Modeling vortex rings dynamics with vortex in cell method

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Abstract. In this article we described Vortex in Cell method. We used it to research influence of the initial vorticity distribution inside the core of a vortex ring on its dynamics. We showed evolution of a vortex ring in time with two different initial distributions: uniform and Gaussian. The graphs for the theoretical and numerical velocities of a vortex ring were created. From presented results one can find out that the evolution of the vorticity around the core of the ring depends on the parameters of the ring and distribution of the vorticity inside the core.

1. Introduction

The vortex ring is a well known and very popular object of fluid dynamics. Inherent beauty of the vortex ring dynamics and interesting phenomena like vortex games, reconnection of the rings have attracted the researchers for a long time. Part of the fascination stems from the compact and persistent nature of the vortex rings. In many experimental pictures of the boundary layer one can observed the presence of the vortex rings. In nature, vortex rings are found to occur in various shapes and size. Their compact nature makes them ideal as simpler building blocks in modeling of more complex flows. Despite of simple geometrical structure the dynamics of the vortex rings are complex enough. In the paper we presented the numerical investigation of the vortex ring dynamics by vortex particles methods. We presented the influence of the vorticity distribution in the core region of the ring on their translation velocity. We compared results with the theoretical formulas. We presented the results how initial distribution of the vorticity inside the core evolved in time and how this influence the stability of the ring. It seems to us that the vortex particle method is very good suited to this study. It stems from the fact that tracing of the positions of the particles we can trace the evolution of the vorticity field. In practices one can distinguish two groups of vortex methods:

- direct methods where velocity of vortex particles are calculated on the basis of the Biot–Savart law by summing of the contribution to the velocities induced by all particles (segments of the vortex lines) in the flow,
- Eulerian-Lagrangian methods, like Vortex–In–Cell method (VIC), where velocity field is determined on a grid by solving Poisson equation for a vector potential.

The direct methods are very attractive from a theoretical point of view. They are based on fundamental law of vector analysis, are grid free and allow to exactly realize far-field
boundary conditions. But direct vortex methods also have a great disadvantage. The amount of a computational time is much larger than for the Eulerian-Lagrangian methods (Cottet & Koumoutsakos, 2000). We used the VIC method.

2. Equations of motion

Equations of incompressible viscous fluid motion have the following form:

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p + \nu \Delta u \]  \hspace{1cm} (1)

\[ \nabla \cdot u = 0 \]  \hspace{1cm} (2)

where \( u = (u, v, w) \) is velocity vector, \( \rho \) – fluid density, \( p \) – pressure, \( \nu \) – kinematic coefficient of viscosity.

The equation (1) can be transformed to the Helmholtz equation for the vorticity evolution:

\[ \frac{\partial \omega}{\partial t} + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u + \nu \Delta \omega \]  \hspace{1cm} (3)

where \( \omega = \nabla \times u \). From incompressibility (2) stems the existence of the vector potential \( A = \nabla \times u \) \hspace{1cm} (4)

Assuming additionally that vector \( A \) is incompressible (\( \nabla \cdot A = 0 \)) its components can be obtained by solution of the Poisson equation

\[ \Delta A_i = -\omega_i, \quad i = 1, 2, 3 \]  \hspace{1cm} (5)

To introduce vorticity into the flow the viscous splitting algorithm can be used. In this algorithm the solution is obtained in two steps. At first one must solve inviscid part of equation, next the viscous part is taken into account. In this work we presented results only for the inviscid flow. So that imply that we will solve only inviscid (Euler) equation of fluid motion.

3. Description of VIC method for three-dimensional case

First we have to discretize our computational domain. To do this we set up a regular 3D grid \((j_1 \Delta x, j_2 \Delta y, j_3 \Delta z)\) \((j_1, j_2, j_3 = 1, 2, \ldots, N)\), where \( \Delta x = \Delta y = \Delta z = h \). The same mesh will be used for solving Poisson equation. The continuous field of vorticity is replaced by a discrete distribution of Dirac delta measures (Cottet & Koumoutsakos, 2000; Kudela & Regucki, 2009)

\[ \omega(x) = \sum_{p=1}^{N} \Gamma_p(x_p) \delta(x - x_p) \]  \hspace{1cm} (6)

where \( \Gamma_p \) means vorticity particle \( \Gamma_p = (\Gamma_{p1}, \Gamma_{p2}, \Gamma_{p3}) \) at position \( x_p = (x_{p1}, x_{p2}, x_{p3}) \). The domain of the flow is covered by numerical mesh \((N_x \times N_y \times N_z)\) with equidistant spacing \( h \), and the \( i \)-th component of the vector particle \( \alpha \) is defined by expression:

\[ \Gamma_i = \int_{V_p} \omega_i(x_1, x_2, x_3), \, dx \approx h^3 \omega_i(x_p), \quad x_p \in V_p, \quad \vert V_p \vert = h^3 \]  \hspace{1cm} (7)

