Mixmaster quantum cosmology 
in terms of physical dynamics

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Abstract
An approach to quantum cosmology, relying on strengths of both canonical and path integral formalisms, is applied to the cosmological model, Bianchi type IX. Physical quantum states are constructed on the maximal slice of the cosmological history. A path integral is derived which evolves observables off the maximal slice. This result is compared a path integral propagator derived earlier with conventional Faddeev-Popov gauge fixing.

I. INTRODUCTION

Previously, we suggested a new approach to the quantum description of cosmology [1]. Relying on the strengths of both canonical and path integral methods, this “composite approach” to quantization offers a real possibility of calculating physical observables for cosmological models. The idea is to define the physical observables and inner product of the canonical theory and then use them to construct a path integral which evolves these physical observables. In the companion paper, we presented a path integral quantization of the cosmological model Bianchi type IX using Faddeev-Popov techniques familiar from gauge theory. Here, we carry out this composite approach for the model cosmology Bianchi type IX and compare our result to that found previously [1].

One of the great difficulties in the canonical quantization of gravity is that, to find the physical quantities in a gauge invariant formalism, it is essentially necessary to solve the theory. However, as pointed out by Vince Moncrief during a discussion three years ago at Santa Barbara, there are cases in which we can identify physical observables and define their algebra. For example, in spatially compact cosmological models, we can define the physical observables to be the metric and extrinsic curvatures of the maximal slice, on which the trace of the extrinsic curvature vanishes. As the metric and extrinsic curvature evaluated there label solutions uniquely, they coordinatize the physical phase space. Moreover, the Poisson brackets are the naive ones, evaluated on that surface.

Alternatively, physical observables can be defined by a complete gauge fixing of the theory. At the classical level, the two procedures must agree. However, in the quantum theory, it is not obvious that either of these procedures must lead to a good quantum theory. Nor is it obvious that they will lead to the same quantum theory. Our main goal in this work is to examine and compare the quantum theories that follow from each of these procedures, for the Bianchi type IX model. In each case we construct the path integral formulation of the quantum theory from the associated canonical theory. We compare the resulting path integrals with each other as well as with the expressions we previously found using the Faddeev-Popov procedure.

We work with the Bianchi type IX cosmological model because it is the simplest model that is, as far as is known, not solvable in closed form. This means that, although the configuration space is two dimensional, it shares with the real theory the property that we know of no procedure to explicitly construct the solutions of the theory. The model describes a cosmological family with homogeneous but anisotropic spatial slices. These anisotropies are the two dynamical degrees of freedom. The model has been studied extensively, especially since Misner expressed the dynamics as single particle mechanics in a time dependent potential [2,3]. A key property, which we shall exploit, is that all of this model’s classical histories recollapse [4]. Recently, quantizations of Bianchi type IX have been proposed by Kodama [5] (see also [6]), Grahm [7], and Marolf [8].

The paper begins with a quick review of the phase space of this model. Once the maximal slice is determined we present the physical quantization and path integral. The third section offers a comparison with our earlier work.
where \( \beta \) physical degrees of freedom are simply the two anisotropies, curvature. Since all Bianchi type IX classical histories end in recollapse \[4\], we can always identify this slice. The maximal \( \mathcal{H} \) Einstein equations.

Writing the structure constants in terms of these pieces, \( C_{ijk} = \epsilon_{jkl} \alpha^l \beta^i \). The action for the model Bianchi type IX in this chart is \[2\]

\[
I = \int p_+ d\beta^+ + p_- d\beta^- + p_\alpha d\alpha - \sqrt{\frac{3\pi}{2}} N e^{-3\alpha} \mathcal{H} dt
\]

in which \( \mathcal{H} = \frac{1}{2} (-p_\alpha^2 + p_+^2 + p_-^2 + e^{3\alpha} \mathcal{U}(\beta^\pm)) \) \[\mathcal{U}(\beta^\pm) = \frac{1}{3} e^{-8\beta^+} - \frac{4}{3} e^{-2\beta^+} \cosh(2\sqrt{3}\beta^-) + \frac{2}{3} e^{4\beta^+} (\cosh(4\sqrt{3}\beta^-) - 1) \].

