ELLiptic Hall algebra on $F_1$

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Abstract. We construct Hall algebra of elliptic curve over $F_1$ using the theory of monoidal scheme due to Deitmar and the theory of Hall algebra for monoidal representations due to Szczesny. The resulting algebra is shown to be a specialization of elliptic Hall algebra studied by Burban and Schiffmann. Thus our algebra is isomorphic to the skein algebra for torus by the recent work of Morton and Samuelson.

1. Introduction

This note is motivated to understand in terms of categorical viewpoint the recent work of Morton and Samuelson [MS17] establishing the algebra isomorphism between the torus skein algebra and the elliptic Hall algebra.

In the late 1980s, Turaev [Tu89, Tu91] introduced the skein algebra $Sk(\Sigma)$ as a $q$-deformation of the Goldman Lie algebra [Go86] on an oriented surface $\Sigma$. The Goldman Lie algebra is the Lie algebra encoding the symplectic structures of character varieties on $\Sigma$. Recently, Morton and Samuelson [MS17] discovered remarkable relationship between skein algebra for a torus $T$ and the Ringel-Hall algebra for an elliptic curve $E$. Let us start with this introduction by explaining Turaev’s skein algebra.

1.1. Skein algebra. Let $R := \mathbb{Z}[s^{\pm 1}, v^{\pm 1}]$. A skein module of an oriented $3$-fold $M$ is the quotient of the $R$-module spanned by the isotopy classes of framed oriented links in $M$ by the skein relation $(sk)$ and the framing relation $(fr)$.

$$S(M) := \left\langle \text{isotopy classes of framed oriented links in } M \right\rangle_{R\text{-lin}} / (sk), (fr)$$

The skein relation $(sk)$ is defined as the following local diagram.

$$(sk) \xrightarrow{-} \xrightarrow{=} = (s - s^{-1})$$

We omit the definition of the framing relation. It concerns with twists of framed links, and the variable $v$ counts the number of twists.

Let $I$ denote the unit interval. The skein algebra of oriented surface $\Sigma$ is $Sk(\Sigma) := S(\Sigma \times I)$ as an $R$-module with the multiplication given by placing one copy of $\Sigma \times I$ on top of another $\Sigma \times I$.

For example, if $\Sigma$ is a torus, we set $L_{0,1}$, $L_{1,1}$, $L_{1,1} \cdot L_{0,1}$ as in the diagrams below. The square shows the

Then the skein relation implies the following equation. The corresponding diagrams are denoted below with orientations on links omitted.

$$L_{1,1} \cdot L_{0,1} = L_{0,1} \cdot L_{1,1} + (s - s^{-1})L_{1,2}.$$

**Fact** (Turaev, [Tu91]). $Sk(\Sigma)$ is an $s$-deformation of the Goldman Lie algebra of $\Sigma$.

Recall that the Goldman Lie algebra [Go86] is the Lie algebra whose underlying vector space is given by the space of isotopy classes of oriented loops on $\Sigma$ and whose Lie bracket is given by

$$[(\alpha), (\beta)]_{\text{Goldman}} = \sum_{p \in \alpha \cap \beta} \pm \langle \alpha_p \beta \rangle.$$

Here $\alpha : S^1 \to \Sigma$ is a loop on $\Sigma$, and $\langle \alpha \rangle \in \pi^1(\Sigma)$ is its class in the fundamental group. $\alpha \cap \beta$ is the set of intersection points, and $\alpha_p \beta$ means a loop obtained by $\alpha$ and $\beta$ joined at $p$. The sign $\pm$ is defined by the orientation of $\Sigma$ and the intersection behavior of $\alpha$ and $\beta$ at $p$.

The result due to Morton and Samuelson [MS17] is that the skein algebra for a torus is isomorphic to a specialization of elliptic Hall algebra. So let us turn to explain the Hall algebras.

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1.2. **Hall algebra.** Let us briefly recall the theory of Hall algebra. We refer [Sc12] as a good review on the subject.

Let \( F_q \) be a finite field. A finitary category \( \mathcal{C} \) is an \( F_q \)-linear abelian category such that hom-spaces are of finite dimensional. Denote by \( \text{Iso}(\mathcal{C}) \) the set of isomorphism classes of objects in \( \mathcal{C} \). The Ringel-Hall algebra [R90] for \( \mathcal{C} \) is the \( \mathbb{Q} \)-vector space

\[
\text{Hall}(\mathcal{C}) := \{ f : \text{Iso}(\mathcal{C}) \to \mathbb{Q} \mid \text{supp}(f) < \infty \}
\]

with the multiplication

\[
f * g([M]) := \sum_{N \leq M} f([M/N])g([N]).
\]

Here we denoted by \([M] \) the isomorphism class of the object \( M \).

If \( \mathcal{C} \) is hereditary, then by Green [Gr95], \( \text{Hall}(\mathcal{C}) \) is a bialgebra with the comultiplication

\[
\Delta(f)([M], [N]) := f([M \otimes N])
\]

together with the Hopf pairing

\[
\langle \delta_{[M]}, \delta_{[N]} \rangle := \delta_{M,N,\mathbb{N}} / \# \text{Aut}_\mathcal{C}(M).
\]

Here we denoted by \( \delta_{[M]} \) the characteristic function of \([M] \in \text{Iso}(\mathcal{C}) \).

