Global well-posedness and zero-diffusion limit of classical solutions to the 3D conservation laws arising in chemotaxis

Hongyun Peng, Huanyao Wen, Changjiang Zhu

Abstract

In this paper, we study the relationship between a diffusive model and a non-diffusive model which are both derived from the well-known Keller-Segel model, as a coefficient of diffusion $\varepsilon$ goes to zero. First, we establish the global well-posedness of classical solutions to the Cauchy problem for the diffusive model with smooth initial data which is of small $L^2$ norm, together with some a priori estimates uniform for $t$ and $\varepsilon$. Then we investigate the zero-diffusion limit, and get the global well-posedness of classical solutions to the Cauchy problem for the non-diffusive model. Finally, we derive the convergence rate of the diffusive model toward the non-diffusive model. It is shown that the convergence rate in $L^\infty$ norm is of the order $O(\varepsilon^{1/2})$. It should be noted that the initial data is small in $L^2$-norm but can be of large oscillations with constant state at far field. As a byproduct, we improve the corresponding result on the well-posedness of the non-diffusive model which requires small oscillations.

Key Words: Conservation laws, chemotaxis, large amplitude solution, convergence rate, zero diffusion limit.

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1 Introduction

In this paper, we are interested in a system of conservation laws arising in chemotaxis

\[
\begin{aligned}
p_{\varepsilon}^t - \nabla \cdot (p_{\varepsilon}^f q_{\varepsilon}^f) &= D \Delta p_{\varepsilon}, \\
q_{\varepsilon}^t + \nabla (\varepsilon |q_{\varepsilon}^f|^2 - p_{\varepsilon}) &= \varepsilon \Delta q_{\varepsilon}^f,
\end{aligned}
\]

with initial data

\[
(p_{\varepsilon}^f, q_{\varepsilon}^f) (x, 0) = (p_0(x), q_0(x)) \to (p_\infty, 0) \text{ as } |x| \to \infty.
\]

The chemotaxis model was proposed by Keller and Segel in [10] to describe the traveling band behavior of bacteria due to the chemotactic response observed in experiments [1] [2].
The following Keller-Segel model has been extensively studied:

\[
\begin{align*}
    u_t &= \nabla \cdot (D \nabla u - \chi u \nabla \phi(c)), \\
    \tau c_t &= \varepsilon \Delta c + g(u, c),
\end{align*}
\]  

(1.3)

where \( u(x, t) \) and \( c(x, t) \) denote the cell density and the chemical concentration, respectively. \( D > 0 \) is the diffusion rate of cells (bacteria) and \( \varepsilon \geq 0 \) is the diffusion rate of chemical substance. \( \tau \geq 0 \) is a relaxation time scale and \( \chi > 0 \) corresponds to attractive chemotaxis. Here \( g(u, c) \) is a kinetic function and \( \phi(c) \) denoting a chemotactical sensitivity function. With different choices of \( g(u, c) \) and \( \phi(c) \), many results have been established in the literatures, cf. [3, 8, 23].

As in [13, 16, 17], if we consider the model (1.3) with \( \tau = 1, \phi(c) = \ln c, g(u, c) = -\alpha uc \), the resulting model reads:

\[
\begin{align*}
    u_t &= \nabla \cdot (D \nabla u - \chi u \nabla \ln c), \\
    c_t &= \varepsilon \Delta c - \alpha uc.
\end{align*}
\]  

(1.4)

The model (1.1) is derived from (1.4) through the Hopf-Cole transformation:

\[
\begin{align*}
    q^\varepsilon &= -\nabla c c = -\nabla \ln c, & p^\varepsilon &= u
\end{align*}
\]  

(1.5)

and scalings

\[
\begin{align*}
    \tilde{t} &= \alpha t, & \tilde{x} &= x \sqrt{\frac{\alpha}{\chi}}, & \tilde{q} &= q \sqrt{\frac{\alpha}{\chi}}, & \tilde{D} &= D \frac{\chi}{\alpha}, & \tilde{\varepsilon} &= \frac{\varepsilon}{\chi},
\end{align*}
\]  

(1.6)

where tilde has been dropped. When the diffusion of chemical substance is so small that it is negligible, i.e., \( \varepsilon \to 0^+ \), then the model (1.4) becomes:

\[
\begin{align*}
    u_t &= \nabla \cdot (D \nabla u - \chi u \nabla \ln c), \\
    c_t &= -\alpha uc.
\end{align*}
\]  

(1.7)

A version of system (1.7) was proposed by Othmer and Stevens in [19] to describe the chemotactic movement of particles where the chemicals are non-diffusible. The models developed in [19] have been studied in depth by Levine and Sleeman in [11]. They gave some heuristic understanding of some of these phenomena and investigated the properties of solutions of a system of chemotaxis equation arising in the theory of reinforced random walks. Y. Yang, H. Chen and W.A. Liu in [25] studied the global existence and blow-up in a finite-time of solutions for the case considered in [11], respectively. For the other results on (1.7), please refer to [7, 12, 26] and references therein.

Similar to the derivation of system (1.1), the system (1.7) can be converted into a system of conservation laws as follows:

\[
\begin{align*}
    p_t - \nabla \cdot (pq) &= D \Delta p, \\
    q_t - \nabla p &= 0,
\end{align*}
\]  

(1.8)

with initial data

\[
(p, q)(x, 0) = (p_0(x), q_0(x)) \to (p_\infty, 0) \text{ as } |x| \to \infty.
\]  

(1.9)

System (1.8) has been studied by several authors. For one dimension, the global well-posedness of smooth solution was obtained in [27] with small initial data and large initial
data, respectively. For high dimensions, the global well-posedness of smooth solution to (1.8) was investigated in [13, 14] for Cauchy problem and initial-boundary value problem, respectively, where the initial data is required to be small at least in $H^2$ norm. For other related results, such as nonlinear stability of waves in one dimension and so on, please refer to [15, 16, 17, 28, 29] and references therein.

Formally, the system (1.1) becomes (1.8) when we take $\varepsilon = 0$. In fact, the investigation of the problem of the zero viscosity limit is one of the challenging topics in fluid dynamics and has been much more extensively investigated for many other models, cf. [4, 5, 9, 21, 22, 24]. However, to our knowledge, there are few results on the system (1.1) in this direction, cf. [20]. Our aim here is to prove accurately that the solutions of (1.8) converge to the solutions of (1.1) as the chemical diffusion $\varepsilon$ goes to zero.

To do this, we first establish the global well-posedness of classical solutions to the Cauchy problem for the diffusive model (1.1) with smooth initial data which is of small $L^2$ norm. Some $a priori$ estimates independent of $t$ and $\varepsilon$ are also obtained. Then, based on these estimates, we get the global existence of classical solutions to the Cauchy problem for the non-diffusive model (1.8) after passing to the limits $\varepsilon \to 0$. Finally, we derive the convergence rate of the diffusive model toward the non-diffusive model.

Before stating the main results, we explain some notations.

Notations: $L^p = L^p(\Omega)$ ($1 \leq p \leq \infty$) denotes usual Lebesgue space with the norm

$$\|f\|_{L^p(\Omega)} = \left( \int_{\Omega} |f(x)|^p \, dx \right)^{\frac{1}{p}}, \quad 1 \leq p < \infty,$$

$$\int f \, dx = \int_{\mathbb{R}^3} f \, dx.$$

$H^l(\Omega)$ ($l \geq 0$) denotes the usual $l$th-order Sobolev space with the norm

$$\|f\|_l = \left( \sum_{j=0}^{l} \|\partial^j_x f\|^2 \right)^{\frac{1}{2}},$$

where $\Omega = \mathbb{R}^3$, and $\| \cdot \| = \| \cdot \|_0 = \| \cdot \|_{L^2}$.

The main results in this paper can be stated as follows:

**Theorem 1.1** For given number $M > 0$ (not necessarily small), assume that initial data $(p_0, q_0)$ satisfy

$$\nabla \times q_0 = 0, \quad \|\nabla p_0\|_{L^2}^2 + \|\nabla \cdot q_0\|_{L^2}^2 \leq M, \quad (p_0 - p_\infty, q_0) \in H^3, \quad p_\infty > 0. \quad (1.10)$$

Then there exists a positive constant $\varepsilon_0$ depending on $M$ and $p_\infty$ such that the Cauchy problem exists a unique global solution in $\mathbb{R}^3 \times (0, \infty)$, which satisfies

$$\begin{cases}
(p^\varepsilon - p_\infty, q^\varepsilon) \in L^\infty([0, \infty), H^3), \\
\nabla p^\varepsilon \in L^2([0, \infty), H^3), \quad \nabla q^\varepsilon \in L^2([0, \infty), H^2), \\
\varepsilon \nabla q^\varepsilon \in L^2([0, \infty), H^3) \quad (1.11)
\end{cases}$$
and
\[
\|q^\varepsilon(t)\|_{H^3}^2 + \|p^\varepsilon(t) - p_\infty\|_{H^3}^2
\]
\[
+ \int_0^t \left( \|\nabla p^\varepsilon(s)\|^2_{H^3} + \|\nabla q^\varepsilon(s)\|^2_{H^3} \right) \, ds
\]
\[
+ \int_0^t \|\nabla q^\varepsilon(s)\|^2_{H^2} \, ds \leq C,
\]
where \(C\) is a positive constant independent of \(\varepsilon\) and \(t\), provided that
\[
\|q_0\|_{L^2}^2 + \|p_0 - p_\infty\|_{L^2}^2 \leq \varepsilon_0.
\]

The last theorem is concerned with the convergence rate as well as the global well-posedness of (1.8)-(1.9).

**Theorem 1.2** Suppose that \((p_0, q_0)\) satisfies the assumptions in Theorem 1.1, then the Cauchy problem (1.8)-(1.9) exists a unique global solution in \(\mathbb{R}^3 \times (0, \infty)\), which satisfies
\[
\begin{cases}
(p - p_\infty, q) \in L^\infty(\mathbb{R}^3), \\
\nabla p \in L^2(\mathbb{R}^3), \quad \nabla q \in L^2(\mathbb{R}^3).
\end{cases}
\]

Furthermore,
\[
\|(p^\varepsilon - p)(t)\|_{H^2} + \|(q^\varepsilon - q)(t)\|_{H^2} \leq C\varepsilon^{1/2}.
\]

In particular,
\[
\|(p^\varepsilon - p)(t)\|_{L^\infty} + \|(q^\varepsilon - q)(t)\|_{L^\infty} \leq C\varepsilon^{1/2}.
\]

Here \(C\) is a positive constant independent of \(\varepsilon\) and \(t\).

