Towards Hodge Theoretic Characterizations
of 2d Rational SCFTs: II

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Abstract

A characterization of rational superconformal field theories (SCFTs) on 1+1 dimensions with Ricci-flat Kähler targets was proposed by S. Gukov and C. Vafa in terms of the Hodge structure of the target space. The article [arXiv:2205.10299] refined this idea and extracted a set of necessary conditions for a $T^4$-target $\mathcal{N} = (1,1)$ SCFT to be rational; only a partial effort was made, however, to study whether it also constitutes a sufficient condition. It turns out that the set of conditions in [arXiv:2205.10299] is not sufficient, and that it becomes a set of necessary and sufficient conditions by adding one more condition in the case of $T^4$. The Strominger–Yau–Zaslow fibration in the mirror correspondence plays an essential role there. At the end, we also propose a refined version of Gukov–Vafa's idea for general Ricci-flat Kähler target spaces.
1 Introduction

A family of \((M, G)\) of a manifold \(M\) of certain topological type that admits a Ricci-flat Kähler metric \(G\) gives rise to a family of \(\mathcal{N} = (1, 1)\) superconformal field theories (SCFTs) on 1+1 dimensions. Which points in the moduli space of those SCFTs correspond to rational CFTs? This article addresses that question.

Based on observations by G. Moore [1, 2] and on examples in Gepner constructions, Gukov and Vafa proposed the following idea [3]:

The SCFTs that are rational may be characterized by CM-type complex structure of \(M\) and CM-type complex structure of the mirror \(W\) of \(M\).

Being \textit{CM-type} is a property of complex structure of a manifold (or of Hodge structure on a cohomology group) that generalizes the notion of complex multiplication of \(M = T^2\). There are in fact a couple of subtle conceptual issues to be resolved when one wants to see if the characterization above works on practical cases; arbitrariness in the choice of complex structure (when \(M\) is a complex torus, K3 surface, or a hyperkähler manifold) is one of them. There is also a concrete example in Meng Chen’s paper [4] that appears to be a counter example, when we read the characterization above naively. So, the characterizations (necessary and sufficient conditions) need to be stated in a way the conceptual issues are resolved, and Meng Chen’s example does not satisfy them. Ref. [5] did both, and presented a set of characterization conditions that are at least necessary in the case \(M = T^4\). One of the key observations in [5] was that (i) the complex structure of \(M\) should be such that \(M\) is regarded as an algebraic variety than a complex analytic manifold, and (ii) the Kähler form of \(M\) should be in the cone generated by divisors (the conditions 1 and 2(b) in §2), for the
corresponding SCFT to be rational; without the observation (ii), one would still be haunted by Meng Chen’s example of non-rational SCFTs.

In this article, we continue along the line of [5]. By adding one more condition to the set of conditions in [5], we obtain—for the family of $M = T^4$—a set of conditions that is necessary and sufficient for the SCFTs to be rational. The Strominger–Yau–Zaslow (SYZ) fibration in the mirror correspondence [6] turns out to play an essential role in the one condition added in this article (the condition 8 in §4). The triple of complex multiplication, mirror symmetry and algebraicity of the target space is the key to the characterization of rational SCFTs. The necessary and sufficient conditions for $M = T^2$ and $T^4$ are stated in a language that makes sense for a family with $M$ that is self-mirror (Conjecture 1 (self-mirror) in §5); we do not have a proof for the cases beyond $M = T^2$ or $T^4$. As a wild speculation, a couple of trial versions of the characterization conditions are presented also for the cases with $M$ that is not self-mirror (Conj. 2 (general) in §5). Busy readers might just have a look at Theorem and Conjectures in section 5.

We will use the same notations and conventions as in [5], whenever possible. Review materials in [5] as well as explanations on notations and conventions may be of some use while reading this article. We did not try very much to make this article self-contained, because this article is an outright continuation of [5]. Various lecture notes and theses on the theory of complex multiplication and abelian varieties are available online (e.g., [7]); they will be useful when following the arguments in this article line by line.

2 Status Summary

Let $(M; G, B)$ be a set of data that consists of a real $2n$-dimensional manifold $M$, a Riemannian metric $G$ on $M$ that is Ricci-flat and can be Kähler for some appropriate choice of a complex structure $I$, and a closed 2-form $B$ on $M$. The non-linear sigma model construction determines an $\mathcal{N} = (1, 1)$ SCFT for the data $(M; G, B)$. Reference [5] used $M = T^4$ as a test case, and extracted several necessary conditions on the data $(M; G, B)$ for the SCFT to be rational.

Thm. 5.8 of [5]: The following is a list of necessary conditions in the case of $M = \mathbb{R}^4/\mathbb{Z}^4 = T^4$ for the $\mathcal{N} = (1, 1)$ SCFT to be rational (so, we should read $n = 2$ here).

1. there exists a polarizable complex structure $I$ on $M$, with which $G$ is compatible and $(M, G, I)$ becomes Kähler, and $B^{\text{transc}} = 0$.

For such a complex structure $I$ $(M_I$ is meant to be the complex manifold $(M, I)$),

2. the horizontal and vertical simple level-$n$ rational Hodge substructures satisfy the following conditions:

(a) The level-$n$ simple Hodge substructure on $[H^n(M; \mathbb{Q})]_{\ell=\infty}$ by $I$ is of CM-type, where the CM field $\text{End}([H^n(M; \mathbb{Q})]_{\ell=\infty})^{\text{Hdg}}$ is denoted by $K'$.

(b) There exists a $[K': \mathbb{Q}]$-dimensional vector subspace of $A(M_I) \otimes \mathbb{Q}$ denoted by $T^\vee_M \otimes \mathbb{Q}$ on which a simple level-$n$ rational Hodge structure of weight-$n$ can be introduced,
with the polarization

\[ H^{\text{even}}(M; \mathbb{Q}) \times H^{\text{even}}(M; \mathbb{Q}) \ni (\psi, \chi) \mapsto (-1)^{\frac{n(n-1)}{2}} \int_M \left( \sum_{k=0}^{n} (-1)^k \Pi_{2k} \psi \right) \wedge \chi \in \mathbb{Q} \]  

(2.1)

where \( \Pi_{2k} \) is the projection \( H^*(M; \mathbb{Q}) \to H^{2k}(M; \mathbb{Q}) \). Its Hodge \((n, 0)\) component is generated by \( \delta := e^{2-1(B+i\omega)} \), where \( \omega = 2^{-1}G(I-, -) \) is the Kähler form, and this polarized rational Hodge structure is of CM-type, with the endomorphism field \( K' \).

(c) There is an isomorphism of polarized rational Hodge structure of weight-\( n \) between the vertical and horizontal simple level-\( n \) components \( T^v_M \otimes \mathbb{Q} \) and \([H^n(M; \mathbb{Q})]_{\ell=n}\).

The rational Hodge substructures other than the weight-\( n \) level-\( n \) components satisfy the following conditions:

(a) All other rational polarizable Hodge substructures on \( H^k(M; \mathbb{Q}) \) by \( I \) are also of CM-type.

(b) There is a filtration \( W^\bullet_v \) on \( H^*(M; \mathbb{Q}) \) so that the data \((\rho_{\text{spin}}(h_\omega, B), W^\bullet_v)\) introduces a generalized rational Hodge structure on \( H^*(M; \mathbb{Q}) \), so that
   i. \( T^v_M \otimes \mathbb{Q} \subset W^n_v \), and
   ii. the rational Hodge structures on \( W^k_v/W^{k+2}_v \) is of CM-type for all \( k \), and the one for \( k = n \) is polarized by the pairing (2.1).

Furthermore,

4. there is a geometric SYZ-mirror\(^2\) to the \( \mathcal{N} = (2, 2) \) SCFT for the data \((M; G, B; I)\), and

5. (weak) at least for some of such geometric SYZ-mirrors, the filtration \( g^*(W_{h_\omega}) \) on \( H^*(M; \mathbb{Q}) \) satisfies the properties of \( W^\bullet_v \) in the conditions 3, 6 and 7.

There is one more condition that makes sense only for a family of \((M; G, B)\) that is self-SYZ-mirror (as in the case of \( M = T^{2n} \) and K3):

6. there is a one-to-one correspondence between the simple rational horizontal Hodge substructures on \( (W^k_h/W^{k+2}_h) \) and vertical Hodge substructures on \( (W^k_v/W^{k+2}_v) \) so that there are Hodge isomorphisms.

Finally, here is one more condition that is stated in a language applicable only to \( M = T^{2n} \):

7. the Hodge isomorphism in the condition 6, when chosen appropriately, can be interpreted as a combination of an isogeny and a mirror map of D-brane charges.

\(^1\) It is implied here that \( W^n_v \subset H^{\text{even}}(M; \mathbb{Q}) \) and that the subspace \( W^{n+2}_v \subset W^n_v \) is orthogonal to any element in \( W^n_v \) under the pairing (2.1).

\(^2\) A complex structure \( I \) specifies one weight-1 operator \( J \) in the left moving sector and one weight-1 operator \( \tilde{J} \) in the right moving sector. In the meanwhile, any T-duality / mirror correspondence is supposed to be an isomorphism of \( \mathcal{N} = (1, 1) \) SCFTs. It is non-trivial whether the isomorphism image of \( J \) and \( \tilde{J} \) can be interpreted by some mirror complex structure \( ^3I_\circ \). Only the SYZ mirrors that pass this test are called geometric SYZ mirrors in this article. See \(^3\) for more information.
One may wonder what happens if the condition 5(weak) above is replaced by

5. (strong) For any geometric-SYZ mirror, the filtration $g^*(W_{ho})$ satisfies the property of $W_{c}^*$ in the conditions 3, 6 and 7.

Although the condition 5 of Thm. 5.8 of [5] is read as 5(weak), one may read the derivation in Ref. [5] and will realize that the condition 5 holds in the stronger version here in the case of $M = T^4$ when the $\mathcal{N} = (1, 1)$ SCFT for $(M = T^4; G, B)$ is rational. All the notations and concepts here are either explained or introduced already in [5].

Let us first leave a few words about how to read the statements of Thm. 5.8 above. Reference [5] worked on the cases where $M = T^4$ and extracted the properties of the data $(M; G, B)$ when the $\mathcal{N} = (1, 1)$ SCFT for $(M; G, B)$ is rational; the properties were stated, however, in languages that make sense also for other Ricci-flat Kähler manifolds $M$. It was for this reason that the way the conditions are listed up is not optimized\(^3\) for the case of $M = T^4$, or for the cases of $M = T^{2n}$. The theorem statement above was not only a confirmed one for the $M = T^4$ case, but also meant to be a trial version for other Ricci-flat Kähler manifolds.

The conditions 1–7 on a set of data $(M; G, B)_{M=T^4}$ are necessary conditions for the corresponding $\mathcal{N} = (1, 1)$ SCFT to be rational. It is a natural question whether the conditions 1–7 on a set of data $(M; G, B)$ are also sufficient conditions for the $\mathcal{N} = (1, 1)$ SCFT to be rational. This converse problem remains to be an open question even in the case $M = T^4$; Ref. [5] made a partial attempt by imposing just the conditions 1–3 on $(M; G, B)_{M=T^4}$, and found that some of such data correspond to $\mathcal{N} = (1, 1)$ SCFTs that are not rational. So, just the set of conditions 1–3 alone does not constitute a sufficient condition for the $\mathcal{N} = (1, 1)$ SCFT to be rational.

Therefore, there are two things we wish to do beyond the work of [5]. One is (i) the converse problem: examine whether the set of all the conditions 1–7 constitutes a sufficient condition for the $\mathcal{N} = (1, 1)$ SCFT to be rational. The other is (ii) to investigate relations among the conditions; some of the conditions might follow from others. There are two motivations to think about (ii). As a preparation for the converse problem (i), we aim to reduce the redundancy among the conditions 1–7 and simplify the problem. As a preparation for application to more general Ricci-flat targets, we also aim to make the trial version of the statement as simple as possible.

We start working on both (i) and (ii) in the next section for the case of $M = T^4$. But there are a few quick comments on (ii) that we can make here.

Whether the condition 2 (a) implies the condition 3 (a) or not: not enough mathematics is known at this moment. Although Rmk. 5.3 of [5] referred to this open question for abelian surfaces, it now turns out that the answer is yes [9]. It is still an open question, however, for general Ricci-flat Kähler manifolds. So, we choose to retain the condition 3 (a) at this

\(^{3}\) For example, an isogeny in the condition 7 implies a Hodge isomorphism in the condition 6 for $k = 1$, from which Hodge isomorphisms in the condition 6 for all other $k$ follow, in the cases of $M = T^{2n}$.

Note that existence of an isogeny $T^*_{2n} \rightarrow T^*_{2n}$ is equivalent to existence of a Hodge isomorphism $H^1(T^{2n}; \mathbb{Q}) \rightarrow H^1(T^*_{2n}; \mathbb{Q})$ or of a Hodge isomorphism $H_1(T^*_{2n}; \mathbb{Q}) \rightarrow H_1(T^{2n}; \mathbb{Q})$; the former follows from the latter because any Hodge isomorphism $H_1(T^{2n}; \mathbb{Q}) \rightarrow H_1(T^{2n}; \mathbb{Q})$ can be multiplied by an appropriate integer to be a linear map $H_1(T^*_{2n}; \mathbb{Z}) \rightarrow H_1(T^{2n}; \mathbb{Z})$. 

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moment in a set of sufficient conditions along with the condition 2 (a). Just as a reminder, the condition 2 (a) is placed before the condition 3 (a) in the statement because the weight-
level-$n$ component $|H^n(M; \mathbb{Q})|_{\ell=n}$ is always present and unique for a Ricci-flat Kähler
manifold $M$ that is not necessarily a complex torus.

As for the condition 7 (with 5(weak) or 5(strong)), the authors realized that the condition can be derived easily from the conditions 3, 4, 5(weak/strong) and 6, when $M = T^{2n}$. This
is because those other conditions already imply that there is a Hodge isomorphism between
$H^1(T^{2n}_I; \mathbb{Q}) \cong W^1_h/W^3_h$ and $W^1_v/W^3_v \cong W^1_h/W^3_h \cong H^1(T^{2n}_I; \mathbb{Q})$; see footnote 3 for more.
This observation (footnote 3 in particular) is elementary; the authors should have been aware
while working for Ref. [5]. Conversely, only the condition 7+5(weak/strong) is enough 4 for the conditions 3(b), 5(weak/strong) and 6, when a set of data $(M; G, B)_{M=T^{2n}}$ satisfies the conditions 1–3(a) and 4. It still makes sense to write down the conditions by using the Hodge isomorphisms of
cohomology groups as in the conditions 3(b)–6 than in terms of geometry (in the condition 7),
because isogenies make sense only for the case of $M = T^{2n}$.

3 Relations among the Conditions

In this section, we show that the conditions 4 and 7+5(strong) on a set of data $(M; G, B)$
(and hence 3(b), 6+5(strong) also) follow from the conditions 1–3(a), when $M = T^4$. So, as
long as the cases with $M = T^4$ are concerned, we may retain only the conditions 1–3(a) and
drop all others from Thm. 5.8 of [5], and yet no information is lost. As is evident from the
following discussions, however, we will resort to a case-by-case analysis for those who do not want to go through [5]. For a
long as the cases with $M = T^4$ are concerned, we may retain only the conditions 1–3(a) and
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following discussions, however, we will resort to a case-by-case analysis for $M = T^4$ in this
section, so we must say that it is not clear whether the conditions 3(b)–7 follow from the
conditions 1–3(a) even when $M = T^{2n}$ with $n > 2$.

To prove that the conditions 4 and 7+5(strong) are satisfied when the conditions 1–3(a) are imposed on a set of data $(T^4; G, B)$, we can take advantage of the analysis of [5, §5.3].
There, a set of data $(T^4; G, B; I)$ satisfying the conditions 1–3(a) is classified into one of the
four cases, (A), (A’), (B and C), based on the complex structure $(T^4, I)$, to get started; when
$(T^4, I)$ is in the case (A), then it was shown in §5.3.3 of [5] that the set of data $(T^4; G, B)$ gives
rise to a rational $\mathcal{N} = (1, 1)$ SCFT, so the data also satisfy the conditions 3(b)–7. When
$(T^4, I)$ is either in the case (A’) or in the case (B, C), however, the complexified Kähler parameter
$(B + i\omega)$ on $(T^4, I)$ is not necessarily that for a rational $\mathcal{N} = (1, 1)$ SCFT.

