Some remarks on the regularity of weak solutions for the stationary Ericksen-Leslie and MHD systems

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Abstract

Abstract. We consider two elliptic coupled systems of relevance in the fluid dynamics. These systems are posed on the whole space $\mathbb{R}^3$ and they consider the action of external forces. The first system deals with the simplified Ericksen-Leslie (SEL) system, which describes the dynamics of liquid crystal flows. The second system is the time-independent magneto-hydrodynamic (MHD) equations. For the (SEL) system, we obtain a new criterion to improve the regularity of weak solutions, provided that they belong to some homogeneous Morrey space. As a bi-product, we also obtain some new regularity criterion for the stationary Navier-Stokes equations and for a nonlinear harmonic map flow. This new regularity criterion also holds true for the (MHD) equations. Furthermore, for this last system we are able to use the Gevrey class to prove that all finite energy weak solutions are analytic functions, provided the external forces belong to some Gevrey class.

Keywords: Coupled systems in fluid mechanics; simplified Ericksen-Leslie system; Magneto-hydrodynamic system; Morrey spaces; the Gevrey class.

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1 Introduction

This note considers with two elliptic coupled systems in the fluid dynamics. The first system arises from the study of the dynamics in liquid crystal flows. This system is posed on the whole space $\mathbb{R}^3$ and strongly couples the incompressible and stationary (time-independent) Navier-Stokes equations with a nonlinear harmonic map flow as follows:

$$\begin{cases}
-\Delta \vec{U} + \text{div}(\vec{U} \otimes \vec{U}) + \text{div}(\vec{\nabla} \otimes \vec{V} \circ \vec{\nabla} \otimes \vec{V}) + \vec{\nabla} P = \text{div}(\mathcal{F}), \\
-\Delta \vec{V} + \text{div}(\vec{V} \otimes \vec{U}) - |\vec{\nabla} \otimes \vec{V}|^2 \vec{V} = \text{div}(\mathcal{G}), \\
\text{div}(\vec{U}) = 0.
\end{cases}$$

Here, $\vec{\nabla} \otimes \vec{V} = (\partial_i V_j)_{1 \leq i,j \leq 3}$, denotes the deformation tensor of the vector field $\vec{V}$ and moreover, for $i = 1, 2, 3$, each component of the vector field $\text{div}(\vec{\nabla} \otimes \vec{V} \circ \vec{\nabla} \otimes \vec{V})$ explicitly writes down as:

$$\left[\text{div}(\vec{\nabla} \otimes \vec{V} \circ \vec{\nabla} \otimes \vec{V})\right]_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \partial_j (\partial_i V_k \partial_j V_k).$$

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The velocity of the fluid $\mathbf{U} : \mathbb{R}^3 \to \mathbb{R}^3$ and the pressure $P : \mathbb{R}^3 \to \mathbb{R}$ are the classical unknowns of the fluid mechanics. Moreover, this system also considers a third unknown $\mathbf{V} : \mathbb{R}^3 \to \mathbb{S}^2$, where $\mathbb{S}^2$ denotes the unit sphere in $\mathbb{R}^3$. The unit vector field $\mathbf{V}$ represents the macroscopic orientation of the nematic liquid crystal molecules [19]. We also take into account the action of external forces, which according to [7] can be written as the divergence of the tensors $\mathbf{F} = (F_{ij})_{1 \leq i,j \leq 3}$ and $\mathbf{G} = (G_{ij})_{1 \leq i,j \leq 3}$, with $F_{ij}, G_{ij} : \mathbb{R}^3 \to \mathbb{R}$. Finally, the equation $\text{div}(\mathbf{U}) = 0$ always represents the fluid’s incompressibility.

The elliptic system (1) is the time-independent counterpart of the following parabolic (time-dependent) system:

$$
\begin{aligned}
\partial_t \mathbf{u} - \Delta \mathbf{u} + \text{div}(\mathbf{v} \otimes \mathbf{u}) + \text{div}(\nabla \otimes \mathbf{v} \otimes \nabla \otimes \mathbf{v}) + \nabla p &= \text{div}(\mathbf{F}), \\
\partial_t \mathbf{v} - \Delta \mathbf{v} + \text{div}(\mathbf{v} \otimes \mathbf{u}) - |\nabla \otimes \mathbf{v}|^2 \mathbf{v} &= \text{div}(\mathbf{G}),
\end{aligned}
$$

(3)

also known as the simplified Ericksen-Leslie system. This parabolic system was proposed by H.F. Lin in [16] as a simplification of the general Ericksen-Leslie system which models the hydrodynamic flow of nematic liquid crystal material [2, 19]. The simplified Ericksen-Leslie system has been successful to model the dynamical behavior of nematic liquid crystals. More precisely, it provides a good macroscopic description of the evolution of the material under the influence of the fluid velocity field, and moreover, it provides a good macroscopic description of rod-like liquid crystals. See the book [7] for more details.

The system (3) has recently attired the interest in the research community of mathematical fluid dynamics. It is worth mentioning one of the major challenges is due, on the one hand, to the presence of the trilinear term $|\nabla \otimes \mathbf{v}|^2 \mathbf{v}$ in the second equations of this system and, on the other hand, to the presence of the super-critical nonlinear term $\text{div}(\nabla \otimes \mathbf{v} \otimes \nabla \otimes \mathbf{v})$ defined in (2). Precisely, the double derivatives in this last term make it more delicate to treat than the classical nonlinear transport term: $\text{div}(\mathbf{u} \otimes \mathbf{u})$. These facts make challenging the study of both (1) and (3). See, for instance, the articles [9, 10, 13, 17, 18, 21] and the references therein.

Some previous works in the homogeneous case. When $\mathbf{F} = 0$ and $\mathbf{G} = 0$, the first works on the mathematically study of the system (3) were devoted to the existence of global in time weak solutions [13, 18]. Thereafter, in the spirit of the celebrated result by H. Koch & D. Tataru [12], the global well-posedness of small solutions in the space $BMO^{-1}(\mathbb{R}^3)$ was proven in [21].

Concerning some regularity issues, T. Huang proved in [5] a regularity criterion on weak solutions of (3) in the framework of the Lebesgue spaces. This result also holds true for the stationary system (1) in the case $\mathbf{F} = \mathbf{G} = 0$. Indeed, first we consider a weak solution of (1) as a couple $(\bar{U}, \bar{V})$ where $(\bar{U}, \bar{\nabla} \otimes \bar{V}) \in H^1(\mathbb{R}^3)$. Thereafter, we obtain $\bar{U} \in C^\infty(\mathbb{R}^3)$ and $\bar{V} \in C^\infty(\mathbb{R}^3)$, provided that $\bar{U} \in L^p(\mathbb{R}^3)$ and $\bar{\nabla} \otimes \bar{V} \in L^p(\mathbb{R}^3)$, with $p > 3$.

Regularity of weak solutions is an important question to a better mathematically comprehension of the system (1), and moreover, it is also one of the key assumptions when studying another relevant problem of this system in the homogeneous case: the uniqueness of weak solutions. Precisely, when $\mathbf{F} = \mathbf{G} = 0$ we always have the trivial solution $(\bar{U}, P, \bar{V}) = (0, 0, 0)$ and we look for some functional spaces in which this solution is the unique one. This problem, also known the Liouville-type problem, was recently studied in [10] where the main interest is the use of more general spaces than the $L^p$ spaces, for instance, the Lorentz and the Morrey spaces. However, to the best of our knowledge, the regularity of weak solutions in these spaces was not studied before and it must be assumed.

A new regularity criterion in the non-homogeneous case. Motivated by this last question, in this note we study some new a priori conditions in the setting of the Morrey spaces to improve the regularity of weak solutions of the system (1). Moreover, we also consider the more general case under the action of the external forces $\text{div}(\mathbf{F})$ and $\text{div}(\mathbf{G})$.

We shall consider here a fairly general notion of weak solutions, which is given in the following:
Definition 1.1 Let $\mathbb{F}, \mathbb{G} \in \mathcal{D}'(\mathbb{R}^3)$. A weak solution of the coupled system (1) is a triplet $(\tilde{U}, P, \tilde{V})$, where: $\tilde{U} \in L^2_{\text{loc}}(\mathbb{R}^3)$, $P \in \mathcal{D}'(\mathbb{R}^3)$, $|\tilde{V}(x)| = 1$ for almost all $x \in \mathbb{R}^3$ and $\nabla \otimes \tilde{V} \in L^2_{\text{loc}}(\mathbb{R}^3)$, such that it verifies (1) in the distributional sense.

It is worth observing we use minimal conditions on the functions $\tilde{U}, \tilde{V}$ and $P$ to ensure that all the terms in (1) are well defined as distributions. Moreover, we let the pressure $P$ to be a very general object as we only have $P \in \mathcal{D}'(\mathbb{R}^3)$.

As $\tilde{U}$ and $\nabla \otimes \tilde{V}$ are locally square integrable functions, in order to improve their regularity we look for some natural conditions on the local quantities $\int_{B(x_0,R)} |\tilde{U}(x)|^2 \, dx$ and $\int_{B(x_0,R)} |\nabla \otimes \tilde{V}(x)|^2 \, dx$, where $B(x_0,R)$ denotes the ball of center $x_0 \in \mathbb{R}^3$ and radius $R > 0$. Thus, the Morrey spaces appear naturally. We recall that for a parameter $2 < p < +\infty$, the homogeneous Morrey space $M^{2,p}(\mathbb{R}^3)$ is the Banach space of functions $f \in L^2_{\text{loc}}(\mathbb{R}^3)$ such that:

$$\|f\|_{M^{2,p}} = \sup_{x_0 \in \mathbb{R}^3, R > 0} R^2 \left( \frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} |f(x)|^2 \, dx \right)^{\frac{1}{2}} < +\infty,$$

(4)

where $|B(x_0,R)| \simeq R^3$ is the Lebesgue measure of the ball. The parameter $p$ measures the decaying rate of the (local) mean quantity $\left( \frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} |f(x)|^2 \, dx \right)^{\frac{1}{2}}$ as $R$ goes to infinity. This is a homogeneous space of degree $-\frac{2}{p}$, and moreover, we have the following chain of continuous embeddings $L^p(\mathbb{R}^3) \subset L^{p,q}(\mathbb{R}^3) \subset M^{2,p}(\mathbb{R}^3)$. Here, $L^{p,q}(\mathbb{R}^3)$ (with $p < q \leq +\infty$) denotes a Lorentz space which describes the decaying properties of functions in a different setting. See the book [1] for a detailed study of these spaces.

Finally, for the parameter $p > 2$ given above, and for the regularity parameter $k \geq 0$, we introduce now the Sobolev-Morrey space

$$\mathcal{W}^{k,p}(\mathbb{R}^3) = \left\{ f \in M^{2,p}(\mathbb{R}^3) : \partial^\alpha f \in M^{2,p}(\mathbb{R}^3), \text{ for all multi-indices } |\alpha| \leq k \right\}.$$

Moreover, we denote by $W^{k,\infty}(\mathbb{R}^3)$ the classical Sobolev space of bounded functions with bounded weak derivatives until the order $k$. Then, our first result reads as follows:

Theorem 1.1 Let $(\tilde{U}, P, \tilde{V})$ be a weak solution of the coupled system (1) given in Definition 1.1. We assume $\tilde{U} \in M^{2,p}(\mathbb{R}^3)$ and $\nabla \otimes \tilde{V} \in M^{2,p}(\mathbb{R}^3)$, with $p > 3$. Then, if for $k \geq 0$:

$$\mathbb{F}, \mathbb{G} \in \mathcal{W}^{k+1,p}(\mathbb{R}^3) \cap \mathcal{W}^{k+1,\infty}(\mathbb{R}^3),$$

(5)

it follows that $\tilde{U} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$, $P \in \mathcal{W}^{k+1,p}(\mathbb{R}^3)$ and $\tilde{V} \in \mathcal{W}^{k+2,p}(\mathbb{R}^3)$. Moreover, for all multi-indices $|\alpha| \leq k + 1$, the functions $\partial^\alpha \tilde{U}$ and $\partial^\alpha \tilde{V}$ are Hölder continuous with exponent $0 < 1 - 3/p < 1$, while for $|\alpha| \leq k$ the function $\partial^\alpha P$ is also Hölder continuous with the same exponent.

