HARD THERMAL LOOPS, CHERN-SIMONS THEORY AND
THE QUARK-GLUON PLASMA

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ABSTRACT

The generating functional for hard thermal loops in Quantum Chromodynamics is important in setting up a resummed perturbation theory. I review how this functional is related to the eikonal for a Chern-Simons gauge theory, and using an auxiliary field, to the gauged Wess-Zumino-Novikov-Witten action. The induced current due to hard thermal loops, properly incorporating damping effects, is also briefly discussed.

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In this talk, I shall be discussing how Chern-Simons theory can help us understand and provide an elegant mathematical framework for the hard thermal loops of Quantum Chromodynamics (QCD) and some of the physical phenomena associated with the quark-gluon plasma. Some of the work I describe was done in collaboration with R.Efraty $^1$ and some in collaboration with Professor R.Jackiw $^2$.

This is an audience that is very familiar with the concept of hard thermal loops; nevertheless, let me begin by briefly recalling what they are. In QCD, hard thermal loops are thermal one-loop Feynman diagrams for which the momenta on the external lines are of the order of $gT$ or less, where $g$ is the coupling constant and $T$ is the temperature and for which the loop-momentum is of the order of $T$ or higher. As is well known, the evaluation of hard thermal loops is necessary for the Braaten-Pisarski resummation procedure $^3$. $^4$. Hard thermal loops are also part of the effective action for long wavelength excitations of the plasma. Let me denote by $\Gamma[A]$ the generating functional for hard thermal loops with external gluon lines, $A_\mu$ being the gauge potential of the external lines. (Hard thermal loops with external fermion lines pose a comparatively simpler problem. In any case, Chern-Simons theory has little to add to our understanding of hard thermal loops with external fermions. I shall therefore confine myself to the case of external gluon lines.) The remarkable fact, which I hope to convince you of, is that $\Gamma[A]$ is essentially given by the eikonal for a Chern-Simons gauge theory. Later, I shall introduce an auxiliary field $G$ and write an action $\Gamma[A,G]$ which is closely related to a gauged Wess-Zumino-Novikov-
Witten (WZNW) theory. The elimination of the auxiliary field $G$ will lead us back to $\Gamma[A]$.

The starting point of my discussion will be the following two key properties of $\Gamma[A]$ which one can show by power-counting analyses of hard thermal loops\(^3\).\(^4\).\(^5\).

1) $\Gamma[A]$ is gauge invariant.
2) $\Gamma[A]$ has the form

$$\Gamma[A] = (N + \frac{1}{2} N_F) \frac{T^2}{12\pi} \left[ \int d^4x \ 2\pi \ A_0^a A_0^b + \int d\Omega \ W(A \cdot Q) \right]$$ \hspace{1cm} (1)

We consider an $SU(N)$ gauge theory with $N_F$ flavors of quarks. The gauge potential $A_\mu = (-it^a)A_\mu^a, \ t^a$ are $N \times N$ traceless hermitian matrices with $\text{Tr}(t^at^b) = \frac{1}{2} \delta^{ab};$ they are a basis of the Lie algebra of $SU(N)$ in the fundamental representation. $Q^\mu$ is a null vector, $Q_\mu Q^\mu = 0; \ $we will parametrize $Q^\mu$ as $(1, \vec{Q})$ with $\vec{Q} \cdot \vec{Q} = 1$. The $d\Omega$-integration in Eq. (1) is over all orientations of the unit vector $\vec{Q}$. The first term in Eq. (1) is a mass term for time-component $A_0^a$; it is essentially the Debye screening effect. The significant feature of Eq. (1) is that the second term of $\Gamma[A]$ is given by a functional of $A \cdot Q;$ there is integration over all the orientations of $\vec{Q}$ but for each $\vec{Q}$, only one component of the potential, viz. $A \cdot Q$, enters. The $d\Omega$-integration is part of the loop-integration; we carry out the $p_\mu$- and $|\vec{p}|$- integrations of the loop-momentum $p_\mu$, leaving the angular integrations.