Thus, mixmaster dynamics may be seen as the dynamics of a single particle in a time dependent potential. The action of Eq. \[2\] can be expressed by fixing the lapse, \( N \), so that \( N = \sqrt{\frac{2}{3\pi}} \exp(3\alpha) \). This allows us to write \[3\]

\[
I = \int p_+ d\beta^+ + p_- d\beta^- + p_\alpha d\alpha - \mathcal{H} d\lambda
\]

This action must be supplemented with the condition \( \mathcal{H} = 0 \). For a fixed value of this potential, the walls contract as the scale factor increases to the maximum volume slice. After this slice, the walls of the potential, for a fixed value, recede back from the center.

To implement the proposed quantization of the theory, we need to determine the slice with vanishing extrinsic curvature. Since all Bianchi type IX classical histories end in recollapse \[4\], we can always identify this slice. The physical degrees of freedom are simply the two anisotropies, \( \beta^+ \) and \( \beta^- \). In addition, on this slice the volume is maximal. Since \( \sqrt{\mathcal{H}} = e^{3\alpha} \), to find this maximum volume slice, we must maximize \( \alpha \). From the action of Eq. \[4\] we find,

\[
\dot{\alpha} = \{ \alpha, \mathcal{H} \} = -p_\alpha;
\]

so that the volume is extremized (for finite volume) when \( p_\alpha \) vanishes. To find the maximum, note that

\[
\dot{\alpha}|_{\alpha=0} = 3\pi N^2 e^\alpha \mathcal{U}(\beta^\pm)
\]

giving the maximum volume, \( e^{3\alpha} \), when the potential is less than zero. This defines an open region \( \mathcal{R} \), \( \mathcal{R} := \{ \beta^\pm : \mathcal{U}(\beta^\pm) < 0 \} \) shown in Fig. 1. This region is finite \[3\].

The kinematical phase space, \( \Gamma \), is \( \mathbb{R}^6 \) coordinatized by \( (\alpha, \beta^+, \beta^-) \) and conjugate momenta \( (p_\alpha, p_+, p_-) \). The physical phase space \( \Gamma \) is a four dimensional submanifold of \( \Gamma \), which is defined as the quotient of \( \Gamma \) by the solution space of \( \mathcal{H} = 0 \). We see that

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*The classification of Bianchi models concerns the irreducible parts of the structure constants of the isometry Lie group. Writing the structure constants in terms of these pieces, \( C^i_{jkl} = \epsilon_{jkl} S^{li} + \delta^i_{jl} V_{ki} \), class A models are those for which \( V_I = 0 \).

†Note that in the literature one often finds written \( V(\beta) := \mathcal{U}(\beta^\pm) + 1 \). This is convenient because \( V(\beta^\pm) \) is positive definite, however what is important to remember is that the actual potential \( \mathcal{U}(\beta^\pm) \) is bounded from below by \(-1\).
FIG. 1. The zero contour for the classical potential for Bianchi IX. Three “channels,” one along the positive $\beta^-$ axis, and the other two sloping diagonally off in the negative $\beta^-$ direction extend to infinity. The area of the interior region, $R$, is finite.

$$\Gamma = T^* R$$

(7)

The coordinates of $\Gamma$ correspond to the metric and extrinsic curvature of solutions on the unique maximal slice $p_\alpha = 0$. The quantities $O = \{\alpha, \beta^\pm, p\pm\}$ on a slice, say $p_\alpha = \tau$, could be found by integrating the solutions from $p_\alpha = 0$ to $p_\alpha = \tau$ using the Hamilton’s equation of motion (See, for example, [10]). We will denote this set of physical observables $O(\tau)$. As all solutions end in singularities in some finite time, the $O(\tau)$ are only defined on a subspace of $\Gamma$ corresponding to data on the surface $p_\alpha = 0$ which specifies non-singular solutions.

