EQUIVALENCE OF STRING AND FUSION LOOP-SPIN STRUCTURES

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Abstract. The importance of the fusion relation of loops was recognized in the context of spin structures on the loop space by Stolz and Teichner and further developed by Waldorf. On a spin manifold $M$ the equivalence classes of ‘fuse’ spin structures on the loop space $\mathcal{L}M$, incorporating the fusion property, strong regularity and reparameterization-invariance, are shown to be in 1-1 correspondence with equivalence classes of string structures on $M$. The identification is through the affine space of ‘string’ cohomology classes considered by Redden.

Introduction

In [14] Stolz and Teichner observed the importance of the notion of a fusion structure on objects over the loop space of a manifold arising from transgression constructions (it is implicit in Barrett [2]). If $M$ is an oriented manifold they exhibit an isomorphism between fusion orientations of the looped frame bundle over the loop space, $\mathcal{L}M = C^\infty(S; M)$, and spin structures on $M$, so refining and extending earlier results of McLaughlin [6] making precise an observation of Atiyah [1]. Using the treatment of bundle 2-gerbes by Stevenson [13] and categorical constructions, Waldorf [16] has shown that the existence of a fusion loop-spin structure, i.e. a spin structure on the loop manifold, is equivalent to the existence of a string structure on the manifold. Redden [11] has identified string structures, up to equivalence, with certain integral cohomology classes on the spin-oriented frame bundle. Here we develop direct forms of Waldorf’s transgression and regression constructions, the former via holonomy of projective unitary bundles and the latter via bundle gerbes in the sense of Murray [8] with a strong emphasis on the smoothness of these objects. In particular it is shown that ‘fusive loop-spin structures’, which are fusion spin structures on the loop space with additional regularity and equivariance properties under reparameterization, and string structures, both up to natural equivalence, are in 1-1 correspondence with the cohomology classes introduced by Redden, denoted by $C(F)$ below. The existence of such loop spin structures which are equivariant under reparameterization by diffeomorphisms of the circle answers a long-standing question, posed for instance by Brylinski [3] in relation to elliptic cohomology. It is expected that the smooth loop-spin structures constructed here can serve as the basis of an analytic discussion of the Dirac-Ramond operator.

If $M$ is a spin manifold of dimension $n \geq 5$ with $\pi_F : F \rightarrow M$ the spin-oriented frame bundle, there is an exact sequence (compare Redden [11])

$$
\begin{align*}
(1) \quad 0 \xrightarrow{} H^3(M; \mathbb{Z}) \xrightarrow{\pi^*} H^3(F; \mathbb{Z}) \xrightarrow{i^*} H^3(\text{Spin}; \mathbb{Z}) \xrightarrow{\frac{1}{2}p_1} \mathbb{Z} \xrightarrow{} 0
\end{align*}
$$
where the second map is restriction to the fibre and the third, transgression, map arises from the construction of the spin-Pontryagin class $\frac{1}{2}p_1 \in H^4(M; \mathbb{Z})$ associated to $F$. The cohomology classes of interest are those in the coset
\[
C(F) = \{ \sigma \in H^3(F; \mathbb{Z}) ; \iota^{\text{fib}}_*(\sigma) = \tau \}
\]
of $H^3(M; \mathbb{Z})$ in $H^3(F; \mathbb{Z})$, where $\tau$ is a chosen generator of $H^3(\text{Spin}; \mathbb{Z}) = \mathbb{Z}$.

A string group in dimension $n$ gives a short exact sequence of topological groups
\[
K \longrightarrow \text{String} \longrightarrow \text{Spin}
\]
where $K$ is a $K(\mathbb{Z}; 2)$ and $\pi_3(\text{String}) = \{0\}$. A string structure on $M$ is a principal bundle for String refining $F$:
\[
\begin{array}{ccc}
K & \longrightarrow & \text{String} \\
\downarrow & & \downarrow \\
\text{Spin} & \longrightarrow & F \\
\downarrow & \downarrow & \downarrow \\
M & & \pi_F \\
\end{array}
\]

In [11], Redden shows that isomorphism classes of string structures are in bijection with classes in $C(F)$.

We refer to [3] for the precise description of a ‘fusive loop-spin structure’, but in brief it is a lifting of the principal $\mathcal{L}\text{Spin}$-bundle $\mathcal{L}F$ over $\mathcal{L}M$ to a principal bundle with structure group the basic central extension of $\mathcal{L}\text{Spin}$ with additional multiplicative structure corresponding to the join of paths to form loops (i.e. fusion), strong equivariance under change of parameterization of loops and strong smoothness properties.

**Theorem.** For a spin manifold $M^n$, $n \geq 5$, there is a bijection from the set of equivalence classes of fusive loop-spin structures on $M$ to $C(F)$, as $H^3(M; \mathbb{Z})$ torsors, given by the Dixmier-Douady class of the associated bundle gerbe on $F$.

Note that $C(F)$ is non-empty if and only if $\frac{1}{2}p_1 = 0$ on $M$; that this is equivalent to the existence of a string structure on $M$ and also to the existence of a fusion loop-spin structure was shown by Waldorf [16].

We do very little here with regard to string structures on $M$, relying on the work of Stolz, Teichner and Redden. The main novelty of our results is therefore the existence of a ‘lithe’ and $\text{Diff}(S)$-equivariant, fusion loop-spin structure corresponding to each class in $C(F)$, where ‘lithe’ is a strong smoothness condition, similar to what was called ‘super-smooth’ by Brylinski in [3]. Combined with the existence results on string structures this realizes the equivalence between a strengthened version of smooth fusion loop-spin structures and smooth string structures as envisaged in [16]. The main reason for our interest in this isomorphism is that it is a first step towards an ongoing careful examination of the Dirac-Ramond operator on the loop manifold, associated to a string structure, with the hope of further elucidating the index theorem of Witten [20] and ideas of Brylinski [4] and Segal [12].

Before approaching the proof of the Theorem above, we first recall in [11] the identification by Stolz and Teichner of spin structures on an oriented manifold with orientations of the loop space (loop-orientations) satisfying the fusion condition. Fusive functions on the loop space valued in $\text{U}(1)$ are analysed in [2] and shown
to give a refinement of the first integral cohomology group making it isomorphic, by regression, to the second cohomology group of the manifold. This is extended to a corresponding result for fusive circle bundles in §3 where a bundle gerbe construction is used to show that these are classified up to equivalence by integral 3-cohomology on the manifold. The classification of string structures, essentially following the work of Redden [11], is recalled in §4. Fusive loop-spin structures, are defined in §5 and equivalence classes of them are shown to map to the integral 3-classes representing string structures. The proof of the Theorem above is completed in §8 by constructing a loop-spin structure corresponding to each string class, using the discussion in §6 of circles bundle over the loop space associated each such class and the description in §7 of the blip map, used to construct the extension of the loop spin action. The three appendices contain material on the notion of lithe regularity introduced here and then applied to loop manifolds and on properties of the basic central extension of the loop group of Spin.

1. Spin and loop-orientation

To serve as a model for our results below, we begin with a proof of the theorem of Stolz and Teichner [14] showing the equivalence between spin structures and fusion orientations of the loop space for a compact oriented manifold $M$. We relate this to the $\mathbb{Z}_2$-fusive 1-cohomology of the loop space, a notion which is generalized substantially below.

Giving $M$ a Riemann metric reduces the oriented frame bundle to the principal $\text{SO}(n)$ bundle of oriented orthonormal frames, denoted $F_{SO}$. Since $\pi_1(\text{SO}) = \mathbb{Z}_2$, the loop space $L:\mathbb{SO}$ has two components and an orientation on it is a reduction of the structure group $L:F_{SO}$ to the component of the identity, which is naturally

$$\text{LSpin} \hookrightarrow \text{LSO}.$$  

Such orientations may be identified with the continuous maps

$$o : \text{LF}_{SO} \longrightarrow \mathbb{Z}_2$$

taking both signs over each loop in $L:M$; the orientation subbundle is then $\{o = +1\}$. The smooth loop space is dense in the energy loop space $L:F_{SO} = H^1(S; F_{SO})$ and the latter retracts onto the former, so each orientation extends uniquely to a continuous function on $L:E:F_{SO}$.

Without the condition on the fibers of $LF_{SO}$, one can simply consider the group of maps $[12]$ for any finite dimensional manifold $Z$. The smooth and energy path spaces $IZ = C^\infty([0, 2\pi]; Z)$, $I_EZ = H^1([0, 2\pi]; Z)$ both fiber over $Z^2$; as discussed in [13]. The $k$-fold fiber product with respect to this fibration, $IZ^{[k]}$, consists of the $k$-tuples of paths having the same endpoints. Then the basic 'fusion map' is

$$\psi : I_E^{[2]}Z \rightarrow L_EZ$$

$$\psi(\gamma_1, \gamma_2) = l, \quad I(t) = \begin{cases} \gamma_1(2t), & 0 \leq t \leq \pi, \\ \gamma_2(4\pi - 2t), & \pi \leq t \leq 2\pi \end{cases}$$

taking a pair of paths with the same endpoints to the loop obtained by joining the first path with the reverse of the second. In general the fusion of smooth paths is not a smooth loop and the image is precisely the space of piecewise smooth loops within the energy space, with possible discontinuities in derivatives at $\{0, \pi\} \in S$.  

The simplicial projections $\pi_{ij} : \mathcal{I}^{[3]}_E \to \mathcal{I}^{[2]}_E$, obtained by keeping the $i$th and $j$th entries for $ij = 12$, $23$ and $13$, generate three lifted maps

$$\psi_{ij} = \psi \circ \pi_{ij} : \mathcal{I}^{[3]}_E \to \mathcal{L}_E Z$$

and then additional fusion condition on a map $o : \mathcal{L}_E Z \to \mathbb{Z}_2$ is the simplicial multiplicativity property (1.4)

$$\psi_{12}^* o \cdot \psi_{23}^* o = \psi_{13}^* o$$

on $\mathcal{I}^{[3]}_E Z$.

In other words, for every triple of paths $(\gamma_1, \gamma_2, \gamma_3) \in \mathcal{I}^{[3]}_E \mathbb{Z}$ with the same endpoints, the function satisfies $o(l_{12}) o(l_{23}) = o(l_{13})$, where $l_{ij} = \psi(\gamma_i, \gamma_j)$ (see Figure 1).

Since $o$ is locally constant, there is no need to consider any refined notions of smoothness. For the same reason such a map is automatically invariant under reparameterization.

**Proposition 1.1.** For any connected compact manifold the group of fusion maps,

$$H_0^{\text{ fus }}(\mathcal{L}_E Z; \mathbb{Z}_2) = \{ \text{continuous } o : \mathcal{L}Z \to \mathbb{Z}_2 \text{ satisfying (1.4)} \},$$

is naturally isomorphic to $H^1(\mathbb{Z}; \mathbb{Z}_2)$.

**Proof.** Extending by continuity to the energy space $\mathcal{L}_E Z$ and pulling back by the fusion map, such a map $o$ defines

$$\tilde{o} : \mathcal{I}^{[2]}_E Z \to \mathbb{Z}_2, \quad \tilde{o} = o \circ \psi.$$

The fusion condition implies that the relation on the trivial $\mathbb{Z}_2$ bundle over $\mathcal{I}_E Z$:

$$\mathcal{I}_E Z \times \mathbb{Z}_2 \ni (\gamma_1, e_1) \sim (\gamma_2, e_2) \iff (\gamma_1, \gamma_2) \in \mathcal{I}^{[2]}_E \mathbb{Z}, e_1 = \tilde{o}(\gamma_1, \gamma_2) e_2$$

is an equivalence relation over the fibers of $\mathcal{I}_E Z \to \mathbb{Z}$. Thus, this trivial bundle descends to a $\mathbb{Z}_2$ bundle $N$ over $\mathbb{Z}$, determined up to isomorphism by a cohomology class in $H^1(\mathbb{Z}; \mathbb{Z}_2) \cong H^1(\mathbb{Z}; \mathbb{Z}_2) \oplus H^1(\mathbb{Z}; \mathbb{Z}_2)$. However since $\mathcal{I}Z$ retracts to the constant paths under path contraction to the midpoint, identified with the subspace $\mathbb{Z} \subset \mathcal{I}Z$ embedded over the diagonal $\text{Diag} \subset \mathbb{Z}_2$, it follows that the pull-back of $N$ to the diagonal is trivial and then it follows that $N$ is isomorphic to $L^{-1} \boxtimes L = \pi_1^* L^{-1} \otimes \pi_2^* L$ for a $\mathbb{Z}_2$-bundle $L \to \mathbb{Z}$ since its cohomology class is of the form $\pi_2^* \alpha - \pi_1^* \alpha$ for a unique $\alpha \in H^1(\mathbb{Z}; \mathbb{Z}_2)$.

If the class $\alpha$ vanishes then $L$ is trivial, and pulling back a global section to $N \cong L^{-1} \boxtimes L$ and then up to $\mathcal{I}Z$ gives a continuous function $o' : \mathcal{I}Z \to \mathbb{Z}_2$.
such that \( o(\gamma_1, \gamma_2) = o^{-1}(\gamma_1)o(\gamma_2) \). However since \( \mathcal{I}Z \) is connected the only such functions are \( o' = \pm 1 \) and then \( o \equiv 1 \). This fixes an injective ‘regression’ map

\[
(1.6) \quad R_g : H^0_{\text{fus}}(\mathcal{L}Z; \mathbb{Z}_2) \to H^1(Z; \mathbb{Z}_2).
\]

To complete the proof of the Proposition it suffices to construct an inverse ‘enhanced transgression’ map. Given a class in \( H^1(Z; \mathbb{Z}_2) \) represented by a \( \mathbb{Z}_2 \)-bundle \( L \to Z \), there is a ‘holonomy’ map

\[
(1.7) \quad L \times_{\text{ev}(0)} \mathcal{L}Z \to \mathbb{Z}_2
\]

defined on \( (e, l) \in L \times_{\text{ev}(0)} \mathcal{L}Z \) by extending the initial value \( e \in L_{t(0)} \) to a continuous section of \( L \) over the path \( [0, 2\pi] \to Z \) covering \( l \) and taking the difference of the endpoints in \( L_{t(2\pi)} = L_{t(0)} \). This map is invariant under the \( \mathbb{Z}_2 \) action on \( L \) so descends to a map \( h : \mathcal{L}Z \to \mathbb{Z}_2 \) which satisfies the fusion condition. The same holonomy map results from isomorphic bundles so the assignment of \( L \) to \( h \) descends to a map

\[
(1.8) \quad T_{g_{\text{fus}}} : H^1(Z; \mathbb{Z}_2) \to H^0_{\text{fus}}(\mathcal{L}Z; \mathbb{Z}_2).
\]

That regression and fusive transgression are inverses follows from the fact that the transgression of \( L \) followed by regression is represented by the class of the bundle \( L^{-1} \boxtimes L \) over \( \mathbb{Z}^2 \). Indeed, using the fact that \( L = L^{-1} \) here due to the structure group, the pull-back of \( L^{-1} \boxtimes L \) to \( \mathcal{I}Z \) is trivialized by taking constant sections. These trivializations of the fibers at \( \gamma_1 \) and \( \gamma_2 \), where \( (\gamma_1, \gamma_2) \in I[2]Z \) have the same end-points, are identified by the holonomy of \( L \). Thus \( R_g \circ T_{g_{\text{fus}}} = \text{Id} \) and both are isomorphisms.

Recall that a spin structure on \( M \) is a lift of \( F_{SO} \) to a principal \( \text{Spin}(n) \) bundle which we denote simply as \( F \), since it is a central object below, giving a commutative diagram

\[
(1.9) \quad \begin{array}{ccc}
\text{Spin} & \longrightarrow & F \\
\downarrow & & \downarrow \\
\text{SO} & \longrightarrow & F_{SO} \end{array} \quad \begin{array}{c}
\downarrow \\
M.
\end{array}
\]

There is an exact sequence

\[
0 \to H^1(M; \mathbb{Z}_2) \to H^1(F_{SO}; \mathbb{Z}_2) \to H^1(SO; \mathbb{Z}_2) \equiv \mathbb{Z}_2 \to w_2(M) \cdot \mathbb{Z}_2 \to 0
\]

obtained for instance from the \( E_2 \) page of the Leray-Serre spectral sequence for the fibration \( F_{SO} \to M \). Spin structures on \( M \) exist if and only if \( w_2(M) = 0 \), and then their equivalence classes are in bijective correspondence with classes in \( H^1(F_{SO}; \mathbb{Z}_2) \) which restrict on fibers to generate \( H^1(SO; \mathbb{Z}_2) \); equivalently, the classes form a torsor over \( H^1(M; \mathbb{Z}_2) \).

**Theorem 1.2** (Stolz and Teichner [14]). *For an oriented compact manifold of dimension \( n > 5 \), fusion orientation structures are in 1-1 correspondence with spin structures.*
Proof. The fusion orientations (1.2) form a torsor over $H^0_{fus}(LM; \mathbb{Z}_2) \cong H^1(M; \mathbb{Z}_2)$. Moreover, Proposition 1.1 associates to each of them a $\mathbb{Z}_2$ bundle, $L$, over $F_{SO}$. The special property of $o$, as opposed to a general element of $H^0_{fus}(LF_{SO}; \mathbb{Z}_2)$, is that it takes both signs over each element of $LM$. By restriction this translates to the condition that $L$ is non-trivial as a bundle over each fiber of $F_{SO}$ which in turn means that the corresponding class in $H^1(F_{SO}; \mathbb{Z}_2)$ restricts to a generator of $H^1(SO; \mathbb{Z}_2)$. Conversely, a spin structure is just such a $\mathbb{Z}_2$ bundle over $F_{SO}$ which is non-trivial over each fiber and which therefore has holonomy which is non-trivial over fiber loops above each loop in $M$. \[ \Box \]

2. Fusive functions

Next we pass from maps into $\mathbb{Z}_2$ as in (1.1) to maps
\[(2.1) \quad f : LEM \rightarrow U(1)\]
satisfying the fusion condition. We will distinguish between the three classes of ‘fusion,’ ‘fusion with figure-of-eight’ and ‘fusive’ functions with increasing restrictions. The third class is closely related to the functions on the quotient of the loop space by ‘thin homotopy equivalence,’ considered by Waldorf [17], since this equivalence implies reparametrization invariance. Variants of the ‘regression’ and ‘enhanced transgression’ maps, but using the pointed path space, are also contained in Waldorf’s work.

First we work in the topological category.

Proposition 2.1. The group $C_{fus}(LM; U(1))$ of continuous functions \((2.1)\) which satisfy the fusion condition
\[(2.2) \quad \psi^*_{12} f \cdot \psi^*_{23} f = \psi^*_{13} f \quad \text{on } \mathcal{I}^{[3]}_E M\]
has path components isomorphic to
\[(2.3) \quad \{ \beta \in H^2(M^2; \mathbb{Z}) ; \beta|_{\text{Diag}} = 0 \} \subset H^2(M^2; \mathbb{Z}) .\]

Proof. As already observed by Barrett, [2], a continuous function \((2.1)\) defines a relation on $I_E M \times U(1)$,
\[(\gamma_1, z_1) \sim (\gamma_2, z_2) \iff (\gamma_1, \gamma_2) \in \mathcal{I}^{[2]}_E M, \ z_1 = f(\gamma_1, \gamma_2)z_2\]
with the fusion condition \((2.2)\) equivalent to this being an equivalence relation. Thus each $f$ determines a circle bundle $L_f$ over $M^2$ and hence there is a map to the Chern class $\beta = [L_f] \in H^2(M^2; \mathbb{Z})$. By construction, $L_f$ lifts to the trivial bundle over $I_E M$. This space retracts onto the constant paths, identified with $M \rightarrow \text{Diag} \hookrightarrow M^2$ embedding as the diagonal. It follows that $L_f$ is trivial over Diag and hence the Chern class lies in the subgroup \((2.3)\).

If two functions are homotopic through fusion functions then the corresponding bundles are isomorphic and hence the map to Chern classes descends to the group of components. Furthermore the product of functions, which descends to a product on path components, maps to the tensor product of bundles so the map is actually a homomorphism of groups.

To see that this map is surjective, observe that a circle bundle over $M^2$ with class in \((2.3)\) is trivial when lifted to $I_E M$ by the end-point fibration since it is isomorphic to its pull-back from the diagonal as constant paths. Choosing a global section $g$
identifies it with $\mathcal{I}_E M \times U(1)$ and taking the difference $g(\gamma_1) = f(\gamma_1, \gamma_2)g(\gamma_2)$ between the values of this section at pairs $(\gamma_1, \gamma_2) \in \mathcal{I}_E^2 M \cong \mathcal{L}_E M$ defines a continuous function (2.1) satsifying the fusion condition, and which generates the original circle bundle.

To see injectivity, suppose $f$ generates a trivial bundle $L_f \to M^2$. Since $L_f = \mathcal{I}_E M \times U(1)/\sim_f$, The pull-back to $\mathcal{I}_E M$ of a global section of the latter may be regarded as a continuous function $g : \mathcal{I}_E M \to U(1)$ which satisfies

\begin{equation}
(2.4) \quad f(\gamma_1, \gamma_2) = g(\gamma_1)g^{-1}(\gamma_2).
\end{equation}

Pulling $g$ back under the retraction of $\mathcal{I}_E M$ to $M$ gives an homotopy $f_t$ through fusion functions with end-point the constant function 1. □

We proceed to identify a group of functions on $\mathcal{L}M$ which are classified up to homotopy by the degree 2 cohomology of $M$. For this we require a second multiplicative condition.

