THE LOCAL GROSS-PRASAD CONJECTURE OVER $\mathbb{R}$: EPSILON DICHOTOMY

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ABSTRACT. Following the work of Jean-Loup Waldspurger, we prove the epsilon dichotomy part of the local Gross-Prasad conjecture over $\mathbb{R}$ for tempered local $L$-parameters.

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1. Introduction

This paper continues our study of the local Gross-Prasad conjecture ([GP92],[GP94]). Let $(W, V)$ be a pair of quadratic spaces defined over a local field $F$ of characteristic zero, where $W$ is a subspace of $V$ and the orthogonal complement $W^\perp$ of $W$ in $V$ is split of odd dimension. Let $H = \text{SO}(W) \ltimes N$ be the subgroup of $G = \text{SO}(W) \times \text{SO}(V)$ where $N$ is the unipotent radical of $\text{SO}(V)$ defined by a full isotropic flag attached to $W^\perp$. Fix a generic character $\xi$ of $N(F)$ which uniquely extends to $H(F)$. For any irreducible admissible representation $\pi$ of $G(F)$, set

\begin{equation}
(1.0.1) \quad m(\pi) = \dim \text{Hom}_H(\pi, \xi).
\end{equation}

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From [AGRS10][GGP12][Wal12d] over $p$-adic fields, and [SZ12][JSZ10] over archimedean fields, it is known that

$$m(\pi) \leq 1.$$  

The local Gross-Prasad conjecture speculates a refinement for the behavior of $m(\pi)$. The pure inner forms of $\text{SO}(W)$ are parametrized by $H^1(F, \text{SO}(W)) \simeq H^1(F, H)$, which classifies quadratic spaces of the same dimension and discriminant as $W$ over $F$. For any $\alpha \in H^1(F, H)$, there is a pair of quadratic spaces $(W_\alpha, V_\alpha = W_\alpha \oplus W')$ that enjoys parallel properties as $(W, V)$. Moreover, the Langlands dual group of $G_\alpha = \text{SO}(W_\alpha) \times \text{SO}(V_\alpha)$, which is a pure inner form of $G$, is isomorphic to the Langlands dual group $^L G$ of $G$. For any local $L$-parameter $\varphi : \mathcal{L}_F \to ^L G$ where $\mathcal{L}_F$ is the local Langlands (Weil-Deligne) group of $F$, let $\Pi^{G_\alpha}(\varphi)$ be the corresponding local $L$-packet of $G_\alpha$ ([Lan70]).

Following the work of D. Vogan ([Vog93]), one can introduce the so called $\text{Vogan } L$-packet attached to the parameter $\varphi$,

$$\Pi^{\text{Vogan}}(\varphi) = \bigsqcup_{\alpha \in H^1(F, G)} \Pi^{G_\alpha}(\varphi).$$

It was conjectured by Vogan ([Vog93]), which is known over archimedean fields ([Vog93, Thm. 6.3]), that after fixing a Whittaker datum for $\{G_\alpha\}_{\alpha \in H^1(F, G)}$, there is a non-degenerate pairing

$$\Pi^{\text{Vogan}}(\varphi) \times S_\varphi \to \{\pm 1\}.$$  

Here $S_\varphi$ is the component group of $S_\varphi$, where $S_\varphi$ is the centralizer of the image $\text{Im}(\varphi)$ in the dual group $\widehat{G}$. In particular, based on the conjecture, every $\pi \in \Pi^{\text{Vogan}}(\varphi)$ corresponds uniquely to a character $\chi_\pi : S_\varphi \to \{\pm 1\}$. On the other hand, it was suggested by Gross and Prasad that one may consider the following relevant Vogan $L$-packet attached to $\varphi$ and pairs of quadratic spaces $(W_\alpha, V_\alpha)_{\alpha \in H^1(F, H)}$,

$$\Pi^{\text{Vogan}}_{\text{rel}}(\varphi) = \bigsqcup_{\alpha \in H^1(F, H)} \Pi^{G_\alpha}(\varphi).$$

The following conjecture was formulated by B. Gross and D. Prasad.

**Conjecture 1.0.1 ([GP92][GP94]).** The following two statements hold.

1. The following identity holds for $\varphi$ a generic local $L$-parameter,

$$\sum_{\pi \in \Pi^{\text{Vogan}}_{\text{rel}}(\varphi)} m(\pi) = 1.$$  

2. Fix the Whittaker datum for $\{G_\alpha\}_{\alpha \in H^1(F, G)}$ as in [GP94, (6.3)]. Based on (1), the unique representation $\pi_\varphi \in \Pi^{\text{Vogan}}_{\text{rel}}(\varphi)$ with the property that $m(\pi_\varphi) = 1$ enjoys the following characterization:

$$\chi_{\pi_\varphi} = \chi_\varphi$$

where $\chi_\varphi$ is defined in (2.3.5).

When $F$ is $p$-adic, Conjecture 1.0.1 for tempered local $L$-parameters was proved by Waldspurger ([Wal10b][Wal12a][Wal12b][Wal12c][Wal12d]). C. Moeglin and Waldspurger proved the conjecture for generic local $L$-parameters based on the tempered case ([MgW12]).

When $F$ is archimedean, Part (1) of Conjecture 1.0.1 for tempered local $L$-parameters was proved by the second author in his thesis ([Luo21]), which follows the strategy of Waldspurger and R. Beuzart-Plessis ([BP20]). In [Che21], the first author reduced the proof of Conjecture
1.0.1 for generic local $L$-parameters to tempered local $L$-parameters over $\mathbb{R}$, following the work of Moeglin and Waldspurger, and H. Xue (Xueb). In particular over $\mathbb{C}$, Conjecture 1.0.1 was fully established in [Che21]. Note that Part (1) of Conjecture 1.0.1 over $\mathbb{C}$ was also established in [Mö17] when the codimension of $W$ in $V$ is one.

Hence to finish the proof of Conjecture 1.0.1, it remains to establish the following theorem, which is the main result of this paper:

**Theorem 1.0.2.** Over $\mathbb{R}$, Part (2) of Conjecture 1.0.1 holds for tempered local $L$-parameters.

The proof follows the strategy of [Wal12b]. First let us briefly recall the proof of [Wal12b] over $p$-adic fields. By Part (1) of Conjecture 1.0.1, to prove Theorem 1.0.2, it suffices to show that for any $s \in S_\varphi$,

\begin{equation}
\sum_{\pi \in \Pi_{\text{tem}}(\varphi)} \chi_\pi(s) \cdot m(\pi) = \chi_\varphi(s).
\end{equation}

In [Wal10b][Wal12c], Waldspurger established a geometric multiplicity formula for any tempered representation $\pi$ of $G(F)$, which expresses the multiplicity $m(\pi)$ via the germ expansion of the distribution character of $\pi$ ([DS99]). In particular (1.0.2) can be replaced as

\begin{equation}
\sum_{\pi \in \Pi_{\text{tem}}(\varphi)} \chi_\pi(s) \cdot m_{\text{geom}}(\pi) = \chi_\varphi(s).
\end{equation}

By ordinary endoscopy ([Art13], see also [Wal12b, §1.6]), for any $s \in S_\varphi$ there is an endoscopic group $G_{1,s} \times G_{2,s}$ of $G$ with $G_{i,s}$ coming from a pair of quadratic spaces $(W_i, V_i)$ in Gross-Prasad conjecture, and $\varphi|_{L(G_1 \times G_2)} = \varphi_1 \times \varphi_2$. Based on the endoscopic character relation,

\begin{equation}
\sum_{\pi \in \Pi_{\text{tem}}(\varphi)} \chi_\pi(s) \cdot m_{\text{geom}}(\pi) = m_{\text{geom}}^{S_1}(\varphi_1) \cdot m_{\text{geom}}^{S_2}(\varphi_2)
\end{equation}

which was established in [Wal12b, Prop. 3.3]. Here $m_{\text{geom}}^{S_i}$ is a stable variant of $m_{\text{geom}}$ ([Wal12b, §3.2]). Then from the theory of twisted endoscopy and the twisted character relation ([DS99][Art13], see also [Wal12b, §1.6,§1.8]), $m_{\text{geom}}^{S_i}$ is connected with $m_{\text{geom}}^{S_i}$ ([Wal12b, Prop. 3.4]) where the latter is the geometric multiplicity attached to a pair of twisted general linear groups of Gross-Prasad type, which turns out to be equal to $\chi_\varphi(s)$ by [Wal12a].

It turns out that over $\mathbb{R}$, by Theorem 5.1.1, besides the situation when the underlying quadratic space is of dimension $\leq 3$, the tempered local $L$-parameters of special orthogonal groups are either of parabolic induction type, or of endoscopic induction type, which is simpler than $p$-adic fields. Hence we do not need to use twisted endoscopy. In particular, we do not need to establish the archimedean analogue of [Wal12a]. However, there are the following two issues: First, there are only two pairs $(W_\alpha, V_\alpha)_{\alpha \in H_1(F,H)}$ over $p$-adic fields, which is not the case over $\mathbb{R}$. Second, the proof of [Wal10b, Lem. 13.4 (ii)], which is a necessary ingredient in the work of Waldspurger, relies on [Wal97, Conj. 1.2]. The parallel result seems unknown over $\mathbb{R}$. We solve the first problem through working with families of pairs $(W_\alpha, V_\alpha)_{\alpha \in H_1(F,H)}$ indexed by the Kottwitz sign of $G_\alpha$ ([Kot83]), and the second by a formula of W. Rossmann ([Ros78]).

Finally it is worth pointing out that as formulated by W. Gan, Gross and Prasad in [GGP12], there are similar conjectures for unitary groups, which was treated by Beuzart-Plessis ([BP16][BP14]) over $p$-adic fields for tempered local $L$-parameters following the work
of Waldspurger, and Gan and A. Ichino over $p$-adic fields for generic local $L$-parameters ([GI16]) following the work of Moeglin-Waldspurger. Beuzart-Plessis solved Part (1) of Conjecture 1.0.1 for tempered local $L$-parameters of unitary groups over $\mathbb{R}$ in [BP20], generalizing the work of Waldspurger. Xue solved Part (2) of Conjecture 1.0.1 for tempered local $L$-parameters of unitary groups over $\mathbb{R}$ in [Xuea], and reduced Conjecture 1.0.1 from generic local $L$-parameters to tempered local $L$-parameters of unitary groups over $\mathbb{R}$ in [Xueb]. The special orthogonal and unitary groups cases discussed above are usually called the Bessel case. There are also parallel conjectures called the Fourier-Jacobi case formulated in [GGP12], which involve Weil representations and treat (skew-hermitian) unitary groups and symplectic-metaplectic groups. Over $p$-adic fields, they were resolved by Gan and Ichino for (skew-hermitian) unitary groups ([GI16]), and H. Atobe for symplectic-metaplectic groups ([Ato18]), using techniques from theta correspondence reducing Fourier-Jacobi case to Bessel case.

Organization. Throughout the paper, we will always work over $\mathbb{R}$.

In section 2, we fix notation and conventions used in the paper, and recall the statement of local Gross-Prasad conjecture formulated in [GP92][GP94].

In section 3, we first review the parameterization of certain regular semi-simple conjugacy classes in special orthogonal groups following [Wal10a, §1.3] and [Wal12b, §1.3 §1.4], then establish Proposition 3.2.4, which is analogous to the description of the fiber of $p_G$ in [Wal12b, §1.4] giving a characterization of the union of these regular semi-simple conjugacy classes over pure inner forms of special orthogonal groups with fixed Kottwitz sign.

In section 4, we first review the geometric multiplicity formula established in [Luo21]. Then we establish Lemma 4.2.2, which is analogous to [Wal10b, Lem. 13.4 (ii)] over $\mathbb{R}$. As a corollary we establish Corollary 4.2.3, which is analogous to [Wal10b, §13.6] expressing the germs of a distribution character by the distribution character itself. Then we recall the stable variant of geometric multiplicity formula introduced in [Wal12b, §3.2], and establish Lemma 4.3.2 describing the union of the support of the geometric multiplicity formula over pure inner forms of Gross-Prasad triples with fixed Kottwitz sign.

In section 5, we first classify the tempered local $L$-parameters of special orthogonal groups over $\mathbb{R}$, then we complete the proof of Theorem 1.0.2 via mathematical induction and [GP94, Prop. 7.4].

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2. Local Gross-Prasad conjecture

In this section, we review the local Gross-Prasad conjecture following [GP92][GP94].

2.1. Gross-Prasad triples. In this subsection, we recall the notion of Gross-Prasad triples following [Luo21, §6].
Let \((W, V)\) be a pair of non-degenerate quadratic spaces over \(\mathbb{R}\). The pair \((W, V)\) is called admissible if there exists an anisotropic line \(D\) and a split non-degenerate quadratic space \(Z\) of dimension \(2r\) over \(\mathbb{R}\) such that \(V \cong W \oplus D \oplus Z\).

Let \(q\) be the quadratic form on \(V\). There exists a basis \(\{z_i\}_{i=1}^{r+1}\) of \(Z\), such that
\[
q(z_i, z_j) = \delta_{i,-j}, \quad \forall \ i, j \in \{\pm 1, \ldots, \pm r\}
\]
Let \(N\) be the unipotent radical of the parabolic subgroup of \(SO(V)\) stabilizing the following totally isotropic flag \(\langle z_r \rangle \subset \langle z_r, z_{r-1} \rangle \subset \langle z_r, \ldots, z_1 \rangle\). Set \(G = SO(W) \times SO(V)\). The group \(SO(W)\) can be identified as an algebraic subgroup of \(G\) via diagonal embedding. In particular \(SO(W)\) acts on \(N\) via adjoint action. Set \(H = SO(W) \ltimes N\). Define a morphism \(\lambda : N \to \mathbb{G}_a\) via
\[
\lambda(n) = \sum_{i=0}^{r-1} q(z_{-i-1}, nz_i), \quad n \in N.
\]
Then \(\lambda\) is \(SO(W)\)-conjugation invariant and hence admits a unique extension to \(H\) trivial on \(SO(W)\), which is still denoted as \(\lambda\). Let \(\lambda_{\mathbb{R}} : H(\mathbb{R}) \to \mathbb{R}\) be the induced morphism on \(\mathbb{R}\)-rational points. Fix an additive character \(\psi\) of \(\mathbb{R}\) and set
\[
\xi(h) = \psi(\lambda_{\mathbb{R}}(h)), \quad h \in H(\mathbb{R}).
\]

**Definition 2.1.1.** The triple \((G, H, \xi)\) is called the Gross-Prasad triple associated to the admissible pair \((W, V)\).

### 2.2. Vogan \(L\)-packets.

In this subsection, we recall the notion of Vogan \(L\)-packets of special orthogonal groups over \(\mathbb{R}\) following [GP92, §3] and [Vog93].

