SPACE-LIKE WEINGARTEN SURFACES IN THE THREE-DIMENSIONAL MINKOWSKI SPACE AND THEIR NATURAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. On any space-like Weingarten surface in the three-dimensional Minkowski space we introduce locally natural principal parameters and prove that such a surface is determined uniquely up to motion by a special invariant function, which satisfies a natural non-linear partial differential equation. This result can be interpreted as a solution to the Lund-Regge reduction problem for space-like Weingarten surfaces in Minkowski space. We apply this theory to linear fractional space-like Weingarten surfaces and obtain the natural non-linear partial differential equations describing them. We obtain a characterization of space-like surfaces, whose curvatures satisfy a linear relation, by means of their natural partial differential equations. We obtain the ten natural PDE’s describing all linear fractional space-like Weingarten surfaces.

1. Introduction

It has been known to Weingarten [21, 22], Eisenhart [7], Wu [23]) that without changing the principal lines on a Weingarten surface, one can find geometric coordinates in which the coefficients of the metric are expressed by the principal curvatures (or radii of curvature).

The geometric parameters on Weingarten surfaces were used in [23] to find the classes of Weingarten surfaces yielding “geometric” so(3)-scattering systems (real or complex) for the partial differential equations, describing these surfaces.

In [10] we have shown that the Weingarten surfaces (W-surfaces) in Euclidean space admit geometrically determined principal parameters (natural principal parameters), which have the following property: all invariant functions on W-surfaces can be expressed in terms of one function \( \nu \), which satisfies one natural partial differential equation. The Bonnet type fundamental theorem states that any solution to the natural partial differential equation determines a W-surface uniquely up to motion.

Thus the description of any class of W-surfaces (determined by a given Weingarten relation) is equivalent to the study of the solution space of their natural PDE. This solves the Lund-Regge reduction problem [15] for W-surfaces in Euclidean space.

In this paper we study space-like surfaces in the three dimensional Minkowski space \( \mathbb{R}^3_1 \).

A space-like surface \( S \) with principal normal curvatures \( \nu_1 \) and \( \nu_2 \) is a Weingarten surface (W-surface) [21, 22] if there exists a function \( \nu \) on \( S \) and two functions (Weingarten functions) \( f, g \) of one variable, such that

\[
\nu_1 = f(\nu), \quad \nu_2 = g(\nu).
\]

A basic property of W-surfaces in Euclidean space is the following theorem of Lie [14]:

The lines of curvature of any W-surface can be found in quadratures.

This remarkable property is also valid for space-like W-surfaces in Minkowski space.

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We use four invariant functions (two principal normal curvatures \(\nu_1, \nu_2\) and two principal geodesic curvatures \(\gamma_1, \gamma_2\)) and divide space-like W-surfaces into two classes with respect to these invariants:

1. The class of *strongly regular* space-like surfaces defined by
   \[(\nu_1 - \nu_2) \gamma_1 \gamma_2 \neq 0;\]

2. The class of space-like surfaces defined by
   \[\gamma_1 = 0, \ (\nu_1 - \nu_2) \gamma_2 \neq 0.\]

The basic tool to investigate the relation between space-like surfaces and the partial differential equations describing them, is Theorem 2.1. This theorem is a reformulation of the fundamental Bonnet theorem for the class of strongly regular space-like surfaces in terms of the four invariant functions. Further, we apply this theorem to space-like W-surfaces.

In Section 3 we prove (Proposition 3.4) that any space-like W-surface admits locally special principal parameters (*natural principal parameters*). Theorem 3.7 is the basic theorem for space-like W-surfaces of type (1):

*Any strongly regular space-like W-surface is determined uniquely up to motion by the functions \(f, g\) and the function \(\nu\), satisfying the natural PDE (3.4).*

Theorem 3.8 is the basic theorem for space-like W-surfaces of type (2):

*Any space-like W-surface with \(\gamma_1 = 0\) is determined uniquely up to motion by the functions \(f, g\) and the function \(\nu\), satisfying the natural ODE (3.6).*

In natural principal parameters the four basic invariant functions, which determine space-like W-surfaces uniquely up to motions in \(\mathbb{R}^3_1\), are expressed by a single function, and the system of Gauss-Codazzi equations reduces to a single partial differential equation (the Gauss equation). Thus, the number of the four invariant functions, which determine space-like W-surfaces, reduces to one invariant function, and the number of Gauss-Codazzi equations reduces to one *natural* PDE. This result gives a solution to the Lund-Regge reduction problem \[15\] for the space-like W-surfaces in \(\mathbb{R}^3_1\). The Lund-Regge reduction problem has been analyzed and discussed from several viewpoints in the paper of Sym \[18\].

The class of space-like W-surfaces contains all classical special surfaces: maximal space-like surfaces (with zero mean curvature \(H = 0\)), space-like CMC-surfaces (with constant mean curvature \(H = \text{const} \neq 0\)), space-like surfaces of constant Gauss curvature \(K = \pm 1\) etc. All these surfaces are examples of space-like W-surfaces with simple Weingarten functions.

A more general class, which contains the above mentioned surfaces, is the class of linear fractional space-like W-surfaces defined by the following relation between the normal curvatures \(\nu_1\) and \(\nu_2\):

\[\nu_1 = \frac{A\nu_2 + B}{C\nu_2 + D}, \quad A, B, C, D - \text{constants}; \quad BC - AD \neq 0.\]

We prove that on any linear fractional space-like W-surface the extrinsic Gauss curvature \(K' = \nu_1 \nu_2\), the mean curvature \(H\) and the curvature \(H' = \frac{\nu_1 - \nu_2}{2}\) satisfy the linear relation

\[\delta K' = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants}; \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0,\]

and vice versa (cf \[11\]).

On space-like surfaces in the three-dimensional Minkowski space the (intrinsic) Gauss curvature \(\bar{K}\) and the extrinsic Gauss curvature \(K'\) are in the relation \(K' = -\bar{K}\).

We note that the class of linear fractional space-like surfaces is invariant under parallel transformations of space-like surfaces.
In Proposition 4.1 we prove that the natural principal parameters of a given space-like $W$-surface $S$ are natural principal parameters for all parallel space-like surfaces $\bar{S}(a)$, $a = \text{const} \neq 0$ of $S$.

In Theorem 4.2 we prove the following essential property of the space-like surfaces, which are parallel to a given space-like surface (cf [11]):

The natural PDE of a given space-like $W$-surface $S$ is the natural PDE of any parallel space-like surface $\bar{S}(a)$, $a = \text{const} \neq 0$, of $S$.

Using this theorem, in Section 5 we classify the natural PDE’s of the linear fractional space-like $W$-surfaces, or equivalently the space-like surfaces satisfying the relation (1.1).

