Mathematical modelling of biology processes based on the table of prime links in the solid torus up to 4 crossings

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Abstract. The motivation to study links in lens spaces can be justified in recent applications to theoretical physics and biology. In this paper, we give a short outline of practical significance and implementation proposals of such links, the study of which begins with the study of links in the solid torus, since a punctured disk diagram of a link in the lens space can be considered as a punctured disk diagram in the solid torus provided with the additional slide move. However, links in the solid torus find their applications themselves as well. In this paper, we propose a method to represent knotted proteins as links in the solid torus. Such a method is based on the existence of correspondence between knotoids and knots in the solid torus using a double branched cover. To this end, the table of links in solid torus is necessary. Therefore, we classify all prime links in the solid torus up to 4 crossings. One of possible future applications of the constructed table is an analysis of the database LinkProt that collects information about protein chains and complexes that form links. Also, our table can be used to construct table of prime links in lens spaces.

1. Introduction
Knot theory is a widespread branch of geometric topology, with many applications to theoretical physics, chemistry and biology. Although the classical theory studies knots and links in the 3-dimensional sphere $S^3$, the cases where the ambient is another 3-dimensional manifolds are widely investigated recently.

In the knot theory, one of the oldest and the most important problems is to recognize a knot (or a link), i.e., to associate the considered object with an unique tabulated one. This problem involves the problem on complete classification of knots and links ordered taking into account some their properties. As regards constructed tables of knots in different 3-dimensional manifolds, see [1, 2, 3] for the 3-dimensional sphere $S^3$, [4] for the solid torus, [5] for thickened Klein bottle, [6] for the lens spaces, [7, 8, 9, 10, 11] for the thickened surfaces. As regards tabulation of links in different 3-dimensional manifolds, see [2, 3] for the 3-dimensional sphere $S^3$, [12] for the projective space, [13, 14, 15, 16] for the thickened surfaces. Obviously, there is a gap between knots and links in the sense of tabulation.

In this paper, we are interested in exploring links in lens spaces and solid torus, which is a relatively young branch of knot theory. The motivation for studying links in lens spaces and solid torus can be justified in recent applications to theoretical physics [17] and biology [18].

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In this paper, we propose a method to represent knotted proteins as links in the solid torus. Such a method is based on the existence of correspondence between knotoids and knots in the solid torus using a double branched cover. To this end, the table of links in solid torus is necessary. Therefore, we classify all prime links in the solid torus up to 4 crossings. Our main result states that there exist no more than 26 pairwise nonequivalent such links. One of possible future applications of the constructed table is an analysis of the database LinkProt that collects information about protein chains and complexes that form links. Also, our table will be used to construct table of prime links in lens spaces. Indeed, following [6], a punctured disk diagram of a link in the lens space can be considered as a punctured disk diagram in the solid torus provided with the additional slide move. Therefore, in order to obtain table of links in the lens space it is enough to consider the corresponding table of links in the solid torus taking into account an additional move.

Here it is necessary to note another point of view for the investigation of links in lens spaces: to consider the lift of such links in the 3-dimensional sphere $S^3$, under the standard universal covering, see [19] for the corresponding algorithm. However, the paper [20] presents examples of different links in lens spaces having equivalent lift. Therefore, the lift provides no one-to-one correspondence between the desired table of links in lens spaces and the existing table of links in the 3-dimensional sphere $S^3$.

Also, as regard to tabulation of links of special type, we mention that the paper [21] defines rational knots and links in the solid torus using rational tangles, proves one-to-one correspondence between such tangles and links, and uses this crucial result together with the theorem that classifies rational tangles to conclude that the fraction of rational tangles classifies rational knots and links in the solid torus. As regards application, rational tangles and rational knots are used in the study of DNA recombination [22].

The paper is organized as follows. In Section 2, we give a short outline of practical significance and implementation proposals of links in lens spaces and solid torus. In particular, in Subsection 2.1, we propose a method to represent knotted proteins as links in the solid torus. In Section 3, we summarize the background material used in the mathematical part of the paper. Section 4 describes tabulation of prime link projections in the annulus and proves that there exist exactly 17 pairwise inequivalent such projections with at most 4 crossings. Finally, Section 5 presents main ideas of the tabulation of prime link projections in the annulus and proves that there exist no more than 26 pairwise inequivalent such links having diagrams with at most 4 crossings.