First let us recall the local Langlands correspondence over \(\mathbb{R}\) following the exposition of [Kna94]. Let \(L_{\mathbb{R}}\) be the Weil group of \(\mathbb{R}\). From [Lan89], for any reductive algebraic group \(G\) over \(\mathbb{R}\), there is a bijective correspondence between local \(L\)-parameters \(\varphi\) for \(G\) and local \(L\)-packets \(\Pi^G(\varphi)\) consisting of a finite set of irreducible Casselman-Wallach ([Cas89][Wal92]) representations of \(G\). A local \(L\)-parameter for \(G\), by definition, is a \(\widehat{G}\)-conjugacy class of admissible homomorphisms
\[
\varphi : L_{\mathbb{R}} \to {}^LG
\]
such that the elements in the image \(\text{Im}(\varphi) \subset {}^LG\) are semi-simple. Here \(\widehat{G}\), resp. \({}^LG\) is the dual, resp. Langlands dual group of \(G\). In particular, \(\varphi\) is called tempered if \(\text{Im}(\varphi)\) is bounded.

The pure inner forms of \(G\) share the same dual, resp. Langlands dual group as \(G\). In particular, for any local \(L\)-parameter \(\varphi\) of \(G\), it can also be viewed as a local \(L\)-parameter of any pure inner forms of \(G\). Following [Vog93], instead of working with a single \(L\)-packet \(\Pi^G(\varphi)\), one should work with the attached Vogan \(L\)-packet, which is the disjoint union of the \(L\)-packets for all pure inner forms of \(G\),
\[
\bigcup_{G'} \Pi^{G'}(\varphi).
\]

Here \(G'\) runs over the isomorphism classes of pure inner forms of \(G\).

Now let us return to special orthogonal groups situation. For a non-degenerate quadratic space \((V, q)\) over \(\mathbb{R}\), the isomorphism classes of pure inner forms of \(SO(V)\) are classified by the set \(H^1(\mathbb{R}, SO(V))\), which in particular classifies quadratic spaces over \(\mathbb{R}\) of the same dimension and discriminant as \(V\) ([GP94, §8]). It is known that quadratic spaces \(V\) over \(\mathbb{R}\)
are classified by their signature \((p, q)\) with \(p, q \in \mathbb{Z}_{\geq 0}\), where \(p = \text{PI}(V)\) is the positive index of \(V\), and \(q = \text{NI}(V)\) is the negative index of \(V\).

The discriminant of the quadratic space \(V\) with signature \((p, q)\) is given by
\[
\text{disc}(V) = (-1)^{\left\lfloor \frac{\dim V}{2} \right\rfloor} \cdot (-1)^q \in \{\pm 1\} \simeq \mathbb{R}^\times / \mathbb{R}^\times 2.
\]
Here \(\left\lfloor \frac{\dim V}{2} \right\rfloor\) is the maximal integer smaller than or equal to \(\frac{\dim V}{2}\). Let the attached special orthogonal group be \(\text{SO}(p, q)\). By calculation, the pure inner forms of \(\text{SO}(p, q)\) are given by
\[
\text{SO}(p_\alpha, q_\alpha) \quad \text{with} \quad p_\alpha + q_\alpha = p + q \quad \text{and} \quad p \equiv p_\alpha \mod 2.
\]

Among all the pure inner forms of \(\text{SO}(V)\), there is a particular class called quasi-split (resp. split) pure inner forms, which are pure inner forms admitting Borel subgroups (resp. maximal split torus) defined over \(\mathbb{R}\). The following statements are not hard to justify:

- **(2.2.3)** When \(\dim V\) is odd, or even \(\text{PI}(V) - \text{NI}(V) \equiv 0 \mod 4\), \(\text{SO}(V)\) has a unique quasi-split pure inner form that is split over \(\mathbb{R}\);
- **(2.2.4)** When \(\dim V\) is even and \(\text{PI}(V) - \text{NI}(V) \equiv 2 \mod 4\), \(\text{SO}(V)\) has two quasi-split pure inner forms \(\text{SO}(p + 2, p)\) and \(\text{SO}(p, p + 2)\) with \(p = \frac{\dim V}{2} - 1\). Note that \(\text{SO}(p + 2, p)\) and \(\text{SO}(p, p + 2)\) are isomorphic as inner forms, but not as pure inner forms.

Two admissible pairs \((W, V)\) and \((W', V')\) are called **relevant** if
\[
\text{dim } W = \text{dim } W', \quad \text{disc}(W) = \text{disc}(W'), \quad \text{dim } V = \text{dim } V', \quad \text{disc}(V) = \text{disc}(V').
\]
Fix an admissible pair \((W, V)\) with attached Gross-Prasad triple \((G, H, \xi)\). For any \(\alpha \in H^1(\mathbb{R}, \text{SO}(W))\) there is a unique relevant admissible pair \((W_\alpha, V_\alpha = W_\alpha \oplus W'_{\perp})\) with Gross-Prasad triple \((G_\alpha, H_\alpha, \xi_\alpha)\) attached to it. For any local \(L\)-parameter \(\varphi : \mathcal{L}_L \to L_G\), define the relevant Vogan \(L\)-packet as follows
\[
\Pi_{\text{rel}}^{\text{Vogan}}(\varphi) = \bigsqcup_{\alpha \in H^1(\mathbb{R}, \text{SO}(W))} \Pi^{G_\alpha}(\varphi).
\]

**Remark 2.2.1.** For an admissible pair \((W, V)\), when \(\dim W\) (resp. \(\dim V\)) is even, as is shown in (2.2.4), it can happen that \(\text{SO}(W)\) (resp. \(\text{SO}(V)\)) has two non-isomorphic quasi-split pure inner forms. But since \(\dim V\) (resp. \(\dim W\)) is odd, there is a unique admissible pair \((W_{qs}, V_{qs})\) that is relevant to \((W, V)\), so that \(\text{SO}(W_{qs}) \times \text{SO}(V_{qs})\) is quasi-split.

## 2.3. The conjecture
In this subsection, we first review the construction of the distinguished character defined in [GP92, §10], then recall the conjecture of Gross and Prasad formulated in [GP92][GP94]. We restrict ourselves to \(\mathbb{R}\).

- **From Vog93, Thm. 6.3**, the following statement holds:

\[
\Pi^{\text{Vogan}}(\varphi) \times \mathcal{S}_\varphi \to \{\pm 1\}.
\]

Returning to special orthogonal groups situation, for a non-degenerate quadratic space \(V\) over \(\mathbb{R}\) with a local \(L\)-parameter \(\varphi_V\) of \(\text{SO}(V)\), as is suggested in [GP92, §6,§7], after composing \(\varphi_V\) with the standard embedding of \(\text{SO}(V)\) into the general linear group \(\text{GL}(M_V)\), one can get a local \(L\)-parameter \(\text{std} \circ \varphi_V : \mathcal{L}_L \to \text{GL}(M_V)\) which determines a non-degenerate
bilinear form $B : M_V \times M_V \to \mathbb{C}$ preserved under the action of $L_\mathbb{R}$ with sign $\epsilon = \{\pm 1\}$. When $\dim V$ is odd (resp. even), the bilinear form $B$ is symplectic (resp. symmetric), and hence $\epsilon = -1$ (resp. $= 1$).

By the semi-simplicity of $\varphi_V$, $M_V = \bigoplus_i m_i \cdot M_{i,V}$ where $M_{i,V}$ are inequivalent irreducible representations of $L_\mathbb{R}$ with multiplicity $m_i$.

(2.3.2) Following [GP92, Prop. 6.5, Prop. 7.6] the irreducible representations of $M_{i,V}$ are classified as follows (here $M_{i,V}^\vee$ is the contragredient of $M_{i,V}$):

(O-type) $M_{i,V} \simeq M_{i,V}^\vee$ where the isomorphism is provided by a non-degenerate pairing with sign $\epsilon$;

(Sp-type) $M_{i,V} \simeq M_{i,V}^\vee$ where the isomorphism is provided by a non-degenerate pairing with sign $-\epsilon$.

Moreover, in this case, $m_i$ is even;

(GL-type) $M_{i,V} \not\simeq M_{i,V}^\vee$, and hence $M_{i,V} \simeq M_{j,V}^\vee$ for some $i \neq j$ and $m_i = m_j$.

Let $I_O$, $I_{Sp}$ and $I_{GL}$ be the index set of of the irreducible (resp. direct sum of two irreducible) representations $M_{i,V}$ (resp. $M_{i,V} \oplus M_{i,V}^\vee$) of O-type or Sp-type (resp. GL-type). Following [GP92, Prop. 6.6, Prop. 7.7], the centralizer of the image of $\varphi_V$ is equal to

\begin{equation}
S_{\varphi_V} = \left( \prod_{i \in I_O} O(m_i, \mathbb{C}) \right) \times \prod_{i \in I_{Sp}} \Sp(m_i, \mathbb{C}) \times \prod_{i \in I_{GL}} \GL(m_i, \mathbb{C})
\end{equation}

where

\begin{equation}
S_{\varphi_V} = \begin{cases} 
(\mathbb{Z}/2\mathbb{Z})^r & \text{if dim } M_{i,V} \text{ are even for all } M_{i,V} \text{ of O-type} \\
(\mathbb{Z}/2\mathbb{Z})^{r-1} & \text{otherwise}
\end{cases}
\end{equation}

where $r = |I_O|$.

In [GP92, §10], Gross and Prasad defined a distinguished character. To make it more precise, let $(G, H, \xi)$ be a Gross-Prasad triple attached to an admissible pair $(W, V)$ over $\mathbb{R}$. From (2.3.1), after fixing a Whittaker datum for $G = \SO(W) \times \SO(V)$, there is a non-degenerate pairing $\Pi_{\text{Vogan}}(\varphi) \times S_\rho \to \{\pm 1\}$ for any local $L$-parameter $\varphi = \varphi_W \times \varphi_V$ of $G$. Therefore for every $\pi \in \Pi_{\text{Vogan}}$, there is a unique character $\chi_\pi : S_\rho \to \{\pm 1\}$ corresponding to it. In [GP92, §10], Gross and Prasad defined the following character $\chi_\rho = \chi_\varphi^W \times \chi_\varphi^V$ of $S_\rho = S_{\varphi_W} \times S_{\varphi_V}$: For every element $s = s_W \times s_V \in S_{\varphi_W} \times S_{\varphi_V}$, set

\begin{equation}
\chi_\varphi^W(s_W) = \det((-\Id_{M_{i,V}^\vee = -1}^{\dim M_W^{i,V} - 1}/2) \cdot \det((-\Id_{M_W^{i,V} = -1})^{\dim M_W^{i,V} - 1}/2) \cdot \varepsilon(\frac{1}{2}, M_W^{i,V = -1} \otimes M_W, \psi)
\end{equation}

\begin{equation}
\chi_\varphi^V(s_V) = \det((-\Id_{M_{i,V}^\vee = -1}^{\dim M_V^{i,V} - 1}/2) \cdot \det((-\Id_{M_V^{i,V} = -1})^{\dim M_V^{i,V} - 1}/2) \cdot \varepsilon(\frac{1}{2}, M_V^{i,W = -1} \otimes M_V, \psi)
\end{equation}

Here $M_{i,V}^{i,V = -1}$ is the $s_V = (-1)$-eigenspace of $M_V$ and $\varepsilon(\ldots)$ is the local root number defined by Rankin-Selberg integral ([Jac09]).

Now let us recall the conjecture of Gross and Prasad. Let $\pi$ be an irreducible Casselman-Wallach representation of $G(\mathbb{R})$. Set

\begin{equation}
m(\pi) = \dim \Hom_{H(\mathbb{R})}(\pi, \xi).
\end{equation}

Following [SZ12][JSZ10], it is known that

$m(\pi) \leq 1$. 

The local Gross-Prasad conjecture studies the refinement behavior of the multiplicity \( m(\pi) \) in a relevant Vogan \( L \)-packet.

**Conjecture 2.3.1.** Let \((G,H,\xi)\) be a Gross-Prasad triple attached to an admissible pair \((W,V)\) over \( \mathbb{R} \). Fix a generic local \( L \)-parameter \( \varphi \) of \( G \). Then the following statements hold:

1. There exists a unique member \( \pi_\varphi \in \Pi_{rel}^{Vogan}(\varphi) \) such that \( m(\pi_\varphi) = 1 \);
2. Fix the Whittaker datum for \( G \) as \([GP94, (6.3)]\). Based on (2.3.1), the character \( \chi_{\pi_\varphi} : \mathcal{S}_\varphi \to \{\pm 1\} \) attached to the unique member \( \pi_\varphi \) is equal to \( \chi_\varphi \) defined in (2.3.5).

When \( \varphi \) is a tempered local \( L \)-parameter, the second author proved Part (1) of the conjecture in [Luo21] following the work of Waldspurger ([Wal10b],[Wal12b]) and Beuzart-Plessis ([BP20]). Following the work of [MgW12], the first author reduced Conjecture 2.3.1 for generic local \( L \)-parameters to tempered local \( L \)-parameters in [Che21]. Therefore in order to finish the proof of Conjecture 2.3.1 over \( \mathbb{R} \), it remains to establish the following theorem, which is the main result of this paper.

**Theorem 2.3.2.** Over \( \mathbb{R} \), Part (2) of Conjecture 2.3.1 holds for tempered local \( L \)-parameters.

3. Some regular semi-simple conjugacy classes

In this section, we review the parameterization of certain regular semi-simple conjugacy classes in special orthogonal groups following [Wal10a, §1.3] and [Wal12b, §1.3 §1.4]. The Lie algebra analogue is also considered in [Wal01] (see also [Luo21, §5.1]). In subsection 3.2, we establish Proposition 3.2.4 which describes the union of the parameterization over pure inner forms of a special orthogonal group with a fixed Kottwitz sign.

3.1. Parameterization. In this subsection, we review the parameterization of certain regular semi-simple conjugacy classes in special orthogonal groups following [Wal10a, §1.3] and [Wal12b, §1.3 §1.4].

(3.1.1) Consider the following datum:

- A finite set \( I \);
- For any \( i \in I \), fix a finite extension \( F_{\pm i} \) of \( \mathbb{R} \) together with a 2-dimensional \( F_{\pm i} \)
- commutative algebra \( F_i \). Let \( \tau_i \) be the unique nontrivial automorphism of \( F_i \) over \( F_{\pm i} \);
- For any \( i \in I \), fix a constant \( u_i \in F_i^\times \) such that \( u_i \cdot \tau_i(u_i) = 1 \).

Let \( \Xi \) be the set of quadruples \( \kappa = (I, (F_{\pm i})_{i \in I}, (F_i)_{i \in I}, (u_i)_{i \in I}) \) satisfying (3.1.1). For two quadruples in \( \Xi \)

\[ \kappa = (I, (F_{\pm i})_{i \in I}, (F_i)_{i \in I}, (u_i)_{i \in I}), \quad \kappa' = (I', (F'_{\pm i})_{i \in I'}, (F'_i)_{i \in I'}, (u'_i)_{i \in I'}) \]

\( \kappa \) is called isomorphic to \( \kappa' \) if there exists a triple \((\iota, (\iota_{\pm i})_{i \in I}, (\iota_i)_{i \in I})\) such that the following properties hold:

- \( \iota : I \to I' \) is a bijective map;
- For any \( i \in I \), \( \iota_{\pm i} : F_{\pm i} \to F'_{\pm i} \) and \( \iota_i : F_i \to F'_i \) are compatible isomorphisms;
- \( \iota_i(u_i) = u'_{\iota(i)} \).