Recall that the join operation on paths is a map

\begin{equation}
(2.5) \quad j : \pi_{12}^*\mathcal{I}_E M \times M^3 \pi_{23}^*\mathcal{I}_E M \to \pi_{13}^*\mathcal{I}_E M
\end{equation}

where $\pi_{ij} : M^3 \to M^2$ are the projections. Pulling back the fiber products $\mathcal{I}_E^2 M \to M^2$, the join defines a map

\begin{equation}
(2.6) \quad j^2 : \pi_{12}^*\mathcal{I}_E^2 M \times M^3 \pi_{23}^*\mathcal{I}_E^2 M \to \pi_{13}^*\mathcal{I}_E^2 M
\end{equation}

which, along with the fusion map $\psi : \mathcal{I}_E^2 M \to \mathcal{L}_E M$ generates the figure-of-eight product on loops:

\begin{equation}
J : \mathcal{L}_E M \times_{ev(0)=\ev(\pi)} \mathcal{L}_E M \to \mathcal{L}_E M
\end{equation}

\begin{equation}
(l_1, l_2) = (\psi(\gamma_1, \gamma_2), \psi(\gamma_1', \gamma_2')) \mapsto \psi(j(\gamma_1, \gamma_1'), j(\gamma_2, \gamma_2'))
\end{equation}

see Figure \ref{fig:figure-of-eight}.

Definition 2.2. A fusion-figure-of-eight function is a continuous map (2.1) which satisfies the fusion condition (2.2) and is multiplicative under the figure-of-eight product:

\begin{equation}
(2.7) \quad \Pi_1^* f \Pi_2^* f = J^* f \text{ on } \mathcal{L}_E M \times_{ev(0)=\ev(\pi)} \mathcal{L}_E M.
\end{equation}

The group of such functions is denoted $C_{foe} (\mathcal{L}M; U(1)) \subset C_{fsn} (\mathcal{L}M; U(1))$.

We define the fusion 1-cohomology of $\mathcal{L}M$ to be

\begin{equation}
(2.8) \quad H^1_{fus} (\mathcal{L}M) = C_{foe} (\mathcal{L}M; U(1))/\text{Fusion homotopy}.
\end{equation}
Equivalence here is homotopy in $C_{\text{fun}}(\mathcal{LM}; U(1))$; we show below that this is equivalent to homotopy in $C_{\text{fus}}(\mathcal{LM}; U(1))$. There is a natural map of abelian groups obtained by ignoring the fusion conditions

$$H^1_{\text{fun}}(\mathcal{LM}) \to H^1(\mathcal{LM}; \mathbb{Z}),$$

which is partial justification for our choice of degree in the notation. It is shown in Theorem 2.11 that fusion 1-cohomology in this sense does realize the idea of Stolz and Teichner that the fusion condition ‘promotes’ transgression to an isomorphism. Proposition 2.11 now leads to the regression map from fusive 1-cohomology to 2-cohomology on $M$.

**Proposition 2.3.** Associated to each element of $C_{\text{fus}}(\mathcal{LM}; U(1))$ is a circle bundle over $M^2$ of the form $L^{-1} \boxtimes L$ for $L \to M$, the Chern class of which induces an injective homomorphism

$$Rg : H^1_{\text{fun}}(\mathcal{LM}) \to H^2(M; \mathbb{Z}).$$

This is shown to be an isomorphism in Theorem 2.11 below.

**Proof.** Consider the circle bundle $L_f$ constructed in the proof of Proposition 2.11 from a function in $C_{\text{fus}}(\mathcal{LM}; U(1))$. The second multiplicativity condition (2.7) gives an isomorphism

$$\theta = \theta_f : \pi^*_{12}L_f \otimes \pi^*_{23}L_f \to \pi^*_{13}L_f$$

over $M^3$.

so the Chern class $\alpha(f) \in H^2(M^2; \mathbb{Z})$ satisfies

$$\pi^*_{23}\alpha(f) - \pi^*_{13}\alpha(f) + \pi^*_{12}\alpha(f) = 0 \in H^2(M^3; \mathbb{Z}).$$

Choose a point $\bar{m} \in M$ and consider the embeddings

$$i_2 : M^2 \to \bar{m} \times M^2 \subset M^3$$

$$i_3 : M \to \bar{m} \times M \subset M^2.$$ 

Pulling back (2.12) to $M^2$ by $i_2^*$ and using the fact that $\pi_{12} \circ i_2 = i_1 \circ \pi_1$ and $\pi_{13} \circ i_2 = i_1 \circ \pi_2$ as maps $M^2 \to \bar{m} \times M \subset M^2$ while $\pi_{23} \circ i_2 = \text{Id}$, it follows that

$$\alpha(f) = \pi^*_{23}\beta(f) - \pi^*_{13}\beta(f), \quad \beta(f) = i_1^*\alpha(f) \in H^2(M; \mathbb{Z}).$$

Proposition 2.1 shows that the regression map is well-defined by

$$Rg(f) = \beta(f)$$

on $H^1_{\text{fun}}(\mathcal{LM})$ with the equivalence relation being homotopy in $C_{\text{fun}}(\mathcal{LM}; U(1))$. \hfill \square

Suppose $f \in C_{\text{fus}}(\mathcal{LM}; U(1))$ satisfies $Rg(f) = 0$, so that the associated bundle $L_f \cong L^{-1} \boxtimes L$ with $L$ trivial. Taking a global section of $L$ and pulling the induced global section of $L_f$ back to $\mathcal{I}_E M$ as in the proof of Proposition 2.1 shows that there is continuous function $e : \mathcal{I}_E M \to U(1)$ such that

$$f(\gamma_1, \gamma_2) = e(\gamma_1)^{-1}e(\gamma_2),$$

$$e(r\gamma) = e(\gamma)^{-1},$$

$$e(j(\gamma_1, \gamma_2)) = e(\gamma_1)e(\gamma_2)$$

where $r : \mathcal{I}_E M \to \mathcal{I}_E M$ denotes path reversal.

**Lemma 2.4.** If $f \in C_{\text{fus}}(\mathcal{LM}; U(1))$ is simplicially trivial in the sense that (2.14) holds for some $e \in C(\mathcal{I}_E M; U(1))$ then $f$ is homotopic to 1 in $C_{\text{fus}}(\mathcal{LM}; U(1))$. 


Proposition 2.3 and the existence of a homotopy in $C$ is deformed through odd functions to its value, necessarily 1, on constant paths. It can therefore be written uniquely in the form
\[
e = \exp(2\pi i \kappa),
\]
(2.15)
\[
\kappa \in C(I_E M; \mathbb{R}), \quad \kappa(r \gamma) = -\kappa(\gamma) \quad \forall \quad \gamma \in I_E M
\]
and $\kappa(j(\gamma_1, \gamma_2)) = \kappa(\gamma_1) + \kappa(\gamma_2) \quad \forall \quad (\gamma_1, \gamma_2) \in I_E M_{\text{ev}(0) = \text{ev}(2\pi)} I_E M$.

The second identity follows from the fact that at constant loops the quotient $e(j(\gamma_1, \gamma_2))/e(\gamma_1)e(\gamma_2)$ is equal to 1 so has a unique continuous logarithm vanishing there.

Now the homotopy $e_t = \exp(2\pi it \kappa)$ to 1 is through odd functions on $I_E M$ which satisfy the identities for $e$ in (2.14) and hence the homotopy $f_t(\gamma_1, \gamma_2) = e_t^{-1}(\gamma_1)e_t(\gamma_2)$ lies in $C_{\text{foo}}(\mathcal{L}E M; U(1))$. \hfill \Box

**Corollary 2.5.** Two elements of $C_{\text{foo}}(\mathcal{L}M; U(1))$ are homotopic if and only if they are homotopic in $C_{\text{foo}}^\infty(\mathcal{L}M; U(1))$.

**Proof.** Using multiplicativity, it suffices to consider homotopies of a point $f \in C_{\text{foo}}(\mathcal{L}M; U(1))$ to 1. Homotopy in $C_{\text{foo}}(\mathcal{L}M; U(1))$ implies that $Rg(f) = 0$ by Proposition 2.3 and the existence of a homotopy in $C_{\text{foo}}(\mathcal{L}M; U(1))$ follows from Lemma 2.4 and the preceding discussion. \hfill \Box

A two-sided inverse to (2.10) is given by holonomy of circle bundles over $M$ and this also gives much smoother representatives of fusive 1-cohomology. We define the corresponding ‘fusive’ functions by abstraction of the properties of holonomy.

**Definition 2.6.** The space $C^\infty_{\text{foo}}(\mathcal{L}M; U(1))$ of fusive functions consists of those maps (2.14) satisfying the following three conditions.

**FF.i)** $f$ is lithe in the sense of [A] so continuously differentiable on the energy space and infinitely differentiable on piecewise smooth and smooth loops with all derivatives in the class of Dirac sections.

**FF.ii)** The fusion identity (2.2) holds.

**FF.iii)** Under reparametrizations of loops (see [E])
\[
\gamma^* f = f^o(\gamma) \quad \forall \quad \gamma \in \text{Rp}(S)
\]
(2.16)
where $\text{Rp}(S)$ is the reparametrization semigroup and $o = \pm 1$ is orientation.

**Lemma 2.7.** The group $\text{Dff}(S)$ is dense in $\text{Rp}(S)$ in the Lipschitz topology and a lithe function on a $\text{Dff}(S)$-invariant open set is invariant if it is annihilated by a dense subset of the smooth vector fields on $S$ in the $L^\infty$ topology.

**Proof.** Making a reflection and rotation it suffices to consider the closure of the subgroup $\text{Dff}^+(0)(S)$ of oriented diffeomorphisms fixing 0. These may be identified with their derivatives, which form the subspace of $C^\infty(S)$ of positive functions with integral $2\pi$. The Lipschitz topology reduces to the $L^\infty$ topology on the derivative so the closure corresponds to non-negative elements of $L^\infty(S)$ of integral $2\pi$ and the smooth subset integrates to $\text{Rp}^+(0)(S)$.

If $v = v(\theta) d\theta$ is a smooth vector field on $S$ then the action of $\exp(itv)$ on lithe functions is
\[
\frac{d}{dt} |_{t=0} f(l \circ \exp(itv)) = \int_S \langle df(l)(s), \nu_I(s) \rangle ds
\]
(2.17)
where $\tau_l(s)$ is the tangent vector field to the loop $l$. The assumption that $f$ is lithe implies in particular that its derivative, interpreted in the sense of the $L^2$ pairing, is a piecewise smooth function on the circle. It follows that if (2.17) vanishes for a dense subset of $v \in L^\infty(\mathbb{S})$ then it vanishes for all $v \in C^\infty(\mathbb{S})$. Although diffeomorphisms close to the identity are not in general given by the exponentiation of a vector field they are given by the integration of curves in the tangent space and hence the vanishing of the differential evaluated on the Lie algebra implies the invariance of the function.

One particular example of a Lipschitz map of the circle of winding number 1 is

$$T(\theta) = \begin{cases} 2\theta & 0 \leq \theta \leq \pi \\ 0 & \pi \leq \theta \leq 2\pi \end{cases}$$

which has the property that the pull-back may be identified with fusion with a trivial path

$$(2.18) \quad T^* l = \psi(l, l(0))$$

where $l(0)$ is the constant path at the initial point and $l$ is interpreted as a path. Thus any fusive function satisfies the ‘trivial’ fusion condition that

$$(2.19) \quad f(\psi(l, l(0))) = f(l)$$

**Lemma 2.8.** The group of fusion figure-of-eight maps contains the fusive maps.

**Proof.** To check that (2.7) holds consider a figure-of-eight loop $l = J(l_1, l_2)$ constructed in (2.6). This is the rotation by $\pi/2$ of the fusion of the two loops

$$\tilde{l}_1 = T^* R(\pi)^* l_1 = \psi(R(\pi)^* l_1, l_1(\pi)), \quad \text{and}$$

$$\tilde{l}_2 = R(\pi)^* T^* l_2 = \psi(l_2(0), l_2).$$

$$l = J(l_1, l_2) = R(\pi/2)^* \psi(R(\pi)^* l_1, l_2).$$

It then follows from the fusion and rotation-invariance of $f$, along with (2.19) that

$$f(l) = f(\psi(R(\pi)^* l_1, l_2)) = f(\psi(R(\pi)^* l_1, l_1(\pi))) f(\psi(l_2(0), l_2)) = f(l_1) f(l_2).$$

**Lemma 2.9.** For elements of $C^\infty_{\text{fus}}(\mathcal{L}M; U(1))$, smooth homotopy in $C^\infty_{\text{fus}}(\mathcal{L}M; U(1))$ is equivalent to homotopy in $C^\infty_{\text{fus}}(\mathcal{S}; U(1))$.

**Proof.** It suffices to show that if $f \in C^\infty_{\text{fus}}(\mathcal{L}M; U(1))$ and 1 are homotopic in $C^\infty_{\text{fus}}(\mathcal{L}M; U(1))$ then they are homotopic in $C^\infty_{\text{fus}}(\mathcal{L}M; U(1))$. Thus there is a continuous map $F : [0, 1] \times \mathcal{E}M \to U(1)$ with $F(t) \in C^\infty_{\text{fus}}(\mathcal{L}M; U(1))$ for all $t$, $F(0) = 1$ and $F(1) = f$ and hence a continuous $G : [0, 1] \times \mathcal{E}M \to \mathbb{R}$ with $G(0) = 0$ and $F = \exp(2\pi i G)$. It follows that $G$ satisfies the additive form of the fusion condition

$$(2.20) \quad \pi_{12}^* G + \pi_{23}^* G = \pi_{13}^* G \quad \text{on} \quad [0, 1] \times T^3_{\mathcal{E}} M$$

since both sides restrict to 1 at $t = 0$ and exponentiate to the same continuous function. Now, the restriction $g = G(1)$, satisfies $\exp(2\pi i t g) = f$ and hence is locally determined up to a constant. It follows that it is lithe and reparameterization invariant. So $\exp(2\pi i t g)$ is a fusive homotopy of $f$ to 1.
If \( L \) is a smooth circle bundle with principal \( \text{U}(1) \)-connection bundle over \( M \), then the holonomy \( h(l) \) of a loop \( l \in \mathcal{L}_E M \) is defined in terms of the paths in the total space, \( \mathcal{T}_E L \), which are parallel. For each point \( \theta_0 \in S \) and each \( e \in L\theta(\theta_0) \) there is a unique such path \( I_{e,\theta} \) with initial point \( e \) which projects to \( l(\theta + \theta_0) \). Its end-point is \( h(l)e \) where \( h(l) \) is independent of the choice of \( \theta_0 \) or \( e \) and so defines the holonomy \( h : \mathcal{L}_E M \to \text{U}(1) \).

**Proposition 2.10.** The holonomy of a smooth circle bundle over \( M \) with a smooth connection is an element of \( C^\infty_{\text{fus}}(\mathcal{L}M; \text{U}(1)) \) and the induced map into \( H^1_{\text{fus}}(\mathcal{L}M) \) descends to a well-defined group homomorphism

\[
T_{\text{fus}} : H^2(M; \mathbb{Z}) \to H^1_{\text{fus}}(\mathcal{L}M).
\]

**Proof.** The holonomy of two isomorphic bundles with related connections is the same. The space of connections on a given circle bundle is affine and homotopic connections give smoothly homotopic holonomies, so the map \( f_{\text{fus}} \) is well-defined. The holonomy of a tensor product with tensor product connection is the product of the holonomies so \( T_{\text{fus}} \) is a group homomorphism. \( \square \)

**Theorem 2.11.** The ‘enhanced transgression map’ \( T_{\text{fus}} \) and the regression map, \( Rg \), are inverses of each other and give a commutative diagram with standard transgression and the forgetful map \( \text{Id} \)

\[
H^2(M; \mathbb{Z}) \xrightarrow{Rg} H^1_{\text{fus}}(\mathcal{L}M) \xrightarrow{T_{\text{fus}}} H^1(\mathcal{L}M; \mathbb{Z}).
\]

**Proof.** To prove that \( Rg \circ T_{\text{fus}} = \text{Id} \), consider a circle bundle \( L \) with connection over \( M \) having Chern class \( \beta \in H^2(M; \mathbb{Z}) \), along with its holonomy function \( h \) representing \( T_{\text{fus}}(\beta) \). We proceed to show that if \( L_h \) is the bundle over \( M^2 \) constructed from \( h \) then its Chern class is \( \pi_2 \beta - \pi_1 \beta \).

Recall that \( L_h \) is constructed as the quotient of the trivial bundle \( \mathcal{T}_E M \times \text{U}(1) \to \mathcal{T}_E M \) by the equivalence relation determined by \( h : \mathcal{T}_E M \to \text{U}(1) \) on the fibers over \( M^2 \). The original bundle \( L \) can be pulled back to the path space under the initial-point map giving the larger fiber space \( \mathcal{T}_E M \times_{\text{ev}(0)} L \to M^2 \), and then taking the quotient of the trivial bundle \( \mathcal{T}_E M \times_{\text{ev}(0)} L \times \text{U}(1) \) by the relation determined by the pull-back of \( h \) to \( (\mathcal{T}_E M \times_{\text{ev}(0)} L)^2 \equiv \mathcal{T}_E M \times_{\text{ev}(0)} L^2 \) defines the same bundle \( L_h \).

Now consider a second bundle on \( M^2 \) arising from the quotient of a trivial bundle, which is related to \( L \) itself. The circle bundle \( L^{-1} \boxtimes L \to M^2 \) has Chern class \( \pi_2 \beta - \pi_1 \beta \) and is defined by the quotient of the trivial bundle \( L \boxtimes L \times \text{U}(1) \to L \boxtimes L \) by the relation determined by

\[
\delta : (L \boxtimes L)^2 \to \text{U}(1),
\]

\[
(e_1 \otimes e_2, e'_1 \otimes e'_2) = (e_1 \otimes e_2, z_1 e_1 \otimes z_2 e_2) \mapsto z_1^{-1} z_2 \in \text{U}(1)
\]
on the fibers over \( M^2 \).

We exhibit a map of fiber spaces over \( M^2 \) with respect to which the two equivalence relations are related by pull-back. The map

\[
\sigma : \mathcal{T}_E M \times_{\text{ev}(0)} L \xrightarrow{\sigma} \mathcal{T}_E L \xrightarrow{\text{Id}} L \boxtimes L
\]
is defined by extending a pair \((\gamma, l)\) to the covariant constant section \(\tilde{\gamma} \in \mathcal{I}_EL\) over \(\gamma\) with initial point \(l\) and mapping it to the pair \((\tilde{\gamma}(0) = l, \tilde{\gamma}(2\pi))\). The pull-back of \([2.23]\) along the induced map

\[
\sigma^{[2]} : (\mathcal{I}_EM \times_{ev(0)} L)^{[2]} = \mathcal{I}_E^{[2]}M \times_{ev(0)} L^{[2]} \longrightarrow (L \boxtimes L)^{[2]}
\]

is well-defined and coincides with the pull-back of the holonomy \(h\) of \(L\). Thus the map \(\sigma \times \text{Id}\) of trivial bundles descends to a bundle isomorphism \(L_h \longrightarrow L^{-1} \boxtimes L\) over \(M^2\), and it follows that \(Rg\) and \(Tg_{\text{can}}\) are inverse isomorphisms.

Finally, it remains to show that the diagram \([2.22]\) commutes. The transgression map is derived from the push-forward map in cohomology. Namely for any \(k\), using the evaluation map

\[
ev : \mathbb{S} \times \mathcal{L}M \ni (\theta, u) \mapsto u(\theta) \in M,
\]

\[
\begin{align*}
H^k(M; \mathbb{Z}) & \xrightarrow{ev^*} H^k(\mathbb{S} \times \mathcal{L}M; \mathbb{Z}) \\
\xrightarrow{Tg} H^{k-1}(\mathcal{L}M; \mathbb{Z}) & \xleftarrow{\text{eval}} H^k(\mathcal{L}M; \mathbb{Z}) + \mathbb{Z} \otimes H^{k-1}(\mathcal{L}M; \mathbb{Z}).
\end{align*}
\]

(2.24)

If a circle bundle \(L\) represents a class in \(H^2(M; \mathbb{Z})\), then \(ev^* L\) represents the pulled back class in \(H^2(\mathbb{S} \times \mathcal{L}M; \mathbb{Z})\). Explicitly, \(ev^* L\) may be trivialized locally by sections \(s_i\) over sets of the form \(\mathbb{S} \times \Gamma_j\), where the contractible sets \(\Gamma_j = \Gamma(l_j, \epsilon)\) as in \([\mathbb{E}]\) form a countable good cover of \(\mathcal{L}M\), and then the Chern class of \(ev^* L\) is represented by the Čech cocycle

\[
\delta_{ij} : \mathbb{S} \times \Gamma_{ij} \longrightarrow U(1)
\]

\[
s_i = \delta_{ij} s_j, \text{ on } \mathbb{S} \times \Gamma_{ij}, \quad \Gamma_{ij} = \Gamma_i \cap \Gamma_j.
\]

The push-forward of this class is represented by the winding number cocycle \(w_{ij} : \Gamma_{ij} \longrightarrow \mathbb{Z}\) of \(\delta_{ij}(\theta, l)\delta_{ij}^{-1}(0, l)\) and for later use we prove this more generally.