To illustrate our investigations, let us consider the space-like surfaces in Minkowski space, which satisfy the linear relation

$$(1.2) \quad \delta K' = \alpha H + \gamma, \quad \alpha^2 + 4\gamma\delta \neq 0.$$ 

Such surfaces are studied in [16, 5, 8, 9] (see also [6]).

All surfaces, satisfying the above linear relation, are linear fractional space-like $W$-surfaces and each of them has a natural PDE.

Milnor has proved in [16] that any space-like surface in $\mathbb{R}^3_1$ satisfying (1.2) is parallel to a surface, satisfying one of the following conditions: $H = 0$, $K = 1$ or $K = -1$.

Here we prove the following result:

- The space-like surfaces, which are parallel to space-like surfaces with $H = 0$, are described by the natural PDE

$\lambda_{xx} + \lambda_{yy} = e^\lambda$.

- The space-like surfaces, which are parallel to space-like surfaces with $H = \text{const}$, are described by the one-parameter family of natural PDE’s

$\lambda_{xx} + \lambda_{yy} = 2|H| \sinh \lambda$.

Up to similarity, the space-like surfaces, which are parallel to space-like surfaces with $H = \text{const}$, are described by the natural PDE of the surfaces with $H = 1/2$.

- The space-like surfaces, which are parallel to space-like surfaces with $K = \text{const} > 0$ ($K' = \text{const} < 0$), are described by the one-parameter family of natural PDE’s

$\lambda_{xx} - \lambda_{yy} = -K^2 \sin \lambda$.

Up to similarity, the space-like surfaces, which are parallel to space-like surfaces with $K = \text{const} > 0$, are described by the natural PDE of the surfaces with $K = 1$ ($K' = -1$).

We call the surfaces with $H = 0$, $H = 1/2$ and $K = -K' = 1$ the basic surfaces in the class of surfaces, determined by the relation (1.2).

Then we have:

Up to similarity, the surfaces, whose curvatures satisfy the linear relation (1.2), are described by the natural PDE’s of the basic surfaces.

We apply the above scheme to obtain all natural PDE’s describing the space-like surfaces, whose curvatures $K'$, $H$ and $H'$ satisfy the linear relation (1.1)

We divide the class of all linear fractional space-like $W$-surfaces into ten subclasses, which are invariant under parallel transformations. For any of these classes we determine the basic surfaces in terms of suitable constants $p$ and $q$. Applying Theorem 4.2, we find the PDE’s which describe all space-like surfaces in Minkowski space, whose curvatures satisfy the linear relation (1.1) (Theorem 5.2).
It is essential to note that the natural PDE’s of the linear fractional space-like W-surfaces are expressed in the form \( \delta \lambda = f(\lambda) \), where \( \delta \) is one of the operators (cf [11]):

\[
\Delta \lambda = \lambda_{uu} + \lambda_{vv}, \quad \bar{\Delta} \lambda = \lambda_{uu} - \lambda_{vv},
\]

\[
\Delta^* \lambda = \lambda_{uu} + \left( \frac{1}{\lambda} \right)_v, \quad \bar{\Delta}^* \lambda = \lambda_{uu} - \left( \frac{1}{\lambda} \right)_v.
\]

In the Euclidean space the intrinsic Gauss curvature \( K \) and the extrinsic Gauss curvature \( K' \) coincide.

Next we draw a parallel between the natural PDE’s of the basic classes of linear fractional W-surfaces in Euclidean space [11] and the natural PDE’s of the corresponding classes of linear fractional space-like surfaces in Minkowski space:

| Nr | Basic surfaces in \( \mathbb{R}^3 \) | Natural PDE in \( \mathbb{R}^3 \) | Natural PDE in \( \mathbb{R}^3 \) |
|----|---------------------------------|---------------------------------|---------------------------------|
| 1  | \( H = 0 \)                     | \( \Delta \lambda = -e^\lambda \) | \( \Delta \lambda = e^\lambda \) |
| 2  | \( H = \frac{1}{2} \)           | \( \Delta \lambda = -\sinh \lambda \) | \( \Delta \lambda = \sinh \lambda \) |
| 3  | \( H' = 1 \)                    | \( \Delta^*(e^\nu) = -2 \nu (\nu + 2) \) | \( \Delta^*(e^\nu) = 2 \nu (\nu + 2) \) |
| 4  | \( H = p H', \quad p^2 > 1 \)  | \( \Delta^*(p^\nu) = -2 \frac{p(p+1)}{(p-1)^2} \nu \) | \( \Delta^*(p^\nu) = 2 \frac{p(p+1)}{(p-1)^2} \nu \) |
| 5  | \( H = p H', \quad p^2 < 1, \quad p \neq 0 \) | \( \bar{\Delta}^*(p^\nu) = -2 \frac{p(p+1)}{(p-1)^2} \nu \) | \( \bar{\Delta}^*(p^\nu) = 2 \frac{p(p+1)}{(p-1)^2} \nu \) |
| 6  | \( H = p H' + 1, \quad p^2 > 1 \) | \( \Delta^*(\lambda^p) = -\frac{p((p-1)\lambda+2)((p+1)\lambda+2)}{2(p-1)\lambda} \) | \( \Delta^*(\lambda^p) = \frac{p((p-1)\lambda+2)((p+1)\lambda+2)}{2(p-1)\lambda} \) |
| 7  | \( H = p H' + 1, \quad p^2 < 1, \quad p \neq 0 \) | \( \bar{\Delta}^*(\lambda^p) = -\frac{p((p-1)\lambda+2)((p+1)\lambda+2)}{2(p-1)\lambda} \) | \( \bar{\Delta}^*(\lambda^p) = \frac{p((p-1)\lambda+2)((p+1)\lambda+2)}{2(p-1)\lambda} \) |
| 8  | \( K' = -1 \)                   | \( \bar{\Delta} \lambda = \sin \lambda \) | \( \bar{\Delta} \lambda = -\sin \lambda \) |
| 9  | \( K' = 2 H' \)                 | \( \Delta^*(e^\lambda) = -2 \) | \( \Delta^*(e^\lambda) = 2 \) |
| 10 | \( K' = p H' - q, \quad p \neq 0, \quad q > 0 \) | \( \Delta^*(e^{p^\mathcal{I}}) = -\frac{pq}{2} \frac{\lambda(p\lambda-2q)}{\lambda^2+q}, \quad \mathcal{I} = \frac{1}{\sqrt{q}} \arctan \frac{\lambda}{\sqrt{q}} \) | \( \Delta^*(e^{p^\mathcal{I}}) = \frac{pq}{2} \frac{\lambda(p\lambda-2q)}{\lambda^2+q}, \quad \mathcal{I} = \frac{1}{\sqrt{q}} \arctan \frac{\lambda}{\sqrt{q}} \) |

Let \( S \) be a linear fractional W-surface in \( \mathbb{R}^3 \) and \( S' \) be its corresponding space-like linear fractional W-surface in \( \mathbb{R}^3_1 \) generated by the same Weingarten functions as \( S \). Then the
natural PDE’s of $S$ and $S'$ are related as follows:

$$\mathbb{R}^3 : \delta \lambda = f(\lambda) \iff \mathbb{R}^3_1 : \delta \lambda = -f(\lambda),$$

where $\delta$ is one of the operators $\Delta, \bar{\Delta}, \Delta^*, \text{ and } \bar{\Delta}^*$.