For example, let us consider the observable for volume of the universe, on the slice where $p_\alpha = \tau$, found by solving the Hamiltonian constraint

$$V(\tau) = e^{3\alpha(\tau)} = \left[\frac{p_+(\tau)^2 + p_-(\tau)^2 - \tau^2}{-U(\beta^\pm(\tau))}\right]^\frac{3}{4}.$$  

(8)

As each of the factors is a complicated function on $\Gamma$, the explicit form of this can only be found by integration of the equation

$$\{V(p_\alpha), \mathcal{H}\} = 0$$

(9)

on the kinematical phase space $\bar{\Gamma}$ with the initial condition,

$$V(0) = \left(\frac{p_+^2 + p_-^2}{-U(\beta^\pm)}\right)^\frac{3}{4},$$

(10)

i.e. one must essentially solve the theory. At each $\tau$ the subspace $\bar{\Gamma}(\tau)$ defined by $V(\tau) > 0$ is the domain on which the $O(\tau)$ are defined. That the physical observables are only defined on subspaces of $\bar{\Gamma}$ is a new feature of cosmology that raises questions such as whether is the Poisson bracket of these functions always defined. Of great interest is the extent to which this causes problems for the quantum theory, in either a canonical or a path integral formalism.

III. MAXIMAL SLICE CANONICAL QUANTUM THEORY

We first quantize the physical phase space $\Gamma = T^* \mathcal{R}$. Since the physical configuration variables, $(\beta^\pm, \beta^-) \in \mathcal{R}$, and the conjugate momenta, $(p_+, p_-) \in \mathcal{R}^2$, are coordinates on the classical phase space, take $C_0^\infty$ functions as
wavefunctions in the Hilbert space. The quantum theory is built from the configuration operators \( \hat{\alpha}^\pm \), which act by multiplication, and the momentum operators \( \hat{\beta}^\pm_0 \) which act by differentiation \( \hat{\beta}^\pm_0 = -i\hbar \partial_{\alpha} \) on the wavefunctions. These operators correspond to measurements of the corresponding quantities defined on the slice \( p_\alpha = 0 \). Hence the subscript and superscript "0."

We have the inner product,

\[
\langle \chi | \xi \rangle = \int_{\mathcal{R}} d^2 \beta^\pm_0 \chi(\beta^\pm_0) \xi(\beta^\pm_0).
\] (11)

If the wavefunctions vanish on the boundary of \( \mathcal{R} \) then \( \hat{\beta}^\pm_0 \) and \( \hat{\beta}^\pm_0 \) are Hermitian in this inner product.

Before considering the problem of observables at times other than \( p_\alpha = 0 \), it is interesting to study the operator that measures the volume on maximal slice. As this is a property of solutions, it is a physical observable, but as it is a function of data on the maximal slice it can be exhibited explicitly. It is convenient to consider the 4/3 power of the volume, which by Eq. (8) is

\[
V^{4/3} = \frac{\hat{p}_i^2 + \hat{p}_j^2}{(-|U|)} \equiv q^{ij}p_ip_j,
\] (12)

written in terms of a metric defined by \( q^{++} = q^{--} = 1/(-|U|) \). This is non-singular in the interior but blows up on the boundary of \( \mathcal{R} \). Quantum mechanically, we want to represent \( V^{4/3} \) as an hermitian operator. This can be done by representing it as the Laplace-Beltrami operator for the metric \( q_{ij} \) on the region \( \mathcal{R} \). We thus defined the corresponding operator to be

\[
\hat{V}^{4/3} = q^{-1/4} \hat{\beta}^0_i q^{ij} q^{1/2} \hat{\beta}^0_j q^{-1/4}.
\] (13)

We note that the form of the inner product Eq. (11) may be interpreted to mean that the states are half-densities on \( \mathcal{R} \).

As this is a hermitian operator acting on a finite area, we conjecture that the volume has a discrete spectra with finite degenerancy. This is a familiar result of one dimensional quantum mechanics.

IV. EVOLUTION IN GAUGE INVARIANT QUANTIZATION

In principle, the entire physical quantum theory is contained in the states and operators on the physical Hilbert space. However, to make predictions we need to find observables \( \mathcal{O}^I(\tau) \) for nonzero \( \tau \). If we were able to do so, all the physical information would be contained in \( N \)-time correlation functions of the form

\[
\langle \beta^\pm_i | \mathcal{O}^I(\tau_1) \ldots \mathcal{O}^N(\tau_N) | \beta^\pm_j \rangle.
\] (14)

