Full automorphism groups of association schemes based on attenuated spaces

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Abstract

The set of subspaces with a given dimension in an attenuated space has a structure of a symmetric association scheme, which is a generalization of both Grassmann schemes and bilinear forms schemes. In [K. Wang, J. Guo, F. Li, Association schemes based on attenuated space, European J. Combin. 31 (2010) 297–305], its intersection numbers were computed. In this paper, we determine its full automorphism group.

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1 Introduction

Let $\mathbb{F}_q$ be a finite field with $q$ elements, where $q$ is a prime power. For two non-negative integers $n$ and $l$, suppose $\mathbb{F}_q^{(n+l)}$ denotes an $(n + l)$-dimensional row vector space over $\mathbb{F}_q$. The set of all matrices over $\mathbb{F}_q$ of the form

$$\begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix},$$

where $T_{11}$ and $T_{22}$ are nonsingular $n \times n$ and $l \times l$ matrices respectively, forms a group under matrix multiplication, called the singular general linear group of degree $n + l$ over $\mathbb{F}_q$ and denoted by $GL_{n+l,n}(\mathbb{F}_q)$.

Let $P$ be an $m$-dimensional subspace of $\mathbb{F}_q^{(n+l)}$, denote also by $P$ an $m \times (n + l)$ matrix of rank $m$ whose rows span the subspace $P$ and call the matrix $P$ a matrix representation of the subspace $P$. The group $GL_{n+l,n}(\mathbb{F}_q)$ acts on $\mathbb{F}_q^{(n+l)}$ by the vector matrix multiplication. This action induces an action on the set of subspaces of $\mathbb{F}_q^{(n+l)}$, i.e., a subspace $P$ is carried by $T \in GL_{n+l,n}(\mathbb{F}_q)$ into the subspace $PT$. The vector space $\mathbb{F}_q^{(n+l)}$ together with this group action is called the $(n + l)$-dimensional singular linear space over $\mathbb{F}_q$. This concept was introduced in [10, 11].

For $1 \leq i \leq n + l$, let $e_i$ be the vector in $\mathbb{F}_q^{(n+l)}$ whose $i$-th coordinate is 1 and all other coordinates are 0. Denote by $E$ the $l$-dimensional subspace of $\mathbb{F}_q^{(n+l)}$ generated by

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Theorem 1.3 Let $X$ of product $(\mathbb{F}_q(n+l))$. Deng and Li determined the full automorphism group of the bilinear forms scheme $\tilde{J}$. Wang et al. [10] proved that the configuration hara [7] computed its character table. If $l$ permutation $\bar{\sigma}$ where

$$R_{i,j-i} = \{(P, Q) \in X \times X | \dim(P' \cap Q') = m - i, \dim(P \cap Q) = m - j\},$$

where

$$P = \begin{pmatrix} n & l \\ P' \\ \hline \end{pmatrix}, \quad Q = \begin{pmatrix} n & l \\ Q' \\ Q'' \end{pmatrix}. \tag{1}$$

Wang et al. [10] proved that the configuration

$$\mathcal{X}_m = (X_m, \{R_{i,j-i} \}_{0 \leq i \leq \min\{m,n-m\}, 0 \leq j-i \leq \min\{m-i,j,l\}})$$

is a symmetric association scheme, and computed its intersection numbers. Recently Kurihara [7] computed its character table. If $l = 0$, the scheme $\mathcal{X}_m$ is the Grassmann scheme $J_q(n,m)$; and if $m = n$, the scheme $\mathcal{X}_m$ is the bilinear forms scheme $H_q(n,l)$. We refer readers to [1, 3] for the general theory of association schemes.