Hu elucidated in [12] the relationship between the PDE’s

\begin{align*}
\alpha_{uu} - \alpha_{uv} &= \pm \sin \alpha \quad \text{(sin – Gordon PDE)}, \\
\alpha_{uu} - \alpha_{vv} &= \pm \sinh \alpha \quad \text{(sinh – Gordon PDE)}, \\
\alpha_{uu} + \alpha_{vv} &= \pm \sin \alpha \quad \text{(sin – Laplace PDE)}, \\
\alpha_{uu} + \alpha_{vv} &= \pm \sinh \alpha \quad \text{(sinh – Laplace PDE)}
\end{align*}

and the construction of various kinds of surfaces of constant curvature in $\mathbb{R}^3$ or $\mathbb{R}^3_1$.

The result of Baran and Marvan in [1] asserts that "the simple relation $\rho_1 - \rho_2 = \text{const}$ between the principal radii of curvature determines an integrable class of Weingarten surfaces in the Euclidean space (the forgotten class)" (see also [10, 17, 19, 20]). The authors associate with the "forgotten class" the nonlinear partial differential equation

$$z_{uu} + \left(\frac{1}{z}\right)_{vv} + 2 = 0$$

and prove that it is integrable. These surfaces correspond to the class 9 in the above table.

In [2] they give classes of Weingarten surfaces in Euclidean space integrable in the sense of soliton theory.

In [13] by using Darboux transformations, from a known solution to the sinh-Laplace (resp. sin-Laplace) equation have been obtained explicitly new solutions of the sin-Laplace (resp. sinh-Laplace) equation.

It is interesting to find the connection between parallel transformations of pseudo-spherical surfaces ($K = -1$) and their Bäcklund transformations. A nice base for further investigations in this direction gives the paper of S. Buyske [5]. This paper shows that many of the problems in Euclidean space have their exact analogues in Minkowski three space.

2. Preliminaries

In this section we introduce the denotations and formulas in the theory of surfaces in Minkowski space, which we use further.

Let $\mathbb{R}_1^3$ be the three dimensional Minkowski space with the standard flat metric $\langle , \rangle$ of signature $(2,1)$. We assume that the following orthonormal coordinate system $Oe_1e_2e_3 : e_1^2 = e_2^2 = -e_3^2 = 1$, $e_i e_j = 0, i \neq j$ is fixed and gives the orientation of the space.

Let $S : z = z(u,v), (u,v) \in \mathcal{D}$ be a space-like surface in $\mathbb{R}^3$ and $\nabla$ be the flat Levi-Civita connection of the metric $\langle , \rangle$. The unit normal vector field to $S$ is denoted by $l$ and $E, F, G; \ L, M, N$ stand for the coefficients of the first and the second fundamental forms, respectively. Then we have

$$E = z_u^2 > 0, \quad F = z_u z_v, \quad G = z_v^2 > 0, \quad EG - F^2 > 0, \quad l^2 = -1;$$

$$L = l z_{uu}, \quad M = l z_{uv}, \quad N = l z_{vv}.$$  

We suppose that the surface has no umbilical points and the principal lines on $S$ form a parametric net, i.e.

$$F(u,v) = M(u,v) = 0, \quad (u,v) \in \mathcal{D}.$$  

Then the principal curvatures $\nu_1, \nu_2$ and the principal geodesic curvatures (geodesic curvatures of the principal lines) $\gamma_1, \gamma_2$ are given by

$$\nu_1 = \frac{L}{E}, \quad \nu_2 = \frac{N}{G}; \quad \gamma_1 = -\frac{E_v}{2E\sqrt{G}}, \quad \gamma_2 = \frac{G_u}{2G\sqrt{E}}.$$
The mean curvature, the (intrinsic) Gauss curvature and the extrinsic Gauss curvature of $S$ are denoted by $H$, $K$ and $K'$, respectively.

We consider the tangential frame field $\{X, Y\}$ defined by

$$X := \frac{z_u}{\sqrt{E}}, \quad Y := \frac{z_v}{\sqrt{G}}$$

and suppose that the moving frame $XYl$ is always positive oriented.

Without loss of generality we consider space-like surfaces satisfying the condition (cf [10])

$$\nu_1(u, v) - \nu_2(u, v) > 0, \quad (u, v) \in D. \quad (2.2)$$

The frame field $XYl$ satisfies the following Frenet type formulas:

\[
\begin{align*}
\nabla_X X &= \gamma_1 Y - \nu_1 l, & \nabla_Y X &= \gamma_2 Y, \\
\nabla_X Y &= -\gamma_1 X, & \nabla_Y Y &= -\nu_2 X - \nu_2 l, \\
\nabla_X l &= -\nu_1 X; & \nabla_Y l &= -\nu_2 Y.
\end{align*}
\]

The Codazzi equations have the form

$$\gamma_1 = \frac{Y(\nu_1)}{\nu_1 - \nu_2} = \frac{(\nu_1)_v}{\sqrt{G}(\nu_1 - \nu_2)}, \quad \gamma_2 = \frac{X(\nu_2)}{\nu_1 - \nu_2} = \frac{(\nu_2)_u}{\sqrt{E}(\nu_1 - \nu_2)}, \quad (2.4)$$

and the Gauss equation can be written as follows:

$$K = Y(\gamma_1) - X(\gamma_2) - (\gamma_1^2 + \gamma_2^2) = -\nu_1\nu_2 = -K', \quad (2.5)$$

The Codazzi equations (2.4) imply the following equivalence

$$\gamma_1 \gamma_2 \neq 0 \iff (\nu_1)_v(\nu_2)_u \neq 0.$$

We consider two types of space-like surfaces parameterized by principal parameters:

- **strongly regular** space-like surfaces determined by the condition (cf [10])

  $$(\nu_1 - \nu_2)\gamma_1(u, v)\gamma_2(u, v) \neq 0, \quad (u, v) \in D,$$

- space-like surfaces satisfying the conditions

  $$\gamma_1(u, v) = 0, \quad (\nu_1 - \nu_2)\gamma_2(u, v) \neq 0, \quad (u, v) \in D.$$

Because of (2.4) the formulas

$$\sqrt{E} = \frac{(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} > 0, \quad \sqrt{G} = \frac{(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} > 0 \quad (2.6)$$

are valid on strongly regular space-like surfaces.

