SUPPORT OF THE BROWN MEASURE OF THE PRODUCT OF A FREE UNITARY BROWNIAN MOTION BY A FREE SELF-ADJOINT PROJECTION

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Abstract. The first part of this paper is devoted to the Brown measure of the product of the free unitary Brownian motion by an arbitrary free non-negative operator. Our approach follows the one recently initiated by Driver-Hall-Kemp though there are substantial differences at the analytical side. In particular, the corresponding Hamiltonian system is completely solvable and the characteristic curve describing the support of the Brown measure has a non-constant (in time) argument. In the second part, we specialize our findings to the product of the free unitary Brownian motion by a free self-adjoint projection and obtain an explicit description of its support.

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1. Introduction

Let \((\mathcal{A}, \tau)\) be a \(W^*\)-probability space, that is, \(\mathcal{A}\) is a von Neumann algebra with a faithful tracial state \(\tau\) and unit \(1\). To an arbitrary \(a \in \mathcal{A}\) is associated its Fuglede-Kadison determinant ([8]):

\[
\Delta(a) := \exp[\tau(\log(|a|))] = \exp\int_{\mathbb{R}} \log(u)\mu_{|a|}(dt) \in [0, +\infty],
\]

where \(|a| = (a^*a)^{1/2}\) is the radial part of \(a\) and \(\mu_{|a|}\) is the spectral measure of \(|a|\). If we set

\[
L(a) := \ln(\Delta(a)) \in [-\infty, +\infty],
\]

then the map \(\lambda \mapsto L(a - \lambda 1)\) is subharmonic on \(\mathbb{C}\) and harmonic on the resolvent set of \(a\) ([4]). As a matter of fact, the Riesz decomposition Theorem gives rise to a probability measure \(\nu_a\) supported in the spectrum of \(a\) and called the Brown measure of \(a\). Concretely, it is given by

\[
\frac{1}{4\pi} \nabla^2 L(a - \lambda 1)
\]

in the distributional sense and is uniquely determined among all compactly-supported measure by the identity

\[
L(a - \lambda 1) = \int_{\mathbb{C}} \ln(|\lambda - z|)\nu_a(dz).
\]

Using a regularization argument for the Fuglede-Kadison determinant, the Brown measure can be computed as (see e.g. [12]):

\[
\frac{1}{4\pi} \nabla^2 \lim_{x \to 0^+} \tau[\log(|a - \lambda 1|^2 + x)] = \frac{1}{4\pi} \lim_{x \to 0^+} \nabla^2 \tau[\log(|a - \lambda 1|^2 + x)],
\]

Key words and phrases. Free unitary Brownian motion; self-adjoint projection; Brown measure; Hamiltonian system.
the free Beta distribution and to the free Jacobi process respectively, and we refer the reader to
Hermitian Jacobi process respectively. These two matrix models converge in the large-size limit to
singular values of (arbitrary) truncations of Haar-unitary matrices and of the unitary Brown
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UP

scription of the support of its Brown measure
PUP
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(t
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As long as the solution of the Hamiltonian system exist, we have

where we simply write
λ
:=
λ(0), x0 := x(0), and H0 is the initial value of the Hamiltonian.

Once this formula is obtained, we specialize our computations to the non normal operator
YtP. Our interest in this particular case is partly motivated by the Brown measure of its (strong) limit as
t → ∞ which was completely determined in [9] using the S-transform of P (see also [3] for further
instances of explicit Brown measures). Another motivation stems from the study of operators of
the form
PYQ, where Q is a self-adjoint projection which is free from Y, which are large-size limits
(in the sense of mixed moments) of truncations of the Brownian motion in the unitary group. In
particular, their limit as
t → ∞ are truncations of Haar unitary matrices whose densities, when they exist, were computed in [14]. Even more, the eigenvalues densities of square truncations were
used in [17] to analyze statistical properties of random quantum channels exhibiting a chaotic
scattering. In this respect, it was further proved in [15] that the empirical measure of any square
truncation converges weakly to the Brown measure of
PUP in the compressed algebra. Up to an atom at zero and a normalizing constant, the latter coincides with the Brown measure of
UP since ker(U P) = ker(P) and this coincidence remains valid at finite time
(t
see [9], p.350). As to the singular values of (arbitrary) truncations of Haar-unitary matrices and of the unitary Brownian motion, they are squares of the eigenvalues of matrices from the Jacobi unitary ensemble and of the Hermitian Jacobi process respectively. These two matrix models converge in the large-size limit to
the free Beta distribution and to the free Jacobi process respectively, and we refer the reader to
the papers [5] and [10] for further details and references.

Coming back to the operator
PYtP, the main result of this paper provides the following description of the support of its Brown measure
μ(PYtP):

\[
x \mapsto S_a(\lambda, x) := \tau[\log(|a - \lambda 1|^2 + x)]
\]
defines an analytic function in the right half-plane and may be expanded for large \(|x|\) into a
power series for the moments of \(|a - \lambda 1|^2\). Actually, if \(a\) is a free Itô process (i.e. solution of a
free stochastic differential equation) then such expansion may be turned into a partial differential
equation (PDE). This is for instance valid for the free multiplicative Brownian motion and for its
additive (circular) counterpart, and led in [6] and in [11] respectively to the full description of the corresponding Brown measures.

In this paper we adapt the approach initiated in [6] to the partial isometry
YtP, \(t \geq 0\). Here,
Y = (Yt)\(_t\geq 0\) is a free unitary free Brownian motion ([1]) and
P is a self-adjoint projection in \((A, \tau)\) with rank \(\tau(P) \in (0, 1)\) and free from \(Y\). Nonetheless, a large part of our computations applies to
operators of the form
YtH where \(h\) is a non negative operator free from \(Y\), which are the natural dynamical analogues of R-diagonal operators ([13]). Using free stochastic calculus, we derive a
nonlinear first-order PDE for the map

\[
(t, \lambda, x) \mapsto S(t, \lambda, x) := S_{YtH}(\lambda, x) = \tau\left[\log((Y_th - \lambda)^*(Y_th - \lambda) + x)\right],
\]
and write down the corresponding Hamiltonian system of coupled ordinary differential equations
(hereafter ode). It turns out that the latter is completely solvable: the characteristic curve

\[
u \mapsto \lambda(u), \quad u \geq 0,
\]
is explicitly determined and allows to solve all the remaining ODEs. However, for ease of reading, we shall only write down those curves needed for the description of the support of the Brown measure. In particular, we determine the blow-up time of the solution of the Hamiltonian system. Compared to the system studied in [6], the angular momentum is still a constant of motion, yet
the argument of the characteristic curve \(u \mapsto \lambda(u)\) is no longer constant and is rather affine in
time. As to its radius, it solves a non-linear second order ODE so that its expression depends also
on its initial speed. The issue of these computations is the following expression of \(S\) along the
characteristic curves \(u \mapsto \lambda(u), u \mapsto x(u)\), valid up to the blow-up time.

**Theorem 1.1.** As long as the solution of the Hamiltonian system exist, we have

\[
S(u, \lambda(u), x(u)) = \tau(\log(|h - \lambda_0|^2 + x_0)) + \left(H_0 - \frac{1}{2}\right) u + \log |\lambda(u)| - \log |\lambda_0|,
\]

where \(\lambda_0 := \lambda(0), x_0 := x(0),\) and \(H_0\) is the initial value of the Hamiltonian.
Theorem 1.2. Set \( \tau(P) := \alpha \in (0, 1) \). Then for any \( t > 0 \), the support of \( \mu_{(PY_tP)} \) is contained in the region enclosed by the Jordan curve \( f_{t,\alpha}(F_{t,\alpha}) \), where

\[
    f_{t,\alpha}(z) := ze^{\frac{2\alpha - 1 + z}{2(1-t^2)}},
\]

and \( F_{t,\alpha} \) is the closure of the set:

\[
    \left\{ \lambda \in \mathbb{C} : |1 - \alpha - \lambda| \neq \alpha, \quad |f_{t,\alpha}(\lambda)|^2 = \frac{\alpha|\lambda|^2}{\alpha|\lambda|^2 + (1 - \alpha)|1 - \lambda|^2} \right\},
\]

and is a Jordan curve.

The proof of this theorem is a straightforward consequence of an explicit expression of the function

\[
    s_t(\lambda) = \lim_{x \to 0} S(t, \lambda, x),
\]

when \( \lambda_0 \) lies in some region outside the closure \( \Sigma_{t,\alpha} \) of the bounded component \( \Sigma_{t,\alpha} \) of the complementary of \( F_{t,\alpha} \). Actually, these initial values of the characteristic curve \( u \mapsto \lambda(u) \) allows to specialize \( x_0 = 0 \) in the expression (1.1), in which case the whole curve \( u \mapsto x(u) \) vanishes and the solution of the Hamiltonian system exists up to time \( t \). On the other hand, we shall retrieve Haagerup-Larsen result for the operator \( PUP \): the boundary of the support of \( \mu_{(PY_tP)} \) approaches the circle \( \mathcal{T}(0, \sqrt{\alpha}) \) as \( t \) approaches infinity. Let us finally point out that the affine time-dependence of the argument of the curve \( u \mapsto \lambda(u) \) makes the description of the non-atomic part of \( \mu_{(PY_tP)} \) far from being accessible. For that reason, we postpone this task to a future research work. We would like also to stress that the rank \( \alpha = \tau(P) \) varies in \( (0, 1) \) since some results proved in the sequel do not extend to the value \( \alpha = 1 \) corresponding to \( Y_t \) whose spectrum is already known ([2]).

