Applications on Cyclic Soft Symmetric Groups

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Abstract. In classical groups, there is a fact state that the symmetric groups are not cyclic in general. So, in this work the notion of cyclic soft symmetric groups using soft set theory is investigated and their orders are given. In this paper, we introduce new classes over non cyclic symmetric groups; these new classes are generated by element in the power set of classical symmetric groups. Furthermore, some applications on these new classes are given.

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1. INTRODUCTION
The soft set theory is investigated by D. Molodtsov [1]. This notion is given to solve complicated problems in our life. He has established the basic results of this notion and successfully applied this new theory into several directions, like game theory, smoothness of functions, Riemann integration, theory of probability operations research, and others. This new theory has a wider application in different fields, see ([2]-[13]).

The set theories in algebraic structures that deal with non-classical sets have been discussed by many authors. Fuzzy set is one of the non-classical sets and fuzzy groups are introduced by Rosenfeld [14] to establish results for the algebraic structures of fuzzy sets. Next, some researches in algebraic structures on non-classical sets have been given, see ([14]-[26]).

In symmetric group $S_n$ ([27],[28]), any pair of permutations are conjugate iff they have the same cycle type. Therefore for any $\beta = (k_1, k_2, \ldots, k_r) \in S_n$ is conjugate to its inverse $(k_1, k_2, \ldots, k_r)^{-1} = (k_r, \ldots, k_2, k_1)$, because they have the same cycle type. A transposition $\beta = (k_1, k_2) \in S_n$ is a permutation of length 2 with identity. Moreover, $(\text{Id} (k_1, k_2)(k_2, k_1) = e$, where $\beta = (k_1, k_2) = (k_2, k_1)$ when $n < \infty$, the order of $S_n$ is $|S_n| = n!$. In this paper, the concept of cyclic soft symmetric groups is investigated and their orders are given. Also, some applications on these new classes are given.

2. Definitions and Notations
In this section, we shall give some basic definitions which are used to obtain our results.
Definition 2.1: ([1], [29])
A pair \( (F, A) \) is called a soft set (over \( G \)) where \( F \) is a multivalued function \( F : A \rightarrow P(G) \), \( A \subseteq E \) and \( E \) is a parameter. For any pair \( (F, A) \) and \( (K, B) \) over the common universe \( G \), we say that \( (F, A) \) is a soft subset of \( (K, B) \) if \( A \subseteq B \) and for all \( e \in A \), \( F(e) \subseteq K(e) \). We write \( (F, A) \subseteq (K, B) \). The family of all soft sets \( (F, A) \) over a universe \( G \) and the parameter set \( E \) is a family of soft sets denoted by \( SS(G, E) \).

Definition 2.2: ([30], [31]) The union of any pair \( (F, A) \) and \( (K, B) \) over \( G \) is the soft set \( (R, T) \), where \( R = (F \cup K)(G) \) and for all \( T \),

\[
T(\sigma) = \begin{cases} 
F(\sigma), & \text{if } \sigma \in A/B \\
K(\sigma), & \text{if } \sigma \in B/BA \\
F(\sigma) \cup K(\sigma), & \text{if } \sigma \in A \cap B 
\end{cases}
\]

We write \( (F, A) \bigcup (K, B) = (T, R) \). Also, we denote to their intersection by \( (F, A) \bigcap (K, B) \) and defined as \( T = A \cap B \), and \( H(\sigma) = F(\sigma) \cap K(\sigma) \) for all \( \sigma \in T \).

Definition 2.3: ([2]) The soft set \( (F, A) \in SS(G, E) \) is called a soft point in \( (K, A) \), denoted by \( e_F \), if for the element \( e \in A \), \( F(e) \neq \emptyset \) and \( F(e') = \emptyset \) for all \( e' \in A \setminus \{e\} \). The soft point \( e_F \) is said to be in the soft set \( (K, A) \), denoted by \( e_F \in (K, A) \), if for the element \( e \in A \) and \( F(e) \subseteq K(e) \).

Definition 2.4: [30] If \( (F, A) \) and \( (K, B) \) are two soft sets, then \( (F, A) \land (K, B) \) denoted by \( (F, A) \land (K, B) \). \( (F, A) \land (K, B) \) is defined as \( (H, A \times B) \), where, for all \( F(\alpha) \cap K(\beta) \) \( H(\alpha, \beta) = (\alpha, \beta) \in A \times B \).

Definition 2.5: [30] If \( (F, A) \) and \( (K, B) \) are two soft sets, then \( (F, A) \lor (K, B) \), denoted by \( (F, A) \lor (K, B) \), is defined by \( (F, A) \lor (K, B) = (H, A \times B) \), where \( H(\alpha, \beta) = F(\alpha) \cup K(\beta) \), \( \forall (\alpha, \beta) \in A \times B \).

