Quadratic Monge-Ampère equation
and Calabi problem for HKT-manifolds

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Abstract
A quaternionic version of the Calabi problem on the Monge-Ampère equation is introduced, namely a quaternionic Monge-Ampère equation on a compact hypercomplex manifold with an HKT-metric. The equation is non-linear elliptic of second order. For a hypercomplex manifold with holonomy in $SL(n, \mathbb{H})$, uniqueness (up to a constant) of a solution is proven, as well as the zero order a priori estimate. The existence of a solution is conjectured, similar to the Calabi-Yau theorem. We reformulate this quaternionic equation as a special case of the complex Hessian equation, making sense on any complex manifold.

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1 Introduction

We introduce a quaternionic version of the Calabi problem on the Monge-Ampère equation. The problem is motivated by close analogy (discussed

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1 Semyon Alesker was partially supported by ISF grant 1369/04.
2 Misha Verbitsky is an EPSRC advanced fellow supported by CRDF grant RM1-2354-MO02 and EPSRC grant GR/R7773/01

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below) with the classical Calabi-Yau theorem [Yau] on complex Monge-Ampère equations on Kähler manifolds. Our version of the Calabi problem is about the quaternionic Monge-Ampère equation on a compact hypercomplex manifold with an HKT-metric. The equation is non-linear elliptic of second order. When a holonomy group of the Obata connection of a hypercomplex manifold lies in $SL(n, \mathbb{H})$, uniqueness (up to a constant) of the solution is proven, as well as the zero order a priori estimate. Under this assumption, we conjecture existence of the solution. We give a reformulation of this quaternionic equation as a special case of the complex Hessian equation, which makes sense on any complex manifold.

**Definition 1.1:** A hypercomplex manifold is a smooth manifold $M$ together with a triple $(I, J, K)$ of complex structures satisfying the usual quaternionic relations:

$$IJ = -JI = K.$$ 

Necessarily the real dimension of a hypercomplex manifold is divisible by 4. The simplest example of a hypercomplex manifold is the flat space $\mathbb{H}^n$.

**Remark 1.2:** (1) In this article we will presume (unlike in much of the literature on the subject) that the complex structures $I, J, K$ act on the right on the tangent bundle $TM$ of $M$. This action extends uniquely to the right action of the algebra $\mathbb{H}$ of quaternions on $TM$.

(2) It follows that the dimension of a hypercomplex manifold $M$ is divisible by 4.

(3) Hypercomplex manifolds were introduced explicitly by Boyer [Bo].

Let $M$ be a hypercomplex manifold, and $g$ a Riemannian metric on $M$. The metric $g$ is called quaternionic Hermitian (or hyperhermitian) if $g$ is invariant with respect to the group $SU(2) \subset \mathbb{H}^*$ of unitary quaternions. Given a quaternionic Hermitian metric $g$ on a hypercomplex manifold $M$, consider the differential form

$$\Omega := -\omega_J + \sqrt{-1}\omega_K$$ 

where $\omega_L(A, B) := g(A, B \circ L)$ for any $L \in \mathbb{H}$ with $L^2 = -1$ and any vector fields $A, B$ on $M$. It is easy to see that $\Omega$ is a $(2, 0)$-form with respect to the complex structure $I$.

**Definition 1.3:** The metric $g$ on $M$ is called an HKT-metric (here
HKT stands for HyperKähler with Torsion) if

$$\partial \Omega = 0$$

where $\partial$ is the usual $\partial$-differential on the complex manifold $(M, I)$.

The form $\Omega$ corresponding to an HKT-metric $g$ will be called an HKT-form.

**Remark 1.4:** HKT-metrics on hypercomplex manifolds first were introduced by Howe and Papadopoulos [HP]. Their original definition was different but equivalent to Definition 1.3 (see [GP]).

HKT-metrics on hypercomplex manifolds are analogous in many respects to Kähler metrics on complex manifolds. For example, it is a classical result that any Kähler form $\omega$ on a complex manifold can be locally written in the form $\omega = dd^c h$ where $h$ is a strictly plurisubharmonic function called a potential of $\omega$, and vice versa (see, e.g. [GH]). Similarly, by [BS] (see also [AV]), an HKT-form $\Omega$ on a hypercomplex manifold locally admits a potential: it can be written as

$$\Omega = \partial \partial J H$$

where $\partial J = J^{-1} \circ \bar{\partial} \circ J$, and $H$ is a strictly plurisubharmonic function in the quaternionic sense. The converse is also true. The notion of quaternionic plurisubharmonicity is relatively new: on the flat space $\mathbb{H}^n$ it was introduced in [A1] and independently by G. Henkin around the same time (unpublished), and on general hypercomplex manifolds in [AV]. More recently the notion of plurisubharmonic functions has been generalized to yet another context of calibrated geometries [HL].

Motivated by the analogy with the complex case, we introduce the following quaternionic version of the Calabi problem. Let $(M^{4n}, I, J, K)$ be a compact hypercomplex manifold of real dimension $4n$. Let $\Omega$ be an HKT-form. Let $f$ be a real-valued $C^\infty$ function on $M$. The quaternionic Calabi problem is to study solvability of the following quaternionic Monge-Ampère equation with an unknown real-valued function $\varphi$:

$$(\Omega + \partial \partial J \varphi)^n = e^f \Omega^n. \tag{1.1}$$

By Lemma 4.9 below, if a $C^\infty$-function $\varphi$ satisfies the Monge-Ampère equation (1.1) then $\Omega + \partial \partial J \varphi$ is an HKT-form, namely it corresponds to a new HKT-metric. This equation is a non-linear elliptic equation of second order. We formulate the following conjecture.
**Conjecture 1.5:** Let us assume that \((M, I)\) admits a holomorphic (with respect to the complex structure \(I\)) non-vanishing \((2n,0)\)-form \(\Theta\). Then the quaternionic Monge-Ampère equation (1.1) has a \(C^\infty\)-solution \(\varphi\) provided the following necessary condition on the initial data is satisfied:

\[
\int_M (e^f - 1) \Omega^n \wedge \overline{\Theta} = 0.
\]

In this article we show that under the condition of existence of such \(\Theta\) a solution of (1.1) is unique up to a constant (Corollary 4.10). Our next main result is a zero order a priori estimate (Corollary 5.7): there exists a constant \(C\) depending on \(M, \Omega, \) and \(||f||_{C^0}\) only, such that the solution \(\varphi\) satisfying the normalization condition \(\int_M \varphi \cdot \Omega^n \wedge \overline{\Theta} = 0\) must satisfy the estimate

\[
||\varphi||_{C^0} \leq C,
\]

where \(||\cdot||_{C^0}\) denotes the maximum norm on \(M\), i.e. \(||u||_{C^0} := \max\{|u(x)| | x \in M\}\). Our proof of this estimate is a modification of Yau’s argument [Yau] in the complex case as presented in [J].

**Remark 1.6:** Let us comment on how restrictive the condition of existence of a form \(\Theta\) is. Recall that a hypercomplex manifold \(M\) carries a unique torsion free connection such that the complex structures \(I, J, K\) are parallel with respect to it. It it called the Obata connection as it was discovered by Obata [O]. It was shown by the second named author [V5] that if \(M\) is a compact HKT-manifold admitting a holomorphic (with respect to \(I\)) \((2n,0)\)-form \(\Theta\), then the holonomy of the Obata connection is contained in the group \(SL_n(\mathbb{H})\) (instead of \(GL_n(\mathbb{H})\)). Conversely, if the holonomy of the Obata connection is contained in \(SL_n(\mathbb{H})\), then there exists a form \(\Theta\) as above which, moreover, can be chosen to be q-positive (in sense of Section 3.2 below).

**Remark 1.7:** The quaternionic Monge-Ampère equation (1.1) can be interpreted in the following geometric way. Assume we are given an HKT-form \(\Omega\), and a strongly q-real \((2n,0)\)-form on \((M, I)\) (see Section 3.2 for the definition) which is nowhere vanishing and hence may be assumed to have the form \(e^f \Omega^n\). We are looking for a new HKT-form of the form \(\Omega + \partial \partial J\varphi\) whose volume form is equal to the prescribed form \(e^f \Omega^n\).
We note that the Calabi problem also has its real version where it becomes a real Monge-Ampère equation on smooth compact manifolds with an affine flat structure. This real Calabi problem was first considered and successfully solved by Cheng and Yau [ChY]. Note also that the classical Dirichlet problem for the Monge-Ampère equation in strictly pseudoconvex domains has its quaternionic version considered and partly solved by the first named author [A2]. We refer to [A2] for the details.

Finally, in this article we present a reformulation of the quaternionic Monge-Ampère equation as a special case of a complex Hessian equation on the complex manifold $X$, $\dim_{\mathbb{C}} X = m$. This equation is:

\begin{equation}
(\omega - \sqrt{-1} \partial \bar{\partial} \varphi)^n \wedge \Phi = e^f \omega^n \wedge \Phi,
\end{equation}

where $n = m - k$, $\Phi \in \Lambda^{k,\bar{k}}(X)$, $\omega \in \Lambda^{1,1}(X)$, $f \in C^\infty(X)$ are fixed, and $\Phi, \omega$ satisfy some positivity assumptions (see Proposition 3.2). Let us state the conditions more explicitly under an additional assumption of existence of the form $\Theta$ as in Conjecture 1.5.