**Remark 1.3** Notice that for the global existence of the solutions to Cauchy problem (1.8)-(1.9), we only assume that the \(L^2\) norm of initial data is small. The initial data can be of large oscillations with constant state at far field. This is an improvement of [13] where the initial data is required to be small in \(H^s\) \((s > \frac{d}{2} + 1)\) norm which implies the oscillations are small.

**Remark 1.4** The power of \(\varepsilon\) in (1.15) could be improved to 1, which needs a slightly modification of the proof of Theorem 1.2 with more regular initial data. But in this case, it seems that the coefficient \(C\) might depend on \(\varepsilon\) and \(t\).

The proofs of Theorems 1.1 and 1.2 are based on the classical energy method. The key point for the proof of Theorem 1.1 is to obtain some \(a\ priori\) estimates independent of \(\varepsilon\) in which the \(L^2\)-bound of \(\nabla \cdot q^\varepsilon\) plays a crucial role. For the proof of Theorem 1.2 some estimates of the order \(O\left(\varepsilon^{\frac{1}{2}}\right)\) are required, which needs some delicate analysis.

The rest of the paper is organized as follows. In Section 2, we study the global unique solvability on the Cauchy problem (1.1)-(1.2). In Section 3, the zero-diffusion limit as well as the global well-posedness of the solutions to (1.8)-(1.9) is considered. We show that the convergence rate in \(L^\infty\)-norm is of the order \(O(\varepsilon^{\frac{1}{2}})\), when diffusion parameter \(\varepsilon \to 0^+\).

Throughout this paper, we denote a generic positive constant by \(C\) which is independent of \(\varepsilon\) and \(t\).
2 Proof of Theorem 1.1

In this section, we are concerned with the global existence of large-oscillations solutions to the Cauchy problem (1.1)-(1.2) when the initial data is sufficiently close to a constant in $L^2$-norm. The global existence follows from a local existence theorem and some a priori estimates globally in time.

The local existence of the solutions could be done by using some arguments similar to [13]. We shall get some a priori estimates globally in time which are also uniform for $\varepsilon$.

More precisely, for any given $T > 0$ and $\varepsilon \geq 0$, suppose $(p^{\varepsilon}(x,t), q^{\varepsilon}(x,t))$ is a smooth solution to the Cauchy problem (1.1)-(1.2) when the initial data is sufficiently close to a constant in $L^2$-norm. The local existence of the solutions could be done by using some arguments similar to [13]. Without loss of generality, we suppose that $D = 1, p_\infty = 1$. Letting $\tilde{p} = p - 1$, we obtain that

\[
\begin{cases}
\tilde{p}^2 - \nabla \cdot (\tilde{p}^2 q^\varepsilon) - \nabla \cdot q^\varepsilon = \Delta \tilde{p}^2 \\
q^\varepsilon + \nabla (\varepsilon |q^\varepsilon|^2 - \tilde{p}^2) = \varepsilon \Delta q^\varepsilon,
\end{cases}
\]  

provided that

\[
\|q^\varepsilon\|_{L^2}^2 \leq \frac{3}{2} \varepsilon_0, \quad \|\nabla \cdot q^\varepsilon\|_{L^2}^2 \leq \frac{3}{2} M,
\]

\[

\text{Proof of Proposition 2.1} \]

The proof of Proposition 2.1 consists of the following Lemmas 2.2 and 2.3.

Lemma 2.2 $L^2$ estimate] Under the conditions of Theorem 1.1, it holds that

\[
\|\tilde{p}^2\| + \|q^\varepsilon\| + \int_0^t \|\nabla \tilde{p}^2\|^2 ds + \varepsilon \int_0^t \|\nabla q^\varepsilon\|^2 ds \leq \frac{3}{2} \varepsilon_0,
\]

provided that $\varepsilon_0$ is small enough.

**Proof.** Multiplying the first equation in (2.3) by $2\tilde{p}^2$ and the second by $2q^\varepsilon$, summing up them and then integrating over $\mathbb{R}^3 \times [0,t]$, one gets after integration by parts that

\[
\begin{align*}
\|\tilde{p}^2\|^2 + \|q^\varepsilon\|^2 + 2 \int_0^t \|\nabla \tilde{p}^2\|^2 ds + 2 \varepsilon \int_0^t \|\nabla q^\varepsilon\|^2 ds \\
= \|\tilde{p}^2_0\|^2 + \|q^\varepsilon_0\|^2 + 2 \int_0^t \tilde{p}^2 q^\varepsilon \cdot (\nabla \tilde{p}^2) dx ds + 4 \varepsilon \int_0^t (q^\varepsilon)^T (\nabla q^\varepsilon \cdot q^\varepsilon) dx ds.
\end{align*}
\]
Next, we shall estimate the last two terms in the right-hand side. By Cauchy inequality, Hölder inequality, Sobolev inequality and Gagliardo-Nirenberg inequality, we obtain

\[ 2 \int_0^t \int \tilde{p}^\varepsilon \mathbf{q}^\varepsilon \cdot (\nabla \tilde{p}^\varepsilon) \, dx \, ds \leq \int_0^t \| \nabla \tilde{p}^\varepsilon \|^2 \, ds + C \int_0^t \| \tilde{p}^\varepsilon \|^2 \, ds \]
\[ \leq \int_0^t \| \nabla \tilde{p}^\varepsilon \|^2 \, ds + C \int_0^t \| \tilde{p}^\varepsilon \|_{L^6}^2 \| \mathbf{q}^\varepsilon \|^2 \, ds \]
\[ \leq \int_0^t \| \nabla \tilde{p}^\varepsilon \|^2 \, ds + C \int_0^t \| \nabla \tilde{p}^\varepsilon \|^2 \| \nabla \mathbf{q}^\varepsilon \| \| \mathbf{q}^\varepsilon \| \, ds \tag{2.7} \]

and

\[ 4\varepsilon \int_0^t \int (\mathbf{q}^\varepsilon)^T \cdot (\nabla \mathbf{q}^\varepsilon \cdot \mathbf{q}^\varepsilon) \, dx \, ds \leq 4\varepsilon \int_0^t \| \mathbf{q}^\varepsilon \|_{L^3} \| \nabla \mathbf{q}^\varepsilon \|_{L^2} \| \mathbf{q}^\varepsilon \|_{L^6} \, ds \]
\[ \leq C\varepsilon \int_0^t \| \nabla \mathbf{q}^\varepsilon \| \frac{4}{3} \| \mathbf{q}^\varepsilon \| \| \nabla \mathbf{q}^\varepsilon \|^2 \, ds. \tag{2.8} \]

Since \( \Delta \mathbf{q}^\varepsilon = \nabla (\nabla \cdot \mathbf{q}^\varepsilon) - \nabla \times (\nabla \times \mathbf{q}^\varepsilon) \), we obtain that
\n\[ \nabla \mathbf{q}^\varepsilon = -\nabla (-\Delta)^{-1} \nabla (\nabla \cdot \mathbf{q}^\varepsilon) + \nabla (-\Delta)^{-1} \nabla \times (\nabla \times \mathbf{q}^\varepsilon). \]

The standard \( L^2 \) estimate shows that
\n\[ \| \nabla \mathbf{q}^\varepsilon \|_{L^2} \leq C (\| \nabla \cdot \mathbf{q}^\varepsilon \|_{L^2} + \| \nabla \times \mathbf{q}^\varepsilon \|_{L^2}). \tag{2.9} \]

Moreover, by taking the curl for (2.3), one has
\n\[ \frac{d}{dt} (\nabla \times \mathbf{q}^\varepsilon) = \varepsilon \Delta (\nabla \times \mathbf{q}^\varepsilon). \tag{2.10} \]

Initial data is given as
\n\[ (\nabla \times \mathbf{q}^\varepsilon) \big|_{t=0} = \nabla \times \mathbf{q}_0 = 0. \tag{2.11} \]

By solving the initial value problem (2.10) and (2.11), one has
\n\[ \nabla \times \mathbf{q}^\varepsilon = 0, \tag{2.12} \]

which implies
\n\[ \Delta \mathbf{q}^\varepsilon = \nabla (\nabla \cdot \mathbf{q}^\varepsilon), \quad \| \nabla \mathbf{q}^\varepsilon \|_{L^2} \leq C \| \nabla \cdot \mathbf{q}^\varepsilon \|_{L^2}. \tag{2.13} \]

The combination of (2.1), (2.7), (2.8) and (2.13) yields
\n\[ 2 \int_0^t \int \tilde{p}^\varepsilon \mathbf{q}^\varepsilon \cdot (\nabla \tilde{p}^\varepsilon) \, dx \, ds \leq \int_0^t \| \nabla \tilde{p}^\varepsilon \|^2 \, ds + C \sqrt{M\varepsilon_0} \int_0^t \| \nabla \tilde{p}^\varepsilon \|^2 \, ds, \tag{2.14} \]

and
\n\[ 4\varepsilon \int_0^t \int (\mathbf{q}^\varepsilon)^T \cdot (\nabla \mathbf{q}^\varepsilon \cdot \mathbf{q}^\varepsilon) \, dx \, ds \leq C\varepsilon (M\varepsilon_0)^{\frac{1}{4}} \int_0^t \| \nabla \mathbf{q}^\varepsilon \|^2 \, ds. \tag{2.15} \]

Substituting (2.14) and (2.15) into (2.6) and setting \( \varepsilon_0 \leq \frac{1}{16C^4M} \), one may arrive at (2.5), where (1.13) has been used. This completes the proof of Lemma 2.2. \( \square \)
Lemma 2.3  \[\text{[First-order energy estimate]}\] Under the conditions of Theorem 2.1, it holds that
\[
\|\nabla \tilde{p}^\varepsilon\|^2 + \|\nabla \cdot q^\varepsilon\|^2 + \int_0^t \|\nabla \cdot q^\varepsilon\|^2 + \int_0^t \left( \|\nabla \tilde{p}^\varepsilon\|^2 \, ds + \varepsilon \|\Delta q^\varepsilon\|^2 \right) \, ds \leq \frac{3M}{2}, \quad (2.16)
\]
provided that $\varepsilon_0$ is small enough.