The gauge invariance and special structure of $\Gamma[A]$ as in Eq. (1) have been analyzed by many people\(^3\).\(^4\).\(^5\). I shall take these properties of $\Gamma[A]$ as the premise of my discussion. Because of the special structure of $\Gamma[A]$, gauge invariance suffices to determine $W(A \cdot Q)$\(^4\). In $\Gamma[A]$, as given by Eq. (1), we can carry out a gauge transformation $A_\mu \rightarrow A_\mu + D_\mu \omega, \ \omega = (-it^a)\omega^a(x)$.

The variation of the terms in Eq. (1) are

$$\delta \int 2\pi A_0^a A_0^b = \int 4\pi \partial_b A_0^a \omega^a = \int d\Omega \ Q \cdot \partial_b A_0^b \omega^a$$ \hspace{1cm} (2a)

$$\delta \int W(A \cdot Q) = - \int \left( Q \cdot \partial \frac{\delta W}{\delta (A \cdot Q)} + [A \cdot Q, \frac{\delta W}{\delta (A \cdot Q)}] \right)^a \omega^a$$ \hspace{1cm} (2b)

The property of gauge invariance of $\Gamma[A]$ then becomes the following equation for $W(A \cdot Q)$.

$$\frac{\partial f}{\partial u} + [A \cdot Q, f] + \frac{1}{2} \frac{\partial (A \cdot Q)}{\partial v} = 0$$ \hspace{1cm} (3)

where $u = \frac{1}{2} Q' \cdot x, \ v = \frac{1}{2} Q \cdot x, \ Q'' = (1, -\vec{Q})$ and

$$f = \frac{1}{2} \frac{\delta W}{\delta (A \cdot Q)} + \frac{1}{2} A \cdot Q$$ \hspace{1cm} (4)

We also define $A_+ = \frac{1}{2} A \cdot Q$. It is also convenient to do a Wick rotation (to the Euclidean space $\mathbb{R}^4$); we then have $u \rightarrow z, \ v \rightarrow \bar{z}, \ A_+ \rightarrow A_z$. Let us also rename $-f$ as $a_{\bar{z}}$. Eq. (3) then becomes

$$\partial_z A_z - \partial_{\bar{z}}a_{\bar{z}} + [a_{\bar{z}}, A_z] = 0$$ \hspace{1cm} (5)
and

\[ a_z \equiv f = \frac{1}{2} \frac{\delta W}{\delta A_z} - A_z \quad (6a) \]

or equivalently

\[ \delta W = 4 \int \text{Tr}(a_z \delta A_z) - \delta \int A_z^a A_z^a \quad (6b) \]

We must solve Eq. (5) for \( a_z \) in terms of \( A_z \), use this in Eq. (6b) and solve for \( W \). If for a moment, we think of \((A_z, a_z)\) as the \( z \)- and \( \bar{z} \)-components of the potential of (some other) gauge theory, we see that Eq. (5) is the statement that the field strength \( F_{z\bar{z}} \) vanishes. This is where Chern-Simons theory can give us some insights, for Chern-Simons gauge theory is the one for which the equations of motion say that the field strength is zero. (The mysterious renaming of variables was to bring out this analogy.) We shall therefore keep aside Eqs. (5,6) for the moment and have a short digression on Chern-Simons gauge theory.

Chern-Simons theory is a gauge theory in three dimensions, i.e. two spatial dimensions. The action for the theory can be written as \(^6\) \(^7\)

\[ S = \frac{k}{4\pi} \int d^3x \text{ Tr} \left( a_\mu \partial_\nu a_\alpha + \frac{2}{3} a_\mu a_\nu a_\alpha \right) \epsilon^{\mu\nu\alpha} \quad (7) \]

(Here \( a_\mu = (-it^a)a^a_\mu \)) We shall use complex coordinates \( z = x + iy, \ \bar{z} = x - iy \) for the spatial dimensions. The equations of motion for the action in Eq. (7) are, as I mentioned before, \( F_{\mu\nu} = 0 \). The gauge choice \( a_0 = 0 \) is best suited to our purposes. In this gauge, the equations of motion \( F_{\mu\nu} = 0 \) become

\[ \partial_0 a_z = 0, \quad \partial_0 a_{\bar{z}} = 0 \quad (8a) \]

\[ \partial_z a_z - \partial_{\bar{z}} a_{\bar{z}} + [a_z, a_{\bar{z}}] = 0 \quad (8b) \]

