Phase transition in generalized inhomogeneous 'cubic' systems

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Phase transition in generalized inhomogeneous ‘cubic’ systems

B R Gadjiev
International University for Nature, Society and Man,
19 Universitetskaya street, Dubna, 141980, Russia

E-mail: gadjiev@uni-dubna.ru

Abstract. In this paper we investigate peculiarities of phase transition in inhomogeneous systems. We consider a case of ‘cubic’ systems with anisotropy invariants in which the distribution of defects generates a small-world property. We define a fractional equation of motion for the order parameter for the systems with a small world property. Linearization of the equation of motion for the order parameter made it possible to define a non-linear dispersion law. A renormalization group analysis of phase transitions in the generalized inhomogeneous “cubic” systems is presented. We discuss the dependence of critical behavior on the nonextensivity parameter of the system.

1. Introduction
During the theoretical description of a spontaneous symmetry breaking in ideal crystals in terms of Landau inverse problem it is supposed that the symmetry groups of high-symmetry $H_0$ and low-symmetry $H_1$ phases are known [1, 2]. The group $H_1$ is a subgroup of the $H_0$, i.e. $H_1 \subset H_0$. If the groups $H_0$ and $H_1$ are known it is possible to determine the finite matrix group $\Xi$, which describes the transformation properties of the major order parameter $\eta$. In this case, in accordance with the symmetry arguments, the expansion of the free energy functional of a crystal depends on the integrity basis of the $\Xi$ matrix group.

It is well known that the major order parameter is a certain linear combination of the system elements’ coordinates. Thus, it could be expected that the equation of the motion of the order parameter is rather complex in the crystals with variable composition. Even if small deviations of the order parameter from its equilibrium value are taken into account, the equation of the motion of the order parameter could include time derivatives of an arbitrarily high order [3, 4].

It is normal for real crystals to have defects and impurities. That is why the issue of a weak disorder influence on critical behaviour of a system appears to be quite natural. The distribution of defects in crystals may be different. For example, a homogeneous random distribution of the defects in a structure may completely destroy large-scale fluctuations. As a result, either singularities of thermodynamic functions disappear or a shift of the critical temperature occurs [5]. However, the conditions of the crystal growth may lead to a inhomogeneous distribution of defects. In this case a new critical regime characterized by new universal critical indices may arise due to the impurities.

The analysis of the weak disorder effect on the system critical behaviour in the frame of the method of renormalization group and $\varepsilon$- expansion is given in [5]. It was shown that the critical indices of the
system do not change (i.e. the universality class remains unchanged) [5]. In the same paper by using a replica symmetry broken method it was shown that in the case of a weak quenched disorder the universality class of the system may change.

In the present paper we study the effect of the weak disorder on the critical behaviour of the system with small-world properties [6] due to the distribution of defects. We study a fractional generalization of the dynamic equation for the major order parameter and obtain a nonlinear dispersion law. First we define a distribution function of the order parameter in the system with a small-world property. Then we introduce a fractional generalization for the equation of motion for the order parameter in such systems and define a non-linear dispersion law. Application of the renormalization group method allows us to show that a new critical regime may appear in the systems with a small-world property.

2. Structure with a small-world property

Usually in the continuous description of the defect influence on the phase transitions in a structure in terms of Ginzburg-Landau Hamiltonian we suppose that the presence of impurities may lead either to the space fluctuations of effective local temperature of the transition or to the space fluctuations inside a crystal field. For defining a distribution function for the order parameter we introduce the notation

$$f(\eta) = -\frac{\delta C}{\delta \eta}.$$ Then the equation of motion for the order parameter $$\eta$$, with a fluctuation of the local temperature and local field may be given in the form

$$\frac{\partial \eta}{\partial t} = f(\eta) + u(\eta)\xi(t) + \zeta(t),$$

where $$\eta(t)$$ is a stochastic variable, $$u(\eta)$$ is an arbitrary function and $$\xi(t)$$ and $$\zeta(t)$$ are uncorrelated noises with a mean value equal to zero

$$\langle \xi(t)\xi(t') \rangle = 2M\delta(t-t'),$$

$$\langle \xi(t)\zeta(t') \rangle = 2A\delta(t-t'),$$

where $$M > 0$$ and $$A > 0$$ are amplitudes of the multiplicative and additive noises, respectively [7].

