PROMOTING ESSENTIAL LAMINATIONS

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ABSTRACT. We show that a co–orientable taut foliation of a closed, orientable, algebraically atoroidal 3–manifold is either the weak stable foliation of an Anosov flow, or else there are a pair of very full genuine laminations transverse to the foliation.

1. INTRODUCTION

A topological manifold is a very flabby object. It has no local internal structure, and except in very special cases, the group of automorphisms is transitive on the set of subsets of a fixed finite cardinality. The same manifold can appear in a myriad of different forms, and the question of recognizing or distinguishing manifolds, or of certifying a useful property, is in the best case very hard, and in the worst (typical) case algorithmically unsolvable.

It is therefore desirable to stiffen or rigidify the structure of a manifold, by introducing geometry in some form or other, in order to reduce this ambiguity of form to a manageable amount. But exactly what sort of geometric constraints are neither too much (so that there are no examples) or too little (so that the geometric structure does not help with the problem of understanding or recognizing the underlying manifold) is very dimension dependent. As a general rule, smaller dimensional objects are easier to understand. Important principles become easier to apply and yield more and richer structure. In this paper, the notion of monotonicity is very important, especially as it relates to natural order or partial-order structures on certain sets. More generally, such order structures provide a bridge from geometric problems to algebraic language, and permit one to perform experiments and construct certificates with the use of computers.

As an organizational tool, monotonicity loses effectiveness as dimension goes up; consequently it is most powerful when used in the context of certain dynamical systems, which can effectively reduce the study of a manifold to two complementary problems of strictly smaller dimension: the study of the orbits of the system, and the study of the parameter space or leaf space of the orbits. Geometric or analytic qualities of the dynamical system are reflected in the properties of the dimensionally reduced systems.

1.1. Foliations and arboreal group theory. In the case of the study of 3–manifolds, a very effective tool for dimensional reduction is the structure of a 2–dimensional foliation, especially a taut foliation \( \mathcal{F} \) of \( M \) which, at least when \( M \) is atoroidal, may be defined as a 2–dimensional foliation without spherical or torus leaves. A basic structure theorem of Novikov implies that the leaf space \( L \) of the universal
cover $\mathcal{F}$ of such a foliation is a typically non-Hausdorff simply-connected 1–manifold, on which $\pi_1(M)$ acts naturally.

Such 1–manifolds are not unlike $\mathbb{R}$–trees in some ways, and many of the tools of arboreal group theory (e.g. [49], [1]) can be used to study the action of $\pi_1(M)$. If $\mathcal{F}$ is co–orientable, the leaf space $L$ is an oriented 1–manifold, and this orientation defines a partial order on the elements of $L$. This global partial order structure adds extra nuances to the arboreal theory, and is the source of many important constructions. For example, in a remarkable tour de force, Roberts, Shareshian and Stein ([17]) recently managed to give examples of an infinite family of hyperbolic 3–manifolds which do not admit taut foliations, simply by studying the action of their fundamental groups on (non–Hausdorff) simply–connected 1–manifolds.

1.2. The classification of surface homeomorphisms. If $L$ and the action of $\pi_1(M)$ are understood, it remains to understand the leaves of $\mathcal{F}$ themselves, and the way they fill out the manifold $M$; the relevant subject is the theory of surface homeomorphisms.

In this subsection we discuss the simplest case of the theory of surface homeomorphisms. We derive Thurston’s famous theorem on the classification of surface homeomorphisms by a route which is nearly the opposite of the historical and, for that matter, the logical direction. The reason is mainly pedagogical: this order of exposition more clearly reveals the order of development of some analogous ideas for more general taut foliations. We will necessarily cover a lot of material very briefly. Most of the details can be found in the papers [55], [60] and [16].

First we recall the statement of the theorem, in its most basic form.

**Theorem 1.2.1** (Thurston, [55] Classification of surface homeomorphisms). Let $\Sigma$ be a closed, orientable surface of genus at least 2, and let $\phi : \Sigma \to \Sigma$ be a homeomorphism. Then one of the following three alternatives holds:

1. $\phi$ is periodic; that is, some finite power of $\phi$ is isotopic to the identity.
2. $\phi$ is reducible; that is, there is some finite collection of disjoint essential simple closed curves in $\Sigma$ which are permuted up to isotopy by $\phi$.
3. $\phi$ is pseudo–Anosov.

For the moment, we postpone the definition of a pseudo–Anosov diffeomorphism of a surface, since this will be the punchline of our revisionist story.

Given the pair $(\Sigma, \phi)$ one forms the mapping torus $M_\phi$ which is the quotient of the product $\Sigma \times I$ by the equivalence relation $(s, 1) \sim (\phi(s), 0)$. $M_\phi$ is a fibration over $S^1$, with fiber $\Sigma$ and monodromy $\phi$. By analogy with the notation for a short exact sequence, we denote this

$$\Sigma \to M_\phi \to S^1$$

and there is a corresponding short exact sequence of groups

$$\pi_1(\Sigma) \to \pi_1(M_\phi) \to \mathbb{Z}$$

which represents $\pi_1(M_\phi)$ as an HNN extension. The automorphism $\phi$ of $\Sigma$ induces an automorphism $\phi_*$ of $\pi_1(\Sigma)$, well–defined up to inner automorphisms. A presentation for $\pi_1(M_\phi)$ is then given by

$$\pi_1(M_\phi) = \langle \pi_1(\Sigma), t \mid t^{-1} \alpha t = \phi_*(\alpha) \text{ for each } \alpha \in \pi_1(\Sigma) \rangle$$

The homeomorphism type of this 3–manifold only depends on the isotopy class of $\phi$. Then the classification of $\phi$ neatly reflects the geometry of the mapping torus.
Recall that for $M$ a closed, topological 3–manifold, and $X$ a simply–connected locally symmetric Riemannian 3–manifold, an $X$ geometry on $M$ is a homeomorphism

$$\varphi : M \to X/\Gamma$$

where $\Gamma$ is a free, discrete, cocompact, properly discontinuous subgroup of $\text{Isom}(X)$. See [55] for more details.

**Theorem 1.2.2** (Thurston, [60] Geometrization of surface bundles). Let $\Sigma$ be a surface of genus at least 2, and let $\phi : \Sigma \to \Sigma$ be a homeomorphism. Then the mapping torus $M_\phi$ satisfies the following:

1. If $\phi$ is periodic, $M_\phi$ admits an $\mathbb{H}^2 \times \mathbb{R}$ geometry.
2. If $\phi$ is reducible, $M_\phi$ has a non–trivial JSJ decomposition.
3. If $\phi$ is pseudo–Anosov, $M_\phi$ admits an $\mathbb{H}^3$ geometry.

From now on we consider the case where $\phi$ is pseudo–Anosov, and therefore $M_\phi$ is hyperbolic, and we can identify its universal cover with hyperbolic 3–space $\tilde{M}_\phi = \mathbb{H}^3$.

The action of $\pi_1(M_\phi)$ on $\tilde{M}_\phi$ extends continuously to an action on the ideal boundary of $\mathbb{H}^3$, which is a topological sphere which we denote by $S^2_\infty$, and the action of $\pi_1(M_\phi)$ on this sphere is by M"obius transformations. We denote the representation inducing this action by

$$\rho_{\text{geo}} : \pi_1(M_\phi) \to \text{Homeo}(S^2_\infty)$$

There is another view of $\tilde{M}_\phi$ which comes from the foliated structure of $M_\phi$. To describe this point of view, we make use of some ideas of coarse geometry from Gromov as developed in [30].

The foliation of $\Sigma \times I$ descends to a (taut) foliation of $M_\phi$ by surfaces which are the fiber of the fibration over $S^1$. This gives $\tilde{M}_\phi$ the structure of an open solid cylinder

$$\tilde{M}_\phi = \tilde{\Sigma} \times \mathbb{R}$$

The universal cover of each fiber $\Sigma_\theta$ is quasi–isometric with its pulled back intrinsic metric to the hyperbolic plane $\mathbb{H}^2$, and can therefore be compactified by its ideal boundary, which is a topological circle $S^1_\infty$.

This circle $S^1_\infty$ can just as well be thought of as the Gromov boundary of the group $\pi_1(\Sigma)$. The group $\pi_1(M_\phi)$ acts on $\pi_1(\Sigma)$ in the obvious way: the subgroup $\pi_1(\Sigma)$ acts on the left by multiplication, and the element $t$ acts by the automorphism $\phi_\ast$. This action on $\pi_1(\Sigma)$ induces an action of $\pi_1(M_\phi)$ on $S^1_\infty(\pi_1(\Sigma))$, and together with the action on $\mathbb{R}$ given by the homomorphism to $\mathbb{Z}$, this gives a (product respecting) action of $\pi_1(M_\phi)$ on $S^1 \times \mathbb{R}$ which partially compactifies the action on the open cylinder $\tilde{\Sigma} \times \mathbb{R}$.

The action of $\pi_1(M_\phi)$ on $\mathbb{R}$ is boring; all the information is already contained in the action on $S^1_\infty$. We denote the representation inducing this action by

$$\rho_{\text{fol}} : \pi_1(M_\phi) \to \text{Homeo}(S^1_\infty)$$

**Theorem 1.2.3** (Cannon–Thurston [16] Continuity of Peano map). Suppose $M_\phi$ is a hyperbolic surface bundle over $S^1$ with fiber $\Sigma$ and monodromy $\phi$. Then there is a continuous, surjective map

$$P : S^1_\infty \to S^2_\infty$$
which is a semiconjugacy between the two actions of $\pi_1(M_\phi)$. That is, for each $\alpha \in \pi_1(M_\phi)$,
\[ P \circ \rho_\phi(\alpha) = \rho_{geo}(\alpha) \circ P \]

Since the image of $S^1_\infty$ under $P$ is closed and invariant under the action of $\pi_1(M_\phi)$, it is equal to the entire sphere $S^2_\infty$; that is, it is a Peano curve, or sphere-filling map.

The fact that $P$ is sphere-filling is disconcerting and beautiful, but it is not the whole story. More interesting is the fact that $P$ can be approximated by embeddings in a natural way.

If $\tilde{\Sigma}_\phi$ denotes the universal cover of a fiber, then $\tilde{\Sigma}_\phi$ is a properly embedded plane in $\tilde{M}_\phi = \mathbb{H}^3$. By theorem 1.2.3, the embedding of $\tilde{\Sigma}_\phi$ extends continuously to the Peano map on the boundary, by the canonical identification of $S^1_\infty(\tilde{\Sigma}_\phi)$ with $S^1_\infty(\pi_1(\Sigma))$. In the unit ball model of $\mathbb{H}^3$, let $p \in \tilde{\Sigma}_\phi$ be a base point at the origin. Let $T_i \subset \Sigma_o$ be a (component of the) intersection of $\tilde{\Sigma}_\phi$ with a family of concentric spheres about $p$. Radial projection from $p$ in $\Sigma_o$ identifies each $T_i$ with $S^1_\infty(\Sigma)$, and radial projection from $p$ in $\mathbb{H}^3$ identifies each $T_i$ with an embedded circle in $S^2_\infty$. We denote the composition of these identifications by
\[ P_i : S^1_\infty \to S^2_\infty \]

which gives a family of maps which converge in the compact-open topology to $P$. Each $P_i$ decomposes the complement of its image into two sides, which we can consistently label as the positive and negative sides, compatibly with an orientation on $S^1_\infty$ and $S^2_\infty$.

Define a positive pair to be a pair of elements $p, q \in S^1$, and a choice, for each $P_i$, of an arc $\gamma_i \subset S^2_\infty$ from $P_i(p)$ to $P_i(q)$ whose interior is disjoint from $P_i(S^1)$ and contained on the positive side, and which satisfies
\[ \lim_{i \to \infty} \text{diameter}(\gamma_i) \to 0 \]

We denote a positive pair by $(p, q, \{\gamma_i\})$.

Now, if $(p_1, p_2, \{\gamma_i\})$ is one positive pair and $(q_1, q_2, \{\delta_i\})$ is another, then either $(p_1, p_2)$ and $(q_1, q_2)$ are unlinked as copies of $S^0$ in $S^1_\infty$, or else all four points are mapped to the same point by $P$. The reason is that if $(p_1, p_2)$ and $(q_1, q_2)$ are linked in $S^1_\infty$, then $\gamma_i$ and $\delta_i$ lying on the same side of the image of $P_i$ must intersect. Since their lengths converge to 0 as $i \to \infty$, the claim follows.

The positive pairs define a subset of $S^1 \times S^1$ which generates a closed equivalence relation, which we denote by $\sim^\dagger$. Similarly, we can define $\sim^\ddagger$ in terms of negative pairs. Note that distinct equivalence classes of $\sim^\dagger$ say have the property that they are unlinked as subsets of $S^1$, in the sense that if $S^0_1, S^0_2$ are two embedded copies of $S^0$ in $S^1$ which are each contained in distinct equivalence classes of $\sim^\dagger$, then the homological linking number of the $S^0$’s is 0.

Applying this fact to the map $P : S^1_\infty \to S^2_\infty$, lets us construct a pair of geodesic laminations $\Lambda^\pm$ of $\Sigma$ as follows (see section §2 for a definition and a discussion of geodesic laminations). The lamination $\Lambda^+$ is the union, over equivalence classes $[p]$ of $\sim^\dagger$, of the boundary of the convex hull of $[p]$, thought of as a subset of $S^1_\infty(\Sigma)$. It is crucial here that distinct equivalence classes are unlinked, so that the result is a lamination, and not merely a collection of geodesics. The action of $\pi_1(\Sigma)$ on $\tilde{\Sigma}$ preserves these laminations, and they descend to geodesic laminations $\Lambda^\pm$ on $\Sigma$. 
which are preserved by the action of (a homeomorphism isotopic to) $\phi$. It is not hard to show that these laminations are transverse, and bind $\Sigma$, in the sense that complementary regions are (compact) finite sided polygons. The usual Perron–Frobenius theory shows that $\Lambda^\pm$ admit transverse measures $\mu^\pm$ which are multiplied by $\lambda, \lambda^{-1}$ respectively by $\phi$, for some $\lambda > 1$. This is one of the definitions of a pseudo–Anosov map, and Thurston’s theorem on the classification of surface homeomorphisms is recovered via a very non–standard route.

Note that a posteriori, it can be seen that the laminations $\tilde{\Lambda}^\pm$ are determined uniquely by the action of $\pi_1(M_\phi)$ on the circle $S^1_\infty$, and can be recovered from the fixed point data of $\phi_*$ and its conjugates.

Of course this is not a logical deduction, since the usual proofs of both theorem 1.2.2 and theorem 1.2.3 depend essentially on theorem 1.2.1.

More useful information can be derived from this picture. Each of the invariant geodesic laminations $\Lambda^\pm$ on $\Sigma$ suspend in the mapping torus to two dimensional laminations. Notice that such laminations have some useful properties. The leaves are covered in $\tilde{M}_\phi$ by planes. The finitely many complementary regions are topologically open solid tori, which have the extra structure of finite sided ideal polygon bundles over $S^1$. They are the prototypical example of very full genuine laminations, a particularly well behaved subclass of the class of genuine laminations, introduced by Gabai and Oertel in [28]. Such laminations certify important properties of their ambient manifold. In [25],[26] and [27] Gabai and Kazez show that an atoroidal 3–manifold $M$ with a genuine lamination has word–hyperbolic fundamental group, has a finite mapping class group, and that every self–homeomorphism homotopic to the identity is isotopic to the identity.

In another direction, the laminations $\Lambda^\pm$ can be used to produce a particularly nice flow $X$ transverse to the fibration. The projectively measured laminations $\tilde{\Lambda}^\pm$ are dual to a pair of topological $\mathbb{R}$–trees $T^\pm$. Then the tautological quotient maps of $\tilde{\Sigma}$ to $T^+$ and $T^-$ define a map to the product $T^+ \times T^-$ whose image is topologically a plane. Moreover, the image inherits a pair of singular foliations $\mathcal{F}^\pm$ by the intersection with factors $T^+ \times$ point and point $\times T^-$ of the product structure. This structure is equivariant, and defines a pair of transversely measured singular foliations on $\Sigma$ which are transverse to each other and invariant by a suitable element in the isotopy class of $\phi$.

The suspension flow $X$ of this homeomorphism is pseudo–Anosov. That is, away from finitely many orbits, there is a decomposition of the tangent space $TM$ into a sum $TX \oplus TE^s \oplus TE^u$ which is preserved by the flow, and where the time $t$ flow multiplies the vectors in the sub–bundles $TE^s$ and $TE^u$ by factors $O(e^{\lambda t})$ and $O(e^{-\lambda t})$ respectively, for some $\lambda > 0$. Moreover, the singular orbits look like branched covers of the ordinary orbits, with branch index $n/2$ for some integer $n \geq 3$. A pseudo–Anosov without such singular orbits is Anosov. This pseudo–Anosov flow has the property of being the minimal entropy flow transverse to the foliation, and it also has the property of being quasigeodesic. That is, flowlines of the lift $\tilde{X}$ in the universal cover are a bounded distance from hyperbolic geodesics in $M_\phi = \mathbb{H}^3$.

1.3. Circle of ideas. This pencil sketch of the theory of surface diffeomorphisms outlines the application of this dimensional reduction idea to 3–manifold theory. A certain kind of foliation — namely a fibration — reduces a 3–manifold $M_\phi$ to
a 2–manifold $\Sigma$ together with some dynamics $\phi$. Ideal geometry reduces the 2–
mantifold $\Sigma$ to a 1–manifold $S_1^\infty$, together with some further dynamics, namely the
action of $\pi_1(\Sigma)$. The relationship between $S_1^\infty$ and $S_2^\infty$ can be encoded in another
pair of 1–dimensional objects, namely the laminations $\Lambda^\pm$, which are actually en-
coded in data living only on $S_1^\infty$, and which can be recovered in principle purely
from the dynamics of $\pi_1(M_\phi)$ on this 1–dimensional object.

The goal of this paper is to reproduce as much of this structure as possible in
the context of a more general kind of foliation, namely a taut foliation. We follow
the principle that smaller is better when it comes to dimension. Accordingly, we
aim to reduce our 3–manifold, via the use of some auxiliary dynamical data, to a
canonical circle $S_1^{\text{univ}}$ called a universal circle, together with a natural representation

$$\rho_{\text{univ}} : \pi_1(M) \to S_1^{\text{univ}}$$

This circle and representation encodes the original dynamical data, or as much of
it as is important. In particular, from $S_1^{\text{univ}}$ and $\rho_{\text{univ}}$ we can reconstruct the origi-
nal 3–manifold $M$ and certify important topological, geometric and dynamical
properties of it.

Loosely speaking, the sources of universal circles are threefold: they arise from
the following three objects, which are all present in the example of surface bundle
over a circle.

1. Taut foliations
2. Very full genuine laminations
3. Quasigeodesic pseudo–Anosov flows

Precise definitions of these structures will be deferred until §3.

In the best situation, all three structures give rise to and can be recovered from
the universal circle, and their interactions are encoded in a uniform way. For de-
tails, consult [8]. In this paper we aim to show how, under suitable circumstances,
one of the structures — a taut foliation — gives rise to another: a (pair of) very full
genuine laminations.

1.4. **Atoroidal versus algebraically atoroidal.** Throughout this paper, we use the
term atoroidal in a slightly nonstandard way as shorthand for algebraically atoroidal.
A 3–manifold $M$ is algebraically atoroidal if there is no $\mathbb{Z} \oplus \mathbb{Z}$ in $\pi_1(M)$. A 3–
manifold $M$ is geometrically atoroidal if every essential embedded torus is bound-
dary parallel. For closed 3–manifolds, the two terms are interchangeable except
when $M$ is a small Seifert fibered space; i.e. a Seifert fibered space over a triangle
orbifold.

In statements of important theorems, in order to minimize confusion, we try to
use the longer term algebraically atoroidal.

1.5. **Statement of results.** In this section we state our results precisely.

§2 gathers basic results and constructions in the point set topology of $S^1$ which
are used again and again throughout the rest of the paper. We define lamina-
tions of $S^1$, laminar relations on $S^1$, and geodesic laminations of the hyperbolic
plane $\mathbb{H}^2$, and we show how to move back and forward between these three kinds
of objects. We also define monotone maps between circles, which are degree one
maps whose point preimages are connected. The most important theorem is the-
orem[2.2.8] which concerns continuous families of monotone maps. This is a tech-
nical theorem which is used in later sections, especially §3.
§3 is mostly expository, being a brief introduction to the theory of taut foliations and their cousins, essential and genuine laminations in 3–manifolds. It is hardly a comprehensive survey, but it gives the definitions of the most important objects and constructions, and gives statements of and references to all the basic foundational results that we make use of in this paper. The subsection §3.3 describes Candel's uniformization theorem §3.3.1 for hyperbolic laminations, and describes how to use this theorem to construct the circle bundle $E_{\infty}$ over the leaf space $L$ of $\mathcal{F}$ for a taut foliation $\mathcal{F}$. The fiber of $E_{\infty}$ over a leaf $\lambda$ of $\mathcal{F}$ is just the ideal boundary of $\lambda$ in the sense of Gromov (see [30]). The more precise details of Candel’s theorem are necessary to define the correct topology on $E_{\infty}$. The circle bundle $E_{\infty}$ is used repeatedly throughout the rest of the paper.

§4 is also expository. We present the outlines of proofs of the Leaf Pocket theorem and the Universal Circle theorem from [8] (theorems 5.2 and 6.2 respectively in [8]). For most of §5 we do not need the details of the proofs of these theorems, and we proceed as far as possible from the axiomatic statements of these theorems. However, later in the paper we need to make use of some of the properties of the universal circles constructed in [8], and therefore it is necessary to explain the constructions in some detail.

§5 contains the really new results in this paper. We construct a pair of laminations $\Lambda_{\text{univ}}^\pm$ of the universal circle $S_{\text{univ}}^1$ constructed in §4 and use these laminations to construct a pair of 2–dimensional laminations $\Lambda_{\text{split}}^\pm$ of $M$ which are transverse to $\mathcal{F}$. We then go on to establish basic properties of these laminations. We say a foliation $\mathcal{F}$ has 2–sided branching if the leaf space $L$ of the pullback foliation $\mathcal{F}$ on the universal cover branches in both the positive and the negative directions. Our main result is the following:

**Theorem A.** Let $\mathcal{F}$ be a co–orientable taut foliation of a closed, orientable algebraically atoroidal 3–manifold $M$. Then either $\mathcal{F}$ has 2–sided branching and is the weak stable foliation of an Anosov flow, or else there are a pair of very full genuine laminations $\Lambda_{\text{split}}^\pm$ transverse to $\mathcal{F}$.

