Conservation Laws with
Coinciding Smooth Solutions but
Different Conserved Variables

Rinaldo M. Colombo\textsuperscript{1}  Graziano Guerra\textsuperscript{2}

October 9, 2018

Abstract
Consider two hyperbolic systems of conservation laws in one space dimension with the same eigenvalues and (right) eigenvectors. We prove that solutions to Cauchy problems with the same initial data differ at third order in the total variation of the initial datum. As a first application, relying on the classical Glimm–Lax result \cite{GlimmLax}, we obtain estimates improving those in \cite{Guerra2} on the distance between solutions to the isentropic and non-isentropic inviscid compressible Euler equations, under general equations of state. Further applications are to the general scalar case, where rather precise estimates are obtained, to an approximation by Di Perna of the \( p \)-system and to a traffic model.

\textbf{Keywords:} Hyperbolic Conservation Laws; Compressible Euler Equations; Isentropic Gas Dynamics

2010 MSC: 35L65, 35Q35, 76N99

1 Introduction
Consider the following Cauchy Problems for \( n \times n \) systems of conservation laws in one space dimension:

\[ \begin{cases} \partial_t g(u) + \partial_x f(u) = 0 \\ u(0, x) = u_o(x) \end{cases} \quad \text{and} \quad \begin{cases} \partial_t \tilde{g}(u) + \partial_x \tilde{f}(u) = 0 \\ u(0, x) = u_o(x) \end{cases} \tag{1.1} \]

where we assume that \( u_o \) varies in a neighborhood of a fixed state \( \bar{u} \). Clearly, the condition

\[ (Dg(u))^{-1} Df(u) = (D\tilde{g}(u))^{-1} \tilde{f}(u) \tag{1.2} \]

ensures that the two systems (1.1) share the same smooth solutions. This paper is devoted to estimate the difference between possibly non smooth solutions to (1.1) yielded by the Standard Riemann Semigroups (\cite{Dafermos, Chapter 9}) generated by these systems.

\textsuperscript{1}INDAM Unit, University of Brescia, Italy. rinaldo.colombo@unibs.it
\textsuperscript{2}Department of Mathematics and its Applications, University of Milano - Bicocca, Italy. graziano.guerra@unimib.it
A first classical situation is the following. Consider the usual Euler equations for an inviscid compressible fluid in one space dimension, which, in Eulerian coordinates, read
\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho v) &= 0, \\
\partial_t (\rho v) + \partial_x (\rho v^2 + p) &= 0, \\
\partial_t \left( \frac{1}{2} \rho v^2 + \rho e \right) + \partial_x \left( \frac{1}{2} \rho v^2 + \rho e + p \right) v &= 0,
\end{aligned}
\] (1.3)
see [6, Formula (3.3.29)], where \( t \) is time, \( x \) is the space coordinate, \( \rho \) is the mass density, \( v \) the speed, \( p = p(\rho, s) \) the pressure, \( e = e(\rho, s) \) and \( s \) are the internal energy and the entropy densities per unit mass. A standard approximation of (1.3) is the so called isentropic \( p \)-system
\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho v) &= 0, \\
\partial_t (\rho v) + \partial_x (\rho v^2 + p(\rho, \bar{s})) &= 0, \\
\partial_t (\rho s) + \partial_x (\rho vs) &= 0,
\end{aligned}
\] (1.4)
see [6, Formula (7.1.12)], where \( \bar{s} \) is a constant entropy density. Below, we provide precise estimates on the distance between solutions to (1.3) and to (1.4), improving those in [11]. This result, Theorem 3.3, actually provides a comparison between solutions to (1.3) and to
\[
\begin{aligned}
\partial_t \rho + \partial_x (\rho v) &= 0, \\
\partial_t (\rho v) + \partial_x (\rho v^2 + p(\rho, s)) &= 0, \\
\partial_t (\rho s) + \partial_x (\rho vs) &= 0.
\end{aligned}
\] (1.5)
Indeed, assigning an initial datum with entropy \( \bar{s} \) constant in space to (1.5) leads to solutions that solve (1.4), too, see Lemma 3.2. Note that (1.5) shares the same smooth solutions with (1.3).

The formulations of (1.3) and (1.5) motivate our choice of presenting general systems of conservation laws in the form (1.1), rather than in the standard form
\[
\begin{aligned}
\partial_t w + \partial_x F(w) &= 0, \\
w(0, x) &= w_0(x)
\end{aligned}
\] and
\[
\begin{aligned}
\partial_t \tilde{w} + \partial_x \tilde{F}(\tilde{w}) &= 0, \\
\tilde{w}(0, x) &= \tilde{w}_0(x).
\end{aligned}
\] (1.6)
Clearly, the connection between (1.1) and (1.6) is given by
\[
\begin{aligned}
w &= g(u) \\
\tilde{w} &= \tilde{g}(u)
\end{aligned}
\] and
\[
\begin{aligned}
F(w) &= f \left( g^{-1}(w) \right), \\
\tilde{F}(\tilde{w}) &= \tilde{f} \left( \tilde{g}^{-1}(\tilde{w}) \right).
\end{aligned}
\] (1.7)
Assume that systems (1.6) generate Standard Riemann Semigroups, see [3, Definition 9.1]. \( S : \mathbb{R}_+ \times D \rightarrow D \) and \( \tilde{S} : \mathbb{R}_+ \times \tilde{D} \rightarrow \tilde{D} \). The distance between the orbits of \( S \) and those of \( \tilde{S} \) is estimated in [11, Theorem 2.1], but only when the physical meanings of the conserved variables are the same, so that \( D = \tilde{D} \). However, \( D \) and \( \tilde{D} \) may well be entirely different since the physical conserved variables \( w = g(u) \) need not be the same as \( \tilde{w} = \tilde{g}(u) \). Therefore, below we aim at the comparison between the semigroups
\[
\begin{aligned}
S_t &= g^{-1} \circ S_t \circ g \\
\tilde{S}_t &= \tilde{g}^{-1} \circ \tilde{S}_t \circ \tilde{g}
\end{aligned}
\] (1.8)
describing the evolutions of the same physical variables \( u \), but with different conserved quantities \( w \) and \( \tilde{w} \). For instance, we have \( u = (\rho, v, s) \) in both (1.3) and (1.5), while the conserved
variables in the two cases are different, since \( w = (\rho, \rho v, \rho s) \) and \( \tilde{w} = (\rho, \rho v, \frac{1}{2} \rho v^2 + \rho e(\rho, s)) \). Clearly, in this case, a direct comparison between \( w \) and \( \tilde{w} \) in (1.6) is inappropriate.

If the initial datum \( u_o \) has sufficiently small total variation, then the weak entropy solutions \( S_t u_o \) and \( \tilde{S}_t u_o \) to (1.1) are known to exist for all times. We prove below sharp estimates (see 3. in Theorem 2.1) that imply the bound

\[
\|S_t u_o - \tilde{S}_t u_o\|_{L^1(\mathbb{R};\mathbb{R})} \leq C \text{TV}(u_o)^3 t.
\] (1.9)

In the case of systems admitting a full set of Riemann coordinates, the above estimate can be improved, so that only the negative total variation appears on the right hand side, (see 4. in Theorem 2.1).

Above, \( C \) is a suitable constant dependent on \( Df, Dg, \tilde{D}f, \tilde{D}g \). A rather careful computation allows to express the leading term in \( C \) by means of \( g \) and \( \tilde{g} \), see Proposition 2.3.

As anticipated above, the present general result, applied to (1.3) and (1.5), allows to improve the estimate obtained in [11] on the distance between solutions to the general inviscid Euler equations (1.3) and to the isentropic \( p \)-system (1.4).

As a further application, we compare the usual \( p \)-system in Eulerian coordinates with the analogous system where speed is conserved, see Section 4.

A specific paragraph is devoted to the scalar case, where rather precise estimates are available. Indeed, the lower order terms in the estimate provided by Theorem 2.3 are third order in the total variation of the initial datum with a coefficient depending on the \( C^0 \) norm of \( f'' g'' - \tilde{f}'' \tilde{g}'' \). This estimate is a counterpart to [1, Theorem 2.6].

Section 2 presents our general result, while applications to gas dynamics are considered in Sections 3 and 4. All technical details are deferred to Section 6.

2 Main Result

For the basic theory of 1D systems of conservation laws, we refer to [3, 6, 12].

Throughout, we fix an open bounded set \( \Omega \) in \( \mathbb{R}^n \), with \( n \in \mathbb{N}, n \geq 1 \). The following assumptions on the functions defining systems (1.1) are of use in the sequel:

(H1) The functions \( f, g, \tilde{f}, \tilde{g} \) are defined in \( \Omega \), attain values in \( \mathbb{R}^n \) and are smooth.

- The functions \( g \) and \( \tilde{g} \) are invertible and admit smooth inverses \( g^{-1} \) and \( \tilde{g}^{-1} \).
- For \( u \in \Omega \), the matrixes \( A(u) = (Dg(u))^{-1} Df(u) \), \( \tilde{A}(u) = (D\tilde{g}(u))^{-1} D\tilde{f}(u) \) admit the real eigenvalues \( \lambda_1(u), \ldots, \lambda_n(u), \tilde{\lambda}_1(u), \ldots, \tilde{\lambda}_n(u) \) with \( \lambda_{i-1}(u) < \lambda_i(u), \tilde{\lambda}_{i-1}(u) < \tilde{\lambda}_i(u) \) for \( i = 2, \ldots, n \), and the right eigenvectors \( r_1(u), \ldots, r_n(u), \tilde{r}_1(u), \ldots, \tilde{r}_n(u) \).