Theorem 1.1. Section 3 is devoted to the particular case \( h = P \). There, we firstly supply a parametrization of \( F_{t,\alpha} \) and prove that its image under \( f_{t,\alpha} \) is a Jordan curve. Afterwards, we prove that any \( \lambda \) lying outside the Jordan domain delimited by \( f_{t,\alpha}(F_{t,\alpha}) \) is attainable by some characteristic curve starting at \( \lambda_0 \in \Sigma_{t,\alpha} \) and derive the explicit expression of \( s_t(\lambda) \). The latter is, up to a linear combination of logarithmic potentials of two dirac measures, the real part of a holomorphic function.

2. Hamiltonian system for \( Y_t h \)

2.1. The PDE for \( S \). For sake of clarity, we introduce the following notations:

\[
    a_t = a_t(h) := Y_t h,
\]

\[
    a_{t,\lambda} = a_{t,\lambda}(h) := |a_t - \lambda|^2 = (a_t - \lambda)(a_t - \lambda) = h^2 - \bar{\lambda}a_t - \lambda a_t^* + |\lambda|^2,
\]

where we omit the dependence on \( h \). Using free stochastic calculus, we prove:
Proposition 2.1. For any $n \geq 1$,
\[
\frac{d}{dt}(a_{t,\lambda}^n) = \frac{n}{2} \sum_{j=1}^{n} (a_{t,\lambda}^{n-1}(\overline{X}_t + \lambda a_t^*) + n|\lambda|^2 \sum_{j=0}^{n-2} \tau(h^2 a_{t,\lambda}^j) \tau(a_{t,\lambda}^{n-2-j})
\]
\[
- \frac{n}{2} \sum_{j=0}^{n-2} X^2 \tau(a_{t,\lambda}^j) \tau(a_{t,\lambda}^{n-2-j}) + \lambda^2 \tau(a_{t,\lambda}^j) \tau(a_{t,\lambda}^{n-2-j}),
\]
where an empty sum is zero.

Proof. Since
\[
da_{t,\lambda} = -\overline{X}_tda_t - \lambda da_t^* = -\overline{X}dY_t - \lambda hdY_t^*,
\]
then the free Itô’s formula entails:
\[(2.1) \quad d(a_{t,\lambda}^n) = \sum_{k=1}^{n} a_{t,\lambda}^{k-1} da_{t,\lambda} a_{t,\lambda}^{n-k} + \sum_{1 \leq j < k \leq n} a_{t,\lambda}^{j-1} \left( da_{t,\lambda} a_{t,\lambda}^{k-j-1} da_{t,\lambda} \right) a_{t,\lambda}^{n-k},
\]
where the term inside the brackets is a free semi-martingale bracket. Moreover, it is known that (see e.g. [1]):
\[dY_t = iY_t dX_t - \frac{1}{2} Y_t dt, \quad dY_t^* = -i dX_t Y_t^* - \frac{1}{2} Y_t^* dt,
\]
where $(X_t)_{t \geq 0}$ is a free additive Brownian motion. Consequently,
\[
da_{t,\lambda} = -\overline{X}(iY_t dX_t - \frac{1}{2} Y_t dt)h - \lambda h(-i dX_t Y_t^* - \frac{1}{2} Y_t^* dt)
\]
\[= i(-\overline{X}Y_t dX_t h + \lambda hdX_t Y_t^*) + \frac{1}{2} (\overline{X}_t + \lambda a_t^*) dt.
\]
Set $dw_t := -\overline{X}Y_t dX_t h + \lambda hdX_t Y_t^*$. Then, for any adapted process $(\kappa_t)_{t \geq 0}$,
\[dw_t \kappa_t dw_t = X^2 \tau(h \kappa_t Y_t) a_t dt + \lambda^2 \tau(Y_t^* \kappa_t h)a_t^* dt - |\lambda|^2 \tau(Y_t^* \kappa_t Y_t) h^2 dt - |\lambda|^2 \tau(h \kappa_t h) dt
\]
\[= X^2 \tau(a_t \kappa_t) a_t dt + \lambda^2 \tau(a_t^* \kappa_t) a_t^* dt - |\lambda|^2 \tau(h \kappa_t h) h^2 dt - |\lambda|^2 \tau(h \kappa_t h) dt
\]

Specializing the last equation to $\kappa_t = a_{t,\lambda}^{k-j-1}$ and taking the trace in both sides of (2.1), we get:
\[
\frac{d}{dt}(a_{t,\lambda}^n) = \frac{1}{2} \sum_{k=1}^{n} \tau(a_{t,\lambda}^{n-1}(\overline{X}_t + \lambda a_t^*) + n|\lambda|^2 \sum_{j=0}^{n-2} \sum_{i=0}^{n-2-i} \tau(h^2 a_{t,\lambda}^j) \tau(a_{t,\lambda}^{n-2-j})
\]
\[+ \tau(h^2 a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j}) - \sum_{i=0}^{n-2-i} \sum_{j=0}^{n-2-i} \tau(X^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j}) + \lambda^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^* a_{t,\lambda}^{n-2-j})
\]
\[= \frac{n}{2} \tau(a_{t,\lambda}^{n-1}(\overline{X}_t + \lambda a_t^*) + n|\lambda|^2 \sum_{j=0}^{n-2} \tau(h^2 a_{t,\lambda}^j) \tau(a_{t,\lambda}^{n-2-j})
\]
\[+ \lambda^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j}) + \sum_{i=0}^{n-2-i} \sum_{j=0}^{n-2-i} \tau(X^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j}) + \lambda^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j})
\]
\[= \frac{n}{2} \tau(a_{t,\lambda}^{n-1}(\overline{X}_t + \lambda a_t^*) + n|\lambda|^2 \sum_{j=0}^{n-2} \tau(h^2 a_{t,\lambda}^j) \tau(a_{t,\lambda}^{n-2-j})
\]
\[+ \lambda^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j}) - \frac{n}{2} \sum_{j=0}^{n-2} X^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j}) + \lambda^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j}) + \sum_{j=0}^{n-2} X^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j}) + \lambda^2 \tau(a_{t,\lambda}^j) a_{t,\lambda}^{n-2-j}).
\]

Write
\[S(t, \lambda, x) := S_{Y_t t}(\lambda, x) = \tau \left[ \log(a_{t,\lambda} + x) \right].
\]
Using the previous proposition, we get
Corollary 2.2. If $\Re(x) > 0$, then the function $S$ satisfies
\[
\frac{\partial S}{\partial t} = \frac{1}{2} \tau((x\lambda_t + \lambda a_t^*)(a_t, \lambda + x)^{-1}) - |\lambda|^2 \tau((h^2(a_t, \lambda + x)^{-1}) \tau((a_t, \lambda + x)^{-1})
\]
\[+ \frac{\lambda^2}{2} |\tau(a_t, \lambda + x)^{-1})|^2 + \frac{\lambda^2}{2} |\tau(a_t^*(a_t, \lambda + x)^{-1})|^2.
\]

Proof. For large $|x|$, we expand $S$ into power series
\[S(t, \lambda, x) = \log x + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{nx^n} \tau[(a_t, \lambda)^n],
\]
and differentiate it termwise with respect to $t$. Using Proposition 2.1, we get
\[
\frac{\partial S}{\partial t} = \frac{1}{x} \frac{d}{dt} \bigg|_{x=1} (\frac{\partial}{\partial x} S(x, \lambda, x)) + \frac{1}{2} \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{nx^n} \tau((a_t, \lambda + x)^{-1}) + |\lambda|^2 \sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{nx^n} \sum_{j=0}^{n-2} \tau(h^2a_t, \lambda_j) \tau((a_t, \lambda_j)^{-1}))
\]
\[+ \frac{\lambda^2}{2} \left(\sum_{j=0}^{\infty} \frac{(-1)^{j}}{x^j} \tau((a_t, \lambda_j)^{-1})\right)^2 + \frac{\lambda^2}{2} \sum_{j=0}^{\infty} \frac{(-1)^{j}}{x^j} \tau((a_t, \lambda_j)^{-1})^2
\]
\[+ \frac{\lambda^2}{2} |\tau(a_t^*(a_t, \lambda + x)^{-1})|^2 + \frac{\lambda^2}{2} |\tau(a_t^*(a_t, \lambda + x)^{-1})|^2.
\]
Since $S$ and the expression displayed in the right-hand side of the last equality are analytic in the right half-plane, the corollary follows. \qed

The analyticity of $x \mapsto S(t, \lambda, x)$ is only needed to prove Corollary 2.2. Henceforth, we shall assume $x > 0$ and derive the following PDE for $S$.