**Lemma 2.12.** If \(X\) is a paracompact manifold and \(\{\Gamma_i\}\) is a good open cover of it then the \(U(1)\)-Čech cohomology of the cover \(\{\mathbb{S} \times \Gamma_i\}\) of \(\mathbb{S} \times X\) is the cohomology \(H^*(X; \mathbb{Z}) \oplus H^{*-1}(X; \mathbb{Z})\) of \(\mathbb{S} \times X\) with the push-forward to \(H^{*-1}(X; \mathbb{Z})\) represented by the winding number of a Čech cocycle

\[
C^\infty(\mathbb{S} \times \Gamma_*; U(1)) \ni u_* \longmapsto \alpha_*(2\pi, x) \in C^\infty(\Gamma_*; \mathbb{Z})
\]

\[
(2.25)
\]

\[
u_* (\theta, x) u_*(0, x)^{-1} = \exp \left(2\pi i \alpha_*(\theta, x)\right) \text{ on } [0, 2\pi] \times \Gamma_*, \quad \alpha_*(0, x) = 0.
\]

**Proof.** The cohomology of the complex maps to the cohomology of \(\mathbb{S} \times X\) through any good refinement, such as the products \(U_a \times \Gamma_i\) for the cover of \(\mathbb{S}\) by the three open intervals \((-\epsilon, 2\pi/3 + \epsilon), (2\pi/3 - \epsilon, 4\pi/3 + \epsilon)\) and \((4\pi/3 - \epsilon, 2\pi + \epsilon)\). The map to \(H^*(X; \mathbb{Z})\) is then given by restriction to \(\{0\} \times \Gamma_*\). This is surjective since a cocycle can be lifted to be constant on \(\mathbb{S}\). The null space of this restriction map consists of the cocycles which are equal to 1 at \(\{0\} \times \Gamma_*\). These have preferred normalized logarithms on the \(U_a \times \Gamma_*\) starting at 0 and the boundary, taken just with respect to the \(\mathbb{S}\) factor, of the resulting real class is the integral winding-number Čech cocycle concentrated in \(U_3 \times \Gamma_*\) and hence projecting to \(H^{*-1}(X; \mathbb{Z})\). This map is also surjective since an integral cocycle \(k_*\) can be lifted to the \(U(1)\) cocycle \(\exp(2\pi ik_*)\).

The null space of the combined map to \(H^*(X; \mathbb{Z}) \oplus H^{*-1}(X; \mathbb{Z})\) consists of the cocycles with global constant logarithms which are therefore exact. \(\square\)
Returning to the proof of Theorem 2.11, the trivializing sections $s_j$ may be compared to the parallel lifts $\tilde{l}$ (as paths) of the loops $l$ with respect to a connection on $L$:

$s_j(\theta, l) = h_j(\theta, l)\tilde{l}_{\gamma_j(0, l)}(\theta) \in L_{l(\theta)}$, $h_j : [0, 2\pi] \times \Gamma_j \rightarrow U(1)$.

Here $h_j(2\pi, l)^{-1}$ is the holonomy of $l$, and since $h_j(0, l) \equiv 1$ there are normalized logarithms $\eta_j : [0, 2\pi] \rightarrow \mathbb{R}$, $\eta_j(0) = 0$ such that $h_j(\theta, l) = \exp(2\pi i\eta_j(\theta, l))$. It follows that the winding number cocycle of

$\delta_{ij}(\theta, l)\delta_{ij}^{-1}(0, l) = \exp(2\pi i(\eta_i(\theta, l) - \eta_j(\theta, l)))$

is given by $w_{ij}(l) = \eta_i(2\pi, l) - \eta_j(2\pi, l)$, which is precisely the $\hat{C}^1(\mathcal{L}M; \mathbb{Z})$ lift of the inverse holonomy $h_{\ast}(2\pi, \cdot)^{-1} \in \hat{C}^0(\mathcal{L}M; U(1))$. □

Note that there are several accounts in the literature of the recovery of an abelian principal bundle with connection on $M$ from its holonomy function, see Teleman, [15], Barrett, [2], and Waldorf [17].

3. Fusive circle bundles

We proceed to the analog of (2.22) in the next topological degree. Using Čech arguments this can be extended to all degrees, but here we are interested in geometric realizations of the fusion classes through circle (or equivalently Hermitian line) bundles. We proceed very much as in the previous section, first examining the fusion condition alone in the topological setting, then adding the figure-of-eight condition to define the notion of fusive 2-cohomology of the loop space in terms of topological circle bundles satisfying these conditions. The regression map is defined through bundle gerbes in the sense of Murray and the inverse, enhanced transgression map is defined via holonomy from principal PU-bundles, again giving smooth and reparametrization invariant representatives.

The notion of fusion for a circle bundle is due to Waldorf, [18, 19]. He obtains an equivalence of categories between bundle gerbes on $M$ and fusion principal bundles on the loop space which are equivariant with respect to thin homotopies via a general transgression functor. We consider bundles with the formal properties of the holonomy bundles of PU bundles to find appropriately regular representatives.

**Definition 3.1.** A fusion circle bundle over $\mathcal{L}M$ is a (locally trivial) topological circle bundle $D \rightarrow \mathcal{L}E_M$ with a continuous fusion isomorphism

$$\Phi : \psi_{12}^* D \otimes \psi_{23}^* D \rightarrow \pi_{13}^* D$$

over $\mathcal{I}^{[3]}_E M$ such that the diagram of circle bundle isomorphisms over $\mathcal{I}^{[4]}_E M$

$$\psi_{12}^* D \otimes \psi_{24}^* D \otimes \psi_{34}^* D \xrightarrow{\Phi \otimes \text{Id}} \psi_{13}^* D \otimes \psi_{34}^* D$$

commutes. A continuous isomorphism between such bundles is fusion if it lifts to intertwine the fusion maps.

An automorphism of a circle bundle is a map into $U(1)$ and such an automorphism of a fusion circle bundle is fusion if and only if this function is a fusion
function. Thus the homotopy classes of fusion automorphisms of any fusion bundle may be identified with the group $\mathbb{Z}^3$. If $J$ is a circle bundle over the path space $\mathcal{I}E^2M$ then $D = \pi_1^*J \otimes \pi_2^*J^{-1}$ over $\mathcal{I}E^2M = \mathcal{L}E^2M$ has the ‘product’ fusion structure given by the pairing

$$\pi_1^*D \otimes \pi_3^*D = \pi_1^*J \otimes \pi_2^*J^{-1} \otimes \pi_3^*J \otimes \pi_3^*J^{-1} \rightarrow \pi_1^*D$$

over $\mathcal{I}E^3M$.

**Proposition 3.2.** Each fusion circle bundle over $\mathcal{L}E^2M$ defines a (topological) bundle gerbe over $M^2$ and is ‘fusion trivial’ i.e. is isomorphic to a trivial bundle with trivial fusion isomorphism, if and only if the bundle gerbe is trivial.

*Proof.* Given a fusion circle bundle $D$, consider the diagram formed from the free path space fibration and fusion map:

$$\begin{array}{ccc}
\mathcal{I}E^2M & \xrightarrow{\pi_1} & \mathcal{I}E^2M \\
\downarrow{\pi_2} & & \downarrow{\pi_1} \\
\mathcal{L}E M & \xrightarrow{\psi} & \mathcal{L}E M \\
\downarrow{\psi} & & \downarrow{\psi} \\
M^2 & & M^2.
\end{array}$$

The fusion conditions (3.3) and (3.2) are equivalent to the condition that $\psi^*D$ defines a bundle gerbe, and this gerbe is (simplicially) trivial if and only if there is a circle bundle $J$ over $\mathcal{I}E^2M$ and an isomorphism

$$D \rightarrow \pi_1^*J \otimes \pi_2^*J^{-1}$$

which induces the fusion isomorphism $\Phi$. Since $\mathcal{I}E^2M$ retracts onto the constant loops $M$ at the initial point, $J$ is isomorphic to the pull-back under $\pi_1$ of a bundle $\tilde{J}$ on $M$. Tensoring $J$ with the pull-back of $\tilde{J}^{-1}$ does not change the isomorphism (3.4) since it cancels in the tensor product. Thus we may assume that $J$ is trivial, and then (3.4) is a fusion isomorphism to a trivial bundle.

Conversely, if $D$ is fusion trivial then $\psi^*D$ is trivial and this implies triviality of the bundle gerbe.

Murray has shown that bundle gerbes up to simplicial triviality are classified by the Dixmier-Douady class. For later reference, we briefly recall how the Dixmier-Douady class is defined, and how the simplicial trivialization is constructed, in the case of the bundle gerbe above.

A cover $\{B_i\}$ of $M$ by small geodesic balls leads to a good open cover $\{B_i \times B_j\}$ of $M^2$. Smooth local sections

$$s_{i,j}: B_i \times B_j \rightarrow \mathcal{T}M \subset \mathcal{I}E^2M$$

can be chosen and then

$$D_{ik,jl} = (s_{i,j}, s_{k,l})^*\psi^*D \rightarrow B_{ik} \times B_{j,l} = (B_i \cap B_j) \times (B_j \cap B_l)$$

are circle bundles over the double intersections. Using the fusion isomorphism, sections $\lambda_{ik,jl}: B_{ik} \times B_{j,l} \rightarrow D_{ik,jl}$ of these may be compared over triple intersections:

$$\Phi(\lambda_{ik,jl}, \lambda_{km,jn}) = \sigma_{ikm,jln} \lambda_{km,jn}, \text{ on } B_{ikm} \times B_{jln}$$

$$\sigma_{ikm,jln}: B_{ikm} \times B_{jln} \rightarrow U(1)$$
resulting in a Čech cocycle $\sigma$. This represents the Dixmier-Douady class of $D$ in $H^3(M^2;\mathbb{Z})$ where the integral 3-cocycle is formed by the Čech boundary of logarithms of the $\sigma_{km,ln}$.

Since the $B_i$ form a good open cover, triviality of this class means that $\sigma$ is a boundary and hence that the sections $\lambda_{ik,jl}$ can be chosen in such a way that (3.7) holds with $\sigma \equiv 1$. Then the bundle $J$ may be defined by gluing the bundles

$$J_{i,j} \to \mathcal{E}_M|_{B_i \times B_j}; \quad (J_{i,j})_\gamma = D_{\psi(\gamma,s_{i,j}(\text{ev}(\gamma)))}$$

using the compatible gluing isomorphisms formed by the $\lambda_{ik,jl}$. This gives the global bundle $J$ over $\mathcal{E}_M$ which trivializes $D$ simplicially.

**Proposition 3.3.** The Dixmier-Douady class of the bundle gerbe (3.3) defined by a fusion circle bundle induces an isomorphism

$$\text{DD} : \{\text{Fusion circle bundles}\}/\text{Fusion isomorphism} \to \{\delta \in H^3(M^2;\mathbb{Z}); \delta|_{\text{Diag}} = 0\}.$$

**Proof.** The Dixmier-Douady class, characterizing the bundle gerbe up to simplicial triviality, lies in $H^3(M^2;\mathbb{Z})$ and behaves naturally under restriction. Over the diagonal, the bundle gerbe has a trivial subgerbe; indeed, the inclusion map defines a bundle gerbe morphism from $D$ restricted to constant paths $M \subset \mathcal{E}_M$ as a trivial fibration over $M = \text{Diag}$. Thus the map has range in the space indicated in (3.9). The injectivity of this map has already been established by Proposition 3.2 since the vanishing of the Dixmier-Douady class implies the simplicial triviality of the bundle gerbe and hence the existence of a fusion isomorphism to a trivial bundle.

The surjectivity of DD is established below in the discussion of the holonomy of principal PU bundles. More directly, it suffices to note that each class in $H^3(M^2;\mathbb{Z})$ is represented by a (topological) principal PU bundle over $M^2$ and if the class lies in the range space in (3.9) then the bundle has a section over the diagonal and the pull-back over $\mathcal{E}_M$ has a global section since the latter retracts to $M = \text{Diag}$ — this is constructed explicitly in the smooth case by parallel transport below. The two values of the section at a point of $\mathcal{E}_M$ are related by an element of $\text{PU}$ mapping the second to the first since they are in the same fiber of the original bundle, and this defines a continuous (essentially holonomy) map

$$h : \mathcal{L}_E M \to \text{PU}$$

which is multiplicative under fusion

$$\pi^*_1 h \cdot \pi^*_2 h = \pi^*_3 h \text{ on } \mathcal{L}^{[3]}_E M.$$

The pull-back of the canonical bundle $U/\text{PU}$ by $h$ is a circle bundle over $\mathcal{L}_E M$ with fusion isomorphism given by the product identification of the canonical bundle, i.e. the product in $U$ and compatibility condition over $\mathcal{L}^{[3]}_E M$. Moreover, the bundle gerbe defined by this bundle is a subgerbe of the inverse of the gerbe with total space the pull-back of the original PU bundle to $\mathcal{L}_E M$. Thus it represents the inverse of the original class and it follows that (3.9) is surjective and hence is an isomorphism.

Proceeding as in §2 to refine (3.9) to a map into the 3-cohomology of $M$ we add the analog of the condition in Definition 2.2 involving the figure-of-eight product (2.6).
Pulling this equation back to (3.12) isomorphism (3.1), (3.2) is an isomorphism of eight structures. Here can be strengthened to isomorphisms intertwining both the fusion and figure-of-eight structures.

In contrast to the case of functions, it is not clear that the equivalence relation which is compatible with the fusion isomorphism in the sense that

\[ \alpha \text{ the join (2.5); thus the Dixmier-Douady class } \]

\[ \left( \gamma_1, \gamma_2, \gamma_3 \right) \in \pi \times \pi \times \pi \times \pi s.t. \gamma_i(2\pi) = \gamma_i(0) \rightarrow \]

\[ D_{\psi}(j(\gamma_1',\gamma_2')) \otimes D_{\psi}(j(\gamma_2,\gamma_3')) \rightarrow D_{\psi}(j(\gamma_1,\gamma_2,\gamma_3')) \]

\[ \gamma \rightarrow D_{\psi}(\gamma_1,\gamma_2,\gamma_3) \otimes D_{\psi}(\gamma_1',\gamma_2',\gamma_3') \]

commutes (see Figure 3); the bundle is then said to be a fusion-figure-of-eight circle bundle.

Again by analogy with the standard case we define fusion 2-cohomology as

\[ H^2_{\text{fun}}(L^2) = \{ \text{Fusion-figure-of-eight circle bundles} \} / \text{Fusion isomorphisms}. \]

In contrast to the case of functions, it is not clear that the equivalence relation here can be strengthened to isomorphisms intertwining both the fusion and figure-of-eight structures.

**Proposition 3.5.** The restriction to fusion-figure-of-eight circle bundles of the map (3.9) takes values in \( \{ \pi^* \beta - \pi^* \beta \in H^3(M^2; \mathbb{Z}); \beta \in H^3(M; \mathbb{Z}) \} \) and the resulting regression map

\[ \text{Rg}: H^2_{\text{fun}}(L^2) \rightarrow H^3(M; \mathbb{Z}), \ Rg(D) = \beta \]

is injective.

It will be proved below that Rg is an isomorphism.

**Proof.** It is only necessary to see that Rg is well-defined, its injectivity then follows from Proposition 3.3.

Let \( D \rightarrow \mathcal{L}_E M \) be a fusion-figure-of-eight circle bundle, and \( \mathcal{G} = (D, \mathcal{L}_E M, M^2) \) its associated bundle gerbe. The figure-of-eight condition implies that there is a gerbe morphism \( \pi_{12} \mathcal{G} \otimes \pi_{23} \mathcal{G} \rightarrow \pi_{13} \mathcal{G} \) over \( M^3 \) where the map on fiber spaces is the join (2.5), thus the Dixmier-Douady class \( \alpha \in H^3(M^2; \mathbb{Z}) \) of \( \mathcal{G} \) satisfies

\[ \pi^* \mathcal{G} \alpha = \pi^* \mathcal{G} \alpha + \pi^* \mathcal{G} \alpha = 0 \in H^3(M^3; \mathbb{Z}). \]

Pulling this equation back to \( H^3(M^2; \mathbb{Z}) \) by the embedding \( i_2 : M^2 \hookrightarrow \mathcal{M} \times M^2 \subset M^3 \) as in the proof of Proposition 2.3 shows that \( \alpha = \pi^* \beta - \pi^* \beta, \) where \( \beta = \)
\(i^* \alpha \in H^3(M; \mathbb{Z})\) is the pull-back of \(\alpha\) to \(H^3(M; \mathbb{Z})\) along the embedding \(i_1 : M \hookrightarrow \bar{m} \times M \subset M^2\).

To complete the proof that the regression map is an isomorphism and find smooth representatives of these fusion classes we consider much more restrictive properties obtained by abstraction from those of the holonomy bundles of principal PU bundles, as shown subsequently in Proposition 3.8.

**Definition 3.6.** A circle bundle \(D\) over \(\mathcal{L}M\) is *fusive* if it satisfies the following four conditions.

- **FB.i)** \(D\) is *lithe* in the sense that it has trivializations as a principal \(U(1)\) bundle over the open sets \(\Gamma(l, \epsilon)\) as in (B.2) for some \(\epsilon > 0\) (3.16)
  
  \[T_l : D \rightarrow \mathbb{R}^p \cdot \Gamma_E(l, \epsilon) \times U(1)\]

  and the transition maps

  \[T_{l'} = T_l \cdot T_{l'}^{-1} : \mathbb{R}^p \cdot \Gamma_E(l, \epsilon) \cap \mathbb{R}^p \cdot \Gamma_E(l', \epsilon) \rightarrow U(1)\]

  are lithe in the sense of [A].

- **FB.ii)** There is a fusion isomorphism for \(D\), a lithe bundle isomorphism

  \[\Phi : \psi_{12}^* D \otimes \psi_{23}^* D \rightarrow \psi_{13}^* D\]

  over \(\mathcal{I}[^3] M\), \(\psi_{ij} = \pi_{ij} \circ \psi\)

  which is associative in the sense that the two iterated maps over \(\mathcal{I}[^4] M\) are equal:

  \[\Phi(\Phi(\cdot, \cdot)) = \Phi(\cdot, \Phi)\].

- **FB.iii)** \(D\) is strongly parameter-independent, in the sense that there is a lithe \(U(1)\) bundle isomorphism

  \[A : \mathbb{R}^s D \rightarrow \pi_2^* D \times \mathbb{R}^1 \times \mathbb{R}^1\]

  over \(\mathbb{R}^p \times \mathcal{L}M\), \(R(f, l) = l \circ f\)

  and this isomorphism is consistent with some choice of trivializations (3.17) in the sense that \(A\) becomes the identity transformation on the trivial bundle, near the identity in \(\text{Diff}^+(\mathbb{S})\).

- **FB.iv)** A consistency condition holds between the fusion and the reparameterization isomorphisms. Namely if \(r_i \in \mathbb{R}^p \cdot \{0, \pi\}\) are, for \(i = 1, 2, 3\), reparametrizations of the interval and \(\chi_{ij}\) are their fusions to elements of \(\mathbb{R}^p \cdot \{0, \pi\}\) (fixing the points 0 and \(\pi\)) then for the fusion of three paths \((\gamma_1, \gamma_2, \gamma_3) \in \mathcal{I}[^3] M\) to loops \(l_{ij}\), the diagram

  \[D_{l_{12} \circ \chi_{12}} \otimes D_{l_{23} \circ \chi_{23}} \xrightarrow{\Phi} \mathbb{D}_{l_{13} \circ \chi_{13}} \xrightarrow{A} \mathbb{D}_{l_{13}} \xrightarrow{A} \mathbb{D}_{l_{13}} \xrightarrow{\Phi} \mathbb{D}_{l_{13}} \]

  commutes.

In brief then, a fusive circle bundle over \(\mathcal{L}M\) is a lithe circle bundle with locally trivial lithe reparameterization and fusion isomorphisms and with the consistency condition holding between reparameterization and fusion.

The tensor product of two fusive circle bundles is again fusive, as is the inverse of a fusive bundle.

We distinguish between two notions of isomorphism between such circle bundles, namely *lithe-fusion* and *fusive* isomorphisms. For the former, which we use initially,
there is merely required to be a lithe isomorphism between the two bundles which intertwines the fusion isomorphisms (3.18). For a fuse isomorphism we require in addition that the isomorphism intertwines the reparameterization isomorphisms. It is shown below that these induce the same equivalence relation on fuse circle bundles.

We believe, but do not show here, that the collection of lithe circle bundles – just satisfying FB.1 above – modulo lithe isomorphisms, may be identified with $H^2(\mathcal{L}M; \mathbb{Z})$ just as in the finite-dimensional case.

Recall that the figure-of-eight product on loops can be written in terms of fusion and reparametrization. The same considerations as in the proof of Lemma 2.8 show

**Lemma 3.7.** Fusive circle bundles are fusion-figure-of-eight and hence there is a natural map

$$\{\text{Fusive circle bundles}\}/\text{Lithe-fusion isomorphisms} \rightarrow H^2_{\text{fus}}(\mathcal{L}M).$$

It is shown below that (3.21) is an isomorphism.

**Proposition 3.8.** The holonomy of a smooth principal $PU$ bundle with connection over $M$ defines a fusive circle bundle $D$ over $\mathcal{L}M$ through the pull-back of the canonical circle bundle over $PU$. This circle bundle is independent of choices up to fuse isomorphism, and induces the enhanced transgression map

$$T_{\text{fus}} : H^3(M; \mathbb{Z}) \rightarrow H^2_{\text{fus}}(\mathcal{L}M).$$

**Proof.** Let $P$ be a smooth principal $PU$ bundle over $M$. Smoothness here means that the bundle has local trivializations with transition maps smooth as maps into the Banach manifold $PU$. A connection on $P$ can then be constructed by summing the local Maurer-Cartan forms over a partition of unity relative to an open cover of $M$ by such trivializations.

Any smooth path $\gamma$ in $\mathcal{L}M$ can be lifted by parallel transport to a unique smooth path $\bar{\gamma}_p \in \mathcal{P}$ covering $\gamma$ and having a given initial point $p \in P_{\gamma(0)}$. For a loop $l \in \mathcal{L}M$ the difference at the endpoints of the lift $\bar{l}_p \in \mathcal{P}$ defines the holonomy

$$H : P \times_{\text{ev}(0)} \mathcal{L}E_{\mathcal{M}} \rightarrow PU$$

by continuous extension to the energy space.