Since we have a bialgebra with a Hopf pairing, we can consider its Drinfeld double (see [Sc12] and [J95] for good accounts). For a hereditary and finitary abelian category \( \mathcal{C} \), the Drinfeld double \( \text{DHall}(\mathcal{C}) \) of \( \text{Hall}(\mathcal{C}) \) is defined to be

\[
\text{DHall}(\mathcal{C}) := \text{Hall}(\mathcal{C}) \otimes \mathbb{Q} \text{Hall}(\mathcal{C})
\]
as a vector space, and the multiplication is given by

\[
(m \otimes 1) \cdot (1 \otimes n) = m \otimes n, \quad \sum (m_{(2)}, n_{(1)}) m_{(1)} \otimes n_{(2)} = \sum (m_{(1)}, n_{(2)}) (1 \otimes n_{(1)}) \cdot (m_{(2)} \otimes 1).
\]

Here we used Sweedler’s notation \( \Delta(m) = \sum m_{(1)} \otimes m_{(2)} \).

Now we can state the result of Burban and Schiffmann [BS12] on the Hall algebra for an elliptic curve.

**Fact 1.1** (Burban-Schiffmann, [BS12]). Let \( E \) be an elliptic curve over \( F_q \) and set \( \mathcal{C} := \text{coh}_E \), the category of coherent sheaves over \( E \). Then \( \text{DHall}(\mathcal{C}) \) has a basis \( \{ u_x \mid x \in \mathbb{Z}^2 \setminus \{(0,0)\} \} \) with relations

(1) \([u_x, u_y] = 0 \text{ if } x, y \text{ parallel}\)

(2) \([u_x, u_y] = \pm \theta_{x+y}/\alpha_1 \text{ if } \Delta(0, x+y) \text{ contains no integral points}\).

Here \( \theta_x \) is defined by

\[
1 + \sum_{k \geq 1} \theta_k u^k = \exp \left( \sum_{n \geq 1} \alpha_n u^n \right)
\]

with \( \alpha_n := (1-q^n)(1-t^{-n})(1-q^n t^{-n})/n \). The symbols \( q, t \) denote the Weil numbers of \( E \).

Recall that the Weil numbers \( q, t^{-1} \) enjoys the properties

\[
q/t = p, \quad \zeta_E(z) := \exp \left( \sum \# E(F_{p^n}) u^n \right) = \frac{(1-qz)(1-z/t)}{(1-z)(1-p)}.
\]

Let us call the algebra \( \text{DHall}(\text{coh}_E) \) the **elliptic Hall algebra**. Burban and Schiffmann also showed in [BS12] that \( \text{DHall}(\text{coh}_E) \) has an integral basis \( \{ w_x \} \), where each \( w_x \) is proportional to \( u_x \), in the following sense.

**Definition.** Let \( q, t \) be indeterminates. Define \( U_{q,t} \) to be the algebra over \( \mathbb{Z}[q^\pm 1/2, t^\pm 1/2] \) generated by \( w_x \)'s with relations (1), (2) in Fact 1.1.

We omit the proportional factor \( w_x/u_x \). The meaning of integral basis is that if we choose \( q \) and \( t \) to be the Weil numbers of \( E \), then we have

\[
\text{DHall}(\text{coh}_E) \otimes \mathbb{Q} \mathbb{C} \simeq U_{q,t} \otimes \mathbb{Z}[q^\pm 1/2, t^\pm 1/2] \mathbb{C}.
\]

1.3. **The Morton-Samuelson isomorphism.** Now we can explain the Morton-Samuelson isomorphism mentioned in the beginning. Denote by \( L_{r,d} \in \text{Sk(torus)} \) the class of loop winding \( r \)-times rightwards and winding \( d \)-times upwards, as in \( L_{0,1} \) and \( L_{1,1} \) in §1.1.

**Fact** (Morton-Samuelson, [MS17]).

\[
\text{Sk(torus)} \xrightarrow{\sim} U_{s^2, s^2} \otimes \mathbb{Z}[s^\pm 1, v^\pm 1],
\]

\[
L_{r,d} \mapsto w_{(r,d)} \text{ if } \gcd(r, d) = 1.
\]

Let us remark that \( U_{q,t} \mathbb{Q} \) is still well-defined since \( \theta_1/\alpha_1 = (\gcd(x)) q^{x/2} u_x \). For \( t = q \), the proportional factor between \( w_x \) and \( u_x \) is given by

\[
w_x = (q^{d/2} - q^{-d/2}) u_x
\]

with \( d := \gcd(x) \).
1.4. The result of this note. Morton and Samuelson showed that the isomorphism between the skein algebra and the elliptic Hall algebra “by hand”. However, as they mentioned in the introduction of their paper [MS17], it is natural to invoke homological mirror symmetry for torus/elliptic curve [Ko95, PZ98]:

\[ D\mathfrak{su}(\text{torus}) \cong D^{b}\mathcal{Coh}(E), \quad L_{1,d} \leftrightarrow L_{d}. \]

Here \( L_{1,d} \) is the Lagrangian submanifold corresponding to the loop \( L_{1,d} \), and \( L_{d} \) is the line bundle over the elliptic curve \( E \) defined over \( \mathbb{C} \). One may guess that the Morton-Samuelson isomorphism comes from equivalence of categories.

