Lax pairs for $N = 2, 3$ supersymmetric KdV equations and their extensions

S. Krivonos$^a,1$, A. Pashnev$^a,2$ and Z. Popowicz$^b,3$

(a) Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow Region, Russia
(b) Institute of Theoretical Physics, University of Wroclaw, pl. M. Borna 9, 50-205 Wroclaw, Poland

Abstract

We present the Lax operator for the $N = 3$ KdV hierarchy and consider its extensions. We also construct a new infinite family of $N = 2$ supersymmetric hierarchies by exhibiting the corresponding super Lax operators. The new realization of $N = 4$ supersymmetry on the two general $N = 2$ superfields, bosonic spin 1 and fermionic spin 1/2, is discussed.

E-Mail:
1) krivonos@thsun1.jinr.dubna.su
2) pashnev@thsun1.jinr.dubna.su
3) ziemek@ift.uni.wroc.pl
1. Introduction. During the last years the description of three different infinite families of $N = 2$ supersymmetric integrable hierarchies in terms of super Lax operators has been proposed in [1, 2, 3, 4]. The generalization to the matrix case has been derived in [3, 4]. All these Lax operators $L$ have the following generic form [3]

$$L_{KP}^{\text{red}} = \partial + b + \mu D + \sum_{j=-\infty}^{-1} \left(a_j \partial - [D a_j] \mathcal{D} + \omega_j D \partial - \frac{1}{2} [D \omega_j] \left[ D, \mathcal{D} \right] \right) \partial^{j-1},$$

(1)

which is the general solution of the reduction constraints $[D, L_{KP}^{\text{red}}] = 0$ [5] for the $N = 2$ supersymmetric KP hierarchy. Here $b (\mu)$ are chiral bosonic (fermionic) $N = 2$ superfields while $a_j (\omega_j)$ are generic bosonic (fermionic) ones. The Lax operator (1) still contains an infinite number of fields and its further reductions [3, 4] are characterized by a finite number of fields and describe three families of $N = 2$ supersymmetric hierarchies with $N = 2$ super $W_n$ algebras as their second Hamiltonian structure.

These three families of hierarchies include the $N = 2$ $a = -2, 4$ KdV equations [6] and two KdV equations with $N = 4$ $SU(2)$ superconformal algebra as the second Hamiltonian structure [6, 8]. However they do not describe neither $N = 2$ $a = 1$ KdV equation [7] nor $N = 3$ KdV ones [11]. The aim of this letter is to present the Lax operator for $N = 3$ KdV equation, which generalize the Lax operator for $N = 2$ $a = 1$ KdV ones [12], and based on it, to construct a new infinite class of further reductions (with a finite numbers of fields) of the Lax operator (1) which gives rise to new integrable $N = 2$ hierarchies.

2. Lax representation for $N = 3$ super KdV equation. $N = 3$ supersymmetric KdV equation with $N = 3$ super conformal algebra as its second Hamiltonian structure has been proposed in [13]. In terms of $N = 2$ superfields it can be written as the following coupled system of evolution equations for the general bosonic spin 1 superfield $J(Z)$ and general fermionic spin 1/2 superfield $g(Z)$:

$$\frac{\partial}{\partial t} J = -J''' - (J^3)' + 3(J [D, \mathcal{D}] J)' + 3(g' [D, \mathcal{D}] g)' - 12(J D g \mathcal{D} g)' + 6(DJ g Dg)' + 6(DJ g \mathcal{D} g)',$$

$$\frac{\partial}{\partial t} g = -g''' + 6J(\mathcal{D} J D g + DJ \mathcal{D} g) + 3(J [D, \mathcal{D}] g)' - 3J^2 g' - 6g' D g \mathcal{D} g - 6 g Dg \mathcal{D} g',$$

(2)

where $Z = (z, \theta, \bar{\theta})$ is a coordinate of the $N = 2$ superspace, $dZ \equiv dzd\theta d\bar{\theta}$ and the fermionic covariant derivatives $D$ and $\mathcal{D}$ are defined as

$$D = \frac{\partial}{\partial \theta} - \frac{1}{2} \bar{\theta} \frac{\partial}{\partial z}, \quad \mathcal{D} = \frac{\partial}{\partial \bar{\theta}} - \frac{1}{2} \theta \frac{\partial}{\partial z}, \quad D^2 = \mathcal{D}^2 = 0, \quad \{D, \mathcal{D} \} = -\frac{\partial}{\partial z} \equiv -\partial.$$

(3)

One can check that the equations (2) are covariant with respect to the following transformations of an extra hidden supersymmetry:

$$\delta J = \varepsilon g', \quad \delta g = \varepsilon J,$$

(4)

where $\varepsilon$ is a Grassmann parameter. Just this additional supersymmetry together with explicit $N = 2$ ones form $N = 3$ supersymmetry.

