Novel results on Hermite–Hadamard kind inequalities for \( \eta \)-convex functions by means of \((k,r)\)-fractional integral operators

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Abstract

We establish new integral inequalities of Hermite–Hadamard type for the recent class of \( \eta \)-convex functions. This is done via generalized \((k,r)\)-Riemann–Liouville fractional integral operators. Our results generalize some known theorems in the literature. By choosing different values for the parameters \( k \) and \( r \), one obtains interesting new results.

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1 Introduction

Throughout this work, \( I \subset \mathbb{R} \) shall denote an interval and \( I^\circ \) the interior of \( I \). We say that a function \( g : I \to \mathbb{R} \) is convex if, for every \( x, y \in I \) and \( \beta \in [0,1] \), one has

\[
g(\beta x + (1-\beta)y) \leq \beta g(x) + (1-\beta)g(y).
\]

Let \( a, b \in I \). For a function \( g \) satisfying (1), the following inequalities hold:

\[
g \left( \frac{a+b}{2} \right) \leq \frac{1}{b-a} \int_a^b g(x) \, dx \leq \frac{g(a) + g(b)}{2}. \tag{2}
\]

Result (2) was proved by Hadamard in 1893 \cite{6} and is celebrated in the literature as the Hermite–Hadamard integral inequality for convex functions \cite{2}. Along the years, it has been extended to different classes of convex functions: see, e.g., \cite{3,8,15} and references therein.

In 2016, the so called \( \varphi \)-convexity was introduced \cite{5}, subsequently denoted as \( \eta \)-convexity \cite{4,12}. Let us recall its definition here.

**Definition 1** (See \cite{5}). A function \( g : I \to \mathbb{R} \) is called convex with respect to \( \eta \) (for short, \( \eta \)-convex), if

\[
g(\beta x + (1-\beta)y) \leq g(y) + \beta \eta(g(x), g(y))
\]

for all \( x, y \in I \) and \( \beta \in [0,1] \).

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By taking \( \eta(x,y) = x - y \), Definition 1 reduces to the classical notion of convexity. It was further shown in [3] that for every convex function \( g \) there exists some \( \eta \), different from \( \eta(x,y) = x - y \), for which the function \( g \) is \( \eta \)-convex. The converse is, however, not necessarily true, that is, there are \( \eta \)-convex functions that are not convex.

**Example 2.** Consider function \( g : \mathbb{R} \to \mathbb{R} \) defined piecewisely by

\[
g(x) = \begin{cases} -x, & x \geq 0, \\ x, & x < 0, \end{cases}
\]

and let \( \eta : [-\infty, 0] \times [-\infty, 0] \to \mathbb{R} \) be given by \( \eta(x,y) = -x - y \). Function \( g \) is clearly not convex but it is easy to see that it is \( \eta \)-convex. Indeed, in [12, Remark 4] it is noted that an \( \eta \)-convex function \( g : [a,b] \to \mathbb{R} \) is integrable if \( \eta \) is bounded from above on \( g([a,b]) \times g([a,b]) \).

For the class of \( \eta \)-convex functions, the following theorem was obtained as an analogue of [2].

**Theorem 3 (See [5]).** Suppose that \( g : I \to \mathbb{R} \) is an \( \eta \)-convex function such that \( \eta \) is bounded from above on \( g(I) \times g(I) \). Then, for any \( a,b \in I \) with \( a < b \),

\[
2g \left( \frac{a + b}{2} \right) - M_\eta \leq \frac{1}{b-a} \int_a^b g(x) \, dx \leq f(b) + \frac{\eta(g(a),g(b))}{2},
\]

where \( M_\eta \) is an upper bound of \( \eta \) on \( g([a,b]) \times g([a,b]) \).

Recently, Rostamian Delavar and De La Sen obtained, among other results, the following theorem associated to \( \eta \)-convex functions [12].

**Theorem 4 (See [12]).** Suppose \( g : [a,b] \to \mathbb{R} \) is a differentiable function and \( |g'| \) is an \( \eta \)-convex function with \( \eta \) bounded from above on \([a,b]\). Then,

\[
\left| \frac{g(a) + g(b)}{2} - \frac{1}{b-a} \int_a^b g(x) \, dx \right| \leq \frac{1}{8}(b-a)K,
\]

where \( K = \min \left\{ |g'(b)| + \frac{\eta(g'(a),g'(b))}{2}, |g'(a)| + \frac{\eta(g'(b),g'(a))}{2} \right\} \).