**Proof.** Notice that
\[
\nabla \cdot q^\varepsilon_t = \Delta \tilde{p}^\varepsilon - \varepsilon \Delta |q^\varepsilon|^2 + \varepsilon \Delta (\nabla \cdot q^\varepsilon)
\]
\[
= \tilde{p}^\varepsilon_t - \nabla \cdot (\tilde{p}^\varepsilon q^\varepsilon) - \nabla \cdot q^\varepsilon - \varepsilon \Delta |q^\varepsilon|^2 + \varepsilon \nabla \cdot (\Delta q^\varepsilon), \quad (2.17)
\]
where we have used (2.13).

Multiplying (2.17) by $2\nabla \cdot q^\varepsilon$ and integrating the resulting equation over $\mathbb{R}^3$, one obtains after integration by parts that
\[
\frac{d}{dt} \|\nabla \cdot q^\varepsilon\|^2 + 2 \|\nabla \cdot q^\varepsilon\|^2 + 2 \varepsilon \|\Delta q^\varepsilon\|^2
\]
\[
= 2 \int \tilde{p}^\varepsilon_t \nabla \cdot q^\varepsilon \, dx - 2 \int \nabla \cdot (\tilde{p}^\varepsilon q^\varepsilon) \nabla \cdot q^\varepsilon \, dx + 2 \varepsilon \int \nabla |q^\varepsilon|^2 (\Delta q^\varepsilon) \, dx. \quad (2.18)
\]

Next, multiplying the first equation in (2.3) by $2\tilde{p}^\varepsilon_t$, integrating the resulting equality over $\mathbb{R}^3$ and using integration by parts, one has
\[
\frac{d}{dt} \|\nabla \tilde{p}^\varepsilon\|^2 + 2 \|\tilde{p}^\varepsilon_t\|^2 = 2 \int \tilde{p}^\varepsilon_t \nabla \cdot q^\varepsilon \, dx + 2 \int \nabla \cdot (\tilde{p}^\varepsilon q^\varepsilon) \tilde{p}^\varepsilon_t \, dx. \quad (2.19)
\]

The combination of (2.18) with (2.19) yields
\[
\frac{d}{dt} \left( \|\nabla \cdot q^\varepsilon\|^2 + \|\nabla \tilde{p}^\varepsilon\|^2 \right) + 2 \|\nabla \cdot q^\varepsilon\|^2 + 2 \|\tilde{p}^\varepsilon_t\|^2 + 2 \varepsilon \|\Delta q^\varepsilon\|^2
\]
\[
= 4 \int \tilde{p}^\varepsilon_t \nabla \cdot q^\varepsilon \, dx + 2 \int \nabla \cdot (\tilde{p}^\varepsilon q^\varepsilon) \tilde{p}^\varepsilon_t \, dx
\]
\[
- 2 \int \nabla \cdot (\tilde{p}^\varepsilon q^\varepsilon) \nabla \cdot q^\varepsilon \, dx + 2 \varepsilon \int \nabla |q^\varepsilon|^2 \cdot (\Delta q^\varepsilon) \, dx
\]
\[
= \sum_{i=1}^4 J_i. \quad (2.20)
\]

For $J_1$, using (2.17), and integration by parts, we have
\[
J_1 = 4 \frac{d}{dt} \int \nabla \cdot q^\varepsilon \tilde{p}^\varepsilon dx - 4 \int \nabla \cdot q^\varepsilon \tilde{p}^\varepsilon dx
\]
\[
= 4 \|\nabla \tilde{p}^\varepsilon\|^2 + 4 \frac{d}{dt} \int \nabla \cdot q^\varepsilon \tilde{p}^\varepsilon dx - 4 \varepsilon \int \nabla |q^\varepsilon|^2 \cdot (\nabla \tilde{p}^\varepsilon) \, dx - 4 \varepsilon \int \Delta q^\varepsilon \cdot (\nabla \tilde{p}^\varepsilon) \, dx
\]
\[
= 4 \|\nabla \tilde{p}^\varepsilon\|^2 + \sum_{i=1}^3 J_i. \quad (2.21)
\]

By Cauchy inequality, Hölder inequality, Sobolev inequality, Gagliardo-Nirenberg inequality and (2.1), we estimate $J_2^2 - J_3^3$ as follows:
\[
J_1^2 = -8\varepsilon \int q^\varepsilon \cdot (\nabla q^\varepsilon) \cdot (\nabla \tilde{p}^\varepsilon) \, dx
\]
\[
\leq C \varepsilon \|q^\varepsilon\|_{L^3} \|\nabla q^\varepsilon\|_{L^6} \|\nabla \tilde{p}^\varepsilon\|_{L^2}
\]
\[
\leq C \varepsilon (M\varepsilon_0)^\frac{1}{2} \|\Delta q^\varepsilon\| \|\nabla \tilde{p}^\varepsilon\|
\]
\[
\leq C \varepsilon (M\varepsilon_0)^\frac{1}{2} \|\Delta q^\varepsilon\|^2 + C \|\nabla \tilde{p}^\varepsilon\|^2 \quad (2.22)
\]
and
\[ J_3^3 \leq \frac{1}{2} \varepsilon \| \triangle q^\varepsilon \|^2 + C \| \nabla \overline{p}^\varepsilon \|^2. \] (2.23)

On the other hand, Cauchy inequality gives
\[ J_2 + J_3 \leq \frac{1}{2} \| \nabla \cdot q^\varepsilon \|^2 + \frac{1}{2} \| \overline{p}^\varepsilon \|^2 + 16 \| \nabla \cdot (\overline{p}^\varepsilon q^\varepsilon) \|^2. \] (2.24)

Similar to (2.22), it is immediate to obtain
\[ J_4 \leq C \varepsilon (M \varepsilon_0)^{\frac{3}{2}} \| \triangle q^\varepsilon \|^2. \] (2.25)

Finally, we estimate the last term on the right hand side of (2.24)
\[ \| \nabla \cdot (\overline{p}^\varepsilon q^\varepsilon) \|^2 \leq C \| \nabla \overline{p}^\varepsilon \cdot q^\varepsilon \|^2 + C \| \overline{p}^\varepsilon \nabla \cdot q^\varepsilon \|^2 = \sum_{i=5}^{6} J_i. \] (2.26)

For \( J_5 \) and \( J_6 \), using Sobolev inequality, Hölder inequality, Gagliardo-Nirenberg inequality, Young inequality, (2.1) and (2.13), we obtain
\[ J_5 \leq C \| \nabla \overline{p}^\varepsilon \|^2_{L^p} \| q^\varepsilon \|^2_{L^q} \]
\[ \leq C \| \nabla^2 \overline{p}^\varepsilon \|^2 \| \nabla q^\varepsilon \| \| q^\varepsilon \| \leq C \sqrt{M \varepsilon_0} \| \nabla^2 \overline{p}^\varepsilon \|^2 \] (2.27) and
\[ J_6 \leq C \left( \| \overline{p}^\varepsilon \|^2_{L^4} + \| \nabla \overline{p}^\varepsilon \|^2_{L^4} \right) \| \nabla \cdot q^\varepsilon \|^2 \]
\[ \leq C \left( \| \overline{p}^\varepsilon \|^2_{L^4} \| \nabla \overline{p}^\varepsilon \|^2_{L^4} + \| \nabla \overline{p}^\varepsilon \|^2 \| \nabla^2 \overline{p}^\varepsilon \|^2 \right) \| \nabla \cdot q^\varepsilon \|^2 \]
\[ \leq C \| \nabla \overline{p}^\varepsilon \|^2 \| \nabla \cdot q^\varepsilon \|^2 + C \varepsilon_0 \| \nabla \cdot q^\varepsilon \|^2 \]
\[ + C \| \nabla \overline{p}^\varepsilon \|^2 \| \nabla \cdot q^\varepsilon \|^2 + C \varepsilon_0 \| \nabla \cdot q^\varepsilon \|^2. \] (2.28)

Substituting (2.27), (2.28) into (2.26) shows that
\[ \| \nabla \cdot (\overline{p}^\varepsilon q^\varepsilon) \|^2 \leq \left( C \sqrt{M \varepsilon_0} + \varepsilon_1 \right) \| \nabla^2 \overline{p}^\varepsilon \|^2 + CM(1 + M^3) \| \nabla \overline{p}^\varepsilon \|^2 + C \varepsilon_0 \| \nabla \cdot q^\varepsilon \|^2. \] (2.29)

Applying the standard \( L^2 \)-estimate for (2.3), one has
\[ \| \nabla^2 \overline{p}^\varepsilon \|^2 \leq C \left( \| \nabla \cdot q^\varepsilon \|^2 + \| \overline{p}^\varepsilon \|^2 + \| \nabla \cdot (\overline{p}^\varepsilon q^\varepsilon) \|^2 \right), \] (2.30)

which together with (2.29) gives
\[ \| \nabla \cdot (\overline{p}^\varepsilon q^\varepsilon) \|^2 \leq \left( C \sqrt{M \varepsilon_0} + C \varepsilon_1 \right) \left( \| \nabla \cdot q^\varepsilon \|^2 + \| \overline{p}^\varepsilon \|^2 + \| \nabla \cdot (\overline{p}^\varepsilon q^\varepsilon) \|^2 \right) \]
\[ + CM(1 + M^3) \| \nabla \overline{p}^\varepsilon \|^2 + C \varepsilon_0 \| \nabla \cdot q^\varepsilon \|^2. \] (2.31)

Taking \( \varepsilon_1 \) suitably small, one can deduce that
\[ \| \nabla \cdot (\overline{p}^\varepsilon q^\varepsilon) \|^2 \leq \left( C \sqrt{M \varepsilon_0} + C \varepsilon_1 \right) \left( \| \nabla \cdot q^\varepsilon \|^2 + \| \overline{p}^\varepsilon \|^2 \right) \]
\[ + CM(1 + M^3) \| \nabla \overline{p}^\varepsilon \|^2 + C \varepsilon_0 \| \nabla \cdot q^\varepsilon \|^2, \] (2.32)
provided \( \varepsilon_0 \leq \frac{(1-2C\varepsilon_1)^2}{4C^2 M} \).

