NON-LOCAL BOUNDARY CONDITIONS FOR

MASSLESS SPIN-$\frac{1}{2}$ FIELDS

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Abstract. This paper studies the 1-loop approximation for a massless spin-1/2 field on a flat four-dimensional Euclidean background bounded by two concentric 3-spheres, when non-local boundary conditions of the spectral type are imposed. The use of $\zeta$-function regularization shows that the conformal anomaly vanishes, as in the case when the same field is subject to local boundary conditions involving projectors. A similar analysis of non-local boundary conditions can be performed for massless supergravity models on manifolds with boundary, to study their 1-loop properties.
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1. Introduction

The quantum theory of fermionic fields can be expressed, following the ideas of Feynman, in terms of amplitudes of going from suitable fermionic data on a spacelike surface $\mathcal{S}_I$, say, to fermionic data on a spacelike surface $\mathcal{S}_F$. To make sure that the quantum boundary-value problem is well posed, one has actually to consider the Euclidean formulation, where the boundary 3-surfaces, $\Sigma_I$ and $\Sigma_F$, say, may be regarded as (compact) Riemannian 3-manifolds bounding a Riemannian 4-manifold. In the case of massless spin-1/2 fields, which are the object of our investigation, one thus deals with transition amplitudes

$$\mathcal{A}[^{\text{boundary data}}] = \int e^{-I_E} \mathcal{D}\psi \mathcal{D}\tilde{\psi}$$

where $I_E$ is the Euclidean action functional, and the integration is over all massless spin-1/2 fields matching the boundary data on $\Sigma_I$ and $\Sigma_F$. The path-integral representation of the quantum amplitude (1.1) is then obtained with the help of Berezin integration rules, and one has a choice of non-local [1] or local [2] boundary conditions. The mathematical foundations of the former lie in the theory of spectral asymmetry and Riemannian geometry [3], and their formulation can be described as follows. In two-component spinor notation, a massless spin-1/2 field in a 4-manifold with positive-definite metric is represented by a pair $\left(\psi^A, \tilde{\psi}^A'\right)$ of independent spinor fields, not related by any spinor conjugation. Suppose now that $\psi^A$ and $\tilde{\psi}^A'$ are expanded on a family of concentric 3-spheres as

$$\psi^A = \frac{1}{2\pi^{\frac{3}{2}}} \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+2)} \alpha_{pq}^{n} \left[ m_{np}(\tau) \rho^{nqA} + \tilde{r}_{np}(\tau) \sigma^{nqA} \right]$$

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\[
\tilde{\psi}' = \frac{1}{2\pi \tau} - \sum_{n=0}^{\infty} \sum_{p,q=1}^{(n+1)(n+2)} \alpha_{pq}^{nq} \left[ \tilde{m}_{np}(\tau) \rho^{nqA'} + r_{np}(\tau) \sigma^{nqA'} \right].
\]

With a standard notation, \(\tau\) is the Euclidean-time coordinate which plays the role of a radial coordinate, and the block-diagonal matrices \(\alpha_{pq}^{nq}\) and the \(\rho\) and \(\sigma\)-harmonics are described in detail in [4]. One can now check that the harmonics \(\rho^{nqA}\) have positive eigenvalues for the intrinsic three-dimensional Dirac operator on \(S^3\):

\[
D_{AB} \equiv e_{nAB'} e^j_B (3) D_j
\]

and similarly for the harmonics \(\sigma^{nqA'}\) and the Dirac operator

\[
D_{A'B'} \equiv e_{nBA'} e^j_B (3) D_j.
\]

With our notation, \(e_{nAB'}\) is the Euclidean normal to the boundary, \(e^j_B\) are the spatial components of the two-spinor version of the tetrad, and \((3) D_j\) denotes three-dimensional covariant differentiation on \(S^3\) [1, 2, 4]. By contrast, the harmonics \(\sigma^{nqA}\) and \(\rho^{nqA'}\) have negative eigenvalues for the operators (1.4) and (1.5) respectively.

