CHEEGER-MÜLLER THEOREM ON MANIFOLDS WITH CUSPS

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ABSTRACT. We prove equality between the renormalized Ray-Singer analytic torsion and the intersection R-torsion on a Witt-manifold with cusps, up to an error term determined explicitly by the Betti numbers of the cross section of the cusp and the intersection R-torsion of a model cone. In the first step of the proof we compute explicitly the renormalized Ray-Singer analytic torsion of a model cusp in general dimension and without the Witt-condition. In the second step we establish a gluing formula for renormalized Ray-Singer analytic torsion on a general class of non-compact manifolds in any dimension that includes Witt-manifolds with cusps, but also scattering manifolds with asymptotically conical ends. In the final step, a Cheeger-Müller theorem on cusps follows by a combination of the previous explicit computation and the gluing formula.

1. INTRODUCTION

One of the fundamental achievements in modern spectral geometry is the proof by Cheeger and Müller of the Ray-Singer conjecture, which asserts equality between the analytic and Reidemeister torsions of a compact smooth odd-dimensional manifold equipped with a flat Hermitian vector bundle, associated to a unitary representation of the fundamental group. Since the analytic torsion is defined in terms of the spectrum of the Hodge Laplace operator, and the Reidemeister torsion is a
purely combinatorial invariant, their equality (along with the Atiyah-Singer index theorem) has many crucial applications in fields including topology, number theory and mathematical physics, notably the Chern-Simons perturbation theory.

The Reidemeister torsion invariant for manifolds which are not simply connected was introduced by Reidemeister in [REI35A, REI35B] and extended to higher dimensions by Franz in [FRA35], as a tool for a full PL-classification of lens spaces. The Reidemeister torsion provided the first example of a topological invariant that distinguished homotopic but not homeomorphic spaces. The definition of Reidemeister and Franz was extended later to smooth manifolds by Whitehead [WHI50] and de Rham [RHA50], who proved that a spherical Clifford-Klein manifold is determined up to isometry by its fundamental group and its Reidemeister torsion.

Analytic torsion was introduced by Ray and Singer in their influential paper [RAS71] as an analytic counterpart to the Reidemeister torsion, and has been equated to the Reidemeister torsion in the setting of lens spaces. This observation has led Ray and Singer to conjecture equality of analytic and Reidemeister torsions on general closed odd-dimensional manifolds, which was proved in the celebrated theorem by Cheeger [CHE79] and Müller [MUE78]. Independent proofs of the Ray-Singer conjecture have been obtained by Bismut and Zhang using the Witten deformation [BIZH92], by Vishik using the gluing principle [VIS95] and Has-sel using analytic surgery [HAS98], to name a few.

The Cheeger-Müller theorem extends to compact manifolds with boundary under product type assumptions on the metric structures, cf. Lück [LU93] and Vishik [VIS95]. In that case, both the analytic Ray-Singer and the combinatorial Reidemeister torsions are equal up to an error term determined explicitly by the Euler characteristic of the boundary. Dependence of analytic torsion on the metric structures near the boundary has been studied by Brüning-Ma [BRMA06] and Dai-Fang [DAFA].

Establishing a Cheeger-Müller type theorem outside the setting of compact smooth manifolds has proven a tedious task with various incremental steps being taken in this direction. We mention here partial results obtained in the setting of spaces with isolated conical and edge singularities, hyperbolic as well as scattering spaces.

On manifolds with isolated conical singularities, the cut and paste property of analytic torsion, established by Lesch [LES12], reduces the analysis to the discussion of a truncated cone. Explicit though intricate formulae for analytic torsion of a truncated cone have been derived using the double summation method of Spreafico [SPR12, SPR05], by Melo-Hartmann-Spreafico [MHS09] and Vertman [VER09]. Further understanding the various terms in the explicit formula for analytic torsion has been obtained by Müller-Vertman [MUVE14] and Hartmann-Spreafico [HASPR10].

On the combinatorial side, Dar [DAR87] has introduced the intersection R-torsion for stratified spaces, computed recently by Dai-Huang [DHA10] in context of truncated cones. The construction is based on the intersection cohomology theory by
Goresky and MacPherson [GoMa80, GoMa80]. The intersection R-torsion of Dar
is defined a priori only for flat vector bundles over the stratified space. However,
Albin, Rochon and Sher [ARS14(a), §8] have extended the combinatorial definition
to a class of flat vector bundles defined over the smooth stratum only.

In view of the gluing formula for analytic torsion by Lesch [Les12], one seeks to
establish a Cheeger-Müller type result by comparing the intersection R-torsion with
the analytic Ray-Singer torsion for model cones, an ansatz which has not yet been
successful due to highly non-trivial spectral contributions on the analytic side. Nev-
evertheless, conjecturing a topological interpretation for the spectral analytic torsion
invariant seems reasonable for a class of singular spaces by a recent observation of
metric independence for manifolds with edges by the author jointly with Mazzeo
in [MaVe12].

In the setting of non-compact hyperbolic spaces, the original definition of ana-
lytic torsion does not make sense due to the continuous spectrum of the Hodge
Laplacian. Still, a renormalized version of analytic torsion exists and the intricate
algebraic structure of the hyperbolic space, equipped with a flat Hermitian vec-
tor bundle that corresponds to a canonical non-unitary unimodular representation
of the fundamental group, allows for a deep analysis of the relation between the
renormalized analytic and Reidemeister torsions by Pfaff [Pfa14] and Müller-Pfaff
[MuPf14a, MuPf14b]. Renormalized analytic torsion has also been discussed in the
setting of non-compact asymptotically conical (scattering) manifolds by Guillarmou
and Sher in [GuSh13], though its relation to the intersection Reidemeister torsion is
still an open question.

Our discussion is organized as follows. §2 is devoted to setting the notation, in-
troducing the fundamental concepts and stating the main results. In §3 we gather
all the relevant facts on Bessel functions. We apply these facts to define zeta-
regularized determinants of certain cusp operators in §4. We establish a variational
formula for that determinant in §5. We then turn to the structure of the de Rham
complex of a model cusp and decompose the complex in §6 into suitable subcom-
plexes. In §7 we establish an integral representation of a zeta function associated
to certain subcomplexes and proceed in §8 with its analytic continuation to zero
using Spreafico’s double summation method. In §9 we compute the renormalized
analytic torsion of the model cusp explicitly in terms of the Euler characteristic and
Betti numbers of the cross section. The gluing formula for the renormalized ana-
lytic torsion on certain classes of complete non-compact manifolds is proved in §10.
These results allow us to deduce a Cheeger-Müller type result for Witt-manifolds
with cusps in §11.

Remark 1.1. Parallel to the present announcement, Albin, Rochon and Sher
[ARS14(A)] have uploaded a very interesting discussion of renormalized torsion
on a general class of manifolds with fibered cusps. Their results use a completely
different ansatz and methodology, and initially require “strong acyclicity” assump-
tions, which we do not pose in the present discussion. The strong acyclicity as-
sumption was relaxed to the Witt condition in the special case of (non-fibered)
cusps in their subsequent paper [ARS14(b)], where the authors obtained our result by a method of degeneration.

**Remark 1.2.** Combination of the gluing formula in §10 and Guillarmou-Sher [GuSh13] allows to compute analytic torsion of a truncated cone in terms of the renormalized analytic torsion of the infinite cone. This leads to potentially new computational results using Lesch [Les97, Chapter II].

## 2. Preliminaries and statement of the main results

In this section we outline some fundamental facts on manifolds with cusps and state our main results.

### 2.1. Riemannian manifolds with cusps.

The present work deals with non-compact Riemannian manifolds with cusps, equipped with a flat Hermitian vector bundle that corresponds to a unitary representation. More precisely, let \((M, g)\) be an oriented complete Riemannian manifold of odd dimension \(\dim M = m\), where \(M = K \cup N \cup \mathcal{U}\) is a union of a compact manifold \(K\) with boundary \(\partial K = N \sqcup N'\) comprised of two boundary components, and \(\mathcal{U} = N \times [1, \infty)\) is a non-compact end glued to \(K\) along \(N = N \times \{1\}\). We assume

\[
g \mid \mathcal{U} = \frac{dx^2 + g_N^N}{x^2}, \quad x \in [1, \infty),
\]

where \(g_N^N\) is a Riemannian metric on the closed manifold \(N\) of dimension \(\dim N = n\).

Fix a base point \(q \in M\) and consider a unitary representation of the fundamental group \(\rho : \pi_1(M, q) \to U(\tau, \mathbb{C})\). The corresponding vector bundle \(E\) is equipped with the canonical Hermitian metric \(h\), induced by the standard Hermitian inner product on \(\mathbb{C}^\tau\), and the canonical flat covariant derivative \(\nabla\), induced by the exterior derivative on the universal cover \(\tilde{M}\). Denote by \(\Omega^k_0(M, E)\) the space of \(E\)-valued differential forms of degree \(k\), compactly supported in the open interior of \(M\). The covariant derivative extends by Leibniz rule to a differential on \(\Omega^*_0(M, E)\) and by flatness defines the corresponding de Rham complex \((\Omega^*_0(M, E), d)\).

We should point out that the condition on the vector bundles to be associated to a unitary representation, is posed so that the induced Hermitian metric structure is product over \(\mathcal{U}\). Equivalently, in our statements we may simply choose a (non-canonical) Hermitian metric that is product over \(\mathcal{U}\) without specifying the underlying representation \(\rho\).

The metric structures \((g, h)\) define an \(L^2\)-scalar product on \(\Omega^*_0(M, E)\) and we denote its completion with respect to the \(L^2\)-scalar product by \(L^2_0(M, E; g, h)\). Let \(d_p^*\) denote the formal adjoint of \(d_p\), acting on \(\Omega^p_0(M, E)\), and consider the Hodge Laplace operator

\[
\Delta_p := d_p^*d_p + d_{p-1}d_{p-1}^* : \Omega^p_0(M, E) \to \Omega^p_0(M, E).
\]
In order to fix a self-adjoint realization of $\Delta$, in $L^2_p(M, E; g, h)$, we recall the notion of the maximal domain for any differential operator $P : \Omega^*_0(M, E) \to \Omega^*_0(M, E)$

\[(\mathcal{D}_{\text{max}}(P)) := \{ \omega \in L^2_p(M, E; g, h) \mid P\omega \in L^2_p(M, E; g, h) \}, \]

where $P\omega \in L^2_p$ is understood in the distributional sense.

We can now introduce self-adjoint domains of $\Delta$, with either relative or absolute boundary conditions at $N' = \partial M$. More precisely, let $\iota : N' \hookrightarrow M$ be the obvious embedding, and $\ast$ be the Hodge star operator on $M$. Then we define two natural geometric self-adjoint extensions $\Delta_{p,\text{rel}}$ and $\Delta_{p,\text{abs}}$ for the Hodge Laplacian $\Delta_p$ by specifying their domains

\[
\mathcal{D}_{\text{rel}}(\Delta_p) := \{ \omega \in \mathcal{D}_{\text{max}}(\Delta_p) \mid \iota^* \omega = 0, \iota^*(d^i \omega) = 0 \},
\]

\[
\mathcal{D}_{\text{abs}}(\Delta_p) := \{ \omega \in \mathcal{D}_{\text{max}}(\Delta_p) \mid \iota^* (\ast \omega) = 0, \iota^*(\ast d\omega) = 0 \},
\]

respectively.

### 2.2. Renormalized analytic torsion on manifolds with cusps.

Consider a family of compact submanifolds $M_\mathbb{R} := \mathbb{K} \cup_N (N \times [1, R]) \subset M$, parametrized by $R \geq 1$. The following observation forms the basis for the general definition of renormalized analytic torsion and is a consequence of explicit computations on the cusp $\mathcal{U}$ and the microlocal heat kernel description by Vaillant [VA101], cf. §10.4.1 for the proof.

**Theorem 2.1.** Let $\text{tr} \mathcal{H}_p$ denote the pointwise trace of the heat kernel $\mathcal{H}_p$ of the Hodge Laplacian $\Delta_p$ with either absolute or relative boundary conditions at $N' = \partial M$.

(i) Then in each degree $p = 0, \ldots, m$, there exists a finite family $(\gamma_j)_j \subset \mathbb{R}$ of positive numbers, and $(k_j)_j \subset \mathbb{N}_0$ such that

\[
\int_{M_\mathbb{R}} \text{tr} \mathcal{H}_p(t, \gamma, \gamma) \text{dvol}_g(\gamma) \sim_{R \to \infty} \sum_{j=0}^{r} \sum_{k=0}^{k_j} a_{jk}(t) R^{\gamma j} \log^k(R) + a_0(t) + o(1).
\]

(ii) The renormalized trace\(^\text{a}\) $\text{Tr}_r \mathcal{H}_p(t)$ is then defined to be the constant term $a_0(t)$ in the asymptotics. There exists a finite family $(\alpha_j)_j \subset \mathbb{R}$ of negative numbers, and $(i_j)_j \subset \mathbb{N}_0$, such that for some $\epsilon > 0$

\[
\text{Tr}_r \mathcal{H}_p(t) \sim_{t \to 0^+} \sum_{j=0}^{\ell} \sum_{i=0}^{i_j} b_{ij} t^{\alpha_j} \log^i(t) + b_0 + O(t^{\ell}).
\]

(iii) Assume the Witt condition $H^{n/2}(N, \mathbb{E}) = 0$. Denote by $\ker \Delta_p$ the finite dimensional subspace of harmonic forms in $L^2_p(M, E; g, h)$ with fixed boundary conditions at $N'$. Then $\text{Tr}_r \mathcal{H}_p(t) - \dim \ker \Delta_p$ is of exponential decay as $t \to \infty$.

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\(^\text{a}\)The restrictions $\iota^* \omega, \iota^*(d\omega), \iota^*(d^i \omega)$ are well-defined for any $\omega \in \mathcal{D}_{\text{max}}(\Delta_p)$ by [PA082, Th. 1.9].

\(^\text{b}\)We point out that the notion of renormalized trace for non-trace class operators strongly depends on the choice of a defining function $x$. 
As a consequence of Theorem 2.1, the zeta function of the Hodge Laplacian \( \Delta_p \) with either either relative or absolute boundary conditions at \( N' = \partial M \)

\[
(2.5) \quad \zeta(s, \Delta_p) := \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} (\text{Tr}_r \mathcal{H}_p(t) - \dim \ker \Delta_p) \, dt, \quad \text{Re}(s) \gg 0,
\]

is well-defined and extends meromorphically to \( \text{Re}(s) > -\varepsilon \), with a simple pole at \( s = 0 \) with residue \( b_0 \). Hence we may define the renormalized analytic torsion as follows.

**Definition 2.2.** The scalar renormalized analytic torsion of \((M, E, g, h)\) with respect to either relative or absolute boundary conditions at \( N' = \partial M \) is denoted by either \( T(M, E, N') \in \mathbb{R}^+ \) or \( T(M, E) \in \mathbb{R}^+ \), respectively, and defined by specifying its logarithmic value as follows\(^3\)

\[
\log T(M, E, N') := \frac{1}{2} \sum_{p=0}^{m} (-1)^p \left. \frac{d}{ds} \right|_{s=0} \left( \zeta(s, \Delta_{p,\text{rel}}) - s^{-1} \text{Res} \zeta(s, \Delta_{p,\text{rel}}) \right),
\]

\[
\log T(M, E) := \frac{1}{2} \sum_{p=0}^{m} (-1)^p \left. \frac{d}{ds} \right|_{s=0} \left( \zeta(s, \Delta_{p,\text{abs}}) - s^{-1} \text{Res} \zeta(s, \Delta_{p,\text{abs}}) \right).
\]

Let \( H^p(M, E) := \ker \Delta_{p,\text{abs}} \) denote the subspace of harmonic forms in \( L^2_p(M, E; g, h) \) with absolute boundary conditions. We write \( \det H^p(M, E) := \Lambda^\top \det H^p(M, E) \). The determinant line of \( L^2\)-cohomology is defined by

\[
\det H^*(M, E) := \bigotimes_{p=0}^{m} \det H^p(M, E)^{(-1)^{p+1}},
\]

where \( V^{-1} \) denotes the dual of a finite-dimensional vector space \( V \). The \( L^2\)-inner product of \( L^2_p(M, E; g, h) \) yields a norm of \( H^*(M, E) \) and \( \det H^*(M, E) \), which we denote by \( \| \cdot \|_{\det H^*(M, E)} \). Analogous construction makes sense for the Hodge Laplacian with relative boundary conditions at \( N' \). In that case we denote the harmonic forms with relative boundary conditions by \( H^*(M, E, N') \), and its determinant line by \( \det H^*(M, E, N') \).

**Definition 2.3.** The renormalized Ray-Singer analytic torsion with either relative or absolute boundary conditions at \( N' = \partial M \), is defined as a norm on the corresponding determinant line of \( L^2\)-cohomology by

\[
\| \cdot \|_{\text{RS}(M, E)} := T(M, E) \| \cdot \|_{\det H^*(M, E)},
\]

\[
\| \cdot \|_{\text{RS}(M, E, N')} := T(M, E, N') \| \cdot \|_{\det H^*(M, E, N')},
\]

2.3. **Brüning-Ma metric anomaly and analytic torsion of a model cusp.** Our first main result identifies \( T(U, E, N) \) explicitly in terms of twisted cohomology of the cross section \((N, E \uparrow N)\) and metric anomaly of analytic torsion at the regular boundary \( N \times \{1\} \). The metric anomaly has been studied by Dai-Fang in [DaFA] and Brüning-Ma [BrMA06]. In [MHS09], de Melo, Hartmann and Spreafico validated

\(^3\)We omit the metric structures \( g, h \) from notation in case they are fixed.
the anomaly formula of Brüning-Ma. We recall the basic facts from [BrMa06] that are needed for the statement of our first main result.

Consider an oriented compact Riemannian manifold \((X, g^X)\) of odd dimensions, with boundary \(\partial X\), equipped with a flat Hermitian vector bundle \((E, \nabla, h)\). Denote by \(\nabla^TX\) the Levi-Civita connection of the Riemannian metric \(g^X\). Brüning and Ma define in [BrMa06, (1.19)] a secondary class \(B(\nabla^TX) \in \Omega^*(\partial X, E_{|\partial X})\), which depends only on the jets of \(g^X\) at \(\partial X\), is trivially zero if \(g^X\) is product in a neighborhood of the boundary, and describes the metric anomaly in the following sense.

Consider two Riemannian metrics \(g^X_1, g^X_2\), on \(X\) and denote by \(\nabla^TX_i\), \(i = 1, 2\), the corresponding Levi-Civita connections. The Ray-Singer analytic torsion norms on \(\det g^X\), corresponding to \(g^X_1, g^X_2\), are denoted by \(\|\cdot\|_{RS(X,E,g^X_i)}\), \(i = 1, 2\), respectively, and are defined similar to Definition 2.3 without the renormalization procedure in Theorem 2.1 for the heat trace. Then

\[
\log \left( \frac{\| \cdot \|_{RS(X,E,g^X_1)}}{\| \cdot \|_{RS(X,E,g^X_2)}} \right) = \frac{\text{rank}(E)}{2} \left[ \int_{\partial X} B(\nabla^TX_2) - \int_{\partial X} B(\nabla^TX_1) \right].
\]

We can now state our first main result.

**Theorem 2.4.** Denote the twisted cohomology of \((N, E | N)\) by \(H^*(N, E)\) and the Euler characteristic by

\[
\chi(N, E) = \sum_{p=0}^{n} (-1)^p \dim H^p(N, E).
\]

Then for \(R > 1\) sufficiently large, the renormalized scalar analytic torsion of the model cusp \((U_R = [R, \infty) \times N, g)\) with respect to relative boundary conditions at \(\partial U_R \equiv N \times \{R\}\) is given by

\[
\log T(U_R, E, N, g) = \frac{\text{rank}(E)}{2} \left[ \int_{\partial U_R} B(\nabla^{TX}_{\nabla^X g}) + \sum_{p \neq n/2} \frac{(-1)^p}{4} \dim H^p(N, E) \log \left| \frac{n}{2} - p \right| \right]
+ \left[ \sum_{p \neq n/2} \frac{(-1)^p}{4} \dim H^p(N, E) \left| \frac{n}{2} - p \right| \log \left( \frac{n}{2} - p \right) \right]
+ \sum_{p=0}^{n} \frac{(-1)^p}{2} \dim H^p(N, E) \left| \frac{n}{2} - p \right| \log R.
\]

**2.4. Gluing formula for analytic torsion on non-compact manifolds.** Consider a non-compact oriented odd-dimensional Riemannian manifold \((M, g)\) with \(M = K \cup \partial N \cup U\), where we do not specify the behavior of \(g\) over \(U\), but pose in view of Theorem 2.1 the following

**Assumption 2.5.** The pointwise trace \(\text{tr} \mathcal{H}_p\) of the heat kernel \(\mathcal{H}_p\) of the Hodge Laplacian \(\Delta_p\) with either relative or absolute boundary conditions at \(N'\) admits in each degree \(p = 0, \ldots, m\), an asymptotic expansion (2.3) and the renormalized
we may define in each degree and 2 we may define the renormalized concerning the 2 of ∆ function of the Hodge Laplacian (for some finite families Tr(\epsilon)2 integrable harmonic forms is finite dimensional.

Under Assumption 2.5 we may define in each degree p = 0, ..., m, the zeta-function of the Hodge Laplacian Δp in terms of regularized integrals

\[ \zeta(s, Δ_p) := \frac{1}{Γ(s)} \int_{0}^{1} t^{s-1}Tr, H_p(t)dt \]

where the regularized integral \( \int_{0}^{1} \) is defined as the constant term in the asymptotic expansion of \( \int_{0}^{1} u \) as \( u \to 0 \). Existence of a partial asymptotic expansion is a consequence of (2.4). The regularized integral \( \int_{1}^{∞} \) is defined similarly as the constant term in the asymptotic expansion of \( \int_{1}^{∞} u \) as \( u \to ∞ \).

The zeta function \( \zeta(s, Δ_p) \) extends meromorphically to an open neighborhood of \( s = 0 \), and following Definitions 2.2 and 2.3 we may define the renormalized Ray-Singer analytic torsion of \( (M, E, g, h) \). A gluing formula for the renormalized Ray-Singer norm \( \| \cdot \|_{RS}^{M, E} \) is established here under the additional two assumptions.

**Assumption 2.6.**

(i) Consider in each degree p a smooth one-parameter family \( Δ_p, θ \in S^1 \), of self-adjoint operators in \( L^2_p(M, E, g, h) \) with \( Δ_p, θ = Δ_p + V_θ \), where the perturbation \( V_θ \) arises in one of the following two ways.

Either \( Δ_p, θ \) is defined by a smooth family of Riemannian metrics \( g_θ \) which coincide outside a compact neighborhood \( K \subset M \). Alternatively, \( V_θ \) commutes with and vanishes on any smooth section that is trivially zero in an open neighborhood of a compact subset \( K \subset U_{R-ε} \setminus U_{R+ε} \cong (R-ε, R + ε) \times N \) for some \( ε > 0 \). We consider the obvious reflection mapping \( S : (R-ε, R) \times N → [R, R + ε) \times N \) and identify \( Ω^*(U_{R-ε} \setminus U_{R+ε}, E) \) with \( Ω^*([R, R + ε) \times N, E ⊕ S^*E) \). We assume that under such identification \( V_θ \) acts as a first order differential operator on \( Ω^*([R, R + ε) \times N, E ⊕ S^*E) \) with compact support \( supp V_θ \subset (R, R + ε) \times N \).

Assume that the corresponding one-parameter family of heat kernels \( H_{p, θ} \) satisfies for each \( θ \in S^1 \) the second part of Assumption 2.5 concerning the
large times asymptotic expansion of the renormalized trace. Assume that the large time asymptotic expansion is differentiable in $\theta$.

Denote by $P_{p,\theta}$ the integral kernel of the orthogonal projection onto the kernel of $\Delta_{p,\theta}$. For any $\phi \in C^\infty_0(M)$ the kernel $\phi H_{p,\theta}$ is trace class and we assume that its trace admits an asymptotic expansion for large times that is stable under $(t \partial_t)$ differentiation

$$\text{Tr} \phi H_{p,\theta} = \text{Tr} \phi P_{p,\theta} + O(t^{-\sigma}), \quad t \to \infty.\quad (2.8)$$

(ii) Consider cutoff functions $\phi, \psi \in C^\infty(M, \mathbb{R})$ with $\text{supp} \phi \subset M$ compact, $\text{supp} \phi \cap \text{supp} \psi = \emptyset$. Denote by $D = d + d^t$ the Gauß Bonnet operator. Then for any $Q \in \mathbb{N}$ we assume that

$$|\phi(q) \text{tr} H_{p,\theta}(t, q, \cdot)\psi(\cdot)| \leq f_1 \cdot t^Q,$$
$$|\phi(q) \text{tr} (D H_{p,\theta}(t, q, \cdot))\psi(\cdot)| \leq f_2 \cdot t^Q,$$

with $f_1, f_2 \in L^2(M, E, g, h)$, uniformly in $t \in (0, t_0]$ and $q \in M$.

Assumption 2.6 is designed specifically to cover relatively compact perturbations of the Hodge Laplacian that appear in the gluing formula for analytic torsion by Lesch [Les12, Section 3.1], as well as compactly supported perturbations of the Riemannian metric $g$. The assumption is satisfied for two fundamental classes of spaces, complete manifolds with a spectral gap around zero, and spaces with a microlocal calculus of the resolvent at low energies, cf. Guillarmou and Sher [GuSh13], which corresponds to a microlocal description of the heat kernel at large times.