A quadruple \( \kappa \in \Xi \) is called regular if the identity map is the only automorphism of \( \kappa \). Let \( \Xi_{\text{reg}} \) be the set of isomorphism classes of regular quadruples.

For any even positive integer \( d \), let \( \Xi_{\text{reg},d} \) be the subset of \( \Xi_{\text{reg}} \) consisting of quadruples \( \kappa = (I, (F_{\pm i})_{i \in I}, (F_i)_{i \in I}, (u_i)_{i \in I}) \in \Xi_{\text{reg}} \) satisfying \( \sum_i |F_i : \mathbb{R}| = d \).
For any $\kappa = (I, (F_{\pm i})_{i \in I}, (F_i)_{i \in I}, (u_i)_{i \in I}) \in \Xi_{\text{reg}}$ and $i \in I$, let $\text{sgn}_{F_i/F_{\pm i}}$ be the quadratic character of $F_{\pm i}$ associated to $F_i$. Let $I^* = I_\kappa^*$ be the subset of $I$ consisting of $i \in I$ such that $F_i$ is a field, i.e. $\text{sgn}_{F_i/F_{\pm i}}$ is nontrivial. Let

$$C(\kappa) = \prod_{i \in I} F_{\pm i}^X/\text{Norm}_{F_i/F_{\pm i}}(F_i^X) \simeq \prod_{i \in I^*} \{\pm 1\}.$$

Fix $c = (c_i)_{i \in I} \in C(\kappa)$. For a pair $(\kappa, c) \in \Xi_{\text{reg}} \times C(\kappa)$, one may associate to it an even dimensional non-degenerate quadratic space $(W_{\kappa, c}, q_{\kappa, c})$ over $\mathbb{R}$ with $W_{\kappa, c} = \bigoplus_{i \in I} F_i$ and

$$(3.1.2) \quad q_{\kappa, c} \left( \sum_{i \in I} w_i, \sum_{i \in I} w'_i \right) = \sum_{i \in I} \text{tr}_{F_i/\mathbb{R}}(\tau_i(w_i)w'_i c_i), \quad w_i, w'_i \in F_i.$$

Here we implicitly identify $c_i \in F_{\pm i}^X/\text{Norm}_{F_i/F_{\pm i}}(F_i^X)$ as one of its representatives in $F_{\pm i}$. It turns out that since $\tau_i(c_i) = c_i$, $q_{\kappa, c}$ is indeed symmetric. Up to isomorphism, the quadratic space $(W_{\kappa, c}, q_{\kappa, c})$ is independent of the choice of the representatives of $c$ ([Wal10a, §1.3]).

It turns out that the positive and negative index of $(W_{\kappa, c}, q_{\kappa, c})$ can be computed as follows.

**Lemma 3.1.1.** Let

$I_\mathbb{C}^+ = \{i \in I \mid F_i \simeq \mathbb{C}, c_i = \pm\}, \quad I_{\mathbb{R} \oplus \mathbb{R}} = \{i \in I \mid F_i \simeq \mathbb{R} \oplus \mathbb{R}\}, \quad I_{\mathbb{C} \oplus \mathbb{C}} = \{i \in I \mid F_i \simeq \mathbb{C} \oplus \mathbb{C}\}.$

Then the following identities hold:

$$\text{Pl}(W_{\kappa, c}) = 2 \cdot |I_\mathbb{C}^+| + |I_{\mathbb{R} \oplus \mathbb{R}}| + 2 \cdot |I_{\mathbb{C} \oplus \mathbb{C}}|,$$

$$\text{Ni}(W_{\kappa, c}) = 2 \cdot |I_\mathbb{C}^-| + |I_{\mathbb{R} \oplus \mathbb{R}}| + 2 \cdot |I_{\mathbb{C} \oplus \mathbb{C}}|.$$

**Proof.** For any $i \in I$, from (3.1.1), there are the following situations:

1. $F_{\pm i} = \mathbb{R}$, $F_i = \mathbb{C}$. The quadratic form is given as follows

$$\text{tr}_{\mathbb{C}/\mathbb{R}}(\tau_i(w_i)w'_i c_i), \quad w_i, w'_i \in \mathbb{C},$$

which has positive index 2 (resp. 0) and negative index 0 (resp. 2) if $c_i = 1$ (resp. $-1$) $\in \{\pm 1\} \simeq F_{\pm i}^X/\text{Norm}(F_i^X);$

2. $F_{\pm i} = \mathbb{R}$, $F_i = \mathbb{R} \oplus \mathbb{R}$. The quadratic form is given as follows

$$\text{tr}_{F_i/\mathbb{R}}(\tau_i(w_i^1, w_i^2) \cdot (w'_i^1, w'_i^2)) = w_i^1 w_i'^1 + w_i^2 w_i'^2, \quad w_i^1, w_i^2, w_i'^1, w_i'^2 \in \mathbb{R},$$

which has positive index 1 and negative index 1;

3. $F_{\pm i} = \mathbb{C}$, $F_i = \mathbb{C} \oplus \mathbb{C}$. The quadratic form is given by

$$\text{tr}_{F_i/\mathbb{R}}(\tau_i(w_i^1, w_i^2) \cdot (w'_i^1, w'_i^2) c_i) = c_i \cdot \text{tr}_{\mathbb{C}/\mathbb{R}}(w_i^1 w_i'^1 + w_i^2 w_i'^2), \quad w_i^1, w_i^2, w_i'^1, w_i'^2 \in \mathbb{C},$$

which has positive index 2 and negative index 2.

It follows that the lemma has been proved. \hfill \Box

**Remark 3.1.2.** From Lemma 3.1.1, for $\kappa = (I, (F_{\pm i})_{i \in I}, (F_i)_{i \in I}, (u_i)_{i \in I}) \in \Xi_{\text{reg, d}}$ and $c \in C(\kappa)$, the isomorphism class of the quadratic space $(W_{\kappa, c}, q_{\kappa, c})$ is essentially determined by the cardinality of $I_\mathbb{C}^+$. Note that $I_\mathbb{C}^+ \bigcup I_\mathbb{C}^- = I^*$. Therefore the cardinality of $I_\mathbb{C}^+$ is uniquely determined by the difference

$$|I_\mathbb{C}^-| - |I_\mathbb{C}^+| = \sum_{i \in I^*} c_i, \quad c = (c_i)_{i \in I} \in C(\kappa) \simeq \prod_{i \in I^*} \{\pm 1\}.$$
For any $\theta \in \{-|I^*|, -|I^*| + 2, \ldots, |I^*| - 2, |I^*|\}$, set
\[
C(\kappa)_\theta = \{c = (c_i) \in C(\kappa) \mid \sum_{i \in I^*} c_i = \theta\}.
\]

Then for $c, c' \in C(\kappa)$, $(W_{\kappa,c}, q_{\kappa,c}) \simeq (W_{\kappa,c'}, q_{\kappa,c'})$ if and only if there exists $\theta$ such that $\sum_{i \in I^*} c_i = \sum_{i \in I^*} c'_i = \theta$. When it is the case, by straightforward calculation,
\[
\text{PI}(W_{\kappa,c}) = \frac{d}{2} + \theta, \quad \text{NI}(W_{\kappa,c}) = \frac{d}{2} - \theta.
\]

Let $x_{\kappa,c} \in \text{GL}(W_{\kappa,c})$ be the element defined by
\[
x_{\kappa,c}(\sum_{i \in I} w_i) = \sum_{i \in I} u_i w_i, \quad w_i \in F_i.
\]

Then one can verify that $x_{\kappa,c} \in \text{SO}(W_{\kappa,c})$ by the identity $u_i \cdot \tau_i(u_i) = 1$.

**Definition 3.1.3.** Let $(V, q)$ be a non-degenerate quadratic space over $\mathbb{R}$ and set $\Delta_V = \text{PI}(V) - \text{NI}(V)$. Define the following notions:

1. When $\dim V$ is even, set
   \[
   \Xi_{\text{reg}, V} = \left\{(\kappa, c) \mid \kappa \in \Xi_{\text{reg}, \dim V}, c \in C(\kappa) \Delta_V \right\};
   \]

2. When $\dim V$ is odd, set
   \[
   \Xi_{\text{reg}, V} = \left\{(\kappa, c) \mid \kappa \in \Xi_{\text{reg}, \dim V - 1}, c \in C(\kappa), \text{ and there exists an anisotropic line } (D_{\kappa,V}, q_{\kappa,V}) \text{ such that } (W_{\kappa,c}, q_{\kappa,c}) \simeq (V, q) \right\}.
   \]

**Remark 3.1.4.** When $\dim V$ is odd, through comparing $\text{disc}(V)$ and $\text{disc}(W_{\kappa,c})$, the signature of $D_{\kappa,c}$ is independent of $c \in C(\kappa)$, and is equal to
\[
i_{V,\kappa} = (-1)^{\frac{\Delta_V + 1}{2} + |I^*|} \quad \text{where } \Delta_V = \text{PI}(V) - \text{NI}(V).
\]

Note that $i_{V,\kappa}$ depends only on $\kappa$ and the pure inner class of $V$.

After comparing $\Delta_{W_{\kappa,c} \oplus D_{\kappa,V}}$ with $\Delta_V$,
\[
\sum_{i \in I^*} c_i = \frac{\Delta_V - i_{V,\kappa}}{2}.
\]

Therefore the following identity holds
\[
\Xi_{\text{reg}, V} = \left\{(\kappa, c) \mid \kappa \in \Xi_{\text{reg}, \dim V - 1}, c \in C(\kappa) \frac{\Delta_V - i_{V,\kappa}}{2} \right\}
\]

Following [Wal10a, §1.3] and [Wal12b, §1.3, §1.4], or mimicking the proofs from [Luo21, §5.1], the following theorem holds.

**Theorem 3.1.5.** Let $(V, q)$ be a non-degenerate quadratic space over $\mathbb{R}$. Let $\text{SO}(V)^{\text{rss}} / \sim$ be the set of regular semi-simple conjugacy classes in $\text{SO}(V)$. When $\dim V$ is even, let $\text{SO}(V)^{\text{rss}}_{\pm 1} / \sim$ be the subset of $\text{SO}(V)^{\text{rss}} / \sim$ without eigenvalue $\pm 1$.

1. When $\dim V$ is even, there is a two-to-one map
   \[
   \text{SO}(V)^{\text{rss}}_{\pm 1} / \sim \rightarrow \Xi_{\text{reg}, V}
   \]
   To make it more precise, for $(\kappa, c) \in \Xi_{\text{reg}, V}$, let $x_{\kappa,c} \in \text{SO}(W_{\kappa,c})$ as in (3.1.4). Note that $x_{\kappa,c}$ does not have eigenvalue $\pm 1$ since $\kappa$ is regular. Through comparing the
signature from (3.1.3), \((W_{\kappa,c}, q_{\kappa,c}) \simeq (V, q)\). Then \(x_{\kappa,c}\) determines an \(O(V)\)-conjugacy class in \(\text{SO}(V)\) which breaks into 2 different \(\text{SO}(V)\)-conjugacy classes consisting of different elements. Denote them as \(x^{\pm}_{\kappa,c}\):

(2) When \(\dim V\) is odd, there is a one-to-one map

\[
\Xi_{\text{reg}, V} \longleftrightarrow \text{SO}(V)^{\text{rss}} / \sim
\]

To make it more precise, for \((\kappa, c) \in \Xi_{\text{reg}, V}\), with \((W_{\kappa,c}, q_{\kappa,c}) \oplus ^{\perp} (D_{\kappa,V}, q_{\kappa,V}) \simeq (V, q)\), let \(x^{D_{\kappa,V}}_{\kappa,c} = \text{Id}_{D_{\kappa,V}} \oplus x_{\kappa,c} \in \text{SO}(D_{\kappa,V} \oplus ^{\perp} W_{\kappa,c})\) where \(x_{\kappa,c}\) is constructed in (3.1.4). Then \(x^{D_{\kappa,V}}_{\kappa,c}\) determines a unique \(\text{SO}(V)\)-conjugacy classes in \(\text{SO}(V)\).

3.2. Union over pure inner forms with fixed Kottwitz sign. In this subsection, we discuss the union of the parameterization established in Theorem 3.1.5 over pure inner forms of a special orthogonal group with a fixed Kottwitz sign. The main result of this subsection is Proposition 3.2.4, which is analogous to the description of the fiber of \(p_G\) considered in [Wal12b, §1.4].

First let us recall the notion of Kottwitz sign defined in [Kot83].

**Definition 3.2.1.** Let \(G\) be a reductive algebraic group over \(\mathbb{R}\) with a fixed maximal compact subgroup \(K\). Let \(G_{qs}\) be the unique quasi-split inner form of \(G\) with a maximal compact subgroup \(K_{qs}\). The Kottwitz sign of \(G\) is defined as

\[
e(G) = (-1)^{\frac{\dim K_{qs} - \dim K}{2}}.
\]

By straightforward calculation, the Kottwitz sign of \(\text{SO}(p,q)\) is given as follows.

**Lemma 3.2.2.**

\[
e(\text{SO}(p,q)) = \begin{cases} 
1 & p + q \text{ even} \\
(-1)^{\frac{(p-q)^2 - 1}{8}} & p + q \text{ odd}
\end{cases}
\]

Based on (2.2.2), the following corollary holds by straightforward calculation.

**Corollary 3.2.3.** For any \(\alpha \in H^1(\mathbb{R}, \text{SO}(V) = \text{SO}(p,q))\) corresponding to a non-degenerate quadratic space \(V_{\alpha}\) with signature \((p_{\alpha}, q_{\alpha})\), the following statements hold.

(1) If \(p + q\) is odd, then

\[
e(\text{SO}(V)) = e(\text{SO}(V_{\alpha})) \iff p \equiv p_{\alpha} \mod 4;
\]

(2) If \(p + q\) is even, then after fixing an anisotropic line \(D\),

\[
e(\text{SO}(V \oplus ^{\perp} D)) = e(\text{SO}(V_{\alpha} \oplus ^{\perp} D)) \iff p \equiv p_{\alpha} \mod 4.
\]

Now we are ready to state the main proposition in this subsection.

**Proposition 3.2.4.** For \(\kappa \in \Xi_{\text{reg}},d\), set \(C(\kappa)^{\pm 1} = \{c \in C(\kappa) \mid \prod_{i \in I} c_i = \pm 1\}\). Then for any non-degenerate quadratic space \(V\) over \(\mathbb{R}\) and \(e_0 = \pm 1\), the following identities hold.