Let $\mathcal{X} = (X, \{R_{i,j} \}_{0 \leq i \leq n})$ be an association scheme. If a permutation $\sigma$ on $X$ induces a permutation $\bar{\sigma}$ on $\{0, 1, \ldots, d\}$ by $\bar{\sigma}(x), \bar{\sigma}(y) \in R_{\bar{\sigma}(i)}$ for $(x, y) \in R_i$, then $\sigma$ is called an automorphism of $\mathcal{X}$. The set of all automorphisms of $\mathcal{X}$ becomes a group, called the full automorphism group of $\mathcal{X}$, denoted by $\text{Aut}(\mathcal{X})$. An automorphism of $\mathcal{X}$ is called an inner automorphism if it induces the identity permutation on $\{0, 1, \ldots, d\}$. Clearly, the set of all inner automorphisms of $\mathcal{X}$ is a normal subgraph of $\text{Aut}(\mathcal{X})$, which is called the inner automorphism group of $\mathcal{X}$, denoted by $\text{Inn}(\mathcal{X})$.

In 1949, Chow determined the full automorphism group of the Grassmann scheme $J_q(n,m)$.

**Theorem 1.1** ([4]) Let $1 < m < n - 1$. Then

$$\text{Aut}(J_q(n,m)) = \begin{cases} \text{PGL}(n, \mathbb{F}_q), & \text{if } n \neq 2m, \\ \text{PGL}(n, \mathbb{F}_q).2, & \text{if } n = 2m. \end{cases}$$

In [5], Fujisaki et al. determined the full automorphism group of the twist Grassmann scheme $J_q(2e+1, e)$. This scheme has the same parameters as $J_q(2e+1, e)$; see [2]. In 1965, Deng and Li determined the full automorphism group of the bilinear forms scheme $H_q(n,l)$.

**Theorem 1.2** ([5]) Let $n$ and $l$ be two integers not less than 2. Then

$$\text{Aut}(H_q(n,l)) = \begin{cases} \text{PGL}(n+l, \mathbb{F}_q)_E, & \text{if } n \neq l, \\ \text{PGL}(n+l, \mathbb{F}_q)_E.2, & \text{if } n = l. \end{cases}$$

Observe that $\mathcal{X}_1$ is an association scheme with two classes. Since one relation graph is $\binom{n}{1}$ copies of the complete graph on $q^j$ vertices, it’s full automorphism group is the wreath product $S_q \wr S_{\binom{n}{1}}$.

Motivated by above results, in this paper we shall determine the full automorphism group of $\mathcal{X}_m$, and obtain the following result.

**Theorem 1.3** Let $1 < m < n - 1$ and $l > 0$. Then $\text{Aut}(\mathcal{X}_m) = \text{PGL}(n+l, \mathbb{F}_q)_E$. 

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2 Proof of Theorem 1.3

In this section we always assume that $1 < m < n - 1$ and $l > 0$. For each integer $k$ with $2 \leq k \leq m$, let $\Gamma^{(k)}$ denote the relation graph $(X_k, R_{1,0})$ of $X_k$. We first determine the full automorphism group of $\Gamma^{(m)}$, then prove Theorem 1.3.

Note that an $m$-dimensional subspace $P$ of form (1) in $\mathbb{F}_q^{(n+l)}$ is a vertex of $\Gamma^{(m)}$ if and only if rank($P^t$) = $m$. Therefore, two vertices $P, Q$ of $\Gamma^{(m)}$ are adjacent if and only if their sum $P + Q$ is a subspace of type $(m + 1, 0)$.

Lemma 2.1 Let $P$ and $Q$ be two vertices as in (1) of $\Gamma^{(m)}$. If dim($P' \cap Q'$) = $m - i$ and dim($P \cap Q$) = $m - j$, then the distance of $P$ and $Q$

$$\partial(P, Q) = \begin{cases} j, & \text{if } i > 0, \\ j + 1, & \text{if } i = 0. \end{cases}$$

Proof. In the Grassmann graph $J_q(n+l, m)$, two vertices $x$ and $y$ are at distance $j$ if and only if dim($x \cap y$) = $m - j$. Since $\Gamma^{(m)}$ is a subgraph of $J_q(n+l, m)$, by dim($P \cap Q$) = $m - j$
we have $\partial(P, Q) \geq j$. Write $P \cap Q = W = (W', W'')$ and

$$P = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \alpha''_1 \\ \vdots & \vdots \\ \alpha'_{j-i} & \alpha''_{j-i} \\ \delta'_1 & \delta''_1 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \quad Q = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \beta''_1 \\ \vdots & \vdots \\ \alpha'_{j-i} & \beta''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \vdots & \vdots \\ \gamma'_i & \gamma''_i \end{pmatrix}.$$ 

