New integral inequalities via $P$-convexity

Wenjun Liu

Abstract. In this note we extend some new estimates of the integral $\int_a^b (x-a)^p(b-x)^q f(x) \, dx$ for functions when a power of the absolute value is $P-$convex.

Mathematics Subject Classification (2010). Primary 26D15; Secondary 33B15, 26D07.

Keywords. Hermite’s inequality, Hölder’s inequality, integral inequality, $P$-convexity.

1. INTRODUCTION

Let $I$ be an interval in $\mathbb{R}$. Then $f : I \rightarrow \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

The notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \rightarrow \mathbb{R}$ is said to be quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}$$

holds for any $x, y \in [a, b]$ and $t \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [8]).

The generalized quadrature formula of Gauss-Jacobi type has the form

$$\int_a^b (x-a)^p(b-x)^q f(x) \, dx = \sum_{k=0}^m B_{m,k} f(\gamma_k) + R_m[f] \quad (1.1)$$

for certain $B_{m,k}, \gamma_k$ and rest term $R_m[f]$ (see [15]).

This work was partly supported by the National Natural Science Foundation of China (Grant No. 40975002) and the Natural Science Foundation of the Jiangsu Higher Education Institutions (Grant No. 09KJB110005).
In [10], Özdemir et al. established several integral inequalities concerning the left-hand side of (1.1) via some kinds of convexity. Especially, they discussed the following result connecting with quasi-convex function:

**Theorem 1.1.** Let \( f: [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]) \), \( 0 \leq a < b < \infty \). If \( f \) is quasi-convex on \([a, b]\), then for some fixed \( p, q > 0 \), we have

\[
\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1} \beta(p+1, q+1) \max\{f(a), f(b)\},
\]

where \( \beta(x,y) \) is the Euler Beta function.

Recently, Liu [9] established some new integral inequalities for quasi-convex functions as follows:

**Theorem 1.2.** Let \( f: [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]) \), \( 0 \leq a < b < \infty \) and let \( k > 1 \). If \( |f|^k \) is quasi-convex on \([a, b]\), for some fixed \( p, q > 0 \), then

\[
\int_a^b (x-a)^p(b-x)^q f(x)dx \\
\leq (b-a)^{p+q+1} \left[ \beta(kp+1, kq+1) \right]^{1/k} \left( \max\{|f(a)|^k, |f(b)|^k\} \right)^{1/k}.
\]

**Theorem 1.3.** Let \( f: [a, b] \to \mathbb{R} \) be continuous on \([a, b]\) such that \( f \in L([a, b]) \), \( 0 \leq a < b < \infty \) and let \( l \geq 1 \). If \( |f|^l \) is quasi-convex on \([a, b]\), for some fixed \( p, q > 0 \), then

\[
\int_a^b (x-a)^p(b-x)^q f(x)dx \\
\leq (b-a)^{p+q+1} \beta(p+1, q+1) \left( \max\{|f(a)|^l, |f(b)|^l\} \right)^{1/l}.
\]

On the other hand, Dragomir et al. in [6] defined the following class of functions.

**Definition 1.** Let \( I \subseteq \mathbb{R} \) be an interval. The function \( f: \to \mathbb{R} \) is said to belong to the class \( P(I) \) (or to be \( P \)-convex) if it is nonnegative and, for all \( x, y \in I \) and \( t \in [0, 1] \), satisfies the inequality

\[
f(tx + (1-t)y) \leq f(x) + f(y).
\]

Note that \( P(I) \) contain all nonnegative convex and quasiconvex functions. Since then numerous articles have appeared in the literature reflecting further applications in this category; see [1, 2, 3, 4, 5, 7, 11, 12, 13, 14, 16] and references therein.

