FOCUSING NLS WITH INVERSE SQUARE POTENTIAL

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Abstract. In this paper, we utilize the method in [4] to establish the radial scattering result for the focusing nonlinear Schrödinger equation with inverse square potential

\[ i \partial_t u - L_a u = -|u|^{p-1}u \]

in the energy space \( H^1_{a}(\mathbb{R}^d) \) in dimensions \( d \geq 3 \), which extends the result of [10, 11] to higher dimensions cases but with radial initial data. The new ingredient is to establish the dispersive estimate for radial function and overcome the weak dispersive estimate when \( a < 0 \).

Key Words: nonlinear Schrödinger equation; scattering; inverse square potential, Morawetz estimate.

AMS Classification: 35P25, 35Q55, 47J35.

1. Introduction

We study the initial-value problem for focusing nonlinear Schrödinger equations of the form

\[
\begin{aligned}
(i \partial_t - L_a)u &= -|u|^{p-1}u, \\
\quad u(0, x) &= u_0(x) \in H^1(\mathbb{R}^d),
\end{aligned}
\]

where \( u : \mathbb{R}^t \times \mathbb{R}^d \to \mathbb{C} \) and \( L_a = -\Delta + \frac{a}{|x|^2} \).

The class of solutions to (1.1) is left invariant by the scaling

\[
u(t, x) \mapsto \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x), \quad \lambda > 0.
\]

Moreover, one can also check that the only homogeneous \( L^2_x \)-based Sobolev space that is left invariant under (1.2) is \( \dot{H}^{s_c}(\mathbb{R}^d) \) with \( s_c := \frac{d}{2} - \frac{2}{p-1} \). Solutions to (1.1) conserve their mass and energy by

\[
\begin{aligned}
M(u(t)) &= \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx, \\
E_a(u(t)) &= \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t, x)|^2 + \frac{a}{2|2|} |u(t, x)|^2 - \frac{1}{p+1} |u(t, x)|^{p+1} \, dx.
\end{aligned}
\]

Initial data belonging to \( H^1_x(\mathbb{R}^d) \) have finite mass and energy. This follows from equivalent of Sobolev norm and the following variant of the Gagliardo-Nirenberg inequality:

\[
\|f\|_{L^{p+1}_{x}(\mathbb{R}^d)}^{p+1} \leq C_a \|f\|_{L^{2}_{x}(\mathbb{R}^d)}^{\frac{d+2-(d-2)p}{2}} \|\sqrt{L_a} f\|_{L^{2}_{x}(\mathbb{R}^d)}^{\frac{d(p-1)}{2}},
\]

where \( C_a \) denotes the sharp constant in the inequality above for radial functions. We will show in Theorem 3.1 that the sharp constant \( C_a \) is attained by a radial solution \( Q_a \) to elliptic equation

\[-L_a Q_a - Q_a + Q_a^p = 0.\]

The functions \( Q_a \) provide examples of non-scattering solutions at the radial threshold via \( u(t, x) = e^{it} Q_a(x) \). We consider the problem of global existence and scattering for (1.1) below threshold. We begin with the following definitions.
Definition 1.1 (Solution, scattering). Let $t_0 \in \mathbb{R}$ and $u_0 \in H_x^1(\mathbb{R}^d)$. Let $I$ be an interval containing $t_0$. A function $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ is a solution to \((1.1)\), if it belongs to $C_t H^1_x \cap L_t^2 H^1_x \cap L^{\frac{4d}{d-4}}(K \times \mathbb{R}^d)$ for any compact $K \subset I$ and obeys the Duhamel formula

$$u(t) = e^{-i(t-t_0)\mathcal{L}_a}u_0 + i \int_{t_0}^{t} e^{-i(t-s)\mathcal{L}_a} \left| |(u(s)|^{p-1} u(s) \right| ds \quad \text{for all} \quad t \in I,$$

where we rely on the self-adjointness of $\mathcal{L}_a$ to make sense of $e^{-it\mathcal{L}_a}$ via the Hilbert space functional calculus. We call $I$ the lifespan of $u$. We call $u$ a maximal-lifespan solution if it cannot be extended to any strictly larger interval. If $I = \mathbb{R}$, we call $u$ global.

Moreover, a global solution $u$ to \((1.1)\) scatters if there exist $u_{\pm} \in H^1_x(\mathbb{R}^d)$ such that

$$\lim_{t \to \pm \infty} \|u(t) - e^{-it\mathcal{L}_a} u_{\pm}\|_{H^1_x(\mathbb{R}^d)} = 0.$$

In this paper, we utilize the method in \[4\] to obtain the following threshold result for the class of radial solutions:

Theorem 1.2 (Radial scattering/blowup dichotomy). Let $(a, d, p)$ satisfy

$$a > \begin{cases} 
-(\frac{d-2}{2})^2 & \text{if } d = 3 \quad \text{and} \quad \frac{4}{3} < p - 1 \leq 2 \\
-(\frac{d-2}{2})^2 + (\frac{d^2 - d - 1}{p-1})^2 & \text{if } d \geq 3 \quad \text{and} \quad \frac{4}{d-2} \vee \frac{4}{d} < p - 1 < \frac{4}{d-2},
\end{cases}$$

(1.4)

where $a \vee b := \max\{a, b\}$. Let $u_0 \in H^1_x(\mathbb{R}^d)$ be radial and satisfy $M(u_0)^{1-s_c} E_a(u_0)^{s_c} < M(Q_a)^{1-s_c} E_a(Q_a)^{s_c}$. Moreover, if

$$\|u_0\|_{L^2_x}^{1-s_c} \|u_0\|_{H_x^1}^{s_c} < \|Q_a\|_{L^2_x}^{1-s_c} \|Q_a\|_{H_x^1}^{s_c},$$

then the solution to \((1.1)\) with initial data $u_0$ is global and scatters.

Remark 1.3. (i) In the case $a = 0$, such result was firstly considered by Holmer and Roudenko \[7\] for the 3D cubic radial and Duyckaerts-Holmer-Roudenko \[6\] for nonradial data. Lately, Killip, Murphy, Visan and the third author \[10\] and Lu, Miao and Murphy \[11\] generalized their result to the focusing Schrödinger equation with inverse square potential, i.e. \((1.1)\). In this paper, we extend the result of \[10\] \[11\] to general nonlinear term in dimensions $d \geq 3$ but with radial initial data. We also refer the reader to the defocusing nonlinear Schrödinger equation with inverse square potential \[9\] \[16\]. The main new ingredient of this paper is to establish the dispersive estimate for radial function and overcome the weak dispersive estimate when $a < 0$.

(ii) The restriction on $(a, d, p)$ stems from the local well-posedness theory in $H^1_x(\mathbb{R}^d)$ for \((1.1)\). While in the proof of local well-posedness, we need to estimate powers of $\mathcal{L}_a$ applied to the nonlinearity term. To obtain the requisite fractional calculus estimates for $\mathcal{L}_a$, we rely on the equivalence of Sobolev spaces to exchange powers of $\mathcal{L}_a$ and powers of $-\Delta$ (for which fractional calculus estimates are known). This argument leads to a restriction on the range of $(a, d, p)$ as in \((1.4)\).

