Uniqueness of charged static asymptotically flat black holes in dynamical Chern-Simons gravity

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I. INTRODUCTION

Gravitational collapse and emergence of black holes is one of the most essential research problems of general relativity and its generalizations. The problem of classification of non-singular black hole solutions was first discussed by Israel [1], Müller zum Hagen et al. [2] and Robinson [3], while the most complete results were proposed in Refs. [4–8]. The classification of static vacuum black hole solutions was finished in [9], where the condition of non-degeneracy of the event horizon was removed. As far as the Einstein-Maxwell (EM) black holes was concerned it was proved that for the static electro-vacuum black holes all degenerate components of the event horizon should have charges of the same signs [10].

The construction of the uniqueness black hole theorem for stationary axisymmetric spacetime turned out to be far more complicated task [11]. However, the complete proof was presented by Mazur [12] and Bunting [13] (see for a review of the uniqueness of black hole solutions story see [14] and references therein).

A different issue, related to the problem of gravitational collapse in generalization of Einstein theory to higher dimensions and emergence of higher dimensional black objects (like black rings, black Saturns) and multidimensional black holes was widely studied. The complete classification of n-dimensional charged black holes both with non-degenerate and degenerate component of the event horizon was proposed in Refs. [15], while partial results for the very nontrivial case of n-dimensional rotating black hole uniqueness theorem were provided in [16]. The problem of the behaviour of matter fields in the spacetime of higher dimensional black hole was studied in Ref. [17].

Due to the attempts of building a consistent quantum gravity theory there was also resurgence of works treating the mathematical aspects of the low-energy string theory black holes. These researches comprise also the case of the low-energy limit of the string theory, like dilaton gravity, Einstein-Maxwell-axion-dilaton (EMAD)-gravity and supergravities theories [18]. On the other hand, the strictly stationary static vacuum spacetimes in Einstein-Gauss-Bonnet theory were discussed in [19].

Black holes and their properties as key ingredients of the AdS/CFT attitude [20] to superconductivity also acquire great attention. Questions about possible matter configurations in AdS spacetime arise naturally during aforementioned researches. In Ref. [21] it was revealed that strictly stationary AdS spacetime could not allow for the existence of nontrivial configurations of complex scalar fields or form fields. The generalization of the aforementioned problem, i.e., strictly stationarity of spacetimes with complex scalar fields in EMAD-gravity with negative cosmological constant was given in [22].

The Chern-Simons modified gravity (CS modified gravity), where the Einstein action is modified by the addition of parity violating Pontryagin term [23] has its roots in particle physics. Namely, the imbalance between left-handed and right-handed fermions induced gravitational anomaly in fermion number current, proportional to the aforementioned Pontryagin term [24]. It also emerges in string theory as an anomaly-canceling term in Green-Schwarz mechanism [25]. Moreover CS-gravity was elaborated in the context of cosmology, gravitational waves, Lorentz invariance [26] (see also references therein). In Ref. [27] it was revealed that a static asymptotically flat black hole solution is unique to be Schwarzschild spacetime in CS modified gravity.

Motivated by the aforementioned problems we shall consider the problem of the uniqueness static asymptotically flat black holes in CS modified gravity with U(1)-gauge field. The basic idea in our treatment of the problem in question will be to implement the conformal positive energy theorem [28].

The paper is organized as follows. In Sec.II we review some basic facts concerning with dynamical CS modified gravity.
Then, applying the conformal positive energy theorem we perform the uniqueness proof of static asymptotically flat electrically charged black hole in CS modified gravity.