Substituting (2.21)-(2.25) and (2.32) into (2.20) and integrating the resulting inequality over \([0,t]\), one gets after using (1.10), (1.13) and (2.5) that

\[
\| \nabla \cdot q^\varepsilon \|^2 + \| \nabla p^\varepsilon \|^2 + 2 \int_0^t \| \nabla \cdot q^\varepsilon \|^2 \, ds + 2 \int_0^t \left( \| \nabla p^\varepsilon \|^2 + \varepsilon \| \Delta q^\varepsilon \|^2 \right) \, ds \\
\leq M + \left( C\sqrt{M\varepsilon_0} + C\varepsilon_1 + C\varepsilon_0 + \frac{1}{2} \right) \int_0^t \left( \| \nabla \cdot q^\varepsilon \|^2 + \| \nabla p^\varepsilon \|^2 \right) \, ds \\
+ C\varepsilon_0 (M + M^4 + 1) + \left( C(M\varepsilon_0)^{\frac{1}{2}} + \frac{1}{2} \right) \varepsilon \int_0^t \| \Delta q^\varepsilon \|^2 \, ds \\
+ 4 \int \nabla \cdot q^\varepsilon p^\varepsilon \, dx - 4 \int \nabla \cdot q_0 p_0 \, dx,
\]

which together with Cauchy inequality and (2.4) deduces

\[
\| \nabla \cdot q^\varepsilon \|^2 + \| \nabla p^\varepsilon \|^2 + 2 \int_0^t \| \nabla \cdot q^\varepsilon \|^2 \, ds + 2 \int_0^t \left( \| \nabla p^\varepsilon \|^2 + \varepsilon \| \Delta q^\varepsilon \|^2 \right) \, ds \\
\leq \frac{5M}{4} + \left( C\sqrt{M\varepsilon_0} + C\varepsilon_1 + C\varepsilon_0 + \frac{1}{2} \right) \int_0^t \left( \| \nabla \cdot q^\varepsilon \|^2 + \| \nabla p^\varepsilon \|^2 \right) \, ds \\
+ C\varepsilon_0 (M + M^4 + 1) + \left( C(M\varepsilon_0)^{\frac{1}{2}} + \frac{1}{2} \right) \varepsilon \int_0^t \| \Delta q^\varepsilon \|^2 \, ds.
\]

Next, choosing \( \varepsilon_1 \) suitably small and taking

\[
\varepsilon_0 \leq \min \left\{ \frac{1}{16C^4 M}, \frac{M}{4C(M + M^4 + 1)}, \frac{1 - 2C\varepsilon_1}{2C(\sqrt{M} + 1)}, 1 \right\},
\]

one can get (2.16). This completes the proof of Lemma 2.3. \( \Box \)

The proof of Proposition 2.1 is complete. To prove Theorem 1.1, we need to get some high order estimates. Before beginning, we give the following corollary.

**Corollary 2.4** Under the conditions of Theorem 1.1, it holds that

\[
\int_0^t \left( \| \nabla^2 p^\varepsilon \|^2 + \| \nabla p^\varepsilon \|_{L^\infty} \right) \, ds \leq C,
\]

provided that \( \varepsilon_0 \) is small enough.

**Proof.** It follows from (2.30), (2.31) and (2.16) that

\[
\int_0^t \| \nabla^2 p^\varepsilon \|^2 \, ds \leq C \left( \int_0^t \| \nabla \cdot q^\varepsilon \|^2 \, ds + \int_0^t \| \nabla p^\varepsilon \|^2 \, ds + \int_0^t \| \nabla (p^\varepsilon q^\varepsilon) \|^2 \, ds \right) \\
\leq C.
\]

By Sobolev’s embedding theorem, combining (2.3) and (2.35), we get (2.34). \( \Box \)

**Lemma 2.5** [Second-order energy estimate] Under the conditions of Theorem 1.1, it holds that

\[
\| \nabla^2 p^\varepsilon \|^2 + \| \nabla (\nabla \cdot q^\varepsilon) \|^2 + \int_0^t \| \nabla (\nabla \cdot q^\varepsilon) \|^2 \, ds \\
+ \int_0^t \left( \| \nabla p^\varepsilon \|^2 + \varepsilon \| \nabla (\Delta q^\varepsilon) \|^2 \right) \, ds \leq C,
\]

provided that \( \varepsilon_0 \) is small enough.
Proof. Differentiating (2.17) yields
\[ \nabla(\nabla \cdot \mathbf{q}^\varepsilon) = \nabla(\Delta \mathbf{p}^\varepsilon) - \varepsilon \nabla(\Delta |\mathbf{q}^\varepsilon|^2) + \varepsilon \nabla(\nabla \cdot \mathbf{q}^\varepsilon) \]
\[ = \nabla \mathbf{p}^\varepsilon_t - \nabla(\nabla \cdot (\mathbf{p}^\varepsilon \mathbf{q}^\varepsilon)) - \nabla(\nabla \cdot \mathbf{q}^\varepsilon) - \varepsilon \nabla(\Delta |\mathbf{q}^\varepsilon|^2) + \varepsilon \nabla(\nabla \cdot (\Delta \mathbf{q}^\varepsilon)). \tag{2.37} \]
Multiplying (2.37) by \(2\nabla(\nabla \cdot \mathbf{q}^\varepsilon)\) and integrating the resulting equation over \(\mathbb{R}^3\), one obtains after integration by parts that
\[ \frac{d}{dt} \|\nabla(\nabla \cdot \mathbf{q}^\varepsilon)\|^2 + 2\|\nabla(\nabla \cdot \mathbf{q}^\varepsilon)\|^2 + 2\varepsilon \|\nabla \cdot (\Delta \mathbf{q}^\varepsilon)\|^2 \]
\[ = 2 \int \nabla(\nabla \cdot \mathbf{q}^\varepsilon) \cdot (\nabla \mathbf{p}^\varepsilon_t) dx - 2 \int \nabla \cdot (\nabla \cdot (\mathbf{p}^\varepsilon \mathbf{q}^\varepsilon)) \cdot (\nabla \nabla \cdot \mathbf{q}^\varepsilon) dx \]
\[ + 2\varepsilon \int \|\Delta \mathbf{q}^\varepsilon\|^2 \nabla \cdot (\Delta \mathbf{q}^\varepsilon) dx. \tag{2.38} \]
Next, applying \(\nabla\) to (2.33), multiplying it by \(2\nabla \mathbf{p}^\varepsilon_t\), integrating the resulting equality over \(\mathbb{R}^3\) and using integration by parts, one has
\[ \frac{d}{dt} \|\Delta \mathbf{p}^\varepsilon\|^2 + 2\|\nabla \mathbf{p}^\varepsilon_t\|^2 = 2 \int \nabla(\nabla \cdot \mathbf{q}^\varepsilon) \cdot (\nabla \mathbf{p}^\varepsilon_t) dx + 2 \int \nabla \cdot (\mathbf{p}^\varepsilon \mathbf{q}^\varepsilon) \cdot (\nabla \mathbf{p}^\varepsilon_t) dx. \tag{2.39} \]
Putting (2.38) and (2.39) together, we have
\[ \frac{d}{dt} \left( \|\nabla(\nabla \cdot \mathbf{q}^\varepsilon)\|^2 + \|\Delta \mathbf{p}^\varepsilon\|^2 \right) + 2\|\nabla(\nabla \cdot \mathbf{q}^\varepsilon)\|^2 + 2\|\nabla \mathbf{p}^\varepsilon_t\|^2 + 2\varepsilon \|\nabla(\Delta \mathbf{q}^\varepsilon)\|^2 \]
\[ = 4 \int \nabla(\nabla \cdot \mathbf{q}^\varepsilon) \cdot (\nabla \mathbf{p}^\varepsilon_t) dx + 2 \int \nabla \cdot (\mathbf{p}^\varepsilon \mathbf{q}^\varepsilon) \cdot (\nabla \mathbf{p}^\varepsilon_t) dx \]
\[ - 2 \int \nabla \cdot (\mathbf{p}^\varepsilon \mathbf{q}^\varepsilon) \cdot (\nabla \nabla \cdot \mathbf{q}^\varepsilon) dx + 2\varepsilon \int \|\Delta \mathbf{q}^\varepsilon\|^2 \nabla \cdot (\Delta \mathbf{q}^\varepsilon) dx \]
\[ = \sum_{i=7}^{10} J_i. \tag{2.40} \]
For \(J_7\), using (2.37), and integration by parts, we have
\[ J_7 = 4 \frac{d}{dt} \int \nabla(\nabla \cdot \mathbf{q}^\varepsilon) \cdot (\nabla \mathbf{p}^\varepsilon_t) dx - 4 \int \nabla \cdot (\nabla \cdot \mathbf{q}^\varepsilon) \cdot (\nabla \mathbf{p}^\varepsilon_t) dx \]
\[ = 4 \|\Delta \mathbf{p}^\varepsilon\|^2 + 4 \frac{d}{dt} \int \nabla(\nabla \cdot \mathbf{q}^\varepsilon) \cdot (\nabla \mathbf{p}^\varepsilon_t) dx \]
\[ - 4\varepsilon \int \|\Delta \mathbf{q}^\varepsilon\|^2 \|\Delta \mathbf{p}^\varepsilon\| dx + 4\varepsilon \int \nabla \cdot (\Delta \mathbf{q}^\varepsilon) \Delta \mathbf{p}^\varepsilon dx \]
\[ = 4 \|\Delta \mathbf{p}^\varepsilon\|^2 + \sum_{i=1}^{3} J_i^2. \tag{2.41} \]
For \(J_2^2\), using Sobolev inequality, Hölder inequality, Gagliardo-Nirenberg inequality, Young inequality, Proposition 2.1 and (2.13), we obtain
\[ J_2^2 = -8\varepsilon \int (\|\nabla \mathbf{q}^\varepsilon\|^2 + \|\Delta \mathbf{q}^\varepsilon \cdot \mathbf{q}^\varepsilon\|) \Delta \mathbf{p}^\varepsilon dx \]
\[ \leq \frac{\varepsilon}{2} \|\Delta \mathbf{p}^\varepsilon\|^2 + C\varepsilon \left( \|\nabla \mathbf{q}^\varepsilon\|^2_{L^4} + \|\mathbf{q}^\varepsilon \Delta \mathbf{q}^\varepsilon\|^2 \right) \]
\[ \leq \frac{\varepsilon}{2} \|\Delta \mathbf{p}^\varepsilon\|^2 + C\varepsilon \left( \|\nabla \mathbf{q}^\varepsilon\|^2 \frac{1}{2} \|\nabla \mathbf{q}^\varepsilon\|^2 \frac{3}{2} + \|\Delta \mathbf{q}^\varepsilon\|^2_{L^6} \|\mathbf{q}^\varepsilon\|^2_{L^3} \right) \]
\[ \leq \frac{\varepsilon}{2} \|\Delta \mathbf{p}^\varepsilon\|^2 + C\varepsilon \|\Delta \mathbf{q}^\varepsilon\|^2 + C\varepsilon \|\nabla \mathbf{q}^\varepsilon\|^2 + C\varepsilon \|\nabla (\Delta \mathbf{q}^\varepsilon)\|^2 \|\mathbf{q}^\varepsilon\| \|\nabla \mathbf{q}^\varepsilon\| \]
\[ \leq \frac{\varepsilon}{2} \|\Delta \mathbf{p}^\varepsilon\|^2 + C\varepsilon \|\Delta \mathbf{q}^\varepsilon\|^2 + C\varepsilon \|\nabla \cdot \mathbf{q}^\varepsilon\|^2 + C\varepsilon \sqrt{M \varepsilon} \|\nabla \cdot (\Delta \mathbf{q}^\varepsilon)\|^2, \tag{2.42} \]
where in the last inequality we have used the following fact
\[
\|\nabla(\Delta q^\varepsilon)\|^2 \leq C \left(\|\nabla \cdot (\Delta q^\varepsilon)\|^2 + \|\nabla \times (\Delta q^\varepsilon)\|^2\right)
\]
\[
= C \left(\|\nabla \cdot (\Delta q^\varepsilon)\|^2 + \|\Delta(\nabla \times q^\varepsilon)\|^2\right)
\]
\[
= C \|\nabla \cdot (\Delta q^\varepsilon)\|^2
\]
due to (2.9) and (2.12). Cauchy inequality gives
\[
J_7 \leq \frac{1}{2} \varepsilon \|\nabla \cdot (\Delta q^\varepsilon)\|^2 + C \|\Delta \tilde{p}\|^2
\]  (2.43)
and
\[
J_8 + J_9 \leq \frac{1}{2} \|\nabla(\nabla \cdot q^\varepsilon)\|^2 + \frac{1}{2} \|\nabla \tilde{p}\|^2 + 16 \|\nabla(\nabla \cdot (\tilde{p} q^\varepsilon))\|^2.
\]  (2.44)

