Bifurcation analysis of a prey-predator model with predator intra-specific interactions and ratio-dependent functional response

Claudio Arancibia–Ibarra\(^1,2\) & José D. Flores\(^3\)

\(^1\)School of Mathematical Sciences, Queensland University of Technology (QUT), Brisbane, Australia.
\(^2\)Facultad de Educación, Universidad de Las Américas (UDLA), Santiago, Chile.
\(^3\)Department of Mathematics, The University of South Dakota (USD), Vermillion, South Dakota, USA

Email: claudio.arancibia@hdr.qut.edu.au, Jose.Flores@usd.edu

August 20, 2019

Abstract

We study a predator-prey model with predator intra-specific interactions and ratio-dependent functional response. We show that the model has at most two equilibrium points in the first quadrant, one is always a saddle point while the other can be a repeller or an attractor. Moreover, we show that when the parameters are varied the model displays a wide range of different bifurcations, such as saddle-node bifurcations, Hopf bifurcations and homoclinic bifurcations. We use numerical simulations to illustrate the impact changing the predator per capita consumption rate, or the efficiency with which predators convert consumed prey into new predators.

Keywords— Predator-prey model, Bifurcations, Ratio-dependent functional response.

1 Introduction

The original Lotka-Volterra predator prey models \([24]\) were straightforward, with simple functional forms for species growth and interactions. Empirical observations required successive changes to these assumptions, leading inter alia to the Bazykin model \([25]\). Ratio-dependent predator-prey models \([17, 19, 27]\) are becoming more interesting in Ecology since this models are better alternative for describing both the theoretical and experimental predator-prey interactions \([10]\). For instance, Jost and Arditi \([21]\) found that the ratio-dependent predator-prey models are more suitable for showing the dynamics between predators and preys. In particular when the predators involve serious hunting processes such as animals searching for animals. The Bazykin model \([25]\) with ratio-dependent functional response is described by an autonomous two-dimensional system of ordinary differential equations, where the equations for the growth of the prey is a logistic-type function \([1, 13, 26]\). The functional response is a ratio-dependent, in which the feeding rate is determined by the ratio of resource biomass to consumer biomass \([23]\). In particular, the model is given by

\[
\begin{align*}
\frac{dx}{dt} &= r x \left(1 - \frac{x}{K}\right) - \frac{q x y}{x + ay}, \\
\frac{dy}{dt} &= \frac{c x y}{x + ay} - \mu_0 y - \mu_1 y^2.
\end{align*}
\]

(1)

Here, \(x(t)\) and \(y(t)\) represent the proportion of the prey respectively predator population at time \(t\); \(r\) is the intrinsic growth rate for the prey; \(K\) is the prey carrying capacity; \(q\) is the per capita predation rate; \(a\) is the amount of prey by which the predation effect is maximum; \(c\) is the efficiency with which predators convert consumed prey into new predators; \(\mu_0\) is the per capita death rate of predators and \(\mu_1\) is predator death rate by density.

The aim of this manuscript is to study the bifurcation dynamics of (1) and, in particular, understanding the change in dynamics the ratio-dependent functional response causes. Moreover, we will show that the model (1) will lead to complex dynamics, and different types of bifurcations such as Hopf bifurcations, homoclinic bifurcations, saddle-node bifurcations and Bogdanov-Takens bifurcations.
The basic properties of the model are briefly described in Section 2. In Section 3 we prove the stability of the equilibrium points for this system and give the conditions for the different types of bifurcations. In addition, we discuss the impact changing the predation rate or the efficiency with which predators convert consumed prey into new predators has on the basins of attraction of the positive equilibrium point in system (1). We further discuss the results and give the ecological implications in Section 4.

2 The model

The ratio-dependent Bazykin predator-prey model is given by (1) and for biological reasons we only consider the model in the domain \( \Omega = \{(x, y) \in \mathbb{R}^2, \; x > 0, \; y > 0\} \). In order to simplify the analysis follow [3, 4, 16] and we introduce the dimensionless variable given by the function

\[
\varphi : \Omega \times \mathbb{R} \rightarrow \Omega \times \mathbb{R}
\]

where \( \varphi (u, v, \tau) = \left( \frac{x}{K}, \frac{ay}{K}, \frac{rKt}{x + ay} \right) \) \( \tau \in \mathbb{R} \).

By substitution \( C := c, M := \mu_0/r, N := \mu_1 K/(ar) \) and \( Q := q/(ar) \) into (1) we obtain

\[
\frac{du}{d\tau} = u(1 - u)(u + v) - Quv,
\]

\[
\frac{dv}{d\tau} = Cu - v(u + v)(M + Nv).
\]

System (3) is defined in \( \Omega = \{(u, v) \in \mathbb{R}^2, \; u \geq 0, \; v \geq 0\} \) and we first recall the stability of the equilibrium points of system (3). Additionally, we have constructed a diffeomorphism \( \varphi (2) \) which preserve the orientation of time since \( \det (\varphi (u, v, \tau)) = K^2 (u^2 + v^2) / (ar) > 0 \) [2, 9]. Moreover, system (3) is of Kolmogorov [12] type since \( du/d\tau = uW(u, v) \) and \( dv/d\tau = vR(u, v) \) with

\[
W(u, v) = (1 - u)(u + v) - Qv \quad \text{and} \quad R(u, v) = Cu - (u + v)(M + Nv).
\]

