Abstract—We consider the ratio of two Gauss hypergeometric functions, in which the parameters of the numerator function differ from the respective parameters of the denominator function by integers. We derive explicit integral representations for this ratio based on a formula for its imaginary part. This work extends our recent results by lifting certain restrictions on parameters. The new representations are illustrated with a few examples and an application to products of ratios.

DOI: 10.1134/S1995080221120118

Keywords and phrases: Gauss hypergeometric function, Integral representation, Runckel’s theorem.

1. INTRODUCTION

Let $(a)_n := a(a+1) \cdots (a+n-1), (a)_0 = 1$, denote the rising factorial. The Gauss hypergeometric function ([8, Chapter II; 13, Chapter 15; 10]) is defined as the analytic continuation of the sum of the power series

$$2F_1(a, b; c; z) = 2F_1\left(\frac{a, b}{c} \mid z\right) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!} z^n. \quad (1)$$

For $a, b \notin -\mathbb{N}_0$ (if this condition is violated, then $2F_1$ reduces to a polynomial), one usually introduces the branch cut $[1, +\infty)$ to make it analytic and single-valued in the rest of the complex plane. The functions $2F_1(a, b; c; z)$ and $2F_1(a + n_1, b + n_2; c + m; z)$, $n_1, n_2, m \in \mathbb{Z}$, are called associated [8, p. 58] or contiguous in a wide sense. Gauss showed that any three functions of this type satisfy a linear relation with coefficients rational in $a, b, c, z$. For $n_1, n_2, m \in \{-1, 0, 1\}$, this relation has coefficients linear in $z$, and the functions are called contiguous (in a narrow sense). In our recent paper [7] we initiated a study of the ratios

$$R_{n_1, n_2, m}(z) = \frac{2F_1(a + n_1, b + n_2; c + m; z)}{2F_1(a, b; c; z)}, \quad (2)$$

with arbitrary integer $n_1, n_2, m$. The particular case $R_{0,1,1}$ was investigated already by Gauss who found a continued fraction for this ratio which, under additional restrictions on parameters, becomes a Stieltjes or $S$-fraction convergent to a Stieltjes transform of a positive measure. Explicit density of this measure was found many decades later by Belevitch [1]. In [7], we extended this integral representation to the general $R_{n_1, n_2, m}(z)$ under the assumptions that $a, b, c \in \mathbb{R}$, $c, c + m \notin -\mathbb{N}_0$ and $R_{n_1, n_2, m}(z)$ has no poles in $\mathbb{C} \setminus [1, +\infty)$. We additionally assumed that the behaviour of $R_{n_1, n_2, m}(z)$ near $z = 1$ is mild.

E-mail: diachenko@sfu.edu
E-mail: dimkrp@gmail.com
so that the singularity at this point is integrable. The main purpose of the present paper is to drop these restrictions and extend the representations given in [7] to a more general setting. Firstly, we will get rid of the assumption of integrability near the point $z = 1$ and allow arbitrary behaviour in the neighbourhood of this point. Secondly, we will remove the assumption that $R_{n_1,n_2,m}(z)$ is free of poles in $\mathbb{C} \setminus [1, +\infty)$. However, the representation we obtain for ratios with poles will depend on rational functions whose numerators and denominators have explicit degrees but unknown coefficients. Calculation of these coefficients requires the knowledge of the zeros of the Gauss hypergeometric function and residues of $R_{n_1,n_2,m}(z)$ at these zeros (for generic values of its parameters). These, of course, cannot be given explicitly, but may, in principle, be computed numerically.

2. ASYMPTOTIC BEHAVIOUR AND BOUNDARY VALUES

It will be convenient to use the following notation: if $a$ is a real number, then

$$(a)_+ := \min(a, 0) \quad \text{and} \quad (a)_- := \max(a, 0).$$

Derivation of the integral representations for the ratio $R_{n_1,n_2,m}$ will require certain estimates of its asymptotic behaviour. The following is a condensed and corrected form of [7, Subsection 2.1]:

**Lemma 1.** Let $c, c + m \notin -N_0$, and let $a, b \in \mathbb{R}$. Then there exist four constants $\varepsilon_1, \varepsilon_\infty \in \{-1, 0, 1\}$ and $L_1, L_\infty \neq 0$ independent of $z$ such that

$$R_{n_1,n_2,m}(z) = L_1(1 - z)^{\eta(a+n_1,b+n_2,c+m)-\eta(a,b,c)} \log(1 - z)^{\frac{1}{\varepsilon_1} (1 + o(1))} \quad \text{as} \quad z \to 1; \quad (3)$$

$$R_{n_1,n_2,m}(-z) = L_\infty z^{\zeta(a+n_1,b+n_2,c+m)-\zeta(a,b,c)} \log(z)^{\varepsilon_\infty (1 + o(1))} \quad \text{as} \quad z \to \infty, \quad (4)$$

where we put

$$\eta(a,b,c) = \begin{cases} (c - a - b)_+ & \text{if} \quad -a, b - c \in \mathbb{N}_0 \quad \text{or} \quad -b, a - c \in \mathbb{N}_0; \\ 0 & \text{if} \quad -a \in \mathbb{N}_0 \quad \text{and/or} \quad -b \in \mathbb{N}_0 \quad \text{while} \quad a - c, b - c \notin \mathbb{N}_0; \\ c - a - b & \text{if} \quad -a, -b \notin \mathbb{N}_0, \quad \text{while} \quad a - c \in \mathbb{N}_0 \quad \text{and/or} \quad b - c \in \mathbb{N}_0; \\ (c - a - b)_- & \text{otherwise} \end{cases}$$

and

$$\zeta(a,b,c) = \begin{cases} -a & \text{if} \quad b - c \in \mathbb{N}_0 \quad \text{and/or} \quad -a \in \mathbb{N}_0; \\ -b & \text{if} \quad a - c \in \mathbb{N}_0 \quad \text{and/or} \quad -b \in \mathbb{N}_0; \\ -\min(a,b), & \text{otherwise}. \end{cases}$$

Note that the above formulae (3), (4) also work for the degenerate cases — i.e. when $2F_1(a, b; c; z)$ is a polynomial or polynomial multiple of power of $1 - z$.

Another important ingredient is the next theorem giving an explicit representation for the imaginary part of $R_{n_1,n_2,m}(z)$ on the banks of the branch cut $[1, +\infty)$. Given $n_1, n_2, m \in \mathbb{Z}$ let us introduce the following related quantities:

$$\underline{n} = \min(n_1, n_2), \quad \overline{n} = \max(n_1, n_2), \quad p = (m - n_1 - n_2)_+, \quad l = (n_1 + n_2 - m)_+,$$

$$r = l + (m)_+ - \underline{n} + 1 = \begin{cases} \max(m - n, \overline{n}) - 1, & m \geq 0, \\ \max(-\underline{n}, \overline{n} - m) - 1, & m \leq 0. \end{cases} \quad (7)$$

Observe that $p - l = m - n_1 - n_2$ and that $r$ is only negative when $n_1 = n_2 = m = 0$, in which case $r = -1$.