2. Applications of links in the lens spaces and solid torus to mathematical modelling of biology processes

In this section, we give a short outline of practical significance and implementation proposals of links in the lens spaces and solid torus.

The motivation for studying links in lens spaces can be justified in recent applications to theoretical physics and biology. We note the following interesting articles explaining applications of knots in lens spaces to other fields of science: [17] exploits the knots to describe topological string theories in theoretical physics and [18] uses them to describe the resolution of a biological DNA recombination problem. More precisely, the paper [18] extends the tangle model to include composite knots and shows that, for any prime tangle, there are no rational tangle attachments of distance greater than one that first yield a 4-plat and then a connected sum of 4-plats. This is done by studying the corresponding Dehn filling problem via double branched covers. In particular, the basis is the results on exceptional Dehn fillings at maximal distance to show that if Dehn filling on an irreducible manifold gives a lens space and then a connect sum of lens spaces, the distance between the slopes must be one. The results are applied to the action of the Hin recombinase on mutated sites. In particular, after solving the tangle equations for
processive recombination, the results are used to give a complete set of solutions to the tangle equations modelling distributive recombination.

These just cited conjectures are the most valuable reasons that lead us to study links in lens spaces. Below we propose an important reason to study links in the solid torus.

2.1. Mathematical model of proteins

Long, flexible physical strands, from macroscopic string to long-chain molecules, are naturally knotted, that determines their configuration and properties. Results obtained in [23] emphasise that virtual knotting is a generic feature of certain geometrical classes of curves, arising from relatively weak geometric constraints even in the absence of the physical complexity of protein chains.

Many of the processes essential to life involve proteins, i.e. long molecules which fold into three-dimensional shapes allowing them to perform their biological role. A folded protein molecule consists of strings of amino acids. According to [23], open protein chains formed from a string of carbon and nitrogen atoms can be considered as long, knotted curves having distinct endpoints. Mathematically, open protein chains can be represented by knotoids proposed in [24]. Here by knotoid we mean an open knot diagram which differs from the classical knot diagram in that the underlying curve is an interval rather than a curve.

Earlier research on knotted proteins attempted to close the protein into a curve using different ways of connection. A method to represent proteins as virtual knots was proposed in [23]. The method corresponds to the virtual closure of the classical knotoid proposed in [24]. The virtual closure determines a well-defined map from knotoids in $S^2$ to virtual knots having genus at most one. In [23], tables of classical knots and virtual knots of genus 1 are used to analyze the database KnotProt 2.0 [25], which collects information about proteins that form knots and knotoids.

We propose research knotted proteins by connection of a protein into a curve using another approach. Namely, to study knotted proteins as knots in the solid torus. Such an approach is based on the existence of correspondence between knotoids and knots in the solid torus defined in [26] using a double branched cover. To this end, the table of knots in solid torus [4] can be used.

However, knots are not enough to describe a structure of all proteins. The database LinkProt [27] collects information about protein chains and complexes that form links and provides an exhaustive list of open linked proteins and topologically linked proteins in the Protein Database [28]. A table of links in solid torus constructed in the present paper can be useful to analyze the database LinkProt using a double branched cover. Therefore, the results of the paper can be introduced into a research on the proteins.

New mathematical techniques for the analysis and exploitation of knots and links from a wide range of complex biology structures need further study.

3. Background material

A direct product of an 1-dimensional sphere $S^1$ and an interval $I = [0, 1]$ is said to be an annulus $A$. In other words, we can consider an annulus $A$ as a 2-dimensional sphere $S^2$ with two holes or a 2-dimensional disk $D^2$ with a hole.

Let us define types of simple closed curves, which can be considered in an annulus $A$.