In contrast to the case of abelian structure groups, the holonomy of a $PU$ bundle is not independent of the initial point $p$, however (3.23) is equivariant relative to the principal action on $P$ and the adjoint action on $PU$ from the fiber equivariance of the connection on a principal bundle. The canonical circle bundle of the unitary group over the projective unitary group for a separable infinite-dimensional Hilbert space, $U \rightarrow PU$, is also equivariant for the conjugation action on $PU$ since this action extends uniquely to the conjugation action on $U$. Thus, pulling back $U$ under $\text{ev}(0)$ gives an equivariant circle bundle and quotient

$$\tilde{D} = H^*U \rightarrow P \times_{\text{ev}(0)} \mathcal{L}E_{\mathcal{M}};$$

by the free $PU$ action on $P \times_{\text{ev}(0)} \mathcal{L}E_{\mathcal{M}}$. We call $D$ (or its restriction to $\mathcal{L}M$) the *holonomy bundle* associated to $P$ and ultimately to its Dixmier-Douady class. We proceed to show that $D$ has the properties required of a fuse circle bundle over $\mathcal{L}M$ and then discuss the dependence on choices.
Lithe regularity of $D$ over $\mathcal{L}M$ is discussed in [A]. Since the reparameterization of a covariant-constant path in $P$ is covariant-constant, the restricted reparameterization semigroup acts trivially on $H$ and this defines an action on $D$:

$$H(l \circ r, p) = H(l, p), \quad r \in R^+_p(\mathcal{S})$$

(3.25)

$$A : R^*D \rightarrow D, \quad R \in R^+_p(\mathcal{S}).$$

Here $R^+_p(\mathcal{S})$ denotes the reparametrizations which fix the initial point $0 \in \mathcal{S}$. For rotations, observe that parallel translation along the path defines an action of the universal cover, $\mathbb{R}$, on $P \times_{ev(0)} \mathcal{L}E: M$ which covers the $U(1)$ rotation action on $\mathcal{L}E: M$. The pull-back of $H$ by this action is invariant up to conjugacy in PU so (3.25) may be extended to an action of the full reparameterization semigroup.

Now consider an open set $\Gamma_E(l, \epsilon)$ consisting of the loops which are $\epsilon$-close in the sense of energy to the loop $l$. The initial point map has image in a small geodesic ball, $ev(0) : \Gamma_E(\epsilon, l) \rightarrow B \subset M$. If $p : B \rightarrow P$ is a section of $P$ over it, then the image of $H$ restricted to $\Gamma_E(\epsilon, l)$ computed at $p \circ ev(0)$ lies in a ball in PU which is small in norm with $\epsilon$. Thus, if $\epsilon$ is small enough, $U/PU$ is trivial over the range and hence so is $D$. The reparameterization isomorphism is the identity in this trivialization.

The properties of parallel transport show that if three loops, $l_{12}$, $l_{23}$ and $l_{13}$ are related by fusion then their holonomies are related by multiplication:

$$H(l_{13}, p) = H(l_{23}, p)H(l_{12}, p), \quad p \in P_m$$

(3.26)

with $m = l_{ij}(0)$ the common initial point. The classifying bundle $U$ pulls back to be equivariantly isomorphic to the product of the classifying bundles under multiplication $PU \times PU \rightarrow PU$, and it follows that $D$ has a fusion isomorphism, the associativity of which follows from associativity of multiplication in PU. Liteness follows from the construction as does the compatibility with restricted reparameterization.

Finally consider the effect on $D$ of altering the choices made. Two PU bundles with the same Dixmier-Douady class are smoothly isomorphic, replacing $P$ by another but transferring the chosen connection leaves the holonomy, and hence $D$ unchanged. Changing the connection on $P$ adds the pull-back from $M$ of a smooth 1-form $\beta : M \rightarrow g(PU) \otimes T^*M$ with values in the Lie algebra of PU; these form an affine space. Such a 1-form can be lifted locally, and hence globally, to a 1-form $\tilde{\beta}$ with values in the Lie algebra of $U$. The holonomy around a curve is shifted by the multiplication by the solution of the exponential equation for $\beta$ along the curve and hence is homotopic to the identity. Integrating $\tilde{\beta}$ instead lifts this factor into $U$ and gives an isomorphism of the pulled back circle bundles. This isomorphism intertwines the structure maps for parameter independence and fusion. □

When we consider fusive isomorphisms below we will make use of the following result.

**Proposition 3.9.** The holonomy bundle $D \rightarrow \mathcal{L}M$ of a PU-bundle on $M$ may be equipped with a lithe connection which is equivariant with respect to fusion and the action of the restricted reparameterization semigroup $R^+_{(0, \pi)}(\mathcal{S})$.

**Proof.** Since PU is a smooth paracompact Banach manifold the locally trivial circle bundle $U$ can be given a smooth connection although this will not be invariant under
the conjugation action of PU. Nevertheless, over each element of the cover of \( M \) by small geodesic balls, \( B_i \), there is a smooth trivializing section \( b_i : B_i \to P \), which gives a local slice for the PU action on \( P \times_{ev(0)} \mathcal{L}M \). Thus over

\[
\Omega_i = \{ t \in \mathcal{L}M; t(0) \in B_i \}, \quad D = H^*_i(U/PU), \quad H_i = H \circ b_i
\]

where \( b_i \) is lifted to a section of the pull-back bundle. In particular, the chosen connection on \( U/PU \) pulls back to give \( D \) a lithe connection, \( \nabla_i \), over each \( \Omega_i \).

Since the map \( H_i \) is restricted-reparameterization-invariant, meaning with respect to \( \text{RP}^{-1}_0(\Sigma) \), so are these connections. Over the intersection of two such open sets

\[
\Omega_{jk} = \Omega_j \cap \Omega_k, \quad \nabla_j - \nabla_k = u_{jk}
\]

the connections differ by a smooth, restricted-reparameterization-invariant, 1-form. Over triple intersections these 1-forms satisfy the cocycle condition. A partition of unity \( \rho_i \) on \( M \) subordinate to the open cover by the \( B_i \) pulls back under \( ev(0) \) to a partition of unity subordinate to the open cover \( \Omega_i \) of \( \mathcal{L}M \) and the 1-forms

\[
v_i = \sum_{j \neq i} \rho_j u_{ij} \text{ on } \Omega_i
\]

are lithe and restricted-reparameterization-invariant. The shifted local connections \( \nabla_i - v_i \) then patch to a global, restricted-reparameterization-invariant, connection on \( D \) over \( \mathcal{L}M \).

This connection need not respect the fusion structure, but we proceed to show that it can be modified to have this property while retaining invariance under the semigroup \( \text{RP}^{-1}_{[0, \pi]}(\Sigma) \) of restricted reparametrizations fixing 0 and \( \pi \).

To do so, consider the bundles

\[
J_{i,j} \to G_{ij} = \{ \gamma \in TM; ev(\gamma) \in B_i \times B_j \}
\]

defined by \( 3.38 \). The restricted-reparameterization-invariant connection on \( D \) induces a connection \( \nabla_{i,j} \) on \( J_{i,j} \). Recall that \( D_{ik,jl} \) defined in \( 3.30 \) functions as a transition bundle, pulled back to \( G_{ij} \cap G_{kl} \), with fusion defining an isomorphism

\[
\beta_{ij,kl} : J_{i,j} \cong J_{k,l} \otimes D_{ik,jl} \text{ over } G_{ij} \cap G_{kl}.
\]

The transition bundles themselves have a cocycle isomorphism

\[
D_{ik,jl} \otimes D_{km,ln} \otimes D_{mi,nj} \simeq U(1), \quad \text{over } B_{ikm} \times B_{jln}
\]

with the consistency condition of a Brylinski-Hitchin gerbe over four-fold intersections of the products \( B_i \times B_j \) in \( M^2 \). It follows that one can choose connections \( \nabla_{ik,jl} \) on the \( D_{ik,jl} \) such that on triple intersections of the sets in \( M^2 \) the tensor product connection is the trivial one, and likewise such that the tensor product is trivial with respect to the isomorphisms \( D_{ik,jl} \otimes D_{kl,ij} \simeq U(1) \) on \( B_{ij} \times B_{kl} \simeq B_{kl} \times B_{ij} \).

The connections on the \( J_{i,j} \) on overlaps \( G_{ij} \cap G_{kl} \) can be compared under the fusion isomorphisms \( 3.30 \) and it follows that

\[
\nabla_{ij} - \beta_{ij,kl}^* (\nabla_{ik,jl} \otimes \nabla_{kl}) = \sigma_{ik,jl}
\]

is a restricted-reparameterization-invariant 1-form on \( G_{ij} \cap G_{kl} \). Moreover, these 1-forms constitute a Čech cocycle on triple intersections and which, using the pull-back of a partition of unity on the \( B_i \times B_j \) in \( M^2 \), is the boundary of a Čech class of restricted-reparameterization-invariant 1-forms on the \( G_{ij} \). Thus in fact the connections on the \( J_{i,j} \) may be modified, remaining restricted-reparameterization-invariant (with respect to \( \text{RP}^{-1}_{[0, \pi]}(\Sigma) \)) so that they are identified with the tensor.
product connections under \((3.30)\). Recalling that \(D = \pi_1^4 J_{i j} \otimes \pi_2^2 J_{i j}^{-1} = J_{i j} \otimes J_{i j}^{-1}\) over loops which have initial point in \(B\), and lie in \(B_j\) at parameter value \(\pi\), this induces a global restricted reparameterization-invariant and fusion connection on \(D\), which is well-defined since the connections induced from \(J_{i j} \otimes J_{i j}^{-1}\) and \(J_{k l} \otimes J_{k l}^{-1}\) agree on overlaps.

**Theorem 3.10.** The map \((3.22)\) is an isomorphism and is the inverse to the regression map \((3.15)\) giving a commutative diagram

\[
\begin{array}{c}
H^3(M; \mathbb{Z}) \xrightarrow{\text{Rg}} H^2_{\text{fus}}(\mathcal{L}M) \\
\downarrow \quad \downarrow \\
H^1(\mathcal{L}M; \mathbb{Z}).
\end{array}
\]

**Proof.** Since the regression map \((3.15)\) has already been shown to be injective, it suffices to show that the fusion transgression map \((3.22)\) satisfies

\[
\text{Rg} \circ \text{Tg}_{\text{fus}} = \text{Id}
\]

to establish that these are isomorphisms inverse to each other.

Proceeding as in the proof of Theorem 2.11 consider a PU bundle \(P\) over \(M\) with Dixmier-Douady class \(\alpha \in H^3(M; \mathbb{Z})\), and its holonomy bundle \(D\). We exhibit an isomorphism of bundle gerbes over \(M^2\); the first is the pull-back of the gerbe \(D \to T[2]M\) to the fiber space \((\mathcal{L}M \times \text{ev}(0), P)[2]\), and the second is a variation on the lifting bundle gerbe of \(P\). More precisely, consider the fibration \((\text{not a PU-bundle})\)

\[
Q = \pi_1^4 P \times \pi_2^2 P \to M^2
\]

with difference map

\[
\delta : Q[2] \to \text{PU}
\]

\[
((p_1, p_2), (q_1, q_2)) = ((p_1, p_2), (a_1 p_1, a_2 p_2)) \mapsto a_1 a_2^{-1}, a_1, a_2 \in \text{PU}.
\]

The pull-back \(\delta^* (U/\text{PU}) \to Q[2]\) defines a U(1) bundle gerbe with Dixmier-Douady class \(\pi_1^4 \alpha - \pi_2^2 \alpha \in H^3(M^2; \mathbb{Z})\).

As in the proof of Theorem 2.11 this gerbe may be pulled back along the map

\[
\sigma[2] : T^2 M \times_{\text{ev}(0)} P[2] \equiv (\mathcal{L}M \times_{\text{ev}(0)} P)[2] \to T[2] P \to Q[2]
\]

of fiber bundles over \(M^2\) induced by the composition \(\sigma = (\text{ev}(0) \times \text{ev}(2\pi)) \circ \text{par}\) of parallel translation and evaluation. In fact the pull-back

\[
(\sigma[2])^* \delta : T^2 M \times_{\text{ev}(0)} P[2] \to \text{PU}
\]

coincides with the holonomy \(H\) evaluated on the first factor of \(P\), which leads to a bundle gerbe isomorphism between \((\pi_1^4 D, \mathcal{L}M \times_{\text{ev}(0)} P, M^2)\) and \((\delta^* U, \pi_1^4 P \times \pi_2^2 P, M^2)\), so these have the same Dixmier-Douady class \(\pi_1^4 \alpha - \pi_2^2 \alpha\) and \((3.34)\) follows.

To prove the commutativity of the diagram \((3.33)\) we apply Lemma 2.12 again. Here the pull-back \(\text{ev}^* P\) of a PU bundle \(P \to M\) to \(S \times \mathcal{L}M\) is trivialized over the cover \(S \times \Gamma_j\) by sections \(s_j : S \times \Gamma_j \to \text{ev}^* P\). The Dixmier-Douady class of \(\text{ev}^* P\) in \(H^3(S \times \mathcal{L}M; \mathbb{Z})\) is then represented as a U(1)-Čech 2-cocycle by

\[
u_{ijk} = \hat{t}_{ij} \hat{r}_{jk} \hat{t}_{ki} : S \times \Gamma_{ijk} \to \text{U}(1)
\]
where $\hat{\tau}_{ij} : S \times \Gamma_{ij} \to U(H)$ are arbitrary unitary lifts of the PU difference classes

\begin{equation}
\tau_{ij} : S \times \Gamma_{ij} \to PU,
\end{equation}

and transgression of $[P] \in H^3(M; \mathbb{Z})$ is represented in $H^2(\mathcal{L}M; \mathbb{Z})$ by the winding number cocycle $w_{ijk} : \Gamma_{ijk} \to \mathbb{Z}$ of $u_{ijk}(\theta, l)u_{ijk}^{-1}(0, l)$.

On the other hand, each section $s_j$ is related to the parallel lift of the initial point of the section via

\begin{equation}
s_j(\theta, l) = h_j(\theta, l)\tilde{s}_{ij}(0, l),
\end{equation}

where $h_j^{-1}(2\pi, l) \in PU$ is the holonomy of $l$ with initial point $s_j(0, l) \in P_{l(0)}$. The $h_j$ also admit unitary lifts $\hat{h}_j : [0, 2\pi] \times \Gamma_i \to U(H)$, and it follows from the construction of the holonomy bundle $D \to \mathcal{L}M$ that $l \mapsto \hat{h}_j^{-1}(2\pi, l)$ is a trivializing section of $D$ over $\Gamma_j$. Hence the Chern class of $D$ is represented by the $U(1)$-Čech 1-cocycle

\begin{equation}
\mu_{ij} : \Gamma_{ij} \to U(1),
\end{equation}

It follows from \text{(3.35)} and \text{(3.36)} that

\begin{equation}
h_{ij}(\theta, l)\tau_{ij}(0, l) = \tau_{ij}(\theta, l)h_j(\theta, l).
\end{equation}

Since $\tau_{ij}(0, l) = \tau_{ij}(2\pi, l)$ this expresses the fact that the holonomies at the different initial points are conjugate: $h_i(2\pi, l) = \tau_{ij}(0, l)h_j(2\pi, l)\tau_{ij}^{-1}(0, l)$. In any case, it follows from \text{(3.37)} that the unitary lifts of the $h_j$ and the $\tau_{ij}$ are related by

\begin{align*}
\sigma_{ij}(\theta, l)\hat{h}_i(\theta, l)\hat{\tau}_{ij}(0, l) &= \hat{\tau}_{ij}(\theta, l)\hat{h}_j(\theta, l),
\sigma_{ij} : [0, 2\pi] \times \Gamma_{ij} \to U(1),
\sigma_{ij}(0, l) &\equiv 1,
\sigma_{ij}(2\pi, l) = \hat{\tau}_{ij}(0, l)\hat{h}_i^{-1}(2\pi, l)\hat{\tau}_{ij}(0, l)\hat{h}_j(2\pi),
\end{align*}

\begin{equation}
u_{ijk}(\theta, l)u_{ijk}^{-1}(0, l) = \sigma_{ij}\sigma_{jk}\sigma_{ki}.
\end{equation}

Note that since $D$ is the quotient of an equivariant bundle by the adjoint action of PU, it follows that $\sigma_{ij}(2\pi) : \Gamma_{ij} \to U(1)$ represents the Chern class of $D$ as well as $\mu_{ij}$. There are normalized logarithms $\eta_{ij} : [0, 2\pi] \times \Gamma_{ij} \to \mathbb{R}$ such that $\sigma_{ij}(\theta, l) = \exp(2\pi i \eta_{ij}(\theta, l))$, and the winding number cocycle is given by

\begin{equation}
w_{ij}(l) = \eta_{ij}(2\pi, l) + \eta_{jk}(2\pi, l) + \eta_{ki}(2\pi, l) \in \mathbb{Z}
\end{equation}

which also represents the $C^2(\mathcal{L}M; \mathbb{Z})$ lift of $\sigma_{ij}(2\pi)$, and hence the Chern class of $D$.

Finally, we consider the strengthening of the equivalence relation on fusive bundles.

**Proposition 3.11.** Two fusive bundles represent the same class in $H^2_{\text{fus}}(\mathcal{L}M)$ if and only if they are fusive isomorphic, i.e. there is a lithe isomorphism between them which intertwines fusion and the actions of the full reparametrization semigroup.

**Proof.** A fusive isomorphism is a lithe-fusion isomorphism, so it suffices to assume that $[D_1] = [D_2] \in H^2_{\text{fus}}(\mathcal{L}M)$ and show that $D = D_1^{-1} \otimes D_2$ has a global section which is lithe and invariant under fusion and the reparametrization action.

By the injectivity of regression, the bundle gerbe defined by $D$ is simplicially trivial, so that

\begin{equation}
\psi^* D \cong \pi_1^* J \otimes \pi_2^* J^{-1},
\end{equation}

trivial.
The lithe, restricted-reparametrization-invariant connection on \( D \) induces connections on each of the bundles \( J_{ij} \) in (3.8), which are invariant with respect to reparametrizations of paths. Moreover by fusion invariance the connection on \( D \) is recovered over \( (ev(0), ev(\pi))^{-1}(B_i \times B_j) \subset \mathcal{L}M \) as the tensor product connection on \( J_{ij} \otimes J_{ij}^{-1} \). In view of the fusion condition, the connections induced on the \( D_{kl,jl} \) in (3.9) are trivial over the canonical trivialization over 3-fold intersections and so correspond to real 1-forms on the double intersections \( B_{ij} \times B_{kl} \) which form a Čech class valued in in 1-forms. Thus there exist smooth 1-forms \( \mu_{ij} \) on the \( B_i \times B_j \) with these as Čech boundary. Shifting the connection on \( J_{ij} \) by the \( \mu_{ij} \) gives a consistent global connection on \( J \) which still induces the original connection on \( D \) and which remains restricted reparameterization-invariant.

Now \( J \) has been arranged to be trivial and over the constant paths \( M \subset \mathcal{I}M \) and has a section \( u : M \to J \). Consider the retraction of \( \mathcal{I}M \) onto the constant paths at the initial points via

\[
\mathcal{I}M \times [0,1] \to \mathcal{I}M, \quad (\gamma, t) \mapsto \gamma_t, \quad \gamma_t(s) = \gamma(ts).
\]

We proceed to show that the parallel translation of \( u \) along the inverse of this retraction results in a global restricted reparameterization invariant section of \( J \) over \( \mathcal{I}F \mathcal{M} \). To see this, first observe that the retraction of a reparameterization of a given path is equivalent to the action by a curve of reparameterizations applied to the retraction of the path. Specifically, if \( r \in \text{Rp}^+([0,2\pi]) \) and \( \gamma \in \mathcal{I}M \), then

\[
(r \cdot \gamma)_t = r_t \cdot \gamma_t, \quad \text{Rp}^+([0,2\pi]) \ni r_t(s) = \begin{cases} r(ts)/t & 0 < t \leq 1, \\ r(0) & t = 0. \end{cases}
\]

Note that the curve of reparameterizations is smooth down to 0 since \( r \) is smooth and fixes 0. If \( u(\gamma_t) \) is the parallel lift of \( u(\gamma_0) \) along the curve \( t \mapsto \gamma_t \) in \( \mathcal{I}M \), it follows from equivariance of the connection that \( t \mapsto r_t \cdot u(\gamma_t) \) is the (necessarily unique) parallel lift of \( u(\gamma_0) \) over \( t \mapsto r_t \cdot \gamma_t = (r \cdot \gamma)_t \), and we conclude that \( u \), evaluated at \( t = 1 \), is indeed a global restricted-reparameterization-equivariant section.

Thus, \( T = \pi^*_1 u \otimes \pi^*_2 u^{-1} \) is a global piecewise lithe section of \( D \) which is invariant under fusion and the action of \( \text{Rp}^+([0,\pi]) \). In fact this section must be invariant under the full reparameterization semigroup. The definition of a fusive bundle includes the requirement of local lithe triviality, simultaneously, of the reparameterization action. So each point is contained in a reparameterization-invariant open set over which there is such a trivialization. Over such an open set there is therefore a fusive function \( f \) which maps \( T \) to section which is invariant under all reparameterizations. Since \( T \) itself is invariant under the restricted group, \( f \) must be invariant under this subgroup. The Lie algebra of this group consists of the vector fields vanishing at 0 and \( \pi \) so is dense in all vector fields on the circle in the \( L^\infty \) topology. Thus Lemma 2.7 implies that it is reparameterization-invariant and hence \( T \) itself is invariant under reparameterization. \(\square\)

**Lemma 3.12.** For a fusive circle bundle the collection of reparameterization actions compatible with the given fusion isomorphism is affine.