(1) If \(\dim V\) is odd, then

\[
\bigsqcup_{\alpha \in H^1(\mathbb{R}, \text{SO}(V)) \atop e(\text{SO}(V_{\alpha})) = e_0} \Xi_{\text{reg}, V_{\alpha}} = \Xi_{\text{reg}, \dim V - 1, e_0}.
\]
Here
\[ \Xi_{\text{reg}, \dim V - 1, c_0} = \{ (\kappa, c) \in \Xi_{\text{reg}, \dim V - 1}, c \in C(\kappa)^{e_0 \cdot \epsilon_{V, \kappa}} \}, \]
\[ \epsilon_{V, \kappa} = (-1)^{-\frac{\dim V + \epsilon_{V, \kappa}}{2}}. \]

\( \text{PI}(V_{qs}) \) is the positive index of the unique quasi-split pure inner form of \( \text{SO}(V) \);
(2) If \( \dim V \) is even, fix an anisotropic line \( D \) with signature \( \epsilon(D) \in \{ \pm 1 \} \). Then
\[ \bigcup_{\alpha \in H^1(\mathbb{R}, \text{SO}(V)) \cap \epsilon(\text{SO}(V_{qs} \oplus D))=e_0} \Xi_{\text{reg}, \alpha} = \Xi_{\text{reg}, \dim V, e_0, D}. \]

Here
\[ \Xi_{\text{reg}, \dim V, e_0, D} = \{ (\kappa, c) \mid \kappa \in \Xi_{\text{reg}, \dim V}, c \in C(\kappa)^{e_0 \cdot \epsilon_{V, \kappa, D}} \}, \]
\[ \epsilon_{V, \kappa, D} = (-1)^{-\frac{\dim V + \epsilon_{V, \kappa, D}}{2}}. \]

\( \text{PI}(V, D) \) is the positive index of the unique quasi-split pure inner form of \( \text{SO}(V_{qs} \oplus \mathbb{R}) \).

**Proof.** First let us assume that \( \dim V \) is odd. Fix \( \alpha \in H^1(\mathbb{R}, \text{SO}(V)) \) and \( e(\text{SO}(V_{\alpha})) = e_0 \). For any \( (\kappa, c) \in \Xi_{\text{reg}, \alpha} \), from Remark 3.1.4 there exists an anisotropic line \( (D_{\kappa, \alpha}, q_{\kappa, \alpha}) \) with signature \( i_{\kappa, \alpha} \) such that \( (V_{\alpha}, q_{\alpha}) \simeq (D_{\kappa, \alpha}, q_{\kappa, \alpha}) \oplus \mathbb{R} (W_{\kappa, \alpha}, q_{\kappa, \alpha}) \). From (3.1.6),
\[ \Xi_{\text{reg}, \alpha} = \{ (\kappa, c) \mid \kappa \in \Xi_{\text{reg}, \dim V - 1}, c \in C(\kappa)\Delta_{V_{qs} - V_{\alpha, \kappa}} \}. \]

In general, for \( \kappa \in \Xi_{\text{reg}, \dim V - 1} \) and \( c \in C(\kappa) \), by direct calculation,
\[ (3.2.1) \quad \prod_{i \in I^*} c_i = 1 \quad \text{(resp. } -1) \iff \left( |I^*_\kappa| - \sum_{i \in I^*_\kappa} c_i \right) \equiv 0 \text{ (resp. } 2) \mod 4. \]

Equivalently,
\[ (3.2.2) \quad \prod_{i \in I^*_\kappa} c_i = (-1)^{-\frac{|I^*_\kappa| - \sum_{i \in I^*_\kappa} c_i}{2}}. \]

In particular, when \( (\kappa, c) \in \Xi_{\text{reg}, \alpha} \), \( \sum_{i \in I^*_\kappa} c_i = \Delta_{V_{\alpha} - V_{\kappa, \alpha}}. \) Hence
\[ \prod_{i \in I^*_\kappa} c_i = (-1)^{-\frac{\Delta_{V_{\alpha} - V_{\kappa, \alpha}}}{2}} \]

Let \( (p_{qs}, q_{qs}) \) be the signature of the unique quasi-split pure inner form \( \text{SO}(V_{qs}) \) of \( \text{SO}(V_{\alpha}) \).

From Corollary 3.2.3, \( e_0 = e(\text{SO}(V_{\alpha})) = (-1)^{-\frac{p_{qs} - q_{qs}}{2}} \) where \( (p_{\alpha}, q_{\alpha}) \) is the signature of \( V_{\alpha} \). Therefore \( e_0 \cdot \prod_{i \in I^*_\kappa} c_i \) is equal to
\[ = (-1)^{-\frac{p_{qs} - q_{qs}}{2}} \cdot (-1)^{-\frac{\Delta_{V_{\alpha} - V_{\kappa, \alpha}}}{2}} = (-1)^{-\frac{p_{qs} + |I^*_\kappa| - \Delta_{V_{qs} - V_{\kappa, \alpha}}}{2}}. \]

Since \( \text{SO}(V_{\alpha}) \) is a pure inner form of \( \text{SO}(V) \), \( \Delta_{V_{\alpha}} \equiv \Delta_{V} \mod 4. \) Therefore
\[ = (-1)^{-\frac{p_{qs} + |I^*_\kappa| - \Delta_{V_{qs} - V_{\kappa, \alpha}}}{2}} = (-1)^{-\frac{\dim V + \epsilon_{V, \kappa}}{2}} = \epsilon_{V, \kappa}. \]
It follows that
\[ \bigcup_{\alpha \in H^1(\mathbb{R}, \text{SO}(V)) \atop e(\text{SO}(V)) = e_0} \Xi_{\text{reg}, V, \alpha} \subset \Xi_{\text{reg}, \dim V - 1, \alpha_0}. \]

Conversely, for \((\kappa, c) \in \Xi_{\text{reg}, \dim V - 1, \alpha_0}\) and \(\alpha \in H^1(\mathbb{R}, \text{SO}(V))\) with \(e(\text{SO}(V)) = e_0\), by (3.2.2)
\[ \sum_{i \in I^*_\kappa} c_i \equiv \frac{\Delta_{V, \alpha} - i_{V, \alpha, \kappa}}{2} \pmod{4}. \]

Equivalently,
\[ (-1)^{\Delta_{V, \alpha} + 1 + |I^*_\kappa|} \cdot \sum_{i \in I^*_\kappa} c_i \equiv \Delta_{V, \alpha} \pmod{8}. \]

Now \(|(-1)^{iv_{\kappa}} + 2 \cdot \sum_{i \in I^*_\kappa} c_i| \leq 1 + 2 \cdot \sum_{i \in I^*_\kappa} 1 \leq \dim V.\) As \(\alpha\) runs over \(H^1(\mathbb{R}, \text{SO}(V))\) with \(e(\text{SO}(V)) = e_0\), \(\Delta_{V, \alpha}\) runs over all the integers between \(-\dim V\) and \(\dim V\) with fixed congruence class modulo 8. Hence there exists \(\alpha_0 \in H^1(\mathbb{R}, \text{SO}(V))\) such that
\[ \Delta_{V, \alpha_0} = (-1)^{iv_{\kappa}} + 2 \cdot \sum_{i \in I^*_\kappa} c_i \iff \sum_{i \in I^*_\kappa} c_i = \frac{\Delta_{V, \alpha} - (-1)^{iv_{\kappa}}}{2}. \]

It follows that
\[ \Xi_{\text{reg}, \dim V - 1, \alpha_0} \subset \bigcup_{\alpha \in H^1(\mathbb{R}, \text{SO}(V)) \atop e(\text{SO}(V)) = e_0} \Xi_{\text{reg}, V, \alpha} \]
and Part (1) of the proposition is established.

It remains to treat the case when \(\dim V\) is even. The proof is similar to odd situation. Fix \(\alpha \in H^1(\mathbb{R}, \text{SO}(V))\) with \(e(\text{SO}(V_\alpha \perp D)) = e_0\). For any \((\kappa, c) \in \Xi_{\text{reg}, V}\), with the same argument as odd case, \(\prod_{i \in I^*_\kappa} c_i = (-1)^{i_{\kappa}} \frac{\Delta_{V, \alpha}}{2}\). Let \(\text{SO}(V_\alpha \perp D) = \text{SO}(p_\alpha, q_\alpha)\) with unique quasi-split pure inner form \(\text{SO}(p_{qs}, q_{qs})\). Then \(e(\text{SO}(V_\alpha \perp D)) = (-1)^{p_{\alpha - p_{qs}}}.\) Using the fact that \(\Delta_{V, \alpha} = p_\alpha - q_\alpha - \text{sig}(D)\), we get
\[ e_0 \cdot \prod_{i \in I^*_\kappa} c_i = (-1)^{i_{\kappa}} \frac{\Delta_{V, \alpha} + p_\alpha - p_{qs}}{2} = (-1)^{i_{\kappa}} \frac{\dim V + 2 + \text{sig}(D) - p_{qs}}{2}. \]

It follows that
\[ \bigcup_{\alpha \in H^1(\mathbb{R}, \text{SO}(V)) \atop e(\text{SO}(V_\alpha \perp D)) = e_0} \Xi_{\text{reg}, V} \subset \Xi_{\text{reg}, \dim V, e_0, D}. \]

For the converse inclusion the same proof as odd case applies verbatim. \(\square\)

**Remark 3.2.5.** From Remark 3.1.4, \(iv_{\kappa}\) depends only on \(\kappa\) and the pure inner class of \(V\). It follows that \(e_{V, \kappa}\) (resp. \(e_{V, \kappa, D}\)) defined in Part (1) (resp. (2)) of Proposition 3.2.4 depends only on \(\kappa\) and Kottwitz sign of \(V\) (resp. \(V \perp D\)).
4. Geometric multiplicity formula

In this section, for a Gross-Prasad triple \((G, H, \xi)\) attached to an admissible pair \((W, V)\) over \(\mathbb{R}\), we recall the geometric multiplicity formula for tempered representations of \(G(\mathbb{R})\) established in [Luo21]. We also establish Lemma 4.2.2 and Corollary 4.2.3, which are analogous to [Wal10b, Lem. 13.4 (ii)] and a formula in [Wal10b, §13.6]. In subsection 4.3 we recall the stable variant of the geometric multiplicity which was first introduced in [Wal12b].

4.1. The formula. In this subsection, we recall the geometric multiplicity formula established in [Luo21] for tempered representations of \(G(\mathbb{R})\). For details see [Luo21, §7.3].

Geometric support. For \(x \in H_{\text{ss}}(\mathbb{R})\) where \(H_{\text{ss}}(\mathbb{R})\) is the set of semi-simple elements in \(H(\mathbb{R})\), up to conjugation by \(H(\mathbb{R})\) there is no harm to assume that \(x \in SO(W)_{\text{ss}}(\mathbb{R})\). Let \(\Gamma(H)\) be the set of semi-simple conjugacy classes in \(H(\mathbb{R})\). Let \(W_x = \ker(1 - x|_W)\), \(V_x = \ker(1 - x|_V)\) and \(W_x'' = \text{Im}(1 - x|_W)\). Then \(W = W_x' \oplus W_x''\) and \(V = V_x' \oplus W_x''\). Moreover \((V_x', W_x'')\) is also an admissible pair over \(\mathbb{R}\). Let \(G_x\) be the connected component of the centralizer of \(x\) in \(G\). Then following [Luo21, §7.3.1],

\[
G_x = G_x' \times G_x'', \quad \text{with} \quad G_x' = SO(W_x') \times SO(V_x'), \quad G_x'' = SO(W_x'') \times SO(W_x'').
\]

Let \(\Gamma(G, H)\) be the subset of \(\Gamma(H)\) consisting of \(x \in \Gamma(G, H)\) satisfying the condition that \(SO(W_x'')\) is an anisotropic torus and \(G_x\) is quasi-split. Following [Luo21, (7.3.3)], this set \(\Gamma(G, H)\) is equipped with topology and measure.

The germ \(c_\theta\). For a reductive algebraic group \(G\) over \(\mathbb{R}\) and \(\Theta\) a quasi-character on \(G(\mathbb{R})\) (see [BP20, §4.4] for the definition), from [BP20, Prop. 4.4 (vi)], the following asymptotic expansion holds for any \(x \in G_{\text{ss}}(\mathbb{R})\) and \(Y \in \mathfrak{g}_x(\mathbb{R}) = (\text{Lie}G_x)(\mathbb{R})\) sufficiently close to 0,

\[
D^G(xe^Y)^{1/2} \cdot \Theta(xe^Y) = D^G(xe^Y)^{1/2} \cdot \sum_{\mathcal{O} \in \text{Nil}_{\text{reg}}(\mathfrak{g}_x)} c_{\Theta, \mathcal{O}}(x) \cdot \widehat{j}(\mathcal{O}, Y) + O(|Y|).
\]

Here \(\text{Nil}_{\text{reg}}(\mathfrak{g}_x)\) is the set of regular nilpotent orbits in \(\mathfrak{g}_x(\mathbb{R})\), \(c_{\Theta, \mathcal{O}}(x) \in \mathbb{C}\), and \(\widehat{j}(\mathcal{O}, \cdot)\) is the Fourier transform of the nilpotent orbital integral attached to \(\mathcal{O}\).

Let \((V, q)\) be a non-degenerate quadratic space over \(\mathbb{R}\). Recall that \((V, q)\) is called quasi-split (see [Luo21, §6.1.1] for instance) if

\[
(V, q) \simeq \mathbb{H}^{n-1} \oplus 1 \begin{cases} (D, q), & \text{dim } V \equiv 1 \mod 2 \\ (E = F(\sqrt{b}), c \cdot N_{E/F}), & \text{dim } V \equiv 0 \mod 2 \end{cases}
\]

for some positive integer \(n\) and \(b, c \in \mathbb{R}^\times\). Here \(\mathbb{H}^{n-1}\) is the unique split quadratic space of dimension \(2n - 2\), \((D, q)\) is an anisotropic line, and \((E = F(\sqrt{b}), c \cdot N_{E/F})\) is the 2-dimensional quadratic space sending \(m \oplus n\sqrt{b} \mapsto c \cdot (m^2 - bn^2)\). Following [Luo21, §6.1.2], the set of \(SO(V)(\mathbb{R})\)-regular nilpotent orbits in \(\mathfrak{so}(V)(\mathbb{R})\), denoted as \(\text{Nil}_{\text{reg}}(\mathfrak{so}(V))\), has the following description:

1. \(\text{Nil}_{\text{reg}}(\mathfrak{so}(V)) \neq \emptyset\) if and only if \((V, q)\) is quasi-split;
2. \(\text{Nil}_{\text{reg}}(\mathfrak{so}(V))\) contains a unique element if
   - \(\text{dim } V\) is odd;
   - \(\text{dim } V \leq 2\);
   - \(\text{dim } V \geq 4\) is even and \((V, q)\) is quasi-split but not split.
(3) When \((V, q)\) is split of even dimension with \(\dim V \geq 4\), \(\text{Nil}^\text{reg}(\mathfrak{so}(V))\) contains two elements \(O_{\pm}\) parametrized by \(\{\pm 1\} \simeq \mathbb{R}^\times / \mathbb{R}^\times 2\). For the explicit parametrization see [Luo21, §6.1.2].

