TWO-BY-TWO UPPER TRIANGULAR MATRICES AND MORREY’S CONJECTURE

TERENCE L. J. HARRIS, BERND KIRCHHEIM, AND CHUN-CHI LIN

Abstract. It is shown that every homogeneous gradient Young measure supported on matrices of the form
\[
\begin{pmatrix}
a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\
0 & \cdots & 0 & a_{2,n}
\end{pmatrix}
\]
is a laminate. This is used to prove the same result on the 3-dimensional nonlinear submanifold of \(M^{2 \times 2}\) defined by \(\det X = 0\) and \(X_{12} > 0\).

1. Introduction and preliminaries

Let \(M^{m \times n}\) be the space of \(m \times n\) matrices with real entries. A function \(f : M^{m \times n} \to \mathbb{R}\) is rank-one convex if
\[
f(\lambda X + (1 - \lambda)Y) \leq \lambda f(X) + (1 - \lambda)f(Y)
\]
for all \(X, Y \in M^{m \times n}\) with \(\text{rank}(X - Y) \leq 1\). A locally bounded Borel measurable function \(f : M^{m \times n} \to \mathbb{R}\) is quasiconvex if for every bounded domain \(\Omega \subseteq \mathbb{R}^n\) and \(X_0 \in M^{m \times n}\),
\[
f(X_0)m(\Omega) \leq \int_{\Omega} f(\bar{X}_0 + \nabla \phi(x)) \, dx,
\]
for every \(\phi \in C_0^\infty(\Omega, \mathbb{R}^m)\), where \(\nabla \phi\) is the derivative of \(\phi\).

In 1952 Morrey conjectured that rank-one convexity does not imply quasiconvexity \(\[8\]\). A counterexample for \(m \geq 3\) and \(n \geq 2\) was given by Sverák in \(\[13\]\), but the question remains open for \(m = 2\) and \(n \geq 2\). Müller proved that rank-one convexity implies quasiconvexity on diagonal matrices (see \(\[9\]\)), but to reformulate the problem on a subspace requires the dual notions for measures.

Throughout, all probability measures are assumed to be Borel. A compactly supported probability measure \(\mu\) on \(M^{m \times n}\) is called a laminate if
\[
f(\mu) \leq \int f(\bar{X}) \, d\mu
\]
for all rank-one convex \(f : M^{m \times n} \to \mathbb{R}\), where \(\bar{X} = \int X \, d\mu(X)\) is the barycentre of \(\mu\). Similarly, \(\mu\) is a called a homogeneous gradient Young measure if the same inequality holds, but with rank-one convex replaced by quasiconvex. The question of whether rank-one convexity implies quasiconvexity is then equivalent to asking whether every homogeneous gradient Young measure is a laminate. In \(\[9\]\) Müller proved that every homogeneous gradient Young measure supported on the \(2 \times 2\) diagonal matrices is a laminate. In \(\[2\]\) and \(\[9\]\) this was extended to the \(n \times n\) diagonal matrices. The purpose of this work is to
generalise the result for \(2 \times 2\) diagonal matrices to the subspace
\[
\mathcal{M}_{2\times n}^{2\times 2} := \left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & \cdots & 0 & a_{2,n} \end{pmatrix} \in \mathcal{M}_{2\times n} \right\}.
\]

When \(n = 2\), \(\mathcal{M}_{2\times 2}^{2\times 2}\) is the space of \(2 \times 2\) upper triangular matrices. Up to linear isomorphisms preserving rank-one directions, the only other 3-dimensional subspace of \(\mathcal{M}_{2\times 2}^{2\times 2}\) is the symmetric matrices (see [3, Corollary 6]).

In Section 3, the result on upper-triangular matrices will be used to prove that rank-one convexity implies quasiconvexity on
\[
\{ X \in \mathcal{M}_{2\times 2} : \det X = 0 \text{ and } X_{12} > 0 \}.
\]

2. Müller’s result

This section contains one particular generalisation of Müller’s result from the \(2 \times 2\) diagonal matrices \(\mathcal{M}_{2\times 2}^{2\times 2}\) to the subspace
\[
\mathcal{M}_{2\times n}^{2\times n} := \left\{ \begin{pmatrix} a_{1,1} & \cdots & a_{1,n-1} & 0 \\ 0 & \cdots & 0 & a_{2,n} \end{pmatrix} \in \mathcal{M}_{2\times n} \right\}, \quad n \geq 2.
\]

This will be used to prove the result for \(\mathcal{M}_{2\times n}^{2\times 2}\). The proof has only minor modifications from the one in [9], but is included for convenience. As in [6], define the elements in the Haar basis for \(L^2(\mathbb{R}^n)\) by

\[
h_{Q}^{(\epsilon)}(x) = \prod_{j=1}^{n} h_{j}^{(\epsilon)}(x_{j}), \quad \text{for } x \in \mathbb{R}^n,
\]

where \(\epsilon \in \{0, 1\}^n \setminus \{(0, \ldots, 0)\}\), \(Q = I_1 \times \cdots \times I_n\) is a dyadic cube in \(\mathbb{R}^n\), and the \(I_j\)'s are dyadic intervals of equal size. A dyadic interval is always of the form \([k \cdot 2^{-j}, (k+1) \cdot 2^{-j})\) with \(j, k \in \mathbb{Z}\). For a dyadic interval \(I = [a, b)\), \(h_I\) is defined by

\[
h_I(x) = h_{[0,1)} \left( \frac{x-a}{b-a} \right) \quad \text{for } x \in \mathbb{R},
\]

where

\[
h_{[0,1)} = \chi_{[0, \frac{1}{2})} - \chi_{[\frac{1}{2}, 1]}.
\]