Taking into account (2.6), for strongly regular space-like surfaces formulas (2.3) become

\[
\begin{align*}
X_u &= \frac{\gamma_1(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} Y - \frac{\nu_1(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} l, \quad Y_u = -\frac{\gamma_1(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} X, \quad l_u = -\frac{\nu_1(\nu_2)_u}{\gamma_2(\nu_1 - \nu_2)} X; \\
X_v &= \frac{\gamma_2(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} Y, \quad Y_v = -\frac{\gamma_2(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} X - \frac{\nu_2(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} l, \quad l_v = -\frac{\nu_2(\nu_1)_v}{\gamma_1(\nu_1 - \nu_2)} Y.
\end{align*}
\]
Now, finding the compatibility conditions for the system (2.7), we can reformulate the fundamental Bonnet theorem for strongly regular space-like surfaces in \( \mathbb{R}^3_1 \) in terms of the invariants of the surface.

**Theorem 2.1.** Let the four functions \( \nu_1(u,v), \nu_2(u,v), \gamma_1(u,v), \gamma_2(u,v) \) be defined in a neighborhood \( \mathcal{D} \) of \((u_0,v_0)\), and satisfy the conditions

1) \( \gamma_1 (\nu_1)_u > 0, \quad \gamma_2 (\nu_2)_u > 0; \)

2.1) \( \left( \frac{\ln (\nu_1)_u}{\gamma_1} \right)_u = \frac{(\nu_1)_u}{\nu_1 - \nu_2}, \quad \left( \frac{\ln (\nu_2)_u}{\gamma_2} \right)_v = -\frac{(\nu_2)_v}{\nu_1 - \nu_2}; \)

2.2) \( \frac{\nu_1 - \nu_2}{2} \left( \frac{\gamma_2^2 u}{(\nu_2)_u} - \frac{\gamma_2^2 v}{(\nu_1)_v} \right) + \gamma_1^2 + \gamma_2^2 = \nu_1 \nu_2. \)

Let \( z_0X_0Y_0l_0 \) be an initial positive oriented orthonormal frame.

Then there exists a unique up to a motion strongly regular space-like surface \( S : z= z(u,v), \ (u,v) \in \mathcal{D}_0 \) \((u_0,v_0) \in \mathcal{D}_0 \subseteq \mathcal{D}) \) with prescribed invariants \( \nu_1, \nu_2, \gamma_1, \gamma_2 \) such that

\( z(u_0,v_0) = z_0, \ X(u_0,v_0) = X_0, \ Y(u_0,v_0) = Y_0, \ l(u_0,v_0) = l_0. \)

Furthermore, \((u,v)\) are principal parameters for the surface \( S \).

### 3. Natural principal parameters on space-like Weingarten surfaces

The main object of our considerations are Weingarten space-like surfaces. We use the usual definition for these surfaces in terms of the principal curvatures \( \nu_1 \) and \( \nu_2 \).

**Definition 3.1.** A space-like surface \( S : z= z(u,v) \) \((u,v) \in \mathcal{D}) \) with principal curvatures \( \nu_1 \) and \( \nu_2 \) is said to be a space-like Weingarten surface (or shortly a space-like W-surface) if there exist two differentiable functions \( f, g \) of one variable, defined in the interval \( \mathcal{T} \subseteq \mathbb{R} \) and a differentiable function \( \nu = \nu(u,v) \in \mathcal{T} \) defined in \( \mathcal{D} \), such that \( f'g' \neq 0, \ f - g \neq 0 \), and the principal curvatures of \( S \) at every point are given by \( \nu_1 = f(\nu), \ \nu_2 = g(\nu) \).

The following functions are essential in the theory of space-like W-surfaces:

\( I := \int_{\nu_0}^{\nu} \frac{f'(\nu) d\nu}{f(\nu) - g(\nu)}, \quad J := \int_{\nu_0}^{\nu} \frac{g'(\nu) d\nu}{g(\nu) - f(\nu)}. \)

The next statement gives the property of space-like Weingarten surfaces, which allows us to introduce special principal parameters on such surfaces (cf [10]).

**Lemma 3.2.** Let \( S : z= z(u,v), \ (u,v) \in \mathcal{D} \) be a space-like W-surface parameterized by principal parameters. Then the function

\( \lambda = \sqrt{E} \exp \left( \int_{\nu_0}^{\nu} \frac{f' d\nu}{f - g} \right) = \sqrt{E} e^I \)

does not depend on \( \nu \), while the function

\( \mu = \sqrt{G} \exp \left( \int_{\nu_0}^{\nu} \frac{g' d\nu}{f - g} \right) = \sqrt{G} e^J \)

does not depend on \( u \).

We define natural principal parameters on a space-like W-surface as follows:
Definition 3.3. Let $S: z = z(u, v), (u, v) \in \mathcal{D}$ be a space-like W-surface parameterized by principal parameters. The parameters $(u, v)$ are said to be natural principal, if the functions $\lambda(u)$ and $\mu(v)$ from Lemma 3.2 are constants.

Proposition 3.4. Any space-like Weingarten surface admits locally natural principal parameters.

Proof: Let $S: z = z(u, v), (u, v) \in \mathcal{D}$ be a space-like W-surface, parameterized by principal parameters. Then $\nu_1 = f(\nu), \nu_2 = g(\nu), \nu = \nu(u, v)$ for some differentiable functions $f(\nu), g(\nu)$ and $\nu(u, v)$ satisfying the condition

$$(f(\nu) - g(\nu)) f'(\nu) g'(\nu) \neq 0, \quad (u, v) \in \mathcal{D}.$$ 

Let $a = \text{const} \neq 0, b = \text{const} \neq 0, (u_0, v_0) \in \mathcal{D}$ and $\nu_0 = \nu(u_0, v_0)$. We change the parameters $(u, v) \in \mathcal{D}$ with $(\bar{u}, \bar{v}) \in \mathcal{D}$ by the formulas

$$\bar{u} = a \int_{u_0}^{u} \sqrt{E} e^J du + \bar{u}_0, \quad \bar{u}_0 = \text{const},$$

$$\bar{v} = b \int_{v_0}^{v} \sqrt{G} e^J dv + \bar{v}_0, \quad \bar{v}_0 = \text{const}.$$ 

According to Lemma 3.2 it follows that $(\bar{u}, \bar{v})$ are again principal parameters and

$$(3.2) \quad \bar{E} = a^{-2} e^{-2J}, \quad \bar{G} = b^{-2} e^{-2J}.$$ 

Then for the functions from Lemma 3.2 we find $\lambda(\bar{u}) = |a|^{-1}, \mu(\bar{v}) = |b|^{-1}$.

Furthermore $a^{-2} = E(u_0, v_0), b^{-2} = G(u_0, v_0)$. \hfill $\square$

Corollary 3.5. The principal parameters on a given space-like Weingarten surface are natural principal if and only if

$$(3.3) \quad \sqrt{E}G(\nu_1 - \nu_2) = \text{const} \neq 0.$$ 

Further we assume that the space-like W-surface $S: z = z(u, v), (u, v) \in \mathcal{D}$ under consideration, is parameterized by natural principal parameters $(u, v)$. It follows from the above proposition that the coefficients $E$ and $G$ (consequently $L$ and $N$) are expressed by the invariants of the surface.