We consider the Monge-Ampère equation (1.2) where the unknown function $\varphi$ belongs to the class of $C^\infty$ functions, such that $\omega - \sqrt{-1} \partial \bar{\partial} \varphi$ lies in the interior of the cone of $\Phi$-positive forms. Theorem 3.9 states that the quaternionic Monge-Ampère equation (1.1) is equivalent to (1.2) for appropriate choices of $\Phi$ and $\omega$ under the assumption of existence of the form $\Theta$ as in Conjecture 1.5. Moreover one may assume $d\Phi = d(\Phi \wedge \omega) = 0$ (see Proposition 3.8). We show that if all these conditions are satisfied on a complex compact manifold $X$, then the complex Hessian equation (1.2) is elliptic, its solution is unique up to a constant, and a necessary condition for solvability is

\[ \int_X (e^f - 1) \omega^n \wedge \Phi = 0. \]

We refer to Theorem 4.7 below for the details.

2 Quaternionic Dolbeault complex

To continue, we need a definition and some properties of the Salamon complex on hypercomplex manifolds. The following Section is adapted from [V6]. The quaternionic cohomology is a well-known subject, introduced by S. Salamon ([CS], [S], [B], [L]). Here we give an exposition of quaternionic cohomology and a quaternionic Dolbeault complex for hypercomplex manifolds.
2.1 Quaternionic Dolbeault complex: the definition

Let $M^{4n}$ be a hypercomplex manifold of real dimension $4n$, and

$$\Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^2(M) \xrightarrow{d} ...$$

its de Rham complex. Consider the natural (left) action of $SU(2)$ on $\Lambda^* M$. Clearly, $SU(2)$ acts on $\Lambda^i(M)$, $i \leq \frac{1}{2} \dim R M$ with weights

$$i, i - 2, i - 4, ...$$

We denote by $\Lambda^i_\pm(M)$ the maximal $SU(2)$-subspace of $\Lambda^i(M)$, on which $SU(2)$ acts (on the left) with weight $i$. We again emphasize that necessarily $i \leq 2n = \frac{1}{2} \dim R M$.

The following linear algebraic lemma allows one to compute $\Lambda^i_+(M)$ explicitly.

**Lemma 2.1:** ([V6, Proposition 2.9]) With the above assumptions, let $I$ be the induced complex structure, and $\mathbb{H}_I$ the quaternion space, considered as a 2-dimensional complex vector space with the complex structure induced by $I$ when $I$ acts on $\mathbb{H}_C$ on the right. Denote by $\Lambda_{I}^{p,0}(M)$ the space of the $(p,0)$-form on $M$, with respect to the Hodge decomposition associated with the complex structure $I$. The space $\mathbb{H}_I$ is equipped with the natural action of $SU(2)$. Consider $\Lambda_{I}^{p,0}(M)$ as a representation of $SU(2)$, with trivial group action. Then, there is a canonical isomorphism

$$\Lambda_{I}^{p,0}(M, \mathbb{C}) \cong S^p_C \mathbb{H}_I \otimes_C \Lambda_{I}^{p,0}(M), \quad (2.1)$$

where $S^p_C \mathbb{H}_I$ denotes a $p$-th symmetric power of $\mathbb{H}_I$ over $\mathbb{C}$. Moreover, the $SU(2)$-action on $\Lambda_{I}^{p,0}(M)$ is compatible with the isomorphism (2.1) when $SU(2)$ acts trivially on $\Lambda_{I}^{p,0}(M)$.

**Proof:** Fix a standard basis $1, I, J, K$ in $\mathbb{H}$. Since $\mathbb{H}$ acts on the tangent bundle $TM$ on the right, $\mathbb{H}$ acts on $\Lambda^1(M)$ on the left, namely we have a canonical map

$$\mathbb{H} \otimes_R \Lambda^1(M) \rightarrow \Lambda^1(M).$$

Taking the complexification, we get a $\mathbb{C}$-linear map

$$\mathbb{H} \otimes_R \Lambda^1(M, \mathbb{C}) \rightarrow \Lambda^1(M, \mathbb{C}).$$
Since $\Lambda^{1,0}(M) \subset \Lambda^1(M, \mathbb{C})$, we get the map
\[ \mathbb{H} \otimes_{\mathbb{R}} \Lambda^{1,0}(M) \to \Lambda^1(M, \mathbb{C}). \tag{2.2} \]

We have a canonical quotient map
\[ \mathbb{H} \otimes_{\mathbb{R}} \Lambda^{1,0}(M) \to \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^{1,0}(M). \]

It is easy to see that the map (2.2) factorizes uniquely via a map
\[ \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^{1,0}(M) \to \Lambda^1(M, \mathbb{C}). \tag{2.3} \]

(It is easy to write down the map (2.3) explicitly. Let $h_1, h_2 \in \mathbb{H}_I$ be the basis in $\mathbb{H}_I$: $h_1 = 1$, $h_2 = J$. Then $h_1 \otimes x \mapsto x$, $h_2 \otimes x \mapsto J(x)$.) Consider the $SU(2)$-equivariant homomorphism
\[ \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^{1,0}(M) \to \Lambda^1(M, \mathbb{C}), \tag{2.4} \]

mapping $h_1 \otimes \eta$ to $\eta$ and $h_2 \otimes \eta$ to $J(\eta)$, where $J$ denotes an endomorphism of $\Lambda^1(M)$ induced by $J$. The isomorphism (2.1) is obvious for $p = 1$:
\[ \Lambda^1(M) = \Lambda^1_+(M) = \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^{1,0}_I(M) \tag{2.5} \]

This isomorphism is by construction $SU(2)$-equivariant. Given two vector spaces $A$ and $B$, we have a natural map
\[ S^i A \otimes \Lambda^j B \to \Lambda^i(A \otimes B), \tag{2.6} \]

given by
\[ (a_1 \otimes \cdots \otimes a_i) \otimes (b_1 \wedge \cdots \wedge b_i) \mapsto \frac{1}{i!} \sum_{\sigma \in \Sigma_i} (a_{\sigma(1)} \otimes b_1) \wedge \cdots \wedge (a_{\sigma(i)} \otimes b_i). \]

From (2.6) and (2.5), we obtain the natural $SU(2)$-equivariant map
\[ S^p \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^{p,0}_I(M) \to \Lambda^p(M, \mathbb{C}). \tag{2.7} \]

Since $S^p \mathbb{H}_I$ has weight $p$, the arrow (2.7) maps $S^p \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^{p,0}_I(M)$ to $\Lambda^p_+(M)$. We have constructed the map
\[ S^p \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^{p,0}_I(M) \xrightarrow{\Psi} \Lambda^p_+(M). \tag{2.8} \]

It remains to show that it is an isomorphism. Let $adI : \Lambda^* M \to \Lambda^* M$ act on the $(p, q)$-forms $ad(\eta) = (p - q)\sqrt{-1} \eta$. Clearly, $-\sqrt{-1} \, adI$ is a root of
the Lie algebra $SU(2)$. It is well known that an irreducible representation of a Lie algebra is generated by a highest weight vector. For the Lie algebra $\mathfrak{su}(2)$, this means that $\Lambda^p_+(M)$ is a subspace of $\Lambda^p(M)$ generated by $SU(2)$ from the subspace $W \subset \Lambda^p_+(M)$ consisting of all vectors on which $-\sqrt{-1}adI$ acts as a multiplication by $p$. On the other hand, $W$ coincides with $\Lambda^{p,0}_I(M)$.

We obtained the following:

The space $\Lambda^p_+(M)$ is generated by $SU(2)$ from its subspace $\Lambda^{p,0}_I(M)$. (2.9)

The image of $\Psi : S^p_C \mathbb{H} \otimes_C \Lambda^{p,0}_I(M) \longrightarrow \Lambda^p_+(M)$ is an $SU(2)$-invariant subspace of $\Lambda^p(M)$ containing $\Lambda^{p,0}_I(M)$. By (2.9), this means that $\Psi$ is surjective. Let $R \subset S^p_C \mathbb{H} \otimes_C \Lambda^{p,0}_I(M)$ be the kernel of $\Psi$. By construction, $R$ is $SU(2)$-invariant, of weight $p$. By the same arguments as above, $R$ is generated by its subspace of highest weight, that is, the vectors of type $h^p_1 \cdot \eta$, where $\eta \in \Lambda^{p,0}_I(M)$ (see (2.4)). On the other hand, on the subspace $h^p_1 : \Lambda^{p,0}_I(M) \subset S^p_C \mathbb{H} \otimes_C \Lambda^{p,0}_I(M)$, the map $\Psi$ is, by construction, injective. Therefore, the intersection $h^p_1 \cdot \Lambda^{p,0}_I(M) \cap R$ is zero. We have proved that $\Psi$ is an isomorphism. [Lemma 2.1] is proven.

Consider an $SU(2)$-invariant decomposition

$$\Lambda^p(M) = \Lambda^p_+(M) \oplus V^p$$

(2.10)

where $V^p$ is the sum of all $SU(2)$-subspaces of $\Lambda^p(M)$ of weight less than $p$. Since $SU(2)$-action is multiplicative on $\Lambda^*(M)$, the subspace $\widetilde{V} := \oplus_p V^p \subset \Lambda^*(M)$ is an ideal. Therefore, the quotient

$$\Lambda^*_+(M) = \Lambda^*(M)/\widetilde{V}$$

is an algebra. Using the decomposition (2.10), we define the quaternionic Dolbeault differential $d_+: \Lambda^*_+(M) \longrightarrow \Lambda^*_+(M)$ as the composition of the de Rham differential and the projection $\Lambda^*(M) \rightarrow \Lambda^*_+(M)$. Since de Rham differential cannot increase the $SU(2)$-weight of a form by more than 1, $d$ preserves the subspace $V^* \subset \Lambda^*(M)$. Therefore, $d_+$ is a differential in $\Lambda^*_+(M)$. 

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Definition 2.2: Let
\[ \Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \Lambda^2_+(M) \xrightarrow{d} \Lambda^3(M) \xrightarrow{d} \cdots \rightarrow \Lambda^{2n}_+(M) \]
be the differential graded algebra constructed above. It is called the quaternionic Dolbeault complex, or Salamon complex.