For \(j \in \mathbb{Z}\) and \(k \in \mathbb{Z}^n\), the notation \(h_{j,k}^{(\epsilon)} = h_{Q}^{(\epsilon)}\) will be used, where

\[
Q = Q_{j,k} = \left[ \frac{k_1}{2^j}, \frac{k_1 + 1}{2^j} \right) \times \cdots \times \left[ \frac{k_n}{2^j}, \frac{k_n + 1}{2^j} \right).
\]

The standard basis vectors in \(\mathbb{R}^n\) or \(\{0, 1\}^n\) will be denoted by \(e_j\). The Riesz transform \(R_{j}\) on \(L^2(\mathbb{R}^n)\) is defined through multiplication on the Fourier side by \(-i\xi_j/|\xi|\). In [8, Theorem 2.1] and [9, Theorem 5] it was shown that if \(\epsilon \in \{0, 1\}^n\) satisfies \(\epsilon_j = 1\), then there is a constant \(C\) such that

\[
\|P^{(\epsilon)} u\|_2 \leq C \|u\|_2^{1/2} \|R_j u\|_2^{1/2} \quad \text{for all } u \in L^2(\mathbb{R}^n),
\]

where \(\epsilon\) is fixed and \(P^{(\epsilon)}\) is the projection onto the closed span of the set

\[
\left\{ h_{Q}^{(\epsilon)} : Q \subseteq \mathbb{R}^n \text{ is a dyadic cube} \right\}.
\]
Lemma 2.1. If $f: M^{2 \times n} \to \mathbb{R}$ is rank-one convex with $f(0) = 0$, and if $u_1, \ldots, u_{n-1}, v_n$ have finite expansions in the Haar basis

$$u_i = \sum_{\epsilon_n=0}^{K} \sum_{j,k \in \mathbb{Z}^n} a_{j,k,i}^{(\epsilon)} h_{j,k}^{(\epsilon)}$$

for $1 \leq i \leq n-1$, and

$$v_n = \sum_{j,k \in \mathbb{Z}^n} b_{j,k} h_{j,k}^{(\epsilon_n)}$$

so that $a_{j,k,i}^{(\epsilon)} = b_{j,k} = 0$ whenever $|k|$ is sufficiently large, then

$$\int_{\mathbb{R}^n} f \begin{pmatrix} u_1 & \cdots & u_{n-1} & 0 \\ 0 & \cdots & 0 & v_n \end{pmatrix} \, dx \geq 0.$$

Proof. The assumption that $a_{j,k,i}^{(\epsilon)} = b_{j,k} = 0$ for $|k|$ sufficiently large means the integral converges absolutely. Let

$$\tilde{u}_i = \sum_{\epsilon_n=0}^{K-1} \sum_{j,k \in \mathbb{Z}^n} a_{j,k,i}^{(\epsilon)} h_{j,k}^{(\epsilon)}$$

for $1 \leq i \leq n-1$, and let

$$\tilde{v}_n = \sum_{j,k \in \mathbb{Z}^n} b_{j,k} h_{j,k}^{(\epsilon_n)}.$$

Then on $Q_{K,k}$, for any $k \in \mathbb{Z}^n$,

$$u_i := u_i - \tilde{u}_i = \sum_{\epsilon_n=0}^{K} a_{K,k,i}^{(\epsilon)} h_{K,k}^{(\epsilon)}, \quad v_n - \tilde{v}_n = b_{K,k} h_{K,k}^{(\epsilon_n)}.$$

and

$$\int_{Q_{K,k}} f \begin{pmatrix} u_1 & \cdots & u_{n-1} & 0 \\ 0 & \cdots & 0 & v_n \end{pmatrix} \, dx = \int_{Q_{K,k}} f \begin{pmatrix} \tilde{u}_1 + u_1' & \cdots & \tilde{u}_{n-1} + u_{n-1}' & 0 \\ 0 & \cdots & 0 & \tilde{v}_n + v_n' \end{pmatrix} \, dx_1 \cdots dx_n.$$

The bottom row is constant in $x_1, \ldots, x_{n-1}$ on $Q_{K,k}$, and so the function is convex for the integration with respect to $x_1, \ldots, x_{n-1}$. The terms $\tilde{u}_i$ and $\tilde{v}_n$ are constant on $Q_{K,k}$, and the $x_1, \ldots, x_{n-1}$ integral of $u_i'$ over the $(n-1)$-dimensional dyadic cube inside $Q_{K,k}$ is zero (for any $x_n$). Hence applying Jensen’s inequality for convex functions gives

$$\int_{Q_{K,k}} f \begin{pmatrix} u_1 & \cdots & u_{n-1} & 0 \\ 0 & \cdots & 0 & v_n \end{pmatrix} \, dx \geq \int_{Q_{K,k}} f \begin{pmatrix} \tilde{u}_1 & \cdots & \tilde{u}_{n-1} & 0 \\ 0 & \cdots & 0 & \tilde{v}_n + v_n' \end{pmatrix} \, dx_1 \cdots dx_n.$$

