} = B(\mathcal{H})$ and $W$ is trim.

**Remark 3.17** Denote the modular groups of $\varphi$ and $\psi$ by $\sigma$ and $\sigma'$ respectively. Then proposition 6.8 of [10] gives us that there exists a strictly positive number $\nu$ such that $\varphi\sigma'_t = \nu^t\varphi$ and $\psi\sigma'_t = \nu^{-t}\psi$ for any $t \in \mathbb{R}$. The number $\nu$ is called the scaling constant. Corollary 2.12 of [12] now says that $\widehat{J}J = \nu^{i/4}J\widehat{J}$. We remarked in proposition 3.14 that $J$ could be rescaled in such a way that it commutes with $\widehat{J}$ and that $(W, J, \widehat{J})$ still is a quantum group frame. But then $J$ is not anymore the anti-unitary obtained naturally as the modular conjugation of $\varphi$. ▲

4 Example: The quantum $az + b$ group

The quantum $az + b$ group was first introduced by Woronowicz in [26]. The second author continued the study of this example in [20]. He constructed right and left Haar weights, and thus showed that it is a locally compact quantum group as in the theory [10] of Kustermans and Vaes. An important feature of this quantum group is that the scaling constant (see remark 3.17) is non-trivial. This feature was foreseen by the theory of Kustermans & Vaes, and the quantum $ax + b$ group (see [25], [20]) was the first example where it really occurred. The quantum $az + b$-group is similar to the $ax + b$-group and has the same property, but is more simple. As we discussed in remark 3.17, in our theory this implies that the anti-unitary operators $J$ and $\widehat{J}$ commute only up to a non-trivial scalar.

Since the quantum $az + b$-group is a locally compact quantum group, it follows from proposition 3.16 that it gives rise to a quantum group frame. The associated multiplicative unitary and the anti-unitary $J$ are found in [20]. In this section, we will explicitly compute the anti-unitary operator $\widehat{J}$ and directly calculate the commutation of $J$ and $\widehat{J}$. 

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For the definition of the quantum $az + b$-group $(A, \Phi)$ and the notations used for its ingredients, we refer to paragraph 3 of [20]. We will need to work with unbounded operators acting on a Hilbert space; we refer to chapter 9 of [15] for the basic concepts and results. We will not always be fully detailed when working with these unbounded operators, but all the arguments of this section can be made rigorous.

Let $a$ and $b$ be as in definition 3.1 of [20], and let $a = u|a|$ and $b = v|b|$ be their polar decompositions. Consider the elements of the form

$$
\sum_{k,\ell=0}^{2n-1} \left( \int f_{k,\ell}(|b|,t)|a|^{it} \, dt \right) v^k u^\ell
$$

where the $f_{k,\ell}$ are continuous complex functions with compact support in $\mathbb{R}^+ \times \mathbb{R}$, such that whenever $k \neq 0$ we have that $f_{k,\ell}(0,t) = 0$ for all $t$ and all $\ell$. These elements form a non-degenerate $^*$-algebra $A_0$, which is norm dense in $A$ ([20] proposition 3.5 and definition 3.6).

The left Hilbert algebra associated to the right invariant weight on $(A, \Phi)$ is constructed as follows (see proposition 4.2 of [20]). The underlying Hilbert space $\mathcal{H}$ is $L^2(\mathbb{R}^+) \otimes L^2(\mathbb{R}) \otimes \mathbb{C}^{2n} \otimes \mathbb{C}^{2n}$, which will be identified with $L^2(\mathbb{R}^+ \times \mathbb{R}, \mathbb{C}^{2n} \times \mathbb{C}^{2n})$. We fix an orthonormal basis $\{e_k \mid k = 0, 1, \ldots, 2n-1\}$ in $\mathbb{C}^{2n}$. The index $k$ will be considered as an element of $\mathbb{Z}_{2n}$; hence e.g. $e_{2n}$ stands for $e_0$. In what follows for $\xi \in \mathcal{H}$ we will use the notation

$$
\xi = \sum_{k,\ell} \xi_{k,\ell} \otimes e_k \otimes e_\ell
$$

with $\xi_{k,\ell} \in L^2(\mathbb{R}^+ \times \mathbb{R})$.

When $x \in A_0$ is given by

$$
\sum_{k,\ell} \left( \int f_{k,\ell}(|b|,t)|a|^{it} \, dt \right) v^k u^\ell
$$

let $\eta(x) \in \mathcal{H}$ be given by

$$
(\eta(x))(r,t) = \sum_{k,\ell} e^{-\frac{\pi i}{n} \frac{r^2}{2}} f_{k,\ell}(r,t) e_k \otimes e_\ell
$$

for $r \in \mathbb{R}^+$ and $t \in \mathbb{R}$. The set of these $\eta(x)$ is made into a left Hilbert algebra by letting

$$
\eta(x)\eta(y) = \eta(xy) \quad \text{and} \quad \eta(x)^* = \eta(x^*)
$$

whenever $x, y \in A_0$.

**Proposition 4.1** The operator $J$ associated to the quantum $az + b$ group is given by $J = J_1 \otimes J_2$, where $J_1$ acts on $L^2(\mathbb{R}^+ \times \mathbb{R})$ as

$$
(J_1 \xi)(r,t) = e^{-\frac{\pi i}{2n}} \xi(e^{-\frac{\pi i}{n} r}, -t),
$$
for \( \xi \in K(\mathbb{R}^+ \times \mathbb{R}) \), \( r \in \mathbb{R}^+ \) and \( t \in \mathbb{R} \), and \( J_2 \) acts on \( \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \) as
\[
J_2(e_k \otimes e_\ell) = q^{k\ell} e_{-k} \otimes e_{-\ell}
\]
for \( k, \ell \in \{0, 1, \ldots, 2n-1\} \). The operator \( \hat{J} \) is given by \( \hat{J} = J_1 \otimes J_2 \) with
\[
\hat{J}_1 = e^{-\frac{nt}{2\pi}} (e^{i\frac{nt}{2}} (\log a_0)^2 \otimes 1) e^{i\frac{nt}{2}} (\log a_0 \otimes \log a_1) C,
\]
where \( C \) denotes the complex conjugation on \( L^2(\mathbb{R}^+ \times \mathbb{R}) \); and \( J_2 \) acts on \( \mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \) as
\[
\hat{J}_2(e_k \otimes e_\ell) = (-1)^k q^{-\frac{1}{2}k^2} e_{-k} \otimes e_{k+\ell}
\]
for \( k, \ell \in \{0, 1, \ldots, 2n-1\} \). Recall that \( a_0 \) and \( a_1 \) are the operators defined as follows (cf. propositions 3.4 and 4.2 of [20]). The operator \( a_0 \) on \( L^2(\mathbb{R}^+) \) is defined by \( (a_0 f)(s) = e^{-\frac{2\pi}{\pi} f} (e^{-\frac{2\pi}{\pi} s}) \) for \( f \in K(\mathbb{R}^+) \), and \( a_1 \) is the operator on \( L^2(\mathbb{R}) \) defined by \( (a_1^0 \xi)(r) = \xi(r - q) \) for \( \xi \in K(\mathbb{R}) \).