The problem is how to construct these operators given that we have neither their classical counterparts in closed form nor the proper subsets of \( \Gamma \). As an example of such an operator, consider an operator \( \hat{V}(\tau) \) corresponding to the volume of the universe on the slice where \( p_\alpha = \tau \). There are two approaches to this problem:

1. In some appropriate approximation procedure, solve Eqs. (9) and (10) (or the appropriate equations for other observables) in the classical theory, and then, term by term in the approximation, define an ordering prescription which realizes the classical expression as a well-defined quantum mechanical operator. (We may note the approximation procedure must be able to keep track consistently of the regions of definition \( \Gamma(\tau) \) which satisfy \( \Gamma(\tau_1) \in \Gamma(\tau_2) \) for \( \tau_2 > \tau_1 \).)

2. Evolve the operators quantum mechanically. As we have defined the quantum theory only for the physical observables, there is no operator that corresponds to \( \mathcal{H} \); we don’t have the quantum mechanical analogue of Eq. (9). We can associate a non-vanishing Hamiltonian to evolution in a physically meaningful time variable, and then use that to define an evolution operator. In the classical theory, this is straightforward.

In the classical phase space, the Hamiltonian which evolves physical variables in \( p_\alpha \) is the conjugate quantity \( \alpha \), which is a time ( \( p_\alpha = \tau \) ) dependent functional on \( \Gamma \). This is given by

\[\footnote{This is delicate. On the boundaries of the non-compact region the metric factors, \( 1/|U| \), diverge. The wavefunctions, interpreted as half-densities, must fall-off faster than the potential.} \]
\[ h(\tau) \equiv \alpha(\tau) = \frac{1}{4} \ln \left[ \frac{p_+(\tau)^2 + p_-(\tau)^2 - \tau^2}{-|\mathcal{U}(\beta^\pm)|} \right]. \] (15)

On \( \bar{\Gamma} \), this satisfies \{h(\tau), p_\alpha\} = 1, so that it generates evolution in \( p_\alpha \) on the constraint surface. We could construct an evolution operator in the physical Hilbert space if we could find the corresponding quantum operator. However, this is not easy as we do not know the operators \( \beta^\pm(\tau) \) and \( \hat{p}_\pm(\tau) \); these themselves are supposed to be found by evolution. In some operator ordering, we would have to find operator solutions to the coupled operator equations,

\[ \dot{h}(\tau) = \frac{1}{4} \ln \left[ \frac{\hat{p}_+(\tau)^2 + \hat{p}_-(\tau)^2 - \tau^2}{-|\mathcal{U}(\beta^\pm)|} \right] \] (16)
and

\[ \frac{d\beta^\pm(\tau)}{d\tau} = [\beta^\pm(\tau), \hat{h}(\tau)] \] (17)
\[ \frac{d\hat{p}_\pm(\tau)}{d\tau} = [\hat{p}_\pm(\tau), \hat{h}(\tau)]. \] (18)

 Needless to say, as the theory cannot be solved at the classical level, this requires some approximation procedure. There are also other issues associated with operator ordering ambiguities such as whether quantum observables should be hermitian, given that each classical universe has a finite lifetime. For, if we find a hermitian ordering for \( \hat{h}(\tau) \) then we ought to be able to evolve an arbitrary quantum state to arbitrarily large times. Given that these practical and conceptual difficulties face any attempt to proceed with this program, we turn to another approach to representing evolution based on the path integral.

V. PATH INTEGRAL REPRESENTATION OF PHYSICAL EVOLUTION

Path integration provides a practical resolution for these problems. While the observables \( \mathcal{O}(\tau) \) cannot be written in closed form on the phase space, they could be computed by summing over histories. That is, as a computer averages the effective action over an ensemble of paths generated by some statistical procedure, it can simply go to the points on the path where \( p_\alpha = \tau \) and then tabulate the other observables at those events. In this way, averages such as Eq. (14) could be evaluated numerically, with an ensemble of paths with appropriate weights or measure. We study one such procedure in which the path integral measure may be derived from the physical quantum theory. In this way, the complementary strengths of the two approaches may be exploited, for a practical approach to calculating physical observables non-perturbatively in quantum cosmology.

