The Locally Fine Coreflection and Normal Covers
in the Products of Partition-complete Spaces

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ABSTRACT. We prove that the countable product of supercomplete spaces having a countable closed cover consisting of partition-complete subspaces is supercomplete with respect to its metric-fine coreflection. Thus, countable products of $\sigma$-partition-complete paracompact spaces are again paracompact. On the other hand, we show (Theorem 7.5) that in arbitrary products of partition-complete paracompact spaces, all normal covers can be obtained via the locally fine coreflection of the product of fine uniformities. These results extend those given in [1], [6], [7], [19], [30], [20], [33].

KEYWORDS. Supercomplete, paracompact, product space, partition-complete, winning strategy, normal cover, locally fine, frame.

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1. Introduction. It was shown by the third author in [30] that locally fine spaces are subspaces of fine spaces. This result was obtained by embedding an arbitrary uniform space into the product of complete metric spaces and by showing that the locally fine coreflection [9] of such a product uniformity contains all normal covers of the product. The important insight that emerged from [30] was the connection of Noetherian trees with the covers of locally fine spaces. The locally fine coreflection $\lambda\mu$ of a given pre-uniformity $\mu$ (i.e., a filter of coverings) may be described as an attempt to obtain the topology of a given space by means of iterated combinatorial refinements (generalized subdivisions) of coverings, starting from $\mu$. The covers obtained in this way are recursively related to $\mu$ in the sense that for each $U \in \lambda\mu$ there is a Noetherian tree $T$ of subsets of $X$, obtained through iterated applications of uniform refinements to the elements of the tree, such that the maximal elements of $T$ form a refinement of $U$. For supercomplete spaces [22], and in particular for complete metric spaces, this procedure reaches all open covers of the space. The complexity of such refinements was studied in [12] and [18]. Related methods were used by the second and third authors to obtain extension theorems for continuous functions defined on subsets of products of metrizable spaces [21].

On the other hand, the condition that the locally fine coreflection of a product contain all open covers was studied by the first author in a series of papers on supercompleteness (see, e.g., [11], [13], [15]). For

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spaces satisfying a structural recursive condition (C-scattered, Čech-scattered) Noetherian trees were applied to show that paracompactness (among others) is preserved in countable products of such spaces. These results extended the initial theorems of Frolík [7] and Arhangel’skii [3] that paracompactness is preserved in countable products of locally compact spaces, and similar results from [36] (scattered spaces) and [6] (C-scattered spaces), including the result of [1] on C-scattered Lindelöf spaces. However, and more importantly, the new results showed that the open covers of such product could be refined by ones generated recursively from open covers of the factors.

For paracompact products, the above condition on the locally fine coreflection is equivalent to the spatiality of the corresponding product of locales, noted by Isbell in [24]. Spatiality was extensively studied by Plewe in [32], who used the productivity of paracompactness in locales and spatiality to extend ([33]) the above topological results to countable products of $C_{\delta}$-absolute spaces. These spaces are equivalent to those with a complete exhaustive sieve studied by Michael in [28] and to the partition-complete spaces of Telgársky and Wicke in [42]. Unfortunately, spatiality does not extend to spaces which are merely countable unions (e.g., the rationals) of closed partition-complete parts. As an extreme case, $\mathbb{Q} \times \mathbb{Q}$ is not spatial.

It has been recently shown by the first author ([16]) that for spaces with pre-uniformities, there is a simple relationship between the ‘covering monoid’ of their localic product and the locally fine coreflection of the product pre-uniformity. The covers consisting of open rectangles of the latter product form a monoid which naturally maps onto a ‘generating monoid’ of the localic product. It follows that for the products of regular spaces with supercomplete pre-uniformities, the locally fine coreflection of the product pre-uniformity is supercomplete provided that the product is spatial. (We call a pre-uniformity supercomplete if it contains every open cover of the underlying space.) On the other hand, the locally fine pre-uniformities are equivalent to the so-called formal spaces, for which the essential questions about covers are related to their effective computability. Thus, we come back to our methods of recursive constructions by means of Noetherian trees. As we cannot rely on the spatiality of the products, we will construct the refinements directly. Moreover, we will handle arbitrary (in general non-paracompact) products of partition-complete regular spaces and show that their locally fine covers by regular open sets belong to (i.e., are refined by members of) the locally fine coreflection of any supercomplete uniformities of the factors.

