Self-similar Blast Wave for A Two-component Fluid with Variable Adiabatic Index

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(Received Date; Revised Date; Accepted Date)

ABSTRACT

We propose a self-similar (SS) solution to hydrodynamic non-relativistic flow behind a spherical strong blast wave (BW) passing through a homogeneous plasma with efficient relativistic particle acceleration at the shock front. The flow is described by an ideal two-fluid model with a relativistic component so that the post-shock gas has an effective SS adiabatic index \( \gamma \) varying from 5/3 to 4/3. This solution is calculated numerically and compared with the standard Sedov solution. We find that the BW center in our solution is dominated by the relativistic component with \( \gamma = 4/3 \) for the divergence of expansion there, and the relativistic component dominates the interior for a moderate acceleration efficiency at the shock front. The overall efficiency of relativistic particle acceleration can be enhanced by a factor of 2 due to the slower adiabatic energy loss rate of the relativistic component during expansion. Tendency of the dominance by the relativistic component may be common in expanding astrophysical two-fluid systems such as supernova remnants, lobes of radio galaxies.

Keywords: shock waves — ISM: supernova remnants — cosmic rays — acceleration of particles

1. INTRODUCTION

Shock waves are common astrophysical phenomena generated by strong energy release processes, e.g., solar activities and supernova (SN) explosions. It has been well studied and some of these shocks may retain self-similarity of the first type (Barenblatt & Zel’dovich 1972), i.e., from all dimensional parameters relevant to the system one can only form a single dimensionless independent variable so that the unsteady flow can be described with ordinary differential equations. The most well known example is the Sedov-Taylor solution to a shocked non-relativistic (NR) fluid behind a spherical strong shock (Sedov 1959; Landau & Lifshitz 1959). It has been widely used to analyze nuclear explosions and adiabatic expansion of young supernova remnants (Reynolds 2008, SNRs). Other self-similar (SS) blast waves (BWs) have also been investigated, e.g., Blandford & McKee (1976) solved the ultra-relativistic (UR) case with generalized energy injection, and a solution of Chevalier (1982) can describe free expansion of even younger SNRs which contain interactions between the stellar ejecta and the external medium. Most of these SS solutions satisfy the Poisson’s adiabatic relation, where the adiabatic index \( \gamma \) is a constant, which may be not appropriate to describe convection of astrophysical plasmas with a relativistic cosmic-ray (CR) component. The fully two-fluid equation of state (EOS) needs to be considered.

There is plenty of observational evidence for efficient CR acceleration by shocks of SNRs (Helder et al. 2012). However, studies of the non-linear effects of CR acceleration on the shock structure have been focused on processes near the shock front, in particular the upstream precursor (Berezko & Ellison 1999; Bell 2004). It is the purpose of this paper to generalize the Sedov solution to allow the shocked gas to have a variable \( \gamma \) ranging from the NR value of 5/3 to the UR value of 4/3. The key requirement is that \( \gamma \) varies with space and time in an SS form. A variable \( \gamma \) allows the freedom to include a relativistic gas component, such as CR produced by diffusive shock acceleration (DSA) at the shock front of SNRs, or relativistic e\( ^\pm \) pairs injected by the central engines of pulsar wind nebulae and/or quasars.

2. TWO-FLUID APPROACH

Although a CR modified fluid is often considered to be diffusive, the diffusion can be suppressed significantly in the presence of strong magneto-hydrodynamic (MHD) turbulence (López-Coto & Giacinti 2018). To have an SS solution of a strong spherical BW, it is also necessary to avoid dependence of the system on the diffusion process. This is because the system already depends on four dimensional...
quantities: the time \( t \), the distance \( r \) to the symmetric center, the total energy \( E \) released from the explosion and the unshocked mass density \( \rho_1 \), where any three quantities have mutually independent dimensions so that there is only one dimensionless SS variable \( \rho_1 r^2 / (E t^2) \) according to the Buckingham \( \pi \) theorem (Sedov 1959). The diffusion coefficient \( \kappa \) introduces an additional dimension which generally is independent of the dimensions of \( E \) and \( \rho_1 \) (unless one introduces additional equation of \( \kappa \), or let \( \kappa \) be a specific combination of some dimensional quantities), leading to a system with two dimensionless independent variables, in violation of the self-similarity. Thus the shock we are looking for has exactly the same evolutionary behavior as the standard Sedov solution, where the shock radius \( R \) is measured with \( t \), \( E \) and \( \rho_1 \), and for a constant \( \rho_1 \) the shock speed is given by