The so-called spectral boundary conditions rely therefore on a non-local operation, i.e. the separation of the spectrum of a first-order elliptic operator (our (1.4) and (1.5)) into a positive and a negative part. They require that half of the spin-1/2 field should vanish on \(\Sigma_F\), where this half is given by those modes \(m_{np}(\tau)\) and \(r_{np}(\tau)\) which multiply harmonics having positive eigenvalues for (1.4) and (1.5) respectively. The remaining half of the field should vanish on \(\Sigma_I\), and is given by those modes \(\tilde{r}_{np}(\tau)\) and \(\tilde{m}_{np}(\tau)\) which
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multiply harmonics having negative eigenvalues for (1.4) and (1.5) respectively. One thus writes [4]

$$\left[ \psi^A_{(+)} \right]_{\Sigma_F} = 0 \implies \left[ m_{np} \right]_{\Sigma_F} = 0 \quad (1.6)$$

$$\left[ \tilde{\psi}^{A'}_{(+)} \right]_{\Sigma_F} = 0 \implies \left[ r_{np} \right]_{\Sigma_F} = 0 \quad (1.7)$$

and

$$\left[ \psi^A_{(-)} \right]_{\Sigma_I} = 0 \implies \left[ \tilde{r}_{np} \right]_{\Sigma_I} = 0 \quad (1.8)$$

$$\left[ \tilde{\psi}^{A'}_{(-)} \right]_{\Sigma_I} = 0 \implies \left[ \tilde{m}_{np} \right]_{\Sigma_I} = 0. \quad (1.9)$$

Massless spin-$\frac{1}{2}$ fields are here studied since they provide an interesting example of conformally invariant field theory for which the spectral boundary conditions (1.6)–(1.9) occur naturally already at the classical level [3].

Section 2 is devoted to the evaluation of the $\zeta(0)$ value resulting from the boundary conditions (1.6)–(1.9). This yields the 1-loop divergence of the quantum amplitude, and coincides with the conformal anomaly in our model. Concluding remarks are presented in section 3.

2. $\zeta(0)$ value with non-local boundary conditions

As shown in [1, 2, 4, 5], the modes occurring in the expansions (1.2) and (1.3) obey a coupled set of equations, i.e.

$$\left( \frac{d}{d\tau} - \frac{(n + \frac{3}{2})}{\tau} \right) x_{np} = E_{np} \tilde{x}_{np} \quad (2.1)$$
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\[ \left(-\frac{d}{d\tau} - \frac{(n + \frac{3}{2})}{\tau}\right) \tilde{x}_{np} = E_{np} x_{np} \tag{2.2} \]

where \( x_{np} \) denotes \( m_{np} \) or \( r_{np} \), and \( \tilde{x}_{np} \) denotes \( \tilde{m}_{np} \) or \( \tilde{r}_{np} \). Setting \( E_{np} = M \) for simplicity of notation one thus finds, for all \( n \geq 0 \), the solutions of (2.1) and (2.2) in the form

\[ m_{np}(\tau) = \beta_{1,n} \sqrt{\tau} I_{n+1}(M\tau) + \beta_{2,n} \sqrt{\tau} K_{n+1}(M\tau) \tag{2.3} \]

\[ r_{np}(\tau) = \beta_{1,n} \sqrt{\tau} I_{n+1}(M\tau) + \beta_{2,n} \sqrt{\tau} K_{n+1}(M\tau) \tag{2.4} \]

\[ \tilde{m}_{np}(\tau) = \beta_{1,n} \sqrt{\tau} I_{n+2}(M\tau) - \beta_{2,n} \sqrt{\tau} K_{n+2}(M\tau) \tag{2.5} \]