Using Kramers-Moyal’s expansion it is easy to show that Fokker-Plank equation for the probability density $$p(\eta,t)$$, connected to the equation (1) in Stratonovich calculus has the form

$$\frac{\partial p(\eta,t)}{\partial t} = -\frac{\partial (f(\eta)p(\eta,t))}{\partial \eta} + M\frac{\partial}{\partial \eta} \left( u(\eta)\frac{\partial}{\partial \eta} (u(\eta)p(\eta,t)) \right) + A\frac{\partial^2 p(\eta,t)}{\partial \eta^2}$$

(4)

Suppose that $$f(\eta) = -u(\eta)\frac{\partial u(\eta)}{\partial \eta}$$. In this case it is easy to show that the stationary solution of this equation takes the form

$$p_{st}(\eta) = p_0 A^{\frac{\tau+M}{2M}} \left( 1 + \frac{M}{A} u^2(\eta) \right)^{-\frac{\tau+M}{2M}},$$

(5)

and, consequently,

$$p_{st}(\eta) = \frac{1}{Z} \left( 1 - (1-q)u^2(\eta) \right)^{-\frac{1}{1-q}},$$

(6)

where $$q = \frac{\tau+3M}{\tau+M}$$ and $$\frac{M}{A} = q - 1.$$
It is clear that \( u^2(\eta) = -\frac{2}{\tau}F[\eta] \). Thus, the multiplicative noise effect leads to the nonextensivity system [8].

Note that by definition \( q = \frac{3M + \tau}{M + \tau} \), and in the case of absence of the multiplicative noise \( M = 0 \) we obtain \( q = 1 \). In this case the stationary distribution \( p_\infty(\eta) \) is Boltzmann-Gibbs distribution and consequently the system nonextensivity is induced by multiplicative noises. Besides, if \( q \neq 1 \) for large \( u^2(\eta) \) we obtain a power law distribution. If \( q \neq 1 \) and \( u^2(\eta) \) are bounded, we have a density of distribution of \( q \)-type which is intermediate between a power and an exponential distribution. It should be stressed that the presence of a multiplicative noise means that some positions in the process of crystal growth are more preferable for the location of impurities. The saturation effect of the chemical bonds leads to the fact that actually we are dealing with a density distribution of \( q \)-type.

It is necessary to emphasize, that in [9] in the frame of a principle maximum entropy it is shown that the distribution (6) describes the structures with a small-world property with a satisfactory accuracy.

3. Derivation equation of motion for the order parameter in compounds with a small-world property

A free energy functional \( F \) is invariant concerning the structure symmetry group \( H_0 \) and consequently depends on the integrity basis of the irreducible representation \( \Xi \) and the corresponding invariant combinations of spatial and time derivatives of the order parameter.

Taking into account the presence of defects in a structure we define a free energy functional as

\[
F[\eta] = F_0[\eta] + F_1[\eta],
\]

where

\[
F_0[\eta] = \int_R \int_R dr'dt' \left\{ \frac{1}{2} \frac{\partial}{\partial t} \eta(r,t) \cdot g_0(r,t,r',t') \cdot \frac{\partial}{\partial t'} \eta(r',t') + \frac{1}{2} \frac{\partial}{\partial r} \eta(r,t) \cdot g_1(r,t,r',t') \cdot \frac{\partial}{\partial r'} \eta(r',t') \right\}
\]

and

\[
F_1[\eta] = -\int_R \int_R \int_R \int_R dr'dt'dt'' V(\eta(r,t),u(r',t'))
\]

