PROPER HOLOMORPHIC EMBEDDINGS OF FINITELY AND SOME INFINITELY CONNECTED SUBSETS OF \( \mathbb{C} \) INTO \( \mathbb{C}^2 \)

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Abstract. We show that any finitely connected domain \( U \subset \mathbb{C} \) can be properly embedded into \( \mathbb{C}^2 \). For some sequences \( \{p_j\} \subset U, \ U \setminus \{p_j\} \) can also be properly embedded into \( \mathbb{C}^2 \).

1. Introduction and Main Result

Let \( \mathcal{R} \) be a noncompact Riemann surface and let \( \phi : \mathcal{R} \to \mathbb{C}^2 \) be a proper holomorphic immersion that is 1-1. In that case we say that \( \phi \) embeds \( \mathcal{R} \) properly into \( \mathbb{C}^2 \). It is known that any \( k \)-dimensional Stein manifold embeds properly into \( \mathbb{C}^{2k+1} \) [Re, Na, Bi], so in particular any noncompact Riemann surface embeds properly into \( \mathbb{C}^3 \). It is however an open question whether any noncompact Riemann surface embeds properly into \( \mathbb{C}^2 \) (it is known that not all compact Riemann surfaces embeds properly into \( \mathbb{C}^2 \), although they do in \( \mathbb{C}^3 \) [GH]). Not much is known even for planar domains. Known results are: The unit disc by Kasahara and Nishino [KN, St], the annulus by Laufer [La], the punctured disc by Alexander [Al], and the most general result so far, due to Globevnik and Stensønes: Any finitely connected bounded domain without isolated points in the boundary. Cerne and Forstnerič have some results regarding bordered Riemann surfaces [CF]. We prove the following theorem:

Theorem 1. Any finitely connected domain \( U \subset \mathbb{C} \) can be properly embedded into \( \mathbb{C}^2 \). Moreover, let \( \{p_j\} \subset U \) be a sequence converging to a point \( p \) in the boundary (we allow \( p = \infty \)), and assume that \( \{p_j\} \) is regular for \( U \). Then \( U \setminus \{p_j\} \) can be embedded properly into \( \mathbb{C}^2 \).

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2. Notation And Preliminaries

Throughout this paper we will use the following notation: For a real number \( R > 0 \), \( B_R \) will denote the open \( R \)-ball centered at the origin in \( \mathbb{C}^2 \). We let \( \Delta_R \) denote the open \( R \)-disc centered at the origin in \( \mathbb{C} \). If there is no subscript \( R \), then we are referring to the unit ball or the unit disc respectively. We will let \( \pi_i \) denote the projection on the \( i \)-th coordinate axis.

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Definition 1. Let $U \subset \mathbb{C}$ be a connected open set such that $U = \mathbb{C} \setminus \bigcup_{i=1}^{n} K_i$ where the $K_i$'s are pairwise disjoint closed connected subset of $\mathbb{C}$. We then say that $U$ is $n$-connected.

Definition 2. Let $U \subset \mathbb{C}$ be a domain, and let $\{p_j\}$ be a sequence in $U$ converging to a boundary point $p$. We say that $\{p_j\}$ is regular for $U$ if there exists a continuous curve $\gamma : [0,1] \to \overline{U}$ such that $\gamma([0,1)) \subset U$, with $\gamma(1) = p$, and $\{p_j\} \subset \gamma$.

Definition 3. Let $L = \{\zeta = x + iy \in \mathbb{C}; x \leq -1, y = 0\}$, let $\Gamma = \{l_j : [0,1] \to \mathbb{C}; j = 1, \ldots, m\}$ be a collection of smooth disjoint curves without self intersections, and let $\{p_j\}$ be a discrete set in $\mathbb{C}$. Assume that these sets are pairwise disjoint. Then we will call the domain $S = \mathbb{C} \setminus (L \cup \Gamma \cup \{p_j\})$ a standard domain. We will also allow a standard domain to lack some of the above components in its complement.

Definition 4. Let $\text{Aut}_p(\mathbb{C}^k)$ denote the group of holomorphic automorphisms of $\mathbb{C}^k$ fixing the point $p \in \mathbb{C}^k$. If all the eigenvalues $\lambda_i$ of $dF(p)$ satisfy $|\lambda_i| < 1$ we say that $F$ is attracting at $p$.

