STRICHARTZ ESTIMATES FOR THE VIBRATING PLATE EQUATION

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Abstract. We study the dispersive properties of the linear vibrating plate (LVP) equation. Splitting it into two Schrödinger-type equations we show its close relation with the Schrödinger equation. Then, the homogeneous Sobolev spaces appear to be the natural setting to show Strichartz-type estimates for the LVP equation.

By showing a Kato-Ponce inequality for homogeneous Sobolev spaces we prove the well-posedness of the Cauchy problem for the LVP equation with time-dependent potentials. Finally, we exhibit the sharpness of our results. This is achieved by finding a suitable solution for the stationary homogeneous vibrating plate equation.

1. Introduction

In this paper we consider the Cauchy problem of the vibrating plate equation

\begin{equation}
\begin{cases}
\partial_t^2 u + \Delta^2 u = F(t, x) \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x)
\end{cases}
\end{equation}

where \( t \in \mathbb{R}, \ x \in \mathbb{R}^d, \ u, F : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, \ \Delta = \sum_{j=1}^d \partial^2_{x_j} \) is the classical Laplace operator and \( \Delta^2 u = \Delta(\Delta u) \). This equation is also known under the name of Germain-Lagrange equation by the ones that first discovered the correct differential equation as a model for the vibration of an elastic surface. It is at the basis of the theory of elasticity and has applications in architecture and engineering, see for example [17]. Related works in the framework of variational calculus can be found in the books of Gelfand and Fomin [11], Gould [14] and Weinstock [27]. Recent studies in Sobolev, Gevrey and modulation spaces by means of pseudo-differential operators and time-frequency techniques are contained in [1, 2, 20] and [9]. The 2-dimensional study of the vibrations of a nonlinear elastic plate (Von Kármán equations) was developed in [25]. For related problems see also [18].

In this paper we focus on dispersive properties of the vibrating plate equation. The study of dispersive properties of evolution equations have become of great importance in PDE, with applications to local and global existence for nonlinear analysis.
well-posedness in Sobolev spaces of lower order, scattering theory and many other topics (see, e.g., Tao’s book [24] and the references therein). The matter of fact is provided by the celebrated Strichartz estimates (see Section 3 below), that play the principal role in the whole study.

The free vibrating plate equation

\[ \partial_t^2 u(t, x) + \Delta^2 u(t, x) = 0 \]

can be factorized as the following product

\[ (\partial_t^2 + \Delta^2)u = (i\partial_t + \Delta)(-i\partial_t + \Delta)u \]

which displays the interesting relation with the Schrödinger equation. Namely, the vibrating plate operator \( P = \partial_t^2 + \Delta^2 \) can be recovered by composing two Schrödinger-type operators: \( S_1 = i\partial_t + \Delta \) and \( S_2 = -i\partial_t + \Delta \). This fact suggests to recover Strichartz estimates for the dispersive problem (1) from the well-known ones for the Schrödinger equation, recalled shortly in Section 3.1. Classical references on the subject are provided by [12] and [16], see also [6]. The \( L^p \) environment of the estimates for the Schrödinger equation yields to homogeneous Sobolev spaces as natural setting for the study of (1).

We shall use the Strichartz estimates for the VP equation to prove a well-posedness result for the Cauchy problem (1) with a time-dependent potential \( V(t, x) \), that is

\[
\begin{aligned}
&\partial_t^2 u + \Delta^2 u + V(t, x)u = F(t, x) \\
u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x).
\end{aligned}
\]

Let us first introduce the notion of admissible pairs.

**Definition 1.1.** We say that the exponent pair \((q, r)\) is admissible if

\[
2 \leq q, r \leq \infty, \quad \frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad d \geq 1, \quad (q, r, d) \neq (2, \infty, 2).
\]

We shall prove that the vibrating plate equation with a time-dependent potential (3) is well-posed in \( \dot{H}^s(\mathbb{R}^d) \times \dot{H}^{s-2}(\mathbb{R}^d) \), with \( s \in [2, d) \), for the following class of potentials

**Theorem 1.2.** Let \( d \geq 3 \), \( I \) be either the interval \([0, T]\), \( T > 0 \), or \([0, +\infty)\), and assume \( V(t, x) \) is a real valued potential such that

\[
V(t, x) \in L^\infty_I \dot{W}^{s-2, \beta} \quad \text{with} \quad \frac{2}{\alpha} + \frac{d}{\beta} = s + 2,
\]

for some fixed \( s \in [2, d] \), \( \alpha \in [1, \infty) \) and \( \beta \in (1, \infty) \). If \( u_0 \in \dot{H}^s \), \( u_1 \in \dot{H}^{s-2} \) and \( F \in L^\infty_I \dot{W}^{s-2, \rho} \), for some admissible pair \((\tilde{q}, \tilde{r})\), then the Cauchy problem (3) has a unique solution \( u \in C(I; \dot{H}^s(\mathbb{R}^d)) \cap L^q(I; \dot{W}^{s, r}(\mathbb{R}^d)) \), for all admissible pairs \((q, r)\), such that

\[
\|u\|_{L^q_I \dot{W}^{s, r}} \leq C_V \|u_0\|_{\dot{H}^s} + C_V \|u_1\|_{\dot{H}^{s-2}} + C_V \|F\|_{L^\tilde{q}_I \dot{W}^{s-2, \rho}}.
\]
The admissibility class of the potentials is represented in Figure 1. The corresponding problem for the Schrödinger equation was studied in [10], see Figure 2.

Differently from what has been achieved in the literature so far (see, e.g. [1, 2, 9]), where the study of the Cauchy problem for the VP equation was only local in time, thanks to the Strichartz estimates, we are able to obtain also a global solution to the problem.

The proof of the theorem (given in Section 4.1) requires an Hölder-type inequality for the homogeneous Sobolev spaces. Indeed, in Section 2, we shall use the Kato-Ponce’s inequality for homogeneous Sobolev spaces and the Sobolev embeddings to obtain the Hölder-type inequality

\[ \| f \cdot g \|_{\dot{W}^{s,r}} \lesssim \| f \|_{\dot{W}^{s_1,r_1}} \| g \|_{\dot{W}^{s_2,r_2}}, \]

with

\[ \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{s - s_1 - s_2}{d}, \quad 0 \leq s \leq \min\{s_1, s_2\}, \quad r, r_1, r_2 \in (1, \infty). \]

The classical results on the existence of solutions to some stationary nonlinear equation (the so-called ground states) [3] will play a central role in the construction of a ground state \( v \) for the stationary vibrating plate equation

\[ \Delta^2 v - v + W(x)v = 0, \]

with \( W \) being a potential in the Schwartz class \( \mathcal{S}(\mathbb{R}^d) \). The ground state \( v \) will be used to build up a suitable solution to (3) which shall be employed to prove the sharpness of (4) (see Sections 4.2 and 4.3).

