The scale of the Fourier transform: a point of view of the fractional Fourier transform

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Abstract. In this paper using the Fourier transform of order fractional, the ray transfer matrix for the symmetrical optical systems type ABCD and the formulae by Collins for the diffraction, we obtain explicitly the expression for scaled Fourier transform conventional; this result is the great importance in optical signal processing because it offers the possibility of scaling the size of output the Fourier distribution of the system, only by manipulating the distance of the diffraction object toward the thin lens, this research also emphasizes on practical limits when a finite spherical converging lens aperture is used. Digital simulation was carried out using the numerical platform of Matlab 7.1.

1. Introduction

The Fractional Fourier transform (FrFT) is a generalization of the classical or conventional Fourier Transform, which has been introduced into the mathematical literature by Namias in 1980 [1]. Later, it was used for optical applications by several authors [2, 3]. Solution of differential equations, quantum mechanics, electromagnetic theory of propagation, interference and diffraction of light, optical system design, image processing, multiple space-variant filtering and study of space or time-frequency distributions are some of the applications of the FrFT.

The purpose of this paper is to find a direct relationship between the Fractional Fourier Transform (FrFT), the Collins diffraction formula and the scaled Fourier transform for the complex amplitude distributions on spherical surfaces. As result, we present the digital computation of the scaled Fourier transform with different values of scaling. The two-dimensional fractional Fourier transform (FrFT) at order $\alpha$, is a linear integral transformation that maps a given function $f(x,y)$ onto a function $f_\alpha(u,v)$, by [2]

$$f_\alpha(u,v) = F^\alpha \{ f(x,y) \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_\alpha(u,v,x,y)f(x,y)dx dy,$$  \hspace{1cm} (1)

with

$$K_\alpha(u,v,x,y) = C_\alpha e^{-i\alpha \left[ \left( u^2 + v^2 + x^2 + y^2 \right) \cot \alpha - 2(xu + vy) \csc \alpha \right]} , \hspace{1cm} C_\alpha = \frac{-ie^{i\alpha}}{\sin \alpha}, \hspace{1cm} -\pi < \alpha \leq \pi,$$  \hspace{1cm} (2)
where $K_\alpha$ is the fractional Fourier kernel. For $\alpha = 0$, it corresponds to the identity transform. For $\alpha = \pi/2$, it reduces to the direct standard Fourier transform. For $\alpha = \pi$, the reverse transform is obtained. For $\alpha = -\pi/2$, it corresponds to the inverse standard Fourier transform. The inverse FrFT corresponds to the FrFT at fractional order $-\alpha$. The FrFT operator is additive with respect to the fractional order, $\Phi^{\alpha} \Phi^{\beta} = \Phi^{\alpha+\beta}$. The properties of the FrFT have been studied in reference [2].

2. Ray transfer matrix
A paraxial ray in a given cross section of an optical system is characterized by its distance $x_1$ from the optic axis and the slope $y_1$; if this slope is assumed small, the ray path through a given structure depends on the optical properties of the structure and the input conditions. In this situation the relation between the input and output parameters of the optical system is given by

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}. \tag{3}$$

The ABCD matrix is called the ray transfer matrix with its determinant equal to one [4].

3. The fractional Fourier transform and the Collins formula
The known Collins formula can be rewritten as [5]

$$U_p(u,v) = \frac{-i}{\lambda B} \exp \left\{ \frac{i\pi D(u^2+v^2)}{\lambda B} \right\} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp \left\{ i\frac{\pi A(\xi^2+\eta^2)}{\lambda B} \right\} \exp \left\{ -i\frac{2\pi(u\xi+v\eta)}{\lambda B} \right\} \times U_A(\xi,\eta) d\xi d\eta. \tag{4}$$

Illuminating with a spherical wave of radius $R/A$ to the input plane $U_A(\xi,\eta)$, introducing appropriate scale parameters as well as input and output coordinates [3, 5], and using the meaning of the FrFT, the previous equation can be given by

$$U_p(u,v) = \frac{e^{-ia}}{\lambda B} \sin \alpha \exp \left\{ \frac{D}{B} - \cos \alpha \frac{\cos \alpha}{A \left( 1 - \frac{B}{R} \right) B} \right\} \int \frac{i\pi(u^2+v^2)}{\lambda} F^{\alpha} \{ U_A(\xi,\eta) \}. \tag{5}$$

An interesting fact is the relationship between a FrFT of order $\alpha$, the output complex field amplitude $U_p(u,v)$ and the input complex field $U_A(\xi,\eta)$. The phase factor is associated in paraxial approximation with a divergent wave [3, 5-7]. Therefore, the output complex field amplitude $U_p(u,v)$ is over a spherical surface with radius $R_2$, which is proportional to the FrFT of order $\alpha$ of the input complex field amplitude $U_A(\xi,\eta)$ [8-10]. The equation (5) corresponds to the Collins Formulae in terms of the fractional Fourier transform and it describes the diffraction in an optical system type ABCD, which is characterized for its ray transfer matrix.

4. Optical systems and propagation
Using the previous results, the relationship between the Collins diffraction formula, the ray transfer matrix and the fractional Fourier transform (FrFT), we developed the diffraction formulae for the well-known optical systems, and then we obtain the resultant expression between the complex amplitude distributions of an object and its propagation for an observation plane. In this case, it is considered the proposed setup of figure 1. This system is composed by an illuminating wave that propagates through the surface of the lens placed at point $(\xi,\eta)$; a pupil is located at point $(u,v)$, and finally, the observation plane is at point $(x,y)$, which is placed in the focal length of the lens; the distance between the pupil and the plane observation is given by $d$. 