The main purpose of this note is to establish some new estimates of the integral \( \int_a^b (x-a)^p(b-x)^q f(x)dx \) for functions when a power of the absolute value is \( P \)-convex. That is, this study is a further continuation and generalization of [10] via \( P \)-convexity.
2. New integral inequalities via $P$-convexity

In this section we generalize Theorems 1.1–1.3 with a $P$-convex function setting. For this purpose, we need the following lemma (see [10, Lemma 2.1]):

**Lemma 2.1.** Let $f : [a, b] \subset [0, \infty) \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b]), a < b$. Then the equality
\[
\int_a^b (x-a)^p(b-x)^q f(x)dx = (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q f(ta + (1-t)b)dt
\] (2.1)
holds for some fixed $p, q > 0$.

The next theorem gives a new result for $P$-convex functions.

**Theorem 2.1.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b]), 0 \leq a < b < \infty$. If $|f|$ is $P$-convex on $[a, b]$, for some fixed $p, q > 0$, then
\[
\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1} \beta(p + 1, q + 1)(|f(a)| + |f(b)|),
\] (2.2)
where $\beta(x, y)$ is the Euler Beta function.

**Proof.** By Lemma 2.1 the Beta function which is defined for $x, y > 0$ as
\[
\beta(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt
\]
and the fact that $f$ is $P$-convex on $[a, b]$, we have
\[
\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q |f(ta + (1-t)b)|dt
\] \[
\leq (b-a)^{p+q+1} \int_0^1 (1-t)^p t^q (|f(a)| + |f(b)|)dt
\] \[
=(b-a)^{p+q+1} \beta(q + 1, p + 1)(|f(a)| + |f(b)|),
\]
which completes the proof. \qed

The corresponding version for powers of the absolute value is incorporated in the following result.

**Theorem 2.2.** Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b]), 0 \leq a < b < \infty$ and let $k > 1$. If $|f|^{\frac{1}{k-1}}$ is $P$-convex on $[a, b]$, for some fixed $p, q > 0$, then
\[
\int_a^b (x-a)^p(b-x)^q f(x)dx \leq (b-a)^{p+q+1} \left[\beta(kp + 1, kq + 1)\right]^\frac{1}{k-1} \left(|f(a)|^{\frac{k}{k-1}} + |f(b)|^{\frac{k}{k-1}}\right)^{\frac{k-1}{k}}.
\] (2.3)
Proof. By Lemma 2.1, Hölder’s inequality, the definition of Beta function and the fact that $|f|^t$ is $P$-convex on $[a, b]$, we have

$$
\int_a^b (x-a)^p (b-x)^q f(x) dx
\leq (b-a)^{p+q+1} \left[ \int_0^1 (1-t)^{kp} t^{kq} dt \right] \left[ \int_0^1 |f(ta + (1-t)b)|^{\frac{k}{p-1}} dt \right]^{\frac{k}{p-1}}
\leq (b-a)^{p+q+1} \left[ \beta(kq + 1, kp + 1) \right]^{\frac{1}{p}} \left[ \int_0^1 \left( |f(a)|^{\frac{k}{p-1}} + |f(b)|^{\frac{k}{p-1}} \right) dt \right]^{\frac{k}{p}}
\leq (b-a)^{p+q+1} \left[ \beta(kq + 1, kp + 1) \right]^{\frac{1}{p}} \left( |f(a)|^{\frac{k}{p-1}} + |f(b)|^{\frac{k}{p-1}} \right)^{\frac{k}{p}},
$$

which completes the proof. \(\square\)

A more general inequality using Lemma 2.1 is as follows:

Theorem 2.3. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ such that $f \in L([a, b])$, $0 \leq a < b < \infty$ and let $l > 1$. If $|f|^t$ is $P$-convex on $[a, b]$, for some fixed $p, q > 0$, then

$$
\int_a^b (x-a)^p (b-x)^q f(x) dx
\leq (b-a)^{p+q+1} \beta(p+1, q+1) \left( |f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}}, \tag{2.4}
$$

where $\beta(x, y)$ is the Euler Beta function.

