ON RANDOM ±1 MATRICES: SINGULARITY AND DETERMINANT

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ABSTRACT. This paper contains two results concerning random \( n \times n \) Bernoulli matrices. First, we show that with probability tending to one the determinant has absolute value \( \sqrt{n!} \exp(O(\sqrt{n})) \). Next, we prove a new upper bound \( .939^n \) on the probability that the matrix is singular.

1. Introduction

Let \( n \) be a large integer parameter, and let \( M_n \) denote a random \( n \times n \) \( \pm 1 \) matrix ("random" meaning with respect to the uniform distribution, i.e., the entries of \( M_n \) are i.i.d. Bernoulli random variables). Throughout the paper, we assume that \( n \) is sufficiently large, whenever needed. We use \( o(1) \) to denote any quantity which goes to zero as \( n \to \infty \), keeping other parameters (such as \( \epsilon \)) fixed.

This model of random matrices is of considerable interest in many areas, including combinatorics, theoretical computer science and mathematical physics. On the other hand, many basic questions concerning this model have been open for a long time. In this paper, we focus on the following two questions:

**Question 1.** What is the typical value of the determinant of \( M_n \)?

**Question 2.** What is the probability that \( M_n \) is singular?

Let us first discuss Question 1. From Hadamard’s inequality, we have the bound \(|\det(M_n)| \leq n^{n/2}\), with equality if and only if \( M_n \) is an Hadamard matrix. However, in general we expect \(|\det(M_n)|\) to be somewhat smaller than \( n^{n/2} \). Indeed, from the second moment

\[
\mathbb{E}((\det M_n)^2) = n!
\]

(first computed by Turán [11]), one is led to conjecture that \(|\det(M_n)|\) should be of the order of \( \sqrt{n!} = e^{-n+o(n)} n^{n/2} \) with high probability. On the other hand, even proving that \( |\det M_n| \) is typically positive (or equivalently, that \( M_n \) is typically non-singular) is already a non-trivial task. This task was first done by Komlós [7] (see Theorem 1.3 below).

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The first main result of this paper shows that with probability tending to one (as \( n \) tends to infinity), the absolute value of the determinant is very close to \( \sqrt{n!} \).

**Theorem 1.1.**

\[
P(|\det M_n| \geq \sqrt{n!} \exp(-24n^{1/2} \ln^{1/2} n)) = 1 - o(1).
\]

The constant 24 is generous but we do not try to optimize it.

Note that from (1) and Chebyshev’s inequality that

\[
P(|\det M_n| \leq \omega(n)\sqrt{n!})) = 1 - o(1)
\]

for any function \( \omega(n) \) which goes to infinity as \( n \to \infty \). Combining this with the preceding Theorem and the observation that \( \det M_n \) is symmetric around the origin, it follows that

**Corollary 1.2.** For each sign \( \pm \), we have

\[
P(\det(M_n) = \pm \sqrt{n!} \exp(O(n^{1/2} \ln^{1/2} n))) = 1/2 - o(1).
\]

Let us now turn to the problem of determining the probability that \( M_n \) is singular. As mentioned above, Komlós showed, in 1967, that

**Theorem 1.3.** \([7]\) \( P(\det M_n = 0) = o(1) \).

The task here is to give a precise formula for \( o(1) \) in the right hand side. Since a matrix \( M_n \) with two identical (or opposite) rows or two identical (or opposite) columns is necessarily singular, it is easy to see that

\[
P(\det M_n = 0) \geq (1 + o(1))n^{21-n}.
\]

It has often been conjectured (see e.g. \([10], [6]\)) that this is the dominant source of singularity. More precisely,

**Conjecture 1.4.**

\[
P(\det M_n = 0) = (1 + o(1))n^{21-n}.
\]

Prior to this paper, the best partial result concerning this conjecture is the following, due to Kahn, Komlós and Szemerédi \([6]\):

**Theorem 1.5.** \([6]\) We have \( P(\det M_n = 0) \leq (1 - \varepsilon + o(1))^n \), where \( \varepsilon := .001 \).

Our second main result is the following improvement of this theorem:

**Theorem 1.6.** We have \( P(\det M_n = 0) \leq (1 - \varepsilon + o(1))^n \), where \( \varepsilon := .06191 \ldots \).

This value of \( \varepsilon \) is the unique solution in the interval \((0, 1/2)\) to the equation

\[
h(\varepsilon) + \frac{\varepsilon}{\log_2 16/15} = 1,
\]

(2)
where \( h \) is the entropy function

\[
h(\varepsilon) := \varepsilon \log_2 \frac{1}{\varepsilon} + (1 - \varepsilon) \log_2 \frac{1}{1 - \varepsilon}.
\]

We prove Theorem 1.6 in Sections 5-7. Our argument uses several key ideas from the original proof of Theorem 1.5 in [6], but invoked in a simpler and more direct fashion. In a sequel to this paper we shall use more complicated arguments to improve this value of \( \varepsilon \) further, to \( \varepsilon = \frac{1}{4} \); see Section 8.13.

This paper is organized as follows. In Section 2 we establish some basic estimates for the distance between a randomly selected point on the unit cube \( \{-1, 1\}^n \) and a fixed subspace, and in Section 3 we obtain similar types of estimates in the case when the subspace is also random. In Section 4 we then apply those estimates to prove Theorem 1.1. As a by-product, we also obtain a short proof of Theorem 1.3. We then give the proof of Theorem 1.6 in Sections 5-7.

In this paper we shall try to emphasize simplicity. Several results obtained in these parts can be extended or refined considerably with more technical arguments. In last part of the paper (Section 8), we will consider some of these extensions/refinements. In particular, we prove an extension of Theorem 1.3 and Theorem 1.1 for more general models of random matrices. We will also sharpen the bound in Theorem 1.1 and mention a stronger version of Theorem 1.6.

2. The distance between a random vector and a deterministic subspace

Let \( X \) be a random vector chosen uniformly at random from \( \{-1, 1\}^n \), thus \( X = (\epsilon_1, \ldots, \epsilon_n) \) where \( \epsilon_1, \ldots, \epsilon_n \) are i.i.d. Bernoulli signs. Let \( W \) be a (deterministic) \( d \)-dimensional subspace of \( \mathbb{R}^n \) for some \( 0 \leq d < n \). In this section we collect a number of estimates concerning the distribution of the distance \( \text{dist}(X, W) \) from \( X \) to \( W \), which we will then combine to prove Theorem 1.3 and Theorem 1.1.

We have the crude estimate

\[
0 \leq \text{dist}(X, W) \leq |X| = \sqrt{n};
\]

later we shall see that \( \text{dist}(X, W) \) is in fact concentrated around \( \sqrt{n - d} \) (see Lemma 2.2). However, we shall first control the distribution of \( \text{dist}(X, W) \) near zero. We begin with a simple observation of Odlyzko.

**Lemma 2.1.** [10] We have \( P(\text{dist}(X, W) = 0) \leq 2^{d-n} \).

**Proof** Since \( W \) has dimension \( d \) in \( \mathbb{R}^n \), there is a set of \( d \) coordinates which determines all other \( n - d \) coordinates of an element of \( W \). But the corresponding \( n - d \) coordinates of \( X \) are distributed uniformly in \( \{-1, 1\}^{n-d} \) (thinking of the other \( k \) coordinates of \( X \) as fixed). Thus the constraint \( \text{dist}(X, W) = 0 \) can only be obeyed with probability at most \( 2^{d-n} \), as desired. \( \blacksquare \)
Lemma 2.2. We have
\[ \mathbb{E}(\text{dist}(X, W)^2) = n - d. \] (4)
Furthermore, for any \( 0 < \gamma < 1 \) and any integer \( l \geq 1 \) we have
\[ \mathbb{P}(|\text{dist}(X, W) - \sqrt{n - d}| \leq \gamma \sqrt{n - d}) \geq 1 - \frac{(2l)^l}{\gamma^{2l}(n - d)^l}. \] (5)

Proof Let \( P = (p_{jk})_{1 \leq j, k \leq n} \) be the \( n \times n \) orthogonal projection matrix from \( \mathbb{R}^n \) to \( W \). Let \( D = \text{diag}(p_{11}, \ldots, p_{nn}) \) be the diagonal component of \( P \), and let \( A := P - D = (a_{jk})_{1 \leq j, k \leq n} \) be the off-diagonal component of \( P \). Since \( P \) is an orthogonal projection matrix, we see that \( A \) is real symmetric with zero diagonal. If we write \( X = (\epsilon_1, \ldots, \epsilon_n) \), then from Pythagoras’s theorem we have
\[ \text{dist}(X, W)^2 = |X|^2 - |PX|^2 \]
\[ = n - \sum_{j=1}^n \sum_{k=1}^n \epsilon_j \epsilon_k p_{jk} \]
\[ = n - \text{tr}(P) - \sum_{j=1}^n \sum_{k=1}^n \epsilon_j \epsilon_k a_{jk} \]
\[ = n - d - \sum_{j=1}^n \sum_{k=1}^n \epsilon_j \epsilon_k a_{jk}. \]
This already gives (4), since \( a_{jk} \) vanishes on the diagonal. We next show the moment estimate
\[ \mathbb{E}(\left| \sum_{j=1}^n \sum_{k=1}^n \epsilon_j \epsilon_k a_{jk} \right|^2) \leq (2l)^l (n - d)^l \] (6)
for any fixed \( l \). This, together with Markov’s inequality, will imply (5).

