Krasovskii and Shifted Passivity Based Output Consensus

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Abstract
Motivated by current sharing in power networks, we consider a class of output consensus (also called agreement) problems for nonlinear systems, where the consensus value is determined by external disturbances, e.g., power demand. This output consensus problem is solved by a simple distributed output feedback controller if a system is either Krasovskii or shifted passive, which is the only essential requirement. The effectiveness of the proposed controller is shown in simulation on an islanded DC power network.

Key words: Nonlinear systems; Krasovskii passivity; shifted passivity; output consensus.

1 Introduction
Steering some variables to a common value is called an agreement problem. Along with massive research attentions of network systems, agreement problems have been studied in various contexts such as distributed optimization [4, 5, 11] and synchronization [17, 18] to name a few. Our interest in this paper is output consensus (also called output agreement) under external disturbances. This problem is motivated by current sharing for balancing demand and supply in power networks [7, 9], where currents and demands are modeled as outputs and external disturbances, respectively.

Various physical systems including the aforementioned power networks possess passivity properties. Passivity and its variant concepts have already been witnessed as useful tools for agreement; see, e.g., [1–3,11,16,19,23]. In particular, the works [2, 3, 16] have studied output consensus problems based on shifted passivity. The common problem formulation in these papers is that passive node

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edges dynamics. Namely, we reveal that imposing special edge dynamics is not an essential requirement to achieve output consensus when one takes control design into account. Moreover, the proposed controller can handle weighted output consensus and partial output consensus problems also.

The main contributions of our approach can be summarized as follows:

1. The main focus of [2, 3, 16] is consensus analysis under special edge dynamics. In contrast, we design an output consensus controller based on Krasovskii or shifted passivity, which can handle a wider class of nonlinear network systems such as the DC microgrids in [7, 9], and weighted output consensus has not been considered in [2, 3, 16].

2. Krasovskii passivity has not been used before for output consensus control or even analysis. The aforementioned papers [2, 3, 16] and a preliminary version [8] for linear port-Hamiltonian systems utilize shifted passivity, but not Krasovskii passivity. One of the advantages of utilizing Krasovskii passivity is that we do not need to assume the existence of an equilibrium point of the closed-loop system beforehand in contrast to shifted passivity.

3. Shifted passivity based output consensus control design for nonlinear networks is also a new contribution of this paper. One of the advantages of utilizing shifted passivity is the ease of dealing with time-varying disturbances. This has been partly observed in [2, 16] for output consensus analysis, under the assumption that disturbances are generated by Sylvester-type equations for output regulation. In this paper, we do not assume this. Namely, we do not require information of disturbances for control design.

Since Krasovskii and shifted passivity are different properties, there is possibility to enlarge the class of systems for which the results in [2, 3, 16] are applicable by revisiting these results from the viewpoint of Krasovskii passivity. As a relevant concept of Krasovskii passivity, differential passivity [10, 22] has been known. The proposed Krasovskii passivity based controller is also applicable to differentially passive systems because differential passivity implies Krasovskii passivity [13, Theorem 2.9].

1.2 Organization

The remainder of this paper is organized as follows. In Section 2, we consider current sharing for a nonlinear islanded DC power network as a motivating example for output consensus under external disturbances. In Section 3, we use linear systems to expose the main ideas of our approaches. In Sections 4 and 5, we show that the proposed controller solves the output consensus problem for Krasovskii and shifted passive systems, respectively.

Also, in Section 6 the proposed controllers are applied to the DC power network in the motivating example and tested in simulation. Finally, Section 7 concludes this paper.

Notation: The set of real numbers is denoted by \( \mathbb{R} \). The \( n \)-dimensional vector whose all components are 1 is denoted by \( \mathbf{1}_n \). The \( n \times n \) identity matrix is denoted by \( \mathbf{I}_n \). For a full column rank real matrix \( A \), its Moore–Penrose inverse is denoted by \( A^+ := (A^T A)^{-1} A^T \). For \( P \in \mathbb{R}^{n \times n} \), \( P \succeq 0 \) \((P \succeq 0 \) means that \( P \) is symmetric and positive (semi) definite). For \( x \in \mathbb{R}^n \), its Euclidean norm weighted by \( P \) is denoted by \( |x|_P := \sqrt{x^T P x} \). If \( P = I_n \), this is simply described by \( |x| \). A continuous function \( \alpha : [0, r) \to [0, \infty) \) is said to be of class \( \mathcal{K} \) if \( \alpha(0) = 0 \) and \( \alpha \) is strictly increasing. Moreover, this is said to be of class \( \mathcal{K}_\infty \) if \( r = \infty \) and \( \lim_{r \to \infty} \alpha(r) = \infty \). For a scalar-valued function \( V : \mathbb{R}^n \to \mathbb{R} \), the column vector-valued function consisting of its partial derivatives is denoted by \( \nabla V := [\partial V / \partial x_1 \cdots \partial V / \partial x_n]^T (x) \).

2 Motivating Example

Consider an islanded DC power network model [9] with \( \nu \) nodes and \( \mu \) edges, described by

\[
\dot{x} = f(x) + gu + d, \tag{1}
\]

\[
x := \begin{bmatrix} \phi & q & \varphi_i \end{bmatrix}^T,
\]

\[
f(x) := (\mathcal{J} - \mathcal{R}) \nabla \mathcal{H}(x) - I_n^* + \text{diag}(C^{-1} q)^{-1} P_L^*, \]

\[
\mathcal{J} := \begin{bmatrix} 0 & -I_n & 0 \\ I_n & 0 & D \end{bmatrix}, \quad \mathcal{R} := \begin{bmatrix} 0 & 0 & G_L^* \\ 0 & 0 & 0 \end{bmatrix}, \quad g := \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
\mathcal{H}(x) := \left( |\varphi|_c^2 + |q|_c^2 + |\varphi_i|_{L^2}^2 \right) / 2,
\]

where \( \phi, q \in \mathbb{R}^\nu \) and \( \varphi_i \in \mathbb{R}^\mu \) are state variables denoting, respectively, the flux and charge of the network’s nodes and the flux associated with the transmission lines interconnecting the nodes, while \( u \in \mathbb{R}^\nu \) denotes the control input. The matrices \( R, R_l, L, L_t, C \succ 0 \) are diagonal and have appropriate dimensions, while \( G_L^* \in \mathbb{R}^{\nu \times \nu} \), \( I_n^* \in \mathbb{R}^\nu \), \( P_L^* \in \mathbb{R}^\nu \), and \( d \in \mathbb{R}^{2\nu + \mu} \) are unknown; see Figure 1 and Table 1 for the meaning of the used symbols. The incidence matrix \( D \in \mathbb{R}^{\nu \times \mu} \) describes the network topology.