We point out that assuming the partial asymptotic expansion (2.8) even without specifying the explicit form of the constant term, the form of the constant term is obtained from a theorem by Chavel and Karp [CHKa91] with an elaboration by Simon [Sim93], where the result by Chavel and Karp was shown to be a straightforward consequence of the spectral theorem and elliptic regularity.

The second part of Assumption 2.6 is a replacement of the classical off-diagonal Gaussian estimates on the heat kernel, with emphasis on integrability of the estimates in the second spacial component. Gaussian upper bounds (in fact for all times) first appeared in the setting of non-compact complete manifolds with bounded sectional curvature in the work of Cheng, Li and Yau [CLY81]. Davies [Dav88] developed an abstract method for the derivation of Gaussian estimates from the log-Sobolev inequality, and established pointwise Gaussian bounds for the spatial and time derivative of the heat kernel in [Dav89]. Sharp estimates have been obtained by Li and Yau [LiYa86] under certain curvature assumptions, to name a few results in this direction.

However, without the assumption of bounded sectional curvature, Gaussian estimates may not hold in general, with certain examples discussed by Barlow and Bass [BaBa99], cf. also Grigor’yan and Telcs [GrTe01]. Moreover, in various microlocal descriptions of the heat kernel asymptotics, see [Vai01], [GuSh13] and [MAVe12], Gaussian estimates are not directly available. Assumption 2.6 allows to encompass these examples and is still sufficient for the analytic arguments here.
A central observation is now invariance of the renormalized Ray-Singer analytic torsion under compactly supported perturbation of the Riemannian metric.

**Theorem 2.7.** Let \((g_\theta)_\theta, \theta \in \mathbb{R}\), denote a smooth family of Riemannian metrics on \(M\), with \(\text{supp} \frac{d}{d\theta} g_\theta\) contained in a compact neighborhood of \(M\). Then under Assumptions 2.5 and 2.6 the renormalized Ray-Singer analytic torsion \(d \cdot \|_{(M,E,g_\theta)}^{\text{RS}}\), defined with respect to \(g_\theta\), is a smooth family of norms on \(\det H^*(M,E)\) such that

\[
\frac{d}{d\theta} \|_{(M,E,g_\theta)}^{\text{RS}} = 0.
\]

We derive a gluing formula for the renormalized Ray-Singer analytic torsion under the third and final assumption.

**Assumption 2.8.** Either the spectrum \(\text{spec} \Delta_*\) of the Hodge Laplacian in all degrees admits a spectral gap around zero, i.e. there exists \(\varepsilon > 0\) such that \((0, \varepsilon) \cap \text{spec} \Delta_* = \emptyset\); or the Hermitian vector bundle \((E, \nabla)\) is acyclic over \(N\), i.e. \(H^*(N,E) = 0\).

Non-compact manifolds \((M,g)\) satisfying these three assumptions include two particular classes of spaces. On one hand, the previously introduced manifolds with cusps and the Witt condition \(H^{n/2}(N,E) = 0\) satisfy Assumptions 2.5, 2.6 and 2.8. These spaces have been studied by Vaillant [VA101] and are closely related to the hyperbolic manifolds with cusps, studied e.g. by Müller-Pfaff in [MuPF14A], [MuPF14B]. On the other hand, a second example comes from scattering manifolds, studied by Guillarmou and Sher [GuSH13], with \(g \upharpoonright \mathcal{U} = dx^2 + x^2 g^N\) and \(H^*(N,E) = 0\).

We now formulate the gluing formula for the renormalized Ray-Singer analytic torsion in terms of canonical isomorphisms between determinants of relative and absolute cohomologies, defined in terms of long exact sequences in cohomology. Consider \(M = K \cup_N \mathcal{U}\), assume \(\partial M = \emptyset\) for notational simplicity\(^4\) and introduce for the obvious inclusion \(\iota\) of \(N = N \times \{1\}\) into either \(K\) or \(\mathcal{U}\) the following complexes

\[
\begin{align*}
\Omega^r_*(\mathcal{U},E) &:= \{\omega \in \Omega^*(\mathcal{U},E) \mid \iota^* \omega = 0\}, \\
\Omega^r_*(K,E) &:= \{\omega \in \Omega^*(K,E) \mid \iota^* \omega = 0\}, \\
\Omega^r_*(M,E) &:= \{\omega \in \Omega^*(\mathcal{U},E) \oplus \Omega^*(K,E) \mid \iota^* \omega_1 = \iota^* \omega_2\}.
\end{align*}
\]

We consider the following short exact sequences of complexes

\[
\begin{align*}
0 &\rightarrow \Omega^r_*(\mathcal{U},E) \xrightarrow{\alpha} \Omega^r_*(M,E) \xrightarrow{\beta} \Omega^*(K,E) \rightarrow 0, \\
0 &\rightarrow \Omega^r_*(\mathcal{U},E) \oplus \Omega^r_*(K,E) \xrightarrow{\gamma} \Omega^r_*(M,E) \xrightarrow{r} \Omega^*(N,E) \rightarrow 0,
\end{align*}
\]

where \(\alpha(\omega) = (\omega, 0)\) is the extension by zero, \(\beta(\omega_1, \omega_2) = \omega_2\) is the restriction to \(K\), \(\gamma\) is the obvious inclusion, and \(r\) the restriction to \(N \times \{1\}\). The harmonic forms of \(\Omega^r_*(M,E)\) and \(\Omega^*(M,E)\) coincide by [Vis95, Proposition 1.1]. Hence, the

\(^4\)Our main results hold also for \(\partial M \neq \emptyset\) with relative or absolute boundary conditions fixed at \(\partial M\).
\[L^2\]-cohomology of the complex \( \Omega^*_p(M, E) \) coincides with \( H^*(M, E) \), and the short exact sequences yield the following long exact sequences in cohomology

\[
\begin{align*}
\mathcal{H}(U, K) : & \cdots \rightarrow H^p(U, E, N) \xrightarrow{\alpha^*} H^p(M, E) \xrightarrow{\beta^*} H^p(K, E) \xrightarrow{\delta^*} H^{p+1}(U, E, N) \rightarrow \cdots, \\
\mathcal{H}(U, K, N) : & \cdots \rightarrow H^p(U, E, N) \oplus H^p(K, E, N) \xrightarrow{\gamma^*} H^p(M, E) \xrightarrow{\delta^*} H^{p+1}(U, E, N) \oplus H^{p+1}(K, E, N) \rightarrow \cdots,
\end{align*}
\]

where \( \delta^* \) denotes the respective connecting homomorphisms.

The long exact sequences in cohomology induce isomorphisms between determinant lines in a canonical way, cf. Nicolaescu [Nico03]

\[
\begin{align*}
\Phi : & \det H^*(U, E, N) \otimes \det H^*(K, E) \rightarrow \det H^*(M, E), \\
\Phi' : & \det H^*(U, E, N) \otimes \det H^*(K, E, N) \otimes \det H^*(N, E) \rightarrow \det H^*(M, E).
\end{align*}
\]

We may now state our second main result.

**Theorem 2.9.** Consider a non-compact oriented odd-dimensional Riemannian manifold \((M, g)\) with \( M = K \cup_{\partial} U \) and a flat Hermitian vector bundle \((E, \nabla, h)\), satisfying Assumptions 2.5, 2.6, 2.8. Let the metric structures \((g, h)\) be product in an open neighborhood of the cut \( N \). Then renormalized Ray Singer analytic torsion obeys the following gluing laws

\[
\| \Phi(\cdot \otimes \cdot) \|_{RS(M, E)} = 2^{\frac{\chi(N, E)}{2}} \| \cdot \|_{RS(U, E, N)} \otimes \| \cdot \|_{RS(K, E)},
\]

\[
\| \Phi'(\cdot \otimes \cdot \otimes \cdot) \|_{RS(M, E)} = \| \cdot \|_{RS(U, E, N)} \otimes \| \cdot \|_{RS(K, E, N)} \otimes \| \cdot \|_{\det H^*(N, E)}.\]

**2.5. Cheeger-Müller theorem for Witt-manifolds with cusps.** Consider the noncompact manifold \( M = K \cup_{\partial} U \), \( \partial K = N \), and its one-point compactification \( M^* = M \cup \{ \infty \} \), which may be viewed as a stratified space with the principal stratum \( M \), a single singular stratum \( \{ \infty \} \) of zero dimension and a conical neighborhood \( U^* = U \cup \{ \infty \} \) with cross section \( N \).

Goresky and MacPherson [GoMa80, GoMa83] have introduced an intersection cohomology theory \( IH^*(M^*, E) \) of stratified spaces by specifying a geometric condition of allowable simplicial chains, the so-called perversity \( p \). Assuming the Witt condition \( H^{n/2}(N, E) = 0 \), the intersection cohomology \( IH^*(M^*, E) \) in middle (upper and lower) perversity of Goresky-MacPherson coincides with the \( L^2\)-cohomology of \( M \) with the cusp metric \( g \), compare for instance the Hodge cohomology theory by Hausel, Hunsicker and Mazzeo [HHMo4], which can be easily extended to the case of flat unitary vector bundles \( E \) and yields

\[
IH^*(M^*, E) \cong H^*(M, E).
\]

Let \( h \) denote a preferred basis on \( IH^*(M, E) \) and consider the (scalar) intersection R-torsion \( \tau(M^*, E, h) \) of \( M^* \) defined with respect to the preferred basis \( h \). The

---

In case \( \partial M \neq \emptyset \) we fix either relative or absolute boundary conditions at the boundary in the combinatorial as well as in the analytic setting. In the combinatorial setting, a cochain satisfying relative boundary conditions is zero on boundary chains, by definition; and absolute boundary conditions pose no restriction.
intersection R-torsion has been introduced by Dar [Dar87] for flat vector bundles over $M^*$, which excludes vector bundles defined over $M$ with non-trivial restriction $E \upharpoonright N$. The extension of the combinatorial construction to flat vector bundles defined in the smooth interior $M \subset M^*$ is due to Albin, Rochon and Sher [ARS14(a), §8].

In [ARS14(a), Proposition 8.14 and 8.15] the authors prove that the intersection R-torsion $\tau(M^*,E,h)$, defined for flat vector bundles over $M$ corresponding to unimodular representations of the fundamental group, is indeed a topological invariant, depending only on the choice of $h$. The basis $h$ yields a norm $\| \cdot \|_{\det IH^*(M^*,E),h}$ on the determinant line bundle of the intersection cohomology and we define the intersection R-torsion norm by

$$\| \cdot \|_{R(M^*,E)}^R = \tau(M^*,E,h) \| \cdot \|_{\det IH^*(M^*,E),h}.$$

As a norm, the intersection R-torsion is independent of the choice of $h$.

In order to state our final result, fix a preferred basis $h_N$ on $H^*(N,E)$ which is orthonormal with respect to $g_N$. Since

$$h_N \text{ yields a preferred basis on the intersection cohomology } IH^*(U^*,E).$$

Let $\tau(U^*,E,h_N)$ be the scalar intersection R-torsion on $U^*$, defined with respect to the preferred basis $h_N$.

Our final main result compares the intersection R-torsion norm of $M^*$ with the renormalized Ray-Singer analytic torsion of $(M,g)$ as norms on the determinant lines $\det IH^*(M,E) \cong \det H^*(M,E)$.

**Theorem 2.10.** Let $(M,g)$ be an odd dimensional non-compact Witt-manifold without boundary and with a cusp end $U = N \times [1,\infty)$ and $g \upharpoonright U = x^{-2}(dx^2 + g_N)$. Let $M$ be equipped with a flat Hermitian vector bundle $E$, induced by a unitary representation of the fundamental group. The intersection R-torsion $\| \cdot \|_{R(M^*,E)}^R$ and the renormalized Ray-Singer analytic torsion $\| \cdot \|_{RS(M,E,g)}^R$, both defined with respect to relative or absolute boundary conditions at $\partial M$, are norms on the determinant line $\det IH^*(M,E) \cong \det H^*(M,E)$ and

$$\log \frac{\| \cdot \|_{RS(M,E,g)}^R}{\| \cdot \|_{R(M^*,E)}^R} = - \log \tau(U^*,E,h_N)$$

$$+ \sum_{p \neq n/2} \frac{(-1)^{p+1}}{2} \dim H^p(N,E) \left| \frac{n}{2} - p \right| \log \left( 2 \left| \frac{n}{2} - p \right| \right).$$

The statement extends in the obvious way to the case of finitely many cusps by the gluing formula for the renormalized and intersection R-torsions.

The basic idea behind the proof of Theorem 2.10 is the use of the gluing formula in Theorem 2.9 to reduce the analysis to the model cusp $U$. On the analytic side we may then apply Theorem 2.4. On the combinatorial side we point out that the
intersection R-torsion of a model cone has been studied by Dai-Huang in [DAHU10].

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3. Asymptotics of modified Bessel functions

In this section we gather all the relevant statements on the asymptotics of the modified Bessel functions of first and second kind, denoted by $I_t(s)$ and $K_t(s)$, respectively. We denote their respective derivatives in $s$ by $I'_t(s)$ and $K'_t(s)$. We distinguish between the following three cases: fixed argument $s$ and large order $t$, large argument and fixed order, as well as large argument and large order. We employ the standard references Abramowitz and Stegun [AbrSt92], Olver [Ol97] as well as Watson [Wat66]. We also refer to Sidi and Hoggan [StHo11] in §3.2.

3.1. Asymptotics for large arguments and fixed order. We infer from [AbrSt92, (9.7.1), (9.7.2)], see also [Wat66, p.202, §7.23 (1)-(2)], that for fixed order $t \in \mathbb{C}$ and large argument $s$ the Bessel functions admit the following asymptotic expansions

$$I_t(s) = \frac{e^s}{\sqrt{2\pi s}} \left(1 + \sum_{k=1}^{\infty} a_k s^{-k}\right),$$

$$K_t(s) = \sqrt{\frac{\pi}{2s}} e^{-s} \left(1 + \sum_{k=1}^{\infty} b_k s^{-k}\right),$$

as $|s| \to +\infty$,

with $|\arg(s)| < \frac{\pi}{2}$ in case of $I_t(s)$, and $|\arg(s)| < \frac{3\pi}{2}$ in case of $K_t(s)$. The expansions hold uniformly if $|\arg(s)| \leq \frac{\pi}{2} - \varepsilon$ in case of $I_t(s)$, and $|\arg(s)| \leq \frac{3\pi}{2} - \varepsilon$ in case of $K_t(s)$, for any $\varepsilon > 0$. As is explained in the last paragraph of [Wat66, p. 199], the asymptotic expansions [Wat66, p. 199, §7.21 (1)-(4)] and hence also the expansions (3.1) here, hold locally uniformly for any order $t \in \mathbb{C}$, modulo an eventual change of coefficients. The coefficients $a_k$ and $b_k$ are polynomials in $4t^2$ of order $k \in \mathbb{N}$.

The derivatives of the Bessel functions also admit an asymptotic expansion of the same structure albeit with different coefficients, cf. [AbrSt92, (9.7.3), (9.7.4)]

$$I'_t(s) = \frac{e^s}{\sqrt{2\pi s}} \left(1 + \sum_{k=1}^{\infty} a'_k s^{-k}\right),$$

$$K'_t(s) = -\sqrt{\frac{\pi}{2s}} e^{-s} \left(1 + \sum_{k=1}^{\infty} b'_k s^{-k}\right),$$

as $|s| \to +\infty$. 
with \(|\arg(s)| < \frac{\pi}{2}\) in case of \(I_t(s)\), and \(|\arg(s)| < \frac{3\pi}{2}\) in case of \(K_t(s)\). The expansions hold uniformly if \(|\arg(s)| \leq \frac{\pi}{2} - \varepsilon\) in case of \(I_t(s)\), and \(|\arg(s)| \leq \frac{3\pi}{2} - \varepsilon\) in case of \(K_t(s)\), for any \(\varepsilon > 0\). As before, the coefficients \(a_j\) and \(b_j\) are polynomials in \(4t^2\) of order \(k \in \mathbb{N}\). The expansions (3.1) and (3.2) hold locally uniformly in \(t \in \mathbb{C}\).

We will also need an expansion of the derivative of \(K_t(s)\) with respect to its order. We quote [Olv85, p. 325 Exercise 1.2 and 1.3] and obtain for fixed \(t\), locally uniformly in \(s\) with \(|\arg(s)| < \frac{3\pi}{2}\)

\[
\frac{d}{dt} K_t(s) \sim \sqrt{\frac{\pi}{2s}} t e^{-s} \left( 1 + \sum_{k=1}^{\infty} c_k s^{-k} \right), \quad |s| \to \infty.
\]

(3.3)

3.2. Asymptotics for fixed arguments and large order. For the asymptotics of modified Bessel functions for large order we refer to Sidi and Hoggan [StHo11]. Even though such a full asymptotic expansion seems not being presented elsewhere, it can also be derived from (3.8) and (3.11) below by taking the argument to zero. Asymptotics of \(I_t(s)\) also follows from the behaviour of the (unmodified) Bessel function [AbSt92, (9.3.1)], compare also [Olv85, p. 374 (7.01)] for the leading order term in the asymptotics. The Stirling formula asymptotics for the Gamma function, see e.g. [AbSt92, (6.1.37)], asserts that for \(|\arg(t)| < \pi\) as \(|t| \to \infty\) we have

\[
\Gamma(t) \sim \left(\frac{t}{e}\right)^t \sqrt{\frac{2\pi}{t}} \left( 1 + \sum_{j=1}^{\infty} \frac{g_j}{t^j} \right).
\]

(3.4)

In view of that expansion, we infer from [StHo11] locally uniformly in \(|\arg(s)| < \pi\)

\[
I_t(s) \sim \frac{1}{t \Gamma(t)} \left(\frac{s}{2}\right)^t \left( 1 + \sum_{j=1}^{\infty} \frac{A_j}{t^j} \right) \sim \frac{1}{\sqrt{2\pi t}} \left(\frac{es}{2t}\right)^t \left( 1 + \sum_{j=1}^{\infty} c_j t^{-j} \right),
\]

(3.5)

\[
K_t(s) \sim \frac{\Gamma(t)}{2} \left(\frac{s}{2}\right)^{-t} \left( 1 + \sum_{j=1}^{\infty} \frac{A_j}{(-t)^j} \right) \sim \sqrt{\frac{\pi}{2t}} \left(\frac{es}{2t}\right)^{-t} \left( 1 + \sum_{j=1}^{\infty} d_j t^{-j} \right),
\]

as \(|t| \to \infty\) with \(|\arg(t)| < \pi\) in case of \(I_t(s)\), and \(|\arg(t)| < \frac{3\pi}{2}\) in case of \(K_t(s)\). The expansions hold uniformly if \(|\arg(s)| \leq \pi - \varepsilon\) in a bounded domain, \(|\arg(t)| \leq \pi - \varepsilon\) in case of \(I_t(s)\), and \(|\arg(t)| \leq \frac{3\pi}{2} - \varepsilon\) in case of \(K_t(s)\), for any \(\varepsilon > 0\). The coefficients \(A_j, c_j\) and \(d_j\) are polynomials in \((s/2)^2\) of degree \(j \in \mathbb{N}\).

Similar expansions hold for the derivatives. Recall the standard recurrence relations for the derivatives of Bessel functions, cf. [AbSt92, (9.6.26)]

\[
I_t'(s) = I_{t+1}(s) + \frac{t}{s} I_t(s), \quad K_t'(s) = -K_{t+1}(s) + \frac{t}{s} K_t(s).
\]

(3.6)
From here and (3.5) we infer directly the following expansions for the derivatives

\[ I'_t(s) \sim \frac{1}{\sqrt{2\pi t}} \left( \frac{es}{2t} \right)^t \left( 1 + \sum_{j=1}^{\infty} c'_j t^{-j} \right), \]

(3.7)

\[ K'_t(s) \sim -\sqrt{\frac{\pi}{2t}} \left( \frac{es}{2t} \right)^{-t} \left( 1 + \sum_{j=1}^{\infty} d'_j t^{-j} \right), \]

as \( |t| \to \infty \), uniformly for \( |\arg(s)| < \pi - \epsilon \), \( |\arg(t)| \leq \pi - \epsilon \) in case of \( I'_t(s) \), and \( |\arg(t)| \leq \frac{\pi}{2} - \epsilon \) in case of \( K'_t(s) \), for any \( \epsilon > 0 \). The coefficients \( c'_j \) are polynomials in \( (s/2) \) of degree 1, the coefficients \( d'_j \) are polynomials in \( (s/2)^2 \) of degree \( j \in \mathbb{N} \).

### 3.3. Asymptotics for large arguments and large order.

We now study asymptotics of Bessel functions, when the argument and the order grow with a fixed ratio. Following Olver [Olve97, p. 377 (7.16), (7.17)], and his extension of validity in [Olve54, Ch. 10 §8], see also [AbSt92, (9.7.7), (9.7.8)], we have for \( t > 0 \) and \( |\arg(s)| < \frac{\pi}{2} \)

\[ I_t(ts) \sim \frac{e^{tv}}{(2\pi t)^{1/4}} \left( 1 + \sum_{k=1}^{\infty} \frac{U_k(p)}{t^k} \right), \]

(3.8)

\[ K_t(ts) \sim \sqrt{\frac{\pi}{2t}} \frac{e^{-tv}}{(1+s^2)^{1/4}} \left( 1 + \sum_{k=1}^{\infty} \frac{U_k(p)}{(-t)^k} \right), \]

where

\[ v = v(s) := \sqrt{1+s^2} + \log(s/(1+\sqrt{1+s^2})), \]

\[ p = p(s) := 1/\sqrt{1+s^2}, \]

and the coefficients \( U_k(p) \) are polynomials in \( p \) of degree 3k. Using Cauchy product formula we find in particular\(^6\)

\[ I_t(ts)K_t(ts) \sim \frac{1}{(2t)^{1/2}} \left( 1 + \sum_{k=1}^{\infty} \frac{U_k(p)}{t^k} \right) \cdot \left( 1 + \sum_{k=1}^{\infty} \frac{U'_k(p)}{(-t)^k} \right), \]

(3.10)

\[ \sim \frac{1}{(2t)^{1/2}} \left( 1 + \sum_{k=1}^{\infty} \frac{U_{2k}(p)}{t^{2k}} \right), \]

as \( t \to \infty \),

where the coefficients \( U_{2k}(p) \) are polynomials in \( p \) of degree 6k. Similar expansions hold for the derivatives and in fact for \( |\arg(s)| < \frac{\pi}{4} \) (cf. [Olve97, p. 378, Ex. 7.2] and [AbSt92, (9.7.9), (9.7.10)])

\[ I'_t(ts) \sim \frac{e^{tv}}{(2\pi t)^{1/2}s(1+s^2)^{-1/4}} \left( 1 + \sum_{k=1}^{\infty} \frac{V_k(p)}{t^k} \right), \]

(3.11)

\[ K'_t(ts) \sim -\sqrt{\frac{\pi}{2t}} \frac{e^{-tv}}{(1+s^2)^{-1/4}} \left( 1 + \sum_{k=1}^{\infty} \frac{V_k(p)}{(-t)^k} \right), \]

as \( t \to \infty \),

\(^6\)pointed out by Luiz Hartmann.
where the coefficients $V_k(p)$ are again polynomials in $p$ of degree $3k$. These asymptotic expansions in fact hold uniformly for $|\arg(s)| \leq \frac{\pi}{2} - \varepsilon$ for any $\varepsilon > 0$.

### 3.4. Extensions of validity for uniform expansions.

Extensions of validity for (3.8) and (3.11) have been studied by Olver [Olv97] Chapter 10 in the Section §8. These validity extensions are obtained as an application of the main theorem in [Olv97, p. 366, Theorem 3.1], where one proceeds as follows. The Bessel equation is transformed as in [Olv97, p. 375 §7.3] to

$$
(3.12) \quad \frac{d^2W}{dv^2} = (t^2 + \psi(v(s))) W, \quad \text{where} \quad \psi(v(s)) = \frac{s^2(4 - s^2)}{4(1 + s^2)^3},
$$

where $v = v(s)$ is defined in (3.9) and the fundamental system is given by

$$
(3.13) \quad \sqrt{s} \left(1 + \frac{s^2}{2^j}ight)^{1/4} I_j(ts), \quad \sqrt{s} \left(1 + \frac{s^2}{2^j}ight)^{1/4} K_j(ts).
$$

Choose any domain $\mathbb{D} \subset \mathbb{C}$ for $s$ and any domain $\mathbb{T} \subset \mathbb{C}$ for $t$. The domain $\mathbb{D}$ for $s$ corresponds to a domain $\Delta \subset \mathbb{C}$ for $v(s)$, which can be constructed out of $\mathbb{D}$ using the transformation described in [Olv97, p. 375 §7.3 (i) - (v)]. The main theorem [Olv97, p. 366, Theorem 3.1] now asserts that the asymptotic expansions of solutions of the form (3.8) and (3.11) hold uniformly for any choice of domains satisfying the following conditions:

(i) the closure of $\mathbb{D}$ does not contain the points $s = \pm i$, which are the pole singularities of $\psi(\nu(s))$,

(ii) there exist reference points $\alpha_1, \alpha_2 \in \Delta$, such that any point in $\Delta$ can be connected inside $\Delta$ to $\alpha_j$ by a continuous path $L_j$ of finitely many straight lines directed from $\alpha_j$ to $\nu$, $j = 1, 2$.

(iii) $\text{Re}(tv)$ is non-decreasing for $\nu$ varying along $L_1$, and non-increasing for $\nu$ varying along $L_2$, with the prescribed orientation. Such paths are called progressive.