Observe that as \( P \) is a projection matrix, the coefficients \( p_{jk} \) are bounded in magnitude by 1, and we have
\[ \sum_{j=1}^n \sum_{k=1}^n p_{jk}^2 = \text{tr}(P^2) = \text{tr}(P) = d. \]
On the other hand
\[ \sum_{j=1}^n p_{jj} = \text{tr}(P) = d \]
so by Cauchy-Schwartz
\[ \sum_{j=1}^n p_{jj}^2 \geq d^2 / n. \]
This implies that
\[
\text{tr}(A^2) = \sum_{j=1}^{n} \sum_{k=1}^{n} p_{jk}^2 - \sum_{j=1}^{n} p_{jj}^2 \leq d - d^2/n \leq \min\{d, n-d\}.
\]

The claim (6) now follows immediately from the following Lemma, which is a consequence of Bonamie’s inequality [2].

**Lemma 2.3.** Let \( A = (a_{jk})_{1 \leq j,k \leq n} \) be a real symmetric matrix with zero diagonal, and let \( \epsilon_1, \ldots, \epsilon_n \in \{-1,+1\} \) be i.i.d. Bernoulli random signs. Then for every \( l \geq 2 \) we have
\[
\mathbb{E}\left( \left| \sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_j \epsilon_k a_{jk} \right|^l \right) \leq l^{l/2} (\text{tr}A^2)^{l/2}.
\]

In Lemma 8.11, we are going to present a direct proof of a more general statement with a slightly weaker bound where \( l^{l/2} \) is replaced by \( 2^{l!} \).

To conclude this section, let us derive a corollary of Lemma 3.1, motivated by the following question of Linial: Given a subspace \( W \) of dimension \( d \), find the smallest neighborhood of \( W \) which contains at least \( 1-\delta \) fraction of the vertices.

Set \( \gamma = \frac{T}{\sqrt{n-d}} \). We have
\[
\frac{(2l)^l}{\gamma^{2l(n-d)^l}} = \left( \frac{2l}{T^2} \right)^l.
\]

A simple calculation shows that right hand side is minimal at \( l = T^2/2e \). Since we need \( l \geq 1 \), we shall assume \( T \geq (2e)^{1/2} \).

**Corollary 2.4.** Let \( W \) be a subspace of dimension \( d \) and \( X \) be a random Bernoulli vector. Then for any \( T \geq (2e)^{1/2} \)
\[
\mathbb{P}\left( |\text{dist}(X,W) - \sqrt{n-d}| \geq T \right) \leq \exp(-T^2/2e).
\]

3. The distance between a random vector and a random subspace

The estimates in the last section are quite accurate when \( n-d \) is sufficiently large, but do not provide much useful information when \( n-d \) is small (e.g. when \( n-d = 3 \)). The goal of this section is to give an estimate for this case, assuming that \( W \) is a subspace spanned by random vectors.
Lemma 3.1. Let $X$ be a random vector in $\{-1, 1\}^n$, let $1 \leq d \leq n - 1$ and $W$ a space spanned by $d$ random vectors in $\{-1, 1\}^n$, chosen independently of each other and with $X$. Then we have
\[
P(\text{dist}(X, W) \leq \frac{1}{4n}) = O(1/\sqrt{\ln n}).
\]

Let $1 \leq l \leq n$. We say that $W$ is $l$-typical if any unit vector $(w_1, \ldots, w_n) \in W^\perp$ has at least $l$ coordinates whose absolute values are at least $\frac{1}{\sqrt{2}n}$. In order to prove Lemma 3.1, we need the following

Lemma 3.2. Let $W$ be a (deterministic) subspace which is $l$-typical for some $1 \leq l \leq n$. Then
\[
P(\text{dist}(X, W) \leq \frac{1}{4n}) \leq O\left(\frac{1}{\sqrt{l}}\right).
\]

Proof By hypothesis and symmetry, we may assume without loss of generality that there is a unit normal $(w_1, \ldots, w_n) \in W^\perp$ such that $|w_1|, \ldots, |w_l| \geq \frac{1}{\sqrt{2}n}$. We then see that
\[
P(\text{dist}(X, W) \leq \frac{1}{4n}) = P(|\epsilon_1 w_1 + \ldots + \epsilon_l w_l| \leq \frac{1}{4n})
\]
\[
\leq \sup_{x \in \mathbb{R}} P(|\epsilon_1 w_1 + \ldots + \epsilon_l w_l - x| \leq \frac{1}{4n})
\]
\[
= \sup_{y \in \mathbb{R}} P(\epsilon_1 2nw_1 + \ldots + \epsilon_l 2nw_l \in [y, y + 1])
\]
where we have made the substitutions $x := \sum_{l<j \leq n} \epsilon_j w_j$ and $y := 2nx - \frac{1}{2}$ respectively. To conclude the claim, we invoke the following variant of the Littlewood-Offord lemma, due to Erdős [3]:

Lemma 3.3. [3] Let $a_1, \ldots, a_k$ be real numbers with absolute values larger than one. Then for any interval $I$ of length at most one
\[
P(\sum_{i=1}^k a_i \epsilon_i \in I) = O(1/\sqrt{k}).
\]

This lemma was proved by Erdős using Sperner’s lemma. The reader may want to check Remark 7.2 for a different argument.

We are now ready to prove Lemma 3.1.

Proof It suffices to prove the extremal case when $W$ is spanned by $n - 1$ random vectors. Set $l := \left\lfloor \frac{n}{\ln n} \right\rfloor$. In light of Lemma 3.2, we see that it suffices to show that
\[
P(W \text{ is not } l \text{-typical}) = O(1/\sqrt{\ln n}).
\]

If $W$ is not $l$-typical, then there exists a unit vector $w$ orthogonal to $W$ with at least $n-l$ co-ordinates which are less than $\frac{1}{2n}$ in magnitude. There are $\binom{n-l}{l}$
such possibilities for these co-ordinates. Thus by symmetry we have

$$P(W \text{ is not } l \text{ typical}) \leq \binom{n}{l} P(W \perp w \text{ for some } w \in \Omega)$$

where $\Omega$ is the space of all unit vectors $w = (w_1, \ldots, w_n)$ such that $|w_j| < \frac{1}{2n}$ for all $l < j \leq n$.

Suppose that $w \in \Omega$ was such that $W \perp w$, then $X_i \perp w$ for all $1 \leq i \leq n - 1$. Write $X_i = (\epsilon_{i,1}, \ldots, \epsilon_{i,n})$, then

$$\sum_{j=1}^{n} \epsilon_{i,j} w_j = 0.$$ 

Since $\epsilon_{i,j} = \pm 1$, and $|w_j| < 1/2n$ for $j > l$, we thus conclude from the triangle inequality that

$$\left| \sum_{j=1}^{l} \epsilon_{i,j} w_j \right| \leq (n-l) \frac{1}{2n} \leq \frac{1}{2}.$$ 

On the other hand, we have

$$\sum_{j=1}^{l} |w_j| \geq \sum_{j=1}^{l} |w_j|^2$$

$$= 1 - \sum_{j=l+1}^{n} |w_j|^2$$

$$\geq 1 - (n-l) \left( \frac{2}{n} \right)^2$$

$$\geq 1 - \frac{4}{n}.$$ 

Comparing these two inequalities, we see that (for $n > 8$; the cases $n \leq 8$ are of course trivial) that for each $1 \leq i \leq n - 1$, at least one of the $\epsilon_{i,j} w_j$ has to be negative. Thus, if we let $\epsilon_{i,1}, \ldots, \epsilon_{i,l}$ be signs such that $\epsilon_{j} w_j$ is positive for all $1 \leq j \leq l$, we thus have

$$\epsilon_{i,j} \neq \epsilon_{j} \text{ for all } 1 \leq i \leq n - 1.$$ 

Thus we have

$$P(W \perp w \text{ for some } w \in \Omega) \leq \sum_{\epsilon_{i,1},\ldots,\epsilon_{i,l} \in \{-1,1\}} P((\epsilon_{i,j})_{1 \leq j \leq l} \neq (\epsilon_{j})_{1 \leq j \leq l} \text{ for all } 1 \leq i \leq n-1).$$