Similar to \(J_7^2\), we estimate \(J_{10}\) as follows:
\[
J_{10} \leq \frac{\varepsilon}{2} \|\nabla \cdot (\Delta q^\varepsilon)\|^2 + C \varepsilon \sqrt{M \varepsilon_0} \|\nabla \cdot (\Delta q^\varepsilon)\|^2
\]
\[
+ C \varepsilon \|\Delta q^\varepsilon\|^2 + C \varepsilon \|\nabla \cdot q^\varepsilon\|^2.
\]  (2.45)

Finally, we estimate the last term on the right hand side of (2.44)
\[
\|\nabla(\nabla \cdot (\tilde{p} q^\varepsilon))\|^2
\]
\[
\leq C \|\nabla^2 \tilde{p} \cdot q^\varepsilon\|^2 + C \|\nabla \tilde{p} \nabla \cdot q^\varepsilon\|^2 + C \|\nabla \tilde{p} \cdot (\nabla q^\varepsilon)\|^2 + C \|\tilde{p} \nabla(\nabla \cdot q^\varepsilon)\|^2
\]
\[
= \sum_{i=11}^{14} J_i.
\]  (2.46)

By Hölder inequality, Sobolev inequality, Gagliardo-Nirenberg inequality, Young inequality, Proposition 2.1 and (2.13), we estimate \(J_{11}-J_{14}\) as follows:
\[
J_{11} \leq C \|\nabla^2 \tilde{p}\|_{L^6}^2 \|q^\varepsilon\|_{L^3}^2
\]
\[
\leq C \|\nabla^3 \tilde{p}\|^2 \|\nabla q^\varepsilon\| \|q^\varepsilon\| \leq C \sqrt{M \varepsilon_0} \|\nabla^3 \tilde{p}\|^2,
\]  (2.47)
\[
J_{12} + J_{13} \leq C \|\nabla \tilde{p}\|_{L^\infty}^2 \left(\|\nabla q^\varepsilon\|^2 + \|\nabla \cdot q^\varepsilon\|^2\right)
\]
\[
\leq C \|\nabla^2 \tilde{p}\| \|\nabla^3 \tilde{p}\| \|\nabla \cdot q^\varepsilon\|^2
\]
\[
\leq C \|\nabla \cdot q^\varepsilon\|^4 \|\nabla^2 \tilde{p}\|^2 + \varepsilon_1 \|\nabla^3 \tilde{p}\|^2
\]  (2.48)
and
\[
J_{14} \leq C \|\tilde{p}\|_{L^\infty}^2 \|\nabla(\nabla \cdot q^\varepsilon)\|^2.
\]  (2.49)

Substituting (2.47)-(2.49) into (2.46) shows that
\[
\|\nabla(\nabla \cdot (\tilde{p} q^\varepsilon))\|^2
\]
\[
\leq \left(C \sqrt{M \varepsilon_0} + \varepsilon_1\right) \|\nabla^3 \tilde{p}\|^2 + C \|\nabla^2 \tilde{p}\|^2 + C \|\tilde{p}\|_{L^\infty}^2 \|\nabla(\nabla \cdot q^\varepsilon)\|^2.
\]  (2.50)
Applying the standard $H^1$-estimate for $2.31$, leads to
\[
\|\nabla^2 \tilde{p}^e\|_{H^1}^2 \leq C \left( \|\nabla \cdot q^e\|_{H^1}^2 + \|\tilde{p}^e\|_{H^1}^2 + \|\nabla \cdot (\tilde{p}^e q^e)\|_{H^1}^2 \right),
\] (2.51)
which together with (2.50) gives
\[
\|\nabla (\nabla \cdot (\tilde{p}^e q^e))\|^2 \leq \left( C\sqrt{M \varepsilon_0 + C \varepsilon_1} \right) \left( \|\nabla \cdot q^e\|_{H^1}^2 + \|\tilde{p}^e\|_{H^1}^2 + \|\nabla \cdot (\tilde{p}^e q^e)\|_{H^1}^2 \right) + C \|\nabla^2 \tilde{p}^e\|^2 + C \|\tilde{p}^e\|^2_{L^\infty} \|\nabla (\nabla \cdot q^e)\|^2.
\] (2.52)
Substituting (2.41)-(2.45) and (2.52) into (2.40) and integrating the resulting inequality over $[0, t)$, one gets after using (1.10), Lemmas 2.2, 2.3 and Corollary 2.4 that
\[
\|\nabla (\nabla \cdot q^e)\|^2 + \|\Delta \tilde{p}^e\|^2 + 2 \int_0^t \|\nabla (\nabla \cdot q^e)\|^2 ds + 2 \varepsilon \int_0^t \|\nabla \cdot (\Delta q^e)\|^2 ds \\
\leq C + \frac{1}{2} \|\nabla (\nabla \cdot q^e)\|^2 + \int_0^t \|\tilde{p}^e\|^2_{L^\infty} \|\nabla (\nabla \cdot q^e)\|^2 ds \\
+ \varepsilon \int_0^t \|\nabla \cdot (\Delta q^e)\|^2 ds + C \varepsilon \sqrt{M \varepsilon_0} \int_0^t \|\nabla \cdot (\Delta q^e)\|^2 ds \\
+ \left( C \sqrt{M \varepsilon_0} + C \varepsilon_1 + \frac{1}{2} \right) \int_0^t \left( \|\nabla (\nabla \cdot q^e)\|^2 + \|\nabla^2 \tilde{p}^e\|^2 \right) ds.
\] (2.53)
One obtains (2.36) by using Corollary 2.4 and Gronwall’s inequality. This completes the proof of Lemma 2.5.

**Corollary 2.6** Under the conditions of Theorem 1.1, it holds that
\[
\int_0^t \left( \|\nabla^2 \tilde{p}^e\|_{H^1}^2 + \|\tilde{p}^e\|_{L^\infty}^2 \right) ds \leq C,
\] (2.54)
provided that $\varepsilon_0$ is small enough.

**Proof.** It follows from (2.51)-(2.52), (2.16) and (2.36) that
\[
\int_0^t \|\nabla^2 \tilde{p}^e\|_{H^1}^2 ds \\
\leq C \left( \int_0^t \|\nabla \cdot q^e\|_{H^1}^2 ds + \int_0^t \|\tilde{p}^e\|_{H^1}^2 ds + \int_0^t \|\nabla \cdot (\tilde{p}^e q^e)\|_{H^1}^2 ds \right) \leq C.
\] (2.55)
By Sobolev’s embedding theorem, we get
\[
\int_0^t \|\tilde{p}^e\|^2_{L^\infty} ds \leq C,
\]
which together with (2.55) leads to (2.54).

**Lemma 2.7** [Higher-order energy estimate] Under the conditions of Theorem 1.1, it holds that
\[
\|\nabla^3 \tilde{p}^e\|^2 + \|\nabla^2 (\nabla \cdot q^e)\|^2 + \int_0^t \|\nabla^2 (\nabla \cdot q^e)\|^2 ds \\
+ \int_0^t \left( \|\nabla^2 \tilde{p}^e\|^2 + \varepsilon \|\nabla^3 (\nabla \cdot q^e)\|^2 \right) ds \leq C,
\] (2.56)
provided that $\varepsilon_0$ is small enough.
Proof. Applying $\triangle$ to (2.17), multiplying it by $\triangle(\nabla \cdot q^\varepsilon)$, taking integrations in $x$ and using integration by parts, one gets

$$
\frac{d}{dt} \|\triangle(\nabla \cdot q^\varepsilon)\|^2 + 2 \|\triangle(\nabla \cdot q^\varepsilon)\|^2 + 2\varepsilon \|\triangle^2 q^\varepsilon\|^2
$$

$$
= 2 \int \triangle \overline{\mu} \triangle(\nabla \cdot q^\varepsilon) dx - 2 \int \triangle(\nabla \cdot (\overline{p}^\varepsilon q^\varepsilon)) \triangle(\nabla \cdot q^\varepsilon) dx
$$

$$
+ 2\varepsilon \int \nabla(\triangle|q^\varepsilon|^2) \cdot (\triangle^2 q^\varepsilon) \, dx.
$$

(2.57)

Similar to (2.57), one has

$$
\frac{d}{dt} \|\nabla(\triangle \overline{p}^\varepsilon)\|^2 + 2 \|\triangle \overline{p}^\varepsilon\|^2 = 2 \int \triangle \overline{\mu} \triangle(\nabla \cdot q^\varepsilon) dx + 2 \int \triangle(\nabla \cdot (\overline{p}^\varepsilon q^\varepsilon)) \triangle \overline{p}^\varepsilon dx.
$$