That is, the axes \( u = 0 \) and \( v = 0 \) are invariant. The \( u \) nullclines are \( u = 0 \) and \( v = u(1 - u) / (Q + 1 + u) \), while the \( v \) nullclines are \( v = 0 \) and \( v = \left\{- (M + Nu) + \sqrt{(M + Nu)^2 + 4Nu(C - M)}\right\} / (2N) \). Hence, the equilibrium points for this system are \((0, 0)\), \((1, 0)\) and up to two interior positive equilibrium points (see Figure 3). \( P_1 = (u_1, v_1) \) and \( P_2 = (u_2, v_2) \), where \( u_1, u_2, v_1 \) and \( v_2 \) are given by

\[
u_{1,2} = \frac{2C(1 - Q) + Q(M + N) \pm Q\sqrt{\Delta}}{2(C + NQ)} \quad \text{and} \quad v_{1,2} = \frac{-(2MNQ + CM + CN(1 - 2Q)) \pm C\sqrt{\Delta}}{2N(C + NQ)}
\]

\[
\Delta = (M - N)^2 + 4N(C - CQ + MQ).
\]

Note that \( u_1 < u_2, \quad v_1 < v_2 \) and \( u_1 > 0 \) if only if \((Q - 1)(C - CQ + MQ) < 0\).

3 Main Results

In this section, we discuss the stability of the equilibrium points of system (3).

Lemma 3.1 The set \( \Gamma = \{(u, v) \in \Omega, \; 0 \leq u \leq 1, \; v \geq 0\} \) is an invariant region and all solutions of (3) which are initiated in the first quadrant are bounded.

Proof. Since the system (3) is of Kolmogorov type the coordinates axes are invariant [15]. Moreover, if \( u \geq 0 \) and \( v = 0 \) then \( du/d\tau = u^2(1 - u) \) and if \( u = 0 \) and \( v \geq 0 \) the \( dv/d\tau = -v^2(M + Nv) \), so that any trajectory with initial point in the positive vertical \( v \)-axis tends to zero and any trajectory with initial point on the positive horizontal \( u \)-axis tend to \( u = 1 \). Next, setting \( u = 1 \) in in the first equation of system (3) (the scaled carrying capacity), we have \( du/d\tau = -Cu < 0 \) and thus for any initial point born along the vertical line \( u = 1 \), the corresponding trajectory enters and remains in \( \Gamma \) for any sign of \( dv/d\tau = Cu - v(u + v)(M + Nv) \).

Now we shall prove that no trajectory in the open region \( \Gamma_1 = \{(x, y) \in \Pi : 0 < x < 1\} \) converges to infinity as \( t \to \infty \). To do so, it suffices to study the behaviour of the solutions in \( \Gamma_1 \) at infinity as \( v \to \infty \) using the Poincaré compactification [13, 22]. Let us consider the the change of variables

\[
(u, v) \rightarrow \left( \frac{x}{y}, \frac{1}{y} \right), \quad x \geq 0, \; y > 0.
\]
Figure 1: Isoclines and equilibria of system (3); several possible scenarios for prey isocline (red curve) and the corresponding equilibria by changing the parameter $Q$. Illustrating that it is possible to have none, one or two interior equilibrium points.

Together with the time rescaling $t \to y^2 t$. This transformation defines the following system:

$$\frac{dx}{dt} = x(1 + x)(N - x) + (1 + M - Q + x - Cx + Mx)xy,$$

$$\frac{dy}{dt} = N(1 + x)y + (M - Cx + Mx)y^2. \tag{6}$$

The origin $O_{xy}(0,0)$ is a critical point of (6) whose Jacobian matrix is

$$J_{O_{xy}}(0,0) = \begin{pmatrix} N & 0 \\ 0 & N \end{pmatrix}$$

indicating that the origin $O_{xy}(0,0)$ is a non-hyperbolic point. Moreover, a neighbourhood of the origin $O_{xy}(0,0)$ presents four types of topologically different structures in the first quadrant of the phase-plane.

(a) A saddle sector and a repelling node (see Figure 3, Region I),
(b) an elliptic sector (see Figure 3, Region II),
(c) a saddle sector (see Figure 3, Region III) and
(d) a saddle sector and an attracting node (see Figure 3, Region IV).

**Proof.** First we observe that in system (3) setting $u = 0$, then $dv/d\tau = -v^2(M + Nv) < 0$ for $v \leq 0$. That is any trajectory starting along the $v-$axis converges to the origin $O = (0,0)$. Also, setting $v = 0$, $du/d\tau = u^2(1 - u)$ and any orbits starting along the $u-$axis near the origin converges to the scale carrying capacity $(1,0)$. Next to analyse the dynamics in a neighbourhood of the origin we consider the vertical blow-up given by the transformation

$$(u, v) \to (xy, y) \text{ and the time rescaling } \tau \to \frac{t}{y}.$$
The transformation (8) ‘blows-up’ the origin of system (3) in the entire \( y-axis \) \[12\]. Our goal is to analyse the equilibria at the positive half axis \( x \leq 0, y \geq 0 \), in the new system, which is given by setting (8) in system (3) as follow \[dx\] 
\[\frac{dx}{dt} = x(x + y)(N - x) + (x + 1)(1 + M) - Q - Cx, \]
\[dy\]
\[\frac{dy}{dt} = y(Cx - (x + 1)(M + Ny)). \]
System (9) has two equilibria in the positive horizontal \( x-axis \) of the form \((x, 0)\) with \( x \leq 0 \). The origin \( O_{xy} = (0, 0) \) and a second equilibrium point \( I_x = (\mu, 0) \) with \( \mu = (1 + M - Q)/(C - M - 1) \). Their corresponding Jacobian matrices at \( O_{xy} = (0, 0) \) is
\[J_{O_{xy}}(0, 0) = \begin{pmatrix} 1 + M - Q & 0 \\ 0 & -M \end{pmatrix}\]
with eigenvalues \( \lambda_1(O_{xy}) = 1 + M - Q \) and \( \lambda_2(O_{xy}) = -M \).