However we have one drawback: the algebra \( \mathbb{U}_{\mathbb{Q},q} \) would correspond to the elliptic curve \( E \) with Weil numbers \( q = t = s^{2} \). Then by (1.1) \( E \) is defined over \( \mathbb{F}_{p} \) with \( p = q/t = 1 \). So in the present situation \( E \) should be “defined over \( \mathbb{F}_{1} \)”. Thus some theory of schemes over \( \mathbb{F}_{1} \) will be needed.

The purpose of this paper is to give a construction of \( \mathbb{U}_{q,q} \) as the Hall algebra of “elliptic curve of \( \mathbb{F}_{1} \)”. Precisely speaking, we argue

B1: Build a category \( \mathcal{B} \) using the monoidal Tate curve \( \hat{E} \). \( \hat{E} \) is a monoidal scheme over the formal monoidal scheme \( \text{Spf}(\mathbb{Q}) \), which is seen as the space of the parameter \( q \). The category \( \mathcal{B} \) will be considered as the category of “coherent sheaves over \( \mathbb{F}_{1} \)”,

B2: The category \( \mathcal{B} \) is not abelian, and it is not even additive. However it is a belian and quasi-exact category in the sense of Deitmar [D11, D06]. Then Szczesny’s construction of Hall algebra [Sz14] can be applied, and we have an algebra \( D\text{Hall}(\mathcal{B}) \).

B3: Check \( D\text{Hall}(\mathcal{B}) \cong \mathbb{U}_{q,q} \)

The step B3, or Theorem 4.2, is the main result of this note.

The steps B1–B3 form the “B-side” of our strategy of the proof of Morton-Samuelson isomorphism in terms of homological mirror symmetry. For the completeness of explanation, let us explain the remaining “A-side”.

A1: Build another category \( \mathcal{A} \) associated to a torus, which may be seen as the “Fukaya category of tori over \( \mathbb{F}_{1} \)”. \( \mathcal{A} \) should have a parameter \( s \) parametrizing the symplectic form, or the area form, on the torus.

A2: Consider Hall algebra \( \text{Hall}(\mathcal{A}) \)

A3: Check \( D\text{Hall}(\mathcal{A}) \cong \text{Sk}(\text{torus}) \).

The Morton-Samuelson isomorphism will be the consequence of

HMS: Show \( \mathcal{A} \cong \mathcal{B} \).

One may prove the step HMS by taking an appropriate \( \mathbb{F}_{1} \)-analogue of the mirror symmetry for torus/elliptic curve over \( \mathbb{Z} \), shown by [G09, LP12].

2. Hall algebra in the monoidal setting

In this section we explain Szczesny’s definition of Hall algebra for monoid representation and restate it in the setting of quasi-exact and belian category in the sense of Deitmar.

2.1. Category of modules over commutative monoid. Let \( A \) be a (multiplicative) commutative monoid with the absorbing element 0, i.e., \( a \cdot 0 = 0 = a \cdot 0 \) for any \( a \in A \).

An \( A \)-module is a pointed set \( (M, \cdot) \) with \( A \)-action \( \cdot : A \times M \to M \) such that \( 0 \cdot m = \cdot \) for any \( m \in M \). We will denote \( M \) for \( (M, \cdot) \) for simplicity.

Morphisms, submodules, images are defined as in the case of modules over a group.

Let \( M \) be an \( A \)-module and \( N \subset M \) be a sub \( A \)-module. The quotient \( M/N \) is defined to be the module with the underlying pointed set \( (M \setminus N) \cup \{ \cdot \} \).

We denote by \( A\text{-mod} \) the category of \( A \)-modules. It is not abelian since \( \text{coim}(f) \to \text{im}(f) \) is not an isomorphism in general. It is not even additive.

\( A\text{-mod} \) has the zero object \( \{ \cdot \} \) in the sense that for every \( A \)-module \( M \) there exists a unique \( A \)-morphism \( M \to \{ \cdot \} \) and a unique \( \{ \cdot \} \to M \). We will denote \( \{ \cdot \} \) by \( 0 \) hereafter. Now we have the notion of kernels and cokernels of morphisms.

For \( A \)-modules \( M \) and \( N \), is defined to be \( M \sqcup N/ \sim \) with \( \sim \) identifying base-points. Here \( \sqcup \) denotes the (set-theoretic) disjoint sum. Then \( M \sqcup N \) is the coproduct in the categorical sense.

The product \( M \times N \) in the categorical sense is defined to be the set-theoretic product with diagonal action.

One can also check that \( A\text{-mod} \) has finite limits and colimits.

Let us state a property of \( A\text{-mod} \) which is necessary to prove the associativity of Hall algebra.

Lemma 2.1. For an \( A \)-module \( M \) and its sub \( A \)-module \( N \subset M \), there is an inclusion-preserving correspondence between \( A \)-modules \( N \subset L \subset M \) and sub \( A \)-modules of \( M/N \) given by \( L \mapsto L/N \).

