Early Universe Evolution in Graviton-Dilaton Models

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ABSTRACT. We present a class of graviton-dilaton models which leads to a singularity free evolution of the universe. We study the evolution of a homogeneous isotropic universe. We follow an approach which enables us to analyse the evolution and obtain its generic features even in the absence of explicit solutions, which are not possible in general. We describe the generic evolution of the universe and show, in particular, that it is singularity free in the present class of models. Such models may stand on their own as interesting models for singularity free cosmology, and may be studied accordingly. They may also arise from string theory. We discuss critically a few such possibilities.
1. Introduction

General Relativity (GR) is a beautiful theory and, more importantly, has been consistently successful in describing the observed universe. But, GR leads inevitably to singularities. In particular, our universe according to GR starts from a big bang singularity. We do not, however, understand the physics near the singularity nor know its resolution. The inevitability and the lack of understanding of the singularities point to a lacuna in our fundamental understanding of gravity itself. By the same token, a successful resolution of the singularities is likely to provide deep insights into gravity.

In order to resolve the singularities, it is necessary to go beyond GR, perhaps to a quantum theory of gravity. A leading candidate for such a theory is string theory. However, despite enormous progress \[1\], string theory has not yet resolved the singularities \[2\].

Perhaps, one may have to await further progress in string theory, or a quantum theory of gravity, to be able to resolve singularities. However, there is still an avenue left open - resolve the singularities, if possible, within the context of a generalised Brans-Dicke model, and then inquire whether such a model can arise from string theory, or a quantum theory of gravity.

Generalised Brans-Dicke model is special for more than one reason. It appears naturally in supergravity, Kaluza-Klein theories, and in all the known effective string actions. It is perhaps the most natural extension of GR \[3\], which may explain its ubiquitous appearance in fundamental theories: Dicke had discovered long ago the lack of observational evidence for the principle of strong equivalence in GR. Together with Brans, he then constructed the Brans-Dicke theory \[4\] which obeys all the principles of GR except that of strong equivalence. This theory, generalised in \[5\], has been applied in many cosmological and astrophysical contexts \[6\]-\[10\].

Brans-Dicke theory contains a graviton, a scalar (dilaton) coupled non minimally to the graviton, and a constant parameter $\omega$. GR is obtained when $\omega = \infty$. In the generalised Brans-Dicke theory, referred to as graviton-dilaton theory, $\omega$ is an arbitrary function of the dilaton \[3: 5\]. Hence, it includes an infinite number of models, one for every function $\omega$. Thus, very likely, it also includes all effective actions that may arise from string theory or a quantum theory of gravity.

This perspective logically suggests an avenue in resolving the singularities: to find, if possible, the class of graviton-dilaton models in which the evolution
of universe is singularity free, and then study whether such models can arise from string theory or a quantum theory of gravity. On the other hand, even if not deriveable from string theory or a quantum theory of gravity, such models may stand on their own as interesting models for singularity free cosmology. In either case, one may study further implications of these models in other cosmological and astrophysical contexts. Such studies are fruitful and are likely to lead to novel phenomena, providing valuable insights.

Accordingly, in this paper, we consider a class of graviton-dilaton models where the function $\omega$ satisfies certain constraints. These constraints were originally derived in [11] in a different context, and some of their generic cosmological and astrophysical consequences were explored in [11, 12]. Here we analyse the evolution of universe in these models which, as we will show, turns out to be singularity free. The essential points of the analysis and the results have been given in a previous letter [13].

We consider a homogeneous isotropic universe, such as our observed one, containing matter. The relevant equations of motion cannot, in general, be solved explicitly except in special cases [10, 12]. Hence, a different approach is needed for the general case which is valid for any matter and for any arbitrary function $\omega$, and which enables one to analyse the evolution and obtain its generic features even in the absence of explicit solutions.

We will present such an approach below. We first present a general analysis of the evolution and then apply it to describe in detail the evolution of universe in the present model. We show, in particular, that the constraints on $\omega$ ensure that the evolution is singularity free. The universe evolves with no big bang or any other singularity and the time continues indefinitely into the past and the future.

An important question to ask, from our perspective, is whether a function $\omega$ as required in the present model, can arise from a fundamental theory. We discuss critically a few such possibilities in string theory. On the other hand, the present model may stand on its own as an interesting model for singularity free cosmology and may be studied accordingly. We mention some issues for future study.

This paper is organised as follows. In section 2, we present our model. In section 3, we write its equations of motion in various forms, so as to facilitate our analysis. In section 4, we present the general analysis of the evolution. In section 5, we illustrate our method by applying the results of section 4 to describe the evolution of toy universes and show that their evolution is
singularity free in the present model. In section 6, we describe the evolution of our observed universe and show that its evolution is singularity free in the present model. In section 7, we discuss further generalisations of our model. In section 8, we give a brief summary, discuss critically a few possibilities of our model arising from string theory, and mention some issues for future study. In the Appendix, we derive the sufficiency conditions for the absence of singularities which are used in the paper.

2. Graviton-Dilaton Model

We consider the following graviton-dilaton action in ‘Einstein frame’:

\[ S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-\bar{g}} \left( -\bar{R} + \frac{1}{2}(\nabla \phi)^2 \right) + S_M(\mathcal{M}, e^{-\psi(\phi)} \bar{g}_{\mu\nu}) , \tag{1} \]

where \( G_N \) is the Newton’s constant, \( \phi \) is the dilaton and \( \psi(\phi) \) is an arbitrary function that characterises the theory. \( S_M \) is the action for matter fields, denoted collectively by \( \mathcal{M} \). They couple to the metric \( \bar{g}_{\mu\nu} \) minimally and to the dilaton \( \phi \) through the function \( \psi(\phi) \). This function cannot be gotten rid of by a redefinition of \( \bar{g}_{\mu\nu} \) except when the matter action \( S_M \) is conformally invariant - in that case, by definition, \( S_M(\mathcal{M}, e^{-\psi(\phi)} \bar{g}_{\mu\nu}) = S_M(\mathcal{M}, \bar{g}_{\mu\nu}) \) - which is assumed to be not the case here.

We can define a metric

\[ g_{\mu\nu} = e^{-\psi} \bar{g}_{\mu\nu} , \tag{2} \]

and write the action (1) in ‘Dicke frame’:

\[ S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} e^{\psi} \left( -R + \frac{1}{2} \left( 3\psi^2 - 1 \right) (\nabla \phi)^2 \right) + S_M(\mathcal{M}, g_{\mu\nu}) , \tag{3} \]

where \( \psi_{\phi} \equiv \frac{d\psi}{d\phi} \). If \( \psi \) has a finite upper bound \( \psi \leq \psi_{\text{max}} < \infty \), then the factor \( e^{\psi_{\text{max}}} \) can be absorbed into \( G_N \) and the range of \( \psi \) can be set to be \( \psi \leq 0 \). Also, we will work in the units where \( G_N = \frac{e^{\psi_{\text{max}}}}{8\pi} \).

Both forms of the action in (1) and (3) are, however, equivalent [4]. In ‘Einstein frame’, the matter fields feel, besides the gravitational force, the dilatonic force also which must be taken into account in obtaining the

\[ \bar{R}_{\mu\nu\lambda\tau} = \frac{1}{2} \frac{\partial^2 \bar{g}_{\lambda\tau}}{\partial x^\mu \partial x^\nu} + \cdots . \]
physical quantities. Whereas, in ‘Dicke frame’, the matter fields feel only the
gravitational force and, hence, the physical quantities are directly obtained
from the metric \( g_{\mu\nu} \). On the other hand, equations of motion are often easier
to solve if \( \bar{g}_{\mu\nu} \) is used.

Defining a new dilaton field \( \chi = e^{\psi} \), the action (3) can be written as

\[
S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left( -\chi R + \frac{\omega(\chi)}{\chi} (\nabla\chi)^2 \right) + S_M(M, g_{\mu\nu}),
\]

where \( \omega(\chi) \) is now the arbitrary function that characterises the theory. This
is the form of the action used more commonly and, hence, we will also use
it in this paper in most of what follows. Note that \( \chi \geq 0 \) since \( \chi = e^{\psi} \).
Furthermore, if \( \chi \) has a finite upper bound \( \chi \leq \chi_{\text{max}} < \infty \), then the factor
\( \chi_{\text{max}} \) can be absorbed into \( G_N \) and the range of \( \chi \) can be set to be \( 0 \leq \chi \leq 1 \).
Also, we will work in the units where \( G_N = \frac{\chi_{\text{max}}}{8\pi} \).

Let \( \Omega \equiv 2\omega + 3 \). Then, \( \phi \) and \( \psi(\phi) \) are related to \( \chi \) and \( \Omega(\chi) \) as follows:

\[
\Omega = \psi_\phi^{-2} \quad \text{or, equivalently,} \quad \phi = \int \frac{d\chi}{\chi} \sqrt{\Omega(\chi)}. \quad (5)
\]

Note that \( \Omega \) must be positive. If one starts from \( \phi \) and \( \psi(\phi) \), then the first
relation gives \( \Omega \) in terms of \( \phi \). To obtain \( \Omega \) in terms of \( \chi \), \( \psi(\phi) \) must be
inverted first to obtain \( \phi \) in terms of \( \psi \) and then \( \psi = \ln \chi \) must be used.
If one starts from \( \chi \) and \( \Omega(\chi) \) and uses \( \chi = e^{\psi} \), then the second relation
gives \( \phi \) in terms of \( \psi \) which must then be inverted to obtain \( \psi \) in terms of \( \phi \).
However, in the following, no inversion is necessary since no explicit forms
for \( \Omega(\chi) \) and \( \psi(\phi) \) will be used.

In the model we study here, the function \( \psi(\phi) \) is required to satisfy the
following constraints:

\[
\lim_{|\phi|\to\infty} \psi = -\frac{\ln |\phi|}{\sqrt{\Omega_0}}, \quad \Omega_0 \leq \frac{1}{3}
\]
\[d^n\psi \quad \text{finite} \quad \forall \ n \geq 1, \quad -\infty \leq \phi \leq \infty
\]
\[d\psi \quad \text{finite} \quad \phi = 0 \text{ only}
\]
\[\lim_{\phi \to 0} \psi(\phi) = -k\phi^{2n}, \quad n \geq 1, \quad (6)
\]
where $\Omega_0 > 0$ and $k > 0$ are constants and $n$ is an integer. The function $\psi(\phi)$ is otherwise arbitrary.

The first two constraints are important and were originally obtained in a different context in [11, 12], where some of their generic consequences were also explored. These constraints imply a finite upper bound on $\psi$, i.e. $\psi(\phi) \leq \psi_{\text{max}} < \infty$. Hence, the factor $e^{\psi_{\text{max}}}$ can be absorbed into $G_\infty$ and the range of $\psi$ can be set to be $\psi \leq 0$. Thus there is at least one critical point where $d\psi/d\phi = 0$, and $\psi = 0$.

The third constraint is not important, but assumed here for the sake of simplicity only. It implies that there is only one critical point, a maximum, of $\psi(\phi)$. As explained above, we can take $\psi_{\text{max}} = 0$. Thus, the range of $\psi$ is given by $-\infty \leq \psi \leq 0$ and, hence, that of $\chi = e^\psi$ by $0 \leq \chi \leq 1$. Also, by adding a suitable constant to $\phi$, we can take the maximum of $\psi(\phi)$ to occur at $\phi = 0$ with no loss of generality.

The fourth constraint follows from the Taylor expansion of $\psi(\phi)$ near its maximum at $\phi = 0$. Note, as follows from (5), that $\Omega \to \infty$ as $\phi \to 0$.

The constraints in (6), expressed equivalently in terms of $\chi$ and $\Omega(\chi)$, become

$$\begin{align*}
\Omega(0) &= \Omega_0 \leq \frac{1}{3} \\
\frac{d^n \Omega}{d\chi^n} &= \text{finite} \quad \forall \ n \geq 1, \quad 0 \leq \chi < 1 \\
\Omega &\to \infty \quad \text{at} \quad \chi = 1 \text{ only} \\
\lim_{\chi \to 1} \Omega &= \Omega_1 (1 - \chi)^{-2\alpha}, \quad 1/2 \leq \alpha < 1,
\end{align*}$$

where $\Omega_0 > 0$ and $\Omega_1 > 0$ are constants, and the range of $\alpha$ follows naturally from (6) and (7). We further assume that $\Omega(\chi)$ is a strictly increasing function of $\chi$. This constraint is not important, but assumed here for the sake of simplicity only. The function $\Omega(\chi)$ is otherwise arbitrary.

Note, for later reference, that $\Omega \to \infty$ and $\frac{1}{\Omega^2} \frac{d\Omega}{d\chi} = -\frac{2\alpha}{\Omega_1^2} (1 - \chi)^{4\alpha - 1}$ in the limit $\chi \to 1$. Hence, for the range of $\alpha$ given in (7), which follows naturally from (6), we have that $\frac{1}{\Omega^2} \frac{d\Omega}{d\chi} \to 0$ in the limit $\chi \to 1$. This ensures, as will become clear later, that the present model satisfies the observational constraints imposed by solar system experiments.