It follows by work of Gabai and Kazez, that a closed 3–manifold with a taut foliation either contains a $\mathbb{Z} \oplus \mathbb{Z}$ in its fundamental group, or contains an Anosov flow whose stable and unstable foliation have 2–sided branching, or else it has word–hyperbolic fundamental group, the mapping class group is finite, and every self–homeomorphism homotopic to the identity is isotopic to the identity.

Finally, in §6 we discuss the dynamics of the laminations $\Lambda_{\text{split}}^\pm$. This section is mainly descriptive, and serves to illustrate some of the structure developed in earlier sections.

1.6. **Notation.** We make use of certain conventions for notation throughout this paper, and try to be consistent throughout. For an object or structure $X$ in a 3–manifold $M$, $\tilde{X}$ will denote the pull back of $X$ to the universal cover $\tilde{M}$, where this makes sense. Surfaces and manifolds will be denoted by upper case Roman letters $S, M, N$ etc. and points by lower case Roman letters $p, q, r$ etc. Foliations will be denoted by script letters $\mathcal{F}, \mathcal{G}$ etc. and laminations by upper case Greek letters $\Lambda$ etc. Leaves will be denoted by lower case Greek letters $\lambda, \mu, \nu$ etc. Guts of genuine laminations will be denoted by Gothic $\mathfrak{G}$ and core circles of interstitial annuli by lower case Gothic letters $\mathfrak{C}$.
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2. **The Topology of $S^1$**

In this section we establish basic properties of the point set topology of $S^1$ which will be used in the rest of the paper. Good general references for point set topology in low dimensions are [35], [4] and [40].

2.1. **Laminations of $S^1$.**

**Definition 2.1.1.** We let $S^0$ denote the 0 sphere; i.e. the discrete, two element set. Two disjoint copies of $S^0$ in $S^1$ are homologically linked, or just linked if the points in one of the $S^0$'s are contained in different components of the complement of the other. Otherwise we say they are unlinked.

Note that the definition of linking is symmetric.

**Definition 2.1.2.** A lamination $\Lambda$ of $S^1$ is a closed subset of the space of unordered pairs of distinct points in $S^1$ with the property that no two elements of the lamination are linked as $S^0$'s in $S^1$. The elements of $\Lambda$ are called the leaves of the lamination.

The space of unordered pairs of distinct points in $S^1$ may be thought of as a quotient of $S^1 \times S^1 \setminus$ diagonal by the $\mathbb{Z}/2\mathbb{Z}$ action which interchanges the two components. Topologically, this space is homeomorphic to a Möbius band.

Most readers will be familiar with the concept of a geodesic lamination on a hyperbolic surface.

**Definition 2.1.3.** A geodesic lamination $\Lambda$ on a complete hyperbolic surface $\Sigma$ is a closed union of disjoint embedded complete geodesics.

For a thorough development of the elementary theory of geodesic laminations, see [17]. A geodesic lamination of $\Sigma$ pulls back to define a geodesic lamination of $\mathbb{H}^2$. Geodesic laminations of $\mathbb{H}^2$ and laminations of $S^1$ are essentially equivalent objects, as the following construction shows:

**Construction 2.1.4.** Let $\Lambda$ be a lamination of $S^1$. We think of $S^1$ as the boundary of $\mathbb{H}^2$ in the unit disk model. Then we construct a geodesic lamination of $\mathbb{H}^2$ whose leaves are just the geodesics whose endpoints are leaves of $\Lambda$. We will sometimes denote this geodesic lamination by $\Lambda_{geo}$. Conversely, given a geodesic lamination $\Lambda$ of $\mathbb{H}^2$, we get a lamination of the ideal boundary $S^1_\infty$ whose leaves are just the pairs of endpoints of the leaves of $\Lambda$. 
There is another perspective on circle laminations, coming from equivalence relations. The correct class of equivalence relations for our purposes are upper semi–continuous decompositions.

**Definition 2.1.5.** A decomposition of a topological space $X$ is a partition into compact subsets. A decomposition $\mathcal{G}$ is upper semi–continuous if for every decomposition element $\zeta \in \mathcal{G}$ and every open set $U$ with $\zeta \subset U$, there exists an open set $V \subset U$ with $\zeta \subset V$ such that every $\zeta' \in \mathcal{G}$ with $\zeta' \cap V \neq \emptyset$ has $\zeta' \subset U$. The decomposition is monotone if its elements are connected.

A proper map from a Hausdorff space $X$ to a Hausdorff space $Y$ induces a decomposition of $X$ by its point preimages which is upper semi–continuous. Conversely, the quotient of a Hausdorff space by an upper semi–continuous decomposition is Hausdorff, and the tautological map to the quotient space is continuous and proper. See e.g. [35].

**Definition 2.1.6.** An equivalence relation $\sim$ on $S^1$ is laminar if the equivalence classes are closed, if the resulting decomposition is upper semicontinuous, and if distinct equivalence classes are unlinked as subsets of $S^1$. That is, if $S^0_1, S^0_2 \subset S^1$ are two $S^0$’s which are contained in distinct equivalence classes, then they are not homologically linked in $S^1$.

We now show how to move back and forth between circle laminations and laminar relations.

**Construction 2.1.7.** Given a laminar equivalence relation $\sim$ of $S^1$, we think of $S^1$ as the ideal boundary of $\mathbb{H}^2$. Then for every equivalence class $[p]$ of $\sim$ we form the convex hull

$$H([p]) \subset \mathbb{H}^2$$

and the boundary of the convex hull

$$\Lambda([p]) = \partial H([p]) \subset \mathbb{H}^2$$

We let $\Lambda$ denote the union over all equivalence classes $[p]$:

$$\Lambda = \bigcup_{[p]} \Lambda([p])$$

Then the fact that the equivalence classes are unlinked implies that the geodesics making up $\Lambda$ are disjoint. Moreover, the fact that $\sim$ is upper semicontinuous implies that $\Lambda$ is closed as a subset of $\mathbb{H}^2$. That is, it is a geodesic lamination, and determines a lamination of $S^1$ by construction 2.1.4.

Conversely, given a lamination $\Lambda$ of $S^1$, we may form the quotient $Q$ of $S^1$ by the smallest equivalence relation which collapses every leaf to a point. This is not necessarily Hausdorff; we let $Q'$ denote the Hausdorffification. Then the map from $S^1$ to $Q'$ induces an upper semi–continuous decomposition of $S^1$. Moreover, this equivalence relation is obviously unlinked; in particular, it is laminar.

We abstract part of construction 2.1.7 to show that every subset $K \subset S^1$ gives rise to a lamination, as follows

**Construction 2.1.8.** Let $K \subset S^1$ be arbitrary. Think of $S^1$ as $\partial \mathbb{H}^2$, and let $H(\overline{K}) \subset \mathbb{H}^2$ be the convex hull of the closure of $K$ in $S^1$. Then the boundary $\partial H(\overline{K})$ is a geodesic lamination of $\mathbb{H}^2$, which determines a lamination of $S^1$ by construction 2.1.4.

We denote this lamination of $S^1$ by $\Lambda(K)$. 


2.2. Monotone maps.

Definition 2.2.1. Let \( S^1_X, S^1_Y \) be homeomorphic to \( S^1 \). A continuous map \( \phi : S^1_X \rightarrow S^1_Y \) is monotone if it is degree one, and if it induces a monotone decomposition of \( S^1_X \), in the sense of definition 2.1.5.

Note that the target and image circle should not be thought of as the same circle.
Equivalently, a map between circles is monotone if the point preimages are connected and contractible. Said yet another way, a map is monotone if it does not reverse the cyclic order on triples of points for some choice of orientations on the target and image circle.

Definition 2.2.2. Let \( \phi : S^1_X \rightarrow S^1_Y \) be monotone. The gaps of \( \phi \) are the maximal open connected intervals in \( S^1_X \) in the preimage of single points of \( S^1_Y \). The core of \( \phi \) is the complement of the union of the gaps.

Note that the gaps of \( \phi \) are the connected components of the set where \( \phi \) is locally constant.

Recall that a set is perfect if no element is isolated.

Lemma 2.2.3. Let \( \phi : S^1_X \rightarrow S^1_Y \) be monotone. Then the core of \( \phi \) is perfect.

Proof. The core of \( \phi \) is closed. If it is not perfect, there is some point \( p \in \text{core}(\phi) \) which is isolated in \( \text{core}(\phi) \). Let \( p^\pm \) be the nearest points in \( \text{core}(\phi) \) to \( p \) on either side, so that the open oriented intervals \( p^-p \) and \( pp^+ \) are gaps of \( \phi \). But then by definition,
\[
\phi(p^-) = \phi(p) = \phi(p^+) = \phi(r)
\]
for any \( r \) in the oriented interval \( p^-p^+ \). So by definition, the interior of this interval is contained in a single gap of \( \phi \). In particular, \( p \) is contained in a gap of \( \phi \), contrary to hypothesis. \( \square \)

It follows that the set of points in \( \text{core}(\phi) \) which are nontrivial limits from both directions is dense in \( \text{core}(\phi) \).

Example 2.2.4 (The Devil’s staircase). Let \( f : [0, 1] \rightarrow [0, 1] \) be the function defined as follows. If \( t \in [0, 1] \), let
\[
0 \cdot t_1t_2t_3 \cdots
\]
denote the base 3 expansion of \( t \). Let \( i \) be the smallest index for which \( t_i = 1 \). Then \( f(t) = s \) is the number whose base 2 expansion is
\[
0 \cdot s_1s_2s_3 \cdots s_is_00 \cdots
\]
where each \( s_j = 1 \) iff \( t_j = 1 \) or \( 2 \) and \( j \leq i \), and \( s_j = 0 \) otherwise. The graph of this function is illustrated in figure [1].

The core of this map is the usual middle third Cantor set.

Definition 2.2.5. Let \( B \) be a topological space, and \( E \) a circle bundle over \( B \). A monotone family of maps is a continuous map
\[
\phi : S^1 \times B \rightarrow E
\]
which covers the identity map on \( B \), and which restricts for each \( b \in B \) to a monotone map of circles
\[
\phi_b = \phi|_{S^1 \times b} : S^1 \times b \rightarrow E_b
\]
We denote a monotone family by the triple \((E, B, \phi)\).
Lemma 2.2.6. Let \((E, B, \phi)\) be a monotone family. Then the family of subsets \(\{\text{core}(\phi_b)\}\) vary lower semicontinuously as a function of \(b \in B\), in the Hausdorff topology. That is, if \(x \in \text{core}(\phi_b)\), then if \(b_i \to b\) in \(B\), there are points \(x_i \in \text{core}(\phi_{b_i})\) such that \(x_i \to x\).

Proof. Let \(x \in \text{core}(\phi_b)\). By lemma 2.2.3, it follows there is a sequence of distinct points \(x_i \to x\) such that \(\phi_b(x_i) \neq \phi_b(x_j)\) for each \(i, j\). It follows that for each \(i\) there is a \(k\) such that \(\phi_{b_K}(x_i) \neq \phi_{b_K}(x_{i+1})\) for all \(K \geq k\). In particular, the core of \(\phi_{b_K}\) contains some point between \(x_i\) and \(x_{i+1}\). The lemma follows.

It follows that the closure of the union of gaps of \(\phi_b\) varies upper semicontinuously as a function of \(b\). An alternate proof of lemma 2.2.6 uses the fact that the closures of gaps are exactly the nontrivial elements in the decomposition of \(S^1 \times B\) induced by \(\phi\).

Definition 2.2.7. Let \((E, B, \phi)\) be a monotone family. Let \(X \subset B\) be a subspace. Define

\[
\text{core}(X) = \bigcup_{b \in X} \text{core}(\phi_b)
\]

Notice that we define \(\text{core}(X)\) to be the closure of the union of the cores of \(\phi_b\) over all \(b \in X\), and not simply the ordinary union. This is important to keep in mind in the sequel; we will refer to this construction on a number of occasions in section 5.

Theorem 2.2.8. Let \((E, B, \phi)\) be a monotone family, and suppose \(X, Y\) are path connected subsets of \(B\). Suppose for each \(x \in X\) and \(y \in Y\) that \(\text{core}(\phi_x)\) and \(\text{core}(\phi_y)\) are unlinked. Then \(\text{core}(X)\) and \(\text{core}(Y)\) are unlinked.

Proof. Since \(\text{core}(\phi_x)\) and \(\text{core}(\phi_y)\) are unlinked for each pair \(x \in X, y \in Y\), it follows that \(\text{core}(\phi_x)\) is contained in the closure of a single gap of \(\text{core}(\phi_y)\), and vice versa.
We claim that for every \( x \in X \), \( \text{core}(\phi_x) \) is contained in the closure of the same gap of \( \text{core}(\phi_y) \). For, let \( g \) be a gap of \( \phi_y \), and let \( T_g \subset X \) be the set of points \( t \) for which \( \text{core}(\phi_t) \subset g \). Since \( g \) is closed, by lemma 2.2.6 the set \( T_g \) is closed. Moreover, by lemma 2.2.3 distinct gaps have disjoint closures, and therefore if \( g_1, g_2 \) are distinct gaps, \( T_{g_1} \) and \( T_{g_2} \) are disjoint. Let \( x_1, x_2 \in X \) be arbitrary, and let \( \gamma \) be a path in \( X \) from \( x_1 \) to \( x_2 \). Then \( \gamma \) is decomposed into closed subsets which are the intersections \( \gamma \cap T_g \) as \( g \) varies over the gaps of \( \phi_y \). But there are only countably many gaps of \( \phi_y \). On the other hand, any decomposition of an interval into countably many closed subsets has only one element, by a beautiful theorem of Sierpinski [50]. It follows that \( T_g = X \), and every \( \text{core}(\phi_x) \) is contained in the same gap \( g \) of \( \phi_y \). We can therefore label \( g \) unambiguously as \( g_{y'} \) and similarly construct \( g_y' \) for every other \( y' \in Y \).

Now, as \( y \) varies in \( Y \), the closures of gaps \( \overline{g_y} \) do not vary continuously, but merely upper semicontinuously. In particular, if \( y_i \to y \) then

\[
\lim_{i \to \infty} \overline{g_y} \subset \overline{g_y}
\]

for every Hausdorff limit. Since each \( g_y \) is a closed arc, the same is true of each Hausdorff limit. For each such discontinuous limit, i.e. where \( \lim_{i \to \infty} \overline{g_y} \neq \overline{g_y} \),

we interpolate a 1-parameter family of closed arcs from \( \lim_{i \to \infty} \overline{g_y} \) to \( \overline{g_y} \) which are all contained in \( \overline{g_y} \). Let \( \mathcal{G} \) denote the union of the set of arcs \( \overline{g_y} \) with \( y \in Y \) and the arcs in the interpolating families. Then \( \mathcal{G} \) is a connected subset of the space of closed arcs in \( S^1 \). It follows that \( \text{core}(X) \) and \( \text{core}(Y) \) are unlinked, as claimed. \( \square \)

2.3. Pushforward of laminations. Laminations of \( S^1 \) can be pushed forward by monotone maps.

**Definition 2.3.1.** Let \( \Lambda \) be a lamination of \( S^1_X \), and \( \phi : S^1_X \to S^1_Y \) a monotone map. Then \( \phi \) induces a map from unordered pairs of points in \( S^1_X \) to unordered pairs of points in \( S^1_Y \). We let \( \phi(\Lambda) \) denote the image of \( \Lambda \) in the complement of the diagonal.

**Lemma 2.3.2.** Let \( \phi : S^1_X \to S^1_Y \) be monotone, and let \( \Lambda \) be a lamination of \( S^1_X \). Then \( \phi(\Lambda) \) is a lamination of \( S^1_Y \).

**Proof.** The map \( \phi \) induces a continuous map from \( S^1_X \times S^1_X \to S^1_Y \times S^1_Y \) which takes the diagonal to the diagonal. It follows that the image of \( \Lambda \) is closed in \( S^1_Y \times S^1_Y \) \( \setminus \) diagonal. It remains to show that it is unlinked. But monotone maps do not reverse the cyclic order of subsets; the claim follows. \( \square \)

Laminations can also be pulled back by monotone maps.

**Definition 2.3.3.** Let \( \Lambda \) be a lamination of \( S^1_Y \), and \( \phi : S^1_X \to S^1_Y \) a monotone map. Then \( \Lambda \) determines a laminar equivalence relation \( \sim_X \) on \( S^1_Y \), by construction 2.1.7.

Let \( \sim_X \) be the equivalence relation on \( S^1_X \) whose equivalence classes are the preimages of equivalence classes in \( \sim_Y \). Then \( \sim_X \) is a laminar relation, and induces a lamination of \( S^1_X \) by construction 2.1.7 which we denote \( \phi^{-1}(\Lambda) \).

The proof that \( \sim_X \) is laminar follows immediately from the fact that \( \phi \) is monotone.
2.4. **Coarse geometry of the hyperbolic plane.** This subsection summarizes some basic facts about coarse geometry, quasigeodesics and geodesics in the hyperbolic plane $\mathbb{H}^2$. We will use the results in this subsection implicitly throughout the rest of the paper, usually without comment. It is included here simply as a service to those readers who might be unfamiliar with or hazy on this material.

The material in this subsection is completely standard; excellent references are [30] and [17].

**Definition 2.4.1.** Let $X$ be a metric space. Let $k > 1$ and $\epsilon > 0$. A $(k, \epsilon)$–quasigeodesic is a map $l : \mathbb{R} \to X$ such that for all $p, q \in \mathbb{R}$,

$$\frac{1}{k}d_X(l(p), l(q)) - \epsilon \leq d(p, q) \leq kd_X(l(p), l(q)) + \epsilon$$

where $d_X(\cdot, \cdot)$ denotes distance in $X$, and $d(\cdot, \cdot)$ denotes Euclidean distance in $\mathbb{R}$.

**Lemma 2.4.2.** Let $\gamma \subset \mathbb{H}^2$ be a $(k, \epsilon)$–quasigeodesic. Then there is a constant $C(k, \epsilon)$ such that there is a complete geodesic $\gamma_s \subset \mathbb{H}^2$ which is $C(k, \epsilon)$ close to $\gamma$ in the Hausdorff metric.

**Lemma 2.4.3.** Let $\gamma_1, \gamma_2$ be two geodesics in $\mathbb{H}^2$ which are Hausdorff distance $C$ apart on subsegments $\gamma_1', \gamma_2'$ of length $t$. Then there is a constant $C_1$ which does not depend on $C$, such that $\gamma_1'$ and $\gamma_2'$ are Hausdorff distance $C/2$ apart on subsegments of length $t - C_1$.

By applying lemma 2.4.3 iteratively, one sees that for any $C$ and any $\epsilon$ there is a $t(C)$ such that two geodesic segments of length $\geq t(C)$ which are Hausdorff distance $C$ apart are Hausdorff distance $\epsilon$ apart on their middle third subsegments. In particular, two bi–infinite geodesics which are a finite Hausdorff distance apart are equal.

**Lemma 2.4.4.** Let $\Lambda$ be a lamination of $S^1_X$, and let $h : S^1_X \to S^1_Y$ be a homeomorphism. Suppose $S^1_X$ and $S^1_Y$ bound copies $\mathbb{H}^2_X, \mathbb{H}^2_Y$ of the hyperbolic plane, and let $\Lambda_{geo}, \Lambda_{geo}$ be the geodesic laminations determined by construction 2.1.4. Then $h$ extends to a homeomorphism

$$H : \mathbb{H}^2_X \to \mathbb{H}^2_Y$$

taking $\Lambda_{geo}$ to $(h(\Lambda))_{geo}$.

See [17] for the proofs of these facts.

**Notation 2.4.5.** If $X, Y$ are Hausdorff distance $\leq C$ apart in some third metric space $Z$, we sometimes abbreviate this by saying $X$ and $Y$ are $C$ close in $Z$, or just $C$ close if $Z$ is understood.

3. **The theory of essential laminations**

In this section we define taut foliations and essential laminations, and present some of their fundamental theory and properties. None of the material in this section is new, but it is presented here for the convenience of the reader. A good reference for the general theory of foliations is [14]. References for basic 3-manifold topology are [34], [32] and [36].
3.1. **Taut foliations.** A 2–dimensional foliation of a 3–manifold is a partition into surfaces called *leaves* whose local connected components have a product structure.

A foliation is *orientable* if the leaves can be oriented in a continuously varying way. It is *co–orientable* if the transverse space can be oriented. By passing to a cover of index at most 4, we can assume that our foliations are orientable and co–orientable. Note that this implies that the ambient manifold is itself orientable. Throughout this paper we assume that all our manifolds are orientable, and all our foliations are orientable and co–orientable.

A basic atomic unit in the theory of foliations of 3–manifolds is the following:

**Definition 3.1.1.** Let $H$ be the closed upper half space in $\mathbb{R}^3$ minus the origin. $H$ is foliated by its intersection with horizontal planes. Every leaf is either a plane or a punctured plane. A nontrivial dilation $\phi$ centered at the origin preserves this foliation of $H$, so it descends to a foliation of the quotient manifold $S = H/\langle \phi \rangle$ which is a solid torus. This foliation is called a *Reeb component*.

See figure 2 for a cutaway of half a Reeb component.

![Figure 2](image)

**Figure 2.** A Reeb component is a foliation of a solid torus by planar leaves. Each leaf is like a sock which is swallowed by the next sock.

Every closed 3–manifold admits a 2–dimensional foliation [39, 53] but such foliations typically contain Reeb components. Notice that the boundary torus of a Reeb component is compressible, by a compression lying entirely within the component. Conversely, foliations with topologically essential leaves are much harder to construct.

**Definition 3.1.2.** A 2–dimensional foliation $\mathcal{F}$ of a closed 3–manifold $M$ is *taut* if no leaf is a sphere or projective plane, and there is a map $\phi : S^1 \to M$ which is transverse to $\mathcal{F}$, and which intersects every leaf.

The condition that $\mathcal{F}$ has no sphere or projective plane leaf is a convention to rule out some very special cases. By the Reeb stability theorem [46] if $\mathcal{F}$ contains
a sphere or projective plane leaf, then $\mathcal{F}$ is finitely covered by a product foliation of $S^2 \times S^1$ by leaves $S^2 \times \text{point}$.

The relationship between taut foliations and Reeb components is complementary, so that one has the following theorem, which is basically due to Novikov [42]:

**Theorem 3.1.3** (Novikov). A foliation $\mathcal{F}$ of an atoroidal 3–manifold $M$ is taut iff it contains no Reeb components.