Here, we give a minimum summary of the results from [5] and a reminder of some no-
tations on the cases (A’) and (B, C) for those who do not want to go through [5]. For a positive rational number $d$, $\sqrt{d}$ stands for a real positive number. For a negative rational

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4 The detailed argument is as follows. As in footnote 3 the condition 6+5 immediately follows, and the condition 3(b)i+5 also follows from 3(a). As for the condition 3(b)i+5, since $\mathfrak{g} \in H^*(T^{2n}; \mathbb{C})$ is the generator of the unique charge-$n$ eigenspace of the U(1) action $\rho_{spin}(h, \omega, B)$, it must be in $W_v^n \otimes \mathbb{C} = g^*(W_h^n \otimes \mathbb{C})$. Furthermore, since the Hodge structure on $H^6_M \otimes \mathbb{Q}$ is of CM-type with the endomorphism field $K^*$ as in the condition 2(b), we know that (a) $\mathfrak{g} \in W_v^n \otimes \mathbb{C}$ is in fact contained within $W_v^n \otimes \tau_{(n,0)}(K^*)$ for a certain embedding $\tau_{(n,0)} : K^* \to \overline{\mathbb{Q}}$, and also that (b) $H^6_M \otimes \mathbb{C}$ is generated by its Galois conjugates
$\{\overline{\sigma}^* | \sigma \in Gal(\overline{\mathbb{Q}}/\mathbb{Q})\}$ (cf. [5, Lemma A.11]). Since all the Galois conjugates $\overline{\sigma}^*$ are in $W_v^n \otimes \mathbb{C}, T_M^6 \otimes \mathbb{Q} \subset W_v^n$, that is, the condition 3(b)i+5 is satisfied.
number $p$, $\sqrt{p} = i \sqrt{-p}$ is a pure imaginary complex number in the upper half complex plane throughout this article. Lemmas A.11 and A.12 of [5] on CM-type rational Hodge structures are elementary, to the extent that math lecture notes do not explain at length, but are still used in this article frequently, sometimes even without a mention.

An abelian surface $T^4_1$ in the case (A') is isogenous to the product $E_1 \times E_2$ of a pair $(E_1$ and $E_2)$ of mutually non-isogenous CM elliptic curves, whose complex structure parameters are $\tau_1 \in \mathbb{Q}(\sqrt{p_1})$ and $\tau_2 \in \mathbb{Q}(\sqrt{p_2})$, respectively (i.e., one may choose $p_1$, $p_2$ to be negative square-free integers such that $p_1 \not\equiv p_2 (Q^\times)^2$). In this case, we have

$$\text{End}(H^1(E_i; \mathbb{Q}))^{\text{Hdg}} \cong \mathbb{Q}(\sqrt{p_i}), \quad \text{End}(H^1(T^4; \mathbb{Q}))^{\text{Hdg}} \cong \mathbb{Q}(\sqrt{p_1}) \oplus \mathbb{Q}(\sqrt{p_2}). \tag{3.1}$$

We can take the rational basis $\{\hat{\alpha}^i, \hat{\beta}_i\}$ of $H^1(E_i; \mathbb{Q})$ and the complex coordinate $z^i$ of $E_i$ so that $dz^i = \hat{\alpha}^i + \sqrt{p_i} \hat{\beta}_i$. The action of $\xi_i \in \mathbb{Q}(\sqrt{p_i})$ as an element of $\text{End}(H^1(E_i; \mathbb{Q}))^{\text{Hdg}}$ is just multiplying $dz^i$ by $\xi_i$. The rational basis of $H^1(T^4, \mathbb{Q})$ obtained by pulling back $\hat{\alpha}^1, \hat{\beta}_1, \hat{\alpha}^2, \hat{\beta}_2$ of $H^1(E_1 \times E_2; \mathbb{Q})$ by an isogeny $T^4 \to E_1 \times E_2$ is also denoted by $\hat{\alpha}^1, \hat{\beta}_1, \hat{\alpha}^2, \hat{\beta}_2$.

When a set of data $(T^4; G, B, I)$ in the case (A') satisfies the conditions 1–3(a), the complexified Kähler parameter $(B + i\omega)/2$ is of the form of either one of

$$(B + i\omega)/2 = (A + C\sqrt{p_1}) \hat{\alpha}^1 \hat{\beta}_1 + (\tilde{A} + \tilde{C}\sqrt{p_2}) \hat{\alpha}^2 \hat{\beta}_2, \quad A, \tilde{A} \in \mathbb{Q}, \ C, \tilde{C} \in \mathbb{Q}_{\not=0}. \tag{3.2}$$

or

$$(B + i\omega)/2 = (A + C\sqrt{p_2}) \hat{\alpha}^1 \hat{\beta}_1 + (\tilde{A} + \tilde{C}\sqrt{p_1}) \hat{\alpha}^2 \hat{\beta}_2, \quad A, \tilde{A} \in \mathbb{Q}, \ C, \tilde{C} \in \mathbb{Q}_{\not=0}. \tag{3.3}$$

A set of data $(T^4; G, B, I)$ that falls into the case (3.2) corresponds to a rational $\mathcal{N} = (1,1)$ SCFT, and a set of data that falls into the case (3.3) to an $\mathcal{N} = (1,1)$ SCFT that is not rational. Derivation is found in §5.3.2 of [5].

An abelian surface $T^4_1$ in the cases (B, C) is a simple abelian variety of CM-type. The endomorphism algebra $\text{End}(H^1(T^4; \mathbb{Q}))^{\text{Hdg}}$ is a degree-4 CM field, which has a structure

$$K \cong \mathbb{Q}[x, y]/(y^2 - d, x^2 - p - qy) \tag{3.4}$$

where $d > 1$ is a positive square-free integer, and $p, q \in \mathbb{Q}$ are subject to the conditions $p < 0$, $q \neq 0$ and $d' := p^2 - q^2d > 0$. The case (B) is when $K$ is Galois over $\mathbb{Q}$, and the case (C) when $K$ is not; the distinction between them is not important in this article, however. The four embeddings $K \hookrightarrow \overline{\mathbb{Q}}$ are denoted by $\tau_{\pm \pm}$, where

$$\tau_{\pm \pm}(y) = \pm \sqrt{d}, \quad \tau_{\pm +}(x) = \sqrt{\mp} := i\sqrt{-p \mp q\sqrt{d}}, \quad \tau_{\pm -}(x) = -\sqrt{\mp} = -i\sqrt{-p \pm q\sqrt{d}}. \tag{3.5}$$

The reflex field $K^r$ of the CM-pair $(K, \{\tau_{++}, \tau_{--}\})$ in the cases (B, C) has the structure

$$K^r \cong \mathbb{Q}[y', \xi]/((y')^2 - d', (\xi')^2 - 2p + 2y'). \tag{3.7}$$
The four embeddings $K^r \hookrightarrow \overline{Q}$ are denoted by $\tau^r_{\pm \pm}$, where

$$\tau^r_{\pm}(y') = \pm \sqrt{d'} = \mp \sqrt{+\pm}, \quad \tau^r_{\pm+}(\xi^r) = \sqrt{+\pm} \sqrt{\pm}, \quad \tau^r_{\pm-}(\xi^r) = -(\sqrt{+\pm} \sqrt{\pm}).$$

The reflex field $K^r$ is also a degree-4 CM field.

When a set of data $(T^4; G, B; I)$ in the cases (B, C) satisfies the conditions 1–3(a), the complexified Kähler parameter $(B + i\omega)/2$ is of the form

$$(B + i\omega)/2 = Z_1\epsilon_1 + Z_2\epsilon_2,$$

where

$$e_1 := \hat{\alpha}^1\hat{\beta}_1 + d\hat{\alpha}^2\hat{\beta}_2, \quad e_2 := \hat{\alpha}^1\hat{\beta}_2 - \hat{\beta}_1\hat{\alpha}_2,$$

$$Z_1 := \tau^r_{++} \left( A + \frac{C'}{2}\xi^r + \frac{D'}{2d}\xi^r \right), \quad Z_2 := \tau^r_{++} \left( \tilde{A} \pm D'\xi^r \pm \frac{C'\xi^r}{2d} \right),$$

$$(A, \tilde{A}, C', D' \in Q, (C', D') \neq (0, 0))$$

as was shown in §5.3.1 of [5]. A set of data that falls into the case (3.10+) corresponds to a rational $\mathcal{N} = (1, 1)$ SCFT. The $\mathcal{N} = (1, 1)$ SCFT that corresponds to a set of data in the case (3.10−), on the other hand, is not rational.

We already know that the conditions 3(b)–7 are satisfied in the cases where the complex structure $(T^4, I)$ and the complexified Kähler parameter $(B + i\omega)/2$ of the data $(T^4; G, B; I)$ are in the case $(A')-(3.3)$ and case $(B, C)-(3.10+)$, because the conditions 3(b)–7 are satisfied by the set of data $(T^4; G, B)$ for a rational $\mathcal{N} = (1, 1)$ SCFT [5]. In the rest of this section, we will show that the conditions 3(b)–7 are satisfied also in the cases $(A')-(3.3)$ and $(B, C)-(3.10−)$.

### 3.1 The Case $(A')$

As a first step, let us list up all the possible geometric SYZ-mirrors of the $\mathcal{N} = (2, 2)$ SCFT for a set of data $(T^4; G, B; I)$ in the case of $(A')-(3.3)$. By finding a non-empty list, we prove that the condition 4 follows automatically in the case $(A')-(3.3)$; the full list of geometric SYZ-mirrors is used to prove the condition 7+5(strong) later in this subsection 3.1.

Recall that, for the T-duality taken along 1-cycles specified by a rank-$n$ primitive subgroup $\Gamma_f \subset H_1(T^{2n}; \mathbb{Z})$, the corresponding geometric SYZ-mirror exists if and only if the following conditions are satisfied [5, Prop. 8]:

$$\omega|_{\Gamma_f \otimes \mathbb{R}} = 0, \quad B|_{\Gamma_f \otimes \mathbb{R}} = 0.$$

To obtain the list of such $\Gamma_f$'s, we first list up all the rank-$n$ subspaces $\Gamma_f \subset H_1(T^{2n}; \mathbb{Q})$ satisfying $\omega|_{\Gamma_f \otimes \mathbb{R}} = 0$ and $B|_{\Gamma_f \otimes \mathbb{R}} = 0$. Then $\Gamma_f \subset H_1(T^{2n}; \mathbb{Z})$ for these $\Gamma_f$ constitute the list of all $\Gamma_f$ satisfying $(3.14)$. 


Let $\Gamma_f = \text{Span}_\mathbb{Q}\{c', d'\}$ be such a rank-$(n = 2)$ subspace of $H_1(T^{2n=4}; \mathbb{Q})$. We parameterize the generators by
\begin{align*}
c' := c_1 \alpha_1 + c_2 \beta_1 + c_3 \alpha_2 + c_4 \beta_2, \\
d' := d_1 \alpha_1 + d_2 \beta_1 + d_3 \alpha_2 + d_4 \beta_2, \\
c_1, \ldots, c_4, d_1, \ldots, d_4 \in \mathbb{Q}.
\end{align*}
(3.15)  
(3.16)  
(3.17)

The above condition $\omega |_{\Gamma_f \otimes \mathbb{R}} = B |_{\Gamma_f \otimes \mathbb{R}} = 0$ for $(A')$ is then equivalent to
\begin{align*}
c_1 : c_2 = d_1 : d_2 \quad \text{and} \quad c_3 : c_4 = d_3 : d_4,
\end{align*}
(3.18)

where we used the positive volume condition $C, \tilde{C} \neq 0$. Therefore, we can reorganize the basis of $\Gamma_f$ so that
\begin{align*}
\Gamma_f = \text{Span}_\mathbb{Q}\{c, d\}, \quad c := c_1 \alpha_1 + c_2 \beta_1, \quad d := c_3 \alpha_2 + c_4 \beta_2, \\
c_1, \ldots, c_4 \in \mathbb{Q}, (c_1, c_2), (c_3, c_4) \neq (0, 0).
\end{align*}
(3.19)  
(3.20)

The $\Gamma_f$’s of our interest are exhausted by those in the form of (3.19) with the parameters scanned as in (3.20) $^5$

Once such an $(n = 2)$-dimensional subspace $\Gamma_f \subset H_1(T^{2n=4}; \mathbb{Q})$ is chosen, the rank-$n$ primitive subgroup $\Gamma_f := \Gamma_f \cap H_1(T^{2n}; \mathbb{Z})$ has been determined. For any such $\Gamma_f$, there is a complement rank-$n$ subgroup $\Gamma_b \subset H_1(T^{2n}; \mathbb{Z})$ so that $\Gamma_f \oplus \Gamma_b \cong H_1(T^{2n}; \mathbb{Z}) \cong \mathbb{Z}^{\oplus 2n}$; there are infinitely many different ways to choose $\Gamma_b$ for a given $\Gamma_f$. Every choice of such $(\Gamma_f, \Gamma_b)$ specifies a geometric SYZ-mirror; $\Gamma_f$ and $\Gamma_b$ are meant to be the 1-cycles along which T-duality is taken and not taken, respectively. This is the end of the process of listing up all the $(\Gamma_f, \Gamma_b)$’s for geometric SYZ mirrors.

Although the condition $4$ has been verified, let us also write down some more explicit information on cohomologies of $T^4$ and $T^4_\circ$ for later convenience. For a given $n$-dimensional subspace $\Gamma_{fQ} \subset H_1(T^{2n}; \mathbb{Q})$, an $n$-dimensional subspace $\Gamma_{bQ}^\vee \subset H^1(T^{2n}; \mathbb{Q})$ is specified as those that vanish on $\Gamma_{fQ}$. In other words, they are cohomologies on $T^{2n}$ that are pulled back from the cohomologies on the base by the SYZ $T^n$ fibration. This subspace $\Gamma_{bQ}^\vee$ is generated by
\begin{align*}
\hat{e} := -c_2 \hat{\alpha}_1 + c_1 \hat{\beta}_1, \quad \hat{f} := -c_4 \hat{\alpha}_2 + c_3 \hat{\beta}_2.
\end{align*}
(3.21)

Note that we may use $\{c, d, \hat{e}, \hat{f}\}$ as a basis of
\begin{align*}
H^1(T^{2n}_\circ; \mathbb{Q}) \cong \Gamma_{fQ} \oplus \Gamma_{bQ}^\vee = \text{Span}_\mathbb{Q}\{c, d, \hat{e}, \hat{f}\};
\end{align*}
(3.22)

\footnote{\begin{itemize}
\item Precisely the same analysis can be repeated for the case $(A')$; one then finds that the $\Gamma_f$’s are parameterized precisely as in (3.19) and (3.20). In Ref. [5] §4.2, only the choice $c_2 = c_4 = 0$ was presented as an example of geometric SYZ-mirrors.
\item The isomorphism $\Gamma_{fQ} \oplus \Gamma_{bQ}^\vee \cong H^1(T^{2n}_\circ; \mathbb{Q})$ is a part of the map $g : H_1(T^{2n}; \mathbb{Z}) \oplus H^1(T^{2n}; \mathbb{Z}) \to H_1(T^{2n}_\circ; \mathbb{Z}) \oplus H^1(T^{2n}_\circ; \mathbb{Z})$ of the winding and Kaluza–Klein charges which identify states on both sides of the T-dual with the same masses. The notation "$g$" of the map is omitted in (3.22) and also in other parts of this article except the appendix [A]. For the spinor representation of this $g$, which is $H^*(T^{2n}; \mathbb{Q}) \to H^*(T^{2n}_\circ; \mathbb{Q})$, however, the notation "$g$" is recycled and used in the main text.
\end{itemize}}

T-dualized 1-cycles of $T^4$ are also regarded as 1-cocycles of the mirror $T^4_\circ$, so we abused notations a little bit here.

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Let us choose a basis \(\{e, f\}\) of \(\Gamma_b \otimes \mathbb{Q}\) so that \(\{e, f\}\) is dual to the basis \(\{\hat{e}, \hat{f}\}\) of \(\Gamma_b^\vee \subset H^1(T^4; \mathbb{Q})\). Then \(e \in (c_1^2 + c_2^2)e' + \Gamma_{fQ}\) and \(f \in (c_3^2 + c_4^2)f' + \Gamma_{fQ}\), where
\[
e' := -c_2\alpha_1 + c_1\beta_1, \quad f' := -c_4\alpha_2 + c_3\beta_2. \tag{3.23}
\]
The choice of a basis \(\{e, d, e, f\}\) of \(H_1(T^4; \mathbb{Q})\) also determines its dual basis of \(H^1(T^4; \mathbb{Q})\), which are denoted by \(\{\hat{e}, \hat{f}, \hat{e}, \hat{f}\}\). For a given \(\Gamma_{fQ}\) (and hence a given \(\Gamma_f\)), we may choose \(\{\hat{e}' + \Gamma_{bQ}, \hat{d}' + \Gamma_{bQ}\}\) as a basis of the coset vector space \(H^1(T^4; \mathbb{Q})/\Gamma_{bQ}\), with
\[
\hat{c}' := c_1\hat{\alpha} + c_2\hat{\beta}_1, \quad \hat{d}' := c_3\hat{\alpha} + c_4\hat{\beta}_2 \tag{3.24}
\]
independent of a choice of \(\Gamma_b\), and yet \(\hat{e} \in (c_1^2 + c_2^2)^{-1}\hat{e}' + \Gamma_{bQ}\) and \(\hat{d} \in (c_3^2 + c_4^2)^{-1}\hat{d}' + \Gamma_{bQ}\). It often happens in the analysis in this article that the basis \(\{\hat{e}, \hat{f}, \hat{e}, \hat{d}\}\) of \(H^1(T^4; \mathbb{Q})\) is more convenient than \(\{\hat{e}, \hat{f}, \hat{e}, \hat{d}\}\) (as we only have to refer to a choice of \(\Gamma_f\), not of \(\Gamma_b\)).