A vast class of functions $\Omega(\chi)$, equivalently $\psi(\phi)$, exists satisfying the above constraints. A simple example is $\Omega = \Omega_0 + \Omega_1 ((1 - \chi)^{-2\alpha} - 1)$. However,
no explicit form for $\Omega(\chi)$ will be used in the following.

3. Equations of Motion

In the following, we take the “matter” to be a perfect fluid with density $\rho$ and pressure $p$, related by $p = \gamma \rho$ where $-1 \leq \gamma \leq 1$. Note that the value of $\gamma$ indicates the nature of “matter”. For example, $\gamma = 0$, $\frac{1}{3}$, or $1$ indicates that the “matter” is dust, radiation, or massless scalar field respectively. Hence, we enclose “matter” within inverted commas where necessary in order to remind the reader that “matter” does not mean dust only. In the following, we study the evolution of a flat homogeneous isotropic universe. (The following analysis can, however, be extended to curved universes also and the main results remain unchanged.) The line element is given by

$$ds^2 = -dt^2 + e^{2A(t)}(dr^2 + r^2 dS_2^2),$$

(8)

where $e^A$ is the scale factor and $dS_2$ is the line element on an unit sphere, and the fields depend on the time coordinate $t$ only. The equation of motion for $A$, following from (4), is given by

$$\dot{A}^2 + \frac{\dot{A}\dot{\chi}}{\chi} - \frac{\omega \chi^2}{\chi^2} = \frac{\rho}{6\chi},$$

(9)

where upper dots denote $t$-derivatives. Solving equation (9) for $\dot{A}$, we obtain

$$\dot{A} = -\frac{\ddot{\chi}}{2\chi} + \epsilon \sqrt{\frac{\rho}{6\chi} + \frac{\Omega \chi^2}{12\chi^2}},$$

(10)

where $\epsilon = \pm 1$ and we have used $\Omega = 2\omega + 3$. The square roots in (10) and in the following are to be taken with a positive sign always. The remaining equations of motion, following from (4), are given by

$$\ddot{\chi} + 3\dot{A}\dot{\chi} + \frac{\dot{\Omega}\dot{\chi}}{2\Omega} = \frac{(1-3\gamma)\rho}{2\Omega}$$

(11)

and

$$\rho = \rho_0 e^{-3(1+\gamma)A},$$

(12)

where we have used $p = \gamma \rho$ and $\rho_0 \geq 0$ is a constant. Equations of motion following from (4) also contain the equation for $\ddot{A}$:

$$\ddot{A} = 2\ddot{\chi} + 3\dot{A}^2 + \frac{2\dot{A}\dot{\chi}}{\chi} + \frac{\omega \chi^2}{2\chi^2} + \frac{\ddot{\chi}}{\chi} + \frac{p}{2\chi} = 0.$$

(13)
In general, this equation follows from (9), (11), and (12) also. But there is a subtlety pointed out in [15]. What one obtains from (9), (11), and (12) is the following:

\[ \left( \dot{A} + \frac{\dot{\chi}}{2\chi} \right) A = 0 , \]  

where the expression for \( A \) is defined in (13). Equation (13) then follows if and only if \( \left( \dot{A} + \frac{\dot{\chi}}{2\chi} \right) \neq 0 \), i.e. if and only if \( e^A \) is not proportional to \( \chi^{-1/2} \). This subtlety should be kept in mind. However, as checked explicitly, the solutions presented in this paper all satisfy equations (9) - (13).

Equations (10) and (11) can also be written in different forms. Equation (11) can be integrated once to obtain

\[ \dot{\chi}(t) = \frac{e^{-3A}}{\sqrt{\Omega}} \left( \sigma(t) + c \right) \]  

where \( \sigma(t) = \frac{(1 - 3\gamma)\rho_0}{2} \int_{t_i}^{t} dt \frac{e^{-3\gamma A}}{\sqrt{\Omega}} \)  

and \( c = \dot{\chi}\sqrt{\Omega}e^{3A} \mid_{t=t_i} \) is a constant. An equation relating \( A \) and \( \chi \) can now be derived. Divide equation (10) by \( \dot{\chi} \chi \) and substitute (15) for \( \dot{\chi} \). We then have

\[ 2\chi \frac{dA}{d\chi} = -1 + \epsilon \text{sign}(\dot{\chi}) \sqrt{K} , \quad K \equiv \frac{\Omega}{3} \left( 1 + \frac{2\rho_0 \chi e^{3(1-\gamma)A}}{(\sigma(t) + c)^2} \right) . \]

All the above equations can be equivalently written in terms of \( \phi \) and \( \psi(\phi) \) also. For example, equations (10) and (11) become

\[ \dot{\phi} + 3\dot{A}\phi + \psi\dot{\phi} = \frac{(1 - 3\gamma)\rho}{2} \psi e^{-\psi} , \]  

\[ \dot{A} + \frac{\dot{\psi}}{2} = \epsilon \sqrt{\frac{\rho e^{-\psi}}{6} + \frac{\bar{\phi}^2}{12}} \]  

Although the derivation of (14) is straightforward, an outline of the steps involved is perhaps helpful. In equation (11), restore the \( \rho \)-dependence by replacing \((1 - 3\gamma)\rho \) by \((\rho - 3p)\). Now, differentiate (3) and use \( \dot{\rho} = -3\dot{A}(\rho + p) \). Then substitute (3) for \( \frac{\rho}{\chi} \), and (11) for \( \dot{\omega} \), equivalently \( \dot{\Omega} \). Again substitute (3) for \( \frac{\rho}{\chi} \). The result is equation (14) - (13).
while equation (15) and (16) become

\[ \dot{\phi}(t) = e^{-3A-\psi} (\sigma(t) + c) \]  (20)

where

\[ \sigma(t) = \frac{(1 - 3\gamma)\rho_0}{2} \int_{t_i}^t dt \, \psi e^{-3\gamma A} \]  (21)

and \( c = \dot{\phi} e^{3A+\psi} \mid_{t=t_i} \) is a constant.

We now make a few remarks. Firstly, under time reversal \( t \to -t, \epsilon \to -\epsilon \) in equations (10) and (18), whereas equations (11) and (19) are unchanged. Thus, evolution for \(-\epsilon\) is same as that for \(\epsilon\), but with the direction of time reversed.

Secondly, in general, there will appear in the solutions positive non zero arbitrary constant factors \( e^{A_0}, \rho_0, \) and \( \chi_0 \) in front of \( e^A, \rho, \) and \( \chi \) respectively. However, it follows from (10) and (11) that they can all be set equal to 1 with no loss of generality if time \( t \) is measured in units of \( \sqrt{\frac{\chi_0 e^{3(1+\gamma)A_0}}{\rho_0}} \). Therefore, in the solutions below we often assume that \( t \) is measured in appropriate units and, hence, set these constants equal to 1. Note, however, that the model dependent constants associated with \( \Omega(\chi) \), such as \( \Omega_0 \) or \( \Omega_1 \) in (7), cannot be set equal to 1.

Thirdly, if \( \rho = 0 \) or \( \gamma = \frac{1}{3} \) then \( \sigma(t) = 0 \) in (13). If it is also possible to perform the integrations given below then solutions to equations (10) and (15) can, in principle, be obtained in a closed form. To see this, define two new variables \( \bar{t} \) and \( \bar{\chi} \) as follows:

\[ d\bar{t} \equiv \sqrt{\chi} dt, \quad e^{\bar{A}} \equiv \sqrt{\chi} e^A. \]  (22)

Note that these variables appear naturally in ’Einstein frame’ given in (1) and (2). After some algebra, equations (14) and (17) become

\[ \frac{d\bar{A}}{dt} = \frac{e\epsilon e^{-3A}}{\sqrt{12}} \sqrt{1 + \frac{2\rho_0 e^{2A}}{e^2}} \]  (23)

\[ \frac{1}{\chi} \frac{d\chi}{dt} = \frac{e\epsilon e^{-3A}}{\sqrt{\Omega}} \]  (24)

from which it follows that

\[ \frac{\chi d\bar{A}}{d\chi} = \epsilon \sqrt{\frac{\Omega}{12}} \left(1 + \frac{2\rho_0 e^{2A}}{e^2}\right). \]  (25)
In the above equations $\epsilon = \pm 1$ as in (10) and $c$ is a constant as in (15).

It turns out that $\bar{A}$-integrations in equations (23) and (25) can be performed explicitly. Omitting the details, which are straightforward, the result is as follows. When $\rho$ and, hence, $\rho_0 = 0$:

$$e^{3\bar{A}} = \frac{ec\sqrt{3}}{2} \int d\bar{t}$$

$$A = \epsilon \int \frac{d\chi}{\chi} \sqrt{\frac{\Omega}{12}}.$$  \hspace{1cm} (26)

When $\rho \neq 0$, define $y \equiv \sqrt{\frac{2\rho_0}{c^2}} e^{\bar{A}}$. Then

$$y\sqrt{1 + y^2} - \ln(\sqrt{1 + y^2} + y) = \frac{ec}{\sqrt{3}} \left( \frac{2\rho_0}{c^2} \right)^{\frac{1}{2}} \int d\bar{t}$$

$$\ln y - \ln(\sqrt{1 + y^2} + 1) = \epsilon \int \frac{d\chi}{\chi} \sqrt{\frac{\Omega}{12}}.$$ \hspace{1cm} (28)

One thus has an explicit relation between $\bar{A}$ and $\bar{t}$, and also between $\bar{A}$ and $\chi$ if one can obtain $\int \frac{d\chi}{\chi} \sqrt{\Omega}$ in a closed form. Thus, one has $\chi(\bar{t})$. If one can obtain $\int \frac{d\bar{t}}{\sqrt{x}}$ also in a closed form then one has an explicit relation between $\bar{t}$ and $t$. Therefore, in principle, one has $A(t)$ and $\chi(t)$ in a closed form.

Evidently, solutions can be obtained thus in a closed form only in special cases. Even then the details of the required integrations and inversions of functions are likely to obscure the general features of the solutions. Such explicit solutions have been studied in [10] for a few specific functions $\Omega(\chi)$, and in [12] for any arbitrary function $\Omega(\chi)$ but with $\rho = 0$. In the present paper, however, we follow a different approach which is valid for any matter and for any arbitrary function $\Omega(\chi)$.

4. Analysis of the Evolution

Our main goal in this paper is to determine whether the evolution of the universe in the present model is singular or not. The task is trivial if one can solve the equations of motion (11) - (12) for arbitrary functions $\Omega(\chi)$ (or $\psi(\phi)$). However, explicit solutions can be obtained only in special cases as
described above. Hence, a different approach is needed for the general case which is valid for any matter and for any arbitrary function $\Omega(\chi)$, and which enables one to analyse the evolution and obtain its generic features even in the absence of explicit solutions.

We will present such an approach below. We analyse the evolution and obtain its generic features for the general case where matter and $\Omega(\chi)$ (or $\psi(\phi)$) are arbitrary. We will show, in particular, that when $\Omega(\chi)$ (or $\psi(\phi)$) satisfies the constraints in (7) (or (9)), the evolution of the universe, such as our observed one, is singularity free at all times.

To prove the presence of a singularity it suffices to show that any one of the curvature invariants, usually the curvature scalar, diverges. However, to prove the absence of a singularity, it is necessary to show that all curvature invariants are finite. As proved in the Appendix, a sufficient condition for all curvature invariants to be finite is that the quantities in (53) be all finite. Some or all of these quantities have a potential divergence when, for example, $e^A$ and/or $\chi$ vanish or when $\Omega \to \infty$. Hence, in the following analysis, we pay particular attention to these quantities, determining their finiteness or otherwise. We perform our analysis, as far as possible, in terms of $\chi$ and $\Omega(\chi)$. In the limit as $\chi \to 1$, however, it is sometimes necessary to perform the analysis in terms of $\phi$ and $\psi(\phi)$, as will become clear below.

To proceed, we need initial conditions at an initial time, say $t = t_i$. That is, we need $\dot{A}(t_i)$, $\dot{\chi}(t_i)$, and $\chi(t_i)$ or equivalently $\Omega(t_i)$. Here and in the following $\Omega(t)$ means $\Omega(\chi(t))$, i.e. the function $\Omega(\chi)$ evaluated at time $t$. Note that if $\dot{\chi}(t_i) = 0$ and $\chi(t_i) = 1$, then $\Omega(t_i) = \infty$ and the evolution is same as in Einstein’s theory. In particular, there is a singularity at a finite time in the past. Therefore, we assume that the initial conditions in the following refer always to generic values of $\dot{A}(t_i)$, $\dot{\chi}(t_i)$, and $\Omega(t_i)$ only, i.e. $\dot{A}(t_i)$ and $\dot{\chi}(t_i)$ are finite and non infinitesimal and $\Omega(t_i) < \infty$ in the following.

