Effective action of quantum fields in the space-time of a cylindrically symmetric spinning body

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Abstract

The aim of this article is to calculate (to first order in $\hbar$) the renormalized effective action of a self interacting massive scalar field propagating in the space-time due to a cylindrically symmetric, rotating body. The vacuum (exterior space-time) contribution is model independent, we also consider the simplest case of a core (interior space-time) model, namely a cylindrical shell. The heat kernel of the system is calculated, and used to obtain an expression for the determinant of the Klein-Gordon operator on the space-time manifold. New ultra-violet poles are discovered, and regularization techniques are then employed to render finite the Klein-Gordon determinant and consequently extract the regularized one loop effective action for a self interacting scalar field theory. The coupling constants of the theory are then renormalized. As a test case a conical singularity with non-zero flux is also considered.

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1 Introduction

The use of manifolds containing conical singularities to model (infinitely) thin line sources has now become quite common. For example, a space-time metric which is flat except for a single conical singularity is a reasonable starting point in the investigation of the physics of cosmic strings - topological defects which may be generated by phase transitions occurring in certain grand unified models [35]. Conical singularities also arise when considering the thermodynamic properties of black holes. By making the integral curves of the (previously time-like) Killing vector in a Euclideanized static black hole metric suitably periodic, it is possible to construct a partition function for a quantum field propagating on the black hole background. The contribution of the quantum field to the black hole entropy can then be easily obtained by standard thermodynamic methods [4] [46] [66]. In addition to these applications in four dimensions, it has been recognized that conical singularities play a crucial role in 2+1 dimensional gravity [9] [24]. The gravitational field of the Einstein theory in 2+1 dimensions has no dynamical degrees of freedom (i.e. there is no graviton in the theory) and gravitational interaction is possible only by the fact that every massive particle has an associated conical singularity. The more massive the particle, the more “pointed” the cone it induces on the space-time manifold. Two particle scattering is then facilitated simply by each particle travelling on the cone generated by the other.

With regard to the conical singularity as a model of cosmic strings, one of the less pleasing features of the infinitely thin idealization is that there is a delta function singularity of Riemannian curvature at the position of the string itself (see [24] or section 12 of this paper). Whilst this is not necessarily fatal, indeed there are now well defined procedures for dealing with this issue, it is more appealing to regard the conical metric as being the exterior space-time to some finite sized cylindrical body, which is suitably well behaved at the origin of coordinates. Naturally appropriate boundary conditions are required to patch together the interior and exterior solutions (B3 or section 12) and the result is a metric describing the space-time of a finitely thick but infinitely long cylindrical body.
Another natural generalization is the space-time of a rotating, cylindrically symmetric body. For example, it has already been shown that in a particular model of a superconducting cosmic string, the angular momentum of the left moving and right moving modes may not cancel, resulting in a string with net angular momentum \[49\]. Another example occurs in Chern-Simons field theory, where string-like objects with angular momentum can also occur \[1\]. Additionally it is hard to imagine a mechanism which would prevent one simply adding angular momentum to a finite thickness cylindrical body by hand, although naturally one must bear the infinite length of such an object in mind.

The natural starting place for this problem would seem to be to generalize the metric describing a simple conical singularity by adding angular momentum to the space-time. This was duly done \[23\], but threw up some surprising results. The constructed metric contained closed time-like curves concentrated in a finite region around the axis of symmetry, and it was soon realized that in addition to the curvature singularity, this model had a torsion singularity on axis as well \[24\] \[63\]. From then on approaches diverged as to how best to treat the problem. With the appearance of torsion, some authors argued that this metric was best considered within the framework of Einstein-Cartan theory, which naturally incorporates torsion as well as curvature \[63\]. Models were constructed within Einstein-Cartan theory to describe the space-time of a conical singularity with spin \[33\], whilst other authors chose a different direction. Discovering that the first quantized scattering problem on the manifold was badly defined within the closed time-like curve region \[23\], it was argued that by restricting the angular momentum of the string to certain values (and hence invoking a quantization condition), it is possible to make the string “transparent” to scattering experiments, and hence remove the difficulty with the closed time-like curve region \[48\]. Others argued for the presence of angular momentum dependence in the first quantum correction to the vacuum energy, with analogy to the case of the spinning circle \[32\], or simply on dimensional grounds \[47\].

In this paper we will take a different point of view. It is clearly the region immediately around the axis which causes the difficulties. The Hamiltonian of scalar quantum fields is non-Hermitian here, both in quantum mechanics
and in quantum field theory (see section 11). More convincingly, the results of Menotti and Seminara ([50] and following papers) indicate that if the energy-momentum tensor of the space-time satisfies the weak energy condition, and if we have no closed time-like curves (CTC’s) at spatial infinity (which must be conical and open in nature), then there are no CTC’s in the space-time at all. In other words, any energy-momentum tensor which we write down as a source of the spinning conical space-time will be non-physical (violate the weak energy condition) if we include the region of closed time-like curves. Thus we will consider the spinning conical space-time only as the exterior solution to some interior space-time, which extends beyond the closed time-like curve region and is torsion free.

There are a variety of models that one can choose for the interior space-time ([33], but note [61]). Since our eventual aim is to calculate the effective action for a massive scalar quantum field propagating on the manifold, the interior contribution to the effective action will be model dependent. Therefore we will initially concentrate on the exterior (model independent) terms, returning to consider the interior metric again in section 12, where we will look at the particular case of the cylindrical shell.

2 The metric of the exterior space-time

The metric describing a simple conical singularity at the origin of co-ordinates can be written in several ways (see, for example, [8]). The representation which will be used in this paper can be expressed in cylindrical polar co-ordinates as follows,

\[ ds^2 = -dt^2 + d\rho^2 + \rho^2 d\phi^2 + dz^2, \]

where \((t, \rho, \phi, z) = (t, \rho, \phi + 2\pi \alpha, z), 0 < \rho < \infty, 0 < \phi < 2\pi \alpha, -\infty < t < \infty, -\infty < z < \infty \) and \(0 < \alpha < 1\). The form of this metric makes it obvious that the space-time is locally flat, but it differs globally from Minkowski space (expressed in cylindrical polar co-ordinates) because of the identification in the angular variable, \(\phi\). This condition means that a “wedge” (or deficit angle) is removed from the range of \(\phi\), and then the edges of the excised region are identified, resulting in a conical manifold. Clearly \(\alpha\) is related to
the magnitude of the deficit angle, and it dictates the degree of “sharpness” of the cone. If \( \alpha = 1 \) then there is no deficit angle and we recover the metric for Minkowski space.

In the above space-time the source is envisaged to be an infinitely long, infinitesimally thin tube along the z-axis of the co-ordinate set. Whilst the space-time is locally flat away from the origin, there is in fact a delta function singularity of curvature at the origin (see [24] or section 12). With regard to the physical parameters relating to the source, \( \alpha \) may be expressed in terms of the mass per unit length \( \mu \) by the relation \( \alpha = 1 - \frac{4G\mu}{c^2} \). From now on factors of \( c, \hbar, G \) and so on will be suppressed in all expressions except where noted.

In order to consider rotating sources, metric (1) needs to be generalized to include an angular momentum parameter. This parameter describes the effect that source spin has on the exterior space-time. The metric for a conical space-time with a spinning source takes the form

\[
\frac{ds^2}{c^2} = -(dt + \frac{S}{\alpha}d\phi)^2 + d\rho'^2 + (\rho' + k)^2d\phi^2 + dz'^2 \\
= -dt^2 - \frac{2S}{\alpha}dtd\phi + d\rho'^2 + \left(\rho' + k\right)^2d\phi^2 + dz'^2 ,
\]

where \( S \) parameterizes the angular momentum and has the dimensions of length [9], \( k \) is a constant [33], \( (t, \rho', \phi, z') = (t, \rho', \phi+2\pi\alpha, z') \), \(-k < \rho' < \infty\), \( 0 < \phi < 2\pi\alpha \), \(-\infty < t < \infty\), \(-\infty < z' < \infty\) and \( 0 < \alpha < 1\). The precise relationship between \( S \) and the physical angular momentum per unit length, \( J \), can be written as \( S = 4GJ/c^3 \).

In addition to the curvature singularity, it can be shown that this metric also contains a torsion delta function singularity at the origin \( \rho' = -k \) [24] [33].

The space-time described by metric (2) clearly possesses closed time-like curves (for example the contour described by \( dt = 0, d\rho' = 0, dz' = 0, d\phi = \text{constant}, (\rho' + k) < \frac{|S|}{\alpha} \) is closed and time-like). Whilst it has been shown that
classically there are no closed time-like geodesics in this space-time \cite{8}, in the path integral formulation of field theory one is supposed to include weighted contributions from all paths, including those non classical paths described by the closed time-like curves. We would therefore expect unitarity problems describing physics on this manifold. As a consequence we restrict the radial co-ordinate $\rho' > \rho_0 > \frac{|S|}{\alpha} - k$. This is sufficient to remove the difficulties with the closed time-like curves and will allow us to discuss quantum field theory on the manifold. The rest of the space-time ($\rho' < \rho_0$) is supposed to be composed of the source core, which is described by another metric and must be matched to this one with appropriate conditions at the boundary (\cite{33} or section 12). The included constant $k$ is determined by correctly matching the components of the two metrics at the boundary $\rho' = \rho_0$.

It should be noted that this metric is stationary, but not static. The applicability of conical methods to stationary but non static space-times has not yet been proved in general, but there is some strong evidence to indicate that these methods hold there too. For example, the first quantum correction to the effective action on the Kerr-Newman black hole background has been obtained using conical methods \cite{46}, after verifying that the induced conical singularity still behaved in a distributional sense. Since it has already been demonstrated that the conical singularity in this space-time behaves in a distribution manner (\cite{24} \cite{53} and section 12) we can have confidence that application of conical methods will yield useful results.

We can define a new time co-ordinate $\bar{\mathcal{T}}$ such that $d\bar{t} = dt + \frac{S}{\alpha} d\phi$ (or alternatively $\bar{t} = t + \frac{S}{\alpha} \phi$). Additionally we can write $\rho'' = \rho' + k$. With these co-ordinates we can re-write metric (2) as

$$ds^2 = -d\bar{t}^2 + d\rho''^2 + \rho''^2 d\phi^2 + dz'^2,$$

where we make the following identifications, $(\bar{t}, \rho'', \phi, z) = (\bar{t} + 2\pi S, \rho'', \phi + 2\pi \alpha, z)$, $\rho_0 + k < \rho'' < \infty$, $0 < \phi < 2\pi \alpha$, $-\infty < \bar{t} < \infty$, $-\infty < z' < \infty$ and $\rho_0 > \frac{|S|}{\alpha} - k$.

The manifold as we have defined it above excludes the origin and the closed time-like curve region. Therefore it has no curvature, no torsion, and quantum theory on this manifold should avoid potential unitarity problems.
It will be convenient in what follows to perform the calculations in a space-
time with a Euclidean signature. By continuing the $\rho''$, $z'$ co-ordinates to
imaginary values, we may rewrite the metric as follows,

$$ds^2 = -d\tau^2 - d\rho^2 - \rho^2 d\phi^2 - dz^2,$$

(4)

where $(\tau, \rho, \phi, z) = (\tau + 2\pi S, \rho, \phi + 2\pi \alpha, z)$, $\rho_0 + k < \rho < \infty$, $0 < \phi < 2\pi \alpha$, 
$-\infty < \tau < \infty$, $-\infty < z < \infty$ and $|\rho_0| > \frac{|S|}{\alpha} - k$.