As an immediate consequence from Proposition 3.4 we get

Corollary 3.6. Let $S$ be a space-like W-surface parameterized by natural principal parameters $(u, v)$. Then any natural principal parameters $(\bar{u}, \bar{v})$ on $S$ are determined by $(u, v)$ up to an affine transformation of one of the types

$$\bar{u} = a_{11} u + b_1, \quad a_{11}a_{22} \neq 0, \quad \bar{u} = a_{12} v + c_1, \quad a_{12}a_{21} \neq 0,$$

$$\bar{v} = a_{22} v + b_2, \quad \bar{v} = a_{21} u + c_2,$$

where $a_{ij}, b_i, c_i; i, j = 1, 2$ are constants.

3.1. Strongly regular space-like W-surfaces. First we consider strongly regular space-like W-surfaces, i.e. W-surfaces, satisfying the condition

$$\nu_u(u, v)\nu_v(u, v) \neq 0, \quad (u, v) \in \mathcal{D}.$$ 

Our main theorem for such surfaces states

Theorem 3.7. Given two differentiable functions $f(\nu), g(\nu); \nu \in \mathcal{I}, f(\nu) - g(\nu) \neq 0, f'(\nu)g'(\nu) \neq 0$ and a differentiable function $\nu(u, v), (u, v) \in \mathcal{D}$ satisfying the conditions

$$\nu_u \nu_v \neq 0, \quad \nu(u, v) \in \mathcal{I}.$$ 

Let \((u_0, v_0) \in \mathcal{D}, v_0 = \nu(u_0, v_0)\) and \(a > 0, b > 0\) be two constants. If
\[
a^2 e^{2J} (J_{uu} + I_u J_u - J_u^2) + b^2 e^{2J} (I_{vv} + I_v J_v - J_v^2) = -f(\nu) g(\nu),
\]
then there exists a unique (up to a motion) strongly regular space-like W-surface \(S : z = z(u, v), (u, v) \in \mathcal{D}_0 \subset \mathcal{D}\) with invariants
\[
\nu_1 = f(\nu), \quad \nu_2 = g(\nu),
\gamma_1 = b e^J (I)_v, \quad \gamma_2 = -a e^J (J)_u.
\]
Furthermore, \((u, v)\) are natural principal parameters on \(S\).

We also have
\[
a \sqrt{E} = e^{-I}, \quad b \sqrt{G} = e^{-J}.
\]

Hence, with respect to natural principal parameters each strongly regular space-like Weingarten surface possesses a natural PDE (3.4).

### 3.2. Space-like W-surfaces with \(\gamma_1 = 0\)

In this subsection we consider space-like W-surfaces with \(\gamma_1 = 0\) and prove the fundamental theorem of Bonnet type for this class.

Let \(S : z = z(u, v), (u, v) \in \mathcal{D}\) be a space-like W-surface, parameterized by natural principal parameters. Then we can assume
\[
\sqrt{E} = \frac{1}{a} e^I, \quad \sqrt{G} = \frac{1}{b} e^J,
\]
where \(I\) and \(J\) are the functions (3.1) and \(a, b\) are some positive constants. We note that under the condition \(\gamma_1 = 0\) it follows that the function \(\nu = \nu(u)\) does not depend on \(v\).

Considering the system (2.3), we obtain that the compatibility conditions for this system reduce to only one - the Gauss equation, which has the form:
\[
X(\gamma_2) + \gamma_2^2 = f(\nu) g(\nu).
\]

Thus we obtain the following Bonnet type theorem for space-like W-surfaces satisfying the condition \(\gamma_1 = 0\):

**Theorem 3.8.** Given two differentiable functions \(f(\nu), g(\nu)\); \(\nu \in \mathcal{I}, f(\nu) - g(\nu) \neq 0, f'(\nu)g'(\nu) \neq 0\) and a differentiable function \(\nu(u, v) = \nu(u), (u, v) \in \mathcal{D}\) satisfying the conditions
\[
\nu_u \neq 0, \quad \nu(u, v) \in \mathcal{I}.
\]

Let \((u_0, v_0) \in \mathcal{D}, v_0 = \nu(u_0, v_0)\) and \(a > 0\) be a constant. If
\[
a^2 e^{2J} (J_{uu} + I_u J_u - J_u^2) = -f(\nu) g(\nu),
\]
then there exists a unique (up to a motion) space-like W-surface \(S : z = z(u, v), (u, v) \in \mathcal{D}_0 \subset \mathcal{D}\) with invariants
\[
\nu_1 = f(\nu), \quad \nu_2 = g(\nu),
\gamma_1 = 0, \quad \gamma_2 = -a e^J (J)_u.
\]
Furthermore, \((u, v)\) are natural principal parameters on \(S\).

Hence, with respect to natural principal parameters each space-like Weingarten surface with \(\gamma_1 = 0\) possesses a natural ODE (3.6).
4. Parallel space-like surfaces in Minkowski space and their natural PDE’s

Let \( S : z = z(u, v), (u, v) \in \mathcal{D} \) be a space-like surface, parameterized by principal parameters and \( l(u, v), l^2 = -1 \) be the unit normal vector field of \( S \). The parallel surfaces of \( S \) are given by

\[
\bar{S}(a) : \bar{z}(u, v) = z(u, v) + a l(u, v), \quad a = \text{const} \neq 0, \quad (u, v) \in \mathcal{D}.
\]

We call the family \( \{\bar{S}(a), a = \text{const} \neq 0\} \) the parallel family of \( S \).

Taking into account (4.1), we find

\[
\bar{z}_u = (1 - a \nu_1) z_u, \quad \bar{z}_v = (1 - a \nu_2) z_v.
\]

Excluding the points, where \((1 - a \nu_1)(1 - a \nu_2) = 0\), we obtain that the corresponding unit normal vector fields \( \bar{l} \) to \( \bar{S}(a) \) and \( l \) to \( S \) satisfy the equality \( \bar{l} = \varepsilon l \), where \( \varepsilon := \text{sign}(1 - a \nu_1)(1 - a \nu_2) \). Hence, the parallel surfaces \( \bar{S}(a) \) of a space-like surface \( S \) are also space-like surfaces.