(2.58)

Putting the above two equalities together, we get

$$
\frac{d}{dt} \left( \|\triangle(\nabla \cdot q^\varepsilon)\|^2 + \|\nabla(\triangle \overline{p}^\varepsilon)\|^2 \right)
$$

$$
+ 2 \|\triangle(\nabla \cdot q^\varepsilon)\|^2 + 2 \|\triangle \overline{p}^\varepsilon\|^2 + 2\varepsilon \|\triangle^2 q^\varepsilon\|^2
$$

$$
= 4 \int \triangle \overline{\mu} \triangle(\nabla \cdot q^\varepsilon) dx + 2 \int \triangle(\nabla \cdot (\overline{p}^\varepsilon q^\varepsilon)) \triangle \overline{p}^\varepsilon dx
$$

$$
- 2 \int \triangle(\nabla \cdot (\overline{p}^\varepsilon q^\varepsilon)) \triangle(\nabla \cdot q^\varepsilon) dx + 2\varepsilon \int \nabla(\triangle|q^\varepsilon|^2) \cdot (\triangle^2 q^\varepsilon) \, dx
$$

$$
= \sum_{i=15}^{18} J_i.
$$

(2.59)

In a manner similar to the estimates of $J_7$-$J_{10}$, $J_{15}$-$J_{18}$ can be bounded as follows:

$$
J_{15} = 4 \frac{d}{dt} \int \triangle(\nabla \cdot q^\varepsilon) \triangle \overline{p}^\varepsilon dx - 4 \int \triangle(\nabla \cdot q^\varepsilon) \triangle \overline{p}^\varepsilon dx
$$

$$
= 4 \|\nabla(\triangle \overline{p}^\varepsilon)\|^2 + 4 \frac{d}{dt} \int \triangle(\nabla \cdot q^\varepsilon) \triangle \overline{p}^\varepsilon dx
$$

$$
- 4\varepsilon \int \triangle^2 q^\varepsilon \cdot (\nabla(\triangle \overline{p}^\varepsilon)) \, dx - 4\varepsilon \int \nabla(\triangle|q^\varepsilon|^2) \cdot (\nabla(\triangle \overline{p}^\varepsilon)) \, dx
$$

$$
= 4 \|\nabla(\triangle \overline{p}^\varepsilon)\|^2 + \sum_{i=1}^{3} J_i.
$$

(2.60)

Cauchy inequality gives

$$
J_{15}^2 \leq \frac{1}{2} \varepsilon \|\triangle^2 q^\varepsilon\|^2 + C \|\nabla(\triangle \overline{p}^\varepsilon)\|^2.
$$

(2.61)

For $J_{15}^3$, we have

$$
J_{15}^3 \leq \frac{\varepsilon}{2} \|\nabla(\triangle \overline{p}^\varepsilon)\|^2 + C\varepsilon \left( \|\nabla q^\varepsilon\|_{L^\infty} \|\triangle q^\varepsilon\|^2 + \|\nabla(\triangle q^\varepsilon)\|_{L^6} \|q^\varepsilon\|_{L^3} \right)
$$

$$
\leq \frac{\varepsilon}{2} \|\nabla(\triangle \overline{p}^\varepsilon)\|^2 + C\varepsilon \|\nabla^2 q^\varepsilon\| \|\nabla^2 q^\varepsilon\| \|\triangle q^\varepsilon\|^2 + C\varepsilon\sqrt{M\varepsilon_0} \|\triangle^2 q^\varepsilon\|^2
$$

$$
\leq \frac{\varepsilon}{2} \|\nabla(\triangle \overline{p}^\varepsilon)\|^2 + C\varepsilon \|\nabla(\triangle \cdot q^\varepsilon)\|^2 + C\varepsilon \|\triangle q^\varepsilon\|^2 + C\varepsilon\sqrt{M\varepsilon_0} \|\triangle^2 q^\varepsilon\|^2.
$$

(2.62)

By Cauchy inequality, we get

$$
J_{16} + J_{17} \leq \frac{1}{2} \|\triangle(\nabla \cdot q^\varepsilon)\|^2 + \frac{1}{2} \|\triangle \overline{\mu}^\varepsilon\|^2 + C \|\triangle(\nabla \cdot (\overline{p}^\varepsilon q^\varepsilon))\|^2.
$$

(2.63)
Substituting (2.66)-(2.69) into (2.65) shows that

\[ J_{18} \leq \frac{\varepsilon}{2} \| \Delta^2 \mathbf{q}^\varepsilon \|^2 + C \varepsilon \| \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2 + C \varepsilon \| \Delta \mathbf{q}^\varepsilon \|^2 + C \varepsilon \sqrt{M \varepsilon_0} \| \Delta^2 \mathbf{q}^\varepsilon \|^2. \]  

(2.64)

Next, we estimate the last term on the right hand side of (2.63)

\[ \| \Delta (\nabla \cdot (\vec{p}^\varepsilon \mathbf{q}^\varepsilon)) \|^2 \leq C \| \nabla (\Delta \vec{p}^\varepsilon) \cdot \mathbf{q}^\varepsilon \|^2 + C \| \nabla^2 \vec{p}^\varepsilon \cdot (\nabla \mathbf{q}^\varepsilon) \|^2 + C \| \Delta \vec{p}^\varepsilon \nabla \cdot \mathbf{q}^\varepsilon \|^2 + C \| \Delta \vec{p}^\varepsilon \cdot (\nabla \mathbf{q}^\varepsilon) \|^2 \]

\[ + C \| \nabla \vec{p}^\varepsilon \cdot (\nabla \mathbf{q}^\varepsilon) \|^2 + C \| \vec{p}^\varepsilon \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2 \]

\[ = \sum_{i=19}^{24} J_i. \]  

(2.65)

We estimate \( J_{19}-J_{24} \) as follows:

\[ J_{19} \leq C \| \nabla (\Delta \vec{p}^\varepsilon) \|_{L^6} \| \mathbf{q}^\varepsilon \|_{L^6} \leq C \| \Delta^2 \vec{p}^\varepsilon \| \| \nabla \mathbf{q}^\varepsilon \| \| \mathbf{q}^\varepsilon \| \leq C \sqrt{M \varepsilon_0} \| \Delta^2 \vec{p}^\varepsilon \|^2, \]

(2.66)

\[ J_{20} + J_{21} \leq C \| \nabla^3 \vec{p}^\varepsilon \| \| \nabla^4 \vec{p}^\varepsilon \| \| \nabla \cdot \mathbf{q}^\varepsilon \| \]

\[ \leq C \| \nabla \cdot \mathbf{q}^\varepsilon \|^4 \| \nabla (\Delta \vec{p}^\varepsilon) \|^2 + C \| \Delta^2 \vec{p}^\varepsilon \|^2 \]

\[ \leq C \| \nabla (\Delta \vec{p}^\varepsilon) \|^2 + \varepsilon_1 \| \Delta^2 \vec{p}^\varepsilon \|^2, \]  

(2.67)

\[ J_{22} + J_{23} \leq C \| \nabla \vec{p}^\varepsilon \|_{L^\infty} \| \nabla (\nabla \cdot \mathbf{q}^\varepsilon) \|^2 \leq C \| \nabla \vec{p}^\varepsilon \|^2_{L^\infty} \]  

(2.68)

and

\[ J_{24} \leq C \| \vec{p}^\varepsilon \|^2_{L^\infty} \| \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2. \]  

(2.69)

Substituting (2.66)-(2.69) into (2.65) shows that

\[ \| \Delta (\nabla \cdot (\vec{p}^\varepsilon \mathbf{q}^\varepsilon)) \|^2 \leq \left( C \sqrt{M \varepsilon_0} + \varepsilon_1 \right) \| \Delta^2 \vec{p}^\varepsilon \|^2 + C \| \nabla (\Delta \vec{p}^\varepsilon) \|^2 \]

\[ + C \| \nabla \vec{p}^\varepsilon \|^2_{L^\infty} + C \| \vec{p}^\varepsilon \|^2_{L^\infty} \| \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2. \]

(2.70)

Applying the standard \( H^2 \)-estimate for (2.31) leads to

\[ \| \Delta \vec{p}^\varepsilon \|^2_{H^2} \leq C \left( \| \nabla \cdot \mathbf{q}^\varepsilon \|^2_{H^2} + \| \vec{p}^\varepsilon \|^2_{H^2} + \| \nabla \cdot (\vec{p}^\varepsilon \mathbf{q}^\varepsilon) \|^2_{H^2} \right), \]

(2.71)

which together with (2.70) gives

\[ C \| \Delta (\nabla \cdot (\vec{p}^\varepsilon \mathbf{q}^\varepsilon)) \|^2 \leq \left( C \sqrt{M \varepsilon_0} + C \varepsilon_1 \right) \left( \| \nabla \cdot \mathbf{q}^\varepsilon \|^2_{H^2} + \| \vec{p}^\varepsilon \|^2_{H^2} + \| \nabla \cdot (\vec{p}^\varepsilon \mathbf{q}^\varepsilon) \|^2_{H^2} \right) \]

\[ + C \| \nabla (\Delta \vec{p}^\varepsilon) \|^2 + C \| \vec{p}^\varepsilon \|^2_{L^\infty} \| \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2 + C \| \nabla \vec{p}^\varepsilon \|^2_{L^\infty}. \]

(2.72)

Substituting (2.66)-(2.69) and (2.72) into (2.59) and integrating the resulting inequality over \([0, t]\), one obtains after using (1.10), Lemmas 2.2, 2.3, Lemma 2.5, Corollary 2.4, and Corollary 2.6 that

\[ \| \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2 + \| \nabla (\Delta \vec{p}^\varepsilon) \|^2 \]

\[ + 2 \int_0^t \| \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2 \, ds + 2 \int_0^t \left( \| \Delta \vec{p}^\varepsilon \|^2 + \varepsilon \| \Delta^2 \mathbf{q}^\varepsilon \| \|^2 \right) \, ds \]

\[ \leq C + \frac{1}{2} \| \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2 + \left( C \sqrt{M \varepsilon_0} + C \varepsilon_1 + \frac{1}{2} \right) \int_0^t \left( \| \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2 + \| \Delta \vec{p}^\varepsilon \|^2 \right) \, ds \]

\[ + C \int_0^t \| \vec{p}^\varepsilon \|^2_{L^\infty} \| \Delta (\nabla \cdot \mathbf{q}^\varepsilon) \|^2 \, ds + \varepsilon \int_0^t \| \Delta^2 \mathbf{q}^\varepsilon \|^2 \, ds + C \varepsilon \sqrt{M \varepsilon_0} \int_0^t \| \Delta^2 \mathbf{q}^\varepsilon \|^2 \, ds. \]  

(2.73)
Using Corollary 2.4, Gronwall's inequality and (2.13), one can immediately get (2.56). This completes the proof of Lemma 2.7.