To construct the path integral corresponding to the matrix element

\[ \langle \beta^\pm_f | \mathcal{O}(\tau_N) \cdots \mathcal{O}(\tau_1) | \beta^\pm_i \rangle \] (19)
where \( \tau_f > \tau_i \); \( I = 1, 2, 3, \ldots, N \), we express the states in terms of physical states, elements of \( L^2(\mathcal{R}) \). These are the time evolved kets

\[ |\beta^\pm_f \tau_f\rangle = \hat{U}(\tau_i, \tau_f) |\beta^\pm_i \tau_i\rangle \] (20)
where \( \hat{U}(\tau_i, \tau_f) \) is the operator for evolution \( \tau \). We derive a path integral expression for Eq.(19) in terms of a quantum effective action \( S = \int_{\tau_i}^{\tau_f} L \), a measure factor \( \mu \), and ranges of integration,

\[ \langle \beta^\pm_f | \mathcal{O}(\tau_N) \cdots \mathcal{O}(\tau_1) | \beta^\pm_i \rangle = \frac{1}{N} \int \left[ d^2\beta d^2p_\mu(\beta, p) \right] \mathcal{O}(\tau_N) \cdots \mathcal{O}(\tau_1) e^{(i/\hbar)S} \] (21)
where the \( \mathcal{O}(\tau_I) \) are the classical expressions, expressed as the value of each observable \( \mathcal{O} \) on the path at the points \( p_\alpha = \tau \), and

\[ N = \int \left[ d^2\beta d^2p_\mu(\beta, p) \right] e^{(i/\hbar)S_\varepsilon} \] (22)
(The brackets a notational device to indicate a factor of $1/2\pi$ for each differential.) To do this we must first find a path integral expression for $\hat{U}(\tau_f, \tau_i)$. The hamiltonian that corresponds to the choice $\tau = p_\alpha$ [given in Eq. (16)], may be expressed as

$$\hat{h}(\tau) = \frac{1}{4} \left[ \ln \left( \frac{\tau^2}{-|\mathcal{U}(\beta^\pm)|} \right) + \ln \left( \frac{-\hat{p}_+^2 - \hat{p}_-^2}{\tau^2} + 1 \right) \right].$$

(23)

The evolution operator is expressed in terms of the corresponding operator as

$$\hat{U}(0, \tau) = T \exp \left[ -\frac{i}{\hbar} \int_0^\tau \hat{h}(\tau')d\tau' \right].$$

(24)

Without the form of $\hat{\beta}^\pm(\tau)$ and $\hat{p}_\pm(\tau)$, we do have the Hamiltonian $\hat{h}(\tau)$ for finite $\tau$ as an operator on the physical Hilbert space. To construct the path integral we make some conjectures about these operators and their spectra. These assumptions cannot be justified directly in the absence of further information about the solutions to Eqs. (16), (17), and (18), but they may be plausible in the light that the resulting path integral agrees with that constructed by the standard method of gauge fixed quantization. We assume that there exist simultaneous operator solutions to Eqs. (16), (17), and (18) such that they are all hermitian operators. As a result time evolution is unitary, so that the expectation values $\langle \beta^\pm(\tau) \rangle$ and $\langle p_\pm(\tau) \rangle$ of the operators $\hat{\beta}^\pm(\tau)$ and $\hat{p}_\pm(\tau)$ satisfy equal time commutation relations, $[\beta^\pm(\tau), \hat{p}_\pm(\tau')] = \delta_{\pm\pm'}$. In this case, for each $\tau$, there must be a complete basis of states made from eigenstates of $\beta^\pm(\tau)$,

$$1 = \int_{\mathcal{R}(\tau)} d\beta^\pm |\beta^\pm(\tau)\rangle \langle \beta^\pm(\tau)|$$

(25)

where the range of the integral, $\mathcal{R}(\tau)$, is the range of the spectra of the operators $\hat{\beta}^\pm(\tau)$. In the absence of a construction of the operators, we do not know this range, but we would like to argue that it is the whole of $R^2$. In the classical theory we have, from the Hamiltonian constraint,

$$\mathcal{U}(\beta^\pm) < \tau^2 e^{-4\alpha}.$$

(26)

At $\tau = 0$, this restricts $\beta^\pm$ to the region $\mathcal{R} = \mathcal{R}(\eta)$. However, there are initial classical configurations for all values of $p_\alpha$; this means that for small $\tau$, the left hand side can be arbitrarily large, which means that there is no limit on how large the anisotropies $\beta^\pm$ can be at any finite $\tau$. As a result, if the theory is to have a good classical limit it seems likely that the spectra must be unbounded.