Applying Jensen’s inequality similarly to the integration in $x_n$, and summing over all $k \in \mathbb{Z}^n$ gives

$$\int_{\mathbb{R}^n} f \begin{pmatrix} u_1 & \cdots & u_{n-1} & 0 \\ 0 & \cdots & 0 & v_n \end{pmatrix} \, dx \geq \int_{\mathbb{R}^n} f \begin{pmatrix} \tilde{u}_1 & \cdots & \tilde{u}_{n-1} & 0 \\ 0 & \cdots & 0 & \tilde{v}_n \end{pmatrix} \, dx.$$

By induction this proves the lemma. □

Theorem 2.2. Let $n \geq 2$. Every homogeneous gradient Young measure supported in $M_{\text{diag}}^{2 \times n}$ is a laminate.

Proof. Let $\mu$ be a homogeneous gradient Young measure supported in $M_{\text{diag}}^{2 \times n}$, and let $f: M^{2 \times n} \to \mathbb{R}$ be a rank-one convex function. It is required to show that

$$\int f \, d\mu \geq f(\mathbb{P}).$$
Without loss of generality it may be assumed that $\overline{\mu} = 0$ and that $f(0) = 0$. After replacing $f$ by an extension of $f$ which is equal to $f$ on $(\text{supp } \mu)^{\infty}$, it can also be assumed that there is a constant $C$ with

$$|f(X)| \leq C(1 + |X|^2) \quad \text{for all } X \in M^{2 \times n}. \tag{2.2}$$

Let $\Omega \subset \mathbb{R}^n$ be the open unit cube, and using the Fundamental Theorem of Young Measures \cite[Theorem 8.16]{12} let $\phi^{(j)} = (\phi_1^{(j)}, \phi_2^{(j)})$ be a sequence in $W^{1, \infty}(\Omega, \mathbb{R}^2)$ whose gradients generate $\mu$, which means that

$$\lim_{j \to \infty} \int_{\Omega} \eta(x) g \left( \nabla \phi^{(j)}(x) \right) \, dx = \int_{\Omega} g \, d\mu \cdot \int_{\Omega} \eta(x) \, dx,$$  

for any continuous $g$ and for all $\eta \in L^1(\Omega)$. In particular $\nabla \phi^{(j)} \to 0$ weakly in $L^2(\Omega, M^{2 \times n})$. By the sharp version of Zhang’s truncation theorem (see \cite{11}) it may be assumed that

$$\left\| \text{dist} \left( \nabla \phi^{(j)}, M^{2 \times n}_{\text{diag}} \right) \right\|_{\infty} \to 0. \tag{2.4}$$

As in Lemma 8.3 of \cite{12}, after multiplying the sequence by cutoff functions in such a way as to not affect \eqref{2.4}, it can additionally be assumed that $\phi^{(j)} \in W^{1, \infty}(\Omega, \mathbb{R}^2)$. Equation \eqref{2.4} gives

$$\left\| \partial_n \phi_1^{(j)} \right\|_{\infty} \to 0 \quad \text{and} \quad \left\| \partial_1 \phi_2^{(j)} \right\|_{\infty}, \ldots, \left\| \partial_{n-1} \phi_2^{(j)} \right\|_{\infty} \to 0. \tag{2.5}$$

Let $P_1 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the projection onto the closed span of

$$\{h_Q^{(\epsilon)} : Q \subset \mathbb{R}^n \text{ is a dyadic cube and } \epsilon_n = 0\},$$

and let $P_2 : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be the projection onto the closed span of

$$\{h_Q^{(\epsilon)} : Q \subset \mathbb{R}^n \text{ is a dyadic cube and } \epsilon = \epsilon_n\}. \tag{2.6}$$

Write $w^{(j)} = \nabla \phi^{(j)}$, so that by \eqref{2.5} and the fact that $R_{ij} \partial_i \phi = \partial_j \phi$, 

$$\left\| R_n w^{(j)}_{1 \to 2}, \ldots, R_n w^{(j)}_{1 \to n} \right\|_2 \to 0, \quad \left\| R_1 w^{(j)}_{2 \to n}, \ldots, R_{n-1} w^{(j)}_{2 \to n} \right\|_2 \to 0. \tag{2.7}$$

Hence by \eqref{2.1} and Pythagoras’ Theorem

$$\left\| w^{(j)}_{1 \to 2}, \ldots, w^{(j)}_{1 \to n-1} - P_1 w^{(j)}_{1 \to n-1} \right\|_2 \to 0, \quad \left\| w^{(j)}_{2 \to n} - P_2 w^{(j)}_{2 \to n} \right\|_2 \to 0. \tag{2.8}$$

The function $f$ is separately convex since it is rank-one convex. Hence by Observation 2.3 in \cite{1} and the quadratic growth of $f$ in \eqref{2.2}, there exists a constant $K$ such that

$$|f(X) - f(Y)| \leq K(1 + |X| + |Y|)|X - Y| \quad \text{for all } X, Y \in M^{2 \times n}. \tag{2.7}$$