---

1 The equivalent reduction $DL_{KP}^{\text{red}} = L_{KP}^{\text{red}} D = 0$ has been proposed in [3, 4].
The $N = 3$ KdV equation (2) can be obtained from the Hamiltonian [11]

$$H = -3 \int dZ \left( J[D,\overline{D}]J + \frac{1}{3}J^3 - g[D,\overline{D}]g' + 2Jgg' + 8JDg\overline{D}g \right),$$

if we use the $N = 3$ superconformal algebra Poisson brackets, which read in terms of $N = 2$ supercurrents $J(Z)$ and $g(Z)$ as follows:

$$\{J(Z_1), J(Z_2)\} = \left( \frac{1}{2}[D,\overline{D}]\partial + \partial J + J\partial + \overline{D}JD + DJ\overline{D} \right) \Delta(Z_1 - Z_2),$$

$$\{J(Z_1), g(Z_2)\} = \left( \partial g + \frac{1}{2}g\partial - \overline{D}gD - Dg\overline{D} \right) \Delta(Z_1 - Z_2),$$

$$\{g(Z_1), g(Z_2)\} = \frac{1}{2} \left( J\partial + [D,\overline{D}] \right) \Delta(Z_1 - Z_2).$$ (6)

Here $\Delta(Z_1 - Z_2) = (\theta_1 - \theta_2)(\overline{\theta}_1 - \overline{\theta}_2)\delta(z_1 - z_2)$ is $N = 2$ superspace delta function and all operators in r.h.s. are evaluated in the second point. Let us stress that in the contrast with two other different $N = 3$ KdV equations [13, 14], the equation (2) and Hamiltonian (5) explicitly break the $SO(3)$ internal symmetry of $N = 3$ superconformal algebra to $U(1)$ symmetry.

It was shown in [11] that the equation (2) possesses first 3 nontrivial conservation laws and therefore it should be integrable one. Now we turn to the basic item of this section, the construction of the Lax operator for the $N = 3$ KdV hierarchy (2). Keeping in mind that in the limit $g \to 0$ the $N = 3$ KdV equation reduces to the $N = 2$ $a = 1$ KdV one, which can be described by the following Lax operator [12]

$$L_{a=1} = \partial - [D,\overline{D}] \partial^{-1} J$$

we propose the following Lax operator for $N = 3$ KdV hierarchy

$$L = \partial - [D,\overline{D}] \partial^{-1} J - [D,\overline{D}] \partial^{-1} g [D,\overline{D}] \partial^{-1} g.$$ (8)

Now, one can check that the Lax operator (8) indeed gives rise to the $N = 3$ KdV flows via the Lax equation

$$\frac{\partial}{\partial t_{2n+1}} L = \left( [L^{2n+1}]^+, L \right),$$

where $\{+\}$ denotes the differential part of the Lax operator. The conserved currents for the Lax operator (8) are defined by the standard $N = 2$ residue form as

$$H_n = \int dZ \text{tr} \left( L^{2n+1} \right),$$ (10)

where $\text{tr}$ denotes the coefficient standing before $[D,\overline{D}]\partial^{-1}$.

Thus, we have proved the integrability of the $N = 3$ KdV equation.