**Proof:** The operator \( J \) is obtained in proposition 4.3 of [20].

The anti-unitary \( \hat{J} \) is to be obtained from the polar decomposition of the closure \( G \) of the map \( \eta(x) \mapsto \eta(\kappa(x)^*) \), where \( \kappa \) is the antipode of \((A, \Phi)\) — see proposition 2.8 and corollary 2.9 of [12].

In proposition 3.12 of [20], the polar decomposition \( \kappa = R\tau_{-\frac{1}{2}} \) of this antipode is obtained. We will consider the map \( \eta(x) \mapsto \eta(\kappa(x)^*) \) as the composition of the map \( \eta(x) \mapsto \eta(\tau_{-\frac{1}{2}}(x)) \) and the map \( \eta(x) \mapsto \eta(R(x)^*) \).

We start with \( \tau \). Let \( s \in \mathbb{R} \). Proposition 3.12 of [20] gives us that \( \tau_s \) is defined by
\[
\tau_s(a) = a \quad \tau_s(b) = e^{\frac{2\pi s}{n}} b.
\]

Hence
\[
\tau_s(|a|) = |a| \quad \tau_s(|b|) = e^{\frac{2\pi s}{n}} |b|
\]
\[
\tau_s(u) = u \quad \tau_s(v) = v.
\]

Let \( x \in A_0 \) be given by (4.8). Then
\[
\tau_s(x) = \sum_{k,\ell} \left( \int f_{k,\ell}(e^{\frac{2\pi s}{n}} |b|, t) |a|^i dt \right) v^k u^\ell
data
\]
and
\[
(\eta(\tau_s(x)))(r, t) = \sum_{k,\ell} e^{-\frac{\pi s}{n}} r^{\frac{1}{2}} f_{k,\ell}(e^{\frac{2\pi s}{n}} r, t) e_k \otimes e_{\ell}
\]
\[
= \sum_{k,\ell} e^{-\frac{2\pi s}{n}} e^{-\frac{\pi s}{n} (e^{\frac{2\pi s}{n}} r)^2} f_{k,\ell}(e^{\frac{2\pi s}{n}} r, t) e_k \otimes e_{\ell}.
\]

Write \( \xi = \eta(x) \) and \( \tilde{\xi} = \eta(\tau_s(x)) \). Then
\[
\tilde{\xi}_{k,\ell}(r, t) = e^{-\frac{2\pi s}{n}} \xi_{k,\ell}(e^{\frac{2\pi s}{n}} r, t)
\]
\[
= e^{-\frac{2\pi s}{n}} e^{\frac{2\pi s}{n}} \xi_{k,\ell}(e^{\frac{2\pi s}{n}} r, t)
\]
\[
= e^{-\frac{2\pi s}{n}} ((a_0^{-2i\pi} \otimes 1) \xi_{k,\ell})(r, t),
\]

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where $a_0$ is given in proposition 3.4 of [20]. We conclude that

$$
\bar{\xi} = e^{-\frac{2n}{\pi}}(a_0^{-2i} \otimes 1 \otimes 1 \otimes 1)\xi,
$$

and so (the closure of) the map $\eta(x) \mapsto \eta(\tau_{-\frac{1}{2}}(x))$ is given by

$$
e^{-\frac{2n}{\pi}}(a_0^{-1} \otimes 1 \otimes 1 \otimes 1).
$$

(4.9)

Now we turn to the unitary antipode $R$. We consider the map $x \mapsto R(x)^*$, which is an anti-linear $^*$-automorphism. Proposition 3.12 of [20] gives us

$$R(a) = a^{-1} \quad R(b) = -e^{\frac{2n}{\pi}}a^{-1}b.
$$

(4.10)

Hence we have that

$$R(a)^* = (a^*)^{-1} = u|a|^{-1}$$

$$R(b)^* = -e^{\frac{2n}{\pi}}b^*(a^*)^{-1} = -e^{\frac{2n}{\pi}}v^* |b| |a|^{-1} = -e^{\frac{2n}{\pi}}v^* u |b| |a|^{-1}.$$

The operators $a$ and $b$ are defined in such a way that $|b| |a|^{-1} = e^{\frac{2n}{\pi}}|a|^{-1}b$ (see definition 3.1 of [20]). Therefore

$$R(b)^* = -v^* u |a|^{-1} |b|.$$

Like in [20], we write $q$ for the scalar $e^{\frac{2n}{\pi}}$. For convenience also the notations $q^\frac{1}{2} = e^{\frac{2n}{\pi}}$ and $q^{-\frac{1}{2}} = e^{-\frac{2n}{\pi}}$ will be used. We have

$$R(b)^* = -q^{-\frac{1}{2}}v^* u q^\frac{1}{2}|a|^{-1} |b|.$$

It follows from lemma 3.15 of [20] that $-q^{-\frac{1}{2}}v^* u$ is unitary. Lemma 3.14 of [20] gives us that (the closure of) $q^\frac{1}{2}|a|^{-1} |b|$ is a self-adjoint positive operator. We obtain

$$R(u)^* = u \quad R(v)^* = -q^{-\frac{1}{2}}v^* u$$

$$R(|a|)^* = |a|^{-1} \quad R(|b|)^* = q^\frac{1}{2}|a|^{-1} |b|.$$

Again let $x$ be as in (4.8). By the above, and since the map $x \mapsto R(x)^*$ is an anti-linear $^*$-automorphism, we have

$$R(x)^* = \sum_{k, \ell} \left( \int \bar{f}_{k, \ell}(q^\frac{1}{2}|a|^{-1} |b|, t)|a|^{it} dt \right) (-q^{-\frac{1}{2}}v^* u)^k u^\ell.
$$

(4.11)

By the definition of $a$ and $b$ we have $uv = qvu$; hence we have the commutation rule

$$(v^* u)^k = v^* u v^* u \ldots v^* u = (q^{-1})^{\frac{1}{2}k(k-1)}(v^*)^k u^k.$$

Therefore

$$(-q^{-\frac{1}{2}}v^* u)^k = (-1)^k q^{-\frac{1}{2}k^2} (v^*)^k u^k.$$

(4.12)
On the other hand, lemma 3.14 of [20] gives us that there is a unitary operator $u_0$ such that $q^\frac{1}{2} |a|^{-1} |b| = u_0 |b| u_0^*$. This unitary $u_0$ is given by $u_0 = \exp(\frac{im}{2\pi} k^2)$ where $h = \log |a|$; hence it commutes with $|a|$. So we have that

$$
\overline{f_{k,\ell}}(q^\frac{1}{2} |a|^{-1} |b|, t) |a|^{it} = u_0 \overline{f_{k,\ell}}(|b|, t) |a|^{it} u_0^*.
$$

