Multiqubit Clifford groups are unitary 3-designs

Huangjun Zhu

Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada and Institute for Theoretical Physics, University of Cologne, Cologne 50937, Germany

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We show that the multiqubit (including qubit) Clifford group in any even prime power dimension is not only a unitary 2-design, but also a unitary 3-design. Moreover, it is a minimal unitary 3-design except for dimension 4. As an immediate consequence, any orbit of pure states of the multiqubit Clifford group forms a complex projective 3-design; in particular, the set of stabilizer states forms a 3-design. By contrast, the Clifford group in any odd prime power dimension is only a unitary 2-design. In addition, we show that no operator basis is covariant with respect to any group that is a unitary 3-design, thereby providing a simple explanation of why no discrete Wigner function is covariant with respect to the multiqubit Clifford group.

The Clifford group plays a fundamental role in various branches of quantum information science, such as quantum computation [1–4], quantum error correction [1–3], quantum tomography [5,17], and randomized benchmarking [8–10]. It is also useful to studying a number of interesting discrete symmetric structures, such as discrete Wigner functions [11–13], mutually unbiased bases, and symmetric informationally complete measurements. Many nice properties of the Clifford group are closely related to the fact that the group is a unitary 2-design [5,6,14–20], which means that random Clifford unitaries give a good approximation to random Haar unitaries. In addition, any orbit of pure states forms a (complex projective) 2-design [21–24] and may be used to approximate random pure states and to construct tight informationally complete measurements [22,25].

In many applications, 2-designs are not sufficient, and 3-designs are desirable. Prominent examples include optimal estimation of pure states [26], construction of isotropic measurements [25,27], derandomization of measurements in quantum state tomography [28,29], quantum state tomography with compressed sensing [7,30], and phase retrieval with the phase lift algorithm [31]. However, examples of unitary and projective 3-designs with $t \geq 3$ are quite rare.

Here we show that the multiqubit (including qubit) Clifford group is not only a unitary 2-design, but also a 3-design. Moreover, it is minimal in the sense that it does not contain any proper subgroup that is also a unitary 3-design except for dimension 4. As an immediate consequence, any orbit of pure states of the multiqubit Clifford group forms a 3-design; in particular, the set of stabilizer states forms a 3-design [24]. By contrast, the Clifford group in any odd prime power dimension is only a unitary 2-design. These results may have wide applications in various research areas, such as quantum computation, quantum tomography, randomized benchmarking, and phase retrieval.

The distinction between discrete Wigner functions in even and odd prime power dimensions has been an elusive question and has profound implications for various interesting subjects, such as computational speedup, contextuality [33,34], and discrete Hudson theorem [12]. In each odd prime power dimension, the discrete Wigner function introduced by Wootters [11] is covariant with respect to the Clifford group [12]; by contrast, no discrete Wigner function is covariant with respect to the multiqubit Clifford group [59]. Here we show that this latter conclusion is a direct consequence of our main result by proving that no operator basis is covariant with respect to any group that is a unitary 3-design. Actually, this work was motivated by an attempt to better understand discrete Wigner functions [13] (which was in turn inspired by Ref. [35]).

A set of pure states $\{|\psi_j\rangle\}$ in dimension $d$ is a $t$-design for a positive integer $t$ if $\sum_j (|\langle \psi_j | \psi_j \rangle|^t)^{\otimes t}$ is proportional to the projector onto the symmetric subspace of the $t$-partite Hilbert space [21,23]. A set of $K$ unitary operators $\{U_j\}$ is a unitary $t$-design [5,16,17] if it satisfies

$$\frac{1}{K} \sum_j U_j^{\otimes t} A(U_j^{\otimes t})^\dagger = \int dU U^{\otimes t} A(U^{\otimes t})^\dagger$$

for any operator $A$ acting on the $t$-partite Hilbert space, where the integral is taken over the whole unitary group with respect to the normalized Haar measure. By definition, a unitary $t$-design is also a $t'$-design for $t' < t$. Note that the above equation remains intact when $U_j$ are multiplied by phase factors, so what we are concerned are actually projective unitary $t$-designs. Alternatively, the set $\{U_j\}$ is a unitary $t$-design if the $t$th frame potential

$$\Phi_t(\{U_j\}) := \frac{1}{K^2} \sum_{j,k} |\text{tr}(U_j^\dagger U_k^\otimes t)|^{2t}$$

attains the minimum $\gamma(t, d) := \int dU |\text{tr}(U)|^{2t}$ [5,17,18]. Here we only need $\gamma(t, d)$ in two special cases [5,30].