2.2. Hall algebra of monoid representations. Let \( A \) be a commutative monoid as in the previous subsection. Since \( A\text{-mod} \) is not abelian, one cannot expect to construct Hall algebra. The idea of Szczesny [Sz14] is to restrict the class of morphisms so that we have a good family of exact sequences.
Definition 2.2. A morphism $f$ in $A\text{-mod}$ is called normal if $\text{coim}(f) \rightarrow \text{im}(f)$ is an isomorphism. Denote by $A\text{-mod}^n$ the subcategory of $A\text{-mod}$ with the same objects and only normal morphisms.

Now in the category $A\text{-mod}$ we have the notion of exact sequence consisting of normal morphisms, and can construct Hall algebra as in the abelian category (recall §1.2).

Fact 2.3 (Szczesny, [Sz14, §3]). Denote by $\text{Iso}(A\text{-mod}^n)$ the set of isomorphism classes of objects in $A\text{-mod}^n$. Then the $\mathbb{Q}$-vector space
\[
\{ f : \text{Iso}(A\text{-mod}^n) \rightarrow \mathbb{Q} \mid \# \text{supp}(f) < \infty \}
\] is a bialgebra with the multiplication $\ast$ and the comultiplication $\Delta$ defined as
\[
f \ast g([M]) := \sum_{N \subseteq M} f([M/N])g([N]), \quad \Delta(f)([M],[N]) := f([M \otimes N]).
\]

Here we denoted by $[M]$ the class of the object $M$.

As mentioned before, the associativity is proved using Lemma 2.1.

Definition 2.4. We denote this bialgebra by $\text{Hall}(A\text{-mod}^n)$.

2.3. Hall algebra for quasi-exact category. According to [Sz14], the category $A\text{-mod}$ is an example of quasi-exact and belian category in the sense of Deitmar [D06, D11], or an example of proto-exact category in the sense of Dyckerhoff-Kapranov. Let us restate the result in the previous subsection in such a categorical setting as we need in the latter argument.

Definition 2.5. Let $\mathcal{C}$ be a small category.

1. $\mathcal{C}$ is called balanced if every morphism which is both monomorphism and epimorphism is an isomorphism.

2. $\mathcal{C}$ is called pointed if it has an object 0 such that for every object $X$ the sets $\text{Hom}_\mathcal{C}(X,0)$ and $\text{Hom}_\mathcal{C}(0,X)$ of morphisms have exactly one element respectively. Such 0 is unique up to unique isomorphism, and called the zero object. The unique element in $\text{Hom}_\mathcal{C}(X,0)$ and $\text{Hom}_\mathcal{C}(0,X)$ are called the zero morphism.

For a pointed category, one can introduce the notion of kernels and cokernels using the zero morphism.

Definition 2.6 (Deitmar [D11]). A belian category is a balanced pointed category which has finite products, kernels and cokernels, and has the property that every morphism with zero cokernel is an epimorphism.

For a commutative monoid $A$, the category $A\text{-mod}$ is an example belian category. In fact, one can check easily that $A\text{-mod}$ is balanced, and the other axioms are already discussed in §2.1.

Next we turn to quasi-exact categories. Let $\mathcal{C}$ be a balanced pointed category. As usual, we have the notion of short exact sequence
\[
0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0
\]
of morphisms in $\mathcal{C}$. We will call $X \rightarrow Y$ the kernel and $Y \rightarrow Z$ the cokernel of the short exact sequence.

Remark. In [D06, D11] Deitmar called our exact sequences strong exact sequences.

Definition 2.7 (Deitmar [D06]). A quasi-exact category is a balanced pointed category $\mathcal{C}$ together with a class $\mathcal{E}$ of short exact sequences such that

- for any two objects $X,Y$, the natural sequence $0 \rightarrow X \rightarrow X \oplus Y \rightarrow Y \rightarrow 0$ belongs to $\mathcal{E}$,
- kernels in $\mathcal{E}$ is closed under composition and base-change by cokernels in $\mathcal{E}$,
- cokernels in $\mathcal{E}$ is closed under composition and base-change by kernels in $\mathcal{E}$.

In the construction of Szczesny’s Hall algebra, we considered the category $A\text{-mod}^n$ of $A$-modules with normal morphisms. One can check that $A\text{-mod}^n$ together with short exact sequences consisting of normal morphisms is a quasi-exact category.

Now we can restate Szczesny’s result (Fact 2.3).

Definition 2.8. A category $\mathcal{C}$ is called finitary if for any two objects $X$ and $Y$, the set $\text{Hom}_\mathcal{C}(X,Y)$ is finite

Let us denote a kernel $N \hookrightarrow M$ in a quasi-exact category by $N \subset M$.

Proposition 2.9. Let $\mathcal{C}$ be a finitary, belian and quasi-exact category with the class $\mathcal{E}$ of short exact sequences. Then the $\mathbb{Q}$-vector space
\[
\{ f : \text{Iso}(\mathcal{C}) \rightarrow \mathbb{Q} \mid \# \text{supp}(f) < \infty \}
\] is a unital associative algebra with the multiplication
\[
f \ast g([M]) := \sum_{N \subseteq M} f([M/N])g([N]),
\]
with $[M]$ the class of the object $M$. 

