Exponential Quantum Speed-ups for Semidefinite Programming with Applications to Quantum Learning

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Abstract

We give semidefinite program (SDP) quantum solvers with an exponential speed-up over classical ones. Specifically, we consider SDP instances with \( m \) constraint matrices of dimension \( n \), each of rank at most \( r \), and assume that the input matrices of the SDP are given as quantum states (after a suitable normalization). Then we show there is a quantum algorithm that solves the SDP feasibility problem with accuracy \( \epsilon \) by using \( \sqrt{m} \log m \cdot \text{poly}(\log n, r, \epsilon^{-1}) \) quantum gates. The dependence on \( n \) provides an exponential improvement over the work of Brandão and Svore [6] and the work of van Apeldoorn et al. [23], and demonstrates an exponential quantum speed-up when \( m \) and \( r \) are small.

We apply the SDP solver to the problem of learning a good description of a quantum state with respect to a set of measurements: Given \( m \) measurements and a supply of copies of an unknown state \( \rho \), we show we can find in time \( \sqrt{m} \log m \cdot \text{poly}(\log n, r, \epsilon^{-1}) \) a description of the state as a quantum circuit preparing a density matrix which has the same expectation values as \( \rho \) on the \( m \) measurements up to error \( \epsilon \). The density matrix obtained is an approximation to the maximum entropy state consistent with the measurement data considered in Jaynes’ principle.

As in previous work, we obtain our algorithm by “quantizing” classical SDP solvers based on the matrix multiplicative weight update method. One of our main technical contributions is a quantum Gibbs state sampler for low-rank Hamiltonians with a poly-logarithmic dependence on its dimension based on the techniques developed in quantum principal component analysis, which could be of independent interest. Our quantum SDP solver is different from previous ones in the following two aspects: (1) it follows from a zero-sum game approach of Hazan [11] of solving SDPs rather than the primal-dual approach by Arora and Kale [5]; and (2) it does not rely on any sparsity assumption of the input matrices.
1 Introduction

Motivation. Semidefinite programming has been a central topic in the study of mathematical optimization, theoretical computer science, and operations research in the last decades. It has become an important tool for designing efficient optimization and approximation algorithms. The power of semidefinite programs (SDPs) lies in their generality (that strengthens the better-known linear programs (LPs)) and the fact that they admit polynomial-time solvers.

It is natural to ask whether quantum computers can have advantage in solving this important optimization problem. Ref. [6] provided an affirmative answer. In particular, Ref. [6] gives a quantum algorithm with worst-case running time $\tilde{O}(\sqrt{mn^2}(Rr/\epsilon)^{32})$ where $n$ and $s$ are the dimension and row sparsity of the input matrices, respectively, $m$ the number of constraints, $\epsilon$ the accuracy of the solution, and $R, r$ upper bounds on the norm of the optimal primal and dual solutions. This is a quadratic speed-up in $m$ and $n$ comparing to the state-of-the-art classical SDP-solver [16] with complexity $\tilde{O}(m(m^2 + n^\omega + mns) \log(1/\epsilon))$ where $\omega$ is the exponent of matrix multiplication. The follow-up work by van Apeldoorn et al. [23] improves the running time of a quantum SDP solver to $\tilde{O}(\sqrt{mn^2}(Rr/\epsilon)^8)$. On the other hand, Ref. [6] proves a quantum lower bound $\Omega(\sqrt{m} + \sqrt{n})$ when $R, r, s, \epsilon$ are constants; stronger lower bounds can be proven if $R$ and/or $r$ scale with $n$ and $m$ [23].

These lower bounds for quantum SDP solvers, however, do not necessarily rule out the possibility of utilizing quantum computers to solve SDPs in a more efficient way. This is because existing lower bounds assume a specific oracle to access SDP instances and deal with general SDPs. Thus, it is still possible to achieve better than quadratic, or even exponential, quantum speed-ups when given different means to access SDP instances, or dealing with special SDPs.

We are motivated by natural SDP instances that could justify the use of different means to access them. For example, consider SDP instances with succinct representations that could arise e.g. from the application of SDPs in quantum complexity theory (e.g., Refs. [12, 9]). In these settings, input matrices of SDP instances, with dimension $2^\ell$, are typically quantum states and/or measurements generated by $\text{poly}(\ell)$-size circuits on $\ell$ qubits. For the sake of these applications, it might be reasonable to equip quantum SDP solvers with the ability to leverage these circuit information, rather than merely allowing access to the entries of the input matrices.

In this paper, we identify a setting in which quantum SDP solvers could achieve exponential speed-ups over classical ones. A subtlety is that we consider a non-standard oracle for the input matrices of the SDP, in which they are given as quantum states. Therefore one might object that it is not clear how to compare this model with the performance of classical solvers. We argue for the relevance of the setting, and for the claim of exponential speed-ups in some cases, by considering applications of the framework to the problem of learning quantum states [1]. We show it leads to an exponential improvement in the gate complexity (when the measurements to be learned are low rank) over the best known classical methods.

1.1 Exponential Quantum Speed-ups for SDPs

We will work with the SDP approximate feasibility problem formulated as follows (see Section 2 for details): Given an $\epsilon > 0$, $m$ real numbers $a_1, \ldots, a_m \in \mathbb{R}$, and Hermitian $n \times n$ matrices

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1. $\tilde{O}$ hides factors that are polynomial in $\log m$ and $\log n$.
2. A similar result was announced in an earlier version of [6], but the proof was unfortunately incorrect.
\[ A_1, \ldots, A_m \text{ where } -I \preceq A_i \preceq I \forall j \in [m], \text{ define the convex region } S_\varepsilon \text{ as all } X \text{ such that} \]
\[ \text{Tr}(A_i X) \leq a_i + \varepsilon \quad \forall i \in [m]; \]
\[ X \succeq 0; \text{ Tr}[X] = 1 \]

For approximate feasibility testing, it is required that either (1) If \( S_0 = \emptyset \), output fail; or (2) If \( S_\varepsilon \neq \emptyset \), output an \( X \in S_\varepsilon \).

It is a routine to reduce any SDP problem to the SDP feasibility problem as described up to some loss in dependence on the norm of optimal solutions. For simplicity in presentation we focus on the feasibility problem only, as our focus will be on the dependence on the input dimension. For the general optimization problem, and for SDPs without the normalization condition \( \text{Tr}[X] = 1 \), the run time will be increased by a factor of \( O((Rr)^8) \), with \( R \) and \( r \) the upper bounds on the norms of the primal and dual optimal solutions, respectively [23, 22].

We imagine a specific setting in which these \( A_i \)'s are “nice” so that the following oracles can be efficiently implemented. We do not aim to justify the relevance of this oracle model in general; they should be addressed in specific applications of SDPs. We will, however, illustrate why these conditions are naturally met in the context of learning quantum states.

**Oracle 1.1** (Oracle for traces of \( A_i \)). A classical oracle, denoted \( O_{\text{Tr}} \), which takes \( i \in [m] \) as input and outputs \( \text{Tr}[A_i^+] \) and \( \text{Tr}[A_i^-] \), where \( A_i^+ \) and \( A_i^- \) are two PSD matrices such that \( A_i = A_i^+ - A_i^- \).

**Oracle 1.2** (Oracle for preparing \( A_i \)). A quantum oracle (unitary), denoted \( O \) (and its inverse \( O^\dagger \)), which acts on \( \mathbb{C}^m \otimes (\mathbb{C}^n \otimes \mathbb{C}^n) \otimes (\mathbb{C}^n \otimes \mathbb{C}^n) \) such that for any \( j \in [m] \),
\[
O|j\rangle \langle j| \otimes |0\rangle \langle 0| \otimes |0\rangle \langle 0|O^\dagger = |j\rangle \langle j| \otimes |\psi^+\rangle \langle \psi^+| \otimes |\psi^-\rangle \langle \psi^-|,
\]
where \( |\psi^+\rangle, |\psi^-\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \) are any purifications of \( \frac{A_i^+}{\text{Tr}[A_i^+]} \), \( \frac{A_i^-}{\text{Tr}[A_i^-]} \), respectively.

Furthermore, assume that each \( A_i \) has rank at most \( r \). Our main result is the following quantum SDP solver.

**Theorem 1.1** (informal; see Corollary 3.5). For any \( \varepsilon > 0 \), there is a quantum algorithm that solves the SDP feasibility problem with success probability at least 0.96 using at most \( \sqrt{m} \log m \cdot \text{poly}(\log n, r, \varepsilon^{-1}) \) quantum gates and queries to **Oracle 1.1** and **Oracle 1.2**.

Our quantum SDP solver has an exponential improvement on the dependence of \( n \) comparing to [6, 23]. In the case of small \( m, r \), it demonstrates an exponential quantum speed-up over the best classical SDP solver [16] (although with the caveat that the input models are different). It is also worth mentioning that our quantum SDP solver does not assume the sparsity condition of \( A_i \)'s which are crucial for quantum SDP solvers in [6, 23]. This is because the assumption on **Oracle 1.1** and **Oracle 1.2** provides an alternative way to address the technical difficulty that was handled in previous approaches by the sparsity condition.

### 1.2 Efficient Learnability of Quantum States

Given many copies of any quantum state \( \rho \), the task of learning an approximate description of \( \rho \) is a fundamental one in quantum information and experimental physics. It refers to quantum state tomography, which has been widely used in experiments to identify quantum systems. However,
to tomograph an \( \ell \)-qubit state \( \rho \), the optimal procedure \cite{19,10} requires \( 2^{\Theta(\ell)} \) number of copies of \( \rho \), which is impractical already for relatively small \( \ell \).

In a sequence of works \cite{1,2}, Aaronson asked whether one can predict information about an \( \ell \)-qubit state with polynomially many copies in \( \ell \) and provided affirmative answers in various settings. In Ref. \cite{1}, he showed that a linear number of copies is sufficient to predict the outcomes of “most” measurements according to some distribution. In Ref. \cite{2}, in turn, he showed that the sample complexity is \( \tilde{O}(\ell^2/\epsilon^6) \) for the so-called “shadow tomography” problem: for any \( \ell \)-qubit \( \rho \) and two-outcome measurements \( E_1, \ldots, E_m \), one should estimate \( \text{Tr}[\rho E_i] \) up to error \( \epsilon \) for each \( i \in [m] \).

Existing results, however, only concern the sample complexity (i.e., the number of copies required), while the gate complexity (i.e., the entire running time of the learning procedure) is rarely discussed. In fact, existing techniques in Refs. \cite{1,2} have a huge gate complexity (roughly \( 2^{O(\ell)} \)) when applied on \( \ell \)-qubit states. This motivates us to seek settings where quantum states can be learnt efficiently in both sample and gate complexity.

Jaynes’ principle \cite{13} from statistical mechanics gives a general form for the solution of the “shadow tomography” learning problem above. It shows that there is always a state of form

\[
\frac{\exp\left(\sum_i \lambda_i E_i\right)}{\text{Tr}\left(\exp\left(\sum_i \lambda_i E_i\right)\right)},
\]

which has the same expectation values on the \( E_i \)'s as the original state \( \rho \), where the \( \lambda_i \)'s are real numbers. In words, there is always a Gibbs state with Hamiltonian given by a linear combination of the \( E_i \)'s which gives the same expectation values. Therefore one can solve the learning problem by finding the right \( \lambda_i \)'s (or even finding a quantum circuit creating the state in Eq. (1.2))

By formulating the learning problem in terms of the SDP feasibility problem (with each \( A_i \) replaced by \( E_i \), elaborated in Section 4.1), we observe that our SDP solver from Theorem 1.1 actually provides a solution to the learning problem. Precisely, we consider we have access to the following oracles:

**Oracle 1.1** for traces of \( E_i \): A classical oracle \( O_{\text{Tr}} \) that takes \( i \in [m] \) as input and outputs \( \text{Tr}[E_i] \).

**Oracle 1.2** for preparing \( E_i \): A unitary \( O \) such that for any \( i \), \( O|i \rangle \langle i| \otimes |0\rangle \langle 0|O^\dagger = |i\rangle \langle i| \otimes |\psi_i\rangle \langle \psi_i| \), where \( |\psi_i\rangle \langle \psi_i| \) is any purification of \( E_i / \text{Tr}[E_i] \).

Then we show:

**Theorem 1.2** (informal; see Corollary 4.3). For any \( \epsilon > 0 \), there is a quantum procedure that solves the “shadow tomography” problem on an \( \ell \)-qubit state \( \rho \) with success probability at least 0.96 and at most \( \sqrt{\ln m} \cdot \log m \cdot \text{poly}(\ell, r, \epsilon^{-1}) \) quantum gates, copies of \( \rho \), and queries to **Oracle 1.1** and **Oracle 1.2**.

The gate complexity of **Theorem 1.2** has a polynomial dependence on \( \ell \), which is an exponential improvement over Refs. \cite{1,2}.