From the Helmholtz theorems (Wu & Ma & Zhou, 2006) we know that the vorticity is carried on by the fluid:

\[ \frac{d x_p}{d t} = u(x_p, t), \]  \hspace{1cm} (8)
We must take into account also the fact that due to three dimensionality of the vorticity field the intensity of the particles are changed by stretching effect

\[
\frac{d\Gamma_p}{dt} = \left[ \nabla \mathbf{u}(\mathbf{x}_p, t) \right] \cdot \Gamma_p
\]  

(9)

Velocity at the grid nodes was obtained by the solution of the Poisson equation (5) by the finite difference method and using (4). The system of equations (8), (9) was solved by Runge–Kutta method of 4-th order.

3.1. Remeshing

In Vortex in Cell method particles have a tendency to gather in regions with high velocity gradients. This can lead to inaccuracies as particles coming too close to one another. To overcome this problem particles have to be remeshed that is distributed back to the nodes of the rectangular mesh. It is done by an interpolation:

\[
\omega_j = \frac{1}{h^3} \sum_p \hat{\Gamma}_{pn} \varphi \left( \frac{x_j - \hat{x}_p}{h} \right)
\]  

(10)

where \( j \) is index of grid node and \( p \) is index of a particle. The quality of that interpolation strongly depend on the properties of the kernel \( \varphi \). At first the kernel should satisfy

\[
\sum_p \varphi \left( \frac{x - x_p}{h} \right) = 1
\]  

(11)

The next the kernel should fulfill the moment conditions:

\[
\sum_p (x - x_p)^\alpha \varphi \left( \frac{x - x_p}{h} \right) = 0, \quad \text{for } 1 \leq |\alpha| \leq m - 1
\]  

(12)

where \( m \) is order of remeshing procedure. The discrepancy between the old and new position of the particle can be measure by the difference (Cottet & Koumoutsakos, 2000)

\[
\sum_p \hat{\Gamma}_{pn} \delta(x - \hat{x}_p) - \sum_p \Gamma_{pn} \delta(x - x_p)
\]  

(13)

If we multiply (13) by a test function \( \phi \) and take into account (11) one get (Cottet & Koumoutsakos, 2000):

\[
E = \sum_p \hat{\Gamma}_{pn} \left[ \phi(\hat{x}_p) - \sum_p \phi(x_j) \varphi \left( \frac{x_j - \hat{x}_p}{h} \right) \right]
\]  

(14)

where \( \hat{\Gamma}_{pn} \) and \( \hat{x}_p \) are values from old distribution. Thus the expression in the brackets characterize the distortion of the redistribution process. Naming the expression the brackets by \( f(x) \), using the Taylor expansion of \( \varphi \) and ones again using the property (11) one can get (Cottet & Koumoutsakos, 2000)

\[
f(x) = \sum_{\alpha} \sum_j [(x_j - x) \cdot \nabla \phi]^{\alpha} \varphi \left( \frac{x - x_j}{h} \right)
\]  

(15)

So

\[
f(x) = O \left( h^m \right).
\]  

(16)
It is regarded that the remeshing procedure is of order \( m \). It means that polynomial functions up to an order \( m \) will be exactly represented by this interpolation. In this work we used following interpolation kernel:

\[
\varphi(x) = \begin{cases} 
(2 - 5x^2 + 3|x|^3)/2 & \text{if } 0 \leq |x| \leq 1 \\
(2 - |x|)^2(1 - |x|)/2 & \text{if } 1 \leq |x| \leq 2 \\
0 & \text{if } 2 \leq |x|
\end{cases}
\]

(17)

Kernel (17) used in this work is of order \( m = 3 \).

4. Tests

For a thin cored ring with uniform circulation \( \Gamma \) formula for the vortex ring velocity is (Green, 1995):

\[
U = \frac{\Gamma}{4\pi R_0} \left[ \ln \left( \frac{8R_0}{\epsilon_0} \right) - \frac{1}{4} + O \left( \epsilon_0/R_0 \right) \right]
\]

(18)

where \( R_0 \) is ring radius and \( \epsilon_0 \) is core radius \( (\epsilon_0/R_0 \ll 1) \).