Taut foliations have several distinct lives: a topological life, a dynamical life, and a geometric life.

Firstly, they certify many useful topological properties of $M$. For instance, there is the following theorem which combines work of Palmeira [44] with earlier important work of Novikov [42] and Rosenberg [48].

**Theorem 3.1.4** (Palmeira, Novikov, Rosenberg). Let $M$ be a 3–manifold which admits a taut foliation $\mathcal{F}$. Then the universal cover $\tilde{M}$ is homeomorphic to $\mathbb{R}^3$, and the leaves of $\tilde{\mathcal{F}}$ are planes. Moreover, there is a foliation $F$ of $\mathbb{R}^2$ by lines such that the pair $(\tilde{M}, \tilde{\mathcal{F}})$ is topologically equivalent to a product

$$(\tilde{M}, \tilde{\mathcal{F}}) = (\mathbb{R}^2, F) \times \mathbb{R}$$

The leaf space of $\tilde{\mathcal{F}}$ is just the quotient space of $\tilde{M}$ by the equivalence relation whose equivalence classes are the leaves of $\tilde{\mathcal{F}}$. We denote this leaf space by $L$. It follows from theorem [3.1.4] that the leaf space of $\tilde{\mathcal{F}}$ is simply connected; on the other hand, it is typically non–Hausdorff. It will become apparent, especially in §5 that this non–Hausdorffness of $L$ is fundamentally the source of much of the structure that we develop, so one should not be too upset to encounter it. In any case, the action of $\pi_1(M)$ on $\tilde{M}$ descends to an action on $L$ by homeomorphisms.

**Definition 3.1.5.** We denote the representation inducing the action of $\pi_1(M)$ on $L$ by

$$\rho_{\text{hol}} : \pi_1(M) \rightarrow \text{Homeo}(L)$$

and call this homomorphism the **holonomy homomorphism**.

Since many readers might not have considered the subject, we should say a few words at this point about non–Hausdorff 1–manifolds. The kinds of non–Hausdorff 1–manifolds we consider are obtained from countably many copies of the open unit interval $I_i$ by identifying open subsets in pairs. In this way, each $I_i$ embeds into the quotient space, and each point in the quotient space is contained in the interior of an embedded interval. One may pass from $L$ to its Hausdorffification, which is just the maximal Hausdorff quotient, and is obtained from $L$ by inductively identifying pairs of points which are not contained in disjoint open subsets; i.e. it is obtained by quotienting out the closed equivalence relation generated by the property of being nonseparated. The Hausdorffification of $L$ is also simply–connected, and is a more familiar kind of object. It is homeomorphic to the underlying topological space of an $\mathbb{R}$–tree (although in fact, the underlying space of an $\mathbb{R}$–tree is more general, and is not typically assumed to be 2nd countable). In practice, one reasons about $L$ by thinking about this Hausdorff quotient, and remembering that branch points of the quotient correspond to unseparable sets of
points in \( L \). The study of the action of \( \pi_1(M) \) on \( L \) by the holonomy homomorphism falls into the domain of arboreal group theory. See [47] for an important example of the kinds of results that may be obtained by such methods.

A co–orientation on \( \mathcal{F} \) pulls back to a co–orientation on \( \mathcal{F} \), and defines an orientation on \( L \). So the Hausdorff quotient should be thought of more as having the structure of an oriented train track (i.e. there is a combing at the branch points into positive and negative directions) than a tree.

Taut foliations can be classified in terms of the kind of branching exhibited by the (Hausdorffified) leaf space \( L \). If \( L \) is Hausdorff, equivalently if its Hausdorffification does not branch at all, then of course it is homeomorphic to \( \mathbb{R} \). In this case \( \mathcal{F} \) is said to be \( \mathbb{R} – \text{covered} \). If \( L \) branches in only one direction (e.g. the positive direction), we say \( \mathcal{F} \) has one sided branching, and otherwise we say \( \mathcal{F} \) has two sided branching. If \( \mathcal{F} \) has one sided branching, then necessarily \( \mathcal{F} \) is co–orientable. In this paper we will concentrate on the case that \( \mathcal{F} \) has two sided branching. Analogous results in the case that \( \mathcal{F} \) is \( \mathbb{R} – \text{covered} \) or has one sided branching are contained in [5] and [6] respectively. See figure 3 for an example of (part of) the universal cover of a foliation with two–sided branching.

![Figure 3](image-url)

**Figure 3.** This foliation of a topological ball by planes exhibits two–sided branching.

The orientation on \( L \) defines a partial order on \( L \), as follows.

**Definition 3.1.6.** The canonical partial order on \( L \) is defined as follows. Let \( \lambda, \mu \) be leaves of \( \mathcal{F} \) If there is a positively oriented transversal to \( \mathcal{F} \) from \( \lambda \) to \( \mu \), then

\[
\mu > \lambda
\]

Similarly, if there is a negatively oriented transversal, then \( \lambda > \mu \). Note that if \( \lambda < \mu \) and \( \mu < \lambda \) then \( \mu = \lambda \), by theorem 3.1.4 If there is no transversal between \( \mu \) and \( \lambda \), we say the leaves are incomparable.

Note that the co–orientation of \( \mathcal{F} \) lets us define unambiguously the positive and negative sides of \( \lambda \) in \( M \). Every leaf of \( \mathcal{F} \setminus \lambda \) is either on the positive or negative side. Moreover, if \( \mu > \lambda \), then \( \mu \) is on the positive side, and if \( \mu < \lambda \), then \( \mu \) is on the negative side, but not conversely. The reader should be careful to distinguish between the two notions.
Taut foliations equally well certify useful topological properties of surfaces. The following theorem amalgamates a theorem of Thurston [54] and a (much harder) theorem of Gabai [21]. Here $\chi(\Sigma)$ denotes the Euler characteristic.

**Theorem 3.1.7** (Gabai, Thurston). Let $M$ be a compact connected irreducible orientable 3-manifold whose boundary $\partial M$ is a (possibly empty) union of tori. A properly embedded homologically essential surface $\Sigma$ is a leaf of a taut foliation of $M$ if and only if it minimizes $-\chi(\Sigma)$ amongst all proper embedded surfaces with no spherical components in its homology class.

**Remark 3.1.8.** Given a surface $S$ in a manifold $M$ satisfying the hypotheses of Theorem 3.1.7, the Thurston norm of $S$ is defined to be the sum of $-\chi(S_i)$ over all non-spherical components $S_i$ of $S$. So this theorem may be restated as saying that a properly embedded surface without spherical components in a manifold $M$ (as above) is a leaf of a taut foliation iff it is Thurston norm minimizing in its homology class. The “if” direction is part of Theorem 5.5, page 445 from [21]; the “only if” is part of Corollary 1, page 118 from [54].

The second life of a taut foliation is dynamical. There is a basic duality between minimal surfaces and volume preserving transverse flows, which in its most rudimentary form is just the min cut — max flow theorem from graph theory. The next theorem of Sullivan [51], [52] makes this precise:

**Theorem 3.1.9** (Sullivan). Let $\mathcal{F}$ be a co–orientable $C^2$ foliation of $M$. The following are equivalent:

1. $\mathcal{F}$ is taut.
2. $\mathcal{F}$ admits a volume preserving transverse flow for some volume form.
3. There is a closed 2–form $\theta$ on $M$ which is positive on $T\mathcal{F}$.
4. There is a Riemannian metric on $M$ for which leaves of $\mathcal{F}$ are (calibrated) minimal surfaces.

If $\mathcal{F}$ is not $C^2$, there is a combinatorial version of this theorem due to Hass, based on an idea of Thurston [51]. We will not make use of the theorem of Sullivan in this paper except as a suggestive analogy.

Finally, foliations reveal geometry of the underlying 3-manifold. One of the main applications of this paper will be to show that the existence of a taut foliation on a 3-manifold allows one to construct auxiliary objects which certify geometric properties of $M$, for instance word hyperbolicity of $\pi_1(M)$. This will be developed in detail in the sequel, and therefore we postpone a discussion until the appropriate time.

### 3.2. Essential laminations.

A 2-dimensional lamination of a 3-manifold is a decomposition of a closed subset into surfaces, which come together locally in product charts. The transverse structure to a lamination is not a manifold in general, but rather an arbitrary (locally compact) space. The class of laminations which most closely resemble taut foliations in their utility are essential laminations.

Essential laminations were introduced into 3-manifold topology in [28] as a simultaneous generalization of the concepts of taut foliations and of incompressible surfaces. Taut foliations can be turned into (nowhere dense) essential laminations in a more or less trivial way by blowing up leaves — i.e. by replacing a leaf with a complementary $I$–bundle. Genuine laminations are those essential laminations which do not arise from this construction. However, it was not until [25] that the
usefulness of the concept of a genuine lamination was realized. Genuine laminations are characterized amongst essential laminations by the property that some complementary region is not an $I$–bundle. Following [28] we make this precise.

**Definition 3.2.1.** The complement of a 2–dimensional lamination $\Lambda$ of a 3–manifold $M$ falls into connected components called *complementary regions*. A lamination is *essential* if it contains no spherical leaf or torus leaf bounding a solid torus, and furthermore, if $C$ is the metric completion of a complementary region (with respect to the path metric on $M$), then $C$ is irreducible, and $\partial C$ is both incompressible and *end incompressible* in $C$. Here an end compressing disk is a properly embedded

$$D^2 - \text{(point in } \partial D^2) \subset C$$

which is not properly isotopic rel. $\partial$ in $C$ to an embedding in a leaf.

![Diagram of an end compressing disk](image)

**Figure 4.** An end compressing disk (boundary in red) is also called an *essential monogon*.

Such an end compressing disk is also called an *essential ideal monogon*, or by abuse of notation, an *essential monogon*. See figure 4 for the justification for this terminology. Note: some authors prefer the term *essential disk-with-end*.

Another way of phrasing the conditions above are that there should be no spherical leaf or torus leaf bounding a solid torus, and complementary regions should contain no essential surfaces (possibly with boundary and/or ideal points) of positive Euler characteristic. The missing point in the boundary of an end compressing disk should be thought of as an ideal point. The Euler characteristic of a polygon with ideal points can be calculated by doubling it: the double of an (ideal) monogon is a punctured sphere, so a monogon has Euler characteristic $1/2$. An ideal bigon has Euler characteristic 0, an ideal triangle (a “trigon”) has Euler characteristic $-1/2$ and so on.

A taut foliation is an example of an essential lamination. An incompressible surface in an irreducible manifold is another example.
A complementary region to a lamination decomposes into a compact *gut* piece and non–compact *interstitial regions* which are $I$–bundles over non–compact surfaces. These interstitial regions are also referred to in the literature as *interstitial $I$–bundles* and *interstices*. These pieces meet along *interstitial annuli*. Formally, the interstitial regions make up the non–compact components of the *characteristic $I$–bundle* of the complementary region (see [36]). For more details, see [25] or [28].

**Definition 3.2.2.** An essential lamination is *genuine* if some complementary region has nonempty gut.

Said another way, an essential lamination is genuine if some complementary region contains some essential surface (possibly with boundary and/or ideal points) of negative Euler characteristic.

The leaf space of $	ilde{\Lambda}$ is in general an *order tree*. Following [28], an order tree can be defined as follows.

**Definition 3.2.3.** An *order tree* is a set $T$ together with a collection $S$ of linearly ordered subsets called *segments*, each with distinct least and greatest elements called the *initial* and *final* ends. If $\sigma$ is a segment, $-\sigma$ denotes the same subset with the reverse order, and is called the *inverse* of $\sigma$. The following conditions should be satisfied:

1. If $\sigma \in S$ then $-\sigma \in S$
2. Any closed subinterval of a segment is a segment (if it has more than one element)
3. Any two elements of $T$ can be joined by a finite sequence of segments $\sigma_i$ with the final end of $\sigma_i$ equal to the initial end of $\sigma_{i+1}$
4. Given a cyclic word $\sigma_0 \sigma_1 \cdots \sigma_{k-1}$ (subscripts mod $k$) with the final end of $\sigma_i$ equal to the initial end of $\sigma_{i+1}$, there is a subdivision of the $\sigma_i$ yielding a cyclic word $\rho_0 \rho_1 \cdots \rho_{n-1}$ which becomes the trivial word when adjacent inverse segments are cancelled
5. If $\sigma_1$ and $\sigma_2$ are segments whose intersection is a single element which is the final element of $\sigma_1$ and the initial element of $\sigma_2$ then $\sigma_1 \cup \sigma_2$ is a segment

If all the segments are homeomorphic to subintervals of $\mathbb{R}$ with their order topology, then $T$ is an $\mathbb{R}$–order tree.

An order tree is topologized by the usual order topology on segments. Order trees are not typically Hausdorff, but even if they are, there are many more possibilities than arise in the case of a foliation.

**Definition 3.2.4.** An essential lamination $\Lambda$ is *tight* if the leaf space of the universal cover $\tilde{\Lambda}$ is Hausdorff.

It follows that a taut foliation is tight iff it is $\mathbb{R}$–covered. Equivalently, a lamination $\Lambda$ is tight if every arc $\alpha$ in $M$ is homotopic rel. endpoints to an efficient arc which is either transverse or tangent to $\Lambda$. Here an arc $\alpha$ is *efficient* if it does not contain a subarc $\beta$ whose interior is disjoint from $\Lambda$, and which cobounds with an arc $\beta'$ in a leaf of $\Lambda$ a disk whose interior is disjoint from $\Lambda$.

If $\Lambda$ has no isolated leaves, then the associated order tree of $\tilde{\Lambda}$ is actually an $\mathbb{R}$–order tree. Any lamination can be transformed into one without isolated leaves by blowing up isolated leaves to foliated interval bundles. It follows that we can always consider $\mathbb{R}$–order trees for our applications.
Moreover, if $\Lambda$ is tight, a Hausdorff $\mathbb{R}$–order tree is just the underlying topological space of an $\mathbb{R}$–tree. We refer to such a space as a topological $\mathbb{R}$–tree to emphasize that the metric is not important. Finally, if $\Lambda$ is a tight 1–dimensional lamination of a surface, so that $\tilde{\Lambda}$ is a tight 1–dimensional lamination of the plane, then the associated order tree $T$ comes with a natural planar embedding, dual to $\tilde{\Lambda}$. See [24] for more details.

Whether or not a lamination $\Lambda$ is tight, the following is true:

**Lemma 3.2.5.** Let $\Lambda$ be an essential lamination of a closed 3–manifold $M$, and give $M$ an arbitrary Riemannian metric. Then there is an $\epsilon > 0$ such that every leaf $\lambda$ of $\tilde{\Lambda}$ is quasi–isometrically embedded in its $\epsilon$–neighborhood, and no two incomparable leaves $\lambda, \mu$ of $\tilde{\Lambda}$ contain points which are closer than $\epsilon$ in $\tilde{M}$.

This lemma is an easy consequence of the compactness of $M$ and the defining property of laminations, that they have local product charts. See [8] for a proof.

Such an $\epsilon$ is called a separation constant for $\Lambda$.

Genuine laminations certify important properties of the ambient manifold $M$. The existence of the interstitial annuli gives a canonical collection of knots in $M$ with important properties. Using these annuli, Gabai and Kazeez prove the following in [26] and [27]. For the definition of word–hyperbolicity of a group, see [30].

**Theorem 3.2.6 (Gabai–Kazeez [26] Word hyperbolicity).** Let $M$ be an atoroidal 3–manifold containing a genuine lamination $\Lambda$. Then $\pi_1(M)$ is word–hyperbolic in the sense of Gromov.

**Theorem 3.2.7 (Gabai–Kazeez [27] Finite MCG).** Let $M$ be an atoroidal 3–manifold containing a genuine lamination $\Lambda$. Then the mapping class group of $M$ is finite.

Laminations come in all degrees of smoothness. Moreover, it is important to distinguish between the smoothness of leaves and the smoothness of the transverse space. For some applications in this paper, it will be important for our laminations to be leafwise smooth. Fortunately, the situation for 2–dimensional laminations in 3–manifolds is as simple as it could be. The main theorem of [7] is the following:

**Theorem 3.2.8.** Let $\Lambda$ be a 2–dimensional lamination in a smooth 3–manifold $M$. Then $\Lambda$ is isotopic to a lamination with smoothly immersed leaves.

**Remark 3.2.9.** 2–dimensional laminations are also sometimes referred to informally as surface laminations.

Amongst all genuine laminations, some are more useful than others. If $M$ is not Haken, then [33] show that the gut regions are all homeomorphic to handlebodies. They call such laminations full, where the terminology is meant to imply that the complementary regions contain no closed incompressible surface. Specializing further, we have the following.

**Definition 3.2.10.** A genuine lamination is very full if all complementary regions are finite sided ideal polygon bundles over $S^1$.

The relationship between the topology of the guts and the topology of the complementary regions is not straightforward in general. However, in the case of a lamination with solid torus guts, the following lemma is proved in [8]:

[24] Reference to a source.
[26] Reference to a source.
[27] Reference to a source.
[30] Reference to a source.
[33] Reference to a source.
Lemma 3.2.11 (Calegari–Dunfield [8] Filling Lemma). Let $\Lambda$ be a genuine lamination of a closed 3–manifold $M$ with solid torus guts. Then $\Lambda$ is a sublamination of a very full genuine lamination $\overline{\Lambda}$. Moreover, if $\Lambda$ is tight, so is $\overline{\Lambda}$.

Very full genuine laminations are particularly nice. There is the following theorem of Gabai and Kazez from [25]:

Theorem 3.2.12 (Gabai–Kazez). Let $M$ be a 3–manifold with a very full genuine lamination $\Lambda$. Then any self–homeomorphism of $M$ homotopic to the identity is isotopic to the identity.

Tight very full genuine laminations have another application, more central to the theme of this paper. In [8] Calegari and Dunfield show that they give rise to a universal circle, as alluded to in §1. In this paper, the very full genuine laminations we produce, although not necessarily tight, already come with the data of a universal circle, so the construction in [8] is superfluous for our purposes.

3.3. Candel’s uniformization theorem. The classical uniformization theorem says that every Riemann surface is conformally equivalent to a surface with a complete metric of constant curvature $1$, $0$ or $-1$. Such a conformal equivalence is called a uniformizing map. If the curvature of the metric is negative, then the surface is said to be of hyperbolic type, and the metric is unique.

Riemann surface laminations can be uniformized leafwise; an important question is to determine when these choices of uniformizing maps are all of the same type, and can be chosen in a continuously varying way.

For Riemann surfaces with all leaves hyperbolic, the complete answer is given by a theorem of Candel [12]:

Theorem 3.3.1 (Candel [12] Uniformization for hyperbolic laminations). Let $\Lambda$ be a 2–dimensional lamination with a leafwise Riemannian metric such that every leaf is of hyperbolic type. Then the leafwise constant curvature hyperbolic metric determined uniquely by the conformal structure of the leaves of $\Lambda$ varies continuously in the transverse direction.

Let $M$ be an atoroidal 3–manifold, and $\Lambda$ an essential lamination. Then the leaves of $\Lambda$ are all of hyperbolic type (see e.g. [5]) and therefore Candel’s theorem applies. If $\Lambda$ is an essential surface lamination with smooth leaves, then there is a $C^0$ Riemannian metric on $M$ (i.e. a continuous section of the bundle of positive definite symmetric 2–tensors) which restricts on each leaf to a Riemannian metric of constant curvature $-1$. In general, if $\Lambda$ is transversely $C^n$, then the Riemannian metric on $M$ can be chosen to be $C^n$.

Using Candel’s theorem, we can define the circle bundle at infinity of an essential lamination.

Definition 3.3.2. Let $\Lambda$ be an essential lamination of $M$, and let $L$ be the leaf space of $\Lambda$. Uniformize $\Lambda$ by theorem 3.3.1 so that every leaf of $\Lambda$ is isometric to $\mathbb{H}^2$.

For each leaf $\lambda$ of $\Lambda$, let $S^1_{\infty}(\lambda)$ denote the ideal boundary of $\lambda$ with respect to this metric. The endpoint map

$$e : UT_p \lambda \to S^1_{\infty}(\lambda)$$

takes a unit vector $v$ in $\lambda$ at $p$ to the endpoint at infinity of the geodesic ray $\gamma_v \subset \lambda$ which emanates from $p$, and satisfies $\gamma_v'(0) = v$. 

Definition 3.3.3. Let $\Lambda$ be an essential lamination of $M$, and let $L$ be the leaf space of the pulled back lamination in the universal cover $\tilde{\Lambda}$. Uniformize $\Lambda$ by theorem 3.3.1 so that every leaf of $\tilde{\Lambda}$ is isometric to $\mathbb{H}^2$. For each leaf $\lambda$ of $\tilde{\Lambda}$, let $S^1_\infty(\lambda)$ denote the circle at infinity of $\lambda$, under its isometric identification with $\mathbb{H}^2$. The circle bundle at infinity is the topological space whose underlying set is the disjoint union
$$E_\infty = \bigcup_{\lambda \in L} S^1_\infty(\lambda)$$
and with the smallest topology so that the endpoint map
$$e : UT\tilde{\Lambda} \to E_\infty$$
is continuous.

With this topology, $E_\infty$ is a circle bundle over $L$, whose fiber over each $\lambda \in L$ is $S^1_\infty(\lambda)$.

Note that by Candel’s theorem 3.3.1 for every efficient transversal $\tau$ to $\tilde{\Lambda}$, the restriction
$$e : UT\tilde{\Lambda}|_{\tau} \to E_\infty|_{\tau}$$
is a homeomorphism.

3.4. Minimal sets. Given a lamination $\Lambda$ a minimal set is a sublamination, defined as follows:

Definition 3.4.1. Let $\Lambda$ be a lamination of a compact manifold. A minimal set $\Lambda_m \subset \Lambda$ is a subset of $\Lambda$ which is minimal with respect to inclusion, and satisfies the following properties:

1. $\Lambda_m$ is nonempty.
2. $\Lambda_m$ is saturated. That is, it is a union of leaves of $\Lambda$.
3. $\Lambda_m$ is closed.

Minimal sets always exist. In fact, a nonempty sublamination $\Lambda_m$ is minimal if and only if every leaf is dense. A lamination which does not satisfy this property contains some leaf $\lambda$ which is not dense; the closure of $\lambda$ is a smaller sublamination. By transfinite induction, the closure of every leaf contains a minimal set. Note that this construction uses the axiom of choice.