The analysis will be carried out for any arbitrary function $\Omega(\chi)$, satisfying only the constraints in (4). Hence, only the generic features of the evolution can be obtained but not the numerical details which require choosing a specific function for $\Omega(\chi)$. Therefore, precise numerical values for $A(t_i)$ and $\dot{\chi}(t_i)$ are not needed. Their signs alone are sufficient. We also need to know whether or not $\Omega(t_i) \lesssim 3$, and also the value of $\epsilon$ in (10). Thus, there are 16 sets of possible initial conditions: 2 each for the signs of $\dot{A}(t_i)$, $\dot{\chi}(t_i)$, and $\epsilon$, and 2 for whether $\Omega(t_i) \lesssim 3$ or not.
However, it is not necessary to analyse all of these 16 sets. The evolution for \(-\epsilon\) is same as that for \(\epsilon\), but with the direction of time reversed. Hence, only 8 sets need to be analysed. Now, let \(\epsilon = 1\). If \(\dot{\chi} < 0\) then \(\dot{A}\) cannot be negative for any value of \(\Omega\), see (10). Similarly, if \(\dot{\chi} > 0\) and \(\Omega > 3\) then also \(\dot{A}\) cannot be negative since 
\[
\sqrt{\frac{\rho}{6\chi} + \frac{12\Omega^2}{12\chi}} \geq \sqrt{\frac{\Omega}{3} \frac{\dot{\chi}}{2\chi}} > \frac{\dot{\chi}}{2\chi}
\]
in equation (10). Thus, there remain only 5 sets of possible initial conditions. We choose these sets so as to be of direct relevance to the evolution of observed universe and analyse each of them for \(t > t_i\). In the course of the analysis, equivalent forms of the equations of motion given in section 3 will become useful. We then use these results in sections 5 and 6 to describe the generic evolution of the universe.

Two remarks are now in order. First, the amount and the nature of the dominant “matter” in the universe are given by \(\rho_0\) and \(\gamma\) where \(-1 \leq \gamma \leq 1\). The later varies as the scale factor \(e^A\) of the universe evolves. If \(e^A\) is increasing, \(i.e.\) if \(\dot{A} > 0\), then the universe is expanding, eventually becoming dominated by “matter” with \(\gamma < \frac{1}{3}\), when such “matter” is present. For the observed universe, which is known to contain dust for which \(\gamma = 0\), one may take \(\gamma \leq 0\). Thus, by choosing \(t_i\) suitably we may assume, with no loss of generality, that \(\gamma < \frac{1}{3}\) (\(\leq 0\) for observed universe) if \(\dot{A} > 0\). Similarly, if \(\dot{A} < 0\) then we may assume, with no loss of generality, that \(\gamma \geq \frac{1}{3}\) when such “matter” is present. This is valid for the observed universe which is known to contain radiation for which \(\gamma = \frac{1}{3}\).

Second, consider \(\sigma(t)\) in (13) or (20). The integrand is positive and \(\propto \frac{1}{\sqrt{\Omega}}\). (The dependence on \(\rho_0\) can be absorbed into the unit of time, as explained in section 3.) Hence, the value of the integral is controlled by \(\Omega\). For example, in the limit \(\chi \rightarrow 1\), it is controlled by the constant \(\Omega_1\) in (7). We will use this fact in the following. Also, since \(t > t_i\), the sign of \(\sigma(t)\) is same as that of \((1 - 3\gamma)\). Moreover, \(\sigma(t) = 0\) if \(\rho_0 = 0\) or \(\gamma = \frac{1}{3}\) corresponding to an universe containing, respectively, no matter or radiation only.

4 a. \(\dot{A}(t_i) > 0, \ \dot{\chi}(t_i) > 0, \ \text{and} \ \Omega(t_i) > 3\)

These conditions imply that \(\epsilon = +1\) in (14) and the constant \(c > 0\) in (15). For \(t > t_i\), the scale factor \(e^A\) increases and, eventually, \(\gamma\) can be taken to be \(< \frac{1}{3}\) when such “matter” is present. In fact, \(\gamma \leq 0\) for the observed universe. Then, \((1 - 3\gamma) > 0\) and, hence, \(\sigma(t)\) increases. Thus, for \(t > t_i\), \(\dot{\chi} > 0\) and \(\chi\) and \(\Omega\) both increase. Since \(\Omega > 3\), it follows from equation
that $\dot{A} > 0$ for $t > t_i$ and, hence, $e^A$ increases. Eventually, $\chi \to 1$ and $\Omega \to \Omega_1(1 - \chi)^{-2\alpha}$.

In the limit as $\chi \to 1$, $(\sigma(t) + c) > 0$ is a constant to an excellent approximation. Then, equation (17) can be solved relating $\chi$ and $e^A$:

$$e^A = (\text{constant}) \left(1 - \chi\right)^{-\frac{2(1-\alpha)}{3(1-\gamma)}} .$$

Substituting this result in equation (15) then yields the unique solution in the limit $\chi \to 1$:

$$e^A = e^{A_0} t^{\frac{2}{3(1-\gamma)}}, \quad \chi = 1 - \chi_0 t^{\frac{4-\gamma}{(1+\gamma)(1-\alpha)}} ,$$

where $A_0$ and $\chi_0 > 0$ are constants and $t$ is measured in appropriate units. Since $\chi \to 1$, it follows from (31) that $t \to \infty$. It can now be seen that $(\sigma(t) + c) > 0$ is indeed a constant to an excellent approximation. Also, the quantities in (32) are all finite, implying that all the curvature invariants are finite. Thus, there is no singularity as $\chi \to 1$ and, thus, for $t > t_i$.

Note that the above relation between $\chi$ and $e^A$ and, hence, the solution (11) is not valid if $\rho = 0$ or $\gamma = 1$ i.e. if the universe contains no matter or “matter” with $\gamma = 1$ only. Although these cases are not relevant for the observed universe, we will now consider them for the sake of completeness. The results, however, are useful in describing the evolution for toy universes in section 5.

$\rho = 0$

$\rho = 0$ implies that $\rho_0 = 0$. In this case, $\sigma(t) = 0$ identically and the analysis was first given in [12], albeit differently. The dynamics turns out to be clear in terms of $\phi$ and $\psi(\phi)$. These variables, although equivalent to $\chi$ and $\Omega(\chi)$, are particularly well suited in describing the evolution near $\phi = 0$, or equivalently $\chi = 1$, where $\Omega \to \infty$. In fact, in this limit, it is not possible sometimes to perform the analysis in terms of $\chi$ and $\Omega(\chi)$.

At $t_i$, we have $\dot{A} > 0$, $\dot{\phi} > 0$ and, with our normalisation, $\phi < 0$, $\psi < 0$ and $\psi_\phi > 0$. Also, $\psi(\phi)$ has a maximum $\psi_{\text{max}} = 0$ at $\phi = 0$. The equations of motion (18) and (19) are now given by

$$\dot{A} = (-\sqrt{3}\psi_\phi + \text{sign}(\dot{\phi})) \frac{\dot{\phi}}{2\sqrt{3}}, \quad \dot{\phi} = ce^{-3A-\psi} ,$$

(32)
where $c > 0$. For $t > t_i$, the scale factor $e^A$ and $\phi$ continue to increase. Eventually, $\phi \to 0$. In this limit, $\psi \to 0$ and $\psi_\phi \to 0$. The solution to (32) is then given by

$$e^A = e^{A_0 t^{\frac{1}{3}}}, \quad \phi = c \ln t,$$

where $t$ is measured in appropriate units. From these expressions, it is clear that the field $\phi$ crosses the value 0 at $t = 1$ (in appropriate units) and becomes positive, while the scale factor $e^A$ remains finite and increasing. For $\phi > 0$, $\psi(\phi)$ decreases and $\psi_\phi < 0$. Equivalently, $\chi$ and $\Omega(\chi)$ both decrease. Thus, we now have $\dot{A} > 0$, $\dot{\chi} < 0$, and $\epsilon = 1$. Further evolution is analysed in section 4b.

Note the crossover at $\phi = 0$ where $\phi$ crosses 0 and becomes positive. In terms of $\chi$ and $\Omega(\chi)$, this amounts to the following: $\chi$ reaches 1, correspondingly $\Omega$ reaches $\infty$, and then $\chi$ “reflects” back and decreases, correspondingly $\Omega$ decreases. This crossover/reflection takes place in a finite time and with $e^A$ remaining finite during the process. It is this phenomenon of crossover that can be seen only in terms of $\phi$ and $\psi(\phi)$, and not $\chi$ and $\Omega(\chi)$.

$$\gamma = 1$$

In this case, the universe contains “matter” with $\gamma = 1$ only. Hence, $\rho = \rho_0 e^{-6A}$ and $\sigma(t)$ is negative. However, as $\chi \to 1$, $\sigma(t) \propto \frac{1}{\sqrt{\Omega_1}}$, see equation (1), and remains finite as can be verified a posteriori. Also, the magnitude of $\sigma(t)$ can be made as small as necessary by choosing the model dependent constant $\Omega_1$ sufficiently large. Hence, as $\chi \to 1$, the factor $(\sigma(t) + c)$ can be taken to remain positive and constant.

Now, $\Omega \chi^2 = (\text{constant}) e^{-6A}$ and its dependence on $e^A$ is same as that of $\rho$. Therefore, in the limit $\chi \to 1$ or equivalently $\Omega \to \infty$, the solution to (15) and (20) and, hence, the evolution turn out to be identical to the $\rho = 0$ case. In particular, the dynamics is clear in terms of $\phi$ and $\psi(\phi)$ only. The field $\phi$ crosses the value 0 (at $t = 1$ in appropriate units) and becomes positive, while the scale factor $e^A$ remains finite and increasing. This crossover cannot be seen in terms of $\chi$ and $\Omega(\chi)$.

For $\phi > 0$, $\psi(\phi)$ decreases and $\psi_\phi < 0$. Equivalently, $\chi$ and $\Omega(\chi)$ both decrease. Thus, we now have $\dot{A} > 0$, $\dot{\chi} < 0$, and $\epsilon = 1$. Further evolution is analysed in section 4b.

4 b. $\dot{A}(t_i) > 0, \dot{\chi}(t_i) < 0, \Omega(t_i) > 3$, and $\epsilon = 1$
The constant $c < 0$ in (15). For $t > t_i$, $\chi$ and $\Omega$ decrease and the scale factor $e^A$ increases. Equation (17) now becomes
\[ 2\chi \frac{dA}{d\chi} = -1 - \sqrt{K} , \]
where $K(t)$ is given in (17).

Consider first the case $\rho = 0$ or $\gamma = 1$, discussed in section 4a and which led to the present phase. Then, $\rho_0 (1 - 3\gamma) \leq 0$ and, therefore, $\sigma(t) \leq 0$. Thus, $(\sigma(t) + c) \leq c < 0$ and, hence, $\dot{\chi}(t)$ is negative and non zero, implying that $\chi(t)$ decreases for $t > t_i$. Also, $K(t)$ remains finite and it follows from equation (17) that $\frac{dA}{d\chi} < 0$. Thus, $\dot{A} > 0$ since $\dot{\chi} < 0$ and, hence, $e^A$ increases for $t > t_i$.

Eventually, as $t$ increases, $\chi \to 0$ and $e^A \to \infty$. Also, $\frac{2\rho \chi e^{3(1-\gamma)A}}{(\sigma(t)+c)^2} \ll 1$ in (17) since $\rho_0 = 0$ or $\gamma = 1$. Equation (17) can then be solved relating $\chi$ and $e^A$: 
\[ e^A = (\text{constant}) \chi^{-\frac{3+\sqrt{3}\Omega_0}{6}} , \]
where $\chi \to 0$ and $\Omega_0$ is given in (7). Substituting this result in equation (13) then yields the unique solution in the limit $\chi \to 0$:
\[ e^A = e^{A_0} (t - t_0)^n , \quad \chi = \chi_0(t - t_0)^m , \]
where $A_0$, $\chi_0$, and $t_0 > t_i$ are some constants and
\[ n = \frac{3 + \sqrt{3}\Omega_0}{3(1 + \sqrt{3}\Omega_0)} , \quad m = \frac{-2}{1 + \sqrt{3}\Omega_0} . \]

Note that $m < 0$. Then, since $\chi \to 0$, it follows from (13) that $t \to \infty$.

Thus, as $\chi \to 0$, $t \to \infty$ and $e^A \to \infty$. It can also be seen that the quantities in (13) are all finite for $t_i \leq t \leq \infty$, implying that all the curvature invariants are finite. Thus, there is no singularity for $t_i \leq t \leq \infty$.