\[ \tilde{r}_{np}(\tau) = \beta_{1,n} \sqrt{\tau} I_{n+2}(M\tau) - \beta_{2,n} \sqrt{\tau} K_{n+2}(M\tau) \tag{2.6} \]

where \( \beta_{1,n} \) and \( \beta_{2,n} \) are some constants. The insertion of (2.3)–(2.6) into the boundary conditions (1.6)–(1.9) leads to the equations (hereafter \( b \) and \( a \) are the radii of the two concentric 3-sphere boundaries, with \( b > a \), and we define \( \beta_n \equiv \beta_{2,n}/\beta_{1,n} \))

\[ I_{n+1}(Mb) + \beta_n K_{n+1}(Mb) = 0 \tag{2.7} \]

for \( m_{np} \) and \( r_{np} \) modes, and

\[ I_{n+2}(Ma) - \beta_n K_{n+2}(Ma) = 0 \tag{2.8} \]

for \( \tilde{m}_{np} \) and \( \tilde{r}_{np} \) modes, with the same value of \( M \) [1]. One thus finds two equivalent formulae for \( \beta_n \):

\[ \beta_n = -\frac{I_{n+1}(Mb)}{K_{n+1}(Mb)} = \frac{I_{n+2}(Ma)}{K_{n+2}(Ma)} \tag{2.9} \]
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which lead to the eigenvalue condition

$$I_{n+1}(Mb)K_{n+2}(Ma) + I_{n+2}(Ma)K_{n+1}(Mb) = 0. \quad (2.10)$$

The full degeneracy is $2(n + 1)(n + 2)$, for all $n \geq 0$, since each set of modes contributes to (2.7) and (2.8) with degeneracy $(n + 1)(n + 2)$ [1].

We can now apply $\zeta$-function regularization to evaluate the resulting conformal anomaly [6], following the algorithm developed in [7, 8] and applied several times in the recent literature [9–16]. The basic properties are as follows. Let us denote by $f_n$ the function occurring in the equation obeyed by the eigenvalues by virtue of boundary conditions, after taking out fake roots (e.g. $x = 0$ is a fake root of order $\nu$ of the Bessel function $I_\nu(x)$). Let $d(n)$ be the degeneracy of the eigenvalues parametrized by the integer $n$. One can then define the function

$$I(M^2, s) \equiv \sum_{n=n_0}^{\infty} d(n)n^{-2s}\log f_n(M^2) \quad (2.11)$$

and the work in [7, 8] shows that such a function admits an analytic continuation to the complex-$s$ plane as a meromorphic function with a simple pole at $s = 0$, in the form

$$"I(M^2, s)" = \frac{I_{\text{pole}}(M^2)}{s} + I^R(M^2) + O(s). \quad (2.12)$$

The function $I_{\text{pole}}(M^2)$ is the residue at $s = 0$, and makes it possible to obtain the $\zeta(0)$ value as

$$\zeta(0) = I_{\log} + I_{\text{pole}}(M^2 = \infty) - I_{\text{pole}}(M^2 = 0) \quad (2.13)$$

where $I_{\log}$ is the coefficient of the log($M$) term in $I^R$ as $M \to \infty$. The contributions $I_{\log}$ and $I_{\text{pole}}(\infty)$ are obtained from the uniform asymptotic expansions of basis functions as
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$M \to \infty$ and their order $n \to \infty$, whilst $I_{\text{pole}}(0)$ is obtained by taking the $M \to 0$ limit of the eigenvalue condition, and then studying the asymptotics as $n \to \infty$. More precisely, $I_{\text{pole}}(\infty)$ coincides with the coefficient of $\frac{1}{n}$ in the expansion as $n \to \infty$ of

$$
\frac{1}{2} d(n) \log \left[ \rho_{\infty}(n) \right]
$$

where $\rho_{\infty}(n)$ is the $n$-dependent term in the eigenvalue condition as $M \to \infty$ and $n \to \infty$.