A simple closed curve $C \subset A$ is called cut, if the complement $A \setminus C$ consists of two components, which are a 2-dimensional disk $D^2$ and an annulus $A$ with a hole, the latter can be considered as a 2-dimensional sphere $S^2$ with three holes.

A simple closed curve $C \subset A$ is called not cut, if the complement $A \setminus C$ consists of two components, which are two copies of an annulus. Note that all not cut curves in the annulus
A are parallel to each other. Indeed, for any two not cut curves $C_1, C_2 \subset A$, the complement $A \setminus (C_1 \cup C_2)$ consists of three components, which are three copies of an annulus.

By a solid torus we mean a direct product of an annulus $A$ and an interval $I = [0, 1]$. In other words, we can consider a solid torus as a direct product of an 1-dimensional sphere $S^1$ and a 2-dimensional disk $D^2$. However, for clearness, in this paper we denote a solid torus by $A \times I$ to show that we consider links in a thickened surface (thickened annulus) and to make definitions given below more visual.

A smooth embedding of a set of $m$ pairwise disjoint closed curves in the interior $\text{Int}(A \times I)$ of a solid torus $A \times I$ is said to be an $m$-component link in $A \times I$ and denoted by $L \subset A \times I$. In particular, a knot in $A \times I$ is obtained, if $m = 1$, i.e. we consider a smooth embedding of an unique curve in $\text{Int}(A \times I)$.

We define the following four types of links in a solid torus $A \times I$ (compare with the types of link projections in an annulus $A$ given below).

(i) A link $L \subset A \times I$ is said to be essential, if every arc that connects two boundaries of the annulus $A$ has nonempty intersection with $L$. In other words, an essential link $L \subset A \times I$ can not be situated in the 3-dimensional sphere $S^3$, i.e. is not a classical link.

(ii) A link $L \subset A \times I$ is said to be composite, if the following condition holds. $L$ is a connected sum of a link $L_1 \subset A \times I$ and a nontrivial link $L_2 \subset S^3$, which is defined by analogy with the classical connected sum of two classical links in the 3-dimensional sphere $S^3$.

Namely, in $A \times I$ (respectively, $S^3$), remove an open 3-dimensional ball $B^3$ that intersects $L_1$ (respectively, $L_2$) by an unknotted arc. As a result, one of components of both links $L_1$ and $L_2$ is transformed to a knotted arc. Then, glue the resulting thickened surfaces (a thickened surface $A^0 \times I$ with a hole and a thickened disk $D^2 \times I$) into one new $A \times I$ by a homeomorphism that identifies the obtained spherical boundaries such that endpoints of different knotted arcs are glued pairwise. Note that one of two terms in the sum can be a knot (see classifications obtained in [1, 2, 3, 4, 9]), since the result is a link anyway.

(iii) A link $L \subset A \times I$ is said to be not split, if there exists a surface (a 2-dimensional sphere $S^2$ or an annulus $A$ parallel to a boundary of $A \times I$) embedded into $A \times I$, which does not intersect $L$ and divides components of $L$.

(iv) A link $L \subset A \times I$ is said to be prime, if $L$ is essential, not composite, not split and contains more than one component.

The natural idea is to tabulate only prime links. Indeed, not essential links correspond to links that can be found in already existing tables of links in the 3-dimensional sphere $S^3$ [2, 3]. In their turn, composite links can be obtained using already known links, i.e. as connected sums of a link in the solid torus and a classical link, recall that one of two terms in the sum can be a knot. Finally, a split link can be considered as a trivial union of already tabulated links, while a link having the unique component is a knot.

