On the two-component generalization of the (2+1)-dimensional Davey-Stewartson I equation

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Abstract. The geometric-gauge equivalent of the famous Ishimori spin equation is the (2+1)-dimensional Davey-Stewartson equation, which in turn is one of the (2+1)-dimensional generalizations of the nonlinear Schrödinger equation. Multicomponent generalization of nonlinear integrable equations attract considerable interest from both physical and mathematical points of view. In this paper, the two-component integrable generalization of the (2+1)-dimensional Davey-Stewartson I equation is obtained based on its one-component representation, and the corresponding Lax representation is also obtained.

1. Introduction
It is known that integrable nonlinear Schrödinger type equations (NSE) are key models in the theory of integrable equations. Recently, their multicomponent generalizations have been actively studied. In the work [1] it is shown that the Manakov two-step system is integrable. The geometrical connection with the last system and the two-layer spin model was established in [2] - [4]. The geometric gauge equivalent of the Ishimori spin equation [5] is a (2+1)-dimensional Davy-Stewartson (DS) equation, which is one of the (2+1) -dimensional generalizations of the NSE [6].

Consider the (2+1)-dimensional DS equation

\[ iq_t + \frac{1}{2}(\sigma^2 q_{xx} + q_{yy}) = (v - qr)q, \]

\[ -ir_t + \frac{1}{2}(\sigma^2 r_{xx} + r_{yy}) = (v - qr)r, \]

\[ v_{xx} - \sigma^2 v_{yy} = 2(qr)_{xx}, \]

where \( r = \pm q^* \), \( q^* \) is the complex conjugate of \( q \). For \( \sigma^2 = 1 \), the system of equations 1-2 is called the Davy-Stewartson Type I equation (DSI), and for \( \sigma^2 = -1 \) – the Davy-Stewartson Type II equation (DSII) [7]. We focus on the (2+1)-dimensional DSI equation, and in the next section we give some well-known data on the single-component (2+1)-dimensional DSI equation. Our goal is to derive a two-component generalization for the last equation.

The result of the study is formed in the form of approval and is proved in the second paragraph.
2. A single-component (2+1)-dimensional DSI equation

As it is known, the standard Lax representation of the DSI equation is

\[ F_y = \sigma_3 F_x + Q F, \]
\[ F_t = A_2 F_{xx} + A_1 F_x + A_0 F, \]

where

\[ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & q \\ -r & 0 \end{pmatrix}, \]

\[ A_0 = i \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad A_1 = 2i \begin{pmatrix} 0 & q \\ -q & 0 \end{pmatrix}, \quad A_2 = 2i\sigma_3. \]

The elements of the matrix \( A_0 \) satisfy the following conditions:

\[ c_{12} = \frac{1}{2}(\partial_x + \partial_y)q, \]
\[ c_{21} = -\frac{1}{2}(\partial_x - \partial_y)r, \]
\[ (\partial_x - \partial_y)c_{11} = -\frac{1}{2}(\partial_x + \partial_y)(qr), \]
\[ (\partial_x + \partial_y)c_{22} = \frac{1}{2}(\partial_x - \partial_y)(qr), \]

here \( \partial_x \equiv \partial/\partial_x, \partial_y \equiv \partial/\partial_y \) and the field \( v \) in the system of DS equations (1)-(2) is defined as

\[ v = -i(c_{22} - c_{11}) + qr. \]

The compatibility condition for the equation (4) and (5) \( F_{yt} = F_{ty} \) implies the following series of equations:

\[ [\sigma_3, A_2] = 0, \quad [\sigma_3, A_1] + [Q, A_2] = 0, \quad [\sigma_3, A_0] + [Q, A_1] - 2AQ_x = 0, \]
\[ \sigma_3 A_{0x} - A_0y + Q_t + [Q, A_0] - 2AQ_{xx} - A_1Q_x = 0. \]

From the system of equations (6)-(9) it is not difficult to obtain a one-component (2+1)-dimensional DSI equation [8].