II. SYSTEM

We commence this section with the action of the CS modified gravity with matter fields provided by the action

\[
I = \kappa \int d^4x \sqrt{-g} R + \frac{\alpha}{4} \int d^4x \sqrt{-g} \theta \ast R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - \frac{\beta}{2} \int d^4x \sqrt{-g} \nabla_\alpha \nabla^\alpha \theta
\]

(1)

where \(\alpha, \beta\) are the dimensional coupling constant, while \(\theta\) (CS coupling field) is scalar field which is a function parameterizing deformation from ordinary Einstein theory. \(* R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}\) is the Pontryagin density, while \(L_{\text{mat}}\) stands for some matter Lagrangian density which does not depend on the scalar field in question. In what follows we assume that \(L_{\text{mat}}\) will constitute matter Lagrangian for \(U(1)\)-gauge fields, given by \(L_{\text{mat}} = -F_{\mu\nu}F^{\mu\nu}\). The dual to Riemannian tensor is defined as

\[
* R_{\alpha\beta\gamma\delta} = \frac{1}{2} \epsilon_{\gamma\delta\psi\zeta} R_{\alpha\beta\psi\zeta}.
\]

(2)

The field equations obtained by variation of the action (2) imply

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \frac{\alpha}{\kappa} C_{\mu\nu} = \frac{1}{\kappa} \left( T_{\mu\nu}(\theta) + T_{\mu\nu}(F) \right),
\]

(3)

\[
\nabla_\alpha \nabla^\alpha \theta = \frac{\alpha}{4 \beta} R_{\alpha\beta\gamma\delta} \ast R^{\alpha\beta\gamma\delta},
\]

(4)

where we have denoted by \(C^{\alpha\beta}\) the following relation:

\[
C^{\alpha\beta} = \nabla_\gamma \theta \epsilon^{\gamma\mu\nu(\alpha} \nabla_\mu R_{\nu)\beta} + \nabla_\gamma \nabla_\delta \theta \ast R^{\delta(\alpha\beta)}\gamma.
\]

(5)

On the other hand, the energy momentum tensor \(T_{\mu\nu} = -\frac{\delta S}{\delta g^{\mu\nu}}\) of matter fields in question yields

\[
T_{\alpha\beta}(\theta) = \frac{\beta}{2} \left( \nabla_\alpha \theta \nabla_\beta \theta - \frac{1}{2} g_{\alpha\beta} \nabla_\gamma \nabla_\gamma \theta \right),
\]

(6)

\[
T_{\alpha\beta}(F) = 2 F_{\alpha\gamma} F_{\beta\gamma} - \frac{1}{2} g_{\alpha\beta} F_{\mu\nu} F^{\mu\nu}.
\]

(7)

The line element of static spacetime subject to the asymptotically timelike Killing vector field \(k_a = \left( \frac{\partial}{\partial t} \right)_a\) and \(V^2 = -k_\mu k^\mu\) can be provided by the relation

\[
ds^2 = -V^2 dt^2 + g_{ij} dx^i dx^j,
\]

(8)

where \(V\) and \(g_{ij}\) are independent of the \(t\)-coordinate as the quantities of the hypersurface \(\Sigma\) of constant \(t\). We assume that on the hypersurface \(\Sigma\) the electromagnetic potential will be of the form \(A_\theta = \psi dt\), i.e., one deals with electrically charged black hole.

In our consideration we shall take into account the asymptotically flat spacetime. Namely, the spacetime in question will contain a data set \((\Sigma_{\text{end}}, g_{ij}, K_{ij})\) with gauge fields of \(R_{\alpha\beta}\) such that \(\Sigma_{\text{end}}\) is diffeomorphic to \(R^3\) minus a ball. The fields \((g_{ij}, K_{ij})\) will satisfy the fall-off condition of the form

\[
| g_{ij} - \delta_{ij} | + r | \partial_a g_{ij} | + \ldots + r^m | \partial_{a_1} \ldots a_m g_{ij} | + r | K_{ij} | + \ldots + r^m | \partial_{a_1} \ldots a_m K_{ij} | \leq \mathcal{O}\left( \frac{1}{r} \right).
\]

(9)

Likewise we require that in the local coordinates as above, defined \(U(1)\)-gauge field fulfills the following fall-off demand:

\[
| F_{\alpha\beta} | + r | \partial_a F_{\alpha\beta} | + \ldots + r^m | \partial_{a_1} \ldots a_m F_{\alpha\beta} | \leq \mathcal{O}\left( \frac{1}{r} \right).
\]

(10)
In the light of these stipulation the hypersurface will be said to be asymptotically flat if it contains an asymptotically flat end.