These conditions have been worked out by Olver in [Olv97, p. 381 §8.3]. The asymptotic expansion of $I_j(ts)$ and hence also of its derivative $I_j'(ts)$ hold uniformly for $|\arg(t)| < \frac{\pi}{2} - \varepsilon$, where $\varepsilon > 0$ is any small positive number, and any $s \in \mathbb{C}$ away from the cuts in the complex plane as depicted in [Olv97, p. 381, Fig. 8.6]. The expansions of $K_j(ts)$ and hence also of its derivative $K_j'(ts)$ hold uniformly for $|\arg(s)| \leq \frac{\pi}{2} - \varepsilon$ and $|\arg(t)| \leq \frac{\pi}{2} - \varepsilon'$ for any $\varepsilon, \varepsilon' > 0$, as explained in the last paragraph of [Olv97, p. 381, §8.3]. In particular, (3.8) and (3.11) hold for the following two choices of domains $(\mathbb{T}_1, \mathbb{D}_1)$ and $(\mathbb{T}_2, \mathbb{D}_2)$

$$
(3.14) \quad \mathbb{T}_1 = \{t \in \mathbb{C} \mid \arg(t) \in [0, \pi/2 - \varepsilon]\},
\mathbb{D}_1 = \{s \in \mathbb{C} \mid \arg(s) \in [-\pi/2 + \varepsilon', 0]\},
\mathbb{T}_2 = \{t \in \mathbb{C} \mid \arg(t) \in [-\pi/2 + \varepsilon, 0]\},
\mathbb{D}_2 = \{s \in \mathbb{C} \mid \arg(s) \in [0, \pi/2 - \varepsilon']\}.
$$

\footnote{these conditions are somewhat stronger than the optimal conditions posed by Olver}
for any small positive numbers $\epsilon, \epsilon' > 0$.

Extending validity of expansions even further is possible along the lines of [Olv97, p. 380, §8.2], but is not worked out explicitly in [Olv97, p. 381, §8.3] for brevity reasons. However, for applications we need to extend the regions of validity to include segments of the imaginary axis. We define for any small positive numbers $\epsilon, \epsilon > 0$

$$T_3 = \{t \in \mathbb{C} \mid \arg(t) \in [\pi/2 - \epsilon, \pi/2]\}$$

$$\Delta_3 = \{\nu \in \mathbb{C} \mid \arg(\nu) \in [-\pi/2, -\pi/2 + \epsilon], \ \Im(\nu) \leq -\frac{i\pi}{2}(1 + \delta)\}.$$ (3.15)

$$T_4 = \{t \in \mathbb{C} \mid \arg(t) \in [-\pi/2, -\pi/2 + \epsilon]\}$$

$$\Delta_4 = \{\nu \in \mathbb{C} \mid \arg(\nu) \in [\pi/2 - \epsilon, \pi/2], \ \Im(\nu) \geq \frac{i\pi}{2}(1 + \delta)\}.$$ We fix the reference points as follows. In case of $(T_3, \Delta_3)$ we set $\alpha_1 = -\frac{i\pi}{2}(1 + \delta)$ and $\alpha_2 = \lim_{u \to +\infty} u e^{(-\pi/2+\epsilon)i}$. In case of $(T_4, \Delta_4)$ we set $\alpha_1 = \frac{i\pi}{2}(1 + \delta)$ and $\alpha_2 = \lim_{u \to +\infty} u e^{(\pi/2-\epsilon)i}$. With these choices, the conditions (i) – (iii) from above are satisfied and hence the theorem [Olv97, p. 366, Theorem 3.1] applies. Transforming the domains $\Delta_3, \Delta_4$ to $\mathbb{D}_3, \mathbb{D}_4$, respectively, we obtain after taking subdomains

$$T_3 = \{t \in \mathbb{C} \mid \arg(t) \in [\pi/2 - \epsilon, \pi/2]\}$$

$$\mathbb{D}_3 = \{s \in \mathbb{C} \mid \arg(s) \in [-\pi/2, -\pi/2 + \epsilon'], \ \Im(s) \leq -i(1 + \delta')(\epsilon)\}.$$ (3.16)

$$T_4 = \{t \in \mathbb{C} \mid \arg(t) \in [-\pi/2, -\pi/2 + \epsilon]\}$$

$$\mathbb{D}_4 = \{s \in \mathbb{C} \mid \arg(s) \in [\pi/2 - \epsilon, \pi/2], \ \Im(s) \geq i(1 + \delta')\}.$$ for some appropriate $\epsilon \in (0, \epsilon)$ and $\delta' > \delta$. We conclude that the expansions (3.8) and (3.11) hold uniformly for the domains $(T_3, \mathbb{D}_3)$ and $(T_4, \mathbb{D}_4)$ as well.

4. The zeta determinant of scalar cuspidal operators

In this section we study scalar Sturm-Liouville operators which will naturally appear in the analysis of the Hodge Laplacian of a manifolds with cusps. We establish existence and a variation formula for the zeta-determinant of a scalar cuspidal Sturm-Liouville operator. More precisely, fix $\mu > 0$ and consider a family of scalar cusp operators

$$D_t := -(x \partial_x)^2 - (x \partial_x) + x^2 \mu^2 + t^2 - \frac{1}{4} : C_0^\infty(R, \infty) \to C_0^\infty(R, \infty), \ (t \geq 0).$$ (4.1)

The second order differential equation $D_t f = 0$ admits a fundamental system of solutions $x^{-1/2}I_t(\mu x), x^{-1/2}K_t(\mu x)$, in terms of modified Bessel functions of first and second order. By the first asymptotic expansion in (3.1), $x^{-1/2}I_t(\mu x)$ does not lie in $L^2((R, \infty), dx)$. Consequently, $D_t$ is in the limit point case at infinity and we a self-adjoint extension of $D_1$ by putting e.g. Dirichlet boundary conditions at $x = R$

$$\mathcal{D}(D_1) = \{f \in \mathcal{D}_{\text{max}}(D_1) \mid f(R) = 0\},$$
where we point out as before, that elements in the maximal domain $D_{\text{max}}(D_t)$ are absolutely continuous and in fact continuously differentiable at $x = \mathbb{R}$. Replacing Dirichlet with generalized Neumann boundary conditions $f'(\mathbb{R}) + \alpha f(\mathbb{R}) = 0$, leads to an analogous discussion which we do not repeat here. If the parameter $t \geq 0$ is allowed to be complex with $|\arg(t)| < \frac{\pi}{2}$, then $D_t$ is not self-adjoint anymore, but a closed operator in $L^2((\mathbb{R}, \infty), dx)$ with an Agmon angle.

The purpose of the present section is the definition of the zeta-regularized determinant of $D_t$.

**Proposition 4.1.** The operator $D_t$ with $|\arg(t)| < \frac{\pi}{2}$ is invertible. The inverse $D_t^{-1}$ is a trace class operator with trace given by the integral of its Schwartz kernel along the diagonal

\[
(4.2) \quad \text{Tr} \, D_t^{-1} \equiv \int_{\mathbb{R}} \mathcal{G}_t(x) = \int_{\mathbb{R}} (\mu x)^{-1} \left( I_t K_t(\mu x) - \frac{I_t(\mu R)}{K_t(\mu R)} K_t^2(\mu x) \right) dx.
\]

**Proof.** Assume first that $t \geq 0$. Denote by $\psi$ a solution to $D_t f = 0$ that is square integrable at infinity, i.e. $\psi \in L^2((\mathbb{R}, \infty), dx)$. Denote by $\phi$ a solution to $D_t f = 0$ satisfying Dirichlet boundary conditions at $x = \mathbb{R}$. Both solutions are uniquely determined up to a multiplicative constant and we put

\[
\psi(x) = x^{-1/2} K_t(\mu x), \quad \phi(x) = x^{-1/2} \left( I_t(\mu x) - \frac{I_t(\mu R)}{K_t(\mu R)} K_t^2(\mu x) \right),
\]

where we point out that $K_t(\mu x) > 0$ is nowhere vanishing for $x > 0$ and $t > -1$, as asserted e.g. in [AbSt92, p. 374]. Hence $\psi(x)$ is strictly positive. Since $I_t(\mu x)$ is growing monotonously, while $K_t(\mu x)$ is falling monotonously, the quotient $I_t(\mu x)/K_t(\mu x)$ is growing monotonously and hence by positivity of $K_t(\mu x)$, the second solution $\phi(x)$ is also strictly positive for $x > R$. The corresponding Wronski determinant is computed as follows

\[
W(\phi, \psi) = (\phi' \psi - \psi' \phi)(x) = x^{-1} \mu \left( I_t' K_t - I_t K_t' \right) (\mu x) = \frac{1}{x^2}.
\]

The Green function $G_t$ of $D_t$ is obtained by the usual ansatz

\[
G_t(x, y) = \begin{cases} 
A \phi(x) \psi(y), & x \leq y, \\
A \psi(x) \phi(y), & x \geq y,
\end{cases}
\]

where $A$ is computed from the condition $D_t G_t(\cdot, y) = \delta(\cdot - y)$ and is given in terms of the Wronski determinant by

\[
A = (x^2 W(\phi, \psi))^{-1} = 1.
\]

In particular we find for the Green function at the diagonal

\[
(4.3) \quad G_t(x) \equiv G_t(x, x) = x^{-1} \left( I_t K_t(\mu x) - \frac{I_t(\mu R)}{K_t(\mu R)} K_t^2(\mu x) \right).
\]

The Green function $G_t(x, y)$ is continuous on $[\mathbb{R}, \infty) \times [\mathbb{R}, \infty)$ and by positivity of solutions $\phi$ and $\psi$, it is non-negative and positive away from $x, y = \mathbb{R}$. Moreover, $G_t$ is integrable on $[\mathbb{R}, \infty)$ along the diagonal by the asymptotic expansion (3.1). Consequently, by the Mercers theorem, as worked out e.g. by Reed and Simon
In particular, we conclude that \( D_t^{-1} \) is a compact self-adjoint operator with discrete spectrum accumulating at zero. Hence the cusp operator \( D_t \) is a self-adjoint operator in \( L^2((\mathbb{R}, \infty), dx) \) with discrete positive spectrum accumulating at infinity. The shifted operator \( (D_t + z^2) \) with \( |\arg(z)| < \frac{\pi}{2} \) still admits a trivial kernel and the Schwartz kernel of its resolvent \( (D_t + z^2)^{-1} \) is given by \( G_{\sqrt{t^2 + z^2}} \), which can be checked to be well-defined and continuous but clearly not positive any longer. That Schwartz kernel is still integrable on \( [\mathbb{R}, \infty) \) along the diagonal by the asymptotic expansion (3.1). Since the spectrum of \( (D_t + z^2) \) and hence also of the inverse \( (D_t + z^2)^{-1} \) is discrete, the integral of \( G_{\sqrt{t^2 + z^2}} \) along the diagonal \( [\mathbb{R}, \infty) \) equals the trace of the resolvent \( (D_t + z^2)^{-1} \).

In particular, we find that for any \( t \in \mathbb{C} \) with \( |\arg(t)| < \frac{\pi}{2} \) we may write

\[
\text{Tr} D_t^{-1} = \int_\mathbb{R} G_t(x) \, dx.
\]

We now study the asymptotic expansion of the resolvent trace.

**Proposition 4.2.** The trace of the resolvent \( D_t^{-1} \) with \( |\arg(t)| < \frac{\pi}{2} \) admits an expansion

\begin{equation}
\text{Tr} D_t^{-1} \sim \sum_{k=0}^{\infty} a_k t^{-1-k} + \sum_{k=0}^{\infty} b_k t^{-1-2k} \log(t), \ |t| \to \infty.
\end{equation}

**Proof.** Let us rewrite the trace as follows

\[
\text{Tr} D_t^{-1} = \int_\mathbb{R} G_t(x) \, dx
\]

\[
= \int_\mathbb{R} G_t(x) \, dx + t \int_{2R/t}^{\infty} G_t(xt) \, dx + t \int_0^\infty G_t(xt) \, dx
\]

\[
=: T_1(t) + T_2(t) + T_3(t),
\]

where we have substituted \( x \) with \( xt \) in \( T_2 \) and \( T_3 \). The asymptotic expansion of \( T_1 \) as \( t \to \infty \) is the simplest of all three and follows directly from (3.5). In particular we obtain after cancellations

\begin{equation}
T_1(t) \sim \sum_{k=0}^{\infty} a'_k t^{-1-k}, \ t \to \infty.
\end{equation}

Let us now study the asymptotics of \( T_2 \), which in view of the representation (4.3) is given explicitly by the following expression

\begin{equation}
T_2(t) = t \int_{2R/t}^{1/\mu} (\mu x t)^{-1} \left( I_t^K_t(\mu xt) - \frac{I_t(\mu R)}{K_t(\mu R)} K^2_t(\mu xt) \right) \, dx.
\end{equation}
We now proceed with estimating the second summand in that last expression, where we denote all uniform constants (possibly dependent on $\mu$) by $C > 0$. In view of the asymptotics (3.5) we find

$$I_t(\mu R) \leq C \left( \frac{e\mu R}{2t} \right)^{2t}.$$

In view of the uniform asymptotics (3.8) we also find for $x < 1/\mu$

$$K_t^2(\mu xt) \leq Ct^{-1} \left( \frac{e\mu x}{1 + \sqrt{1 + (\mu x)^2}} \right)^{-2t} \leq Ct^{-1} \left( \frac{e\mu x}{3} \right)^{-2t}.$$ 

From here we conclude after cancellations for any $N \in \mathbb{N}$

$$\int_{2R/t}^{1/\mu} (\mu x)^{-1} \left| \frac{I_t(\mu R)}{K_t(\mu R)} K_t^2(\mu xt) \right| \leq Ct^{-1} \left( \frac{3R}{2t} \right)^{2t} \int_{2R/t}^{1/\mu} x^{-2t-1} dx \leq C t^{-2} \left( \frac{3R}{2t} \right)^{2t} \left( \frac{2R}{t} \right)^{-2t} \leq C t^{-2} \left( \frac{3}{4} \right)^{2t} = O(|t|^{-N}), \text{ as } |t| \to \infty,$$

From here and in view of the uniform expansion (3.10) we conclude

$$T_2(t) = \int_{2R/t}^{1/\mu} (\mu x)^{-1} I_t(\mu xt) dx + O(|t|^{-N}),$$

$$\sim \sum_{k=0}^{\infty} a_k t^{-1-k} + \sum_{k=0}^{\infty} b_k t^{-1-2k} \log(t), \text{ as } |t| \to \infty,$$

For $T_3$ we compute similarly, using (4.7) and (3.8)

$$T_3(t) = \int_{1/\mu}^{\infty} (\mu x)^{-1} I_t(\mu xt) dx + O(|t|^{-N}) \sim \sum_{k=0}^{\infty} a_k'' t^{-1-k}, \text{ as } |t| \to \infty,$$

Summarizing the expansions for $T_1, T_2$ and $T_3$, we conclude with an asymptotic expansion for the resolvent trace, as stated.

Note that the resolvent trace asymptotic expansion as established in Proposition 4.2 does not admit terms of the form $t^{-2} \log^k(t), k \in \mathbb{N}$. Hence we may now define the zeta-regularized determinant of $D_t$ using the notion of regularized integrals as Lesch [Les98].

**Definition 4.3.** Denote by the regularized limit $\lim_{\varepsilon \to 0}$ the constant term in the asymptotic expansion as $\varepsilon \to 0$, and by $\lim_{\delta \to \infty}$ the constant term in the asymptotic...
expansion as $\delta \to \infty$. Then the zeta-regularized determinant $\det_\xi D_t$ is defined by\footnote{The zeta-regularized determinant can be equivalently defined using the zeta-function $\zeta(s, D_t)$ introduced in (2.5), by extending $\zeta(s, D_t)$ meromorphically to $\mathbb{C}$ with a regular point at $s = 0$, and setting $\log \det_\xi D_t := -\zeta'(0, D_t)$.}

$$\log \det_\xi D_t := -2 \lim_{\delta \to \infty} \lim_{\epsilon \to 0} \int_\epsilon^\delta z \operatorname{Tr} (D_t + z^2)^{-1} dz$$

$$= -2 \int_0^\infty z \operatorname{Tr} (D_t + z^2)^{-1} dz.$$

5. Variation of the zeta determinant of a cuspidal operator

In this section we establish a variational formula for $\log \det_\xi D_t$ for variable parameter $t$ in the spirit of Lesch [Les98, Prop. 3.4]. Variation with respect to other terms in the differential expression for $D_t$ is an interesting question in itself, which is addressed in a forthcoming project jointly with Hartmann and Lesch [Hale16].

**Proposition 5.1.** The zeta-regularized determinant of $D_t$ is differentiable in $t \in \mathbb{R}$ with

$$\left. \frac{d}{dt} \right|_{t=t_0} \log \det_\xi D_t = -2t_0 \operatorname{Tr} (D_{t_0})^{-1}.$$

**Proof.** By a Neumann series argument we find (we write $I$ for the identity operator)

$$(D_t + z^2)^{-1} = (D_{t_0} + z^2 + (t^2 - t_0^2))^{-1}$$

$$= \left( I + (t^2 - t_0^2) (D_{t_0} + z^2)^{-1} \right)^{-1} (D_{t_0} + z^2)^{-1}$$

$$= \sum_{n=0}^\infty (-1)^n \left( (t^2 - t_0^2) (D_{t_0} + z^2)^{-1} \right)^n (D_{t_0} + z^2)^{-1}$$

(5.1)

By an argument similar to Proposition 4.2, the trace norm of $(D_{t_0} + z^2)^{-1}$ is of the order $O(z^{-1}\log(z))$ as $z \to \infty$. Moreover, the operator norm of $(D_{t_0} + z^2)^{-1}$ is of the order $O(z^{-2})$ as $z \to \infty$. Taking trace norms of the expression in (5.1), we obtain for some uniform constant $C > 0$ and $z > 1$

$$\| (D_t + z^2)^{-1} - (D_{t_0} + z^2)^{-1} \|_{\operatorname{tr}}$$

$$\leq \| (D_{t_0} + z^2)^{-1} \| \cdot \sum_{n=0}^\infty (-1)^n \| t^2 - t_0^2 \|^n \| (D_{t_0} + z^2)^{-1} \|_{\operatorname{tr}}^n$$

(5.2)

$$\leq C \frac{z^2}{z} \sum_{n=1}^\infty \left( |t^2 - t_0^2| z^{-1}\log(z) \right)^n \leq C |t^2 - t_0^2| z^{-3}\log(z).$$

Next, we consider the difference of logarithmic zeta-determinants

$$\log \det_\xi D_t - \log \det_\xi D_{t_0} = -2 \int_0^\infty z \left( \operatorname{Tr} (D_t + z^2)^{-1} - \operatorname{Tr} (D_{t_0} + z^2)^{-1} \right) dz.$$




From the estimate (5.2) we easily conclude

\[ \frac{d}{dt} (D_t + z^2)^{-1} \bigg|_{t=0} = \frac{d}{dt} \left( \text{Tr} (D_t + z^2)^{-1} - \text{Tr} (D_t_0 + z^2)^{-1} \right) \bigg|_{t=0} = O(z^{-3} \log(z)), \quad z \to \infty. \]

Hence the regularized integral in (5.3) may be replaced by the standard integral, and we can differentiate under the integral to obtain

\[
\frac{d}{dt} \bigg|_{t=t_0} \left( \log \det D_t - \log \det D_{t_0} \right)
= -2 \int_0^\infty z \left( \text{Tr} (D_t + z^2)^{-1} - \text{Tr} (D_{t_0} + z^2)^{-1} \right) \frac{d}{dt} (D_t + z^2) \bigg|_{t=t_0} dz
= -2 \int_0^\infty z \text{Tr} (D_t + z^2)^{-2} \frac{d}{dt} (D_{t_0} + z^2)^{-1} dz
= -4t_0 \int_0^\infty z \text{Tr} (D_{t_0} + z^2)^{-2} = 2t_0 \int_0^\infty \frac{d}{dz} \text{Tr} (D_{t_0} + z^2)^{-1} dz = -2t_0 \text{Tr} (D_{t_0})^{-1}.
\]

We can now prove the main result of this subsection.

**Theorem 5.2.** Consider solutions \( \phi, \psi \) of \( D_t \), where \( \psi(x) = x^{-1/2} K_t(\mu x) \), and \( \phi \) satisfies Dirichlet boundary conditions at \( x = R \), normalized such that \( \phi'(R) = 1 \). Then

\[
\frac{d}{dt} \log \det D_t = \frac{d}{dt} \log (x^2 W(\phi, \psi))
= -\frac{d}{dt} \log \left( I_t'(\mu R) - \frac{I_t(\mu R)}{K_t(\mu R)} K_t'(\mu R) \right).
\]

**Proof.** The solutions \( \phi, \psi \) satisfy the following relations

\[
((x \partial_x)^2 + (x \partial_x)) \phi = \left( x^2 \mu^2 + t^2 - \frac{1}{4} \right) \phi,
((x \partial_x)^2 + (x \partial_x)) \partial_t \psi = 2t \psi + \left( x^2 \mu^2 + t^2 - \frac{1}{4} \right) \partial_t \psi.
\]

Hence we compute for the Wronskian of \( \phi \) and \( \partial_t \psi \)

\[
(x \partial_x + 1)[x W(\phi, \partial_t \psi)]
= \partial_t \psi \left( (x \partial_x)^2 + (x \partial_x) \right) \phi - \phi \left( (x \partial_x)^2 + (x \partial_x) \right) \partial_t \psi
= \partial_t \psi \left( x^2 \mu^2 + t^2 - \frac{1}{4} \right) \phi - \phi \left( 2t \psi + \left( x^2 \mu^2 + t^2 - \frac{1}{4} \right) \partial_t \psi \right)
= -2t \phi \psi.
\]
By (3.3), $\partial_t \psi = O(e^{-\mu x}/x^2)$ and hence $W(\phi, \partial_t \psi) = O(x^{-2})$, as $x \to \infty$. Hence we may compute using integration by parts and (5.5)

$$-2t(x^2W(\phi, \psi))\text{Tr}D_t^{-1} = \int_{-\infty}^{\infty} (x\partial_x + 1)[xW(\phi, \partial_t \psi)]dx$$

$$= x^2W(\phi, \partial_t \psi)|_{-\infty}^{\infty} = R^2W(\phi, \partial_t \psi)(R).$$

Under the normalization of $\phi$, $W(\partial_t \phi, \psi)(R) = 0$ and hence

$$\partial_t W(\phi, \psi) = W(\phi, \partial_t \psi)(R).$$

Using Proposition 5.1 and the relations (5.6), (5.7) we find

$$\frac{d}{dt} \log \text{det}_t D_t = -2t \text{Tr}D_t^{-1} = \frac{\partial_t([x^2W(\phi, \psi)](R))}{(x^2W(\phi, \psi))(R)}.$$
6. The de Rham complex of a model infinite cusp

Let \((N, g^N)\) be a closed even-dimensional oriented Riemannian manifold, \(\dim N = n\), and consider the model cusp \(U_R = N \times [R, \infty), R > 0\), with the cusp metric

\[
g = \frac{dx^2 + g^N}{x^2}, \quad x \in [R, \infty).
\]

Fix a base point \(q = (y_0, x_0) \in U_R\) and consider a unitary representation \(\rho : \pi_1(U_R, q) \to U(\tau, \mathbb{C})\) of the fundamental group \(\pi_1(U_R, q)\). The corresponding flat Hermitian vector bundle \((E, \nabla, h)\) over \(U_R\) is equipped with the canonical Hermitian metric \(h\), and the canonical flat covariant derivative \(\nabla\), with the former induced from the standard Hermitian inner product on \(\mathbb{C}^r\) and the latter induced from the exterior derivative on the universal cover of \(U_R\).

By the product structure of \(U_R\), \(\pi_1(U_R, q) \cong \pi_1(N, y_0)\). Hence the unitary representation \(\rho\) also defines a flat Hermitian vector bundle \((E_N, \nabla_N, h_N)\) over \(N\), related to the vector bundle over \(U_R\) as follows. Let \(\pi : U_R = N \times [R, \infty) \to N\) be the projection onto the first factor. Then for compactly supported sections \(s \in \Gamma_0(E) \cong C_0^\infty([R, \infty), \Gamma(E_N))\)

\[
\pi^*E_N = E_N \times [R, \infty) \cong E,
\]

\[
\pi^*h_N = h, \quad \nabla_s = \frac{ds}{dx} \otimes dx + \nabla_N s.
\]

Denote by \(\Omega^p_0(U_R, E)\) the space of \(E\)-valued differential forms of degree \(p\), compactly supported in the open interior of \(U_R\). The flat covariant derivative \(\nabla\) extends by Leibniz rule to a differential operator on \(\Omega^*_0(U_R, E)\) and gives rise to the twisted de Rham complex \((\Omega^*_0(U_R, E), d_s)\). Similarly, \(\nabla_N\) extends by Leibniz rule to a differential operator on twisted differential forms \(\Omega^*(N, E_N)\) over \(N\) and gives rise to the twisted de Rham complex \((\Omega^*(N, E_N), d_{N,*})\).