- In both systems, each characteristic field is either genuinely nonlinear or linearly degenerate, see [3, Definition 5.2].

(H2) For all \( u \in \Omega \), (1.2) holds, namely \( A(u) = \tilde{A}(u) \).

(H3) The integral curves of the right eigenvectors define a full set of Riemann coordinates.
Below, we choose \( D\lambda_i(u) \cdot r_i(u) \geq 0 \) and \( D\tilde{\lambda}_i(u) \cdot \tilde{r}_i(u) \geq 0 \) for \( i = 1, \ldots, n \) and for all \( u \in \Omega \).

When necessary, a specific normalization of the \( r_i \), respectively \( \tilde{r}_i \), in (H1) is adopted and, consequently, a particular parametrization of the Lax curves is selected. Here and in what follows, we assume that the left eigenvectors of \( A \), namely \( l_1, \ldots, l_n \), are normalized so that

\[
l_i(u) \cdot r_j(u) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}
\]

with \( r_j \) as in (H1). Concerning (H3), see also the definition of rich systems in [12] § 5.9.

It is well known, see [3, chapters 7 and 8], that under assumption (H1) and by (1.8) both systems (1.1) generate Standard Riemann Semigroups (SRS) \( S \) and \( \tilde{S} \) defined on domains \( D \) and \( \tilde{D} \) containing all \( L^1_{\text{loc}} \) functions with sufficiently small total variation.

With reference to the Riemann Problems

\[
\begin{align*}
\begin{cases}
\partial_t g(u) + \partial_x f(u) = 0 \\
u(0, x) = \begin{cases} u^l & \text{if } x < 0 \\ u^r & \text{if } x > 0 \end{cases}
\end{cases}
\end{align*}
\]

introduce the notation, see [3, Chapter 7],

\[
\begin{align*}
\sigma &\to \psi_j(\sigma)(u) & \text{j-th Lax curve of system (1.1), left, exiting } u, \ j = 1, \ldots, n. \\
\tilde{\sigma} &\to \psi_j(\tilde{\sigma})(u) & \text{j-th Lax curve of system (1.1), right, exiting } u, \ j = 1, \ldots, n. \\
(\sigma_1, \ldots, \sigma_n) &\to E(u_1, u_r) & \text{waves' sizes in the solution to Riemann problem (2.1), left.} \\
(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n) &\to E(u_1, u_r) & \text{waves' sizes in the solution to Riemann problem (2.1), right.}
\end{align*}
\]

We set \( \sigma_i = E_i(u_1, u_r) \) and \( \tilde{\sigma}_i = \tilde{E}_i(u_1, u_r) \), for \( i = 1, \ldots, n. \)

By possibly changing the values of a function \( u \in BV(\mathbb{R}; \Omega) \) at countably many points, we assume that \( u \) is right continuous. The distributional derivative \( \mu \) of \( u \) is then a vector measure that can be decomposed into a continuous part \( \mu_c \) and an atomic part \( \mu_a \). Following [4, Formula (4.1)], for \( i = 1, \ldots, n \), we consider the wave measure \( \mu_i \) defined by

\[
\mu_i(B) = \int_B l_i(u) \, d\mu_c + \sum_{x \in B} E_i(u(x-), u(x+)) ,
\]

for any Borel set \( B \subseteq \mathbb{R} \). Let \( \mu^+_i \) and \( \mu^-_i \) be the positive and negative, respectively, parts of \( \mu_i \) and \( |\mu_i| = \mu^+_i + \mu^-_i \) be the total variation of \( \mu_i \). When necessary, to remind the connection between the measure \( \mu \) and the function \( u \), we denote below the left hand side in (2.3) by \( \mu_i(u, B) \), setting also \( \mu_i(u, B) = \mu^+_i(u, B) + \mu^-_i(u, B) \).

**Theorem 2.1.** Let \( f, \bar{f}, g, \bar{g} \) satisfy (H1). Fix \( \bar{u} \in \Omega \). Let \( \hat{\lambda} \) be an upper bound for all characteristic speeds of both systems (1.1) and define for \( a, b \in \mathbb{R} \) with \( a < b \)

\[
\begin{align*}
I_t &= [a + \hat{\lambda} t, b - \hat{\lambda} t] \\
T_t &= \{ (\tau, x) \in \mathbb{R}^+ \times \mathbb{R} : \tau \in [0, t] \text{ and } x \in I_\tau \} \text{ for } t > 0 \text{ and } t < (b - a)/\hat{\lambda}.
\end{align*}
\]

Then,

1. The two systems (1.1) generate the SRSs \( S : \mathbb{R}^+ \times D \to D \) and \( \tilde{S} : \mathbb{R}^+ \times \tilde{D} \to \tilde{D} \).
2. There exists a positive $\delta$ such that
\[
\left( \mathcal{D} \cap \tilde{\mathcal{D}} \right) \supseteq \left\{ u \in L^1_{\text{loc}}(\mathbb{R}; \Omega) : \| u - \bar{u} \|_{L^\infty(\mathbb{R}; \mathbb{R}^n)} < \delta \text{ and } TV(u) < \delta \right\}.
\]

3. If moreover (H2) holds, there exists a positive constant $C$ such that for all $u_0 \in (\mathcal{D} \cap \tilde{\mathcal{D}})$,
\[
\int_{I_t} \| (S_t u_0)(x) - (\tilde{S}_t u_0)(x) \| \, dx \leq C t TV(u_0; I_0) \left( \text{diam } u(T_i) \right)^2 \tag{2.5}
\]
where $u(t, x) = (S_t u_0)(x)$.

4. If (H2) and (H3) hold, then we have the improved estimate
\[
\int_{I_t} \| (S_t u_0)(x) - (\tilde{S}_t u_0)(x) \| \, dx \leq C t \left( \sum_{i=1}^n \mu_i^{-1}(u_0; I_0) \right) \left( \text{diam } u(T_i) \right)^2. \tag{2.6}
\]

The proof is deferred to Section 6. As a corollary, since $\text{diam } u(T_i) \leq \mathcal{O}(1) TV(u_0; I_0)$, we immediately obtain the following result.

**Corollary 2.2.** Let $f$, $\tilde{f}$, $g$, $\tilde{g}$ satisfy (H1) and (H2). With the same notation as in Theorem 2.1,
\[
\int_{I_t} \| (S_t u_0)(x) - (\tilde{S}_t u_0)(x) \| \, dx \leq C t TV(u_0; I_0)^3 \tag{2.7}
\]

Throughout this paper, $C$ and $\mathcal{O}(1)$ are constants that depends on norms of $f$, $g$, $\tilde{f}$, $\tilde{g}$ computed on $\Omega$. More detailed information on the constant $C$ appearing in (2.5), (2.6) and (2.7) are provided by the following Proposition.

**Proposition 2.3.** Let $f$, $\tilde{f}$, $g$, $\tilde{g}$ satisfy (H1) and (H2). With the same notation as in Theorem 2.1 define
\[
\Delta \left( (f, g), (\tilde{f}, \tilde{g}) \right) = \sup_{u \in \Omega} \max_{i=1, \ldots, n} \left\| (D\lambda_i(u) r_i(u)) \left( (D\tilde{g}(u))^{-1} D^2 \tilde{g}(u) - (Dg(u))^{-1} D^2 g(u) \right) (r_i(u), r_i(u)) \right\|. \tag{2.8}
\]
Then, the constant $C$ appearing in (2.5), (2.6) and (2.7) satisfies
\[
C = \mathcal{O}(1) \left( \Delta \left( (f, g), (\tilde{f}, \tilde{g}) \right) + \text{diam } u(T_i) \right). \tag{2.9}
\]

**Remark 2.4.** Proposition 2.3 implies that if $\Delta \left( (f, g), (\tilde{f}, \tilde{g}) \right) = 0$ in (2.9), then the bounds (2.5), (2.6) and (2.7) provide fourth order estimates.

### 2.1 The Scalar Case

Consider the Cauchy problems (1.1) in the scalar $(n = 1)$ case, so that the characteristic speed is $\lambda(u) = f'(u)/g'(u)$. Now, condition (H2) takes the form
\[
\frac{f'(u)}{g'(u)} = \frac{f'(u)}{g'(u)}. \tag{2.10}
\]
Rather precise estimates are now available, as shown in the next result.
Lemma 3.1. From the analytic point of view, formula (2.5.14), the second states that pressure is an increasing function of the density at constant entropy. Above, condition (H1) and (1.5) satisfy (1.35). Moreover, conditions (H1) and (H2) hold, with

\[ A = \begin{bmatrix} \frac{1}{\rho} \partial_\rho p & v & 0 \\ v & \frac{1}{\rho} \partial_\rho s & 0 \\ 0 & 0 & v \end{bmatrix}, \quad \lambda_1 = v - \sqrt{\frac{\partial_\rho p}{\rho}}, \quad \lambda_2 = v, \quad \lambda_3 = v + \sqrt{\frac{\partial_\rho p}{\rho}}. \]

Theorem 2.5. In the scalar case, assume that conditions (H1) and (H2) hold. Then, for any \( u_h \in BV(\mathbb{R}; \Omega), \)

\[ \int_{\Omega} \left| f'(u) - f'(\tilde{u}_h) \right| dx \leq \frac{2}{(\inf \{|g'|)(\inf |\tilde{g}'|)})} \left\| f'' - \tilde{f}'' \right\|_{C^0(\Omega; \mathbb{R})} TV^-(u_h) \left( \text{diam} u_h(\mathbb{R}) \right)^2 t \]

Above, by TV\(^-\)(u) we denote the negative total variation. The proof is deferred to Section 6.