Remark 2.3: The isomorphism (2.1) is clearly multiplicative:
\[ \bigoplus_p S^p_\mathbb{C}(\mathbb{H}_I) \otimes \Lambda^p_+(M) \simeq \bigoplus_p \Lambda^p_+(M, \mathbb{C}). \]

Notice that, in the course of the proof of Lemma 2.1, we have proven the following result (see 2.9).

Claim 2.4: For any \( p \geq 0 \)
\[ \Lambda^p_+I(M) \subset \Lambda^p_+(M). \]

2.2 Hodge decomposition for the quaternionic Dolbeault complex

Let \( M \) be a hypercomplex manifold and \( I \) an induced complex structure. As usual, we have the operator \( adI : \Lambda^*(M) \rightarrow \Lambda^*(M) \) mapping a \((p, q)\)-form \( \eta \) to \( \sqrt{-1} (p-q)\eta \). By definition, \( adI \) belongs to the Lie algebra \( \mathfrak{su}(2) \) acting on \( \Lambda^*(M) \) in the standard way. Therefore, \( adI \) preserves the subspace \( \Lambda^*_+(M) \subset \Lambda^*(M) \). We obtain the Hodge decomposition
\[ \Lambda^*_+(M) = \bigoplus_{p+q \leq 2n} \Lambda^{p,q}_+(M). \]

Definition 2.5: The decomposition
\[ \Lambda^*_+(M) = \bigoplus_{p+q \leq 2n} \Lambda^{p,q}_+(M) \]
is called the Hodge decomposition for the quaternionic Dolbeault complex.

\footnote{We identify \( \Lambda^0 M \) and \( \Lambda^0_+ M \), \( \Lambda^1 M \) and \( \Lambda^1_+ M \).}
2.3 The Dolbeault bicomplex and quaternionic Dolbeault complex

Let $M^{4n}$ be a hypercomplex manifold of real dimension $4n$ and $I, J, K \in \mathbb{H}$ the standard triple of induced complex structures. Clearly, $J$ acts on the complexified cotangent space $\Lambda^1(M, \mathbb{C})$ mapping $\Lambda^{0,1}_I(M)$ to $\Lambda^{1,0}_I(M)$. Consider a differential operator
\[
\partial_J : C^\infty(M) \longrightarrow \Lambda^{1,0}_I(M),
\]
mapping $f$ to $J^{-1}(\partial f)$, where $\overline{\partial} : C^\infty(M) \longrightarrow \Lambda^{0,1}_I(M)$ is the standard Dolbeault differential on the complex manifold $(M, I)$. We extend $\partial_J$ to a differential
\[
\partial_J : \Lambda^{p,0}_I(M) \longrightarrow \Lambda^{p+1,0}_I(M),
\]
using the Leibnitz rule. Then $\partial_J = J^{-1} \circ \overline{\partial} \circ J$.

**Proposition 2.6:** (see also [V6, Theorem 2.10]) Let $M^{4n}$ be a hypercomplex manifold, $I$ an induced complex structure, $I, J, K$ the standard basis in quaternion algebra, and
\[
\Lambda^*_+(M) = \oplus_{p+q \leq 2n} \Lambda^{p,q}_I(M)
\]
the Hodge decomposition of the quaternionic Dolbeault complex (Subsection 2.2). Then there exists a canonical isomorphism
\[
\Lambda^{p,q}_I(M) \cong \Lambda^{p+q,0}_I(M). \tag{2.11}
\]
Under this identification, the quaternionic Dolbeault differential
\[
d_+ : \Lambda^{p,q}_I(M) \longrightarrow \Lambda^{p+1,q}_I(M) \oplus \Lambda^{p,q+1}_I(M)
\]
corresponds to the sum
\[
\partial \oplus \partial_J : \Lambda^{p+q,0}_I(M) \longrightarrow \Lambda^{p+q+1,0}_I(M) \oplus \Lambda^{p+q+1,0}_I(M).
\]
**Proof:** Consider the isomorphisms (2.1)
\[
\Lambda^*_+(M, \mathbb{C}) \cong S^*_C \mathbb{H}_I \otimes \mathbb{C} \Lambda^{0,0}_I(M). \tag{2.12}
\]
The Hodge decomposition of (2.12) is induced by the $SU(2)$-action, as follows. Let $\rho_I : U(1) \longrightarrow SU(2)$ be the group homomorphism defined by
\[ \rho_I(e^{\sqrt{-1} \theta}) = e^{I \theta} \text{ for any } \theta \in \mathbb{R}/2\pi \mathbb{Z}. \] From the definition of the SU(2)-action, it follows that the Hodge decomposition of \( \Lambda^*(M) \) coincides with the weight decomposition under the action of \( \rho_I : U(1) \longrightarrow \text{End}(\Lambda^*(M)) \). The SU(2)-action on \( S^p_C \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^p_I(M) \) is trivial on the second component.

Consider the weight decomposition

\[ S^i_C \mathbb{H}_I \cong \bigoplus_{p+q=i} S^p_q C\mathbb{H}_I \]

associated with \( \rho_I \). Then (2.12) translates to the isomorphism

\[ \Lambda^p_q(M) \cong S^p_q \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^q_{p,0}(M). \]

Let \( h_1, h_2 \) be the basis in \( \mathbb{H}_I \) defined as in the proof of Lemma 2.1, i.e. \( h_1 = 1, h_2 = J \). An elementary calculation shows that \( h_1 \) has weight \((1,0)\), and \( h_2 \) has weight \((0,1)\). Therefore, the space \( S^p_q \mathbb{H}_I \) is 1-dimensional and generated by \( h_1^p h_2^q \). We have obtained an isomorphism

\[ \Lambda^p_q(M) \cong h_1^p \cdot h_2^q \cdot \Lambda^q_{p,0}(M). \quad (2.13) \]

This proves (2.11). The isomorphism (2.13) is multiplicative by Remark (2.1). Consider the differential

\[ \hat{d}_+ = h_1 \partial + h_2 \partial_J : S^p_q \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^q_{p,0}(M) \longrightarrow S^{p+1}_C \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^{q+1}_{p,0}(M) \]

To prove our proposition, we need to show that the quaternionic Dolbeault differential \( d_+ \) coincides with \( \hat{d}_+ \) under the identification (2.13). The isomorphism (2.13) is multiplicative, and the differentials \( d_+ \) and \( \hat{d}_+ \) both satisfy the Leibnitz rule. Therefore, it is sufficient to show that

\[ d_+ = \hat{d}_+ \]

(2.14)

on \( C^\infty(M) = \Lambda^0_+(M) \). On functions, the equality (2.14) is immediately implied by the definition of the isomorphism

\[ \Lambda^1_+(M) \cong \mathbb{H}_I \otimes_{\mathbb{C}} \Lambda^1_0(M). \]

Proposition 2.6 is proven. \( \blacksquare \)

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\footnote{The differential \( \hat{d}_+ \) satisfies the Leibnitz rule, because \( \partial \partial_J = -\partial_J \partial \). The last equation is clear: the differentials \( d, d_I := -IdI, d_J := -JdJ, d_K := -KdK \) anticommute because of integrability of \( I, J, K \).}
The statement of Proposition 2.6 can be represented by the following diagram:

\[
\begin{array}{ccc}
\Lambda^0(M) & \xrightarrow{d''} & \Lambda^0,0(M) \\
\Lambda^0,1(M) & \xrightarrow{d'} & \Lambda^0,1(M) \\
\Lambda^2,0(M) & \xrightarrow{d''} & \Lambda^2,0(M) \\
\Lambda^2,1(M) & \xrightarrow{d'} & \Lambda^2,1(M)
\end{array}
\]

(2.15)

where \(d_+ = d'_+ + d''_+\) is the Hodge decomposition of the quaternionic Dolbeault differential.

Definition 2.7: With the above assumptions, the bicomplex (2.15) is called the quaternionic Dolbeault bicomplex.

Lemma 2.8: The projection of \(\eta \in \Lambda^{1,1}_I(M)\) to the \(SU(2)\)-invariant part of \(\Lambda^2(M)\) is given by the map

\[
\eta \mapsto \frac{1}{2} (\eta(\cdot, \cdot) + \eta(\cdot \circ J, \cdot \circ J)).
\]

(2.16)

Proof: It is easy to see that the 2-form (2.16) is invariant under \(I\) and \(J\), hence under \(K\). This implies that this 2-form is \(SU(2)\)-invariant. Also if \(\eta\) was already \(SU(2)\)-invariant then (2.16) is equal to \(\eta\). ■

Lemma 2.9: Let \(g\) be a quaternionic Hermitian metric on a hypercomplex manifold \((M, I, J, K)\). Define

\[
\omega_I(X, Y) := g(X, Y \circ I).
\]

Then \(\omega_I \in \Lambda^{1,1}_I(M)\).

Proof: It is clear that \(\omega_I \in \Lambda^{1,1}_I(M)\). Since

\[
\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_{SU(2)}(M),
\]

to prove the lemma we have to check that the projection of \(\omega_I\) to the \(SU(2)\)-invariant forms vanishes. By Lemma 2.8 this projection is equal to

\[
\frac{1}{2} (g(X, Y \circ I) + g(X \circ J, Y \circ JI)) = 0.
\]
The lemma is proven. ■

**Lemma 2.10:** Let $g$ be a quaternionic Hermitian metric on a hypercomplex manifold $(M, I, J, K)$. Define $\omega_I \in \Lambda^{1,1}_+(M)$ as in [Lemma 2.9] and

$$\Omega(X, Y) := -(g(X, Y \circ J) - \sqrt{-1}g(X, Y \circ K)).$$

Then under the isomorphism (2.1) $(h_1 \cdot h_2) \otimes \Omega$ corresponds to $\sqrt{-1}\omega_I$.

