Skewness Dependence of GPD / DVCS, Conformal OPE and AdS/CFT Correspondence II: — a holographic model of GPD —

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Abstract

Traditional idea of Pomeron/Reggeon description for hadron scattering is now being given theoretical foundation in gravity dual descriptions, where Pomeron corresponds to exchange of spin-\(j \in 2\mathbb{Z}\) states in the graviton trajectory. Deeply virtual Compton scattering (DVCS) is essentially a 2 to 2 scattering process of a hadron and a photon, and hence one should be able to study non-perturbative aspects (GPD) of this process by the Pomeron/Reggeon process in gravity dual. We find, however, that even one of the most developed formulations of gravity dual Pomeron (Brower–Polchinski–Strassler–Tan (BPST) 2006) is not able to capture skewness dependence of GPD properly. In Part I (arXiv:1212.3322), therefore, we computed Reggeon wavefunctions on AdS\(_5\) so that the formalism of BPST can be generalized. In this article, Part II, we use the wavefunctions to determine the DVCS amplitude, bring it to the form of conformal OPE/collinear factorization, and extract a holographic model of GPD, which naturally fits into the framework known as “dual parametrization” or “(conformal) collinear factorization approach”.

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Introduction to Part II

This preprint is a continuation of the study in another preprint [1]; those two preprints share the same title and are regarded as part II and part I, respectively. They will be combined to be a single article when submitted to a journal. Since Part II has to refer to equations in Part I many times, the full text of Part I [1] (except Introduction) is included as a part of this preprint for convenience of the readers, after minimum corrections are made. Sections 2.1–5.3 and the appendices A.1–A.4 have appeared already in Part I [1]; the new material in part II is found in sections 5.4–7 and the appendices A.5–B.

We found that interesting preprints [2, 3] cover a subject that is closely related to our study in sections 5–6 and the appendix A. References [2, 3] mainly deal with correlation functions of CFT's as functions of spacetime coordinates, whereas we deal with them in this article and in [1] as functions of incoming/outgoing momenta, and confinement effects are also implemented, so that we can study hadron scattering processes.

1 Introduction: journal-article version

Scattering processes of hadrons involve non-perturbative information of QCD. When it comes to scattering with the center of mass energy higher than the QCD scale, lattice computation will not have enough computation power in a near future, yet perturbative QCD is able to say something only about hard components involved in the scattering. This is where holographic descriptions of strongly coupled gauge theories may find a role to play. Although we cannot expect gravitational “dual” descriptions to be both calculable and perfectly equivalent to the QCD of the real world at the same time, we still hope to be able to learn non-perturbative aspects of hadrons at qualitative level, using calculable holographic dual descriptions of nearly conformal strongly coupled gauge theories.
String theory started out as the dual resonance model describing scattering amplitudes of hadrons. One of its major problems as a theory of hadrons was a “prediction” that the amplitude of the elastic scattering of two hadrons falls off exponentially $e^{Bt}$ in the momentum transfer squared $t$ for some $B > 0$, although the amplitude is known in reality to fall off in a power-law in $|−t|$ for hard scattering. The “prediction,” however, is now understood as that of string theory with a flat background metric; the amplitude of elastic scattering turns into such a power-law indeed, when the target space of string theory has a warped metric. At the qualitative level, string theory on a warped spacetime—holographic (gravitational dual) descriptions—can be a viable theory of hadron scattering [4, 5, 6].

Holographic technique can be used to study not just amplitudes of hadron scattering as a whole, but also to extract information of partons within hadrons [5]. Parton distribution functions (PDFs) are defined by the inverse Mellin transformation of hadron matrix elements of gauge singlet parton-bilinear operators in QCD, and gravity dual descriptions can be used to determine matrix elements of the gauge singlet operators. The PDF extracted in this way satisfies DGLAP ($q^2$-evolution) and BFKL ($\ln(1/x)$-evolution) equations (e.g., [7, 8, 9, 10]); just like in perturbative QCD [11], those two evolution equations follow from how the saddle point $j^*$ moves in the complex angular momentum $j$-plane integral (inverse Mellin transform). The holographic description for the PDF and the generalized parton distribution (GPD) also shows crossover transition between this DGLAP/BFKL behavior and the Regge behavior [6] (see also [10]). Thus, the parton information studied in this way may be used to understand non-perturbative issues associated with partons in a hadron at qualitative level.

In this article, we study 2-body–2-body scattering between a hadron and a photon (that is possibly virtual) in gravitational dual descriptions; $\gamma^*(q_1) + h(p_1) \rightarrow \gamma^*(q_2) + h(p_2)$. A special case of this scattering—the forward scattering with $q_1 = q_2$ and $p_1 = p_2$—has been studied extensively in the literature (e.g., [5, 7, 8, 9, 10]) for study of DIS and PDF, and some references also cover the case of non-forward elastic scattering ($(q_1)^2 = (q_2)^2, (q_1 - q_2)^2 \neq 0$). This article extends the analysis so that all kinds of skewed $(q_1^2 \neq q_2^2)$ cases are covered. In hadron physics, therefore, the kinematics needed for deeply virtual Compton scattering, hard exclusive vector meson production and time-like Compton scattering processes [12] is covered in this analysis. With the full skewness dependence included in this analysis, it is also possible to use the result of this study to bridge a gap between data in such scattering processes at non-zero skewness [13] and the transverse profile of partons in a hadron, which is encoded by GPD at zero skewness [14].

From theoretical perspective, the task of this article is to generalize the formalism of [5, 6] (see also [7, 9, 10]), so that it can be used for 2-body-to-2-body scattering that is not necessarily elastic. Pomeron/Reggeon propagators have been treated as if it were for a scalar
field in [5, 6, 10], but they correspond to exchange of stringy states with non-zero (arbitrarily high) spins; for the study of scattering with non-zero skewness, the polarization of higher spin state propagator should also be treated with care (see also the approach in [2, 3]).

It is notoriously a difficult problem to compute scattering of strings on a curved background geometry. We do not pretend that the generalization of the formalism in this article is something derived from string theory without a flaw. This is rather an attempt at capturing an approximately right picture of non-perturbative aspects in hadron scattering that string theory would predict in a distant future. We are forced to rely sometimes on physics intuition, and to ignore subtleties or corrections that are not under control, when we face situations where not enough techniques have been developed in string theory at the moment.

This article is organized as follows. We begin in section 2.1 with a review of parameterization of GPD in terms of conformal OPE, because the expansion in a series of conformal primary operators becomes the key concept in using AdS/CFT correspondence (cf [8]). After plainly stating what needs to be done in the gravity dual approach in section 2.2, we proceed to explain our basic gravity dual setting and idea of how to construct a scattering amplitude of our interest by using string field theory in sections 3 and 4. Section 5 shows the results of computing wavefunctions of spin-\(j\) fields on AdS\(_5\), while more detailed account of derivation of the wavefunctions is given in the appendix A. Classification of eigenmodes that turn out to be relevant for the “twist-2” operators in later sections is given in section 5.1 and wavefunctions are presented as analytic functions of the complex spin (angular momentum) variable \(j\) in section 5.2. Those wavefunctions are organized into irreducible representations of conformal algebra in section 5.3, the representation for spin-\(j\) primary operators contain more eigenmode components than those treated by the Pomeron exchange amplitude in the formalism of [6], indicating that more contributions are needed in the scattering amplitude with non-vanishing skewness than in the formalism of [6]. These wavefunctions (and propagators) are used in section 6 in organizing scattering amplitude on AdS\(_5\). The amplitude obtained in this way can be cast into the form of conformal OPE, from which one can also extract GPD as a function of kinematical variables. We are not committed to a particular form of implementing confining effects in the holographic description, as discussed in section 5.4. Some qualitative aspects of the GPD profile are examined in section 7.

Not surprisingly, holographic models of GPD so obtained provide a special subclass of GPD models that have been called “dual parametrization” or “(conformal) collinear factorization approach” in QCD/hadron community [15, 16, 17, 18]. After all, it is the combination of the dual resonance model and the AdS/CFT correspondence that are being used.
2 Our Approach: Conformal OPE and Gravity Dual

2.1 Review: Conformal OPE of DVCS Amplitude

2.1.1 Notation and Conventions

Deeply virtual Compton scattering $\gamma^* + h \to h + \gamma$ (DVCS), hard exclusive vector meson production $e + h \to e + h + V$ (VMP) and time-like Compton scattering processes $e + h \to e + h + e^+e^-$ (TCS) are shown in Figure 1(a), (c) and (d), respectively. As a part of all these processes, the photon–hadron 2-body to 2-body scattering amplitude,

$$\mathcal{M}(\gamma^* h \to \gamma^{(*)} h) = (\epsilon_1^\gamma T_{\mu\nu}(\epsilon_2^{(*)})^*$$,  \hspace{1cm} (1)

is involved. This 2-body to 2-body scattering amplitude with this exclusive choice of the final states (Figure 2) is truly non-perturbative information, and this is the subject of this

2 There are two contributions from (a) the $\gamma^* + h \to \gamma + h$ virtual Compton scattering and (b) Bethe–Heitler process in the leptoproduction process of a photon on a target hadron $h$: $\ell + h \to \ell + \gamma + h$, and
Because the “final state” photon is required to be on-shell $q_2^2 = 0$ in DVCS and time-like $q_2^2 < 0$ in VMP and TCS, we are interested in developing a theoretical framework to calculate this non-perturbative amplitude in the case $q_2^2$ is different from space-like $q_1^2 > 0$.

Just like in QCD/hadron literature, we use the following notation for Lorentz invariant kinematical variables:

$$
p^\mu = \frac{1}{2} (p_1^\mu + p_2^\mu), \quad q^\mu = \frac{1}{2} (q_1^\mu + q_2^\mu), \quad \Delta^\mu = p_2^\mu - p_1^\mu = q_1^\mu - q_2^\mu, \quad (2)
$$

$$
x = -\frac{q^2}{2p \cdot q}, \quad \eta = \frac{-\Delta \cdot q}{2p \cdot q}, \quad s = W^2 = -(p + q)^2, \quad |t| = -\Delta^2. \quad (3)
$$

$q_2^2$ is called skewness; in the scattering process of our interest, $q_1^2 = q^2 + \Delta^2/4 + q \cdot \Delta$ and $q_2^2 = q^2 + \Delta^2/4 - q \cdot \Delta$ are not the same, and hence the skewness does not vanish. We will focus on high-energy scattering; for typical energy scale of hadron masses/confined scale $\Lambda$, we assume that

$$
\Lambda^2 \ll q_1^2, W^2, \quad \text{while} \quad |t| \lesssim O(\Lambda). \quad (4)
$$

The photon–hadron scattering amplitude (Figure 2) in the real-world QCD (where all charged partons are fermions), the Compton tensor is given by the hadron matrix element with insertion of two QED currents $J^{\mu}$:

$$
T^{\mu \nu} = i \int d^4x e^{-iq \cdot x} \langle h(p_2)|T\{J^\nu(x/2)J^{\mu}(-x/2)\}|h(p_1)\rangle. \quad (5)
$$

For simplicity, we assume that the target hadron is a scalar, and further pay attention only to the structure function $V_1$ appearing in the gauge-invariant decomposition of the Compton tensor:

$$
T^{\mu \nu} = V_1 P[q_1]^{\mu \rho} P[q_2]^{\nu \rho} + V_2 (p \cdot P[q_1])^{\mu \nu} (p \cdot P[q_2])^{\rho \sigma} + V_3 (q_2 \cdot P[q_1])^{\mu \nu} (q_1 \cdot P[q_2])^{\rho \sigma} + V_4 (p \cdot P[q_1])^{\mu \nu} (q_1 \cdot P[q_2])^{\rho \sigma} + V_5 (q_2 \cdot P[q_1])^{\mu \nu} (p \cdot P[q_2])^{\rho \sigma} + A\epsilon^{\mu \nu \rho \sigma} q_1^\rho q_2^\sigma. \quad (6)
$$

Those structure functions, $V_{1,2,3,4,5}(x, \eta, t, q^2)$, should be expressed in terms of the kinematical variables $x, \eta$ and $t$, and one of our primary purposes of this article is to study how the structure functions depend on the skewness $\eta$. They interfere. They can be separated experimentally, however, by exploiting kinematical dependence and polarization $^{[19]}$. It thus makes sense to focus only on the amplitude (a).

$^3$We use the $(-, +, +)$ metric throughout this paper.

$^4$Here, we introduced a convenient notation

$$
P[q]_{\mu \nu} = \left[ \eta_{\mu \nu} - \frac{q_\mu q_\nu}{q^2} \right]. \quad (6)$$
2.1.2 Light-cone Operator Product Expansion

The light-cone operator product expansion (OPE) can be applied to the product of currents $T \{ J^\nu J^\mu \}$, before evaluating it as a hadron matrix element. Let the expansion be

$$i \int d^4x e^{-iq \cdot x} T \{ J^\nu (x/2) J^\mu (-x/2) \} = \sum_I C_{I \rho_1 \cdots \rho_{jI}}^{\mu\nu} (q) O_{I \rho_1 \cdots \rho_{jI}} (0; q^2)$$

for some basis of local operators $O_{I \rho_1 \cdots \rho_{jI}}$ renormalized at $\mu^2 = q^2$. $C_{I \rho_1 \cdots \rho_{jI}}^{\mu\nu}$'s are the corresponding Wilson coefficients renormalized at $\mu^2 = q^2$. If we were to evaluate these local operators on the right-hand side with the same state for both bra and ket, $\langle h(p_2) |$ and $| h(p_1) \rangle$ with $p_2^\mu = p_1^\mu$, then the Compton tensor and its structure functions do not receive non-zero contributions from local operators that are given by total derivative of some other local operators. In the case of our interest, however, such operators do contribute.

Let us take a series of operators in QCD that are called twist-2 operators in the weak coupling limit. The twist-2 operators in the flavor non-singlet sector are labeled by two integers, $j, l$,

$$O_{j,l}^\alpha := \left[ (-i)^{j+l-1} \partial^{\mu_{j+1}} \cdots \partial^{\mu_{j+l}} \bar{\Psi}_a \gamma^{\mu_1} \left( \overleftarrow{D} \right)^{\mu_2} \cdots \left( \overleftarrow{D} \right)^{\mu_j} \lambda_{ab}^\alpha \Psi_b \right]_{\text{t.s.t.l.}} (0; q^2),$$

with an $N_F \times N_F$ flavor matrix $(\lambda^\alpha)_{ab}$. Similarly, in the flavor-singlet sector, there are two series of twist-2 operators with the label $j, l$, given by quark bilinear and gluon bilinear. Here, these operators are made totally symmetric and traceless (t.s.t.l) in the $j + l$ Lorentz indices so that they transform in irreducible representations of the Lorentz group $\text{SO}(3, 1)$. $\overleftarrow{D} := \overrightarrow{D} - \overleftarrow{D}$. 

Figure 2: photon–hadron 2-body to 2-body scattering amplitude
Suppose that the hadron matrix element of the operator $O_{j,l}^\alpha$ is given by
\begin{equation}
\langle h(p_2)|O_{j,l}^\alpha|h(p_1)\rangle = \sum_{k=0}^j [\Delta^{j-k+1} \cdot \Delta^{j-k+1} \cdots \Delta^{j-k+1}]_{t,s,t,l} A_{j,k}(t; q^2)(-2)^{j-k},
\end{equation}
the reduced matrix element $A_{j,k}^\alpha(t)$ is non-perturbative information and cannot be determined by perturbative QCD. If we pay attention only to Wilson coefficients $C_{j,l,\alpha; \mu_1 \cdots \mu_{j+l}}^{\mu \nu}$ that are proportional to $\eta^{\mu \nu}$, and are to write them as
\begin{equation}
\eta^{\mu \nu} C_{j,l}^{\alpha} q_{\rho_1} \cdots q_{\rho_{j+l}} \frac{1}{(q^2)^{j+l}},
\end{equation}
then the twist-2 flavor non-singlet contribution to the structure function $V_1$ becomes
\begin{equation}
V_1 \simeq \sum_{j,l} C_{j,l}^{\alpha} \frac{1}{x^j} \sum_{k=1}^j A_{j,k}^\alpha(t; q^2)\eta^{k+l} := \sum_{j} C_{j}^{\alpha}(\vartheta) \frac{1}{x^j} A_j^\alpha(\eta; t; q^2),
\end{equation}
where $\vartheta := (\eta/x)$. $C_{j}^{\alpha}(\vartheta)$ and $A_j^\alpha(\eta; t)$ are now meant to be holomorphic functions on $j$ (possibly with some poles and cuts) that coincide with the original ones at $j \in 2\mathbb{Z}$. Precisely the same story holds true also for flavor-singlet sector.

Because the structure function is given by the inverse Mellin transform of a product of three factors, namely, (a) the signature factor $\mp [1 \pm e^{-\pi i j}] / \sin(\pi j)$, (b) Wilson coefficients $C_{j}^{\alpha}$ and (c) hadron matrix elements $A_j^\alpha$, it can be regarded as a convolution of inverse Mellin transforms of those three factors. The inverse Mellin transform of the signature factor becomes
\begin{equation}
\int \frac{dj}{2\pi i} x^j \frac{1}{\sin(\pi j)} (\mp [1 \pm e^{-\pi i j}] / \sin(\pi j)) = -\frac{1}{2} \left[ \frac{1}{1 - x + i\epsilon} \mp \frac{1}{1 + x} \right],
\end{equation}
which corresponds to propagation of the parton in perturbative calculation \cite{20}, and the inverse Mellin transform of the matrix element is called the generalized parton distribution:
\begin{equation}
H^\alpha(x, \eta, t; \mu^2 = q^2) = \int \frac{dj}{2\pi i} x^j A_j^\alpha(\eta; t; \mu^2 = q^2).
\end{equation}

\textsuperscript{5} In the leading order of QCD perturbation, $C_{j,0}^{\alpha} = -[1 + (-1)^j]$ for $j = 2, 4, \cdots$ and $(\lambda^\alpha)_{ab} = \{\text{diag}(4/9, 1/9, 1/9)\}_{l.l.}$.
Generalized parton distribution (GPD) $H^\alpha(x, \eta, t; \mu^2)$ of a hadron $h$ is a non-perturbative information, just like the ordinary PDF, which is obtained by simply setting $\eta = 0$ and $t = 0$. For phenomenological fit of experimental data of DVCS and VMP, some function form of the GPD needs to be assumed, because of the convolution involved in the scattering amplitude \[13\]. Setting up a model (and assuming a function form) for the non-perturbative information in terms of $A_j(\eta, t; q^2)$ rather than the GPD itself $H(x, \eta, t; q^2)$ is called dual parameterization \[13\] \[16\] \[17\] \[18\], and some phenomenological ansätze have been proposed. In this article, we aim at deriving qualitative form of $A_j(\eta, t)$ by using gravitational dual (that is analytic in $j$), instead of assuming the form of $A_j(\eta, t)$ by hand.

2.1.3 Renormalization and OPE in dilatation eigenbasis

Remembering that the distinction between the $\gamma^* + h \rightarrow \gamma + h$ scattering amplitude and GPD originates from the factorization into the Wilson coefficients and local operators (and their matrix elements), one will notice that the GPD defined in this way should depend on the choice of the basis of local operators. Although the choice of operators $O^\alpha_{j,l}$ with $j \geq 1$ and $l \geq 0$ in \[2\] appears to be the most natural (and intuitive) one for the twist-2 operators in the flavor non-singlet sector, there is nothing wrong to take a different linear combinations of these operators as a basis, when the corresponding Wilson coefficients also become linear combinations of what they are for $O^\alpha_{j,l}$.

Given the fact that the operators $O^\alpha_{j,l}$ mix with one another under renormalization, it should not be compulsory for us to stick to the basis $O^\alpha_{j,l}$.

Under the perturbation of QCD, the flavor non-singlet twist-2 operators are renormalized under

\[
\mu \frac{\partial}{\partial \mu} \left[ O_{j-m,m}(0; \mu^2) \right] = -[\gamma^{(j)}]_{mm'} \left[ O_{j-m',m'}(0; \mu^2) \right];
\]  

because operators can mix only with those with the same number of Lorentz indices, the anomalous dimension matrix $[\gamma]$ is block diagonal in the basis of $O^\alpha_{j,l}$; the $j \times j$ matrix for the operators $O^\alpha_{j-m,m}$ ($m = 0, \cdots, j-1$) is denoted by $[\gamma^{(j)}]$. This matrix is upper triangular in this basis, and the diagonal entries are given by the anomalous dimensions of the twist-2 spin-$j$ operators without a total derivative:

\[
[\gamma^{(j)}]_{mm} = \gamma(j - m).
\]

Therefore, the eigenvalues of the anomalous dimension matrix is $\{\gamma(j - m)\}_{m=0,\cdots,j-1}$ in this diagonal block, and the corresponding operator $\overline{O}^{\alpha}_{j-m-1,m}$ is a linear combination of operators $O^\alpha_{j-m',m'}$ with $m' = m, \cdots, j - 1$ \[21\]. The corresponding Wilson coefficient $\overline{C}^{\alpha}_{j-m-1,m}$ for
such an operator is a linear combination of $C^\alpha_{j-m',m'}$ with $m' = m, \cdots, 0$. In this operator basis, matrix elements and Wilson coefficients renormalize multiplicatively, without mixing.

In this new basis of local operators, the structure function becomes

$$V_1 \simeq \sum_{n,K} C^\alpha_{n,K} \frac{1}{x^{n+K}} \sum_k A^\alpha_{n+1,k}(t; \mu^2) \eta^{K+k} =: \sum_n C^\alpha_n(\vartheta) \frac{1}{x^n} \bar{A}^\alpha_{n+1}(\eta; t; \mu^2),$$

where

$$C^\alpha_n(\vartheta) = \sum_{K=0}^\infty C^\alpha_{n,K} \vartheta^K,$$

and $\bar{A}_{n+1,k}(t; \mu^2)$ is the reduced matrix element of the operator $\bar{C}^\alpha_{n,0}(0; \mu^2)$. The structure function is therefore written as yet another inverse Mellin transform

$$V_1 \simeq -\int \frac{dj}{4i} \frac{1 + e^{-\pi ij}}{\sin(\pi j)} C^\alpha_{j-1}(\vartheta) \frac{1}{x^j} \bar{A}^\alpha_j(\eta, t; \mu^2).$$

Yet another GPD can also be defined by using $\bar{A}^\alpha_j$, instead of $A^\alpha_j(\eta, t; q^2)$:

$$\bar{H}^\alpha(x, \eta, t; \mu^2) = \int \frac{dj}{2\pi i x^j} \frac{1}{x^j} \bar{A}^\alpha_j(\eta, t; \mu^2).$$

When it comes to the description of the $\gamma^* + h \rightarrow \gamma^* + h$ scattering amplitude as a whole, it does not matter which operator basis is used. Although we need GPD rather than the scattering amplitude in order to talk about the distribution of partons in the transverse directions in a hadron, yet we only need GPD at $\eta = 0$. Thus, the newly defined GPD $\bar{H}$ does just as good a job as $H$ defined in (15); they are the same at $\eta = 0$.

Even within the dual parameterization approach, it has been advantageous to use this operator basis, because it becomes much easier to implement a phenomenological assumption (function form) of $\bar{A}^\alpha_j(\eta, t; \mu^2)$ that is consistent with renormalization group flow.

### 2.1.4 Conformal OPE

Although the hadron matrix element is essentially non-perturbative, and is not calculable within perturbative QCD, more discussion has been made on the Wilson coefficients $C^\alpha_{n,K}$.

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6In reality, the anomalous dimension matrix depends on the coupling constant $\alpha_s$, and $\alpha_s$ changes over the scale. Thus, the eigenoperator of the renormalization / dilatation also changes over the scale. In scale invariant theories (and in theories only with slow running in $\alpha_s$), however, this multiplicative renormalization is exact or a good approximation. (c.f. [22])

7Just like $O_{j,l} = (-i\partial)^j O_{j,0}$, there is a relation $\bar{C}_{n,K} = (-i\partial)^K \bar{C}_{n,0}$ in the new basis. This is why all the hadron matrix elements of $\bar{C}_{n,K}$ can be parameterized by $A_{n+1,k}$, just like those of $O_{j,l}$ are by $A_{j,k}$. Here, $n$ corresponds to the conformal spin, which is sometimes denoted by $j$ in the literature. In this article, however, we maintain $j = n + 1$. 

They still have to be calculated order by order in perturbation theory, if one is interested strictly in the QCD of the real world. If one is interested in gauge theories that are more or less “similar” to QCD, however, stronger statements can be made for a system with higher symmetry: conformal symmetry. One can think of $\mathcal{N} = 4$ super Yang–Mills theory or $\mathcal{N} = 1$ supersymmetric $SU(N) \times SU(N)$ gauge theory of [23] as an example of theories with exact (super) conformal symmetry. The QED probe in the real world QCD can be replaced by gauging global symmetries (such as (a part of) $SU(4)$ R-symmetry of $\mathcal{N} = 4$ super Yang–Mills theory and $SU(2) \times SU(2) \times U(1)$ symmetry of [23]). By applying the conformal symmetry, one can derive stronger statements on the Wilson coefficients of primary operators appearing in the OPE.

Suppose that we are interested in the OPE of two primary operators, $A$ and $B$, that are both scalar under $SO(3, 1)$. If we take the basis of local operators for the expansion to be primary operators $\mathcal{O}_n$ (with $j_n$ Lorentz indices and $l_n$ scaling dimension) and their descendants $\partial^K \mathcal{O}_n$ (with $j_n + K$ Lorentz indices), then in the OPE,

$$T \{A(x)B(0)\} = \sum_n \left( \frac{1}{x^2} \right)^{\frac{1}{2}(l_A + l_B - l_n + j_n)} \sum_{K=0}^{\infty} c_{n,K} \frac{x^{\rho_1} \cdots x^{\rho_{j_n} + K}}{(x^2)^{j_n + K}} [\partial^K \mathcal{O}_n(0)]_{\rho_1 \cdots \rho_{j_n}},$$

the conformal symmetry determines all the coefficients of the descendants $c_{n,K}$ ($K \geq 1$) in terms of that of the primary operator, $c_{n,0} =: c_n$. Ignoring the mixture of non-traceless contributions, one finds that [24]

$$T \{A(x)B(0)\} \simeq \sum_n \left( \frac{1}{x^2} \right)^{\frac{1}{2}(l_A + l_B - l_n + j_n)} x^{\rho_1} \cdots x^{\rho_{j_n}} c_n \, _1F_1 \left( \frac{l_n - l_B + l_n + j_n}{2}, l_n + j_n; x \cdot \partial \right) [\mathcal{O}_n(0)]_{\rho_1 \cdots \rho_{j_n}}.$$

(22)

Questions of real interest to us is the OPE of conserved currents $J^\nu$ and $J^\mu$. They are not scalars of $SO(3, 1)$, but the same logic as in [24] can be used also to show that, in the terms with Wilson coefficients proportional to $\eta^{\mu\nu}$,

$$T \{J^\nu(x)J^\mu(0)\} \simeq \eta^{\mu\nu} \sum_n \left( \frac{1}{x^2} \right)^{\frac{3}{2}} x^{\rho_1} \cdots x^{\rho_{j_n}} c_n \, _1F_1 \left( \frac{l_n + j_n}{2}, l_n + j_n; x \cdot \partial \right) [\mathcal{O}_n(0)]_{\rho_1 \cdots \rho_{j_n}} + \cdots,$$

(23)

where $\tau_n := l_n - j_n$ is the twist, mixture of the non-traceless (and hence higher twist) contributions are ignored, and terms with Wilson coefficients without $\eta^{\mu\nu}$ are all omitted here. The scaling dimension of conserved currents $l_A = l_B = 3$ have been used. The
momentum space version of the OPE is \[25\]

\[
\int d^4x \ e^{-iq_2 \cdot x} \ T \{J^\nu(x), J^\mu(0)\} \approx \eta^{\mu\nu} \sum_n \frac{(2\pi)^2 \Gamma \left(\frac{l_n + j_n - 2}{2}\right)}{4^{2 - \frac{2}{d}} \Gamma \left(3 - \frac{n}{2}\right)} \frac{(-2i)j_n q_2^{\rho_1} \cdots q_2^{\rho_{j_n}}}{(q_2^2)^{\frac{d}{2} - 1}(q_2^2)^{j_n}} \left(\frac{l_n + j_n}{2} - 1, l_n + j_n; -\frac{2iq_2 \cdot \partial}{q_2^2}\right) \bar{O}_n(0) + \cdots , \tag{25}\]

or equivalently \[18\],

\[
\int d^4(x - y) e^{-iq_2 \cdot (x - y)} \ T \{J^\nu(x), J^\mu(y)\} \approx \eta^{\mu\nu} \sum_n \frac{(2\pi)^2 \Gamma \left(\frac{l_n + j_n - 2}{2}\right)}{4^{2 - \frac{2}{d}} \Gamma \left(3 - \frac{n}{2}\right)} \frac{(-2i)j_n q_2^{\rho_1} \cdots q_2^{\rho_{j_n}}}{(q_2^2)^{\frac{d}{2} - 1}(q_2^2)^{j_n}} \left(\frac{l_n + j_n - 2}{4}, \frac{l_n + j_n}{4}, \frac{l_n + j_n}{2}; -\frac{iq_2 \cdot \partial}{q_2^2}\right) \bar{O}_n \left(\frac{x + y}{2}\right) + \cdots . \tag{26}\]

Either in the form of (25) or (26), the primary operators \(\bar{O}_n\) and corresponding coefficients \(c_n\) are renormalized multiplicatively.

### 2.2 AdS/CFT Approach

In AdS/CFT correspondence, Type IIB string theory on AdS\(_5 \times W\) with a 5-dimensional Einstein manifold \(W\) corresponds to a gauge theory on \(\mathbb{R}^{3,1}\) with an exact conformal symmetry; theories with an exact conformal symmetry, however, are qualitatively different from the QCD in the real world. But the Type IIB string on a geometry that is close to AdS\(_5 \times W\), but with confining end in the infrared, may be used to extract qualitative lesson on strongly coupled gauge theories with confinement, which are not qualitatively different from the QCD.

In a dual pair of a CFT and a string theory on a background AdS\(_5 \times W\), primary operators of the CFT are in one to one correspondence with string states on AdS\(_5\), and their correlation functions can be calculated by using the wavefunctions of the string states on AdS\(_5\). When the background geometry is changed from AdS\(_5 \times W\) to some warped geometry that is nearly AdS\(_5\) with an end in the infrared, then the wavefunctions might be used to calculate matrix elements of the corresponding “primary” operators in an almost conformal theory. The correspondence between the operators and string states can be made precise, because they are both classified in terms of representation of the conformal algebra, which is shared by both of the dual theories.

In order to determine GPD \(\overline{H}\) in gravitational dual descriptions, it is therefore sufficient to determine wavefunctions of string states corresponding to the “primary” operators of interest. Although there are plenty of literature discussing the correspondence between the
(superconformal) primary operators and string states at the supergravity level, it is known that the flavor-singlet twist-2 operators (labeled by the spin \( j \)) correspond to the stringy excitations with arbitrary high spin \( j \) that are in the same trajectory as graviton \([26, 6]\). Our task is therefore to determine the wavefunctions of such string states. Needless to say, one has to fix all the gauge degrees of freedom associated with string component fields (not just the general coordinate invariance associated with the graviton) before working out the mode decomposition. Furthermore, wavefunctions need to be grouped together properly so that they form an irreducible representation of the conformal group, in order to establish correspondence with a primary operator of the gauge theory side, which also forms an irreducible representation of the conformal group along with its descendants.

It will be clear by the end of this article that all of such technical works is necessary and essential for the purpose of extracting skewness dependence of GPD.

There are two different (but equivalent) ways to study the DVCS \( \gamma^* + h \rightarrow \gamma(\ast) + h \) amplitude and GPD in gravitational dual descriptions. One is to determine the hadron matrix elements of spin-\( j \) primary operators by using appropriate wavefunctions; GPD \( \bar{H} \) is obtained by the inverse Mellin transform of the matrix elements. Using the Wilson coefficients that are governed by the conformal symmetry (see (26)), the DVCS amplitude will also be obtained. Conversely, the other way is to calculate disc/sphere amplitude directly, with the vertex operators given (approximately) by using the wavefunctions associated with the target hadron (see sections 3 and 4). We will identify the structure of conformal OPE in the expression for the \( \gamma^* + h \rightarrow \gamma(\ast) + h \) scattering amplitude in gravity dual (see (160, 163, 178)), with the Wilson coefficient for the “twist-2” operators precisely as predicted by conformal symmetry (26). That makes it also possible to read out hadron matrix elements, and to extract the GPD. In these approaches, one can hope to work also for higher twist contributions, in principle, but we are not ambitious enough to do that in this article. In this article, we will proceed in the latter approach.

3  Gravity Dual Settings

A number of warped solutions to the Type IIB string theory has been constructed, and they are believed to be dual to some strongly coupled gauge theories. When the geometry is close to \( \text{AdS}_5 \times W \) with some 5-dimensional Einstein manifold \( W \), with weak running of the AdS radius along the holographic radius, the corresponding gauge theory will also have approximate conformal symmetry, and the gauge coupling constant runs slowly. If the “AdS\(_5\) \times W” geometry has a smooth end at the infrared as in \([27]\), then the dual gauge
theory will end up with confinement. Gravitational backgrounds in the Type IIB string theory with the properties we stated above all provide a decent framework of studying qualitative aspects of non-perturbative information associated with gluons/Yang–Mills theory on 3+1 dimensions.

In studying the $h + \gamma^* \rightarrow h + \gamma$ scattering process in gravitational dual, we need a global symmetry to be gauged weakly, just like QED for QCD. In Type IIB D-brane constructions of gauge theories that have gravity dual, U(1) subgroups of an R-symmetry or a flavor symmetry on D7-branes can be used as the models of the electromagnetic U(1) symmetry. Therefore, we have in mind gravity dual models on a background that is approximately “AdS$^5 \times W$” with a non-trivial isometry group on $W$, or with a D7-brane configuration on it, as in [5].