Moreover, it turns out that our quantum SDP solver is closely connected to the preparation of Gibbs states. As a bonus, our algorithm also outputs each of the \( \lambda_i \)'s (one can show that \( \text{poly}(\log(nm))/\epsilon^2 \) non-zero of them suffices for a solution with error \( \epsilon \)), as well as a circuit description of the Gibbs state in Eq. (1.2) achieving the same expectation values as \( \rho \) up to error \( \epsilon \). In this sense our result can be seen as an algorithmic efficient version of Jaynes’ principle (for low rank measurements).
We can further justify our assumption on Oracle 1.1 and Oracle 1.2 in this context. For example, suppose the measurement operators \( E_i \)'s are of the form
\[
E_i = V_i P_i V_i^\dagger
\]
for polynomial time circuits \( V_i \) and projectors \( P_i \) of rank at most \( r \) diagonal in the computational basis. Then for Oracle 1.1, we just need to compute the trace of \( P_i \)'s (which can be done efficiently for low \( r \)), while Oracle 1.2 can be implemented efficiently (for low \( r \)) first by creating a maximally entangled state between the subspace spanned by \( P_i \) and a purification and applying \( V_i \) to one half of it.

### 1.3 Techniques

At a high level, and in similarity to Refs. [6, 23], our quantum SDP solver can be seen as a “quantized” version of classical SDP solvers based on the matrix multiplicative weight update (MMWU) method [4]. In particular, we will leverage quantum Gibbs samplers as the main source of quantum speed-ups. In Refs. [6, 23], quantum Gibbs samplers with quadratic speed-ups (e.g., [20, 8]) have been exploited to replace the classical Gibbs state calculation step. Because the number of iterations in MMWU is poly-logarithmic in terms of the input size, the use of quantum Gibbs samplers, together with a few other tricks, leads to the overall quadratic quantum speed-up.

To achieve exponential speed-ups, we need to work with a much more efficient quantum Gibbs sampler. To that end, as our main technical contribution, we construct the following quantum Gibbs sampler of low-rank Hamiltonians when given Oracles 1.1 and 1.2:

**Theorem 1.3** (informal; see Theorem 5.4). Assume the \( n \times n \) matrix \( K = K^+ - K^- \) and PSD matrices \( K^+, K^- \) have rank at most \( r \). Given quantum oracles that prepare copies of \( \rho^+ = K^+ / \text{Tr}(K^+) \), \( \rho^- = K^- / \text{Tr}(K^-) \) and estimates of \( \text{Tr}(K^+) \), \( \text{Tr}(K^-) \), there is a quantum Gibbs sampler that prepares the Gibbs state \( \rho_G = \exp(-K) / \text{Tr}(\exp(-K)) \) to precision \( \epsilon \) in trace distance, using \( \text{poly}(\log n, r, \epsilon^{-1}) \) quantum gates.

Our quantum Gibbs sampler has a poly-logarithmic dependence on \( n \) which is an exponential improvement comparing to [20, 8]. This is also the core source of our exponential speed-ups. Our construction deviates significantly from [20, 8]. Because of the existence of copies of \( \rho^+ \) and \( \rho^- \), we instead rely on efficient Hamiltonian simulation techniques developed in quantum principle component analysis (PCA) [17] and its follow-up work in [15]. As a result, we can also get rid of the sparsity assumption which is crucial for [20, 20].

Our quantum SDP solver also distinguishes from [6, 23] in that we do not follow the primal-dual approach in Arora-Kale’s SDP solver [5]. Instead, we follow a zero-sum game framework to solve SDP feasibility problems, which is also based on the MMWU method (details in Section 2). This framework has appeared in the classical literature (e.g., [11]) and has already been used in solving semidefinite programs in the context of quantum complexity theory (e.g., [24, 9]). We observe that the technique to quantize Arora-Kale’s primal-dual approach [5] for solving SDPs in [6, 23] readily extends to the zero-sum game approach, except for solving a different oracle problem. Specifically, this oracle seeks to find a violation \( i^* \) such that \( \text{Tr}(A_i X) > a_i + \epsilon \) on

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\(^3\)A similar observation was also independently made in a joint work of one of us (Brandão) with van Apeldoorn, Gilyen, Gribling, and de Wolf, using properties of the quantum relative entropy to show the convergence of the method.
input $X$ (see Oracle 2.1). One motivation to deal with this oracle is because it admits a rather straightforward implementation in the context of learning quantum states.

We also use the existence of Oracle 1.1 and Oracle 1.2 in finding such a violation $i^*$. Specifically, by using SWAP tests with the help of $A_i^+ / \text{Tr}[A_i^+]$ and $A_i^- / \text{Tr}[A_i^-]$ states, we can estimate $\text{Tr}(A_i X)$ without the sparsity assumption of $A_i$.

**Organization.** We will formulate the SDP feasibility problem and prove the correctness of the basic framework in Section 2. We prove our main result, i.e., the exponential quantum speed-up of SDP solvers, in Section 3. The application of our exponential quantum speed-up in learning quantum states is illustrated in Section 4. In Section 5 (with full details in Appendix A) we demonstrate how to sample from the Gibbs state of low-rank Hamiltonians.

**Notation.** Throughout the paper, we denote $n$ to be the dimension of matrices, $m$ to be the number of constraints, $\epsilon$ to be the error of the solution. For Hermitian matrices $A$ and $B$, we denote $A \preceq B$ if $B - A$ is positive semidefinite, and $A \succeq B$ if $A - B$ is positive semidefinite. We denote $I_n$ to be the $n \times n$ identity matrix.

## 2 Feasibility of SDPs

In this section, we formulate the feasibility problem of SDPs. It is a standard fact that one can use binary search to reduce any optimization problem to a feasibility one. One can further assume the following specific form up to some loss of the running time in terms of the dependence on the size of optimal solution. Since our main motivation is to achieve the exponential speed-up in terms of the dependence on $n$, we have not tried to optimize the dependence on other parameters.

**Definition 2.1 (Feasibility).** Given an $\epsilon > 0$, $m$ real numbers $a_1, \ldots, a_m \in \mathbb{R}$, and Hermitian $n \times n$ matrices $A_1, \ldots, A_m$ where $-I \preceq A_j \preceq I \forall j \in [m]$, define the convex region $S_\epsilon$ as all $X$ such that

\[
\text{Tr}(A_i X) \leq a_i + \epsilon \quad \forall i \in [m];
\]

\[
X \succeq 0;
\]

\[
\text{Tr}[X] = 1.
\]

For approximate feasibility testing, it is required that:

- If $S_0 = \emptyset$, output fail;
- If $S_\epsilon \neq \emptyset$, output an $X \in S_\epsilon$.

Both [6] and [23] follow the framework of [5] which uses a primal-dual approach to solve SDPs. We adopt a different approach that does not leverage the dual program but rather relies on the following oracle:

**Oracle 2.1 (Search for Violation).** Input a density matrix $X$, output an $i \in [m]$ such that Eq. (2.1) is violated. If no such $i$ exists, output “FEASIBLE”.
This oracle helps establish a zero-sum game view to solve any SDP feasibility problem. Imagine Player 1 who wants to provide a feasible \( X \in S_\epsilon \). Player 2, on the other side, wants to find any violation of any proposed \( X \). (This is exactly the function of \textbf{Oracle 2.1}. If the original problem is feasible, there exists a feasible point \( X_0 \) (provided by Player 1) such that there is no violation of \( X_0 \) that can be found by Player 2 (i.e., \textbf{Oracle 2.1}). This actually refers to an 
\textit{equilibrium} point of the zero-sum game, which can be approximated by the multiplicative weight update method \cite{4}.

This game view of solving the SDP feasibility problem has appeared in the classical literature (e.g., \cite{11}) and has already been used in solving semidefinite programs in the context of quantum complexity theory (e.g., \cite{24, 9}). We observe that the technique to quantize Arora-Kale’s primal-dual approach \cite{5} for solving SDPs in Refs. \cite{6, 23} readily extends to the zero-sum game approach. \footnote{A similar observation has also been independently made in joint work with one of us (Brandão) and van Apeldoorn, Gilyen, Gribling, and de Wolf.}

To quantize this zero-sum game approach, one needs to implement \textbf{Oracle 2.1} rather the one involving the dual program in \cite{6, 23}, which is part of our technical contribution. Another reason to deal with \textbf{Oracle 2.1} is it admits a rather straightforward implementation in the context of quantum learning.

We present a master algorithm that solves the SDP feasibility problem with the help of \textbf{Oracle 2.1}. It should be understood that the master algorithm is \textit{not} the final quantum algorithm, where a few steps will be replaced by their quantum counterparts. However, the master algorithm helps demonstrate the correctness of the algorithm and the number of oracle queries.

Our algorithm heavily relies on the matrix multiplicative weight method given in Algorithm 1.

\begin{algorithm}
\caption{Matrix multiplicative weights algorithm (Figure 3.1 of \cite{14}).}
\label{alg:matrix_multiplicative_weights}
\begin{algorithmic}[1]
\State \textbf{Initialization:} Fix a \( \delta \leq 1/2 \). Initialize the weight matrix \( W^{(1)} = I_n \);
\For {\( t = 1, 2, \ldots, T \)}
\State Compute the density matrix \( \rho^{(t)} = \frac{W^{(t)}}{\text{Tr}[W^{(t)}]} \);
\State Observe the gain matrix \( M^{(t)} \);
\State Update the weight matrix: \( W^{(t+1)} = \exp[\delta \sum_{\tau=1}^{t} M^{(\tau)}] \);
\EndFor
\end{algorithmic}
\end{algorithm}

\textbf{Proposition 2.2} (Corollary 4 of \cite{14}). Algorithm 1 guarantees that after \( T \) rounds, for any density matrix \( \rho \), we have
\[ (1 - \delta) \sum_{t: M^{(t)} \preceq 0} \text{Tr}(M^{(t)} \rho^{(t)}) + (1 + \delta) \sum_{t: M^{(t)} \succeq 0} \text{Tr}(M^{(t)} \rho^{(t)}) \geq \sum_{t=1}^{T} \text{Tr}(M^{(t)} \rho) - \frac{\ln n}{\delta}. \] (2.4)

We use Algorithm 1 and Proposition 2.2 to test the feasibility of SDPs.

\textbf{Theorem 2.3} (Master Algorithm). Assume we are given \textbf{Oracle 2.1}. Then for any \( \epsilon > 0 \), feasibility of the SDP in (2.1), (2.2), and (2.3) can be tested by Algorithm 2 with at most \( \frac{8 \ln n}{\epsilon^2} \) queries to the oracle.
Proof of Theorem 2.3. For all $j \in [m]$, denote $M_j = \frac{1}{2}(I_n - A_j)$; note that $0 \preceq M_j \preceq I \forall j \in [m]$. In round $t$, after computing the density matrix $\rho^{(t)}$, equivalently speaking, Oracle 2.1 checks whether there exists an $j \in [m]$ such that $\text{Tr}(M_j \rho^{(t)}) < \frac{1}{2} - \frac{\alpha_{j(0)} + \epsilon}{2}$. If not, then $\text{Tr}(M_j \rho^{(t)}) \geq \frac{1}{2} - \frac{\alpha_{j(0)} + \epsilon}{2} \forall j \in [m]$, $\text{Tr}(A_j \rho^{(t)}) \leq \alpha_j + \epsilon \forall j \in [m]$, and hence $\rho^{(t)} \in S_{\epsilon}$.

Otherwise, the oracle outputs an $M_{j(1)} \in \{M_j\}_{j=1}^m$ such that $\text{Tr}(M_{j(1)} \rho^{(t)}) < \frac{1}{2} - \frac{\alpha_{j(1)} + \epsilon}{2}$. After $T = \frac{8 \ln n}{\epsilon^2}$ iterations, by Proposition 2.2, this matrix multiplicative weights algorithm promises that for any density matrix $\rho$, we have

$$
\left(1 + \frac{\epsilon}{2}\right) \sum_{t=1}^{T} \text{Tr}(M_{j(0)} \rho^{(t)}) \geq \sum_{t=1}^{T} \text{Tr}(M_{j(0)} \rho) - \frac{2 \ln n}{\epsilon}.
$$

(2.5)

If $S_0 \neq \emptyset$, there exists a $\rho^* \in S_0$ such that $\text{Tr}(M_{j(0)} \rho^*) \geq \frac{1}{2} - \frac{\alpha_{j(0)}}{2}$ for all $t \in [T]$. On the other hand, $\text{Tr}(M_{j(1)} \rho^{(t)}) < \frac{1}{2} - \frac{\alpha_{j(1)} + \epsilon}{2}$ for all $t \in [T]$. Plugging these two inequalities into (2.5), we have

$$
\frac{T}{2} \left(1 + \frac{\epsilon}{2}\right) (1 - \epsilon) > \frac{T}{2} - \frac{2 \ln n}{\epsilon},
$$

(2.6)

which gives

$$
T < \frac{8 \ln n}{\epsilon^2 (1 + \epsilon)} < \frac{8 \ln n}{\epsilon^2},
$$

(2.7)

contradiction! Therefore, if $\text{Tr}(M_{j(0)} \rho^{(t)}) < \frac{1}{2} - \frac{\alpha_{j(0)} + \epsilon}{2}$ happens for at least $\frac{8 \ln n}{\epsilon^2}$ times, it must be the case that $S_0 = \emptyset$. 