A well known possible application of Theorem 2.5 is given by the various versions of Burgers’ scalar equation

\[ u_t + \left( \frac{u^m}{m+1} \right)_x = 0, \quad m \in \{1, 2, \ldots \}, \text{ see [10, Formulæ (11.34) and (11.35)]}. \]

3 The Isentropic Approximation of Euler Equations

On equations (1.3) and (1.5) we assume throughout that:

(e) The internal energy \( e \) is a real valued smooth function defined on \([0, +\infty[ \times \mathbb{R}\) and satisfies
\[ \partial_s e(\rho, s) > 0. \]

(p) The pressure \( p \) is a real valued smooth function defined on \([0, +\infty[ \times \mathbb{R}\) and satisfies
\[ p(\rho, s) = \rho^2 \partial_\rho e(\rho, s), \quad \partial_\rho p(\rho, s) > 0, \quad \partial_{\rho\rho} (\rho p(\rho, s)) > 0. \]

Above, condition (e) ensures that the absolute temperature \( \vartheta = \partial_s e \) is positive, see [6, Formula (7.1.10)]. In (p), the former condition follows from Gibbs relation, see [6, Formula (2.5.14)], the second states that pressure is an increasing function of the density at constant entropy. From the analytic point of view, (e) and (p) ensure that both systems (1.3) and (1.5) satisfy (H1) and (H2).

Lemma 3.1. Let (e) and (p) hold. Then, systems (1.3) and (1.5) fit into (1.1) setting

\[ g(u) = \begin{bmatrix} \rho \\ \rho v \\ \rho s \end{bmatrix}, \quad f(u) = \begin{bmatrix} \rho v \\ \rho v^2 + p(\rho, s) \\ \rho v s \end{bmatrix}, \quad \tilde{f}(u) = \begin{bmatrix} \rho v \\ \rho v^2 + p(\rho, s) \\ \left( \frac{1}{2} \rho v^2 + \rho e(\rho, s) + p(\rho, s) \right) v \end{bmatrix}. \]

Moreover, conditions (H1) and (H2) hold, with

\[ A = \begin{bmatrix} v & \rho & 0 \\ \frac{1}{\rho} \partial_\rho p & v & 0 \\ 0 & 0 & v \end{bmatrix}, \quad \lambda_1 = v - \sqrt{\frac{\partial_\rho p}{\rho}}, \quad \lambda_2 = v, \quad \lambda_3 = v + \sqrt{\frac{\partial_\rho p}{\rho}}. \]

r_1 = \begin{bmatrix} -1 \\ \sqrt{\frac{\partial_\rho p}{\rho}} \\ 0 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 \\ 0 \\ -\frac{\partial_\rho p}{\rho} \end{bmatrix}, \quad r_3 = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial_\rho p}{\rho} \end{bmatrix}. \]
The proof is obtained through elementary computations.

We now check that solutions to the classical p-system also solve (1.3) as soon as the initial datum has constant entropy.

**Lemma 3.2.** Let (e) and (p) hold. Fix a constant state \((\bar{\rho}, \bar{\nu}, \bar{s}) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}\). Call \(S^{2 \times 2}: \mathbb{R}^+ \times \mathcal{D}^{2 \times 2} \to \mathcal{D}^{2 \times 2}\) the SRS generated by (1.4) and \(S^{3 \times 3}: \mathbb{R}^+ \times \mathcal{D}^{3 \times 3} \to \mathcal{D}^{3 \times 3}\) the SRS generated by (1.5), with

\[
\mathcal{D}^{2 \times 2} \supseteq \left\{(\rho, v) \in L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R}) : \frac{\| (\rho, v) - (\bar{\rho}, \bar{\nu}) \|^2_{L^\infty(\mathbb{R}; \mathbb{R}^2)}}{TV(\rho, v)} < \delta \right\}
\]

\[
\mathcal{D}^{3 \times 3} \supseteq \left\{(\rho, v, s) \in L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}) : \frac{\| (\rho, v, s) - (\bar{\rho}, \bar{\nu}, \bar{s}) \|^2_{L^\infty(\mathbb{R}; \mathbb{R}^3)}}{TV(\rho, v, s)} < \delta \right\}
\]

for a positive \(\delta\). For any \((\rho_0, v_0)\) such that \(\| (\rho_0, v_0) - (\bar{\rho}, \bar{\nu}) \|^2_{L^\infty(\mathbb{R}; \mathbb{R}^2)} < \delta\) and \(TV(\rho_0, v_0) < \delta\), then

\[
(\rho_0, v_0, \bar{s}) \in \mathcal{D}^{2 \times 2}
\]

and

\[
(\rho_0, v_0, \bar{s}) \in \mathcal{D}^{3 \times 3}
\]

We are now ready to estimate the distance between solutions to (1.3) and (1.5).

**Theorem 3.3.** Let (e) and (p) hold. Fix a constant state \((\bar{\rho}, \bar{\nu}, \bar{s}) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}\). Call \(S^{2 \times 2}: \mathbb{R}^+ \times \mathcal{D}^{2 \times 2} \to \mathcal{D}^{2 \times 2}\) the SRS generated by (1.4) and \(S^{3 \times 3}: \mathbb{R}^+ \times \mathcal{D}^{3 \times 3} \to \mathcal{D}^{3 \times 3}\) the SRS generated by (1.5), with

\[
\mathcal{D}^{2 \times 2} \supseteq \left\{(\rho, v) \in L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R}) : \frac{\| (\rho, v) - (\bar{\rho}, \bar{\nu}) \|^2_{L^\infty(\mathbb{R}; \mathbb{R}^2)}}{TV(\rho, v)} < \delta \right\}
\]

\[
\mathcal{D}^{3 \times 3} \supseteq \left\{(\rho, v, s) \in L^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}) : \frac{\| (\rho, v, s) - (\bar{\rho}, \bar{\nu}, \bar{s}) \|^2_{L^\infty(\mathbb{R}; \mathbb{R}^3)}}{TV(\rho, v, s)} < \delta \right\}
\]

for a positive \(\delta\). If \((\rho_0, v_0)\) is such that \(\| (\rho_0, v_0) - (\bar{\rho}, \bar{\nu}) \|^2_{L^\infty(\mathbb{R}; \mathbb{R}^2)} < \delta\) and \(TV(\rho_0, v_0) < \delta\), then \((\rho_0, v_0) \in \mathcal{D}^{2 \times 2}\), \((\rho_0, v_0, \bar{s}) \in \mathcal{D}^{3 \times 3}\) and, for a suitable positive \(C\),

\[
\int_{I_1} \left\| \tilde{S}_t^{2 \times 2}(\rho_0, v_0)(x), \bar{s} \right\| \tilde{S}_t^{3 \times 3}(\rho_0, v_0, \bar{s})(x) \right) \, dx \leq C \left( \frac{\mu_1^+(\rho_0, v_0; I_0) + \mu_2^-(\rho_0, v_0; I_0)}{\mu_1^+(\rho_0, v_0; I_0) + \mu_2^-(\rho_0, v_0; I_0)} \right)(\text{diam}(\rho_0, v_0)(I_0))^2 \tag{3.1}
\]

where \((\mu_1, \mu_2)\) are the measures \((2.3)\) referred to \((1.4)\).

The proof is deferred to Section 6. The present theorem improves the analogous result in [11] Theorem 1.2] in the following aspects:

1. Only the negative variation is present here, so that the estimate (3.1) is optimal whenever no shock arises from the initial datum.

2. The diameter of the initial datum in (3.1) provides an estimate significantly better than its total variation.

3. Theorem 3.3 applies to any equation of state satisfying (e) and (p).

With reference to 1. above, note that (3.1) is localized, hence the initial datum need not be in \(L^1\) and the case of solutions consisting of only rarefactions is included in Theorem 3.3.
4 Speed Conservation vs. Momentum Conservation

In [7, Section 5], the following system is considered:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + \frac{\partial (\rho v^2 + P(\rho))}{\partial x} &= 0
\end{align*}
\]

(4.1)

and \( p = p(\rho) \) is the pressure law for a polytropic gas, i.e., \( p(\rho) = (k^2/\gamma) \rho^\gamma \) with \( k > 0 \) and \( \gamma > 1 \). Theorem 2.1 allows to estimate the difference between solutions to (4.1) and those to the classical \( p \)-system

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} &= 0 \\
\frac{\partial v}{\partial t} + \frac{\partial (\rho v^2 + p(\rho))}{\partial x} &= 0
\end{align*}
\]

(4.2)

Below, we require the pressure law only to satisfy the standard assumption

\( \text{(P)} \quad p \in C^2(\mathbb{R}^+; \mathbb{R}^+) \) is such that for all \( \rho > 0 \), \( p'(\rho) > 0 \), \( \frac{\partial^2 p}{\partial \rho^2}(\rho p(\rho)) > 0 \).

We refer to [8] for a further result on the estimates of the difference of solutions to systems of the type (4.1) and (4.2) in the case, with source terms, motivated by ducts with slowly varying section.

The comparison between (4.1) and (4.2) fits within the scope of Theorem 2.1.