Since the $\epsilon_{i,j}$ are i.i.d. Bernoulli variables, we have

$$P((\epsilon_{i,j})_{1 \leq j \leq l} \neq (\epsilon_{j})_{1 \leq j \leq l} \text{ for all } 1 \leq i \leq n - 1) = (1 - 2^{-l})^{n-1}.$$ 

Putting this all together, we obtain

$$P(W \text{ is not } l \text{ typical}) \leq \binom{n}{l} 2^l (1 - 2^{-l})^{n-1} \leq n^{l+1} 2^l e^{-2^l(n-1)},$$

and (7) follows by choice of $l$. This proves Lemma 3.1.
As a consequence of this lemma, we derive a short proof\(^1\) of Theorem 1.3. Let \(X_1, \ldots, X_n\) be the row vectors of \(M_n\) and \(W_j\) be the subspace spanned by \(X_1, \ldots, X_j\). Observe that if \(M_n\) is singular, then \(X_1, \ldots, X_n\) are linearly dependent, and thus we have \(\text{dist}(X_{j+1}, W_j) = 0\) for some \(1 \leq j \leq n - 1\). Thus we have

\[
P(\det(M_n) = 0) \leq \sum_{j=1}^{n-1} P(\text{dist}(X_{j+1}, W_j) = 0)
= \sum_{j=1}^{n-1} P(\text{dist}(X_j, W_j) = 0).
\]

From Lemma 2.1 we have \(P(\text{dist}(X_j, W_j) = 0) \leq 2^{-j}\). Since \(P(\text{dist}(X_j, W_j) = 0)\) is clearly monotone increasing in \(j\), we obtain the inequality

\[
P(\det(M_n) = 0) \leq 2^{-k} + k P(\text{dist}(X, W_{n-1}) = 0)
\]

for any \(1 \leq k < n\). By Lemma 3.1, \(P(\text{dist}(X, W_{n-1}) = 0) = O(1/\sqrt{\ln n})\). By choosing \(k = \ln^{1/4} n\)

\[
2^{-k} + O(k/\sqrt{\ln n}) = o(1)
\]

completing the proof.

4. Proof of Theorem 1.1

For an \(n \times n\) matrix \(A\), \(|\det A|\) is the volume of the parallelepiped spanned by the row vectors of \(A\). If one instead expresses this volume in terms of base times height, we obtain the factorization

\[
|\det(M_n)| = \prod_{0 \leq j \leq n-1} \text{dist}(X_{j+1}, W_j).
\]

To estimate this quantity, we shall simply control each of the factors \(|\text{dist}(X_{j+1}, W_j)|\) separately, using the estimates obtained in the previous two sections.

We may assume \(n\) is large. Set \(d_0 = n - \ln^{1/4} n\). For \(1 \leq j \leq d_0\), set \(l_j := 3 \ln(n - j)\) and

\[
\gamma_j := (2l_j)^{1/2}(n - j)^{-1/2 + 1/l_j}.
\]

It is trivial that all \(\gamma_j\) are bounded from above by \(1/2\) if \(n\) is large. By definition

\(^1\)Some further estimates on the quantity \(\text{P}(d(X, W) = 0)\) were obtained in [12], and it is likely that those arguments could also be used to provide yet another proof of Theorem 1.3.
\[ \frac{(2l_j)^{l_j}}{\gamma_j^{2l_j}(n-d)^{l_j}} = (n-j)^{-2}. \]

This, together with Lemma 2.2 implies that for each \( j \leq d_0 \), we have with probability at least

\[ 1 - (n-j)^{-2} \]

that the distance \( \text{dist}(X_{j+1}, W_j) \) is at least \( (1 - \gamma_j) \sqrt{n-j} \). This implies that with probability at least

\[ 1 - \sum_{j=1}^{d_0} (n-j)^{-2} = 1 - o(1) \]

the distance \( \text{dist}(X_{j+1}, W_j) \) is at least \( (1 - \gamma_j) \sqrt{n-j} \), for every \( 1 \leq j \leq d_0 \).

For \( d_0 < j \leq n-1 \), we are going to use Lemma 3.1 to estimate the distances. By this lemma, we have that with probability at least

\[ 1 - \sum_{d_0 < j \leq n} O\left( \frac{1}{\sqrt{\ln n}} \right) = 1 - o(1) \]

the distance \( \text{dist}(X_{j+1}, W_j) \) is at least \( \frac{1}{4n} \) for every \( d_0 < j \leq n-1 \). (In fact, the bound holds for all \( 1 \leq j \leq n-1 \).)

Combining the two estimates on distances, we see that with probability \( 1 - o(1) \),

\[ \prod_{0 \leq j \leq n-1} \text{dist}(X_{j+1}, W_j) \geq \frac{\sqrt{n!}}{\sqrt{(n-d_0)!}} \left( \frac{1}{4n} \right)^{n-d_0} \prod_{j=0}^{d_0} (1 - \gamma_j). \]

Since \( n - d_0 = o(\ln n) \), the error term \( \frac{1}{\sqrt{(n-d_0)!}} (\frac{1}{4n})^{n-d_0} \) is only \( \exp(-o(\ln^2 n)) \).

The main error term comes from the product \( \prod_{j=0}^{d_0} (1 - \gamma_j) \). By the definition of \( \gamma_j \) and the fact that all \( \gamma_j \) are less than \( 1/2 \), we have

\[ \prod_{j=1}^{d_0} (1 - \gamma_j) \geq \exp(-2 \sum_{j=1}^{d_0} \gamma_j) \geq \exp(-8 \sum_{j=1}^{d_0} \frac{2^{1/2} \ln^{1/2}(n-j)}{(n-j)^{1/2}}) \leq \exp(-24n^{1/2} \ln^{1/2} n). \]

Putting these together, we obtain, with probability \( 1 - o(1) \), that
\[
\prod_{0 \leq j \leq n-1} \text{dist}(X_{j+1}, W_j) \geq \sqrt{n!} \exp(-24n^{1/2} \ln^{1/2} n)
\]
proving the claim. \hfill \Box

5. Proof of Theorem 1.6

In this section, we denote \( N := 2^n \). Our goal is to prove that \( \mathbf{P}(\det M_n = 0) \leq N^{-\epsilon} \), where \( \epsilon \) is as in Theorem 1.6.

Notice that if \( M_n \) is singular, then \( X_1, \ldots, X_n \) span a proper subspace \( V \) of \( \mathbb{R}^n \). The first (fairly simple) observation is that we can restrict to the case \( V \) is a hyperplane, thanks to the following lemma:

**Lemma 5.1.** [6] We have
\[
\mathbf{P}(X_1, \ldots, X_n \text{ linearly dependent}) \leq N^{o(1)} \mathbf{P}(X_1, \ldots, X_n \text{ span a hyperplane}).
\]

**Proof** If \( X_1, \ldots, X_n \) are linearly dependent, then there must exist \( 0 \leq d \leq n-1 \) such that \( X_1, \ldots, X_{d+1} \) span a space of dimension exactly \( d \). Since the number of possible \( d \) is at most \( n = N^{o(1)} \), it thus suffices to show that
\[
\mathbf{P}(X_1, \ldots, X_{d+1} \text{ span a space of dimension exactly } d) \\
\leq \text{const} \times \mathbf{P}(X_1, \ldots, X_n \text{ span a hyperplane})
\]
for each fixed \( d \). However, from Lemma 2.1 we see that
\[
\mathbf{P}(X_1, \ldots, X_{d+2} \text{ span a space of dimension exactly } d+1) \\
|X_1, \ldots, X_{d+1} \text{ span a space of dimension exactly } d| \geq 1 - 2^{n-d},
\]
and so the claim follows from \( n - d - 1 \) applications of Bayes’ identity. \hfill \Box

In view of this lemma, it suffices to show
\[
\sum_{V, V \text{ hyperplane}} \mathbf{P}(X_1, \ldots, X_n \text{ span } V) \leq N^{-\epsilon + o(1)}.
\]

Clearly, we may restrict our attention to those hyperplanes \( V \) which are spanned by their intersection with \( \{-1,1\}^n \). Let us call such hyperplanes non-trivial. Furthermore, we call a hyperplane \( H \) degenerate if there is a vector \( v \) orthogonal to \( H \) and at most \( \log \log n \) coordinates of \( v \) are non-zero.