### 3 Exponential speed-up of solving SDP feasibility

In this section, we illustrate our exponential quantum speed-up for solving the SDP feasibility problem. To that end, we first formulate the precise technical scenario that allows such exponential speed-ups. We then demonstrate how to implement Oracle 2.1 in such scenario and how the actually quantum algorithm works.
The Setting

As mentioned in the introduction, we would like to equip the quantum SDP solver with some extra power beyond only accessing the entries of the input matrices (i.e., $A_i$, $i = 1, \cdots, m$, each of $n \times n$ size) of any instance. We imagine the scenario where these $A_i$s are nice so that the following oracles, representing various means to access $A_i$s, can be efficiently implemented. 

**Oracle 3.1** (Oracle for traces of $A_i$). A classical oracle, denoted $O_{Tr}$, which takes $i \in [m]$ as input and outputs $\text{Tr}[A_i^+]$ and $\text{Tr}[A_i^-]$, where $A_i^+$ and $A_i^-$ are two PSD matrices such that $A_i = A_i^+ - A_i^-$. 

**Oracle 3.2** (Oracle for preparing $A_i$). A quantum oracle (unitary), denoted $O$ (and its inverse $O^+$), which acts on $\mathbb{C}^n \otimes (\mathbb{C}^n \otimes \mathbb{C}^n) \otimes (\mathbb{C}^n \otimes \mathbb{C}^n)$ such that for any $j \in [m]$,  
\begin{equation}
O|j\rangle \langle j| \otimes |0\rangle \otimes |0\rangle = |j\rangle \langle j| \otimes |\psi^+\rangle \langle \psi^+| + |\psi^-\rangle \langle \psi^-|,
\end{equation}
where $|\psi^+\rangle, |\psi^-\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n$ are any purifications of $\frac{A_j^+}{\text{Tr}[A_j^+]}$, $\frac{A_j^-}{\text{Tr}[A_j^-]}$, respectively. 

With the access to **Oracle 3.1** and **Oracle 3.2**, the following lemma shows how to prepare two normalized quantum states $K^\pm / \text{Tr}[K^\pm]$ where $K^\pm = \sum_{i \in S} c_i A_i^\pm$, $c_i > 0$ and $A_i^\pm$ refers to either $A_i^+$ or $A_i^-$. 

**Lemma 3.1.** Both $K^+ / \text{Tr}[K^+]$ and $K^- / \text{Tr}[K^-]$ can be prepared by $|S|$ samples to **Oracle 3.1** and one sample to **Oracle 3.2**. 

**Proof.** Consider the following protocol, where we choose all $\pm$ to be $+$ when preparing $K^+ / \text{Tr}[K^+]$, and choose all $\pm$ to be $-$ when preparing $K^- / \text{Tr}[K^-]$: 

1. For all $i \in S$, sample **Oracle 3.1** to obtain $\text{Tr}[A_i^+]$; 

2. To prepare $K^\pm / \text{Tr}[K^\pm]$, toss a coin $j \in S$ such that $\text{Pr}[j = i] = \frac{c_i \text{Tr}[A_i^+]}{\sum_{k \in S} c_k \text{Tr}[A_k^+]}$, take one sample of **Oracle 3.2** to obtain $A_j^\pm / \text{Tr}[A_j^\pm]$, and output this state.

By symmetry, we only consider the preparation of $K^\pm / \text{Tr}[K^\pm]$. With probability $\frac{c_i \text{Tr}[A_i^+]}{\sum_{k \in S} c_k \text{Tr}[A_k^+]}$, the output state is $A_j^\pm / \text{Tr}[A_j^\pm]$; therefore, in average the density matrix prepared is 
\begin{equation}
\sum_{i \in S} \frac{c_i \text{Tr}[A_i^+]}{\sum_{k \in S} c_k \text{Tr}[A_k^+]}, \frac{A_i^\pm}{\text{Tr}[A_i^\pm]} = \sum_{i \in S} \frac{c_i A_i^\pm}{\sum_{k \in S} c_k \text{Tr}[A_k^+]}, \frac{K^\pm}{\text{Tr}[K^\pm]}.
\end{equation}

Furthermore, Step 1 takes $|S|$ samples to **Oracle 3.1** and Step 2 takes one sample to **Oracle 3.2**; this exactly matches the sample complexity claimed in **Lemma 3.1**. 

Combining **Lemma 3.1** and **Theorem 5.4**, leads to a lemma that generates the Gibbs state in **Line 3** of **Algorithm 2**.

**Lemma 3.2.** Suppose $K = K^+ - K^-$, where $K^\pm = \sum_{i \in S} c_i A_i^\pm$, $c_i > 0$ and $A_i^\pm$ refers to either $A_i^+$ or $A_i^-$. Moreover, assume that $\text{Tr}(K^+) + \text{Tr}(K^-) \leq B$ for some bound $B$, and that $K^+, K^-$ have rank at most $r$. Then it is possible to prepare the Gibbs state $\rho_G = \exp(-K) / \text{Tr}(\exp(-K))$ to $\epsilon$ precision in trace distance, with $|S| \cdot \text{poly} \log n, r, B, \epsilon^{-1}$ quantum gates and queries to **Oracle 3.1** and **Oracle 3.2**.

5 We adopt that convention that one oracle query is assumed to cost unit time.

6 By tracing out the extra space, one can easily obtain states $A_j^+ / \text{Tr}[A_j^+]$, $A_j^- / \text{Tr}[A_j^-]$. 


Implementation of \textbf{Oracle 2.1}

Now we turn to the implementation of \textbf{Oracle 2.1} with the help of \textbf{Oracle 3.1} and \textbf{Oracle 3.2}. The high-level idea is rather straightforward: given (many copies of) any density operator $\rho$, we can estimate $\text{Tr}(A_i \rho)$ for each $i$ and then check whether there is a violation for some $i$. We accelerate the brute force search quadratically by quantum search.

There is an important distinction from the technique in \cite{6,23} for estimating $\text{Tr}(A_i \rho)$ where a sparsity assumption about $A_i$ is necessary. We, however, rely on the help of \textbf{Oracle 3.1} and \textbf{Oracle 3.2} and perform a simple SWAP-test \cite{7} to estimate $\text{Tr}(A_i \rho)$\footnote{It is possible to further improve our efficiency by using the quantum speed-up of classical concentration bounds (e.g., Ref. \cite{18}).}. This does not imply we have a way to replace the sparsity assumption in the setting of Refs. \cite{6, 23}, but rather means the sparsity assumption is no longer necessary with the existence of \textbf{Oracle 3.1} and \textbf{Oracle 3.2}.

A subtle issue arises when estimating $\text{Tr}(A_i \rho)$ because the complexity could be different for each $i$. Normal quantum search does not handle this case and we instead rely on the quantum search technique with variable costs developed in \cite{3}.

\textbf{Proposition 3.3} (\cite{3}, Theorem 3). Assume that for any $j \in [m]$, there exists a quantum algorithm $A_j$ which is initialized in the state $|0\rangle$ and, after $t_j$ steps, outputs the final state $|x_j\rangle|\psi_j^{\text{anc}}\rangle$ where $x_j \in \{0, 1\}$ and $|\psi_j^{\text{anc}}\rangle$ is the state in the ancilla register. Then there exists a quantum algorithm $A$ that finds an $j \in [m]$ such that $x_j = 1$ within time $O(\sqrt{t_1^2 + \cdots + t_m^2})$; if no such $j \in [m]$ exists, $A$ rejects in time $O(\sqrt{t_1^2 + \cdots + t_m^2})$.

Using \textbf{Oracle 3.1}, \textbf{Oracle 3.2}, and \textbf{Proposition 3.3} we can implement \textbf{Oracle 2.1} for finding a violation constraint:

\textbf{Lemma 3.4}. Assume we are given \textbf{Oracle 3.1}, \textbf{Oracle 3.2}, and $1200 \sqrt{m} \log m \cdot \frac{B^2}{\epsilon^2} \log \frac{1.1 m n}{\epsilon^2}$ copies of a state $\rho \in C^n$, where $\text{Tr}[A_j^+] + \text{Tr}[A_j^-] \leq B$ for some bound $B$ for all $j \in [m]$. Then for any $\epsilon > 0$, \textbf{Algorithm 3} finds a $j \in [m]$ such that $\text{Tr}(A_j \rho) > a_j + \epsilon$ with success probability at least $1 - \frac{0.005 n^2}{\ln n}$, using $\sqrt{m} \log m \cdot \text{poly}(\log n, B, \epsilon^{-1})$ quantum gates and queries to \textbf{Oracle 3.1} and \textbf{Oracle 3.2}.

\textbf{Algorithm 3}: Implementation of the oracle that finds a violation constraint.

1. Use \textbf{Proposition 3.3} to search for a $j \in [m]$ such that $\text{Tr}(A_j \rho) > a_j + \epsilon$. For each $j$, this is achieved within poly$(\log n, B, \epsilon^{-1})$ steps by:

2. Using \textbf{Oracle 3.2}, apply the SWAP test on $\rho$ and $\frac{A_j^+}{\text{Tr}[A_j^+]}$, for $600 \log m \frac{B^2}{\epsilon^2} \log \frac{8 \ln n}{\epsilon^2}$ times. Denote the frequency of getting 1 to be $\tilde{p}_{j,+}$.

3. Using \textbf{Oracle 3.2}, apply the SWAP test on $\rho$ and $\frac{A_j^-}{\text{Tr}[A_j^-]}$, for $600 \log m \frac{B^2}{\epsilon^2} \log \frac{8 \ln n}{\epsilon^2}$ times. Denote the frequency of getting 1 to be $\tilde{p}_{j,-}$.

4. Apply \textbf{Oracle 3.1} to compute $\text{Tr}[A_j^+]$ and $\text{Tr}[A_j^-]$. Claim that $\text{Tr}(A_j \rho) > a_j + \epsilon$ if $\left(2\tilde{p}_{j,+} - 1\right) \text{Tr}[A_j^+] - \left(2\tilde{p}_{j,-} - 1\right) \text{Tr}[A_j^-] > a_j + \epsilon / 2$;
Proof. We first establish the correctness. Recall that the SWAP test on $\rho$ and $\frac{A_j^+}{\text{Tr}[A_j^+]}$ outputs 1 with probability $\frac{1}{2} + \frac{\text{Tr}(A_j^+ \rho)}{2 \text{Tr}[A_j^+]}$, and the SWAP test on $\rho$ and $\frac{A_j^-}{\text{Tr}[A_j^-]}$ outputs 1 with probability $\frac{1}{2} + \frac{\text{Tr}(A_j^- \rho)}{2 \text{Tr}[A_j^-]}$. Therefore, by Chernoff’s bound and the fact that $\text{Tr}[A_j^+], \text{Tr}[A_j^-] \leq B$, we have

$$\Pr \left[ |p_{j,+} - \left( \frac{1}{2} + \frac{\text{Tr}(A_j^+ \rho)}{2 \text{Tr}[A_j^+]} \right) | \geq \frac{\epsilon}{8 \text{Tr}[A_j^+]} \right] \leq \Pr \left[ |p_{j,+} - \left( \frac{1}{2} + \frac{\text{Tr}(A_j^+ \rho)}{2 \text{Tr}[A_j^+]} \right) | \geq \frac{\epsilon}{8B} \right]$$

(3.3)

$$\leq 2e^{-600 \log m \frac{\epsilon^2}{88B^2}}$$

(3.4)

$$< 0.02 \cdot \frac{\epsilon^2}{8m \ln n}.$$  

(3.5)

Similarly,

$$\Pr \left[ |p_{j,-} - \left( \frac{1}{2} + \frac{\text{Tr}(A_j^- \rho)}{2 \text{Tr}[A_j^-]} \right) | \geq \frac{\epsilon}{8 \text{Tr}[A_j^-]} \right] < 0.02 \cdot \frac{\epsilon^2}{8m \ln n}.$$  

(3.6)

In other words, with probability at least $1 - 0.02 \cdot \frac{\epsilon^2}{8m \ln n}$,

$$\left| (2p_{j,+} - 1) \text{Tr}[A_j^+] - \text{Tr}(A_j^+ \rho) \right| \leq \frac{\epsilon}{4}, \quad \left| (2p_{j,-} - 1) \text{Tr}[A_j^-] - \text{Tr}(A_j^- \rho) \right| \leq \frac{\epsilon}{4}.$$  

(3.7)

Therefore, if $\text{Tr}(A_j \rho) = \text{Tr}(A_j^+ \rho) - \text{Tr}(A_j^- \rho) > a_j + \epsilon$, then with probability at least $1 - 0.04 \cdot \frac{\epsilon^2}{8m \ln n}$,

$$(2p_{j,+} - 1) \text{Tr}[A_j^+] - (2p_{j,-} - 1) \text{Tr}[A_j^-] > a_j + \epsilon / 2,$$  

(3.8)

which is exactly Line 4 in Algorithm 3. In all, by Proposition 3.3, with probability at least $1 - 0.04 \cdot \frac{\epsilon^2}{8m \ln n} \cdot \sqrt{m} \geq 1 - \frac{0.005 \epsilon^2}{\ln n}$, Algorithm 3 is correct.