Eqs. (8a) say that the fields are independent of time; the dynamics of the Chern-Simons theory (without sources) is trivial. The (equal-time) constraint (8b) thus defines the theory. There are many ways to solve Eq. (8b). We shall first take \( a_z \) as the independent variable and solve for \( a_{\bar{z}} \). The solution is

\[ a_{\bar{z}} = \sum (-1)^{n-1} \int \frac{d^2z_1}{\pi} \cdots \frac{d^2z_n}{\pi} \frac{a_z(1) \cdots a_z(n)}{\bar{z} - \bar{z}_1} \cdots (\bar{z} - \bar{z}_n) \quad (9) \]

\((a_z(1) = a_z(z_1, \bar{z}_1), \text{etc.})\) It is easy to check that this is indeed a solution to Eq. (8b) using the identity \( \partial_{\bar{z}} \frac{1}{z - \bar{z}} = \pi \delta^{(1)}(x) \). (This latter identity is essentially Cauchy’s integral formula. We can also interpret this as saying that \( \frac{1}{(z - \bar{z})^2} \) is the Green’s function for \( \partial_z \), convert Eq. (8b) into an integral equation and iteratively solve it. This will lead us back to Eq. (9).)

We now define a quantity \( I(a_z) \) by the equation \(^1\)

\[ \delta I = \frac{i}{\pi} \int d^2x \text{ Tr}(a_z \delta a_z) \quad (10) \]
It is trivial to check that $I(a_z)$ is given by

$$I(a_z) = i \sum \frac{(-1)^n}{n} \int \frac{d^2 z_1}{\pi} \cdots \frac{d^2 z_n}{\pi} \frac{\text{Tr}(a_z(1) \cdots a_z(n))}{\bar{z}_{12} \bar{z}_{23} \cdots \bar{z}_{n-1n} z_n} \tag{11}$$

where $\bar{z}_{ij} = \bar{z}_i - \bar{z}_j$.

This quantity $I$, mathematically defined by Eq. (10), has a very nice interpretation in the language of analytical dynamics. In the $a_0 = 0$ gauge, the action becomes

$$S = \frac{i}{\pi} \int d^3 x \, \text{Tr}(a_z \partial_0 a_z), \tag{12}$$

which shows that $a_\bar{z}$ and $a_z$ are canonically conjugate variables. Thus $\delta I$ is the analogue of $p \, dx$ of point-particle mechanics. We can integrate $p \, dx$ if we express $p$ in terms of $x$. In one dimension, a familiar constraint is the condition of fixed energy, say $\frac{\dot{x}^2}{2m} + V(x) = E$ or $p = \sqrt{2m(E - V(x))}$. The integral of $p \, dx$ is the eikonal or Hamilton’s characteristic function for the system, familiar as the exponent of the WKB wave functions in quantum mechanics. For us, Eq. (8b) is the constraint relating the conjugate variables $a_z$ and $a_\bar{z}$ and $I$, as defined by Eq. (10), is thus the eikonal of the Chern-Simons theory. (The wave function of the Chern-Simons theory will be $\Psi \sim e^{iI}$; cf. the lectures by R. Jackiw.)