We sketch the idea and argument for the proof here. First, by variational analysis and blowup criterion, we derive that the solution $u$ is global. And then, by radial Sobolev embedding and dispersive estimate, we establish a scattering criterion as the case $a = 0$ \[13\]. Here we should be careful in the case $a < 0$, since we have
only the weak dispersive estimate, see Lemma 2.3. Finally, using Virial argument, radial Sobolev embedding and variational analysis, we prove the above scattering criterion.

We conclude the introduction by giving some notations which will be used throughout this paper. To simplify the expression of our inequalities, we introduce some symbols \( \lesssim, \sim, \ll \). If \( X, Y \) are nonnegative quantities, we use \( X \lesssim Y \) or \( X = O(Y) \) to denote the estimate \( X \leq CY \) for some \( C \), and \( X \sim Y \) to denote the estimate \( X \lesssim Y \lesssim X \). We use \( X \ll Y \) to mean \( X \leq cY \) for some small constant \( c \). We use \( C \gg 1 \) to denote various large finite constants, and \( 0 < c \ll 1 \) to denote various small constants. For any \( r, 1 \leq r \leq \infty \), we denote by \( \| \cdot \|_r \) the norm in \( L^r = L^r(\mathbb{R}^d) \) and by \( r' \) the conjugate exponent defined by \( \frac{1}{r} + \frac{1}{r'} = 1 \).

\[ \text{Lemma 2.2 (Equivalence of Sobolev spaces, [8])} \]

2. Preliminaries

2.1. Harmonic analysis for \( \mathcal{L}_a \). In this section, we collect some harmonic analysis tools adapted to the operator \( \mathcal{L}_a \). The primary reference for this section is [8].

For \( 1 < r < \infty \), we write \( \dot{H}^1_r(\mathbb{R}^d) \) and \( H^1_r(\mathbb{R}^d) \) for the homogeneous and inhomogeneous Sobolev spaces associated with \( \mathcal{L}_a \), respectively, which have norms

\[ \| f \|_{\dot{H}^1_r(\mathbb{R}^d)} = \| \sqrt{\mathcal{L}_a} f \|_{L^r(\mathbb{R}^d)} \quad \text{and} \quad \| f \|_{H^1_r(\mathbb{R}^d)} = \| 1 + \mathcal{L}_a f \|_{L^r(\mathbb{R}^d)}. \]

When \( r = 2 \), we simply write \( \dot{H}^1_2(\mathbb{R}^d) = \dot{H}^1(\mathbb{R}^d) \) and \( H^1_2(\mathbb{R}^d) = H^1(\mathbb{R}^d) \).

By the sharp Hardy inequality, the operator \( \mathcal{L}_a \) is positive precisely for \( a \geq -(\frac{d-2}{2})^2 \). Denote

\[ \sigma := \frac{d-2}{2} - (\frac{d-2}{2})^2. \tag{2.1} \]

Estimates on the heat kernel associated to the operator \( \mathcal{L}_a \) were found by Liskevich–Sobol [12] and Milman–Semenov [13].

Lemma 2.1 (Heat kernel bounds, [12, 13]). Let \( d \geq 3 \) and \( a \geq -(\frac{d-2}{2})^2 \). There exist positive constants \( C_1, C_2 \) and \( c_1, c_2 \) such that for any \( t > 0 \) and any \( x, y \in \mathbb{R}^d \setminus \{ 0 \} \),

\[ C_1 (1 + \sqrt{\frac{t}{|y|^2}})^\sigma (1 + \sqrt{\frac{t}{|y|^2}})^{\frac{d}{2} - 2} e^{-\frac{|x-y|^2}{4t}} e^{-\frac{|x-y|^2}{c_2t^2}} \leq e^{-t \mathcal{L}_a(x, y)} \leq C_2 (1 + \sqrt{\frac{t}{|y|^2}})^\sigma (1 + \sqrt{\frac{t}{|y|^2}})^{\frac{d}{2} - 2} e^{-\frac{|x-y|^2}{c_2t^2}}. \]

As a consequence, we can obtain the following equivalence of Sobolev spaces.

Lemma 2.2 (Equivalence of Sobolev spaces, [8]). Let \( d \geq 3 \), \( a \geq -(\frac{d-2}{2})^2 \), and \( 0 < s < 2 \). If \( 1 < q < \infty \) satisfies \( \frac{d-s}{q} < \frac{d}{p} < \min\{ 1, \frac{d-s}{2} \} \), then

\[ \| |\nabla|^s f \|_{L^q_x} \lesssim c_{d,p,s} \| (\mathcal{L}_a)^\frac{s}{2} f \|_{L^p_x} \text{ for all } f \in C^\infty_c(\mathbb{R}^d \setminus \{ 0 \}). \]

If \( \max\{ \frac{d}{2}, \frac{d-s}{2} \} < \frac{d}{p} < \min\{ 1, \frac{d-s}{2} \} \), then

\[ \| (\mathcal{L}_a)^\frac{s}{2} f \|_{L^p_x} \lesssim c_{d,p,s} \| |\nabla|^s f \|_{L^q_x} \text{ for all } f \in C^\infty_c(\mathbb{R}^d \setminus \{ 0 \}). \]

We will make use of the following fractional calculus estimates due to Christ and Weinstein [3]. Combining these estimates with Lemma 2.2, we can deduce analogous statements for the operator \( \mathcal{L}_a \) (for restricted sets of exponents).

Lemma 2.3 (Fractional calculus).

(i) Let \( s \geq 0 \) and \( 1 < r, r_j, q_j < \infty \) satisfy \( \frac{1}{r} = \frac{1}{r_j} + \frac{1}{q_j} \) for \( j = 1, 2 \). Then

\[ \| |\nabla|^s (fg) \|_{L^r_x} \lesssim \| f \|_{L^r_{x_1}} \| |\nabla|^s g \|_{L^{r_j}_{x_2}} + \| |\nabla|^s f \|_{L^{r_j}_{x_2}} \| g \|_{L^2_{x_2}}. \]
(ii) Let $G \in C^1(\mathbb{C})$ and $s \in (0, 1]$, and let $1 < r_1 \leq \infty$ and $1 < r, r_2 < \infty$ satisfy \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \). Then
\[
\|\nabla^s G(u)\|_{L^r_x} \lesssim \|G'(u)\|_{L^r_x}^s \|u\|_{L^r_x^2}.
\]

We will need the following radial Sobolev embedding from \([15]\).

**Lemma 2.4** (Radial Sobolev embedding). Let $d \geq 3$. For radial $f \in H^1(\mathbb{R}^d)$, there holds
\[
\|x^s f\|_{L^2_x(\mathbb{R}^d)} \lesssim \|f\|_{H^1(\mathbb{R}^d)},
\]
for $\frac{1}{2} - 1 \leq s \leq \frac{d-1}{2}$.