For \( I_x \), it is given by setting (10) in system (3)\[\frac{dx}{dt} = \begin{pmatrix} -1 - M + Q \\ 0 \end{pmatrix}, \]
\[\begin{pmatrix} C - Q \left(1 + M - Q\right)(N(C - 1) - (M + N + Q - 1) \\ (C - Q)(1 + M - Q)(C - Q + MQ) \end{pmatrix}\]
with eigenvalues \( \lambda_1(I_x) = -1 - M + Q \) and \( \lambda_2(I_x) = \frac{C - MQ + MQ}{C - M - 1} \).

Depending on the parameters \( C, Q \) and \( M \) (see Table 2), we summarise the the stability of \( O_{xy} \) and \( I_x \) as follows
(a) \( O_{xy} \) is a saddle if \( Q < M + 1 \), see Regions I and II in Figure 2
(b) \( O_{xy} \) is a stable node if \( Q > M + 1 \), see Regions III and IV in Figure 2
(c) \( I_x \) is a saddle if \( Q < M + 1, C > M + 1 \) and \( C - MQ + MQ > 0 \), see Regions I and IV in Figure 2
(d) \( I_x \) is an unstable node if \( Q > M + 1, C < M + 1 \) and \( C - MQ + MQ < 0 \), see Region II in Figure 2
(e) \( I_x \) does not exist in regions II and III in Figure 2

Next, we repeat the blow-up procedure to analyse the behaviour of system (7) near the \( v-axis \) shows the curves \( C - MQ + MQ = 0, C = M + 1 \) and \( Q = M + 1 \) in the \((Q, C)\) parameter-space together with the different open regions, bounded 1-4, respectively (see Figure 2). Thus we consider the horizontal blow-up given by the transformation \[(u, v) \rightarrow (x, xy) \text{ and the time rescaling } \tau \rightarrow \frac{\tau}{x} \]

The goal now is to analyse the equilibria at the positive half axis \( x = 0, y \leq 0 \), in the new system which is given by setting (10) in system (3)\[\frac{dx}{dt} = x(1 - x + y(1 - Q - x)), \]
\[\frac{dy}{dt} = y(C - 1 + y(Q - 1) + (M + x)(1 + y)(1 - Ny)). \]
System (11) has two equilibria in the positive vertical \( y-axis \) of the form \((0, y)\) with \( y \leq 0 \). The equilibrium \( O_{xy} = (0, 0) \) is new (which is not in the map \((xy, y)\)) and a second equilibrium point \( I_y^* = (0, \mu) \) with \( \mu = (C - M - 1)/(1 + M - Q) \) which corresponds to the point \( I_y \) and it does not need to be analysed again. The Jacobian matrix at the origin \( O_{xy} = (0, 0) \) is given by
\[J_{O_{xy}}(0, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \]
with eigenvalues \( \lambda_1(O_{xy}) = 1 \) and \( \lambda_2(O_{xy}) = C - 1 - M \).

Hence the the stability \( O_{xy} \) depend on the parameters \( C \) and \( M \) (see Table 3). Therefore,\[O_{xy} \text{ is a saddle if } C < M + 1, \text{ see Regions II, III and IV in Figure 2}\]
Figure 2: Parametric diagram with the different structures in a neighbourhood of the origin $O = (0,0)$ in the $(Q,C)$-parameter-space of system (3).
Eigenvalues | Value | Regions in Figure 2
--- | --- | --- | --- | --- | --- | --- | ---
| | | I | II | III | IV |
\(\lambda_1(O_{xy})\) | \(1 + M - Q\) | + | + | − | − | + | − |
\(\lambda_2(O_{xy})\) | \(-M\) | − | − | − | − | − | − |
\(\lambda_1(I_x)\) | \(-1 - M + 1\) | − | − | * | + | * | + |
\(\lambda_2(I_x)\) | \(C - CQ + MQ\) | + | − | * | + | * | − |
\(\lambda_1(O_{xy}^*)\) | \(1\) | + | + | + | + | + | + |
\(\lambda_2(O_{xy}^*)\) | \(C - 1 - M\) | + | + | − | − | − | − |

Table 1: Signs of the eigenvalues according to the expressions \(C - 1 - M, 1 + M - Q\) and \(C - CQ + MQ\) in the open regions in Figure 2. Note that (*) means the point \(I_x\) does not exist.

Figure 3: For \(Q = 3.05, C = 10.05, M = 1.05,\) and \(N = 10.00,\) such that \(\Delta < 0\) [3]. The two nullclines do not intersect and the two positive equilibrium points disappear. The origin \((0, 0)\) is a global attractor with two attracting sectors; one parabolic and the other one elliptical, see Figure 2. The blue (red) curve represents the prey (predator) nullcline and the orange region represent the basin of attraction of \((0, 0)\).

(b) \(O_{xy}^*\) is a stable node if \(C > M + 1,\) see Regions I, II(i) and II(ii) in Figure 2.

To finalise the proof we note that the three phase portrait of the origin \(O = (0, 0)\) for \(C - CQ + MQ < 0\) are topologically equivalent and they only present different asymptotes for the characteristic trajectories, see Figure 2.