**Lemma 4.1.** Let (P) hold. Then, systems (4.1) and (4.2) fit into (1.1) setting

\[
\begin{align*}
\lambda_1 &= v - \sqrt{p'(\rho)} \\
\lambda_2 &= v + \sqrt{p'(\rho)}
\end{align*}
\]

(4.3)

Moreover, conditions (H1), (H2) and (H3) hold, with

\[
A = \begin{bmatrix}
\frac{v}{p'(\rho)} & \rho \\
\frac{v}{p'(\rho)} & v
\end{bmatrix}, \quad r_1 = \begin{bmatrix} -\rho \\ \sqrt{p'(\rho)} \end{bmatrix}, \quad r_2 = \begin{bmatrix} \rho \\ \sqrt{p'(\rho)} \end{bmatrix}.
\]

The proof is immediate and hence omitted.

Theorem 2.1 applied to the case of (4.1) and (4.2) yields the following estimate.

**Corollary 4.2.** Let (P) hold. Fix a constant state \( (\bar{\rho}, \bar{v}) \in \mathbb{R}^+ \times \mathbb{R} \). Call \( S : \mathbb{R}^+ \times \mathcal{D} \to \mathcal{D} \) the semigroup generated by (4.1) and \( \bar{S} : \mathbb{R}^+ \times \mathcal{D} \to \mathcal{D} \) the semigroup generated by (4.2) with

\[
\mathcal{D} \cap \bar{\mathcal{D}} \supseteq \left\{ (\rho, v) \in L^1_{\text{loc}}(\mathbb{R}; B((\bar{\rho}, \bar{v}), \delta)) : \| (\rho, v) - (\bar{\rho}, \bar{v}) \|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} < \delta \right\}
\]

for \( \delta > 0 \). Then, for any \( (\rho_o, v_o) \) such that \( \| (\rho_o, v_o) - (\bar{\rho}, \bar{v}) \|_{L^\infty(\mathbb{R}; \mathbb{R}^2)} < \delta \) and \( TV(\rho_o, v_o) < \delta \),

\[
\int_0^T \left\| (S_t(\rho_o, v_o))(x) - (\bar{S}_t(\rho_o, v_o))(x) \right\| \, dx \\
&\leq C t \left( \mu_1((\rho_o, v_o); I_0) + \mu_2((\rho_o, v_o); I_0) \right) (\text{diam}(\rho_o, v_o)(I_0))^2
\]

where \( (\mu_1, \mu_2) \) are the measures [2.3] referred to system (4.1).

The proof is slightly simpler than that of Theorem 3.3 and, hence, it is omitted.
5 A Traffic Model

As a final example, we consider the traffic model [5, Formula (1.2)], which reads

\[
\begin{align*}
\partial_t \rho + \partial_x \left( \rho w \psi(\rho) \right) &= 0 \\
\partial_t w + \psi(\rho) \partial_x w &= 0
\end{align*}
\]

(5.1)

where \( \rho \) is the car density and \( v = w \psi(\rho) \) is the traffic speed at density \( \rho \), the Lagrangian variable \( w \) describing the maximal speed of drivers. For any smooth invertible function \( q = q(w) \), system (5.1) can be put in the conservative form

\[
\begin{align*}
\partial_t \rho + \partial_x \left( \rho w \psi(\rho) \right) &= 0 \\
\partial_t (\rho q(w)) + \partial_x (\rho w \psi(\rho) q(w)) &= 0
\end{align*}
\]

(5.2)

With the notation in Section 2, we have

\[
 u = \begin{bmatrix} \rho \\ w \end{bmatrix}, \quad g(u) = \begin{bmatrix} \rho \\ \rho q(w) \end{bmatrix}, \quad f(u) = \begin{bmatrix} \rho w \psi(\rho) \\ \rho w \psi(\rho) q(w) \end{bmatrix}.
\]

Elementary computations yield

\[
A(u) = (Dg(u))^{-1} Df(u) = \begin{bmatrix} (\psi(\rho) + \rho \psi'(\rho)) & \rho w \psi(\rho) \\ 0 & w \psi(\rho) \end{bmatrix}.
\]

Note that the matrix \( A \) is independent of the choice of \( q \). Moreover, Remark 2.4 applies, coherently with the fact that all systems of the form (5.2) share the same weak as well as strong solutions, whatever the function \( q \), see [5, Remark 5.3].

6 Proofs

Throughout, by \( O(1) \) we denote a quantity dependent only on norms of \( Df, Dg, D\tilde{f} \) and \( D\tilde{g} \) computed on a fixed neighborhood of \( \tilde{u} \) in \( \Omega \).

Lemma 6.1. Let \( f, g \) satisfy (H1). Then, the function \( F \) defined by \( F = f \circ g^{-1} \) is smooth and for all \( w \in g(\Omega) \) the matrix \( DF(w) \) admits the eigenvalues \( \Lambda_1(w), \ldots, \Lambda_n(w) \) and the eigenvectors \( R_1(w), \ldots, R_n(w) \), with

\[
\begin{align*}
\Lambda_i(w) &= \lambda_i \left( g^{-1}(w) \right) \\
R_i(w) &= Dg \left( g^{-1}(w) \right) r_i \left( g^{-1}(w) \right) \quad \text{for } i = 1, \ldots, n.
\end{align*}
\]

The proof is immediate and hence omitted.

Proof of Theorem 2.1. The proof is divided into several steps. We use throughout the notation (2.2).
Step 1: Let (H1) hold. Fix a positive $\varepsilon$ and let $u^\varepsilon$ be an $\varepsilon$-approximate front tracking solution in the sense of [3, Definition 7.1] to the Cauchy problem in (1.1), left. Then, for any $T > 0$ such that $a + \tilde{\lambda}T < b - \tilde{\lambda}T$,

$$
\int_{I_T} \left\| u^\varepsilon(T, x) - \left( \tilde{S}_T \left( u^\varepsilon(0) \right) \right)(x) \right\| \, dx 
\leq \tilde{L} \int_0^T \sum_{y \in I \cap J^*_T} \int_{-\tilde{\lambda}}^{\tilde{\lambda}} \left\| U(t, y, \xi) - \left( \tilde{R} \left( u(t, y^-, u(t, y^+) \right) \right)(\xi) \right\| \, d\xi \, dt
$$

(6.1)

where

$$
\tilde{L} = \text{L}^1 \text{-Lipschitz constant of } t \to \tilde{S}_t u_0,
$$

$$
J^*_T = \{ y \in \mathbb{R} : u^\varepsilon(t, y^-) \neq u^\varepsilon(t, y^+) \}
$$

$$
U(t, y, \xi) = u^\varepsilon(t, y^-) \chi_{\mathbb{R} \setminus [\tilde{\lambda}(t, y), +\infty)}(\xi) + u^\varepsilon(t, y^+) \chi_{(-\infty, \tilde{\lambda}(t, y))}(\xi)
$$

$$
\lambda(t, y) = \text{speed of the jump in } u^\varepsilon(t) \text{ at time } t \text{ at point } y \text{ with } y \in J^*_T
$$

$$
\xi \to \left( \tilde{R} (u_l, u_r) \right)(\xi) = \text{Lax solution to } \begin{cases}
\partial_t \tilde{g}(u) + \partial_\xi \tilde{f}(u) = 0 \\
u(0, \xi) = \begin{cases}
u_l & \xi < 0 \\
u_r & \xi > 0
\end{cases}
\end{cases} \text{ at } t = 1.
$$

Indeed, by the finite propagation speed of (1.1), we can apply [3, Theorem 2.9] on the set

$$
\{(t, x) \in \mathbb{R}_+ \times \mathbb{R} : t \in [0, (b - a)/\tilde{\lambda}] \text{ and } x \in I_t\}
$$

obtaining

$$
\int_{I_T} \left\| u^\varepsilon(T, x) - \left( \tilde{S}_T \left( u^\varepsilon(0) \right) \right)(x) \right\| \, dx 
\leq \tilde{L} \int_0^T \liminf_{h \to 0^+} \frac{1}{h} \int_{I_t} \left\| u^\varepsilon(t + h, x) - \left( \tilde{S}_h \left( u^\varepsilon(t) \right) \right)(x) \right\| \, dx \, dt
$$

$$
\leq \tilde{L} \int_0^T \sum_{y \in I \cap J^*_T} \liminf_{h \to 0^+} \frac{1}{h} \int_{y - \tilde{\lambda}h}^{y + \tilde{\lambda}h} \left\| u^\varepsilon(t + h, x) - \left( \tilde{S}_h \left( u^\varepsilon(t) \right) \right)(x) \right\| \, dx \, dt
$$

$$
\leq \tilde{L} \int_0^T \sum_{y \in I \cap J^*_T} \liminf_{h \to 0^+} \frac{1}{h} \int_{-\tilde{\lambda}h}^{\tilde{\lambda}h} \left\| u^\varepsilon(t + h, y + \xi) - \left( \tilde{S}_h \left( u^\varepsilon(t) \right) \right)(\xi) \right\| \, d\xi \, dt
$$

which gives the desired estimate, thanks to the hyperbolic rescaling

$$
\left[ \frac{h}{\xi} \right] \to \left[ \frac{1}{\xi/h} \right].
$$

Step 2: Assume that $u_l = \psi_1(\sigma)(u_r)$ and define $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n) = \tilde{E}(u_l, u_r)$. Then,

$$
\sum_{j \neq i} |\tilde{\sigma}_j| + |\rho - \tilde{\sigma}_i| \leq O(1) \left\| \psi_i(\sigma)(u_l) - \tilde{\psi}_i(\sigma)(u_l) \right\|.
$$