Let $\mathcal{F}$ be a taut foliation of $M$. If $\mathcal{F}$ is not minimal, by the definition of essentiality, every minimal set $\Lambda$ is an essential lamination. Such a lamination is either genuine, or else all complementary regions are products. In this case, either $\Lambda$ is a single fiber of a fibration of $M$ over $S^1$, or else such complementary regions can be collapsed to give a new taut foliation $\mathcal{F}'$ of $M$ which is minimal.

This collapsing procedure is the converse of the operation of blowing-up or Denjoying a leaf. A thorough discussion is contained in [14], so we do not elaborate here. We summarize the discussion above in the following lemma:

Lemma 3.4.2. Let $\mathcal{F}$ be a taut foliation of a 3–manifold $M$. Then either $\mathcal{F}$ contains a genuine sublamination, or a fiber of a fibration over $S^1$, or else $M$ contains a taut foliation $\mathcal{F}'$ with every leaf dense.

4. Universal circles for taut foliations

In this section we define a universal circle for a taut foliation, and give an outline of the construction of a universal circle in [8].
4.1. Definition of a universal circle.

**Definition 4.1.1.** Let $\mathcal{F}$ be a taut foliation of an atoroidal 3–manifold $M$. A *universal circle* for $\mathcal{F}$ is a circle $S^1_{\text{univ}}$, together with the following data:

1. There is a faithful representation
   \[ \rho_{\text{univ}} : \pi_1(M) \to \text{Homeo}^+(S^1_{\text{univ}}) \]
2. For every leaf $\lambda$ of $\widetilde{\mathcal{F}}$ there is a monotone map
   \[ \phi_\lambda : S^1_{\text{univ}} \to S^1_\infty(\lambda) \]
   Moreover, the map
   \[ \phi : S^1_{\text{univ}} \times L \to E_\infty \]
   defined by $\phi(\cdot, \lambda) = \phi_\lambda(\cdot)$ is continuous. That is, $(E_\infty, L, \phi)$ is a monotone family.
3. For every leaf $\lambda$ of $\widetilde{\mathcal{F}}$ and every $\alpha \in \pi_1(M)$ the following diagram commutes:
   \[ \begin{array}{ccc}
   S^1_{\text{univ}} & \xrightarrow{\rho_{\text{univ}}(\alpha)} & S^1_{\text{univ}} \\
   \phi_\lambda \downarrow & & \phi_\lambda(\cdot) \downarrow \\
   S^1_\infty(\lambda) & \xrightarrow{\alpha} & S^1_\infty(\alpha(\lambda)) 
   \end{array} \]
4. If $\lambda$ and $\mu$ are incomparable leaves of $\widetilde{\mathcal{F}}$ then the core of $\lambda$ is contained in the closure of a single gap of $\mu$ and vice versa.

**Theorem 4.1.2** (Thurston, Calegari–Dunfield [8] Universal circles for foliations). Let $\mathcal{F}$ be a co–oriented taut foliation of an atoroidal, oriented 3–manifold $M$. Then there is a universal circle for $\mathcal{F}$.

In section §5 we will see how the axiomatic definition of a universal circle lets us construct transverse very full genuine laminations. However, to analyze the properties of these laminations in more detail, we need to know more about the construction of the universal circle.

4.2. Markers.

**Definition 4.2.1.** Let $\Lambda$ be an essential lamination of $M$ with hyperbolic leaves. A *marker* for $\Lambda$ is a map
\[ m : I \times \mathbb{R}^+ \to \widetilde{M} \]
with the following properties:

1. There is a closed set $K \subset I$ such that for each $k \in K$, the image of $k \times \mathbb{R}^+$ in $\widetilde{M}$ is a geodesic ray in a leaf of $\Lambda$. Further, for $k \in I \setminus K$,
   \[ m(k \times \mathbb{R}^+) \subset \widetilde{M} \setminus \widetilde{\Lambda} \]
   We call these rays the *horizontal rays* of the marker.
2. For each $t \in \mathbb{R}^+$, the interval $m(I \times t)$ is a tight transversal. Further, there is a separation constant $\epsilon$ for $\Lambda$, such that
   \[ \text{length}(m(I \times t)) < \epsilon/3 \]
   We call these intervals the *vertical intervals* of the marker.
For a marker $m$, a horizontal ray $m(k \times \mathbb{R}^+) \subset \Lambda$ is asymptotic to a unique point in $S^2_{\infty}(\lambda)$, which we call the $\text{endpoint}$ of $m(k \times \mathbb{R}^+)$. By abuse of notation, we call the union of such endpoints, as $k$ varies over $K$, the $\text{endpoints of the marker } m$.

If $\mathcal{F}$ is a foliation, then $K = I$ for each marker $m$, and the set of endpoints of $m$ define an embedded interval in $E_\infty$ transverse to the foliation by circles.

Markers are related to, and arise in practice from sawblades, defined as follows:

**Definition 4.2.2.** Let $\Lambda$ be an essential lamination of $M$ with hyperbolic leaves. An $\epsilon$-$\text{sawblade}$ for $\mathcal{F}$ is an embedded polygonal surface $P \subset M$ obtained from a square by gluing the right hand edge to a subset of the left hand edge in such a way that the lowermost vertices are identified. In co–ordinates: if we parameterize the square as $[0,1] \times [0,1]$ and denote the image of $P$ as $P([0,1],[0,1])$, then under the identification, $P(1,[0,1])$ gets identified with a subset $P(0,[0,t])$ with $0 < t \leq 1$. Moreover, $P$ must satisfy the following properties:

1. There is a closed subset $K \subset I$ including the endpoints of $I$, such that for each $t \in K$, the subset $P([0,1],t) \subset \mathcal{F}$ is a geodesic arc in a leaf $\lambda_t$ of $\Lambda$. If $t = 0$, the subset $P([0,1],0) \subset \mathcal{F}$ closed up to a geodesic loop $\gamma \subset \lambda_0$.
2. For each $t \in [0,1]$, the subset $P(t,[0,1])$ is an embedded, tight transversal to $\Lambda$ of length $\leq \epsilon$. The transversal $P(1,[0,1])$ is contained in the image of $P(0,[0,1])$, and the corresponding geodesic segments $P([0,1],t_1)$ and $P([0,1],t_2)$, where $P(1,t_1) = P(0,t_2)$ for $t_1 \in K$, join up to a geodesic segment in the corresponding leaf of $\Lambda$; i.e. there is no corner along $P(0,[0,1])$.

If $\epsilon$ is understood, we just say a $\text{sawblade}$. Note that holonomy transport of the transversal $P(0,[0,1])$ around $\gamma$ induces an embedding $K \to K$ taking one endpoint to itself. Here $\gamma$ is oriented compatibly with the usual orientation on $I = [0,1]$. We call the positive direction on $\gamma$ the $\text{contracting direction}$ for the sawblade, and the negative direction the $\text{expanding direction}$.

We show how to construct a marker from a sawblade.

**Construction 4.2.3.** Let $P$ be a sawblade, and let $\bar{P}$ be a component of the preimage in $\tilde{M}$. $\bar{P}$ is the universal cover of $P$, and the deck group of the cover is $\pi_1(P) = \mathbb{Z}$, generated by the closed geodesic $\gamma$ as in definition 4.2.2.

Let $\tau$ be a lift of $P(0,I)$, and let $K \subset I$ be as in definition 4.2.2. Parameterize $\tau$ as $\tau(t)$ where $\tau(t)$ corresponds to the lift of $P(0,t)$. Then for each $k \in K$, let $\lambda_k$ denote the leaf of $\Lambda$ containing $\tau(k)$. By the second property of a sawblade, the intersection $\lambda_k \cap \bar{P}$ contains an entire geodesic ray starting from $\tau(k)$. Together with complementary strips of $\bar{P}$, the union of these rays are a marker for $\Lambda$.

Notice that the union of the markers constructed in construction 4.2.3 over all lifts $\tau$ of $P(0,I)$, is exactly the preimage $\bar{P}$.

By abuse of notation, we refer to the union of the endpoints of the markers associated to $\bar{P}$ in construction 4.2.3 as the $\text{endpoints of } \bar{P}$.

Every closed geodesic $\gamma$ contained in a non–simply connected leaf $\lambda$ of $\Lambda$ is the boundary geodesic of some $\epsilon$-sawblade, for any positive $\epsilon$. If $M$ is a closed 3–manifold containing an essential lamination $\Lambda$ with every leaf simply connected, then $M$ is $T^3$ (see [37] for an elegant proof). It follows that if $M$ is atoroidal, many sawblades can be constructed.
Let $P$ be a sawblade, and suppose $\Lambda$ is minimal. Then there is a uniform constant $C$ such that for every leaf $\lambda$ of $\tilde{\Lambda}$, and for every point $q \in \lambda$ within distance $C$ in the path metric on $\lambda$, such that $q$ is contained in a lift of $P$. It follows that $q$ is contained in a marker, and there is a geodesic ray $r$ through $q$ such that holonomy transport of a sufficiently short transversal $\tau(q)$ through $q$ along $r$ keeps the length of the transversal smaller than $\epsilon/3$ for all time. It is not hard from this to conclude the following lemma, proved in §5.6 of [8]:

**Lemma 4.2.4** (Calegari–Dunfield). \textit{Let $\Lambda$ be a minimal essential lamination of an atoroidal 3–manifold $M$, and let $P$ be an $\epsilon$–sawblade for $\Lambda$. Then the set of endpoints of lifts $\tilde{P}$ of $P$ is dense in $S^1_{\infty}(\lambda)$ for every leaf $\lambda$ of $\tilde{\Lambda}$.}

From this lemma, it is not hard to conclude the following theorem, called the **Leaf Pocket Theorem**:

**Theorem 4.2.5** (Calegari–Dunfield [8] Leaf Pocket theorem). \textit{Let $\Lambda$ be an essential lamination on an atoroidal 3–manifold $M$. Then for every leaf $\lambda$ of $\tilde{\Lambda}$, and every $\epsilon > 0$, the set of endpoints of $\epsilon$–markers is dense in $S^1_{\infty}(\lambda)$.}

As explained above, each marker $M$ defines by the endpoint map, an embedded interval $e(M) \subset E_\infty$ transverse to the foliation by circles.

The following lemma is a restatement of lemma 6.11 in [8]:

**Lemma 4.2.6** (Calegari–Dunfield). \textit{Let $e(m_1), e(m_2)$ be two endpoint intervals of markers $m_1, m_2$. Then these intervals are either disjoint, or else their union is an embedded, ordered interval transverse to the foliation of $E_\infty$ by circles.}

It follows that distinct markers whose endpoints intersect can be amalgamated, and the unions give a $\pi_1(M)$–invariant family of disjoint, embedded intervals in $E_\infty$ transverse to the foliation by circles. We denote this family of intervals by $\mathcal{M}$, and denote a typical element of $\mathcal{M}$ by $M$ or $m_i$ for some index $i$.

### 4.3. Special sections

In this section we indicate how to go from theorem 4.2.5 to a proof of theorem 4.1.2.

**Definition 4.3.1.** \textit{Let $p \in S^1_{\text{univ}}$. The special section associated to $p$ is a section $\sigma_p : L \to E_\infty$ defined by}

$$\sigma_p(\lambda) = \phi_\lambda(p)$$

The strategy in [8] in constructing $S^1_{\text{univ}}$ is to construct sufficiently many non–crossing sections $L \to E_\infty$, show that there is a natural $\pi_1(M)$–invariant circular ordering on this set $\mathcal{S}$ of sections, and define $S^1_{\text{univ}}$ to be the order completion of $\mathcal{S}$.

For each leaf $\lambda$ of $\tilde{\mathcal{F}}$, and each point $p \in S^1_{\infty}(\lambda)$, we construct a section $s_p : L \to E_\infty$ satisfying

$$s_p(\lambda) = p$$

The combinatorics of the construction are somewhat complicated, and involve detailed care in orientations of transversals and circles in the oriented circle bundle $E_\infty$.

Some simple cases, which nevertheless give an idea of what is going on, are illustrated in §4.4. We suggest that the reader unfamiliar with [8] go back and forth
between that subsection and this, in order to get a clear idea of our conventions. Of course, for details, consult [3].

**Definition 4.3.2.** Let \( I \subset L \) be an embedded interval. A section \( \tau : I \to E_\infty|_I \) is admissible if its image does not cross any element \( m \in \mathcal{M} \) transversely.

Now suppose \( I \) is an oriented interval in \( L \), and let \( \lambda_t \) with \( t \in I \) denote the corresponding leaves of \( \mathcal{F} \). Given \( p \in S_\infty^1(\lambda_t) \), a section \( \tau : I \to E_\infty|_I \) with \( \tau(0) = p \) is leftmost if it is never to the right of any other admissible section \( \tau' \) with \( \tau'(0) = p \).

Here the orientation on the interval \( I \) and on the cylinder \( E_\infty|_I \) determine the meaning of the “left” and “right” sides of \( \tau(I) \) and \( \tau'(I) \). Note that if the orientation on \( I \) is reversed, then the left and right sides are reversed too.

Since they agree in \( S_\infty^1(\lambda_0) \), where they first start to diverge, it makes sense to say that \( \tau(I) \) stays to the left of \( \tau'(I) \), and it makes sense to say that thereafter \( \tau(I) \) never crosses \( \tau'(I) \) from the left side. That is, if \( t \) is a local maximum for the subset of \( I \) for which \( \tau(t) = \tau'(t) \), then for all \( s > t \) with \( s - t \) sufficiently small, \( \tau(s) \) must be to the left of \( \tau'(s) \).

Notice that if the orientation on \( I \) agrees with the orientation on \( L \) coming from the co-orientation of \( \mathcal{F} \), then leftmost admissible sections over \( I \) are clockwisestart. Conversely, if the orientation on \( I \) disagrees with the orientation on \( L \), then leftmost admissible sections over \( I \) are anticlockwisemost.

Now, let \( l \subset L \) be a properly embedded copy of \( \mathbb{R} \), intersecting \( \lambda \), and let \( p \in S_\infty^1(\lambda) \). \( l \) consists of two rays \( l^\pm \). Give \( l^+ \) the usual orientation, agreeing with the order structure on \( L \), but reverse the orientation on \( l^- \), so that it points in the negative direction. Then define \( s_p|_{l^\pm} \) to be leftmost admissible section with \( s_p(\lambda) = p \). Notice that \( s_p|_l \) is clockwisemost and \( s_p|_{l^-} \) is anticlockwisemost amongst admissible sections, with respect to the global order structure on \( L \) and \( E_\infty \).

This defines the section \( s_p \) over the union of leaves which are comparable with \( \lambda \). Now, suppose \( \mu_1, \mu_2 \) are two leaves such that \( \mu_1 < \lambda \), and with the additional property that there is a 1-parameter family of leaves \( \nu_t \) with \( t \in [0, 1) \), such that \( \nu_t < \nu_s \) for \( t < s \), and both \( \mu_1 \) and \( \mu_2 \) are positive limits (in \( L \)) of \( \nu_t \) as \( t \to 1 \).

Consider the union of the intervals \( m_{11}, m_{21} \) in \( \mathcal{M} \) which intersect \( S_\infty^1(\mu_1) \) and \( S_\infty^1(\mu_2) \) respectively. As \( t \to 1 \), more and more of the \( m_{11}, m_{21} \) intersect \( S_\infty^1(\nu_t) \). Since they are disjoint, they inherit a circular ordering as follows: if \( a, b, c \) are three such elements of \( \mathcal{M} \), then there is a \( t \) such that all three intersect \( S_\infty^1(\nu_t) \). Then the cyclic order on \( a, b, c \) is just the cyclic order of the intersections \( a, b, c \cap S_\infty^1(\nu_t) \).

Moreover, it is not hard to show that for any choice of \( m_{11}, m_{12} \) and \( m_{21}, m_{22} \), the two unordered pairs cannot link in any \( S_\infty^1(\nu_t) \), and therefore all the \( m_{11} \) are contained in a “gap” of the circularly ordered set of \( m_{2j} \). Since the \( m_{2j} \) are dense in \( S_\infty^1(\mu_2) \), this gap defines a unique point \( q \in S_\infty^2(\mu_2) \). We can now define \( s_p(\mu_2) = q \). We call this the method of turning corners, since it shows how to continue a leftmost section \( s_p \) across nonseparated leaves in \( L \).

Since \( L \) is simply connected, the Hausdorffification is a (topological) \( \mathbb{R} \)-tree \( T \). Given a point \( \lambda \in L \), there is a unique embedded segment in \( T \) from \( \lambda \) to any other point. Back in \( L \), this defines a sequence of leaves

\[
\lambda = \lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n = \mu
\]

for every pair \( \lambda, \mu \), where \( \lambda_{2i} \) and \( \lambda_{2i+1} \) are comparable, and \( \lambda_{2i-1} \) and \( \lambda_{2i} \) are incomparable and nonseparated.
We define \( s_p \) inductively, by leftmost sections over the oriented subintervals \([\lambda_{2i}, \lambda_{2i+1}]\) of \( L \), and by the method of turning corners to go from the section at \( \lambda_{2i-1} \) to the section at \( \lambda_{2i} \). Notice, in fact, that the value of \( s_p(\lambda_{2i}) \) does not depend at all on the values of \( s_j(\lambda_j) \) with \( j < 2i \), and depends only on the leaf \( \lambda_{2i-1} \).

We let \( \mathcal{S} \) denote the union of all the special sections \( s_p \) as above, as \( p \) ranges over \( E_\infty \), and we identify \( s_p \) and \( s_q \) if they define the same section \( L \to E_\infty \).

For each leaf \( \lambda \) of \( \tilde{F} \), let \( \mathcal{S}(\lambda) \) denote the set of special sections \( s_p \) where \( p \in S^1_{\text{univ}}(\lambda) \).

Finally, in lemma 6.25 of [8] it is shown that the set \( S \) of sections \( s_p \) as above is naturally circularly ordered. It follows that we can take the order completion \( \overline{\mathcal{S}} \), which is homeomorphic with the order topology to a closed subset of a circle. By collapsing the complementary gaps in this image, we get a universal circle, which we call \( S^1_{\text{univ}} \). That is, we have

\[
\mathcal{S} \to \overline{\mathcal{S}} \to S^1_{\text{univ}}
\]

where the first map is an inclusion, and the second is a surjection. Notice that the map \( \overline{\mathcal{S}} \to S^1_{\text{univ}} \) is at most 2–1, and is 1–1 away from countably many points. It follows that the natural inclusion \( \mathcal{S}(\lambda) \to \mathcal{S} \) extends to an inclusion \( \mathcal{S}(\lambda) \to S^1_{\text{univ}} \).

The map \( \phi_\lambda \) is defined on \( s_p \in S^1_{\text{univ}} \) by

\[
\phi_\lambda(s_p) = s_p(\lambda)
\]

and these maps are collated, by varying over \( \lambda \in L \), to \( \phi \). It is clear that \( \phi : S^1_{\text{univ}} \times L \to E_\infty \) defined in this way is continuous, and that \( \phi_\lambda \) is monotone for each \( \lambda \).

Moreover, it is clear that the natural action of \( \pi_1(M) \) on \( E_\infty \) induces an action on \( \mathcal{S} \) preserving the circular order, and therefore induces a representation

\[
\rho_{\text{univ}} : \pi_1(M) \to \text{Homeo}^+(S^1_{\text{univ}})
\]

4.4. Examples of universal circles. In this subsection we give some idea of the combinatorics of universal circles.

Example 4.4.1 (Linear segment). Let \( I \subset L \) be a closed interval, with lowest leaf \( \lambda \) and highest leaf \( \lambda' \). Leftmost trajectories can run into each other, but not cross. A leftmost ascending trajectory can coalesce with a leftmost descending trajectory. The set of special sections gives the cylinder \( E_\infty|_I \) the structure of a (1–dimensional) branched lamination; see definition 5.3.2 for a general definition.

In the universal circle, the set of special sections which intersect \( \lambda \) at \( x \) and \( \lambda' \) at \( x' \) is an interval, running positively from \( s_x \) to \( s_{x'} \).

Here is another way to see the circular order on special sections in \( I \). Lift to the universal cover of the cylinder \( E_\infty|_I \). Each special section lifts to \( \mathbb{Z} \) copies in the cover. In the cover, two sections \( s_y, s_z \) satisfy \( s_y < s_z \) iff there is a nontrivial positive transversal from \( s_y \) to \( s_z \). This defines a total order upstairs, which is evidently order isomorphic to \( \mathbb{R} \). The action of the deck group on the cover of the cylinder induces an action on the ordered set of lifts of special sections, inducing a circular order on their quotient.

Example 4.4.2 (Nonseparated leaves). The next example incorporates positive branching. Let \( \lambda, \mu \) be two incomparable leaves which are nonseparated, and such that
there is a 1–parameter family of leaves \( \nu_t \) with \( t \in [0, 1) \), satisfying \( \nu_t < \lambda, \mu \) for all \( t \), and converging to both \( \lambda \) and \( \mu \) as \( t \to 1 \).

Every marker which intersects \( \lambda \) or \( \mu \) will intersect \( \nu_t \), for sufficiently large \( t \). As described in the previous subsection, this induces a circular order on the union of a dense subset of \( S^1_\infty(\lambda) \) and \( S^1_\infty(\mu) \), and by comparing special sections in \( S^1_\infty(\nu_t) \) for sufficiently large \( t \), these can be completed to a circular order on the disjoint union of all special sections \( s_x \) where

\[
x \in S^1_\infty(\lambda) \cup S^1_\infty(\mu)
\]

In this circularly ordered set the set of special sections \( s_x \) with \( x \in S^1_\infty(\lambda) \) is a half–open interval, containing a (locally) clockwisemost point, but not a (locally) anticlockwisemost point, and similarly for the \( s_y \) with \( y \in S^1_\infty(\mu) \).

Notice that if \( \lambda, \mu \) were nonseparated, but the approximating sequence \( \nu_t \) satisfied \( \nu_t > \lambda, \mu \) then the half–open intervals of special sections would contain (locally) anticlockwisemost points instead.

---

**Figure 5.** The special sections might coalesce, but they don’t cross.

**Figure 6.** The special sections coming from each of the two nonseparated leaves determine a half–open interval in the circular order on the union. Here, the point \( x \) is in \( S^1_\infty(\lambda) \), and the point \( y \) is in \( S^1_\infty(\mu) \).
Example 4.4.3 (More branching). The next example includes both positive and negative branching. In this case, we have nonseparating leaves \( \mu, \lambda \) exhibiting positive branching, nonseparating leaves \( \nu, \lambda' \) exhibiting negative branching, where \( \lambda' > \lambda \).

Let \( x \in S^1_{\infty}(\lambda) \) be the point determining the locally clockwisemost segment \( s_x \) in the previous example, and let \( x' \) be the corresponding point (determining the locally anticlockwisemost segment) in \( S^1_{\infty}(\lambda') \).

