Fisher information of a single qubit interacts with a spin-qubit in the presence of a magnetic field

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Abstract: In this contribution, quantum Fisher information is utilized to estimate the parameters of a central qubit interacting with a single-spin qubit. The effect of the longitudinal, transverse and the rotating strengths of the magnetic field on the estimation degree is discussed. It is shown that, in the resonance case, the number of peaks and consequently the size of the estimation regions increase as the rotating magnetic field strength increases. The precision estimation of the central qubit parameters depends on the initial state settings of the central and the spin-qubit, either encode classical or quantum information. It is displayed that, the upper bounds of the estimation degree are large if the two qubits encode classical information. In the non-resonance case, the estimation degree depends on which of the longitudinal/transverse strength is larger. The coupling constant between the central qubit and the spin-qubit has a different effect on the estimation degree of the weight and the phase parameters, where the possibility of estimating the weight parameter decreases as the coupling constant increases, while it increases for the phase parameter.

For large number of spin-particles, namely, we have a spin-bath particles, the upper bounds of the Fisher information with respect to the weight parameter of the central qubit decreases as the number of the spin particle increases. As the interaction time increases, the upper bounds appear at different initial values of the weight parameter.

1 Introduction

Fisher information plays an important role in the context of quantum metrology [1] and quantum information processing [2, 3, 4, 5, 6]. Quantum Fisher information (QFI) quantifies the information that can be elicited about a parameter. In other words, QFI is used as an estimation tool of parameters that contained in the quantum system during its evolution [7]. Due to its importance, there are some efforts that has been done to quantify QFI in different quantum systems.

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Recently, quantifying quantum Fisher information in open quantum systems has paid attentions by some authors. For example, Zheng et. al [8] investigated the dynamics of QFI for a two-qubit system, where each qubit interacts with its own Markovian environment. Ozaydin [9] has quantified the QFI analytically for the W-state in the presence of different noisy channels. The effect of the Markovian reservoirs on the dynamics of the quantum Fisher information of a two-level system is discussed by G.-You et.al [10]. Quantum Fisher information for a noisy open quantum system and initially prepared in a steady state is quantified by Altinats [11].

The central-spin system represents one of the most important models of decoherence[12]. Rao [13] discussed the dynamics of one and two-qubit systems interacting with a spin-qubit. The coherent and the non-coherent properties of a single and a maximum entangled two-qubit systems interact with a spin-qubit are discussed by Metwally et. al [14]. The dynamics of quantum Fisher information for a spin-boson model is investigated by Hao et. al [15]. The dynamics of the quantum Fisher information for a qubit system and initially prepared in a coherent spin-squeezing state is investigated by Zhong et. al. [16].

In the present work, we quantify the quantum Fisher information of a single central qubit interacting with a single spin-qubit in the presence of a magnetic field. The QFI is utilized to estimate the weight and the phase parameters which describe the initial state of the central qubit. The paper is organized as follows. In Sec.2, we introduce the suggested model, where an analytical solution in terms of the Bloch vector is given. A brief description of quantum Fisher information is given in Sec.3. The effect of longitudinal and transverse strengths of the magnetic field on the estimation degree of the weight parameter is given in Sec.3. In Sec.4, the behavior of the quantum Fisher information with respect to the phase parameter is described. In Sec. 5, we investigate the behavior of the Fisher information with respect to the weight parameter for large numbers of spin-bath particles. Finally, we draw our conclusion in Sec.5.

