Abstract

It is proved that an element $r$ in the center of a coherent ring $\Lambda$ annihilates $\Ext^n_\Lambda(M, N)$, for some positive integer $n$ and all finitely presented $\Lambda$-modules $M$ and $N$, if and only if the bounded derived category of $\Lambda$ is an extension of the subcategory consisting of complexes annihilated by $r$ and those obtained as $n$-fold extensions of $\Lambda$. This has applications to finiteness of dimension of derived categories.

1. Introduction

Let $\Lambda$ be a right coherent ring, mod $\Lambda$ the category of finitely presented right $\Lambda$-modules, and $\mathcal{D}^b(\Lambda)$ its bounded derived category. The purpose of this note is to prove the result below that reveals a close link between the existence of uniform annihilators of Ext-modules, as modules over the center $\Lambda^c$ of $\Lambda$, and a kind of decomposition of the derived category. In the statement, $\mathcal{G}$ is the class of morphisms in $\mathcal{D}^b(\Lambda)$ that induce the zero map in cohomology, $r$ is an element in $\Lambda^c$, and $\mathcal{D}^b(\Lambda)_r$ consists of complexes $X$ with $r \Ext^0_\Lambda(X, X) = 0$, while $\mathcal{C} \circ \mathcal{D}$ is the subcategory of complexes obtained as extensions of complexes in $\mathcal{C}$ and $\mathcal{D}$; see 2.1.

Theorem 1.1. Fix a non-negative integer $n$ and an element $r$ in $\Lambda^c$. The following conditions on $\mathcal{D}^b(\Lambda)$ are equivalent.

1. $r^G^n = 0$;
2. $\mathcal{D}^b(\Lambda) = \mathcal{D}^b(\Lambda)_r \circ \{\Lambda\}^n$;
3. $\mathcal{D}^b(\Lambda) = \{\Lambda\}^n \circ \mathcal{D}^b(\Lambda)_r$.

When they hold, $r \Ext^n_\Lambda(\mod \Lambda, \mod \Lambda) = 0$. Conversely, the latter condition gives $r^3G^{2n} = 0$.

This result is a consequence of Theorem 2.10, which applies to abelian categories with enough projectives. In fact, the equivalence of conditions (1)--(3), and the proofs,
carry over verbatim to generating projective classes in triangulated categories, in the  

sense of Christensen [1]; with Ext as in Section 4 of op. cit., the entire statement  

carries over.

Here is one application (see Corollary 2.12) of the theorem above: If \( r \in \Lambda \) is a  

non-zerodivisor on \( \Lambda \) and satisfies \( rG^n = 0 \), then there is an inequality  
\[
\dim \text{D}^b(\Lambda) \leq \dim \text{D}^b(\Lambda/r\Lambda) + n
\]

concerning dimensions of the appropriate triangulated categories, in the sense of  

Rouquier [4]. This inequality gives a way to deduce the finiteness of the dimension  

of the derived category of \( \Lambda \) from that of the derived category of \( \Lambda/r\Lambda \). The point is  

that the ring \( \Lambda/r\Lambda \) is “smaller” than \( \Lambda \); for example, the Krull dimension of \( (\Lambda/r\Lambda)^c \)  
is strictly smaller than that of \( \Lambda \). This approach is predicated on the existence of  

non-zerodivisors that annihilate Ext-modules. For results in this direction, see [2,  

Section 7].

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2. Decompositions

We deduce the statement in the Introduction from Theorem 2.10 below that concerns  

derived categories of abelian categories.

Definition 2.1. Let \( T \) be a triangulated category, and \( \Sigma \) its suspension functor; soon  

we will focus on the derived category of an abelian category.

Let \( C \) be a subcategory (always assumed to be full) of \( T \). We write add(\( C \)) for the  

smallest subcategory of \( T \) containing \( C \) and closed under finite direct sums, retracts,  

and shifts. Given a subcategory \( D \) of \( T \), the subcategory consisting of objects \( E \) that  

appear in exact triangles of the form  
\[
C \to E \to D \to \Sigma C \quad \text{with} \quad C \in C \quad \text{and} \quad D \in D
\]
is denoted \( C \ast D \). It is convenient to introduce also the following notation:  
\[
C \circ D := \text{add}(C \ast D).
\]

It is a consequence of the octahedral axiom that there are equalities  
\[
(B \ast C) \ast \ast D = B \ast (C \ast D) \quad \text{and} \quad (B \circ C) \circ \circ D = B \circ (C \circ D).
\]

In particular, we may denote them \( B \ast C \ast D \) and \( B \circ C \circ D \), respectively.

Throughout the rest of this section, \( R \) will be a commutative ring.

