THE FACE VECTOR OF A HALF-OPEN HYPERSIMPLEX

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ABSTRACT. The half-open hypersimplex $\Delta_{n,k}'$ consists of those $x = (x_1, \ldots, x_n) \in [0,1]^n$ with $k-1 < x_1 + \cdots + x_n \leq k$, where $0 < k \leq n$. The $f$-vector of a half-open hypersimplex and related generating functions are explicitly studied.

INTRODUCTION

The hypersimplex is one of the most basic polytopes and has been well studied. In [Sta], Stanley gave geometric proof that the volume of the hypersimplex is the Eulerian number. De Loera, Sturmfels and Thomas [DST] studied a natural connection of the hypersimplex with Gröbner bases. Lam and Postnikov [LP] studied four triangulations of the hypersimplex and showed that these triangulations are identical. People have also looked at the Ehrhart $h^*$-vectors of the hypersimplex, for example [Kat] and [Li]. The half-open hypersimplex is introduced in [Li], where Li proved a conjecture of Stanley on nice combinatorial description of the Ehrhart $h^*$-vectors. In this paper, we study the $f$-vectors of the hypersimplex and the half-open hypersimplex.

Let $k$ and $n$ be integers with $0 < k \leq n$. Recall that the hypersimplex $\Delta_{n,k}$ and the half-open hypersimplex $\Delta_{n,k}'$ are defined as follows:

$$\Delta_{n,k} = \{ x = (x_1, \ldots, x_n) \in [0,1]^n : k-1 \leq x_1 + \cdots + x_n \leq k \},$$

$$\Delta_{n,k}' = \{ x = (x_1, \ldots, x_n) \in [0,1]^n : k-1 < x_1 + \cdots + x_n \leq k \}.$$ 

Let $f_j = f_{j}^{(n,k)}$ denote the number of $j$-faces of $\Delta_{n,k}$ and $f_j' = f_{j'}^{(n,k)}$ those of $\Delta_{n,k}'$, where $j = 0, 1, \ldots, n$. The $f$-vector of $\Delta_{n,k}$ is $f(\Delta_{n,k}) = (f_0, f_1, \ldots, f_n)$ and that of $\Delta_{n,k}'$ is $f(\Delta_{n,k}') = (f_0', f_1', \ldots, f_n')$. The computation of the $f$-vector of $\Delta_{n,k}$ is discussed in [Zie, Exercise 38]. In the present paper we are interested in the $f$-vector of $\Delta_{n,k}'$.

First, in Section 1, a formula to compute $f(\Delta_{n,k}')$ is obtained (Theorem 1.2). The formula yields easily a formula to compute $f(\Delta_{n,k})$ (Corollary 1.4). Section 2 is devoted to the study of generating functions related to $f(\Delta_{n,k}')$ and $f(\Delta_{n,k})$. More precisely, we show that

$$\sum_{n=1}^\infty \left( \sum_{k=1}^n \left( \sum_{j=1}^n f_{j}^{(n,k)} t_j \right) x^k y^{n-k} \right) = \frac{tx(1-x)}{(1-x-y)(1-x-y-ty)(1-x-y-ty)},$$

$$\sum_{n=1}^\infty \left( \sum_{k=1}^n \left( \sum_{j=1}^n f_{j'}^{(n,k)} t_j \right) x^k y^{n-k} \right) = \frac{tx}{(1-x-y)(1-x-y-ty)(1-x-y-ty)},$$

$$\sum_{k=1}^n f_{j'}^{(n,k)} = j \cdot 2^{n-j-1} \frac{n+j+2}{n+1} \cdot \binom{n+1}{j+1}.$$
\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} f_j^{(n,k)} \right) x^n = \frac{jx^j(1-x)}{(1-2x)^{j+2}}.
\]

Finally, in Section 3, we propose two open questions.

1. Face numbers of \( \Delta'_{n,k} \)

We study the \( f \)-vector of \( \Delta'_{n,k} \).