**Theorem 1 [7, Theorem 2.11].** Suppose that $n_1, n_2, m \in \mathbb{Z}$. On the banks of the branch cut $x > 1$, the following expression holds

$$\Im[R_{n_1,n_2,m}(x \pm i0)] = \pm \pi B_{n_1,n_2,m}(a,b,c) \frac{x^{l-n-c}(x-1)^{c-a-b-l}P_r(1/x)}{|2F_1(a,b;c;x)|^2}, \quad (8)$$

where

$$B_{n_1,n_2,m}(a,b,c) = \frac{\Gamma(c)\Gamma(c + m)}{\Gamma(a)\Gamma(b)\Gamma(c - a + m - n_1)\Gamma(c - b + m - n_2)} \quad (9)$$
and $P_r(t)$ is a polynomial of degree $r$ ($P_{-1} = 0$).

Note that the coefficients of $P_r$ depend on the parameters, so the whole expression (8) may remain nonzero even when $B_{n_1,n_2,m}(a,b,c)$ vanishes. The polynomial $P_r(t)$ can be computed via the Taylor expansion of the underlying hypergeometric identity [7, eq. (2.19)] multiplied by $t^{r/2}(1-t)^p$, in which $\alpha = a$, $\beta = 1-c+a$, $\gamma = 1-b+a$. Alternatively, one can write $P_r(t)$ explicitly in the following form:

**Lemma 2** [4, Lemma 2]. For $r \geq 0$, the polynomial $P_r(t)$ in (8) is given by

$$P_r(t) = (-1)^r \sum_{k=0}^{r} \frac{(-t)^k}{k!} \sum_{j=k-p}^{k-p} (-1)^j \binom{p}{k-j} K_j,$$

where, with the convention $1/(-i) = 0$ for $i \in \mathbb{N}$,

$$K_j = \frac{(1-a)_j(c-a)_{m+j}}{(b-a)_{n_2+j+1}(j+n_1)!} F_3 \left( \begin{array}{c} -j-n_1, a+1-a-c, a-b-n_2-j \\ a-j, 1+a-c-m-j, 1+a-b \end{array} \right),$$

$$+ \frac{(1-b)_j(c-b)_{m+j}}{(a-b)_{n_1+j+1}(j+n_2)!} F_3 \left( \begin{array}{c} -j-n_2, b+1+b-c, b-a-n_1-j \\ b-j, 1+b-c-m-j, 1+b-a \end{array} \right).$$

At least one of the numbers $a, b, c-a, c-b$ lies in $-\mathbb{N}_0$ exactly when $2F_1(a,b;c;z)$ degenerates to a polynomial, possibly times a fractional power of $(1-z)$. In this case the denominator in (8) in Theorem 1 may vanish for a certain $x > 1$: for instance,

$$2F_1 \left( 1, -2; \frac{4}{5}, \frac{6}{5} \right) = 2F_1 \left( -\frac{1}{5}, \frac{14}{5}; \frac{4}{5}, \frac{6}{5} \pm i0 \right) = 0.$$

Runckel showed that this situation is impossible in the non-degenerate case. More specifically, the following fact is a direct corollary of [14, Lemma 2]:

**Lemma 3.** If $a, b, c-a, c-b \notin -\mathbb{N}_0$ and $x > 1$, then $2F_1(a,b;c;x \pm i0) \neq 0$.

**Proof.** The case $c-a - b \geq 0$ and $b \geq a$ is the first assertion of [14, Lemma 2]. Due to the symmetry with respect to exchanging $a \leftrightarrow b$, the lemma also holds for the case $c-a - b \geq 0$ and $b \leq a$. Now, if $c-a - b \leq 0$ and $x > 1$, the right hand side of Euler’s identity

$$2F_1(a,b;c;z) = (1-z)^{c-a-b} 2F_1(c-a,c-b;c;z)$$

(10) does not vanish for $z = x \pm i0$, and hence neither does the left hand side when $x > 1$. 

\[ \square \]

### 3. INTEGRAL REPRESENTATIONS

Our goal is to derive explicit integral representations for the ratio $R_{n_1,n_2,m}(z)$. This ratio is known to have at most finitely many poles in the cut plane $\mathbb{C} \setminus [1, +\infty)$ and on both banks of the branch cut $(1, +\infty)$, see Theorem 3 below. It may also have at most a polynomial growth near the branch points $z = 1$ and $z = \infty$. There are two options for dealing with the poles and the growth: to subtract the corresponding rational correction term, or to multiply $R_{n_1,n_2,m}(z)$ by a specially tailored polynomial. We are going to deal with both options via an adapted version of the Schwarz formula stated here as Lemma 4.

A function meromorphic in the (open) upper half of the complex plane is called real if it extends as a meromorphic function to the (open) lower half of the complex plane according to the rule $f(\overline{z}) = \overline{f(z)}$ wherever $f(z)$ is defined. If $f(z)$ is analytic near the origin, then $f(z)$ is real precisely when all coefficients of its Taylor expansion at the origin are real.

**Lemma 4.** Let $f(z)$ be a real function meromorphic in the cut plane $\mathbb{C} \setminus [1, +\infty)$ and analytic near the origin. Suppose that there exists a real polynomial $q(z)$ of degree $M$, for which $q(z)f(z)$ is analytic in $\mathbb{C} \setminus [1, +\infty)$ and $q(x)u(x)$ is continuous for $x \in (1, +\infty)$, where $u(x) := \frac{1}{\pi} \Im f(x \pm i0)$. If $N \in \mathbb{N}_0$ is such that

$$\lim_{|z-1| \to 1} \left| q(z)(1-z)f(z) \right| = \lim_{|z| \to \infty} \left| f(z)z^{M-N} \right| = 0$$

(11)

LOBACHEVSKII JOURNAL OF MATHEMATICS Vol. 42 No. 12 2021
and \( q(x)u(x)/x^{N+1} \) is absolutely integrable on \((1, +\infty)\), then

\[
f(z) = \sum_{k=0}^{N-1} \frac{z^k}{q(z)} \sum_{j=0}^{k} \frac{q^{(k-j)}(0)f^{(j)}(0)}{(k-j)!j!} + \frac{z^N}{q(z)} \int_{1}^{+\infty} \frac{q(x)u(x)dx}{(x-z)x^N}.
\]  

(12)

**Proof.** Let \( C \) be the closed contour consisting of a small circle around the point \( z = 1 \) of radius \( \epsilon < 1/2 \), then (the upper bank of) the interval \((1 + \epsilon + i0, 1/\epsilon + i0)\) followed by a large circle \(|z| = 1/\epsilon\) and (the lower bank of) the interval \((1 + \epsilon - i0, 1/\epsilon - i0)\). The contour is traversed so that the bounded domain inside it is on the left (in particular, the large circle is traversed in the anticlockwise direction). Given \( k \in \mathbb{N}_0 \), the Cauchy formula for the \((N+k)\)th Taylor coefficient of \( g(z) := q(z)f(z) \) reads

\[
2\pi i \frac{g^{(N+k)}(0)}{(N+k)!} = \oint_{C} \frac{g(z)}{z^{N+k+1}}dz
= \oint_{|z-1|=\epsilon} \frac{g(z)dz}{z^{N+k+1}} + \frac{1}{1+\epsilon} \int_{1+\epsilon}^{+\infty} \frac{g(x+i0) - g(x-i0)}{x^{N+k+1}}dx + \oint_{|z|=1/\epsilon} \frac{g(z)dz}{z^{N+k+1}},
\]  

(13)

note also that

\[
g(x+i0) - g(x-i0) = 2i\Im g(x+i0) = 2\pi iq(x)u(x).
\]

