FROM STATE INTEGRALS TO \(q\)-SERIES

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Abstract. It is well-known to the experts that multi-dimensional state integrals of products of Faddeev’s quantum dilogarithm which arise in Quantum Topology can be written as finite sums of products of basic hypergeometric series in \(q = e^{2\pi i \tau}\) and \(\tilde{q} = e^{-2\pi i /\tau}\). We illustrate this fact by giving a detailed proof for a family of one-dimensional integrals which includes state-integral invariants of \(4_1\) and \(5_2\) knots.

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1. Introduction

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1.1. **State-integrals and their \( q \)-series.** Multi-dimensional state integrals of products of Faddeev’s quantum dilogarithm appear in abundance in Quantum Topology, and were studied among others by Hikami [Hik01], Dimofte–Gukov–Lennels–Zagier [DGLZ09], Andersen–Kashaev [AK], and Kashaev–Luo–Vartanov [KLV12]. It is well-known to the experts that such state-integrals can be written as finite sums of products of pairs of \( q \)-series and \( \tilde{q} \)-series. The reason for this is a factorized structure of Faddeev’s quantum dilogarithm, the structure of the set of its poles, and the specific form of exponential factors of the integrand of the state-integrals, while its derivation is based on an application of the residue theorem. Instead of formulating a general theorem for multi-dimensional integrals which obscures the principle, we will give a detailed proof for the case of a family of 1-dimensional integrals and illustrate it with some concrete examples taken from [AK, KLV12].

To state our results, recall that Faddeev’s quantum dilogarithm function \( \Phi_b(x) \) is given by [Fad95]

\[
\Phi_b(x) = \frac{(e^{2\pi i(b+c_b)}; q)_\infty}{(e^{2\pi b^{-1}(x-c_b)}; \tilde{q})_\infty},
\]

where

\[
q = e^{2\pi ib^2}, \quad \tilde{q} = e^{-2\pi i b^{-2}}, \quad c_b = \frac{i}{2}(b + b^{-1}), \quad \Re(b^2) > 0.
\]

Remarkably, this function admits an extension to all values of \( b \) with \( b^2 \notin \mathbb{R}_{\leq 0} \). \( \Phi_b(x) \) is a meromorphic function of \( x \) with

poles: \( c_b + iN + iN b^{-1} \),

zeros: \( -c_b - iN b - iN b^{-1} \).

The functional equation

\[
\Phi_b(x)\Phi_b(-x) = e^{\pi ix^2} \Phi_b(0)^2, \quad \Phi_b(0) = q^{\frac{1}{24}} \tilde{q}^{-\frac{1}{24}}
\]

allows us to move \( \Phi_b(x) \) from the denominator to the numerator of the integrand of a state-integral.

For natural numbers \( A, B \) with \( B > A > 0 \), we consider the absolutely convergent integral

\[
\mathcal{I}_{A,B}(b) = \int_{\mathbb{R}+i\epsilon} \Phi_b(x)^B e^{-A\pi ix^2} dx
\]

with small positive \( \epsilon \). The condition \( B > A > 0 \) ensures not only the convergence of \( \mathcal{I}_{A,B}(b) \) for \( \Re(b^2) > 0 \), but also the convergence of the \( q \)-series and the \( \tilde{q} \)-series (for \( |q|, |\tilde{q}| < 1 \)) that appear in Theorem 1.1 below.

To express the above state-integral in terms of series, consider the generating series

\[
F_{A,B}(q, x) = \sum_{m=0}^{\infty} \frac{(-1)^A q^{A(m+1)} x^m}{(q)_m^B}, \quad \tilde{F}_{A,B}(q, x) = F_{B-A,B}(q, x).
\]

Consider the operators \( \delta \) and \( \delta_k \) (for \( k \) a positive natural number) which act on the space of functions of \( x \) as follows

\[
(\delta F)(x) = x \partial_x F(x), \quad (\delta_k F)(x) = \sum_{s=1}^{\infty} \frac{s^{k-1} q^s}{1 - q^s} F(q^s x).
\]
Likewise, there are operators $\tilde{\delta}$ and $\tilde{\delta}_k$ which act on the space of functions of $\tilde{x}$ and with $q$ replaced by $\tilde{q}$. It is easy to see that any two of the operators $\delta$, $\delta_k$, $\tilde{\delta}$, $\tilde{\delta}_k$ commute and they freely generate over $\mathbb{Q}$ a commutative ring $\mathcal{D} \otimes \tilde{\mathcal{D}}$, where

$$\mathcal{D} = \mathbb{Q}[\delta, \delta_1, \delta_2, \ldots], \quad \tilde{\mathcal{D}} = \mathbb{Q}[\tilde{\delta}, \tilde{\delta}_1, \tilde{\delta}_2, \ldots].$$

Let

$$\mathcal{D}_b = \mathcal{D}[(2\pi i)^{-1}, b^{\pm 1}, e_2, e_4, e_6, \ldots], \quad \tilde{\mathcal{D}}_b = \tilde{\mathcal{D}}[(2\pi i)^{-1}, b^{\pm 1}, e_2, e_4, e_6, \ldots],$$

where $e_l = e_l(\tilde{q}) = \tilde{\delta}_l(1) \in \mathbb{Z}[[\tilde{q}]]$. Consider the following operator valued polynomial:

$$P_{A,B} = \text{Res}_{w=0} \left( e^{\frac{1}{4\pi i}w^2 + Aw(b(\delta + \frac{1}{2}) + b^{-1}(\delta + \frac{1}{2}))} \right)^A \left( \frac{\phi(bw, \delta_*)}{b(1 - e^{b^{-1}w})} \right)^B \in \mathcal{D}_b \otimes \tilde{\mathcal{D}}_b,$$

where

$$\phi(w, \delta_*) = \exp \left( -\sum_{l=1}^{\infty} \frac{\delta_l}{l!} w^l \right)$$

and

$$\tilde{\phi}(w, \tilde{\delta}_*) = \exp(-\tilde{\delta}w) \exp \left( 2 \sum_{l=\text{even} \geq 0} e_l(\tilde{q}) \frac{w^l}{l!} \right) \exp \left( -\sum_{l=1}^{\infty} \frac{\tilde{\delta}_l}{l!} (-w)^l \right).$$

