MAX-PLUS DEFINITE MATRIX CLOSURES AND THEIR EIGENSACES

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Abstract. In this paper we introduce the definite closure operation for matrices with finite permanent, reveal inner structures of definite eigenspaces, and establish some facts about Hilbert distances between these inner structures and the boundary of the definite eigenspace.

1. Introduction

This paper is a contribution to geometrical understanding of some algebraic results on max-plus eigenspaces that were obtained by P. Butković in [2] (see also [3]). The sources of geometrical inspiration for our work are [5] and [13], as well as [8] and [9]. Our approach is closer to that of [8] and [9], since we think of max-plus semiring with its simplifying total order rather than of generalized algebraic structures of idempotent analysis. However, our viewpoint differs from that of [8] and [9] in that we use basic tools of max-algebra instead of the more sophisticated machinery of convex geometry.

The paper is organized as follows. In Sect. 2 we recall the basic tools of max-algebra that we need. In Sect. 3 we introduce definite forms for max-plus matrices with nonzero (finite) permanent and prove that closures of all definite forms of a given matrix coincide. Thus we can introduce the ‘definite closure’ of any max-plus matrix with nonzero permanent. We also introduce definite eigenspaces, make some observations on systems of inequalities that define them, and consider an application to the cellular decomposition introduced in [9]. In Sect. 4 we use a representation, due to [2], of the definite eigenspace that reveals some inner structures. Then we establish some facts about Hilbert distances between these inner structures and the boundary of the eigenspace.

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2. Some tools of max-algebra

In its ordinary setting, max-algebra is linear algebra over the semiring $\mathbb{R}_{max}$. This semiring is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the operations of ‘addition’ $\oplus = \max$ and ‘multiplication’ $\odot = +$. Its ‘zero’ 0 is equal to $-\infty$ and its ‘unity’ 1 is equal to 0.

The semiring $\mathbb{R}_{max}$ resembles $\mathbb{R}^+$, the positive part of the field of real numbers: in both structures multiplication admits inverses and enjoys distributivity over...
addition, but subtraction is not allowed. In \( \mathbb{R}^{\max} \), however, the addition \( \oplus \) is idempotent, i.e. for any \( \alpha \in \mathbb{R}^{\max} \) we have \( \alpha \oplus \alpha = \alpha \). In \( \mathbb{R}^{+} \) this is certainly not the case. The semiring \( \mathbb{R}^{\max} \) can be seen as an idempotent ‘dequantization’ of \( \mathbb{R}^{+} \) (see, e.g., [12] and [11]).

In max-algebra, it is also possible to exponentiate and to take roots. These operations are nothing but conventional multiplication and conventional division, respectively. Indeed, for any \( \alpha \neq 0 \) one has \( \alpha^{n} = \alpha \times n \) and \( \alpha^{\frac{1}{n}} = \alpha / n \) (for the remaining case \( 0^{n} = 0^{\frac{1}{n}} = 0 \) since we assume that \( \alpha \oplus 0 = 0 \) for any \( \alpha \)).

One of the principal objects of max-algebra is \( \mathbb{R}^{\max}_{n} \), the set of \( n \)-component vectors with components in \( \mathbb{R}^{\max} \). This set is equipped with ‘addition’ \( (x \oplus y)_{i} = x_{i} \oplus y_{i} \) and with ‘multiplication’ by any max-plus scalar (i.e. by any element of \( \mathbb{R}^{\max} \)) \( (\alpha \odot y)_{i} = \alpha \odot y_{i} \). The set \( \mathbb{R}^{\max}_{n} \) equipped with these operations, as well as subsets of \( \mathbb{R}^{\max}_{n} \) closed under these operations, will be called max-plus spaces. Below the notation \( \odot \) will be frequently omitted.

These max-plus spaces resemble linear spaces in that the laws of associativity and distributivity hold, but again there is no subtraction and there is idempotency of addition. Structures of this kind are called idempotent semimodules and are a central object of the study in the idempotent analysis, see [11].

A max-plus space \( S \subset \mathbb{R}^{\max}_{n} \) is said to be finitely generated if there is a set of vectors \( \{v_{1}, \ldots, v_{s}\} \) such that for any \( y \in S \) one can find scalars \( \alpha_{1}, \ldots, \alpha_{s} \) such that \( y = \bigoplus_{i=1}^{s} \alpha_{i}v_{i} \). The set \( \{v_{1}, \ldots, v_{s}\} \) is the generating set of \( S \). It is minimal if no \( v_{i} \) can be expressed as a linear combination of the other generators, i.e. if there are no equalities of the form

\[
(1) \quad v_{i} = \bigoplus_{j \neq i} \alpha_{j}v_{j}.
\]

The minimal generating sets will be also called bases.

The following crucial result is due to Moller [15] and to Wagneur [17, 18] (it is also contained in [9] and [10]).

**Proposition 1.** If \( \{u_{1}, \ldots, u_{s}\} \) and \( \{v_{1}, \ldots, v_{t}\} \) are two bases of a max-plus space, then \( s = t \) and there is a permutation \( \sigma \) and a set of nonzero scalars \( \{\alpha_{1}, \ldots, \alpha_{s}\} \) such that \( u_{i} = \alpha_{i}v_{\sigma(i)} \) for all \( i = 1, \ldots, s \).

Prop. 1 means that if we have found a finite base for a max-plus space, then it is in some sense unique: we can only multiply the vectors of the base by nonzero scalars. For more information on max-plus bases we refer the reader to [7].

The max-plus matrix algebra is formally analogous to the conventional matrix algebra (minus subtraction and plus idempotency): \( (A \oplus B)_{ij} = A_{ij} \oplus B_{ij} \) and \( (A \odot B)_{ij} = \bigoplus_{k} A_{ik} \odot B_{kj} \).

Let us introduce two important characteristics of max-plus matrices. The first characteristic deals with the cyclic permutations. Let \( A \) be an \( n \times n \) max-plus matrix. Here and below \( N \) will stand for the \( n \)-element set \( \{1, \ldots, n\} \). Denote by \( C_{n} \) the set of all cyclic permutations \( \tau \) that act on the subsets of the set \( N \). For \( \tau \in C_{n} \) denote by \( K(\tau) \) the subset on which \( \tau \) acts and by \( |K(\tau)| \) the number of elements in this subset. Then

\[
\lambda(A) = \bigoplus_{\tau \in C_{n}} (\odot \ A_{i\tau(i)})^{\frac{|K(\tau)|}{n}}
\]
is the maximal cycle mean of the matrix \( A \) (the notation \( \lambda(A) \) for the maximal cycle mean of \( A \) will be used throughout the paper). The summand \( \bigoplus_{\tau \in K(\tau)} A_{\tau(i)} \) is called the cycle mean of the cyclic permutation \( \tau \).

The cyclic permutation whose cycle mean is maximal will be called critical.

The second characteristic deals with the permutations of \( N \). For any square \( n \times n \) max-plus matrix \( A \), its permanent is defined as

\[
\text{per}(A) = \bigoplus_{\sigma \in S_n} \bigoplus_{i=1}^{n} A_{\sigma(i)}.
\]

Here \( S_n \) is the group of all permutations of \( N \).