We plan further investigations on this topic, concerning the extension of the results attained to homogeneous Triebel-Lizorkin spaces and the study of Strichartz estimates in Wiener amalgam spaces for the LVP equation, by exploiting the recent results for the Schrödinger one in [8].

**Figure 1.** Admissibility conditions on the potentials for the VP equation.

**Figure 2.** Comparison between Schrödinger and VP equations (\( s = 2 \)).
Further, observe that the Cauchy problem
\[
\begin{aligned}
\frac{\partial^2}{\partial t^2}u + \Delta^2 u + V(x)u &= F(t, x) \\
u(0, x) &= u_0(x), \quad \partial_t u(0, x) = u_1(x)
\end{aligned}
\]
with the time independent potentials \(V(x)\), is not allowed in Theorem 1.2 (since \(\alpha < \infty\)). However, this topic for the Schrödinger propagator has been extensively studied in the literature. For instance, by combining the arguments in [22] with the counterexamples given in [13], it is possible to prove that the decay \(|x|^{-2-\epsilon}\) for \(\epsilon > 0\) and \(|x| >> 1\) is the sharp decay to be required to the potential \(V(x)\), in order to guarantee the validity of Strichartz estimates for the corresponding perturbed Schrödinger propagator \(e^{it(\Delta+V(x))}\).

In view of Theorem 1.2, taking formally \(\alpha = \infty\), it seems that the natural (and eventually sharp) decay to be imposed on \(V(x)\) is \(|x|^{-4-\epsilon}\) in order to guarantee Strichartz estimates (for instance similar to the ones in Theorem 1.2 with \(s = 2\)). This topic will be also developed in the future.

We conclude by fixing some notations. For \(1 \leq p \leq \infty\), let \(p'\) be the conjugate exponent of \(p\) (\(1/p + 1/p' = 1\)). We shall use \(A \lesssim B\) to mean that there exists a constant \(c > 0\) such that \(A \leq cB\) and \(A \asymp B\) means that \(A \lesssim B \lesssim A\). For any subinterval \(I\) of \(\mathbb{R}\) (bounded or unbounded), we define the mixed space-time norms
\[
\|u\|_{L^q_I X} := \left( \int_I \|u(t, \cdot)\|_X dt \right)^{1/q}
\]
with \(X\) being either \(L^r(\mathbb{R}^d)\) or \(\dot{W}^{s,r}(\mathbb{R}^d)\), and, similarly, for the spaces \(C_I X\). When \(I = [0, +\infty)\) we write simply \(L^q X\) in place of \(L^q_I X\) and, similarly, when \(L^q_I\) is replaced by \(C_I\). The space \(S(\mathbb{R}^d)\) denotes the Schwartz class and \(S'(\mathbb{R}^d)\) is its dual (the space of tempered distributions). We write \(xy = x \cdot y\), \(x, y \in \mathbb{R}^d\), for the scalar product on \(\mathbb{R}^d\). The Fourier transform is given by \(\hat{f}(\xi) = \mathcal{F} f(\xi) = \int e^{-it\xi} f(t) dt\).

2. Kato-Ponce’s inequality

In this section we shall present the Kato-Ponce’s inequality for the homogeneous Sobolev spaces. We first recall the definitions of (potential) Sobolev spaces.

2.1. Sobolev spaces. Let \(s \in \mathbb{R}, 1 \leq r \leq \infty\) and \(f \in S'(\mathbb{R}^d)\). The (potential) Sobolev spaces \(W^{s,r}(\mathbb{R}^d)\) and the (potential) homogeneous Sobolev spaces \(\dot{W}^{s,r}(\mathbb{R}^d)\) are defined by
\[
W^{s,r}(\mathbb{R}^d) = \left\{ f \in S'(\mathbb{R}^d) : \|f\|_{W^{s,r}} = \|\langle D \rangle^s f\|_{L^r} < \infty \right\}
\]
\[
\dot{W}^{s,r}(\mathbb{R}^d) = \left\{ f \in S'(\mathbb{R}^d) : \|f\|_{\dot{W}^{s,r}} := \|\langle D \rangle^s f\|_{L^r} < \infty \right\}.
\]
where the fractional differentiation operators $\langle D \rangle^s$ and $|D|^s$ are the Fourier multipliers defined by

\[
\langle D \rangle^s f(\xi) := \langle \xi \rangle^s \hat{f}(\xi) \quad \text{and} \quad |D|^s f(\xi) := |\xi|^s \hat{f}(\xi).
\]

In particular, if $s = 2$ then $\langle D \rangle^2 = I - \Delta$, where $I$ is the identity operator, and $|D|^2 = -\Delta$. The operators $\langle D \rangle^{-s}$ with $s > 0$ and $|D|^{-s}$ with $0 < s < d$ are also known as the Bessel and Riesz potentials of order $s$, respectively (see, e.g., [23]). If $r = 2$ these spaces are also denoted by $H^s(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d)$. For $s \geq 0$, $W^{s,r}$ and its homogeneous counterpart $\dot{W}^{s,r}$ are related as follows

\[
W^{s,r} = L^r \cap \dot{W}^{s,r}, \quad \text{with} \quad \| \cdot \|_{W^{s,r}} \approx \| \cdot \|_{L^r} + \| \cdot \|_{\dot{W}^{s,r}}.
\]

Recall that the homogeneous Sobolev space $\dot{W}^{s,r}$ is a seminormed space and $\| f \|_{\dot{W}^{s,r}} = 0$ if and only if $f$ is a polynomial.

Consider the dilation operator $S_\lambda f(x) = f(\lambda x)$ for $\lambda > 0$, then the dilation properties for homogeneous Sobolev spaces read

\[
\| S_\lambda f \|_{\dot{W}^{s,r}} = \lambda^{s-d/r} \| f \|_{\dot{W}^{s,r}}, \quad s \in \mathbb{R}, \quad 1 \leq r \leq \infty.
\]

Finally, the following homogeneous Sobolev embeddings [4] will be useful in the sequel.