The parameters are related to each other by $\rho'' = i\rho$, $z' = iz$, $k = ik$, $\rho_0 = i\rho_0$.

3 Green’s function and heat kernel

We define the differential operator $\Box_x$ on our manifold by the expression

$$\Box_x = -\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial z^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) - \frac{1}{\rho^2} \frac{\partial^2}{\partial \phi^2},$$

(5)

where $x = (\tau, \rho, \phi, z)$, and we define the Euclidean Feynman Propagator $G^E_F(x, x')$ for a scalar field of mass $M$ by the equation

$$\left( \Box_x + M^2 \right) G^E_F(x, x') = -\delta^{(4)}(x - x').$$

(6)

We may regard $\Box_x + M^2$ as the Euclidean Klein-Gordon operator on the
manifold, the four dimensional delta function can be expressed as follows,

$$\delta^{(4)}(x - x') = \frac{1}{\rho} \delta \left( \tau - \tau' - S \frac{(\phi - \phi')}{\alpha} \right) \delta(\rho - \rho') \delta(\phi - \phi') \delta(z - z').$$

(7)

The eigenfunctions of the operator $\Box_x + M^2$ with the correct cylindrical
symmetry and angular behaviour may be written as follows,

$$\Psi_{\omega, k, k_3, m}(x) = \frac{1}{(2\pi)^{\frac{3}{2}} \alpha^\frac{3}{2}} J_{m + \omega S \frac{1}{\alpha}}(k \rho) e^{i(k_3 z - \omega \tau + (m + \omega S) \frac{\phi}{\pi})},$$

(8)

with eigenvalues
\[ E_{\omega,k,k_3,m} = \omega^2 + k_3^2 + k^2 + M^2 \]  \quad (9)

\( J_\nu(z) \) is a Bessel function of the first kind. Many of the properties of the special functions used in this paper can be located either in [1] or [26]. By standard Sturm-Liouville theory we may immediately write down the Euclidean Feynman propagator for massive scalar fields on this space-time as

\[ G^E_F(x,x') = -\int_0^\infty d\tau \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} dk_3 \int_0^\infty dk \ e^{-\tau E_{\omega,k,k_3,m}\Psi^*_{\omega,k,k_3,m}(x)}\Psi_{\omega,k,k_3,m}(x'), \quad (10) \]

where \( \Psi^* \) denotes the complex conjugate of \( \Psi \) and \( \tau \) takes the role of a Schwinger parameter.

There is a relationship between the Euclidean heat kernel \( F^E(x,x',\tau) \) for the diffusion problem on this manifold and the Euclidean Green’s function, namely

\[ G^E_F(x,x') = -\int_0^\infty d\tau F^E_E(x,x',\tau) \quad . \quad (11) \]

If we note the following integral identities,

\[ \int_{-\infty}^{\infty} dk_3 e^{-\tau k_3^2 + ik_3 (z-z')} = \sqrt{\frac{\pi}{\tau}} e^{-\frac{(z-z')^2}{4\tau}} , \quad (12) \]

\[ \int_0^\infty kdkJ_{\frac{m+\omega S}{\alpha}}(k\rho)J_{\frac{m+\omega S}{\alpha}}(k\rho')e^{-\tau k^2} = \frac{1}{2\tau} e^{-\frac{(\rho^2+\rho'^2)}{4\tau}} I_{\frac{m+\omega S}{\alpha}}(\frac{\rho\rho'}{2\tau}) \]  \quad , (13)

where \( I_\nu(z) \) is a modified Bessel function of the first kind, we may write an expression for the Euclidean heat kernel of this problem as follows,

\[ F^E_E(x,x',\tau) = \frac{1}{(2\pi)^3 \alpha(2\tau)} \sqrt{\frac{\pi}{\tau}} e^{-\frac{(\rho^2+\rho'^2)}{4\tau} - \frac{(z-z')^2}{4\tau} - M^2 \tau} \int_{-\infty}^{\infty} d\omega e^{-\tau \omega^2 - i\omega(1-\tau) - \frac{\omega^2}{8} (\phi - \phi')} \sum_{m=-\infty}^{\infty} I_{\frac{m+\omega S}{\alpha}}(\frac{\rho\rho'}{2\tau}) e^{im(\phi - \phi')} . \quad (14) \]
4 Simplification of the heat kernel

Having found an expression for the Euclidean heat kernel of the system, our intention is to use it to find a regularized expression for the determinant of the Klein-Gordon operator $\Box_x + M^2$ on this manifold. This in turn will allow us to write down the first quantum correction term in the effective action (see section 11).

Before we can proceed with this operation, we need to find a simpler expression for the heat kernel, which can be more easily manipulated. In order to proceed further we must invoke a complex representation for the modified Bessel function $I_{\nu}(x)$,

$$I_{\frac{|m + \omega S|}{\alpha}}\left(\frac{\rho^2}{2\tau}\right) = \frac{1}{2\pi} \int_{C_1} e^{iz\left(\frac{|m + \omega S|}{\alpha}\right) + \frac{\rho^2}{2\tau} \cos z} dz,$$

where $C_1$ is the contour depicted above in Figure 1.

We now gather together terms in the expression for the heat kernel which depend on $m$ and perform the summation over $m$, i.e. we evaluate the following sum,
\[ \sum_{m=-\infty}^{\infty} e^{iz+\frac{m+\omega S}{\alpha} + \frac{\phi - \phi'}{\alpha}}. \] 

Let us define the following functions, \( \text{Int}(X) \) and \( \text{Frac}(X) \) for real \( X \). They are defined such that \( X = \text{Int}(X) + \text{Frac}(X) \), \( \text{Int}(X) \) gives the integer part of \( X \) and \( \text{Frac}(X) \) gives the fractional part of \( X \), where \( 0 < \text{Frac}(X) < 1 \) for both positive and negative \( X \). So for example \( \text{Frac}(-5.2) = 0.8 \) and \( \text{Int}(-5.2) = -6 \). It is then clear that the above sum can be re-written (using \( m \mapsto m - \text{Int}(\omega S) \)) as

\[ e^{-i\text{Int}(\omega S) \frac{(\phi - \phi')}{\alpha}} \sum_{m=-\infty}^{\infty} e^{iz+\frac{m+\text{Frac}(\omega S)}{\alpha} + \frac{\phi - \phi'}{\alpha}}. \] 

This sum can be evaluated by standard means, and gives the result

\[ e^{-i\text{Int}(\omega S) \frac{(\phi - \phi')}{\alpha}} \left( \frac{e^{i\frac{\text{Frac}(\omega S)}{\alpha} \frac{(\phi - \phi')}{\alpha}}}{1 - e^{\frac{z}{\alpha}(\phi - \phi')}} + \frac{e^{-i\frac{\text{Frac}(\omega S)}{\alpha} \frac{(\phi - \phi')}{\alpha}}}{e^{-\frac{z}{\alpha}(\phi - \phi')} - 1} \right). \] 

The two terms resulting from the above summation are very similar, and if we let \( z \mapsto -z \) in one of the terms, it becomes the negative of the other. This suggests altering the contour \( C_1 \) in the integral over \( z \) to the contour \( C_2 \), which is depicted in Figure 2. This enables us to write the heat kernel in the following form,

\[ F^E(x, x', \tau) = \frac{1}{(2\pi)^4 \alpha(2\tau)} \sqrt{\frac{\pi}{\tau}} e^{-\frac{(\mu^2 + \phi^2)}{4\tau}} - \frac{(z-x')^2}{4\tau} - M^2 \tau \int_{C_2} e^{z' \cos z - i\frac{\text{Int}(\omega S) \frac{(\phi - \phi')}{\alpha}}{\alpha} - \frac{i}{\alpha} \text{Frac}(\omega S) \frac{(\phi - \phi')}{\alpha}} d\omega \]

\[ \int_{-\infty}^{\infty} d\omega e^{-\tau \omega^2} \frac{z-\tau \frac{z}{\alpha} - i\frac{\text{Frac}(\omega S) \frac{(\phi - \phi')}{\alpha}}{\alpha}}{e^{-\frac{z}{\alpha}(\phi - \phi')} - 1}. \] 

5 MANIPULATION OF THE CONTOURS

Having recast the problem into one involving contour integration, we can now manipulate the contour to make evaluation of the integral easier. In particular, it will be seen that it is easy to pick out the (rotating, Euclideanized) Minkowski space contribution to the heat kernel (in other words terms which
one would expect in flat, non-conical, but rotating space). The remainder should be considered as a correction to the rotating Minkowski space result caused by a combination of the conical and spinning nature of the space.

Examining the integrand of (19) carefully, we see that there are poles all along the real axis, at \( z = (\phi - \phi') + 2\pi m\alpha \). By manipulating the contour \( C_2 \) into the contour \( C_3 \), as shown in Figure 3, it is easy to pick out the contribution of the \( m = 0 \) pole. The contours in the right hand diagram which do not encircle the pole at \( m = 0 \) are designated \( C_3 \). The curves constituting \( C_3 \) can always be chosen so they pass arbitrarily close to the \( m = 0 \) pole, and hence we pick up no other polar contribution inside our contours of integration. The residue of this pole (remembering the integration is clockwise), gives a contribution to the heat kernel \( F_{pE}^E(x, x', \tau) \), explicitly

\[
F_{pE}^E(x, x', \tau) = \frac{\sqrt{\pi} \sinh \tau}{(2\pi)^3(2\tau)} e^{-\frac{(x-x')^2}{4\tau}} \left[ e^{-\frac{\tau M^2}{4\tau}} \int_{-\infty}^{\infty} d\omega e^{-\tau \omega^2 - i\omega(\tau - \tau')} \right]
= \frac{1}{(4\pi\tau)^2} e^{-\frac{(r-r')^2 + (z-z')^2 + M^2}{4\tau}} \left( \frac{\tau M^2}{4\tau} \right),
\]

where \( r = (\rho, \phi) \). Thus we can write the total result for the heat kernel as
Figure 3: Manipulation of the contours in the complex $z$ plane

\[ F^E(x, x', \tau) = F^E_p(x, x', \tau) + F^E_\alpha(x, x', \tau) \quad , \quad (21) \]

where

\[
F^E_\alpha(x, x', \tau) = \frac{1}{(2\pi)^4 \alpha (2\tau)} \sqrt{\frac{\pi}{\tau}} e^{-\frac{(\rho-\rho')^2}{4\tau} + \frac{(z-z')^2}{4\tau} + M^2\tau} \\
\int_{-\infty}^{\infty} d\omega e^{-\tau \omega^2 - i\omega(\frac{\phi-\phi'}{\alpha})} \int_{C_3} e^{-\frac{\omega d}{\alpha}(\frac{\sin^2(\frac{\phi}{4})}{2} - i\text{Int}(\omega S)(\frac{(\phi-\phi')}{\alpha} - i\frac{\text{Frac}(\omega S)}{\alpha}) - 1) dz} \quad . \quad (22) \]