To improve the generation efficiency, it is generally desired in DC microgrids that the total current demand is shared among all the nodes (current sharing) [7], i.e., for
Remark 2.1 For the DC network (1), the results in [2,3,16] for output consensus analysis are not applicable. These papers impose special edge dynamics such that the interconnected system naturally possesses a consensus property. However, in the DC network (1), the interconnection structure is determined by the physical couplings of the circuit components, and the interconnected system (1) does not have an output consensus property (2) in itself. Therefore, we need to design the controller (3) to enforce output consensus.

3 Output Consensus for Linear Systems

In this section, to expose the main ideas of this paper, we focus on linear systems, described by

\[
\begin{align*}
\dot{x} &= Ax + Bu + d \\
y &= Hx,
\end{align*}
\]  

where \(x \in \mathbb{R}^n\) and \(u, y \in \mathbb{R}^m\) denote the state, input, and output, respectively, and \(d \in \mathbb{R}^n\) denotes a constant disturbance.

Suppose that \(B\) is of full column rank, and that there exist \(P, Q > 0\) such that for \(d = 0\) and \(V(x) := |x|^2_P\),

\[
\dot{V}(x) \leq -|x|^2_Q + y^\top u.
\]

Note that this requirement can be relaxed into passivity and output detectability properties. Under (6), the controller (3) achieves output consensus, i.e., for some \(\alpha \in \mathbb{R}\),

\[
\lim_{t \to \infty} y = \alpha I_n.
\]

In the following two subsections, we show this by using two different approaches based on Krasovskii passivity and shifted passivity.

3.1 Krasovskii Passivity Based Output Consensus

First, we use Krasovskii passivity. Thus, instead of \(x\), we consider the dynamics of its time derivative \(\dot{x}\), and from (5) we obtain

\[
\begin{align*}
\dot{x} &= A\dot{x} + B\dot{u} \\
\dot{y} &= H\dot{x}.
\end{align*}
\]

Recall that \(d\) is a constant, i.e., \(d = 0\). Then, comparing (5) with (8), one can observe from (6) that

\[
\dot{V}(\dot{x}) \leq -|\dot{x}|^2_Q + \dot{y}^\top \dot{u}.
\]

This property is called strict Krasovskii passivity, which will be formally introduced in Definition 4.1 in Section 4.

To study output consensus, we propose to use \(\dot{V}_K(\dot{x}, y) := \dot{V}(\dot{x}) + |\dot{y}|^2_Q/2\) as an energy function. Then, it follows from (3) and (9) that

\[
\dot{V}_K(\dot{x}, y) \leq -|\dot{x}|^2_Q + \dot{y}^\top \dot{u} + \dot{y}^\top Ly = -|\dot{x}|^2_Q.
\]
Since \( Q > 0 \), and \( B \) is of full column rank, it holds from (3) and (8) that
\[
\lim_{t \to \infty} \dot{x}(t) = 0 \quad \Rightarrow \quad \lim_{t \to \infty} \dot{\bar{u}}(t) = 0
\]
\[
\Rightarrow \lim_{t \to \infty} \dot{E}\bar{\xi}(t) = 0 \quad \Rightarrow \lim_{t \to \infty} E\bar{\xi}(t) = 0 \quad \Rightarrow \lim_{t \to \infty} Ly(t) = 0.
\]
From \( \ker(L) = \text{span}\{m\} \), output consensus (7) is achieved.

3.2 Shifted Passivity Based Output Consensus

Next, we show output consensus from the viewpoint of shifted passivity by rewriting the controller (3) as
\[
\begin{aligned}
\dot{\xi} &= E^\top y \\
u &= -E\bar{\xi},
\end{aligned}
\tag{10}
\]
where \( \xi \in \mathbb{R}^N \), and \( EE^\top = L \). Computing \( \dot{u} \), we obtain indeed (3).

In the previous subsection, we have shown that \( \lim_{t \to \infty} x = 0 \) and \( \lim_{t \to \infty} \bar{u} = 0 \). This implies that the closed-loop system consisting of (5) and (10) has an equilibrium point \((x^*, \xi^*) \) \( \in \mathbb{R}^n \times \mathbb{R}^m \). Define \( u^* := -E\bar{\xi}^* \) and \( y^* := Hx^* \). Then, the first equation of (10) implies \( E^\top y^* = 0 \). From (6), one can verify that \( V(x-x^*) \) satisfies
\[
\dot{V}(x-x^*) \leq -|x-x^*|^2_Q + (y-y^*)^\top(u-u^*). \tag{11}
\]
This property is called strict shifted passivity, which will be formally introduced in Definition 5.2 in Section 5.