Lemma 3.2 The equilibrium \((1, 0)\) (the rescaled carrying capacity) is a saddle node if \(C > M\).

Proof. The result follows by analysing the signs of the eigenvalues of the Jacobian matrix [4]. The Jacobian matrix evaluated at \((1, 0)\) is

\[ J(1, 0) = \begin{pmatrix} -1 & -Q \\ 0 & C - M \end{pmatrix} \]

Lemma 3.3 If \(\Delta < 0\) [5] then there are no positive equilibrium points in the first quadrant and therefore \((0, 0)\) is globally asymptotically stable in the first quadrant for system [3].

Proof. Finally, by Theorem 3.1 we have that solutions starting in the first quadrant are bounded and eventually end up in the invariant region \(\Gamma\). Moreover, the equilibrium point \((1, 0)\) is a saddle point and, if \(\Delta < 0\) [5], there are no equilibrium points in the interior of the first quadrant. Thus, by the Poincaré–Bendixson Theorem the unique \(\omega\)-limit of all the trajectories is the origin, see Figure 3.

Next, we consider the stability of the two positive equilibrium points \(P_{1,2}\) of system \([3]\) in the interior of \(\Gamma\). These equilibrium points lie on the curve \(u = v\) such that \(W(u, v) = 0\) [4], and they only exist if the system parameters are such that \(\Delta > 0\) [5]. The Jacobian matrix of system \((4)\) at these equilibrium points becomes

\[ J(u, v) = \begin{pmatrix} u(1 - 2u - v) & u(1 - Q - u) \\ v(C - M - Nv) & -v(M + N(u + 2v)) \end{pmatrix} \] (12)
Therefore, we have that the determinant and the trace of the Jacobian matrix \((12)\) are given by

\[
\det(J(u,v)) = CN(Q - 1)(u + v)(3(u + v) - 2) + M(2(u + v) - Q) \quad \text{and} \\
\text{tr}(J(u,v)) = u(1 - 2u - v) - v(M + N(u + 2v)).
\]

This gives the following results.

**Theorem 3.2** The equilibrium point \(P_1 = (u_1, v_1)\) is a saddle point for all parameter values whenever it exists.

**Proof.** Evaluating the determinant \((13)\) at \(P_1 = (u_1, v_1)\) gives

\[
\det(P_1) = -\frac{\sqrt{\Delta} (N - M - \sqrt{\Delta})}{2N} < 0.
\]

Since

\[
u_1 + v_1 = N - M - \sqrt{(N - M)^2 + 4N(C - CQ + MQ)} = N - M - \sqrt{\Delta} > 0,
\]

with \(u_1, v_1 > 0\) and \(\Delta\) defined in \((5)\), then the result follows. It is necessary to remark the following: Since we are working under the assumption that \(u_1, v_1 > 0\) and therefore \(u_1 + v_1 > 0\), we must have \(N > M\) and \(C - CQ + MQ < 0\).

**Theorem 3.3** Defining the function \(T(u,v) = \text{tr}(J(u,v)) = -2u^2 - 2Nv^2 - (1 + N)uv - u - Mv\) the stability of equilibrium point \(P_2 = (u_2, v_2)\) is as follows:

(a) asymptotically stable, if and only if, \(T(u_2, v_2) < 0\),

(b) unstable, if and only if, \(T(u_2, v_2) < 0\) and

(c) a weak-focus, if and only if, \(T(u_2, v_2) = 0\).

**Proof.** Evaluating the determinant \((13)\) at \(P_2 = (u_2, v_2)\) gives

\[
\det(P_2) = \frac{\sqrt{\Delta} (N - M + \sqrt{\Delta})}{2N} > 0.
\]

Since

\[
u_2 + v_2 = N - M + \sqrt{(N - M)^2 + 4N(C - CQ + MQ)} = N - M + \sqrt{\Delta} > 0,
\]

with \(\Delta\) defined in \((5)\) and since \(u_2, v_2 > 0\) the results follows by analysing the trace \((14)\) of the Jacobian matrix \((12)\) evaluated at \((u_2, v_2)\). The trace of the Jacobian matrix at a positive equilibrium point is given by \(\text{tr}(J(u,v)) = T(u,v)\) which is an elliptical paraboloid open downward, with an absolute maximum at the point \((\bar{u}, \bar{v})_{\max}\) and maximum value at \(T(\bar{u}, \bar{v})_{\max}\) which are given by

\[
(\bar{u}, \bar{v}) = \left(\frac{M + N + 4N}{1 + N(N - 14)}, \frac{1 + 4M + N}{1 + N(N - 14)}\right)
\]

and

\[
T(\bar{u}, \bar{v})_{\max} = \frac{2N + M(1 + 2M + N)}{1 + N(N - 14)}.
\]

Note that \(T(\bar{u}, \bar{v})_{\max} > 0\) if only if \(1 + N(N - 14)\) and hence \(N\) must be between the two real roots of the quadratic equation \(N^2 - 14N + 1 = 0\). That is \(7 < N < 7 + 4\sqrt{3}\) in which case \(u > 0\) and \(v < 0\). By continuity of \(T(u,v)\) and by the fact that \(T(0,0) = T(0.5,0) = 0\) a sector of the rotated paraboloid \((0\text{-contour of } T(u,v))\) \(T(u,v) = 0\) has a sector on the first quadrant, see Figure \((4)\). Hence there are positive values \((u_2, v_2)\) satisfying either (a), (b) or (c).