\[
u_1 = \frac{dR}{dt} = \frac{2R}{5t} = \frac{2\beta}{5} \left( \frac{E}{\rho_1 r^2} \right)^{\frac{3}{5}},
\]

where the dimensionless scaling constant \( \beta \) should be modified by existence of CRs.

Therefore, we shall assume the shocked gas to be an ideal fluid without any diffusion which may be appropriate for the dynamics of large-scale flows. Classical ideal hydrodynamics with spherical symmetry is given by the following conservation equations of energy, momentum, and mass,

\[
\frac{\partial}{\partial t} \left( \rho u^2 + U \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho u^2 + \rho \right) u = 0,
\]

\[
\frac{\partial}{\partial t} (\rho u) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho u^2) + \frac{\partial P}{\partial r} = 0,
\]

\[
\frac{\partial P}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \rho u) = 0,
\]

respectively, where \( u \) is NR flow speed, \( U \), \( P \) and \( \rho \) are internal energy density, gas pressure and mass density contributed by all particles in a local fluid element, respectively. Although these classical forms of conservation laws are valid only if \( u \ll c \) and \( P \ll \rho c^2 \) with \( c \) being the speed of light, they do correctly show the large-scale behavior of the SS BW whose \( u \) remains NR but \( P \) is relativistic (see Section 4).

In view of the assumption of a cold upstream matter and the self-similarity, the shocked gas has a conserved total energy

\[
E = \int_0^R \left( \frac{\rho u^2}{2} + U \right) 4\pi r^2 dr,
\]

and the energy conservation can be integrated to an SS algebraic equation (Landau & Lifshitz 1959)

\[
\left( \frac{\rho u^2}{2} + U \right) \frac{2r}{5t} = \left( \frac{\rho u^2}{2} + U + P \right) u.
\]

On the other hand, Equations (2)–(4) naturally lead to the first law of thermodynamics

\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} \right) U + \frac{U + P}{r^2} \frac{\partial}{\partial r} (r^2 u) = 0,
\]

implying that variation of \( U \) can only come from work arising from the flow velocity divergence. Obviously, the three conservation laws along with the self-similarity can be satisfied with Equations (4), (6) and (7), which will be considered as the basic equations for solving the overall profile of the SS BW.

It is common practice to describe the transport of HE particles (which travel much faster than the flow) over hydrodynamic scale via a convection-diffusion equation with adiabatic energy change of the particles, i.e., via Parker’s transport equation (Parker 1965). Generally, these HE particles are highly ionized so that their scattering centers may be considered as plasma (Alfvén) waves which travel at different speed from the overall flow, leading to a different convection velocity for the HE particles. However, we shall ignore this difference for the sake of simplicity. Taking energy moment of the transport equation of HE particles, and dropping the averaged diffusion term for a convection-dominated BW \( \kappa \ll u_1 R \), one can readily obtain the first law of thermodynamics (Drury 1983)

\[
\left( \frac{\partial}{\partial t} + u \frac{\partial}{\partial r} \right) U_c + \frac{U_c + P_c}{r^2} \frac{\partial}{\partial r} (r^2 u) = 0
\]

for the HE component, where \( U_c \) and \( P_c \) are internal energy density and gas pressure of HE particles in the local fluid element, respectively. We assume a UR EOS \( P_c = U_c / 3 \) for the HE component, and an NR EOS \( P = 2 \left( U - U_c \right) / 3 \) to the remaining gas. Next we define an effective \( \gamma \) via classical EOS \( P = (\gamma - 1) U \). We obtain

\[
\gamma = 1 + \frac{2}{3} \frac{P + P_c}{U_c}.
\]

Poisson’s adiabatic law, i.e., invariant \( P/\rho^\gamma \) for the motion of a given fluid element, can be derived with Equations (4), (7) and the classical EOS only if \( \gamma \) is a constant. However, Equation (8) implies that \( U_c/\rho^{3/5} \) is invariant, leading to

\[
U = C_1 \rho^4 + C_2 \rho^5,
\]

where \( C_1 \) and \( C_2 \) are advective constants determined by initial conditions. Note that this still describes an adiabatic system due to the lack of dissipation.