Definition 5. Let $\{F_j\} \subset \text{Aut}_p(\mathbb{C}^k)$. We let $F(j)$ denote the composition map $F_j \circ \cdots \circ F_1$, and we define the basin of attraction of $p$ by

$$\Omega^0_{\{F_j\}} = \{z \in \mathbb{C}^k; \lim_{j \to \infty} F(j)(z) = p\}.$$

The construction of certain Fatou-Bieberbach domains will be an integral part of our proof of Theorem 1, and we will use the following theorem and proposition:

Theorem 2. [Wo] Let $0 < s < r < 1$ such that $r^2 < s$, let $\delta > 0$, and let $\{F_j\} \subset \text{Aut}_0(\mathbb{C}^k)$ with $s\|z\| \leq \|F_j(z)\| \leq r\|z\|$ for all $z \in B_{\delta}$, and for all $j \in \mathbb{N}$. Then there exists a biholomorphic map

$$\Phi : \Omega^0_{\{F_j\}} \to \Phi(\Omega^0_{\{F_j\}}) = \mathbb{C}^k.$$

Proposition 1. [Wo] Let $K \subset \mathbb{C}^k$ be polynomially convex, let $V \subset \mathbb{C}^k$ be a closed subvariety, and let $K' \subset V$ be compact set such that $K \cap V \subset K'$. Then $K \cap \overline{K'}_{C(V)} = K \cap \\overline{K'}_{C(V)} = K \cap \overline{K'}_{C(V)}$.

3. A Classification of Some Infinitely Connected Domains in $\mathbb{C}$

Recall how one can use the existence of a Runge Fatou-Bieberbach domain $\Omega$ together with The Riemann Mapping Theorem to embed the unit disc in $\mathbb{C}$ properly into $\mathbb{C}^2$. We may assume that the intersection between $\Omega$ and the $z$-plane is not the whole plane, and it follows from the Runge property that all the connected components of this intersection have to be simply connected. Let $U$ be such a component. Now the Riemann Mapping Theorem tells us that there is a biholomorphic map $\phi$ mapping $\Delta$ onto $U$. So if $\psi : \Omega \to \mathbb{C}^2$ is the associated Fatou-Bieberbach map, then $\psi \circ \phi$ will map $\Delta$ properly into $\mathbb{C}^2$.

Now, this method fails if one wants to embed something that is not simply connected. This is because intersections between Runge Fatou-Bieberbach domains and embedded complex curves are simply connected (it is an open question whether all Fatou-Bieberbach domains are Runge). Moreover, two multiply connected domains are not automatically biholomorphically equivalent. Our approach will be
the following: First we map the domain into \(\mathbb{C}^2\) such that the image \(V\) is Runge.

We will prove that any standard domain can be embedded properly into \(\mathbb{C}^2\), and we begin by investigating which subsets of \(\mathbb{C}\) are biholomorphically equivalent to a standard domain.

Define the following map \(\mu: \Delta \to \mathbb{C} \setminus L:\)

\[
\mu(\zeta) = \frac{\zeta + 1}{\zeta - 1}^2 - 1
\]

For a sequence \(p_j\) in \(\Delta\) such that \(\lim p_j = 1\) we have that \(\lim |\mu(p_j)| = \infty\). In particular, both \(\mu(\Delta)\) and \(\mu(\Delta \setminus \{p_j\})\) are standard domains.

**Remark 1.** The Riemann Mapping Theorem states that for any simply connected domain \(U \subset \mathbb{C}\) which is not the whole of \(\mathbb{C}\), there exists a biholomorphism \(\phi: U \to \Delta\) that is onto. Let \(p \in \partial U\). If \(\{p_j\} \subset U\) is a sequence converging to \(p\) and if \(\{p_j\}\) is regular for \(U\), then we can assume that \(\lim_{j \to \infty} \phi(p_j) = 1\).

We will use the following theorem from [Go] to show that all the domains in Theorem 1 are in fact standard domains.