(6.3)
Indeed, use the Lipschitz continuity of $\tilde{E}_j$ and recall that $\tilde{E}_j \left( u_i, \tilde{\psi}_i(\sigma)(u_i) \right) = 0$ for $j \neq i$ and $\tilde{E}_i \left( u_i, \tilde{\psi}_i(\sigma)(u_i) \right) = \sigma$, to estimate

$$
\sum_{j \neq i} |\tilde{\sigma}_j| + |\sigma - \tilde{\sigma}_i| = \sum_{j=1}^n \left| \tilde{E}_j \left( u_i, \psi_i(\sigma)(u_i) \right) - \tilde{E}_j \left( u_i, \tilde{\psi}_i(\sigma)(u_i) \right) \right| 
\leq O(1) \left\| \psi_i(\sigma)(u_i) - \tilde{\psi}_i(\sigma)(u_i) \right\|.
$$

**Step 3:** Let $\sigma \to S_i(\sigma)(u)$, respectively $\sigma \to \tilde{S}_i(\sigma)(u)$, be the $i$–shock curve for system \ref{eq:system}, left, respectively, right, exiting $u$ parametrized by $\sigma$. Similarly, $\Lambda_i(u, \sigma)$, respectively $\tilde{\Lambda}_i(u, \sigma)$ is the corresponding Rankine–Hugoniot speed. Then, if \ref{ass:H2} holds, by possibly reducing $\Omega$, the quantity

$$
\kappa_V = \sup \left\{ \left\| S_i(\sigma)(u) - \tilde{S}_i(\sigma)(u) \right\| \frac{|\sigma|^2}{\sigma^2} : \begin{array}{c}
\tilde{S}_i(\sigma)(u) \in V \\
\Lambda_i(u, \sigma) - \tilde{\Lambda}_i(u, \sigma) \\
\sigma \in \{1, \ldots, n\}
\end{array} \right\}
$$

is bounded, where $V$ is any open set with $\nabla V \subset \Omega$. The proof of this boundedness is a consequence of Lemma \ref{lem:boundedness}.

**Step 4** Under assumption \ref{ass:H2} if the open set $V$ is such that $V \supseteq \tilde{u}(\mathbb{R}^+, \mathbb{R}) \cup \tilde{u}(\mathbb{R}^+, \mathbb{R})$, we also have

$$
\int_{-\lambda}^{\lambda} \left\| U(t, y, \xi) - \left( \tilde{R} \left( u(t, y-), u(t, y+) \right) \right)(\xi) \right\| \, d\xi \leq \begin{cases}
O(1) \left( \kappa_V |\sigma|^3 + \varepsilon |\sigma| \right) & \sigma \text{ is a shock}, \\
O(1) \varepsilon |\sigma| & \sigma \text{ is a rarefaction}, \\
O(1) |\sigma| & \sigma \text{ is non–physical}.
\end{cases}
$$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The Riemann problem with data $u_l, u_r$ is solved by a single (physical) $i$-wave $\sigma$ of the first system in \ref{eq:system} and from the waves $(\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ of the second system in \ref{eq:system}. Note that $u_r = \psi_i(\sigma)(u_l)$.}
\end{figure}

Indeed, let $y \in J^i_t$ and call $u_l = u(t, y-)$ and $u_r = u(t, y+)$. 

11
Assume first that \( \sigma \geq 0 \), so that \( \sigma = O(1) \varepsilon \). Then, \((H2)\) ensures that rarefaction curves of the two systems coincide and hence \( u_r = \psi_i(\sigma)(u_l) = \tilde{\psi}_i(\sigma)(u_l) \). By \( [3, (ii) \text{ in Lemma 9.1}] \),
\[
\int_{-\lambda}^{\lambda} \left\| U(t, y, \xi) - \left( \tilde{R}(u_l, u_r) \right)(\xi) \right\| d\xi = \int_{-\lambda}^{\lambda} \left\| U(t, y, \xi) - (R(u_l, u_r))(\xi) \right\| d\xi \leq O(1) \sigma^2 \leq O(1) \varepsilon \sigma.
\]
On the other hand, assume \( \sigma < 0 \). Applying \((6.3)\) and \((6.4)\),
\[
|\tilde{\sigma}_i - \sigma| \leq O(1) \left| \tilde{\psi}_i(\sigma)(u_l) - \psi_i(\sigma)(u_l) \right| = O(1) \left| \tilde{S}_i(\sigma)(u_l) - S_i(\sigma)(u_l) \right| \leq O(1) \kappa V |\sigma|^3
\]
which ensures that \( \tilde{\sigma}_i < 0 \). Define \( \lambda_i = \lambda(t, y) \), \( \hat{u}_{i-1} = \tilde{\psi}_i(\sigma_{i-1}) \circ \ldots \circ \tilde{\psi}_1(\sigma)(u_l) \) and \( \hat{\lambda}_i = \Lambda_i(\hat{u}_{i-1}, \hat{\sigma}) \). Assume that \( \hat{\lambda}_i \leq \lambda_i \), the other case being analogous.
\[
\int_{-\lambda}^{\lambda} \left\| U(t, y, \xi) - \tilde{R}(u_l, u_r)(\xi) \right\| d\xi \leq \int_{-\lambda}^{\lambda_i} \left\| u_l - \tilde{R}(u_l, u_r)(\xi) \right\| d\xi + \int_{\lambda_i}^{\lambda} \left\| u_r - \tilde{R}(u_l, u_r)(\xi) \right\| d\xi
\]
Compute the three terms above separately. For \( \xi < \lambda_i \), using \((6.3)\) and \((6.4)\),
\[
\left\| u_l - \tilde{R}(u_l, u_r)(\xi) \right\| \leq O(1) \sum_{j<i} |\tilde{\sigma}_j| \leq O(1) \left\| \tilde{S}_i(\sigma)(u_l) - S_i(\sigma)(u_l) \right\| \leq O(1) \kappa V |\sigma|^3.
\]
As a particular case of the above estimate, note that \( \left\| u_l - \hat{u}_{i-1} \right\| \leq O(1) \kappa V |\sigma|^3 \). Hence, to estimate the middle summand in the right hand side of \((6.7)\), use \((6.6)\) to obtain
\[
\int_{\lambda_i}^{\lambda} \left\| u_l - \tilde{R}(u_l, u_r)(\xi) \right\| d\xi \leq O(1) \left( \left| \tilde{\lambda}_i(\hat{u}_{i-1}, \tilde{\sigma}_i) - \Lambda_i(u_l, \sigma) \right| + \varepsilon \right) |\sigma|
\]
\[
\leq O(1) \left( \left| \tilde{\lambda}_i(\hat{u}_l, \sigma) - \Lambda_i(u_l, \sigma) \right| + |\hat{u}_{i-1} - u_l| + |\tilde{\sigma}_i - \sigma + \varepsilon| \right) |\sigma|
\]
\[
\leq O(1) \left( \kappa V \sigma^2 + \kappa V |\sigma|^3 + \kappa V |\sigma|^3 + \varepsilon \right) |\sigma|
\]
\[
\leq O(1) \left( \kappa V \sigma^2 + \varepsilon \right) |\sigma|.
\]
The third summand in \((6.7)\), for \( \xi > \lambda_i \), is treated similarly to \((6.8)\):
\[
\sum_{j>i} |\tilde{\sigma}_j| \leq O(1) \kappa V |\sigma|^3.
\]
Therefore, \((6.7)\) yields
\[
\int_{-\lambda}^{\lambda} \left\| U(t, y, \xi) - \tilde{R}(u_l, u_r)(\xi) \right\| d\xi \leq O(1) \left( \kappa V \sigma^2 + \varepsilon \right) |\sigma|.
\]
Finally, the case of a non–physical wave follows from \([3, (i) \text{ in Lemma 9.1}]\), completing the proof of Step 4.
Step 5: The previous steps directly imply that
\[
\int_{I_T} \left\| u^\varepsilon(T,x) - \tilde{S}_T u^\varepsilon(0,x) \right\| \, dx
\leq \mathcal{O}(1) \int_0^T \left( \kappa_V \sum_{y \in I_t \cap J^i_t: \sigma_y < 0} |\sigma_y|^3 + \varepsilon |\sigma_y| \right) \, dt
\]
where, as usual, with \( \mathcal{N} \mathcal{P} \) we denote the set of non-physical waves, see [3, Paragraph 7.1].

Step 6: Proof of 3. in Theorem 2.1
Below, we exploit the fact that the total variation of the wave front tracking approximate solution at time \( t \) is bounded by a constant times the total variation of the initial datum. By (6.9),
\[
\int_{I_T} \left\| u^\varepsilon(T,x) - \tilde{S}_T u^\varepsilon(0,x) \right\| \, dx
\leq \mathcal{O}(1) \int_0^T \left( \kappa_V \sum_{y \in I_t \cap J^i_t: \sigma_y < 0} |\sigma_y|^3 + \varepsilon \text{TV}(u^\varepsilon(t); I_t) + \varepsilon \right) \, dt
\leq \mathcal{O}(1) \int_0^T \left( \kappa_V \max_{y \in I_t \cap J^i_t: \sigma_y < 0} |\sigma_y|^2 \sum_{y \in I_t \cap J^i_t: \sigma_y < 0} |\sigma_y| + \varepsilon \text{TV}(u^\varepsilon(t); I_t) + \varepsilon \right) \, dt
\leq \mathcal{O}(1) \int_0^T \left( \kappa_V (\text{diam } V)^2 \sum_{y \in I_t \cap J^i_t: \sigma_y < 0} |\sigma_y| + \varepsilon \text{TV}(u^\varepsilon(t); I_t) + \varepsilon \right) \, dt
\leq \mathcal{O}(1) \int_0^T \left( \kappa_V (\text{diam } V)^2 \text{TV}(u^\varepsilon(t); I_t) + \varepsilon \text{TV}(u^\varepsilon(t); I_t) + \varepsilon \right) \, dt
\]
in the limit \( \varepsilon \to 0 \) we obtain (2.5), thanks to the arbitrariness of \( V \), provided \( V \supset u(\mathbb{R}^+, \mathbb{R}) \).