It remains to count the total number of quantum gates applied in Line 14. To apply the SWAP tests in Line 2 and Line 3, we need $\log m \cdot \text{poly}(\log n, B, \epsilon^{-1})$ copies of $\rho$ as well as $\frac{A_j^+}{\text{Tr}[A_j^+]}$ and $\frac{A_j^-}{\text{Tr}[A_j^-]}$; the former one is implemented by Line 3 and the latter two are implemented by querying Oracle 3.2. It also takes $\Theta(\log n)$ quantum gates to implement the SWAP test. In total, for each $j \in [m]$, the gate cost of running Line 14 is at most $\log m \cdot \text{poly}(\log n, B, \epsilon^{-1})$. By Proposition 3.3, the total complexity (both gate cost and queries to Oracle 3.1 and Oracle 3.2) of the quantum search is at most $\sqrt{m} \log m \cdot \text{poly}(\log n, B, \epsilon^{-1})$.

**Quantum Algorithm for SDP feasibility**

We now expand Algorithm 2 to the fully quantum version (Algorithm 4). A key difference is that we use Lemma 3.2 to generate (many copies) of the Gibbs state $\rho^{(i)}$ and rely on Lemma 3.4 to implement Oracle 2.1. At a high-level, the correctness of Algorithm 4 still roughly comes from Theorem 2.3 as well as Lemma 3.2 and Lemma 3.4. However, its gate complexity will be efficient because of the help of Oracle 3.1 and Oracle 3.2.
Corollary 3.5. Assume we are given Oracle 3.1 and Oracle 3.2. Furthermore, assume $\text{Tr}[A_j^+] + \text{Tr}[A_j^-] \leq B$ for some bound $B$, and $A_j$ have rank at most $r$ for all $j \in [m]$. Then for any $\epsilon > 0$, feasibility of the SDP in (2.1), (2.2), and (2.3) can be tested by Algorithm 4 with success probability at least 0.96 and at most $\sqrt{m} \log m \cdot \text{poly}(\log n, r, B, \epsilon^{-1})$ quantum gates and queries to Oracle 3.1 and Oracle 3.2.

Algorithm 4: Efficiently testing the feasibility of SDPs: fulfilled version of Algorithm 2

1. Initialize the weight matrix $W^{(1)} = I_n$, and $T = \frac{8 \ln n}{\epsilon^2}$;
2. for $t = 1, 2, \ldots, T$
   3. Prepare $1200 \sqrt{m} \log m \cdot \frac{n^2}{\epsilon^2} \log \frac{8 \ln n}{\epsilon^2}$ samples of the Gibbs state $\rho^{(t)} = \frac{W^{(t)}}{\text{Tr}[W^{(t)}]}$ by Lemma 3.2;
   4. Using these $1200 \sqrt{m} \log m \cdot \frac{n^2}{\epsilon^2} \log \frac{8 \ln n}{\epsilon^2}$ copies of $\rho^{(t)}$, search for a $j^{(t)} \in [m]$ such that $\text{Tr}(A_{j^{(t)}} \rho^{(t)}) > a_{j^{(t)}} + \epsilon$ by Algorithm 3. Take $M^{(t)} = \frac{1}{2}(I_n - A_{j^{(t)}})$ if such $j^{(t)}$ is found; otherwise, claim that $S_\epsilon \neq \emptyset$, output $\rho^{(t)}$ as a feasible solution, and terminate the algorithm;
   5. Update the weight matrix: $W^{(t+1)} = \exp[\frac{t}{2} \sum_{t=1}^{t} M^{(t)}]$;
6. Claim that $S_0 = \emptyset$ and terminate the algorithm.

Proof of Corollary 3.5. The correctness of Algorithm 4 is automatically established by Theorem 2.3 if it suffices to analyze the gate cost of Algorithm 4.

In Line 3 of Algorithm 4, we apply Lemma 3.2 to compute the Gibbs state $\rho^{(t)}$. In round $t$, because $M_j = \frac{1}{2}[I_n - (A_j^- - A_j^+)] = \frac{1}{2}I_n + \frac{1}{2}A_j^- - \frac{1}{2}A_j^+ \forall j \in [m]$, we take $K_j^+ = \frac{t}{2} \sum_{t=1}^{t} \frac{1}{2}A_{j^{(t)}}$ and $K_j^- = \frac{t}{2} \sum_{t=1}^{t} \frac{1}{2}A_{j^{(t)}}$. Because $t \leq \frac{8 \ln n}{\epsilon^2}$, $K_j^+$, $K_j^-$ have rank at most $t \cdot r = O(\log n \cdot r / \epsilon^2)$, and $\text{Tr}[K_j^+]$, $\text{Tr}[K_j^-]$ are at most $\frac{e^2}{4} \cdot B = O(\log n \cdot B / \epsilon)$, Lemma 3.2 guarantees that

$$\frac{8 \ln n}{\epsilon^2} \cdot \text{poly}\left(\log n, \frac{r \log n}{\epsilon^2}, \frac{B \log n}{\epsilon}, \epsilon^{-1}\right) = \text{poly}(\log n, r, B, \epsilon^{-1})$$

(3.9)

quantum gates and queries to Oracle 3.1 and Oracle 3.2 suffice to prepare the Gibbs state $\rho^{(t)}$. Because there are at most $\frac{8 \ln n}{\epsilon^2} \cdot \text{poly}(\log n, \epsilon^{-1})$ iterations and in each iteration $\rho^{(t)}$ is prepared for $1200 \sqrt{m} \log m \cdot \frac{n^2}{\epsilon^2} \log \frac{8 \ln n}{\epsilon^2} = \sqrt{m} \log m \cdot \text{poly}(\log n, B, \epsilon^{-1})$ copies, in total the gate cost for Gibbs state preparation is still $\sqrt{m} \log m \cdot \text{poly}(\log n, r, B, \epsilon^{-1})$.

Furthermore, by Lemma 3.4, Line 4 finds a $j^{(t)} \in [m]$ such that $\text{Tr}(A_{j^{(t)}} \rho^{(t)}) > a_{j^{(t)}} + \epsilon$ with success probability at least $1 - \frac{0.008 \epsilon^2}{\ln n}$, using $\sqrt{m} \log m \cdot \text{poly}(\log n, B, \epsilon^{-1})$ quantum gates and queries to Oracle 3.1 and Oracle 3.2. Because Algorithm 4 has at most $\frac{8 \ln n}{\epsilon^2}$ iterations, with success probability at least $1 - \frac{8 \ln n}{\epsilon^2} - \frac{0.008 \epsilon^2}{\ln n} = 0.96$ we can assume that the quantum search in Algorithm 3 works correctly, and the total gate cost of calling Algorithm 4 is still $\sqrt{m} \log m \cdot \text{poly}(\log n, B, \epsilon^{-1})$.

In conclusion, $\sqrt{m} \log m \cdot \text{poly}(\log n, r, B, \epsilon^{-1})$ is an upper bound on the total number of quantum gates and queries to Oracle 3.1 and Oracle 3.2.

Remark 3.1. In Corollary 3.5, the only restriction on the decomposition $A_j = A_j^+ - A_j^-$ for all $j \in [m]$ is that $\text{Tr}[A_j^+] + \text{Tr}[A_j^-] \leq B$. If we assume this decomposition to be the eigen-decomposition, i.e., $A_j^+$ represents the subspace spanned by the eigenvectors of $A_j$ with positive eigenvalues, and
\( A_j^- \) represents the subspace spanned by the eigenvectors of \( A_j \) with negative eigenvalues, then by the low-rank assumption and \(-I \preceq A_j \preceq I, \Tr[A_j^+] + \Tr[A_j^-] \leq r\). In this case, Corollary 3.5 takes at most \( \sqrt{m} \log m \cdot \text{poly}(\log n, r, \epsilon^{-1}) \) quantum gates and queries to \text{Oracle 3.1} and \text{Oracle 3.2}; this result gives an exponential speed-up in \( n \) compared to the previous quantum algorithms in Refs. \([6]\) and \([23]\), while keeping the square-root dependence in \( m \) and only having polynomial dependence in \( r \).

## 4 Efficient learnability of quantum states

We consider the following question about learning quantum states via measurements:

**Question 4.1.** Let \( \rho \) be an unknown quantum state in an \( n \)-dimensional Hilbert space, \( E_1, \ldots, E_m \) be known two-outcome POVMs, and \( 0 < \epsilon < 1 \). Given independent copies of \( \rho \), one wants to estimate \( \Tr[\rho E_i] \) within additive error \( \epsilon \) for all \( i \in [m] \), with success probability at least \( 2/3 \). What is the sample complexity (i.e., the number of required copies of \( \rho \)) and gate complexity (i.e., the total running time) of the best such procedure?

In Ref. \([2]\), Aaronson named Question 4.1 as “shadow tomography”, and proved the sample complexity of \( \rho \) to be \( O(\log^4 m \cdot \log^2 n \cdot \log \log n / \epsilon^6) \). In this section we show that, for low rank matrices and small \( m \), we can also make the learning process \textit{computationally efficient}, by using our previous result on speeding up solutions to SDPs.

### 4.1 Reduction of Question 4.1 to SDP feasibility

We start with a simple explanation of using the solution to SDP feasibility to address Question 4.1. Given (many copies of) any unknown quantum state \( \rho \) and two-outcome POVM \( E_1, \ldots, E_m \), in order to estimate \( \Tr[\rho E_i] \), it suffices to find a state \( \sigma \) that is the solution to the following SDP feasibility problem:

\[
\begin{align*}
\Tr[\sigma E_i] &\leq \Tr[\rho E_i] + \epsilon & \forall i \in [m]; \\
\Tr[\sigma E_i] &\geq \Tr[\rho E_i] - \epsilon & \forall i \in [m]; \\
\Tr[\sigma] &= 1; \\
\sigma &\succeq 0.
\end{align*}
\]

Any feasible solution \( \sigma \) satisfies that \( |\Tr[\sigma E_i] - \Tr[\rho E_i]| < \epsilon \) for all \( i \in [m] \). Of course, we don’t know \( \Tr[\rho E_i] \), and hence the constraints of the SDP feasibility problem, in advance. The key observation is that it suffices to implement \text{Oracle 2.1} to solve the feasibility problem, which could be implemented without knowing \( \Tr[\rho E_i] \) for each \( i \) explicitly.

Recall that \text{Oracle 2.1} asks to find such \( i \) that \( |\Tr[\sigma E_i] - \Tr[\rho E_i]| > \epsilon \). To implement such an oracle, we need to operate on copies of \( \rho \). For sample complexity, we only care about the number of copies of \( \rho \) used for the purpose, while ignoring all other computation cost. For gate complexity, we do care about the entire computation cost as well as the number of copies of \( \rho \).
4.2 Gate complexity

Similar to Section 3, we assume the existence of Oracle 3.1 and Oracle 3.2 to achieve exponential speed-ups of dependence on n. Specifically, for the feasibility problem (4.1)-(4.4), we have:

Oracle 3.1 for traces of $E_i$: A classical oracle $O_{T r}$ that takes $i \in [m]$ as input and outputs $\text{Tr}[E_i]$.

Oracle 3.2 for preparing $E_i$: A unitary $O$ (and its inverse $O^\dagger$) acting on $C^n \otimes (C^n \otimes C^n)$ such that for any $i \in [m]$,

$$O|i\rangle \langle i| \otimes |0\rangle \langle 0|O^\dagger = |i\rangle \langle i| \otimes |\psi_i\rangle \langle \psi_i|,$$

(4.5)

where $|\psi_i\rangle \langle \psi_i|$ is any purification of $\frac{E_i}{\text{Tr}[E_i]}$. Furthermore, we assume that the POVM operator $E_i$ has rank at most $r$, for all $i \in [m]$.

Implementation of Oracle 2.1. We can prove the following lemma by directly using Lemma 3.4

Lemma 4.2. Assume we are given Oracle 3.1, Oracle 3.2, and $2400\sqrt{m} \log m \cdot \frac{2^2}{\epsilon^2} \log \frac{8\ln n}{\epsilon^2}$ copies of two states $\rho, \sigma \in \mathbb{C}^n$. Then for any $\epsilon > 0$, Algorithm 3 finds a $i \in [m]$ such that $\left| \text{Tr}[\sigma E_i] - \text{Tr}[\rho E_i] \right| \geq \epsilon$ with success probability at least $1 - \frac{0.005\epsilon^2}{\ln n}$, using $\sqrt{m} \log m \cdot \text{poly}(\log n, r, \epsilon^{-1})$ quantum gates and queries to Oracle 3.1 and Oracle 3.2

Proof. Lemma 3.4 directly implies Lemma 4.2 with the following minor differences:

- The trace of $E_i$ is at most $r$: This is because rank($E_i$) = $r$ and $0 \leq E_i \leq I$. This modifies all $B$ in Lemma 3.4 to $r$.