Definition 2.7: ([32], [33])
Each \( \lambda \in S_n \) can be written as \( \gamma_1, \gamma_2, ..., \gamma_{c(\lambda)} \). With \( \gamma_i \) disjoint cycles of length \( \alpha_i \) and \( c(\lambda) \) is the number of disjoint cycle factors including the 1-cycle of \( \lambda \). A partition \( \alpha = \alpha(\lambda) = (\alpha_1(\lambda), \alpha_2(\lambda), ..., \alpha_{c(\lambda)}(\lambda)) \) is called cycle type of \( \lambda \).

Definition 2.6: ([34]) The order of any finite symmetric group \( S_n \) is \( |S_n| = n! \), and each permutation \( \lambda \in S_n \) has finite order \( |\lambda| = L.C.M(\alpha_1, \alpha_2, ..., \alpha_{c(\lambda)}) \).

3. The Soft Symmetric Groups (SSG)

Definition 3.1:
Let $(SSG)$ be a soft symmetric group $(\psi_m, A)$. We say $A \subseteq S_n$ be a symmetric group and $S_n$ over $S_n$, where $\psi_m(\beta) = \{\beta, \beta^{-1}, \beta^2, \beta^{-2}, \ldots, \beta^{n-1}, \beta^{-(n-1)}\} \ \forall \beta \in A$, and $m = \text{Max}\{\vert \beta \vert; \beta \in A\}$.

Example 3.2:
Let $S_3$ be a symmetric group and $S_3$ over $S_3$, where $\{1, 2, 3\} = \{1, 2, 3\}$. Then we consider that $\{1, 2, 3\}$ is a (SSG) over $S_3$.

Definition 3.3:
(1) $(\psi_m, A)$ is called an identity soft symmetric group (ISSG) over $S_n$ if $\psi_m(A) = \{e\}$.
(2) $(\psi_m, A)$ is called an absolute soft symmetric group (ASSG) over $S_n$ if $\psi_m(A) = S_n$.

Lemma 3.4:
Let $(\psi_m, A)$ be a (SSG) over $S_n$. If $m = 1$, then $(\psi_m, A)$ is (ISSG).

Proof:
Assume that $m = 1$, but $m = \text{Max}\{\vert \beta \vert; \beta \in A\}$, and $A \subseteq S_n$. Then $A = \{e\}$, since only the identity element $e$ of $S_n$ satisfies $\vert e \vert = 1$ and hence $\psi_m(A) = \{e\}$. Thus $(\psi_m, A)$ is (ISSG).

Remarks 3.5:
(1) It is clearly, if $A = S_n$. Then $(\psi_m, A)$ is (ASSG).
(2) For any $\beta \in A$, we have $\psi_m(\beta)$ is a subgroup of $S_n$.
(3) Every soft set $(F, A) \in \mathbb{SS}(G_n)$ is (SSG) if and only if $G = E = S_n$ with $F = \psi_m$.

Lemma 3.6:
Let $(\psi_m, A)$ be a (SSG) over $S_n$. If $m < n$, then $(\psi_m, A)$ is not (ASSG).

Proof:
Assume that $m < n$, but $\lambda = (a_1, a_2, \ldots, a_n) \in S_n$ with $\vert \lambda \vert = n$. Thus $\lambda \notin A$, since $\vert \lambda \vert > m = \text{Max}\{\vert \beta \vert; \beta \in A\}$. This implies that $A \neq S_n$ and hence $\psi_m(A) \neq S_n$ because there is not exist permutation $\beta \in A$ such that $\psi_m(\beta) = \lambda$ with $\vert \lambda \vert = m$.

Definition 3.7:
Let $(\psi_m, A)$ and $(\psi_k, B)$ be two soft symmetric groups (SSGs) over $S_n$. We say $(\psi_k, B)$ is a soft symmetric subgroup of $(\psi_m, A)$ and written as $(\psi_k, B) \overset{\sim}{\subseteq} (\psi_m, A)$, if

(1) $B \subseteq A$,
(2) $\psi_k(\beta)$ is a subgroup of $\psi_m(\beta)$ for all $\beta \in B$.

Definition 3.8:
Let \((\psi_m, A)\) and \((\psi_k, B)\) be two (SSGs) over \(S_n\) and \(S_q\), respectively, and let \(f: S_p \rightarrow S_q\) and \(g: A \rightarrow B\) be two maps. We say \((f, g)\) is a soft symmetric homomorphism (SSH) and \((\psi_m, A)\) is soft symmetric homomorphic to \((\psi_k, B)\), denoted by \((\psi_m, A) \cong (\psi_k, B)\), if the following are hold:

1. \(f\) is a homomorphism from \(S_p\) onto \(S_q\),
2. \(g\) is a mapping from \(A\) onto \(B\),
3. \(f(\psi_m(\beta)) = \psi_k(g(\beta)), \forall \beta \in A\).