**Lemma 2.1.** Assume $f \in \mathcal{S}(\mathbb{R}^d)$, $s \in \mathbb{R}$ and $1 < r < \infty$, then

\[
\| f \|_{\dot{W}^{s,r}} \lesssim \| f \|_{\dot{W}^{s_1,r_1}}
\]

where $s \leq s_1$ and $1 < r_1 \leq r < \infty$ are such that $s - d/r = s_1 - d/r_1$.

For further properties of these spaces we address the reader to, e.g., [4].

**2.2. Kato-Ponce’s inequality.** The Kato-Ponce’s inequality, established in [21] by combining the original argument in [15] with the general version of the Coifman and Meyer’s result in [7], reads as follows

**Theorem 2.2.** Suppose $f, g \in \mathcal{S}(\mathbb{R}^d)$, $s \geq 0$ and $1 < r < \infty$, then

\[
\| \langle D \rangle^s (f \cdot g) \|_{L^r} \lesssim \| f \|_{L^{r_1}} \| \langle D \rangle^s g \|_{L^{r_2}} + \| \langle D \rangle^s f \|_{L^{r_3}} \| g \|_{L^{r_4}}
\]

with

\[
\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4}, \quad r_2, r_3 \in (1, \infty).
\]

Observe that this implies $r_1, r_4 \in (1, \infty)$. Combining dilation properties of the operator $S_\lambda$ with the Lebesgue’s dominated convergence Theorem, it follows corresponding inequality for the homogeneous Sobolev spaces.

**Theorem 2.3.** Suppose $f, g \in \mathcal{S}(\mathbb{R}^d)$, $s \geq 0$ and $1 < r < \infty$, then

\[
\| |D|^s (f \cdot g) \|_{L^r} \lesssim \| f \|_{L^{r_1}} \| |D|^s g \|_{L^{r_2}} + \| |D|^s f \|_{L^{r_3}} \| g \|_{L^{r_4}}
\]

with the indices’ relations (7).
Proof. The inequality is well-known. However, we detail the proof for the sake of clarity. We apply the Kato-Ponce’s inequality \( (8) \) to the rescaled product \( S_\lambda(f \cdot g) = S_\lambda(f)S_\lambda(g) \)

\[
\| \langle D \rangle^s (S_\lambda(f \cdot g)) \|_{L^r} \lesssim \| S_\lambda f \|_{L^{r_1}} \| S_\lambda g \|_{L^{r_2}} + \| \langle D \rangle^s S_\lambda f \|_{L^{r_3}} \| S_\lambda g \|_{L^{r_4}},
\]

where \( r_2, r_3 \in (1, \infty) \) are such that \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{r_3} + \frac{1}{r_4} \). Combining the commutativity property

\[
\langle D \rangle^s S_\lambda(f)(x) = \lambda^s S_\lambda((\lambda^{-2} - |D|^2)^{s/2} f)(x), \quad \lambda > 0
\]

with \( \| S_\lambda f \|_{L^r} = \lambda^{-d/r} \| f \|_{L^r} \), \( \lambda > 0 \), the equation \((9)\) can be rewritten as

\[
\| (\lambda^{-2} - |D|^2)^{s/2}(f \cdot g) \|_{L^r} \lesssim \| f \|_{L^{r_1}} \| (\lambda^{-2} - |D|^2)^{s/2} g \|_{L^{r_2}} + \| (\lambda^{-2} - |D|^2)^{s/2} f \|_{L^{r_3}} \| g \|_{L^{r_4}}.
\]

We reach the claim if we show that

\[
\lim_{\lambda \to +\infty} \| (\lambda^{-2} - |D|^2)^{s/2} f \|_{L^r} = \| D^{s/2} f \|_{L^r}.
\]

But this fact is simply a consequence of the Lebesgue’s dominated convergence Theorem applied to the functions

\[
f_\lambda := |(\lambda^{-2} - |D|^2)^{s/2} f|^r.
\]

\[
\text{Remark 2.4. In the case } d = 2 \text{ the above inequality } (8) \text{ was already mentioned in [19].}
\]

From this result we derive the following Hölder-type inequality

Corollary 2.5. Suppose \( f, g \in \mathcal{S}(\mathbb{R}^d) \), \( s \geq 0 \) and \( 1 < r < \infty \), then

\[
\| f \cdot g \|_{\dot{W}^{s,r}} \lesssim \| f \|_{\dot{W}^{s_1,r_1}} \| g \|_{\dot{W}^{s_2,r_2}}
\]

with

\[
\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{s - s_1 - s_2}{d}, \quad 0 \leq s \leq \min\{s_1, s_2\}, \quad r, r_1, r_2 \in (1, \infty).
\]

Proof. Using the estimate \((8)\) and the embeddings of Lemma 2.1 we attain the desired estimate. \( \square \)

Remark 2.6. The Kato-Ponce’s inequality in the case \( s = s_1 = s_2 = 0 \) reduces to the classical Hölder’s inequality.

3. Strichartz estimates

In this section, we first recall the well-known Strichartz estimates for the Schrödinger equation. Then, we use them to derive the corresponding ones for the Cauchy problem \((1)\).
3.1. Schrödinger equation. Consider the linear Schrödinger equation
\[ i\partial_t u(t, x) + \Delta u(t, x) = F(t, x), \]
with initial data \( u(0, x) = u_0(x) \). The solution \( u(t, x) \) can be formally written in the integral form
\[ u(t, \cdot) = e^{it\Delta} u_0 + \int_0^t e^{i(t-s)\Delta} F(s) \, ds, \]
where, for every fixed \( t \), the Schrödinger propagator \( e^{it\Delta} \) is a Fourier multiplier with symbol \( e^{-it|\xi|^2}, \, \xi \in \mathbb{R}^d \). We recall that, given a function \( \sigma \) on \( \mathbb{R}^d \) (the so-called symbol of the multiplier), the corresponding Fourier multiplier operator \( H_{\sigma} \) is formally defined by
\[ H_{\sigma} f(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} \sigma(\xi) \hat{f}(\xi) \, d\xi. \]

So, continuity properties for multipliers in suitable spaces yield estimates for the solution to the equation (12). From the previous formula, when \( F(t, x) = 0 \), the following fixed-time estimates can be obtained
\[ \|e^{it\Delta} u_0\|_{L^r} \lesssim |t|^{-d\left(\frac{1}{2} - \frac{1}{r}\right)} \|u_0\|_{L^{r'}}, \quad 2 \leq r \leq \infty. \]

Indeed, if the initial data \( u_0 \) displays a suitable integrability in space, then the evolution will have a power-type decay in time. In the particular case \( r = 2 \), we have the conservation law of energy \( \|e^{it\Delta} u_0\|_{L^2} = \|u_0\|_{L^2} \).