Our interest, however, is not so much in writing down an exact mathematical expression based on a particular gravity dual model that is equivalent to a particular strongly coupled gauge theory, but more in extracting qualitative information of partons in hadrons of confining gauge theories in general. It is therefore more suitable for this purpose to use a simplified set-up that carries common (and essential) features of the Type IIB models that we described above. Throughout this article, we assume pure AdS$^5 \times W$ metric background,

$$ds^2 = G_{MN} dx^M dx^N = g_{mn} dx^m dx^n + R^2 (g_W)_{ab} d\theta^a d\theta^b,$$

$$g_{mn} dx^m dx^n = e^{2A(z)} (\eta_{\mu\nu} dx^\mu dx^\nu + dz^2), \quad e^{2A(z)} = \frac{R^2}{z^2};$$

that is, we ignore the running effect, and we do not specify the 5-dimensional manifold $W$. The dilaton vev is simply assumed to be constant, $e^\phi = g_s$. Confining effect—the infra-red end of this geometry—can be introduced, for example, by sharply cutting off the AdS$^5$ space at $z = \Lambda^{-1}$ (hard wall models), or by similar alternatives (soft wall models). We are not committed to a particular implementation of the infra-red cut-off in this article (see discussion in section 5.4), except in a couple of places where we write down some concrete expressions for illustrative purposes (sections 7.1 and 7.3). The energy scale $\Lambda$ associated with (any form of implementation of) the infra-red cut-off corresponds to the confining energy scale in the dual gauge theories. When we consider (simplified version of the) models with D7-branes for flavor, we assume that the D7-brane worldvolume wraps on a 3-cycle on $W$, and extends all the way down to the infra-red end of the holographic radius $z$; i.e., all of $0 \leq z \leq \Lambda^{-1}$. This corresponds to assuming massless quarks. In this article, we will not pay attention to physics where spontaneous chiral symmetry breaking is essential.

As we stated earlier, we would like to work out the $h + \gamma^* \rightarrow h + \gamma^{(*)}$ scattering amplitude by using the gravity dual models. This is done by summing up sphere / disc amplitudes, along with those with higher genus worldsheets. We will restrict our attention to kinematical regions where saturation is not important (i.e., large $q^2$ and/or not too small $x$, and large
That allows us to focus only on sphere / disc amplitudes, with insertion of four vertex operators corresponding to the incoming and outgoing hadron \( h \) and (possibly virtual) photon \( \gamma \).

As a string-based model of the target hadron \( h \) (that is SO(3,1) scalar), we have in mind either a scalar “glueball”\(^8\) that has non-trivial R-charge, or a scalar meson made of matter fields. The former corresponds to a vertex operator (in the \((-1,-1)\) picture)

\[
V(p) =: e^{ip_{\mu} X_{\hat{\mu}}} \psi_m \tilde{\psi}^n g_{mn} \Phi(Z; m_n) Y(\Theta) :,
\]

where \( Y(\Theta) \) is a “spherical harmonics” on \( W \), and the latter to

\[
V(p) =: e^{ip_{\mu} X_{\hat{\mu}}} \psi \Phi(Z; m_n) :,
\]

with \( \psi \) corresponding to the D7-brane fluctuations in its transverse directions. \( \Phi(Z) \) is the wavefunction on AdS\(_5\), with the argument promoted to the field on the world sheet \( \Sigma \). Vertex operators above are approximate expressions in the \( (\alpha'/R^2) \sim 1/\sqrt{\lambda} \) expansion (e.g., \cite{28}) in a theory formulated with a non-linear sigma model given by \( (27) \). If we are to employ the hard-wall implementation of the infra-red boundary, with the AdS\(_5\) metric in the bulk without modification, then the wavefunction \( \Phi(Z; m_n) \) is of the form

\[
\sqrt{t_h} \Phi(z; m_n) = 2 \Lambda z^2 J_{\Delta - 2}(j_{\Delta - 2,n} \Lambda z) J'_{\Delta - 2}(j_{\Delta - 2,n} \Lambda z).
\]

This wavefunction is that of the \( n \)-th lightest hadron corresponding to some scalar operator with conformal dimension \( \Delta = \Delta_\Phi \); the hadron mass \( m_n = j_{\Delta - 2,n} \Lambda \) is given by the \( n \)-th zero of the Bessel function \( J_{\Delta - 2} \). We will comment on the normalization factor \( \sqrt{t_h} \) in later sections, though it disappears from the expression for physical observables.

The “photon” current in the correlation function/matrix element \( T^{\nu \mu} \) in the gauge theory description corresponds to insertion of vertex operators associated with non-normalizable wavefunctions, rather than with the normalizable wavefunctions \( (31) \) for the target hadron state. If we are to employ an R-symmetry current as the string-based model of the QED current, then the corresponding closed string vertex operator is

\[
V(q) =: e^{iq_{\mu} X_{\hat{\mu}}} v_a(\Theta) A_m(Z; q)(\psi^a \tilde{\psi}^m + \psi^m \tilde{\psi}^a) :,
\]

with some Killing vector \( v_a \partial/\partial \theta^a \) on \( W \). The vertex operator in the case of D7-brane U(1) current is

\[
V(q) =: e^{iq_{\mu} X_{\hat{\mu}}} A_m(Z; q) \psi^m :
\]
The wavefunction $A_m(Z; q)$ on AdS$_5$ is of the form

$$A_\mu(z; q) = \left[ \delta_\mu^\kappa - \frac{q_\mu q_\kappa}{q^2} \right] \epsilon_\kappa(q)(qz) K_1(qz) + q_\mu \frac{q_\kappa}{2q^2} (qz)^2 K_2(qz), \quad (34)$$

$$A_z(z; q) = -i \partial_z q_\kappa \frac{q_\kappa(qz)}{2q^2} (qz)^2 K_2(qz). \quad (35)$$

Rationale for our choice of the terms proportional to $(q \cdot \epsilon)$ will be explained later on in the appendix A.4, but those terms should not be relevant in the final result, because of the gauge invariance of $T^{\nu\mu}$. When the infra-red boundary is implemented by the hard wall, $K_1(qz)$ should be replaced by $K_1(qz) + [K_0(q/\Lambda)/I_0(q/\Lambda)]I_1(qz)$, and $K_2(qz)$ by arbitrary linear combination of $K_2(qz)$ and $I_2(qz)$.

It is not as easy to calculate the sphere/disc amplitudes in practice, however. It has been considered that the parton contributions to $\gamma^* + h \to \gamma^{(*)} + h$ scattering is given by amplitude with states in the leading trajectory with arbitrary high spin being exchanged [6]. Those fields are not scalar on AdS$_5$ but come with multiple degree of freedom associated with polarizations. Such polarization of higher spin fields propagating on AdS$_5$ needs to be treated properly—including such issues as covariant derivatives and kinetic mixing among different polarizations (diagonalization of the Virasoro generator $L_0$)—in gravity dual descriptions, in order to be able to discuss skewness dependence of GPD / DVCS amplitude. Direct impact of the curved background geometry can be implemented through the non-linear sigma model on the world sheet, but one has to define the vertex operators as a composite operator properly in such an interacting theory. Ramond–Ramond background is an essential ingredient in making the warped background metric stable, yet non-zero Ramond–Ramond background cannot be implemented in the NSR formalism.

Instead of world-sheet calculation in the NSR formalism in implementing the effect of curved background [27], we use string field theory action on flat space in this article, and make it covariant. Because the gravity dual set-up of our interest is in the Type IIB string theory, we are thus supposed to use superstring field theory for closed string and open string. In order to avoid technical complications associated with the interacting superstring field theories, however, we employ a sort of toy-model approach by using the cubic string field theory for bosonic string theory.

In our toy-mode approach, we deal with the cubic string field theory on AdS$_5$ ($\times$ some internal compact manifold), and ignore instability of the background geometry. The probe photon in this toy-model gravity dual set-up will be the massless vector state of the bosonic string theory with the wavefunction $[31,33]$. The target hadron can be any scalar states, say, the tachyon, with the wavefunction $[31]$. We are to construct a toy-model amplitude of the $h + \gamma^* \to h + \gamma^{(*)}$ scattering, by using the 2-to-2 scattering of the massless photon and
some scalar in the bosonic string on the AdS$_5$ background. In short, this is to maintain the spirit of the set-up in [5, 6], use the bosonic cubic string field theory to compute and obtain something concrete, from which qualitative lessons are to be extracted for the set-up of our interest.

Clearly one of the cost of this approach (without technical complexity of interacting superstring field theory) is that we have to restrict our attention to the Reggeon exchange (flavor non-singlet) amplitude. The amplitude constructed in this way is certainly not faithful to the equations of the Type IIB string theory, either. Since our motivation is not in constructing yet another exact solution to superstring theory, however, we still expect that this (flavor non-singlet) toy-mode amplitude in bosonic string still maintains some fragrance of hadron scattering amplitude to be calculated in superstring theory. This discussion continues to section 7.3.

4 Cubic String Field Theory

Section 4.1 summarizes technical details of cubic string field theory that we need in later sections. We then proceed in section 4.2 to explain an idea of how to reproduce disc amplitude only from string-field-theory $t$-channel amplitude, using photon–tachyon scattering on a flat spacetime background as an example. This idea of constructing amplitude is generalized in section 6 for scattering on a warped spacetime, and we will see that this construction of the amplitude allows us to cast the amplitude almost immediately into the form of conformal OPE (25, 26).

4.1 Action of the Cubic SFT on a Flat Spacetime

The action of the cubic string field theory (cubic SFT) is given by [29]

$$ S = -\frac{1}{2\alpha'} \int \left( \Phi \ast Q_B \Phi + \frac{2}{3} g_o \Phi \ast \Phi \ast \Phi \right), $$

(36)

$$ = -\frac{1}{2\alpha'} \left( \Phi \cdot Q_B \Phi + \frac{2g_o}{3} \Phi \cdot \Phi \ast \Phi \right), $$

(37)

where $g_o$ is a coupling constant of mass dimension $(1 - D/2)$, where $D = 26$ is the spacetime dimensions of the bosonic string theory. The string field $\Phi$ is, as a ket state, expanded in

---

$^9$ The sign of the interaction term is just a matter of convention, because field redefinition for all the component fields $\Phi \rightarrow -\Phi$ is always possible. Under this redefinition, however, covariant derivative can be
terms of the Fock states as in

\[ \Phi = |\Phi\rangle = \phi(x)|\downarrow\rangle + (A_M(x)\alpha^M_{-1} + C(x)b_{-1} + \bar{C}(x)c_{-1})|\downarrow\rangle + \left( f_{MN}(x)\frac{1}{\sqrt{2}}\alpha^M_{-1}\alpha^N_{-1} + ig_M(x)\frac{1}{\sqrt{2}}\alpha^M_{-2} + h(x)b_{-1}c_{-1} + \cdots \right) |\downarrow\rangle, \] (38)

with component fields \( \phi, A_M, C, \bar{C}, f_{MN}, g_M, h, \cdots \); we have already chosen the Feynman–Siegel gauge here. We will eventually be interested only in the states with vanishing ghost number, \( N_{\text{gh}} = 0 \), because states with non-zero ghost number do not appear in the \( t \)-channel / \( s \)-channel exchange for the disc amplitude.

The Hilbert space of one string state is spanned by the Fock states given (in this gauge)
by

\[ \prod_{a=1}^{h_a} \alpha^{M_a}_{-n_a} \prod_{b=1}^{h_b} b_{-l_b} \prod_{c=1}^{h_c} c_{-m_c} |\downarrow\rangle, \] (39)

with \( 1 \leq n_1 \leq n_2 \leq \cdots \leq n_{h_a}, 1 \leq l_1 < l_2 < \cdots < n_{h_b} \) and \( 1 \leq m_1 < m_2 < \cdots < m_{h_c} \). Let us use \( Y := \{n_a\}'s, \{l_b\}'s, \{m_c\}'s \) as the label distinguishing different Fock states of string on a flat spacetime. Mass of these Fock states are determined by

\[ \alpha' k^2 + (N(Y) - 1) = 0, \quad N(Y) = \sum_{a=1}^{h_a} n_a + \sum_{b=1}^{h_b} l_b + \sum_{c=1}^{h_c} m_c. \] (40)

A component field corresponding to a Fock state may be further decomposed into multiple irreducible representation of the Lorentz group, but at least, the rank-\( h_a \) totally symmetric traceless tensor representation is always contained. Fock states of particular interest to us are the ones in the leading trajectory: \( Y = \{1^N, 0, 0\} \), so that all \( n_a \)'s are 1, \( h_b = h_c = 0 \), and \( N(Y) = h_a \). The totally symmetric traceless tensor component field of these states are denoted by \( (N!)^{-1/2} A^{(Y)}_{M_1 \cdots M_{h_a}} \).

The kinetic term—the first term of (36, 37)—is written down in terms of the component fields as follows:

\[ -\frac{1}{2\alpha'} \Phi \cdot Q_B \Phi = \frac{1}{2} \int d^{26}x \text{tr} \left[ \phi(x) \left( \partial^2 + \frac{1}{\alpha'} \right) \phi(x) + A_M(x)\partial^2 A^M(x) + f_{MN}(x) \left( \partial - \frac{1}{\alpha'} \right) f^{MN}(x) + g_M(x) \left( \partial^2 - \frac{1}{\alpha'} \right) g^{M}(x) - h(x) \left( \partial^2 - \frac{1}{\alpha'} \right) h(x) + \cdots \right] \] (41)

either \( \partial_m - i\rho(A_m) \) or \( \partial_m + i\rho(A_m) \). The sign convention above is for \( \partial_m - i\rho(A_m) \), following the convention of section 6.5 of Polchinski’s textbook.
The totally symmetric tensor component field of the Fock states in the leading trajectory $Y = \{1^N, 0, 0\}$ has a kinetic term

$$\frac{1}{2} \int d^{26}x \, \text{tr} \left[ A_{M_1\ldots M_j}^M \left( \partial^2 - \frac{N-1}{\alpha'} \right) A_{M_1\ldots M_j} \right].$$

(42)

The cubic string field theory action in the Feynman–Siegel gauge has two nice properties; first, the kinetic terms of those Fock states do not mix in the flat spacetime background, and second, the second derivative operators are simply given by d’Alembertian, without complicated restrictions or mixing among various polarizations in the component fields.

The second term of the action (36, 37) gives rise to interactions involving three component fields. Interactions involving Fock states with small excitation level $N$ are

$$- \frac{1}{2\alpha'} \frac{2g_o}{3} \Phi \cdot \Phi \ast \Phi = - \int d^{26}x \, \frac{g_o \lambda_{\text{sf}}}{3\alpha'} \hat{E} \left( \text{tr} \left[ \phi^3(x) \right] + \sqrt{\frac{8\alpha'}{3}} \text{tr} \left[ (-i A_M) \left( \phi \overset{\rightarrow}{\partial}^M \phi \right) \right] 
- \frac{8\alpha'}{9\sqrt{2}} \text{tr} \left[ f_{MN} \left( \phi \overset{\rightarrow}{\partial}^M \overset{\rightarrow}{\partial}^N \phi \right) \right] - \frac{5}{9\sqrt{2}} \text{tr} \left[ f_M^M \phi^2 \right] 
+ \frac{2\sqrt{\alpha'}}{3} \text{tr} \left[ (\partial_M g^M) \phi^2 \right] - \frac{11}{9} \text{tr} \left[ h\phi^2 \right] \right) + \cdots,$$

(43)

where $\lambda_{\text{sf}} = 3^{9/2} / 2^6$ \cite{31}, $\overset{\rightarrow}{\partial}^M = \left( \overset{\rightarrow}{\partial}^M - \overset{\leftarrow}{\partial}^M \right)$, and

$$\hat{E} = \exp \left[ -2\alpha' \ln \left( \frac{2}{3^{3/4}} \right) (\partial^2_{(1)} + \partial^2_{(2)} + \partial^2_{(3)}) \right].$$

(44)

The $\partial^2_{(1,2,3)}$ means taking derivatives of the 1st, 2nd, and 3rd field\footnote{Concretely,}

$$\hat{E} A(x) B(x) C(x) = \left[ \left( \frac{27}{16} \right)^{\frac{\alpha'}{3\alpha'^2}} A(x) \right] \left[ \left( \frac{27}{16} \right)^{\frac{\alpha'}{3\alpha'^2}} B(x) \right] \left[ \left( \frac{27}{16} \right)^{\frac{\alpha'}{3\alpha'^2}} C(x) \right].$$

The interactions involving totally symmetric leading trajectory states are also of interest to us. The tachyon–tachyon–$Y = \{1^N, 0, 0\}$ cubic coupling with $N$-derivatives is given by

$$- \frac{g_o \lambda_{\text{sf}}}{\alpha'} \int d^{26}x \, \hat{E} \text{tr} \left[ A_{M_1\ldots M_N}^{(Y)} \left( \phi (-i \overset{\rightarrow}{\partial}^M_1) \cdots (-i \overset{\rightarrow}{\partial}^M_N) \phi \right) \right] \left( \frac{8\alpha'}{27} \right)^{\frac{N}{3\sqrt{2}}} \frac{1}{\sqrt{N!}}.$$ 

(45)
in the interaction part of the action. The photon (the level-1 state)–photon–$Y = \{1^N, 0, 0\}$
coupling in the cubic string field theory includes
\[
-\frac{g_0\lambda_{\text{stf}}}{\alpha'} \int d^{26}x \hat{E} \text{tr} \left[ \left( A_{M_1\cdots M_N}^{(N)}(A_L(-i \partial^M_{M_1}) \cdots (-i \partial^M_{M_N})A_K) \right) \left( \frac{8\alpha'}{27} \frac{\eta^{KL} \eta_{16}}{\sqrt{N!}} \right) + \cdots \right],
\]
where we kept only the terms that have $N$-derivatives and are proportional to $\eta^{KL}$, as they
are necessary in deriving (61).

4.2 Cubic SFT Scattering Amplitude and $t$-Channel Expansion

Before proceeding to study the $h + \gamma^* \rightarrow h + \gamma^{(s)}$ scattering amplitude by using the cubic
string field theory on the warped spacetime background, let us remind ourselves how to obtain $t$-channel operator product expansion from the amplitude calculation based on string
field theory, by using tachyon–photon scattering on the flat spacetime as an example.

Let us consider the disc amplitude of tachyon–photon scattering. The vertex operators
labeled by $i = 1, 2, V_i = \epsilon_i e^{ik \cdot X}$, are for photon incoming ($i = 1$) and outgoing ($i = 2$)
states, which come with Chan–Paton matrices $\lambda^a_i$. Tachyon incoming ($i = 3$) and outgoing
($i = 4$) states correspond to vertex operators $V_i = \epsilon_i e^{ik \cdot X}$ with Chan–Paton matrices $\lambda^a_i$.
The photon–tachyon scattering amplitude $A + \phi \rightarrow A + \phi$ in bosonic open string theory
(Veneziano amplitude) is given by
\[
\mathcal{M}_{\text{Ven}}(s, t) = -\frac{g_0^2}{\alpha'} \frac{\Gamma(-\alpha' t - 1)\Gamma(-\alpha' s - 1)}{\Gamma(-\alpha'(s + t) - 1)} \epsilon_M(k_2)\epsilon_N(k_1)
\times \left\{ \left[ \eta^{MN} - \frac{k_1^M k_1^N}{k_1 \cdot k_2} \right] (\alpha's + 1) + 2\alpha' \left( \left[ p^M - \frac{k_1^M k_2^N}{k_1 \cdot k_2} \right] - \frac{k_2^M}{2} \right) \left( \left[ p^N - \frac{k_2^N k_1^M}{k_2 \cdot k_1} \right] - \frac{k_1^N}{2} \right) (\alpha't + 1) \right\}.
\]
which is to be multiplied by the Chan–Paton factor Tr $[\lambda^{a_2} \lambda^{a_4} \lambda^{a_3} \lambda^{a_1} + \lambda^{a_4} \lambda^{a_2} \lambda^{a_1} \lambda^{a_3}]$. (see
Figure 3 (a, b).) If the Chan–Paton matrices of a pair of incoming and outgoing vertex
operators, $\lambda^{a_1}$ and $\lambda^{a_2}$, commute with each other,\(^\text{12}\) then the Chan–Paton factors from the
diagrams Figure 3 (c, d) are the same, and the total kinematical part of the amplitude for
this Chan–Paton factor becomes $\mathcal{M}_{\text{Ven}}(s, t) + \mathcal{M}_{\text{Ven}}(u, t)$.

\(^\text{11}\) Here, $p := (k_3 - k_4)/2$, averaged momentum of tachyon before and after the scattering, just like in \(^\text{2}\).
\(^\text{12}\) Just like in the case both $\lambda^{a_1}$ and $\lambda^{a_2}$ are an $N_F \times N_F$ matrix diag$(2/3, -1/3, -1/3)$. 

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Let us stay focused on $\mathcal{M}_{\text{V}}(s,t)$ alone for now. The amplitude proportional to $\eta^{MN}$ can be expanded, as is well-known, as a sum only of $t$-channel poles:\footnote{It is also possible to expand this as a sum only of $s$-channel poles; that’s the celebrated $s$-$t$ duality of the Veneziano amplitude.}

$$
\frac{g_s^2 \Gamma(-\alpha' t - 1) \Gamma(-\alpha' s)}{\alpha'} = \frac{g_s^2}{\alpha'} \int_0^1 dx \, x^{-\alpha' t - 2(1-x) \alpha' s - 1},
$$

$$
= \frac{g_s^2}{\alpha'} \sum_{N=0}^{\infty} \frac{-1}{\alpha' t - (N+1)} (\alpha' s + N \cdots (\alpha' s + N)!.
$$

The Veneziano amplitude $^{17}$ can also be obtained in cubic string field theory $^{32}$. In the cubic SFT, the scattering amplitude consists of two pieces, a collection of $t$-channel exchange diagrams and that of $s$-channel diagrams (Figure $\ref{fig:amplitudes}$).

$$
\mathcal{M}_{\text{V}}(s,t) = \sum_Y \mathcal{M}_{Y}^{(t)}(s,t) + \sum_Y \mathcal{M}_{Y}^{(s)}(s,t).
$$

Infinitely many one string states $^{39}$ with zero ghost number ($h_b = h_c$)—labeled by $Y$—can be exchanged in the $t$-channel or in the $s$-channel, and the corresponding contributions are in the form of

$$
\mathcal{M}_{Y}^{(t)} = \frac{f_Y^{(t)}(s,t)}{-\alpha' t - 1 + N(Y)}, \quad \mathcal{M}_{Y}^{(s)} = \frac{f_Y^{(s)}(t,s)}{-\alpha' s - 1 + N(Y)},
$$

where $f_Y^{(t)}$ and $f_Y^{(s)}$ are regular function at finite $s$ and $t$; $N(Y)$ is the excitation level $^{40}$ of a component field $A^{(Y)}$.\footnote{It is also possible to expand this as a sum only of $s$-channel poles; that’s the celebrated $s$-$t$ duality of the Veneziano amplitude.}

Figure 3: Disc amplitudes with two photon vertex operators ($V_1$ and $V_2$) and two tachyon vertex operators ($V_3$ and $V_4$) inserted. Kinematical amplitudes given by the disc amplitudes above are multiplied by the Chan–Paton factors $\text{tr}[\lambda^{a_1} \lambda^{a_3} \lambda^{a_4} \lambda^{a_2}]$ in (a), $\text{tr}[\lambda^{a_1} \lambda^{a_2} \lambda^{a_4} \lambda^{a_3}]$ in (b), $\text{tr}[\lambda^{a_1} \lambda^{a_4} \lambda^{a_3} \lambda^{a_2}]$ in (c) and $\text{tr}[\lambda^{a_1} \lambda^{a_2} \lambda^{a_3} \lambda^{a_2}]$ in (d), respectively. The two disc amplitudes (a, b) become $\mathcal{M}_{\text{V}}(s,t)$, while (c, d) $\mathcal{M}_{\text{V}}(u,t)$.\footnote{It is also possible to expand this as a sum only of $s$-channel poles; that’s the celebrated $s$-$t$ duality of the Veneziano amplitude.}
Because both the world-sheet calculation \cite{47,49} and the cubic SFT calculation \cite{50,51} are the same thing, \( \mathcal{M}_{\text{Ven}}(s,t) \) in both approaches should be completely the same functions of \((s,t)\). Therefore, for an arbitrary given value of \(s\), the residue of all the poles in the complex \(t\)-plane should be the same. We also know that the Veneziano amplitude can be expanded purely in the infinite sum of \(t\)-channel poles with \(t\)-independent residues. This means that the full Veneziano amplitude \cite{47} can be reproduced just from the \(t\)-channel cubic SFT amplitude \cite{14}, \[ \sum_Y \mathcal{M}_Y^{(t)}(s,t), \] by the following procedure:

\[
\sum_Y \frac{f_Y^{(t)}(s,t)}{-\alpha' t - 1 + N_Y} \rightarrow \sum_Y \frac{f_Y^{(t)}(s,(N_Y - 1)/\alpha')}{-\alpha' t - 1 + N_Y} = \mathcal{M}_{\text{Ven}}(s,t).
\]  

(52)

To see that this prescription really works, let us take a look at the amplitudes of \(t\)-channel exchange of one string states with small excitation level \(N_Y = 0,1,2\). Focusing on the amplitude of \(A + \phi \rightarrow A + \phi\) proportional to \(\eta^{MN}\), we find that the tachyon exchange in the \(t\)-channel (Figure 5 (a)) gives rise to the amplitude \cite{33}

\[
\mathcal{M}_Y^{(t)}(s,t) = \left( \frac{g_\alpha \lambda_{\text{sft}}}{\alpha'} \right)^2 \left( \frac{2}{3^{3/4}} \right)^{-2\alpha' t - 2\alpha' t + 4} \frac{-1}{t + 1/\alpha'} = \frac{g_\alpha^2}{\alpha'} \left( \frac{27}{16} \right)^{\alpha' t + 1} \frac{-1}{\alpha' t + 1},
\]  

which is obtained simply by using the \(\phi-\phi-\phi\) vertex rule \cite{43} and \(A-A-\phi\) vertex rule \cite{15}. The prescription (52) turns this amplitude into

\[
\rightarrow \mathcal{M}_\phi(s,t) = \frac{g_\alpha^2}{\alpha'} \frac{-1}{\alpha' t + 1},
\]  

(54)

\footnote{The \(t\)-channel and \(s\)-channel amplitudes of the cubic SFT, \(\sum_Y \mathcal{M}_Y^{(t)}\) and \(\sum_Y \mathcal{M}_Y^{(s)}\) correspond to the integration over \([0,1/2]\) and \([1/2,1]\) in \cite{48}, respectively \cite{32}. Thus, \(\sum_Y \mathcal{M}_Y^{(s)}\) does not contain a pole in \(t\).}
Figure 5: $t$-channel exchange diagrams for $A + \phi \rightarrow A + \phi$ scattering in the cubic string field theory. The tachyon ($N = 0$), photon ($N = 1$) and level-2 states are exchanged in the diagrams (a), (b) and (c), respectively.

which reproduces the $N = 0$ term of (49).

The $t$-channel exchange of level $N^{(Y)} = 1$ excited states can also be calculated in the cubic string field theory (Figure 5 (b)). The amplitude proportional to $\eta^{MN}$ is

$$
\mathcal{M}_A^{(t)}(s, t) = \frac{g_o^2}{\alpha'} \left( \frac{27}{16} \right)^{\alpha'/t - 1} \left( \frac{\alpha'(s - u)}{2} \right),
$$

where $(s - u) = (k_{(1)} - k_{(2)}) \cdot (k_{(4)} - k_{(3)})$. Using the relation $\alpha'(s + t + u) = -2$ in the tachyon–photon scattering to eliminate $u$ in favor of $s$ and $t$, and following the prescription (52)—which is to exploit $\alpha' t = 0$ in the numerator, this amplitude is replaced by

$$
\rightarrow \mathcal{M}_A(s, t) = \frac{g_o^2}{\alpha'} \left( \alpha' s + 1 \right).
$$

Once again, this reproduces the level $N = 1$ contribution to the Veneziano amplitude (49).

Similar calculation for level-2 state exchange can be carried out (Figure 5 (c)). Using the vertex rule in (43) for the $[\text{level-2}]-\phi-\phi$ couplings, and also the interactions among $[\text{level-2}]-A-A$ coupling in the literature, the cubic SFT $t$-channel amplitude is given by

$$
\mathcal{M}_f^{(t)}(s, t) = \frac{g_o^2}{\alpha'} \left( \frac{27}{16} \right)^{\alpha'/t - 1} \left( \frac{(\alpha'(s - u))^2}{8} - \frac{5(\alpha't + 2)}{16 \cdot 2} + \frac{490}{16^2 \cdot 2} \right),
$$

$$
\mathcal{M}_g^{(t)}(s, t) = \frac{g_o^2}{\alpha'} \left( \frac{27}{16} \right)^{\alpha'/t - 1} \left( \frac{36 \alpha't}{16^2} \right),
$$

$$
\mathcal{M}_h^{(t)}(s, t) = \frac{g_o^2}{\alpha'} \left( \frac{27}{16} \right)^{\alpha'/t - 1} \left( \frac{112}{16^2} \right).
$$

After using $\alpha'u = -\alpha'(s + t) - 2$ to eliminate $u$ in favor of $s$ and $t$, and further following the prescription (52) [$\alpha' t \rightarrow 1$ in the numerator], one will see that the level $N^{(V)} = 2$ amplitude
turns into
\[
\rightarrow (\mathcal{M}_f + \mathcal{M}_g + \mathcal{M}_h) (s, t) = \frac{g_6^2}{\alpha' \alpha't} - 1 \left[ \frac{(\alpha' s)^2 + 3(\alpha' s) + 2}{2} \right].
\]
(60)

Once again, this is precisely the same as the $N = 2$ contribution to the Veneziano amplitude.

Contributions from the $t$-channel exchange of states in the leading trajectory can also be examined systematically. Using the vertex rule involving the states in the leading trajectory ($Y = \{1^N, 0, 0\}$), one finds that the amplitude proportional to $\eta^{MN}$ is
\[
\mathcal{M}^{(t)}_{\{1^N, 0, 0\}} \approx \frac{g_6^2}{\alpha'} \left( \frac{27}{16} \right) \frac{(\alpha' s + \alpha' t + \alpha' u + 2)}{\alpha' t - (N - 1)} \frac{(\alpha' (s - u)/2)^N}{N!},
\]
where we maintained only the terms with the highest power of either $s$ or $u$. After using the kinematical relation $\alpha' (s + t + u) + 2 = 0$ to eliminate $u$ in favor of $s$ and $t$, and following the prescription $[t \rightarrow (N - 1)$ in the numerator], we obtain the large-$\alpha'$ leading power contribution to the $N$-th term of (49) with the correct coefficient.

We have therefore seen that the prescription allows us to use the $t$-channel exchange amplitude in the cubic string field theory to construct the full disc scattering amplitude. In section this prescription is extended for the disc scattering amplitudes on a spacetime with curved background metric, which is the situation of real interest in the context of hadron scattering.

## 5 Mode Decomposition on AdS$_5$

Let us now proceed to work out mode decomposition of the totally symmetric (traceless) component field on the warped spacetime. The correspondence between the primary operators of the conformal field theory on the (UV) boundary and wavefunctions on AdS$_5$ is made clear in this section. The Pomeron/Reggeon wavefunctions are obtained as a holomorphic function of the spin variable $j$, since we need to do so for the further inverse Mellin transformation. The wavefunctions will then be used also to construct the scattering amplitude of $h + \gamma^* \rightarrow h + \gamma^{(*)}$ and GPD in sections and [7].
Let the bilinear (free) part of the (bulk) action of a rank-$j$ tensor field on AdS$_5$ to be

$$S_{\text{eff. kin.}} = -\frac{1}{2} t_A y \int d^4 x \int d z \sqrt{-g(z)} g^{m_1 n_1} \cdots g^{m_j n_j}$$

\[ g^{m_0 n_0} (\nabla_{m_0} A^{(y)}_{n_1 \cdots n_j})(\nabla_{n_0} A^{(y)}_{m_1 \cdots m_j}) + \left( \frac{c_y}{R^2} + \frac{N_{\text{eff}}^{(y)}}{\alpha'} \right) A^{(y)}_{m_1 \cdots m_j} A^{(y)}_{n_1 \cdots n_j}, \]  

(62)

where we assume that kinetic mixing between different fields is either absent or sufficiently small. Here, the dimensionless parameter $N_{\text{eff}}^{(y)}$ is $N^{(y)} - 1$ for an $N^{(y)} \in \mathbb{Z}_{\geq 0}$ for bosonic open string ($j \leq N^{(y)}$), which would be $4(N^{(y)} - 1)$ for an $N^{(y)} \in \mathbb{Z}_{\geq 1}$ for closed string ($j \leq 2N^{(y)}$). This field is regarded as a reduction of some field with some “spherical harmonics” on the internal manifold and hence $j \leq h_a$ in general. Another dimensionless coefficient $c_y$ may contain a contribution from the “mass” associated with the “spherical harmonics” over the internal manifold, and also include ambiguity (which is presumably of order unity) associated with making d’Alembertian of the flat metric background covariant.

The equation of motion (in the bulk part)\(^{15}\) then becomes

$$g^{m_1 m_2} (\nabla_{m_1} \nabla_{m_2} A^{(y)}_{n_1 \cdots n_j}) - \left( \frac{c_y}{R^2} + \frac{N_{\text{eff}}^{(y)}}{\alpha'} \right) A^{(y)}_{n_1 \cdots n_j} = 0. \quad (64)$$

Solutions to this equation of motion can be obtained from solutions of the following eigenmode equation\(^{16}\)

$$\nabla^2 A_{m_1 \cdots m_j} = -\frac{\mathcal{E}}{R^2} A_{m_1 \cdots m_j}, \quad (65)$$

by imposing the on-shell condition

$$\left( \frac{\mathcal{E}}{R^2} + c_y \right) + N_{\text{eff}}^{(y)} = 0. \quad (66)$$

\(^{15}\) The dimensionless constant $t_A y$ is something like $N_c^2$ for a mode obtained by reduction of closed string component fields in higher dimensions. More comment on $t_A y$ for open string is found in footnote \(^{20}\).