Case 1 $i > 0$. If $j = i$, write

$$P_1 = \begin{pmatrix} W' & W'' \\ \gamma'_1 & \gamma''_1 \\ \delta'_2 & \delta''_2 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \quad P_2 = \begin{pmatrix} W' & W'' \\ \gamma'_1 & \gamma''_1 \\ \delta'_2 & \delta''_2 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \quad \ldots, P_{j-1} = \begin{pmatrix} W' & W'' \\ \gamma'_{i-1} & \gamma''_{i-1} \\ \delta'_i & \delta''_i \end{pmatrix},$$

then $(P, P_1, \ldots, P_{j-1}, Q)$ is a path of length $j$ from $P$ to $Q$. Therefore $\partial(P, Q) = j$.

If $j > i$, write

$$P_1 = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \alpha''_1 \\ \vdots & \vdots \\ \alpha'_{j-i-2} & \alpha''_{j-i-2} \\ \alpha'_{j-i-1} & \alpha''_{j-i-1} \\ \gamma'_1 & \gamma''_1 \\ \delta'_1 & \delta''_1 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \quad P_2 = \begin{pmatrix} W' & W'' \\ \alpha'_1 & \alpha''_1 \\ \vdots & \vdots \\ \alpha'_{j-i-2} & \alpha''_{j-i-2} \\ \alpha'_{j-i} & \beta''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \delta'_1 & \delta''_1 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix}, \quad \ldots, P_{j-i} = \begin{pmatrix} W' & W'' \\ \alpha'_{j-i-1} & \beta''_{j-i-1} \\ \alpha'_{j-i} & \beta''_{j-i} \\ \gamma'_1 & \gamma''_1 \\ \delta'_1 & \delta''_1 \\ \vdots & \vdots \\ \delta'_i & \delta''_i \end{pmatrix},$$


then \( (P, P_1, \ldots, P_{j-i}) \) is a path of length \( j-i \). Since \( \dim(P_{j-i} \cap Q') = \dim(P_{j-i} \cap Q) = m-i \), we have \( \partial(P_{j-i}, Q) = i \). It follows that \( \partial(P, Q) = j \).

**Case 2** \( i = 0 \).

The neighborhood of \( P \) consists of vertices

\[
R = \left( \begin{array}{cc}
P' & \tilde{P}' \\
\xi' & \tilde{\xi}''
\end{array} \right),
\]

where \( \tilde{P} = \left( \begin{array}{cc}
P' & \tilde{P}' \\
\xi' & \tilde{\xi}''
\end{array} \right) \) is an \((m-1)\)-dimensional subspace of \( P \) and \( \xi' \in \mathbb{F}(n) \setminus P' \).

If \( W \subseteq P \), then \( \dim(R \cap Q) = m - j \) and \( \dim\left( \left( \begin{array}{cc}
P' & \tilde{P}' \\
\xi' & \tilde{\xi}''
\end{array} \right) \cap Q' \right) = m - 1 \). By Case 1, one gets \( \partial(R, Q) = j \). If \( W \not\subseteq \tilde{P} \), similarly we have \( \partial(R, Q) = j + 1 \). Hence, \( \partial(P, Q) = j + 1 \). \( \Box \)

Next we shall study the first and second subconstituents of \( \Gamma^{(m)} \). Lemma 2.1 in [10] implies that \( PTL(n + l, \mathbb{F}_q) \) acts transitively on each \( R_{i,j} \), in particular the graph \( \Gamma^{(m)} \) is vertex-transitive. So we only need to consider the subconstituents with respect to the vertex

\[
M = \left( \begin{array}{ccc}
m & n - m & l \\
I & 0 & 0 \end{array} \right) = \left( \begin{array}{ccc}
M' & M'' \end{array} \right).
\]

