Interference of probabilities and number field structure of quantum models

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March 31, 2022

Abstract

We study the probabilistic consequences of the choice of the basic number field in quantum formalism. We demonstrate that by choosing a number field for a linear space representation of quantum model it is possible to describe various interference phenomena. We analyse interference of probabilistic alternatives induced by real, complex, hyperbolic (Clifford) and $p$-adic representations.

1 Introduction

In this paper we study the role of the basic number field, namely number field that can be used for the linear space representation, in quantum formalism. Since the creation of quantum formalism, there are continuous debates on the role of the number field structure of quantum theory, see, for example, [1]-[14]. There is the large diversity of opinions on this problem.

We just point out that many consider the use of complex numbers as the convenient mathematical description of two dimensional real quantities. From this point of view $z = x + iy \in \mathbb{C}$ is just the vector $v = (x, y) \in \mathbb{R}^2$. This viewpoint justifies the reduction of complex Shrödinger’s equation to the pair of real equations, in hydrodynamic models of quantum theory as well
as the pilot wave theory [15], [16]. It is important to note that, in fact, the real hydrodynamic model was one of the first quantum models. In principle, we can consider Shrödinger’s wave mechanics as the complexification of the hydrodynamic model. In this context it is natural to ask ourselves: What was the main advantage of this complexification? I think it was the linearization of nonlinear hydrodynamic equations.

Our investigation on the role of number field in quantum formalisms is strongly based on famous Dirac’s analysis [1] of the foundations of quantum theory. Regarding to the appearance of complex numbers in quantum formalism, Dirac noted that by using only real coefficients for superpositions of states we could not describe all superpositions that can be produced in experiments with elementary particles. This remark is also very important for us.

In the present paper we find probabilistic roots of the appearance of the complex structure. It seems that the main aim of the introduction of complex numbers was the linearization (in a \(\mathbb{C}\)-linear space) of quantum rule for interference of probabilistic alternatives. We also consider the experimental domain that can be described by the \(\mathbb{R}\)-linearization. We found that the latter linearization describes only maximal interference (the \(\cos \theta = \pm 1\)) and trivial interference (\(\cos \theta = 0\)).

Then we study the ‘quantum formalism’ for the algebra of so called hyperbolic numbers \(\mathbb{G}\), namely two dimensional Clifford algebra\(^1\) (here \(z = x + jy, x, y \in \mathbb{R}\) and \(j^2 = 1\), see, e.g. [6], [14]). Suddenly we discovered the possibility of a new type of interference, namely hyperbolic interference. In the opposite to the standard quantum formalism in that the conventional probabilistic rule for the addition of alternatives is perturbed by \(\cos \theta\)–factor, in the hyperbolic case we get \(\cosh \theta\)–perturbation. It may be that such an ‘interference’ could be observed in some experiments.

Finally, we provide probabilistic analysis of ‘quantum formalism’ for the fields of \(p\)-adic numbers \(\mathbb{Q}_p\), see, e.g. [5], [7]–[14] for \(p\)-adic physical models. We recall that first \(p\)-adic models appeared in string theory as an attempt to modify the description of space-time on Planck distances\(^2\). There are two

\(^1\)We call this algebra hyperbolic algebra, because hyperbolic functions are naturally related to the algebraic structure of \(\mathbb{G}\) through a hyperbolic generalization of Euler’s formula for the complex numbers.

\(^2\)Of course, we did not forget about the work [5]. However, Beltrametti and Cassinelli obtained the following negative result: \(p\)-adic linear space could not be the used for quantization in the framework of quantum logic. As a consequence, this paper did not stimulate
main types of $p$-adic quantum models:

(1) wave functions $\varphi : \mathbb{Q}_p \to \mathbb{C}$;

(2) wave function $\varphi : \mathbb{Q}_p \to \mathbb{Q}_p$ or into one of extensions of $\mathbb{Q}_p$.

The first model is based on the representation in the standard $\mathbb{C}$-linear space, see [7]-[10]. There is nothing special (compare to the conventional quantum theory) from the probabilistic viewpoint. The second model is based on the representation in linear spaces over $\mathbb{Q}_p$ or its extensions, see [11]. We consider probabilistic consequences of such representations.

We found that from the probabilistic viewpoint the $p$-adic quantum formalism describes a rather special sub-domain of standard quantum theory (over the field of complex numbers). For some physical models it might be more fruitful to use $p$-adic quantization, because the $p$-adic geometry might be more natural to describe such phase behaviour. From this point of view we discuss a $p$-adic analogue of the interference experiment. Such an experiment would exhibit unusual fluctuations of the spatial distribution of probability: it should fluctuate discontinuously with respect to the ordinary Euclidean geometry (but continuously with respect to the $p$-adic topology), see Example on the $p$-adic interference model.