3. SELF-SIMILAR SOLUTION
Figure 1. Flow speed profile of the SS BW passing through a cold homogeneous media, where all the solid and the dashed lines refer to the two-fluid solutions given by Equations (20)–(22) and the Sedov’s cases with \( \gamma \equiv \Gamma \), respectively. The red, orange, cyan and blue colors refer to \( \Gamma = 4.01/3, 4.3/3, 4.7/3 \) and 4.99/3, respectively. The same line type is used in the following figures.

According to Equation (1), one may take the SS variable \( \xi = r/R \propto r/\tau^2 \), and assume

\[
\begin{align*}
\rho &= \rho_1 G(\xi), \quad u = \frac{2r}{\tau l} V(\xi), \\
U_c &= \left(\frac{2r}{\tau l}\right)^2 X(\xi), \quad \frac{U}{\rho} = \left(\frac{2r}{\tau l}\right)^2 Y(\xi),
\end{align*}
\]

where \( \xi < 1 \). For strong shocks, the corresponding boundary conditions can be given by the Rankine-Hugoniot relations (Landau & Lifshitz 1959)

\[
\begin{align*}
G(1) &= \frac{\Gamma + 1}{\Gamma - 1}, \quad V(1) = \frac{2}{\Gamma + 1}, \\
X(1) &= 2(5 - 3\Gamma)/(\Gamma + 1)^2, \quad Y(1) = \frac{2}{(\Gamma + 1)^2},
\end{align*}
\]

where the constant \( \Gamma \) refers to \( \gamma \) at the shock front. This may partially characterize a DSA with a constant CR acceleration efficiency at the shock front, e.g., \( \Gamma = 4.7/3 \) corresponds to an efficiency \( P_c/(\rho_1 u_1^2) \sim 10\% \) at the strong shock (which is different from the overall efficiency \( E_c/E \) calculated over the whole BW, see the last paragraph of Section 3), where the value 10% is usually considered as a reasonable lower limit for the CR acceleration efficiency of SNRs in the hypothesis of SNR origin of galactic CRs. Due to implicit dependence of the system on the space-time coordinate, this SS approach is only compatible with a non-evolutionary boundary condition at the shock, although the CR acceleration efficiency has been suggested to be evolving in SNRs (Zhang & Liu 2019b). In other words, the self-similarity can be approximately valid at most within a range where the shock acceleration efficiency does not evolve significantly. Note that we recover the classical Sedov solution with \( U_c \) and/or \( U - U_c \equiv 0 \) for the limiting cases with \( \Gamma = 5/3 \) and/or 4/3, respectively, according to Equations (7) and (8).

Then, Equations (4)–(9) can be put into the SS form:

\[
\begin{align*}
(V - 1) \frac{d \ln G}{d \ln \xi} + \frac{d V}{d \ln \xi} + 3V &= 0, \\
\frac{1}{\beta^2} &= \left(\frac{4}{5}\right)^2 \pi \int_0^1 G \left(\frac{V^2}{2} + Y\right) \xi^2 d\xi, \\
\frac{3}{2} V^2 (1 - V) + VX + (3 - 5V) Y &= 0, \\
\frac{1}{3} \left(2 - \frac{X}{Y}\right) \frac{dV}{d \ln \xi} + (V - 1) \frac{d \ln Y}{d \ln \xi} + V \left(4 - \frac{X}{Y}\right) - 5 &= 0,
\end{align*}
\]