The $I_{\text{pole}}(0)$ value is instead obtained as the coefficient of $\frac{1}{n}$ in the expansion as $n \to \infty$ of

$$
\frac{1}{2} d(n) \log \left[ \rho_{0}(n) \right]
$$

where $\rho_{0}(n)$ is the $n$-dependent term in the eigenvalue condition as $M \to 0$ and $n \to \infty$ [7, 8, 14].

In our problem, using the limiting form of Bessel functions when the argument tends to zero [17], one finds that the left-hand side of (2.10) is proportional to $M^{-1}$ as $M \to 0$. Hence one has to multiply by $M$ to get rid of fake roots. Moreover, in the uniform asymptotic expansion of Bessel functions as $M \to \infty$ and $n \to \infty$, both $I$ and $K$ functions contribute a $\frac{1}{\sqrt{M}}$ factor. These properties imply that $I_{\text{log}}$ vanishes:

$$
I_{\text{log}} = \frac{1}{2} \sum_{l=1}^{\infty} 2l(l+1) \left( 1 - \frac{1}{2} - \frac{1}{2} \right) = 0. \quad (2.14)
$$

Moreover,

$$
I_{\text{pole}}(\infty) = 0 \quad (2.15)
$$
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since there is no $n$-dependent coefficient in the uniform asymptotic expansion of (2.10) [7–16]. Last, one finds

$$I_{\text{pole}}(0) = 0$$

(2.16)

since the limiting form of (2.10) as $M \to 0$ and $n \to \infty$ is

$$\frac{2}{Ma}(b/a)^{n+1}.$$

The results (2.14)–(2.16), jointly with the general formula (2.13), lead to a vanishing value of the 1-loop divergence:

$$\zeta(0) = 0.$$  

(2.17)

3. Concluding remarks

To our knowledge, the analysis leading to (2.17) in the spectral case, had not been performed in the current literature. Our detailed calculation shows that, in flat Euclidean 4-space, the conformal anomaly for a massless spin-1/2 field subject to non-local boundary conditions of the spectral type on two concentric 3-spheres vanishes, as in the case when the same field is subject to the local boundary conditions

$$\sqrt{2} e n_A A'^A \bar{\psi}^A = \pm \bar{\psi} A'^A \text{ on } \Sigma_I \text{ and } \Sigma_F.$$  

(3.1)

If (3.1) holds and the spin-1/2 field is massless, the work in [10] shows in fact that $\zeta(0) = 0$.

Backgrounds given by flat Euclidean 4-space bounded by two concentric 3-spheres are not the ones occurring in the Hartle-Hawking proposal for quantum cosmology, where the initial 3-surface $\Sigma_I$ shrinks to a point [18]. Nevertheless, they are relevant for the
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quantization programme of gauge fields and gravitation in the presence of boundaries [11, 12]. In particular, similar techniques have been used in section 5 of [16] to study a two-boundary problem for simple supergravity subject to spectral boundary conditions in the axial gauge. One then finds the eigenvalue condition

$$I_{n+2}(Mb)K_{n+3}(Ma) + I_{n+3}(Ma)K_{n+2}(Mb) = 0$$  \hspace{1cm} (3.2)

for all $n \geq 0$. The analysis of (3.2) along the same lines of section 2 shows that transverse-traceless gravitino modes yield a vanishing contribution to $\zeta(0)$, unlike transverse-traceless modes for gravitons, which instead contribute $-5$ to $\zeta(0)$ [12, 16].

Thus, the results in [16] seem to show that, at least in finite regions bounded by one 3-sphere or two concentric 3-spheres, simple supergravity is not one-loop finite in the presence of boundaries. Of course, more work is in order to check this property, and then compare it with the finiteness of scattering problems suggested in [19]. Further progress is thus likely to occur by virtue of the fertile interplay of geometric and analytic techniques [20–24] in the investigation of heat-kernel asymptotics and (1-loop) quantum cosmology.

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