2 Presentation of the model

The Hamiltonian operator which describes the interaction of a central single 2-level qubit with upper and ground states \( |±\rangle \) with a single- spin \( \frac{1}{2} \) particle is given by [13],

\[
\hat{H} = \hbar \omega_0 \hat{\sigma}_z + \hbar \omega_1 (\hat{S}_+ e^{-i\omega t} + \hat{S}_- e^{i\omega t}) + \hat{S}_z g, \tag{1}
\]
where the spin operators $\hat{S}_{\pm,z}$ satisfy $[\hat{S}_z, \hat{S}_\pm] = \pm \hat{S}_\pm$, $[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z$. The parameters $\omega_0$ and $\omega_1$, represent the longitudinal and the transverse components of the magnetic field, respectively, $\omega_1$ is the rotating magnetic field strength and $g$ is the coupling constant between the spin-qubit and the central qubit. In this treatment, it is assumed that, the initial state of the system is described by the product state $\rho_s(0) = \rho_q(0) \otimes \rho_b(0)$, where $\rho_q(0) = |\psi_q\rangle \langle \psi_q|$ and $\rho_b(0) = |\chi_b\rangle \langle \chi_b|$ represent the states of the central qubit and the spin-qubit, respectively. In the computational basis $\{|0\rangle, |1\rangle\}$, the states $|\psi_q\rangle$ and $|\chi_b\rangle$ may be written as,

$$
|\psi_q(0)\rangle = \cos \theta_1 |0\rangle + \sin \theta_1 e^{-i\phi_1} |1\rangle, \\
|\chi_b(0)\rangle = \cos \theta_2 |0\rangle + \sin \theta_2 e^{-i\phi_2} |1\rangle,
$$

(2)

where $(\theta_1, \phi_1)$ and $(\theta_2, \phi_2)$ are the weight and the phase angles of the central and the spin-qubits, respectively. The Bloch vectors of these initial states are given by,

$$
s_{x_i}(0) = \sin \theta_i \cos \phi_i, \quad s_{y_i}(0) = \sin \theta_i \sin \phi_i, \quad s_{z_i}(0) = \cos \theta_i, \quad i = 1, 2.
$$

(3)

At any $t > 0$, the final state of the total system is given by,

$$
|\psi_{gb}(t)\rangle = \mu(t) |\psi_{gb}(0)\rangle, \quad \mu(t) = e^{-i\hat{H}t}.
$$

(4)

In the computational basis set, $\{00, 01, 10, 11\}$, the unitary operator $\mu(t)$ may be described by the following $4 \times 4$ matrix,

$$
\mu(t) = \\
\begin{pmatrix}
\mu_{11} & 0 & \mu_{13} & 0 \\
0 & \mu_{22} & 0 & \mu_{24} \\
\mu_{31} & 0 & \mu_{33} & 0 \\
0 & \mu_{42} & 0 & \mu_{44}
\end{pmatrix},
$$

(5)

where

$$
\begin{align*}
\mu_{11} &= \frac{2i\omega_1}{\gamma_2} e^{i\omega t/2} \sin \gamma_2 t, \\
\mu_{13} &= e^{-i\omega t/2} (\cos \gamma_2 t - 2i \frac{\Delta}{\gamma_2} \sin \gamma_1 t), \\
\mu_{22} &= e^{-i\omega t/2} (\cos \gamma_1 t - 2i \frac{\Delta}{\gamma_1} \sin \gamma_1 t), \\
\mu_{24} &= \frac{2i\omega_1}{\gamma_1} e^{-i\omega t/2} \sin \gamma_1 t, \\
\mu_{31} &= \frac{2i\omega_1}{\gamma_2} e^{i\omega t/2} \sin \gamma_2 t, \\
\mu_{33} &= e^{i\omega t/2} (\cos \gamma_2 t + 2i \frac{\Delta}{\gamma_2} \sin \gamma_1 t).
\end{align*}
$$
\[ \mu_{42} = \frac{2i\omega_1}{\gamma_1} e^{i\omega t/2} \sin \gamma_1 t, \quad \mu_{44} = e^{i\omega t/2} (\cos \frac{2\Delta}{\gamma_1} \sin \gamma_1 t), \] (6)

and \( \Delta_{\pm} = \Delta \pm \frac{\Delta_0}{2}, \gamma_1 = \sqrt{\omega_1^2 + \Delta_+^2}, \gamma_2 = \sqrt{\omega_1^2 + \Delta_-^2} \) and \( \Delta = \omega - \omega_0 \) is the detuning parameter.