By eliminating \( \xi \) and \( Y \) in Equations (17)–(19) (and the total differential of Equation (17)), one has

\[
\begin{align*}
(3 - 5V) \left[2(4V - 3) + \frac{3(V - 1)(4V - 5)}{X}\right] dV + \frac{2(40V^2 - 127V + 81)}{3(V - 1)} X \\
+ 100V^3 - 30V^2 + 279V - 90 &= 0,
\end{align*}
\]

which is better solved for \( V(X) \) since \( X(V) \) generally is not a single-valued function under the boundary condition Equations (13) and (14). This solution of \( V(X) \) can be further used to determine \( Y \) via Equation (17), and Equations (15) and (19) give

\[
\begin{align*}
G &= \frac{(\Gamma + 1)^{\frac{\Gamma}{\Gamma - 1}}}{\Gamma - 1} \sqrt{\frac{3V - 5 - \Gamma}{1 - 3\Gamma}} \left(\frac{1 - \Gamma}{V - 1}\right)^\frac{3}{2}, \\
\xi &= \left[\frac{1 - 5\Gamma}{(\Gamma + 1)(3V - 5)}\right]^\frac{1}{2} \exp \int_{\frac{3(V - 5)}{5 - 3V(X')}}^X \frac{3V(X')}{5 - 3V(X')} dX'.
\end{align*}
\]
Finally, the constant $\beta$ of Equation (1) is determined by Equation (16).

Numerical solution of the above model is shown and compared with the standard Sedov solution with $\gamma \equiv \Gamma$ (Sedov 1959; Landau & Lifshitz 1959) in Figures 1–6. Most dramatically, we find that $V_{\xi=0}$ is modified from Sedov’s $1/\Gamma$ to a constant $3/4$ unless $\Gamma = 5/3$, as indicated by the right panel of Figure 1. This is because, as $X \to \infty$, Equations (20)–(22) show that

$$X^3 \sim \frac{8 (5 - 3\Gamma) \Gamma^4}{(\Gamma + 1)(4V - 3)^3} \left( \frac{5V - 3}{7 - 3\Gamma} \right)^{16} \left( \frac{1 - \Gamma}{V - 1} \right)^3,$$

$$G \sim 4V^3 \left( \frac{5V - 3}{5 - 3\Gamma} \right)^3,$$

$$\xi^3 \sim \frac{4V^3}{(\Gamma + 1)(5 - 3\Gamma)} \left( \frac{7 - 3\Gamma}{5V - 3} \right)^2.$$

Therefore, we obtain $3/4 - V \propto \xi^3, G \propto \xi^9$ and $Y \to X \propto \xi^{-11}$ as $\xi \to 0$. This modification implies that the gas pressure at symmetric center of the spherical BW is dominated by the UR component, as clearly shown by Figures 2 and 3, where $\gamma$ increases from $4/3$ to $\Gamma$ as $\xi$ varies from 0 to 1. The right panel of Figure 2 also shows that for a moderate CR acceleration efficiency at the shock front, the pressure of the UR component can be higher than that of the NR component over the bulk of the BW.

Since the relativistic gas cools down with a slower rate compared with the NR one in the process of expansion (see Equations (7) and (8)), it is not surprising that the UR component will dominate the BW center as the expansion diverges with $\rho \to 0$ toward the center (see Equation (10)).

Furthermore, the overall cooling down ability is also slightly reduced by the presence of the relativistic component. The left panel of Figure 3 shows that the two-fluid SS solutions have a slightly higher overall pressure. Since the relativistic gas carries more internal energy per particle than the NR one, the UR modified center does not need as much matter as the Sedov’s one, as shown by Figure 4, to maintain the overall pressure magnitude. Then the modified matter needs to flow faster (see Figure 1) to balance the overall pressure gradient. An effective ideal-gas temperature can be
A cosmic-ray modified shock is defined with $T \equiv mP/\rho$, where $m$ is the rest mass of the gas particle. The profile of $T$ is given in Figure 5. As $\xi \to 0$, one has

$$
\rho \propto \xi^0, \quad u \to \frac{3}{4} u_1 \xi, \quad P - P_c \propto \xi^3, \quad T \propto \xi^{-9}.
$$

These results can be used to understand the asymptotic behaviors in Figures 1–5.