For a series $F(x, \tilde{x})$, we define:

$$\langle F(x, \tilde{x}) \rangle = F(1, 1).$$

**Theorem 1.1.** We have:

$$\mathcal{I}_{A,B}(b) = \left( \frac{\tilde{q}}{q} \right)^{\frac{B-3A}{24}} e^{\frac{\pi i}{4}\frac{B+2(A+1)}{2}} \left\langle P_{A,B} \left( F_{A,B}(q, x) \tilde{F}_{A,B}(\tilde{q}, \tilde{x}) \right) \right\rangle.$$

**Corollary 1.2.** Writing $P_{A,B} = \sum_k p_k P_k$ (a finite sum), for $p_k \in \mathcal{D}_b$ and $P_k \in \tilde{\mathcal{D}}_b$, it follows that

$$\mathcal{I}_{A,B}(b) = \left( \frac{\tilde{q}}{q} \right)^{\frac{B-3A}{24}} e^{\frac{\pi i}{4}\frac{B+2(A+1)}{2}} \sum_k g_k(q) G_k(\tilde{q})$$

where

$$g_k(q) = \langle p_k F_{A,B} \rangle, \quad G_k(\tilde{q}) = \langle P_k \tilde{F}_{A,B} \rangle.$$

**Remark 1.3.** The left hand side of Equation (8) has analytic continuation to the cut plane $\mathbb{C} \setminus \{b^2 \mid b^2 < 0\}$ whereas each of the series $g_k$ and $G_k$ is only well-defined in the upper-half plane $\{b^2 \mid 3(b^2) > 0\}$.

**Remark 1.4.** $P_{A,B}$, as a polynomial in the variables $e_2, e_4, \ldots$ has degree $B - 1$, where the degree of $e_l$ is $l$. $P_{A,B}$ as a Laurent polynomial in $b$ has $b$-monomials of degrees in $\{-B+1, -B+3, \ldots, B-3, B-1\}$.
1.2. \textit{q-difference equations.} Next we describe a linear \(q\)-difference equation of \(F_{A,B}(q,x)\). Consider the operators \(\hat{x}\) and \(\hat{E}\) which act on \(f(x) \in \mathbb{Q}(q)[[x]]\) by:

\[
(\hat{E}f)(x) = f(qx), \quad (\hat{x}f)(x) = xf(x).
\]

Observe that \(\hat{E}\hat{x} = q\hat{x}\hat{E}\).

**Lemma 1.5.** (a) We have:

\[
F_{A,B}(q^{-1}, x) = \tilde{F}_{A,B}(q, x).
\]

(b) \(F_{A,B}\) satisfies the linear \(q\)-difference equation

\[
\left( (1 - \hat{E})^B - (-1)^A q^A x \hat{E}^A \right) F_{A,B}(q, x) = 0.
\]

**Corollary 1.6.** (a) If we define \(\omega(q,x) = F_{A,B}(q,qx)/F_{A,B}(q,x)\) and \(\omega(q,x)_n = \prod_{j=1}^n \omega(q,q^jx)\), then \(\omega\) satisfies the nonlinear equation

\[
\sum_{j=0}^B (-1)^j \binom{B}{j} \omega(q,x)_j - (-1)^A q^A x \omega(q,x)_A = 0.
\]

(b) \(F\) is an admissible power series in the sense of Kontsevich-Soibelman [KS11, Sec.6], the limit \(\lim_{q \to 1} \omega(q,x) = \omega(x) \in \mathbb{Q}[[x]]\) exists and satisfies the algebraic equation (also known as the Nahm equation or the gluing equation)

\[
(1 - \omega(x))^B = (-1)^A x \omega(x)^A.
\]

The Nahm equation has been studied by several authors including [Zag07, Sec.3], [Vla, VZ11], [RV, Sec.4].

1.3. \textbf{The case of the 4_1 knot.} We now specialize Corollary 1.2 to the invariant of the 4_1 and 5_2 knots is given by [KLV12, AK]

\[
\mathcal{I}_{1,2} = \mathcal{I}_{4_1}, \quad \mathcal{I}_{2,3} = \mathcal{I}_{5_2}.
\]

In this section, let

\[
F(q,x) = F_{1,2}(q,x) = \sum_{n=0}^\infty (-1)^n q^{\frac{n(n+1)}{2}} (q)_n^2 x^n.
\]

**Corollary 1.7.** (a) We have:

\[
\mathcal{I}_{4_1}(b) = -\frac{i}{2} \left( \frac{q}{\bar{q}} \right)^{\frac{1}{24}} (b G(q)g(\bar{q}) - b^{-1} G(\bar{q})g(q))
\]

where

\[
g(q) = \sum_{n=0}^\infty (-1)^n q^{\frac{n(n+1)}{2}} (q)_n^2
\]

\[
G(q) = \sum_{m=0}^\infty \left( 1 + 2m - 4 \sum_{s=1}^\infty \frac{q^{s(m+1)}}{1-q^s} \right) (-1)^m q^{\frac{1}{2}m(m+1)} (q)_m^2
\]
(b) The series $g(q)$ and $G(q)$ are given in terms of $F(q, x)$ by:

\begin{align}
(16a) & \quad g(q) = \langle F \rangle \\
(16b) & \quad G(q) = \langle (2 + 2\delta - 4\delta_1)F \rangle
\end{align}