Theorem 2.3. The function $S$ satisfies:
\[
\frac{\partial S}{\partial t} = x |\lambda|^2 \left(\frac{\partial S}{\partial x}\right)^2 + \frac{\lambda^2}{2} \left(\frac{\partial S}{\partial \lambda}\right)^2 + \frac{\lambda^2}{2} \left(\frac{\partial S}{\partial \lambda}\right)^2 - \frac{\lambda}{2} \frac{\partial S}{\partial \lambda} - \frac{\partial S}{\partial \lambda}
\]
with the initial value:
\[S(0, \lambda, x) = \tau(\log(|h - \lambda|^2 + x)).
\]
Equivalently, if $\lambda = a + ib$ then this PDE reads:
\[
(2.2) \quad \frac{\partial S}{\partial t} = x |\lambda|^2 \left(\frac{\partial S}{\partial x}\right)^2 - a \frac{a}{2} \frac{\partial S}{\partial a} + b \frac{b}{2} \frac{\partial S}{\partial b} + a_0 \frac{\partial S}{\partial a} \frac{\partial S}{\partial b} + a^2 - b^2 \left(\frac{\partial S}{\partial a} \right)^2 - \frac{\partial S}{\partial b} \right)^2.
\]
Proof. We appeal to the following identities:
\[
(2.3) \quad \frac{\partial S}{\partial a} = \tau((a_t, \lambda + x)^{-1}),
\]
\[
(2.4) \quad \frac{\partial S}{\partial \lambda} = -\tau((a_t - \lambda)^*(a_t, \lambda + x)^{-1}),
\]
\[
(2.5) \quad \frac{\partial S}{\partial \lambda} = -\tau((a_t - \lambda)(a_t, \lambda + x)^{-1}),
\]
which are straightforward consequences of [4, Lemma 1.1]. It follows that:
\[
\tau((a_t, \lambda + x)^{-1}) = \lambda \frac{\partial S}{\partial x} - \frac{\partial S}{\partial \lambda}, \quad \tau((a_t^*(a_t, \lambda + x)^{-1}) = \lambda \frac{\partial S}{\partial x} - \frac{\partial S}{\partial \lambda}.
\]
Besides, writing \( h^2 = a_t^*a_t = (a_t - \lambda + \lambda^*)(a_t - \lambda + \lambda) \), we get
\[
\tau(h^2(a_t, \lambda+x) - 1) = \tau(a_t(a_t, \lambda+x) - 1) + \tau((a_t - \lambda)^*(a_t, \lambda+x) - 1)
\]
\[
+ \lambda \tau((a_t - \lambda)(a_t, \lambda+x) - 1) + |\lambda|^2 \tau((a_t, \lambda+x) - 1)
\]
\[
= 1 + |\lambda|^2 \frac{\partial S}{\partial x} - \lambda \frac{\partial S}{\partial \lambda} - \lambda \frac{\partial S}{\partial \lambda}.
\]
Together with Corollary 2.2, we end up with:
\[
\frac{\partial S}{\partial t} = \frac{\lambda}{2} \left[ \frac{\partial S}{\partial x} - \frac{\partial S}{\partial \lambda} \right]^2 + \frac{\lambda}{2} \left[ |\lambda| \frac{\partial S}{\partial x} - \frac{\partial S}{\partial \lambda} \right] + \frac{\lambda}{2} \frac{\partial S}{\partial \lambda} - |\lambda|^2 \frac{\partial S}{\partial x} \left[ 1 + |\lambda|^2 \right] \frac{\partial S}{\partial \lambda} - \lambda \frac{\partial S}{\partial \lambda} + \lambda \frac{\partial S}{\partial \lambda}
\]
as desired. □

2.2. Hamilton equations. Since (2.2) is a nonlinear and first-order PDE, then it is natural to appeal to the method of characteristics. Even more, it turns out that the Hamiltonian formalism is well-suited to our situation as it was in [6], which amounts to make a coupling between space variables \( a, b, x, \) and the partial derivatives of \( S \) with respect to them (momenta \( p_a, p_b, p_x \)) such that the right-hand side of (2.2) (the Hamiltonian) viewed as a function of these six variables is constant along the characteristics. More precisely, the Hamiltonian corresponding to the PDE (2.2) is (up to a sign) given by:
\[
H(a, b, x, p_a, p_b, p_x) = -x(a^2 + b^2)p_x^2 + \frac{a}{2}p_a + \frac{b}{2}p_b - ap_b p_o - \frac{a^2 - b^2}{4} (p_a^2 - p_b^2).
\]
If we require that \( a, b, x, p_a, p_b, p_x \), evolve along curves, then the Hamilton’s equations are given by:
\[
\frac{da}{du} = \frac{\partial H}{\partial p_a}; \quad \frac{db}{du} = \frac{\partial H}{\partial p_b}; \quad \frac{dx}{du} = \frac{\partial H}{\partial p_x};
\]
\[
\frac{dp_a}{du} = -\frac{\partial H}{\partial a}; \quad \frac{dp_b}{du} = -\frac{\partial H}{\partial b}; \quad \frac{dp_x}{du} = -\frac{\partial H}{\partial x}.
\]
(2.6)

and ensure that
\[
\partial_u[H(a(u), b(u), x(u), p_a(u), p_b(u), p_x(u)))] = 0.
\]
Besides, the initial conditions of the momenta curves are deduced from (2.3), (2.4) and (2.5):
\[
p_a(0) = \frac{\partial S}{\partial a}(a_0, b_0, x_0) = -2\tau(q_0(h - a_0)),
\]
\[
p_b(0) = \frac{\partial S}{\partial b}(a_0, b_0, x_0) = 2b_0\tau(q_0),
\]
\[
p_x(0) = \frac{\partial S}{\partial x}(a_0, b_0, x_0) = \tau(q_0),
\]
where we simply write
\[
a(0) = a_0; \quad b(0) = b_0; \quad x(0) = x_0;
\]
and we set
\[
q_0 = ((h - a_0)^2 + b_0^2 + x_0)^{-1}.
\]
Consequently, the value of the Hamiltonian along the characteristic curves is
\[
H_0 := -x_0(a_0^2 + b_0^2)p_x(0)^2 - a_0\tau(q_0)(h - a_0) + b_0^2\tau(q_0) + 4a_0b_0^2\tau(q_0(h - a_0))\tau(q_0)
\]
\[
- (a_0^2 - b_0^2) ([\tau(p_0 h - a_0 q_0)]^2 - b_0^2\tau(q_0)^2)^2.
\]
One also checks by direct computations using (2.6) that the following are also constant of motions:

**Proposition 2.4.** Along any solution of (2.6), the following quantities remains constant in time:
- \( K_1 := ap_a - bp_b \) (angular momentum).
- \( K_2 := xp_x + \frac{1}{2}(ap_a + bp_b) \).
- \( xp_x^2 \).
2.3. Solving the equations. Let us write the odes in (2.6) more explicitly:

\( \dot{a} = -\frac{a^2 - b^2}{2} p_a - a(b p_b - \frac{1}{2}) \);  
\( \dot{b} = \frac{a^2 - b^2}{2} p_b - b(a p_a - \frac{1}{2}) \);  
\( \dot{x} = -2x(a^2 + b^2)p_x \);  
\( \dot{p}_a = \frac{a}{2}(p_a^2 - p_b^2) + p_a(b p_b - \frac{1}{2}) + 2a x_0 p_a(0)^2 \);  
\( \dot{p}_b = -\frac{b}{2}(p_a^2 - p_b^2) + p_b(a p_a - \frac{1}{2}) + 2b x_0 p_x(0)^2 \);  
\( \dot{p}_x = (a^2 + b^2)p_x^2 \).

From (2.12), we obtain

\[ p_x(u) = \frac{p_x(0)}{1 - p_x(0) \int_0^u \lambda(s)^2 ds}, \quad u \geq 0, \]

up to the blow-up time, or equivalently

\[ \int_0^u \lambda(s)^2 ds = \frac{1}{p_x(0)} - \frac{1}{p_x(u)}. \]

Besides, since \( x p_x^2 \) is a constant of motion then

\[ x(u) = x_0 \left(1 - p_x(0) \int_0^u \lambda(s)^2 ds \right)^2. \]

Now, we come to the following key result:

**Proposition 2.5.** Provided that \( \lambda \) does not vanish, it satisfies the following differential equation:

\[ x_0 p_x^2(0) |\lambda|^2 + \frac{\tilde{\lambda} \lambda - (\tilde{\lambda})^2}{\lambda^2} = 0. \]

**Proof.** Let \( \tilde{\lambda}(\cdot) := \lambda(2\cdot) \), then the sum of (2.7) and (2.8) gives:

\( \dot{\lambda} = -p_\lambda \tilde{\lambda}^2 + \tilde{\lambda} \Leftrightarrow p_\lambda = \frac{\tilde{\lambda} - \hat{\lambda}}{\lambda^2}. \)

where we set

\[ p_\lambda(u) := (p_a - i p_b)(2u) \]

while subtracting (2.11) from (2.10) gives:

\( \dot{p}_\lambda = (p_\lambda)^2 \tilde{\lambda} - p_\lambda + 4\tilde{\lambda} x_0 p_x^2(0). \)

Differentiating (2.13) and comparing the resulting equation with (2.14), we further get:

\[-\frac{\dot{\lambda}}{\lambda^2} - \frac{\tilde{\lambda} \dot{\lambda}^2 - 2 \tilde{\lambda} \dot{\lambda}^2}{\lambda^4} = \frac{\tilde{\lambda}^2 + \dot{\lambda}^2 - 2 \tilde{\lambda} \dot{\lambda}}{\lambda^4} \quad \frac{\dot{\lambda}}{\lambda} + \frac{\tilde{\lambda} - \hat{\lambda}}{\lambda^2} + 4\tilde{\lambda} x_0 p_x^2(0). \]

Multiplying both sides of the last equation by \( \tilde{\lambda}^4 \), we equivalently get:

\[-\dot{\lambda} \tilde{\lambda}^2 - \tilde{\lambda} \dot{\lambda}^2 + 2 \tilde{\lambda} \dot{\lambda}^2 = \tilde{\lambda}(\tilde{\lambda}^2 + \dot{\lambda}^2 - 2 \tilde{\lambda} \dot{\lambda}) + (\dot{\hat{\lambda}} - \hat{\lambda}) \tilde{\lambda}^2 + 4|\tilde{\lambda}|^2 \tilde{\lambda}^3 x_0 p_x^2(0). \]

which reduces after some simplifications to:

\[ 4x_0 p_x^2(0) |\tilde{\lambda}(u)|^2 + \frac{\tilde{\lambda} \lambda - (\tilde{\lambda})^2}{\lambda^2} = 0. \]

