Meteor Stream Membership Criteria

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Abstract. Criteria for the membership of individual meteors in meteor streams are discussed from the point of view of their mathematical and also physical properties. Discussion is also devoted to the motivation. It is shown that standardly used criteria have unusual mathematical properties in the sense of a term “distance”, physical motivation and realization for the purpose of obtaining their final form is not natural and correct, and, moreover, they lead also to at least surprising astrophysical results. A new criterion for the membership in meteor streams is suggested. It is based on probability theory. Finally, a problem of meteor orbit determination for known parent body is discussed.

Key words: interplanetary medium - meteoroids – meteor streams

1. Introduction

Meteor streams are composed of meteors originating from a parent body (comet, asteroid) during an ejection process. It is very important to know the properties of a meteor stream also in order to make some possible conclusions about physical properties of the parent body, ejection process(es) and time of possible ejection(s). One of the most fundamental feature of a meteor stream is the membership of individual meteors, i. e., if an individual meteor belongs to the given meteor stream or if it is a background meteor. Meteor stream membership criteria were suggested for this purpose.

Fundamental characteristics of a body in the Solar System is its orbit characterized by orbital elements. This holds also for meteors. Orbital elements of meteors originating
from one parent body are similar. Thus, it is natural that the most simple meteor stream membership criteria are based on investigation of meteor orbits and their orbital similarity. Some other criteria may be also used, if it is possible: e. g., physical composition of individual meteors, based on the observed spectra.

D-criterion of Southworth and Hawkins (1963) is the most frequent meteor membership criterion used in literature. It measures the orbital similarity of two individual meteors and thus it enables to find complete meteor streams. Any meteor is represented by a point in a phase space of orbital elements. The phase space is a “five-dimensional orthogonal coordinate system” (in this section quotation marks are used for the text taken from Southworth and Hawkins, 1963) and each element is considered as a coordinate. Southworth and Hawkins take the following orbital elements: $q$ (perihelion distance), $e$ (eccentricity), $i$ (inclination), $\Omega$ (longitude of the ascending node), $\omega$ (argument of perihelion). According to the authors, “the distance between two points is a natural measure of the difference between the two corresponding orbits”. The orbital similarity function – “distance between two points” (meteors) $A$ and $B$ – $D(A, B)$ is then defined as

$$D(A, B)^2 = (q_A - q_B)^2 + (e_A - e_B)^2 + \left(2 \sin \frac{I_{AB}}{2}\right)^2 + \left(\frac{e_A + e_B}{2}\right)^2 \left(2 \sin \frac{\pi_{AB}}{2}\right)^2,$$

where $I_{AB}$ is the angle made by the orbital planes given by the formula

$$\left(2 \sin \frac{I_{AB}}{2}\right)^2 = \left(2 \sin \frac{i_A - i_B}{2}\right)^2 + \sin i_A \sin i_B \left(2 \sin \frac{\Omega_A - \Omega_B}{2}\right)^2,$$

and $\pi_{AB}$ is the difference between the longitudes of perihelion measured from the point of the orbits intersection

$$\pi_{AB} = \omega_A - \omega_B + 2 \arcsin\left\{\cos \left(\frac{i_A + i_B}{2}\right) \sin \left(\frac{\Omega_A - \Omega_B}{2}\right) \sec \frac{I_{AB}}{2}\right\},$$

or, to a sufficient approximation, when $i_A$ and $i_B$ are small,

$$\pi_{AB} = (\Omega_A + \omega_A) - (\Omega_B + \omega_B).$$

This D-criterion is used for identification of meteor streams: if $D(A, B) < D_c$, where $D_c$ is a constant assumed as a threshold value, then orbits of meteors $A$ and $B$ are similar and both meteors may be members of the same meteor stream.

A little modified D-criteria were suggested by Drummond (1981) and Jopek (1993).

The aim of this article is to analyze mathematical and physical properties of Southworth-Hawkins’ D-criterion (properties of Drummond’s and Jopek’s D-criteria are analogous). Finally, we present a new meteor stream membership criterion.
2. Mathematics and D-criterion

In the construction of their D-criterion, Southworth and Hawkins were originally inspired by the well-known definition of the distance in Euclidean space. As a consequence they have obtained D-criterion in the form (1) (and also Eqs. (2)-(4)). As it is considered, D-criterion measures the distance between orbits of two meteoroids (meteors). However, if it is so, then D-criterion (1) must fulfill properties required for a quantity called distance. The standard properties of a distance are closely connected with the so-called metric space. Definition states:

Let $X$ be a set with elements $u, v, w, \ldots$. A nonnegative function $\rho$ defined on the Cartesian product $X \times X$ is called a metric if it satisfies the following axioms:

(i) $\rho(u, v) = 0$ if and only if $u = v$;

(ii) $\rho(u, v) = \rho(v, u)$;

(iii) $\rho(u, v) \leq \rho(u, w) + \rho(w, v)$.