The summand \( \bigoplus_{i=1}^{n} A_{\sigma(i)} \) is called the weight of the permutation \( \sigma \), so the max-plus permanent is the maximal weight of all permutations of \( N \). The permanent of \( A \) is said to be strong (following [3]), if \( A \) has only one maximal permutation, i.e., only one permutation with maximal weight.

In the papers [8] and [9] the max-plus permanent is called the ‘tropical determinant’. To some extent the max-plus permanent can overtake the role that the usual determinant plays, see [3] and the just mentioned papers for details. We also refer the reader to [1] and [10], for a symmetrized version of max-algebra, which admits subtraction and determinants.

An \( n \times n \) max-plus matrix \( A \) is called invertible, if there is another \( n \times n \) max-plus matrix \( B \) such that the products \( AB \) and \( BA \) are both equal to the max-plus identity matrix \( I \). Prop. [11] implies that \( A \) is invertible iff there is a permutation \( \sigma \) and a set of nonzero scalars \( \alpha_1, \ldots, \alpha_n \) such that

\[
A_{ij} = \begin{cases} 
\alpha_i, & \text{if } j = \sigma(i); \\
0, & \text{otherwise.}
\end{cases}
\]

So the class of invertible matrices in max-algebra is very small. But it makes sense to calculate the series

\[
A^* = I \oplus A \oplus A^2 \oplus \ldots,
\]

where \( I \) is the max-plus identity matrix. If the sum of this series exists, then it is called the max-plus algebraic closure of \( A \). It is an obvious analogue of \( (I - A)^{-1} \).

Powers of max-plus matrices and max-plus closures play important role in optimization on graphs. Indeed, if we associate an \( n \)-node graph with an \( n \times n \) matrix \( A \) and let \( A_{ij} \) be the weight of the edge \((i,j)\) of this graph, then the entry \((A^m)_{ij}\) of the matrix \(A^m\) will represent the maximal weight of paths of length \( m \) running from \( i \) to \( j \). The entry \((A^m)_{ii}\) is the maximal weight of all cyclic paths, i.e. cycles, of length \( m \) that traverse \( i \). In other words, it is the maximal weight of the cyclic permutations \( \tau \) such that \( i \in K(\tau) \) and \( |K(\tau)| = m \). Analogously, \((A^*)_{ij}\), for \( i \neq j \), is the maximal weight of all paths of any length running from \( i \) to \( j \). The path from \( i \) to \( j \) whose weight equals \((A^*)_{ij}\) is called optimal.

The following proposition, due to Carré [4], solves the problem of existence of the closure.

**Proposition 2.** The closure of the matrix \( A \) exists if and only if \( \lambda(A) \leq 1 \).

Max-plus closures enjoy the property

\[
(A^*)^2 = A^*.
\]

Otherwise stated, the inequality \( A^*_{ij} A^*_{jk} \leq A^*_{ik} \) holds for all \( i,j, \) and \( k \).
As $A^*$ is an equivalent of $(I - A)^{-1}$, some algorithms of linear algebra can be adapted to calculate closures (that is, all optimal paths on a graph) in max-algebra, see [3], [4], and [16].

Another important tool is the max-plus spectral theory. For $A$, an $n \times n$ max-plus square matrix, the max-plus spectral problem consists in finding an $n$-element vector $x$, such that not all of its components are 0, and a scalar $\lambda$ such that

$$Ax = \lambda x.$$  

The scalar $\lambda$ is a max-plus eigenvalue, and the vector $x$ is a max-plus eigenvector. The set of max-plus eigenvectors associated with $\lambda$ is closed under addition and multiplication by any nonzero scalar. Therefore it is called the max-plus eigenspace (associated with $\lambda$). The eigenspace associated with the maximal eigenvalue of $A$ will be denoted by eig($A$), and the space generated by the columns of $A$ will be denoted by span($A$).

One of the main results on max-plus spectral theory is the following.

**Proposition 3.** The maximal eigenvalue of any max-plus square matrix is equal to its maximal cycle mean.

The eigenspace associated with the maximal eigenvalue is easy to describe.

**Proposition 4.** Let $A$ be an $n \times n$ square max-plus matrix such that $\lambda(A) = 1$. Then

1) eig($A$) is generated by the columns $A^*_j$ of $A^*$ such that $AA^*_j = A^*_j$;
2) $AA^*_j = A^*_j$ iff there is a critical cyclic permutation $\tau$ such that $j \in K(\tau)$.

Columns of $A^*$ corresponding to vertices of the same cycle are proportional (in the max-plus sense) to each other.

**Proposition 5.** Let $A$ be an $n \times n$ max-plus matrix such that $\lambda(A) = 1$ and let $\tau$ be a critical cyclic permutation. Then for any $l \in N$ and $i \in K(\tau)$ we have

$$A^*_il = A_{\tau(i)}^*A_{\tau(i)}^{-1},$$
$$A^*_li = A_{\tau^{-1}(i)}^*A_{\tau^{-1}(i)}^{-1}.$$  

If the graph associated with $A$ is strongly connected (i.e. for any $i$ and $j$ there is a path with nonzero weight running from $i$ to $j$), then $A$ is said to be irreducible. In this case the maximal cycle mean of $A$ is known to be its only eigenvalue. In the reducible case this eigenvalue need not be unique. If another eigenvalue exists, then it is the (only) eigenvalue of some maximal irreducible submatrix of $A$. However, not all maximal irreducible submatrices of $A$ yield eigenvalues for $A$.

For more details on max-plus spectral theory, as well as for the proofs of the above propositions, we refer the reader to [1], [6], [10], and [19].

3. **Definite eigenspaces and definite closures**

We begin this section with the following important proposition. The proof makes use of the uniqueness of the base (Prop. [1]).

**Proposition 6.** Let $A$ and $B$ be square max-plus matrices such that $\lambda(A) \leq 1$, and $\lambda(B) \leq 1$. Then span($A^*$) = span($B^*$) if and only if $A^* = B^*$.
Proof. First let us prove that whenever
\[ A^*_i = \bigoplus_{j \neq i} \alpha_j A^*_j, \]
the column \( A^*_i \) is proportional to \( A^*_j \) for some \( j \neq i \).

Suppose (1) holds. Then there is an \( l \) such that \( A^*_i = 1 = \alpha_l A^*_l \) and that \( A^*_l \geq \alpha_l \). Combining this we obtain that \( A^*_l A^*_l \geq 1 \), hence \( A^*_l A^*_l = 1 \) (otherwise there is a cycle whose weight exceeds 1). This means that there is a critical cycle with weight 1 traversing \( i \) and \( l \), and due to Prop. (2) \( A^*_i = \alpha_l A^*_l \). We conclude that no column of \( A^* \) can be expressed as linear combination (1) without being proportional to some of the columns involved in this combination.

Further let \( \{ A^*_{r_1}, \ldots, A^*_{r_k} \} \) be the base of \( \text{span}(A^*) = \text{span}(B^*) \). If we use columns of \( B^* \) to form the base, then, due to Prop. (3) it must be of the form \( \{ B^*_{s_1}, \ldots, B^*_{s_k} \} \), so that \( B^*_{s_i} = \alpha_i A^*_{r(i)} \) for \( i = 1, \ldots, k \) and some nonzero \( \alpha_i \). All remaining columns of \( A^* \) are proportional to base columns of \( A^* \) of \( B^* \). So every column of \( A^* \) is proportional to some column of \( B^* \), and vice versa. This implies that the rows \( A^*_i \) and \( A^*_j \) are proportional iff the rows \( B^*_i \) and \( B^*_j \) are proportional.