To close this section, let us note that the field $g$ appears in the Lax operator (8) only in pair. Therefore, one can immediately generalize the Lax operator (8) as follows

$$L = \partial - [D,\overline{D}] \partial^{-1} J - \sum_{i=1}^M [D,\overline{D}] \partial^{-1} g_i [D,\overline{D}] \partial^{-1} g_i - \sum_{j=1}^K [D,\overline{D}] \partial^{-1} b_j [D,\overline{D}] \partial^{-1} b_j$$ (11)
where we introduced $M$ general $N = 2$ fermionic superfields $g_i$ and $K$ general $N = 2$ bosonic superfields $b_j$. The resulting coupled system of evolution equations look a bit complicated and we write here the third flow equations only for the case $b_j = 0$

\[
\frac{\partial}{\partial t} J = -J''' - (J^3)' + 3(J [D, \overline{D}] J)' - 12(J D g_i \overline{D} g_i)' + \\
6(\overline{D} J g_i D g_i)' + 6(D J g_i \overline{D} g_i)' - 3([D, \overline{D}] g_i g_i)',
\]

\[
\frac{\partial}{\partial t} g_i = -g_i''' + 6J(\overline{D} J D g_i + DJ \overline{D} g_i) + 3(J [D, \overline{D}] g_i)' - 3J^2 g_i - \\
6(g_i \overline{D} g_i)' D g_i - 6(g_i D g_i)' \overline{D} g_i + 3[D, \overline{D}] g_i [D, \overline{D}] g_i g_i - 3g_i' g_i g_i
\]  

(12) (summation over repeated indices is understood). Thus we have got new integrable extensions of $N = 3$ KdV hierarchy.

3. New Lax representation for $N = 4$ KdV. In some cases the known Lax operators could give hints how to construct their generalization or ever how to construct new Lax operators. For example, the $N = 4$ KdV Lax operator [4] has been constructed as junction of two known Lax operators. As an another example, the $N = 2 a = 4$ KdV lax operator [7] can be constructed from the $N = 2 a = 1$ one [2] as follows [13]:

\[
L_{a=4} = L_{a=1} - L_{a=1}^*,
\]  

(13)

where star means formal operator conjugation. Since the $N = 3$ KdV Lax operator [8] is a generalization of the $N = 2 a = 1$ KdV one, the analogous procedure in this case might yield new integrable systems. The basic aim of this section is to demonstrate that this is indeed the case.

Thus, based on the above consideration, let us present the following Lax operator

\[
L = \partial - J - D \partial^{-1} [D, J] + \partial^{-1} [\overline{D} D g] \overline{D} \partial^{-1} [D g] + \partial^{-1} \overline{D} g [D g] + g \overline{D} \partial^{-1} [D g].
\]  

(14)

Here $J$ and $g$ are the general $N = 2$ bosonic and fermionic superfields correspondingly, and the square brackets mean that the relevant operators act only on the superfields inside the brackets. We have checked that the Lax operator (14) gives rise, through the Lax equation

\[
\frac{\partial}{\partial t_n} L = \left((L^n)_{\geq 1}, L\right)
\]  

(15)

to the self-consistent hierarchy of the evolution equations (here subscript $\{\geq 1\}$ denotes the purely differential part of the Lax operator). For the Lax operator (14) the Hamiltonians $H_n$ are obtained from the constant term of $L^n$ [14], that is

\[
H_n = \int dZ (L^n)_0,
\]  

(16)

where subscripts 0 means the constant part of an operator.

Explicitly, the second flow reads:

\[
\frac{\partial}{\partial t_2} J = [D, \overline{D}] J' + 2J' J + 2(D g \overline{D} g)',
\]

\[
\frac{\partial}{\partial t_2} g = [D, \overline{D}] g' - 2\overline{D} J D g - 2D J \overline{D} g + 2g' J.
\]  

(17)
An interesting peculiarity of the Lax operator (14) is that it can be rewritten in terms of \( J \) and bosonic chiral-antichiral superfields \( G, \overline{G} \), defined as
\[
G \equiv Dg, \quad \overline{G} \equiv \overline{Dg}.
\] (18)
In this new basis the Lax operator (14) reads as
\[
L = \partial - J - D\partial^{-1}(DJ) + \partial^{-1}G\overline{G} - \partial^{-1}(D\overline{G})\partial^{-1}\overline{DG},
\] (19)
while the second flow equations take the form
\[
\frac{\partial}{\partial t}J = [D, \overline{D}]J' + 2(G \overline{G})' + 2J'J,
\]
\[
\frac{\partial}{\partial t}\overline{G} = \overline{G}'' - 2D\overline{D}(J \overline{G}), \quad \frac{\partial}{\partial t}G = -\overline{G}'' - 2D\overline{D}(J G).
\] (20)
In the equations (20) one can immediately recognize the second flow equations of the \( N = 4 \) KdV hierarchy [8, 17]. Thus, the Lax operators (14), (19) provide the new description of the \( N = 4 \) KdV hierarchy.

