ALGORITHMIC COMPUTATION OF LOCAL COHOMOLOGY MODULES AND THE LOCAL COHOMOLOGICAL DIMENSION OF ALGEBRAIC VARIETIES

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Abstract. In this paper we present algorithms that compute certain local cohomology modules associated to a ring of polynomials containing the rational numbers. In particular we are able to compute the local cohomological dimension of algebraic varieties in characteristic zero. Our approach is based on the theory of $D$-modules.

1. Introduction

1.1. Let $R$ be a commutative Noetherian ring, $I$ an ideal in $R$ and $M$ an $R$-module. The $i$-th local cohomology functor with respect to $I$ is the $i$-th right derived functor of the functor $H^0_I(-)$ which sends $M$ to the $I$-torsion $\bigcup_{k=1}^{\infty} (0 : M I^k)$ of $M$ and is denoted by $H^i_I(-)$. Local cohomology was introduced by Grothendieck as an algebraic analog of (classical) relative cohomology. A brief introduction to local cohomology may be found in appendix 4 of [3].

The cohomological dimension of $I$ in $R$, denoted by $\text{cd}(R,I)$, is the smallest integer $c$ such that the local cohomology modules $H^q_I(M) = 0$ for all $q > c$ and all $R$-modules $M$. If $R$ is the coordinate ring of an affine variety $X$ and $I \subseteq R$ is the defining ideal of the Zariski closed subset $V \subseteq X$ then the local cohomological dimension of $V$ in $X$ is defined as $\text{cd}(R,I)$. It is not hard to show that if $X$ is smooth, then the integer $\dim(X) - \text{cd}(R,I)$ depends only on $V$ but neither on $X$ nor on the embedding $V \hookrightarrow X$.

1.2. Knowledge of local cohomology modules provides interesting information, illustrated by the following three situations. Let $I \subseteq R$ and $c = \text{cd}(R,I)$. Then $I$ cannot be generated by fewer than $c$ elements. In fact, no ideal $J$ with the same radical as $I$ will be generated by fewer than $c$ elements.

Let $H^i_{dR}$ stand for the $i$-th de Rham cohomology group. A second application is a family of results commonly known as Barth theorems.
which are a generalization of the classical Lefschetz theorem that states that if \( Y \subseteq \mathbb{P}^n \) is a hypersurface then \( H^{i}_{C}x(\mathbb{P}^n) = H^{i}_{C}x(Y) \) is an isomorphism for \( i < \dim(Y) - 1 \) and injective for \( i = \dim(Y) \). For example, let \( Y \subseteq \mathbb{P}^n \) be a closed subset and \( I \subseteq R = \mathbb{C}[x_0, \ldots, x_n] \) the defining ideal of \( Y \). Then \( H^{i}_{C}x(\mathbb{P}^n) = H^{i}_{C}x(Y) \) is an isomorphism for \( i \leq \text{depth}_{O_{\mathbb{P}^n}}(O_Y) - \text{cd}(R, I) \) (compare [15], 4.7 and [8], the theorem after III.7.6).

Finally, it is also a consequence of the work of Ogus and Hartshorne ([13], 2.2, 2.3 and [8], IV.3.1) that if \( I \subseteq R = \mathbb{C}[x_0, \ldots, x_n] \) is the defining ideal of a complex smooth variety \( V \subseteq \mathbb{P}^n \) then, for \( i < n - \text{codim}(V) \), \( \dim_{\mathbb{C}} \text{soc}_{R}(H^{n-i}_{m}(H^{n-i}(R))) = \dim_{\mathbb{C}} H^{i}_{x}(V, \mathbb{C}) \) where \( H^{i}_{x}(V, \mathbb{C}) \) stands for the \( i \)-th singular cohomology group of the affine cone \( \tilde{V} \) over \( V \) with support in the vertex \( x \) of \( \tilde{V} \) and with coefficients in \( \mathbb{C} \) (\( \text{soc}_{R}(M) \) denotes the socle \( (0 :_{M} m) \subseteq M \) for any \( R \)-module \( M \)).

1.3. The cohomological dimension has been studied by many authors, for example R. Hartshorne ([13]), A. Ogus ([15]), R. Hartshorne and R. Speiser ([9]), C. Peskine and L. Szpiro ([16]), G. Faltings ([4]), C. Huneke and G. Lyubeznik ([8]). Yet despite this extensive effort, the problem of finding an algorithm for the computation of cohomological dimension remained open. For the determination of \( \text{cd}(R, I) \) it is in fact enough to find an algorithm to decide whether or not the local cohomology module \( H^{i}_{C}x(R) = 0 \) for given \( i, R, I \). This is because \( H^{q}_{C}x(R) = 0 \) for all \( q > c \) implies \( \text{cd}(R, I) \leq c \) (see [8], section 1).

In [14] G. Lyubeznik gave an algorithm for deciding whether or not \( H^{i}_{C}x(R) = 0 \) for all \( I \subseteq R = K[x_1, \ldots, x_n] \) where \( K \) is a field of positive characteristic. One of the main purposes of this work is to produce such an algorithm in the case where \( K \) is a field containing the rational numbers and \( R = K[x_1, \ldots, x_n] \).

Since in such a situation the local cohomology modules \( H^{i}_{C}x(R) \) have a natural structure of finitely generated left \( D(R, K) \)-modules ([1]), \( D(R, K) \) being the ring of \( K \)-linear differential operators of \( R \), explicit computations may be performed. Using this finiteness we employ the theory of Gröbner bases to develop algorithms that give a representation of \( H^{i}_{C}x(R) \) and \( H^{i}_{m}(H^{n-j}(R)) \) for all triples \( i, j, n \in \mathbb{N}, I \subseteq R \) in terms of generators and relations over \( D(R, K) \) (where \( m = (x_1, \ldots, x_n) \)). This also leads to an algorithm for the computation of the invariants \( \lambda_{i,j}(R/I) = \dim_{K} \text{soc}_{R}(H^{n-j}(R)) \) introduced in [14].

We remark that if \( R \) is an arbitrary finitely generated \( K \)-algebra and \( I \) is an ideal in \( R \) then, if \( R \) is regular, our algorithms can be used to determine \( \text{cd}(R, I) \) for all ideals \( I \) of \( R \), but if \( R \) is not regular, then
1.4. The outline of the paper is as follows. The next section is devoted to a short survey of results on local cohomology and $D$-modules as they apply to our work, as well as their interrelationship.

In section 2 we review the theory of Gröbner bases as it applies to $A_n$ and modules over the Weyl algebra. Most of that section should be standard and readers interested in proofs and more details are encouraged to look at the book by D. Eisenbud ([3], chapter 15 for the commutative case) or the fundamental article [9] (for the more general situation).

In section 4 we generalize some results due to B. Malgrange and M. Kashiwara on $D$-modules and their localizations. The purpose of sections 4 and 5 is to find a representation of $R_f \otimes N$ as a cyclic $A_n$-module if $N$ is a given holonomic $D$-module (for a definition and some properties of holonomic modules, see subsection 2.3 below). Many of the essential ideas in section 5 come from T. Oaku’s work [14].

In section 6 we describe our main results, namely algorithms that for arbitrary $i, j, k, I$ determine the structure of $H^k_I(R), H^i_m(H^j_I(R))$ and find $\lambda_{i,j}(R/I)$. Some of these algorithms have been implemented in the programming language C and the theory is illustrated with examples. The final section is devoted to comments on implementations, effectivity and examples.

It is a pleasure to thank my advisor Gennady Lyubeznik for suggesting the problem of algorithmic computation of cohomological dimension to me and pointing out that the theory of $D$-modules might be useful for its solution.

2. Preliminaries

2.1. Notation. Throughout we shall use the following notation: $K$ will denote a field of characteristic zero, $R = K[x_1, \ldots, x_n]$ the ring of polynomials over $K$ in $n$ variables, $A_n = K\langle x_1, \partial_1, \ldots, x_n, \partial_n \rangle$ the Weyl algebra over $K$ in $n$ variables, or, equivalently, the ring of $K$-linear differential operators on $R$, $\mathfrak{m}$ will stand for the maximal ideal $(x_1, \ldots, x_n)$ of $R$, $\Delta$ will denote the maximal left ideal $(\partial_1, \ldots, \partial_n)$ of $A_n$ and $I$ will stand for the ideal $(f_1, \ldots, f_r)$ in $R$.

All tensor products in this work will be over $R$ and all $A_n$-modules (resp. ideals) will be left modules (resp. left ideals).

2.2. Local cohomology. It turns out that $H^k_I(M)$ may be computed as follows. Let $C^\bullet(f_i)$ be the complex $0 \to R \xrightarrow{1} R_{f_i} \to 0$. Then
$H^i_f(M)$ is the $k$-th cohomology group of the Čech complex defined by $C^\bullet(M; f_1, \ldots, f_r) = \bigotimes_i C^\bullet(f_i) \otimes M$. Unfortunately, explicit calculations are complicated by the fact that $H^i_f(M)$ is rarely finitely generated as $R$-module. This difficulty disappears for $H^i_f(R)$ if we enlarge the ring to $A_n$, in essence because $R_f$ is finitely generated over $A_n$ for all $f \in R$.

2.3. $D$-modules. A good introduction to $D$-modules is the book by Björk, [1].