Consider now the case where $\rho$ and $\gamma$ are non zero and have generic values. We have $\dot{A}(t_i) > 0$ and $\dot{\chi}(t_i) < 0$. Hence, $e^A$ increases and $\chi$ decreases. As $e^A$ increases, $\gamma$ can eventually be taken to be $< \frac{1}{3}$, when such “matter” is present. In fact, $\gamma \leq 0$ for the observed universe. In such cases where $\gamma \leq 0$, $\sigma(t)$ grows faster than $t$ as can be seen from (16). Then, as $t$ increases, the factor $(\sigma(t) + c)$ becomes positive.
This implies that \( \dot{\chi} \), initially negative, passes through zero and becomes positive. We then have \( \dot{A} > 0 \) and \( \dot{\chi} > 0 \). As \( t \) increases further, \( \chi \) and, hence, \( \Omega(\chi) \) increase. If \( \Omega > 3 \) then further evolution proceeds as described in section 4a. If \( \Omega < 3 \) then, upon making a time reversal, the initial conditions become identical to the one described in section 4e. The evolution in the present case then proceeds as in section 4d for the case which leads to the initial conditions of section 4e.

4 c. \( \dot{A}(t_i) < 0, \dot{\chi}(t_i) < 0, \text{and } \Omega(t_i) > 3 \)

These conditions imply that \( \epsilon = -1 \) in (10) and the constant \( c < 0 \) in (15). For \( t > t_i \), the scale factor \( e^A \) decreases and, eventually, \( \gamma \) can be taken to be \( \geq \frac{1}{3} \) when such “matter” is present. In fact, this is the case for the observed universe. Then \( (1 - 3\gamma) \leq 0 \) and \( \sigma(t) \) decreases or remains constant. Therefore, the factor \( \sigma(t) + c \leq c < 0 \) and, since \( e^A \) decreases, we have that \( \dot{\chi}(t) < \dot{\chi}(t_i) < 0 \) for \( t > t_i \). Thus \( \chi \) and, hence, \( \Omega \) decrease for \( t > t_i \). Also, \( \frac{dA}{d\chi} > 0 \) since \( \ddot{\chi} < 0 \) and \( \dot{A} < 0 \).

In (17), \( K(t_i) > 1 \) since \( \Omega > 3 \). From the behaviour of \( e^A \), \( \chi \), \( \Omega(\chi) \), and \( \sigma(t) + c \) described above, it follows that \( K \) decreases monotonically. The lowest value of \( K \) is \( \frac{\Omega_0}{3} \leq \frac{1}{9} \), achievable when \( \chi \) vanishes. This, together with \( K(t_i) > 1 \) and the monotonic behaviour of \( K(t) \), then implies that there exists a time, say \( t = t_m > t_i \) where \( K(t_m) = 1 \) with \( \chi(t_m) > 0 \). Hence, \( \frac{dA}{d\chi}(t_m) = 0 \). Therefore, \( A(t_m) = 0 \) since \( \dot{\chi}(t_m) \) is non zero. This is a critical point of \( e^A \) and is a minimum. Also, equation (17) can be written as

\[
A(t_i) - A(t_m) = \int_{\chi(t_m)}^{\chi(t_i)} \frac{d\chi}{2\chi} (-1 + \sqrt{K}) = \text{finite}, \tag{37}
\]

where the last equality follows because both the integrand and the interval of integration are finite. Therefore, \( A(t_m) \) is finite and \( > -\infty \) and, hence, \( e^{A(t_m)} \) is finite and non vanishing.

Thus, for \( t > t_i \), the scale factor \( e^A \) continues to decrease and reaches a non zero minimum at \( t = t_m \). The precise values of \( t_m, A(t_m), \chi(t_m), \text{and } \Omega(t_m) \) are model dependent. Hence, nothing further can be said about them.

\[3\text{The existence of such a non vanishing minimum of } e^A \text{ has also been deduced in the variable mass theories of } [6]. \text{ However, singularities may subsequently arise in these theories. See section 7 also.}\]
except, as follows from $K(t_m) = 1$, that $\Omega(t_m) \lessgtr 3$ in general and $= 3$ when $\rho_0 = 0$.

However, the above information suffices for our purposes. It can now be seen that the quantities in (53) are all finite, implying that all the curvature invariants are finite. Thus, there is no singularity for $t_i \leq t \leq t_m$.

For $t > t_m$, one has $\dot{A}(t) > 0$, $\dot{\chi}(t) < 0$, and $\Omega(\chi(t_m)) \lessgtr 3$ by continuity. Further evolution is analysed below.

\[4 \text{ d. } \dot{A}(t_i) > 0, \quad \dot{\chi}(t_i) < 0, \quad \Omega(t_i) \lessgtr 3, \text{ and } \epsilon = -1\]

This implies that the constant $c < 0$ in (15) and that $\frac{dA}{d\chi}(t_i) < 0$ in (17). Hence, $K(t_i) < 1$. For $t > t_i$, the scale factor $e^A$ increases and, eventually, $\gamma$ can be taken to be $< \frac{1}{3}$ when such “matter” is present. In fact, $\gamma \leq 0$ for the observed universe. Then, $(1 - 3\gamma) > 0$ and, hence, $\sigma(t)$ increases. Thus, $e^A$ is increases, $\chi$ decreases, and $(\sigma(t) + c)$ increases. Now, depending on $\rho_0$, $\gamma$, and the details of $\Omega(\chi)$, $K(t)$ in (17) may or may not remain $< 1$ for $t > t_i$.

Consider first the case where $K(t)$ remains $< 1$ for $t > t_i$. It then follows that $(\sigma(t) + c)$ cannot have a zero for $t > t_i$ and, since $c < 0$, must remain negative and non infinitesimal. Note that $\gamma$ must be $> 0$ for otherwise $\sigma(t)$ grows as fast as or faster than $t$, making $(\sigma(t) + c)$ vanish at some finite time. Let

$$\lim_{t \to \infty} (\sigma(t) + c) = c_e$$

where $c_e$ is a negative, non infinitesimal constant. We then have for $t_i \leq t \leq \infty$, $c \leq (\sigma(t) + c) \leq c_e < 0$ and, hence, $\dot{\chi}(t) < 0$. Therefore, eventually $\chi(t) \to 0$. Since $K(t) < 1$ and $\dot{\chi}(t) < 0$, it also follows from (17) that $\frac{dA}{d\chi} < 0$ and, hence, $\dot{A}(t) > 0$. We will now consider the limit $\chi \to 0$.

$K(t) < 1$ implies that as $\chi \to 0$, $K \to \frac{\Omega}{\frac{2}{3}} < 1$ where

$$\Omega_c \equiv \Omega_0 \left(1 + \frac{2\rho_0}{c_e^2} \lim_{\chi \to 0} \chi e^{3(1-\gamma)A}\right)$$

is a constant and $\Omega_0$ is given in (7). Now, as $\chi \to 0$, equation (17) can be solved relating $\chi$ and $e^A$:

$$e^A = (\text{constant}) \chi^{-\frac{2\Omegac}{\sqrt{3c}}}.$$
Note that \((3 - \sqrt{3\Omega_e}) > 0\) since \(\Omega_e < 3\). Hence, \(e^A \to \infty\) as \(\chi \to 0\).

Substituting \((40)\) in equation \((15)\) then yields the unique solution in the limit \(\chi \to 0\): For \(\Omega_e \neq \frac{1}{3}\),

\[
e^A = e^{A_0} (t_0 - \text{sign}(m)t)^n, \quad \chi = \chi_0 (t_0 - \text{sign}(m)t)^m,
\]

where \(t\) is measured in appropriate units, \(A_0, \chi_0, \) and \(t_0 > t_i\) are some constants, and

\[
n = \frac{3 - \sqrt{3\Omega_e}}{3(1 - \sqrt{3\Omega_e})}, \quad m = \frac{-2}{1 - \sqrt{3\Omega_e}}.
\]

For \(\Omega_e = \frac{1}{3}\),

\[
e^A = e^{A_0} e^{-\frac{c_c(t-t_0)}{\chi}}, \quad \chi = \chi_0 e^{c_c(t-t_0)},
\]

where \(c_c\) is defined in \((38)\).

Let \(\Omega_e > \frac{1}{3}\). Hence, \(m > 0\). Also, \(n < 0\) because \(\Omega_e < 3\). Since \(\chi \to 0\), it follows from \((11)\) that \(t \to t_0 > t_i\), which implies that \(\chi\) vanishes at a finite time \(t_0\). In this limit, the scale factor \(e^A \to \infty\) since \(n < 0\). The quantities in \((53)\), for example \(\frac{\dot{\chi}}{\chi}\), also diverge, implying that the curvature invariants, including the Ricci scalar, diverge. Thus, for \(\Omega_e > \frac{1}{3}\), there is a singularity at a finite time \(t_0\).

Let \(\Omega_e \leq \frac{1}{3}\), which is the case in our model, see \((7)\). When \(\Omega_e < \frac{1}{3}\), \(m < 0\). Also, \(n > 0\) because \(\Omega_e < 3\). Since \(\chi \to 0\), it follows from \((11)\) that \(t \to \infty\). In this limit, the scale factor \(e^A \to \infty\) since \(n > 0\) now. The same result holds also for the solution in \((43)\) with \(\Omega_e = \frac{1}{3}\).

For \(K(t)\) to be \(< 1\), \((\sigma(t) + c)\) must not be too small. Since \(c < 0\) and \(\sigma(t) > 0\), it necessarily implies that \(\sigma(t)\) must remain finite. It follows from the above solutions that \(\sigma(t)\) can remain finite only if \(\sqrt{3\Omega_e} \leq \frac{1-\gamma}{1-\gamma}\), equivalently, \(\gamma > \frac{1-\sqrt{3\Omega_e}}{3-\sqrt{3\Omega_e}}\). Under this condition, it can be checked easily that

\[\lim_{\chi \to 0} \chi e^{3(1-\gamma)A} = 0.\]

Hence, \(\Omega_e = \Omega_0\) as follows from \((39)\).

It can now be seen, for \(\Omega_e = \Omega_0 \leq \frac{1}{3}\), that the quantities in \((53)\) are all finite for \(t_i \leq t \leq \infty\), implying that all the curvature invariants are finite. Thus, there is no singularity for \(t_i \leq t \leq \infty\) when \(\Omega_0 \leq \frac{1}{3}\).

Consider the second case where \(K(t)\) in \((17)\) may not remain \(< 1\) for all \(t > t_i\). This is the case for the observed universe where \(\gamma \leq 0\) and, hence, \(\sigma(t)\) grows as fast as or faster than \(t\). Then there is a time, say \(t = t_1\), when
the value of $K = 1$. It follows that $c < (\sigma(t_1) + c) < 0$ and, hence, $\chi(t) < 0$ for $t_i \leq t \leq t_1$. Also, $\chi(t)$ is non-vanishing since the case of $\chi(t_1) \to 0$ is same as the case described above. Therefore, we have that $\frac{dA}{d\chi}(t_1) = 0$ which implies, since $\dot{\chi}(t_1) \neq 0$, that $\dot{A}(t_1) = 0$. This is a critical point of $e^A$ and is a maximum.

For $t > t_1$, $K(t)$ becomes $> 1$ and, hence, $\dot{A}(t) < 0$. Also, $\dot{\chi} < 0$ and $\Omega(t) < 3$. Then $\epsilon = -1$ necessarily. Further evolution is analysed below.

4 e. $\dot{A}(t_i) < 0$, $\dot{\chi}(t_i) < 0$, and $\Omega(t_i) < 3$

The above conditions imply that $\epsilon = -1$, the constant $c < 0$ in (15), and $\frac{dA}{d\chi}(t_i) > 0$. Hence, $K(t_i) > 1$. For $t > t_i$, $(\sigma(t) + c)$ which is negative at $t_i$ may or may not vanish for non zero $\chi$.

Consider the case where $(\sigma(t) + c)$ does not vanish and remains negative for non zero $\chi$. Therefore, $\dot{\chi}(t) < 0$ and, hence, $\chi(t)$ decreases. For $t > t_i$, the scale factor $e^A$ decreases and, eventually, $\gamma$ can be taken to be $\geq \frac{1}{3}$ when such “matter” is present. In fact, this is the case for the observed universe. Then, $(1 - 3\gamma) < 0$ and $\sigma(t)$ decreases. Hence, $(\sigma(t) + c)$ also decreases.

It follows from (17) that $K$, initially $> 1$, is now decreasing since $\chi$ and $e^A$ are decreasing and $(\sigma(t) + c)^2$ is increasing. Its lowest value $= \frac{\Omega_0}{3} \leq \frac{1}{9}$, achievable at $\chi = 0$. Therefore, there exists a time, say $t = t_M$, where $K = 1$ and, hence, $\frac{dA}{d\chi} = 0$. Clearly, $\chi(t_M)$ and $\dot{\chi}(t_M)$ are non zero for the same reasons as in section 4c, implying that $\dot{A}(t_M) = 0$. This is a critical point of $e^A$ and is a minimum.