**Example 1.1.** Since the hyperplanes of \( \Delta'_{n,k} \) are

\[
h : x_1 + \cdots + x_n = k
\]

and

\[
H = \{ x_i = 0, x_i = 1 : i = 1, \ldots, n \},
\]

it follows that \( f'_{n-1} = 2n + 1 \). Let us then see an example of \( f'_{n-2} \) for \( 2 = k \leq n \). This is equivalent to computing pairs of hyperplanes \( h_1, h_2 \) which has \((n-2)\)-dimensional intersection with \( \Delta'_{n,k} \). Here is the enumeration:

1. \( h \not\in \{ h_1, h_2 \} \):
   (a) \( \{ h_1, h_2 \} = \{ x_i = 1, x_j = 1 \} \): this kind of pairs do not count, since the intersection with \( \Delta'_{n,2} \) is
   \[
   \left\{ x \in [0,1]^n : \sum_{m \in [n] \setminus \{i,j\}} x_m = 0 \right\},
   \]
   which is 0-dimensional for all \( n \).
   (b) \( \{ h_1, h_2 \} = \{ x_i = 1, x_j = 0 \} \): this kind of pairs counts for all \( n > 2 \), since the intersection with \( \Delta'_{n,2} \) is
   \[
   \left\{ x \in [0,1]^n : 0 < \sum_{m \in [n] \setminus \{i,j\}} x_m \leq 1 \right\},
   \]
   which is \((n-2)\)-dimensional for \( n > 2 \), and 0-dimensional for \( n = 2 \). Therefore there are \( 2 \binom{n}{2} \) such pairs for \( n > 2 \), and 0 for \( n = 2 \).
   (c) \( \{ h_1, h_2 \} = \{ x_i = 0, x_j = 0 \} \): this kind of pairs counts for all \( n > 3 \), since the intersection with \( \Delta'_{n,2} \) is
   \[
   \left\{ x \in [0,1]^n : 1 < \sum_{m \in [n] \setminus \{i,j\}} x_m \leq 2 \right\},
   \]
   which is \((n-2)\)-dimensional for \( n > 3 \), and 0-dimensional for \( n \leq 3 \). Therefore there are \( \binom{n}{3} \) such pairs for \( n > 3 \), and 0 for \( n \leq 3 \).

2. \( h \in \{ h_1, h_2 \} \).
   (a) \( \{ h_1, h_2 \} = \{ h, x_i = 1 \} \): the intersection with \( \Delta'_{n,2} \) is
   \[
   \left\{ x \in [0,1]^n : \sum_{m \in [n] \setminus \{i\}} x_m = 1 \right\},
   \]
   which is \((n-2)\)-dimensional for \( n > 2 \), and 0-dimensional for \( n = 2 \).
In conclusion, for \( k = 2 \) and \( n \geq 2 \), one has

\[
f'_{{n-2}} = 2 \binom{n}{2} + n \quad \text{if} \quad n = 3,
\]

and

\[
f'_{{n-2}} = 3 \binom{n}{2} + 2n \quad \text{if} \quad n > 3.
\]

Following the above enumeration method, one has the following general formula.

**Theorem 1.2.** Let \( 0 < k \leq n \) and \( f(\Delta'_{n,k}) = (f'_0, f'_1, \ldots, f'_n) \) the \( f \)-vector of the half-open hypersimplex \( \Delta'_{n,k} \). Then one has

\[
f'_j = \binom{n + 1}{j + 1} \sum_{s = \max\{0, k - j \}}^{k - 1} \binom{n - j}{s} \frac{n - s + 1}{n + 1}
\]

for \( j = 1, 2, \ldots, n \).

**Proof.** Similar as in the above example for \( f'_{{n-2}} \) when \( k = 2 \), here for general \( k \), \( f'_{{n-i}} \) is counting the \( i \)-set \( h_1, \ldots, h_i \) which has \((n - i)\)-dimensional intersection with \( \Delta'_{n,k} \). There are again two cases: the part of \( \binom{n}{i} \) in the above formula deals with the case \( h \) is not included in these \( i \) hyperplanes, and the part of \( \binom{n}{i-1} \) is counting the case when \( h \) is one of the \( i \) hyperplanes.

Let us look at the first case carefully, and the second case is enumerated similarly. In the first case, the \( i \)-tuple looks like

\[
\{h_1, \ldots, h_i\} = \{x_{i \in I} = 0, x_{j \in J} = 1 : I \cap J = \emptyset, \#\{I \cup J\} = i, I, J \in [n]\}.
\]

Let \( s = \#J \). Then there are in total \( \binom{i}{s} \) such \( i \)-tuples. Now the key point is whether the intersection of all these \( i \) hyperplanes with \( \Delta'_{n,k} \) is \((n - i)\)-dimensional or not. Here is their intersection:

\[
x \in [0, 1]^n : k - s - 1 < \sum_{m \in [n] \setminus (I \cup J)} x_m \leq k - s \}
\]