Let $f \in R$. Then the $R$-module $R_f$ has a structure as left $A_n$-module: $x_i(\frac{g}{f^n}) = \frac{x_i g}{f^n}, \partial_i(\frac{g}{f^n}) = \frac{\partial_i(g) f^{-k} \partial_i(f) g}{f^{k+1}}$. This may be thought of as a special case of localizing an $A_n$-module: if $M$ is an $A_n$-module and $f \in R$ then $R_f \otimes_R M$ becomes an $A_n$-module via $\partial_i(\frac{g}{f^n} \otimes m) = \partial_i(\frac{g}{f^n}) \otimes m + \frac{g}{f^n} \partial_i m$. Localization of $A_n$-modules lies at the heart of our arguments.

Of particular interest are the holonomic modules which are those finitely generated $A_n$-modules $N$ for which $\text{Ext}^j_{A_n}(N, A_n)$ vanishes unless $j = n$. Holonomic modules are always cyclic and of finite length over $A_n$. Besides that, if $N = A_n/L$, $f \in R$, $s$ is an indeterminate and $g$ is some fixed generator of $N$, then there is a nonzero polynomial $b(s)$ in $K[s]$ and an operator $P(s) \in A_n[s]$ such that $P(s)(f \cdot f^s \otimes g) = b(s) \cdot f^s \otimes g$. The unique monic polynomial that divides all other polynomials satisfying an identity of this type is called the Bernstein polynomial of $L$ and $f$ and denoted by $b_f^L(s)$. Any operator $P(s)$ that satisfies $P(s)f^{s+1} \otimes g = b_f^L(s)f^s \otimes g$ we shall call a Bernstein operator and refer to the roots of $b_f^L(s)$ as Bernstein roots of $f$ on $A_n/L$.

Localizations of holonomic modules at a single element (and hence at any finite number of elements) of $R$ are holonomic (see [1], section 5.9) and in particular cyclic over $A_n$, generated by $f^{-a}g$ for sufficiently large $a \in \mathbb{N}$ (see also our proposition 4.2). So the complex $C^\bullet(N; f_1, \ldots, f_r)$ consists of holonomic $A_n$-modules whenever $N$ is holonomic. This facilitates the use of Gröbner bases as computational tool for maps between holonomic modules and their localizations. As a special case we note that localizations of $R$ are holonomic, and hence finite, over $A_n$ (since $R = A_n/\Delta$ is holonomic).

2.4. The Čech complex. In [1] it is shown that local cohomology modules are not only $D$-modules but in fact holonomic: we know already that the modules in the Čech complex are holonomic, it suffices to show that the maps are $A_n$-linear. All maps in the Čech complex are direct sums of localization maps. Suppose $R_f$ is generated by $f^s$ and $R_{fg}$ by $(fg)^t$. We may replace $s, t$ by their minimum $u$ and then we see that the inclusion $R_f \to R_{fg}$ is nothing but the map $A_n/\text{ann}(f^u) \to$
\(A_n/\text{ann}((fg)^n)\) sending the coset of the operator \(P\) to the coset of the operator \(P \cdot g^n\). So \(C^i(N; f_1, \ldots, f_r) \rightarrow C^{i+1}(N; f_1, \ldots, f_r)\) is an \(A_n\)-linear map between holonomic modules for every holonomic \(N\). One can prove that kernels and cokernels of \(A_n\)-linear maps between holonomic modules are holonomic. Holonomicity of \(H^i_k(I(R))\) follows.

In the same way it can be seen that \(H^i_j(I(R))\) is holonomic for \(i, j \in \mathbb{N}\) (since \(H^j_k(I(R))\) is holonomic).

3. Gröbner bases of modules over the Weyl algebra

In this section we review some of the concepts and results related to the Buchberger algorithm in modules over Weyl algebras. It turns out that with a little care many of the important constructions from the theory of commutative Gröbner bases carry over to our case. For an introduction into non-commutative monomial orders and related topics, [9] is a good source.

Let us agree that every time we write an element in \(A_n\), we write it as a sum of terms \(c_{\alpha\beta}x^\alpha\partial^\beta\) in multi-index notation. That is, \(\alpha, \beta \in \mathbb{N}^n\), \(c_{\alpha\beta}\) are scalars, \(x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}\), \(\partial^\beta = \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}\) and in every monomial we write first the powers of \(x\) and then the powers of the differentials. Further, if \(m = c_{\alpha\beta}x^\alpha\partial^\beta, c_{\alpha\beta} \in K\), we will say that \(m\) has degree \(\deg m = |\alpha + \beta|\) and an operator \(P \in A_n\) has degree equal to the largest degree of any monomial occurring in \(P\).

Recall that a monomial order \(<\) in \(A_n\) is a total order on the monomials of \(A_n\), subject to \(m < m' \Rightarrow mm'' < m'm''\) for all nonzero monomials \(m, m', m''\). Since the product of two monomials in our notation is not likely to be a monomial (as \(\partial_i x_i = x_i\partial_i + 1\)) it is not obvious that such orderings exist at all. However, the commutator of any two monomials \(m_1, m_2\) will be a polynomial of degree at most \(\deg m_1 + \deg m_2 - 2\). That means that the degree of an operator and its component of maximal degree is independent of the way it is represented. Thus we may for example introduce a monomial order on \(A_n\) by taking any monomial order on \(\tilde{A}_n = K[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n]\) (the polynomial ring in \(2n\) variables) that refines the partial order given by total degree, and saying that \(m_1 > m_2\) in \(A_n\) if and only if \(m_1 > m_2\) in \(\tilde{A}_n\).

Let \(<\) be a monomial order on \(A_n\). Let \(G = \bigoplus^d_i A_n \cdot \gamma_i\) be the free \(A_n\)-module on the symbols \(\gamma_1, \ldots, \gamma_d\). We define a monomial order on \(G\) by \(m_i\gamma_i > m_j\gamma_j\) if either \(m_i > m_j\) in the order on \(A_n\), or \(m_i = m_j\) and \(i > j\). The largest monomial \(m\gamma\) in an element \(g \in G\) will be denoted by \(\text{in}(g)\). Of fundamental importance is
Algorithm 3.1 (Remainder). Let \( h \) and \( g = \{g_i\}_1^s \) be elements of \( G \). Set \( h_0 = h, \sigma_0 = 0, j = 0 \) and let \( \varepsilon_i = \varepsilon(g_i) \) be symbols. Then

Repeat

\[
\text{If } \text{in}(g_i) \mid \text{in}(h_j) \text{ set}
\]
\[
\{ h_{j+1} := h_j - \frac{\text{in}(h_j)}{\text{in}(g_i)} g_i, \] \\
\sigma_{j+1} := \sigma_j + \frac{\text{in}(h_j)}{\text{in}(g_i)} \varepsilon_i, \\
\} \\
j := j + 1
\]

Until No \( \text{in}(g_i) \mid \text{in}(h_j) \).

The result is \( h_a \), called a remainder \( R(h,g) \) of \( h \) under division by \( g \), and an expression \( \sigma_a = \sum_{i=1}^s a_i \varepsilon_i \) with \( a_i \in A_n \). By Dickson’s lemma ([8], 1.1) the algorithm terminates. It is worth mentioning that \( R(h,g) \) is not uniquely determined, it depends on which \( g_i \) we pick amongst those whose initial term divides the initial term of \( h_j \).

Note that if \( h_a \) is zero, \( \sigma_a \) tells us how to write \( h \) in terms of \( g \). Such a \( \sigma_a \) is called a standard expression for \( h \) with respect to \( \{g_1, \ldots, g_s\} \).

Definition 3.2. If \( \text{in}(g_i) \) and \( \text{in}(g_j) \) involve the same basis element of \( G \), then we set \( s_{ij} = m_{ji}g_i - m_{ij}g_j \) and \( \sigma_{ij} = m_{ji}\varepsilon_i - m_{ij}\varepsilon_j \) where \( m_{ij} = \frac{\text{lcm}(\text{in}(g_j), \text{in}(g_i))}{\text{in}(g_j)} \). Otherwise, \( \sigma_{ij} \) and \( s_{ij} \) are defined to be zero. \( s_{ij} \) is the Schreyer-polynomial to \( g_i \) and \( g_j \).

Suppose \( R(s_{ij}, g) \) is zero for all \( i, j \). Then we call \( g \) a Gröbner basis for the module \( A_n \cdot (g_1, \ldots, g_s) \).

The following proposition ([8], Lemma 3.8) indicates the usefulness of Gröbner bases.

Proposition 3.3. Let \( g \) be a finite set of elements of \( G \). Then \( g \) is a Gröbner basis if and only if \( h \in A_n g \) implies \( \exists i : \text{in}(g_i) \mid \text{in}(h) \). \( \square \)

Computation of Gröbner bases over the Weyl algebra works just as over polynomial rings:

Algorithm 3.4 (Buchberger). Input: \( g = \{g_1, \ldots, g_s\} \subseteq G \).
Output: a Gröbner basis for \( A_n \cdot (g_1, \ldots, g_s) \).

Begin.

\[
\text{Repeat}
\]
\[
\text{If } h = R(s_{ij}, g) \neq 0 \text{ add } h \text{ to } g
\]

Until all \( R(s_{ij}, g) = 0 \).