Of course, in the present note we do not plan to provide extended analysis of the use of various algebraic structures in physics (especially quantum). We would like just to mention investigations on quaternionic models, e.g., [17], [18], [3], [6], $p$-adic models, e.g. [5], [7]–[14], models over supercommutative Banach (and more general topological) superalgebras, e.g. [12]–[14], applications of Clifford algebra, e.g. [6].

### 2 Probabilistic roots of complex and real linear representations

It is well known that the creation of quantum formalism was, in fact, induced by two kinds of experiments (see, for example, the introductory chapters of Dirac’s or Heisenberg's books, [1] or [19]):

1) **Corpuscular**: Black body radiation, photoelectric effect,...

2) **Wave**: Diffraction, two slit experiment for elementary (especially massive) particles,...

[3] further investigations in $p$-adic physics.
The first class of experiments showed that light has the corpuscular structure. Corpuscles of light, photons, demonstrated discrete behaviour similar to behaviour of macro objects, for examples, balls. In such a situation it was natural to assume that statistical properties of these corpuscular objects would be described by the conventional probability theory.\footnote{We remark that, in fact, in 20th the mathematical foundations of probability theory (Kolmogorov’s axiomatics, 1933, \cite{20}) were not yet created. There was the large diversity of approaches to the notion of probability in the mathematical as well physical communities.} In particular, we should get the conventional rule for the addition of probabilities of alternatives:

\[ P = P_1 + P_2. \]  

However, the two slit experiment implies that conventional rule \((1)\) is violated. The right hand side of quantum rule for addition of alternatives contains a perturbation term:

\[ P = P_1 + P_2 + 2\sqrt{P_1 P_2} \cos \theta. \] \(2\)

Rules \((1)\) and \((2)\) are used as prediction rules (especially in more general situation related to the formula of total probability, e.g., \cite{21}, see further considerations): if we know probabilities \(P_1\) and \(P_2\), then we can find (predict) probability \(P\).

We recall that trigonometric behaviour of ‘quantum probabilities’ was first observed in interference experiments. One of the aims of quantum formalism was to derive probabilistic transformation \((4)\). In all textbooks on quantum mechanics transformation \((2)\) is obtained by using calculations in the complex Hilbert space. Well, such an approach reproduce the true experimental result, namely \((2)\). However, this C-Hilbert space derivation of the ‘quantum rule’ of the addition of probabilities does not explain the origin of the linear space structure. In particular, in such a way we could never get the answer to the following fundamental question:

"Why do we use complex numbers as the basis of linear calculus?"

Moreover, the derivation of \((2)\) on the basis of the C-linear calculus and not on the basis of the conventional probability theory (Kolmogorov’s axiomatics, 1933, \cite{20}) induces a rather common opinion that there is the crucial difference between the conventional (‘classical’) and quantum probabilities. In any case the majority of quantum and probabilistic communities is sure
that interference rule (2) could never be derived in the conventional proba-
bilistic framework.

So, we shall study simultaneously two closely connected problems:

(A) Is it possible to derive quantum rule (2) for addition of probabilities
of alternatives in the conventional probabilistic framework?

(B) Is it possible to derive the \( \mathbb{C} \)-linear space structure starting with
quantum rule (2)?

The problem (A) was investigated in author’s works [22], [23]. It was
demonstrated that (2) could be obtained in the conventional probabilistic
framework. We shall briefly discuss this derivation in section 3. We now
start to investigate the problem (B). At the moment we are not interested in
the ‘classical’ theoretical derivation of (2). Quantum rule (2) is considered
as just an experimental fact. We would like to derive the \( \mathbb{C} \)-linear structure
(\( \mathbb{C} \)-linear space probabilistic calculus based on the superposition principle)
starting with (4). In particular, such a derivation will provide the clear
probabilistic explanation of the choice of the field \( \mathbb{C} \) as the basic number
filed for quantization.

We note that, in the opposite to the standard rule, the quantum rule is
given by the \textbf{nonlinear} transformation:

\[
f_{sq}(P_1, P_2) = P_1 + P_2 + 2 \sqrt{P_1 P_2} \cos \theta .
\]

(for \( \cos \theta \neq 0 \).) We can ask:

"Is it possible to linearize this transformation?"

We first consider the case of maximal interference, namely \( \lambda = \cos \theta = \pm 1 \). Here

\[
f_{sq}(P_1, P_2) = P_1 \pm 2 \sqrt{P_1 P_2} + P_2 = |\sqrt{P_1} + \epsilon \sqrt{P_2}|^2, \epsilon = \pm 1 .
\]

Therefore, instead of the nonlinear transformation, we can work with the
linear transformation for square roots of probabilities:

\[
g(x_1, x_2) = x_1 + \epsilon x_2, \text{ where } x_1 = \sqrt{P_1}, x_2 = \sqrt{P_2}.
\]

It seems that there is no special physical meaning in this linearization.
It is just a question of better mathematical representation of the quantum
probabilistic transformation (in the particular case, \( \lambda = \pm 1 \)).