**Corollary 2.8** Under the conditions of Theorem 1.1, it holds that
\[
\int_0^t \| \nabla^2 \tilde{p} \|^2_{H^2} ds \leq C, \tag{2.74}
\]
provided that \( \varepsilon_0 \) is small enough.

**Proof.** It follows from (2.16), (2.36), (2.56) and (2.71) that
\[
\int_0^t \| \nabla^2 \tilde{p} \|^2_{H^2} ds \leq C \left( \int_0^t \| \nabla \cdot q \|_{H^2}^2 ds + \int_0^t \| \tilde{p} \|^2_{H^2} ds + \int_0^t \| \nabla \cdot (\tilde{p} q) \|^2_{H^2} ds \right) \leq C.
\]

**Proof of Theorem 1.1:**
As a consequence of (2.5), (2.16), (2.36), (2.56) and (2.74), one obtains
\[
\| q(x) \|_{H^2}^2 + \| \tilde{p} \|^2_{H^2} + \int_0^t \| \nabla \tilde{p} \|^2_{H^2} ds \leq C \left( \int_0^t \| \nabla \cdot q \|_{H^2}^2 ds + \int_0^t \| \nabla \times q \|_{H^2}^2 ds \right) + \varepsilon \int_0^t \| \nabla q \|^2_{H^2} ds \leq C,
\]
where we have used (2.12) and the following fact:
\[
\| \nabla q \|_{H^s} \leq C \left( \| \nabla \cdot q \|_{H^s} + \| \nabla \times q \|_{H^s} \right), \quad \text{for } s = 1, 2, 3. \tag{2.75}
\]

The uniqueness result can be proved by the method used in [14], we thus omit the details for brevity. This completes the proof of Theorem 1.1.

### 3 Proof of Theorem 1.2

From (1.12), one can easily get a unique smooth solution \((p, q)\) to (1.8)-(1.9) with regularities as in Theorem 1.2 after passing to the limits \( \varepsilon \to 0 \) (take subsequence if necessary).

We will give the proof of the last part of Theorem 1.2, i.e., the convergence rate as \( \varepsilon \to 0 \). To do this, it suffices to show the following Proposition 3.1.

**Proposition 3.1** Assume that the assumptions listed in Theorem 1.1 are satisfied. Then there exists a positive constant \( C \), independent of \( \varepsilon \) and \( \varepsilon \), such that
\[
\| q^\varepsilon - q \|^2_{H^2} + \| \tilde{p}^\varepsilon - p \|^2_{H^2} + \varepsilon \int_0^t \| \nabla (q^\varepsilon - q) \|^2_{H^2} ds \leq C \varepsilon. \tag{3.1}
\]

**Proof.** Denote \( D = 1, \quad \tilde{p} = p - 1 \). Then (1.8)-(1.9) can be translated into the following system
\[
\begin{cases}
\tilde{p}_t - \nabla \cdot (\tilde{p} q) - \nabla \cdot q = \triangle \tilde{p}, \\
q_t - \nabla \tilde{p} = 0
\end{cases} \tag{3.2}
\]
with initial data
\[
(\tilde{p}, q)(x, 0) = (\tilde{p}_0(x), q_0(x)) \to (0, 0) \quad \text{as } |x| \to \infty. \tag{3.3}
\]
Setting
\[ \psi^\varepsilon = q^\varepsilon - q, \quad \theta^\varepsilon = p^\varepsilon - p = \tilde{p}^\varepsilon - \tilde{p}. \] (3.4)

Then we deduce from (3.2)-(3.3) and (2.3)-(2.4) that \((\psi^\varepsilon, \theta^\varepsilon) (x,t)\) satisfy the following Cauchy problem:
\[
\begin{align*}
\theta_t^\varepsilon &- \nabla \cdot (\psi^\varepsilon \tilde{p} + q^\varepsilon \theta^\varepsilon) - \nabla \cdot \psi^\varepsilon = \Delta \theta^\varepsilon, \\
\psi_t^\varepsilon + \nabla \left( \varepsilon (q^\varepsilon)^2 - \theta^\varepsilon \right) &= \varepsilon \Delta \psi^\varepsilon + \varepsilon \Delta q
\end{align*}
\] (3.5)

with initial data
\[ (\psi^\varepsilon, \theta^\varepsilon) (x,0) = (0,0). \] (3.6)

**Step 1.**

Multiplying the first and second equations of (3.5) by \(2 \theta^\varepsilon\) and \(2 \psi^\varepsilon\) respectively, integrating the adding result with respect \(x\) and \(t\) over \(\mathbb{R}^3 \times [0,t]\), we have
\[
\|\psi^\varepsilon\|^2 + \|\theta^\varepsilon\|^2 + 2 \int_0^t \left( \varepsilon \|\nabla \psi^\varepsilon\|^2 + \|\nabla \theta^\varepsilon\|^2 \right) ds
\]
\[= -4\varepsilon \int_0^t \int q^\varepsilon \cdot (\nabla q^\varepsilon) \cdot \psi^\varepsilon dx ds + 2\varepsilon \int_0^t \int \psi^\varepsilon \cdot (\Delta q) dx ds
\]
\[-2 \int_0^t \int \tilde{p} \nabla \theta^\varepsilon \cdot \psi^\varepsilon dx ds - 2 \int_0^t \int q^\varepsilon \cdot (\nabla \theta^\varepsilon) \theta^\varepsilon dx ds
\]
\[= \sum_{i=1}^{4} K_i. \] (3.7)

By Cauchy inequality, Hölder inequality, Sobolev inequality, Gagliardo-Nirenberg inequality and Theorem 1.1, we obtain
\[
K_1 \leq 2\varepsilon \int_0^t \|\nabla q^\varepsilon\|^2 ds + 2\varepsilon \int_0^t \|q^\varepsilon \psi^\varepsilon\|^2 ds
\]
\[\leq C\varepsilon + C \varepsilon \int_0^t \|q^\varepsilon\|^2_{L^3} \|\psi^\varepsilon\|^2_{L^6} ds
\]
\[\leq C\varepsilon + C \varepsilon \sqrt{M\varepsilon_0} \int_0^t \|\nabla \psi^\varepsilon\|^2 ds. \] (3.8)

Integration by parts and Cauchy inequality implies
\[
K_2 \leq \varepsilon \int_0^t \|\nabla q^\varepsilon\|^2 ds + \varepsilon \int_0^t \|\nabla \psi^\varepsilon\|^2 ds
\] (3.9)

and
\[
K_3 + K_4 \leq 2 \int_0^t \left( \|\tilde{p}\|^2_{L^\infty} + \|q^\varepsilon\|^2_{L^\infty} \right) \left( \|\psi^\varepsilon\|^2 + \|\theta^\varepsilon\|^2 \right) ds
\]
\[+ \frac{3}{2} \int_0^t \|\nabla \theta^\varepsilon\|^2 ds. \] (3.10)

Substituting (3.8)-(3.10) into (3.7), we get
\[
\|\psi^\varepsilon\|^2 + \|\theta^\varepsilon\|^2 + \int_0^t \left( \varepsilon \|\nabla \psi^\varepsilon\|^2 + \|\nabla \theta^\varepsilon\|^2 \right) ds
\]
\[\leq C\varepsilon + C \int_0^t \left( \|\tilde{p}\|^2_{L^\infty} + \|q^\varepsilon\|^2_{L^\infty} \right) \left( \|\psi^\varepsilon\|^2 + \|\theta^\varepsilon\|^2 \right) ds. \] (3.11)
Step 2.

Multiplying the first and second equations of (3.5) by $-2\Delta \theta^\varepsilon$ and $-2\Delta \psi^\varepsilon$ respectively, integrating the adding result with respect $x$ and $t$ over $\mathbb{R}^3 \times [0, t]$, we have

$$
\|\nabla \psi^\varepsilon\|^2 + \|\nabla \theta^\varepsilon\|^2 + 2 \int_0^t \left( \varepsilon \|\Delta \psi^\varepsilon\|^2 + \|\Delta \theta^\varepsilon\|^2 \right) ds
$$

$$
= 4\varepsilon \int_0^t \int \mathbf{q}^\varepsilon \cdot (\nabla \mathbf{q}^\varepsilon) \cdot \Delta \psi^\varepsilon dx ds - 2\varepsilon \int_0^t \int \Delta \psi^\varepsilon \cdot (\Delta \mathbf{q}) dx ds
$$

$$
- 2\int_0^t \int \nabla \cdot (\psi^\varepsilon \widetilde{p} + \mathbf{q}^\varepsilon \theta^\varepsilon) \Delta \theta^\varepsilon dx ds
$$

$$
= \sum_{i=5}^7 K_i. \tag{3.12}
$$

Cauchy inequality leads to

$$
K_5 \leq C\varepsilon \int_0^t \|\mathbf{q}^\varepsilon\|^2_{L^\infty} \|\nabla \mathbf{q}^\varepsilon\|^2 ds + \frac{\varepsilon}{2} \int_0^t \|\nabla \psi^\varepsilon\|^2 ds \tag{3.13}
$$

and

$$
K_6 \leq \varepsilon \int_0^t \|\Delta \mathbf{q}^\varepsilon\|^2 ds + \varepsilon \int_0^t \|\Delta \psi^\varepsilon\|^2 ds. \tag{3.14}
$$