Remembering the definition \( \tilde{\lambda}(u) = \lambda(2u) \), we are done. \( \square \)

Writing

\[ \frac{\tilde{\lambda} \lambda - (\tilde{\lambda})^2}{\lambda^2} = (\ln(\tilde{\lambda})) \]

where \( \ln \) is any determination of the logarithm which coincides with the real logarithm on the positive half-line, and setting \( \lambda(u) = r(u)e^{i \theta(u)} \), we readily get:

\[ \frac{\dot{\theta}}{\theta} = 0 \]

\[ x_0 p_x^2(0) r^2 + (\ln(r)) = 0. \]
Setting further \( r = e^v \), it follows that:

\[
\theta(u) = \dot{\theta}(0)u + \theta_0, \quad \theta_0 := \theta(0),
\]

\[
\dot{v} + x_0p_x^2(0)e^{2v} = 0.
\]

In particular, \( \dot{v} \) is decreasing and as such, it is either non positive on the whole interval where it is defined or there exists a time \( u_1 \) after which it remains non positive. Moreover, \[
(\dot{v})^2(u) = (\dot{v}(0))^2 + x_0p_x^2(0)(e^{2v(0)} - e^{2v(u)}),
\]

and using (2.7) and (2.8), we have

\[
\dot{v}(0) = \frac{\dot{\theta}(0)}{r(0)} = \frac{a_0\dot{\theta}(0) + b_0\ddot{\theta}(0)}{a_5^2 + b_5^2} = \frac{1}{2}[1 - a_0\dot{p}_0(0) - b_0\dot{p}_0(0)].
\]

Note also that

\[
(\dot{v}(0))^2 + x_0p_x^2(0)(e^{2v(0)} - e^{2v(t)}) \geq 0 \iff 0 \leq |\lambda(t)| \leq \sqrt{|\lambda_0|^2 + \frac{(\dot{v}(0))^2}{x_0p_x^2(0)}}.
\]

where \( \lambda_0 = a_0 + ib_0 \), and that \( u_1 \) satisfies

\[
|\lambda(u_1)| = \sqrt{|\lambda_0|^2 + \frac{(\dot{v}(0))^2}{x_0p_x^2(0)}}.
\]

We shall distinguish two cases:

- \( \dot{v} \) is non positive (\( \dot{v}(0) \leq 0 \)): \( v \) is decreasing and \( |\lambda(t)| \leq |\lambda_0| \) on the whole time-interval where it is defined. Precisely, we have

\[
\dot{v}(u) = -\sqrt{(\dot{v}(0))^2 + x_0p_x^2(0)(e^{2v(0)} - e^{2v(u)})}.
\]

Solving this equation leads to the following result:

**Proposition 2.6.** Assume \( \dot{v}(0) \leq 0 \). Then, for any \( u \geq 0 \),

\[
|\lambda(u)| = \frac{2|\lambda_0|e^{\sqrt{C_1}u}(1 + \sqrt{1 - C_2|\lambda_0|^2})}{C_2|\lambda_0|^2 + e^{2\sqrt{C_1}u}(1 + \sqrt{1 - C_2|\lambda_0|^2})^2},
\]

where

\[
C_1 := (\dot{v}(0))^2 + x_0p_x^2(0)e^{2v(0)} > 0, \quad C_2 := \frac{x_0p_x^2(0)}{C_1} \in (0, 1).
\]

**Proof.** We need to compute the primitive:

\[
F(g) := \int_0^g \frac{dy}{\sqrt{(\dot{v}(0))^2 + x_0p_x^2(0)(e^{2v(0)} - e^{2y})}}, \quad g < v(0) = \ln(r_0),
\]

\[
= \int_0^{e^g} \frac{dy}{\sqrt{C_1y\sqrt{1 - C_2y^2}}} = \int_{\arcsin(\sqrt{C_2}e^g)} dy = \frac{1}{\sqrt{C_1}} \ln \left( \tan \left( \frac{1}{2} \arcsin(\sqrt{C_2}e^g) \right) \right)
\]

where the last equality follows from the trigonometric identity:

\[
\tan \left( \frac{u}{2} \right) = \frac{\sin(u)}{1 + \cos(u)}.
\]

As a result,

\[
F(v(u)) = F(v(0)) - u \geq 0.
\]

But \( F \) is invertible and its inverse reads

\[
F^{-1}(g) = \ln \frac{2e^{-\sqrt{C_1}g}}{\sqrt{C_2(1 + e^{-2\sqrt{C_1}g})}}.
\]
Hence

\[ v(u) = F^{-1}[F(v(0)) - u] = \ln(r(u)), \]

or equivalently

\[ |\lambda(u)| = \frac{2e^{\sqrt{C_1}(u-F(v_0))}}{\sqrt{C_2(1+e^{2\sqrt{C_1}(u-F(v_0))})}}. \]

Finally,

\[ e^{-\sqrt{C_1}F(v_0)} = \frac{1 + \sqrt{1 - C_2e^{2v_0}}}{\sqrt{C_2e^{2v_0}}}, \quad e^{v_0} = |\lambda_0|, \]

whence

\[ |\lambda(u)| = \frac{2e^{\sqrt{C_1}u+v_0}(1 + \sqrt{1 - C_2e^{2v_0}})}{C_2e^{2v_0} + e^{2\sqrt{C_1}u}(1 + \sqrt{1 - C_2e^{2v_0}})} \]

\[ = \frac{2|\lambda_0|e^{\sqrt{C_1}u}(1 + \sqrt{1 - C_2|\lambda_0|^2})}{C_2|\lambda_0|^2 + e^{2\sqrt{C_1}u}(1 + \sqrt{1 - C_2|\lambda_0|^2})} \]

The proposition is proved. \( \square \)

- \( \dot{v} \) is positive (\( \dot{v}(0) > 0 \)): \( v \) is increasing on \((0, u_1)\) and

\[ |\lambda_0| \leq |\lambda(u)| \leq \sqrt{\frac{\dot{r}(0)^2}{|\lambda_0|^2|\dot{r}_0|^2}} \]

Moreover, we have on that interval:

\[ \dot{v}(u) = \sqrt{(\dot{v}(0))^2 + x_0p_x^2(0)(e^{2v(0)} - e^{2v(u)})}. \]

Similar computations as above yield:

**Proposition 2.7.** Assume \( \dot{v}(0) > 0 \). Then, for any \( u \in (0, u_1) \),

\[ |\lambda(t)| = \frac{2|\lambda_0|e^{-\sqrt{C_1}t}(1 + \sqrt{1 - C_2|\lambda_0|^2})}{C_2|\lambda_0|^2 + e^{-2\sqrt{C_1}t}(1 + \sqrt{1 - C_2|\lambda_0|^2})^2} \]

For \( u \geq u_1 \),

\[ \dot{v} = -\sqrt{(x_0p_x^2(0)(e^{2v(t_1)} - e^{2v(t)})}, \]

and we are led to:

\[ \int_0^u \frac{dy}{\sqrt{(x_0p_x^2(0)(e^{2v(u_1)} - e^{2y})}}, \quad g < v(u_1), \]

On this interval, we get:

**Proposition 2.8.** Assume \( \dot{v}(0) > 0 \). Then, for any \( u \geq u_1 \),

\[ |\lambda(u)| = \frac{2|\lambda(u_1)|e^{\sqrt{C_1}(u-u_1)}}{1 + e^{2\sqrt{C_1}(u-u_1)}}, \]

where

\[ C_3 := x_0p_x^2(0)e^{2v(t_1)} - x_0p_x^2(0)|\lambda(t_1)|^2 = x_0p_x^2(0)|\lambda_0|^2 + \dot{v}(0)^2 = C_1, \]

**Remark 2.9.** If we let \( x_0p_x^2(0) \to 0^+ \) (note that \( p_x(0) > 0 \) by the faithfulness of \( \tau \)), then the first two expressions of \( |\lambda(u)| \) reduce to:

\[ |\lambda(u)| = e^{\dot{v}(0)u} = |\lambda_0|e^{\dot{v}(0)u}, \]

giving the solution to the equation \( \dot{v} = 0 \). As we shall see below, this limit plays a key role in the proof of the main result of the paper.

As to the angular part of the curve \( u \mapsto \lambda(u) \), it admits the following expression:

**Proposition 2.10.** As long as the solutions of (2.6) exist, we have:

\[ \theta(u) = \theta(0) + K_1u/2, \]

where we recall \( K_1 = a_0p_0(0) - b_0p_0(0) \) is the angular momentum.
Proof. Since \( \theta \) is a linear map, then
\[
\frac{d}{du} \arctan \left( \frac{b}{a} \right)(u) = \frac{b - \dot{b}}{a^2 + b^2}(u)
\]
\[
= \frac{[(a^2 - b^2)(ap_a + bp_b) - 2ab(ap_a - bp_b)]}{2(a^2 + b^2)}(u)
\]
\[
= \frac{ap_a - bp_b}{2}(u).
\]
Since \( ap_a - bp_b \) is a constant of motion, the proposition follows. \(\square\)

2.4. Proofs of Theorem 1.1. In this paragraph, we proceed to the proof of Theorem 1.1 which provides an expression of \( S \) along characteristic curves.