In the case \( \lambda = \cos \theta \neq 0, \pm 1 \) we cannot provide the linearization over \( \mathbb{R} \).
However, such a linearization can be easily performed over \( \mathbb{C} \):

\[
f_{sq}(P_1, P_2) = |\sqrt{P_1} + \epsilon \sqrt{P_2}|^2, \text{ where } \epsilon = e^{i\theta} . \quad (3)
\]
This explains why we use $\mathbb{C}$-linear formalism and not $\mathbb{R}$-linear. This also implies that it seems that the only source of the appearance of complex numbers in quantum formalism was the linearization of the transformation $f_{\text{sq}}(P_1, P_2)$.

This is the good place to present the following (rather long) citation from Dirac’s book [1]: ‘... from the superposition of two given translation states for a photon a twofold infinity of translational states may be obtained, the general one of which is described by two parameters, which may be taken to be the ratio of the amplitudes of the two wave functions that are added together and their phase relationship. This confirmation shows the need for allowing complex coefficients in equation (1). \footnote{If these coefficients were restricted to be real, then, since only their ratio is of importance for determining the direction of the resultant ket vector $|R>$ when $|A>$ and $|B>$ are given, there would be only a simple infinity of states obtainable from the superposition.}

Our considerations imply that we have to observe (in some experiment) a new transformation of probabilities different from quantum rule (2) (for the addition of probabilistic alternatives) to start to use new linear representations over fields, algebras, rings that are different from the field of complex numbers.

However, we can also use another way for investigations. We can study different nonconventional linear representations (over fields, algebras, rings) and probabilistic transformations induced by such representations. Then we should look for experiments that could produce such new probabilistic transformations.

Before to go to such nonconventional models, we demonstrate why the linearization of transformation (2) is so useful for quantum calculations.

Typically in statistical experiments in classical physics we do not use just the trivial addition of alternatives, (4). We apply so called \textit{formula of total probability} based on Bayes’ formula for conditional probabilities, see, e.g., [21]. We present this formula in the situation that is natural for further quantum considerations.

Let $a = a_1, a_2$ and $b = b_1, b_2$ be two dichotomic ("classical") physical variables. Then we have:

$$P(a = a_j) = P(b = b_1)P(a = a_j | b = b_1) + P(b = b_2)P(a = a_j | b = b_2), \quad (4)$$

where conditional probabilities are given by Bayes’ formula ($i, j = 1, 2$):

\footnote{The equation for the superposition of two states: $c_1|A> + c_2|B> = |R>$.}
The variable \( b \) is divided into two domains, \( D \) discretized position of a particle on the screen that is used for the registration: the screen the second one is closed; distribution of slits with respect to the source of particles. In the symmetric case formulas of total probability, we have to use another formula\(^5\):

\[
P(a = a_j | b = b_1) = \frac{P((a = a_j) \land (b = b_1))}{P(b = b_1)}.
\]

In experiments with elementary particles, instead of the conventional formula of total probability, we have to use another formula\(^6\):

\[
p_j^a = p_{i1} p_{ij}^a + p_{22} p_{ij}^b + 2 \sqrt{p_{i1} p_{ij}^a p_{22} p_{ij}^b} \cos \theta_j.
\]

Here \( p_j^a \) and \( p_j^b \) are probabilities to observe \( a = a_j \) and \( b = b_j \), respectively, for some state \( \varphi; p_{ij}^{a/b} \) are probabilities to observe \( a = a_j \) for the state determined by the condition \( b = b_i \). It must be noticed that formula (5) is just the experimental rule that had been found on the basis of the two slit experiment.

We now would like to linearize transformation (3). We do this as usual:

\[
\begin{align*}
p_j^a &= \left| \sqrt{p^b_j} \sqrt{p_{ij}^{a/b}} + e^{i\theta_j} \sqrt{p^b_j} \sqrt{p_{ij}^{a/b}} \right|^2. \\
\end{align*}
\]

We can observe that this is a linear transformation with respect to square roots of probabilities:

\[
y_j = \sum_i x_i d_{ij},
\]

where \( x_j = \sqrt{p^b_j}, y_j = \sqrt{p^b_j} \) and the matrice \( D = (d_{ij} = e^{i\theta_j} \sqrt{p_{ij}^{a/b}}) \).