\(^{16}\) The internal manifold would be a five-dimensional one, $W$, for closed string modes in Type IIB, and a three-cycle for open string states on the flavor D7-branes. For sufficiently small $x$, however, amplitudes of exchanging modes with non-trivial “spherical harmonics” on these internal manifolds are relatively suppressed, and we are not so much interested.

\(^{17}\) The ambiguity in $c_y/R^2$ includes insertion of the curvature tensor,

$$\left([\nabla_M, \nabla_N]\right)_{Q'} = -\Gamma^Q_{Q,M,N} + \Gamma^Q_{Q,M,N} + \Gamma^L_{Q,M} \Gamma^Q_{L,N} - \Gamma^L_{Q,N} \Gamma^Q_{L,M} = \frac{\delta^Q_M g_{Q,N} - \delta^Q_N g_{Q,M}}{R^2}, \quad (63)$$

which vanishes in flat space. Depending on details of how it is inserted, the value of $c_y$ may not be the same for all the individual irreducible components of $SO(4,1)$ in a rank-$j$ tensor field $A_{m_1 \cdots m_j}$.

\(^{18}\) There is also IR boundary part of the equation motion. We will come back to this issue in section \(^{24}\).

\(^{19}\) The differential operator $\nabla^2 := g^{m n} \nabla_m \nabla_n$ is Hermitian under the measure $d^4 x dz \sqrt{-g(z)} g^{m_1 n_1} \cdots g^{m_j n_j}$. 

26
We will work out the eigenmode decomposition for rank-\(j\) tensor fields in the following, where we only have to work for separate \(j\), without referring to the mass parameter.\(^{20}\)

The eigenmode wavefunctions are used not just for construction of solutions to the equation of motions, but also in constructing the Reggeon exchange contributions to the \(h + \gamma^* \rightarrow h + \gamma(\gamma)\) scattering amplitude. The propagator is proportional to

\[
\frac{-i}{\sqrt{\lambda + N_{\text{eff}}}} + N_{\text{eff}}^{-1} \frac{\alpha' R^3}{t_A y}.
\]

The mode equation for a rank-\(j\) tensor field \(A_{m_1 \cdots m_j}\) on AdS\(_5\) is further decomposed into those of irreducible representations of SO\((4,1)\). For simplicity of the argument, we only deal with the mode equations for the totally symmetric (and traceless) rank-\(j\) tensor fields. Namely,

\[
A_{m_1 \cdots m_j} = A_{m_{\sigma(1)} \cdots m_{\sigma(j)}} \quad \text{for } \forall \sigma \in \mathfrak{S}_j.
\]

We call them spin-\(j\) fields.

The eigenmode equation \((65)\) for a totally symmetric spin \(j\) field can be decomposed into \(j+1\) pieces, labeled by \(k = 0, \cdots, j\):

\[
((R^2 \Delta_j) - [(2k+1)j - 2k^2 + 3k]) \, A_{z^k \mu_1 \cdots \mu_{j-k}} + 2z k \partial^\rho A_{z^k \mu_1 \cdots \mu_{j-k}} + k(k-1) A_{z^{k+2} \mu_1 \cdots \mu_{j-k}} - 2z (D[A_{z^{k+1} \cdots \mu_{j-k}}])_{\mu_1 \cdots \mu_{j-k}} + (E[A_{z^{k+2} \cdots \mu_{j-k}}])_{\mu_1 \cdots \mu_{j-k}} = -\mathcal{E} A_{z^k \mu_1 \cdots \mu_{j-k}}.
\]

Here,

\[
A_{z^k \mu_1 \cdots \mu_{j-k}} := A_{z^{k+1} \cdots z \mu_1 \cdots \mu_{j-k}},
\]

and can be regarded as a rank-(\(j-k\)) totally symmetric tensor of SO\((3,1)\) Lorentz group.\(^{21}\) \(D[a]\) and \(E[a]\) are operations creating totally symmetric rank-(\(r+1\)) and rank-(\(r+2\)) tensors of SO\((3,1)\), respectively, from a totally symmetric rank-\(r\) tensor of SO\((3,1)\), \(a\);

\[
(D[a])_{\mu_1 \cdots \mu_{r+1}} := \sum_{i=1}^{r+1} \partial_{\mu_i} a_{\mu_1 \cdots \mu_{r+1}},
\]

\[
(E[a])_{\mu_1 \cdots \mu_{r+2}} := 2 \sum_{p<q} \eta_{\mu_p \mu_q} a_{\mu_1 \cdots \mu_{p} \cdots \mu_{q} \cdots \mu_{r+2}}.
\]

\(^{20}\)There are many states with the same value of \(j\), but with different \(c_y\) and \(N(y)_{\text{eff}}\).

\(^{21}\) The SO\((3,1)\) indices with \(^\wedge\) in the superscript, such as \(\partial^\rho\), are raised by the 4D Minkowski metric \(\eta^{\rho\sigma}\) from a subscript \(\sigma\), not by the 5D warped metric \(g^{mn}\).
The differential operator $\Delta_j$ in the first term is defined, as in [6], by

$$ R^2 \Delta_j := R^2 z^{-j} \left[ \left( \frac{z}{R} \right)^5 \frac{\partial}{\partial z} \left[ \left( \frac{R}{z} \right)^3 \frac{\partial}{\partial z} \right] \right] z^j + R^2 \left( \frac{z}{R} \right)^2 \partial^2, $$

$$ = z^2 \partial_z^2 + (2j - 3)z \partial_z + j(j - 4) + z^2 \partial^2. \quad (73) $$

The eigenmode equation [65, 69] is a generalization of the “Schrödinger equation” of [6] determining the Pomeron wavefunction. As we will see, the single-component Pomeron wavefunction discussed in [6] etc. corresponds to $(93)$—that of $(n, l, m) = (0, 0, 0)$ eigenmode in our language, and the Schrödinger equation to $(90, 212)$; there are other eigenmodes, whose wavefunctions are to be determined in the following.

In the following sections 5.1–5.2, we simply state the results of the eigenmode decomposition of [65, 69] for spin-$j$ fields. More detailed account is given in the appendix A.

### 5.1 Eigenvalues and Eigenmodes for $\Delta^\mu = 0$

Because of the 3+1-dimensional translational symmetry in $\nabla^2$, solutions to the eigenmode equations can be classified by the eigenvalues of the generators of translation, $(-i\partial_\mu)$. Until the end of section 5.2, we will focus on eigenmodes in the form of

$$ A_{m_1 \cdots m_j}(x, z) = e^{i\Delta \cdot x} A_{m_1 \cdots m_j}(z; \Delta), \quad (74) $$

and study the eigenmode equation [65] separately for different eigenvalues $\Delta^\mu$.

The eigenmode equation for $\Delta^\mu = 0$ and that for $\Delta^\mu \neq 0$ are qualitatively different, and need separate study. The eigenmodes for $\Delta^\mu \neq 0$ will be presented in section 5.2 (and appendix A.2); we begin in section 5.1 (and appendix A.1) with the eigenmode equation for $\Delta^\mu = 0$, which is also regarded as an approximation of the eigenmode equation for $\Delta^\mu \neq 0$ in the asymptotic UV boundary region (at least $\Delta z \ll 1$, and maybe $z$ is as small as $R$).

For now, we relax the traceless condition on the spin-$j$ field $A_{m_1 \cdots m_j}$ ($m_i = 0, 1, \cdots, 3, z$), and we just assume that the rank-$j$ tensor field $A_{m_1 \cdots m_j}$ is totally symmetric. Consider the following decomposition of the space of $z$-dependent field configuration $A_{m_1 \cdots m_j}(z; \Delta = 0)$:

$$ A_{z^{\mu_1 \cdots \mu_j-k}}(z; \Delta^\mu = 0) = \sum_{N=0}^{[j-k]/2} \left( E^N [a^{(k,N)}] \right)_{\mu_1 \cdots \mu_j-k}; \quad (75) $$

22This only makes the following presentation more far reaching; in the end, it is quite easy to identify which eigenmodes fall into the traceless part within $A_{m_1 \cdots m_j}$. See [52, 54] at the end of section 5.1.
here, \(a^{(k,N)}(z; \Delta^\mu = 0)\) is a rank-\((j-k-2N)\) totally symmetric tensor of SO(3, 1), and satisfies the 4D-traceless condition,

\[
\eta^{\mu_1\mu_2} a^{(k,N)}_{\mu_1\cdots\mu_{j-k-2N}} = 0.
\]

Thus, the field configuration can be described by \(a^{(k,N)}\)'s with \(0 \leq k \leq j\), \(0 \leq N \leq [(j-k)/2]\). These components form groups labeled by \(n = 0, \cdots, j\), where the \(n\)-th group consists of \(a^{(k,N)}\)'s with \(k + 2N = n\); they are all rank-\((j-n)\) totally symmetric tensors of SO(3, 1); let us call the subspace spanned by the components in this \(n\)-th group as the \(n\)-th subspace. The eigenmode equation for \(\Delta^\mu = 0\) becomes block diagonal under the decomposition into these subspaces labeled by \(n = 0, \cdots, j\). (See (205) in the appendix.) Therefore, the eigenmode equation for \(\Delta^\mu = 0\) can be studied separately for the individual diagonal blocks.

The \(n\)-th diagonal block contains \([n/2] + 1\) components, and hence there are \([n/2] + 1\) eigenmodes. Let \(\mathcal{E}_{n,l}\) \((l = 0, \cdots, [n/2])\) be the eigenvalues in the \(n\)-th diagonal block. The corresponding eigenmode wavefunction is of the form

\[
(a^{(k,N)}(z; \Delta^\mu = 0))_{\mu_1\cdots\mu_{j-n}} = c_{k,l,n} \left(\epsilon^{(n,l)}_{\mu_1\cdots\mu_{j-n}} \right) z^{2-j-i\nu},
\]

where \(\epsilon^{(n,l)}\) is a \(z\)-independent \(k\)-independent rank-\((j-n)\) tensor of SO(3, 1). \(c_{k,l,n} \in \mathbb{R}\). In the eigenmode equation for \(\Delta^\mu = 0\), the eigenmode wavefunctions are all in a simple power of \(z\), and the power is parameterized by \(i\nu\) \((\nu \in \mathbb{R})\). The eigenvalues \(\mathcal{E}_{n,l}\) are functions of \(\nu\); once the mass-shell condition \((\text{66})\) is imposed, the eigenmodes turn into solutions of the equation of motion, and \(i\nu\) is determined by the mass parameter.

The eigenmodes with smaller \((n, l)\) are as follows:

\[
\begin{align*}
\mathcal{E}_{0,0} &= (j + 4 + \nu^2), \\
\mathcal{E}_{1,0} &= (3j + 5 + \nu^2), \\
\mathcal{E}_{2,0} &= (5j + 4 + \nu^2), \\
\mathcal{E}_{2,1} &= (j + 2 + \nu^2),
\end{align*}
\]

Empirically, the \(j\)-dependence of the eigenvalues in the \(n\)-th diagonal block appear to be \(\mathcal{E}_{n,l} = ((2n+1-4l)j + \nu^2 + O(1)) \ (l = 0, \cdots, [n/2])\). [see (214) (229) in the appendix for more samples of the eigenvalues] and we promote this \(j\)-dependence to a rule of the labeling of the eigenmodes with \(l\).

The eigenmode with \(l = 0\) is found in any one of the diagonal blocks \((n = 0, \cdots, j)\). Its eigenvalue is

\[
\mathcal{E}_{n,0} = (2n + 1)j + 2n - n^2 + 4 + \nu^2,
\]

29
and
\[
c_{2\tilde{k},0,2n} = (-)^{\tilde{k}} k \frac{\bar{n}!}{(\bar{n} - k)! (j - \bar{n} - k + 1)!}, \quad (n = 2\bar{n}, \tilde{k} = 0, \ldots, \bar{n}), \tag{83}
\]
\[
c_{2\tilde{k}+1,0,2n+1} = (-)^{\tilde{k}} k \frac{\bar{n}!}{(\bar{n} - k)! (j - \bar{n})!}, \quad (n = 2\bar{n} + 1, \tilde{k} = 0, \ldots, \bar{n}). \tag{84}
\]

These \((n, l) = (n, 0)\) eigenmodes are characterized by the 5D-traceless condition
\[
g^{m_1m_2} A_{m_1 \cdots m_j} = 0.
\]

Thus, the eigenmodes within the 5D-traceless (and totally symmetric) component—spin-\(j\) field—for \(\Delta^\mu = 0\) are labeled simply by \(n = 0, \ldots, j\).

### 5.2 Mode Decomposition for non-zero \(\Delta^\mu\)

#### 5.2.1 Diagonal Block Decomposition for the \(\Delta^\mu \neq 0\) Case

The eigenmode equation (65, 69) is much more complicated in the case of \(\Delta^\mu \neq 0\), because of the 2nd and 4th terms in (69). The eigenmode equation is still made block diagonal for an appropriate decomposition of the space of field \(A_{m_1 \cdots m_j}(z; \Delta^\mu)\).

Consider a decomposition
\[
A_{z\mu_1 \cdots \mu_j} (z; \Delta^\mu) = \sum_{s=0}^{j-k} \sum_{N=0}^{[s/2]} \left( \bar{E}^N D^{s-2N} [a^{(k,s,N)}] \right)_{\mu_1 \cdots \mu_j-k}, \tag{85}
\]
where a new operation \(a \mapsto \bar{E}[a]\) on a totally symmetric \(\text{SO}(3,1)\) tensor \(a\),
\[
\left( \bar{E}[a] \right)_{\mu_1 \cdots \mu_{r+2}} := 2 \sum_{p < q} \left( \eta_{\mu_p \mu_q} - \frac{\partial_{\mu_p} \partial_{\mu_q}}{\partial^2} \right) a_{\mu_1 \cdots \mu_p \cdots \mu_q \cdots \mu_{r+2}}, \tag{86}
\]
is used. \(a^{(k,s,N)}\)'s are totally symmetric 4D-traceless (i.e. (76)) rank-\((j - k - s)\) tensor fields of \(\text{SO}(3,1)\) that satisfies an additional condition, the 4D-transverse condition:
\[
\partial^\mu \left( a^{(k,s,N)} \right)_{\rho \mu_2 \cdots \mu_{j-k-s}} = i \Delta^\mu \left( a^{(k,s,N)} \right)_{\rho \mu_2 \cdots \mu_{j-k-s}} = 0. \tag{87}
\]

The space of field configuration \(A_{m_1 \cdots m_j}(z; \Delta^\mu)\) is now decomposed into \(a^{(k,s,N)}\)'s with \(0 \leq k \leq j, 0 \leq s \leq j - k, 0 \leq N \leq [s/2]\); these components form groups labeled by \(m = 0, \ldots, j\), where the \(m\)-th group consists of \(a^{(k,s,N)}\)'s with \(k + s = m\); they are all rank-\((j - m)\) totally
symmetric 4D-traceless and 4D-transverse tensors of $\text{SO}(3, 1)$; let us call the subspace spanned by the components in this $m$-th group as the $m$-th subspace. The eigenmode equation for $\Delta^\mu \neq 0$ becomes block diagonal under the decomposition into these subspaces labeled by $m = 0, \cdots, j$. The eigenmode equation for the $m$-th sector is given by (241) in the appendix A.2. The $m$-th subspace should have

$$\sum_{s=0}^{m} ([s/2] + 1)$$

eigenmodes.

Eigenvalues $\mathcal{E}$ are determined in terms of the characteristic exponent in the expansion of the solution in power series of $z$; let the first term in the expansion be $z^{2-j-iv}$; the eigenvalues are functions of $\nu$ then. Because the indicial equation at the regular singular point $z \approx 0$ allows us to determine the eigenvalues in terms of $\nu$, the eigenvalues in the case of $\Delta^\mu \neq 0$ cannot be different from the ones we have already known in the $\Delta^\mu = 0$ case. In the $m$-th diagonal block, the eigenvalues consist of $\mathcal{E}_{n,l}$ with $0 \leq n \leq m$, $0 \leq l \leq [n/2]$.

To summarize, the eigenmodes in the totally symmetric rank-$j$ tensor field of $\text{SO}(4, 1)$ are labeled by $(n, l, m)$ and $\Delta^\mu$ and $\nu$. Their eigenvalues $\mathcal{E}_{n,l}$ depend only on $n$ and $l$ (with $0 \leq n \leq j$ and $0 \leq l \leq [n/2]$) and $\nu$. Corresponding eigenmodes are denoted by

$$A(x, z)_{\nu;\Delta^\mu}^{n,l,m} = e^{i\Delta \cdot x} A_{z^{k}_{\mu_{1} \cdots \mu_{j-k}}}^{n,l,m}(z; \Delta^\mu, \nu) = e^{i\Delta \cdot x} \sum_{N=0}^{[s/2]} \tilde{E}^{N} D^{s-2N} [\epsilon^{(n,l,m)}] b_{s,N}^{(j-m)} \psi_{s,N}^{(j-s),n,l,m} (\Delta^2, z).$$

$\epsilon^{(n,l,m)}$ is a ($z$-independent) totally symmetric 4D-traceless 4D-transverse rank-$(j-m)$ tensor of $\text{SO}(3, 1)$, and all the $s$’s appearing in the expression above are understood as $s = m - k$. $b_{s,N}^{(r)}$ is a constant whose definition is given in (240) in the appendix.

### 5.2.2 Single Component Pomeron Wavefunction

The Pomeron wavefunction that has been discussed in the literature (e.g. [9]) does not look as awful as (89). To our knowledge, the Pomeron wavefunction in the literature in the context of hadron high-energy scattering has been a single component one, $\Psi_{\nu}(t, z)$. How is $A_{m_{1} \cdots m_{j}}^{n,l,m}(z; \Delta^\mu, \nu)$ related to $\Psi_{\nu}(\Delta^2, z)$?

In the block diagonal decomposition of the eigenmode equation, there is only one subspace where the diagonal block is $1 \times 1$. That is the $m = 0$ subspace, which consists only of
The eigenmode equation is
\[
\Delta_j - \frac{j}{R^2} a^{(0,0,0)}(z; \Delta^\mu) = -\frac{\mathcal{E}}{R^2} a^{(0,0,0)}(z; \Delta^\mu).
\]  
(90)

This equation, as well as (212) in the \(\Delta^\mu = 0\) case, corresponds to the “Schrödinger equation” in [6] determining the Pomeron wavefunction. It should be noted, however, that we consider that \(\nabla^2\) is the operator relevant to the eigenmode decomposition \(^{23}\) rather than \(\Delta_j\); furthermore the operator \(\nabla^2\) and \(\Delta_j\) has a simple relation \(\nabla^2 = \Delta_j - j/R^2\) only on this \(m = 0\)-th subspace of a totally symmetric rank-\(j\) tensor field of SO(4, 1).

The eigenmode equation is
\[
\mathcal{E}_{0,0} = (j + 4 + \nu^2),
\]  
(91)

when we define the first term in the power series expansion of \(z\) to be \(z^{2-j-i\nu}\). The eigenmode wavefunction is
\[
a^{(0,0,0)}(z; \Delta^\mu)_{\mu_1\ldots\mu_j} = \epsilon_{\mu_1\ldots\mu_j}^{(0,0,0)} \Psi_{i\nu}^{(j)}(-\Delta^2, z),
\]  
(92)

\[
\Psi_{i\nu}^{(j)}(-\Delta^2, z) := \frac{2}{\pi} \sqrt{\nu \sinh(\pi \nu)} \frac{2}{2R} e^{(j-2)A} K_{i\nu}(\Delta z).
\]  
(93)

The normalization factor is determined \(^{24}\) so that it satisfies the normalization condition \(^{25}\)
\[
\int d^4x \int dz \sqrt{-g(z)} e^{-2jA} \left[ e^{i\Delta x \Psi_{i\nu}^{(j)}(-\Delta^2, z)} \right] \left( \Psi_{i\nu}^{(j)}(-\Delta^2, z) \right) e^{-i\Delta' x} = (2\pi)^4 \delta^4(\Delta - \Delta') \delta(\nu - \nu').
\]  
(94)

The single component Pomeron/Reggeon wavefunction \(\Psi_{i\nu}^{(j)}(-\Delta^2, z)\) is now understood as \(\Psi_{i\nu;0,0,0}^{(j)}(-\Delta^2, z)\).

### 5.2.3 5D-Traceless 5D-Transverse Modes

The eigenmode equation (65) for a totally symmetric rank-\(j\) tensor field of SO(4, 1) should be closed within its 5D-traceless component. The subspace of 5D-traceless component is characterized by the 5D-traceless condition
\[
g^{m_1m_2} A_{m_1\ldots m_j}(z; \Delta^\mu) = 0.
\]  
(95)

\(^{23}\)Thus, the propagator (67) uses the eigenvalue of \(\nabla^2\), rather than that of \(\Delta_j\). The eigenvalue \(\mathcal{E}\) of \(\nabla^2\) in the \(m = 0\)-th subspace is \((j + 4 + \nu^2)\) as in (211), instead of \((4 + \nu^2)\). Reference [4] uses a mode \(h_{mn} \propto z^{-2}(\eta_{\mu\nu}, \delta_{zz})\) of the spin-2 field to fix the details of (40) and (41). This \(h_{mn} \propto z^{-2}(\eta_{\mu\nu}, \delta_{zz})\) mode, however, corresponds to the \((n, l) = (2, 1)\) mode of the spin-\(j = 2\) field in (219), rather than the 5D-traceless 5D-transverse mode \((n, l) = (0, 0)\). The eigenvalue \(\mathcal{E}_{2,1} = (2 + j + \nu^2)\) with \(j = 2\) becomes \((4 + \nu^2)\), though.

\(^{24}\)The Pomeron wavefunction in [10] was of the form (124), which becomes (93) in the limit of \(\Lambda \to 0\), while keeping \(z\) and \(\Delta^\mu\) fixed.

\(^{25}\)The normalization condition is generalized to (99) later on.
The fact that the Hermitian operator $\nabla^2$ maps this subspace to itself implies that the eigenmode equation of $\nabla^2$ is block diagonal, when the space of (not-necessarily 5D-traceless) $A_{m_1 \cdots m_j}$ is decomposed into the sum of the 5D-traceless subspace and its orthogonal complement. Collection of the eigenmodes with $l = 0$ correspond to the subspace of 5D-traceless field configuration.

Similarly, one can think of a subspace of field configuration satisfying both the 5D-traceless condition \((95)\) and the 5D-transverse condition

$$g^{n_1 \cdots n_j} \nabla_n A_{m_1 \cdots m_j} = 0. \quad (96)$$

Obviously this is a subspace of the subspace of 5D-traceless modes we discussed above. Since the Hermitian operator $\nabla^2$ on AdS\(_5\) maps this new subspace also to itself, the eigenmode equation of $\nabla^2$ should also become block diagonal, when the subspace of 5D-traceless modes is decomposed into this new subspace and its orthogonal complement.

As we will see in the appendix \[A.3\] there is only one such mode satisfying this set of conditions \((95, 96)\) in each one of the $m$-th diagonal block. Thus, the combination of the 5D-traceless and 5D-transverse conditions allows us to determine an eigenmode completely. This mode turns out to be \((n, l, m) = (0, 0, m)\) (for $0 \leq m \leq j$). Put differently, the eigenmodes with the eigenvalue $E_{n,l} = E_{0,0} = (j + 4 + \nu^2)$ are characterized by the traceless and transverse conditions on AdS\(_5\).

The eigenmode wavefunctions of the 5D-traceless transverse modes \((n, l, m) = (0, 0, m)\) are (see the appendix \[A.3\])

$$\Psi^{(j);s,N}_{n,0,0,m}(\Delta^2, z) = \sum_a^N (-)^a N_a C_a \left( \frac{z^3 \partial_z z^{-3}}{\Delta} \right)^{s-2a} (z \Delta)^m \Psi^{(j);0,0}_{n,0,0,0}(\Delta^2, z) \times N_{j,m}. \quad (97)$$

$N_{j,m}$ is a dimensionless normalization constant. We choose it to be\[26\]

$$N_{j,m}^{-1} = j C_m \frac{\Gamma(j + 1 - i\nu)}{\Gamma(j + 1 - m - i\nu)} \frac{\Gamma(j + 1 + i\nu)}{\Gamma(j + 1 - m + i\nu)} \frac{\Gamma(3/2 + j - m)}{2^m \Gamma(3/2 + j)} \frac{\Gamma(2 + j)}{\Gamma(2 + 2j - m)}, \quad (98)$$

so that the eigenmode wavefunctions are normalized as in

$$\int d^4x \int_0^\infty \int dz \sqrt{-g(z)} g^{m_1 n_1} \cdots g^{m_j n_j} A^{n_1 \cdots m_j; \Delta; \nu}(x, z) A^{n'_1 \cdots m'_j; \Delta'; \nu'}(x, z) = (2\pi)^4 \delta(\Delta + \Delta') \delta(\nu - \nu') \delta_n \delta_{m,m'} \left[ \epsilon^{(n,l,m)}(\Delta) \right] \cdot \left[ \epsilon^{(n',l',m')}(\Delta') \right]. \quad (99)$$

Here, $[\epsilon^{(n,l,m)}, \epsilon^{(n',l',m')} := \epsilon_{\mu_1 \cdots \mu_j \cdots \mu_j}^{(n,l,m)} \epsilon_{\nu_1 \cdots \nu_j \cdots \nu_j}^{(n',l',m')} \eta^{\mu_1 \nu_1} \cdots \eta^{\mu_j \nu_j}$. \[26\]

Note that $N_{j,m} = 1$, if $m = 0$. 33
5.2.4 Propagator

The propagator of the totally symmetric rank-\(j\) tensor field [resp. spin-\(j\) field] on AdS\(_5\) is given by summing up propagators of the \((n, l, m)\) modes [resp. \((n, l, m)\) modes with \(l = 0\)]. For the purpose of writing down the propagator of a given \((n, l, m)\) eigenmode, it is convenient to introduce the following notation:

\[
A_{m_1 \cdots m_j}^{n,l,m; \Delta, \nu}(x, z) = \left[ A_{m_1 \cdots m_j}^{n,l,m; \Delta, \nu}(x, z) \right]_{\kappa_1 \cdots \kappa_{j-m}}^{\kappa_1 \cdots \kappa_{j-m}} = e^{i \Delta \cdot x} A_{m_1 \cdots m_j}^{n,l,m}(z; \Delta, \nu)_{\kappa_1 \cdots \kappa_{j-m}}^{\kappa_1 \cdots \kappa_{j-m}}. \tag{100}
\]

With this notation, the propagator of the \((n, l, m)\) mode is given by

\[
G(x, z; x', z')_{m_1 \cdots m_j n_1 \cdots n_j}^{(n,l,m)} = \int \frac{d^4 \Delta}{(2\pi)^4} \int_0^\infty d\nu \frac{-i P^{(j-m)}_{\rho_1 \cdots \rho_{j-m}; \sigma_1 \cdots \sigma_{j-m}} \alpha' R^3}{\sqrt{\lambda} + N_{\text{eff}} - i \epsilon} \frac{1}{t_y}
\left[ A_{m_1 \cdots m_j}^{n,l,m; \Delta, \nu}(x, z) \right]_{\rho_1 \cdots \rho_{j-m}}^{\rho_1 \cdots \rho_{j-m}} \left[ A_{m_1 \cdots m_j}^{n,l,m; -\Delta, \nu}(x', z') \right]_{\sigma_1 \cdots \sigma_{j-m}}^{\sigma_1 \cdots \sigma_{j-m}}. \tag{101}
\]

Here, \(P^{(j-m)}_{\rho_1 \cdots \rho_{j-m}; \sigma_1 \cdots \sigma_{j-m}}\) is a polarization tensor generalizing \(\eta_{\rho \sigma} - \partial_\rho \partial_\sigma / \partial^2\); when an orthogonal basis \(\epsilon_a(q) \cdot \epsilon_b(-q) = \delta_{a,b} D_a\) of rank-\(r\) 4D-traceless 4D-transverse tensors is given,

\[
P^{(r)}_{\mu_1 \cdots \mu_r; \nu_1 \cdots \nu_r} := \sum_a \frac{1}{D_a} \epsilon_a(q)_{a; \mu_1 \cdots \mu_r} \epsilon(-q)_{a; \nu_1 \cdots \nu_r}. \tag{102}
\]

An alternative characterization of this \(P^{(r)}_{\mu_1 \cdots \mu_r; \nu_1 \cdots \nu_r}\) is given by a combination of the two following conditions: one is

\[
P^{(r)}_{\mu_1 \cdots \mu_r; \nu_1 \cdots \nu_r} \epsilon_a^{\hat{\nu}_1 \cdots \hat{\nu}_r} = \epsilon_{a; \mu_1 \cdots \mu_r}, \tag{103}
\]

and the other is that \(P^{(r)}_{\mu_1 \cdots \mu_r; \nu_1 \cdots \nu_r}\) be also a totally symmetric 4D-transverse 4D-traceless tensor with respect to \((\mu_1 \cdots \mu_r)\) for any choice of \((\nu_1 \cdots \nu_r)\). Its explicit form \(\text{(270)}\) given in the appendix is useful for practical computations.

5.3 Representation in the Dilatation Eigenbasis

It is an essential process in the application of AdS/CFT correspondence to classify solutions to the equation of motions on the gravity dual background (AdS\(_5\)) into irreducible representations of the conformal group SO(4,2) (or possibly its supersymmetric extension). In the CFT description, primary operators are in one to one correspondence with (highest weight) irreducible representations of the conformal group, and it is believed that one can establish an one-to-one correspondence between i) a primary operator in the CFT description and ii)
a group of solutions to the equation of motion forming an irreducible representation in the gravity dual description. Once this correspondence is given, then hadron matrix elements of the primary operators in a (nearly conformal) field theory can be calculated by using the corresponding solutions to the equation of motions (wavefunctions) on AdS$_5$. Note that the hadron matrix elements of primary operators are all that remains unknown in the formulation of conformal operator product expansion [26].

Let $P_\mu$, $K_\mu$, $L_{\mu\nu}$ and $D$ denote the generators of the unitary operators of the conformal group transformation on the Hilbert space. They satisfy the following commutation relations:

$$[D, P_\mu] = iP_\mu, \quad [P_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \quad (104)$$

$$[D, K_\mu] = -iK_\mu, \quad [K_\rho, L_{\mu\nu}] = i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu), \quad (105)$$

$$[P_\mu, K_\nu] = -2i(\eta_{\mu\nu}D + L_{\mu\nu}), \quad (106)$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = i(\eta_{\nu\rho}L_{\mu\sigma} - \eta_{\nu\sigma}L_{\mu\rho} - \eta_{\mu\rho}L_{\nu\sigma} + \eta_{\mu\sigma}L_{\nu\rho}). \quad (107)$$

When such a conformal symmetry exists in a conformal field theory on 3+1 dimensions, those generators have a representation as differential operators on fields on $\mathbb{R}^{3,1}$; those differential operators are denoted by $P_\mu$, $K_\mu$, $L_{\mu\nu}$ and $D$. The generators and the differential operators on a CFT are in the following relation:

$$[\mathcal{O}(x), P_\mu] = \mathcal{P}_\mu \mathcal{O}(x), \quad [\mathcal{O}(x), K_\mu] = \mathcal{K}_\mu \mathcal{O}(x), \quad [\mathcal{O}(x), D] = \mathcal{D} \mathcal{O}(x), \cdots, \quad (108)$$

and those differential operators acts on primary operators as follows:

$$\mathcal{D} \mathcal{O}_n(x) = -i(x \cdot \partial + l_n) \mathcal{O}_n(x), \quad (109)$$

$$\mathcal{L}_{\mu\nu} \mathcal{O}_n(x) = (i(x_\mu \partial_\nu - x_\nu \partial_\mu) + [S_{\mu\nu}]) \mathcal{O}_n(x), \quad (110)$$

$$\mathcal{P}_\mu \mathcal{O}_n(x) = -i\partial_\mu \mathcal{O}_n(x), \quad (111)$$

$$\mathcal{K}_\mu \mathcal{O}_n(x) = (-2i(x_\mu \partial_\rho - x_\rho \partial_\mu) - i2l_n x_\mu - x^\nu[S_{\mu\nu}]) \mathcal{O}_n(x), \quad (112)$$

where $[S_{\mu\nu}]$ is a finite dimensional representation of SO(3,1) generators satisfying the same commutation relation as $L_{\mu\nu}$’s. Thus, for a primary operator $\mathcal{O}_n(x)$, $\mathcal{O}_n(x = 0)$ plays the role of the highest weight state

$$[\mathcal{O}_n(0), K_\mu] = 0, \quad [\mathcal{O}_n(0), D] = -i l_n \mathcal{O}_n(0); \quad (113)$$

all other states in the highest weight state representation—descendants—are generated by applying $[\bullet, P_\mu]$ multiple times; the whole representation, therefore, is spanned by a collection of

$$\{ \mathcal{O}_n(0), \partial_\mu \mathcal{O}_n(0), \partial_\mu \partial_\nu \mathcal{O}_n(0), \cdots \}; \quad (114)$$
it is also equivalent to a collection of $\mathcal{O}(x = x_0)$ with arbitrary $x_0 \in \mathbb{R}^{3,1}$.

In the preceding sections, we have worked on solutions to the eigenmode equation on AdS$_5$; once the mass-shell condition $[66]$ is imposed, they become solutions to the equation of motion. They are obtained as an eigenmode of the spacetime translation in 3+1 dimensions, $(-i\partial^\mu) = \Delta^\mu$. Under the conformal group SO(4,2), which contains Lorentz SO(3,1) symmetry, however, an irreducible representation has to include solutions with all kinds of eigenvalues $\Delta^\mu$.

