The processing of saturation function in saturation control system

Lijie Jia

1 College of Mathematics and Systems Science, Shandong University of Science and Technology, Qingdao, Shandong, 266590, China

* Corresponding author’s e-mail: jljmath@163.com

Abstract. This article mainly for linear systems with actuator saturation as the research object, in order to further expand the saturated attraction domain of linear system and the estimate for the main purpose, starting from the process of saturation function, enumerates the convex hull representation and the conditions of local sectors two methods, and then get bigger attraction domain estimation. Finally, an example is given to compare the two methods.

1. Introduction
Actuator saturation phenomenon exists widely in practical engineering systems. Actuator saturation phenomenon exists widely in practical engineering systems. In recent decades, scholars in the field of control have paid extensive attention to saturated constrained control, and systematically studied various problems of global and local stability of saturated systems, and obtained considerable research results. In the study of local stability, there is some conservatism in dealing with saturation function and selecting Lyapunov function.

Compared with the global and local sector conditions, the convex hull representation is less conservative when dealing with saturated linear feedback, so a looser set of LMI[1] conditions can be obtained to ensure the invariance of ellipsoid contraction. Taking this set of inequality conditions as constraint conditions, we can establish a convex optimization problem to obtain maximum contraction invariant ellipsoids. These two methods and their applications in attractor domain estimation are described respectively. This paper focuses on a class of saturation constrained control system, namely, linear system with saturated input as shown below:

\[ \dot{x} = Ax + Bsat(Kx) \]  

Estimating the absorption domain of the control system with saturation constraint involves two aspects: the processing of saturation constraint and the selection of Lyapunov function. Next, we mainly discuss the processing of saturation[2] function.

2. Saturation Constraint Processing

2.1 Convex Hull Representation
Define a set of matrices, let \( S \) be a set of \( m \times m \) diagonal matrices, where the diagonal elements of each diagonal matrix are either 1 or 0. Take \( m = 2 \) as an example, there are

\[ S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \]
Let $S_i=I_m-S_i$, then $S_i^* \in S$. The convex hull representation is shown below.

**Lemma 1**  
Let $K$, $H \in \mathbb{R}^{m \times n}$ for each of these states $x \in L(H)$, then
\[
\text{sat}(Kx) \in \text{co}(S_iKx + S_i^*Hx : i \in I[1,2^n])
\]  
where $\text{co}\{\}$ is the convex hull formed by a set of vectors.

There exists a set of non-negative scalar functions of $\alpha_i(x)$ that satisfy $\sum_{i=1}^{n} \alpha_i(x)=1$ such that
\[
\text{sat}(Kx) = \sum_{i=1}^{n} \alpha_i(x)(S_iK + S_i^*H)
\]

Consider the quadratic Lyapunov function, for each $x \in L(H)$, there is
\[
\dot{V}(x) = 2x^TP(Ax + B\text{sat}(Kx)) = 2x^T \sum_{i=1}^{2^n} \alpha_i(x)(P(A + B(S_iK + S_i^*H)))x.
\]

**Theorem 1** [3]  
Given the positive definite matrix $P \in \mathbb{R}^{n \times n}$, if there is a matrix $H \in \mathbb{R}^{m \times n}$, that satisfies $He(P(A + BS_i,K + BD_iH)) < 0$, $i \in I[1,2^n]$, and $\Omega(P,1) \subseteq L(H)$, the ellipsoid $\Omega(P,1)$ is the contractive invariant set of the system.

For the convex hull representation, we can understand it through figure 1. Taking $m=2$ as an example, $k_j$ and $h_j$ are the $j$th row of matrix $K$ and $H$ respectively, $j = 1,2$, the shaded region in the figure represents the linear region, that is $L(H) = \{k_{ij} x \in \mathbb{R}^n: L(x) \subseteq \text{sat}(Hx)\}$

**Figure 1. Convex hull representation**

2.2 Local Sector Conditions

The local sector condition is to place the saturation function in the linear sector, so as to obtain the sector condition in the form of inequality, given a matrix $H \in \mathbb{R}^{m \times n}$, write for $L(H) = \{x \in \mathbb{R}^n: \|Hx\|_\infty \leq 1\}$

Among them, $L(H)$ is the region where $Hx$ is not saturated, which is the linear region of $\text{sat}(Hx)$.