**Theorem 3.** (Hilbert) Every \(n\)-connected domain in the \(z\)-plane can be mapped univalently onto the \(\zeta\)-plane with \(n\) parallel finite cuts of inclination \(\Theta\) with the real axis in such a way that a given point \(z = a\) is mapped into \(\zeta = \infty\), and the expansion of the mapping function about \(z = a\) has the form

\[
\frac{1}{z - a} + \alpha_1(z - a) + \cdots \quad \text{or} \quad z + \frac{\alpha_1}{z} + \cdots
\]

according as \(a\) is finite or not. Some of the cuts referred to may consist of single points.

**Proposition 2.** Let \(U \subset \mathbb{C}\) be \(n\)-connected, and let \(\{p_j\} \subset U\) be a sequence converging to a point \(p\) in the boundary. Assume that \(\{p_j\}\) is regular for \(U\). Then \(\tilde{U} = U \setminus \{p_j\}\) is biholomorphically equivalent to a standard domain \(S\).

**Proof.** Write \(U = \mathbb{C} \setminus \bigcup_{i=1}^{n} K_i\), and write \(K = \bigcup_{i=1}^{n} K_i\). In the case of \(K\) being unbounded, we will assume that \(K\) has only got one unbounded component. It will be clear that the proof will work also if we have several unbounded components. We will have to look at the different possibilities for the limit point \(p\) of the sequence \(\{p_j\}\).

Case 1 - \(K\) is bounded, and \(p = \infty\): By Theorem 3 there is a biholomorphism \(\phi\) mapping \(U\) onto \(\mathbb{C}\) minus a finite number of cuts. The domain \(\phi(U) \setminus \{\phi(p_i)\}\) is a standard domain.

Case 2 - \(K\) is bounded, and \(p = K_k\): By Theorem 3 there is a biholomorphism \(\phi\) mapping \(U \cup \{p\}\) onto a domain with \(n - 1\) cuts. Define \(\varphi(\zeta) = \frac{1}{\zeta - \phi(p)}\), and \(\varphi \circ \phi\) maps \(\tilde{U}\) onto a standard domain.

Case 3 - \(K\) is bounded, and \(p \in K_k\): By Theorem 3 there is a biholomorphism \(\phi\) mapping \(U \cup K_k\) onto a domain with \(n - 1\) cuts. Define \(\varphi(\zeta) = \frac{1}{\zeta - \phi(p)}\). We have that \(\varphi\) maps \(\mathbb{C} \setminus \phi(K_k)\) onto a simply connected domain \(W\) take away zero. There
is a biholomorphism $f: W \to \triangle$, such that $\lim_{j \to \infty} f(\varphi(\phi(p_j))) = 1$. So the map $\mu \circ f \circ \varphi$ maps $\hat{U}$ onto a standard domain.

Case 4 - $K$ is unbounded, and $p = \infty$: Let $K_k$ be the unbounded component of $K$. By Theorem 2.2 in [BF] there is a biholomorphism $\phi$ mapping $U \cup K_k$ onto a domain with $n - 1$ cuts. Let $V = \phi(U \cup K_k) \setminus \phi(K_k)$. $V$ is simply connected, so there is a biholomorphism $f: V \to \triangle$ such that $\lim_{j \to \infty} f(\phi(p_j)) = 1$, so the map $\mu \circ f \circ \phi$ maps $\hat{U}$ onto a standard domain.

Case 5 - $K$ is unbounded, and $p = K_k$: Define $\varphi(\zeta) = \frac{1}{\zeta - p}$. The map $\varphi$ maps $\hat{U}$ onto a domain that falls under Case 1.

Case 6 - $K$ is unbounded, and $p \in K_k$: The map $\varphi(\zeta) = \frac{1}{\zeta - p}$ maps $\hat{U}$ onto a domain falling under Case 4.

4. Proper Holomorphic Embeddings

In the proof of the following lemma, we use an idea from the proof of Lemma 2.2 in [BF].

Lemma 1. Let $K \subset \mathbb{C}^2$ be a polynomially convex compact set, let $\epsilon > 0$, and let $\Gamma = \{\gamma_j(t); j = 1, \ldots, m, t \in [0, \infty)\}$ be a collection of disjoint smooth curves in $\mathbb{C}^2 \setminus K$ without self-intersection, such that $\lim_{t \to \infty} |\pi_1(\gamma_j(t))| = \infty$ for all $j$. Assume that there exists an $M \in \mathbb{R}$ such that $\mathbb{C} \setminus (\overline{\Delta}_R \cup \pi_1(\Gamma))$ does not contain any relatively compact components for $R \geq M$. Let $p \in K$. Then for any $R \in \mathbb{R}$ there exists an automorphism $\varphi \in Aut(\mathbb{C}^2)$ such that the following is satisfied:

(i) $\|\phi(x) - x\| < \epsilon$ for all $x \in K$,
(ii) $\phi(\Gamma) \subset \mathbb{C}^2 \setminus \overline{B}_R$,
(iii) $\phi(p) = p$.