By using Eqs.(2-6), the state (4) may be written as,

\[ |\psi_{qb}(t)\rangle = B_1 |00\rangle + B_2 |01\rangle + B_3 |10\rangle + B_4 |11\rangle, \] (7)

where,

\[ B_1 = \mu_{11}c_1c_2 + \mu_{13}s_1c_2 e^{-i\phi_1}, \quad B_2 = \mu_{22}s_1s_2 e^{-i\phi_2} + \mu_{24}s_1s_2 e^{-i\phi_{12}}, \]
\[ B_3 = \mu_{31}c_1c_2 + \mu_{33}s_1c_2 e^{-i\phi_1}, \quad B_4 = \mu_{42}s_1s_2 e^{-i\phi_2} + \mu_{44}s_1s_2 e^{-i\phi_{12}}, \] (8)

and \( c_i = \cos \theta_i, s_i = \sin \theta_i, i = 1, 2 \) and \( \phi_{12} = \phi_1 + \phi_2. \) Since we are interested in investigating the dynamics of the Fisher information with respect to the central qubit, we trace out the state of the spin- qubit, where \( \rho_q(t) = tr_b\{\rho_s(t)\} \) and \( \rho_q(t) = |\psi_{qb}(t)\rangle \langle \psi_{qb}(t)|. \)

In the Bloch vector representation, the state \( \rho_q(t) \) can be written as,

\[ \rho_q(t) = \frac{1}{2} (1 + s_{xq}(t)\sigma_x + s_{yq}(t)\sigma_y + s_{zq}(t)\sigma_z), \] (9)

where

\[ s_{xq}(t) = B_1 B_3^* + B_3 B_1^* + B_2 B_4^* + B_4 B_2^*, \]
\[ s_{yq}(t) = i(B_3 B_1^* + B_2 B_4^* - B_1 B_3^* - B_4 B_2^*), \]
\[ s_{zq}(t) = |B_1|^2 + |B_2|^2 - |B_3|^2 - |B_4|^2, \] (10)

where \( B_i, i = 1...4 \) are given by (8).

3 Quantum Fisher Information

In the following subsection, we review the mathematical form of the quantum Fisher information for a single qubit in the Bloch vector representations. Moreover, we quantify numerically the QFI corresponding to the weight and the phase parameters of the central qubit.
3.1 Mathematical form

The density operator of a single qubit is given by,
\[ \rho = \frac{1}{2}(I + \sum_{i} s_i \sigma_i), \]  
(11)

where \( I \) is a unit matrix of size \( 2 \times 2 \), \( \vec{s} = (s_1, s_2, s_3) \) is the Bloch vector and \( \sigma_i, i = x, y, z \) are the Pauli matrices. In terms of the Bloch vector, the QFI with respect to a parameter \( \eta \) is defined as \[ 17, 18 \],
\[ F_{\eta} = \left\{ \begin{array}{ll} \frac{1}{1 - |\vec{s}(\eta)|} \left[ \vec{s}(\eta) : \frac{\partial \vec{s}(\eta)}{\partial \eta} \right] + \left( \frac{\partial \vec{s}(\eta)}{\partial \eta} \right)^2, & \text{for mixed state, } |\vec{s}(\eta)| < 1, \\
\left| \frac{\partial \vec{s}(\eta)}{\partial \eta} \right|^2, & \text{for pure state, } |\vec{s}(\eta)| = 1, \end{array} \]  
(12)

where \( \eta \) is the parameter to be estimated. In the following subsections, we shall estimate the weight and the phase parameters of the central qubit, where it is assumed that the spin qubit is polarized in \( z \)-direction, namely, \( \rho_b = \frac{1}{2}(1 + \sigma_z) \). For this aim, we calculate the quantum Fisher information, \( F_{\eta} \) corresponding to the weight and the phase parameters, i.e., \( \eta = \theta_1, \phi_1 \).

