Seperability of Tripartite Quantum Systems

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Abstract

We investigate the separability of arbitrary dimensional tripartite systems. By introducing a new operator related to transformations on the subsystems a necessary condition for the separability of tripartite systems is presented.

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Quantum entanglement has been recently recognized as the most essential ingredient in the quantum information technology. One of the important problems in the theory of quantum entanglement is the separability. A multipartite state is called fully separable if and only if the density matrix $\rho_{AB\cdots C}$ can be written as:

$$\rho_{AB\cdots C} = \sum_i p_i \rho_i^A \otimes \rho_i^B \otimes \cdots \otimes \rho_i^C,$$

where $\rho_i^A, \rho_i^B, \cdots, \rho_i^C$ are density matrices associated with the subsystems $A, B, \cdots, C$, and $0 < p_i \leq 1$, $\sum_i p_i = 1$.

Many separability criteria have been found in recent years. For pure states, the problem is completely solved, e.g., by using the Schmidt decomposition \cite{1}. For mixed states, there are separability criteria such as PPT, reduction, majorization, realignment etc. \cite{2, 3, 4, 5, 6}. In \cite{7} the authors have given a lower bound of concurrence for tripartite quantum states which can be used to detect entanglement. In \cite{8} the authors have provided a numerically computable criterion which can detect PPT entangled states for three qubits systems. The efficient criterion is then generalized to tripartite systems with arbitrary dimensions \cite{9}. In \cite{10} some nice results show that some quantity related to Hermitian matrix is positive for quantum mixed states in $2 \times N$ systems, which was further discussed in \cite{11}. These results were generalized to arbitrary dimensional bipartite systems (or $2 \times 2 \times N$ quantum systems) in \cite{12}. In this paper, we study arbitrary tripartite systems in analogue to the approach used in \cite{10, 11, 12}. The properties of tripartite density matrices are studied in terms of the Bloch representations. A necessary condition for the separability of tripartite states has been obtained. These results are non-trivial when they are reduced to bipartite systems discussed in \cite{10, 11, 12} and the separability criterion do detect some entanglements.
Any Hermitian operator on an $N$-dimensional Hilbert space $\mathcal{H}_N$ can be expressed according to the generators of the special unitary group $SU(N)$ [13]. The generators of $SU(N)$ can be introduced according to the transition-projection operators

$$P_{jk} = |j\rangle\langle k|,$$

where $|i\rangle$, $i = 1, ..., N$, are the orthonormal eigenstates of a linear Hermitian operator on $\mathcal{H}_N$. Set

$$\omega_l = -\sqrt{\frac{2}{l(l+1)}}(P_{11} + P_{22} + \cdots + P_{ll} - lP_{l+1,l+1}),$$

$$u_{jk} = P_{jk} + P_{kj},$$

$$v_{jk} = i(P_{jk} - P_{kj}),$$

where $1 \leq l \leq N - 1$ and $1 \leq j < k \leq N$. We get a set of $N^2 - 1$ operators

$$\Gamma \equiv \{\omega_1, \omega_2, \cdots, \omega_{N-1}, u_{12}, u_{13}, \cdots, v_{12}, v_{13}, \cdots\},$$

which satisfy the relations

$$\text{Tr}\{\lambda_i\} = 0, \quad \text{Tr}\{\lambda_i\lambda_j\} = 2\delta_{ij}, \quad \forall \lambda_i \in \Gamma$$

and thus generate the $SU(N)$ [14].

Any Hermitian operator $\rho$ in $\mathcal{H}_N$ can be represented in terms of these generators of $SU(N)$,

$$\rho = \frac{1}{N}I_N + \frac{1}{2} \sum_{j=1}^{N^2-1} r_j \lambda_j,$$  \hspace{1cm} (1)

where $I_N$ is a unit matrix and $r = (r_1, r_2, \cdots, r_{N^2-1}) \in \mathbb{R}^{N^2-1}$. $r$ is called Bloch vector. The set of all the Bloch vectors that constitute a density operator is known as the Bloch vector space $B(\mathbb{R}^{N^2-1})$.