3. A two-component (2+1)-dimensional DSI equation

**Theorem.** If the matrices \( \Sigma, Q \) and \( A \) belong to the group \( SU(3) \), then a two-component (2+1)-dimensional DSI equation has the following form:

\[ iq_{1t} + q_{1xx} + q_{1yy} - v_1q_1 - w_1q_2 = 0, \]
\[ iq_{2t} + q_{2xx} + q_{2yy} - w_2q_1 - v_2q_2 = 0, \]
\[ -ir_{1t} + r_{1xx} + r_{1yy} - v_1r_1 - w_1r_2 = 0, \]
\[ -ir_{2t} + r_{2xx} + r_{2yy} - w_2r_1 - v_2r_2 = 0, \]
\[ v_{1xx} - v_{1yy} = (2r_1q_1 + r_2q_2)_{xx} + 2(r_2q_2)_{xy} + (2r_1q_1 + r_2q_2)_{yy}, \]
\[ v_{2xx} - v_{2yy} = (r_1q_1 + 2r_2q_2)_{xx} + 2(r_1q_1)_{xy} + (r_1q_1 + 2r_2q_2)_{yy}, \]
\[ w_{1xx} - w_{1yy} = (q_1r_2)_{xx} - 2(q_1r_2)_{xy} + (q_1r_2)_{yy}, \]
\[ w_{2xx} - w_{2yy} = (q_2r_1)_{xx} - 2(q_2r_1)_{xy} + (q_2r_1)_{yy}. \]
where \(q\) and \(r\) are complex-valued functions, and \(v_j\) and \(w_j\) are real functions.

Proof. For proof, we require that the column matrix \(F\) satisfies the following Lax representation

\[
\begin{align*}
F_y &= \Sigma F_x + PF, \\
F_t &= B_2 F_{xx} + B_1 F_x + B_0 F,
\end{align*}
\]

where

\[
\Sigma = \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1
\end{pmatrix},
\]

the remaining matrices belong to the group \(su(3)\):

\[
P = \begin{pmatrix}
0 & q_1 & q_2 \\
-r_1 & 0 & 0 \\
-r_2 & 0 & 0
\end{pmatrix},
\]

\[
B_0 = \begin{pmatrix}
c_{11} & c_{12} & c_{13} \\
c_{21} & c_{22} & c_{23} \\
c_{31} & c_{32} & c_{33}
\end{pmatrix},
\]

\[
B_1 = \begin{pmatrix}
b_{11} & b_{12} & b_{13} \\
b_{21} & b_{22} & b_{23} \\
b_{31} & b_{32} & b_{33}
\end{pmatrix},
\]

\[
B_2 = \begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}.
\]

Then, from the compatibility condition \(F_{yt} = F_{ty}\) of the system (18) and (19), we obtain

\[
\begin{align*}
[\Sigma, B_2] &= 0, \\
\Sigma B_{2x} - B_{2y} + [\Sigma, B_1] + [P, B_2] &= 0, \\
\Sigma B_{1x} - B_{1y} + [\Sigma, B_0] + [P, B_1] - 2B_2 P_x &= 0, \\
\Sigma B_{0x} - B_{0y} + P_t + [P, B_0] - B_2 P_{xx} - B_1 P_x &= 0.
\end{align*}
\]

Now we define elements of the matrices \(B_0, B_1\) and \(B_2\). From the equation (20) it is determined that

\[a_{12} = a_{21} = a_{13} = a_{31} = 0.\]

Therefore, we have

\[
B_2 = \begin{pmatrix}
a_{11} & 0 & 0 \\
0 & a_{22} & a_{23} \\
0 & a_{32} & a_{33}
\end{pmatrix}.
\]

Similarly, from the equations (21) and (22) we get a number of restrictions for the elements of the matrices \(B_0, B_1\) and \(B_2\).

Namely, for the elements \(a_{ij}\) \((i, j = 1, 2, 3)\) of the matrix \(B_2\):

\[
\begin{align*}
a_{11x} + a_{11y} &= 0, \\
&
\]

\[
\begin{align*}
a_{22x} + a_{22y} &= 0, \\
&
\]

\[
\begin{align*}
a_{33x} + a_{33y} &= 0, \\
&
\]

\[
\begin{align*}
a_{23x} + a_{23y} &= 0, \\
&
\]

\[
\begin{align*}
a_{32x} + a_{32y} &= 0,
\end{align*}
\]

for the elements \(b_{ij}\) \((i, j = 1, 2, 3)\) of the matrix \(B_1\):