Now that the condition 4 has been verified, let us move on to show that the condition 7+5(strong) is satisfied. That is to verify\(^7\) the existence of an isogeny \(T_{f_1}^4 \to T_{f_0}^4\) for all the geometric SYZ-mirrors.

Recall that \(T_{f_1}^4\) in the case \((A')\) is isogenous to the product \(E_1 \times E_2\) of CM elliptic curves \(E_1\) and \(E_2\), whose complex structure parameters are \(\tau_1 \in \mathbb{Q}(\sqrt{p_1})\) and \(\tau_2 \in \mathbb{Q}(\sqrt{p_2})\), respectively. In addition, in the case of \((A')\)\(^8\), their complexified Kähler parameters are \(\rho_1 \in \mathbb{Q}(\sqrt{p_2})\) and \(\rho_2 \in \mathbb{Q}(\sqrt{p_1})\), respectively. For any \(\Gamma_{fQ}\) in \((3.19)\), as shown in the appendix A, the geometric SYZ-mirror \(T_{f_0}^4\) is isogenous to \(E_1^\circ \times E_2^\circ\), where \(E_i^\circ\) is the T-dual of \(E_i\) along \(c \in H_1(E_i; \mathbb{Q})\), and \(E_2^\circ\) that of \(E_2\) along \(d \in H_1(E_2; \mathbb{Q})\). Here, the complex structure parameters \(\tau_1^\circ\) of \(E_1^\circ\) and \(\tau_2^\circ\) of \(E_2^\circ\) are in the imaginary quadratic fields \(\mathbb{Q}(\rho_1) = \mathbb{Q}(\sqrt{p_1})\) and \(\mathbb{Q}(\rho_2) = \mathbb{Q}(\sqrt{p_2}) = \mathbb{Q}(\tau_2)\), respectively. Therefore, there exist isogenies \(E_1 \to E_2^\circ\) and \(E_2 \to E_1^\circ\), and hence also an isogeny \(T_{f_1}^4 \to T_{f_0}^4\).

This argument for the existence of an isogeny holds for any choice of geometric SYZ-mirror, so we have verified the condition 7+5(strong). This completes the proof of the conditions 4 and 7+5(strong) (and hence 3(b)-7) for the case \((A')\)\(^[(3.3)]\).

### 3.2 The Case (B, C)

Let us move on to the proof of the conditions 4 and 7+5(strong) for the case \((B,C)\)\(^{(3.10)}\). The outline of the proof is the same as in section 3.1.

Our first task in this section \((3.2)\) is to list up all the rank-\( n \) subspaces \(\Gamma_{fQ} \subset H_1(T^{2n}; \mathbb{Q})\) satisfying \(\omega|_{\Gamma_{fQ} \otimes \mathbb{R}} = 0\) and \(B|_{\Gamma_{fQ} \otimes \mathbb{R}} = 0\). Let \(\Gamma_f = \text{Span}_\mathbb{Q}\{c', d'\}\) be such a rank-\( n \) subspace of \(H_1(T^{2n}; \mathbb{Q})\). The condition \(\omega|_{\Gamma_{fQ} \otimes \mathbb{R}} = B|_{\Gamma_{fQ} \otimes \mathbb{R}} = 0\) is then equivalent to
\[
e_1(c', d') = e_2(c', d') = 0; \tag{3.25}
\]

\(^7\) If we are to choose \(e \in (c_1^2 + c_2^2)^{-1}e' + \Gamma_{fQ}\) and \(f \in (c_3^2 + c_4^2)^{-1}f' + \Gamma_{fQ}\) arbitrarily, however, there is no guarantee that the primitive subgroup \(\Gamma_b := \text{Span}_\mathbb{Q}\{e, f\} \cap H_1(T^4; \mathbb{Z})\) satisfies \(\Gamma_f \oplus \Gamma_b = H_1(T^4; \mathbb{Z})\).

\(^8\) Although the following proof for the existence of an isogeny exploits properties very specific to the case \((A')\), which allows us to make the proof simple, we can also find isogenies by direct calculations of the rational Hodge structures just like we will do for the cases \((B, C)\) in section \((3.2)\).
we have used the fact that the volume of $T^4$ must be positive ($\omega \neq 0$, i.e. $(C', D') \neq (0, 0)$), and that the integer $d$ is square-free.

Let us rewrite the condition (3.25) in a way useful for later analysis, by parametrizing the generators $c', d'$ of $\Gamma_{f'Q}$ as

$$c' := c_1 \alpha_1 + c_2 \beta^1 + c_3 \alpha_2 + c_4 \beta^2,$$
$$d' := d_1 \alpha_1 + d_2 \beta^1 + d_3 \alpha_2 + d_4 \beta^2,$$  
(3.26)  
(3.27)  
$$c_1, \ldots, c_4, d_1, \ldots, d_4 \in \mathbb{Q}.  
(3.28)$$

To proceed, note that there are linear combinations of $e_1$ and $e_2$ that are decomposable:

$$e_1 + \sqrt{d}e_2 = (\hat{\alpha}^1 + \sqrt{d}\hat{\alpha}^2)(\hat{\beta}_1 + \sqrt{d}\hat{\beta}_2) = \hat{\gamma}^1 \hat{\delta}_1,$$
$$e_1 - \sqrt{d}e_2 = (\hat{\alpha}^1 - \sqrt{d}\hat{\alpha}^2)(\hat{\beta}_1 - \sqrt{d}\hat{\beta}_2) = \hat{\gamma}^2 \hat{\delta}_2,$$  
(3.29)  
(3.30)

where

$$\hat{\gamma}^1 := \hat{\alpha}^1 + \sqrt{d}\hat{\alpha}^2, \quad \hat{\delta}_1 := \hat{\beta}_1 + \sqrt{d}\hat{\beta}_2,$$  
(3.31)  
$$\hat{\gamma}^2 := \hat{\alpha}^1 - \sqrt{d}\hat{\alpha}^2, \quad \hat{\delta}_2 := \hat{\beta}_1 - \sqrt{d}\hat{\beta}_2.$$  
(3.32)

The condition (3.25) is therefore equivalent to $\hat{\gamma}^1 \hat{\delta}_1(c', d') = \hat{\gamma}^2 \hat{\delta}_2(c', d') = 0$. This translates to

$$\left\{ \begin{array}{l}
(c_1 + \sqrt{d}c_3) : (c_2 + \sqrt{d}c_4) = (d_1 + \sqrt{d}d_3) : (d_2 + \sqrt{d}d_4) \\
(c_1 - \sqrt{d}c_3) : (c_2 - \sqrt{d}c_4) = (d_1 - \sqrt{d}d_3) : (d_2 - \sqrt{d}d_4).
\end{array} \right.$$  
(3.33)

The generator $d'$ for a given $c'$ is therefore constrained by the relation (3.33); let us see how, by translating the relation (3.33) further. An immediate consequence of the relation (3.33) is that there exist $k_1, k_2 \in \mathbb{C}$ such that $d'$ can be rewritten as

$$d' = \frac{1}{2} \left( (c_1 + \sqrt{d}c_3)k_1 + (c_1 - \sqrt{d}c_3)k_2 \right) \alpha_1 + \frac{1}{2\sqrt{d}} \left( (c_1 + \sqrt{d}c_3)k_1 - (c_1 - \sqrt{d}c_3)k_2 \right) \alpha_2 + \frac{1}{2} \left( (c_2 + \sqrt{d}c_4)k_1 + (c_2 - \sqrt{d}c_4)k_2 \right) \beta^1 + \frac{1}{2\sqrt{d}} \left( (c_2 + \sqrt{d}c_4)k_1 - (c_2 - \sqrt{d}c_4)k_2 \right) \beta^2.$$  
(3.34)

Since the generator $d'$ is an element of $H_1(T^4; \mathbb{Q})$, the coefficients in (3.34) must be rational numbers. It follows from this fact and some easy calculation that $k_1, k_2 \in \mathbb{Q}(\sqrt{d})$, and moreover

$$k_1 = a + b\sqrt{d}, \quad k_2 = a - b\sqrt{d}.$$  
(3.35)

$$\left( a, b \in \mathbb{Q}, (a, b) \neq (0, 0) \right).$$  
(3.36)

So, the generator $d'$ has to be of the form

$$d' = (ac_1 + bdc_3)\alpha_1 + (bc_1 + ac_3)\alpha_2 + (ac_2 + bdc_4)\beta^1 + (bc_2 + ac_4)\beta^2.$$  
(3.37)
for the conditions (3.25, 3.33) to be satisfied. This (3.37) is also sufficient.

This brings us to the conclusion\(^9\) that the \(\Gamma_f\)’s of our interest (the \(n\) 1-cycles along which T-dualities are taken to be geometric SYZ-mirrors) are exhausted by those of the form\(^10\)

\[
\Gamma_{fQ} = \text{Span}_\mathbb{Q}\{c, d\}, \quad c := c_1\alpha_1 + c_2\beta^1 + c_3\alpha_2 + c_4\beta^2, \\
d := dc_3\alpha_1 + dc_4\beta^1 + c_1\alpha_2 + c_2\beta^2, \\
c_1, \ldots, c_4 \in \mathbb{Q}, c_1^2 + c_2^2 + c_3^2 + c_4^2 \neq 0
\]

(3.38)

with the parameters scanned as in (3.39). The directions \(\Gamma_b \subset H_1(T^4; \mathbb{Z})\) in which T-duality is not taken are subject only to the condition that \(\Gamma_f \oplus \Gamma_b \cong H_1(T^4; \mathbb{Z})\).

Now that we have found a non-empty list of geometric SYZ-mirrors, the condition 4 is verified. For later convenience, however, let us leave some more explicit information on cohomologies of \(T^4\) and \(T^4_\alpha\).

Think of a \(\Gamma_{fQ}\) with \((c_1, c_3) \neq (0, 0)\), first, for simplicity. The \(n\)-dimensional subspace \(\Gamma_{bQ}^\vee \subset H^1(T^{2n}; \mathbb{Q})\) that vanishes on \(\Gamma_{fQ}\) is generated by

\[
\hat{e} := (c_1^2 - dc_3^2)\hat{\beta}_1 - (c_1c_2 - dc_3c_4)\hat{\alpha}_1 - d(c_1c_4 - c_2c_3)\hat{\alpha}_2, \\
\hat{f} := (c_1^2 - dc_3^2)\hat{\beta}_2 - (c_1c_4 - c_2c_3)\hat{\alpha}_1 - (c_1c_2 - dc_3c_4)\hat{\alpha}_2
\]

(3.40, 3.41)

and we may use \(\{c, d, \hat{e}, \hat{f}\}\) as a basis of

\[
H^1(T^4; \mathbb{Q}) \cong \Gamma_{fQ} \oplus \Gamma_{bQ}^\vee = \text{Span}_\mathbb{Q}\{c, d, \hat{e}, \hat{f}\};
\]

(3.42)

T-dualized 1-cycles of \(T^4\) are also regarded as 1-cocycles of the mirror \(T^4_\alpha\), so we abused notations a little bit here again.

Let us choose a basis \(\{e, f\}\) of \(\Gamma_b \otimes \mathbb{Q}\) so that \(\{e, f\}\) is dual to the basis \(\{\hat{e}, \hat{f}\}\) of \(\Gamma_{bQ}^\vee \subset H^1(T^4; \mathbb{Q})\). Then \(e \in (c_1^2 - dc_3^2)^{-1}e' + \Gamma_{fQ}\) and \(f \in (c_1^2 - dc_3^2)^{-1}f' + \Gamma_{fQ}\), where

\[
e' := \beta^1, \quad f' := \beta^2.
\]

(3.43)

The choice of a basis \(\{c, d, e, f\}\) of \(H_1(T^4; \mathbb{Q})\) determines its dual basis of \(H^1(T^4; \mathbb{Q})\), denoted by \(\{\hat{c}, \hat{d}, \hat{e}, \hat{f}\}\). While \(\{\hat{c}, \hat{d}\}\) generates the subspace \(\Gamma_{fQ}^\vee \subset H^1(T^4; \mathbb{Q})\) that vanishes on \(\Gamma_{bQ}\), the coset space \(H^1(T^4; \mathbb{Q})/\Gamma_{bQ}^\vee\) has a basis \(\{\hat{c}' + \Gamma_{bQ}^\vee, \hat{d}' + \Gamma_{bQ}^\vee\}\), where

\[
\hat{c}' := c_1\hat{\alpha}_1 - dc_3\hat{\alpha}_2, \quad \hat{d}' := -c_3\hat{\alpha}_1 + c_1\hat{\alpha}_2
\]

(3.44)

and \(\hat{c} \in (c_1^2 - dc_3^2)^{-1}\hat{c}' + \Gamma_{bQ}^\vee\) and \(\hat{d} \in (c_1^2 - dc_3^2)^{-1}\hat{d}' + \Gamma_{bQ}^\vee\).

Think of a general \(\Gamma_{fQ}\) now, when there is no guarantee that \((c_1, c_3) \neq 0\). Because \(c \neq 0\), however, \((c_2, c_4) \neq 0\) then. So the \(\Gamma_{fQ}\)’s that are not covered by the argument above is

\(^9\)Precisely the same analysis can be repeated for the case of (3.10); one then finds that the \(\Gamma_{fQ}\)’s are parameterized precisely as in (3.33). In Ref. [5 §4.2], only the choice \(c_2 = c_3 = c_4 = 0\) was presented as an example of geometric SYZ-mirrors.

\(^10\)apologies for our poor choice of notations: \(d \in \Gamma_{fQ}\) here and a positive integer \(d\) that generates \(\mathbb{Q}(\sqrt{d})\)

\(^11\)The remark in footnote \(^9\) also holds true here.
covered by $\Gamma_{f\mathcal{O}}$’s where $(c_2, c_4) \neq 0$. Suppose, now, that $(c_2, c_4) \neq 0$. The same discussion as above holds for this case, when

\[ \hat{\epsilon} := (c_2^2 - dc_4^2)\hat{\alpha}^1 - (c_1c_2 - dc_3c_4)\hat{\beta}_1 - d(c_2c_3 - c_1c_4)\hat{\beta}_2, \quad (3.45) \]

\[ \hat{f} := (c_2^2 - dc_4^2)\hat{\alpha}^2 - (c_2c_3 - c_1c_4)\hat{\beta}_1 - (c_1c_2 - dc_3c_4)\hat{\beta}_2, \quad (3.46) \]

and

\[ \epsilon' := \alpha_1, \quad f' := \alpha_2, \quad \hat{\epsilon}' := c_2\hat{\beta}_1 - dc_4\hat{\beta}_2, \quad \hat{d}' := -c_4\hat{\beta}_1 + c_2\hat{\beta}_2; \quad (3.47) \]

\[ \hat{c} \in (c_2^2 - dc_4^2)^{-1}\hat{c}' + \Gamma_{b\mathcal{O}}^\vee \text{ and } \hat{d} \in (c_2^2 - dc_4^2)^{-1}\hat{d}' + \Gamma_{b\mathcal{O}}^\vee \text{ now.} \]

We could write down an expression for $\hat{\epsilon}$, $\hat{f}$ for a $\Gamma_{f\mathcal{O}}$ with a fully general $c \neq 0$, it is much easier in the analysis in this article to work on the cases with $(c_1, c_3) \neq 0$, and on those with $(c_2, c_4) \neq 0$ because the expressions of those generators remain as simple as above then. In the rest of this article, we will work only on the cases with $(c_1, c_3) \neq 0$, and will not repeat the same discussions for the cases with $(c_2, c_4) \neq 0$.

The remaining task is to verify the condition 7+5(strong): the existence of an isogeny $T^4 \to T^4_0$ for all the geometric SYZ-mirrors. So, we will show that $W^3_v (\cong_{by \ g^*} H^3(T^4_0; \mathbb{Q}))$ is Hodge isomorphic to $H^1(T^4; \mathbb{Q})$ in the following (cf. footnote 3).