The first step of the study of links in a solid torus is to find a suitable representation. Links in a solid torus can be represented using different techniques such as Gauss diagrams [29], Gauss words [4], punctured disk diagrams [6], band diagrams [30], and others. In this paper, we prefer to use band diagrams. To this end we note that, as in the classical case, a link $L$ in a solid torus $A \times I$ can be represented by its diagram, which is defined by analogy with a classical link diagram except that $L$ is projected into the annulus $A$ instead of a plane. We represent the annulus $A$ by a square with identified opposite sides forming one pair (top and bottom), while opposite sides forming the second pair (left and right) are not identified. In this way the 3-dimensional equivalence problem of links in reduced to a 2-dimensional equivalence problem of diagrams. Reidemeister proved that two links are equivalent if any of their diagrams can be connected by a finite sequence of three local moves, called Reidemeister moves. Diagrams also are necessary to calculate invariants of links.
A link projection $G$ in the annulus $A$ is a diagram such that the crossings of the diagram contain no information about under/over-crossings. Hence, a link projection $G$ can be considered as an embedding of a regular graph of degree 4, i.e., valence of each vertex of the graph is equal to 4. Vertices of the graph are said to be crossings of $G$, connected components of the complement $A \setminus G$ are said to be faces of $G$, while each projection of a component of the link is said to be a component of $G$.

Two link projections $G$ and $G'$ in the annulus $A$ are called equivalent, if there exists a homeomorphism $f : A \to A$ such that $f(G) = G'$.

We define the following four types of link projections in an annulus $A$ (compare with the types of links in a solid torus $A \times I$ given above).

(i) The projection $G \subset A$ is said to be essential, if there exists no face of $G$ homeomorphic to an annulus $A$ with a hole. Namely, there exists the unique face of $G$, which is homeomorphic to an annulus, while each of the rest faces of $G$ is homeomorphic to a 2-dimensional disk $D^2$.

(ii) The projection $G \subset A$ is said to be composite, if the following condition holds. There exists a 2-dimensional disk $D^2 \subset A$ such that the boundary $\partial D^2$ intersects $G$ transversally exactly in two points, which are internal for edges of $G$, and at least one vertex of $G$ is inside $D^2$.

(iii) The projection $G \subset A$ is said to be not split, if each component of $G$ contains at least two crossings.

(iv) The projection $G \subset A$ is said to be prime, if $G$ is essential, not composite, not split and contains more than one component.

In this article, we consider only prime projections, i.e., only those that correspond to links, which can not be obtained by some known operations from already tabulated ones. Indeed, a projection having at least one of the properties (not essential, composite or split) correspond to a link having the same property.

4. Classification of projections

**Theorem 1** In the annulus $A$, there exist exactly 17 pairwise nonequivalent prime link projections with no more than 4 crossings. The projections are shown in figure 1.

We prove Theorem 1 by three steps presented in Subsections 4.1–4.3. First, Subsection 4.1 presents some auxiliary statements. Then, Subsection 4.2 enumerates all possible embeddings of the graphs into the annulus $A$ giving prime projections. Finally, Subsection 4.3 shows that all obtained projections are pairwise nonequivalent.
4.1. Some auxiliary statements

Lemma 1  Let $G \subset A$ be a prime link projection. Then

(i) if $G$ contains a loop, then the loop is parallel to a not cut curve,
(ii) $G$ contains at most 2 loops,
(iii) if $G$ contains three edges incident to the same pair of vertices $v_1, v_2$, then two of these three edges form a not cut curve.

Proof. Assume that Statement (i) is not fulfilled, i.e. a loop of $G$ bounds a 2-dimensional disk $D^2$. Then we arrive at contradiction: $G$ is composite, i.e. not prime, since there exists a 2-dimensional disk $D^2 \subset A$ such that the boundary $\partial D^2$ intersects $G$ transversally exactly in two points, which are internal for edges of $G$, and the vertex of $G$ associated with the loop is inside $D^2$.

In order to prove Statement (ii), it is enough to take into account Statement (i) and the fact that if a loop is parallel to a not cut curve, then presence of 3 or more such loops contradicts with the fact that $G$ contains exactly one face homeomorphic to an annulus.

If Statement (iii) is not fulfilled, i.e. two of three edges form a cut curve, then we arrive at contradiction: $G$ is composite, i.e. not prime, since there exists a 2-dimensional disk $D^2 \subset A$ such that the boundary $\partial D^2$ intersects $G$ transversally exactly in two points, which are internal for edges of $G$, and both vertices $v_1, v_2$ are inside $D^2$.

Lemma 2 In the annulus $A$, all prime link projections with no more than 4 crossings can be obtained as embeddings of the graphs $a \rightarrow o$ given in Fig. 2.