A set $X$ with a metric $\rho$ is called a metric space.

(Metric – distance. The property (iii) is called the triangle inequality.)

Now, question is, if these properties are fulfilled also for D-criterion (1). One can easily verify that triangle inequality is violated. It means that triangle inequality, which is an evident property of a distance, is not fulfilled in the case of measuring “distances” between meteor (meteoroid) orbits.

As an evident property of the D-criterion we introduce the following one. If $D(u, v)$ is smaller than $D(u, w)$, then orbits $u$ and $v$ are more similar than the orbits $u$ and $w$. However, due to the violation of the triangle inequality, the orbits $v$ and $w$ may be more similar than one would expect on the basis of general conception about distance:

$$0 < D(v, w) < D(u, w) - D(u, v).$$

We can formulate this in terms of meteor orbits: the distance between meteors $u$ and $v$ is small, the distance between meteors $u$ and $w$ is large, but the distance between meteors $v$ and $w$ may be small.

From the mathematical point of view it would be useful to have a D-criterion which fulfills the properties (i)-(iii). It may have also other forms than presented by Eq. (1), i.e., it may contain not only terms of the form $(\alpha_j - \alpha_k)^2$, but also, e.g., terms of the
form $|\alpha_j - \alpha_k|$, and so on. We know from mathematics that the number of possible (from a mathematical point of view) metrics is infinity: if $\rho(x, y)$ is a metric, then $\rho_1(x, y) = \rho(x, y) / \{1 + \rho(x, y)\}$ is also a metric. One of the most simple modifications of Eq. (1) satisfying properties (i)-(iii) may be obtained by substituting the term $(e_A + e_B)/2$ by any constant; the consequence of the modification $(e_A + e_B)/2 \rightarrow 1$ is, e. g., that the group of Cyldids (Southworth and Hawkins, 1963, p. 271) consists of two meteor streams ($D_{SH} = 0.06, 0.10, 0.10, 0.10, 0.14; D_{new} = 0.24, 0.19, 0.59, 0.59, 0.77$).

As for meteors, standard procedure is to choose one orbit as a reference orbit. The reference orbit may be given by mean values of the known members of a given meteor stream. Any meteor $A$ is a member of the stream if $D(A, M) < D_c$, where $M$ represents an orbit defined by mean values of orbital elements and $D_c$ is a constant assumed to be a threshold value. Since $D(A, M)$ is given by Eqs. (1) – (4), we may have $D(A, M) < D_c$ and $D(B, M) < D_c$ for some two meteors $A$ and $B$ of the stream, but $D(A, B) > 2D_c$. In other words: the first two inequalities assert that meteors $A$ and $B$ have similar orbits but the last inequality states that their orbits are not similar! (Points $A$ and $B$ are situated inside a sphere with a centre $M$ and a radius $r$, but the distance between $A$ and $B$ may be greater than $2r$!) This is the consequence of the triangle inequality violation. (If $D \rightarrow 0$ and, moreover, $\Delta e/e \ll 1$, $\sin \Delta \pi/2 \ll 1$ (i. e., orbits are identical or almost identical), then the violation of the triangle inequality plays no role. However, this may not be the case occurring in astronomical applications (see Klačka and Vološin 1996).)

Those, who are interested in mathematical properties of semi-metric (or, even in a more general case of semi-pseudometric) defined by Eq. (1), we refer to section 18 in Čech (1966).

3. Physics and D-criterion

Southworth and Hawkins (1963) present also physical arguments for the choice of D-criterion in the form of Eqs. (1)-(4). However, their arguments are not convincing and thus one should take Eq. (1) as an empirical criterion. We will justify this statement now.