- The number of constraints in Lemma 4.2 is doubled: This is because $\left| \text{Tr}[\sigma E_i] - \text{Tr}[\rho E_i] \right| \geq \epsilon$ happens either when $\text{Tr}[\sigma E_i] - \text{Tr}[\rho E_i] \geq \epsilon$ or $\text{Tr}[\sigma E_i] - \text{Tr}[\rho E_i] \leq -\epsilon$. This also doubles the sample complexity to $2400\sqrt{m} \log m \cdot \frac{2^2}{\epsilon^2} \log \frac{8\ln n}{\epsilon^2}$.

\[ \square \]

Similar to Corollary 3.5, we solve Question 4.1 by using Lemma 3.2 to generate (many copies) of the Gibbs state $\rho^{(t)}$ and relying on Lemma 4.2 to implement Oracle 2.1

Corollary 4.3. Assume we are given Oracle 3.1 and Oracle 3.2. Then for any $\epsilon > 0$, Question 4.1 can be solved by Algorithm 5 with success probability at least 0.96 and at most $\sqrt{m} \log m \cdot \text{poly}(\log n, r, \epsilon^{-1})$ quantum gates, copies of $\rho$, and queries to Oracle 3.1 and Oracle 3.2

Proof. Similar to Corollary 3.5, the correctness of Algorithm 5 is automatically established by Theorem 2.3; it suffices to analyze the gate cost of Algorithm 5.

In Line 8 of Algorithm 5, we apply Lemma 3.2 to compute the Gibbs state $\rho^{(t)}$. In round $t$, because either

$$M^{(t)} = \frac{1}{2} (I_n - (-E^{(t)}_i + \text{Tr}[\rho E^{(t)}_i] I_n)) = \frac{1 - \text{Tr}[\rho E^{(t)}_i]}{2} I_n + \frac{1}{2} E^{(t)}_i$$

(4.6)

when Line 6 executes, or

$$M^{(t)} = \frac{1}{2} (I_n - (E^{(t)}_i - \text{Tr}[\rho E^{(t)}_i] I_n)) = \frac{1 + \text{Tr}[\rho E^{(t)}_i]}{2} I_n - \frac{1}{2} E^{(t)}_i$$

(4.7)
Algorithm 5: Efficiently learn a quantum state via measurements.

1. Initialize the weight matrix \( W^{(1)} = I_n \), and \( T = \frac{8\ln n}{\epsilon^2} \).
2. for \( t = 1, 2, \ldots, T \) do
   3. Prepare 2400 \( \sqrt{m} \log m \cdot \frac{r^2}{\epsilon^2} \log \frac{8\ln n}{\epsilon^2} \) samples of the Gibbs state \( \rho^{(t)} = \frac{W^{(t)}}{T r[W^{(t)}]} \) by Lemma 3.2 and take 2400 \( \sqrt{m} \log m \cdot \frac{r^2}{\epsilon^2} \log \frac{8\ln n}{\epsilon^2} \) copies of \( \rho \).
   4. Using these 2400 \( \sqrt{m} \log m \cdot \frac{r^2}{\epsilon^2} \log \frac{8\ln n}{\epsilon^2} \) copies of \( \rho^{(t)} \) and \( \rho \), apply Lemma 4.2 to search for a \( i^{(t)} \in [m] \) such that \( | T r[\rho E_{i^{(t)}}] - T r[\rho^{(t)} E_{i^{(t)}}] | \geq \epsilon \). if such \( i^{(t)} \) is found then
      5. if \( (T r[\rho E_{i^{(t)}}] - T r[\rho^{(t)} E_{i^{(t)}}]) \geq \epsilon \) then
         6. Take \( M^{(t)} = \frac{1}{2}(I_n - (E_{i^{(t)}} - T r[\rho E_{i^{(t)}]} I_n]) \);
      else \( (T r[\rho E_{i^{(t)}}] - T r[\rho^{(t)} E_{i^{(t)}}]) \leq -\epsilon \) then
         7. Take \( M^{(t)} = \frac{1}{2}(I_n - (E_{i^{(t)}} - T r[\rho E_{i^{(t)}]} I_n]) \);
   8. else (no such \( i^{(t)} \) exists)
      9. Claim \( \rho^{(t)} \) is the solution, and terminate the algorithm;
   10. Update the weight matrix: \( W^{(t+1)} = \exp[\frac{1}{t} \sum_{\tau=1}^{t} M^{(\tau)}] \).

when Line 8 executes, we can take \( K_+ = \frac{\epsilon}{2} \sum_{t=1}^{T} E_{i^{(t)}}^+ \) and \( K_- = \frac{\epsilon}{2} \sum_{t=1}^{T} E_{i^{(t)}}^- \), where \( E_{i^{(t)}}^+ = E_{i^{(t)}} \), \( E_{i^{(t)}}^- = 0 \) when (4.6) holds for round \( \tau \), and \( E_{i^{(t)}}^+ = 0, E_{i^{(t)}}^- = E_{i^{(t)}} \) when (4.7) holds for round \( \tau \). Because \( t \leq \frac{8\ln n}{\epsilon^2} \), \( K_+ \), \( K_- \) have rank at most \( t \cdot r = O(\log n \cdot r / \epsilon^2) \) and \( T r[K_+], T r[K_-] \) are at most \( \frac{\epsilon^2}{r} \cdot r = O(\log n \cdot r / \epsilon) \), Lemma 3.2 guarantees that

\[
\frac{8\ln n}{\epsilon^2} \cdot \text{poly} \left( \log n, \frac{r \log n}{\epsilon^2}, \frac{r \log n}{\epsilon}, \epsilon^{-1} \right) = \text{poly}(\log n, r, \epsilon^{-1}) \tag{4.8}
\]

quantum gates and queries to Oracle 3.1 and Oracle 3.2 suffice to prepare the Gibbs state \( \rho^{(t)} \). Because there are at most \( \frac{8\ln n}{\epsilon^2} = \text{poly}(\log n, \epsilon^{-1}) \) iterations and in each iteration \( \rho^{(t)} \) is prepared for 2400 \( \sqrt{m} \log m \cdot \frac{r^2}{\epsilon^2} \log \frac{8\ln n}{\epsilon^2} \) copies of \( \rho \) in the total gate cost for Gibbs state preparation is still \( \sqrt{m} \log m \cdot \text{poly}(\log n, r, \epsilon^{-1}) \).

Furthermore, by Lemma 4.2 Line 3 finds an \( i^{(t)} \in [m] \) such that \( | T r[\rho E_{i^{(t)}}] - T r[\rho^{(t)} E_{i^{(t)}}] | \geq \epsilon \) with success probability at least \( 1 - 0.005 \epsilon^2 \), using \( \sqrt{m} \log m \cdot \text{poly}(\log n, r, \epsilon^{-1}) \) quantum gates, copies of \( \rho \), and queries to Oracle 3.1 and Oracle 3.2. Because Algorithm 5 has at most \( \frac{8\ln n}{\epsilon^2} \) iterations, with success probability at least \( 1 - \frac{8\ln n}{\epsilon^2} \cdot 0.005 \epsilon^2 = 0.96 \) we can assume that the quantum search in Lemma 4.2 works correctly, and the total gate cost of calling Algorithm 5 is still \( \sqrt{m} \log m \cdot \text{poly}(\log n, r, \epsilon^{-1}) \).

In conclusion, \( \sqrt{m} \log m \cdot \text{poly}(\log n, r, \epsilon^{-1}) \) is an upper bound on the total number of quantum gates, copies of \( \rho \), and queries to Oracle 3.1 and Oracle 3.2.

5 Gibbs sampling of low-rank Hamiltonians

In this section we demonstrate how to sample from the Gibbs state of low-rank Hamiltonians given a quantum oracle generating desired states. We repeatedly use the following result of [17] (with a straightforward generalization in [15]):
Lemma 5.1 ([17, 15]). Suppose we are given a quantum oracle that prepares copies of two unknown (normalized) n-qubit quantum states $\rho^+$ and $\rho^-$, and we wish to evolve under the Hamiltonian $H = a_+\rho^+ - a_-\rho^-$ for some nonnegative numbers $a_+, a_- \geq 0$. Then we can approximately implement the unitary $\exp(iHt)$ up to diamond-norm error $\delta$, using $O(a^2 t^2 / \delta)$ copies of $\rho^+$ and $\rho^-$ and $O(na^2 t^2 / \delta)$ other 1- or 2-qubit gates, where $a = a_+ + a_-$. By using phase estimation on the operator $\exp(iHt)$ with $t = O(1/a)$, we have

Lemma 5.2. Under the same assumptions as Lemma 5.1, we can perform eigenvalue estimation of $H$: given an eigenstate of $H$, we can estimate its eigenvalue up to precision $\epsilon$, with probability $1 - \xi$, using $O(a^2 e^{-2}\xi^{-2})$ copies of $\rho^+$ and $\rho^-$ and $O(na^2 e^{-2}\xi^{-2})$ other 1- or 2-qubit gates, where $a = a_+ + a_-$. This procedure disturbs the input state by at most a trace distance error of $O(\sqrt{\xi})$.

In the following proof, we instead assume that eigenvalue estimation of $H$ can be done exactly. This assumption is not true, but it helps to simplify the exposition; the assumption will be removed in Appendix A.

5.1 Computing the partition function

As a warmup, we start with the following lemma:

Lemma 5.3. Suppose $K = K^+ - K^-$, where $K^+$ and $K^-$ are $n \times n$ PSD matrices, and there is a quantum oracle that prepares copies of the states $\rho^+ = K^+ / \text{Tr}(K^+)$, $\rho^- = K^- / \text{Tr}(K^-)$, and an oracle for the numbers $\text{Tr}(K^+), \text{Tr}(K^-)$. Moreover, assume that $\text{Tr}(K^+) + \text{Tr}(K^-) \leq B$ for some bound $B$, and that $K^+, K^-$ have rank at most $r$. Then it is possible to estimate the partition function $Z = \text{Tr}(\exp(-K))$ to multiplicative error $\epsilon$ with success probability at least $1 - \xi$, with $\text{poly}(\log n, r, B, \epsilon^{-1}, \xi^{-1})$ quantum gates.

Proof Sketch. As mentioned above, we assume that we can implement the unitary evolution $\exp(iKt)$ as well as the phase estimation protocol, perfectly to infinite precision. This idealization is made here for the sake of flashing out the core ideas behind the proposed protocol. These assumptions will be lifted in Lemma A.3, where a careful error analysis of this scheme is presented.

Under these assumptions, let us first consider the estimation of $Z_{\text{supp}} \equiv \sum_{|\lambda| \geq \delta} e^{-\lambda}$, where $0 < \delta < 1$ is a small threshold and $\lambda$'s are eigenvalues of $K$. Since $\delta$ is a small strictly-positive parameter, $Z_{\text{supp}}$ is the partition function when considering the approximated support of $K$.

The main idea in the estimation of $Z_{\text{supp}}$ is to perform phase estimation of the unitary operator $\exp(2\pi i K)$ on $\rho^+$ and $\rho^-$, after which we obtain

$$\rho^\pm = \frac{K^\pm}{\text{Tr}(K^\pm)} \rightarrow \rho^\pm \frac{1}{\text{Tr}(K^\pm)} \sum_{\lambda} \Pi_{\lambda} K^\pm \Pi_{\lambda} \otimes |\lambda\rangle \langle \lambda|,$$

where $\Pi_{\lambda}$ is the projection onto the $\lambda$-eigenspace of $K$, and $\lambda$ is any eigenvalue of $K$. Let us define

$$K^\pm_\lambda := \Pi_{\lambda} K^\pm \Pi_{\lambda}, \quad K^-_\lambda := \Pi_{\lambda} K^- \Pi_{\lambda}.\quad (5.2)$$

Then,

$$K^+_\lambda - K^-_\lambda = \Pi_{\lambda} K \Pi_{\lambda} = \lambda \Pi_{\lambda},$$

$$\lambda_\lambda = \sum_{|\lambda| \geq \delta} e^{-\lambda},$$

$$\sum_{\lambda \left| \lambda \right| \geq \delta} e^{-\lambda} = Z_{\text{supp}} \approx \sum_{\lambda \left| \lambda \right| \geq \delta} e^{-\lambda},$$

$\text{Tr}(K^\pm) \leq B$.
and therefore $K^+_\lambda$ and $K^-_{\lambda}$ differ by a multiple of the identity in their support space (the $\lambda$-eigenspace of $K$). Hence $K^+_\lambda$ and $K^-_{\lambda}$ are simultaneously diagonalizable, and their corresponding eigenvalues differ by exactly $\lambda$. In other words, there exists an eigenbasis of $K$, which we call $\{|v_i\}_i$, with corresponding eigenvalues $\lambda_i$, such that $K^+_\lambda$ and $K^-_{\lambda}$ are diagonal in this eigenbasis for all $\lambda$. We can therefore write

$$
K^+_\lambda = \sum_{\|v_i\|=\lambda} \lambda_i^+ |v_i\rangle \langle v_i|,
K^-_{\lambda} = \sum_{\|v_i\|=\lambda} \lambda_i^- |v_i\rangle \langle v_i|,
$$

(5.4)

for some nonnegative numbers $\lambda_i^+$, $\lambda_i^-$ satisfying $\lambda_i^+ - \lambda_i^- = \lambda_i$. Combining Eqs. (5.1) and (5.4), we obtain that $\tilde{\rho}^+ (\tilde{\rho}^-)$ – the state after performing phase estimation of the unitary operator $e^{2\pi i K}$ on $\rho^+ (\rho^-)$ is given by

$$
\tilde{\rho}^\pm = \frac{1}{\text{Tr}(K^\pm)} \sum_{\lambda_i} \lambda_i^\text{sgn} |v_i\rangle \langle v_i| \otimes |\lambda_i\rangle \langle \lambda_i|.
$$

(5.5)

Now consider the following procedure, and let its output be the random variable $X$:

**Algorithm 6: Estimation of $Z_{\text{supp}}$**

1. Let $\text{sgn} = +$ with probability $\text{Tr}(K^+)/[\text{Tr}(K^+) + \text{Tr}(K^-)]$, and $\text{sgn} = -$ otherwise.