In the case of a scalar field on AdS$_5$, one can think of the following linear combination $G(x, z; x_0; R_0)$ (for some $R_0 \ll \Delta^{-1}$):

$$G(x, z; x_0) = \frac{i}{\pi^2} \frac{\Gamma(l_n)}{\Gamma(l_n - 2)} R_0^{l_n-4} \left(\frac{z}{z^2 + (x - x_0)^2}\right)^{l_n} = \int \frac{d^4(\Delta z)}{(2\pi)^4} e^{i(\Delta z)(x - x_0)} \frac{(\Delta z)^2 K_{l_n-2}(\Delta z)}{(\Delta R_0)^2 K_{l_n-2}(\Delta R_0)}. \quad (115)$$

The factor $[e^{i\Delta x_0((\Delta z)^2 K_{l_n-2}(\Delta z))}]$ in the integrand on the right hand side is a solution to the equation of motion of a scalar field on AdS$_5$ whose mass-square $M_{\text{eff}}^2$ is given by $l_n - 2 = i\nu = \sqrt{4 + M_{\text{eff}}^2 R^2}$. The coefficient of the linear combination, $e^{-i\Delta x_0((\Delta R_0)^2 K_{l_n-2}(\Delta R_0))^{-1}}$, is chosen so that the integrand behaves as

$$e^{i(\Delta x_0)} \left(\frac{z}{R_0}\right)^{4-l_n} \quad (116)$$

at $0 \leq z \ll \Delta^{-1}$. The space of solutions to the equation of motion $G(x, z; x_0)$ parameterized by $x_0 \in \mathbb{R}^{3,1}$ is alternatively spanned by derivatives of $G(x, z; x_0)$ with respect to $x_0$ at $x_0 = 0$. It is easy to see that this basis

$$\{ G(x, z; 0), \, \partial_{\mu}^{(x_0)} G(x, z; 0), \, \partial_{\mu}^{(x_0)} \partial_{\nu}^{(x_0)} G(x, z; 0), \, \cdots \} \quad (117)$$

is an eigenbasis under the action of dilatation, $\mathcal{D} := i(z \partial_z + x \cdot \partial)$, and their weights are $-il_n$, $-i(l_n + 1)$, $-i(l_n + 2)$, $\cdots$, respectively. Correspondence between scalar field wavefunctions on AdS$_5$ and scalar primary operators of the dual CFT is established in this way $[34]$.

Let us now generalize the discussion above slightly, to construct an analogue of $G(x, z; x_0)$ for a spin-$j$ field $A_{m_1 \cdots m_j}$ on AdS$_5$, from which the dilatation eigenbasis is constructed. To this end, note that all the $(0, 0, m)$-modes ($m = 0, \cdots, j$) have the leading $z^{2-j-i\nu}$ term in the power series expansion only in the $A_0^{\mu_1 \cdots \mu_j}$ component, not in any other $A_{\pm}^{\mu_1 \cdots \mu_{j-k}}$ components$^{27}$ with $k > 0$. It is possible to choose $e^{(0,0,m)}(\Delta^\mu)$ properly so that

$$\sum_{m=0}^j \left[A_0^{\mu_1 \cdots \mu_j}(x, z)\right]_{\nu_1 \cdots \nu_j} e^{i\Delta x_0} \simeq e^{i(\Delta x_0)} \left(\frac{z}{R_0}\right)^{2-j-i\nu} \epsilon_{\mu_1 \cdots \mu_j} \quad (118)$$

$^{27}$ Use $[17]$.
in the region near the UV boundary \( z \ll \Delta^{-1} \), where \( \epsilon_{\mu_1 \cdots \mu_j} \) is a \( \Delta^\mu \)-independent 4D-traceless totally symmetric rank-\( j \) tensor of \( \text{SO}(3,1) \); the condition on \( \epsilon^{(0,0,m)}(\Delta^\mu) \) is

\[
\epsilon_{\mu_1 \cdots \mu_j} = \left( \frac{R_0}{R} \right)^{2-j} K_{i\nu}(\Delta R_0) \frac{2}{\pi} \sqrt{\nu \sinh(\pi \nu)} \sum_{m=0}^{j} \frac{N_{i,m} \Gamma(m - j - i \nu)}{\Gamma(-j - i \nu)} \sum_{N=0}^{[m/2]} \frac{b_{m,N}^{(j-m)}}{\Delta^{m-2N}} \left( \tilde{E}^N D^{m-2N} \right) \epsilon^{(0,0,m)}(\Delta)_{\mu_1 \cdots \mu_j}.
\]

(119)

It is possible to invert this relation by using (239) and write down \( \epsilon^{(0,0,m)}(\Delta^\mu) \) in terms of \( \epsilon_{\mu_1 \cdots \mu_j} \), though we will not present the result here. What really matters to us is that \( \epsilon^{(0,0,m)}(\lambda \Delta) = \epsilon^{(0,0,m)}(\Delta) \lambda^{i \nu} \). With \( \epsilon^{(0,0,m)} \)'s satisfying the condition above, one can see that the following linear combination of solutions to the equation of motion,

\[
G_{m_1 \cdots m_j}(x, z; x_0) := \int \frac{d^4 \Delta}{(2\pi)^4} \sum_{m=0}^{j} \left[ A_{m_1 \cdots m_j}^{0,0,m;\Delta^\nu}(x, z) \right]^{\hat{k}_1 \cdots \hat{k}_j = -m} \epsilon^{(0,0,m)}(\Delta)_{\kappa_1 \cdots \kappa_j = -m} e^{-i \Delta x_0},
\]

(120)

has a property

\[
G_{m_1 \cdots m_j}(\lambda x, \lambda z; \lambda x_0) = \lambda^{-(2+j+i \nu)} G_{m_1 \cdots m_j}(x, z; x_0).
\]

(121)

\( i \nu \) is determined by the mass parameter on \( \text{AdS}_5 \), once the mass shell condition \([56]\) is imposed. Therefore, \( G_{m_1 \cdots m_j}(x, z; 0) \) is an eigenstate of dilatation, and so are the derivatives of \( G_{m_1 \cdots m_j}(x, z; x_0) \) with respect to \( x_0^\mu \) at \( x_0^\mu = 0 \). All the derivatives combined forms a dilatation eigenbasis in the space of solutions to the equation of motion of a spin-\( j \) field.

It is now clear that the eigenmodes with \((n, l, m) = (0, 0, m) \) \((0 \leq m \leq j) \) and arbitrary \( \Delta^\mu \) as a whole—modes that satisfy the 5D-traceless and 5D-transverse conditions \([95, 96]\) forms an irreducible representation of the conformal group. If one is interested purely in the matrix element of a spin-\( j \) primary operator \( \mathcal{O}_n(x_0 = 0) \) of an approximately conformal gauge theory, then the matrix element can be calculated by using the wavefunction \( G_{m_1 \cdots m_j}(x, z; 0) \). Note that the \( m = 0 \) mode alone, where the Pomeron/Reggeon wavefunction has a single component as in \([9]\), cannot reproduce all the matrix element associated with matrix elements of spin-\( j \) primary operators.

### 5.4 Confinement Effect

**Top down approach:** QCD in the real world is not a conformal gauge theory, but it has a mass gap in the hadron spectrum due to confinement. Confinement of a nearly conformal strongly coupled gauge theory is realized in its gravitational dual description in the form of a nearly \( \text{AdS} \) geometry with a minimum value in the warp factor.
Klebanov–Strassler geometry of Type IIB string theory \cite{27} will be one of the most popular background geometries of that kind. The Klebanov–Strassler geometry is not dual to a confining gauge theory that is asymptotically free, however; it is dual to a gauge theory that is confining in the infrared, but its 't Hooft couplings become stronger and stronger toward ultraviolet. Such geometries as Klebanov–Strassler are not truly dual to the QCD of the real world, but still one will be able to learn a lot from studying the mode-decomposition on such geometries.

Mode decomposition can be carried out, once we know the background configuration and the action of the bilinear fluctuations around the background; we do not need interactions of stringy fields. Thus, it will be a doable task, at least at the supergravity level. Reduction over the \( W_5 = T^{1,1} \) geometry has been worked out in the literature, and one is left to translate the smoothness condition of mode functions at the tip of the deformed warped conifold into the language of boundary conditions on a warped 4+1-dimensional spacetime.\footnote{Such geometries typically are often in the form of \( \mathbb{R}^{3,1} \times W_5' \) which nearly remains constant around the tip of the throat \( r = 0 \), and a shrinking \((5 - n)\)-cycle with the metric \( ds^2 = dr^2 + r^2 (d\Omega_{5-n})^2 \). For simplicity, let \( n = 4 \) and \( d\Omega_1 = d\theta \). A scalar field \( \phi(r, \theta) \) with smooth configuration in the coordinate \((r \cos \theta, r \sin \theta)\) is decomposed into \( \sum_k e^{ik\theta} \phi_k(r) \), when the mode \( \phi_k(r) \) needs to be in the form of \( r^k \times \text{fcn}(r^2) \). Thus, \( \partial_r [r^{-k} \phi_k(r)] = 0 \) at \( r = 0 \). The authors do not find a reason not to work on it, except that it will take extra time to do so.}

In this article, however, we set higher priority in getting a broader perspective on the subject ranging from string theory to hadron physics, and avoid taking too much time to solve technical problems in string theory. Instead, we discuss in the following, two temporary approaches of implementing the confinement effects; one is an effective-theory model building approach, and the other is phenomenological approach. We will proceed with the phenomenological approach in the following sections, although we understand that the top-down approach above will eventually replace/backup/verify the phenomenological approach to be adopted in this article. The following “effective theory model building approach” is not used in this article, but we present it here, because it helps us understand physical meaning (hidden assumptions) of the phenomenological approach.

**Effective Theory Model Building Approach:** The hard-wall model and its variations are introduced in order to mimic the presence of minimum value of the warped factor, mass gap and nearly AdS background geometry. It remains simple enough so that analytic results are obtained in a relatively short amount of time, though we cannot discuss stability of the geometry or theoretical consistency of string theory.

With this philosophy in mind, one could think of implementing the confining effect in
the form of
\[ S = \int d^4 x \int_0^{1/\Lambda} dz \sqrt{-g(z)} \mathcal{L}_{\text{bulk}} + \int d^4 x \sqrt{-g}|_{z=1/\Lambda} \mathcal{L}_{\text{bdry}}, \]
where the background geometry remains to be AdS$_5$, and the holographic radius $z$ is cut-off at $z = \Lambda^{-1}$. Note that different choices of $\mathcal{L}_{\text{bdry}}$ lead to different physics; to be more precise, different choices of $(\mathcal{L}_{\text{bulk}}, \mathcal{L}_{\text{bdry}})$ modulo partial integration should be regarded as different models. It is reasonable to have such freedom in the choice of effective-theory models, because we know that there are more than one holographic backgrounds of Type IIB string theory that are dual to confining gauge theories. Such constraints as SO(3,1) symmetry unbroken global symmetry of a strongly coupled gauge theory, however, are very weak in constraining $\mathcal{L}_{\text{bdry}}$.

Once a model is fixed, then the Euler–Lagrange equation of this theory not only includes the equation of motion in the bulk $[\mathcal{E}] = [\mathcal{L}] + [\mathcal{E}]_{\text{bdry}}$ but also boundary conditions at $z = 1/\Lambda$. Different models (i.e., different $\mathcal{L}_{\text{bdry}}$) predict different Pomeron/Reggeon wavefunctions.

We require that SO(3,1) symmetry is preserved even in $\mathcal{L}_{\text{bdry}}$. Boundary conditions might introduce mixing between the eigenmode decomposition determined in the bulk, in principle, but the unbroken SO(3,1) symmetry excludes mixing between SO(3,1)-irreducible tensors of different ranks. This observation still does not exclude mixing among $(n, l, m)$-modes of a spin-$j$ totally symmetric field on AdS$_5$ with a common $m$, but different $(n, l)$’s.

**Phenomenological Approach:** As another alternative approach, one can think of a phenomenological approach, which is to start from a small number of parameters, and let the physical consequences constrain those parameters. When one finds that reasonable physical consequences cannot be available under a given set of parameters, then a little more parameters will be introduced so that more freedom is available.

As one of the simplest trial parametrizations of the confining effect, we make a following changes in the mode functions $\Psi_{i\nu,0,0,m}^{(j)}(\Delta, z)$:

\[ K_{i\nu}(\Delta z) \longrightarrow \left[ K_{i\nu}(\Delta z) + \frac{\pi}{2} \frac{c_{i\nu,0,0,m}^{(j)}}{\sin(\pi i \nu)} I_{i\nu}(\Delta z) \right] =: "K_{i\nu}(\Delta z)"'. \]  

$c_{i\nu,0,0,m}^{(j)}$’s, which may depend on $\Delta^2$ and $\Lambda$, are the parameters we introduce. An implicit assumption here is that the confining effect does not introduce mixing among modes with different $(n, l, m)$’s. Under this assumption, however, the parametrization above does not lose any generality; once the ratio between the $K_{i\nu}(\Delta z)$ wave and $I_{i\nu}(\Delta z)$ is given for $\Psi_{i\nu,0,0,m}^{(j)}(-\Delta^2, z)$, there is no freedom left for the other $\Psi_{i\nu,s,N}^{(j)}(\Delta^2, z)$ functions ($(s, N) \neq (0, 0)$) of the same $(n, l, m) = (0, 0, m)$ mode, because the relation among them is completely fixed by the equa-
tion of motion in the bulk. In section 7.1 we will carry out a test of whether this simple parametrization works well or not.

When the infrared boundary is introduced in the holographic background geometry, the normalization of the Pomeron/Reggeon wavefunction also needs to be changed. In the case of \((n, l, m) = (0, 0, 0)\) mode, with the Dirichlet boundary condition at the IR boundary \(z = 1/\Lambda\), for example, the wavefunction \(\Psi^{(j)0,0,0}_{iv}(z) = \Psi^{(j)}_{iv}(-\Delta^2, z)\) was given the following normalization [6, 10]:

\[
\Psi^{(j)}(x_0, \nu) = e^{(j-2)A} \frac{2}{\pi} \sqrt{\frac{\nu \sinh(\pi \nu)}{2R}} \sqrt{\frac{I_{iv}(x_0)}{I_{-iv}(x_0)}} \left[ K_{iv}(\Delta z) - \frac{K_{iv}(x_0)}{I_{iv}(x_0)} I_{iv}(\Delta z) \right],
\]

with an extra factor \(\sqrt{I_{iv}(x_0)/I_{-iv}(x_0)}\), where \(x_0 := \Delta/\Lambda\). This result is generalized as follows. By repeating the same argument as in the appendix A.3.1, one finds that the normalization factor \(N_{j,m}\) should be replaced by

\[
N_{j,m} \rightarrow N_{j,m} \times \frac{1}{\sqrt{1 - c^{(j)}_{iv,0,0,m}}}.
\]

The Dirichlet boundary condition for the \(m = 0\) mode above corresponds to \((1 - c^{(j)}_{iv,0,0,0}) = [I_{iv}(x_0)/I_{iv}(x_0)]\); the modified normalization (124) is a special case of (125). The mode functions are defined, so far, for \(\nu \geq \mathbb{R}\), since the eigenvalue \(E_{0,0} = 4 + j + \nu^2\) depends only on \(\nu^2\). When the modefunction is analytically continued to the \(\nu < 0\) region, the mode function for \(-\nu\) should be the same as \(+\nu\). From this observation, it follows that

\[
(1 - c^{(j)}_{-iv,0,0,m}) = (1 - c^{(j)}_{iv,0,0,m})^{-1}.
\]

6 Organizing the Scattering Amplitude on AdS$_5$

6.1 “Effective” String Field Action on AdS$_5$

If we are to start from Type IIB string theory on 10-dimensions with a background that is approximately AdS$_5 \times W_5$ (except near the infrared boundary), one can think of an effective theory on AdS$_5$ after carrying out “spherical harmonics” mode decomposition on $W_5$. As we have already discussed in section 5 how to construct propagators in such an effective theory, we would now like to construct the scattering amplitude.

For this purpose, we need interaction among string fields, and we turn to the cubic string field theory, which we reviewed already in section 4. This allows us to write down a concrete
expression for the scattering amplitude. Clearly the biggest drawback of this approach is in the fact that no stable background geometry \( \text{AdS}_5 \times W_{21} \) is known in bosonic string theory for some 21-dimensional internal manifold \( W_{21} \). In the following, we will construct an “effective” action on \( \text{AdS}_5 \) by carrying out dimensional reduction of the cubic string field theory action, as if there exists an \( \text{AdS}_5 \times W_{21} \) solution to the bosonic string theory. This is not meant to claim that we obtain such an action as an effective theory of the bosonic string theory, but to use it as a starting point in constructing a toy-model scattering amplitude of a hadron and a (virtual) photon that may still carry some fragrance of interaction structure in superstring theory.

Let us start off by clarifying the relation between the normalization of string component fields in (38, 41, 42) and that of the component fields in (62). All the component fields in (38) are normalized so that they have canonically normalized kinetic terms in the action in the 26-dimensional spacetime. Now, we make them dimensionless by redefinition \( \phi \rightarrow g_o^{-1} \phi \), \( A_M \rightarrow g_o^{-1} A_M \), etc. All the terms in the cubic string field theory—both the kinetic terms and interactions—will then have \( 1/g_o^2 \) as an overall factor. When a mode decomposition of the following form is assumed for the component in this new normalization,

\[
\phi(x, z, \theta) = \sum_y \phi^{(y)}(x, z) Y_y(\theta), \quad A_M(x, z, \theta) = \left\{ \begin{array}{ll}
\sum_y A_{m}^{(y)}(x, z) Y_y(\theta) & M = m = 0, \ldots, 3, z \\
0 & M = 5, \ldots, 25.
\end{array} \right.
\]

(127)

Similarly decomposition holds for spin-\( h_a \) fields \( A_{M_1 \ldots M_{h_a}}(x, z, \theta) \); we take spherical harmonics \( Y_y(\theta) \) (labeled by \( y \)) to be dimensionless, so that the component fields on \( \text{AdS}_5 \) such as \( \phi^{(y)}(x, z), A_m^{(y)}(x, z), A_{m_1 \ldots m_{h_a}}^{(y)}(x, z) \) are also dimensionless.

The overall coefficient of the “effective” action on \( \text{AdS}_5 \) then becomes a dimension-(+3) parameter

\[
\frac{\text{vol}(W_{21})}{2(g_o)^2} \times O(1),
\]

(128)

which is to be identified with the overall coefficient \( t_y/(2R^3) \) in (62). Reduction of interaction terms (13, 15, 16) also yield the same overall factor (128) apart from possibly order one factor coming from overlap integration of spherical harmonics over the internal manifold. Because the amplitudes from exchanging states with higher spherical harmonics are suppressed in small-\( x \) DIS and DVCS (e.g. [10]), we will be interested only in the interactions involving \( \phi^{(y)} - \phi^{(y)} \) [intermediate-states] and \( A_{m}^{(y)} - A_{m}^{(y)} \) [intermediate-states] cubic couplings, with the intermediate states having spherical harmonics \( Y(\theta) = 1 \). The overall factor of the cubic interactions then becomes precisely the same as that of the kinetic terms of \( \phi^{(y)} \) and \( A_{m}^{(y)} \).

For this reason, we write down the following interaction terms for the “effective” action
on AdS$_5$: 
\[
S_{\text{eff. int.}} = -\frac{t_{\phi y} \lambda_{\text{soft}}}{3\alpha' R^3} \int d^4xdz \sqrt{-g(z)} \hat{E} \left(3 \text{tr} \left[\phi_y^2 \phi_y\right] + \sqrt{\frac{8\alpha'}{3}} \text{tr} \left[-iA_m \left(\phi_y \nabla^m \phi_y\right)\right] \right)
\]
\[
- \frac{8\alpha'}{9\sqrt{2}} \text{tr} \left[f_{mn} \left(\phi_y \nabla^m \nabla^n \phi_y\right)\right] - \frac{5}{9\sqrt{2}} \text{tr} \left[f_m^2 \phi_y^2\right]
\]
\[
+ \frac{2\sqrt{\alpha'}}{3} \text{tr} \left[\left(\nabla_m g^m\right)\phi_y^2\right] - \frac{11}{9} \text{tr} \left[h\phi_y^2\right] + \ldots.
\]

(129)

Fields without a label $y$ are to be used for the intermediate states exchanged in the $t$-channel (in the sense that we explained in section 4.2); $\phi_y$ are for the incoming and outgoing states. Partial derivatives have been replaced by covariant derivatives on AdS$_5$. Similarly, all other interactions such as (45, 46) in 26-dimensions also give rise to their corresponding cubic interactions on AdS$_5$. Certainly such a choice of “effective” action on AdS$_5$ will be one of the most likely (and simple enough) set-ups that may still maintain some aspects of scattering amplitude in string theory, although top-down justification is not given.

We will only sum up $t$-channel amplitudes where $Y_y(\theta) = 1$ modes of the stringy states in the leading Reggeon/Pomeron trajectory are exchanged, because that constitutes the dominant contribution in the small $x$ scattering. Thus, three point interactions of such modes with incoming and outgoing tachyon states are necessary, which we write down as follows,

\[
\Delta S_{\text{eff. int.}} = -\frac{t_h \lambda_{\text{soft}}}{R^3 \alpha'} \int d^4xdz \sqrt{-g(z)} \hat{E} \text{tr} \left[A_{m_1 \ldots m_N} \left(\phi \nabla^{m_1} \ldots \nabla^{m_N} \phi\right)\right] \left(\frac{8\alpha'}{27}\right)^N \frac{(-i)^N}{\sqrt{N!}},
\]

by keeping only the $Y_y(\theta) = 1$ modes and replacing derivatives in (45) by covariant derivatives. The normalization constant $t_{\phi y}$ for the target hadron kinetic term is now simply written as $t_h$, as we will only have to pay attention to individual choices of target hadrons (individual choices of $Y_y(\theta)$) in the external states. Similarly, we also need interaction of the same group of modes with the incoming and outgoing photon states, which we write down as follows:

\[
\Delta S_{\text{eff. int.}} = -\frac{t_{\gamma} \lambda_{\text{soft}}}{R^3 \alpha'} \int d^4xdz \sqrt{-g(z)} \hat{E} \text{Tr} \left[A_{m_1 \ldots m_N} \left(\phi \nabla^{m_1} \ldots \nabla^{m_N} \phi\right)\right] \left(\frac{8\alpha'}{27}\right)^N \frac{g^{k l 16}}{\sqrt{N!}} \frac{16}{27} + \ldots\]

(131)

following the same procedure by starting from (46). We have retained only the terms that have $N$-derivatives and are proportional to $\eta^{k l}$, as they are necessary in determining the...
“twist-2” contributions to the structure function $V_1$. Since we only need the normalization constant $t_{A_y}$ of the kinetic term of the external state only for the spherical harmonics $Y(\theta) = 1$, we no longer need to refer to the choice of spherical harmonics; $t_{A_y}$ is therefore rewritten as $t_\gamma$.

### 6.2 External States Wavefunction

The vertex operator insertions in the world-sheet calculation are replaced by appropriate external state wavefunctions in amplitude calculations based on string field theories.

First, the insertions of vertex operator of the form (33) for the U(1) currents on flavor D7-branes are replaced by wavefunctions for the massless vector field in the bosonic string theory. We use the wavefunctions for the incoming state $\gamma^* (q_1)$ and outgoing state $\gamma^* (q_2)$

\[
A_{m_{in}}^{in} (x, z, \gamma) = R \int \frac{d^4 q_1}{(2\pi)^4} e^{iq_1 \cdot (x - (\Delta x)/2)} A_m (z, q_1), \tag{132}
\]

\[
A_{m_{out}}^{out} (x, z, \gamma) = R \int \frac{d^4 q_2}{(2\pi)^4} e^{-iq_2 \cdot (x + (\Delta x)/2)} A_m (z, q_2), \tag{133}
\]

where $A_m (z, q)$ on the right-hand sides are the wavefunctions given in (31). A factor $R$ is inserted here, because we adopted a normalization convention so that $A_m^{(in/out)}(x, z)$ on AdS$_5$ is dimensionless.\(^{29}\) The arguments of the electromagnetic current insertions $T \{ J^\nu (x) J^\mu (y) \}$—coordinates in the boundary theory $x$ and $y \in \mathbb{R}^{3,1}$—are now denoted by $\bar{x} + (\Delta x)/2$ and $\bar{x} - (\Delta x)/2$, respectively.

The vertex operators (30) for the target hadron are replaced by wavefunctions of the form

\[
\phi_{in}^{in} (x, z, h) = e^{ip_1 \cdot x_h} \Phi (z_h; m_n), \quad \phi_{out}^{out} (x, z, h) = e^{-ip_2 \cdot x_h} \Phi (z_h; m_n), \tag{135}
\]

where $\Phi (z, m)$’s on the right-hand sides are the wavefunction given by (31). The first one is for the incoming state, and the second for the outgoing hadron.

\(^{29}\) $A_m(x, z)$ is often normalized so that it has mass dimension (+1), and hence this factor $R$ is not necessary then. In the case the gauging of a global symmetry of a strongly coupled gauge theory is realized in the form of flavor D7-brane, the natural reduction of the 7-brane action on a three-cycle leads to the form of

\[
S_{\text{eff}} \sim - \frac{N_c}{R} \int d^4x dz \sqrt{-g(z)} F_{mn} F^{mn}; \tag{134}
\]

the external state wavefunction without the factor $R$ can be used in such cases. In the presentation adopted in this section, where bosonic string is used and the gauge field is assigned zero mass dimension (like other higher spin fields), the factor $R$ is included in (132, 133), and the kinetic term of $F_{mn} F^{mn}$ has the coefficient $t_\gamma / R^3$ instead. Thus, we can think of $t_\gamma$ as something like $N_c$. 

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6.3 Leading Trajectory Contribution to the Compton Tensor

When the target hadron is to be identified with some Kaluza–Klein state of the tachyon of the bosonic string theory, then $\Delta_\phi - 2 = \sqrt{4 + c - \sqrt{\lambda}}$ is not real valued for $\lambda \gg 1$. We treat this $\Delta_\phi - 2$ as if it were real valued, until last moment. Since our true interest is in scattering amplitude in Type IIB string theory, or in hadron scattering in the real world, this problem is absent in such situations, and we do not bother about this issue.

Let us combine all the pieces together to organize an amplitude of photon-tachyon scattering given by $t$-channel exchange of leading trajectory spin-$j$ state reduced to AdS$_5$ with $Y_\gamma(\theta) = 1$. Such an amplitude—denoted by $iM^{(t)}_{\gamma\gamma}(N_{\text{eff}}=j,j)$—consists of $t$-channel exchange of all the eigenmodes labeled by $(n,l,m)$. We will further focus on contributions from $(n,l,m) = (0,0,m)$. It is given by

$$iM^{(t)}_{(j,j);(0,0,m)} \simeq \frac{-it_\gamma}{R^3} \int d^4x dz \sqrt{-g(z_\gamma)J_{\gamma_1\cdots k_j;pq}^\gamma(g^{k_{1}r_1}\cdots g^{k_jr_j})(z_\gamma) \left(\frac{\alpha'}{2}\right)^{j/2} e^{-2A(z_\gamma)}$$

$$\frac{-ih}{R^3} \int d^4x dz \sqrt{-g(z_h)J_{l_1\cdots l_j}^{hh}(g^{l_{1}s_1}\cdots g^{l_j s_j})(z_h) \left(\frac{\alpha'}{2}\right)^{j/2} e^{-2A(z_h)}}$$

$$\frac{1}{j!} \left(\frac{27}{16}\right)^{\alpha'-j-1} \langle e^{2A(z_\gamma)} e^{2A(z_h)} G(0,0,m)(x_\gamma, z_\gamma; x_h, z_h)_{r_1\cdots r_s; s_1\cdots s_j} \rangle; \quad (136)$$

just like in the amplitude calculation in section 4.2, this amplitude is meant to be the coefficient of $\text{Tr}[\lambda^2 \lambda^1 \lambda^1 \lambda^2]$. $J_{\gamma\gamma}$ and $J_{hh}$ above are given by the external state wavefunctions as follows:

$$J_{k_1\cdots k_j;pq}^{\gamma}(x_\gamma, z_\gamma) = (-i)^j \left[ A_{p}^{\text{out}} \bigtriangledown_{k_1} \cdots \bigtriangledown_{k_j} A_{q}^{\text{in}} \right] (x_\gamma, z_\gamma), \quad (137)$$

$$J_{l_1\cdots l_j}^{hh}(x_h, z_h) = (-i)^j \left[ \phi_{l_1}^{\text{in}} \bigtriangledown_{l_1} \cdots \bigtriangledown_{l_j} \phi_{l_1}^{\text{out}} \right] (x_h, z_h). \quad (138)$$

Here, $\phi_{\text{in/out}}(x_h, z_h)$ are both of mass dimension $(-1)$, and $A_{m}^{\text{in/out}}(x_\gamma, z_\gamma)$ of mass dimension $(+3) + \dim[\epsilon_\mu]$. From this expression, one can see that the first line has mass dimension $(+6) + 2 \times \dim[\epsilon_\mu]$, the second line $(-2)$, and the last line 0. Thus, $iM^{(t)}_{(j,j);(0,0,m)}$ is a function of $p_1^\mu$, $p_2^\mu$, $\vec{\epsilon}$, and $\Delta x^\mu$ of mass dimension $4 + 2 \times \dim[\epsilon_\mu]$. This is precisely the property expected for

$$(i)^2 \langle h(p_2)|T \{J''(\vec{x} + (\Delta x)/2) J'(\vec{x} - (\Delta x)/2) \} |h(p_1)\rangle \epsilon_\mu \epsilon_\nu^{2*}. \quad (139)$$

Its Fourier transform with respect to $(\Delta x)^\mu$ becomes $iT^{\mu\nu} \times e^{-i\vec{x}(p_2 - p_1)}$.

If we carry out integration over $d^4x_\gamma$, $d^4x_h$ and $d^4(\Delta x)$ first, then the three integration variables $\Delta^\mu$ in (101) and $q_{1,2}$ in (132, 133) are determined in terms of the input $p_{1,2}^\mu$ and $q^\mu$;
we have $\Delta^\mu := (p_2 - p_1)^\mu$, $q_1^\mu = (q - \Delta/2)^\mu$ and $q_1 := (q + \Delta/2)$. As a result, it follows that

$$[T^\nu_{\mu'} \epsilon_{\mu
u}^{2*}]^{(t)} = \int d^4(\Delta x) e^{-i\vec{q} \cdot (\Delta x)} M^{(t)}_{(j,j);(0,0,m)\bar{x}=0}$$

$$\simeq \frac{t_n}{R^3 \alpha'} \int dz_i \sqrt{-g(z_i)} \tilde{J}^{\gamma}_{k_1 \cdots k_j \rho q} R^2 g^{pq} (g^{k_1 r_1} \cdots g^{k_j r_j}) \left( \frac{\alpha'}{2} \right)^{j/2}$$

$$+ \frac{t_h}{R^3 \alpha'} \int dz_h \sqrt{-g(z_h)} \tilde{J}^{h}_{l_1 \cdots l_j} (g^{l_1 s_1} \cdots g^{l_j s_j}) \left( \frac{\alpha'}{2} \right)^{j/2}$$

$$\frac{1}{j!} \left( \frac{27}{16} \right)^{\alpha'-(j-1)} R^3 \alpha' \int_0^\infty dv \frac{P_{\rho_1 \cdots \rho_j}^{(j-m)} \Phi(z_h)_{\rho_1 \cdots \rho_j \gamma} (z_h; \Delta, v)}{z_h + c v} + N_{\text{eff}} - i \epsilon$$

$$\left[ A^{0,0,m}_{(x_1 \cdots x_j \rho)} (\Delta, \nu) \right]_{\rho_1 \cdots \rho_j \gamma} (z_h; \Delta, \nu)$$

$$\left[ A^{0,0,m}_{(x_1 \cdots x_j \rho)} (\Delta, \nu) \right]_{\rho_1 \cdots \rho_j \gamma} (z_h; \Delta, \nu) \right. \right) \right]_{\rho_1 \cdots \rho_j \gamma}, \quad (140)$$

where

$$\tilde{J}^{\gamma}_{k_1 \cdots k_j \rho q} (z_i) = (-i)^j \left[ A_p (z_i)_{\rho q} \bigtriangledown_{k_1} \cdots \bigtriangledown_{k_j} A_q (z_i)_{\rho q} \right] , \quad (141)$$

$$\tilde{J}^{h}_{l_1 \cdots l_j} (z_h) = (-i)^j \left[ \Phi (z_h)_{\rho q} \bigtriangledown_{l_1} \cdots \bigtriangledown_{l_j} \Phi (z_h)_{\rho q} \right] \right) , \quad (142)$$

Although momentum vectors are used in the second arguments of the external state wavefunctions $A$ and $\Phi$ here, instead of their Lorentz-invariant momentum square, this is only to remind ourselves of the sign when $\bigtriangledown$’s act on the wavefunctions.

The expression (140) is meant to be a part of the $t$-channel contribution to the Compton tensor, $[T^\nu_{\mu'} \epsilon_{\mu
u}^{2*}]^{(t)}$, and we should obtain the full contribution to the Compton tensor $[T^\nu_{\mu'} \epsilon_{\mu
u}^{2*}]$ after employing the prescription (52). At least this prescription tells us to set the factor $(27/16)^{\alpha'-(j-1)}$ in the 4th line to $(27/16)^{\alpha'-(j-1)} \simeq 1$. Now, we claim that this is the only necessary change under this prescription, so far as the amplitude of $(0, 0, m)$-mode exchange is concerned.