$$\gamma(t, d) = \begin{cases}
\frac{(2t)!}{(t!^t)!}, & d = 2, \\
\frac{(2t)!}{(t!^t)!}, & d \geq t.
\end{cases}$$
Most examples of unitary designs are constructed from subgroups of the unitary group, which are referred to as (unitary) group designs. Given a group $G$ of unitary operators, in most cases we are only concerned with the quotient $\overline{G}$ of $G$ over the phase factors. The frame potential of $\overline{G}$ takes on the form
\[
\Phi_t(\overline{G}) := \frac{1}{|\overline{G}|} \sum_{U \in \overline{G}} |\text{tr}(U)|^{2t} = \frac{1}{|G|} \sum_{U \in G} |\text{tr}(U)|^{2t},
\]
which coincides with the sum of squared multiplicities of irreducible components of $\overline{G}(t) := \{U^t \mid U \in \overline{G}\}$. The group $\overline{G}$ is a unitary $t$-design if and only if $\overline{G}(t)$ has the same number of irreducible components as $U(t)$, where $U$ denotes the group of all unitary operators $[17]$. For example, the group $\overline{G}$ is a unitary 1-design if and only if it is irreducible. It is a 2-design if $G(t)$ has two irreducible components, which correspond to the symmetric and antisymmetric subspaces of the bipartite Hilbert space. Prominent examples of unitary group 2-designs include Clifford groups and restricted Clifford groups in prime power dimensions $[6, 14–17]$. Not much is known about unitary $t$-designs with larger $t$.

Before presenting our main results, we need to introduce the (multipartite) Heisenberg-Weyl (HW) group. In prime dimension $p$, the HW group $D$ is generated by the phase operator $Z$ and the cyclic-shift operator $X$,
\[
Z |u\rangle = \omega^u |u\rangle, \quad X |u\rangle = |u + 1\rangle,
\]
where $\omega = e^{2\pi i/p}$, $u \in \mathbb{F}_p$, and $\mathbb{F}_p$ is the field of integers modulo $p$. In prime power dimension $q = p^n$, the HW group $D$ is the tensor power of $n$ copies of the HW group in dimension $p$. The elements in the HW group are called displacement operators (or Weyl operators). Up to phase factors, they can be labeled by vectors in $\mathbb{F}_p^{2n}$ as $D_{\mu} = \tau \mathbf{1}_{p^n} \otimes |\mu\rangle \langle \mu|$, where $\tau = e^{2\pi i/p}$, and $Z_j$ and $X_j$ are the phase operator and cyclic shift operator of the $j$th party. These operators satisfy the commutation relation $D_{\mu} D_{\nu} D_{\mu}^\dagger D_{\nu}^\dagger = \omega^{(\mu, \nu)}$, where $J = (\mathbf{1}_{p^n}^T \otimes \mathbf{1}_{p^n}^T)$. The symplectic group Sp(2n,$p$) is the group of linear transformations on $\mathbb{F}_p^{2n}$ that preserves the symplectic product.

The (full) Clifford group $\overline{C}$ is composed of all unitary operators that map displacement operators to displacement operators up to phase factors $[1, 12, 13, 37, 39]$. It is referred to as the multiqubit Clifford group when the dimension is a power of 2 (including 2). Any Clifford unitary $U$ induces a symplectic transformation $F$ on the symplectic space $\mathbb{F}_p^{2n}$ that labels the displacement operators. Conversely, given any symplectic matrix $F$, there exist $q^2$ Clifford unitaries (up to phase factors) that induce $F$ $[37, 39]$. The quotient $\overline{C}/\overline{D}$ can be identified with the symplectic group Sp(2n,$p$) $[37, 38]$.