Proof. Let us denote the coordinates on $\mathbb{C}^2$ by $x = (z, w)$, and the curves by $\gamma_i(t) = (z_i(t), w_i(t))$ (hence $z_i(t) = \pi_1(\gamma_i(t))$). Choose a larger polynomially convex compact set $K'$ containing an $\epsilon$-neighborhood of $K$. We may assume that $R > M$ and that $K' \subset \overline{\Delta}_R \times \mathbb{C}$.

By Theorem 2.1 in [FL] there exists a $\varphi \in Aut(\mathbb{C}^2)$ such that

(a) $\|\varphi(x) - x\| < \frac{\epsilon}{4}$ for all $x \in K'$,
(b) $\varphi(\Gamma') \subset \mathbb{C}^2 \setminus \overline{B}_R$,
(c) $\varphi(p) = p$.

To see this, define an isotopy of diffeomorphisms removing $\Gamma'$ from $\overline{B}_R$. Since the union of a polynomially convex compact set and finitely many disjoint smooth compact curves in its complement is polynomially convex [Sb], the theorem applies. Lastly, a small translation gives us (c).

Fix such $\varphi$ and set $\Gamma_R = \{x \in \Gamma; \varphi(x) \in \overline{B}_R\} = \Gamma \cap \varphi^{-1}(\overline{B}_R)$.

By the construction the complement of $\overline{\Delta}_R \cup \pi_1(\Gamma)$ does not contain any bounded components and $\pi_1(\Gamma_R) \subset \pi_1(\Gamma) \setminus \overline{\Delta}_R$.

For $i = 1, \ldots, m$ let $t_i^0 \in \mathbb{R}^+$ be such that $z_i(t_i^0) \in \partial \Delta_R$ and $z_i(t) \in \mathbb{C} \setminus \overline{\Delta}_R$ for $t > t_i^0$. Define $L_i = \{z_i(t_i^0)\} \times \mathbb{C}$,
and let $K_i$ be the intersection

$$K_i = L_i \cap \varphi^{-1}(B_R).$$

Since $\varphi^{-1}(B_R)$ is polynomially convex, the complement of these sets in $L_i$ is connected. So for any $T \in \mathbb{R}$, for each $i$ there is a continuous curve $c_i : [0, 1] \to \mathbb{C}$ such that $c_i(0) = 0$, and such that

$$\tilde{c}_i(t) = (z_i(t_0^i), w_i(t_0^i) + c_i(t))$$

is a curve in $L_i \setminus K_i$ with $|w_i(t_0^i) + c_i(1)| > T$. There is a neighborhood $U_i$ of $\tilde{c}_i$ in $\mathbb{C}^2$ such that $U_i \subset \mathbb{C}^2 \setminus \varphi^{-1}(B_R)$. For any $\delta > 0$ we may now define curves

$$l_i(t) = (z_i(t_0^i + \delta t), w_i(t_0^i + \delta t) + c_i(t)).$$

If we let $\delta$ be small enough, the entire curve $l_i$ will be contained in $U_i$, and $\pi_2(l_i(1)) = T_i$ satisfies $|T_i| > T$.

Define the following function $f$ on a subset of $\mathbb{C}$:

(i) $f \equiv 0$ on $\triangle_R$,
(ii) $f(z_i(t_0^i + t)) = c_i(t)$ for $t \in (0, \delta)$,
(iii) $f(z_i(t_0^i + t)) = c_i(1)$ for $t \geq \delta$.