On letting \( \epsilon \to +0 \), the first and the last integrals on the right-hand side of (13) vanish due to (11). Hence,

\[
\frac{g^{(N+k)}(0)}{(N+k)!} = \frac{1}{\pi} \int_{1}^{+\infty} \Im g(x+i0)dx = \int_{1}^{+\infty} \frac{q(x)u(x)dx}{x^{N+k+1}} =: c_k, \quad k \in \mathbb{N}_0.
\]

The power series \( \sum_{k=0}^{+\infty} c_k z^k \) uniformly converges on compact subsets of the unit disc (in fact, \( c_k \to 0 \) as \( k \to \infty \) since for each \( x > 1 \) the integrand monotonically tends to zero). Therefore, if \(|z| < 1\) we have

\[
\sum_{k=0}^{\infty} c_k z^k = \int_{1}^{+\infty} \left( \sum_{k=0}^{\infty} \frac{z^k}{x^{k+1}} \right) \frac{q(x)u(x)dx}{x^N} = \int_{1}^{+\infty} \frac{q(x)u(x)dx}{(x-z)x^N}.
\]  

(14)

A comparison between the left-hand side and the Taylor expansion of \( g(z) \) at the origin shows that

\[
z^N \sum_{k=0}^{\infty} c_k z^k = g(z) - \sum_{k=0}^{N-1} \frac{z^k}{k!} g^{(k)}(0)
\]

for \( z \) inside the unit disc. At the same time, the ratio \( x/(x-z) \) is bounded in \( z \) on compact subsets of \( \mathbb{C} \setminus [1, +\infty) \) uniformly in \( x > 1 \), so the integral on the right-hand side of (14) uniformly converges there to an analytic function. Thus, for all \( z \in \mathbb{C} \setminus [1, +\infty) \)

\[
g(z) = \sum_{k=0}^{N-1} \frac{z^k}{k!} g^{(k)}(0) + z^N \int_{1}^{+\infty} \frac{q(x)u(x)dx}{(x-z)x^N}.
\]

This expression yields (12) after division by \( q(z) \) and the substitutions \( g(z) = q(z)f(z) \) and

\[
g^{(k)}(0) = \sum_{j=0}^{k} \binom{k}{j} q^{(k-j)}(0)f^{(j)}(0).
\]

**Corollary 1.** Let \( f(z) \) be a real analytic function defined in the cut plane \( \mathbb{C} \setminus [1, +\infty) \) such that \( u(x) := \frac{1}{\pi} \Im f(x+i0) \) is continuous on \((1, +\infty)\). Suppose also that there exist \( M, N \in \mathbb{N}_0 \) for which

\[
\lim_{|z|-1 \to 1} |f(z)(1-z)^{M+1}| = \lim_{|z| \to \infty} |f(z)z^{M-N}| = 0
\]

(15)
and \( u(x)(x - 1)^M / x^{N + 1} \) is absolutely integrable over \((1, +\infty)\). Then
\[
f(z) = \sum_{k=0}^{N-1} \frac{z^k}{(1 - z)^M} \sum_{j=\max\{k-M,0\}}^{k} (-1)^{k-j} \binom{M}{k-j} \frac{f^{(j)}(0)}{j!}
\]
\[
+ \frac{z^N}{(1 - z)^M} \int_{1}^{\infty} \frac{(1 - x)^M u(x)dx}{(x-z)x^N}.
\]

(16)

Proof. Put \( q(z) := (1 - z)^M \) in Lemma 4. Then, due to
\[
\frac{z^k}{q(z)} \sum_{j=0}^{k} q^{(k-j)}(0) \frac{f^{(j)}(0)}{(k-j)!j!} = \frac{z^k}{(1 - z)^M} \sum_{j=\max\{k-M,0\}}^{k} \binom{M}{k-j} \frac{f^{(j)}(0)}{(M-(k-j))!(k-j)!j!},
\]
the expression in (12) becomes (16). 

One can also reshape the sum in formula (16) as follows:
\[
\sum_{k=0}^{N-1} \frac{z^k}{(1 - z)^M} \sum_{j=\max\{k-M,0\}}^{k} (-1)^{k-j} \binom{M}{k-j} \frac{f^{(j)}(0)}{j!}
\]
\[
= \sum_{j=0}^{N-1} \frac{f^{(j)}(0)}{j!} \sum_{k=0}^{\min\{N-j-1,M\}} \binom{M}{k} \frac{(-1)^k z^{k+j}}{(1 - z)^M}.
\]

Theorem 2. Suppose that \( a, b, c \in \mathbb{R} \) and \( n_1, n_2, m \in \mathbb{Z} \), where \(-c, -c - m \notin \mathbb{N}_0\). Choose\(^1\) a real rational function \( Q(z) \) and a real polynomial \( T(z) \) such that \( T(z)\left(R_{n_1,n_2,m}(z) - Q(z)\right) \) is analytic in \( \mathbb{C} \setminus [1, +\infty) \) and
\[
\lim_{y \to +0} \left| T(x + iy)\left|R_{n_1,n_2,m}(x + iy) - Q(x + iy)\right| < \infty \text{ for each } x \in (1, +\infty).
\]

(17)

Let also \( Q(z) \) be analytic near the origin. Take numbers \( M, N \in \mathbb{N}_0 \) such that
\[
R_{n_1,n_2,m}(z) - Q(z) = o\left((1 - z)^{-M-1}\right) \text{ as } z \to 1 \text{ and}
\]
\[
R_{n_1,n_2,m}(-z) - Q(-z) = o\left(z^{N-M}\right) \text{ as } z \to \infty,
\]

(18)

and denote \( d = \deg T \). Then the following representation holds
\[
R_{n_1,n_2,m}(z) = Q(z)
\]
\[
+ \sum_{k=0}^{N+d-1} \frac{z^k}{(1 - z)^M T(z)} \sum_{j=0}^{k} \frac{R_{n_1,n_2,m}^{(j)}(0) - Q^{(j)}(0)}{j!} \sum_{h=0}^{\min\{k-j,M\}} \binom{M}{h} \frac{(-1)^h T(k-j-h)(0)}{(k-j-h)!}
\]
\[
+ \frac{B_{n_1,n_2,m}(a,b,c) z^{N+d}}{(z - 1)^M T(z)} \int_{1}^{\infty} \frac{x^r - \frac{\alpha - c - N - d}{z} (x - 1)^{M+c - \alpha - b - l} T(x) P_r(1/x)}{|2 F_1(a,b;c;x)|^2 (x - z)} dx,
\]

(19)

where \( r, l, n \) and \( B(a,b,c) \) are the same as in Theorem 1 and \( P_r \) is defined in (9). In the case \( N = d = 0 \), the sum in \( k \) on the right-hand side of (19) is void.

As no poles can be produced by the Cauchy-type integral in (19), each pole of \( R_{n_1,n_2,m}(z) \) excluding a possible one at \( z = 1 \) must be a pole of \( Q(z) \) or a zero of \( T(z) \) of the same multiplicity. In fact, one can always eliminate one of the functions \( Q(z), T(z) \) by setting either \( Q(z) \equiv 0 \) or \( T(z) \equiv 1 \), so that the other function absorbs the poles of \( R_{n_1,n_2,m}(z) \).