The symplectic space $\mathbb{F}_p^{2n}$ can also be identified with a two-dimensional vector space over $\mathbb{F}_q$. The special linear group SL(2, $q$) on this space is an extension-field-type subgroup of Sp(2n,$p$). The restricted Clifford group $\overline{C}_t$ (coinciding with the full Clifford group when $q$ is prime) is the subgroup of $\overline{C}$ whose quotient $\overline{C}_t/\overline{D}$ corresponds to SL(2, $q$); see Refs. $[12, 17, 24, 40]$ for more details.

Theorem 1. The multiqubit Clifford group is a unitary 3-design but not a 4-design. The Clifford group in any odd prime power dimension is only a unitary 2-design. The restricted Clifford group in any prime power dimension is only a unitary 2-design except for dimension 2.

To prove Theorem 1 we shall compute the frame potentials of the Clifford group up to fourth order and that of the restricted Clifford group for any order, from which the theorem follows. Our approach also applies to subgroups of the Clifford group that contain the HW group.

Lemma 1. Suppose $\overline{G} \geq \overline{D}$ is a subgroup of the Clifford group $\overline{C}$ in dimension $q = p^n$ and $R = \overline{C}/\overline{D}$ (taken as a subgroup of Sp(2n,$p$)). Then
\[
\Phi_t(\overline{G}) = \frac{1}{|R|} \sum_{F \in R} f(F)^{t-1},
\]
where $f(F)$ is the number of fixed points of $F$ in $\mathbb{F}_p^{2n}$. Moreover, $\Phi_t(\overline{C})$ is equal to the number of orbits of $R$ on $(\mathbb{F}_p^{2n})^\times (t-1)$. The group $\overline{G}$ is a unitary 2-design if and only if $R$ is transitive on $\mathbb{F}_p^{2n}$. It is a unitary 3-design if and only if $R$ is 2-transitive when $n = 1$ and is a rank-3 permutation group when $n \geq 2$.

Remark 1. $\mathbb{F}_p^{2n}$ is the set of nonzero vectors in $\mathbb{F}_p^{2n}$. A subgroup of Sp(2n,$p$) is transitive if it can map any nonzero vector to any other and 2-transitive or doubly transitive if it can map any pair of distinct nonzero vectors to any other pair. It is a rank-3 permutation group if it is transitive and each point stabilizer has three orbits on $\mathbb{F}_p^{2n}$ including the orbit of the fixed point $[35, 41, 42]$.

Proof. Let $F \in R$ and $U_F$ be a Clifford unitary that induces the transformation $F$; then $U_F D_{\mu}$ induces the same transformation for all $\mu \in \mathbb{F}_p^{2n}$. According to a similar argument as in the proof of Theorem 2.34 in Zauner’s thesis $[43]$ (see also Ref. $[27]$), $|\text{tr}(U_F D_{\mu})|^2$ is either zero or equal to the number of displacement operators that commute with $U_F$, which is in turn equal to the number $f(F)$ of fixed points of $F$ in $\mathbb{F}_p^{2n}$. On the other hand, we have $\sum_{\mu\in\mathbb{F}_p^{2n}} |\text{tr}(U_F D_{\mu})|^2 = q^2$ given that the HW group is a unitary error basis. It follows that $|\text{tr}(U_F D_{\mu})|^2 = f(F)$ for $q^2/p f(F)$ of displacement operators $D_{\mu}$.

\[
\Phi_t(\overline{G}) = \frac{1}{q^2 |R|} \sum_{F \in R} \sum_{\mu} |\text{tr}(U_F D_{\mu})|^{2t} = \frac{1}{q^2 |R|} \sum_{F \in R} f(F)^t q^2 f(F) = \frac{1}{|R|} \sum_{F \in R} f(F)^{t-1}. \tag{7}
\]
According to the orbit-stabilizer relation, \( \Phi_t(G) \) is equal to the number of orbits of \( R \) on \((\mathbb{F}_p^{2n}) \times (t-1)\).