We discuss the structure of \((\Omega^*_0(U_R, E), d_s)\) under the transformation

\[
\Phi : \Omega^*_0(U_R, E) \to C_0^\infty([R, \infty), \Omega^{p-1}(N, E_N) \oplus \Omega^p(N, E_N)),
\]

\[
\Phi(\omega_p + \omega_{p-1} \wedge dx) = x^{-\frac{n+1}{2}+p}(\omega_{p-1}, \omega_p).
\]

\(\Phi\) extends to a unitary transformation on the \(L^2\)-completions

\[
\Phi : L^2_0(U_R, g, h) \to L^2([R, \infty), dx; L^2_{p-1}(N, E_N, g^N, h_N) \oplus L^2_p(N, E_N, g^N, h_N)).
\]

Under this unitary transformation, \(d_s\) acts as follows

\[
d_p \left( \begin{array}{c} \omega_{p-1} \\ \omega_p \end{array} \right) = \left( \begin{array}{cc} 0 & (-1)^p x \partial_x \\ 0 & 0 \end{array} \right) + \left( \begin{array}{cc} x d_{N,p-1} & (-1)^p \left( \frac{n+1}{2} - p \right) \end{array} \right) \left( \begin{array}{c} \omega_{p-1} \\ \omega_p \end{array} \right).
\]

Very much in the spirit of [Ver09] and [MuVe14], we decompose the de Rham complex into harmonic and non-harmonic subcomplexes. The non-harmonic subcomplexes are obtained as follows. Let \(\psi \in \Omega^p(N, E_N)\) be a coclosed \(\eta\)-eigenform,
$\eta > 0$, of the twisted Laplacian $\Delta_N$ of the de Rham complex $(\Omega^*(N, E_N), d_{N,*})$. We write

$$\xi_1 = \begin{pmatrix} 0 \\ \psi \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \psi \\ 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{\eta}} d_N \psi \end{pmatrix}, \quad \xi_4 = \begin{pmatrix} \frac{1}{\sqrt{\eta}} d_N \psi \end{pmatrix}.$$ 

Repeated application of $d_*$ shows that the subspace $C^\infty_0((R, \infty), (\xi_1, \xi_2, \xi_3, \xi_4))$ is preserved under the action of $d_*$ and in fact defines a non-harmonic subcomplex

$$0 \to C^\infty_0((R, \infty), (\xi_1)) \xrightarrow{d^0} C^\infty_0((R, \infty), (\xi_2, \xi_3)) \xrightarrow{d^1} C^\infty_0((R, \infty), (\xi_4)) \to 0,$$

where $d^0, d^1$ are the restrictions of $d_*$, given with respect to the basis $(\xi_1, \xi_2, \xi_3, \xi_4)$ by the following matrix representations

$$d^0 = \begin{pmatrix} (-1)^p x \partial_x + (-1)^p \left( \frac{n+1}{2} - p \right) \end{pmatrix},$$

$$d^1 = \begin{pmatrix} x \sqrt{\eta} \left( -1 \right)^{p+1} x \partial_x + (-1)^p \left( \frac{n-1}{2} - p \right) \end{pmatrix}.$$

The Laplacians of the non-harmonic subcomplex are given by the following scalar actions\(^{10}\)

$$\Delta_0 = (d^0)^* d^0 = -(x \partial_x)^2 - (x \partial_x) + x^2 \eta + \left( \frac{n}{2} - p \right)^2 - \frac{1}{4},$$

$$\Delta_1 = d^1 (d^1)^* = -(x \partial_x)^2 - (x \partial_x) + x^2 \eta + \left( \frac{n}{2} - p - 1 \right)^2 - \frac{1}{4}.$$

As a consequence of Poincare duality on $N$, non-harmonic subcomplexes come in pairs. The twin subcomplex is obtained by replacing $\psi$ by $\psi' := \frac{1}{\sqrt{\eta}} d_N^* \psi$, where $^*$ is the Hodge star operator of $N$. $\psi' \in \Omega^{n-p-1}(N, E)$ is again an $\eta$-eigenform of the twisted Laplacian $\Delta_N$ and we may repeat the construction of the associated subcomplex as above, denoting the corresponding operators with an additional apostrophe. The resulting Laplacians of the twin non-harmonic subcomplex are given by\(^{11}\)

$$\Delta'_0 = \Delta_1, \quad \Delta'_1 = \Delta_0.$$

The harmonic subcomplexes are constructed as follows. Let $u \in H^p(N, E)$ be a $\Delta_N$-harmonic twisted differential form of degree $p$. The subspace $C^\infty_0((R, \infty), (u \oplus 0, 0 \oplus u))$ is again invariant under the action of $d_*$ and defines a subcomplex

$$0 \to C^\infty_0((R, \infty), (0 \oplus u)) \xrightarrow{d_{H,1}} C^\infty_0((R, \infty), (u \oplus 0)) \to 0,$$

where $d_{H,1}$ is the restriction of $d_*$, given with respect to the basis $(u \oplus 0, 0 \oplus u)$ by

$$d_{H,1} = (-1)^p \left( x \partial_x + \left( \frac{n+1}{2} - p \right) \right).$$

\(^{10}\) Note that the formal adjoint of $(x \partial_x)$ in $L^2((R, \infty), dx)$ is given by $(-x \partial_x - 1)$.

\(^{11}\) In contrast to the setting of isolated conical singularities in [VER09], the twin subcomplexes do not lead to simplifying cancellations.
The corresponding Laplacians of the harmonic subcomplex are given by the following scalar action

\[ \Delta^0_i := d_i^* d_i = -(x \partial_x)^2 - (x \partial_x) + \left( \frac{n}{2} - p \right)^2 - \frac{1}{4} = d_i^* d_i^1 =: \Delta^1_i. \]

By the Hodge de Rham decomposition of \( \Omega^*(N, E) \), the de Rham complex \( (\Omega^*_0(\mathcal{U}_R, E), d_*) \) decomposes completely into a direct sum of harmonic and non-harmonic subcomplexes above.

We close the subsection with a discussion of relative boundary conditions for the Hodge Laplacian \( \Delta_* \) of \( (\Omega^*_0(\mathcal{U}_R, E), d_*) \) at the regular end \( x = R \). The Hodge Laplacian is essentially self-adjoint at \( x = \infty \) and the boundary conditions at the infinite cusp end of \( \mathcal{U}_R \) amount only to the \( L^2 \)-integrability condition.

Consider the inclusion \( i : N \times \{ R \} \rightarrow \mathcal{U}_R \) with the pullback \( \iota^* : \Omega^p(\mathcal{U}_R, E) \rightarrow \Omega^p(N, E) \) given by \( \iota^*(\omega_p + \omega_{p-1} \wedge dx) = \omega_p(x = R) \). The self-adjoint domain of \( \Delta_p \) with relative and absolute boundary conditions is then given by (cf. (2.2))

\[
\mathcal{D}_{rel}(\Delta_p) = \{ \omega \in \mathcal{D}_{max}(\Delta_p) \mid \iota^* \omega = 0, \iota^*(d^i \omega) = 0 \} = \{ (\omega_{p-1}, \omega_p) \in \mathcal{D}_{max}(\Delta_p) \mid \omega_p(R) = 0, (\partial_x \omega_{p-1})(R) - \frac{1}{R} \left( \frac{n+1}{2} - p \right) \omega_{p-1}(R) = 0 \},
\]

\[
\mathcal{D}_{abs}(\Delta_p) = \{ \omega \in \mathcal{D}_{max}(\Delta_p) \mid \iota^*(\ast \omega) = 0, \iota^*(\ast d \omega) = 0 \} = \{ (\omega_{p-1}, \omega_p) \in \mathcal{D}_{max}(\Delta_p) \mid \omega_{p-1}(R) = 0, (\partial_x \omega_p)(R) + \frac{1}{R} \left( \frac{n+1}{2} - p \right) \omega_p(R) = 0 \}.
\]

The relative and absolute boundary conditions are compatible with the decomposition of the de Rham complex into harmonic and non-harmonic subcomplexes, and induce self-adjoint extensions of the Laplacians \( \Delta_i, \Delta'_i, \Delta^1_i, i = 0, 1 \), which we make explicit in case of relative boundary conditions:

\[
\begin{align*}
\mathcal{D}_{rel}(\Delta_0) &= \{ f \in \mathcal{D}_{max}(\Delta_0) \mid f(R) = 0 \}, \\
\mathcal{D}_{rel}(\Delta'_0) &= \{ f \in \mathcal{D}_{max}(\Delta'_0) \mid f(R) = 0 \}, \\
\mathcal{D}_{rel}(\Delta^0_i) &= \{ f \in \mathcal{D}_{max}(\Delta^0_i) \mid f(R) = 0 \}, \\
\mathcal{D}_{rel}(\Delta_1) &= \{ f \in \mathcal{D}_{max}(\Delta_1) \mid (\partial_x f - x^{-1}((n-3)/2 - p)f)(R) = 0 \}, \\
\mathcal{D}_{rel}(\Delta'_1) &= \{ f \in \mathcal{D}_{max}(\Delta'_1) \mid (\partial_x f + x^{-1}((n+1)/2 - p)f)(R) = 0 \}, \\
\mathcal{D}_{rel}(\Delta^1_i) &= \{ f \in \mathcal{D}_{max}(\Delta^1_i) \mid (\partial_x f - x^{-1}((n-1)/2 - p)f)(R) = 0 \},
\end{align*}
\]

where we point out that elements in the maximal domains of the Laplacians \( \Delta_i, \Delta'_i, \Delta^1_i, i = 0, 1 \), are continuously differentiable at \( x = R \) by standard arguments.

Comparing relative and absolute boundary conditions for the individual scalar operators of the harmonic and non-harmonic subcomplexes, we find by Poincare duality on the even-dimensional cross section \( N \) as expected

\[ T(\mathcal{U}_R, E, N, g) = T(\mathcal{U}_R, E, g). \]
Remark 6.1. A minor extension of the arguments by Lax and Phillips [LaPh76] on the cutoff Laplacian asserts that for the space
\[ D_p^\perp := \{ f \in D_{rel}(\Delta_p) \mid \forall \omega \in C_0^\infty([N,E]) : (f, \omega)_{L^2([N,E])} = 0 \}, \]
the resolvent of the cutoff Laplacian \( \Delta_p \upharpoonright D_p^\perp \), which is precisely the union of all Laplacians of the non-harmonic subcomplexes, is compact and hence admits a discrete spectrum. The spectrum is strictly positive, since each \( D_t \) with parameter \( \mu > 0 \) and \( t \geq 0 \) are invertible.

7. An integral representation for infinite sums of zeta functions

In this section we establish an integral representation for an infinite sum of zeta functions associated to cuspidal operators in the form as they enter the definition of the analytic torsion of a model cusp. We are concerned with zeta-functions associated to cuspidal operators in the form as they enter the definition of the analytic torsion of a model cusp. We have seen in Section 4 that the operators \( D_c(\mu)^{-1}, D_c(\mu, \alpha)^{-1} \) are trace class with discrete spectrum. The corresponding observation for \( D_c'(\mu)^{-1}, D_c'(\mu, \alpha)^{-1} \) is classical\(^\text{12}\). Hence for any
\[ D_c \in \{ D_c(\mu), D_c(\mu, \alpha), D_c'(\mu), D_c'(\mu, \alpha) \}, \]
we may enumerate its eigenvalues \( \text{spec } D_c = \{ \lambda_k \mid k \in \mathbb{N}_0 \} \subset (0, \infty) \) in the ascending order. Denote by \( m(\lambda_k) \) the multiplicity of the eigenvalue \( \lambda_k, k \in \mathbb{N}_0 \). Then for \( \text{Re}(s) > 1 \) (note that \( D_c^{-1} \) is trace class) we may write
\[ \zeta(s, D_c) := \sum_{k=0}^{\infty} m(\lambda_k) \lambda_k^{-s} = \mu^{-2s} \sum_{k=0}^{\infty} m(\lambda_k)(\lambda_k/\mu^2)^{-s} = -\frac{\mu^{-2s}}{\Gamma(s)} \int_0^{\infty} t^{s-1} \frac{1}{2\pi i} \int_{\Lambda} e^{-\lambda t} \frac{d}{d\lambda} t(\lambda, D_c) d\lambda dt, \]

\(^\text{12}\)We assume \( D_c'(\mu)^{-1} \) and \( D_c'(\mu, \alpha)^{-1} \) to be invertible, which is the case in the applications below. In fact, since the \( L^2 \)-cohomology of the model cusp is entirely determined by the cohomology of \( N \), the rescaling assumption of Remark 5.4 is obsolete in the spectral geometric applications below.
where
\[
t(\lambda, D_c) = -\sum_{k=0}^{\infty} m(\lambda_k) \log \left( 1 - \frac{\lambda \mu^2}{\lambda_k} \right), \quad \frac{d}{d\lambda} t(\lambda, D_c) = -\text{Tr} (\lambda - \mu^{-2} D_c)^{-1},
\]
and \( \Lambda = \{ \lambda \in \mathbb{C} \mid \arg(\lambda - \gamma) = \pi/4 \} \) is a counter-clockwise oriented integration contour for some \( \gamma \in (0, \lambda_0) \).

This integral representation of \( \zeta(s, D_c) \) is a consequence of absolute convergence of sums for \( \text{Re}(s) > 1 \). Integrating by parts first in \( \lambda \in \Lambda \) and then in \( t \in (0, \infty) \) yields, cf. Spreafico [Spr05, Lemma 1]

\[
(7.4) \quad \zeta(s, D_c) = \frac{s^2 \mu^{2s}}{\Gamma(s + 1)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_\Lambda \frac{e^{-\lambda t}}{-\lambda} t(\lambda, D_c) \frac{d\lambda}{dt} dt.
\]

In view of [Les98, Proposition 4.6], which is a general result on zeta-determinants of scalar operators with discrete spectrum, we find for \(^{13}\) \( z = \sqrt{-\lambda} \) and \( c(\mu z) = \sqrt{c^2 + (\mu z)^2} \)

\[
(7.5) \quad t(\lambda, D_c) = -\log \prod_{k=0}^{\infty} \left( 1 + \frac{(\mu z)^2}{\lambda_k} \right)^{m(\lambda_k)} = -\log \frac{\det_c D_c(\mu z)}{\det_c D_c}.
\]

As a direct consequence of Theorem 5.2 and Theorem 5.3 we obtain

**Proposition 7.1.**

\[
t(-z^2, D_c(\mu)) = \log \left( I_c' \mu - I_c \right) \frac{K_c'}{K_c(\mu z)} \left( \mu R \right) - \log \left( I_c' - \frac{I_c}{K_c} \right) \left( \mu R \right),
\]

\[
t(-z^2, D_c(\mu, \alpha)) = \log \left( I_c(\mu z) - \mu I_c' + \frac{2\alpha - 1}{R} I_c(\mu z) K_c(\mu z) \right) \left( \mu R \right) - \log \left( I_c' - \mu K_c' + \frac{2\alpha - 1}{R} K_c(\mu z) \right) \left( \mu R \right).
\]

Similar representations hold for the operators \( D_c'(\mu), D_c'(\mu, \alpha) \).

---

\(^{13}\)We define square roots using the main branch of the logarithm in \( \mathbb{C} \setminus \mathbb{R}^- \).
Proposition 7.2.

\[ t(-z^2, D'_c(\mu)) = t(-z^2, D_c(\mu)) \leq \log \left( \frac{I_c(\mu, \alpha)}{K_c(\mu, \alpha)} \right) (\mu R') + \log \left( \frac{I_c(\mu R)}{K_c(\mu R)} \right) (\mu R'), \]

\[ t(-z^2, D'_c(\mu, \alpha)) = t(-z^2, D_c(\mu, \alpha)) \leq \log \left( \frac{\mu K'_c(\mu, \alpha)}{\mu K_c(\mu, \alpha) + \frac{2\alpha - 1}{R} K_c(\mu R)} \right) (\mu R') \]

\[ + \log \left( \frac{\mu I'_c(\mu, \alpha)}{\mu I_c(\mu, \alpha) + \frac{2\alpha - 1}{R} I_c(\mu R)} \right) (\mu R') - \frac{2\alpha - 1}{R} I_c(\mu R) \]
As a corollary of Proposition 7.1 and 7.2 we obtain an integral representation for a combination of zeta functions, which will become relevant in the analysis of the contribution of non-harmonic sub complexes from §6 to the analytic torsion of the model cusp.

**Corollary 7.3.** For any $c, c_0 > 0$ we obtain

\[
\zeta(s, D'_c(\mu, \alpha)) - \zeta(s, D_c(\mu, \alpha)) - \zeta(s, D'_{c_0}(\mu)) - \zeta(s, D_{c_0}(\mu)) = \frac{s^2 \mu^{-2s}}{\Gamma(s + 1)} \int_0^\infty t^{s-1} \frac{1}{2\pi i} \int_{\Lambda} \frac{e^{-\lambda t}}{-\lambda} t(\lambda, \mu) d\lambda dt,
\]

The term $t(\lambda, \mu)$ is given explicitly by $t(-z^2, \mu) = F_{\mu z} - F_0$, where we write

\[
F_u := -\log \frac{I_{c(u)}(\mu R', \alpha)}{I_{c(u)}(\mu R')} + \log \frac{I_{c_0(u)}(\mu R', \alpha)}{I_{c_0(u)}(\mu R')} - \log \left( 1 - \frac{I_{c(u)}(\mu R')}{K'_{c(u)}(\mu R')} \frac{K_{c(u)}'(\mu R', \alpha)}{I_{c(u)}(\mu R', \alpha)} \right) + \log \left( 1 - \frac{I_{c_0(u)}(\mu R')}{K_{c_0(u)}(\mu R')} \frac{K_{c_0(u)}'(\mu R', \alpha)}{I_{c_0(u)}(\mu R', \alpha)} \right).
\]

**Remark 7.4.** The operators $\{D_c(\mu, \alpha), D_{c_0}(\mu)\}$ and $\{D'_c(\mu, \alpha), D'_{c_0}(\mu)\}$, parametrized by $\mu^2 \in \text{spec} \Delta_{p,cld,N}\setminus\{0\}$, clearly model the Laplacians of the non-harmonic subcomplexes in §6 for $U_R$ and $\text{U}_R \setminus \text{U}_R^0 \cong [R, R'] \times N$. Hence, by Remark 6.1 we may assume that the union of spectra of all these operators is discrete and fix $\gamma > 0$ to be smaller than the smallest non-zero spectral element. This fixes the contour $\Lambda = \{\lambda \in \mathbb{C} \mid \arg(\lambda - \gamma) = \pi/4\}$.

We will prove below in Proposition 8.5 and Proposition 8.6 by a detailed analysis of Bessel functions that $t(-z^2, \mu)$ admits a uniform asymptotic expansion for $\mu$ going to infinity, with terms of the form $a_j(z)(\mu z)^{-j+1}$ and $b_j\mu^{-j}$, $j \in \mathbb{N}$ and with coefficients $a_j(z)$ bounded as $|z| \to \infty$. Assuming existence of $\gamma > 0$, bounding all eigenvalues from below, we conclude that the infinite sum

\[
\zeta(s) \equiv \zeta(s)[c, c_0, \alpha, R', \Delta_{p,cld,N}] := \sum_{\mu} \zeta(s, D'_c(\mu, \alpha)) - \zeta(s, D_c(\mu, \alpha)) - \sum_{\mu} \zeta(s, D'_{c_0}(\mu)) - \zeta(s, D_{c_0}(\mu)),
\]

for $\text{Re}(s) > \frac{\dim N + 1}{2}$,

where we sum over $\mu^2 \in \text{spec} \Delta_{p,cld,N}\setminus\{0\}$ is well-defined for $\text{Re}(s)$ sufficiently large. We will see below that the difference between logarithms of scalar analytic torsions of the model cusp $U_R$ and the cylinder $U_R \setminus U_R^0 \cong [R, R'] \times N$ is given in terms

\[^{14}\text{U}_R^0 \text{ denotes the open interior of } U_R.\]
of derivatives of such combinations, see (9.1) below. Similar to (7.4) we have an integral representation

\begin{equation}
\zeta(s) = \frac{s^2}{\Gamma(s+1)} \int_0^\infty \frac{t^{s-1}}{2\pi i} \int_{\Lambda} \frac{e^{-\lambda t}}{-\lambda} \sum_{\mu} t(\lambda, \mu) \mu^{-2s} d\lambda \, dt.
\end{equation}

8. Spreafico’s double summation method applied to cusps

Spreafico’s double summation method, cf. [SPR05] and [SPR12], provides a powerful tool for studying zeta-functions of infinite sums of scalar operators. The following theorem is proved in [SPR05, p. 364 (2)] and in a more general setting in Hartmann-Spreafico [HASP10, Theorem 3.2], cf. also Spreafico [SPR12, Theorem 2.12].

**Theorem 8.1.** Consider for \( \text{Re}(s) > \frac{\dim N + 1}{2} \) the following holomorphic function

\begin{equation}
\zeta(s) = \frac{s^2}{\Gamma(s+1)} \int_0^\infty \frac{t^{s-1}}{2\pi i} \int_{\Lambda} \frac{e^{-\lambda t}}{-\lambda} \sum_{\mu} t(\lambda, \mu) \mu^{-2s} d\lambda \, dt.
\end{equation}

(i) Assume \( t(\lambda, \mu) \) admits an asymptotic expansion

\begin{equation}
t(\lambda, \mu) \sim \sum_{j=0}^\infty c_{j-1}(\lambda) \mu^{-j+1}, \quad \mu \to \infty,
\end{equation}

uniformly in \( \lambda = (-z^2) \in \Lambda \) with coefficients and the error term growing at most polynomially as \(|z| \to \infty\). We subtract a finite part of the asymptotics to define

\begin{equation}
p(\lambda, \mu) := t(\lambda, \mu) - \sum_{j=1}^{n/2} c_{2j}(\lambda) \mu^{-2j}, \quad P(\lambda, s) := \sum_{\mu} p(\lambda, \mu) \mu^{-2s}.
\end{equation}

(ii) Assume \( p(\lambda, \mu) \) admits an asymptotic expansion as \(|\lambda| \to \infty\),

\[ p(\lambda, \mu) \sim a_\mu \log(-\lambda) + b_\mu + O(\lambda^{-1/2}), \quad |\lambda| \to \infty, \]

Assume that for \( \text{Re}(s) > 0 \) sufficiently large the sums

\begin{equation}
A(s) := \sum_{\mu} a_\mu \mu^{-2s}, \quad B(s) := \sum_{\mu} b_\mu \mu^{-2s}
\end{equation}

are absolutely convergent holomorphic series and define meromorphic functions on \( \mathbb{C} \) with \( A(s) \) and \( sB(s) \) regular at \( s = 0 \).

---

15Uniform asymptotics of \( t(\lambda, \mu) \) as \( \mu \to \infty \) justifies integral representation of \( \zeta(s) \) for \( \text{Re}(s) \gg 0 \).

16We define square roots using the main branch of the logarithm in \( \mathbb{C} \setminus \mathbb{R}^+ \).
Then $P(\lambda, s)$ and $\zeta(s)$ are both regular at $s = 0$ and we have the following analytic continuation of the latter to a neighborhood of $s = 0$

$$\zeta(s) = \frac{s}{\Gamma(s + 1)} \left( \gamma A(s) - \frac{1}{s} A(s) - B(s) + P(0, s) \right)$$

\begin{equation}
\begin{aligned}
&+ \frac{s^2}{\Gamma(s + 1)} \sum_{j=1}^{n/2} \zeta(s + j, \Delta_{p,cl,N}) \int_{0}^{\infty} \frac{t^{s-1}}{2\pi i} \int_{\Lambda} e^{-t\lambda} c_{2j}(\lambda) d\lambda dt \\
&+ \frac{s^2}{\Gamma(s + 1)} h(s),
\end{aligned}
\end{equation}

where $h$ is analytic at $s = 0$ and $\gamma$ denotes the Euler-Mascheroni constant.

Note that the assumptions of Theorem 8.1 amount to the condition that the (double) sequence of eigenvalues for the operators in (7.3) employed in the definition of $\zeta(s)$ are spectrally decomposable over the sequence $\{\mu\}$ in the sense of Spreafico [Spr12].

**Remark 8.2.** The referee justly questions the fact that seemingly the sequence $\{\mu\} = \text{spec} \Delta_{p,cl,N} \setminus \{0\}$ can be replaced by a sequence of eigenvalues of any discrete positive self-adjoint operator $L$. This of course cannot be the case. The first and obvious condition is that $L$ admits a well-defined zeta-function, which is an absolutely convergent series for Re($s$) > 0 sufficiently large, and admits a meromorphic continuation to $\mathbb{C}$ with $s = 0$ being a regular point and simple pole singularities at integer locations. Under this condition, $P(\lambda, s)$ is regular at $s = 0$. This of course is the generic setting for self-adjoint differential operators on closed manifolds.

The other restriction comes from the fact that the eigenvalues of the operators in (7.3), employed in the definition of $\zeta(s)$ and indexed by the eigenvalues of $L$, cannot accumulate at zero. This would make a choice of a contour $\Lambda \subset \mathbb{C}$ impossible and $\zeta(s)$ would not be well-defined any longer.

Finally, the assumptions of Theorem 8.1 specify that the (double) sequence of eigenvalues for the operators in (7.3) employed in the definition of $\zeta(s)$ are spectrally decomposable over the sequence $\{\mu\}$ in the sense of Spreafico [Spr12]. This conditions encodes the necessary setup for the assumptions of Theorem 8.1 to be satisfied. Thus, the choice of $\{\mu\}$ is far from arbitrary.

### 8.1. Asymptotic expansion of $t(-z^2, \mu)$ as $\mu \to \infty$.

Write for any $(-z^2) \in \Lambda$

$$t \equiv t(\mu) = c(\mu z), \quad s \equiv s(\mu) = \frac{\mu R'}{c(\mu z)},$$

\begin{equation}
\nu(s) \equiv \nu(s)(\mu) = \log \frac{s(\mu)}{1 + \sqrt{1 + s(\mu)^2}} + \sqrt{1 + s(\mu)^2}.
\end{equation}

where we used the notation $c(\mu z) := \sqrt{c^2 + (\mu z)^2}$ from above.