Of course, this linear transformation for square roots of probabilities can be easily derived by using linear space calculation (see any textbook on quantum mechanics); by considering the following expansions of state vectors:

\[
\begin{align*}
\varphi &= e^{ik_1} \sqrt{p^b_{i1}} |b_1 > + e^{ik_2} \sqrt{p^b_{i2}} |b_2 >; \\
\varphi &= e^{im_1} \sqrt{p^a_{i1}} |a_1 > + e^{im_2} \sqrt{p^a_{i2}} |a_2 >; \\
|b_1 > &= e^{ik_{11}} \sqrt{p_{i11}^{a/b}} |a_1 > + e^{ik_{12}} \sqrt{p_{i12}^{a/b}} |a_2 > \\
|b_2 > &= e^{ik_{21}} \sqrt{p_{i21}^{a/b}} |a_1 > + e^{ik_{22}} \sqrt{p_{i22}^{a/b}} |a_2 >
\end{align*}
\]

\(^5\)First this was observed as just the experimental fact in interference experiments.

\(^6\)We consider the discrete version of the two slit experiment. The variable \( a \) describes discretized position of a particle on the screen that is used for the registration: the screen is divided into two domains, \( D_1 \) and \( D_2 \) and \( a = a_j \) if a particle is found in the domain \( D_j \). The variable \( b \) describes the following selections (filters): \( b = b_1 \) : the first slit is open and the second one is closed; \( b = b_2 \) : the inverse case. So, for example, the probability \( p_{i1j}^{a/b} \) is the probability to find \( a = a_j \) if the first slit open and the second closed. Probabilities \( p_i^b, i = 1, 2, \) gives the probability for a particle to choose the \( i \)th slit (in the case in that both slits are open). In fact, we have to be more careful: by the standard interpretation of quantum mechanics we could not determine the concrete slit (when both slits are open) that is passed by an individual particle. We can say that probabilities \( p_i^b \) describe the distribution of slits with respect to the source of particles. In the symmetric case \( p^b_1 = p^b_2 = 1/2. \)
(here phases $\theta_j$ are easily computed via phases $\xi_{ij}$ and $\xi_j$).

It seems that quantum formalism, namely $C$-linear space representation of probabilistic transformation (5), was created in such a way (probably this pathway was not recognized consciously). We underline again that (5) was first found in interference experiments and only then the $C$-linear representation of interference transformation (‘quantization’) was created to justify (5).

3 Classical probabilistic derivation of the interference rule; trigonometric and hyperbolic interference

We start with the following contextual explanation of the violation of the classical rule of the addition of probabilistic alternatives in quantum experiments. In fact, the classical rule for addition of probabilities (1) (as well as the formula of total probability (4)) can be only derived if all probabilities are defined for one fixed complex of conditions (context), $C$ (one fixed Kolmogorov probabilistic space, see, e.g., [21]). So, we must have $P = P(A /C)$ and $P_j = P(A_j /C)$, $j = 1, 2$, where $C$ is one fixed context. However, in quantum experiments corresponding to the superposition of states we use a few distinct contexts. Here $P = P(A /C)$ and $P_j = P(A_j /C_j)$, $j = 1, 2$, where $C$ and $C_j$ are distinct complexes of physical conditions corresponding to three different experimental frameworks. Thus there is nothing surprising that in such a framework we get the violation of (1). This explanation of the violation of (1) could be found already in the book of Heisenberg [19], see also Accardy [24], Ballentine [25], De Muynck [26]. Unfortunately, these works do not provide the classical probabilistic derivation of ‘quantum rule’ (2). Such a derivation was presented in author’s papers [22], [23]. Here we consider briefly the main points of this derivation.

We always can write:

7For example, in the two slit experiment we have: $C$ - both slits are open, $C_j$ only jth slit is open.

8For example, Heisenberg explained in the details (and Accardy, Ballentine, De Muynck and many other ‘contextual people’ even in more details) experimental dependence of probabilities obtained via the superposition principle. However, we could not find even a trace of the classical probabilistic derivation. Moreover, it seems that Heisenberg was sure that it would be impossible to derive (2) without wave particle dualism.

8
\[ P = P_1 + P_2 + \delta, \text{ where } \delta = P - (P_1 + P_2). \]

We make the following normalization of the perturbation \( \delta \):

\[ \delta = 2\sqrt{P_1 P_2} \lambda, \]

where

\[ \lambda = \delta / 2\sqrt{P_1 P_2} \]

describes normalized statistical deviations induced by the transition from the original context \( C \) to contexts \( C_j, j = 1, 2, \ldots, \text{ see } [22], [23] \text{ for the details}. \]

Thus, in general,

\[ P = P_1 + P_2 + 2\sqrt{P_1 P_2} \lambda \quad (6) \]

We make the trivial (but fundamental) remark. There are only two possibilities:

1) \(|\lambda| \leq 1\) and 2) \(|\lambda| \geq 1\)

(it is convenient to include \(|\lambda| = 1\) both in 1) and 2) to get the possibility of the continuous transition from 1) to 2) and vice versa.

In the first case we can always represent

\[ \lambda = \cos \theta \quad (7) \]

where \( \theta \) is a kind of a phase parameter. So we get the trigonometric interference (2). To derive the standard quantum formalism, we have to linearize probabilistic transformation (2) in the complex Hilbert space, see the previous section.