In view of Eqs. (3) and (4), the group \( \overline{G} \) is a unitary 2-design if and only if \( R \) has two orbits on \( \mathbb{F}_p^{2n} \) and is transitive on \( \mathbb{F}_p^{2n} \). \( \overline{G} \) is a unitary 3-design if and only if \( R \) has five orbits on \( \mathbb{F}_p^{2n} \times 2 \) when \( n = 1 \) and six orbits when \( n \geq 2 \); that is, \( R \) has two orbits on \( \mathbb{F}_p^{2n} \times 2 \) and is 2-transitive when \( n = 1 \), and it has three orbits and rank-3 when \( n \geq 2 \).

**Proof of Theorem 1.** According to Lemma 1, the frame potential \( \Phi_t(G) \) is equal to the number of orbits of \( \text{Sp}(2n, p) \) on \((\mathbb{F}_p^{2n}) \times (t-1) \). The number is 2 when \( t = 2 \) given that \( \text{Sp}(2n, p) \) is transitive [41, 42]. When \( t = 3 \), let \( 0 = (0,0,\ldots,0)^t \), \( 1 = (1,0,\ldots,0)^t \in \mathbb{F}_p^{2n} \). Then any orbit on \( \mathbb{F}_p^{2n} \times 2 \) contains one of the following elements \((0,0),(0,1),(1,0),(1,1)\), and \((1,\mu)\), where \( \mu \neq 0,1 \). The vector \( \mu \) is a fixed point of the stabilizer of \( 1 \) if and only if it is proportional to \( 1 \); there are \( p - 2 \) such fixed points excluding \( 0,1 \). Suppose \( \mu, \nu \in \mathbb{F}_p^{2n} \) are not proportional to \( 1 \), then \((1,\mu)\) and \((1,\nu)\) are on the same orbit if and only if the symplectic products \((1,\mu)\) and \((1,\nu)\) are equal by Witt’s lemma [41]. When \( n > 1 \), \((1,\mu)\) may take on any value in \( \mathbb{F}_p \), while it is nonzero when \( n = 1 \). So there are \( 2p + 1 \) orbits in total when \( n = 1 \) and \( 2p + 2 \) orbits when \( n > 1 \). Similar reasoning applies to frame potentials of higher orders. Here are those up to order 4,

\[
\Phi_2(G) = 2; \\
\Phi_3(G) = \begin{cases} 2p + 1, & n = 1, \\ 2p + 2, & n \geq 2; \end{cases} \\
\Phi_4(G) = \begin{cases} p^3 + p^2 + p + 1, & n = 1, \\ 2p^3 + 2p^2 + 2p + 1, & n = 2, \\ 2(p^3 + p^2 + p + 1), & n \geq 3. \end{cases}
\]

The same result also applies to other subgroups of the Clifford groups (the restricted Clifford group for example) whose quotients over the HW group are isomorphic to \( \text{Sp}(2n, p^k) \) with \( mk = n \) if we replace \( n \) with \( m \) and \( p \) with \( p^k \). Now Theorem 1 can be verified by comparing the above frame potentials with the minimum potentials in Eq. (3). In addition, these frame potentials show that the Clifford group in dimension 2 is quite close to a unitary 4-design. The one in dimension 3 is quite close to a unitary 3-design; the larger the prime \( p \), the farther away is the Clifford group from being a unitary 3-design.

For the restricted Clifford group, the summation over \( f(F)^{t-1} \) in Eq. (4) can be evaluated directly, note that \( f(F) = q \) for the \( q^2 - 1 \) order-\( p \) elements in \( \text{SL}(2, q) \) and \( f(F) = 1 \) for other \( q^3 - q^2 - q \) nonidentity elements (see Refs. [15, 47] for the conjugacy classes of \( \text{SL}(2, q) \)),