Straightforward calculations show that:

$$
K_7 = -2 \int_0^t \int \nabla \cdot \psi^\varepsilon \widetilde{p} \Delta \theta^\varepsilon dx ds - 2\int_0^t \psi^\varepsilon \cdot (\nabla \widetilde{p}) \Delta \theta^\varepsilon dx ds
$$

$$
- 2\int_0^t \int \nabla \cdot \mathbf{q}^\varepsilon \theta^\varepsilon \Delta \theta^\varepsilon dx ds - 2\int_0^t \int \mathbf{q} \cdot (\nabla \theta^\varepsilon) \Delta \theta^\varepsilon dx ds
$$

$$
= \sum_{i=1}^4 K_i^7. \tag{3.15}
$$

Here $K_7^1 - K_7^4$ are estimated as follows:

$$
K_7^1 \leq \frac{1}{4} \int_0^t \|\Delta \theta^\varepsilon\|^2 ds + C \int_0^t \|\widetilde{p}\|^2_{L^\infty} \|\nabla \psi^\varepsilon\|^2 ds, \tag{3.16}
$$

$$
K_7^2 \leq \frac{1}{4} \int_0^t \|\Delta \theta^\varepsilon\|^2 ds + C \int_0^t \|\nabla \mathbf{q}^\varepsilon\|^2_{L^\infty} \|\psi^\varepsilon\|^2 ds \tag{3.17}
$$

and

$$
K_7^3 + K_7^4 \leq \frac{1}{4} \int_0^t \|\Delta \theta^\varepsilon\|^2 ds + C \int_0^t \|\nabla \cdot \mathbf{q}^\varepsilon\|^2_{L^\infty} \|\theta^\varepsilon\|^2 ds
$$

$$
+ C \int_0^t \|\mathbf{q}\|^2_{L^\infty} \|\nabla \theta^\varepsilon\|^2 ds. \tag{3.18}
$$

Substituting (3.13)-(3.18) into (3.12), we get

$$
\|\nabla \psi^\varepsilon\|^2 + \|\nabla \theta^\varepsilon\|^2 + \int_0^t \left( \varepsilon \|\Delta \psi^\varepsilon\|^2 + \|\Delta \theta^\varepsilon\|^2 \right) ds
$$

$$
\leq C\varepsilon + C \int_0^t \left( \|\widetilde{p}\|^2_{W^{1,\infty}} + \|\mathbf{q}\|^2_{W^{1,\infty}} \right) \left( \|\psi^\varepsilon\|^2_{H^1} + \|\theta^\varepsilon\|^2_{H^1} \right) ds. \tag{3.19}
$$
Step 3.

Differentiating (3.5) yields

\[
\begin{aligned}
\begin{cases}
\nabla \theta_t^\varepsilon - \nabla \left( \nabla \cdot (\psi^\varepsilon \tilde{p} + q^{\theta^\varepsilon}) \right) - \Delta \psi^\varepsilon = \nabla (\Delta \theta^\varepsilon), \\
\nabla \cdot \psi_t^\varepsilon + \Delta \left( \varepsilon \left( q^{\psi^\varepsilon} \right)^2 - \theta^\varepsilon \right) = \varepsilon \nabla \cdot (\Delta \psi^\varepsilon) + \varepsilon \nabla \cdot (\Delta q),
\end{cases}
\end{aligned}
\]  

(3.20)

Multiplying the first and second equations of (3.20) by \(-2\nabla (\Delta \theta^\varepsilon)\) and \(-2\nabla \cdot (\Delta \psi^\varepsilon)\) respectively, integrating the adding result with respect \(x\) and \(t\) over \(\mathbb{R}^3 \times [0, t]\), we have

\[
\| \Delta \psi^\varepsilon \|^2 + \| \Delta \theta^\varepsilon \|^2 + 2 \int_0^t \left( \varepsilon \| \nabla \cdot (\Delta \psi^\varepsilon) \|^2 + \| \nabla (\Delta \theta^\varepsilon) \|^2 \right) ds
\]

\[
= 2\varepsilon \int_0^t \int \nabla \cdot (\Delta \psi^\varepsilon) \nabla |q^{\psi^\varepsilon}|^2 ds dt - 2\varepsilon \int_0^t \int \nabla \cdot (\Delta \psi^\varepsilon) \nabla \cdot (\Delta q) ds dt
\]

\[
- 2\varepsilon \int_0^t \int \nabla (\nabla \cdot (\psi^\varepsilon \tilde{p} + q^{\theta^\varepsilon})) \cdot (\nabla (\Delta \theta^\varepsilon)) ds dt
\]

\[
= \sum_{i=8}^{10} K_i.
\]  

(3.21)

Then, it follows from Cauchy inequality that

\[
K_8 = 4\varepsilon \int_0^t \int \nabla \cdot (\Delta \psi^\varepsilon) |\nabla q^{\psi^\varepsilon}|^2 ds dt + 4\varepsilon \int_0^t \int \nabla \cdot (\Delta \psi^\varepsilon) q^{\theta^\varepsilon} \cdot (\Delta q^{\theta^\varepsilon}) ds dt
\]

\[
\leq C\varepsilon \int_0^t \left( \| \nabla q^{\psi^\varepsilon} \|^2_{L^4} + \| q^{\theta^\varepsilon} \Delta q^{\theta^\varepsilon} \|^2 \right) ds + \frac{\varepsilon}{2} \int_0^t \| \nabla \cdot (\Delta \psi^\varepsilon) \|^2 ds
\]

\[
\leq C\varepsilon \int_0^t \left( \| q^{\psi^\varepsilon} \|^2_{L^\infty} + \| \nabla q^{\psi^\varepsilon} \|^2_{L^\infty} \right) \left( \| \nabla q^{\psi^\varepsilon} \|^2 + \| \Delta q^{\theta^\varepsilon} \|^2 \right) ds
\]

\[
+ \frac{\varepsilon}{2} \int_0^t \| \nabla \cdot (\Delta \psi^\varepsilon) \|^2 ds
\]

\[
\leq C\varepsilon + \frac{\varepsilon}{2} \int_0^t \| \nabla \cdot (\Delta \psi^\varepsilon) \|^2 ds,
\]  

(3.22)

\[
K_9 \leq \varepsilon \int_0^t \| \nabla \cdot (\Delta q^{\theta^\varepsilon}) \|^2 ds + \varepsilon \int_0^t \| \nabla \cdot (\Delta \psi^\varepsilon) \|^2 ds
\]  

(3.23)

and

\[
K_{10} \leq \int_0^t \| \nabla (\Delta \theta^\varepsilon) \|^2 ds + C \int_0^t \| \Delta \psi^\varepsilon \tilde{p} \|^2 ds + C \int_0^t \| \nabla \psi^\varepsilon \cdot (\nabla \tilde{p}) \|^2 ds
\]

\[
+ C \int_0^t \| \nabla^2 \tilde{p} \cdot \psi^\varepsilon \|^2 ds + C \int_0^t \| \Delta q^{\theta^\varepsilon} \|^2 ds
\]

\[
+ C \int_0^t \| \nabla^2 \theta^\varepsilon \cdot q \|^2 ds + C \int_0^t \| \nabla q \cdot (\nabla \theta^\varepsilon) \|^2 ds
\]

\[
\leq \int_0^t \| \nabla (\Delta \theta^\varepsilon) \|^2 ds + \sum_{i=1}^{6} K_{10}^i.
\]  

(3.24)

Here \(K_{10}^i - K_6^i\) are estimated as follows:

\[
K_{10}^1 + K_{10}^2 \leq C \int_0^t \left( \| \tilde{p} \|^2_{L^\infty} + \| \nabla \tilde{p} \|^2_{L^\infty} \right) \left( \| \nabla \psi^\varepsilon \|^2 + \| \Delta \psi^\varepsilon \|^2 \right) ds
\]

\[
\leq C \int_0^t \left( \| \nabla \tilde{p} \|^2_{H^1} + \| \nabla^2 \tilde{p} \|^2_{H^1} \right) \left( \| \nabla \psi^\varepsilon \|^2 + \| \Delta \psi^\varepsilon \|^2 \right) ds,
\]  

(3.25)
\[ K_{10}^3 + K_{10}^4 \leq C \int_0^t \left( \| \psi^\varepsilon \|_{L^\infty}^2 + \| \theta^\varepsilon \|_{L^\infty}^2 \right) \left( \| \Delta \tilde{p} \|^2 + \| \Delta q \|^2 \right) \, ds \]
\[
\leq C \int_0^t \left( \| \nabla \psi^\varepsilon \|_{H^1}^2 + \| \nabla \theta^\varepsilon \|_{H^1}^2 \right) \left( \| \Delta \tilde{p} \|^2 + \| \Delta q \|^2 \right) \, ds \quad (3.26)
\]

and
\[ K_{10}^5 + K_{10}^6 \leq C \int_0^t \left( \| q \|_{L^\infty}^2 + \| \nabla q \|_{L^\infty}^2 \right) \left( \| \nabla \theta^\varepsilon \|^2 + \| \Delta \theta^\varepsilon \|^2 \right) \, ds \]
\[
\leq C \int_0^t \left( \| \nabla q \|_{H^1}^2 + \| \nabla^2 q \|_{H^1}^2 \right) \left( \| \nabla \theta^\varepsilon \|^2 + \| \Delta \theta^\varepsilon \|^2 \right) \, ds. \quad (3.27)
\]

Plugging the estimates for \( K_8 \sim K_{10} \) into (3.21), we get
\[
\| \Delta \psi^\varepsilon \|^2 + \| \Delta \theta^\varepsilon \|^2 + \int_0^t \left( \varepsilon \| \nabla \cdot (\Delta \psi^\varepsilon) \|^2 + \| \nabla^3 \theta^\varepsilon \|^2 \right) \, ds \]
\[
\leq C \varepsilon + C \int_0^t \left( \| \nabla \tilde{p} \|_{H^2}^2 + \| \nabla q \|_{H^2}^2 \right) \left( \| \psi^\varepsilon \|_{H^2}^2 + \| \theta^\varepsilon \|_{H^2}^2 \right) \, ds. \quad (3.28)
\]

Combination of (3.11), (3.19) and (3.28) yields
\[
\| \psi^\varepsilon \|_{H^2}^2 + \| \theta^\varepsilon \|_{H^2}^2 + \int_0^t \left( \varepsilon \| \nabla \psi^\varepsilon \|_{H^2}^2 + \| \nabla \theta^\varepsilon \|_{H^2}^2 \right) \, ds \]
\[
\leq C \varepsilon + C \int_0^t \left( \| \nabla \tilde{p} \|_{H^2}^2 + \| \nabla q \|_{H^2}^2 \right) \left( \| \psi^\varepsilon \|_{H^2}^2 + \| \theta^\varepsilon \|_{H^2}^2 \right) \, ds. \quad (3.29)
\]

which, together with Theorem 1.1 and Gronwalls inequality gives (3.1). This completes the proof of Lemma 3.1.

From Lemma 3.1, we get (1.15). Using Sobolev inequality and (1.15), we get (1.16). This completes the proof of Theorem 1.2.

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