The slightly higher energy density arising from the slower overall cooling down also leads to a slightly smaller value for $\beta$ in view of the total energy conservation, implying a slightly smaller shock radius and slower shock speed compared with the Sedov’s case for a given $t, E$ and $\rho_1$ according to Equation (1). Numerical results of $\beta$ corresponding to different values of $\Gamma$ in Figures 1–5 are calculated via Equation (16) and given in Table 1. One can see that modification of $\beta$ is within 3%.

Figure 6 shows that for a given CR acceleration efficiency at the shock front the overall CR acceleration efficiency $E_c/E$ may be enhanced by a factor of 2 for $\Gamma \gtrsim 4.7/3$ in the two-fluid scenario compared with the Sedov solution, where

$$
E_c = \int_0^R U_c 4\pi r^2 dr, \quad E_g = \int_0^B U_c 4\pi r^2 dr.
$$

We note that in the case of very efficient relativistic particle acceleration at the shock front so that $\Gamma = 4/3$, $E_c/E$ is still less than about 80% for the kinetic energy carried by the NR gas.

4. DISCUSSION

The SS energy conservation Equation (6) is valid only for $u \ll c$ and $P \ll \rho c^2$, while Figure 5 shows $P \gg \rho c^2 (u_1/c)^2$ near the BW center which generally violates the latter constraint. Although it is possible to obtain a set of quasi-NR equations describing an ideal fluid which flows with $u \ll c$ but is compatible with an arbitrarily high pressure, the unavoidable introduction of the dimensional constant $c$ will break down the self-similarity. Starting with the fully relativistic ideal hydrodynamics (Landau & Lifshitz 1959), the energy and the momentum conservation of the quasi-NR ap-
provision with spherical symmetry can be found as follows
\[ \frac{\partial}{\partial t} \left( \rho \left( \frac{\rho + U + P}{c^2} \right) \frac{u^2}{r^2} \right) + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \left( \rho + \frac{U + P}{c^2} \right) u \right) = 0, \]  
(28)
\[ \frac{\partial}{\partial t} \left( \rho + \frac{U + P}{c^2} \right) + \frac{\partial}{\partial r} \left( \frac{1}{r^2} \frac{\partial}{\partial r} \left( \rho + \frac{U + P}{c^2} \right) u^2 \right) + \frac{\partial P}{\partial r} = 0, \]  
(29)

where \( \rho \) still satisfies the mass conservation Equation (4).

Three conservation laws directly lead to the first law of thermodynamics, which has the same form as Equation (7) except for higher order \( O(u/c)^2 \) correction terms. Obviously, Equation (8) should also be held since it comes from a transport equation without any restriction on the particle energy. In addition, Equation (28) implies the same asymptotic behavior as \( \xi \rightarrow 0 \) just as that given by Equation (6) for \( U \) and \( P \gg \rho c^2 \gg \rho u^2 \). Therefore, the two-fluid SS solution is still valid around the center even with an arbitrarily high effective temperature. Moreover, an NR shock is capable of heating the incoming gas only to \( P \sim \rho c^2 \) \( O(u/c)^2 \) as indicated by the Taub jump conditions (Taub 1948; Zhang & Liu 2019b), validating the classical NR hydrodynamics near the shock front. Therefore, the classical NR fluid equations used in this work give globally a good description of adiabatic BWs with NR flow speed and relativistic pressure.

On the other hand, one should notice that the divergence of the effective temperature at the center is the remnant of the unphysical initial condition of the explosion with a divergent energy per gas particle. In actual physical systems, modification is expected. Large gradient caused by high central temperature necessarily gives rise to thermal conduction, which arises from particle diffusion and in turn can remove the center singularity (Sedov 1959). In view of dominance of the UR gas at the center, the thermal conduction should be largely attributed to the ignored UR diffusive energy flux \( \kappa \partial U_\perp / \partial r \) (Drury 1983) in Equations (7) and (8). Consequently, the UR dominance of the center pressure may disappear if the thermal conduction is significant enough that excessive UR gases diffuse out of the center. In the case of extremely efficient heat transfer, i.e., an isothermal BW with \( \partial T / \partial r = 0 \) (Solinger et al. 1975), the adiabatic index actually no longer appears in the Euler equation due to \( \partial P / \partial \rho \rangle = T / \rho \), thus the adiabatic index can get into the solution only via boundary conditions, resulting in a constant percentage of the UR component energy density. Furthermore, in general, introduction of a realistic thermal conductivity is not compatible with an SS solution, since the self-similarity may break down with diffusion (Shestakov 1999). Moreover, with more realistic initial conditions, such as the presence of ejected materials in SNRs, the structure of the central region can be modified significantly.