Let the columns \( A^*_i \) and \( A^*_j \) be proportional. Then \( A^*_i A^*_j = 1 \), hence there is a critical cycle containing \( i \) and \( j \). Due to Prop. (2) the rows \( A^*_i \) and \( A^*_j \) are proportional, so are the rows \( B^*_i \) and \( B^*_j \), and, again due to Prop. (3) so are the columns \( B^*_i \) and \( B^*_j \). We conclude that the columns \( A^*_i \) and \( A^*_j \) are proportional iff so are the columns \( B^*_i \) and \( B^*_j \).

Now it is clear that \( \{ A^*_{r_1}, \ldots, A^*_{r_k} \} \) is the base of \( \text{span}(A^*) \) iff \( \{ B^*_{r_1}, \ldots, B^*_{r_k} \} \) is also the base, so that \( B^*_{r_i} = \alpha_i A^*_{r(i)} \). We can assume w.l.o.g. that \( r_1 = i \). Consider the decomposition of \( \sigma \) into cyclic permutations and let \( \tau \) be one of them. Then \( B^*_{r(i)} = \alpha_i A^*_{r(i)} = \alpha_i^{-1} \) for all \( i \in K(\tau) \). If \( \bigcup_{i \in K(\tau)} \alpha_i > 1 \), then \( B^* \) has a cycle with weight greater than 1, and if \( \bigcup_{i \in K(\tau)} \alpha_i < 1 \), then so does \( A^* \). The only remaining possibility \( \bigcup_{i \in K(\tau)} \alpha_i = 1 \) implies that all columns of \( A^* \) and \( B^* \) with indices belonging to \( K(\tau) \) must be a singleton for any \( \tau \), otherwise the minimality of the bases is violated. This implies that \( \sigma \) is the identity permutation, and \( A^*_i = B^*_i \) for any \( i = 1, \ldots, k \). Taking into account that there is a one-to-one correspondence between the sets of proportional columns of \( A^* \) and \( B^* \), and that all columns of \( A^* \) and \( B^* \) are proportional to some base columns, we conclude that \( A^* = B^* \).

Now consider matrices with maximal cycle mean equal to 1 and with all diagonal entries equal to 1. Following (4), we call such matrices definite. The following proposition contains some simple facts on eigenspaces of such matrices. The third statement is an easy consequence of (5), Ch. IV, Th. 2.2.4, and its proof is recalled here for convenience of the reader. We consider the general reducible case, in which the eigenvectors may have zero entries. For any \( y \in \mathbb{R}_{\max}^n \), the index set \( K \) such that \( y_i \neq 0 \) iff \( i \in K \) is called the support of \( y \) and is denoted by \( \text{supp}(y) \).

Proposition 7. If \( A \) is a definite matrix, then

1) it has a unique eigenvalue equal to 1;
2) \( \text{eig}(A) = \text{span}(A^*) \);
3) an eigenvector with the support \( K \subset N \) exists iff \( A_{ij} = 0 \) for all \( i \in N \setminus K \) and \( j \in K \).
Proof. 1) Any eigenvalue of $A$ is the maximal cycle mean of some of its submatrices, and the maximal cycle mean of any submatrix of $A$ is equal to 1.

2) follows from Prop. 4.

3) If an eigenvector $x$ such that supp$(x) = K$ exists, then $A_{ij}x_j = 0$ for all $i \in N \setminus K$ and all $j \in K$. Hence $A_{ij} = 0$ for all such $i$ and $j$.

Conversely, if the set $K$ satisfies the condition, we look for eigenvectors $x$ such that supp$(x) = K$. We may reduce the system $Ax = x$ to the system $A_{KK}x_K = x_K$, where $A_{KK}$ is a submatrix of $A$ standing on the rows and columns with indices belonging to $K$, and $x_K$ is a vector with $|K|$ nonzero components. The space eig$(A_{KK})$ is generated by the columns $(A_{KK}^*)_j$, where $j \in K$. Taking any combination of all these generators with all coefficients not equal to 0 we obtain an eigenvector with the support $K$.

In the case of definite matrices we have one more implication of the uniqueness of the base. The proof is similar to that of Prop. 6.

Proposition 8. If $A$ is definite and $\text{span}(A) = \text{eig}(A)$, then $A = A^*$. 

Proof. Let $\{A_{s_1}, \ldots, A_{s_k}\}$ be the base of $\text{span}(A) = \text{eig}(A) = \text{span}(A^*)$. Due to Prop. 4 if we use columns of $A^*$ to form the base, then it must be of the form $\{A_{t_1}, \ldots, A_{t_k}^*\}$ so that $A_{s_i} = \alpha_i A_{t_i}^*$ for $i = 1, \ldots, k$ and some nonzero $\alpha_i$. More precisely, $\alpha_i = A_{t_i,s_i}$, and this implies $A_{t_i,s_i} A_{s_i,t_i} = 1$. Then there is a critical cycle traversing $s_i$ and $t_i$, so $A_{s_i} = A_{s_i} A_{t_i}^*$ according to Prop. 5. So $A_{s_i} = A_{s_i}^*$ for all $i = 1, \ldots, k$.

Now assume that there are columns $A_{s_i}$ and scalars $\alpha_i$ such that

$$A_j = \sum_{i=1}^{k} \alpha_i A_{s_i},$$

and that no column of the base is proportional to $A_j$. It follows from 7 that there is an index $m$ such that $A_{jj} = 1 = \alpha_m A_{js_m}$. The columns $A_{s_j}$ and $A_{s_m}$ are not proportional, hence there is an $l$ such that $A_{lj} > \alpha_m A_{ls_m}$. This implies $A_{lj} A_{js_m} > A_{ls_m}$ and $A_{js_m}^* > A_{ls_m}$. This is a contradiction, since we have proved that $A_{s_i}^* = A_{s_i}$ for any $i = 1, \ldots, k$. So any column of $A$ is proportional to some column of the base. But if $A_{s_i}$ and $A_{s_j}$ are proportional, then $A_{s_i} A_{s_j} = 1$, hence $A_i^*$ and $A_j^*$ are also proportional with the same coefficient. This implies $A = A^*$. 

Let us now introduce a definite form of a matrix. Consider an $n \times n$ max-plus matrix $A$ that has nonzero permanent, i.e., at least one permutation whose weight is not equal to 0. Let $\sigma$ be one of the maximal permutations of $A$. If this permutation is not the identity permutation, then we turn this permutation into the identity permutation by rearranging the columns of $A$. Then we divide (in the max-plus sense) all columns by the corresponding diagonal entries, thus obtaining a matrix $A'$ with entries

$$A'_{ij} = A_{\sigma(i)j} A_{j\sigma(j)}^{-1}.$$

Obviously, passing to a definite form of a matrix does not alter its span: $\text{span}(A') = \text{span}(A)$. 

It is clear that $A'$ is definite, hence we call it the definite form of $A$ corresponding to the permutation $\sigma$. For example, if

$$A = \begin{pmatrix} 1 & 5 & -\infty \\ 2 & 1 & 7 \\ 6 & -\infty & 2 \end{pmatrix},$$

then

$$A' = \begin{pmatrix} 0 & -\infty & -5 \\ -4 & 0 & -4 \\ -\infty & -5 & 0 \end{pmatrix}$$

is the definite form of $A$ corresponding to $\sigma = (231)$. (Here and in the sequel we prefer not to use the symbols 0 and 1 in numerical examples.)