The vector subspace $W^3_v$ in $H^*(T^4; \mathbb{Q})$ is determined by pulling back

\[ W^3_{h\mathcal{O}} := H^3(T^4_0; \mathbb{Q}) \cong \text{Span}_\mathbb{Q}\{ce\hat{e}, cd\hat{f}, c\hat{e}\hat{f}, d\hat{e}\hat{f}\} \quad (3.48) \]

by the map of D-brane charges $g^* : H^*(T^4_0; \mathbb{Q}) \to H^*(T^4; \mathbb{Q})$, which is

\[ W^3_v = g^*(W^3_{h\mathcal{O}}) = \text{Span}_\mathbb{Q}\{\hat{\epsilon}, \hat{f}, \hat{d}\hat{f}, \hat{c}\hat{e}\hat{f}\} \quad (3.49) \]

\[ = \text{Span}_\mathbb{Q}\{\hat{\epsilon}, \hat{f}, e_1\hat{f}, e_2\hat{f}\}. \quad (3.50) \]

We might (or might not) use in the rest of this article that $e_1\hat{f} = -e_2\hat{e}$ and $e_2\hat{f} = -e_1\hat{e}/d$.

The vertical rational Hodge structure on $W^3_v$—equivalent to the rational Hodge structure on $H^3(T^4_0; \mathbb{Q})$ with respect to the complex structure of the mirror—is determined by recalling the following fact: its Hodge $(1, 0)$ component is $\mathcal{U} \wedge (\Gamma_{b\mathcal{O}}^\vee \otimes \mathbb{C})$, which has two generators

\[ \mathcal{U}\hat{\epsilon} = \hat{\epsilon} + Z_1e_1\hat{\epsilon} + Z_2e_2\hat{\epsilon} \quad \text{and} \quad \mathcal{U}\hat{f} = \hat{f} + Z_1e_1\hat{f} + Z_2e_2\hat{f}. \quad (3.51) \]

We claim that this rational Hodge structure on $W^3_v$ is of CM-type. An easy way to see this is to reorganize the two generators into the form where [3] Lemma A.12 can be used (ignore the sign choice — [resp. +] in $\mp$ [resp. $\pm$] here [13]).

\[ \begin{pmatrix} \mathcal{U}\hat{\epsilon} & \mathcal{U}\hat{f} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pm\sqrt{d} & \mp\sqrt{d} \end{pmatrix} = \begin{pmatrix} \hat{\epsilon} & \hat{f} & e_1\hat{f} & e_2\hat{f} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \tau_{++}(\mp y) & \tau_{+-}(\mp y) \\ \tau_{+-}(\mp z) & \tau_{--}(\mp z) \\ \tau_{++}(\pm z y) & \tau_{+-}(\pm z y) \end{pmatrix}, \quad (3.52) \]

\[ \text{See [10] §3, §9.3] or [3] footnote 13]. \]

The sign written below $(+ \text{ in } \mp \text{ and } - \text{ in } \pm)$ is for the case $(B, C) - \mp \pm -$), while the sign written above is for the case $(B, C) - \mp \pm +$.

The eigenbasis of the action of the CM field on $W^3_v = g^*(W^3_{h\mathcal{O}})$ was worked out for both the case $(B, C) - \mp \pm -$ and $(B, C) - \mp \pm +$, for the choice $(c_1, c_2, c_3, c_4) = (1, 0, 0, 0)$ already in [3] eq. (5.40). The present version is for a general $\Gamma_{f\mathcal{O}}$ of a geometric SYZ mirror.
where
\[ \Xi_\pm := \tilde{A} \pm Ay \pm D'x \pm C'xy \in K; \] (3.53)

it is now easy to see from the way the two elements of the basis in (3.52) are related to a rational basis (see, e.g., [5, Lemma A.12]) that the weight-1 vertical rational Hodge structure on \( W^3_v \) (horizontal rational Hodge structure on \( H^3(T^4_\circ; \mathbb{Q}) \)) is of CM-type. The CM-pair is 
\( (K, \{\tau^+\}, \tau^- \}) \), which is identical to that of \( H_1(T^4; \mathbb{Q}) \). This means that there is a Hodge isometry between \( H_1(T^4; \mathbb{Q}) \) and \( H_1(T^4_\circ; \mathbb{Q}) \).

This is what we needed to show the existence of an isogeny \( T^4 \to T^4_\circ \). The argument holds for any choice of geometric SYZ-mirror, so the condition 7+5(strong) is now verified. This completes the proof of the conditions 4 and 7+5(strong) (and hence 3(b)–7) for the case (B, C)–(3.10;).

4 One More Condition

The conditions (with 5(strong)) listed up in Thm. 5.8 of [5] are necessary conditions for the \( \mathcal{N} = (1, 1) \) SCFT of a set of data \( (T^4; G, B) \) to be rational (as shown in [5]), but fails to be a sufficient condition as a whole. This is because Ref. [5] already exploited the conditions 1–3(a) and found that the \( \mathcal{N} = (1, 1) \) SCFT is not rational in the cases of (A')–(3.3) and (B, C)–(3.10;–); we have now seen in the previous section that the properties 3(b), 4, 5(strong), 6 (and 7) follow from the conditions 1–3(a), and hence are satisfied by those cases, although the SCFTs are not rational.

We claim here, and will prove in this section 4, that one can add one more condition to the conditions 1–6 and turn them into a set of necessary and sufficient conditions on a set of data \( (T^4; G, B) \) for the corresponding \( \mathcal{N} = (1, 1) \) SCFT to be rational. There are two versions in doing so. One is to add to the conditions 1–6 with 5(weak);

8. (weak) there exists a Hodge isomorphism \( \phi^* : H^1(T^4; \mathbb{Q}) \to H^1(T^4_\circ; \mathbb{Q}) \) for some geometric SYZ-mirror satisfying the conditions 5(weak), 3 and 6 such that
\[ \phi^*|_{\Gamma_{\mathbb{Q}}} = \text{id}_{\Gamma_{\mathbb{Q}}}. \] (4.1)

The other is to add to the conditions 1–6 with 5(strong);

8. (strong) there exists a Hodge isomorphism \( \phi^* : H^1(T^4; \mathbb{Q}) \to H^1(T^4_\circ; \mathbb{Q}) \) satisfying (4.1) for any geometric SYZ-mirror.

Obviously a set of data satisfying the condition 8(strong) satisfies the condition 8(weak); a set of data not satisfying the condition 8(weak) does not satisfy the condition 8(strong) either.

In the rest of this section 4, we will see that the condition 8(weak) is not satisfied by a set of data in the cases (A')–(3.3) and (B, C)–(3.10;–). It will also turn out that the condition 8(strong) is satisfied in the cases (A')–(3.2), (B, C)–(3.10;+) and (A). This means that the set of conditions 1, 2, 3(a) and 8 (either weak or strong is fine) on a set of data \( (T^4; G, B) \) is necessary and sufficient for the corresponding \( \mathcal{N} = (1, 1) \) SCFT to be rational.
4.1 The Case (A’)

First, let us show that the condition 8(weak) is not satisfied by a set of data \((T^4; G, B; I)\) in the case \((A')-\text{(3.3)}\); we do so in a proof by contradiction.

Assume that a set of data \((T^4; G, B; I)\) in the case \((A')-\text{(3.3)}\) satisfies the condition 8(weak); let \((\Gamma_f, \Gamma_b)\) be the choice for the geometric SYZ-mirror and \(\phi^* : H^1(T^4; \mathcal{Q}) \rightarrow H^1(T^4_0, \mathcal{Q})\) be the Hodge isomorphism in the condition. We will write down explicitly (i) what is implied by the condition \(\phi^*|_{\Gamma_{kq}} = \text{id}_{\Gamma_{kq}}\) and (ii) what is implied by \(\phi^*\) being a Hodge isomorphism, and then show that there cannot be such a \(\phi^*\) satisfying both (i) and (ii).

(i) The condition \(\phi^*|_{\Gamma_{kq}} = \text{id}_{\Gamma_{kq}}\) can be expressed in the form of

\[
\phi^* (c' \ d' \ \hat{e} \ \hat{f}) = (c \ d \ \hat{e} \ \hat{f}) \left( \begin{array}{cc} r_{kl} & 0 \\ 0 & 1 \end{array} \right),
\]

by using the basis \(\{c', d', \hat{e}, \hat{f}\}\) of \(H^1(T^4; \mathcal{Q})\) and the basis \(\{c, d, \hat{e}, \hat{f}\}\) of \(H^1(T^4_0; \mathcal{Q})\) explained in \((3.22)\) and \((3.24)\), respectively. Here, \((r_{kl})\) is a \(4 \times 2\) \(\mathcal{Q}\)-valued matrix.

Now, the condition \((4.2)\) implies that the holomorphic basis

\[
((c_1^2 + c_2^2)dz^1 \ (c_3^2 + c_4^2)dz^2) = (c' \ d' \ \hat{e} \ \hat{f}) \left( \begin{array}{cc} c_1 + c_2 \sqrt{p_1} \\ c_3 + c_4 \sqrt{p_2} \\ -c_2 + c_1 \sqrt{p_1} \\ -c_4 + c_3 \sqrt{p_2} \end{array} \right)
\]

should be mapped by \(\phi^*\) into

\[
\phi^*_{c}((c_1^2 + c_2^2)dz^1 \ (c_3^2 + c_4^2)dz^2) = (c \ d \ \hat{e} \ \hat{f}) \left( \begin{array}{cc} r_{kl} & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} c_1 + c_2 \sqrt{p_1} \\ c_3 + c_4 \sqrt{p_2} \\ -c_2 + c_1 \sqrt{p_1} \\ -c_4 + c_3 \sqrt{p_2} \end{array} \right),
\]

\[(4.4)\]

(ii) Next, we claim that \(\phi^*\)'s being a Hodge isomorphism is equivalent to the presence of \(\theta_1 \in \mathcal{Q}(\sqrt{p_1})^\times\) and \(\theta_2 \in \mathcal{Q}(\sqrt{p_2})^\times\) such that

\[
\phi^*_{c}((c_1^2 + c_2^2)dz^1 \ (c_3^2 + c_4^2)dz^2) = (dz_5^1 \ dz_5^2) \left( \begin{array}{cc} \theta_1 \\ \theta_2 \end{array} \right);
\]

\[(4.5)\]

here, \(dz_5^1\) and \(dz_5^2\) are holomorphic 1-forms on \(E_1^0\) and \(E_2^0\) (pulled back to \(T^4_0\)) respectively, normalized so that they return a rational value for some rational element in \(H_1(E_1^0; \mathcal{Q})\) and \(H_1(E_2^0; \mathcal{Q})\), respectively. This claim is based on the following two observations; first, there is an isogeny \(\psi : T^4 \sim E_1 \times E_2 \leftrightharpoons E_2 \times E_1 \sim T^4_0\) in the case \((A')-\text{(3.3)}\), as we have seen in section \(3.1\); this means that the map \(\psi^*\) has the form of \((4.5)\) with \(\theta_1, \theta_2\) replaced by some elements \(\theta_1^\psi \in \mathcal{Q}(\sqrt{p_1})^\times\) and \(\theta_2^\psi \in \mathcal{Q}(\sqrt{p_2})^\times\) for \(\psi\). Second, any Hodge isomorphism \(\phi^* : H^1(T^4; \mathcal{Q}) \rightarrow\)

\[\text{\footnotesize\underline{15}}\] The normalization factor \((c_1^2 + c_2^2)\) and \((c_3^2 + c_4^2)\) are inserted on the left hand side, only to make sure that \(dz^1\) and \(dz^2\) have the same normalization as those in \([5]\). They are not essential at all.
\(H^1(T^4_0; \mathbb{Q})\) can differ from \(\psi^*\) only by an additional action by \([\text{End}(H^1(T^4_0; \mathbb{Q}))^{\text{Hdg}}]^\times \cong \mathbb{Q}(\sqrt{p_2})^\times \oplus \mathbb{Q}(\sqrt{p_1})^\times\).

Here, the holomorphic 1-forms on the mirror, \(dz_1^0\) and \(dz_2^0\), must be in the form of

\[
(\lambda_1' d z^2_0 \quad \lambda_2' d z^1_0) = \begin{pmatrix} d & f & c & \hat{c} \end{pmatrix} \begin{pmatrix} 1 \\ \rho_1' \\ \rho_2' \end{pmatrix}
\]  

(4.6)

**Footnote 17:** Some \(\lambda_1', \rho_1' \in \mathbb{Q}(\sqrt{p_1})^\times\) and \(\lambda_2', \rho_2' \in \mathbb{Q}(\sqrt{p_2})^\times\). Therefore, (4.5) becomes

\[
\phi^*_c ((c_1^2 + c_2^2)dz^1 \quad (c_3^2 + c_4^2)dz^2) = \begin{pmatrix} c_1 + c_2 \sqrt{p_1} \\ -c_2 + c_1 \sqrt{p_2} \\ -c_1 + c_3 \sqrt{p_2} \\ -c_2 + c_3 \sqrt{p_2} \end{pmatrix} = \begin{pmatrix} 1 \\ \rho_1' \\ \rho_2' \end{pmatrix} \begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix},
\]

(4.7)

where \(\theta_1' := \theta_1/\lambda_1' \in \mathbb{Q}(\sqrt{p_1})^\times\) and \(\theta_2' := \theta_2/\lambda_2' \in \mathbb{Q}(\sqrt{p_2})^\times\).

Now that the properties (i) and (ii) in the condition 8 have been paraphrased as (4.4) and (4.7) respectively, let us see that there is no common solution \((r_{kl}, \theta_1')\) to (4.4) and (4.7) indeed, or equivalently, no solution to

\[
\begin{pmatrix} r_{kl} & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} c_1 + c_2 \sqrt{p_1} \\ -c_2 + c_1 \sqrt{p_2} \\ -c_1 + c_3 \sqrt{p_2} \\ -c_2 + c_3 \sqrt{p_2} \end{pmatrix} = \begin{pmatrix} 1 \\ \rho_1' \\ \rho_2' \end{pmatrix} \begin{pmatrix} \theta_1' \\ \theta_2' \end{pmatrix},
\]

(4.8)

for any \((c_{1-4}, \rho_{1,2}')\). To see this, note that the both sides of (4.8) are \(4 \times 2\) matrices after multiplication, and look at the \((3,1)\) entry:

\[
r_{31}(c_1 + c_2 \sqrt{p_1}) + (-c_2 + c_1 \sqrt{p_1}) = 0.
\]

(4.9)

Such \(r_{31} \in \mathbb{Q}\) does not exist because of \((c_1, c_2) \neq (0, 0)\) (see (3.20)). Similarly, we can also see that there is no way this equality holds for the \((4,2)\) entry.

The remaining task is to prove that the condition 8(strong) is satisfied in the case \((\Lambda') - (3.2)\). That is to construct a Hodge isomorphism \(\phi^* : H^1(T^4; \mathbb{Q}) \to H^1(T^4_0; \mathbb{Q})\) satisfying \(\phi^*|_{\Gamma_{\lambda^\prime}} = \text{id}_{\Gamma_{\lambda^\prime}}\) for each one of geometric SYZ-mirrors for a set of data \((T^4; G, B; I)\) in the case of \((\Lambda') - (3.2)\).

To do so, think of a geometric SYZ-mirror for \((\Gamma_f, \Gamma_b)\), and derive a condition for such a Hodge isomorphism \(\phi^*\), first. The derivation goes parallel to that for (4.8), except for the

\footnote{Note that \(\phi^* = (\phi^* \circ (\psi^*)^{-1}) \circ \psi^*\), and that \((\phi^* \circ (\psi^*)^{-1}) \in [\text{End}(H^1(T^4_0; \mathbb{Q}))^{\text{Hdg}}]^\times\).}

\footnote{Just as a reminder, these \(\lambda_1', \rho_1', \lambda_2', \rho_2'\) depend on the isogeny \(T^4_{\Gamma} \to E_1^0 \times E_2^0\), whose pullback is used to identify \(H^1(T^4_0; \mathbb{Q})\) and \(H^1(E^0_1; \mathbb{Q}) \oplus H^1(E^0_2; \mathbb{Q})\), and hence depend on \((\Gamma_f, \Gamma_b)\) for the geometric SYZ-mirror we chose at the beginning of the proof.}
fact that \(E_1\) and \(E_2\) are isogenous to \(E_1^\circ\) and \(E_2^\circ\), respectively. Consequently we obtain

\[
\begin{pmatrix}
r_{kl} & 0 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
c_1 + c_2 \sqrt{p_1} \\
-c_2 + c_1 \sqrt{p_1} \\
-c_4 + c_3 \sqrt{p_2}
\end{pmatrix}
= \begin{pmatrix}
1 \\
\rho'_1 \\
\rho'_2
\end{pmatrix}
\begin{pmatrix}
\theta'_1 \\
\theta'_2
\end{pmatrix},
\]

where \(r_{kl} \in \mathbb{Q}\) and \(\theta'_1, \theta'_2 \in \mathbb{Q}(\sqrt{p_1})^\times\) are parameters for \(\phi^*\), while \(c_{1-4} \in \mathbb{Q}\) and \(\rho'_1, \rho'_2 \in \mathbb{Q}(\sqrt{p_1})\) depend on \((\Gamma_f, \Gamma_b)\) (see footnote 17).