**Lemma 8.3.** For $R' \gg 0$ and $\mu \gg 0$ both sufficiently large, the real part of $tv(s) = (tv(s))(\mu, R')$ is a strictly increasing function of $R'$. 
Proof. We compute (we work with the variable $R$ instead of $R'$ here)

\[
\frac{d}{dR} (t_{\nu}(s))(\mu, R) = \frac{\mu R}{\sqrt{1 + s(\mu, R)^2}} \left( 1 - \frac{1}{1 + \sqrt{1 + s(\mu, R)^2}} \right) + \frac{c(\mu z)}{R}.
\]

For a given $\delta > 0$ with $|\arg(\lambda)| \geq \delta$, there exists $\delta' > 0$ such that

\[
|\arg \sqrt{1 + s(\mu, R)^2} - 1| \leq \pi - \delta'.
\]

Moreover, the following qualitative Figure 1 describes the graph of $\sqrt{1 + s(\mu, R)^2} - 1$ as $(-z^2)$ varies along $\Lambda$. Here, $\tilde{a}(\mu, R) = \sqrt{\frac{\gamma - (c/\mu)^2}{R}}$.

\[
\begin{align*}
\tilde{a}(\mu, R) & \quad 1 \quad -\tilde{a}(\mu, R) \\
\end{align*}
\]

**Figure 1.** The graph of $\sqrt{1 + s(\mu, R)^2} - 1$ as $(-z^2)$ varies along $\Lambda$.

Consequently, we may assume

\[
|\arg \left( 1 - \sqrt{1 + s(\mu, R)^2} \right) | \leq \frac{\delta'}{2},
\]

for $R \gg 0$ sufficiently large. From here we easily conclude by studying the phase of $\frac{d}{dR} (t_{\nu}(s))(\mu, R)$ that the real part of $\frac{d}{dR} (t_{\nu}(s))(\mu, R)$ is strictly positive for $R \gg 0$ sufficiently large and $|\arg(\lambda)| \geq \delta$. This proves the statement for $|\arg(\lambda)| \geq \delta$.

It remains to study the case $|\arg(\lambda)| < \delta$. Note, as $\mu \to \infty$

\[
(t_{\nu}(s))(\mu, R) = t(\mu) \log \frac{s(\mu)}{1 + \sqrt{1 + s(\mu)^2}} + t(\mu) \sqrt{1 + s(\mu)^2}
\]

\[
= - (\mu z) \sqrt{1 + \left( \frac{c}{\mu z} \right)^2} \log \frac{z}{R} \left( \sqrt{1 + \left( \frac{c}{\mu z} \right)^2} + \sqrt{z^2 + R^2} + \left( \frac{c}{\mu z} \right)^2 \right)
\]

\[
+ (\mu z) \sqrt{z^2 + R^2} \sqrt{1 + \frac{z^2}{z^2 + R^2}} \left( \frac{c}{\mu z} \right)^2 \sim \sum_{j=0}^{\infty} a_j(z, R, c)(\mu z)^{-2j+1},
\]

for certain coefficients $a_j(z, R, c)$ that are bounded as $|z| \to \infty$ The leading order term in the expansion (8.9) will be important for the argument below and is given
explicitly after cancellations by

\[ a_0(z, R, c) = \sqrt{\frac{z^2 + R^2}{z^2}} + \log R - \log \left( z + \sqrt{z^2 + R^2} \right). \]  

One checks easily that \( a_0(z, R, c) \) is a strictly increasing function of \( R \) for \( R \gg 0 \) sufficiently large and bounded \( z \). This proves the statement for \( |\arg(\lambda)| < \delta \) and \( \mu, R \gg 0 \) both sufficiently large.

When \((-z^2)\) varies along the contour \( \Lambda \), and \( \gamma \mu^2 > c^2 \), we find that \( t \) varies along the contour \( T \) and \( s \) along the contour \( D \), both of which are illustrated in Figure 2 below.

![Figure 2. The domains T and D.](image)

In that figure, \( \hat{t}(\mu) = i\sqrt{\gamma \mu^2 - c^2} \) and \( \hat{s}(\mu) = i\left( \frac{R'}{\sqrt{\gamma}} + O(\mu^{-1}) \right) \) as \( \mu \to \infty \). Note that for \( \text{Im}(t(\mu)) > 0 \) we have \( \text{Im}(s(\mu)) \leq 0 \), and for \( \text{Im}(t(\mu)) < 0 \) we have \( \text{Im}(s(\mu)) \geq 0 \). We find that for \( R' \gg 0 \) sufficiently large, the domains \( T \) and \( D \) are covered by the domains of the type \( (T_j, D_j), j = 1, 2, 3, 4 \), defined in (3.14) and (3.16). Consequently, the uniform expansions (3.8) and (3.11) for these particular choices of \( t \in T \) and \( s \in D \), when \( \gamma \mu^2 > c^2 \) and \( R' \gg 0 \) is sufficiently large.

We henceforth assume \( R' \gg R \gg 0 \). We simplify the presentation below using a notion of a polyhomogeneity order for polynomials.

**Definition 8.4.** We say that a polynomial \( M \) of degree \( \deg M = Q \) is of polyhomogeneity order \( \rho \), if there exist coefficients \( \left( g_j \right)_{j=\rho}^Q \), such that \( M(x) = \sum_{j=\rho}^Q g_j x^j \).

**Proposition 8.5.** There exist polynomials \( \{M_j\}_{j \in \mathbb{N}} \) with each polynomial \( M_j \) of polyhomogeneity order \( j \), and coefficients \( a_j(z, R', c), a_j(z, R', c_0) \) that are bounded as \( |z| \to \infty \), such
that for each fixed \( R, R' \) with \( R' \gg R \gg 0 \)

\[
F_{\mu z} \equiv F_{\mu z}(\mu, z, R, R') \sim \log \frac{R'}{z} \sqrt{\frac{z^2}{z^2 + R'^2}} + \sum_{j=1}^{\infty} \mathcal{M}_j \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right) (\mu z)^{-j} + \sum_{j=0}^{\infty} (a_j(z, R', c_0) - a_j(z, R', c)) (\mu z)^{-2j+1}, \quad \mu \to \infty,
\]

uniformly in \( z \) with \((-z^2) \in \Lambda\).

**Proof.** We discuss the individual terms in the last expression of (7.7) with \( u = \mu z \). Write \( t_0 = c_0(\mu z) \) and \( s_0 = \mu R'/t_0 \). We find in view of (3.8) as \( \mu \to \infty \)

\[
\log \frac{I_{c_0(\mu z)}(\mu R')}{I_{c(\mu z)}} \sim \frac{1}{2} \log \frac{t \sqrt{1 + s_0^2}}{t_0 \sqrt{1 + s_0^2}} + t_0 v(s_0) - tv(s) + \log \left( 1 + \sum_{k=1}^{\infty} \frac{U_k(p(s))}{t^k} \right) - \log \left( 1 + \sum_{k=1}^{\infty} \frac{U_k(p(s))}{t^k} \right).
\]

Note as \( \mu \to \infty \)

\[
\log \frac{t \sqrt{1 + s^2}}{t_0 \sqrt{1 + s_0^2}} = \log \frac{1 + \sum_{j=1}^{\infty} a_j \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \frac{c_0}{\mu z} \right)^{2j}}{1 + \sum_{j=1}^{\infty} a_j \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \frac{c_0}{\mu z} \right)^{2j}} - \sum_{j=1}^{\infty} b_j(c, c_0) \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right)^{2j} (\mu z)^{-2j},
\]

for some coefficients \( a_j, b_j(c, c_0), j \in \mathbb{N} \). Note also

\[
p(s) = \frac{1}{1 + s^2} = t(\mu) \sqrt{\frac{z^2}{z^2 + R'^2}} (\mu z)^{-1} = t(\mu) \sum_{j=0}^{\infty} N_j \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right) (\mu z)^{-2j-1},
\]

for some polynomials \((N_j)_{j \in \mathbb{N}}\) with each polynomial \(N_j\) of polyhomogeneity order \(2j + 1\). We infer from [Olv97, (7.10)] that the polynomials \(U_k(p)\) are of the structure

\[
U_k(p) = \sum_{b=0}^{k} a_{kb} p^{k+2b},
\]

Hence using the expansion of \(p(s)\) above we find

\[
\frac{U_k(p(s))}{t^k} = \sum_{b=0}^{k} a_{kb} t^{2b} \left( \frac{p(s)}{t} \right)^{k+2b} \sim \sum_{j=0}^{\infty} W_j \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right) (\mu z)^{-2j-k}, \quad \mu \to \infty,
\]

for some polynomials \(W_j\) with each polynomial \(W_j\) of polyhomogeneity order \(2j+k\). Note also the expansion obtained in (8.9). Consequently there exist polynomials
(Q_j)_{j \in \mathbb{N}}, with each polynomial Q_j of polyhomogeneity order j, such that

\[
\log \frac{I_{c(\mu z)}(\mu R')}{I_{c(\mu z)}}(\mu R') \sim \sum_{j=1}^{\infty} Q_j \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right) (\mu z)^{-j}
\]

(8.12)

\[
\quad + \sum_{j=0}^{\infty} (a_j(z, R', c_0) - a_j(z, R', c)) \mu^{-2j+1}, \quad \mu \to \infty.
\]

Using (3.8), (3.11) and (8.11) (which holds for U_s replaced by V_s) we find as \( \mu \to \infty \)

\[
\log \frac{I_{c(\mu z)}}{I_{c(\mu z)}}(\mu R') \sim \log \frac{R'}{z} \sqrt{\frac{z^2}{z^2 + R'^2}} - \log \sqrt{1 + \frac{z^2}{z^2 + R'^2}} \left( \frac{c}{\mu z} \right) \]

(8.13)

\[
\quad + \log \left( 1 + \sum_{k=1}^{\infty} \frac{U_k(p(s))}{t^k} \right) - \log \left( 1 + \sum_{k=1}^{\infty} \frac{V_k(p(s))}{t^k} \right)
\]

\[
\quad \sim \log \frac{R'}{z} \sqrt{\frac{z^2}{z^2 + R'^2}} + \sum_{j=1}^{\infty} B_j \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right) (\mu z)^{-j},
\]

for some polynomials \((B_j)_{j \in \mathbb{N}}\) with each polynomial \(B_j\) of polyhomogeneity order \(j\). Similarly, there exist polynomials \((L_j)_{j \in \mathbb{N}}\) with each polynomial \(L_j\) of polyhomogeneity order \(j\), such that

(8.14) \[ \log I_{c(\mu z)}(\mu R', \alpha) \sim \sum_{j=1}^{\infty} L_j \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right) (\mu z)^{-j}, \quad \mu \to \infty. \]

Similar expansion holds for \( \log K_{c(\mu z)}(\mu R', \alpha) \). In view of the previous Lemma 8.3, we may choose \( R' \gg R \gg 0 \) sufficiently large such that as \( \mu \to \infty \)

(8.15) \[ \frac{I_{c(\mu z)}(\mu R)}{I_{c(\mu z)}} \frac{K_{c(\mu z)}(\mu R')}{I_{c(\mu z)}}(\mu R') = O(\mu^{-\infty}), \quad \frac{I'_{c(\mu z)}(\mu R)}{K_{c(\mu z)}} \frac{K'_{c(\mu z)}(\mu R')}{I'_{c(\mu z)}} = O(\mu^{-\infty}). \]

Plugging the expansions (8.13), (8.14), (8.15), as well as the expansion obtained in (8.12) into the expression (7.7), we arrive at the statement of the proposition. \( \square \)

**Proposition 8.6.** There exist \((m_j)_{j \in \mathbb{N}}\) such that for any \( Q \in \mathbb{N} \) and fixed \( R \gg 0 \)

\[ F_0 \equiv F_0(\mu, R, R') = \sum_{j=1}^{Q-1} m_j(\mu R')^{-j} + O(\mu R')^{-Q}, \quad (\mu R') \to \infty. \]
Proof. We study the individual summands in the last expression of (7.7) with \( u = 0 \). Using the expansions (3.1) and (3.2) we find as \( \mu \to \infty \)
\[
\log \frac{I_\mu}{I_c}(\mu R') - \sum_{j=1}^{\infty} q_j(\mu R')^{-j}, \quad \log \frac{I_\mu}{I_c}(\mu R') \sim \sum_{j=1}^{\infty} p_j(\mu R')^{-j},
\]
for certain coefficients \((q_j)_{j \in \mathbb{N}}, (p_j)_{j \in \mathbb{N}}, (p_j)_{j \in \mathbb{N}}\). Similar expansion holds for \( \log K_c(\mu R', \alpha) \). Moreover, as \( \mu \to \infty \)
\[
\frac{I_\mu}{K_c}(\mu R') \approx O(e^{-u(R'-R)}), \quad \frac{I'_\mu}{K'_c}(\mu R') \approx O(e^{-u(R'-R)}).
\]
This proves the statement with \( m_j = q_j + p_j - r_j, j \in \mathbb{N} \), since \( R' > R \).

We summarize the results of the both preceeding propositions.

Corollary 8.7. Assume \( R' \gg R \gg 0 \) are fixed. Then there exist polynomials \((M_j)_{j \in \mathbb{N}}\) with each polynomial \( M_j \) of polyhomogeneity order \( j \), coefficients \((m_j)_{j \in \mathbb{N}}\) and coefficients \( a_j(z, R', c), a_j(z, R', c_0) \) that are bounded as \( |z| \to \infty \), such that
\[
t(-z^2, \mu) \sim \log \frac{R'}{z} \sqrt{\frac{z^2}{z^2 + R'^2}} + \sum_{j=1}^{\infty} \left( M_j \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right) (\mu z)^{-j} + m_j(\mu R')^{-j} \right)
\]
\[
\quad + \sum_{j=0}^{\infty} (a_j(z, R', c_0) - a_j(z, R', c)) \mu^{-2j+1}, \quad \mu \to \infty,
\]
uniformly in \( z \) with \( (-z^2) \in \Lambda \).

The zeta function \( \zeta(s, \Delta_{p,cd,N}) = \sum \mu^{-2s}, \text{Re}(s) > n/2 \), where we sum over \( \mu^2 \in \text{spec}\Delta_{p,cd,N}\setminus\{0\} \) according to their multiplicity, extends meromorphically to \( \mathbb{C} \) with pole singularities at integer locations \( s = n/2 - k, k \in \mathbb{N}_0 \setminus \{n/2\} \). Since \( n = \text{dim} \mathbb{R} \) is even, the asymptotic terms \( \mu^{-2j}, j = 1, \ldots, n/2 \), in the expansion of \( t(-z^2, \mu) \) as \( \mu \to \infty \), lead after summation in \( \mu^2 \in \text{spec}\Delta_{p,cd,N}\setminus\{0\} \) to an irregular behaviour of \( \zeta(s) \) and hence we define
\[
(8.16) \quad p(-z^2, \mu) \defeq t(-z^2, \mu) - \sum_{j=1}^{n/2} \left( M_{2j} \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right) (\mu z)^{-2j} + m_{2j}(\mu R')^{-2j} \right).
\]

8.2. Asymptotic expansion of \( p(\lambda, \mu) \) as \( \lambda \to \infty \).

Proposition 8.8.
\[
p(\lambda, \mu) \sim a_\mu \log(-\lambda) + b_\mu + O(\lambda^{-1/2}), \quad z \to \infty,
\]
where \( a_\mu = -1/2 \) and \( b_\mu = \log(2/\mu) - F_0 - \sum_{j=1}^{n/2} m_{2j}(\mu R')^{-2j} \).
Proof. We study the asymptotic expansion of the individual terms in
\[
(8.17) \quad p(-z^2, \mu) = F_{\mu z} - F_0 - \sum_{j=1}^{n/2} \left( M_{2j} \left( \sqrt{\frac{z^2}{z^2 + R'^2}} \right) (\mu z)^{-2j} + m_{2j}(\mu R'^{-2j}) \right).
\]

Applying (3.5) and (3.7) with either \( t = c(\mu z) \) or \( t = c_0(\mu z) \), we find
\[
F_{\mu z} \sim -\log(z) + \log(2/\mu) + O(z^{-1}), \quad z \to \infty.
\]

The statement is now straightforward, since \( F_0 \) is independent of \( z \). \( \square \)

8.3. Analytic continuation of \( \zeta(s) \) near \( s = 0 \). Following Theorem 8.1 we put
\[
P(\lambda, s) := \sum_{\mu} p(\lambda, \mu) \mu^{-2s}, \quad A(s) := \sum_{\mu} a_\mu \mu^{-2s} = -\frac{1}{2} \zeta(s, \Delta_{p,cl,N}),
\]
\[
B(s) := \sum_{\mu} b_\mu \mu^{-2s} = \zeta(s, \Delta_{p,cl,N}) \log 2 + \frac{1}{2} \zeta'(s, \Delta_{p,cl,N})
\]
\[
- \sum_{j=1}^{n/2} \zeta(s + j, \Delta_{p,cl,N}) m_{2j}(R'^{-2j}) - \sum_{\mu} F_0(\mu) \mu^{-2s}.
\]

we sum over \( \mu^2 \in \text{spec} \Delta_{p,cl,N} \setminus \{0\} \) according to their multiplicity, and indicate the dependence of \( F_0 \) on \( \mu \) by \( F_0 = F_0(\mu) \). Then Spreafico’s double summation method in Theorem 8.1 yields the following analytic continuation of \( \zeta(s) \) to \( s = 0 \)
\[
(8.19) \quad \zeta(s) = \frac{s}{\Gamma(s+1)} \left( \gamma A(s) - \frac{1}{s} A(s) - B(s) + P(0, s) \right)
\]
\[
+ \frac{s^2}{\Gamma(s+1)} \sum_{j=1}^{n/2} \zeta(s + j, \Delta_{p,cl,N}) \int_0^\infty \frac{t^{s-1}}{2\pi i} \int_{\Lambda} \frac{e^{-t\lambda}}{(-\lambda)^{1+j}} M_{2j} \left( \sqrt{\frac{(\lambda)}{R'^2 - \lambda}} \right) d\lambda dt
\]
\[
+ \frac{s^2}{\Gamma(s+1)} \sum_{j=1}^{n/2} \zeta(s + j, \Delta_{p,cl,N}) \int_0^\infty \frac{t^{s-1}}{2\pi i} \int_{\Lambda} \frac{e^{-t\lambda}}{(-\lambda)} m_{2j}(R'^{-2j}) d\lambda dt
\]
\[
+ \frac{s^2}{\Gamma(s+1)} h(s),
\]

where \( h \) is analytic at \( s = 0 \) and \( \gamma \) denotes the Euler-Mascheroni constant. We may now prove the following central result of this section.

**Theorem 8.9.** Let \( R \gg 0 \). Then\(^{18}\)
\[
\lim_{R' \to \infty} \zeta'(0) \equiv \lim_{R' \to \infty} \zeta'(0) [c, c_0, \alpha, R', \Delta_{p,cl,N}] = -\zeta(0, \Delta_{p,cl,N}) \log 2.
\]

\(^{17}\)We point out that \( t = c(\mu z) \sim \mu z + O(z^{-1}) \) as \( z \to \infty \).

\(^{18}\)We point out that the proof of this limit does not require the expansion of Corollary 8.7 to hold uniformly in \( R, R' > 0 \), but only for any fixed \( R' \gg R \gg 0 \) sufficiently large. The only crucial uniformity requirement with respect to \( R' \) is the statement of Proposition 8.6.
Proof. We find for any $k \in \mathbb{N}_0$

$$
\int_0^\infty \frac{t^{s-1}}{2\pi i} \int_\Lambda \frac{e^{-t\lambda}}{(-\lambda)^{1+j}} \left( \sqrt{\frac{(-\lambda)}{R'^2 - \lambda}} \right)^{2j+k} d\lambda \, dt
= (R')^{-2s-2j} \int_0^\infty \frac{t^{s-1}}{2\pi i} \int_\Lambda \frac{e^{-t\lambda}}{(-\lambda)^{1+j}} \left( \sqrt{\frac{(-\lambda)}{1 - \lambda}} \right)^{2j+k} d\lambda \, dt
$$

In particular, for $j \geq 1$, these terms and their derivatives, both evaluated at $s = 0$, vanish in the limit $R' \to \infty$. Moreover we note

$$
\int_0^\infty \frac{t^{s-1}}{2\pi i} \int_\Lambda \frac{e^{-t\lambda}}{(-\lambda)^{1+j}} d\lambda \, dt = 0,
$$

since the contour $\Lambda$ does not encircle the origin $\lambda = 0$. Consequently, the expression (8.19) reduces in the limit to

$$
\lim_{R' \to \infty} \zeta'(0) = \lim_{R' \to \infty} \frac{d}{ds} \bigg|_{s=0} \frac{s}{\Gamma(s+1)} \left( \frac{\gamma A(s) - \frac{1}{s} A(s) - B(s) + P(0,s)}{s} \right).
$$

In view of Proposition 8.6 we find for any integer $Q > n$

$$
\sum_{\mu} F_0(\mu) \mu^{-2s} = \sum_{j=1}^Q n_j (R')^{-j} \zeta(s + j/2, \Delta_{p,ccl,N}) + (R')^{-Q+1} \omega(s, R'),
$$

where we sum over $\mu^2 \in \text{spec} \Delta_{p,ccl,N} \setminus \{0\}$ according to their multiplicity, and $\omega(s, R')$ is analytic at $s = 0$ with $\omega(0, R'), \omega'(0, R')$ both bounded as $R' \to \infty$. Consequently, in view of (8.18)

$$
\lim_{R' \to \infty} \frac{d}{ds} \bigg|_{s=0} \frac{sB(s)}{\Gamma(s+1)} = \lim_{R' \to \infty} \frac{d}{ds} \bigg|_{s=0} \frac{s}{\Gamma(s+1)} \left( \frac{\zeta(s, \Delta_{p,ccl,N}) \log 2 + \frac{1}{2} \zeta'(s, \Delta_{p,ccl,N})}{s} \right)
= \zeta(0, \Delta_{p,ccl,N}) \log 2 + \frac{1}{2} \zeta'(0, \Delta_{p,ccl,N}).
$$

Denote for each polynomial $M_{2j}$ the coefficient of the lowest degree term by $M_{2j,0}$. Then, by construction we find

$$
P(0,s) = - \sum_{j=1}^{n/2} (M_{2j,0} + m_{2j})(R')^{-2j} \zeta(s + j, \Delta_{p,ccl,N}).
$$

Hence

$$
\lim_{R' \to \infty} \frac{d}{ds} \bigg|_{s=0} \frac{sP(0,s)}{\Gamma(s+1)} = 0.
$$

Finally, a straightforward computation yields

$$
\lim_{R' \to \infty} \frac{d}{ds} \bigg|_{s=0} \frac{sA(s)}{\Gamma(s+1)} \left( \frac{\gamma - \frac{1}{s}}{s} \right) = -A'(0) = \frac{1}{2} \zeta'(0, \Delta_{p,ccl,N}).
$$

These identities lead in view of (8.19) to the statement of the theorem. \qed
9. Renormalized Ray-Singer analytic torsion on a model cusp

In this section we establish Theorem 2.4.

9.1. Comparison of analytic torsions on a truncated and a full model cusp. Consider the model cusp $\mathcal{U}_R$, $R \gg 1$, and the truncated cusp $\mathcal{U}_R \setminus \mathcal{U}_R^\circ$, $R' \gg R$, which is a finite cylinder $[R, R'] \times \mathbb{N}$. The Riemannian metric $g$ and the flat Hermitian vector bundle $(E, \nabla, h)$ over $\mathcal{U}_R$, restrict to metric structures over $\mathcal{U}_R \setminus \mathcal{U}_R^\circ$, and the Definitions 2.2 and 2.3 carry over to the special case of a compact Riemannian manifold $(\mathcal{U}_R \setminus \mathcal{U}_R^\circ, g)$ equipped with a flat Hermitian vector bundle.