Let $f$ be Schwartz function defined on $\mathbb{R}^d$, we define the Hankel transform of order $\nu$:
\[
(\mathcal{H}_\nu f)(\xi) = \int_0^\infty (\rho \omega)^{-\frac{d+2}{2}} J_\nu(\rho \omega) f(\rho) \rho^{d-1} dr,
\]
where $\rho = |\xi|$, $\omega = \xi/|\xi|$ and $J_\nu$ is the Bessel function of order $\nu$ defined by the integral
\[
J_\nu(\rho) = \frac{(\rho/2)^\nu}{\Gamma(\nu + \frac{1}{2})\Gamma(1/2)} \int_{-1}^1 e^{\rho r} (1 - s^2)^{(\nu-1)/2} ds \text{ with } \nu > -\frac{1}{2} \text{ and } r > 0.
\]
Specially, if the function $f$ is radial, then
\[
(\mathcal{H}_\nu f)(\rho) = \int_0^\infty (\rho \omega)^{-\frac{d+2}{2}} J_\nu(\rho \omega) f(\rho) \rho^{d-1} dr.
\]

The following properties of the Hankel transform are obtained in \([1]\):

**Lemma 2.5.** Let $\mathcal{H}_\nu$ be defined above and $A_\nu := -\partial_x^2 + \frac{d-1}{r} \partial_r + \left[u^2 - \left(\frac{d-2}{2}\right)^2 r^{-2}\right]$. Then
(i) $\mathcal{H}_\nu = \mathcal{H}_\nu^{-1}$,
(ii) $\mathcal{H}_\nu$ is self-adjoint, i.e. $\mathcal{H}_\nu = \mathcal{H}_\nu^*$,
(iii) $\mathcal{H}_\nu$ is an $L^2$ isometry, i.e. $\|\mathcal{H}_\nu \phi\|_{L^2_x} = \|\phi\|_{L^2_x}$,
(iv) $\mathcal{H}_\nu(A_\nu \phi)(\xi) = |\xi|^2 (\mathcal{H}_\nu \phi)(\xi)$, for $\phi \in L^2$.

### 2.2. Strichartz estimates and dispersive estimate

Strichartz estimates for the propagator $e^{-it\mathcal{L}_\alpha}$ were proved by Burq, Planchon, Stalker, and Tahvildar-Zadeh in \([1]\). Combining these with the Christ–Kiselev Lemma \([2]\), we obtain the following Strichartz estimates:

**Proposition 2.6** (Strichartz estimate, \([1\ [17]\). Let $d \geq 3$, and fix $a > -\left(\frac{d-2}{2}\right)^2$. The solution $u$ to $(i\partial_t - \mathcal{L}_\alpha)u = F$ on an interval $I \ni t_0$ obeys
\[
\|u\|_{L^q_t L^\infty_x(I \times \mathbb{R}^d)} \lesssim \|u(t_0)\|_{L^2_x(\mathbb{R}^d)} + \|F\|_{L^q_t L^{\infty}_x(I \times \mathbb{R}^d)}
\]
for any $2 \leq q, \tilde{q} \leq \infty$ with $\frac{3}{q} + \frac{3}{\tilde{q}} = 1 + \frac{3}{d}$ including $(q, \tilde{q}) = (2, 2)$.

As a consequence of Strichartz estimate, we obtain the local well-posedness theory in $H^1(\mathbb{R}^d)$.

**Theorem 2.7** (Local well-posedness, \([1\ [11]\). Let $(a, d, p)$ satisfy the condition \([1\ [15]\). Assume $u_0 \in H^1_a(\mathbb{R}^d)$, and $t_0 \in \mathbb{R}$. Then the following hold:
(i) There exist $T = T(\|u_0\|_{H^1_a}) > 0$ and a unique solution $u : (t_0 - T, t_0 + T) \times \mathbb{R}^d \to \mathbb{C}$ to \((1.1)\) with $u(t_0) = u_0$. In particular, if $u$ remains uniformly bounded in $H^1_a$ throughout its lifespan, then $u$ extends to a global solution.
There exists $\eta_0 > 0$ such that if
\[ \|e^{-i(t-t_0)\mathcal{L}_a}u_0\|_{L^{p-1}_{t,x}((t_0,\infty)\times\mathbb{R}^d)} < \eta \quad \text{for some} \quad 0 < \eta < \eta_0, \]
then the solution $u$ to (1.1) with data $u(t_0) = u_0$ is forward-global and satisfies
\[ \|u\|_{L^{p-1}_{t,x}((t_0,\infty)\times\mathbb{R}^d)} \lesssim \eta. \]
The analogous statement holds backward in time (as well as on all of $\mathbb{R}$).

For any $\psi \in H^1_a$, there exist $T > 0$ and a solution $u : (T, \infty) \times \mathbb{R}^d \to \mathbb{C}$ to (1.1) such that
\[ \lim_{t \to \infty} \|u(t) - e^{-it\mathcal{L}_a}\psi\|_{H^1_a} = 0. \]
The analogous statement holds backward in time.

Next, we prove the key estimate (dispersive estimate) which will be useful in the proof of scattering criterion (Lemma 4.1 below).

**Theorem 2.8** (Dispersive estimate). Let $f$ be radial function.

(i) If $a \geq 0$, then we have
\[ \|e^{it\mathcal{L}_a}f\|_{L^{\infty}(\mathbb{R}^d)} \leq C|t|^{-\frac{d}{2}}\|f\|_{L^1(\mathbb{R}^d)}. \] (2.5)

(ii) If $-\frac{(d-2)^2}{4} < a < 0$, then there holds
\[ \|(1 + |x|^{-\sigma})^{-1}e^{it\mathcal{L}_a}f\|_{L^{\infty}(\mathbb{R}^d)} \leq C\frac{1 + |t|^\sigma}{|t|^{\frac{d}{2}}}\|(1 + |x|^{-\sigma})f\|_{L^1(\mathbb{R}^d)}, \] (2.6)
with $\sigma$ being as in (2.1).