Now, it is easy to see that a solution \((r_{kl}, \theta'_i)\) to (4.10) exists. The equation (4.10) at six out of the 4 \(\times\) 2 entries imposes

\[
\begin{align*}
r_{12} &= r_{21} = r_{32} = r_{41} = 0, \\
\theta'_1 &= r_{11}(c_1 + c_2 \sqrt{p_1}), \\
\theta'_2 &= r_{22}(c_3 + c_4 \sqrt{p_2}),
\end{align*}
\]

and the remaining variables \(r_{11}, r_{31}, r_{22}, r_{42} \in \mathbb{Q}\) are subject

\[
\begin{align*}
-c_2 + c_1 \sqrt{p_1} &= (\rho'_1 r_{11} - r_{31})(c_1 + c_2 \sqrt{p_1}), \\
-c_4 + c_3 \sqrt{p_2} &= (\rho'_2 r_{22} - r_{42})(c_3 + c_4 \sqrt{p_2})
\end{align*}
\]

because of (4.10) at the two remaining entries. A solution \((r_{11}, r_{31}, r_{22}, r_{42})\) is determined by elementary algebra in the imaginary quadratic fields \(\mathbb{Q}(\sqrt{p_1})\) and \(\mathbb{Q}(\sqrt{p_2})\). Moreover, \(r_{11}\) and \(r_{22}\) turn out to be non-zero, which means that \(\phi^*\) is invertible. Since the above argument holds for arbitrary choice of \((\Gamma_f, \Gamma_b)\), this completes the proof of the condition 8(strong) for the case \((A')-(3.2)\).

### 4.2 The Case \((B, C)\)

Let us show first that the condition 8(weak) is not satisfied by a set of data \((T^4; G, B; I)\) in the case \((B, C)-(3.10)\); we do so in a proof by contradiction. The outline of the proof is the same as in section 4.1; it just takes a little more time to do so.

Assume that a set of data \((T^4; G, B; I)\) in the case \((B, C)-(3.10)\) satisfies the condition 8(weak); let \((\Gamma_f, \Gamma_b)\) be the choice for the geometric SYZ-mirror, and \(\phi^*: H^1(T^4; \mathbb{Q}) \to H^1(T^4; \mathbb{Q})\) the Hodge isomorphism in the condition. We will write down explicitly (i) what is implied by the condition \(\phi^*|_{\Gamma_b^\circ} = id_{\Gamma_b^\circ}\) and (ii) what is implied by \(\phi^*\) being a Hodge isomorphism, and then show that there cannot be such \(\phi^*\) satisfying both (i) and (ii).

(i) The condition \(\phi^*|_{\Gamma_b^\circ} = id_{\Gamma_b^\circ}\) can be expressed in the form of

\[
\phi^* \left( \begin{pmatrix} c' & \hat{d}' & \hat{e} & \hat{f} \end{pmatrix} \right) = \left( \begin{pmatrix} c & d & e & f \end{pmatrix} \right) \left( r_{kl} \middle| 0 \right),
\]

by using the basis \(\{c', \hat{d}', \hat{e}, \hat{f}\}\) of \(H^1(T^4; \mathbb{Q})\) and the basis \(\{c, d, e, f\}\) of \(H^1(T^4; \mathbb{Q})\) explained in (3.42) and (3.44); here, \((r_{kl})\) is a 4 \(\times\) 2 \(\mathbb{Q}\)-valued matrix. We only deal with the cases with \((c_1, c_3) \neq (0, 0)\) here, as we did in section 3.2; the following discussion can be repeated for the case of \((c_2, c_4) \neq (0, 0)\) easily.
Now, the condition (4.15) implies that the holomorphic basis (cf. footnote 15)

\((c_1^2 - dc_3^2)dz^1 \quad (c_1^2 - dc_3^2)dz^2 = (c' \, \hat{d}' \, \hat{e} \, \hat{f}) \begin{pmatrix} c_1 & c_3 & c_2 & c_4 \\ dc_3 & c_1 & dc_4 & c_2 \\ 1 & 1 & \tau_{++}(y) & \tau_{++}(x) \\ \tau_{++}(xy) & \tau_{++}(xy) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \)

(4.16)

should be mapped by \(\phi^*\) into

\(\phi^*_C \left((c_1^2 - dc_3^2)dz^1 \quad (c_1^2 - dc_3^2)dz^2\right) = (c \quad d \quad \hat{e} \quad \hat{f}) \begin{pmatrix} r_{kl} \\ 1 \end{pmatrix} \begin{pmatrix} \tau_{++}(\Gamma) & \tau_{--}(\Gamma) \\ \tau_{++}(\Gamma)y & \tau_{--}(\Gamma)y \\ \tau_{++}(x) & \tau_{--}(x) \\ \tau_{++}(xy) & \tau_{--}(xy) \end{pmatrix} \),

(4.17)

where \(\Gamma := c_1 + c_3y + c_2x + c_4xy \leq K\).

(ii) Next, we will translate the condition that \(\phi^*\) is a Hodge isomorphism. The images of the holomorphic basis elements

\(\phi^*_C \left((c_1^2 - dc_3^2)dz^1 \quad (c_1^2 - dc_3^2)dz^2\right) = (dz_1^1 \quad dz_2^2) \)

(4.18)

are eigenvectors of the action of the endomorphism field \(\text{End}(H^1(T^4; \mathbb{Q}))^{\text{Hdg}}\), and generate the Hodge (1,0) component of \(H^1(T^4; \mathbb{C})\); when the endomorphism fields are identified through \(\text{Ad}_{\phi^*} : K \cong \text{End}(H^1(T^4; \mathbb{Q}))^{\text{Hdg}} \ni \xi \mapsto \phi^* \circ \xi \circ (\phi^*)^{-1} \in \text{End}(H^1(T^4; \mathbb{Q}))^{\text{Hdg}}\), the eigenvalues of the endomorphisms on \(dz_1^1\) and \(dz_2^2\) are given by the embeddings \(\tau_{++}\) and \(\tau_{--}\), respectively. We note also that both \(dz_1^1\) and \(dz_2^2\) are normalized so that they return a rational value for some rational elements in \(H_1(T^4; \mathbb{Q})\).

On the other hand, there is an easy way to describe how an \(\text{End}(H^1(T^4; \mathbb{Q}))^{\text{Hdg}}\)-diagonal basis \(\{dz_1', dz_2'\}\) of \(H^{1,0}(T^4; \mathbb{C})\) is related to the rational basis \(\{c, d, \hat{e}, \hat{f}\}\) of \(H^1(T^4; \mathbb{Q})\) (as reviewed and then used heavily in [5]). This is done, in principle, by working out the complex structure of the mirror \(T^4_o\) (cf. (A.12)). Instead, we will use an easy way available for complex tori to identify the (1,0) component with respect to the vertical Hodge structure on \(W_1/W_3\)—like we did in (3.51)—and translate the information to the mirror side by using the map of D-brane charges (cohomology groups). This latter strategy enables us to save time a little more, because we have done the latter calculation already in a special case in [3] §5.3.1] (see footnote 19).

As a first step, let us identify the subspace \(W_1' \subset H^*(T^4; \mathbb{Q})\) and introduce a rational basis of \(W_1/W_3\). The basis elements \(\{c, d, \hat{e}, \hat{f}\}\) of \(H^1(T^4; \mathbb{Q}) \cong W_1/W_3\) on the mirror side are identified with the followings by the mirror map of the D-brane charges:

\(g^*(c) \in s' \hat{d}' + W_3', \) \hspace{1cm} (4.19)
\(g^*(d) \in -s' \hat{c}' + W_3', \) \hspace{1cm} (4.20)
\(g^*(\hat{e}) \in s'' \hat{c}' \hat{d}' \hat{e} + W_3', \) \hspace{1cm} (4.21)
\(g^*(\hat{f}) \in s'' \hat{c}' \hat{d}' \hat{f} + W_3', \) \hspace{1cm} (4.22)

where \(s := [\Gamma_f : (Zc \oplus Zd)] \in \mathbb{Q}, \ s' := s/(c_1^2 - dc_3^2)\) and \(s'' := s/(c_1^2 - dc_3^2)^2\).
The next step is to identify the Hodge (1,0) component of \((W_{v}^{1}/W_{v}^{3}) \otimes \mathbb{C}\) in terms of the rational basis \(\{c', d', c'd\hat{e}, \hat{c}' d\hat{f}\}\) of \(W_{v}^{1}/W_{v}^{3}\). The (1,0) component is given by \(U \wedge g^*(H^{1}(T_{0}^{4}; \mathbb{C})/(\Gamma_{bQ}^{v} \otimes \mathbb{C}))\), which is generated by
\[
Uc' = \hat{c}' + Z_{1}e_{1}c' + Z_{2}e_{2}c' \quad \text{and} \quad Ud' = \hat{d}' + Z_{1}e_{1}d' + Z_{2}e_{2}d'.
\] (4.23)

It is not difficult\(^\text{19}\) to reorganize the basis elements \(\{Uc', Ud'\}\) to find eigenvectors of the endomorphisms of the vertical Hodge structure on \(W_{v}^{1}/W_{v}^{3}\).

\[
(g^*(dz_{0}^{V}), g^*(dz_{0}^{2})) := (Uc' - Ud') \begin{pmatrix} 1 & 1 \\ \mp \sqrt{d} & \pm \sqrt{d} \end{pmatrix}
\] (4.25)

\[
= (\hat{c}' \hat{d}' - e_{1}d' - e_{2}d') \begin{pmatrix} 1 & 1 \\ \tau_{++}(\mp y) & \tau_{++}(\mp y) \\ \tau_{+-}(\mp y) & \tau_{+-}(\mp y) \end{pmatrix}
\] (4.26)

\[
= \left( \begin{array}{c} c' \\ d' \end{array} \right) = 1 \\ \frac{c' \hat{e}}{(c_{1} - dc_{3})} + \frac{c' \hat{f}}{(c_{1} - dc_{3})} \begin{pmatrix} 1 & 1 \\ c_{1} & c_{3} \\ dc_{3} & c_{1} \end{pmatrix} \begin{pmatrix} \tau_{++}(\mp y) & \tau_{++}(\mp y) \\ \tau_{+-}(\mp y) & \tau_{+-}(\mp y) \end{pmatrix}
\] (4.27)

where \(\Xi_{\pm} \in K\) was introduced in (3.53); it is obvious from this expression (cf. [5, Lemma A.11]) that they are eigenstates of \(K \cong \text{End}(W_{v}^{1}/W_{v}^{3})^{Hdg}\). Although we are working over the case \((B, C) = (3.10, -)\) at this moment, we will have to repeat the same computation for the case \((B, C) = (3.10, +)\) later in this section 4.2. So, we have done the computation for both cases above; the sign choice below (in \(\pm\) and \(+\) in \(\mp\)) is for the case \((3.10, -)\) and the sign choice above for the case \((3.10, +)\).

The final step is to compute the endomorphism-eigenstates \(\{dz_{0}^{V}, dz_{0}^{2}\}\) of \(H^{1}(T_{0}^{4}; \mathbb{Q}) \otimes \mathbb{C}\). Using (4.19)–(4.22), we obtain

\[
(dz_{0}^{V}, dz_{0}^{2}) = (-\tilde{s}'d \quad \tilde{s}'c \quad \tilde{s}'e \quad \tilde{s}'f) \begin{pmatrix} 1 & 1 \\ c_{1} & c_{3} \\ dc_{3} & c_{1} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \tau_{++}(\mp y) & \tau_{++}(\mp y) \\ \tau_{+-}(\mp y) & \tau_{+-}(\mp y) \end{pmatrix}
\] (4.28)

\[
= (c \quad d \quad \hat{e} \quad \hat{f}) \begin{pmatrix} \frac{-\tilde{s}'}{\tau_{++}(\tilde{s}')y} & \frac{\tau_{+-}(\tilde{s}')y}{-\tilde{s}'} \\ -\tilde{s}' & -\tilde{s}' \end{pmatrix}
\] (4.29)

where \(\tilde{s}' := s'^{-1}\), \(\tilde{s} := s^{-1} \in \mathbb{Q}^{x}\).

The diagonal basis \(\{dz_{0}^{1}, dz_{0}^{2}\}\) in (4.18) and another basis \(\{dz_{0}^{V}, dz_{0}^{2}\}\) here are not guaranteed to be identical, but \(dz_{0}^{1}\) [resp. \(dz_{0}^{2}\)] should be a complex multiple of \(dz_{0}^{V}\) [resp. \(dz_{0}^{2}\)].

\(^{18}\)We use the same notation in this article for both an element of \(W_{v}^{1}\) and its equivalence class in \(W_{v}^{1}/W_{v}^{3}\).

\(^{19}\)When \((c_{1}, c_{2}, c_{3}, c_{4}) = (1, 0, 0, 0)\), eq. (4.26) coincides with [5, eq. (3.38)].

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Furthermore, the proportionality constant should be found within \( \tau_{++}(K) \) [resp. \( \tau_{--}(K) \)], which follows from the fact that both \( dz^1 \) and \( dz^1' \) return (not necessarily identical) rational numbers to (not necessarily identical) rational elements in \( H_1(T^4; \mathbb{Q}) \). So, to conclude, the condition (ii) that \( \phi^* \) is a Hodge isomorphism is translated to the existence of elements \( \theta_+, \theta_- \in K^\times \) so that

\[
\begin{align*}
\phi^*_C ((c_1^2 - dc_3^2)dz^1 & \quad (c_1^2 - dc_3^2)dz^2) \\
= (c & \quad d \hat f) \left( \begin{array}{cc}
\tau_{++}(\mp \tilde{s}'y) & \tau_{--}(\mp \tilde{s}'y) \\
\tau_{++}(\tilde{s}(c_1 \pm c_3 y)\Xi_\pm) & \tau_{--}(\tilde{s}(c_1 \pm c_3 y)\Xi_\pm)
\end{array} \right) \left( \begin{array}{c}
\tau_{++}(\theta_+) \\
\tau_{--}(\theta_-)
\end{array} \right).
\end{align*}
\]

Now that the properties (i) and (ii) in the condition 8 have been paraphrased as (4.17) and (4.31), respectively, let us see that there is no common solution \((r_{kl}, \theta_+, \theta_-)\) to (4.17) and (4.31) for any \((c_1, c_3, \tilde{s}, s')\); this then implies that there is no choice \((\Gamma_f, \Gamma_b)\) for a geometric SYZ-mirror where the condition 8(weak) holds true in the case of (B,C)–(3.10).

Indeed, the compatibility condition on \((r_{kl}, \theta_+, \theta_-)\) is

\[
\begin{pmatrix}
(r_{kl} & 0) \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\tau_{++}(\Gamma) & \tau_{--}(\Gamma) \\
\tau_{++}(\Gamma y) & \tau_{--}(\Gamma y) \\
\tau_{++}(x) & \tau_{--}(x) \\
\tau_{++}(xy) & \tau_{--}(xy)
\end{pmatrix}
\begin{pmatrix}
\tau_{++}(\mp \tilde{s}'y) & \tau_{--}(\mp \tilde{s}'y) \\
\tau_{++}(s(c_1 \pm c_3 y)\Xi_\pm) & \tau_{--}(s(c_1 \pm c_3 y)\Xi_\pm)
\end{pmatrix}
\begin{pmatrix}
\tau_{++}(\theta_+) \\
\tau_{--}(\theta_-)
\end{pmatrix}
\]

when \((c_1, c_3) \neq 0\). The two elements \(\theta_+, \theta_- \in K^\times\) have to be identical for the relations in the second row to hold. Now, the \(4 \times 2\) relations among algebraic numbers in (4.32) are regarded as four relations among the elements in the CM field \(K\). To see that there is no solution \((r_{kl}, \theta)\) to (4.32), it suffices to focus on the lower two relations:

\[
\begin{align*}
\frac{r_{31}\Gamma + r_{32}\Gamma y + x = \tilde{s}(c_1 \pm c_3 y)\Xi_\pm \theta}{r_{41}\Gamma + r_{42}\Gamma y + xy = \tilde{s}(dc_3 \pm c_1 y)\Xi_\pm \theta}.
\end{align*}
\]

Comparing (4.33) \times \((\pm y)\) and (4.34), we can eliminate \(\theta\) and obtain

\[
[\pm dr_{32} - r_{41} + (\pm r_{31} - r_{42})y] \Gamma = a xy,
\]

\[
a = \begin{cases} 0 & \text{in the case (B,C)–(3.10)+}, \\ 2 & \text{in the case (B,C)–(3.10)--}. \end{cases}
\]

Recalling that \(\Gamma = c_1 + c_3 y + c_2 x + c_4 xy\) and \((c_1, c_3) \neq (0, 0)\), some easy algebra in the number field \(K\) shows that this equation (4.35) \(a = 2\) for the case (B,C)–(3.10)-- does not have a solution \((r_{31}, r_{32}, r_{41}, r_{42})\).
The remaining task is to prove that the condition 8(strong) is satisfied in the case (B, C)–(3.10+). That is to construct a Hodge isomorphism \( \phi^* : H^1(T_4^4; \mathbb{Q}) \to H^1(T_0^4; \mathbb{Q}) \) satisfying \( \phi^*|_{\Gamma_{00}} = \text{id}|_{\Gamma_{00}} \) for each one of geometric SYZ-mirrors for a set of data \((T^4; G, B; I)\) in the case (B, C)–(3.10+).