We write in each degree $p = 0, \ldots, n$

$$\Delta_{H,p} := -(x \partial_x)^2 - (x \partial_x) + \left( \frac{n}{2} - p \right)^2 - \frac{1}{4} : C^\infty_0(R, \infty) \to C^\infty_0(R, \infty),$$

$$\Delta'_{H,p} := -(x \partial_x)^2 - (x \partial_x) + \left( \frac{n}{2} - p \right)^2 - \frac{1}{4} : C^\infty_0(R, R') \to C^\infty_0(R, R'),$$

and their self-adjoint extensions with Dirichlet and Neumann boundary conditions

$$\mathcal{D}(\Delta_{H,p}) = \{ f \in \mathcal{D}_{\max}(\Delta_{H,p}) \mid f(R) = 0 \},$$

$$\mathcal{D}(\Delta_{H,p,\text{Neu}}) = \{ f \in \mathcal{D}_{\max}(\Delta_{H,p}) \mid f'(R) = f(R)((n - 1)/2 - p)/R \},$$

$$\mathcal{D}(\Delta'_{H,p}) = \{ f \in \mathcal{D}_{\max}(\Delta'_{H,p}) \mid f(R) = f(R') = 0 \}.$$

Then, in full analogy with [VER09, Remark 4.5 and (4.4)], where a similar decomposition of the de Rham complex into harmonic and non-harmonic subcomplexes has been employed in the setup of conical singularities, we have the following identity

$$\log T(\mathcal{U}_R \setminus \mathcal{U}_R^\circ, E, N^2, g) - \log T(\mathcal{U}_R, E, N, g)$$

$$= \frac{1}{2} \sum_{p=0}^{n} (-1)^p \zeta(s) \left[ \left| \frac{n}{2} - p - 1 \right|, \left| \frac{n}{2} - p \right|, \left( p - \frac{n - 3}{2} \right), R', \Delta_{p,\text{cll},N} \right]$$

$$+ \frac{1}{2} \sum_{p=0}^{n} (-1)^{p+1} \dim H^p(N, E) \left( \zeta'(0, \Delta_{H,p}') - \zeta'(0, \Delta_{H,p}) \right)$$

$$- \frac{1}{2} \sum_{p=0}^{n} (-1)^{p+1} (p + 1) \dim H^p(N, E) \left( \zeta'(0, \Delta_{H,p,\text{Neu}}') - \zeta'(0, \Delta_{H,p}) \right).$$

(9.1)

Proposition 9.1.

$$\sum_{p=0}^{n} (-1)^{p+1} \dim H^p(N, E) \zeta'(0, \Delta_{H,p}')$$

$$\sim \sum_{p \neq n/2} (-1)^p \dim H^p(N, E) \left( \log(R'/R)^{\frac{n}{2} - p} - \log \left| \frac{n}{2} - p \right| \right)$$

$$+ (-1)^{n/2} \dim H^{\frac{n}{2}}(N, E)(\log 2 + \log \log R') + o(1), \text{ as } R' \to \infty.$$
Proof. Consider the rescaling \( f \mapsto x^{1/2}f \) that extends to a unitary transformation \( L^2([R', R'], dx) \to L^2([R, R'], x^{-1}dx) \). Under that transformation, \( \Delta_{H,p}' \) is unitarily equivalent to a self-adjoint extension of \(- (x\partial_x)^2 + (n/2 - p)^2\) in \( L^2([R, R'], x^{-1}dx) \) with Dirichlet boundary conditions. Under the change of coordinates \( x = e^r \) we obtain that \( \Delta_{H,p}' \) is spectrally equivalent to a self-adjoint extension of

\[
D_{H,p}' = -\frac{d^2}{dr^2} + \left(\frac{n}{2} - p\right)^2 \colon C_0^\infty(\log R, \log R') \to C_0^\infty(\log R, \log R'),
\]

in \( L^2([\log R, \log R'], dr) \) with Dirichlet boundary conditions. Let \( \phi \) and \( \psi \) be solutions to \( D_{H,p}' f = 0 \), where \( \phi \) satisfies the Dirichlet boundary conditions at \( r = \log R \), normalized such that \( \phi'(\log R) = 1 \); and \( \psi \) satisfies the Dirichlet boundary conditions at \( r = \log R' \), normalized such that \( \psi'(\log R') = -1 \). Then

\[
\phi(r) = \begin{cases} 
|n - 2p|^{-1} \left( e^{\frac{(R\log R')}{2} + \frac{(R'/R)^{1/2}}{2}} - e^{-\frac{|n - 2p|}{2} R} \right), & p \neq \frac{n}{2}, \\
\frac{R - \log R}{R}, & p = \frac{n}{2}.
\end{cases}
\]

The solution \( \psi \) is obtained similarly by replacing \( R \) with \( R' \) and multiplying the expression with \((-1)\). Then, by [BFK95] (recall \( W(\phi, \psi) \) denotes the Wronskian determinant of the fundamental system \( \phi, \psi \))

\[
\zeta'(0, D_{H,p}') = \zeta'(0, \Delta_{H,p}') = -\log \frac{\pi W(\phi, \psi)}{2\Gamma(3/2)^2} = -\log 2\psi(\log R)
\]

\[
= \begin{cases} 
\log \left| \frac{n}{2} - p \right| - \log \left( (R'/R)^{1/2} - (R/R')^{1/2} \right), & p \neq \frac{n}{2}, \\
- \log 2 - \log(\log R' - \log R), & p = \frac{n}{2}.
\end{cases}
\]

The statement is now obvious \( \square \)

**Proposition 9.2.**

\[
\sum_{p=0}^{n} (-1)^{p+1} \dim H^p(N, E) \zeta'(0, \Delta_{H,p})
\]

\[
= \sum_{p \neq n/2} (-1)^{p+1} \dim H^p(N, E) \left( \left| \frac{n}{2} - p \right| \log R + \frac{1}{2} \log \left| \frac{n}{2} - p \right| \right).
\]

**Proof.** As in the proof of Proposition 9.1, similar to [Pfa13, Lemma 10.6], we transform \( \Delta_{H,p} \) to a self-adjoint extension of

\[
D_{H,p} = -\frac{d^2}{dr^2} + \left(\frac{n}{2} - p\right)^2 \colon C_0^\infty(\log R, \infty) \to C_0^\infty(\log R, \infty),
\]

in \( L^2([\log R, \infty), dr) \) with Dirichlet boundary conditions. Its resolvent kernel is given explicitly by

\[
(D_{H,p} + z^2)^{-1}(r, r') = \frac{1}{2k} e^{-k|r - r'|} - \frac{1}{2k} e^{-k(r + r' - 2\log R)},
\]
where we have set $k = \sqrt{\mu_p^2 + z^2}$, $\mu_p = (n/2 - p)$ and fixed $z > 0$. We compute
\[
\int_{\log R}^{\log R'} (D_{H,p} + z^2)^{-1}(r, r) \, dr = \frac{1}{2k} \log(R'/R) + \frac{1}{4k^2}((R'/R)^{-2k} - 1).
\]
The renormalized trace $\text{Tr}_r(D_{H,p} + z^2)^{-1}$ is defined as the constant term in the expansion of the above expression as $R' \to \infty$. Hence we find
\[
\text{Tr}_r(D_{H,p} + z^2)^{-1} = -\frac{\log R}{2k} - \frac{1}{4k^2}.
\]
According to the Definition 4.3 we have
\[
\zeta'(0, \Delta_{H,p}) \equiv \zeta'(0, D_{H,p}) = 2 \int_0^\infty z \text{Tr}_r(D_{H,p} + z^2)^{-1} \, dz.
\]
Straightforward computations lead to the following formulæ
\[
\zeta'(0, \Delta_{H,p}) = \begin{cases} \frac{n}{2} - p \left| \log R + \frac{1}{2} \log \left| \frac{n}{2} - p \right| \right|, & p \neq \frac{n}{2}, \\ 0, & p = \frac{n}{2}. \end{cases}
\]

**Proposition 9.3.**
\[
\sum_{p=0}^{n} (-1)^{p+1}(p + 1) \dim H^p(N, E) \left( \zeta'(0, \Delta_{H,p,Neu}) - \zeta'(0, \Delta_{H,p}) \right)
\]
\[
= \sum_{p \neq n/2} (-1)^{p+1} \dim H^p(N, E) \left| \frac{n}{2} - p \right| \log \left( 2 \left| \frac{n}{2} - p \right| \right).
\]

**Proof.** We continue under the notation of Proposition 9.2. The Laplacian $\Delta_{H,p,Neu}$ with generalized Neumann boundary conditions transforms to a self-adjoint extension of $D_{H,p}$ in $L^2([\log R, \infty), dr)$ with boundary conditions $f'(\log R) = \mu_p f(\log R)$. Its resolvent kernel is given explicitly by, cf. [Der07, Theorem 5.3]
\[
(D_{H,p,Neu} + z^2)^{-1}(r, r') = \frac{1}{2k} e^{-k|r-r'|} + \frac{1}{2k} \frac{k - \mu_p}{k + \mu_p} e^{-k(r + r' - 2 \log R)}.
\]
As in the previous proposition we find
\[
\text{Tr}_r(D_{H,p,Neu} + z^2)^{-1} = -\frac{\log R}{2k} + \frac{1}{4k^2} \frac{k - \mu_p}{k + \mu_p}.
\]
Straightforward computations lead to the following formulæ
\[
\zeta'(0, \Delta_{H,p,Neu}) - \zeta'(0, \Delta_{H,p}) = \begin{cases} -\log 2|\mu_p|, & p < \frac{n}{2}, \\ \log 2|\mu_p|, & p > \frac{n}{2}, \\ 0, & p = \frac{n}{2}. \end{cases}
\]
Corollary 9.4.

\[ \log T(\mathcal{U}_R \mathcal{U}_{R'}, E, N^2, g) - \log T(\mathcal{U}_R, E, N, g) - \frac{1}{2} \dim H^2(N, E)(\log 2 + \log \log R') \]

\[ + \sum_{p \neq n/2} \frac{(-1)^p}{2} \dim H^p(N, E) \left( \left\lfloor \frac{n}{2} - p \right\rfloor \log R' - \frac{1}{2} \log \left\lfloor \frac{n}{2} - p \right\rfloor \right) \]

\[ + \sum_{p \neq n/2} \frac{(-1)^p}{2} \dim H^p(N, E) \left\lfloor \frac{n}{2} - p \right\rfloor \log \left( 2 \left\lfloor \frac{n}{2} - p \right\rfloor \right) \]

\[ + o(1), \quad R' \to \infty. \]

**Proof.** The statement follows from plugging in the results from the Propositions 8.9, 9.1 and 9.2 into (9.1), and the fact that for an even dimensional Riemannian manifold \((N, g^N)\)

\[ \sum_{p=0}^{n} (-1)^p \zeta(s, \Delta_{p, c_{cl}, N}) \equiv 0. \]

\[ \square \]

9.2. Metric anomaly and final result for analytic torsion on a model cusp. The final step in our argument leading up to a formula for the renormalized analytic torsion on a model cusp is the explicit computation of the Ray Singer analytic torsion on \((\mathcal{U}_R \mathcal{U}_{R'}, E, g)\) in terms of the Brüning-Ma metric anomaly. Consider a Riemannian metric on \(\mathcal{U}_R \mathcal{U}_{R'} \cong [R, R'] \times N\)

\[ g_0 = dx^2 + g^N, \quad x \in [R, R']. \]

Then, by [BrMa06, Theorem 0.1] we find as in\(^{19}\) (2.6)

\[ \log \left( \frac{\| \cdot \|_{RS(\mathcal{U}_R \mathcal{U}_{R'}, E, N^2, g_0)}^2}{\| \cdot \|_{RS(\mathcal{U}_R \mathcal{U}_{R'}, E, N^2, g_0)}^2} \right) = \left\lfloor \frac{\text{rank}(E)}{2} \left[ \int_{N \times [R]} B \left( \nabla_g^{T\mathcal{U}_R \mathcal{U}_{R'}} \right) + \int_{N \times [R']} B \left( \nabla_g^{T\mathcal{U}_R \mathcal{U}_{R'}} \right) \right] \right. \]

\[ = \left\lfloor \frac{\text{rank}(E)}{2} \left[ \int_{N \times [R]} B \left( \nabla_g^{T\mathcal{U}_R \mathcal{U}_{R'}} \right) + \int_{N \times [R']} B \left( \nabla_g^{T\mathcal{U}_R \mathcal{U}_{R'}} \right) \right] \right. \]

where the subindex indicates dependence on the cusp metric \(g\). The metric anomaly term \(B(\nabla_g^{T\mathcal{U}_R \mathcal{U}_{R'}})\) is invariant under scaling of the Riemannian metric, cf. [MuVe14, Proposition 4.1]. Hence we may study \(B(\nabla_g^{T\mathcal{U}_R \mathcal{U}_{R'}})\) at the boundary component at \(N \times [R']\) using the rescaled metric (we write \(y = R' - x\))

\[ (R')^2g = \left( \frac{R'}{R' - y} \right)^2 (dy + g^N) = f(y, R')(dy^2 + g^N), \quad y \in [0, \varepsilon), \]

where \(f(0, R') = 1\) and \(f'(0, R') = 2(R')^{-1}\).

Our next argument requires some additional notation. Let \(A\) and \(B\) be two \(\mathbb{Z}_2\)-graded algebras with identity and denote by \(A \hat{\otimes} B\) their \(\mathbb{Z}_2\)-graded tensor product.\(^{19}\)

---

\(^{19}\)On oriented compact Riemannian manifolds of odd dimension, the Ray-Singer analytic torsions for relative and absolute boundary conditions coincide by Poincare duality.
We identify $\mathcal{A}$ with $\mathcal{A} \otimes I$ and write $\widehat{\mathcal{B}} := I \otimes \mathcal{B}$. Moreover we put $\wedge := \otimes$ so that $\mathcal{A} \otimes \mathcal{B} = \mathcal{A} \wedge \mathcal{B}$.

Let $R^{TN}$ be the curvature tensor of $(N, g^N)$ and denote by $(e_k)_{k=1}^{m-1}$ a local orthonormal frame field on $(N, g^N)$. Let $e_m$ denote the inward-pointing unit normal vector at every boundary point of $N \times \{R\}$. Let $(\hat{e}_k)_{k=1}^{m-1}$ be the dual orthonormal frame field of $T^*U_R \cup U^0_R$, and let $\hat{e}_m^*$ be the canonical identification with the element $e_k^*$ of $\Lambda T^*U_R \cup U^0_R$. Let $j: N = N \times \{R\} \hookrightarrow U_R \cup U^0_R$ be the canonical embedding. Then [BrMa06, (1.15)] defines

$$
\hat{R}^{TN} := \frac{1}{2} \sum_{1 \leq k, j \leq m-1} (e_k, R^{TN} e_j) \hat{e}_k^* \wedge \hat{e}_j^* \in \Lambda T^*N \otimes \Lambda T^*N
$$

(9.2)

$$
\hat{S} := \frac{1}{2} j^* \nabla_{U_R \cup U^0_R} e_m^* \in \Lambda T^*N \otimes \Lambda T^*N.
$$

$\hat{R}^{TN}$ and $\hat{S}^2$ are both homogeneous of degree two. While $\hat{R}^{TN}$ encodes the curvature of $(N, g^N)$, $\hat{S}$ measures the deviation of $g$ from a metric product structure near the boundary. By [BrMa06, (4.39)] $\hat{S}$ is given explicitly by

$$
\hat{S} = \frac{1}{4} f'(0, R') \sum_k e_k^* \wedge \hat{e}_k^* = -\frac{1}{4 R'} \sum_k e_k^* \wedge \hat{e}_k^* =: -\frac{\hat{S}_g}{4 R'},
$$

(9.3)

where we used the fact that $f'(0, R') = -2(R')^{-1}$. The final ingredient in the construction is the Berezin integral (see [BrMa06, Section 1.1])

$$
\int_{B_N}^{B_N} : \Lambda T^*N \otimes \Lambda T^*N \rightarrow \Lambda T^*N,
$$

which is non-trivial only on elements which are homogeneous of degree $(m - 1)$. The secondary class $B(\nabla_{U_R \cup U^0_R})$, introduced in [BrMa06, (1.17)] is then defined by

$$
B(\nabla_{U_R \cup U^0_R}) := \int_{B_N}^{B_N} \exp \left(-\frac{1}{2} R^{TN}\right) \sum_{k=1}^{\infty} \frac{(-\hat{S}_g)^{k}}{4k!(k + 1)}.
$$

(9.5)

Recall that the Berezin integral is non-trivial only on elements of degree $(m - 1)$. Hence, despite an infinite sum in the formula (9.5), only a finite number of summands yield a non-trivial contribution to $B(\nabla_{U_R \cup U^0_R})$. Consequently, in view of the $(R'^{-1})$ factor in (9.3), and the fact that the sum in (9.5) starts with $k = 1$, the anomaly term $B(\nabla_{U_R \cup U^0_R})$ vanishes in the limit as $R' \rightarrow \infty$ and hence

$$
\lim_{R' \rightarrow \infty} \log \left(\frac{\| \cdot \|_{T(g^N)} \cdot \left\| \nabla_{U_R \cup U^0_R} \right\|_{T(g^N)}}{\| \cdot \|_{T(g^N)}}\right) = \frac{\text{rank}(E)}{(-2)} \int_{N \times \{R\}} B(\nabla_{g^R}).
$$

(9.6)
Theorem 9.5.

\[
\log T(\mathcal{U}_R, E, N, g) = \frac{\text{rank}(E)}{(-2)} \int_{N \times [R]} B(\nabla^T g) + \sum_{p \neq n/2} \frac{(-1)^{p+1}}{2} \dim H^p(N, E) \log \left| \frac{n}{2} - p \right|
\]

\[
+ \sum_{p \neq n/2} \frac{(-1)^{p+1}}{2} \dim H^p(N, E) \left| \frac{n}{2} - p \right| \log R
\]

\[
+ \sum_{p \neq n/2} \frac{(-1)^{p+1}}{2} \dim H^p(N, E) \left| \frac{n}{2} - p \right| \log \left( 2 \left| \frac{n}{2} - p \right| \right).
\]

Proof. For any \( \omega \in H^p(N, E) \cong H^p(\mathcal{U}_R \backslash \mathcal{U}_{R'}, E) \) we compute

\[
(9.7) \quad \frac{\|\omega\|^2_{g_0}}{\|\omega\|^2_{g}} = \int_{R'} (\omega \cdot \omega)_{g_0} - (\omega \cdot \omega)_{g} \, dx = \begin{cases} 
\frac{(R')^{2p-n} - R^{2p-n}}{(2p-n)(R'-R)}, & p \neq \frac{n}{2}, \\
\log R' - \log R, & p = \frac{n}{2}.
\end{cases}
\]

Consequently

\[
(9.8) \quad \log \frac{\| \cdot \|_{\det H^*(\mathcal{U}_k, E, N^2, g)}}{\| \cdot \|_{\det H^*(\mathcal{U}_k, E, N^2, g_0)}} = \sum_{p=0}^{n} \frac{(-1)^{p+1}}{2} \left( \left| \frac{n}{2} - p \right| - 1 \right) \dim H^p(N, E) \log R'
\]

\[
+ \sum_{p=0}^{n} \frac{(-1)^p}{2} \dim H^p(N, E) \left| \frac{n}{2} - p \right| \log R
\]

\[
+ \sum_{p \neq n/2} \frac{(-1)^{p+1}}{2} \dim H^p(N, E) \log \left( 2 \left| \frac{n}{2} - p \right| \right)
\]

\[
+ \frac{(-1)^{n+1}}{2} \dim H^2(N, E) \log \log R' + o(1), \quad R' \to \infty.
\]

The product rule for the scalar analytic torsion implies

\[
(9.9) \quad \log T(\mathcal{U}_R \backslash \mathcal{U}_{R'}, E, N^2, g_0) = \frac{1}{2} \chi(N, E) \log 2(R' - R).
\]

Note that

\[
\log \frac{\| \cdot \|_{\det H^*(\mathcal{U}_k, E, N^2, g)}}{\| \cdot \|_{\det H^*(\mathcal{U}_k, E, N^2, g_0)}} - \log T(\mathcal{U}_R, E, N, g)
\]

\[
= \log T(\mathcal{U}_R \backslash \mathcal{U}^0_{R'}, E, N^2, g) - \log T(\mathcal{U}_R, E, N, g)
\]

\[
- \log T(\mathcal{U}_R \backslash \mathcal{U}^0_{R'}, E, N^2, g_0) + \log \frac{\| \cdot \|_{\det H^*(\mathcal{U}_k, E, N^2, g)}}{\| \cdot \|_{\det H^*(\mathcal{U}_k, E, N^2, g_0)}}
\]
Hence by Corollary 9.4, (9.6), (9.8) and (9.9) we find
\[
\log T(\mathcal{U}_R, E, N, g) \sim \frac{\text{rank}(E)}{2} \int_{N \times [R]} B(\nabla_{g}^{\text{T}_{\mathcal{U}_R}}) + \sum_{p \neq n/2} \frac{(-1)^{p+1}}{2} \dim H^{p}(N, E) \log \left| \frac{n}{2} - p \right| \\
+ \sum_{p \neq n/2} \frac{(-1)^{p+1}}{2} \dim H^{p}(N, E) \left| \frac{n}{2} - p \right| \log R \\
+ \sum_{p \neq n/2} \frac{(-1)^{p+1}}{2} \dim H^{p}(N, E) \left| \frac{n}{2} - p \right| \log \left( 2 \left| \frac{n}{2} - p \right| \right) \\
+ o(1), \quad R' \to \infty.
\]

Since \( \log T(\mathcal{U}_R, E, N, g) \) is independent of \( R' \), the statement follows by taking the limit as \( R' \to \infty \). \( \square \)

10. Gluing formula for analytic torsion on non-compact manifolds

In this section we consider a non-compact orientied odd-dimensional Riemannian manifold \((M, g)\) with \( M = K \cup_{N} \mathcal{U} \). We do not assume a specific structure of \( g \) here, but pose the Assumptions 2.5, 2.6 and 2.8 instead.

The fundamental proof strategy is due to Lesch [Les12] and Pfaff [Pfa13], where the former proved a gluing formula on compact possibly singular manifolds, and the latter extended the argument to non-compact hyperbolic spaces using a sequence of compact manifolds that in some sense approximates the non-compact space.

Certain aspects of their argument need to be reproved here due to different assumptions, which apply to a larger class of non-compact spaces. Moreover, in contrast to [Les12] and [Pfa13] we need formulate the gluing formula in terms of the Ray-Singer norms on the determinant lines.

10.1. Stability of analytic torsion under compactly supp. metric variations. For any \( R > 1 \), we embed \( \mathcal{U}_R \subset M \) in an obvious way and write \( M_R := M \setminus \mathcal{U}_R^c \). Below we omit the lower index \( p \) for the Hodge Laplacian and the heat kernel, if we refer to their actions on differential forms in all degrees.

Consider now for \( R > 1 \) the closed double manifold \( \widetilde{M}_R := M_{R+1} \cup_{N} (-M_{R+1}) \), where \((-M_{R+1})\) denotes a second copy of \( M_{R+1} \) with reversed orientation. Choose a metric on \( \widetilde{M}_R \) that coincides with \( g \) on both copies of \( M \subset \widetilde{M}_R \) and is product in a tubular neighborhood \( N \times (R+1/2, R+3/2) \subset \widetilde{M}_R \) of the join \( N \times \{ R+1 \} = \partial M_{R+1} \). The vector bundle \((E, \nabla, h)\) yields a flat Hermitian vector bundle over \( \widetilde{M}_R \) in a canonical way, which we denote by the same letter again. We denote the heat kernel of the Hodge Laplacian on \( \widetilde{M}_R \) by \( H(\widetilde{M}_R) \).

The following result, due to Pfaff [Pfa13, Proposition 11.2] in the context of hyperbolic manifolds, does not use any specific geometry of \( M \) and holds in the general setting of the present discussion.
Proposition 10.1. For any differential operator \( P \) acting on sections of \( \Lambda^* T^* M \otimes E \) and a fixed \( R > 1 \) there exist constants \( C, c > 0 \), such that for all \( k \in \mathbb{N}_0 \), \( R' > R \), \( (t, q, q') \in \mathbb{R}^+ \times M^2_R \) and \( P \) acting on the first spacial variable \( q \in M_R \)
\[
\| \partial_t^k P(\mathcal{H} - \mathcal{H}(\tilde{M}_R'))(t, q, q') \| \leq C e^{-c |R' - R|/t}.
\]

We can now prove invariance of the renormalized Ray-Singer analytic torsion norm under compactly supported metric variations, as stated in Theorem 2.7. We employ a special case of Proposition 10.3 which we state and prove in the next subsection.

Corollary 10.2. Let \( g_\theta, \theta \in S^1 \), denote a smooth family of Riemannian metrics on \( M \), with closure of \( \text{supp} \partial_\theta g_\theta \) compact in \( M \). Then
\[
\frac{d}{d\theta} \| \cdot \|_{\mathcal{RS}(M, E, g_\theta)} = 0.
\]

Proof. Denote the Hodge Laplacian corresponding to the Riemannian metric \( g_\theta \) by \( \Delta_\theta \). The one-parameter family \( \Delta_\theta \) fits into the framework of Assumption 2.6 and hence Proposition 10.3 applies to the associated heat kernel family, which we denote by \( \mathcal{H}_\theta \). We find
\[
\frac{d}{d\theta} \text{Tr}_r \mathcal{H}_\theta = \text{Tr} \frac{d}{d\theta} \mathcal{H}_\theta = -t \text{Tr} \left( \left( \frac{d}{d\theta} \Delta_\theta \right) \mathcal{H}_\theta \right),
\]
where the second equality follows by a standard argument, cf. [Pfa13, (14.3)] and also [MuVe14, MAVe12]. Indicate the action of the Hodge Laplacian and the heat kernel on differential forms in degree \( p \) by the lower index \( p \) again. Then we find by Proposition 10.1 and the classical argument of Ray-Singer [RaSi71] on closed manifolds \( (\alpha_{p, \theta} = *_{\theta} \frac{d}{d\theta} *_{\theta} | \Omega^*(\tilde{M}_R, E)) \)
\[
\frac{d}{d\theta} \sum_{p=0}^m (-1)^p \text{Tr}_r \mathcal{H}_{p, \theta} = -t \sum_{p=0}^m (-1)^p \text{Tr} \left( \left( \frac{d}{d\theta} \Delta_\theta \right) \mathcal{H}_{p, \theta} \right)\]
\[
= -t \sum_{p=0}^m (-1)^p \lim_{R \to \infty} \text{Tr} \left( \left( \frac{d}{d\theta} \Delta_\theta \right) \mathcal{H}_{p, \theta}(\tilde{M}_R) \right)\]
\[
= -t \frac{d}{dt} \sum_{p=0}^m (-1)^p \lim_{R \to \infty} \text{Tr} \left( \alpha_{p, \theta} \mathcal{H}_{p, \theta}(\tilde{M}_R) \right)\]
\[
= -t \frac{d}{dt} \sum_{p=0}^m (-1)^p \text{Tr} \left( \alpha_{p, \theta} \mathcal{H}_{p, \theta} \right)\]
Using the notation set in Proposition 10.3 with additional upper indices $p$ indicating the degree, we write

$$2 \frac{d}{d\theta} \log T(M, E, g_\theta)$$

$$= \frac{d}{d\theta} \left| \frac{d}{ds} \int_0^t \frac{1}{\Gamma(s)} \sum_{p=0}^m (-1)^p \left( \text{Tr}_r \mathcal{H}_{p, \theta} - \sum_{j=0}^\ell \sum_{i=0}^{\ell_j} b^p_{ij}(\theta) t^{s+\alpha_j} \log^i(t) - b^p_0(\theta) \right) dt \right|$$

$$+ \frac{d}{d\theta} \frac{d}{ds} \int_0^1 \frac{1}{\Gamma(s)} \sum_{p=0}^m (-1)^p \left( \sum_{j=0}^\ell \sum_{i=0}^{\ell_j} b^p_{ij}(\theta) t^{s+\alpha_j} \log^i(t) \right) dt$$

$$+ \frac{d}{d\theta} \frac{d}{ds} \sum_{p=0}^m (-1)^p \left( \sum_{j=0}^\ell \sum_{i=0}^{\ell_j} c^p_{ij}(\theta) t^{s+\alpha_j} \log^i(t) - c^p_0(\theta) \right) dt$$

Using differentiability of the asymptotic expansions in Proposition 10.3 with respect to the parameter $\theta$, we may pass differentiation in $\theta$ past the first and third integrals above and find as in [Les12, Proposition 2.4]

$$\frac{d}{d\theta} \log T(M, E, g_\theta) = \frac{d}{ds} \int_0^\infty \frac{t^{s-1}}{\Gamma(s)} \sum_{p=0}^m (-1)^p \left( \frac{d}{d\theta} \sum_{j=0}^\ell \sum_{i=0}^{\ell_j} c^p_{ij}(\theta) t^{s+\alpha_j} \log^i(t) \right) dt$$

$$= \frac{d}{ds} \int_0^\infty \frac{t^{s-1}}{\Gamma(s)} \sum_{p=0}^m (-1)^{p+1} \text{Tr} (\alpha_{p, \theta} \mathcal{H}_{p, \theta}) dt$$

$$= \frac{d}{ds} \int_0^\infty \frac{t^{s-1}}{\Gamma(s)} \sum_{p=0}^m (-1)^p \text{Tr} (\alpha_{p, \theta} \mathcal{H}_{p, \theta}) dt$$

$$= \frac{1}{2} \lim_{t \to \infty} \sum_{p=0}^m (-1)^p \text{Tr} (\alpha_{p, \theta} \mathcal{H}_{p, \theta}) - \frac{1}{2} \lim_{t \to 0^+} \sum_{p=0}^m (-1)^p \text{Tr} (\alpha_{p, \theta} \mathcal{H}_{p, \theta})$$

$$= \frac{1}{2} \sum_{p=0}^m (-1)^p \text{Tr} (\alpha_{p, \theta} | \text{ker } \Delta),$$

where in the last equality we used (2.8) in Assumption 2.6 and the fact that by the Duhamel principle the short time asymptotics of $\text{Tr} (\alpha_{p, \theta} \mathcal{H}_{p, \theta})$ does not admit a constant term, since $\text{supp } \alpha_{p, \theta}$ is compact in the interior of $M$ and $m = \dim M$ is odd.