Proof of Theorem 1.1. Set \( J(u) := (\lambda(u), x(u)) = (a(u), b(u), x(u)) \) and \( P(u) := (p_a(u), p_b(u), p_x(u)) \). Then, we compute the inner product:
\[
P \cdot \frac{dJ}{du} = P \cdot \nabla_P H = 2H - \frac{a}{2}p_a - \frac{b}{2}p_b = 2H_0 - \frac{a}{2}p_a - \frac{b}{2}p_b.
\]
whence (see e.g. [6], Proposition 6.3),
\[
S(u, J(u)) = S(0, J_0) + H_0u - \frac{1}{2} \int_0^u [a(s)p_a(s) + b(s)p_b(s)] ds
\]
\[
= \tau(\log(|\lambda_0|^2 + x_0)) + H_0u - \frac{1}{2} \int_0^u [a(s)p_a(s) + b(s)p_b(s)] ds.
\]
Moreover, using Proposition 2.4 together with
\[
x(s)p_x(s) = x_0p_x(0) \left( 1 - p_x(0) \int_0^u |\lambda(s)|^2 ds \right),
\]
it follows that:
\[
\frac{1}{2} \int_0^u a(s)p_a(s) + b(s)p_b(s) ds = \left(x_0p_x(0) + \frac{1}{2}a_0p_a(0) + \frac{1}{2}b_0p_b(0)\right) u
\]
\[
- \int_0^u x(s)p_x(s) ds = \left(x_0p_x(0) + \frac{1}{2}a_0p_a(0) + \frac{1}{2}b_0p_b(0)\right) t - x_0p_x(0) u
\]
\[
+ x_0p_x^2(0) \int_0^s |\lambda(y)|^2 dy = \left(\frac{1}{2}a_0p_a(0) + \frac{1}{2}b_0p_b(0)\right) t + x_0p_x^2(0) \int_0^s |\lambda(y)|^2 dy.
\]
Finally, recall that \( |\lambda| \) satisfies
\[
x_0p_x^2(0)|\lambda|^2 + (\ln(\lambda))) = 0,
\]
which implies that
\[
x_0p_x^2(0) \int_0^s |\lambda(y)|^2 dy = - \ln |\lambda(u)| + \ln |\lambda_0| + \dot{v}(0) u
\]
\[
= - \ln |\lambda(u)| + \ln |\lambda_0| + \frac{1}{2} [1 - a_0p_a(0) - b_0p_b(0)] u.
\]
The sought expression follows then after straightforward computations. \(\square\)

2.5. Blow-up time. We close this section with the explicit formula for the time
\[
t_* := t_* (\lambda_0, x_0)
\]
defined by:
\[
\int_0^{t_*} |\lambda(s)|^2 ds = \frac{1}{p_x(0)} = \frac{1}{r(\theta_0)}.
\]
which is the first time when \( p_x \) blows-up or equivalently the curve \( t \mapsto x(t) \) attains zero. In order to unify both cases corresponding to \( \dot{v}(0) > 0 \) and \( \dot{v}(0) \leq 0 \), we shall use the left-continuous sign function:
\[
\text{sgn}(u) = \begin{cases} 
1; & u > 0 \\
-1; & u \leq 0
\end{cases}.
\]
and assume that $x_0p_x^2(0)$ is small enough so that $t_1 \geq t_*$.

**Proposition 2.11.** The blow-up time $t_*$ is given by

$$\frac{2|\lambda_0|^2}{C_2|\lambda_0|^2 + e^{-2\text{sgn}(\dot{\psi}(0))\sqrt{C_1 t_*}}(1 + \sqrt{1 - C_2|\lambda_0|^2})^2} = \frac{2|\lambda_0|^2}{C_2|\lambda_0|^2 + (1 + \sqrt{1 - C_2|\lambda_0|^2})^2} + \frac{\sqrt{C_1}}{\tau(q_0)} \text{sgn}(\dot{\psi}(0)).$$

In particular, if $x_0p_x^2(0) \to 0^+$ then

$$t_* = \frac{1}{2\dot{\psi}(0)} \ln \left(1 + \frac{2\dot{\psi}(0)}{|\lambda_0|^2 \tau(q_0)}\right).$$

**Proof.** Recall that

$$|\lambda(u)| = \frac{2|\lambda_0|e^{-\text{sgn}(\dot{\psi}(0))\sqrt{C_1 t_*}}(1 + \sqrt{1 - C_2|\lambda_0|^2})^2}{C_2|\lambda_0|^2 + e^{-2\text{sgn}(\dot{\psi}(0))\sqrt{C_1 t_*}}(1 + \sqrt{1 - C_2|\lambda_0|^2})^2}.$$

Then

$$\int_0^u |\lambda(s)|^2 ds = \int_0^u \frac{4|\lambda_0|^2 e^{-2\text{sgn}(\dot{\psi}(0))\sqrt{C_1 t_*}}(1 + \sqrt{1 - C_2|\lambda_0|^2})^2}{(C_2|\lambda_0|^2 + e^{-2\text{sgn}(\dot{\psi}(0))\sqrt{C_1 t_*}}(1 + \sqrt{1 - C_2|\lambda_0|^2})^2) ds}.$$

Performing the variable change:

$$y = e^{-2\text{sgn}(\dot{\psi}(0))\sqrt{C_1 t_*}},$$

then the equation satisfied by $t_*$ and its limit as $x_0p_x^2(0) \to 0^+$ follow from straightforward computations. \qed

**Remark 2.12.** The equation satisfied by $t_*$ may be rewritten as:

$$|\lambda(t_*)|e^{2\text{sgn}(\dot{\psi}(0))\sqrt{C_1 t_*}} = |\lambda_0| + \frac{\sqrt{C_1}(1 + \sqrt{1 - C_2|\lambda_0|^2})}{\tau(q_0)|\lambda_0|} \text{sgn}(\dot{\psi}(0)).$$

Remember that in the Hamiltonian picture, $p_x$ is the partial derivative $\partial_x S$ along the characteristic curves $u \mapsto (\lambda(u), x(u))$. Therefore, if the curve $u \mapsto \lambda(u)$ starts in the resolvent set of $Y_t h$ then the identity (2.3) shows that $\lambda(t_*)$ lies in the spectrum of $Y_t h$. This observation raises the following problem: given a time $t > 0$, find a curve $\lambda$ (i.e., initial conditions) such that $t_* = t$? As suggested by our previous result, such curves are easy to find whenever the limit $x_0p_x^2(0) \to 0^+$ is allowed. For instance, if $|h - \lambda_0|^2$ is invertible with bounded or integrable inverse then $p_x(0) < \infty$ for all $x_0 \geq 0$ and we can let $x_0 \to 0^+$ in which case the whole curve $u \mapsto x(u)$ vanishes. Otherwise, letting $x_0 \to 0^+$ yields $p_x(0) = +\infty$ in which case $t_* = 0$.

### 3. The Support of the Brown Measure of $Y_t P$

In this section, we deal with the special case $h = P$ where we recall that $P$ is a selfadjoint projection with rank $\tau(P) = \alpha \in (0, 1)$. Define:

$$T_\alpha(\lambda_0) := t_*(\lambda_0, 0) = \frac{1}{2\dot{\psi}(0)} \ln \left(1 + \frac{2\dot{\psi}(0)}{|\lambda_0|^2 \tau(q_0)}\right), \quad \dot{\psi}(0) \neq 0, \lambda_0 \neq 1,$$

$$= \frac{1}{|\lambda_0|^2 \tau(q_0)} \dot{\psi}(0) = 0, \lambda_0 \neq 1,$$

$$= 0, \lambda_0 = 1.$$

Since

$$2\dot{\psi}(0) = 2\alpha - 1 + 2\alpha a_0 - \frac{|\lambda_0|^2}{|\lambda_0 - 1|^2} = -\frac{(a_0 - (1 - \alpha))^2 + b_0^2 - \alpha^2}{|\lambda_0 - 1|^2} = \alpha(1 - |\lambda_0|^2) - (1 - \alpha)|1 - \lambda_0|^2,$$

and

$$|\lambda_0|^2 \tau(q_0) = 1 + \alpha \frac{2a_0 - 1}{|\lambda_0 - 1|^2} = \frac{\alpha|\lambda_0|^2 + (1 - \alpha)|\lambda_0 - 1|^2}{|\lambda_0 - 1|^2}.$$
it follows that:

\[
T_{\alpha}(\lambda_0) = \begin{cases} 
\frac{|1 - \lambda_0|^2}{\alpha(1 - |\lambda_0|^2) - (1 - \alpha)|1 - \lambda_0|} \log \left( \frac{\alpha|\lambda_0|^2 + (1 - \alpha)|1 - \lambda_0|^2}{\alpha|\lambda_0|^2 + (1 - \alpha)|1 - \lambda_0|^2} \right), & |1 - \alpha - \lambda_0| \neq \alpha, \\
\frac{|1 - \lambda_0|^2}{\alpha|\lambda_0|^2 + (1 - \alpha)|1 - \lambda_0|^2} = \frac{|\lambda_0 - 1|^2}{\alpha}, & |1 - \alpha - \lambda_0| = \alpha, 
\end{cases}
\]

In particular, \(T_{\alpha}, \alpha \in (0, 1)\), is a real analytic function on the whole plane. Note that this description of \(T_{\alpha}(\lambda_0)\) extends continuously to \(\alpha = 1\) since:

\[
T_1(\lambda_0) = \begin{cases} 
\frac{|1 - \lambda_0|^2}{|\lambda_0|^2 - 1} \log \left( \frac{|\lambda_0|^2}{|\lambda_0|^2 - 1} \right), & |\lambda_0| \neq 1, \\
|1 - \lambda_0|^2, & |\lambda_0| = 1,
\end{cases}
\]

However, \(T_1\) is real analytic on the punctured plane \(\mathbb{R}^2 \setminus \{0\}\).