To see this, remember that, prior to applying the prescription (52), we need to rewrite the residues of the $t$-channel poles in terms only of Mandelstam variables $s$ and $t$, not of $u$. Let us take an expression $[\Phi_h \bigtriangledown_{m} \Phi_{h}] g^{mn} [A_j \bigtriangledown_{n} A_j]$ as an example, which captures the feature of contraction of $\text{SO}(4,1)$ indices in (140). In the scattering $\Phi(P_1) + A(Q_1) \rightarrow \Phi(P_2) + A(Q_2)$ with $P_{1,2}$ and $Q_{1,2}$ “momenta” $\sim$ derivatives in 5-dimensions, $(s - u) \sim (P_1 + P_2) \cdot (Q_1 + Q_2)$ is converted to $(2s + t)$ in the following steps:

$$(P_1 + P_2) \cdot (Q_1 + Q_2) = (2P_1 + (P_2 - P_1)) \cdot (Q_1 + Q_2),$$

$$= (2P_1) \cdot (Q_1 + Q_2) + (Q_1 - Q_2) \cdot (Q_1 + Q_2) = (2P_1) \cdot (2Q_1 + (Q_2 - Q_1)) + (Q_1)^2 - (Q_2)^2,$$

$$= (2P_1) \cdot (2Q_1 + (P_1 - P_2)) + (Q_1)^2 - (Q_2)^2 = (4P_1 \cdot Q_1) + (-2P_1 \cdot P_2) + 2(P_1)^2 + (Q_1)^2 - (Q_2)^2 ;$$

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each one of the steps above is regarded as either one of partial integration in $dx_γdz_γ$, one in $dx_hdz_h$, or rewriting $(P_2 - P_1)$ by $(Q_1 - Q_2)$ or vice versa. The last procedure is to pass a derivative on one side of the propagator to the other. Because of the 5D-transverse condition characterizing the $(0, 0, m)$ modes, such terms proportional to $∇$ drop out from the amplitudes exchanging the $(0, 0, q)$ while the ratio among $(q · Δ)$ passing a derivative $∇$ generalizes Bjorken scaling, where $\text{in holographic models.}$

6.3.1 Casting the Amplitude into the form of OPE

So far, the (virtual) photon and target hadron have been treated equally in the scattering amplitude. We are interested, however, in the $h + γ^* → h + γ(\ast)$ scattering in the regime of generalized Bjorken scaling, where

$$|(q^2)|, (q · p)|, |(q_1 · Δ)|, |(q_2 · p)| \gg |Δ^2|, m_h^2, Δ^2, \quad (143)$$

while the ratio among $(q · p), (q^2)$ and $(q · Δ)$, namely, $x$ and $η$, are kept finite. It is, thus, desirable to rewrite the scattering amplitude (structure functions) in a form that fits to the conformal OPE. To do this, we follow a prescription that has been used in the study of DIS in holographic models.

Let us focus on the following factors that appear in the 3rd and 4th lines of (140):

$$\int_0^∞ dν \left[ A^{0,0,m}_{r_1...r_j,0,0} (z_γ, −Δ, ν) \right] \tilde{ρ}_1...\tilde{ρ}_j−m \left[ A^{0,0,m}_{0,0,0}(z_h, Δ, ν) \right] \delta_1...\delta_j−m \times [\cdots]. \quad (144)$$

The last factor $[\cdots]$ denotes the remaining $ν$-dependence (denominator) in the integrand; we only need to remember that $E_{0,0} = (4 + j + ν^2)$, and hence it is even under the change $ν → −ν$.

We begin with the case $m = 0$. The expression (144) for the $m = 0$ case becomes

$$\int_0^∞ dν \left[ Ψ^{(j):0,0}_{iv,0,0,0} (−Δ^2; z_γ) \right] \left[ Ψ^{(j):0,0}_{iv,0,0,0}(−Δ^2; z_h) \right] × [\cdots], \quad (145)$$

multiplied by a factor $[δ_1...ρ_1...δ_{s_1}...ρ_{s_1}...δ_{s_2}...ρ_{s_2}]$. Using the fact that $K_{iv}(x) = iπ/2 × (I_{iv}(x) − I_{−iv}(x))/[\sinh(πν)]$, the $ν$-integral above can be rewritten as

$$\frac{1}{πR} \int_{−∞}^{+∞} dν \left[ e^{(j−2)A(z_γ)} I_{iv}(Δz_γ) \right] \left[ e^{(j−2)A(z_h)} e^{(j−2)A(z_h)} \right] × [\cdots], \quad (146)$$
where we used the relation \([126]\). This expression is more convenient than \([145]\); this is because i) the \(z_γ\) integration is dominated in the region \(q z_γ \ll 1\), due to the photon external state wavefunctions containing \(K_1(q, z)\), ii) \(I_{I\nu}(\Delta z_γ)\) decreases rapidly toward positive \(i\nu\), for \(q z_γ \ll 1\) and \(q \gg \Delta\) (generalized Bjorken scaling \([143]\)), and iii) the rapidly decreasing \(I_{I\nu}(\Delta z_γ)\) in the lower half of the complex \(\nu\)-plane allows us to close the \(\nu\)-integration contour through the large-radius lower half complex \(\nu\)-plane (see \([10]\) and literatures therein).

It is straightforward to generalize this treatment for all other \(m \neq 0\) modes. Note that the Pomeron/Reggeon wavefunction \([\mathcal{A}^{0,0;\nu}_{m_1\cdots m_j}(z; \Delta, \nu)]\hat{J}_1\cdots\hat{J}_m\) for \(m \neq 0\) is obtained from that of \(m = 0\) by multiplying \((\Delta z)^m\) and \(N_{j,m}\) (which is even in \(\nu\)), applying differential operators in \(z\) and manipulating Lorentz indices. Obviously the order of such manipulations on the wavefunction and the procedure from \([145]\) to \([146]\) can be exchanged.

Therefore, the contribution to the Compton tensor from the leading trajectory spin-\(j\) state \((0, 0, m)\) mode is

\[
(T^{\mu\nu} \epsilon_\mu^1 \epsilon_\nu^2)_{(j,j);(0,0,m)} \simeq \frac{1}{j!} \frac{t_y}{t_y \pi} \left( \frac{\alpha'}{2} \right)^j \int_{-\infty}^{+\infty} d\nu \frac{P_{\rho_1\cdots\rho_j\cdots \nu}(\Delta, \nu) \hat{J}_1^{\rho_1\cdots \rho_j\cdots \nu} \hat{J}_m^{\rho_1\cdots \rho_j\cdots \nu}}{\varepsilon_0 + c \nu} + N_{\text{eff.}} - i\epsilon
\]

\[
\frac{R^2}{R^3} \int dz_\gamma \sqrt{-g(z_\gamma)} J^{\gamma\nu}_{\mu_1\cdots \mu_j\cdots \nu} \left[ \mathcal{A}^{0,0,\nu}_{(0,0,\nu)}(z_\gamma; -\Delta, \nu) \right] \hat{J}_1^{\mu_1\cdots \nu} \cdots \hat{J}_m^{\mu_1\cdots \nu},
\]

where \(\mathcal{A}\) and \(\bar{\mathcal{A}}\) are obtained from \(A\) by removing the factor \((2/\pi)^{\alpha'}/\sqrt{\nu \sinh(\pi \nu)/2R}\) in \([93]\) first, and then replacing \(K_{I\nu}(\Delta z_\gamma)\) by "\(\bar{K}_{I\nu}(\Delta z_\gamma)\)" in \(\bar{A}(z_\gamma)\), while replacing \(K_{I\nu}(\Delta z_\gamma)\) by \(I_{I\nu}(\Delta z_\gamma)\) in \(\bar{A}(z_\gamma)\). Short distance (stringy) parameters such as AdS radius \(R\) and string length \(\sqrt{\alpha'}\) can be eliminated from this expression of the Compton tensor, so that it is written purely in terms of parameters of strongly coupled gauge theory / hadron physics;

\[
(T^{\mu\nu} \epsilon_\mu^1 \epsilon_\nu^2)_{(j,j);(0,0,m)} \simeq \frac{1}{j!} \frac{t_y}{t_y \pi} \left( \frac{1}{2\sqrt{\lambda}} \right)^j \int_{-\infty}^{+\infty} d\nu \frac{P_{\rho_1\cdots\rho_j\cdots \nu}(\Delta, \nu) \hat{J}_1^{\rho_1\cdots \rho_j\cdots \nu} \hat{J}_m^{\rho_1\cdots \rho_j\cdots \nu}}{\varepsilon_0 + c \nu} - i\epsilon
\]

\[
\int_0^{dz_\gamma} \mathcal{A}_{\mu}(z_\gamma; q_2) \left( -i \nabla \right)_{\mu_1\cdots \mu_j} A_{\nu}(z_\gamma; q_1) \delta^{\nu\nu} \hat{J}_1^{\mu_1\cdots \mu_j} \cdots \hat{J}_m^{\mu_1\cdots \mu_j} \left[ \epsilon^{(2-j)A} \mathcal{A}^{0,0,\nu}_{(0,0,\nu)}(z_\gamma; -\Delta, \nu) \right]^{\rho_1\cdots \nu},
\]

\[
\frac{t_y}{t_y} \int_0^{dz_\gamma} \Phi(-i \nabla)_{\mu_1\cdots \mu_j} \Phi^\dagger \left[ \epsilon^{(2-j)A} \mathcal{A}^{0,0,\nu}_{(0,0,\nu)}(z_\gamma; -\Delta, \nu) \right]^{\rho_1\cdots \nu}.
\]

Each line of this expression has zero mass-dimension, and hence \(T^{\mu\nu}\) is also of zero mass-dimension, as expected from the Fourier transform of the matrix element \([133]\).
The leading twist contribution to the Compton tensor $T^{\mu \nu}$ should be obtained by summing up the amplitudes of exchanging the spin-$j$ field in the leading trajectory, with $m = 0, \cdots, j$ also being summed up. It is known in the literature that, for each spin-$j$, the second line of (149) becomes something close to the Wilson coefficient of the OPE, and the third line of (149) something close to the operator matrix element. We will elaborate more on it, with a particular emphasis on the role played by the summation over $m$. For now, we define

$$C^{0,0,m} := \int_0^1 \frac{dz}{z} \left[ A_p(z; -q_2)(-i \nabla)^j_{k_1 \cdots k_j} A_q(z; q_1) \right] \times \left[ \frac{(2\Lambda)^{\nu-j}}{\Delta^\nu} \Gamma(i\nu + 1) \right] \delta^{\rho_1 \cdots \rho_j} \delta^{\sigma_1 \cdots \sigma_j}$$

$$\left[ \delta^{\rho_1 \cdots \rho_j} \delta^{\sigma_1 \cdots \sigma_j} \right] \left[ e^{(2-j)A} \bar{A}^{0,0,m}_{s_1 \cdots s_j} (z^-\Delta, \nu) \right] \delta^{\rho_1 \cdots \rho_j} \delta^{\sigma_1 \cdots \sigma_j}$$

and

$$\Gamma^{0,0,m} := t_h \int_0^{1/\Lambda} \frac{dz}{z^3} \left[ \Phi(-i \nabla)^j_{l_1 \cdots l_j} \Phi \right] z^j \times \left( \frac{\Delta}{2\Lambda} \right)^{i\nu} \frac{A^j}{\Gamma(i\nu)}$$

separately. The factor $[\Gamma(i\nu + 1)(2\Lambda)^{\nu-j}/\Delta^\nu]$ in $C^{0,0,m}$ and a similar factor in $\Gamma^{0,0,m}$ are introduced so that $C^{0,0,m}$ and $\Gamma^{0,0,m}$ correspond to the OPE Wilson coefficients and hadron matrix elements, respectively, renormalized at $\mu_F \sim \Lambda$, as we will see later.

We will focus on the spin-even contribution to a flavor-non-singlet component of the structure function $V_1$ in (17). The $V_1$ structure function is picked up here, only because it is computed a little more easily than other structure functions. We will not touch flavor-singlet components in this article, apart from a brief discussion in section 7.3; this is because the cubic SFT with Chan–Paton factor in section 4 is not the adequate tool to study the singlet components. The coefficient $C^{0,0,0}$ above is decomposed, just like $T^{\mu \nu} e^1 \times e^2$ is; the spin-$j$ (with $j \in 2\mathbb{Z}$) contribution to the structure function $V^{+,\alpha}_1$—spin-even (+) and flavor non-singlet ($\alpha$)—is denoted by $C^{0,0,m}_{V^{+,\alpha}_1}$.

### 6.3.2 Amplitude of the $(m = 0)$-Mode Exchange

We first study $V^{+,\alpha}_1$ from the $m = 0$ mode exchange. With the Reggeon wavefunction given by $\Psi^{(j),0,0}_{\nu,0,0}(t, z) = \Psi^{(j)}_{\nu}(t, z)$ in (93), this $m = 0$ contribution is expected to be the closest to what has been studied in the literatures such as [3, 7, 9, 10]. Indeed, we reproduce the expression known in the literature, but with a little refinement in (163).

Note first that the Reggeon wavefunctions $\bar{A}^{0,0,m=0}_{r_1 \cdots r_j}$ and $\bar{A}^{0,0,m=0}_{s_1 \cdots s_j}$ are non-zero only when all the $r_i$’s and $s_i$’s are in the 3+1 Minkowski directions, $(r_1 \cdots r_j) = (\rho_1 \cdots \rho_j)$ and
appearing in $C$ confinement effects do not play a significant role quantitatively for most of kinematical region.

This makes it much easier to evaluate the matrix element $\Gamma^{0,0,m=0}$. Because

$$\left(\nabla^k \Phi\right)_{\sigma_1 \cdots \sigma_k} = \partial_{\sigma_1} \cdots \partial_{\sigma_k} \Phi + \text{[terms proportional to } \eta_{\sigma_a \sigma_b}], \quad (150)$$

only

$$\left[ \Phi(z; p_1)(-i \nabla)^j \Phi(z; -p_2) \right]_{\sigma_1 \cdots \sigma_j}$$

$$:= \sum_{k=0}^j j C_k \left[ (i \nabla)^j (-k \Phi(z; p_1) \right]_{\sigma_k+1 \cdots \sigma_j} \left[ (-i \nabla)^k \Phi(z; -p_2) \right]_{\sigma_1 \cdots \sigma_k}$$

$$\rightarrow (-1)^j (p_1 + p_1)_{\sigma_1} \cdots (p_1 + p_2)_{\sigma_j} \Phi(z; p_1) \Phi(z; -p_2) \quad (151)$$

contributes to $\Gamma^{0,0,m=0}$:

$$\Gamma^{0,0,m=0} = \left[ \epsilon^{(0,0,0)}_{\sigma_1 \cdots \sigma_j} (-1)^j (p_1 + p_2)^{\sigma_1} \cdots (p_1 + p_2)^{\sigma_j} \right] \bar{g}^{0,0}(j, i\nu, \Delta), \quad (152)$$

$$\bar{g}^{0,0}(j, i\nu, \Delta) := \int_0^{1/\Lambda} \frac{dz}{z^3} (\Lambda z)^j t_h(\Phi(z; m_h))^2 \frac{\left[ \left\{ K_{\mu\nu}(\Delta z) = \frac{1}{(\Lambda^2)^-i\nu} \right\} \right]}{(153)}$$

note here, that the confinement effect has been included in the form of i) introducing a cut in the holographic radius $z_h \leq 1/\Lambda$, and ii) $K_{\mu\nu}(\Delta z) \mu \nu$ modified to “$K_{\mu\nu}(\Delta z)$” in (123). The expression of $\bar{g}^{0,0}$ here, or that of $\Gamma^{0,0,m}$ in (149) implicitly ignores a possibility of $\mathcal{L}_{\text{bdy}} \neq 0$. For practical purposes, though, this may not be a big deal, since Ref. [9] reports that such confinement effects do not play a significant role quantitatively for most of kinematical region.

Let us also evaluate the Wilson coefficient $C^{0,0,m=0}$. The expression

$$\left[ A_p(z; -q_2) (-i \nabla)^j A_q(z; q_1) \right]_{\rho_1 \cdots \rho_j} \delta^{\rho \bar{\rho}}$$

$$:= \sum_{k=0}^j j C_k \left[ (i \nabla)^j K(z; -q_2) \right]_{\rho_{k+1} \cdots \rho_k} \left[ (i \nabla)^k A(z; q_1) \right]_{\rho_1 \cdots \rho_k} \delta^{\rho \bar{\rho}} \quad (154)$$

appearing in $C^{0,0,m=0}$ can be evaluated by using the fact that

$$(\nabla^k A)_{\rho_1 \cdots \rho_k} \equiv (\partial_{\rho_1} \cdots \partial_{\rho_k} A_k) - \frac{\eta^{\mu\kappa}}{z} \left( \partial_{\rho_1} \cdots \partial_{\rho_a} \cdots \partial_{\rho_k} A_z \right)$$

$$- \sum_{1 \leq a < b \leq k} \frac{\eta^{\rho_\mu\kappa}}{z^2} \left( \partial_{\rho_1} \cdots \partial_{\rho_a} \cdots \partial_{\rho_b} \cdots \partial_{\rho_k} A_{\rho_b} \right), \quad (155)$$

$$(\nabla^k A)_{\rho_1 \cdots \rho_k} \equiv (\partial_{\rho_1} \cdots \partial_{\rho_k} A_z) + \frac{1}{z} \sum_{a=1}^k (\partial_{\rho_1} \cdots \partial_{\rho_a} \cdots \partial_{\rho_k} A_{\rho_a}) \quad (156)$$

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modulo terms proportional to $\eta_{\mu \nu \rho_4}$. As we will focus only on the structure function $V_1^{+, \alpha}$, we can further drop the terms with $A_z$ in (155, 156). Then the expression above becomes

$$\left[\eta^{\mu \nu} \epsilon_1^{2 \nu} \epsilon_1^{2 \nu} \right](q_1 + q_2)_{\rho_1} \cdots (q_1 + q_2)_{\rho_j} + \frac{2}{\eta^{2 \rho}} \sum_{\alpha \neq \beta} \epsilon(-q_2)_{\rho_\alpha} \epsilon(q_1)_{\rho_\beta} (q_1 + q_2)_{\rho_1} \cdots \rho_\alpha \rho_\beta \cdots (q_1 + q_2)_{\rho_j}$$

multiplied by $[(q_1 z)K_1(q_1 z)][(q_2 z)K_1(q_2 z)]$.

There are two remaining tasks in evaluating the $(m = 0)$-mode contribution to the $V_1^{+, \alpha}$ structure function; a) one is to carry out the $z_\gamma$ integral, and b) the other is to sum up $C_0^{0,0} \Gamma^{0,0,0}$ for different polarizations of $\epsilon^{(0,0,0)}$. As for the $z_\gamma$ integral, the integrand sharply falls off at $z_\gamma \approx q^{-1}$ because of the photon wavefunctions of the form $[(q_i z)K_{1\nu}(q_i z)]$. The $z_\gamma$ integral in $C_{0,0,m}$ over the holographic radius $z_\gamma \in [0, \Lambda^{-1}]$ therefore comes mainly from a very small fraction of it, $\Lambda/q \ll 1$, in the regime of generalized Bjorken scaling (143). It is then all right to make an approximation that

$$I_{i\nu}(\Delta z_\gamma) \approx \frac{1}{\Gamma(i\nu + 1)} \left(\frac{\Delta z_\gamma}{2}\right)^{i\nu} [1 + O(\Delta/q)] \quad \text{when } (\Delta z_\gamma) \lesssim \Delta/q \ll 1,$$

and also to replace the range of integral $z_\gamma \in [0, \Lambda^{-1}]$ to $[0, +\infty)$, as in the literature; the error due to this approximation is only in the higher order in $(\Delta/q)$, and the twist-$(2 + \gamma(j))$ contribution is still obtained properly. The integral is then cast into the form of (285) with $\delta = j + i\nu$ for the first line of (157) [resp. $\delta = j + i\nu - 2$ for the second line of (157)], and $\vartheta = \eta/x$; thus we can use the analytic expression (287, 289) in the appendix.

The other task, b) tensor computations, is carried out in the appendix A.6. Using the results of (279) and (283), one finds that the contribution to $(C_{V_1^{+, \alpha}}^{0,0})_{m=0}$ from the second line of (157) is roughly

$$\frac{q^2 \Lambda^2}{(q \cdot \Delta)(\eta \cdot q)} \ll 1$$

times smaller than the contribution from the first line of (157) in the generalized Bjorken scaling regime (143), and is hence ignored, when only the twist-$(2 + \gamma(j))$ contributions are retained.

Combining all above, the spin-$j \in 2\mathbb{Z}$ contribution is

$$(V_1^{+, \alpha})_{j,m=0} \approx \frac{\sqrt{\lambda}}{\Gamma(j+1) \pi} \int_{-\infty}^{+\infty} d\nu \frac{1}{\nu} \int_{-\infty}^{+\infty} d\nu \frac{1}{\nu^2 + \nu^2 + \epsilon_\nu} \frac{1}{\nu^2 + \nu^2 + \epsilon_\nu} \frac{\lambda^{j-1}}{\eta}$$

$$C_1(j + i\nu, \vartheta) \left(\frac{\Lambda}{q}\right)^{i\nu-j} \left(\frac{1}{\sqrt{\lambda x}}\right)^j \tilde{g}^{0,0,0}(j, i\nu, \Delta)\tilde{d}_j(\eta),$$

(160)
where \( C_1 \) is given in (289), and \( \hat{d}_j \) is a polynomial of degree \( j \) in the argument

\[
[\eta] := \eta \times \sqrt{-\frac{4p^2}{\Delta^2}} = \eta \sqrt{\frac{4m^2_+ + \Delta^2}{\Delta^2}},
\]

and is given in terms of Legendre polynomial, as in (281).

Now that all the factors of the spin-\( j \) contribution to \( V_{1+}^{(\alpha)} \) are given as analytic functions of \( j \), it is possible to convert the sum over the (spin-\( j \in 2\mathbb{N} \)) string states in the leading trajectory to a contour integral in the complex angular momentum plane;

\[
(V_{1+}^{(\alpha)})_{m=0} = -\int \frac{dj}{4i \sin(\pi j)} (V_{1+}^{(\alpha)})_{j,m=0},
\]

with the contour in the \( j \) plane moving just below the real positive axis toward the left, and then just above the real positive axis toward the right. The integration contour in the \( \nu \) plane is deformed so that it picks up the residue of the pole in the lower complex \( \nu \) plane coming from the \( t \)-channel propagation of strings. Thus,

\[
(V_{1+}^{(\alpha)})_{m=0} \approx \int \frac{dj}{4i \sin(\pi j)} \frac{\lambda}{(j+1) i\nu_j} \frac{t_{\gamma/2}}{\Gamma(j+1)} \frac{\Lambda}{q} \gamma(j) \left( \frac{1}{\sqrt{\lambda x}} \right)^j \hat{g}^{0,0,0}(j,i\nu_j,\Delta) \hat{d}_j([\eta]),
\]

where \( \gamma(j) = i\nu_j - j \), and \( i\nu_j \geq 0 \) is a function of \( j \) determined by the on-shell condition

\[
j - 1 + \frac{4 + j + \nu^2 + c_j}{\sqrt{\lambda}} = 0.
\]

This is the result known in [5, 6, 7, 8, 9, 10] etc.; under an assumption that \( \hat{g}^{0,0,0,0}(j,i\nu_j,\Delta) \) does not grow too rapidly for large \( \text{Re}(j) \) to cancel the large factor \( \Gamma(j+1) \) in the denominator, the integration contour in the \( j \)-plane can be deformed toward the left in the \( j \)-plane, as in the classical Watson–Sommerfeld transformation; this is how the non-converging \( j \in 2\mathbb{N} \) sum of the OPE is rendered well-defined for physical kinematics \( x < 1 \). The integrand forms a saddle point due to the two factors \((1/x)^j\) and \((\Lambda/q)^{\gamma(j)}\); let \( j_\ast \) in the complex \( j \)-plane be where the saddle point is.\(^{30}\) The integrand also has poles in the \( j \)-plane. The hadron matrix element \( \hat{g}^{0,0,0,0} \) contains \( c^{(j)}_{\nu j,0,0,0} \) in its definition, and \( c^{(j)}_{\nu j,0,0,0} \) may have a pole in the \( j \)-plane [9, 11]. The saddle point value \( j_\ast \) has larger real part than any one of the poles, when

\(^{30}\)The saddle point value \( j_\ast \) is determined by \( \frac{\partial \gamma(j)}{\partial j} \bigg|_{j=j_\ast} = \frac{n(1/x)}{\ln(q/\Lambda)} \).

\(^{31}\)For example, imagine a case \( (1 - c^{(j)}_{\nu j,0,0,0}) = \frac{1}{I_{\nu}(\Delta/\Lambda)/I_{\nu}(\Delta/\Lambda)} \); the factor \( c^{(j)}_{\nu j,0,0,0} \) has poles \( j = a_{\gamma,n}(t) \) \((n = 1, 2, \cdots) \) in the \( j \)-plane given by the condition \( j_{\nu j,n} = \sqrt{\gamma/\Lambda} \); \( j_{\mu,n} \)'s are the \( n \)-the zero of the Bessel function \( J_{\mu} \).
\(\ln(q/\Lambda)\) is large relatively to \(\ln(1/x)\); the \(j\)-integral is well-approximated by the saddle point value of the integrand, and yields the DGLAP regime. When \(\ln(1/x)\) is large relatively to \(\ln(q/\Lambda)\), however, one of the poles may have a real part larger than \(\text{Re}(j_n)\). Then the integral is approximated by the residue at such leading pole. In this way, the sting-theory result \((V_1^{\perp,\alpha})_{m=0}\) goes back and forth between the DGLAP phase and Regge phase, depending on the kinematical variables \(x, (q^2/\Lambda^2)\) and \(t = -\Delta^2\) \([6, 10]\).

The derivation of (163) was not just a review of preceding works, however. First, the integration over \(z\) yields a function \(C_1(j + i\nu_j, \eta/x)\), which has precisely the same form as the one expected from the conformal OPE; comparing (25, 26) and (287, 289), one finds that they agree, under the identification 
\[
\left(\ln + j_n - 2\right) = 2j_n + \left(\tau_n - 2\right) \iff \left(j + i\nu_j\right) = 2j + \gamma(j).
\]

The expression (163) is indeed regarded as conformal OPE contributions from twist-\(\tau_n = (2 + \gamma(j))\) operators.

Secondly, the \(\eta\)-dependence of the \(m = 0\) contribution is worked out, now. As we will see later in section 7, it comes in a form that fits very well with what has been known as “dual parametrization” of GPD \([15]\). One will also notice that the argument of the degree-\(j\) polynomial \(d_j(\eta)\) is \([\eta]\) in (161), rather than \(\eta\). This means that the coefficients of the \(\eta^2\) term and higher diverge in the \(t = -\Delta^2 \to 0\) limit. This indicates that it is essential to sum up the \(m \neq 0\) modes to obtain results that are physically sensible. We will address this issue in section 7.1.

### 6.3.3 Preparation

Let us move on to the amplitudes of \(m \geq 1\)-mode exchange. We begin with deriving a few general properties of those amplitudes, which make the subsequent computations less tedious.

First, we observe that the hadron matrix element \(\Gamma^{0,0,m}\) vanishes for any odd value of \(m\). To see that this statement is true, we use a following property of \(J_{i_1 \cdots i_j}^{hh}\):
\[
\Phi(z, p_1) \bigtriangledown_{i_1} \cdots \bigtriangledown_{i_j} \Phi(z, -p_2) = (-1)^j \Phi(z, -p_2) \bigtriangledown_{i_1} \cdots \bigtriangledown_{i_j} \Phi(z, p_1);
\]
this is true in a process where the initial state hadron \(h(p_1)\) remains to be the same hadron \(h(p_2)\) in the final state, so that \(-(p_1)^2 = -(p_2)^2 = m_h^2\). This property is used below, to study when \(J_{z^k \lambda_{k+1} \cdots \lambda_j}^{hh} A_{z^k \lambda_{k+1} \cdots \lambda_j}\) vanishes for various \(k = 0, \cdots, m\).

For an even \(j\), the SO(3,1)-indices of \(J_{z^k \lambda_{k+1} \cdots \lambda_j}^{hh}\) are provided by an even number of \((p_1 + p_2)_\lambda\)'s and even [resp. odd] number of \(\Delta_\lambda\)'s when \(k\) is even [resp. odd]. The hadron
matrix element $\Gamma^{0,0,m}$ receives non-vanishing contribution from $J^{hh}_{z^k \lambda_{k+1} \cdots \lambda_j} \tilde{A}^{z^k \lambda_{k+1} \cdots \lambda_j}$ (no sum in $k$), only when the $D$ operator $\{71\}$ is used for even [resp. odd] number of times in the Reggeon wavefunction $\{89\}$. This means that $s$ is even [resp. odd], and hence $\Gamma^{0,0,m}$ can be non-zero only when $m = k + s$ is even.

For an odd $j$, the $\text{SO}(3,1)$-indices of $J^{hh}_{z^k \lambda_{1} \cdots \lambda_{j}}$ are provided by an odd number of $(p_1 + p_2)_\lambda$’s and an even [resp. odd] number of $\Delta$’s when $k$ is even [resp. odd]. Thus, the matrix element $\Gamma^{0,0,m}$ receives non-zero contribution only when an even [resp. odd] number of the $D$ operator is used in $\{89\}$. This means, once again, that $s$ is even [resp. odd], and hence $\Gamma^{0,0,m}$ can be non-zero only when $m = k + s$ is even. This statement for an odd $j$ is not more than a side remark though, since we focus on the spin-even contribution $\propto [1 + e^{-\pi ij}] / \sin(\pi j)$ in this article.

Secondly, $\Gamma^{0,0,m}$ can always be written in the form of

$$\Gamma^{0,0,m} = \left[(-2)^{j-m} (p^{\bar{s}_1} \cdots p^{\bar{s}_{j-m}}) \cdot e^{0,0,m}_{\sigma_1 \cdots \sigma_{j-m}}\right] \times \bar{g}^{0,0,m}(j, i
u, \Delta^2), \quad (167)$$

and $\bar{g}^{0,0,m}$ is an $\text{SO}(3,1)$-scalar of mass-dimension $m$; we have encountered a special case of this statement in $\{152\}$ $\{153\}$. This statement itself is understood as follows. When we write down the covariant derivatives in $\bar{J}^{hh}_{z^k \lambda_{1} \cdots \lambda_{j-k}}$ explicitly, the $\text{SO}(3,1)$-indices—there are $(j - k)$ of them—are either one of $p_\lambda$, $\Delta_\lambda$ and $\eta_\lambda$; $\eta_\lambda$ can be further rewritten as $\eta_\lambda - \Delta_\lambda \Delta_{\lambda'} / \Delta^2$ and $\Delta_\lambda \Delta_{\lambda'}$. Suppose that there are $Np$ of the $\text{SO}(3,1)$ indices from $\{p_\lambda\}$’s, $N\Delta$ indices from $\{\Delta\}$’s and $N\eta$ from $\bar{\eta}_\lambda$’s in a given term; $N_p + N_{\Delta} + 2N_{\eta} = (j - k)$. When such an $\text{SO}(3,1)$ tensor is contracted with $\sum_N^{(m-k)/2} \bar{E}^N D^{m-k-2N} [\epsilon(0,0,m)]$ in the Reggeon wavefunction $\bar{A}^{z^k \lambda_{1} \cdots \lambda_{j-k}}$, it remains non-zero only when $(m - k - 2N) = N_\Delta$ and $N \geq N_{\eta}$, because of the relation $\{239\}$. It is not hard now to see that all the remaining terms are proportional to the prefactor of $\bar{g}^{0,0,m}$ in $\{167\}$; the mass dimension of the remaining scalar factor (reduced matrix element) $\bar{g}^{0,0,m}$ follows from the fact that $\Gamma^{0,0,m}$ is defined to be of mass dimension $j$.

Finally, we note that the twist-$\{2 + \gamma(j)\}$ contribution to the coefficient $C^{0,0,m}$ arises only from the contraction $\bar{J}^{\gamma}_{z^k \kappa_{k+1} \cdots \kappa_j} \bar{A}^{z^k \kappa_{k+1} \cdots \kappa_j}$ with $k = 0$. We have already seen an example of this in the $m = 0$ amplitude; the first term of $\{157\}$ contributes to $\{163\}$, while the second term does not because of $\{159\}$, and the first term came from the $k = 0$ contraction.

In order to verify the claim above, note first that both an extra $\partial_z$ and an extra power of $1/z$ virtually change the integral of $C^{0,0,m}$ by about an extra power of $q \sim q_1 \sim q_2 \gg \Lambda, \Delta$. Explicitly writing down covariant derivatives in $\bar{J}^{\gamma}_{z^k \kappa_{k+1} \cdots \kappa_j}$, and evaluating the integrals only
by the order of magnitudes, one can see that

\[
(C^{0,0,m}_{V_1^+,\alpha})_k \sim \sum_{j} \left( \frac{\Lambda}{q} \right)^{iv-j} q^{j-2M} \left( \frac{q}{\eta \kappa q^2} \right)^M \cdot \sum_{M} \frac{1}{\Delta^{s-2N}} \tilde{E}^N D^{s-2N}[e^{(0,0,m)}].
\]  

(168) 

The \( M = 0 \) contribution is further evaluated by using the definition of \( \tilde{E} \) and \( D \) operators. Details of computation is found partially in [283]; we find that

\[
(C^{0,0,m}_{V_1^+,\alpha})_{k,M=0} \sim \left( \frac{\Lambda}{q} \right)^{iv-j} (q \cdot p)^{j-m} \left( \frac{q \cdot \Delta}{\Delta} \right)^s q^k.
\]  

(169) 

Keeping the relation \( m = k + s \) and also the result (167) in mind, we obtain

\[
C^{0,0,m}_{k,M=0} \cdot \Gamma^{0,0,m} \sim \left( \frac{\Lambda}{q} \right)^{iv-j} \left( \frac{1}{x} \right)^j \eta^{m-k} \left( \frac{q \cdot \Delta}{\Delta} \right)^s \frac{1}{\Delta^m}.
\]  

(170) 

Therefore, this is regarded as a twist-(2 + \( \gamma(j) + k/2 \)) contribution in the generalized Bjorken scaling regime. Thus, only the \( k = 0 \) term remains a twist-(2 + \( \gamma(j) \)) contribution, and the terms with \( k > 0 \) are irrelevant to GPD.