10.2. **Proof of a gluing formula following Lesch and Pfaff.** The following result follows the outline of [Pfa13, Proposition 14.1], cf. also the parametrix construction in [Don79], where however the Gaussian estimate is replaced by Assumption 2.6.
**Proposition 10.3.** Consider the one-parameter family $\mathcal{H}_0$ of heat kernels, introduced in Assumption 2.6. Then the difference $(\mathcal{H}_0 - \mathcal{H})$ is trace class, the renormalized trace of $\mathcal{H}_0$ is differentiable in $\theta \in S^1$ and admits asymptotic expansions

$$
\operatorname{Tr}_{\mathcal{H}_0}(t) \sim_{t \to 0^+} \sum_{j=0}^{\ell} \sum_{i=0}^{j} b_{ij}(\theta) t^{\delta j} \log^i(t) + b_0(\theta) + O(t^\delta),
$$

$$
\operatorname{Tr}_{\mathcal{H}_0}(t) \sim_{t \to \infty} \sum_{j=0}^{d} \sum_{i=0}^{k_j} c_{ij}(\theta) t^{-\beta_j} \log^i(t) + c_0(\theta) + O(t^{-\delta}),
$$

which are differentiable in $\theta$. Moreover we have\(^{20}\)

$$
\frac{d}{d\theta} \operatorname{Tr}_{\mathcal{H}_0} = \operatorname{Tr}_{\mathcal{H}_0}. \frac{d}{d\theta}.
$$

*Proof.* Consider cutoff functions $\phi_1, \psi_1 \in C_0^\infty(M)$, where $\phi_1, \psi_1 \equiv 1$ over $M_{R+1}$, supp $\phi_1, \text{supp } \psi_1 \subset M_{R+2}$, $\phi_1 \equiv 1$ over $\text{supp } \psi_1$ and supp $d\phi_1 \cap \text{supp } \psi_1 = \emptyset$. Put $\psi_2 := 1 - \psi_1$ and fix some cutoff function $\phi_2 \in C^\infty(M)$ with $\phi_2 \equiv 0$ on an open neighborhood of $M_R$, such that $\phi_2 \equiv 1$ over $\text{supp } \psi_2$ and supp $d\phi_2 \cap \text{supp } \psi_2 = \emptyset$. We define for $(t, q, q') \in \mathbb{R}^+ \times M^2$

$$
P(t, q, q'; \theta) := \phi_1(q) \mathcal{H}_0(t, q, q') \psi_1(q') + \phi_2(q) \mathcal{H}(t, q, q') \psi_2(q') :=: P_1 + P_2
$$

We assume without loss of generality that the compact subset $\mathcal{K} \subset M$ in the notation of Assumption 2.6 is contained in the open interior of $M_R$. Then $\Delta_0 \circ \phi_2 = \Delta \circ \phi_2$ since $\Delta_0 \equiv \Delta$ over $\mathcal{U}_R$. Moreover, $V_0$ commutes with $\phi_1$ by assumption and hence, writing $\delta$ for some first order derivatives and $D$ for the Gauß Bonnet operator, we compute

$$(\partial_t + \Delta_0) P = \left( (\delta^2 \phi_1) \mathcal{H}_0 + 2(\delta \phi_1) D \mathcal{H}_0 \right) \psi_1$$

$$+ \left( (\delta^2 \phi_2) \mathcal{H} + 2(\delta \phi_2) D \mathcal{H} \right) \psi_2 := Q_1 + Q_2 := Q.$$

We now define inductively for each $k \in \mathbb{N}$

$$Q^{k+1}(t, q, q'; \theta) := Q \ast Q^k(t, q, q'; \theta)$$

$$= \int_0^t \int_M Q(t - \tilde{t}, q, \tilde{q}; \theta) Q^k(\tilde{t}, \tilde{q}, q'; \theta) d\tilde{t} d\text{vol}_g(\tilde{q}),$$

where in each step the spacial integration is over a compact region $M_{R+2}$, since supp $\delta \phi_{1,2} \subset M_{R+2}$. By Assumption 2.6 we find for the pointwise traces and any $S \in \mathbb{N}$

$$\operatorname{tr} Q^{k+1}(t, q, \cdot; \theta) \leq t^{k+S} \frac{\text{vol}_g(M_{R+2})^k}{k!} f \|Q\|_k^{k} \left| \left| \left( t, 0, t_0 \right) \times M_{R+2} \right. \right|,$$

where $f \in L^2(M, E, g, h)$ and the estimate holds uniformly in $(t, q, \theta) \in (0, t_0] \times M \times S^1$. Consequently the Volterra series

$$\hat{Q}(t, q, q'; \theta) := \sum_{k=0}^{\infty} (-1)^k Q^k(t, q, q'; \theta),$$

\(^{20}\)The trace on the right hand side of the equality is defined without the regularization.
converges and admits an $L^2_t(M,E,g,h)$-integrable majorant in $q' \in M$, times a factor $t^S$ for any $S \in \mathbb{N}$, uniformly in $t \in (0,t_0), \theta \in S^1$ and $q \in M$. Note that in fact $\text{supp} \hat{Q}(t,\cdot,q';\theta) \subset M_{R+2}$. Similar argument applies to the Volterra series with $Q$ replaced by $\partial_\theta Q$, and hence $\hat{Q}$ is differentiable in $\theta$ and $\partial_\theta \hat{Q}$ admits a square-integrable majorant, uniformly in the parameters $(t,\theta,q) \in [0,t_0] \times S^1 \times M$.

The heat kernel $\mathcal{H}_\theta$ is then recovered by (cf. [Pfa13, pp. 48-50])

$$\mathcal{H}_\theta = P + P \ast \hat{Q} = (P_1 + P_2) + (P_1 \ast \hat{Q} + P_2 \ast \hat{Q}),$$

where by existence of a square integrable majorant for $\hat{Q}(t,q',\cdot;\theta)$, uniformly $(t,\theta,q) \in (0,t_0] \times S^1 \times M$, as well as Assumption 2.6 (ii) applied\(^{21}\) to $P_1, P_2$, the integral kernels $(P_j \ast \hat{Q})$ and $(P_j \ast \partial_\theta \hat{Q}), j = 1,2$, are trace class, and their traces vanish to infinite order as $t \to 0^+$. In particular we can interchange differentiation and integration and find

$$\frac{d}{d\theta} \text{Tr}(P_2 \ast \hat{Q}) = \text{Tr}\left( P_2 \ast \frac{d}{d\theta} \hat{Q} \right) = O(t^\infty), \quad t \to 0^+.$$

$$\frac{d}{d\theta} \text{Tr}(P_1 \ast \hat{Q}) = \text{Tr}\left( P_1 \ast \frac{d}{d\theta} \hat{Q} \right) = O(t^\infty), \quad t \to 0^+.$$

In particular $(\mathcal{H}_\theta - \mathcal{H})$ is indeed trace class. Hence the renormalized trace exists by Assumption 2.5 and its differentiability in the parameter follows from smoothness of the one-parameter family of kernels $\mathcal{H}_\theta$.

The statement on existence and differentiability of the asymptotic expansion of the renormalized trace as $t \to 0^+$ now follows from Assumption 2.5 and the fact that $P_1$ by Assumption 2.6 fits into the interior elliptic parametric calculus, cf. Shubin [Shu01] and hence its trace admits a classical short time asymptotic expansion, differentiable in $\theta$ and $t$. The corresponding statement on the large times asymptotics follows from Assumptions 2.5 and 2.6.

Since $P_2$ does not depend on $\theta$ we find

$$\frac{d}{d\theta} \text{Tr}_\theta \mathcal{H}_\theta = \frac{d}{d\theta} \text{Tr} \left( P_1 + P_1 \ast \hat{Q} + P_2 \ast \hat{Q} \right)$$

$$= \text{Tr} \frac{d}{d\theta} \left( P_1 + P_1 \ast \hat{Q} + P_2 \ast \hat{Q} \right) = \text{Tr} \frac{d}{d\theta} \partial_\theta \mathcal{H}_\theta.$$

Assume now that the Riemannian metric $g$ is product in an open neighborhood of $\mathbb{N} \times \{1\}$, which we may do without loss of generality by Corollary 10.2. Consider the cut manifold $M^{\text{cut}} := K \sqcup \mathcal{U}$ with $\partial M^{\text{cut}} = \mathbb{N}^2$, obtained from $(M,g)$ by cutting along the $\mathbb{N} \times \{1\}$ separating hypersurface. The Riemannian metric $g$ induces a Riemannian metric on $M^{\text{cut}}$, which we denote by the same letter again. Similarly, the flat

\(^{21}\)Recall, $\text{supp} \hat{Q}(t,\cdot,q';\theta) \subset M_{R+2}$ is bounded.
Hermitian vector bundle \((E, \nabla, h)\) over \(M\) gives rise to the corresponding flat Hermitian vector bundle over \(M^{\text{cut}}\), denoted by the same letter again. By assumption, the metric \(g\) on \(M^{\text{cut}}\) is product near the boundary.

The main ingredient in the proof of the gluing formula is a family of boundary conditions on \(M^{\text{cut}}\), introduced by Vishik [Vis95]. Denote by \(\iota\) the obvious embedding of \(N \times \{1\}\) into \(K\) and \(U\). We define in each degree \(p\) for any \(\theta \in (0, \pi/2)\)

\[
D_\theta^p := \{(\omega_1, \omega_2) \in \Omega^p(K, E) \oplus \Omega^p(U, E) \mid \cos \theta \iota^* \omega_1 = \sin \theta \iota^* \omega_2\}.
\]

The corresponding exterior derivative \(D\) with domain \(D(D_\theta) := D_\theta\) is then gauge transformed to a family of operators with constant domain. More precisely, consider a cutoff function \(\phi \in C_0^\infty(M)\) with \(\text{supp } \phi \subset N \times (1 - 2\varepsilon, 1 + 2\varepsilon)\) and \(\phi \equiv 1\) over \(N \times (1 - \varepsilon, 1 + \varepsilon)\) for \(\varepsilon > 0\) sufficiently small such that \(g\) is product over \(N \times (1 - 2\varepsilon, 1 + 2\varepsilon) \subset M\). We introduce a reflection map across \(N \times \{1\}\)

\[
S : N \times (1 - 2\varepsilon, 1 + 2\varepsilon) \rightarrow N \times (1 - 2\varepsilon, 1 + 2\varepsilon), \quad (q, x) \mapsto (q, 2 - x)
\]

Consider an open neighborhood

\[
W := N \times (1 - 2\varepsilon, 1] \sqcup N \times [1, 1 + 2\varepsilon) \subset M^{\text{cut}},
\]

of the boundary of the cut manifold. The cutoff function \(\phi\) and the action \(S\) lift to \(W\) and hence we may define in each degree \(p\)

\[
T : \Omega^p(W, E) \rightarrow \Omega^p(W, E), \quad T(\omega_1, \omega_2) := (S^* \omega_2, S^* \omega_1),
\]

\[
\Phi_\theta : \Omega^p(W, E) \rightarrow \Omega^p(W, E), \quad \Phi_\theta := \cos(\theta \phi) \text{Id} + \sin(\theta \phi) T.
\]

Since \(\Phi_\theta \omega = \omega\) for \(\omega \in \Omega^p(W, E)\) with \(\text{supp } \omega \subset W\setminus \text{supp } \phi\), \(\Phi_\theta\) extends in an obvious way to \(\Omega^p(M^{\text{cut}}, E)\) and defines a unitary transformation on the corresponding \(L^2\) completion. As explained in Lesch [LES12, Lemma 5.1 and (5.8)], the gauge transformed family of exterior derivatives

\[
\tilde{D}_\theta := \Phi_\theta \circ D_{\theta + \pi/4} \circ \Phi_\theta^* = D_{\pi/4} + \theta e^{\phi T},
\]

is defined on a fixed domain \(D_{\pi/4}\). We obtain a complex \((D_\theta^p, \tilde{D}_\theta)\) and denote by \(\Delta_{p, \theta}\) the corresponding family of Hodge Laplace operators acting on \(D_{\pi/4}^p\) in each degree \(p\). As before, we denote in each degree \(p\) the corresponding family of heat kernels by \(\mathcal{K}_{p, \theta}\). As explained in [LES12, page 26], \(\Delta_{p, \theta} = \Delta_p + V_\theta\), where for each \(s \in \mathbb{R}\)

\[
V_\theta : H^{s}_{\text{loc}}(M, \Lambda^p T^* M \otimes E) \rightarrow H^{s-1}_{\text{comp}}(M, \Lambda^p T^* M \otimes E),
\]

is a smooth family of symmetric operators that map sections that are locally of Sobolev class \(s\) into the space of compactly supported sections of Sobolev class \((s - 1)\). The operator family \((V_\theta)\) fits into the framework of Assumption 2.6 with \(\mathcal{K} = \text{supp } d\phi\). In particular, the corresponding analytic torsion \(T_\theta(M, E)\) in terms of \(\Delta_\theta\) is well-defined by Assumption 2.6.

The next result is obtained as a consequence of Propositions 10.1 and 10.3 by an ad verbatim repetition of Pfaff’s argument in [PFA13, Proposition 14.3], where variations of renormalized traces on \(M\) are approximated by the corresponding
variations on a sequence of compact manifolds \( \tilde{M}_R \) (cf. notation in Proposition 10.2) with \( R \to \infty \), and [Les12, Theorem 5.3] is applied for each finite \( R > 1 \).

**Theorem 10.4.** For \( \theta \in (0, \pi/2) \) we have
\[
\frac{d}{d\theta} \sum_{p=0}^{m} (-1)^p \text{Tr}_{\text{reg}}(\mathcal{H}_{p,\theta}(t)) = -t \frac{4}{\sin 2\theta} \frac{d}{dt} \sum_{p=0}^{m} (-1)^p \text{Tr}(\beta_0 \mathcal{H}_{p,\theta}(t)),
\]
\[
\sum_{p=0}^{m} (-1)^p \text{Tr}(\beta_0 \mathcal{H}_{p,\theta}(t)) = \chi(K, E) - \sin^2 \theta \chi(N, E) + O(t^\infty), \ t \to 0+,
\]
where \( \beta_0 : D_0^* \to \Omega^*(K, E) \), \( \beta(\omega_1, \omega_2) = \omega_2 \) is the obvious restriction.

The final step in the derivation of a gluing formula is the analysis of certain long exact sequences in cohomology. We write
\[
\Omega^*_c(\mathcal{U}, E) := \{ \omega \in \Omega^*(\mathcal{U}, E) \mid \iota^* \omega = 0 \},
\]
\[
\Omega^*_c(K, E) := \{ \omega \in \Omega^*(K, E) \mid \iota^* \omega = 0 \}.
\]
We consider the following short exact sequences of complexes
\[
0 \to \Omega^*_c(\mathcal{U}, E) \xrightarrow{\alpha_0} D_0^* \xrightarrow{\beta_0} \Omega^*(K, E) \to 0,
\]
\[
0 \to \Omega^*_c(\mathcal{U}, E) \oplus \Omega^*_c(K, E) \xrightarrow{\gamma_0} D_0^* \xrightarrow{r_0} \Omega^*(N, E) \to 0,
\]
where \( \alpha_0 \omega = (\omega, 0) \) is an extension by zero, \( \beta_0(\omega_1, \omega_2) = \omega_2 \) is the restriction to \( K \), \( \gamma_0(\omega_1, \omega_2) = (\omega_1, \omega_2) \) is the inclusion and \( r_0(\omega_1, \omega_2) = \sin \theta \iota^* \omega_1 + \cos \theta \iota^* \omega_2 \). We denote the corresponding long exact cohomology sequences by \( \mathcal{H}_c^0(\mathcal{U}) \) and \( \mathcal{H}_c^0(\mathcal{U}, K, N) \), respectively. The torsions of these long exact sequences \( \mathcal{H}_c^0 \) are defined combinatorially, see for instance [Les12, §2.2], in terms of the induced \( L^2 \)-Hilbert space structure and are denoted by \( \tau(\mathcal{H}_c^0) \). The following theorem is due to Lesch [Les12, Theorem 4.1] with minor adaptations to the present setup.

**Theorem 10.5.** For \( \theta \in (0, \pi/2) \)
\[
\frac{d}{d\theta} \log T_0(M, E) = \frac{d}{d\theta} \log \tau(\mathcal{H}_c^0(\mathcal{U}, K, N)),
\]
Moreover, \( \log T_0(M, E) - \log \tau(\mathcal{H}_c^0) \) is differentiable at \( \theta \in (0, \pi/2) \), where \( \mathcal{H}_c^0 \) stands for either \( \mathcal{H}_c^0(\mathcal{U}) \) or \( \mathcal{H}_c^0(\mathcal{U}, K, N) \).

**Proof.** Following the proof of [Les12, Theorem 4.1] in [Les12, §5.2] it remains in view of Theorem 10.4 to rule out jumps in the dimensions of the cohomology groups in \( \mathcal{H}_c^0 \) at \( \theta = 0 \), cf. [Les12, §5.2.3]. Here we employ Assumption 2.8.

If the spectrum \( \text{spec} \Delta_p \setminus \{0\} \) of the Hodge Laplacian \( \Delta_p \) admits a spectral gap around zero in all degrees \( p \), then same holds for the family \( \Delta_{p,\theta} \), since \( \Delta_{p,\theta} \) is a relatively compact perturbation of \( \Delta_p \) and essential spectrum is stable under relatively compact perturbations. In this case, the argument of [Les12, §5.2.3] carries over ad verbatim.
In case of no spectral gap around zero, Assumption 2.8 requires $H^r(N, E) = \{0\}$. Then exactness of $\mathcal{H}^0(\mathcal{U}, K, N)$ implies

$$H^r(\Delta_\delta^*, \tilde{D}_\delta) \cong H^r(\Omega_\delta^* (\mathcal{U}, E)) \oplus H^r(\Omega_\delta^* (K, E))$$

$$= H^r(\mathcal{U}, E, N) \oplus H^r(K, E, N),$$

which forces the dimension of $H^r(\Delta_\delta^*, \tilde{D}_\delta)$ to be independent of $\theta \in [0, \pi/2)$. $\Box$

As a consequence of Theorem 10.5, the gluing formula [Les12, Theorem 6.1] follows ad verbatim and we state it here as follows.

**Theorem 10.6.** Writing $\mathcal{H} := \mathcal{H}^{\delta/4}$ we find

$$\log T(M, E) = \log T(K, E) + \log T(\mathcal{U}, E, N) + \log \tau(\mathcal{H}(\mathcal{U}, K)) - \chi(N, E) \log \sqrt{2},$$

$$\log T(M, E) = \log T(K, N, E) + \log T(\mathcal{U}, E, N) + \log \tau(\mathcal{H}(\mathcal{U}, K, N)).$$

It will become convenient below to rewrite the gluing formula of Theorem 10.6 in terms of renormalized Ray-Singer analytic torsion norms. Consider the long exact sequences in cohomology

$$\mathcal{H}(\mathcal{U}, K) : \ldots H^p(\mathcal{U}, E, N) \xrightarrow{\alpha^*} H^p(M, E) \xrightarrow{\beta^*} H^p(K, E) \xrightarrow{\delta^*} H^{p+1}(\mathcal{U}, E, N) \ldots,$$

$$\mathcal{H}(\mathcal{U}, K, N) : \ldots H^p(\mathcal{U}, E, N) \oplus H^p(K, E, N) \xrightarrow{\gamma^*} H^p(M, E) \xrightarrow{\gamma^*} H^p(N, E)$$

$$\xrightarrow{\delta^*} H^{p+1}(\mathcal{U}, E, N) \oplus H^{p+1}(K, E, N) \ldots,$$

where $\delta^*$ denotes the respective connecting homomorphisms. These sequences induce isomorphisms on determinant lines in a canonical way, cf. [Nico3]

$$\Phi : \det H^r(\mathcal{U}, E, N) \otimes \det H^r(K, E) \to \det H^r(M, E),$$

$$\Phi' : \det H^r(\mathcal{U}, E, N) \otimes \det H^r(K, E, N) \otimes \det H^r(N, E) \to \det H^r(M, E).$$

A careful combinatorial analysis, carried out e.g. in [Ver08, Theorem 7.12], implies

$$\log \| \cdot \|_{\det H^r(\mathcal{U}, E, N) \otimes} \| \cdot \|_{\det H^r(K, E)} = \log \| \Phi(\cdot \otimes \cdot) \|_{\det H^r(M, E)} + \log \tau(\mathcal{H}(\mathcal{U}, K)),$$

$$\log \| \cdot \|_{\det H^r(\mathcal{U}, E, N) \otimes} \| \cdot \|_{\det H^r(K, E, N)} \otimes \| \cdot \|_{\det H^r(N, E)} = \log \| \Phi'(\cdot \otimes \cdot \otimes) \|_{\det H^r(M, E)}$$

$$+ \log \tau(\mathcal{H}(\mathcal{U}, K, N)),$$

where the Hilbert structures on the corresponding cohomologies are induced by the $L^2$-structure defined by the metrics $g$ and $h$. In combination with Theorem 10.6 we arrive at the following result:\textsuperscript{22}

**Corollary 10.7.**

$$\| \Phi(\cdot \otimes \cdot) \|_{RS}^{\mathcal{U}, E, N} = 2^{-\frac{\chi(N, E)}{2}} \| \cdot \|_{RS(\mathcal{U}, E, N) \otimes} \| \cdot \|_{RS(K, E),}$$

$$\| \Phi'(\cdot \otimes \cdot \otimes) \|_{RS}^{\mathcal{U}, E, N} = \| \cdot \|_{RS(\mathcal{U}, E, N) \otimes} \| \cdot \|_{RS(K, E, N) \otimes} \| \cdot \|_{\det H^r(N, E)}.$$

\textsuperscript{22}This is the statement of Theorem 2.9.
10.3. Brüning-Ma metric anomaly on non-compact manifolds. A particular consequence of the gluing formula in terms of analytic torsion norms, as obtained in Corollary 10.7, is the application of the Brüning-Ma metric anomaly result [BrMA06] to the non-compact setting.

Assume that $M$ has non-empty smooth compact boundary $\partial M$, which is contained in the open interior of $K$. Consider a pair of Riemannian metrics $g_1$ and $g_2$ over $M$, which satisfy the conditions of Assumptions 2.5, 2.6, 2.8 and coincide over the infinite end $\mathcal{U}$. By Corollary 10.2 we assume without loss of generality that $g_1$ and $g_2$ are both product over an open neighborhood of the separating hyper surface $N \times 1$. Then by (2.6)

$$
\log \frac{\| \cdot \|_{RS(M,E,g_1)}}{\| \cdot \|_{RS(M,E,g_2)}} = \log \frac{\| \cdot \|_{RS(U,N,g_1)}}{\| \cdot \|_{RS(U,N,g_2)}} \otimes \frac{\| \cdot \|_{RS(K,E,g_1)}}{\| \cdot \|_{RS(K,E,g_2)}} = \log \frac{\| \cdot \|_{RS(K,E,g_1)}}{\| \cdot \|_{RS(K,E,g_2)}}
$$

$$
= \frac{\text{rank}(E)}{2} \left[ \int_{\partial M} B(\nabla_{g_2}^{TM}) - \int_{\partial M} B(\nabla_{g_1}^{TM}) \right].
$$

We have thus extended the Brüning-Ma metric anomaly result to non-compact manifolds subject to Assumptions 2.5, 2.6 and 2.8.