\[
b_{12} = -\frac{1}{2}(a_{2211}q_1 + a_{32q_2}),
\]
\[ b_{21} = \frac{1}{2} (a_{2211} r_1 + a_{23} r_2), \]
\[ b_{13} = -\frac{1}{2} (a_{3311} q_2 + a_{23} q_1), \]
\[ b_{31} = \frac{1}{2} (a_{3311} r_2 + a_{32} r_1), \]
\[ b_{11x} - b_{11y} = 0, \]
\[ b_{22x} + b_{22y} = -\frac{1}{2} (a_{23} q_1 r_2 - a_{32} q_2 r_1), \]
\[ b_{33x} + b_{33y} = \frac{1}{2} (a_{23} q_1 r_2 - a_{32} q_2 r_1), \]
\[ b_{23x} + b_{23y} = \frac{1}{2} (a_{3322} q_2 r_1 + a_{23} (q_1 r_1 - q_2 r_2)), \]
\[ b_{32x} + b_{32y} = -\frac{1}{2} (a_{3322} q_1 r_2 + a_{32} (q_1 r_1 - q_2 r_2)), \]

where \( a_{ijj} = a_{ii} - a_{jj} \ (i > j) \). And for the elements \( c_{ij} \ (i, j = 1, 2, 3) \) of the matrix \( B_0 \), we have

\[ c_{12} = \frac{1}{4} [(3a_{11} + a_{22}) q_1 x - a_{2211} q_1 y + a_{32} q_2 x - a_{32} q_2 y + (a_{22} - a_{22} - 2b_{2211}) q_1 + (a_{32} - a_{32} - 2b_{32}) q_2], \]
\[ c_{21} = \frac{1}{4} [(3a_{11} + a_{22}) r_1 x - a_{2211} r_1 y + 3a_{32} q_2 x - 3a_{32} q_2 y + (a_{11} + a_{11} + 2b_{2211}) r_1 + 2b_{32} r_2), \]
\[ c_{13} = \frac{1}{4} [a_{23} q_1 x - a_{23} q_1 y + (3a_{11} + a_{33}) q_2 x - 3a_{311} q_2 y + (a_{23} - a_{23} - 2b_{12}) q_1 + (a_{33} - a_{33} - 2b_{331}) q_2], \]
\[ c_{31} = \frac{1}{4} [a_{33} r_1 x - a_{33} r_1 y + (a_{11} + a_{33}) r_2 x - a_{3311} r_2 y + 2b_{32} r_1 + (a_{11} + a_{11} + 2b_{331}) r_2]. \]

From the equation (23) we get the following system of equations:

\[ q_{1t} = a_{11} q_{1x} + b_{11} q_{1x} - c_{12} x + c_{12} y - c_{2211} q_1 - c_{32} q_2, \]  
\[ q_{2t} = a_{11} q_{2x} + b_{11} q_{2x} - c_{13} x + c_{13} y - c_{3311} q_1 - c_{32} q_2, \]  
\[ r_{1t} = a_{22} r_{1x} + a_{23} r_{2x} + b_{22} r_{1x} + b_{23} r_{2x} - c_{21} x - c_{21} y + 2b_{2211} r_1 + c_{23} r_2, \]  
\[ r_{2t} = a_{33} r_{1x} + a_{33} r_{2x} + b_{33} r_{1x} + b_{33} r_{2x} - c_{31} x - c_{31} y + 2b_{3311} r_1 + c_{33} r_2, \]  
\[ c_{11x} + c_{21} q_1 + c_{31} q_2 + c_{12} r_1 + c_{13} r_2 + b_{12} r_1 + b_{13} r_2 - c_{11y} = 0, \]  
\[ -c_{22} x - c_{21} q_1 - b_{21} q_1 - c_{22} y = 0, \]  
\[ -c_{23} x - c_{13} r_1 - c_{21} q_2 - b_{21} q_2 - c_{23} y = 0, \]  
\[ -c_{32} x - c_{12} r_2 - c_{31} q_1 - c_{32} y = 0, \]  
\[ -c_{33} x - c_{13} r_2 - c_{31} q_2 - c_{33} y = 0, \]

Then, taking into account the above results, the equation (10) can be rewritten as

\[ i q_{1t} + \frac{i}{4} a_{2211} q_{1x} + \frac{i}{4} a_{2211} q_{1y} + \frac{i}{4} a_{32} q_{2x} + \frac{i}{4} a_{32} q_{2y} + \frac{i}{2} (-a_{22} - a_{11}) q_{1x} - \frac{i}{2} a_{32} q_{2xy} + \frac{i}{2} (a_{22} - a_{22} - b_{22} - b_{11}) q_{1y} + \frac{i}{2} (a_{22} - a_{22} + b_{22} - b_{11}) q_{1y} + \frac{i}{4} [a_{22} q_{2x} - 2a_{22} q_{2x} + a_{22} q_{2y} - 2(b_{22} - b_{22}) + 4c_{2211}] q_{1y} + \frac{i}{4} [a_{32} q_{2x} - 2a_{32} q_{2x} + a_{32} q_{2y} - 2(b_{32} - b_{32}) + 4c_{32}] q_{2y} = 0. \]
Assuming that the coefficients of the second derivative of $q_1$ with respect to $x$ and $y$ in the equation (33) are equal to unity, i. e. $\frac{1}{4}a_{2211} = 1$, we obtain $a_{2211} = a_{22} - a_{11} = -4i$. Without loss of generality, take $a_{11} = 2i$ and $a_{22} = -2i$.