We carried out computational experiments. First we did calculations for constant vorticity distribution in the core. The parameters of the vortex ring \( R_0 \) and \( \epsilon_0 \) were constant, \( R_0 = 1.5, \epsilon_0 = 0.3 \), but \( \Gamma \) was changed, \( \Gamma \in [0.23, 0.94] \). Computational domain was \( 2\pi \times 2\pi \times 2\pi \) and a numerical grid had 257 nodes in each direction. It was assumed the periodic boundary conditions in all directions.

Initial condition was given in a form that every node which fulfills equation:

\[
r_p = \left( \sqrt{x^2 + y^2} - R_0 \right)^2 + z^2 < \epsilon_0^2
\]

(19)

obtain initial vorticity. The initial number of particles used for initial approximation of the ring was 179780.

Time step for iteration of equation (8) was equal to \( \Delta t = 0.01 \).

Next we did calculations for Gauss vorticity distribution in the core. The parameters of the vortex ring \( R_0 \) and \( \epsilon_0 \) were constant, \( R_0 = 1.5, \epsilon_0 = 0.3 \), but \( \Gamma \) was changed, \( \Gamma \in [0.23, 0.94] \). Computational domain was \( 2\pi \times 2\pi \times 2\pi \) and a numerical grid had 257 nodes in each direction. It was assumed the periodic boundary conditions in all directions.

Initial condition was given in a form that every node which fulfills equation:

\[
r_p = \left( \sqrt{x^2 + y^2} - R_0 \right)^2 + z^2 < \epsilon_0^2
\]

(20)

and was equal to:

\[
\omega_p = c \frac{1}{\sigma \sqrt{2\pi}} e^{-r_p^2/2\sigma^2}
\]

(21)

where \( c \) was a known coefficient used in order to get the desired circulation and \( \sigma = \frac{\epsilon_0}{3} \). The initial number of particles used for initial approximation of the ring was 179780.

In Figure 1 it was presented the translation velocity of the ring for uniform distribution of vorticity inside the core versus circulation of the core. For the constant distribution of vorticity the discrepancy between the results from analytical Kelvin formula (18) and our numerical calculation may result from the fact that formula (18) was derived for the vortex ring with the uniform distribution of the vorticity inside of the vortex core. During the motion of the ring the vorticity distribution in core changes in time and is not uniform (see Figure 3). It is worth to
noticed that our results better fit to the formula derived by Hicks for stagnant fluids inside of the core (Saffman, 1992). In formula of Hicks the coefficient $1/4$ in (18) is replaced by $1/2$. In Figure 1 the velocity calculated from Hicks formula was drawn by dashed line.

In Figure 2 it was presented the translation velocity of the ring for Gaussian distribution of vorticity inside the core. The velocity now is a bit greater (approximately the coefficient $1/4$ in formula (18) one should change to $1/8$). On the other side has a strong influence on the behavior of circumference of the core. In Figures 4 and 5 one can see that behavior the core is more smooth and distribution remains Gaussian. One can conclude that the distribution of the vorticity inside the core influences on stability of the vortex ring.

In Figures 4 and 5 we can see the development of the ring in time. The evolution of the vorticity around the core of the ring depend on the parameter of the ring and distribution of the vorticity inside of the core. It is well known that the vortex ring is unstable structure (Green, 1995) and we can expect that a ring will take the fuzzy shape (Alekseenko & Kuibin & Okulov, 2007). It is clearly visible in the Figure 4.

During computation the invariants of motion were checked. Kinetic energy was calculated from two different equations:

$$ E_k = \int_{\Omega} u^2 \, dV $$

(22)

$$ E_k = \int_{\Omega} \omega \, dV $$

(23)
After 600 time steps kinetic energy calculated from equations (22), (23) dropped down by less than 2%. Divergences of vector potential ($\mathbf{A}$), vorticity ($\mathbf{\omega}$) and velocity ($\mathbf{u}$) were all lower then $1.0E-10$ during whole calculations.

5. Conclusions

From presented results we can conclude that Vortex Particle Methods are very good suited for numerical studies of 3D vortex structures. In our work we use the multicore architecture for parallel computing. It speeds up the calculations about 15 times with respect to a single processor. Distribution of vorticity inside the core is important element of the behavior of the vortex ring. The initial uniform vorticity distribution evolve in time approaching non-uniform distribution resembling the Gaussian one.

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