3.2 Numerical results

We assume that, the phase angle \( \phi_1 = \pi \), namely, the initial state of the central qubit is prepared in the state \( |\psi_q\rangle = \cos \theta_1 |0\rangle - \sin \theta_1 |1\rangle \), while the spin-qubit is prepared in the state \( |\chi(0)\rangle = -|0\rangle \), namely \( \theta_2 = \pi \) and the phase \( \phi_2 \) is arbitrary.

Fig.(1) describes the dynamics of the quantum Fisher information \( F_{\theta_1} \) with respect to the weight parameter \( (\theta_1) \). The effect of the rotating magnetic field strength \( (\omega_1) \) on the quantum Fisher information \( F_{\theta_1} \) is investigated in the resonance case \( (\Delta = 0) \) and for a fixed value of the coupling constant between the central and spin qubits \( (g = 0.5) \). The general behavior shows that, the maximum values of Fisher information depend on the initial weight angle \( (\theta_1) \) and the interaction time. As it is displayed in Figs.(1a) and (1b), the Fisher information \( F_{\theta_1} \) increases gradually to reach its maximum value at \( t = 0.7 \). For further values of the interaction time, \( F_{\theta_1} \) decreases gradually. The effect of the initial weight settings shows that, Fisher information decreases gradually to its...
minimum value at $\theta_1 = \pi/2$, namely the initial state of the central qubit is papered in the state $|\psi_q(0)\rangle = |1\rangle$. However, the behavior changes for $\theta_1 \in [\pi/2, \pi]$, where the Fisher information increases to reach its maximum bounds at $\theta_1 = \pi$.

The effect of larger values of the rotating magnetic field strength $\omega_1$ is depicted in

Figure 1: Fisher information for resonance case, namely, $\Delta = \omega - \omega_0 = 0$, where $g = 0.5$, $\phi = \pi$ and $\omega_1 = 0.1, 0.5, 7$, for (a,b),(c,d) and (e,f), respectively.
Fig.s(1c-1f), where the number of peaks increases as $\omega_1$ increases and the maximum bounds appear at larger values of $\theta_1$. On the other hand, the larger values of $\omega_1$ has no effect on the upper bounds of $F_{\theta_1}$. Moreover, as the strength of the rotating magnetic field strength ($\omega_1$) increases, the quantum Fisher information increases suddenly to reach its maximum values. The contour description (Figs.1(b,d,f)) shows that, the size of bright regions in which one may estimate the weight parameter, decreases as the initial weight parameter $\theta_1$ increases. As the rotating field strength ($\omega_1$) increases, the bright regions are shifted as $\theta_1$ increases. The number of peaks increases at the expense of the bright size regions.

From Fig.(1), one may conclude that, at small values of the rotating field strength, ($\omega_1$) Fisher information has periodic time behavior, while at larger values of $\omega_1$, the phenomena of the sudden changes are displayed. The number of peaks and consequently the numbers of estimation regions increase as the rotating field strength increases.

![Figure 2](image_url)

Figure 2: For the non-resonance case where $\Delta = -0.08, 0.08$ namely we set $\omega_0 = 0.01, \omega = 0.09$ in (a,b) and $\omega_0 = 0.09, \omega = 0.01$ in (c,d), where $\omega = 0.1, g = 0.5, \phi = \pi$. 
The behavior of Fisher information, $({\mathcal F}_{\theta_1})$ in the non-resonance case is shown in Fig.(2), where we fix the strength of the rotating field and the coupling constant. Two different cases are considered; in the first case, it is assumed that, the longitudinal component of the field ($\omega_0$) is smaller than the transverse component ($\omega$), while for the second case we assume that, ($\omega_0 > \omega$). Our results display that, for the first case ($\omega_0 < \omega$) the upper bounds of the Fisher information are smaller than those depicted for the second case ($\omega_0 > \omega$). The minimum values of the Fisher information $({\mathcal F}_{\theta_1})$ are observed in the interval $\theta_1 \in [\pi/2, 3\pi/4]$, where the longitudinal component is larger than the transverse component of the field, while it is maximum for the second case, namely ($\omega_0 > \omega$). The interaction time plays an important role in both cases, where $({\mathcal F}_{\theta_1})$ increases as soon as the interaction is switched on, while it takes longer time to increase for the second case. i.e., ($\omega > \omega_0$).