Notice that the intersection is \((n - i)\)-dimensional if and only if \( \#\{[n] \setminus \{I \cup J\}\} = n - i > k - s \) and \( s < k \). This is exactly why we have

\[
\max\{0, k + i - n\} \leq s \leq \min\{k - 1, i\}
\]

in the summand when counting the above \( i \)-tuples. Thus, we have

\[
f'_{{n-i}} = \binom{n}{i} \left( \sum_{s = \max\{0, k+i-n\}}^{\min\{k-1,i\}} \binom{i}{s} \right) + \binom{n}{i-1} \left( \sum_{s = \max\{0, k+i-n\}}^{\min\{k-1,i-1\}} \binom{i-1}{s} \right).
\]
Hence,
\[
f'_j = \binom{n}{j} \left( \sum_{s=\max\{0,k-j\}}^{k-1} \binom{n-j}{s} \right) + \binom{n}{j+1} \left( \sum_{s=\max\{0,k-j\}}^{k-1} \binom{n-j-1}{s} \right)
\]
\[
= \sum_{s=\max\{0,k-j\}}^{k-1} \left( \binom{n}{j} \binom{n-j}{s} + \binom{n}{j+1} \binom{n-j-1}{s} \right)
\]
\[
= \sum_{s=\max\{0,k-j\}}^{k-1} \binom{n+1}{j+1,s,n-j-s} \frac{n-s+1}{n+1}
\]
\[
= \binom{n+1}{j+1} \sum_{s=\max\{0,k-j\}}^{k-1} \binom{n-j}{s} \frac{n-s+1}{n+1}.
\]

Example 1.3. Let us take \( k = i = 2 \) in Theorem 1.2

- For \( n = 3 \), we have \( \max\{0, k+i-n\} = 1 \) and \( \min\{k-1, i\} = \min\{k-1, i-1\} = 1 \). So \( f_{n-2} = \binom{n}{\frac{n}{2}} \left( \binom{3}{0} \right) + \binom{n}{\frac{n}{2}} \left( \binom{3}{1} \right) \) for \( n = 3 \).
- For \( n > 3 \), we have \( \max\{0, k+i-n\} = 0 \) and \( \min\{k-1, i\} = \min\{k-1, i-1\} = 1 \). So \( f_{n-2} = \binom{n}{\frac{n}{2}} \left( \binom{3}{0} + \binom{3}{1} \right) + \binom{n}{\frac{n}{2}} \left( \binom{3}{0} + \binom{3}{1} \right) \) for \( n > 3 \).

This matches Example 1.1 we computed.

Using the above method, it is not hard to get the \( f \)-vector of the hypersimplex \( \Delta_{n,k} \) from the \( f \)-vector of the hypersimplex \( \Delta'_{n,k} \).

Corollary 1.4. Let \( f(\Delta_{n,k}) = (f_0, f_1, \ldots, f_n) \) be the \( f \)-vector of the hypersimplex \( \Delta_{n,k} \) and \( f(\Delta'_{n,k}) = (f'_0, f'_1, \ldots, f'_n) \) that of the half-open hypersimplex \( \Delta'_{n,k} \). Then one has

\[
f_j = f'_j + \binom{n}{j+1} \sum_{s=\max\{0,k-1-j\}}^{k-2} \binom{n-j-1}{s} = \binom{n+1}{j+1} \sum_{s=\max\{0,k-j\}}^{k-1} \binom{n-j}{s}
\]

for \( j = 1, 2, \ldots, n \).

Proof. The only difference with the half-open hypersimplex is that there is now one more case for the \( i \)-tuple \( \{h_1, \ldots, h_i\} \), which is when one of them is the hyperplane \( x_1 + \cdots + x_n = k-1 \). And the set of such \( i \)-tuples \( \{h_1, \ldots, h_i\} \) with \( (n-i) \)-dimensional intersection with \( \Delta_{n,k} \) is exactly the same as the \( i \)-tuples \( \{h_1, \ldots, h_i\} \) in the second case of Theorem 1.2 for the half-open hypersimplex \( \Delta'_{n,k-1} \), i.e., when the hyperplane \( x_1 + \cdots + x_{n-1} = k-1 \in \{h_1, \ldots, h_i\} \). The number of such \( i \)-tuples is enumerated by the second summand in Theorem 1.2, replacing \( k \) by \( k-1 \). Therefore, we obtain the formula