In this definition, if \(f\) is an isomorphism from \(S_p\) to \(S_q\) and \(g\) is a one-to-one mapping from \(A\) onto \(B\), then we say that \((f, g)\) is a soft symmetric isomorphism and \((\psi_m, A)\) is soft symmetric isomorphic to \((\psi_k, B)\) which is denoted by \((\psi_m, A) \cong (\psi_k, B)\). The image of soft symmetric group \((\psi_m, A)\) under soft symmetric homomorphism \((f, g)\) will be denoted by \((f(\psi_m), g(A))\).

Definition 3.9:
Let \((\psi_m, A)\) and \((\psi_k, B)\) be two (SSGs) over \(S_n\) and \(S_q\), respectively. The product of \((\psi_m, A)\) and \((\psi_k, B)\) is defined as \((\psi_m, A) \times (\psi_k, B) = (\psi_{m,k}, A \times B)\), where \(\psi_{m,k}(\lambda, \beta) = \psi_m(\lambda) \times \psi_k(\beta)\), for all \((\lambda, \beta) \in A \times B\).

4. The Order of Soft Symmetric Groups
In this section, we define order of the (SSG) and investigate their properties.

Definition 4.1:
Let \((\psi_m, A)\) be a (SSG) over \(S_n\) and let \(\psi_m(\beta) \in (\psi_m, A)\) for some \(\beta \in A\). We say \(\psi_m(\beta)^d = \{\alpha^d | \alpha \in \psi_m(\beta); d \in \mathbb{Z}\}\) is \(d\) – power of \(\psi_m(\beta)\).

Example 4.2: Let \((\psi_m, A)\) be a (SSG) over \(S_3\), where \(A = \{(13), (312)\}\). Hence \(m = 3\) and \((\psi_m, A) = \{\psi_m(13) = \{e, (13)\}, \psi_m(312) = \{e, (312), (213)\}\}\). Now, for example the third power of \(\psi_m(312) = (\psi_m(312))^3 = \{e, (312)^3, (213)^3\} = \{e\}\).

Theorem 4.3: Let \((\psi_m, A)\) be a (SSG) over \(S_n\) and \(\psi_m(\beta), \psi_m(\alpha) \in (\psi_m, A)\) for \(\alpha, \beta \in A\). Then,

1. \((\psi_m(\beta) \cap \psi_m(\alpha))^d = \psi_m(\beta)^d \cap \psi_m(\alpha)^d\), for all \(d \in \mathbb{Z}\),
2. \((\psi_m(\beta) \cup \psi_m(\alpha))^d = \psi_m(\beta)^d \cup \psi_m(\alpha)^d\), for all \(d \in \mathbb{Z}\),
3. \((\psi_m(\beta) \times \psi_m(\alpha))^d = \psi_m(\beta)^d \times \psi_m(\alpha)^d\), for all \(d \in \mathbb{Z}\).

Proof: Suppose that \(\lambda^d \in (\psi_m(\beta) \cap \psi_m(\alpha))^d\), for \(d \in Z\). Then from Definition (4.1), we consider that:

\[
\lambda^d \in (\psi_m(\beta) \cap \psi_m(\alpha))^d \iff \lambda \in \psi_m(\beta) \cap \psi_m(\alpha) \iff \lambda \in \psi_m(\beta) \& \lambda \in \psi_m(\beta).
\]

This completes the proof. Also, by \(\lambda^d \in (\psi_m(\beta) \cup \psi_m(\alpha))^d\), \(\lambda^d \in (\psi_m(\beta) \times \psi_m(\alpha))^d\) using Definition (4.1), we can prove (2) and (3).

Definition 4.4:
Let \((\psi_m, A)\) be a (SSG) over \(S_n\) and \(\psi_m(\beta) \in (\psi_m, A)\). If \(d\) is a least positive integer satisfies \(\psi_m(\beta)^d = \{e\}\), then \(d\) is said to be the order of \(\psi_m(\beta)\). Also, in a special case in infinite symmetric
group $S_n$, we have $\psi_m(\beta)$ has infinite order, if no such $d$ exists. The order of $\psi_m(\beta)$ is denoted by $|\psi_m(\beta)|$.

**Example 4.5:** In Example (4.2) the order of element $\psi_j(3 \ 1 \ 2)$ in (SSG). $|\psi_j(3 \ 1 \ 2)| = 3$ is $S_n$

**Theorem 4.6:** Let $(\psi_m, A)$ be a (SSG) over $S_n$. Then, the orders of elements of $(\psi_m, A)$ are finite.

**Proof:** It is straightforward.

**Theorem 4.7:** Let $(\psi_m, A)$ be a (SSG) over $S_n$. Then, the orders of elements of $(\psi_m, A)$ are finite.

**Proof:** Let $d$ be the order of $\psi_m(\beta)$. Then $\psi_m(\beta)^d = \{e\}$. This means that $\alpha^d = e$ for all $\alpha \in \psi_m(\beta)$. We know $\psi_m(\beta)$ is a subgroup of $S_n$. So, from classical group theory we have $|\alpha^d| = d$, namely, $\psi_m(\beta)$ is common multiple of elements $d$. Hence, $\alpha \in \psi_m(\beta)$ for all $d$ divides $\alpha$. Let $k$ be another common multiple of elements of $\psi_m(\beta)$. Then, by reason of $\alpha^k = e$ for all $\alpha \in \psi_m(\beta)$, $\psi_m(\beta)^k = \{e\}$. Furthermore, since $d$ is the least number that satisfies the condition $\psi_m(\beta)^d = \{e\}$, hence $d \mid k$. This completes the proof.