2. Perform phase estimation of the operator $e^{2\pi i K}$ on $\rho^\text{sgn}$; Let the output state be

$$
\tilde{\rho}^\text{sgn} = \frac{1}{\text{Tr}(K^\text{sgn})} \sum_{\lambda_i} \lambda_i^\text{sgn}^\text{sgn} |v_i\rangle \langle v_i| \otimes |\lambda_i\rangle \langle \lambda_i|.
$$

Measure the second register and let the obtained eigenvalue of $K$ be $\lambda_i$.

3. If $|\lambda_i| < \delta$ output 0; else if $\text{sgn} = +$ output $\lambda_i^{-1} e^{-\lambda_i}$; else output $-\lambda_i^{-1} e^{-\lambda_i}$.

Then, under the assumption of perfect phase estimation, we have

$$
E[X] = \frac{\text{Tr}(K^+)}{\text{Tr}(K^+) + \text{Tr}(K^-)} \sum_{|\lambda_i| \geq \delta} \frac{\lambda_i^+}{\text{Tr}(K^+)} e^{-\lambda_i} - \frac{\text{Tr}(K^-)}{\text{Tr}(K^+) + \text{Tr}(K^-)} \sum_{|\lambda_i| \geq \delta} \frac{\lambda_i^-}{\text{Tr}(K^-)} e^{-\lambda_i}
$$

$$
= \frac{1}{\text{Tr}(K^+) + \text{Tr}(K^-)} \sum_{|\lambda_i| \geq \delta} e^{-\lambda_i} = \frac{Z_{\text{supp}}}{\text{Tr}(K^+) + \text{Tr}(K^-)}.
$$

(5.6)

where $\lambda_i^\pm$ are the eigenvalues of $K_i^\pm$, satisfying $\lambda_i^+ - \lambda_i^- = \lambda_i$. Therefore $E[X]$ is proportional to $Z_{\text{supp}}$, and obtaining a multiplicative estimate of $E[X]$ gives us a multiplicative estimate of $Z_{\text{supp}}$.

The second moment of $X$ reads

$$
E[X^2] = \frac{1}{\text{Tr}(K^+) + \text{Tr}(K^-)} \sum_{|\lambda_i| \geq \delta} (\lambda_i^+ + \lambda_i^-) e^{-2\lambda_i} \frac{\lambda_i^2}{\lambda_i^2} \leq \max_{|\lambda_i| \geq \delta} |\lambda_i|^{-2} e^{-2\lambda_i} \leq \delta^{-2} Z_{\text{supp}}^2.
$$

(5.7)

We see that $E[X^2] \leq B^2 \delta^{-2} E[X]^2$, and therefore by Chebyshev’s inequality we can obtain, with constant probability, an $\epsilon$-error multiplicative estimate of $E[X]$, hence of $Z_{\text{supp}}$, by running the above procedure $O(B^2 \delta^{-2} \epsilon^{-2})$ times and taking the mean.

We still need to calculate $Z$, the full partition function including small eigenvalues of $K$. Let $R$ denote the number of eigenvalues of $K$ (including degeneracy) with absolute value at least $\delta$, and note that $R \leq 2r$, where recall that $r$ upper bounds the rank of $K^+$ and $K^-$. Define the following approximation of $Z$:

$$
Z' \equiv Z_{\text{supp}} + (n - R) = \sum_{|\lambda_i| \geq \delta} e^{-\lambda_i} + \sum_{|\lambda_i| < \delta} e^0.
$$

(5.8)
Using \( e^\delta \leq 1 + 2\delta \) and \( e^{-\delta} \geq 1 - \delta \), we get that
\[
|Z - Z'| \leq 2\delta(n - R).
\] (5.9)

Therefore if we make \( \delta \) small enough, say \( \delta = O(\varepsilon) \), \( Z' \) gives a good multiplicative estimate for \( Z \).

To compute \( Z' \), we need a good multiplicative estimate of \( n - R \). This can essentially be done by estimating the probability of a random state having eigenvalue smaller than \( \delta \). Let the output of the following procedure be \( Y \):

**Algorithm 7:** Estimation of \( n - R \)

1. Perform phase estimation of the operator \( e^{2\pi iK} \) on the uniformly random state \( I/n \); let the output eigenvalue be \( \lambda \).
2. If \( |\lambda| < \delta \) output 1; otherwise output 0.

\( Y \) is a Bernoulli random variable with mean \( \mathbb{E}[Y] = (n - R)/n \) and variance \( \text{Var}[Y] = R(n - R)/n^2 \geq \mathbb{R}[Y]^2 \). By Chebyshev’s inequality, \( O(\varepsilon^{-2}) \) repetitions of the above procedure gives us an \( \varepsilon \)-error multiplicative estimate of \( \mathbb{E}[Y] \), and thus of \( n - R \).

Putting everything together, we see that \( O(B^2\varepsilon^{-4} + \varepsilon^{-2}) \) uses of (perfect) phase estimation of \( e^{2\pi iK} \) suffices to get a \( O(\varepsilon) \)-error multiplicative estimate of \( Z \), completing the proof.

5.2 Sampling from the Gibbs state

**Theorem 5.4** (Full proof deferred to Appendix A). Suppose \( K = K^+ - K^- \), where \( K^+ \) and \( K^- \) are \( n \times n \) PSD matrices, and there is a quantum oracle that prepares copies of the states \( \rho^+ = K^+ / \text{Tr}(K^+) \), \( \rho^- = K^- / \text{Tr}(K^-) \), and an oracle for the numbers \( \text{Tr}(K^+) \), \( \text{Tr}(K^-) \). Moreover, assume that \( \text{Tr}(K^+) + \text{Tr}(K^-) \leq B \) for some bound \( B \), and that \( K^+ \), \( K^- \) have rank at most \( r \). Then it is possible to prepare the Gibbs state \( \rho_G = \exp(-K) / \text{Tr}(\exp(-K)) \) to \( \varepsilon \) precision in trace distance, with \( \text{poly}(\log n, r, B, \varepsilon^{-1}) \) quantum gates.

**Proof Sketch.** Similar to the proof sketch of the partition function, here as well we assume an infinite precision implementation of the unitary evolution operator \( \exp(ikt) \) as well as of the phase estimation protocol. In addition we assume that quantum principal component analysis can be implemented perfectly. These assumptions will be lifted in Appendix A (Theorem A.4), where a complete proof is presented.

The procedure is somewhat similar to that of calculating the partition function above. We pick \( \delta = O(\varepsilon) \), a small threshold, and first consider a procedure to sample from
\[
\rho_{\text{supp}} \equiv \sum_{|\lambda_i| \geq \delta} e^{-\lambda} |v_i\rangle \langle v_i| / Z_{\text{supp}},
\] (5.10)
where \( \lambda_i \)'s and \( |v_i\rangle \)'s are eigenvalues and eigenstates of \( K \). (In the case that \( \rho_{\text{supp}} \) is undefined, i.e. that all eigenvalues of \( K \) have magnitude less than \( \delta \), it is easy to see that the uniformly mixed state \( I/n \) is already an \( O(\varepsilon) \)-trace distance error approximation to \( \rho_G \). This is the case when \( Z_{\text{supp}} = 0 \).) \( \rho_{\text{supp}} \) is the Gibbs state when considering only the (approximated) support of \( K \). Consider the procedure in Algorithm 8.

Note that \( \frac{Z_{\text{supp}}}{\sum_{|\lambda| \geq \delta} (1 - \varepsilon) e^{-\lambda_i}} \leq 1 \) since \( \lambda^\pm \geq 0 \) and by assumption \( |\lambda| = |\lambda^+ - \lambda^-| \geq \delta \), and \( Z'_{\text{supp}} \geq (1 - \varepsilon) Z_{\text{supp}} \geq e^{-\lambda_i} \) for all \( i \). Moreover assuming that \( K \) has at least one eigenvalue
Algorithm 8: Estimation of $\rho_{\text{supp}}$

1. Let $\text{sgn} = +$ with probability $\text{Tr}(K^+)/[\text{Tr}(K^+) + \text{Tr}(K^-)]$, and $\text{sgn} = -$ otherwise.

2. Perform phase estimation of the unitary operator $e^{2\pi i K}$ on $\rho_{\text{sgn}}$; let the output state be $\hat{\rho}_{\text{sgn}} = \frac{1}{\text{Tr}(K^{\supp})} \sum_{\lambda_i:|\lambda_i| \geq \delta} \lambda_i^{\text{sgn}} |v_i\rangle \langle v_i| \otimes |\lambda_i\rangle \langle \lambda_i|$.

3. Project $\hat{\rho}_{\text{sgn}}$ onto $\hat{\rho}_{\text{sgn}}' = \frac{1}{\text{Tr}(K^+) + \text{Tr}(K^-)} \sum_{\lambda_i:|\lambda_i| \geq \delta} (\lambda_i^+ + \lambda_i^-) |v_i\rangle \langle v_i| \otimes |\lambda_i\rangle \langle \lambda_i|$.

4. The average state at this stage is

$$\bar{\rho} = \frac{1}{\text{Tr}(K^+) + \text{Tr}(K^-)} \sum_{\lambda_i:|\lambda_i| \geq \delta} (\lambda_i^+ + \lambda_i^-) |v_i\rangle \langle v_i| \otimes |\lambda_i\rangle \langle \lambda_i|.$$

Perform quantum principal component analysis of $\bar{\rho}$; let the measured eigenvalue be $\mu = \lambda^+ + \lambda^-$, and the resulting state be $\rho_{\mu}$.

5. Accept the state $\rho_{\mu}$ with probability $\frac{\delta}{\mu} \frac{(1-e^{-\lambda_i})}{Z_{\text{supp}}'}$, for $Z_{\text{supp}}'$ a $\epsilon$-multiplicative error approximation of $Z_{\text{supp}}$.

with magnitude at least $\delta$, the overall success probability of the the above protocol can be lower bounded by

$$\frac{\delta}{\text{Tr}(K^+) + \text{Tr}(K^-)} \sum_{\lambda_i:|\lambda_i| \geq \delta} (1-e^{-\lambda_i}) Z_{\text{supp}}' = \frac{\delta(1-e)}{B(1+e)} \tag{5.11}$$

and therefore we can output $\rho_{\text{supp}}$ efficiently by repeating this algorithm until success, which takes $O(B/\delta)$ trials in expectation.

Accounting for the randomness in Step 1, at the end of step 3, we obtain the mixed state $\bar{\rho}$. However, for Gibbs sampling, we should have factors of the form $e^{-\lambda_i^+} |v_i\rangle \langle v_i|$ instead of $(\lambda_i^+ + \lambda_i^-) |v_i\rangle \langle v_i|$ that appear in $\bar{\rho}$. Therefore, at this stage of the protocol, to accept $|v_i\rangle \langle v_i|$ with probability proportional to $e^{-\lambda_i^+}/(\lambda_i^+ + \lambda_i^-)$, but for that we need to measure $\lambda_i^+ + \lambda_i^-$. This is done in steps 4 and 5 of the above procedure, which is equivalent to applying $\sum_{\lambda_i:|\lambda_i| \geq \delta} \frac{\delta}{\lambda_i^+ + \lambda_i^-} e^{-\lambda_i} Z_{\text{supp}}' |v_i\rangle \langle v_i| \otimes |\lambda_i\rangle \langle \lambda_i|$ to $\bar{\rho}$. Upon keeping only the first register we obtain

$$\frac{\delta}{\text{Tr}(K^+) + \text{Tr}(K^-)} \sum_{|\lambda_i| \geq \delta} e^{-\lambda_i} Z_{\text{supp}}' |v_i\rangle \langle v_i| \otimes |\lambda_i\rangle \langle \lambda_i| \equiv \rho_{\text{supp}}, \tag{5.12}$$

where $\rho_{\text{supp}}$ is the Gibbs state when considering only the (approximated) support of $K$.

We still need to calculate $\rho_G$, the full Gibbs state including small eigenvalues of $K$. Recall that $R$ denotes the number of eigenvalues (including degeneracy) of $K$ with absolute value at least $\delta$, and note that $R \leq 2r$, where $r$ upper bounds the rank of $K^+$ and $K^-$. Define the following approximation of $\rho_G$:

$$\rho'_G \equiv \frac{Z_{\text{supp}}}{Z'} \rho_{\text{supp}} + \frac{n-R}{Z'} \rho_{\text{ker}} = \frac{1}{Z'} \left( \sum_{|\lambda_i| \geq \delta} e^{-\lambda_i} |v_i\rangle \langle v_i| + \sum_{|\lambda_i| < \delta} |v_i\rangle \langle v_i| \right), \tag{5.13}$$
where $\rho_{\ker} = \frac{1}{Z'} \sum_{|\lambda_i| < \delta} |v_i\rangle \langle v_i|$, the uniformly random state on the orthogonal complement of the (approximate) support of $K$. Then

$$\|\rho_G - \rho'_G\|_{\text{Tr}} = \left|\frac{1}{Z} - \frac{1}{Z'}\right| \sum_{|\lambda_i| \geq \delta} e^{-\lambda_i} + \sum_{|\lambda_i| < \delta} \left|\frac{1}{Z} - \frac{1}{Z'}\right|$$

(5.14)

$$\leq \left|\frac{1}{Z} - \frac{1}{Z'}\right| \sum_{|\lambda_i| \geq \delta} e^{-\lambda_i} + \left[\sum_{|\lambda_i| < \delta} \left|\frac{1}{Z} - \frac{1}{Z'}\right| + 2\delta e^{-\lambda_i}\right]$$

(5.15)

$$\leq \left|\frac{1}{Z} - \frac{1}{Z'}\right| Z' + 2\delta \leq 4\delta.$$  

(5.16)

Therefore if we make $\delta$ small enough, $\rho'_G$ gives a good estimate (in trace distance) for $\rho_G$.