Before starting our preliminary section for necessary definitions, we state our two main results:

**Theorem A (Theorem 6.4):** Let $(X_i : i \in \mathbb{N})$ be a countable family of $\sigma$-partition-complete, regular spaces, and let $\mu_i$ be a supercomplete pre-uniformity for $X_i$. Then $\lambda m \Pi \mu_i$ is a supercomplete pre-uniformity for the topological product $\Pi X_i$.

**Theorem B (Theorem 7.5):** Let $(X_i : i \in I)$ be a family of partition-complete, completely regular spaces, and for each $i \in I$, let $\mu_i$ be a supercomplete uniformity for $X_i$. Then every normal cover $V$ of $X = \Pi I X_i$ belongs to $\lambda \Pi_I \mu_i$. 
2. Preliminaries. Here we take up definitions of several tools needed in the main part of this paper. The locally fine coreflections of uniformities were introduced in [9], and they were constructed internally by means of consecutive ‘derivatives’ closing the given (pre-) uniformity under the combinatorial condition of local fineness. In the theory of ‘formal spaces’, this condition is referred to as composition ([5]) or transitivity ([37]) of the related covering relation. It may be regarded as an attempt to obtain the underlying topology of the space through iterated applications of uniform subdivisions of covers. We recall the definition of a tree from [11] by means of slowed-down Ginsburg-Isbell derivatives. Let $\mu$, $\nu$ be pre-uniformities on a set $X$. Then $[34]) \mu/\nu$ denotes the pre-uniformity generated by all covers of $X$ of the form $\{U_i \cap V^i_j\}$, where $\{U_i\} \in \mu$ and for all $i$, $\{V^i_j\} \in \nu$. Let $\mu^{(0)} = \mu$, $\mu^{(\alpha+1)} = \mu^{(\alpha)}/\mu$, and $\mu^{(\beta)} = \bigcup(\mu^{(\alpha)} : \alpha < \beta)$ for a limit ordinal $\beta$. The first $\mu^{(\alpha)}$ unchanged under this derivation is denoted by $\lambda \mu$ and called the locally fine coreflection of $\mu$. It is the minimal pre-uniformity closed under the operation $\nu \mapsto \nu/\nu$ finer than $\mu$. If $V \in \lambda \mu$, we call $V$ a $\lambda$-cover (with respect to $\mu$) or directly a $\lambda \mu$-uniform cover. For a subset $A \subset X$, a $\lambda$-neighbourhood (with respect to a given pre-uniformity $\mu$) is a set $N \supset A$ for which there is $V \in \lambda \mu$ such that $\text{St}(A, V) \subset N$. We note here that if the $\mu_i$ are pre-uniformities and $V \in \lambda \Pi \mu_i$, then $V$ has a refinement $W \in \lambda \Pi \mu_i$ consisting of basic rectangles, i.e., sets of the form $B = \cap \{\pi_i^{-1}[B_i] : i \in F\}$, where $F$ is a finite set. This result can be easily proved by using the inductive definition of the operation $\lambda$.

**Noetherian trees.** Well-founded trees in the context of locally fine refinements were first used in [31] and in an early version of [30]. They were consequently applied in several papers: [18], [19], [20], and in [17] in the context of well-founded cubical triangulations refining open covers of infinite-dimensional cubes.

In this paper, a tree is a set $T$ with a partial order $\leq$ such that $T$ has a unique minimal element (root) with respect to $\leq$ and for each $p \in T$, the set of $\leq$-predecessors of $p$ is linearly ordered and finite. We call a tree Noetherian if each of its linearly ordered parts is finite. To fix some notation, we denote by $S(p)$ the set of all immediate successors of $p \in T$. The symbol $\text{End}(T)$ denotes the set of all maximal elements of $T$. Thus, if $T$ is Noetherian, then not only is $\text{End}(T)$ non-empty, but each maximal linear part (branch) of $T$ meets $\text{End}(T)$ in a unique element.