**Theorem 4.8:** Let $(\psi_m, A)$ be a (SSG) over $S_n$, and $\psi_m(\beta) \in (\psi_m, A)$ for $\alpha, \beta \in A$. Then, the following hold:

1. $|\psi_m(\beta) \cap \psi_m(\alpha)| \leq \text{GCD}(|\psi_m(\beta)|, |\psi_m(\alpha)|)$.
2. $|\psi_m(\beta) \cup \psi_m(\alpha)| = \text{LCM}(|\psi_m(\beta)|, |\psi_m(\alpha)|)$.
3. $|\psi_m(\beta) \times \psi_m(\alpha)| = |\psi_m(\beta)| \cdot |\psi_m(\alpha)|$.

**Proof:** We consider the following.

1. Let $|\psi_m(\beta) \cap \psi_m(\alpha)|$ is subgroup of $|\psi_m(\beta)|$ and $|\psi_m(\alpha)|$, so $|\psi_m(\beta) \cap \psi_m(\alpha)| \mid |\psi_m(\beta)|$ and $|\psi_m(\beta) \cap \psi_m(\alpha)| \mid |\psi_m(\alpha)|$. It follows $|\psi_m(\beta) \cap \psi_m(\alpha)| \leq \text{GCD}(|\psi_m(\beta)|, |\psi_m(\alpha)|)$.

2. Let $|\psi_m(\beta) \cup \psi_m(\alpha)| = k$, $|\psi_m(\beta)| = h$, and $|\psi_m(\alpha)| = t$. From Theorem (4.3). This follows $|k \mid h \cdot k \cdot (\psi_m(\beta) \cup \psi_m(\alpha))^d = \psi_m(\beta)^d \cup \psi_m(\alpha)^d = \{e\}$ Thus $k$ is a common multiple of $h$ and $t$. Let $d$ be another common multiple of $h$ and $t$. Consider $(\psi_m(\beta) \cup \psi_m(\alpha))^d = \psi_m(\beta)^d \cup \psi_m(\alpha)^d = \{e\}$ for all $\beta \in A$. Hence $k$ is LCM order of $\psi_m(\beta)$ and $\psi_m(\alpha)$. This completes the proof.

**Definition 4.9:** Let $(\psi_m, A)$ be a (SSG) over $S_n$. The set $(\psi_m, A)^d = \{\psi_m(\beta)^d : \beta \in A, n \in \mathbb{Z}\}$ is said to be $n$th power of (SSG) $(\psi_m, A)$. 

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Example 4.10: Let $(\psi_m, A)$ be a (SSG) over $S_n$ defined in Example (4.3). Then, the second power of $(\psi_m, A)$ is that $(\psi_m, A)^2 = \{e\}, \psi_m(312)^2 = \{213\}.$

Theorem 4.11: Let $(\psi_m, A)$ and $(\psi_k, B)$ be two (SSGs) over $S_n$. Then,

1. $((\psi_m, A) \cup (\psi_k, B))^d = (\psi_m, A)^d \cup (\psi_k, B)^d$
2. if $A \subseteq B$ and, for all $\alpha \in A$, $\psi_m(\alpha)$ and $\psi_k(\alpha)$ are identical approximations, then $((\psi_m, A) \setminus (\psi_k, B))^d = (\psi_m, A)^d \setminus (\psi_k, B)^d$.

Proof: We consider the following.

1. Suppose that $(\psi_m, A) \cup (\psi_k, B) = (H, A \times B)$ and $(\psi_m, A)^d \cup (\psi_k, B)^d = (T, A \times B)$. We can write $((\psi_m, A) \cup (\psi_k, B))^d = (H, A \times B)^d$. Using Definition (4.9) and Theorem (4.3) we have

   $$(H, A \times B)^d = \{H(\alpha, \beta)^d; (\alpha, \beta) \in A \times B\}$$

   $$= \{(\psi_m(\alpha) \cup \psi_k(\beta))^d; (\alpha, \beta) \in A \times B\}$$

   $$= \{\psi_m(\alpha)^d \cup \psi_k(\beta)^d; (\alpha, \beta) \in A \times B\}$$

   $$= \{T(\alpha, \beta); (\alpha, \beta) \in A \times B\}$$

   $$= (\psi_m, A)^d \cup (\psi_k, B)^d.$$

2. Suppose that $(\psi_m, A) \setminus (\psi_k, B) = (H, A \times B)$ and $(\psi_m, A)^d \setminus (\psi_k, B)^d = (T, A \times B)$. Using the same arguments in (1), we have

   $$(H, A \times B)^d = \{H(\alpha, \beta)^d; (\alpha, \beta) \in A \times B\}$$

   $$= \{(\psi_m(\alpha) \setminus \psi_k(\beta))^d; (\alpha, \beta) \in A \times B\}$$

   $$= \{\psi_m(\alpha)^d \setminus \psi_k(\beta)^d; (\alpha, \beta) \in A \times B\}$$

   $$= \{T(\alpha, \beta); (\alpha, \beta) \in A \times B\}$$

   $$= (\psi_m, A)^d \setminus (\psi_k, B)^d.$$

Remarks 4.12: The order of a symmetric group is defined as the number of elements it contains. But in (SSG), it differs from classical groups.