Still in the same spirit, Khan et al. established in 2017 the following result for \( \eta \)-convex functions via Riemann–Liouville fractional integral operators [9].

**Theorem 5 (See [9]).** Let \( g : [a,b] \to \mathbb{R} \) be a differentiable function on \((a,b)\) with \( a < b \). If \( |g'| \) is an \( \eta \)-convex function on \([a,b]\), then for \( \alpha > 0 \) the inequality

\[
\left| \frac{g(a) + g(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(b-a)^\alpha} \left[ J_a^\alpha g(b) + J_b^\alpha g(a) \right] \right| \\
\leq \frac{b-a}{2(\alpha + 1)} \left( 1 - \frac{1}{2^\alpha} \right) \left( 2|g'(b)| + \eta(|g'(a)|,|g'(b)|) \right)
\]

holds, where

\[
J_a^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} g(t) \, dt
\]

is the left Riemann–Liouville fractional integral and

\[
J_b^\alpha g(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} g(t) \, dt
\]

is the right Riemann–Liouville fractional integral.
Fractional calculus is an area under strong development \[11\] and in \[13\] Sarikaya et al. proposed the following broader definition of the Riemann–Liouville fractional integral operators.

**Definition 6** (See \[13\]). The \((k,r)\)-Riemann–Liouville fractional integral operators \(\mathcal{I}_a^{\alpha+k}\) and \(\mathcal{I}_b^\alpha\) of order \(\alpha > 0\), for a real valued continuous function \(g(x)\), are defined as
\[
\mathcal{I}_a^{\alpha+k}g(x) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^x (x^{-r+1} - t^{-r+1})^{\frac{\alpha}{k} - 1} t^r g(t) \, dt, \quad x > a, 
\]
and
\[
\mathcal{I}_b^\alpha g(x) = \frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_x^b (t^{-r+1} - x^{-r+1})^{\frac{\alpha}{k} - 1} t^r g(t) \, dt, \quad x < b, 
\]
where \(k > 0, \ r \in \mathbb{R}\setminus\{-1\}\), and \(\Gamma_k\) is the \(k\)-gamma function given by
\[
\Gamma_k(x) := \int_0^\infty t^{r-1}e^{-\frac{t}{k}} \, dt, \quad Re(x) > 0,
\]
with the properties \(\Gamma_k(x+k) = x\Gamma_k(x)\) and \(\Gamma_k(k) = 1\).

For some results related to the operators \(3\) and \(4\), we refer the interested readers to \[7, 10, 13\]. Using these operators, Agarwal et al. established the following Hermite–Hadamard type result for convex functions \[1\].

**Theorem 7** (See \[1\]). Let \(\alpha > 0\) and \(r \in \mathbb{R}\setminus\{-1\}\). If \(g\) is a convex function on \([a, b]\), then
\[
g\left(\frac{a + b}{2}\right) \leq \frac{(r+1)^{\alpha + 1}}{4(a^{r+1} - b^{r+1})^\frac{\alpha}{k}} \left[\mathcal{I}_a^{\alpha+k}g(b) - \mathcal{I}_a^{\alpha+k}G(b) + \mathcal{I}_b^\alpha G(a) - \mathcal{I}_b^\alpha g(a)\right] \leq \frac{g(a) + g(b)}{2},
\]
where function \(G\) is defined by \((5)\) below.

Inspired by the above works, it is our purpose to obtain here more general integral inequalities associated to \(\eta\)-convex functions via the \((k,r)\)-Riemann–Liouville fractional operators. Theorems \[8\] and \[12\] generalize Theorems \[7\] and \[9\] respectively (see Remarks \[9\] and \[13\]). In addition, two more fractional Hermite–Hadamard type inequalities are also established (see Theorems \[14\] and \[15\]).