It is possible to write the potential $a_z$ as $-\partial_2 M M^{-1}$, where $M$ is a complex $N \times N$ matrix of unit determinant. (It is not necessarily unitary.) From Eq. (8b), we then have $a_\bar{z} = -\partial_\bar{z} M M^{-1}$. The eikonal $I(a_z)$ of Eq. (11) can be written in a ‘summed-up’ version as $I = -i S_{WZW}(M)$ where

$$S_{WZW} = \frac{1}{2\pi} \int_{\mathcal{M}^2} \text{Tr}(\partial_2 M \partial_\bar{z} M^{-1}) - \frac{i}{12\pi} \int_{\mathcal{M}^3} d^3 x \, \text{Tr}(\partial_\mu MM^{-1} \partial_\nu MM^{-1} \partial_\alpha MM^{-1}) e^{i\mu\alpha} \tag{13}$$

The action (13) defines the Wess-Zumino-Novikov-Witten theory used extensively in studies of two-dimensional conformal field theories. The second term in Eq. (13) involves an extension of $M$ into a three-dimensional space $\mathcal{M}^3$, the boundary of which is the two-dimensional world of interest (spacetime). Eventually, physical results are independent of how this extension is carried out. The action (13) obeys the composition rule, as can be verified directly,

$$S_{WZW}(hg) = S_{WZW}(h) + S_{WZW}(g) - \frac{1}{\pi} \int_{\mathcal{M}^2} \text{Tr}(h^{-1} \partial_\bar{z} h \partial_z g g^{-1}) \tag{14}$$

The infinitesimal version of this rule gives

$$\delta S_{WZW}(M) = \frac{1}{\pi} \int \text{Tr} \left[ (\partial_\bar{z} MM^{-1}) D_z (\delta MM^{-1}) \right] \tag{15}$$

with $D_z \xi = \partial_z \xi + [a_z, \xi]$. This gives the identification $I = -i S_{WZW}(M)$ using Eq. (10) and the expressions for $a_z$, $a_\bar{z}$ in terms of $M$.

We can now return to the plasma problem. From comparing Eqs. (5,6) to Eqs. (8b) and (10), we see that $A_z$ now plays the role of the $a_z$ of the Chern-Simons theory and $W$ is essentially $I$. The potential $A_z$ (and so $a_\bar{z}$) in Eq. (5) depend on all
four coordinates. However, the coordinates $x^T$ transverse to $\bar{Q}$ do not appear in Eq. (5); they just play the role of parameters on which $A_z$ depends. This dependence is carried over to $W$ with an integration over all $x^T$. From Eq. (11), we thus write down $W$ and eventually $\Gamma[A]$ as

$$\Gamma[A] = (N + \frac{1}{2} N_F) \frac{T^2}{12 \pi} \int d^4 x \left\{ 2 \pi A_0^a A_0^a - \int d\Omega (A_z^a A_z^a) \right\} - 4 \pi i \int d\Omega \ d^2 x^T \ I(A_z)$$

(16a)

$$I(A_z) = i \sum \frac{(-1)^n}{n} \left[ \frac{d^2 z_1}{\pi} \cdots \frac{d^2 z_n}{\pi} \frac{\text{Tr}(A_z(z_1, \bar{z}_1, x^T) A_z(z_2, \bar{z}_2, x^T) \cdots A_z(z_n, \bar{z}_n, x^T))}{\bar{z}_{12} \bar{z}_{23} \cdots \bar{z}_{n-1n} \bar{z}_{n1}} \right]$$

(16b)

(Strictly speaking $a_z$ and $A_z$ are not complex conjugates; the analogy holds better with a Chern-Simons theory of complex gauge group. However, we are only using the Chern-Simons analogy to obtain Eq. (16). The expression for $\Gamma[A]$ can also be directly checked to be a solution to Eq. (5).)

The $n$-point functions can be directly evaluated in the kinematic regime appropriate to hard thermal loops, at least for $n = 2, 3, 4$. Needless to say, the results agree with the $n$-point functions as given by Eq. (16).