Now we discuss the stable manifold of the saddle point \(P_1\), \(W^s(P_1)\), often acts as a separatrix curve between the basins of attraction of the equilibrium points \((0,0)\) and \(P_2\). Moreover, by continuation of the variation of the stable manifold we can proof the conditions for the existence of an homoclinic curve and homoclinic bifurcation. Let \(W^u_{\text{un}}(P_1)\) be the (un)stable manifold of \(P_1\) that goes up to the right (from \(P_1\)) and let \(W^w_{\text{un}}(P_1)\) be the (un)stable manifold of \(P_1\) that goes down to the left (from \(P_1\)) \((5)\). Following \((14)\), we get

**Lemma 3.4** There exist conditions on the parameter values for which

(a) there is a homoclinic curve determined by the stable and unstable manifold of equilibrium point \(P_1 = (u_1, v_1)\) and
Figure 4: The interior of the rotated paraboloid $T(u, v) = 0$, defined in Theorem (3.3), intersecting the positive first quadrant.

Figure 5: For $C = 0.363$, $M = 0.16$, and $N = 0.25$, such that $\Delta > 0$ [3]. In the left panel if $Q = 1.695$ then the origin $(0, 0)$ and $P_2$ are attractors and in the right panel if $Q = 1.8$ then the origin $(0, 0)$ is quasi-global-attractor since $P_2$ is unstable [20]. The blue (red) curve represents the prey (predator) nullcline and the orange (grey) region represent the basin of attraction of $(0, 0)$ ($P_2$). Observe that the same colour conventions are used in the upcoming figures.
(a) a saddle-node attractor if $0 < C < \frac{-A + \sqrt{16MN^3(M-N)^2 + A^2}}{8N^2}$, in which case the equilibrium point $P_3$ is stable surrounded by an unstable limit cycle which acts as a separatrix between the basins of attraction of $(0,0)$ and $P_2$. The blue (red) curve represents the prey (predator) nullcline (See Figure 5 for the colour conventions).

(b) there is a limit cycle that bifurcates from the homoclinic which is surrounded the equilibrium point $P_2 = (u_2, v_2)$.

**Proof.** As $\Gamma$ is an invariant region and by the Theorem of existence and uniqueness the trajectory determined by the unstable manifold $W^s_\omega(P_1)$ cannot intersect the trajectories determined by the superior stable manifold $W^s_\alpha(P_1)$. Moreover, the $\omega$-limit of the $W^s_\alpha(P_1)$ can lie at the point $(1,0)$ or infinity in the direction of $u$-axis. Therefore, the $\omega$-limit of the right unstable manifold $W^u_\omega(P_1)$ must be the equilibrium point $P_2$, when this is an attractor, a stable limit cycle if $P_2$ is a repeller the equilibrium point $(0,0)$. Then, there exists a subset on the parameter space for which $W^u_\omega(P_1)$ intersects $W^s_\alpha(P_1)$ and therefore an homoclinic curve is obtained, see Figure 6.

Additionally, when the point $P_2$ is an attractor and the $\omega$-limit of the right unstable manifold $W^u_\omega(P_1)$ is the point $(0,0)$, there exists an unstable limit cycle which acts as a separatrix between the basins of attraction of the equilibrium points $P_3$ and $(0,0)$.

Note that when the homoclinic curve breaks which is determined by the intersection of the stable and unstable manifolds ($W^s_\alpha(P_1)$ and $W^u_\omega(P_1)$) of the equilibrium point $P_1$ it generates a non-infinitesimal limit cycle (originating a homoclinic bifurcation), which could coincide with other limit cycle obtained via Hopf bifurcation (infinitesimal limit cycle), when $P_2$ is a center-focus, see Theorem 3.3. Additionally, we observe that the non-infinitesimal limit cycle can increase until coincides with the homoclinic curve, then when it breaks the point $P_2$ becomes an unstable node and so the equilibrium point $(0,0)$ is a global attractor.

Finally, if $\Delta = 0$ in equation (3) the equilibrium points $P_1$ and $P_2$ collapse such that $u_3 = u_2 = u_3 = (2(1-Q) + Q(M+N))/(2(C+Q))$ and $v_3 = v_2 = v_3 = (CN-2Q-1-2MNQ-CM)/(2N(C+Q))$, see Figure 7. Therefore, system (3) has one equilibrium point of order two in the first quadrant given by $P_3 = (u_3, v_3)$.

**Theorem 3.4** Defining $A = -8MN^2 - (1-N)(M+N)^2$ then the stability of equilibrium point $P_3 = (u_3, v_3)$ is as follows:

(a) a saddle-node attractor if $0 < C < \frac{-A + \sqrt{16MN^3(M-N)^2 + A^2}}{8N^2}$, 

(b) a saddle-node repeller if $C > \frac{-A + \sqrt{16MN^3(M-N)^2 + A^2}}{8N^2}$.