However, in general there are many maximal permutations and many definite forms corresponding to them. The fact that we want to prove is that closures of all definite forms coincide. It is convenient to pose the problem as follows. Let $A$ be definite, let it have maximal permutations different from the identity permutation, and let $\sigma$ be one of them. Let $A'$ be the definite form of $A$ corresponding to $\sigma$, then its entries are defined according to (10). $A'$ also has maximal permutations different from the identity permutation, and $\sigma^{-1}$ is one of them, since

$$A'_{\sigma^{-1}(i)} = A_{\sigma^{-1}(i)i}^{-1}.$$

First we prove the following proposition.

**Proposition 9.** Let $A$ be a definite matrix and let $\sigma$ be one of its maximal permutations. For $A'$ the definite form of $A$ corresponding to $\sigma$, $\text{eig}(A') = \text{eig}(A)$.

**Proof.** Consider the decomposition of $\sigma$ into cyclic permutations. Let $r$ be the number of these permutations and denote these permutations by $\tau_l$ (where $l = 1, \ldots, r$). Denote by $K(\tau_l)$ the index set on which $\tau_l$ acts. The sets $K(\tau_l)$ are pairwise disjoint and $\bigcup_{l=1}^r K(\tau_l) = N$.

We prove the inclusion $\text{eig}(A) \subset \text{eig}(A')$. Let $y$ be an eigenvector of $A$ with the support $M \subset N$. Due to the third statement of Prop. 7 the set $M$ must be of the form $\bigcup_{l \in L} K(\tau_l)$ for some $L \subset \{1, \ldots, r\}$, and we have $A_{ij} = 0$ for all $i \in N \setminus M$ and $j \in M$.

Now note that every cyclic permutation $\tau_l$ is critical, hence for any $\tau_l$ such that $K_l \in M$ we have

$$(12) \quad \bigcirc_{i \in K(\tau_l)} A_{\tau_l(i)y_{\tau_l(i)}y_{\tau_l(i)^{-1}}} = 1.$$

On the other hand we have $A_{\tau_l(i)y_{\tau_l(i)}y_{\tau_l(i)^{-1}}} \leq y_i$. This inequality is an equality, otherwise the violation of (12) occurs. Thus we obtain

$$(13) \quad A_{\tau_l(i)y_{\tau_l(i)}} = y_i$$

for all $i \in M$.

Now note that, since $\sigma(j) \in M$ for all $j \in M$, we have that $A'_{ij} = 0$ for all $i \in N \setminus M$ and $j \in M$, the same as for $A_{ij}$. Therefore it suffices to prove that if $A_{MM}y_M = y_M$, then $A'_{MM}y_M = y_M$. In other words, we want to show that
Proposition 11. Closures of all definite forms of any matrix with nonzero permanent coincide.

We have just proved that closures of all definite forms coincide. Now we can define the definite closure of any $n \times n$ max-plus matrix with nonzero permanent to be the closure of any of its definite forms. Due to the second statement of Prop. 7 for any definite form $A'$ of $A$ we have $\text{eig}(A') = \text{eig}((A')^*) = \text{span}((A')^*)$. So the eigenspace of the definite closure of $A$ coincides with the eigenspace of any of its definite forms (they are all the same), and is generated by columns of the definite closure. The eigenspace of definite closure will be called the definite eigenspace.

The third statement of Prop. 7 suggests that, if we want to work with the space of eigenvectors of definite closure of non-full support, then we must confine ourselves to the corresponding submatrix.

Further we always assume that the eigenvectors considered have full support, i.e., that we study the eigenvectors with certain support and have passed to the corresponding submatrix.

Let us show that the definite eigenspace can be described by some system of inequalities.

Proposition 10. Let $A$ be a definite matrix. Then its eigenspace (and the eigenspace of its closure) is the set $X = \{ x \mid A_{ij} \leq x_i x_j^{-1}, i \neq j, A_{ij} = A_{ij}^* \}$

Proof. Let $x$ be an eigenvector of $A$ corresponding to its maximal eigenvalue. It satisfies the system $Ax = x$, therefore it satisfies all inequalities of the form $A_{ij} \leq x_i x_j^{-1}$. Hence $x \in X$.

Conversely, if all inequalities $A_{ij} \leq x_i x_j^{-1}$, for $i \neq j$ and $A_{ij} = A_{ij}^*$, are satisfied, then absolutely all inequalities $A_{ij} \leq x_i x_j^{-1}$ are satisfied. Indeed, if $A_{ij} < A_{ij}^*$, then the optimal path from $i$ to $j$ is not the edge $(i, j)$, but it traverses other nodes, say, $i_1, \ldots, i_k$. Then $A_{ij}^* = A_{i_1 i_{i_1}} \cdots A_{i_k j}$ where $A_{i_1 i_{i_1}} = A_{i_k i_{i_k}}, \ldots, A_{i_k j} = A_{i_k j}^*$ (all edges $(i, i_1), \ldots, (i_k, j)$ are optimal paths). The inequality $A_{ij} \leq x_i x_j^{-1}$ is now an easy consequence of the inequalities $A_{i_1 i_{i_1}} \leq x_i x_{i_1}^{-1}, \ldots, A_{i_k j} \leq x_i x_j^{-1}$ that are satisfied. So all inequalities $A_{ij} \leq x_i x_j^{-1}$ are satisfied and this implies $Ax = x$. □

An inverse problem can also be posed. Suppose that there is a system of inequalities $\{ a_{ij} \leq x_i x_j^{-1} \}$, with at most one inequality per each pair $(i, j)$, and the set of vectors with full support defined by this system is not empty. Is it a full-support subspace of an eigenspace of a definite matrix?
To answer this question, consider the matrix $A$ whose entries $A_{ij}$ are equal either to 1, if $i = j$, or to $a_{ij}$, if there is an inequality of the form $a_{ij} \leq x_i x_j^{-1}$, or to 0, if there is no such inequality. Then we have the following proposition.

**Proposition 12.** The set $X = \{ x \mid A_{ij} \leq x_i x_j^{-1} \}$ is nonempty if and only if $A$ is definite.

**Proof.** If $A$ is definite then the set of its eigenvectors associated with the eigenvalue 1 is nonempty and, due to Prop. 11, it is precisely the set $X$ (some of inequalities being redundant).

Conversely, let $X$ be nonempty, then there exists $x \in X$. Take an arbitrary cyclic permutation $\tau$. As a consequence of all inequalities $A_{i\tau(i)} \leq x_i x_{\tau(i)}^{-1}$, where $i \in K(\tau)$, we obtain that $\bigoplus_{i \in K(\tau)} A_{i\tau(i)} \leq 1$. Hence all cycle means of $A$ are not greater than 1, and $A$ is definite. $\Box$

Now consider the following application.