**Proof:** Under the isomorphism (2.1) the form $(h_1 \otimes h_2) \otimes \Omega$ corresponds to the form $\zeta$ given by

$$\zeta(X, Y) = \frac{1}{2}(\Omega(X, Y \circ J) + \Omega(X \circ J, Y)) =$$

$$- \frac{1}{2} \left( g(X, Y \circ J^2) - \sqrt{-1}g(X, Y \circ JK) + g(X \circ J, Y \circ J) - \sqrt{-1}g(X \circ J, Y \circ K) \right) =$$

$$= \sqrt{-1}g(X, Y \circ I) = \sqrt{-1}\omega_I(X, Y).$$

The lemma is proven. ■

**3 Quaternionic Monge-Ampère equation**

**3.1 First reformulation of quaternionic Monge-Ampère equation.**

Let $(M, I, J, K)$ be a hypercomplex manifold. Let $g$ be a quaternionic Hermitian Riemannian metric. Define as in Section 2.3 the 2-forms

$$\omega_I(X, Y) := g(X, Y \circ I) \in \Lambda^{1,1}_+(M),$$

$$\Omega(X, Y) := -(g(X, Y \circ J) - \sqrt{-1}g(X, Y \circ K)).$$

We want to rewrite the quaternionic Monge-Ampère equation

$$(\Omega + \partial \partial_I \varphi)^n = Ae^f \Omega^n$$

in terms of the $\Lambda^{*,*}_+(M)$-bicomplex. Let us multiply both sides of (3.1) by $h_1^n \cdot h_2^\alpha$ and apply the isomorphism (2.1). We get

$$P_+((\sqrt{-1}\omega_I + d'_+d''_+\varphi)^n) = P_+(Ae^f(\sqrt{-1}\omega_I)^n)$$

(3.2)
where $P_+ : \Lambda^*(M) \to \Lambda_+^*(M)$ is a natural $SU(2)$-invariant projection. Indeed the isomorphism (2.1) is multiplicative and, by Lemma 2.10, carries $h_1 \cdot h_2 \cdot \Omega$ to $\sqrt{-1} \omega_I$ and $\partial \bar{\partial} I$ to $d'_+ d''_+$. But it is easy to see that $d'_+ d''_+ = P_+ \circ \partial \bar{\partial}$. Hence equation (3.1) is equivalent to the equation

\[ P_+((\omega_I - \sqrt{-1} \partial \bar{\partial} \varphi)^n) = Ae^f \omega_I^n. \]  

(3.3)

**Lemma 3.1:** For any $\eta \in \Lambda_+^{2n}(M)$, any non-negative integer $m$, and any $\xi \in \Lambda^m(M)$,

\[ \eta \wedge \xi = \eta \wedge P_+(\xi). \]

**Proof:** It is enough to show that if $\xi \in \Lambda^m(M)$ belongs to a subspace of $SU(2)$-weight at most $m-1$ then $\eta \wedge \xi = 0$. In this case, the Clebsch-Gordan formula implies that $\eta \wedge \xi$ belongs to a subspace of $\Lambda^{2n+m}(M)$ generated by $SU(2)$-weights $2n+m-1, 2n+m-3, \ldots, 2n-m+1$. But $\Lambda^{2n+m}(M)$ has no vectors for these weights because it is dual to $\Lambda^{2n-m}(M)$, all of the $SU(2)$-weights of the latter space are less than or equal to $2n-m$.

**Proposition 3.2:** Let $M$ be a hypercomplex quaternionic Hermitian manifold, and

\[ (\Omega + \partial \bar{\partial} \varphi)^n = Ae^f \Omega^n, \]

(3.4)

the quaternionic Monge-Ampère equation. Then (3.4) is equivalent to the following equation

\[ (\omega_I - \sqrt{-1} \partial \bar{\partial} \varphi)^n \wedge P_+(\omega_I^n) = Ae^f \omega_I^n \wedge P_+(\omega_I^n). \]

(3.5)

**Proof:** We need to check that (3.5) is equivalent to (3.3). Both sides of (3.3) belong to $\Lambda^{2n}(M)$. However, the vector space $\Lambda^{2n}(M) \cong \Lambda^{2n,0}(M)$ is clearly 1-dimensional, and generated by $P_+(\omega_I^n)$. By Lemma 3.1, for any $\eta, \xi \in \Lambda^{2n}(M)$, one has $P(\eta) \wedge \xi = \eta \wedge P(\xi) = P(\eta) \wedge P(\xi)$. This implies that the equation (3.3) is equivalent to this equation multiplied by $\omega_I^n$, and the latter is equivalent to

\[ (\omega_I - \sqrt{-1} \partial \bar{\partial} \varphi)^n \wedge P_+(\omega_I^n) = Ae^f \omega_I^n \wedge P_+(\omega_I^n). \]

(3.6)

**Remark 3.3:** Equation (3.5) is a special case of the so-called complex Hessian equation. More generally, a generalized complex Hessian equation...
is written as $\Psi(\sqrt{-1} \partial \bar{\partial} u) = f$, where $\Psi$ is a symmetric polynomial in the eigenvalues of the $(1,1)$-form $\sqrt{-1} \partial \bar{\partial} u$.

3.2 Operators $R$ and $V$

We would like to present yet another reformulation of the quaternionic Monge-Ampère equation. For this we introduce, in this section, two operators $R$ and $V$ on differential forms. Denote by

$$\tilde{R} : \Lambda^{p,q}_{i,+}(M) = h_1^p h_2^q \Lambda^{p+q,0}(M) \longrightarrow \Lambda^{p+q,0}(M)$$

the isomorphism constructed in Proposition 2.6. Let

$$R : \Lambda^{p,q}(M) \longrightarrow \Lambda^{p+q,0}(M)$$

be the composition of the standard projection

$$\Lambda^{p,q}(M) \overset{P_*}{\longrightarrow} \Lambda^{p,q}_{i,+}(M)$$

with $\tilde{R}$. In [AV], we defined a real structure on $\Lambda^{2p,0}_{i}(M)$, that is, an anticomplex involution mapping $\lambda \in \Lambda^{2p,0}_{i}(M)$ into $J\lambda$ (since $I$ and $J$ anticommute, $J$ maps $(p,q)$-forms into $(q,p)$-forms). Forms fixed under this involution we call $q$-real ($q$ stands for quaternions). We also define a notion of positivity: a real $(2,0)$-form $\eta$ is $q$-positive if $\eta(X, X \circ J) \geq 0$ for any real vector field $X$. A strongly $q$-positive cone is the cone of $q$-real $(2p,0)$-forms which is generated by the products of positive forms with non-negative coefficients (this definition is parallel to one given in complex analysis - see e.g. [D]). It can be shown that this convex cone is closed and has non-empty interior. A $q$-real $(2p,0)$-form $\eta$ is called weakly $q$-positive if for any strongly $q$-positive $(2n-2p,0)$-form $\xi$ the product $\eta \wedge \xi \in \Lambda^{2n,0}_{i}(M)$ is strongly $q$-positive. The set of weakly $q$-positive forms is a closed convex cone with non-empty interior. Note that any strongly $q$-positive form is weakly $q$-positive, and the notions of weak and strong $q$-positivity coincide for $(0,0)$, $(2n,0)$, $(2,0)$, and $(2n-2,0)$-forms (see [AM], Propositions 2.2.2 and 2.2.4, where the $q$-positivity is called just positivity, and only the flat space $M = \mathbb{H}^n$ is considered). The map $R$ satisfies the following properties.

**Theorem 3.4:** (see also [V7, Claim 4.2, Claim 4.5]) Let $M$ be a hypercomplex manifold and

$$R : \oplus_{p,q} \Lambda^{p,q}_{i}(M) \longrightarrow \oplus_{p,q} (h_1^p h_2^q) \otimes \Lambda^{p+q,0}_{i}(M) = \oplus_r S^r_c(\mathbb{H}_I) \otimes \Lambda^{r,0}_{i}(M)$$

the map constructed above. Then
(i) $R$ is multiplicative: $R(x \wedge y) = R(x) \wedge R(y)$.

(ii) $R$ is related to the real structures as follows:

$$R(\bar{\lambda}) = (-1)^p J(R(\lambda)),$$

for any $\lambda \in \Lambda^p_q(M)$ where the action of $J$ on $S^c_\xi(\mathbb{H}_I)$ is identical.

(iii) We have

$$R(\partial \lambda) = \partial R(\lambda), \quad R(\bar{\partial} \lambda) = \partial J R(\lambda).$$

(iv) $(\sqrt{-1})^p R$ maps strongly positive $(p, p)$-forms (in the complex sense) to strongly positive $(2p, 0)$-forms (in the quaternionic sense).

**Proof:** [Theorem 3.4 (i)](3.4) is clear from the construction.

Let us prove part (ii). Due to the multiplicativity of $R$ it is enough to check the statement for $\lambda \in \Lambda^1_1(M)$. Set $\eta := R(\lambda) \in \Lambda^1_1(I(M))$, $\xi := R(\bar{\lambda}) \in \Lambda^1_1(M)$. First assume that $\lambda \in \Lambda^1_1(M)$. Then

$$\lambda = h_1 \eta = \eta, \quad \bar{\lambda} = h_2 \xi = J(\xi). \quad (3.7)$$

We have to show that $\xi = -J(\eta)$ which is obvious by (3.7). Let us assume now that $\lambda \in \Lambda^{0,1}$. We have

$$\lambda = h_2 \eta = J(\eta), \quad \bar{\lambda} = h_1 \xi = \xi. \quad (3.8)$$

We have to show that $\xi = J(\eta)$ which is obvious by (3.8).