Proof. Any link projection can be considered as an embedding of a regular graph of degree 4, while, following Statement (ii) of Lemma 1, any prime link projection contains at most 2 loops. Following Lemma 2 in [9], there exist exactly 15 regular graphs of degree 4 with at most 4 vertices and at most 2 loops (the given above number of regular graphs includes a curve without vertices, see figure 2).

4.2. Construction of prime projections

Lemma 3 All projections shown in figure 1 can be obtained as embeddings of the graphs $d, f, i, j, m, n$ given in figure 2. Namely, the graph $d$ gives the projection $2_1$, the graph $f$ gives the projections $3_1$ and $3_2$, the graph $i$ gives the projections $4_1 - 4_5$, the graph $j$ gives the projections $4_6 - 4_9$, the graph $m$ gives the projections $4_{10} - 4_{12}$, the graph $m$ gives the projections $4_{13}$ and $4_{14}$.
Proof. First of all, following complete enumeration given in [9], the graphs \(a, b, c, e, h, g, l, o\) involve only knot projections in the annulus \(A\). Also, we note that the graph \(k\) involve no prime link projections. Indeed, taking into account Statements (i) and (iii) of Lemma 1, we arrive at contradiction with the fact that prime link projection contains exactly one face homeomorphic to an annulus.

In order to obtain all prime link projections for the rest graphs, we represent an embedding of each graph as an union of a number of curves and enumerate all possible combinations of types of the curves.

**Graph d.** Assume that the projection \(G\) is an embedding of the graph \(d\) in the annulus \(A\). Since the graph \(d\) contains three edges incident to the same pair of vertices, then, following Statement (iii) of Lemma 1, two of these three edges form a not cut curve. The rest two edges also form a not cut curve, since otherwise \(G\) is composite. Therefore, we obtain the projection 2₁.

**Graph f.** Assume that the projection \(G\) is an embedding of the graph \(f\) in the annulus \(A\), then \(G\) can be considered as an union of three curves such that the curves \(C₁\) and \(C₂\) have no intersection points, while the curve \(C₃\) has 1 intersection point with the curve \(C₁\) and 2 intersection points with the curve \(C₂\). Following Statements (i) and (iii) of Lemma 1, the curves \(C₁\) and \(C₂\) are not cut. We arrive at the projections 3₁ and 3₂, if the curve \(C₃\) is cut and not cut, respectively.

**Graph i.** Assume that the projection \(G\) is an embedding of the graph \(i\) in the annulus \(A\), then \(G\) can be considered as an union of four curves such that each of the curves \(C₁\) and \(C₄\) have exactly 1 intersection point with the curve \(C₃\), while the curves \(C₂\) and \(C₃\) have exactly 2 intersection points. Following Statement (i) of Lemma 1, the curves \(C₁\) and \(C₄\) are not cut.

**Case i.1.** Let the curve \(C₂\) be cut, then we arrive at the projections 4₁ and 4₃, if the curve \(C₃\) is also cut and touch the curve \(C₂\) either by another side as both curves \(C₁\) and \(C₄\) or by the same side, respectively. Otherwise, i.e. if the curve \(C₃\) is not cut, we have the projection 4₂.

**Case i.2.** Let the curve \(C₂\) be not cut, then we arrive at the projections 4₄ and 4₅, if the curve \(C₃\) is cut and not cut, respectively.

**Graph j.** Assume that the projection \(G\) is an embedding of the graph \(j\) in the annulus \(A\), then \(G\) can be considered as an union of four curves as follows. The curve \(C₃\) has exactly 1 intersection point with each of the curves \(C₂\) and \(C₄\), while the curves \(C₁\) and \(C₂\) have exactly 2 intersection points. Following Statements (i) and (iii) of Lemma 1, the curves \(C₁\) and \(C₄\) are not cut.

**Case j.1.** Let the curve \(C₂\) be cut, then we arrive at the projections 4₆ and 4₈, if the curve \(C₃\) is cut and not cut, respectively.