There are two topologically distinct cases to consider: in the first, the special sections \( s_x \) and \( s_x' \) do not agree on the entire interval \( [\lambda, \lambda'] \), although they might agree on some closed subset of this interval, which might include either or both of the endpoints. In the second, the sections \( s_x, s_x' \) do agree on the entire interval, and therefore agree on all of \( E_{\infty} \).

These examples contain all the necessary information to show how to go from a finite union \( K \) of ordered subsegments in \( L \), whose image in the Hausdorffification of \( L \) is connected, to a circle \( S^1(K) \) which realizes the circular order on the set of special sections associated to points in leaves \( \lambda \) in \( K \). By following the model of example 4.4.2, one can amalgamate the circles associated to a pair of ordered segments whose endpoints are nonseparated. Given \( K_i, K_j \) disjoint, finite connected unions, we get circles \( S^1(K_i) \) and \( S^1(K_j) \); if \( K_i \) and \( K_j \) contain a pair of nonseparated leaves, we can follow example 4.4.2 to amalgamate \( S^1(K_j) \) and \( S^1(K_i) \) into \( S^1(K_i \cup K_j) \), completing the induction step. One must verify that the result does not depend on the order in which one constructs \( K \) from ordered subsegments; implicitly, this is a statement about the commutativity of the amalgamating operation in example 4.4.2. This commutativity is evident even in example 4.4.3, where one may choose to amalgamate the segment \( [\lambda, \lambda'] \) with \( \mu \) first and then \( \nu \), or the other way around.

![Figure 7](image-url)

**Figure 7.** In case 1, \( s_x \) and \( s_x' \) differ somewhere on \( E_{\infty}[\lambda, \lambda] \). In case 2, they are equal on all of \( E_{\infty} \).

### 4.5. Special sections and cores.

Recall the notation \( \mathcal{S}(\lambda) \) to denote the set of special sections associated to points \( x \in S^1_{\infty}(\lambda) \). In this subsection we describe the relationship between \( \mathcal{S}(\lambda) \) and the core of \( \phi_\lambda \).

**Lemma 4.5.1.** Let \( \lambda \) be a leaf of \( \hat{\mathcal{T}} \). Then \( \text{core}(\phi_\lambda) \) is contained in the closure \( \overline{\mathcal{S}(\lambda)} \subset S^1_{\text{minor}} \) and the difference \( \mathcal{S}(\lambda) \setminus \text{core}(\phi_\lambda) \) consists of at most countably many isolated points, at most one in each gap of \( \phi_\lambda \).
Proof. Given \( p, q \in \text{core}(\phi_\lambda) \), either \( p \) and \( q \) are the boundary points of the closure of some gap, or else \( \phi_\lambda(p) \neq \phi_\lambda(q) \), and therefore there are \( p', q' \in \mathcal{E}(\lambda) \) which link \( p, q \). It follows that every accumulation point of \( \text{core}(\phi_\lambda) \) is an accumulation point of \( \mathcal{E}(\lambda) \). Since \( \text{core}(\phi_\lambda) \) is perfect, it follows that \( \text{core}(\phi_\lambda) \subset \mathcal{E}(\lambda) \).

Conversely, given \( p, q \in S^1_{\infty}(\lambda) \) distinct points, we have \( \phi_\lambda(p) = p \neq q = \phi_\lambda(q) \), and therefore there are points \( p', q' \in \mathcal{E}(\lambda) \) which link \( p, q \). In particular, \( p \) and \( q \) are not both in the same gap region of \( \phi_\lambda \), and therefore there is at most one such point in each gap. Since \( \phi_\lambda \) has only countably many gaps, the lemma follows. \( \square \)

An example where \( \mathcal{E}(\lambda) \setminus \text{core}(\phi_\lambda) \) might contain isolated points is illustrated in figure 2.

Now, if \( \lambda \) and \( \mu \) are incomparable leaves, then \( \phi_\mu(\mathcal{E}(\lambda)) \) is a single point of \( S^1_{\infty}(\mu) \), and similarly for \( \phi_\lambda(\mathcal{E}(\mu)) \). Since \( \phi_\lambda \) is 1–1 on \( \mathcal{E}(\lambda) \), it follows that \( \mathcal{E}(\lambda) \) and \( \mathcal{E}(\mu) \) are not linked as subsets of \( S^1_{\text{univ}} \), and therefore the same is true of \( \text{core}(\phi_\lambda) \) and \( \text{core}(\phi_\mu) \), by lemma 4.5.1. This is the last defining property of a universal circle, and completes the sketch of the argument of theorem 4.1.2.

5. CONSTRUCTING INVARIANT LAMINATIONS

This section contains the first important new results in this paper. Given a taut foliation \( \mathcal{F} \) of an atoroidal 3–manifold \( M \), we construct a pair of essential laminations \( \lambda_{\text{split}} \) of \( M \) transverse to \( \mathcal{F} \) and describe their properties.

5.1. Minimal quotients. New universal circles can be obtained from old in an uninteresting way: given a point \( p \in S^1_{\text{univ}} \), we can blow up the orbit of \( p \) to obtain a new universal circle \( S^1_{\text{univ}} \) and a monotone map to \( S^1_{\text{univ}} \) whose gaps are the interiors of the preimages of the points in the orbit of \( p \).

These blown up universal circles have the property that there are distinct points \( p, q \in S^1_{\text{univ}} \) whose images are identified under every map \( \phi_\lambda \). We make the following definition:

Definition 5.1.1. A universal circle is minimal if for any distinct \( p, q \in S^1_{\text{univ}} \) there is some \( \lambda \) such that \( \phi_\lambda(p) \neq \phi_\lambda(q) \).

In the next lemma, we show that any universal circle which is not minimal is obtained from a minimal universal circle by blow up.

Lemma 5.1.2. Let \( S^1_{\text{univ}} \) be a universal circle for \( \mathcal{F} \). Then there is a minimal universal circle \( S^1_{\text{univ}} \) for \( \mathcal{F} \) with monotone maps \( \phi^m_\lambda : S^1_m \to S^1_{\infty}(\lambda) \) and a monotone map \( m : S^1_{\text{univ}} \to S^1_m \) such that for all \( \lambda \in L \)

\[
\phi^m_\lambda \circ m = \phi_\lambda
\]

Proof. If \( S^1_{\text{univ}} \) is not minimal, define an equivalence relation on \( S^1_{\text{univ}} \) by \( p \sim q \) if \( \phi_\lambda(p) = \phi_\lambda(q) \) for all \( \lambda \in L \). Let \( \gamma^p \subset S^1_{\text{univ}} \) be the interiors of the two closed arcs from two such distinct \( p, q \) with \( p \sim q \). Then for each \( \lambda \in L \), either \( \gamma^+ \) is contained in a single gap of \( \phi_\lambda \), or \( \gamma^- \) is. Moreover, if both \( \gamma^- \) and \( \gamma^+ \) were contained in gaps of \( \phi_\lambda \), the map \( \phi_\lambda \) would be constant, which is absurd.

Now, by lemma 2.2.6 closures of gaps of \( \phi_\lambda \) vary upper semicontinuously as a function of \( \lambda \in L \). It follows that the subset of \( \lambda \in L \) for which \( \gamma^+ \) is contained in a gap of \( \phi_\lambda \) is closed, and similarly for \( \gamma^- \). But \( L \) is path connected, so either \( \gamma^+ \) is contained in a gap of \( \phi_\lambda \) for every \( \lambda \), or \( \gamma^- \) is.
It follows that the equivalence classes of \( \sim \) are a \( \rho_{\text{univ}}(\pi_1(M)) \)-equivariant collection of closed disjoint intervals of \( S^1_{\text{univ}} \), and single points, and therefore the quotient space of \( S^1_{\text{univ}} \) by this decomposition defines a new circle with a \( \pi_1(M) \) action induced by the quotient map

\[
m : S^1_{\text{univ}} \to S^1_m
\]

By construction, for each \( \lambda \in L \) the equivalence relation on \( S^1_{\text{univ}} \) defined by \( \phi_\lambda \) is coarser than the equivalence relation defined by \( m \), and therefore \( \phi_\lambda \) factors through \( m \) to a unique map \( \phi_\lambda^m : S^1_m \to S^1_\infty(\lambda) \) satisfying

\[
\phi_\lambda = \phi_\lambda^m \circ m
\]

Note that the construction of a universal circle in §4 produces a minimal circle.

5.2. Laminations of \( S^1_{\text{univ}} \). The main purpose of this section is to prove that a minimal universal circle for a taut foliation with 2–sided branching admits a pair of nonempty laminations \( \Lambda_{\text{univ}}^\pm \) which are preserved by the action of \( \pi_1(M) \), acting via the representation \( \rho_{\text{univ}} \).

**Construction 5.2.1.** Let \( \lambda \in L \). Let \( L^+(\lambda), L^-(\lambda) \) denote the two connected components of \( L \setminus \lambda \) where the labelling is such that \( L^+(\lambda) \) consists of the leaves on the positive side of \( \lambda \), and \( L^-(\lambda) \) consists of the leaves on the negative side.

Recall that for \( X \subset L \), the set \( \text{core}(X) \) denotes the union, over \( \lambda \in X \), of the sets \( \text{core}(\phi_\lambda) \). As in construction 2.1.8, we can associate to the subset \( \text{core}(X) \) the lamination of \( \mathbb{H}^2 \) which is the boundary of the convex hull of the closure of \( \text{core}(X) \), and thereby construct the corresponding lamination \( \Lambda(\text{core}(X)) \) of \( S^1 \).

Then define

\[
\Lambda^+(\lambda) = \Lambda(\text{core}(L^+(\lambda)))
\]

and

\[
\Lambda_{\text{univ}}^+ = \bigcup_{\lambda \in L} \Lambda^+(\lambda)
\]

and similarly for \( \Lambda^-(\lambda) \) and \( \Lambda_{\text{univ}}^- \), where the closure is taken in the space of unordered pairs of distinct points in \( S^1_{\text{univ}} \).

Observe the following property of \( \Lambda^+(\lambda) \).

**Lemma 5.2.2.** Let \( \lambda, \mu \) be leaves of \( \tilde{\mathcal{F}} \). Then \( \phi_\mu(\Lambda^+(\lambda)) \) is trivial unless \( \mu < \lambda \).

**Proof.** If \( \mu \in L^+(\lambda) \) then by definition, \( \text{core}(\mu) \subset \text{core}(L^+(\lambda)) \) and therefore every leaf of \( \Lambda^+(\lambda) \) is contained in the closure of a gap of \( \mu \). If \( \mu \in L^-(\lambda) \) but \( \mu \) is incomparable with \( \lambda \), then \( \mu \) is incomparable with every element of \( L^+(\lambda) \), and therefore by theorem 2.2.8 \( \text{core}(L^+(\lambda)) \) is contained in the closure of a single gap of \( \mu \), and therefore \( \phi_\mu(\Lambda^+(\lambda)) \) is trivial in this case too. \( \square \)

We are now ready to establish the key property of \( \Lambda_{\text{univ}}^\pm \): that they are laminations of \( S^1_{\text{univ}} \).

**Theorem 5.2.3.** Let \( \mathcal{F} \) be a taut foliation of an atoroidal 3–manifold \( M \), and let \( S^1_{\text{univ}} \) be a minimal universal circle for \( \mathcal{F} \). Then \( \Lambda_{\text{univ}}^\pm \) are laminations of \( S^1_{\text{univ}} \) which are preserved by the natural action of \( \pi_1(M) \). Furthermore, if \( L \) branches in the positive direction, then \( \Lambda_{\text{univ}}^+ \) is nonempty, and if \( L \) branches in the negative direction, then \( \Lambda_{\text{univ}}^- \) is.
Proof. We first show that no leaf of $\Lambda^+(\lambda)$ links any leaf of $\Lambda^+(\mu)$, for $\mu, \lambda \in L$. There are three cases to consider

Case (i): $\lambda \in L^-(\mu)$ and $\mu \in L^-(\lambda)$

In this case, $L^+(\lambda)$ and $L^+(\mu)$ are disjoint, and moreover they are incomparable. That is, for every $\nu_1 \in L^+(\lambda)$ and $\nu_2 \in L^+(\mu)$ the leaves $\nu_1$ and $\nu_2$ are incomparable. It follows from the definition of a universal circle that for all such pairs, the core of $\phi_{\nu_1}$ is contained in the closure of a single gap of $\phi_{\nu_2}$, and vice versa. Since $L^+(\lambda)$ and $L^+(\mu)$ are path connected, theorem 2.2.8 implies that $\text{core}(L^+(\mu))$ and $\text{core}(L^+(\lambda))$ are unlinked. It follows that no leaf of $\Lambda^+(\lambda)$ links any leaf of $\Lambda^+(\mu)$, as claimed.

Case (ii): $\lambda \in L^-(\mu)$ and $\mu \in L^+(\lambda)$

In this case, we have $L^+(\lambda) \subset L^+(\mu)$ and therefore

$$\text{core}(L^+(\lambda)) \subset \text{core}(L^+(\mu))$$

so the claim is proved in this case too.

Case (iii): $\lambda \in L^+(\mu)$ and $\mu \in L^+(\lambda)$

In this case, observe that $L^-(\lambda) \subset L^+(\mu)$ and $L^-(\mu) \subset L^+(\lambda)$, and therefore

$$L = L^+(\mu) \cup L^+(\lambda)$$

Since $S_{\text{univ}}^1$ is minimal, every point in $S_{\text{univ}}^1$ is a limit of a sequence of points in $\text{core}(\phi_{\lambda_i})$ for some sequence $\lambda_i$. It follows that $\text{core}(L)$ is all of $S_{\text{univ}}^1$, and therefore $\text{core}(L^+(\lambda)) \cup \text{core}(L^+(\mu)) = S_{\text{univ}}^1$.

Now, if two subsets $X, Y \subset S^1$ satisfy $\overline{X} \cup \overline{Y} = S^1$, then the boundaries of the convex hulls of $X$ and $Y$ do not cross in $\mathbb{H}^2$. For, if $l, m$ are boundary geodesics of $H(X)$ and $H(Y)$ respectively which cross in $\mathbb{H}^2$, then $l, m$ both bound open half spaces $l^+, m^+$ which are disjoint from $H(X)$ and $H(Y)$ respectively. Moreover, since $l, m$ are transverse, the intersection $l^+ \cap m^+$ contains an open sector in $\mathbb{H}^2$, which limits to some nonempty interval in $S^1$ which by construction is disjoint from both $X$ and $Y$. But this contradicts the defining property of the pair $X, Y$. This contradiction proves the claim in this case too.

It remains to show that $\Lambda^+_{\text{univ}}$ is nonempty when $L$ branches in the positive direction. Now, for any $\lambda \in L$, $\text{core}(\phi_{\lambda})$ is perfect by lemma 2.2.8. It suffices to show $\text{core}(L^+(\lambda))$ is not equal to $S_{\text{univ}}^1$.

If we can find another leaf $\mu$ with $\lambda \in L^-(\mu)$ and $\mu \in L^-(\lambda)$, then as above, $\text{core}(L^+(\lambda))$ and $\text{core}(L^+(\mu))$ are unlinked as subsets. It follows that the subset $\text{core}(L^+(\lambda))$ is contained in the closure of a single interval in the complement of $\text{core}(L^+(\mu))$ and conversely, and therefore neither core is dense. To see that such a $\mu$ exists, note that if there is $\nu$ with $\nu < \mu$ and $\nu < \lambda$ but $\mu, \lambda$ incomparable, then $\mu$ will have the desired properties.

Since $L$ branches in the positive direction, there is $\nu$ and some leaves $\lambda', \mu$ with $\nu < \mu, \lambda'$ and $\lambda', \mu$ incomparable. Since $F$ is taut, if $\pi(\lambda')$ and $\pi(\lambda)$ denote the projections of $\lambda, \lambda'$ to $M$, there is some transverse positively oriented arc $\gamma$ from $\pi(\lambda')$ to $\pi(\lambda)$. Lifting to $\tilde{M}$, we see there is some $\alpha \in \pi_1(M)$ such that $\alpha(\lambda') < \lambda$. Then $\alpha(\mu)$ is the desired leaf.

The corresponding properties for $\Lambda^-_{\text{univ}}$ are proved by reversing the orientation on $L$. \qed

5.3. Branched surfaces and branched laminations.
Construction 5.3.1. Let $\Lambda^\pm_{\text{univ}}$ be the invariant laminations of $S^1_{\text{univ}}$ provided by theorem 5.2.3. For each $\lambda \in L$, there are lamination of $S^1_{\infty}(\lambda)$ given by the pushforward $\phi_\lambda(\Lambda^\pm_{\text{univ}})$. By construction 2.1.4, these laminations of $S^1_{\infty}(\lambda)$ span geodesic laminations of $\lambda$, which we denote by $\Lambda^\pm_{\text{geo}}(\lambda)$. Then define

$$\tilde{\Lambda}^\pm_{\text{geo}} = \bigcup_{\lambda \in L} \Lambda^\pm_{\text{geo}}(\lambda)$$

Note the tilde notation to be consistent with the convention that $\tilde{\Lambda}^\pm_{\text{geo}}$ covers an object in $M$. The objects $\tilde{\Lambda}^\pm_{\text{geo}}$ are not yet necessarily 2–dimensional laminations; rather they are branched laminations, to be defined shortly. On the other hand, they have the important property that the branch locus of each leaf is a 1–manifold (that is, there are no double points of the branch locus) and moreover, the sheets come with a parameterization by leaves of $\Lambda^\pm_{\text{univ}}$ that lets us split them open in a canonical way to a lamination.

The definition we give here of a branched lamination is not the most general possible, since for us, every branched lamination comes together with an ordinary lamination which it fully carries. Branched laminations are a generalization of branched surfaces; see [43] for a definition and basic properties of branched surfaces.

Definition 5.3.2. A branched lamination fully carrying a lamination $K \subset M$ is given by the following data:

1. An open submanifold $N \subset M$
2. A 1–dimensional foliation $X_V$ of $N$
3. A lamination $\Lambda$ of $N$ transverse to $X_V$, intersecting every leaf of $X_V$
4. A surjective map $\psi : N \to N$ from $N$ to itself which is monotone on each leaf $x$ of $X_V$

The underlying space of the branched lamination itself is the image $K = \psi(\Lambda)$, thought of as a subset of $M$. We say that the lamination $\Lambda$ is fully carried by $K$, and is obtained by splitting $K$ open.

Notice that with this definition, we allow the possibility that $K = N = M$, which would happen for instance if $\Lambda$ is a foliation.

Let us describe our strategy to realize $\tilde{\Lambda}^\pm_{\text{geo}}$ as branched laminations, which fully carry split open laminations $\tilde{\Lambda}^\pm_{\text{split}}$.

Firstly, observe that we can define in generality a branched lamination as a structure on $M$ which is locally modelled on the structure in definition 5.3.2, and for which the 1–dimensional foliations $X_V$ in local charts are required to piece together to give a global transverse 1–dimensional foliation, but for which the laminations $\Lambda$ and the map $\psi$ are only defined locally, with no conditions on how they might piece together globally. General branched laminations do not always fully carry laminations.

Another way of thinking of a branched lamination is as the total space of a distribution defined on a closed subset of $M$ which is integrable, but not uniquely. That is, through every point, there is a complete integral submanifold tangent to the distribution, but such submanifolds might not be disjoint. The branch locus of the branched lamination consists of the union of the boundaries of the subsets where such distinct integral submanifolds agree. In particular, the branch locus has the structure of a union of 1–manifolds. In the case of a branched surface, this
Given a branched lamination $K$, one can always find an abstract lamination “carried” by $K$ which consists of the disjoint union of some collection of integral submanifolds, topologized leafwise with the path topology, and as a lamination by the compact open topology. The difficulty is in embedding this abstract lamination in $N$ transverse to the foliation $X_V$. This amounts to finding a local order structure on the leaf space of this abstract lamination. Once this order structure is obtained, the process of recovering $\psi$ from $K$ is more or less the same as the usual process of blowing up some collection of leaves of a foliation or lamination, as described in [14]. In our case, the leaves of the abstract laminations carried by $\tilde{\Lambda}^{\pm}_{\text{geo}}$ are the unions $\bigcup_{\lambda \in L} \phi_{\lambda}(l)$ where $l$ is a leaf of $\Lambda^{\pm}_{\text{univ}}$, and we are implicitly thinking of $\phi_{\lambda}(l)$ as a geodesic in $\lambda$ by construction 2.1.4. The desired local order structure on the leaves of these abstract laminations comes from the local order structure on the order trees which are the leaf spaces of the geodesic laminations of $\mathbb{H}^2$ constructed from $\Lambda^{\pm}_{\text{univ}}$. In this way, the abstract laminations may be realized as laminations in $\tilde{M}$ fully carried by $\tilde{\Lambda}^{\pm}_{\text{geo}}$. This is the summary of our strategy. Now we go into detail.

To establish the desired properties of $\tilde{\Lambda}^{\pm}_{\text{geo}}$, we must first understand how the laminations $\Lambda^{\pm}_{\text{geo}}(\lambda)$ vary as a function of $\lambda$.

Let $\tau$ be a transversal to $\tilde{\mathcal{F}}$. The cylinder $UT\tilde{\mathcal{F}}|_{\tau}$, thought of as a circle bundle over $\tau$, carries two natural families of sections. The first family of sections comes from the structure maps $e$ and $\phi$.

**Construction 5.3.3.** Let $\tau$ be a transversal to $\tilde{\mathcal{F}}$. The endpoint map defines an embedding $e : UT\tilde{\mathcal{F}}|_{\tau} \rightarrow E_{\infty}$. The structure map of the universal circle $\phi : S^1_{\text{univ}} \times L \rightarrow E_{\infty}$, composed with $e^{-1}$, defines a canonical collection of sections of the circle bundle

$$UT\tilde{\mathcal{F}}|_{\tau} \rightarrow \tau$$

as follows. If we let $\iota : \tau \rightarrow L$ denote the embedding induced by the quotient map $\tilde{M} \rightarrow L$, then the arcs $p \times \iota(\tau)$ with $p \in S^1_{\text{univ}}$ map to a family of arcs in $E_{\infty}|_{\iota(\tau)}$. In the case of the universal circles constructed in §4 these are the restriction of the special sections to $\iota(\tau)$. Then $e^{-1}$ pulls these back to define a family of sections of $UT\tilde{\mathcal{F}}|_{\tau}$, which by abuse of notation we call the special sections over $\tau$. If $p \in S^1_{\text{univ}}$, we denote by $\sigma(p)|_{\tau}$ the special section corresponding to $p$ over $\tau$.