## 2 Main results

We establish four new results. For this, we start by making the following observations. Let \(g\) be a function defined on \(I\) with \([a, b] \subset I^0\) and define functions \(G, \tilde{g} : [a, b] \rightarrow \mathbb{R}\) by
\[
\tilde{g}(x) := g(a + b - x) \quad \text{and} \quad G(x) := g(x) + \tilde{g}(x).
\]
For the fractional operators to be well defined, we shall assume \(g \in L_\infty[a, b]\). By making use of the substitutions \(w = \frac{t-a}{a-x}\) and \(w = \frac{b-t}{b-x}\) in \[3\] and \[4\], respectively, one gets that
\[
\mathcal{I}_a^{\alpha+k}g(x) = (x-a)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^1 \frac{(wx + (1-w)a)^r g(wx + (1-w)a)}{[x^{r+1} - (wx + (1-w)a)^{r+1}]^{\frac{\alpha}{k}} - \frac{\alpha}{k}} \, dw.
\]
and
\[
\mathcal{I}_b^\alpha g(x) = (b-x)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(wx + (1-w)b)^r g(wx + (1-w)b)}{[(wx + (1-w)b)^{r+1} - x^{r+1}]^{\frac{\alpha}{k}} - \frac{\alpha}{k}} \, dw.
\]
Noting that \(\tilde{g}((1-w)a + wb) = g(\omega a + (1-w)b)\), we also obtain
\[
\mathcal{I}_a^{\alpha+k}\tilde{g}(x) = (x-a)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_a^1 \frac{(wx + (1-w)a)^r g((1-w)x + wa)}{[x^{r+1} - (wx + (1-w)a)^{r+1}]^{\frac{\alpha}{k}} - \frac{\alpha}{k}} \, dw \quad \text{and}
\]
\[
\mathcal{I}_b^\alpha \tilde{g}(x) = (b-x)\frac{(r+1)^{1-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)} \int_0^1 \frac{(wx + (1-w)b)^r g((1-w)x + wb)}{[(wx + (1-w)b)^{r+1} - x^{r+1}]^{\frac{\alpha}{k}} - \frac{\alpha}{k}} \, dw.
\]
We are now ready to formulate and prove our first result.
Theorem 8. Let \( \alpha, k > 0, r \in \mathbb{R} \setminus \{-1\} \), and \( g : I \to \mathbb{R} \) be a positive function on \([a, b] \subset I^\circ\) with \( a < b \). If, in addition, \( g \) is \( \eta \)-convex on \([a, b] \) with \( \eta \) bounded on \( g([a, b]) \times g([a, b]) \), then the \((k, r)\)-fractional integral inequality

\[
\frac{(r+1)^{\frac{\alpha}{2}}}{4(b^{r+1} - a^{r+1})^{\frac{\alpha}{2}}} \left[ kJ^\alpha_a G(b) + \frac{r}{k}J^\alpha_b G(a) \right] \leq g(b) + \frac{\eta(g(a), g(b))}{2}
\]

holds.

Proof. Function \( g \) is \( \eta \)-convex on \([a, b] \), which implies, by definition, the following inequalities for \( t \in [0, 1] \):

\[
g(ta + (1-t)b) \leq g(b) + t\eta(g(a), g(b)) \tag{10}
\]

and

\[
g((1-t)a + tb) \leq g(b) + (1-t)\eta(g(a), g(b)). \tag{11}
\]

Adding inequalities (10) and (11), we get

\[
g(ta + (1-t)b) + g((1-t)a + tb) \leq 2g(b) + \eta(g(a), g(b)). \tag{12}
\]

Multiplying both sides of (12) by

\[
(b-a)(r+1)^{\frac{1-\frac{\alpha}{2}}{k\Gamma_k}} \frac{(tb + (1-t)a)^r}{[b^{r+1} - (tb + (1-t)a)^{r+1}]^{1-\frac{\alpha}{2}}},
\]