The analysis becomes a little more complicated when \( M > 0 \) terms are also included, but not in an essential way. Contributions with some \( (k, M) \) correspond to twist-(2 + \( \gamma + M + k/2 \)), and only the \( k = M = 0 \) terms contribute to GPD. This means that \( C^{0,0,m} \) can be evaluated under the following approximation:

\[
[A_{\mu}(z; -q_2)(-i \nabla)^j A_{\nu}(z; q_1)]^{m_1 \cdots m_j} \delta^{\mu \nu} \bar{A}_{m_1 \cdots m_j} \rightarrow [A_{\mu}(z; -q_2)(-i \nabla)^j A_{\nu}(z; q_1)]^{m_1 \cdots m_j} \delta^{\mu \nu} \bar{A}_{m_1 \cdots m_j}.
\]  

(171) 

### 6.3.4 Wilson Coefficients, Conformal OPE and Hadron Matrix Elements

The twist-(2 + \( \gamma(j) \)) contribution to \( C^{0,0,m}_{V_1^+,\alpha} \) can be determined completely, using the approximations above.

\[
C^{0,0,m}_{V_1^+,\alpha} \sim \left( \frac{2\Lambda}{\Delta} \right)^{iv} \left( \frac{\Gamma(2iv + 1)}{(2\Lambda)^j} \right) \int \frac{dz}{z} [(q_1 z)K_1(q_1 z)][(q_2 z)K_1(q_2 z)] z^j
\]

\[
\sum_{N=0} \left[ 2^j (q^{\rho_1} \cdots q^{\rho_j}) \cdot \tilde{E}^N D^{m-2N}[e^{(0,0,m)}]_{\rho_1 \cdots \rho_j} \right] \frac{b^{j-m}_{m,N}}{\Delta^{m-2N}} \left[ e^{(2-j)A_{\rho_1 \cdots \rho_j}} A_{\rho_1 \cdots \rho_j} \right].
\]  

(172)
The product of rank-\( j \) SO(3, 1) tensors in the second line is reduced to a product of rank-\((j - m)\) tensors by the computation in (284). The Reggeon wavefunction \( \Psi \) is also rewritten by using the small \((\Delta z_\gamma) \lesssim (\Delta/q)\) approximation (158):

\[
[e^{(2-j)A} \overline{\Psi}_{ivr,0,0,m}] \simeq \sum_{a=0}^{N} (-1)^a N_C a \left( \zeta^{j+1} \partial^m \zeta^{-2a} \left( \zeta^{-1-j+m} (\zeta/2)^{iv} \right) \right)_{\zeta \rightarrow (\Delta z)} \frac{N_{j,m}}{\Gamma(i\nu + 1)}. \tag{173}
\]

The \( a = 0 \) term in this expression has the lowest dimension in \( \zeta = \Delta z_\gamma \lesssim (\Delta/q) \), and hence we only need to retain the \( a = 0 \) term for a given \( N \) for the twist-(2 + \( \gamma(j) \)) contribution. Thus,

\[
[e^{(2-j)A} \overline{\Psi}_{ivr,0,0,m}] \simeq 2^{-i\nu} (-1)^m \Gamma(j + 1 - i\nu) \frac{(\Delta z_\gamma)^{iv}}{\Gamma(j + 1 - i\nu - m)} \frac{N_{j,m}}{\Gamma(i\nu + 1)}. \tag{174}
\]

Using this expression and (284) in (172), we obtain

\[
C_{V_{1+},\alpha}^{0,0,m} \simeq \frac{\Lambda}{q^{iv-j}} \frac{j!}{(j-m)!} \sum_{N=0}^{[m/2]} \left[ \frac{(q \cdot \Delta)^2}{\Delta^2} \right]^N (-i)^m \frac{(q \cdot \Delta)^{m-2N}}{\Delta^{m-2N}} \frac{b^{(j-m)}}{b_{m,N}} \left[ (q_{\mu_1} \cdots q_{\mu_{j-m}}) \cdot \epsilon^{(0,0,m)} \right]
\]

\[
\int \frac{dz}{z} \left[ (q_1 z) K_1(q_1 z) \right] \left[ (q_2 z) K_1(q_2 z) \right] \frac{(q z)^{j+i\nu}}{\Gamma(j + 1 - i\nu - m)} N_{j,m}, \tag{175}
\]

\[
= \left( \frac{\Lambda}{q} \right)^{iv-j} \left[ (q_{\mu_1} \cdots q_{\mu_{j-m}}) \cdot \epsilon^{(0,0,m)} \right] \frac{(q \cdot \Delta)^m}{\Delta^m} C_1 \frac{j+i\nu}{x}, \tag{176}
\]

\[
\int \frac{dz}{z} \left[ (q_1 z) K_1(q_1 z) \right] \left[ (q_2 z) K_1(q_2 z) \right] \frac{(q z)^{j+i\nu}}{\Gamma(j + 1 - i\nu - m)} N_{j,m}, \tag{177}
\]

where (285-289) is used for the equality in the middle, while (269) is for the last one.

Repeating the same argument as in section 6.3.2 we thus arrive at

\[
(V_{1+},\alpha)_m \simeq - \int \frac{dj}{4i \sin(\pi j)} \frac{t_y}{\Gamma(j+1)} \frac{\lambda}{i\nu_j} C_1(j+i\nu_j) \left( \frac{\Lambda}{q} \right)^{\gamma(j)} \frac{1}{\sqrt{\lambda x}} \eta^{m} d_{j-m}(\eta) \frac{g_{0,0,m}}{\Delta^m} \frac{\Gamma(j + 1 + i\nu - m)}{N_{j,m}} \frac{\Gamma(j + 1 + i\nu)}{\Gamma(j + 1 + i\nu)}, \tag{178}
\]

the computation in (279-281) for an even \( j \) and \( m \) was used once again. Similarly to the case of \( m = 0 \) amplitude, this expression is in the form of conformal OPE and inverse Mellin transformation in (20). It should be noted that the integrand can be defined as a holomorphic
function of \( j \) (apart from poles and cuts), using the definition of \( C_1 \) in (285), and that of \( \hat{d}_{j-m} \) in (281), not just for integer-valued \( j \); at the same time, \( \eta^m \hat{d}_{j-m}([\eta]) \) becomes a polynomial of \( \eta \) of degree \( j \) for \( j \in 2\mathbb{N} \), which is one of the important properties expected for the hadron matrix element [12].

The integration contour of (178) is chosen so that it circles around the pole at \( j = m \) after running just below the real positive axis in the \( j \)-plane and before running just above the real positive axis. Only spin-\( j \) stringy states with \( m \leq j \) contribute then. It is not obvious whether the contour can be deformed so that it encircles \( j = 0, 2, \ldots, m \) without changing \( (V_1^{+,\alpha})_m \), and we leave it an open question. \( \hat{d}_{j-m} \) in (178) is given by a Legendre polynomial of degree \( j - m \) when \( j - m \) is an even positive integer, but otherwise it is defined by the hypergeometric function as in (281), and it may or may not have a zero at negative even integer \( j - m \). The authors have not found a reason to believe that they have a zero, but we may be wrong.

The twist-(2+\( \gamma(j) \)) contribution to the structure function \( V_1^{+,\alpha} \) is obtained by summing \( (V_1^{+,\alpha})_m \) from the \((n, l, m) = (0, 0, m)\) modes with \( m = 0, 2, \ldots \):

\[
V_1^{+,\alpha} = \sum_{m=0,2,\ldots} (V_1^{+,\alpha})_m. \tag{179}
\]

Combining (163, 178) with (179), a holographic version of (20) is obtained. It is not obvious, though, whether or not the integration variable \( j \) in (178) for all the different \( m \)'s should be identified. If we are to define \( j' := (j - m) \) and use it as a new variable of integration, then the integration contour of (178) would be the same for all different \( m \)'s; the cost of doing so, however, is in this:

\[
\hat{d}_{j'} \text{ still remains to be a polynomial of degree at most } j', \text{ but the expression no longer fits into the form of conformal OPE. For this reason, we identify the integration variable } j \text{ in (178) for all } m = 0, 2, \ldots \text{ with that (complex angular momentum) of the inverse Mellin transformation (20). This implies that the reduced hadron matrix element of the spin-} j \text{ primary operator is given a holographic expression}
\]

\[
\overline{A}_j^{+,\alpha}(\eta, t) \propto \sum_{m=0}^{j} \frac{(-1)^{m/2}}{\sqrt{\lambda}} \frac{\tilde{g}^{0,0,0}(j, \nu_j, \Delta)}{\Gamma(j + 1 + \nu_j - m)} \frac{\tilde{g}^{0,0,0}(j, \nu_j, \Delta^2)}{\Gamma(j + 1 + \nu_j)} \times \eta^m \hat{d}_{j-m}([\eta]). \tag{181}
\]
6.3.5 The \((m = 2)\)-Mode Hadron Matrix Element

Most aspects of the expression \([178]\) are dictated by basic principles of field theory such as (conformal) OPE. Additional information from the holographic set-up is found primarily in the hadron matrix element \(\bar{g}^{0,0,m}(j, i\nu, \Delta)\), apart from the anomalous dimension \(\gamma(j) = i\nu_j - j\) of the twist-(2+\(\gamma(j)\)) operators. Now we have seen that \(\bar{g}^{0,0,0}(j, i\nu_j, \Delta)\) is not the only hadron matrix element contributing to the non-perturbative information of \(h + \gamma^* \rightarrow h + \gamma^*\); let us take a moment here to have a closer look at one of the new hadron matrix elements we encounter, \(\bar{g}^{0,0,2}(j, i\nu, \Delta)\).

The hadron matrix element \(\Gamma^{0,0,m}\) receives contributions from \(\bar{J}_{hh} z \lambda \lambda_k \lambda_{k+1} \cdots \lambda_j \hat{A} \hat{z} \hat{\lambda}_{k+1} \cdots \hat{\lambda}_j\)'s with \(k = 0, 1, \cdots, m\). The contribution from each \(k\) can be written in the form of \([167]\), and hence \((\bar{g}^{0,0,m}(j, i\nu, \Delta))_k\) is defined \((k \leq m)\). We compute \((\bar{g}^{0,0,2})_k\) explicitly for \(k = 0, 1, 2\).

For this purpose, we need the following technical results:

\[
(\nabla^i \Phi)_{\lambda_1 \cdots \lambda_l} \equiv (\partial_{\lambda_1} \cdots \partial_{\lambda_l} \Phi) = \sum_{1 \leq a < b \leq l} \eta_{\lambda_a \lambda_b} \left( \partial_{\lambda_1} \cdots \hat{\lambda}_a \hat{\lambda}_b \cdots \partial_{\lambda_l} \left( \partial_z + \frac{l - a - 1}{z} \right) \Phi \right) \qquad (182)
\]

modulo terms proportional to \(\eta_{\lambda_a \lambda_b} \eta_{\lambda_c \lambda_d}\) instead of \([150]\), and

\[
(\nabla^i \Phi)_{\lambda_1 \cdots z \cdots \lambda_l} \equiv \left( \partial_z + \frac{l - 1}{z} \right) \partial_{\lambda_1} \cdots \hat{\lambda}_a \cdots \partial_{\lambda_l} \Phi, \qquad (183)
\]

\[
(\nabla^i \Phi)_{\lambda_1 \cdots z \cdots \cdots \lambda_l} \equiv \left[ \left( \partial_z + \frac{l - 1}{z} \right) \left( \partial_z + \frac{l - 2}{z} \right) + \frac{a - 1}{z^2} \right] \partial_{\lambda_1} \cdots \hat{\lambda}_a \hat{\lambda}_b \cdots \partial_{\lambda_l} \Phi,
\]

modulo terms proportional to \(\eta_{\lambda_c \lambda_d}\).

It is now a straightforward computation to use the relations above as well as the explicit
Reggeon wavefunctions $\tilde{A}$ determined in section 5, to derive the following:

$$\frac{\tilde{g}_{k=2}^{0,0,2}}{N_{j,2}\Delta^2} = \frac{j(j-1)}{2} \int_0^{1/\Lambda} \frac{dz}{z^3} (\Lambda z)^j \frac{\{z^{2\nu}K_{\nu}(\Delta z)^n\}}{[(\frac{\Lambda}{2\Lambda})^{-i\nu} \Gamma(i\nu)]} \times \left[t_h \left\{2\Phi(\partial^2 \Phi) + (\partial_z \Phi)^2\right\} - \frac{2}{z} \Phi(\partial_z \Phi) - \frac{4(j-2)}{3z^2} \Phi^2\right],$$

$$\frac{\tilde{g}_{k=1}^{0,0,2}}{N_{j,2}\Delta^2} = \frac{j(j-1)}{2} \int_0^{1/\Lambda} \frac{dz}{z^3} (\Lambda z)^j \frac{\{z^{j+1}\partial_z (\Delta z)\}^{2\nu}K_{\nu}(\Delta z)^n\}}{[(\frac{\Lambda}{2\Lambda})^{-i\nu} \Gamma(i\nu)]} \times \left[-\frac{2t_h}{z} \Phi^2\right],$$

$$\frac{\tilde{g}_{k=0}^{0,0,2}}{N_{j,2}\Delta^2} = \frac{j(j-1)}{2} \int_0^{1/\Lambda} \frac{dz}{z^3} (\Lambda z)^j \times \left\{\frac{\{z^{1-j}\partial_z z^{1-j} - (z\Delta)^2\}^{2\nu}K_{\nu}(\Delta z)^n\}}{[(\frac{\Lambda}{2\Lambda})^{-i\nu} \Gamma(i\nu)]} \times \left[\frac{p^2}{(j-\frac{1}{2})\Delta^2} t_h \Phi^2\right] + \frac{\{z^{1-j}\partial_z z^{1-j} - (z\Delta)^2\}^{2\nu}K_{\nu}(\Delta z)^n\}}{[(\frac{\Lambda}{2\Lambda})^{-i\nu} \Gamma(i\nu)]} \times t_h \left[\frac{1}{z} \Phi(\partial_z \Phi) + \frac{j-2}{3z^2} \Phi^2\right]\right\].$$

These results are used in the study below.

### 7 A Holographic Model of GPD

The differential cross section of DVCS process involves integral of GPD; GPD needs to be parametrized first, and then the parameters are determined by fitting the data [13]. The idea of dual parametrization of GPD [15]—also known as collinear factorization approach [17, 18]—is to expand the reduced hadron matrix element $A_j^{+\alpha}(\eta,t)$ as

$$A_j^{+\alpha}(\eta,t) = \sum_{m=0}^{j} \Gamma_m^{+\alpha}(j,t) \eta^m \times [\eta^{j-m} d_{j-m}(1/\eta)],$$

where $d_\ell(\cos \theta)$'s are polynomials of degree $\ell$ in the argument $\cos \theta$; Legendre polynomials, Gegenbauer polynomials or Jacobi polynomials are used depending on the helicity change of the target hadron $h$ in the scattering process [12]. When the target hadron is a scalar, as in the study of this article, Legendre polynomial is chosen for $d_\ell [13]$. With no ambiguity introduced in the polynomials $d_{j-m}(x)$, $\Gamma_m^{+\alpha}(j,t)$'s are the fully general, yet non-redundant parametrization for the reduced hadron matrix element for GPD.

At the end of the study in the preceding sections, we arrived at a holographic model of GPD, with the reduced hadron matrix element given by [18] for the flavor-non-singlet
sector. String theory—the descendant of the dual resonance model—yields a result that fits straightforwardly with the format of the dual parametrization (185); this should not be a surprise, but must be something the authors of [15] have anticipated. With the string-theory implementation provided, one can now move forward; now

\[ \Gamma_m^{+, \alpha}(j, t) \sim (-1)^{m/2} \tilde{g}^{0,0,m}(j, i\nu_j, \Delta) N_{j,m} / \Delta^m \]  

(186)
can be computed using holographic backgrounds, independently from experimental data. Certainly the matrix elements \([\tilde{g}^{0,0,m}/\Delta^m]\) will depend on holographic backgrounds to be used for computation, and predictions from individual holographic backgrounds should not be taken seriously at the quantitative level. But it is still worth looking closely into qualitative features of the holographic hadron matrix elements \(\tilde{g}^{0,0,m}/\Delta^m\) to learn non-perturbative aspects of \(\Gamma_m^{+, \alpha}(j, t)\).

### 7.1 \(\Delta^2 \longrightarrow 0\) Limit

As we have already remarked earlier in this article, the holographic result (181) is not precisely in the same form of parametrization as in (185); the argument of the polynomial \(\tilde{d}_{j-m}\) is \([\eta]\) defined in (161), rather than \(\eta\). This difference itself does not raise an issue immediately; \([\eta]\) is the same as \(\eta\) in the hard scattering regime, \(\Delta^2 \gg m_h^2\).

Let us study how the hadron matrix element behaves in the \(t = -\Delta^2 \longrightarrow 0\) limit, however. The matrix element \(\tilde{g}^{0,0,0}(j, i\nu_j, \Delta)\) has already been studied in the literature, and is known not to diverge or vanish in the \(\Delta^2 \longrightarrow 0\) limit. The polynomial \(\tilde{d}_j([\eta])\) to be multiplied with this \(\tilde{g}^{0,0,0}(j, i\nu_j, \Delta)\), however, has diverging coefficients in all of the terms \(\eta^2, \eta^4, \cdots\) except the \(\eta^0\) term. Therefore, the \(m = 0\) contribution (163) alone does not have a physically reasonable behavior in the \(\Delta^2 \longrightarrow 0\) limit. A natural expectation will be that the hadron matrix element \(\tilde{A}_j^{+, \alpha}(\eta, t)\) still has a reasonable behavior, after summing up \(m = 0, 2, \cdots, j\).

To get started, we focus on the \(\eta^2\) term. It is generated from the \(m = 0\) mode exchange, and also from the \(m = 2\) mode exchange. There is a \((p^2)/\Delta^2\) factor both in \(\tilde{g}^{0,0,0} \times \tilde{d}_j([\eta])|_{\eta^2}\) and \(\tilde{g}^{0,0,2}/\Delta^2 \times \eta^2\), and hence both diverge in the \(\Delta^2 \longrightarrow 0\) limit. When they are summed up, however, the divergence may cancel, as we see in the following. Let us study the coefficient of the \(\eta^2\) term

\[ -\int \frac{dj}{4i} e^{-\pi ij} \left( \frac{\Lambda}{q} \right)^{iv_j} \left( \frac{1}{\sqrt{\lambda x}} \right)^j C_1 \left( j + iv_j, \frac{\eta}{x} \right) \frac{\lambda}{iv_j} \Gamma(j+1) \times \eta^2 \]  

(187)
in the $\Delta^2 \rightarrow 0$ limit, picking up contribution to the integral $\bar{g}^{0,0,0}$ and $\bar{g}^{0,0,2}$ from the $I_{-i\nu}(\Delta z_h)$ component in “$K_{i\nu}(\Delta z)$” first. Then in that limit, the coefficient of the expression (187) becomes

$$
\frac{p^2}{\Delta^2} \lim_{\Delta^2 \rightarrow 0} \left[ \bar{g}^{0,0,0}(j, i\nu_j, \Delta) \left( j - \frac{1}{2} \right) - \frac{\bar{g}^{0,0,2}(j, i\nu_j, \Delta)}{(j - 1 + i\nu_j)(j + i\nu_j)} \right] + O(\Delta^0). \quad (189)
$$

The two terms in $\lim_{\Delta^2 \rightarrow 0}[\cdots]$ cancel each other, as one can see by using the approximation in footnote 32. Thus, the $\eta^2$ term in $\overline{A}_j^{+,\alpha}(\eta, t)$ also has a finite limit value in the $\Delta^2 \rightarrow 0$ limit.

It is quite likely, however, that the $I_{i\nu}(\Delta z)$ component in “$K_{i\nu}(\Delta z)$” has just as important contribution as the $I_{-i\nu}(\Delta z)$ component does in the $\Delta^2 \rightarrow 0$ limit to the hadron matrix elements $\bar{g}^{0,0,0}$ and $\bar{g}^{0,0,2}$; the coefficient $(1 - c_{i\nu;0,0,m}^{(j)})$ may behave as $(\Delta/\Lambda)^{-2i\nu}$ in the $\Delta^2 \rightarrow 0$ limit. Because we have seen above that the divergence $(p^2/\Delta^2)$ cancels when only the $I_{-i\nu}(\Delta z)$ component is taken into account, the contributions from the $I_{i\nu}(\Delta z)$ should also have some cancellation mechanism. Using an approximation for the $I_{i\nu}(\Delta z)$ components in “$K_{i\nu}(\Delta z)$” similar to the one in footnote 32, one finds that the $(p^2/\Delta^2)$ divergence cancels in the $\eta^2$ coefficient, if and only if

$$
\lim_{\Delta^2/\Delta^2 \rightarrow 0} \left[ \left( \frac{\Delta}{2\Lambda} \right)^{2i\nu} \left\{ (1 - c_{i\nu;0,0,0}^{(j)}) - (1 - c_{i\nu;0,0,0,2}^{(j)}) \frac{(j - 1 - i\nu_j)(j - i\nu_j)}{(j - 1 + i\nu_j)(j + i\nu_j)} \right\} \right] = 0. \quad (190)
$$

The coefficients $c_{i\nu;0,0,m}^{(j)}$ are functions of $\Delta/\Lambda$, rather than complex numbers. The discussion above shows that physically sensible implementations of the confining effect require one condition above between the two functions $c_{i\nu;0,0,0}$ and $c_{i\nu;0,0,2}$.

The $\eta^{2M}$ term with $M = 2, \cdots$, instead of the $\eta^2$ term in (187), also receives divergent contributions from amplitudes of the $m = 0, 2, \cdots, 2M$ mode exchange. There will be apparent divergence of order $(p^2/\Delta^2)^M$, $(p^2/\Delta^2)^{M-1}$, $\cdots$, $(p^2/\Delta^2)$. The cancellation of divergence in the $\Delta^2 \rightarrow 0$ limit will set $M$ conditions on the $\Delta^2/\Lambda^2 \rightarrow 0$ limit of $(1 - c_{i\nu;0,0,2M})$.

In a phenomenological approach of implementing the confining effect, that is all we can say for now. With a little more model-building mind set, however, we can find some solutions to the conditions above. It is not hard to verify that the combination of

$$
\left[ \partial_z \left( \Psi_{i\nu;0,0,0}^{(j);0,0}(t, z) \right) \right] \big|_{z=1} = 0, \quad \left[ \partial_z \left( \Psi_{i\nu;0,0,0,2}^{(j);2,0}(t, z) \right) \right] \big|_{z=1} = 0 \quad (191)
$$

32 The leading divergence in the $\Delta^2 \rightarrow 0$ limit comes from

$$
K_{i\nu}(\Delta z) \sim \left( \frac{\pi}{2} \right) \frac{I_{-i\nu}(\Delta z)}{\sin(\pi i\nu)} \sim \left( \frac{\pi}{2} \right) \frac{(\Delta z/2)^{-i\nu}}{\sin(\pi i\nu)\Gamma(-i\nu + 1)} = \frac{\Gamma(i\nu)}{2} (\Delta z/2)^{-i\nu}. \quad (188)
$$
results in \( c_{i\nu,0,0,0}^{(j)} \) and \( c_{i\nu,0,0,2}^{(j)} \) satisfying the condition \((190)\). It is tempting to generalize this and impose the boundary condition \( \partial_z [\Psi_{i\nu,0,0,2M}^{(j)}] = 0 \) to determine \( c_{i\nu,0,0,2M}^{(j)} \), though we do not know whether all the \( m_2^2/\Delta^2 \) divergences above are removed under this boundary condition. The top-down approach is much more authentic and well-motivated than such a hand-waving and wishful approach, and we do not try to speculate beyond that; we use this implementation of the confining effect, \((191)\), only to “get the feeling” in the numerical presentation in section \([7.4]\).

### 7.2 Large \( \Delta^2 \) Behavior

Certainly the holographic model of GPD yields a result of the reduced hadron matrix element that fits perfectly with the dual parametrization. The holographic result, however, turns out to be a little more complicated than the models that have often been explored for the purpose of phenomenological fit of the DVCS data. An example of model for phenomenological fit (see e.g., \([18]\)) was to introduce an ansatz that

\[
\Gamma_m^{+,\alpha}(j,t) = f_{j,m} \Sigma_{j-m}(t),
\]

where only one \((t = -\Delta^2)\)-dependent function is involved in the form of a “form factor” \( \Sigma_{j-m}(t) \) for some “spin \((j - m)\)”, and all the remaining non-perturbative information is reduced to some numbers \( f_{j,m} \in \mathbb{R} \). The function \( \Sigma_J(t) \) may also be parametrized by an ansatz like

\[
\Sigma_J(t) = \frac{1}{J - \alpha_0 - \alpha'_\text{eff} t} \left[ \frac{1}{m^2(J)} \right]^p,
\]

in order to implement both the Regge behavior and the power-law form factor in the hard regime \( 1 \ll -t/\Lambda^2 \). To fit the data in practice, it is certainly unavoidable to reduce the unknown information into a finite set of real numbers.

A theoretical picture based on the holographic model, on the other hand, suggests that the \( t = -\Delta^2 \) dependence is more complicated than this. If we strictly stick to the expansion \((185)\), then individual \( \Gamma_{m}^{+,\alpha}(j,t) \)'s may diverge at \( t = -\Delta^2 = 0 \), as we have seen above, and are not like form factors. The \( \Gamma_m^{+,\alpha}(j,t) \) would not depend only on the difference \((j - m)\) as in \((192)\) either; we have already seen that \( \Gamma_{m=2}^{+,\alpha}(j,t) \propto \bar{g}^{0,0,m=2}/\Delta^2 \) diverges at \( t = -\Delta^2 \to 0 \) for arbitrary \( j \), but there is no such divergence in \( \Gamma_{m=0}(j,t) \propto \bar{g}^{0,0,0} \), for example. Therefore, holographic models of GPD might be used as a theoretical guide to think of parametrization (for fitting) that is different from \((192)\).

The holographic model provided by the calculation in the previous section involves infinitely many spin-dependent form factors, \( \bar{g}^{0,0,m}(j,i\nu_j,\Delta)/\Delta^m \). We can still find that they
share a common behavior at large $\Delta^2 = -t$. To see this, note that “$K_{i\nu}(\Delta z_h)$” in the Reggeon wavefunction effectively cuts off the integral over the holographic radius $z_h$ at $z_h \lesssim 1/\Delta$ in the regime

$$
\Lambda^2, m_h^2 \ll \Delta^2 \ll |q^2|, (p \cdot q), |(q \cdot \Delta)|.
$$

(194)

The explicit form of $\Psi^{(j)\mathbf{s}N}_{i\nu\mathbf{0},0,0,m}(z; \Delta)$ in (97) is not more than modification “$K_{i\nu}(\Delta z)$” by a function of $\Delta z_h$, and hence they still play just the role of cutting-off the integral at $z_h \Delta \lesssim 1$.

The “current” $\bar{\mathcal{J}}_{hh} z_{k+1}^{\lambda_k \lambda_k+1} \cdots \lambda_j$ provides extra $m$-th powers of either $1/z$ or $\partial_z$ and $(j-m)$ momenta $p_{\lambda}$, in addition to $[\Phi]^2$, which behaves like

$$
[\Phi] \sim z (\Lambda z)^{\Delta_{\phi} - 1}
$$

(195)
in the region $z \lesssim 1/\Delta \ll 1/\Lambda$; $\Delta_{\phi}$ is the conformal dimension of an operator in a strongly coupled gauge theory dual to the holographic model, which is a property of the target hadron $h$. The $\tilde{E}^N D^{(2N-2N)}[\epsilon]/\Delta^{2-2N}$ operation on the SO(3,1) tensor in (89) does not introduce any power of $(\Delta/\Lambda)$ or $(\Lambda z)$. Therefore, we find in the hard scattering regime (194) that

$$
\frac{\tilde{g}^{0,0,m}}{\Delta^m} \sim \left(\frac{\Delta}{\Lambda}\right)^{i\nu} \times (\Lambda/\Delta)^{j+2(\Delta_{\phi} - 1)} \times \Delta^m/\Delta^m \sim \frac{1}{(\Delta/\Lambda)^{2\Delta_{\phi} - 2 - \gamma(j)}}.
$$

(196)

Interestingly, the reduced hadron matrix elements $\tilde{g}^{0,0,m}/\Delta^m$ for $(j, m)$ have the large $\Delta^2$ power-law behavior that is independent of $m$; $2\Delta_{\phi}$ reflects a property of the target hadron $h$, and $-(2 + \gamma(j)) = -\tau_n$ is $j$-dependent, but the power does not depend on $m$. Holographic models suggest this $j$-dependent $p = \text{const.} - \gamma(j)/2$ scaling behavior as an alternative to the fixed-power $p = \text{const.}$ scaling of (193).

We have chosen a factorization into the Wilson coefficient and the matrix element that corresponds to renormalization at $\mu = \Lambda$; this choice was made implicitly when we chose a factor $[\Delta^{i\nu}/\Lambda^{i\nu-j}]^{j+1}$ at the time the amplitude was factorized into $C^{0,0,m}$ and $\Gamma^{0,0,m}$ in (149). When we keep the renormalization scale $\mu$ to be arbitrary (e.g., taking $\mu$ higher than $\Delta$ when $\Delta \gg \Lambda$), the Wilson coefficient contains a factor $(\mu/q)^{\gamma(j)}$ instead of $(\Lambda/q)^{\gamma(j)}$, and the reduced matrix element also has the following large $\Delta^2$ behavior,

$$
\frac{\tilde{g}^{0,0,m}}{\Delta^m} \sim \frac{1}{(\Delta/\Lambda)^{2\Delta_{\phi} - 2}} \times \frac{1}{(\mu/\Delta)^{\gamma(j)}}.
$$

(197)

### 7.3 Pomeron and Superstring

We have so far talked about Reggeon and the flavor-non-singlet sector in sections 6–7, instead of Pomeron. Since flavor-singlet sector ($\approx$ gluon) dominates in the small-$x$ physics, that was not a desired choice.

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33 This scaling was known already for $\tilde{g}^{0,0,0}$ [10].
This is due to technical limitation in string theory at this moment. In order to deal with propagation of string states on a curved spacetime, vertex operators and $L_0$ (Virasoro generator) need to be defined properly as composite operators; the non-linear sigma model for AdS$_5 \times$ S$^5$ on the world-sheet becomes conformal and the renormalization of the composite operators well-defined, however, only after the Ramond–Ramond background is also implemented (e.g., [36]). Presumably an option in the future will be to implement the Klebanov–Strassler model and its variations in the Green–Schwarz formalism. One then computes the spectrum of stringy excited states, and further works out the world-sheet OPE, in the form of

$$V^{(q_1)}(z)V^{(-q_2)}(-z) \sim \sum I C_I(z)O_I(0)$$

using operators $O_I(0)$ at the middle point, where $V^{(q_1)}$ and $V^{(-q_2)}$ are the vertex operators corresponding to the incoming and outgoing photons [32]. In this way, we would not have to use string field theory.

It may also be possible to use the bosonic string field theory for closed string, instead of the bosonic open string field theory we used in section 4 of this article. Bosonic closed string field theory is also well-understood already [37]. Certainly the bosonic closed string field theory is not for Type IIB superstring, but it will still allow us to get the feeling of how much open string (flavor non-singlet sector) and closed string (flavor singlet sector) are different, from theoretical perspectives, as well as in phenomenological consequences. At least it is known that the Virasoro–Shapiro amplitude is generated, not just by the 1-string exchange in the $t$-channel, $s$-channel and $u$-channel, but also a four-point contact interaction vertex in the string field theory [38]. The Virasoro–Shapiro amplitude does not have a simple $s$–$t$ duality of the Veneziano amplitude, either. Certainly it is possible to write it down in the form of “$t$-channel” expansion only (cf [6] and [10]), but we also need to be aware that the discussion in these two references did not use the OPE at the middle point as in (198), but used an OPE of the form $V(z)V(0) \sim \sum I C_I(z)O_I(0)$. To get the skewness-dependence right, this difference really matters. Thus, an analogue of the prescription [52] needs to be worked out separately for the closed string amplitude.

Orthodox approaches such as those above are way beyond the scope of this article. One can hardly overestimate importance of such a solid approach, but at the same time, very few would find that the following guess would not be terribly off the mark. For practical purposes, therefore, one can live with that for the time being. First of all, the on-shell relation for the bosonic open string in (164) will be replaced by

$$j^2/2 - 1 + 4 + j + v^2 + c_j/4\sqrt{\lambda} = 0,$$

with the constraint $c_{j=1} = -4$ for bosonic open string replaced by $c_{j=2} = -2$. Interaction
vertices should also be different; looking at the difference between the Veneziano amplitude and the Virasoro–Shapiro amplitude, one finds that the following replacements should be made:

\[
\frac{t_\gamma}{t_y} \rightarrow \frac{t_\gamma}{\Gamma(j+1)} \quad \frac{1}{\Gamma(j/2)}^2, \quad \left( \frac{1}{\sqrt{\lambda x}} \right)^j \rightarrow \left( \frac{1}{4\sqrt{\lambda x}} \right)^j.
\]

(200)

The overall normalization \(t_\gamma/t_y\) is like \(N_c/N_c^{-2} \sim N_c^{-1}\) now, when the Pomeron (closed string) contribution is used in the t-channel, and the source field for the “QED current” is implemented in the form of D7-brane gauge field; the \(1/N_c\) scaling (see footnotes 15 and 29) is also the natural expectation in the large \(N_c\) argument.