Fix a hyperplane \( V \). Clearly we have
\[
\mathbf{P}(X_1, \ldots, X_n \text{span } V) \leq \mathbf{P}(X_1, \ldots, X_n \in V) = \mathbf{P}(X \in V)^n.
\]
The contribution of the degenerate hyperplanes is negligible, thanks to the following easy lemma (cf. the proof of (7)):

**Lemma 5.2.** The number of degenerate non-trivial hyperplanes is at most \( N^{o(1)} \).

**Proof** If \( V \) is degenerate, then there is an integer normal vector \( v = (v_1, \ldots, v_n) \) with at most \( \log \log n \) non-zero entries. There are \( \sum_{k \leq \log \log n} \binom{n}{k} \leq \log \log nn^{\log \log n} \leq N^{o(1)} \) possible places for the non-zero entries. By relabeling if necessary we may assume that it is \( v_1, \ldots, v_k \) which are non-zero for some \( 1 \leq k \leq \log \log n \). Let \( \pi : \{-1,1\}^n \to \{-1,1\}^k \) be the obvious projection map. Then \( V \) is then determined by the projections \( \{\pi(X_1), \ldots, \pi(X_n)\} \), which are a subset of \( \{-1,1\}^k \). The number of such subsets is at most \( 2^{2^k} \leq 2^{2\log \log n} = N^{o(1)} \), and the claim follows. \( \square \)

By Lemma 2.1, \( P(X \in V) \) is at most \( 1/2 \) for any hyperplane \( V \), so the contribution of the degenerate non-trivial hyperplanes to \( P(\det M_n = 0) \) is only \( N^{-1+o(1)} \).

Following [6], it will be useful to specify the magnitude of \( P(X \in V) \). For each non-trivial hyperplane \( V \), define the discrete codimension \( d(V) \) of \( V \) to be the unique integer multiple of \( 1/n \) such that

\[
N^{-\frac{d(V)}{n}} < P(X \in V) \leq N^{-\frac{d(V)}{n}}. \tag{9}
\]

We define by \( \Omega_d \) the set of all non-degenerate, non-trivial hyperplanes with codimension \( d \). It is simple to see that \( 1 \leq d(V) \leq n \) for all non-trivial \( V \). In particular, there are at most \( O(n^2) = N^{o(1)} \) possible values of \( d \), so to prove our theorem it suffices to prove that

\[
\sum_{V \in \Omega_d} P(X_1, \ldots, X_n \text{ span } V) \leq N^{-\varepsilon + o(1)} \tag{10}
\]

for all \( 1 \leq d \leq n \).

We first handle the (simpler) case when \( d \) is large. Note that if \( X_1, \ldots, X_n \text{ span } V \), then some subset of \( n-1 \) vectors already spans \( V \). By symmetry, we have

\[
\sum_{V \in \Omega_d} P(X_1, \ldots, X_n \text{ span } V) \leq n \sum_{V \in \Omega_d} P(X_1, \ldots, X_{n-1} \text{ span } V)P(X_n \in V) \\
\leq nN^{-\frac{d}{n}} \sum_{V \in \Omega_d} P(X_1, \ldots, X_{n-1} \text{ span } V) \\
\leq nN^{-\frac{d}{n}} = N^{-\frac{d}{n} + o(1)}
\]

\( \text{The above estimates were extremely crude. In fact, as shown in [6], one can replace } \log \log n \text{ with a quantity as high as } n - 3 \log_2 n \text{ and still achieve the same result. However, for our argument we only need the quantity } \log \log n \text{ to grow very slowly in } n. \)
This disposes of the case when \( d \geq (\varepsilon - o(1))n \). Thus to prove Theorem 1.6 it will now suffice to prove

**Lemma 5.3.** If \( d \) is any integer such that

\[
1 \leq d \leq (\varepsilon - o(1))n
\]  

(11)

then we have

\[
\sum_{V \in \Omega_d} \mathbb{P}(X_1, \ldots, X_n \text{ span } V) \leq N^{-\varepsilon + o(1)}.
\]

This is the objective of the next section.

6. **Proof of Lemma 5.3**

The key idea in [6] is to find a new kind of random vectors which are more concentrated on hyperplanes in \( \Omega_d \) (with small \( d \)) than \((\pm 1)\) vectors. Roughly speaking, if we can find a random vector \( Y \) such that for any \( V \in \Omega_d \)

\[
\mathbb{P}(X \in V) \leq c\mathbb{P}(Y \in V)
\]

for some \( 0 < c < 1 \), then, intuitively, one may expect that

\[
\mathbb{P}(X_1, \ldots, X_n \text{ span } V) \leq c^n\mathbb{P}(Y_1, \ldots, Y_n \text{ span } V)
\]

(12)

where \( X_i \) and \( Y_i \) are independent samples of \( X \) and \( Y \), respectively.

While (12) may be too optimistic (because the samples of \( Y \) on \( V \) may be too linearly dependent), it has turned out that something little bit weaker can be obtained, with a proper definition of \( Y \). We next present this important definition.

**Definition 6.1.** For any \( 0 \leq \mu \leq 1 \), let \( \eta^{(\mu)} \in \{-1, 0, 1\} \) be a random variable which takes \(+1\) or \(-1\) with probabilities \( \mu \) and \( 0 \) with probability \( 1 - \mu \). Let \( X^{(\mu)} \in \{-1, 0, 1\}^n \) be a random variable of the form \( X^{(\mu)} = (\eta^{(\mu)}_1, \ldots, \eta^{(\mu)}_n) \), where the \( \eta^{(\mu)}_j \) are iid random variables with the same distribution as \( \eta^{(\mu)} \).

Thus \( X^{(1)} \) has the same distribution as \( X \), while \( X^{(0)} \) is concentrated purely at the origin. We shall work with \( X^{(\mu)} \) for \( \mu := 1/16 \); this is not the optimal value of \( \mu \) but is the cleanest to work with. For this value of \( \mu \) we have the crucial inequality, following an argument of Halász [5] (see also [6]).

**Lemma 6.2.** Let \( V \) be a non-degenerate non-trivial hyperplane. Then we have

\[
\mathbb{P}(X \in V) \leq \left(\frac{1}{2} + o(1)\right)\mathbb{P}(X^{(1/16)} \in V).
\]
Remark 6.3. One can obtain similar results for smaller values of \( \mu \) than 1/16; for instance this was achieved in [6] for the value \( \mu := \frac{1}{108} e^{-1/108} \), eventually resulting in their final gain \( \varepsilon := \cdot001 \) in Theorem 1.5. However the smaller one makes \( \mu \), the smaller the final bound on \( \varepsilon \); indeed, most of the improvement in our bounds over those in [6] comes from increasing the value of \( \mu \). One can increase the 1/16 parameter somewhat at the expense of worsening the 1/2 factor; in fact one can increase 1/16 all the way to 1/4 but at the cost of replacing 1/2 with 1. This shows that \( (3/4 + o(1))^n \) is the limit of our method. We have actually been able to attain this limit; see Section 8.13.

Let \( V \) be a hyperplane in \( \Omega_d \) for some \( d \) obeying the bound in Lemma 5.3. Let \( \gamma \) denote the quantity

\[
\gamma := \frac{d}{n \log_2 16/15};
\]

(13)

note from (2) and (11) that \( 0 < \gamma < 1 \). Let \( \varepsilon' := \min(\varepsilon, \gamma) \).

Consider the event that \( X_1, \ldots, X_{(1-\gamma)n}, X'_1, \ldots, X'_{(\gamma-\varepsilon')n} \) are linearly independent in \( V \). One can lower bound the probability of this event by the probability that all \( X_i \) and all \( X'_j \) belong to \( V \), which is

\[
P(X \in V)^{(1-\varepsilon')n} = N^{-(1-\varepsilon')d-o(1)}.\]

Let us replace \( X_j \) by \( X^{(1/16)} \) for \( 1 \leq j \leq (1-\gamma)n \) and consider the event \( A_V \) that \( X^{(1/16)}_1, \ldots, X^{(1-\gamma)n}_1, X'_1, \ldots, X'_{(\gamma-\varepsilon')n} \) are linearly independent in \( V \). Using Lemma 6.2, we are able to give a much better lower bound for this event:

\[
P(A_V) \geq N^{(1-\gamma)-(1-\varepsilon')d-o(1)}.\]

(14)

The critical gain is the term \( N^{(1-\gamma)} \). In a sense, this gain is expected since \( X^{(1/16)} \) is much more concentrated on \( V \) than \( X \). We will prove (14) at the end of the section. Let us now use it to conclude the proof of Lemma 5.3.