For \( \dot{V}_s(x-x^*, \xi-\xi^*) := V(x-x^*) + |\xi-\xi^*|^2/2 \), it follows from (10), (11), \( u^* = -E\bar{\xi}^* \), and \( E^\top y^* = 0 \) that
\[
\dot{V}_s(x-x^*, \xi-\xi^*) \leq -|x-x^*|^2_Q + (y-y^*)^\top(u-u^*) + (y-y^*)^\top E(\xi-\xi^*)
\]
\[
= -|x-x^*|^2_Q.
\]
Since \( Q > 0 \), and \( B \) is of full column rank, it holds from (5), (10), and \( EE^\top = L \) that
\[
\lim_{t \to \infty} x(t) = x^* \quad \Rightarrow \quad \lim_{t \to \infty} \dot{x}(t) = 0
\]
\[
\Rightarrow \lim_{t \to \infty} E\bar{\xi}(t) = B^+(Ax^* + d)
\]
\[
\Rightarrow \lim_{t \to \infty} E\bar{\xi}(t) = 0 \quad \Rightarrow \lim_{t \to \infty} Ly(t) = 0.
\]
That is, output consensus (7) is achieved.

In the linear case, it is possible to obtain the consensus value \( \alpha \) and \( x^* \) explicitly as
\[
\alpha = -\frac{1}{1^\top(HA^{-1}B)^{-1}HA^{-1}d}
\tag{12}
\]
\[
x^* = A^{-1}B(HA^{-1}B)^{-1}(\alpha 1 + HA^{-1}d) - A^{-1}d,
\]
and \( \xi^* \) satisfies
\[
E\xi^* = (HA^{-1}B)^{-1}(\alpha 1 + HA^{-1}d).
\]
One notices that \( \alpha \) and \( x^* \) are determined by \( d \) only. In other words, the consensus value \( \alpha \) is independent from the controller dynamics (3) or (10).

In the above, we use the inverse of \( HA^{-1}B \). Its nonsingularity can be shown from (6), i.e.,
\[
PA + A^\top P \preceq -Q, \quad PB = H^\top \tag{13}
\]
by contradiction. If \( HA^{-1}B \) is singular, there exists \( v \in \mathbb{R}^m \setminus \{0\} \) such that \( HA^{-1}Bv = 0 \). It follows from (13) that
\[
v^\top B^\top QA^{-1}Bv \preceq -v^\top B^\top A^{-1}(PA + A^\top P)A^{-1}Bv
\]
\[
= -2v^\top B^\top PA^{-1}Bv
\]
\[
= -2v^\top HA^{-1}Bv = 0.
\]
This contradicts with \( Q > 0 \) or \( B \) being of full column rank.

**Remark 3.1** The consensus value \( \alpha \) can be shifted by adding a constant input \( \bar{u} \) to (10) as follows:
\[
\begin{aligned}
\dot{\xi} &= E^\top y \\
u &= \bar{u} - E\bar{\xi}.
\end{aligned}
\]
From (12), the new consensus value \( \alpha \) becomes
\[
\alpha = -\frac{1}{1^\top(HA^{-1}B)^{-1}HA^{-1}(B\bar{u} + d)}
\]
That is, by a simple modification of the controller, one can control the consensus value \( \alpha \) also.

Now, we have shown that the controller (3) or equivalently (10) achieves output consensus based on Krasovskii passivity or shifted passivity. As illustrated above, for Krasovskii passivity based output consensus, we do not assume that the closed-loop system has an equilibrium point beforehand. This is an advantage in comparison with shifted passivity based output consensus, especially in the nonlinear case. The shifted passivity based approach has also an own advantage that the results can readily be generalized to time-varying cases, as will be explained later.
4 Krasovskii Passivity Based Output Consensus

In this subsection, we generalize the Krasovskii passivity based approach to nonlinear systems, described by

\[
\begin{aligned}
\dot{x} &= f(x, u, d) \\
y &= h(x, d),
\end{aligned}
\]  

(14)

where \( f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^n \) and \( h : \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^m \) are of class \( C^1 \), and \( \partial f(x, u, d)/\partial u \) is of full column rank at each \((x, u, d) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \). We recall that \( d \in \mathbb{R}^r \) is a constant disturbance, i.e., \( \dot{d} = 0 \).

As in the linear case, we use the extended system of (14):

\[
\begin{aligned}
\dot{x} &= f(x, u, d) \\
\frac{d}{dt} \dot{x} &= \frac{\partial f(x, u, d)}{\partial x} \dot{x} + \frac{\partial f(x, u, d)}{\partial u} \dot{u} \\
y &= h(x, d).
\end{aligned}
\]  

(15)

This can be understood as a system with the state \((x, \dot{x}, u, \dot{u})\), input \( \dot{u} \), and output \( y \). Focusing on the dynamics of \( \dot{x} \), we define Krasovskii passivity as a variant of the original definition [13, Definition 2.8].

**Definition 4.1** Given \( d \in \mathbb{R}^r \), the system (14) is said to be strictly Krasovskii passive on \( \mathcal{D} \subset \mathbb{R}^n \times \mathbb{R}^m \) if for its extended system (15), there exist \( V_K : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) of class \( C^1 \) and continuous \( W_K : \mathcal{D} \times \mathbb{R}^n \to \mathbb{R} \) such that

\[
\begin{aligned}
V_K(x, u, \dot{x}) &\geq 0, \\
W_K(x, u, \dot{x}) &\geq 0 \quad \text{and} \quad W_K(x, u, \dot{x}) = 0 \iff \dot{x} = 0 \\
\dot{V}_K(x, u, \dot{x}) &\leq -W_K(x, u, \dot{x}) + \dot{y}^T \dot{u},
\end{aligned}
\]  

(16) (17) (18)

for all \((x, u) \in \mathcal{D} \) and \((\dot{x}, \dot{u}) \in \mathbb{R}^n \times \mathbb{R}^m \).

To deal with weighted output consensus, we generalize the controller (4) as

\[
\dot{u} = -M^T \mathcal{L} y,
\]  

(19)

where \( \mathcal{L} \geq 0 \) is such that \( \ker(\mathcal{L}) = \text{span}\{1_m\} \), and \( M \in \mathbb{R}^{m \times m} \) is a matrix representing the weights associated with the output, which can be singular. For example, in DC microgrids it is generally desired that the total current demand is shared among the various nodes proportionally to the generation capacity of their corresponding energy sources (proportional current sharing).