A matrix of the form (1) is of unit trace and Hermitian, but it might not be positive. To guarantee the positivity restrictions must be imposed on the Bloch vector. It is shown that $B(\mathbb{R}^{N^2-1})$ is a subset of the ball $D_R(\mathbb{R}^{N^2-1})$ of radius $R = \sqrt{2(1 - \frac{1}{N})}$, which is the minimum ball containing it, and that the ball $D_r(\mathbb{R}^{N^2-1})$ of radius $r = \sqrt{\frac{2}{N(N-1)}}$ is included in $B(\mathbb{R}^{N^2-1})$ [15], that is,

$$D_r(\mathbb{R}^{N^2-1}) \subseteq B(\mathbb{R}^{N^2-1}) \subseteq D_R(\mathbb{R}^{N^2-1}).$$

Let the dimensions of systems A, B and C be $N_1, N_2$ and $N_3$ respectively. Any tripartite quantum states $\rho_{ABC} \in \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2} \otimes \mathcal{H}_{N_3}$ can be written as:

$$\rho_{ABC} = I_{N_1} \otimes I_{N_2} \otimes M_0 + \sum_{i=1}^{N_1^2-1} \lambda_i(1) \otimes I_{N_2} \otimes M_i + \sum_{j=1}^{N_2^2-1} I_{N_1} \otimes \lambda_j(2) \otimes \tilde{M}_j$$
Noticing that $ho$ is a tripartite quantum state, where $\lambda_i(1)$, $\lambda_j(2)$ are the generators of $SU(N_1)$ and $SU(N_2)$; $M_i$, $\tilde{M}_j$ and $M_{ij}$ are operators of $\mathcal{H}_{N_3}$.

[Theorem 1] Let $\mathbf{r} \in \mathbb{R}^{N_1^2-1}$, $\mathbf{s} \in \mathbb{R}^{N_2^2-1}$ and $|\mathbf{r}| \leq \sqrt{\frac{2}{N_1(N_1-1)}}$, $|\mathbf{s}| \leq \sqrt{\frac{2}{N_2(N_2-1)}}$. For a tripartite quantum state $\rho \in \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2} \otimes \mathcal{H}_{N_3}$ with representation (2), we have

$$M_0 - \sum_{i=1}^{N_1^2-1} r_i M_i - \sum_{j=1}^{N_2^2-1} s_j \tilde{M}_j + \sum_{i=1}^{N_1^2-1} \sum_{j=1}^{N_2^2-1} r_i s_j M_{ij} \geq 0. \quad (3)$$

[Proof] Since $\mathbf{r} \in \mathbb{R}^{N_1^2-1}$, $\mathbf{s} \in \mathbb{R}^{N_2^2-1}$ and $|\mathbf{r}| \leq \sqrt{\frac{2}{N_1(N_1-1)}}$, $|\mathbf{s}| \leq \sqrt{\frac{2}{N_2(N_2-1)}}$, we have that $A_1 \equiv \frac{1}{2}(\frac{2}{N_1^2} I - \sum_{i=1}^{N_1^2-1} r_i \lambda_i(1))$ and $A_2 \equiv \frac{1}{2}(\frac{2}{N_2^2} I - \sum_{j=1}^{N_2^2-1} s_j \lambda_j(2))$ are positive Hermitian operators. Let $A = \sqrt{A_1} \otimes \sqrt{A_2} \otimes I_{N_3}$. Then $A\rho A \geq 0$ and $(A\rho A)^\dagger = A\rho A$. The partial trace of $A\rho A$ over $\mathcal{H}_{N_1}$ (and $\mathcal{H}_{N_2}$) should be also positive. Hence

$$0 \leq Tr_{AB}(A\rho A)$$

$$= Tr_{AB}(A_1 \otimes A_2 \otimes M_0 + \sum_i \sqrt{A_1} \lambda_i(1) \sqrt{A_1} \otimes A_2 \otimes M_i + \sum_j A_1 \otimes \sqrt{A_2} \lambda_j(2) \sqrt{A_2} \otimes \tilde{M}_j)$$

$$+ \sum_{ij} \sqrt{A_1} \lambda_i(1) \sqrt{A_1} \otimes \sqrt{A_2} \lambda_j(2) \sqrt{A_2} \otimes M_{ij}).$$

$$= M_0 - \sum_{i=1}^{N_1^2-1} r_i M_i - \sum_{j=1}^{N_2^2-1} s_j \tilde{M}_j + \sum_{i=1}^{N_1^2-1} \sum_{j=1}^{N_2^2-1} r_i s_j M_{ij}.$$