It is easy to check the covariance of the second flow equations (20) under the transformations of an extra hidden \( N = 2 \) supersymmetry [17]
\[
\delta J = \epsilon D\overline{G} + \overline{\epsilon}DG, \quad \delta G = \epsilon DJ, \quad \delta \overline{G} = \overline{\epsilon}D\overline{J},
\] (21)
where \( \epsilon, \overline{\epsilon} \) are mutually conjugated Grassmann parameters. Together with explicit \( N = 2 \) supersymmetry they form \( N = 4 \) supersymmetry. It is somewhat surprising that the \( N = 4 \) supersymmetry can be realized on the superfields \( J \) and \( g \), but in a non-local way
\[
\delta J = \overline{\epsilon}D\overline{D}g + \epsilon D\overline{D}g, \quad \delta g = \overline{\epsilon}D\overline{D}\partial^{-1}J + \epsilon D\overline{D}\partial^{-1}J.
\] (22)
Thus the pair of general superfields \( J, g \) also forms the multiplet of \( N = 4 \) supersymmetry.

Let us stress ones more that all flow equations of \( N = 4 \) KdV hierarchy together with all Hamiltonians can be rewritten in terms of general superfields \( J \) and \( g \). We suspect the \( N = 4 \) \( SU(2) \) superconformal algebra itself can be rewritten in terms of superfields \( J \) and \( g \).

4. New reduction of \( N = 2 \) KP hierarchy. In the previous section we constructed the new Lax operator for the \( N = 4 \) KdV hierarchy. It turns out that this Lax operator (14) commutes with spinor covariant derivative \( D \) and then it belongs to the same class of Lax operators as (1). Moreover, it looks like \( N = 2 \ a = 4 \) KdV Lax operator plus extra pieces including only additional field \( g \) (or \( G \) and \( \overline{G} \)). Therefore, it is natural to consider the following extension of this Lax operator (3):
\[
L_s = \partial^s + \sum_{j=1}^{s-1} \left( J_{s-j} \partial - [DJ] \overline{D} \right) \partial^{j-1} - \sum_{j=1}^{s-1} J_s - \overline{D}\partial^{-1} [DU] - F_A F_A - F_A \overline{D}\partial^{-1} [DF_A]
\]
\[
+ \partial^{-1} [Dg_B] \overline{D}\partial^{-1} [Dg_B] + (-1)^{d_g B} \partial^{-1} g_B [Dg_B] + g_B \overline{D}\partial^{-1} [Dg_B]
\] (23)
(summation over repeated indices is understood). Here \( d_g B \) is a grassmann parity of superfield \( g_B, d_g = 1 \ (d_g = 0) \) for fermionic (bosonic) superfields, \( J_s \) are general bosonic \( N = 2 \) superfields, \( F_A \) and \( \overline{F}_A \) are \( (n + m) \) pairs of chiral and antichiral \( N = 2 \) superfields
\[
DF_A = \overline{DF}_A = 0
\] (24)
which are fermionic for \( A = 1, \ldots, n \) and bosonic for \( A = n+1, \ldots, n+m \) and \( g_B \) are \( k+l \) pairs of general \( N = 2 \) superfields, fermionic for \( B = 1, \ldots, k \) and bosonic for \( B = k+1, \ldots, k+l \). Such operator provides the consistent flows

\[
\frac{\partial}{\partial t_n} L = \left[ (L^n)_{\geq 1}, L \right], \tag{25}
\]

and the infinite number of Hamiltonians can be obtained in a standard way:

\[
H_n = \int dZ (L^n)_0 \tag{26}
\]

To better understand what kind of hierarchies we have proposed, let us consider explicitly the first simplest hierarchies corresponding to the values \( s = 1, 2 \) and \( s = 3 \) in the Lax operator \( L_s \) (23), with single fermionic superfield \( g \) (\( k = 1, l = 0 \)) and one pair of \( F, \overline{F} \) superfields (\( n = 1, m = 0 \)).