### 7.4 Numerical Results

At the end of this article, we leave a few plots of numerical evaluation of various results that have been obtained. We do not intend to provide a quantitative (precise) prediction from holography, as we have repeatedly emphasized our perspective on this issue in this article; the holographic approach to GPD will provide at best a qualitatively new way to think of how to parametrize the matrix elements for GPD. Having said that, it is still desirable to grasp various expressions in a more intuitive form and bring them down to more practical situations. This section 7.4 serves for this purpose.

There are a couple of parameters that need to be specified, in order to obtain numerical outputs in a few summary plots. We used the on-shell relation (164), which means that we should understand the numerical results to be that of Reggeon contribution. We adopted \(c_j + 4 = 0\) for all \(j\), although there is no rationale to specify the \(j\)-dependence in this way (see [39] and literatures therein for how to work out the \(j\)-dependence of \(c_j\)). The confining effect was implemented in the form of the boundary condition (191) for the Reggeon wavefunction. As for the target hadron, we set the mass term of the scalar field to be \(5/R^2\) (i.e., \(c_y = 5\)), just like the lowest non-trivial spherical harmonics on \(W_5 = S_5\) for the Type IIB dilaton field [35]. The operator dimension in the dual CFT becomes \(\Delta_\phi = 2 + \sqrt{4 + c_y} = 5\).

Figure 6 shows the reduced matrix element \(\bar{g}^{0,0,0}(j, i\nu_j, \Delta)\) for the \(m = 0\)-mode exchange; the results for different values of spin \(j = 1, 1.5, 2, 2.5\) are shown in the figure. Lattice computation can be used to determine matrix elements at integer valued spins, but the analytic expression (153) allows us to determine the matrix element even for non-integer spin, so that the inverse Mellin transformation is possible, and we can also talk of the matrix elements evaluated at the saddle point value of spin \(j = j^*\). The panel (b) in Figure 6 is essentially the same as that of Fig. 5 in [10], while the panel (a) shows \(\bar{g}^{0,0,0}\) without normalizing the matrix element by its value at \(t = -\Delta^2 = 0\). Since they are not the
Figure 6: The panel (a) shows $\bar{g}^{0,0,0}(j, i\nu, \Delta)$ as a function of $\Delta^2/\Lambda^2$. The curve at the bottom is for $j = 1$, while the one at the top is for $j = 2.5$; two in the middle correspond to $j = 1.5$ and $j = 2$. The panel (b) shows $\bar{g}^{0,0,0}(\Delta)/\bar{g}^{0,0,0}(\Delta = 0)$, i.e., $\bar{g}^{0,0,0}(j, i\nu, \Delta)$ normalized at the value of $\Delta^2 = 0$. The curve at the bottom is for $j = 1$, and the curve goes up for $j = 1.5$, 2 and 2.5; this softer behavior for larger $j$ is consistent with (196).

matrix element of a “conserved current” for $j \neq 1$, the matrix element does not necessarily approach 1 in the $\Delta^2 \to 0$ limit. The panel (b) has a property that $\bar{g}^{0,0,0}$ is soft ($\bar{g}^{0,0,0}$ gets smaller slowly in $\Delta^2$) for larger $j$; this is consistent with the observation in (196), because $\partial \gamma(j)/\partial j > 0$.

A numerical result for the $\eta^2$ term in $\bar{A}_j^{+\alpha}$, which is proportional to

$$
\bar{g}^{0,0,0}(j, i\nu, \Delta) \times \left[ \frac{p^2 j(j-1)}{\Delta^2} \right] + \frac{\bar{g}^{0,0,2}(j, i\nu, \Delta)}{N_{j,2}\Delta^2} \times \frac{-1}{(j + i\nu)(j - 1 + i\nu)},
$$

(201)
is shown in Figure 7 using $j = 2$. The first and second terms of (201) both diverge at the $\Delta^2 \to 0$ limit, as we have seen in section 7.1, but their sum has a finite value at $\Delta^2 = 0$, as one can see in the figure. It is worth mentioning that this finite limit value $\approx -700$ is much larger than that of $\bar{g}^{0,0,0}$. This is likely to be due, at least partially, to the hadron mass $m_h$ value in this case; for the value of parameters we chose, $m_h = j\Delta_h - 2,1, j, 3, 1 \approx 6.4$, and $m_h^2/\Lambda^2 \approx 40$. An extra derivative $\partial \gamma$ in the matrix elements $\bar{g}^{0,0,2}$ is more like $m_h$ than $\Lambda$, and hence the second term can be larger than the first term by about $(m_h/\Lambda)^2$. The factor $(m_h/\Lambda)^2 \approx 40$ does not explain all of the moderately large value $-700$, however. The $t = -\Lambda^2$-dependence of the $\eta^2$ term (i.e., $\bar{g}^{0,0,0}(j, i\nu, \Delta)$) is quite different from that of the coefficient of the $\eta^2$, at least at small $\Delta^2$.

In the DGLAP phase, a crude approximation to the GPD is given by

$$
\bar{H}^{+\alpha}(x, \eta, t; q^2) \approx \left( \frac{1}{x} \right)^{j*} \left( \frac{\Lambda}{q} \right)^{\gamma(j*)} \bar{A}_j^{+\alpha}(\eta, t),
$$

(202)
Figure 7: The first and second term of (201) are plotted in (a) and (b), respectively, as functions of $\Delta^2/\Lambda^2$; parameters are set to the values described in the text, and we used $j = 2$ in these figures. Although both (a) and (b) diverge at $\Delta^2 \rightarrow 0$, they add up to be (c), where the $\Delta \rightarrow 0$ limit is finite. The large $\Delta^2$ behavior is seen better in the panel (d).
Figure 8: The coefficient of the $\eta^2$ term of $A_j^{+,\alpha}(\eta, t)$ to that of the $\eta^0$ term, as a function of $-t/\Lambda^2 = \Delta^2/\Lambda^2$. We used $j = 2$ and other parameters described in the text. This is the ratio of Figure 7 (d) to Figure 6 (a).

where $j^*$ is the saddle point value of $j$ depending primarily on $\ln(1/x)$, $\ln(q/\Lambda)$ and $t = -\Lambda^2$. Apart from applications to the time-like Compton scattering with very large (positive) lepton invariant mass-square, relevant range of $|\eta|$ is not much more than $x$ in such processes as TCS, DVCS and VMP. Suppose, in the power series expansion of $A_j^{+,\alpha}$ in $\eta$, that all the terms with different power of $\eta$ have a ($t$-dependent) coefficient at most of $O(1)$. Then the GPD (or $A_j^{+,\alpha}(\eta, t)$) in the small-$x$ regime would not have skewness dependence very much in the range of interest, $|\eta| \lesssim x$, because $\eta^2$ and higher-order terms are small relatively to the $\eta^0$ term. The coefficient of the $\eta^2$ term, however, turns out to be of $O(-700)$ for $\Delta^2 \approx 0$, which at least contains a factor $m_h^2/\Lambda^2$. Thus, for the range of moderately small $x$, such as $x \sim 10^{-1}$ and $|\eta| \lesssim x$, the $\eta^2$ term in $A_j^{+,\alpha}(\eta, t)$ can be just as important as the $\eta^0$ term for small $\Delta^2$. Consequently the prediction/fit of the slope parameter ($t$-dependence) may also be affected, since the $\eta^2$ term with a steeper $t$-dependence is involved. Toward higher $\Delta^2$, however, the ratio of the coefficient of the $\eta^2$ term to that of the $\eta^0$ term changes as in a numerical computation shown in Figure 8. Since the $\eta^2$-term coefficient becomes not more than 10 times the $\eta^0$ term for $5\Lambda^2 \lesssim (\Delta^2 = -t)$ at $j = 2$ in this numerical computation, the $\eta^0$-term alone will become a good enough approximation in this range of $t$ even for moderately small $|\eta| \lesssim x \approx O(10^{-1})$; for an even smaller $x$, the $\eta^2$-term can be negligible for a broader range of $t = -\Lambda^2$. We have nothing more to say about the $\eta^4$ term and higher at this moment, or whether this moderately large value $\approx 700$ is an artifact of a specific implementation of confining effects we adopted for the numerical presentation in this section. If this relatively large coefficient of the $\eta^2$ term (and also higher order terms) turns out to be a robust consequence of holographic models, that may be regarded as an unexpected lesson from holography to phenomenology.
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A More on the Mode Decomposition on AdS$_5$

For convenience, let us copy here the eigenmode equation (65) for a totally symmetric rank-$j$ tensor field on AdS$_5$; the equation consists of the following equations labeled by $k = 0, \ldots, j$:

$$
\left( (R^2 \Delta_j) - \left[ (2k + 1)j - 2k^2 + 3k \right] \right) A_{z^k \mu_1 \cdots \mu_j - k} \\
+ 2z k \partial^\rho A_{z^{k-1} \mu_1 \cdots \mu_j - k} + k(k - 1) A_{z^{k-2} \mu_1 \cdots \mu_j - k} \\
- 2z (D[A_{z^{k+1} \ldots}])_{\mu_1 \cdots \mu_j - k} + (E[A_{z^{k+2} \ldots}])_{\mu_1 \cdots \mu_j - k} = -E A_{z^k \mu_1 \cdots \mu_j - k}.
$$

(203)

A.1 Eigenvalues and Eigenmodes for $\Delta^{\mu} = 0$

block diagonal decomposition

In the main text, we considered a decomposition of the rank-$j$ totally symmetric tensor field with $(-i \partial_\mu) = \Delta_\mu = 0$ in the form of

$$
A_{z^k \mu_1 \cdots \mu_j - k} (z; \Delta^{\mu} = 0) = \sum_{N=0}^{[(j-k)/2]} (E^N [a^{(k, N)}])_{\mu_1 \cdots \mu_j - k},
$$

where $a^{(k, N)}$'s are $z$-dependent rank-$(j - k - 2N)$ totally symmetric tensor field of SO(3, 1), satisfying the 4D-traceless condition. This is indeed a decomposition, in that all the degrees of freedom in $A_{z^k \mu_1 \cdots \mu_j - k} (z; \Delta^{\mu} = 0)$ are described by $a^{(k, N)} (z)_{\mu_1 \cdots \mu_j - k - 2N}$ with $0 \leq N \leq [(j-k)/2]$ without redundancy. To see this, one only needs to note that there is a relation that, for a totally symmetric 4D-traceless rank-$r$ SO(3, 1)-tensor $a$,

$$
\eta^{\rho \delta} E^N [a]_{\rho \sigma \mu_1 \cdots \mu_{r+2N-2}} = 4N (r + N + 1) E^{N-1} [a]_{\mu_1 \cdots \mu_{r+2N-2}}.
$$

(204)

This relation can be verified recursively in $N$. 

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Using this relation, \( a_{\mu_1 \cdots \mu_j \cdots k-2N}^{(k,N)} \) can be retrieved from \( A_{z^k \mu_1 \cdots \mu_j \cdots k} \), starting from ones with larger \( N \) to ones with smaller \( N \).

Let us now see that the eigenmode equation (203) can be made block diagonal by using this decomposition. The eigenmode equation (203) with the label \( k \) for \( \Delta^\mu = 0 \) can be rewritten by using this relation (204) as follows:

\[
\sum_N \left( R^2 \Delta_j - \left[ (2k+1)j - 2k^2 + 3k \right] + \mathcal{E} \right) E^N[a^{(k,N)}] + k(k-1)[4(N+1)(j-k-N+2)]E^N[a^{(k-2,N+1)}] + E[E^{N-1}[a^{(k+2,N-1)}]] = 0.
\]

Although this equation has to hold only after the summation in \( N \), it actually has to be satisfied separately for different \( N \)'s. To see this, let us first multiply \( \eta^{\dot{\rho}\dot{a}} \) for \( [(j-k)/2] \) times and contract indices just like in (204); we obtain an equation that involves only \( a^{(k,[(j-k)/2])} \), \( a^{(k-2,[(j-k)/2]+1)} \) and \( a^{(k+2,[(j-k)/2]-1)} \). Next, multiply \( \eta^{\dot{\rho}\dot{a}} \) for \( [(j-k)/2] - 1 \) times, to obtain another equation involving \( a^{(k,[(j-k)/2]-1)} \), \( a^{(k-2,[(j-k)/2])} \) and \( a^{(k+2,[(j-k)/2]-2)} \). In this way, we obtain

\[
\left( R^2 \Delta_j - \left[ (2k+1)j - 2k^2 + 3k \right] + \mathcal{E} \right) a^{(k,N)} + k(k-1)[4(N+1)(j-k-N+2)]a^{(k-2,N+1)} + a^{(k+2,N-1)} = 0. \quad \text{(for } \forall k, N) \tag{205}
\]

Fields \( a^{(k,N)} \)'s with the same \( k + 2N = n \) form a system of coupled equations, but those with different \( n = k + 2N \) do not mix. Thus, the eigenmode equation for \( \Delta^\mu = 0 \) is decomposed into sectors labeled by \( n \). The \( n \)-th sector consists of \( z \)-dependent fields that are all in the rank-\( (j-n) = (j-k-2N) \) totally symmetric tensor of \( \text{SO}(3, 1) \).

### Classification of Eigenmodes for \( \Delta^\mu = 0 \)

Let us now study the eigenmode equations more in detail for the separate diagonal blocks we have seen. Simultaneous treatment is possible for all the \( n \)-th sectors with even \( n \), and for all the sectors with odd \( n \).

Let us first look at the \( n \)-th sector of the eigenmode problem for an \( n = 2\bar{n} \leq j \). In the eigenmode equation of \( \Delta^\mu = 0 \), we can assume\(^{35} \) the same \( z \)-dependence for all the fields in this diagonal block:

\[
a^{(k,N)}(z)_{\mu_1 \cdots \mu_j \cdots n} = \bar{a}^{(k,N)}_{\mu_1 \cdots \mu_j \cdots n} z^{2-j-\nu}, \quad k + 2N = n, \tag{206}
\]

where \( \bar{a}^{(k,N)} \)'s are \((x, z)\)-independent 4D-traceless rank-\((j-n)\) tensor of \( \text{SO}(3, 1) \). The eigenmode equations with the label \( (k, N) = (2\bar{k}, \bar{n} - \bar{k}) \) with \( \bar{k} = 0, \ldots, \bar{n} \) are relevant to the

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\(^{35} \)This is because, in the absence of \( z^2 \partial^2 \) term, the operator \( \Delta_j \) becomes a constant multiplication when it acts on a simple power of \( z \). Upon \( z^{2-j-\nu} \), for example, \( R^2 \Delta_j \) returns \( -(4 + \nu^2) \).
\textit{n = 2\bar{n}} sector, and are now written in a matrix form:

\begin{equation}
\sum_{k'=0}^{\bar{n}} D_{2k,2k'} \tilde{a}^{(2k',\bar{n}-\bar{k}') \bar{n}} = \left((4 + \nu^2) - \mathcal{E}\right) \tilde{a}^{(2\bar{k},\bar{n}-\bar{k})},
\end{equation}

where

\begin{itemize}
  \item diagonal \((k, k') = (2\bar{k}, 2\bar{k})\) entry: \(D_{2\bar{k},2\bar{k}} = -[(2\bar{k} + 1)j - 2k^2 + 3k],\)
  \item diagonal\(^+\) \((k, k') = (2\bar{k}, 2\bar{k} + 2)\) entry: \(D_{2\bar{k},2\bar{k}+2} = 1,\)
  \item diagonal\(^-\) \((k, k') = (2\bar{k}, 2\bar{k} - 2)\) entry: \(D_{2\bar{k},2\bar{k}-2} = k(k - 1) \times 4(\bar{n} - \bar{k} + 1)(j - \bar{n} - \bar{k} + 2).\)
\end{itemize}

There must be \((\bar{n}+1)\) independent eigenmodes in this \((\bar{n}+1) \times (\bar{n}+1)\) matrix equation. Let \(\mathcal{E}_{n,l}\) denote the collection of eigenvalues in this \(n = 2\bar{n}\)-th diagonal block, and \(l = 0, \cdots, \bar{n} = n/2\) label distinct eigenmodes. Corresponding eigenmode wavefunction for the \((n = 2\bar{n}, l)\) mode is in the form of

\begin{equation}
a^{(k,N)}(z; \Delta^\mu = 0) = a^{(2\bar{k},\bar{n}-\bar{k})} = c_{2\bar{k},\bar{k}+j}^{} n \epsilon^{(n,l)} e^{2j-i\nu},
\end{equation}

where \(\epsilon^{(n,l)}\) is an \((x, z)\)-independent 4D-traceless totally symmetric rank-\((j-n) = (j-2\bar{n})\) tensor of \(\text{SO}(3,1)\), and \(c_{2\bar{k},\bar{k}+j}^{}\) are \((x, z)\)-independent constants determined as the eigenvector corresponding to the eigenvalue \(\mathcal{E}_{n,l}\).

Similarly, in the \(n = 2\bar{n} + 1 \leq j\)-th sector of the eigenmode problem, with an odd \(n\), we can assume a simple power law for all the component fields involved in this sector;

\begin{equation}
a^{(k,N)}(z)_{\mu_1 \cdots \mu_{j-n}} = a_{(k,N)}^{(\bar{k},\bar{n}-\bar{k})} z^{2-j-i\nu}, \quad k + 2N = n,
\end{equation}

where \(a^{(k,N)}\) are \((x, z)\)-independent 4D-traceless totally symmetric tensor of \(\text{SO}(3,1)\). The eigenmode equation with the label \((k, N) = (2\bar{k} + 1, \bar{n} - \bar{k})\) with \(\bar{k} = 0, \cdots, \bar{n}\) are relevant to this sector, and in the matrix form, the eigenmode equation now looks

\begin{equation}
\sum_{\bar{k}'=0}^{\bar{n}} D_{2\bar{k}+1,2\bar{k}'+1} \tilde{a}^{(2\bar{k}'+1,\bar{n}-\bar{k})} = \left((4 + \nu^2) - \mathcal{E}\right) \tilde{a}^{(2\bar{k}+1,\bar{n}-\bar{k})},
\end{equation}

where

\begin{itemize}
  \item diagonal \((k, k') = (2\bar{k} + 1, 2\bar{k} + 1)\) entry: \(D_{2\bar{k}+1,2\bar{k}+1} = -[(2\bar{k} + 1)j + (2k^2 + 3k)],\)
  \item diagonal\(^+\) \((k, k') = (2\bar{k} + 1, 2\bar{k} + 3)\) entry: \(D_{2\bar{k}+1,2\bar{k}+3} = 1,\)
  \item diagonal\(^-\) \((k, k') = (2\bar{k}+1, 2\bar{k}-1)\) entry: \(D_{2\bar{k}+1,2\bar{k}-1} = k(k-1) \times 4(\bar{n} - \bar{k} + 1)(j - \bar{n} - \bar{k} + 1).\)
\end{itemize}
From here, \( \tilde{n} + 1 \) independent modes arise; their eigenvalues are denoted by \( E_{n,l} \), and \( l = \{0, \cdots, \tilde{n} \} \) is the label distinguishing different modes. The eigenmode labeled by \( (n = 2\tilde{n} + 1, l) \) has a wavefunction

\[
a^{(k,N)}(z; \Delta^\mu = 0) = a^{(2k+1,\tilde{n}-\bar{k})} = c_{2\bar{k}+1,l,n} \epsilon^{(n,l)} z^{2-j-i\nu},
\]

where \( \epsilon^{(n,l)} \) is an \((x,z)\)-independent 4D-traceless rank-\((j-n)\) totally symmetric tensor of SO\((3,1)\), and \( c_{2\bar{k}+1,l,n} \) is the eigenvector for the \((n,l)\) eigenmode determined in the matrix equation above.

**Explicit Examples**

Let us take a moment to see how the general theory above works out in practice.

The easiest of all is the \( n = 0 \) sector, which contains only one rank-\( j \) 4D-traceless field, \( a^{(0,0)} \). The eigenmode equation is

\[
\left[ \Delta_j - \frac{[(2k + 1)j - 2k^2 + 3k]|_{k=0}}{R^2} \right] a^{(0,0)} = -\frac{\mathcal{E}_{0,0}}{R^2} a^{(0,0)}.
\]

The eigenmode wavefunction has the form of

\[
a^{(0,0)}(z)_{\mu_1 \cdots \mu_j} = \epsilon^{(0,0)}_{\mu_1 \cdots \mu_j} z^{2-j-i\nu},
\]

and the eigenvalue \( \mathcal{E}_{n,l} \) is

\[
\mathcal{E}_{0,0} = (j + 4 + \nu^2).
\]

Also to the \( n = 1 \) sector, only one rank-\((j-1)\) 4D-traceless tensor field contributes. That is \( a^{(1,0)} \). The eigenmode equation becomes

\[
\left[ R^2 \Delta_j - \left( (2k + 1)j - 2k^2 + 3k \right)|_{k=1} \right] a^{(1,0)} = -\frac{\mathcal{E}_{1,0}}{R^2} a^{(1,0)}.
\]

The solution is

\[
a^{(1,0)}(z)_{\mu_1 \cdots \mu_{j-1}} = \epsilon^{(1,0)}_{\mu_1 \cdots \mu_{j-1}} z^{2-j-i\nu}, \quad \mathcal{E}_{1,0} = (3j + 5 + \nu^2).
\]

In the \( n = 2 \) sector, two rank-\((j-2)\) 4D-traceless fields are involved. They are \( a^{(0,1)} \) and \( a^{(2,0)} \). After introducing the \( z \)-dependence \( \propto z^{2-j-i\nu} \), the eigenmode equation in the \( n = 2 \) sector becomes

\[
\begin{bmatrix}
-j & 1 \\
8j & -(5j - 2)
\end{bmatrix}
\begin{bmatrix}
\tilde{a}^{(0,1)} \\
\tilde{a}^{(2,0)}
\end{bmatrix} = (4 + \nu^2) - \mathcal{E}
\begin{bmatrix}
\tilde{a}^{(0,1)} \\
\tilde{a}^{(2,0)}
\end{bmatrix}.
\]
One of the two eigenmodes is

$$\mathcal{E}_{2,0} = (4 + 5j + \nu^2), \quad \begin{pmatrix} a^{(0,1)}(z)_{\mu_1 \cdots \mu_{j-2}} \\ a^{(2,0)}(z)_{\mu_1 \cdots \mu_{j-2}} \end{pmatrix} = \begin{pmatrix} 1 \\ -4j \end{pmatrix} \epsilon^{(2,0)}_{\mu_1 \cdots \mu_{j-2}} z^{-j-\nu},$$

(218)

and the other

$$\mathcal{E}_{2,1} = (2 + j + \nu^2), \quad \begin{pmatrix} a^{(0,1)}(z)_{\mu_1 \cdots \mu_{j-2}} \\ a^{(2,0)}(z)_{\mu_1 \cdots \mu_{j-2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \epsilon^{(2,1)}_{\mu_1 \cdots \mu_{j-2}} z^{-j-\nu}.$$  

(219)

In the \textit{n = 3 sector}, two rank-(\textit{j} − 3) 4D-traceless tensor fields are involved: \( a^{(1,1)} \) and \( a^{(3,0)} \). The eigenmode equations \((210)\) become

$$\begin{pmatrix} -(3j + 1) \\ 24(j - 1) - (7j - 9) \end{pmatrix} \left( \begin{array}{c} \bar{a}^{(1,1)}(z) \\ \bar{a}^{(3,0)}(z) \end{array} \right) = ((4 + \nu^2) - \mathcal{E}) \left( \begin{array}{c} \bar{a}^{(1,1)} \\ \bar{a}^{(3,0)} \end{array} \right).$$

(220)

So, one of the two eigenmodes is

$$\mathcal{E}_{3,0} = (7j + 1 + \nu^2), \quad \begin{pmatrix} a^{(1,1)}(z)_{\mu_1 \cdots \mu_{j-3}} \\ a^{(3,0)}(z)_{\mu_1 \cdots \mu_{j-3}} \end{pmatrix} = \begin{pmatrix} 1 \\ -4(j - 1) \end{pmatrix} \epsilon^{(3,0)}_{\mu_1 \cdots \mu_{j-3}} z^{-j-\nu},$$

(221)

and the other one

$$\mathcal{E}_{3,1} = (3j - 1 + \nu^2), \quad \begin{pmatrix} a^{(1,1)}(z)_{\mu_1 \cdots \mu_{j-3}} \\ a^{(3,0)}(z)_{\mu_1 \cdots \mu_{j-3}} \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \epsilon^{(3,1)}_{\mu_1 \cdots \mu_{j-3}} z^{-j-\nu}.$$  

(222)

Finally, in the \textit{n = 4 sector}, the eigenmode equation \((207)\) is given by

$$\begin{pmatrix} -j \\ 16(j - 1) - (5j - 2) \\ 0 \end{pmatrix} \begin{pmatrix} \bar{a}^{(0,2)} \\ \bar{a}^{(2,1)} \\ \bar{a}^{(4,0)} \end{pmatrix} = ((4 + \nu^2) - \mathcal{E}) \begin{pmatrix} \bar{a}^{(0,2)} \\ \bar{a}^{(2,1)} \\ \bar{a}^{(4,0)} \end{pmatrix}.$$  

(223)

There are three solutions. First,

$$\mathcal{E}_{4,0} = (9j - 4 + \nu^2), \quad (a^{(0,2)}, a^{(2,1)}, a^{(4,0)}) = (1, -8(j - 1), 32(j - 1)(j - 2)) \epsilon^{(4,0)} z^{-j-\nu},$$

(224)

second,

$$\mathcal{E}_{4,1} = (5j - 6 + \nu^2), \quad (a^{(0,2)}, a^{(2,1)}, a^{(4,0)}) = (1, -(j - 10), -48(j - 2)) \epsilon^{(4,1)} z^{-j-\nu},$$

(226)

(227)
and finally,
\[
\mathcal{E}_{4,2} = (j + \nu^2),
\]
\[
(a^{(0,2)}, a^{(2,1)}, a^{(4,0)}) = (1, 4, 24) \epsilon^{(4,2)} z^{2-j-\nu}.
\]

An empirical relation is observed in the \(j\)-dependence of the eigenvalues we have worked out so far. The eigenvalues in the \(n\)-the sector are in the form of
\[
E_{n,l} = \nu^2 + (2n+1-4l)j + \mathcal{O}(1)
\]
for \(0 \leq l \leq [n/2] \).

**5D-traceless modes: the \(l = 0\) modes**

Although the precise expressions for the eigenvalues \(\mathcal{E}_{n,l}\) and eigenvectors \(c_{k,l,n}\) are not given for all the eigenmodes, there is a class of eigenmodes whose eigenvalues and eigenvectors (wavefunctions) are fully understood.

As we discussed in p. 32, it is possible to require both a field is an eigenmode and satisfies the 5D-traceless condition \((95)\) at the same time. In the \(n = (k + 2N)\)-th sector, the 5D-traceless condition becomes
\[
0 = (E^N[a^{(k,N)}])_{\mu_3 \cdots \mu_{k+3}} + (E^{N-1}[a^{(k+2,N-1)}])_{\mu_3 \cdots \mu_{k+3}},
\]
\[
= E^{N-1} \left[ 4N(j - n + N + 1)a^{(k,N)} + a^{(k+2,N-1)} \right] \begin{cases} k = 0, 2, \cdots, 2(\bar{n} - 1) & \text{(even } n) , \\ k = 1, 3, \cdots, 2\bar{n} - 1 & \text{(odd } n) . \end{cases}
\]

Thus, the 5D-traceless condition uniquely determines one eigenmode in each one of the \(n\)-th sector.
\[
\mathcal{E}_{n,0} = (2n+1)j + 2n - n^2 + 4 + \nu^2,
\]
and
\[
c_{2k,0,2n} = (-)^k 4^k \frac{\bar{n}!}{(\bar{n} - k)!} \frac{(j - \bar{n} + 1)!}{(j - \bar{n} - k + 1)!},
\]
\[
c_{2k+1,0,2n+1} = (-)^k 4^k \frac{\bar{n}!}{(\bar{n} - k)!} \frac{(j - \bar{n})!}{(j - \bar{n} - k)!}.
\]

**A.2 Mode Decomposition for non-zero \(\Delta_\mu\)**

**A.2.1 Diagonal Block Decomposition for the \(\Delta_\mu \neq 0\) Case**

Let us now turn our attention to the eigenmode equation \((65, 69)\) with \(\Delta_\mu \neq 0\). Because of the 2nd and 4th terms in \((69)\), the eigenmode problem becomes much more complicated. We begin by finding diagonal block decomposition suitable for the case with \(\Delta_\mu \neq 0\).
In the main text, we introduced a decomposition of a totally symmetric rank-\( j \) tensor field \( A_{m_1\cdots m_j} \) of \( \text{SO}(4, 1) \) into a collection of totally symmetric 4D-traceless 4D-transverse tensor fields of \( \text{SO}(3, 1) \). Instead of (85), a new decomposition is given by (85–233):

\[
A_{a_{k+1}\cdots a_{j-k}}(z; \Delta^\mu) = \sum^{j-k}_{s=0} \sum^{|s/2|}_{N=0} D^N D^{s-2N}[a(k,s,N)]_{\mu_1\cdots \mu_{j-k}},
\]

where \( a(k,s,N) \) are totally symmetric 4D-traceless 4D-transverse rank-(\( j-k-s \)) tensor fields of \( \text{SO}(3, 1) \). An operation \( a \mapsto \tilde{E}[a] \) on a totally symmetric \( \text{SO}(3, 1) \) tensor \( a \) is given by (86).

In order to see that the parameterization of \( A_{a_{k+1}\cdots a_{j-k}} \) by \( (a(k,s,N))_{\mu_1\cdots \mu_{j-k-s}} \)'s above is indeed a decomposition, one needs to see that \( a(k,s,N) \)'s can be retrieved from \( A_{a_{k+1}\cdots a_{j-k}} \), so that the degrees of freedom \( a(k,s,N) \) are independent. For this purpose, it is convenient to derive some relations analogous to (204). First of all, note that \( E[D[a]] = D[E[a]] \) and \( \tilde{E}[D[a]] = D[\tilde{E}[a]] \) for a totally symmetric \( \text{SO}(3, 1) \) tensor \( a \). If the rank-\( r \) tensor \( a \) is also 4D-transverse and 4D-traceless, then one can derive the following relations:

\[
\begin{align*}
\partial^\hat{\rho}(E^r D^{s-2t}[a])_{\rho_{\mu_2\cdots \mu_{r+s}}} &= -\Delta^2(s-2t)E^r D^{s-2t-1}[a] + (2t)E^{t-1} D^{s-2t+1}[a], \\
\eta^\hat{\rho}\hat{\sigma}(E^r D^{s-2t}[a])_{\rho\sigma_{\mu_3\cdots \mu_{r+s}}} &= -\Delta^2(s-2t)(s-2t-1)E^r D^{s-2t-2}[a] \\
&+ 4t(r+s-t+1)E^{t-1} D^{s-2t}[a].
\end{align*}
\]

With the relations above, it is now possible to compute

\[
\begin{align*}
\left( \eta^\hat{\rho}_1 \hat{\mu}_2 - \frac{\partial^\hat{\rho}_1}{\partial^2} \hat{\mu}_2 \right) \cdots \left( \eta^\hat{\rho}_{2p-1} \hat{\mu}_{2p} - \frac{\partial^\hat{\rho}_{2p-1}}{\partial^2} \hat{\mu}_{2p} \right) \frac{\partial^\hat{\rho}_{2p+1}}{\partial^2} \cdots \frac{\partial^\hat{\rho}_{2p+q}}{\partial^2} (\tilde{E}^N D^{s-2N}[a]_{\mu_1\cdots \mu_{r+s}})
\end{align*}
\]

\[
= \begin{cases} 
\frac{b^{(r)}_{s,N}}{b^{(r)}_{s,N}} (\tilde{E}^N D^{s-2N-q}[a]_{\mu_2p+1\cdots \mu_{r+s}}) & \text{if } p \leq N \text{ and } q \leq s-2N, \\
0 & \text{otherwise},
\end{cases}
\]

where we assume that \( a \) is a totally symmetric 4D-traceless 4D-transverse rank-\( r \) tensor of \( \text{SO}(3, 1) \). In the last line,

\[
b^{(r)}_{s,N} := \frac{1}{4^N N!(s-2N)!} \frac{\Gamma(r+3/2)}{\Gamma(r+N+3/2)}.
\]

\[
E^r D^{s-2t}[a] = \sum \eta_{\mu_1\mu_2} \cdots \eta_{\mu_{2p-1}\mu_{2p}} \partial_{\mu_{2p+1}} \cdots \partial_{\mu_{r+s}} [a]_{\mu_1\cdots \mu_{r+s}},
\]

where the sum is taken over all possible ordered choices of \( p_1, p_2, \ldots, p_s \in \{1, \ldots, j\} \) such that \( p_i \neq p_j \) for \( i \neq j \).
It is now clear how to retrieve $a^{(k,s,N)}$ from $A_{z^k \mu_1 \cdots \mu_{j-k}}$ given by \((85=233)\). First, one has to multiply $\hat{\eta} \rho \hat{\sigma} - \partial \hat{\rho} / \partial^2$ and $\partial \hat{\sigma} / \partial^2$ to $A_{z^k \mu_1 \cdots \mu_{j-k}}$ as many times as possible in order to obtain $a^{(k,s,N)}$ with larger $N$ and $(s - 2N)$. Then $a^{(k,s,N)}$'s with smaller $N$ or $(s - 2N)$ can be determined by multiplying $\hat{\eta} \rho \hat{\sigma} - \partial \hat{\rho} / \partial^2$ and $\partial \hat{\sigma} / \partial^2$ fewer times.