Fix \( V \in \Omega_d \). Let us denote by \( B_V \) the event that \( X_1, \ldots, X_n \) span \( V \). Since \( A_V \) and \( B_V \) are independent, we have, by (14) that

\[
P(B_V) \leftarrow \frac{P(A_V \land B_V)}{P(A_V)} \leq N^{-(1-\gamma)+(1-\varepsilon')d+o(1)}P(A_V \land B_V).\]

Consider a set

\[
X^{(1/16)}_1, \ldots, X^{(1/16)}_{(1-\gamma)n}, X'_1, \ldots, X'_{(\gamma-\varepsilon')n}, X_1, \ldots, X_n
\]

of vectors satisfying \( A_V \land B_V \). Then there exists \( \varepsilon'n-1 \) vectors \( X_{j_1}, \ldots, X_{j_{n-1}} \) inside \( X_1, \ldots, X_n \) which, together with \( X^{(1/16)}_1, \ldots, X^{(1/16)}_{(1-\gamma)n}, X'_1, \ldots, X'_{(\gamma-\varepsilon')n} \), span
V. Since the number of possible indices \((j_1, \ldots, j_{\epsilon' n - 1})\) is \(N^{h(\epsilon') + o(1)}\), by conceding a factor of \(N^{h(\epsilon') + o(1)}\), we can assume that \(j_i = i\) for all relevant \(i\). Let \(C_V\) be the event that \(X^{(1/16)}_1, \ldots, X^{(1/16)}_{(1-\gamma)n}, X'_1, \ldots, X'_{(\gamma-\epsilon')n}, X_{1}, \ldots, X_{\epsilon' n - 1}\) span \(V\). Then we have

\[
P(B_V) \leq N^{-(1-\gamma) + (1-\epsilon')d + h(\epsilon') + o(1)} P\left( C_V \land \left( X_{\epsilon' n}, \ldots, X_n \text{ in } V \right) \right).
\]

On the other hand, \(C_V\) and the event \((X_{\epsilon' n}, \ldots, X_n \text{ in } V)\) are independent, so

\[
P\left( C_V \land \left( X_{\epsilon' n}, \ldots, X_n \text{ in } V \right) \right) = P(C_V) P(X \in V)^{(1-\epsilon')^{n+1}}.
\]

Putting the last two estimates together we obtain

\[
P(B_V) \leq N^{-(1-\gamma) + (1-\epsilon')d + h(\epsilon') + o(1)} N^{-(1-\epsilon')^{n+1}} d/n P(C_V)
\]

Since any set of vectors can only span a single space \(V\), we have \(\sum_{V \in \Omega_d} P(C_V) \leq 1\). Thus, by summing over \(\Omega_d\), we have

\[
\sum_{V \in \Omega_d} P(B_V) \leq N^{-(1-\gamma) + h(\epsilon') - \epsilon + o(1)}
\]

We can rewrite the right hand side using (13) as \(N^{h(\epsilon') + \frac{1}{\log 2^{-1/15}} - 1 + o(1)}\). Since \(\frac{1}{\log 2^{-1/15}} - 1 > 0\), \(d/n \leq \epsilon\), and \(h\) is monotone in the interval \(0 < \epsilon' \leq \epsilon < 1/2\) we obtain

\[
\sum_{V \in \Omega_d} P(B_V) \leq N^{h(\epsilon') + \frac{1}{\log 2^{-1/15}} - 1 + o(1)}
\]

and the claim follows from the definition of \(\epsilon\) in (2). \(\square\)

In the rest of this section, we prove (14). The proof of Lemma 6.2, which uses entirely different arguments, will be presented in the next section.

To prove (14), first notice that the right hand side is the probability of the event \(A'_V\) that \(X^{(1/16)}_1, \ldots, X^{(1/16)}_{(1-\gamma)n}, X'_1, \ldots, X'_{(\gamma-\epsilon)n}\) belong to \(V\). Thus, by Bayes’ identity it is sufficient to show that

\[
P(A'_V | A_V) = N^{o(1)}
\]
From (9) we have
\[ P(X \in V) = (1 + O(1/n))2^{-d} \]  
and hence by Lemma 6.2
\[ P(X^{(1/16)} \in V) \geq (2 + O(1/n))2^{-d}. \]  
On the other hand, by a trivial modification of the proof of Lemma 2.1 we have
\[ P(X^{(1/16)} \in W) \leq (15/16)^{-\dim(W)} \]  
for any subspace \( W \). By Bayes’ identity we thus have the conditional probability bound
\[ P(X^{(1/16)} \in W|X^{(1/16)} \in V) \leq (2 + O(1/n))2^d(15/16)^{n-\dim(W)}. \]  
This is non-trivial when \( \dim(W) \leq (1 - \gamma)n \) thanks to (13).

Let \( E_k \) be the event that \( X^{(1/16)}_1, \ldots, X^{(1/16)}_k \) are independent. The above estimates imply that
\[ P(E_{k+1}|E_k \wedge A'_V) \geq 1 - (2 + O(1/n))2^d(15/16)^{n-k}. \]
for all \( 0 \leq k \leq (1 - \gamma)n \). Applying Bayes’ identity repeatedly (and (13)) we thus obtain
\[ P(E_{(1-\gamma)n}|A'_V) \geq N^{-o(1)}. \]
If \( \gamma \leq \varepsilon \) then we are now done, so suppose \( \gamma > \varepsilon \) (so that \( \varepsilon' = \varepsilon \)). From Lemma 2.1 we have
\[ P(X \in W) \leq (1/2)^{-\dim(W)} \]  
for any subspace \( W \), and hence by (15)
\[ P(X \in W|X \in V) \leq (1 + O(1/n))2^d(1/2)^{n-\dim(W)}. \]
Let us assume \( E_{(1-\gamma)n} \) and denote by \( W \) the \( (1-\gamma)n \)-dimensional subspace spanned by \( X^{(1/16)}_1, \ldots, X^{(1/16)}_{(1-\gamma)n} \). Let \( U_k \) denote the event that \( X'_1, \ldots, X'_k, W \) are independent. We have
\[ p_k = P(U_{k+1}|U_k \wedge A'_V) \geq 1 - (1 + O(1/n))2^d(1/2)^{n-k-(1-\gamma)n} \geq 1 - \frac{1}{100}2^{(k+\varepsilon-\gamma)n} \]
for all \( 0 \leq k < (\gamma - \varepsilon)n \), thanks to (11). Thus by Bayes’ identity we obtain
\[ P(A_V|A'_V) \geq N^{-o(1)} \prod_{0 \leq k < (\gamma - \varepsilon)n} p_k = N^{-o(1)} \]
as desired. \( \square \)
7. Halász-type arguments

We now prove Lemma 6.2. The first step is to use Fourier analysis to obtain usable formulae for \( P(X \in V) \) and \( P(X^{(\mu)} \in V) \). Let \( v \in \mathbb{Z}^n \setminus \{0\} \) be an normal vector to \( V \) with integer co-efficients (such a vector exists since \( V \) is spanned by the integer points \( V \cap \{-1,1\}^n \)). By hypothesis, at least \( \log \log n \) of the co-ordinates of \( v \) are non-zero.