By combining the above dispersive estimates with some duality arguments, one can obtain the celebrated Strichartz estimates \([6, 12, 16]\).

**Theorem 3.1.** Let \( I \subseteq \mathbb{R} \), \((q, r)\) and \((\tilde{q}, \tilde{r})\) be admissible pairs. Then, for all initial data \( u_0 \in L^2(\mathbb{R}^d) \) and forcing term \( F \in L^q_I L^{r'} \), we have the homogeneous Strichartz estimates
\[ \|e^{it\Delta} u_0\|_{L^q_I L^r} \lesssim \|u_0\|_{L^2} \]
and the inhomogenous Strichartz estimates
\[ \left\| \int_0^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{L^q_I L^r} \lesssim \|F\|_{L^q_I L^{r'}}, \]
with constants that do not depend on the length of \( I \).

3.2. Vibrating plate equation. In order to find a solution to the homogeneous VP equation (1), (e.g., with \( F = 0 \)), we compute the Fourier transform of the problem (2), obtaining the following differential equation
\[ \left\{ \begin{aligned} \partial_t^2 \hat{u} + |\xi|^4 \hat{u} &= 0 \\ \hat{u}(0, \xi) &= \hat{u}_0(\xi), \quad \partial_t \hat{u}(0, \xi) = \hat{u}_1(\xi). \end{aligned} \right. \]
It is straightforward to solve it and, taking the inverse Fourier transform of the solution, we can express the solution to \( (2) \) in the form
\[
(17) \quad u(t, \cdot) = K'(t) u_0 + K(t) u_1
\]
where,
\[
(18) \quad K'(t) = \cos(t \Delta), \quad K(t) = \frac{\sin(t \Delta)}{\Delta}.
\]
Here, for every fixed \( t \), the propagators \( K(t), K'(t) \) are Fourier multipliers with symbols \( \cos(t|\xi|^2), \sin(t|\xi|^2)/|\xi|^2, \xi \in \mathbb{R}^d \).

Next, by Duhamel’s formula, the solution to the inhomogeneous equation \( (1) \) can be formally written in the integral form
\[
(19) \quad u(t, \cdot) = \cos(t \Delta) u_0 + \sin(t \Delta) u_1 + \int_0^t \sin((t-s) \Delta) F(s) \, ds.
\]

The simply but efficacious formulas
\[
\cos(t \Delta) = \frac{e^{it \Delta} + e^{-it \Delta}}{2}, \quad \sin(t \Delta) = \frac{e^{it \Delta} - e^{-it \Delta}}{2i \Delta}
\]
show that estimates for \( K' \) follow directly from the ones for the Schrödinger equation, whereas estimates on \( K \) can be obtained by the continuity properties of the propagator \( e^{it \Delta} \).

The commutativity property of the Fourier multipliers \( e^{it \Delta} \) and \( \Delta \), combined with \( (13) \) yields the fixed-time estimates
\[
\| e^{it \Delta} u_0 \|_{W^{s, r}} \lesssim \| u_0 \|_{H^s}, \quad \left\| \frac{e^{it \Delta}}{\Delta} u_1 \right\|_{W^{s, r}} \lesssim |t|^{-d \left( \frac{1}{r} - \frac{1}{2} \right)} \| u_1 \|_{W^{s-2, r}},
\]
with \( 2 \leq r \leq \infty, s \in \mathbb{R} \). Moreover, the sharpness of these estimates follows from the optimality of \( (13) \). Then, using the Strichartz estimates for the Schrödinger equation and the commutativity of Fourier multipliers we obtain the following estimates

**Theorem 3.2.** Let \( d \geq 1 \) and \( s \in \mathbb{R} \). Then, the following homogeneous Strichartz estimates
\[
(20) \quad \| e^{it \Delta} u_0 \|_{L_t^q W^{s, r}} \lesssim \| u_0 \|_{H^s}, \quad \left\| \frac{e^{it \Delta}}{\Delta} u_1 \right\|_{L_t^q W^{s, r}} \lesssim \| u_1 \|_{H^{s-2}}
\]
and inhomogeneous Strichartz estimates
\[
(21) \quad \left\| \int_0^t \frac{e^{i(t-s) \Delta}}{\Delta} F(s) \, ds \right\|_{L_t^\bar{q} W^{s, r}} \lesssim \| F \|_{L_s^\bar{q}' W^{s-2, r'}}.
\]

**Proof.** Estimates \( (20) \) and \( (21) \) follow from the the commutativity property of the Fourier multipliers \( e^{it \Delta} \) and \( \Delta \) and the Strichartz estimates \( (14) \) and \( (15) \), respectively. \[ BOX \]
Observe that the admissible pairs for the VP equation are the admissible Schrödinger couples.

**Corollary 3.3** (Strichartz estimates for the vibrating plate). Let \( d \geq 1 \), \( s \in \mathbb{R} \) and \((q, r), (\tilde{q}, \tilde{r})\) be admissible pairs. If \( u \) is a solution to the Cauchy problem (1), then
\[
\|u\|_{L_t^q W^{s,r}} \lesssim \|u_0\|_{H^s} + \|u_1\|_{\dot{H}^{s-2}} + \|F\|_{L^q_t W^{s-2,r'}}.
\]

4. **Application to vibrating plate equations with time-dependent potentials**

The Strichartz estimates obtained above are a fundamental tool to obtain the well-posedness of the Cauchy problem (3). First, we present the proof of the main theorem stated in the introduction. Secondly, we prove the sharpness of the admissibility conditions for the potentials in (4). The basic idea is to use suitable rescaling arguments for a standing wave, solution to the homogeneous equation
\[
\partial_t^2 u + \Delta^2 u + V(t, x)u = 0.
\]

4.1. **Proof of the Theorem 1.2.**

**Proof.** The proof uses the arguments of [10, Theorem 1.1] (see also [8, Theorem 6.1]), based on the Banach-Caccioppoli contraction theorem. Strichartz estimates represent the key to achieve the contraction.