Taking the form of static metric into account, the corresponding equations of motion yield

\[ V^{(g)} \nabla_i^{(g)} \nabla^i V = \frac{1}{\kappa} V^{(g)} \nabla_i \psi^{(g)} \nabla^i \psi, \]

\[ (g) \nabla_i \psi = \frac{1}{V} (g) \nabla_i \psi^{(g)} \nabla^i V, \]

\[ V^{(g)} \nabla_i (g) \nabla^i \theta + (g) \nabla_i \theta (g) \nabla^i V = 0, \]

\[ (g) R_{ij} - \frac{(g) \nabla_i (g) \nabla_j V}{V} = \frac{1}{\kappa} \left( \frac{\beta}{2} (g) \nabla_i \theta (g) \nabla_j \theta + g_{ij} \frac{(g) \nabla_i \psi^{(g)} \nabla^i \psi}{V^2} - 2 \frac{(g) \nabla_i \psi^{(g)} \nabla^i \psi}{V^2} \right). \]

In the above relations covariant derivative with respect to the metric tensor \( g_{ij} \) is denoted by \( (g) \nabla \), while \( (g) R_{ij}(g) \) is the Ricci tensor defined on the hypersurface \( \Sigma \). Furthermore, let us suppose that for each of the quantity in question, i.e., \( V, \psi, \phi \), there is a standard coordinate system in which they have usual asymptotic expansion.

To proceed further, let us introduce the definitions of the crucial quantities in the the proof of the uniqueness. Namely, they can be written as follows:

\[ \Phi_1 = \frac{1}{2} \left[ V + \frac{1}{2} V \right], \]

\[ \Phi_0 = i \sqrt{\frac{\beta}{2 \kappa}} \theta, \]

\[ \Phi_{-1} = \frac{1}{2} \left[ V - \frac{1}{2} V \right], \]

\[ \Psi_1 = \frac{1}{2} \left[ V + \frac{1}{2} V - \sqrt{\frac{2}{\kappa}} \psi^2 \right], \]

\[ \Psi_0 = \frac{2}{\kappa} \psi, \]

\[ \Psi_{-1} = \frac{1}{2} \left[ V - \frac{1}{2} V - \sqrt{\frac{2}{\kappa}} \psi^2 \right]. \]

It worth pointing out that defining the metric tensor \( \eta_{AB} = diag(1, -1, -1) \), it can be achieved that \( \Phi_A \Phi^A = \Psi_A \Psi^A = -1 \), where \( A = -1, 0, 1 \). Having in mind the conformal transformation provided by

\[ \tilde{g}_{ij} = V^2 g_{ij}, \]

one can introduce the symmetric tensors written in terms of \( \Phi_A \) in the following form:

\[ \tilde{G}_{ij} = \tilde{\nabla}_i \Phi_{-1} \tilde{\nabla}_j \Phi_{-1} - \tilde{\nabla}_i \Phi_0 \tilde{\nabla}_j \Phi_0 - \tilde{\nabla}_i \Phi_1 \tilde{\nabla}_j \Phi_1, \]

and similarly for the potential \( \Psi_A \)

\[ \tilde{H}_{ij} = \tilde{\nabla}_i \Psi_{-1} \tilde{\nabla}_j \Psi_{-1} - \tilde{\nabla}_i \Psi_0 \tilde{\nabla}_j \Psi_0 - \tilde{\nabla}_i \Psi_1 \tilde{\nabla}_j \Psi_1, \]

where by \( \tilde{\nabla}_i \) we have denoted the covariant derivative with respect to the metric \( \tilde{g}_{ij} \). Consequently, according to the relations (22) and (23), the field equations may be cast in the forms

\[ \tilde{\nabla}^2 \Phi_A = \tilde{G}_i^i \Phi_A, \]

\[ \tilde{\nabla}^2 \Psi_A = \tilde{H}_i^i \Psi_A. \]