**Proof.** Since $u_3 + v_3 = (N - M)/(2N)$ and $\Delta = 0$ then $Q = ((M-N)^2 + 4CN)/(4N(C-M))$. Therefore, the Jacobian matrix of (3) at the equilibrium point $P_3$ becomes

$$J(u_3, v_3) = \frac{M - N}{2N^2(2C - M + N)^2} \begin{pmatrix}
-C(M-N)^2 & \frac{(4CN+(M-N)^2)(M+N)^2}{4(C-M)} \\
4CN(C-M)^2 & N(C-N)(4CN+(M-N)^2)
\end{pmatrix}$$

Figure 6: For $C = 0.363$, $M = 0.16$, and $N = 0.25$, such that $\Delta > 0$. In the left panel if $Q = 1.7$ then the origin $(0,0)$ and $P_2$ are attractors and there exist a homoclinic curve. Moreover, in the right panel if $Q = 1.705$ then the equilibrium point $P_2$ is stable surrounded by an unstable limit cycle which act as a separatrix between the basins of attraction of $(0,0)$ and $P_2$. The blue (red) curve represents the prey (predator) nullcline (See Figure 5 for the colour conventions).
Figure 7: For $Q = 1.826$, $C = 0.363$, $M = 0.16$, and $N = 0.25$, such that the two nullclines intersect in one point in the first quadrant, i.e., for $\Delta < 0$ [5], then $P_1 = P_2 = P_3$. The blue (red) curve represents the prey (predator) nullcline (See Figure 5 for the colour conventions).

Then the determinant of the Jacobian matrix [15] is $\det J(P_3) = 0$. Next, the trace of the Jacobian matrix (15) is given by

$$
\text{tr}(J(P_3)) = 4\left(\frac{M - N}{N^2}\right)(C - M)\left((4N^2)C^2 + (N(M^2 - 6MN + N^2) - (M + N)^2)C - MN(M - N)^2\right)
$$

$$
= \frac{4(M - N)(C - M)}{N^2(2C - M + N)^2}\left(\alpha C^2 + \beta C - \gamma\right)
$$

(16)

Note that $\text{tr}(J(P_3)) = 0$ if only if

$$
C = -\frac{A + \sqrt{16MN^3(M - N)^2 + A^2}}{8N^2}.
$$

with $A = -8MN^2 - (1 - N)(M + N)^2$ and hence $C$ must be between the two real roots of the quadratic equation $\alpha C^2 + \beta C - \gamma = 0$, but $C > 0$. Therefore, there are positive values $(u_3, v_3)$ satisfying either (a) and (b), see Figure 7.

Now we can discuss some of the possible bifurcation scenarios of system (3). Observe that the stability of $(0,0)$, $(1,0)$ and $P_1$ do not change the stability. Additionally, we can see that the stability of the equilibrium points $P_2$ and $P_3$ depend on the system parameter $C$. Therefore, $C$ can be one of the natural candidates to act as bifurcation parameter.

**Remark 3.1** In the proof of Theorem (3.4) under the assumption that $\Delta = 0$ in (5) and in addition considering the condition $\text{tr}(J(P_3)) = 0$ in (16), that is when

$$
C = -\frac{A + \sqrt{16MN^3(M - N)^2 + A^2}}{8N^2}.
$$

The Jacobian matrix evaluated at $P_3$, $J(P_3)$, has Jordan canonical form:

$$
J = \begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}.
$$

Hence, the equilibrium $P_3$ has a single (repeated) eigenvalue $\lambda_1 = \lambda_2 = 0$ with algebraic multiplicity 2. This is a necessary condition for system (3) to undergo a Bogdano-Taken bifurcation [25]. One needs the variation of two parameters in order to encounter the bifurcation in a structurally stable way and to describe all possible qualitative behaviours nearby [5]. Nowadays, there are several computational methods to find Bogdano-Taken points in vector fields to high accuracy [22]. These methods are implemented in software packages such as MATCONT [17]. Figure 8 illustrates the Bogdano-Taken bifurcation which was detected with [17] in the $(Q,C)$-plane with parameter values $M = 0.16$ and $N = 0.25$.

**Theorem 3.5** If $\Delta = 0$ [5], then system (3) experiences a saddle-node bifurcation at the equilibrium point $P_3$ (for changing $C$).
has two positive equilibrium points namely (see Lemma 3.3). If \((Q,C)\) then system (3) does not have equilibrium points in the first quadrant and therefore \((0,0)\) is a saddle point. The bifurcation curves obtained from Theorems 3.3, 3.5 and Lemma 3.4 divide the \((Q,C)\) parameter space into four parts. When \((Q,C)\) is located in the black region the equilibrium point \(P_3\) is stable. Moreover, \(P_2\) is stable surrounded by an unstable limit cycle when \((Q,C)\) are located in the yellow region, while the equilibrium point \(P_2\) is unstable when \((Q,C)\) are located in the black region (see Figure 8).

**Proof.** The proof of this theorem is based on Sotomayor’s Theorem [25]. For \(\Delta = 0\), there is only one equilibrium point \(P_3 = (u_3, v_3)\) in the first quadrant, with 
\[
  u_3 = (2C(1 - Q) + Q(M + N))/((2C + NQ))
\]
and 
\[
  v_3 = (CN(2Q - 1) - 2MNQ - CM)/(2N(C + NQ)).
\]
From the proof of Theorem 3.4 we know that 
\[
  \det(J(P_3)) = 0 \text{ if } \Delta = 0.
\]
Additionally, let \(U = (1, 1)^T\) the eigenvector corresponding to the eigenvalue \(\lambda = 0\) of the Jacobian matrix \(J(P_3)\), and let
\[
  W = \left(\frac{-4N(C - M)^2}{(M + N)^2}, 1\right)^T
\]
be the eigenvector corresponding to the eigenvalue \(\lambda = 0\) of the transposed Jacobian matrix \(J(P_3)^T\).