**Case j.2.** Let the curve \(C₂\) be not cut, then we arrive at the projections 4₇ and 4₉, if the curve \(C₃\) is cut and not cut, respectively.

**Graph m.** Assume that the projection \(G\) is an embedding of the graph \(m\) in the annulus \(A\), then \(G\) can be considered as an union of four curves such that for each curve there exist exactly two curves with each of which the curve has exactly 1 intersection point and has no intersection points with the rest curve. Note that the number of not cut curves is at most two, since otherwise we arrive at contradiction with the fact that prime link projection contains exactly one face homeomorphic to an annulus.

**Case m.1.** Assume that there are no not cut curves, then we have the projection 4₁₂.

**Case m.2.** Assume that there exists exactly one not cut curve, then we arrive at composite projection.

**Case m.3.** Assume that there exist exactly two not cut curves, then we have the projection 4₁₀, if the cut curves have no intersection points, and the projection 4₁₁ otherwise.

**Graph n.** Assume that the projection \(G\) is an embedding of the graph \(n\) in the annulus \(A\). Since the graph \(n\) twice contains three edges incident to the same pair of vertices, then, following Statement (iii) of Lemma 1, in each case two of these three edges form a not cut
curve. We obtain the projections $4_{13}$ and $4_{14}$, if the rest four edges form a cut and not cut curve, respectively.

4.3. Proof of the fact that all obtained projections are pairwise nonequivalent

**Lemma 4** All 17 projections shown in figure 1 are pairwise nonequivalent.

**Proof.** Let us associate each face of a projection with a natural number equal to the number of edges forming boundary of the face. For any prime link projection $G \subset A$, there exists the unique face of $G$, which is homeomorphic to an annulus, while each of the rest faces of $G$ is homeomorphic to a 2-dimensional disk $D^2$. Associate each projection $G$ shown in figure 1 with an ordered set of the form $\{(m)\ [k\ i_1\ i_2 \ldots\ i_n\ x\}\$, where $n$ and $m$ are the numbers of crossings and components of $G$, respectively, $k$ is a natural number associated with the face of $G$ homeomorphic to an annulus, $i_j$ is a natural number associated with the $j$-th face of $G$ homeomorphic to a 2-dimensional disk $D^2$ (numbers of the form $i_j$ are taking in nondecreasing order, $j = 1, 2, \ldots, n$), and $x$ is a graph such that the projection $G$ is an embedding of $x$ in the annulus $A$, where $x \in \{d, f, i, j, m, n\}$.

Here we note that the projections in figure 1 are ordered with respect to increase in their numerical characteristics considered with the following priority:

(i) number of crossings $n$,
(ii) number of components $m$,
(iii) number $k$ associated with the face homeomorphic to an annulus,
(iv) numbers $i_1\ i_2 \ldots\ i_n$ associated with the faces homeomorphic to a 2-dimensional disk $D^2$,
(v) type $x$ of the graph.

Such ordered sets of the form $\{(m)\ [k\ i_1\ i_2 \ldots\ i_n\ x\}\$ are sufficient to prove that all projections shown in figure 1 are pairwise nonequivalent except for the following 6 pairs: $(3_1, 3_2)$, $(4_6, 4_7)$, $(4_8, 4_9)$, $(4_{10}, 4_{11})$, and $(4_{13}, 4_{14})$.

We say that an edge $e$ is of the type $(i, j)$, if $e$ is a common edge of $i$-gonal and $j$-gonal faces, each of which is homeomorphic to a 2-dimensional disk $D^2$.

(i) Projections $(3_1, 3_2)$ are nonequivalent, because only the first projection contains the edge of the type $(2, 3)$. The same is true for $(4_6, 4_7)$ and $(4_8, 4_9)$.

(ii) Projections $(4_4, 4_5)$ are nonequivalent, because only the first projection contains the edge of the type $(3, 3)$.

(iii) Projections $(4_{13}, 4_{14})$ are nonequivalent, because both edges of the type $(2, 4)$ belong to the same 4-gonal face in the first projection only.

(iv) Projections $(4_{10}, 4_{11})$ are nonequivalent, because biangle faces have no common points in the first projection only.