Similarly, we consider the equation (25). We get that

$$iq_{2t} + \frac{i}{4}a_{3311}q_{2xx} + \frac{i}{4}a_{3311}q_{2yy} + \frac{i}{4}a_{23}q_{1xx} + \frac{i}{4}a_{23}q_{1yy} + \frac{i}{2}(-a_{33} - a_{11})q_{2xy} - \frac{i}{2}a_{23}q_{1xy} + \frac{i}{2}(a_{23x} - a_{23y} - b_{32})q_{1x} + \frac{i}{2}(-a_{23x} + a_{23y} + b_{32})q_{1y} + \frac{i}{2}(a_{33x} - a_{33y} - b_{33} - b_{11})q_{2x} + \frac{i}{2}(a_{33x} + a_{33y} + b_{33} - b_{11})q_{2y} + \frac{i}{4}[a_{33xx} - 2a_{33xy} + a_{33yy} - 2(b_{33x} - b_{33y}) + 4c_{3311}]q_{2} + \frac{i}{4}[a_{23xx} - 2a_{23xy} + a_{23yy} - 2(b_{23x} - b_{23y}) + 4c_{23}]q_{1} = 0. \tag{34}$$

We assume that in the equation $34\frac{1}{4}a_{3311} = 1$. Then we determine that $a_{33} = -2i$. Given the above, the matrix $B_2$ takes the form

$$B_2 = \begin{pmatrix} 2i & 0 & 0 \\ 0 & -2i & a_{23} \\ 0 & a_{32} & -2i \end{pmatrix} = 2i\Sigma + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{pmatrix}.$$

In the case when $a_{23} = a_{32} = 0$, the matrix $B_2$ takes the form

$$B_2 = 2i\Sigma.$$

Now we list all the equations obtained from 20 - 23 for the elements $b_{ij}$ and $c_{ij}$. We have

$$b_{12} = 2iq_1,$$
$$b_{13} = 2iq_2,$$
$$b_{21} = -2ir_1,$$
$$b_{31} = -2ir_2,$$
$$b_{11x} - b_{11y} = 0,$$
$$b_{22x} + b_{22y} = 0,$$
$$b_{33x} + b_{33y} = 0,$$
$$b_{23x} + b_{23y} = 0,$$
$$b_{32x} + b_{32y} = 0.$$

The last five equations have a solution

$$b_{11} = b_{22} = b_{23} = b_{32} = b_{33} = 0.$$

Thus, for the elements of the matrix $B_1$ we get

$$B_1 = \begin{pmatrix} 0 & 2iq_1 & 2iq_2 \\ -2ir_1 & 0 & 0 \\ -2ir_2 & 0 & 0 \end{pmatrix} = 2iP.$$
Similarly, we define the expressions for the elements $c_{ij}$ of the matrix $B_0$ in the form

- $c_{12} = i(q_1x + q_1y)$,
- $c_{13} = i(q_2x + q_2y)$,
- $c_{21} = -i(r_1x - r_1y)$,
- $c_{31} = -i(r_2x - r_2y)$.

The remaining five elements $c_{11}, c_{22}, c_{33}, c_{23}, c_{32}$ satisfy the following nontrivial equations:

\begin{align*}
    c_{11}x - c_{11}y &= -i(r_1q_1 + r_2q_2)x - i(r_1q_1 + r_2q_2)y, \
    c_{22}x + c_{22}y &= i(r_1q_1)x - i(r_1q_1)y, \
    c_{33}x + c_{33}y &= i(r_2q_2)x - i(r_2q_2)y, \
    c_{23}x + c_{23}y &= i(r_1q_2)x - i(r_1q_2)y, \
    c_{32}x + c_{32}y &= i(r_2q_1)x - i(r_2q_1)y.
\end{align*}