1. The \( s = 1 \) case.

For this simplest case the Lax operator (23) has the following form (\( J_1 \equiv J \)):

\[
L_1 = -J - \overline{D} \partial^{-1} [DJ] - FF - \overline{F} D \partial^{-1} [DF] + \partial^{-1} \left[ \overline{DD}g \right] \overline{D} \partial^{-1} [Dg] + \partial^{-1} \overline{D}g [Dg] + g \overline{D} \partial^{-1} [Dg], \tag{27}
\]

while the second flow equations read

\[
\begin{align*}
\frac{\partial}{\partial t_2} J &= [D, \overline{D}] J' + 2J'J + 2 (Dg \overline{D}g)' - 2\overline{D}JFDF - 2F J' + 2DJ \overline{F} \overline{D}F, \\
\frac{\partial}{\partial t_2} g &= [D, \overline{D}] g' - 2\overline{D}JDg - 2DJ\overline{D}g - 2g'J - 2gFF - 2\overline{F} Dg \overline{D}F + 2\overline{F} \overline{D}g \overline{D}F, \\
\frac{\partial}{\partial t_2} F &= -F'' + 2F'J + 2J'F + 2F D\overline{F} \overline{D}F - 2DJ \overline{D}F, \\
\frac{\partial}{\partial t_2} \overline{F} &= \overline{F}'' + 2\overline{F}'J - 2J' \overline{F} \overline{D}F - 2\overline{F}' \overline{D}F - 2\overline{D}JDF. \tag{28}
\end{align*}
\]

From these expressions we can easily see that in the both limits \( F = \overline{F} = 0 \) and \( g = 0 \) they coincide with the corresponding flows

of the \( N = 4 \) KdV hierarchy [8, 17] in the different bases while at \( F = \overline{F} = g = 0 \) we obtain the \( N = 2 \ a = 4 \) KdV hierarchy. Thus, our family of \( N = 2 \) hierarchies includes the well-known \( N = 2 \ a = 4 \) and \( N = 4 \) KdV hierarchies (in two different bases) and possesses the new Lax-pair representation for their extensions.

2. The \( s = 2 \) case.

For this case the Lax operator reads as \( (J_1 \equiv J, J_2 \equiv W) \)

\[
L = \partial^2 + J \partial - [DJ] \overline{D} - W - \overline{D} \partial^{-1} [DW] - FF - \overline{F} D \partial^{-1} [DF] + \partial^{-1} \left[ \overline{DD}g \right] \overline{D} \partial^{-1} [Dg] + \partial^{-1} \overline{D}g [Dg] + g \overline{D} \partial^{-1} [Dg], \tag{29}
\]
providing the following second flows equations

\[
\frac{\partial}{\partial t_2} J = 2W' - 2(FF)' \quad \frac{\partial}{\partial t_2} W = [D, \overline{D}] W' - \overline{D} W DJ - DW \overline{D} J + 2(DgDg)' - W'J ,
\]

\[
\frac{\partial}{\partial t_2} F = -F'' - F'J + DJDF \quad \frac{\partial}{\partial t_2} F = \overline{F}'' - \overline{F}'J + \overline{D} DJ \overline{F} ,
\]

\[
\frac{\partial}{\partial t_2} g = [D, \overline{D}] g' + \overline{D} JDg + DJDg - g'J .
\]

(30)

With \( g = 0 \) this Lax operator has been considered in [15]. With the superfield \( g \) it describes new extension of the \( N = 2 \) \( a = -\frac{3}{2} \) super Boussinesq hierarchy (\( F = \overline{F} = g = 0 \) limit).