The second family of sections comes from the geometry of $\tilde{M}$.

**Construction 5.3.4.** A Riemannian metric on $M$ pulls back to a Riemannian metric on $\tilde{M}$. Parallel transport with respect to the Levi–Civita connection does not preserve the 2–dimensional distribution $T\tilde{\mathcal{F}}$, but the combination of the Levi–Civita connection of the metric on $\tilde{M}$ together with orthogonal projection to $T\tilde{\mathcal{F}}$ defines an orthogonal (i.e. metric preserving) connection on $T\tilde{\mathcal{F}}$.

If $\tau$ is a transversal, this connection defines a trivialization of $UT\tilde{\mathcal{F}}|_{\tau}$ by parallel transport along $\tau$. We call the fibers of this trivialization the geometric sections over $\tau$. 

branch locus is a finite union of circles, and one typically requires this union of circles to be in general position with respect to each other.
Lemma 5.3.6. There is a uniform modulus $f : \mathbb{R}^+ \to \mathbb{R}^+$ with
\[
\lim_{t \to 0} f(t) = 0
\]
such that for any $p \in S^1_{\text{univ}}$, any $q \in \tilde{M}$ and $\tau(t)$ any integral curve of $\tilde{v}$ through $q$ parameterized by arclength, then if
\[
r = \sigma(p)|_{\tau(0)} \in UT_q \tilde{\mathcal{F}}
\]
and $\sigma'(\cdot)$ denotes the geometric section over $\tau$ obtained by parallel transporting $r$, we have
\[
\text{arcwise distance from } \sigma(p)|_{\tau(t)} \text{ to } \sigma'(t) \text{ in } UT_{\tau(t)} \tilde{\mathcal{F}} \leq f(t)
\]
Proof. This just follows from the compactness of $UT \tilde{\mathcal{F}}$ and the continuity of $e$ and $\phi$. □

Said another way, lemma 5.3.5 says that a geometric section and a special section which agree at some point cannot move apart from each other too quickly. Since the geometric sections are defined by an orthogonal connection, it follows that if $\sigma(p)$ and $\sigma(q)$ are two special sections, the angle between them cannot vary too quickly. This lets us prove the following.

Lemma 5.3.6. The laminations $\Lambda_{\text{geo}}^\pm(\lambda)$ vary continuously on compact subsets of $\tilde{M}$, as a function of $\lambda \in L$. Moreover, the sets $\Lambda_{\text{geo}}^\pm$ are closed as subsets of $\tilde{M}$.

Proof. The continuity of $\Lambda_{\text{geo}}^\pm(\lambda)$ on compact subsets of $\tilde{M}$ follows from the fact that the leaves $\lambda$ themselves vary continuously on compact subsets, together with the continuity of $e$ and $\phi$.

Now we show that the unions $\Lambda_{\text{geo}}^\pm$ are closed. Let $\lambda_i \to \lambda$ and $p_i \in \lambda_i \to p \in \lambda$ be a sequence of leaves of $\tilde{\mathcal{F}}$ and points in those leaves. Since $p_i \to p$, it follows that for sufficiently large $i$, the leaves $\lambda_i$ are all comparable, and contained in an interval $I \subset L$, so without loss of generality, we can assume that all $\lambda_i$ are contained in $I$. Let $\tau$ be an orthogonal trajectory to $\tilde{\mathcal{F}}$ through $p$, parameterized by arclength, and let $q_i \in \lambda_i$ be equal to $\tau \cap \lambda_i$. Suppose that $p_i \in \Lambda_{\text{geo}}^\pm(\lambda_i)$ for each $i$. We must show that $p \in \Lambda_{\text{geo}}^\pm(\lambda)$. Now, since $p_i \in \Lambda_{\text{geo}}^\pm(\lambda_i)$, there is a leaf $l_i$ of $\Lambda_{\text{geo}}^\pm(\lambda_i)$ with $p_i \in l_i$. Geometrically, $l_i$ is just a geodesic in $\lambda$ with respect to its hyperbolic metric. Let $l_i^\pm \in S^1_{\text{univ}}$ be a pair of points which span a leaf $l_i'$ of $\Lambda_{\text{geom}}^\pm$ which maps $l_i$ under $\phi_\lambda$ by the pushforward construction and construction 2.1.4. Then by lemma 5.3.5, the angle between the special sections over $\tau$ defined by $l_i^\pm$ cannot vary too quickly. But in $UT_{\lambda_i} \tau$, the angle between the endpoints of $l_i$ is $\pi$, since $p_i$ lies on the geodesic $l_i$. It follows that as $i \to \infty$, the angle between the special sections over $\tau$ defined by $l_i^\pm$ converges to $\pi$, and therefore the pushforward leaves $\phi(l_i')$ span geodesics in $\Lambda_{\text{geo}}^\pm(\lambda)$ which contain points converging to $p$. Since $\Lambda_{\text{geo}}^\pm(\lambda)$ is closed in $\lambda$, the point $p \in \Lambda_{\text{geo}}^\pm(\lambda)$, as claimed. □

The next lemma shows, as promised, that $\Lambda_{\text{geo}}^\pm$ are branched laminations which can be split open. The following lemma is somewhat ad hoc. However, the basic idea is very simple, and is precisely as described in the paragraphs following definition 5.3.2. Namely, the branched laminations $\Lambda_{\text{geo}}^\pm$ can be split open because
they are parameterized by abstract laminations whose leaf spaces already have well–defined local order structures.

**Lemma 5.3.7.** \( \Lambda_{\text{geo}}^\pm \) are branched laminations of \( \tilde{M} \), fully carrying laminations \( \tilde{\Lambda}_{\text{split}}^\pm \) which are preserved by the action of \( \pi_1(M) \).

**Proof.** For the sake of notation, we restrict to \( \Lambda_{\text{geo}}^+ \).

Fix some small \( \epsilon \), and for each leaf \( \lambda \) of \( L \) let \( N(\lambda) \) be the subset of points in \( \lambda \) which are distance \( < \epsilon \) from \( \Lambda_{\text{geo}}^+(\lambda) \), and let

\[
\tilde{N} = \bigcup_\lambda N(\lambda)
\]

The nearest point map (in the path metric on \( \lambda \)) defines a retraction from \( N(\lambda) \) to \( \Lambda_{\text{geo}}^+(\lambda) \), away from the set of points which are equally close to two leaves; call these *ambivalent* points. The preimages of this retraction, together with the points equally close to two leaves, give a 1–dimensional foliation of \( N(\lambda) \setminus \Lambda_{\text{geo}}^+(\lambda) \) by open intervals, with at most one ambivalent point on each open interval, as the midpoint.

If \( \Lambda_{\text{geo}}^+ \) foliates some region, then the integral curves of the orthogonal distribution define a foliation of the foliated region of \( \Lambda_{\text{geo}}^+ \). Together, this defines a 1–dimensional foliation of \( N(\lambda) \). By lemma 5.3.6 these foliations vary continuously from leaf to leaf of \( \lambda \), and define a 1–dimensional foliation \( X_{\text{geo}} \) of \( \tilde{N} \), which is an open neighborhood of \( \Lambda_{\text{geo}}^+ \).

If \( l, m \) are leaves of \( \Lambda_{\text{geo}}^+(\lambda) \) which are both in the closure of the same complementary region, and which contain points which are \( < 2\epsilon \) apart, then there are at most two points \( p, q \) in this complementary region which are distance exactly \( \epsilon \) in \( \lambda \) to both \( l \) and \( m \). Call such points *cusps*. The set of cusps in each leaf \( \lambda \) of \( \tilde{\mathcal{F}} \) are isolated; furthermore, by lemma 5.3.6 the set of cusps in \( \lambda \) varies continuously as a function of \( \lambda \), thereby justifying the notation \( p(\lambda) \) for a family of leafwise cusps, with the possibility of birth–death pairs in the sense of Morse theory when two distinct cusp points \( p(\lambda), q(\lambda) \) coalesce at some leaf \( \lambda_0 \) and disappear for nearby leaves on one side. It follows that the union of all cusps defines a locally finite collection \( \tilde{c} \) of properly embedded lines in \( \tilde{M} \) which covers a link \( c \subset M \). By abuse of notation, we call \( \tilde{c} \) the *cusps* of \( N \). Observe that the cusps parameterize the branching of the leaf space of \( X_{\text{geo}} \), as follows. For each point \( p \in \tilde{c} \) there is a 1–parameter family \( \gamma_t \) of leaves of \( X_{\text{geo}} \), with \( t \in [0, 1) \), such that the limit of the as \( t \to 1 \) is a union of two leaves \( \gamma_1^+ \) together with the point \( p_t \), which is in the closure of both \( \gamma_1^+ \) and \( \gamma_1^- \). We refer to such a family of leaves of \( X_{\text{geo}} \) as a *bifurcating family*.

To show that \( \Lambda_{\text{geo}}^+ \) is a branched lamination fully carrying a lamination, we must first define a map \( \psi : \tilde{N} \to \tilde{N} \) which is monotone on each leaf of \( X_{\text{geo}} \). For convenience, we use construction 2.1.4 to think of \( \Lambda_{\text{univ}}^+ \) as a geodesic lamination of a copy \( \mathbb{H}^2_{\text{univ}} \) of the hyperbolic plane bounded by \( S_{\text{univ}} \). Notice that each leaf \( \gamma \) of \( X_{\text{geo}} \) is contained in a leaf \( \lambda \) of \( \tilde{\mathcal{F}} \). The leaf \( \gamma \) might be bounded or unbounded in \( \lambda \), the latter case occurring for instance if \( \Lambda_{\text{geo}}^+(\lambda) \) is a foliation. A bounded endpoint of \( \gamma \) determines a complementary region to \( \Lambda_{\text{univ}}^+ \) in \( \mathbb{H}^2_{\text{univ}} \). Pick a point in such a complementary region. An unbounded end determines an endpoint in \( S_{\text{univ}}(\lambda) \), which determines its preimage under \( \phi_{\lambda}^{-1} \) in \( S_{\text{univ}} \). This preimage might be a point or an
interval; for concreteness, if it is an interval, pick its anticlockwisemost point. Span
the two points constructed in this way by a geodesic $\gamma_{\text{univ}}$. This geodesic $\gamma_{\text{univ}}$ can
be thought of as a “preimage” to $\gamma$. Note that by our choice of ideal endpoints for $\gamma_{\text{univ}}$
that $\gamma_{\text{univ}}$ does not cross any leaves of $\Lambda_{\text{univ}}^+$ whose endpoints are identified
by $\phi_\lambda$. It follows that $\gamma_{\text{univ}}$ crosses exactly those leaves of $\Lambda_{\text{univ}}^+$ which correspond
to leaves of $\Lambda^+(\lambda)$ crossed by $\gamma$. We define a monotone map $\psi : \gamma_{\text{univ}} \to \gamma$ which
takes each intersection $\gamma_{\text{univ}} \cap \Lambda_{\text{univ}}^+$ to the corresponding intersection $\gamma \cap \Lambda_{\text{geo}}^+(\lambda)$,
and takes complementary intervals either to the corresponding intervals, or collapses them to points if the corresponding leaves in $\Lambda_{\text{univ}}^+$ are identified in $\Lambda_{\text{geo}}^+(\lambda)$.

We want to make the assignment $\gamma \to \gamma_{\text{univ}}$ continuously as a function of $\gamma$,

at least away from the cusps $\partial \gamma$. This amounts to choosing the endpoints of $\gamma_{\text{univ}}$
in complementary regions to $\Lambda_{\text{univ}}^+$ in $\mathbb{H}^2_{\text{univ}}$ continuously as a function of $\gamma$. Since
the complementary regions are all homeomorphic to disks, and are therefore contractible, there is no obstruction to making such a choice. It is clear that this construction can be done in a $\pi_1(M)$ equivariant manner, where we think of $\pi_1(M)$
acting on the leaves of $\Lambda_{\text{univ}}^+$, and permuting the complementary regions as sets.

Along the cusps $\partial \gamma$, one must be slightly more careful. If $\gamma_t$ with $t \in [0, 1)$ limiting
to $\gamma_1^\pm$ is a bifurcating family, we must choose $(\gamma_t)_{\text{univ}}$ and $(\gamma_1^\pm)_{\text{univ}}$ so that there is
an equality

$$(\gamma_1^-)_{\text{univ}} \cup \{p\} \cup (\gamma_1^+)_{\text{univ}} = \lim_{t \to 1} (\gamma_t)_{\text{univ}}$$

for some $p$ in a complementary region to $\mathbb{H}^2_{\text{univ}}$. Again, the contractibility of complementary
regions implies this can be done, even equivariantly.

For each $\gamma$, the graph of $\psi : \gamma_{\text{univ}} \to \gamma$ defines an interval $\psi(\gamma_{\text{univ}})$ in the product
$\mathbb{H}^2_{\text{univ}} \times \tilde{N}$. The disjoint union of intervals $\psi(\gamma_{\text{univ}})$ as $\gamma$ varies over leaves of $X_V$
is itself an open 3-manifold $\tilde{N}'$ homeomorphic to $\tilde{N}$ as a subspace of $\mathbb{H}^2_{\text{univ}} \times \tilde{N}$.
Moreover, the intersections of the geodesics $\gamma_{\text{univ}}$ with leaves of $\Lambda_{\text{univ}}^+$ defines a
lamination $\Lambda_{\text{split}}^+$ of $\tilde{N}'$ that maps by $\psi$ to $\Lambda_{\text{geo}}^+$.

The action of $\pi_1(M)$ on the base 3-manifold $N$ induces an action on $\tilde{N}'$ as follows.
Since we want the actions on $\tilde{N}'$ and $\tilde{N}$ to be semiconjugate under the monotone map $\psi$, we must just decide how an element $\alpha \in \pi_1(M)$ should act on point preimages of $p \in \tilde{N}$. Now, for each $p \in \tilde{N}$, either $\psi^{-1}(p)$ is a point, or an interval in a complementary region of $\Lambda_{\text{univ}}^+$ with endpoints on distinct leaves $l, m$ of $\Lambda_{\text{univ}}^+$ which map to the same leaf of some $\Lambda_{\text{geo}}^+(\lambda)$. But then for $\alpha \in \pi_1(M)$, the preimage
of $\psi^{-1}(\alpha(p))$ is also a complementary interval, with endpoints on leaves $\alpha(l), \alpha(m)$
of $\Lambda_{\text{univ}}^+$. The interval we define $\alpha : \psi^{-1}(\alpha(p)) \to \psi^{-1}(\alpha(p))$ to be the unique affine homeomorphism which takes the endpoint on $l$ to the endpoint on $\alpha(l)$, and the endpoint on $m$ to the endpoint on $m$. Here we mean affine with respect to the length induced as a geodesic segment in $\mathbb{H}^2$. This is the desired action. \qed

Notice that we can choose $\psi : N \to N$ to have point preimages which are as small as desired. It follows that the laminations $\tilde{\Lambda}_{\text{split}}^\pm$ can be chosen to intersect leaves of $\tilde{\mathcal{F}}$ in lines which are uniformly $(k, \epsilon)$ quasigeodesic, for any choice of $k > 1, \epsilon > 0$.

Define $\Lambda_{\text{split}}^\pm$ to be the laminations of $M$ covered by $\tilde{\Lambda}_{\text{split}}^\pm$. 

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Notice too that if $\Lambda_{\text{split}}^+$ for instance is a genuine lamination, there is a choice of partition into guts and interstitial regions for which the cores of the interstitial annuli are exactly the cusps $c$.

**Theorem 5.3.8.** Let $\mathcal{F}$ be a taut foliation of an atoroidal 3–manifold $M$. Suppose $\mathcal{F}$ has two–sided branching. Then $M$ admits laminations $\Lambda_{\text{split}}^\pm$ which are essential laminations of $M$, which are transverse to $\mathcal{F}$, and which intersect the leaves of $\mathcal{F}$ in curves which are uniformly $(k, \epsilon)$ quasigeodesic, for any $k > 1, \epsilon > 0$.

**Proof.** We construct $\tilde{\Lambda}_{\text{split}}^\pm$ as in lemma 5.3.7 covering laminations $\Lambda_{\text{split}}^\pm$ in $M$.

By construction, the leaves of $\tilde{\Lambda}_{\text{split}}^+$ are all planes, so $\Lambda_{\text{split}}^\pm$ do not contain any spherical leaves or torus leaves bounding a solid torus, and complementary regions admit no compressing disks. Moreover, since $M$ admits a taut foliation $\mathcal{F}$ by hypothesis, $\tilde{M}$ is homeomorphic to $\mathbb{R}^3$, so complementary regions admit no essential spheres. It remains to show that there are no compressing monogons.

If $D$ is a compressing monogon for $\Lambda_{\text{split}}^+$, there are points $p, q$ in $\partial D$ contained in a leaf $\lambda$ of $\Lambda_{\text{split}}^+$ which are arbitrarily close together in $D$ but arbitrarily far apart in $\lambda$. Lift $D, p, q, \lambda$ to $\tilde{M}$, where by abuse of notation we refer to them by the same names. Since $p, q$ are arbitrarily close in $\tilde{M}$, they are contained in comparable leaves $\mu_1, \mu_2$ of $\tilde{\mathcal{F}}$. Suppose $p \in \mu_1 \cap \lambda$. Let $\tau$ be a short orthogonal trajectory from $\mu_1$ to $\mu_2$. The endpoints of the quasigeodesic $\lambda \cap \mu_1$ determine a leaf of $\Lambda_{\text{univ}}^+$, which determines a pair of special sections of $\text{UT}_\tau \tilde{\mathcal{F}}$. By lemma 5.3.5 and the uniformity of $k, \epsilon$, the angle between these special sections stays close to $\pi$ along $\tau$, for $\tau$ sufficiently short. It follows that there is a short path in $\lambda$, starting from $p$, from $\mu_1$ to some $p' \in \mu_2$. But $\tilde{\Lambda}_{\text{split}}^+ \cap \mu_2$ is a $(k, \epsilon)$–quasigeodesic lamination, so $p'$ and $q$ can be joined by a short path in $\lambda \cap \mu_2$, and therefore $p$ and $q$ are close in $\lambda$, contrary to the definition of $D$.

It follows that no such compressing monogon $D$ exists, and the laminations $\Lambda_{\text{split}}^\pm$ are essential, as claimed. $\square$

### 5.4. Straightening interstitial annuli

In this subsection we show that each complementary region to $\Lambda_{\text{split}}^+$ can be exhausted by a sequence of guts, for some partition into guts and $I$–bundles, such that the interstitial annuli are transverse to $\mathcal{F}$. This implies that complementary regions are solid tori. Note that this does not address the question, left implicit in the last subsection, of whether or not the laminations $\Lambda_{\text{split}}^\pm$ are genuine; but it does show that if they are genuine, then they are very full.

Each leaf of $\tilde{\Lambda}_{\text{split}}^\pm$ is transverse to the foliation $\tilde{\mathcal{F}}$, and therefore it inherits a codimension one foliation, whose leaves are the intersection with leaves of $\tilde{\mathcal{F}}$. We show that this foliation branches in at most one direction.

**Lemma 5.4.1.** Let $\pi$ be a leaf of $\tilde{\Lambda}_{\text{split}}^+$. The induced foliation $\pi \cap \mathcal{F}$ of $\pi$ does not branch in the positive direction, and similarly for leaves of $\tilde{\Lambda}_{\text{split}}^-$. 

**Proof.** Let $l$ be a leaf of $\Lambda^+(\lambda)$. That is, a leaf of $\Lambda(\text{core}(L^+(\lambda)))$, thought of as an unordered pair of distinct points in $S^1_{\text{univ}}$. By lemma 5.2.2 the image $\phi_\mu(l)$ is trivial unless $\mu < \lambda$. 


The subset of $L$ consisting of leaves $\mu$ with $\mu < \lambda$ does not branch in the positive direction. Consider the union

$$
\Pi(l) = \bigcup_{\lambda \in L} \phi_\lambda(l) \subset \tilde{M}
$$

where for each $\lambda$, we think of $\phi_\lambda(l)$ as a leaf of $\Lambda^+_{\text{geo}}(\lambda)$. Let $H \subset L$ be the subset of leaves $\lambda$ with $\phi_\lambda(l)$ nontrivial. Then $H$ does not branch in the positive direction. Moreover, $\Pi(l)$ is carried by the branched lamination $\tilde{\Lambda}^+_{\text{geo}}$, and naturally embeds into the split open lamination $\tilde{\Lambda}^+_{\text{split}}$ as a union of leaves $\pi_1, \pi_2, \cdots$, corresponding to the connected components $H_1, H_2, \cdots$ of $H$. Moreover, for each leaf $\lambda \in H_i$, the intersection $\pi_i \cap \lambda = \phi_\lambda(l)$ is a single line. It follows that the induced foliation of each $\pi_i$ does not branch in the positive direction.

Now, the leaves $l$ of laminations $\Lambda^+(\lambda)$ with $\lambda$ in $\tilde{F}$ are dense in $\Lambda^+_{\text{univ}}$. If $\pi$ is a limit of leaves $\pi_i$ where the induced foliation of $\pi_i$ does not branch in the positive direction, the same is true for $\pi$. To see why this is true, let $J$ be the subset of $L$ which $\pi$ intersects. Lemma 5.3.6 implies that the set of leaves of $\tilde{F}$ which $\pi$ intersects in a single component is both open and closed in $J$, and is therefore equal to $J$. It follows that if $\pi$ branches in the positive direction, then $J$ branches in the positive direction. In this case, $\pi$ intersects leaves $\mu_1, \mu_2$ of $\tilde{F}$ which are incomparable but satisfy $\mu_1 > \lambda, \mu_2 > \lambda$ for some third leaf $\lambda$ of $\tilde{F}$. But this means that $\pi_i$ intersects both $\mu_1$ and $\mu_2$ for sufficiently big $i$, contrary to the fact that $H_i$ does not branch in the positive direction.

This contradiction proves the claim, and the lemma follows.

**Lemma 5.4.2.** Let $\Lambda^+_{\text{split}}$ be the laminations constructed in theorem [5.3.8] Then there is a system of interstitial annuli $A^+_i$ for $\Lambda^+_{\text{split}}$, such that, (supressing the superscript $\pm$ for the moment) each $A_i$ satisfies the following properties:

1. The intersection of $A_i$ with the foliation $\mathcal{F}$ induces a nonsingular product foliation of $A_i = S^1 \times I$ by intervals point $\times I$.
2. There is a uniform $\epsilon$, which may be chosen as small as desired, such that each leaf of the induced foliation of each $A_i$ has length $\leq \epsilon$. Moreover, every point $p$ in an interstitial region can be connected to a point in the foliation by an arc contained in a leaf of $\mathcal{F}$ of length $\leq \epsilon/2$.