**Proof.** By assumption the bundle \( D \) satisfies FB.i)–FB.vi) but initially we ignore the fusion conditions. The reparameterization action \( A \) gives a \( U(1) \)-equivariant isomorphism \( A_r : D_l \to D_{l\circ r} \) for each \( r \in \text{Diff}^+(\mathbb{S}) \) and each \( l \in \mathcal{L}M \). Thus if \( A \) is
a second such action then
\[ A_r^{-1} \hat{A}_r : D_l \to D_l \]
defines
\[ g : \text{Diff}^+(S) \times LM \to U(1) \text{ s.t.} \]
\[ \hat{A}_r = A_r g(r, l) \text{ on } D_l. \]

If \( r_1, r_2 \in \text{Diff}^+(S) \) then at \( l \in LM, \)
\[ \hat{A}_{r_1} \hat{A}_{r_2} = \hat{A}_{r_1} A_{r_2} g(r_2, l) = A_{r_1} A_{r_2} g(r_1, l \circ r_2) g(r_2, l) \]
and conversely this condition ensures that \( \hat{A} \) is a reparameterization action if \( A \) is one.

Thus \( g(r, l) \) is a twisted character on \( \text{Diff}^+(S) \times LM \), in particular this fixes a class in \( H^1(\text{Diff}^+(S); \mathbb{Z}) + H^1(LM; \mathbb{Z}) \). Restricted to the identity \( g \equiv 1 \) so in fact the class descends to \( H^1(\text{Diff}^+(S); \mathbb{Z}) \).

To compute this class, consider the restriction of \( g \) to a constant loop, a fixed point of the reparameterization group. Then (3.39) reduces to the condition that \( g \) be a character, a homomorphism \( \tilde{g} : \text{Diff}^+(S) \to U(1) \). However all such lithe maps are trivial. Indeed, the differential \( f \) of \( \tilde{g} \) at the identity is a linear map on the tangent space \( C^\infty(S; TS) \cong C^\infty(S) \) of \( \text{Diff}^+(S) \) which vanishes on commutators.

That is, using the assumed liteness of the action, \( f \in C^\infty(S) \) satisfies
\[ \int_S f(ab' - a'b)d\theta = 0 \forall a, b \in C^\infty(S). \]

Taking \( b = 1 \) it follows that \( f' = 0 \) and \( f \) is constant. Then (3.40) reduces to
\[ \int_S ab' = 0 \forall a, b \in C^\infty(U(1)), \]
unless \( f \equiv 0 \). However, the latter identity cannot hold in general, for example \( a \) can be chosen to be a positive function of compact support in a small interval where \( b \) is linear.

The differential of \( \tilde{g} \) therefore vanishes, and by exponentiating (parameter dependent) vector fields it follows that \( \tilde{g} \equiv 1 \) as claimed.

Since the cohomology class of \( g \) is zero it has a global logarithm, \( g = \exp(2\pi i\eta) \) which is unique when normalized to vanish at \( \text{Id} \times LM \). The multiplicative condition (3.39) for \( g \) then reduces to the corresponding additive condition (given the connectedness of the space)
\[ \eta(r_1 r_2, l) = \eta(r_1, l \circ r_2) + \eta(r_2, l). \]

Thus \( g_t = \exp(2\pi it\eta), t \in [0, 1] \) defines a 1-parameter family of reparameterization actions connecting the two actions and all actions form an affine space modelled on this linear space of functions.

The condition that \( \hat{A} \) and \( \hat{A} \) are both compatible with a fixed fusion structure implies precisely that \( g \) itself is fusive. Since the logarithm then satisfies the corresponding linear condition it follows that the reparameterization actions compatible with a fixed fusion structure also form an affine space modelled on the lithe functions satisfying (3) and the corresponding fusion condition. \( \square \)
4. String structures

In this section we briefly review the notion of string structures and summarize the conclusions of Redden [11]; in particular how these are related to the exact sequence (1).

By definition, a string group is a topological group \( \text{String}(n) \) with a surjective homomorphism

\[
\text{String}(n) \to \text{Spin}(n)
\]

the kernel of which is a \( K(\mathbb{Z}, 2) \) and such that this bundle is classified by a map \( f : \text{Spin}(n) \to BK(\mathbb{Z}, 3) = K(\mathbb{Z}, 4) \). Thus a string structure exists on \( M \) if and only if the induced class, which is \( \frac{1}{2}p_1(M) \in H^3(M, \mathbb{Z}) \) vanishes, and then homotopy classes of maps to \( B\text{String}(n) \) form a torsor over \( H^3(M, \mathbb{Z}) \).

On the other hand, a string structure induces a \( K(\mathbb{Z}, 2) \) bundle over \( F = F_{\text{Spin}} \) with restriction to each fiber equivalent to the \( K(\mathbb{Z}, 2) \) bundle (4.1); such bundles on \( F \) are in bijective correspondence with the set

\[
C(F) = \{ \gamma \in H^3(F, \mathbb{Z}); i^*\gamma = \gamma_{\text{can}} \in H^3(\text{Spin}, \mathbb{Z}) \}.
\]

Conversely an element of \( C(F) \) corresponds to an equivalence class of string structures since there is an exact sequence

\[
0 \to H^3(M, \mathbb{Z}) \to H^3(F, \mathbb{Z}) \to H^3(\text{Spin}, \mathbb{Z}) \to \frac{1}{2}p_1(M)\mathbb{Z} \to 0
\]

which follows directly from the Leray-Serre spectral sequence for \( F \) and the fact that \( \text{Spin} \) is 2-connected.

In particular, \( C(F) \) is empty unless \( \frac{1}{2}p_1(M) = 0 \), in which case it is an \( H^3(M, \mathbb{Z}) \) torsor and so equivalence classes of string structures are in bijection with \( C(F) \).

5. Fusive loop-spin structures

From now on we assume that \( M \) is a spin manifold, with a given lift of the orthonormal frame bundle to a spin-oriented frame bundle \( F \). By a loop-spin structure we mean an extension of the \( \mathcal{L}\text{Spin} \) bundle \( \mathcal{L}F \) to a principal bundle with structure group \( E\mathcal{L}\text{Spin} \), the basic central extension, with the additional properties listed below. The total space of such an extended bundle is a circle bundle, \( D \), over \( \mathcal{L}F \) and we state conditions in terms of this.

LS.i) (Fusive bundle) First, we require that \( D \) be a fusive circle bundle over \( \mathcal{L}F \) in the sense of §3 Thus it is lithe, has an action of the reparameterization semigroup which is the trivial action in appropriate local trivializations, has a fusion isomorphism and satisfies the compatibility conditions between these.

LS.ii) (Principal action) The principal bundle structure on the total space of \( D \) corresponds to a twisted equivariance condition covering the action of \( \mathcal{L}\text{Spin} \).
on $\mathcal{L}F$, and we therefore demand that there be a lithe circle bundle isomorphism (‘multiplication’)

\[ M : \pi_1^* E \otimes \pi_2^* D \longrightarrow m^* D \text{ over } \mathcal{L}\text{Spin} \times \mathcal{L}F, \]

\[ m : \mathcal{L}\text{Spin} \times \mathcal{L}F \ni (\gamma, l) \longmapsto \gamma l \in \mathcal{L}F. \]

To induce a group action this needs further to satisfy the associativity condition that the pointwise diagram involving the product on $\mathcal{E}_L \text{Spin}^3$

\[ E_{g_1} \otimes E_{g_2} \otimes D_l \xrightarrow{M \otimes 1} E_{g_1} \otimes D_l \]

\[ \xrightarrow{1 \otimes M} E_{g_2} \otimes D_{g_1 l} \]

\[ \xrightarrow{M} D_{g_1 g_2 l} \]

commute for each $g_1, g_2 \in \mathcal{L}\text{Spin}$ and $l \in \mathcal{L}F$.

LS.iii) (Compatibility) It is further required that $M$ be compatible with the fusion isomorphism in the sense that the diagram

\[ E_{g_{12}} \otimes E_{g_{23}} \otimes D_{l_{12}} \otimes D_{l_{23}} \xrightarrow{\Phi \otimes \Phi} E_{g_{13}} \otimes D_{l_{13}} \]

\[ \xrightarrow{M \otimes M} E_{g_{12} l_{13}} \otimes D_{g_{23} l_{23}} \xrightarrow{\Phi} D_{g_{13} l_{13}} \]

commutes for $(\gamma_1, \gamma_2, \gamma_3) \in \mathcal{T}^3[3]F$, with $l_{ij} = \psi(\gamma_i, \gamma_j)$ and $(h_1, h_2, h_3) \in \mathcal{T}\text{Spin}$ with $g_{ij} = \psi(h_i, h_j)$.

There is compatibility condition between the reparameterization semigroup action $A$ and $M$ corresponding to the known action $A$ of $\mathbb{R}^p$ on $\mathcal{E}_L \text{Spin}$ and so reducing to the commutativity of

\[ E_{g \circ r} \otimes D_{l \circ r} \xrightarrow{A \otimes A} E_g \otimes D_l \]

\[ \xrightarrow{M} E_g \otimes D_l \]

\[ \xrightarrow{A} D_{g \circ l \circ r} \]

\[ \xrightarrow{M} D_{g \circ l} \]

for each $r \in \mathbb{R}^p(S)$ and $l \in \mathcal{L}F$.

Two fusive loop-spin structures are equivalent if and only if the two circle bundles are lithe-isomorphic over $\mathcal{L}F$ by an isomorphism that intertwines the structure isomorphism $M$ and $\Phi$.

We denote the set of equivalence classes of fusive loop-spin structures by $C_{\text{fus}}(F)$.

**Proposition 5.1.** If $D_i, i = 1, 2$, are fusion loop-spin structures associated to the same spin structure on $M$ then $D_1 \otimes D_2$ is the pull-back of a fusive circle bundle, $K$, over $\mathcal{L}M$, and $D_1$ and $D_2$ are equivalent if and only if $K$ is lithe-fusion trivial. Conversely, for any fusive circle bundle $K$ on $\mathcal{E}_L M$, and fusion loop-spin bundle $D$ the product $D \otimes K$ has a fusion loop-spin structure.

It follows that, if non-empty, the collection $C_{\text{fus}}(F)$ of equivalence classes of fusive loop-spin structures is a torsor over $H^3(M; \mathbb{Z})$. 
Proof. If $D_i$, $i = 1, 2$ are fusible loop-spin structures in the sense described above, then

$$K = D_1 \otimes D_2^{-1}$$

is a fusible circle bundle over $\mathcal{L}F$. The tensor product of the two multiplicative isomorphisms gives an untwisted lithe isomorphism

$$\tilde{M} : m^* \tilde{K} \rightarrow \pi_2^* \tilde{K} \text{ over } \mathcal{L}\text{Spin} \times \mathcal{L}F.$$  

The associativity conditions ensure that this gives an action of $\mathcal{L}\text{Spin}$ on $\tilde{K}$ which covers the action on $\mathcal{L}F$ and hence is free. Thus $\tilde{K}$ descends to a circle bundle $K$ over $LM$. By restricting to local sections this bundle can be seen to be lithe and the consistency conditions ensures that the induced fusion isomorphism on $\tilde{K}$ descends to a fusion isomorphism on $K$. Thus $K$ is indeed a fusible circle bundle over $LM$ and hence defines an element of $H^3_{\text{fus}}(LM) = H^3(M; \mathbb{Z})$ the vanishing of which is equivalent to fusible triviality.

An isomorphism between two fusible loop-spin structures, $D_1$ and $D_2$, is a fusible section of $\tilde{K}$ with equivariance properties corresponding to the intertwining conditions. In particular it descends to a lithe section of $K$ which commutes with the induced fusion isomorphism and so implies its fusible triviality. Conversely, a fusible section of $K$ lifts to generate an isomorphism of the two loop-spin structures. The map to $H^3(M; \mathbb{Z})$ is surjective since if $K$ is a fusible circle bundle over $LM$ and $D_2$ is a loop-spin structure then $D_1 = K \otimes D_2$ has a natural loop-spin structure giving $K$ as difference bundle. □

Next we show that a fusible loop-spin structure defines an element of the set $C(\tilde{F})$ in (4) and that this induces a map

$$g: C_{\text{fus}}(F) \rightarrow C(\tilde{F}).$$

**Proposition 5.2.** The Dixmier-Douady class in $H^3(F; \mathbb{Z})$ assigned to a loop-spin structure by Proposition lies in $C(\tilde{F})$ and this induces a map consistent with the torsor structure of Proposition 5.1.

Proof. To show that the element of $H^3(F; \mathbb{Z})$ assigned to the bundle gerbe associated to a loop-spin structure lies in $C(\tilde{F})$ it is only necessary to show that restricted to any one fibre of $F$ it is a basic gerbe for Spin, i.e. the 3-class is the chosen generator of $H^3(\text{Spin})$.

Thus consider a fiber $F_m \subset F$. The pull-back of the bundle gerbe over $F^2$ to $F_m^2$ has fiber space consisting of paths with end-points in $F_m$. This contains the sub-gerbe with fiber $\mathcal{L}F_m$ consisting of paths entirely contained in the fiber. The inclusion map is a bundle gerbe map and hence the gerbes have the same Dixmier-Douady invariants. On the other hand, restricted to the fiber loops $\mathcal{L}_{\text{fil}}F_m$, the fusive bundle $D$ is isomorphic to the circle bundle $\mathcal{E}\mathcal{L}\text{Spin} \rightarrow \mathcal{L}\text{Spin}$, which has Dixmier-Douady class the chosen generator of $H^3(\text{Spin}; \mathbb{Z})$.

As discussed in Proposition 3.3 the $H^3(M; \mathbb{Z})$-torsor structure on fusive loop-spin structures is generated by the tensor difference. Tensoring with a fusive circle bundle pulled back from $LM$ is equivalent to taking the gerbe product with the pull-back of the corresponding gerbe over $M$ and so shifts the Dixmier-Douady class by the pull-back into $H^3(F; \mathbb{Z})$ of the corresponding element of $H^3(M; \mathbb{Z})$. Thus the induced map commutes with the torsor actions. □
6. The holonomy bundle for a string class

If $\alpha \in C(F)$ is a string class then there is a principal PU bundle $Q_F = Q_F(\alpha)$, smooth in the norm topology, over $F$ with Dixmier-Douady class $\alpha$. Although the class is invariant under the action of Spin on $F$ the bundle cannot be equivariant with respect to this free action, since it would then descend to $M$. Nevertheless, we need to construct such a Spin action on the holonomy bundle of $Q_F$ as a preliminary step to the construction of the full loop spin action.

**Proposition 6.1.** To each class $\alpha \in C(F)$ there corresponds an equivalence class in $H^2_{\text{top}}(\mathcal{L}F)$ of fusive circle bundles over $\mathcal{L}F$, containing the holonomy bundle of $Q_F(\alpha)$, and each of these bundles can be given an equivariant, lithe, fusion- and reparameterization-invariant action of Spin covering the action by constant loops on $\mathcal{L}F$.

**Proof.** Given the results of §3, the only issue remaining is the construction of a compatible equivariant Spin action. To do so we pass from $F$ to the product $F^2$ and the quotient $A(F) = F^2/\text{Spin}$ by the diagonal Spin action. Since $F^2$ is a principal Spin bundle over $A(F)$ there is an isomorphism given by pull back

$$H^3(A(F); \mathbb{Z}) \rightarrow \{ \beta \in H^3(F^2; \mathbb{Z}) \text{ fiber trivial} \}.$$  

In particular each difference class $\pi_1^1 \alpha - \pi_1^2 \alpha$ is fiber trivial, so is the pull-back of a unique class $\beta(\alpha) \in H^3(A(F); \mathbb{Z})$.

Thus, rather than $Q_F$, we may consider a smooth PU bundle, $Q$, over $A(F)$ with Dixmier-Douady class $\beta(\alpha)$. With each path in $F$ there is an associated path in $F^2$ given by the constant path at the initial point in the left factor and the path itself in the right factor. This projects to $A(F)$ giving a map

$$\mathcal{I}_E F \ni \mu \mapsto (\mu(0), \mu) \in \mathcal{I}_E F^2 \rightarrow \mathcal{I}_E A(F).$$

The range consists precisely of the smooth paths in $A(F)$ which cover a path in $M^2$ which is constant in the first factor, starts at the diagonal, and has initial value the identity automorphism of the fibre of $F$ at the initial point. Moreover, (6.2) is a principal bundle over the range for the action of Spin on paths and so (6.2) gives an embedding of paths which extends to an embedding of loops:

$$\mathcal{I}_E F/\text{Spin} \rightarrow \mathcal{I}_E A(F), \ \mathcal{L}_E F/\text{Spin} \rightarrow \mathcal{L}_E A(F).$$

This embedding maps smooth loops to smooth loops and the fusion product in $F$ to the corresponding product for $A(F)$. It also identifies restricted reparameterization for loops in $F$ with restricted reparameterization for loops in $A(F)$. However the dependence of the embedding on the initial point of the path in $F$ means that it does not map the rotation action on loops to the corresponding rotation action.

The discussion in §3 applies to $Q$ as a bundle over $A(F)$. The restriction of $A(F)$ to the diagonal in $M^2$ is the automorphism bundle of $F$ itself, i.e. automorphisms of fixed fibers of $F$. Restricting $Q$ to the identity section over this submanifold and then pulling back to $\mathcal{L}F/\text{Spin}$, via the embedding (6.3) and the map to the initial point, parallel translation of $Q$ around the image loop in $A(F)$ and comparison to
the initial point gives the holonomy map and bundle as in (3.23)

\[ H : Q \times \text{ev}(0) (\mathcal{L}F/\text{Spin}) \to \text{PU} \]

(6.4)

\[ \tilde{D}(A) = H^* U, \quad D(A) = \pi^* \tilde{D}(A)/\text{PU} \to \mathcal{L}F, \quad \pi : \mathcal{L}F \to \mathcal{L}F/\text{Spin}. \]

Thus \( D(A) \) is a circle bundle over \( \mathcal{L}F \) which is restricted fusive, in the sense that only the semigroup of reparameterizations fixing \( 0 \in S \) acts, and has a compatible equivariant Spin action covering the action by constant loops. On the other hand we know that the gerbe induced by \( D(A) \) over \( F^2 \) has Dixmier-Douady class \( \pi^{-1} \alpha - \pi^{-2} \alpha \) since restricted to pointed paths it has class \( \alpha \).

The discussion in §3 shows that \( D(A) \) and \( D \), corresponding to the same class \( \alpha \in C(F) \) are fusive isomorphic. This allows the equivariant Spin structure on \( D(A) \) to be transferred to \( D \).

However, as noted above, this need not be compatible with the full reparameterization action. Rather, the action of Spin on \( D \) induces a multiplicative map of Spin into the affine space of fusion-compatible reparameterization actions, as discussed in Lemma 3.12. This allows the reparameterization action to be averaged over Spin to produce a compatible (and fusion-compatible) action. Thus if \( s \in \text{Spin} \) then \( s^* A = A + a_s \), where \( a_s \in C^\infty(\text{Spin} \times \mathcal{L}F) \) is in the linear space of twisted characters discussed in Lemma 3.12 and satisfies

\[ a_{st} = a_s + s^* a_t, \quad s, \ t \in \text{Spin}. \]

Then setting \( \bar{a} = \int_{\text{Spin}} a_s ds \) the action \( A + \bar{a} \) is compatible with Spin. \( \square \)

The fusive circle bundle \( D = D_\alpha \), with its compatible Spin structure will be called the \textit{holonomy bundle of the string class} \( \alpha \).

Below we construct a loop-spin structure on \( D \). First consider its restriction to fiber loops in \( F \). Each \( l \in \mathcal{L}_{\text{fib}}F \) is determined by its initial point in \( F \) and the ‘difference loop’ giving the shift in the fiber and conversely each such loop is given by shifting a constant loop. Thus there is an isomorphism

\[ \mathcal{L}_{\text{fib}}F \to F \times \mathcal{L}\text{Spin}. \]

The Spin action on the left corresponds to the principal Spin action on \( F \) and the adjoint action of Spin on \( \mathcal{L}\text{Spin} \). Thus the quotient by this free action is a bundle of groups over \( M \), modelled on \( \mathcal{L}\text{Spin} \) and with structure group Spin. Since the central extension of \( \mathcal{L}\text{Spin} \) is equivariant for the conjugation action, the pull-back of \( E \) is a well-defined bundle over \( \mathcal{L}_{\text{fib}}F/\text{Spin} \).

**Proposition 6.2.** The holonomy bundle \( D \) of a string class, given by (6.4), restricted to fiber paths is Spin-equivariant and fusive isomorphic to the pull-back of the basic central extension of \( E\mathcal{L}\text{Spin} \).