The physical model of Southworth and Hawkins is based on the idea that the change of meteoroid’s orbital elements with respect to those of the parent body may be represented as an average value of the changes (weighted by velocity) during one period of the parent body. However, this idea has no advantage. Nor it is a simple idea (hypothesis), nor does correspond to real processes: the meteoroid is ejected at once (at one moment) from the parent body.
Moreover, mathematical realization of the physical model is incorrect and some other physical inconsistencies arise in the process of mathematical calculations. According to the idea, we should write for the change of any orbital element \( G \)

\[
\Delta G = (\nabla_v G) \cdot (\Delta v)
\]

However, the authors use in their calculations

\[
\Delta G = |\nabla_v G| |\Delta v|
\]

which is different from the correct value, since, in general, \( \Delta v \) is not in the direction of \( \nabla_v G \):

\[
-1 \leq \{(\nabla_v G) \cdot (\Delta v)\} / \{|\nabla_v G| |\Delta v|\} \leq +1.
\]

The consequence of these three equations is that even if the model of the authors would be correct, its realization is incorrect – they add together not changes of orbital elements, but their extremal values with positive signs at each instant. Although this would seem acceptable in the sense of calculations of maximum possible changes – which is not the author’s interpretation, and, even if it would be –, it is incorrect: for the same meteoroid, \( \Delta v_R \) may be dominant for one orbital element, \( \Delta v_S \) may be dominant for another orbital element (see Eqs. (A8) in Southworth and Hawkins (1963); we used the same notation as Southworth and Hawkins). The other important nonphysical result rests in the assumption that \( |\Delta v| \) is proportional to circular velocity at the distance \( r \):

\[
|\Delta v| \propto U, \text{i.e.,} \quad |\Delta v| \propto r^{-1/2}.
\]

(In the appendix of Southworth and Hawkins, there should be

\[
\Delta G \propto \int r^{-1/2} |\nabla_v G| \, dt.
\]

Fortunately, calculations of the authors correspond to this result.)

I am not sure about the correctness of the final results of Southworth and Hawkins (their appendix and Fig. 1), e. g., my result is: \( \lim_{e \to 0} \left\{ (1/a) \, Grad \, q \right\} = 0.64 \), which seems to be not consistent with results depicted on Fig. 1 in Southworth and Hawkins’ article. (The mean values calculated in appendix of the authors are calculated from absolute values of quantities, so as the mathematical lemma could be used.)

According to Fig. 1 in Southworth and Hawkins (1963), the authors make a conclusion that their D-criterion may be applied for \( e < 0.85 \). In reality, it is used even for \( e \approx 1 \). Individual terms of the sum in Eq. (1) are comparable in their values also for \( e > 0.85 \), which is not consistent with Fig. 1 in Southworth and Hawkins. This also shows that Eq. (1) is not consistent with the physics suggested by the authors.
4. Astronomy and D-criteria

On the basis of the previous two sections we can conclude that meteor stream membership criteria (Southworth and Hawkins 1963, Drummond 1981, Jopek 1993) have two unpleasant properties, as for mathematics and physics:
i) triangle inequality does not hold for arbitrary orbits,
ii) physics of the criteria is unknown.

If we want to make a simple physical model, we may imagine that meteoroids are ejected only at parent body’s perihelion and calculate the change of meteoroid’s orbital elements with respect to those of the parent body. One can obtain

\[
\begin{align*}
\left( \frac{\Delta v}{v_{PB}} \right)^2 &= \frac{(p_A - p_M)^2}{8p_M^2} + \frac{(e_A - e_M)^2}{8(1 + e_M)^2} + (\sin I_{AM})^2 + \left( \frac{e_M}{1 + e_M} \right)^2 (\sin \pi_{AM})^2 ,
\end{align*}
\]

(5)
where \( v_{PB} \) is parent body’s speed at perihelion (ejection of the meteoroid \( A \)), subscript \( M \) corresponds to the parent body; \( p = a(1 - e) \), \( a \) – semimajor axis. We may define, on the basis of Eq. (5) the following quantity:

\[
[D(A, M)]^2 = \alpha (p_A - p_M)^2 + \beta (e_A - e_M)^2 + (\sin I_{AM})^2 + \gamma (\sin \pi_{AM})^2 ,
\]

(6)
where \( \alpha, \beta \) and \( \gamma \) are numerical constants for a given meteor stream: \( \alpha = 1/(8p_M^2) \), \( \beta = 1/(8(1 + e_M)^2) \), \( \gamma = e_M^2/(1 + e_M)^2 \). The first two terms of the sum are equal in the problem of two bodies. In reality, however, they are often not comparable (some terms of the sum are often negligible in comparison with the others) for real meteor streams. This shows that although physics of Eq. (6) is simple, it does not correspond to useful meteor stream membership criterion.