### 3.1. The regions \(\Sigma_{t,\alpha}\)

Recall the map:

\[f_{t,\alpha}(z) := ze^{\frac{2\alpha - 1 + \alpha}{1 - \alpha}}, \quad t \geq 0, z \in \mathbb{C} \setminus \{1\}.
\]

We can easily see that the equality

\[|f_{t,\alpha}(z)|^2 = \frac{\alpha|z|^2}{\alpha|z|^2 + (1 - \alpha)|1 - z|^2}
\]

is satisfied on the circle \(T(1 - \alpha, \alpha)\) (we take a tangential limit at \(z = 1\)). However, it is also satisfied outside this circle and we are led to consider the following set:

\[G_{t,\alpha} := \left\{ z \in \mathbb{C} : |1 - \alpha - z| \neq \alpha, \quad |f_{t,\alpha}(z)|^2 = \frac{\alpha|z|^2}{\alpha|z|^2 + (1 - \alpha)|1 - z|^2} \right\}.
\]

In this respect, recall that \(F_{t,\alpha}\) is the closure of \(G_{t,\alpha}\). Then

**Lemma 3.1.** Let \(\alpha \in (0, 1)\). Then, \(F_{t,\alpha}\) is a Jordan curve for each \(t > 0\). Moreover, \(f_{t,\alpha}\) is a one-to-one map there.

**Proof.** Equivalently, we shall consider the image of \(G_{t,\alpha}\) under the Mobius transformation:

\[z \mapsto w = \frac{2\alpha - 1 + z}{1 - z}
\]

since clearly \(G_{t,\alpha}\) does not contain \(z = 1\). Note also that the circle

\(T(1 - \alpha, \alpha) = \{ z, |1 - \alpha - z| = \alpha \}\)

is mapped onto \(\{\Re(w) = 0\} \cup \{\infty\}\). Now, write \(w = x + iy\) then

\[|f_{t,\alpha}(z)|^2 = \frac{\alpha|z|^2}{\alpha|z|^2 + (1 - \alpha)|1 - z|^2} \Leftrightarrow \left| \frac{x + 1 - 2\alpha + iy}{x + 1 + iy} \right|^2 e^{tx} = \frac{\alpha(x + 1 - 2\alpha)^2 + \alpha y^2}{\alpha(x + 1 - 2\alpha)^2 + 4\alpha y^2 + 4\alpha^2(1 - \alpha)} \Leftrightarrow \frac{x^2 + 2x(1 - 2\alpha) + 1 + y^2}{(x + 1)^2 + y^2} e^{tx} = 1.
\]

Consequently,

\[
y^2 = \frac{\phi_{t,\alpha}(x)}{e^{tx} - 1}, \quad x \neq 0,
\]

where

\[
\phi_{t,\alpha}(x) = (x + 1)^2 - (x^2 + 2x(1 - 2\alpha) + 1) e^{tx},
\]

while

\[y^2 = \frac{4\alpha}{t} - 1; \quad x = 0,
\]

provided that \(t \leq 4\alpha\). The map \(\phi_{t,\alpha}\) is smooth and satisfies \(\phi_{t,\alpha}(0) = 0, \phi_{t,\alpha}(-1) = -4\alpha e^{-t} < 0\) together with the limits:

\[
\lim_{x \to +\infty} \phi_{t,\alpha}(x) = +\infty, \quad \lim_{x \to -\infty} \phi_{t,\alpha}(x) = -\infty.
\]

Moreover its zero set coincides with the number of the roots of the equation:

\[1 - e^{-tx} = \frac{4\alpha x}{(x + 1)^2}.
\]

A quick inspection shows then that there is a unique negative root \(x_{t,\alpha}^{-} < -1\) for which \(\phi_{t,\alpha}(x) \geq 0\) on \((-\infty, x_{t,\alpha}^{-}]\). Moreover, there is at most one positive solution and at most another negative
solution in $(-1,0)$. In particular, there is no negative solution in $(-1,0)$ when $t \leq 4\alpha$. To see this last fact, we note that for any $x \in (-1,0)$,
\[
e^{-tx} - 1 + \frac{4\alpha x}{(1+x)^2} \leq e^{-tx} - 1 + \frac{tx}{(1+x)^2}.
\]
Then it suffices to prove that
\[
(3.2)\quad e^{-tx} - 1 + \frac{tx}{(1+x)^2} < 0, \quad x \in (-1,0).
\]
To this end, we differentiate the LHS of this inequality with respect to $t \in (0,4)$ to get:
\[
-x[(1+x)e^{-tx/2} + 1][(1+x)e^{-tx/2} - 1]
\]
\[
(1+x)^2
\]
But, the variations of the real function:
\[
x \mapsto (1+x)e^{-tx/2} - 1
\]
on the interval $(-1,0)$ show that it is negative when $t < 2$ while its sign changes exactly once from negative to positive when $t > 2$. If $t < 2$, then the inequality $(3.2)$ is clear since the function
\[
t \mapsto e^{-tx} - 1 + \frac{tx}{(1+x)^2}
\]
is decreasing and vanishes at $t = 0$. Otherwise, by continuity at $t = 2$, it only remains to prove that the value of
\[
e^{-tx} - 1 + \frac{tx}{(1+x)^2}
\]
at $t = 4$ is negative. Explicitly,
\[
e^{-4x} - 1 + \frac{4x}{(1+x)^2} = \left(e^{-2x} - \frac{1-x}{1+x}\right)\left(e^{-2x} + \frac{1-x}{1+x}\right) < 0, \quad x \in (-1,0),
\]
or equivalently:
\[
(3.3)\quad e^{-2x} - \frac{1-x}{1+x} < 0, \quad x \in (-1,0).
\]
Since the derivative of the LHS of this inequality is given by:
\[
2[1 + (1+x)e^{-x}][(1+x)e^{-x} - 1]
\]
\[
(1+x)^2
\]
and since $1 - (1+x)e^{-x} > 0$ on $(-1,0)$ then $(3.3)$ holds true. Now, since $\phi_{t,\alpha}'(0) = 4\alpha - t$, then we shall then distinguish separately the three cases $t < 4\alpha$, $t = 4\alpha$ and $t > 4\alpha$.

- $t < 4\alpha$: in this case $\phi_{t,\alpha}'(0) > 0$ therefore there exists a unique $\tilde{x}_{t,\alpha}^+ > 0$ such that $\phi_{t,\alpha}(x) \geq 0$ on $[0,\tilde{x}_{t,\alpha}^+]$. Consequently, the image of $F_{t,\alpha}$ under the Mobius transformation above is parametrized by:
\[
x \in [x_{t,\alpha}^-, \tilde{x}_{t,\alpha}^+] \setminus \{0\}, \quad y_{t,\alpha}(x) = \pm \sqrt{\frac{\phi_{t,\alpha}(x)}{tx - 1}},
\]
and
\[
x = 0, \quad y_{t,\alpha}(0) = \left\{\pm \sqrt{\frac{4\alpha}{\alpha - 1}}\right\},
\]
which is clearly a Jordan curve.

- $t = 4\alpha$: since $\phi_{t,\alpha}'(0) = 0$ then $\phi_{t,\alpha}(x) \leq 0$ for all $x \geq 0$. As a matter of fact, the parametrization of the image of $F_{t,\alpha}$ is given by:
\[
x \in [x_{t,\alpha}^-, 0), \quad y_{t,\alpha}(x) = \pm \sqrt{\frac{\phi_{t,\alpha}(x)}{tx - 1}},
\]
and $x = 0, \quad y_{t,\alpha}(0) = 0$, therefore the image of $F_{t,\alpha}$ is a Jordan curve as well.

- $t > 4\alpha$: this case is similar to the previous one since $\phi_{t,\alpha}'(0) < 0$ which forces the existence of a negative root $\tilde{x}_{t,\alpha}^-$ in $(-1,0)$. We then have $\phi_{t,\alpha}(x) > 0$ on $(\tilde{x}_{t,\alpha}^-, 0)$ and $\phi_{t,\alpha}(x) \leq 0$ for $x \geq 0$. The corresponding parametrization is then given by:
\[
x \in [\tilde{x}_{t,\alpha}^-, \tilde{x}_{t,\alpha}^-], \quad y_{t,\alpha}(x) = \pm \sqrt{\frac{\phi_{t,\alpha}(x)}{tx - 1}},
\]
which clearly yields a Jordan curve.
Finally, take \( z_1 \neq z_2 \) lying on \( F_{t,\alpha} \) such that \( f_{t,\alpha}(z_1) = f_{t,\alpha}(z_2) \). Then
\[
\frac{1 - z_1}{z_1} = \frac{1 - z_2}{z_2}.
\]
Setting:
\[
w_1 = \frac{2\alpha - 1 + z_1}{1 - z_1}, \quad w_2 = \frac{2\alpha - 1 + z_2}{1 - z_2},
\]
we get the equivalent condition:
\[
|w_1 + 1 - 2\alpha| = |w_2 + 1 - 2\alpha|.
\]
Write \( w_1 = x_1 + iy_1, w_2 = x_2 + iy_2 \) then the last condition reads:
\[
(x_1 + 1 - 2\alpha)^2 + \frac{\phi_{t,\alpha}(x_1)}{e^{tx_1} - 1} = (x_2 + 1 - 2\alpha)^2 + \frac{\phi_{t,\alpha}(x_2)}{e^{tx_2} - 1}
\]
which reduces after some simplifications to:
\[
\frac{e^{tx_1} - 1}{x_1} = \frac{e^{tx_2} - 1}{x_2}.
\]
Since the map \( u \mapsto (e^u - 1)/u \) is invertible on \( \mathbb{R} \) then \( x_1 = x_2 \) and in turn \( z_2 = z_1 \) or \( z_2 = \overline{z} \). The second alternative is only possible when \( z_1 \) is real since \( f_{t,\alpha}(\overline{z}) = \overline{f_{t,\alpha}(z)} \), in which case \( z_1 = z_2 \) as well.