Remark 1.1 Recall that the external forces acting on the system (1) are given by $\text{div}(\mathbb{F})$ and $\text{div}(\mathbb{G})$. Then, by (5) we have $\text{div}(\mathbb{F}), \text{div}(\mathbb{G}) \in \mathcal{W}^{k,p}(\mathbb{R}^3)$ which yields a gain of regularity of weak solutions of the order $k + 2$. This (expected) maximum gain of regularity is given by the effects of the Laplacian operator in the system (1). We refer to Remark 3.1 for the technical details.

Remark 1.2 For the particular homogeneous case when $\mathbb{F} = \mathbb{G} = 0$, we obtain that weak solutions of the system (1) verify $(\tilde{U}, P, \tilde{V}) \in C^\infty(\mathbb{R}^3)$, provided that $\tilde{U} \in M^{2,p}(\mathbb{R}^3)$ and $\nabla \otimes \tilde{V} \in M^{2,p}(\mathbb{R}^3)$, with $p > 3$. As explained, this particular result is of interest in connection to the Liouville-type problem for (1) in the Morrey spaces [10].
Mathematically, the coupled system (1) is also of interests as it contains two relevant equations. On the one hand, by setting \( \vec{V} \) a constant unitary vector we get the stationary and forced Navier-Stokes equations:

\[
- \Delta \vec{U} + \text{div}(\vec{U} \otimes \vec{U}) + \vec{\nabla} P = \text{div}(\vec{F}), \quad \text{div}(\vec{U}) = 0. \tag{6}
\]

On the other hand, by setting now \( \vec{U} = 0 \) we obtain the following harmonic map flow:

\[
- \Delta \vec{V} - |\vec{\nabla} \otimes \vec{V}|^2 \vec{V} = \text{div}(\vec{G}). \tag{7}
\]

As a direct consequence of this result we obtain a new regularity criterion for these equations:

**Corollary 1.1**

1. Let \( (\vec{U}, P) \in L^2_{\text{loc}}(\mathbb{R}^3) \times \mathcal{D}'(\mathbb{R}^3) \) be a weak solution of the equation (6). If \( \vec{U} \in \dot{M}^{2,p}(\mathbb{R}^3) \), with \( p > 3 \), and \( \vec{F} \) verifies (5) with \( k \geq 0 \), then we have \( \vec{U} \in W^{k+2,p}(\mathbb{R}^3) \) and \( P \in W^{k+1,p}(\mathbb{R}^3) \). Furthermore, for \( |\alpha| \leq k + 1 \) the function \( \partial^\alpha \vec{U} \) is \((1 - 3/p)\)–Hölder continuous, while this holds true for \( \partial^\alpha P \) with \( |\alpha| \leq k \).

2. Let \( \vec{V}(x) \in S^2 \) with \( \vec{\nabla} \otimes \vec{V} \in L^2_{\text{loc}}(\mathbb{R}^3) \) be a weak solution of the equation (7). If \( \vec{\nabla} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3) \), \( p > 3 \) and \( \vec{G} \) verifies (5) with \( k \geq 0 \), then we have \( \vec{V} \in W^{k+2,p}(\mathbb{R}^3) \). In addition, \( \partial^\alpha \vec{V} \) is a \((1 - 3/p)\)–Hölder continuous function for all \( |\alpha| \leq k + 1 \).

**Remark 1.3** Of course, Remark 1.2 holds true for the equations (6) and (7) in the homogeneous case \( \vec{F} = \vec{G} = 0 \).

Let us briefly explain the general strategy in the proof of Theorem 1.1. The proof bases on two key ideas. First, by assuming \( \vec{U}, \vec{\nabla} \otimes \vec{V} \in M^{2,p}(\mathbb{R}^3) \) and by using the framework of an auxiliary parabolic system (9), we prove that \( \vec{U} \) and \( \vec{\nabla} \otimes \vec{V} \) are bounded functions on \( \mathbb{R}^3 \). Thereafter, we use a bootstrap argument to show that \( \vec{U} \in W^{k+2,p}(\mathbb{R}^3) \) and \( \vec{V} \in W^{k+2,p}(\mathbb{R}^3) \).

These ideas can also be extended to other relevant coupled system of the fluid dynamics. This system is the time-independent magneto-hydrodynamic equations which describe the steady state of the magnetic properties of electrically conducting fluids, including plasma and liquid metals [20]:

\[
\begin{align*}
-\Delta \vec{U} + \text{div}(\vec{U} \otimes \vec{U}) - \text{div}(\vec{B} \otimes \vec{B}) + \vec{\nabla} P &= \text{div}(\vec{F}), \quad \text{div}(\vec{U}) = 0, \\
-\Delta \vec{B} + \text{div}(\vec{B} \otimes \vec{U}) - \text{div}(\vec{U} \otimes \vec{B}) &= \text{div}(\vec{G}), \quad \text{div}(\vec{B}) = 0,
\end{align*} \tag{8}
\]

Here, \( \vec{U} : \mathbb{R}^3 \to \mathbb{R}^3 \) and \( P : \mathbb{R}^3 \to \mathbb{R} \) always denote the velocity and the pressure of the fluid respectively. Moreover, \( \vec{B} : \mathbb{R}^3 \to \mathbb{R}^3 \) is the magnetic field. Furthermore, \( \text{div}(\vec{F}) \) and \( \text{div}(\vec{G}) \) are the external forces acting on this system.

Our second result states as follows:

**Theorem 1.2** Let \( \vec{U} \in L^2_{\text{loc}}(\mathbb{R}^3) \), \( \vec{B} \in L^2_{\text{loc}}(\mathbb{R}^3) \), \( P \in \mathcal{D}'(\mathbb{R}^3) \) be a weak solution of the system (8). We assume \( \vec{U}, \vec{B} \in \dot{M}^{2,p}(\mathbb{R}^3) \) with \( p > 3 \). If for \( k \geq 0 \) the functions \( \vec{F} \) and \( \vec{G} \) verify (5) then we have \( \vec{U} \in W^{k+2,p}(\mathbb{R}^3) \), \( \vec{B} \in W^{k+2,p}(\mathbb{R}^3) \) and \( P \in W^{k+1,p}(\mathbb{R}^3) \). Moreover, for \( |\alpha| \leq k + 1 \), \( \partial^\alpha \vec{U} \) and \( \partial^\alpha \vec{B} \) are Hölder continuous functions with exponent \( 1 - 3/p \), while this fact holds true for \( \partial^\alpha P \) with \( |\alpha| \leq k \).

The system (8) actually has a simpler structure than the system (1) as the four nonlinear terms have the same writing. Consequently, they are treated similarly provided that \( \vec{U} \) and \( \vec{B} \) have the same properties. Thus, we are able to adapt the ideas above to obtain this new criterion for weak solutions in the setting of the Morrey spaces.

When \( F, G \in L^2(\mathbb{R}^3) \), and consequently the external forces verify \( \text{div}(\vec{F}), \text{div}(\vec{G}) \in \dot{H}^{-1}(\mathbb{R}^3) \), it is well-known that the system (8) has finite energy weak solutions \( \vec{U}, \vec{B} \in \dot{H}^1(\mathbb{R}^3) \) and \( P \in \dot{H}^{1/2}(\mathbb{R}^3) + L^2(\mathbb{R}^3) \).
Some well-known results

where, the unknown \( \theta \mid R \in R \) be also adapted to prove analogous theorems for other coupled systems of the fluid dynamics. For instance, the function \( F \) associated to \( \phi \mid R \in R \)

Corollary 1.2 Let \( F, G \in L^2(R^3) \) and let \( (\vec{U}, \vec{B}) \in H^1(R^3) \) be a weak solution of the system (8). If for \( k \geq 0 \) the functions \( F \) and \( G \) verify (5) then we have \( \vec{U} \in W^{k+2,6}(R^3), \vec{B} \in W^{k+2,6}(R^3) \) and \( P \in W^{k+1,6}(R^3) \). Moreover, if \( F = G = 0 \) we have \( (\vec{U}, \vec{B}, \vec{P}) \in C^\infty(R^3) \).

For these finite energy weak solutions we are able to go further in the study of their regularity. We recall that for a parameter \( b > 0 \) we define the weighted exponential operator \( e^{b \sqrt{\text{V}}} \) as \( F(e^{b \sqrt{\text{V}}} \varphi)(\xi) = e^{b |\xi|} \varphi(\xi) \), for \( \varphi \in S(R^3) \). Thereafter, for a parameter \( s \in R \) we define the Gevrey class

\[
G_b^s(R^3) = \left\{ f \in \mathcal{H}^s(R^3) : e^{b \sqrt{\text{V}}} f \in \mathcal{H}^s(R^3) \right\}.
\]

For \( |s| < 3/2 \), \( G_b^s(R^3) \) is a Banach space with the norm \( ||e^{b \sqrt{\text{V}}} (\cdot)||_{\mathcal{H}^s} \). Moreover, for \( s \geq 0 \) the functions in \( G_b^s(R^3) \) are analytic. Thus, our third result writes down as follows:

Theorem 1.3 Let \( F, G \in L^2(R^3) \). For \( b > 0 \) we assume \( F, G \in G_0^0(R^3) \). Then, there exists \( b_1 > 0 \) such that all the finite energy weak solutions \( (\vec{U}, \vec{B}) \in H^{1/2}(R^3) \) and \( P \in H^{1/2}(R^3) + L^2(R^3) \) of the system (8) associated to \( F \) and \( G \) verify \( \vec{U} \in G_{b_1}^{1/2}(R^3), \vec{B} \in G_{b_1}^{1/2}(R^3) \) and \( P \in G_{b_1}^{1/2}(R^3) + G_0^0(R^3) \).

Consequently, we obtain that \( \vec{U}, \vec{B} \) and \( \vec{P} \) are analytic functions and they admit holomorphic extensions to the strip \( \left\{ (x + iz) \in C^3 : |z| < \min(b, b_1) \right\} \). Moreover, in the homogeneous case \( F = G = 0 \) we have:

Corollary 1.3 All finite energy solutions \( \vec{U}, \vec{B} \in H^{1/2}(R^3) \) and \( P \in H^{1/2}(R^3) \) of the homogeneous (MHD) system verify \( \vec{U}, \vec{B} \in G_{b_1}^1(R^3) \) and \( P \in G_{b_1}^{1/2}(R^3) \), for any \( b_1 > 0 \).

To close this section, let us mention that both results given in Theorems 1.2 and 1.3 respectively could be also adapted to prove analogous theorems for other coupled systems of the fluid dynamics. For instance, the stationary Boussinesq system:

\[
\begin{cases}
-\Delta \vec{U} + \text{div}(\vec{U} \otimes \vec{U}) + \nabla P = \theta \vec{e}_3 + \text{div}(F), & \text{div}(\vec{U}) = 0, \\
-\Delta \theta + \text{div}(\theta U) = \text{div}(G),
\end{cases}
\]

where, the unknown \( \theta : R^3 \rightarrow R \) is the temperature of the fluid and \( \vec{e}_3 \) denotes the third vector of the canonical basis in \( R^3 \). On the other hand, the stationary tropical climate model:

\[
\begin{cases}
-\Delta \vec{U} + \text{div}(\vec{U} \otimes \vec{U}) + \text{div}(\vec{B} \otimes \vec{D}) + \nabla P = \text{div}(F), \\
-\Delta \vec{D} + \text{div}(\vec{U} \otimes \vec{D}) + \text{div}(\vec{B} \otimes \vec{U}) + \nabla \theta = \text{div}(G), \\
-\Delta \theta + \vec{U} \cdot \nabla \theta + \text{div}(\vec{D}) = 0, & \text{div}(\vec{U}) = 0,
\end{cases}
\]

where \( \vec{D} : R^3 \rightarrow R^3 \) stands for the baroclinic mode of the velocity field \( \vec{U} \).