For the sake of simplicity, we consider a one-dimensional case. Here \( r \) is a coordinate and \( t \) is time and functions \( g_0(r,t;r',t') \) and \( g_1(r,t;r',t') \) describe the effect of small-world properties on the critical property of the system. The integration is carried out over a region \( R \) in the two dimensional space \( R^2 \) to which \( (r,t) \) belongs. Generalization of the obtained in this chapter results for high dimensional space is evident. The equation of motion for the field \( \eta(r,t) \) is derived by using Gateaux derivative of the functional \( F[\eta(r,t)] \), which is defined as

\[
\delta F[\eta,h] = \left[ \frac{d}{d\epsilon} F[\eta + \epsilon h] \right]_{\epsilon=0} = \lim_{\epsilon \to 0} \frac{F[\eta + \epsilon h] - F[\eta]}{\epsilon},
\]

where \( h(r) = \delta \eta(r) \) is a smooth integrable function. The functional \( F[\eta(r,t)] \) for \( \eta + \epsilon h(r,t) \) is
\[
F[\eta + \epsilon h] = \int dr \int dt \int dr' \int dt' \left[ \frac{1}{2} \frac{\partial}{\partial r}(\eta(r, t) + \epsilon h(r, t)) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial r'}(\eta(r', t') + \epsilon h(r', t')) \right] + \\
\int dr \int dt \int dr' \int dt' \left[ \frac{1}{2} \frac{\partial}{\partial t}(\eta(r, t) + \epsilon h(r, t)) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial t'}(\eta(r', t') + \epsilon h(r', t')) \right] \\
\int dr \int dt \int dr' \int dt' \int V'(\eta(r, t) + \epsilon h(r, t), \eta(r', t') + \epsilon h(r', t')) \\
\int dr \int dt \int dr' \int dt' \int dr'' \int dt'' \left[ \frac{1}{2} \frac{\partial}{\partial t}(\eta(r, t) + \epsilon h(r, t)) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial t''} h(r', t'') \right] + \\
\epsilon \int dr \int dt \int dr' \int dt' \int V'[\eta(r, t), \eta(r', t')] + \epsilon \int dr \int dt \int dr' \int dt' \int V'[\eta(r, t), \eta(r', t')] + \\
\epsilon \int dr \int dt \int dr' \int dt' \left[ \frac{1}{2} \frac{\partial}{\partial t} h(r, t) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial t'} \eta(r', t') + \frac{1}{2} \frac{\partial}{\partial t} h(r, t) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial t'} \eta(r', t') \right] \\
\epsilon \int dr \int dt \int dr' \int dt' \left[ \frac{1}{2} \frac{\partial}{\partial r} \eta(r, t) \cdot g_i(r, t, r', t') \cdot \frac{\partial}{\partial r'} h(r', t') + \frac{1}{2} \frac{\partial}{\partial r} \eta(r, t) \cdot g_i(r, t, r', t') \cdot \frac{\partial}{\partial r'} h(r', t') \right] \\
\epsilon \int dr \int dt \int dr' \int dt' \left[ - \frac{\partial V'(\eta(r, t), \eta(r', t'))}{\partial \eta(r, t)} h(r, t) - \frac{\partial V'(\eta(r, t), \eta(r', t'))}{\partial \eta(r', t')}, h(r', t') \right].
\]