Return \( g \).

End.
3.1. Now we shall describe the construction of kernels of $A_n$-linear maps using Gröbner bases. Again, this is similar to the commutative case and we first consider the case of a map between free $A_n$-modules. Let $E = \bigoplus_s A_n e_i, G = \bigoplus_r A_n \gamma_j$ and $\phi : E \to G$ be an $A_n$-linear map. Assume $\phi(e_i) = g_i$. Suppose that in order to make $g$ a Gröbner basis we have to add $g_1', \ldots, g_s'$ to $g$ which satisfy $g_i' = \sum_{k=1}^s a_{ik} g_k$. We get an induced map
\[
\begin{array}{c}
\bigoplus^s A_n e_i \\
\downarrow \pi \\
\bigoplus^s A_n e_i \\
\downarrow \phi \\
\bigoplus^r A_n \gamma_j
\end{array}
\]
where $\pi$ is the identity on $e_i$ for $i \leq s$ and sends $e_{s+1}$ into $\sum_{k=1}^s a_{ik} e_k$. Of course, $\tilde{\phi} = \phi \pi$.

The kernel of $\phi$ is just the image of the kernel of $\tilde{\phi}$ under $\pi$. So in order to find kernels of maps between free modules one may assume that the generators of the source are mapped to a Gröbner basis and replace $\phi$ by $\tilde{\phi}$. Recall from definition 3.2 that $\sigma_{ij} = m_{ji} e_i - m_{ij} e_j$ or zero, depending on the leading terms of $g_i$ and $g_j$.

Proposition 3.5. Assume that $\{g_1, \ldots, g_s\}$ is a Gröbner basis. Let $s_{ij} = \sum d_{ijk} g_k$ be standard expressions for the Schreyer polynomials. Then $\{\sigma_{ij} - \sum_{k}^s d_{ijk} e_k\}_{1 \leq i < j \leq s}$ generate the kernel of $\phi : \bigoplus^s A_n e_i \to \bigoplus^r A_n \gamma_j$, sending $e_i$ to $g_i$.

The proof proceeds exactly as in the commutative case, see for example [3], section 15.10.8.

3.2. We explain now how to find a set of generators for the kernel of an arbitrary $A_n$-linear map. Let $E, G$ be as in subsection 3.1 and suppose $A_n(p_1, \ldots, p_a) = P \subseteq E, A_n(q_1, \ldots, q_b) = Q \subseteq G$ and $\phi : \bigoplus_i^s A_n e_i / P \to \bigoplus_j^r A_n \gamma_j / Q$. It will be sufficient to consider the case $P = 0$ since we may lift $\phi$ to the free module $E$ surjecting onto $E/P$.

Let as before $\phi(e_i) = g_i$. A kernel element in $E$ is a sum $\sum_{a} a_i e_i, a_i \in A_n$, which if $e_i$ is replaced by $g_i$ can be written in terms of the generators $q_j$ of $Q$. Let $\beta = \{\beta_1, \ldots, \beta_c\}$ be such that $q \cup q \cup \beta$ is a Gröbner basis for $A_n(q, q)$. We may assume that the $\beta_i$ are the results of applying algorithm 3.4 to $q \cup q$. Then from algorithm 3.1 we have expressions
\[
\beta_i = \sum_j c_{ij} g_j + \sum_k c'_{ik} g_k,
\] (3.1)
with \( c_{ij}, c'_{ik} \in A_n \). Furthermore, by proposition 3.5, algorithm 3.4 returns a generating set \( \sigma \) for the syzygies on \( g \cup q \cup \beta \). Write
\[
\sigma_i = \sum_j a_{ij} \varepsilon_{g_j} + \sum_k a'_{ik} \varepsilon_{q_k} + \sum_l a''_{il} \varepsilon_{\beta_l}
\]
and eliminate the last sum using the relations (3.1) to obtain syzygies
\[
\tilde{\sigma}_i = \sum_j a_{ij} \varepsilon_{g_j} + \sum_k a'_{ik} \varepsilon_{q_k} + \sum_l a''_{il} \left( \sum_v c_{lv} \varepsilon_{g_v} + \sum_w c'_{lw} \varepsilon_{q_w} \right)
\]
These will then form a generating set for the syzygies on \( g \cup q \). Cutting off the \( q \)-part of these syzygies we get a set of forms
\[
\left\{ \sum_j a_{ij} \varepsilon_{g_j} + \sum_l a''_{il} \left( \sum_v c_{lv} \varepsilon_{g_v} \right) \right\}
\]
which generate the kernel of the map \( E \to G/Q \).

3.3. The comments in this subsection will find their application in algorithm 6.2 which computes the structure of \( H^i_m(H^j_I(R)) \) as \( A_n \)-module. Let
\[
\begin{array}{c}
M'_3 \xrightarrow{\alpha} M_3 \xrightarrow{\alpha'} M''_3 \\
\uparrow \phi' \quad \uparrow \psi' \quad \uparrow \rho' \\
M'_2 \xrightarrow{\beta} M_2 \xrightarrow{\beta'} M''_2 \\
\uparrow \phi \quad \uparrow \psi \quad \uparrow \rho \\
M'_1 \xrightarrow{\gamma} M_1 \xrightarrow{\gamma'} M''_1
\end{array}
\]
be a commutative diagram of \( A_n \)-modules. Note that the row cohomology of the column cohomology at \( N \) is given by
\[
\left[ \ker(\psi') \cap \beta'^{-1}(\im \rho) + \im(\psi) \right] / \left[ \beta(\ker(\phi')) + \im(\psi) \right].
\]
In order to compute this we need to be able to find:
- preimages of submodules,
- kernels of maps,
- intersections of submodules.

It is apparent that the second and third calculation is a special case of the first: kernels are preimages of zero, intersections are images of preimages (if \( A_n^r \xrightarrow{\phi} A_n^s/M \xleftarrow{\psi} A_n^t \) is given, then \( \im(\phi) \cap \im(\psi) = \psi(\psi^{-1}(\im(\phi))) \)).

So suppose in the situation \( \phi : A_n^r/M \to A_n^s/N, \psi : A_n^t/P \to A_n^s/N \) we want to find the preimage under \( \psi \) of the image of \( \phi \). We
may reduce to the case where $M$ and $P$ are zero and then lift $\phi, \psi$ to maps into $A_n^s$. The elements $x$ in $\psi^{-1}(\im \phi) \subseteq A_n^t$ are exactly the elements in $\ker(A_n^t \rightarrow A_n^s/N \rightarrow A_n^s/(N + \im \phi))$ and this kernel can be found according to the comments in \[3.2\].

4. $D$-modules after Kashiwara and Malgrange

The purpose of this and the following section is as follows. Given $f \in R$ and an ideal $L \subseteq A_n$ such that $A_n/L$ is holonomic and $L$ is $f$-saturated (i.e. $f \cdot P \in L$ only if $P \in L$), we want to compute the structure of the module $R_f \otimes A_n/L$. It turns out that it is useful to know the ideal $J^L(f^s)$ which consists of the operators $P(s) \in A_n[s]$ that annihilate $f^s \otimes \overline{T} \in M := R_f[s]f^s \otimes A_n/L$ where the bar denotes cosets in $A_n/L$. In order to find $J^L(f^s)$, we will consider the module $M$ over the ring $A_{n+1} = A_n(t, \partial_t)$. It will turn out in \[4.2\] that one can easily compute the ideal $J^L_{n+1}(f^s) \subseteq A_{n+1}$ consisting of all operators that kill $f^s \otimes \overline{T}$. In section \[4\] we will then show how to compute $J^L(f^s)$ from $J^L_{n+1}(f^s)$.

The second basic fact in this section (proposition \[4.2\]) shows how to compute the structure of $R_f \otimes A_n/L$ as $A_n$-module once $J^L(f^s)$ is known.

4.1. Consider $A_{n+1} = A_n(t, \partial_t)$, the Weyl algebra in $x_1, \ldots, x_n$ and the new variable $t$. B. Malgrange has defined an action of $t$ and $\partial_t$ on $M = R_f[s] \cdot f^s \otimes_R A_n/L$ by $t(g(x, s) \cdot f^s \otimes \overline{P}) = g(x, s + 1) f \cdot f^s \otimes \overline{P}$ and $\partial_t(g(x, s) \cdot f^s \otimes \overline{P}) = \overline{t}g(x, s - 1) \cdot f^s \otimes \overline{P}$ for $\overline{P} \in A_n/L$. $A_n$ acts on $M$ as expected, the variables by multiplication on the left, the $\partial_i$ by the product rule.

One checks that this actually defines an structure of $M$ as a left $A_{n+1}$-module and that $-\partial_t t$ acts as multiplication by $s$.

We denote by $J^L_{n+1}(f^s)$ the ideal in $A_{n+1}$ that annihilates the element $f^s \otimes \overline{T}$ in $M$. Then we have an induced morphism of $A_{n+1}$-modules $A/J^L_{n+1}(f^s) \rightarrow M$ sending $P + J^L_{n+1}(f^s)$ to $P(f^s \otimes \overline{T})$.

The operators $t$ and $\partial_t$ were introduced in \[4.3\]. The following lemma generalizes lemma 4.1 in \[4.3\] (as well as part of the proof given there) where the special case $L = (\partial_1, \ldots, \partial_n), A_n/L = R$ is considered.