Let \( \Gamma_{i,j-1}(M) \) be the set of vertices \( P \) of \( \Gamma^{(m)} \) satisfying \( (M, P) \in R_{i,j-1} \). By Lemma 2.2, the second subconstituent \( \Gamma_2(M) \) has a partition

\[
\Gamma_{0,1}(M) \cup \Gamma_{1,1}(M) \cup \Gamma_{2,0}(M).
\]

For simplicity, write

\[
P(U; \alpha, \beta, \gamma) = \left( \begin{array}{ccc}
m & n - m & l \\
U & 0 & 0 \end{array} \right) \alpha \beta \gamma t,
\]

\[
P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2) = \left( \begin{array}{ccc}
m & n - m & l \\
U & 0 & 0 \end{array} \right) \alpha_1 \beta_1 \gamma_1 \alpha_2 \beta_2 \gamma_2 t,
\]

where \( \operatorname{rank}(U) = t \), \( \operatorname{rank}(\beta \gamma) = 1 \) and \( \operatorname{rank}\left( \begin{array}{cc}
\beta_1 & \gamma_1 \\
\beta_2 & \gamma_2
\end{array} \right) = 2 \). Then

\[
\Gamma_1(M) = \{ P(W; \alpha, \beta, \gamma) \mid \operatorname{rank}(W) = m - 1, \beta \neq 0 \},
\]
\[
\Gamma_{0,1}(M) = \{ P(W; \alpha, 0, \gamma) \mid \operatorname{rank}(W) = m - 1, \alpha \notin W, \gamma \neq 0 \},
\]
\[
\Gamma_{1,1}(M) = \{ P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \mid \operatorname{rank}(U) = m - 2, \alpha_2 \notin U, \beta_1 \neq 0, \gamma_2 \neq 0 \},
\]
\[
\Gamma_{2,0}(M) = \{ P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2) \mid \operatorname{rank}(U) = m - 2, \operatorname{rank} \left( \begin{array}{c}
\beta_1 \\
\beta_2
\end{array} \right) = 2 \}.
\]

**Lemma 2.2** ([4] Corollary 1.9) Let \( 0 \leq k \leq m \leq n \). Then the number of \( m \)-dimensional subspaces containing a given \( k \)-dimensional subspace in \( \mathbb{F}^n \) is equal to \( \left[ \frac{n-k}{m-k} \right] \).

**Lemma 2.3** (i) Let \( P(W; \alpha, 0, \gamma) \in \Gamma_{0,1}(M) \) and \( P(W'; \alpha', \beta', \gamma') \in \Gamma_1(M) \). Then the two vertices are adjacent in \( \Gamma^{(m)} \) if and only if \( W = W' \). In particular, each vertex in \( \Gamma_{0,1}(M) \) has \( q^l + 1 \left[ \frac{n-m}{1} \right] \) neighbors in \( \Gamma_1(M) \);

(ii) Each vertex in \( \Gamma_{1,1}(M) \) has \( q^2 \) neighbors in \( \Gamma_1(M) \);

(iii) Each vertex in \( \Gamma_{2,0}(M) \) has \( (q+1)^2 \) neighbors in \( \Gamma_1(M) \).
Proof. (i) Suppose \( P(W; \alpha, 0, \gamma) \) is adjacent to \( P(W'; \alpha', \beta', \gamma') \). If \( W \neq W' \), by \( \beta' \neq 0 \) and \( \gamma \neq 0 \) the dimension of \( P(W; \alpha, 0, \gamma) + P(W'; \alpha', \beta', \gamma') \) is \( m + 2 \), a contradiction. The converse is immediate from \( \alpha \notin W \) and \( \beta' \neq 0 \). Therefore, the first statement is valid.