6 Evaluating the trace of the heat kernel

The expression for the determinant of the Klein-Gordon operator on this manifold requires only the trace of the heat kernel (see section 10). Therefore we will henceforth specialize to the case $x = x'$. This simplifies matters considerably, and allows us to write the heat kernel in the following form,

\[
F^E(x, x, \tau) = \frac{1}{(4\pi \tau)^2} e^{-M^2\tau} + \]

12
\[
\sqrt{\frac{\tau}{\pi}} e^{-M^2 \tau / (2\pi)^4 \alpha (2\tau)} \int_{-\infty}^{\infty} d\omega e^{-\tau \omega^2} \int_{C_3} e^{-\frac{\omega^2}{\tau} \sin^2 \left(\frac{\tau}{\omega}\right) - i \frac{\alpha}{\tau} \text{Frac}(\omega S)} dz. \tag{23}
\]

The first term of this expression is exactly the same as one would obtain in flat, non-rotating Minkowski space, the rest is a correction due to the conical and spinning nature of the space. If we make the following definition,

\[
I(z, S) = \int_{-\infty}^{\infty} d\omega e^{-\tau \omega^2 - i \frac{\alpha}{\tau} \text{Frac}(\omega S)} , \tag{24}
\]

we can write the expression thus,

\[
F^E(x, x, \tau) = \frac{e^{-M^2 \tau}}{(4\pi \tau)^2} \left[ 1 + \frac{1}{2\pi \alpha} \sqrt{\frac{\tau}{\pi}} \int_{C_3} I(z, S) e^{-\frac{\omega^2}{\tau} \sin^2 \left(\frac{\tau}{\omega}\right)} dz \right]. \tag{25}
\]

7 The conical space-time with flux

Our next task will be to integrate over the position variable \(x\) in order to obtain the trace of the heat kernel. After this we will then perform the contour integral over \(z\), writing the heat kernel in terms of a single integration variable \(\omega\). The procedure used is somewhat lengthy, so in order to demonstrate the technique it seems appropriate that we first consider the simpler case of a conical singularity with non-zero flux (but zero rotation).

In the case of non-zero flux, the metric is the same as for an infinitesimally thin line source at the origin (1), but the novel feature is that the fields upon this space-time are “twisted”, the amount of twisting being dictated by a parameter \(\sigma\) (see [51]). After finding the eigenfunctions in this space-time which satisfy the cylindrical symmetry and twisted field requirements, we can go through exactly the same procedure as above to derive an expression for the heat kernel.

For example, the Green’s function for this problem is required to display the following symmetry,

\[
G^E_F(t, \rho, \phi + 2\pi \alpha, z) = e^{2\pi i \sigma} G^E_F(t, \rho, \phi, z) . \tag{26}
\]
The eigenfunctions of the Klein-Gordon operator on the conical manifold satisfying this twisted boundary condition are

\[ \Psi_{\omega,k,k_3,m}(x) = \frac{1}{(2\pi)^{\frac{3}{2}} \alpha^2} J_{|m+\sigma|}(k\rho) e^{i(k_3z-\omega t+(m+\sigma)\phi)} \] . \quad (27)

This system bears considerable similarity to the case we are interested in, but because \( \sigma \) is a constant and not a function of \( \omega \), it is somewhat simpler. We can proceed through analogous steps to those above, to obtain the following expression,

\[ F_E(x,x',\tau) = \frac{1}{(2\pi)^{\frac{3}{2}} \alpha(2\tau)} \sqrt{\frac{\pi}{\tau}} e^{-\frac{(\rho^2+\rho'^2)}{4\tau}} - M^2 \tau \]

\[ \int_{-\infty}^{\infty} d\omega e^{-\tau \omega^2 - i\omega(t-t') + i\frac{\omega}{\alpha}(\phi-\phi')} \int_{C_2} \frac{e^{\frac{\omega'}{2\tau} \cos z - i\text{Int}(\sigma) \frac{\omega'}{\alpha} - i\frac{\omega}{\alpha} \text{Frac}(\sigma) dz}}{e^{-\frac{1}{\alpha}(z-(\phi-\phi'))} - 1} . \quad (28) \]

As before, we can distort the contour to pick out the Minkowski space contribution, but because \( \sigma \) is not a function of \( \omega \), we can now perform the \( \omega \) integration as well, which is simply Gaussian. Thus if we set the variable \( x = x' \), we obtain the following,

\[ F_E(x,x,\tau) = \frac{e^{-M^2 \tau}}{(4\pi \tau)^2} \left[ 1 + \frac{1}{2\pi \alpha} \int_{C_3} \frac{e^{-\frac{\omega}{\alpha} \sin^2 \left( \frac{z}{\alpha} \right) - i\frac{\omega}{\alpha} \text{Frac}(\sigma) \, dz}}{e^{-\frac{1}{\alpha}z} - 1} \right] . \quad (29) \]

We now take the trace of the heat kernel by integrating over \( x \). Naturally this generates the usual volume divergences in the expression, but any local divergences are where the interest lies. Let us write \( \int dz dt = V_2 \), and \( \int \rho d\rho d\phi = \alpha U_2 \) where we integrate over the (infinite) volume of the space. On this manifold there is no problem with closed time-like curves, because we have no rotation, so the space-time is well defined everywhere. Therefore after integration we obtain the following expression,

\[ \text{Tr}(F_E(x,x',\tau)) = \frac{e^{-M^2 \tau}}{(4\pi \tau)^2} \left[ \alpha V_2 U_2 + \frac{V_2 \tau}{2} \int_{C_3} \frac{e^{-i\frac{\omega}{\alpha} \text{Frac}(\sigma) \, dz}}{(e^{-\frac{1}{\alpha}z} - 1) \sin^2 \left( \frac{z}{\alpha} \right)} \right] . \quad (30) \]
The only thing left to do is to evaluate the contour integral. It should be noted that the integrand is well behaved (goes to zero) at positive and negative imaginary infinity. Therefore we can close the contour $C^3$ at positive and negative imaginary infinity, thus encircling the only pole in the integrand, which is at the origin. Therefore we need merely to find the residue of the pole at the origin.

The easiest way to evaluate this is to Taylor expand the denominator for small $z$. This gives the result

$$\frac{1 - \cos z}{(e^{-\frac{iz}{2\alpha}} - 1)} = \frac{i}{2\alpha} z^3 \left(1 - \frac{iz}{2\alpha} - \frac{z^2}{12} - \frac{z^2}{6\alpha^2}\right) + O(z^6) \quad (31)$$

Inverting the denominator, we arrive at the result

$$\frac{1}{(1 - \cos z)(e^{-\frac{iz}{2\alpha}} - 1)} = 2i\alpha \left(1 + \frac{iz}{2\alpha} + \frac{z^2}{12} \left(1 - \frac{1}{\alpha^2}\right)\right) \left(1 - \frac{1}{\alpha^2}\right) + O(1) \quad (32)$$

Since only the $z^{-1}$ terms contribute to the integral, it is necessary to Taylor expand the numerator too. If we call the numerator $J(z)$, then

$$J(z) = J(0) + zJ'(0) + \frac{z^2}{2}J''(0) + O(z^3) \quad (33)$$

so we obtain

$$Tr(F^E(x, x', \tau)) = \frac{e^{-M^2\tau}}{(4\pi \tau)^2} \left[\alpha V_2 U_2 + \right.$$

$$\left.2i\alpha V_2 \tau \int \frac{J(0)}{12z} \left(1 - \frac{1}{\alpha^2}\right) + \frac{iJ'(0)}{2\alpha} + \frac{J''(0)}{2}\right] \quad (34)$$

and finally

$$Tr(F^E(x, x', \tau)) = \frac{e^{-M^2\tau}}{(4\pi \tau)^2} \left[\alpha V_2 U_2 - 4\pi\alpha V_2 \tau \left(\frac{J(0)}{12} \left(1 - \frac{1}{\alpha^2}\right) \right.ight.$$

$$\left.\left.+ \frac{iJ'(0)}{2\alpha} + \frac{J''(0)}{2}\right)\right] \quad . \quad (35)$$
In this case, \( J(0) = 1, J'(0) = -\frac{i}{\alpha} \text{Frac}(\sigma) \) and \( J''(0) = \frac{-(\text{Frac}(\sigma))^2}{\alpha^2} \), so we can write

\[
Tr \left( F^E(x, x', \tau) \right) = e^{-M^2 \tau} \left[ \alpha V_2^2 - 4\pi \alpha V_2 \tau \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) + \text{Frac}(\sigma)(1 - \text{Frac}(\sigma)) \right) \right].
\]

It should be noted that in addition to the usual Minkowski space volume divergence \( V_2^2 \) (which is here multiplied by a factor of \( \alpha \) to compensate for the volume of a cone compared with globally flat space) there is an area divergence \( V_2 \) which is the contribution from the conical singularity. The standard result for a zero flux conical space-time is recovered from this result by setting \( \sigma = 0 \). It is interesting to note that the heat kernel trace depends only on the fractional part of \( \sigma \), something which is alluded to in \[51\].

8 Application to the rotating case

If we compare equation (29) with (25), we see that in the rotating space we have a very similar situation to the non-zero flux case, where \( I(z, S) \) plays an analogous role to \( J(z) \). There is an additional complication though, in that we want to restrict our manifold to radii \( \rho > \rho_0 + k \), to avoid the closed time-like curves and potential unitarity problems. Therefore when we take the trace of the heat kernel, we should be careful to only integrate over the relevant region.

Bearing this in mind, it is easy to obtain the following expression for the trace of the heat kernel,

\[
Tr \left( F^E(x, x', \tau) \right) = \frac{e^{-M^2 \tau}}{(4\pi \tau)^2} \left[ \alpha V_2^2 \left( U_2 - \pi (\overline{\rho}_0 + \overline{k})^2 \right) + \right.
\]

\[
\left. \frac{\sqrt{\pi} V_2 \tau}{2} \int_{C_3} I(z, S) e^{-\overline{\rho}_0 z \overline{k} \sin^2(\zeta) d\zeta} \right].
\]

\[
(37)
\]
Thus if we identify $J(z) = \sqrt{\frac{2}{\pi}} I(z, S)e^{-\frac{(\rho_0 + k)^2}{2\tau} \sin^2(\frac{\pi}{2})}$ we clearly have an analogous situation to the case of the conical singularity with non-zero flux. This gives us the following results,

\begin{align*}
J(0) &= 1, \quad (38) \\
J'(0) &= -\frac{i}{\alpha} \sqrt{\frac{\tau}{\pi}} \int_{-\infty}^{\infty} \text{Frac}(\omega S)e^{-\tau\omega^2}, \quad (39) \\
J''(0) &= -\frac{1}{\alpha^2} \sqrt{\frac{\tau}{\pi}} \int_{-\infty}^{\infty} \text{Frac}^2(\omega S)e^{-\tau\omega^2} - \frac{(\rho_0 + k)^2}{2\tau}. \quad (40)
\end{align*}

Consequently we can write the trace of the heat kernel in the following form,

\begin{equation}
Tr \left( F^E(x, x', \tau) \right) = \frac{e^{-M^2\tau}}{(4\pi\tau)^2} \left[ \alpha V_2 U_2 \\
-4\pi\alpha V_2 \tau \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) + \frac{K(S, \tau)}{2\alpha^2} \right) \right], \quad (41)
\end{equation}

where

\begin{equation}
K(S, \tau) = \sqrt{\frac{\tau}{\pi}} \int_{-\infty}^{\infty} d\omega e^{-\tau\omega^2} \left( \text{Frac}(\omega S) - \text{Frac}^2(\omega S) \right). \quad (42)
\end{equation}

It should be noted that the introduction of a radial boundary at $\rho' = \rho_0$ to cut out the closed time-like curve region has had little impact on expression for the trace of the heat kernel. This is not entirely surprising, as it has already been noted that the mathematics of manifolds with conical singularities bears strong resemblance to the mathematics of manifolds with boundaries. In effect, in the simple conical singularity case described by metric (1) there is a boundary with zero radius, and shifting it to a finite distance away from the origin does nothing to change the terms which arise purely from its existence.