Note that $J^L_{n+1}(f^s)$ makes perfect sense even if $L$ is not holonomic.

Lemma 4.1. Suppose that $L = A_n \cdot (P_1, \ldots, P_r)$ is $f$-saturated (i.e., if $f \cdot P \in L$, then $P \in L$). With the above definitions, $J^L_{n+1}(f^s)$ is the ideal generated by $f - t$ together with the images of the $P_j$ under the automorphism $\phi$ of $A_{n+1}$ induced by $x \rightarrow x, t \rightarrow t - f$. 
4.2. Let \( J = \text{terms that contain } \partial \) write operators \( Q \).

Again, we may talk about \( \phi \) \([10], \text{proposition 6.2.}\).

Proof. We mimick the proof given by Kashiwara, who proved the proposition for the case \( L = (\partial_1, \ldots, \partial_n), A_n/L = R \) \([10]\), proposition 6.2.

\[ \text{Proof.} \] The automorphism sends \( \partial_t \) to \( \partial_t + f_i \partial_t \) and \( \partial_t \) to \( \partial_t \). So if we write \( P_j = P_j(\partial_1, \ldots, \partial_n), \) then \( \phi P_j = P_j(\partial_1 + f_i \partial_t, \ldots, \partial_n + f_i \partial_t) \).

One checks that \((\partial_t + f_i \partial_t)(f^s \otimes \overline{T}) = f^s \otimes \overline{T}\) for all differential operators \( Q \) so that \( \phi(P_j(\partial_1, \ldots, \partial_n))(f^s \otimes \overline{T}) = f^s \otimes \overline{T} \).

By definition, \( f \cdot (f^s \otimes \overline{T}) = t \cdot (f^s \otimes \overline{T}) \). So \( t - f \in J_{n+1}^L(f^s) \) and \( \phi(P_j) \in J_{n+1}^L(f^s) \).

Conversely let \( P(f^s \otimes \overline{T}) = 0 \). We may assume, that \( P \) does not contain any \( t \) since we can eliminate \( t \) using \( f - t \). Now rewrite \( P \) in terms of \( \partial_t \) and the \( \partial_t + f_i \partial_t \). Say, \( P = \sum c_{\alpha \beta}^t x^\beta Q_{\alpha \beta}(\partial_t + f_1 \partial_t, \ldots, \partial_n + f_n \partial_t) \).

We show that the sum of terms that contain \( \partial_t \) is in \( A_{n+1}, \phi(L) \) as follows. In order for \( P(f^s \otimes \overline{T}) \) to vanish, the sum of terms with the highest \( s \)-power, namely \( s^\alpha \), must vanish, and so \( \sum_{c_{\alpha \beta}^t} x^\beta Q_{\alpha \beta}(\partial_t + f_1 \partial_t, \ldots, \partial_n + f_n \partial_t) \in R_{f^s} \).

Let \( \overline{T} \) be the largest \( \alpha \in \mathbb{N} \) for which there is a nonzero \( c_{\alpha \beta}^t \) occurring in \( P = \sum c_{\alpha \beta}^t x^\beta Q_{\alpha \beta}(\partial_t + f_1 \partial_t, \ldots, \partial_n + f_n \partial_t) \).

We show that the sum of terms that contain \( \partial_t \) is in \( A_{n+1}, \phi(L) \) as follows. In order for \( P(f^s \otimes \overline{T}) \) to vanish, the sum of terms with the highest \( s \)-power, namely \( s^\alpha \), must vanish, and so \( \sum_{c_{\alpha \beta}^t} x^\beta Q_{\alpha \beta}(\partial_t + f_1 \partial_t, \ldots, \partial_n + f_n \partial_t) \in R_{f^s} \).

We will in the next section show how the lemma can be used to determine \( J^L(f^s) \). Now we show why \( J^L(f^s) \) is useful, generalizing \([10]\), proposition 6.2.

Recall that the Bernstein polynomial \( b^t_{J^s}(s) \) is defined to be the monic generator of the ideal of polynomials \( b(s) \in K[s] \) for which there exists an operator \( P(s) \in A_n[s] \) such that \( P(s)(f^{s+1} \otimes \overline{T}) = b(s)f^s \otimes \overline{T} \) \([10]\), chapter 1), and that \( b^t_{J^s}(s) \) will exist for example if \( L \) is holonomic.

**Proposition 4.2.** If \( L \) is holonomic and \( a \in \mathbb{Z} \) is such that no integer root of \( b^t_{J^s}(s) \) is smaller than \( a \), then we have isomorphisms

\[ R_f \otimes A_n/L \cong A_n[s]/J^L(f^s)|_{s=a} \cong A_n \cdot f^a \otimes \overline{T}. \] (4.1)

**Proof.** We mimick the proof given by Kashiwara, who proved the proposition for the case \( L = (\partial_1, \ldots, \partial_n), A_n/L = R \) \([10]\), proposition 6.2.
Let us first prove the second equality. Certainly \( J^L(f^s) \big|_{s=a} \) kills \( f^a \otimes \overline{T} \). So we have to show that if \( P(f^a \otimes \overline{T}) = 0 \) then \( P \in J^L(f^s) + A_n[s] \cdot (s-a) \). To that end note that \( st \) acts as \( t(s-1) \) which means that \( t \cdot (A_n[s]/J^L(f^s)) \) is a left \( A_n[s] \)-module. Identify \( A_n[s]/J^L(f^s) \) with \( N^L_f := A_n[s] \cdot (f^s \otimes \overline{T}) \). By definition, \( b_f^L(s) \) is the minimal polynomial for which there is \( P(s) \) with \( b_f^L(s)(f^s \otimes \overline{T}) = P(s)f^{s+1} = t \cdot P(s-1)(f^s \otimes \overline{T}) \).

So \( b_f^L(s) \) multiplies \( A_n[s] \cdot (f^s \otimes \overline{T}) \) into \( t \cdot A_n[s](f^s \otimes \overline{T}) \) and whenever the polynomial \( b(s) \in K[s] \) is relatively prime to \( b_f^L(s) \) its action on \( N^L_f / t \cdot N^L_f \) is injective.

Since by hypothesis \( s-a+j \) is not a divisor of \( b_f^L(s) \) for \( 0 < j \in \mathbb{N} \),

\[
(s-a+j)N^L_f \cap t \cdot N^L_f \subseteq (s-a+j)t \cdot N^L_f. \tag{4.2}
\]

So \( (s-a+m)N^L_f \cap t^mN^L_f \subseteq (s-a+m)tN^L_f \cap t^mN^L_f = t[(s-a+m-1)N^L_f \cap t^{m-1}N^L_f] \) whenever \( m \geq 1 \).

We show now by induction on \( m \) that \( (s-a+m)N^L_f \cap t^mN^L_f \subseteq (s-a+m)t^mN^L_f \) for \( m \geq 1 \). The claim is clear for \( m = 1 \) from equation (4.2). So let \( m > 1 \). The inductive hypothesis states that \( (s-a+m-1)N^L_f \cap t^{m-1}N^L_f \subseteq (s-a+m-1)t^{m-1}N^L_f \). The previous paragraph shows that \( (s-a+m)N^L_f \cap t^mN^L_f \subseteq t[(s-a+m-1)N^L_f \cap t^{m-1}N^L_f] \).

Combining these two facts we get

\[
(s-a+m)N^L_f \cap t^mN^L_f \subseteq t(s-a+m-1)t^{m-1}N^L_f
= (s-a+m)t^{m}N^L_f.
\]

Now if \( P(s) \in A_n[s] \) is of degree \( m \) in the \( \partial_i \) and \( P(a)(f^a \otimes \overline{T}) = 0 \), then \( P(s+m) \cdot f^m + J^L(f^s) \in (s-a+m) \cdot N^L_f \) because we can interpete \( P(s+m)(f^{s+m} \otimes \overline{T}) \) as a polynomial in \( s+m \) with root \( a \). But then \( P(s+m)(f^{s+m} \otimes \overline{T}) = P(s+m)(f^m f^s \otimes \overline{T}) \) is in

\[
(s-a+m)N^L_f \cap t^mN^L_f \subseteq (s-a+m)t^mN^L_f,
\]

implying \( P(s+m)(f^{s+m} \otimes \overline{T}) = (s-a+m)Q(s)(f^{s+m} \otimes \overline{T}) \) for some \( Q(s) \in A_n[s] \) (note that \( J^L(f^s) \) kills \( f^s \otimes \overline{T} \)). In other words, \( P(s)-(s-a)Q(s-m) \in J^L(f^s) \).

For the first isomorphism we have to show that \( A_n \cdot (f^a \otimes \overline{T}) = R_f \otimes A_n/L \). It suffices to show that every term of the form \( f^m f^a \otimes \overline{Q} \) is in the module generated by \( (f^a \otimes \overline{T}) \) for all \( m \in \mathbb{Z} \). Furthermore, we may assume that \( Q \) is a monomial in \( \partial_1, \ldots, \partial_n \).