The Ginsburg-Isbell derivatives are related with Noetherian trees as follows. Let $\mu$ be a pre-uniformity on a set $X$. We say that a mapping $\varphi : T \to 2^X$ satisfies the uniform mapping condition with respect to $\mu$ if for each $p \in T$, the family $\{\varphi(q) : q \in S(p)\}$ is a $\mu$-uniform cover of $\varphi(p)$, i.e., there is a cover $U \in \mu$ such that $U \upharpoonright \varphi(p) \prec \varphi(S(p))$, or in still other words, if $\varphi(S(p)) \in \mu \upharpoonright \varphi(p)$. The principle introduced by the third author in [30] states that $U \in \lambda \mu$ if, and only if, there is a Noetherian tree $T$ and a mapping $\varphi : T \to 2^X$ satisfying the uniform mapping condition with respect to $\mu$, such that $\varphi(\text{End}(T)) \prec U$. We may delimit our discussion to trees consisting of subsets of $X$, and define the uniform refinement condition with respect to the associated inclusion of the tree in $2^X$.

**Perverse products.** We will define special ‘weak’ subproducts of direct products of trees with the property that (under suitable conditions) infinite chains project onto infinite chains in the factor trees. In this paper, a perversity is a decreasing sequence $p : N \to Z$ which is eventually zero. (This concept is modified from that used in intersection homology theory, cf. [27].) Sets of perversities, ordered by the coordinatewise partial order of functions $N \to Z$, are used to reduce a given direct product of countably many trees. Let $(T_i : i \in N)$ be a countable family of trees, and let $\mathcal{P}$ be a set of perversities.

The perverse product $\Pi_{P \in \mathcal{P}} T_i$ is the subset of the direct product $\Pi_{i \in I} T_i$, consisting of all elements $x$ for which there is a perversity $p \in \mathcal{P}$ such that for each $i \in N$, the level of $x_i$ in $T_i$, i.e., the number of predecessors in the tree, equals $p(i)$. In addition to countable products, we define the finite perverse products in the same way.
way, by considering the restriction $P \upharpoonright F = \{ p \upharpoonright F : p \in P \}$ of perversities to the given finite set $F \subseteq \mathbb{N}$. The set-theoretic direct product of the $T_i$ has as its elements the Cartesian products $\Pi_i p_i$ (where each factor $p_i$ is considered a set), and the set-theoretic perverse product is the corresponding subproduct.

The perverse product of countably many trees is a tree, provided that the associated family of perversities is itself a tree under its natural partial order. The perverse product of trees is a partially ordered set such that infinite chains project onto infinite chains in the factors, provided that the set of perversities has the same property.

We now define a standard set $P = \{ p_n : n \in \mathbb{N} \}$ of perversities to be used in the sequel. Let $p_1 = (0,0,0,\ldots)$. Suppose that $p_n = (i_1,\ldots,i_m,0,0,\ldots)$ has been defined. Put

$$k = \min\{ j \in \mathbb{N} : i_j = i_{j+1} \}.$$

Then define $p_{n+1} = (i_1,\ldots,i_k+1,i_{k+1},\ldots,i_m,0,0,\ldots)$. The elements of this sequence are $(\cdot)$, $(1)$, $(1,1)$, $(2,1)$, $(2,1,1)$, $(2,2,1)$ etc., where we have indicated the non-zero entries only. The set $P$ is linearly ordered with respect to the coordinatewise order, and hence any perverse product of trees with respect to $P$ is again a tree. To give an example of its use, let $T$ be a tree, and let $T' = (\Pi_{i\in\mathbb{N}} T_p)$ be the perverse set-theoretic power of $T$ with respect to $P$ as explained above. The root of $T'$ (the unique element of level 0) is the Cartesian power $R^\mathbb{N}$ of the root of $T$, and the elements of level 3, say, are products $P_1 \times P_2 \times R \times R \times \ldots$,
3. Partition-completeness and stationary winning strategies. This section, we consider a condition that generalizes both C-scattered and Čech-complete spaces. Recall that a sequence $U = (U_n)$ of covers of a space $X$ is complete if any filter base $F$ ‘controlled’ by $U$ has a cluster point, in other words if any filter $F$ with $F \cap U_n \neq \emptyset$ for all $n$ satisfies $\bigcap \{F : F \in F\} \neq \emptyset$. The classical characterization from Frolík[8] and Arhangel’skii[2] describes the Čech-complete spaces as the completely regular spaces with a complete sequence of left-open partitions.