Similarly, we assume that there is a complete set of states

$$1 = \int_{R^2} d^2\beta |p_\pm(\tau)\rangle \langle p_\pm(\tau)|.$$

(27)

Given all these assumptions we may deduce

$$\langle p_\pm(\tau) | \beta^\pm(\tau) \rangle = e^{i(\beta^+ p_+ + \beta^- p_-)}.$$

(28)

We use standard techniques to construct the path integral. Inserting complete sets of states in the usual convenient ways, we evaluate

$$h(\tau) = \langle \beta^\pm(\tau) | \hat{h}(\tau) | p_\pm(\tau) \rangle = \frac{1}{4} \left[ \ln \left( \frac{\tau^2}{-|\mathcal{U}(\beta^\pm)|} \right) + \ln \left( \frac{-\hat{p}_+^2 - \hat{p}_-^2}{\tau^2} + 1 \right) \right].$$

(29)

to find that,

$$\hat{U}(0, \tau) = \int [d^2\beta^\pm d^2p_\pm] \exp \left[ -\frac{i}{\hbar} \int_0^\tau \left( p_+\beta^+ + p_-\beta^- - h(p_\pm, \beta^\pm, \tau) \right) d\tau \right].$$

(30)

where the ranges of the integrals are, given our assumptions, unbounded. Comparing to Eq. (21), we then see that in these coordinates the measure is trivial, the integration regions are unbounded and the effective lagrangian is given simply by $L = p_+\beta^+ + p_-\beta^- - h(p_\pm, \beta^\pm, \tau)$. 

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VI. SCALE FACTOR TIME

To compare this result to the earlier derivation we'll change from an extrinsic curvature time gauge to a gauge related to the volume ($\tau = \ln \sqrt{h} = \alpha$). Geometrically, the measure transfers from one slice to another along the orbits of the gauge transformation. Unlike $p_\alpha$, $\alpha$ is not monotonic on the whole history of the cosmology. When the gauge transformation is made we will choose which half of the history to consider. Expanding the phase space to include time and the Hamiltonian, so that

$$
\hat{U}(0, \tau) = \int \left[ \left. d^2 \beta^\pm d^2 p_\pm d\alpha \delta (\tau - p_\alpha) \delta (\alpha - h(\tau)) \right] \exp \left[ \frac{i}{\hbar} \int p_+ \dot{\beta}^+ + p_- \dot{\beta}^- + p_\alpha \dot{\alpha} d\tau \right].
$$

The new gauge, $\tau = \alpha$, may be introduced as

$$
1 = \Delta_\alpha \int d\lambda \delta (\tau - \alpha^\lambda)
$$

where $\alpha^\lambda$ is the gauge transformed $\alpha$. Under small gauge transformations,

$$
\alpha^\lambda = \alpha + \lambda \{ \alpha, H \} + O(\lambda^2).
$$

Thus, $\Delta_\alpha = |\{\alpha, H\}| = |p_\alpha|$. Finally, we express the delta function involving $\alpha$ in terms of the Hamiltonian constraint as $\delta (\alpha - h(\tau)) = \delta(H) |\{\alpha, H\}| |\{\alpha, p_\alpha\}|$. Inserting these identities into the propagator of Eq. (31), gives,

$$
\hat{U}(0, \tau) = \int \left[ \left. d^2 \beta^\pm d^2 p_\pm d\alpha d\lambda \delta (\tau - p_\alpha) \delta (\lambda) \delta (\tau - \alpha^\lambda) \right| \{\alpha, H\} |\{\alpha, p_\alpha\}| \right] \\
\times \exp \left[ \frac{i}{\hbar} \int p_+ \dot{\beta}^+ + p_- \dot{\beta}^- + p_\alpha \dot{\alpha} d\tau \right].
$$

Observe that, as before with the new time choice, the extrinsic time choice satisfies

$$
1 = \Delta_{p_\alpha} \int d\lambda \delta (\tau - \alpha^{\lambda_\alpha}).
$$