Step 7: Proof of 4. in Theorem 2.1
By (H3), we may measure sizes of \( i \)-waves through the variation in the \( i \)-th Riemann coordinate. Introduce the following functional defined on the wave front tracking approximate solutions \( u^\varepsilon = u^\varepsilon(t,x) \) to (1.1):
\[
\Upsilon^\varepsilon(t) = \sum_{y \in I_t \cap J^i_t: \sigma_y < 0} |\sigma_y| + C \sum_{(\sigma_y, \sigma_y') \in \mathcal{A}^*} |\sigma_y \sigma_y'| \quad (6.11)
\]
where \( \mathcal{A}^* \) is the set of pairs of approaching waves in \( u^\varepsilon \) at time \( t \) (see [3, § 7.3]), that we modify excluding all pairs of rarefaction waves, also those belonging to different families.

The map \( t \to \Upsilon^\varepsilon(t) \) is non increasing. Indeed, assume that two waves interact at time \( \tilde{t} \). Whenever the interacting waves are not both rarefactions, the standard interaction estimates apply, see [3, § 7.3]. In interactions involving two rarefactions, (H3) ensures that \( \Delta \Upsilon^\varepsilon(u^\varepsilon(\tilde{t})) = 0 \), since rarefactions simply cross each other and their sizes measured by means of Riemann coordinates remain constant.
We now have:

\[ \sum_{y \in I_0 \cap J^+_0 : \sigma_y < 0} |\sigma_y| \leq \chi(t) \leq \chi(0) = \sum_{y \in I_0 \cap J^+_0 : \sigma_y < 0} |\sigma_y| + C \sum_{(\sigma_y, \sigma_y') \in \mathcal{A}^+(0)} |\sigma_y \sigma_y'| \]

\[ \leq (1 + C \text{TV}(u_0, I_0)) \sum_{y \in I_0 \cap J^+_0 : \sigma_y < 0} |\sigma_y| . \]

Denote by \( \mu^-_i \) the negative part of the measure \( \mathcal{H} \) constructed from the initial datum \( u_0 \). Similarly, denote by \( \mu^-_{i,\varepsilon} \) the analogous measure constructed from the \( \varepsilon \)-approximate piecewise constant initial datum \( u^\varepsilon \). Note that by \([4, \text{Formula (4.8) in Lemma 4.2}]\), we can choose the piecewise constant initial datum \( u^\varepsilon(0, \cdot) \) such that \( \mu^-_{i,\varepsilon}(I_0) \leq \mu^-_i(I_0) + \varepsilon \).

\[ \sum_{y \in I_0 \cap J^+_0 : \sigma_y < 0} |\sigma_y| = \sum_{y \in I_0 \cap J^+_0 : \sigma_y < 0} \sum_{i=1}^n \left[ E_i (u^\varepsilon(0, y-), u^\varepsilon(0, y+)) \right]^- = \sum_{i=1}^n \mu^-_{i,\varepsilon}(u^\varepsilon(0); I_0) \leq \sum_{i=1}^n \mu^-_i(u_0; I_0) + n \varepsilon . \]

Summarizing, starting from \([6,10])\), the above inequalities yield

\[ \int_{I_T} \left\| u^\varepsilon(T, x) - S_T u^\varepsilon(0, x) \right\| dx \]

\[ \leq O(1) T \left( \kappa_V \left( \sum_{i=1}^n \mu^-_i(u_0; I_0) + n \varepsilon \right) (\text{diam } V)^2 + \varepsilon \text{TV}(u_0; I_0) + \varepsilon \right) dt \]

and passing to the limit \( \varepsilon \to 0 \) the proof is completed. \( \square \)

**Lemma 6.2.** Under assumptions \( \textbf{(H1)} \) and \( \textbf{(H2)} \), the following bound on \( \kappa_V \) as defined in \([6,1])\) holds:

\[ \kappa_V \leq O(1) \left( \Delta \left( f, g, (\tilde{f}, \tilde{g}) \right) + \text{diam } V \right) , \]

where \( V \) is an open subset of \( \mathbb{R}^n \) and \( \Delta \left( f, g, (\tilde{f}, \tilde{g}) \right) \) is defined in \([2,8])\).

**Proof.** Using the ideas in \([3, \text{Theorem 5.2}]\), we proceed obtaiing higher order estimates.

Note that at the zero-th order, by \( \textbf{(H1)} \), we have

\[ S_i(0)(u) - \tilde{S}_i(0)(u) = 0 \quad \text{and} \quad \Lambda_i(u, 0) - \tilde{\Lambda}_i(u, 0) = 0 . \]

To simplify the notation in the computations below, we keep \( i \in \{1, \ldots, n\} \) and \( u \in \Omega \) fixed and set

\[ \lambda_i(u) \to \lambda, \quad \Lambda_i(u, \sigma) \to \Lambda(\sigma), \quad S_i(\sigma)(u) \to S(\sigma) \]

so that the Rankine–Hugoniot conditions now read

\[ \Lambda(\sigma) \left( g(S(\sigma)) - g(u) \right) = f(S(\sigma)) - f(u) . \]
A derivative in the direction $r = r_i(u)$ of the eigenvalues–eigenvector relation

$$\lambda_i(u) \, Dg(u) \, r_i(u) = Df(u) \, r_i(u)$$

shortened as $\lambda \, Dg(u) \, r = Df(u) \, r$

yields:

$$(D\lambda \, r) \, Dg(u) \, r + \lambda \, D^2 g(u)(r, r) + \lambda \, Dg(u) \, Dr \, r = D^2 f(u) \, (r, r) + Df(u) \, Dr \, r \quad (6.14)$$

where $D^2 g(u)(\cdot, \cdot)$ and $D^2 f(u)(\cdot, \cdot)$ are bilinear forms. A further derivative in the direction $r$ yields:

$$D^2 \lambda(r, r) \, Dg(u) \, r + (D\lambda \, Dr) \, Dg(u) \, r + 2 \,(D\lambda \, r) \, D^2 g(u)(r, r) + 2 \,(D\lambda \, r) \, Dg(u) \, Dr \, r
+ \lambda \, D^3 g(u)(r, r, r) + 3 \, \lambda \, D^2 g(u)(Dr \, r, r) + \lambda \, Dg(u) \, D^2 r \,(r, r) + \lambda \, Dg(u) \, Dr \, Dr \, r 
\quad (6.15)$$

$$= D^3 f(u) \,(r, r, r) + 3 \, D^2 f(u)(Dr \, r, r) + Df(u) \, D^2 r \,(r, r) + Df(u) \, Dr \, Dr \, r,$$

here, $D^3 g(u)(\cdot, \cdot, \cdot)$ and $D^3 f(u)(\cdot, \cdot, \cdot)$ are trilinear forms, while $D^2 \lambda(\cdot, \cdot)$ and $D^2 r(\cdot, \cdot)$ are bilinear ones. A first differentiation of (6.13) with respect to $\sigma$ and denoting the differentiation with respect to $\sigma$ with a dot, yields:

$$\dot{\Lambda} \,(g(S) - g(u)) + \Lambda \, Dg(S) \, \dot{S} = Df(S) \, \dot{S}$$

Setting $\sigma = 0$, we obtain

$$\Lambda(0) \, Dg(u) \, \dot{S}(0) = Df(u) \, \dot{S}(0)$$

which implies that $\dot{S}(0) = r$ and $\Lambda(0) = \lambda$. The same result holds for the “tilde” system, hence

$$\dot{S}(0) - \dot{\tilde{S}}(0) = 0.$$

Computing the second derivative of (6.13), we obtain:

$$\ddot{\Lambda} \,(g(S) - g(u)) + 2 \, \dot{\Lambda} \, Dg(S) \, \dot{S} + \Lambda \, D^2 g(S)(\dot{S}, \dot{S}) + \Lambda \, Dg(S) \, \ddot{S}
= D^2 f(S)(\dot{S}, \dot{S}) + Df(S) \, \ddot{S} \quad (6.16)$$

and setting $\sigma = 0$ we obtain

$$2 \, \dot{\Lambda}(0) \, Dg(u) \, r + \lambda \, D^2 g(u)(r, r) + \lambda \, Dg(u) \, \ddot{S}(0) = D^2 f(u)(r, r) + Df(u) \ddot{S}(0). \quad (6.17)$$

Subtract now term by term (6.14) from (6.17) and obtain

$$\left(2 \, \dot{\Lambda}(0) - D\lambda \, r \right) \, Dg(u) \, r + \lambda \, Dg(u) \, \left(\dot{S}(0) - Dr \, r \right) = Df(u) \left(\dot{S}(0) - Dr \, r \right)
\quad (2 \, \dot{\Lambda}(0) - D\lambda \, r) \, r = \left(Dg^{-1}(u) \, Df(u) - \lambda \right) \left(\dot{S}(0) - Dr \, r \right) \quad (6.18)$$

Multiply now both terms in the latter expression (6.18) by the $i$-th left eigenvector $l = l_i$ of

$$A(u) = Dg^{-1}(u) \, Df(u)$$

to obtain

$$\dot{\Lambda}(0) = \frac{1}{2} \,(D\lambda \, r). \quad (6.19)$$

We thus proved that

$$\dot{\Lambda}(0) - \dot{\Lambda}(0) = 0.$$
The left hand side in (6.18) vanishes, implying that \( \tilde{S}(0) - Dr r \) is a right eigenvector of \( Dg^{-1}(u) Df(u) \) corresponding to the eigenvalue \( \lambda \), so that, for a \( \beta \in \mathbb{R} \),

\[
\tilde{S}(0) - Dr r = \beta r .
\]  