Let $V$ be an $m \times n$ max-plus matrix with at least one nonzero entry in each column, and let $y \in \mathbb{R}_\text{max}^m$ have full support. Then, following [9], we can define the *combinatorial type* of $y$ with respect to $V$. It is an $m$-tuple $S = (S_1, \ldots, S_m)$ of subsets $S_j$ of \{1, \ldots, n\}, such that $i \in S_j$ whenever $\bigoplus_{k=1}^m V_{ki} y_k^{-1} = V_{ji} y_j^{-1}$. It is proved in [9] that the collection of the sets

$$(14) \quad X_S = \{ y \mid V_{ki} V_{ji}^{-1} \leq y_k y_j^{-1} \text{ for } j, k = 1, \ldots, m \text{ and } i \in S_j \}$$

defines a cellular decomposition of the full-support subspace of $\mathbb{R}_\text{max}^m$.

Consider an $m \times m$ matrix $V^S$ whose columns are defined by

$$(15) \quad V_{ij}^S = \bigoplus_{i \in S_j} V_{ji}^{-1} V_i,$$

if $S_j$ is not empty, and by

$$(16) \quad V_{ij}^S = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j. \end{cases}$$

if $S_j$ is empty. Then it is clear that

$$(17) \quad X_S = \{ y \mid V_{ij}^S \leq y_i y_j^{-1} \text{ for } i, j = 1, \ldots, m \},$$

and we immediately have the following proposition.

**Proposition 13.**

1) The cell $X_S$ is nonempty if and only if $\lambda(V^S) \leq 1$;

2) If $\lambda(V^S) \leq 1$, then $X_S = \{ y \in \text{span}((V^S)^*) \mid \text{supp}(y) = N \}$;

3) If $\lambda(V^S) \leq 1$, then $X_S = \{ x \mid V_{ij}^S \leq x_i x_j^{-1}, i \neq j, V_{ij}^S = (V^S)^*_{ij} \}$.

We close this section with three examples.

**Example 1** Consider the matrix

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 0 \\ 0 & -1 & -5 \end{pmatrix}. $$
The only maximal permutation of $A$ is (231), and the only definite form is

$$A' = \begin{pmatrix} 0 & 0 & 1 \\ -3 & 0 & 2 \\ -4 & -5 & 0 \end{pmatrix}.$$  

The definite closure of $A$ is

$$(A')^* = \begin{pmatrix} 0 & 0 & 2 \\ -2 & 0 & 2 \\ -4 & -4 & 0 \end{pmatrix}.$$  

Fig. 1 displays the cross section by $z = 0$ of $\text{span}(A)$ (left) and $\text{span}((A')^*) = \text{eig}(A')$ (right).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The definite closure operation for $A$}
\end{figure}

Note that $\text{eig}(A')$ is the set

$$\{ (x, y, z) \mid xy^{-1} \geq 0, yz^{-1} \geq 2, zx^{-1} \geq -4 \},$$  
in accordance with Prop. 11.

\textit{Example 2} Consider the matrix

$$B = \begin{pmatrix} 2 & 0 & 2 \\ 1 & 1 & 3 \\ 0 & -3 & -2 \end{pmatrix}.$$  

It has two maximal permutations: (13)(2) and (231). Therefore it has two definite forms, namely

$$B' = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ -4 & -4 & 0 \end{pmatrix}$$

and

$$B'' = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 1 \\ -3 & -5 & 0 \end{pmatrix}.$$  

But, in accordance with Prop. 11, the definite closure

$$(B')^* = (B'')^* = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & 3 \\ -3 & -4 & 0 \end{pmatrix}.$$
is unique.

Fig. 2 displays the cross section by \( z = 0 \) of \( \text{span}(B) \) (left) and \( \text{span}((B')^*) = \text{eig}(B') \) (right).

![Figure 2. The definite closure operation for \( B \)](image)

In accordance with Prop. 11, the space \( \text{eig}(B') \) is the set

\[
\{ (x, y, z) \mid yx^{-1} \geq 1, xy^{-1} \geq -1, xz^{-1} \geq 2, zy^{-1} \geq -4 \},
\]
or, equivalently, the set

\[
\{ (x, y, z) \mid yx^{-1} \geq 1, xy^{-1} \geq -1, xz^{-1} \geq 2, zx^{-1} \geq -3 \}.
\]
The first system of inequalities corresponds to the definite form \( B' \) and the second one corresponds to \( B'' \).

Comparing Fig. 1 with Fig. 2 we see that \( \text{eig}(A') \) has ‘interior’ whereas \( \text{eig}(B') \) does not have ‘interior’ (for the exact meaning of the term ‘interior’ see Sect. 4 below). As a consequence of [2], Th. 4.2, or [8], Th. 4.2, one can obtain that \( \text{eig}(A') \), for \( A' \) a definite form of \( A \), has ‘interior’ if and only if \( A \) (or equivalently \( A' \)) has strong permanent. This fact will be revisited in Prop. 14 of this paper.

**Example 3** Consider the matrix

\[
V = \begin{pmatrix} 1 & 4 & 6 & 7 \\ 4 & 1 & 5 & 8 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
\]

Let \( S = (\{2, 3\}, \{4\}, \{1\}) \), \( P = (\emptyset, \{4\}, \{1, 2, 3\}) \), \( U = (\{3\}, \{1, 3, 4\}, \{2\}) \), and \( W = (\{1, 4\}, \{2\}, \{3\}) \) be four combinatorial types. Do they exist in the cellular decomposition? If they do, what vectors generate the respective cells \( X_S, X_P, X_U, \) and \( X_W \)?

For \( S \):

\[
V^S = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 4 \\ -4 & -8 & 0 \end{pmatrix}, \quad (V^S)^* = \begin{pmatrix} 0 & -1 & 3 \\ 0 & 0 & 4 \\ -4 & -5 & 0 \end{pmatrix}.
\]

Hence \( X_S \) exists and is generated by \([4 \ 4 \ 0]^T\), \([4 \ 5 \ 0]^T\), and \([3 \ 4 \ 0]^T\).

For \( P \):

\[
V^P = \begin{pmatrix} 0 & -1 & 6 \\ -\infty & 0 & 5 \\ -\infty & -8 & 0 \end{pmatrix} = (V^P)^*.
\]
Hence $X_P$ is generated by $[0 - \infty - \infty]^T$, $[7 \ 8 \ 0]^T$ and $[6 \ 5 \ 0]^T$. However, $[0 - \infty - \infty]^T$ does not have full support and does not belong to $X_P$.

For $U$:

$$V^U = \begin{pmatrix} 0 & 1 & 4 \\ -1 & 0 & 1 \\ -6 & -4 & 0 \end{pmatrix}, \quad (V^U)^* = \begin{pmatrix} 0 & 1 & 4 \\ -1 & 0 & 3 \\ -5 & -4 & 0 \end{pmatrix}.$$ 

Hence $X^U$ exists and is generated by $[5 \ 4 \ 0]^T$ and $[4 \ 3 \ 0]^T$.

For $W$:

$$V^W = \begin{pmatrix} 0 & 3 & 6 \\ 3 & 0 & 5 \\ -1 & -1 & 0 \end{pmatrix}.$$ 

We have $\lambda(V^W) = 3 > 1$, hence $W$ does not exist in the cellular decomposition.

Fig. 3 displays $\text{span}(V)$ (blue), $X_S$ (red), $X_P$ (light grey), and $X_U$ (dark green), projected onto $z = 0$. The generators of $\text{span}(V)$ (the larger circles) and the generators of $X_S$ and $X_U$ (the smaller squares) are also shown.