Using the properties of the $d\Omega$-integration, we can write Eq. (16) as

$$\Gamma = k \int d\Omega \ K(A_z, A_{\bar{z}})$$

(17a)

$$K(A_z, A_{\bar{z}}) = - \left[ \frac{1}{\pi} \int d^4 x \ \text{Tr}(A_z A_{\bar{z}}) + i \int d^2 x^T \{ I(A_z) + \bar{I}(A_{\bar{z}}) \} \right]$$

(17b)

$$= \int d^2 x^T S_{WZNW}(M^1 M)$$

(17c)

where $k = (N + \frac{1}{2} N_F) \frac{T^2}{12 \pi}$. Since the Chern-Simons action is not even under parity, its relevance to QCD might be, at first sight, worrisome. But actually, from Eqs. (17a) and (17b), we see that the result is parity even. ($A_z \rightarrow A_+, A_{\bar{z}} \rightarrow A_-$ when we continue to Minkowski space.) We use Eq. (14) to obtain Eq. (17c); the requirement that physics be independent of the extension of $M$ into $M^3$ usually requires that the coefficient $k$ multiplying the action must be quantized. This is for groups with nontrivial third homotopy group. Eq. (17c) shows that the hermitian matrix $M^1 M$ is what is relevant; for hermitian matrices, the homotopy groups are trivial and there is no requirement of quantization of $k$. We see that QCD neatly avoids what may appear, a priori, as difficulties in using Chern-Simons or WZNW theories. We can also write

$$K(A_z, A_{\bar{z}}) = \int d^2 x^T \log \det(D_z D_{\bar{z}})$$

(17d)

$D_z, D_{\bar{z}}$ are chiral Dirac operators, which continue to $D_+, D_-$ respectively. For $U(1)$ gauge theory, the logarithm of the Dirac determinant, i.e. $\log \det(D_+ D_-)$ is the mass term for the gauge boson, as we know from the Schwinger model. Eq. (17d) thus displays hard thermal loops or $\Gamma[A]$, which is after all the electric mass term made gauge invariant, as a non-Abelian generalization of Schwinger’s result. (Of course,
In writing the above formulae in Minkowski space, most of the changes to be made are obvious. The subtlety is in choosing the physically correct \( i\epsilon \)-prescription in writing down the inverses of \( \partial \pm \). A simple way to understand this is by writing the equations of motion (for a \( U(1) \) gauge theory) as

\[
\partial_\nu F^{\nu\mu} = iS^{-1} \frac{\delta S}{\delta A^\mu_{\nu}} \tag{18}
\]

where \( S \) is the scattering operator as a function of the in-fields. This is a standard formula that goes back to the LSZ formulation of field theory. If we expand out the current on the right hand side of Eq. (18), we see that it involves multiple retarded commutators. Thus the \( i\epsilon \)-prescription we choose must give a current which has the same retardation properties as the multiple retarded commutators. With this understanding, we can write the equations for the evolution of field configurations with soft momenta as

\[
D_\nu F^{\nu\mu,a} = J^{\mu,a} \tag{19a}
\]

\[
J^{\mu,a} = \sum_{n=1}^{\infty} \int \frac{d^4k_1}{(2\pi)^4} \ldots \frac{d^4k_n}{(2\pi)^4} e^{-i\sum k_j \cdot x} J^{\mu,a}_n(k) \tag{19b}
\]

\[
J^{\mu,a}_n(k) = \frac{k}{\pi} \int d\Omega \left[ \text{Tr} \left( -\frac{i\alpha Q^\mu}{2} A_-(k_1) + A_+(k_1)(-\frac{i\alpha Q^\mu}{2}) \right) \delta_{n,1} \right.
\]

\[
\left. + \left\{ -(2i)^{n-1} \text{Tr} \left( -\frac{i\alpha Q^\mu}{2} A_+(k_1) \ldots A_+(k_n) \right) F(k_1, \ldots, k_n) + (Q \leftrightarrow Q') \right\} \right] \tag{19c}
\]

where

\[
F(k_1, \ldots, k_n) = \sum_{i=0}^{n} \frac{-q_i}{(q_0 - q_i)(q_1 - q_i) \ldots (q_{i-1} - q_i)(q_{i+1} - q_i) \ldots (q_n - q_i)} \tag{20a}
\]

\[
\tilde{q}_i = \sum_{j=1}^{i} (k_j \cdot Q + i\epsilon_j), \quad q_i = \sum_{j=1}^{i} k_j \cdot Q' \tag{20b}
\]

We may reexpress the current as

\[
J^{\nu a} = -\frac{k}{2\pi} \int d\Omega \text{ Tr} \left[ (-it^a) \{ (a_+ - A_+)Q^\nu + (Q' \leftrightarrow Q) \} \right] \tag{21}
\]

where \( a_+ \) is defined by

\[
\partial_- a_+ - \partial_+ A_- + [A_-, a_+] = 0 \tag{22}
\]

The kinetic theory calculation recently carried out by Blaizot and Iancu also gives these equations\(^{10}\).