\[
f_{n-i} = f'_{n-i} + \binom{n}{i-1} \sum_{s=\max\{0,k-1+i-n\}}^{\min\{k-2,i-1\}} \binom{i-1}{s}.
\]
Hence

\[
\begin{align*}
    f_j & = f'_j + \binom{n}{j+1} \sum_{s=\max\{0,k-1-j\}}^{k-2} \binom{n-j-1}{s} \\
                      & = \sum_{s=\max\{0,k-j\}}^{k-1} \binom{n+1}{j+1,s,n-j-s} \frac{n-s+1}{n+1} \binom{n}{j+1,s,n-1-j-s} \\
                      & + \sum_{s=\max\{0,k-j\}}^{k-1} \binom{n+1}{j+1,s,n-j-s} \frac{n-s+1}{n+1} \binom{n}{j+1,s-1,n-j-s} \\
                      & = \sum_{s=\max\{0,k-j\}}^{k-1} \binom{n+1}{j+1,s,n-j-s} \\
                      & + \sum_{s=\max\{0,k-j\}}^{k-1} \left( \binom{n}{j+1,s-1,n-j-s} - \binom{n+1}{j+1,s,n-j-s} \frac{s}{n+1} \right) \\
                      & = \binom{n+1}{j+1} \sum_{s=\max\{0,k-j\}}^{k-1} \binom{n-j}{s},
\end{align*}
\]

as desired. \[\square\]

2. Generating functions

We are now discuss generating functions which are related to the \(f\)-vectors of half-open hypersimplices.

**Theorem 2.1.** Let \(f_j^{(n,k)}\) denote the number of \(j\)-faces of \(\Delta_{n,k}'\). Then, we have

\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \sum_{j=1}^{f_j^{(n,k)}} x^k y^{n-k} \right) = \frac{tx(1-x)}{(1-x-y)(1-x-y-tx)(1-x-y-ty)}.
\]
Proof. The coefficient of \(x^ky^{n-k}\) in

\[
\frac{tx(1-x)}{(1-x-y)(1-x-y-tx)(1-x-y-ty)}
\]

\[
= \frac{tx(1-x)}{(1-x-y)^3} \cdot \frac{1}{1-\frac{tx}{1-x-y}} \cdot \frac{1}{1-\frac{tx}{1-x-y}}
\]

\[
= \frac{tx(1-x)}{(1-x-y)^3} \left( \sum_{p=0}^{\infty} \left( \frac{tx}{1-x-y} \right)^p \right) \left( \sum_{q=0}^{\infty} \left( \frac{tx}{1-x-y} \right)^q \right)
\]

\[
= \frac{tx(1-x)}{(1-x-y)^3} \left( \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left( \frac{t}{1-x-y} \right)^{p+q} x^p y^q \right)
\]

\[
= \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \frac{t^{p+q+1}}{(1-x-y)^{p+q+3}} (1-x)x^{p+1}y^q
\]

is equal to the coefficient of \(x^ky^{n-k}\) in

\[
\sum_{p=0}^{j-1} \frac{(1-x)x^{p+1}y^{j-p-1}}{(1-x-y)^{j+2}}
\]

(since \(p+q+1 = j\) if and only if \(q = j-p-1 \geq 0\)). It then follows that

\[
\sum_{p=0}^{j-1} \frac{(1-x)x^{p+1}y^{j-p-1}}{(1-x-y)^{j+2}} = \sum_{p=0}^{j-1} (1-x)x^{p+1}y^{j-p-1} \sum_{r=0}^{\infty} \binom{j+r+1}{j+1} (x+y)^r
\]

\[
= \sum_{p=0}^{j-1} \sum_{r=0}^{\infty} \sum_{u=0}^{r} (1-x)x^{p+1}y^{j-p-1} \binom{j+r+1}{j+1} \binom{r}{u} x^u y^{r-u}
\]

\[
= \sum_{p=0}^{j-1} \sum_{r=0}^{\infty} \sum_{u=0}^{r} \binom{j+r+1}{j+1} \binom{r}{u} x^{p+u+1}y^{j-p+r-u-1}
\]

\[
- \sum_{p=0}^{j-1} \sum_{r=0}^{\infty} \sum_{u=0}^{r} \binom{j+r+1}{j+1} \binom{r}{u} x^{p+u+2}y^{j-p+r-u-1}.
\]