Let us prove part (iii). We have

$$R(\partial \lambda) = \bar{R}(P_+(\partial \lambda)) = \bar{R}(P_+(\partial (P_+ \lambda)))$$

$$= \bar{R}(d'_+(P_+ \lambda)) \quad \text{Proposition 2.6} \quad \partial (\bar{R}(P_+ \lambda)) = \partial (R(\lambda)).$$

Similarly one proves the equality $R(\bar{\partial} \lambda) = \partial J R(\lambda)$. Let us prove part (iv). Again due to the multiplicativity of $R$ it is enough to prove it for 2-forms, i.e. $p = 1$. First recall that $\Lambda^{1,1}_I(M) = \Lambda^{1,1}_{I,+}(M) \oplus \Lambda^{2}_{SU(2)}(M)$ where $\Lambda^{2}_{SU(2)}(M)$ denotes the space of $SU(2)$-invariant 2-forms (which are necessarily of type $(1,1)$ on $(M, I)$). Let $\omega \in \Lambda^{1,1}_I(M)$. By [Lemma 2.8](3.8) its projection $P_{SU(2)}(\omega)$ to $\Lambda^{2}_{SU(2)}(M)$ is equal to

$$P_{SU(2)}(\omega)(X, Y) = \frac{1}{2} (\omega(X, Y) + \omega(XJ, YJ)).$$
Then the projection $P_+(\omega)$ to $\Lambda^{1,1}_{I,+}(M)$ is equal to
\[
P_+(\omega)(X,Y) = \frac{1}{2}(\omega(X,Y) - \omega(XJ,YJ)).
\]
Hence
\[
P_+(\omega)(X,XI) = \frac{1}{2}(\omega(X,XI) - \omega(XJ,XIJ)) = \frac{1}{2}(\omega(X,XI) + \omega(XJ,(XJ)I)).
\]

It follows that if $\omega$ is positive then $P_+(\omega)$ is positive. Next we have the equality $\Lambda^{1,1}_{I,+}(M) = h_1h_2\Lambda^{2,0}_I(M)$. It remains to show that $\Omega \in \Lambda^{2,0}_I(M)$ is positive provided $\sqrt{-1}h_1h_2\Omega \in \Lambda^{1,1}_{I,+}(M)$ is positive. We have
\[
((h_1h_2) \cdot \Omega)(X, X \circ I) = \frac{1}{2}(\Omega(X, X \circ IJ) + \Omega(X \circ J, X \circ I)) = \Omega(X, X \circ K).
\]
But for any $\Omega \in \Lambda^{2,0}_I(M)$ and any vector field $X$ one has $\Omega(X, X \circ K) = -\sqrt{-1}\Omega(X, X \circ J)$. Hence
\[
\sqrt{-1}((h_1h_2) \cdot \Omega)(X, X \circ I) = \Omega(X, X \circ J).
\]
Part (iv) is proven. ■

We will need also the following lemma.

**Lemma 3.5:** Let $\eta \in \Lambda^{1,1}_{I}(M)$ be positive (in the complex sense). If $\eta$ is $SU(2)$-invariant then $\eta = 0$.

**Proof:** For any real vector field $X$ we have $\eta(X, X \circ I) \geq 0$. Due to the $J$-invariance of $\eta$ we have
\[
0 \leq \eta(X, X \circ I) = \eta(X \circ J, (X \circ I) \circ J) = -\eta(X \circ J, (X \circ J) \circ I) \leq 0.
\]
Hence $\eta(X, X \circ I) = 0$ for any real vector field $X$. But since $\eta$ has type $(1,1)$ this implies that $\eta = 0$. ■

Fix a non-vanishing holomorphic section $\Theta \in \Lambda^{2n,0}_I(M)$ of the canonical class. Assume moreover that $\Theta$ is q-real and q-positive. We define a map
\[
V : \Lambda^{2p,0}_I(M) \longrightarrow \Lambda^{n+p,n+p}_I(M)
\]
by the following relation

\[ V(\eta) \wedge \xi = \eta \wedge R(\xi) \wedge \overline{\Theta}, \]  

(3.9)

where \( \xi \in \Lambda^{n-p,n-p}_I(M) \) is an arbitrary test form, and \( \eta \in \Lambda^{2p,0}_I(M) \).

**Theorem 3.6:** Let \((M, I, J, K)\) be a hypercomplex manifold equipped with a non-vanishing holomorphic section \( \Theta \in \Lambda^{2p,0}_I(M) \) of the canonical class. Assume that \( \Theta \) is q-real and q-positive. Then

\[ V : \Lambda^{2p,0}_I(M) \longrightarrow \Lambda^{n+p,n+p}_I(M) \]

satisfies the following properties:

(i) For any \( \eta \in \Lambda^{2p,0}_I(M) \), one has

\[ V(J\eta) = V(\eta). \]

In particular \( V \) maps q-real \((2p,0)\)-forms to real (in the complex sense) \((n+p,n+p)\)-forms.

(ii) A form \( \eta \in \Lambda^{2p,0}_I(M) \) is \( \partial \)-exact (\( \partial \)-closed, \( \partial_J \)-exact, \( \partial_J \)-closed) if and only if \( V(\eta) \) is \( \overline{\partial} \)-exact (\( \overline{\partial} \)-closed, \( \overline{\partial_J} \)-exact, \( \overline{\partial_J} \)-closed respectively).

(iii) \( V \) maps weakly q-positive forms to weakly positive (in the complex sense) forms.

(iv) \( V : \Lambda^{2p,0}_I(M) \longrightarrow \Lambda^{n+p,n+p}_I(M) \) is injective.

**Proof:** Theorem 3.6 follows from Theorem 3.4 by duality. To see that \( V \) maps q-real forms to real forms, we use

\[ V(J\eta) \wedge \xi = J\eta \wedge R(\xi) \wedge \overline{\Theta} = \eta \wedge J(R(\xi) \wedge \overline{\Theta}). \]

(The last equation is true, because \( J \) acts on volume forms trivially.) Since \( \Theta \) is q-real, the last expression is equal to

\[ \eta \wedge J(R(\xi)) \wedge \Theta \overset{\text{Theorem 3.4 (ii)}}{=} \eta \wedge R(\xi) \wedge \overline{\Theta} = \eta \wedge R(\xi) \wedge \overline{\Theta} = \overline{V(\eta)} \wedge \xi. \]

Thus we have shown that \( V(J\eta) \wedge \xi = \overline{V(\eta)} \wedge \xi \) for any \( \xi \). This proves Theorem 3.6 (i). To check positivity of \( V(\eta) \), we use Theorem 3.4 (iv)
Calabi problem for HKT-manifolds

S. Alesker, M. Verbitsky, July 27, 2008

(Strongly positive forms are dual to weakly positive). To show that $V$ maps $\partial$-closed forms to $\partial$-closed ones, we use

$$
\int_M V(\partial \eta) \wedge \xi = \int_M \partial \eta \wedge R(\xi) \wedge \overline{\Theta} = - \int_M \eta \wedge \partial R(\xi) \wedge \overline{\Theta} = - \int_M \eta \wedge R(\partial \xi) \wedge \overline{\Theta}
$$

(the last equation follows from Theorem 3.4 (iii)). Then, for any $\partial$-closed $\xi$, $\int_M V(\partial \eta) \wedge \xi = 0$, hence $V(\partial \eta)$ is exact. The converse is also true, because $R$ is injective (Proposition 2.6). In a similar way one deduces the rest of statements of (ii) from Theorem 3.4 (iii) and injectivity of $R$. Let us prove (iv). Assume that $\varphi \in \Lambda^p(M)$ belongs to the kernel of $V$. Then for any $\xi \in \Lambda^{n-p}(M)$ we have

$$
0 = V(\varphi) \wedge \xi = \varphi \wedge R(\xi) \wedge \Theta.
$$

But since $R: \Lambda^{n-p}(M) \to \Lambda^{2(n-p),0}(M)$ is onto, and $\Theta \in \Lambda^{2n,0}(M)$ is non-vanishing this implies that $\varphi = 0$.

The following trivial lemma is used later on in this paper.

**Lemma 3.7:** In assumptions of Theorem 3.6, the following formula is true

$$
V(R(\eta \wedge \nu)) = V(R(\eta)) \wedge \nu,
$$

for all $\eta \in \Lambda^p(M), \nu \in \Lambda^q(M)$.

**Proof:** Since $R$ is multiplicative, we have

$$
V(R(\eta \wedge \nu)) \wedge \xi = R(\eta \wedge \nu) \wedge R(\xi) \wedge \overline{\Theta} = R(\eta) \wedge R(\nu \wedge \xi) \wedge \overline{\Theta} = V(R(\eta)) \wedge \nu \wedge \xi
$$

proving Lemma 3.7.

Let us define now

$$
\Phi := V(1) \in \Lambda^{n,n}(M).
$$

The following proposition summarizes the main properties of $\Phi$.

**Proposition 3.8:** The form $\Phi$ satisfies the following properties:

(i) $\Phi \in \Lambda^{n,n}(M)$;

(ii) $\Phi$ is real in the complex sense, i.e. $\overline{\Phi} = \Phi$;
(iii) $\Phi$ is weakly positive.

(iv) $d\Phi = 0$.

(v) For any Hermitian form $\omega \in \Lambda_{1,1}^{1,1}(M)$, the product $\Phi \wedge \omega^{n-1}$ belongs to the interior of the cone of strongly (= weakly) positive $(2n-1, 2n-1)$-forms.

(vi) A Hermitian form $\omega \in \Lambda_{1,1}^{1,1}(M)$ is HKT if and only if $\Phi \wedge \omega$ is closed. In this case $\Phi \wedge \omega^j$ is closed for any $j$.

Proof: To prove (i) it is enough to show that for any $\xi \in \Lambda^{n,n}(M)$ which belongs to the subspace of $(n,n)$-forms generated by $SU(2)$-weights at most $2n-1$, one has $\Phi \wedge \xi = 0$. But $\Phi \wedge \xi = R(\xi) \wedge \Theta$, and $R(\xi) = 0$. Thus (i) is proven.