This expression up to the order \( \epsilon \) has the form

\[
F[\eta + \epsilon h] = \int dr \int dt \int dr' \int dt' \left[ \frac{1}{2} \frac{\partial}{\partial r} \eta(r, t) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial r'} \eta(r', t') \right] + \\
\int dt \int dr \int dr' \int dt' \left[ \frac{1}{2} \frac{\partial}{\partial t} \eta(r, t) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial t'} h(r', t') \right] + \\
\epsilon \int dr \int dt \int dr' \int dt' \int dr'' \int dt'' \left[ \frac{1}{2} \frac{\partial}{\partial t} h(r, t) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial t''} h(r', t'') \right] + \\
\epsilon \int dr \int dt \int dr' \int dt' \int V'[\eta(r, t), \eta(r', t')] + \epsilon \int dr \int dt \int dr' \int dt' \int V'[\eta(r, t), \eta(r', t')] + \\
\epsilon \int dr \int dt \int dr' \int dt' \left[ \frac{1}{2} \frac{\partial}{\partial t} h(r, t) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial t'} \eta(r', t') + \frac{1}{2} \frac{\partial}{\partial t} h(r, t) \cdot g_o(r, t, r', t') \cdot \frac{\partial}{\partial t'} \eta(r', t') \right] \\
\epsilon \int dr \int dt \int dr' \int dt' \left[ \frac{1}{2} \frac{\partial}{\partial r} \eta(r, t) \cdot g_i(r, t, r', t') \cdot \frac{\partial}{\partial r'} h(r', t') + \frac{1}{2} \frac{\partial}{\partial r} \eta(r, t) \cdot g_i(r, t, r', t') \cdot \frac{\partial}{\partial r'} h(r', t') \right] \\
\epsilon \int dr \int dt \int dr' \int dt' \left[ - \frac{\partial V'(\eta(r, t), \eta(r', t'))}{\partial \eta(r, t)} h(r, t) - \frac{\partial V'(\eta(r, t), \eta(r', t'))}{\partial \eta(r', t')}, h(r', t') \right].
\]

It is convenient to introduce the functions:

\[
K_0(r, t, r', t') = \frac{1}{2} (g_o(r, t, r', t') + g_o(r', t', r, t)), \\
K_1(r, t, r', t') = \frac{1}{2} (g_i(r, t, r', t') + g_i(r', t', r, t)).
\]

The dynamical equation for the order parameter follows from the stationary principle \( \partial F[\eta, h] = 0 \), for any \( h(r, t) \). For the symmetric function \( U(\eta(r, t), \eta(r', t')) = U(r, t) \delta(r-r') \delta(t-t') \) the equation of motion for the order parameter has the form

\[
\int dt' \int dr' \frac{\partial}{\partial t} K_0(r, t, r', t') \cdot \frac{\partial}{\partial r'} \eta(r', t') + \int dt' \int dr' \frac{\partial}{\partial r} K_1(r, t, r', t') \cdot \frac{\partial}{\partial r'} \eta(r', t') + \frac{\partial U(\eta(r, t))}{\partial \eta(r, t)} = 0 .
\]

We suppose that \( K_0(r, t, r', t') = \delta(r-r') \tilde{K}_0(t, t') \), \( K_1(r, t, r', t') = \delta(t-t') \tilde{K}_1(r', r') \).

Introducing the notations

\[
Z_t = \int dt' \frac{\partial}{\partial t} \tilde{K}_0(t, t') \cdot \frac{\partial}{\partial t} \eta(t, t') = Z_t(r, t), \\
Z_r = \int dr' \frac{\partial}{\partial r} \tilde{K}_1(r, r') \cdot \frac{\partial}{\partial r} \eta(r, t') = Z_r(r, t),
\]

we can write the equation of motion for the order parameter \( \eta(r, t) \) as

\[
Z_t + Z_r + \frac{\partial U(\eta(r, t))}{\partial \eta(r, t)} = 0 .
\]
It is suitable to represent the equation of motion for the order parameter in the form
\[ \int_0^1 G(t-t') \frac{\partial \eta(r,t')}{\partial t'} dt' = -\int_0^1 dr' V(r-r') \frac{\partial^2 \eta(r-r')}{\partial r'^2} dr' - \alpha \eta(r',t) - \beta \eta^3(r',t) = 0. \quad (17) \]