Definition 2.2. An additive category \( A \) is said to be \emph{\( R \)-linear} if for each \( A \) in \( A \) there  

are homomorphisms of rings  
\[
\eta_A : R \to \text{End}_A(\Lambda)
\]

with the property that the action of \( R \) on \( \text{Hom}_A(A, B) \) induced by \( \eta_A \) and \( \eta_B \) coincide,  

for all \( A, B \) in \( A \). Said otherwise, \( \text{Hom}_A(A, B) \) is an \( R \)-module and this structure is  

compatible with compositions in \( A \).
Let $A$ be an $R$-linear Abelian category. The category of complexes over $A$ inherits an $R$-linear structure, as does the bounded derived category, $D^b(A)$, of $A$. In either case, the action is compatible with the suspension, in that the morphisms $\Sigma(X \overset{r}{\to} X)$ and $\Sigma X \overset{L}{\to} \Sigma X$ coincide for all $r \in R$ and complexes $X$. What is used repeatedly in the sequel is that for any $r \in R$ and morphism $f : X \to Y$, in either category, there is an induced commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{r} & & \downarrow{r} \\
X & \xrightarrow{f} & Y
\end{array}
$$

Henceforth, we assume that $A$ has enough projective objects, and write $\text{proj } A$ for the corresponding subcategory. For ease of notation, we abbreviate

$$
T := D^b(A) \\
P_n := \underbrace{\text{proj } A \circ \cdots \circ \text{proj } A}_{n \text{ copies}} \quad \text{for each } n \geq 0.
$$

Recall that ghost in $T$ is a morphism $f : X \to Y$ such that

$$\text{Hom}_T(\Sigma^n P, f) = 0 \quad \text{for all } P \in \text{proj } A \text{ and } n \in \mathbb{Z}.$$ 

In what follows, we write $G$ for the class of ghosts; it is an ideal in $T$. For any integer $n$, the ideal $G^n$ consists of morphisms that are $n$-fold compositions of ghosts.

**Remark 2.3.** For each non-negative integer $n$, one has

$$\text{Hom}_T(P, g) = 0 \quad \text{for all } P \in P_n \text{ and } g \in G^n.$$ 

This is the well-known Ghost Lemma; for a proof, see, for example, [3, Theorem 3].

**Remark 2.4.** For each complex $X$ in $T$ and integer $n \geq 1$, there is an exact triangle

$$P \xrightarrow{P} X \xrightarrow{q} Y \xrightarrow{} \Sigma P$$

with $P$ in $P_n$ and $q$ in $G^n$; one can get this, for instance, from the construction of an Adams resolution of $X$; see [1, Section 4]. When $X$ is in $A$, such a triangle exists with $\Sigma^{-n}Y$ in $A$.

**Definition 2.5.** For $r \in R$, let $T_r$ denote the subcategory of $T$ consisting of complexes $X$ such that the multiplication morphism $X \overset{r}{\to} X$ is zero in $T$; in other words, $r$ is in the kernel of the natural map $R \to \text{End}_T(X)$.

**Remark 2.6.** Let $r, s$ be elements of $R$. In any exact triangle $X \to Y \to Z \to \Sigma X$ in $T$, if $X \in T_r$ and $Z \in T_s$, then $Y \in T_{rs}$ holds.

Indeed, this is a well-known argument (analogous to one for the Ghost Lemma).
contained in the commutative diagram below:

\[
\begin{array}{ccc}
Y & \xrightarrow{g} & Z \\
\downarrow{r} & & \downarrow{r} \\
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma X \\
\end{array}
\]

The squares in the diagram are commutative by the definition of the \( R \)-action on \( T \). The morphism \( Y \to X \) exists because \( gs = sg = 0 \); the second equality holds since \( Z \) is in \( T_s \). The morphism \( Y \xrightarrow{r} Y \) thus factors as \( Y \to X \xrightarrow{r} X \xrightarrow{f} Y \) and hence is zero, since \( X \) is in \( T_r \).

In what follows, given a morphism \( f : X \to Y \) of complexes over \( A \), its mapping cone is denoted \( \text{cone}(f) \); thus

\[
\text{cone}(f)^n := Y^n \bigoplus X^{n+1} \quad \text{with differential} \quad \begin{bmatrix} d_Y & f \\ 0 & -d_X \end{bmatrix}
\]

The canonical exact sequence of complexes

\[
0 \longrightarrow Y \longrightarrow \text{cone}(f) \longrightarrow \Sigma X \longrightarrow 0
\]

gives rise to an exact triangle \( X \xrightarrow{f} Y \to \text{cone}(f) \to \Sigma X \) in \( T \).