**Proof.** The Dixmier-Douady invariant of the bundle gerbe induced by \( D \) is an element of

\[ H^3(F^2; \mathbb{Z}) = H^3(F; \mathbb{Z}) \oplus H^3(\text{Spin}; \mathbb{Z}). \]

The naturality properties of the invariant show that the first term is the Dixmier-Douady invariant of the restriction to \( F \to F^2 \) as the fiber diagonal. Over this submanifold there is a natural section of \( \mathcal{I}_{\text{fib}}F \) given by the constant path at each point. This gives a trivial subgerbe and shows that the restriction to this subspace is itself trivial. On the other hand, restricted to one fiber of Spin it is the holonomy of the restricted PU bundle which by definition corresponds to the chosen generator of \( H^3(\text{Spin}; \mathbb{Z}) \). Thus the gerbes induced by \( E \) and \( D \) over \( F^2 \) do have the same
Dixmier-Douady invariants. Moreover, both are lifts of bundle gerbes over $M \times \text{Spin}$ with the same 3-class so Proposition 3.8 gives a fusive isomorphism between them there, and hence lifted to $F[2]$ where it intertwines the Spin actions. \[\square\]

7. **Blips**

Although most of the discussion of loops has been relegated to an appendix, we describe here a crucial component of our construction of fusive loop-spin structures in the next section. This is a map from smooth loops in the spin frame bundle to special piecewise smooth loops that we call ‘blips’.

The basic form of the blip construction can be carried out on free paths in $F$. We associate to a given path $\gamma \in IF$ two paths. The first, horizontal path, is determined by three conditions. It has the same projection into $M$ as $\gamma$ and it has the same initial point in $F$ as $\gamma$ but as a section of the pull-back of $F$ to the base curve it is covariant constant with respect to the pull-back of the Levi-Civita connection:

\begin{equation}
(7.1) \quad h_\gamma \in IF, \quad \pi(h_\gamma) = \pi(\gamma), \quad h_\gamma(0) = \gamma(0), \quad h_\gamma^* \nabla = 0.
\end{equation}

Thus, $h_\gamma$ is obtained from $\gamma$ by replacing the given section of $F$ over the projection of the curve into $M$ by the covariant constant, which is to say horizontal, section of $F$ with the same initial point as $\gamma$.

Since $\gamma$ and $h_\gamma$ cover the same path in $M$, there is a well-defined path in Spin shifting $h_\gamma$ to $\gamma$:

\[\gamma = s_\gamma h_\gamma, \quad s_\gamma \in \mathcal{I}_{\text{Spin}}, \quad s_\gamma(0) = \text{Id}.\]

Then the ‘vertical’ path associated to $\gamma$ is in the fiber of $F$ through the end-point of $\gamma$ and is given by applying this path to the constant path

\[v_\gamma \in \mathcal{I}_{\text{fib}} F, \quad v_\gamma = s_\gamma h_\gamma(2\pi).\]

Thus the initial point of $v_\gamma$ is the end-point of $h_\gamma$, and we define the ‘blip’ path associated to $\gamma$ to be the join of these:

\[\text{Bl} : IF \to I_{(\pi)} F, \quad \text{Bl}(\gamma) = j(h_\gamma, v_\gamma).\]

Observe that $\text{Bl}$ is a bijection onto its image which consists of all paths, with one ‘kink’ at $\pi$, which are covariant constant on $[0, \pi]$ and are confined to a fiber over $[\pi, 2\pi]$; $\text{Bl}(\gamma)$ has the same initial and end points as $\gamma$. So in particular the operation extends to the fibre products over $IF \to F^2$:

\begin{equation}
(7.2) \quad \text{Bl}^{[j]} : I^{[j]} F \to I^{[j]}_{(\pi)} F, \quad j = 2, 3.
\end{equation}

There is an even simpler, fiber, form of the blip map, corresponding to the base being a point, which can be written explicitly as

\begin{equation}
(7.3) \quad \text{bl} : \mathcal{I}_{\text{Spin}} \to \mathcal{I}_{(\pi)} \text{Spin}, \quad \text{bl}(\mu) = j(\mu(0), \mu),
\end{equation}

and is just given by adjoining to a path in Spin an initial constant path with the same initial value as the given path. Again it preserves endpoints and has the same simplicial properties as $\text{Bl}$ above, of which it is a special case.

In terms of this map the relation between the blip construction and the action of $\mathcal{I}_{\text{Spin}}$ is

\begin{equation}
(7.4) \quad \text{Bl}(\mu \gamma) = \text{bl}(\mu) \text{Bl}(\gamma), \quad \mu \in \mathcal{I}_{\text{Spin}}, \quad \gamma \in IF.
\end{equation}
Similarly, there is a simple relationship between reparameterization of paths and blips. If \( r \in \text{Diff}^+(\mathbb{R}) \) is an oriented diffeomorphism then for any \( \gamma \in \mathcal{I}F \)

\[
    h_\gamma \circ r = h_{\gamma \circ r}, \quad s_\gamma \circ r = s_{\gamma \circ r}.
\]

Thus

\[
    \text{Bl}(\gamma \circ r) = \text{Bl}(\gamma) \circ j(r, r)
\]

where the reparametrization is rescaled and applied on each of the intervals \([0, \pi]\) and \([\pi, 2\pi]\).

Since the paths, forming \( \mathcal{L}(2)F \), with kinks at 0 and \( \pi \) may be identified with \( \mathcal{I}^{[2]}F \) by the fusion map, \( \psi \), the blip construction extends to loops:

\[
    \text{Bl} : \mathcal{L}(2)F \longrightarrow \mathcal{L}(4)F
\]

(7.6)

\[
    \text{Bl}(\psi(p_1, p_2)) = \psi(\text{Bl}^{[2]}(p_1, p_2)), \quad (p_1, p_2) \in \mathcal{I}^{[2]}F.
\]

The resulting blip loop consists of a horizontal segments for \( t \in [0, \pi/2] \cup [3\pi/2, 2\pi] \) and vertical path for \( t \in [\pi/2, 3\pi/2] \). Applied to a smooth loop the blip construction yields a loop which is smooth, rather than only piecewise smooth, at 0 and at \( \pi \), but not in general at the points \( \pi/2 \) and \( 3\pi/2 \). Thus in fact if we ‘rotate’ the loop by \( \pi/2 \) it has only kink points at 0 and \( \pi \):

(7.8)

\[
    R(\pi/2) \text{Bl} : \mathcal{L}F \longrightarrow \mathcal{L}(2)F.
\]

**Lemma 7.1.** The blip map preserves the fusion relation.

**Proof.** This is immediate from the definitions.

From the discussion above, similar relationships between the blip construction on loops and the \( \mathcal{LS}_{\text{spin}} \) and (restricted) reparameterization actions follow. Thus

(7.9)

\[
    \text{Bl}(\mu l) = \text{bl}(\mu) \text{Bl}(l), \quad l \in \mathcal{L}F, \quad \mu \in \mathcal{LS}_{\text{spin}}.
\]

The action of a constant loop in Spin commutes with the blip map. On the other hand the action of an element of \( \mathcal{I}_{\text{spin}} \), a loop with initial value \( \text{Id} \), is ‘compressed’ into the fiber action on the vertical part of the blipped loop. Thus (7.9) may be rewritten

(7.10)

\[
    R(\pi/2) \text{Bl}(\lambda) = \psi(V_\lambda, H_\lambda), \quad (V_\lambda, H_\lambda) \in \mathcal{I}^{[2]}F, \quad \lambda \in \mathcal{L}F,
\]

\[
    \lambda = \psi(\gamma_1, \gamma_2), \quad V_\lambda = \psi(v_{\gamma_1}, v_{\gamma_2}), \quad H_\lambda = \psi(rh_{\gamma_1}, rh_{\gamma_2})
\]

and then the action of \( \phi \in \mathcal{LS}_{\text{spin}} \) is again through the vertical part

(7.11)

\[
    R(\pi/2) \text{Bl}(\phi \gamma) = \psi(\phi V_\gamma, H_\gamma)
\]

where on the right \( \phi \) is identified as a path rather than a loop. The significance of (7.11) is that

(7.12)

\[
    (\phi V_\gamma, V_\gamma, H_\gamma) \in \mathcal{I}^{[3]}F
\]

with the first two paths in the same fiber. This allows the application of the fusion isomorphism below.

There is a similar property for restricted reparameterization. If \( r \in \text{Diff}^+([0, \pi]) \), then

(7.13)

\[
    R(\pi/2) \text{Bl}(\gamma \circ r) = \psi(V_\gamma \circ r', H_\gamma)
\]
where $r' \in \text{Diff}^+(\lbrack 0,2\pi \rbrack)$ is simply $r$ transferred to the interval. Then again there is a fusion triple
\begin{equation}
(7.14) \quad (V_\gamma \circ r', V_\gamma, H_\gamma) \in I^3F
\end{equation}
with the first two paths in one fiber.

We also need to consider the regularity of the blip map. Consider a coordinate patch $\Gamma(l, \epsilon) \subset IM$. Over the geodesic ball $B(l(0), \epsilon) \subset M$ the spin frame bundle is trivialized by the section $f_0(x)$ obtained by choosing an element
\begin{equation}
(7.15) \quad f_0 = f_0(l(0)) \in F_{l(0)}
\end{equation}
which is then parallel-translated along radial geodesic to each $x \in B(l(0), \epsilon)$. Then each $l' \in \Gamma(l, \epsilon)$ has a determined lift to $IF$ given by parallel translation of $f_0(l'(0))$ along $l'$. This identifies the preimage in $IF$ as
\begin{equation}
(7.16) \quad \pi^{-1}\Gamma(l, \epsilon) = \Gamma(l, \epsilon) \times I\text{Spin}
\end{equation}
where on the right each element $l'$ is identified with $L(l', f_0)$, the horizontal lift described above.

If $\gamma \in IF$ and $\pi(\gamma) \in \Gamma(l, \epsilon)$ then under $7.15$
\begin{equation}
\gamma = S \cdot L(\pi(\gamma), f_0), \quad S \in I\text{Spin},
\end{equation}
\begin{equation}
(7.16) \quad h_\gamma = S(0)L(\pi(\gamma), f_0), \quad v_\gamma = S \circ S(0)^{-1}.
\end{equation}
It follows directly that Bl extends to finite energy paths.

Next we show that the blip map is boundary-lithe on paths in the sense of $\lfloor 14 \rfloor$ The tangent bundle of $F$ decomposes according to the Levi-Civita connection
\begin{equation}
(7.17) \quad TF \cong VF \oplus HF.
\end{equation}
Pulling this back, any section over a path be decomposed into horizontal and vertical sections
\begin{equation}
(7.18) \quad C^\infty([0, 2\pi]; \gamma^*TF) \ni u \longrightarrow \nu_V u + \nu_H u \in C^\infty([0, 2\pi]; \gamma^*VF) \oplus C^\infty([0, 2\pi]; \gamma^*HF)
\end{equation}
In addition the initial value of the vertical section can be extended uniquely along the curve to be covariant constant to give a third ‘flat’ section:
\begin{equation}
(7.19) \quad C^\infty([0, 2\pi]; \gamma^*TF) \ni u \longrightarrow \nu_F u \in C^\infty([0, 2\pi]; \gamma^*VF).
\end{equation}
The two vertical sections, $\nu_V u$ and $\nu_F u$ may be identified naturally with maps into the Lie algebra of Spin. The derivative of Bl can be evaluated from the definition above.

**Lemma 7.2.** The blip map is boundary-lithe in the sense that its derivative is given in terms of the rescaling of the induced section in (7.18), (7.19) by
\begin{equation}
(7.20) \quad d\text{Bl}_\gamma : C^\infty([0, 2\pi]; \gamma^*TF) \ni u \longrightarrow (\nu_H u + \nu_F u)(\cdot/2) \oplus \nu_V u(\pi + \cdot/2) \in C^\infty([0, \pi]; h_\gamma^*TF) \times C^\infty([\pi, 2\pi]; T\gamma(2\pi)F).
\end{equation}

In the discussion of the regularity of the loop-spin structure below, it is important that the boundary part here, which is to say $\nu_F u$, appears in both terms in terms in (7.20), which can instead be written as the sum of $\nu_F u$ in both factors, the horizontal section in the first and the ‘reduced vertical section’ $\nu_V u - \nu_F u$ in the second.
8. Construction of fusive loop-spin structures

To complete the proof of the Main Theorem in the Introduction it remains only to show the existence of a fusive loop-spin structure whenever \( \frac{1}{2}p_1 = 0 \). Under this condition Waldorf in [16] shows the existence of a loop-spin structure which satisfies the fusion condition, but without lite regularity. A circle bundle with the correct topological type to correspond to a loop-spin structure is constructed in §6 above.

**Theorem 8.1.** There is a fusive loop-spin structure on the loop space of a spin manifold of dimension \( n \geq 5 \) corresponding to each element of \( C(F) \) and giving a right inverse to \( (5.7) \).

The short exact sequence (1) shows that the vanishing of \( \frac{1}{2}p_1 \) is equivalent to \( C(F) \) being non-empty.

**Proof.** Consider a circle bundle \( D \) as discussed in §6. The loop-spin structure will be constructed through the pull-back of this bundle under the blip map (7.8): \( T = \text{Bl}^* \, D, \, \text{Bl} : LF \rightarrow R(\pi/2)L(2)F. \)

It follows directly from Lemma 7.1 that the fusion structure on \( D \) pulls back to a fusion structure \( \Phi_T = (\text{Bl}^{[3]})^* \Phi_D \) on \( T \).

The fundamental property of \( T \) is that it carries a multiplication isomorphism. If \( \gamma \in LF \) and \( \lambda \in L\text{Spin} \) then \( \lambda \gamma = \lambda(0) \phi \gamma \) where \( \phi \in L\text{Spin} \). Thus (7.11) applies and

\[
\text{Bl}(\lambda \gamma) = \lambda(0) R(\pi/2) \psi(\phi V_\gamma, H_\gamma).
\]

The fusion triple (7.12) gives the desired multiplication identification, \( M \), as the long composite in

\[
(8.2) \quad T \lambda \gamma = D_{\text{Bl}(\lambda \gamma)} \xrightarrow{M} T_\gamma \otimes E_\lambda \xrightarrow{D_{\text{Bl}(\gamma)} \otimes E_{\text{Bl}(\phi)}} D_{\text{Bl}(\gamma)} \otimes E_{\text{Bl}(\phi)} \xrightarrow{\lambda(0)^{-1} A(3\pi/2) \otimes \mu} D_{\text{Bl}(\phi)} \otimes E_{\text{Bl}(\phi)} \xrightarrow{\lambda(0)^{-1} A(3\pi/2) \otimes \mu} D_{\text{Bl}(\phi)} \otimes E_{\text{Bl}(\phi)}.
\]

Here the map on the left is the Spin action on \( D \), the first map on the lower line is given by the rotation action on \( D \) and the identification (7.11), the second is the the fusion map on \( D \) corresponding to (7.12). The map on the right is the opposite rotation action on \( D \) tensored by the identification \( \mu \) of \( D \) with \( E \) over fiber loops, followed by the product decomposition of \( E \) using the triviality of the loop \( \psi(V, V) \).

The map on the upper right is reparameterization for \( E \) – excising the constant segment – followed by the action of the constant loop \( \lambda(0) \) and the definition of \( T \). That this map is piecewise lite is essentially a consequence of the fact that it is well-defined.

We need to check that this has the properties required above of a loop-spin structure as a circle bundle over \( LF \).

First consider the associativity condition for \( M \). This involves comparing the action of the product \( \Lambda = \lambda_1 \lambda_2 \) of two loops in \( L\text{Spin} \) with the composite of the actions. The only complication here involves the removal of the constant loop, which is necessary to make the paths in (7.12) have the same end points. Thus, for the product loop the normalized loop involves conjugation:

\[
\phi = (\lambda_2(0)^{-1} \phi_1 \lambda_2(0)) \phi_2.
\]
However the fibers of $E$ over constant loops are canonically trivial, since these have natural lifts to invertible operators on the Hardy space, so the action is associative. Thus the conditions in LS.iii, excepting light smoothness, have been verified.

Next we lift the restricted reparameterization action to $T$. Thus, suppose $r \in \text{Diff}_{[0, \pi]}(U(1))$ lies in the subgroup of leaving both 0 and $\pi$ fixed. Thus $r = r_+ r_-$ is the composite of diffeomorphisms which fix one of the two intervals on the circle bounded by these two points. It therefore suffices to consider just $r = r_+$ since the behaviour of $r_-$ is essentially the same. This reparameterization of the loop corresponds to the reparameterization of the path forming the outgoing half of the loop, with the end points and the incoming half unaffected. The properties of parallel translation ensure that for a path $p$, the horizontal path of the reparameterization is the reparameterization of $h_p$ and the same is then true of $s_p$. Thus the blipped outgoing path is

\[(8.3) \quad \text{Bl}(\gamma \circ r) = (h_p \circ r, s_p \circ rp(2\pi)).\]

As for the group action discussed above, this allows the rotation of the blipped loop, where the return path is unchanged, to be written in terms of the outer two of the triple of paths

\[(8.4) \quad (H \circ r', V, V \circ r') \in T_{\pi}^3 F\]

where $r'$ is the reverse of $r$. The other two fusion loops are then the rotation of the reparameterization of $\text{Bl}(\gamma)$ by $r$ compressed to a quarter circle and the rotation of the fiber loop which is given by the loop in $\text{Spin}$ formed from $s_p$, $s_p \circ r$ and the same return path twice. Using the fusion isomorphism and rotation invariance of $D$, the fiber isomorphism to $E$ and the reparameterization action on $E$ give a natural identification

\[(8.5) \quad A(r) : T_{\gamma} \rightarrow T_{\gamma'}, \ r \in \text{Diff}_{(0, \pi)}(\mathbb{S}).\]

This again can be seen to have the associativity condition required of a reparameterization action and also to satisfy the compatibility conditions with fusion for this subgroup and the multiplication action in LS.iii.

Now the two bundles $T$ and $D$ are, by construction, homotopic through fusion bundles and hence, following Proposition 3.11, are isomorphic through a piecewise light, fusion preserving and restricted-reparameterization invariant bundle map. This allows all the structures above to be transferred back to $D$. Now, unlike $T$, $D$ has a full reparameterization action with the subgroup $\text{Diff}_{[0, \pi]}(\mathbb{S})$ compatible with the action $\mathcal{L}\text{Spin}$. In fact the smoothness of the action ensures that the full reparameterization action is compatible, following the arguments of Lemma 2.7.

The commutativity (5.4) is the triviality of piecewise light function $f : R^+p(\mathbb{S}) \times \mathcal{L}\text{Spin} \times \mathcal{L}_E F$ such that

\[(8.6) \quad A(r) M(\lambda) l = f(r, \lambda, l) M(A(r) \lambda) A(r) l, \ l \in \mathcal{L}_E F, \ \lambda \in \mathcal{L}\text{Spin}, \ r \in R^+p(\mathbb{S}).\]

Since this is an action, $f(r' r, \lambda, l) = f(r', A(r) \lambda, A(r) l) f(r, \lambda, l)$. It is shown above that $f(r', \lambda', l) = 1$ for $r' \in \text{Diff}_{[0, \pi]}(\mathbb{S})$ and hence for each $r \in R^+p(\mathbb{S})$ and $\lambda \in \mathcal{L}\text{Spin}$ fixed, the differential of $f(r, \lambda, l)$ with respect to $r$ vanishes on the Lie algebra of $\text{Diff}_{[0, \pi]}(\mathbb{S})$ and hence vanishes identically by the density with respect to $L^\infty(\mathbb{S})$.

Finally then it remains to examine the regularity of $T$. The construction of the blip curve extends by continuity to the energy space, so the trivialization of $D$ discussed in the proof of Theorem 3.10 pulls back to give trivializations of $T$ over...
tubular neighborhoods of each loop in the energy topology. The transition maps for $D$ are lithe in the sense that the derivative at a piecewise smooth curve, such as $\text{Bl}(\gamma), \gamma \in LF$, are piecewise smooth sections of the tangent bundle pulled back to the curve. The regularity of the blip map is such that the pulled-back transition maps for $T$ are therefore lithe in the weaker sense that the derivatives are piecewise smooth over $\gamma$ with the possibility also of delta functions over 1 corresponding to the ‘boundary term’ $\nu_F$. We proceed to show that this boundary term is in fact absent and that the transition maps have smooth derivatives, without discontinuities in the derivatives at 0. □

Appendix A. Lithe regularity

The regularity of objects, particularly circle bundles, over the loop space $LM = \mathcal{C}^\infty(S; M)$ of a finite-dimensional oriented, connected and compact manifold $M$ ultimately reduces to the regularity of functions. Since loop spaces are modelled on $\mathcal{C}^\infty(S; \mathbb{R}^n)$ we first consider appropriate notions of smoothness for functions on $\mathcal{C}^\infty(Z)$ for a finite-dimensional manifold $Z$ and then generalize to infinite-dimensional manifolds modelled on $\mathcal{C}^\infty(Z)$.

The Fréchet topology on $\mathcal{C}^\infty(Z)$ arises from the standard Sobolev, or equivalently $C^k$, norms relative to a partition of unity and is given by the metric

$$d(f, g) = \sum_{i=1}^{\infty} 2^{-k} \frac{\|f - g\|_k}{1 + \|f - g\|_k}.$$  

By continuity of a function on an open set of $O \subset \mathcal{C}^\infty(Z)$ we will mean continuity in this sense, so with respect to uniform convergence of all derivatives.

For such a function the assumption of directional differentiability at each point leads to a derivative

$$F' : O \times \mathcal{C}^\infty(Z) \rightarrow \mathbb{R}.$$  

Standard notions of continuous differentiability would require this to be continuous, as a continuous linear functional in the second variable. For general Fréchet spaces it is difficult to refine this, but in the case of $\mathcal{C}^\infty(Z)$ the pointwise derivative becomes a distributional density on $Z$ and there are many subspaces of distributions. In particular the smooth densities themselves form a subspace. We define $F$ to be ‘$C^1$-lithe’ if its (weakly defined) derivative arises from a continuous map

$$O \rightarrow \mathcal{C}^\infty(Z; \Omega)$$  

through the integration, or ‘$L^2$’, pairing $\mathcal{C}^\infty(Z; \Omega) \times \mathcal{C}^\infty(Z) \rightarrow \mathbb{R}$.