We underline one important consequence of our classical probabilistic derivation of the ‘quantum probabilistic rule’. This rule should be obtained automatically in all experiments that produce relatively small, \(|\lambda| \leq 1\), statistical deviations due to the transition from one context to another. In particular, if perturbations are negligibly small from the statistical viewpoint, namely \( \lambda = 0 \), we get ‘classical statistical physics’ (one fixed context statistical physics).

After the above theoretical (classical probabilistic) derivation of quantum transformation (2), we could repeat considerations of section 2 and \( C \)-linearize (2), the special case of the general transformation (3).

Trigonometric parameterization (7) of the normalized statistical deviation is fruitful for a large class of experiments. It seems that it is a consequence of the use of the Euclidean metric for the description of such experiments.
In experiments with phase-shifts, e.g., in neuron interferometry [27], the parameter $\theta$ is the real Euclidean angle.

In principle, it may be that some experimental framework induces another natural parameterization

$$\lambda = u(s)$$

where $|u(s)| \leq 1$ and $s$ (generalized phase) belong to some domain $O$. In general it need not be a domain in the Euclidean space. However, we can always use parameterization (7) with

$$\theta(s) = \arccos u(s)$$

and $\mathbf{C}$-linearize the probabilistic transformation:

$$P = P_1 + P_2 + 2\sqrt{P_1 P_2} u(s) = |\sqrt{P_1} + e^{i\theta(s)} \sqrt{P_2}|^2$$

**Remark.** The previous consideration of the possibility of the trigonometric parameterization of general transformation (7) implies that if the statistical deviation $\lambda$ is relatively small, $|\lambda| \leq 1$, we can always use quantization in a $\mathbf{C}$-linear space. However, it may occur in some experiments that parameterization $\theta(s)$, see (8), is not naturally related to this experiment. In that case it would be practically impossible to found the $\theta$-picture that gives usual trigonometric oscillations (for example, $u(s)$ is a continuous parameterization on some space of parameters $s \in O$ and $\theta(s)$ is a discontinues function on $O$). In such a case it would be more natural to work in the $s$-picture. It would be fruitful to construct new quantization (a linear calculus over noncomplex space) to describe this phenomenon.

In case 2) we can always represent

$$\lambda = \pm \cosh \theta$$

Here we get hyperbolic probabilistic rule:

$$P = P_1 + P_2 \pm 2\sqrt{P_1 P_2} \cosh \theta$$

We observe that this rule can be linearized over the (commutative) algebra of so called hyperbolic numbers, see appendix 1:

$$P = |\sqrt{P_1} \pm e^{i\theta} \sqrt{P_2}|^2, \text{ where } j^2 = 1.$$
This $G$-linearization is based on the hyperbolic analogue of the complex formula:
\[ a^2 + b^2 \pm 2ab \cos \theta = |a \pm be^{i\theta}|^2, \]
that we have used to linearize (2). The corresponding hyperbolic formula is
\[ a^2 + b^2 \pm 2ab \cosh \theta = |a \pm be^{j\theta}|^2. \]

In case 2) the conventional formula of total probability is also perturbed hyperbolically:
\[ p_j^a = p_1 p_{1,j}^{a/b} + p_2 p_{2,j}^{a/b} + 2\epsilon_j \sqrt{p_1 p_2 p_{1,j}^{a/b} p_{2,j}^{a/b}} \cosh \theta_j, \epsilon_j = \pm 1. \]

In the same way as in the trigonometric case (by using hyperbolic numbers instead of complex) this transformation can be realized as a $G$-linear transformation with respect to square roots of probabilities.

**Remark.** (Geometric images of trigonometric and hyperbolic interferences)

(T): Let $P(\theta) = P_1 + P_2 \pm 2\sqrt{P_1 P_2} \cos \theta$. Suppose that $P(\theta)$ gives the probability of a particle to be registered on the circle of radius $r = \theta$. This is a kind of idealized two slit experiment (in the real experiment probabilities $P_j$ also depend on $r = \theta$). We get the following geometric image of such an interference. There are circles of maximal, $r_k = 2\pi k$, and minimal $r_k = \pi k$, brightness. Brightness decreases and increases rather slowly (trigonometrically) between these maximums and minimums.

(Ha): Let $P_+(\theta) = P_1 + P_2 + 2\sqrt{P_1 P_2} \cosh \theta$ with the same as in (T) assumption: $r = \theta$. Here brightness is minimal in the center and increases very quickly (exponentially) for $\theta$ varying from $\theta = 0$ to $\theta = \theta_{\text{max}}$, where the value $\theta_{\text{max}}$ of the hyperbolic phase parameter is determined by the condition:
\[ P_1 + P_2 + 2\sqrt{P_1 P_2} \cosh \theta_{\text{max}} = 1. \]
So, $\theta_{\text{max}} = \ln[q_+ + \sqrt{q_+^2 - 1}]$, where $q_+ = \frac{1-P_1-P_2}{2\sqrt{P_1 P_2}}$.