As in the case of the conical singularity with flux, the pure conical singularity result is regained when we set the rotation parameter $S$ to zero.
9 Simplifying the result

In order to be able to do calculations with the quantity we have just obtained, it would be preferable if we could evaluate our expression for $K(S, \tau)$ in a somewhat more palatable form. The key to doing this is noticing that part of the integrand in $K(S, \tau)$, which we will call $f(\omega, S)$, is a continuous, even periodic function, with period $\frac{1}{S}$. This allows us to make a useful expansion of the integrand as a Fourier Series.

If we define $K(S, \tau) = \sqrt{\frac{\tau}{\pi}} \int d\omega f(\omega, S)e^{-\tau\omega^2}$, we can write $f(\omega, S)$ as

$$f(\omega, S) = \frac{1}{6} - \sum_{k=1}^{\infty} \frac{\cos(2k\pi S\omega)}{(k\pi)^2} \quad . \tag{43}$$

This then allows us to write down a more convenient expression for $K(S, \tau)$,

$$K(S, \tau) = \frac{1}{6} - \sum_{k=1}^{\infty} e^{-\frac{k^2\pi^2 S^2}{\tau}} \frac{\omega^2}{(k\pi)^2} \quad . \tag{44}$$

It is obvious from this expression for $K(S, \tau)$ that $K(0, \tau) = 0$, and so our result reproduces the simple conical singularity case when the rotation parameter $S$ is set to zero. It is also the case that the result depends only on $S^2$, so there will be no distinction in the effective action between rotations in opposite senses.

10 Evaluating the determinant of the Klein-Gordon operator

Having obtained the trace of the heat kernel in a useful form, it is now fairly straightforward to obtain an expression for the determinant of the Klein-Gordon operator on this manifold \[18\, 19\, 20\, 21\]. Since the eigenvalues of this operator increase without bound, the value of the determinant itself is infinite. As is standard, we invoke the following De Witt-Schwinger proper time representation for the determinant, which allows us to isolate (and subsequently remove) this divergence more easily,
\[
\log \det (\Box_x + M^2) = Tr \log (\Box_x + M^2) = -Tr \int_0^\infty \frac{d\tau}{\tau} e^{-\tau(\Box_x + M^2)} = -\int_0^\infty \frac{d\tau}{\tau} Tr e^{-\tau(\Box_x + M^2)}.
\]

It should now be noted that the expression \( e^{-\tau(\Box_x + M^2)} \) satisfies the following differential equation

\[
\frac{\partial}{\partial \tau} F_E(x, x', \tau) = -(\Box_x + M^2) F_E(x, x', \tau),
\]

which is simply the heat equation for the differential operator \((\Box_x + M^2)\). Therefore \( e^{-\tau(\Box_x + M^2)} \) is a representation of the heat kernel, and we arrive at the expression

\[
\log \det (\Box_x + M^2) = -\int_0^\infty \frac{d\tau}{\tau} Tr F_E(x, x', \tau).
\]

Since we have an expression for \( Tr (F_E(x, x', \tau)) \) it is now a simple matter to calculate \( \log \det (\Box_x + M^2) \), explicitly

\[
-\int_0^\infty \frac{d\tau}{\tau} Tr F_E(x, x', \tau) = -\int_0^\infty \frac{d\tau}{\tau} e^{-M^2\tau} \left[ \alpha V_2 U_2 
\right.
\]

\[
-4\pi\alpha V_2 \tau \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) + \frac{K(S, \tau)}{2\alpha^2} \right) \right] \] (48)

Some of the integrals in this expression are divergent, so we need to find a suitable regularization method to make sense of the result. Dimensional regularization proves useful, where we extend the dimensionality of space-time to \( d \) dimensions. The space-time is considered to have \( d - 2 \) globally flat dimensions, with the other 2 dimensions forming the truncated cone. In other words \( d = 4 \) in all above expressions. The expression for \( Tr (F(x, x', \tau)) \) then becomes

\[
Tr (F^E_d (x, x', \tau)) = \frac{e^{-M^2\tau}}{(4\pi \tau)^{\frac{d-2}{2}}} \left[ \alpha V_{d-2} U_2 \right]
\]
\[-4\pi \alpha V_{d-2} \tau \left( \frac{1}{12} \left(1 - \frac{1}{\alpha^2}\right) + \frac{K(S, \tau)}{2\alpha^2}\right) \right] . \quad (49)\]

If we then let \( d = 4 - \epsilon \), we can write \( V_{d-2} = \mu^2 V_2 \), where \( \mu \) is just
an arbitrary parameter with the dimensions of mass, included to keep the
overall dimensions correct. This allows us to write
\[
\text{Tr} \left( F_d^E(x, x', \tau) \right) = (4\pi \tau \mu^2)^{\frac{\epsilon}{2}} \text{Tr} \left( F_4^E(x, x', \tau) \right) \quad . \quad (50)
\]
The intention is now to calculate integral (48) with \( F_d^E(x, x', \tau) \) instead of
\( F_4^E(x, x', \tau) \), and then afterwards let \( \epsilon \to 0 \). Concentrating initially on the
first two terms in (48), we obtain after integration (and some initial regula-
tion),
\[
\begin{align*}
-\alpha V_2 U_2 \left( \frac{4\pi \mu^2}{M^2} \right)^{\frac{\epsilon}{2}} M^4 \Gamma(-2 + \frac{\epsilon}{2}) \\
+ \frac{4\pi \alpha V_2}{16\pi^2} \left( \frac{1}{12} \left(1 - \frac{1}{\alpha^2}\right) \right) \left( \frac{4\pi \mu^2}{M^2} \right)^{\frac{\epsilon}{2}} M^2 \Gamma(-1 + \frac{\epsilon}{2}) 
\end{align*}
\]
\quad . \quad (51)

The Gamma function has poles along the negative real axis for inte-
ger values, which are regulated by the \( \epsilon \) factors. The \( \epsilon \) poles resulting from this
need to be isolated and included in counter-terms if we intend to consider
field theory on the manifold (see section 14).

It is easy to extract the finite part of these terms by using an expansion
of the Gamma function. Explicitly
\[
\begin{align*}
\Gamma(-2 + \frac{\epsilon}{2}) &= \frac{1}{2} \left[ \frac{2}{\epsilon} + \frac{3}{2} - \gamma \right] + O(\epsilon) \\
\Gamma(-1 + \frac{\epsilon}{2}) &= - \left[ \frac{2}{\epsilon} + 1 - \gamma \right] + O(\epsilon)
\end{align*}
\]
\quad , \quad (52)
\quad , \quad (53)

where \( \gamma \) is the Euler-Mascheroni constant. The finite pieces of the first two
terms of the logarithm of the determinant of the Klein-Gordon operator are then
\[- \frac{\alpha V_2 U_2 M^4}{16\pi^2} \left[ \frac{3}{4} - \frac{\gamma}{2} + \frac{1}{2} \log \left( \frac{4\pi \mu^2}{M^2} \right) \right] + \frac{4\pi\alpha V_2 M^2}{16\pi^2} \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \right) \left[ \gamma - 1 - \log \left( \frac{4\pi \mu^2}{M^2} \right) \right] \quad .\]  

(54)

The integration of the final term of (48) is more interesting. It should first be noted that \( K(S, \tau) \) contains a constant term which is independent of \( S \), and also a sum of terms which are \( S \) dependent. Since \( K(0, \tau) = 0 \), this implies that if we consider the integration of \( K(S, \tau) \) in (48), the resultant (correctly regularized) terms should all vanish when \( S = 0 \). In other words, the \( S \) dependent pieces of the result should still cancel with the \( S \) independent term in the limit \( S \to 0 \). However, the integral of the \( S \) independent term is formally infinite, for all \( S \), for the same reason as the terms we dealt with above. Since it must cancel with the \( S \) dependent terms as \( S \to 0 \), this means that the \( S \) dependent pieces must also become infinite in this limit.

If we regulate the \( S \) independent piece in the above manner to control its divergence, (which is enough to make the last term of integral (48) finite for all non-zero \( S \)), then it will no longer cancel with the \( S \) dependent terms in the limit \( S \to 0 \), since these terms still diverge. Therefore we must find a way to regulate the \( S \) dependent pieces too, so that they will be finite in the \( S \to 0 \) limit, and properly cancel with the regulated \( S \) independent piece. Bearing this in mind, and noting the following integral identity,

\[
\int_0^\infty x^{\nu-1} e^{-\frac{\beta}{x}} e^{-\gamma x} dx = 2 \left( \frac{\beta}{\gamma} \right)^{\frac{\nu}{2}} K_{\nu}(2\sqrt{\beta \gamma}) \quad ,
\]

(55)

where \( \beta \) and \( \gamma \) are real and positive and \( K_{\nu}(z) \) is a modified Bessel function of the second kind, we can write the integral of one of the \( S \) dependent terms in the sum as follows,

\[
\int_0^\infty \frac{d\tau}{\tau^2} e^{-\frac{k^2 x^2 \tau^2}{\tau^2}} e^{-M^2 \tau} = \frac{2M}{k\pi S} K_1(2k\pi SM) \quad .
\]

(56)
As expected, this expression diverges in the limit $S \to 0$. Consequently the integral needs to be regulated and we must remove terms in $S$ which are potentially divergent as $S \to 0$. These subtractions will be absorbed in the redefinitions of the bare coupling constants of field theories on the manifold. We can be guided by the following power series expansion,

$$K_1(z) = \frac{1}{z} + \log \left( \frac{z}{2} \right) I_1(z) - \frac{z}{4} \sum_{m=0}^{\infty} \left[ \Psi(m+1) + \Psi(m+2) \right] \frac{(\frac{z^2}{2})^m}{m!(1+m)!}, \quad (57)$$

where $\Psi(z)$ is the Psi (di-gamma) function.