2 Some well-known results

For the reader’s convenience, we summarize here some well-known results which will be useful in the sequel. For \( 1 < r < p \) and \( 1 < p < +\infty \), we consider the homogeneous Morrey space \( \dot{M}^{r,p}(R^3) \), which is defined as in (4) with \( r \) instead of 2.
Lemma 2.1 (Page 169 of [15]) The space $\widetilde{M}^{r,p}(\mathbb{R}^3)$ is stable under convolution with functions in the space $L^1(\mathbb{R}^3)$ and we have $\|g*f\|_{\widetilde{M}^{r,p}} \leq c\|g\|_{L^1} \|f\|_{\widetilde{M}^{r,p}}$.

Lemma 2.2 Let $f \in \widetilde{M}^{r,p}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Then, for all $1 \leq \sigma < +\infty$ we have $f \in \widetilde{M}^{r,\sigma,\sigma}(\mathbb{R}^3)$ and the following estimate holds: $\|f\|_{\widetilde{M}^{r,\sigma,\sigma}} \leq c \|f\|_{\widetilde{M}^{r,p}}^{1/\sigma} \|f\|_{L^\infty}^{1-1/\sigma}$.

**Proof.** By the interpolation inequalities with parameter $\frac{1}{\sigma}$ we write:

$$\left(\int_{B(x_0,R)} |f(x)|^{r \sigma} \, dx\right)^{1/r} \leq c \left(\int_{B(x_0,R)} |f(x)|^r \, dx\right)^{1/r} \|f\|_{L^\infty}^{1-1/\sigma},$$

then we multiply each side by $R^{\frac{3}{r} - \frac{1}{\sigma}}$ to obtain:

$$R^{\frac{3}{r} - \frac{1}{\sigma}} \left(\int_{B(x_0,R)} |f(x)|^{r \sigma} \, dx\right)^{1/r} \leq c \left[R^{\frac{3}{r} - \frac{1}{\sigma}} \left(\int_{B(x_0,R)} |f(x)|^r \, dx\right)^{1/r}\right] \|f\|_{L^\infty}^{1-1/\sigma}.$$

Finally, as $|B(x_0,R)| \simeq R^3$ we obtain $\|f\|_{\widetilde{M}^{r,\sigma,\sigma}} \leq c \|f\|_{\widetilde{M}^{r,p}} \|f\|_{L^\infty}^{-\frac{1}{\sigma}}$. 

Lemma 2.3 (Page 171 of [15]) Let $t > 0$ and let $h_t$ be the heat kernel. The following estimate holds $t^{\frac{3}{r} - \frac{1}{\sigma}} \|h_t*f\|_{L^\infty} \leq c \|f\|_{\widetilde{M}^{r,p}}$.

This estimate is a direct consequence of the continuous embedding $\widetilde{M}^{r,p}(\mathbb{R}^3) \subset \dot{B}^{-\frac{3}{r}}_{\infty,\infty}(\mathbb{R}^3)$. We recall that the homogeneous Besov space $\dot{B}^{-\frac{3}{r}}_{\infty,\infty}(\mathbb{R}^3)$ can be characterized as the space of temperate distributions $f \in S'(\mathbb{R}^3)$ such that $\sup_{t>0} t^{\frac{3}{r} - \frac{1}{\sigma}} \|h_t*f\|_{L^\infty} < +\infty$.

Lemma 2.4 (Lemme 4.2 of [11]) For $i = 1, 2, 3$ let $\mathcal{R}_i = \frac{\partial}{\partial x_i} \sqrt{-\Delta}$ be the $i$-th Riesz transform. Then, for $i, j = 1, 2, 3$ the operator $\mathcal{R}_i \mathcal{R}_j$ is continuous in the space $\widetilde{M}^{r,p}(\mathbb{R}^3)$ and we have $\|\mathcal{R}_i \mathcal{R}_j (f)\|_{\widetilde{M}^{r,p}} \leq c \|f\|_{\widetilde{M}^{r,p}}$.

Finally, we shall use the following result linking the Morrey spaces and the Hölder regularity of functions.

Lemma 2.5 (Proposition 3.4 of [8]) Let $f \in S'(\mathbb{R}^3)$ such that $\nabla f \in \dot{M}^{1,p}(\mathbb{R}^3)$, with $p > 3$. There exists a constant $C > 0$ such that for all $x, y \in \mathbb{R}^3$ we have $|f(x) - f(y)| \leq C \|\nabla f\|_{\dot{M}^{1,p}} |x - y|^{1-3/p}$. Recall that the Morrey space $\dot{M}^{1,p}(\mathbb{R}^3)$ is defined as the space of locally finite Borel measures $d\mu$ such that

$$\sup_{x_0 \in \mathbb{R}^3, R > 0} R^3 \left(\frac{1}{|B(x_0,R)|} \int_{B(x_0,R)} d\mu(x)\right) < +\infty.$$

### 3 Proof of the Theorem 1.1

For the sake of clearness, we shall divide the proof into three main steps.

**Step 1. The auxiliary parabolic system.** Our starting point is the study of the following auxiliary parabolic system. Let $\vec{V} : \mathbb{R}^3 \to \mathbb{S}^2$ be the vector field given in Definition 1.1. Moreover, $\mathbb{P}$ stands for the the Leray’s projector. We consider the initial value problem for the parabolic coupled system:

$$\begin{cases}
\partial_t \vec{u} - \Delta \vec{u} + \mathbb{P}(\text{div}(\vec{u} \otimes \vec{u})) + \mathbb{P}(\text{div}(\vec{V} \otimes \vec{V})) = \mathbb{P}(\text{div}(F)) , & \text{div}(\vec{u}) = 0 , \\
\partial_t \vec{V} - \Delta \vec{V} + \vec{V} \otimes (\vec{u} \vec{V}) - \vec{V} \otimes (|\vec{V}|^2 \vec{V}) = \vec{V} \otimes (\text{div}(G)) , & \\
\vec{u}(0,\cdot) = \vec{u}_0 , & \vec{V}(0,\cdot) = \vec{V}_0 ,
\end{cases}$$

(9)
where, the vector field $\vec{u} = (u_1, u_2, u_3)$ and the matrix $V = (v_{i,j})_{1 \leq i,j \leq 3}$ are the unknowns. We emphasize that in the second equation the vector field $\vec{V}$ is given.

For a time $0 < T < +\infty$, we denote $C_*([0, T], \dot{M}^{2,p}(\mathbb{R}^3))$ the functional space of bounded and weak--* continuous functions from $[0, T]$ with values in the Morrey space $\dot{M}^{2,p}(\mathbb{R}^3)$. We prove now the following:

**Proposition 3.1** Consider the initial value problem (9) where $F$ and $G$ verify (5). Let $p > 3$ and let $\vec{u}_0 \in \dot{M}^{2,p}(\mathbb{R}^3)$ and $V_0 \in \dot{M}^{2,p}(\mathbb{R}^3)$ be the initial data. There exists a time $T_0 > 0$, depending on $\vec{u}_0$, $V_0$, $F$ and $G$; and there exist $(\vec{u}, V) \in C_*([0, T_0], \dot{M}^{2,p}(\mathbb{R}^3))$, which is the unique solution of (9). Moreover this solution verifies:

$$\sup_{0 < t < T_0} \int_{\mathbb{T}^3} (\|\vec{u}(t, \cdot)\|_{L^\infty} + \|V(t, \cdot)\|_{L^\infty}) < +\infty. \tag{10}$$

**Proof.** Mild solutions of the system (9) write down as the integral formulation:

$$\vec{u}(t, \cdot) = e^{t\Delta} \vec{u}_0 + \int_0^t e^{(t-s)\Delta} \mathbb{P}(\text{div}(F)) \, ds + \int_0^t e^{(t-s)\Delta} \mathbb{P}(\text{div}(\vec{u} \otimes \vec{u}))(s, \cdot) \, ds$$

$$+ \int_0^t e^{(t-s)\Delta} \mathbb{P}(\text{div}(V \otimes V))(s, \cdot) \, ds,$$

and

$$V(t, \cdot) = e^{t\Delta} V_0 + \int_0^t e^{(t-s)\Delta} \mathbb{V}(\text{div}(G)) \, ds + \int_0^t e^{(t-s)\Delta} \mathbb{V}(\text{div}(\vec{u} V))(s, \cdot) \, ds$$

$$- \int_0^t e^{(t-s)\Delta} \mathbb{V}(\text{div}(\vec{V}^2))(s, \cdot) \, ds.$$ \tag{11}

By the Picard’s fixed point argument, we will solve both problems (11) and (12) in the Banach space

$$E_T = \left\{ f \in C_*([0, T], \dot{M}^{2,p}(\mathbb{R}^3)) : \sup_{0 < t < T} \int_{\mathbb{T}^3} \|f(t, \cdot)\|_{L^\infty} < +\infty \right\},$$

with the norm

$$\|f\|_{E_T} = \sup_{0 < t < T} \|f(t, \cdot)\|_{\dot{M}^{2,p}} + \sup_{0 < t < T} \int_{\mathbb{T}^3} \|f(t, \cdot)\|_{L^\infty}.$$

Let us mention that for $f_1, f_2 \in E_T$, for the sake of simplicity, we shall write $\|(f_1, f_2)\|_{E_T} = \|f_1\|_{E_T} + \|f_2\|_{E_T}$.