Definition 4.13 Let $(\psi_m, A)$ be a (SSG) over $S_n$, the least common multiple of orders of elements of $(\psi_m, A)$ is called order of $(\psi_m, A)$. The order of soft group $(\psi_m, A)$ is denoted by $|\psi_m, A|.$

Lemma 4.14

Let $(\psi_m, A)$ be a (SSG) over $S_n$. Then $|\psi_m(\bar{A})| = n!$, if $(\psi_m, A)$ is surjective.

Proof: Since $(\psi_m, A)$ is surjective, then by definition 3.3 we have $\psi_m(\bar{A}) = S_n$, but $|S_n| = n!$, thus $|\psi_m(\bar{A})| = n!.$

Now, we will show some results on (SSG) similar to Lagrange Theorem.

Theorem 4.15: Let $(\psi_m, A)$ be a (SSG) over $S_n$ and $\psi_m(\beta) \in (\psi_m, A).$ Then;

1. $|\psi_m(\beta)|$ divides $|(\psi_m, A)|$.
2. $|\psi_m(\beta)|$ divides $n!.$

Proof:
(1) Since $|\langle \psi_m \rangle, A \rangle|_3$ is the least common multiple of orders of elements of $\langle \psi_m \rangle, A \rangle$, then $\psi_m(\beta)^{|\langle \psi_m \rangle, A \rangle|} = \psi_m(\beta)^{|\langle \psi_m \rangle, A \rangle|}$. This implies that $|\langle \psi_m \rangle, A \rangle| = \text{LCM} \{ |\langle \psi_m \rangle, A \rangle|, |\langle \psi_m \rangle, A \rangle| \in |\langle \psi_m \rangle, A \rangle| \}$. $|\langle \psi_m \rangle, A \rangle|$ divides $|\psi_m(\beta)|$ we have $|\psi_m(\beta)| \leq |\langle \psi_m \rangle, A \rangle|$ and since $\{ \psi_m \}$

(2) Since $\psi_m(\beta)$ is a subgroup of $S_n$, for all $\beta \in A$. Then by Lagrange Theorem in group theory we have $|\psi_m(\beta)|$ divides $|S_n| = n!$.

Theorem 4.16: Let $\langle \psi_m, A \rangle$ and $\langle \psi_k, B \rangle$ be two (SSG) over $S_n$. Then $|\langle \psi_m, A \rangle \cap \langle \psi_k, B \rangle| \leq |\langle \psi_m, A \rangle|$ and $|\langle \psi_m, A \rangle \cap \langle \psi_k, B \rangle| \leq |\langle \psi_k, B \rangle|$.

Proof: Suppose that $|\langle \psi_m, A \rangle \cap \langle \psi_k, B \rangle| = \text{LCM}(H(\alpha, \beta))$, for $(\alpha, \beta) \in A \times B$

\[
|\langle \psi_m, A \rangle \cap \langle \psi_k, B \rangle| = \text{LCM}(\psi_m(\alpha) \cap \psi_k(\beta))
\]

\[
|\langle \psi_m, A \rangle \cap \langle \psi_k, B \rangle| \leq \text{LCM}(\psi_m(\alpha)) = |\langle \psi_m, A \rangle|
\]

The other inequality is shown similarly.

5. Cyclic Soft Symmetric Groups

In group theory, we know the symmetric groups are not cyclic in general. However, the cyclic groups are important type in group theory. Therefore, in this section the notions of cyclic soft symmetric groups using soft set theory is investigated and prove some of their properties.

Definition 5.1: (CSSG)

Let $F$ if and only if $G$ is a soft cyclic group over $(F, A)$. We say $G$ over be a soft set $(F, A)$ power $P(G)$ is element in $X$, where $\alpha \in X$ and $\beta \in A$ for all $F(\beta) = \{ \alpha \}$ with $G$ is subgroup of set of group $G$ and $\{ \alpha \} = \{ \alpha^k \ | \ n \in \mathbb{Z} \}$. Also, if $G = S_n$ and $\psi_m : A \rightarrow P(G)$ satisfies $\psi_m(\beta) = F(\beta)$, $\forall \beta \in A$. We say $(F, A)$ is a cyclic soft symmetric group (CSSG).