**Proof.** Since $f(x)$ is radial, $u(t,x) := e^{it\mathcal{L}_a}f$ solve
\[ \begin{cases} 
  i\partial_t u - A_\nu u = 0, \\
  u(0,r) = f(r),
\end{cases} \] (2.7)
where the operator $A_\nu$ is defined as in Lemma 2.5 with $\nu = \frac{d-2}{2} - \sigma$. Applying the Hankel transform to the equation (2.7), by (iv) in Lemma 2.5 we have
\[ \begin{cases} 
  i\partial_\rho \tilde{u} - \rho^2 \tilde{u} = 0 \\
  \tilde{u}(0,\rho) = (\mathcal{H}_\nu f)(\rho),
\end{cases} \] (2.8)
where $\tilde{u}(t,\rho) = (\mathcal{H}_\nu u)(t,\rho)$. Solving this ODE and inverting the Hankel transform, we obtain
\[ u(t,r) = \int_0^\infty \frac{(\rho)}{2} J_\nu(\rho r)e^{-it\rho^2}(\mathcal{H}_\nu f)(\rho)^\rho_{d-1} d\rho = \int_0^\infty \frac{(s)}{2} J_\nu(s r)e^{-it\rho^2}s_{d-1} \int_0^\infty (\rho)\frac{d-2}{2} J_\nu(s \rho)f(s)s_{d-1} d s d\rho = \int_0^\infty f(s)s_{d-1}K(t,r,s) ds, \]
with the kernel
\[
K(t, r, s) = (rs)^{-\frac{d-2}{2}} \int_0^\infty J_\nu(rp) J_\nu(sp) e^{-it\rho^2} \rho \, d\rho
\]
\[
= (rs)^{-\frac{d-2}{2}} e^{-\frac{\nu \pi i}{2it}} e^{-\frac{2s^2}{4\pi t} J_\nu \left( \frac{rs}{2t} \right)},
\]
where we used the analytic continuation as in \[6\] in the second equality. Thus,
\[
\text{Therefore by collecting all of them, we conclude the proof of Theorem 2.8.}
\]
\[
\text{On the other hand, by } |J_\nu(r)| \lesssim r^{-\frac{d-2}{2}} \text{ with } r \geq 1, \text{ we obtain}
\]
\[
|II| \lesssim Ct^{-\frac{d}{4}} \int_0^\infty |f(s)| s^{d-1} \left( \frac{rs}{2t} \right)^{-\frac{d-2}{4}} \, ds \lesssim Ct^{-\frac{d}{4}} \|f\|_{L_1^2(\mathbb{R}^d)}.
\]
\[
\text{On the other hand, by } |J_\nu(r)| \lesssim r^{-\nu} \text{ with } r \leq 1, \text{ we get for } a \geq 0
\]
\[
|I| \lesssim Ct^{-\frac{d}{2}} \int_0^{2t} |f(s)| s^{d-1} \left( \frac{rs}{2t} \right)^{-\frac{d-2}{4}} \nu \, ds
\]
\[
\lesssim Ct^{-\frac{d}{2}} \int_0^{2t} |f(s)| s^{d-1} \left( \frac{rs}{2t} \right)^{-\sigma} \, ds
\]
\[
\lesssim Ct^{-\frac{d}{2}} \|f\|_{L_1^2(\mathbb{R}^d)},
\]
while for \( a < 0 \)
\[
(1 + r^{-\sigma})^{-1} |I| \lesssim Ct^{-\frac{d}{2}} \int_0^{2t} (1 + s^{-\sigma}) |f(s)| s^{d-1} \frac{\nu^\sigma}{(1 + r^\sigma)(1 + s^\sigma)} \, ds
\]
\[
\lesssim Ct^{-\frac{d}{2}} \nu^\sigma \|1 + |x|^{-\sigma}\|f\|_{L_1^2(\mathbb{R}^d)}.
\]
Therefore by collecting all of them, we conclude the proof of Theorem 2.8.

3. Variational analysis

In this section, we carry out the variational analysis for the sharp Gagliardo–Nirenberg inequality, which leads naturally to the thresholds appearing in Theorem 1.2.

**Theorem 3.1** (Sharp Gagliardo–Nirenberg inequality). Fix \( a > \frac{(d-2)^2}{4} \) and define
\[
C_a := \sup \{ \|f\|_{L_1^{p+1}}^{\frac{d+2-(d-2)p}{2}} \|f\|_{H_1^d}^{\frac{(d-1)p}{2}} : f \in H_1^d \setminus \{0\}, \text{ } f \text{ } \text{radial} \}.
\]
Then \( C_a \in (0, \infty) \) and the Gagliardo–Nirenberg inequality for radial functions
\[
\|f\|_{L_1^{p+1}}^{\frac{d+2-(d-2)p}{2}} \|f\|_{H_1^d}^{\frac{(d-1)p}{2}} \leq C_a \|f\|_{L_1^{p+1}}^{\frac{d+2-(d-2)p}{2}} \|f\|_{H_1^d}^{\frac{(d-1)p}{2}}
\]
(3.1)
is attained by a function \( Q_a \in H^1_x \), which is a non-zero, non-negative, radial solution to the elliptic problem

\[
-\mathcal{L}_a Q_a - Q_a + Q_a^p = 0. \tag{3.2}
\]

**Proof.** Define the functional

\[
J_a(f) := \frac{\|f\|^{p+1}_{L^{p+1}_x}}{\|f\|^{d+2-(d-2)p}_{L^2_x}} \quad \text{so that} \quad C_a = \sup\{J_a(f) : f \in H^1_x \setminus \{0\}, f \text{ radial} \}.
\]

Note that the standard Gagliardo–Nirenberg inequality and the equivalence of Sobolev spaces imply \( 0 < C_a < \infty \).

We prove by mimicking the well-known proof for \( a = 0 \) and Theorem 3.1 in [10]. Take the sequence of radial functions \( \{f_n\} \subset H^1_x \setminus \{0\} \) such that \( J_a(f_n) \nearrow C_a \). Choose \( \mu_n \in \mathbb{R} \) and \( \lambda_n \in \mathbb{R} \) so that \( g_n(x) := \mu_n f_n(\lambda_n x) \) satisfy \( \|g_n\|_{L^2_x} = \|g_n\|_{H^1_x} = 1 \). Note that \( J_a(f_n) = J_a(g_n) \). As \( H^1_x \rightarrow L^{p+1}_x \) compactly, passing to a subsequence we may assume that \( g_n \) converges to some \( g \in H^1_x \) strongly in \( L^{p+1}_x \) as well as weakly in \( H^1_x \). As \( g_n \) is an optimizing sequence, we deduce that \( C_a = \|g\|_{L^{p+1}_x}^p \). We also have that \( \|g\|_{L^2_x} = \|g\|_{H^1_x} = 1 \), or else \( g \) would be a super-optimizer. Thus \( g \) is an optimizer.

The Euler–Lagrange equation for \( g \) is given by

\[
-\frac{d(p-1)}{2} C_a \mathcal{L}_a g - \frac{d+2-(d-2)p}{2} C_a g + (p+1)g^p = 0.
\]

Thus, if we define \( Q_a \) via

\[
g(x) = \alpha Q_a(\lambda x), \quad \text{with} \quad \alpha = \left(\frac{d+2-(d-2)p}{2(p+1)} C_a\right)^{-\frac{1}{p-1}} \quad \text{and} \quad \lambda = \sqrt{\frac{d+2-(d-2)p}{3(p-1)}},
\]

then \( Q_a \) is an optimizer of (3.1) that solves (3.2).

\[\square\]

By integration by part, we easily get

**Lemma 3.2.** Assume \( \phi \in \mathcal{S}(\mathbb{R}^d) \), then

\[
\begin{align*}
\int_{\mathbb{R}^d} \Delta \phi x \cdot \nabla \phi dx &= \frac{d-2}{2} \int_{\mathbb{R}^d} |\nabla \phi|^2 dx, \\
\int_{\mathbb{R}^d} \phi x \cdot \nabla \phi dx &= -\frac{d}{2} \int_{\mathbb{R}^d} |\phi|^2 dx, \\
\int_{\mathbb{R}^d} x \cdot \nabla \phi \phi^p dx &= -\frac{d}{p+1} \int_{\mathbb{R}^d} |\phi|^{p+1} dx.
\end{align*}
\]