Figure 3: The effect of different phases of the initial state of the central qubit on the Fisher information in the resonance case where $\omega_0 = \omega_1 = 0.1$ and $\omega = 0.1, g = 0.5$ and $\phi = \pi/4, \pi/2$. 
Figure 4: The same as Fig1.(a,b) but for different values of the coupling constant, where we set \( g = 0.3, 0.7 \) for (a,b) and (c,d), respectively.

Fig.(3) displays the behavior of the Fisher information \( F_{\theta_1} \) for different phases, where two values of \( \phi_1 \) are considered and have used the same values of the parameters as in Fig.1(a,b). It is shown that, the upper bounds of \( F_{\theta_1} \) increase as the phase angle increases. Moreover, these upper bounds of the Fisher information are reached at small values of the interaction time with a large phase angle. The size of estimation areas of the weight parameter increases at the expense of the precision degree of estimation.

From Fig.(3), one may conclude that, the possibility of estimating the weight parameter \( (\theta_1) \) increases if the initial central qubit encodes only classical information, while this possibility decreases if it encodes quantum information.

Fig.(4) displays the effect of different values of the coupling constant, \((g)\) on the behavior of the Fisher information \( F_{\theta_1} \), and consequently on the estimation degree of the weight parameter, \( \theta_1 \). It is clear that, at small values of the coupling constant \((g)\), the maximum values of \( F_{\theta_1} \) are larger than those displayed at larger values of \( g \). The interac-
tion time plays an important role on the Fisher information’s behavior, where, at small values of the coupling constant, $F_{\theta_1} \simeq 0$ for $t \in [0, 2]$, then it increases for further values of interaction time $t$. For larger values of the coupling constant, the maximum peaks of the Fisher information are shifted, and consequently the estimation areas are shifted at $\theta_1 = 0$ and $\pi$. The size of the estimation areas increase at smaller values of $g$, while the number of maximum peaks increases at larger values of the coupling constant.

From Fig.(4), it is clear that, the estimation degree of the weight parameter may be maximized at smaller values of the coupling constant and the initial weight parameter, $\theta_1$ of the central qubit is chosen in the interval $\theta_1 \in [\pi/4, 3\pi/4]$ i.e, the central qubit codes quantum information. The other strategy, the estimation degree may be maximized at larger values of the coupling constant and the initial weight $\theta_1 \in \left\{ [0, \pi/4] \text{ or } [3\pi/4, \pi] \right\}$.

Fig.(5), displays the effect of different initial state settings of the spin-qubit on the quantum Fisher information with respect to the weight parameter ($\theta_1$) of the central qubit, where different values of the weight parameter ($\theta_2$) are considered, while the phase parameter is fixed, namely, $\phi_2 = \pi$. The general behavior shows that, the upper bounds of $F_{\theta_1}$ decreases as $\theta_2$ decreases. Also, as soon as the interaction is switched on, the Fisher information increases gradually to reach its maximum value. For further time $F_{\theta_1}$ decreases to reach its minimum values. This behavior is repeated as the interaction time increases to appear another peak. The areas in which one can estimate the weight parameter with high precision increase as $\theta_2$ decreases.