Hence applying \eqref{2.3} with $\eta = \chi_{\Omega}$ gives

$$\int f \, d\mu = \lim_{j \to \infty} \int_{\Omega} f \left( w^{(j)} \right) \, dx \tag{2.8}$$

(by \eqref{2.5}, \eqref{2.6}, \eqref{2.7} and the Cauchy-Schwarz inequality. The functions $w^{(j)}$ are supported in $\Omega$ and satisfy $\int_{\Omega} w^{(j)} \, dx = 0$, by the definition of weak derivative. Hence the $L^2(\mathbb{R}^n)$ inner product satisfies $\left\langle w^{(j)}, h_Q^{(\epsilon)} \right\rangle = 0$ whenever $Q$ is a dyadic.
cube not contained in $\overline{\Omega}$. This implies that $P_1 w^{(j)}$ and $P_2 w^{(j)}$ are supported in $\overline{\Omega}$. The integrand in (2.8) therefore vanishes outside $\overline{\Omega}$, and so

$$\int f \, d\mu = \lim_{j \to \infty} \int_{\mathbb{R}^n} f \begin{pmatrix} P_1 w^{(j)}_{11} & \cdots & P_1 w^{(j)}_{1,n-1} & 0 \\ 0 & \cdots & 0 & P_2 w^{(j)}_{2,n} \end{pmatrix} \, dx \geq 0$$

by (2.7) and Lemma 2.1. This finishes the proof. \qed

3. The Linear Space

To show a homogeneous gradient Young measure $\mu$ supported in $M_{2\times n}^\tri$ is a laminate, the argument consists of two steps. The projection $P_\# \mu$ onto $M_{2\times n}^\di$ is shown to be a gradient Young measure, and therefore a laminate by Theorem 2.2. It is then shown that since $P_\# \mu$ is a laminate, $\mu$ is also a laminate. Some of the arguments are similar to those in 2.

The proof requires a few extra definitions, which give a more constructive characterisation of laminates (see also 12).

Definition 3.1. A set $\{ (t_1, Y_1), \ldots, (t_l, Y_l) \} \subseteq (0, 1] \times M^{m \times n}$ with $\sum_{i=1}^l t_i = 1$ satisfies the $H_l$ condition if:

- i) $l = 2$ and $\text{rank}(Y_1 - Y_2) \leq 1$, or;
- ii) $l > 2$ and after a permutation of the indices, $\text{rank}(Y_1 - Y_2) \leq 1$ and the set

$$\left\{ \left( t_1 + t_2, \frac{t_1 Y_1 + t_2 Y_2}{t_1 + t_2} \right), (t_3, Y_3), \ldots, (t_l, Y_l) \right\},$$

satisfies the $H_{l-1}$ condition.

A convex combination of Dirac measures $\mu = \sum_{i=1}^N \lambda_i \delta_{X_i}$ is called a prelamine if the set $\{ (\lambda_1, X_1), \ldots, (\lambda_N, X_N) \}$ satisfies the $H_N$ condition. This definition essentially says that the class of prelaminates is the smallest class of probability measures that contains the Dirac masses and is closed under rank-one splitting of its atoms.

The following theorem is a special case of Theorem 4.12 in 6, see also Theorem 3.1 in 11.

Theorem 3.2. Let $\mu$ be a laminate with support inside a compact set $K \subseteq M_{2\times n}^\di$, and let $U \subseteq M_{2\times n}^\di$ be any relatively open neighbourhood of $K^{co}$. There exists a sequence $\mu^{(n)}$ of prelaminates supported in $U$, with common barycentre, such that $\mu^{(n)} \rightharpoonup \mu$.

In what follows, $P : M_{2\times n}^\tri \to M_{2\times n}^\tri$ will be the projection onto $M_{2\times n}^\di$. Given a probability measure $\mu$ on $M_{2\times n}^\tri$, $P_\# \mu$ will denote the pushforward measure of $\mu$ by $P$, given by

$$(P_\# \mu)(E) = \mu(P^{-1}(E)),$$

for any Borel set $E$.

Lemma 3.3. If $\mu$ is a homogeneous gradient Young measure supported in $M_{2\times n}^\tri$, then $P_\# \mu$ is a homogeneous gradient Young measure.

Proof. It is first shown that if $T : M_{2\times n}^\tri \to M_{2\times n}^\tri$ is defined by $T(X) = AXB$ where $A \in M_{2\times 2}$ and $B \in M_{n\times n}$ is invertible, then $T_\# \nu$ is a homogeneous gradient Young measure whenever $\nu$ is a homogeneous gradient Young measure.
To show this, let $f : \mathbb{M}^{2 \times n} \to \mathbb{R}$ be a quasiconvex function and let $g = f \circ T$. Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty bounded domain and let $\phi \in C_0^\infty(\Omega, \mathbb{R}^2)$. Define $\psi \in C_0^\infty(B^{-1}(\Omega), \mathbb{R}^2)$ by $\psi(x) = A\phi(Bx)$, and let $X_0 \in \mathbb{M}^{2 \times n}$. Then $\nabla \psi(x) = A\nabla \phi(Bx)B$, and hence

$$
\int_\Omega g(X_0 + \nabla \phi(x)) \, dx = \int_\Omega f(AX_0B + A\nabla \phi(x)B) \, dx
$$

$$
= |\det B| \int_{B^{-1}(\Omega)} f(AX_0B + \nabla \psi(y)) \, dy
$$

$$
\geq |\det B| m(B^{-1}(\Omega)) g(X_0) \quad \text{since } f \text{ is quasiconvex},
$$

$$
= m(\Omega) g(X_0).
$$

This shows that $g$ is quasiconvex, which implies that

$$
\int g \, d(T\# \nu) = \int g \, d\nu \geq g(\overline{\nu}) = f(\overline{T\# \nu}),
$$

and therefore $T\# \nu$ is a homogeneous gradient Young measure.