Existence and definition of \( b_f^L(s) \) provides an operator \( P(s) \) with \( [b_f^L(s-1)]^{-1}P(s-1)(f^s \otimes \overline{T}) = f^{-1}f^s \otimes \overline{T} \). As \( b_f^L(a-m) \neq 0 \) for all \( 0 < m \in \mathbb{N} \) we have \( f^m f^a \otimes \overline{T} \in A_n \cdot (f^a \otimes \overline{T}) \) for all \( m \). Now let \( Q \) be a monomial in \( \partial_1, \ldots, \partial_n \) of \( \partial \)-degree \( j > 0 \) and assume that
\( f^m f^a \otimes Q' \in A_n \cdot (f^a \otimes \mathbb{T}) \) for all \( m \) and all operators \( Q' \) of \( \partial \)-degree lower than \( j \). Then \( Q = \partial_i Q' \) for some \( 1 \leq i \leq n \). Fix \( m \in \mathbb{Z} \). By assumption on \( j \), for some \( P' \) we have \( P'(f^a \otimes \mathbb{T}) = f^m f^a \otimes Q' \). So

\[
    f^m f^a \otimes Q = \partial_i P'(f^a \otimes \mathbb{T}) - f_i \cdot (a + m) f^{m-1} f^a \otimes Q' \in A_n \cdot (f^a \otimes \mathbb{T}).
\]

The claim follows by induction. This completes the proof of the proposition. \( \square \)

We remark that if any \( a \in \mathbb{Z} \) satisfies the conditions of the proposition, then so does every integer smaller than \( a \).

5. An algorithm of Oaku

The purpose of this section is to review and generalize an algorithm due to Oaku. In \([4, 5]\) (algorithm 5.4.), Oaku showed how to construct a generating set for \( J^L(f^a) \) in the case where \( L = (\partial_1, \ldots, \partial_n) \). According to \([4, 2]\), \( J^L(f^a) \) is the intersection of \( J^L_{n+1}(f^a) \) with \( A_n[-\partial_t t] \). We shall explain how one may calculate \( J \cap A_n[-\partial_t t] \) whenever \( J \subseteq A_{n+1} \) is any given ideal and as a corollary develop an algorithm that for \( f \)-saturated \( A_n/L \) computes \( J^L(f^a) \). The proof follows closely Oaku’s argument.

On \( A_{n+1}[y_1, y_2] \) define weights \( w(t) = w(y_1) = 1, w(\partial_t) = w(y_2) = -1, w(x_i) = \partial_i = 0 \). If \( P = \sum_i P_i \in A_{n+1}[y_1, y_2] \) and all \( P_i \) are monomials, then we will write \((P)^h \) for the operator \( \sum_i P_i \cdot y_i^d \) where \( d_i = \max_j (w(P_j)) - w(P_i) \) and call it the \( y_1 \)-homogenization of \( P \).

Note that the Buchberger algorithm preserves homogeneity in the following sense: if a set of generators for an ideal is given and these generators are homogeneous with respect to the weights above, then any new generator for the ideal constructed with the classical Buchberger algorithm will also be homogeneous. (This is a consequence of the facts that the \( y_i \) commute with all other variables and that \( \partial_t t = t \partial_t + 1 \) is homogeneous of weight zero.)

**Proposition 5.1.** Let \( J = A_{n+1} \cdot (Q_1, \ldots, Q_r) \) and let \( y_1, y_2 \) be two new variables. Let \( \tilde{I} \) be the left ideal in \( A_{n+1}[y_1] \) generated by the \( y_1 \)-homogenizations \((Q_i)^h \) of the \( Q_i \), relative to the weight \( w \) above, and let \( \tilde{J} = A_{n+1}[y_1, y_2] \cdot (I, 1 - y_1 y_2) \). Let \( G \) be a Gröbner basis for \( \tilde{I} \) under a monomial order that eliminates \( y_1, y_2 \). For each \( P \in G \) set \( P' = t^{-w(P)} P \) if \( w(P) < 0 \) and \( P' = \partial_t^{w(P)} P \) if \( w(P) > 0 \) and let \( G' = \{ P' : P \in G \} \). Then \( G_0 = G' \cap A_n[-\partial_t t] \) generates \( J \cap A_n[-\partial_t t] \).

**Proof.** Note first that \( G \) consists of \( w \)-homogeneous operators and so \( w(P) \) is well defined for \( P \in G \).
Suppose $P \in G_0$. Hence $P \in \tilde{I}$. So $P = Q_{-1} \cdot (1-y_1y_2) + \sum a_i \cdot (Q_i)^h$ where the $a_i$ are in $A_{n+1}[y_1, y_2]$. Since $P \in A_n[-\partial t]$, the substitution $y_1 \to 1$ shows that $P = \sum a_i(1, 1) \cdot (Q_i)^h(1, 1) = \sum a_i(1, 1) \cdot Q_i \in J$. Therefore $G_0 \subseteq J \cap A_n[-\partial t]$.

Now assume that $P \in J \cap A_n[-\partial t]$. So $P$ is $w$-homogeneous of weight $0$. Also, $P \in J$ and $J$ is contained in $I(1)$, the ideal of operators $Q(1) \subseteq A_{n+1}$ for which $Q(y_1) \in I$. By lemma 5.2 below (taken from [13]), $y_1^aP \in I$ for some $a \in \mathbb{N}$. Therefore $P = (1-(y_1y_2)^a)P + (y_1y_2)^aP \in \tilde{I}$.

Let $G = \{P_1, \ldots, P_b, P_{b+1}, \ldots, P_c\}$ and assume that $P_i \in A_{n+1}$ if and only if $i \leq b$. Buchberger algorithm gives a standard expression $P = \sum a_i P_i$ with all $\text{in}(a_i P_i) \leq \text{in}(P)$. That implies that $a_{b+i}$ is zero for positive $i$ and $a_i$ does not contain $y_1, y_2$ for any $i$.

Since $P, P_i$ are $w$-homogeneous, the same is true for all $a_i$, from Buchberger algorithm. In fact, $w(P) = w(a_i) + w(P_i)$ for all $i$. As $w(P) = 0$ (and $t, \partial_t$ are the only variables with nonzero weight that may appear in $a_i$) we find $a_i' \in A_n$ with $a_i = a_i' \cdot t^{-w(P_i)}$ or $a_i = a_i' \cdot \partial_t^{w(P_i)}$, depending on whether $w(P_i)$ is negative or positive.

It follows that $P = \sum_{i=1}^b a_i P_i = \sum_{i=1}^b a_i' P_i' \in A_n[-\partial_t] \cdot G_0$. \qed

Lemma 5.2. Let $I$ be a $w$-homogeneous ideal in $A_{n+1}[y_1]$ with respect to the weights introduced before the proposition and $I(1)$ defined as in the proof of the proposition. Assume $P \in A_{n+1}$ is a $w$-homogeneous operator. Then $P \in I(1)$ implies $y_1^aP \in I$ for some $a$.

Proof. Note first that $y_1 \to 1$ will not lead to cancellation of terms in any homogeneous operator as $w(y_1) \neq 0$.

If $P \in I(1)$, $P = \sum Q_i(1)$, with all $Q_i$ $w$-homogeneous in $I$. Then the $y_1$-homogenization of $Q_i(1)$ will be a divisor of $Q_i$ and the quotient will be some power of $y_1$, say $y_1^{\eta_i}$. Homogenization of the equation $P = \sum Q_i(1)$ results in $y_1^\eta P = \sum Q_i(1)^h$ (since $P$ is homogeneous) so that

$$y_1^{\eta + \text{max}(\eta_i)}P = \sum y_1^{\text{max}(\eta_i) - \eta_i}Q_i \in I.$$ \qed

So we have

Algorithm 5.3. Input: $f \in R, L \subseteq A_n$ such that $L$ is $f$-saturrated.
Output: Generators for $J^L(f^*)$.

Begin

1. For each generator $Q_i$ of $L$ compute the image $\phi(Q_i)$ under $x_i \to x_i, t \to t - f, \partial_t \to \partial_t + f, \partial_i \to \partial_i$. Add $t - f$ to the list.
2. Homogenize all $\phi(Q_i)$ with respect to the new variable $y_1$ relative to the weight $w$ introduced before proposition 5.1.
3. Compute a Gröbner basis for the ideal generated by \((\phi(Q_1))^h, \ldots, (\phi(Q_r))^h, 1 - y_1y_2, t - y_1f\) in \(A_{n+1}[y_1, y_2]\) using an order that eliminates \(y_1, y_2\).

4. Select the operators \(\{P_j\}_1^b\) in this basis which do not contain \(y_1, y_2\).

5. For each \(P_j, 1 \leq j \leq b\), if \(w(P_j) > 0\) replace \(P_j\) by \(P'_j = \partial^w(P_j)\).

   Otherwise replace \(P_j\) by \(P'_j = t^{-w(P_j)}P_j\).

6. Return the new operators \(\{P'_j\}_1^b\).

   End.

In order to guarantee existence of the Bernstein polynomial \(b_f^L(s)\) we assume for our next result that \(L\) is holonomic.

**Corollary 5.4.** Suppose \(L\) is a holonomic ideal. If \(J^L(f^s)\) is known or it is known that \(L\) is \(f\)-saturated, then the Bernstein polynomial \(b_f^L(s)\) of \(R_f \otimes_R A_n/L\) can be found from \((b_f^L(s)) = A_n[s] \cdot (J^L(f^s), f) \cap K[s]\).