(c) $F$ satisfies the linear $q$-difference equation

\begin{equation}
F(q, q^{-1}x) + F(q, qx) = (2 - x)F(q, x)
\end{equation}

The series $g(q)$ that appears in Theorem 1.7 was known to the first author and Zagier to be closely related to the $4_1$ knot. For a detailed discussion of experimental facts below, see [GZ]. Empirically, it appears that

- the pair $(g(q), G(q))$ is related to the 3D index of the $4_1$ knot,
- the radial asymptotics of the pair $(g(q), G(q))$ are related to the asymptotics of the Kashaev invariant of the $4_1$ knot,
- the above observations for $4_1$ also hold for the case of $5_2$ knot discussed below.

Recall that the index of an ideal triangulation was introduced in [DGGB, DGGA], necessary and sufficient conditions for its convergence was established in [Gar] and its topological invariance was proven in [GHRS]. For a detailed discussion of the above experimental facts, see [GZ].

1.4. The case of the $5_2$ knot. In this section, let

$$F(q, x) = F_{2,3}(q, x) = \sum_{m=0}^{\infty} t_m(q)x^m, \quad \tilde{F}(q, \bar{x}) = F_{1,3}(q, \bar{x}) = \sum_{n=0}^{\infty} T_n(q)\bar{x}^n$$

where

$$t_m(q) = \frac{q^{m+1}}{(q^3)^m}, \quad T_n(q) = (-1)^n q^{\frac{2n+1}{3}} = t_n(q^{-1}).$$

Let

$$R_{m,n}(q, \bar{q}) = -\frac{b^2}{2} \left( 1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_1^{(m)^2}(q) - 3E_2^{(m)}(q) \right)$$

$$- \frac{1}{2\pi i} + \frac{1}{2} \left( 1 + 2m - 3E_1^{(m)}(q) \right) \left( 1 + 2n - 6E_1^{(n)}(\bar{q}) \right)$$

$$+ \frac{b^2}{2} \left( -n - n^2 - 6E_2^{(0)}(\bar{q}) + 3E_1^{(n)}(\bar{q}) + 6nE_1^{(n)}(\bar{q}) - 9E_1^{(n)^2}(\bar{q}) + 3E_2^{(n)}(\bar{q}) \right),$$

where $E_i^{(m)}(q)$ are defined in Equation (29a). For $k = 1, \ldots, 4$ let

\begin{equation}
g_k(q) = \sum_{m=0}^{\infty} p_k(m)t_m(q), \quad G_k(\bar{q}) = \sum_{n=0}^{\infty} P_k(n)T_n(\bar{q}),
\end{equation}

where

\begin{align}
(19a) & \quad p_{1,m}(q) = 1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_1^{(m)^2}(q) - 3E_2^{(m)}(q) \\
(19b) & \quad p_{2,m}(q) = 1 + 2m - 3E_1^{(m)}(q) \\
(19c) & \quad p_{3,m}(q) = 1
\end{align}
and
\begin{align}
(20a) \quad P_{1,m}(q) &= 1 \\
(20b) \quad P_{2,m}(q) &= 1 + 2n - 6E_1^{(n)}(\tilde{q}) \\
(20c) \quad P_{3,m}(q) &= -n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) - 9E_1^{(n)^2}(\tilde{q}) + 3E_2^{(n)}(\tilde{q}).
\end{align}

**Corollary 1.8.** (a) We have:
\begin{equation}
\mathcal{I}_{2,3}(q) = -e^{\frac{3\pi i}{4}} \left( \sum_{m,n=0}^{\infty} R_{m,n}(q, \tilde{q}) t_m(q) T_n(\tilde{q}) \right) = -e^{\frac{3\pi i}{4}} \left( \frac{b^2}{2} g_1(q) G_1(\tilde{q}) - \frac{1}{2\pi i} g_3(q) G_1(\tilde{q}) + \frac{1}{2} g_2(q) G_2(\tilde{q}) + \frac{b^2}{2} g_3(q) G_3(\tilde{q}) \right)
\end{equation}

(b) \( F \) and \( \tilde{F} \) satisfy the linear \( q \)-difference equations
\begin{align*}
F(q, q^3 x) - (3 - q^2 x) F(q, q^2 x) + 3F(q, qx) - F(q, x) &= 0 \\
\tilde{F}(q, q^3 x) - 3\tilde{F}(q, q^2 x) + (3 - q^2 x) \tilde{F}(q, qx) - \tilde{F}(q, x) &= 0.
\end{align*}

**Remark 1.9.** A computation gives that \( P(A, B) = P(B - A, B) \) for \( (A, B) = (1, 2) \) and \( (A, B) = (2, 3) \) corresponding to the invariants of the 4_1 and 5_2 knots. In all other cases that we tried, we found that \( P(A, B) \) is not equal to \( P(B - A, B) \).