We first observe that the probability \( P(X^{(\mu)} \in V) \) can be computed using the Fourier transform:

\[
P(X^{(\mu)} \in V) = P(X^{(\mu)} \cdot v = 0) = \mathbf{E}\left( \int_0^1 e^{2\pi i \xi X^{(\mu)} \cdot v} \, d\xi \right)
= \int_0^1 \mathbf{E}(e^{2\pi i \xi \sum_{j=1}^n \epsilon_j^{(\mu)} v_j}) \, d\xi
= \int_0^1 \prod_{j=1}^n ((1 - \mu) + \mu \cos(2\pi \xi v_j)) \, d\xi.
\]

Applying this with \( \mu = 1/16 \) we obtain

\[
P(X^{(1/16)} \in V) = \int_0^1 \prod_{j=1}^n \left( \frac{15}{16} + \frac{1}{16} \cos(2\pi \xi v_j) \right) \, d\xi.
\]

Applying instead with \( \mu = 1 \), we obtain

\[
P(X \in V) = \int_0^1 \prod_{j=1}^n \cos(2\pi \xi v_j) \, d\xi
\leq \int_0^1 \prod_{j=1}^n |\cos(2\pi \xi v_j)| \, d\xi
= \int_0^1 \prod_{j=1}^n |\cos(\pi \xi v_j)| \, d\xi,
\]

where the latter identity follows from the change of variables \( \xi \mapsto \xi/2 \) and noting that \( |\cos(\pi \xi v_j)| \) is still well-defined for \( \xi \in [0,1] \). Thus if we set

\[
F(\xi) := \prod_{j=1}^n |\cos(\pi \xi v_j)|; \quad G(\xi) := \prod_{j=1}^n \left( \frac{15}{16} + \frac{1}{16} \cos(2\pi \xi v_j) \right), \quad (17)
\]

it will now suffice to show that

\[
\int_0^1 F(\xi) \, d\xi \leq (\frac{1}{2} + o(1)) \int_0^1 G(\xi) \, d\xi. \quad (18)
\]

We now observe three estimates on \( F \) and \( G \).

**Lemma 7.1.** For any \( \xi, \xi' \in [0,1] \), we have the pointwise estimates

\[
F(\xi) \leq G(\xi)^4 \quad (19)
\]
and

$$F(\xi)F(\xi') \leq G(\xi + \xi')^2$$ (20)

and the crude integral estimate

$$\int_0^1 G(\xi) \, d\xi \leq o(1)$$ (21)

Of course, all operations on $\xi$ and $\xi'$ such as $(\xi + \xi')$ in (20) are considered modulo 1.

**Proof of Lemma 7.1.** We first prove (19). From (17) it will suffice to prove the pointwise inequality

$$| \cos \theta | \leq \left[ \frac{15}{16} + \frac{1}{16} \cos(2\theta) \right]^4$$

for all $\theta \in \mathbb{R}$. Writing $\cos 2\theta = 1 - 2x$ for some $0 \leq x \leq 1$, then $| \cos(\theta) | = (1 - x)^{1/2}$ and the inequality becomes

$$(1 - x)^{1/2} \leq (1 - x/8)^4.$$ 

Introducing the function $f(x) := \log(\frac{1}{1 - x^2})$, this inequality is equivalent to

$$\frac{f(x) - f(0)}{x - 0} \geq \frac{f(x/8) - f(0)}{x/8 - 0}$$

but this is immediate from the convexity of $f$.

Now we prove (20). It suffices to prove that

$$| \cos \theta || \cos \theta' | \leq \left[ \frac{15}{16} + \frac{1}{16} \cos(2(\theta + \theta')) \right]^2$$

for all $\theta, \theta' \in \mathbb{R}$. As this inequality is periodic with period $\pi$ in both $\theta$ and $\theta'$ we may assume that $|\theta|, |\theta'| < \pi/2$ (the cases when $\theta = \pi/2$ or $\theta' = \pi/2$ being trivial). Next we observe from the concavity of $\log \cos(\theta)$ in the interval $(-\pi/2, \pi/2)$ that

$$\cos \theta \cos \theta' \leq \cos^2 \frac{\theta + \theta'}{2} = \frac{1}{2} + \frac{1}{2} \cos(\theta + \theta').$$

Writing $\cos(\theta + \theta') = 1 - 2x$ for some $0 \leq x \leq 1$, then $\cos(2(\theta + \theta')) = 2(1 - 2x)^2 - 1 = 1 - 8x + 8x^2$, and our task is now to show that

$$1 - x \leq (1 - (x - x^2)/2)^2 = 1 - x + x^2 + (x - x^2)^2/4,$$

but this is clearly true.

Now we prove (21). We know that at least $\log \log n$ of the $v_j$ are non-zero; without loss of generality we may assume that it is $v_1, \ldots, v_K$ which are non-zero for some
Then we have by Hölder’s inequality, followed by a rescaling by $v_j$

$$
\int_0^1 G(\xi) \ d\xi \leq \int_0^1 \prod_{j=1}^K \left( \frac{15}{16} + \frac{1}{16} \cos(2\pi \xi v_j) \right) \ d\xi \\
\leq \prod_{j=1}^{\log \log n} \left( \int_0^1 \left( \frac{15}{16} + \frac{1}{16} \cos(2\pi \xi v_j) \right)^{\log \log n} \ d\xi \right)^{1/\log \log n} \\
= \prod_{j=1}^K \left( \int_0^1 \left( \frac{15}{16} + \frac{1}{16} \cos(2\pi \xi v_j) \right)^K \ d\xi \right)^{1/K} \\
= \int_0^1 \left( \frac{15}{16} + \frac{1}{16} \cos(2\pi \xi) \right)^K \ d\xi \\
= o(1)
$$
as desired, since $K \geq \log \log n$. \hfill \Box

Now we can quickly conclude the proof of (18). From (20) we have the sumset inclusion

$$
\{ \xi \in [0, 1] : F(\xi) > \alpha \} + \{ \xi \in [0, 1] : F(\xi) > \alpha \} \subseteq \{ \xi \in [0, 1] : G(\xi) > \alpha \}
$$

for any $\alpha > 0$. Taking measures of both sides and applying the Mann-Kneser-Macbeath “$\alpha + \beta$ inequality” $|A + B| \geq \max(|A| + |B|, 1)$ (see [9]), we obtain

$$
\max(2|\{ \xi \in [0, 1] : F(\xi) > \alpha \}|, 1) \leq |\{ \xi \in [0, 1] : G(\xi) > \alpha \}|.
$$

But from (21) we see that $|\{ \xi \in [0, 1] : G(\xi) > \alpha \}|$ is strictly less than 1 if $\alpha > o(1)$. Thus we conclude that

$$
|\{ \xi \in [0, 1] : F(\xi) > \alpha \}| \leq \frac{1}{2} |\{ \xi \in [0, 1] : G(\xi) > \alpha \}|
$$

when $\alpha > o(1)$. Integrating this in $\alpha$, we obtain

$$
\int_{[0,1]:F(\xi) > o(1)} F(\xi) \ d\xi \leq \frac{1}{2} \int_0^1 G(\xi) \ d\xi.
$$

On the other hand, from (19) we see that when $F(\xi) \leq o(1)$, then $F(\xi) = o(F(\xi)^{1/4}) \leq G(\xi)$, and thus

$$
\int_{[0,1]:F(\xi) \leq o(1)} F(\xi) \ d\xi \leq o(1) \int_0^1 G(\xi) \ d\xi.
$$

Adding these two inequalities we obtain (18) as desired. This proves Lemma 6.2. \hfill \Box

**Remark 7.2.** A similar Fourier-analytic argument can be used to prove Lemma 3.3. To see this, we first recall Essén’s concentration inequality [4]

$$
P(X \in I) \leq C \int_{|t| \leq 1} \mathbf{E}(e^{itX}) \ dt
$$

for any random variable $X$ and any interval $I$ of length at most 1. Thus to prove Lemma 3.3 it would suffice to show that

$$
\int_{|t| \leq 1} \mathbf{E}(\exp(it \sum_{j=1}^k a_j \xi_j)) \ dt = O(1/\sqrt{k}).
$$
But by the independence of the $\epsilon_j$, we have

$$E(\exp(it \sum_{j=1}^{k} a_j \epsilon_j)) = \prod_{j=1}^{k} E(e^{ita_j \epsilon_j}) = \prod_{j=1}^{k} \cos(ta_j)$$

and hence by Hölder’s inequality

$$\int_{|t| \leq 1} E(\exp(it \sum_{j=1}^{k} a_j \epsilon_j)) \, dt \leq \prod_{j=1}^{k} \left( \int_{|t| \leq 1} |\cos(ta_j)|^k \, dt \right)^{1/k}.$$

But since each $a_j$ has magnitude at least 1, it is easy to check that $\int_{|t| \leq 1} |\cos(ta_j)|^k \, dt = O(1/\sqrt{k})$, and the claim follows.

8. Extensions and Refinements

8.1. Singularity of more general random matrices. In [8], Komlós extended Theorem 1.3 by showing that the singularity probability is still $o(1)$ for a random matrix whose entries are i.i.d. random variables with non-degenerate distribution. By slightly modifying our proof of Theorem 1.3, we are able to prove a different extension.