We start by studying the linear terms in (11) and (12). As $\vec{u}_0 \in \dot{M}^{2,p}(\mathbb{R}^3)$ and $V_0 \in \dot{M}^{2,p}(\mathbb{R}^3)$ by Lemma 2.1 we have $\|(e^{t\Delta} \vec{u}_0, e^{t\Delta} V_0)\|_{\dot{M}^{2,p}} \leq c\|\vec{u}_0, V_0\|_{\dot{M}^{2,p}}$, hence we obtain $e^{t\Delta} \vec{u}_0 \in C_*([0, T], \dot{M}^{2,p}(\mathbb{R}^3))$ and $e^{t\Delta} V_0 \in C_*([0, T], \dot{M}^{2,p}(\mathbb{R}^3))$. On the other hand, by Lemma 2.3 the following estimate directly follows

$$\|e^{t\Delta} \vec{u}_0, e^{t\Delta} V_0\|_{L^\infty} \leq c\|\vec{u}_0, V_0\|_{\dot{M}^{2,p}}.$$ Thus, we have $e^{t\Delta} \vec{u}_0 \in E_T$ and $e^{t\Delta} V_0 \in E_T$, and moreover, the following estimate holds:

$$\|(e^{t\Delta} \vec{u}_0, e^{t\Delta} V_0)\|_{E_T} \leq c\|(\vec{u}_0, V_0)\|_{\dot{M}^{2,p}}. \tag{13}$$

Thereafter, as $F, G$ are time independent tensors, and moreover, as we assume (5), we write:

$$\left\| \int_0^t e^{(t-s)\Delta} (\text{div}(F), \text{div}(G)) \, ds \right\|_{\dot{M}^{2,p}} \leq \int_0^t \left\| e^{(t-s)\Delta} (\text{div}(F), \text{div}(G)) \right\|_{\dot{M}^{2,p}} \, ds$$

$$\leq c \left\| (\text{div}(F), \text{div}(G)) \right\|_{\dot{M}^{2,p}} \left( \int_0^t ds \right),$$

7
\[
\begin{align*}
\text{we have} & \quad \| e^{(t-s)\Delta} (\text{div}(F), \text{div}(G)) \|_{L^\infty} \leq c (t-s)^{-\frac{3}{p}} \| (\text{div}(F), \text{div}(G)) \|_{M^{2,p}}. \\
\text{On the other hand, we remark that by Lemma 2.3 we have} & \quad \| e^{(t-s)\Delta} (\text{div}(F), \text{div}(G)) \|_{L^\infty} \leq c (t-s)^{-\frac{3}{p}} \| (\text{div}(F), \text{div}(G)) \|_{M^{2,p}}, \\
\text{and then we can write} & \quad t^{\frac{3}{p}} \left\| \int_0^t e^{(t-s)\Delta} (\text{div}(F), \text{div}(G)) \, ds \right\|_{L^\infty} \leq t^{\frac{3}{p}} \int_0^t \left\| e^{(t-s)\Delta} (\text{div}(F), \text{div}(G)) \right\|_{L^\infty} \, ds \\
& \quad \leq t^{\frac{3}{p}} \int_0^t (t-s)^{-\frac{3}{p}} \| (\text{div}(F), \text{div}(G)) \|_{M^{2,p}} \, ds \\
& \quad \leq c t^{\frac{3}{p}} \| (\text{div}(F), \text{div}(G)) \|_{M^{2,p}} \left( \int_0^t (t-s)^{-\frac{3}{p}} \, ds \right) \\
& \quad \leq c t \| (\text{div}(F), \text{div}(G)) \|_{M^{2,p}}.
\end{align*}
\]

We thus obtain
\[
\begin{align*}
\sup_{0 < t < T} t^{\frac{3}{p}} \left\| \int_0^t e^{(t-s)\Delta} (\text{div}(F), \text{div}(G)) \, ds \right\|_{L^\infty} & \leq c T \| (\text{div}(F), \text{div}(G)) \|_{M^{2,p}}. \\
\text{By the estimates (14) and (15) we get} & \quad \left\| \int_0^t e^{(t-s)\Delta} (\text{div}(F), \text{div}(G)) \, ds \right\|_{E_T} \leq c T \| (\text{div}(F), \text{div}(G)) \|_{M^{2,p}}. \\
\text{We study now the bilinear terms in (11) and (12). First, the terms } & \quad B_1(\vec{u}, \vec{u}) \text{ and } B_2(\vec{V}, \vec{V}) \text{ in (11) are estimated as follows:}
\end{align*}
\]

\[
\begin{align*}
\sup_{0 \leq t \leq T} \| B_1(\vec{u}, \vec{u}) + B_2(\vec{V}, \vec{V}) \|_{M^{2,p}} & \leq c T^{\frac{1}{2} - \frac{3}{p}} \| (\vec{u}, \vec{V}) \|_{E_T}^2, \\
\end{align*}
\]

where, as } p > 3 \text{ then we have } \frac{1}{2} - \frac{3}{p} > 0. \text{ Indeed, by Lemma 2.1, by the well-known estimate on the heat kernel: } \| \nabla h_{(t-s)}(\cdot) \|_{L^1} \leq \frac{c}{(t-s)^{1/2}}, \text{ and moreover, by Lemma 2.4 (hence } P \text{ is continuous in } M^{2,p}(\mathbb{R}^3) \text{) we have:}
\[
\begin{align*}
\sup_{0 \leq t \leq T} \| B_1(\vec{u}, \vec{u}) + B_2(\vec{V}, \vec{V}) \|_{M^{2,p}} & = \sup_{0 \leq t \leq T} \left\| \int_0^t e^{(t-s)\Delta} P(\text{div}(\vec{u} \otimes \vec{u}))(s, \cdot) ds + \int_0^t e^{(t-s)\Delta} P(\text{div}(\vec{V} \otimes \vec{V}))(s, \cdot) ds \right\|_{M^{2,p}} \\
& \leq c \sup_{0 \leq t \leq T} \left\| \int_0^t e^{(t-s)\Delta} (\text{div}(\vec{u} \otimes \vec{u}))(s, \cdot) + e^{(t-s)\Delta} (\text{div}(\vec{V} \otimes \vec{V}))(s, \cdot) \right\|_{M^{2,p}} ds \\
& \leq c \sup_{0 \leq t \leq T} \int_0^t \frac{1}{(t-s)^{1/2}} \left( \| \vec{u}(s, \cdot) \otimes \vec{u}(s, \cdot) \|_{M^{2,p}} + \| \vec{V}(s, \cdot) \otimes \vec{V}(s, \cdot) \|_{M^{2,p}} \right) ds \\
& \leq c \sup_{0 \leq t \leq T} \int_0^t \frac{1}{(t-s)^{1/2} s^{3/2}} \left( \| \vec{u}(s, \cdot) \|_{L^\infty} \| \vec{u}(s, \cdot) \|_{M^{2,p}} + \| \vec{V}(s, \cdot) \|_{L^\infty} \| \vec{V}(s, \cdot) \|_{M^{2,p}} \right) ds \\
& \leq c T^{\frac{1}{2} - \frac{3}{p}} \| (\vec{u}, \vec{V}) \|_{E_T}^2.
\end{align*}
\]
We will prove now the following estimate:

\[
\sup_{0 \leq t \leq T} t^{\frac{3}{2p}} \|B_1(\bar{u}, \bar{u}) + B_2(V, V)\|_{L^\infty} \leq c T^{\frac{1}{2} - \frac{3}{2p}} \|\bar{u}, V\|_{E_T}^2.
\]

We write:

\[
\sup_{0 \leq t \leq T} t^{\frac{3}{2p}} \|B_1(\bar{u}, \bar{u}) + B_2(V, V)\|_{L^\infty} = \sup_{0 < t < T} t^{\frac{3}{2p}} \int_0^t e^{(t-s)\Delta} P(div(\bar{u} \otimes \bar{u}))(s, \cdot) ds + \int_0^t e^{(t-s)\Delta} P(div(V \otimes V))(s, \cdot) ds \bigg|_{L^\infty} \leq \sup_{0 < t < T} t^{\frac{3}{2p}} \int_0^t \left\|e^{(t-s)\Delta} P div(\bar{u} \otimes \bar{u} + V \otimes V)(s, \cdot)\right\|_{L^\infty} ds = (a).
\]

Here, we recall that the operator \(e^{(t-s)\Delta} P(div(\cdot)) \) writes down as a matrix of convolution operators (in the spatial variable) whose kernels \(K_{ij} \) verify \(|K_{ij}(t-s, x)| \leq \frac{c}{((t-s)^{1/2} + |x|)^4}\), see Proposition 11.1 of [14].

Then, we have \(\|K(t-s, \cdot)\|_{L^1} \leq \frac{c}{(t-s)^{1/2}}\); and we can write:

\[
(a) \leq c \sup_{0 \leq t \leq T} t^{\frac{3}{2p}} \int_0^t \frac{1}{(t-s)^{1/2}} \left(\|\bar{u}(s, \cdot) \otimes \bar{u}(s, \cdot)\|_{L^\infty} + \|V(s, \cdot) \otimes V(s, \cdot)\|_{L^\infty}\right) ds
\]

\[
\leq c \sup_{0 \leq t \leq T} t^{\frac{3}{2p}} \int_0^t \frac{ds}{(t-s)^{1/2} s^{\frac{3}{p}}} \left(\left(\frac{s^{\frac{3}{p}}}{s^{\frac{3}{p}}} M(s, \cdot)\right)^{1/2} + \left(\frac{s^{\frac{3}{p}}}{s^{\frac{3}{p}}} M(s, \cdot)\right)^{2/2}\right) ds
\]

\[
\leq c \left(\sup_{0 \leq t \leq T} t^{\frac{3}{2p}} \int_0^t \frac{ds}{(t-s)^{1/2} s^{\frac{3}{p}}}\right)^{1/2} \|\bar{u}, V\|_{E_T}^2.
\]

We study now the bilinear terms \(B_3(\bar{u}, V)\) and \(B_4(V, V)\) in (12). Precisely, we shall prove the estimates:

\[
\sup_{0 \leq t \leq T} \|B_3(\bar{u}, V) + B_4(V, V)\|_{M^{2, p}} \leq c T^{\frac{1}{2} - \frac{3}{2p}} \|\bar{u}, V\|_{E_T}^2.
\]

and

\[
\sup_{0 \leq t \leq T} t^{\frac{3}{2p}} \|B_3(\bar{u}, V) + B_4(V, V)\|_{L^\infty} \leq c T^{\frac{1}{2} - \frac{3}{2p}} \|\bar{u}, V\|_{E_T}^2.
\]

In order to prove the estimate (20), we recall that the vector field \(\vec{V}\) verifies \(\vec{V}(x) = 1\) for a.e. \(x \in \mathbb{R}^3\) (see Definition 1.1) and then we have \(\|\vec{V}\|_{L^\infty} = 1\). With this information at hand, for the first in the norm \(\| \cdot \|_{E_T}\)
we are able to write the following estimates:

\[
\begin{align*}
&\sup_{0 \leq t \leq T} \|B_3(\vec{u}, \vec{V}) + B_4(\vec{V}, \vec{V})\|_{M^{2,p}} \\
= &\sup_{0 \leq t \leq T} \left\| \int_0^t e^{(t-s)\Delta} \vec{V} \otimes (\vec{u}(\vec{V})(s, \cdot))ds - \int_0^t e^{(t-s)\Delta} \vec{V} \otimes (|\vec{V}|^2 \vec{V})(s, \cdot)ds \right\|_{M^{2,p}} \\
\leq &\ c \sup_{0 \leq t \leq T} \int_0^t \frac{1}{(t-s)^{1/2}} \left( \|\vec{u}\vec{V}(s, \cdot)\|_{M^{2,p}} + \|\vec{V}(s, \cdot)\|^2_{M^{2,p}} \right) ds \\
\leq &\ c \sup_{0 \leq t \leq T} \int_0^t \frac{1}{(t-s)^{1/2}} \left( \|\vec{u}(s, \cdot)\|_{M^{2,p}} \|\vec{V}(s, \cdot)\|_{L^\infty} + \|\vec{V}(s, \cdot)\|^2_{M^{2,p}} \|\vec{V}\|_{L^\infty} \right) ds \\
\leq &\ c \sup_{0 \leq t \leq T} \left[ \int_0^t \frac{ds}{(t-s)^{1/2} s^{3/2p}} \right] \cdot \|\vec{u}, \vec{V}\|^2_{E_T} \leq c T^{\frac{1}{2} - \frac{3}{2p}} \|\vec{u}, \vec{V}\|^2_{E_T}.
\end{align*}
\]

In order to prove the estimate (21), we essentially follow the estimates performed in (19) to obtain:

\[
\begin{align*}
&\sup_{0 \leq t \leq T} \int_0^t e^{(t-s)\Delta} \vec{V} \otimes (\vec{u}(\vec{V})(s, \cdot))ds - \int_0^t e^{(t-s)\Delta} \vec{V} \otimes (|\vec{V}|^2 \vec{V})(s, \cdot)ds \right\|_{L^\infty} \\
= &\ c \sup_{0 \leq t \leq T} \left[ \int_0^t \frac{ds}{(t-s)^{1/2} s^{3/2p}} \right] \cdot \|\vec{u}, \vec{V}\|^2_{E_T}.
\end{align*}
\]

Summarizing, we obtain the following estimate for the four bilinear terms in the equations (11) and (12):

\[
\|B_1(\vec{u}, \vec{u})\|_{E_T} + \|B_2(\vec{V}, \vec{V})\|_{E_T} + \|B_3(\vec{u}, \vec{V})\|_{E_T} + \|B_4(\vec{V}, \vec{V})\|_{E_T} \leq c T^{\frac{1}{2} - \frac{3}{2p}} \|\vec{u}, \vec{V}\|^2_{E_T}, \quad \frac{1}{2} - \frac{3}{2p} > 0.
\]

(24)

Once we have the estimates (13), (16) and (24), for a time \(0 < T_0 = T_0(\vec{u}_0, \vec{V}_0, \text{div}(P), \text{div}(G)) < +\infty\) small enough, the existence and uniqueness of a solution \((\vec{u}, \vec{V})\) for the equations (11) and (12) follow from standard arguments. Proposition 3.1 is proven.