It is tempting to merely subtract all the terms, using the above power series expansion, which diverge as $S \to 0$. However, such subtractions should not spoil the large $S$ behaviour of the $S$ dependent terms (they go to zero). We therefore regulate expression (56) by replacing the right hand side with the following expression

$$M^2 \Upsilon(k, S, M, \delta) = M^2 \left[ \frac{2K_1(2k\pi SM)}{k\pi SM} - \frac{1}{k^2\pi^2 S^2 M^2} \right. \left. \frac{2K_1(2k\pi S\delta)}{k\pi S\delta} + \frac{1}{k^2\pi^2 S^2 \delta^2} \right], \quad (58)$$

where $\delta$ is another parameter with the dimensions of mass, in the same fashion as $\mu$. Its exact value will be determined by renormalization equations in section 14. Instead of diverging in the limit $S \to 0$, $\Upsilon(k, S, M, m)$ tends to the finite value $\log(M^2/\delta^2)$. The terms we have added to the result of (56) in order to achieve this can be regarded as redefinitions of the coupling constants of field theories on our manifold (see section 14).

The $S$ independent piece also needs regulation, but as noted earlier it is merely another Gamma function-like term and can be regulated as follows,

$$\int_0^\infty \frac{d\tau}{\tau^2} e^{-M^2\tau} = \frac{M^2}{6} \left[ \gamma - 1 - \log \left( \frac{4\pi\mu^2}{M^2} \right) \right]. \quad (59)$$

This result relates the arbitrary parameter $\mu$ to the arbitrary parameter $\delta$, since we require that this expression cancels with the $S$ dependent terms.
when $S \to 0$. Hence $\mu^2 = \frac{\delta^2}{4\pi} e^{1/2}$. Therefore we can write the completely regulated integral of expression (44) as

$$
\int_0^\infty \frac{d\tau}{\tau^2} K(S, \tau) = \frac{M^2}{6} \log \left( \frac{M^2}{\delta^2} \right) - \sum_{k=1}^\infty \frac{M^2}{k^2 \pi^2} \Upsilon(k, M, \delta, S). 
$$

(60)

Combining all our results together, we obtain that the correct, regularized result for the log det($\Box_x + M^2$) on our conical, spinning manifold is

$$
\log \det(\Box_x + M^2) = \frac{\alpha V_2 U_2 M^4}{16\pi^2} \left[ -\frac{1}{4} + \frac{1}{2} \log \left( \frac{M^2}{\delta^2} \right) \right] 
+ \frac{4\pi \alpha V_2 M^2}{16\pi^2} \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \log \left( \frac{M^2}{\delta^2} \right) \right) 
+ \frac{4\pi \alpha V_2 M^2}{16\pi^2} \left[ \frac{1}{12} \log \left( \frac{M^2}{\delta^2} \right) - \sum_{k=1}^\infty \frac{M^2}{2k^2 \pi^2} \Upsilon(k, M, \delta, S) \right]. 
$$

(61)

11 The physics of the space-time

It is well known from study of thermodynamics that a complete description of the thermal behaviour of a physical system can be obtained if one knows the partition function for that system. It is also well known that one can obtain results about quantum systems in which temperature plays no part by using similar mathematical techniques. This methodology is known as the path integral formalism of quantum field theory, a useful reference being [54]. If one considers a self interacting scalar field theory with potential $V(\phi(x))$, the action for the theory can be written

$$
S(\phi) = \int \sqrt{-g} d^4x \left( \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right).
$$

(62)

It should be noted that if we wish to use metric (2) as the space-time background $g_{\mu\nu}$ for our quantum fields, $g = \det g_{\mu\nu}$ changes sign for $\rho' > \frac{|S|}{\alpha} - k$ and $\rho' < \frac{|S|}{\alpha} - k$. This means that the Hamiltonian of a scalar field on this background would not be Hermitian in the region $\rho' < \frac{|S|}{\alpha} - k$, which is another reason why it was excluded from the manifold.
With Euclideanization, and an integration by parts the above expression becomes


tilde{S}(\phi) = iS_E(\phi) = i\int \sqrt{g} d^4x \left( \frac{1}{2} \phi \Box \phi + V(\phi) \right) . \quad (63)

The partition function (or transition amplitude as it is known in field theory) is then defined as

\[ Z = \int D\phi e^{iS(\phi)} = \int D\phi e^{-S_E(\phi)} . \quad (64) \]

If we assume \( V(\phi) \) has a minimum at \( \phi(x) = \phi_0 \), where \( \phi_0 \) is a constant (not a function of \( x \)), we can expand the action around this minimum point to obtain a first order (saddle point) approximation to the partition function \( Z \). For example, if \( V'(\phi_0) = 0 \) and \( V''(\phi_0) = M^2 \), we can write

\[ S_E(\phi_0 + \phi) = S_E(\phi_0) + \frac{1}{2} \int \sqrt{g} d^4x \phi \left[ \Box + M^2 \right] \phi . \quad (65) \]

This gives as a first order approximation to the partition function

\[ Z_1 = e^{-X_4V(\phi_0) - \frac{\hbar}{2} \log \det(\Box + M^2)} , \quad (66) \]

where \( X_4 \) is the total volume of the space, and we have temporarily re-inserted the factor of \( \hbar \).

The saddle-point first order approximation to the effective action is then defined as the negative of the logarithm of the above partition function, namely

\[ W_1(\phi_0) = X_4V(\phi_0) + \frac{\hbar}{2} \log \det(\Box + M^2) . \quad (67) \]

A term proportional to classical potential at its minimum value is clearly included in this effective action, along with a term proportional to \( \hbar \). Now the reason for concentrating on evaluating the logarithm of the determinant of the Klein-Gordon operator on our manifold becomes clear - it allows us to directly calculate the first quantum correction term to the classical action.
Considering now the classical field $\phi_c(x)$ [54], we can expand perturbatively around its minimum value at $\phi_0$ to obtain the first order ($\hbar$) approximation to the effective action for classical fields on our spinning manifold. We consider a typical self interacting potential, $V(\phi_c) = \frac{m^2}{2} \phi_c^2 + \frac{\lambda \phi_c^4}{24}$, so consequently, inserting $M^2 = m^2 + \frac{\lambda \phi_0^2}{2}$ and defining $r = (\rho', \phi)$ we obtain as the first order approximation to the effective action with Minkowskiian signature

$$\Gamma(\phi_c) = \int_{\rho' > \rho_0} \sqrt{-g} d^4x \left[ \frac{1}{2} \partial_\mu \phi_c \partial^\mu \phi_c - \frac{m^2 \phi_c^2}{2} - \frac{\lambda \phi_c^4}{24} \right]$$

$$+ \int_{\rho' > \rho_0} \sqrt{-g} d^4x (1 + \alpha \pi \rho_0^2 \delta(2)(r - r_0)) \frac{M^4}{32\pi^2} \left( -\frac{1}{4} + \frac{1}{2} \log \left( \frac{M^2}{\delta^2} \right) \right)$$

$$+ \int_{\rho' > \rho_0} \sqrt{-g} d^4x \delta(2)(r - r_0) \frac{4\alpha M^2}{32\pi^2} \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \right) \left[ \log \left( \frac{M^2}{\delta^2} \right) \right]$$

$$+ \int_{\rho' > \rho_0} \sqrt{-g} d^4x \delta(2)(r - r_0) \frac{4\alpha M^2}{32\pi^2} \left[ \frac{1}{12} \log \left( \frac{M^2}{\delta^2} \right) \right]$$

$$\sum_{k=1}^{\infty} \frac{1}{2k^2\pi^2} \Upsilon(k, M, \delta, S) \right]. \quad (68)$$

The effective action is expressed entirely in terms of the classical field $\phi_c$, coupling constants, physical variables and the (as yet) undetermined parameter $\delta$. It has been successfully regularized, but the coupling constants of the theory still need to be renormalized to provide us with a complete description of the system. This will be considered in section 14. It needs to be borne in mind that the minimum $\phi_0$ of the potential we have chosen is either zero if $m^2$ is positive, or $\phi_0 = \sqrt{\frac{-6m^2}{\lambda}}$ if $m^2$ is considered to be negative. Thus we will continue to denote the minimum point simply by $\phi_0$ to accommodate the possibility of spontaneous symmetry breaking.

12 The interior metric : an example

So far we have only considered the properties of the exterior metric, i.e. the one which is supposed to describe the space-time at radii $\rho' > \rho_0$. There are a variety of models that one could choose for the interior metric [33] [61] [62], all varying in terms of ease of manipulation and realism in representing
a supposed physical system. In order to demonstrate how they can be used, and consequently the complete manifold be considered, we will look at the simplest case here.

The simplest model to consider is that of a infinitely thin rotating shell of matter at radius $\rho' = \rho_0$. The metric inside this shell is then accurately described by the metric of a rotating disk, namely

$$ds^2 = -\left(\sqrt{1 + \frac{S^2(\rho_0^2 - \rho'^2)}{\rho_0^4}} dt + \frac{S \left(\frac{\rho'}{\rho_0}\right)^2 d\phi}{\alpha \sqrt{1 + \frac{S^2(\rho_0^2 - \rho'^2)}{\rho_0^4}}} \right) + dz'^2 + d\rho'^2$$

$$+ \left(\frac{\rho'}{\alpha^2} + \frac{S^2 \left(\frac{\rho'}{\rho_0}\right)^4}{\alpha^2(1 + \frac{S^2(\rho_0^2 - \rho'^2)}{\rho_0^4})}\right) d\phi^2 \quad , (69)$$

where $(t, \rho', \phi, z') = (t, \rho', \phi + 2\pi\alpha, z')$, $0 < \rho' < \rho_0$, $0 < \phi < 2\pi\alpha$, $-\infty < t < \infty$, $-\infty < z' < \infty$ and $0 < \alpha < 1$.

Although the above metric has terms containing the constant $\alpha$, it should be noted that there is no deficit angle in this interior space-time. It should also be noted that when $\rho' = \rho_0$ this metric becomes

$$ds^2 = -(dt + \frac{S}{\alpha} d\phi)^2 + dz'^2 + d\rho'^2 + \left(\frac{\rho_0^2 + S^2}{\alpha^2}\right) d\phi^2 \quad . (70)$$

Since we require the metric components to patch up at $\rho' = \rho_0$, this determines for us the value of the constant $k$ in equation (2) for this interior model, explicitly $k = -\rho_0 + \sqrt{\frac{\rho_0^2 + S^2}{\alpha}}$.

If we now examine the condition on $\rho_0$ from section 1 which prevents the occurrence of closed time-like curves in the space-time, $(\rho_0 > \frac{|S|}{\alpha} - k)$, we see that this choice of interior space-time and hence constant $k$ satisfies the inequality. Thus we can be assured that causality, unitarity and such things are all intact on the complete manifold.
Despite looking rather complicated, the metric (69) is indeed flat, as can be verified by making the change of variables \(d\phi = d\phi' + \frac{2S}{\rho_0^2}dt\). The metric then becomes

\[
ds^2 = - \left(1 + \frac{S^2}{\rho_0^2}\right) dt^2 + \frac{\rho'^2}{\alpha^2} d\phi'^2 + d\rho'^2 + dz'^2.
\] (71)

Since \(0 < \phi' < 2\pi\alpha\) it is clear that the original metric (69) is merely a flat disc viewed from a rotating co-ordinate system. This fact allows us to place an upper bound on the variable \(S\). We cannot consider a rotating cylinder which has superluminal velocity, so this constrains \(S < \rho_0\).