Now for a Gross-Prasad triple \((G, H, \xi)\) attached to an admissible pair \((W, V)\) over \(\mathbb{R}\), and \(\Theta\) a quasi-character on \(G(\mathbb{R})\), by (4.1.1) and the definition of \(\Gamma(\mathbb{R}, H)\), for any \(x \in \Gamma(\mathbb{R}, H)\), \(\text{Nil}^\text{reg}(g_x) = \text{Nil}^\text{reg}(g'_x)\) with \(g'_x = \mathfrak{so}(W'_x) \times \mathfrak{so}(V'_x)\). Following [Luo21, §7.3.2], for any quasi-character \(\Theta\) on \(G(\mathbb{R})\), define the following function \(c_{\Theta}(x)\) on \(\Gamma(\mathbb{R}, H)\):

\[
\Theta \quad \text{(Special values of } a \text{ over } R) \quad \text{Theorem 4.1.1.} 
\]

For any tempered representation \((4.1.3) (1) \quad \text{If } \text{Nil}^\text{reg}(g_x) \text{ is singleton with unique member } O_{\Theta, \text{reg}}(x) \quad \text{set } c_{\Theta}(x) = c_{\Theta, O_{\text{reg}}}(x); \\
(2) \quad \text{If } \text{Nil}^\text{reg}(g_x) \text{ has two elements (in particular both } W'_x \text{ and } V'_x \text{ are split), write the decomposition } V'_x \simeq W'_x \oplus D'_x \oplus Z'_x \text{ from the definition of admissible pair. Let } \\
\text{sig}(D'_x) \in \{\pm 1\} \text{ be the signature of the line } D'_x. \\
\bullet \quad \text{If dim } V'_x \text{ is even and } \geq 4, \text{ choose } O_{\pm \text{sig}(D'_x)}(x) \leftrightarrow \text{Nil}^\text{reg}(\mathfrak{so}(V'_x)) \quad \text{and set } c_{\Theta}(x) = c_{\Theta, O_{\pm \text{sig}(D'_x)}}(x); \\
\bullet \quad \text{If dim } W'_x \text{ is even and } \geq 4, \text{ choose } O_{-\text{sig}(D'_x)}(x) \leftrightarrow \text{Nil}^\text{reg}(\mathfrak{so}(W'_x)) \quad \text{and set } c_{\Theta}(x) = c_{\Theta, O_{-\text{sig}(D'_x)}}(x). \\
\]

**Integral formula.** With the above notation, for \(x \in \Gamma(\mathbb{R}, H)\) set

\[
\Delta(x) = |\det(1 - x)_{|W'_x}|, \\
D^G(x) = |\det(1 - \text{Ad}(x))_{\mathfrak{g}/g_x}|.
\]

For any quasi-character \(\Theta\) on \(G(\mathbb{R})\), consider the following integral

\[
m_{\text{geom}}(\Theta) := \int_{\Gamma(\mathbb{R}, H)} D^G(x)^{1/2} c_{\Theta}(x) \Delta(x)^{-1/2} \, dx.
\]

From [Luo21, Prop. 7.3.3.3] the integral (4.1.4) is absolutely convergent.

For any tempered representation \(\pi\) of \(G(\mathbb{R})\), let \(\Theta_{\pi}\) be the distribution character of \(\pi\) ([HC63]) which in particular is a quasi-character ([BV80]). Set \(m_{\text{geom}}(\pi) := m_{\text{geom}}(\Theta_{\pi})\). The following theorem is established in [Luo21].

**Theorem 4.1.1. For any tempered representation \(\pi\) of \(G(\mathbb{R})\),**

\[
m_{\text{geom}}(\pi) = m(\pi).
\]

4.2. Special values of \(\hat{j}(O, \cdot)\). In this subsection, we establish the analogue of [Wal10b, Lem. 13.4 (ii)] over \(\mathbb{R}\). The result will be used to express \(c_{\Theta}(x)\) via the original quasi-character \(\Theta\).

Let \((V, q)\) be a split quadratic space of even dimension \(\geq 4\) (so that \(\text{Nil}^\text{reg}(\mathfrak{so}(V))\) has two elements). Consider the following regular semi-simple element in \(\mathfrak{so}(V)(\mathbb{R})\) (which is also considered in [Luo21, §5.3] following [Wal01]):

(2.1) Fix two nonzero purely imaginary numbers \(a_1, a_2 \in i\mathbb{R}^\times\) such that \(a_1 \neq \pm a_2\). Fix an isomorphism of split quadratic spaces

\[
(V, q) \simeq (\mathbb{C}, c \cdot \text{Nr}) \oplus^\perp (\mathbb{C}, -c \cdot \text{Nr}) \oplus^\perp (\tilde{Z}, q)
\]

where \(c = \pm 1\), \(\text{Nr} = \text{Norm}_{\mathbb{C}/\mathbb{R}}\) and \((\tilde{Z}, q)\) is a split quadratic space of dimension \(\dim V - 4\). Let \(\tilde{T}\) be a fixed maximal split torus of \(\text{SO}(V)\) with Lie algebra \(\tilde{t}\), and fix a regular semi-simple element \(\tilde{S} \in \tilde{t}(\mathbb{R})\). Consider the element \(X_{a_1, a_2, \tilde{S}} \in \mathfrak{so}(V)\) acting on \((\mathbb{C}, c \cdot \text{Nr})\) via \(a_1\), \((\mathbb{C}, -c \cdot \text{Nr})\) via \(a_2\) and \((\tilde{Z}, q)\) via \(\tilde{S}\). From [Luo21, Lem. 5.1.0.5] the conjugacy classes of \(X_{a_1, a_2, \tilde{S}}\) within the stable conjugacy class is
determined by \( c \in \mathbb{R}^x/\mathbb{R}^x \). Let \( X^\pm_{a_1,a_2,S} \) be the element corresponding to \( c = c^\pm \) with \( \text{sgn}_{C/\mathbb{R}}(c^\pm) = \pm \text{sgn}_{C/\mathbb{R}}(\text{Nr}(a_1) - \text{Nr}(a_2)) \).

Let \( \widehat{j}(X^\pm_{a_1,a_2,S}, \cdot) \) be the Fourier transform of the Lie algebra orbital integral at \( X^\pm_{a_1,a_2,S} \) ([BP20, §1.9]). Following [BP20, Lem. 4.3.1], the following limit formula holds

\[
\lim_{t \to 0^+} \frac{D^G(tY)^{1/2} \cdot j(X^\pm_{a_1,a_2,S}, tY)}{\sum_{O \in \text{Nil}_{\text{reg}}(\mathfrak{so}(V))} \Gamma_O(X^\pm_{a_1,a_2,S}) \cdot \widehat{j}(O, Y)} = 0.
\]

Here \( \Gamma_O(X^\pm_{a_1,a_2,S}) \in \mathbb{C} \) is the regular germs attached to \( O \in \text{Nil}_{\text{reg}}(\mathfrak{so}(V)) \) and \( X^\pm_{a_1,a_2,S} \), whose explicit formula is established in [Luo21, Thm. 4.2.0.1].

The following lemma can be found in [Luo21, Lem. 5.3.0.1].

**Lemma 4.2.1.**

(1) For \( O_\pm \in \text{Nil}_{\text{reg}}(\mathfrak{so}(V)) \) with the parameterization in [Luo21, §6.1.2],

\[
\Gamma_{O_\pm}(X^+_{a_1,a_2,S}) - \Gamma_{O_\pm}(X^-_{a_1,a_2,S}) = \pm 1.
\]

(2) For any \( X_{ld} \in \mathfrak{t}_V \) that is regular semi-simple, where \( \mathfrak{t}_V \) is the Lie algebra of a maximal split torus in \( SO(V) \),

\[
\Gamma_{O_\pm}(X_{ld}) = 1.
\]

The main result of this subsection is the following lemma, which is analogues to [Wal10b, Lem. 13.4 (ii)] over \( \mathbb{R} \).

**Lemma 4.2.2.** With the above notation, for \( \nu = \pm \),

\[
\widehat{j}(O_\nu, X^+_{a_1,a_2,S}) = -\widehat{j}(O_\nu, X^-_{a_1,a_2,S}) = \nu \cdot \frac{|W_{T_{\text{cpt}}}|}{2} \cdot D^G(X^+_{a_1,a_2,S})^{-1/2}
\]

where \( T_{\text{cpt}} \) is the centralizer of \( X^+_{a_1,a_2,S} \) in \( SO(V) \) and \( W_{T_{\text{cpt}}} = N(T_{\text{cpt}})/T_{\text{cpt}} \).

**Proof.** By (4.2.2),

\[
\lim_{t \to 0^+} D^G(tY)^{1/2} \cdot (\widehat{j}(X^+_{a_1,a_2,S}, tY) - \widehat{j}(X^-_{a_1,a_2,S}, tY)) = D^G(Y)^{1/2} \cdot \sum_{\pm} (\Gamma_{O_\pm}(X^+_{a_1,a_2,S}) - \Gamma_{O_\pm}(X^-_{a_1,a_2,S})) \cdot \widehat{j}(O_\pm, Y).
\]

By Lemma 4.2.1, the above formula is equal to

\[
D^G(Y)^{1/2} \cdot (\widehat{j}(O_+, Y) - \widehat{j}(O_-, Y)).
\]

From [BP20, (3.4.6)], since \( X^\pm_{a_1,a_2,S} \) are not split,

\[
\widehat{j}(O_+, X^\pm_{a_1,a_2,S}) - \widehat{j}(O_-, X^\pm_{a_1,a_2,S}) = 2 \cdot \widehat{j}(O_+, X^\pm_{a_1,a_2,S}).
\]

Hence

\[
\lim_{t \to 0^+} \frac{D^G(tX^\pm_{a_1,a_2,S})^{1/2}}{D^G(X^\pm_{a_1,a_2,S})^{1/2}} \cdot (\widehat{j}(X^+_{a_1,a_2,S}, tX^\pm_{a_1,a_2,S}) - \widehat{j}(X^-_{a_1,a_2,S}, tX^\pm_{a_1,a_2,S})) = 2 \cdot \widehat{j}(O_+, X^\pm_{a_1,a_2,S}.
\]

On the other hand, following the last part of the proof of [Wal10b, Lem. 13.4] and [BP20, (3.4.4)], as a distribution, \( \widehat{j}(X^\pm_{a_1,a_2,S}, \cdot) \) is the parabolic induction from \( \widehat{j}(X^\pm_{a_1,a_2}, \cdot) \) where \( \widehat{j}(X^\pm_{a_1,a_2}, \cdot) \) is the Fourier transform of the Lie algebra orbital integral at \( X^\pm_{a_1,a_2} \) on the Levi
subgroup \( \text{SO}(2, 2) \times \widetilde{T} \) of \( \text{SO}(V) \) (where \( \text{SO}(2, 2) \cong \text{SO}(\mathbb{C}, \text{Nr}) \oplus \mathbb{C}, \text{Nr}) \)). Using the same argument as the last paragraph of [Wal10b, Lem. 13.4], we can reduce the proof to \( \dim V = 4 \) (and hence \( \widetilde{Z} = 0 \)) situation. Hence it suffices to evaluate the following limit when \( \dim V = 4 \)

\[
\lim_{t \in \mathbb{R}^{2}, t \to 0^{+}} \frac{\dim \text{SO}(V) - \dim T_{\text{cpt}}}{2} \cdot \left( \widetilde{j}(X_{a_{1}, a_{2}}, tX_{a_{1}, a_{2}}) \right).
\]

This can be achieved by the formula of Rossmann in [Ros78, p.217 (15)]. Notice that as a real Lie group, \( T_{\text{cpt}}(\mathbb{R}) \cong \text{SO}(2)(\mathbb{R}) \times \text{SO}(2)(\mathbb{R}) \) which is a maximal compact Cartan subgroup in \( \text{SO}(2, 2)(\mathbb{R}) \) containing both \( X_{a_{1}, a_{2}}^{\pm} \). In particular, since the maximal compact subgroup of \( \text{SO}(2, 2)(\mathbb{R}) \) can be identified with \( \text{SO}(2)(\mathbb{R}) \times \text{O}(2)(\mathbb{R}) \), the Weyl group

\[
W_{T_{\text{cpt}}} = N_{\text{SO}(2, 2)(\mathbb{R})}(T_{\text{cpt}}(\mathbb{R}))/T_{\text{cpt}}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}.
\]

Let \( W_{T_{\text{cpt}, C}} \) be the Weyl group attached to \( (t_{\text{cpt}, C}, \mathfrak{so}(4, \mathbb{C})) \) where \( t_{\text{cpt}} \) is the Lie algebra of \( T_{\text{cpt}} \) and \( t_{\text{cpt}, C} \) is its complexification. Then \( W_{T_{\text{cpt}}} \) embeds diagonally into the complex Weyl group \( W_{T_{\text{cpt}, C}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Following [Ros78, p.208], fix once for all a system of positive roots for \( (t_{\text{cpt}, C}, \mathfrak{so}(4, \mathbb{C})) \) and let \( \pi : t_{\text{cpt}, C} \to \mathbb{C} \) be the product of these positive roots. Notice that \( \pi \) is equal to the square root of Weyl discriminant up to \( \frac{\pi(w)}{\pi(\alpha)} \). In particular, one should note that the Lie algebra orbital integral defined in [Ros78, (1)] differs from ours (following [BP20, §1.8]) by \( \frac{\pi(w)}{\pi(\alpha)} \).

Now by [Ros78, p.217 (15)], after straightforward calculation, the following identity holds

\[
\frac{\pi(X_{a_{1}, a_{2}}^{\pm})}{\pi(X_{a_{1}, a_{2}}^{\ast})} \cdot \widetilde{j}(X_{a_{1}, a_{2}}^{\ast}, tX_{a_{1}, a_{2}}^{\pm}) = -\frac{1}{W_{T_{\text{cpt}}}} \cdot \pi(tX_{a_{1}, a_{2}}^{\pm})^{-1} \cdot \sum_{w \in W_{T_{\text{cpt}}}} (-1)^{\ell(w)} \cdot e^{i(\omega \cdot tX_{a_{1}, a_{2}}^{\pm}, X_{a_{1}, a_{2}}^{\ast})}.
\]

where \( \ast = \pm \) and \( \ell(w) \) is the length of \( w \). Since \( W_{T_{\text{cpt}}} \cong \mathbb{Z}/2\mathbb{Z} \) embeds into \( W_{T_{\text{cpt}, C}} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) diagonally, we get

\[
\frac{\pi(X_{a_{1}, a_{2}}^{\ast})}{\pi(X_{a_{1}, a_{2}}^{\ast})} \cdot \lim_{t \in \mathbb{R}^{2}, t \to 0^{+}} \frac{\dim \text{SO}(2, 2) - \dim T_{\text{cpt}}}{2} \cdot \widetilde{j}(X_{a_{1}, a_{2}}^{\ast}, tX_{a_{1}, a_{2}}^{\pm}) = -\frac{1}{\pi(X_{a_{1}, a_{2}}^{\pm})}.
\]

Hence the limit (4.2.3) becomes

\[
= \left( \frac{\pi(X_{a_{1}, a_{2}}^{\ast})}{\pi(X_{a_{1}, a_{2}}^{\ast})} - \frac{\pi(X_{a_{1}, a_{2}}^{\pm})}{\pi(X_{a_{1}, a_{2}}^{\pm})} \right)^{-1} \cdot -\frac{1}{\pi(X_{a_{1}, a_{2}}^{\pm})}.
\]

By straightforward calculation, \( \frac{\pi(X_{a_{1}, a_{2}}^{\ast})}{\pi(X_{a_{1}, a_{2}}^{\ast})} = -\frac{\pi(X_{a_{1}, a_{2}}^{\ast})}{\pi(X_{a_{1}, a_{2}}^{\ast})} \). Moreover, using the isomorphism \( \mathfrak{so}(2, 2) \cong \mathfrak{sl}_{2} \times \mathfrak{sl}_{2} \), the number \( \pi(X_{a_{1}, a_{2}}^{\pm}) \) is purely imaginary. Using the fact that \( \pi(X_{a_{1}, a_{2}}^{\pm}) = \frac{\pi(X_{a_{1}, a_{2}}^{\ast})}{\pi(X_{a_{1}, a_{2}}^{\ast})} \cdot D_{G}(X_{a_{1}, a_{2}}^{\pm})^{1/2} \), the limit (4.2.3) eventually becomes

\[
2 \cdot \pm \cdot D_{G}(X_{a_{1}, a_{2}}^{\pm})^{1/2}.
\]

It follows that

\[
\widetilde{j}(O_{+}, X_{a_{1}, a_{2}}^{\pm}) = \pm \cdot D_{G}(X_{a_{1}, a_{2}}^{\pm})^{1/2}.
\]

With the aid of [BP20, (3.4.6)] the lemma is established. \( \square \)

Based on Lemma 4.2.2, the following corollary holds.
Corollary 4.2.3. Let $(G, H, \xi)$ be a Gross-Prasad triple attached to an admissible pair $(W, V)$ over $\mathbb{R}$, and $\Theta$ a quasi-character on $G(\mathbb{R})$. For any $x \in \Gamma(G, H)$, fix a Borel pair $(B_x, T_{qd, x})$ for $G_x$ over $\mathbb{R}$. Let $t_{qd, x} = \text{Lie}T_{qd, x}$, and fix a regular semi-simple element $X_{qd, x} \in t_{qd, x}(\mathbb{R})$. Set $D^{G_x}(tX_{qd, x}) = |\text{det} \exp(tX_{qd, x})|_{G_x/T_{qd, x}}$ and $W_{T_{qd, x}} = W(G_x, T_{qd, x})$. Then the following limit formulas hold for $c_\Theta(x)$ defined in (4.1.3).