Then $f$ is continuous on $S = \overline{\triangle_R} \cup \pi_1(\Gamma)$, and holomorphic on $\triangle_R$. For any $C, \rho > 0$, by Mergelyan’s Theorem [Ru] there exists a holomorphic function $g \in O(\mathbb{C})$ such that $||g - f||_{S \cap \triangle_c} < \rho$. We may also assume that $g(0) = 0$. Define an automorphism

$$\psi(z, w) = (z, w + g(z)).$$

If $\rho$ is chosen small enough, and if $T$ and $C$ are chosen big enough, then each $\psi(\gamma_i)$ is close to $l_i$ over $z_i(t_0^i + t)$ for $0 \leq t \leq \delta$, and $\psi |_{\overline{\triangle_R} \times \mathbb{C}} \approx \text{id}$, such that $\psi(\Gamma) \cap \varphi^{-1}(B_R) = \emptyset$. If we also have that $\rho < \frac{\varepsilon}{2}$ then $\phi = \varphi \circ \psi$ satisfies the claims of the lemma.

**Remark 2.** It is clear from the proof that the corresponding formulation of Lemma 1 for $\mathbb{C}^k$ ($k \geq 2$) is also true.

**Lemma 2.** Let $\gamma : [0, 1] \to \mathbb{C}$ be a $\mathcal{C}^2$ curve, and let $\epsilon(\zeta)$ be holomorphic on an open set $U$ containing $\gamma$. Let $a \in \mathbb{C}$, and define the following function $\varphi : U \to \mathbb{C}^2$:

$$\varphi(\zeta) = (\zeta, \frac{a}{\zeta - \gamma(0)} + \epsilon(\zeta))$$

Let $\tilde{\gamma}(t) = \varphi(\gamma(t))$ for $t \in (0, 1]$. Let $\pi(t)$ be the projection of the curve $\tilde{\gamma}$ on the complex line $L = \{z = w\}$. Let $c$ be the complex number corresponding to $\gamma'(0)$, and let $l(t) = \frac{\sqrt{2}}{\gamma(0)}(\gamma(0) + \epsilon(\gamma(0)) + \frac{a}{c})$. There is a $\delta > 0$ and an $R \in \mathbb{R}$ such that the following hold: (i) $|\pi(t) - l(t)| < R$ for $t < \delta$, and (ii) $|\pi(t)|$ is strictly decreasing for $t < \delta$.

**Proof.** $\pi(t)$ is given by

$$\pi(t) = \frac{\sqrt{2}}{2}(\gamma(t) + \epsilon(\gamma(t)) + \frac{a}{\gamma(t) - \gamma(0)}).$$

We have to show that the last term gets “close” to $\frac{\sqrt{2}}{2} - \frac{a}{c}$ as $t$ gets small. Now $\gamma(t) = \gamma(0) + ct + h(t)$ where $h(t) = O(||t||^2)$, so we have that

$$\|\frac{a}{\gamma(t) - \gamma(0)} - \frac{a}{ct}\| = \|\frac{a}{ct+h(t)} - \frac{a}{ct}\| = \|\frac{ah(t)}{ct^2(1+\frac{h(t)}{ct})}\|.$$
Since there is a constant $C$ such that $\lim_{t \to 0} \frac{b(t)}{t} = C$, (i) follows.

Write $x(t) = \frac{2(\gamma(t) - \gamma(0))}{\sqrt{2a}}$, and $y(t) = \sqrt{2} (\gamma(t) + \epsilon(\gamma(t)))$. Now, $\pi(t) = \frac{1}{x(t)} + y(t)$, so $|\pi(t)|^2$ is given by

$$m(t) = |\pi(t)|^2 = \frac{1}{x(t) \cdot x(t)} + 2 Re \left( \frac{y(t)}{x(t)} \right) + y(t) \cdot \frac{y(t)}{x(t)}.$$ 

This can be rewritten as

$$m(t) = \frac{1}{t^{2k} \cdot g(t)} + \frac{h(t)}{t^k \cdot g(t)} + v(t),$$

where $h, g, v$ are differentiable real valued functions, $k \in \mathbb{N}^+$, $g(0) > 0$. Differentiating $m(t)$ gives (ii). \hfill $\square$

**Theorem 4.** Let $U$ be a standard domain as defined in Definition 3. Then $U$ can be embedded properly into $\mathbb{C}^2$.