Now, we present the first main result of this paper.

**Theorem 4.2** Given \( d \in \mathbb{R}^r \), suppose that the closed-loop system consisting of a strictly Krasovskii passive system (14) on \( \mathcal{D} \) and a controller (19) is positively invariant on a compact set \( \Omega \subset \mathcal{D} \). Then, there exists some \( \alpha : \mathbb{R} \to \mathbb{R} \) such that

\[
\lim_{t \to \infty} (My(t) - \alpha(t)1_m) = 0
\]  

(20)

for each \((x(0), u(0)) \in \Omega \).

**PROOF.** Define \( \bar{V}_K(x, u, \dot{x}) := V_K(x, u, \dot{x}) + |My|_2^2/2 \). Then, it follows from (18) and (19) that

\[
\dot{\bar{V}}_K(x(t), u(t), \dot{x}(t)) + \int_0^t W_K(x(\tau), u(\tau), \dot{x}(\tau)) d\tau \leq \bar{V}_K(x(0), u(0), \dot{x}(0)).
\]  

(21)

Since the closed-loop system is positively invariant on the compact set \( \Omega \), \( \int_0^t W_K(x(\tau), u(\tau), \dot{x}(\tau)) d\tau \) exists for any \((x(0), u(0)) \in \Omega \) and \( \dot{x}(0) \in \mathbb{R}^n \). Also, this is upper bounded and increasing with respect to \( t \geq 0 \), which implies that \( \lim_{t \to \infty} \int_0^t W_K(x(\tau), u(\tau), \dot{x}(\tau)) d\tau \) exists and is finite. Thus, it follows from (17), Barbalat’s lemma [14, Lemma 8.2], and the uniform continuity of \( \dot{x} \) and \( \dot{\dot{x}} \) on \( \Omega \) that

\[
\lim_{t \to \infty} W_K(x(\tau), u(\tau), \dot{x}(\tau)) = 0 \iff \lim_{t \to \infty} \dot{x}(t) = 0 \iff \lim_{t \to \infty} \ddot{x}(t) = 0
\]  

for any \((x(0), u(0)) \in \Omega \) and \( \dot{x}(0) \in \mathbb{R}^n \). Therefore, it holds from (15), (19), and \( \partial f/\partial u \) being of full column rank that

\[
\lim_{t \to \infty} \dot{u}(t) = 0 \iff \lim_{t \to \infty} M^T \mathcal{L} My(t) = 0
\]  

for any \((x(0), u(0)) \in \Omega \). That is, we have (20). \( \square \)

**Remark 4.3** In Theorem 4.2, \( \alpha : \mathbb{R} \to \mathbb{R} \) is not necessarily to be a constant. To guarantee that the consensus value \( \alpha \) is constant, we need an additional assumption such as the convergence of \( x(t) \). Such an assumption holds if \( V_K(x, u, \dot{x}) = |\dot{x}|_P^2/2 \), \( P \succ 0 \) and \( W(x, u, \dot{x}) = |\dot{x}|_Q^2/2 \), \( Q \succ 0 \). In this case, (21) becomes

\[
\frac{d(|\dot{x}(t)|_P^2 + |My|^2)}{dt} \leq -|\dot{x}(t)|_Q^2.
\]
Taking the time integration yields
\[ |\dot{x}(t)|_p^2 \leq |\dot{x}(0)|_p^2 + |My(0)|_L^2 \]
\[ \leq |\dot{x}(0)|_p^2 + |My(0)|_L^2 - \int_0^t |\dot{x}(\tau)|_p^2 d\tau \]
\[ \leq |\dot{x}(0)|_p^2 + |My(0)|_L^2 - c \int_0^t |\dot{x}(\tau)|_p^2 d\tau \]
for some \( c > 0 \). Then, the Gronwall-Bellman inequality, e.g., [14, Lemma A.1] leads to
\[ |\dot{x}(t)|_p^2 \leq e^{-ct} (|\dot{x}(0)|_p^2 + |My(0)|_L^2). \]

Therefore, the convergence speed of \( \dot{x}(t) \) is exponential, which implies that \( x(t) \) converges to a constant. Accordingly, \( y(t) \) converges to a constant, and the consensus value \( \alpha \) is constant.

\begin{remark}
\begin{align*}
\dot{V}_K(x, u, \dot{x}) & \leq \dot{y}^\top \dot{u} \\
\text{instead of (18). Then, the controller } u &= -Ky + v \text{ with } K > 0 \text{ achieves, for the closed-loop system,} \\
\dot{V}_K(x, u, \dot{x}) & \leq -|\dot{y}|_K^2 + \dot{y}^\top \dot{v}.
\end{align*}
If the system (14) has a kind of output detectability property:
\[ \lim_{t \to \infty} \ddot{u}(t) = 0 \quad \text{and} \quad \lim_{t \to \infty} \ddot{y}(t) = 0 \implies \lim_{t \to \infty} \dot{x}(t) = 0, \]
then one can achieve weighted output consensus (20) by \( \dot{v} = -M^\top \mathcal{L}My \). This can be shown in a similar manner as the proof of Theorem 4.2 based on the fact that \( \dot{y} = 0 \) implies \( \ddot{u} = -K\dot{y} - M^\top \mathcal{L}My = 0 \).
\end{remark}

Theorem 4.2 illustrates that a simple controller (19) achieves weighted output consensus. However, its tuning parameter is only \( \mathcal{L} \), and there is no enough freedom to improve control performances. Regarding this aspect, one can utilize the following dynamic extension:
\[ \begin{cases} 
\dot{\rho} = -\rho + y \\
\ddot{\rho} = -M^\top \mathcal{L}My + K(\rho - y),
\end{cases} \tag{22} \]
where \( K > 0 \) is a new tuning parameter. Following similar steps as Theorem 4.2, we can confirm that this new controller achieves weighted output consensus also.