To estimate $\rho_{\ker}$ we consider the output of the following procedure:

**Algorithm 9: Estimation of $\rho_{\ker}$**

1. Perform phase estimation of the operator $e^{2\pi i K}$ on the uniformly random state $I/n$; let the output eigenvalue be $\lambda$ and the resulting state be $\Pi_\lambda$.
2. If $|\lambda| \geq \delta$ abort; otherwise, accept the state.

Finally, $\rho_G$ is generated by running the Algorithm 8 with probability $\frac{Z_{\text{supp}}}{Z}$ until we accept $\rho_{\text{supp}}$ and running the Algorithm 9 with probability $\frac{n-R}{Z}$ until we accept $\rho_{\ker}$.

In the previous Section we proved that, upon setting $\delta = O(\epsilon)$ we can obtain $Z_{\text{supp}}$, $Z$ and $n - R$ up to an $O(\epsilon)$ multiplicative error with poly($\log n, r, B, \epsilon^{-1}$) quantum gates. Therefore, Using Lemma 7 of [23] we obtain $\frac{Z_{\text{supp}}}{Z}$ and $\frac{n-R}{Z}$ to $O(\epsilon)$ multiplicative error. This, in turns, implies the with the above procedure we prepare the Gibbs state $\rho_G$ up to error $O(\epsilon)$ in trace distance, with poly($\log n, r, B, \epsilon^{-1}$) quantum gates.

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In this section we provide a complete proof of Lemma 5.3 and Theorem 5.4 given in Section 5.

A.1 Preliminaries

Before we start the proof, we first gather some preliminary facts that we need. First of all, the output of the phase estimation protocol is probabilistic and depends on the measurement. This is a problem since we often want the output of phase estimation to be consistent across multiple runs of the protocol. For example, when the eigenvalue is to our cutoff $\delta$, the phase estimation protocol may not be able to consistently decide whether it was greater than or less than $\delta$. To overcome this problem, in the proofs below we use Ta-Shma’s consistent phase estimation algorithm instead:

**Lemma A.1** ([21]). Let $U$ be a $D$-dimensional unitary matrix, and $\delta, \xi > 0$. There is a quantum algorithm that first chooses a random shift $s$, such that with probability at least $1 - D\xi$, the following holds for all eigenstates $|v_\lambda\rangle$ of $U$ (where $U|v_\lambda\rangle = e^{2\pi i \lambda}|v_\lambda\rangle$) (in this case we call $s$ a good shift):

- On input $|v_\lambda\rangle|0\rangle$, where $|0\rangle$ is a fixed reference state, the algorithm outputs a state $O(\sqrt{\xi})$-close to $|v_\lambda\rangle|f(s, \lambda)\rangle$ in trace distance.
- $f(s, \lambda)$ is a function only of $s$ and $\lambda$, and $|f(s, \lambda) - \lambda| < \delta$.

This algorithm requires $\text{poly}(\xi^{-1}, \delta^{-1})$ uses of the controlled-$U$ operation and other quantum gates.

The essential idea of this algorithm is to choose a random shift $s$, and perform phase estimation on $e^{is}U$ instead. If the precision $\delta$ is small enough, then with high probability over $s$, the eigenvalue $s + \lambda$ will always be far away from any half-multiple of $\delta$ (i.e. a number of the form $(z + 0.5)\delta$, $z \in \mathbb{Z}$), for all $\lambda$. The result of phase estimation will therefore (with high probability) depend only on $\lambda$, and not on the measurement.
Using the consistent phase estimation together with Lemma 5.1 we can straightforwardly derive the following lemma:

**Lemma A.2.** Suppose we are given a quantum oracle that prepares copies of two unknown (normalized) $n$-qubit quantum states $\rho^+$ and $\rho^-$, and define the Hamiltonian $H = a_+ \rho^+ - a_- \rho^-$. Also assume the ranks of $\rho^+$ and $\rho^-$ are upper bounded by $r$. Then for $\delta, \zeta > 0$, there is a quantum algorithm that first chooses a random shift $s$, such that with probability at least $1 - 2r\zeta^2$ the following holds for all eigenstates $|v_i\rangle$ of $H$, where $H|v_i\rangle = \lambda_i|v_i\rangle$ (we call such a $s$ a good shift):

- On input $|v_i\rangle |0\rangle$, where $|0\rangle$ is a fixed reference state, the algorithm outputs a state $O(\sqrt{\zeta})$-close to $|v_i\rangle |f(s, \lambda_i)\rangle$ in trace distance.
- $f(s, \lambda_i)$ is a function only of $s$ and $\lambda_i$, and $|f(s, \lambda_i) - \lambda_i| < \delta$.

This algorithm requires $\text{poly}(a^+ + a^-, \zeta^{-1}, \delta^{-1})$ copies of $\rho^+$ and $\rho^-$, and $\text{poly}(n, a^+ + a^-, \zeta^{-1}, \delta^{-1})$ 1- and 2-qubit quantum gates.

This, in particular, allows us to consistently estimate eigenvalues of $K = K^+ - K^-$ using $\text{poly}(\log n, B, \zeta^{-1}, \delta^{-1})$ operations in total.

For technical reasons, we will also need an approximation of the minimum eigenvalue of $K$ (possibly ignoring eigenvalues less than a threshold $\delta$). We use the following procedure:

**Algorithm 10: Estimation of minimum eigenvalue of $K$**

1. **Input:** Quantum oracles for $\rho^+, \rho^-$. A random good shift $s$ for eigenvalue estimation of $K$. Numbers $\delta, \gamma > 0$.
2. Use consistent phase estimation to estimate the eigenvalue of $K$ on $\rho^+$ and $\rho^-$, with precision $\delta$ and error probability $\zeta = O(B^{-1} \delta / \log \gamma^{-1})$. Discard the estimate if its absolute value is less than $\delta$.
3. Repeat Step 2 $\Theta(B \delta^{-1} \log \gamma^{-1})$ times, and output the minimum.

It is straightforward to see that with probability $1 - O(\gamma)$ the above algorithm outputs the minimum number $\lambda_{\min}$ such that $|\lambda_{\min}| \geq \delta$ and $\lambda_{\min} = f(s, \lambda)$ for some eigenvalue $\lambda$ of $K$.

Finally, for operators $A$ and $B$, we will use $A \approx_{O(\epsilon)} B$ to denote that $A$ is $O(\epsilon)$-close to $B$ in trace distance.

### A.2 Computing the partition function

We will prove the following lemma, using consistent phase estimation protocol:

**Lemma A.3.** Suppose $K = K^+ - K^-$, where $K^+$ and $K^-$ are $n \times n$ PSD matrices, and there is a quantum oracle that prepares copies of the states $\rho^+ = K^+ / \text{Tr}(K^+)$, $\rho^- = K^- / \text{Tr}(K^-)$, and an oracle for the numbers $\text{Tr}(K^+)$, $\text{Tr}(K^-)$. Moreover, assume that $\text{Tr}(K^+) + \text{Tr}(K^-) \leq B$ for some bound $B$, and that $K^+$, $K^-$ have rank at most $r$. Then it is possible to estimate the partition function $Z = \text{Tr}(\exp(-K))$ to multiplicative error $\epsilon$ with success probability at least $1 - \zeta$, with $\text{poly}(\log n, r, B, \epsilon^{-1}, \zeta^{-1})$ quantum gates.
Proof. As stated previously, we are using consistent phase estimation to unambiguously decide whether to keep an eigenvector in our approximate support. To be precise, choose \( \delta = O(\epsilon) \), \( \zeta = O(\epsilon^2 \delta^2 B^{-2} r^{-1}) \), and pick a random shift \( s \) - assume that this \( s \) is a good shift (this happens with probability \( 1 - O(\epsilon^2 \delta^2 B^{-2}) \)). Define \( \tilde{Z}_{\text{supp}} = \sum_{|f(s,\lambda)| \geq \delta} \epsilon^{-f(s,\lambda)} \), and consider Algorithm 11 for estimating \( \tilde{Z}_{\text{supp}} \) (let its output be \( \tilde{X} \)). In Step 3 we need to discard eigenvalues smaller than the approximate minimum eigenvalue \( \tilde{\lambda}_{\text{min}} \) to keep the expectation of \( \tilde{X} \) well-bounded. This is one consequence of possible error due to the application of the phase estimation procedure.

Another consequence, is that if some eigenvalues of \( K \) are close enough, they could be mapped to the same approximation \( \tilde{\lambda} \), and are therefore treated as degenerate. We will redo the analysis to illustrate this fact: Recall that \( \Pi_\lambda \) be the projection onto the \( \lambda \)-eigenspace of \( K \). Define the projector
\[
\tilde{\Pi}_\lambda = \sum_{\lambda : f(s,\lambda) = \lambda} \Pi_\lambda
\]  
(A.1)
to be the projector that projects onto the set of eigenvectors of \( K \), with eigenvalues that get mapped to \( \tilde{\lambda} \) under our consistent phase estimation procedure. We note that if we define the unnormalized states (here \( \text{id} \) stands for “ideal”)
\[
\tilde{K}_{\lambda,\text{id}}^+ = \tilde{\Pi}_\lambda K^+ \tilde{\Pi}_\lambda, \quad \tilde{K}_{\lambda,\text{id}}^- = \tilde{\Pi}_\lambda K^- \tilde{\Pi}_\lambda
\]  
(A.2)
then
\[
\tilde{K}_{\lambda,\text{id}}^+ - \tilde{K}_{\lambda,\text{id}}^- = \tilde{\Pi}_\lambda K \tilde{\Pi}_\lambda = \sum_{i : f(s,\lambda) = \lambda} \lambda |v_i\rangle \langle v_i| \approx \zeta \tilde{\lambda} \sum_{i : f(s,\lambda) = \lambda} |v_i\rangle \langle v_i| = \tilde{\lambda} \tilde{\Pi}_\lambda. \tag{A.3}
\]
Note that consistent phase estimation implements an operation \( O(\sqrt{\zeta}) \)-close to the operation \( \sum_{\lambda} \tilde{\Pi}_\lambda \otimes |\tilde{\lambda}\rangle \langle \tilde{\lambda}| \), and therefore
\[
\tilde{\rho}_{\text{sgn}} = \frac{1}{\text{Tr}[\tilde{K}_{\text{sgn}}]} \sum_{\lambda} \tilde{K}_{\lambda,\text{id}}^\text{sgn} \otimes |\tilde{\lambda}\rangle \langle \tilde{\lambda}| \approx O(\sqrt{\zeta}) \quad \frac{1}{\text{Tr}[\tilde{K}_{\text{sgn}}]} \sum_{\lambda} \tilde{\Pi}_\lambda K_{\text{sgn}} \tilde{\Pi}_\lambda \otimes |\tilde{\lambda}\rangle \langle \tilde{\lambda}| = \frac{1}{\text{Tr}[\tilde{K}_{\text{sgn}}]} \sum_{\lambda} \tilde{K}_{\lambda,\text{id}}^\text{sgn} \otimes |\tilde{\lambda}\rangle \langle \tilde{\lambda}|. \tag{A.4}
\]
Thus \( \tilde{K}_{\lambda,\text{id}}^\text{sgn} \approx O(B \sqrt{\zeta}) \tilde{K}_{\lambda,\text{id}}^\text{sgn} \), and hence
\[
\tilde{K}_{\lambda}^+ - \tilde{K}_{\lambda}^- \approx O(B \sqrt{\zeta}) \tilde{\lambda} \tilde{\Pi}_\lambda. \tag{A.5}
\]