Practical advantage of the Southworth and Hawkins’ criterion in comparison with the criterion defined in Eq. (6) is that the individual terms of the sum are more comparable than it is in the case of Eq. (6). However, if this is the only requirement for the choice of the criterion for practical usage, then we can write better criterion at once (numerical factor \( \xi = e_M \)):

\[
D(A, B) = |q_A - q_B| + |e_A - e_B| + 2 \left| \sin \frac{i_A - i_B}{2} \right| + 2 \xi \left| \sin \frac{\pi_A - \pi_B}{2} \right| .
\]

(7)
In practice, the quantity \( e_M \) should be calculated as the mean value of eccentricities of bodies forming given meteor stream. Individual terms of the sum of Eq. (7) are more comparable than it is in Eq. (1) (moreover, triangle inequality is also fulfilled; remark:
Eq. (6) does not represent any distance – the index M cannot be changed into B!). Many modifications of Eq. (7) may be used, e.g., \( |q_A - q_B| \rightarrow |q_A - q_B|/(q_A + q_B) \), the same substitution for eccentricities, their various combinations, etc..

There is no reason for the fact that individual terms of the sum in Eqs. (1) or (7) should be statistically comparable for a given meteor stream. On the contrary, one should expect that some terms are dominant for some meteor streams, other terms may be more important for other meteor streams. *Terms of the sum should be weighted.*

There are other unpleasant properties of the membership criteria of Southworth and Hawkins (1963), Drummond (1981), Jopek (1993). All of them use perihelion distance \( q \) as an orbital element. The consequence is that meteors with large dispersions in semi-major axis \( a \) may be classified as members of a given stream, which does not seem to be real – there is very small probability that one or several meteoroids can be ejected with velocities much greater (even in orders of magnitude) than the other meteoroids from the same parent body (or bodies of comparable properties). But, when we would like to make a substitution \( q \rightarrow a \), or, \( p \rightarrow a \) (e.g., in Eqs. (1) and (7)) the individual terms in \( D(A, B) \) are much less comparable and usually the term with \( a \) is dominant (probably, weighting coefficients might solve this situation). The last remark: many people using D-criteria incorrectly calculate the mean values of angular quantities.

In conclusion of this section we can make a statement that it is not possible to give a simple D-criterion based on physical arguments. Thus, one can write many forms of D-criteria. If we would apply various D-criteria on meteor streams (and their background) containing several tens of meteors, we should expect more than 80 % (or, perhaps, even more than 90 %) coincidence between them. If we would compare them with the results obtained by Tisserand parameter, the coincidence may even decrease to about 50 – 70 %. (Of course, it has no sense to use various D-criteria for meteor streams containing less than 10 known members – various D-criteria lead to completely different results.)

All statements made in the last paragraph should be understand in the way that it is not very useful to use D-criteria based only on the problem of two bodies. Tisserand parameter, invariant of the motion in the restricted three-body problem, may be a better criterion. However, one must bear in mind that meteoroids are perturbed on their orbits not only by gravitational forces. Nongravitational forces may also be very important. And some of them may be of stochastic nature. This is the reason of our suggestion presented in the following section.
5. Probability, Statistical Mathematics and D-criteria

Since we do not know any simple physics which can define in a simple way a meteor stream, we take the data (set of orbital elements for various meteors) as a random sample.

Since four orbital elements completely define a meteor stream (intersecting the orbit of the Earth), we will take four orbital elements $Q_1, Q_2, Q_3, Q_4$. Let the real distribution of orbital elements of meteors in the stream may be approximated by density function $f(Q_1, Q_2, Q_3, Q_4) \equiv f(X)$. If we define the meteor stream as a set of bodies with $X \in \Omega$, then there is a probability less than $1 - \alpha$ that objects with $X \in \Omega'$ belong to the stream.

6. Meteor Orbit Determination

Another problem concerning the orbits of meteors is the problem of finding radiants for given parent bodies. In determining orbital elements for a meteor corresponding to the meteoroid $A$ initially ejected from some parent body $B$, we must use the following sets of equations:

$$\frac{\partial F}{\partial \beta_{iA}} + \lambda \frac{\partial}{\partial \beta_{iA}} \left\{ \frac{p_A}{1 + e_A \cos \omega_A} \right\} = 0, \ i = 1 \text{ to } 5, \ \frac{p_A}{1 + e_A \cos \omega_A} = 1,$$