Let us now return to the eigenmode equation for the cases with $\Delta^\mu \neq 0$. Following precisely the same argument as in section \[A.1\], one can see that the eigenmode equation can be separated into the following independent equations labeled by $k, s$ and $N$:

\[
\begin{align*}
\left[ R^2 \Delta_j - \left( (2k + 1)j - 2k^2 + 3k \right) + \mathcal{E} \right] a^{(k,s,N)} \\
+ 2zk(s + 1 - 2N)(\partial^2) a^{(k-1,s+1,N)} \\
+ k(k-1)(s + 2 - 2N)(s + 1 - 2N)(\partial^2) a^{(k-2,s+2,N)} \\
+ 4k(k-1)(N + 1)(j - m + N + 3/2) a^{(k-2,s+2,N+1)} \\
- 2z a^{(k+1,s-1,N)} + a^{(k+2,s-2,N-1)} + (\partial^2)^{-1} a^{(k+2,s-2,N)} = 0 
\end{align*}
\]

for $\forall k, s, N$.

The relations \((235, 236)\) were used to evaluate the 2nd–4th terms of \((203)\). One can see that $a^{(k,s,N)}$'s with a common value of $m := k + s$ form a coupled eigenmode equation, but those with different $m$’s do not. Thus, $a^{(k,s,N)}(z; \Delta^\mu)$’s with $k + s = m$ form the $m$-th subspace of $A_{m_1 \cdots m_j}(z; \Delta^\mu)$, and the eigenmode equation becomes block diagonal in the decomposition into the subspaces labeled by $m = 0, \cdots, j$.

The eigenmode equation on the $m$-th subspace is given by the equation above with $0 \leq k = (m - s) \leq m$, and $0 \leq N \leq [s/2]$. Thus, the total number of equations is

\[
\sum_{s=0}^{m} ([s/2] + 1),
\]

and the same number of eigenvalues should be obtained from the $m$-th sector.

**A.2.2 Examples**

**The sector** $m = 0$: There is only one field $a^{(0,0,0)}$ in this sector, and the eigenmode equation is

\[
\left[ \Delta_j - \frac{j}{R^2} \right] a^{(0,0,0)}(z; \Delta^\mu) = -\mathcal{E} \frac{R^2}{R^2} a^{(0,0,0)}(z; \Delta^\nu).
\]

Assuming a power series expansion for the solution to this equation, beginning with some power $z^{2-j-\nu}$, the eigenvalue is determined as a function of $(\nu)$:

\[
\mathcal{E}_{0,0} = (j + 4 + \nu^2),
\]

75
and the wavefunction can be chosen as
\[
a^{(0,0,0)}(z; \Delta^\mu)_{\mu_1 \cdots \mu_j} = \epsilon^{(0,0,0)}_{\mu_1 \cdots \mu_j} \psi^{(j)}(\Delta^2, z),
\]
(244)
\[
\psi^{(j)}(\Delta^2, z) := \frac{2}{\pi} \sqrt{\nu \sinh(\pi \nu)} \epsilon^{(0,0,0)}_{\mu_1 \cdots \mu_j} \Psi^{(j)}(z),
\]
(245)

The sector \( m = 1 \): The eigenmode equation in this sector becomes
\[
\begin{bmatrix}
R^2 \Delta_j - j & -2z \\
-2z \Delta^2 & R^2 \Delta_j - (3j + 1)
\end{bmatrix}
\begin{bmatrix}
\frac{1}{a^{(0,1,0)}} \\
\frac{1}{a^{(1,0,0)}}
\end{bmatrix} = -E \begin{bmatrix}
a^{(0,1,0)} \\
a^{(1,0,0)}
\end{bmatrix}.
\]
(246)
Assuming the power series expansion in \( z \), beginning with \( z^{2-j-i\nu} \) terms, we obtain two eigenvalues depending on \( i\nu \). They are given by evaluating \( R^2 \Delta_j - j \) and \( R^2 \Delta_j - (3j + 1) \) on \( z^{2-j-i\nu} \):
\[
E_{0,0} = (j + 4 + \nu^2), \quad \text{and} \quad E_{1,0} = (3j + 5 + \nu^2).
\]
(247)

The sector \( m = 2 \): The eigenmode equation becomes
\[
\begin{pmatrix}
(R^2 \Delta_j + E)1_{4 \times 4} + \\
\begin{bmatrix}
-j & -2z & 1/\partial^2 \\
-j & -2z & 1/\partial \\
4z \partial^2 & -2z & 1/\partial^2 \\
4 \partial^2 & 1/\partial & 8j - 4 \\
4 \partial^2 & 1/\partial & 5j - 2 \\
\end{bmatrix}
\end{pmatrix}
\begin{bmatrix}
a^{(0,2,0)} \\
a^{(0,2,1)} \\
a^{(1,1,0)} \\
a^{(2,0,0)}
\end{bmatrix} = 0.
\]
(248)

The indicial equation relating the exponent \( (2 - j - i\nu) \) at \( z = 0 \) and the eigenvalues split into two parts; three eigenvalues of this matrix
\[
\begin{pmatrix}
-j & 1 \\
-j & 1 \\
4 & 1 \\
8j - 4 & 5j - 2
\end{pmatrix},
\]
(249)
determine \(-E - (4 + \nu^2)\) for the three eigenmodes, and \(-E - (4 + \nu^2)) = -(3j + 1)\) for the last eigenmode. Therefore, the four eigenvalues in the \( m = 2 \) sector are
\[
E_{0,0} = (j + 4 + \nu^2), \quad E_{1,0} = (3j + 5 + \nu^2), \quad E_{2,0} = (5j + 4 + \nu^2), \quad E_{2,1} = (j + 2 + \nu^2).
\]
(250)

In all the examples above, the \( m \)-th sector consists of eigenmodes with eigenvalues \( E_{n,l} \) for \( 0 \leq n \leq m, \ 0 \leq l \leq [n/2] \). The number of eigenmodes is, of course, the same as \( 2m \).
A.3 Wavefunctions of 5D-Traceless 5D-Transverse Modes

As we discussed toward the end of section 5.2, it is possible to require for a rank-\( j \) totally symmetric tensor field configuration \( A_{m_1\cdots m_j}(z; \Delta^\mu) \) to be an eigenmode and to be 5D-traceless 5D-transverse (95, 96) at the same time. We will see in the following that these two extra conditions (95, 96) leave precisely one eigenmode in each one of the block-diagonal sectors labeled by \( m = 0, \cdots, j \). We will further determine the wavefunction profile of such eigenmodes.

Let us first rewrite the 5D-traceless condition (95) in a more convenient form.

\[
\eta^{\hat{\rho}\hat{\sigma}} A_{z^k\rho\sigma\mu_1\cdots\mu_{j-k}} + A_{z^k\mu_1\cdots\mu_{j-k}} = 0,
\]  

(251)

which, in the \( m \)-th sector, means

\[
a^{(k,s,N)} = (s+2-2N)(s+1-2N)\Delta^2 a^{(k-2,s+2,N)} + 4(N+1)(j-m+N+3/2) a^{(k-2,s+2,N+1)}
\]

(252)

for \( N = 0, \cdots, [s/2] \); \( k + s = m \) is understood. Under the 5D-traceless condition, the 5D-transverse condition

\[
(k-1)\eta^{\hat{\rho}\hat{\sigma}} A_{z^k\rho\sigma\mu_1\cdots\mu_{j-k}} + z\partial^\hat{\rho} A_{z^{k-1}\rho\mu_1\cdots\mu_{j-k}} + (z\partial_z + (k-4)) A_{z^k\mu_1\cdots\mu_{j-k}} = 0,
\]

(253)

becomes

\[
z\partial^\hat{\rho} A_{z^{k-1}\mu_1\cdots\mu_{j-k}} + (z\partial_z - 3) A_{z^k\mu_1\cdots\mu_{j-k}} = 0.
\]

(254)

In the \( m \)-th sector \( (k + s = m) \), therefore,

\[
(s+1-2N)\Delta^2 a^{(k-1,s+1,N)} = z^3\partial_z z^{-3} a^{(k,s,N)}
\]

(255)

for \( N = 0, \cdots, [s/2] \). Hereafter, we use a simplified notation \( \mathcal{D} := z^3\partial_z z^{-3} \). One can see that all of \( a^{(k,s,N)} \)'s with \( k + s = m \) and \( N \leq [s/2] \) can be determined from \( a^{(m,0,0)} \) by using the relations (252, 255). This observation already implies that there can be at most one eigenmode in a given \( m \)-th sector that satisfies both the 5D-traceless and 5D-transverse conditions.

For now, let us assume that there is one, and proceed to determine the wavefunction. The wavefunction—\( z \)-dependence—of \( a^{(m,0,0)}(z; \Delta) \) can be determined from the eigenmode equation (241) with \( k = m, s = N = 0 \). Using (252) and (255), we can rewrite the equation as

\[
[ R^2\Delta_j \{ (2m+1)j - m^2 + 2m \} - 2m (z\partial_z - 3) + \mathcal{E} ] a^{(m,0,0)}(z; \Delta) = 0.
\]

(256)

For this equation,

\[
(a^{(m,0,0)}(z; \Delta))_{\mu_1\cdots\mu_{j-m}} = \epsilon_{\mu_1\cdots\mu_{j-m}} \left(\frac{z}{R}\right)^{2-j} (\Delta z)^m K_\nu(\Delta z), \quad \mathcal{E} = (j + 4 + \nu^2),
\]

(257)
is a solution, where $\epsilon_{\mu_1\cdots\mu_{j-m}}$ is a $z$-independent 4D-traceless 4D-transverse totally symmetric rank-$(j - m)$ tensor of SO$(3,1)$. From the value of the eigenvalue, it turns out that the 5D-traceless 5D-transverse mode in the $m$-th sector corresponds to the $(n, l, m) = (0, 0, m)$ mode. The $z$-dependence we determined above implies that

$$
\Psi^{(j);0,0}_{i\nu;0,0,m}(-\Delta^2, z) \propto (\Delta z)^m \Psi^{(j);0,0}_{i\nu;0,0,0}(-\Delta^2, z).
$$

(258)

This result corresponds to the $(s, N) = (0, 0)$ case of (97). The normalization constant $N_{j,m}$ is determined later in this section.

Let us now proceed to determine other $\Psi^{(j);s,N}_{i\nu;0,0,m}$, not just for $(s, N) = (0, 0)$. Using the 5D-transverse condition, (255), $a^{(m-1,1,0)}(z; \Delta)$ can be determined from $a^{(m,0,0)}(z; \Delta)$.

$$
a^{(m-1,1,0)} = \frac{\mathcal{D}}{\Delta^2} a^{(m,0,0)}, \quad \Psi^{(j);1,0}_{i\nu;0,0,m} = \frac{\mathcal{D}}{\Delta} \Psi^{(j);0,0}_{i\nu;0,0,m}.
$$

(259)

In order to determine the $s = 2$ components $a^{(m-2,2,N)}(N = 0, 1)$ of the $(n, l) = (0, 0)$ mode in the $m$-th sector, one has to use both the 5D-transverse condition and 5D-traceless condition:

$$
2\Delta^2 a^{(m-2,2,0)} = \mathcal{D} a^{(m-1,1,0)},
$$

(260)

$$
2\Delta^2 a^{(m-2,2,0)} - 4(j - m + 3/2)a^{(m-2,2,1)} = a^{(m,0,0)}.
$$

(261)

Therefore,

$$
a^{(m-2,2,0)} = \frac{1}{2\Delta^2} \left( \frac{\mathcal{D}}{\Delta} \right)^2 a^{(m,0,0)}, \quad a^{(m-2,2,1)} = \frac{1}{4(j - m + 3/2)} \left\{ \left( \frac{\mathcal{D}}{\Delta} \right)^2 - 1 \right\} a^{(m,0,0)}.
$$

(262)

After factoring out the normalization factor $(b^{(j-m)}_{s,N})/\Delta^{s-2N}$ and the common 4D-tensor $\epsilon^{(0,0,m)}$, we obtain

$$
\Psi^{(j);2,0}_{i\nu;0,0,m} = \left( \frac{\mathcal{D}}{\Delta} \right)^2 \Psi^{(j);0,0}_{i\nu;0,0,m}, \quad \Psi^{(j);2,1}_{i\nu;0,0,m} = \left\{ \left( \frac{\mathcal{D}}{\Delta} \right)^2 - 1 \right\} \Psi^{(j);0,0}_{i\nu;0,0,m}.
$$

(263)

The 5D-transverse conditions (255) determine the $s = 3$ components $a^{(m-3,3,N)}(z; \Delta)$ $(N = 0, 1)$ from the $s = 2$ components.

$$
a^{(m-3,3,0)} = \frac{1}{6\Delta^3} \left( \frac{\mathcal{D}}{\Delta} \right)^3 a^{(m,0,0)}, \quad a^{(m-3,3,1)} = \frac{1}{4(j - m + 3/2)\Delta} \left\{ \left( \frac{\mathcal{D}}{\Delta} \right)^3 - \left( \frac{\mathcal{D}}{\Delta} \right)^2 \right\} a^{(m,0,0)},
$$

(264)
and after factoring out the normalization factor \( b_{s,N}^{(j-m)} / \Delta^{s-2N} \) and \( \epsilon^{(0,0,m)} \) as before, we obtain

\[
\Psi_{iv;0,0,m}^{(j);3,0} = \left( \frac{\mathcal{D}}{\Delta} \right)^3 \Psi_{iv;0,0,m}^{(j);0,0}, \quad \Psi_{iv;0,0,m}^{(j);3,1} = \left\{ \left( \frac{\mathcal{D}}{\Delta} \right)^3 - \left( \frac{\mathcal{D}}{\Delta} \right) \right\} \Psi_{iv;0,0,m}^{(j);0,0}. \tag{265}
\]

The \( s = 3 \) components determined purely by the conditions (255) satisfy the 5D-traceless condition (252) with the \( s = 1 \) component:

\[
6\Delta^2 a^{(m-3,3,0)} - 4(j - m + 3/2)a^{(m-3,3,1)} = \frac{\mathcal{D}}{\Delta^2} a^{(m,0,0)} = a^{(m-1,1,0)}. \tag{266}
\]

In this way, the wavefunctions \( \Psi_{iv;0,0,m}^{(j);s,N}(-\Delta^2, z) \) for all \((s, N)\) are determined, and the result is

\[
\Psi_{iv;0,0,m}^{(j);s,N}(-\Delta^2, z) = \sum_{a=0}^N (-)^a NC_a \left( \frac{\mathcal{D}}{\Delta} \right)^{s-2a} \left[ (z\Delta)^m \Psi_{iv;0,0,0}^{(j);0,0}(-\Delta^2, z) \right] \times N_{j,m}. \tag{267}
\]

The only remaining concern was that there are more conditions from (252, 255) than the number of components \( a^{(k,s,N)} \) in the \( m \)-th sector; there can be at most one eigenmodes satisfying these 5D-traceless 5D-transverse conditions, as we stated earlier, but there may be no eigenmode left, if the conditions are overdetermining. We have confirmed, however, that the wavefunctions (267) satisfy all of the relations given by (252, 255).

### A.3.1 Normalization

We have yet to determine the normalization factor \( N_{j,m} \); as in the main text, we choose (99) to be the normalization condition. Orthogonal nature among the eigenmodes is guaranteed because of the Hermitian nature of the operator \( \alpha' (\nabla^2 - M^2) \). It is thus sufficient to focus only on the divergent part of the integral in the normalization condition in order to determine \( N_{j,m} \).

The divergent part of the integral in (99) comes only from terms with \( s = m, k = 0, \ldots \)
(0 \leq N \leq [m/2]) and a = 0. For a given m,
\[ \langle \epsilon \cdot \epsilon' \rangle \delta(\nu - \nu') \approx N_{j,m}^2 \int_0^1 dz \sqrt{-g(z)} e^{-2jA} \]
\[ \left( \sum_{N=0}^{[m/2]} \tilde{E}^N D^{m-2N}[\epsilon'(0,0,m)] \frac{b_{m,N}^{(j-m)}}{\Delta^{m-2N}} \frac{z^3 \partial_z z^{-3}}{\Delta^m} (z\Delta)^m \Psi_{j,0,0,m}^{(j,0,0)}(\Delta^2, z) \delta_{\mu_1...\mu_j \mu_1...\mu_j} \right) \]
\[ \left( \sum_{M=0}^{[m/2]} \tilde{E}^M D^{m-2M}[\epsilon'(0,0,m)] \frac{b_{m,M}^{(j-m)}}{\Delta^{m-2M}} \frac{z^3 \partial_z z^{-3}}{\Delta^m} (z\Delta)^m \Psi_{j,0,0,m}^{(j,0,0)}(\Delta^2, z) \right) \]
Divergent part of the integral in this expression comes from
\[ \frac{2}{\pi} \frac{\nu \sinh(\pi \nu)}{2} \int dx x^{2j-5} \left[ x^3 \partial_x x^{-1-j+m} K_{\nu}(x) \right] \left[ x^3 \partial_x x^{-1-j+m} K_{\nu'}(x) \right] \]
\[ \approx \prod_{p=1}^{m} [(j - p + 1)^2 + \nu^2] \delta(\nu - \nu') = \frac{\Gamma(j + 1 - i\nu)\Gamma(j + 1 + i\nu)}{\Gamma(j + 1 - m - i\nu)\Gamma(j + 1 - m + i\nu)} \delta(\nu - \nu'). \]
Noting that
\[ \left( \sum_{N=0}^{[m/2]} \tilde{E}^N D^{m-2N}[\epsilon'(0,0,m)] \frac{b_{m,N}^{(j-m)}}{\Delta^{m-2N}} \right) \left( \sum_{M=0}^{[m/2]} \tilde{E}^M D^{m-2M}[\epsilon'(0,0,m)] \frac{b_{m,M}^{(j-m)}}{\Delta^{m-2M}} \right) = \frac{j!}{(j-m)!} \sum_{N=0}^{[m/2]} b_{m,N}^{(j-m)}, \]
we find that (268) implies
\[ N_{j,m}^{-2} = \frac{\Gamma(j + 1 - i\nu)\Gamma(j + 1 + i\nu)}{\Gamma(j + 1 - m - i\nu)\Gamma(j + 1 - m + i\nu)} \frac{j!}{(j-m)!} \left( \sum_{N=0}^{[m/2]} b_{m,N}^{(j-m)} \right), \]
\[ = \frac{\Gamma(j + 1 - i\nu)\Gamma(j + 1 + i\nu)}{\Gamma(j + 1 - m - i\nu)\Gamma(j + 1 - m + i\nu)} j^m \sum_{m=0}^{[m]} \Gamma(3/2 + j - m) \Gamma(2 + 2j) \Gamma(2 + 2j - m). \]

A.4 A Note on Wavefunction of Massless Vector Field

For a rank-1 tensor (vector) field on AdS_5, we can determine the wavefunction of the \((n, l, m) = (1, 0, 1)\) eigenmode, not just for the \((n, l, m) = (0, 0, m)\) modes with \(m = 0, 1\). With the eigenvalue \(\mathcal{E}_{1,0} = (3j + 5 + \nu^2)|_{j=1},\)
\[ a^{(0,1,0)} = \epsilon^{(1,0,1)} z^2 K_{\nu}(\Delta z), \quad a^{(1,0,0)} = \epsilon^{(1,0,1)} \partial_z (z^2 K_{\nu}(\Delta z)) \]
(270)
is the eigenvector solution to (246).

The \((n,l,m) = (0,0,1)\) mode and \((n,l,m) = (1,0,1)\) mode are independent, even after the mass-shell condition \((66)\) for generic vector fields in the bosonic string theory. However, for the massless vector field \(A_m\) obtained by simple dimensional reduction of the massless vector field \(A^Y_M\) with \(Y = \{1,0,0\}\), those two modes become degenerate. To see this, note that \(c_y = -4\) for this mode, so that the mass-shell condition \((66)\) implies,

\[
(j + 4 + \nu^2 + c_y)|_{j=1} = 0 \quad (0,0,1) \text{ mode}, \quad (3j + 5 + \nu^2 + c_y)|_{j=1} = 0 \quad (1,0,1) \text{ mode},
\]

or equivalently, \(i\nu = 1\) and \(i\nu = 2\), respectively, for these two modes. It is now obvious that the terms proportional to \((\epsilon \cdot q)\) in (35) are in the form of this \((n,l,m) = (1,0,1)\) mode.

With the relations

\[
\begin{align*}
 x^3 \partial_x [x^{-3+2}K_1(x)] &= -x^3 [x^{-1}K_2(x)] \\
 \partial_x [x^2 K_2(x)] &= -x^2 K_1(x),
\end{align*}
\]

one can also see that the wavefunction for the \((n,l,m) = (0,0,1)\) mode is also proportional to the form given in (35) when the on-shell condition is imposed.

### A.5 Projection operator of SO(3,1) tensors

Note first that

\[
a = \sum_{s=0}^{r} \sum_{N=0}^{[s/2]} \bar{E}^N D^{-2N}_\Delta [a^{(s,N)}]
\]

is an orthogonal decomposition of a totally symmetric SO(3,1) tensor \(a\) of rank-\(r\) into totally symmetric 4D-traceless 4D-transverse SO(3,1) tensors \(a^{(s,N)}\) of rank-\((r-s)\). Here, the metric is given by

\[
[b(-\Delta)] \cdot [a(\Delta)] := [b(-\Delta)]_{\rho_1 \ldots \rho_r} [a(+\Delta)]_{\sigma_1 \ldots \sigma_r} \eta^{\rho_1 \sigma_1} \ldots \eta^{\rho_r \sigma_r}
\]

as in the main text. To see that the decomposition is orthogonal under this metric, one only needs to use (239) to verify that

\[
\left[ \bar{E}^M D^{-2M}_\Delta [b^{(t,M)}] \right] \cdot \left[ \bar{E}^N D^{-2N}_\Delta [a^{(s,N)}] \right] = \delta_{M,N} \delta_{t-2M,s-2N} \frac{\Delta^{2(s-2N)}_{2N}}{b^{(r-s)}_{s,N}} [b^{(t,M)}] \cdot [a^{(s,N)}].
\]

Using the fact that (272) is an orthogonal decomposition, let us construct projection operators \(\bar{P}^{(r;s,N)}\) that extract various components \(a^{(s,N)}\) from a totally symmetric SO(3,1) tensor \(a\) of rank-\(r\). We introduced an operator \(P^{(r)}\) in (102), which acts on rank-\(r\) SO(3,1) tensors. From what we have seen above, it can be used to extract the \(a^{(s,N)=(0,0)}\) component from a rank-\(r\) tensor \(a\). That is, \(\bar{P}^{(r;0,0)} = P^{(r)}\). It is straightforward to see that the projection
operator for other components $a^{(s,N)}$ with general $(s,N)$ is given by

$$
\hat{P}^{(r;s,N)} := \sum_{a} \frac{b_{r;N}^{(s)}(r)}{\Delta_{2(s-2N)}} \frac{1}{D_a} \left( \hat{E}^N D_{s-2N}^{\sigma_1 \cdots \sigma_r} \left[ \epsilon_{a} \right] \right)_{\rho_1 \cdots \rho_r} \cdot \left( \hat{E}^N D_{s-2N}^{\sigma_1 \cdots \sigma_r} \left[ \epsilon_{a} \right] \right)_{\rho_1 \cdots \rho_r},
$$

where $\epsilon_{a}$'s are an orthogonal basis of totally symmetric 4D-traceless 4D-transverse SO(3,1) tensors of rank-$(r-s)$.

It is also useful to have a concrete form of the projection operator $P^{(r)}$, not just its abstract definition in (102). We find that it is given by

$$
P^{(r)} \cdot a = \sum_{M=0}^{[\frac{r}{2}]} \frac{(-1)^M \Gamma \left( r + \frac{1}{2} - M \right)}{4^M M! \Gamma \left( r + \frac{1}{2} \right)} \sum_{k=0}^{r-2M} \frac{(-1)^k}{k!} [\hat{E}^M D^k OP_{(p,q)=(M,k)}] \cdot a,
$$

where $OP_{(p,q)}$ is the operator given in (239). A totally symmetric rank-$r$ tensor $a$ is converted once into rank-$(r-2M-k)$ tensors, and then they are converted back to a rank-$r$ tensor under the operator $P^{(r)}$. To see that all the $\hat{E}^N D^{s-2N}[a^{(s,N)}]$ components are projected out by $P^{(r)}$, one only needs to use the following formula [40]

$$
\sum_{M=0}^{N} (-1)^M N C_M \frac{\Gamma \left( r - N + \frac{3}{2} - M \right)}{\Gamma \left( r - N + \frac{3}{2} - M \right)} \frac{\Gamma \left( r + \frac{1}{2} - M \right)}{\Gamma \left( r + \frac{1}{2} \right)} = \frac{\Gamma \left( \frac{1}{2} - r \right)}{\Gamma \left( \frac{1}{2} - r + N \right) \Gamma(1 - N)},
$$

which vanishes for an integer $N \geq 0$.

A.6 Some Tensor Computations

Let us derive a more concrete expression for the product $(q^{\mu_1} \cdots q^{\mu_r}) \cdot [P^{(r)}]_{\mu_1 \cdots \mu_r}^{\nu_1 \cdots \nu_r} \cdot (p_{\nu_1} \cdots p_{\nu_r})$, by using the explicit expression for the projection operator $P^{(r)}$ to the SO(3,1)-transverse SO(3,1)-traceless rank-$r$ tensor.

$$
(q^{\mu_1} \cdots q^{\mu_r}) \cdot [P^{(r)}]_{\mu_1 \cdots \mu_r}^{\nu_1 \cdots \nu_r} \cdot (p_{\nu_1} \cdots p_{\nu_r})
$$

$$
= \sum_{M=0}^{[\frac{r}{2}]} \frac{(-1)^M \Gamma \left( j + \frac{1}{2} - M \right)}{4^M M! \Gamma \left( j + \frac{1}{2} \right)} \frac{r!}{(r-2M)!} \left[ q^2 - \frac{(q \cdot \Delta)^2}{\Delta^2} \right] M \left( p^2 \right)^M \left( q \cdot p \right)^{r-2M},
$$

where we used that $p \cdot \Delta = 0$. Within the regime of $q^2, (p \cdot q), (q \cdot \Delta) \gg \Lambda^2, \Delta^2, p^2$ we have been interested in in this article, $(q \cdot \Delta)^2/\Delta^2 \gg q^2$. Thus, after ignoring $q^2$,

$$
(q^{\mu_1} \cdots q^{\mu_r}) \cdot [P^{(r)}]_{\mu_1 \cdots \mu_r}^{\nu_1 \cdots \nu_r} \cdot (p_{\nu_1} \cdots p_{\nu_r})
$$

$$
\approx (p \cdot q)^r \sum_{M=0}^{[\frac{r}{2}]} \frac{\Gamma \left( r + \frac{1}{2} - M \right)}{4^M M! \Gamma \left( r + \frac{1}{2} \right)} \frac{r!}{(r-2M)!} \left[ \left( \frac{q \cdot \Delta}{q \cdot p} \right)^2 \frac{p^2}{\Delta^2} \right] M =: (p \cdot q)^r \times d_\ast (\eta, \Delta^2).
$$
This introduces \( \hat{d}_r \), which is a polynomial of skewness \((q \cdot \Delta)/(p \cdot q) = -2\eta\) of degree \(2[r/2]\).

When \( r \) is even, this polynomial of \( \eta \) can also be rewritten by using Legendre polynomial, \( P_\ell(x) \), which is defined by (p.82, [42])

\[
P_\ell(x) = \binom{2\ell}{\ell} \frac{(1-x^2)^{\ell-1}}{\ell!} \cdot \binom{-\ell-1}{\ell} F_2 \left( -\frac{\ell}{2}, \frac{1}{2}; \frac{1}{2}; 1-x \right).
\]

(280)

For an even \( r \),

\[
\hat{d}_r(\eta, \Delta^2) = \sum_{M=0}^{r/2} \frac{\left( -\frac{3}{2} \right)_M}{M!} \left( \frac{1-r}{2} \right)_M \left( -4p^2 \Delta^2 \eta^2 \right)^M = 2 \binom{-r}{\ell} \binom{-r}{\ell} F_2 \left( -\frac{r}{2}, \frac{1}{2}; \frac{1}{2}; \frac{1}{2} - \eta; \frac{1}{2} - \eta \right).
\]

(281)

where we used the kinematical relation \(4p^2 = -(4m_h^2 + \Delta^2)\).

Similarly, it is also necessary to compute the following expression in order to study the \( m = 0 \) exchange amplitude in section [6.3.2]

\[
\sum_{a \neq b} \epsilon_{\rho_a} \epsilon_{\rho_b} \epsilon_\mu q_{j_1} \cdots q_{j_n} \cdot [P^{(j)}]_{\sigma_1 \cdots \sigma_j} \cdot [P^{\hat{\sigma}_1 \cdots \hat{\sigma}_j} q_{\hat{j}1} \cdots q_{\hat{j}m}],
\]

(282)

which is also evaluated as above. The term proportional to \( \eta^{\hat{\rho}^\mu} \epsilon_\rho \epsilon_{\mu} \) (contribution to the structure function \( V_1 \)) is

\[
2^2 \sum_{M=1}^{[j/2]} \frac{(-1)^M \Gamma(j+\frac{1}{2}-M)}{4M! \Gamma(j+\frac{1}{2})} \frac{j!}{(j-2M)!} \left[ q^2 - \frac{(q \cdot \Delta)^2}{\Delta^2} \right]^{M-1} (p^2)^M (q \cdot p)^{j-2M}
\]

\[
\approx -2 \frac{\Delta^2}{(q \cdot \Delta)^2} \times (q \cdot p)^{j/2} \sum_{M=1}^{[j/2]} \frac{\Gamma(j+\frac{1}{2}-M)}{4M! \Gamma(j+\frac{1}{2})} \frac{j!}{(j-2M)!} \left[ \frac{(q \cdot \Delta)^2}{\Delta^2} \right]^{M-1} \left[ \frac{p^2}{q \cdot p} \right]^{2M}.
\]

(283)

This expression is once again a polynomial of \( \eta \) of degree \(2[j/2] - 2\), and is roughly of order \( \Delta^2/(q \cdot p)^2 \) times the expression (279).

We will also need the following computation in sections [6.3.3] and [6.3.4]

\[
(q_{\mu_1} \cdots q_{\mu_{j-k}}) \cdot \left( \bar{E}^N D_{-\Delta}^{s-2N}[\epsilon^{(0,0,m)}] \right)^{\hat{\mu}_1 \cdots \hat{\mu}_{j-k}}
\]

\[
= \frac{(j-k)!}{(j-m)!} \left[ q^2 - \frac{(q \cdot \Delta)^2}{\Delta^2} \right]^N (-iq \cdot \Delta)^{s-2N} [(q_{\mu_1} \cdots q_{\mu_{j-m}}) \cdot \epsilon^{(0,0,m)}].
\]

(284)
\section*{B Conformal OPE Coefficients from AdS Integrals}

Let us introduce an integral
\[ C_1(\delta, \vartheta) := (1 - \vartheta^2)^{1/2} \int_0^\infty dy \, y^{1+\delta} \, K_1(y\sqrt{1+\vartheta}) \, K_1(y\sqrt{1-\vartheta}), \]
(285)
which we encounter as the photon–photon–Pomeron/Reggeon vertex on \( AdS_5 \). \( \vartheta = \eta/x \) and \( \delta = j + i\nu \) in that context.

It is known (p.101, \cite{41}), if \( \text{Re}(\alpha + \beta) > 0 \) and \( \text{Re}(1 \pm \nu \pm \mu - \rho) > 0 \), that
\[
\left. \int_0^\infty dtt^{-\rho}K_{\mu}(\alpha t)K_{\nu}(\beta t) = 2^{-\rho-2}(-1)^{\rho-1}\beta^\nu[\Gamma(1-\rho)]^{-1} \times \Gamma\left(\frac{1+\nu+\mu-\rho}{2}\right)\Gamma\left(\frac{1-\nu+\mu-\rho}{2}\right)\Gamma\left(\frac{1-\nu-\mu-\rho}{2}\right)\right. \\
\left. \times \right. \left. \right)_2 F_1\left(\frac{1+\nu+\mu-\rho}{2},\frac{1+\nu-\mu-\rho}{2};1-\rho;1-\frac{\beta^2}{\alpha^2}\right). \quad (286) \]

Substituting \( \rho = -1 - \gamma, \mu = 1, \nu = -1, \alpha = \sqrt{1-\vartheta}, \) and \( \beta = \sqrt{1+\vartheta}, \) we obtain
\[
C_1(\delta, \vartheta) = \frac{\Gamma(\delta/2)(\Gamma(\delta/2+1)^2\Gamma(\delta/2+2)}{\Gamma(\delta+2)}2^{\delta-1}(1-\vartheta)^{-\delta} _2 F_1\left(\frac{\delta}{2},\frac{\delta+2}{2};\delta+2;\frac{2\vartheta}{\vartheta-1}\right) \quad (287)
\]

An equivalent, but a little different expression is also obtained by using the following relation (p.60, \cite{42})
\[
_2 F_1(\alpha, \beta, 2\beta; 2z) = (1-z)^{-\alpha} _2 F_1\left(\frac{\alpha}{2},\frac{\alpha+1}{2},\beta+\frac{1}{2};\left(\frac{z}{1-z}\right)^2\right); \quad (288)
\]
namely,
\[
C_1(\delta, \vartheta) = 2^{\delta-1} \frac{\delta + 2 (\Gamma(\delta/2+1)^4)}{\delta \Gamma(\delta+2)} _2 F_1\left(\frac{\delta}{4},\frac{\delta}{4}+\frac{1}{2},\frac{\delta}{2}+\frac{3}{2};\vartheta^2\right). \quad (289)
\]
As a function of \( \vartheta = \eta/x, \) \( (287) \) and \( (289) \) are precisely of the form \( (25) \) and \( (26) \), respectively, required in the conformal OPE coefficients.

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