To do so, we will find a solution \((r_{kl}, \theta)\) to the equation (4.32+) for a given \((c_{1-4}, \tilde{s}, \tilde{s}')\). Just as a reminder, \((r_{kl}, \theta)\) are parameters for \(\phi^*\), whereas \((c_{1-4}, \tilde{s}, \tilde{s}')\) are determined by a given \((\Gamma_f, \Gamma_b)\) for a geometric SYZ-mirror.

In fact, we can solve the equation (4.32+) as follows. As already discussed above, the lower two rows of the equation (4.32+) lead to equations (4.33+), (4.34+), and (4.35a = 0). The last one (4.35a = 0) is equivalent to

\[
\begin{align*}
\tilde{s}'r_{31} + \tilde{s}'r_{32}y + r_{21}\Xi'_+ + r_{22}\Xi'_+y &= -\tilde{s}'x\Gamma^{-1},
\end{align*}
\]

where \(\Xi'_+ := \tilde{s}(c_1 + c_3y)\Xi_+\). Since 1, \(y, \Xi'_+, \Xi'_+y \in K\) are linearly independent over \(\mathbb{Q}\), this equation (4.40) determines \((r_{31}, r_{32}, r_{21}, r_{22})\) uniquely.

It is easy to confirm that the equation (4.32+) is satisfied whenever the parameters \((r_{kl}, \theta)\) are subject to the relations:

\[
\begin{align*}
(4.40) & : \tilde{s}'r_{31} + \tilde{s}'r_{32}y + r_{21}\Xi'_+ + r_{22}\Xi'_+y = -\tilde{s}'x\Gamma^{-1}, \\
(4.37) & : r_{42} = r_{31}, \quad r_{41} = dr_{32}, \\
(4.38) & : r_{11} = dr_{22}, \quad r_{12} = r_{21}, \\
(4.39) & : \tilde{s}'\theta = -(r_{21}\Gamma + r_{22}\Gamma y).
\end{align*}
\]

So, indeed, there is a solution \((r_{kl}, \theta)\) common to both (i) and (ii). Furthermore, the Hodge morphism \(\phi^*\) specified by this \((r_{kl}, \theta)\) is an isomorphism; if \(\theta\) were zero (and hence \(\phi^*\) were not invertible), then \(\Gamma, \Gamma y, \) and \(x\) would not be linear independent over \(\mathbb{Q}\), according to (4.33). Since the above argument holds for arbitrary choice of a geometric SYZ-mirror, this completes the proof of the condition 8(strong) for the case (B,C)–(3.10+).

### 4.3 The Case (A)

We have already dealt with the cases (A’) and (B, C) earlier in this section, but there is one more case we should consider: the case (A). As reviewed briefly at the beginning of section 3 a set of data \((T^4; G, B; I)\) satisfying 1–3(a) is classified into one of the four cases (A), (A’),
(B and C) based on its complex structure \((T^4; I)\), and such a set of data in the case (A) always gives rise to a rational \(\mathcal{N} = (1, 1)\) SCFT. The case (A) is when the abelian surface \((T^4, I)\) is isogenous to the product \(E \times E\) of CM elliptic curves (cf [5] Lemma 2.22).

Therefore, to affirm that the set of conditions 1, 2, 3(a) and 8 (either weak or strong) on a set of data \((T^4; G, B; I)\) is necessary and sufficient for the corresponding \(\mathcal{N} = (1, 1)\) SCFT to be rational, we also have to prove that the condition 8(strong) is always satisfied in the case (A). That is to show the existence of a Hodge isomorphism \(\phi^*: H^1(T^4; \mathbb{Q}) \to H^1(T^4_0; \mathbb{Q})\) satisfying \(\phi^*|_{\Gamma_{fQ}^\vee} = \text{id}|_{\Gamma_{fQ}^\vee}\) for any choice of \((\Gamma_f, \Gamma_b)\) for a geometric SYZ-mirror in the case (A).

To get started, let us have a few words about rational and holomorphic bases of \(H^1(T^4)\) and \(H^1(T^4_0)\) to be used in the analysis. Let \(\{c, d\}\) be a basis of \(\Gamma_{fQ}\), and \(\{e, f\}\) that of \(\Gamma_{bQ}\); the subspaces \(\Gamma_{fQ}^\vee\) and \(\Gamma_{bQ}^\vee\) of \(H^1(T^4_0; \mathbb{Q})\) are generated by \(\{\hat{c}, \hat{d}\}\) and \(\{\hat{e}, \hat{f}\}\), respectively; here, \(\{\hat{c}, \hat{d}, \hat{e}, \hat{f}\}\) is the dual basis of \(\{c, d, e, f\}\).

An easiest basis of \(H^{1,0}(T^4; \mathbb{C})\) over \(\mathbb{C}\) one thinks of is obtained by choosing a basis of \(H^{1,0}(E; \mathbb{C})\) of the two CM elliptic curves, \(dz^1\) and \(dz^2\), and pull them back to \(H^{1,0}(T^4; \mathbb{C})\). When \(dz^1\) and \(dz^2\) are normalized so that they return rational values to a rational cycle of the two \(E\)'s, their pullbacks on \(T^4\)—denoted by the same \(dz^1\) and \(dz^2\)—are related to the rational basis by

\[
(dz^1\ dz^2) = (\hat{c} \ \hat{d} \ \hat{e} \ \hat{f}) \begin{pmatrix}
\lambda_{11} & \lambda_{12} \\
\vdots & \vdots \\
\lambda_{41} & \lambda_{42}
\end{pmatrix},
\]

with the coefficients \(\lambda_{ij}\) in the imaginary quadratic field \(\mathbb{Q}(\sqrt{p})\) of the complex multiplication of \(E\).

We may change the holomorphic basis and rational basis a little bit so that the analysis later in this section [4.3] is easier. Note first that the upper 2 \(\times\) 2 block of the coefficient matrix in (4.41) is invertible. This is because the holomorphic \(n\)-form on the \(T^n\) fiber should have a non-zero period in the SYZ mirror correspondence. So, one may change the basis \(\{dz^1, dz^2\}\) by a \(\text{GL}_2(\mathbb{Q}(\sqrt{p}))\) transform so that the upper 2 \(\times\) 2 block in (4.41) is the identity matrix. Furthermore, it is possible to rearrage the rational basis of \(\Gamma_{fQ}^\vee\), denoted by \(\{\hat{c}', \hat{f}'\}\) now, so that the coefficient \(\lambda_{31}\) becomes rational. To summarize, there are a rational basis \(\{\hat{c}, \hat{d}\}\) of \(\Gamma_{fQ}^\vee\), \(\{\hat{e}', \hat{f}'\}\) of \(\Gamma_{bQ}^\vee\), and a holomorphic basis \(\{dz^1, dz^2\}\) of \(H^{1,0}(T^4; \mathbb{C})\) so that

\[
(dz^1\ dz^2) = (\hat{c} \ \hat{d} \ \hat{e}' \ \hat{f}') \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\hat{\lambda}'_{31} & \hat{\lambda}'_{32} \\
\hat{\lambda}'_{41} & \hat{\lambda}'_{42}
\end{pmatrix}, \quad \text{where} \quad \hat{\lambda}'_{31} \in \mathbb{Q},
\]

and \(\hat{\lambda}'_{32}, \hat{\lambda}'_{41}, \hat{\lambda}'_{42} \in \mathbb{Q}(\sqrt{p})\).

On the mirror side, we can regard \(\{c, d, e', f'\}\) as a rational basis of \(H^1(T^4_0; \mathbb{Q})\) (see [3.22] and footnote[4]). Since a set of data \((T^4; G, B; I)\) satisfying 1–3(a) in the case (A) always gives rise to a rational \(\mathcal{N} = (1, 1)\) SCFT, we can freely use the properties 1–7; in particular, \(T^4_{bQ}\) is isogenous to \(T^4_I\) (see [4 Prop. 3.10]), and hence to \(E \times E\). Repeating the same argument as
in the case of $T^4$, one can find a basis $\{d\bar{z}^1, d\bar{z}^2\}$ of $H^{1,0}(T^4_o; \mathbb{C})$ so that

$$(d\bar{z}^1_o \ d\bar{z}^2_o) = (c \ d \ \check{c}' \ \check{f}') \begin{pmatrix} 1 & 0 \\ \rho'_{31} & \rho'_{32} \\ \rho'_{41} & \rho'_{42} \end{pmatrix}$$

(4.43)

for some coefficients $\rho'_{ij}$ in $\mathbb{Q}(\sqrt{p})$.

Having done a preparation, we start showing that the condition 8 (strong) is satisfied. As we have done earlier, we will translate (i) the condition (4.41) and (ii) the Hodge-ness condition of the isomorphism $\phi^*: H^1(T^4_o; \mathbb{Q}) \rightarrow H^1(T^4_o; \mathbb{Q})$, and then find a common solution.

(i) The condition $\phi^*|_{T^4_o} = \text{id}_{T^4_o}$ is equivalent to the existence of a $4 \times 2 \mathbb{Q}$-valued matrix $(r_{kl})$ such that

$$\phi^*_C(d\bar{z}^1 \ d\bar{z}^2) = (c \ d \ \check{c}' \ \check{f}') \begin{pmatrix} 0 & 0 \\ r_{kl} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \hat{\lambda}'_{31} & \hat{\lambda}'_{32} \\ \hat{\lambda}'_{41} & \hat{\lambda}'_{42} \end{pmatrix}.$$  

(4.44)

(ii) We will write down the condition implied by $\phi^*$ being a Hodge isomorphism. In the case (A), we have $\text{End}(H^1(T^4_o; \mathbb{Q}))^{\text{Hdg}} \cong M_2(\mathbb{Q}(\sqrt{p}))$, where $M_2(\mathbb{Q}(\sqrt{p}))$ is the $2 \times 2$ matrix algebra with the matrix entries in $\mathbb{Q}(\sqrt{p})$. Moreover, all the Hodge isomorphisms $H^1(T^4_o; \mathbb{Q}) \rightarrow H^1(T^4_o; \mathbb{Q})$ form a set that is one to one with $\text{End}(H^1(T^4_o; \mathbb{Q}))^{\text{Hdg}} \cong M_2(\mathbb{Q}(\sqrt{p}))^\times$. Therefore, the condition that $\phi^*$ is a Hodge isomorphism is equivalent to the existence of $(\theta_{ij}) \in M_2(\mathbb{Q}(\sqrt{p}))^\times$ such that

$$\phi^*_C(d\bar{z}^1 \ d\bar{z}^2) = (c \ d \ \check{c}' \ \check{f}') \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}.$$ 

(4.45)

Now that the properties (i) and (ii) in the condition 8 have been paraphrased as (4.44) and (4.45), respectively, let us see that there exists a common solution $(r_{kl}, \theta_{ij})$ to (4.44) and (4.45) for a given $(\hat{\lambda}'_{ij}, \rho'_{ij})$. Just as a reminder, $(r_{kl}, \theta_{ij})$ are parameters of $\phi^*$, whereas $(\hat{\lambda}'_{ij}, \rho'_{ij})$ depend on a given $(\Gamma_f, \Gamma_b)$ for a geometric SYZ-mirror.

The compatibility condition of (4.44) and (4.45) is

$$
\begin{pmatrix} r_{kl} \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \hat{\lambda}'_{31} & \hat{\lambda}'_{32} \\ \hat{\lambda}'_{41} & \hat{\lambda}'_{42} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \rho'_{31} & \rho'_{32} \\ \rho'_{41} & \rho'_{42} \end{pmatrix} \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix}.
$$

(4.46)

---

20 Recall that $T^4_o$ is always isogenous to $T^4_o$ in the case (A). Let $\psi: T^4_o \rightarrow T^4_o$ be an isogeny. The one-to-one correspondence in the main text assigns $\phi \circ \psi^* \in [\text{End}(H^1(T^4_o; \mathbb{Q}))^{\text{Hdg}}]^\times$ to a Hodge isomorphism $\phi: H^1(T^4_o; \mathbb{Q}) \rightarrow H^1(T^4_o; \mathbb{Q})$. 

22
This equation (4.46) is read as the following relations in the $4 \times 2$ matrix entries:

$$\begin{align*}
\theta_{ij} &= r_{ij} \quad (i, j = 1, 2), \\
r_{31} + \tilde{\lambda}'_{31} &= \tilde{\rho}'_{31} r_{11} + \tilde{\rho}'_{32} r_{21}, \\
r_{41} + \tilde{\lambda}'_{41} &= \tilde{\rho}'_{41} r_{11} + \tilde{\rho}'_{42} r_{21}, \\
r_{32} + \tilde{\lambda}'_{32} &= \tilde{\rho}'_{31} r_{12} + \tilde{\rho}'_{32} r_{22}, \\
r_{42} + \tilde{\lambda}'_{42} &= \tilde{\rho}'_{41} r_{12} + \tilde{\rho}'_{42} r_{22}.
\end{align*}$$

(4.47)  

(4.48)  

(4.49)  

(4.50)  

(4.51)

The parameters $\theta_{ij}$ are fixed relatively to $r_{kl} \in \mathbb{Q}$. The parameters $(r_{11}, r_{21}, r_{31}, r_{41})$ are determined from the equations (4.48) and (4.49) as follows. From the imaginary parts of (4.48) and (4.49), we find

$$\begin{align*}
\tilde{\rho}'_{31} r_{11} + \tilde{\rho}'_{32} r_{21} &= \tilde{\lambda}'_{31} (= 0), \\
\tilde{\rho}'_{41} r_{11} + \tilde{\rho}'_{42} r_{21} &= \tilde{\lambda}'_{41},
\end{align*}$$

(4.52)

where we introduced a notation $\mu = \mu^{(1)} + \sqrt{p} \mu^{(2)}$ ($\mu^{(1)}, \mu^{(2)} \in \mathbb{Q}$) for $\mu \in \mathbb{Q}((\sqrt{p})$. This system of equations has a unique solution $(r_{11}, r_{21})$, because $\left(\tilde{\rho}'_{31} \quad \tilde{\rho}'_{32}\right)$ and $\left(\tilde{\rho}'_{41} \quad \tilde{\rho}'_{42}\right)$ are linearly independent; if not, it contradicts the fact that the $4 \times 4$ matrix $\left(\begin{array}{cc} 1 & 1 \\ \tilde{\rho}'_{ij} & \tilde{\rho}'_{ij}^{\text{c.c.}} \end{array}\right)$ is a change-of-basis matrix from the rational one to the complex one of $H^1(T^4; \mathbb{C})$, and hence invertible. Now that we have obtained $(r_{11}, r_{21})$, we can also determine the remaining $r_{31}$ and $r_{41}$ from (4.48) and (4.49), respectively. The parameters $(r_{12}, r_{22}, r_{32}, r_{42})$ are also determined in the same way from (4.50) and (4.51).

This solution $(r_{kl}, \theta_{ij})$ specifies a Hodge morphism $\phi^* : H^1(T^4; \mathbb{Q}) \to H^1(T^4; \mathbb{Q})$ satisfying $\phi^* |_{\gamma^{q}} = \text{id} |_{\gamma^{q}}$. To make sure that this $\phi^*$ is an isomorphism, we have to show that $\left(\begin{array}{cc} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{array}\right)$ or equivalently $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ for this solution is invertible. We will do this by contradiction as follows.

Assume that the $2 \times 2$ $\mathbb{Q}$-valued matrix $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ is not invertible. This implies that the right-hand sides of the equations (4.48) and (4.50) coincide up to multiplication by rational constant, and therefore we find that $\tilde{\lambda}'_{32}$ can be written as a $\mathbb{Q}$-linear combination of $r_{31}, r_{32}$ and $\tilde{\lambda}'_{31}$. Since $\tilde{\lambda}'_{31}$ is a rational number as in (4.42), both $\tilde{\lambda}'_{31}$ and $\tilde{\lambda}'_{32}$ are real valued, in particular. This contradicts the fact that the $4 \times 4$ matrix $\left(\begin{array}{cc} 1 & 1 \\ \tilde{\lambda}'_{ij} & \tilde{\lambda}'_{ij}^{\text{c.c.}} \end{array}\right)$ is invertible. This ends the proof of $\phi^*$ being an isomorphism.