Since \( P(W; \alpha', \beta', \gamma') \) is of type \((m, 0)\), the rank of \( \begin{pmatrix} W & 0 \\ \alpha' & \beta' \end{pmatrix} \) is \( m \). By Lemma 2.2 the subspace \( \begin{pmatrix} W & 0 \\ \alpha' & \beta' \end{pmatrix} \) with \( \beta' \neq 0 \) has \( q^\left[\frac{n-m}1\right] \) choices. For a given subspace \( \begin{pmatrix} W & 0 \\ \alpha' & \beta' \end{pmatrix} \), there are \( q \) choices for \( \gamma' \). Hence, (i) holds.

(ii) Given a vertex \( P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \in \Gamma_{1,1}(M) \). We claim that this vertex is adjacent to \( P(W; \alpha, \beta, \gamma) \in \Gamma_1(M) \) if and only if \( U \subseteq W \) and \( P(W; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \) is a subspace of type \((m + 1, 0)\) containing \( P(W; \alpha, \beta, \gamma) \). Suppose \( P(W; \alpha, \beta, \gamma) \) is adjacent to \( P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \). If \( U \not\subseteq W \), by \( \beta_1 \neq 0 \) and \( \gamma_2 \neq 0 \), the dimension of \( P(W; \alpha, \beta, \gamma) + P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \) is at least \( m + 2 \), a contradiction. It follows that \( U \subseteq W \). Then \( P(W; \alpha, \beta, \gamma) + P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) = P(W; \alpha, \beta, \gamma) + P(U; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \) and so \( P(W; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \) is a subspace of type \((m + 1, 0)\) containing \( P(W; \alpha, \beta, \gamma) \). The converse is immediate. Hence, our claim is valid.

By Lemma 2.2 the number of \((m - 1)\)-subspaces \( W \) in \( \mathbb{F}^{(m)} \) containing \( U \) is \( q + 1 \). Observe that \( P(W; \alpha_1, \beta_1, \gamma_1; \alpha_2, 0, \gamma_2) \) is of type \((m + 1, 0)\) if and only if \( W \neq \begin{pmatrix} U \\ \alpha_2 \end{pmatrix} \). Then \( W \) has \( q \) choices. For a fixed \( W \), by Lemma 2.2 again \( P(W; \alpha, \beta, \gamma) \) has \( q \) choices. Therefore, (ii) holds.

The proof of (iii) is similar to that of (ii), and will be omitted.

For a subspace \( W \) of type \((m - 1, 0)\) in \( \mathbb{F}_q^{(n+l)} \), let \( C(W) \) be the set of all vertices of \( \Gamma^{(m)} \) containing \( W \). For convenience we denote by \( \Delta \) the induced subgraph of \( \Gamma \) on \( \Gamma^{(m)} \).

**Lemma 2.4** Let \( W \) be a subspace of type \((m - 1, 0)\) in \( \mathbb{F}_q^{(n+l)} \). Then \( C(W) \) is isomorphic to the complete multipartite graph \( K^{(n-m+1)}(q^l) \).

**Proof.** Let \( W = \begin{pmatrix} W' & W'' \end{pmatrix} \). Then \( C(W) \) consists of vertices

\[
\begin{pmatrix}
W' & W'' \\
\alpha' & \alpha''
\end{pmatrix},
\]  

where \( \text{rank} \left( \begin{pmatrix} W' \\ \alpha' \end{pmatrix} \right) = m \). By Lemma 2.2 the number of \( m \)-dimensional subspaces \( \begin{pmatrix} W' \\ \alpha' \end{pmatrix} \) in \( \mathbb{F}_q^{(n)} \) is \( \left[\frac{n-m+1}1\right] \). Since \( \alpha'' \) has \( q^l \) choices, the subgraph \( C(W) \) has \( q^l \left[\frac{n-m+1}1\right] \) vertices.

For a given \( m \)-dimensional subspace \( \begin{pmatrix} W' \\ \alpha' \end{pmatrix} \) in \( \mathbb{F}_q^{(n)} \), the vertices of form (2) form an independent set with \( q^l \) vertices. Note that all these independent sets form a partition of \( C(W) \). Since each vertex in an independent set is adjacent to any vertex in the remaining independent sets, the desired result follows.