We say that such an interstitial system is horizontally foliated.

**Proof.** We do the construction upstairs in $\tilde{M}$. For convenience, we concentrate on $\tilde{\Lambda}^+_{\text{split}}$. By abuse of notation, we denote $\tilde{\Lambda}^+_{\text{split}} \cap \lambda$ by $\Lambda^+_{\text{split}}(\lambda)$, for $\lambda$ a leaf of $\tilde{F}$. We suppose that we have performed the splitting in such a way that the geodesic curvatures of the leaves of $\Lambda^+_{\text{split}}(\lambda)$ are uniformly pinched as close to 0 as we like.

Recall that we can split open $\tilde{\Lambda}^+_{\text{geo}}$ so that the laminations $\Lambda^+_{\text{split}}(\lambda)$ for a leaf of $\tilde{F}$ are as close as desired to geodesic laminations. We define the interstitial regions to be precisely the set of points $p$ in each leaf $\lambda$ of $\tilde{F}$, not in $\Lambda^+_{\text{split}}(\lambda)$, and which are contained in an arc in $\lambda$ of length $\leq \epsilon$ between two distinct boundary leaves. This obviously satisfies the desired properties. □
Lemma 5.4.3. Let $A_i$ be a horizontally foliated system of interstitial annuli for $\Lambda^+_{\text{split}}$. Then, after possibly throwing away annuli bounding compact interstitial regions, the system $A_i$ can be isotoped so that at the end of the isotopy, each annulus is transverse or tangent to $\mathcal{F}$.

Proof. We assume before we start that we have thrown away annuli bounding compact interstitial regions. The key to the proof of this lemma is the fact that the foliation of leaves of $\Lambda^+_{\text{split}}$ by the intersection with $\mathcal{F}$ does not branch in the positive direction. This lets us inductively push local minima in the positive direction, until they cancel local maxima. The first step is to describe a homotopy from each $A_i$ to some new $A_i$ which is either transverse or tangent to $\mathcal{F}$. At each stage of this homotopy, we require that the image of $A_i$ be foliated by arcs of its intersection with leaves of $\mathcal{F}$. It is clear that there is no obstruction to doing this. We fix notation: let $C$ be a complementary region, and $C_i$ the interstitial $I$–bundle bounded by the $A_i$

Let $c_i$ be the core of an interstitial annulus $A_i$. Suppose $c_i$ is not transverse or tangent to $\mathcal{F}$. Then $c_i$ must have at least one local maximum and one local minimum, with respect to the foliation $\mathcal{F}$. Either $A_i$ bounds a compact $I$ bundle over a disk, or else the universal cover $\tilde{A}_i$ is noncompact, and $c_i$ has infinitely many local maxima and minima. By hypothesis, we have already thrown away compact $I$–bundles, so we may assume $\tilde{A}_i$ is noncompact. Let $p$ be a local minimum on $\tilde{c}_i \cap \lambda$, and $p^\pm$ neighboring local maxima on $\tilde{c}_i \cap \lambda^\pm$ for leaves $\lambda, \lambda^\pm$ of $\tilde{\mathcal{F}}$. Then by construction, $\lambda < \lambda^+, \lambda^-$ in the partial order on $L$. (It should be remarked that the $\pm$ notation reflects the order of the points $p^-, p, p^+$ in the arc $\tilde{c}_i$, and not the order structure of the related leaves $\lambda^-, \lambda, \lambda^+$ in $L$.)

By lemma 5.4.1 $\lambda^+$ and $\lambda^-$ are comparable; without loss of generality, we can assume $\lambda^- \leq \lambda^+$. Then there is $q$ on $\tilde{c}_i$ between $p$ and $p^+$ with $q \in \tilde{c}_i \cap \lambda^-$. The points $q$ and $p^-$ are contained in arcs $I_q, I_p^-$ of $\tilde{A}_i$ which bound a rectangle $R \subset \tilde{A}_i$. The arcs $I_q, I_p^-$ also bound a rectangle $R' \subset \lambda^-$ of a complementary region to $\Lambda^+(\lambda^-)_{\text{split}}$. The union $R \cup R'$ is a cylinder which bounds an interval bundle over a disk $D \times I$ in a complementary region. We push $R$ across this $D \times I$ to $R'$, and then slightly in the positive direction, cancelling the local minimum at $p$ with the local maximum at $p'$. Do this equivariantly with respect to the action of $\pi_1(A_i)$ on the lift $\tilde{A}_i$. After finitely many moves of this kind, all maxima and minima are cancelled, and we have produced new immersed annuli $A_i'$ either transverse or tangential to $\mathcal{F}$, and homotopic to the original $A_i$. If $A_i'$ is tangent to $\mathcal{F}$, it finitely covers some annular complementary region to $\Lambda^+_{\text{split}} \cap \lambda$ for some leaf $\lambda$ of $\mathcal{F}$. Since it is homotopic to an embedded annulus, by elementary 3–manifold topology the degree of this covering map must be one, and $A_i'$ must be embedded. See for example chapter 13 of [34].

If $A_i'$ is transverse to $\mathcal{F}$, it is either embedded, or cuts off finitely many bigon $\times I$ where the edges of the bigons are transverse to $\mathcal{F}$. By inductively pushing arcs across innermost embedded bigons, we can reduce the number of bigons by two at a time. We can do this unless there is a single arc of self–intersection of $A_i'$ which corresponds to both cusps of a (non–embedded) bigon. But, since $A_i'$ is homotopic to $A_i$, which is embedded, the number of arcs of intersection must be even, for homological reasons. It follows that all bigons can be cancelled, and $A_i'$ is homotopic to $A_i''$ which is embedded. By further cancelling bigons of intersection of $A_i''$ with
$A''_i$ for distinct indices $i, j$ we can assume the union of the $A''_i$ are disjointly embedded. Let $C''$ be the $I$–bundle bounded by the $A''_i$. By construction, $C_i$ and $C''_i$ are homotopy equivalent in $C$. Since $C_i, C''_i$ and $C$ are all Haken, again, by standard 3–manifold topology, $C_i$ and $C''_i$ are isotopic in $C$, and therefore the system $A_i$ is isotopic to the system $A''_i$.

Compare lemma 2.2.2 of [6].

**Theorem 5.4.4.** Every complementary region to $\Lambda^\pm_{\text{split}}$ is a finite–sided ideal polygon bundle over $S^1$. Moreover, after possibly removing finitely many isolated leaves and collapsing bigon bundles over $S^1$, the laminations $\Lambda^\pm_{\text{split}}$ are minimal.

**Proof.** By lemma 5.4.2 and lemma 5.4.3 we can exhaust each complementary region by a sequence of guts $\Theta_i$ bounded by interstitial annuli transverse to $\mathcal{F}$. It follows that the boundary of each $\Theta_i$ is foliated by the intersection with $\mathcal{F}$, and therefore each boundary component is a torus. Since $M$ is irreducible and atoroidal, these tori either bound solid tori, or are contained in balls. But by construction, the core of each essential annulus is transverse to $\mathcal{F}$, and is therefore essential in $\pi_1(M)$ by theorem 3.1.4. So every torus bounds a solid torus, which is necessarily on the $\Theta_i$ side, and therefore each $\Theta_i$ is a solid torus.

For distinct $i, j$, the core of a complementary annulus to the interstitial annulus is a longitude contained in both $\partial\Theta_i$ and $\partial\Theta_j$. It follows that the inclusion of each $\Theta_i$ into $\Theta_{i+1}$ is a homotopy equivalence, and therefore the union is an open solid torus. Moreover, this inclusion takes interstitial regions to interstitial regions, and therefore each interstitial region is of the form $S^1 \times I \times \mathbb{R}^+$. It follows that each complementary region is a finite–sided ideal polygon bundle over $S^1$, as claimed.

If some complementary region is actually a bigon bundle over $S^1$, after lifting to $\tilde{M}$, the boundary leaves intersect each leaf of $\tilde{\mathcal{F}}$ in quasigeodesics which are asymptotic at infinity. It follows that such regions arose by unnecessarily splitting open a leaf of $\tilde{\Lambda}_{\text{geo}}^\pm$; we blow such regions down, identifying their boundary leaves.

To see that $\Lambda^\pm_{\text{split}}$ are minimal, after possibly removing finitely many isolated leaves, observe that if $\Lambda$ is a minimal sublamination of $\Lambda^+_{\text{split}}$, then the construction in lemma 5.4.2 still applies, and therefore by lemma 5.4.3 the complementary regions of this minimal sublamination are also finite sided ideal polygon bundles over $S^1$. It follows that every leaf of $\Lambda^+_{\text{split}} \setminus \Lambda$ must be a suspension of one of the finitely many diagonals of one of the finitely many ideal polygons. The theorem follows. \qed

5.5. **Anosov flows.** In this subsection we address the question of when the laminations $\Lambda^\pm_{\text{split}}$ are genuine, and not merely essential, at least when $\mathcal{F}$ has 2–sided branching.

**Lemma 5.5.1.** Let $S^1_{\text{univ}}$ be a minimal universal circle. The endpoints of leaves of $\Lambda^+_{\text{univ}}$ are dense in $S^1_{\text{univ}}$, and similarly for $\Lambda^-_{\text{univ}}$.

**Proof.** Suppose not. Then there is some interval $I \subset S^1_{\text{univ}}$ which does not intersect any leaf of $\Lambda^+_{\text{univ}}$. Since $S^1_{\text{univ}}$ is minimal, there is some leaf $\lambda$ of $\tilde{\mathcal{F}}$ such that core($\phi_\lambda$) intersects the interior of $I$. It follows that $\phi_\lambda(I)$ is an interval in $S^1_{\text{univ}}(\lambda)$ which does not intersect a leaf of $\phi_\lambda(\Lambda^+_{\text{univ}})$. But this is contrary to the fact from theorem 5.4.4.
that complementary regions to $\Lambda^+_\text{split}$ are finite sided ideal polygon bundles over $S^1$.

It is clear from theorem 5.4.3 that $\Lambda^+_\text{split}$ are genuine iff for some leaf $\lambda$ of $\mathcal{F}$, the geodesic laminations $\Lambda^+_{\text{geo}}(\lambda)$ are not foliations.

**Example 5.5.2.** Suppose $p \in S^1_{\text{univ}}$ is invariant under the action of $\pi_1(M)$ on $S^1_{\text{univ}}$. We let $\Lambda_p$ be the lamination of $S^1_{\text{univ}}$ consisting of all unordered pairs $p, q$ where $q \in S^1_{\text{univ}} \setminus p$. By construction 2.1.4 this corresponds to the geodesic lamination of $\mathbb{H}^2$ by all geodesics with one endpoint at $p$.

Note that for each leaf $\lambda$ of $\mathcal{F}$, the pushforward lamination $\phi_\lambda(\Lambda_p)_{\text{geo}}$ consists of the geodesic lamination of $\lambda$ by all geodesics with one endpoint at $\phi_\lambda(p)$.

We call the foliation of $\mathbb{H}^2$ by all geodesics with one endpoint at some $p \in S^1_\infty$ the geodesic fan centered at $p$. By abuse of notation, we also refer to the corresponding lamination of $S^1_\infty$ as the geodesic fan centered at $p$.

At first glance, it appears as though example 5.5.2 is the typical example of a taut foliation for which $\Lambda^+_\text{split}$ is essential but not genuine. However, this is somewhat misleading, as we will shortly see.

**Lemma 5.5.3.** Suppose for some $\mathcal{F}$ that there is a point $p \in S^1_{\text{univ}}$ which is invariant under the action of $\pi_1(M)$ on $S^1_{\text{univ}}$. Then $\mathcal{F}$ does not have two–sided branching.

**Proof.** Let $\lambda$ be some leaf of $\mathcal{F}$, and let $\mu_1 > \lambda$ and $\nu_2, \nu_3 > \lambda$ leaves such that $\mu_1$ is incomparable with $\nu_2, \nu_3$ and $\nu_2, \nu_3$ are incomparable with each other. Such leaves can certainly be found if $\mathcal{F}$ branches in the positive direction. Since $\mathcal{F}$ is taut, there is a positive transversal from the projection to $M$ of each $\nu_i$ to the projection of $\mu_1$. Lifting to $\widetilde{M}$, there exist elements $\alpha_2, \alpha_3 \in \pi_1(M)$ such that $\alpha_i(\mu_1) = \mu_i > \nu_i$ for $i = 2, 3$. Then the $\mu_i$ are all translates of each other, are mutually incomparable, and are all $> \lambda$ for some $\lambda$.

It follows that $L^+(\mu_i)$ are disjoint and incomparable for $i = 1, 2, 3$ and therefore $\text{core}(L^+(\mu_i))$ is contained in the closure of a single gap of $\text{core}(L^+(\mu_j))$ for $i \neq j$. But this implies that the sets $\text{core}(L^+(\mu_i))$ do not contain a common point of intersection. Since $p$ is preserved by the action of $\pi_1(M)$, if it were contained in $\text{core}(L^+(\mu_i))$ for some $i$, it would be contained in $\text{core}(L^+(\mu_j))$ for all three, contrary to what we have just shown. It follows that $p$ is not contained in $\text{core}(L^+(\mu_i))$.

But if $\mathcal{F}$ branches in the negative direction, by the tautness of $\mathcal{F}$ we can find an element $\beta \in \pi_1(M)$ such that $\beta_1(\mu_1)$ and $\mu_1$ are incomparable, and both satisfy $\mu_1, \beta(\mu_1) < \lambda'$ for some $\lambda'$. But then the union of $L^+(\mu_1)$ and $L^+(\beta(\mu_1))$ is all of $L$, and therefore $\text{core}(L^+(\mu_1)) \cup \text{core}(L^+(\beta(\mu_1))) = S^1_{\text{univ}}$, so $p$ is contained in one of them, which is a contradiction.

It follows that $\mathcal{F}$ does not have 2–sided branching, as claimed. □

**Construction 5.5.4.** Let $\mathcal{G}$ be a foliation of $\mathbb{H}^2$ by geodesics, and suppose $\mathcal{G}$ is not a geodesic fan. Then $\mathcal{G}$ does not branch, and the leaf space of $\mathcal{G}$ is homeomorphic to $\mathbb{R}$. Corresponding to the two ends of $\mathbb{R}$ there are exactly two points in $S^1_\infty$ which are not the endpoints of any leaf of $\mathcal{G}$. Call these the ideal leaves of $\mathcal{G}$.
**Lemma 5.5.5.** Let $\mathcal{F}$ be a taut foliation of an atoroidal 3–manifold $M$, and suppose for every leaf $\lambda$ of $\mathcal{F}$ that $\Lambda^+_{\text{geo}}(\lambda)$ is a foliation. Then every foliation $\Lambda^+_{\text{geo}}(\lambda)$ is a geodesic fan centered at some unique $s(\lambda) \in S^1_{\infty}(\lambda)$.

**Proof.** Let $J \subset L$ be the leaves of $\tilde{\mathcal{F}}$ for which $\Lambda^+_{\text{geo}}(\lambda)$ is not a geodesic fan. Then by construction 5.5.4, to each $\lambda \in J$ we can associate two points $p^\pm(\lambda) \in S^1_{\infty}(\lambda)$ which are the ideal leaves of the foliation $\Lambda^+_{\text{geo}}(\lambda)$. Let $\gamma_\lambda$ be the geodesic spanned by $p^\pm(\lambda)$.

By lemma 5.3.6, $J$ is open as a subset of $L$, and the union $\tilde{G} = \bigcup_{\lambda \in J} \gamma_\lambda$ is a locally finite union of complete planes transverse to $\tilde{\mathcal{F}}$. This union covers some compact surface $G \subset M$ transverse to $\mathcal{F}$, and the intersection with leaves of $\mathcal{F}$ defines a foliation of $G$. It follows that $G$ consists of a union of incompressible tori and Klein bottles. But $M$ is atoroidal, so $J$ is empty. The lemma follows. $\square$

To characterize those taut foliations for which $\Lambda^\pm_{\text{split}}$ are essential but not genuine, we must introduce the notion of an Anosov flow.

**Definition 5.5.6.** An Anosov flow $\phi_t$ on a 3–manifold $M$, with orbit space the 1–dimensional foliation $X$, is a flow which preserves a decomposition of the tangent bundle $TM = E^s \oplus E^u \oplus TX$ and such that the time $t$ flow uniformly expands $E^u$ and contracts $E^s$. That is, there are constants $\mu_0 \geq 1, \mu_1 > 0$ so that

$$\|d\phi_t(v)\| \leq \mu_0 e^{-\mu_1 t} \|v\|$$

for any $v \in E^s, t \geq 0$ and

$$\|d\phi_{-t}(v)\| \leq \mu_0 e^{-\mu_1 t} \|v\|$$

for any $v \in E^u, t \geq 0$.

The 1–dimensional foliations obtained by integrating $E^s$ and $E^u$ are called the strong stable and strong unstable foliations, and we denote them $X^s, X^u$ respectively. Furthermore, the bundles $TX \oplus E^s$ and $TX \oplus E^u$ are integrable, by a theorem of Anosov [2], and are tangent to 2–dimensional foliations called the weak stable and weak unstable foliations, denoted $\mathcal{F}^{ws}, \mathcal{F}^{wu}$ respectively.

See [18] and [19] for more details, and some important results in the theory of Anosov flows on 3–manifolds.

**Example 5.5.7.** Let $\mathcal{F}$ be $\mathcal{F}^{ws}$ for some Anosov flow $X$ on $M$. Then every leaf of $\mathcal{F}$ is foliated by flowlines of $X$. Suppose flowlines of $X$ are quasigeodesic in leaves of $\mathcal{F}$. Then after straightening flowlines leafwise, the foliation of each leaf $\lambda$ by $X$ is a geodesic fan, asymptotic to some $p \in S^1_{\infty}(\lambda)$.

The main result of this subsection is that such examples are the only possibility, when $\Lambda^\pm_{\text{split}}$ are essential but not genuine.

**Theorem 5.5.8.** Let $\mathcal{F}$ be a taut foliation of $M$, and suppose that $\Lambda^+_{\text{split}}$ is essential but not genuine. Then there is an Anosov flow $\phi_t$ of $M$ such that $\mathcal{F}$ is the weak stable foliation of $\phi_t$, and $\Lambda^+_{\text{split}}$ is the weak unstable foliation.
Proof. Constructing the flow is easy; most of the proof will be concerned with verifying that it has the requisite properties.

By lemma 5.5.5 for every leaf $\lambda$ of $\hat{F}$, the lamination $\Lambda^+_\text{geo}(\lambda)$ is a geodesic fan, asymptotic to some unique $p(\lambda) \in S^1(\lambda)$.

Let $\tilde{Y}$ be the unit length vector field on $\tilde{M}$ contained in $T\hat{F}$ which on a leaf $\lambda$ points towards $p(\lambda) \in S^1(\lambda)$. Here we are identifying $UT\hat{F}_p\lambda$ with $S^1(\lambda)$ for each $p \in \lambda$ by the endpoint map $e$. Then $\tilde{Y}$ descends to a nowhere vanishing leafwise geodesic vector field $Y$ on $\mathcal{F}$. We will show that if $\phi_t$ denotes the time $t$ flow generated by $Y$, then $\phi_t$ is an Anosov flow, and that $\mathcal{F}$ is the weak stable foliation for $\phi_t$.

We define $E^u$ as follows. Each point $q \in S^1\text{univ}$ determines a geodesic $\gamma_q(\lambda)$ in each leaf $\lambda$ of $\hat{F}$ by setting $\gamma_q(\lambda)$ equal to the unique geodesic from $\phi_\lambda(q)$ to $p(\lambda)$. As we let $\lambda$ vary but fix $q$, the $\gamma_q(\lambda)$ sweep out a (possibly disconnected) union of planes transverse to $\hat{F}$, whose leaves intersect leaf of $\hat{F}$ exactly in the flowlines of $\tilde{Y}$. We define $E^u$ to be the orthogonal distribution to $\tilde{Y}$ in the tangent space to these leaves.

For simplicity, we first treat the case that $\mathcal{F}$ is minimal, and then show how to modify the argument for general $\mathcal{F}$.

Recall definition 4.2.2 of a sawblade from §4.2 and the definition of the contracting and expanding directions.

Let $\gamma$ be a closed embedded geodesic contained in a leaf $\lambda$ of $\mathcal{F}$. Let $\tilde{\lambda}$ be a covering leaf of $\lambda$ in $\tilde{M}$, stabilized by the corresponding element $[\gamma] \in \pi_1(M)$. Since $p(\tilde{\lambda})$ is defined intrinsically by the foliation $\Lambda^+_\text{geo}(\lambda)$, it follows that $[\gamma]$ fixes $p(\lambda)$, and $\gamma$ is a closed orbit of $Y$. Let $\tilde{\gamma}$ be the corresponding axis of $[\gamma]$ on $\tilde{\lambda}$. Then one endpoint of $\tilde{\gamma}$ is $p(\lambda)$; let $r \in S^1(\tilde{\lambda})$ be the other endpoint of $\tilde{\gamma}$. By hypothesis, $\tilde{\gamma}$ is equal to $\phi_{\gamma}(l)_{\text{geo}}$ for some leaf $l$ of $\Lambda^+_{\text{univ}}$.

Let $\tau$ is an embedded interval in $L$ containing $\tilde{\lambda}$ as an endpoint. For sufficiently short $\tau$, the intervals $\tau$ and $[\gamma](\tau)$ are completely comparable; moreover, for some choice of orientation on $\gamma$, we can assume $[\gamma](\tau) \subset \tau$

Then the set of leaves $\phi_{\gamma}(l)_{\text{geo}}$ for $\nu \in \tau$ is an embedded rectangle $R$ in $\tilde{M}$, such that $\gamma(R) \subset R$, and we can find an embedded $\epsilon$-sawblade $P$ for $\mathcal{F}$ in $M$ with $\gamma$ as a boundary circle. Notice that $R$ is tangent to $E^u \oplus TY$. Since $P$ is embedded, there is a lower bound on the length of an arc in $M$ from $P$ to itself which is not homotopic into $P$. By minimality of $\mathcal{F}$, there is a uniform $R$ such that for any leaf $\lambda$ of $\mathcal{F}$, and every point $p \in \lambda$, the ball of radius $R$ about $p$ in $\lambda$ (in the path metric) intersects $P$.