We say that a random variable $\xi$ has $(c, \rho)$-property if

$$\min\{P(\xi \geq c), P(\xi \leq -c)\} \geq \rho.$$

Let $\xi_{ij}$, $1 \leq i, j \leq n$ be independent random variables. Assume that there are positive constants $c$ and $\rho$ (not depending on $n$) such that for all $1 \leq i, j \leq n$, $\xi_{ij}$ has $(c, \rho)$-property. The new feature here is that we do not require $\xi_{ij}$ to be identical.

**Theorem 8.2.** Let $\xi_{ij}$, $1 \leq i, j \leq n$ be as above. Let $M_n$ be the random matrix with entries $\xi_{ij}$. Then

$$P(\det M_n = 0) = o(1).$$

We only sketch the proof, which follows the proof of Theorem 1.3 very closely and uses the same notation: $X_1, \ldots, X_n$ are the row vectors of $M_n$ and $W_j$ is the subspace spanned by $X_1, \ldots, X_j$. We will show

$$\sum_{j=1}^{n-1} P(X_{j+1} \in W_j) = o(1). \quad (22)$$

This estimate is a consequence of the following two lemmas, which are generalization of Lemmas 2.1 and 3.1.
Lemma 8.3. Let $W$ be a $k$ dimensional subspace of $\mathbb{R}^n$. Then for any $1 \leq j \leq n$

$$\mathbf{P}(X_j \in W) \leq (1 - \rho)^{n-k}.$$  

Lemma 8.4. For any $n/2 \leq j \leq n$

$$\mathbf{P}(X_j \in W_{j-1}) = O(1/\sqrt{\ln n}).$$

The proof of Lemma 8.3 is the same as that of Lemma 2.1. The only information we need is that for any fixed number $x$ and any plausible $i, j$, $\mathbf{P}(\xi_{ij} = x) \leq 1 - \rho$.

To prove Lemma 8.4, let us consider the case $j = n$ (the proof is the same for other cases). We need to modify the definition of universality as follows.

We call a subset $V$ of $n$-dimensional vectors $k$-universal if for any set of $k$ indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and any sign sequence $\epsilon_1, \ldots, \epsilon_k$, one can find a vector $v \in V$, such that the $i_j$ coordinate of $v$ has sign $\epsilon_j$ and absolute value at least $c$.

In what follows, we set $l = \ln n/10$. We first show $X_1, \ldots, X_n$ is very likely to be $l$-universal. (Notice that the $X_j$ have different distribution.)

Lemma 8.5. With probability $1 - o(1/n)$, $X_1, \ldots, X_n$ is $l$-universal.

Proof of Lemma 8.5. Fix a set of indices and a sequence of signs. For any $1 \leq j \leq n$, the probability that $X_j$ fails is at most $1 - \rho^l$. The rest of the proof is the same. \hfill \Box

It follows that

Corollary 8.6. Let $H$ be a subspace spanned by $n - 1$ random vectors. Then with probability $1 - o(1/n)$, any unit vector perpendicular to $H$ has at least $l + 1$ coordinates whose absolute values are at least $1/Kn$, where $K$ is a constant depending on $c$.

The last ingredient is the following generalization of Lemma 3.3.

Lemma 8.7. Let $a_1, \ldots, a_k$ be real numbers with absolute values larger than one and $\epsilon_1, \ldots, \epsilon_k$ be independent random variables satisfying the $(c, \rho)$-property. Then for any interval $I$ of length one

$$\mathbf{P}\left(\sum_{i=1}^k a_i \epsilon_i \in I\right) = O(1/\sqrt{k}).$$

Theorem 8.2 follows from Corollary 8.6 and Lemma 8.7. To conclude, let us remark that statements more accurate than Lemma 8.7 are known (see e.g. [5]). However, this lemma can be proved using an argument similar to the one in Remark 7.2.
8.8. Determinants of more general random matrices. We say this set of random variables is $l$-regular if the random variables are independent, symmetric random variables whose 2nd, 4th, ..., 2$^l$th moments all equal one and there is a constant $K$ such that $|\xi_{ij}| \leq K$ with probability 1, for all $i, j$. Notice that again we do not require the entries to have identical distributions.

**Theorem 8.9.** Let $\epsilon$ be an arbitrary positive constant. With probability $1 - o(1)$, the $n \times n$ random matrix $M_n$ whose entries form a set of $l$-regular random variables satisfies the bound of Theorem 1.1

$$|\det M_n| \geq \sqrt{n!} \exp(-8ln^{1/2+\frac{\epsilon}{2l^2}}).$$

Notice that $l = 2$ already gives a non-trivial bound.

Let $\xi_{ij}$ be the entries of $M_n$. It is easy to see that the random variables in a $l$-regular set have the $(c, \rho)$ property for some small values of $c$ and $\rho$ (say $c = \rho = 1/10$). Thus, Lemma 2.1 holds for this model of random matrices. The only place where we need some modification is Lemma 2.2. Consider a row vector, say, $X = (\xi_1, \ldots, \xi_n)$ and a fixed subspace $W$ of dimension $d$. Again, we have (with the same notation as in Section 2)

$$\text{dist}(X, W)^2 = |X|^2 - |PX|^2 = |X|^2 - \sum_{j=1}^{n} \sum_{k=1}^{n} \xi_{ij} \xi_{ik} p_{jk}. $$

However, it is no longer the case that the last formula equals

$$n - d - \sum_{j=1}^{n} \sum_{k=1}^{n} \xi_{ij} \xi_{ik} a_{jk}$$

since $\xi_{ij}$ are not Bernoulli random variables. On the other hand, we can have something similar with a small extra error term.

**Lemma 8.10.** Let $l$ be a fixed positive integer. There is a constant $C$ depending on $K$ such that the following holds. For any positive number $\gamma < 1$ (which may depend on $n$) we have

$$\mathbb{P}(\text{dist}(X, W) \geq (1 - \gamma)\sqrt{n - d - Cn^{1/2} \ln n}) \geq 1 - \left(\frac{4^l(2l)!}{\gamma^{2l}(n - d)^l} + \frac{1}{n^2}\right).$$

(23)

**Proof.** It is easy to show, using Chernoff’s bound, that

$$|X|^2 = \sum_{j=1}^{n} \xi_{ij}^2 \geq n - \frac{C}{2} n^{1/2} \ln n$$
holds with probability at least $1 - 1/2n^2$, for some sufficiently large $C$. Similarly,

$$\sum_{j=1}^{n} \xi_{ij}^2 p_{jj} \leq d - \frac{C}{2} n^{1/2} \ln n$$

holds with probability at least $1 - 1/2n^2$. (The use of Chernoff’s bound requires of random variables be bounded. One can of course, use some other method to remove this assumption.) It suffices to show that with probability at least $1 - C_l \gamma^{2l} (n - d)^l$,

$$|\sum_{j=1}^{n} \sum_{k=1}^{n} \xi_{ij} \xi_{ik} a_{jk}| \leq \gamma \sqrt{n - d}.$$ 

In this situation, we cannot use Bonamie’s inequality, but a similar statement can be proved directly using moment estimates. In fact, this was our original approach for Theorem 1.1.

Lemma 8.11. Let $A = (a_{jk})_{1 \leq j, k \leq n}$ be a real symmetric matrix with zero diagonal, and let $\epsilon_1, \ldots, \epsilon_n$ be a set of $l/2$-regular random variables for some even integer $l$. Then

$$E(|\sum_{j=1}^{n} \sum_{k=1}^{n} \epsilon_j \epsilon_k a_{jk}|^l) \leq 2^l l! (\text{tr} A^2)^{l/2}. \quad (24)$$

Proof Consider the matrix $A$ together with the complete graph $K_n$ on $\{1, \ldots, n\}$. The entry $a_{ij}$ is associated with the edge between $i$ and $j$ in the graph. Let $\Omega_l$ be the set of closed walks of length $l$ in $K_n$. Associate to each walk $W$ in $\Omega_l$ the product $p_W$ of the entries corresponding to the edges in the walk. It is well known (and fairly easy to verify directly) that

$$\sum_{W \in \Omega_l} p_W = \text{tr}(A^l). \quad (25)$$

Now let us focus on the matrix $A$ in question and express the left hand side of (24) as

$$\sum_{1 \leq j_1, \ldots, j_l, k_1, \ldots, k_l \leq n} a_{j_1 k_1} \cdots a_{j_l k_l} E(\epsilon_{j_1} \epsilon_{k_1} \cdots \epsilon_{j_l} \epsilon_{k_l}).$$

The expectation vanishes unless each variable $\epsilon_i$ occurs an even number of times, in which case the expectation is one.