Substituting (4.12) and (4.13) in (4.11), and taking into account that $u_0$ commutes with $u$ and (by the definition of $a$ and $b$) also with $v$, we obtain that

$$
R(x)^* = u_0 \left( \sum_{k,\ell} \left( \int \overline{f_{k,\ell}}(|b|, t) |a|^i t \, dt \right) (-1)^k q^{-\frac{1}{2} k^2} v^{-k} u^{k+\ell} \right) u_0^*.
$$

For $x \in A$, let $\pi(x)$ and $\pi'(x)$ be the GNS-operators given by $\pi(x) \eta(y) = \eta(xy)$ and $\pi'(x) \eta(y) = \eta(yx)$ if $y \in A_0$. Then

$$
\eta(R(x)^*) = \pi(u_0) \eta \left( \sum_{k,\ell} \left( \int \overline{f_{k,\ell}}(|b|, t) |a|^i t \, dt \right) (-1)^k q^{-\frac{1}{2} k^2} v^{-k} u^{k+\ell} \right) u_0^*
$$

$$
= \pi(u_0) \pi'(u_0^*) \left( \sum_{k,\ell} \left( \int \overline{f_{k,\ell}}(|b|, t) |a|^i t \, dt \right) (-1)^k q^{-\frac{1}{2} k^2} v^{-k} u^{k+\ell} \right).
$$

Hence the map $\eta(x) \mapsto R(x)^*$ is given by

$$
\pi(u_0) \pi'(u_0^*) (C \otimes G_2)
$$

where $C$ is the complex conjugation on $L^2(\mathbb{R}^+ \times \mathbb{R})$, and $G_2$ is the anti-linear operator on $L^2(\mathbb{C}^{2n} \times \mathbb{C}^{2n})$ given by

$$
G_2(e_k \otimes e_\ell) = (-1)^k q^{-\frac{1}{2} k^2} e_{-k} \otimes e_{k+\ell}.
$$

We calculate $\pi(u_0)\pi'(u_0^*)$. In proposition 4.2 of [20], the formula

$$
\pi(a) = a_0 \otimes a_1 \otimes m \otimes s
$$

is obtained. Here $m$ and $s$ are unitary, and $a_0$ and $a_1$ are positive operators. Therefore, since $\pi$ is a $^*$-representation, it follows that

$$
\pi(|a|) = a_0 \otimes a_1 \otimes 1 \otimes 1.
$$

The situation for $\pi'$ is slightly more complicated, since it is an anti-representation but not a $^*$-anti-representation. Lemma 4.13 of [20] gives us the formulas

$$
\pi'(a) = q(1 \otimes a_1 \otimes 1 \otimes s)
$$

$$
\pi'(a^*) = q(1 \otimes a_1 \otimes 1 \otimes s^*).$$
It follows that
\[ \pi'(|a|) = q(\mathbb{1} \otimes a_1 \otimes \mathbb{1} \otimes \mathbb{1}). \]

Since there is no contribution in the last two legs of the tensor product, we forget about these for a moment. With \( h = \log |a| \) we then have
\[
\begin{align*}
\pi(h) &= \log a_0 \otimes \mathbb{1} + \mathbb{1} \otimes \log a_1 \\
\pi'(h) &= \frac{\pi i}{n} \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \log a_1.
\end{align*}
\]

Therefore,
\[
\begin{align*}
\pi(h^2) &= (\log a_0)^2 \otimes \mathbb{1} + \mathbb{1} \otimes (\log a_1)^2 + 2 \log a_0 \otimes \log a_1 \\
\pi'(h^2) &= -\frac{\pi^2}{n^2} \mathbb{1} \otimes \mathbb{1} + \frac{2\pi i}{n} (\mathbb{1} \otimes \log a_1) + \mathbb{1} \otimes (\log a_1)^2.
\end{align*}
\]

and with \( u_0 = \exp \left( \frac{in}{2\pi} h^2 \right) \) we obtain
\[
\begin{align*}
\pi(u_0) &= (e^{\frac{in}{2\pi}(\log a_0)^2} \otimes e^{\frac{in}{2\pi}(\log a_1)^2}) e^{\frac{in}{\pi}(\log a_0 \otimes \log a_1)} \\
\pi'(u_0^*) &= e^{-\frac{in}{\pi}(\frac{\pi^2}{n^2}) (\mathbb{1} \otimes a_1) (\mathbb{1} \otimes e^{-\frac{2in}{\pi}(\log a_1)^2})}.
\end{align*}
\]

Hence
\[
\pi(u_0)\pi'(u_0^*) = e^{\frac{ni}{2\pi} \left( \frac{\pi^2}{n^2} \otimes a_1 \right) \left( \frac{\pi^2}{n^2} \otimes a_1 \right)} e^{\frac{2ni}{\pi} (\log a_0 \otimes \log a_1)}.
\]

We now bring together (4.9), (4.14) and (4.15). We obtain
\[
G = e^{\frac{2ni}{\pi} (\frac{\pi^2}{n^2} \otimes a_1)} e^{\frac{2ni}{\pi} (\log a_0 \otimes \log a_1)} C e^{\frac{2ni}{\pi} (a_0^{-1} \otimes \mathbb{1})} \otimes G_2.
\]

We see that the operators \( \hat{J} \) and \( |G| \) in the polar decomposition \( G = \hat{J} |G| \) are given as follows:

- \( \hat{J} = \hat{J}_1 \otimes \hat{J}_2 \) with
  \[
  \hat{J}_1 = e^{-\frac{2ni}{\pi} \left( \frac{\pi^2}{n^2} \otimes \mathbb{1} \right)} e^{\frac{2ni}{\pi} (\log a_0 \otimes \log a_1)} C,
  \]
  and \( \hat{J}_2 = G_2; \)
- \( |G| = a_0^{-1} \otimes a_1 \otimes \mathbb{1} \otimes \mathbb{1}. \)

\[\blacksquare\]

**Proposition 4.2** \( \hat{J} \hat{J} = e^{-\frac{2ni}{\pi}} \hat{J} \hat{J} \).
Proof: We start with the anti-unitary operators $J_2$ and $\hat{J}_2$. For $k, \ell \in \{0,1,\ldots,2n-1\}$ we have
\[
J_2 \hat{J}_2(e_k \otimes e_\ell) = J_2((-1)^k q^{-\frac{1}{2}k^2} e_{-k} \otimes e_{k+\ell}) = (-1)^k q^{\frac{1}{2}k^2} q^{(-k)(k+\ell)} e_k \otimes e_{-k-\ell} = (-1)^k q^{-\frac{1}{2}k^2} q^{-k\ell} e_k \otimes e_{-k-\ell}.
\]
On the other hand, we have
\[
\hat{J}_2 J_2(e_k \otimes e_\ell) = \hat{J}_2(q^{k\ell} e_{-k} \otimes e_{-\ell}) = q^{-k\ell} (-1)^k q^{-\frac{1}{2}(k-\ell)^2} e_k \otimes e_{-k-\ell}.
\]
So the operators $J_2$ and $\hat{J}_2$ commute.