Note that all tabulated projections are prime by construction.

This completes the proof of both Lemma 4 and Theorem 1.

5. Classification of links

**Theorem 2** In the solid torus, there exist no more than 26 pairwise inequivalent prime links having diagrams with at most 4 crossings, see figure 3.

Theorem 2 is proved by three steps described in Subsections 5.1 – 5.3.
5.1. Construction of a preliminary list of diagrams on prime projections

Let us convert each projection constructed in Theorem 1 and given in figure 1 to the set of corresponding diagrams. To this end, enumerate all possible ways to consider each crossing of a projection to be either an over- or undercrossing of a diagram. Obviously, there are \(2^n\) diagrams on each projection with \(n\) crossings. Therefore, direct construction by tabulated 17 projections leads to \(2^2 + 2 \cdot 2^3 + 14 \cdot 2^4 = 244\) diagrams. However, we can significantly reduce this procedure by the following ideas [10, 13].

First, the simultaneous switching of all crossings convert any diagram to the equivalent one. Therefore, we can fix the type of a crossing of each projection and, consequently, to halve the set of diagrams on the projection.

Let a fragment \(F\) of a projection be of the types \(A_1, A_2,\) or \(A_3\), see figure 4, then \(F\) can be converted into a fragment of the corresponding diagram only in two ways such that each arc of the converted \(F\) must be alternating, i.e. overcrossings must alternate with undercrossings as we go around the arc. Otherwise the corresponding diagram allows to reduce number of crossings.

Finally, each component should go both over and under the union of all the rest components, since otherwise the link is split, i.e. not prime.

5.2. Formation of equivalence classes of the constructed diagrams

In order to compare the obtained diagrams, we use the software ”Wolfram Mathematica” to calculate the Kauffman bracket polynomials [13] for all diagrams constructed in Subsection 5.1, and then we reduce each of the constructed polynomials to the Kauffman bracket skeleton [14]. Therefore, we find more then ten groups formed by diagrams having the same Kauffman bracket skeletons. Each group includes from 2 to 3 diagrams, and there exist groups having diagrams on inequivalent projections. Then, by hand, we construct sequences of Reidemeister moves and
simultaneous switching of all crossings in order to show that diagrams having the same Kauffman bracket skeletons are equivalent. The list of the Kauffman bracket skeletons presented in Table 1 is enough to prove that all tabulated links are pairwise inequivalent except for only two pairs: \((2_1, 4_{18})\) and \((3_1, 4_{17})\). In order to show that diagrams that form each pair are inequivalent, we present their Kauffman bracket polynomials below:

\[
K(2_1) = (a^{-12} - a^{-4}) + a^{-4}x^2, \quad K(3_1) = (-a^{-4} + a^4) + (1 - a^4 + a^8)x^2, \\
K(4_{18}) = (a^{-24} - a^{-8}) + a^{-8}x^2, \quad K(4_{17}) = (a^{-20} - a^{-4}) + (a^{-12} - a^{-8} + a^{-4})x^2.
\]

5.3. On primarily of the tabulated links
In order to prove that a link is prime, it is enough to show that the link is essential, not composite and not split. As regards to the latter property, we note that when constructing the table, we remove obviously split links, while the first two properties can be shown as follows.

Taking into account that the Kauffman bracket skeleton of each tabulated link contains variable \(x\) corresponding to a not cut curve, it is easy to see that each tabulated link can not be situated in the 3-dimensional sphere \(S^3\), i.e. is not a classical link, and, therefore, is essential.

In order to prove that all 26 tabulated knots are noncomposite, it is enough to show that each link can not be represented as a connected sum under the hypothesis that the complexity of a connected sum is not less than the sum of complexities of the terms that form the sum. More precisely, we assume that there exists no a pair of nontrivial links such that the connected sum of these links admits a diagram having number of crossings, which is smaller than a minimal sum of numbers of crossings of the diagrams corresponding to both links formed the pair. Within the considered problem on tabulation of links having diagrams with at most 4 crossings, the impossibility of representation as a connected sum is obvious, see given below enumeration of all possible cases to represent a link \(L_{\text{conn}} \subset A \times I\) having diagrams with at most 4 crossings as a connected sum of a link \(L_1 \subset A \times I\) and a nontrivial link \(L_2 \subset S^3\), where \(L_1\) or \(L_2\) can be a knot.