Now let

$$
A_k = \begin{pmatrix}
1 & 0 \\
0 & k
\end{pmatrix}, \quad B_k = \begin{pmatrix}
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & 1/k
\end{pmatrix},
$$

and define $P^{(k)} : \mathbb{M}^{2 \times n} \to \mathbb{M}^{2 \times n}$ by $X \mapsto A_k X B_k$, so that for $X \in \mathbb{M}^{2 \times 2}$,

$$
P^{(k)}(X) = P^k \begin{pmatrix}
x_{1,1} & \cdots & x_{1,n-1} & x_{1,n} \\
0 & \cdots & 0 & x_{2,n}
\end{pmatrix} = \begin{pmatrix}
x_{1,1} & \cdots & x_{1,n-1} & x_{1,n} \\
0 & \cdots & 0 & x_{2,n}
\end{pmatrix}.
$$

Let $\mu$ be a homogeneous gradient Young measure supported on $\mathbb{M}^{2 \times n}_{tril}$. Then $P^{(k)} \to P$ uniformly on compact subsets of $\mathbb{M}^{2 \times n}_{tril}$, and hence for any continuous function $f : \mathbb{M}^{2 \times n} \to \mathbb{R}$,

$$
\lim_{k \to \infty} \int f \, d(P^{(k)} \mu) = \int \lim_{k \to \infty} (f \circ P^{(k)}) \, d\mu = \int f \, d(P\# \mu).
$$

This shows that $P^{(k)} \mu \rightharpoonup P\# \mu$, and since the measures $P^{(k)} \mu$ have common compact support, $P\# \mu$ is a homogeneous gradient Young measure. \qed

**Lemma 3.4.** Let $\mu = \sum_{i=1}^N \lambda_i \delta_{X_i}$ be a convex combination of Dirac measures supported in $\mathbb{M}^{2 \times n}_{tril}$. If $P\# \mu$ is a prelaminate then $\mu$ is a prelaminate.

**Proof.** Given $\mu = \sum_{i=1}^N \lambda_i \delta_{X_i}$, it can be assumed that each $\lambda_i > 0$. By induction it may also be assumed that a set of $(N - 1)$ pairs $\{(t_i, Y_i) : 1 \leq i \leq N - 1\}$ satisfies the $H_{N-1}$ condition whenever $\sum_{i=1}^{N-1} t_i \delta_{P(Y_i)}$ is a prelaminate.

By assumption the set of pairs $\{(\lambda_i, P(X_i)) : 1 \leq i \leq N\}$ satisfies the $H_N$ condition. This means that after a permutation of indices, $\text{rank}(P(X_1) - P(X_2)) \leq 1$ and the set

$$
\left\{ \left( \lambda_1 + \lambda_2, P \left( \frac{\lambda_1 X_1 + \lambda_2 X_2}{\lambda_1 + \lambda_2} \right) \right), (\lambda_3, P(X_3)), \ldots, (\lambda_N, P(X_N)) \right\}
$$
satisfies the $H_{N-1}$ condition. Hence rank$(X_1 - X_2) \leq 1$ and by the inductive assumption the set
\[
\left\{ \left( \frac{\lambda_1 + \lambda_2, \lambda_1 X_1 + \lambda_2 X_2}{\lambda_1 + \lambda_2}, (\lambda_3, X_3), (\lambda_N, X_N) \right) \right\}
\]
satisfies the $H_{N-1}$ condition. By definition this shows that $\{(\lambda_i, X_i) : 1 \leq i \leq N\}$ satisfies the $H_N$ condition, and therefore $\mu$ is a prelaminate. □

**Lemma 3.5.** Let $\mu$ be a probability measure with compact support in $\mathbb{M}^2_{\text{tri}}$. If $P#\mu$ is a laminate then $\mu$ is a laminate.