We now look at the operators $J_1$ and $\hat{J}_1$. Write $J_1 = KC$ and $\hat{J}_1 = \hat{K}C$, where $C$ is the complex conjugation. So $K$ is given by
\[
(K\xi)(r,t) = e^{-\frac{\pi i}{2n} \xi(e^{-\frac{\pi i}{2n}} r, -t)}
\]
for $\xi \in L^2(\mathbb{R}^+ \times \mathbb{R})$ and $r \in \mathbb{R}^+$, $t \in \mathbb{R}$. Note that $KC = CK$. Write $K = (\mathbb{1} \otimes K')K''$, where $K'$ is the unitary operator on $L^2(\mathbb{R})$ given by $(K'\xi)(t) = \xi(-t)$, and $K''$ is the unitary operator on $L^2(\mathbb{R}^+ \times \mathbb{R})$ given by
\[
(K''\xi)(r,t) = e^{\frac{\pi i}{2n} \xi(e^{\frac{\pi i}{2n}} r, t)} = ((a_0^{-it} \otimes \mathbb{1})\xi)(r,t).
\]
The operator $\hat{K}$ is given by
\[
\hat{K} = e^{-\frac{\pi i}{2n} (e^{\frac{\pi i}{2n} (\log a_0)^2} \otimes \mathbb{1})} e^{\frac{\pi i}{2n} (\log a_0 \otimes \log a_1)}.
\]
We calculate $C\hat{K}C$. Since $Ca_0C = a_0^{-1}$ and $Ca_1C = a_1^{-1}$, we have
\[
C\hat{K}C = e^{\frac{\pi i}{2n}(e^{\frac{\pi i}{2n} (\log a_0)^2} \otimes \mathbb{1})} e^{-\frac{\pi i}{2n} (\log a_0 \otimes \log a_1)}.
\]
We know that $J_1$ and $\hat{J}_1$ commute up to a scalar factor. Since these operators have their first leg in the commutative C*-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^+))$ generated by $\log(a_0)$, we can find this scalar factor by evaluating the first leg of $J_1 \hat{J}_1$ and $\hat{J}_1 J_1$ in any point of the spectrum of $a_0$. Formally, this corresponds to replacing $\log a_0$ by a scalar $s$. Then $\hat{K}$ corresponds to $e^{-\frac{\pi i}{2n} e^{\frac{\pi i}{2n} a_0 \frac{\pi i}{2n} a_1 \frac{\pi i}{2n}}}$, and $C\hat{K}C$ corresponds to $e^{\frac{\pi i}{2n} e^{-\frac{\pi i}{2n} a_0 \frac{\pi i}{2n} a_1 \frac{\pi i}{2n}}}$. Let $k$ denote the self-adjoint operator on $L^2(\mathbb{R})$ given by $k\xi(t) = t\xi(t)$ for $\xi \in K(\mathbb{R})$ and $t \in \mathbb{R}$. Then $K''$ corresponds to $e^{-isk}$ and $K = (\mathbb{1} \otimes K')K''$ corresponds to $K' e^{-isk}$. Hence $J_1 \hat{J}_1 = K(C\hat{K}C)$ corresponds to
\[
K' e^{-isk} e^{\frac{\pi i}{2n} e^{-\frac{\pi i}{2n} a_0 \frac{\pi i}{2n} a_1 \frac{\pi i}{2n}}}.
\]
(4.16)
The operator $\hat{J} J_1 = \hat{K} C K C = \hat{K} K$ corresponds to
\[
e^{-\frac{a_1^2 \pi}{2n}} K' e^{-isk} = e^{-\frac{a_1^2 \pi}{2n}} K' a_1^{-\frac{1}{n}} e^{-isk} = K' e^{-\frac{a_1^2 \pi}{2n}} a_1^{-\frac{1}{n}} e^{-isk},
\]
where we used $a_1^{-\frac{1}{n}} K' = K' a_1^{-\frac{1}{n}}$. Now for $p \in \mathbb{R}$ we have
\[
(a_1^{ip} e^{-isk} f)(t) = (e^{-isk} f)(t - p) = e^{-i\pi(t-p)} f(t - p),
\]
while
\[
(e^{-isk} a_1^{ip} f)(t) = (e^{-ist} f)(t - p);
\]
so $a_1^{ip} e^{-isk} = e^{isp} e^{-isk} a_1^{ip}$. Using this in (4.17) we obtain that $\hat{J} J_1$ corresponds to
\[
K' e^{-\frac{a_1^2 \pi}{2n}} e^{i\pi(-\frac{n \pi}{2})} e^{-isk} a_1^{-\frac{1}{n}}.
\]
Comparing (4.16) and (4.18) we obtain that $\hat{J} J_1 = e^{-\frac{a_1^2 \pi}{2n}} J_1 \hat{J}$. Since $J_2$ and $J_2$ commute, this gives us the result.

**Remark 4.3** In remark 3.3.3, we mentioned that in the setting of the left regular representation $\hat{J} J = \nu^{i/4} J \hat{J}$, where $\nu$ is the scaling constant. Recall how the translation between left and right approach can be made. The $(W, J, \hat{J})$ associated to the right Haar measure we considered in this example, is in fact the "left" quantum group frame corresponding to the dual of the quantum $az + b$ group with the opposite comultiplication. Passing to the dual with the opposite comultiplication does not alter the scaling constant. The first author showed in [20] that this scaling constant equals $e^{-\frac{a_1^2 \pi}{2n}}$. So proposition 4.2 corresponds to the result predicted by the theory.

\[\Box\]

\section{Crossed product with a locally compact group}

In classical group theory, the construction of the semi-direct product is a powerful tool to construct classes of examples of groups. In this section we generalize this procedure, to obtain the crossed product of a quantum group frame with a locally compact group.

Let $G, H$ be locally compact groups, and let $\alpha$ be a continuous action of $G$ on $H$. We look at this situation on the group von Neumann algebra level. The group action gives rise to an action of $G$ on $\mathcal{M}(H)$, which we will denote by $\tilde{\alpha}$. It is the homomorphism of $G$ into the group of *-automorphisms of $\mathcal{M}(H)$ given by
\[
\tilde{\alpha}_g(\lambda_h) = \lambda_{\alpha} \lambda_g(h)
\]

\[\Box\]
if \( g \in G \) and \( h \in H \). One can then construct a unitary representation \( u \) of \( G \) on \( L^2(H) \), which implements this action, i.e.

\[
\tilde{\alpha}_g(m) = u_g m u_g^*
\]

for any \( g \in G \) and \( m \in \mathcal{M}(\mathcal{H}) \).

We denote the quantum group frame corresponding to the group von Neumann algebra \( \mathcal{M}(\mathcal{H}) \) by \((W^H, J^H, \hat{J}^H)\). Note that this is the situation dual to the one discussed in example 3.3; we use superscripts here. The fact that \( \alpha \) acts as automorphisms of \( H \), translates into the property that for any \( g \in G \), the operator \( u_g \otimes u_g \) commutes with \( W^H \), and \( u_g \) commutes with \( J^H \) and \( \hat{J}^H \).