(6.20)

We parameterize the Lax curves by means of the arc–length in the physical variable \( u \), obtaining

\[
\sigma \text{ arc–length } \Rightarrow \begin{cases} 
|\dot{S}| = 1 \Rightarrow \dot{S}^T \tilde{S} = 0 \\
|r| = 1 \Rightarrow r^T Dr r = 0
\end{cases}
\]  

(6.21)

so that multiplying both sides of (6.20) by \( r^T = \dot{S}(0)^T \), we obtain \( \beta = 0 \) and hence

\[
\tilde{S}(0) = Dr r .
\]  

(6.22)

Since we expressed \( \tilde{S} \) by means of only the vector field \( r \), we also obtained

\[
\tilde{S}(0) - \ddot{S}(0) = 0 .
\]

(6.23)

Differentiate now (6.16) with respect to \( \sigma \):

\[
\ddot{\tilde{S}}(0) = -\dddot{\tilde{S}}(0).
\]

(6.24)

Compute the above terms in \( \sigma = 0 \), using (6.19) and (6.22), to obtain

\[
3 \ddot{\Lambda}(0) Dg(u) r + \frac{3}{2} (D\lambda r) D^2g(u)(r, r) + \frac{3}{2} (D\lambda r) Dg(u) Dr r
\]

\[
+ \lambda D^3g(u)(r, r, r) + 3 \lambda D^2g(u)(r, Dr r) + \lambda Dg(u) \dddot{S}(0)
\]

\[
= D^3f(u)(r, r, r) + 3 D^2f(u)(r, Dr r) + Df(u) \dddot{S}(0).
\]

Subtract now the latter relation from (6.19) to obtain

\[
\left( D^2\lambda(r, r) + (D\lambda Dr r) - 3 \ddot{\Lambda}(0) \right) Dg(u) r + \frac{1}{2} (D\lambda r) D^2g(u)(r, r)
\]

\[
+ \frac{1}{2} (D\lambda r) Dg(u) Dr r + \lambda Dg(u) \left( D^2r(r, r) + Dr Dr r - \dddot{S}(0) \right)
\]

\[
= Df(u) \left( D^2r(r, r) + Dr Dr r - \dddot{S}(0) \right).
\]

Multiply now on the left by \( (Dg(u))^{-1} \):

\[
\left( D^2\lambda(r, r) + (D\lambda Dr r) - 3 \ddot{\Lambda}(0) \right) r + \frac{1}{2} (D\lambda r) \left( Dg(u) \right)^{-1} D^2g(u)(r, r)
\]

\[
+ \frac{1}{2} (D\lambda r) Dr r + \lambda \left( D^2r(r, r) + Dr Dr r - \dddot{S}(0) \right) \right)
\]

\[
= (Dg(u))^{-1} Df(u) \left( D^2r(r, r) + Dr Dr r - \dddot{S}(0) \right).
\]  

(6.25)

The same computations leading to (6.23) can now be repeated with the “\( \tilde{\text{tilde}} \)” system, yielding an expression analogous to (6.25), which, subtracted from (6.25), yields:

\[
3 \left( \ddot{\Lambda}(0) - \tilde{\Lambda}(0) \right) r - \frac{1}{2} (D\lambda r) \left( (D\tilde{g}(u))^{-1} D^2\tilde{g}(u) - (Dg(u))^{-1} D^2g(u) \right)(r, r)
\]

\[
= \left( \lambda \text{Id} - (Dg(u))^{-1} Df(u) \right) \left( \tilde{S}(0) - \dddot{S}(0) \right).
\]  

(6.26)
and multiplying on the left by the $i$-th left eigenvector $l = l_i$ gives
\[
3 \left( \tilde{\Lambda}(0) - \hat{\Lambda}(0) \right) = \frac{1}{2} (D\lambda r) l \left( (D\tilde{g}(u))^{-1} D^2\tilde{g}(u) - (Dg(u))^{-1} D^2g(u) \right) (r, r).
\]
This ensures that $\left| \tilde{\Lambda}(0) - \hat{\Lambda}(0) \right| = O(1) \Delta((f, g), (\tilde{f}, \tilde{g}))$, so that
\[
\frac{\Lambda_i(u, \sigma) - \hat{\Lambda}_i(u, \sigma)}{\sigma^2} \leq O(1) \left( \Delta((f, g), (\tilde{f}, \tilde{g})) + |\sigma| \right)
\]
and moreover, by (6.24),
\[
\left( \lambda I_d - (Dg(u))^{-1} Df(u) \right) \left( \ddot{\tilde{S}}(0) - \ddot{S}(0) \right) = O(1) \Delta((f, g), (\tilde{f}, \tilde{g}))
\]
Write $\ddot{\tilde{S}}(0) - \ddot{S}(0) = \sum_j \alpha_j r_j$. Then, multiplying the latter expression above by $l = l_j$ on the left, we have, for $j \neq i$,
\[
\alpha_j = O(1) \Delta((f, g), (\tilde{f}, \tilde{g})). \tag{6.25}
\]
On the other hand, by the choice (6.24) of the parameterization
\[
\begin{align*}
\ddot{S}^T \ddot{S} &= 0 \Rightarrow \dddot{\tilde{S}}^T \dddot{S} + \ddot{\tilde{S}}^T \ddot{S} = 0 \\
\ddot{S}^T \ddot{S} &= 0 \Rightarrow \dddot{S}^T \ddot{S} + \ddot{\tilde{S}}^T \ddot{\tilde{S}} = 0 \Rightarrow (\ddot{\tilde{S}}(0) - \ddot{S}(0))^T r = 0
\end{align*}
\]
which ensures that $\left( \sum_{j=1}^n \alpha_j r_j \right)^T r = 0$ and hence
\[
\alpha_i = -\sum_{j \neq i} \alpha_j r_j = O(1) \Delta((f, g), (\tilde{f}, \tilde{g}))
\]
which, together with (6.25), ensures that
\[
\left| S_i(\sigma)(u) - \tilde{S}_i(\sigma)(u) \right| \leq O(1) \left( \Delta((f, g), (\tilde{f}, \tilde{g})) + |\sigma| \right)
\]
completing the proof.

**Proof of Proposition 2.3.** Is a direct consequence of Lemma 6.2. 

**Proof of Theorem 2.5.** Note that by (H1) we can assume that $\frac{d}{da} \left( \tilde{f}(u) \right) > 0$.

We follow the same lines of the proof of Theorem 2.1 using as wave front tracking solutions those constructed in [3, Section 6.1]. By (H1), if $U$ is a single shock, respectively a rarefaction, then $\tilde{R}(u_l, u_r)$ also consists of a shock, respectively a rarefaction.

In the scalar case, we have now an estimate different from (6.5). While rarefactions are treated entirely in the same way, there are no non–physical waves and in the case of shocks a finer estimates is available. Indeed, shock curves in the two equations in (1.1) coincide so that
We are thus lead to find a general bound on the quantity
\[
\int_{-\lambda}^{\lambda} \left| U(t, y, \xi) - \left( R(u^c(t, y-)), u^c(t, y+) \right) (\xi) \right| \, d\xi = \int_{-\lambda}^{\lambda} \left| U(t, y, \xi) - \left( R(u, u + \sigma) \right) (\xi) \right| \, d\xi \leq |\lambda - \lambda| |\sigma|.
\]

We are thus lead to find a general bound on the quantity
\[
\left| \lambda - \lambda \right|
= \left| \frac{f(u + \sigma) - f(u)}{g(u + \sigma) - g(u)} - \frac{\tilde{f}(u + \sigma) - \tilde{f}(u)}{\tilde{g}(u + \sigma) - \tilde{g}(u)} \right|
= \left| \frac{(f(u + \sigma) - f(u)) (\tilde{g}(u + \sigma) - \tilde{g}(u)) - (\tilde{f}(u + \sigma) - \tilde{f}(u)) (g(u + \sigma) - g(u))}{(g(u + \sigma) - g(u)) (\tilde{g}(u + \sigma) - \tilde{g}(u))} \right|
\leq \frac{1}{\sigma^2} \frac{1}{\inf_{\Omega} |g'|} \left| \frac{f(u + \sigma) - f(u)}{g(u + \sigma) - g(u)} - \frac{\tilde{f}(u + \sigma) - \tilde{f}(u)}{\tilde{g}(u + \sigma) - \tilde{g}(u)} \right|
\times \left| \frac{f(u + \sigma) - f(u)}{g(u + \sigma) - g(u)} (\tilde{g}(u + \sigma) - \tilde{g}(u)) - (\tilde{f}(u + \sigma) - \tilde{f}(u)) (g(u + \sigma) - g(u)) \right|
\]