(Hb): Let $P_-(\theta) = P_1 + P_2 - 2\sqrt{P_1 P_2} \cosh \theta$. Here brightness is maximal in the center and decreases very quickly (exponentially) for $\theta$ varying from $\theta = 0$ to $\theta = \theta_{\text{min}}$, where the value $\theta_{\text{min}}$ of the hyperbolic phase parameter is determined by the condition:
\[ P_1 + P_2 - 2\sqrt{P_1 P_2} \cosh \theta_{\text{min}} = 0. \]
So, $\theta_{\text{min}} = \ln[q_- + \sqrt{q_-^2 - 1}]$, where $q_- = \frac{P_1+P_2}{2\sqrt{P_1 P_2}}$.

(Ha)+(Hb): By considering $P(\theta) = P_+(\theta)$ on some intervals and $P(\theta) = P_-(\theta)$ on others we get an interference picture of circles of different brightness that is similar to the standard interference. The main difference is that
in the hyperbolic case brightness increases and decreases very rapidly (exponentially).

4 $p$-adic probabilistic transformations

Elementary facts about the fields $\mathbb{Q}_p$ of $p$-adic numbers (where $p$ is a prime number) can be found in appendix 2, see also [21]. Let $x \in \mathbb{Q}_p$ and $|x|_p \leq 1$. Then $x$ can be uniquely represented in the form:

$$x = p^l \epsilon, \ |\epsilon|_p = 1, \ l = 0, 1, 2, ...$$

The $\epsilon$ is called a $p$-adic unit. Here $|x|_p = p^{-l}$. This representation is an analog of the representation of $x \in \mathbb{R}$ as $x = |x| \epsilon, \ \epsilon = \pm 1$, or $x \in \mathbb{C}$ as $x = |x|\epsilon, \ |\epsilon| = 1$ (so $\epsilon = e^{i \theta}$). The calculus in the $\mathbb{Q}_p$-linear space produces the following rule for the addition of probabilistic alternatives:

$$P = |\alpha_1 + \epsilon \alpha_2|_p^2, \ P_1 = |\alpha_1|_p^2, \ P_2 = |\alpha_2|_p^2.$$

A). Let $P_1 > P_2$. By using the strong triangle inequality (see appendix 2) we get:

$$P = P_1 + P_2 + 2\sqrt{P_1 P_2} \lambda, \ \text{where} \ \lambda = -\frac{1}{2} \sqrt{\frac{P_2}{P_1}}.$$

Thus

$$-\frac{1}{2} < \lambda \leq 0. \quad (11)$$

B). Let $P_1 < P_2$. Here $\lambda = -\frac{1}{2} \sqrt{\frac{P_1}{P_2}}$, so $\lambda$ satisfies to (11).

C). Let $P_1 = P_2$. So $\alpha_i = p^l \epsilon_i, \ i = 1, 2, |\epsilon_i|_p = 1$. So $P_1 = P_2 = p^{-2l}$. Here $P = cP_1$, where $c = |\epsilon_1 + \epsilon \epsilon_2|_p^2$. We remark that (as a consequence of the strong triangle inequality)

$$0 \leq c \leq 1.$$

As

$$\lambda = \frac{c}{2} - 1,$$

we get

$$-1 \leq \lambda \leq -\frac{1}{2}.$$
Therefore:

\textit{p-adic quantum mechanics induces the standard quantum probabilistic rule, (2), for the restricted range of angles, }\frac{\pi}{2} \leq \theta \leq \pi.\textit{ }

So, by using p-adic linear representations we do not have something new from the probabilistic viewpoint. However, by using p-adic numbers we can get the description of rather special behaviour of phase. It may that such a phase behaviour might be observed in some experiments.

\textbf{Example.} (p-adic two slit experiment). In such an experiment nonsymmetrical location of slits with respect to the "source" would imply just the reductions of probability \( P \rightarrow P_1 \) (for \( P_1 > P_2 \)) and \( P \rightarrow P_2 \) (for \( P_2 > P_1 \)).

In the symmetric case, \( P_1 = P_2 = A \), we would observe the following discrete fluctuations depending on the parameter \( \epsilon \) belonging to the unit sphere \( S_1(0) = \{ x \in \mathbb{Q}_p : |x|_p = 1 \} \). In practical consideration the sphere \( S_1(0) \) is reduced to the set \( N_p = \{ 1, 2, \ldots, p - 1, p + 1, \ldots \} \) of natural numbers that are not divisible by \( p \).