The existence of this critical point of $e^A$ can be seen in another way also. As $(\sigma(t) + c) < 0$ grows in magnitude, it follows from equation (15) that $\Omega \chi^2 \propto e^{-6A}$, whereas $\rho$ is given by (12). Note that $\dot{\chi}(t) < 0$ and, hence, $\chi(t)$ decreases and eventually $\to 0$. Then, in the limit $\chi \to 0$, the $\frac{\Omega_0}{\chi^2}$ term in equation (10) dominates $2\chi$ term. Hence, in this limit where $\Omega \to \Omega_0 \leq \frac{1}{3} < 3$, we have that $\dot{A} > 0$. Since $\dot{A} < 0$ initially, this implies the existence of a zero of $\dot{A}$. This is a critical point of $e^A$ and is a minimum.

Note that all quantities remain finite for $t_i \leq t \leq t_M$. In particular, the quantities in (53) are all finite, implying that all the curvature invariants are finite. Thus, there is no singularity for $t_i \leq t \leq t_M$.

For $t > t_M$, we have $\dot{A} > 0$, $\dot{\chi} < 0$, $\Omega < 3$, and $\epsilon = -1$. Further evolution then proceeds as described in section 4d. The evolution can thus become repetitive and oscillatory.
Consider now the case where \((\sigma(t) + c)\) vanishes, say at \(t = t_m'\), for non-zero \(\chi\). Hence, \(\dot{\chi}(t_m')\) vanishes and this is a minimum of \(\chi\). For \(t > t_m'\), \((\sigma(t) + c)\) and \(\dot{\chi}\) are positive by continuity and, hence, \(\chi\) increases. Also, \(\dot{A} < 0\) and the scale factor \(e^A\) decreases. Eventually, \(\gamma\) can be taken to be \(\geq \frac{1}{3}\) when such “matter” is present. In fact, this is the case for the observed universe. Then, \((1 - 3\gamma) < 0\) and \(\sigma(t)\) decreases. Hence, \((\sigma(t) + c)\) also decreases.

Now, as \(\chi\) increases and \(\to 1\), \((\sigma(t) + c)\) may or may not vanish with \(\chi < 1\). If \((\sigma(t) + c)\) vanishes with \(\chi < 1\), then \(\dot{\chi}\) vanishes. This is a critical point of \(\chi\) and is a maximum, beyond which we have \(\dot{A} < 0\), \(\dot{\chi} < 0\), and \(\epsilon = -1\). Further evolution then proceeds as described in section 4c if \(\Omega > 3\), and as in section 4e if \(\Omega < 3\). The evolution can thus become repetitive and oscillatory.

If \((\sigma(t) + c)\) does not vanish and remain positive for \(\chi \leq 1\) then, eventually, \(\chi \to 1\). It follows from equation (15) that \(\Omega \dot{\chi}^2 \propto e^{-6A}\), whereas \(\rho\) is given by (12). Note that \(\dot{A} < 0\) and \(e^A\) is decreasing. Then, in this limit, the \(\frac{\Omega \dot{\chi}^2}{\chi^2}\) term in equation (14) dominates or, if \(\gamma = 1\), is of the same order as the \(\frac{\dot{A}}{\chi}\) term.

Thus we have \(\dot{A} < 0\), \(\dot{\chi} > 0\), \(\chi \to 1\), equivalently \(\Omega \to \infty\), and \(\epsilon = -1\). Also, \(\frac{\Omega \dot{\chi}^2}{\chi^2}\) term in equation (14) dominates or, if \(\gamma = 1\), is of the same order as the \(\frac{\dot{A}}{\chi}\) term. But this is precisely the time reversed version of the evolution analysed in section 4b and in section 4a for the case which led to the initial conditions of section 4a, where the dynamics is clear in terms of \(\phi\) and \(\psi(\phi)\). Applying the results of sections 4a and 4b, it follows that \(\phi\) will cross the value 0 after which \(e^\psi\) and, hence, \(\chi\) begins to decrease. We then have \(\dot{A} < 0\), \(\dot{\chi} < 0\), and \(\Omega > 3\). Also, \(\epsilon = -1\) necessarily. Further evolution then proceeds as described in section 4c. The evolution can thus become repetitive and oscillatory.

### 5. Evolution of toy Universes

We now use the results of the analysis in section 4 to describe the generic evolution of universe in the present model. For the purpose of illustration we first describe in this section the generic evolution of three toy universes: One, where the universe contains no matter, \(i.e.\ \rho = 0\). Two, where the universe contains radiation only, \(i.e.\ \rho \neq 0\) but \(\gamma = \frac{1}{3}\). And three, where the universe contains massless scalar field only, \(i.e.\ \rho \neq 0\) but \(\gamma = 1\). Clearly, however,
none of these toy models are realistic. The generic evolution of a realistic universe, such as our observed one, will be described in the next section.

Note that $\sigma(t) = 0$ in the first two models. As described in section 3, it may then be possible to obtain closed form solutions, although only in special cases [10, 12]. Moreover, the details of the specific calculations obscure the general features of the solutions. Hence, and also for the purpose of illustration, we describe the evolution of the toy universes within the framework of our analysis. However, the explicit evolutions described in [10, 12] for specific cases all conform to the ones described by the present general analysis.

We first describe the evolution for $t > t_i$ taking, as initial conditions $\dot{A}(t_i) > 0$, $\dot{\chi}(t_i) > 0$, and $\Omega(t_i) > 3$. Then, $\epsilon = 1$ necessarily. To describe the evolution for $t < t_i$, we reverse the direction of time and take, as initial conditions, $\dot{A}(t_i) < 0$, $\dot{\chi}(t_i) < 0$, and $\Omega(t_i) > 3$. Then, $\epsilon = -1$ necessarily. The required evolution is that for $t > t_i$ in terms of the reversed time variable, which is also denoted as $t$. The results of section 4 can then be applied directly.

When the initial conditions are different, the evolution can again be analysed along similar lines as follows. However, the main result that the evolution is singularity free remains unchanged.

5 a. $\rho = 0, \ t > t_i$

The evolution for this case was first described in [12] where explicit solutions to the equations of motion were obtained for any arbitrary function $\Omega(\chi)$ in terms of the variables defined in (22). We will now describe this evolution within the framework of the analysis presented in section 4 which is valid for any matter and for any arbitrary function $\Omega(\chi)$.

Initially, we have $A(t_i) > 0$, $\dot{\chi}(t_i) > 0$, and $\Omega(t_i) > 3$. Then, $\epsilon = 1$ necessarily. For $t > t_i$, the evolution proceeds as described in section 4a. Both $e^A$ and $\chi$ increase. Hence, $\Omega$ increases. Eventually, $\chi \to 1$ and $\Omega \to \infty$. The dynamics near $\chi = 1$, equivalently $\phi = 0$ in our normalisation, is clear in terms of $\phi$ and $\psi(\phi)$. It follows from the analysis of section 4a that $\phi$ crosses 0 and continues to increase. Then, $\psi(\phi)$ and, hence, $\Omega$ decrease.

We then have $\dot{A} > 0$, but $\ddot{\chi} < 0$. The evolution then proceeds as described in section 4b. $e^A$ increases whereas $\chi$ and, hence, $\Omega$ decrease. As $t \to \infty$, $e^A$ and $\chi$ evolve as given in (35) and (36).
In particular, it can be seen that the quantities in (53) are all finite for $t_i \leq t \leq \infty$, implying that all the curvature invariants are finite. Hence, the evolution is singularity free.

$t < t_i$, equivalently $(-t) > (-t_i)$

To describe the evolution for $t < t_i$, we reverse the direction of time. Then, in terms of the reversed time variable also denoted as $t$, we have $\dot{A}(t_i) < 0$, $\dot{\chi}(t_i) < 0$, and $\Omega(t_i) > 3$. Then, $\epsilon = -1$ necessarily. For $t > t_i$, the evolution proceeds as described in section 4c. Both $e^A$ and $\chi$ decrease. Hence, $\Omega$ decreases. Then, as shown in section 4c, there exists a time, say $t = t_m > t_i$ where $K(t_m) = 1$ in equation (47), $e^A(t_m) > 0$, $\chi(t_m) > 0$, and $\dot{\chi}(t_m) < 0$. Note that, in this case where $\rho$ and, hence, $\rho_0$ vanish, we have $\Omega(t_m) = 3$.

As shown in section 4c, $K(t_m) = 1$ implies that $\dot{A}(t_m) = 0$. Hence, the scale factor $e^A$ reaches a minimum. For $t > t_m$, we then have $\dot{A} > 0$, $\dot{\chi} < 0$, $\Omega < 3$, and $\epsilon = -1$. The evolution proceeds as described in section 4d.

It follows that $\sigma(t) = 0$ since $\rho$ and, hence, $\rho_0$ vanishes. Hence, in equation (38), $c_e = c$ which is negative and non infinitesimal. Therefore, as described in section 4d, $e^A$ increases whereas $\chi$ and, hence, $\Omega$ decrease. As $t \to \infty$, $e^A$ and $\chi$ evolve as given in (41) - (43).

In particular, it can be seen that, for $\Omega_0 \leq \frac{1}{3}$, the quantities in (53) are all finite for $t_i \leq t \leq \infty$, implying that all the curvature invariants are finite. Hence, the evolution is singularity free.

Thus, the evolution of the toy universe with $\rho = 0$ and with the initial conditions $\dot{A}(t_i) > 0$, $\dot{\chi}(t_i) > 0$, and $\Omega(t_i) > 3$ proceeds in the present model as follows. For $t > t_i$, the scale factor $e^A$ increases continuously to $\infty$. The field $\chi$ increases, reaches the value 1, and then decreases continuously to 0. Correspondingly, $\Omega$ increases, reaches $\infty$, and then decreases continuously to $\Omega_0$, given in (7).

For $t < t_i$, the scale factor $e^A$ decreases, reaches a non zero minimum, and then increases continuously to $\infty$. The field $\chi$ decreases continuously to 0. Correspondingly, $\Omega$ decreases continuously to $\Omega_0$, given in (4).

Also, the curvature invariants all remain finite for $-\infty \leq t \leq \infty$. Hence, in the present model, the evolution of the toy universe with $\rho = 0$ is singularity free.
When the initial conditions are different, the evolution can again be analysed along similar lines. However, the main result that the evolution is singularity free remains unchanged.

5 b. \( \gamma = \frac{1}{3}, \ t > t_i \)

The evolution for this case was first described in [10] where explicit solutions to the equations of motion were obtained for a few specific functions \( \Omega(\chi) \). We will now describe this evolution within the framework of the analysis presented in section 4 which is valid for any matter and for any arbitrary function \( \Omega(\chi) \).

Initially, we have \( \dot{A}(t_i) > 0, \ \dot{\chi}(t_i) > 0, \) and \( \Omega(t_i) > 3. \) Then, \( \epsilon = 1 \) necessarily. For \( t > t_i, \) the evolution proceeds as described in section 4a. Both \( e^A \) and \( \chi \) increase. Hence, \( \Omega \) increases. As \( t \to \infty, \ e^A \) and \( \chi \) evolve as given in (53) where \( \gamma = \frac{1}{3} \) now.

In particular, it can be seen that the quantities in (53) are all finite for \( t_i \leq t \leq \infty, \) implying that all the curvature invariants are finite. Hence, the evolution is singularity free.

\( t < t_i, \) equivalently \( (\text{\textit{t}}) > (\text{\textit{t}}) \)

To describe the evolution for \( t < t_i, \) we reverse the direction of time. Then, in terms of the reversed time variable also denoted as \( t, \) we have \( \dot{A}(t) < 0, \ \dot{\chi}(t) < 0, \) and \( \Omega(t) > 3. \) Then, \( \epsilon = -1 \) necessarily. For \( t > t_i, \) the evolution proceeds as described in section 4c. Both \( e^A \) and \( \chi \) decrease. Hence, \( \Omega \) decreases. Then, as shown in section 4c, there exists a time, say \( t = t_m > t_i \) where \( K(t_m) = 1 \) in equation (17), \( e^{A(t_m)} > 0, \chi(t_m) > 0, \) and \( \dot{\chi}(t_m) < 0. \) Note that \( \Omega(t_m) < 3. \)

As shown in section 4c, \( K(t_m) = 1 \) implies that \( \dot{A}(t_m) = 0. \) Hence, the scale factor \( e^A \) reaches a minimum. For \( t > t_m, \) we then have \( \dot{A} > 0, \ \dot{\chi} < 0, \) \( \Omega < 3, \) and \( \epsilon = -1. \) The evolution proceeds as described in section 4d.

It follows that \( \sigma(t) = 0 \) since \( \gamma = \frac{1}{3}. \) Hence, in equation (53), \( c_e = c \) which is negative and non infinitesimal. Therefore, as described in section 4d, \( e^A \) increases whereas \( \chi \) and, hence, \( \Omega \) decrease. As \( t \to \infty, \ e^A \) and \( \chi \) evolve as given in (41) - (43).