By a simple computation, we have

**Lemma 3.3.** Let \( Q_a \) be the solution to \(-\mathcal{L}_a Q_a - Q_a + Q_a^p = 0\). Then

\[
\|Q_a\|^2_{L^2_x} = \frac{d+2-(d-2)p}{2(p+1)} \|Q_a\|^{p+1}_{L^{p+1}_x}, \quad \|Q_a\|^2_{H^1_x} = \frac{d(p-1)}{2(p+1)} \|Q_a\|^{p+1}_{L^{p+1}_x}, \tag{3.3}
\]

and

\[
E_a(Q_a) = \frac{dp-(d+4)}{2d(p-1)} \|Q_a\|^2_{H^1_x} = \frac{dp-(d+4)}{4(p+1)} \|Q_a\|^{p+1}_{L^{p+1}_x}. \tag{3.4}
\]
Moreover,
\[
C_a = \frac{\|Q_a\|^{p+1}_{L^{p+1}}}{\|Q_a\|_{L^2}} \|Q_a\|_{H^1_a}^{\frac{p(1-\nu)}{2}} = \left( \frac{d+2-(d-2)p}{2(p+1)} \right)^{-\frac{d(p-1)}{4}} \|Q_a\|_{L^{p+1}}^{\frac{d(p-1)}{4}}.
\]
and
\[
C_a \|Q_a\|^{(1-s_c)(p-1)}_{L^2} \|Q_a\|^{s_c(p-1)}_{H^1_a} = \frac{2(p+1)}{d(p-1)}.
\]

**Proposition 3.4** (Coercivity). Fix \( a \geq -\frac{(d-2)^2}{4} \). Let \( u : I \times \mathbb{R}^d \to \mathbb{C} \) be the maximal-lifespan solution to (1.1) with \( u(t_0) = u_0 \in H^1_a \setminus \{0\} \) for some \( t_0 \in I \).

Assume that
\[
M(u_0)^{1-s_c} E_a(u_0)^{s_c} \leq (1-\delta) M(Q_a)^{1-s_c} E_a(Q_a)^{s_c} \quad \text{for some} \quad \delta > 0.
\]
Then there exist \( \delta' = \delta'(\delta) > 0 \), \( c = c(\delta, a, \|u_0\|_{L^2}) > 0 \), and \( \varepsilon = \varepsilon(\delta) > 0 \) such that: If \( \|u_0\|^{1-s_c}_{L^2} \|u_0\|^{s_c}_{H^1_a} \leq \|Q_a\|^{1-s_c}_{L^2} \|Q_a\|^{s_c}_{H^1_a} \), then for all \( t \in I \),

(i) \( \|u(t)\|^{1-s_c}_{L^2} \|u(t)\|^{s_c}_{H^1_a} \leq (1-\delta') \|Q_a\|^{1-s_c}_{L^2} \|Q_a\|^{s_c}_{H^1_a} \),

(ii) \( \|u(t)\|^{2}_{H^1_a} - \frac{d(p-1)}{2(p+1)} \|u(t)\|^{p+1}_{L^{p+1}} \geq c \|u(t)\|^{2}_{H^1_a} \),

(iii) \( \frac{(d-p-4)}{2}\|Q_a\|^{1-s_c}_{L^2} \|Q_a\|^{s_c}_{H^1_a} \leq E_a(u) \leq \frac{1}{2} \|u(t)\|^{2}_{H^1_a} \).

**Proof.** By the sharp Gagliardo–Nirenberg inequality, conservation of mass and energy, and (3.7), we may write
\[
(1-\delta) M(Q_a)^{1-s_c} E_a(Q_a)^{s_c} \geq M(u)^{1-s_c} E_a(u)^{s_c} \geq \frac{1}{2} \|u(t)\|^{2}_{H^1_a} - \frac{1}{p+1} C_a \|u(t)\|^{\frac{d+2-(d-2)p}{4}}_{L^2} \|u(t)\|^{\frac{d(p-1)}{4}}_{H^1_a}
\]
for any \( t \in I \). Using (3.2) and (3.6), this inequality becomes
\[
(1-\delta) \frac{1}{2} \geq \frac{d(p-1)}{dp - (d+4)} \left( \frac{\|u(t)\|^{1-s_c}_{L^2} \|u(t)\|^{s_c}_{H^1_a}}{\|Q_a\|^{1-s_c}_{L^2} \|Q_a\|^{s_c}_{H^1_a}} \right)^{\frac{1}{s_c}} - \frac{2}{dp - (d+4)} \left( \frac{\|u(t)\|^{1-s_c}_{L^2} \|u(t)\|^{s_c}_{H^1_a}}{\|Q_a\|^{1-s_c}_{L^2} \|Q_a\|^{s_c}_{H^1_a}} \right)^{\frac{1}{s_c}(p-1)}.
\]
Claims (i) now follow from a continuity argument, together with the observation that
\[
(1-\delta) \geq \frac{d(p-1)}{dp - (d+4)} \geq \frac{2}{dp - (d+4)} \implies |y - 1| \geq \delta' \quad \text{for some} \quad \delta' = \delta'(\delta) > 0.
\]

For claim (iii), the upper bound follows immediately, since the nonlinearity is focusing. For the lower bound, we again rely on the sharp Gagliardo–Nirenberg inequality. Using (i) and (3.6) as well, we find
\[
E_a(u) \geq \frac{1}{2} \|u(t)\|^{2}_{H^1_a} \left[ 1 - \frac{2}{p+1} C_a \|u(t)\|^{(1-s_c)(p-1)}_{L^2} \|u(t)\|^{s_c(p-1)}_{H^1_a} \right]
\]
for all \( t \in I \). Thus (iii) holds.

We turn to (ii). We begin by writing
\[
\|u(t)\|^{2}_{H^1_a} - \frac{d(p-1)}{2(p+1)} \|u(t)\|^{p+1}_{L^{p+1}} = \frac{d(p-1)}{2} E_a(u) - \frac{dp - (d+4)}{4} \|u(t)\|^{2}_{H^1_a},
\]
and
\[
(1+\varepsilon) \|u(t)\|^{2}_{H^1_a} - \frac{d(p-1)}{2(p+1)} \|u(t)\|^{p+1}_{L^{p+1}} = \frac{d(p-1)}{2} E_a(u) - (\frac{dp - (d+4)}{4} - \varepsilon) \|u(t)\|^{2}_{H^1_a},
\]
where
for \( t \in I \). Thus (ii) follows from (iii) by choosing any \( 0 < c \leq \delta' \).

\[ \square \]

**Remark 3.5.** Suppose \( u_0 \in H^1_0 \setminus \{0\} \) satisfies \( M(u_0)^{1-s_c} E_a(u_0)^{s_c} < M(Q_a)^{1-s_c} E_a(Q_a)^{s_c} \) and \( \|u_0\|_{L^2}^{1-s_c} \|u_0\|_{H^1_a}^{s_c} \leq \|Q_a\|_{L^2}^{1-s_c} \|Q_a\|_{H^1_a}^{s_c} \). Then by continuity, the maximal-lifespan solution \( u \) to (1.1) with initial data \( u_0 \) obeys \( \|u(t)\|_{L^2}^{1-s_c} \|u(t)\|_{H^1_a}^{s_c} < \|Q_a\|_{L^2}^{1-s_c} \|Q_a\|_{H^1_a}^{s_c} \) for all \( t \) in the lifespan of \( u \). In particular, \( u \) remains bounded in \( H^1_0 \) and hence is global.