Let \( \Sigma_{t,\alpha} \) be the bounded component of the complement of \( F_{t,\alpha} \). By the Jordan curve Theorem, we have:
\[
\partial \Sigma_{t,\alpha} = F_{t,\alpha}.
\]
Then, \( \Sigma_{t,\alpha} \) may be characterized as follows.

**Proposition 3.2.** For any \( t > 0 \) and \( \alpha \in (0, 1) \), we have
\[
\Sigma_{t,\alpha} := \{ \lambda_0 \in \mathbb{C} : T_\alpha(\lambda_0) < t \}.
\]

In particular, \( 1 \in \Sigma_{t,\alpha} \).

**Proof.** We first recall the following expressions valid for \( x_0 = 0 \):
\[
|\lambda(t)| = |\lambda_0|e^{\dot{\varphi}(0)t} = |\lambda_0|e^{-\frac{1}{2} - \alpha t \frac{(\alpha - 1)}{|\lambda_0 - 1|^2}}, \quad \theta(t) = \alpha t \frac{b_0}{|\lambda_0 - 1|^2} + \theta_0.
\]
Equivalently
\[
\lambda(t) = \lambda_0 e^{\frac{\alpha}{2} (-1 + 2\alpha / (1 - \lambda_0))} = f_{t,\alpha}(\lambda_0).
\]
On the other hand, since
\[
2\dot{\varphi}(0) = \frac{\alpha (1 - |\lambda_0|^2) - (1 - \alpha) |1 - \lambda_0|^2}{|\lambda_0 - 1|^2},
\]
then the circle \( \mathbb{T}(1 - \alpha, \alpha) \) is exactly the zero set of \( \dot{\varphi}(0) \). Hence, if \( \lambda_0 \) is such that \( \dot{\varphi}(0) \neq 0 \), then
\[
|f_{t,\alpha}(\lambda_0)|^2 = |\lambda_0|^2 e^{2\varphi(0)} = \frac{\alpha |\lambda_0|^2}{|\lambda_0 - 1|^2 + (1 - \alpha) |1 - \lambda_0|^2}.
\]
\[
\iff t = \alpha (1 - |\lambda_0|^2) - (1 - \alpha) |1 - \lambda_0|^2 \frac{\log \left( \frac{\alpha |\lambda_0|^2}{|\lambda_0 - 1|^2 + (1 - \alpha) |1 - \lambda_0|^2} \right)}{\alpha (1 - |\lambda_0|^2) - (1 - \alpha) |1 - \lambda_0|^2}
\]
Thus, the set \( G_{t,\alpha} \), which coincides with the Jordan curve \( F_{t,\alpha} \) except possibly at one or two points, is exactly the set where \( T_\alpha(\lambda_0) = t \). But, the set
\[
\{ \lambda_0, T_\alpha(\lambda_0) > t \},
\]
is unbounded since
\[
\lim_{|\lambda_0| \to +\infty} T_\alpha(\lambda_0) = +\infty
\]
while
\[
\{ \lambda_0, T_\alpha(\lambda_0) < t \},
\]
is bounded, then the first statement of the proposition is clear. As to the second one, it follows from the first statement together with \( T_\alpha(1) = 0 \). \( \square \)
Remark 3.3. When $\alpha = 1$, the map

$$f_{t,1}(z) = ze^{\frac{1+\alpha}{2t}}$$

describes the spectrum of $Y_t$ ([2]) and encodes the support $\Sigma_{t,1}$ of the Brown measure of the free multiplicative Brownian motion ([6]). Writing

$$f_{t,\alpha}(z) = e^{(\alpha-1)t/2}f_{t,1}(z)$$

we readily deduce from [2] (see paragraph 4.2.3) that $f_{t,\alpha}$ is a one-to-one map from the Jordan domain

$$\Gamma_{t,\alpha} := \{z \in \mathbb{D}, |f_{t,\alpha}(z)| < e^{(\alpha-1)t/2}\} = \{z \in \mathbb{D}, |f_{t,1}(z)| < 1\}$$

onto the open disc $\mathbb{D}(0, e^{(\alpha-1)t/2})$ and from a neighborhood of infinity (the image of $\Gamma_{t,\alpha}$ under the inversion $z \mapsto 1/z$) onto $\mathbb{C} \setminus \overline{\mathbb{D}(0, e^{(\alpha-1)t/2})}$. In both cases, it extends to a homeomorphism between the corresponding boundaries. These properties satisfied by $f_{t,\alpha}$ will be used to prove Proposition 3.4 below. Note also that $\Sigma_{t,1}$ may be doubly-connected in contrast to $\Sigma_{t,\alpha}, \alpha \in (0,1)$ (see [2]).
3.2. Proof of Theorem 1.2. From Lemma 3.1, we deduce that \( f_{t,\alpha}(F_{t,\alpha}) \) is a Jordan curve for any \( t > 0 \) and any \( \alpha \in (0, 1) \). Denote \( \Omega_{t,\alpha} \) the region enclosed by this curve. The last ingredient needed for proving Theorem 1.2 is the following proposition which shows that any complex number lying outside \( \overline{\Omega}_{t,\alpha} \) may be reached by a characteristic curve \( u \mapsto \lambda(u) \) exactly at time \( t \).

**Proposition 3.4.** Let \( \lambda \) outside \( \overline{\Omega}_{t,\alpha}, t > 0, \alpha \in (0, 1) \). Then there exists \( \lambda_0 \) outside \( \Sigma_{t,\alpha} \) so that the solution to the system (2.6) is well defined up to time \( t \) and \( \lambda(u) \) approaches \( \lambda \) as \( u \) approaches \( t^- \). Moreover,

\[
\lambda_0 = f_{t,\alpha}^{-1}(\lambda) \in \Gamma_{t,\alpha} \cup 1/\Gamma_{t,\alpha}
\]

provided that \( |\lambda| \neq e^{(\alpha-1)t/2} \) in which case it is unique.

**Proof.** Fix \( \lambda \notin \overline{\Omega}_{t,\alpha} \). Then, the second statement of the proposition follows readily from Remark 3.3. Moreover, if \( |\lambda| = e^{(\alpha-1)t/2} \) then the same remark shows that there are two inverse images of \( \lambda \) by \( f_{t,\alpha} \) and lying respectively on the boundaries of \( \Gamma_{t,\alpha} \) and of \( 1/\Gamma_{t,\alpha} \). As a matter of fact, it only remains to prove that if \( \lambda = f_{t,\alpha}(\lambda_0) \notin \overline{\Omega}_{t,\alpha} \) then \( \lambda_0 \notin \Sigma_{t,\alpha} \).

To this end, consider the circle \( C(\lambda) \) of radius \( |\lambda| \) and centered at the origin. If \( |\lambda| \leq e^{(\alpha-1)t/2} \) then the inverse image of \( C(\lambda) \) under \( f_{t,\alpha} \) is a Jordan curve \( C_{t,\alpha}(\lambda_0) \) around zero lying in \( \Gamma_{t,\alpha} \) and intersects \( \partial \Sigma_{t,\alpha} \) in no more than two points (the last fact may be seen along the same lines written to prove that \( f_{t,\alpha} \) is injective on \( \partial \Sigma_{t,\alpha} \)). Moreover, the fact that \( f_{t,\alpha} \) is a homeomorphism on \( C_{t,\alpha}(\lambda_0) \) and on \( \partial \Sigma_{t,\alpha} \) implies that the images of their intersection points (if there are any) are exactly the intersection points of \( C(\lambda) \) and \( \partial \Sigma_{t,\alpha} \). Since arg\((f_{t,\alpha}(C_{t,\alpha}(\lambda_0))) \in [0, 2\pi) \) is a monotone function (because it is continuous invertible), then \( \lambda_0 = f_{t,\alpha}^{-1}(\lambda) \) lies outside \( \Sigma_{t,\alpha} \). Replacing \( \Gamma_{t,\alpha} \) with \( 1/\Gamma_{t,\alpha} \), similar arguments lead to the same conclusion when \( |\lambda| \geq e^{(\alpha-1)t/2} \), proving the proposition.