\begin{corollary}
Given \( d \in \mathbb{R}^r \), suppose that the closed-loop system consisting of a strictly Krasovskii passive system (14) on \( \mathcal{D} \) and the controller (22) is positively invariant on a compact set \( \Omega \subset \mathcal{D} \times \mathbb{R}^m \). Then, there exists some \( \alpha : \mathbb{R} \to \mathbb{R} \) such that (20) holds for each \((x(0), u(0), \rho(0)) \in \Omega\).
\end{corollary}

\begin{proof}
Define \( \tilde{V}_K(x, u, \dot{x}, \rho) := V_K(x, u, \dot{x}) + |My|^2 / 2 + |\rho - y|^2 / 2 \). Then, it follows from (18) and (22) that
\[ \dot{\tilde{V}}_K(x, u, \dot{x}, \rho) \leq -W_K(x, u, \dot{x}) + \dot{y}^\top \dot{u} + \dot{y}^\top M^\top \mathcal{L}My + (\dot{\rho} - \dot{y})^\top K(\rho - y) = -W_K(x, u, \dot{x}) - |\rho - y|^2 / K \]
for all \((x, u) \in \mathcal{D}, (\dot{x}, \dot{u}) \in \mathbb{R}^n \times \mathbb{R}^m, \rho \in \mathbb{R}^m \). The rest of the proof is similar to that of Theorem 4.2.
\end{proof}

Theorem 4.2 and Corollary 4.5 can be generalized to impose partial weighted output consensus. To achieve consensus among \( y_j, j = i_1, \ldots, i_m, \bar{m} \leq m \), one only has to implement the following controller:
\[ \begin{align*}
\ddot{u} &= -M^\top \mathcal{L}My, \\
\ddot{y} &= [y_{i_1} \cdots y_{i_m}]^\top, \\
\ddot{\rho} &= [u_{i_1} \cdots u_{i_m}]^\top,
\end{align*} \]
where \( \mathcal{L} \geq 0 \) is such that \( \ker(\mathcal{L}) = \text{span}\{1_n\} \), and \( M \in \mathbb{R}^{m \times m} \) can be singular. It is further possible to achieve weighted consensus among some of \( y_j, j \neq i_1, \ldots, i_m \).

At the end of this section, we revisit the motivating example introduced in Section 2.

\begin{example}
The DC network (1) satisfies
\[ \frac{d\dot{x}}{dt} = (J - R)\nabla^2 \mathcal{H}\dot{x} + \begin{bmatrix} 0 \\
\text{diag}\{C_1 \frac{q_1}{q_1}, \ldots, C_n \frac{q_n}{q_n}\} \mathcal{P}^r_L \\
0 \\
\end{bmatrix} \]
\[ + g\ddot{u}, \]
where note that the Hessian matrix \( \nabla^2 \mathcal{H} \) is constant. Strict Krasovskii passivity can be shown by using \( V_K(\dot{x}) = |\dot{x}|_{\nabla^2 \mathcal{H}}^2 / 2 \). Indeed, it follows that
\[ \dot{\tilde{V}}_K(\dot{x}) = \dot{\dot{x}}^\top \nabla^2 \mathcal{H}(J - R)\nabla^2 \mathcal{H}\dot{x} + \dot{\dot{x}}^\top \nabla^2 \mathcal{H} \text{diag}\{C_1 \frac{q_1}{q_1}, \ldots, C_n \frac{q_n}{q_n}\} \mathcal{P}^r_L \]
\[ + \dot{\dot{x}}^\top \nabla^2 \mathcal{H} g\ddot{u} = -W(\dot{x}, q) + \dot{\dot{x}}^\top \ddot{u}, \]
where
\end{example}
where $y$ is defined in (4), and

$$W(\dot{x}, q) := \dot{x}^T \nabla^2 H \dot{x} - \dot{x}^T \nabla^2 H \dot{x} \begin{bmatrix} 0 \\ \text{diag}\{C_1 \frac{q_1}{q_1}, \ldots, C_n \frac{q_n}{q_n}\} P_L \end{bmatrix}.$$ 

Let $D_q \subset \mathbb{R}^n$ and $\Gamma > 0$ be such that

$$G_L^T - \text{diag}\left( \frac{C_1^2 P_{L,1}}{q_1}, \ldots, \frac{C_n^2 P_{L,n}}{q_n} \right) \geq \Gamma, \quad \forall q \in D_q,$$

where $\Gamma$ represents a suitable lowerbound for the so-called equivalent conductance associated with the microgrid’s loads. Then, for $Q = \text{diag}\{L^{-1} R L^{-1}, \Gamma, L^{-1} R_L L^{-1}\}$, the DC network (1) is strictly Krasovskii passive on $D = \mathbb{R}^n \times D_q \times \mathbb{R}^n \times \mathbb{R}^n$.

From Theorem 4.2, the controller (19) achieves weighted current sharing (20) under the positive invariance assumption that is common in the literature on DC microgrids with constant power loads (see e.g. [9] and the references therein). Furthermore, from Remark 4.3, the consensus value constant $\alpha$ is constant. We note that Krasovskii-like passivity has been already used for the design and analysis of voltage controllers for electric circuits and grids (see e.g. [6, 15]). However, to the best of our knowledge, Krasovskii passivity has never been exploited before for achieving current sharing and, more generally, output consensus.

5 Shifted Passivity Based Output Consensus

5.1 Converging Disturbances

In Section 3, it has been mentioned that an advantage of shifted passivity based approach is the ease of dealing with a time-varying case. To see this, we consider nonlinear time-varying systems, described by

$$\begin{cases} \dot{x} = f(t, x, d) + g(t, x, d)u \\ y = h(t, x, d), \end{cases} \quad (23)$$

where $d : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function of $t \in \mathbb{R}$. Also, $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$, $g : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^{n \times m}$, and $h : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}$. $f$ and $h$ are continuous in $(t, d)$ and locally Lipschitz in $x$ on $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r$, and $g(t, x, d)$ is of full column rank at each $(t, x, d) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^r$.