and integrating over \([0, 1]\) with respect to \( t \), we get

\[
\begin{align*}
&(b-a)(r+1)^{\frac{1-\frac{\alpha}{2}}{k\Gamma_k}} \int_0^1 \frac{(tb + (1-t)a)^r g((1-t)b + ta)}{[b^{r+1} - (tb + (1-t)a)^{r+1}]^{1-\frac{\alpha}{2}}} dt \\
&\quad + (b-a)(r+1)^{\frac{1-\frac{\alpha}{2}}{k\Gamma_k}} \int_0^1 \frac{(tb + (1-t)a)^r g((1-t)a + tb)}{[b^{r+1} - (tb + (1-t)a)^{r+1}]^{1-\frac{\alpha}{2}}} dt \\
&\leq [2g(b) + \eta(g(a), g(b))] (b-a) \frac{(r+1)^{\frac{1-\frac{\alpha}{2}}{k\Gamma_k}}}{k\Gamma_k} \int_0^1 \frac{(tb + (1-t)a)^r}{[b^{r+1} - (tb + (1-t)a)^{r+1}]^{1-\frac{\alpha}{2}}} dt.
\end{align*}
\]

Now, using (7) and (8) in the above inequality, we get

\[
\frac{r}{k}J^\alpha_a G(b) + kJ^\alpha_b G(b) \leq \frac{(s+1)^{\frac{\alpha}{2}}}{(s+1)\alpha\Gamma_\alpha} \frac{(b^{r+1} - a^{r+1})^{\frac{\alpha}{2}}}{2g(b) + \eta(g(a), g(b))},
\]

that is,

\[
\frac{r}{k}J^\alpha_a G(b) \leq \frac{(b^{r+1} - a^{r+1})^{\frac{\alpha}{2}}}{(s+1)\alpha\Gamma_\alpha} \frac{(s+1)^{\frac{\alpha}{2}}}{2g(b) + \eta(g(a), g(b))}. \tag{13}
\]

Similarly, multiplying again both sides of (13) by

\[
(b-a)(r+1)^{\frac{1-\frac{\alpha}{2}}{k\Gamma_k}} \frac{(tb + (1-t)a)^r}{[(tb + (1-t)a)^{r+1} - a^{r+1}]^{1-\frac{\alpha}{2}}}
\]

and integrating with respect to \( t \) over \([0, 1]\), we obtain that

\[
\frac{r}{k}J^\alpha_b G(a) \leq \frac{(b^{r+1} - a^{r+1})^{\frac{\alpha}{2}}}{(s+1)\alpha\Gamma_\alpha} \frac{(s+1)^{\frac{\alpha}{2}}}{2g(b) + \eta(g(a), g(b))}. \tag{14}
\]

Hence, the intended inequality follows by adding (13) and (14). \( \square \)

Remark 9. By taking \( \eta(x, y) = x - y \) in our Theorem 8, we recover the right-hand side of the inequalities in Theorem 7.
For the rest of our results, we will need the following two lemmas.

**Lemma 10** (See [1]). Let \( \alpha, k > 0 \) and \( r \in \mathbb{R} \setminus \{-1\} \). If \( g : I \to \mathbb{R} \) is differentiable on \( I^\circ \) and \( a, b \in I^\circ \) such that \( g' \in L[a, b] \) with \( a < b \), then the following identity holds:

\[
\frac{g(a) + g(b)}{2} - \frac{(r+1) \Gamma_k(\alpha + k)}{4(b^{r+1} - a^{r+1})^\frac{1}{r}} \left[ \kappa \mathcal{J}_\alpha^a G(b) + \kappa \mathcal{J}_\alpha^b G(a) \right] = \frac{b-a}{4(b^{r+1} - a^{r+1})^\frac{1}{r}} \int_0^1 \Theta_{\alpha, r}(t)g'(ta + (1-t)b) \, dt,
\]

where \( \Theta_{\alpha, r} : [0, 1] \to \mathbb{R} \) is defined by

\[
\Theta_{\alpha, r}(t) := \left[ (ta + (1-t)b)^{r+1} - a^{r+1} \right]^\frac{1}{r} - \left[ (tb + (1-t)a)^{r+1} - a^{r+1} \right]^\frac{1}{r}
+ \left[ b^{r+1} - (tb + (1-t)a)^{r+1} \right]^\frac{1}{r} - \left[ b^{r+1} - (ta + (1-t)b)^{r+1} \right]^\frac{1}{r}.
\]