These data can be generalized to the quantum group frame setting as follows.

**Definition 5.1** Let \( \mathcal{H} \) be a Hilbert space and \((W, J, \hat{J})\) a quantum group frame on \( \mathcal{H} \). Let \( G \) be a locally compact group and

\[
u : G \to \mathcal{B}(\mathcal{H}) : p \mapsto u_p
\]

a (strongly continuous) unitary representation of \( G \) on \( \mathcal{H} \). We say that \( u \) is an action of \( G \) on \((W, J, \hat{J})\) if for any \( p \in G \), we have that

1. \( W(u_p \otimes u_p) = (u_p \otimes u_p)W \);
2. \( J u_p = u_p J \) and \( \hat{J} u_p = u_p \hat{J} \).

\[\Box\]

The group \( G \) will come into play in its form of group von Neumann algebra. We will denote the associated quantum group frame by \((W^0, J^0, \hat{J}^0)\). For further reference, we write down the formulas once again:

\[
W^0\xi(p, q) = \xi(qp, q) \quad (5.19)
\]
\[
J^0\eta(p) = \Delta(p)^{-\frac{1}{2}} \eta(p^{-1}) \quad (5.20)
\]
\[
\hat{J}^0\eta(p) = \eta(p) \quad (5.21)
\]

whenever \( \xi \in L^2(G \times G) \), \( \eta \in L^2(G) \) and \( p, q \in G \).

We lift the action \( u \) to the level of the operator algebras. Recall (see e.g. [19]) that the space \( K(G, \mathcal{H}) \) of \( \mathcal{H} \)-valued continuous functions on \( G \) with compact support can be identified with a dense subspace of \( \mathcal{H} \otimes L^2(G) \). For \( \xi \in K(G, \mathcal{H}) \), let \( U \xi : G \to \mathcal{H} \) be given by

\[
(U \xi)(p) = u_{p^{-1}} \xi(p)
\]

if \( p \in G \). Note that \( U \xi \) is again an element of \( K(G, \mathcal{H}) \), which has the same norm as \( \xi \). Hence \( U \) extends to a unitary operator on \( \mathcal{H} \otimes L^2(G) \).
**Definition 5.2** Define the operators $W^1 \in B(\mathcal{H} \otimes L^2(G) \otimes \mathcal{H} \otimes L^2(G))$ and $J^1, \widehat{J}^1$ on $\mathcal{H} \otimes L^2(G)$ by

- $W^1 = W_0^0 U_{14} W_{13} = U_{14} W_0^0 W_{13}$,
- $J^1 = (J \otimes J^0)U = U^*(J \otimes J^0)$,
- $\widehat{J}^1 = \widehat{J} \otimes \widehat{J}^0$.

**Remark 5.3** The equality of operators

$$W_0^0 U_{13} = U_{13} W_0^0.$$  \hspace{1cm} (5.22)

is easily checked on the continuous $\mathcal{H}$-valued functions on $G$ with compact support. Since these functions form a dense subspace of the Hilbert space $\mathcal{H} \otimes L^2(G)$, we conclude that $W_0^0$ and $U_{13}$ indeed commute. Similarly, one can verify the relation

$$(J \otimes J^0)U = U^*(J \otimes J^0).$$ \hspace{1cm} (5.23)

Note that in the group example the formulas of the definition above give us the quantum group frame of (the group von Neumann algebra of) the semi-direct product of $H$ and $G$ by the action $\alpha$.

The formulas of definition 5.2 also correspond to the formulas of the crossed product of a Kac system with an automorphism group, which Baaj & Skandalis construct in proposition 8.16 of [2]. Those formulas are based on the right regular representation; the translation to the "left regular" setting is made by noting that when $W$ is the left regular representation of a quantum group, the unitary $\Sigma W^* \Sigma$ corresponds to the right regular representation of the quantum group with the opposite comultiplication.

In general, we have the following result. The necessary background on the crossed product of a von Neumann algebra with a locally compact group can be found in e.g. [19].

**Proposition 5.4** Let $(W, J, \widehat{J})$ be a quantum group frame on a Hilbert space $\mathcal{H}$. Let $G$ be a locally compact group, and $u$ an action of $G$ on $(W, J, \widehat{J})$. Let $W^1, J^1$ and $\widehat{J}^1$ be defined as in definition 5.2. Then $(W^1, J^1, \widehat{J}^1)$ is a quantum group frame on $\mathcal{H} \otimes L^2(G)$. The corresponding von Neumann algebra $M^1$ is the crossed product $M \rtimes G$, where $\alpha$ is the action of $G$ on $M$ given by $\alpha_p(m) = u_p m u_p^*$ if $m \in M$ and $p \in G$. The dual von Neumann algebra $\widehat{M}^1$ is the (ordinary) von Neumann algebra tensor product $\widehat{M} \otimes L^\infty(G)$. 

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Proof:

(1) In the same way as those in remark 5.3, the following commutation relations are proved:

\[ W_{23} U_{12} = U_{12} W_{23}^0 ; \]  
\[ U_{13} U_{23} = U_{23} U_{13} ; \]  
\[ U_{13} U_{23} W_{12} = W_{12} U_{13} U_{23} ; \]  
\[ (\tilde{J} \otimes J^0) U = U^* (\tilde{J} \otimes J^0) . \]

(2) Clearly, \( W^I \) is unitary. When we refer to \((\mathcal{H} \otimes L^2(G))^\otimes 3\) as a three-fold tensor product, we will use brackets for the leg numbering. When we consider it as a six-fold tensor product we use the leg numbering notation as before. By the equalities (5.22), (5.26) and again (5.22), we have that

\[ W^I_{(12)} W^I_{(13)} W^I_{(23)} = W_{46} U_{14} W^0_{24} U_{36} W_{35} W_{13} = W^I_{(23)} W^I_{(12)}. \]

So \( W^I \) is multiplicative.