First, we need to choose the suitable space and the mapping to apply the contraction theorem. Let \( J = [0, \delta] \) be a small interval and, for any admissible pair \((q, r)\), \( s \in [2, d) \), set
\[
Z_{q,r}^s = L^q(J; W^{s,r}(\mathbb{R}^d))
\]
and
\[
Z = C(J; \dot{H}^s(\mathbb{R}^d)) \cap Z_{2,2d/(d-2)}^s \quad \text{with} \quad \|v\|_Z = \max \left\{ \|v\|_{L^q_t \dot{H}^s}, \|v\|_{Z_{2,2d/(d-2)}^s} \right\}.
\]
Notice that, by complex interpolation (see [26, Parag. 5.2.5] or [4, Parag. 6.4]),
\[
[C(J; \dot{H}^s(\mathbb{R}^d)) ; Z_{2,2d/(d-2)}^s]_{\theta} = Z_{q,r}^s \quad \text{with} \quad \frac{1}{q} = \frac{\theta}{2} \quad \text{and} \quad \frac{1}{r} = \frac{1 - \theta}{2} + \frac{\theta(d - 2)}{2d}.
\]
This gives \( 2/q + d/r = d/2 \), i.e., the admissibility condition; so the space \( Z \) is embedded in all admissible spaces \( Z_{q,r}^s \).

Consider now, for any \( v \in Z \), the mapping
\[
\phi(v) = \cos(t\Delta)u_0 + \frac{\sin(t\Delta)}{\Delta}u_1 + \int_0^t \frac{\sin(t\Delta)}{\Delta} [F(s) - V(s)v(s)]ds.
\]
From Corollary 3.3 it is easy to see that the subsequent estimates hold:
\[
\|\phi(v)\|_{Z_{q,r}^s} \leq C_0 \|u_0\|_{\dot{H}^s} + C_0 \|u_1\|_{\dot{H}^{s-2}} + C_0 \|F\|_{Z_{q',r'}^{s-2}} + C_0 \|Vv\|_{Z_{q',r'}^{s-2}}.
\]
for all \((q, r), (\tilde{q}, \tilde{r})\) and \((q_0, r_0)\) admissible. In order to estimate the product \(Vv\) appropriately we shall prove the case \(2 \leq \alpha < \infty\) and \(1 \leq \alpha < 2\) separately.

Starting with \(2 \leq \alpha < \infty\), among such admissible pairs, there is a pair \((q_0, r_0)\) such that
\[
\frac{1}{q_0} = \frac{1}{2} - \frac{1}{\alpha} \quad \text{and} \quad \frac{1}{r_0} = -\frac{1}{\beta} + \frac{1}{2} + \frac{s + 1}{d}.
\]
In fact, from \(\alpha \geq 2\) and (4), it follows \(q_0 \geq 2\) and
\[
\frac{2}{q_0} + \frac{d}{r_0} = 2\left(\frac{1}{2} - \frac{1}{\alpha}\right) + d\left(-\frac{1}{\beta} + \frac{1}{2} + \frac{s + 1}{d}\right) = \frac{d}{2}.
\]
We choose such a pair in (24). Then, we apply Hölder’s inequality and the estimate (10), obtaining
\[
\|\phi(v)\|_{Z^s_{q, r}} \leq C_0\|u_0\|_{\dot{H}^s} + C_0\|u_1\|_{\dot{H}^{s-2}} + C_0\|F\|_{Z^{\alpha-2}_{q, r}} + C_0\|V\|_{Z^{\alpha-2}_{s, \beta}}\|v\|_{Z^s_{2d/(d-2)}}.
\]
By taking \((q, r) = (2, 2d/(d-2))\) and \((\tilde{q}, \tilde{r}) = (\infty, 2)\) one deduces that \(\phi : Z \to Z\) (the fact that \(\phi(u)\) is continuous in \(t\) when valued in \(\dot{H}^s\) follows from a classical limiting argument, see [10, Theorem 1.1, Remark 1.3]). Also, since \(\alpha < \infty\), if \(J\) is small enough, we get \(C_0\|V\|_{L_{\tilde{J}}^\infty \dot{W}_{s-2, \beta}} < 1/2\), hence \(\phi\) is a contraction. The contraction theorem then states the existence and the uniqueness of the solution in \(J\). By iterating this argument a finite number of times one obtains a solution in \([0, T]\) if \(T < \infty\) or in \(\mathbb{R}\) if \(T = \infty\) (again, see [10, Theorem 1.1]).

In the case \(1 \leq \alpha < 2\), there is a pair \((q_0, r_0)\) such that
\[
\frac{1}{q_0} = 1 - \frac{1}{\alpha} \quad \text{and} \quad \frac{1}{r_0} = -\frac{1}{\beta} + \frac{1}{2} + \frac{s}{d}.
\]
In fact, from \(1 \leq \alpha < 2\) and (4), it follows \(q_0 > 2\) and
\[
\frac{2}{q_0} + \frac{d}{r_0} = 2\left(1 - \frac{1}{\alpha}\right) + d\left(-\frac{1}{\beta} + \frac{1}{2} + \frac{s}{d}\right) = \frac{d}{2}.
\]
We choose such a pair in (24), in order to apply Hölder’s inequality and the estimate (10) again. We obtain
\[
\|\phi(v)\|_{Z^s_{q, r}} \leq C_0\|u_0\|_{\dot{H}^s} + C_0\|u_1\|_{\dot{H}^{s-2}} + C_0\|F\|_{Z^{\alpha-2}_{q, r}} + C_0\|V\|_{Z^{\alpha-2}_{s, \beta}}\|v\|_{Z^s_{\infty, 2}}.
\]
Finally, the same arguments as for the previous case yield the desired result. \(\square\)

**Remark 4.1.** Let us make a few comments about the constraints on the indices \(d, s, \alpha, \beta\). They are mainly due to the need of applying the Kato-Ponce’s estimate to the product \(\|Vv\|\) in (24). First, we must exclude the dimension \(d = 2\), since it forces \(s = 2, \alpha = \beta = 1\) and Kato-Ponce’s inequality does not hold. So, we start with the dimension \(d \geq 3\). Similarly, Kato-Ponce does not hold in the cases: (i) \(s < 2\), (ii) \(\beta = 1\). If \(s \geq d\) there are no pairs \((\alpha, \beta)\) \(\in [1, \infty) \times (1, \infty)\) that satisfy (4) and the same holds when \(\beta = \infty\), for all \(\alpha\). Finally, observe that the condition
on $\alpha \in [1, \infty)$, forces $\beta$ to live in the bounded range $\left(\frac{d}{s+2}, \frac{d}{s}\right]$, see Figure 3 and also Figure 1.