Moreover, if \(K \subseteq \mathbb{C}\), suppose \(b_f^L(s) = s^d + b_{d-1}s^{d-1} + \ldots + b_0\) and define \(B = \max\{\{b_i\}^{1/(d-i)}\}\). In order to find the smallest integer root of \(b_f^L(s)\), one only needs to check the integers between \(-2B\) and \(2B\).

If in particular \(L = (\partial_1, \ldots, \partial_n)\), it suffices to check the integers between \(-b_{d-1}\) and \(b_{d-1}\).

**Proof.** If \(L\) is \(f\)-saturated, propositions \([1.1]\) and \([5.1]\) enable us to find \(J^L(f^s)\). The first part follows then easily from the definition of \(b_f^L(s)\): as \((b_f^L(s) - P_f^L \cdot f)(f^s \otimes T) = 0\) it is clear that \(b_f^L(s)\) is in \(K[s]\) and in \(A_n[s](J^L(f^s), f)\). Using an elimination order on \(A_n[s]\), \(b_f^L(s)\) will be (up to a scalar factor) the unique element in the reduced Gröbner basis for \(J^L(f^s) + (f)\) that contains no \(x_i\) nor \(\partial_i\).

Now suppose \(K \subseteq \mathbb{C}\), \(|s| = 2B\rho\) where \(B\) is as defined above and \(\rho > 1\). Assume also that \(s\) is a root of \(b_f^L(s)\). We find

\[
(2B\rho)^d = |s|^d = \left| - \sum_{0}^{d-1} b_is^i \right| \leq \sum_{0}^{d-1} B^{d-i}|s|^i \leq B^d \sum_{0}^{d-1} (2\rho)^i \leq B^d((2\rho)^d - 1),
\]

using \(\rho \geq 1\). By contradiction, \(s\) is not a root.

The final claim is a consequence of Kashiwara’s work \([10]\) where it is proved that if \(L = (\partial_1, \ldots, \partial_n)\) then all roots of \(b_f^L(s)\) are negative and hence \(-b_{d-1}\) is a lower bound for each single root. \(\square\)

For purposes of reference we also list algorithms that compute the Bernstein polynomial to a holonomic module and the localization of a holonomic module.
Algorithm 5.5. Input: $f \in R, L \subseteq A_n$ such that $A_n/L$ is holonomic and $f$-torsionfree.
Output: The Bernstein polynomial $b_f^L(s)$.
Begin
1. Determine $J^L(f^s)$ following algorithm 5.3.
2. Find a reduced Gröbner basis for the ideal $J^L(f^s) + A_n[s] \cdot f$ using an elimination order for $x$ and $\partial$.
3. Pick the unique element in that basis contained in $K[s]$ and return it.
End.

Algorithm 5.6. Input: $f \in R, L \subseteq A_n$ such that $A_n/L$ is holonomic and $f$-torsionfree.
Output: Generators for an ideal $J$ such that $R \otimes f A_n/L \cong A_n/J$.
Begin
1. Determine $J^L(f^s)$ following algorithm 5.3.
2. Find the Bernstein polynomial $b_f^L(s)$ using algorithm 5.5.
3. Find the smallest integer root $a$ of $b_f^L(s)$ (using corollary 5.4, if $K \subseteq \mathbb{C}$).
4. Replace $s$ by $a$ in all generators for $J^L(f^s)$ and return these generators.
End.

The algorithms 5.3 and 5.5 appear already in [14] in the special case $L = (\partial_1, \ldots, \partial_n), A_n/L = R$.

6. LOCAL COHOMOLOGY AS $A_n$-MODULE

In this section we will combine the results from the previous sections to obtain algorithms that compute for given $i, j, k \in \mathbb{N}, I \subseteq R$ the local cohomology modules $H^k_I(R), H^i_m(H^j_I(R))$ and the invariants $\lambda_{i,j}(R/I)$ associated to $I$.

6.1. Computation of $H^k_I(R)$. Here we will describe an algorithm that takes in a finite set of polynomials $\bar{f} = \{f_1, \ldots, f_r\} \subseteq R$ and returns a presentation of $H^k_I(R)$ where $I = (f_1, \ldots, f_r)$. In particular, if $H^k_I(R)$ is zero, then the algorithm will return the zero presentation.

Consider the Čech complex associated to $f_1, \ldots, f_r$ in $R$,

$$0 \to R \to \bigoplus_{i=1}^r R_{f_i} \to \bigoplus_{1 \leq i < j \leq r} R_{f_i f_j} \to \cdots \to R_{f_1 \cdots f_r} \to 0.$$  (6.1)
Its $k$-th cohomology group is the local cohomology module $H^k_I(R)$. The map

$$C^k = \bigoplus_{1 \leq i_1 < \cdots < i_k \leq r} R_{f_{i_1} \cdots f_{i_k}} \to \bigoplus_{1 \leq j_1 < \cdots < j_{k+1} \leq r} R_{f_{j_1} \cdots f_{j_{k+1}}} = C^{k+1}$$

(6.2)

is the sum of maps

$$R_{f_{i_1} \cdots f_{i_k}} \to R_{f_{j_1} \cdots f_{j_{k+1}}}$$

(6.3)

which are either zero (if $\{i_1, \ldots, i_k\} \not\subseteq \{j_1, \ldots, j_{k+1}\}$) or send $\frac{1}{f_i}$ to $\frac{1}{f_j}$, up to sign. Recall that $A_n/\Delta = A_n/\Delta_n \cdot (\partial_1, \ldots, \partial_n) \cong R$ and identify $R_{f_{i_1} \cdots f_{i_k}}$ with $A_n/J^{\Delta}((f_{i_1} \cdots f_{i_k})^s)|_{s=a}$ and $R_{f_{j_1} \cdots f_{j_{k+1}}}$ with $A_n/J^{\Delta}((f_{j_1} \cdots f_{j_{k+1}})|_{s=b}$ where $a, b$ are sufficiently small integers. By proposition 4.2 we may assume that $a = b \leq 0$. Then the map (6.2) is in the nonzero case multiplication from the right by $(f_1)^{-a}$ where $l = \{j_1, \ldots, j_{k+1}\} \setminus \{i_1, \ldots, i_k\}$, again up to sign.

It follows that the matrix representing the map $C^k \to C^{k+1}$ in terms of $A_n$-modules is very easy to write down once the annihilator ideals and Bernstein polynomials for all $k$- and $(k+1)$-fold products of the $f_i$ are known: the entries are 0 or $\pm f_i^{-a}$ where $f_i$ is the new factor.

Let $\Theta'_k$ be the set of $k$-element subsets of $1, \ldots, r$ and for $\theta \in \Theta'_k$ write $F_\theta$ for the product $\prod_{i \in \theta} f_i$. We have demonstrated the correctness and finiteness of the following algorithm.

**Algorithm 6.1.** Input: $f_1, \ldots, f_r \in R; k \in \mathbb{N}$.

Output: $H^k_I(R)$ in terms of generators and relations as finitely generated $A_n$-module.

Begin

1. Compute the annihilator ideal $J^{\Delta}((F_\theta)^s)$ and the Bernstein polynomial $b_{F_\theta}(s)$ for all $(k-1)$-, $k$- and $(k+1)$-fold products of $f_1^s, \ldots, f_r^s$ as in 5.3 and 7.3 (so $\theta$ runs through $\Theta_{k-1} \cup \Theta'_k \cup \Theta'_{k+1}$).
2. Compute the smallest integer root $a_\theta$ for each $b_{F_\theta}^\Delta(s)$, let $a$ be the minimum and replace $s$ by $a$ in all the annihilator ideals.
3. Compute the two matrices $M_{k-1}, M_k$ representing the $A_n$-linear maps $C^{k-1} \to C^k$ and $C^k \to C^{k+1}$ as explained in subsection 6.1.
4. Compute a Gröbner basis $G$ for the kernel of the map

$$\bigoplus_{\theta \in \Theta'_k} A_n \to \bigoplus_{\theta \in \Theta'_k} A_n/J^{\Delta}((F_\theta)^s)|_{s=a} \xrightarrow{M_k} \bigoplus_{\theta \in \Theta'_{k+1}} A_n/J^{\Delta}((F_\theta)^s)|_{s=a}$$

as in 1.2.
5. Compute a Gröbner basis $G_0$ for the module

$$\text{im}(M_{k-1}) + \bigoplus_{\theta \in \Theta'_k} J^{\Delta}((F_\theta)^s)|_{s=a} \subseteq \bigoplus_{\theta \in \Theta'_k} A_n/J^{\Delta}((F_\theta)^s)|_{s=a}.$$
6. Compute the remainders of all elements of \( G \) with respect to lifts of \( G_0 \) to \( \bigoplus_{\theta \in \Theta} A_n \).

7. Return these remainders and \( G_0 \).

End.

The nonzero elements of \( G \) generate the quotient \( G/G_0 \cong H^k_j(R) \) so that \( H^k_j(R) = 0 \) if and only if all returned remainders are zero.

6.2. Computation of \( H^m_i(H^j_1(R)) \). As a second application of Gröbner basis computations over the Weyl algebra we show now how to compute \( H^m_i(H^j_1(R)) \). Note that we cannot apply lemma 4.1 to \( A_n/L = H^j_1(R) \) since \( H^j_1(R) \) may well contain some torsion.