We consider the simplest case \( \epsilon_1 = \epsilon_2 = 1 \). So \( c = |1 + \epsilon|^2 \). Let \( A = 1/p^{2l} \). We see that \( P = cA = A \) for \( \epsilon = 1, \ldots, p - 2 \); then \( P = A/p^l \) for \( \epsilon = p - 1 \); then again \( P = A \) for \( p = p + 1, \ldots, 2p - 2 \) and \( P = A/p^l \) for \( \epsilon = 2p - 1 \) (if \( p \neq 2 \)).

Such fluctuations will be observed until \( \epsilon \) approaches \( p^2 - 1 \). For \( \epsilon = p^2 - 1 \), we get \( P = A/p^l \). Then again \( P = A \) for \( \epsilon = p^2 + 1, \ldots, p^2 + p - 2 \), and \( P = A/p^l \) for \( p = p^2 + p - 1 \) and so on.

Thus there exist some exceptional values of the parameter \( \epsilon \) that reduces the probability. Roughly speaking we have the following p-adic interference picture: there are portions of concentric circles (e.g., of radii \( r_k = 1 + \epsilon, \epsilon = k = 1, 2, \ldots \)) having the same brightness, between these portions periodically appear circles having essentially lower brightness; brightness is reduced due to divisibility of \( r_k \) by powers of \( p \). The main difference from the ordinary trigonometric interference is that brightness varies discontinuously depending on radius. However, we notice that this is discontinuity with respect to the Euclidean metric. In p-adic metric brightness behave continuously.

We would like to mention one distinguishing feature of the p-adic quantum mechanics, \( \mathbb{Q}_p \)-linear probabilistic calculus. There exist representations of canonical commutation relations by bounded operators in the \( \mathbb{Q}_p \)-adic Hilbert space, see [29], [30]. Therefore it might more convenient to provide \( \mathbb{Q}_p \)-linear calculations, instead of standard \( \mathbb{C} \)-linear calculations. In this way we could escape to work with unbounded operators. Moreover, if the hypothesis on the p-adic structure of space time on Planck distances be confirmed by further
investigations (in the framework of string theory, [7]-[10] or general gravity, see, e.g., [31]), then the p-adic space representation of the corresponding probabilistic calculus would be very natural.

5 Appendix 1: hyperbolic algebra

A hyperbolic algebra \( G \) is a two dimensional real algebra with the basis \( e_0 = 1 \) and \( e_1 = j \), where \( j^2 = 1 \). Elements of \( G \) have the form \( z = x + yj \), \( x, y \in \mathbb{R} \). We have \( z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2) \) and \( z_1 z_2 = (x_1 x_2 + y_1 y_2) + j(x_1 y_2 + x_2 y_1) \). This algebra is commutative. We introduce the involution in \( G \) by setting \( \bar{z} = x - yj \). We set \( |z|^2 = z\bar{z} = x^2 - y^2 \). We remark that \( |z| = \sqrt{x^2 - y^2} \) is not well defined for an arbitrary \( z \in G \). We set \( G^+ = \{ z \in G : |z|^2 \geq 0 \} \). We remark that \( G^+ \) is the multiplicative semigroup: \( z_1, z_2 \in G^+ \rightarrow z = z_1 z_2 \in G^+ \). It is a consequence of the equality

\[
|z_1 z_2|^2 = |z_1|^2 |z_2|^2.
\]

Thus, for \( z_1, z_2 \in G^+ \), we have \( |z_1 z_2| = |z_1| |z_2| \). We introduce

\[
e^{j\theta} = \cosh \theta + j \sinh \theta, \quad \theta \in \mathbb{R}.
\]

We remark that

\[
e^{j\theta_1} e^{j\theta_2} = e^{j(\theta_1 + \theta_2)}, \quad e^{j\theta} = e^{-j\theta}, \quad |e^{j\theta}|^2 = \cosh^2 \theta - \sinh^2 \theta = 1.
\]

Hence, \( z = \pm e^{j\theta} \) always belongs to \( G^+ \). We also have

\[
cosh \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}, \quad \sinh \theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}.
\]

We set \( G^*_+ = \{ z \in G^+_+ : |z|^2 > 0 \} \). Let \( z \in G^*_+ \). We have

\[
z = |z| (\frac{x}{|z|} + j \frac{y}{|z|}) = \text{sign } x \ |z| (\frac{x \text{sign } x}{|z|} + j \frac{y \text{sign } y}{|z|}).
\]

As \( \frac{x^2}{|z|^2} - \frac{y^2}{|z|^2} = 1 \), we can represent \( x \text{ sign } x = \cosh \theta \) and \( y \text{ sign } x = \sinh \theta \), where the phase \( \theta \) is unequally defined. We can represent each \( z \in G^*_+ \) as

\[
z = \text{sign } x \ |z| e^{j\theta}.
\]

By using this representation we can easily prove that \( G^*_+ \) is the multiplicative group. Here \( \frac{1}{z} = \frac{\text{sign } x}{|z|} e^{-j\theta} \). The unit circle in \( G \) is defined as \( S_1 = \{ z \in G : |z|^2 = 1 \} = \{ z = \pm e^{j\theta}, \theta \in (-\infty, +\infty) \} \). It is a multiplicative subgroup of \( G^*_+ \).