Consider the complete graph $K_n$. The set of edges corresponding to a non-vanished expectation covers each vertex an even number of times. By Euler’s theorem, such
a set can be decomposed into a union of vertex disjoint closed walks. Applying (25), it follows that the expectation in question is a linear combination of terms of the form

$$\prod_{i=1}^{s} \text{tr}(A^{h_i})$$

where $1 \leq s \leq l$ and $\sum_{i=1}^{s} h_i = l$. Since $\text{tr}(A) = 0$, we can assume that all $h_i$ are at least 2. Moreover, for any $h \geq 2$

$$\text{tr}(A^h) \leq \text{tr}(A^{2^{h/2}}),$$

which yields that every term is at most $\text{tr}(A^2)^{l/2}$. To conclude the proof, notice that the coefficients are at most $l!$ and the number of terms is at most the number of ways to represent $l$ as a sum of positive integers, which is (very crudely) at most $2^l$.

To prove Theorem 8.9, first notice that the extra probability $1/n^2$ is negligible. The difference is that we have to multiply $(1 - \gamma_j)(n - j) - Cn^{1/2} \ln n$ (over $j$) instead of $(1 - \gamma_j)(n - j)$ and also the definition of $\gamma_j$ is different and $l$ is fixed. Fix a small constant $\epsilon > 0$ and define

$$\gamma_j = 2l(n - j)^{-1/2 + \frac{(1+\epsilon)}{2l}}.$$

This definition of $\gamma_j$ guarantees that

$$\sum_{j} 4^{l}(2l)! \gamma_j^{2l}(n - j)^{l} = \sum_{j} (n - j)^{-1-\epsilon}$$

converges. With this definition of $\gamma_j$, the main error term

$$\exp(-2 \sum_{j=1}^{n-d_0} \gamma_j)$$

can be bounded from below by

$$\exp(-8l^{1/2 + \frac{1+\epsilon}{2l}}).$$

The error term arising from using $(1 - \gamma_j)(n - j) - Cn^{1/2} \ln n$ instead of $(1 - \gamma_j)(n - j)$ is
\[
\prod_{j=1}^{d_0} \frac{(1 - \gamma_j)(n - j) - Cn^{1/2} \ln n}{(1 - \gamma_j)(n - j)}
\]

which can be bounded from below by

\[
\exp(-2Cn^{1/2} \ln n \sum_{j=1}^{d_0} (n - j)^{-1}) \leq \exp(-4Cn^{1/2} \ln n),
\]

which is negligible and this concludes the proof.

In certain situations, we do not have the assumption that \(|\xi_{ij}|\) are bounded from above by a constant. We are going to consider the following model. Let \(\xi_{ij}, 1 \leq i, j \leq n\) be i.i.d. random variables with mean zero and variance one. Assume furthermore that their fourth moment is finite. Consider the random matrix \(M_n\) with \(\xi_{ij}\) as its entries.

By using Lemmas 2.1 and 3.1 and replacing Lemma 2.2 by a result of Bai and Yin [1], which asserts that the volume of the \((1 - \gamma)n\)-dimensional parallelepiped spanned by the first \((1 - \gamma)n\) row vectors is at least \(n(1/2 - \gamma/2 - o(1))n\) with probability \(1 - o(1)\) for any fixed \(\gamma > 0\), we can prove

**Theorem 8.12.** We have, with probability \(1 - o(1)\), that

\[
|\det M_n| \geq n^{(1/2 - o(1))n}.
\]

**8.13. A better bound for the singularity probability.** The limit of the approach used to prove Theorem 1.6 is \(\varepsilon = 1/4\), which would give an upper bound of \((3/4 + o(1))^n\) for \(P(\det(M_n) = 0)\). It has turned out that we can actually achieve this limit. However, the proof needs several new ideas and is far more complicated than the one presented here. The details will appear in a future paper; we give only a brief sketch of the ideas here.

There are two main hurdles to overcome in order to improve the value of \(\varepsilon\) in Theorem 1.6 to 1/4. The first is to eliminate the loss arising from the entropy function \(h(\varepsilon)\), which prevents us from taking \(\varepsilon = \log_2 16/15\). The second is to increase the \(\mu\) parameter from 1/16 to 1/4 - \(o(1)\) without affecting the remainder of the argument.

We first discuss the loss arising from the entropy function. This loss arises in computing \(P(A_V \land B_V)\), when considering which \(n - 1\) vectors in

\[
X_1, \ldots, X_n, X_1^{(1/16)}, \ldots, X_{(1-\gamma)n}^{(1/16)}, X_1', \ldots, X_{(\gamma - \varepsilon)n}.'\]
span $V$. On the other hand, if the points in $\{-1,1\}^n \cap V$ were spread around in “general position”, then we would expect almost every $n-1$-tuple of points in this set to span $V$, in which case we would not suffer the entropy loss. This general position hypothesis may not be true for some spaces $V$, because most of the points in $\{-1,1\}^n \cap V$ could in fact be concentrated in a lower-dimensional subspace $W \subset V$. However, in that case the probability that $X_1, \ldots, X_n$ lie in $V$ is to a large extent controlled by the probability that $X_1, \ldots, X_n$ lie in $W$, and so we should somehow pass from “irregular” subspaces $V$ (in which $\{-1,1\}^n \cap V$ is not well distributed) to “regular” subspaces $W$, in which the set $\{-1,1\}^n \cap W$ is not concentrated in any smaller subspace. It turns out that this reduction to regular subspaces can be achieved fairly easily; the only catch is that one now has to consider the contribution of spaces $W$ of codimension larger than one. However, while this higher codimension serves to make the arguments messier, it does not introduce any significant new obstacles to the proof. Pursuing this idea one can already improve $\varepsilon$ to $\log_2 16/15 = 0.0931 \ldots$

To improve $\varepsilon$ further, to $1/4$, we must first improve $\mu$ to $1/4 - o(1)$. The starting point is the trigonometric inequality

$$|\cos(x)| \leq \frac{3}{4} + \frac{1}{4} \cos 2x$$

which is sharp in the sense that one cannot replace $\frac{3}{4}$ by $\frac{3}{4} - \varepsilon$ and $\frac{1}{4} + \varepsilon$ for any $\varepsilon$. If we set $\mu$ to equal $1/4 - o(1)$ instead of $1/16$, then in the notation of Section 7 we can then show that $F(\xi) \leq G(\xi)^{1+o(1)}$. This is enough to prove that

$$\mathbf{P}(X \in V) \leq (1 - o(1))\mathbf{P}(X^{(1/4-o(1))} \in V),$$

but this is not directly useful for the above arguments (we now do not obtain the crucial gain of $N^{1-\gamma}$ in (14)). Part of the problem is that the Mann-Kneser-Macbeath inequality is too weak to be useful for this large value of $\mu$. However, we can do better using inverse theorems in additive number theory such as Freiman’s theorem. Such inverse theorems can be viewed as a dichotomy: if $A$ is a subset of the unit circle $\mathbb{R}/\mathbb{Z}$, then either $|A+A|$ is much larger than $|A|$ (as opposed to the bound of $\max(2|A|, 1)$ given by Mann-Kneser-Macbeath) or $A$ has substantial arithmetic structure (roughly speaking, it can be efficiently contained inside a “generalized arithmetic progression”). Inserting this type of result into the above Halasz-type analysis, we can thus conclude a bound of the form

$$\mathbf{P}(X \in V) \leq \delta \mathbf{P}(X^{(1/4-o(1))} \in V)$$

for any small $\delta$ we please, unless $V$ (or more precisely the coefficients of the unit normals of $V$) has substantial arithmetic structure. This $\delta$ gain can then be used in much the same way the $\frac{1}{2}$ gain in Lemma 6.2 was used to show that the contribution of the “non-arithmetic” spaces $V$ to $\mathbf{P}(\det M_n = 0)$ is at most $(C\delta)^n$, at least for spaces of codimension at most $(\log_2 \frac{4}{3} + o(1))n$; the spaces of higher codimension cannot take advantage of the random variable $X^{(1/4-o(1))}$ (whose presence in such spaces may come entirely from the origin), and we can only crudely bound the contribution of those spaces by $(3/4 + o(1))^n$ by the same argument used to prove (10) when $d$ is large. This leaves us with the arithmetic spaces, but these can be
counted directly (by a straightforward but messy argument) and end up contributing an even more negligible contribution to $P(\det M_n = 0)$, namely $O(n^{-n/2+o(n)})$, where the constants in $o(n)$ depend on the choice of $\delta$.

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