On the other hand, $\mathcal{K}$-scattered spaces have canonical exhaustion into left-open partitions of subsets $S$ such that $S$ is a $\mathcal{K}$-subset. A partition $\mathcal{P}$ of a space $X$ is called left-open if $\mathcal{P}$ admits a well-ordering $\{P_\alpha : \alpha < \beta\}$ such that $\bigcup \{P_\alpha : \alpha \leq \gamma\}$ is open for all $\gamma < \beta$. It was proved in [26] that if $\mathcal{K}$ is closed-hereditary, then $X$ is $\mathcal{K}$-scattered if and only if $X$ has a left-open partition $\mathcal{P}$ such that $P$ is a $\mathcal{K}$-subset for all $P \in \mathcal{P}$. In [42], Telgársky and Wicke studied spaces which have a complete sequence of left-open partitions: A sequence $\mathcal{P} = (\mathcal{P}_n)$ of left-open partitions is called complete if 1) $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for all $n$, 2) the well-order of $\mathcal{P}_{n+1}$ is compatible with that of $\mathcal{P}_n$ in the sense that the relation $Q \subset P$ between elements $P \in \mathcal{P}_n, Q \in \mathcal{P}_{n+1}$ preserves the order, and 3) any filter base controlled by $\mathcal{P}$ has a cluster point.

**Definition 3.1 ([42], p. 737):** A space is called partition-complete if it has a complete sequence of left-open partitions.

We call a space $\sigma$-partition-complete if it is the union of countably many partition-complete, closed subspaces.

The results important for us were already obtained by E. Michael [28]. Recall that a space is $\mathcal{K}$-scattered if every non-empty closed subset of the space contains a point with a $\mathcal{K}$-neighbourhood in the subset. Michael called a cover $\mathcal{U}$ of $X$ exhaustive if every non-empty subset $S$ of $X$ has a non-empty relatively open subset of the form $U \cap S$, where $U \in \mathcal{U}$. (Thus, for C-scattered spaces, the interiors of compact subsets form an exhaustive cover.) The following result of Michael gives us the definition of partition-completeness we will use in this paper. (The result follows directly from [28], Prop. 4.1 and [42], Prop. 1.2.)

**Theorem 3.2 ([28], Prop. 4.1):** A space $X$ is partition-complete if, and only if, it has a complete sequence of exhaustive covers.

We will use this condition in terms of a topological game $G(X)$ related to such sequences of covers. Players I and II choose in alternating steps non-empty subsets $S_1 \supset T_1 \supset S_2 \supset \cdots$ of $X$ such that Player I chooses the sets $S_i$, Player II chooses the sets $T_i$, and $T_i$ is relatively open in $S_i$ for all $i$. Player II wins if any filter base finer than $\{T_n : n \in \mathbb{N}\}$ has a cluster point. A stationary winning strategy for Player II in $G(X)$ is simply a function $\phi : P(X) \to P(X)$ giving the relatively open choices $\phi(S) \subset S$ ($\phi(S) \neq \emptyset$) of Player II for all $S \neq \emptyset$. Michael proved that a space $X$ has a complete sequence of exhaustive covers if and only if Player II has such a stationary winning strategy in the game $G(X)$ ([28], Prop. 4.1 and Thm. 7.3).

Notice the connection between $\mathcal{K}$-scattered spaces and stationary winning strategies in the above sense. By providing for each non-empty closed subset $F \subset X$ an open set $\emptyset \neq \phi(F) \subset F$, the winning strategy $\phi$ yields a decreasing decomposition of $X$ into ‘derivative’ subsets and thus (inverting the order) into a Noetherian decomposition tree. The $\phi$-derivative of a non-empty subset $S \subset X$ is simply defined as the relatively closed set $S \setminus \phi(S)$. The consecutive derivatives are then defined in a complete analogy with the definition given in the previous section. We denote the corresponding decomposition tree by $T_{\phi}(X)$.