**Lemma 2.5** If an induced subgraph \( \Delta \) of \( \Gamma^{(m)} \) is isomorphic to \( K^{(n-m+1)}(q^l) \), then \( \Delta \) is a subgraph \( C(W) \), where \( W \) is a subspace of type \((m - 1, 0)\) in \( \mathbb{F}_q^{(n+l)} \).

**Proof.** Since \( \Gamma^{(m)} \) is vertex-transitive, we may assume that \( \Delta \) contains \( M \). Pick \( X \in \Delta \cap \Gamma_1(M) \), and write \( M \cap X = W \). Then

\[
W = \begin{pmatrix} W' & 0 & 0 \\
\alpha' & \beta' & \gamma'
\end{pmatrix}, \quad X = P(W'; \alpha', \beta', \gamma').
\]
Now we shall show that $\Delta = C(W)$. Suppose $\Delta \cap (\Gamma_{1,1}(M) \cup \Gamma_{2,0}(M)) \neq \emptyset$. Pick a vertex $P$ from this set. Then the vertices $P$ and $M$ have $q^{i+1\lfloor \frac{n-m}{i}\rfloor}$ common neighbors in $\Delta$, a contradiction to Lemma 2.3 (ii), (iii). So $\Delta \cap \Gamma_{0,1}(M) \neq \emptyset$. Pick a vertex $P(W''; a', 0, \gamma') \in \Delta \cap \Gamma_{0,1}(M)$. Since $\Delta$ is a complete multipartite graph, the vertices $X$ and $P(W''; a', 0, \gamma')$ are adjacent. By Lemma 2.3 (i), one gets $W' = W''$. It follows that $\Delta \cap \Gamma_{0,1}(M) \subseteq C(W)$. Similarly, we have $\Delta \cap \Gamma_1(M) \subseteq C(W)$. Hence, $\Delta \subseteq C(W)$. By Lemma 2.4 the subgraphs $\Delta$ and $C(W)$ have the same number of vertices, so $\Delta = C(W)$.

**Lemma 2.6** $\text{Aut}(\Gamma^m) = \text{PGL}(n+l, \mathbb{F}_q)E$.

**Proof.** We first prove that the result holds for $m = 2$. Pick any automorphism $\tau$ of $\Gamma^{(2)}$. Then $\tau$ is a permutation on the set of lines and permutes the points of the attenuated space $A_q(n,l)$. By Deng and Li’s result in [3], the automorphism $\tau$ can be extended to a collineation fixing $E$ of the projective space $PG(n+l, \mathbb{F}_q)$. By the fundamental theorem of the projective geometry [3, Theorem 2.23], we have $\tau \in \text{PGL}(n+l, \mathbb{F}_q)E$. Thus $\text{Aut}(\Gamma^{(2)}) \subseteq \text{PGL}(n+l, \mathbb{F}_q)E$.

On the other hand, $\text{PGL}(n+l, \mathbb{F}_q)E \leq \text{Aut}(\Gamma^{(2)})$. Hence, $\text{Aut}(\Gamma^{(2)}) = \text{PGL}(n+l, \mathbb{F}_q)E$.

Now let $m \geq 3$ and $\sigma_m$ be an automorphism of the graph $\Gamma^m$. By Lemmas 2.3 and 2.5 the automorphism $\sigma_m$ induces a permutation on the set $\{C(W) \mid W \in X_{m-1}\}$, and further induces a permutation $\sigma_{m-1}$ on $X_{m-1}$. For any two adjacent vertices $W_{m-1}, W'_{m-1}$ of $\Gamma^{(m-1)}$, we have $C(W_{m-1}) \cap C(W'_{m-1}) \neq \emptyset$. Therefore $\sigma_{m}(C(W_{m-1}) \cap C(W'_{m-1})) = C(\sigma_{m-1}(W_{m-1})) \cap C(\sigma_{m-1}(W'_{m-1})) \neq \emptyset$, which implies that the two vertices $\sigma_{m-1}(W_{m-1})$ and $\sigma_{m-1}(W'_{m-1})$ are adjacent in $\Gamma^{(m-1)}$. Hence, $\sigma_{m-1} \in \text{Aut}(\Gamma^{(m-1)})$.