Case 1. Assume that both terms, \(L_1 \subset A \times I\) and \(L_2 \subset S^3\), are links, then diagrams of both of them have at least 2 crossings, and we arrive at the unique case: \(L_1\) is \(2_1\) given in figure 3 and \(L_2\) is the Hopf link.

Case 2. Assume that only one of the terms, \(L_1 \subset A \times I\) or \(L_2 \subset S^3\), is a link.

| $S(2_1)$ | $S(4_{10})$ | $S(4_{11})$ | $S(4_{12})$ | $S(4_{13})$ | $S(4_{14})$ | $S(4_{15})$ | $S(4_{16})$ | $S(4_{17})$ | $S(4_{18})$ | $S(4_{19})$ | $S(4_{20})$ | $S(4_{21})$ | $S(4_{22})$ |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $(1, -1) + x^2$ | $(-1, 1) + (-2, -1)x^2 + x^4$ | $(-1, 1) + (1, -2, -2)x^2 + x^4$ | $(-1, 1, -1, -1)x^3$ | $(-1, 1) + (1, -2, -1)x^2 + x^4$ | $(-1, 1) + (1, -1, -2)x^2 + x^4$ | $(1, 1, 1, 1) + (-1, -1, -1)x^3 + x^4$ | $(1, 1, 2) + (1, -2, 1)x^2$ | $(-1, 1, -1, 1)x + x^3$ | $(1, -1, 1, 1) + (-1, -3)x^2 + x^4$ | $(1, -1, 1, 1) + (-1, -2, -1)x^3$ | $(1, -1, 1, 1) + (-1, -2, -1, -1)x^3$ | $(1, -1) + 3x^2 + x^4$ | $(1, -1, -2) + (1, -2, 1)x + (1, -1, -1)x^3$ | $(1, -1, -2) + (1, -2, 1)x + x^3$ | $(1, -1) - 3x^2 + x^4$ |
Case 2.1. Assume that $L_1 \subset A \times I$ is a link and $L_2 \subset S^3$ is a knot. Since any diagram of nontrivial classical knot contains at least 3 crossings, then $L_1$ can be only trivial, i.e. an union of curves. But in this case the resulting $L$ is a split link, and we do not consider this case.

Case 2.2. Assume that $L_1 \subset A \times I$ is a knot and $L_2 \subset S^3$ is a link. Since any diagram of nontrivial classical link contains at least 2 crossings, then $L_1$ can be either trivial, i.e. not cut curve (the case of cut curve leads to not essential link), or one of $1_1$, $2_1$–$2_3$ given in figure 1 of the paper [9]. If $L_1$ is trivial, then $L_2$ can be either the Hopf link or the unique classical link having 4 crossings. If $L_1$ is one of $1_1$, $2_1$–$2_3$ given in figure 1 of the paper [9], then $L_2$ can be the Hopf link only.

In order to calculate the Kauffman bracket polynomial of a connected sum, it is enough to multiply the Kauffman bracket polynomials of the terms. Therefore, we use the software "Wolfram Mathematica" to factor the Kauffman bracket polynomials and show that each of the polynomials can not be represented as a product of polynomials of the knots and links described above as possible terms.

6. Conclusion
In this paper, we study links in lens spaces and solid torus, that can be justified in recent applications to theoretical physics and biology. We propose a method to represent knotted proteins as links in the solid torus based on the existence of correspondence between knotoids and knots in the solid torus using a double branched cover. To this end, the table of links in solid torus is necessary. Therefore, we classify all prime links in the solid torus up to 4 crossings. Our main result states that there exist no more than 26 pairwise nonequivalent such links. One of possible future applications of the constructed table is an analysis of the database LinkProt that collects information about protein chains and complexes that form links. Also, our table can be used to construct table of prime links in lens spaces.

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