**Step 2. The global boundedness of \(\vec{U}\) and \(\vec{V} \otimes \vec{V}\).** With the help of Proposition 3.1, we are able to prove the following:

**Proposition 3.2** Let \((\vec{U}, P, \vec{V})\) be a weak solution of the system (1) given in Definition 1.1. If \(\vec{U} \in \dot{M}^{2,p}(\mathbb{R}^3)\) and \(\vec{V} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3)\), with \(p > 3\), then we have \(\vec{U} \in L^\infty(\mathbb{R}^3)\) and \(\vec{V} \otimes \vec{V} \in L^\infty(\mathbb{R}^3)\).

**Proof.** In the initial value problem (9), we set the initial data \((\vec{u}_0, \vec{V}_0) = (\vec{U}, \vec{V} \otimes \vec{V})\). Then, by Proposition 3.1 there exists a time \(0 < T_0\) and there exists a unique solution \((\vec{u}, \vec{V})\) \(\in C([0, T], \dot{M}^{2,p}(\mathbb{R}^3))\) of (9) arising from \((\vec{U}, \vec{V} \otimes \vec{V})\).

On the other hand, we have the following key remark. First, we apply the Leray’s projector \(P\) in the first equation of the system (1). Newt, we apply the operator \(\vec{V} \otimes (\cdot)\) in the second equation of this system. Moreover, as \(\vec{U}\) and \(\vec{V}\) are time-independent functions we have \(\partial_t \vec{U} = 0\) and \(\partial_t (\vec{V} \otimes \vec{V}) = 0\). Thus, the couple \((\vec{U}, \vec{V} \otimes \vec{V})\) is also a solution of the initial value problem (9) with the initial data \((\vec{u}_0, \vec{V}_0) = (\vec{U}, \vec{V} \otimes \vec{V})\), and moreover, we have \((\vec{U}, \vec{V} \otimes \vec{V}) \in C([0, T], \dot{M}^{2,p}(\mathbb{R}^3))\).
Consequently, in the space $C([0, T], \dot{M}^{2,p}(\mathbb{R}^3))$ we have two solutions of (9) with the same initial data: on the one hand, the solution $(\vec{u}, \vec{V})$ given by Proposition 3.1 and, on the other hand, the solution $(\vec{U}, \vec{\nabla} \otimes \vec{V})$. By uniqueness we have the identity $(\vec{u}, \vec{V}) = (\vec{U}, \vec{\nabla} \otimes \vec{V})$ and by (10) we can write

$$\sup_{0 < t < T} t^{\frac{2}{27}} (\|\vec{U}\|_{L^\infty} + \|\vec{\nabla} \otimes \vec{V}\|_{L^\infty}) < +\infty.$$  

But, as the solution $(\vec{U}, \vec{\nabla} \otimes \vec{V})$ does not depend on the time variable we have $\vec{U} \in L^\infty(\mathbb{R}^3)$ and $\vec{\nabla} \otimes \vec{V} \in L^\infty(\mathbb{R}^3)$. Proposition 3.2 is now proven. 

**Step 3. Estimates on high order derivatives in the Morrey spaces.** The global boundness of $\vec{U}$ and $\vec{\nabla} \otimes \vec{V}$ obtained in the previous step is the key tool to prove the following:

**Proposition 3.3** We assume that $\mathcal{F}$ and $\mathcal{G}$ verify (5) for $k \geq 0$, and moreover, we assume $\vec{U}, \vec{\nabla} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3)$ with $p > 3$. Then we have $\vec{U} \in W^{k+2,p}(\mathbb{R}^3)$, $\vec{V} \in W^{k+2,p}(\mathbb{R}^3)$ and $P \in W^{k+1,p}(\mathbb{R}^3)$.

**Proof.** We will study first the functions $\vec{U}$ and $\vec{V}$. For this, we get rid (temporally) of the pressure term by applying the Leray’s projector $\mathcal{P}$ in the first equation of the system (1). Then, we shall consider the following coupled system:

$$\begin{cases} -\Delta \vec{U} + \mathcal{P}(\text{div}(\vec{U} \otimes \vec{U})) + \mathcal{P}(\text{div}(\vec{\nabla} \otimes \vec{V} \otimes \vec{\nabla} \otimes \vec{V})) = \mathcal{P}(\text{div}(\mathcal{F})), \\
-\Delta \vec{V} + \text{div}(\vec{V} \otimes \vec{U}) - |\vec{\nabla} \otimes \vec{V}|^2 \vec{V} = \text{div}(\mathcal{G}), \\
\text{div}(\vec{U}) = 0, \end{cases} \quad (25)$$

As $\vec{U}$ and $\vec{V}$ solve this system they verify the following (equivalent) integral formulations:

$$\begin{align} \vec{U} &= -\frac{1}{-\Delta} \left( \mathcal{P}(\text{div}(\vec{U} \otimes \vec{U})) \right) - \frac{1}{-\Delta} \left( \mathcal{P}(\text{div}(\vec{\nabla} \otimes \vec{V} \otimes \vec{\nabla} \otimes \vec{V})) \right) + \frac{1}{-\Delta} \left( \mathcal{P}(\text{div}(\mathcal{F})) \right), \quad (26) \\
\vec{V} &= -\frac{1}{-\Delta} \left( \text{div}(\vec{V} \otimes \vec{U}) \right) + \frac{1}{-\Delta} \left( |\vec{\nabla} \otimes \vec{V}|^2 \vec{V} \right) + \frac{1}{-\Delta} \left( \text{div}(\mathcal{G}) \right). \quad (27) \end{align}$$

By using these integral formulations, we will show that $\partial^\alpha \vec{U} \in \dot{M}^{2,p}(\mathbb{R}^3)$ and $\partial^\alpha \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3)$ for all multi-index $|\alpha| \leq k + 2$. We shall prove this fact by iteration respect to the order of the multi-indices $\alpha$, which we will denote as $|\alpha|$. For the reader’s convenience, in the following couple of technical lemmas we prove each step in the iterative argument separately.

**Lemma 3.1 (The case initial case)** Recall that by Proposition 3.2 we have $\vec{U}, \vec{\nabla} \otimes \vec{V} \in L^\infty(\mathbb{R}^3)$. Then, for $|\alpha| \leq 2$ and for all $1 \leq \sigma < +\infty$ we have $\partial^\alpha \vec{U} \in \dot{M}^{2\sigma,\sigma}(\mathbb{R}^3)$ and $\partial^\alpha \vec{V} \in \dot{M}^{2\sigma,\sigma}(\mathbb{R}^3)$.

**Proof.** Due to the coupled structure of the equations (26) and (27), we must study first the function $\vec{V}$ and then we study the function $\vec{U}$.

- Let $|\alpha| = 1$. As $\vec{\nabla} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, by Lemma 2.2 we have $\partial^\sigma \vec{V} \in \dot{M}^{2\sigma,\sigma}(\mathbb{R}^3)$ for all $1 \leq \sigma < +\infty$. On the other hand, for the function $\vec{U}$, by (26) we have the identity:

$$\partial^\sigma \vec{U} = -\frac{1}{-\Delta} \left( \mathcal{P}(\partial^\sigma \text{div}(\vec{U} \otimes \vec{U})) \right) - \frac{1}{-\Delta} \left( \mathcal{P}(\partial^\sigma \text{div}(\vec{\nabla} \otimes \vec{V} \otimes \vec{\nabla} \otimes \vec{V})) \right) + \frac{1}{-\Delta} \left( \mathcal{P}(\text{div}(\partial^\sigma \mathcal{F})) \right), \quad (28)$$

where we shall verify that each term on the right-hand side belong to the space $\dot{M}^{2\sigma,\sigma}(\mathbb{R}^3)$, for all $1 \leq \sigma < +\infty$. For the first term, as $\vec{U} \in \dot{M}^{2,p}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$ by Lemma 2.2 we have $\vec{U} \in \dot{M}^{r\sigma,\sigma}(\mathbb{R}^3)$, for all $1 \leq \sigma < +\infty$. Moreover, by the Hölder inequalities we also have $\vec{U} \otimes \vec{U} \in \dot{M}^{r\sigma,\sigma}(\mathbb{R}^3)$ (for all $1 \leq \sigma < +\infty$). Then, as $|\alpha| = 1$ the operator $\frac{1}{-\Delta} (\mathcal{P}(\partial^\sigma \text{div} (\cdot)))$ writes down as
a linear combination of the Riesz transforms $\mathcal{R}_i \mathcal{R}_j$ with $i, j = 1, 2, 3$; and by Lemma 2.4 we obtain
$$\left\| \frac{1}{-\Delta} \left( \mathcal{P}(\partial^p \text{div}(\vec{U} \otimes \vec{U})) \right) \right\|_{M^{2,p},\sigma} \leq c \left\| \vec{U} \otimes \vec{U} \right\|_{M^{2,\sigma, p}} < +\infty.$$ 

The second term is similarly estimated by using now the information $\vec{\nabla} \otimes \vec{V} \in M^{\sigma, p\sigma}(\mathbb{R}^3)$.