**Proof.** We will prove the theorem in the case that $U$ contains all the sets mentioned in Definition 3 in its boundary, but it will be clear that the proof works for all standard domains. Let $W = U \cup L \cup \{L_j(t); t \in (0, 1]\}$, and write $q_j = l_j(0)$. We then have that $W = C \setminus \{p_j, q_j\}$. We start by mapping $W$ properly into $\mathbb{C}^2$. Define a function $f \in \mathcal{O}(W)$ as follows:

$$f(\zeta) = \sum_{j=1}^{m} \frac{a_j}{\zeta - q_j} + \sum_{j=1}^{\infty} \frac{b_j}{\zeta - p_j}.$$ 

The coefficients $\{b_j\}$ are chosen such that this converges uniformly on compact sets in $W$, and such that $f$ is bounded over $L$. We will specify conditions on $a_1, ..., a_m$ later on. Now, we define a map $\omega: W \to \mathbb{C}^2$ by:

$$\omega(\zeta) = (\zeta, f(\zeta)).$$

It is clear that $\|\omega(\zeta)\| \to \infty$ as $\zeta \to q_j$ for $j = 1, ..., m$, or $\zeta \to p_j$ for any $j \in \mathbb{N}$. We are going to complete the proof by constructing a Fatou-Bieberbach domain $\Omega$ that intersects the manifold $\omega(W)$ exactly in $\omega(U)$. If $\psi: \Omega \to \mathbb{C}^2$ is the corresponding Fatou-Bieberbach map, the composition $\psi \circ \omega$ will map $U$ properly into $\mathbb{C}^2$.

Define curves $\gamma_j: (0, 1] \to \mathbb{C}^2$ by $\gamma_j(t) = \omega(l_j(t))$. Let $L: [0, \infty) \to \mathbb{C}$ be the curve $L(t) = -1 - t$, and let $\gamma_{m+1}(t) = \omega(L(t))$. Now, since $f$ was chosen to be bounded over $L$, the projection of $\gamma_{m+1}$ on the plane $\{z = w\}$ is contained in some $s$-strip around the negative real numbers. Further, Lemma 2 tells us that we can choose $a_1, ..., a_m$ in the definition of $f$ such that the projections of the $\gamma_j$’s all point in different directions. By the same lemma, none of the projections of the $\gamma_j$’s are self-intersecting when $t$ is outside of a compact subset of the open unit interval, so by a change of coordinates, the conditions on the curves in Lemma 1 are satisfied.

We will need a polynomially convex compact exhaustion of $\omega(U) = \omega(W) \setminus \{\gamma_i\}$. For an $\epsilon > 0$, let $S_\epsilon$ denote $\{\zeta \in \mathbb{C}; \text{dist}(\zeta, L \cup \{p_j\} \cup \{l_j\}) < \epsilon\}$. It is clear that if we for an appropriate sequence $\{e_j\}$ converging to zero, define $C_j = \sum_j \setminus S_{e_j}$, the sequence $C_1 \subset C_2 \subset \ldots$ is a compact exhaustion of $U$. The sets $C_i$ are not polynomially convex, but the sets $K_i = \omega(C_i)$ are.

We will make one last observation before we start constructing the Fatou-Bieberbach domain: $\omega(W)$ is a closed submanifold of $\mathbb{C}^2$, so by Proposition 1, if $K \subset \mathbb{C}^2$ is
a polynomially convex compact set such that \( K \cap \omega(W) \subset \omega(U) \), then there exists an \( N \in \mathbb{N} \) such that \( K \cup K_i \) is polynomially convex for all \( i \geq N \).

We may assume that the origin is not contained in any of the \( \gamma_i \)'s or in the set \( \{ p_j \} \), so we can choose a \( \rho > 0 \) such that \( \overline{B}_\rho \) does not intersect these sets either.

Define a linear map
\[
A: (z, w) \rightarrow \left( \frac{z}{2}, \frac{w}{2} \right).
\]

Theorem \( \ref{thm:hyperbolic} \) tells us that we can choose a \( \delta > 0 \) such that if we have a sequence of automorphisms \( \{ \sigma_k \} \subset Aut_0(\mathbb{C}^2) \) such that

\[
(*) \quad \| \sigma_k - A \|_{\mathcal{P}_\rho} < \delta
\]

for all \( k \in \mathbb{N} \), then the basin of attraction to zero of the sequence \( \sigma(k) \) is biholomorphic to \( \mathbb{C}^2 \). What we will do is to construct a sequence of automorphisms that attracts \( \omega(U) \), but not any of the \( \gamma_i \)'s.