### 2. Proofs

#### 2.1. A residue computation.
To relate the state-integral \( \mathcal{I}_{A,B} \) to a sum, we will apply the residue theorem on a semicircle \( \gamma_R \) with center 0 and radius \( R \), oriented counterclockwise in the upper half-plane:

\[ \gamma_R \]

Then, we will take the limit \( R \to \infty \). To compute the residue of the integrand, we need to expand \( \Phi_b(x) \) near the pole
\[ x_{m,n} = c + ibm + ib^{-1}n \]
for natural numbers \( m \) and \( n \). Let
\begin{align}
\phi_m(x) &= \frac{(q^{m+1}e^x; q)_\infty}{(q^{m+1}; q)_\infty} \\
\tilde{\phi}_n(x) &= \frac{(\tilde{q}; \tilde{q})_\infty}{(\tilde{q}e^x; \tilde{q})_\infty} \frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^x; \tilde{q}^{-1})_n}
\end{align}

**Lemma 2.1.** We have:
\begin{equation}
\Phi_b(x + x_{m,n}) = \frac{(q; q)_\infty}{(\tilde{q}; \tilde{q})_\infty} \frac{1}{(q; q)_m} \frac{1}{(\tilde{q}^{-1}; \tilde{q}^{-1})_n} \frac{\phi_m(2\pi b x) \tilde{\phi}_n(2\pi b^{-1} x)}{1 - e^{2\pi b^{-1} x}}.
\end{equation}
Proof. Equation (1) implies the functional equations
\[ \frac{\Phi_b(x + c_b + ib)}{\Phi_b(x + c_b)} = \frac{1}{1 - qe^{2\pi bx}} \]
\[ \frac{\Phi_b(x + c_b + ib^{-1})}{\Phi_b(x + c_b)} = \frac{1}{1 - \tilde{q}^{-1}e^{2\pi b^{-1}x}} \]
which give
\[ \Phi_b(x + x_{m,n}) = \Phi_b(x + c_b) \frac{1}{(qe^{2\pi bx}; q)_m} \frac{1}{(\tilde{q}^{-1}e^{2\pi b^{-1}x}; \tilde{q}^{-1})_n} \]
\[ \Phi_b(x + c_b) = \frac{1}{1 - e^{2\pi b^{-1}x}} \frac{1}{(qe^{2\pi bx}; q)_\infty} \frac{1}{(\tilde{q}^{-1}e^{2\pi b^{-1}x}; \tilde{q}^{-1})_\infty}. \]
Thus,
\[ \Phi_b(x + x_{m,n}) = \frac{(q; q)_\infty}{(q^{n+1}; q)_\infty} \frac{1}{(\tilde{q}; q)_\infty} \frac{1}{(\tilde{q}^{-1}; q)_\infty}. \]

The result follows.  

The decoupling of \((m, n)\) in the quadratic form comes as follows: since \(A, m, n\) are integers, \(e^{\pi imn} = 1\) and a computation gives
\[ e^{-A\pi i(x+x_{m,n})^2} = i^A \left( \frac{q}{\tilde{q}} \right)^{\frac{1}{2}} t_m^A(q) \tilde{t}_n^A(\tilde{q}) e^{-A\pi ix^2 + 2A\pi x(b(m+\frac{1}{2}) + b^{-1}(n+\frac{1}{2}))} \]
where
\[ t_m^A(q) = (-1)^{Am} q^{\frac{A(m+1)}{2}}, \quad \tilde{t}_n^A(\tilde{q}) = (-1)^{An} \tilde{q}^{-\frac{A(n+1)}{2}}. \]

The Dedekind function \(\eta(\tau) = q^{1/24}(q; q)_\infty\) (with \(q = e^{2\pi i\tau}\)) satisfies the modular equation \(\eta(-\tau^{-1}) = \sqrt{-i}\eta(\tau) [\text{And76}].\) It follows that
\[ (26) \quad \frac{(q; q)_\infty}{(q; q)_\infty} = e^{\frac{\pi}{4}} \left( \frac{\tilde{q}}{q} \right)^{\frac{1}{24}} b^{-1}. \]

After we set \(w = x/(2\pi)\), the above discussion implies that
\[ (27) \quad I_{A,B}(b) = \left( \frac{q}{\tilde{q}} \right)^{\frac{B+3A}{24}} e^{\pi i B(4 + A + 1)} \sum_{m,n=0}^{\infty} (\text{Res}_{w=0} F_{A,B,m,n}(w)) \frac{t_m^A(q)}{(q; q)_m} \tilde{t}_n^A(\tilde{q}) \frac{1}{(\tilde{q}^{-1}; \tilde{q}^{-1})_n}. \]
where

\[
F_{A,B,m,n}(w) = e^{\frac{A - e^{B}}{2} + A w (b + \frac{1}{2}) + b^{-1} (n + \frac{1}{2})} \left( \frac{\phi_m(bw) \tilde{\phi}_n(b^{-1}w)}{b(1 - e^{b^{-1}w})} \right)^B. 
\]

2.2. The Taylor series of \( \phi_m(x) \) and \( \tilde{\phi}_n(x) \). In this section we express the Taylor series of \( \phi_m(x) \) and \( \tilde{\phi}_n(x) \) in terms of the \( q \)-series \( E_i^{(m)}(q) \) and \( \tilde{E}_i^{(m)}(\tilde{q}) \) defined by:

\[
E_i^{(m)}(q) = \sum_{s=1}^{\infty} \frac{s^{l-1} q^{s(m+1)}}{1 - q^s} = \langle \delta_i(x^m) \rangle 
\]

\[
\tilde{E}_i^{(m)}(\tilde{q}) = \begin{cases} 
- n + E_1^{(n)}(\tilde{q}) & \text{if } l = 1 \\
E_i^{(m)}(\tilde{q}) & \text{if } l > 1 \text{ is odd} \\
2 E_i^{(0)}(\tilde{q}) - E_i^{(m)}(\tilde{q}) & \text{if } l > 1 \text{ is even} 
\end{cases} 
\]