Then,

- \(x^{p+u+1}y^{j-p+r-u-1} = x^ky^{n-k}\) if and only if \(p+u+1 = k\) and \(j+r = n\);
- \(x^{p+u+2}y^{j-p+r-u-1} = x^ky^{n-k}\) if and only if \(p+u+2 = k\) and \(j+r+1 = n\).
Thus, the coefficient of \( x^k y^{n-k} \) is

\[
\sum_{p=0}^{j-1} \sum_{r=n-j} \sum_{u=k-p-1} \left( \begin{array}{c} j+r+1 \\ n \\ j+1 \end{array} \right) \left( \begin{array}{c} r \\ u \\ j+1 \end{array} \right) - \sum_{p=0}^{j-1} \sum_{r=n-j} \sum_{u=k-p-2} \left( \begin{array}{c} j+r+1 \\ n \\ j+1 \end{array} \right) \left( \begin{array}{c} r \\ u \\ j+1 \end{array} \right)
\]

\[
= \sum_{p=\max\{0,k+1-n+j\}}^{\min\{j-1,k-1\}} (n+1) \left( \begin{array}{c} n-j \\ j+1 \end{array} \right) (k-p-1) - \sum_{p=\max\{0,k+1-n+j\}}^{\min\{k-1,n-j\}} (n) \left( \begin{array}{c} n-j \\ j+1 \end{array} \right) (k-p-2)
\]

\[
= \sum_{s=\max\{0,k-j\}}^{\min\{k-1,n-j\}} (n+1) \left( \begin{array}{c} n-j \\ j+1 \end{array} \right) (s) - \sum_{s=\max\{0,k-j\}}^{\min\{k-1,n-j\}} (n) \left( \begin{array}{c} n-j \\ j+1 \end{array} \right) (s+1)
\]

\[
= \sum_{s=\max\{0,k-j\}}^{\min\{k-1,n-j\}} (n) \left( \begin{array}{c} n-j \\ j+1 \end{array} \right) \frac{n-s+1}{n+1},
\]

as desired. 

By the proof of Theorem 2.1, it is also shown that

**Corollary 2.2.** Let \( f_j^{(n,k)} \) denote the number of \( j \)-faces of \( \Delta_{n,k} \). Then, we have

\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} \left( \sum_{j=1}^{n} f_j^{(n,k)} t^j \right) \right) x^k y^{n-k} = \frac{tx}{(1-x-y)(1-x-y-ty)(1-x-y-ty)}. \]

On the other hand, by Theorem 2.1, we have the \( f \)-vector of the hypersimplicial decomposition of the unit cube, and its generating function.

**Theorem 2.3.** Let \( f_j^{(n,k)} \) denote the number of \( j \)-faces of the half-open hypersimplex \( \Delta_{n,k}' \). Then, we have

\[
\sum_{k=1}^{n} f_j^{(n,k)} = j \cdot 2^{n-j-1} \frac{n+j+2}{n+1} \cdot \frac{(n+1)}{j+1}
\]

and

\[
\sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} f_j^{(n,k)} \right) x^n = \frac{jx^j(1-x)}{(1-2x)^{j+2}}.
\]

**Proof.** By substituting \( x \) for \( y \) in the equation of Theorem 2.1, we have

\[
\sum_{n=1}^{\infty} \sum_{k=1}^{n} \sum_{j=1}^{n} f_j^{(n,k)} x^k y^{n-k} = \frac{tx(1-x)}{(1-2x)(1-2x-tx)^2}
\]

\[
= \frac{tx(1-x)}{(1-2x)^3(1-x-2tx+x^2)^2}
\]

\[
= \frac{tx(1-x)}{(1-2x)^3} \sum_{j=0}^{\infty} (j+1) \left( \frac{x}{1-2x} \right)^j t^j.
\]
Thus, we have
\[ \sum_{n=1}^{\infty} \left( \sum_{k=1}^{n} f_j^{(n,k)} \right) x^n = \frac{x(1-x)}{(1-2x)^3} \cdot j \left( \frac{x}{1-2x} \right)^{j-1} = \frac{jx^j(1-x)}{(1-2x)^{j+2}}. \]
Moreover, the coefficient of \( x^n \) in
\[ \frac{jx^j(1-x)}{(1-2x)^{j+2}} = jx^j(1-x) \sum_{m=0}^{\infty} \binom{m+j+1}{j+1} 2^m x^m \]
\[ = j \sum_{m=0}^{\infty} \binom{m+j+1}{j+1} 2^m x^{m+j} - j \sum_{m=0}^{\infty} \binom{m+j+1}{j+1} 2^m x^{m+j+1} \]
\[ = j \sum_{m=0}^{\infty} \binom{m+j+1}{j+1} 2^m x^{m+j} - j \sum_{m=0}^{\infty} \binom{m+j}{j+1} 2^{m-1} x^{m+j} \]
\[ = j \sum_{m=0}^{\infty} \left( 2 \binom{m+j+1}{j+1} - \binom{m+j}{j+1} \right) 2^{m-1} x^{m+j} \]
\[ = j \sum_{m=0}^{\infty} \frac{m+2j+2}{m+j+1} \binom{m+j+1}{j+1} 2^{m-1} x^{m+j} \]
is
\[ j \frac{(n-j) + 2j + 2}{(n-j) + j + 1} \binom{n-j+1}{j+1} 2^{(n-j)-1} = j \cdot 2^{n-j-1} \frac{n+j+2}{n+1} \cdot \frac{(n+1)}{j+1}. \]