The only non-trivial point is the associativity of the multiplication, which is proved by using Lemma 2.1. Let us restate it in the present context. For a quasi-exact category \( \mathcal{C} \) with the class \( \mathcal{E} \) of short exact sequences, we denote a kernel \( X \to Y \) appearing in a sequence of \( \mathcal{E} \) by \( X \hookrightarrow Y \), and denote a cokernel \( Y \to Z \) appearing in a sequence of \( \mathcal{E} \) by \( Y \twoheadrightarrow Z \).

**Lemma 2.10.** Let \( \mathcal{C} \) a belian and quasi-exact category. Suppose that we are given a commutative exact diagram with solid arrows in the next line.

\[
\begin{array}{ccc}
L & \rightarrow & M \\
\downarrow & & \downarrow \\
L' & \rightarrow & N \\
\downarrow & & \downarrow \\
Z & \rightarrow & Z
\end{array}
\]

Then there are dotted morphisms making the whole diagram commutative and exact.

The proof of the lemma can be done by a standard diagram chasing.

Next we turn to coalgebra and bialgebra structures. The coalgebra structure is defined in a usual way.

**Lemma 2.11.** Let \( \mathcal{C} \) a finitary belian quasi-exact category. The vector space (2.2) has a structure of counital coassociative coalgebra with the comultiplication

\[
\Delta(f)([M], [N]) := f([M \oplus N]).
\] (2.4)

Now to construct a bialgebra, we need the hereditary condition on our category \( \mathcal{C} \).

**Definition 2.12.** A belian quasi-exact category \( \mathcal{C} \) is called hereditary if \( \text{Ext}^n(X, Y) = 0 \) for any objects \( X, Y \) and any \( n \in \mathbb{Z}_{\geq 2} \).

Here the higher extension \( \text{Ext}^n \) is defined by the derived functor of \( \text{Hom} \). The theory of derived functors for a belian category is given in [D11, §1.6], and we will not repeat it.

Now the argument [Gr95] (see also [Sc12, §1.5]) on the Hall bialgebra for abelian category can be applied to our setting, and we have

**Theorem 2.13.** Let \( \mathcal{C} \) a belian quasi-exact category which is finitary and hereditary. The vector space (2.2) has a structure of bialgebra with the multiplication (2.3) and the comultiplication (2.4).

### 3. Monoidal schemes, monoidal Tate curve and Hall algebra

In this section we introduce the monoidal Tate curve \( \hat{E} \). The desired category \( \mathfrak{B} \) mentioned in §1.4 will be the constructed as a category of sheaves over \( \hat{E} \). Our monoidal Tate curve is a counter part of the standard Tate curve in the \( \mathbb{F}_1 \)-scheme setting. We will use Deitmar’s theory [D05] of monoid schemes as \( \mathbb{F}_1 \)-scheme theory.

#### 3.1. Monoidal schemes

We follow Deitmar [D05] and use monoidal schemes as the theory of schemes over \( \mathbb{F}_1 \).

For a commutative monoid \( A \), an ideal \( a \) is a subset such that \( aA \subseteq a \). One can define prime ideals and localizations as in the commutative ring case.

Now an affine monoidal scheme \( \text{Spec}^{\text{mon}}(A) \) is the set of prime ideals of \( A \) with the Zariski topology. A monoidal schemes is a topological space \( X \) with a sheaf \( \mathcal{O}^{\text{mon}}_{\text{X}} \) of monoids, locally isomorphic to some \( \text{Spec}^{\text{mon}}(A) \).

Given a monoidal scheme \( X \), we can define \( \mathcal{O}^{\text{mon}}_{\text{X}} \)-modules, (quasi-)coherent \( \mathcal{O}^{\text{mon}}_{\text{X}} \)-modules and locally free \( \mathcal{O}^{\text{mon}}_{\text{X}} \)-modules in a quite similar was as the usual scheme case. We also have the notion of a perfect complex of \( \mathcal{O}^{\text{mon}}_{\text{X}} \)-modules, i.e., a cohomologically bounded complex of coherent \( \mathcal{O}^{\text{mon}}_{\text{X}} \)-modules which is locally quasi-isomorphic to a bounded complex of locally free \( \mathcal{O}^{\text{mon}}_{\text{X}} \)-modules of finite ranks.

Relative notions can also be introduced to monoidal schemes. In particular, we have the notion of flat family of monoidal schemes, i.e., a flat morphism \( X \to S \) of monoidal schemes. Let us explain the definition of flatness: A commutative monoid \( B \) over another \( A \) is flat if the tensor product functor \( B \otimes_A (-) : A-\text{mod}^n \to B-\text{mod}^n \) is exact.

We also have a functor

\[
X \mapsto X_Z
\] (3.1)

mapping a monoidal scheme \( X \) to a scheme \( X_Z \) over \( Z \). It is induced by the functor \( A \mapsto \mathbb{Z}[A] \) from a commutative monoid \( A \) to the monoidal ring \( \mathbb{Z}[A] \) of \( A \).
3.2. Monoidal Tate curve. Now let us recall the Tate curve $\tilde E_{\text{Tate}}$ in the sense of usual scheme, following [G09, §8.4.1] and [LP12, §9.1].