From Fig.5, it is clear that if the spin-qubit encodes classical information i.e., ($|\psi_b\rangle = -i|1\rangle$), the possibility of estimating the weight parameter is larger than that depicted if it encodes quantum information, where we set $|\psi_q\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$, $|\psi_q\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{i}{2}|1\rangle$ in Figs.(5c,cd) and Figs.(5e,5f), respectively. Moreover, if the spin-qubit encodes quantum information, the upper bounds depend on the weight of the superposition of $|0\rangle$ and $|1\rangle$.

4 Estimation the phase parameter

Fig.(6) shows the behavior of the Fisher information ($F_{\phi_1}$) in the resonance case with respect to the phase parameter $\phi_1$. Two different initial state settings are considered, where we set the weight parameter $\theta_1 = \pi/2, \pi/4$, namely, the initial state of the central qubit is prepared in the state, $|\psi_q\rangle = e^{-i\phi_1}|1\rangle$ and $|\psi_q\rangle = \frac{1}{\sqrt{2}}(|0\rangle + e^{-i\phi_1}|1\rangle)$, respectively. The results show that, the behavior of the Fisher information is similar for the two different initial states, but only the upper bounds at $\theta_1 = \pi/2$ are larger than those displayed at $\theta_1 = \pi/4$. In the non-resonance case, the behavior depends on which component is larger.
Figure 5: Fisher information $F_{\theta_1}$ against for different initial state settings of the spin qubit where (a,b) $\theta_2 = \pi/2$, (c,d) $\theta_2 = \pi/3$ and (e,f) $\theta_2 = \pi/4$, (g,h) $\theta_2 = \pi/6$, while $\phi_2 = \pi, g = 0.1$ and $\omega_1 = 0.5$.

However, if the longitudinal component, $\omega_0$ is larger than the transverse component $\omega$, then the upper bounds are smaller than those displayed for $\omega > \omega_0$. Moreover, the upper and lower bounds of $F_{\phi_1}$ appear at different values of interaction time. These results
Figure 6: Fisher information $\mathcal{F}_{\phi_1}$ against the interaction time for resonant case and $\omega_1 = g = 0.5$ where (a) at $\theta_1 = \pi/2$ and (b) $\theta_1 = \pi/4$.

Figure 7: Fisher information $\mathcal{F}_{\phi_1}$ against the interaction time for the non-resonant case with $\omega_1 = g = 0.5$ where (a) $\omega_0 = 0.09$ and $\omega_1 = 0.01$ and (b) $\omega_0 = 0.01$ and $\omega_1 = 0.09$. 
are displayed in Fig.(7) where it is assumed that, the initial state of the central qubit is prepared by setting $\theta_1 = \pi/2$.

The effect of the coupling constant $g$ is displayed in Fig.(8), where we consider the non-resonance case and a fixed value of the rotating field component. It is clear that, at smaller values of the coupling constant the upper bounds of the Fisher information are smaller than those displayed at large coupling constant. On the other hand, the estimated areas are larger for smaller values of the coupling constant. The estimating area for larger values of the coupling constant appears at smaller values of the interaction time, while it takes longer time to appear at small values of the coupling constant, $g$.

From Figs.(6-8), one may conclude that, for the resonance case the precision of estimating the phase parameter depends on the initial state settings, where if the central qubit is prepared in a state encodes classical information, the possibility of the estimation degree of the phase parameter is larger than that depicted if the central qubit encodes quantum information. For the non-resonance case, the estimation degree depends on which component (longitudinal/ transverse) of the field is larger. Larger values of the coupling constant increase the size of the estimation area, in which one may estimate the phase parameter with high degree of estimation.