Since the above argument holds for arbitrary choice of a geometric SYZ-mirror, this completes the proof of the condition 8(strong) for the case (A).
5 Towards Complete Characterization of Rational SCFTs

As is evident from what we wrote in Introduction, this section is still about 2d SCFTs that are interpreted as non-linear sigma models with the target spaces and Ricci-flat Kähler metrics. The section title has been trimmed down to fit within a single line.

Let us first write down an updated version of Thm. 5.8 of [5], verified for the case of \(M = T^4\), in a language applicable to any self-mirror family of Ricci-flat Kähler manifolds \(M\).

**Theorem** (for \(M = T^4\) / **Conjecture 1** (for self-mirror \(M\)):

Let \(M\) be a real \(2n\)-dimensional manifold which admits a Ricci-flat Kähler metric \(G\), and \(B\) a closed 2-form on \(M\). Suppose, further, that the family of such \((M;G,B)\) is self-mirror in that \(h^{p,q}(M) = h^{n-p,q}(M)\). The non-linear sigma model \(\mathcal{N} = (1,1)\) SCFT associated with the data \((M;G,B)\) is a rational SCFT if and only if the following conditions are satisfied:

1. there exists a polarizable complex structure \(I\) with which the metric \(G\) is compatible and \((M;G,I)\) becomes Kähler, and \(B^{(2,0)} = 0\);
2. the complexified Kähler parameter \((B+i\omega)\), where \(\omega(-,-) = 2^{-1}G(I-,\cdot)\) is the Kähler form, is in the algebraic part \((H^2(M;\mathbb{Q}) \cap H^{1,1}(M;\mathbb{R})) \otimes \mathbb{C}\);
3. there exists a geometric SYZ-mirror of the \(\mathcal{N} = (1,1)\) SCFT; there may be more than one; the data of such a mirror is denoted by \((W;G^o,B^o,I^o)\);
4. the rational Hodge structure on \(H^*(M;\mathbb{Q})\) is of CM-type;
5. (strong) for any one of the geometric SYZ-mirror SCFTs, there is a Hodge isomorphism \(\phi^* : H^*(M;\mathbb{Q}) \to H^*(W;\mathbb{Q})\) such that \(\phi^*\) is the identity map on the vector subspaces \(\pi_M^* : H^*(B;\mathbb{Q}) \hookrightarrow H^*(M;\mathbb{Q})\) and \(\pi_W^* : H^*(B;\mathbb{Q}) \hookrightarrow H^*(W;\mathbb{Q})\); here, \(\pi_M : M \to B\) and \(\pi_W : W \to B\) are the SYZ \(T^n\)-fibrations over a common base manifold \(B\) of real dimension \(n\).
6. (weak) there exists a geometric SYZ-mirror SCFT for which there is a Hodge isomorphism \(\phi^* : H^*(M;\mathbb{Q}) \to H^*(W;\mathbb{Q})\) such that \(\phi^*\) is the identity map on the vector subspaces \(\pi_M^* : H^*(B;\mathbb{Q}) \hookrightarrow H^*(M;\mathbb{Q})\) and \(\pi_W^* : H^*(B;\mathbb{Q}) \hookrightarrow H^*(W;\mathbb{Q})\); here, \(\pi_M : M \to B\) and \(\pi_W : W \to B\) are the SYZ \(T^n\)-fibrations over a common base manifold \(B\) of real dimension \(n\).

The set of conditions 1–8 (strong/weak) in the earlier sections is equivalent to the set of conditions 1–5 (strong/weak) here (Thm. 1) in the case \(M = T^4\) (as we argue shortly). In the case of a more general self-mirror \(M\), the set of conditions 1–5 here (Conj. 1) is meant as a proposal/guess for how to generalize.

Let us first confirm in the case \(M = T^4\) that the conditions 1–5 here are equivalent to the conditions 1–8 earlier (so that Thm. 1 holds for \(M = T^4\)). The conditions 2–6 of Thm. 5.8 of [5] are captured by the conditions 2–4 and the existence of \(\phi^*\) in the condition 5 here. It may appear that the condition 5 here demands more properties on \(\phi^*\) than the condition 8 in section [4] does; \(\phi^*\) in Thm. 1 is required to be defined on the whole \(H^*(M;\mathbb{Q})\) here than on \(H^1(M;\mathbb{Q})\), first of all, and to be the identity on the whole \(H^*(B;\mathbb{Q})\), not just on \(\Gamma_{\mathbb{Q}} \subset H^1(B;\mathbb{Q}) \subset H^*(B;\mathbb{Q})\), secondly. In fact, a Hodge isomorphism on \(H^1(M;\mathbb{Q})\) as in
the condition 8 can be extended by the wedge product (cup product) to a Hodge isomorphism on \( H^*(M; \mathbb{Q})_{M=T^4} \); the property that \( \phi^* \) is identity on the subspace \( H^*(B; \mathbb{Q})_{B=T^2} \) also follows automatically from that for \( H^1(B; \mathbb{Q})_{B=T^2} \).

The conditions are reorganized as above for the following reason. We have found in this article that the condition 8 in section 4 involves the SYZ torus fibration. Now that the SYZ torus fibration plays an essential role, it is more natural to use the horizontal Hodge structure on the mirror \( H^*(W; \mathbb{Q}) \) rather than the vertical Hodge structure on \( H^*(M; \mathbb{Q}) \). This is also why the conditions above are stated without referring to \( T_M \otimes \mathbb{Q} \subset A(M) \otimes \mathbb{Q} \) as in Thm. 5.8 of [3].

Conjecture 1 in the case of a self-mirror \( M \) is still stated in the language of a classical geometry, such as \((M; G, B; I)\). That is not particularly an issue when \( M \sim T^{2n} \) and \( M \sim K3 \); for a more general self-mirror family, however, treated data and conditions on them should be not in terms of classical geometry, but in terms of \( \mathcal{N} = (1, 1) \) SCFT. A set of data \((M; G, B)\) corresponds to a point (one SCFT) in the moduli space of \( \mathcal{N} = (1, 1) \) SCFTs of the target space \( M \), and the homology groups of \( H_*(M; \mathbb{Z}) \) to D-brane charges in the \( \mathcal{N} = (1, 1) \) SCFT. A choice of a complex structure \( I \) must be encoded in a choice of one weight-1 operator in the left moving sector and one weight-1 operator in the right moving sector that enhances the \( \mathcal{N} = 1 \) superconformal algebra in each sector to an \( \mathcal{N} = 2 \) superconformal algebra; this choice determines spectral flow operators in the \( \mathcal{N} = (1, 1) \) SCFT, and supersymmetry charges of the effective field theory after the compactification on “\((M; G, B)\)” The Hodge structures on \( H^*(M; \mathbb{Z}) \) appearing in the conditions of Conjecture 1 should be phrased in terms of the central charges of the effective field theory. The pull back \( \pi^*_M : H^*(B; \mathbb{Q}) \to H^*(M; \mathbb{Q}) \) of an SYZ torus fibration is captured by introducing a filtration in the space of D-brane charges. That will be an outline; details and gaps between the lines should still be filled here and there in writing down Conjecture 1 in the abstract language of 2d SCFT.

Conjecture 1 is certainly a natural extrapolation from the case \( M = T^4 \) to the cases of \( M \)'s that forms a self-mirror family. It makes sense, and there is no obvious evidence that it is wrong at this moment, at least in the eyes of the present authors. What if we think of a more general class of families of Ricci-flat Kähler manifolds \( M \) then?

When a family of Ricci-flat Kähler manifolds \( M \) is not self-mirror, the conditions 1–3 in Conj. 1 still make sense. The condition 5 should be modified, however, because examples of rational SCFTs are known for non-self-mirror \( M \)'s (e.g., Gepner constructions), and yet a Hodge isomorphism \( \phi^* \) in the condition 5 would imply that \( b_k(M) = b_k(W) \) for \( k = 0, 1, \cdots, 2n \). Relying on a random guess, we propose to modify the conditions as follows:

**Conjecture 2** (general): Let \( M \) be a real 2\( n \)-dimensional manifold which admits a Ricci-flat Kähler metric \( G \), and \( B \) a closed 2-form on \( M \). The non-linear sigma model \( \mathcal{N} = (1, 1) \)

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21 As a part of necessary conditions, the CM-ness and the Hodge isomorphism of the vertical Hodge structure on \( T_M^v \otimes \mathbb{Q} \) is still useful.

22 When we think of \( M \) with \( h^{2,0}(M) = 0 \) (for example, when \( M \) is a Calabi–Yau \( n \)-fold with \( n > 2 \)), the conditions 1 and 2 in Conj. 1 are automatically satisfied.
SCFT associated with the data \((M; G, B)\) is conjectured to be a rational SCFT if and only if the following conditions are satisfied:

The conditions 1–3 remain the same as in the case of self-mirror \(M\)'s.

4. (gen./strong) for any one of the SYZ mirrors, one can find a rational Hodge substructure \(V^{(k)}_M \subset H^k(M; \mathbb{Q})\) and also a rational Hodge substructure \(V^{(k)}_W \subset H^k(W; \mathbb{Q})\) for each \(0 \leq k \leq n\) that satisfy the following conditions (a–c); let \(\Pi^{(k)}_M\) and \(\Pi^{(k)}_W\) be the projection from \(H^k(M; \mathbb{Q})\) to \(V^{(k)}_M\) and \(H^k(W; \mathbb{Q})\) to \(V^{(k)}_W\), respectively; first, \(\Pi^{(k)}_M \circ \pi^{*}_M : H^k(B; \mathbb{Q}) \rightarrow V^{(k)}_M\) and \(\Pi^{(k)}_W \circ \pi^{*}_W : H^k(B; \mathbb{Q}) \rightarrow V^{(k)}_W\) should be injective (a); furthermore,

(b) the rational Hodge structure on \(V^{(k)}_M\) is of CM-type,

(c) there exists a Hodge morphism \(\phi^* : V^{(k)}_M \rightarrow V^{(k)}_W\) such that \(\phi^* \circ (\Pi^{(k)}_M \circ \pi^{*}_M) = \Pi^{(k)}_W \circ \pi^{*}_W\).

The conditions written above are only meant to be trial versions. One may think of changing the condition 4 from the one for arbitrary geometric SYZ mirrors as above, to the one for some geometric SYZ mirrors, when the condition is referred to as the condition 4(gen./weak). Besides this strong vs weak variation, there is still a wide variety in modifying the conditions; one might demand that the entire \(H^k(M; \mathbb{Q})\) is of CM-type instead of the condition 4(b) in Conj. 2, or demand a Hodge morphism \(\phi^* : H^k(M; \mathbb{Q}) \rightarrow H^k(W; \mathbb{Q})\) such that \(\phi^* \circ \pi^{*}_M = \pi^{*}_W\) on \(H^k(B; \mathbb{Q})\) instead of the condition 4(c) in Conj. 2. At this moment, there is not much evidence to pin down the right characterization conditions from study of SCFTs.

One may still run a few tests on formal aspects of Conjecture 2 to access its credibility. We do so in the rest of this article. The condition 4 in Conj. 2 as it stands passes the two tests below; reference to a substructure \(V^{(k)}_M\) is motivated in that context.

Let us take a moment before jumping into the first test, to prepare notations and cultivate intuitions on what the substructure \(V^{(k)}_M\) should be like. Suppose that

\[
H^k(M; \mathbb{Q}) \cong \bigoplus_{\alpha \in A^k(M)} \bigoplus_{a \in \alpha} V^{(k)}_a =: \bigoplus_{\alpha \in A^k(M)} V^{(k)}_{a_{\alpha}} \tag{5.1}
\]

is a decomposition into simple rational Hodge substructures\(^{23}\) simple rational Hodge substructures labeled by \(a \in A^k(M)\) are grouped together by Hodge isomorphisms among them\(^{24}\) Hodge-isomorphism classes are labeled by \(\alpha \in A^k(M)\). First, whenever \(V^{(k)}_a\) of one \(a \in \alpha \in A^k(M)\) is of CM-type, all the other \(V^{(k)}_{a'}\) with \(a' \in \alpha\) are of CM-type. So,

\(^{23}\)There exists a simple substructure decomposition, because the rational Hodge structure is polarizable (the condition 1).

\(^{24}\) Take the Fermat quintic Calabi–Yau threefold \((M; I)\) as an example (it extracts the complex structure information of the Gepner construction \((3^\otimes 5) / \mathbb{Z}_5\)), and focus on \(H^3(M; \mathbb{Q})\). There are two Hodge-isomorphism classes (i.e., \(#[A^3(M)] = 2\)), denoted by \(\alpha_3\) and \(\alpha_1\). The level-3 component \(V^{(3)}_{\alpha_3}\) is of 4-dimensions over \(\mathbb{Q}\), which contains the Hodge \((3,0)\) and \((0,3)\) components. The level-1 component \(V^{(3)}_{\alpha_1}\) consists of 50 copies of simple rational Hodge substructures, each of which is of 4-dimensions over \(\mathbb{Q}\) \(\Pi\). The authors are curious, in an SYZ \(T^3\)-fiberation of a Fermat quintic \(\pi_M : M \rightarrow B = S^3\), whether the Poincaré dual of the \(T^3\) fiber (which is the pull-back image of the generator of \(H^3(B; \mathbb{Q})\)) is entirely within the level-3 component \(V^{(3)}_{\alpha_3}\) or not.
whether a rational Hodge substructure of $H^k(M; \mathbb{Q})$ is of CM-type or not can be asked for individual Hodge-isomorphism classes in $\mathcal{A}^k(M)$. Secondly, one can see that the subspace $V^{(k)}_\alpha \subset H^k(M; \mathbb{Q})$ for $\alpha \in \mathcal{A}^k(M)$ does not depend on a choice of a simple substructure decomposition. To see this, suppose that there is another simple substructure decomposition, $H^k(M; \mathbb{Q}) \cong \oplus_{\beta \in \mathcal{A'}} \oplus_{b \in \beta} U_b^{(k)}$; the set of Hodge-isomorphism classes $\mathcal{A}'$ should be the same as $\mathcal{A}^k(M)$, and the Hodge isomorphism between $\oplus_\alpha \oplus_{a \in \alpha} V_a^{(k)}$ and $\oplus_\beta \oplus_{b \in \beta} U_b^{(k)}$ should be block-diagonal with respect to $\alpha, \beta \in \mathcal{A}^k(M)$. The subspace $\oplus_{b \in \alpha} U_b^{(k)} \subset H^k(M; \mathbb{Q})$ is therefore identical to $V^{(k)}_\alpha$. As a candidate of a rational Hodge substructure of $H^k(M; \mathbb{Q})$ in the condition 4 of Conj. 2, it is enough to think of the form $\oplus_{\alpha \in \mathcal{A}^k(M)_{\text{sub}}} V^{(k)}_\alpha$, with the Hodge-isomorphism classes running over some subset $\mathcal{A}^k(M)_{\text{sub}}$ of $\mathcal{A}^k(M)$.

There is a subset $\mathcal{A}^k_B(M) \subset \mathcal{A}^k(M)$ characterized as the set of those where the image of $p^{(k)}_\alpha \circ \pi_M^* : H^k(B; \mathbb{Q}) \to V^{(k)}_\alpha$ is non-zero; $p^{(k)}_\alpha : H^k(M; \mathbb{Q}) \to V^{(k)}_\alpha$ is the projection. The same set of notations (such as $\mathcal{A}^k(W)$, $\mathcal{A}^k(W)_{\text{sub}}$, $\mathcal{A}^k_B(W)$) is introduced for the SYZ torus fibration $\pi_W : W \to B$. For the purpose of the condition 4(a) in Conj. 2, it is enough to choose $\mathcal{A}^k(M)_{\text{sub}} \subset \mathcal{A}^k(M)$ as large as $\mathcal{A}^k_B(M)$. The conditions 4(a) does not necessarily require that $\mathcal{A}^k(M)_{\text{sub}}$ should contain all of $\mathcal{A}^k_B(M)$, because the images of $H^k(B; \mathbb{Q})$ in $V^{(k)}_\alpha$'s with different $\alpha$'s may be correlated in general.

Here, we begin with the first test. The conditions 4(b) and 4(c) do not treat $M$ and the mirror $W$ democratically, at least at first sight. If the Conj. 2 is to provide a necessary and sufficient condition for a $\mathcal{N} = (1, 1)$ SCFT to be rational, then all of its mirror SCFTs must be rational. So, we have to make sure that the condition 4 implies the condition $4[M \leftrightarrow W]$.