It is possible to re-write this metric with a Euclidean signature by using the same transformations as in section 2, \(\rho' = i\rho, z' = iz\). We then obtain

\[
ds^2 = - \left(1 + \frac{S^2}{\rho_0^2}\right) dt^2 - \frac{\rho'^2}{\alpha^2} d\phi'^2 - d\rho'^2 - dz'^2.
\] (72)

Although both the interior and exterior space-times can be seen to be locally flat, this does not imply that there is no curvature at all in the complete space-time - curvature may still exist at the boundary between the two space-times, \(\rho' = \rho_0\). Although the interior and exterior metric coefficients have the same value at the boundary, the derivatives of those coefficients are not the same there. This means that the extrinsic curvature of the boundary is different for the two sides, and thus the net curvature is the difference between the interior and exterior extrinsic curvatures at the boundary.

The energy-momentum tensor of the singular boundary surface was calculated by this method in [33]. The non-zero components of the four dimensional energy-momentum tensor describing the cylinder at \(\rho' = \rho_0\) can be written (using metric (69)) as

\[
8\pi GT^0_0 = \frac{2\pi\alpha\rho_0}{\sqrt{\rho_0^2 + S^2}} \left(\alpha - \frac{(\rho_0^2 + S^2)^{\frac{3}{2}}}{\rho_0^3}\right) \delta^{(2)}(r - r_0).
\]
where $r = (\rho', \theta)$ and $0 < \theta < 2\pi\alpha$. The curvature is easily obtained if we note the Einstein equation for general relativity and take the trace, i.e. $R = -Tr(8\pi GT_{\mu\nu})$. Hence

$$R(\alpha, S) = \frac{4\pi\alpha}{\sqrt{\rho_0^2 + S^2}} \left( \sqrt{\rho_0^2 + S^2} - \alpha\rho_0 \right) \delta^{(2)}(r - r_0) \quad . \quad (74)$$

Expressions for the mass and angular momentum of the cylinder can be simply obtained from (73). To contrast with the infinitesimally thin conical singularity case (1), it can be seen that the expression for the source mass will involve both the spin parameter $S$ and the deficit angle parameter $\alpha$. What this means is that for a cylinder with given physical mass and angular momentum, we must compute $\alpha$ and $S$ using expressions for the physical quantities derived from the energy momentum tensor. This system also allows the possibility of a cylinder with non-zero mass and angular momentum, but with the deficit parameter $\alpha = 1$. This appears to be a model of a domain wall, which separates two topologically distinct regions of Minkowski space.

It should also be noted that when $S = 0$ and $\rho_0 = 0$ we return to the expression for curvature for the simple conical singularity obtained in [24], [46] and other papers.

## 13 Heat kernel in the interior region

In order to complete the calculation of the effective action on the whole space-time, we need to consider the expression for the heat kernel in the
interior region. Fortunately this is fairly straightforward, since the interior region is flat and has no deficit angle. If we consider the metric

\[ ds^2 = -dt_i^2 - \rho^2 d\phi^2 - dz^2 - d\rho^2, \]  

(75)

where \( t_i = \sqrt{1 + \frac{s^2}{\rho_0^2} t} \), and \( 0 < \rho < \rho_0 \), this is clearly a special case of metric (1), where we have changed the variables and let \( \alpha = 1 \). We would therefore expect the heat kernel to be written

\[ F^E(x, x', \tau) = e^{-M^2 \tau} \left[ \frac{1}{4\pi \tau} \int_{C_3} \frac{e^{-\frac{\rho_2^2}{\tau} \sin^2(\frac{z}{2})}}{e^{-iz} - 1} \right]. \]  

(76)

Taking the trace of this equation gives

\[ Tr \left( F^E(x, x', \tau) \right) = \frac{e^{-M^2 \tau} \sqrt{1 + \frac{s^2}{\rho_0^2} V_2}}{(4\pi \tau)^2} \left[ \frac{1}{2} \int_{C_3} \frac{1 - e^{-\frac{\rho_2^2}{\tau} \sin^2(\frac{z}{2})}}{\sin^2(\frac{z}{2}) (e^{-iz} - 1)} \right]. \]  

(77)

Inspection of this result and reviewing section 7 convinces us that this is zero. Therefore the only internal contribution to the effective action in this model comes from the classical terms

\[ \Gamma(\phi_c) = \int_{\rho_0'} \sqrt{-g} d^4x \left[ \frac{1}{2} \partial_\mu \phi_c \partial^\mu \phi_c - \frac{m^2 \phi_c^2}{2} - \frac{\lambda \phi_c^4}{24} \right]. \]  

(78)

14 Renormalization of the coupling constants

Now that we have an expression for the effective action on the complete manifold, we can discuss the renormalization of the coupling constants which occur in the theory. The approach used here is based on the method discussed in [7]. All the coupling constants and parameters of the bare Lagrangian need renormalization since we are dealing with an interacting field theory. The typical terms which arise in the Lagrangian for a self interacting scalar field on curved background (including a term for the gravitational Lagrangian itself) are [4]
\[ \mathcal{L}(\phi_{c,B}, R) = \frac{1}{2} \phi_{c,B} \Box_x \phi_{c,B} + \frac{m_B^2}{2} \phi_{c,B}^2 + \frac{\lambda_B \phi_{c,B}^4}{4!} + \varepsilon_B R + \frac{1}{2} \xi_B R \phi_{c,B}^2, \quad (79) \]

where the subscript B denotes a bare parameter. The first part of the renormalization procedure is to redefine the bare parameters so that they remove the divergent pieces in the effective action, calculated in section 10. These may be written

\[ -\frac{\alpha V_2 U_2}{16\pi^2} M^4 \left( \frac{1}{\epsilon} \right) + \frac{4\pi\alpha V_2}{16\pi^2} \frac{M^2}{12} \left( 1 - \frac{1}{\alpha^2} \right) \left( \frac{-2}{\epsilon} \right) \]
\[ + \frac{4\pi\alpha V_2}{16\pi^2} \frac{M^2}{12} \left( \frac{-2}{\epsilon} \right) - \sum_{k=1}^{\infty} \frac{1}{2k^2\pi^2} \left( \frac{1}{k^2\pi^2 S^2} \right) \]
\[ + \left[ \frac{2M^2 K_1(2k\pi S\delta)}{k\pi S\delta} - \frac{M^2}{k^2\pi^2 S^2 \delta^2} \right] \quad , \quad (80) \]

where as before \( M^2 = m^2 + \frac{\lambda}{2} \phi_c^2 \). By redefining the (physically meaningless) bare parameters to cancel these terms, we can obtain the finite effective action derived in section 10. Explicitly, by comparing powers of \( \phi_c \) in the expressions, we are lead to the following results, where we have written \( \frac{1}{2} \xi_B R = \sigma_B \delta^{(2)}(r - r_0) \) and \( \varepsilon_B R = \nu_B \delta^{(2)}(r - r_0) \),

\[ \phi_{c,B} = \mu^{-\frac{1}{2}} \phi_c \quad \quad (81) \]
\[ m_B^2 = m^2 \left( 1 + \frac{\hbar \lambda}{16\pi^2 \epsilon} \right) + \mathcal{O}(\hbar^2) \quad \quad (82) \]
\[ \lambda_B = \mu^\epsilon \left( \lambda + \frac{3\lambda^2 \hbar}{16\pi^2 \epsilon} \right) + \mathcal{O}(\hbar^2) \quad \quad (83) \]
\[ \nu_B = \mu^{-\epsilon} \left[ \nu + \frac{4\pi\alpha m^2}{16\pi^2} \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \right) \frac{\hbar}{\epsilon} + \frac{4\pi\alpha m^2}{16\pi^2} \frac{1}{12} \frac{\hbar}{\epsilon} + \mathcal{O}(\hbar^2) \right] \]
\[ + \frac{4\pi\alpha \hbar}{32\pi^2 \alpha^2} \sum_{k=1}^{\infty} \frac{1}{2k^2\pi^2} \left[ \frac{1}{k^2\pi^2 S^2} (1 - m^2) \frac{1}{\delta^2} + \frac{2m^2 K_1(2k\pi S\delta)}{k\pi S\delta} \right] \quad (84) \]
\[ \sigma_B = \sigma + \frac{4\pi\alpha}{16\pi^2} \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \right) \frac{\hbar \lambda}{\epsilon^2} + \frac{4\pi\alpha}{16\pi^2} \frac{1}{12} \frac{\hbar \lambda}{\epsilon^2} \]
\[ + \frac{4\pi\alpha \hbar}{32\pi^2 \alpha^2} \sum_{k=1}^{\infty} \frac{1}{2k^2\pi^2} \left[ \lambda K_1(2k\pi S\delta) - \lambda \frac{2m^2 K_1(2k\pi S\delta)}{k\pi S\delta} \right] + \mathcal{O}(\hbar^2) \quad . \quad (85) \]
It should be noted that because of the particular regularization scheme that we have chosen to employ, the structure of the ultra-violet poles in the theory when spin is included has been somewhat obscured. We will address these issues in the final part of this paper, section 15. However, it is the finite part of the effective action that is the more important piece, certainly as far as physical applications are concerned, and it is correctly given by expression (60).

Although we have now successfully removed all divergent pieces of the effective action, additional redefinitions of the coupling constants are required to ensure that the physical parameters (e.g. the mass $m$) have the same values (obtained by differentiating the Lagrangian) after the quantum corrections have been included. To achieve this we must add further counterterms to the Lagrangian.

We can now work with a completely finite effective potential for the our theory, inferred from the expressions for the effective action in both the interior and exterior regions and expressed in terms of the renormalized couplings $m, \lambda, \text{etc.}$, namely

$$V(\phi_c, \alpha, S) = \left[ \frac{m^2 \phi_c^2}{2} + \frac{\lambda \phi_c^4}{24} \right] +$$

$$\frac{M^4}{32\pi^2} \left[ -\frac{1}{4} + \frac{1}{2} \log \left( \frac{M^2}{\delta^2} \right) \right] \left( 1 - \frac{\pi \rho_0^2}{\alpha} \left( \sqrt{1 + \frac{S^2}{\rho_0^2}} - \alpha^2 \right) \delta^{(2)}(r - r_0) \right)$$

$$+ \frac{4\pi\alpha M^2}{32\pi^2} \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \log \left( \frac{M^2}{\delta^2} \right) \delta^{(2)}(r - r_0) \right) +$$

$$\frac{4\pi\alpha}{32\pi^2 \alpha^2} \frac{M^2}{12} \log \left( \frac{M^2}{\delta^2} \right) - \sum_{k=1}^{\infty} \frac{M^2}{k^2 \pi^2} \left[ \left( \frac{K_1(2k\pi SM)}{k\pi SM} - \frac{1}{2k^2 \pi^2 S^2 M^2} \right) \frac{1}{k\pi \delta S} - \frac{1}{2k^2 \pi^2 \delta^2 S^2} \right] \delta^{(2)}(r - r_0). \quad (86)$$