Figs.(9), display the behavior of $\mathcal{F}_{\phi_1}$ for different initial states of the spin-qubit. It is clear that, for larger values of $\theta_2$ the upper bounds are larger than those displayed at smaller values of $\theta_2$. It is clear that, $\mathcal{F}_{\phi_1}$ increases as $\theta_1$ increases to reach its maximum values at $\theta_2 = \frac{\pi}{4}$. On the other hand, the minimum values of $\mathcal{F}_{\phi_1}$ are depicted at $\theta_1 = \frac{\pi}{2}$. Moreover, the number of peaks and the estimated areas increases as the weight parameter of the spin-qubit increaser.
Figure 9: Fisher information $\mathcal{F}_{\phi_1}$ against the interaction time at $\Delta = 0$ and $\omega_1 = 0.1, g = 0.5, \theta_1 = \frac{\pi}{2}, \phi_2 = \pi$, where (a,b) $\theta_2 = \frac{\pi}{2}$ (c,d) $\theta_2 = \frac{\pi}{4}$ (e,f) $\theta_2 = \frac{\pi}{6}$.

From Figs. (5) and (9), it is clear that the initial state settings of the spin-qubit has a clear effect on the precision degree of estimating the weight and the phase parameters of the central qubit.
5 Spin Bath problem

In this context, it is important to mention that in the previous sections, we consider one central qubit interacting with a minimum dimension of the bath, namely we consider a single spin-qubit, which is polarized in $z-$ direction. In this case, the state of the spin particle is defined by $\rho_s = \frac{1}{2} (1 \pm \tau_z)$. If we have a bath of large number of particles polarized in $z$-direction, then their initial state is defined by $\rho_s(N) = \frac{1}{2^N} (I_{2 \times 2} \pm \tau_z)^{\otimes N}$, which is more complicated to solve. Therefore, to simplify the calculations, we assume that the $N$ spin particles are unpolarized. Under this condition, the states of the $N$-bath unpolarized particles reduces to be $\rho_s(N) = \frac{1}{2^N} I_{2 \times 2}$. However, a similar calculation may be done, where the detuning parameter is affected to be $-\frac{1}{2} gN \leq \Delta \leq \frac{1}{2} gN$ and consequently $\Delta_{\pm} = \Delta \pm \frac{gN}{2}$, where it is assumed that $g_1 = g_2 = \ldots = g_N = g$.

In Fig.(10), we consider the central qubit interacts with larger number of unpolarized particles.
spin-bath particles. The general behavior shows that, the upper bounds decrease as the numbers of spin-bath particle increase, where we consider that \( N = 5, 7 \) particles in Figs.(10a,10b) and (10c,10d,) respectively. As the initial weight parameter increases, \( F_{\theta_1} \) increases to reach its maximum value at \( \theta_1 = \pi/2 \), then decays gradually to reach its minimum value at \( \theta_1 = \pi \). Moreover, the upper bounds are shifted as the interaction time increases to appear at a different value of the initial weight. These results are clearly described on Figs.(10a-10d).

6 Conclusion

In this contribution, Fisher information is used as an estimator of the weight and the phase parameters of a central qubit interacts with a single-qubit in the presence of a magnetic field. The effect of the magnetic field parameters on the degree of estimating the weight and phase parameters is investigated. For resonance case, the possibility of estimating the weight parameter may be achieved with high degree of precision at large values of the rotating field strength. Also, the size of the areas, in which one may estimate the weight parameter, decreases at the expense of their numbers, where the numbers of these areas increase as the rotating field strength increases. For non-resonance case, the upper bounds depend on the strength of the field longitudinal/transverse parameters. It is clear that, the upper bounds are larger if the transverse strength is stronger than the longitudinal strength. The possibility of estimating the weight parameter increases if it encodes only classical information. The coupling constant plays an important role in controlling the estimation precision, where estimation degree is large at smaller values of the coupling constant. On the other hand, a larger interaction time is required in order to estimate the weight parameter. Further, a shorter time of interaction is required if the coupling constant is larger.

The behavior of Fisher information with respect to the phase parameter is discussed for different values of the initial state setting of the central qubit. It is shown that, for the resonance case, the possibility of estimating the phase parameter increases if the initial state of the central qubit encodes quantum information. For the non-resonance case, the estimation degree of the phase parameter depends on the components of the magnetic field. For large values of the coupling constant, the upper bounds of estimation degree of the phase parameter, increases at the expense of the size of the estimation areas.