In particular, it can be seen that, for \( \Omega_0 \leq \frac{1}{3}, \) the quantities in (53) are all finite for \( t_i \leq t \leq \infty, \) implying that all the curvature invariants are finite. Hence, the evolution is singularity free.
Thus, the evolution of the toy universe containing only radiation, i.e. \( \rho \neq 0 \) but \( \gamma = \frac{1}{3} \), and with the initial conditions \( \dot{A}(t_i) > 0 \), \( \dot{\chi}(t_i) > 0 \), and \( \Omega(t_i) > 3 \) proceeds in the present model as follows. For \( t > t_i \), the scale factor \( e^A \) increases continuously to \( \infty \). The field \( \chi \) increases continuously to 1. Correspondingly, \( \Omega \) increases continuously to \( \infty \).

For \( t < t_i \), the scale factor \( e^A \) decreases, reaches a non zero minimum, and then increases continuously to \( \infty \). The field \( \chi \) decreases continuously to 0. Correspondingly, \( \Omega \) decreases continuously to \( \Omega_0 \), given in (4).

Also, the curvature invariants all remain finite for \( -\infty \leq t \leq \infty \). Hence, in the present model, the evolution of the toy universe with \( \rho \neq 0 \) but \( \gamma = \frac{1}{3} \) is singularity free.

When the initial conditions are different, the evolution can again be analysed along similar lines. However, the main result that the evolution is singularity free remains unchanged.

5 c. \( \gamma = 1, \ t > t_i \)

Initially, we have \( \dot{A}(t_i) > 0 \), \( \dot{\chi}(t_i) > 0 \), and \( \Omega(t_i) > 3 \). Then, \( \epsilon = 1 \) necessarily. For \( t > t_i \), the evolution proceeds as described in section 4a. Both \( e^A \) and \( \chi \) increase. Hence, \( \Omega \) increases.

The constant \( c > 0 \) in (13). However, \( (1 - 3\gamma) < 0 \) and, hence, \( \sigma(t) < 0 \). Therefore, \( (\sigma(t) + c) \) can become negative for \( \chi < 1 \), or remain positive as \( \chi \to 1 \). In the case where \( (\sigma(t) + c) \) becomes negative for \( \chi < 1 \), it must pass through a zero where \( \dot{\chi} \) vanishes. This critical point is a maximum of \( \chi(t) \) beyond which we have \( \dot{A} > 0 \) and \( \dot{\chi} < 0 \), and \( \Omega > 3 \).

Consider the case where \( (\sigma(t) + c) \) remains positive as \( \chi \to 1 \). Furthermore, as described in section 4, it can be taken to be non infinitesimal by choosing the model dependent constant \( \Omega_1 \) in (7) large enough. The dynamics near \( \chi = 1 \), equivalently \( \phi = 0 \) in our normalisation, is clear in terms of \( \phi \) and \( \psi(\phi) \). It follows from the analysis of section 4a that \( \phi \) crosses 0 and continues to increase. Then, \( \psi(\phi) \) and, hence, \( \Omega \) decrease. We then have \( \dot{A} > 0 \), but \( \dot{\chi} < 0 \).

Thus, in both of the above cases, we now have \( \dot{A} > 0 \) and \( \dot{\chi} < 0 \). The evolution then proceeds as described in section 4b. \( e^A \) increases whereas \( \chi \) and, hence, \( \Omega \) decrease. As \( t \to \infty \), \( e^A \) and \( \chi \) evolve as given in (33) and (36).
In particular, it can be seen that the quantities in (53) are all finite for \( t_i \leq t \leq \infty \), implying that all the curvature invariants are finite. Hence, the evolution is singularity free.

\( t < t_i \), equivalently \((-t) > (-t_i)\)

To describe the evolution for \( t < t_i \), we reverse the direction of time. Then, in terms of the reversed time variable which is also denoted as \( t \), we have \( \dot{A}(t_i) < 0 \), \( \dot{\chi}(t_i) < 0 \), and \( \Omega(t_i) > 3 \). Then, \( \epsilon = -1 \) necessarily. For \( t > t_i \), the evolution proceeds as described in section 4c. Both \( e^A \) and \( \chi \) decrease. Hence, \( \Omega \) decreases. Then, as shown in section 4c, there exists a time, say \( t = t_m > t_i \) where \( K(t_m) = 1 \) in equation (17), \( e^A(t_m) > 0 \), \( \chi(t_m) > 0 \), and \( \dot{\chi}(t_m) < 0 \). Note that \( \Omega(t_m) < 3 \).

As shown in section 4c, \( K(t_m) = 1 \) implies that \( \dot{A}(t_m) = 0 \). Hence, the scale factor \( e^A \) reaches a minimum. For \( t > t_m \), we then have \( \dot{A} > 0 \), \( \dot{\chi} < 0 \), \( \Omega < 3 \), and \( \epsilon = -1 \). The evolution proceeds as described in section 4d.

It follows that \( \sigma(t) = 0 \) since \( \gamma = 1 \). Note that the constant \( c < 0 \) in (15). Thus, \( (\sigma(t) + c) < c < 0 \) and, hence, \( \dot{\chi} < 0 \) for \( t > t_i \). Therefore, \( \chi \) decreases. Eventually, \( \chi \to 0 \). Moreover, and as can be verified a posteriori, \( \sigma(t) \to (\text{constant}) \) as \( \chi \to 0 \). Hence, in equation (18), the constant \( c_e \) is negative and non infinitesimal. Therefore, as described in section 4d, \( e^A \) increases whereas \( \chi \) and, hence, \( \Omega \) decrease. As \( t \to \infty \), \( e^A \) and \( \chi \) evolve as given in (11) - (13).

In particular, it can be seen that, for \( \Omega_0 \leq \frac{1}{3} \), the quantities in (23) are all finite for \( t_i \leq t \leq \infty \), implying that all the curvature invariants are finite. Hence, the evolution is singularity free.

Thus, the evolution of the toy universe containing only massless scalar field, i.e. \( \rho \neq 0 \) but \( \gamma = 1 \), and with the initial conditions \( \dot{A}(t_i) > 0 \), \( \dot{\chi}(t_i) > 0 \), and \( \Omega(t_i) > 3 \) proceeds in the present model as follows. For \( t > t_i \), the scale factor \( e^A \) increases continuously to \( \infty \). The field \( \chi \) increases, reaches a maximum \( \leq 1 \), and then decreases continuously to 0. Correspondingly, \( \Omega \) increases, reaches a maximum \( \leq \infty \), and then decreases continuously to \( \Omega_0 \), given in (7).

For \( t < t_i \), the scale factor \( e^A \) decreases, reaches a non zero minimum, and then increases continuously to \( \infty \). The field \( \chi \) decreases continuously to 0. Correspondingly, \( \Omega \) decreases continuously to \( \Omega_0 \), given in (6).
Also, the curvature invariants all remain finite for $-\infty \leq t \leq \infty$. Hence, in the present model, the evolution of the toy universe with $\rho \neq 0$ but $\gamma = 1$ is singularity free.

When the initial conditions are different, the evolution can again be analysed along similar lines. However, the main result that the evolution is singularity free remains unchanged.

6. Evolution of observed Universe

We now use the results of the analysis in section 4 to describe the generic evolution of a realistic universe, such as our observed one, in the present model. A realistic universe may contain different types of “matter” described by different values of $\gamma$, but with $-1 \leq \gamma \leq 1$. For example, $\gamma = -1$, 0, $\frac{1}{3}$, and 1 respectively for vacuum energy, dust, radiation, and massless scalar field. Our observed universe is certainly known to contain dust and radiation. However, in a variety of models, vacuum energy and/or massless scalar field are often assumed to be present. Our assumption that $-1 \leq \gamma \leq 1$ then covers all these cases. As follows from (12), “matter” with larger $\gamma$ dominates the evolution for smaller value of $e^A$ and vice versa. Therefore, when $e^A$ is decreasing we take $\gamma \geq \frac{1}{3}$ eventually, and when $e^A$ is increasing we take $\gamma \leq 0$ eventually.

In considering the evolution of the universe, we start with an initial time $t_i$, corresponding to a temperature, say $\approx 10^{16}$ GeV, such that GUT symmetry breaking, inflation, and other (matter) model dependent phenomena may occur for $t > t_i$ only. Our observed universe is expanding at $t_i$ and, hence, $\dot{A}(t_i) > 0$. For the sake of definiteness, we take $\dot{\chi}(t_i) > 0$ and $\Omega(t_i) > 3$, equivalently $\omega(t_i) > 0$, as commonly assumed.

We first describe the evolution for $t > t_i$ taking, as initial conditions $\dot{A}(t_i) > 0$, $\dot{\chi}(t_i) > 0$, and $\Omega(t_i) > 3$. Then, $\epsilon = 1$ necessarily. To describe the evolution for $t < t_i$, we reverse the direction of time and take, as initial conditions, $\dot{A}(t_i) < 0$, $\dot{\chi}(t_i) < 0$, and $\Omega(t_i) > 3$. Then, $\epsilon = -1$ necessarily. The required evolution is that for $t > t_i$ in terms of the reversed time variable, which is also denoted as $t$. The results of section 4 can then be applied directly.

When the initial conditions are different, the evolution can again be analysed along similar lines as follows. However, the main result that the evolution is singularity free remains unchanged.
Initially, we have $\dot{A}(t_i) > 0$, $\dot{\chi}(t_i) > 0$, and $\Omega(t_i) > 3$. Then, $\epsilon = 1$ necessarily. For $t > t_i$, the evolution proceeds as described in section 4a. Both $e^A$ and $\chi$ increase. Hence, $\Omega$ increases. Eventually, as $e^A$ increases, we can take $\gamma \leq 0$. Then, as $t \to \infty$, $e^A$ and $\chi$ evolve as given in (11).

In particular, it can be seen that the quantities in (53) are all finite for $t_i \leq t \leq \infty$, implying that all the curvature invariants are finite. Hence, the evolution is singularity free.

Thus, in the present day universe in our model, $\chi(t_{today}) \to 1$ and $\Omega(t_{today}) \to \infty$. Also, as follows from (7), $\frac{1}{\Omega^2} \frac{d\Omega}{d\chi}(today) = -\frac{2a}{\Omega^3} (1 - \chi)^{4a-1} \to 0$. Therefore, our model satisfies the observational constraints imposed by solar system experiments, namely $\Omega(t_{today}) > 2000$ and $\frac{1}{\Omega^2} \frac{d\Omega}{d\chi}(today) < 0.0002$ (see [3] pg. 117, 124-5, and 339).

For $t < t_i$, equivalently $(-t) > (-t_i)$

To describe the evolution for $t < t_i$, we reverse the direction of time. Then, in terms of the reversed time variable, also denoted as $t$, we have $\dot{A}(t_i) < 0$, $\dot{\chi}(t_i) < 0$, and $\Omega(t_i) > 3$. Then, $\epsilon = -1$ necessarily. For $t > t_i$, the evolution proceeds as described in section 4c. Both $e^A$ and $\chi$ decrease. Hence, $\Omega$ decreases. Then, as shown in section 4c, there exists a time, say $t = t_m > t_i$ where $K(t_m) = 1$ in equation (17), $e^{A(t_m)} > 0$, $\chi(t_m) > 0$, and $\dot{\chi}(t_m) < 0$. Note that $\Omega(t_m) < 3$.

As shown in section 4c, $K(t_m) = 1$ implies that $\dot{A}(t_m) = 0$. Hence, the scale factor $e^A$ reaches a minimum. For $t > t_m$, we then have $\dot{A} > 0$, $\dot{\chi} < 0$, $\Omega < 3$, and $\epsilon = -1$. The evolution proceeds as described in section 4d. Now, however, the evolution for $t > t_m$ is complicated since the universe is known to contain “matter” with $\gamma = 0$. Nevertheless, all qualitative features of its evolution can be obtained using the results of section 4.

For $t \geq t_m$, $\dot{A} > 0$, $\dot{\chi} < 0$, and $(\sigma(t) + c) < 0$. Hence, $e^A$ increases and $\chi$ decreases and, eventually, “matter” with $\gamma \leq 0$ dominates the evolution. Then $(1 - 3\gamma) > 0$ and $\sigma(t)$ begins to increase. Hence $(\sigma(t) + c)$, which is negative, also begins to increase. For $\gamma \leq 0$, $\sigma(t)$ grows atleast as fast as $t$, as follows from (16). Then eventually, as described in section 4d, $e^A$ reaches a maximum eventually at time, say $t = t_1$. Also, $(\sigma(t_1) + c) < 0$ and, hence, $\dot{\chi}(t_1) < 0$. Note that this critical point of $e^A$ exists independent of the details.
of the model, as long as “matter” with $\gamma \leq 0$ is present. Further evolution then proceeds as described in section 4e.

For $t > t_1$, we then have $\dot{A} < 0$, $\ddot{\chi} < 0$, and $\Omega < 3$. Also, $(\sigma(t) + c) < 0$ but $\sigma(t)$ remains increasing. Hence, depending now on the details of the model, $(\sigma(t) + c)$ may remain negative for all $t > t_1$, or it may reach a zero and become positive.