In the limit case $s = 2$ condition (4) becomes

$$V(t, x) \in L^\alpha L^\beta \quad \text{with} \quad \frac{2}{\alpha} + \frac{d}{\beta} = 4,$$

for some fixed $\alpha \in [1, \infty)$ and $\beta \in (\frac{d}{4}, \frac{d}{2}]$. In this case the potentials are reduced to the usual Lebesgue spaces in the space variable, and we can compare the results for the VP equation with the corresponding ones for Schrödinger equation, exhibited in [10, Theorem 1.1]. Indeed, the class of admissible potentials for the Schrödinger equation is given by

$$V(t, x) \in L^\alpha L^\beta \quad \text{with} \quad \frac{2}{\alpha} + \frac{d}{\beta} = 2,$$

for some fixed $\alpha \in [1, \infty)$ and $\beta \in (\frac{d}{2}, \frac{d}{2}]$. This means that the effect of composing two Schrödinger-type propagators reduces notably the class of admissible potentials: the exponent $\beta$ goes from the unbounded region $(\frac{d}{2}, \infty]$ to the bounded one $(\frac{d}{4}, \frac{d}{2}]$. Considering the reciprocal $\frac{1}{\beta}$, we see that the class of admissible potentials is shifted by a factor $2/d$, as shown in Figure 2.

**Remark 4.2.** If $I = [0, T]$ is a bounded set, by Hölder’s inequality in time, assumption (4) can be relaxed to

$$V(t, x) \in L^\alpha \dot{W}^{s-2, \beta} \quad \text{with} \quad \frac{2}{\alpha} + \frac{d}{\beta} \leq s + 2.$$  

In this case the exponent $\beta$ is free to stay on the entire unbounded range $(\frac{d}{s+2}, \infty)$.

---

**Figure 3.** The black vertical segment shows the admissibility range for the exponent $\beta$. For every fixed dimension $d \geq 3$, if the regularity $s$ increases the range decreases. On the other hand, for every fixed regularity $s$, if the dimension $d$ grows the range increases as well.
Remark 4.3. Condition (4) is quite natural in view of the following argument: the standard rescaling \( u_\epsilon(t,x) = u(\epsilon^2 t, \epsilon x) \) takes equation (3) into the equation
\[
\partial_t u_\epsilon + \Delta^2 u_\epsilon + V_\epsilon(t,x)u_\epsilon = F(\epsilon^2 t, \epsilon x), \quad V_\epsilon(t,x) = \epsilon^4 V(\epsilon^2 t, \epsilon x)
\]
and we have
\[
\|V_\epsilon\|_{L^\alpha \dot{W}^{s-2,\beta}} = \epsilon^{-\frac{2}{\alpha} - \frac{d}{\beta} + s + 2} \|V\|_{L^\alpha \dot{W}^{s-2,\beta}}
\]
so that the \( L^\alpha \dot{W}^{s-2,\beta} \) norm of \( V_\epsilon \) is independent of \( \epsilon \) precisely when \( \alpha, \beta \) satisfy (4).

In the following two subsections we shall prove that condition (4) is compulsory for the global Strichartz estimate to be true. The most delicate part will be the construction of a standing wave for the equation (23). Hence, we now focus on this topic.

4.2. Existence of a ground state. We shall prove the existence of a solution for the corresponding stationary equation of (23), that is
\[
\Delta^2 v - v + W(x)v = 0,
\]
for some potential \( W(x) \). Even though the one dimensional case is not covered by our results, it is useful to start from it, in order to understand and solve the problem for any dimension. So, for the moment, let \( d = 1 \) and consider the Newton equation
\[
-v'' + v - v^2 = 0.
\]
This classical equation comes out in the examination of solitary waves for the Klein-Gordon and the Schrödinger equations. In this specific example the explicit form of the solution is well-known, precisely
\[
v(x) = \frac{3/2}{\cosh^2(x/2)}, \quad \text{where} \quad \cosh t = \frac{e^t + e^{-t}}{2}.
\]
Notice that \( v(x) > 0 \), for every \( x \in \mathbb{R} \). Now, differentiating two times this equation, one obtains
\[-v^{iv} + v'' - 2(v')^2 - 2vv'' = 0.
\]
The key idea is to replace the second derivative \( v'' \) with its expression in (27), so that
\[v^{iv} - v + 3v^2 - 2v^3 + 2(v')^2 = 0.
\]
Finally, using the explicit form of the solution \( v \) in (28), we can compute \((v')^2/v\) and show the existence of a solution \( v \) for the following fourth order equation
\[
v^{iv} - v + W(x)v = 0 \quad \text{where} \quad W(x) = -\frac{10}{3} v^2 + 5v.
\]
In this particular example the solution is known from the beginning, given by (28), and we determine the potential \( W(x) \) accordingly. Let us underline some properties of this solution we would like to have in any dimension \( d \): i) \( v > 0 \); ii) \( v \) is even: \( v(x) = v(-x) \); iii) \( v \in C^\infty(\mathbb{R}) \); iv) \( v \) and all its derivatives have exponential decay

\[\cosh t = \frac{e^t + e^{-t}}{2}.
\]
at $\pm\infty$ (in particular, $v \in S(\mathbb{R})$). Moreover, we observe that $W(x)$ belongs to the Schwartz class $S(\mathbb{R})$.

Let us come to the $d$-dimensional case ($d \geq 3$). Inspired by this simple example we shall prove the existence of a solution to (26), arguing as follows: i) prove the existence of a solution for a well-known second order PDE, ii) apply the operator $\Delta$ to it and use the original second order PDE to eliminate remaining second order terms, iii) study the regularity and the decay of the solution. The natural generalization from dimension 1 to $d$ would be to substitute the power nonlinearity $v^2$ by $|v|^{p-1}v$, with $p > 1$. But this does not work for any dimension, as we shall see presently. We shall make use of the following classical result by Berestycki and Lions [3].