If we represent (3) by its vector form 
\[
  F(u, v; C) = \begin{pmatrix}
    u(1 - u)(u + v) - Quv \\
    Cuv - v(u + v)(M + Nv)
  \end{pmatrix},
\]
then differentiating \(F\) at \(P_3\) with respect to the bifurcation parameter \(C\) gives
\[
  FC(u_3, v_3; C) = \begin{pmatrix}
    (CM + CN - 2CNQ + 2MNQ)(-2C + 2CQ - MQ - NQ) \\
    4N(C + NQ)^2
  \end{pmatrix}.
\]
Therefore,
\[
  \det(FC(u_3, v_3; C)) = \frac{(2C - 2CQ + MQ + NQ)(CM + CN - 2CNQ + 2MNQ)}{4N(C + NQ)^2} \neq 0.
\]
Note that \(\det(FC(u_3, v_3; C)) \neq 0\) under some conditions in the parameters \((C, Q, N, M)\). Next, we analyse the expression \(W \cdot |D^2FC(u_3, v_3; C)(U, U)|\). Therefore, we first compute the Hessian matrix
\[
  D^2F(u, v; C)(v, v) = \frac{\partial^2 F(u, v; C)}{\partial u \partial v} v_1 v_1 + \frac{\partial^2 F(u, v; C)}{\partial u \partial v} v_1 v_2 + \frac{\partial^2 F(u, v; C)}{\partial v \partial u} v_2 v_1
\]
\[
  + \frac{\partial^2 F(u, v; C)}{\partial v \partial u} v_2 v_2.
\]
At the equilibrium point \(P_3\) and \(V = U\), this simplifies to
\[
  D^2F(u_3, v_3; C)(U, U) = \begin{pmatrix}
    -(Q + 7) \\
    C - 3M - 10N
  \end{pmatrix}.
\]
Therefore, if
\[
  W \cdot |D^2F(u_3, v_3; C)(U, U)| = C - 3M - 10N + \frac{4N(Q + 7)(C - M)^2}{(M + N)^2},
\]
then, \(W \cdot |D^2F(u_3, v_3; C)(U, U)| \neq 0\) under some conditions in the parameters \((C, Q, N, M)\). Therefore, by Sotomayor’s Theorem [25] it now follows that system (3) has a saddle-node bifurcation at the equilibrium point \(P_3\).

In order to get the bifurcation diagram of system (3) for the parameters \(M\) and \(N\) fixed we follow [4] and we use the numerical bifurcation package MATCONT [11]. We observe that if \((Q, C)\) are located in the SN curve (see Figure 8), then system (3) has only one positive equilibrium point which is the collision of \(P_1\) and \(P_2\) (see Theorem 3.4); while if \((Q, C)\) are located in the green region (see Figure 8), then system (3) does not have equilibrium points in the first quadrant and therefore \((0, 0)\) is global attractor (see Lemma 3.3). If \((Q, C)\) are located in the black, yellow or brown regions in Figure 8, then system (3) has two positive equilibrium points namely \(P_1 = (u_1, v_1)\) and \(P_2 = (u_2, v_2)\) with \(u_1 < u_2\) and \(v_1 < v_2\). In these regions the equilibrium point \(P_1\) is always a saddle point. The bifurcation curves obtained from Theorems 3.3, 3.5 and Lemma 3.4 divide the \((Q, C)\) parameter space into four parts. When \((Q, C)\) are located in the brown region the equilibrium point \(P_2\) is stable. Moreover, \(P_2\) is stable surrounded by an unstable limit cycle when \((Q, C)\) are located in the yellow region, while the equilibrium point \(P_2\) is unstable when \((Q, C)\) are located in the black region (see Figure 8).
4 Conclusions

In this manuscript, the Bazykin predator-prey model with predator intra-specific interactions and ratio-dependent functional response was studied. Using a diffeomorphism we analysed a topologically equivalent system (3). This system has four system parameters which determine the number and the stability of the equilibrium points. We showed that the equilibrium points (1, 0) which correspond to the rescaled carrying capacity and \( P_1 \) are always saddle points. We showed in Theorem 3.1 the origin has a complex dynamics and by using vertical and horizontal blow-up we showed the dynamic in the neighbourhood of the origin. Furthermore, for some sets of parameters values the stable manifold of \( P_1 \) determines a separatrix curve which divides the basins of attraction of (0, 0) and \( P_2 \). As a result, the equilibrium point \( P_2 \) can be stable, stable surrounded by unstable limit cycle or unstable, depending on the trace of its Jacobian matrix, see Theorem 3.3. Moreover, the equilibrium points \( P_1 \) and \( P_5 \) collapse for \( \Delta = 0 \) and system (3) experiences a saddle-node bifurcation [25], see Theorem 3.5. We can also conclude that a modification of the parameters \( Q \) and \( C \) changes the location of the equilibrium points \( P_1 \) and \( P_2 \) and this variation also changes the behaviour of the equilibrium point (0, 0). Therefore, the basins of attraction of the equilibrium points (0, 0) and \( P_2 \) depend on the parameters \( Q \) and \( C \).

Since the function \( \varphi \) is a diffeomorphism preserving the orientation of time, the dynamics of system (3) is topologically equivalent to system (1). Therefore, we can conclude that for certain population sizes, there exists self-regulation in system (1), that is, the species can coexist. However, system (1) is sensitive to disturbances of the parameters and also in the initial population size. We can see this impact in the size of the basins of attraction of the equilibrium points (0, 0) and \( P_5 \) in Figures 3, 5, 6 and 7. In these figures the orange region represent the extinction of both population and the grey region represent the stabilisation of both population over the time. In addition, we showed that the stabilisation of the predator and the prey depends on the values of the parameters \( Q \) and \( C \) by taking the parameters \( M \) and \( N \) fixed. Note that the parameter \( Q \) correspond to the rescaled per capita predation rate \( q \) and the parameter \( C \) correspond to the rescaled efficiency with which predators convert consumed prey into new predators \( c \).