Part (ii) follows immediately from Theorem 3.6 (i). Part (iii) follows from Theorem 3.6 (iii). Let us prove (iv). Proposition 3.8 (iv) is clear from Theorem 3.6 (ii), because $\Phi = V(1)$, and 1 is closed.

Let us prove (v). To prove that $\Phi \wedge \omega^{n-1}$ lies in the interior of the cone of positive elements, let us suppose to the contrary that it lies on the boundary. Since the cone of (strongly) positive $(1,1)$-forms is closed there exists $\eta \in \Lambda_{1,1}^{1,1}(M)$ such that $\eta \geq 0$, $\eta \neq 0$, and $\omega^{n-1} \wedge \Phi \wedge \eta = 0$.

But by (3.9)

$$\omega^{n-1} \wedge \Phi \wedge \eta = R(\omega^{n-1} \wedge \eta) \wedge \Theta = (R(\omega^{n-1}))^{n-1} \wedge R(\eta) \wedge \Theta = 0.$$ 

Hence $(R(\omega^{n-1}))^{n-1} \wedge R(\eta) = 0$. Set $\Omega := \sqrt{-1}R(\omega^{n-1})$ be the corresponding HKT $(2,0)$-form. Then $\Omega$ belongs to the interior of the cone of strongly positive $(2,0)$-forms in the quaternionic sense. This fact together with the equality $\Omega^{n-1} \wedge R(\eta) = 0$ and the inequality $\sqrt{-1}R(\eta) \geq 0$ (the latter holds by Theorem 3.3), imply that $R(\eta) = 0$. But this means that $\eta$ is an $SU(2)$-invariant 2-form on $M$. But, since $\eta \geq 0$, Lemma 3.5 implies that $\eta = 0$. This contradiction finishes the proof of (v). Let us prove (vi). Recall that $\omega$ is HKT if and only if $R(\omega)$ is $\partial$-closed. By Theorem 3.6 (ii), this is equivalent to $\partial V(R(\omega)) = 0$. However, $V(R(\omega)) = \omega \wedge V(1)$, by Lemma 3.7 and $\omega \wedge V(1)$ is a real $(n+1,n+1)$-form, hence $\omega$ is HKT if and only if $\omega \wedge V(1)$ is closed. Then $\omega^k \wedge V(1) = V(R(\omega)^k)$ is also closed, because $R(\omega)^k$ is a power of an HKT-form $\Omega$, and

$$\partial \Omega^k = k\Omega^{k-1} \wedge \partial \Omega = 0,$$
by the Leibnitz identity. ■

Now we are ready to give yet another reformulation of the quaternionic Monge-Ampère equation in complex terms under the additional assumption that we are given a non-vanishing holomorphic q-real q-positive form $\Theta \in \Lambda^2_{I,0}(M)$. Let us fix an HKT-metric on $M$. Let $\Omega \in \Lambda^2_{I,0}(M)$ and $\omega_I \in \Lambda^1_{I,+}(M)$ be the corresponding forms. Namely

$$\omega_I(X,Y) = g(X, Y \circ I), \quad \Omega = \sqrt{-1} R(\omega_I).$$

As previously we denote $\Phi := V(1) \in \Lambda^1_{I,+}$. Then we have

**Theorem 3.9:** Let $(M^{4n}, I, J, K, g)$ be an HKT-manifold of real dimension $4n$. Consider the quaternionic Monge-Ampère equation

$$(\Omega + \partial \partial J \varphi)^n = e^f \Omega^n. \quad (3.11)$$

Then (3.11) is equivalent to the following equation

$$(\omega_I - \sqrt{-1} \partial \partial J \varphi)^n \wedge \Phi = e^f \omega_I^n \wedge \Phi. \quad (3.12)$$

**Proof:** It is easy to see that

$$(\Omega + \partial \partial J \varphi)^n = e^f \Omega^n$$

is equivalent to

$$(\Omega + \partial \partial J \varphi)^n \wedge \overline{\Phi} = e^f \Omega^n \wedge \overline{\Phi}.$$ 

However, $R(\sqrt{-1} \omega + \partial \partial J \varphi) = \Omega + \partial \partial J \varphi$ as follows from Theorem 3.4. Therefore

$$(\Omega + \partial \partial J \varphi)^n \wedge \overline{\Phi} = R(\sqrt{-1} \omega + \partial \partial J \varphi)^n \wedge \overline{\Phi} = (\sqrt{-1})^n (\omega - \sqrt{-1} \partial \partial J \varphi)^n \wedge \Phi$$

by definition of $\Phi$. On the other hand

$$\Omega^n \wedge \overline{\Phi} = R((\sqrt{-1} \omega)^n) \wedge \overline{\Phi} = (\sqrt{-1})^n \omega^n \wedge \Phi.$$ 

The result follows. ■
4 Complex Hessian equation.

The goal of this section is to propose a generalization of the quaternionic Monge-Ampère equation written in the form (3.12) for any complex manifold $X$. Then, under appropriate assumptions, satisfied in the HKT-case, we prove ellipticity of the equation and uniqueness of the solution. The main results of the section are Theorem 4.7 and Corollary 4.10. Throughout this section, we fix a complex manifold $X$ of complex dimension $m$.

**Definition 4.1:** Let $\Phi \in \Lambda^{k,k}(X)$. A form $\eta \in \Lambda^{p,p}(X)$ is called $\Phi$-positive if, for any $\nu \in \Lambda^{q,q}(X)$ such that $\Phi \wedge \nu$ is weakly positive, the form $\Phi \wedge \nu \wedge \eta$ is weakly positive.

**Lemma 4.2:** If $\Phi$ is weakly positive, and $\kappa \in \Lambda^{p,p}(X)$ is strongly positive then $\kappa$ is $\Phi$-positive.

**Proof** is obvious.

**Lemma 4.3:** (i) The set of $\Phi$-positive $(p,p)$-forms is a convex cone.

(ii) If $\Phi$ is weakly positive then the cone of $\Phi$-positive $(p,p)$-forms has a non-empty interior.

**Proof:** Part (i) is obvious. Part (ii) follows from Lemma 4.2 because the sub-cone of strongly positive forms has a non-empty interior.

**Lemma 4.4:** Let $\Phi \in \Lambda^{k,k}(X)$ be a weakly positive form. Assume that $\omega_1, \ldots, \omega_r$ are $\Phi$-positive. Then $\omega_1 \wedge \ldots \wedge \omega_r \wedge \Phi$ is also weakly positive.

**Proof:** Since $\Phi = \Phi \wedge 1$ is weakly positive and $\omega_1$ is weakly positive then $\Phi \wedge 1 \wedge \omega_1 = \Phi \wedge \omega_1$ is weakly positive. Then continue by induction.

**Lemma 4.5:** Let $X$ be a compact complex manifold of complex dimension $m$. Let $\Phi \in \Lambda^{k,k}(X)$ be a weakly positive form. Let $f \in C^\infty(X)$, $\omega \in \Lambda^{1,1}(X)$ be real. Denote $n := m - k$. Consider the Monge-Ampère equation

\[
(\omega - \sqrt{-1} \partial \bar{\partial} \varphi)^n \wedge \Phi = e^f \omega^n \wedge \Phi.
\]

(i) Assume that the form $(\omega - \sqrt{-1} \partial \bar{\partial} \varphi)^{n-1} \wedge \Phi \in \Lambda^{m-1,m-1}(X)$ belongs to the interior of the cone of (strongly=weakly) positive $(m-1,m-1)$-forms. Then the Monge-Ampère equation (4.1) is elliptic at $\varphi$. (ii) The Monge-Ampère equation (4.1) has at most unique (up to a constant) solution in the class of $C^\infty$ functions $\varphi$ satisfying the following two conditions:
(\omega - \sqrt{-1} \partial \overline{\partial} \varphi)^{n-1} \wedge \Phi \in \Lambda^{m-1,m-1}(X) \text{ belongs to the interior of the cone of positive } (m-1,m-1)\text{-forms (strongly or weakly they are the same);}

- \omega - \sqrt{-1} \partial \overline{\partial} \varphi \text{ is } \Phi\text{-positive.}

**Proof:** (i) The linearization of the equation is

$$
\psi \mapsto \psi \wedge (\omega - \sqrt{-1} \partial \overline{\partial} \varphi)^{n-1} \wedge \Phi,
$$

where \( \psi \in \Lambda^{1,1}(X) \). This operator is obviously elliptic. (ii) Let \( \varphi_1, \varphi_2 \) be two solutions as in (ii). Then they satisfy

$$
dd^c(\varphi_1 - \varphi_2) \wedge \left( \sum_{k=0}^{n-1} (\omega - \sqrt{-1} \partial \overline{\partial} \varphi_1)^k \wedge (\omega - \sqrt{-1} \partial \overline{\partial} \varphi_2)^{n-1-k} \wedge \Phi \right) = 0
$$

By Lemma 4.4, the form \( (\omega - \sqrt{-1} \partial \overline{\partial} \varphi_1)^k \wedge (\omega - \sqrt{-1} \partial \overline{\partial} \varphi_2)^{n-1-k} \wedge \Phi \) is weakly positive for each \( k \). Moreover, for \( k = 0 \), this form belongs to the interior of the cone of (strongly=weakly) positive \( (m-1,m-1)\)-forms. Then the function \( \varphi_1 - \varphi_2 \) satisfies the linear elliptic equation of second order on the compact manifold \( X \). Hence it must be constant by the strong maximum principle ([GT]).