**Lemma 2** [4]  
Let $K$, $H \in \mathbb{R}^{m \times n}$ for each state $x \in L(H)$, and any positive definite diagonal
moment \( Q \in \mathbb{R}^{m \times m} \), the following inequality is true:

\[
(Kx - \text{sat}(Kx))^T Q (\text{sat}(Kx) - Hx) \geq 0 \quad (3)
\]

![Figure 2. Geometric interpretation of local sector conditions](image)

Figure 2 shows the geometric interpretation of the local sector conditions in the case of \( m=1 \) and \( m=2 \). When the \( x \in [-x_0, x_0] \), saturated linear feedback \( \text{sat}(Kx) \) is located in sectors of lines \( F_x \) and \( K_x \). It is worth noting that inequality (2) is only valid if \( x \) is on local region \( L(H) \). If set to \( H = 0 \), the local sector condition becomes the (global) sector condition, i.e.

\[
(Kx - \text{sat}(Kx))^T Q (\text{sat}(Kx) - Kx) T \geq 0 \quad (4)
\]

The general use of local sector conditions is, the non-negative term \( (Kx - \text{sat}(Kx))^T Q (\text{sat}(Kx) - Hx) \) is added to the time derivative of Lyapunov function along the trajectory of the system to form a quadratic function of augmented state vector composed of system state and saturation function.

Consider the quadratic Lyapunov function \( V(x) = x^T P x \) and its level set ellipsoid \( \Omega(P,1) = \{ x \in \mathbb{R}^n : V(x) \leq 1 \} \), if \( \Omega(P,1) \subseteq L(H) \),

\[
\dot{V}(x) = 2 x^T P x \leq 2 x^T P (Ax + B \text{sat}(Kx)) + 2 (Kx - \text{sat}(Kx))^T Q (\text{sat}(Kx) - Hx)
\]

\[
= \eta^T \begin{bmatrix} H e(PA - K^T QH) & PB + K^T Q + H^T Q \\ * & -2Q \end{bmatrix} < 0
\]

where ellipsoid \( \Omega(P,1) \) is the systolic invariant set of the system.

According to the above analysis, theorem 2 can be obtained.

**Theorem 2** Given the positive definite matrix \( P \in \mathbb{R}^{m \times m} \), if there is a matrix \( H \in \mathbb{R}^{m \times m} \) and a positive definite diagonal matrix \( Q \in \mathbb{R}^{m \times m} \), it satisfies

\[
\begin{bmatrix} H e(PA - K^T QH) & PB + K^T Q + H^T Q \\ * & -2Q \end{bmatrix} < 0 \quad (5)
\]

and \( \Omega(P,1) \subseteq L(H) \), the ellipsoid \( \Omega(P,1) \) is the systolic invariant set of the system. In this way, ellipsoid \( \Omega(P,1) \) satisfying theorem 2 can be used as an attractor domain estimation for the system.

### 3. Model optimization

By using convex hull representation and local sector conditions to deal with saturation constraints, theorems 1 and 2 give sufficient conditions to ensure the invariance of ellipsoid[5] \( \Omega(P,1) \) contraction, so that an attractor domain estimation of the system can be obtained. In the estimation of the attractor domain, we aim to find a maximum contraction invariant ellipsoid. First, we need a standard to measure the size of ellipsoid \( \Omega(P,1) \), and let \( X_R \subset \mathbb{R}^n \) be a bounded convex set[6].
There's a positive real number $\alpha$, $\alpha X_R = \{\alpha x : x \in X_R\}$. We can get the maximum estimation of the attractive domain by maximizing $\alpha$. Take the convex hull representation as an example to construct the corresponding optimization problem:

$$\max_{\alpha \geq 0} \alpha, \quad \text{s.t.} \ (a) \alpha X_R \subseteq \Omega(P,1),$$

$$(b) He\left(P\left(A + BS_i K + BS_i^T H\right)\right) \leq 0, \quad i = 1, 2^n,$$

$$(c) E(P) \subseteq L(H).$$

By transforming the set inclusion constraints (a) and (b) into matrix inequality conditions [7], the optimization problem of the new linear matrix inequality form can be obtained:

$$\min_{Q \geq 0, Z} \gamma, \quad \text{s.t.} \ (a) \begin{bmatrix} \gamma & x_i^T \\ x_i & P^{-1} \end{bmatrix} \geq 0, \ i \in [1, p],$$

$$(b) He\left(AP^{-1} + BS_i K P^{-1} + BS_i Z\right) \leq 0, \quad i \in [1, 2^n],$$

$$(c) \begin{bmatrix} 1 & z_j \\ z_j^T & P^{-1} \end{bmatrix} \geq 0, \ j \in [1, m].$$

Similarly, according to local sector conditions, the corresponding optimization problem is constructed and the corresponding maximum shrinkage invariant ellipsoid is solved.

4. Numerical examples

Example. Consider the saturation constraint control system $\dot{x} = Ax + B_{sat}(Kx)$,

$$A = \begin{bmatrix} 0 & 2 \\ -3 & 0.4 \end{bmatrix}, \quad B = \begin{bmatrix} 1.4 & 3 \\ 0 & -0.7 \end{bmatrix}, \quad K = \begin{bmatrix} -0.4698 & -0.0770 \\ -0.8318 & 0.7640 \end{bmatrix}. $$

Let $X_R = \{x_i\}$, where $x_i = [0 \ 1]^T$, Matlab to solve the optimization problem, get

Convex hull representation: $\alpha = 7.1024$, $P = \begin{bmatrix} 0.0102 & -0.0024 \\ -0.0024 & 0.0121 \end{bmatrix}$;

Local sector conditions: $\alpha = 6.3167$, $P = \begin{bmatrix} 0.0198 & -0.0031 \\ -0.0031 & 0.0134 \end{bmatrix}$;

From the point of view of $\alpha$, using the convex hull representation to deal with the saturation constraint can obtain a larger shrinkage invariant ellipsoid than the local sector condition.

5. Summary

In this paper, several approaches to deal with saturation constraints are introduced, and their applications in attractor region estimation of saturated control systems are summarized. However, there is a big gap between the estimation of attraction field obtained by quadratic Lyapunov function and that obtained by quadratic Lyapunov function.

References

[1] Zhou, X.F.,Wei, J.,Hu, L.G. (2013) Controllability of a fractional linear time-invariant neutral dynamical system.Appl.Math.Lett., 26: 418-424.

[2] Sabatier, J.,Moze, M.,Farges, M.C. (2009) LMI stability conditions for fractional order
systems. Comput. Math. Appl., 59: 1-16.

[3] Tingshu, H., Zongli, L., Ben, C., (2002) An analysis and design method for linear systems subject to actuator saturation and disturbances. J. Automatica., 38(2): 351-359.

[4] Tingshu, H., Zongli, L. (2001) Control systems with actuator saturation: Analysis and design. In: William S. L., (Eds.) Birkhauser, Boston. 55-67.

[5] Blanchini, F. (1999) Set invariance in control. J. Automatica., 35(11): 1747-1767.

[6] Fiacchini, M., Tarbouriech, S., Prieur, C. (2012) Quadratic stability for hybrid systems with nested saturations. J. IEEE Trans on Automatic Control., 57(7): 1832-1838.

[7] Li, Y., Lin, Z. (2014) Saturation-based switching anti-windup design for linear systems with nested input saturation. J. Automatica., 50(11): 2888-2896.