In the first case, $(\sigma(t) + c)$ remains negative for all $t > t_1$. Hence, $\dot{\chi} < 0$ and $\chi$ continues to decrease. The scale factor $e^A$, as shown in section 4e, reaches a minimum at time, say $t = t_M > t_1$, with $\chi(t_M) > 0$ non vanishing. For $t > t_M$, $e^A$ increases, and we have $\dot{A} > 0$, $\ddot{\chi} < 0$, and $\Omega < 3$. Further evolution then proceeds as described in section 4d.

Note that this means that the scale factor increases and reaches a maximum, then decreases and reaches a minimum, then increases and so on. The existence of maxima of $e^A$ is model independent as long as “matter” with $\gamma \leq 0$ is present, which is the case for the observed universe. The minima of $e^A$ are all non zero for the reasons described in section 4c. Their existence depends on whether $(\sigma(t) + c)$ remains negative or not, but is otherwise model independent as long as “matter” with $\gamma \geq \frac{1}{3}$ is present, which is the case for the observed universe. The evolution can thus become repetitive and oscillatory.

In the second case, $(\sigma(t) + c)$ reaches a zero at time, say $t = t_{m'} > t_1$ and becomes positive. It then follows that $\dot{\chi}(t_{m'}) = 0$ and that this is a minimum of $\chi$. For $t > t_{m'}$, we have $\dot{A} < 0$, $\ddot{\chi} > 0$, and $\Omega < 3$. Hence, $e^A$ decreases and $\chi$ increases. Eventually, we can take $\gamma \geq \frac{1}{3}$.

Now, $(1 - 3\gamma) \leq 0$ and $\sigma(t)$ begins to decrease or remains constant. Hence $(\sigma(t) + c)$, which is positive, also begins to decrease or remains constant. Thus, depending on the details of $\rho_0$ and $\gamma$, $(\sigma(t) + c)$ may or may not vanish with $\chi < 1$. If $(\sigma(t) + c)$ vanishes with $\chi < 1$, then $\dot{\chi}$ vanishes. This critical point is a maximum of $\chi$, beyond which we have $\dot{A} < 0$, $\ddot{\chi} < 0$, and $\epsilon = -1$. Further evolution then proceeds as described in section 4e if $\Omega > 3$ and as in section 4e if $\Omega < 3$. The evolution can thus become repetitive and oscillatory.

If $(\sigma(t) + c)$ does not vanish and remain positive for $\chi < 1$ then, eventually, $\chi \to 1$. It follows from equation (15) that $\Omega \dot{\chi}^2 \propto e^{-6A}$, whereas $\rho$ is given by (12). Note that $\dot{A} < 0$ and $e^A$ is decreasing. Then, in this limit, the $\frac{\Omega \dot{\chi}^2}{\chi}$ term in equation (10) dominates or, if $\gamma = 1$, is of the order of the $\frac{\rho}{\chi}$ term.

Thus we have $\dot{A} < 0$, $\ddot{\chi} > 0$, $\chi \to 1$, equivalently $\Omega \to \infty$, and $\epsilon = -1$. 28
Also, $\frac{\Omega \dot{\chi}^2}{\chi}$ term in equation (11) dominates or is of the order of the $\chi^2$ term. But this is precisely the time reversed version of the evolution analysed in section 4b, where the dynamics is clear in terms of $\phi$ and $\psi(\phi)$. Applying the results of section 4b, it then follows that $\phi$ will cross the value 0 after which $e^\psi$ and, hence, $\chi$ will begin to decrease. We then have $\dot{A} < 0$, $\dot{\chi} < 0$, $\Omega > 3$. The evolution then proceeds as described in section 4c. The evolution can thus become repetetive and oscillatory.

Thus it is clear that depending on the details of the model, the universe undergoes oscillations, perhaps infinitely many. As follows from the above description, the oscillations can stop, if at all, only in the limit $\chi \to 0$. The solutions then are given by (41) - (43). In particular, however, the quantities in (53) all remain finite during the oscillations, implying that all the curvature invariants remain finite. They also remain finite in the solutions (41) - (43) if $\Omega_0 \leq \frac{1}{3}$ in (7).

Thus, it follows that if $\Omega_0 \leq \frac{1}{3}$ then the quantities in (53) are all finite for $t_i \leq t \leq \infty$, implying that all the curvature invariants are finite. Hence, the evolution is singularity free.

In summary, the evolution of a realistic universe, such as our observed one, with the initial conditions $\dot{A}(t_i) > 0$, $\dot{\chi}(t_i) > 0$, and $\Omega(t_i) > 3$ proceeds in the present model as follows. For $t > t_i$, the scale factor $e^A$ increases continuously to $\infty$. The field $\chi$ increases continuously to 1. Correspondingly, $\Omega$ increases continuously to $\infty$.

For $t < t_i$, the scale factor $e^A$ decreases and reaches a non zero minimum. It then increases and reaches a maximum, then decreases and reaches a minimum, then increases and so on, perhaps ad infinitum. The oscillations can stop, if at all, only in the limit $\chi \to 0$. The solutions then are given by (41) - (43). The field $\chi$ may, depending on the details of model, continuously decrease to 0, or undergo oscillations, its maxima always being $\leq 1$.

Also, the curvature invariants all remain finite for $-\infty \leq t \leq \infty$. Hence, the evolution, although complicated and model dependent in details, is completely singularity free. Thus we have that a homogeneous isotropic universe, such as our observed one, evolves in the present model with no big bang or any other singularity. The time continues indefinitely into the past and the future, without encountering any singularity.

When the initial conditions are different, the evolution can again be analysed along similar lines. However, the main result that the evolution is singularity free remains unchanged.
7. Generalisations of the model

The analysis and the results of sections 4 - 6 are generic and valid for any function $\Omega$ satisfying only the constraints (7). However, these constraints can be relaxed further thus generalising the model, but still maintaining the evolution singularity free. We now study these generalisations.

We first summarise the essential features of the constraints (7) on $\Omega(\chi)$ by noting the following points. First, as evident from the results of the previous sections, any singularity is likely to arise, if at all, only in the limit $\chi \to 0$. It then follows from the relevant solutions (41) - (43) that this potential singularity is avoided by the constraint $\Omega_0 \leq \frac{1}{3}$ in (7).

Second, we used only the sufficiency condition given in the Appendix to determine the absence of singularity. Hence, the requirement of finiteness of $\frac{d^n\Omega}{d\chi^n}$, $\forall n \geq 1$.

Third, there is no singularity in the limit $\chi \to 1$, although $\Omega \to \infty$. The evolution of the present day universe corresponds to this limit. Also, the behaviour of $\Omega(\chi)$ in this limit, in particular the constraint on $\alpha$ given in (7) which follows naturally from (8), ensures that our model satisfies the observational constraints imposed by solar system experiments. Note, however, that the observational constraints are satisfied for any $\alpha > \frac{1}{4}$.

Fourth, the monotonicity of $\Omega(\chi)$ for $0 < \chi < 1$ is not required either to avoid the singularity or to conform with the observations. It was invoked only to keep the analysis simple and transparent. Without this requirement, the evolution would be more complicated, its history possibly bearing the imprints of the ups and downs of $\Omega(\chi)$. The analysis in section 4 would then involve further subclasses not directly relevant for the issue of singularity.

It is now clear how the present model, equivalently the constraints (7) on $\Omega(\chi)$, can be generalised further. In the generalised model also we take $\chi$ to have a finite upper bound $\chi \leq \chi_{\text{max}} < \infty$ which was naturally the case in the previous model. Then, as in section 2, the factor $\chi_{\text{max}}$ can be absorbed into the Newton’s constant $G_N$ and the range of $\chi$ can set to be $0 \leq \chi \leq 1$. The generalised model is now given by the function $\Omega(\chi) > 0$ satisfying the following more general constraints:

$$\Omega(0) \leq \frac{1}{3}$$
$$\frac{d^n\Omega}{d\chi^n} = \text{finite} \quad \forall n \geq 1, \quad 0 \leq \chi < 1$$
\[ \Omega \rightarrow \infty \text{ at } \chi = 1 \text{ only} \]

\[ \lim_{\chi \to 1} \Omega = \Omega_1 (1 - \chi)^{-2\alpha}, \quad \alpha > \frac{1}{4}, \quad (44) \]

where \( \Omega_1 > 0 \) is a constant. The function \( \Omega(\chi) \) is arbitrary otherwise.

Thus, in the generalised model, \( \Omega(\chi) \) in the limit \( \chi \to 0 \) need not necessarily be a constant, but can be a function of \( \chi \). For example, one may have \( \Omega(\chi) = \Omega_0 \chi^\beta \) as \( \chi \to 0 \) where \( \beta > 0 \) and \( \Omega_0 > 0 \) are constants. Such functions do arise in the variable mass theories of [6] also. Note that the position of the pole, namely \( \chi = 1 \), is not a constraint but only a choice of normalisation of \( \chi \). However, the order of the pole \( \alpha \) can now be, for example, \( \geq 1 \).

We have taken \( \frac{d^n \Omega}{d\chi^n}, \forall n \geq 1 \) for the sake of simplicity. This requirement can perhaps be relaxed if one uses only the appropriate necessary conditions to determine the absence of singularity instead of the sufficiency conditions given in the Appendix. However, we will not pursue it further in this paper.

The evolution of the universe in the generalised model can be analysed along the same lines as in sections 4 - 6. It is clear that the only important limits are \( \chi \to 0 \) and, perhaps, \( \chi \to 1 \). Hence, we now analyse only these two limits in the generalised model. To be definite, we consider the following example:

\[ \lim_{\chi \to 0} \Omega = \Omega_0 \chi^\beta, \quad \beta > 0 \]

\[ \lim_{\chi \to 1} \Omega = \Omega_1 (1 - \chi)^{-2\alpha}, \quad \alpha \geq 1, \quad (45) \]

where \( \Omega_0 > 0 \) and \( \Omega_1 > 0 \) are constants. Note that the analysis in section 4 applies directly with no modification in the limit \( \chi \to 1 \) if \( \frac{1}{4} \leq \alpha < 1 \). Hence, we have taken \( \alpha \geq 1 \) in (45). Using (5), the constraints (45) can be expressed equivalently in terms of \( \phi \) and \( \psi(\phi) \) as follows:

\[ \lim_{\psi \to -\infty} \psi = \frac{2}{\beta} \ln(\phi - \phi_0) \]

\[ \lim_{\psi \to 0} \psi = \psi_0 e^{-\phi}, \quad \text{if } \alpha = 1 \]

\[ = \psi_0 \phi^{-\frac{1}{\alpha-1}}, \quad \text{if } \alpha > 1, \quad (46) \]

where \( \phi_0 \) and \( \psi_0 \) are constants. The range of \( \psi = \ln \chi \) is given by \( -\infty \leq \psi \leq 0 \). Hence, the constant \( \psi_0 < 0 \) in (10). Also, the limit \( \chi \to 0 \) corresponds
to $\psi \to -\infty$ and, hence, to $\phi \to \phi_0$ whereas the limit $\chi \to 1$ corresponds to $\psi \to 0$ and, hence, to $\phi \to \infty$.

$$\chi \to 0$$

In this limit, the analysis is similar to that given in section 4d. Under the conditions of section 4b, equation (17) can be solved in the limit $\chi \to 0$ relating $\chi$ and $e^A$:

$$e^A = (\text{constant}) \chi^{-\frac{1}{2}}. \quad (47)$$

Hence, $e^A \to \infty$ as $\chi \to 0$. Substituting (17) in equation (15) then yields the unique solution in the limit $\chi \to 0$: For $\beta \neq 1$,

$$e^A = e^{A_0} (t_0 - \text{sign}(m)t)^{-\frac{m}{2}}, \quad \chi = \chi_0 (t_0 - \text{sign}(m)t)^m, \quad (48)$$

where $t$ is measured in appropriate units, $A_0$, $\chi_0$, and $t_0 > t_i$ are constants and $m = \frac{2}{\beta - 1}$. For $\beta = 1$,

$$e^A = e^{A_0} e^{-\frac{c_e(t-t_0)}{2}}, \quad \chi = \chi_0 e^{c_e(t-t_0)} \quad (49)$$

where $c_e < 0$ is a constant. Note, as pointed out in [15] and as discussed in section 3, that when $\chi$ and $e^A$ satisfies equation (17) the field equation (13) is not guaranteed to follow from (10) - (12). Nevertheless, the above solution satisfies equation (13) also, as checked explicitly.

Let $\beta > 1$. Then, $m > 0$. Since $\chi \to 0$, it follows from (18) that $t \to t_0 > t_i$, implying that $\chi$ vanishes at a finite time $t_0$. In this limit, the scale factor $e^A \to \infty$ as follows from (18). The quantities in (53), for example $\dot{\chi}$, also diverge implying that the curvature invariants, including the Ricci scalar, diverge. Thus, for $\beta > 1$, there is a singularity at a finite time $t_0$. This implies, for example, that the variables mass theories of [6] which exhibit a minimum of $e^A$ can nevertheless have singularities because $\beta$ can be $> 1$ in these theories.