Consider the following toric data.
\[
\begin{align*}
\rho_i &:= \mathbb{Q}_{\geq 0}(i,1) \subset \mathbb{Q}^2 \ (i \in \mathbb{Z}), \quad \rho_i^\vee := \{ x \in \mathbb{Q}^2 \mid \langle x, \rho_i \rangle \subset \mathbb{Q}_{\geq 0} \} = \{(x_1, x_2) \mid ix_1 + x_2 \geq 0\}, \\
\sigma_{i+1/2}^\vee &:= \rho_i^\vee \cap \rho_i^\vee.
\end{align*}
\] (3.2)

For a commutative ring $R$, set $U_{i+1/2} := \text{Spec} R[\sigma_{i+1/2}^\vee \cap \mathbb{Z}^2]$. These affine schemes glue to give a scheme $E$ over $R$ (see [LP12, §9.1.1] for the explicit form of this gluing map). The map $\sigma_{i+1/2}^\vee \to \mathbb{Q}$, $(x,y) \mapsto y$ gives a morphism $E_{\text{Tate}} \to \text{Spec} R[q]$. The Tate curve $\tilde E_{\text{Tate}} \to \text{Spf} R[[q]]$ is a formal thickening of this morphism.

The Tate curve $\tilde E_{\text{Tate}}$ can be considered as a family of elliptic curves over $R$ in the following sense. $Z$ acts on $E_{\text{Tate}}$ in the way that $1 \in Z$ corresponds to the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) acting on $(\mathbb{Q}^2)\vee$. Taking $R = \mathbb{C}$, we can describe this action on the big torus $(\mathbb{C}^*)^2 \subset E_{\text{Tate}}$ as $(z,q) \mapsto (zq,q)$ with $z,q$ corresponding to the basis $(1,0), (0,1)$ of $\mathbb{Q}^2$. If we fix $q \in \mathbb{C} \setminus \{0\}$ and consider the fiber of the map $E_{\text{Tate}} \to \text{Spec} \mathbb{C}[q]$, then the $Z$-action on this fiber is generated by $z \mapsto zq$, and the quotient is identified with $(\mathbb{C}^*)^2 / q^\mathbb{Z}$.

Now we construct the monoidal Tate curve $\tilde E$ defined by replacing rings in the construction of $\tilde E_{\text{Tate}}$ by monoids. So we consider the toric data (3.2) and set $U_{i+1/2}^{\text{mon}} := \text{Spec}^\text{mon}(\sigma_{i+1/2}^\vee \cap \mathbb{Z}^2)$. Gluing and formal thickening give $\tilde E \to \text{Spf}^\text{mon}(\mathbb{C}[q])$, where $\text{Spf}^\text{mon}(\mathbb{C}[q])$ is the (formal) monoidal scheme of $\mathbb{C}[q] := q^\mathbb{Z}$. As mentioned above in the case of $\tilde E_{\text{Tate}}$, the monoidal Tate curve $\tilde E$ has a $\mathbb{Z}$-action.

Recall the functor (3.1) mapping each monoidal scheme to a scheme over $\mathbb{Z}$. By construction we have
\[
\tilde E_{\mathbb{Z}} \simeq \tilde E_{\text{Tate}}
\] (3.3)

For later use we consider the zeta function of $\tilde E$ using Deitmar’s result [D06]. The (local) zeta function of a monoidal scheme $X$ is defined to be $\zeta_X(z) := \exp(\sum_{n \geq 1} \# X_{\mathbb{Q}(\mathbb{F}_p^n)^{\text{gal}}}/n)$ with $p$ a prime number. Then by (3.3) and by the fact that the Tate curve is an elliptic curve over $\mathbb{Z}$, we have
\[
\zeta_{\tilde E/\mathbb{Q}[q]}(z) = \frac{(1 - qz)(1 - z/q)}{(1 - z)^2}.
\] (3.4)

4. Hall algebra of monoidal Tate curve

We now introduce the category $\mathfrak{B}$ outlined in §1.4, and define the Hall algebra associated to it using the general theory in §2.

4.1. Category of sheaves over monoidal scheme. We want to consider the category of $\mathcal{O}^\text{mon}_E$-modules, i.e. sheaves over the monoidal Tate curve. An $\mathcal{O}^\text{mon}_E$-module consists of the modules $M_{i+1/2}$ over the commutative monoids $(\sigma_{i+1/2}^\vee \cap \mathbb{Z}^2)$ and of the gluing data.

As in the case of the category $\mathcal{A}$-mod of monoid representations, the category of $\mathcal{O}^\text{mon}_E$-modules with normal morphisms is a belian quasi-exact category. We can restrict the consideration to coherent $\mathcal{O}^\text{mon}_E$-modules, and denote by $\mathcal{Coh}_E$ the belian quasi-exact category of $\mathcal{O}^\text{mon}_E$-modules with normal morphisms, which is obviously finitary and hereditary.

4.2. Fourier-Mukai transform. Before introducing some $\mathcal{O}^\text{mon}_E$-modules, let us recall the standard facts on the sheaves over elliptic curves in the usual scheme theory. Hereafter the word “a sheaf over a scheme $X$” means a $\mathcal{O}_X$-module, and the word “a vector bundle” means a locally free sheaf.