Finally, we showed that the intra-specific interactions [6] and ratio-dependent functional response in the Bazykin predator-prey model [1] modified the dynamics of the original Bazykin predator-prey model studied in [15]. Haque showed that the ratio-dependent predator-prey models are more appropriate for predator-prey interactions when the predators involve serious hunting processes. This manuscript extend the analysis in the neighbourhood of the origin showed in [15] and it also provide new graphic explanation about the different behaviour around this point.

References

[1] P. Aguirre, E. González-Olivares, and E. Sáez. Two limit cycles in a Leslie–Gower predator–prey model with additive Allee effect. Nonlinear Analysis: Real World Applications, 10:1401–1416, 2009.

[2] A. Andronov. Qualitative theory of second-order dynamic systems, volume 22054. Halsted Press, 1973.
[3] C. Arancibia-Ibarra. The basins of attraction in a Modified May-Holling-Tanner predator-prey model with Allee effect. *Nonlinear Analysis*, 185:15–28, 2019.

[4] C. Arancibia-Ibarra, M. Bode, J. Flores, G. Pettet, and P. van Heijster. A May-Holling-Tanner predator-prey model with multiple Allee effects on the prey and an alternative food source for the predator. *arXiv preprint arXiv:1904.02886*, 2019.

[5] C. Arancibia-Ibarra, J. Flores, G. Pettet, and P. van Heijster. A Holling-Tanner predator-prey model with strong Allee effect. *accepted for publication in International Journal of Bifurcation and Chaos*, April 2019.

[6] R. Arditi and L. Ginzburg. *How species interact: altering the standard view on trophic ecology*. Oxford University Press, 2012.

[7] F. Berezovskaya, G. Karev, and R. Arditi. Parametric analysis of the ratio-dependent predator–prey model. *Journal of Mathematical Biology*, 43:221–246, 2001.

[8] F. Berezovskaya, A. Novozhilov, and G. Karev. Population models with singular equilibrium. *Mathematical biosciences*, 208:270–299, 2007.

[9] C. Chicone. *Ordinary Differential Equations with Applications*, volume 34 of *Texts in Applied Mathematics*. World Scientific, Springer-Verlag New York, 2006.

[10] C. Cosner, D. Angelis, J. Ault, and D. Olson. Effects of spatial grouping on the functional response of predators. *Theoretical Population Biology*, 56:65–75, 1999.

[11] A. Dhooge, W. Govaerts, and Y. Kuznetsov. Matcont: a matlab package for numerical bifurcation analysis of odes. *ACM Transactions on Mathematical Software (TOMS)*, 29:141–164, 2003.

[12] F. Dumortier, J. Llibre, and J. Artés. *Qualitative theory of planar differential systems*. Springer Berlin Heidelberg, Springer-Verlag Berlin Heidelberg, 2006.

[13] J. Flores and E. González-Olivares. Dynamics of a predator–prey model with allee effect on prey and ratio–dependent functional response. *Ecological Complexity*, 18:59–66, 2014.

[14] J. Flores and E. González-Olivares. A modified Leslie—Gower predator—prey model with ratio-dependent functional response and alternative food for the predator. *Mathematical Methods in the Applied Sciences*, 40:2313–2328, 2017.

[15] H. Freedman. *Deterministic mathematical models in population ecology*. Pure and applied mathematics (Dekker); 57. Wiley, New York, 1980.

[16] E. González-Olivares, C. Arancibia-Ibarra, A. Rojas-Palma, and B. González-Yañez. Bifurcations and multistability on the May-Holling-Tanner predation model considering alternative food for the predators. *Mathematical Biosciences and Engineering*, 16:4274–4298, 2019.

[17] D. Greenhalgh and M Haque. A predator–prey model with disease in the prey species only. *Mathematical Methods in the Applied Sciences*, 30:911–929, 2007.

[18] M. Haque. Ratio-dependent predator-prey models of interacting populations. *Bulletin of mathematical biology*, 71:430–452, 2009.

[19] M. Haque and E. Venturino. An ecoepidemiological model with disease in predator: the ratio-dependent case. *Mathematical methods in the Applied Sciences*, 30:1791–1809, 2007.

[20] M. Hurley. Attractors: persistence, and density of their basins. *Transactions of the American Mathematical Society*, 269:247–271, 1982.

[21] C. Jost and R. Arditi. Identifying predator–prey processes from time-series. *Theoretical population biology*, 57:325–337, 2000.

[22] Y. Kuznetsov. *Elements of applied bifurcation theory*, volume 112. Springer Science & Business Media, 2013.

[23] R. Liu, D. DeAngelis, and J Bryant. Ratio-dependent functional response emerges from optimal foraging on a complex landscape. *Ecological modelling*, 292:45–50, 2014.
[24] A. Lotka. Contribution to the theory of periodic reactions. *The Journal of Physical Chemistry*, 14:271–274, 1910.

[25] L. Perko. *Differential Equations and Dynamical Systems*. Springer New York, 2001.

[26] P. Turchin. *Complex population dynamics: a theoretical/empirical synthesis*, volume 35 of *Monographs in population biology*. Princeton University Press, Princeton, N.J., 2003.

[27] Y. Xiao and L. Chen. A ratio-dependent predator–prey model with disease in the prey. *Applied Mathematics and Computation*, 131:397–414, 2002.