Nevertheless, for a “collisionless” plasma dominated by long-range electromagnetic interaction, it has been argued that the thermal conduction may be strongly suppressed due to magnetic turbulence, where motions of the charged particles are mainly bounded along the wandering magnetic field lines leading to significant lengthening of the conduction paths (Cox et al. 1999) and reduction of thermal conductivity (López-Coto & Giacinti 2018). Since magnetic trapping requires that the gyro-radius of the charged particles should be much smaller than the fluid scale, which can be easily satisfied in most astrophysical environments, it is reasonable to expect that domination of the UR over the NR gas pressure around the BW center is not overturned by the magnetic energy sharing. On the other hand, scattering centers of particles are usually considered as plasma waves, e.g., Alfvén waves that travel along magnetic field with a speed of \( \sqrt{2P_m/\rho} \) relative to the flow, where \( P_m \) is the magnetic pressure. For the NR and the UR component and the waves to move with the same speed \( u \), \( P_m \) needs to be much less than \( \rho u^2 \). If the magnetic field is close to energy partition with the UR component, these constraints likely place a necessary condition \( P_e < \rho u^2 \) to the present SS solution, by which applicability of the solution is mainly confined to vicinity of the shock (see the right panel of Figure 3). However, this restriction may be largely relaxed when considering an isotropic turbulence with very weak large-scale magnetic field, where

| Table 1. Numerical results of \( \beta \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \Gamma \)    | 4.01/3          | 4.30/3          | 4.70/3          | 4.99/3          |
| Two-fluid       | 0.996           | 1.030           | 1.090           | 1.149           |
| Sedov           | 0.997           | 1.050           | 1.112           | 1.150           |

Figure 6. Dependence of the overall CR acceleration efficiency \( E_v/E \) on \( \Gamma \) in the SS BW, where the green and the purple colors refer to \( E_v/E \) and \( E_v/E \), respectively. We note that the line type still follows the caption of Figure 1, i.e., the solid and the dashed lines refer to the two-fluid and the Sedov solutions, respectively.
the magnetic field is randomized enough that propagation of the plasma waves on average has no contribution to convection of the scattered particles over a larger but still macroscopically small scale. Meanwhile, a constant partition between the magnetic and the UR gas pressure may be realized in ideal MHD, since the turbulent magnetic field behaves the same way as the non-diffusive UR gas. Intuitively, the UR component and magnetic turbulence may be confined by NR gas pressure wherever $P_c < P - P_e$ (see the right panel of Figure 2). The above discussion suggests that validity of our SS solution holds even when UR gas dominates with $P_c > P - P_e$. The right panel of Figure 3 shows that the UR pressure also exceeds the ram pressure of the background flow near the shock front for a moderate CR acceleration efficiency.

Due to time reversal symmetry of the adiabatic system, not only for the expansion cooling but also for compression heating, the UR pressure always changes slower than the NR one. Since shock is a compressional discontinuity, dissipation processes are essential for efficient CR acceleration at the shock front which also reduces the adiabatic index of the flow near the shock. Recent observations of SNRs have shown that CR precursor in the upstream can be extremely thin compared with radius of the SNR (Katsuda et al. 2016; H. E. S. Collaboration et al. 2018), implying a weak diffusion caused presumably by magnetic turbulence amplified by the particle escape. This weak diffusion can increase the probability of particles crossing the shock front to gain energies, giving rise to efficient CR acceleration (Zhang et al. 2017) which can dramatically extend the relativistic component dominant regime. The right panel of Figure 3 shows that a CR acceleration efficiency $P_c / \langle \rho_1 u_1^2 \rangle \sim 10\%$ at the strong shock front, i.e., the case of $\Gamma = 4.7/3$, can in principal extend the relativistic component dominant region to about half of the shock radius.