Formula (3) is valid for any tripartite states. By setting $\mathbf{s} = 0$ in (3), one can get a result for bipartite systems:

[Corollary 1] Let $\rho_{AB} \in \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}$ which can be generally written as $\rho_{AB} = I_{N_1} \otimes M_0 + \sum_{j=1}^{N_2^2-1} \lambda_j \otimes M_j$, then for any $\mathbf{r} \in \mathbb{R}^{N_2^2-1}$ with $|\mathbf{r}| \leq \sqrt{\frac{2}{N_1(N_1-1)}}$, $M_0 - \sum_{j=1}^{N_2^2-1} r_j M_j \geq 0$.

In [12], a separability criterion for $N_1 \times N_2$ systems has been obtained: if $\rho_{AB}$ is separable, then $M_0 - \sum_{j=1}^{N_2^2-1} \frac{4}{3N_1 - 2} d_j M_j$ is positive for any vector $\vec{d} = (d_1, d_2, \ldots, d_{N_2^2-1})$ with $|\vec{d}| \leq 1$. Noticing that $\frac{4}{3N_1 - 2} \leq \sqrt{\frac{2}{N_1(N_1-1)}}$ for any $N_1 \geq 2$, we get from our corollary that this criterion can not recognize any bipartite entangled states, as it is true for both entangled and separable states.

A separable state $\rho_{ABC}$ can be written as

$$\rho_{ABC} = \sum_i p_i |\psi_i^A\rangle \langle \psi_i^A| \otimes |\phi_i^B\rangle \langle \phi_i^B| \otimes |\omega_i^C\rangle \langle \omega_i^C|.$$
From (1) it can also be represented as:

\[
\rho_{ABC} = \sum_i p_i \left( \frac{1}{2} N_1^i I_{N_1} + \sum_{k=1}^{N_1^2-1} a^{(k)}_i \lambda_k(1) \right) \otimes \frac{1}{2} \left( \sum_{l=1}^{N_2^2-1} b^{(l)}_i \lambda_l(2) \right) \otimes |\omega_i^C \rangle \langle \omega_i^C |
\]

\[
= I_{N_1} \otimes I_{N_2} \otimes \frac{1}{N_1 N_2} \sum_i p_i |\omega_i^C \rangle \langle \omega_i^C | + \sum_{k=1}^{N_1^2-1} \lambda_k(1) \otimes I_{N_2} \otimes \frac{1}{2 N_2} \sum_i a^{(k)}_i p_i |\omega_i^C \rangle \langle \omega_i^C | + \sum_{l=1}^{N_2^2-1} b^{(l)}_i \lambda_l(2) \otimes |\omega_i^C \rangle \langle \omega_i^C |
\]

\[
+ \sum_{i=1}^{N_1^2-1} I_{N_1} \otimes \lambda_i(2) \otimes \frac{1}{2 N_1} \sum_i a^{(k)}_i b^{(l)}_i p_i |\omega_i^C \rangle \langle \omega_i^C |,
\]

\[
+ \sum_{k} \sum_{l} \lambda_k(1) \otimes \lambda_l(2) \otimes \frac{1}{4} \sum_i a^{(k)}_i b^{(l)}_i p_i |\omega_i^C \rangle \langle \omega_i^C |,
\]

(4)

where \((a^{(1)}_i, a^{(2)}_i, \ldots, a^{(N_1^2-1)}_i)\) and \((b^{(1)}_i, b^{(2)}_i, \ldots, b^{(N_2^2-1)}_i)\) are real vectors on the Bloch sphere satisfying \(|\overline{\alpha_i}|^2 = \sum_{j=1}^{N_2^2-1} (a^{(j)}_i)^2 = 2(1 - \frac{1}{N_1})\) and \(|\overline{\beta_i}|^2 = \sum_{j=1}^{N_2^2-1} (b^{(j)}_i)^2 = 2(1 - \frac{1}{N_2})\).