**Figure 3.** Three cells of a cellular decomposition

4. **INNER STRUCTURES OF DEFINITE EIGENSPACES**

We have introduced the definite closure operation and have given an external description of a definite eigenspace in terms of a system of inequalities. In this section we give an internal description of definite eigenspace and measure Hilbert distances between the structures involved in this description and the boundary.

For further considerations we need the following notions and notation. Let $A$ be a definite matrix. The sets $X_{ij} = \{x \mid A_{ij} = x_i x_j^{-1}\}$ will be called the *supporting planes* of $\text{eig}(A)$, and the sets $\Gamma_{ij} = X_{ij} \cap \text{eig}(A)$ will be called the *faces* of $\text{eig}(A)$. The *boundary*, i.e., the union of all faces of $\text{eig}(A)$ will be denoted by $\Gamma(\text{eig}(A))$. The set of eigenvectors not belonging to the boundary will be called the *interior* of $\text{eig}(A)$ and denoted by $\text{int}(\text{eig}(A))$.

Also, denote by $\tilde{A}$ the matrix obtained from $A$ by replacing the diagonal $1$’s by $0$’s, and denote by $A_\mu$ the matrix obtained from $\mu^{-1}\tilde{A}$ by replacing the diagonal
0's by 1's. For example, if

\[ A = \begin{pmatrix} 0 & -4 & 1 \\ 1 & 0 & 1 \\ -5 & -7 & 0 \end{pmatrix} \]

then

\[ \tilde{A} = \begin{pmatrix} -\infty & -4 & 1 \\ 1 & -\infty & 1 \\ -5 & -7 & -\infty \end{pmatrix} \]

and, e.g.,

\[ A_{-1} = \begin{pmatrix} 0 & -3 & 2 \\ 2 & 0 & 2 \\ -4 & -6 & 0 \end{pmatrix} \]

Also, the maximal cycle mean of \( \tilde{A} \) is \( \lambda(\tilde{A}) = -1.5 \), and

\[ A_{-\lambda} = \begin{pmatrix} 0 & -2.5 & 2.5 \\ 2.5 & 0 & 2.5 \\ -3.5 & -5.5 & 0 \end{pmatrix} \]

(we write \( A_{\lambda} \) instead of \( A_{\lambda(\tilde{A})} \) for the sake of simplicity).

The following crucial proposition can be derived from [2].

**Proposition 14.** Let \( A \) be a definite matrix such that \( \lambda(\tilde{A}) \neq 0 \).

1) If \( A \) does not have strong permanent then eig(\( A \)) does not have interior;
2) If \( A \) has strong permanent then eig(\( A \)) has interior, and

\begin{equation}
\text{int(eig}(A)) = \bigcup_{\lambda(\tilde{A}) \leq \mu < 1} \text{eig}(A_{\mu}).
\end{equation}

**Proof.** 1) If \( A \) does not have strong permanent, then it has a maximal permutation that differs from the identity permutation. Let \( \sigma \) be such permutation, and assume that there is a \( y \) belonging to int(eig(\( A \))). Then \( A_{i\sigma(i)}y_{\sigma(i)} < y_i \) for all \( i \). After multiplying all these inequalities and then cancelling the product \( \bigodot y_i \) one obtains \( \bigodot A_{i\sigma(i)} < 1 \). This implies that \( \sigma \) is not maximal, a contradiction.

2) \( y \) belongs to int(eig(\( A \))) if and only if \( \bigoplus_{j \neq i} A_{ij}y_j < y_i \) for all \( i \). This takes place if and only if there is a \( \mu < 1 \) such that \( \bigoplus_{j \neq i} A_{ij}y_j \leq \mu y_i \), or, equivalently, \( A_{\mu}y = y \). If we take \( \mu \geq \lambda(\tilde{A}) \), then 1 is the maximal cycle mean of \( A_{\mu} \), hence eig(\( A_{\mu} \)) exists. The proof is complete. \( \square \)

If \( \lambda(\tilde{A}) = 0 \), i.e. if the graph associated with \( \tilde{A} \) is acyclic, then representation (18) is replaced by the representation

\begin{equation}
\text{int(eig}(A)) = \bigcup_{0 < \mu < 1} \text{eig}(A_{\mu}).
\end{equation}

In the sequel, we always assume that \( A \) is definite and has at least one off-diagonal entry not equal to 0.

According to Prop. 14, the eigenspace eig(\( A \)) is the set

\[ X = \{ x \mid A_{ij} \leq x_i x_j^{-1}, i \neq j, A_{ij} = A_{ij}^* \} \]

Analogously, the eigenspace eig(\( A_{\mu} \)), for any \( \mu \) involved in (18) or (19), is the set
Proposition 15. Let $\mu$ be a scalar such that $\lambda(\bar{A}) \leq \mu < 1$, if $\lambda(\bar{A}) > 0$, or such that $0 < \mu < 1$, if $\lambda(\bar{A}) = 0$. If $(A_\mu)_{ij} = (A_{\mu})_{ij}$, then $A_{\mu}^* = A_{ij}$.

Proof. In both cases considered the maximal cycle mean of $A_\mu$ is equal to 1, hence $A_\mu^*$ exists. Let $A_{ij}^* > A_{ij}$, then $A_{ij}^* = A_{i_1i_2} \cdots A_{i_kj}$ for some $i_1, \ldots, i_k$ not equal to $i$ or $j$. Since $\mu < 1$, we have $\mu^{-1}A_{ii_1} \cdots \mu^{-1}A_{i_kj} > \mu^{-1}A_{ij}$, hence $(A^*_\mu)_{ij} > (A_\mu)_{ij}$.

The eigenspaces $\text{eig}(A_\mu)$ are the inner structures mentioned above. Now we are going to measure the Hilbert distances between these inner structures and the boundary $\Gamma(\text{eig}(A))$.

The Hilbert distance between the two vectors $x$ and $y$ both having support $K$ is defined to be

$$d_H(x, y) = \bigoplus_{i,j \in K} x_ix_j^{-1}y_j^{-1}y_j.$$  

(20)

Note that in [5] the Hilbert distance is defined as an inverse of the quantity $d_H(x, y)$. If the supports of $x$ and $y$ differ, then we assume the Hilbert distance between $x$ and $y$ to be infinite.

It can be easily verified (see also [5], Th. 17) that the following properties hold:

1) $d_H(x, y) \geq 1$, and $d_H(x, y) = 1$ iff $x = \lambda y$, where $\lambda$ is a nonzero scalar;
2) $d_H(x, y) = d_H(y, x)$;
3) $d_H(x, y)d_H(y, z) \geq d_H(x, z)$.

In fact these properties show that $d_H$ is a semidistance (recall that $1 = 0$ and $\odot = +$). Indeed, $d_H(x, y) = 1 = 0$ whenever $x$ is equivalent to $y$ modulo $x \sim y \Leftrightarrow \exists \lambda \neq 0 : x = \lambda y$.

This semidistance is induced by the range seminorm

$$||x|| = \bigoplus_{i,j \in K} x_ix_j^{-1},$$  

(21)

introduced in [6], see also [7]. However, by a slight abuse of language we will refer to $d_H$ as to a distance.