A CR acceleration efficiency of about 10% is needed to attribute the CR flux observed at earth to shocks of SNRs. We therefore predict a CR pressure dominant interior for SNRs. For $P_c / \langle \rho_1 u_1^2 \rangle \sim 10\%$ at the shock front with $\Gamma = 4.7/3$, Figure 6 shows that the overall CR acceleration efficiency can reach more than 30% for our two-fluid SS solutions. While the classical Sedov solution gives an overall efficiency of about 20%. Considering modification to the flow structure by ejecta of the SNe, the model can readily explain TeV bright shell-type SNRs (Yang et al. 2014, 2015; Zhang & Liu 2019a), where it is found that the total energy of energetic electrons and magnetic fields can reach $10^{39}$ erg. The total energy of accelerated ions could be even higher. For older SNRs, considering the diffusive escape of CRs near the shock front, we expect a steep CR pressure gradient, which may cause the slow diffusion of energetic particles in TeV halos (Evoli et al. 2018; Di Mauro et al. 2019). Moreover, there are a few TeV halos with no spectral cutoff toward high energies, indicating a hadronic origin of the TeV emission (Fujinaga et al. 2011; Sakai et al. 2011; Xin et al. 2019). Such halos may be remnants of our SS solutions for SNRs. Our study focuses on the large-scale fluid properties of BW. For more quantitative comparison with observations of SNRs, one needs to study the evolution of HE particle distribution, which is beyond the scope of this paper.

Note that with the adiabatic two-fluid model, Equations (7) and (8) show that the UR component always dominates with $\gamma = 4/3$ in the limit of infinite expansion, while the NR component dominates with $\gamma = 5/3$ when the system is extremely compressed, except for an initially strict UR or NR EOS. It is easy to derive the evolution of the adiabatic index observed in a comoving frame

$$\gamma = \frac{4 (5 - 3 \gamma_0) + 5 (3 \gamma_0 - 4) \exp \int_0^\tau \frac{H(\rho)}{\gamma} \, d\tau'}{3 \left[ 5 - 3 \gamma_0 + (3 \gamma_0 - 4) \exp \int_0^\tau \frac{H(\rho)}{\gamma} \, d\tau' \right]}, \quad (30)$$

where $\gamma_0$ refers to $\gamma$ at the time $t_0$. $H$ is expansion or compression rate given by positive or negative divergence of the flow velocity, respectively. For example, a cosmology application may be simply regarding $H$ as the Hubble parameter $\sim k/t$ with $k$ being a positive constant, then one obtains a decreasing $\gamma \sim 4/3 + (t_0/t)^{4/3} (\gamma_0 - 4/3) / (5 - 3 \gamma_0)$ for $t \gg t_0$, implying a tendency of UR pressure dominance of the universe under the matter-dominated adiabatic expansion phase.

Another possible application of the two-fluid SS solution presented here is a radiation-dominated, optically thick, BW, such as the early phase of a BW propagating inside a supermassive stellar envelope. If the optical depth is sufficiently large, the post-shock radiation can approach a quasi-blackbody with the radiation field being in thermal equilibrium with relativistic $e^\pm$ pairs while the ions are NR. Hence the radiation pressure may become a substantial fraction of the total pressure behind the shock. To the extent that the radiation fluid stays tightly coupled to the gas during the expansion due to large optical depths, the adiabaticity of the radiation can be maintained, leading to the domination of radiation pressure in the deep interior of the BW. In this case, the central temperature remains finite with the Stefan-Boltzmann law. Of course, the SS and adiabatic assumptions eventually break down once radiation diffusion becomes important. However, for a very large supermassive star, the adiabatic SS regime may indeed be realized over a finite radius and temporal regime.
ACKNOWLEDGMENTS

This work is supported in part by National Key Research and Development Program of China (No. 2018YFA0404203), National Natural Science Foundation of China (Nos. U1738122, 11761131007 and U1931204), International Partnership Program of Chinese Academy of Sciences (No. 114332KYSB20170008), and scholarships from China Scholarship Council (No. 201806340077).

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