To deal with the time-varying case, we assume that the system (23) admits an equilibrium trajectory. Namely, given $d(t), t \in \mathbb{R}$, there exists a class $C^1$ bounded trajectory $(x^*(t), u^*(t)), t \in \mathbb{R}$, such that

$$\dot{x}^*(t) = f(t, x^*(t), d(t)) + g(t, x^*(t), d(t))u^*(t), \quad \forall t \in \mathbb{R}.$$

As a technical assumption, in this subsection, we assume a sort of convergence properties of $d$. This is relaxed in the next subsection.

**Assumption 5.1** Both $\lim_{t \to \infty} f(t, x^*(t), d(t))$ and $\lim_{t \to \infty} g(t, x^*(t), d(t))$ exist and are finite. \(<\)

To extend the concept of shifted passivity, e.g., [13, Definition 2.14] to the time-varying case, we use the error dynamics $e := x - x^*$:

$$\begin{cases} \dot{e} = f(t, e + x^*, d) + g(t, e + x^*, d)u \\ -f(t, x^*, d) + g(t, x^*, d)u^* \\ \dot{y} = h(t, e + x^*, d). \end{cases} \quad (24)$$

**Definition 5.2** The system (23) is said to be strictly shifted passive along $(x^*(t), u^*(t))$ on $D \subset \mathbb{R}^n$ if for the error dynamics (24), there exist $V_e : \mathbb{R} \times D \to \mathbb{R}$ of class $C^1$ and continuous positive definite $W_e : D \to \mathbb{R}$ such that

$$\begin{cases} V_e(t, e) \geq 0 \\ \dot{V}_e(t, e) \leq -W_e(e) + (y - y^*)^T (u - u^*) \end{cases} \quad (25, 26)$$

for all $(t, e) \in \mathbb{R} \times D$ and $u \in \mathbb{R}^m$, where $y^* := h(t, x^*(t), d(t))$. When $W_e(\cdot) = 0$, we simply say that the system is shifted passive. \(<\)

As in the linear case, we use another representation of the controller (19):

$$\begin{cases} \dot{\xi} = E^T M y \\ u = -M^T E \xi, \end{cases} \quad (27)$$

where $\xi \in \mathbb{R}^N$, and $EE^T = L$. We show that this controller achieves weighted output consensus for a strictly shifted passive system.

**Theorem 5.3** Given $d(t), t \in \mathbb{R}$, suppose that

1) the closed-loop system consisting of a system (23) and a controller (27) admits a class $C^1$ bounded trajectory $(x^*(t), \xi^*(t)), t \geq 0$, such that $\dot{\xi}^* = E^T M y^* = 0$, i.e., $\xi^*$ is constant;
2) for $(x^*(t), \xi^*)$ in item 1), the system (23) satisfies Assumption 5.1 and is strictly shifted passive along $(x^*(t), u^*)$ on $D$, where $u^* := -M^T E \xi^*$;
3) when rewriting the system (23) as the error dynamics (24), the closed-loop system is positively invariant on some compact set $\Omega \subset D \times \mathbb{R}^N$ (for any initial time $t_0 \in \mathbb{R}$), where the projection of $\Omega$ onto the $e$-space contains the origin.

Then, there exists some $a : \mathbb{R} \to \mathbb{R}$ such that (20) holds for any $t_0 \in \mathbb{R}$ and $(e(t_0), \xi(t_0)) \in \Omega$. 


PROOF. Define \( \bar{V}_s(t, e, \xi) := V_s(t, e) + |\xi - \xi^*|^2/2 \). Then, it follows from (26), (27), \( \dot{\xi}^* = E^T My^* = 0 \) in item 1), and \( u^* = -M^T E \xi^* \) in item 2) that

\[
\dot{V}_s(t, e, \xi) \leq -W_s(e) + (y - y^*)(u - u^*) + (y - y^*)^T M^T E (\xi - \xi^*) = -W_s(e)
\]

for all \( t \in \mathbb{R} \) and \((e, \xi) \in \Omega\). Taking the time integration yields

\[
\bar{V}_s(t, e(t), \xi(t)) + \int_{t_0}^{t} W_s(e(\tau))d\tau \leq \bar{V}_s(t_0, e(t_0), \xi(t_0)).
\]

Since \( \Omega \) is positively invariant with respect to \((e(t), \xi(t))\), and since from item 3), the projection of \( \Omega \) onto the \( e \)-space contains the origin, taking \( t \to \infty \) and using Barbalat’s lemma leads to

\[
\lim_{t \to \infty} W_s(e(t)) = 0 \iff \lim_{t \to \infty} e(t) = 0 \iff \lim_{t \to \infty} \dot{e}(t) = 0
\]

for any \( t_0 \in \mathbb{R} \) and \((x(t_0), \xi(t_0)) \in \Omega\).