Comparing (2) with (4), we have

\[
M_0 = \frac{1}{N_1 N_2} \sum_i p_i |\omega_i^C \rangle \langle \omega_i^C |,
\]

\[
M_k = \frac{1}{2 N_2} \sum_i a^{(k)}_i p_i |\omega_i^C \rangle \langle \omega_i^C |,
\]

\[
\tilde{M}_l = \frac{1}{2 N_1} \sum_i b^{(l)}_i p_i |\omega_i^C \rangle \langle \omega_i^C |,
\]

\[
M_{kl} = \frac{1}{4} \sum_i a^{(k)}_i b^{(l)}_i p_i |\omega_i^C \rangle \langle \omega_i^C |.
\]

(5)

For any \((N_1^2 - 1) \times (N_1^2 - 1)\) real matrix \(R(1)\) and \((N_2^2 - 1) \times (N_2^2 - 1)\) real matrix \(R(2)\) satisfying \(\frac{1}{(N_1^2 - 1)^2} I - R(1)^T R(1) \geq 0\) and \(\frac{1}{(N_2^2 - 1)^2} I - R(2)^T R(2) \geq 0\), we define a new matrix

\[
\mathcal{R} = \begin{pmatrix}
R(1) & 0 & 0 \\
0 & R(2) & 0 \\
0 & 0 & T
\end{pmatrix},
\]

(6)

where \(T\) is a transformation acting on an \((N_2^2 - 1) \times (N_2^2 - 1)\) matrix \(M\) by

\[
T(M) = R(1) M R^T(2).
\]

Using \(\mathcal{R}\) we define a new operator \(\gamma_{\mathcal{R}}\),

\[
\gamma_{\mathcal{R}}(\rho_{ABC}) = I_{N_1} \otimes I_{N_2} \otimes M'_0 + \sum_{i=1}^{N_1^2-1} \lambda_i(1) \otimes I_{N_2} \otimes \tilde{M}'_i + \sum_{j=1}^{N_2^2-1} I_{N_1} \otimes \lambda_j(2) \otimes \tilde{M}'_j + \sum_{i=1}^{N_1^2-1} \sum_{j=1}^{N_2^2-1} \lambda_i(1) \otimes \lambda_j(2) \otimes M'_{ij},
\]

(7)

where \(M'_0 = M_0\), \(M'_k = \sum_{m=1}^{N_1^2} R_{km}(1) M_m\), \(\tilde{M}'_i = \sum_{m=1}^{N_2^2} R_{im}(2) \tilde{M}_m\) and \(M'_{ij} = (T(M))_{ij} = (R(1) M R^T(2))_{ij}\).
[Theorem 2] If $\rho_{ABC}$ is separable, then $\gamma_R(\rho_{ABC}) \geq 0$.

[Proof] From (5) and (7) we get

$$ M'_0 = M_0 = \frac{1}{N_1N_2} \sum_i p_i |\omega_i^C\rangle \langle \omega_i^C|, \quad M'_k = \frac{1}{2N_2} \sum_{mi} R_{km}(1) a_i^{(m)} p_i |\omega_i^C\rangle \langle \omega_i^C|, $$

$$ \tilde{M}'_j = \frac{1}{2N_1} \sum_{ni} R_{tn}(2) b_i^{(n)} p_i |\omega_i^C\rangle \langle \omega_i^C|, \quad M'_{kl} = \frac{1}{4} \sum_{mni} R_{km}(1) a_i^{(m)} b_j^{(n)} R_{ln}(2) R_{nm}(1) |\omega_i^C\rangle \langle \omega_i^C|. $$

A straightforward calculation gives rise to

$$ \gamma_R(\rho_{ABC}) = \sum_i p_i \frac{1}{2} \left( \frac{2}{N_1} I_{N_1} + \sum_{k=1}^{N_2-1} \sum_{m=1}^{N_2-1} R_{km}(1) a_i^{(m)} \lambda_k(1) \right) $$

$$ \otimes \frac{1}{2} \left( \frac{2}{N_2} I_{N_2} + \sum_{i=1}^{N_2-1} \sum_{n=1}^{N_2-1} R_{in}(2) b_i^{(n)} \lambda_i(2) \right) \otimes |\omega_i^C\rangle \langle \omega_i^C|. $$

As $\frac{1}{(N_1-1)^2} I - R(1)^T R(1) \geq 0$ and $\frac{1}{(N_2-1)^2} I - R(2)^T R(2) \geq 0$, we get

$$ |a_i|^2 = |R(1) a_i|^2 \leq \frac{1}{(N_1-1)^2} |a_i|^2 = \frac{2}{N_1(N_1-1)}, $$

$$ |b_i|^2 = |R(2) b_i|^2 \leq \frac{1}{(N_2-1)^2} |b_i|^2 = \frac{2}{N_2(N_2-1)}. $$

Therefore $\gamma_R(\rho_{ABC})$ is still a density operator, i.e. $\gamma_R(\rho_{ABC}) \geq 0$.