By induction, for each $3 \leq k \leq m$, the map $f_k : \sigma_k \mapsto \sigma_{k-1}$ is a homomorphism from $\text{Aut}(\Gamma^{(k)})$ to $\text{Aut}(\Gamma^{(k-1)})$. We claim that $f_k$ is injective. Suppose $\sigma_{k-1} = i$, the identity permutation on $X_{k-1}$. For each vertex $W_k$ of $\Gamma^{(k)}$, there exist two vertices $W_{k-1}$ and $W'_{k-1}$ of $\Gamma^{(k-1)}$ such that $W_k = W_{k-1} + W'_{k-1}$. Since $\{W_k\} = C(W_{k-1}) \cap C(W'_{k-1})$, we have $\{\sigma_k(W_k)\} = C(\sigma_{k-1}(W_{k-1})) \cap C(\sigma_{k-1}(W'_{k-1})) = \{W_k\}$. Hence our claim is valid. It follows that $|\text{Aut}(\Gamma^m)| \leq |\text{Aut}(\Gamma^{(2)})|$. Since $\text{PGL}(n+l, \mathbb{F}_q)E$ is a subgroup of $\text{Aut}(\Gamma^m)$, the desired result follows.

**Proof of Theorem 1.3.** The fact that $\text{PGL}(n+l, \mathbb{F}_q)E$ acts transitively on each $R_{i,j-i}$ implies $\text{Im}(\mathbb{X}_m) = \text{Aut}(\Gamma^m)$. In order to show our result, it suffices to prove that all valencies $n_{i,j-i}$’s of $\mathbb{X}_m$ are pairwise distinct.

By [10] we have

$$n_{i,j-i} = q^{a^2 + il + \frac{i(i+1)(j-i)}{2}} \frac{n-m}{i} \frac{m-m}{j} \frac{m-i}{j-i} \prod_{s=l-(j-i)+1}^{l} (q^s - 1).$$

Suppose $n_{a,b} = n_{a',b'}$. Since $q$ is a prime power, we obtain

$$a^2 + al + \frac{(b-a)(b-a-1)}{2} = a'^2 + a'l + \frac{(b'-a')(b'-a'-1)}{2}.$$  \hspace{1cm} (3)

$$\prod_{s=(b-a)+1}^{l} (q^s - 1) = \prod_{s=(b'-a')+1}^{l} (q^s - 1).$$  \hspace{1cm} (4)

Simplifying (3), we have

$$2(a-a')(a+a'+l) = ((b'-a')-(b-a))(b'-a'+b-a-1).$$  \hspace{1cm} (5)
Let

\[ f_{i,j}(x) = \prod_{s=1}^{i} (x^s - 1)^2 \prod_{s=1}^{j-i} (x^s - 1). \]

\[ g_{i,j}(x) = \prod_{s=n-m-i+1}^{n-m} (x^s - 1) \prod_{s=m-i+1}^{m} (x^s - 1) \prod_{s=m-j+1}^{m-i} (x^s - 1) \prod_{s=l-j+i+1}^{l} (x^s - 1). \]

The equality (4) implies that

\[ f_{a,b}(x) g_{a',b'}(x) = f_{a',b'}(x) g_{a,b}(x) \]

for all prime powers \( q \), so \( f_{a,b}(x) g_{a',b'}(x) = f_{a',b'}(x) g_{a,b}(x) \). Since 1 is a root with multiplicity \( 2a + (b - a) + 2a' + 2(b' - a') \) of \( f_{a,b}(x) g_{a',b'}(x) \) and 1 is a root with multiplicity \( 2a + 2(b - a) + 2a' + (b' - a') \) of \( f_{a',b'}(x) g_{a,b}(x) \), we obtain \( b' - a' = b - a \). By (5) one gets \( a' = a \), and so \( b' = b \). Hence the desired result follows.

\[ \blacksquare \]

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