---

**Algorithm 11: Estimation of \( \tilde{Z}_{\text{supp}} \)**

1. Let \( \text{sgn} = + \) with probability \( \text{Tr}(K^+)/(\text{Tr}(K^+) + \text{Tr}(K^-)) \), and \( \text{sgn} = - \) otherwise.
2. Use consistent phase estimation to perform eigenvalue estimation of \( K \) on \( \rho_{\text{sgn}} \); let the output be the normalized state \( \tilde{\rho}_{\text{sgn}} = \frac{1}{\text{Tr}[\tilde{K}_{\text{sgn}}]} \sum_{\tilde{\lambda}} \tilde{K}_{\lambda,\text{id}}^\text{sgn} \otimes |\tilde{\lambda}\rangle \langle \tilde{\lambda}| \) for some unnormalized states \( \tilde{K}_{\lambda,\text{id}}^\text{sgn} \). Measure the obtained eigenvalue to obtain some \( \tilde{\lambda} \). With probability at least \( 1 - O(\zeta) \), \( \tilde{\lambda} = f(s,\lambda) \) for some eigenvalue \( \lambda \) of \( K \).
3. If \( |\tilde{\lambda}| < \delta \) or \( \tilde{\lambda} < \tilde{\lambda}_{\text{min}} \) output 0; else if \( \text{sgn} = + \) output \( \tilde{\lambda} e^{-\tilde{\lambda}} \), else output \( -\tilde{\lambda} e^{-\tilde{\lambda}} \).
We see that the consistent phase estimation of Step 2 serves to approximately project $\rho^+$ or $\rho^-$ onto the span of eigenvectors of $K$ with eigenvalue approximately equal to some $\lambda$; and on this space, the unnormalized output states at Step 2 approximately differ only on by a multiple of the identity on their support. There is therefore a basis of vectors $\{|\tilde{v}_i\rangle\}$ where $K^+_\lambda$ and $K^-_\lambda$ are approximately diagonal for all $\lambda$. These vectors are approximate eigenvectors of $K$, i.e.

$$\|K|\tilde{v}_i\rangle - \lambda_i|\tilde{v}_i\rangle\| = O(\xi)$$

(A.6) for some numbers $\lambda_i$.\(^8\) Working in the approximate eigenbasis basis, we can write

$$\tilde{K}^+_\lambda \approx_{O(\sqrt{\xi})} \sum_i \tilde{\lambda}^+_i |\tilde{v}_i\rangle \langle \tilde{v}_i|, \quad \tilde{K}^-_\lambda \approx_{O(\sqrt{\xi})} \sum_i \tilde{\lambda}^-_i |\tilde{v}_i\rangle \langle \tilde{v}_i|$$

(A.7) for nonnegative $\lambda^+_i - \lambda^-_i = \lambda$. This gives the following approximation for $\tilde{\rho}^{\text{sgn}}$:

$$\tilde{\rho}^{\text{sgn}} \approx_{O(\sqrt{\xi})} \frac{1}{\text{Tr}[K^{\text{sgn}}]} \sum_i \tilde{\lambda}^{\text{sgn}}_i |\tilde{v}_i\rangle \langle \tilde{v}_i| \otimes |\tilde{\lambda}_i\rangle \langle \tilde{\lambda}_i|.$$  

(A.8)

Therefore the expectation of $\tilde{X}$ is upper bounded by

$${\mathbb{E}}[\tilde{X}] \leq \frac{\sum_i e^{-\lambda_i}}{\text{Tr}[K^+] + \text{Tr}[K^-]} + O(\sqrt{\xi}) \max_{i:\tilde{\lambda}_i \geq \lambda_{\min}} \lambda_i^{-1} e^{-\lambda}$$

(A.9)

\leq (1 + \delta) \frac{\tilde{Z}_{\text{supp}}}{\text{Tr}[K^+] + \text{Tr}[K^-]} + O(\sqrt{\xi} \delta^{-1}) \tilde{Z}_{\text{supp}}$$

(A.10)

\leq (1 + \delta + \sqrt{\xi} \delta^{-1}B) \frac{\tilde{Z}_{\text{supp}}}{\text{Tr}[K^+] + \text{Tr}[K^-]}$$

(A.11)

\leq (1 + O(\epsilon)) \frac{\tilde{Z}_{\text{supp}}}{\text{Tr}[K^+] + \text{Tr}[K^-]}.$$  

(A.12)

A similar bound holds for lower bounding $\mathbb{E}[\tilde{X}]$, showing that knowing $\mathbb{E}[\tilde{X}]$ would give a $O(\epsilon)$-multiplicative error approximation to $\tilde{Z}_{\text{supp}}$. Just as in the ideal case, we can simply repeat our procedure $O(\eta^{-1}B^2\delta^{-2}e^{-2})$ times and take the mean to obtain a $O(\epsilon)$-multiplicative error approximation of $\mathbb{E}[\tilde{X}]$, and hence of $\tilde{Z}_{\text{supp}}$.

As before, we also need to estimate the number of eigenvalues $\lambda$ (including degeneracy) with $|f(s, \lambda)| < \delta$, i.e. the number of $i$'s with $|\tilde{\lambda}_i| < \delta$. Let this number be $n - \tilde{R}$. Let the output of the following procedure be $\tilde{Y}$:

**Algorithm 12: Estimation of $n - \tilde{R}$**

1. Perform consistent phase estimation to estimate eigenvalues of $K$ on the uniformly random state $I/n$; let the output eigenvalue be $\lambda$.

2. If $|\lambda| < \delta$ output 1; otherwise output 0.

\(^8\)Note that the basis $\{|\tilde{v}_i\rangle\}$ and exact eigenbasis of $K$, $\{|v_i\rangle\}$, are not necessarily equivalent, because the vectors in the former are only approximate eigenvectors of $K$. 

24
It is clear that $n \mathbb{E}[\bar{Y}]$ is an $O(r\sqrt{\xi})$-multiplicative error approximation of $n - \bar{R}$, and it can be proven as before that $O(re^{-2})$ repetitions of the above procedure suffice to give an $O(\epsilon)$-error multiplicative stimate of $(n - \bar{R})/n$. It can again be argued that $O(re^{-2})$ repetitions suffice to estimate $\mathbb{E}[\bar{Y}]$, and thus $n - \bar{R}$, to $O(\epsilon)$-multiplicative error.

Finally, to estimate the full partition function we merely note that $\bar{Z}_{\text{supp}} + (n - \bar{R})$ is an $O(\delta + \epsilon) = O(\epsilon)$-multiplicative error estimate of the partition function $Z$; we can therefore estimate $Z$ by estimating both terms separately and taking the sum. \hfill \square

### A.3 Computing the Gibbs function

In this section we prove the following result:

**Theorem A.4.** Suppose $K = K^+ - K^-$, where $K^+$ and $K^-$ are $n \times n$ PSD matrices, and there is a quantum oracle that prepares copies of the states $\rho^+ = K^+ / \text{Tr}(K^+)$, $\rho^- = K^- / \text{Tr}(K^-)$, and an oracle for the numbers $\text{Tr}(K^+), \text{Tr}(K^-)$. Moreover, assume that $\text{Tr}(K^+) + \text{Tr}(K^-) \leq B$ for some bound $B$, and that $K^+, K^-$ have rank at most $r$. Then it is possible to prepare the Gibbs state $\rho_G = \exp(-K) / \text{Tr}(\exp(-K))$ up to error $\epsilon$ in trace distance, with poly($\log n, r, B, e^{-1}$) quantum gates.

**Proof.** The procedure will the sketch given in [Section 5.2](#) but using consistent phase estimation rather than the naïve protocol. We again assume we chose a good shift $s$ for the operator $K$, and first consider a procedure to sample from

$$
\bar{\rho}_{\text{supp}} \equiv \sum_{i: |\lambda_i| \geq \delta} e^{-\lambda_i} |\phi_i\rangle \langle \phi_i| / \bar{Z}_{\text{supp}},
$$

where $\delta > 0$ is a small threshold and $\lambda_i, |\phi_i\rangle$ were defined previous in [A.6](#). $\rho_{\text{supp}}$ is the Gibbs state when considering only the space spanned by approximate eigenvectors of $K$ whose eigenvalues estimates (under consistent phase estimation) are at least $\delta$ in absolute value. Again choose $\delta = O(\epsilon)$, $\xi = O(\epsilon^2 \delta^2 B^{-2} e^{-1})$, and pick a good random shift $s$ - assume that this $s$ (this happens with probability $1 - O(\epsilon^2 \delta^2 B^{-2})$). Consider [Algorithm 13](#) below.

**Algorithm 13** will give us a good approximation for $\bar{\rho}_{\text{supp}}$, the Gibbs state on the approximate support of $K$ (ignoring small eigenvalues). As before, we will need to approximate the Gibb state on the approximate kernel of $K$ as well, which we define as

$$
\bar{\rho}_{\text{ker}} = \frac{1}{n - \bar{R}} \sum_{i, \lambda_i} |\psi_i\rangle \langle \psi_i|.
$$

This state can easily be approximated by starting with the completely mixed state $I/n$ and performing consistent phase estimation to estimate eigenvalues of $K$, postselecting on the case that the measured estimate is less than $\delta$ in magnitude.

To complete our estimation for the full Gibbs state, we see that $\rho_G = \exp(-K) / \text{Tr}(\exp(-K))$...
Algorithm 13: Estimation of $\hat{\rho}_{\text{supp}}$

1. **Input:** A good random shift $s$, an $O(\epsilon)$-multiplicative error estimate $\hat{Z}_{\text{supp}}'$ of $\hat{Z}_{\text{supp}}$, quantum oracles for $\rho^+, \rho^-$. 

2. Let $\text{sgn} = +$ with probability \( \frac{\text{Tr}(K^+)}{\text{Tr}(K^+) + \text{Tr}(K^-)} \) and $\text{sgn} = -$ otherwise. 

3. Use consistent phase estimation to perform eigenvalue estimation of $K$ on $\rho^\text{sgn}$; let the output be the normalized state $\tilde{\rho}^\text{sgn} = \frac{1}{\text{Tr}[K^\text{sgn}]} \sum_i \tilde{K}_\lambda^\text{sgn} \otimes |\tilde{\lambda}_i\rangle \langle \tilde{\lambda}_i| $ for unnormalized states $\tilde{K}_\lambda^\text{sgn}$. Including the randomness on choosing sgn, we have the state

$$\tilde{\rho} = \frac{\text{Tr}[K^+]}{\text{Tr}[K^+] + \text{Tr}[K^-]} \hat{\rho}^+ + \frac{\text{Tr}[K^-]}{\text{Tr}[K^+] + \text{Tr}[K^-]} \hat{\rho}^-$$  \hspace{1cm} (A.14) 

$$\approx O(\sqrt{\xi}) \frac{1}{\text{Tr}[K^+] + \text{Tr}[K^-]} \sum_i (\tilde{\lambda}_i^+ + \tilde{\lambda}_i^-) |\tilde{\lambda}_i\rangle \langle \tilde{\lambda}_i| \otimes |\tilde{\lambda}_i\rangle \langle \tilde{\lambda}_i|.$$  \hspace{1cm} (A.15) 

Here in the second line we used the approximation (A.8).

4. Apply the projection $I \otimes \sum_{|\tilde{\lambda}| < \delta, \tilde{\lambda} \geq \tilde{\lambda}_{\text{min}}} |\tilde{\lambda}\rangle \langle \tilde{\lambda}|$. In other words, measure the second register to make sure that $|\tilde{\lambda}| \geq \delta$ and $\tilde{\lambda} \geq \tilde{\lambda}_{\text{min}}$, and reject otherwise.

5. Apply the measurement operator

$$\sum_{i:|\tilde{\lambda}_i| \geq \delta, \tilde{\lambda}_i \geq \tilde{\lambda}_{\text{min}}} \frac{\delta}{\tilde{\lambda}^+ + \tilde{\lambda}^-} e^{-\tilde{\lambda}} |\tilde{\lambda}_i\rangle \langle \tilde{\lambda}_i| \otimes |\tilde{\lambda}_i\rangle \langle \tilde{\lambda}_i|,$$  \hspace{1cm} (A.16) 

up to $O(\sqrt{\xi})$ error, to the state. We can do this by first estimating $\tilde{\lambda}^+ + \tilde{\lambda}^-$ by quantum principal analysis (i.e. phase estimation of $\hat{\rho}$), to precision $O(\sqrt{\xi} \delta)$ and error probability $O(\sqrt{\xi})$; then accept the resulting state with probability approximately $\frac{\delta}{\tilde{\lambda}^+ + \tilde{\lambda}^-} e^{-\tilde{\lambda}}$.
can be approximated by
\[ \rho_G \approx O(\delta) \frac{1}{Z} \sum_i e^{-\lambda_i |\tilde{v}_i|} \langle \tilde{v}_i \rangle \]  
(A.18)

\[ = \frac{\tilde{Z}_{\text{supp}}}{Z_{\text{supp}}} \frac{1}{Z} \sum_{i:|\lambda_i| \geq \delta} e^{-\lambda_i |\tilde{v}_i|} \langle \tilde{v}_i \rangle + \frac{n - \tilde{R}}{Z} \sum_{i:|\lambda_i| < \delta} e^{-\lambda_i |\tilde{v}_i|} \langle \tilde{v}_i \rangle \]  
(A.19)

\[ \approx O(\epsilon) \frac{\tilde{Z}_{\text{supp}}}{Z} \tilde{\rho}_{\text{supp}} + \frac{n - \tilde{R}}{Z} \tilde{\rho}_{\text{ker}}. \]  
(A.20)

Thus by Lemma 7 of [23], it suffices to have O(\epsilon)-multiplicative error estimates for \( \tilde{Z}_{\text{supp}}, n - \tilde{R} \), and Z, and O(\epsilon)-trace distance error approximations for \( \tilde{\rho}_{\text{supp}} \) and \( \tilde{\rho}_{\text{ker}} \). We have already shown how to achieve all of this. \( \square \)