The relations between the principal curvatures \( \nu_1(u, v), \nu_2(u, v) \) of \( S \) and \( \bar{\nu}_1(u, v), \bar{\nu}_2(u, v) \) of its parallel space-like surface \( \bar{S}(a) \) are

\[
\bar{\nu}_1 = \varepsilon \frac{\nu_1}{1 - a \nu_1}, \quad \bar{\nu}_2 = \varepsilon \frac{\nu_2}{1 - a \nu_2}; \quad \nu_1 = -\varepsilon \bar{\nu}_1, \quad \nu_2 = \varepsilon \bar{\nu}_2.
\]

Let \( K' = \nu_1 \nu_2, H = \frac{1}{2}(\nu_2 + \nu_2), H' = \frac{1}{2}(\nu_2 - \nu_2) \) be the three invariants of the space-like surface \( S \). The equalities (4.3) imply the relations between the invariants \( \bar{K}', \bar{H} \) and \( \bar{H}' \) of \( \bar{S}(a) \) and the corresponding invariants of \( S \):

\[
\bar{K}' = \frac{K'}{1 + 2a \varepsilon H + a^2 K'}, \quad H = \frac{\varepsilon H + a \bar{K}'}{1 + 2a \varepsilon H + a^2 K'}, \quad \bar{H}' = \frac{\varepsilon \bar{H}'}{1 + 2a \varepsilon H + a^2 K'}.
\]

Now let \( S : z = z(u, v), (u, v) \in \mathcal{D} \) be a space-like Weingarten surface with Weingarten functions \( f(\nu) \) and \( g(\nu) \). We suppose that \( (u, v) \) are natural principal parameters for \( S \).

We show that \( (u, v) \) are also natural principal parameters for any parallel space-like surface \( \bar{S}(a) \).

**Proposition 4.1.** The natural principal parameters \((u, v)\) of a given space-like \( W \)-surface \( S \) are natural principal parameters for all parallel space-like surfaces \( \bar{S}(a), a = \text{const} \neq 0 \) of \( S \).

**Proof:** Let \((u, v) \in \mathcal{D}\) be natural principal parameters for \( S \), \((u_0, v_0)\) be a fixed point in \( \mathcal{D} \) and \( \nu_0 = \nu(u_0, v_0) \). The coefficients \( E \) and \( G \) of the first fundamental form of \( S \) are given by (3.2). The corresponding coefficients \( \bar{E} \) and \( \bar{G} \) of \( \bar{S}(a) \) in view of (4.2) are

\[
\bar{E} = (1 - a \nu_1)^2 E, \quad \bar{G} = (1 - a \nu_2)^2 G.
\]

Equalities (4.3) imply that \( \bar{S}(a) \) is again a Weingarten surface with Weingarten functions

\[
\bar{\nu}_1(u, v) = \bar{f}(\nu) = \frac{\varepsilon f(\nu)}{1 - a f(\nu)}, \quad \bar{\nu}_2(u, v) = \bar{g}(\nu) = \frac{\varepsilon g(\nu)}{1 - a g(\nu)}.
\]

Using (4.6), we compute

\[
\bar{f} - \bar{g} = \frac{\varepsilon (f - g)}{(1 - a f)(1 - a g)},
\]

which shows that \( \text{sign} (\bar{f} - \bar{g}) = \text{sign} (f - g) \).

Further, we denote by \( f_0 := f(\nu_0), g_0 := g(\nu_0) \) and taking into account (3.3) and (4.5), we compute

\[
\sqrt{\bar{E} \bar{G} (\bar{f} - \bar{g})} = \sqrt{E G (f - g)} = \text{const},
\]
which proves the assertion.

Using the above statement, we prove the following theorem.

**Theorem 4.2.** The natural PDE of a given space-like W-surface $S$ is the natural PDE of any parallel space-like surface $\bar{S}(a)$, $a = \text{const} \neq 0$, of $S$.

**Proof.** We have to express the equation (3.4) (resp. (3.6)) in terms of the Weingarten functions of the parallel space-like surface $\bar{S}(a)$. Using (3.1) and (4.6), we compute

$$I = I - \ln \frac{1 - af}{1 - af_0}, \quad J = J - \ln \frac{1 - ag}{1 - ag_0}.$$ 

Putting

$$\bar{E}_0 = (1 - a\nu_1(u_0, v_0))^2 E_0 = a^{-2}(1 - af_0)^2 =: \bar{a}^{-2},$$ 

$$\bar{G}_0 = (1 - a\nu_2(u_0, v_0))^2 G_0 = b^{-2}(1 - ag_0)^2 =: \bar{b}^{-2},$$

we obtain

$$a^2 e^{2\bar{f}}(\bar{J}_{uu} + \bar{I}_u \bar{J}_u - \bar{J}_u^2) + \bar{b}^2 e^{2\bar{f}}(\bar{I}_{vv} + \bar{I}_v \bar{J}_v - \bar{J}_v^2) + \bar{f}(\nu) \bar{g}(\nu)$$

$$= a^2 e^{2f} (J_{uu} + I_u J_u - J_u^2) + b^2 e^{2f} (I_{vv} + I_v J_v - J_v^2) + f(\nu) g(\nu).$$

Hence, the natural PDE of $\bar{S}(a)$ in terms of the Weingarten functions $\bar{f}(\nu)$, $\bar{g}(\nu)$ coincides with the natural PDE of $S$ in terms of the Weingarten functions $f(\nu)$ and $g(\nu)$. \qed

**5. Linear fractional space-like Weingarten surfaces**

In this section we study special classes of space-like Weingarten surfaces with respect to the Weingarten functions $f$ and $g$.

A space-like W-surface with principal curvatures $\nu_1$ and $\nu_2$ is said to be **linear fractional** if

$$\nu_1 = \frac{A\nu_2 + B}{C\nu_2 + D}, \quad A, B, C, D \text{ constants, } \quad BC - AD \neq 0.$$ 

We exclude the case $A = D$, $B = C = 0$, which characterizes the umbilical points.

**Lemma 5.1.** Any linear fractional space-like Weingarten surface determined by (5.1) is a surface whose invariants $K', H, H'$ satisfy the linear relation

$$\delta K' = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants, } \quad \alpha^2 - \beta^2 + 4\gamma\delta \neq 0$$

and vice versa.

The relations between the constants $A, B, C, D$ in (5.1) and $\alpha, \beta, \gamma, \delta$ in (5.2) are given by the equalities:

$$\alpha = A - D, \quad \beta = -(A + D), \quad \gamma = B, \quad \delta = C.$$ 

We denote by $\mathcal{K}$ the class of all space-like surfaces, free of umbilical points, whose curvatures satisfy (5.2) or equivalently (5.1).

The aim of our study is to classify all natural PDE’s of the surfaces from the class $\mathcal{K}$.

The parallelism between two surfaces given by (4.1) is an equivalence relation. On the other hand, Theorem 4.2 shows that the surfaces from an equivalence class have one and the same natural PDE. Hence, it is sufficient to find the natural PDE’s of the equivalence classes. For any equivalence class, we use a special representative, which we call a **basic class**. Thus the classification of the natural PDE’s of the surfaces in the class $\mathcal{K}$ reduces to the natural PDE’s of the basic classes.