**Theorem 4.4.** Suppose $d \geq 3$ and $g : \mathbb{R} \to \mathbb{R}$ be an odd continuous function such that $g(0) = 0$. If the function $g$ satisfies the following conditions:

\[
-\infty < \lim\inf_{s \to 0^+} \frac{g(s)}{s} \leq \lim\sup_{s \to 0^+} \frac{g(s)}{s} = -m < 0, \tag{30}
\]

\[
-\infty \leq \lim\sup_{s \to +\infty} \frac{g(s)}{s^l} \leq 0, \quad \text{where} \quad l = \frac{d + 2}{d - 2}, \tag{31}
\]

\[
\text{there exists } \zeta > 0 \text{ such that } G(\zeta) = \int_0^\zeta g(s)ds > 0, \tag{32}
\]

then the problem

\[
- \Delta v = g(v) \text{ in } \mathbb{R}^d, \quad u \in H^1(\mathbb{R}^d), \tag{33}
\]

possesses a non trivial solution $v$ such that

i) $v > 0$ on $\mathbb{R}^d$.

ii) $v$ is spherically symmetric: $v(x) = v(r)$, where $r = |x|$ and $v$ decreases with respect to $r$.

iii) $v \in C^2(\mathbb{R}^d)$.

iv) $v$ together with its derivatives up to order 2 have exponential decay at infinity, i.e.

\[
|D^\alpha v(x)| \leq Ce^{-\delta|x|}, \quad x \in \mathbb{R}^d,
\]

for some $C, \delta > 0$ and for $|\alpha| \leq 2$.

Condition (31) tells that the existence of a ground state might depend on the dimension $d$ involved. Indeed, if one consider the case $g(v) = -v + |v|^{p-1}v$ and $p > 1$, then the problem (33) possesses a solution if and only if $1 < p < \frac{d + 2}{d - 2}$. Since we are interested in finding a solution for every $d \geq 3$, we select a different nonlinearity, precisely the difference $|v|^{p_1-1}v - |v|^{p_2-1}v$, with $p_1 < p_2$ chosen conveniently, as explained in the following result (see also [3, Example 2]).
Theorem 4.5. Let $d \geq 3$. Then there exist a function $v \in \mathcal{S}(\mathbb{R}^d)$ and a potential $W(x) \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\Delta^2 v - v + W(x)v = 0.$$  

(34)

The potential has the following explicit representation in terms of the solution $v$

$$W(x) = -5v^8 + 24v^6 - 33v^4 + 12v^2|\nabla v|^2 + 18|\nabla v|^2.$$  

(35)

Proof. We shall follow the plan above. Consider the nonlinearity $g(v) = -v + 3v^3 - v^5$ (here $p_1 = 3$ and $p_2 = 5$). We look for a solution to

$$-\Delta v + v - 3v^3 + v^5 = 0.$$  

(36)

It’s easy to verify that assumptions (30), (31) and (32) of Theorem 4.4 are satisfied by $g$. Observe that with this particular choice of $g$, condition (31) is satisfied in any dimension $d \geq 3$. Then, we come to the fourth order PDE by applying the Laplacian operator to (36)

$$\Delta^2 v - \Delta v + 18v|\nabla v|^2 + 9v^2\Delta v - 20v^3|\nabla v|^2 - 5v^4\Delta v = 0.$$  

Finally, we replace $\Delta v$ by its expression in (36) and obtain the desired equation (34), with the potential $W(x)$ given by (35). The regularity and the exponential decay of the solution and its derivatives can be proved by a bootstrapping argument starting from (34) and using the properties of the solution $v$ in Theorem 4.4, Items iii) and iv). Thus we have proved the existence of a Schwartz solution $v$ to (34) that decay exponentially. Alternatively, the Schwartz and the exponential decay properties, or, more precisely, the Gelfand-Shilov class $\mathcal{S}^{1,1}(\mathbb{R}^d)$ property of the solution $v$ can be obtained by using [5, Theorem 4.3].

Remark 4.6. Observe that a variational argument like the one developed in [10] is not sufficient to prove Theorem 4.5 for a given potential $W \in \mathcal{C}^\infty_0(\mathbb{R}^d)$. In fact, one would consider (34) as the Euler-Lagrange equation of an associated minimization problem that doesn’t have a minimum by the following consideration. Let $w(x)$ be a smooth compactly supported function such that $w(x_0) > 0$ at least in one point $x_0$. Then, the minimization problem

$$\min_{f \in M} \int_{\mathbb{R}^d} (|\Delta f|^2 - |f|^2) dx \quad \text{on} \quad M = \left\{ f \in H^2 : \int_{\mathbb{R}^d} w(x)|f|^2 dx = 1 \right\}.$$  

admits a minimum, so the problem (34) does. But this problem has no minimum. It is sufficient to take a function $f_0$ with compact support that satisfies the constraint and a function $u_1 \in H^2$ such that $\int_{\mathbb{R}^d} (|\Delta u_1|^2 - |u_1|^2) dx < 0$ and having disjoint support from that of $f_0$. Then the functions $f_t = f_0 + tu_1$, with $t > 0$, belongs to $M$ and the functional $\langle F, f \rangle = \int_{\mathbb{R}^d} (|\Delta f|^2 - |f|^2) dx$, applied to $f = f_t$, goes to $-\infty$ as $t \to +\infty$. 


4.3. Sharpness of the global Strichartz estimates. Now we have all the instruments to prove the sharpness of Theorem 1.2. We shall detail the global case, the local one can be obtained by using similar arguments, so we omit the proof.