**Lemma 11.** Under the conditions of Lemma 10, we have that

\[
\int_0^1 |\Theta_{\alpha, r}(t)| \, dt = \frac{1}{b-a} \left( \mathcal{R}_1 + \mathcal{R}_2 + \mathcal{R}_3 + \mathcal{R}_4 \right),
\]

where

\[
\mathcal{R}_1 = \int_{\frac{a+b}{2}}^b \left[ w^{r+1} - a^{r+1} \right]^\frac{1}{r} \, dw - \int_a^{\frac{a+b}{2}} \left[ w^{r+1} - a^{r+1} \right]^\frac{1}{r} \, dw,
\]

\[
\mathcal{R}_2 = \int_{\frac{a+b}{2}}^b \left[ (b+a-w)^{r+1} - a^{r+1} \right]^\frac{1}{r} \, dw - \int_a^{\frac{a+b}{2}} \left[ (b+a-w)^{r+1} - a^{r+1} \right]^\frac{1}{r} \, dw,
\]

\[
\mathcal{R}_3 = \int_a^{\frac{a+b}{2}} \left[ (b+a-w)^{r+1} - w^{r+1} \right]^\frac{1}{r} \, dw - \int_{\frac{a+b}{2}}^b \left[ (b+a-w)^{r+1} - w^{r+1} \right]^\frac{1}{r} \, dw,
\]

and

\[
\mathcal{R}_4 = \int_a^{\frac{a+b}{2}} \left[ (b+a-w)^{r+1} - a^{r+1} \right]^\frac{1}{r} \, dw - \int_{\frac{a+b}{2}}^b \left[ (b+a-w)^{r+1} - a^{r+1} \right]^\frac{1}{r} \, dw.
\]

**Proof.** Using the substitution \( w = ta + (1-t)b \), we get

\[
\int_0^1 |\Theta_{\alpha, r}(t)| \, dt = \frac{1}{b-a} \int_a^b |\varphi(w)| \, dw,
\]

where

\[
\varphi(w) = \left( w^{r+1} - a^{r+1} \right)^\frac{1}{r} - \left[ (b+a-w)^{r+1} - a^{r+1} \right]^\frac{1}{r}
+ \left[ b^{r+1} - (b+a-w)^{r+1} \right]^\frac{1}{r} - \left( b^{r+1} - w^{r+1} \right)^\frac{1}{r}.
\]

The required result follows from (15) and by observing that \( \varphi \) is a non-decreasing function on \( [a, b] \), \( \varphi(a) = -2(b^{r+1} - a^{r+1})^\frac{1}{r} < 0 \), \( \varphi \left( \frac{a+b}{2} \right) = 0 \), and thus

\[
\begin{align*}
\varphi(w) &\leq 0 \quad \text{if} \quad a \leq w \leq \frac{a+b}{2}, \\
\varphi(w) &> 0 \quad \text{if} \quad \frac{a+b}{2} < w \leq b.
\end{align*}
\]

This concludes the proof. \( \square \)
**Theorem 12.** Let $\alpha, k > 0$, $r \in \mathbb{R} \setminus \{-1\}$, $g : I \to \mathbb{R}$ be a differentiable function on $I^\circ$ and $a, b \in I^\circ$ with $a < b$. Suppose $|g'|$ is $\eta$-convex on $[a, b]$ with $\eta$ bounded on $|g'|([a, b]) \times |g'|([a, b])$. Then the following $(k, r)$-fractional integral inequality holds:

$$
\left| \frac{g(a) + g(b)}{2} - \frac{(r + 1)\frac{\pi}{2} \Gamma_k(\alpha + k)}{4(b^{r+1} - a^{r+1})} \left[ \frac{k}{r} J^{a+}_k G(b) + \frac{r}{k} J^{b+}_k G(a) \right] \right| \leq \frac{1}{4(b^{r+1} - a^{r+1})} \left[ \Re|g'(b)| + \frac{\Xi}{b - a}\eta(|g'(a)|, |g'(b)|) \right].
$$

where $\Re = R_1 + R_2 + R_3 + R_4$ (see Lemma 11) and $\Xi = \xi_1 + \xi_2 + \xi_3 + \xi_4$ with