If a structure possesses a small-world property it is possible to assume that the functions \( G(t-t') \) and \( V(r-r') \) are Tsallis distributions and then for the large values of arguments it is possible to write
\[ G(t-t') = \frac{1}{\Gamma(1-\nu)} \left( t-t' \right)^{-\nu}, \quad (18) \]
and
\[ V(r-r') = \frac{1}{\cos \left( \frac{\pi \nu}{2} \right)} \frac{1}{\Gamma(1-\nu)} \left( r-r' \right)^{-\nu}, \quad (19) \]
where \( 0 < \nu < 1 \) and \( 1 < \epsilon < 2 \). After substituting (18) and (19) into the equation (17) we obtain
\[ \frac{\partial \epsilon \eta(r,t)}{\partial r^\epsilon} = -\kappa \frac{\partial \eta(r,t)}{\partial r^\epsilon} - \alpha \eta(r,t) - \beta \eta^3(r,t), \quad (20) \]
where for \( \frac{\partial}{\partial r^\epsilon} \) we used Caputo definition of the fractional derivative [9]. Using Fourier transformation we obtain the nonlinear dispersion law
\[ (\iota \omega)^\epsilon = \left( k^2 \right)^\epsilon + \alpha. \quad (21) \]

Hence, if a structure has a small-world property then the system exhibits a nonlinear dispersion law.

**4. Renormalization group theory of the phase transition** \( P4_2/mnm \Rightarrow P2_1/c \)

Consider the phase transition from high-symmetry phase with a space group symmetry \( H_0 \equiv P4_2/mnm \) to low symmetric phase with a space group symmetry \( H_1 \equiv P2_1/c \). The Hamiltonian of Ginzburg – Landau \( H \) depends on the integrity basis of invariants of the matrix group \( \Xi \) (the elements of \( \Xi \) are given in [10]). The thermodynamic theory of Landau type describing this phase transition is constructed in [10]. The corresponding irreducible representation \( \Xi \) of the space group \( H_0 \equiv P4_2/mnm \) of high-symmetry phase has the fourth order invariants in the following form
\[ \psi_1 = \eta_1^4 + \eta_2^4 + \eta_3^4 + \eta_4^4, \quad \psi_2 = \eta_1^2 \eta_2^2 + \eta_3^2 \eta_4^2, \]
\[ \psi_3 = \eta_1^2 \eta_3^2 + \eta_2^2 \eta_4^2, \quad \psi_4 = \eta_1^2 \eta_4^2 + \eta_2^2 \eta_3^2. \quad (22) \]

The renormalization group calculations start with the classical Hamiltonian \( H \), which is a functional of the “spins” \( \eta_i \), where \( i = 1,2,3,4 \).

After using Fourier transformation the effective Hamiltonian of the system is represented by the expression
\[
H = -\frac{1}{2} \sum_{i=1}^{d} \left[ r + q^\gamma \right] \eta_i(\tilde{q}) \mu_i(-\tilde{q}) d^d q
\]

where we are used an arbitrariness in the choice of invariants and here \( \omega(q) = \gamma + q^\gamma \) is taken into account.

The effective Hamiltonian contains a number of anisotropic invariants and that is a distinctive peculiarity of ‘cubic’ systems. Therefore the derivation of the renormalization group equations is much more complicated. Application of the symmetry properties of the renormalization group transformation greatly simplifies a derivation of renormalization group equations. The relationship between the properties of transformations of the renormalization group and the symmetry of the Hamiltonian is investigated in [12]. A detailed study of this problem and a specific algorithm necessary for calculations are described in [13]. In [13] the \( (m+1) \)-dimensional parameter space is introduced:

\[
\Pi = \left\{ (u,u_0,u_1,\ldots,u_m) \mid \lim_{n \to +\infty} \sum_{j=0}^{m} u_j I_j = +\infty \right\},
\]

where \( R \) is a real line and we denote by \( u \) a point \( (u_0,u_1,\ldots,u_m) \). The definition (24) provides a normalization condition of the distribution function \( \exp[H(\eta)] \).