By explicitly changing gauge one can check that $\Delta_{p_\alpha}$ is gauge independent and equals $|\{H, p_\alpha\}|$ (for small gauge transformations). Performing an inverse gauge transformation to change gauge we have

$$
\hat{U}(0, \tau) = \int \left[ \left. d^2 \beta^\pm d^2 p_\pm d\alpha d\lambda \delta (\tau - \alpha^{\lambda_\alpha^{-1}}) \delta (H) \delta (\tau - \alpha^\lambda) \right| \{\alpha, H\} |\{H, \alpha\} |\Delta_{p_\alpha}\right] \\
\times \exp \left[ \frac{i}{\hbar} \int p_+ \dot{\beta}^+ + p_- \dot{\beta}^- + p_\alpha \dot{\alpha} d\tau \right].
$$

Performing the integration over the gauge parameter, allowing a delta to eat up the integration over $\alpha$, and exponentiating the Hamiltonian constraint we find,

$$
\hat{U}(0, \tau) = \int \left[ \left. d^2 \beta^\pm d^2 p_\pm \right| \{p_\alpha|dN \right] \\
\times \exp \left[ \frac{i}{\hbar} \int_0^\tau p_+ \dot{\beta}^+ + p_- \dot{\beta}^- + p_\alpha - NH d\tau \right],
$$

the propagator in the physical phase space. This is the path integral for the canonical ADM Hamiltonian under the choice $\tau = \alpha$. With this choice the Hamiltonian is given by $-p_\alpha$ (the last term in Eq. (31)) and the lagrange multiplier or lapse is determined so that $\dot{\alpha} = 1$. The propagator of Eq. (37) can be integrated to a path integral in the phase space $\hat{\Gamma}$, as in $\Gamma$, the integration over $p_\alpha$ may be performed yielding

$$
\hat{U}(0, \tau) = \int_\hat{\Gamma} \left[ \left. d^2 \beta^\pm d^2 p_\pm \frac{dN}{N} \right| \exp \left[ \frac{i}{\hbar} \int_0^\tau p_+ \dot{\beta}^+ + p_- \dot{\beta}^- - NH d\tau \right].
$$
with \( H = [p_1^2 + p_2^2 + e^{4\alpha}U(\beta^\pm)]^{1/2} \). This can be directly compared with the propagator in Section III of [1]. These propagators are identical up to a measure factor \( \mu(\beta^\pm) \). There is a simple reason for this: in the previous paper [1] we began with a path integral which included an integration over all the components of the frame fields and connections, where as here we specified the theory in Eqs. (1-4) directly in terms of diagonal gauge. Thus, the path integral in [1] has an additional factor in the measure that came from the gauge fixing down to diagonal gauge. That factor is non-trivial, as can be seen directly from Eq. (40) of [1]. Had we begun the treatment of [1] with the model defined in diagonal gauge, the results would have been identical to those found here. We conclude that the definition of the path integral depends on the point at which the reduced theory is taken to define the quantum theory; reduction does not commute with quantization.

This measure factor is not relevant for the main question of interest here, which is whether this “composite” approach for defining the physical theory leads in the end to the same path integral as did the Faddeev-Popov ansatz. The answer is that, at least in this case, if one starts from the same model, they are the same.

VII. CONCLUSIONS

We have implemented this new approach to quantum cosmology in the restricted, but non-trivial, case of the Bianchi type IX model. By first, identifying the physical degrees of freedom on a slice and then evolving physical observables off this slice by a sum over histories approach, we have shown how it is possible to learn physics without “solving the theory.” We hope that this method will aid calculation of observables in other models and even the full theory. Given the physical theory on one slice, physical observables could be computed numerically.

This work suggests several directions for further study. While the technology appears robust enough to handle the non-linear behavior of Bianchi type IX, it would be interesting if a full quantization could be carried out for one of the more simple Bianchi cosmological models. This composite approach also holds some hope for numerical work in that operators at arbitrary times may be found by computing the propagator for the cosmology. Expressed in terms of a path integral, this could be the basis for a numerical approach to quantum cosmology that has a physical interpretation grounded in the canonical theory.

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