\(^1\)Such rational functions \( Q(z) \) and \( T(z) \) always exist, as \( R_{n_1,n_2,m}(z) \) may have at most finitely many poles, see Theorem 3.
Proof. Apply Lemma 4 with \(f(z) = R_{n_1,n_2,m}(z) - Q(z)\) and \(q(z) = (1 - z)^{M} T(z)\), then observe that
\[
\frac{d^j}{dz^j} (1 - z)^{M} \bigg|_{z=0} = \begin{cases} (-1)^j M!/(M-j)! & \text{if } j \leq M; \\ 0 & \text{if } j > M, \end{cases}
\]
and hence
\[
q^{(k-j)}(0) = \frac{d^{k-j}}{dz^{k-j}} (1 - z)^{M} T(z) \bigg|_{z=0} = \frac{1}{(k-j)!} \sum_{h=0}^{\min(k-j,M)} \binom{k-j}{h} (-1)^h M! (M-h)! T^{(k-j-h)}(0)
\]
\[
= \min(k-j,M) \binom{M}{h} (-1)^h T^{(k-j-h)}(0)/(k-j-h)!
\]
The only detail we have to take care of is that \(\Re (f(x+i0))(1 - x)^{M} T(x)/x^{N+d+1}\) must be absolutely integrable on \((1, +\infty)\).

Due to (17), it is enough to check the integrability of \(\Re (f(x+i0))(1 - x)^{M} /x^{N+1}\) as \(x \to +\infty\) and \(x \to +\infty\). For \(x \to +\infty\) this property holds: the leading term of \(f(x+i0)x^{M-N} = o(1)\) is no worse than \(O(1/ \log(x))\) yielding \(\Re (f(x+i0))(1 - x)^{M} /x^{N+1} = O(x^{-1} \log(x))^{-2}\) which is integrable. Details can be found in [7, Proof of Theorem 2.12].

A similar reasoning applies for \(x \to +\infty\): from [7, Lemma 2.5] one obtains that the leading term of \(f(x+i0)(1 - x)^{M} /x^{N+1}\) is no worse than \(O((1 - x)^{-1} / \log(x-1))\), which implies \(\Re (f(x+i0))(1 - x)^{M} /x^{N+1} = O((x-1)^{-1} \log(x-1))^{-2}\) confirming integrability.

\[
\hat{R}_{n_1,n_2,m}(z) = \frac{T(z)}{z^d} R_{n_1,n_2,m}(z),
\]
where
\[
T(z) = (z - \alpha_1)(z - \overline{\alpha_1}) \cdots (z - \alpha_k)(z - \overline{\alpha_k})(z - \beta_1) \cdots (z - \beta_l) = z^d + \gamma_d z^{d-1} + \cdots + \gamma_1 z + \gamma_0,
\]
has the same asymptotics at \(z = \infty\) as \(R_{n_1,n_2,m}(z)\) and has no poles other then the pole of order \(d\) at \(z = 0\). Hence, this pole can be removed by subtracting the principal part of the Laurent series of \(\hat{R}_{n_1,n_2,m}(z)\) at \(z = 0\). This leads to the definition:
\[
\tilde{R}_{n_1,n_2,m}(z) = \hat{R}_{n_1,n_2,m}(z) - L(z), \quad \text{where} \quad L(z) = \frac{A_1}{z} + \frac{A_2}{z^2} + \cdots + \frac{A_d}{z^d}.
\]
Then, in view of \(R_{n_1,n_2,m}(0) = 1\), we will have
\[
A_d = \lim_{z \to 0} \frac{d^j}{dz^j} [\tilde{R}_{n_1,n_2,m}(z) z^d] = \gamma_0 = (-1)^{2k+l} |\alpha_1|^2 \cdots |\alpha_k|^2 |\beta_1| \cdots |\beta_l|,
\]
\[
A_{d-1} = \lim_{z \to 0} \frac{d}{dz} [\tilde{R}_{n_1,n_2,m}(z) z^d] = [T(z) \frac{d}{dz} R_{n_1,n_2,m}(z)]|_{z=0} + [R_{n_1,n_2,m}(z) \frac{d}{dz} T(z)]|_{z=0}
= \gamma_1 + \gamma_0 \left( \frac{(a+n_1)(b+n_2)}{(c+m)} - \frac{ab}{c} \right),
\]
\[
A_{d-j} = \lim_{z \to 0} \frac{d^j}{dz^j} [\tilde{R}_{n_1,n_2,m}(z) z^d] = \sum_{n=0}^{j} \binom{j}{n} \frac{R^{(n)}_{n_1,n_2,m}(0)}{n!} \gamma_{j-n}, \quad j = 2, \ldots, d - 1,
\]

LOBACHEVSKII JOURNAL OF MATHEMATICS Vol. 42 No. 12 2021
where $\gamma_j$ are the coefficients of the polynomial $T(z)$.

There are several ways to compute the $n$th derivative of the quotient $R_{n_1,n_2,m}(z)$ based on Faà di Bruno’s formula. One particularly simple version suggested by Al-Jamal [9] reads

$$R_{n_1,n_2,m}^{(n)}(z) = \sum_{k=0}^{n} (-1)^k \binom{n+1}{k+1} \left( \frac{(2F_1(a+n_1,b+n_2;c+m;z)[2F_1(a,b;c;z)]^k}{[2F_1(a,b;c;z)]^{k+1}} \right).$$

This can be combined with Leibnitz’s rule [3, (1.1.5)] to get

$$= n! \sum_{i_0+i_1+\cdots+i_k=n} \frac{(a+n_1)_{i_0}(b+n_2)_{i_0}}{i_0!(c+m)_{i_0}} 2F_1(a+n_1+i_0,b+n_2+i_0;c+m+i_0;z) \times \prod_{h=1}^{k} \frac{(a)_{i_h}(b)_{i_h}}{i_h!(c)_{i_h}} 2F_1(a+i_h,b+i_h;c+i_h;z).$$

Next we note that the asymptotic behaviour of $\tilde{R}_{n_1,n_2,m}(z)$ in the neighbourhood of the point $z = 1$ is the same as of $R_{n_1,n_2,m}(z)$, namely

$$|\tilde{R}_{n_1,n_2,m}(z)| \sim |R_{n_1,n_2,m}(z)| \sim |L_1| |1-z|^{\eta(a+n_1,b+n_2,c+m)-\eta(a,b,c)}|\log(1-z)|^{\varepsilon_1} \text{ as } z \to 1$$

with $\eta$ defined by (5), $\varepsilon_1 \in \{-1,0,1\}$ and $L_1 \neq 0$, see Lemma 1. Assume that $\eta(a+n_1,b+n_2,c+m) > \eta(a,b,c) - 1$, so that a possible singularity at $z = 1$ is integrable. In the neighbourhood of $z = \infty$, let us assume that for some $\tau > 0$ and $C \in \mathbb{R}$

$$R_{n_1,n_2,m}(z) = Q(z) + \frac{C}{\log z} \left( 1 + O\left( |\log z|^{-1} \right) \right) + O(z^{-\tau}), \quad z \to \infty.$$ 

If $Q(z) = r_s z^s + \cdots + r_1 z + r_0$, then

$$\frac{Q(z)T(z)}{z^d} = r_s z^s + (r_{s-1} + r_s \gamma_{d-1}) z^{s-1} + \cdots + r_0 \gamma_0 z^{-d} = \hat{Q}(z) + O(z^{-1}),$$

so that

$$\tilde{R}_{n_1,n_2,m}(z) = \hat{Q}(z) + \frac{C}{\log z} \left( 1 + O\left( |\log z|^{-1} \right) \right) + O(z^{-\tau}), \quad z \to \infty.$$ 