(9)

where $\beta_{iA}$ correspond to orbital elements of the meteoroid (body $A$), the constraint corresponds to the fact that meteoroid may strike the Earth (circular orbit of the Earth is supposed, for the sake of simplicity). The quantity $F$ may be of the type

$$F = \alpha_1 (E_A - E_B)^2 + \alpha_2 (L_A - L_B)^2 +$$

$$+ \alpha_3 (v_{TA} - v_{TB})^2 + \alpha_4 (v_{RA} - v_{RB})^2 +$$

$$+ \alpha_5 (v_{TfA} - v_{TfB})^2,$$

(11)

where

$$(E_A - E_B)^2 = \left( \frac{1 - e_A^2}{p_A} - \frac{1 - e_B^2}{p_B} \right)^2,$$

$$(L_A - L_B)^2 = p_A + p_B - 2 \sqrt{p_A p_B} \times$$

$$\{ \cos (\Omega_A - \Omega_B) \sin i_A \sin i_B + \cos i_A \cos i_B \}.$$
\[
(v_{TA} - v_{TB})^2 = \frac{(1 + e_A)^2}{p_A} + \frac{(1 + e_B)^2}{p_B} - 2 \frac{(1 + e_A)(1 + e_B)}{\sqrt{p_A p_B}} X,
\]
\[
X = \cos(\Omega_B - \Omega_A) \{ \sin \omega_A \sin \omega_B + \cos \omega_A \cos \omega_B \cos i_A \cos i_B \} + \sin(\Omega_B - \Omega_A) \{ \sin \omega_A \cos \omega_B \cos i_B - \sin \omega_B \cos \omega_A \cos i_A \} + \cos \omega_A \cos \omega_B \sin i_A \sin i_B,
\]
\[
(v_{RA} - v_{RB})^2 = \frac{e_A^2}{p_A} + \frac{e_B^2}{p_B} - 2 \frac{e_A e_B}{\sqrt{p_A p_B}} Y,
\]
\[
Y = \cos(\Omega_B - \Omega_A) \{ \cos \omega_A \cos \omega_B + \sin \omega_A \sin \omega_B \cos i_A \cos i_B \} + \sin(\Omega_B - \Omega_A) \{ \sin \omega_A \cos \omega_B \cos i_A - \sin \omega_B \cos \omega_A \cos i_B \} + \sin \omega_A \sin \omega_B \sin i_A \sin i_B,
\]
\[
(v_{fA} - v_{fB})^2 = \frac{(1 - e_A / \sqrt{2})^2}{p_A} + \frac{(1 - e_B / \sqrt{2})^2}{p_B} - \frac{(1 - e_A / \sqrt{2})(1 - e_B / \sqrt{2})}{\sqrt{p_A p_B}} Z,
\]
\[
Z = \cos(\Omega_B - \Omega_A) \times \{ (\sin \omega_A - \cos \omega_A) (\sin \omega_B - \cos \omega_B) + (\sin \omega_A + \cos \omega_A) (\sin \omega_B + \cos \omega_B) \times \cos i_A \cos i_B \} + \sin(\Omega_B - \Omega_A) \times \{ (\sin \omega_A - \cos \omega_A) (\sin \omega_B + \cos \omega_B) \times \cos i_B - (\sin \omega_A + \cos \omega_A) (\sin \omega_B - \cos \omega_B) \times \cos i_A \} + (\sin \omega_A + \cos \omega_A) (\sin \omega_B + \cos \omega_B) \times \sin i_A \sin i_B.
\]

One of the coefficients \(\alpha_1 - \alpha_5\) may be put equal to 1 (e. g., \(\alpha_1 = 1\)). \(\alpha\)-coefficients are functions of orbital elements and their values must be determined from known pairs.
“parent body – meteor stream”. As for the physical sense of the individual terms of the sum in Eq. (11), they correspond to: energy, angular momentum, perihelion velocity and radial velocity of maximum value, transversal velocity for true anomaly \( f = 3 \pi / 4 \) in the problem of two bodies. The \( \alpha \)-coefficients guarantee that Eqs. (9)-(12) can be applied on real situations. The advantage of the form of Eq. (11) is that it contains known physical quantities. In principle, we can choose other forms. However, the requirement that they must contain all five orbital elements in an independent way may not suffice in obtaining good coincidence between theoretical and observed radiants. Local and global minima may be important, in general.

7. Conclusion

We have shown that standardly used method for determining meteor stream membership is not correct from the point of view of mathematics, physics and astronomy. We have presented correct method, based on probability theory and statistical mathematics.

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