### 4. Proof of Theorem 1.2

In this section, we turn to prove Theorem 1.2. Assume that \( u \) is a solution to (1.1) satisfying the hypotheses of Theorem 1.2. It follows from Remark 3.5 that \( u \) is global and satisfies the uniform bound

\[ \|u_0\|_{L^2}^{1-s_c} \|u(t)\|_{H^1_a}^{s_c} < (1 - \delta') \|Q_a\|_{L^2}^{1-s_c} \|Q_a\|_{H^1_a}^{s_c}. \]  \hspace{1cm} (4.1)

To show Theorem 1.2, we first establish a scattering criterion by following the argument as in [4, 15].

#### 4.1. Scattering criterion.

**Lemma 4.1** (Scattering criterion). Suppose \( u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) is a radial solution to (1.1) satisfying

\[ \|u\|_{L^\infty_t(\mathbb{R}, H^1(\mathbb{R}^d))} \leq E. \]  \hspace{1cm} (4.2)

There exist \( \epsilon = \epsilon(E) > 0 \) and \( R = R(E) > 0 \) such that if

\[ \liminf_{t \to \infty} \int_{|x| < R} |u(t, x)|^2 \, dx \leq \epsilon^2, \]  \hspace{1cm} (4.3)

then, \( u \) scatters forward in time.

**Proof.** First, by interpolation with \( \|u\|_{L^\infty_t(\mathbb{R}, H^1(\mathbb{R}^d))} \), we only need to show that

\[ \|u\|_{L^r_t([0, \infty), L^2(\mathbb{R}^d))} < +\infty, \]  \hspace{1cm} (4.4)

with \( r = \frac{2d}{d-2} \). By Hölder’s inequality, Sobolev embedding and (4.2), we have for any finite interval \( I \),

\[ \|u\|_{L^r_t(I, L^2)} \leq C|I|^\frac{1}{r} \|u\|_{L^\infty_t H^1} \leq C|I|^\frac{1}{r}. \]

Thus, we are reduced to show for some \( T > 0 \)

\[ \|u\|_{L^r_t([T, \infty), L^2(\mathbb{R}^d))} < +\infty. \]  \hspace{1cm} (4.5)

By continuity argument, Strichartz estimate and Sobolev embedding, we are further reduced to show

\[ \|e^{i(T-t)\mathcal{L}_a} u(T)\|_{L^r_t([T, \infty), L^2(\mathbb{R}^d))} < 1. \]  \hspace{1cm} (4.6)

Now, let \( 0 < \epsilon < 1 \) and \( R \geq 1 \) to be determined later. Using Duhamel formula, we can write

\[ e^{i(t-T)\mathcal{L}_a} u(T) = e^{it\mathcal{L}_a} u_0 + F_1(t) + F_2(t), \]  \hspace{1cm} (4.7)

where

\[ F_1(t) = i \int_{I_1} e^{i(t-s)\mathcal{L}_a} (|u^{p-1} u|) (s) \, ds, \hspace{0.5cm} I_1 = [0, T - \epsilon^\theta], \hspace{0.5cm} I_2 = [T - \epsilon^{-\theta}, T], \]
where $0 < \theta < 1$ to be determined later. Using Sobolev embedding, Strichartz estimate, we can pick $T_0$ sufficiently large such that
\[
\|e^{it\mathcal{L}_u}u_0\|_{L^1_t([T_0,\infty),L^2_x)} < \epsilon. \quad (4.8)
\]

**Estimate the term $F_1(t)$:** We can rewrite $F_1(t)$ as
\[
F_1(t) = e^{it(T+\epsilon^\theta)}\mathcal{L}_u[u(T-\epsilon^\theta)] - e^{it\mathcal{L}_u}u_0. \quad (4.9)
\]

Using Strichartz estimate, we have
\[
\|F_1\|_{L^p_t([T,\infty),L^q_x)} \lesssim 1. \quad (4.10)
\]

On the other hand, by Lemma 2.8, we get for $p \geq 2$
\[
\|F_1(t)\|_{L^p_t(|x| \leq R_1)} \lesssim \int_{|s|} \|e^{i(t-s)}\mathcal{L}_u(|u|^{p-1}u)(s)\|_{L^q_x(|x| \leq R_1)} \, ds
\]
\[
\lesssim \int_{|s|} \|(1 + |x|^{-\alpha_1})^{-1}e^{i(t-s)}\mathcal{L}_u(|u|^{p-1}u)(s)\|_{L^q_x} \, ds \cdot \|(1 + |x|^{-\alpha_1})\|_{L^{2p/(d-2)}(|x| \leq R_1)}
\]
\[
\lesssim R_1^{d-2} \int_{|s|} |t-s|^{-\frac{d}{2}+\alpha_1}(1 + |x|^{-\alpha_1})(|u|^{p-1}u)\|_{L^q_x} \, ds
\]
\[
\lesssim R_1^{d-2} |t-T + \epsilon^\theta|^{-\frac{d}{2}+\alpha_1}, \quad t > T, \quad (4.11)
\]

where
\[
\alpha_1 = \begin{cases} 0 & \text{if } a \geq 0 \\ \sigma & \text{if } a < 0, \quad (4.12) \end{cases}
\]

and we have used the estimate for $a \geq 0$
\[
\| |u|^{p-1}u\|_{L^1_x} \leq \|u\|_{L^p_x} \leq C\|u\|_{L^\infty H^1_x} < \infty,
\]
while for $a < 0$
\[
\| |x|^{-\sigma} |u|^{p-1}u\|_{L^1_x} \leq \| |x|^{-\sigma} u\|_{L^p_x} \|u|^{p-1}u\|_{L^2_{p-1}} \lesssim \| |x|^{-\sigma} u\|_{L^p_x} \|u|^{p-1}u\|_{H^1_x} \lesssim \|u\|_{H^1_x} < \infty,
\]

since $\sigma < 1$ by the assumption (1.3). When $p < 2$, we have
\[
\|F_1(t)\|_{L^p_t(|x| \leq R_1)} \lesssim \int_{I_2} \|(1 + |x|^{-\alpha_1})^{-1}e^{i(t-s)}\mathcal{L}_u(|u|^{p-1}u)(s)\|_{L^{2p/(d+2)}} \, ds
\]
\[
\times \|(1 + |x|^{-\alpha_1})^{-1}\|_{L^{2p/(d+2)}(|x| \leq R_1)}
\]
\[
\lesssim R_1^{\frac{pd-(d+2)}{2}} \int_{I_2} |t-s|^{-\frac{d}{2}+\alpha_1}(1 + |x|^{-\alpha_1})^{-1}(|u|^{p-1}u)\|_{L^{2p/(d+2)}} \, ds
\]
\[
\lesssim R_1^{\frac{pd-(d+2)}{2}} |t-T + \epsilon^\theta|^{-\frac{d}{2}+\alpha_1}(1 + |x|^{-\alpha_1})^{-1}\|u|^{p-1}u\|_{L^{2p/(d+2)}}, \quad t > T, \quad (4.13)
\]

where $\alpha_1$ is as in (4.12) and we have used the estimate for $a \geq 0$
\[
\| |u|^{p-1}u\|_{L^p_x} \leq \|u\|_{L^p_x} < \infty
\]
and for $\alpha < 0$

$$\|x|^{-(p-1)}|u|^{p-1}u\|_{L^2_T} \lesssim \|x|^{-\sigma}u\|_{L^p_T} \lesssim \|u\|_{L^\infty_T H^1_x} < +\infty.$$  