$$
\begin{align*}
\xi_1 &= \int_a^{a+b} (b - w)(b^{r+1} - w^{r+1})^{\frac{\alpha}{b}} dw - \int_a^b (b - w)(b^{r+1} - w^{r+1})^{\frac{\alpha}{b}} dw, \\
\xi_2 &= \int_a^b \frac{b - w}{2} (b^{r+1} - w^{r+1})^{\frac{\alpha}{b}} dw - \int_a^b (b - w)(b^{r+1} - w^{r+1})^{\frac{\alpha}{b}} dw, \\
\xi_3 &= \int_a^{a+b} \frac{b - w}{2} (b^{r+1} - w^{r+1})^{\frac{\alpha}{b}} dw - \int_a^b (b - w)(b^{r+1} - w^{r+1})^{\frac{\alpha}{b}} dw, \\
\xi_4 &= \int_a^b \frac{b - w}{2} (b^{r+1} - (b + a - w)^{r+1})^{\frac{\alpha}{b}} dw - \int_a^{a+b} \frac{b - w}{2} (b^{r+1} - (b + a - w)^{r+1})^{\frac{\alpha}{b}} dw.
\end{align*}
$$

Proof. Since $|f'|$ is $\eta$-convex, it follows, by definition, that

$$
|g'(t(a + (1 - t)b))| \leq |g'(b)| + t\eta(|g'(a)|, |g'(b)|)
$$

for $t \in [0, 1]$. From [1] p. 9, we have

$$
\int_0^1 t|\Theta_{\alpha,r}(t)| dt = \frac{\xi_1 + \xi_2 + \xi_3 + \xi_4}{(b - a)^2}.
$$

Using Lemmas 10 and 11, inequality (16), identity (17), and properties of the modulus, we obtain

$$
\begin{align*}
\left| \frac{g(a) + g(b)}{2} - \frac{(r + 1)\frac{\pi}{2} \Gamma_k(\alpha + k)}{4(b^{r+1} - a^{r+1})} \left[ \frac{k}{r} J^{a+}_k G(b) + \frac{r}{k} J^{b+}_k G(a) \right] \right| &\leq \frac{b - a}{4(b^{r+1} - a^{r+1})} \int_0^1 |\Theta_{\alpha,r}(t)||g'(t(a + (1 - t)b))| dt \\
&\leq \frac{b - a}{4(b^{r+1} - a^{r+1})} \int_0^1 |\Theta_{\alpha,r}(t)||g'(b)| + t\eta(|g'(a)|, |g'(b)|)) dt \\
&= \frac{b - a}{4(b^{r+1} - a^{r+1})} \left( |g'(b)| \int_0^1 |\Theta_{\alpha,r}(t)| dt + \eta(|g'(a)|, |g'(b)|) \int_0^1 t|\Theta_{\alpha,r}(t)| dt \right) \\
&= \frac{b - a}{4(b^{r+1} - a^{r+1})} \left[ |g'(b)| \frac{1}{b - a} (R_1 + R_2 + R_3 + R_4) + \eta(|g'(a)|, |g'(b)|) \frac{\xi_1 + \xi_2 + \xi_3 + \xi_4}{(b - a)^2} \right].
\end{align*}
$$

The desired result follows.

**Remark 13.** By taking $r = 0$ and $k = 1$ in Theorem 12, we recover Theorem 5. In this case,

$$
\Re = \frac{4}{\alpha + 1} (b - a)^{\alpha+1} \left( 1 - \frac{1}{2^\alpha} \right)
$$

and

$$
\Xi = \frac{2}{\alpha + 1} (b - a)^{\alpha+2} \left( 1 - \frac{1}{2^\alpha} \right).
$$
Theorem 14. Let \( g \) be differentiable on \( I^\circ \) with \( a, b \in I^\circ \). If \( |g'|^q \) is \( \eta \)-convex on \([a, b]\) and \( q > 1 \) with \( \eta \) bounded on \([g'^q][a, b] \times |g'|^q([a, b])\), then the \((k, r)\)-fractional integral inequality

\[
\frac{|g(a) + g(b)|}{2} - \frac{(r + 1)^{\frac{\eta}{q}} \Gamma_k(\alpha + k)}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left[ \frac{r}{k} \mathcal{J}_a^\alpha G(b) + \frac{r}{k} \mathcal{J}_b^\alpha G(a) \right]
\leq \frac{b - a}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left( \left| g(b)' \right|^q + \eta(\left| g(a)' \right|^q, |g(b)'|^q) \right)^\frac{1}{q} \left\| \Theta_{\alpha, r} \right\|_p
\]

holds, where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \left\| \Theta_{\alpha, r} \right\|_p = \left( \int_0^1 |\Theta_{\alpha, r}(t)|^p \, dt \right)^\frac{1}{p} \).