Return to the universal cover. Then preimages of $P$ intersect every leaf $\lambda$ of $\tilde{F}$ in a union of bi–infinite geodesics and geodesic rays, contained in flowlines of $\tilde{Y}$, which intersect the ball of radius $R$ about any point in $\lambda$, as measured in $\lambda$. If $\tilde{P}$ is one component of the preimage, then $\tilde{P} \cap \lambda$ contains a geodesic ray $\delta$ in the contracting direction. Let $q \in \lambda$ be a point far from the geodesic containing $\delta$. Then there is a translate $\alpha(\tilde{P})$ with $\alpha \in \pi_1(M)$ which intersects the ball in $\lambda$ of radius $R$ about $q$, and whose intersection with $\lambda$ contains a geodesic ray $\delta'$. By the choice of $q$, the rays $\delta$ and $\delta'$ are not contained in the same geodesic. Moreover, since $P$ is compact and embedded in $M$, it does not accumulate on itself, so the rays $\delta$ and
δ′ are not asymptotic to the same point in $S_{\infty}^1(\lambda)$. On the other hand, δ and δ′ are both contained in flowlines of $\tilde{Y}$, which are asymptotic in the positive direction; it follows that the contracting direction of $\gamma$ is the negative direction. A priori, holonomy around the sawblade $P$ is merely non-increasing for some nearby leaf. But in fact, this argument shows that the holonomy is actually strictly decreasing for all leaves sufficiently close to $\gamma$. The same argument shows that there is another sawblade $P′$ on the other side of $\gamma$, for which $\gamma$ is also the contracting direction.

We show now that flow along $\tilde{Y}$ eventually strictly increases the length of any integral curve of $E^u$. Let $\tau$ be a short integral curve, and let $\tilde{\tau}$ be a lift to $\tilde{M}$. Let $\lambda$ be a leaf of $\tilde{F}$ which intersects $\tilde{\tau}$ and also some lift $\tilde{\gamma}$ of $\gamma$. Then the flowline of $\tilde{Y}$ through $\tau \cap \lambda$ is eventually asymptotic to the (negatively oriented) $\tilde{\gamma}$. But we have just seen that holonomy around $\gamma$ is strictly decreasing, so flow along $\tilde{Y}$ eventually blows up any arbitrarily short transversal to $\gamma$ to a transversal of definite size. If follows that flow along $\tilde{Y}$ eventually blows up the length of any arbitrarily short $\tilde{\tau}$, as claimed. By covering an integral curve with such short curves, and using the compactness of $M$, we can find uniform estimates for the rate of this blow up, as required.

If $F$ is not minimal, the argument is basically the same, except that we must use the fact that almost every geodesic ray in a leaf of $F$ is asymptotic to some minimal set to extend the arguments to all of $F$.

It is now easy to deduce our main theorem.

**Theorem A.** Let $F$ be a co-orientable taut foliation of a closed, orientable algebraically atoroidal 3–manifold $M$. Then either $F$ has 2–sided branching and is the weak stable foliation of an Anosov flow, or else there are a pair of very full genuine laminations $\Lambda_{\text{split}}^\pm$ transverse to $F$.

**Proof.** If $F$ has two–sided branching, then this follows from theorem 5.3.8 theorem 5.4.4 and theorem 5.5.8

If $F$ has one–sided branching, this follows from theorem 4.1.1 of [6]. If $F$ is $\mathbb{R}$–covered, this follows from theorem 5.3.13 of [5].

**Corollary 5.5.9.** Let $M$ be a closed 3–manifold which admits a taut foliation. Then either $M$ is toroidal, or admits an Anosov flow whose weak stable and unstable foliations have two–sided branching, or else $\pi_1(M)$ is word hyperbolic, the mapping class group of $M$ is finite, and every self–homeomorphism of $M$ homotopic to the identity is isotopic to the identity.

**Proof.** This follows from theorem A together with theorem 3.2.6 theorem 3.2.7 and theorem 3.2.12 all due to Gabai–Kazez.

### 6. The Dynamics of $\Lambda_{\text{split}}^\pm$

This section is basically descriptive. No important lemmas or theorems are proved, but we try to describe in some geometric detail the interaction of the foliation $F$ with the laminations $\Lambda_{\text{split}}^\pm$. Finally, in §6.3 we pose some questions, and paint an optimistic and conjectural picture of the interaction of $\Lambda_{\text{split}}^\pm$ and $F$, at least in the best case.
6.1. The structure of gut regions. In this subsection we describe the structure of a gut region of $\Lambda^\pm_{\text{split}}$. This subsection is basically descriptive; no theorems are proved here, and only one straightforward lemma. For convenience of notation, we concentrate on $\Lambda^+_{\text{split}}$; of course, the case of $\Lambda^-_{\text{split}}$ is completely analogous.

By theorem 5.4.4 we know that complementary regions are finite sided ideal polygon bundles over $S^1$, and therefore gut regions are finite sided neutered ideal polygon bundles over $S^1$. This terminology is standard: an ideal polygon is neutered by removing neighborhoods of its ideal points. The usual neighborhoods one takes are the intersections with small horoballs centered at the points.

Recall that the collection of cusps constructed in lemma 5.3.7 are the cores of a system of interstitial annuli $A_i$ for a decomposition of complementary regions of $\Lambda^\pm_{\text{split}}$ into guts and interstitial $I$–bundles, and that there is a corresponding decomposition of $\Lambda^\pm_{\text{geo}}$ into guts and interstitial $I$–bundles, where some of the $I$–bundles might limit on branch circles or lines. Throw away annuli bounding compact $I$–bundles. Now observe that each $\tilde{A}_i$ intersects each leaf $\lambda$ of $\tilde{\mathcal{F}}$ in at most two intervals, and therefore by lemma 5.4.1 we can conclude that the $A_i$ are actually transverse to $\mathcal{F}$. Note a posteriori that we can conclude from theorem 5.4.4 that every interstitial annular system can be isotoped to be transverse to $\mathcal{F}$; that is, we need to allow components which are tangential to $\mathcal{F}$, as is proved in lemma 5.4.3.

Recall further that the branch locus of $\Lambda^\pm_{\text{geo}}$ consists of a collection of geodesic lines and circles in leaves of $\mathcal{F}$. Some of these might be in the closure of an interstitial region of $\Lambda^\pm_{\text{geo}}$.

Let $\mathfrak{G}$ be a gut region of $\Lambda^\pm_{\text{split}}$. As remarked above, it is bounded by a system of interstitial annuli which are transverse to $\mathcal{F}$. Let $\mathfrak{G}$ be a cover of $\mathfrak{G}$ in $\mathbb{M}$. Then $\mathfrak{G}$ is also the universal cover of $\mathfrak{G}$. Topologically, $\mathfrak{G}$ is a solid torus, and $\mathfrak{G}$ is a solid cylinder. If $\gamma$ denotes the core circle of $\mathfrak{G}$, we can also think of $\gamma$ by abuse of notation as the generator of $\pi_1(\mathfrak{G}) = \mathbb{Z}$ which acts on $\mathfrak{G}$ by deck transformations. Let $\lambda_t$, with $t \in (-\infty, \infty)$ parameterize the leaves of $\tilde{\mathcal{F}}$ which intersect $\mathfrak{G}$, and suppose this parameterization is chosen so that the action of $\gamma$ on $L$ satisfies

$$
\gamma(\lambda_t) = \lambda_{t+1}
$$

The boundary $\partial \mathfrak{G}$ consists of two parts: the annular components $A_t \subset \partial \mathfrak{G}$, and the laminar boundary $\partial \mathfrak{G} \cap \Lambda^\pm_{\text{split}}$. Note that if $\Lambda^\pm$ are co-orientable, this decomposition defines the structure of a sutured manifold on $\mathfrak{G}$, in the sense of Gabai; see [21] for a definition and basic properties of sutured manifolds. We denote these subsets of $\partial \mathfrak{G}$ by $\partial_v \mathfrak{G}$ and $\partial_h \mathfrak{G}$, consistent with the usual notation from [21]. These lift to $\partial_v \tilde{\mathfrak{G}}$ and $\partial_h \tilde{\mathfrak{G}}$ in the obvious way. The boundary $\partial \mathfrak{G}$ is foliated by circles of intersection with leaves of $\tilde{\mathcal{F}}$.

There are three distinct classes of interstitial regions. Recall the map $\psi$ from lemma 5.3.7.

(1) If an interstitial region $R$ of $\Lambda^+_{\text{geo}}$ contains no branch locus, that is, if the corresponding interstitial region $R'$ of $\Lambda^+_{\text{split}}$ maps homeomorphically to $R$ by $\psi$, then the foliation of $\partial \mathfrak{G}$ by circles transverse to $\mathcal{F}$ can be extended to the entire interstitial region. The lift of the interstitial region intersects exactly the leaves $\lambda_t$ of $\tilde{\mathcal{F}}$; i.e. the same set of leaves that $\mathfrak{G}$ intersects. We call this kind of interstitial region a cusp.
(2) If an interstitial region $R$ of $\Lambda^+_{\text{geo}}$ contains a circle branch component $c$, this circle gets split open to a tangential interstitial annulus in the corresponding interstitial region $R'$ of $\Lambda^+_{\text{split}}$. The leaves of $F \cap R'$ spiral around this annulus and limit on to it. In $\tilde{M}$, the annulus is covered by a rectangle contained in a leaf $\nu$ of $\tilde{F}$ which is a limit of $\lambda_t$ as either $t \to -\infty$ or $t \to \infty$. In the figure, the spiralling is from the positive side, and $\nu$ is a limit of $\lambda_t$ as $t \to -\infty$. We call this kind of interstitial region a (positive or negative) annular spiral.

(3) If an interstitial region $R$ of $\Lambda^+_{\text{geo}}$ contains a line branch component, it might conceivably contain infinitely many such components, which we denote $\gamma_i$. Each of these gets split open to a rectangle contained in the interior of an interstitial region $R'$ of $\Lambda^+_{\text{split}}$ bounded by a transverse interstitial annulus. Moreover, the leaves $F \cap R$ spiral out to fill all of the preimage of $R$ under the collapsing map $\psi$, and limit on the union of split open rectangles in $R'$ contained in distinct leaves $\nu_i$ of $\tilde{F}$ which are all limits of $\lambda_t$ as $t \to -\infty$. Note that there is no claim that the $\nu_i$ fall into finitely many orbit classes under the action of $\gamma$. Note that since there are at least infinitely many $\nu_i$ which are limits of $\lambda_t$, the spiralling in this case must be from the positive side, by lemma 5.4.1. We call this kind of interstitial region a positive rectangular spiral.

These three classes of interstitial region are illustrated in figure 8.

We say that two interstitial regions bounding a gut component of $\Lambda^+_{\text{split}}$ are adjacent if they have a common boundary leaf.

**Lemma 6.1.1.** Adjacent interstitial regions bounding a gut component of $\Lambda^+_{\text{split}}$ do not both contain negative annular spirals.

**Proof.** If so, then there are at least two distinct leaves $\nu, \nu'$ of $\tilde{F}$ which are positive limits of $\lambda_t$ as $t \to \infty$, and which intersect the same leaf of $\Lambda^+_{\text{split}}$, contrary to lemma 5.4.1. \hfill \Box

6.2. **Dynamics of $\Lambda^+_{\text{split}}$.** We continue to use the setup and notation from subsection 6.1. Throughout this subsection, for convenience, we work with $\Lambda^+_{\text{geo}}$ instead
of $\Lambda^+_{\text{split}}$ since the relationship with the geometry of leaves of $\mathcal{F}$ is clearer. By abuse of notation, we refer to the leaves of $\Lambda^+_{\text{geo}}$ by which we mean the images of leaves of $\Lambda^+_{\text{split}}$ under the monotone map $\psi$ from lemma 5.3.7. The relationship between leaves of $\Lambda^+_{\text{geo}}$ and leaves of $\Lambda^+_{\text{univ}}$ is more straightforward: each leaf $l_{\text{univ}}$ of $\Lambda^+_{\text{univ}}$ determines a geodesic $l_{\text{univ}}(\lambda)$ in each leaf $\lambda$ of $\mathcal{F}$, as a leaf of $\Lambda^+_{\text{geo}}(\lambda) = (\phi_\lambda(\Lambda^+_{\text{univ}}))_{\text{geo}}$. The union of the geodesics $l_{\text{univ}}(\lambda)$ as $\lambda$ varies over $L$ is a union of leaves of $\Lambda^+_{\text{geo}}$. Moreover, every leaf of $\Lambda^+_{\text{geo}}$ arises this way, although possibly not uniquely. For a leaf $l$ of $\Lambda^+_{\text{geo}}$ we let $l_{\text{univ}}$ denote a leaf of $\Lambda^+_{\text{uni}}$ associated to $l$ by this construction. In particular,

$$l \cap \kappa = (\phi_\kappa(l_{\text{univ}}))_{\text{geo}}$$

for every leaf $\kappa$ of $\mathcal{F}$ which $l$ intersects.

Suppose $l, m$ are boundary leaves of $\Lambda^+_{\text{geo}}$ which both bound a common annular interstitial region after splitting, and let $\bar{c}$ denote the corresponding branch circle of $\Lambda^+_{\text{geo}}$. Let $l_{\text{univ}}, m_{\text{univ}}$ denote associated leaves of $\Lambda^+_{\text{univ}}$. Let $l_i, m_i$ be leaves of $\Lambda^+_{\text{geo}}$ which accumulate on $l, m$ respectively, with associated leaves $(l_i)_{\text{univ}}, (m_i)_{\text{univ}}$ of $\Lambda^+_{\text{uni}}$, and let $\nu$ be the leaf of $\mathcal{F}$ containing the lift $\bar{c}$, which is a geodesic in $\nu$, and a leaf of $\Lambda^+_{\text{geo}}(\nu)$. Note that we have an equality

$$\bar{c} = l \cap \nu = m \cap \nu$$

The element $\gamma \in \pi_1(M)$ stabilizes $\nu$, and therefore acts as a translation on $\bar{c}$. Since $l_i \to l$, we must have $l_i \cap \nu \to l \cap \nu$. On the other hand, $\gamma$ stabilizes $\nu$, and permutes $l_i \cap \nu$. Since this sequence of geodesics accumulates on $l \cap \nu$, they must all share a common endpoint with $\bar{c}$, or else some $l_i \cap \nu$ would cross $\gamma(l_i \cap \nu)$ transversely, contrary to the definition of a lamination. There are two possibilities: either the $l_i \cap \nu$ share the attracting fixed point of $\gamma$ with $l \cap \nu$, or else they share the repelling fixed point. We call these type 1 and type 2 respectively; these types are illustrated in figure [9].

The projection $\pi(l)$ of the leaf $l$ to $M$ is an annulus which is a boundary leaf of $\Lambda^+_{\text{geo}}$. The projection of the leaf $l_i$ spirals around this annulus under holonomy transport around the core of the annulus. In type 1, the projection of $l_i$ accumulates on the projection of $l$ in the positive direction, and in type 2, the accumulation is in the negative direction. That is, holonomy transport around the core of the annulus $\pi(l)$ is contracting in type 1, and expanding in type 2. Here we are implicitly orienting the core of the annulus so that the positive direction agrees with the co-orientation of $\mathcal{F}$.

As described in §6.1, the intersections $\lambda_\ell \cap \bar{c}$ spiral around $\nu$, accumulating on the geodesic $\bar{c}$. It follows that if our annular spiral is positive, then in type 2, the geodesics $l_i \cap \lambda_\ell$ and $l \cap \lambda_\ell$ are asymptotic. Similarly, if the annular spiral is negative, then in type 1, the geodesics $l_i \cap \lambda_\ell$ and $l \cap \lambda_\ell$ are asymptotic. This is illustrated in figure [10].

Figure [10] illustrates types 1 and 2 for a positive annular spiral. Notice that although $l \cap \lambda_\ell$ and $l_i \cap \lambda_\ell$ must be asymptotic in type 2, they are not necessarily asymptotic in type 1.
6.3. An optimistic picture. The constructions and results above lead to many other natural questions and potential connections with other areas of foliation theory and 3–manifold topology. We discuss some of these connections now.

Universal circles for $\Lambda_{\text{split}}^\pm$: If $\Lambda_{\text{split}}^\pm$ are genuine and tight, then theorem 3.8 of [8] shows that there are a pair of circles $(S_{\text{univ}}^1)^\pm$ and laminations $\mathcal{L}^\pm$ of $(S_{\text{univ}}^1)^\pm$ respectively whose leaves are in bijective correspondence with the leaves of $\tilde{\Lambda}_{\text{split}}^\pm$. Moreover, there is a natural action of $\pi_1(M)$ on either circle, preserving the laminations $\mathcal{L}^\pm$. In this case, what is the relationship between the three circles $S_{\text{univ}}^1$, $(S_{\text{univ}}^1)^+$, $(S_{\text{univ}}^1)^-$?

Good dynamical pairs for $\Lambda_{\text{split}}^\pm$: If $\Lambda_{\text{split}}^\pm$ are carried by a pair of branched surfaces which are a good dynamic pair in the sense of Mosher (see [41] for a definition), then by theorem 4.10.4 of [41], $M$ admits a pseudo–Anosov flow $\psi_t$. Can $\psi_t$ be made transverse or almost transverse to $\mathcal{F}$? Corollary 3.9 of [8] constructs a universal circle associated to a pseudo–Anosov flow; what is the relationship of this universal circle with the other universal circles above?

Finite depth foliations: If $\mathcal{F}$ is finite depth, Mosher and Gabai independently construct a pseudo–Anosov flow on $M$ almost transverse to $\mathcal{F}$. The leaves of the singular stable and unstable foliations of the flow can be split open to a pair of genuine laminations. What is the relationship of these laminations to $\Lambda_{\text{split}}^\pm$?

Anosov flows: If $\mathcal{F}$ is the weak stable foliation of an Anosov flow, is there nevertheless a universal circle for $\mathcal{F}$ for which $\Lambda_{\text{split}}^\pm$ are genuine? If $\Lambda_{\text{split}}^\pm$ is (monotone equivalent to ) a taut foliation, what is the relationship between its universal circle, and the universal circle of $\mathcal{F}$?

In the most optimistic picture for the structure of $\mathcal{F}$, $\Lambda_{\text{split}}^\pm$, all circles and structures constructed by various means are compatible to the extent that this makes sense. This motivates the definition of the following object, called a pseudo–Anosov package for $M$. 

Figure 9. Either the attracting or the repelling fixed point of $\gamma$ on $\nu$ is asymptotic to nearby leaves of $\Lambda^+(\nu)$ on either side.
Figure 10. In type 1, nearby leaves $l_i$ to $l$ spiral around $l$ in the positive direction; in type 2, they spiral around in the negative direction. This determines the geometry of $l_i \cap \lambda_t$ for large $\lambda_t$. In particular, $l_i \cap \lambda_t$ is asymptotic to $l \cap \lambda_t$ in type 2, but not necessarily in type 1.

**Definition 6.3.1.** Let $M$ be an algebraically atotoroidal 3–manifold, and $\mathcal{F}$ a taut foliation. A pseudo–Anosov package $\Psi$ for $M$ consists of the following structure:

1. The universal cover $\tilde{M}$ has the structure of a product $D^2_{\text{univ}} \times \mathbb{R}$ in such a way that the action of $\pi_1(M)$ descends to an action on the $D^2_{\text{univ}}$ factor. The action of $\pi_1(M)$ on $\tilde{M}$ extends to a continuous action on the product of the closed disk with $\mathbb{R}$. Moreover, there should be a natural identification $\partial D^2_{\text{univ}} = S^1_{\text{univ}}$, compatible with the representation $\rho_{\text{univ}}$.

2. The laminations $\Lambda^\pm_{\text{univ}}$ determine geodesic laminations $(\Lambda^\pm_{\text{univ}})_{\text{geo}}$ of $D^2_{\text{univ}}$ with finite sided complementary regions which are transverse to each other and bind $D^2_{\text{univ}}$; that is, complementary regions to the union of the two geodesic laminations are finite sided compact polygons.

3. For each leaf $\lambda$ of $\mathcal{F}$, the projection of $\lambda$ to $D^2_{\text{univ}}$ is convex; i.e. it is the interior of a region bounded by an embedded union of complete geodesics of $(\Lambda^\pm_{\text{univ}})_{\text{geo}}$.

4. The laminations $\Lambda^\pm_{\text{split}}$ can be blown down to the stable/unstable foliations of a pseudo–Anosov flow $\psi_t$ transverse, or almost transverse to $\mathcal{F}$.
(5) The quotient of the circle \( S^1_{\text{univ}} \) by the laminar relations determined by \( \Lambda^\pm_{\text{univ}} \) is topologically a sphere \( S^2_{\text{univ}} \), which is naturally homeomorphic to the ideal boundary \( S^2_{\infty} \) of \( \tilde{M} \), and determines a Peano map \( P : S^1_{\text{univ}} \to S^2_{\text{univ}} \) which can be approximated by embeddings.

![Figure 11](image)

**Figure 11.** Leaves of \( \Lambda^\pm_{\text{split}} \) (red and blue) run up seams in either direction, and bind the leaves of \( \tilde{F} \) (green).

Properties 3 and 5 above imply that for each leaf \( \lambda \) of \( \tilde{F} \), the Peano map \( P : S^1_{\text{univ}} \to S^2_{\text{univ}} \) factors through the monotone map \( \phi_\lambda : S^1_{\text{univ}} \to S^2_{\infty}(\lambda) \) and induces a continuous map from \( S^1_{\infty}(\lambda) \) to \( S^2_{\infty}(\pi_1(M)) \), continuously extending the inclusion \( \lambda \to \tilde{M} \). This so-called continuous extension property is actually established by Fenley for many classes of taut foliations, including all those with quasigeodesic transverse (or almost transverse) pseudo–Anosov flows. See [20] for a detailed discussion.

**Pseudo–Anosov packages:** Given a pseudo–Anosov package \( \Psi \), all the data of the package can be recovered from the representation \( \rho_{\text{univ}} : \pi_1(M) \to \text{Homeo}^+(S^1_{\text{univ}}) \) except for the foliation \( F \). When are distinct taut foliations compatible with the same pseudo–Anosov package? Are there only finitely many pseudo–Anosov packages (up to isotopy and the ambiguity of \( \tilde{F} \)) on a fixed 3–manifold?

Figure 11 shows how leaves of \( \Lambda^\pm_{\text{split}} \) and of \( \tilde{F} \) should interlock. Notice how the blue leaf branches in the negative direction, and the red leaves branch in the positive direction. Notice too how the leaves of the laminations are asymptotic into the “seams” of \( \tilde{F} \).
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