It follows from (24), (27), Assumption 5.1, \( \dot{\xi}^* = 0 \) in item 1), the continuity of \( f \) and \( g \), \( g \) being of full column rank, and \( E E^T \mathcal{L} \) that

\[
\lim_{t \to \infty} (f(t, e + x^*, d) - g(t, e + x^*, d)M^T E \xi - (f(t, x^*, d) - g(t, x^*, d)M^T E \xi^*)) = \lim_{t \to \infty} g(t, x^*, d)M^T E(\xi - \xi^*) = 0
\]

\[
\implies \lim_{t \to \infty} M^T E \xi(t) \lim_{t \to \infty} M^T \mathcal{L} My(t) = 0,
\]

which completes the proof. \( \square \)

Remark 5.4 The controller dynamics (27) look similar to the edge dynamics imposed in [2, 3, 16]. However, ours is more general in the sense that \( E \) is not necessarily to be an incidence matrix, and ours is weighted by \( M \) to achieve weighted output consensus. More importantly, we only require shifted or Krasovskii passivity of the system and can handle a wider class of systems than [2, 3, 16]. Namely, we reveal that imposing special edge dynamics is not an essential requirement to achieve output consensus when one takes control design into account. \(< \)

Remark 5.5 A similar remark as Remark 4.4 holds. Namely, in Theorem 5.3, we can relax strict shifted passivity into shifted passivity and a kind of output detectability. \(< \)

Remark 5.6 The positive invariance assumption, i.e., item 3), can be removed from Theorem 5.3 if there exist class \( \mathcal{K} \) functions \( \alpha_1, \alpha_2 \) such that

\[
\alpha_1(|e|) \leq V_s(t, e) \leq \alpha_2(|e|), \quad \forall t \in \mathbb{R}, \forall e \in \mathcal{D}.
\]

Furthermore, global output consensus can be achieved if \( \mathcal{D} = \mathbb{R}^n \), and \( \alpha_1, \alpha_2 \) are class \( \mathcal{K}_\infty \) functions. \(< \)

In the proof of Theorem 5.3, we show \( \lim_{t \to \infty} e(t) = 0 \), i.e., \( \lim_{t \to \infty} (x(t) - x^*(t)) = 0 \). This implies that the consensus value (more precisely, the valued-function):

\[
\alpha(t) = \frac{1}{m} \frac{\mathbb{1}_n M h(t, x^*, d)}{m_m}
\]

does not depend on the initial state \((x(t_0), \xi(t_0))\) (or initial time \( t_0 \in \mathbb{R} \)). This further implies that if \( x^* \) is constant, and \( h \) is time-invariant, \( \alpha \) is constant as well.

5.2 More General Disturbances

To deal with non-converging disturbances, we modify the controller (27) as follows:

\[
\begin{cases}
\dot{\xi} = E^T My \\
u = -M^T E(\xi + GE^T My)
\end{cases}
\]

where \( G > 0 \) is a tuning parameter.

Then, we have weighted output consensus without Assumption 5.1 for the convergence of the disturbance.

Theorem 5.7 Given \( d(t), t \in \mathbb{R} \), suppose that

1) the closed-loop system consisting of a system (23) and a controller (28) admits a class \( \mathcal{C}^1 \) bounded trajectory \((x^*(t), \xi^*(t))\), \( t \geq 0 \) such that \( \xi^* = E^T My^* \), i.e., \( \xi^* \) is constant;
2) for \((x^*(t), \xi^*)\) in item 1), the system (23) is shifted passive along \((x^*(t), u^*(t))\) on \( \mathcal{D} \), where \( u^*(t) := -M^T E(\xi^* + GE^T My^*) \);
3) item 3) of Theorem 5.3 holds.

Then, there exists some \( \alpha : \mathbb{R} \to \mathbb{R} \) such that (20) holds for any \((e(t_0), \xi(t_0)) \in \Omega\) and \( t_0 \in \mathbb{R} \).

PROOF. Define \( \bar{V}_s(t, e, \xi) := V_s(t, e) + |\xi - \xi^*|^2/2 \). Then, it follows from (26), (28), \( \dot{\xi}^* = E^T My^* = 0 \) in item 1), and \( u^* = -M^T E(\xi^* + GE^T My^*) \) in item 2) that

\[
\dot{V}_s(t, e, \xi) \leq (y - y^*)^T (u - u^*) + (y - y^*)^T M^T E(\xi - \xi^*) - |E^T My^*_G - (y - y^*)^T M^T E(\xi - \xi^*) + (y - y^*)^T M^T E(\xi - \xi^*) - |E^T My^*_G
\]
for all $t \in \mathbb{R}$ and $(\xi, \zeta) \in \Omega$. In a similar manner as the proof of Theorem 5.3, we have

$$\lim_{t \to \infty} E^\top My(t) = 0$$

for any $(x(t_0), \xi(t_0)) \in \Omega$ and $t_0 \in \mathbb{R}$.  \(\square\)

Theorem 5.7 can be generalized to non-input-affine systems because we directly prove the convergence of $E^\top My(t)$ without using the input-affine structure.

**Remark 5.8** A similar structure as (28) is found in [2, 16], and a similar remark as Remark 5.4 holds. Namely, our controller works for more general systems and for weighted output consensus. Moreover, we do not require that disturbance $d(t)$ is generated by an ecosystem, nor the controller (28) satisfies the Sylvester-type equation for output regulation.

**Example 5.11** Consider the system (1) and controller (28). From the convexity of $\mathcal{H}(x)$, it follows that

$$\mathcal{H}_s(e, x^*) := \mathcal{H}(e + x^*) - \mathcal{H}(x^*) - \nabla^\top \mathcal{H}(x^*) e \geq 0$$

for any $(e, x^*) \in \mathbb{R}^{2v + \mu}$; this is a standard technique for shifting a storage function as found in [12]. Moreover, $\mathcal{H}$ in (1) satisfies $\mathcal{H}_s(e, x^*) = 0$ if and only if $e = 0$. It follows from $x = e + x^*$, (30), and the structure of $\mathcal{H}$ in (1) that

$$\dot{\mathcal{H}}_s(e, x^*) = (\nabla^\top \mathcal{H}(e + x^*) - \nabla^\top \mathcal{H}(x^*)) (\dot{e} + \dot{x}^*) - e^\top \nabla^2 \mathcal{H} \dot{x}^* = e^\top \nabla^2 \mathcal{H} \dot{e} = e^\top \nabla^2 \mathcal{H} (f(e + x^*) - f(x^*) + g(u - u^*)) = -W_s(e, x^*) + (y - y^*)^\top (u - u^*),$$

where $y$ is defined in (4), and

$$W_s(e, x^*) := -e^\top \nabla^2 \mathcal{H} (f(e + x^*) - f(x^*)) = e^\top \nabla^2 \mathcal{H} R \nabla^2 \mathcal{H} e + e^\top \nabla^2 \mathcal{H} (\text{diag}(C^{-1} q)^{-1} - \text{diag}(C^{-1} q^*)^{-1}) P_L^*.$$