Consider now the term in the latter modulus above and compute its derivatives using (2.10):
\[
k_u(\sigma) = (f(u + \sigma) - f(u)) (\tilde{g}(u + \sigma) - \tilde{g}(u)) - (\tilde{f}(u + \sigma) - \tilde{f}(u)) (g(u + \sigma) - g(u))
= \int_0^\sigma \int_0^\sigma \left( f'(u + \xi) \tilde{g}'(u + \eta) - \tilde{f}'(u + \xi) \tilde{g}'(u + \eta) \right) \, d\xi \, d\eta.
\]
\[
k'_u(\sigma) = \int_0^\sigma \left( f'(u + \xi) \tilde{g}'(u + \sigma) - \tilde{f}'(u + \xi) \tilde{g}'(u + \sigma) \right) \, d\xi \quad + \quad \int_0^\sigma \left( f'(u + \sigma) \tilde{g}'(u + \eta) - \tilde{f}'(u + \sigma) \tilde{g}'(u + \eta) \right) \, d\eta.
\]
\[
k''_u(\sigma) = \int_0^\sigma \left( f''(u + \xi) \tilde{g}''(u + \sigma) - \tilde{f}''(u + \xi) \tilde{g}''(u + \sigma) \right) \, d\xi \quad + \quad \int_0^\sigma \left( f''(u + \sigma) \tilde{g}''(u + \eta) - \tilde{f}''(u + \sigma) \tilde{g}''(u + \eta) \right) \, d\eta.
\]
\[
k'''_u(\sigma) = \int_0^\sigma \left( f'''(u + \xi) \tilde{g}'''(u + \sigma) - \tilde{f}'''(u + \xi) \tilde{g}'''(u + \sigma) \right) \, d\xi \quad + \quad \int_0^\sigma \left( f'''(u + \sigma) \tilde{g}'''(u + \eta) - \tilde{f}'''(u + \sigma) \tilde{g}'''(u + \eta) \right) \, d\eta.
\]
\[
k''''_u(\sigma) = \int_0^\sigma \left( f'''''(u + \xi) \tilde{g}'''''(u + \sigma) - \tilde{f}'''''(u + \xi) \tilde{g}'''''(u + \sigma) \right) \, d\xi \quad + \quad \int_0^\sigma \left( f'''''(u + \sigma) \tilde{g}'''''(u + \eta) - \tilde{f}'''''(u + \sigma) \tilde{g}'''''(u + \eta) \right) \, d\eta.
\]
\[
k'''''_u(\sigma) = \int_0^\sigma \left( f'''''(u + \xi) \tilde{g}'''''(u + \sigma) - \tilde{f}'''''(u + \xi) \tilde{g}'''''(u + \sigma) \right) \, d\xi \quad + \quad \int_0^\sigma \left( f'''''(u + \sigma) \tilde{g}'''''(u + \eta) - \tilde{f}'''''(u + \sigma) \tilde{g}'''''(u + \eta) \right) \, d\eta.
\]
Note that \( k(0) = k'(0) = k''(0) = k'''(0) = 0 \), hence a Taylor expansion yields
\[
|k_u(\sigma)| \leq \sup_w \sup_s |k'''_w(s)| \sigma^4
\]
where
\[
\sup_w \sup_s |k'''_w(s)| \leq 2 \left\| f''(\tilde{\rho}' - \tilde{f}'\tilde{\rho}'') - g''' \tilde{\rho}'' \right\|_{C^0(\Omega; \mathbb{R})}
+ O(1) \left( \left\| f' - \tilde{f} \right\|_{C^0(\Omega; \mathbb{R})} + \left\| g' - \tilde{g} \right\|_{C^0(\Omega; \mathbb{R})} \right) |\sigma|
+ O(1) \left( \left\| f''' - \tilde{f}''' \right\|_{C^0(\Omega; \mathbb{R})} + \left\| g''' - \tilde{g}''' \right\|_{C^0(\Omega; \mathbb{R})} \right) |\sigma|^4.
\]
Therefore,
\[
\int_{\tilde{\lambda}}^{\lambda} \left| U(t, y, \xi) - (\mathcal{R}(u, u + \sigma))(\xi) \right| \, d\xi
\leq \frac{2}{(\inf_{\Omega}|g'|)(\inf_{\Omega}|\tilde{g}'|)} \left\| f''(\tilde{\rho}' - \tilde{f}'\tilde{\rho}'') - g''' \tilde{\rho}'' \right\|_{C^0(\Omega; \mathbb{R})} |\sigma|^3
+ O(1) \left( \left\| f' - \tilde{f} \right\|_{C^0(\Omega; \mathbb{R})} + \left\| g' - \tilde{g} \right\|_{C^0(\Omega; \mathbb{R})} \right) |\sigma|^4
+ O(1) \left( \left\| f''' - \tilde{f}''' \right\|_{C^0(\Omega; \mathbb{R})} + \left\| g''' - \tilde{g}''' \right\|_{C^0(\Omega; \mathbb{R})} \right) |\sigma|^4,
\]
and the proof is completed as in Step 4 in the proof of Theorem 2.1 using the Maximum Principle for scalar conservation laws.

**Proof of Lemma 3.2.** Lemma 3.1 ensures the existence of the semigroups \( S^{3 \times 3} \), while that of \( S^{2 \times 2} \) follows from (p) through well-known arguments.

By the properties of the SRSs \( \mathcal{L} \), it is sufficient to compare the solutions to Riemann problems for (1.5), with constant entropy, and (1.4). Since a constant entropy \( \bar{s} \) factorizes in the third equation (1.5), the Lax curves for the \( 2 \times 2 \) system (1.4) are the Lax curves for the \( 3 \times 3 \) system (1.5) corresponding to the first and third families. Therefore, an entropy solution to the Riemann Problem for (1.5) is an entropy solution to the Riemann Problem for (1.4) provided the data has constant entropy \( \bar{s} \). This concludes the proof.

**Proof of Theorem 3.3.** Let \( t \to (\rho^\varepsilon(t), v^\varepsilon(t)) \) be an \( \varepsilon \)-approximate wave front tracking solution to (1.4), see [3, Definition 7.1]. Since the \( 2 \times 2 \) system (1.4) satisfies (H3), we parametrize \( i \)-Lax curve through the variation in the \( i \)-th Riemann coordinate.

Then, \( t \to (\rho^\varepsilon(t), v^\varepsilon(t), \bar{s}) \) is an \( \varepsilon \)-approximate wave front tracking solution to (1.5). Follow Steps 1–5 in the proof of Theorem 2.1 comparing \( t \to (\rho^\varepsilon(t), v^\varepsilon(t), \bar{s}) \) to the orbit \( t \to \tilde{S}_t^{3 \times 3} (\rho^0(0), v^\varepsilon(0), \bar{s}) \) and obtain (6.9). Apply Step 7 to system (1.5). Therefore, the total size of negative waves in the \( \varepsilon \)-approximate solution to (1.5) at time \( t \) is bounded, as \( \varepsilon \to 0 \), by a constant times the total size of negative waves in the initial datum, obtaining the estimate
\[
\int_{\mathcal{L}} \left\| \left( \tilde{S}_t^{2 \times 2}(\rho_o, v_o)(x), \bar{s} \right) - \tilde{S}_t^{3 \times 3}(\rho_o, v_o, \bar{s})(x) \right\| dx
\leq C t \left( \mu^1_1((\rho_o, v_o); I_0) + \mu^2_2((\rho_o, v_o); I_0) \right) \left( \text{diam}(\rho, v)(\mathcal{T}_t) \right)^2.
\]
Recall now [2, Theorem 3.12], which extends the classical result [9], that ensures the estimate
\[
\text{diam}(\rho, v)(T_t) \leq \mathcal{O}(1) \text{diam}(\rho_o, v_o)(I_0),
\]
completing the proof.

Acknowledgment: The present work was supported by the PRIN 2015 project Hyperbolic Systems of Conservation Laws and Fluid Dynamics: Analysis and Applications, by the GNAMPA 2017 project Conservation Laws: from Theory to Technology and by the Simons – Foundation grant 346300 together with the Polish Government MNiSW 2015-2019 matching fund.

References

[1] S. Bianchini and R. M. Colombo. On the stability of the Standard Riemann Semigroup. Proc. Amer. Math. Soc., 130(7):1961–1973 (electronic), 2002.

[2] S. Bianchini, R. M. Colombo, and F. Monti. 2 × 2 systems of conservation laws with $L^\infty$ data. J. Differential Equations, 249(12):3466–3488, 2010.

[3] A. Bressan. Hyperbolic systems of conservation laws, volume 20 of Oxford Lecture Series in Mathematics and its Applications. Oxford University Press, Oxford, 2000. The one-dimensional Cauchy problem.

[4] R. M. Colombo and G. Guerra. On the stability functional for conservation laws. Nonlinear Anal., 69(5-6):1581–1598, 2008.

[5] R. M. Colombo, F. Marcellini, and M. Rasce. A 2-phase traffic model based on a speed bound. SIAM J. Appl. Math., 70(7):2652–2666, 2010.

[6] C. M. Dafermos. Hyperbolic conservation laws in continuum physics, volume 325 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, third edition, 2010.

[7] R. J. DiPerna. Global solutions to a class of nonlinear hyperbolic systems of equations. Comm. Pure Appl. Math., 26:1–28, 1973.

[8] J. Geng and Y. Zhang. Irrotational approximation to the quasi-1-d gas flow. Z. Angew. Math. Phys., 60(6):1053–1073, 2009.

[9] J. Glimm and P. D. Lax. Decay of solutions of systems of nonlinear hyperbolic conservation laws. Memoirs of the American Mathematical Society, No. 101. American Mathematical Society, Providence, R.I., 1970.

[10] R. J. LeVeque. Finite volume methods for hyperbolic problems. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2002.

[11] L. Saint-Raymond. Isentropic approximation of the compressible Euler system in one space dimension. Arch. Ration. Mech. Anal., 155(3):171–199, 2000.

[12] D. Serre. Systems of conservation laws. 1 & 2. Cambridge University Press, Cambridge, 1999. Translated from the 1996 French original by I. N. Sneddon.