**Proof.** Suppose $\mu$ satisfies the assumptions of the lemma. Let $N$ be such that $\text{supp} \mu \subseteq B(0,N)$, and identify $\mathbb{M}^2_{\text{tri}}$ with $\mathbb{M}^2_{\text{diag}} \times \mathbb{R}$. By Theorem 2.28 in \[1\] there exist probability measures $\lambda_X$ on $[-N,N]$ such that
\[
\int f \, d\mu = \int \int f(X, t) \, d\lambda_X(t) \, d(P#\mu)(X),
\]
for all continuous $f : \mathbb{M}^2_{\text{tri}} \to \mathbb{R}$. In particular $X \mapsto \int f(X, t) \, d\lambda_X(t)$ is a Borel measurable function of $X \in \mathbb{M}^2_{\text{diag}}$ whenever $f$ is continuous. Hence the function
\[
g(X) = (X, \overline{\lambda_X})
\]
is Borel measurable and bounded on $\mathbb{M}^2_{\text{tri}} \cap B(0,N)$. By Lusin’s Theorem (see [1, Theorem 1.45]) applied to $X \mapsto \overline{\lambda_X}$, there exists a uniformly bounded sequence of continuous functions $g^{(k)}(X) = (X, h^{(k)}(X))$ such that
\[
(P#\mu) \left\{ X \in B(0,N) \cap \mathbb{M}^2_{\text{diag}} : g^{(k)}(X) \neq g(X) \right\} < \frac{1}{k}.
\]
It follows that for any continuous function $f$,
\[
\lim_{k \to \infty} \int (f \circ g^{(k)}) \, d(P#\mu) = \int (f \circ g) \, d(P#\mu), \quad \text{and thus } \lim_{k \to \infty} g^{(k)}#P#\mu = g#P#\mu.
\]
Using Theorem 3.2 let $\nu^{(j)}$ be a sequence of prelaminates supported in a common compact subset of $\mathbb{M}^2_{\text{diag}}$ such that $\nu^{(j)} \rightharpoonup P#\mu$. If $f$ is rank-one convex, then
\[
\int f \, d\mu = \int \int f(X, t) \, d\lambda_X(t) \, d(P#\mu)(X)
\]
\[
\geq \int (f \circ g) \, d(P#\mu)(X) \quad \text{by Jensen’s inequality},
\]
\[
= \lim_{k \to \infty} \int (f \circ g^{(k)}) \, d(P#\mu)
\]
\[
= \lim_{k \to \infty} \lim_{j \to \infty} \int f \, d\left( g^{(k)}#\nu^{(j)} \right)
\]
\[
\geq \lim_{k \to \infty} \lim_{j \to \infty} \int f \left( g^{(k)}#\nu^{(j)} \right) \quad \text{by Lemma 3.3}
\]
\[
= \lim_{k \to \infty} \int f \left( g^{(k)}#P#\mu \right) \quad \text{by continuity of } f \text{ and } g^{(k)},
\]
\[
= f \left( g#P#\mu \right) = f(\overline{\nu}).
\]
This shows that $\mu$ is a laminate. □
**Theorem 3.6.** Every homogeneous gradient Young measure supported in \( M_{2\times n}^{2\times n} \) is a laminate.

**Proof.** If \( \mu \) is a homogeneous gradient Young measure supported in \( M_{2\times 3}^{2\times n} \), then \( P_\# \mu \) is a homogeneous gradient Young measure by Lemma 3.3 and therefore a laminate by Theorem 2.2. The fact that \( P_\# \mu \) is a laminate then implies that \( \mu \) is a laminate by Lemma 3.5. \( \square \)

4. The 3-dimensional nonlinear space

Let

\[ M_{2\times 2}^+ = \{ X \in M_{2\times 2} : \det X = 0 \}, \quad M_{2\times 2}^+ = \{ X \in M_{2\times 2} : X_{12} > 0 \}. \]

The previous result will be used to prove that every homogeneous gradient Young measure on \( M_{2\times 2}^+ \cap M_{2\times 2}^{2\times 2} \) is a laminate. This is done via a change of variables used in [2] and [4], which will now be described.

Given an open set \( \Omega \subseteq \mathbb{R}^2 \) and a smooth function \( u = (u_1, u_2) : \Omega \to \mathbb{R}^2 \), consider the functions

\[ T_1(x) = (x_1, u_1(x)), \quad T_2(x) = (x_2, u_2(x)). \]

If \( T_1 \) is invertible with nonvanishing Jacobian, define the function \( v = (v_1, v_2) : T_1(\Omega) \to \mathbb{R}^2 \) by \( v \circ T_1 = T_2 \), that is

\[ v(x_1, u_1(x)) = (x_2, u_2(x)) \quad \text{for all } x \in \Omega. \]

This implies that

\[ u(x_1, v_1(x)) = (x_2, v_2(x)) \quad \text{for all } x \in T_1(\Omega). \]

which can be checked by substituting \( x = T_1(y) \). If \( u \) has gradient \( X \in M_{2\times 2} \) at some point \( x \in \Omega \), then by the chain rule \( v \) has gradient

\[ \Psi(X) = \frac{1}{X_{12}} \begin{pmatrix} -X_{11} & -1 \\ X_{22} & \det X \end{pmatrix}, \]

at the point \( T_1(x) \), where \( \Psi \) is defined on \( M_{2\times 2}^+ \). If \( S_1(x) = (x_1, v_1(x)) \), then \( S_1(T_1(x)) = x \) by Lemma 4.1, and therefore \( \Psi \) is a self-inverse mapping of \( M_{2\times 2}^+ \) onto itself.

Given a function \( h : M_{2\times 2}^+ \to \mathbb{R} \), define the dual function \( \tilde{h} : M_{2\times 2}^+ \to \mathbb{R} \) by

\[ \tilde{h}(X) = X_{12} h(\Psi(X)). \]

The term \( X_{12} \) corresponds to the determinant of \( T_1 \), which will later simplify the change of variables in integration.