**Proposition 2.2.** We have:

\[
\phi_m(x) = \exp \left( -\sum_{l=1}^{\infty} \frac{1}{l!} E_i^{(m)}(q)x^l \right) 
\]

\[
\tilde{\phi}_n(x) = \exp \left( \sum_{l=1}^{\infty} \frac{1}{l!} \tilde{E}_i^{(m)}(\tilde{q})x^l \right). 
\]

The proof of this proposition is given in Section 2.6. The first few terms in Equations (30a)-(30a) are given by:

\[
\phi_m(x) = \exp \left( -E_1^{(m)}x - \frac{1}{2}E_2^{(m)}x^2 - \frac{1}{6}E_3^{(m)}x^3 - \frac{1}{24}E_4^{(m)}x^4 - \ldots \right) 
\]

\[
= 1 - E_1^{(m)}x + \frac{1}{2}(E_1^{(m)} - E_2^{(m)})x^2 + \frac{1}{6}(E_1^{(m)} + 3E_2^{(m)} - E_3^{(m)})x^3 + \frac{1}{24}(E_1^{(m)} + 6E_2^{(m)} + 3E_3^{(m)} + E_4^{(m)})x^4 + \ldots 
\]

\[
\tilde{\phi}_n(x) = \exp \left( \tilde{E}_1^{(m)}x + \frac{1}{2}\tilde{E}_2^{(m)}x^2 + \frac{1}{6}\tilde{E}_3^{(m)}x^3 + \frac{1}{24}\tilde{E}_4^{(m)}x^4 - \ldots \right) 
\]

\[
= 1 + \tilde{E}_1^{(m)}x + \frac{1}{2}(\tilde{E}_1^{(m)} + \tilde{E}_2^{(m)})x^2 + \frac{1}{6}(\tilde{E}_1^{(m)} + 3\tilde{E}_2^{(m)} + \tilde{E}_3^{(m)})x^3 + \frac{1}{24}(\tilde{E}_1^{(m)} + 6\tilde{E}_2^{(m)} + 3\tilde{E}_3^{(m)} + \tilde{E}_4^{(m)})x^4 + \ldots 
\]

where \( E_i^{(m)}(q) \) and \( \tilde{E}_i^{(m)}(\tilde{q}) \) are given by

2.3. The connection with the differential operators \( \delta_i \) and \( \tilde{\delta}_i \). In this section we connect the series \( E_i^{(m)}(q) \) and \( \tilde{E}_i^{(m)}(\tilde{q}) \) with the action of the differential operators \( \delta_i \) and \( \tilde{\delta}_i \) on a series \( F(x) \) and \( \tilde{F}(\tilde{x}) \) respectively. Consider formal power series

\[
F(x) = \sum_{m=0}^{\infty} t(m)x^m 
\]

\[
\tilde{F}(\tilde{x}) = \sum_{m=0}^{\infty} \tilde{t}(m)\tilde{x}^m. 
\]
Lemma 2.3. We have:

\(\sum_{m=0}^{\infty} \left( \prod_{j=1}^{r} E_{i_j}^{(m)}(q) \right) t(m) = \langle \prod_{j=1}^{r} \delta_{i_j} F \rangle \) \hspace{1cm} (32)

\(\sum_{m=0}^{\infty} m^r t(m) = \langle \delta^r F \rangle \) \hspace{1cm} (33)

and

\(\sum_{n=0}^{\infty} \left( \prod_{j=1}^{r} \tilde{E}_{i_j}^{(n)}(\tilde{q}) \right) \tilde{t}(n) = \langle \prod_{j=1}^{r} \tilde{\delta}_{i_j} \tilde{F} \rangle \) \hspace{1cm} (34)

\(\sum_{n=0}^{\infty} n^r \tilde{t}(n) = \langle d^r \tilde{F} \rangle \). \hspace{1cm} (35)

Proof. For a positive natural number \(l\) we have:

\(\sum_{m=0}^{\infty} E_{i}^{(m)}(q)t(m) = \sum_{m=0}^{\infty} \langle \delta_{i}(x^m) \rangle t(m) = \left\langle \delta_{i} \left( \sum_{m=0}^{\infty} t(m)x^m \right) \right\rangle = \langle \delta_{i} F \rangle .\)

Moreover, for positive natural numbers \(l, l'\) we have:

\(\sum_{m=0}^{\infty} E_{i}^{(m)}(q)E_{i'}^{(m)}(q)t(m) = \sum_{m=0}^{\infty} \langle \delta_{i}(x^m) \rangle \langle \delta_{i'}(x^m) \rangle t(m) = \left\langle \delta_{i} \left( \sum_{m=0}^{\infty} \langle \delta_{i'}(x^m) \rangle t(m)x^m \right) \right\rangle .\)

Now,

\(\langle \delta_{i'}(x^m) \rangle t(m)x^m = \sum_{s=1}^{\infty} \frac{s^{l'-1}q^s}{1-q^s q^{s^m} t(m)x^m} = \delta_{i'}(x^m)t(m)\)

and summing up over \(m\), we obtain that

\(\sum_{m=0}^{\infty} \langle \delta_{i'}(x^m) \rangle t(m)x^m = \delta_{i'} F(q, x) .\)

It follows that

\(\sum_{m=0}^{\infty} E_{i}^{(m)}(q)E_{i'}^{(m)}(q)t(m) = \langle \delta_{i} \delta_{i'} F \rangle .\)