Let $D_q \subset \mathbb{R}^v$ and $\Gamma > 0$ be such that

$$G_L^* - \text{diag}\left( \frac{C^2 P_{L,1}^* e_1 + q^*_1 q_1}{(e_1 + q^*_1) q_1}, \ldots, \frac{C^2 P_{L,v}^* e_v + q^*_v q_v}{(e_v + q^*_v) q_v} \right) > \Gamma,$$

forall $t \geq t_0$, $\forall t_0 \in \mathbb{R}$, $\forall e \in D_q$, where $e = q^*_i - q^*_i$, $i = 1, \ldots, v$. Then, the DC network (1) is strictly shifted passive on $D = \mathbb{R}^u \times D_q \times \mathbb{R}^u$, where

$$Q = \text{diag}\{ L^{-1} R L^{-1}, \Gamma, L_i^{-1} R_i L_i^{-1} \}.$$  

From Theorem 5.7, the controller (28) achieves weighted current sharing under the positive invariance assumption. When $q^*_i$, $i = 1, \ldots, n$ are constant, we can use the controller (27). In this case, the output consensus on $D$ is guaranteed by the reasoning mentioned in Remark 5.6.  \(\square\)

**6 Simulations**

In this section, the proposed controllers are verified in simulation. We consider an islanded DC microgrid composed of 4 nodes in ring topology as shown in Figure 2, where the dashed blue lines represent the communication network. As mentioned above, this system is both Krasovskii and shifted passive. The values of the parameters of each node and line are mainly taken from [7, Tables II, III], while those of the nominal loads are reported in Table 2. Note that we consider also load variations, which are gathered into the disturbance $d$. For the sake of notational simplicity, let $V_i := q_i / C_i$ and $I_i := \varphi_i / L_i$ denote respectively the voltage and the generated current associated with node $i = 1, \ldots, 4$. The desired voltage value at each node is chosen equal to $V_i^* = 380$ V for all $i$. The controller parameters in (22) and (29) are chosen as $M = 100 I_4$, $K = 0.2 I_4$, $G = I_3$. Moreover, for the considered application, we select $\bar{u}$...
Fig. 2. Scheme of the considered microgrid composed of 4 nodes. The black solid arrows indicate the positive direction of the currents through the power lines. The dashed blue lines represent the communication network.

Table 2

| Node | 1  | 2  | 3  | 4  |
|------|----|----|----|----|
| $P_{L_i}$ (kW) | 1  | 2.5 | 1.5 | 5  |
| $G_{L_i}$ (S)  | 0.08 | 0.04 | 0.02 | 0.08 |
| $I_{L_i}$ (A)  | 12.5 | 7.5 | 5.0 | 15.0 |
| $\Delta P_{L_i}$ (kW) | 4  | 1  | 1  | -4 |

in Remark 3.1 as $\bar{u} = C^{-1}q^* + RL^{-1}\varphi$. This simply allows us to shift the system equilibrium such that voltage average ($V_{av}$) is equal to the voltage reference (see, e.g., [21]). Four different scenarios are investigated in the following.

**Scenario 1.** In this scenario, we show current sharing with constant loads. Let the system initially be at the equilibrium. Then, at the time instant $t = 5$ s a step variation equal to $\Delta P_{L_i}$ (see Table 2) occurs in the $P$ loads. From Fig. 3, we can observe that both voltages and currents converge to a constant equilibrium, where the voltage average converges to the voltage reference, and output consensus (i.e., current sharing) is achieved. Specifically, we can observe that since the loads are constant, then the consensus value is constant as well.

**Scenario 2.** In this scenario, we show current sharing with converging time-varying loads. Consider Scenario 1. At the time instant $t = 5$ s, we add to the $P$ load of node 3 a converging time-varying component equal to $0.1e^{-0.25(t-5)}\sin(4t)$ kW. From Fig. 4, we can observe that both voltages and currents converge to a constant equilibrium, where the voltage average converges to the voltage reference, and output consensus (i.e., current sharing) is achieved. Specifically, we can observe that since the loads are time-varying, then the consensus value depends on time.

**Scenario 3.** In this scenario, we show current sharing with non-converging time-varying loads. Consider Scenario 1. At the time instant $t = 5$ s, we add to the $P$ load of node 3 a non-converging time-varying component equal to $0.1\sin(4t)$ kW. From Fig. 5, we can observe that both voltages and currents converge to an equilibrium trajectory, where the voltage average is stabilized around the voltage reference, and output consensus (i.e., current sharing) is achieved. Specifically, we can observe that since the loads are time-varying, then the consensus value depends on time.

**Scenario 4.** In this scenario, we show weighted current sharing with constant loads. Consider Scenario 1 with
$M_{33} = 80$, which implies that node 3 is required to generate a current that is 25% higher than the current generated by each of the other nodes. From Fig. 6, we can observe that both voltages and currents converge to a constant equilibrium, where the voltage average converges to the voltage reference, and weighted output consensus (i.e., proportional current sharing) is achieved. Specifically, we can observe that $I_1 = I_2 = I_4$ and $I_3 = 1.25I_1$.

7 Conclusion

In this paper, we have studied an output consensus problem for nonlinear systems under external disturbances. As the main contribution, we have proposed a simple distributed output feedback controller that achieves output consensus, based on Krasovskii or shifted passivity. An advantage of Krasovskii passivity based approach is that we do not need to assume the existence of an equilibrium point for the closed-loop system. An advantage of shifted passivity based approach is the ease of dealing with time-varying cases. The utility of the proposed controller has been illustrated by an isolated DC power network which is both Krasovskii and shifted passive.

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