\[
\Phi_t(G) = \frac{q(q^{2t-4} - 1)}{q^2 - 1} + q^{t-2} + 1. 
\]
Theorem 1 also has a profound implication for studying discrete Wigner functions. In particular, it implies that no discrete Wigner function is covariant with respect to the multiqubit Clifford group, which complements our earlier result in Ref. [13]. To elucidate this point, it suffices to show that no operator basis is covariant with respect to the multiqubit Clifford group, note that any Clifford covariant discrete Wigner function determines a Clifford covariant operator basis. For example, in each odd prime power dimension, the discrete Wigner function introduced by Wootters [11] determines the phase point basis (the operator basis composed of phase point operators), and vice versa [12] [13]. Here an operator basis \{\mathcal{L}_j\} is covariant with respect to the group \(G\) of unitary operators if \(G\) leaves this basis invariant and acts transitively on the basis operators. So any \(U \in G\) induces a permutation \(\sigma\) as \(UL_jU^{-1} = L_{\sigma(j)}\).

**Theorem 4.** No operator basis is covariant with respect to a unitary group 3-design. No discrete Wigner function is covariant with respect to the multiqubit Clifford group.

**Proof.** In view of Theorem 1, it suffices to prove the first statement. Suppose on the contrary that \(\{\mathcal{L}_j\}\) is an operator basis that is covariant with respect to the unitary group 3-design \(\mathcal{G}\) in dimension \(d\). Then \(\Phi_2(\mathcal{G}) = 2\) and \(\Phi_3(\mathcal{G}) = 6\) (\(\Phi_3(\mathcal{G}) = 5\) when \(d = 2\)). Note that \(\{\mathcal{L}_j \otimes \mathcal{L}_k\}\) and \(\{\mathcal{L}_j \otimes \mathcal{L}_k \otimes \mathcal{L}_l\}\) form operator bases for the bipartite and tripartite Hilbert spaces, respectively. According to Lemma 1 in Ref. [13] (cf. Lemma 7.2 in Ref. [27]), \(\mathcal{G}\) acts transitively on ordered pairs of distinct operators in \(\{\mathcal{L}_j\}\) and has two orbits (one orbit when \(d = 2\)) on ordered triples. The triple products \(\text{tr}(\mathcal{L}_j \mathcal{L}_k \mathcal{L}_l)\) for distinct \(j, k, l\) must all be equal and thus real when \(d = 2\), and they can take on at most two different values when \(d \geq 3\). However, these triple products cannot all be real, since, otherwise, the basis operators would commute with each other and thus cannot form an operator basis. When \(d = 2\), this contradiction confirms the theorem. When \(d \geq 3\), these triple products must take on two distinct values, which are complex conjugates of each other. Consequently, \(\mathcal{G}\) acts transitively on unordered triples; that is, \(\mathcal{G}\) is 3-homogeneous in the language of permutation groups [35] [11] [12]. According to Theorem 1 of Kantor [52] (see also Theorem 9.4B in Ref. [11] and Lemma 2 in Ref. [35]), any 3-homogeneous permutation group on \(m\) objects with \(m \geq 9\) a perfect square is 3-transitive. Therefore, \(\mathcal{G}\) acts transitively on ordered triples, so that all triple products \(\text{tr}(\mathcal{L}_j \mathcal{L}_k \mathcal{L}_l)\) are real, in contradiction with the previous observation.

In summary, we showed that the multiqubit Clifford group is a unitary 3-design. It is also a minimal 3-design except for dimension 4. As a consequence, any orbit of pure states of the multiqubit Clifford group forms a 3-design; in particular, the set of multiqubit stabilizer states forms a 3-design. We also showed that no operator basis is covariant with respect to any group that is a unitary 3-design, thereby providing a simple explanation of why no discrete Wigner function is covariant with respect to the multiqubit Clifford group. Our study is of broad interest to researchers on quantum computation, quantum tomography, randomized benchmarking, and signal processing, etc.

Note added: upon completion of this work, we noticed a comprehensive (also long and technically demanding) paper by Robert M. Guralnick and Pham Huu Tiep [53], from which it is possible to deduce our Theorems 1 and 2 with some additional work. However, this paper mentions neither t-designs nor the Clifford group, though the latter appears in a disguised form. In addition, their results rely on Hering’s theorem, which relies on the classification of finite simple groups (CFSG). Our proofs are completely independent of the CFSG and are thus simpler and more transparent. Recently (Sep 2015), unaware of our work (our draft without Theorem 2 was completed in May 2015 and shared with a number of experts in the field), Zak Webb also proved that the multiqubit Clifford group is a unitary 3-design, which offers a complementary perspective to our approach [54].