Remarks 5.2:

(1) If $G$ is a cyclic group and $(F, A)$ is a soft set over $G$ with for all $G$ is subgroup of $F(\beta)$ cyclic group since each subgroup of cyclic group are cyclic and hence is a soft $(F, A)$, then $\beta \in A$ for any $\beta \in A$ there exists $\alpha \in X \subseteq G$ satisfies $F(\beta) = \{ \alpha \}$.

(2) If $G$ is non-cyclic group and $(F, A)$ is a soft set over $G$ with for $G$ is subgroup of $F(\beta)$ all $\beta \in A$, then $(F, A)$ is not necessary to be soft cyclic group or not soft cyclic group.

Now, we can consider some properties of cyclic soft groups in the following theorem

Theorem 5.3: Let $\langle \psi_m, A \rangle$ be a (SSG) over $S_n$. Then;

(1) If $\langle \psi_m, A \rangle$ is a (CSSG) generated by $X$, then $|\langle \psi_m, A \rangle| = \text{LCM} \{ \alpha \ | \ \alpha \in X \}$.

(2) If $\langle \psi_m, A \rangle$ is an (ISSG), then it is a (CSSG) generated by $\{ e \}$.

Proof: It is easily seen from Definitions (3.3),(4.13) and Remark (3.5).

Theorem 5.4: Let $\langle \psi_m, A \rangle$ be an (ASSG) over $S_n$. Then, $\langle \psi_m, A \rangle$ is a (CSSG) if $n \leq 3$.

Proof:
Suppose that \( n < 3 \). In classical group \( S_n \), we have \( S_n \) is a cyclic group if \( n < 3 \). Then by Remark (5.2-1) we consider that \((\psi_m, A)\) is a (CSSG).

**Example 5.5:** Let \((T, A)\) be a soft set over \( S_4 \) and \( A = \{ e, (2, 4), (1, 4, 3), (4, 3, 1) \} \) be the set of parameters. Define \( T : A \rightarrow P(S_n) \) as \( T(\beta) = \{ \alpha \in S_n | \alpha = \beta^d, d \in \mathbb{Z} \} \) for all \( \beta \in A \), hence \((T, A)\) is a cyclic soft symmetric group over \( S_4 \) however \( S_4 \) is not a cyclic group.

**Theorem 5.6:** Let \((f, g)\) be a (SSH) of (SSG) \((\psi_m, A)\) over \( S_p \) in to (SSG) \((\psi_1, B)\) over \( S_q \). If \((\psi_m, A)\) is a (CSSG) over \( S_p \), then \((f(\psi_m), g(A))\) is a (CSSG) over \( S_q \).

**Proof:** We need in the first to show that \((f(\psi_m), g(A))\) is a (CSSG) over \( S_q \). Since \( f \) is a homomorphism from \( S_p \) to \( S_q \). So, \( f(\psi_m(\beta)) = \psi_1(g(\beta)) \) is a subgroup of \( S_q \) for all \( g(\beta) \in (g(A)) \). Thus \((f(\psi_m), g(A))\) is a (SSG) over \( S_q \). Since \( \psi_m(\beta) \) is cyclic subgroup, \( \forall \beta \in A \), image of \( \psi_m(\beta) \) under \( f \) is cyclic; that means \( S_q \) is cyclic subgroup of \( \psi_1(g(\beta)) \) for all \( g(\beta) \in (g(A)) \). Consequently, \((f(\psi_m), g(A))\) is a (CSSG) over \( S_q \).

**Theorem 5.7:** Letwith , respectively \( S_k \) and \( S_p \) be two (SSGs) over \((\psi_k, B)\) and \((\psi_m, A)\). \((\psi_k, B)\) is a (CSSG), then so is \((\psi_m, A)\). If \((\psi_k, B)\) \((\psi_m, A)\) \(\equiv\)

**Proof:** Since \((\psi_m, A) \equiv (\psi_k, B)\), then there is an homomorphism \( f : S_p \rightarrow S_q \) satisfies \( f(\psi_m(\beta)) = \psi_1(g(\beta)) \) for all \( g(\beta) \in (g(A)) \), where \( g \) is a one-to-one mapping from \( A \) onto \( B \). Since \( \psi_m(\beta) \) is a cyclic subgroup for any \( \beta \in A \) and \( f \) is a homomorphism, then \( \psi_1(g(\beta)) \) is a cyclic subgroup of \( S_q \). Hence, \( f(\psi_m(\beta)) \) is a cyclic subgroup of \( S_q \). Thus, for any member in \((\psi_k, B)\) is cyclic. Then \((\psi_k, B)\) is a (CSSG).

**Theorem 5.8:** If \((\psi_m, A)\) and \((\psi_k, B)\) are two (CSSGs) over \( S_n \), then \((\psi_m, A) \wedge (\psi_k, B)\) is a (CSSG) over \( S_n \).