**Remark 2.7.** For \( r \in R \) and complex \( X \) over \( A \), set \( X\parallel r := \text{cone}(X \xrightarrow{r} X) \). Observe that \( X\parallel r \) is in \( T_r \), because the map

\[
\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} : X\parallel r \longrightarrow X\parallel r
\]

defines a homotopy between multiplication by \( r \) and the zero morphism.

**Lemma 2.8.** For each subcategory \( C \) of \( T \) and element \( r \in R \) there are inclusions

\[
T_r \ast C \subseteq C \ast T_{r^2} \quad \text{and} \quad C \ast T_r \subseteq T_{r^2} \ast C.
\]

**Proof.** We verify the first inclusion; the second one can be checked along the same lines.

Fix an \( X \) in \( T_r \ast C \). Thus, there exist \( T \in T_r \) and \( C \in C \) and an exact triangle in the top row of the following diagram:

\[
\begin{array}{ccc}
T & \longrightarrow & X \\
\downarrow{r} & & \downarrow{r} \\
X & \xleftarrow{f} & C \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \xrightarrow{g} \Sigma T \\
& & \downarrow{r} \\
& & \Sigma T
\end{array}
\]

The map \( h \) exists because \( gr = rg = 0 \), where the second equality holds because \( T \) is in \( T_r \). By the octahedral axiom, the factorization \( r = fh \) gives rise to an exact
triangle

\[ T \rightarrow \text{cone}(h) \rightarrow C \rightarrow r \]

It follows from Remarks 2.6 and 2.7 that \( r^2 \) annihilates \( \text{cone}(h) \). It remains to notice the exact triangle \( C \rightarrow X \rightarrow \text{cone}(h) \rightarrow \Sigma C \). \( \square \)

**Definition 2.9.** For an element \( r \in R \) and an integer \( n \geq 0 \) we consider the following four conditions on the triangulated category \( T := D^b(A) \).

\[ D_{r,n} T = T_r \circ P_n, \quad \text{and} \quad E_{r,n} \Ext^0_A(A, A) = 0, \]
\[ D'_{r,n} T = P_n \circ T_r, \quad \text{and} \quad G_{r,n} r \mathcal{G}^n = 0. \]

The statement from the introduction is a consequence of the following theorem.

**Theorem 2.10.** The following implications hold

\[ D'_{r,n} \iff D_{r,n} \iff G_{r,n} \iff E_{r,n} \iff D_{r^3,2n} \]

**Proof.** (\( D'_{r,n} \Rightarrow G_{r,n} \)): Fix \( f : X \rightarrow Y \) to be in \( \mathcal{G}^n \), and \( P \xrightarrow{p} X \xrightarrow{q} T \rightarrow \Sigma P \) the exact triangle provided by the hypothesis. Consider the commutative diagram below where the morphism \( X \rightarrow P \) is induced by the fact the \( qr = rq = 0 \), since \( T \) is in \( T_r \).

\[
\begin{array}{ccc}
X & \xrightarrow{q} & T \\
\downarrow & & \downarrow \\
P & \xrightarrow{p} & X \\
\downarrow & \downarrow & \downarrow \\
0 & \xrightarrow{f} & Y.
\end{array}
\]

It remains to note that the composition \( fp = 0 \), by Remark 2.3.

\( (D_{r,n} \Rightarrow G_{r,n}) \) can be verified by an argument analogous to the one above.

\( (G_{r,n} \Rightarrow D'_{r,n}) \) and \( (G_{r,n} \Rightarrow D_{r,n}) \): Fix \( X \) in \( T \) and \( P \xrightarrow{p} X \xrightarrow{q} Y \rightarrow \Sigma P \) the exact triangle from Remark 2.4. By hypothesis \( rq = 0 \), so the octahedral axiom applied to the composition \( rq \) gives rises to an exact triangle

\[ \Sigma P \rightarrow Y \bigoplus \Sigma X \rightarrow Y/r \rightarrow \Sigma^2 P. \]

It remains to recall that \( Y/r \) is in \( T_r \), by Remark 2.7, so that property \( D_{r,n} \) holds. Applying the octahedral axiom to the map \( qr \), which is also zero, shows that \( D_{r,n} \) holds as well.

\( (G_{r,n} \Rightarrow E_{r,n}) \): This holds because any morphism \( f : A \rightarrow \Sigma^n B \), with \( A, B \) in \( A \), is in \( \mathcal{G}^n \); see Remarks 2.3 and 2.4.