Let $\beta \leq 1$. When $\beta < 1$, $m < 0$. Since $\chi \to 0$, it follows from (18) that $t \to t_0 > t_i$, implying that $\chi$ vanishes at a finite time $t_0$. In this limit, the scale factor $e^A \to \infty$ as follows now from (18). The same result holds when $\beta = 1$ also, as follows from (19).

If $\rho \neq 0$ then the consistency of the above solutions requires that $\gamma \geq \frac{1}{3}$. This is analogous to the condition $\gamma > 0$ required in section 4d.

It can now be seen, for $\beta \leq 1$, that the quantities in (53) are all finite implying that all the curvature invariants are finite. Thus, there is no singularity in the above solutions when $\beta \leq 1$. 

32
\(\chi \to 1\)

In this limit, the analysis is similar to that given in section 4a. For \(\rho \neq 0\) and \(\gamma \neq 1\), the solution is given uniquely by

\[
\begin{align*}
  e^A &= e^{A_0} t^{\frac{2}{3(1+\gamma)}}, \\
  \chi &= 1 - \chi_0 e^{kT}, \quad \text{if } \alpha = 1 \\
  &= 1 - \chi_0 (1 - kT)^{-\frac{1}{\alpha - 1}}, \quad \text{if } \alpha > 1, \\
\end{align*}
\]

(50)

where \(\chi_0 > 0\) and \(k > 0\) are constants, \(t\) is measured in appropriate units, and we have defined \(T \equiv t^{-\frac{1}{1+\gamma}}\) which \(\to 0\) in the limit \(t \to \infty\). The above solution then implies that \(e^A \to \infty\) and \(\chi \to (1 - \chi_0)\). The precise value of \(\chi_0\) is model dependent. In particular, it also depends on \(\Omega_1\) and can be made as small as necessary by choosing a sufficiently large \(\Omega_1\). Equivalently, as \(t \to \infty\), \(\Omega\) can be made as large as necessary.

This will imply, as in section 6, that in the present day universe in the generalised model, \(\chi(\text{today}) \to 1 - \chi_0\), \(\Omega(\text{today}) \to \Omega_1 \chi_0^{-2\alpha}\), and \(\frac{1}{H^2} \frac{d\Omega}{d\chi}(\text{today}) \to -\frac{2\alpha}{\Omega_1^2} \chi_0^{4\alpha - 1}\). Therefore, by choosing \(\Omega_1\) sufficiently large one can ensure that the generalised model also satisfies the observational constraints imposed by solar system experiments, namely \(\Omega(\text{today}) > 2000\) and \(\frac{1}{H^2} \frac{d\Omega}{d\chi}(\text{today}) < 0.0002\) (see [3] pg. 117, 124-5, and 339).

For \(\rho = 0\), the solution is obtained easily in terms of \(\phi\) and \(\psi(\phi)\), and is given by

\[
\begin{align*}
  e^A &= e^{A_0} t^{\frac{2}{3}} \\
  e^{\phi} &= (\text{constant}) t^{\frac{2}{3}}, \\
\end{align*}
\]

(51)

where \(A_0\) is a constant. \(\psi(\phi)\) and, hence, \(\chi = e^{\psi}\) are then obtained from (46). Note that, as in section 4a, these solutions also apply to the case when \(\rho \neq 0\), but \(\gamma = 1\). As \(t \to \infty\) in the above solution, we have \(e^A \to \infty\) and \(\phi \to \infty\). Thus, as follows from (49), \(\psi \to 0\). Hence, \(\chi \to 1\) and \(\Omega \to \infty\).

It can be seen that the quantities in (53) are all finite implying that all the curvature invariants are finite. Thus, there is no singularity in the above solutions.

Now the evolution of the universe in the generalised model can be analysed as in sections 4 - 6. For the three toy universes of section 5, the evolution
proceeds as described in section 5. For a realistic universe, such as our observed one, the evolution is complicated but proceeds as described in section 6. In particular, it follows that the evolution of the universe is singularity free in the generalised model when the function \( \Omega(\chi) \) satisfies the constraints (45). Although not proved here, the evolution is likely to be singularity free when \( \Omega(\chi) \) satisfies the constraints (44) only.

Before concluding this section, we would like to point out an interesting aspect of the generalised models, where \( \Omega(\chi) \rightarrow 0 \), equivalently \( \omega \rightarrow -\frac{3}{2} \), in the limit \( \chi \rightarrow 0 \). When \( \omega = -\frac{3}{2} \), the graviton-dilaton part of the action in (4) acquires a conformal symmetry under which

\[
g_{\mu\nu} \rightarrow e^{\eta(x)} g_{\mu\nu} \quad \text{and} \quad \chi \rightarrow e^{-\eta(x)} \chi,
\]

where \( \eta(x) \) is an arbitrary function of space time coordinates. In general, the matter action \( S_M \) breaks this symmetry. However, it can be seen from the solutions presented in section 4 and the present one that precisely in the limit \( \chi \rightarrow 0 \) do the matter terms become negligible compared to the dilatonic terms. This suggests that the matter action \( S_M \) may be neglected and that the conformal symmetry of the action (4) may become exact as \( \chi \rightarrow 0 \). A possible implication of the appearance of this conformal symmetry is mentioned in the conclusion.

8. Summary and Conclusion

To summarise, we considered the evolution of a flat homogeneous isotropic universe in a graviton-dilaton model. The model is specified by the function \( \Omega(\chi) \) which satisfies the constraints given in (7), or more generally (44), but is arbitrary otherwise. We studied the evolution of universe in this model with the main purpose of determining whether it is singular or not.

Assuming generic initial conditions where \( \dot{A}(t_i) \) and \( \dot{\chi}(t_i) \) are finite and non infinitesimal and \( \Omega(t_i) < \infty \), we presented a general analysis which is valid for any matter and for any arbitrary function \( \Omega(\chi) \), and which enables one to analyse the evolution and obtain its generic features even in the absence of explicit solutions.

We illustrated our method by applying it to describe the evolution of three toy universes. We showed, in particular, that their evolution in the present model is singularity free. We then described the evolution of a universe, such as our observed one. We showed, in particular, that its evolution, although
rich and complicated, is singularity free. This is a generic result valid for any 
$\Omega(\chi)$ satisfying only the constraints given in (7) or (44).

An important question to ask now, from our perspective, is whether a 
function $\Omega(\chi)$ as required in the present model can arise from a fundamental 
theory. We now discuss critically a few such possibilities in string theory.

In the string theoretic context, Einstein metric formulation given in (1) 
is more natural where $\phi$ is the dilaton and $\chi(\phi)$ the arbitrary function. 
The strong coupling limit in string theory, relevant for singularities, corresponds 
to diverging effective Newton’s constant ($= \frac{1}{8\pi\chi}$) and, thus, to the limit 
$\chi(\phi) \to 0$. In this limit $\ln \chi \to \frac{\phi}{\sqrt{4 \Omega_0}}$ in our model (see equation (7) and 
(1)).

(1) Note that, in string theory, $\ln(\chi(\phi))$ can be thought of as Kahler 
potential for $\phi$, which is expected to be modified at strong coupling by non 
perturbative effects [17]. But, such modifications are usually exponential in 
nature and, hence, are unlikely to lead to a $\chi(\phi)$ as required here.

(2) However, at strong coupling, $\phi$ here may not be the stringy dilaton 
directly, but instead a combination of the stringy dilaton and other compact-
ification dependent moduli. In that case, after writing the relevant effective 
action as given in (1), the Kahler potential $\ln \chi$ for this ‘effective dilaton’ $\phi$ 
may well turn out to be of the required form.

(3) Another, perhaps more promising, possibility is the following which 
arises when $\Omega_0 = \frac{1}{3}$ (see equation (7) and (1)). This case corresponds to a five 
dimensional space time compactified to four dimensions on a circle [18]. In 
recent developments in string theory at strong coupling, analogous phenom-
ena relating a $d$-dimensional theory and a $(d+1)$-dimensional theory on a 
circle are found to occur: Using S-duality symmetries, Witten has dis-
covered that the ten dimensional string at strong coupling is an eleven dimen-
sional (M-)theory compactified on a circle [19, 1]. A similar phenomenon, that the 
three dimensional string at strong coupling is a four dimensional theory com-
 pactified on a circle, is at the heart of Witten’s novel proposal for solving 
the cosmological constant problem [20]; for its possible stringy realisation see 
[21]. A similar phenomenon in four/five dimensions, if exists, may possibly 
lead to a $\chi(\phi)$ as required here.

(4) As discussed in section 7, the graviton-dilaton action in (4) acquires 
a conformal symmetry in the limit $\chi \to 0$ where $\Omega(\chi) \to 0$, equivalently 
$\omega(\chi) \to -\frac{3}{2}$. Note that this is also the limit where singularities can poten-
tially arise, but are avoided in the present model. The appearance of this conformal symmetry is, perhaps, a hint of a connection between the present model and string theory, known to lead generically to conformal field theories, but which must now be in the strong coupling limit since $\chi \to 0$.

Admittedly, at present, these are plausibility arguments only. Nevertheless, given the elegant way the present model resolves the big bang singularity it is worthwhile to derive it, perhaps along the above lines, from a fundamental theory such as string theory.

On the other hand, however, such a model, even if not derivable from string theory, may stand on its own as an interesting model for singularity free cosmology. One may then study its implications in other cosmological and astrophysical contexts. There are many issues that can be studied further. We mention some of them here.

(1) One can study the evolution of a more complicated universe, e.g. an inhomogeneous anisotropic universe, and determine whether it is singularity free or not.

(2) In section 6 we found that the evolution of a realistic universe can be oscillatory. Such early universe oscillations have significant implications and may also lead to observable predictions. It is therefore important to study this issue in more detail.

(3) In the astrophysical context, one may study the evolution and/or collapse of stars. In fact, in [11] where the present model was originally derived in a different context, we found that our model suggests a novel scenario for stellar collapse in which a black hole is unlikely to form. Such a phenomenon, if established rigorously, is likely to have far reaching consequences.

(4) The present model is characterised by one arbitrary function $\Omega(\chi)$, required to satisfy the constraints given in (7) or (44). These constraints, however, do not fix $\Omega(\chi)$ uniquely. The consequent arbitrariness in the model greatly diminishes its predictive power. It is therefore desirable to find the criteria which can fix $\Omega(\chi)$ uniquely.

In this regard, note that the present model naturally incorporates, as described in [12], an ingredient crucial for the success of hyperextended inflation [8]. Therefore, with a view to fix $\Omega(\chi)$ uniquely, one may require the present model to lead to a successful inflation also. However, our preliminary calculations [22] indicate that this requirement is not strong enough to fix $\Omega(\chi)$ uniquely. Hence, more criteria are needed.

Considering the simplicity of the present model and the novel conse-
quences that follow from it generically, we believe that its further study is fruitful and is likely to lead to interesting phenomena.

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APPENDIX

Finiteness of Curvature Invariants

In this Appendix, we derive a sufficient condition for all curvature invariants to be finite. Curvature invariants are constructed using metric tensor, Riemann tensor, and covariant derivatives. When the metric is diagonal, every term in any curvature invariant can be grouped into three types of factors: (A) \( \sqrt{g_{\mu\nu}g_{\lambda\tau}} \, R_{\mu\nu\lambda\tau} \), (B) \( \sqrt{g_{\mu\nu}g_{\lambda\tau}} \, \Gamma^\lambda_{\mu\nu} \), and (C) \( \sqrt{g_{\mu\nu}} \, \partial_{\mu} \) acting multiply on (A) and (B) type factors (no summation over repeated indices).

Using the metric given by (8) and evaluating the above factors explicitly, one obtains that (A) and (B) type factors are functions of \( \ddot{A}, \dot{A} \), and \( e^{-A} \). The action of (C) produces extra time derivatives. Thus, it follows that the curvature invariants are functions of \( e^{-A} \) and \( \frac{d^n A}{dt^n} \), \( n \geq 1 \).

By repeated use of equations (10)-(12), \( \frac{d^n A}{dt^n} \) for any \( n \geq 1 \) and, hence, any curvature invariant, can be expressed in terms of the following quantities:

\[
e^{-A}, \quad \frac{\rho}{\chi}, \quad \frac{\rho}{\chi\Omega}, \quad \frac{\dot{\chi}}{\chi}, \quad \frac{\Omega\chi^2}{\chi^2}; \quad \text{and} \quad \frac{\chi^n}{\Omega} \frac{d^n\Omega}{d\chi^n} \left( \frac{\dot{\chi}}{\chi} \right)^n, \quad \forall n \geq 1.
\]

The resultant expressions are finite if the above quantities are finite. The algebra is straightforward and, hence, we omit the details here.

Hence, a sufficient condition for all curvature invariants to be finite is that the quantities in (53) be all finite.

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