In the space \( \Pi \) a nonlinear renormalization group transformation is defined as

\[
R : u \to \bar{u} = Ru,
\]

Since the transformation \( R \) forms a semigroup, we can also write

\[
R_s R_t = R_{s+t}, \text{ for all } s \text{ and } t \text{ from } [1, \infty).
\]

In the fixed-point the equation (25) takes the form

\[
R u^* = u^*.
\]

In [13] it was shown that every diffeomorphism \( T \) in \( \Pi \) can generate the transformation \( R^T \) equivalent to the transformation \( R \)

\[
R^T = T^{-1} R T.
\]

From the equation (28) follows that the symmetry properties of the renormalization group transformation \( R \) are defined by the operations that commute with \( T \). These \( T \)-elements form a group which is denoted by \( G_T \):
Knowledge of the fourth-order invariants contained in the Hamiltonian, allows to determine the group $G_T$. Indeed, consider the transformation

$$\eta_i \Rightarrow V\eta_i,$$  

(30)

where $V$ is the orthogonal matrix $\mathbb{I} \times \mathbb{I} : V \in O(l)$, where $O(l)$ is a group of rotations in the $l$ dimensional space. In general under the action of $V$ other forms of the fourth order invariants are generated. The requirement of absence of these additional invariants leads to the equations for $u$. In the most general form $V(\theta_1,\ldots) \in O(l)$, where $(\theta_1,\ldots)$ are parameters. In the general case

$$V(\theta_1,\ldots)\Pi \subseteq \Pi$$  

(31)

defines a system of equations for the parameters $(\theta_1,\ldots)$. Therefore the solution of the equation (31) defines the group

$$G_g = \{V; V \in O(l), V \subseteq \Pi\}.$$  

(32)

Further, it is easy to prove [13], that $G_T$ is a linear representation of the group $G_g$, which commutes with the $R$ transformation. Thus, we have

$$RTu = TRu \quad \text{for any} \; T \in G_T.$$  

(33)

Consequently, $Ru$ must be a covariant with respect to $G_T$. To introduce an equation of the renormalization group we can use the following theorem:

Let $\Gamma_j$ be an irreducible representation (other than identity) of a point group, and the functions $\psi^{ja}_\mu(\vec{r})$ ($\mu = 1,2,\ldots,l_j$ and $a$ is fixed) are the basis functions of $\Gamma_j$. Then for the arbitrary function $\psi^{ja}_\mu(\vec{r})$ we can write the expansion

$$\psi^{ja}_\mu(\vec{r}) = \sum_{a=1}^{l_j} c^{ja}_\mu(\vec{r})\psi^{ja}_\mu(\vec{r}),$$  

(34)

where $\mathcal{H}(\nu)$ is an integrity basis of the group $G_T$, $c_{\nu}(u)$ is a linearly independent covariant function of $G_T$, $f_\nu$ are arbitrary functions[13]. Note that if $G_T$ is a point group, the integrity basis forms a finite set. Thus, $Ru$ is defined by the finite number of constants which can be determined from Feynman diagram expansion.

Define a group which commutes with the renormalization group operator. Let $(\Phi_1,\Phi_2,\Phi_3,\Phi_4)$ are transformed in accordance with a vector representation of $O(4)$. Then the requirement for the invariants $\{\Psi_1,\Psi_2,\Psi_3,\Psi_4\}$ to be expressed linearly via each other leads to a system of equations for the parameters of the $O(4)$ group the solution of which makes it possible to define the group $G_T$

The $G_T$ group consists of the following elements

$$e = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$
\[
\begin{align*}
&A_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
&B_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
&C_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\
&D_f = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.
\end{align*}
\] (35)

Thus, the finite matrix group \( G_T = \{e, f_1, f_2, f_3, f_4, f_5\} \) is a group which commutes with a transformation of the renormalization group.

According to the symmetry arguments the equations of the renormalization group in the order \( O(\varepsilon^2) \) presented in the form:

\[
\begin{align*}
&\bar{u}_1 = b^{-3d} \varepsilon^4 \left[ u_1 + \alpha_2 u_1^2 + \alpha_4 \left( u_1^2 + u_2^2 + u_4^2 \right) \right], \\
&\bar{u}_2 = b^{-3d} \varepsilon^4 \left[ u_2 + \beta u_2 u_4 + \mu u_2^2 + \Delta u_2 u_2 \right], \\
&\bar{u}_3 = b^{-3d} \varepsilon^4 \left[ u_3 + \beta u_3 u_4 + \mu u_3^2 + \Delta u_3 u_3 \right], \\
&\bar{u}_4 = b^{-3d} \varepsilon^4 \left[ u_4 + \beta u_4 u_4 + \mu u_4^2 + \Delta u_4 u_4 \right].
\end{align*}
\] (36)