As $\tilde{R}_{n_1,n_2,m}(z)$ has no singularities in $C \setminus [1, \infty)$ and is continuous on the branch cut, we can apply the Schwarz formula (e.g. in the form of Lemma 4) to the difference $\tilde{R}_{n_1,n_2,m}(z) - \hat{Q}(z)$. The coefficients of $\hat{Q}$ and the numbers $A_1, \ldots, A_d$ are real, so the boundary values of the imaginary parts of $\tilde{R}_{n_1,n_2,m}(z)$ and $R_{n_1,n_2,m}(z)$ are related by

$$\Im \tilde{R}_{n_1,n_2,m}(x \pm i0) = \Re(T(x)/x^d) \Im R_{n_1,n_2,m}(x \pm i0),$$

where for $x > 1$

$$\Re(T(x)) = T(x) = |x - \alpha_1|^2 \cdots |x - \alpha_k|^2 |x - \beta_1| \cdots |x - \beta_l|.$$ 

Then the Schwarz formula and Theorem 1 applied to $\tilde{R}_{n_1,n_2,m}(z)$ lead to the following representation

$$R_{n_1,n_2,m}(z) = \frac{z^d}{T(z)} (\tilde{R}_{n_1,n_2,m}(z) + L(z))$$

$$= \frac{z^d \hat{Q}(z) + z^d L(z)}{T(z)} + \frac{B_{n_1,n_2,m}(a,b,c)z^d}{T(z)} \int_1^\infty \frac{T(x)x^{l-n-c-d}(x-1)^{c-a-b-l}P_r(1/x)}{|2F_1(a,b;c;x)|^2(x-z)} \, dx,$$

where $l, n, r, B_{n_1,n_2,m}(a,b,c)$ and $P_r$ retain their meanings from Theorem 1.
4. POLE-FREE CASE

Given \( \xi \in \mathbb{R} \), let \( \lfloor \xi \rfloor \) be the maximal integer number \( \leq \xi \). Note that if \( \xi \) is non-integer, then \( -\lfloor \xi \rfloor = \lfloor -\xi \rfloor - 1 \). The number of zeros of the Gauss hypergeometric function may be calculated according to Runckel’s theorem [14, Theorem], which we present here in an extended form:

**Theorem 3** (Runckel). Given \( a, b, c \in \mathbb{R} \), where \( -\xi \not\in \mathbb{N}_0 \), let \( \xi_1, \ldots, \xi_4 \) be the numbers \( a, b, c - a, c - b \) taken in non-decreasing order:

\[
\min(a, b, c - a, c - b) = \xi_1 \leq \xi_2 \leq \xi_3 \leq \xi_4 = \max(a, b, c - a, c - b).
\]

Denote by \( \nu(a, b, c) \) the number of zeros of \( {}_2F_1(a, b; c; z) \) in \( \mathbb{C} \setminus [1, +\infty) \), as well as on the upper bank of the branch cut (1, +\infty).

If \( \{-a, -b, a - c, b - c\} \cap \mathbb{N}_0 \neq \emptyset \), then

\[
\nu(a, b, c) = \xi \quad \text{with} \quad \xi = \min\{(\{-a, -b, a - c, b - c\} \cap \mathbb{N}_0) = \min\left(4 \bigcup_{j=1}^{4} (\xi_j) \cap \mathbb{N}_0 \right); \]

otherwise

\[
\nu(a, b, c) = \begin{cases} 0, & \text{if } \xi_1 > 0; \\ \lfloor -\xi_1 \rfloor + \frac{1 + \text{sign}(\Gamma(a)\Gamma(b)\Gamma(c - a)\Gamma(c - b))}{2}, & \text{if } \xi_1 < 0 \text{ and } \xi_4 > 0; \\ \lfloor -\xi_1 \rfloor + \frac{1 + \text{sign}(\Gamma(a)\Gamma(b)\Gamma(c - a)\Gamma(c - b))}{2} + S[1 - \xi_4], & \text{if } \xi_1 < 0 \text{ and } \xi_4 < 0, \\ \end{cases}
\]

where \( S = \text{sign}(\Gamma(a)\Gamma(b)\Gamma(c - a)\Gamma(c - b)) = \text{sign} \prod_{j=1}^{4} \Gamma(\xi_j) \).

**Proof.** Consider the degenerate case when \( \bigcup_{j=1}^{4} (\xi_j) \cap \mathbb{N}_0 \neq \emptyset \). If \( \xi = -a \) or \( \xi = -b \), then the function \( {}_2F_1(a, b, c, z) \) reduces to a polynomial of degree \( \xi \) that has precisely \( \xi \) zeros in \( \mathbb{C} \setminus \{1\} \): Lemma 1 shows that \( {}_2F_1(a, b, c, 1) \neq 0 \). If \( \xi = a - c \) or \( \xi = b - c \), the function \( {}_2F_1(a, b, c, z) \) is a (fractional or integer) power of \((1 - z)\) times a polynomial of degree \( \xi \) that similarly has precisely \( \xi \) zeros in \( \mathbb{C} \setminus \{1\} \).

Now, consider the non-degenerate case \( \bigcup_{j=1}^{4} (\xi_j) \cap \mathbb{N}_0 = \emptyset \). Observe that \( \xi_1 + \xi_4 = \xi_2 + \xi_3 \in a + (c - a) = c = b + (c - b) \). When \( \xi_1 = a \), we automatically obtain \( \xi_4 = c - a \), and hence \( c - a \geq b \geq a \). The last condition allows us to use [14, Theorem].

If \( \xi_1 = b \), then \( \xi_4 = c - b \), as well as \( \nu(a, b, c) = \nu(b, a, c) \). Therefore, application of [14, Theorem] to \( {}_2F_1(b, a; c; z) \) furnishes the proof. Analogously, if \( \xi_1 = c - a \) or \( \xi_1 = c - b \), then we employ Euler’s identity (10) to see that

\[
\nu(a, b, c) = \nu(c - a, c - b, c) = \nu(c - b, c - a, c).
\]

So, it is enough to apply [14, Theorem] to, respectively, \( {}_2F_1(c - a, c - b; c; z) \) or \( {}_2F_1(c - b, c - a; c; z) \).

Recall that the banks of the branch cut may only contain zeros in the degenerate case \( \{-a, -b, a - c, b - c\} \cap \mathbb{N}_0 \neq \emptyset \), see Lemma 3. The function \( {}_2F_1(a, b, c, z) \) is then a polynomial, possibly multiplied by a (fractional or integer) power of \((1 - z)\). So, Theorem 3 mentions “the upper bank of the branch cut” in order to count each zero of that polynomial in \((1, +\infty)\) exactly one time (the situation on both banks of the branch cut is the same, as \( {}_2F_1(a, b, c, z) \) is a real function). This is different to the statement of [14, Theorem] that speaks about both banks of the branch cut together with the branch point \( z = 1 \). The reason is that [14] only touches upon the special non-degenerate case, where this distinction disappears.

One can also count the number of real zeros of \( {}_2F_1(a, b; c; z) \) for \( z \in (0, 1) \) and (via Pfaff’s transformation [8, p. 64, Eq. (22)]) for \( z < 0 \) by applying the results of [11, 12], see also [6] for the polynomial case.