Given a probability measure \( \mu \) on \( M_{2\times 2}^+ \), define the dual probability measure \( \tilde{\mu} \) on \( M_{2\times 2}^+ \) by

\[ \int f \, d\tilde{\mu}(X) = \frac{1}{\mu_{12}} \int f \, d\mu. \]

for all Borel measurable \( f : M_{2\times 2}^+ \to [0, \infty] \). The basic properties are summarised in the following proposition.

**Proposition 4.1.** Let \( h : M_{2\times 2}^+ \to \mathbb{R} \) be a function, and let \( \mu \in \mathcal{M}(M_{2\times 2}^+) \). Then:

(i) \( \Psi^2 = \text{id}_{M_{2\times 2}^+} \);

(ii) \( \tilde{\tilde{h}} = h \);

(iii) \( \tilde{\mu} = \mu \) and \( \text{supp} \tilde{\mu} = \Psi(\text{supp} \mu) \);
Theorem 4.2. The function $\mu \mapsto \bar{\mu}$ if and only if $\mu$ is polyconvex.

Proof. Part (i) has been shown, and (ii) follows from (i). For (iii), the fact that supp $\bar{\mu} = \Psi(\mu)$ follows directly from the definition of $\bar{\mu}$ and the support. The barycentre of $\bar{\mu}$ is

\begin{equation}
\bar{\mu} = \int X \ d\bar{\mu} = \frac{1}{|T|_2^2} \int X_{12} \Psi(X) \ d\mu(X) = \frac{1}{|T|_2^2} \left(-\int \det(X) \ d\mu + \frac{1}{|T|_{22}}\right).
\end{equation}

This shows that $\bar{\mu} \in \frac{|\Omega|_2}{|T|_2}$, and thus

$$\int f \ d\bar{\mu} = \bar{\mu} \int f \ d\mu = \int f \ d\mu \quad \text{by (ii)}.$$ 

Hence $\bar{\mu} = \mu$, and therefore (iii) holds. For (iv), the measure $\mu$ is polyconvex if and only if $\int \det X \ d\mu = \det(\bar{\mu})$, and hence (4.2) gives (iv). $\blacksquare$

Let $\mathcal{M}_{pc}(\mathbb{M}^{2\times 2}_+)$ be the set of polyconvex measures with support in $\mathbb{M}^{2\times 2}_+$, and define $\mathcal{M}_{qc}(\mathbb{M}^{2\times 2}_+)$ and $\mathcal{M}_{rc}(\mathbb{M}^{2\times 2}_+)$ similarly.

Theorem 4.2. The function $\mu \mapsto \bar{\mu}$ maps

(i) $\mathcal{M}_{pc}(\mathbb{M}^{2\times 2}_+)$ bijectively onto $\mathcal{M}_{pc}(\mathbb{M}^{2\times 2}_+)$,

(ii) $\mathcal{M}_{qc}(\mathbb{M}^{2\times 2}_+)$ bijectively onto $\mathcal{M}_{qc}(\mathbb{M}^{2\times 2}_+)$, and

(iii) $\mathcal{M}_{rc}(\mathbb{M}^{2\times 2}_+)$ bijectively onto $\mathcal{M}_{rc}(\mathbb{M}^{2\times 2}_+)$. 

Proof. If $\mu \in \mathcal{M}_{pc}(\mathbb{M}^{2\times 2}_+)$ then by Proposition 4.1

\[ \hfill \] 

and therefore $\bar{\mu} \in \mathcal{M}_{pc}(\mathbb{M}^{2\times 2}_+)$. This proves (i).

For (ii), the result will be proven for functions first, and then for measures. Let $h$ be quasiconvex in $\mathbb{M}^{2\times 2}_+$, let $\Omega \subseteq \mathbb{R}^2$ be a nonempty bounded domain, let $A \in \mathbb{M}^{2\times 2}_+$ and $\phi \in C_0^\infty(\Omega, \mathbb{R}^2)$ be such that $\text{ran}(A+\nabla \phi) \subseteq \mathbb{M}^{2\times 2}_+$. Then $A \in \mathbb{M}^{2\times 2}_+$.

Let $\psi(x) = Ax + \phi(x)$, and write $\psi = (\psi_1, \psi_2)$. Then $\frac{\partial \psi}{\partial x_2}$ is bounded below by a positive constant. This implies that $T_1$ is injective on $\Omega$, where

\[ T_1(x) = (x_1, \psi_1(x)), \quad T_2(x) = (x_2, \psi_2(x)). \]

The set $T_1(\Omega)$ is bounded since $\psi$ is Lipschitz, and $T_1$ is a diffeomorphism from $\Omega$ onto the bounded domain $T_1(\Omega)$ by the Inverse Function Theorem. Hence the Lipschitz map $g : T_1(\Omega) \to \mathbb{R}^2$ can be defined by $g \circ T_1 = T_2$. Let

\[ g_0(x) = g(x) - \Psi(A)x, \]

so that $g_0 \in C_0^\infty(T_1(\Omega), \mathbb{R}^2)$, and

\[ (\nabla g)(T_1(x)) = (\nabla \psi)(x), \]