The general case of Equation (32) follows by induction on \(r\). Equation (33) is obvious. \(\square\)
2.4. Proof of Theorem 1.1. Fix natural numbers $A$ and $B$ with $B > A \geq 1$, and let
\[ t(m) = \frac{(-1)^{A m} A_{m}^{2m+1}}{(q)_{m}^{B}}, \quad F(q, x) = \sum_{m=0}^{\infty} t(m)x^{m} \]
and
\[ \tilde{t}(n) = \frac{(-1)^{(B-A) n} A_{n+1}^{(B-A) n}}{(\tilde{q})_{n}^{B}}, \quad \tilde{F}(\tilde{q}, \tilde{x}) = \sum_{n=0}^{\infty} \tilde{t}(n)x^{n}. \]
Use Equations (27) and (28) and Proposition 2.2 to expand $F_{A, B, m, n}(w)$ as a power series with coefficients polynomials in the variables $m, E_{l}(m)$, $n, \tilde{E}_{l}(n)$ and $2 \pi i$. Now apply Lemma 2.3 to convert the variables $m, E_{l}(m), n, \tilde{E}_{l}(n)$ in terms of the action of the operators $\delta, \delta_{l}, \tilde{\delta}, \tilde{\delta}_{l}$ respectively. This concludes the proof of Theorem 1.1. □

2.5. Some auxiliary power series. Consider the auxiliary series
\[ \frac{1}{ae^{x} - 1} = \sum_{n=0}^{\infty} p_{n}(a)x^{n} \]
where
\[ p_{n}(a) = -a \frac{1}{n!(1-a)^{n+1}} \sum_{m=0}^{n-1} A_{n,m} a^{m}, \quad p_{0}(a) = -\frac{1}{1-a} \]
and $A_{n,m}$ are Euler triangular numbers (sequence A008292 in the online encyclopedia of integer sequences [Slo]) that satisfy the recursion
\[ A_{n,m} = (n-m)A_{n-1,m-1} + (m+1)A_{n-1,m} \]
and also given by the sum
\[ A_{n,m} = \sum_{k=0}^{m} (-1)^{k} \binom{n+1}{k} (m+1-k)^{n}. \]
For a detailed discussion on this subject, see [FS70]. A table of the first few numbers $A_{n,m}$ is given by

| $n \backslash m$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----------------|---|---|---|---|---|---|---|---|---|
| 1               | 1 |   |   |   |   |   |   |   |   |
| 2               | 1 | 1 |   |   |   |   |   |   |   |
| 3               | 1 | 4 | 1 |   |   |   |   |   |   |
| 4               | 1 | 11| 11| 1 |   |   |   |   |   |
| 5               | 1 | 26| 66| 26| 1 |   |   |   |   |
| 6               | 1 | 57| 302|302|57|1 |   |   |   |
| 7               | 1 | 120|1191|2416|1191|120|1 |   |   |
| 8               | 1 | 247|4293|15619|15619|4293|247|1 |   |
| 9               | 1 | 502|14608|88234|156190|88234|14608|502|1 |
Lemma 2.4. For \( l \geq 1 \), we have:

\[
\frac{d^l}{dx^l} \log(1 - q^k e^{bx}) |_{x=0} = b^l p_{l-1}(q^k) + b \delta_{l,1}
\]

Proof. It follows from

\[
\frac{d}{dx} \log(1 - q^k e^{bx}) = b \left( 1 + \frac{1}{q^k e^{bx} - 1} \right)
\]

and Equation (36). \( \square \)

For positive natural numbers \( l, r \) with \( l \geq r \) and \( m \) consider the \( q \)-series \( E_{l,r}^{(m)}(q) \) defined by

\[
E_{l,r}^{(m)}(q) = \sum_{k=m+1}^{\infty} \frac{q^{kr}}{(1 - q^k)^l}
\]

Lemma 2.5. (a) We have

\[
E_{l,r}^{(m)}(q) = \sum_{s=r}^{\infty} a_{l,s} \frac{q^{s(m+1)}}{1 - q^s}
\]

where

\[
\frac{x^r}{(1 - x)^l} = \sum_{s=r}^{\infty} a_{l,s} x^s
\]

(b) It follows that

\[
\sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{(m)}(q) = E_{l}^{(m)}(q)
\]

Proof. For (a), interchange \( k \) and \( s \) summation:

\[
E_{l,r}^{(m)}(q) = \sum_{k=m+1}^{\infty} \sum_{s=r}^{\infty} a_{l,s} q^{sk} = \sum_{s=r}^{\infty} \sum_{k=m+1}^{\infty} a_{l,s} q^{sk} = \sum_{s=r}^{\infty} \sum_{k=0}^{(m+1)s} a_{l,s} q^{sk} = \sum_{s=r}^{\infty} a_{l,s} \frac{q^{(m+1)s}}{1 - q^s}
\]

(b) follows from (a) and the fact that

\[
\sum_{r=0}^{l-1} A_{l-1,r} x^r = \sum_{s=1}^{\infty} s^{l-1} x^s.
\]

Lemma 2.6. We have:

\[
\phi_m(x) = \exp \left( - \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} E_{l,r+1}^{(m)}(q)x^l \right)
\]

Proof. It follows from Lemma 2.4 combined with

\[
\log(\phi_m(x)) = \log \left( \frac{(q^{m+1} e^x; q)_{\infty}}{(q^m; q)_{\infty}} \right) = \sum_{l=m+1}^{\infty} (\log(1 - q^l e^x) - \log(1 - q^l))
\]

\( \square \)
2.6. Proof of Proposition 2.2. Part (a) of Proposition 2.2 follows from Lemma 2.5 and Lemma 2.6. For part (b), we will use the series

\[ E_i^{[m]}(q) = \sum_{s=1}^{\infty} \frac{s^{k-1}q^{s(m+1)}}{1-q^s} \]