**Proof:**
Let \((\psi_m, A) \wedge (\psi_k, B) = (T, A \times B)\), where \( T(\beta, \rho) = \psi_m(\beta) \wedge \psi_k(\rho) \), \( \forall (\beta, \rho) \in A \times B \). For any \( \beta \in A \) and \( \rho \in B \) we have \( \psi_m(\beta) \wedge \psi_k(\rho) \) is a subgroup of both \( \psi_m(\beta) \) and \( \psi_k(\rho) \) (since \( \psi_m(\beta) \) and \( \psi_k(\rho) \) are cyclic subgroups of \( S_n \). Thus, for all \( (\beta, \rho) \in A \times B \) we have \( \psi_m(\beta) \wedge \psi_k(\rho) \) is a cyclic subgroup of \( S_n \). Hence, \( (T, A \times B) \) is (CSSG) over \( S_n \).

**Theorem 5.8:** Let \((\psi_m, A)\) and \((\psi_k, B)\) be two (CSSGs) over \( S_n \), and \( A \cap B = \phi \). Then, \((\psi_m, A)\) \(\wedge\) \(S_n\) is a (CSSGs) over \((\psi_k, B)\)

**Proof:** It is trivial

**Theorem 5.9:** Let \((\psi_m, A)\) and \((\psi_k, B)\) be two (CSSGs) over \( S_p \) and \( S_q \), respectively. of orders \( d \) and \( h \). If \( d \) and \( h \) are relatively prime, then the product \((\psi_m, A) \times (\psi_k, B)\) is a (CSSG).

**Proof:** Let \((d, h) = 1\). Then by (Lagrange Theorem) we have \( |\psi_m(\beta)| \) divides \( d \) and \( |\psi_k(\rho)| \) divides \( h \) for all \( \beta \in A, \rho \in B \). Since \((d, h) = 1\), \( |\psi_m(\beta)| \) and \( |\psi_k(\rho)| \) are relatively prime. So \( \psi_m(\beta) \times \psi_k(\rho) \) is cyclic group for all \( (\beta, \rho) \in A \times B \). Hence \((\psi_m, A) \times (\psi_k, B)\) is a (CSSG).
6. Conclusion

In this work, the soft set theory is applied on symmetric groups and we investigated and discussed new classes of groups like (SSG) and (CSSG). Also, their orders are given and some applications on these new classes are shown. The algebraic properties of our notions with respect to a group structure are given. In future work, we will solve equations \( \psi_m(x)^d = \alpha \) in \((\psi_m, A)\) (SSG) for some \( d \in \mathbb{Z} \) and \( \alpha \in A \). That means we will find the solution set \( S = \{ x \mid \psi_m(x)^d = \alpha \} \).