Just the Ricci curvature tensor with respect to the conformally rescaled metric \( \tilde{g}_{ij} \) implies

\[ \tilde{R}_{ij} = \tilde{G}_{ij} + \tilde{H}_{ij}. \]
In general, as far as the conformal positive energy theorem is concerned, one assumes that we have to do with two asymptotically flat Riemannian \((n-1)\)-dimensional manifold \((\Sigma^\Phi, \Phi, g_{ij})\) and \((\Sigma^\Psi, \Psi, g_{ij})\). Moreover the conformal transformation of the form \(\Phi g_{ij} = \Omega^2 \Phi g_{ij}\). Then, it implies that the corresponding masses satisfy \(m + \beta \Omega^2 R \geq 0\), for some positive constant \(\beta\). The aforementioned inequalities are fulfilled it the \((n-1)\)-dimensional Riemannian manifolds are flat \[28\].

To proceed further, due to the requirement of the conformal positive energy theorem, we introduce conformal transformations obeying the relations

\[
\Phi g_{ij}^\pm = \Phi \omega_\pm^2 \, \hat{g}_{ij}, \quad \Psi g_{ij}^\pm = \Psi \omega_\pm^2 \, \hat{g}_{ij}.
\]

On the other hand, their conformal factors are subject to the relations

\[
\Phi \omega_\pm = \frac{\Phi_1 \pm 1}{2}, \quad \Psi \omega_\pm = \frac{\Psi_1 \pm 1}{2}.
\]

Thus, the above conformal transformations enable one to obtain four manifolds \((\Sigma^\Phi_+, \Phi, g_{ij}^+), (\Sigma^\Phi_-, \Phi, g_{ij}^-), (\Sigma^\Psi_+, \Psi, g_{ij}^+), (\Sigma^\Psi_-, \Psi, g_{ij}^-)\). The standard procedure of pasting \((\Sigma^\Phi_+, \Phi, g_{ij}^+)\) and \((\Sigma^\Psi_-, \Psi, g_{ij}^-)\) across the surface \(V = 0\) endues to construct a regular hypersurfaces \(\Sigma^\Phi = \Sigma^\Phi_+ \cup \Sigma^\Phi_-\) and \(\Sigma^\Psi = \Sigma^\Psi_+ \cup \Sigma^\Psi_-\). If \((\Sigma, g_{ij}, \Phi, \Psi)\) are asymptotically flat solution of \[24\] and \[25\] with non-degenerate black hole event horizon, our next task will be to check that total gravitational mass on hypersurfaces \(\Sigma^\Phi\) and \(\Sigma^\Psi\) is equal to zero. In order to do this we shall implement the conformal positive mass theorem presented in Ref.\[28\]. Hence, using another conformal transformation given by

\[
\hat{g}_{ij}^\pm = \left(\Phi \omega_\pm \right)^2 \left(\Psi \omega_\pm \right)^2 \, \hat{g}_{ij},
\]

it follows that the Ricci curvature tensor on the space yields

\[
2 \hat{R} = \left[\Phi \omega_\pm^2, \Psi \omega_\pm^2\right]^{-\frac{1}{2}} \left(\Phi \omega_\pm^2 \Phi R + \Psi \omega_\pm^2 \Psi R\right) + \left(\hat{\nabla}_i \ln \Phi \omega_\pm - \hat{\nabla}_i \ln \Psi \omega_\pm\right) \left(\hat{\nabla}_i \ln \Phi \omega_\pm - \hat{\nabla}_i \ln \Psi \omega_\pm\right).
\]

The close inspection of the first term in relation \[29\] reveals that it is non-negative. Namely one can establish that it may be written in the form as follows:

\[
\Phi \omega_\pm^2 \Phi R + \Psi \omega_\pm^2 \Psi R = 2 \left| \frac{\Phi_0 \hat{\nabla}_i \Phi_{-1} - \Phi_{-1} \hat{\nabla}_i \Phi_0}{\Phi_1 \pm 1} \right|^2 + 2 \left| \frac{\Psi_0 \hat{\nabla}_i \Psi_{-1} - \Psi_{-1} \hat{\nabla}_i \Psi_0}{\Psi_1 \pm 1} \right|^2.
\]