The effect of different initial state settings of the spin-qubit on the estimation degree of the weight and the phase parameters is discussed. It is shown that, Fisher information
corresponding to these two parameters decreases as the weight parameter of the spin-qubit decreases.

The Fisher information with respect to the weight parameter of the central qubit interacts with unpolarized large number of spin-bath particles is discussed. We show that, the upper pounds of the Fisher information decreases as the bath-spin numbers increase. As the interaction time increases, the upper bounds appear at larger values of initial weight of the central qubit.

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References

[1] A. S. Holevo, ”Probabilistic and statistical Aspects of Quantum Theory”, North-Holland, Amsterdam (1982).

[2] Z. Sun, J. Ma, X. M. Lu and X. G. Wang, phys Rev. A82 022306 (2010).

[3] X.-Ming Lu, X. Wang, and C. P. Sun, ”Quantum Fisher information flow and non-Markovian processes of open system”, Phys Rev. A82042103 (2010).

[4] Nan Li and S. Luo,”Entanglement detection via quantum Fisher information”, Phys, Rev. A 88 014301 (2013).

[5] N. Metwally,”Estimation of teleported and gained parameters in a non-inertial frame”, Laser Phys. Lett. 14 045202 (2017).

[6] N. Metwally and S. S. Hassan,”Estimation of pulsed driven qubit parameters via quantum Fisher information”, Laser Phys. Lett. 14 115204 (2017).

[7] J. Ma, Yi-X. Huang, X. Wang and C. P. Sun, ”Quantum Fisher information of the Greenberger-Horne-Zeilinger state in decoherence channels”, Phys. Rev. A. 84 022302 (2011).

[8] Q. Zheng, Y. Uao and Y. Li,”Optimal quantumchannel estimation of two interacting qubit subject to decoherence” Eur Phys. J. D. 68 170 (2014).
[9] F. Ozaydin,” Quantum Fisher Information of W States in Decoherence Channels”, Phys. Lett. A 378, 3161 (2014).

[10] W. G.-You, G. Y.-Neng and Z. Ke.” Dynamics of quantum Fisher information in a two-level system coupled to multiplied bosonic reservoirs” Chin Phys. B 24 114201 (2015).

[11] A. A. Altinats,”Quantum Fisher information of an open and noisy system in the steady state”, Annals of Physics, 376 192 (2016).

[12] H.-P. Breuer, D. Burgarth, and F. Petruccione” Non-Markovian dynamics in a spin star system: Exact solution and approximation techniques”, Phys. Rev. B 70, 045323 (2004).

[13] D. B. Rao, ” Controlled dynamics of qubits in the presence of decoherence”, Phys. Rev. A 76 042312 (2007).

[14] N. Metwally, M. Abdel-Aty and A.-S. Obada,” Coherent and incoherent behaviors of qubits interacting with a spin-bath particle”, Int. J. Mod. Phys. B 27 1350076 (2013).

[15] X. Hao, N.-H. Tong, and S. Zhu,” Dynamics of Quantum Fisher information in a spin-boson model”, J. Phys. A : Math. Theor. 46, 355302(2013).

[16] W. Zhong, J. Liu, J. Ma and X. Wang, ”Quantum Fisher information and spin squeezing in one-axis twisting model”, Chin. Phys. B, 23 060302 (2014).

[17] W. Zhong, Z. Sun, J. Ma, X. Wang, and F. Nori,” Fisher information under decoherence in Bloch representation”, Phys. Rev. A 87, 022337 (2013).

[18] X. Xiao, Y. Yao, W.-J. Zhong, Y.-Ling and Y.-Mao Xie,” Enhancing teleportation of quantum Fisher information by measurements”, Phys. Rev. A 93012307 (2016).