In view of Theorem 4.2, we prove the following classification Theorem.
Theorem 5.2. Up to similarity, the space-like surfaces in Minkowski space, whose curvatures \(K', H\) and \(H'\) satisfy the linear relation
\[
\delta K' = \alpha H + \beta H' + \gamma, \quad \alpha, \beta, \gamma, \delta - \text{constants; } \alpha^2 - \beta^2 + 4 \gamma \delta \neq 0,
\]
are described by the natural PDE’s of the following basic surfaces:

1. \(H = 0\) : \(\nu = -e^\lambda, \quad \Delta \lambda = e^\lambda,\)

2. \(H = \frac{1}{2}\) : \(\nu = \frac{1}{2}(1 - e^\lambda), \quad \Delta \lambda = \sinh \lambda,\)

3. \(H' = 1\) : \(\Delta^*(\nu^\rho) = 2 \nu (\nu + 2),\)

4. \(H = p H', \quad p^2 > 1\) : \(\Delta^*(\nu^\rho) = 2 \frac{p(p + 1)}{(p - 1)^2} \nu,\)

5. \(H = p H', \quad p^2 < 1\) : \(\Delta^*(\nu^\rho) = 2 \frac{p(p + 1)}{(p - 1)^2} \nu,\)

6. \(H = p H' + 1, \quad p^2 > 1\) : \(\nu = \frac{(p - 1) \lambda + 2}{2}, \quad \Delta^*(\lambda^\rho) = \frac{p((p - 1)\lambda + 2)((p + 1)\lambda + 2)}{2(p - 1) \lambda},\)

7. \(H = p H' + 1, \quad p^2 < 1\) : \(\nu = \frac{(p - 1) \lambda + 2}{2}, \quad \Delta^*(\lambda^\rho) = \frac{p((p - 1)\lambda + 2)((p + 1)\lambda + 2)}{2(p - 1) \lambda},\)

8. \(K' = -1\) : \(\nu = \tan \lambda, \quad \bar{\Delta} \lambda = -\sin \lambda,\)

9. \(K' = 2 H'\) : \(\nu = \frac{\lambda - 4}{\lambda - 2}, \quad \Delta^*(e^\lambda) = 2,\)

10. \(K' = p H' - q, \quad p \neq 0, q > 0\) : \(\nu = \frac{\lambda + \frac{p}{2}}{\sqrt{q}}, \quad \mathcal{I} = \frac{1}{\sqrt{q}} \arctan \frac{\lambda}{\sqrt{q}}, \quad \Delta^*(e^\rho \mathcal{I}) = \frac{p q \lambda}{2} \frac{(p \lambda - 2 q)}{\lambda^2 + q}.

Proof: According to the constant \(C\) in (5.1), the linear fractional space-like W-surfaces are divided into two classes: linear fractional space-like W-surfaces, determined by the condition \(C = 0\) and linear fractional space-like W-surfaces, determined by the condition \(C \neq 0\).

I. Linear fractional space-like Weingarten surfaces with \(C = 0\). This class is determined by the equality

\(\alpha H + \beta H' + \gamma = 0, \quad (\alpha, \gamma) \neq (0, 0), \quad \alpha^2 - \beta^2 \neq 0.\)

For the invariants of the space-like parallel surface \(\bar{S}(a)\) of \(S\), because of (4.4), we get the relation

\(\varepsilon (\alpha + 2 a \gamma) \bar{H} + \varepsilon \beta \bar{H}' + \gamma = -a (\alpha + a \gamma) \bar{K}'.\)

Let \(\eta := \text{sign} (\alpha^2 - \beta^2)\). Each time choosing appropriate values for the constants \(a, b\) and \(\nu_0\) in (3.4), we consider the following subclasses and their natural PDE’s:
1) $\alpha = 0$, $\beta \neq 0$, $\gamma \neq 0$. Assuming that $\gamma = 1$, the relation (5.4) becomes

$$\beta H' + 1 = 0.$$ 

The natural PDE for these W-surfaces is

$$(5.6) \quad (e^\beta \nu)_{vv} + (e^{-\beta} \nu)_{uu} = \frac{2}{\beta} \nu (\beta \nu - 2).$$

Up to similarities these W-surfaces are generated by the basic class $H' = 1$ with

$$H + \beta H' = 0.$$ 

2) $\alpha \neq 0$, $\gamma = 0$. Assuming that $\alpha = 1$, the relation (5.4) becomes

$$H + \gamma = 0.$$ 

2.1) $\beta \neq 0$, $\eta = -1$ ($\beta^2 - 1 > 0$). Choosing $b^2 \frac{\beta - 1}{\beta + 1} v_0^{-\beta+1} = 1$, $a^2 v_0^{\beta-1} = 1$, the natural PDE becomes

$$(5.7) \quad \left(\nu^\beta\right)_{vv} + \left(\nu^{-\beta}\right)_{uu} = 2 \frac{\beta(\beta - 1)}{(\beta + 1)^2} \nu,$$

which is the case (4) in the statement of the theorem.

2.2) $\beta \neq 0$, $\eta = 1$ ($\beta^2 - 1 < 0$). Choosing $b^2 \frac{\beta - 1}{\beta + 1} v_0^{-\beta+1} = -1$, $a^2 v_0^{\beta-1} = 1$, the natural PDE becomes

$$(5.8) \quad \left(\nu^\beta\right)_{vv} - \left(\nu^{-\beta}\right)_{uu} = -2 \frac{\beta(\beta - 1)}{(\beta + 1)^2} \nu,$$

which is the case (5) in the statement of the theorem.

2.3) $\beta = 0$. Putting $\nu = e^\lambda$, we get the natural PDE for space-like surfaces with $H = 0$:

$$(5.9) \quad \Delta \lambda = e^\lambda,$$

which is the case (1) in the statement of the theorem.

3) $\alpha \neq 0$, $\beta = 0$, $\gamma \neq 0$. Assuming that $\alpha = 1$, the relation (5.4) becomes

$$H + \gamma = 0.$$ 

Putting $|H| e^\lambda := H - \nu = H' > 0$, we get the one-parameter system of natural PDE’s for CMC space-like surfaces with $H = -\gamma$:

$$(5.10) \quad \Delta \lambda = 2 |H| \sinh \lambda.$$ 

Up to similarities these W-surfaces are generated by the basic class $|H| = \frac{1}{2}$ with

$$(5.10^*) \quad \Delta \lambda = \sinh \lambda,$$

which is the case (2) in the statement of the theorem.
4) \( \alpha \neq 0, \beta \neq 0, \gamma \neq 0 \).

Assuming that \( \alpha = 1 \) we have

\[
H + \beta H' + \gamma = 0, \quad \beta^2 - 1 \neq 0.
\]

Let \( \lambda := 2H = \frac{-2}{\beta + 1}(\nu + \gamma) > 0 \).