We study the third term. By Lemma 2.4 we write
$$\left\| \frac{1}{-\Delta} \left( \mathcal{P}(\text{div}(\partial^p \mathcal{F})) \right) \right\|_{M^{2,p},\sigma} \leq c \left\| \mathcal{F} \right\|_{M^{2,\sigma, p}}.$$ Then, by (5) we have $\mathcal{F} \in M^{2,p} \cap L^\infty(\mathbb{R}^3)$, and by Lemma 2.2 we get
$$\left\| \mathcal{F} \right\|_{M^{2,p}} \leq c \left\| \mathcal{F} \right\|_{\frac{2}{p}}^{\frac{1}{2}} \left\| \mathcal{F} \right\|_{L^\infty}^{\frac{1}{2}} < +\infty.$$

Let $|\alpha| = 2$. By (27) we write
$$\partial^\alpha \vec{V} = -\frac{1}{-\Delta} \left( \partial^\alpha \text{div}(\vec{V} \otimes \vec{U}) \right) + \frac{1}{-\Delta} \left( \partial^\alpha \left( |\vec{\nabla} \otimes \vec{V}|^2 \vec{V} \right) \right) + \frac{1}{-\Delta} \left( \partial^\alpha \text{div}(\vec{G}) \right). \quad (29)$$

As before, we will prove that each term on the right-hand side belong to the space $\dot{M}^{2, p\sigma}(\mathbb{R}^3)$. For the first term, for $i, j = 1, 2, 3$ we write $\partial_i (V_i U_j) = \partial_i V_i U_j + V_i \partial_j U_j$. By recalling that $\vec{\nabla} \otimes \vec{V} \in M^{2, p\sigma}(\mathbb{R}^3)$, $\vec{U} \in L^\infty(\mathbb{R}^3)$, and moreover, by recalling that $\vec{V} \in L^\infty(\mathbb{R}^3)$ and $\partial_i \vec{U} \in \dot{M}^{2, p\sigma}(\mathbb{R}^3)$, we directly have $\text{div}(\vec{V} \otimes \vec{U}) \in \dot{M}^{2, p\sigma}(\mathbb{R}^3)$. Thereafter, as $|\alpha| = 2$, by Lemma 2.4 the operator $-\frac{1}{-\Delta} (\partial^\alpha (\cdot))$ is continuous in the space $\dot{M}^{2, p\sigma}(\mathbb{R}^3)$. We thus have
$$\frac{1}{-\Delta} \left( \partial^\alpha \text{div}(\vec{V} \otimes \vec{U}) \right) \in \dot{M}^{2, p\sigma}(\mathbb{R}^3).$$

For the second term, by recalling that $||\vec{V}||_{L^\infty} = 1$, by using the information $\vec{\nabla} \otimes \vec{V} \in \dot{M}^{2,p}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$, and moreover, by Lemma 2.2, we can write:

$$\left\| \vec{\nabla} \otimes \vec{V} \right\|_{M^{2,2},\sigma} \leq \left\| \vec{\nabla} \otimes \vec{V} \right\|_{M^{2,p},\sigma} \left\| \vec{\nabla} \otimes \vec{V} \right\|_{L^\infty} \left\| \vec{V} \right\|_{L^\infty}$$
$$\leq c \left\| \vec{\nabla} \otimes \vec{V} \right\|_{\frac{2}{p}} \left\| \vec{\nabla} \otimes \vec{V} \right\|_{L^\infty}^{\frac{1}{2}} \left\| \vec{\nabla} \otimes \vec{V} \right\|_{L^\infty} \left\| \vec{V} \right\|_{L^\infty}$$
$$\leq c \left\| \vec{\nabla} \otimes \vec{V} \right\|_{\frac{2}{p}} \left\| \vec{\nabla} \otimes \vec{V} \right\|_{L^\infty}^{\frac{2}{p}} < +\infty.$$

Consequently, by Lemma 2.4 we have
$$\frac{1}{-\Delta} \left( \partial^\alpha \left( |\vec{\nabla} \otimes \vec{V}|^2 \vec{V} \right) \right) \in \dot{M}^{2, p\sigma}(\mathbb{R}^3).$$

For the third term, always by Lemma 2.4 and by (5), we obtain
$$\left\| \frac{1}{-\Delta} \left( \mathcal{P}(\text{div}(\mathcal{G})) \right) \right\|_{M^{2,2},\sigma} \leq c \left\| \text{div}(\mathcal{G}) \right\|_{M^{2,p},\sigma} \leq c \left\| \text{div}(\mathcal{G}) \right\|_{M^{2,2,p}} \left\| \text{div}(\mathcal{G}) \right\|_{L^\infty}^{1-\sigma} < +\infty. \quad (30)$$

We study now the function $\partial^\alpha \vec{U}$ given in the expression (28). For the first term on the right-hand side we write
$$\frac{1}{-\Delta} \left( \mathcal{P}(\partial^p \text{div}(\vec{U} \otimes \vec{U})) \right) = \frac{1}{-\Delta} \left( \mathcal{P}(\partial^{p_1} \text{div} \partial^{p_2} (\vec{U} \otimes \vec{U})) \right), \quad \text{where } |\alpha_1| = 1 \text{ and } |\alpha_2| = 1.$$ Here, we must verify that $\partial^{p_1} (\vec{U} \otimes \vec{U}) \in \dot{M}^{2,p\sigma}(\mathbb{R}^3)$. Indeed, for $i, j = 1, 2, 3$ we write $\partial^{p_1} (U_i U_j) = (\partial^{p_2} U_i) U_j + U_i (\partial^{p_2} U_j)$. Then, as we have $\partial^{p_2} \vec{U} \in M^{2, p\sigma}(\mathbb{R}^3)$ and $\vec{U} \in L^\infty(\mathbb{R}^3)$, we obtain $\partial^{p_1} (\vec{U} \otimes \vec{U}) \in \dot{M}^{2, p\sigma}(\mathbb{R}^3)$. With this information, and the fact that by Lemma 2.4 the operator $\frac{1}{-\Delta} (\mathcal{P}(\partial^{p_1} \text{div}(\cdot)))$ is continuous in the space $\dot{M}^{2, p\sigma}(\mathbb{R}^3)$, we finally get
$$\frac{1}{-\Delta} \left( \mathcal{P}(\partial^p \text{div}(\vec{U} \otimes \vec{U})) \right) \in \dot{M}^{2, p\sigma}(\mathbb{R}^3).$$

The second term on the right-hand side of (28) follows the same ideas above (with $\vec{\nabla} \otimes \vec{V}$ instead of $\vec{U}$) and we have
$$\frac{1}{-\Delta} \left( \mathcal{P}(\partial^p \text{div}(\vec{V} \otimes \vec{V} \otimes \vec{V} \otimes \vec{V})) \right) \in \dot{M}^{2, p\sigma}(\mathbb{R}^3).$$

The third term on the right-hand side is similarly estimated as in (30).
Lemma 3.2 (The iterative process) For all \(1 \leq m \leq k\) and for all \(|\alpha| \leq m\) we assume \(\partial^\alpha \tilde{V} \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\) and \(\partial^\alpha \tilde{U} \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\), for all \(1 \leq \sigma < +\infty\). Then, it holds true for all \(|\alpha| = k + 2\).

Proof. We shall follow the main ideas in the proof of the previous lemma.

- Let \(|\alpha| = k + 1\). We start by studying the function \(\partial^\alpha \tilde{V}\) given in the identity (29). Let us verify that each term on the right-hand side belong to the space \(\dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\).

  For the first term, we split \(\alpha = \alpha_1 + \alpha_2\), with \(|\alpha_1| = 1\) and \(|\alpha_2| = k\). Then, we write

  \[-\frac{1}{\Delta} \left( \partial^\alpha \text{div}(\tilde{V} \otimes \tilde{U}) \right) = -\frac{1}{\Delta} \left( \partial^{\alpha_1} \text{div} \partial^{\alpha_2} (\tilde{V} \otimes \tilde{U}) \right).\]

  In the last expression, we verify that \(\partial^{\alpha_2} (\tilde{V} \otimes \tilde{U}) \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\). For \(i,j = 1,2,3\), by the Leibniz rule we write \(\partial^{\alpha_2} (\tilde{V}_i \tilde{U}_j) = \sum_{|\beta| \leq k} c_{\alpha_2, \beta} \partial^\beta \tilde{V}_i \partial^{\alpha_2 - \beta} \tilde{U}_j\), where \(c_{\alpha_2, \beta} > 0\) is a constant depending on the multi-indices \(\alpha_2\) and \(\beta\). Then, by the recurrence hypothesis we have \(\partial^{\alpha_2 - \beta} \tilde{V}_i \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\) and \(\partial^{\alpha_2} \tilde{U}_j \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\), and moreover, by applying the Hölder inequalities, we get \(\partial^{\alpha_2} (\tilde{V} \otimes \tilde{U}) \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\).

  On the other hand, recall that as \(|\alpha_1| = 1\) then by Lemma 2.4 the operator \(-\frac{1}{\Delta} (\partial^{\alpha_1} \text{div}(\cdot))\) is continuous in the space \(\dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\). Finally, for \(|\alpha| = k + 1\) we obtain \(-\frac{1}{\Delta} (\partial^{\alpha_1} \text{div}(\tilde{V} \otimes \tilde{U})) \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\).

  For the second term, we split now \(\alpha = \alpha_1 + \alpha_2\), with \(|\alpha_1| = 2\) and \(|\alpha_2| = k - 1\). Then we write

  \[-\frac{1}{\Delta} \left( \partial^\alpha \left( |\tilde{V} \otimes \tilde{V}|^2 \tilde{V} \right) \right) = -\frac{1}{\Delta} \left( \partial^{\alpha_1} \left( \partial^{\alpha_2} \left( |\tilde{V} \otimes \tilde{V}|^2 \tilde{V} \right) \right) \right).\]

  As before, we must verify that \(\partial^{\alpha_2} \left( |\tilde{V} \otimes \tilde{V}|^2 \tilde{V} \right) \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\). Always by the Leibniz rule\(^1\), for \(i,j,k = 1,2,3\) we have the following identity \(\partial^{\alpha_2} \left( |\partial_i V_j|^2 V_k \right) = \sum_{|\beta| \leq k-1} \partial^\beta (|\partial_i V_j|^2) \partial^{\alpha_2 - \beta} V_k\). Moreover, in order to compute the term \(\partial^\beta (|\partial_i V_j|^2)\), we make use again of the Leibniz rule to write \(\partial^\beta (|\partial_i V_j|^2) = \sum_{|\gamma| \leq |\beta|} (\partial^\gamma \partial_i V_j) (\partial^\beta - \gamma \partial_i V_j)\). Thus, by gathering these identities we obtain

  \[\partial^{\alpha_2} \left( |\tilde{V} \otimes \tilde{V}|^2 \tilde{V} \right) = \sum_{|\beta| \leq k-1} \sum_{|\gamma| \leq |\beta|} \partial^\gamma (\partial_i V_j) \partial^\beta - \gamma (\partial_i V_j) (\partial^{\alpha_2 - \beta} V_k).\]

  Once we have this identity, by using the recurrence hypothesis and the Hölder inequalities we get \(\partial^{\alpha_2} \left( |\tilde{V} \otimes \tilde{V}|^2 \tilde{V} \right) \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\). Finally, as we have \(|\alpha_2| = 2\), by Lemma 2.4 the operator \(-\frac{1}{\Delta} (\partial^{\alpha_1} (\cdot))\) is continuous in \(\dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\) and for \(|\alpha| = k + 1\) we get \(-\frac{1}{\Delta} \left( \partial^\alpha \left( |\tilde{V} \otimes \tilde{V}|^2 \tilde{V} \right) \right) \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\).

  For the third term, we split again \(\alpha = \alpha_1 + \alpha_2\) with \(|\alpha_1| = 1\) and \(|\alpha_2| = k\). Then, by Lemma 2.4 and by (5) we have:

  \[\left\| \frac{1}{\Delta} \partial^\alpha \text{div}(\tilde{G}) \right\|_{\dot{M}^{2\sigma, p\sigma}} = \left\| \frac{1}{\Delta} \partial^{\alpha_1} \text{div} \partial^{\alpha_2} (\tilde{G}) \right\|_{\dot{M}^{2\sigma, p\sigma}} \leq c \left\| \partial^{\alpha_2} G \right\|_{\dot{M}^{2\sigma, p\sigma}} \left\| \partial^{\alpha_2} G \right\|_{L^{1-\sigma}_w} < +\infty.\]

  Once we have \(\partial^\alpha \tilde{V} \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\) for all \(|\alpha| \leq k + 1\), by using the identity (28) and by following the same estimates above, we obtain \(\partial^\alpha \tilde{U} \in \dot{M}^{2\sigma, p\sigma} (\mathbb{R}^3)\) for \(|\alpha| = k + 1\).

\(^1\)For the sake of simplicity, we omit the constants.
Let $|\alpha| = k + 2$. Once we have $\partial^\alpha \vec{V} \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$ and $\partial^\alpha \vec{U} \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$ for all $|\alpha| \leq k + 1$, we just repeat again the estimates above to obtain $\partial^\alpha \vec{U}, \partial^\alpha \vec{V} \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$ for $|\alpha| = k + 2$. ■

**Remark 3.1** By (5) and by the estimate (31) (a similar estimate holds for $\mathbb{F}$), $k + 2$ is the maximum gain of regularity expected for $\partial^\alpha \vec{U}$ and $\partial^\alpha \vec{V}$ given in (28) and (29) respectively.