There are two related but different contexts in which we need the expression for \( \Gamma[A] \). The first is in setting up thermal perturbation theory. The version of \( \Gamma[A] \)
as given in Eqs. (17a,b) is probably best suited for this purpose. We can also use \( \Gamma[A] \) added to the usual Yang-Mills action as an effective action for the soft modes. (This is the spirit of Eqs. (19).) The nonlocality of \( \Gamma[A] \) makes it somewhat difficult to handle in this context. It is useful to rewrite \( \Gamma[A] \) using an auxiliary field which makes the equations of motion local \(^{11}\). The auxiliary field we use will be an \( SU(N) \)-matrix field \( G(x, \vec{Q}) \) which is a function of \( x \) and \( \vec{Q} \), i.e. defined on \( \mathcal{M}^4 \times S^2, \mathcal{M}^4 \) being Minkowski space. Further \( G(x, \vec{Q}) \) must satisfy the condition \( G^\dagger(x, \vec{Q}) = G(x, -\vec{Q}) \). The action is given by

\[
S = \int \left[ -\frac{1}{4} F^2 + k \int d\Omega \left[ d^2x^T \, S_{WZNW}(G) + \frac{1}{\pi} \int d^4x \, \text{Tr}(G^{-1} \partial_- G A_+) 
- A_- \partial_+ G \, G^{-1} A_+ - A_+ \partial_+ G \, G^{-1} A_- - A_+ A_- \right] \right] (23)
\]

where \( S_{WZNW}(G) \) is the WZNW action, Eq. (13), for \( G \). The quantity in the square brackets in Eq. (23) is the gauged WZNW action \(^{12}\). It is invariant under gauge transformations with \( G \) transforming as \( G \rightarrow G' = h(x)G \, h^{-1}(x), \, h(x) \in SU(N) \). The equations of motion are

\[
\partial_+ A_- - \partial_- A_+ + [a_+, A_-] = 0 \tag{24a}
\]

\[
a_+ \equiv GA_+ G^{-1} - \partial_+ G \, G^{-1} \tag{24b}
\]

\[
(D_\mu F^{\mu\nu})^a - J^\nu a = 0 \tag{25a}
\]

\[
J^\nu a = -\frac{k}{2\pi} \int d\Omega \, \text{Tr}[\{(\tilde{Q}^a)^\nu (a_+ - A_+ + Q^\nu) + (Q^\nu \leftrightarrow Q^\nu)\}]
= -\frac{k}{2\pi} \int d\Omega \, \text{Tr}[-it^a \{G^{-1} D_\nu G \, Q^\nu - D_\nu G \, G^{-1} Q^\nu\}] \tag{25b}
\]

Clearly these equations are equivalent to Eqs. (19a,21,22); the only difference is that the equation defining \( a_+ \) in Eq. (22) is now obtained as the equation of motion (24a) for \( G \). Notice that the current in Eq. (25c) looks like the current of a matter field; thus except for the fact that \( G \) depends on \( \vec{Q} \), QCD, with hard thermal loops added, is no stranger than Yang-Mills theory coupled to a matter field. (One might wonder if there are solutions to Eq. (24a) which exist even if \( A_\mu = 0 \). These might give give extra degrees of freedom which would vitiate the equivalence with Eqs. (19,20). Actually, from the Minkowski version of Eq. (14) we see that \( S_{WZNW}(G) \) has an additional gauge symmetry \( G \rightarrow B(u)G \, C(v) \). This is the Kac-Moody symmetry of the WZNW-theory. This gauge symmetry eliminates the possible ‘extra’ solutions.)