The sequence of automorphisms will be constructed inductively, and we make the following induction hypothesis \( I_j \): We have constructed automorphisms \( \{ F_1, \ldots, F_j \} \subset Aut_0(\mathbb{C}^2) \) such that the following is satisfied:

1. Each \( F_i \) is a finite composition of maps \( \sigma_k \) satisfying \( (*) \),
2. \( F(j)(K_j) \subset B_\rho \),
3. \( F(j)(\gamma_i) \subset \mathbb{C}^2 \setminus \overline{B}_\rho \) for \( i = 1, \ldots, m + 1 \).

If we define \( K_1 = \{ 0 \} \) and choose \( \rho \) small enough, then \( I_1 \) is satisfied by letting \( F_1 = A \). We will now show how to construct \( F_{j+1} \) so as to ensure \( I_{j+1} \).

\( \hat{K} = F(j)^{-1}(\overline{B}_\rho) \) is a polynomially convex compact set satisfying \( \hat{K} \cap \omega(W) \subset \omega(U) \), so there exists an \( r \geq j + 1 \) such that \( K = \hat{K} \cup K_r \) is polynomially convex.

Choose a polynomially convex compact set \( K' \) containing a neighborhood of \( K \) such that \( K' \) does not intersect any of the \( \gamma_i \)'s.

Choose an \( s \in \mathbb{N} \) such that

1. \( A^s(F(j)(K')) \subset B_\rho \),

and then choose an \( R \in \mathbb{R}^+ \) such that

2. \( A^s(F(j)(x)) \notin \overline{B}_\rho \) for all \( \| x \| \geq R \).

For any \( \epsilon > 0 \) by Lemma 1 there is a \( \phi \in Aut_0(\mathbb{C}^2) \) such that \( \| \phi(x) - x \| < \epsilon \) for all \( x \in K' \), and such that

3. \( \phi(\gamma_i) \subset \mathbb{C}^2 \setminus B_R \) for \( i = 1, \ldots, m \).

Define
\[
F_{j+1} = A^s \circ F(j) \circ \phi \circ F(j)^{-1}.
\]

If \( \epsilon \) is chosen small enough to ensure that \( \phi(K) \subset K' \) then (i) gives us that

\[
F(j + 1)(K) = A^s(F(j)(\phi(K))) \subset A^s(F(j)(K')) \subset B_\rho,
\]

which ensures (2). For each \( \gamma_i \) by (ii) and (iii) we have that

\[
F(j + 1)(\gamma_i) = A^s(F(j)(\phi(\gamma_i))) \subset A^s(F(j)(\mathbb{C}^2 \setminus B_R)) \subset \mathbb{C}^2 \setminus \overline{B}_\rho,
\]

which ensures (3). Lastly, the map \( A \circ F(j) \circ \phi \circ F(j)^{-1} \) can be made arbitrarily close to \( A \) on \( \overline{B}_\rho \) by choosing \( \epsilon \) small enough, which means that \( F_{j+1} \) can also be assumed to satisfy (1).
We have now constructed a sequence of automorphisms \( \{ F_j \} \subset \text{Aut}_0(\mathbb{C}^2) \), and by (1) and Theorem 2, we have that \( \Omega = \Omega^0_{\{ F_j \}} \) is biholomorphic to \( \mathbb{C}^2 \). By (2) and (3), we have that \( \Omega \cap \omega(W) = \omega(U) \).

Let \( \psi \) be the Fatou-Bieberbach map \( \psi : \Omega \to \mathbb{C}^2 \), and define \( \Psi = \psi \circ \omega \). This map is a proper holomorphic embedding of \( U \) into \( \mathbb{C}^2 \), and the proof is finished. \( \square \)

Now our main theorem follows easily:

**Proof of Theorem 1:** By Proposition 2 we have that \( U \setminus \{ p_j \} \) is biholomorphically equivalent to a standard domain \( S \). By Theorem 1, \( S \) can be embedded properly into \( \mathbb{C}^2 \).

Our method has also given us a rather interesting result regarding Fatou-Bieberbach domains, namely that there exists a Fatou-Bieberbach domain whose intersection with a complex line \( \mathbb{C} \) is exactly \( \mathbb{C} \setminus L \).

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