Using (4.19), Lemma 2.4, we obtain

$$\|F_1(t)\|_{L^p_T(|x| \geq R_1)} \lesssim \|F_1\|_{L^p_T} \|F_1\|_{L^\infty_T} \lesssim R_1^{-\frac{d-2}{p}}.$$  

Therefore, by taking $R_1 \frac{d-2}{p} = \epsilon^{-\frac{(d-2)}{2} - \alpha_1}$ for $p \geq 2$, and $R_1 \frac{d-2}{p} = \epsilon^{-\frac{(d-2)}{2} + \alpha_1}$ for $p < 2$, we get by Hölder’s inequality and (4.10)

$$\|F_1\|_{L^p_T(T, \infty), L^2_x} \lesssim \|F_1\|_{L^\infty_T(T, \infty), L^2_x} \|F_1\|_{L^\infty_T(T, \infty), L^2_x} \lesssim \epsilon^\beta,$$  

where

$$\beta = \begin{cases} \frac{2(d-1)}{d} \left( \frac{d-2}{2} - \alpha_1 \right) & \text{if } p \geq 2 \\ \frac{2(d-1)}{d} \left( 1 - \left( \frac{d-2}{2} + \alpha_1 \right) (p-1) \right) & \text{if } p < 2. \end{cases}$$

**Estimate the term** $F_2(t)$. First, by (4.3), we may choose $T > T_0$

$$\int \chi_R(x)|u(T, x)|^2 \, dx \leq \epsilon^2,$$  

where $\chi_R(x) \in C_0^\infty(\mathbb{R}^d)$ and

$$\chi_R(x) = \begin{cases} 1 & \text{if } |x| \leq R, \\ 0 & \text{if } |x| \geq 2R. \end{cases}$$

On the other hand, combining the identity $\partial_t |u|^2 = -2\nabla \cdot \text{Im}(\bar{u}\nabla u)$ and integration by parts, Hölder’s inequality, we obtain

$$\left| \partial_t \int \chi_R(x)|u(t, x)|^2 \, dx \right| \lesssim \frac{1}{R}.$$  

Hence, choosing $R \gg \epsilon^{-2-\theta}$, we get by (4.14)

$$\|\chi_R u\|_{L^{\infty}_T L^{2}_x(I_2 \times \mathbb{R}^d)} \lesssim \epsilon.$$  

And so, by Hölder’s inequality, Sobolev embedding and Lemma 2.4, we have for

$$q = \frac{2(d+1)}{d}$$

$$\|u\|_{L^q_{t,x}(I_2 \times \mathbb{R}^d)} \lesssim \epsilon^{-\frac{q}{2}} \|u\|_{L^2_{t,x}(I_2, L^2_x)}$$

$$\lesssim \epsilon^{-\frac{q}{2}} \left( \|\chi_R u\|_{L^2_{t,x}(I_2, L^2_x)} \|u\|_{L^\infty_{t,x}} \|u\|_{L^q_{t,x}} \right)$$

$$\lesssim \epsilon^{-\frac{q}{2}} \left( \epsilon^{-\frac{4}{p-2}} + R^{-\frac{d-2}{2}} \right) \lesssim \epsilon^{-\frac{4}{p-2}}.$$

On the other hand, using Strichartz estimate and continuous argument, we have

$$\|\mathcal{L} u\|_{L^q_{t,x}(I_2 \times \mathbb{R}^d)} \lesssim 1 + |J_3|.$$  

and for $\alpha < 0$
Thus, we use Sobolev embedding, Strichartz estimate, equivalence of Sobolev spaces (Lemma 2.2) to get
\[
\|F_2\|_{L^4_t((T,\infty),L^6_x)} \lesssim \|\nabla \|^{\frac{1}{2}} F_2\|_{L^\infty_t((T,\infty),L^6_x)} \\
\lesssim \|\mathcal{L}_\alpha^{\frac{1}{2}} (|u|^{p-1} u)\|_{L^6_t(I_2,L^\infty_x)} \\
\lesssim \|u\|_{L^{\frac{d+2}{d+2}(p-1)}(I_2 \times \mathbb{R}^d)}^{p-1} \|L_{t,x}^\alpha u\|_{L^{\frac{d+2}{d+2}(p-1)}(I_2,L^\infty_x)} \\
\lesssim |J_2|^{\frac{d+2}{d+2}(p-1)} \|u\|_{L^{\frac{d+2}{d+2}(p-1)}(I_2 \times \mathbb{R}^d)}^{p-1} |u|_{L^{\frac{d+2}{d+2}(p-1)}(I_2 \times \mathbb{R}^d)} \\
\lesssim \|e^{i(t-T)\mathcal{L}_\alpha u(T)}\|_{L^6_t((T,\infty),L^\infty_x(\mathbb{R}^d))} \lesssim \epsilon + \epsilon^{2+} + \epsilon^{4+6(1-s_c)(p-1)}.
\]
by taking \(\epsilon^{4+6(1-s_c)(p-1)} = \epsilon^{4+6(1-s_c)(p-1)}\). This together with (4.17), (4.18), and (4.14) yields that
\[
\|e^{i(t-T)\mathcal{L}_\alpha u(T)}\|_{L^6_t((T,\infty),L^\infty_x(\mathbb{R}^d))} \lesssim \epsilon + \epsilon^{2+} + \epsilon^{4+6(1-s_c)(p-1)}.
\]
And so (4.6) follows. Therefore, we conclude the proof of Lemma 4.1.

4.2. Virial identities. In this section, we recall some standard virial-type identities. Given a weight \(w : \mathbb{R}^d \to \mathbb{R}\) and a solution \(u\) to (1.1), we define
\[
V(t,w) := \int |u(t,x)|^2 w(x) \, dx.
\]
Using (1.1), one finds
\[
\partial_t V(t;w) = \int 2 \text{Im} \bar{u} \nabla u \cdot \nabla w \, dx, \hspace{1cm} (4.18)
\]
\[
\partial_t V(t;w) = \int (-\Delta \Delta w)|u|^2 + 4 \text{Re} \bar{u}_j u_k w_{jk} + 4|u|^2 \frac{\partial w}{|x|} \cdot \nabla w - \frac{2(p-1)}{p+1} |u|^{p+1} \Delta w \, dx.
\]

The standard virial identity makes use of \(w(x) = |x|^2\).