References

[1] D. Molodtsov, Soft set theory—First results, Comput. Math. Appl. 37(4–5) (1999), 19–31.
[2] I. Zorlutuna, M. Akdag, K. W. Min, and S. Atmaca, Remarks on soft topological spaces, Annals of Fuzzy Math. and Info., 3(2) (2012), 171-185.
[3] S. Mahmood, Decision making using algebraic operations on soft effect matrix as new category of similarity measures and study their application in medical diagnosis problems, Journal of Intelligent & Fuzzy Systems, DOI: 10.3233/JIFS-179249, IOS Press, (2019).
[4] S. Mahmood, Soft Regular Generalized b-Closed Sets in Soft Topological Spaces, Journal of Linear and Topological Algebra, 3(4),(2014), 195–204.
[5] S. Mahmood, On intuitionistic fuzzy soft b-closed sets in intuitionistic fuzzy soft topological spaces, Annals of Fuzzy Mathematics and Informatics, 10(2), (2015), 221-233.
[6] S. Mahmood, Dissimilarity Fuzzy Soft Points and their Applications, Fuzzy Information and Engineering, 8(3), (2016), 281-294
[7] S. Mahmood, Soft Sequentially Absolutely Closed Spaces, International Journal of Applications of Fuzzy Sets and Artificial Intelligence, 7 (2017), 73-92.
[8] S. Mahmood, Tychonoff Spaces in Soft Setting and their Basic Properties, International Journal of Applications of Fuzzy Sets and Artificial Intelligence, 7 (2017), 93-112
[9] S. Mahmood, M. Abd Ulrazaq, New Category of Soft Topological Spaces, American Journal of Computational and Applied Mathematics, 8(1), (2018), 20-25.
[10] N. M. Ali Abbas, S. Mahmood, H. J. Sadiq, A Note on the New Closed Classes and GP-Regular Classes in Soft System, Journal of Southwest Jiaotong University, 54(1), (2019)-01-0091-06, (doi : 10.3969/j. issn. 0258-2724.2019.155).
[11] A. Firas Muhamad Al-Musawi, S. Mahmood, M. Abd Ulrazaq, Soft (1,2)-Strongly Open Maps in Bi-Topological Spaces, IOP Conference Series: Materials Science and Engineering, 571 (2019) 012002, doi:10.1088/1757-899X/571/1/012002.
[12] S. Mahmood, S. A. Abdul-Ghani, Soft M-Ideals and Soft S-Ideals in Soft S-Algebras, J. Phys.: Conf. Ser., 1234 (2019) 012100, doi:10.1088/1742-6596/1234/1/012100.
[13] S. Mahmood and N. M. A. Abbas, New Classes in Soft Closed Setting Using Soft Bi-topological Spaces, International Journal of Applications of Fuzzy Sets and Artificial Intelligence, 9(2019), 109 – 117.
[14] A. Rosenfeld, “Fuzzy groups,” Journal of Mathematical Analysis and Applications, vol. 35, pp. 512–517, 1971.
[15] S. Mahmood Khalil, M. Ulrazaq, S. Abdul-Ghani, Abu Firas Al-Musawi, \( \sigma \)-Algebra and \( \sigma \)-Baire in Fuzzy Soft Setting, Advances in Fuzzy Systems, Volume 2018, Article ID 5731682, 10 pages.
[16] S. Mahmood and Z. Al-Batat, Intuitionistic Fuzzy Soft LA-Semigroups and Intuitionistic Fuzzy Soft Ideals, International Journal of Applications of Fuzzy Sets and Artificial Intelligence, 6 (2016), 119-132.
[17] S. Mahmood and M. A. Alradha, Soft Edge \( \rho \)-Algebras of the power sets, International Journal of Applications of Fuzzy Sets and Artificial Intelligence, 7, (2017), 231-243
[18] S. Mahmood and A. Muhamad, Soft BCL-Algebras of the Power Sets, International Journal of Algebra, 11(7),(2017), 329 – 341.
[19] S. Mahmood, M. Abd Ulrazaq, Soft BCH-Algebras of the Power Sets, American Journal of Mathematics and Statistics, 8(1), (2018), 1-7.
[20] S. Mahmood, F. Hameed, An algorithm for generating permutation algebras using soft spaces, Journal of Taibah University for Science, 12(3), (2018), 299-308.
[21] S. Mahmood, Generating BCK-Fuzzy Topological Spaces Using Γ – Fuzzy BCK-Algebras, International Journal of Applications of Fuzzy Sets and Artificial Intelligence, Vol. 9 (2019), 97-10.
[22] S. Mahmood and F. Hameed, An algorithm for generating permutations in symmetric groups using soft spaces with general study and basic properties of permutations spaces. J Theor Appl Inform Technol, 96(9) (2018), 2445-2457.
[23] S. Adnan Abdul-Ghani, S. M. Khalil, Mayadah Abd Ulrazaq, and Abu Firas Muhammad Jawad Al-Musawi, New Branch of Intuitionistic Fuzzification in Algebras with Their Applications, International Journal of Mathematics and Mathematical Sciences, Volume 2018, Article ID 5712676, 6 pages.
[24] N. M. Ali Abbas and S. M. Khalil, On α∗ – Open Sets in Topological Spaces, IOP Conference Series: Materials Science and Engineering, 571 (2019) 012021, doi:10.1088/1757-899X/571/1/012021.
[25] S. Mahmood, New category of the fuzzy d-algebras, Journal of Taibah University for Science, 12(2), (2018), 143-149.
[26] S. Mahmood and F. Hameed, Applications of Fuzzy ρ-Ideals in ρ-Algebras. Soft Computing, (2020), To appear.
[27] S. M. Khalil and N. M. Ali Abbas, Characteristics of the Number of Conjugacy Classes and P-Regular Classes in Finite Symmetric Groups, IOP Conference Series: Materials Science and Engineering, 571 (2019) 012007, doi:10.1088/1757-899X/571/1/012007
[28] D. Zeindler, Permutation matrices and the moments of their characteristic polynomial, Electronic Journal of Probability, 15(34), (2010),1092-1118.
[29] C. Yang, A not on soft set theory, Computers and Mathematics with Applications, 56 (2008) 1899–1900.
[30] P. K. Maji, R. Biswas and A. R. Roy, Soft Set Theory, Comput. Math. Appl., 45 (2003), 555-562.
[31] F. Feng, Y. B. Jun, and X. Zhao, Soft semirings, Computers and Mathematics with Applications, 56 (10) (2008), 2621-2628.
[32] S. Mahmood and A. Rajah, Solving Class Equation \( x^d = \beta \) in an Alternating Group for each \( \beta \in H \bigcap C^\alpha \) and \( n \notin \emptyset \), journal of the Association of Arab Universities for Basic and Applied Sciences, 10(1), (2011), 42-50.
[33] S. Mahmood and A. Rajah, Solving Class Equation \( x^d = \beta \) in an Alternating Group for all \( n \in \emptyset \) & \( \beta \in H_n \bigcap C^\alpha \), journal of the Association of Arab Universities for Basic and Applied Sciences, 16, (2014), 38–45.
[34] G. D. James and A. Kerber, The Representation Theory of the Symmetric Group. Addison-Wesley Publishing, Cambridge University Press (1984).