Theorem 2 gives a necessary separability criterion for general tripartite systems. The result can be also applied to bipartite systems. Let $\rho_{AB} \in \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}$, $\rho_{AB} = I_{N_1} \otimes M_0 + \sum_{j=1}^{N_2-1} \lambda_j \otimes M_j$. For any real $(N_1^2 - 1) \times (N_2^2 - 1)$ matrix $R$ satisfying $\frac{1}{(N_1-1)^2} I - R^T R \geq 0$ and any state $\rho_{AB}$, we define

$$ \gamma_R(\rho_{AB}) = I_{N_1} \otimes M_0 + \sum_{j=1}^{N_2-1} \lambda_j \otimes M_j', $$

where $M'_j = \sum_k R_{jk} M_k$.

[Corollary 2] For $\rho_{AB} \in \mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}$, if there exists an $R$ with $\frac{1}{(N_1-1)^2} I - R^T R \geq 0$ such that $\gamma_R(\rho_{AB}) < 0$, then $\rho_{AB}$ must be entangled.

For $2 \times N$ systems, our corollary is reduced to the results in [10]. Generally this criterion do detect certain entanglement of $\mathcal{H}_{N_1} \otimes \mathcal{H}_{N_2}$ systems. As an example we consider the $3 \times 3$ Iostropic states,

$$ \rho_i = \frac{1-p}{9} I_3 \otimes I_3 + \frac{p}{3} \sum_{i,j=1}^3 |ii\rangle \langle jj| $$

$$ = I_3 \otimes \left( \frac{1}{9} I_3 \right) + \sum_{i=1}^5 \lambda_i \otimes \left( \frac{p}{6} \lambda_i \right) - \sum_{i=6}^8 \lambda_i \otimes \left( \frac{p}{6} \lambda_i \right). $$
If we choose \( \mathcal{R} \) to be \( \text{Diag}\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \), we get when \( 0.5 < p \leq 1 \), \( \rho_I \) is entangled. For tripartite case, we take the following 3 \( \times \) 3 mixed state as an example:

\[
\rho = \frac{1-p}{27}I_{27} + p|\psi\rangle\langle\psi|,
\]

where \( |\psi\rangle = \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)(|000\rangle + |111\rangle + |222\rangle) \). Taking \( R(1) = R(2) = \text{Diag}\{\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\} \), we have that \( \rho \) is entangled for \( 0.6248 < p \leq 1 \).

In fact the criterion for \( 2 \times N \) systems [10] is equivalent to the PPT criterion [5]. Our theorem 2 is also equivalent to the PPT criterion for \( 2 \times 2 \times N \) systems. This can be seen from the followings. Let us choose \( R(1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \) and \( R(2) = I_3 \). A separable state \( \rho_{ABC} \in \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_N \) can be represented as:

\[
\rho_{ABC} = \sum_i p_i \frac{1}{2}(I + \sum_{j=x,y,z} r_i^{(j)} \sigma_j) \otimes \frac{1}{2}(I + \sum_{j=x,y,z} s_i^{(j)} \sigma_j) \otimes |\omega_i^C\rangle\langle\omega_i^C|.
\]

By the definition we get

\[
\gamma_\mathcal{R}(\rho_{ABC}) = \sum_i p_i \frac{1}{2}(I + r_i^{(x)} \sigma_x - r_i^{(y)} \sigma_y + r_i^{(z)} \sigma_z) \otimes \frac{1}{2}(I + \sum_{j=x,y,z} s_i^{(j)} \sigma_j) \otimes |\omega_i^C\rangle\langle\omega_i^C| = \rho_{ABC}^{T_A}.
\]

For \( \rho_{ABC}^{T_B} \) and \( \rho_{ABC}^{T_{AB}} \), we can similarly choose

\[
R(1) = I_3, \quad R(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad R(1) = R(2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

If \( \gamma_\mathcal{R}(\rho_{ABC}) \geq 0 \), we get that \( \rho^{T_A}, \rho^{T_B} \) and \( \rho^{T_{AB}} \) are also positive operators.