As in the previous sections, \( C^j(R; f_1, \ldots, f_r) \) denotes the \( j \)-th module in the Čech complex to \( R \) and \( \{ f_1, \ldots, f_r \} \). Let \( C^{i,j} \) be the double complex with \( C^{i,j} = C^i(R; x_1, \ldots, x_n) \otimes_R C^j(R; f_1, \ldots, f_r) \), the vertical maps \( \phi^{i,j} \) induced by the identity on the first factor and the usual Čech maps on the second, whereas the horizontal maps \( \xi^{i,j} \) are induced by the Čech maps on the first factor and the identity on the second. Since \( C^i(R; x_1, \ldots, x_n) \) is \( R \)-projective, the column cohomology of \( C^{i,j} \) at \((i, j)\) is \( C^i(R; x_1, \ldots, x_n) \otimes_R H^j_1(R) \) and the induced horizontal maps in the \( j \)-th row are simply the maps in the Čech complex \( C^i_* (H^j_1(R); x_1, \ldots, x_n) \). It follows that the row cohomology of the column cohomology at \((i_0, j_0)\) is \( H^m_i(H^j_0_1(R)) \).

Now note that \( C^{i,j} \) is a direct sum of modules \( R_g \) where \( g = x_{\alpha_1} \cdot \ldots \cdot x_{\alpha_i} \cdot f_{\beta_1} \cdot \ldots \cdot f_{\beta_j} \). So the whole double complex can be rewritten in terms of \( A_n \)-modules and \( A_n \)-linear maps using 5.6:

\[
\begin{array}{ccc}
C^{i-1,j+1} & \xrightarrow{\phi^{i-1,j}} & C^{i,j+1} & \xrightarrow{\phi^{i,j}} & C^{i+1,j+1} \\
\downarrow & & \downarrow & & \downarrow \\
C^{i-1,j} & \xrightarrow{\phi^{i-1,j}} & C^{i,j} & \xrightarrow{\phi^{i,j}} & C^{i+1,j} \\
\downarrow & & \downarrow & & \downarrow \\
C^{i-1,j-1} & \xrightarrow{\phi^{i-1,j-1}} & C^{i,j-1} & \xrightarrow{\phi^{i,j-1}} & C^{i+1,j-1}
\end{array}
\]

Using the comments in subsection 3.3, we may now compute the modules \( H^m_i(H^j_1(R)) \). More concisely, we have the following

**Algorithm 6.2.** Input: \( f_1, \ldots, f_r \in R; i_0, j_0 \in \mathbb{N} \).

Output: \( H^m_i(H^j_1(R)) \) in terms of generators and relations as finitely generated \( A_n \)-module.

Begin.
1. For \( i = i_0 - 1, i_0, i_0 + 1 \) and \( j = j_0 - 1, j_0, j_0 + 1 \) compute the annihilators \( J^\lambda((F_\theta \cdot X_{\theta'})) \) and Bernstein polynomials \( b^\lambda_{F_\theta \cdot X_{\theta'}}(s) \) of \( F_\theta \cdot X_{\theta'} \) where \( \theta \in \Theta^\lambda_j, \theta' \in \Theta^\lambda_i \) and \( X_{\theta'} \) denotes in analogy to \( F_\theta \) the product \( \prod_{a \in \theta} x_a \).
2. Let \( a \) be the minimum integer root of the product of all these Bernstein polynomials and replace \( s \) by \( a \) in all the annihilators computed in the previous step.
3. Compute the matrices to the \( A_n \)-linear maps \( \phi^{i,j} : C^{i,j} \to C^{i,j+1} \) and \( \xi^{i,j} : C^{i,j} \to C^{i+1,j} \), again for \( i = i_0 - 1, i_0, i_0 + 1 \) and \( j = j_0 - 1, j_0, j_0 + 1 \).
4. Compute Gröbner bases for the modules
   \[
   G = \ker(\phi^{i_0,j_0}) \cap \left( \left( \xi^{i_0,j_0} \right)^{-1}(\text{im}(\phi^{j_0+1,j_0-1})) \right) + \text{im}(\phi^{i_0,j_0-1})
   \]
   and \( G_0 = \xi^{i_0,j_0}(\ker(\phi^{i_0-1,j_0})) + \text{im}(\phi^{i_0,j_0-1}) \).
5. Compute the remainders of all elements of \( G \) with respect to \( G_0 \) and return these remainders together with \( G_0 \).

End.

The elements of \( G \) will be generators for \( H^m_i(H^0_1(R)) \) and the elements of \( G_0 \) generate the relations that are not syzygies.

6.3. Computation of \( \lambda_{i,n-j}(R/I) \). In [1] it has been shown that \( H^1_i(H^0_1(R)) \) is an injective \( m \)-torsion \( R \)-module of finite socle dimension \( \lambda_{i,n-j} \) (which depends only on \( i, j \) and \( R/I \)) and so isomorphic to \( (E_R(K))^{\lambda_{i,n-j}} \) where \( E_R(K) \) is the injective hull of \( K \) over \( R \). We are now in a position that allows computation of these invariants of \( R/I \). For, let \( H^m_i(H^0_1(R)) \) be generated by \( g_1, \ldots, g_t \in A_n \) modulo the relations \( h_1, \ldots, h_e \in A_n \). Let \( H \) be the module generated by the \( h_i \). We know that \( (A_n \cdot g_1 + H)/H \) is \( m \)-torsion and so it is possible (with trial and error) to find a multiple of \( g_1 \), say \( mg_1 \) with \( m \) a monomial in \( R \), such that \( (A_n \cdot mg_1 + H)/H \) is nonzero but \( x;mg_1 \in H \) for all \( 1 \leq i \leq n \). Then the element \( mg_1 + H/H \) has annihilator equal to \( m \) and hence generates an \( A_n \)-module isomorphic to \( A_n/A_n \cdot m \cong E_R(K) \). The injection \( A_n \cdot mg_1 + H/H \hookrightarrow A_n \cdot (g_1, \ldots, g_t) + H/H \) splits as map of \( R \)-modules because of injectivity and so the cokernel \( A_n(g_1, \ldots, g_t)/A_n(mg_1, h_1, \ldots, h_e) \) is isomorphic to \( (E_R(K))^{\lambda_{i,n-j}-1} \).

Reduction of the \( g_i \) with respect to a Gröbner basis of the new relation module and repetition of the previous will lead to the determination of \( \lambda_{i,n-j} \).

6.4. Local cohomology in ambient spaces different from \( \mathbb{A}^n_K \).
If \( A \) equals \( K[x_1, \ldots, x_n], I \subseteq A \), \( X = \text{Spec}(A) \) and \( V = \text{Spec}(A/I) \), knowledge of \( H^i_j(A) \) for all \( i \in \mathbb{N} \) answers of course the question about
the local cohomological dimension of $V$ in $X$. It is worth mentioning, that if $W \subseteq X$ is a smooth variety containing $V$ then our algorithm 6.1 for the computation of $H^i_I(A)$ also leads to a determination of the local cohomological dimension of $V$ in $W$. Namely, if $J$ stands for the defining ideal of $W$ in $X$ so that $R = A/J$ is the affine coordinate ring of $W$ and if we set $c = \text{ht}(J)$, then it can be shown that $H^i_{I-c}(R) = \text{Hom}_A(R,H^i_I(A))$ for all $i \in \mathbb{N}$. As $H^i_I(A)$ is $I$-torsion (and hence $J$-torsion), $\text{Hom}_A(R,H^i_I(A))$ is zero if and only if $H^i_I(A) = 0$. It follows that the local cohomological dimension of $V$ in $W$ equals $\text{cd}(A,I) - c$ and $\{q \in \mathbb{N} : H^q_I(A) \neq 0\} = \{q \in \mathbb{N} : H^q_{I-c}(R) \neq 0\}$.

If however $W$ is not smooth, no algorithms for the computation of either $H^i_I(R)$ or $\text{cd}(R,I)$ are known, irrespective of the characteristic of the base field.

7. Implementation and examples

Some of the algorithms described above have been implemented as C-scripts and tested on some examples.

7.1. The algorithm 5.3 with $L = \Delta$ has been implemented by Oaku using the package Kan (see [17]) which is a postscript language for computations in the Weyl algebra and in polynomial rings. An implementation for general $L$ is written by the current author and part of a program that deals exclusively with computations around local cohomology ([18]). [18] is theoretically able to compute $H^i_I(R)$ for arbitrary $i, R = \mathbb{Q}[x_1, \ldots, x_n], I \subseteq R$ in the above described terms of generators and relations, using algorithm 6.1. It is expected that in the near future [18] will work for $R = K[x_1, \ldots, x_n]$ where $K$ is an arbitrary field of characteristic zero and also algorithms for computation of $H^t_m(H^i_I(R))$ and $\lambda_{i,j}(R)$ will be implemented, but see the comments in 7.2 below.