\( (E_{r,n} \Rightarrow D_{r^3,2n}) \): For a start observe that \( A \subseteq T_r \circ P_n \); this follows by an argument along the lines of the one for \( G_{r,n} \Rightarrow D'_{r,n} \) above. For a complex \( X \) over \( A \) let \( Z^*(X) \) and \( B^*(X) \) denote the cycles and boundaries of \( X \), respectively. There are canonical
exact triangles

\[ Z^*(X) \to X \to \Sigma B^*(X) \to \Sigma Z^*(X) \]
\[ B^*(X) \to Z^*(X) \to H^*(X) \to \Sigma B^*(X). \]

As \( Z^*(X) \) and \( B^*(X) \) are in \( \text{add}(A) \), one gets the first of the following chain of inclusions:

\[
\begin{align*}
T \subseteq A \circ A \\
\subseteq (T_r \circ P_n) \circ (T_r \circ P_n) \\
\subseteq T_r \circ T_r \circ P_n \circ P_n \\
\subseteq T_{r^3} \circ P_{2n}.
\end{align*}
\]

The third inclusion holds by the associativity of \( \circ \) and Lemma 2.8. The last one holds by Remark 2.6 and the definition of the \( P_n \). This is the desired implication.

Non-zerodivisors

Now let \( \Lambda \) be a right coherent ring and \( r \in \mathcal{R} \) a non-unit element in the center of \( \Lambda \). The homomorphism of rings \( \Lambda \to \Lambda/r\Lambda \) then induces, by restriction of scalars, an exact functor of triangulated categories

\[ D^b(\Lambda/r\Lambda) \to D^b(\Lambda). \]

Evidently, its image lies in the subcategory \( D^b(\Lambda)_r \).

**Lemma 2.11.** When \( r \) is a non-zerodivisor on \( \Lambda \), the functor \( D^b(\Lambda/r\Lambda) \to D^b(\Lambda)_r \) is dense up to direct summands.

**Proof.** Since \( r \) is a non-zerodivisor on \( \Lambda \), the canonical map \( \Lambda \to H^0(\Lambda//r) \cong \Lambda/r\Lambda \) is a quasi-isomorphism in \( D^b(\Lambda) \). This gives rise to an exact triangle

\[ \Lambda \xrightarrow{r} \Lambda \to \Lambda/r\Lambda \to \Sigma \Lambda. \]

For any \( X \in D^b(\Lambda)_r \), applying \( X \otimes^L_{\Lambda} \) yields an exact triangle

\[ X \xrightarrow{r} X \to X \otimes^L_{\Lambda} (\Lambda/r\Lambda) \to \Sigma X. \]

Since the first morphism in this triangle is zero, one gets an isomorphism

\[ X \otimes^L_{\Lambda} (\Lambda/r\Lambda) \cong X \oplus \Sigma X. \]

Note that \( X \otimes^L_{\Lambda} (\Lambda/r\Lambda) \) is in the image of the functor \( D^b(\Lambda/r\Lambda) \to D^b(\Lambda) \).

**Dimension**

Recall that the *dimension* of a triangulated category \( T \), denoted \( \dim T \), is the least non-negative integer \( d \) for which there exists an object \( G \) such that \( \{G\}^{d+1} = T \); see [4, Definition 3.2].

The result below justifies the inequality stated in the introduction. Recall that \( \mathcal{G} \) denotes the class of ghosts in \( D^b(\Lambda) \).

**Corollary 2.12.** Let \( \Lambda \) be a right coherent ring. If \( r \in \mathcal{R} \) is a non-zerodivisor on \( \Lambda \) and satisfies \( r^n \mathcal{G} = 0 \) for some non-negative integer \( n \), then there is an inequality

\[ \dim D^b(\Lambda) \leq \dim D^b(\Lambda/r\Lambda) + n \]
Proof. Part of the hypothesis is that $\mathbb{D}^b(\Lambda)$ satisfies condition $G_{r,n}$, in the notation of Theorem 2.10. Keeping in mind Lemma 2.11 and that $\text{proj} \Lambda = \text{add} \Lambda$, op. cit. yields

$$\mathbb{D}^b(\Lambda) = \mathbb{D}^b(\Lambda/r\Lambda) \circ \{\Lambda\}^{n \circ}.$$  

We have identified $\mathbb{D}^b(\Lambda/r\Lambda)$ with its image in $\mathbb{D}^b(\Lambda)$. If for some complex $F$ and integer $d$ one has $\mathbb{D}^b(\Lambda/r\Lambda) = \{F\}^{(d+1) \circ}$, then the equality above yields

$$\mathbb{D}^b(\Lambda) = \{F \bigoplus \Lambda\}^{(d+n+1) \circ}.$$  

This implies the desired inequality.  

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