3. The \( s = 3 \) case.

For this more complicated case we present only the third–flow equations in the limit \( F = \overline{F} = 0 \) which give the new supersymmetric extension of the \( N = 2 \) \( a = -1/2 \) Boussinesq equation (again we redefined the fields as \( J_1 \equiv J, J_2 \equiv W, J_3 \equiv U \)):

\[
\frac{\partial}{\partial t_3} J = 3U' ,
\]

\[
\frac{\partial}{\partial t_3} W = 3U'' - 3\frac{[D, \overline{D}] U'}{2} + \overline{D} UDJ + DU \overline{D} J - 3Dg' \overline{D} g + 2U'J - 3Dg \overline{D} g' ,
\]

\[
\frac{\partial}{\partial t_3} U = -U'' - \overline{D} UD J + DW \overline{D} U - \frac{1}{2} \overline{D} J [D, \overline{D}] g Dg + \frac{1}{2} [D J Dg]' + DJDg + D UD J
\]

\[
+ \frac{3}{2} ([D, \overline{D}] gg)' + DU \overline{D} J' + DW \overline{D} U + DJ[D, \overline{D}] g Dg + DJDg + DJ \overline{D} g
\]

\[
+ [D, \overline{D}] U' J + J'DgDg + \frac{1}{2} [D, \overline{D}] U' J' + \frac{1}{2} U' J' - WU' + 2J(Dg \overline{D} g)' ,
\]

\[
\frac{\partial}{\partial t_3} g = -g''' - (\overline{D} JDg)' + [D, \overline{D}] g' J + \frac{1}{2} [D, \overline{D}] g J + \overline{D} WDg + DW \overline{D} g
\]

\[
+ \frac{1}{2} g' J' - g'W + DJDg' .
\]

(31)

The extension of this system by the superfields \( F \) and \( \overline{F} \) can be straightforwardly derived from the Lax-pair representation (23) and we do not present it here.

5. Conclusion. In this letter we present the Lax operator for \( N = 3 \) KdV hierarchy and thus proved the integrability of the latter. We also constructed a new infinite family of \( N = 2 \) supersymmetric hierarchies by exhibiting the corresponding super Lax operators. As a by-product we find the new realization of \( N = 4 \) supersymmetry on the two general superfields – bosonic spin 1 \( J \) and fermionic spin 1/2 \( g \). We suspect that the constructed Lax operators can be extended to the matrix case, like in [3, 4] (at least for the Lax operators from the sections 4,5). It is rather interesting problem to extend the present consideration to the case of third KP reduction hierarchies [1]. The detailed analysis of these problems is under way.

Acknowledgments. S.K. and A.P. thank Institute of Theoretical Physics in Wroclaw for hospitality during the course of this work. This work was partially supported by the Russian Foundation for Basic Research, Grant No. 96-02-17634, RFBR-DFG Grant No. 96-02-00180, INTAS Grants 93-127 ext, 96-0308 and 96-0538.
References

[1] L. Bonora, S. Krivonos and A. Sorin, Nucl. Phys. B 477 (1996) 835.

[2] F. Delduc and L. Gallot, Comm. Math. Phys. 190 (1997) 395.

[3] L. Bonora, S. Krivonos and A. Sorin, The $N = 2$ supersymmetric matrix GNLS hierarchies, SISSA 142/97/EP; solv-int/9711009.

[4] S. Krivonos and A. Sorin, Extended $N = 2$ supersymmetric matrix $(1, s)$-KdV hierarchies, JINR E2-97-365; solv-int/9712002.

[5] A. Sorin, The discrete symmetries of the $N = 2$ supersymmetric GNLS hierarchies, JINR E2-97-37, solv-int/9701020.

[6] Z. Popowicz, Phys. Lett. B 319 (1993) 478.

[7] P. Labelle and P. Mathieu, J. Math. Phys. 32 (1991) 923.

[8] F. Delduc and E. Ivanov, Phys. Lett. B 309 (1993) 312.

[9] E. Ivanov and S. Krivonos, Phys. Lett. A 231 (1997) 75.

[10] F. Delduc, L. Gallot and E. Ivanov, Phys. Lett. B 396 (1997) 122.

[11] C.M. Yung, Mod. Phys. Lett. A 8 (1993) 1161.

[12] Z. Popowicz, Phys. Lett. A 174 (1993) 411.

[13] M. Chaichian and J. Lukierski, Phys. Lett. B 212 (1988) 461.

[14] S. Bellucci, E. Ivanov and S. Krivonos, J. Math. Phys. 34 (1993) 3087.

[15] Z. Popowicz, Phys. Lett. A 236 (1997) 455.

[16] S. Krivonos, A. Sorin and F. Toppan, Phys. Lett. A 206 (1995) 146.

[17] F. Delduc, E. Ivanov and S. Krivonos, J. Math. Phys. 37 (1996) 1356.