Example 7.1. Let $I$ be the ideal in $R = K[x_1, \ldots, x_6]$ that is generated by the $2 \times 2$ minors $f, g, h$ of the matrix

$$
\begin{pmatrix}
x_1 & x_2 & x_3 \\
x_4 & x_5 & x_6
\end{pmatrix}
$$

Then $H^i_I(R) = 0$ for $i < 2$ and $H^2_I(R) \neq 0$ because $I$ is a height 2 prime and $H^3_I(R) = 0$ for $i > 3$ because $I$ is 3-generated, so the only remaining case is $H^2_I(R)$. This module in fact does not vanish, but until the discovery of our algorithm, its non-vanishing was a rather non-trivial fact. Our algorithm provides the first known proof of this fact by direct calculation.

Previously, Hochster pointed out that $H^2_I(R)$ is nonzero, using the fact that the map $K[f, g, h] \rightarrow R$ splits (compare [3], Remark 3.13) and Bruns and Schwänzl ([2], the corollary to Lemma 2) provided a
topological proof of the nonvanishing of $H^3_1(R)$ via étale cohomology. Both of these proofs are quite ingenious and work only in very special situations.

Using the program [18], one finds that $H^3_1(R)$ is isomorphic to a cyclic $A_6$-module generated by $1 \in A_6$ subject to relations $x_1 = \ldots = x_6 = 0$. This is a straightforward computational proof of the non-vanishing of $H^3_1(R)$. Of course this proof gives more than simply the non-vanishing. Since the quotient of $A_6$ by the left ideal generated by $x_1, \ldots, x_6$ is known to be isomorphic as an $R$-module to $E_R(R/(x_1, \ldots, x_6))$, the injective hull of $R/(x_1, \ldots, x_6) = K$ in the category of $R$-modules, our proof implies that $H^3_1(R) \cong E_R(K)$.

7.2. Computation of Gröbner bases in many variables is in general a time- and space consuming enterprise. Already in (commutative) polynomial rings the worst case performance for the number of elements in reduced Gröbner bases is doubly exponential in the number of variables and the degrees of the generators. In the (relatively small) example above $R$ is of dimension 6, so that the intermediate ring $A_{n+1}[y_1, y_2]$ contains 16 variables. In view of these facts the following idea has proved useful.

The general context in which lemma 4.1 and proposition 4.2 were stated allows successive localization of $R_{fg}$ in the following way. First one computes $R_f$ according to algorithm 7.1 as quotient of $A_n$ by a certain holonomic ideal $L = J^\Delta(f^s)|_{s=a}, a \ll 0$. Then $R_{fg}$ may be computed using 4.2 again since $R_{fg} \cong R_g \otimes A_n/L$. (Note that all localizations of $R$ are automatically $f$-torsion free for $f \in R$ as $R$ is a domain.) This process may be iterated for products with any finite number of factors. Note also that the exponents for the various factors might be different. This requires some care as the following situations illustrate. Assume first that $-1$ is the smallest integer root of the Bernstein polynomials of $f$ and $g$ (both in $R$) with respect to the holonomic module $R$. Assume further that $R_{fg} \cong A_n \cdot f^{-2} g^{-1} \supset A_n \cdot (fg)^{-1}$. Then $R_f \to R_{fg}$ can be written as $A_n/\text{ann}(f^{-1}) \to A_n/\text{ann}(f^{-2} \cdot g^{-1})$ sending $P \in A_n$ to $P \cdot f \cdot g$.

Suppose on the other hand that we are interested in $H^3_1(R)$ where $I = (f, g, h)$ and we know that $R_f = A_n \cdot f^{-2} \supset A_n \cdot f^{-1}, R_g = A_n \cdot g^{-2}$ and $R_{fg} = A_n \cdot f^{-1} g^{-2}$. (In fact, the degree 2 part of the Čech complex of example 7.1 consists of such localizations.) It is tempting to write the embedding $R_f \to R_{fg}$ with the use of a Bernstein operator (if $P_f(s)f^{s+1} = b^\Delta_f(s)f^s$ then take $s = -2$) but as $f^{-1}$ is not a generator for $R_f$, $b^\Delta_f(-2)$ will be zero. In other words, we must write $R_{fg}$ as $A_n/\text{ann}((fg)^{-2})$ and then send $P \in \text{ann}(f^{-2})$ to $P \cdot g^2$. 
The two examples indicate how to write the Čech complex in terms of generators and relations over $A_n$ while making sure that the maps $C^k \to C^{k+1}$ can be made explicit using the $f_i$: the exponents used in $C^i$ have to be at least as big as those in $C^{i-1}$ (for the same $f_i$).

**Remark 7.2.** We suspect that for all holonomic $fg$-torsionfree modules $M = A_n/L$ we have (with $R_g \otimes M \cong A_n/L'$):

$$\min\{s \in \mathbb{Z} : b_f^i(s) = 0\} \leq \min\{s \in \mathbb{Z} : b_f^{i'}(s) = 0\}.$$

This would have two nice consequences.

First of all, it would guarantee, that successive localization at the factors of a product does not lead to matrices in the Čech complex with entries of higher degree than localization at the product at once.

Secondly, if 7.2 were known to be true, we could proceed as follows for the computation of $C^i(R; f_1, \ldots, f_r)$. First compute $J^\Delta((f_i)^s)$ for all $i$, find all minimal integer Bernstein roots $\beta_i$ of $f_i$ on $R$ and substitute them into the appropriate annihilator ideals. If from now on we want to use algorithm 5.6 in order to compute $R_{f_1 \cdot \ldots \cdot f_k \cdot f_{k+1}}$ from $R_{f_1 \cdot \ldots \cdot f_k}$ then we can skip steps 2 and 3 of 5.4 as the remark gives us a lower bound for the minimal integer Bernstein root of $f_{ik+1}$ on $R_{f_1 \cdot \ldots \cdot f_k}$. (From the comments before 7.2 it is also clear that we cannot hope to use a larger value.)

The advantage of localizing $R_{fg}$ as $(R_f)_g$ is twofold. For one, it allows the exponents of the various factors to be distinct which is useful for the subsequent cohomology computation: it helps to keep the degrees of the maps small. (So for example $R_{x \cdot (x^2+y^2)}$ can be written as $A_n \cdot x^{-1}(x^2+y^2)^{-2}$ instead of $A_n \cdot (x^{-2} \cdot (x^2+y^2)^{-2}$). On the other hand, since the computation of Gröbner bases is doubly exponential it seems to be advantageous to break a big problem (localization at a product) into many “easy” problems (successive localization).

An extreme case of this behaviour is our example 7.1: if we compute $R_{fgh}$ as $((R_f)_g)_h$, the calculation uses approximately 2.5 kB and lasts 32 seconds on a Sun workstation using [18]. If one tries to localize $R$ at the product of the three generators at once, [18] crashes after about 30 hours having used up the entire available memory (1.2 GB).

**References**

[1] J.-E. Björk. *Rings of Differential Operators*. North Holland, 1978.

[2] W. Bruns and R. Schwänzl. The number of equations defining a determinantal variety. *Bull. London Math. Soc.*, 22:439–445, 1990.

[3] D. Eisenbud. *Commutative Algebra with a View toward Algebraic Geometry*. Springer Verlag, 1995.
[4] G. Faltings. Über lokale Kohomologiegruppen hoher Ordnung. *Journal für die Reine und Angewandte Mathematik*, 313:43–51, 1980.

[5] R. Hartshorne. Cohomological Dimension of Algebraic Varieties. *Ann. of Math.*, 88:403–450, 1968.

[6] R. Hartshorne. On the de Rham Cohomology of Algebraic Varieties. *Publ. Math. Inst. Hautes Sci.*, 45:5–99, 1975.

[7] R. Hartshorne and R. Speiser. Local cohomological dimension in characteristic $p$. *Ann. Math.(2)*, 105:45–79, 1977.

[8] C. Huneke and G. Lyubeznik. On the vanishing of local cohomology modules. *Invent. Math.*, 102:73–93, 1990.

[9] A. Kandri-Rody and V. Weispfenning. Non-commutative Gröbner bases in algebras of solvable type. *J. Symb. Comp.*, 9:1–26, 1990.

[10] M. Kashiwara. $B$-functions and holonomic systems, rationality of $B$-functions. *Invent. Math.*, 38:33–53, 1976.

[11] G. Lyubeznik. Finiteness Properties of Local Cohomology Modules: an Application of $D$-modules to Commutative Algebra. *Invent. Math.*, 113:41–55, 1993.

[12] G. Lyubeznik. $F$-modules: applications to local cohomology and $D$-modules in characteristic $p > 0$. *To appear in "Journal für die Reine und Angewandte Mathematik"*, 1997.

[13] B. Malgrange. Le polynôme de Bernstein d’une singularité isolée. *Lecture Notes in Mathematics*, Springer Verlag, 459:98–119, 1990.

[14] T. Oaku. An algorithm for computing $B$-functions. *Duke Math. Journal*, 87:115–132, 1997.

[15] A. Ogus. Local cohomological dimension of algebraic varieties. *Ann. of Mathematics*, 98:327–365, 1973.

[16] C. Peskine and L. Szpiro. Dimension projective finie et cohomologie locale. *Inst. Hautes Études Sci. Publ. Math.*, 42:47–119, 1973.

[17] N. Takayama. Kan, a computer package for symbolic computations in Weyl algebras. *www.math.s.kobe-u.ac.jp/KAN/*, 1991–.

[18] U. Walther. Local, a computer program for symbolic computations in local cohomology. *walther@math.umn.edu*.

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