It remains to study the pressure term in the first equation of the system (1). For this, we apply the divergence operator in this equation to obtain that $P$ is related to $\vec{U}$ and $\vec{V} \otimes \vec{V}$ through the expression:

$$
P = \frac{1}{-\Delta} \left( \text{div} \left( \text{div}(\vec{U} \otimes \vec{U}) \right) \right) + \frac{1}{-\Delta} \left( \text{div} \left( \text{div}(\vec{\nabla} \otimes \vec{V} \circ \vec{\nabla} \otimes \vec{V}) \right) \right) + \frac{1}{-\Delta} \left( \text{div} \left( \text{div}(\mathbb{F}) \right) \right). \quad (32)
$$

By following the same estimates in the proof of the previous lemmas, we obtain:

**Lemma 3.3** For all $|\alpha| \leq k + 1$ we have $\partial^\alpha P \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$.

**Proof.** We study each term on the right-hand side in the identity (32). For the first term, as we have $\partial^\alpha \vec{U} \in \dot{M}^{2\sigma, p\sigma}$ for all $|\alpha| \leq k + 2$, and as we have $\vec{U} \in L^\infty(\mathbb{R}^3)$, then we get $\frac{1}{-\Delta} \left( \text{div} \left( \text{div}(\vec{U} \otimes \vec{U}) \right) \right) \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$ for $|\alpha| \leq k + 1$. Similarly, for the second term, as $\partial^\alpha \vec{V} \in \dot{M}^{2\sigma, p\sigma}$ for all $|\alpha| \leq k + 2$, and as $\vec{\nabla} \otimes \vec{V} \in L^\infty(\mathbb{R}^3)$, we get $\frac{1}{-\Delta} \left( \text{div} \left( \text{div}(\vec{\nabla} \otimes \vec{V} \circ \vec{\nabla} \otimes \vec{V}) \right) \right) \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$ for $|\alpha| \leq k + 1$. Finally, by (5) for the third term we have $\frac{1}{-\Delta} \left( \text{div} \left( \text{div}(\mathbb{F}) \right) \right) \in \dot{M}^{2\sigma, p\sigma}(\mathbb{R}^3)$, for all $|\alpha| \leq k + 1$. ■

We are able to finish the proof of Theorem 1.1. Recall that we have the embedding $\dot{M}^{2, p}(\mathbb{R}^3) \subset \dot{M}^{1, p}(\mathbb{R}^3)$. Then, for all $|\alpha| \leq k + 1$, by Lemma 2.5 the functions $\partial^\alpha \vec{U}$ and $\partial^\alpha \vec{V}$ are Hölder continuous with exponent $0 < 1 - 3/p < 1$. Moreover, for $|\alpha| \leq k$ the function $\partial^\alpha P$ is also Hölder continuous with the same exponent. Theorem 1.1 is now proven. ■

### 4 Proof of Theorem 1.2

The proof of this theorem essentially follows the same lines in the proof of Theorem 1.1. Consequently, we will give a sketch of the main steps. We consider first the evolution problem for the (MHD) equations:

$$
\begin{align*}
\partial_t \vec{u} - \Delta \vec{u} + \mathbb{P}(\text{div}(\vec{u} \otimes \vec{u})) - \mathbb{P}(\text{div}(\vec{b} \otimes \vec{b})) &= \mathbb{P}(\text{div}(\mathbb{F})), \\
\partial_t \vec{b} - \Delta \vec{b} + \text{div}(\vec{b} \otimes \vec{u}) - \text{div}(\vec{u} \otimes \vec{b}) &= \text{div}(\mathbb{G}), \\
\text{div}(\vec{u}) &= \text{div}(\vec{b}) = 0, \\
\vec{u}(0, \cdot) &= \vec{u}_0, \quad \vec{b}(0, \cdot) = \vec{b}_0.
\end{align*}
$$

(33)

By following the same computations done in the proof of Proposition 3.1, for the initial data $(\vec{u}_0, \vec{b}_0) \in \dot{M}^{2, p}(\mathbb{R}^3)$ (with $p > 3$) there exists a time $T_0 > 0$ and there exists $(\vec{u}, \vec{b}) \in \mathcal{C}_s([0, T_0], \dot{M}^{2, p}(\mathbb{R}^3))$ a unique solution of (33) which verifies:

$$
\sup_{0 < t < T} t^{\frac{3}{p'}} \left( ||\vec{u}(t, \cdot)||_{L^\infty} + ||\vec{b}(t, \cdot)||_{L^\infty} \right) < +\infty.
$$

(34)

As above, the key idea is the fact that the time-independent functions $(\vec{U}, \vec{B}) \in \dot{M}^{2, p}(\mathbb{R}^3)$, which solve the system (8), also belong to the space $\mathcal{C}_s([0, T], \dot{M}^{2, p}(\mathbb{R}^3))$ and they solve the evolutionary system (33) with the initial data $\vec{u}_0 = \vec{U}$ and $\vec{b}_0 = \vec{B}$. Then, by uniqueness of solutions in this space we obtain that $\vec{U}$ and $\vec{B}$ verify (34), hence, we have that $\vec{U} \in L^\infty(\mathbb{R}^3)$ and $\vec{B} \in L^\infty(\mathbb{R}^3)$. 

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On the other hand, \((\vec{U}, \vec{B}) \in \tilde{M}^{2,p}(\mathbb{R}^3)\) verify the following integral equations:

\[
\begin{align*}
\vec{U} &= -\frac{1}{-\Delta} \left( \mathbb{P}(\text{div}(\vec{U} \otimes \vec{U})) \right) - \frac{1}{-\Delta} \left( \mathbb{P}(\text{div}(\vec{B} \otimes \vec{B})) \right) + \frac{1}{-\Delta} \left( \mathbb{P}(\text{div}(\vec{F})) \right), \\
\vec{B} &= -\frac{1}{-\Delta} \left( \text{div}(\vec{B} \otimes \vec{U}) \right) - \frac{1}{-\Delta} \left( \text{div}(\vec{U} \otimes \vec{B}) \right) + \frac{1}{-\Delta} \left( \text{div}(\vec{G}) \right), \\
P &= \sum_{i,j=1}^{3} \mathcal{R}_i \mathcal{R}_j \left( U_i U_j + B_i B_j + F_{i,j} \right), \quad \mathcal{R}_i = \frac{\partial_i}{\sqrt{-\Delta}}, \ i = 1, 2, 3.
\end{align*}
\]

(35)

As we have \(\vec{U} \in L^{\infty}(\mathbb{R}^3)\) and \(\vec{B} \in L^{\infty}(\mathbb{R}^3)\), and moreover, by following the computations performed in the proof of Proposition 3.3, for all \(|\alpha| \leq k+2\) we have \(\partial^\alpha \vec{U} \in \tilde{M}^{2,p}(\mathbb{R}^3)\), \(\partial^\alpha \vec{B} \in \tilde{M}^{2,p}(\mathbb{R}^3)\); and for all \(|\alpha| \leq k+1\) we have \(\partial^\alpha P \in \tilde{M}^{2,p}(\mathbb{R}^3)\). Theorem 1.2 is proven.

5 Estimates in the Gevrey class

5.1 Proof of Theorem 1.3

We consider the evolution problem for the (MHD) equations given in (33), but with external forces \(\vec{f}\) and \(\vec{g}\) in the first and the second equation respectively:

\[
\begin{cases}
\partial_t \vec{u} - \Delta \vec{u} + \mathbb{P}(\text{div}(\vec{u} \otimes \vec{u})) - \mathbb{P}(\text{div}(\vec{b} \otimes \vec{b})) = \vec{f}, \\
\partial_t \vec{b} - \Delta \vec{b} + \text{div}(\vec{b} \otimes \vec{u}) - \text{div}(\vec{u} \otimes \vec{b}) = \vec{g}, \\
\text{div}(\vec{u}) = \text{div}(\vec{b}) = 0, \\
\vec{u}(0,\cdot) = \vec{u}_0, \quad \vec{b}(0,\cdot) = \vec{b}_0.
\end{cases}
\]

(36)

Then, we recall the following classical result:

**Lemma 5.1** Let \((\vec{u}_0, \vec{b}_0) \in \dot{H}^1(\mathbb{R}^3)\) be a divergence-free initial data. Moreover, for \(0 < T < +\infty\) we assume \(\vec{f}, \vec{g} \in C([0,T], \dot{H}^1(\mathbb{R}^3))\). Then, there exists a time \(0 < T_0 < T\), depending on \(\vec{u}_0, \vec{b}_0\), \(\vec{f}\) and \(\vec{g}\), and there exists \((\vec{u}, \vec{b}) \in C([0,T_0], \dot{H}^1(\mathbb{R}^3))\) which is the unique solution of (36).

We refer to Theorem 7.1 of the book [15] for a proof in the case of the Navier-Stokes equations (with \(\vec{b} = 0\)). However, this proof can be easily adapted for the (MHD) system (36).

We recall now that for \(b > 0\) and for \(t > 0\), the weighted exponential operator \(e^{b \sqrt{t} \sqrt{-\Delta}}\) is defined in the Fourier variable as

\[
\mathcal{F} \left( e^{b \sqrt{t} \sqrt{-\Delta}} \varphi(t,\cdot) \right)(\xi) = e^{b \sqrt{t} |\xi|} \hat{\varphi}(t,\xi), \quad \text{for all} \ \varphi \in \mathcal{S}([0, +\infty) \times \mathbb{R}^3).
\]

In the following result we prove that the solution \((\vec{u}, \vec{b})\) obtained above belong to the Gevrey class, provided that

\[
e^{b \sqrt{t} \sqrt{-\Delta}} \vec{f} \in C([0,T_0], \dot{H}^1(\mathbb{R}^3)) \quad \text{and} \quad e^{b \sqrt{t} \sqrt{-\Delta}} \vec{g} \in C([0,T_0], \dot{H}^1(\mathbb{R}^3)).
\]

(37)

**Proposition 5.1** Assume that \(\vec{f}\) and \(\vec{g}\) verify (37). Then, there exists a time \(0 < T_1 < T_0\) such that the unique solution \((\vec{u}, \vec{b}) \in C([0,T_1], \dot{H}^1(\mathbb{R}^3))\) of (36) obtained in Lemma 5.1 verifies:

\[
e^{b \sqrt{t} \sqrt{-\Delta}} \vec{u} \in C([0,T_1], \dot{H}^1(\mathbb{R}^3)) \quad \text{and} \quad e^{b \sqrt{t} \sqrt{-\Delta}} \vec{b} \in C([0,T_1], \dot{H}^1(\mathbb{R}^3)).
\]

(38)

**Proof.** We split the proof in two main steps.
• For a time $0 < T_1 < T_0$ (which we shall set small enough) we consider the space

$$E = \left\{ f \in C([0, T_1], H^1(\mathbb{R}^3)) : e^{b \sqrt{t} \sqrt{-\Delta}} f \in C([0, T], H^1(\mathbb{R}^3)) \right\},$$

which is a Banach space with the norm $\| \cdot \|_E = \left\| e^{b \sqrt{t} \sqrt{-\Delta}} (\cdot) \right\|_{L_t^\infty H_x^1}$. For the initial data $(\vec{u}_0, \vec{b}_0) \in \dot{H}^1(\mathbb{R}^3)$, we will construct a mild solution $(\vec{u}_1, \vec{b}_1)$ of the equation (36) in the space $E$. We recall that this mild solution solves the equations:

$$\vec{u}_1(t, \cdot) = e^{t \Delta} \vec{u}_0 + \int_0^t e^{(t-s)\Delta} \mathcal{P} \text{div} (\vec{u}_1 \otimes \vec{u}_1) (s, \cdot) ds$$

$$- \int_0^t e^{(t-s)\Delta} \mathcal{P} \text{div} (\vec{b}_1 \otimes \vec{b}_1) (s, \cdot) ds, \tag{39}$$

$$\vec{b}_1(t, \cdot) = e^{t \Delta} \vec{b}_0 + \int_0^t e^{(t-s)\Delta} \vec{g} ds + \int_0^t e^{(t-s)\Delta} \text{div} (\vec{b}_1 \otimes \vec{u}_1) (s, \cdot) ds$$

$$- \int_0^t e^{(t-s)\Delta} \text{div} (\vec{u}_1 \otimes \vec{b}_1) (s, \cdot) ds, \tag{40}$$