Now we measure the distance between an arbitrary $y \in \text{eig}(A)$ and the supporting plane $X_{ij} = \{ x \ | x_ix_j^{-1} = A_{ij} \}$, i.e., the minimal distance between $y$ and $x \in X_{ij}$. From [5], Th. 18 it follows that this minimum is attained at the maximal vector of $X_{ij}$ not greater than $y$. Denote this vector by $y^{ij}$. Its coordinates are very easy to find:

$$y^{ij}_l = \bigoplus \{ x_l \ | x_l \leq y_l \} = y_l \text{ for } l \neq i, j;$$  

(22)

$$y^{ij}_i = \bigoplus \{ x_i \ | x_i \leq y_i, A^{-1}_{ij}x_i \leq y_j \} = A_{ij}y_j;$$

$$y^{ij}_j = \bigoplus \{ x_j \ | x_j \leq y_j, A_{ij}x_j \leq y_i \} = y_j.$$

The distance (20) between $y$ and $X_{ij}$ is then equal to

$$d_H(y, X_{ij}) = A^{-1}_{ij}y_j^{-1}y_i.$$  

(23)
However, what we need is the distance between \( y \) and \( \Gamma(\text{eig}(A)) \), i.e., the minimal distance between \( y \) and \( \Gamma_{ij} \). The following proposition makes our life simpler.

**Proposition 16.** The distance between \( y \in \text{eig}(A) \) and the boundary \( \Gamma(\text{eig}(A)) \) is equal to the minimal distance between \( y \) and supporting planes.

**Proof.** Clearly the minimal distance between \( y \) and supporting planes is not greater than the distance between \( y \) and \( \Gamma(\text{eig}(A)) \). Suppose \( X_{ij} \) is the supporting plane such that the distance between \( y \) and \( X_{ij} \) is minimal. This distance is equal to the distance between \( y \) and \( y^{ij} \). If \( y^{ij} \) belongs to \( \text{eig}(A) \) and hence to \( \Gamma_{ij} \) then we are done. Suppose not; then the system of equalities \( \bigoplus_{l} A_{kl}y_{kl}^{ij} = y_{k}^{ij} \) must be violated for some \( k \in N \). Note that, if \( k \neq i \), then \( y_{k}^{ij} = y_{k} \) (see (22)), and there is no violation. So the violation must take place for \( k = i \). There must be an \( l \) such that \( A_{il}y_{l}^{ij} > y_{l}^{ij} \), i.e., such that \( A_{il}y_{l} > A_{ij}y_{j} \). Now consider \( z \) such that \( z_{i} = A_{il}y_{l} \) and \( z_{k} = y_{k} \) for any \( k \neq i \). Then \( z \) belongs to \( X_{il} \) and \( d_{H}(y, z) = A_{il}^{-1}y_{l}^{-1}y_{l} \). This distance is strictly less than the distance between \( y \) and \( y^{ij} \), a contradiction. □

Consequently,

\[
d_{H}(y, \Gamma(\text{eig}(A))) = \bigwedge_{i \neq j, A_{ij} \neq 0} A_{ij}^{-1}y_{j}^{-1}y_{i}.
\]

The key idea of Prop. 17 below is that \( \lambda(\tilde{A})^{-1} \), if \( \lambda(\tilde{A}) \) is invertible, is the largest radius of Hilbert balls contained in \( \text{eig}(A) \). It can be said that \( \lambda(\tilde{A})^{-1} \) is the radius of inscribed Hilbert balls, as depicted on Fig. 5.

Let \( \tau \) be any critical cyclic permutation of \( \tilde{A} \).

**Proposition 17.**
1) In the case \( \lambda(\tilde{A}) > 0 \) for any \( y \in \text{eig}(A) \) the distance between \( y \) and \( \Gamma(\text{eig}(A)) \) is not greater than \( \lambda(\tilde{A})^{-1} \);
2) Let \( \mu \) be such that \( \lambda(\tilde{A}) \leq \mu < 1 \), if \( \lambda(\tilde{A}) > 0 \), or such that \( 0 < \mu < 1 \), if \( \lambda(\tilde{A}) = 0 \). Then for any \( y \in \text{eig}(A_{\mu}) \) the distance between \( y \) and \( \Gamma(\text{eig}(A)) \) is not less than \( \mu^{-1} \);
3) In the case \( \lambda(\tilde{A}) > 0 \), for any \( i, j \in K(\tau) \) such that \( j = \tau(i) \), and any \( y \in \text{eig}(A_{\lambda}) \), the distance between \( y \) and the face \( \Gamma_{ij} \) is equal to \( \lambda(\tilde{A})^{-1} \).

**Proof.**
1) The distance between \( y \) and \( \Gamma(\text{eig}(A)) \) does not exceed the minimal distance between \( y \) and supporting planes \( X_{ij} \) that correspond to the edges \((i, j)\) of the cyclic path determined by \( \tau \), and this minimal distance is not greater than \( \lambda(\tilde{A})^{-1} \):

\[
d_{H}(y, \Gamma(\text{eig}(A))) \leq \bigwedge_{i \in K(\tau)} A_{\tau(i)}^{-1}y_{\tau(i)}^{-1}y_{i} \leq \\
\leq (\bigcap_{i \in K(\tau)} A_{\tau(i)}^{-1})^{\frac{1}{\lambda(\tilde{A})}} = \lambda(\tilde{A})^{-1}.
\]

2) If \( y \in \text{eig}(A_{\mu}) \) then, since \( \text{eig}(A_{\mu}) = \text{span}(A_{\mu}^{*}) \), we have

\[
y = \bigoplus_{k \in M} \alpha_{k}(A_{\mu}^{*})._{k},
\]

where \( M = \text{supp}(\alpha) \). Substituting (25) into (24), we get

\[
d_{H}(y, \Gamma(\text{eig}(A))) = \bigwedge_{i \neq j, A_{ij} \neq 0} A_{ij}^{-1} \bigwedge_{k \in M_{j}} \alpha_{k}(A_{\mu}^{*})_{j,k} \bigoplus_{l} \alpha_{l}(A_{\mu}^{*})_{l}.
\]
Here by $M_j$ we denote the set $M \cap \text{supp}((A^*_\lambda)_{ij})$. Now we estimate \( d_H(y, \Gamma(\text{eig}(A))) \) from below and use the inequalities $A_{ij} \leq \mu(A^*_\lambda)_{ij}$ and $(A^*_\mu)_{ik} (A^*_\lambda)_{jk} \leq (A^*_\mu)_{ik}$ (see (15)):

$$d_H(y, \Gamma(\text{eig}(A))) \geq \wedge_{i \neq j, A_{ij} \neq 0} \wedge_{k \in M_j} A_{ij}^{-1} (A^*_\mu)_{jk} (A^*_\lambda)_{ik} \geq$$

$$\geq \wedge_{i \neq j, A_{ij} \neq 0} \wedge_{k \in M_j} \mu^{-1} (A^*_\mu)_{ij}^{-1} (A^*_\lambda)_{jk}^{-1} (A^*_\lambda)_{ik} \geq \mu^{-1}.$$

3) The distance between $y \in \text{eig}(A) = \text{span}(A^*_\lambda)$ and the supporting plane $X_{ij}$ is equal to

$$(27) \quad d_H(y, X_{ij}) = A_{ij}^{-1} \bigwedge_{k \in M_j} \alpha_k^{-1} (A^*_\lambda)_{jk}^{-1} \bigoplus_{l \in M_i} \alpha_l (A^*_\lambda)_{il}.$$