Recall that for a smooth projective curve $C$ over a field any coherent $\mathcal{O}_C$-module has the Harder-Narasimhan filtration, and the associated factors are semi-stable sheaves. Each semi-stable sheaf has a Jordan-Hölder filtration and the associated factors are stable sheaves. Let us denote by $\mathcal{Coh}_c^{ss}(r,d)$ the category of coherent semi-stable $\mathcal{O}_C$-modules with rank $r$ and degree $d$. Denote also by $\mathcal{St}_c(d)$ the category of skyscraper sheaves on $C$ of length $d$. We have $\mathcal{St}_c(d) = \mathcal{Coh}_c^{ss}(0,d)$.

In the case of an elliptic curve $C = E$, the Fourier-Mukai transform gives the equivalence of categories
\[
\mathcal{Coh}_c^{ss}(r,d) \simeq \mathcal{Coh}_c^{ss}(d,-r) \quad \text{if} \quad d > 0, \quad \mathcal{Coh}_c^{ss}(r,0) \simeq \mathcal{St}_c(r) \quad \text{if} \quad r > 0.
\] (4.1)

The tensor product functor $- \otimes \mathcal{O}_E L$ with a degree one line bundle $L$ gives the equivalence
\[
\mathcal{Coh}_c^{ss}(r,d) \simeq \mathcal{Coh}_c^{ss}(r,d + r).
\] (4.2)

These equivalences gives a complete picture of the category $\mathcal{Coh}_E$ of coherent sheaves over $E$. We refer [BBH, Chap. 3] as a nice account. Let us recall a part of the description of $\mathcal{Coh}_E$. All stable vector bundles are simple semi-homogeneous vector bundles. Here a semi-homogeneous vector bundle is a sheaf of the form $\pi_* L$ with some isogeny $\pi : E' \to E$ and a line bundle $L$ on $E'$. 


4.3. Fourier-Mukai transform in the monoidal setting. Now let us turn to the monoidal Tate curve \( \hat{E} \). The notions of relatively locally free sheaves is well-defined in the monoidal setting. We also have the notion of relative divisors, so that the rank and degree of relative sheaves are well-defined. As in the case of the usual elliptic scheme, we have the relative Jacobian monoidal scheme \( \text{Jac}_{E/\mathbb{Q}(q)} \) parametrizing the degree 0 line bundles over \( E/\mathbb{Q}(q) \) with the universal family \( \mathcal{P} \) over \( E \times \text{Jac}_{E/\mathbb{Q}(q)} \). The sheaf \( \mathcal{P} \) is nothing but the Poincaré bundle. As in the usual case, we have a natural isomorphism \( \text{Jac}_{E/\mathbb{Q}(q)} \simeq \hat{E} \).

Thus we have the Fourier-Mukai transform \( \mathfrak{S}(-) := (\mathbb{R} \pi_1)_* (\pi_2^*(-) \otimes \mathcal{P}) \) with \( \pi_1 \), the projection from \( E \times \text{Jac}_{E/\mathbb{Q}(q)} \) to \( i \)-th factor. Here we used the derived functors in the monoidal scheme setting [D11].

We also denote by \( \mathfrak{T}(-) := L \otimes (-) \) the tensor product functor with the relative line bundle of degree 1 on \( E/\mathbb{Q}(q) \).

The functors \( \mathfrak{S} \) and \( \mathfrak{T} \) are derived equivalences, and those actions are similar as in the usual scheme case (4.1) and (4.2) if we appropriately change the definition of the subcategories \( \text{Coh}_{E/\mathbb{Q}(q)}^\alpha \). In particular, the rank and degree of sheaves, denoted by \( (r, d) \), are changed to \( (d, -r) \) and \( (r, d + r) \) respectively. In other words, the group generated by \( \mathfrak{S} \) and \( \mathfrak{T} \) acts on the lattice \( \mathbb{Z}^2 \) of ranks and degrees by the natural \( \text{SL}(2, \mathbb{Z}) \) action.

4.4. The category \( \mathcal{B} \) and Hall algebra. Let us recall the Fourier-Mukai transform \( \mathfrak{S} \) introduced in the previous subsection. We denote by \( \text{St}_n^*(-) \) the full subcategory of \( \text{Coh}_{E/\mathbb{Q}(q)}^\alpha \) consisting of \( \text{Coh}_{E/\mathbb{Q}(q)}^\alpha \)'s for all \( d \in \mathbb{Z}_{\geq 0} \) and normal morphisms.

Definition 4.1. Define \( \mathcal{B} \) to be the subcategory of \( \mathbb{Q}^\mathfrak{S}_E \)-modules consisting of the image of \( \text{St}_n^*(-) \) under the repetitions of \( \mathfrak{S}^\pm 1 \) and \( \mathfrak{T}^\pm 1 \) with normal morphisms.

One can apply Szczesny’s construction of Hall algebra (Definition 2.4) to \( \mathcal{B} \). Denote by Hall\((\mathcal{B})\) the resulting algebra. It is a bialgebra with Hopf pairing, so we can consider its Drinfeld double, denoted by DHall\((\mathcal{B})\).