Proposition 10.8. Assume that $\partial M \neq \emptyset$ is contained in the open interior of $K$. Consider a pair of Riemannian metrics $g_1$ and $g_2$ over $M$, which satisfy the conditions of Assumptions 2.5, 2.6 and 2.8, and moreover coincide over $\mathcal{U}$. Then

$$
\log \frac{\| \cdot \|_{RS(M,E,g_1)}}{\| \cdot \|_{RS(M,E,g_2)}} = \frac{\text{rank}(E)}{2} \left[ \int_{\partial M} B(\nabla_{g_2}^{TM}) - \int_{\partial M} B(\nabla_{g_1}^{TM}) \right].
$$

10.4. Examples of manifold classes satisfying Assumptions 2.5, 2.6, 2.8. Our discussion requires the notion of polyhomogeneous distributions on a manifold with corners, introduced by Melrose cf. [MEL92]. Let $\mathfrak{M}$ be a manifold with corners, modelled over open neighborhoods of $(\mathbb{R}^+)^k \times \mathbb{R}^l$, and embedded boundary faces $\{(H_1, \rho_1)\}_{i=1}^N$ where $\{\rho_i\}$ denote the corresponding boundary defining functions. We adopt the multi-index notation and for any multi-index $b = (b_1, \ldots, b_N) \in \mathbb{C}^N$ we write $\rho^b = \rho_1^{b_1} \cdots \rho_N^{b_N}$. Consider the space $\mathcal{V}(\mathfrak{M})$ of smooth b-vector fields on $\mathfrak{M}$ which by definition are tangent to all boundary faces.

Definition 10.9. We say that a distribution $w$ on $\mathfrak{M}$ is conormal if $w \in \rho^b L^\infty(\mathfrak{M})$ for some $b \in \mathbb{C}^N$ and its regularity is stable under b-vector fields, i.e. $V_1 \cdots V_\ell w \in \rho^b L^\infty(\mathfrak{M})$ for all $V_j \in \mathcal{V}(\mathfrak{M})$ and for every $\ell \geq 0$. A collection $E_i = \{ (\gamma, p) \} \subset C \times \mathbb{N}$ is said to be an index set if it satisfies the following hypotheses:

(i) $\text{Re}(\gamma)$ accumulates only at $+\infty$,
(ii) if $(\gamma, p) \in E_i$, then $(\gamma + j, p') \in E_i$ for all $j \in \mathbb{N}_0$ and $0 \leq p' \leq p$,
(iii) for each $\gamma$ there exists $P_\gamma \in \mathbb{N}_0$ such that $(\gamma, p) \in E_i$ for every $0 \leq p \leq P_\gamma < \infty$.

We define an index family $E = (E_1, \ldots, E_N)$ to be an $N$-tuple of index sets, a call a conormal distribution $w$ polyhomogeneous on $\mathfrak{M}$ with index family $E$, denoted $w \in A^E_{\text{phg}}(\mathfrak{M})$, if $w$ is conormal and expands near each $H_i$ as $w \sim$
\[ \sum_{(\gamma, p) \in E_i} a_{\gamma, p} \rho_i^\gamma (\log \rho_i)^p, \]
when \( \rho_i \to 0 \), where the coefficients \( a_{\gamma, p} \) are required to be conormal distributions on \( H_i \) and polyhomogeneous with index \( E_j \) at any \( H_i \cap H_j \).

We turn to (parabolic) blowups now. The notion of a blowup has been introduced by Melrose cf. [Mel93] to capture the nonuniform behavior of Schwartz kernels of certain integral operators. Consider \( \mathbb{R}^+ \times \mathbb{R}^+ \) as a fundamental example of a manifold with corner at the origin 0. The blowup \([\mathbb{R}^+ \times \mathbb{R}^+, 0]\) is defined as a disjoint union of \( \mathbb{R}^+ \times \mathbb{R}^+ \setminus 0 \) with the interior spherical normal bundle of 0 in \( \mathbb{R}^+ \times \mathbb{R}^+ \). The blowup \([\mathbb{R}^+ \times \mathbb{R}^+, 0]\) is by definition equipped with the unique minimal differential structure, with respect to which smooth functions on \( \mathbb{R}^+ \times \mathbb{R}^+ \) and polar coordinates around 0 are smooth. The blowup is illustrated in Figure 3.

![Figure 3. \( \mathbb{R}^+ \times \mathbb{R}^+ \) and its blowup \([\mathbb{R}^+ \times \mathbb{R}^+, 0]\)](image)

The front boundary face ff illustrates the interior spherical normal bundle of \( 0 \in \mathbb{R}^+ \times \mathbb{R}^+ \). In applications it is often convenient to work with locally defined projective coordinates instead of globally defined polar coordinates. Projective coordinates near the front face ff, near its lower corner and away from the left boundary face ff, are given by

\[ \rho_{lf} = \frac{x}{\bar{x}}, \quad \rho_{ff} = \bar{x}, \]
where \( \rho_{lf} \) is a boundary defining function of rf, and \( \rho_{ff} \) a boundary defining function of ff. Projective coordinates near the front face ff, near its upper corner and away from the right boundary face rf, are given by

\[ \rho_{rf} = \frac{\bar{x}}{x}, \quad \rho_{ff} = x, \]
where \( \rho_{rf} \) is a defining function of lf, and \( \rho_{ff} \) a defining function of ff.

Similar construction makes sense in case of \( X \) and \( Y \) being manifolds with boundary and \([X \times Y, \partial X \times \partial Y]\) is defined as the blowup of \( X \times Y \) at the highest codimension corner \( \partial X \times \partial Y \). Locally, around each point \( q \in \partial X \times \partial Y \) the blowup reduces to the model situation \([\mathbb{R}^+ \times \mathbb{R}^+, 0]\), where \( q \) corresponds to the origin and both copies of \( \mathbb{R}^+ \) correspond to the boundary defining functions of \( X \) and \( Y \). We refer for details to [Mel93] and [Maz91].

10.4.1. First example: Witt manifolds with cusps. Assume that \( g \mid U = x^{-2}(dx^2 + g^N) \) and the vector bundle \((E, \nabla, h)\) is induced by a unitary representation of the fundamental group of \( M \). We begin with a proof of Theorem 2.1.
Proof of Theorem 2.1. In his dissertation Vaillant derives the microlocal description for the heat kernel of the square of the Dirac operator, and hence in particular for the heat kernel $\mathcal{H}$ of the full Hodge Laplacian on all degrees. From that microlocal description one concludes that at the diagonal the pointwise trace of $\mathcal{H}$ is a polyhomogeneous distribution on the parabolic blowup $[M \times \mathbb{R}^+, \partial M \times \{0\}], \partial M = N$. Here "parabolic" refers to the fact that on the second component $\mathbb{R}^+$, the parameter space for time $t$, $\sqrt{t}$ is viewed as a smooth coordinate. Now, a general observation, e.g. Sher [SH13, Theorem 13], implies that the regularized integral of $tr \mathcal{H}$ exists and admits an asymptotic expansion as $t \to 0$.

However, such an asymptotic expansion is needed for the heat kernel $\mathcal{H}_p$ in each degree $p$ individually, rather than for the whole heat kernel $\mathcal{H}$. A priori we cannot conclude an asymptotic expansion for the regularized integral of $tr \mathcal{H}_p$, from the expansion of the regularized integral of $tr \mathcal{H}$, since there might be cancellations. However, the heat kernel $\mathcal{H} = \bigoplus_p \mathcal{H}_p$ is a direct sum of individual heat kernels, where each heat kernel $\mathcal{H}_p$ takes values in endomorphisms of $\Lambda^p T^*M$. These vector bundles are orthogonal to each other for different degrees $p$ and hence the asymptotics of the individual heat kernel $\mathcal{H}_p$ in each degree $p$ cannot cancel in its contribution to the microlocal description of the full $\mathcal{H} = \bigoplus_p \mathcal{H}_p$ before taking traces. We conclude that the microlocal description of Vaillant for the full heat kernel $\mathcal{H}$ holds for the individual heat kernels $\mathcal{H}_p$ as well. Thus the regularized integral of $tr \mathcal{H}_p$ exists for any fixed degree $p$ and admits an asymptotic expansion as $t \to 0$. This proves Theorem 2.1 (i) and (ii).

It remains to prove Theorem 2.1 (iii). Assume the Witt condition $\mathcal{H}^{n/2}(N, E) = 0$. By Remark 6.1, the continuous spectrum of the Hodge Laplacian $\Delta_{p, cut}$ on $\mathcal{U}$ comes from the Laplacians associated to the harmonic sub-complexes in the decomposition of the de Rham complex in §6. The continuous spectrum of these harmonic Laplacians in (6.1), resulting from the cohomology $H^p(N, E)$ of the cross section, with either Dirichlet or Neumann boundary conditions, is given by $[(n/2 - p)^2, \infty)$. This can be read off directly after the transformation in the proof of Proposition 9.2. Hence the Witt condition $\mathcal{H}^{n/2}(N, E) = \{0\}$ yields a spectral gap at zero, i.e. there exists $\epsilon > 0$ sufficiently small such that $(0, \epsilon) \cap \text{spec} \Delta_\epsilon = \emptyset$.

One may now consider the Hodge Laplacian $\Delta_{p, cut}$ on $M^{cut} := K \sqcup \mathcal{U}$ with relative boundary conditions at the boundary $N \sqcup N$. By arguments similar to Proposition 10.3, the difference of heat kernels of $\Delta_{p, cut}$ and the Hodge Laplacian $\Delta_p$ on $M$ is trace class and hence by [Mue98, Lemma 2.2] $\Delta_p$ admits a spectral gap at zero as well and moreover there exist constants $c, C > 0$ such that for $t > 0$ large enough

$$| \text{Tr} (e^{-t\Delta_p} - e^{-t\Delta_{p, cut}}) + \dim \ker \Delta_{p, cut} - \dim \ker \Delta_p | < C e^{-ct}. $$

The direct sum component of $\Delta_{p, cut}$ corresponding to the non-harmonic sub-complexes, cf. §6, has discrete spectrum, trivial kernel and the corresponding heat kernel is trace class with exponentially decaying heat trace.

The direct sum component of $\Delta_{p, cut}$ that corresponds to harmonic sub-complexes has been studied in Proposition 9.2 and 9.3, and is given by $\Delta_{H, p} \oplus \Delta_{H, p, Neu}$. While
Propositions 9.2 and 9.3 compute their regularized resolvent traces explicitly, their regularized heat traces may be computed in a similar manner. The heat kernel of \( \Delta_{H,p} \) is given under the transformation \( L^2([R, \infty), dx) \to L^2([R, \infty), \chi^{-1} dx), f \mapsto \chi^{1/2} f \) and a change of variables \( x = e^r \) by

\[
\mathcal{H}(t, r, r') = \frac{e^{-t(\frac{n}{2} - p)^2}}{\sqrt{4\pi t}} \left( e^{-\frac{(r-r')^2}{4t}} - e^{-\frac{(r-r')^2 + 2\log R)^2}{4t}} \right).
\]

We compute

\[
\int_{\log R}^{\log R'} \mathcal{H}(t, r, r') dr = \frac{e^{-t(\frac{n}{2} - p)^2}}{\sqrt{4\pi t}} \int_0^{\log R'/R} \left( 1 - e^{-y^2/t} \right) dy = e^{-t(\frac{n}{2} - p)^2} \left( \frac{\log R'}{\sqrt{4\pi t}} - \frac{\log R}{\sqrt{4\pi t}} - \frac{1}{4} \right) + o(1), \quad R' \to \infty.
\]

The regularized trace \( Tr\mathcal{H} \) is defined as the constant term in the expansion above and hence we find for \( p \neq n/2 \)

\[
(10.3) \quad Tr\mathcal{H} = e^{-t(\frac{n}{2} - p)^2} \left( -\frac{\log R}{\sqrt{4\pi t}} - \frac{1}{4} \right) = O(e^{-ct}), \quad t \to \infty,
\]

for some constant \( c > 0 \). Similarly, one finds by explicit computations

\[
(10.4) \quad Tr(\mathcal{H}_{Neu} - \mathcal{H}) - \dim \ker \Delta_{p,\mu} = O(e^{-ct}), \quad t \to \infty.
\]

Consequently, in view of (10.2), (10.3) and (10.4), the regularized trace \( Tr e^{-t\Delta_p} \) exists and there exist constants \( c', C' > 0 \) such that for \( t > 0 \) sufficiently large

\[
| Tr e^{-t\Delta_p} - \dim \ker \Delta_p | < C' e^{-c't}.
\]

By Theorem 2.1 and the spectral gap observation, Assumptions 2.5 and 2.8 are satisfied. By a similar argument, the family \( \Delta_{p,\theta} \) in the notation of Assumption 2.6 admits a spectral gap at zero. As in [Les12], its kernel is independent of \( \theta \) and hence as before, by [Mue98, Lemma 2.2], the renormalized heat trace of \( \Delta_{p,\theta} \) admits a large time asymptotic expansion that is differentiable in the parameter. The full statement of Assumption 2.6 (i) now follows from e.g. [Les97, Proposition 3.3.1].

Assumption 2.6 (ii) is a direct consequence of the microlocal asymptotic description of the heat kernel on manifolds with cusps by Vaillant [Vai01]. More precisely, consider the heat kernel \( \mathcal{H}_{p,\theta}(t, q, q') \) of \( \Delta_{p,\theta} \). If \( q, q' \in \mathcal{U} = [1, \infty) \times \mathbb{N} \), we write \( q = (x, y), q' = (x', y') \) with \( x, x' \in [1, \infty) \). By composition formulae of Vaillant \( \mathcal{H}_{p,\theta} \) lies again in his heat calculus, which in particular asserts that the heat kernel is polyhomogeneous at the origin of \( \mathbb{R}^{+}_{\sqrt{t}} \times \mathbb{R}^{+}_{1/x'} \), vanishing to infinite order as \( t \to 0 \) and as \( x' \to \infty \), uniformly in other variables as long as \( 1/x \geq \delta > 0 \). This yields the estimates of Assumption 2.6 (ii).
10.4.2. Second example: Scattering manifolds. \( g \mid \mathcal{U} = dx^2 + \chi^2 g^N \) and the vector bundle \((E, \nabla, h)\) is induced by a unitary representation of the fundamental group of \( M \), such that \( H^*(N, E) = 0 \). Assumption 2.8 is trivially satisfied, note however that in contrast to the previous example there is no spectral gap at zero here.

The asymptotic properties of the heat kernel have been studied in this setting by Guillarmou and Sher [GuSh13]. In particular their construction asserts that at the diagonal the pointwise trace of \( \Delta_p \) is a polyhomogeneous distribution on the parabolic blowup of \([0, \infty]_t \times M\) illustrated in Figure 4.

\[ \begin{align*}
\begin{array}{ccc}
1/\chi & 1/\chi \\
\downarrow & \downarrow \\
0 & 0
\end{array}
\end{align*} \]

\( \mathbb{R} \leftarrow t \quad t \rightarrow 0 \)

Figure 4. \([0, \infty]_t \times M\) and its blowup \([0, \infty]_t \times M, \{t = \infty\} \times \partial M\).

Now, the general observation of Sher [Sh13, Theorem 13] implies that the regularized integral of \( \text{tr} \mathcal{H}_p \) exists and admits an asymptotic expansion as \( t \to 0 \) and as \( t \to \infty \). Consequently, Assumption 2.5 is satisfied.

Consider the family \( \Delta_{p, \theta} \) in the notation of Assumption 2.6, and the corresponding heat kernels \( \mathcal{H}_{p, \theta} \). Since \( \Delta_{p, \theta} \) differs from the Hodge Laplacian \( \Delta_p \) only on a compact subset, by the composition formulae of [GuSh13], the kernels \( \mathcal{H}_{p, \theta} \) lie again in the calculus and Assumption 2.6 (i) follows. Assumption 2.6 (ii) is satisfied by exactly the same argument as in the previous example, since the heat calculi of Vaillant and Guillarmou-Sher differ only in their behaviour at the corners of highest codimension.

11. Cheeger-Müller Theorem for Witt-manifolds With Cusps

Consider an odd-dimensional oriented Riemannian manifold \((M, g)\) with

\[ M = K \cup_{\partial K} \mathcal{U}_R, \]

where \( K \) is a compact manifold with smooth boundary \( \partial K = N \times \{\mathbb{R}\} \), and \( \mathcal{U} = N \times [\mathbb{R}, \infty) \). Consider a flat Hermitian vector bundle \((E, \nabla, h)\) induced by a unitary representation of the fundamental group \( \pi_1(M) \). Assume that \( M \) satisfies the Witt condition, which is a condition on the middle degree cohomology \( H^{n/2}(N, E) = 0, n = \dim N \). Let \( g^N \) be a Riemannian metric on the closed manifold \( N \). We consider Riemannian metrics \( g \) such that

\[ g \mid \mathcal{U}_R = \frac{dx^2 + g^N}{\chi^2}, \quad \chi \in [\mathbb{R}, \infty). \]

Consider also a Riemannian metric \( g' \) on \( M \) that is product in an open neighborhood of \( N \times \{\mathbb{R}\} \) and coincides with \( g \) on \( \mathcal{U}_{R+1} = N \times [\mathbb{R} + 1, \infty) \).
We continue in the notation of §2.5. We write $M^*$ for the one-point compactification of $M$ at infinity. Recall from above that the intersection cohomology of Goresky-MacPherson for the Witt space $M^*$ with values in $E$ is denoted by $IH^*(M^*, E)$, and $H^*(M, E)$ refers to the $L^2$-cohomology of the cusp manifold $(M, g)$ with values in $E$. Due to the Witt condition $H^{n/2}(N, E) = 0$, both cohomologies coincide, compare for instance the Hodge cohomology theory by Hausel, Hunsicker and Mazzeo [HHM04].

The flat Hermitian vector bundle $(E, \nabla, h)$ need not arise from a unitary representation of the fundamental group $\pi_1(M^*)$ and hence the definition of intersection $R$-torsion $\| \cdot \|_{(M^*, E)}$ as provided by Dar [Dar87], does not apply in this setting. We use the definition as provided by Albin, Rochon and Sher in [ARS14A]. Their definition of intersection $R$-torsion in fact applies to a more general class of flat vector bundles $E$ arising from unimodular representations $\rho : \pi_1(M) \cong \pi_1(K) \to \text{GL}(d, \mathbb{R})$. It uses a specific cochain complex $R^*(Y, \rho)$ associated to $\rho$ and a triangulation $X$ of the stratified space $M^*$. If $Y \subset X$ is a sub-complex triangulating $K$, then the standard cochain complex $C^*(Y, \rho) = C^*(\tilde{Y}) \otimes_{\rho} \mathbb{R}^d$ is a sub-complex of $R^*(X, \rho)$, where $\tilde{Y}$ denotes the universal cover of $Y$. This yields a short exact sequence [ARS14A, (8.15)] of complexes and the corresponding splitting formula [ARS14A, (8.16)] by a formula of Milnor. We refer to [ARS14A] for more details on the definition of $\| \cdot \|_{(M^*, E)}$ which we use henceforth.

The intersection $R$-torsion $\| \cdot \|_{(M^*, E)}$ defines a norm on the determinant line of the middle perversity intersection cohomology $\det IH^*(M^*, E) \cong \det H^*(M, E)$. The combinatorial gluing formula by Milnor [Mil66], see also Vishik [Vis95, Proposition 1.3 and 1.4], yields a combinatorial analogue of Corollary 10.7

\begin{equation}
\| \Phi(\cdot \otimes \cdot) \|_{(M^*, E)} = \| \cdot \|_{(U^*_R, E)} \otimes \| \cdot \|_{(K, E)},
\end{equation}

where $\| \cdot \|_{(U^*_R, E)}$ denotes the intersection $R$-torsion on relative cohomology $\det IH^*(U^*_R, E, N) \cong \det H^*(U_R, E, N)$, and $\| \cdot \|_{(K, E)}$ coincides with the classical Reidemeister torsion as a norm on the determinant line $\det H^*(K, E)$.

The celebrated theorem by Cheeger [Che79] and Müller [Mue78] has been extended to manifolds with boundary by Lück [Lue93] and Vishik [Vis95]. By their results we find

\begin{equation}
\log \frac{\| \cdot \|_{(K, E, g')}^{RS}}{\| \cdot \|_{(K, E)}^{RS}} = \frac{\chi(N, E)}{4} \log 2.
\end{equation}

Combining invariance of the torsion norms under compactly supported metric variations with Corollary 10.7, (11.1) and (11.2), we compute

\begin{equation}
\log \frac{\| \cdot \|_{(M^*, E, g')}^{RS}}{\| \cdot \|_{(M^*, E)}^{RS}} = \log \frac{\| \cdot \|_{(U^*_R, E)}^{RS}}{\| \cdot \|_{(M^*, E)}^{RS}} = \log \frac{\| \cdot \|_{(U^*_R, E, N, g')}^{RS}}{\| \cdot \|_{(U^*_R, E, N)}^{RS}} = \frac{\chi(N, E)}{4} \log 2.
\end{equation}
By Proposition 10.8

\[
\log \frac{\| \cdot \|_{\mathcal{U}_R, E, N, g'}}{\| \cdot \|_{\mathcal{U}_R, E, N, g}} = \frac{\text{rank}(E)}{2} \int_{\partial \mathcal{U}_R} B(\nabla g_{\mathcal{U}_R}).
\]

Consider an orthonormal basis \( h_g \) of \( \mathcal{I}H^*(\mathcal{U}_R, E, N) \), with the Hilbert structure on cohomology induced by the isomorphism \( \mathcal{I}H^*(\mathcal{U}_R, E, N) \cong H^*(\mathcal{U}_R, E, N) \), where the right hand side is equipped with the Hilbert structure induced by the Riemannian metric \( g \). Then we arrive at the following relation

\[
\log \frac{\| \cdot \|_{\mathcal{U}_R, E, N, g'}}{\| \cdot \|_{\mathcal{U}_R, E, N, g}} = \log \frac{\| \cdot \|_{\mathcal{U}_R, E, N, g'}}{\| \cdot \|_{\mathcal{U}_R, E, N, g}} - \frac{\chi(N, E)}{4} \log 2 + \frac{\text{rank}(E)}{2} \int_{\partial \mathcal{U}_R} B(\nabla g_{\mathcal{U}_R})
\]

(11.4)

where \( \tau(\mathcal{U}_R, E, N, h_g) \) is the scalar intersection R-torsion, introduced by Dar [DAR87] and defined with respect to the preferred basis \( h_g \). Let an orthonormal basis of the absolute intersection cohomology \( \mathcal{I}H^*(\mathcal{U}_R, E) \cong H^*(\mathcal{U}_R, E) \) with the Hilbert structure induced by the Riemannian metric \( g \) as above, be denoted by \( h_g \) again. Then by Poincare duality and the Witt condition (cf. also Hartmann-Spreafico [HASP14])

\[
\tau(\mathcal{U}_R^*, E, N, h_g) = \tau(\mathcal{U}_R^*, E, h_g).
\]

Consider an orthonormal basis \( h_N \) of \( H^*(N, E) \), with the Hilbert structure on cohomology induced by the Riemannian metric \( g^N \). We obtain a preferred basis on \( \mathcal{I}H^*(\mathcal{U}_R^*, E) \) by extending the basis to \( \mathcal{U}_R \), constant in \( x \)-direction. We denote such a basis by \( h_N \) again. We may compare intersection R-torsions on \( \mathcal{U}_R^* \) defined with respect to the preferred bases \( h_g \) and \( h_N \). Using a computation similar to (9.7) we compute

\[
\log \tau(\mathcal{U}_R^*, E, h_g) = \log \tau(\mathcal{U}_R^*, E, h_N)
+ \sum_{p < n/2} \frac{(-1)^p}{2} \dim H^p(N, E) \left( (2p - n) \log R - \log(n - 2p) \right)
\]

(11.5)

\[
= \log \tau(\mathcal{U}_R^*, E, h_N) + \sum_{p=0}^{n} \frac{(-1)^{p+1}}{2} \dim H^p(N, E) \left| \frac{n}{2} - p \right| \log R
+ \sum_{p=0}^{n} \frac{(-1)^{p+1}}{4} \dim H^p(N, E) \log \left( 2 \left| \frac{n}{2} - p \right| \right).
\]

Note that \( \tau(\mathcal{U}_R^*, E, h_N) \equiv \tau(\mathcal{U}_R^*, E, h_N) \) and that \( T(\mathcal{U}_R^*, E, N, g) \) has been computed explicitly in our first main Theorem 2.4. Hence plugging the formula of Theorem 2.4 as well as (11.5) into (11.4) we arrive at the following final result.
Theorem 11.1. Let $(M, g)$ be a complete Witt manifold with a cusp end and without boundary. Then
\[
\log \left( \frac{\| \cdot \|_{[M,E,g]}^{RS}}{\| \cdot \|_{[M^*,E,g]}^{RS}} \right) = -\log \tau(U^*, E, h_N) \\
+ \sum_{p \neq n/2} \frac{(-1)^{p+1}}{2} \dim H^p(N, E) \left| \frac{n}{2} - p \right| \log \left( 2 \left| \frac{n}{2} - p \right| \right).
\]

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