1. If $\text{Nil}_{\text{reg}}(g_x)$ is singleton, then
   $$c_\Theta(x) = |W_{T_{qd, x}}|^{-1} \lim_{t \to 0^+} D^{G_x}(tX_{qd, x})^{1/2} \cdot \Theta(x \cdot \exp(tX_{qd, x}));$$

2. If $\text{Nil}_{\text{reg}}(g_x)$ has two elements, as in Part (3) of (4.1.3), fix an admissible pair decomposition $V'_x \cong W'_x \oplus D'_x \oplus Z'_x$ and let $\text{sig}(D'_x) \in \{-1, 1\}$ be the signature of $D'_x$. Then there exists a pair of regular semi-simple elements $X'_x \in g_x$ with centralizer $T_{X'_x} \subset G_x$ that are stably conjugate but not conjugate to each other, such that the following identity holds:

$$c_\Theta(x) = \begin{cases} \frac{\text{dim} V'_x}{2} & \text{if } \text{sig}(D'_x) = 1, \\
\frac{\text{dim} V'_x}{2} - 1 & \text{if } \text{sig}(D'_x) = -1, \\
\end{cases} \lim_{t \to 0^+} D^{G_x}(tX_{qd, x})^{1/2} \cdot \Theta(x \cdot \exp(tX_{qd, x}));$$

Proof. By basic calculus, for any $X \in g_x$

$$\lim_{t \to 0} \frac{D^G(x \cdot \exp(tX))}{D^{G_x}(tX)} = D^G(x).$$

From [BP20, Prop. 4.4.1 (vi)], [BP20, (1.8.5)] and (4.2.4), for $Y \in g_x(\mathbb{R})$ that is regular semi-simple

$$\lim_{t \to 0^+} D^{G_x}(tY)^{1/2} \cdot \Theta(x \cdot \exp(tY)) = D^{G_x}(Y)^{1/2} \cdot \sum_{O \in \text{Nil}_{\text{reg}}(g_x)} c_\Theta(O) \cdot \tilde{j}(O, Y).$$

In particular, by [BP20, (3.4.7)], for $X_{qd, x} \in t_{qd, x}(\mathbb{R})$ that is regular semi-simple,

$$\lim_{t \to 0^+} D^{G_x}(tX_{qd, x})^{1/2} \cdot \Theta(x \cdot \exp(tX_{qd, x})) = \frac{|W_{T_{qd, x}}|}{|\text{Nil}_{\text{reg}}(g_x)|} \cdot \sum_{O \in \text{Nil}_{\text{reg}}(g_x)} c_\Theta(O)$$

Hence Part (1) holds.

For Part (2), by assumption, $\text{Nil}_{\text{reg}}(g_x)$ has two elements. Following (4.1.3), let us consider the case when $\text{dim} V'_x$ is even and $\geq 4$. The other case is similar. Fix the unique member $O_{W'_x} \in \text{Nil}_{\text{reg}}(\mathfrak{so}(W'_x))$ and set $\text{Nil}_{\text{reg}}(\mathfrak{so}(V'_x)) = \{O_{+, V'_x}, O_{-, V'_x}\}$ parametrized as [Luo21, §6.1.2]. Then $\text{Nil}_{\text{reg}}(g_x) = \{O_{W'_x} \times O_{+, V'_x}, O_{W'_x} \times O_{-, V'_x}\}$. From (4.1.1),

$$G_x = T_x \times G'_x, \quad G''_x = T''_x \times T''_x, \quad G'_x = \text{SO}(W'_x) \times \text{SO}(V'_x)$$

with $T''_x$ being anisotropic torus from the definition of $\Gamma(G, H)$.

From Lemma 4.2.2, we can choose regular semi-simple elements $X^\pm_{V'_x} \in \mathfrak{so}(V'_x)$ with centralizer torus $T_{\text{cpt}, V'_x}$ such that for $\nu = \pm$,

$$\tilde{j}(O_{+, V'_x}, X^+_x) = -\tilde{j}(O_{-, V'_x}, X^-_x) = \nu \cdot \frac{|W_{T_{\text{cpt}, V'_x}}|}{2} \cdot D^{	ext{SO}(V'_x)}(X^+_x)^{1/2}. $$
Similarly, from [BP20, (3.4.7)], we may fix a regular semi-simple element $X_{qd,W'} \in \mathfrak{t}_{qd,W'} \subset \mathfrak{so}(W'_x)$ where $\mathfrak{t}_{qd,W'}$ is the Lie algebra of a fixed split torus $T_{qd,W'}$ in $SO(W'_x)$, such that
\[
\hat{j}(\mathcal{O}_{W'_x}, X_{qd,W'}) = |W_{T_{qd,W'}}| \cdot D^{SO(W'_x)}(X_{qd,W'})^{-1/2}.
\]

Fix $X^\pm_x = (X_{qd,W'}, X_{1/2}, X_{2/3}) \in \mathfrak{g}_x$ where $X_{g_k'}$ is an arbitrary fixed regular semisimple element in the Lie algebra of $G'_x$. After plugging $X^\pm_x$ into the limit formula (4.2.5) and using (4.2.4),
\[
\lim_{t \to \mathbb{R}^+} D^{G_x}(tX^\pm_x)^{1/2} \cdot \Theta(x \exp(tX^\pm_x)) = D^{G_x}(X^\pm_x)^{1/2} \cdot \{c_\Theta, \mathcal{O}_{W'_x} \times \mathcal{O}_{+,V'_x}, X^\pm_x\}
\]
\[
+ c_\Theta, \mathcal{O}_{W'_x} \times \mathcal{O}_{-,V'_x}(x) \cdot \hat{j}(\mathcal{O}_{W'_x} \times \mathcal{O}_{+,V'_x}, X^\pm_x)
\]
\[
= c_\Theta, \mathcal{O}_{W'_x} \times \mathcal{O}_{+,V'_x}(x) \cdot \pm \cdot \frac{|W_{T_{ctpt,V'_x}}| \cdot |W_{T_{qd,W'_x}}|}{2}
\]
\[
- c_\Theta, \mathcal{O}_{W'_x} \times \mathcal{O}_{-,V'_x}(x) \cdot \pm \cdot \frac{|W_{T_{ctpt,V'_x}}| \cdot |W_{T_{qd,W'_x}}|}{2}
\]

For convenience, write $D^{G_x}(tX^\pm_x)^{1/2} = D^{G_x}(tX_x)$. Set $T_{X_x} = T_{ctpt,V'_x} \times T_{qd,W'_x} \times G''_x$ so that $W_{T_{X_x}} = W_{T_{ctpt,V'_x}} \times W_{T_{qd,W'_x}}$. Then we eventually arrive at the following limit formula:
\[
\lim_{t \to \mathbb{R}^+} D^{G_x}(tX_x)^{1/2} \cdot \Theta(x \exp(tX_{qd,x})) = c_\Theta, \mathcal{O}_{W'_x} \times \mathcal{O}_{+,V'_x}(x) - c_\Theta, \mathcal{O}_{W'_x} \times \mathcal{O}_{-,V'_x}(x).
\]

But from (4.2.6),
\[
\frac{2}{|W_{T_{qd,x}}|} \cdot \lim_{t \to \mathbb{R}^2} D^{G_x}(tX_{qd,x})^{1/2} \cdot \Theta(x \exp(tX_{qd,x})) = c_\Theta, \mathcal{O}_{W'_x} \times \mathcal{O}_{+,V'_x}(x) + c_\Theta, \mathcal{O}_{W'_x} \times \mathcal{O}_{-,V'_x}(x).
\]

Therefore
\[
c_\Theta, \mathcal{O}_{W'_x} \times \mathcal{O}_{+,V'_x}(x) = \frac{1}{|W_{T_{qd,x}}|} \cdot \lim_{t \to \mathbb{R}^2} D^{G_x}(tX_{qd,x})^{1/2} \cdot \Theta(x \exp(tX_{qd,x}))
\]
\[
+ \frac{1}{2 \cdot |W_{T_{X_x}}|} \cdot \lim_{t \to \mathbb{R}^2} D^{G_x}(tX_x) \cdot \{\Theta(x \exp(tX^+)) - \Theta(x \exp(tX^-))\}.
\]

It follows that Part (2) has been proved. \qed

4.3. Stable variant. In this subsection, we recall the stable variant of geometric multiplicity introduced in [Wal12b, §3.2]. We follow the notation from subsection 4.1.

Recall that $x \in \Gamma(G, H) \subset \Gamma(H)$ if and only if $SO(W''_x)$ is an anisotropic torus and $G_x$ (equivalently $SO(W'_x)$) is quasi-split. In particular, following the parametrization as Part (1) of Theorem 3.1.5 (notice that $W''_x = \text{Im}(1 - x|W)$ is always of even dimension), $x$ corresponds uniquely to a pair $(\kappa'', c'') \in \Xi_{\text{reg}, W''_x}$. Moreover, following the same argument as [Luo21, Lem. 5.1.0.4], $I_{\kappa''} = I_{\kappa''}$, if and only if $SO(W''_x)$ is anisotropic.

Let $\mathcal{C}(V, W)$ be the set of pairs $(\kappa'', c'')$ satisfying the condition that there exists an orthogonal embedding $(W_{\kappa''}, c'', q_{\kappa'', c''}) \hookrightarrow (W, q_W)$ whose complement $W_{\kappa''}^{\perp}$ is quasi-split, and $I_{\kappa''} = I_{\kappa''}$. Similarly, set $\Xi(d_V, d_W) = \{\kappa'' \in \Xi_{\text{reg}} \mid I_{\kappa} = I_{\kappa}; 2 \cdot I_{\kappa} \leq \min\{d_W, d_V\}\}$ where $d_V$ (resp. $d_W$) is the dimension of $V$ (resp. $W$).

By Part (1) of Theorem 3.1.5 and the above discussion, any $(\kappa'', c'') \in \mathcal{C}(V, W)$ determines uniquely a pair $x_{\kappa'', c''}^\pm \in \Gamma(G, H)$. Up to a set of measure zero, there is a two-to-one map
\(\Gamma(G, H) \to \mathcal{C}(V, W)\). Similarly, there is a natural projection map \(\mathcal{C}(V, W) \to \Xi(d_V, d_W)\). Let us equip \(\mathcal{C}(V, W)\) and \(\Xi(d_V, d_W)\) with the push-forward measure. The following lemma, which characterizes the fiber of \(\mathcal{C}(V, W) \to \Xi(d_V, d_W)\), will be useful in the final proof of Theorem 2.3.2.

**Lemma 4.3.1.** The fiber over \(\kappa'' \in \Xi(d_V, d_W)\) under the natural map \(\mathcal{C}(V, W) \to \Xi(d_V, d_W)\) can be identified with

\[
\begin{cases}
    C(\kappa'') & 2 \cdot \dim W \text{ odd} \\
    C(\kappa'') & 2 \cdot \dim W \text{ even}
\end{cases}
\]

**Proof.** By definition, \((\kappa'', c'') \in \mathcal{C}(V, W)\) if and only if \((W_{\kappa'', c''}, q_{\kappa'', c''}) \hookrightarrow (W, q_W)\) with \(W_{\kappa'', c''}\) quasi-split. By (3.1.3), it happens if and only if

\[
2 \cdot \sum_{i \in I_{\kappa''}^*} c''_i = \Delta_{W_{\kappa'', c''}} = \Delta_W - \Delta_{W_{\kappa''}}.
\]

Let us first consider the case when \(\dim W\) (and hence \(\dim W_{\kappa'', c''}\)) is odd. In this case, \(\Delta_{W_{\kappa'', c''}} = \pm 1\) is equal to the discriminant of \(W_{\kappa', c'}\), defined in (2.2.1). Since \(I_{\kappa''}^* = I_{\kappa''}\), based on Lemma 3.1.1, the discriminant of \(W_{\kappa'', c''}\) is equal to \((-1)^{\dim W_{\kappa'', c''}}\) = (-1)^{|I_{\kappa''}|}, and the discriminant of \(W\) is equal to \((-1)^{\Delta_W}\). It follows that \(\Delta_{W_{\kappa'', c''}} = i_{W, \kappa''}\) and hence

\[
\sum_{i \in I_{\kappa''}^*} c''_i = \frac{\Delta_W - i_{W, \kappa''}}{2}.
\]

In particular the fiber is contained in \(C(\kappa'') \Delta_{W_{\kappa'', c''}}\). For the converse inclusion, it suffices to notice that \(\Delta_{W_{\kappa'', c''}} = \pm 1\) if and only if \(W_{\kappa'', c''}\) is quasi-split.

Now let us consider the case when \(\dim W\) is even. In this case, we can embed \((W_{\kappa'', c''}, q_{\kappa'', c''})\) into \((V, q_V)\), and by definition \((\kappa'', c'') \in \mathcal{C}(V, W)\) if and only if the orthogonal complement of \(W_{\kappa'', c''}\) in \(V\), which is of odd dimension, is quasi-split. Then following the same argument as the odd case,

\[
\sum_{i \in I_{\kappa''}^*} c''_i = \frac{\Delta_V - i_{V, \kappa''}}{2}
\]

and the result follows. \(\square\)

Combining Lemma 4.3.1 with the notation and the proof from Proposition 3.2.4, the following lemma can be established by tedious calculation. We omit the proof.