*Geometric proof of Theorem 2.3.* The first equation of Theorem 2.3 gives the \( f \)-vector of the hypersimplicial decomposition of the unit cube. Here we compute this \( f \)-vector directly. Similar as the proof of Theorem 1.2 for \( f^{(n,k)}_j \), we count the \( j \)-dimensional intersections obtained by \((n-j)\) hyperplanes of the unit cube. There are again two types of such \( j \)-faces:

(1) obtained by intersections of \((n-j)\) hyperplanes of the form \( x_i = 0 \) or \( 1 \). There are \( 2^{n-j} \binom{n}{n-j} \);

(2) obtained by intersections of \((n-j-1)\) hyperplanes of the form \( x_i = 0 \) or \( 1 \) together with one more hyperplane of the form \( \sum_{i=1}^{n} x_i = k \). Similar to the argument in the proof of Theorem 1.2 we can see that each combination of the \((n-j-1)\) hyperplanes intersects nontrivially (i.e., get \( j \)-dimensional intersection) with exactly \( j \) hyperplanes of the form \( \sum_{i=1}^{n} x_i = k \). In fact, for a given choice of \((n-j-1)\) hyperplanes of the form \( x_i = 0 \) or \( 1 \), let \( s \) be the number of indices \( m \) with \( x_m = 1 \), then above \( j \) hyperplanes corresponds to \( k = s+1, \ldots, s+j \). Therefore, there are \( j \cdot 2^{n-j-1} \binom{n}{n-j-1} \) such \( j \)-faces.

Now comes the tricky part: the correct number of first type \( j \)-face should be \( 2^{n-j} \binom{n}{n-j} \) times \( j \). This is because there are exactly \((j-1)\) \((j-1)\)-faces in the interior of each such \( j \)-face, resulting in \( j \) \( j \)-faces. Consider \( f_2 \) for the three dimensional cube for a visual help. Therefore, there are in total
\[ 2^{n-j} \binom{n}{n-j} \cdot j + j \cdot 2^{n-j-1} \binom{n}{n-j-1} = j \cdot 2^{n-j-1} \frac{n+j+2}{n+1} \cdot \frac{(n+1)}{j+1} \]
\( j \)-faces in the hypersimplicial decomposition of the unit cube. □
3. Two open questions

In this section, we present two open questions related with Theorem 2.3.

First, notice that in the above geometric proof for Theorem 2.3, we get the result as a sum of two parts. Since the result of Theorem 2.3 is neat and simple, it will be very nice to have a direct combinatorial proof avoiding sums.

**Question 3.1.** Fine combinatorial proof for Theorem 2.3.

We also observe a relation between Theorem 2.3 and the coefficients of Chebyshev polynomials for \( j = 2 \). In this case \( \frac{1}{j} \sum_{k=1}^{n} f'_{k} = 2^{n-j-1} \frac{n+j+2}{n+1} \cdot \binom{n+1}{j+1} \), which is 1, 7, 32, 120, 400, 1232, 3584, . . . (for \( n = 2, 3, 4, 5, 6, 7, 8, \ldots \)). According to OEIS, this sequence appears in the triangle table of coefficients of Chebyshev polynomials of the first kind. There are some known combinatorial models for Chebyshev polynomials such as [BW], [Sha] and [Mun], but their connection with \( f \)-vectors studied here is not clear to us.

**Question 3.2.** Find combinatorial connection between Chebyshev polynomials and the sum of \( f \)-vectors for the half-open hypersimplex, or equivalently, the \( f \)-vectors of the hypersimplicial decomposition of the unit cube.

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