Using

\[ \log(\tilde{\phi}_n(x)) = \log\left(\frac{(\tilde{q}; \tilde{q})_{\infty}}{(\tilde{q}e^x; \tilde{q})_{\infty}}\right) + \log\left(\frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^x; \tilde{q}^{-1})_n}\right) \]

and the proof of part (a) of Proposition 2.2, it follows that

\[ \log(\tilde{\phi}_n(x)) = \log\left(\frac{(\tilde{q}; \tilde{q})_{\infty}}{(\tilde{q}e^x; \tilde{q})_{\infty}}\right) + \log\left(\frac{(\tilde{q}^{-1}; \tilde{q}^{-1})_n}{(\tilde{q}^{-1}e^x; \tilde{q}^{-1})_n}\right) \]

\[ = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} E_{l+1}^{(0)}(\tilde{q})x^l + \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} E_l^{[n]}(\tilde{q}^{-1})x^l \]

\[ = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} \left(E_{l+1}^{(0)}(\tilde{q}) + E_l^{[n]}(\tilde{q}^{-1})\right)x^l \]

where

\[ E_l^{[n]}(q) = \sum_{k=1}^{n} \frac{q^{kr}}{(1-q^k)^l}. \]

Let

\[ \tilde{E}_{l,r}^{(n)}(\tilde{q}) = \begin{cases} -n + E_{1,1}^{(n)}(\tilde{q}) & \text{if } l = r = 1 \\ E_{l,r}^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is odd} \\ 2E_{l,r}^{(0)}(\tilde{q}) - E_{l,r}^{(n)}(\tilde{q}) & \text{if } l > 1 \text{ is even} \end{cases} \]

We claim that

\[ E_{l,r}^{(0)}(\tilde{q}) + E_{l,l-r}^{[n]}(\tilde{q}^{-1}) = \tilde{E}_{l,r}^{(n)}(\tilde{q}) \]

for \( l > r \geq 1 \) and

\[ E_{1,1}^{(0)}(\tilde{q}) + E_{1,1}^{[n]}(\tilde{q}^{-1}) = \tilde{E}_{1,1}^{(n)}(\tilde{q}) \]

Assuming Equations (44) and (45), it follows that

\[ \log(\tilde{\phi}_n(x)) = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{r=0}^{l-1} A_{l-1,r} \tilde{E}_{l,r+1}^{(n)}(\tilde{q})x^l \]

\[ = \sum_{l=1}^{\infty} \frac{1}{l!} E_l^{(n)}(\tilde{q})x^l \]

where the last step follows from part (b) of Lemma 2.5.
It remains to prove Equations (44) and (45). Equation (44) follows from the definition of $\tilde{E}^{(n)}_{1,1}(\bar{q})$ and

$$
E^{(0)}_{l,r}(\bar{q}) + E^{[n]}_{l,l-r}(\bar{q}^{-1}) = \sum_{k=1}^{\infty} \frac{\bar{q}^{kr}}{(1 - \bar{q}^k)^l} + \sum_{k=1}^{n} \frac{\bar{q}^{-k(l-r)}}{(1 - \bar{q}^{-k})^l} \\
= \sum_{k=1}^{\infty} \frac{\bar{q}^{kr}}{(1 - \bar{q}^k)^l} + (-1)^l \sum_{k=1}^{n} \frac{\bar{q}^{kr}}{(1 - \bar{q}^k)^l} \\
= (1 + (-1)^l) \sum_{k=1}^{n} \frac{\bar{q}^{kr}}{(1 - \bar{q}^k)^l} + \sum_{k=n+1}^{\infty} \frac{\bar{q}^{kr}}{(1 - \bar{q}^k)^l} 
$$

Equation (45) follows from

$$
E^{(0)}_{1,1}(\bar{q}) + E^{[n]}_{1,1}(\bar{q}^{-1}) = \sum_{k=1}^{\infty} \frac{\bar{q}^k}{1 - \bar{q}^k} + \sum_{k=1}^{n} \frac{\bar{q}^{-k}}{1 - \bar{q}^{-k}} \\
= \sum_{k=1}^{\infty} \frac{1 - \bar{q}^k}{1 - \bar{q}^k} - \sum_{k=1}^{n} \frac{1}{1 - \bar{q}^k} = -n + \sum_{k=n+1}^{\infty} \frac{\bar{q}^k}{1 - \bar{q}^k} 
$$

This completes the proof of Proposition 2.2. □

2.7. Proof of Lemma 1.5. Part (a) of Lemma 1.5 follows from the definition of $F_{A,B}$ and $\tilde{F}_{A,B}$.

Part (b) follows from an application of Zeilberger’s creative telescoping [Zei91]. To apply the method, define

$$
t(m, x) = \frac{(-1)^Am q^{A(m+1)}_m}{(q)_m^B} x^m
$$

Then, observe that $t$ satisfies the recursions with respect to $m$ and $x$:

$$(1 - q^{m+1})^B t(m + 1, x) = (-1)^A q^{A(m+1)} t(m, x) \quad t(m, qx) = q^m t(m, x).$$

Now, we eliminate $q^m$ from the above equations as follows. The second equation implies that $t(m + 1, q^j x) = q^{j(m+1)} t(m + 1, x)$. Expanding the first equation, it follows that

$$
\sum_{j=0}^{B} (-1)^j \binom{B}{j} t(m + 1, q^j x) = (-1)^A q^A xt(m, q^A x)
$$