Higher derivatives, defined by successive weak differentiability, lead to maps

$$F^{(k)} : O \times \mathcal{C}^\infty(Z) \times \cdots \times \mathcal{C}^\infty(Z) \rightarrow \mathbb{R}$$  

which are multilinear at each point of $O$. Given some continuity these define, by the Schwartz kernel theorem and its extensions, distributional densities on $Z^k$ symmetric under factor-exchange. One immediate generalization of (A.3) would be to require that these derivatives factor through smooth densities on $Z^k$ but this is much too restrictive as can be seen in the case $Z = S$ by choosing a smooth function $f \in \mathcal{C}^\infty(\mathbb{R})$ and setting

$$F(u) = \int_S f(u(\theta))d\theta, \ u \in \mathcal{C}^\infty(S),$$
just the integral of the pull-back. The first derivative at a point \( u \) is

\[
(A.6) \quad F'(u; v) = \int_S f'(u(\theta))v(\theta)d\theta
\]

which is indeed given by pairing with a smooth function on the circle at each point. However the second derivative is

\[
(A.7) \quad F''(u; v, w) = \int_S f''(u(\theta))v(\theta)w(\theta)d\theta.
\]

As a distribution on \( S^2 \), the torus, this is given by a distributional density supported on the diagonal, with a smooth coefficient. It is this property that we generalize.

For any closed embedded submanifold \( N \subset Z \) and vector bundle \( W \) over \( Z \) the space of ‘Dirac sections of \( W \)’ with support \( N \) is well-defined by reference to local coordinates and local trivializations. A distribution supported on \( N \) is locally a finite sum of normal derivatives (with respect to \( N \)) of the Dirac delta distribution along \( N \) with distributional coefficients on \( N \). By a Dirac section of order \( m \) we mean that there are at most \( m \geq 0 \) normal derivatives and that all the coefficients are smooth along \( N \). This space is isomorphic to the space of smooth sections of a bundle over \( N \) (without being one) and this is the topology we take. There are other somewhat larger spaces one could allow here corresponding to various classes of distributions conormal with respect to \( N \).

The multidiagonals in \( Z^k \) are the the embedded submanifolds which are the fixed sets of some element of the permutation group acting by factor-exchange. There are all diffeomorphic to \( Z^l \) for some \( l \leq k \).

**Definition A.1.** For any compact smooth manifold \( Z \) the space of ‘Dirac distributional sections’ of a vector \( W \) over \( Z^k \) (for any \( k \)) is the direct sum of the Dirac sections of \( W \), as discussed above, with respect to the multi-diagonals of \( Z^k \).

So for any \( k \) this space is topologically the direct sum of copies of \( C^\infty(Z^j; U_j) \) for \( j \leq k \) and vector bundles \( U_j \).

An important property of these Dirac distributional sections is that they pullback under factor exchange to the corresponding space of sections of the pulled back bundle. Moreover they are preserved under exterior product, interpreted as multilinear functionals the product of Dirac distributions on \( Z^l \) and \( Z^k \) is Dirac on \( Z^{l+k} \). They are also preserved by global diffeomorphisms of \( Z \).

**Definition A.2.** A \( C^\infty \)-lithe function on an open subset \( O \subset C^\infty(Z) \) is a function with weak derivatives of all orders which are given by continuous maps \( F^{(k)} \) from \( O \) into the sum of the spaces of Dirac distributional densities on \( Z^k \), supported on the multidiagonals, for each \( k \).

This definition can be refined by fixing the number of ‘normal derivatives’ \( p_k \) which can appear in the derivatives of order \( k \). We do not discuss this below, except the important case of \( k = 1 \). The notion of ‘\( C^\infty \)-lithe’ is somewhat strengthened in the case of loop manifolds below by demanding that the functions also be \( C^1 \) on the associated energy Hilbert manifold.

For a finite-dimensional manifold this notion of \( C^\infty \)-liteness can be construed as reducing to smoothness in the usual sense. In consequence there is an immediate extension of the notion of a \( C^\infty \)-lithe function to maps from the product \( U \times O \) of a finite dimensional manifold and an open subset of \( C^\infty(Z) \) into a finite-dimensional manifold \( N \) by simply requiring the same condition of all derivatives.
in both variables in coordinate charts on \( N \), or equivalently considering smooth maps from \( U \) into \( C^\infty \)-lithe maps on \( O \).

The \( C^\infty \)-lithe functions on \( O \) form a \( C^\infty \) algebra. That is, not only are linear combinations and products of such functions again \( C^\infty \)-lithe, but if \( u_i, i = 1, \ldots, k \), are real-valued \( C^\infty \)-lithe functions and \( G \) is a \( C^\infty \) function on \( \mathbb{R}^k \) then \( G(u_1, \ldots, u_k) \) is also \( C^\infty \)-lithe.

In the body of the paper, heavy use is made of the construction of loops from paths by fusion. In view of this we need to allow the model space to be a compact manifold with boundary; in fact we really need (but do not develop here) as small a part of the theory of ‘articulated manifolds’ in which manifolds with corners are joined to various orders of smoothness along their boundaries. To keep the discussion within reasonable bounds we only consider the two cases of a compact manifold with boundary \( H \) and also that of a compact manifold \( Z \), without boundary, but with an interior dividing hypersurface \( H \subset Z \). In these cases in the products \( Z^k \) it is possible to weaken Definition A.1, and hence Definition A.2, by allowing ‘Dirac sections’ over not just the multi-diagonals of \( Z^k \) but of submanifolds which are in the same sense products of \( H \) and the various multi-diagonals. We distinguish between these boundary and separating hypersurface cases and weaken Definition A.1 as follows.

**Definition A.3.** In the case that \( Z \) is a compact manifold with boundary the space of Dirac distributional sections of a vector \( W \) over \( Z^k \) is the direct sum of the Dirac sections of \( W \), as discussed above, with respect to the multi-diagonals of \( Z^k \) and all their boundary faces, with smooth coefficients on the as manifolds with corners. For a compact manifold with interior hypersurface \( H \) no terms supported on factors of \( H \) are permitted, and all coefficients are required to be smooth up to \( H \) (separately from both sides).

Definition A.2 is then extended by using this notion of Dirac section.

To extend this definition to a Fréchet manifold modelled on \( C^\infty(Z) \), an appropriate \( ‘C^\infty\)-lithe’ structure is needed. With loop spaces in mind, consider the open sets in \( C^\infty(Z; \mathbb{R}^n) \) which are determined by a corresponding open subset \( Y \subset Z \times \mathbb{R}^n \) which is tubular, in the sense that it fibers over \( Z \) and then \( O = O_Y \subset C^\infty(Z; \mathbb{R}^n) \) consists of all the maps with \( f(m) \in \{m\} \cap Y \) for all \( m \), i.e. all the sections of this trivial bundle taking values in \( Y \).

**Proposition A.4.** If \( Y_i \subset Z \times \mathbb{R}^n \ i = 1, 2 \) are tubular open subsets and \( O_i = C^\infty(Z; Y_i) \subset C^\infty(Z; \mathbb{R}^n) \) is the corresponding open subset with values in \( Y_i \) then any fiber-preserving diffeomorphism \( T : O_1 \rightarrow O_2 \) induces a bijection between the spaces of \( C^\infty \)-lithe functions on \( O_1 \) and \( O_2 \).

**Proof.** If \( F \) is \( C^\infty \)-lithe on \( O_2 \) then the pull-back is weakly differentiable and the derivatives are

\[
(T^*F)^{(k)}(u \circ T; v_1, \ldots, v_k) = F^{(k)}(u, T_*v_1, \ldots, T_*v_k)
\]

where \( T_* \) is the differential of \( T \) as a bundle isomorphism along the section \( u \circ T \), i.e. at each point in \( O_1 \) it is a smooth family, parameterized by \( Z \), of invertible linear transformations of \( \mathbb{R}^n \). The space of Dirac distributions is preserved by such bundle maps so the invariance follows. \( \square \)
Appendix B. Loop manifolds

If $M$ is a finite dimensional oriented compact manifold we consider here some of the basic properties of the loop space $\mathcal{L}M = C^\infty(S; M)$ of $M$. We also need to consider some related manifolds, in particular the ‘energy’ loop space $\mathcal{L}_E M = H^1(S; M)$ and the path space $C^\infty([0, 2\pi]; M)$. If $S \subset \mathbb{S}$ is finite will also consider the space, $\mathcal{L}_S M$, of continuous loops which are piecewise smooth, i.e. smooth on each of the closed intervals into which $S$ divides $\mathbb{S}$, the case $S = \{0, \pi\}$ is particularly important. For any $S$,

\[ \mathcal{L}M \rightarrow \mathcal{L}_E M \rightarrow \mathcal{L}_S M \]

are dense subsets of the energy space (but smooth loops are not dense in piecewise smooth loops in the standard topology).

For a given metric on $M$ let $\epsilon_0$ be the injectivity radius. For $u \in \mathcal{L}M$ and $0 < \epsilon < \epsilon_0$ the sets

\[
\Gamma(u, \epsilon) = \{ v \in \mathcal{L}M; d_M(v(t), u(t)) < \epsilon \ \forall \ t \in S \}
\]

are identified by the exponential map at points along $u$ with a tubular neighbourhood of the zero section of the pull back to $S$ of the tangent bundle $TM$ under $u$. Since this bundle is trivial over $u$ it may be identified with $C^\infty(S; \mathbb{R}^n)$ or the corresponding piecewise smooth space $\mathcal{C}^\infty_S(S; \mathbb{R}^n)$. These sets form coordinate covers of $\mathcal{L}M$ and $\mathcal{L}_S M$ and for any non-trivial intersection the coordinate transformation is a loop into the groupoid of local diffeomorphisms, $\text{Diff}(\mathbb{R}^n)$, of $\mathbb{R}^n$. As such it is closely related to the corresponding structure groupoid of a finite dimensional smooth fiber bundle as noted for example by Omori [9]. The convexity properties of small geodesic balls show that this is actually a good open cover of $\mathcal{L}M$ as a Fréchet manifold locally modelled on $\mathcal{L}\mathbb{R}^n = C^\infty(S)^n$.

The sets $\Gamma(u, \epsilon)$ are open with respect to the supremum topology on continuous loops and since $H^1(S) \subset C(S)$ there are similar open sets in the finite-energy loop space fixed by the $H^1$ norm of the section of the tangent bundle over the base loop representing a nearby loop

\[
\Gamma_E(u, \epsilon') = \{ v \in H^1(S; M); \|v_u(t)\|_{H^1} < \epsilon' \}.
\]

Since $\mathcal{L}_E M$ is a real Hilbert manifold, the derivative of a function, thought of as a linear functional on the tangent space which is $H^1(S, u^*TM)$ at $u \in \mathcal{L}_E M$, would usually be interpreted as acting through the Riesz identification with the dual. However, in terms of functions on the circle it is appropriate to interpret the duality through $L^2$, so here the derivative $f'(u)$ always acts on the tangent space through

\[
f'(u)(v) = \int_S \langle f'(u), v \rangle_{u^*g} d\theta.
\]

This identifies a bounded linear functional, with respect to the energy norm, on the tangent space as an element of $H^{-1}(S, u^*TM)$. Thus the following definition
requires enhanced regularity of the derivative even as a once-differentiable function on $\mathcal{L}_EM$.

**Definition B.1.** By a lithe function on the loop space $\mathcal{L}M$ is meant a $C^\infty$-lithe function in the sense of Definition A.2 which has a $C^1$ extension to $\mathcal{L}_EM$ with derivative an element of $L^2(\mathbb{S}, u^*TM)$ depending continuously on $u \in \mathcal{L}_EM$. By an $S$-lithe function, for $S \subset \mathbb{S}$ finite, we mean a $C^\infty$-lithe function on $\mathcal{L}_SM$ (see Definition A.3) with such a $C^1$ extension to $\mathcal{L}_EM$.

**Proposition B.2.** The condition that a continuous function be lithe on each of the open subsets $\Gamma(u, \epsilon)$ forming an open cover of $\mathcal{L}M$ is independent of the open cover and these functions form a $C^\infty$ algebra on $\mathcal{L}M$.

**Proof.** This is a direct consequence of Proposition A.4 above.

**Lemma B.3.** The holonomy of a smooth circle bundle with connection over $M$ is a lithe function on $\mathcal{L}M$ or any $\mathcal{L}_SM$.

**Proof.** Consider a smooth circle bundle $D$, with connection, over $M$. Over any one loop, $u \in \mathcal{L}M$, $D$ is trivial (since the structure bundle is oriented) i.e. has a smooth section. Pulling $D$ back to the tubular neighborhood $N_\epsilon(u)$ of the zero section of $u^*TM$ via the exponential map at each point gives a trivial circle bundle to which this section is extended by parallel transport along the radial paths from each point $u(t)$. This induces a section of the pull-back of $D$ to each element of $\Gamma(u, \epsilon)$ which factors through the pull-back to $N_\epsilon(u)$. The pulled back connection is then represented by a smooth 1-form on $N_\epsilon(u)$ and the holonomy of $D$ around each loop $\gamma \in \Gamma(u, \epsilon)$ is given by the integral of this 1-form pulled back from $N_\epsilon(u)$:

\[
(B.5) \quad h(\gamma) = \exp\left(i \int_{U(1)} \alpha(\gamma(\theta)) \cdot \dot{\gamma}(\theta) d\theta\right)
\]

using the pairing of a smooth 1-form pulled back to the curve and the derivative of the curve. It follows from this that the holonomy is lithe on each of these open sets and therefore globally on $\mathcal{L}M$. Note that the only derivative within the integral involves the tangent vector field, $\tau_u(\theta)$, to the curve, used to pull back the 1-form. This is in $L^2(\mathbb{S}, u^*TM)$ and by integration by parts the derivative can always be thrown onto the background tangent vector so the regularity required in Definition B.1 follows.

The group of oriented diffeomorphisms of the circle, $\text{Diff}^+(\mathbb{S})$ is an open subset of $\mathcal{L}_S$. Thus it inherits a lithe structure. It acts on $\mathcal{L}M$ by reparameterization and one of the basic ideas in the study of the loop space is to produce objects which are equivariant with respect to this action. Observe first that the action, viewed as a map

\[
(B.6) \quad \text{Diff}^+(\mathbb{S}) \times \mathcal{L}M \rightarrow \mathcal{L}M
\]

is itself lithe, i.e. the pull-back of a lithe function under it is lithe.

If $r(s) \in \text{Diff}^+(\mathbb{S})$ is a smooth curve with $r(0) = \text{Id}$ then the derivative of the pulled back action on lithe functions is readily computed in terms of the tangent vector field $v = v(\theta) d/d\theta = dr/ds(0) \in \mathcal{V}(\mathbb{S})$ as

\[
(B.7) \quad v \cdot f(u) = \frac{d}{ds}\bigg|_{s=0} f(u \circ r(s)) = \int_{\mathbb{S}} v(\theta) \langle f'(u)(\theta), \tau_u(\theta) \rangle u^*g d\theta, \; u \in \mathcal{L}_EM,
\]

where $\tau_u(\theta) \in L^2(\mathbb{S}, u^*TM)$ is again the tangent vector field to $u$. 
If \( S \subset \mathcal{S} \) is finite consider the subgroup \( \text{Diff}_S^+(S) \subset \text{Diff}^+(\mathcal{S}) \) of the diffeomorphisms which fix each point of \( S \). The whole group \( \text{Diff}^+(\mathcal{S}) \) has the homotopy type of the circle, in the \( C^\infty \) topology while these subgroups are contractible. Although \( \text{Diff}_S^+(S) \) is a closed Fréchet subgroup, with Lie algebra \( \mathcal{V}_S(S) \subset \mathcal{V}(S) \) consisting of the vector fields vanishing at \( S \) it is, in much a weaker sense, dense in the whole group. This is one reason to insist on high regularity for functions.

**Lemma B.4.** If \( f : \mathcal{L}_EM \to \mathbb{C} \) is \( S \)-lithe then for each smooth vector \( v \in \mathcal{V}(S) \) and fixed finite set \( S \) there is a sequence \( v_n \in \mathcal{V}_S(S) \) such that

\[
\text{(B.8)} \quad vf(u) = \lim_{n \to \infty} v_n f(u) \quad \forall \ u \in \mathcal{L}_EM.
\]

**Proof.** This follows directly from (B.7) since one only needs to choose a sequence \( v_n \in \mathcal{V}_S(S) \) bounded in supremum norm such that \( v_n \to v \) almost everywhere and then the integrand in (B.7) converges in \( L^1 \). \( \square \)

This leads to a regularity result which is important in the analysis of circle bundles.

**Lemma B.5.** If \( f \) is an \( S \)-lithe function on \( O \subset \mathcal{L}_SM \) for some finite set \( S \) and some open set \( O \) which is invariant under \( \text{Diff}_S^+(S) \) and \( f \) is invariant under the action of \( \text{Diff}_S(S) \) then it is lithe and invariant under the action of \( \text{Diff}^+(S) \).

**Proof.** Invariance under the action of \( \text{Diff}_S^+(S) \) implies that \( v \cdot f = 0 \) on \( O \). Then Lemma [B.4] implies that \( v \cdot f = 0 \) for all \( v \in \mathcal{V}(S) \). Now, it is not the case these vector fields exponentiate to a neighborhood of the identity of \( \text{Diff}_S(S) \) but any such diffeomorphism is given by integration of the action of a parameter-dependent vector field, and hence the triviality of the action extends to the whole of \( \text{Diff}^+(S) \). \( \square \)

As well as lithe functions over \( \mathcal{L}M \) we consider circle, i.e. principal \( U(1) \), bundles.

**Definition B.6.** A circle bundle \( D \) over \( \mathcal{L}M \) is \((S)\)-lithe if it has trivializations over a covering of \( \mathcal{L}M \) by open sets \( \mathcal{B} \) with lithe transition maps on intersections.

The inverse of a lithe circle bundle is then lithe, as is the tensor product of two lithe circle bundles. The notion of a lithe section over an open set is also well defined. Then a \( U(1) \) map from one lithe circle bundle to another will be considered lithe if it corresponds to a lithe section of the tensor product of the image bundle with the inverse of the source bundle. Applying Lemma B.5 to the transition maps of a lithe \( U(1) \) bundle proves

**Lemma B.7.** A lithe action of \( \text{Diff}_S^+(S) \) on a lithe circle bundle \( D \) may be extended to a lithe action of \( \text{Diff}^+(S) \).

We also consider the path space in \( M, \mathcal{I}M = C^\infty([0, \pi]; M) \) although the choice of interval is somewhat arbitrary. Evaluation at the two end-point maps gives a fibration fibers over \( M^2 \). Indeed, local coordinates based at a point \( \bar{m} \in U \subset M \) can be used to construct a family of diffeomorphisms

\[
\text{(B.9)} \quad \psi_{m,s} = \text{Id} s > \frac{1}{2}, \quad \psi_{m,0} = \text{Id}, \quad \psi_{m,1}(\bar{m}) = m.
\]
and then 
(B.10) \[ u \rightarrow u'(t) = \psi_{m,t}(u(t)) \]
is an isomorphism of paths with initial point at \( \tilde{m} \) to paths with initial point at \( m \) giving a local section of \( \mathcal{I}M \) with respect to one end-point and the other end can be treated similarly.

We leave the properties of litheness for functions on the path space as an exercise, but notice that the appropriate extension of Definition [15,1] should permit single Dirac delta terms at the boundary of the interval. For instance, the pull-back to \( \mathcal{I}M \) of a function on \( M \) by mapping to the initial point as derivative given by such a delta function.

Appendix C. Basic central extension

We consider here properties of the basic central extension of \( \mathcal{L}\text{Spin} \), the loop group on Spin in dimensions \( n \geq 5 \). Waldorf [16] has shown that it satisfies a form of the fusion condition. Here we show that the central extension is ‘fusive’ in that it is also lithe and exhibits reparameterization equivariance. These results are proved using the realization of the central extension obtained via the Toeplitz algebra.

Consider the Hardy space of smooth functions on the circle with values in the complexified Clifford algebra; it is the image of the projection \( P_H \) which deletes negative Fourier coefficients

\[ H = \{ u \in C^\infty(S; \mathbb{C}\ell) : u = \sum_{k \geq 0} u_k e^{ik\theta} \} \subset C^\infty(S; \mathbb{C}\ell). \]

Then \( \mathcal{L}\text{Spin} \) is embedded as a subspace of the pseudodifferential operators by compression to \( H \)

\[ \mathcal{L}\text{Spin} \ni l \mapsto P_H l P_H \in \Psi^0(S; \mathbb{C}\ell). \]

This map is injective since principal symbol \( \sigma(P_H l P_H) = l \) on the positive cosphere bundle of the circle. In fact \( P_H \) is microlocally equal to the identity on the positive side and equal to 0 on the negative side from which it follows commutators with multiplication operators are smoothing \([P_H, l] \in \Psi^{-\infty} = C^\infty(S^2; \text{End}(\mathbb{C}\ell))\), and hence that

(C.1) \[ \Psi_H(S; \mathbb{C}\ell) = P_H \mathcal{L}\text{Spin} P_H + P_H \Psi^{-\infty} P_H \xrightarrow{\sigma} \mathcal{L}\text{Spin} \]

forms a ring of operators on \( H \) with surjective (symbol) homomorphism back to the group algebra \( \mathcal{L}\text{Spin} \).