(3) Now we will show that \( M^I \) is the crossed product \( M \rtimes_G \). Let \( \omega \in \mathcal{B}(\mathcal{H})_* \) and \( \omega^0 \in \mathcal{B}(L^2(G))_* \). Then, if we denote \((\iota \otimes \omega)(W) \) by \( m \), we have that

\[ (\iota \otimes (\omega \otimes \omega^0))(W^I) = (\iota \otimes \iota \otimes \omega \otimes \omega^0)(U_{14} W^0_{24} W_{13}) = (\iota \otimes \iota \otimes \omega^0)(U_{13} W^0_{23} (m \otimes \mathbb{1}). \]

Denoting \((\iota \otimes \omega^0))(W^0) \) by \( m^0 \), we obtain

\[ (\iota \otimes \iota \otimes \omega^0)(U_{13} W^0_{23}) = U^* ((\iota \otimes \iota \otimes \omega^0)(U_{12} U_{13} W^0_{23})) = U^* (\mathbb{1} \otimes m^0) U, \]

where we used equality (5.24) in the last step. Observe that for \( p \in G \), we have that \( U^* (\mathbb{1} \otimes \lambda_p) U = u_p \otimes \lambda_p \). So \( M^I \) is the von Neumann algebra generated by the sets

\[ \{ m \otimes \mathbb{1} \mid m \in M \} \text{ and } \{ u_p \otimes \lambda_p \mid p \in G \}. \]

Hence \( M^I \) is spatially isomorphic to the crossed product \( M \rtimes_G \): in fact \( M^I = U^* (M \ltimes_G) U \) (see e.g. proposition 2.12 of [19]).
(4) In this part we show that $\hat{M}^1 = \hat{M} \otimes L^\infty(G)$. We will for a moment work on the $C^*$-algebraic level. When $A$ is a $C^*$-algebra, we denote its multiplier algebra (see e.g. [22]) by $M(A)$. Recall that $W^0 \in M(C^*_r(G) \otimes C_0(G))$, where $C^*_r(G)$ denotes the reduced group $C^*$-algebra of $G$ and $C_0(G)$ denotes the $C^*$-algebra of continuous complex functions on $G$ vanishing at infinity. Therefore the operator $W^0$ can be considered as a strictly continuous bounded $M(C^*_r(G))$-valued function on $G$. We will denote the $C^*$-algebra of compact operators on a Hilbert space $\mathcal{H}$ as $\mathcal{K}(\mathcal{H})$. The unitary operator $U$ belongs to the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes C_0(G))$, and hence can be considered as a strictly continuous bounded function from $G$ to $M(\mathcal{K}(\mathcal{H}))$.

First note that using the fact that $W^0 \in \mathcal{M}(G) \otimes L^\infty(G)$, that $U \in M(\mathcal{K}(\mathcal{H}) \otimes C_0(G)) \subseteq \mathcal{B}(\mathcal{H}) \otimes L^\infty(G)$ and that $W \in M \otimes \hat{M}$, we obtain

$$W^1 = W^0_{14} U_{11} W_{13} \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{M}(G) \otimes \hat{M} \otimes L^\infty(G),$$

and hence $\hat{M} \subseteq \hat{M} \otimes L^\infty(G)$. We need to show that this last inclusion is an equality.

On the $C^*$-algebraic level, from the observations above, it follows that

$$W^1 \in M(\mathcal{K}(\mathcal{H}) \otimes C^*_r(G) \otimes \mathcal{K}(\mathcal{H}) \otimes C_0(G)).$$

Hence $\hat{S}^1 \subseteq M(\mathcal{K}(\mathcal{H}) \otimes C_0(G))$, and we can consider elements of $\hat{S}^1$ as $\mathcal{B}(\mathcal{H})$-valued functions on $G$. Under this identification, we have for $\theta \in \mathcal{B}(\mathcal{H})_*$ and $\theta^0 \in \mathcal{B}(L^2(G))_*$ that

$$((\theta \otimes \theta^0 \otimes \iota \otimes \iota)(W^1))(p) = ((1 \otimes (\theta^0 \otimes \iota)(W^0))(\theta \otimes \iota \otimes \iota)(U_{13} W_{12}))(p)$$

$$= ((\theta^0 \otimes \iota)(W^0))(p) \phi(u_{p^{-1}} \iota) \iota(W)$$

(5.28)

for any $p \in G$.

Let $\xi, \eta \in \mathcal{H}$ and $\xi^0, \eta^0 \in K(G)$ with support in a compact $K \subseteq G$. Consider the vector functionals $\omega = \omega_{\xi, \eta}$ on $\mathcal{H}$ and $\omega^0 = \omega_{\xi^0, \eta^0}$ on $L^2(G)$. We will prove that

$$(\omega \otimes \iota)(W) \otimes (\omega^0 \otimes \iota)(W^0) \in \hat{M}^1,$$

and as linear combinations of such elements are dense in $\hat{M} \otimes L^\infty(G)$, this will provide the inclusion $\hat{M} \otimes L^\infty(G) \subseteq \hat{M}^1$.

Let $\varepsilon > 0$ be arbitrary. For each $q \in K$ we fix an open neighbourhood $E_q$ of $q$ such that

$$\|u_{p^{-1}} \eta - u_{q^{-1}} \eta\| < \varepsilon \|\xi^0\| \|\eta^0\| \|\xi\|$$

for any $p \in E_q$. This is possible because the map $G \to \mathcal{H} : p \mapsto u_{p^{-1}} \eta$ is continuous. In this way we obtain an open cover $\{E_q \mid q \in K\}$ of $K$. Let $\mathcal{E} = \{E_{p_1}, E_{p_2}, \ldots, E_{p_n}\}$ be a finite subcover.
Let $A(G)$ denote the Fourier algebra of the group $G$ (see [5] chapitre 3). In lemma 5.5 below we will show that it is possible to take a partition of unity $\{h_1, h_2, \ldots, h_n\}$ of $K$ subordinate to $E$, in such a way that $h_i \in K(G) \cap A(G)$ for every $i \in \{1, 2, \ldots, n\}$. Recall ([5] théorème 3.10 and following comments) that $A(G)$ is isomorphic to the von Neumann algebra predual of $\mathcal{M}(G)$. We will identify these two algebras, and for $h \in A(G)$ denote the corresponding $\sigma$-weakly continuous linear functional on $\mathcal{M}(G)$ again by $h$. The duality is given by

$$h \left( \int f(p) \lambda_p \; dp \right) = \int h(p) f(p) \; dp$$

whenever $h \in A(G)$ and $f \in L^1(G)$. For $i \in \{1, 2, \ldots, n\}$, define $\omega_i$ to be $h_i \cdot \omega^0$ as an element of $\mathcal{M}(G)_\sigma$. So $\omega_i$ is the linear functional on $\mathcal{M}(G)$ given by

$$\omega_i(x) = (h_i \cdot \omega^0)(x) = (h_i \otimes \omega^0)(\Phi^0(x))$$

if $x \in \mathcal{M}(G)$, where $\Phi^0$ is the comultiplication on $\mathcal{M}(G)$ induced by $W^0$. As a strictly continuous $M(C^*_r)$-valued function on $G$, we have that $W^0$ is given by $W^0(p) = \lambda_p$. Hence

$$(\omega \otimes \iota)(W^0)(p) = (h_i \cdot \omega^0)(\lambda_p) = h_i(p) \omega^0(\lambda_p) = h_i(p)((\omega^0 \otimes \iota)(W^0))(p). \tag{5.29}$$

As both $h_i$ and $\omega^0$ are $\sigma$-weakly continuous linear functionals on $\mathcal{M}(G)$, so will be their product $\omega_i$. Therefore it can be extended to a $\sigma$-weakly continuous linear functional on $\mathcal{B}(L^2(G))$, which we again will denote by $\omega_i$. From equality (5.28), it follows for $p \in G$ that

$$(\omega(u_{p_1} \cdot) \otimes \omega_i \otimes \iota \otimes \iota)(W^1)(p) = ((\omega \otimes \iota)(W^0))(p) (\omega(u_{p_1}u_{p-1} \cdot) \otimes \iota)(W). \tag{5.30}$$