Proof. By Lemma 2.1, Hölder’s inequality, the definition of Beta function and the fact that $|f|^t$ is $P$-convex on $[a, b]$, we have

$$
\int_a^b (x-a)^p (b-x)^q f(x) dx
\leq (b-a)^{p+q+1} \left[ \int_0^1 (1-t)^{kp} t^{kq} \right]^{\frac{1}{p-1}} \left[ \int_0^1 |f(ta + (1-t)b)|^l dt \right]^{\frac{1}{l}}
\leq (b-a)^{p+q+1} \left[ \beta(q+1, p+1) \right]^{\frac{1}{p}} \left( |f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}}
\leq (b-a)^{p+q+1} \beta(p+1, q+1) \left( |f(a)|^l + |f(b)|^l \right)^{\frac{1}{l}},
$$

which completes the proof. \(\square\)

References

[1] M. Alomari, M. Darus and S.S. Dragomir, Inequalities of Hermite-Hadamard’s type for functions whose derivatives absolute values are quasi-convex, RGMIA Res. Rep. Coll., 12 (2009), Supp., Article 14, 11 pp.
New integral inequalities via $P$-convexity

[2] A. O. Akdemir and M. E. Özdemir, some Hadamard-type inequalities for co-ordinated $P$-convex functions and Godunova-Levin functions, AIP Conf. Proc. 1309 (2010), 7–15.

[3] S. S. Dragomir, On some new inequalities of Hermite-Hadamard type for $m$-convex functions, Tamkang J. Math. 33 (2002), no. 1, 55–65.

[4] S. S. Dragomir and C.E.M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[5] S. S. Dragomir and C. E. M. Pearce, Quasi-convex functions and Hadamard’s inequality, Bull. Austral. Math. Soc. 57 (1998), no. 3, 377–385.

[6] S. S. Dragomir, J. Pečarić and L. E. Persson, Some inequalities of Hadamard type, Soochow J. Math. 21 (1995), no. 3, 335–341.

[7] V. N. Huy and N. T. Chung, Some generalizations of the Fejér and Hermite-Hadamard inequalities in Hölder spaces, J. Appl. Math. Inform. 29 (2011), no. 3-4, 859–868.

[8] D. A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, An. Univ. Craiova Ser. Mat. Inform. 34 (2007), 83–88.

[9] W. J. Liu, New integral inequalities via $(\alpha, m)$-convexity and quasi-convexity, arXiv:1201.6226v1 [math.FA]

[10] M. E. Özdemir, E. Set and M. Alomari, Integral inequalities via several kinds of convexity, Creat. Math. Inform. 20 (2011), no. 1, 62–73.

[11] M. E. Özdemir and C. Yıldız, New inequalities for Hermite-Adamard and Simpson type and applications, arXiv:1103.1965v1 [math.CA]

[12] M. Z. Sarikaya, E. Set and M. E. Özdemir, On some new inequalities of Hadamard type involving $h$-convex functions, Acta Math. Univ. Comenian. (N.S.) 79 (2010), no. 2, 265–272.

[13] E. Set, M. E. Özhemir and S. S. Dragomir, On Hadamard-type inequalities involving several kinds of convexity, J. Inequal. Appl. 2010, Art. ID 286845, 12 pp.

[14] E. Set, M. Sardari, M. E. Özdemir and J. Rooin, On generalizations of the Hadamard inequality for $(\alpha, m)$-convex functions, RGMIA Res. Rep. Coll., 12 (2009), no. 4, Article 4, 10 pp.

[15] D. D. Stancu, G. Coman and P. Blaga, Numerical analysis and approximation theory. Vol. II (Romanian), Presa Universitară Clujeană, Cluj, 2002.

[16] K. L. Tseng, G. S. Yang and S. S. Dragomir, On quasiconvex functions and Hadamard’s inequality, RGMIA Res. Rep. Coll. 6 (2003), no. 3, Article 1, 12 pp.

Wenjun Liu
College of Mathematics and Statistics
Nanjing University of Information Science and Technology
Nanjing 210044, China
e-mail: wjliu@nuist.edu.cn