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**Appendix: Proof of Theorem 3**

To prove Theorem 3 we need to introduce several auxiliary results following Refs. [49] [50]. A subgroup of \(\text{Sp}(2n, 2)\) is primitive if it acts transitively on nonzero vectors and preserves no nontrivial partition. It is anti-flag transitive if it acts transitively on all pairs \((\mu, A)\) of nonzero vector \(\mu\) and hyperplane \(A\) such that \(\mu \notin A\).

**Lemma 2.** Any rank-3 subgroup of \(\text{Sp}(2n, 2)\) for \(n \geq 2\) is primitive.

**Proof.** Let \(R\) be a rank-3 subgroup, then \(R\) is transitive and the point stabilizer of any nonzero vector \(\mu\) partitions the remaining nonzero vectors into two orbits according to their symplectic products with \(\mu\). Therefore, the stabilizer has two orbits on \(\mathbb{F}_2^{2n}\) of lengths \(2^{2n-1} - 2\) and...
$2^{2n-1}$. If $R$ is not primitive, then any block in a nontrivial partition has size either $2^{2n-1} - 1$ or $2^{2n-1} + 1$. On the other hand, the size must be a divisor of $2^{2n-1} - 1$. This contradiction shows that $R$ is primitive.

**Lemma 3.** Any rank-3 subgroup of $\text{Sp}(2n, 2)$ for $n \geq 2$ is antiflag transitive.

**Proof.** Let $R$ be a rank-3 subgroup, then $R$ is transitive and the point stabilizer of any nonzero vector $\mu$ has two orbits on the remaining nonzero vectors. Denote by $\mu^\perp$ the hyperplane composed of all vectors that are orthogonal to $\mu$ with respect to the given symplectic product. Then the map $\mu \mapsto \mu^\perp$ sets a one-to-one correspondence between vectors and hyperplanes, which is preserved by the symplectic group, that is, $g\mu^\perp = (g\mu)^\perp$ for any $g \in \text{Sp}(2n, 2)$. Let $\nu_1^\perp, \nu_2^\perp$ be two hyperplanes that do not contain $\mu$, that is $\langle \mu, \nu_1 \rangle = \langle \mu, \nu_2 \rangle = 1$. Then the point stabilizer of $\mu$ within $R$ can map $\nu_1$ to $\nu_2$ and, accordingly, $\nu_1^\perp$ to $\nu_2^\perp$. So $R$ is antiflag transitive.

**Proof of Theorem 5** Let $R$ be a proper rank-3 subgroup of $\text{Sp}(2n, 2)$ with $n \geq 2$, then $R$ is primitive, antiflag transitive, but not 2-transitive. So one of the following three cases holds according to Theorem 2.2 in Ref. [50].

1. $n = 2$ and $R$ is $A_6$ inside $\text{Sp}(4, 2)$;
2. $n = 3$ and $R$ is isomorphic to $\text{PSU}(3, 3)$ (also written as $\text{PSU}(3, 3^2)$ depending on the convention);
3. $n = 3$ and $G_2(2) \leq R \leq \text{Aut}(G_2(2))$.

To complete the proof, it suffices to rule out cases 2 and 3. Recall that the point stabilizer within $R$ of any nonzero vector has two orbits of lengths $2^{2n-1} - 2$ and $2^{2n-1}$, respectively. The order of the stabilizer is divisible by the least common multiple of $2^{2n-1} - 2$ and $2^{2n-1}$, that is, $2^{2n-1}(2^{2n-2} - 1)$. So the order of $R$ is divisible by $2^{2n-1}(2^{2n-2} - 1)(2^{n} - 1)$, which is equal to 30240 when $n = 3$. Cases 2 and 3 cannot happen given that $\text{PSU}(3, 3)$ has order 6048, and that $G_2(2)$ has order 12096 and trivial outer automorphism. [51].
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