**Lemma 4.2** (Standard virial identity). Let \(u\) be a solution to (1.1). Then
\[
\partial_t V(t;|x|^2) = 8 \left[ \|u(t)\|_{L^4}^2 - \frac{d(p-1)}{2(p+1)} \|u(t)\|_{L^{p+1}}^2 \right].
\]

In general, we do not work with solutions for which \(V(t;|x|^2)\) is finite. Thus, we need a truncated version of the virial identity (cf. [14], for example). For \(R > 1\), we define \(w_R(x)\) to be a smooth, non-negative radial function satisfying
\[
w_R(x) = \begin{cases} |x|^2 & |x| \leq \frac{R}{2} \\ R^2 & |x| > R, \end{cases}
\]
with
\[
\partial_s w_R \geq 0, \ \partial_s^2 w_R \geq 0, \ |\partial_\alpha w_R(x)| \lesssim R|x|^{-|\alpha|+1}, \ |\alpha| \geq 1.
\]
In this case, we use (4.18) to deduce the following:
Lemma 4.3 (Truncated virial identity). Let \( u \) be a radial solution to \((1.1)\) and let \( R > 1 \). Then

\[
\partial_t V(t; w_R) = 8 \int_{|x| \leq \frac{R}{2}} \left[ |\nabla u(t)|^2 + a \frac{|u|^2}{|x|^2} - \frac{d(p-1)}{2(p+1)} |u(t)|^{p+1} \right] dx \\
+ \int_{|x| > R} \left[ 4aR \frac{|u|^2}{|x|^2} - \frac{2(d-1)(p-1)}{p+1} \frac{R}{|x|} |u|^p + \frac{4R}{|x|} (|\nabla u|^2 - |\partial_r u|^2) \right] dx \\
+ \int_{\frac{R}{2} \leq |x| \leq R} \left[ 4\text{Re}\partial_{jk}w_R \bar{u}_j \partial_k u + O \left( \frac{R}{|x|} |u|^p + \frac{R}{|x|^p} |u|^2 \right) \right] dx.
\]

Furthermore, by \((4.20)\), we have that

\[
\int_{\frac{R}{2} \leq |x| \leq R} \left[ 4\text{Re}\partial_{jk}w_R \bar{u}_j \partial_k u \right] dx \geq 0.
\]

4.3. Proof of Theorem 1.2 By the scattering criterion (Lemma 4.1) and Hölder’s inequality, Theorem 1.2 follows from the following lemma.

Lemma 4.4. There exists a sequence of times \( t_n \to \infty \) and a sequence of radii \( R_n \to \infty \) such that

\[
\lim_{n \to \infty} \int_{|x| \leq R_n} |u(t_n, x)|^{p+1} dx = 0. \tag{4.21}
\]

It is easy to see that the above lemma can be derived by the following proposition (choosing \( T \) sufficiently large and \( R = \max \{ T^{1/3}, T^{1/p} \} \)).

Proposition 4.5 (Morawetz estimate). Let \( T > 0 \). For \( R = R(\delta, M(u), Q_a) \) sufficiently large, we have

\[
\frac{1}{T} \int_0^T \int_{|x| \leq R} |u(t, x)|^{p+1} dx dt \leq \frac{R}{T} + \frac{1}{R^2} + \frac{1}{R^{p-1}}. \tag{4.22}
\]

Proof. First, by Lemma 4.3, we have

\[
\partial_t V(t; w_R) = 8 \int_{|x| \leq \frac{R}{2}} \left[ |\nabla u(t)|^2 + a \frac{|u|^2}{|x|^2} - \frac{d(p-1)}{2(p+1)} |u(t)|^{p+1} \right] dx \\
+ \int_{|x| > R} \left[ 4aR \frac{|u|^2}{|x|^2} - \frac{2(d-1)(p-1)}{p+1} \frac{R}{|x|} |u|^p + \frac{4R}{|x|} (|\nabla u|^2 - |\partial_r u|^2) \right] dx \\
+ \int_{\frac{R}{2} \leq |x| \leq R} \left[ 4\text{Re}\partial_{jk}w_R \bar{u}_j \partial_k u + O \left( \frac{R}{|x|} |u|^p + \frac{R}{|x|^p} |u|^2 \right) \right] dx.
\]

We define \( \chi \) to be a smooth cutoff to the set \( \{|x| \leq 1\} \) and set \( \chi_R(x) = \chi(x/R) \). Note that

\[
\int \chi_R^2 |\nabla u|^2 dx = \int \left[ |\nabla (\chi_R u)|^2 + \chi_R \Delta (\chi_R) |u|^2 \right] dx, \tag{4.26}
\]
On the other hand, by radial Sobolev embedding, one has
which is accepted. Hence we conclude the proof of Proposition 4.5.

Combining the above together, we obtain

\[ 3.4 \text{(ii)}. \]

\[ \| \chi R u \|_{H^2}^2 - \frac{d(p-1)}{2(p+1)} \| \chi R u \|_{L^{p+1}}^{p+1} \geq c \| \chi R u \|_{L^{p+1}}^{p+1}. \quad (4.27) \]

Indeed, by (4.26), we have

\[ \| \chi R u \|_{H^2}^2 \leq \| u \|_{H^2}^2 + O\left( \frac{M(u)}{R^2} \right), \]

and \( \| \chi R u \|_{L^2} \leq \| u \|_{L^2} \). Then, (4.26) follows by the same argument as Proposition 3.4 (ii).

Now, applying the fundamental theorem of calculus on an interval \([0, T]\), discarding the positive terms, using (4.23)-(4.27), we obtain

\[ \int_0^T \int_{\mathbb{R}^d} |\chi R u|^{p+1} \, dx \leq \sup_{t \in [0, T]} |V(t, w_R)| + \int_0^T \int_{|x| \geq R} |u(t, x)|^{p+1} \, dx \, dt + \frac{T}{R^2} M(u). \]

From Hölder’s inequality, (4.18) and (4.4), we get

\[ \sup_{t \in \mathbb{R}} |\partial_t V(t, w_R)| \lesssim R. \]

On the other hand, by radial Sobolev embedding, one has

\[ \int_{|x| \geq R} |u(t, x)|^{p+1} \, dx \lesssim \frac{1}{R^{p-1}} \| u \|_{L^p \cap H^1}^{p-1} M(u). \quad (4.28) \]

Combining the above together, we obtain

\[ \frac{1}{T} \int_0^T \int_{|x| \leq R} |u(t, x)|^{p+1} \, dx \, dt \lesssim \frac{R}{T} + \frac{1}{R^2} + \frac{1}{R^{p-1}}, \]

which is accepted. Hence we conclude the proof of Proposition 4.5.

Therefore, we complete the proof of Theorem 1.2

\[ \square \]

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