On the other hand, if there is an entangled PPT state \( \rho_{ABC} \in \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \mathcal{H}_N \), \( \rho_{ABC}^{T_A} \geq 0 \), \( \rho_{ABC}^{T_B} \geq 0 \) and \( \rho_{ABC}^{T_{AB}} \geq 0 \), but \( \gamma_\mathcal{R}(\rho_{ABC}) < 0 \) for some real \( 3 \times 3 \) matrices \( R(1) \) and \( R(2) \) such that \( R^T(1)R(1) \leq I \) and \( R^T(2)R(2) \leq I \), with \( \mathcal{R} \) being defined in (6). Then one can define, for all \( \vec{\beta} = (\beta_1, \beta_2, \beta_3) \in \mathbb{C}^3 \), \( \vec{\beta} = R(1)\vec{\beta} \) with \( \vec{\beta}' = (\beta'_1, \beta'_2, \beta'_3) \), and \( \Lambda_{R(1)}(\alpha I + \sum_{i=1}^3 \beta_i \sigma_i) = \alpha I + \sum_{i=1}^3 \beta'_i \sigma_i \). Obviously \( \Lambda_{R(1)} \) would map the Bloch sphere to itself. Hence \( \Lambda_{R(1)} \) is a positive map. From [16] it follows that \( \Lambda_{R(1)} \) can be expressed as \( \Lambda_{R(1)} = \Lambda_{R(1)}^{CP}(1) + \Lambda_{R(1)}^{CP}(2) \circ T \), where \( \Lambda_{R(1)}^{CP}(1) \) and \( \Lambda_{R(1)}^{CP}(2) \) denote completely positive maps, and \( T \) the transpose. Similar result can be obtained for \( \Lambda_{R(2)} \). A straightforward calculation shows that

\[
\gamma_\mathcal{R}(\rho_{ABC}) = (\Lambda_{R(1)} \otimes \Lambda_{R(2)} \otimes I)(\rho_{ABC})
\]

\[
= ((\Lambda_{R(1)}^{CP}(1) + \Lambda_{R(1)}^{CP}(2) \circ T) \otimes (\Lambda_{R(2)}^{CP}(1) + \Lambda_{R(2)}^{CP}(2) \circ T) \otimes I)(\rho_{ABC})
\]

\[
= (\Lambda_{R(1)}^{CP}(1) \otimes \Lambda_{R(2)}^{CP}(1) \otimes I)(\rho_{ABC}) + (\Lambda_{R(1)}^{CP}(2) \otimes \Lambda_{R(2)}^{CP}(1) \otimes I)(\rho_{ABC})^{T_A}
\]

\[
+ (\Lambda_{R(1)}^{CP}(1) \otimes \Lambda_{R(2)}^{CP}(2) \otimes I)(\rho_{ABC})^{T_B} + (\Lambda_{R(1)}^{CP}(2) \otimes \Lambda_{R(2)}^{CP}(2) \otimes I)(\rho_{ABC})^{T_{AB}}.
\]

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Since that the tensor product of two completely positive maps is still a completely positive map and \( \rho_{T_A}^{ABC} \geq 0 \), \( \rho_{T_B}^{ABC} \geq 0 \) and \( \rho_{T_A}^{T_B A^B} \geq 0 \), this implies that \( \gamma_R(\rho_{ABC}) \geq 0 \). This is a contradiction. Hence the theorem 2 is equivalent to the PPT criterion for \( 2 \times 2 \times N \) systems.

We have studied the separability of tripartite quantum systems. In terms of the Bloch representation of density matrices, a necessary condition for the separability of tripartite states has been obtained. Our approach gives a new way of separability investigation. For \( 2 \times 2 \times N \) systems our criterion is equivalent to PPT, namely PPT criterion can be also understood according to the Bloch representation approach. Nevertheless it is rather complicated to compare our criterion with PPT generally for higher dimensional tripartite systems. Moreover as the PPT and realignment separability criteria can give rise to lower bonds of entanglement of formation and concurrence [17], one could also discuss the possible relations between the lower bounds of entanglement and the separability criterion in this letter. The approach can be also generalized to arbitrary multipartite systems.

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