The condition 4(b)$[M \leftrightarrow W]$ follows from the condition 4(a,b,c) in fact. To see this, think of the subset $\mathcal{A}^k(W)_{\text{sub}} \subset \mathcal{A}^k(W)_{\text{sub}}$ where the image of $\phi^*$ in the condition 4(c) is non-zero. Then we may replace $V^{(k)}_W = \oplus_{\alpha \in \mathcal{A}^k(W)_{\text{sub}}} V^{(k)}_\alpha$ by $V^{(k)}_{W_{\text{new}}} = \oplus_{\alpha \in \mathcal{A}^k(W)_{\text{sub}}} V^{(k)}_\alpha$. This rational Hodge substructure of $H^k(W; \mathbb{Q})$ still satisfies the condition 4(a). Because they are in the non-zero image from CM-type simple rational Hodge substructures (the condition 4(b)), $V^{(k)}_{W_{\text{new}}}$ is also of CM-type.

The condition 4(c)$[M \leftrightarrow W]$ also follows from the conditions 4(a,b,c). Think of the subset $\mathcal{A}^k(M)_{\text{sub}} \subset \mathcal{A}^k(M)_{\text{sub}}$ where $\phi^*$ in the condition 4(c) is non-zero. Then there is one-to-one correspondence between $\mathcal{A}^k(M)_{\text{sub}}$ and $\mathcal{A}^k(W)_{\text{sub}}$. We may replace $V^{(k)}_M = \oplus_{\alpha \in \mathcal{A}^k(M)_{\text{sub}}} V^{(k)}_\alpha$ by $V^{(k)}_{M_{\text{new}}} = \oplus_{\alpha \in \mathcal{A}^k(M)_{\text{sub}}} V^{(k)}_\alpha$, and yet the condition 4(a) is satisfied. Now, we construct a Hodge morphism $\psi^* : V^{(k)}_{W_{\text{new}}} \to V^{(k)}_{M_{\text{new}}}$ as follows, by focusing on each one-to-one correspondence pair $\alpha \in \mathcal{A}^k(M)_{\text{sub}}$ and $\beta \in \mathcal{A}^k(W)_{\text{sub}}$. The vector space $V^k_\beta \subset H^k(W; \mathbb{Q})$ over $\mathbb{Q}$ can also be seen as a vector space over the CM field $K$ of the simple rational Hodge structures in $\alpha$ and $\beta$; choose any decomposition $V^{(k)}_\beta = \text{Im}(\phi^*) \oplus [\text{Im}(\phi^*)]^c$ as a vector space over $K$, and fix it. The vector space $V^{(k)}_\alpha$ also has a decomposition $\text{Ker}(\phi^*) \oplus [\text{Ker}(\phi^*)]^c$ as a vector space over $K$; choose this decomposition in a way $[\text{Ker}(\phi^*)]^c$ contains the image $p^{(k)}_\alpha \circ \pi_M^*(H^k(B; \mathbb{Q}))$. Then the Hodge morphism $\phi^* : [\text{Ker}(\phi^*)]^c \to \text{Im}(\phi^*)$ is invertible; the inverse $\psi^*$ may be extended to $[\text{Im}(\phi^*)]^c \subset V^{(k)}_\beta$ by zero, so we have a Hodge morphism $\psi^* : V^{(k)}_\beta \to V^{(k)}_\alpha$. This completes the construction of a Hodge morphism $\psi^* : V^{(k)}_{W_{\text{new}}} \to V^{(k)}_{M_{\text{new}}}$. By construction, $\phi^*$ is an isomorphism between the injective images of $H^k(B; \mathbb{Q})$ within
$V^{(k)}_{M^{\text{new}}}$ and $V^{(k)}_{W^{\text{new}}}$, and $\psi^*$ gives the inverse of $\phi^*$ on those injective images. So, the condition 4(c) $[M \leftrightarrow W]$ follows indeed.

The other test is to see if Conjecture 2 (gen.) is consistent with Conjecture 1 (self-mirror); the latter is more reliable because it has been tested by the case $M = T^4$. When we read the conditions 4 of Conjecture 2 for a self-mirror manifold $M$, they appear to be weaker than the conditions 4 and 5 of Conj. 1. This raises a concern that the conditions in Conj. 2 might not be strong enough to be a sufficient condition for an SCFT to be rational.

It is beyond the scope of this article to run this test on all the self-mirror families, but we can do it on the family $M = T^4$ here. To see that the conditions 4 of Conjecture 2 for a self-mirror manifold $M$, they appear to be weaker than the conditions 4 and 5 of Conj. 1. This raises a concern that the co conditions in Conj. 2 might not be strong enough to be a sufficient condition for an SCFT to be rational.

Here are two remarks. First, note that the first test motivated an idea of restricting the range (i.e., $V^{(k)}_{M} \subset H^k(M;\mathbb{Q})$) in which the Hodge morphism $\phi^*$ is defined. We could replace the condition 4(c) by that—4(c')—of a Hodge morphism $\phi^* : H^1(M;\mathbb{Q}) \rightarrow H^1(W;\mathbb{Q})$; we have not found how to prove existence of $\psi^* : H^k(W;\mathbb{Q}) \rightarrow H^k(M;\mathbb{Q})$ by relying on the conditions 4(a,b,c), however, in the absence of an extra observation in either variations of Hodge structures, SYZ $T^n$-fibration or SCFTs (cf. footnote 24).

Second, there is an alternative to Conjecture 2, which is to replace the condition 4(b) by 4(b') the rational Hodge structures on the entire $H^k(M;\mathbb{Q})$ and $H^k(W;\mathbb{Q})$ are of CM-type. The alternative version, Conjecture 2' also passes the first test (as well as the second one). The conditions 4(b) and 4(b') are likely to be different. The Borcea–Voisin orbifold Calabi–Yau threefolds will be good test cases in finding out which is right (cf. footnote 6 of [5]).

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A  Isogenous Tori Have Isogenous Geometric SYZ-mirrors

We will prove that isogenous complex tori also have isogenous geometric SYZ-mirrors. The precise statement is as follows. Suppose that two complex tori \((T^{2n}_1; I_1)\) and \((T^{2n}_2; I_2)\) are isogenous, and let \(\psi : (T^{2n}_1; I_1) \rightarrow (T^{2n}_2; I_2)\) be an isogeny. Suppose further that the \(\mathcal{N} = (2, 2)\) SCFT of a set of data \((T^{2n}_2; G_2, B_2; I_2)\) has a geometric SYZ-mirror with \((T^{2n}_2; G_0^0, B_2^0; I_2)\) for the T-dual along a rank-\(n\) primitive subgroup \(\Gamma(2)^2 \subset H_1(T^{2n}_2; \mathbb{Z})\). Then we claim that the set of geometric data

\[
(T^{2n}_1; G_1, B_1; I_1) \quad \text{where} \quad G_1 := \psi^*(G_2), \quad B_1 := \psi^*(B_2)
\]  

also has a geometric SYZ-mirror when the T-duality is taken along \(\Gamma_f^{(1)} := \psi^{-1}_* (\Gamma_f^{(2)} \otimes \mathbb{Q}) \cap H_1(T^{2n}_1; \mathbb{Z})\); the geometric data for the mirror is denoted by \((T^{2n}_1; G_0^1, B_1^0; I_1^1)\). Furthermore, the mirror complex tori \((T^{2n}_2; I_0^1)\) and \((T^{2n}_2; I_2^2)\) are isogenous. The rest of this appendix is very elementary and straightforward; we leave this note in this preprint because this might still save a little bit of time of some readers.

Let us first show that \((T^{2n}_1; G_1, B_1; I_1)\) has a geometric SYZ-mirror for the T-dual along \(\Gamma_f^{(1)}\). That can be done by checking the properties that \([8, \text{Prop. } 8]\)

\[
\omega_1|_{\Gamma_f^{(1)} \otimes \mathbb{R}} = B_1|_{\Gamma_f^{(1)} \otimes \mathbb{R}} = 0,
\]

where \(\omega_1(-, -) := \frac{1}{2}G_1(I_1 -, -)\) is the Kähler form. In fact, \((A.2)\) immediately follows from the existence of a geometric SYZ-mirror for \((T^{2n}_2; G_2, B_2; I_2)\) (i.e., \(\omega_2|_{\Gamma_f^{(2)} \otimes \mathbb{R}} = 0\) and \(B_2|_{\Gamma_f^{(2)} \otimes \mathbb{R}} = 0\)), and the fact that

\[
B_1(-, -) = \psi^*(B_2)(-,-) = B_2(\psi_*, -), \quad (A.3)
\]

\[
\omega_1(-, -) = \frac{1}{2}G_2(\psi_* I_1-, -) = \frac{1}{2}G_2(I_2 \psi_*-, -) = \omega_2(\psi_*-, -), \quad (A.4)
\]

in the latter, we used the fact that the isogeny \(\psi\) is a holomorphic map and hence

\[
\psi_* I_1 = I_2 \psi_*.
\]  

Let us move on to the proof of the existence of an isogeny \(\psi_0 : (T^{2n}_{I_0}; I_1^0) \rightarrow (T^{2n}_{I_2}; I_2^0)\). That is (see footnote \([3]\) to construct a linear isomorphism \(\psi_*^0 : H_1(T^{2n}_1; \mathbb{Q}) \rightarrow H_1(T^{2n}_2; \mathbb{Q})\) satisfying

\[
\psi_*^0 I_1^0 = I_2^0 \psi_*^0, \quad (A.6)
\]

\(^{25}\)To be precise, there is no such notion as “the” T-dual of a torus-target (S)CFT along a set of 1-cycles \(\Gamma_f \subset H_1(T^{2n}_1; \mathbb{Z})\); we also have to specify the directions \(\Gamma_k \subset H_1(T^{2n}_1; \mathbb{Z})\) in which the T-dual is not taken, to specify the dual (S)CFT. We claim that the statement here holds true for any \((\Gamma_f^{(2)}, \Gamma_f^{(2)}_e)\) and \((\Gamma_f^{(1)}, \Gamma_f^{(1)}_e)\), for the corresponding T-dualities, and for the corresponding sets of geometric data \((T^{2n}_1, I_1^0)\) and \((T^{2n}_2, I_0^0)\).

\(^{26}\)In section \([3, \text{II}]\), we use this statement for \(T^{2n}_1 \rightarrow T^{2n}_2\) and \(\Gamma_f = \Gamma^{(2)} \otimes \mathbb{Q} = \text{Span}_\mathbb{Q}\{\psi_*(c), \psi_*(d)\}\). The T-dual of \(E_1 \times E_2\) is \(E_2^* \times E_1^*\) when we choose \(\Gamma_f^{(2)}\) so that it is rank-1 in \(H_1(E_2; \mathbb{Z})\) and also rank-1 in \(H_1(E_1; \mathbb{Z})\); this choice is always possible because \(\Gamma_f^{(2)} \otimes \mathbb{Q} = \Gamma_f\) is of the form \([3, \text{II}]\).
where $\psi_\circ \otimes \mathbb{R} : H_1(T^{2n}_i; \mathbb{R}) \to H_1(T^{2n}_{i_0}; \mathbb{R})$ is also denoted simply by $\psi_\circ$.

To do so, let us recall how the complex structure $I_i^\circ$ of the mirror is determined by $\omega_i$ and $B_i$ ($i = 1, 2$) (see [8, §§2.4–2.5] and references therein). First, the horizontal and vertical generalized complex structures $\mathcal{I}_i$ and $\mathcal{J}_i$ are linear operators on $H_1(T^{2n}_i; \mathbb{R}) \oplus H^1(T^{2n}_{i_0}; \mathbb{R})$,

$$
\mathcal{I}_i : (\partial_{X^I} \, dX^I) \mapsto (\partial_{X^J} \, dX^J) \begin{pmatrix}
(I_i)_J^I & 0 \\
(B_i)_{JK} (I_i)_K^J + (I_{i_0})_J^I (B_i)_{KI} - (I_{i_0})_J^I 
\end{pmatrix},
$$

(A.7)

$$
\mathcal{J}_i : (\partial_{X^I} \, dX^I) \mapsto (\partial_{X^J} \, dX^J) \begin{pmatrix}
(\omega_i^{-1})^{JK}(B_i)_{KI} & - (\omega_i^{-1})^{JI} \\
(\omega_i)_{JI} + (B_i)_{JK}(\omega_i^{-1})^{KL}(B_i)_{LI} & - (B_i)_{JK}(\omega_i^{-1})^{KI}
\end{pmatrix}.
$$

(A.8)

Second, the winding and Kaluza–Klein charges of the SYZ-mirror pairs of SCFTs are identified through the lattice isometry

$$
g : H_1(T^{2n}_i; \mathbb{Z}) \oplus H^1(T^{2n}_i; \mathbb{Z}) \to H_1(T^{2n}_{i_0}; \mathbb{Z}) \oplus H^1(T^{2n}_{i_0}; \mathbb{Z}) \quad \text{such that}
$$

(A.9)

$$
g((\Gamma_b^{(i)})^\vee \oplus (\Gamma_f^{(i)})^\vee) = H_1(T^{2n}_{i_0}; \mathbb{Z}), \quad g((\Gamma_b^{(i)})^\vee \oplus (\Gamma_f^{(i)})^\vee) = H^1(T^{2n}_{i_0}; \mathbb{Z});
$$

(A.10)

its matrix form is

$$
g = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
$$

(A.11)

when presented in the basis of $\Gamma_b^{(i)} \oplus \Gamma_f^{(i)} \oplus (\Gamma_b^{(i)})^\vee \oplus (\Gamma_f^{(i)})^\vee$ (cf. footnote [9]. Now, the horizontal generalized complex structure of the mirror (i.e., $\mathcal{I}_i^\circ$) is given by the vertical generalized complex structure before the mirror (i.e., $\mathcal{J}_i^\circ$) through

$$
\mathcal{I}_i^\circ = g\mathcal{J}_i g^{-1}.
$$

(A.12)

The complex structure $I_i^\circ$ of the mirror is extracted from the upper-left $2n \times 2n$ block of $\mathcal{I}_i^\circ$, which is further given in terms of $\omega_i$ and $B_i$.

A relation between the complex structures $I_i^\circ$ ($i = 1, 2$) of the mirrors can be derived from the relations between the $\omega_i$s and $B_i$s of the isogenous pair before the mirror. The relations between the latter, (A.3) and (A.4), are written in the matrix form as

$$
(B_1)_{JK} = (\psi_s^T)_j^j (B_2)_{jK} (\psi_s)_K^K \quad \text{and} \quad (\omega_1)_{JK} = (\psi_s^T)_j^j (\omega_2)_{jK} (\psi_s)_K^K.
$$

(A.13)

Therefore, we have

$$
\Psi \mathcal{J}_1 = \mathcal{J}_2 \Psi, \quad \text{where} \quad \Psi := \begin{pmatrix}
(\psi_s^T)_j^j & 0 \\
0 & (\psi_{s^{-1}}^T)_j^j
\end{pmatrix}.
$$

(A.14)

This means, because of (A.12), that

$$
g\Psi g^{-1} \mathcal{I}_1^\circ = \mathcal{I}_2^\circ g\Psi g^{-1}.
$$

(A.15)
The $4n \times 4n$ matrix $g \Psi g^{-1}$ is not a general $\mathbb{Q}$-valued matrix, but has a structure. Since $\Gamma^{(1)}_f$ was defined to be $\psi^{-1}_*(\Gamma^{(2)}_f \otimes \mathbb{Q}) \cap H_1(T^{2n}_1; \mathbb{Z})$, the isogeny $\psi_* : H_1(T^{2n}_1; \mathbb{Q}) \to H_1(T^{2n}_2; \mathbb{Q})$ has the following structure:

$$
\psi_* \left( \begin{array}{cc} \Gamma^{(1)}_b & \Gamma^{(1)}_f \\ \Gamma^{(2)}_b & \Gamma^{(2)}_f \end{array} \right) = \left( \begin{array}{cc} P_{11} & 0 \\ P_{21} & P_{22} \end{array} \right). 
$$

(A.16)

As a result, the upper-right $2n \times 2n$ block of

$$
g \Psi g^{-1} = \begin{pmatrix}
P_{11} & 0 & 0 & 0 \\
0 & P_{22}^{T-1} & 0 & 0 \\
0 & -P_{22}^{-1}P_{21}P_{11}^{-1}^{T} & P_{11}^{T-1} & 0 \\
P_{21} & 0 & 0 & P_{22}
\end{pmatrix}
$$

(A.17)

vanishes.

We are now ready to read out from the upper-left $2n \times 2n$ block of the equation (A.15) that

$$
\psi^\circ := \begin{pmatrix} P_{11} & 0 \\ 0 & P_{22}^{T-1} \end{pmatrix} : H_1(T^{2n}_{10}; \mathbb{Q}) \to H_1(T^{2n}_{20}; \mathbb{Q})
$$

(A.18)

is a Hodge isomorphism (i.e., the condition (A.6) is satisfied). Isogenies $T^{2n}_{10} \to T^{2n}_{20}$ are obtained as appropriate integer multiples of $\psi^\circ$ above (cf. footnote 3).

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