**Lemma 4.3.2.** Fix \(e_0 = \pm 1\).

1. When \(\dim W\) is odd, the fiber over \(\kappa'' \in \Xi(d_V, d_W)\) under the natural map

\[
\bigcup_{\alpha \in H^1(B, \text{SO}(W))} \mathcal{C}(V_\alpha, W_\alpha) \to \Xi(d_V, d_W)
\]

is given by

\[
C(\kappa'')^{e_0 \cdot W_{\kappa''}}.
\]
(2) When \( \dim W \) is even, fix an anisotropic line \( D \) with signature \( \text{sig}(D) \in \{\pm 1\} \). Then the fiber over \( \kappa'' \in \Xi(d_V,d_W) \) under the natural map

\[
\bigcup_{\alpha \in H^1( \mathbb{R}, \text{SO}(W \oplus D)) \setminus \epsilon(\text{SO}(W \oplus D)) = e_0} \mathcal{C}(V_\alpha, W_\alpha) \to \Xi(d_V,d_W)
\]

is given by

\[
\mathcal{C}(\kappa'')^e \epsilon_{V,W,\kappa'',D} = (-1)^{\frac{|I_{\kappa''}^*|}{2} - \frac{\dim W + \frac{1}{2}\dim \Pi(W,D) + \frac{1}{2} \dim \Pi(W,D) + \|W,D\|}{2}}.
\]

Let \( \Theta \) be a stable quasi-character on \( G(\mathbb{R}) \) in the sense of [BP20, §12.1]. In particular, either by [Luo21, (10.1.2)], or from Corollary 4.2.3 (using the fact that \( X_\pm x \) are stably conjugate to each other),

\[
c_\Theta(x) = |\text{Nil}_{\text{reg}}(g_x)|^{-1} \cdot \sum_{O \in \text{Nil}_{\text{reg}}(g_x)} c_{\Theta,O}(x), \quad x \in G_{\text{ss}}(\mathbb{R}).
\]

Following [Wal12b, §3.2], set

\[
(4.3.1) \quad n_{\text{geom}}^S(\Theta) = \int_{\kappa'' \in \Xi(d_V,d_W)} 2|I_{\kappa''}^*| \cdot D^G(\kappa'')^{1/2} c_{\Theta}(\kappa'') \Delta(\kappa'')^{-1/2} \, d\kappa''
\]

where \( D^G(\kappa'') = D^G(x_{\kappa'', c''}, \Delta(\kappa''), \Delta(x_{\kappa'', c''})) \) and

\[
c_{\Theta}(\kappa'') = \begin{cases} 
    c_{\Theta}(x_{\kappa'', c''}) = c_{\Theta}(x_{\kappa'', c''}^\pm) & (\kappa'', c'') \in \mathcal{C}(V,W) \text{ and } 2 \cdot |I_{\kappa''}^*| < \dim W \\
    \Theta(x_{\kappa'', c''}) & (\kappa'', c'') \in \mathcal{C}(V,W) \text{ and } 2 \cdot |I_{\kappa''}^*| = \dim W
  \end{cases}
\]

5. Reduction to endoscopic situation and the proof

In this section, we establish Theorem 2.3.2, and hence complete the proof of Conjecture 2.3.1. In subsection 5.1, we establish the structure theorem for tempered local \( L \)-parameters of special orthogonal groups over \( \mathbb{R} \). In subsection 5.2, based on Theorem 5.1.1, we deduce Theorem 2.3.2 by mathematical induction and Proposition 5.2.3, where the latter is analogous to [Wal12b, Prop. 3.3].

5.1. Classification of \( L \)-parameters. Let \( V \) be a non-degenerate quadratic space over \( \mathbb{R} \). In this subsection, we classify the tempered local \( L \)-parameters of \( \text{SO}(V) \).

**Theorem 5.1.1.** For every tempered local \( L \)-parameter \( \varphi_V \) of \( \text{SO}(V) \) with \( \text{std} \circ \varphi_V : \mathcal{L}_R \to \text{GL}(M_V) \), one of the following statements holds:

- (B) \( \dim V \leq 3 \);
- (P) There exists a decomposition \( \varphi_V = (\varphi_V^{GL}) \oplus \varphi_{V,0} \oplus (\varphi_V^{GL})^\vee \) such that \( \varphi_V^{GL} \neq 0 \);
- (E) \( \varphi_V \) is not of type (B) or (P), and \( S_{\varphi_V} \not\subset \text{Z}_{\text{GL}(M_V)} \cdot S_{\varphi_V}^{\circ} \).

**Remark 5.1.2.** In the above notions, (B) refers to the base case; (P) refers to the “parabolic induction” case, i.e. every representation in the Vogan \( L \)-packet arises from a non-trivial parabolic induction; (E) refers to the “endoscopic induction”, i.e. for \( s_V \in (S_{\varphi_V} \setminus \text{Z}_{\text{GL}(M_V)} \cdot S_{\varphi_V}^{\circ})/S_{\varphi_V}^{\circ} \), the elliptic endoscopic group attached to \( s_V \) is “smaller” than \( \text{SO}(V) \).
Based on Theorem 5.1.1, we can reduce the proof of Theorem 2.3.2 via parabolic and endoscopic induction to base case, which was known from [GP94, Prop. 7.4].

First, from [Kna94], the following lemma is known.

**Lemma 5.1.3.** For \( \varphi_V \) tempered with the decomposition \( \text{std} \circ \varphi_V = \bigoplus_i m_i \cdot M_{i,V} \) as (2.3.2), the following statements hold.

1. \( M_{i,V} \) is either of dimension one or two;
2. The only isomorphism classes of one-dimensional representations \( M_{i,V} \) are the trivial representation or sign representation.

**Proof.** From [Kna94, p.403], the first statement is clear. When \( \dim M_{i,V} = 1 \), \( M_{i,V} \) cannot be of GL-type. Hence \( M_{i,V} \simeq M_{i,V}^\vee \) and the second result follows. \( \square \)

**Lemma 5.1.4.** If \( \varphi_V \) is not of type (P), then all \( M_{i,V} \) are of O-type and \( m_{i,V} = 1 \).

**Proof.** Suppose that there exists an \( M_{i,V} \) of O-type with \( m_i \geq 2 \), or of Sp-type (in this case \( m_i \) is even and hence \( \geq 2 \)), or of GL-type. Let \( \varphi_{i,V} \) be the irreducible component of \( M_{i,V} \) in \( \varphi_V \), then there exists a decomposition

\[
\varphi_V = (\varphi_{V,i}) \oplus \varphi_{V,0} \oplus (\varphi_{V,i})^\vee
\]

for some \( \varphi_{V,0} \neq \varphi_V \). But then \( \varphi_V \) is of type (P), which is a contradiction. \( \square \)

Since \( \dim M_V \) is even, from Lemma 5.1.4, the following corollaries holds.

**Corollary 5.1.5.** If \( \varphi_V \) is not of type (P), one of the statements hold:

1. \( \dim M_{V,i} = 2 \) for any \( i \in I_O \);
2. There exists \( i_1, i_2 \in I_O \) such that \( M_{V,i_1} \) is the trivial representation, \( M_{V,i_2} \) is the sign representation, and \( \dim M_{V,i} = 2 \) for any \( i \in I_O \setminus \{i_1, i_2\} \).

**Corollary 5.1.6.** If \( \varphi_V \) is not of type (P), for every \( s_V \in S_{\varphi_V} \), the \( s_V \)-eigenspaces \( M_V^{s_V = 1} \) and \( M_V^{s_V = -1} \) are of even dimension.

**Proof.** Since \( \varphi_V \) is not of type (P), based on Lemma 5.1.4, Corollary 5.1.5 and (2.3.3), the centralizer of \( \varphi_V \) is of the following form

\[
S_{\varphi_V} \simeq \begin{cases} 
O(1, \mathbb{C})^{|I_O|}, & \text{dim } M_{V,i} = 2 \text{ for any } i \in I_O \\
O(1, \mathbb{C})^{|I_O| - 2} \times S(O(1, \mathbb{C}) \times O(1, \mathbb{C})), & \text{otherwise}
\end{cases}
\]

with each \( O(1, \mathbb{C}) = \{\pm 1\} \) corresponds to \( M_{i,V} \) of dimension 2, and \( S(O(1, \mathbb{C}) \times O(1, \mathbb{C})) = \{(1,1),(-1,-1)\} \) corresponds to trivial \( \oplus \) sgn. It follows that the corollary holds. \( \square \)

Now we are ready to prove Theorem 5.1.1. Suppose that \( \varphi_V \) is not of type (P) or (E), then

\[
S_{\varphi_V} \subset Z_{\text{GL}(M_V)} \cdot S_{\varphi_V}^0.
\]

But following (5.1.1), it happens only when \( \dim M_V \leq 2 \), from which we immediately deduce that \( \dim V \leq 3 \). It follows that we have established Theorem 5.1.1.
5.2. The proof. In this subsection, for an admissible pair \((W, V)\) over \(\mathbb{R}\), through mathematical induction on the dimension of \((W, V)\), we establish Theorem 2.3.2.

In Theorem 5.1.1, we have proved that any tempered local \(L\)-parameters of a special orthogonal group over \(\mathbb{R}\) are of type (B), (P) or (E). From [GP94, Prop. 7.4], Theorem 2.3.2 is known when \(\dim V \leq 3\). Therefore we can make the following induction hypothesis:

(HYP) Theorem 2.3.2 holds for any admissible pairs \((W', V')\) with the following conditions:

1. \(\dim V' < \dim V\), or
2. \(\dim V = \dim V'\) and \(\dim W < \dim W\).

Fix a tempered local \(L\)-parameter \(\varphi = \varphi_W \times \varphi_V\) of \(G = SO(W) \times SO(V)\). Based on (HYP) and Theorem 5.1.1, to prove Theorem 2.3.2, it suffices to treat the following cases:

(I) \(\varphi_V\) or \(\varphi_W\) is of type (P);
(II) \(\varphi_V\) is of type (E) and \(\varphi_W\) is not of type (P).

Case (I). Fix a tempered local \(L\)-parameter \(\varphi = \varphi_W \times \varphi_V\) of \(G = SO(W) \times SO(V)\). Assume that there exists a decomposition \(\varphi_V = (\varphi_{V}^{GL} + \varphi_{V}^{GL})_{\psi}\) and \(\varphi_W = (\varphi_{W}^{GL} + \varphi_{W}^{GL})_{\psi}\) such that at least one of the parameters \(\varphi_W^{GL}\), \(\varphi_W^{GL}\) is non-trivial. In this case, we can apply the induction hypothesis (HYP) to assume that Theorem 2.3.2 holds for the admissible pair \((W_0, V_0)\) or \((V_0, W_0)\), depending on \(\dim W_0 \leq \dim V_0\) or \(\dim V_0 \leq \dim W_0\).

Set \(\varphi_0 = \varphi_{W_0} \times \varphi_{V_0}\). Let \(i\) be the natural embedding from \(GL(M_{V_0}) \times GL(W_{W_0}) \hookrightarrow GL(M_V) \times GL(W_W)\). Then \(\varphi = i \circ \varphi_0\). Note that the embedding \(i\) induces an injective map

\[
i_{\text{Vogan}} : \Pi_{\text{Vogan}}(\varphi_0) \rightarrow \Pi_{\text{Vogan}}(\varphi).
\]

More precisely, by the local Langlands correspondence for general linear groups over \(\mathbb{R}\) ([Lan89]), let \(\sigma\) be the unique tempered representation in the local \(L\)-packet \(\Pi(\varphi_{W}^{GL}) \times \Pi(\varphi_{V}^{GL})\). Then for any \(\pi_0 \in \Pi_{\text{Vogan}}(\varphi_0)\),

\[
i_{\text{Vogan}}(\pi_0) = \pi_0 \times \sigma
\]

where \(\pi_0 \times \sigma\) is the normalized parabolic induction from the parabolic subgroup of \(SO(V) \times SO(W)\) with Levi \(SO(V_0) \times GL_{\dim \varphi_{V}^{GL}} \times (SO(W_0) \times GL_{\dim \varphi_{W}^{GL}})\). Since

\[
|\Pi_{\text{Vogan}}(\varphi_0)| = |S_{\varphi_0}| = |S_\varphi| = |\Pi_{\text{Vogan}}(\varphi)|,
\]

we deduce that \(i_{\text{Vogan}}\) provides a bijection between \(\Pi_{\text{Vogan}}(\varphi_0)\) and \(\Pi_{\text{Vogan}}(\varphi)\). By (2.3.4), the embedding \(i\) also induces an isomorphism of component groups

\[
i_{V_0} : S_{\varphi_{V_0}} \simeq S_{\varphi_V}, \quad i_{W_0} : S_{\varphi_{W_0}} \simeq S_{\varphi_W}.
\]

By [Luo21, Cor. 7.3.1],

\[
m(\pi_{\varphi_0}) = 1 \implies m(\pi_\varphi \times \sigma) = 1.
\]

Hence \(\pi_{\varphi} = \pi_{\varphi_0} \times \sigma = i_{\text{Vogan}}(\pi_{\varphi_0})\). Therefore, in order to accomplish the proof of Theorem 2.3.2 in Case (I), it suffices to show that

\[
(5.2.1) \quad \chi_\varphi \circ (i_{W_0} \times \iota_{V_0}) = \chi_{\varphi_0}.
\]

But it follows from the same argument as [GP92, p.988].
Case (II). Fix a tempered local L-parameter \( \varphi = \varphi_W \times \varphi_V \) of \( G = \text{SO}(W) \times \text{SO}(V) \). Assume that \( \varphi_V \) is of type (E) and \( \varphi_W \) is not of type (P). In this case, \( S_{\varphi_V} \not\subseteq Z_{GL(M_V)} \cdot S_{\varphi_V}^0 \). In particular \( S_{Z_V} := (Z_{GL(M_V)} \cdot S_{\varphi_V}^0 \cap S_{\varphi_V})/S_{\varphi_V}^0 \) is a non-trivial subgroup of \( S_{\varphi_V} \). Using the fact that both \( \chi_\varphi \) and \( \chi_{\pi_s} \) are characters of \( S_{\varphi} \), and the fact that \( S_{Z_V} \) can be identified as the subgroup \( \{(1,1,\ldots,1),(-1,-1,\ldots,-1)\} \) of \( S_{\varphi} \), based on [GP92, (10.4)] and [GP92, Prop. 10.5], we only need to prove

\[
\chi_{\pi_s} = \chi_\varphi \quad \text{on} \quad (S_{\varphi_V} \setminus S_{Z_V}) \times S_{\varphi_W}.
\]

Since there is a unique representation \( \pi_s \in \Pi_{\text{Vogan}}^{\text{rel}}(\varphi) \) with \( m(\pi) = 1 \), for any \( s \in S_{\varphi} \), we have the following identity

\[
\chi_{\pi_s}(s) = \sum_{\pi \in \Pi_{\text{Vogan}}^{\text{rel}}(\varphi)} \chi_\pi(s) \cdot m(\pi).
\]

It follows that to prove Theorem 2.3.2 in Case (II), we only need to prove the following identity for any \( s \in S_{\varphi} \),

\[
(5.2.2) \quad \sum_{\pi \in \Pi_{\text{Vogan}}^{\text{rel}}(\varphi)} \chi_\pi(s) \cdot m(\pi) = \chi_\varphi(s).
\]

Let us introduce the following notation.