Using (5.30), we obtain that

$$\left\| \left( \sum_{i=1}^n (\omega(u_{p_1} \cdot) \otimes \omega_i \otimes \iota \otimes \iota)(W^1) \right)(p) - (\omega \otimes \iota)(W) ((\omega^0 \otimes \iota)(W^0))(p) \right\|
= \left| (\omega \otimes \iota)(W^0)(p) \right| \left\| \sum_{i=1}^n h_i(p)((\omega(u_{p_1} \cdot) \otimes \iota)(W) - (\omega \otimes \iota)(W)) \right\|
\leq ||\xi^0|| \cdot ||\eta^0|| \cdot \sum_{i=1}^n h_i(p) \|\omega(u_{p_1}u_{p-1} \cdot) - \omega\|
\leq ||\xi^0|| \cdot ||\eta^0|| \cdot \sum_{i=1}^n h_i(p) \|\xi\| \|u_{p}u_{p-1} \eta - \eta\|
\leq ||\xi^0|| \cdot ||\eta^0|| \cdot ||\xi|| \cdot \sum_{i=1}^n h_i(p) \|u_{p}\| \|u_{p-1} \eta - u_{p-1} \eta\|. \tag{5.31}$$

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Now \( h_i(p) = 0 \) unless \( p \in E_i \), in which case
\[
\| u_{p^{-1} \eta} - u_{p_i^{-1} \eta} \| < \frac{\varepsilon}{\| \xi^0 \| \| \eta^0 \| \| \xi \|}.
\]
So the above norm is less than
\[
\| \xi^0 \| \| \eta^0 \| \| \xi \| \sum_{i=1}^{n} h_i(p) \frac{\varepsilon}{\| \xi^0 \| \| \eta^0 \| \| \xi \|} = \varepsilon.
\]
We have obtained that \((\omega \otimes \iota)(W) \otimes (\omega_0 \otimes \iota)(W^0)\), considered as a \( B(\mathcal{H}) \)-valued functions on \( G \), can be uniformly approximated by elements of \( \hat{S}^1 \), where \( B(\mathcal{H}) \) is considered with its norm topology. This uniform topology is stronger than the \( \sigma \)-weak topology on \( \hat{S}^1 \). Therefore we can conclude that \((\omega \otimes \iota)(W) \otimes (\omega_0 \otimes \iota)(W^0)\) belongs to the \( \sigma \)-weak closure of \( \hat{S}^1 \), and hence to \( \hat{M}^1 \).

(5) From parts (3) and (4) of this proof, it follows that the \( \sigma \)-weak closures of \( S^1 \) and \( \hat{S}^1 \) are self-adjoint. Therefore, these \( \sigma \)-weak closures are \( M^1 \) and \( \hat{M}^1 \) respectively. We have obtained that
\[
M^1 = \{ m \otimes 1, u_p \otimes \lambda_p \mid m \in M, p \in G \}'' \supseteq \hat{M} \otimes L^\infty(G).
\]
It follows that \( M^1\hat{M}^1 \) contains \( \hat{M}M \otimes 1 \). Since \( W \) is trim, \( M^1\hat{M}^1 = B(\mathcal{H}) \) and it follows that \( M^1\hat{M}^1 \) also contains \( 1 \otimes M(G) L^\infty(G) \). Hence from the fact that \( W \) and \( W^0 \) are trim, we obtain that also \( W^1 \) is trim.

(6) It is clear that \( J^1 = (J^1)^* \), \( (J^1)^2 = 1 \), \( \tilde{J}^1 = (\tilde{J}^1)^* \) and \( (\tilde{J}^1)^2 = 1 \). We will show that \((\tilde{J}^1 \otimes J^1)W^1(\tilde{J}^1 \otimes J^1) = (W^1)^* \). By equality (5.27), we have
\[
(\tilde{J}^1 \otimes J^1)W^1(\tilde{J}^1 \otimes J^1) = U^*_{34}U^*_{14}(\tilde{J} \otimes J^0 \otimes J \otimes J^0)W^0_{24}W^0_{13}(\tilde{J} \otimes J^0 \otimes J \otimes J^0)U_{34} = U^*_{34}U^*_{14}(W^0_{24})^*W^0_{13}U_{34}.
\]
Using the commutation relations (5.26) and (5.25), we obtain that this is
\[
W^*_0U^*_3U^*_4U^*_1U^*_3W^*_0U^*_4^* = W^*_1U^*_4(W^0_{24})^* = (W^1)^*.
\]
(7) The von Neumann algebra
\[
J^1M^1J^1 = U^*(J \otimes J^0)M^1(J \otimes J^0)U
\]
is generated by the elements
- \( U^*(J \otimes J^0)(m \otimes 1)(J \otimes J^0)U = U^*(JmJ \otimes 1)U \) where \( m \in M \), and
\[
(J \otimes J^0)UU^* (1 \otimes \lambda_p)U \ast (J \otimes J^0) = 1 \otimes J^0 \lambda_p J^0 \text{ where } p \in G.
\]

By proposition 3.11 of [19] and equality (5.23), we have that
\[
1 \otimes J^0 \lambda_p J^0 = (J \otimes J^0) U (u_p \otimes \lambda_p) U^* (J \otimes J^0)
= U^* (Ju_p J \otimes J^0 \lambda_p J^0) U = U^* (1 \otimes \rho_p) U,
\]
where \( \rho_p \) denotes the operator of right translation given by
\[
(\rho_p \xi)(q) = \Delta(p)^{\frac{1}{2}} \xi(qp)
\]
for \( \xi \in L^2(G) \) and \( q \in G \).

Using the fact that \( JMJ \subseteq M' \), we obtain that \( J^1 M^1 J^1 \) is contained in the von Neumann algebra generated by
\[
\{ U^* (m' \otimes 1) U, U^* (u_p \otimes \rho_p) U \mid m' \in M', p \in G \}
\]
which (by e.g. theorem 3.12 of [19]) is exactly \( U^* (M \rtimes_{\alpha} G)' U = (M^1)' \). With \( \hat{M}^1 = \hat{M} \otimes \hat{M}^0 \)
we immediately have that \( \hat{J}^1 \hat{M}^1 \hat{J}^1 \subseteq \hat{M}' \otimes (\hat{M}^0)' = (\hat{M}^1)' \).

\[ \square \]

We still need to show that a partition of unity on a locally compact group \( G \) can be chosen in \( K(G) \cap A(G) \).