One can also check easily that the Hamiltonian corresponding to the action (23) is

\[
\mathcal{H} = \int d^3x \left[ \frac{1}{2} (E^2 + B^2) + \frac{k}{8\pi} \int d\Omega \, \text{Tr}\left\{(D_0 GD_0 G^{-1}) + (\vec{Q} \cdot \vec{D} G \, \vec{Q} \cdot \vec{D} G^{-1})\right\} - A_0^a G^a \right] \tag{26}
\]

where \( G^a \) is the Gauss law function, i.e. \( G^a = 0 \) for all physical field configurations \(^{11}\). We see that \( \mathcal{H} \geq 0 \) for such configurations.
The auxiliary field and the positive Hamiltonian help in the analysis of classical solutions. Clearly \( G = 1, \ E = B = 0 \) is the vacuum solution with \( \mathcal{H} = 0 \). Certainly, wavelike solutions also exist. This is seen by taking, as an ansatz, the fields to be in a \( U(1) \) subgroup of \( SU(N) \), whereupon the standard plasma wave solutions of electrodynamics are reproduced. Presumably more general solutions, capturing the full non-Abelian features of the theory also exist. These will be the genuine non-Abelian plasmons of the quark-gluon plasma. Other physically interesting possibilities are static (perhaps unstable) solutions corresponding to local minima of \( \mathcal{H} \) and nontopological solitons.

References

1. R.Efraty and V.P.Nair, *Phys.Rev.Lett.* 68 (1992) 2891; *Phys.Rev.* D47 (1993) 5601.
2. R.Jackiw and V.P.Nair, Columbia-MIT Report CU-TP 594, CTP# 2205 (to be published in *Phys.Rev.* D).
3. R.Pisarski, *Physica A* 158 (1989) 246; *Phys.Rev.Lett.* 63 (1989) 1129; E.Braaten and R.Pisarski, *Phys.Rev.* D42 (1990) 2156; *Nucl.Phys.* B337 (1990) 569; *ibid.* B339 (1990) 310; *Phys.Rev.* D45 (1992) 1827.
4. J.Frenkel and J.C.Taylor, *Nucl.Phys.* B334 (1990) 199; J.C.Taylor and S.M.H.Wong, *Nucl.Phys.* B346 (1990) 115.
5. R.Kobes, G.Kunstatter and A.Rebhan, *Nucl.Phys.* B355 (1991) 1.
6. R.Jackiw and S.Templeton, *Phys.Rev.* D23 (1981) 2291; J.Schonfeld, *Nucl.Phys.* B185 (1981) 157; S.Deser, R.Jackiw and S.Templeton, *Phys.Rev.Lett.* 48 (1982) 975; *Ann.Phys.* 140 (1982) 372.
7. E.Witten, *Commun.Math.Phys.* 121 (1989) 351.
8. E.Witten, *Commun.Math.Phys.* 92 (1984) 455; V.G.Knizhnik and A.B.Zamolodchikov, *Nucl.Phys.* B247 (1984) 83; D.Gepner and E.Witten, *Nucl.Phys.* B278 (1986) 493.
9. A.Polyakov and P.Wiegmann, *Phys.Lett.* 141B (1984) 223; D.Gonzales and A.N.Redlich, *Ann.Phys.* 169 (1986) 104; G.Dunne, R.Jackiw and C.A.Trugenberger, *Ann.Phys.* 194 (1989) 194.
10. J.P.Blaizot and E.Iancu, *Phys.Rev.Lett.* 70 (1993) 3376; Saclay Report SPHT-93-064.
11. V.P.Nair, Columbia Report CU-TP 601 (to be published in *Phys.Rev.* D).
12. R.I.Nepomechie, *Phys.Rev.* D33 (1986) 3670; D.Karabali, Q-H.Park, H.J.Schnitzer and Z.Yang, *Phys.Lett.* 216B (1989) 307; D.Karabali and H.J.Schnitzer, *Nucl.Phys.* B329 (1990) 649; K.Gawedzki and A.Kupianen, *Phys.Lett.* B15B (1988) 119; *Nucl.Phys.* B320 (1989) 649.