**Definition 5.2.1.** Let \( V \) be a non-degenerate quadratic space over \( \mathbb{R} \). Let \( \varphi_V \) be a local L-parameter of \( \text{SO}(V) \). Set the following virtual representation for any \( s \in S_{\varphi_V} \),

\[
\Sigma_{V,\varphi_V}^s = \sum_{\pi \in \Pi_{\text{SO}(V)}^\text{rel}(\varphi_V)} \chi_\pi(s) \cdot \pi.
\]

When \( \varphi_V \) is clear, we abbreviate it as \( \Sigma_V^s \).

**Definition 5.2.2.** Let \((W, V)\) be an admissible pair over \( \mathbb{R} \) with Gross-Prasad triple \((G, H, \xi)\). Let \( \varphi = \varphi_W \times \varphi_V \) be a local L-parameter of \( G = \text{SO}(W) \times \text{SO}(V) \). Define

\[
m_{W,V,\varphi}^s = \sum_{\alpha \in H^1(\mathbb{R} \rtimes \text{SO}(W))} m(\Sigma_{W,\alpha}^{-1}\varphi_W, \Sigma_{V,\alpha}^{1}\varphi_V)
\]

Here

\[
m(\Sigma_{W,\alpha}^{-1}\varphi_W, \Sigma_{V,\alpha}^{1}\varphi_V) = \sum_{\pi \in \Pi_{\text{SO}(V)}^\text{rel}(\varphi)} \chi_\pi(-1\varphi_W, 1\varphi_V) \cdot m(\pi).
\]

When \( \varphi_\ast \) is clear, we abbreviate \( 1\varphi_\ast \) as \( 1 \).

For \( s_V \in S_{\varphi_V} \setminus S_{Z_V} \), from Corollary 5.1.6, the eigenspaces

\[
M_{V,\pm} = M_{V,\varphi_V}^{s_V \pm 1}, \quad M_{W,\pm} = M_{W,\varphi_W}^{s_W \pm 1}
\]

are both of even dimension. Following [Wal10a, §1.8], there exists an elliptic endoscopic group \( \text{SO}(V_+) \times \text{SO}(V_-) \) (resp. \( \text{SO}(W_+) \times \text{SO}(W_-) \)) of \( \text{SO}(V) \) (resp. \( \text{SO}(W) \)) attached to the eigenspace decomposition

\[
M_V = M_{V,\varphi_V} \oplus M_{V,-}, \quad \varphi_V = \varphi_{V,\varphi_V} \oplus \varphi_{V,-}, \quad \varphi_W = \varphi_{W,\varphi_W} \oplus \varphi_{W,-}.
\]

Moreover, since \( s_V \notin S_{Z_V} \), both \( \dim M_{V,\varphi_V} \) and \( \dim M_{V,-} \) are strictly smaller than \( \dim M_V \). It follows that

\[
\dim V_+ < \dim V, \quad \dim V_- < \dim V.
\]
Moreover, by the description of endoscopic groups ([Wal12b, §1.7]), up to permutation, 
\((W_\pm, V_\pm)\) are still admissible pairs.

It turns out that the following proposition holds.

**Proposition 5.2.3.** Fix an admissible pair \((W, V)\) over \(\mathbb{R}\) with Gross-Prasad triple \((G, H, \xi)\). Let \(\varphi = \varphi_W \times \varphi_V\) be a tempered local \(L\)-parameter of \(G\), and \(s = (s_V, s_W) \in S_\varphi = S_{\varphi_V} \times S_{\varphi_W}\). For \(e_0 \in \{\pm 1\}\), the following identity holds

\[
\sum_{\alpha \in H^1(\mathbb{R}, SO(W))} \left( e_0 \cdot m_{sW,V_+}^S \cdot m_{sW,V_-}^S \right) = \frac{1}{2} \left( e_0 \cdot m_{W_+,V_+}^S \cdot m_{W_+,V_-}^S + m_{W_+,V_-}^S \cdot m_{W_-,V_+}^S \right).
\]

Here

\[
m_{sW,V_+}^S = \sum_{\pi \in \Pi_{\text{rel}}(\varphi)} \chi_\pi(s) \cdot m(\pi), \quad m_{W_+,V_+}^S = m_{W_+,V_-}^S \cdot m_{W_-,V_+}^S.
\]

Let us postpone the proof of Proposition 5.2.3 to last subsubsection. Based on Proposition 5.2.3, we are ready to prove (5.2.2).

Adding (5.2.3) for \(e_0 = \pm 1\),

\[
\sum_{\alpha \in H^1(\mathbb{R}, SO(W))} m_{sW,V_+}^S \cdot m_{sW,V_-}^S = m_{W_+,V_+}^S \cdot m_{W_-,V_-}^S.
\]

Now LHS is equal to

\[
\sum_{\alpha \in H^1(\mathbb{R}, SO(W))} \sum_{\pi \in \Pi_{\text{rel}}(\varphi)} \chi_\pi(s) \cdot m(\pi) = \sum_{\pi \in \Pi_{\text{rel}}(\varphi)} \chi_\pi(s) \cdot m(\pi).
\]

Therefore in order to establish (5.2.2), it suffices to show that

\[
m_{W_+,V_-}^S \cdot m_{W_-,V_+}^S = \chi_\varphi(s).
\]

By Definition 5.2.2,

\[
m_{W_+,V_-}^S = \sum_{\pi \in \Pi_{\text{rel}}(\varphi_W \times \varphi_V)} \chi_\pi(-1, 1) \cdot m(\pi).
\]

It follows that by (HYP), we have

\[
m_{W_+,V_-}^S = \chi_{\varphi_W \times \varphi_V}(-1, 1), \quad m_{W_-,V_+}^S = \chi_{\varphi_W \times \varphi_V}(1, -1).
\]

Hence we only need to prove the following identity

\[
\chi_{\varphi_W \times \varphi_V}(-1, 1) \cdot \chi_{\varphi_W \times \varphi_V}(1, -1) = \chi_\varphi(s).
\]

From [GP92, (10.4)] and [GP92, Prop. 10.5], it suffices to show the following identity

\[
\chi_{\varphi_W \times \varphi_V}(1, -1) \cdot \chi_{\varphi_W \times \varphi_V}(1, -1) = \chi_\varphi(s).
\]

Now from (2.3.5),

\[
\chi_{\varphi_W \times \varphi_V}(1, -1) = \det(-\text{Id}_{M_{M_W}^V = -1}^{\dim M_{M_W}^V = 1}) \cdot \det(-\text{Id}_{M_{M_W}^V = -1}^{\dim M_{M_W}^V = 1})
\]

\[
\cdot \varepsilon\left(\frac{1}{2}, M_{M_W}^V = -1 \otimes M_{M_W}^V = 1, \psi\right)
\]
and
\[
\chi_{\varphi^W \times \varphi^V}(1, -1) = \det(-\text{Id}_{W^V}^{\dim M^W_V}) \cdot \det(-\text{Id}_{V^V}^{\dim M^W_V}) = \xi\left(\frac{1}{2}, M^W_V = -1, M^W_V = -1, \psi\right).
\]

Comparing with the definition of \(\chi_\varphi\), we only need to show the following identity
\[
\xi\left(\frac{1}{2}, M^W_V = -1, M^W_V = -1, \psi\right)^2 \cdot \det(-\text{Id}_{W^V}^{\dim M^W_V}) \cdot \det(-\text{Id}_{V^V}^{\dim M^W_V}) = 1.
\]

But this follows from the fact that \(\xi(... \in \{\pm 1\}\). It follows that we completed the proof of Theorem 2.3.2 in Case (II).

**Proof of Proposition 5.2.3.** Let \(\Theta_{\Sigma^W_V} = \sum_{\pi \in \Pi(\varphi(\varphi))} \chi_\pi(s_V) \cdot \Theta_\pi\) be the virtual distribution character attached to \(\Sigma^W_V\). Based on the work of Shelstad, the following statements hold.

(5.2.4) (1) From [She79a, Lem. 5.2], the distribution characters \(\Theta_{\Sigma^W_V}\) and \(\Theta_{\Sigma^W_V}\) are stable.

(2) From [She79b][She79a][She81][She08a][She08b], \(\epsilon(SO(V)) \cdot \Theta_{\Sigma^W_V}\) (resp. \(\epsilon(\text{SO}(V)) \cdot \Theta_{\Sigma^W_V}\)) is the endoscopic transfer (see [Wal12b, §1.6] for instance) of \(\Theta_{\Sigma_{W^V}^V} \times \Theta_{\Sigma_{V^V}^V}\) (resp. \(\Theta_{\Sigma_{W^V}^V} \times \Theta_{\Sigma_{V^V}^V}\)) for any \(\alpha \in H^1(\mathbb{R}, \text{SO}(V))\) (resp. \(H^1(\mathbb{R}, \text{SO}(W))\)).

The following proposition analogous to [Wal12b, Prop. 3.3] holds.

**Proposition 5.2.4.** For \(e_0 = \pm 1\),
\[
\sum_{\alpha \in H^1(\mathbb{R}, \text{SO}(W)) \atop \epsilon(G_\alpha) = e_0} m_{\text{geom}}(\Sigma^W, \Sigma^V) = \frac{1}{2}(e_0 \cdot m_{\text{geom}}(\Sigma^W, \Sigma^V) + m_{\text{geom}}(\Sigma^V, \Sigma^V)) + m_{\text{geom}}(\Sigma^W, \Sigma^V) \\
\]
Here
\[
m_{\text{geom}}(\Sigma^W, \Sigma^V) = m_{\text{geom}}(\Theta_{\Sigma^W_V} \times \Theta_{\Sigma^W_V})
\]
where the latter is defined in (4.3.1).

**Proof.** Based on (5.2.4), Lemma 4.3.1, Lemma 4.3.2 and Corollary 4.2.3, the proof of [Wal12b, Prop. 3.3] works verbatim. More precisely, let us for convenience assume that \(\dim W \) is odd, the other case is similar. By (4.1.4) and Lemma 4.3.2,
\[
\sum_{\alpha \in H^1(\mathbb{R}, \text{SO}(W)) \atop \epsilon(G_\alpha) = e_0} m_{\text{geom}}(\Sigma^W, \Sigma^V) = \int_{\Xi(dV, dW)} f_1, e_0(k'') \, dk''
\]
where
\[
f_1, e_0(k'') = \sum_{e'' \in C(k'')^{e_0} \cdot W, e''} C_{\Sigma^W_V}^{\Sigma^V}(x, e'') D^G(x, e'') \Delta(x, e'')^{-1/2}
\]
and \( c_{\Sigma_1^W, \Sigma_1^V} = c_{\Sigma_1^W} \times c_{\Sigma_1^V} \). Notice that from Remark 3.2.5, \( \epsilon_{W, \kappa''} \) depends only on \( \kappa'' \) and \( \epsilon_0 \). Similarly, let \( G_{\pm\pm} = SO(W_{\pm}) \times SO(V_{\pm}) \). By (4.3.1) and direct computation,

\[
\frac{1}{2} (\epsilon_0 \cdot m_{geom}^S (\Sigma_1^{W_+}, \Sigma_1^{V_+}) \cdot m_{geom}^S (\Sigma_1^{W_-}, \Sigma_1^{V_-}) + m_{geom}^S (\Sigma_1^{W_+}, \Sigma_1^{V_+}) \cdot m_{geom}^S (\Sigma_1^{W_-}, \Sigma_1^{V_-}))
\]

\[
= \int_{\Xi(d_V, d_W)} f_{2, \epsilon_0}(\kappa'') d\kappa''
\]

where \( f_{2, \epsilon_0}(\kappa'') \) is equal to

\[
2^{I_{\kappa''}^* \mid -1} \epsilon_0 \sum_{(I_1''', I_2'') \in I^+(\kappa'')} \left\{ c_{\Sigma_1^{W_+}, \Sigma_1^{V_+}} (\kappa''(I_1''')) \Delta G^{++} (\kappa''(I_1'''))^{1/2} (\kappa''(I_1'''))^{1/2} \right\}
\]

\[
+ 2^{I_{\kappa''}^* \mid -1} \sum_{(I_1''', I_2'') \in I^-(\kappa'')} \left\{ c_{\Sigma_1^{W_+}, \Sigma_1^{V_+}} (\kappa''(I_1''')) \Delta G^{+-} (\kappa''(I_1'''))^{1/2} (\kappa''(I_1'''))^{1/2} \right\}
\]

Here for \( \kappa'' \in \Xi(d_V, d_W) \), the pair \((I_1'', I_2'') \in I^+(\kappa'') \) (resp. \((I_1'', I_2'') \in I^-(\kappa'') \)) if and only if

- \( I_{\kappa''} = I_1'' \sqcup I_2'' \),
- \( \kappa''(I_1'') = (I_1'', (F_{\pm\pm})_{i \in I_1''}, (F_i)_{i \in I_1''}, (u_i)_{i \in I_1''}) \in \Xi(d_{V_+}, d_{W_+}) \) (resp. \( \Xi(d_{V_+}, d_{W_-}) \)) and
- \( \kappa''(I_2'') \in \Xi(d_{V_-}, d_{W_-}) \) (resp. \( \Xi(d_{V_-}, d_{W_+}) \)).

Hence it suffices to show that \( f_{1, \epsilon_0} = f_{2, \epsilon_0} \), which follows from [Wal12b, Prop. 3.3] verbatim. \( \Box \)

We are ready to deduce Proposition 5.2.3.

By Corollary 5.2.5 and Theorem 4.1.1, \( m(\Sigma_{W_{\alpha}, V_{\alpha}}) = m_{geom}(\Sigma_{W_{\alpha}, V_{\alpha}}) \). In Proposition 5.2.4, taking \( s_V = 1 \), \( s_W = -1 \), we get

\[
V_+ = V_{qs}, \quad V_- = 0, \quad W_+ = 0, \quad W_- = W_{qs}
\]

where \((W_{qs}, V_{qs})\) is introduced in Remark 2.2.1.

Therefore Proposition 5.2.4 becomes

\[
\sum_{\alpha \in H^1(\mathbb{R}, SO(W))} m_{geom}(\Sigma_1^{W_{\alpha}, V_{\alpha}}) = \frac{1}{2} (\epsilon_0 \cdot m_{geom}^S (0, \Sigma_1^{V_{\alpha}}) \cdot m_{geom}^S (\Sigma_1^{W_{\alpha}, 0}) + m_{geom}^S (\Sigma_1^{W_{\alpha}, \Sigma_1^{V_{\alpha}}}))
\]

Taking the sum over \( \epsilon_0 = \pm 1 \), and applying Theorem 4.1.1, we get the following corollary.

**Corollary 5.2.5.** The following identity holds

\[
m_{geom}^S (\Sigma_1^{W_{qs}, V_{qs}}) = \sum_{\alpha \in H^1(\mathbb{R}, SO(W))} m_{geom}(\Sigma_1^{W_{\alpha}, V_{\alpha}}) = m_{V_{qs}, W_{qs}}^S.
\]

It follows that Proposition 5.2.3 holds.
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