by the definition of $\Psi$. The area of $T_1(\Omega)$ is

\[ m(T_1(\Omega)) = \int_{\Omega} \det(T_1(x)) \ dx = \int_{\Omega} A_{12} + (\nabla \phi)_{12}(x) \ dx = m(\Omega)A_{12}. \]
Hence
\[
\int_{\Omega} \tilde{h}(A + \nabla \phi(x)) \, dx = \int_{\Omega} (\nabla \psi)_{12}(x) h(\Psi(\nabla \psi(x))) \, dx \\
= \int_{\Omega} \det T_1(x) h((\nabla g)(T_1(x))) \, dx \\
= \int_{T_1(\Omega)} h((\nabla g)(x)) \, dx \\
= \int_{T_1(\Omega)} h(\Psi(A) + \nabla g_0(x)) \, dx \\
\geq m(T_1(\Omega)) h(\Psi(A)) \\
= m(\Omega) \tilde{h}(A).
\]
This shows that \( \tilde{h} \) is quasiconvex on \( M_+^{2 \times 2} \).

By Theorem 1.6 in [5], a compactly supported probability measure \( \mu \) on \( M_+^{2 \times 2} \) is a homogeneous gradient Young measure if and only if it satisfies Jensen’s inequality for all quasiconvex \( f : U \to \mathbb{R} \), where \( U \) is any open neighbourhood of \( (\text{supp} \mu)^{\mathcal{C}} \).

Hence if \( \mu \in \mathscr{M}_{qc}(M_+^{2 \times 2}) \) and \( f : M_+^{2 \times 2} \to \mathbb{R} \) is quasiconvex on \( M_+^{2 \times 2} \), then by Proposition 141,
\[
\int f \, d\tilde{\mu} = \frac{1}{\mu_{12}} \int \tilde{f} \, d\mu \geq \frac{1}{\mu_{12}} \tilde{f}(\tilde{\mu}) = f(\tilde{\mu}),
\]
and therefore \( \tilde{\mu} \in \mathscr{M}_{qc}(M_+^{2 \times 2}) \). This proves (ii). Part (iii) can also be done by duality; it suffices to show that \( \tilde{f} \) is rank-one convex whenever \( f \) is. This follows from part (ii) since a function is rank-one convex if and only if it satisfies Jensen’s inequality for all homogeneous gradient Young measures supported on two points. \( \square \)

**Theorem 4.3.** Every homogeneous gradient Young measure supported on \( M_+^{2 \times 2} \) is a laminate.

**Proof.** This follows from Theorem [3] and Theorem [4], since if \( \mu \) is a homogeneous gradient Young measure supported in \( M_+^{2 \times 2} \cap M_+^{2 \times 2} \), then \( \mu \) is a homogeneous gradient Young measure supported on \( M_+^{2 \times 2} \cap M_+^{2 \times 2} \), and therefore a laminate. This implies that \( \mu = \tilde{\mu} \) is a laminate. \( \square \)

**References**

[1] Ambrosio, L., Fusco, N., Pallara, D.: Functions of bounded variation and free discontinuity problems. The Clarendon Press, Oxford University Press, New York (2000)

[2] Chaudhuri, N., Müller, S.: Rank-one convexity implies quasi-convexity on certain hypersurfaces. Proc. Roy. Soc. Edinburgh Sect. A 133, 1263–1272 (2003)

[3] Conti, S., Faraco, D., Maggi, F., Müller, S.: Rank-one convex functions on \( 2 \times 2 \) symmetric matrices and laminates on rank-three lines. Calc. Var. Partial Differential Equations 24, 479–493 (2005)

[4] Evans, L. C., Gariepy, R. F.: On the partial regularity of energy-minimizing, area-preserving maps. Calc. Var. Partial Differential Equations 9, 357–372 (1999)
Kirchheim, B.: Rigidity and geometry of microstructures. Habilitation thesis, University of Leipzig (2003)

Lee, J., Müller, P. F. X., Müller, S.: Compensated compactness, separately convex functions and interpolatory estimates between Riesz transforms and Haar projections. Comm. Partial Differential Equations 36, 547–601 (2011)

Matoušek, J., Plecháč, P.: On functional separately convex hulls. Discrete Comput. Geom. 19, 105–130 (1998)

Morrey Jr, C. B.: Quasi-convexity and the lower semicontinuity of variational integrals. Pacific J. Math. 2, 25–53 (1952)

Müller, S.: Rank-one convexity implies quasiconvexity on diagonal matrices. Int. Math. Res. Not. 20, 1087–1095 (1999)

Müller, S.: A sharp version of Zhang’s theorem on truncating sequences of gradients. Trans. Amer. Math. Soc. 351, 4585–4597 (1999)

Müller, S., Šverák, V.: Convex integration with constraints and applications to phase transition and partial differential equations. J. Eur. Math. Soc. 1, 393–422 (1999)

Pedregal, P.: Parametrized measures and variational principles. Birkhäuser Verlag, Basel, (1997)

Šverák, V.: Rank-one convexity does not imply quasiconvexity. Proc. Roy. Soc. Edinburgh Sect. A 120, 185–189 (1992)

Department of Mathematics, University of Illinois, Urbana, IL 61801, U.S.A.
E-mail address: terence2@illinois.edu

Department of Mathematics, University of Leipzig
E-mail address: bernd.kirchheim@math.uni-leipzig.de

Department of Mathematics, National Taiwan Normal University, Taipei, 116 Taiwan
E-mail address: chunlin@math.ntnu.edu.tw