Summing for $m \geq 0$ implies (b). □

Proof. (of Corollary 1.6) The admissibility of $F$ in the sense of Kontsevich-Soibelman, follows from [KS11, Sec.6.1] and [KS11, Thm.9]. Given this, the Nahm Equation (12) for $\omega$ follows easily from part (b) of Lemma 1.5. □
3. AN APPLICATION: STATE-INTEGRALS OF THE 4_1 AND 5_2 KNOTS

3.1. Proof of Corollary 1.7. Assume now that \((A, B) = (1, 2)\). Then,

\[
\frac{1}{(b(1 - e^{b^{-1}w}))^{2}} = \frac{1}{w^2} - \frac{b^{-1}}{w} + O(1)
\]

\[
(\phi_m(bw))^2 = 1 - 2E_1^{(m)}(q)bw + O(w^2)
\]

\[
(\tilde{\phi}_n(b^{-1}w))^2 = 1 + 2\tilde{E}_1^{(n)}(\tilde{q})b^{-1}w + O(w^2)
\]

\[
e^{\frac{1}{4m}w^2 + w(b(m+1/2)+b^{-1}(n+1/2))} = 1 + \left(\frac{1}{2} + m\right)bw + \left(\frac{1}{2} + n\right)b^{-1}w + O(w^2)
\]

Combined with \(\tilde{E}_1^{(n)}(\tilde{q}) = -n + E_1^{(n)}(\tilde{q})\), it follows that the residue \(R = \text{Res}_{w=0}(F_{1,2,m,n}(w))\) is given by

\[
R = \left(b\left(\frac{1}{2} + m - 2E_1^{(m)}(q)\right) - b^{-1}\left(\frac{1}{2} + n - 2\tilde{E}_1^{(n)}(\tilde{q})\right)\right)
\]

The above, together with the fact that \(t_n(q) = (-1)^n\frac{2^{n(n+1)}}{\pi q^{3/2}}\) satisfies \(t_n(q^{-1}) = t_n(q)\) implies Equation (14). Equation (17) follows from Equation (11) for \((A, B) = (1, 2)\).

3.2. Proof of Corollary 1.8. Assume now that \((A, B) = (2, 3)\). Then,

\[
\frac{1}{(b(1 - e^{b^{-1}w}))^{3}} = -\frac{1}{w^3} + \frac{3b^{-1}}{2w^2} - \frac{b^{-2}}{w} + O(1)
\]

\[
(\phi_m(bw))^3 = 1 - 3E_1^{(m)}(q)bw + \frac{3}{2}\left(3E_1^{(m)}(q) - E_2^{(m)}(q)\right)b^2w^2 + O(w^3)
\]

\[
(\tilde{\phi}_n(b^{-1}w))^3 = 1 + 3\tilde{E}_1^{(n)}(\tilde{q})b^{-1}w + \frac{3}{2}\left(3\tilde{E}_1^{(n)}(\tilde{q}) - \tilde{E}_2^{(n)}(\tilde{q})\right)b^{-2}w^2 + O(w^3)
\]

\[
e^{\frac{2}{4m}w^2 + 2w(b(m+1/2)+b^{-1}(n+1/2))} = 1 + \left(\left(1 + 2m\right)b + (1 + 2n)b^{-1}\right)w +
\]

\[
\left(1 + \frac{b^2 + b^{-2}}{2} + \frac{1}{2}\pi i + 2b^2 m^2 + 2b^{-2}n^2 + 4mn\right)
\]

\[+ 2(1 + b^2)m + 2(1 + b^{-2})n\] w^2 + O(w^3)
\]

If \(R = \text{Res}_{w=0}(F_{2,3,m,n}(w))\), then

\[
R_{m,n} = -\frac{b^2}{2} \left(1 + 4m + 4m^2 - 6E_1^{(m)}(q) - 12mE_1^{(m)}(q) + 9E_2^{(m)}(q) - 3E_2^{(m)}(q)\right)
\]

\[+ \frac{1}{2}\pi i + \frac{1}{2}\left(1 + 2m - 3E_1^{(m)}(q)\right)\left(1 + 2n - 6E_1^{(n)}(\tilde{q})\right)
\]

\[+ \frac{b^{-2}}{2} \left(-n - n^2 - 6E_2^{(0)}(\tilde{q}) + 3E_1^{(n)}(\tilde{q}) + 6nE_1^{(n)}(\tilde{q}) - 9E_2^{(n)}(\tilde{q}) + 3E_2^{(n)}(\tilde{q})\right),
\]

This proves part (a) of Corollary 1.8. Part (b) follows from Equation (11) for \((A, B) = (2, 3)\) and \((A, B) = (1, 3)\). Note that Theorem 1.1 states that

\[
\mathcal{I}_{2,3}(q) = -e^{\frac{3\pi i}{4}} \langle P_{2,3}(F \tilde{F}) \rangle
\]
where
\[
P_{2,3} = -\frac{b^2}{2} \left(1 + 4\delta + 4\delta^2 - 6\delta_1 - 12\delta\delta_1 + 9\delta^2 - 3\tilde{\delta}^2\right)
+ \frac{1}{2} \left(1 + 2\delta + \frac{i}{\pi} + 2\tilde{\delta} + 4\delta\tilde{\delta} - 3\delta_1 - 6\delta\delta_1 - 6e_2(q) - 6\delta_1 - 12\delta\delta_1 + 18\delta_1\delta_1\right)
+ \frac{b^{-2}}{2} \left(-\tilde{\delta} - \tilde{\delta}^2 + 3\tilde{\delta}_1 + 6\delta\delta_1 - 9\delta^2 + 3\tilde{\delta}^2\right).
\]

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