\footnote{So, it is the two dimensional Clifford algebra.}
6 Appendix 2: $p$-adic numbers

The field of real numbers $\mathbb{R}$ is constructed as the completion of the field of rational numbers $\mathbb{Q}$ with respect to the metric $\rho(x, y) = |x - y|$, where $|\cdot|$ is the usual valuation of distance given by the absolute value. The fields of $p$-adic numbers $\mathbb{Q}_p$ are constructed in a corresponding way, but using other valuations. For a prime number $p$, the $p$-adic valuation $|\cdot|_p$ is defined in the following way. First we define it for natural numbers. Every natural number $n$ can be represented as the product of prime numbers, $n = 2^{r_2}3^{r_3}\cdots p^{r_p}\cdots$, and we define $|n|_p = p^{-r_p}$, writing $|0|_p = 0$ and $|-n|_p = |n|_p$. We then extend the definition of the $p$-adic valuation $|\cdot|_p$ to all rational numbers by setting $|n/m|_p = |n|/|m|_p$ for $m \neq 0$. The completion of $\mathbb{Q}$ with respect to the metric $\rho_p(x, y) = |x - y|_p$ is the locally compact field of $p$-adic numbers $\mathbb{Q}_p$.

The number fields $\mathbb{R}$ and $\mathbb{Q}_p$ are unique in a sense, since by Ostrovsky’s theorem (see [28], [10], [11]) $|\cdot|$ and $|\cdot|_p$ are the only possible valuations on $\mathbb{Q}$, but have quite distinctive properties. The field of real numbers $\mathbb{R}$ with its usual valuation satisfies $|n| = n \to \infty$ for valuations of natural numbers $n$ and is said to be Archimedean. By a well known theorem of number theory, [28], [10], [11], the only complete Archimedean fields are those of the real and the complex numbers. In contrast, the fields of $p$-adic numbers, which satisfy $|n|_p \leq 1$ for all $n \in \mathbb{N}$, are examples of non-Archimedean fields.

Unlike the absolute value distance $|\cdot|$, the $p$-adic valuation satisfies the strong triangle inequality

$$|x + y|_p \leq \max(|x|_p, |y|_p), \quad x, y \in \mathbb{Q}_p.$$  \hfill (12)

Consequently the $p$-adic metric satisfies the strong triangle inequality

$$\rho_p(x, y) \leq \max(\rho_p(x, z), \rho_p(z, y)), \quad x, y, z \in \mathbb{Q}_p,$$  \hfill (13)

which means that the metric $\rho_p$ is an ultrametric.

Write $U_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\}$ and $U_r^{-}(a) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}$ where $r = p^n$ and $n = 0, \pm 1, \pm 2, \ldots$. These are the “closed” and “open” balls in $\mathbb{Q}_p$, while the sets $S_r(a) = \{x \in K : |x - a|_p = r\}$ are the spheres in $\mathbb{Q}_p$ of such radii $r$. These sets (balls and spheres) have a somewhat strange topological structure from the viewpoint of our usual Euclidean intuition: they are both open and closed at the same time, and as such are called clopen sets. Another interesting property of $p$-adic balls is that two balls...
have nonempty intersection if and only if one of them is contained in the other. Also, we note that any point of a \( p \)-adic ball can be chosen as its center, so such a ball is thus not uniquely characterized by its center and radius. Finally, any \( p \)-adic ball \( U_r(0) \) is an additive subgroup of \( \mathbb{Q}_p \), while the ball \( U_1(0) \) is also a ring, which is called the ring of \( p \)-adic integers and is denoted by \( \mathbb{Z}_p \).

Any \( x \in \mathbb{Q}_p \) has a unique canonical expansion (which converges in the \( | \cdot |_p \)-norm) of the form

\[
x = a_{-n}/p^n + \cdots a_0 + \cdots + a_k p^k + \cdots
\]

where the \( a_j \in \{0, 1, \ldots, p-1\} \) are the “digits” of the \( p \)-adic expansion.

This paper was prepared during my visit to University of Havanna. I would like to thank Professor Lillian Alvarez for her hospitality.

I would like to thank S. Albeverio, L. Accardy, L. Ballentine, E. Beltrametti, G. Cassinelli, W. De Baere, W. De Myunck, D. Greenberger, C. Fuchs, L. Hardy, T. Hida, P. Lahti, D. Mermin, A. Peres, I. Pitowsky, A. Shiryaev, J. Summhammer, L. Vaidman and I. Volovich for fruitful discussions on probabilistic foundations of quantum mechanics.

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