Theorem 4.2. As associative algebras over \( \mathbb{Z}[q^\pm 1] \), we have the isomorphism
\[
\text{DHall}(\mathcal{B}) \simeq U_{q, q'}.
\]

One remark is in order. The algebra DHall\((\mathcal{B})\) is defined over \( \mathbb{Q} \). But in the following argument we show that it has an integral basis and can be seen defined over \( \mathbb{Z}[q^\pm 1] \).

Our argument is the same as [BS12], except the point that we use \( \mathfrak{S} \) and \( \mathfrak{T} \) instead of the usual Fourier-Mukai transform and the tensor product functor with line bundle. We will not repeat the detailed discussion, but explain some key steps.

First we consider the grading and commutative subalgebras of DHall\((\mathcal{B})\) arising naturally from the geometric setting. Consider the subalgebra generated by the class of objects in \( \text{St}_n^*(-) \). We denote by \( S(0, d) \) the class of the relative skyscraper sheaf with length \( d \). Recall also that \( \delta_L \) is the characteristic function of \( L \).

Proposition 4.3. The subalgebra Hall\((\text{St}_n^*(-))\) of DHall\((\mathcal{B})\) is isomorphic to the polynomial algebra with infinite variables \( \delta_{S(0, d)} \), \( d \in \mathbb{Z}_{\geq 1} \).

Proof. The exact sequences in \( \text{St}_n^*(-) \) are all split, so that the algebra Hall\((\text{St}_n^*(-))\) is commutative. It is also clear that \( S(0, d) \)'s generate Hall\((\text{St}_n^*(-))\), so the statement holds.

The argument of [BS12, §4.1] or [M95, Chap. III] gives an explicit description of the commutative subalgebra Hall\((\text{St}_n^*(-))\). In fact, it is a bialgebra under the comultiplication (2.4), and it is isomorphic to the classical Hall bialgebra. In the present context, we have

Proposition 4.4. The subalgebra Hall\((\text{St}_n^*(-))\) is isomorphic to the subalgebra of \( U_{q, q'} \) generated by \( w_{(0, d)} \), \( d \in \mathbb{Z}_{\geq 1} \).

By the description of \( U_{q, q'} \), this isomorphism enables us to consider the subalgebra Hall\((\text{St}_n^*(-))\) to be defined over \( \mathbb{Z}[q^\pm 1] \) with basis \( \left(q^{d^2}/2 - q^{-d^2}/2\right)\delta_{S(0, d)} \).

Next, we note the following properties of our Hall algebra.

Proposition 4.5. (1) DHall\((\mathcal{B})\) has a \( \mathbb{Z}^2 \)-grading. We will denote the associated decomposition as DHall\((\mathcal{B})\) = \( \oplus_{(r, d) \in \mathbb{Z}^2} H^{r, d} \).

(2) The group \( \text{SL}(2, \mathbb{Z}) \) acts on DHall\((\mathcal{B})\) as algebra automorphisms, which induces the natural \( \text{SL}(2, \mathbb{Z}) \)-action on the grading \( \mathbb{Z}^2 \).

(3) For a coprime pair \( (r_0, d_0) \in \mathbb{Z}^2 \), the submodule \( \oplus_{n \in \mathbb{Z}} H^{nr_0, nd_0} \) is a commutative algebra isomorphic to the Drinfeld double of Hall\((\text{St}_n^*(-))\).

Proof. (1) The rank and degree of a sheaf give the grading.

(2) This is a consequence of derived equivalences \( \mathfrak{S} \) and \( \mathfrak{T} \).
The functors $\mathcal{S}$ and $\mathcal{T}$ give the standard $\text{SL}(2, \mathbb{Z})$-action on the grading $\mathbb{Z}^2$. Note also that the Drinfeld double $\text{DHall}(\mathfrak{g}_0)$ is the same as $\oplus_{d \in \mathbb{Z}} H^{0,d}$. Using the actions of $\mathcal{S}$ and $\mathcal{T}$, one can map $\oplus_{d \in \mathbb{Z}} H^{0,d}$ to a given $\oplus_{n \in \mathbb{Z}} H^{n_0,n_0}$. Since $\mathcal{S}$ and $\mathcal{T}$ preserve exact triangles, we find that $\oplus_{n \in \mathbb{Z}} H^{n_0,n_0}$ is isomorphic to $\oplus_{d \in \mathbb{Z}} H^{0,d}$. □

The remaining part of the proof of Theorem 4.2 is the construction of the algebraic homomorphism $\varphi : U_{q,q} \to \text{DHall}(\mathfrak{g})$ sending generator $w(r,d)$ with coprime $(r,d) \in \mathbb{Z}^2$ to $\delta S(r,d)$, where $S(r,d)$ is the class of a stable sheaf with rank $r$ and degree $d$. In fact, the correspondence $w(r,d) \mapsto \delta S(r,d)$ determines the vector space homomorphism $\varphi$ uniquely, since each object of $\mathfrak{g}$ has the Harder-Narasimhan filtration and the Jordan-Hölder filtration. Then the argument in [BS12, §5] can be applied to our situation, showing that $\varphi$ is an algebra homomorphism. Thus we have the isomorphism $U_{q,q} \cong \text{DHall}(\mathfrak{g})$.

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