For definition of this equation parameters it is necessary to calculate the integral

\[
C(r) = K_d \int_{\Lambda/b}^{\Lambda} \frac{q^{d-1} \, dq}{(q^2 - \gamma)^2} = K_d \ln b(1 + 2\gamma \ln \Lambda - \gamma \ln b).
\] (37)

Note that the integrals are calculated in the three dimensional case \( d = 3 \). We define the parameter \( \xi \) from the condition that after transformation the coefficient \( q^\gamma \) in \( v(q) = r + q^\gamma \) is equal to unity. It gives \( \varepsilon^2 b^{-d} b^{-\gamma} = 1 \) and therefore \( \varepsilon^2 b^{-d} = b^\gamma \). In this case we obtain \( b^{-3d} \varepsilon^4 = b^{-d+2\gamma} \). Assume that \( \gamma = \frac{3}{2} + \frac{\varepsilon}{2} \). Then we come to the result \( b^{-3d} \varepsilon^4 = b^\xi \). Assuming that the parameter \( b^{-1} \) is small and leaving only the terms proportional to \( \ln b \), we obtain

\[
\begin{align*}
&\bar{u}_1 = b^\xi \left[ u_1 - \frac{1}{2} \left( k_1 u_1^2 + k_2 \left( u_1^2 + u_2^2 + u_4^2 \right) \right) K_d \ln b \right], \\
&\bar{u}_2 = b^\xi \left[ u_2 - \frac{1}{2} \left( l u_2^2 + m u_2 u_4 + n u_2 u_2 \right) K_d \ln b \right], \\
&\bar{u}_3 = b^\xi \left[ u_3 - \frac{1}{2} \left( l u_3^2 + m u_3 u_4 + n u_3 u_3 \right) K_d \ln b \right], \\
&\bar{u}_4 = b^\xi \left[ u_4 - \frac{1}{2} \left( l u_4^2 + m u_4 u_4 + n u_4 u_4 \right) K_d \ln b \right],
\end{align*}
\] (38)

where \( k_1 = 72, k_2 = 2, l = 16, m = 4, n = 48 \). After the expansion \( b^\xi = 1 + \varepsilon \ln b \) and introducing the notation \( u_i = \frac{\varepsilon}{K_d} x_i \) in the fixed point \( \bar{x}_i = x_i = x_i^* \) we obtain:

8
\[ x_1 = 36x_1^2 + x_2^2 + x_3^2 + x_4^2, \]
\[ x_2 = 8x_2^2 + 2x_3x_4 + 24x_1x_2, \]  
(39)
\[ x_3 = 8x_3^2 + 2x_2x_4 + 24x_1x_3, \]
\[ x_4 = 8x_4^2 + 2x_2x_3 + 24x_1x_4. \]

Note that we are interested in the real solutions of these equations only. Complex solutions are not considered because the initial Hamiltonian, being real, under the action of renormalization group transformations approaches the fixed points and remains real. The real solutions for the system renormalization group equations at the fixed points are given in table 1.

| Table 1. Fixed points of the renormalization group equations |
|-----------------|-----------------|-----------------|-----------------|
| \( x_1 \)       | \( x_2 \)       | \( x_3 \)       | \( x_4 \)       |
| 0               | 0               | 0               | 0               |
| 1/36            | 1/20            | 0               | 0               |
| 1/40            | 1/12            | 0               | 0               |
| 1/72            | 0               | 1/20            | 0               |
| 1/40            | 0               | 0               | 1/20            |
| 1/72            | 0               | 1/12            | 0               |