Let \( \mathcal{G}_H = \{ B \in \Psi_H : B^* = B^{-1} \} \subset \Psi_H(S; \mathbb{C}\ell) \)

consist of those invertible elements which are unitary with respect to the \( L^2 \) inner product on \( H \). It follows from the Toeplitz index theorem and the simple connectedness of Spin that any element of \( \Psi_H \) with symbol \( l \in \mathcal{L}\text{Spin} \) is Fredholm with index 0 (the winding number of \( l \)), hence has an invertible perturbation by a smoothing operator. Since the symbol \( l \) itself is unitary, the radial part of the polar decomposition of this invertible operator is of the form \( \text{Id} + A \) with \( A \) smoothing and it follows that the lift of \( l \) can be deformed to be unitary. Thus

(C.2) \[ \mathcal{G}_H^{-\infty} \rightarrow \mathcal{G}_H \xrightarrow{\sigma} \mathcal{L}\text{Spin} \]
is a short exact sequence of groups where the kernel is the normal subgroup
\[ G_H^{-\infty} = \{ B = \text{Id} + A ; \quad B^* = B^{-1}, \quad A \in P_H \Psi^{-\infty} P_H \} \]
consisting of all unitary smoothing perturbations of the identity on \( H \); in particular these differ from \text{Id} by trace class operators on \( H \).

The Fredholm determinant on \( H \)
\[ (C.3) \quad \det : G_H^{-\infty} \rightarrow \text{U}(1) \]
is a group homomorphism and moreover is invariant under conjugation by the full group:
\[ (C.4) \quad \det(UBU^{-1}) = \det(B) \forall U \in G_H. \]
It follows that \( G_H^{-\infty}/K \cong \text{U}(1) \), where
\[ (C.5) \quad K = \{ B \in G_H^{-\infty} : \det(B) = 1 \} \]
is also a normal subgroup of \( G_H \), and therefore passing to quotients in \[ (C.3) \] gives a central extension
\[ (C.6) \quad \text{U}(1) \rightarrow \text{ELSpin} \rightarrow \text{LSpin}, \quad \text{ELSpin} = G_H/K \]
Viewed as a principal \( \text{U}(1) \) bundle over \( \text{LSpin} \), the group multiplication on \( E = \text{ELSpin} \) becomes an isomorphism of \( \text{U}(1) \)-principal bundles over \( \text{LSpin} \):
\[ (C.7) \quad M : \pi_1^*E \otimes \pi_2^*E \rightarrow m^*E, \quad m : \text{LSpin}^2 \ni (l_1, l_2) \mapsto l_1l_2 \in \text{LSpin} \]
being the multiplication map on \( \text{LSpin} \) and \( \pi_i \) the two projections. The group condition then reduces to associativity over \( \text{LSpin} \) whereby the two maps
\[ (C.8) \quad \pi_1^*E \otimes \pi_2^*E \otimes \pi_3^*E \rightarrow m_3^*E, \quad m_3(l_1, l_2, l_3) = m_2(m_2(l_1, l_2), l_3) = m_2(l_1, m_2(l_2, l_3)) = l_1l_2l_3 \]
obtained by applying \( M \) first in the left two factor and then again, or in the right two factors and then again, are the same.

**Theorem C.1.** The central extension \( (C.6) \) is the basic central extension of \( \text{LSpin} \), corresponding to the generator of \( H^3(\text{Spin}; \mathbb{Z}) \cong \mathbb{Z} \). Moreover it is fusive as a circle bundle over \( \text{LSpin} \), for which the multiplication map \( (C.7) \) is lithe.

The remainder of this section is devoted to a proof of Theorem \( (C.1) \) which is split into a number of separate results.

**Proposition C.2.** The basic central extension of \( \text{LSpin} \) is lithe.

**Proof.** Recall that the loop space of any manifold is covered by contractible ‘geodesic tubes’ \( (B.3) \) for \( \epsilon < \epsilon_0 \), the injectivity radius for a left-invariant metric on \( \text{Spin} \). If \( U_l = P_H l P_H + A \) is a unitary lift of \( l \) into \( G_H \) then for \( 0 < \epsilon \) sufficiently small \( P_H l' P_H + A \) is invertible for all \( l' \in \Gamma(l, \epsilon) \) since in particular the norm on \( L^2 \) of the difference \( P_H (l^{-1}l) P_H - P_H \) vanishes uniformly with \( \epsilon \). Thus replacing \( P_H l' P_H + A \) by the unitary part of its radial decomposition gives a section of \( G_H \) over \( \Gamma(l, \epsilon) \) and hence a trivialization of \( E \) there.
The transition map between these trivializations over the intersection $\Gamma(l_1, \epsilon_1) \cap \Gamma(l_2, \epsilon_2)$ is given by the circular part of the determinant of the relative factors (C.9)

$$\det \left( (P_H l P_H + A_1 )^{-1} (P_H l P_H + A_2) \right) = \det \left( P_H + (P_H l P_H + A_1 )^{-1} (A_2 - A_1) \right).$$

Using the standard formula for the derivative of the (entire) function given by the determinant

(C.10) \[ d_B \det(P_H + B) = \det(P_H + B) \Tr \]

as a linear functional on the Toeplitz smoothing operators at $B$, it follows that (C.10) is infinitely differentiable with high derivatives coming from repeated differentiation of the determinant and of the inverse in (C.9). Thus as a polynomial on the k-fold tensor product of the tangent space $L$ spin to $L\text{Spin}$, this is the symmetrization of a sum of products of terms each of which is the trace of a repeated product

(C.11) \[ \Tr \left( (P_H l P_H + A_1 )^{-1} L_1 (P_H l P_H + A_1 )^{-1} (A_2 - A_1) \right) \cdots \left( P_H l P_H + A_1 )^{-1} L_j (P_H l P_H + A_1 )^{-1} (A_2 - A_1) \right) \]

where the $L_i \in L$ spin are the tangent vectors. These are of the form as required in the ‘lithe’ condition on functions in $\mathcal{A}$ as are the imaginary parts, so $E$ is a lithe circle bundle.

To classify the central extension as an element of $H^2(\text{Spin}; \mathbb{Z}) \cong H^2(L\text{Spin}; \mathbb{Z})$, we construct a natural connection on $EL\text{Spin}$ as a principal $\text{U}(1)$-bundle. First consider the Maurer-Cartan form

$$g^{-1} dg \in \Omega^1(\mathcal{G}_H; T_{\text{id}} \mathcal{G}_H),$$

which is well-defined as $\mathcal{G}_H$ is an open subset of the algebra $\Psi_H$. Here $dg$ takes values in the Lie algebra $T_{\text{id}} \mathcal{G}_H = \Psi_H$. Restricted to $\mathcal{G}_H^{-\infty}$, for which the Lie algebra $T_{\text{id}} \mathcal{G}_H^{-\infty} = P_H \Psi^{-\infty} P_H$ consists of trace-class operators on $H$, the trace

$$\Tr(g^{-1} dg) \in \Omega^1(\mathcal{G}_H^{-\infty}; \mathbb{C})$$

is well-defined, and from the fact that the trace is the logarithmic derivative of the determinant it follows that $\Tr(g^{-1} dg)$ vanishes on $\mathcal{K} = \{ B \in \mathcal{G}_H^{-\infty} : \det(B) = 1 \}$, and hence descends to the Maurer-Cartan form on

$$\text{U}(1) \cong \mathcal{G}_H^{-\infty}/\mathcal{K}.$$ 

To extend the form $\Tr(g^{-1} dg)$ to the whole group $\mathcal{G}_H$, we use the regularized trace and its relation to the residue trace of Wodzicki [21] and Guillemin [3]. For $A \in \Psi^2(\mathcal{S}; \mathbb{C})$, the regularized trace is defined to be

$$\overline{\Tr}(A) = \lim_{z \to 0} \left( \Tr \left( (1 + D^{2z}_{\theta})^{-z/2} A \right) - \frac{1}{z} \Tr_R(A) \right),$$

$$\Tr_R(A) = \lim_{z \to 0} z \Tr \left( (1 + D^{2z}_{\theta})^{-z/2} A \right).$$

Here $\Tr_R$ is the residue trace, which vanishes on trace-class operators and satisfies $\Tr_R([A, B]) = 0$, whereas the regularized trace $\overline{\Tr}$ extends the trace from trace-class operators, but is not a trace, rather satisfies the trace defect formula

(C.12) \[ \overline{\Tr}(\{ A, B \}) = \Tr_R \left( [B, \log((1 + D^{2z}_{\theta})^{-z/2})] A \right). \]
**Proposition C.3.** *(See also [1]) The one form \( \overline{\text{Tr}} (g^{-1} \, dg) \in \Omega^1(G_H; \mathbb{C}) \) descends to a connection form

\[
\text{(C.13)} \quad \omega = [\overline{\text{Tr}} (g^{-1} \, dg)] \in \Omega^1(E\text{Spin}; \mathbb{C})
\]

which has curvature

\[
\text{(C.14)} \quad d\omega(a, b) = - \frac{1}{2\pi i} \int_S \text{tr} \left( a(\theta) b'(\theta) \right) d\theta \quad a, b \in \mathcal{L}_{\text{Spin}}.
\]

In particular \([E] \in H^2(\mathcal{L}_{\text{Spin}}; \mathbb{Z}) \cong \mathbb{Z}\), is a generator and hence the central extension \( E\text{Spin} \) is basic.

**Proof.** It follows from the above discussion above that \( \overline{\text{Tr}} (g^{-1} \, dg) \) is an extension of \( \text{Tr}(g^{-1} \, dg) \) from \( G_H^{-\infty} \) to \( G_H \), and descends to a 1-form on \( E\text{Spin} \) whose restriction to the fibers is the Maurer-Cartan form on \( U(1) \). The invariance of \( g^{-1} \, dg \) also implies the invariance of this form the left action of an element of \( E\text{Spin} \). Equivariance with respect to the action by \( U(1) \) follows from the invariance of the regularizing operator \((1 + D^2_0)^{-z/2}\) under rotation and the invariance of the (analytic continuation) of the trace, so \( \omega \) is indeed a connection on \( E\text{Spin} \).

Now \( d\overline{\text{Tr}} (g^{-1} \, dg) = - \text{Tr}(g^{-1} \, dgg^{-1} \, dg) \) so at the identity, using \( \text{(C.12)} \)

\[
\text{(C.15)} \quad d\omega(a, b) = - \overline{\text{Tr}}([A, B]) = - \text{Tr}_R([B, \log(1 + D^2_0)^{-z/2}] A), \quad \forall \ a, b \in \mathcal{L}_{\text{Spin}}
\]

here \( A, B \in \Psi_H \) are any self-adjoint lifts of \( a, b \); we take \( A = P_H a P_H \) and \( B = P_H b P_H \). It suffices to compute \( \text{(C.15)} \) in terms of the coordinate \( \theta \in (0, 2\pi) \) with cotangent variable \( \xi = (d\theta)^+ \). With this convention the symbol of the regularizer \((1 + D^2_0)^{-z/2}\) is the boundary defining function \( \rho = (1 + \xi^2)^{-z/2} \) for the radial compactification of \( T^*\mathbb{S} \).

The residue trace of any element \( P_H P P_H \in \Psi^k(\mathbb{S}), \ k \in \mathbb{Z} \) is

\[
\text{Tr}_R(P_H P P_H) = \frac{1}{2\pi} \int_{S^+} \text{tr} \sigma^{-1}(P)
\]

where \( \sigma^{-1}(P) \) denotes coefficient of \( \rho^z \) in the expansion of the full symbol. Since

\[
P = [B, \log((1 + D^2_0)^{-1/2})] \in \Psi^{-1} \quad \text{only the principal symbol}
\]

\[
\sigma^{-1}(B, \log((1 + D^2_0)^{-1/2})) = -i\partial_b \sigma(B)(\theta)
\]

is involved, so \( \text{(C.14)} \) follows. This is 2-cocycle generating \( H^2(\mathcal{L}_{\text{Spin}}; \mathbb{Z}) \) — see for instance [10].

**Lemma C.4.** As a \( U(1) \)-bundle over \( \mathcal{L}_{\text{Spin}} \), \( E\text{Spin} \) has an equivariant action of \( \text{Diff}^+(\mathbb{S}) \)

\[
A : a^* E \rightarrow \pi^*_{\mathbb{S}} E, \ a : \text{Diff}^+(\mathbb{S}) \times \mathcal{L}_{\text{Spin}} \rightarrow \mathcal{L}_{\text{Spin}}
\]

under which the connection \( \text{(C.13)} \) is invariant.

**Proof.** The group \( \text{Diff}^+(\mathbb{S}) \) acts on \( C^\infty(\mathbb{S}); \mathcal{C}f \) by pullback, though not preserving \( H \) in general nor is the action unitary. To make it unitary we use the action on half-densities, rather than introduce them formally this simply means twisting the action on sections of the Clifford bundle to

\[
\text{(C.16)} \quad F^\# u(\theta) = u(F(\theta))|F'(\theta)|^{\frac{1}{2}}
\]

where of course \( f' > 0 \) on \( \text{Diff}^+(\mathbb{S}) \). Nonetheless, for any \( F \in \text{Diff}^+(\mathbb{S}) \), \( P_H F^\# P_H \) is Fredholm with index 0 since \( \text{Diff}^+(\mathbb{S}) \) is connected to the identity and \( P_H F^\# (\text{Id} -
$P_H$) and $(\text{Id} - P_H)F^\# P_H$ are smoothing operators. Thus $P_H F^\# P_H$ can be perturbed by a smoothing operator to a unitary operator, $U_F$, acting on $H$.

Now, the action of a diffeomorphism $F \in \text{Diff}^+(S)$ on $LS\text{Spin}$ is by composition and it follows that if $L \in G_H$ is a lift of $l \in L\text{Spin}$, so $\sigma(L) = l$, then

\begin{equation}
\sigma(U_F L U_F^{-1}) = l \circ F.
\end{equation}

Changing $U_F$ to another unitary extension $U'_F = BU_F$, $B \in G_H^{\infty}$ changes the lift of $l \circ F$ to

\begin{equation}
U'_F L U_F^{-1} = (B G B^{-1} G^{-1}) U_F L U_F^{-1},
\end{equation}

and $B G B^{-1} G^{-1} \in K$ since it has determinant 1. It follows that there is a natural isomorphism of $E$ covering the action of $F$:

\begin{equation}
\tilde{F} : F^* E \to E
\end{equation}

which commutes with the group action, and is itself multiplicative, giving an action of $\text{Diff}^+(S)$.

To see that this action on $E$ leaves the connection form $\omega$ it suffices to examine the infinitesimal action. Since the span the Lie algebra, it is enough to consider vector fields on the circle $v(\theta) \frac{d}{d\theta}$ with $0 < v \in \mathcal{C}^\infty(S)$. The infinitesimal generator of the half-density action \((C.16)\) is the selfadjoint first order differential operator

\begin{equation}
V = v^2 D_\theta v \frac{\partial}{\partial v}.
\end{equation}

For small $s$, $P_H \exp(isV) P_H$ is invertible and unitary, and the variation

\[
\frac{d}{ds} \big|_{s=0} \text{Tr}(P_H \exp(-isV) P_H g^{-1}dg P_H \exp(isV) P_H) = \text{Tr}([g^{-1}dg,iV])
\]

is given by residue trace

\[
\text{Tr}_R([iV, \log((1 + D_\theta^2)^{-\frac{1}{2}})] g^{-1}dg).
\]

Since $V$ is the Weyl quantization of the symbol $v(\theta)\xi$, the symbolic expansion of the commutator has no term of order $-1$, and this residue trace vanishes. \hfill \Box

Note that the construction of $E L\text{Spin}$ could as well have been carried out using any subspace of $\mathcal{C}^\infty(S; \mathbb{C})$ with projection operator differing from $P_H$ by a smoothing operator. In particular this applies to the shifted Hardy spaces $H_k = \{u \in \mathcal{C}^\infty \mid u = \sum_{n \geq k} u_n e^{in\theta}, \ k \in \mathbb{Z} \}$ resulting in bundles $E_k$. There are explicit isomorphisms $G_{H_k} \to G_{H_{k'}}$ given by conjugation by the unitary multiplication operators $e^{i(k' - k)\theta}$ and the identity $e^{i(k' - k)\theta} P_{H_k} = P_{H_{k'}} e^{i(k' - k)\theta}$ descend to natural isomorphisms

\begin{equation}
E_k L\text{Spin} \cong E_{k'} L\text{Spin}, \quad \forall \ k \in \mathbb{Z}.
\end{equation}

On the other hand, the use of a complementary subspace leads naturally to the opposite central extension. This can either be seen by going through the above construction and noting that the integral in the proof of Proposition $\text{(C.3)}$ is over $S^-$ instead of $S^+$, hence has the opposite orientation, or by the following argument. Denoting the complementary subspace to $H_k$ by

\[
\overline{H}_k = \{u \in \mathcal{C}^\infty \mid u = \sum_{n < k} u_n e^{in\theta} \},
\]
consider the direct sum algebra $\Psi_{H_k} \oplus \Psi_{\overline{H}_k}$, with the unitary subgroup
\[
\mathcal{G}_{H_k} \times \mathcal{L}_{\text{Spin}} \overline{\mathcal{G}}_{\overline{H}_k} = \{ (B, \overline{B}) \in \mathcal{G}_{H_k} \times \mathcal{G}_{\overline{H}_k} : \sigma(B) = \sigma(\overline{B}) \in \mathcal{L}_{\text{Spin}} \}.
\]
The kernel of the symbol map is $\mathcal{G}_{H_k}^{-\infty} \times \mathcal{G}_{\overline{H}_k}^{-\infty}$, and taking the quotient by the kernel of $\det_{H_k} \times \det_{\overline{H}_k}$ leads to the product $E_k \times \overline{E}_k$ as a $U(1) \times U(1)$ bundle, while taking the quotient by the kernel of the full Fredholm determinant leads to the tensor product
\[
E_k \mathcal{L}_{\text{Spin}} \otimes \overline{E}_k \mathcal{L}_{\text{Spin}} \to \mathcal{L}_{\text{Spin}}
\]
as $U(1)$-bundles over $\mathcal{L}_{\text{Spin}}$. However this has a canonical section, given by the fact that any $l \in \mathcal{L}_{\text{Spin}}$ is already a unitary operator on the space $H_k \oplus \overline{H}_k = C^\infty$, which leads to a canonical isomorphism
\[
E_k \mathcal{L}_{\text{Spin}} \otimes \overline{E}_k \mathcal{L}_{\text{Spin}} = (E_k \mathcal{L}_{\text{Spin}})^{-1} \quad \forall k \in \mathbb{Z}.
\]

**Lemma C.5.** $E \mathcal{L}_{\text{Spin}} \to \mathcal{L}_{\text{Spin}}$ is equivariant with respect to the full diffeomorphism group $\text{Diff}(\mathbb{S})$, that is
\[
A : \alpha^* E \xrightarrow{\sim} \pi_2^* E^{\pm 1},
\]
\[
a : \text{Diff}(\mathbb{S}) \times \mathcal{L}_{\text{Spin}} \to \mathcal{L}_{\text{Spin}}
\]
with sign $-1$ over the component $\text{Diff}^{-}(\mathbb{S}) \times \mathcal{L}_{\text{Spin}}$ and $+1$ over $\text{Diff}^+(\mathbb{S}) \times \mathcal{L}_{\text{Spin}}$.

**Proof.** In light of Lemma C.4 it suffices to check that $\alpha^* E \cong E^{-1}$ with respect to any one $\alpha \in \text{Diff}^{-}(\mathbb{S})$ which reverses orientation, an obvious choice being the involution $\alpha(\theta) = 2\pi - \theta$. Observe that pull-back by $\alpha$ interchanges $e^{ik\theta}$ with $e^{-ik\theta}$ and hence $\alpha^* : H_0 \to \overline{H}_1$ is a unitary isomorphism; here $\overline{H}_1 = C^\infty \cap \text{span} \{ e^{ik\theta} : k \leq 0 \}$.

If $L \in \mathcal{G}_H$ is a unitary lift of $l \circ \alpha \in \mathcal{L}_{\text{Spin}}$, it follows from the above discussion that $\alpha^* L \alpha^* \in \mathcal{G}_H$ is a unitary operator on $\overline{H}_1$, with symbol $l \circ \alpha^2 = l$. Using (C.21) and (C.20) leads to the canonical isomorphism
\[
\alpha^* E \xrightarrow{\sim} E^{-1}
\]
which concludes the proof. \hfill \Box

The fusion property now follows from equivariance and the multiplicativity (C.7).

**Corollary C.6.** $E \mathcal{L}_{\text{Spin}} \to \mathcal{L}_{\text{Spin}}$ has the fusion property as a $U(1)$-bundle.

**Proof.** For any $l \in \mathcal{L}_{\text{Spin}}$ which is invariant under the action of the loop reversal involution $\alpha$,
\[
l \circ \alpha = l \implies E_l \cong U(1)
\]
since the fiber $E_l$ is isomorphic to its inverse $E_l^{-1}$. In particular $E$ is naturally trivial, in a way that is consistent with the group multiplication, over ‘there-and-back’ paths
\[
E_{\psi(\gamma, \gamma)} \equiv U(1) \quad \forall \gamma \in I_{\text{Spin}}.
\]
This in turn leads to the full fusion condition for $E$. Namely if $(\gamma_1, \gamma_2, \gamma_3) \in I_{\text{Spin}}^3$, then for the three loops $l_{12} = \psi(\gamma_1, \gamma_2), l_{23} = \psi(\gamma_2, \gamma_3)$ and $l_{23} = \psi(\gamma_1, \gamma_3)$, then
\[
l_{12} l_{23}^{-1} l_{23} = \psi(\gamma_2, \gamma_2) \in \mathcal{L}_{\text{Spin}}
\]
from which it follows that
\[
E_{l_{12}} \otimes E_{l_{23}} \simeq E_{l_{13}}, \quad \Box
\]
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