Figure 1. Proposed optical system.

Using a paraxial approach to a spherical wave that illuminates the input, the amplitude of the transmitted wave by the input and which reaches to the pupil can be written as

$$U_0(u,v) = \frac{Af}{d} P\left(u\frac{f}{d}, v\frac{f}{d}\right) \exp\left\{-\frac{i\pi(u^2 + v^2)}{\lambda d} \right\} t_A(u,v). \quad (6)$$

This paraxial approach corresponds to an spherical convergent wave towards the focal plane of radius $R_1 = -d$; applying the FrFT to the wave propagation from the coordinate plane to the point $U_0(u,v)$, where the pupil is found until the coordinate plane to the point $U_S(x,y)$, where the following observation is made and replacing the values of the ray transfer matrix elements for free space propagation ($A = D = 1, B = d$, and $C = 0$) in the complex amplitude distribution for the observation plane and $R_1 = -d$, we get

$$U_S(x,y) = e^{-i\alpha} \frac{\sin \alpha}{\lambda d} \exp\left\{i\frac{\pi(x^2 + y^2)}{\lambda R_2} \right\} F^\alpha \{U_0(u,v)\}, \quad (7)$$

where

$$R_2 = \frac{d}{1 - \cos^2 \alpha}. \quad (8)$$

Taking into account that

$$F^\alpha \{U_0(u,v)\} = \frac{-ie^{i\alpha}}{\lambda d \sin \alpha} \exp\left\{i\frac{\pi(x^2 + y^2)}{\lambda d \tan \alpha} \right\} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-\frac{i2\pi(xu + yv)}{\lambda d \sin \alpha} \right\}$$

$$\times \frac{Af}{d} P\left(u\frac{f}{d}, v\frac{f}{d}\right) t_A(u,v) du dv. \quad (9)$$

In the particular case if $\alpha = \pi/2$

$$F^{\pi/2} \{U_0(u,v)\} = \frac{1}{\lambda d} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{Af}{d} P\left(u\frac{f}{d}, v\frac{f}{d}\right) t_A(u,v) \exp\left\{-\frac{i2\pi(xu + yv)}{\lambda d} \right\} du dv, \quad (10)$$

this result corresponds to the classical Fourier transform $F^{\pi/2} = F$

$$F^{\pi/2} \{U_0(u,v)\} = F \left\{\frac{Af}{d} P\left(u\frac{f}{d}, v\frac{f}{d}\right) t_A(u,v) \right\}. \quad (11)$$
Finally, replacing the previous result into the equation (7), we obtain

\[
U_s(x, y) = \frac{A}{i \lambda d} \frac{1}{d} \exp \left\{ i \pi (x^2 + y^2) \right\} F \left\{ P \left( \frac{u}{d}, \frac{v}{d} \right) t_d(u, v) \right\}.
\]  \hspace{1cm} (12)

As one can notice from the configuration appears the obtained parameter of flexibility described in [4], that corresponds to the scaling factor of the Fourier Transform, which is under control by the researcher. When \( d \) increases, it means that the scaling factor in the direct representation (spatial) produces a reduction of the diffraction object; consequently the size of the transform is bigger, which corresponds a zoom of the pupil (where the object of the transmission is) directly to the lens, until the overhead transparency is directly against the lens; when the distance \( d \) decreases, it means that the scaling factor in the direct representation produces an expansion of the diffraction object; consequently the size of the transform is smaller, which corresponds a zoom of the pupil in direction of the plane of observation. These considerations are very important in spatial filtering applications.

5. Digital simulations

The results of figure 2 represent how the variety of size of the diffraction object, that is on the pupil, works when a displacement is produced, zooming or moving away the pupil of the surface observation and/or of the pupil. With the purpose of showing the behavior of the proposed optical system of the figure 1, it is used the diffraction object of figure 2(a), which has into its structure a chessboard. The figure 2(b) shows its Fourier transform of the image of figure 2(a) without any scaling; the figure 2(c) shows the Fourier transform when an increase of the diffraction object is produced with a scaling factor of 1.25. The figure 2(d) corresponds to the Fourier transform of a reduced object with a factor 0.5.

![Figure 2. Fourier transform obtained by digital simulation of the proposed optical system.](image)

(a) Diffraction object. (b) Fourier transform of the diffraction object. Fourier transform of the diffraction object with a scaling factor of: (c) 1.25, and (d) 0.5.

6. Conclusions

In the present paper was examined the behavior of the scaled Fourier transform (SFT) for the two-dimensional case, in the particular situation proposed by Goodman when the optical setup corresponds to the case of scaling factor using Fourier transform. We obtain the same result as Goodman, only with the help of the ray transfer matrix, the Collins formula and the fractional Fourier transform. In this analysis we assume that the object is totally illuminated by a spherical beam perfectly convergent and it has been demonstrated that the Fractional Fourier transform allows to interpret the same details of the scaling factors for the diffraction object as well as the conventional Fourier Transform. We proved that the scaling factor obtained is expressed in terms of the distance between the lens position and diffraction object, when the plane of observation is placed in the back focal plane of the lens.

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