**Theorem 4.7.** Let \( d \geq 3 \) and \( I = [0, \infty) \). For every \( s \in [2, d) \), \( \alpha \in [1, \infty) \) and \( \beta \in (1, \infty) \), such that
\[
\frac{2}{\alpha} + \frac{d}{\beta} \neq s + 2
\]
there exist a potential \( V(t, x) \in L^\alpha \dot{W}^{s-2, \beta} \) and a sequence of solutions \( u_k(t, x) \in C_t^1 H^s \) to the equation (23) such that
\[
\lim_{k \to +\infty} \|u_k\|_{L^q \dot{W}^{s,r}} = +\infty, \quad \text{for every admissible pair } (q, r) \neq (\infty, 2).
\]

**Proof.** The pattern is provided by [10, Theorem 1.3]. Of course, working with a higher order equation and with Sobolev spaces instead of simply Lebesgue spaces, some suitable modifications are needed. We detail the case \( \frac{2}{\alpha} + \frac{d}{\beta} < s + 2 \), the case \( \frac{2}{\alpha} + \frac{d}{\beta} > s + 2 \) is obtained similarly.

Considering the functions \( v \) and \( W \) of Theorem 4.5 we define the standing wave \( u(t, x) = e^{it} v(x) \in C([0, \infty); L^2(\mathbb{R}^d)) \) and observe that, by (34), this function solves the equation (23) with
\[
V(t, x) = W(x) \in L^\infty([0, \infty); \dot{W}^{s-2, \beta}),
\]
for all \( s \in [2, \infty) \) and \( \beta \in (1, \infty) \). From Remark 4.3 we see that the rescaled function
\[
(37) \quad u_\epsilon(t, x) = e^{it\epsilon} v(\epsilon x)
\]
solves globally
\[
\partial_t^2 u_\epsilon + \Delta^2 u_\epsilon + W_\epsilon(x)u_\epsilon = 0, \quad \text{where } W_\epsilon(x) = e^t W(\epsilon x).
\]
Choose two monotone sequences of positive real numbers
\[
0 = T_0 < T_1 < \cdots < T_k \to +\infty, \quad 0 < \epsilon_k \to 0, \quad k \in \mathbb{N}
\]
and define a potential \( V(t, x) \) on \([0, \infty) \times \mathbb{R}^d\) as follows
\[
V(t, x) = W_\epsilon_k(x) \quad \text{for } t \in [T_k, T_{k+1}), \quad k \in \mathbb{N}.
\]
With this choice, \( u_{\epsilon_k} \) solves the equation
\[
(38) \quad \partial_t^2 u + \Delta^2 u + V(t, x)u = 0
\]
on \([T_k, T_{k+1})\). Select now \( \alpha \) and \( \beta \) such that
\[
(39) \quad \frac{2}{\alpha} + \frac{d}{\beta} < s + 2
\]
and assume we can choose the parameters $T_k$ and $\epsilon_k$ in such a way that
\[(40) \quad \| V \|_{L^\alpha W^{s-2,\beta}} \lesssim \| W \|_{W^{s-2,\beta}} \sum_{k=0}^\infty \epsilon_k^{s+2-\frac{d}{q}} (T_{k+1} - T_k)^{1/\alpha} < +\infty.
\]

Then $V \in L^\alpha([0, \infty), W^{s-2,\beta}(\mathbb{R}^d))$. On the other hand, Theorem 1.2 allows to extend uniquely the solutions $u_{\epsilon_k}$ to a global one of \[(38)\] in $C([0, \infty); H^s(\mathbb{R}^d))$ which we shall denote by $u_k(t, x)$. Observe that we have
\[
\| u_k(t, \cdot) \|_{H^s} \equiv \text{const.} \equiv \| u_{\epsilon_k}(T_k, \cdot) \|_{H^s} = \| u \|_{H^s},
\]
where in the last equality we have used the dilation properties for homogeneous Sobolev spaces. Now, the Strichartz estimates \[(22)\] are violated when
\[(41) \quad \frac{\| u_k(0) \|_{L^\alpha W^{s,r}}}{\| u_k(0) \|_{H^s} + \| \partial_t u_k(0) \|_{H^{s-2}}} \text{ is unbounded.}
\]

The numerator can be estimated from below by
\[
\| u_k \|_{L^\alpha W^{s,r}} \geq \epsilon_k^{s-d/r} \| v \|_{W^{s,r}} (T_{k+1} - T_k)^{1/q}
\]
while the denominator is estimated from above by
\[
\| u_k(t, \cdot) \|_{H^s} + \| \partial_t u_k(t, \cdot) \|_{H^{s-2}} = \| v(\epsilon_k \cdot) \|_{H^s} + \| i \epsilon_k^2 v(\epsilon_k \cdot) \|_{H^{s-2}} = \epsilon_k^{s-d/2} (\| v \|_{H^s} + \| v \|_{H^{s-2}}).
\]

Thus, the claim \[(41)\] holds provided that $T_k$, $\epsilon_k$ satisfy the condition
\[(42) \quad \frac{\| u_k(0) \|_{L^\alpha W^{s,r}}}{\| u_k(0) \|_{H^s} + \| \partial_t u_k(0) \|_{H^{s-2}}} \geq \epsilon_k^{s-d/r} \| v \|_{W^{s,r}} (T_{k+1} - T_k)^{1/q} \leq \epsilon_k^{s-d/2} (\| v \|_{H^s} + \| v \|_{H^{s-2}})
\]

Finally, by taking
\[T_0 = 0, \quad T_{k+1} = T_k + k^a \quad \text{and} \quad \epsilon_0 = 1, \quad \epsilon_k = k^{-b/2}, \quad k \in \mathbb{N}
\]
for some $a, b > 0$, the conditions \[(10)\] and \[(12)\] reduce to
\[
\frac{a}{\alpha} + \frac{b d}{2 \beta} < \frac{s b}{2} + b - 1, \quad \frac{a}{q} + \frac{b d}{2 r} > \frac{b d}{4}.
\]

Since $(q, r)$ is admissible, the second condition becomes $a > b$, hence, using $a > b$ and the assumptions \[(39)\], the first one becomes $\frac{1}{\alpha} + \frac{d}{2 \beta} < \frac{s}{2} + 1$. Thus, the conditions \[(40)\] and \[(12)\] are satisfied for
\[
a > b > 2 \left[ 2 - \left( \frac{2}{\alpha} + \frac{d}{\beta} - s \right) \right]^{-1}.
\]

Note that the term in the square bracket is never zero by the particular assumption on $\alpha, \beta$, see \[(39)\].
ACKNOWLEDGEMENTS

The authors would like to thank Professors Piero D’Ancona, Fabio Nicola, Luigi Rodino and Paolo Tilli for fruitful conversation and comments. We are grateful to the anonymous referee for his valuable comments.

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