The following corollary is an extended and improved version of [7, Theorem 2.1].

**Corollary 2.** Suppose \( c \neq 0 \). Then \( {}_2F_1(a, b; c; z) \) does not vanish for \( z \in \mathbb{C} \setminus [1, +\infty) \) as well as on the banks of the branch cut if and only if any of the following conditions is true:

(I) \(-1 < \min(a, b) \leq c \leq \max(a, b) \leq 0;\)

(II) \(-1 < \min(a, b) \leq 0 \leq \max(a, b) \leq c;\)

LOBACHEVSKII JOURNAL OF MATHEMATICS Vol. 42 No. 12 2021
if we had one of the equalities

(V) $a, b, c, c - a, c - b$ are non-integer negative numbers, such that $|\xi_1| + 1 = |\xi_4|$ and $|\xi_2| = |\xi_3|$, where $\xi_1, \ldots, \xi_4$ are the numbers $a, b, c - a, c - b$ taken in non-decreasing order:

$$\min(a, b, c - a, c - b) = \xi_1 \leq \xi_2 \leq \xi_3 \leq \xi_4 = \max(a, b, c - a, c - b);$$

(VI) $0 \in \{a, b, c - a, c - b\}$.

In the notation of condition (V), one necessarily has $c - \xi_4 = \xi_1$ and $c - \xi_2 = \xi_3$. Indeed: $\xi_1 + \xi_4 = c = \xi_2 + \xi_3$ in view of $a + (c - a) = c = b + (c - b)$. Moreover, condition (V) implies $\xi_1 < \xi_2$ and $\xi_3 < \xi_4$; if we had one of the equalities $\xi_1 = \xi_2$ and $\xi_3 = \xi_4$, we automatically had the other, but the last two equalities cannot hold simultaneously due to $|\xi_1| + 1 = |\xi_4|$ and $|\xi_2| = |\xi_3|$.

**Proof.** On assuming that $\{-a, -b, a - c, c - b\} \cap \mathbb{N}_0 \neq \emptyset$ holds, the condition $0 \in \{a, b, c - a, c - b\}$ becomes equivalent to that $\zeta_1(a, b, c, z) \neq 0$ for $z \in \mathbb{C} \setminus [1, +\infty)$ as well as on the banks of the branch cut $[1, +\infty)$, see Theorem 3.

Let us show now that for $\{-a, -b, a - c, c - b\} \cap \mathbb{N}_0 = \emptyset$ the necessity part holds, namely: if $\zeta_1(a, b, c, z) \neq 0$ for $z \in \mathbb{C} \setminus [1, +\infty)$ as well as on the banks of the branch cut $(0, 1)$, then at least one of the conditions (I)–(V) is satisfied. By Theorem 3, either $\xi_1 > 0$, or simultaneously $\xi_1 < 0$, $\prod_{j=1}^{4} \Gamma(\xi_j) < 0$ and $|\xi_4| - |\xi_1| = 1$. If $\xi_1 > 0$, then $0 < \min(a, b) < c$ and $\max(a, b) < c$, so we obtain the condition (VI).

Let $-1 < \xi_1 < 0$, then $\xi_4 > 0$ and, due to $\prod_{j=1}^{4} \Gamma(\xi_j) < 0$, additionally $\xi_2 \xi_3 > 0$. If $\xi_1 = \min(a, b)$, then $\xi_4 = c - \xi_1 = c - \min(a, b)$ as remarked above, and hence $(c - \max(a, b)) \max(a, b) > 0$. The case $-1 < \xi_1 < \min(a, b) < 0$ and $c - \max(a, b) > 0$ yields $\max(a, b) > 0$, and hence that condition (II) holds. The case $-1 < \xi_1 = \min(a, b) < 0$ and $c - \max(a, b) < 0$ yields $\max(a, b) < 0$, and hence that the condition (I) holds.

If $\xi_1 = \min(c - a, c - b) = c - \max(a, b)$, then automatically $\xi_4 = c - \xi_1 = \max(a, b)$, and hence $(c - \min(a, b)) \min(a, b) > 0$. So, the assumptions $-1 < c - \max(a, b) < 0$ and $\min(a, b) > 0$ yield $c - \min(a, b) > 0$, which leads to condition (VI). In turn, the assumptions $-1 < c - \max(a, b) < 0$ and $\min(a, b) < 0$ yield $c - \min(a, b) < 0$, and hence condition (III) is satisfied.

Now, let $\xi_1 < -1$. The equality $(|\xi_4|) - |\xi_1| = 1$ given by Theorem 3 then implies

$$|\xi_4| = 1 + |\xi_1| < 0.$$  

In particular, $\xi_1, \xi_2, \xi_3, \xi_4$ are non-integer negative numbers. Now, the additional inequality $|\xi_2| < |\xi_3|$ is impossible, since it leads to the contradiction $\prod_{j=1}^{4} \Gamma(\xi_j) > 0$. Therefore, for $\xi_1 < -1$ condition (V) is satisfied.

The sufficiency part for $\{-a, -b, a - c, c - b\} \cap \mathbb{N}_0 = \emptyset$ is precisely [7, Theorem 2.1].

The next Corollary aims at lifting the integrability condition $\nu > -1$ of [7, Theorem 2.12]. This corollary turns into a complete generalization of [7, Theorem 2.12] if the rational function $Q(z)$ from Theorem 2 is retained (instead of letting $Q(z) \equiv 0$) and if $Q(z)$ is analytic in $\mathbb{C} \setminus \{1\}$: this rational function then generalizes the polynomial $\zeta_{a,b,c}(z)$ from [7, Theorem 2.12].

**Corollary 3.** Suppose that $a, b, c \in \mathbb{R}$ and $n_1, n_2, m \in \mathbb{Z}$, where $-c, -c - m \notin \mathbb{N}_0$. Let any of the conditions (I)–(VI) in Corollary 2 be satisfied, so that $\zeta_1(a, b, c, z) \neq 0$ for all $z$ in $\mathbb{C} \setminus [1, +\infty)$ and on the banks of the branch cut $(1, +\infty)$. Take numbers $M, N \in \mathbb{N}_0$ such that

$$R_{n_1, n_2, m}(z) = o((1 - z)^{M - 1}) \text{ as } z \to 1 \quad \text{and} \quad R_{n_1, n_2, m}(z) = o(z^{N - M}) \text{ as } z \to \infty,$$

(20)
or, equivalently, in terms of $\eta(a, b, c)$ and $\zeta(a, b, c)$ defined in (5)–(6),

$$M > \eta(a, b, c) - \eta(a + n_1, b + n_2, c + m) - 1 \quad \text{and} \quad N > M + \zeta(a + n_1, b + n_2, c + m) - \zeta(a, b, c).$$

LOBACHEVSKII JOURNAL OF MATHEMATICS Vol. 42 No. 12 2021
Then

\[ R_{n_1, n_2, m}(z) = \sum_{k=0}^{N-1} \frac{z^k}{(1-z)^M} \sum_{j=0}^{\min\{M, k\}} (-1)^j \binom{M}{j} \frac{R_{n_1, n_2, m}^{(k-j)}}{(k-j)!} \]

+ \frac{z^N}{(z-1)^M} B_{n_1, n_2, m}(a, b, c) \int_{1}^{\infty} \frac{x^{1-a-c-N(x-1)}^{M+a-b-1} P_r(1/x)}{|2F_1(a, b; c; x)|^2 (x-z)} dx, \quad (21)

where \( r, l, n \) and \( B_{n_1, n_2, m}(a, b, c) \) are the same as in Theorem 1 and \( P_r \) is defined in (9).