4.1) If \( \eta = -1 (\beta^2 - 1 > 0) \), and choosing

\[
b^2 = \frac{\beta + 1}{\beta - 1} \left( \frac{-2}{\beta + 1}(\nu_0 + \gamma) \right)^{\beta + 1}, \quad a^2 = \left( \frac{-2}{\beta + 1}(\nu_0 + \gamma) \right)^{-(\beta - 1)},
\]

the natural PDE becomes

\[
(\lambda^\beta)_{vv} + (\lambda^{-\beta})_{uu} = \frac{\beta}{2(\beta + 1)} \frac{((\beta + 1)\lambda + 2\gamma)((\beta - 1)\lambda + 2\gamma)}{\lambda}.
\]

Up to similarities these W-surfaces are generated by the basic class \( H = \beta H' + 1, \beta^2 > 1 \) with the natural PDE

\[
(\lambda^\beta)_{uu} + (\lambda^{-\beta})_{vv} = \frac{\beta}{2(\beta - 1)} \frac{((\beta + 1)\lambda + 2\gamma)((\beta - 1)\lambda + 2\gamma)}{\lambda},
\]

which is the case (6) in the statement of the theorem.

4.2) If \( \eta = 1 (\beta^2 - 1 < 0) \), and choosing

\[
b^2 = -\frac{\beta + 1}{\beta - 1} \left( \frac{-2}{\beta + 1}(\nu_0 + \gamma) \right)^{\beta + 1}, \quad a^2 = \left( \frac{-2}{\beta + 1}(\nu_0 + \gamma) \right)^{-(\beta - 1)},
\]

the natural PDE becomes

\[
(\lambda^\beta)_{vv} - (\lambda^{-\beta})_{uu} = \frac{-\beta}{2(\beta + 1)} \frac{((\beta + 1)\lambda + 2\gamma)((\beta - 1)\lambda + 2\gamma)}{\lambda}.
\]

Up to similarities these W-surfaces are generated by the basic class \( H = \beta H' + 1, \beta^2 < 1 \) with the natural PDE

\[
(\lambda^\beta)_{uu} - (\lambda^{-\beta})_{vv} = \frac{\beta}{2(\beta - 1)} \frac{((\beta + 1)\lambda + 2\gamma)((\beta - 1)\lambda + 2\gamma)}{\lambda},
\]

which is the case (7) in the statement of the theorem.

II. Linear fractional space-like Weingarten surfaces with \( C \neq 0 \). Let \( C = 1 \). The equality (5.2) gets the form

\[
K' = \alpha H + \beta H' + \gamma, \quad \alpha^2 - \beta^2 + 4\gamma \neq 0.
\]

The corresponding relation for the parallel surface \( \tilde{S}(a) \) is

\[
\varepsilon(\alpha + 2a\gamma) \tilde{H} + \varepsilon \beta \tilde{H}' + \gamma = (1 - a\alpha - a^2\gamma) \tilde{K}'.
\]

Each time choosing appropriate values for the constants \( a, b \) and \( \nu_0 \) in (3.4), we consider the following subclasses and their natural PDE’s:

5) \( \alpha = \gamma = 0, \beta \neq 0 \). The relation (5.13) becomes

\[
K' = \beta H' \iff \rho_1 - \rho_2 = -\frac{2}{\beta},
\]

where \( \rho_1 = \frac{1}{\nu_1}, \rho_2 = \frac{1}{\nu_2} \) are the principal radii of curvature of \( S \).
Putting $\lambda := 4 \frac{\nu - \beta}{2 \nu - \beta^2}$ and choosing $\nu_0 = \beta$, the natural PDE of these space-like surfaces gets the form

\begin{equation}
(e^\lambda)_{uu} + (e^{-\lambda})_{vv} - \frac{\beta^4}{8} = 0.
\end{equation}

(5.15)

Up to similarities these W-surfaces are generated by the basic class $K' = 2 H'$ with the natural PDE

\begin{equation}
(e^\lambda)_{uu} + (e^{-\lambda})_{vv} - 2 = 0,
\end{equation}

(5.15*)

which is the case (9) in the statement of the theorem.

**Remark 5.3.** A. Ribaucour [17] has proved that a necessary condition for the curvature lines of the first and second focal surfaces of $S$ to correspond to each other resp. to a conjugate parametric lines on $S$ is $\rho_1 - \rho_2 = \text{const resp. } \rho_1 \rho_2 = \text{const}$.

Von Lilienthal [19] (cf [20, 3, 4, 7]) has proved in $\mathbb{R}^3$ that a surface with $\rho_1 - \rho_2 = 1/R$, $R = \text{const} \neq 0$, has first and second focal surfaces of constant Gauss curvature $-R^2$ and vice versa.

In $\mathbb{R}^3$ one can prove in a similar way the corresponding property: The first and second focal surfaces of a space-like surface with $K' = \beta H'$ are time-like of constant Gauss curvature $\beta^2/4$, and vice versa.

6) $(\alpha, \gamma) \neq (0, 0)$, $\alpha^2 + 4\gamma \geq 0$. The relation (5.14) implies that there exists a space-like surface $\bar{S}(a)$, parallel to $S$, which satisfies the relation (5.4). Hence the natural PDE of $S$ is one of the PDE’s (5.6) - (5.12).

7) $\alpha^2 + 4 \gamma < 0$. It follows that $\gamma < 0$. The relation (5.14) implies that there exists a surface $\bar{S}(a)$ parallel to $S$, which satisfies the relation

\begin{equation}
K' = \beta H' + \gamma.
\end{equation}

(5.16)

7.1) $\beta = 0$. The relation (5.16) becomes $K' = \gamma < 0$, i.e. $\bar{S}$ is of constant negative extrinsic sectional curvature $\gamma$ (of constant positive intrinsic sectional curvature $K = -\gamma$). Putting $\lambda := 2 \arctan \frac{\nu}{\sqrt{-\gamma}}$, we get the natural PDE of this surface

\begin{equation}
\Delta \lambda = -K^2 \sin \lambda.
\end{equation}

(5.17)

Up to similarities these W-surfaces are generated by the basic class $K' = -1$ ($K = 1$) with the natural PDE

\begin{equation}
\Delta \lambda = - \sin \lambda,
\end{equation}

(5.17*)

which is the case (8) in the statement of the theorem.

7.2) $\beta \neq 0$, $\gamma < 0$. Choosing $\nu_0 = \frac{\beta}{2}$, the natural PDE of $S$ becomes

\begin{equation}
(\exp (\beta I))_{uu} + (\exp (-\beta I))_{vv} = -\frac{\beta}{2} \frac{\lambda (\beta \lambda + 2 \gamma)}{\lambda^2 - \gamma},
\end{equation}

(5.18)

where

\[ I = \frac{1}{\sqrt{-\gamma}} \arctan \frac{\lambda}{\sqrt{-\gamma}}, \quad \lambda := \nu - \frac{\beta}{2}, \]

which is the case (10) in the statement of the theorem.

\[ \square \]
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