The cyclic permutation $\tau$ of $A^*_\lambda$ has the weight $1$. Hence for all $i, j \in K(\tau)$ such that $j = \tau(i)$ we have, according to Prop. 5 that $A_{ij} = \lambda(\tilde{A})(A^*_\lambda)_{ij}$ and $(A^*_\lambda)_{ij} (A^*_\lambda)_{ji} = (A^*_\lambda)_{ij}$. Note that $(A^*_\lambda)_{ij} \neq 0$ for all $i, j \in K$ and therefore $(A^*_\lambda)_{ij} = 0$ if and only if $(A^*_\lambda)_{ji} = 0$, i.e. $M_i$ and $M_j$ coincide. Making use of all this we write the upper estimate for $d_H(y, X_{ij})$:

$$d_H(y, X_{ij}) \leq \bigoplus_{l \in M_i} (A^*_\lambda)_{ij}^{-1} (A^*_\lambda)_{ji}^{-1} (A^*_\lambda)_{il} =$$

$$= \bigoplus_{l \in M_i, \lambda(\tilde{A})^{-1} (A^*_\lambda)_{ij}^{-1} (A^*_\lambda)_{ji}^{-1} (A^*_\lambda)_{il} = \lambda(\tilde{A})^{-1}.$$  

We also have $d_H(y, \Gamma(\text{eig}(A))) \geq \lambda(\tilde{A})^{-1}$ and therefore (see Prop. 16) $d_H(y, X_{ij}) = d_H(y, \Gamma(\lambda)) = \lambda(\tilde{A})^{-1}$. □

The sets $\Gamma(\text{eig}(A_\mu))$, for $\mu < 1$, are the subsets of $\text{eig}(A)$ equidistant from $\Gamma(\text{eig}(A))$, as Prop. 18 suggests.

**Proposition 18.** For all $\mu$ such that $\lambda(\tilde{A}) \leq \mu < 1$, if $\lambda(\tilde{A}) > 0$, or such that $0 < \mu < 1$, if $\lambda(\tilde{A}) = 0$, the distance $d_H(y, \Gamma(\text{eig}(A)))$ is equal to $\mu^{-1}$ if and only if $y \in \Gamma(\text{eig}(A_\mu))$.

**Proof.** If $\lambda(\tilde{A}) > 0$ and $\mu = \lambda(\tilde{A})$ then the statement readily follows from the observation that $A_\lambda$ does not have strong permanent and therefore (see Prop. 12) $\text{eig}(A_\lambda)$ does not have interior.

Let us consider $\mu > \lambda(\tilde{A})$. First, the equality

$$\bigoplus_{i \neq j, A_{ij} \neq 0} A_{ij} y_j y_i^{-1} = \mu$$

implies $A_{ij} y = y$. So, if $d_H(y, \Gamma(\text{eig}(A))) = \mu^{-1}$ then $y \in \text{eig}(A_\mu)$. Assume that $y$ belongs to the interior of $\text{eig}(A_\mu)$. Since $A_\mu$ is definite and has strong permanent, we can use representation (13) or (14) and obtain $\kappa < 1$ such that $y \in \text{eig}(A_\mu_\kappa)$. Now statement 2) of Prop. 17 implies that $d_H(y, \Gamma(\text{eig}(A))) \geq (\mu \kappa)^{-1} > \mu$. This is a contradiction, so $y \in \Gamma(\text{eig}(A_\mu))$.

Suppose now that $y \in \Gamma(\text{eig}(A_\mu))$. It means that there are $i \neq j$ such that $y_i y_j^{-1} = (A_\mu)_{ij}$, where $(A_\mu)_{ij} = (A_\mu)_{ji}$. According to Prop. 15 this face corresponds to the face of $\text{eig}(A)$ determined by the entry $A_{ij}$, and the distance between these two faces is clearly $\mu^{-1}$. □

Throughout this section we dealt with eigenvectors having full support. But let us recall the third statement of Prop. 7. It says that there might be eigenvectors with nontrivial support $K$. The distance between these eigenvectors and part of any face with full support would be infinite. Also, these eigenvectors are eigenvectors of
the submatrix $A_{KK}$. Therefore it is presumable, in this case, to pose the problem of finding $d_H(y, \Gamma(\text{eig}(A_{KK})))$.

We conclude this section with two examples.

*Example 1* In the beginning of this section we considered the definite matrix

$$A = \begin{pmatrix} 0 & -4 & 1 \\ 1 & 0 & 1 \\ -5 & -7 & 0 \end{pmatrix},$$

with the maximal cycle mean of $\tilde{A}$ equal to $\lambda = -1.5$. Now we pick the following three members of the $\{A_\mu\}$ family:

$$A_{-0.5} = \begin{pmatrix} 0 & -3.5 & 1.5 \\ 1.5 & 0 & 1.5 \\ -4.5 & -6.5 & 0 \end{pmatrix},$$
$$A_{-1} = \begin{pmatrix} 0 & -3 & 2 \\ 2 & 0 & 2 \\ -4 & -6 & 0 \end{pmatrix},$$
$$A_\lambda = \begin{pmatrix} 0 & -2.5 & 2.5 \\ 2.5 & 0 & 2.5 \\ -3.5 & -5.5 & 0 \end{pmatrix}.$$
Figure 5. Two Hilbert balls inscribed in $\Gamma(\text{eig}(A))$

eig(A) \mid yx^{-1} = 1 \}$ and \{ \( (x, y, z) \in \text{eig}(A) \mid xy^{-1} = -4 \) \}, in accordance with the third statement of Prop. 17.

Example 2 Consider a Hilbert ball with radius $d$ centered at \{ $\lambda x$ \} ($\lambda$ is any nonzero scalar). It is the set

\[ Y = \{ y \mid \bigoplus_{i,j} x_{i}y^{-1}_{i}y_{j}x^{-1}_{j} \leq d \}, \]

or, equivalently,

\[ Y = \{ y \mid y_{i}y^{-1}_{j} \geq d^{-1}x_{i}x^{-1}_{j} \}. \]

Denote by $D$ the matrix with entries $d_{ij} = d^{-1}x_{i}x^{-1}_{j}$. It is easily verified that $D = D^*$. Then it follows from Prop. 14 that the Hilbert ball is the eigenspace of $D$ and the columns of this matrix are its generators. The maximal cycle mean of $\tilde{D}$ is clearly $d^{-1}$. The eigenspaces $\text{eig}(D_{\mu})$ where $d^{-1} < \mu \leq 1$ are Hilbert balls with radii $(\mu d)^{-1}$ centered at \{ $\lambda x$ \}, and $\text{eig}(D_{d^{-1}})$ is precisely \{ $\lambda x$ \}.

For a three-dimensional example, set $x = [5 \ 4 \ 0]^T$ and $d = 3$. Then

\[ D = \begin{pmatrix} 0 & -2 & 2 \\ -4 & 0 & 1 \\ -8 & -7 & 0 \end{pmatrix}. \]
Fig. 6 displays the sets $\Gamma(\text{eig}(D_\mu))$ for $\mu = 0$ (dark blue), $\mu = -1$ (green), and $\mu = -2$ (brown) with the same convention about the weight of lines as in Fig. 4. These sets are concentric spheres centered at $\text{eig}(D_{-3}) = \{Ax\}$ (the large red circle in the center of Fig. 6).

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