In particular, if (20) holds with \( N = M = 0 \) we obtain

\[ R_{n_1, n_2, m}(z) = B_{n_1, n_2, m}(a, b, c) \int_{1}^{\infty} \frac{x^{1-a-c}(x-1)^{c-a-b-1} P_r(1/x)}{|2F_1(a, b; c; x)|^2 (x-z)} dx. \quad (22) \]

Proof. According to Corollary 2, the function \( R_{n_1, n_2, m}(z) \) is analytic in \( \mathbb{C} \setminus [1, +\infty) \) and on the banks of the branch cut \( (1, +\infty) \). Therefore, one can apply Theorem 2 assuming \( Q(z) \equiv 0 \) and \( T(z) \equiv 1 \). \( \square \)

5. EXAMPLES AND APPLICATION

An interesting application is to plug the integral representations provided by Theorem 2 and Corollary 3 into various hypergeometric expressions: for instance, into those given in [5].

By taking derivatives on both sides of formula [5, Eq. (2.9)] and changing \( c \to c + 1 \) we get:

\[ z^{c-1} \frac{2F_1(a+1, b+1; c+1; z) \int_{1}^{\infty} \frac{x^{1-a-c-N(x-1)}^{M+a-b-1} P_r(1/x)}{|2F_1(a, b; c; x)|^2 (x-z)} dx}{[2F_1(a, b; c; z)]^2} = \frac{c^2 z^{c-1}}{ab} (1 - R_{0,0,1}(z)) - \frac{ce}{ab} \frac{d}{dz} R_{0,0,1}(z) \]

On the other hand, according to [7, Example 8] we have

\[ R_{0,0,1}(z) = Q_{a,b,c} - \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b)\Gamma(c-a+1)\Gamma(c-b+1)} \int_{0}^{1} \frac{t^{a+b-1}(1-t)^{c-a-b}dt}{(1-zt)[2F_1(a, b; c; 1/t)]^2}, \quad (23) \]

where

\[ Q_{a,b,c} = \frac{c}{c - \text{min}(a, b)}. \]

Substituting and simplifying we arrive at the representation

\[ z R_{1,1,1}(z) R_{0,0,1}(z) = z \frac{2F_1(a+1, b+1; c+1; z) \int_{1}^{\infty} \frac{x^{1-a-c-N(x-1)}^{M+a-b-1} P_r(1/x)}{|2F_1(a, b; c; x)|^2 (x-z)} dx}{[2F_1(a, b; c; z)]^2} \]

\[ = \frac{c^2}{ab} (1 - Q_{a,b,c}) + B \int_{0}^{1} \frac{t^{a+b-1}(1-t)^{c-a-b}(c+zt(1-c))dt}{(1-zt)^2[2F_1(a, b; c; 1/t)]^2}, \]

where

\[ B = \frac{[\Gamma(c+1)]^2}{\Gamma(a+1)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b+1)}. \]

This representation holds under the assumptions of Corollary 2.

Even more surprising result follows by application of [5, eq. (2.10)] which after differentiation and changing \( c \to c + 1 \) reads:

\[ \frac{2F_1(a, b; c-1; z) \int_{1}^{\infty} \frac{x^{1-a-c-N(x-1)}^{M+a-b-1} P_r(1/x)}{|2F_1(a, b; c; x)|^2 (x-z)} dx}{[2F_1(a, b; c; z)]^2} = \frac{1}{c-1} (c - R_{0,0,1}(z)) - \frac{z}{c - 1} \frac{d}{dz} R_{0,0,1}(z). \]
Combining this with (23) we obtain the representation containing the so-called generalized Stieltjes transform of order 2 of a positive measure:

\[ R_{0,0,-1}(z)R_{0,0,1}(z) = \frac{2F_1(a, b; c - 1; z)\, 2F_1(a, b; c + 1; z)}{[2F_1(a, b; c; z)]^2} \]

\[ = \frac{c(c - \min(a, b) - 1)}{(c - 1)(c - \min(a, b))} + \frac{\Gamma(c - 1)\Gamma(c + 1)}{\Gamma(a)\Gamma(b)\Gamma(c - a + 1)\Gamma(c - b + 1)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}dt}{(1-zt)^2[2F_1(a, b; c; 1/t)]^2}. \]

This representation also holds under the assumptions of Corollary 2. In particular, if parameters are positive and \( c > 1, c - a, c - b > -1 \), this representation shows that the function on the left is monotonically increasing on \((-\infty, 1)\). In fact, a much stronger claim holds: if \( c \geq \min(a, b) + 1 \) and the constant in front of the integral is positive, then the function \( x \rightarrow R_{0,0,-1}(x)R_{0,0,1}(x) \) is logarithmically completely monotonic on \([0, \infty)\). This follows from the highly non-trivial inclusion of the Stieltjes class of order 2 into the class of logarithmically completely monotonic functions. Details and history can be found in [2, Theorem 2.1]. Logarithmic complete monotonicity of this function may be very hard to establish by other means.

It is clear that other formulae from [5, Section 2] may also be treated in the same way.

**Example 1.** To illustrate Corollary 3, let us modify the integral expression for \( R_{1,1,1}(z) \) constructed in [7, Example 3], so that the result becomes applicable under milder conditions. Suppose that any of the conditions (I)-(V) in Corollary 2 is satisfied. For simplicity, restrict ourselves to the non-degenerate case \( a, b, c - a, c - b \notin -N_0 \).

Lemma 1 shows that, for \( \tau := (c - a - b - 1)_- - (c - a - b)_- \) and \( \varepsilon_1, \varepsilon_\infty \in \{-1, 0, 1\} \),

\[ R_{1,1,1}(z) = L_1(1 - z)\tau^\varepsilon_1 [\log(1 - z)]^\varepsilon_\infty (1 + o(1)) \text{ as } z \to 1, \]

\[ R_{1,1,1}(-z) = L_\infty z^{-\varepsilon_1} [\log z]^\varepsilon_\infty (1 + o(1)) \text{ as } z \to \infty. \]

Observe also that \( \tau \geq (c - a - b)_- - 1 - (c - a - b)_- = -1, \) and that \( \tau > -1 \) is equivalent to \( c > a + b \).

In [7, Example 3], we required \( c > a + b \) to make \( R_{1,1,1}(z) \) integrable near \( z = 1 \). Now, let us also allow the reverse inequality \( c \leq a + b \) that implies \( \tau = -1 \). Corollary 3 with \( N = M = 1 \) in this case yields

\[ R_{1,1,1}(z) = \frac{1}{1 - z} + \frac{z}{z - 1} \frac{\Gamma(c)\Gamma(c + 1)}{\Gamma(a + 1)\Gamma(b + 1)\Gamma(c - a)\Gamma(c - b)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}dt}{(1-zt)^2[2F_1(a, b; c; 1/t)]^2}, \]

which holds true for both \( c > a + b \) and \( c \leq a + b \).