It will be more helpful to rewrite this expression in terms of the space-time curvature $R,$
\[ V(\phi_c, \alpha, S) = \left[ \frac{m^2 \phi_c^2}{2} + \frac{\lambda \phi_c^4}{24} \right] + \]

\[ \frac{M^4}{32\pi^2} \left[ -\frac{1}{4} + \frac{1}{2} \log \left( \frac{M^2}{\delta^2} \right) \right] \left( 1 - \frac{\rho_0 \sqrt{\rho_0^2 + S^2}}{4\alpha^2} \left( \frac{\sqrt{1 + \frac{S^2}{\rho_0^2} - \alpha^2}}{\sqrt{1 + \frac{S^2}{\rho_0^2} - \alpha}} \right) R \right) \]

\[ + \frac{M^2}{32\pi^2} \frac{\sqrt{\rho_0^2 + S^2}}{\sqrt{\rho_0^2 + S^2 - \alpha \rho_0}} \left( \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \right) \log \left( \frac{M^2}{\delta^2} \right) R \]

\[ + \frac{M^2}{32\pi^2} \frac{\sqrt{\rho_0^2 + S^2}}{\sqrt{\rho_0^2 + S^2 - \alpha \rho_0}} \left[ \frac{1}{12} \log \left( \frac{M^2}{\delta^2} \right) - \sum_{k=1}^{\infty} \frac{1}{k^2 \pi^2} \left( K_1(2k\pi SM) \right) \right] R. \quad (87) \]

We now impose the following renormalization condition,

\[ m^2 = \left. \frac{\partial^2 V}{\partial \phi_c^2} \right|_{\phi_c = 0, R = 0}. \quad (88) \]

With some algebra we obtain the expression

\[ \frac{m^2 \lambda (- \log \left( \frac{\delta^2}{m^2} \right))}{32\pi^2} = 0. \quad (89) \]

This result enables us to fix the parameter \( \delta \) in our previous expressions, giving \( \delta^2 = m^2 \). We are then left with the effective potential

\[ V(\phi_c, \alpha, S) = \left[ \frac{m^2 \phi_c^2}{2} + \frac{\lambda \phi_c^4}{24} \right] + \]

\[ \frac{M^4}{64\pi^2} \left[ -\frac{1}{2} + \log \left( \frac{M^2}{m^2} \right) \right] \left( 1 - \frac{\rho_0 \sqrt{\rho_0^2 + S^2}}{4\alpha^2} \left( \frac{\sqrt{1 + \frac{S^2}{\rho_0^2} - \alpha^2}}{\sqrt{1 + \frac{S^2}{\rho_0^2} - \alpha}} \right) R \right) \]

\[ + \frac{M^2}{32\pi^2} \frac{\sqrt{\rho_0^2 + S^2}}{\sqrt{\rho_0^2 + S^2 - \alpha \rho_0}} \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \log \left( \frac{M^2}{m^2} \right) R \]

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$$+ \frac{M^2}{32\pi^2\alpha^2} \sqrt{\rho_0^2 + S^2} \left[ \frac{1}{12} \log \left( \frac{M^2}{m^2} \right) - \sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \right] \left( \frac{K_1(2k\pi SM)}{k\pi SM} - \frac{1}{2k^2\pi^2 S^2 M^2} \right) \right] \right) R. \ (90)$$

Although we have now fixed all the arbitrary parameters in the effective potential, we have yet to impose several renormalization conditions on the other coupling constants in the theory. We satisfy these conditions by adding further counterterms, effectively re-defining the coupling constants once again, although this time not by an infinite amount. The next condition we impose is

$$\frac{\partial^4 V}{\partial \phi_c^4} \bigg|_{\phi_c=\phi_0, R=0} = \lambda \quad . \quad (91)$$

This gives the counter-term

$$\frac{\lambda^2}{64\pi^2} \left( \frac{8m^4}{M_0^4} + \frac{8m^2}{M_0^2} - 22 - 6 \log \left( \frac{M_0^2}{m^2} \right) \right) \frac{\phi_c^4}{4!} \quad , \quad (92)$$

where $M_0^2 = m^2 + \frac{\lambda}{2} \phi_0^2$. We can view this term as arising from a redefinition of $\lambda$.

We must now look at the counter-terms which arise from terms proportional to $R$. This is more complicated than in the non-spinning case, because $R$ depends on two variables here, not one. However when $R$ is small the two variables $(\alpha, S)$ decouple from one another ($\delta R = \frac{\partial R}{\partial \alpha} \big|_{\alpha=1, S=0} \ d\alpha + \frac{\partial R}{\partial S} \big|_{\alpha=1, S=0} \ dS + ...$), and we can treat the two limiting cases $(\alpha = 1, S = 0)$ separately.

The next renormalization condition can be (somewhat symbolically) written as

$$\frac{\partial V}{\partial R} \bigg|_{\phi_c=0, R=0} = 0 \quad . \quad (93)$$
When $\phi_c = 0$ then $M = m$, so we merely need consider the expression $V(0, \alpha, S)$,

$$V(0, \alpha, S) = -\frac{m^4}{64\pi^2} \left( \frac{1}{2} \right) \left( 1 - \frac{\rho_0 \sqrt{\rho_0^2 + S^2}}{4\alpha^2} \left( \sqrt{\frac{1 + \rho_0^2}{\rho_0^2} - \alpha^2} \right) - R \right). \quad (94)$$

If we set $\alpha = 1$ and differentiate, we obtain

$$\left. \frac{\partial V(0,1,S)}{\partial (R(1,S))} \right|_{S=0} = \frac{m^4}{64\pi^2} \left( \frac{1}{2} \right) \frac{\rho_0^2}{4}. \quad (95)$$

Considering now the other variable, set $S = 0$ and differentiate,

$$\left. \frac{\partial V(0,\alpha,0)}{\partial (R(\alpha,0))} \right|_{\alpha=1} = \frac{m^4}{64\pi^2} \left( \frac{1}{2} \right) \frac{2\rho_0^2}{4}. \quad (96)$$

Thus we derive the counterterm

$$- \frac{m^4}{64\pi^2} \frac{1}{2} \frac{\rho_0^2}{4} (R(1,S) + 2R(\alpha,0)) \quad (97)$$

The next renormalization condition to consider is

$$\left. \frac{\partial^3 V}{\partial R \partial \phi^2} \right|_{\phi = \phi_0, R = 0} = 0 \quad (98)$$

Once again we regard the $S = 0$ and $\alpha = 1$ cases separately, and in the former case we obtain the counterterm

$$\left[ \frac{\lambda}{64\pi^2} \left( 4(M_0^2 - m^2) - (6M_0^2 - 4m^2) \log \left( \frac{m^2}{M_0^2} \right) \right) \frac{2\rho_0^2}{4} + \right.$$  
$$\left. \frac{\lambda}{64\pi^2} \left( 3M_0^2 - 2m^2 + M_0^2 \log \left( \frac{M_0^2}{m^2} \right) \right) \right] \frac{1}{2} R(\alpha,0) \phi_c^2. \quad (99)$$

The $\alpha = 1$ case is slightly more complicated, because the corresponding expression diverges as $S \to 0$. Therefore instead of subtracting a constant
part of $\xi$, we must subtract an $S$ dependent piece from $\xi$ in order to remove this divergence and satisfy (98). This leads us to the following expression for the counterterm,

$$
\frac{\rho_0^2}{4} \frac{\lambda}{64\pi^2} \left( 4(M_0^2 - m^2) + (6M_0^2 - 4m^2) \log\left(\frac{M_0^2}{m^2}\right) \right) \frac{1}{2} R(1, S) \phi_c^2 - \frac{\rho_0^3}{32\pi^2 S^2} \frac{\partial^2}{\partial \phi^2} \left[ \frac{M^2}{6} \log\left(\frac{M^2}{m^2}\right) - \sum_{k=1}^{\infty} \frac{M^2}{k^2\pi^2} Y(k, M, \delta, S) \right] \phi_c^2 \phi_c \left( \frac{1}{2} R(1, S) \phi_c^2 \right). (100)
$$

The term in the square brackets in the previous expression has the form

$$
C(S, M_0, m, \lambda) = \sum_{k=1}^{\infty} \frac{\lambda}{k^2\pi^2} \left[ \log\left(\frac{M_0^2}{m^2}\right) + \frac{2\lambda}{M_0^2} (M_0^2 - m^2) + 2K_0(2k\pi SM_0) + 1 - 4\lambda \frac{k\pi S(M_0^2 - m^2)}{M_0^2} K_1(2k\pi SM_0) - \frac{2K_1(2k\pi Sm)}{k\pi Sm} \right] + \frac{1}{k^2\pi^2 S^2 m^2}. (101)
$$

The counterterms can then be included in the effective action as follows,

$$
V(\phi_c, \alpha, S) = \left[ \frac{m^2 \phi_c^2}{2} + \frac{\lambda \phi_c^4}{24} \right] + \frac{\lambda^2}{64\pi^2} \left( \frac{8m^4}{M_0^4} + \frac{8m^2}{M_0^2} - 22 - 6 \log\left(\frac{M_0^2}{m^2}\right) \right) \phi_c^4 + \frac{M^4}{64\pi^2} \left[ \frac{1}{2} + \log\left(\frac{M^2}{m^2}\right) \right] \left( 1 - \frac{\rho_0 \sqrt{\rho_0^2 + S^2}}{4\alpha^2} \left( \frac{1 + \frac{S^2}{\rho_0^2} - \alpha^2}{\sqrt{1 + \frac{S^2}{\rho_0^2} - \alpha}} \right) \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \right) \log\left(\frac{m^2}{M^2}\right) R
$$

$$
- \frac{M^2}{32\pi^2} \sqrt{\rho_0^2 + S^2} - \alpha \rho_0 \frac{1}{12} \left( 1 - \frac{1}{\alpha^2} \right) \log\left(\frac{m^2}{M^2}\right) R
$$

$$
+ \frac{M^2}{32\pi^2 \alpha^2} \sqrt{\rho_0^2 + S^2} \left[ \frac{1}{12} \log\left(\frac{M^2}{m^2}\right) - \sum_{k=1}^{\infty} \frac{M^2}{k^2\pi^2} \left( \frac{K_1(2k\pi SM)}{k\pi SM} - \frac{1}{2k^2\pi^2 S^2 M^2} \right) - \frac{1}{2k^2\pi^2 S^2 M^2} \right] R
$$

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\[ \begin{align*}
&- \frac{m^4}{64\pi^2} \left[ \frac{1}{2} \rho_0^2 (R(1, S) + 2R(\alpha, 0)) \right] \\
&+ \left[ \frac{\lambda}{64\pi^2} \left( 4M_0^2 - 4m^2 + (6M_0^2 - 4m^2) \log \left( \frac{M_0^2}{m^2} \right) \right) + \frac{\lambda^2}{64\pi^2} \left( \frac{3M_0^2 - 2m^2 + M_0^2 \log \left( \frac{M_0^2}{m^2} \right)}{3M_0^2} \right) \right] \frac{2\rho_0^2}{4} + \\
&+ \frac{\lambda}{64\pi^2} \left( 4M_0^2 - m^2 \right) + (6M_0^2 - 4m^2) \log \left( \frac{\lambda}{64\pi^2} \right) \frac{3M_0^2}{2} R(1, S) \phi_c^2 \\
&+ \frac{\rho_0^2}{32\pi^2 S^2} C(S, M_0, m, \lambda) \frac{1}{2} R(1, S) \phi_c^2.
\end{align*} \] (102)