After linearization of the renormalization group equations we obtain

\[ \Delta \bar{\eta}_i = M_{ij}\Delta u_j, \]  
(40)

where

\[
M_{ij} = \begin{bmatrix}
1 + \varepsilon \ln b - k_1u_{1i}^* K_d \ln b; & -k_2 K_d \ln bu_2^*; & -k_2 K_d \ln bu_3^*; & -k_2 K_d \ln bu_4^*\\
-\frac{n}{2}u_2^* K_d \ln b; & 1 + \varepsilon \ln b - lu_{2i}^* K_d \ln b; & -\frac{m}{2} K_d \ln bu_4^*; & -\frac{m}{2} K_d \ln bu_3^* \\
-\frac{n}{2}u_3^* K_d \ln b; & -\frac{m}{2} K_d \ln bu_3^*; & 1 + \varepsilon \ln b - lu_{3i}^* K_d \ln b; & -\frac{m}{2} K_d \ln bu_2^* \\
-\frac{n}{2}u_4^* K_d \ln b; & -\frac{m}{2} K_d \ln bu_4^*; & -\frac{m}{2} K_d \ln bu_2^*; & 1 + \varepsilon \ln b - lu_{4i}^* K_d \ln b - \frac{n}{2}u_i^* K_d \ln b
\end{bmatrix}
\]  
(41)

Now, it is necessary to calculate the eigenvalues of the matrix \( M_{ij} \) at the fixed points presented in table 1.

I. The fixed point \( x_1 = x_2 = x_3 = x_4 = 0 \). The eigenvalues of the matrix \( M_{ij} \) are equal to

\[ \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = b^\varepsilon. \]

II. The fixed point \( x_1 = 1/36, x_2 = x_3 = x_4 = 0 \). The eigenvalues of the matrix \( M_{ij} \) have the form:

\[ \lambda_1 = b^{-\varepsilon}; \quad \lambda_2 = \lambda_3 = \lambda_4 = b^{\varepsilon/3}. \]

III. The fixed point \( x_1 = 1/40, x_2 = 1/20, x_3 = x_4 = 0 \). The eigenvalues of the matrix \( M_{ij} \) have the form:
\[ \lambda_1 = b^{-0.8\varepsilon}, \quad \lambda_2 = b^{-0.4\varepsilon}, \quad \lambda_3 = \lambda_4 = b^{0.6\varepsilon}. \]

IV. The fixed point \( x_1 = 1/72, x_2 = 1/12, x_3 = x_4 = 0 \). The eigenvalues of the matrix \( M \) are defined as:

\[ \lambda_1 = b^0; \quad \lambda_2 = b^{-(2/3)\varepsilon}; \quad \lambda_3 = \lambda_4 = b^{(2/3)\varepsilon}. \]

Thus the stability of the fixed points is determined by a sign of \( \varepsilon \). If \( \varepsilon \) is positive \( \varepsilon > 0 \), all the fixed points are unstable whereas if \( \varepsilon \) is negative \( \varepsilon < 0 \), the fixed point \( x_1 = x_2 = x_3 = x_4 = 0 \) is stable, whereas the other fixed points are unstable.

5. Conclusion

In the framework of the standard method of the renormalization group and \( \varepsilon \)-expansion the analysis of the phase transitions with the effective Hamiltonian (25) is considered in [14, 15]. It is shown, that all the fixed points of the system are unstable. Consequently the phase transition is a first order transition in spite of the fact that the analysis in the framework of Landau theory leads to a phase transition of the second order.

If the presence of defects in such structures generates a small-world property the system spectrum has the form \( \omega(q) = r + q^\gamma \). In this case if the parameter \( \varepsilon = 2\gamma - 3 > 0 \) then the fixed points of the system are unstable and we have a transition of the first order.

However if \( \varepsilon = 2\gamma - 3 < 0 \) the fixed point \( x_1 = x_2 = x_3 = x_4 = 0 \) of the system is stable, therefore the phase transition in this case is a second order one. Note that in the case \( \varepsilon = 2\gamma - 3 > 0 \) the other fixed points of the system are unstable, so the corresponding transitions are of the first order. Thus a small-world property of systems taken into account leads to a new class of universality.

6. References

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