**Lemma 4.6:** Let \( X \) be a complex manifold of complex dimension \( m \). Let \( \Phi \in \Lambda^{k,k}(X) \) be weakly positive. Denote as previously \( n = m - k \). Assume moreover that there exists a (strongly) positive form \( \gamma \in \Lambda^{1,1}(X) \) such that \( \gamma^{n-1} \wedge \Phi \in \Lambda^{m-1,m-1}(X) \) belongs to the interior of the cone of positive \( (m-1,m-1)\)-forms (weakly or strongly they are the same). Let \( \eta \in \Lambda^{1,1}(X) \) belong to the interior of the cone of \( \Phi \)-positive forms. Then \( \eta^{n-1} \wedge \Phi \) belongs to the interior of the cone of positive \( (m-1,m-1)\)-forms.

**Proof:** Multiplying \( \gamma \) by a small \( \varepsilon > 0 \), we may assume that \( \eta - \gamma \) is \( \Phi\)-positive. We have

$$
\eta^{n-1} \wedge \Phi = (\gamma + (\eta - \gamma))^{n-1} \wedge \Phi = \gamma^{n-1} \wedge \Phi + \sum_{j=0}^{n-2} \gamma^j \wedge (\eta - \gamma)^{n-1-j} \wedge \Phi.
$$

Every summand in the second sum is (weakly) positive by Lemma 4.4, while \( \gamma^{n-1} \wedge \Phi \) belongs to the interior of positive \( (m-1,m-1)\)-forms. Hence the whole sum also belongs to the interior of positive \( (m-1,m-1)\)-forms.

As a corollary we deduce the main result of this section.

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**Theorem 4.7:** Let $X$ be a compact complex manifold of complex dimension $m$. Let $\Phi \in \Lambda^{k,k}(X)$ be a weakly positive form such that there exists a (strongly) positive form $\gamma \in \Lambda^{1,1}(X)$ with the property that, for $n := m - k$, the form $\gamma^{n-1} \wedge \Phi$ belongs to the interior of the cone of positive $(m-1,m-1)$-forms (weakly or strongly they are the same). Let $f \in C^\infty(X)$, $\omega \in \Lambda^{1,1}(X)$ be real. Consider the Monge-Ampère equation

$$ (\omega - \sqrt{-1} \partial \bar{\partial} \varphi)^n \wedge \Phi = e^f \omega^n \wedge \Phi \quad (4.2) $$

where the unknown function $\varphi$ belongs to the class of $C^\infty$ functions such that $\omega - \sqrt{-1} \partial \bar{\partial} \varphi$ lies in the interior of the cone of $\Phi$-positive forms. (i) Then on this class of functions the Monge-Ampère equation (4.2) is elliptic, and its solution is unique up to a constant. (ii) If moreover the forms $\Phi$ and $\omega \wedge \Phi$ are closed, then a necessary condition of the solvability of (4.2) is

$$ \int_X (e^f - 1) \omega^n \wedge \Phi = 0. $$

**Proof:** Part (i) follows immediately from [Lemma 4.5] and [Lemma 4.6]. Let us prove part (ii). It is enough to show that, for any $j$, one has

$$ \int_X (\partial \bar{\partial} \varphi)^j \wedge \omega^{n-j} \wedge \Phi = 0. $$

This equality will follow from Stokes’ formula if we prove that $d(\omega^j \wedge \Phi) = 0$ for any $j$. But

$$ d(\omega^j \wedge \Phi) = j \omega^{j-1} \wedge d\omega \wedge \Phi = j \omega^{j-1} \wedge d(\omega \wedge \Phi) = 0. $$

Theorem is proven. \(\blacksquare\)

**Lemma 4.8:** Let $\left( M^{4n}, I, J, K \right)$ be a hypercomplex manifold. Let $\Theta \in \Lambda^{2n,0}(M)$ be q-real, q-positive, non-vanishing holomorphic form. Let $\Phi = V(1) \in \Lambda^{n,n}(M)$ be as in (3.10). Let $\omega \in \Lambda^{1,1}(M)$ be a positive form (in the complex sense). Then $\omega$ belongs to the interior of the cone of $\Phi$-positive $(1,1)$-forms.

**Proof:** This follows from [Lemma 4.2] because $\Phi$ is weakly positive by [Proposition 3.8] and $\omega$ is strongly positive by assumption. \(\blacksquare\)

**Lemma 4.9:** Assume that $\varphi$ satisfies the quaternionic Monge-Ampère equation

$$ (\Omega + \partial \bar{\partial} \varphi)^n = e^f \Omega^n $$
on a compact manifold \( M \). Then the form \( \Omega + \partial \partial J \varphi \) belongs to the interior of the cone of (strongly=weakly) \( q \)-positive \((2,0)\)-forms. Hence \( \Omega + \partial \partial J \varphi \) is an HKT-form.

**Proof:** Let \( x \in M \) be a point where \( \varphi \) achieves its minimum. One has \( \partial \partial J \varphi(x) \geq 0 \). Hence \( (\Omega + \partial \partial J \varphi(x)) \geq 0 \). But the top power \( (\Omega + \partial \partial J \varphi)^n \) is nowhere vanishing and continuous. Hence, everywhere \( \Omega + \partial \partial J \varphi \) belongs to the interior of the cone of \( q \)-positive elements. \( \blacksquare \)

**Corollary 4.10:** Let \((M^{4n}, I, J, K)\) be a compact hypercomplex manifold. Let \( \Omega \in \Lambda_I^{2,0}(M) \) be an HKT-form. Let us assume moreover that \( M \) admits a non-vanishing holomorphic \( q \)-positive form \( \Theta \in \Lambda_I^{2n,0}(M) \). Fix a real-valued smooth function \( f \). Consider the quaternionic Monge-Ampère equation

\[
(\Omega + \partial \partial J \varphi)^n = e^f \Omega^n
\]

on the class of \( C^\infty \)-smooth functions \( \varphi \). Then the quaternionic Monge-Ampère equation is elliptic and the solution is unique up to a constant. Moreover a necessary condition for solvability of this equation is

\[
\int_M (e^f - 1) \Omega^n \wedge \overline{\Theta} = 0.
\]

**Proof:** The proof follows immediately from Theorem 4.7, Lemma 4.9 and the properties of the form \( \Phi \) given in Proposition 3.8. \( \blacksquare \)

**Remark 4.11:** As we have already mentioned in the introduction, it was shown in [5] that if \( M \) is a compact HKT-manifold admitting a holomorphic (with respect to \( I \)) \((2n,0)\)-form \( \Theta \) then the holonomy of the Obata connection is contained in the group \( SL_n(\mathbb{H}) \) (instead of \( GL_n(\mathbb{H}) \)). Conversely, if the holonomy of the Obata connection is contained in \( SL_n(\mathbb{H}) \) then there exists a form \( \Theta \) as above which moreover can be chosen to be \( q \)-positive (in sense of Section 3.2 below).

5 Zero-order estimates for the quaternionic Monge-Ampère equation.

In this section we will make the following assumption on an HKT-manifold \( M^{4n} \). We assume that \( M \) is compact, connected, and there exists a non-vanishing \( q \)-positive holomorphic section \( \Theta \in \Lambda_I^{2n,0}(M) \). The main result of
this section is Corollary 5.7. Recall that we study the quaternionic Monge-Ampère equation
\[(\Omega_0 + \partial J \varphi)^n = e^f \Omega_0^n, \tag{5.1}\]
where \(\varphi\) is a real valued \(C^\infty\)-smooth function. By Lemma 4.9, \(\Omega_0 + \partial J \varphi\) is an HKT-form.

Let us formulate a conjecture which is a quaternionic version of the Calabi conjecture.

**Conjecture 5.1:** Let \((M^{4n}, I, J, K)\) be a compact hypercomplex manifold with an HKT-form \(\Omega_0 \in \Lambda_2^{I,0}(M)\). Assume, in addition, that there exists a non-vanishing holomorphic q-positive form \(\Theta \in \Lambda_2^{I,0}(M)\). Then the Monge-Ampère equation (5.1) has a \(C^\infty\)-solution provided the following necessary condition is satisfied:
\[\int_M (e^f - 1) \Omega_0^n \wedge \Theta = 0.\]

Let \(\varphi \in C^2(M, \mathbb{R})\) be a solution of the Monge-Ampère equation
\[(\Omega_0 + \partial J \varphi)^n = e^f \Omega_0^n\]
satisfying the normalization condition
\[\int_M \varphi \cdot \Omega_0^n \wedge \Omega_0^i = 0. \tag{5.2}\]

For brevity, we will denote \(\Omega := \Omega_0 + \partial J \varphi\). Let us normalize the form \(\Omega_0\) such that \(\text{vol}_{g_0}(M) = 1\) where \(g_0\) is the HKT-metric corresponding to \(\Omega_0\). The next lemma is essentially linear algebraic.

**Lemma 5.2:** For any smooth function \(\psi\) one has pointwise
\[|\nabla \psi|_{g_0}^2 = 4n \cdot \frac{\partial \psi \wedge \partial J \psi \wedge \Omega_0^n - \Omega_0^n}{\Omega_0^n},\]
where \(| \cdot |_{g_0}\) denotes the norm on \(TM\) with respect to \(g_0\).