Let us introduce the notations $v_1 = -ic_{2211}$, $v_2 = -ic_{3311}$, $w_1 = -ic_{32}$ and $w_2 = -ic_{23}$. Acting on $v_1$ with operators $D^+ = \partial_x + \partial_y$ and $D^- = \partial_x - \partial_y$, we get

\begin{equation}
    D^- D^+ (-ic_{22}) = -i(c_{22xx} - c_{22yy}),
\end{equation}

\begin{equation}
    D^+ D^- (ic_{22}) = (r_1q_1)_{xx} - (r_1q_1)_{yx} - (r_1q_1)_{xy} + i(r_1q_1)_{yy} = (r_1q_1)_{xx} - 2(r_1q_1)_{xy} + i(r_1q_1)_{yy},
\end{equation}

\begin{equation}
    D^+ D^- v_1 = (2r_1q_1 + r_2q_2)_{xx} + 2(r_2q_2)_{xy} + (2r_1q_1 + r_2q_2)_{yy}.
\end{equation}

As a result, we have

\begin{equation}
    v_{1xx} - v_{1yy} = (2r_1q_1 + r_2q_2)_{xx} + 2(r_2q_2)_{xy} + (2r_1q_1 + r_2q_2)_{yy},
\end{equation}

which in turn gives the equation (14).

Similarly, acting by the operators $D^+ y$ and $D^-$ on $v_2$, $w_1$ and $w_2$, we obtain the following equations, respectively:

\begin{align*}
    v_{2xx} - v_{2yy} &= (r_1q_1 + 2r_2q_2)_{xx} + 2(r_1q_1)_{xy} + (r_1q_1 + 2r_2q_2)_{yy}, \
    w_{1xx} - w_{1yy} &= (q_1r_2)_{xx} - 2(q_1r_2)_{xy} + (q_1r_2)_{yy}, \
    w_{2xx} - w_{2yy} &= (q_2r_1)_{xx} - 2(q_2r_1)_{xy} + (q_2r_1)_{yy}.
\end{align*}

As we can see, the last three equations obtained above turn out to be the equivalent equations (15)-(17). Furthermore, using the above notation (35)-(39), we can write for equations for $q_{1t}$, $q_{2t}$, $r_{1t}$ and $r_{2t}$ in the form

\begin{align*}
    iq_{1t} + q_{1xx} + q_{1yy} - v_{1q} - w_{1q} &= 0, \
    iq_{2t} + q_{2xx} + q_{2yy} - w_{2q1} - v_{2q2} &= 0, \
    -ir_{1t} + r_{1xx} + r_{1yy} - v_{1r1} - w_{1r2} &= 0, \
    -ir_{2t} + r_{2xx} + r_{2yy} - w_{2r1} - v_{2r2} &= 0.
\end{align*}

Thus, we obtained the system of equations (10)-(12), which was required to prove.
4. Conclusion
In conclusion, we note that the obtained two-component generalization of the DSI (10)-(17) equation and its Lax representation (18)-(19) are new. A detailed study of the algebraic and geometric properties of the (10)-(17) system is the subject of our further research.

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References
[1] Kostov N., Dandoloff R., Gerdjikov V., Grahovski G. The Manakov System as Two Moving Interacting Curves, Proc.Int.Works. "Complex structures and vector fields", Sofia, Bulgaria. Eds.: K.Sekigawa, S.Dimiev. World Scientific (2007).
[2] Myrzakul Akbota and Myrzakulov Ratbay. Darboux transformations exact soliton solutions of integrable coupled spin systems related with the Manakov system, [arXiv:1607.08151]
[3] Myrzakul Akbota and Myrzakulov Ratbay. Integrable geometric flows of interacting curves/surfaces, multilayer spin systems and the vector nonlinear Schrodinger equation, [arXiv:1608.08553]
[4] Nugmanova G., Myrzakul A. Integrability of the two-layer spin system, Proc.of Twentieth Int. Conf. "Geometry, Integrability and Quantization", Varna, Yoshioka (2018).
[5] Ishimori Y. Multi-Vortex solutions of the a two-dimensional nonlinear wave equation, Prog. Theor. Phys. 1984. 72, N1. pp. 33-37.
[6] Davey A., Stewartson K. On three-dimensional packets of surface waves, Proc. of the Royal Society of London Series A, 1974. 338, pp. 101–110.
[7] Ablowitz M. A., Clarkson P. A. Solitons, it Nonlinear Evolution Equations and Inverse Scattering London Mathematical Society Lecture Note Series Book 149, p. 516 ISBN 0521387302
[8] Zhou Z., Ma W X., Zhou R. Finite-dimensional integrable systems associated with Davey-Stewartson I equation Nonlinearity 14 (2001) pp.701-717.