Applying the conformal energy theorem we draw a conclusion that \((\Sigma^\Phi, \Phi, g_{ij}), (\Sigma^\Psi, \Psi, g_{ij})\) and \((\hat{\Sigma}, \hat{g}_{ij})\) are flat and it in turns implies that the conformal factors \(\Phi \omega = \Psi \omega\) and \(\Phi_1 = \Psi_1\). Furthermore \(\Phi_0 = const\Phi_{-1}\) and \(\Psi_0 = const\Psi_{-1}\). Just the above potentials are functions of a single variable. Moreover, the manifold \((\Sigma, g_{ij})\) is conformally flat. We can rewrite \(\hat{g}_{ij}\) in a conformally flat form, i.e., we define a function

\[
\hat{g}_{ij} = U^4 \Phi g_{ij},
\]

where one sets \(U = (\Phi \omega_\pm V)^{-1/2}\). Because of the fact that the Ricci scalar in \(\hat{g}_{ij}\) metric is equal to zero, equations of motion of the system in question reduce to the Laplace equation on the three-dimensional Euclidean manifold

\[
\hat{\nabla}_i \hat{\nabla}^i U = 0,
\]

where \(\hat{\nabla}\) is the connection on a flat manifold. The above equation implies that the following expression for the flat base space is valid. Namely, one gets

\[
\Phi g_{ij}dx^i dx^j = \rho^2 d\hat{U}^2 + \hat{h}_{AB} dx^A dx^B.
\]

Then, the event horizon will be located at some constant value of \(\hat{U}\). The radius of the black hole event horizon can be terminated at fix value of \(\rho\)-coordinate \[24\], which in turn can be introduced on the hypersurface \(\Sigma\) by the relation

\[
\hat{g}_{ij}dx^i dx^j = \rho^2 dV^2 + h_{AB} dx^A dx^B.
\]
Moreover, a connected component of the event horizon can be identified at fixed value of $\rho$.

Proceeding further, let us assume that $\mathcal{U}_1$ and $\mathcal{U}_2$ consist two solutions of the boundary value problem of the system in question. Using Green identity and integrating over the volume element, we arrive at the relation

$$
\left( \int_{r \rightarrow \infty} - \int_{H} \right) \left( \mathcal{U}_1 - \mathcal{U}_2 \right) \frac{\partial}{\partial r} \left( \mathcal{U}_1 - \mathcal{U}_2 \right) dS = \int_{\Omega} \left| \nabla \left( \mathcal{U}_1 - \mathcal{U}_2 \right) \right|^2 d\Omega.
$$

(34)

In view of the last equation, the surface integrals disappear due to the imposed boundary conditions. On the other hand, by virtue of the above relation one finds that the volume integral must be identically equal to zero. To summarize we have established the conclusion of our investigations.

**Theorem:**

Let us consider a static four-dimensional solution to equation of motion in Chern-Simons modified gravity with $U(1)$-gauge field. Suppose that one has an asymptotically timelike Killing vector field $k_\mu$ orthogonal to the connected and simply connected spacelike hypersurface $\Sigma$. The topological boundary $\partial \Sigma$ of $\Sigma$ is a nonempty topological manifold with $g_{ij}k^ik^j = 0$ on $\partial \Sigma$. It yields the following:

If $\partial \Sigma$ is connected, then there exist a neighbourhood of the hypersurface $\Sigma$ which is diffeomorphic to an open set of Reissner-Nordsrød non-extreme solution with electric charge.

### III. CONCLUSIONS

In our paper we prove the uniqueness of four-dimensional static black hole being the solution of Chern-Simons modified gravity with $U(1)$-gauge field. Assuming the existence of an asymptotically timelike Killing vector field orthogonal to the simply connected spacelike hypersurface with topological boundary, it turns out that if the boundary in question is connected, then there is a neighbourhood of the hypersurface which is diffeomorphic to an open set Reissner-Nordsrød non-extreme solution with electric charge.

It may be interesting to generalize the proof to the case of both degenerate and nondegenerate components of the event horizon of the black hole in question. On the other hand, stationary axisymmetric case as well as the Chern-Simons modified gravity with cosmological constant are challenges for the future investigations. We hope to return to these problems elsewhere.

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