We have now renormalized all the parameters present in the bare free Lagrangian and also the coupling \( \lambda \). The final counter-term is generated by the constraint \( V(\phi_0, 1, 0) = 0 \) and can be written

\[ \begin{align*}
&- \frac{m^2 \phi_0^2}{2} - \frac{\lambda \phi_0^4}{24} - \frac{\lambda^2}{64\pi^2} \left( \frac{8m^4}{M_0^4} + \frac{8m^2}{M_0^2} \right) - 22 - 6 \log \left( \frac{M_0^2}{m^2} \right) \frac{\phi_0^4}{4!} + \\
&+ \frac{M_0^4}{64\pi^2} \left[ - \frac{1}{2} - \log \left( \frac{M_0^2}{m^2} \right) \right] \frac{1}{2} R(1, S) \phi_c^2.
\end{align*} \] (103)

Having now calculated all the necessary counterterms, we can write the final expression for the effective potential as follows,

\[ V(\phi_c, \alpha, S) = \left[ \frac{m^2 \phi_c^2}{2} + \frac{\lambda \phi_c^4}{24} \right] - \left[ \frac{m^2 \phi_0^2}{2} - \frac{\lambda \phi_0^4}{24} \right] + \frac{\lambda^2}{64\pi^2} \left( \frac{8m^4}{M_0^4} + \frac{8m^2}{M_0^2} \right) - 22 - 6 \log \left( \frac{M_0^2}{m^2} \right) \frac{\phi_0^4}{4!} + \\
+ \frac{M_0^4}{64\pi^2} \left[ - \frac{1}{2} - \log \left( \frac{M_0^2}{m^2} \right) \right] \left( 1 - \left( \frac{\pi \rho_0^2}{\alpha} \sqrt{1 + \frac{S^2}{\rho_0^2} - \alpha^2} \right) \delta(2)(r - r_0) \right) + \\
+ \frac{M_0^4}{64\pi^2} \left[ - \frac{1}{2} - \log \left( \frac{M_0^2}{m^2} \right) \right] \left( 1 - \left( \frac{1}{\alpha} \right) \log \left( \frac{M_0^2}{m^2} \right) \delta(2)(r - r_0) \right) + \\
+ \frac{4\pi \alpha M_0^2}{32\pi^2 \alpha^2} \left[ - \frac{1}{12} \log \left( \frac{M_0^2}{m^2} \right) - \sum_{k=1}^{\infty} \frac{M^2}{k^2 \pi^2} \left( \frac{K_1(2k\pi SM^2)}{k\pi SM^2} - \frac{1}{2k^2 \pi^2 S^2 M^2} \right) \right] .
\]
\[ - \left( \frac{K_1(2k\pi Sm)}{k\pi Sm} - \frac{1}{2k^2\pi^2 S^2 m^2} \right) \delta^{(2)}(r - r_0) \]

\[ - \frac{4\pi m^4 \rho_0^2}{64\pi^2} \frac{1}{2} \frac{\rho_0^2}{4} \left( \frac{\sqrt{\rho_0^2 + S^2} - \rho_0}{\sqrt{\rho_0^2 + S^2}} + 2\alpha(1 - \alpha) \right) \delta^{(2)}(r - r_0) \]

\[ + \left[ \frac{\lambda}{64\pi^2} \left( 4M_0^2 - 4m^2 + (6M_0^2 - 4m^2) \log \left( \frac{M_0^2}{m^2} \right) \right) \right] \frac{2\rho_0^2}{4} + \]

\[ \frac{\lambda}{64\pi^2} \left( \frac{3M_0^2 - 2m^2 + M_0^2 \log \left( \frac{M_0^2}{m^2} \right) \right)}{3M_0^2} \left[ \frac{1}{2} 4\pi\alpha(1 - \alpha)\phi_c^2 \delta^{(2)}(r - r_0) \right] \]

\[ + \frac{\rho_0^2}{4} \frac{\lambda}{64\pi^2} \left( 4(M_0^2 - m^2) + \right) \]

\[ (6M_0^2 - 4m^2) \log \left( \frac{M_0^2}{m^2} \right) \right) 2\pi \left( 1 - \frac{\rho_0}{\sqrt{\rho_0^2 + S^2}} \right) \phi_c^2 \delta^{(2)}(r - r_0) \]

\[ - \frac{\rho_0^2}{32\pi^2 S^2} C(S, M_0, m, \lambda) 2\pi \left( 1 - \frac{\rho_0}{\sqrt{\rho_0^2 + S^2}} \right) \phi_c^2 \delta^{(2)}(r - r_0) \] . (104)

15 Conclusions and discussion

In the course of this article we have employed a variety of mathematical techniques in order to obtain the renormalized first order effective action for massive scalar field theory with a self interacting \( \lambda \phi^4 \) type potential on the space-time generated by a rotating cylindrical shell. We have demonstrated that this theory is well defined, and is not troubled by the closed time-like curves and unitarity problems that have hindered previous attempts at quantization on such a space-time. Having obtained the effective potential \( V(\phi_c, \alpha, S) \), it can then be used to model quantum physics on the manifold, which behaves as if we were dealing with a classical field theory with field variable \( \phi_c(x) \) and potential \( V(\phi_c, \alpha, S) \). It remains to be seen whether there are any interesting phenomenological consequences of the new terms in the effective action, particularly with regard to the physics of cosmic strings and astrophysics in general, and also in electromagnetic theory, where there is an analogous system in the Aharonov-Bohm effect [32].
Since many of the novel features in our effective action first arose in section 10, where we dealt with the regularization and renormalization of the theory, it seems appropriate to expand a little on the discussion of the procedure which was used there to regulate the $S$ dependent terms in the effective action. A superior and more illuminating way in which this regulation can be achieved will now be suggested.

It will be recalled that in section 10 dimensional regularization was used to regulate the integral of the $S$ independent term of $K(S, \tau)$ (expression (59)) and then terms were subtracted from the $S$ dependent parts of $K(S, \tau)$ so that they would cancel with the regulated $S$ independent term in the limit $S \to 0$. I would like to suggest an alternative regularization procedure, in which integral (59) is rendered ultra-violet finite in the following manner,

$$\int_0^{\infty} \frac{d\tau}{\tau^2} e^{-M^2 \tau} e^{-\frac{\eta^2}{\tau}} = \frac{1}{6} \frac{2M\eta}{K_1(2\eta M)}$$

(105)

$\eta$ being a regulation parameter in the same manner as $\epsilon$, which was employed previously in section 10. The added exponential term in the integrand has the effect of supressing the $\tau \to 0$ (small time, high frequency) contribution to the integral and hence provides an ultra-violet cut off, making the expression finite for non-zero $\eta$. The $S$ dependent terms of $K(S, \tau)$ may also be regulated in this way (they are potentially divergent at $S=0$) and this gives the result

$$\int_0^{\infty} \frac{d\tau}{\tau^2} e^{-M^2 \tau} e^{-\frac{k^2\pi^2 S^2}{\tau}} e^{-\frac{\eta^2}{\tau}} = 2M \frac{K_1(2M\sqrt{\eta^2 + k^2\pi^2 S^2})}{\sqrt{\eta^2 + k^2\pi^2 S^2}}$$

(106)

The equivalent of expression (60), although still including the regulation parameter $\eta$, can therefore be written

$$\frac{1}{6} \frac{2M\eta}{K_1(2\eta M)} - 2M \sum_{k=1}^{\infty} \frac{1}{(k\pi)^2} \frac{K_1(2M\sqrt{\eta^2 + k^2\pi^2 S^2})}{\sqrt{\eta^2 + k^2\pi^2 S^2}}$$

(107)

It is now conventional to redefine the bare coupling constants of the theory to absorb the poles in $\eta$ of the above expression as $\eta \to 0$. To that end I introduce the following counterterms,
The first three terms subtract from expression (105), whilst the second three terms subtract from (106). It should be noted that if \( S = 0 \), then all the above terms cancel precisely with one another. This implies that in the non-spinning limit \( (S = 0) \) of our spinning space-time, there are no poles in the theory that need removing above and beyond those found in physics on the simple conical singularity manifold described by metric (1). However, for all \( S \neq 0 \) the terms in the previous expression no longer all cancel, and there are additional ultra-violet poles in the theory, which must be removed.

The nature of these poles is easily deduced by examining the equations appearing earlier in this article. In mode expansion (8), for example, the mode frequency \( \omega \) is seen to couple to the spin \( S \) in the combination \( \omega S \). Since quantum theory involves an integration over all possible mode frequencies, this means that for any \( S \) which is non-zero, a sufficiently high frequency mode will exist which makes \( \omega S \) of a significant size. Examining equation (42), which describes the integration over the mode frequencies, it can be seen that because part of the integrand is periodic, and does not fall off at high frequencies, the integrand is bounded by a gaussian envelope, which does not go to zero fast enough at small \( \tau \) to suppress the energetic modes and make the contribution to the effective action finite. Additional suppression (and hence regularization) is required to obtain a finite answer. The physical origin of the required additional suppression is also easy to understand. Whilst it is reasonable to insist that the low frequency modes should couple with the spin like \( \omega S \), any spinning object has an angular frequency associated with its rotation, and one would not expect modes which have frequency much larger than the angular frequency to be greatly affected by the object’s rotation. In other words, the interaction between the spin and the modes should fall off as the mode frequency increases, and this is precisely what the regularization procedure accomplishes. The simple boundary condition which allowed us to include spin in the conical space-time in the classical theory is therefore seen
to be too powerful a constraint for the high frequency modes in the quantum theory, resulting in the appearance of additional ultra-violet divergences.

Combining expressions (105) and (106) with terms (108), and setting $\eta = 0$, we then obtain the finite piece of the integral of $K(S, \tau)/\tau^2$, which agrees with the result quoted in (60). This type of regulation improves upon that used in section 10, because it is clear from the argument presented here that there are no additional poles beyond those in the standard non spinning theory as $S \to 0$. This is not readily apparent from expressions (84) and (85). Effectively in section 10 we neglected the occurrence of $\eta^2$ in the denominators of several terms in (84) and (85), because for $S \neq 0$ $\eta^2$ is negligible compared to $k^2\pi^2S^2$. However if $S = 0$ then the $\eta^2$ factors come into play, and prevent the occurrence of $S$ poles in the theory, replacing them with $\eta$ poles instead. These eta poles then cancel with the pole resulting from the $S$ independent term, demonstrating that there are no additional divergences in the $S = 0$ theory.

In conclusion, we note that the use of the more physical cylindrical shell source has successfully resolved most of the difficulties that have been traditionally associated with this space-time. Models involving closed time-like curves always seem to contain a non-physical component, be it constituent matter which does not satisfy the energy conditions, or sources with unphysical amounts of angular momentum. Once a more physically realistic system is considered, quantization proves possible, and new ultra-violet divergences are discovered, associated with the spin, which do not occur in the simple conical singularity case. These divergences are related to the interaction between the high energy modes of quantum fields propagating on the space-time and the source spin, but it proves possible to regulate them and renormalize the coupling constants of the theory so that a consistent description of the theory in all regimes of $S$ is possible.

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