**Example 2.** In a similar way, let us modify the integral expression for \( R_{0,0,-1}(z) \) constructed in [7, Example 9]. Suppose that at least one of the conditions (I)-(V) in Corollary 2 is satisfied, and that \( a, b, c - a, c - b \notin -N_0 \). From [7, Subsection 2.1] one can see that

\[ R_{0,0,-1}(z) = L_1(1 - z)\tau^\varepsilon_1 [\log(1 - z)]^\varepsilon_\infty (1 + o(1)) \text{ as } z \to 1, \]

\[ R_{0,0,-1}(-z) = Q + o(1) \text{ as } z \to \infty \]

hold for some numbers \( \varepsilon_1 \in \{-1, 0, 1\} \) and \( L_1 \neq 0 \), where

\[ Q = (c - \min(a, b) - 1)/(c - 1) \text{ and } \tau = (c - a - b - 1)_- - (c - a - b)_-. \]

Observe that \( \tau \geq -1 \), and that the strict inequality \( \tau > -1 \) is equivalent to \( c > a + b \). Now, we remove the assumption \( c > a + b \) of [7, Example 9] introduced there to make \( R_{0,0,-1}(z) - Q \) integrable near \( z = 1 \). Theorem 2 with \( N = M = 1 \) and \( T(z) \equiv 1 \) yields

\[ R_{0,0,-1}(z) = Q + \frac{1 - Q}{1 - z} + \frac{z}{z - 1} \frac{\Gamma(c)\Gamma(c - 1)}{\Gamma(a)\Gamma(b)\Gamma(c - a)\Gamma(c - b)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}dt}{(1-zt)^2[2F_1(a, b; c; 1/t)]^2}. \]

From here, one can also immediately derive an analogous expression for \( R_{0,1,0}(z) \) by applying the formula [10, Eq. (12)]

\[ R_{0,1,0}(z) = \frac{c - 1}{b} R_{0,0,-1}(z) - \frac{c - b - 1}{b}. \]
Example 3. For the Gauss ratio $R_{0,1,1}(z)$, Lemma 2 and definition (9) imply:

$$B_{0,1,1} P_0(t) = \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-a+1)\Gamma(c-b)}.$$ 

So, due to $\lim_{z\to\infty} R_{0,1,1}(z) = [c(b-a)_+]/[b(c-a)]$ Corollary 3 yields (cf. [7, Example 1]):

$$R_{0,1,1}(z) = \frac{c(b-a)_+}{b(c-a)} + \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-b)\Gamma(c-a+1)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}dt}{(1-zt)^2 F_1(a,b;c; t^{-1})}. \quad (24)$$

In order for this representation to hold we need to assume that any of the conditions (1)–(VI) of Corollary 2 is satisfied (for condition (VI), we additionally require $a \neq c$ to exclude a non-integrable case). Then verification of the formulae (3) and (5) shows that $|R_{0,1,1}(z)|$ is integrable near $z = 1$.

Let, for instance, $0 < c < a < c+1$ and $-1 < b < 0$, so conditions of Corollary 2 are violated. Theorem 3 shows that $R_{0,1,1}(z)$ actually has a unique simple pole denoted further by $\beta_1 \neq 1$. Since $R_{0,1,1}(z)$ is a real function, this pole is necessarily real. Lemma 3 implies that $\beta_1 \notin [1,\infty)$. According to Theorem 2,

$$R_{0,1,1}(z) = \frac{c(b-a)_+}{b(c-a)} + \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-b)\Gamma(c-a+1)} \int_0^1 \frac{t^{a+b-1}(1-t\beta_1)(1-t)^{c-a-b}dt}{(1-zt)^2 F_1(a,b;c; t^{-1})}.$$ 

At the origin, the left-hand side in Pfaff’s identity [8, p. 64, Eq. (22)]

$$2 F_1(a,c-b;c;z) = (1-z)^{-a} 2 F_1 \left( a, b; c; \frac{z}{z-1} \right)$$

has a Taylor expansion with only positive coefficients, so it cannot vanish in $(0,1)$. Consequently, the only option is $\beta_1 \in (0,1)$.

Alternatively, the choice $T(z) \equiv 1$ in Theorem 2 gives another expression:

$$R_{0,1,1}(z) = \frac{c(b-a)_+}{b(c-a)} + \frac{A_1}{z - \beta_1} + \frac{\Gamma(c)\Gamma(c+1)}{\Gamma(a)\Gamma(b+1)\Gamma(c-b)\Gamma(c-a+1)} \int_0^1 \frac{t^{a+b-1}(1-t)^{c-a-b}dt}{(1-zt)^2 F_1(a,b;c; t^{-1})},$$ 

where the residue $A_1$ at $z = \beta_1$ may be computed through the formula

$$A_1 = \frac{c}{ab} 2 F_1(a+1,b+1,c+1,\beta_1).$$

REFERENCES

1. V. Belevitch, “The Gauss hypergeometric ratio as a positive real function,” SIAM J. Math. Anal. 13, 1024–1040 (1982).
2. Ch. Berg, S. Koumandos, and H. L. Pedersen, “Nielsen’s beta function and some infinitely divisible distributions,” Math. Nachr. 294, 426–449 (2021).
3. Yu. A. Brychkov, Handbook of Special Functions: Derivatives, Integrals, Series and other Formulas (CRC, London, 2008).
4. A. Çetinkaya, D. B. Karp, and E. G. Prilepkina, “Hypergeometric functions at unit argument: Simple derivation of old and new identities.” arXiv: 2105.05196 (2021).
5. J. T. Conway, “Indefinite integrals of quotients of Gauss hypergeometric functions,” Integr. Transf. Spec. Funct. 29, 417–430 (2018).
6. K. Driver and K. H. Jordaan, “Zeros of the hypergeometric polynomial $F(-n,b;c;z)$,” in Algorithms for Approximation IV, Ed. by J. Levesley, I. J. Anderson and J. C. Mason (Univ. Huddersfield, Huddersfield, 2002), pp. 436–444.
7. A. Dyachenko and D. Karp, “Ratios of the Gauss hypergeometric functions with parameters shifted by integers: Part I,” arXiv: 2103.13312v1 (2021).
8. A. Erdélyi, *Higher Transcendental Functions*, Vol. 1: Bateman Manuscript Project (McGraw-Hill, New York, 1953).
9. M. F. Al-Jamal, “Quotient rule for higher order derivatives,” Discussion at Physics Forums, 2009–2012. https://www.physicsforums.com/threads/quotient-rule-for-higher-order-derivatives.289320/. Accessed 2021.
10. C. F. Gauss, “Disquisitiones generales circa seriem infinitam...,” Comm. Soc. R. Sci. Gottingensis Recent. 2, 1–46 (1812); C. F. Gauß, *Werke, Band III* (Königl. Gesellsch. Wissensch. Göttingen, Göttingen, 1876), pp. 123–162.
11. A. Hurwitz, “Ueber die Nullstellen der hypergeometrischen Reihe,” Gött. Nachr., 557–564 (1890); Math. Ann. 38, 452–458 (1891).
12. F. Klein, “Ueber die Nullstellen der hypergeometrischen Reihe,” Math. Ann. 37, 573–590 (1890).
13. *NIST Digital Library of Mathematical Functions*, release 1.0.19, Ed. by F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller, and B. V. Saunders (2018). http://dlmf.nist.gov/. Accessed 2021.
14. H.-J. Runckel, “On the zeros of the hypergeometric function,” Math. Ann. 191, 53–58 (1971).