**Proof:** The proof is elementary and is left to a reader. \(\blacksquare\)

**Proposition 5.3:** Let \(p > 1\). Then the solution \(\varphi\) satisfies the following estimate
\[\|\nabla |\varphi|^{p/2}\|^2_{L^2} \leq \frac{1}{16n} \cdot \frac{p^2}{(p - 1)} \int_M (1 - e^f) \varphi |\varphi|^{p-2} \Omega_0^n \wedge \Theta.\]

\[\text{This is equivalent to } \text{Hol}(M) \subset SL(n, \mathbb{H}), \text{ see Remark 4.11}\]
Proof: We have
\[
\int_M (1 - e^f) \varphi \varphi^{p-2} \Omega_0^n \wedge \Theta = \int_M \varphi \varphi^{p-2} (\Omega_0^n - \Omega^n) \wedge \Theta = \\
- \int_M \varphi \varphi^{p-2} \partial \partial J \varphi \wedge \left( \sum_{l=0}^{n-1} \Omega_0^l \wedge \Omega^{n-1-l} \right) \wedge \Theta = \\
\int_M \partial (\varphi \varphi^{p-2}) \wedge \partial J \varphi \wedge \left( \sum_{l=0}^{n-1} \Omega_0^l \wedge \Omega^{n-1-l} \right) \wedge \Theta = \\
(p - 1) \int_M \varphi \varphi^{p-2} \partial \varphi \wedge \partial J \varphi \wedge \left( \sum_{l=0}^{n-1} \Omega_0^l \wedge \Omega^{n-1-l} \right) \wedge \Theta.
\]
Since \(\Omega_0, \Omega,\) and \(\Theta\) are positive the last expression is at least
\[
(p - 1) \int_M |\varphi|^{p-2} \partial \varphi \wedge \partial J \varphi \wedge \Omega_0^{n-1} \wedge \Theta.
\]
But
\[
|\varphi|^{p/2} = \frac{p}{2} |\varphi|^{p/2-1} \partial \varphi, \quad \partial J |\varphi|^{p/2} = \frac{p}{2} |\varphi|^{p/2-1} \partial J \varphi.
\]
Thus we get
\[
\int_M (1 - e^f) \varphi |\varphi|^{p-2} \Omega_0^n \wedge \Theta \geq (p - 1) \int_M |\varphi|^{p-2} \partial \varphi \wedge \partial J \varphi \wedge \Omega_0^{n-1} \wedge \Theta \geq \\
(p - 1) \frac{4}{p^2} \int_M \partial |\varphi|^{p/2} \wedge \partial J |\varphi|^{p/2} \wedge \Omega_0^{n-1} \wedge \Theta \overset{\text{Lemma 5.2}}{\geq} 16n \cdot \frac{p - 1}{p^2} |\nabla |\varphi|^{p/2}|_{g_0}^2.
\]
This implies Proposition 5.3.}

In this section we use the notation \(\kappa := \frac{2n}{2n-1}\). Let us denote by \(L^2_\kappa(M)\) the Sobolev space of functions on \(M\) such that all partial derivatives up to order 1 are square integrable.

Lemma 5.4: There exists a constant \(C_1\) depending on \(M\) and \(\Omega_0\) only, such that, for any function \(\psi \in L^2_\kappa(M)\),
\[
||\psi||_{L^{2n}} \leq C_1 (||\nabla \psi||_{L^2}^2 + ||\psi||_{L^2}^2).
\]
Moreover, if \(\psi\) satisfies \(\int_M \psi \cdot \Omega_0^n \wedge \Omega_0^0 = 0\), one has
\[
||\psi||_{L^{2n}}^2 \leq C_1 ||\nabla \psi||_{L^2}^2.
\]
Proof: By the Sobolev imbedding theorem there exists a constant $C'$ such that, for any function $\psi \in L^2(M)$, one has

$$||\psi||^2_{L^{2\kappa}} \leq C'(||\nabla \psi||^2_{L^2} + ||\psi||^2_{L^2}).$$

If the function $\psi$ satisfies $\int_M \psi \cdot \Omega_0^n \wedge \overline{\Omega}_0^n = 0$ then one has $||\psi||^2_{L^2} \leq \tilde{C}||\nabla \psi||^2_{L^2}$ since the second eigenvalue of the Laplacian on $M$ is strictly positive. Thus Lemma 5.4 is proven. 

Lemma 5.5: There exists a constant $C_2$ depending on $M, g_0$, and $||f||_{C^0}$ only such that if $p \in [2, 2\kappa]$ then $||\varphi||_{L^p} \leq C_2$.

Proof: Let us put $p = 2$ in Proposition 5.3. We get

$$||\nabla \varphi||^2_{L^2} \leq 4 \text{const} \exp(||f||_{C^0}) ||\varphi||_{L^1}
\leq 4 \text{const} \cdot \text{vol}_{g_0}(M)^{1/2} \exp(||f||_{C^0}) ||\varphi||_{L^2}$$

where the second inequality follows from the Hölder inequality. Since

$$\int_M \varphi \cdot \Omega_0^n \wedge \overline{\Omega}_0^n = 0,$$

we have

$$||\varphi||_{L^2} \leq C||\nabla \varphi||_{L^2}.$$

Hence $||\nabla \varphi||_{L^2} \leq C' \cdot 4 \text{const} \cdot \text{vol}_{g_0}(M)^{1/2} \exp(||f||_{C^0})$. Therefore by Lemma 5.4 there exists a constant $C'_2$ depending on $M, g_0$, and $||f||_{C^0}$ only such that

$$||\nabla \varphi||_{L^{2\kappa}} \leq C'_2.$$

Hence by the Hölder inequality $||\nabla \varphi||_{L^p} \leq C''_2$ for $p \in [2, 2\kappa]$. 

Proposition 5.6: There exist constants $Q_1, C_3$ depending on $M, g_0$, $||f||_{C^0}$ only such that for any $p \geq 2$

$$||\varphi||_{L^p} \leq Q_1(C_3p)^{-\frac{2\kappa}{p}}.$$ 

Proof: Define $C_3 = C_1(2 \cdot \text{const} \cdot e^{||f||_{C^0}} + 1) \cdot \kappa^{(2n-1)}$ where const is from Proposition 5.3. Choose $Q_1$ so that $Q_1 > C_2(C_3p)^\frac{2\kappa}{p}$ for $2 \leq p \leq 2\kappa$ and $Q_1 > (C_3p)^\frac{2\kappa}{p}$ for $2 \leq p < \infty$. We will prove the result by induction on $p$. By Lemma 5.5 if $2 \leq p \leq 2\kappa$ then $||\varphi||_{L^p} \leq C_2 \leq Q_1(C_3p)^{-\frac{2\kappa}{p}}$. For the inductive step, suppose that

$$||\varphi||_{L^p} \leq Q_1(C_3p)^{-\frac{2\kappa}{p}}$$

for $2 \leq p \leq k$, where $k \geq 2\kappa$ is a real number.
We will show that, for
\[ ||\varphi||_{L^q} \leq Q_1(C_3q)^{-\frac{2n}{q}} \text{ for } 2 \leq q \leq \kappa k; \]
and therefore by induction Proposition 5.6 will be proved. Let \( p \in [2, k] \).
By Proposition 5.3 we get
\[ \frac{2}{p} \leq Q_1(C_3q)^{-\frac{2n}{q}} \]
(5.3)
Applying Lemma 5.4 to \( \psi = ||\varphi||_{L^p}^{p/2} \) we get
\[ ||\varphi||_{L^p}^{p/2} \leq C_1(||\nabla||_{L^2}^{p/2}||\varphi||_{L^p}^{p-1} + ||\varphi||_{L^p}^p). \]  
(5.4)
Combining (5.4) and (5.3) we obtain
\[ ||\varphi||_{L^{p/2}}^{p/2} \leq C_1(2p \cdot \text{const} \cdot e^{||f||_{C^0}}||\varphi||_{L^{p-1}}^{p-1} + ||\varphi||_{L^p}^p). \]
Let \( q = \kappa p \). Since \( 2 \leq p \leq k \) we have \( ||\varphi||_{L^p} \leq Q_1(C_3p)^{-\frac{2n}{p}} \). Since \( ||\varphi||_{L^{p-1}} \leq ||\varphi||_{L^p} \) we get
\[ ||\varphi||_{L^{p/2}}^{p/2} \leq C_1 \left( 2p \cdot \text{const} \cdot e^{||f||_{C^0}}||\varphi||_{L^{p-1}}^{p-1} + \left( Q_1(C_3p)^{-\frac{2n}{p}} \right)^p \right) \]
\[ = C_1 \left( 2p \cdot \text{const} \cdot e^{||f||_{C^0}} \left( Q_1(C_3p)^{-\frac{2n}{p}} \right)^p \right). \]
But \( Q_1(C_3p)^{-\frac{2n}{p}} \geq 1 \). Hence
\[ ||\varphi||_{L^{p/2}}^{p/2} \leq C_1 Q_1^p(C_3p)^{-2n}(2 \cdot \text{const} \cdot pe^{||f||_{C^0}} + 1). \]
It remains to show that the last expression is at most \( Q_1^p(C_3q)^{-\frac{2n}{q}} \). It is enough to check that
\[ C_1(C_3p)^{-2n}(2p \cdot \text{const} \cdot e^{||f||_{C^0}} + 1) \leq (C_3q)^{-\frac{2n}{q}} = (C_3\kappa p)^{-2n-1}. \]
The left-hand side is at most \( C_1(C_3p)^{-2n} \cdot p(2 \cdot \text{const} \cdot e^{||f||_{C^0}} + 1) \). Hence it is enough to check that
\[ C_1 C_3^{-2n}(2 \cdot \text{const} \cdot e^{||f||_{C^0}} + 1) \leq (C_3\kappa)^{-(2n-1)}. \]
Namely
\[ C_1(C_3p)^{-2n}(2 \cdot \text{const} \cdot e^{||f||_{C^0}} + 1) \leq (C_3\kappa)^{-(2n-1)}. \]
But this holds by the definition of \( C_3 \).
The following corollary is the main result of this section.

**Corollary 5.7:** The exists a constant $C_4$ depending on $M, g_0, \|f\|_{C^0}$ only, such that

$$\|\varphi\|_{C^0} \leq C_4,$$

for any solution of quaternionic Calabi-Yau equation (5.1) which satisfies the normalization condition (5.2).

**Proof:** We have

$$\|\varphi\|_{C^0} = \lim_{p \to \infty} \|\varphi\|_{L^p} \leq Q_1$$

where the last inequality follows from Proposition 5.6.

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