**Lemma 5.5** Let \( G \) be a locally compact group. Let \( K \) be a compact subset of \( G \) and \( \mathcal{E} = \{E_1, E_2, \ldots, E_n\} \) a finite open cover of \( K \). Then there are positive functions \( \{h_1, h_2, \ldots, h_n\} \) in \( A(G) \cap K(G) \) such that

1. \( 0 \leq \sum_{i=1}^{n} h_i \leq 1 \),
2. \( \sum_{i=1}^{n} h_i(p) = 1 \) for any \( p \in K \),
3. For any \( i \in \{1, 2, \ldots, n\} \), the function \( h_i \) has its support in \( E_i \).

**Proof:** We denote the support of a function \( f \) by \( \text{supp}(f) \). Let \( D \) be an open neighbourhood of the unit \( e \) of \( G \) with compact closure \( \overline{D} \), such that \( K \overline{D} \subseteq E_1 \cup E_2 \cup \ldots \cup E_n \). Let \( \{f_1, f_2, \ldots, f_n\} \) be a partition of unity of \( K \overline{D} \) subordinate to \( \mathcal{E} \). Fix an open neighbourhood \( E \) of \( e \) such that \( E \subseteq D \) and \( (\text{supp}(f_i))^{-1} \subseteq E_i \) for all \( i \). Let \( g \in K(G) \) such that \( g \) has its support in \( E \), \( g \geq 0 \) and \( \int g(p) \, dp = 1 \).

For \( i \in \{1, 2, \ldots, n\} \) and \( p \in G \), let
\[
h_i(p) = \int g(p^{-1}q)f_i(q) \, dq.
\]
For \( f, g \in K(G) \), we denote by \( \tilde{f} \) the function in \( K(G) \) given by
\[
\tilde{f}(p) = \overline{f(p^{-1})},
\]
and by \( f \ast g \) the function given by
\[
(f \ast g)(p) = \int f(q)g(q^{-1}p)\,dq.
\]
Then we can write the above equality as
\[
h_i(p) = \int f_i(q)\tilde{g}(q^{-1}p)\,dq = (f_i \ast \tilde{g})(p).
\]
Since for \( p \in G \), we have that \( (f_i \ast \tilde{g})(p) = \langle \lambda_{p^{-1}} f_i \mid g \rangle \), the functions \( f_i \ast \tilde{g} \) will belong to the predual \( A(G) \) of \( M(G) \). Hence the functions \( h_i \) are in \( A(G) \). Since both \( g \) and \( f_i \) are positive elements of \( K(G) \), also \( h_i \in K(G) \) and is positive.

(1) We have that
\[
0 \leq \sum_{i=1}^{n} h_i(p) = \int g(p^{-1}q) \sum_{i=1}^{n} f_i(q)\,dq \\
\leq \int g(p^{-1}q)\,dq = \int g(q)\,dq = 1.
\]

(2) Let \( p \in K \). We have that
\[
\sum_{i=1}^{n} h_i(p) = \int g(p^{-1}q) \sum_{i=1}^{n} f_i(q)\,dq.
\]
Now \( g(p^{-1}q) \neq 0 \) is only possible if \( p^{-1}q \in E \), and hence \( q \in pE \subseteq p\overline{D} \). Since \( \{f_1, f_2, \ldots, f_n\} \) is a partition of unity of \( K\overline{D} \), it follows that \( \sum_{i=1}^{n} f_i(q) = 1 \). So we have that \( g(p^{-1}q) \sum_{i=1}^{n} f_i(q) = g(p^{-1}q) \) for any \( q \in G \). We conclude that \( \sum_{i=1}^{n} h_i(p) = 1 \).

(3) Let \( i \in \{1, 2, \ldots, n\} \) and \( p \in K \) such that \( h_i(p) \neq 0 \). Then there must be a \( q \in G \) such that \( g(p^{-1}q) \neq 0 \) and \( f_i(q) \neq 0 \). So \( p^{-1}q \) must be in \( E_i \) and \( q \) must be an element of \( \text{supp}(f_i) \).

Remark 5.6 Let us take a closer look at the von Neumann algebras \( M^1 \) and \( \hat{M}^1 \). The algebra structure of \( M^1 \) is a crossed product. Both \( M \) and \( M^0 = \mathcal{M}(G) \) are embedded in \( M^1 \), but not in the trivial way as in \( M \otimes M^0 \). We could say that the algebra structure of \( M \otimes M^0 \) has been twisted by the given action. On the other hand, the coalgebra structure is trivially given by the
embeddings $m \mapsto m \otimes \mathbb{1}$ of $M$ and $\lambda_p \mapsto U^*(\mathbb{1} \otimes \lambda_p)U$ of $\mathcal{M}(G)$ in $M^1$. For $m \in M$ we indeed have

$$
\Phi^1(m \otimes \mathbb{1}) = (W^1)^*(\mathbb{1} \otimes \mathbb{1} \otimes m \otimes \mathbb{1})W^1
= W_{13}(W^0)_{24}^*U_{14}^*(\mathbb{1} \otimes \mathbb{1} \otimes m \otimes \mathbb{1})U_{14}W_{24}^*W_{13}
= \Phi(m)_{13},
$$

and for $p \in G$ we have

$$
\Phi^1(U^*(\mathbb{1} \otimes \lambda_p)U) = (W^1)^*U_{34}^*(\mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \lambda_p)U_{34}W^1
= W_{13}(W^0)_{24}^*U_{14}^*(\mathbb{1} \otimes \mathbb{1} \otimes \lambda_p)U_{34}U_{14}W_{24}^*W_{13}
= U_{14}U_{34}^*(W^0)_{24}^*(\mathbb{1} \otimes \mathbb{1} \otimes \lambda_p)W_{24}^*W_{13}U_{34}U_{14}
= U_{14}U_{34}^*(\mathbb{1} \otimes \lambda_p \otimes \mathbb{1} \otimes \mathbb{1})U_{34}U_{14}
= (U^* \otimes U^*)\Phi(\lambda_p)_{24}(U \otimes U),
$$

where in the third step we used the commutation relations (5.26) and (5.22). For the dual $\hat{M}^1$, we have that the algebra structure trivially is the tensor product $\hat{M} \otimes \hat{M}^0$. But now the coalgebra structure is deformed: For $\hat{m} \in \hat{M}$ and $f \in \hat{M}^0 = L^\infty(G)$, we have that

$$
\hat{\Phi}^1(\hat{m} \otimes f) = (\hat{W}^1)^*(\mathbb{1} \otimes \mathbb{1} \otimes \hat{m} \otimes f)\hat{W}^1
= U_{32}(\hat{W}^0)_{24}^*\hat{W}_{13}^*(\mathbb{1} \otimes \mathbb{1} \otimes \hat{m} \otimes f)\hat{W}_{13}(\hat{W}^0)_{24}^*U_{32}
= (\iota \otimes \tau \otimes \iota)(\hat{\Phi} \otimes \hat{\Phi}^0)(\hat{m} \otimes f),
$$

where $\tau$ denotes the twisted flip

$$
\tau : B(\mathcal{H} \otimes L^2(G)) \to B(L^2(G) \otimes \mathcal{H}) : x \otimes x^0 \mapsto \sigma(U(x \otimes x^0)U^*).
$$

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