Proof. Function \( |g'|^q \) is \( \eta \)-convex, which implies

\[
|g'(t(a + (1 - t)b)|^q \leq |g'(b)|^q + t\eta(|g'(a)|^q, |g'(b)|^q),
\]

for \( t \in [0, 1] \). Using Lemma 10 inequality 18, Hölder’s inequality and the properties of modulus, we get

\[
\frac{|g(a) + g(b)|}{2} - \frac{(r + 1)^{\frac{\eta}{q}} \Gamma_k(\alpha + k)}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left[ \frac{r}{k} \mathcal{J}_a^\alpha G(b) + \frac{r}{k} \mathcal{J}_b^\alpha G(a) \right]
\leq \frac{b - a}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left( \int_0^1 |\Theta_{\alpha, r}(t)|^p \, dt \right)^\frac{1}{p} \left( \int_0^1 |g'(t(a + (1 - t)b)|^q \, dt \right)^\frac{1}{q}
\]

\[
\leq \frac{b - a}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left( \int_0^1 |\Theta_{\alpha, r}(t)|^p \, dt \right)^\frac{1}{p} \left( \int_0^1 \left| g(b)' \right|^q + t\eta(\left| g(a)' \right|^q, |g(b)'|^q) \, dt \right)^\frac{1}{q}
\]

\[
= \frac{b - a}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left( \int_0^1 |\Theta_{\alpha, r}(t)|^p \, dt \right)^\frac{1}{p} \left( \left| g(b)' \right|^q + \eta(\left| g(a)' \right|^q, |g(b)'|^q) \right)^\frac{1}{q}.
\]

This completes the proof. \(\square\)

Theorem 15. Let \( g \) be differentiable on \( I^\circ \) with \( a, b \in I^\circ \). If \( |g'|^q \) is \( \eta \)-convex on \([a, b]\) and \( q > 1 \) with \( \eta \) bounded on \([g'^q][a, b] \times |g'|^q([a, b])\), then the \((k, r)\)-fractional integral inequality

\[
\frac{|g(a) + g(b)|}{2} - \frac{(r + 1)^{\frac{\eta}{q}} \Gamma_k(\alpha + k)}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left[ \frac{r}{k} \mathcal{J}_a^\alpha G(b) + \frac{r}{k} \mathcal{J}_b^\alpha G(a) \right]
\leq \frac{\Re^p}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left[ \left| g(b)' \right|^q + \frac{\Xi}{b - a} \eta(\left| g(a)' \right|^q, |g(b)'|^q) \right]^\frac{1}{q}
\]

holds, where \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( \Re \) and \( \Xi \) are defined as in Theorem 12.

Proof. Following a similar approach as in the proof of Theorem 14, we have, by using Lemmas 10 and 11 combined with the power mean inequality plus inequality 18, that

\[
\frac{|g(a) + g(b)|}{2} - \frac{(r + 1)^{\frac{\eta}{q}} \Gamma_k(\alpha + k)}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left[ \frac{r}{k} \mathcal{J}_a^\alpha G(b) + \frac{r}{k} \mathcal{J}_b^\alpha G(a) \right]
\leq \frac{b - a}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left( \int_0^1 |\Theta_{\alpha, r}(t)| \, dt \right)^{1 - \frac{1}{q}} \left( \int_0^1 |\Theta_{\alpha, r}(t)| \left| g(t(a + (1 - t)b)|^q \, dt \right)^\frac{1}{q}
\]

\[
\leq \frac{b - a}{4^{(b^r + 1 - a^r + 1)^{\frac{1}{q}}}} \left( \int_0^1 |\Theta_{\alpha, r}(t)| \, dt \right)^{1 - \frac{1}{q}} \left( \int_0^1 \left| \Theta_{\alpha, r}(t) \right| \left| g(b)' \right|^q + t\eta(\left| g(a)' \right|^q, |g(b)'|^q) \, dt \right)^\frac{1}{q}.
\]

The required inequality follows. \(\square\)
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