Combining Theorem 1.1 and Proposition 3.4, we get:

**Corollary 3.5.** For any \( t > 0 \) and any \( \alpha \in (0, 1) \), if \( \lambda \neq 0 \) lies outside \( \overline{\Omega}_{t,\alpha} \) and \( |\lambda| \neq e^{(\alpha-1)t/2} \) then,

\[
s_t(\lambda) = \alpha \log |1 - \lambda_0|^2 + (1 - \alpha) \log |\lambda_0|^2 \\
+ \frac{1}{2} \left( \frac{\alpha}{1 - \lambda_0} + \frac{\alpha}{1 - \overline{\lambda_0}} - \frac{\alpha^2}{(1 - \lambda_0)^2} - \frac{\alpha^2}{(1 - \overline{\lambda_0})^2} - 1 \right) t + \log \left| \frac{\lambda}{\lambda_0} \right|
\]

where \( \lambda_0 = f_{t,\alpha}^{-1}(\lambda) \).
Proof. Recall from (1.1) the expression of $S$ along curves $(u, \lambda(u), x(u))$ when $x_0 = 0$: 

$$S(u, \lambda(u), x(u)) = \tau(\log(|P - \lambda_0|^2)) + \left(H_0 - \frac{1}{2}\right) u + \ln|\lambda(u)| - \ln|\lambda_0|.$$ 

By Remark 3.3, we have $\lambda_0 \in \Gamma_{t,\alpha} \cup 1/\Gamma_{t,\alpha}$, in particular $\lambda_0 \neq 1$. Besides, we have $\lambda_0 \neq 0$ since $\lambda \neq 0$. Consequently, the spectral Theorem allows to write: 

$$\tau[\log((P - \lambda_0)^2)] = \alpha \log|1 - \lambda_0|^2 + (1 - \alpha) \log|\lambda_0|^2.$$ 

On the other hand, if $x_0 = 0$ then the Hamiltonian reduces to: 

$$H_0 = K_2(1 - K_2) + \frac{K_4^2}{4} = \left(\frac{a_0p_0(0) + b_0p_0(0)}{4}(2 - a_0p_0(0) - b_0p_0(0)) + \frac{a_0p_0(0) - b_0p_0(0)}{4}\right)^2 = \left(1 - 2\hat{v}(0)(1 + 2\hat{v}(0)) + (a_0p_0(0) - b_0p_0(0))^2\right).$$ 

Using the formulas, 

$$1 - 2\hat{v}(0) = 2 - 2\alpha - 2\alpha \frac{a_0 - |\lambda_0|^2}{|\lambda_0 - 1|^2},$$ 

$$1 + 2\hat{v}(0) = 2 + 2\alpha - 2\alpha \frac{a_0 - |\lambda_0|^2}{|\lambda_0 - 1|^2},$$ 

$$\frac{a_0p_0(0) - b_0p_0(0))^2}{4} = \frac{\alpha^2k_0^2}{|\lambda_0 - 1|^4},$$ 

we end up with: 

$$H_0 - \frac{1}{2} = \frac{1}{2} \left(\frac{\alpha}{1 - \lambda_0} + \frac{\alpha}{1 - \lambda_0} - \frac{\alpha^2}{(1 - \lambda_0)^2} - \frac{\alpha^2}{(1 - \lambda_0)^2} - 1\right).$$ 

Together with Proposition 3.4 yield: for any $\lambda \in \mathbb{C} \setminus \overline{\Omega_{t,\alpha}}, \lambda \neq 0, \lambda_0 \notin \{0,1\}$ then the linear combination 

$$s_t(\lambda) = \tau[\log((P - \lambda_0)^2)] + \left(H_0 - \frac{1}{2}\right) t + \frac{1}{2} \left(\frac{\alpha}{1 - \lambda_0} + \frac{\alpha}{1 - \lambda_0} - \frac{\alpha^2}{(1 - \lambda_0)^2} - \frac{\alpha^2}{(1 - \lambda_0)^2} - 1\right) t$$ 

$$+ \log\frac{\lambda}{\lambda_0}. \quad \Box$$ 

We are now ready to prove our main result which asserts that $\Delta s_t(\lambda) = 0$ in distributional sense for $\lambda \in \mathbb{C} \setminus \overline{\Omega_{t,\alpha}} \cup \{0\}$.

Proof of Theorem 1.2. From Proposition 3.5, the function 

$$\lambda \mapsto s_t(\lambda) - (1 - \alpha) \log|\lambda_0|^2 - \alpha \log|1 - \lambda_0|^2$$ 

is the real part of a holomorphic function in $\mathbb{C} \setminus \overline{\Omega_{t,\alpha}} \cup \{0\}$, and is therefore harmonic there. Moreover, since $\lambda_0 \notin \{0,1\}$ then the linear combination 

$$\lambda \mapsto (1 - \alpha) \log|\lambda_0|^2 + \alpha \log|1 - \lambda_0|^2$$ 

is also harmonic in the same domain. Since the circle of radius $e^{(\alpha - 1)t/2}$ has (two-dimensional) zero Lebesgue measure, then the Theorem is proved. \hfill $\Box$

Remark 3.6. When $\alpha = 1$, the region $\Omega_{t,1}$ becomes the closed unit disc since the boundary of $\Sigma_{t,1}$ maps to the unit circle under $f_{t,1}$. In particular, $\Omega_{t,1}$ contains the support of $Y_t$. On the other hand, if $|\lambda| > 1 > e^{(\alpha - 1)t}$ then $\lambda_0 \in 1/\Gamma_{t,\alpha}$ and $\Delta s_t(\lambda) = 0$ in the strong sense. This is in agreement with the general theory since the spectrum of $Y_tP$ is contained in the closed unit disc.

We close the paper by the following result showing that $\partial\Omega_{t,\alpha}$ approaches the circle $T(0, \sqrt{\alpha})$ when $t$ approaches infinity. This in agreement with Haagerup and Larsen result on the Brown measure of $PUP$.

Proposition 3.7. For fixed $t > 0$ and $\alpha \in (0, 1)$, let $z_{t,\alpha}$ denote a boundary point of $\Sigma_{t,\alpha}$. Then, 

$$\lim_{t \to \infty} |f_{t,\alpha}(z_{t,\alpha})| = \sqrt{\alpha}.$$
Figure 5. The region $\Omega_{t,\alpha}$ for $(t,\alpha) = (8, 0.25)$ with the circle $C(0, 0.5)$ (dashed) indicated for comparison.

Proof. By definition, $z_{t,\alpha} \neq 1$ and satisfies

$$t = T_\alpha(z_{t,\alpha}) = \frac{|1 - z_{t,\alpha}|^2}{\alpha(|z_{t,\alpha}|^2 - 1) + (1 - \alpha)|z_{t,\alpha}|^2} \log \left( \frac{\alpha|z_{t,\alpha}|^2 + (1 - \alpha)|1 - z_{t,\alpha}|^2}{\alpha} \right).$$

But, since for all $x > 0$

$$\log(x) \leq x - 1,$$

it follows that,

$$\alpha t \leq |1 - z_{t,\alpha}|^2 \leq 1 + |z_{t,\alpha}|^2.$$

Hence, for $t$ approaching infinity we have

$$|1 - z_{t,\alpha}|^2 \sim |z_{t,\alpha}|^2 \to \infty,$$

and therefore, we obtain

$$|f_{t,\alpha}(z_{t,\alpha})|^2 = \frac{\alpha|z_{t,\alpha}|^2}{\alpha|z_{t,\alpha}|^2 + (1 - \alpha)|1 - z_{t,\alpha}|^2} \sim \alpha.$$

□

References

[1] P. Biane. Free Brownian motion, free stochastic calculus and random matrices. Free probability theory (Waterloo, ON, 1995), 1-19, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, 1997.

[2] P. Biane. Segal-Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems, J. Funct. Anal., 144 (1997), 232-286.

[3] P. Biane, F. Lehner. Computation of some examples of Brown’s spectral measure in free probability. Colloq. Math. 90 (2001), no. 2, 181-211.

[4] L.G. Brown. Lidskii’s theorem in the type II case. H. Araki, E. Effros (Eds.), Geometric Methods in Operator Algebras, Pitman Res. Notes Math. Ser., vol. 123, Kyoto, 1983, Longman Sci. Tech. (1986), 1-35.

[5] B. Collins. Product of random projections, Jacobi ensembles and universality problems arising from free probability. Probab. Theory Related Fields. 133, (2005), no. 3, 315-344.

[6] Bruce K. Driver, Brian C. Hall and Todd Kemp. The Brown measure of the free multiplicative Brownian motion. arXiv:1903.11015.

[7] L. C. Evans. Partial differential equations. Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010. xxii+749 pp.

[8] B. Fuglede, R. V. Kadison. Determinant theory in finite factors. Ann. of Math. (2), 55 (1952), 520-530.

[9] U. Haagerup, F. Larsen. Brown’s spectral distribution measure for $R$-diagonal elements in finite von Neumann algebras. J. Funct. Anal. 176 (2000), no. 2, 331-367.

[10] T. Hamdi. Spectral distribution of the free Jacobi process, revisited. Anal. PDE. 11, (2018), no. 8, 2137-2148.

[11] C. Ho, P. Zhong. Brown Measures of Free Circular and Multiplicative Brownian Motions with Probabilistic Initial Point. Available on arXiv.

[12] J. A. Mingo, R. Speicher. Free probability and random matrices. Fields Institute Monographs, 35. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, 2017.
[13] A. Nica, R. Speicher. Lectures on the Combinatorics of Free Probability. London Mathematical Society Lecture Note Series, vol. 335, 2006.

[14] G. I. Olshanski. Unitary representations of infinite-dimensional pairs (G,K) and the formalism of R. Howe. Representation of Lie groups and related topics, 269-463, Adv. Stud. Contemp. Math., 7, Gordon and Breach, New York, 1990.

[15] D. Petz, J. Réffy. Large deviation for the empirical eigenvalue density of truncated Haar unitary matrices. Probab. Theory Related Fields. 133, (2005), no. 2, 175-189.

[16] Rains, E. M. Combinatorial properties of Brownian motion on the compact classical groups. J. Theoret. Probab. 10 (1997), 659-679.

[17] K. Zyczkowski, H. J. Sommers. Truncations of random unitary matrices. J. Phys. A 33, (2000), no. 10, 2045-2057.

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