The linear terms in the system (39)-(40) are easy to estimate and for a constant $c > 0$, depending on $b$ and $T$, we have:

$$\left\| \left( e^{t \Delta} \vec{u}_0 + \int_0^t e^{(t-s)\Delta} \vec{f} ds, e^{t \Delta} \vec{b}_0 + \int_0^t e^{(t-s)\Delta} \vec{g} ds \right) \right\|_E \leq c \left( \| \vec{u}_0 \|_{\dot{H}^1} + \| \vec{b}_0 \|_{\dot{H}^1} + \| e^{b \sqrt{\xi} \sqrt{-\Delta}} \vec{f} \|_{L_t^\infty H_x^1} + \| e^{b \sqrt{\xi} \sqrt{-\Delta}} \vec{g} \|_{L_t^\infty H_x^1} \right). \tag{41}$$

On the other hand, as the four nonlinear terms $\mathcal{B}_i(\cdot, \cdot)$, with $i = 1, \ldots, 4$, in the system (39)-(40) have the same structure, we shall only estimate the first one. By the Plancherel’s formula we can write:

$$\left\| \mathcal{B}_1(\vec{u}_1, \vec{u}_1) \right\|_E = \sup_{0 \leq t \leq T} \left\| e^{b \sqrt{\xi} \sqrt{-\Delta}} \left( \int_0^t e^{(t-s)\Delta} \mathcal{P}(\text{div}(\vec{u}_1 \otimes \vec{u}_1))(s, \cdot) ds \right) \right\|_{H^1}$$

$$\leq \sup_{0 \leq t \leq T} \left\| \xi \| e^{b \sqrt{\xi} \xi} \int_0^t e^{-(t-s)\xi^2} |\xi| \left( \vec{u}_1 \ast \vec{u}_1 \right)(s, \cdot) ds \right\|_{L_x^2} \leq \sup_{0 \leq t \leq T} \left\| \xi \|^2 \int_0^t e^{-(t-s)\xi^2} e^{b \sqrt{\xi} \xi} \left( \vec{u}_1 \ast \vec{u}_1 \right)(s, \cdot) ds \right\|_{L_x^2}. \tag{42}$$

We must study now the expression $e^{b \sqrt{\xi} \xi} \left( \vec{u}_1 \ast \vec{u}_1 \right)(s, \xi)$. We remark that for all $\xi, \eta \in \mathbb{R}^3$ we have $e^{b \sqrt{\xi} \xi} \leq e^{b \sqrt{\xi} \xi - e^{b \sqrt{\xi} \xi}}$. Then, we obtain the following pointwise estimate:

$$\left| e^{b \sqrt{\xi} \xi} \left( \vec{u}_1 \ast \vec{u}_1 \right)(s, \xi) \right| \leq \left( \left( e^{b \sqrt{\xi} \xi} \left| \vec{u}_1 \right| \right) \ast \left( e^{b \sqrt{\xi} \xi} \left| \vec{u}_1 \right| \right) \right)(s, \xi).$$
By getting back to the estimate (42) we write
\[
\begin{align*}
\sup_{0 \leq t \leq T} \left\| |\xi|^2 \int_0^t e^{-(t-s)|\xi|^2} e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)} (s,\cdot) ds \right\|_{L^2} \\
\leq \sup_{0 \leq t \leq T} \left\| |\xi|^2 \int_0^t e^{-(t-s)|\xi|^2} \left( (e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)} + (e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)}) \right) (s,\cdot) ds \right\|_{L^2} \\
\leq \sup_{0 \leq t \leq T} \int_0^t \left\| |\xi|^3/2 e^{-(t-s)|\xi|^2} |\xi|^{1/2} \left( (e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)} + (e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)}) \right) (s,\cdot) ds \right\|_{L^2} = (a).
\end{align*}
\]

By applying again the Plancherel’s formula we get back to the spatial variable. Moreover, by the well-known properties of the heat kernel and by the product laws in the homogeneous Sobolev spaces, we have:
\[
(a) \leq \sup_{0 \leq t \leq T} \int_0^t \left\| (-Delta)^{3/4} h_{t-s}(\cdot) \right\|_{L^1} \left\| (-Delta)^{1/2} \left( (e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)}) \times (e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)}) \right) (s,\cdot) \right\|_{L^2} ds \\
\leq c T^{1/4} \sup_{0 \leq s \leq T} \left\| (e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)}) \times (e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)}) \right\|_{H^{1/2}} \leq c T^{1/4} \sup_{0 \leq s \leq T} \left\| (e^{b\sqrt{\Gamma\xi}|(\vec{\eta}_1^2 + \vec{\eta}_1^1)}) \right\|_{H^1} \leq c T^{1/4} \|\vec{\eta}_1\|_E \|\vec{u}_1\|_E.
\]

The other nonlinear terms are treated in the same way; and we can write:
\[
\|B_1(\vec{\eta}_1, \vec{u}_1)\|_E + \|B_2(\vec{b}_1, \vec{b}_1)\|_E + \|B_3(\vec{b}_1, \vec{u}_1)\|_E + \|B_4(\vec{u}_1, \vec{b}_1)\|_E \leq c T^{1/4} \|(\vec{u}_1, \vec{b}_1)\|^2_E. \tag{44}
\]

Thus, by (41) and (44), we fix a time $0 < T_1 < T_0$ small enough such that we can apply the Picard’s iterative schema to obtain a solution $(\vec{u}_1, \vec{b}_1) \in E$ of the system (39)-(40).

- By recalling that $E \subset C([0, T_0], H^1(\mathbb{R}^3))$, and moreover, by uniqueness of solutions in the space $C([0, T_0], H^1(\mathbb{R}^3))$, for all $0 < t < T_1$ we have the identity $(\vec{u}_1(t, \cdot), \vec{b}_1(t, \cdot)) = (\vec{u}(t, \cdot), \vec{b}(t, \cdot))$. Consequently, the solution $(\vec{u}, \vec{b})$ obtained in Lemma 5.1 verifies (38). Proposition 5.1 is proven.

We are able to finish the proof of Theorem 1.3. We observe that the time-independent solution $(\vec{U}, \vec{B}) \in H^1(\mathbb{R}^3)$ of the system (8) verifies $(\vec{U}, \vec{B}) \in C([0, T_1], H^1(\mathbb{R}^3))$. Moreover, this solution also solves the evolutionary problem (36) with initial data $(\vec{u}_0, \vec{b}_0) = (\vec{U}, \vec{B})$ and with the external forces $\vec{f} = \text{div} (\mathbb{F})$ and $\vec{g} = \text{div} (\mathbb{G})$.

As $\mathbb{F}, \mathbb{G} \in C_0^1(\mathbb{R}^3)$ with $b > 0$, we define the parameter $\beta = \frac{2b}{\sqrt{\Gamma\varepsilon}} > 0$, and then we rewrite the external forces $\vec{f}$ and $\vec{g}$ as follows:
\[
\vec{f} = e^{-\beta \sqrt{\Gamma\varepsilon} - \Delta} \left( e^{\beta \sqrt{\Gamma\varepsilon} - \Delta} \text{div} (\mathbb{F}) \right), \quad \vec{g} = e^{-\beta \sqrt{\Gamma\varepsilon} - \Delta} \left( e^{\beta \sqrt{\Gamma\varepsilon} - \Delta} \text{div} (\mathbb{G}) \right). \tag{45}
\]

Then, we shall prove that $\vec{f}, \vec{g} \in C([0, T_0], H^1(\mathbb{R}^3))$, and moreover, we shall prove that $\vec{f}$ and $\vec{g}$ verify (37).
Indeed, we write:

\[
\sup_{0 < t < T_0} \left\| \left( e^{\beta \sqrt{\nabla} \cdot \cdot \cdot \frac{\partial}{\partial t} f, e^{\beta \sqrt{\nabla} \cdot \cdot \cdot \frac{\partial}{\partial t} g} \right) \right\|_{H^1}^2
\]

\[
\leq \sup_{0 < t < T_0} \int_{\mathbb{R}^3} |\xi|^2 e^{2\beta \sqrt{\nabla} |\xi|} \left( |\hat{f}(t, \xi)|^2 + |\hat{g}(t, \xi)|^2 \right) d\xi
\]

\[
\leq \sup_{0 < t < T_0} \int_{\mathbb{R}^3} |\xi|^2 e^{2\beta \sqrt{\nabla} |\xi|} \left( |\text{div}(\hat{F})(\xi)|^2 + |\text{div}(\hat{G})(\xi)|^2 \right) d\xi
\]

\[
\leq \sup_{0 < t < T_0} \int_{\mathbb{R}^3} |\xi|^4 e^{2\beta \sqrt{\nabla} |\xi|} \left( |\hat{F}(\xi)|^2 + |\hat{G}(\xi)|^2 \right) d\xi
\]

\[
\leq \frac{1}{(\sqrt{T_0} \beta)^4} \int_{\mathbb{R}^3} (\sqrt{T_0} \beta |\xi|)^4 e^{4\beta (\sqrt{T_0} \beta |\xi|)} \left( |\hat{F}(\xi)|^2 + |\hat{G}(\xi)|^2 \right) d\xi
\]

\[
\leq c \int_{\mathbb{R}^3} e^{2b|\xi|} \left( |\hat{F}(\xi)|^2 + |\hat{G}(\xi)|^2 \right) d\xi < +\infty.
\]

By Lemma 5.1 we have that \((\hat{U}, \hat{B})\) is the unique solution of the system (36) in the space \(C([0, T_0], \mathcal{H}^1(\mathbb{R}^3))\). Thereafter, by Proposition 5.1 we have \(e^{\beta \sqrt{\nabla} \cdot \cdot \cdot \frac{\partial}{\partial t} \hat{U} \in C([0, T_1], \mathcal{H}^1(\mathbb{R}^3))\) and \(e^{\beta \sqrt{\nabla} \cdot \cdot \cdot \frac{\partial}{\partial t} \hat{B}_t \in C([0, T_1], \mathcal{H}^1(\mathbb{R}^3))\). We thus set \(b_1 = \beta \frac{T_1}{T_0} > 0\) and we get \(\hat{U} \in G^{1/2}_b(\mathbb{R}^3)\) and \(\hat{B} \in G^1_b(\mathbb{R}^3)\). Finally, by the third identity in (35) and by the product laws in the homogeneous Sobolev spaces we have \(P \in G^{1/2}_b(\mathbb{R}^3) + G^0_b(\mathbb{R}^3)\). Theorem 1.3 is proven.

5.2 Proof of Corollary 1.3
In the case \(F = G = 0\), the external forces \(\hat{f}\) and \(\hat{g}\) given in (45) are null and evidently they verify (37) for any \(b > 0\). Then, the subsequent parameters \(\beta = \frac{3b}{T_0^2} > 0\) and \(b_1 = \beta \frac{T_1}{T_0} > 0\) can be fixed arbitrary. \(\blacksquare\)

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Data availability statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.