If $X$ is partition-complete with respect to a winning strategy $\phi$, we extend its decomposition tree $T_{\phi}(X)$ to a (in general) non-Noetherian ‘game tree’ as follows. For each $U \in \text{End}(T_{\phi}(X))$, let $\mathcal{A}(U)$ be the collection
of all closed $F \subset U$ of the form $\tilde{V}$, where $V \subset U$ is non-empty and open. In order to ensure that $U$ is the union of such closures, we assume that $X$ is regular. Denote by $\phi_S$ the restriction of $\phi$ to the subsets of $S$, for any subset $S \subset X$. Extend $T_1 = T_\phi(X)$ to $T_2$ by declaring the $\tilde{V} \in A(U)$ as the immediate successors of $U$, and hanging the trees $T_{\phi_n}(V)$ below $V$. This operation is repeated countably many times; we obtain an increasing sequence $T_1, T_2, \ldots$ of Noetherian trees, and define $T(X)$ as their union. Although $T(X)$ in general is not well-founded, it has the following Noetherian property with respect to open covers. If $\mathcal{G}$ is an open cover of $X$, then there is a Noetherian subtree $T'$ of $T(X)$ such that $\text{End}(T') \prec \mathcal{G}^{<\omega}$, where $\mathcal{G}^{<\omega}$ denotes the directed open cover obtained as the finite unions of elements in $\mathcal{G}$. This follows from the following general (and simple) lemma:

**Lemma 3.3:** Let $X$ be a topological space, and let $\mathcal{F} = (F_n : n \in \mathbb{N})$ be a decreasing sequence of closed subsets such that any filter finer than $\mathcal{F}$ has a cluster point. Then $\bigcap \{F_n : n \in \mathbb{N}\}$ is compact.

4. The $\lambda$-neighbourhood induction lemma. Our main method for proving that a given cover $\mathcal{G}$ belongs to the locally fine coreflection applies the following ‘Noetherian’ induction: There is a Noetherian tree $T$ such that each maximal element $E$ of $T$ has a $\lambda$-neighbourhood $U$ with the property that $\mathcal{G} \upharpoonright U$ is a $\lambda$-cover of $U$. Then the construction of $T$ as a (subtree of) perverse power and the induction lemma given below enable us to conclude the validity of this property for the top element of $T$. This was stated and proved in [20] for uniform spaces, but it is valid for pre-uniformities (covering monoids).

**Lemma 4.1:** Let $X$ be a set, and let $\mu$ be a pre-uniformity on $X$. Let $A \subset X$, let $V \in \lambda\mu \upharpoonright A$ and for each $V \in \mathcal{V}$, let $V'$ be a $\lambda\mu$-uniform neighbourhood of $V$ in $X$ with respect to $\mu$. Then $\bigcup\{V' : V \in \mathcal{V}\}$ is a $\lambda\mu$-uniform neighbourhood of $A$ in $X$. Moreover, if $\mathcal{G}$ is a $\lambda\mu$-uniform cover of each $V'$, then $A$ has a $\lambda\mu$-uniform neighbourhood $N$ such that $\mathcal{G}$ is a $\lambda\mu$-uniform cover of $N$.

5. The Metric-fine Coreflection. We call a cover $\mathcal{G}$ of a uniform space $\mu X$ $\sigma$-uniform if there is a countable closed cover $\{F_n : n \in \mathbb{N}\}$ of $X$ such that each restriction $\mathcal{G} \upharpoonright F_n$ is a uniform cover of the subspace $F_n$. This property can be formulated in several other settings. One calls [10] a uniform space $\mu X$ metric-fine, if for any metric uniform space and any uniformly continuous map $f : \mu X \to \rho M$, $f$ remains uniformly continuous when the target $\rho M$ is changed to $\mathcal{F}M$, i.e., the fine space associated with $M$. The metric-fine coreflection $m\mu$ is the weakest metric-fine uniformity stronger than $\mu$. We note that $m\mu$ is the collection of all $\sigma$-uniform covers of $X$. Each $\sigma$-uniform (open) cover has a $\sigma$-uniformly discrete (open) refinement. A cover $\mathcal{G}$ is $\sigma$-uniform if, and only if, there is a sequence $\{U_n : n \in \mathbb{N}\}$ of uniform covers of $\mu X$ such that for each $x \in X$, there is $n(x) \in \mathbb{N}$ with $\text{St}(x, U_{n(x)}) \prec \mathcal{G}$. (Notice the relation with the so-called $\theta$-refinability of (topological) spaces). The metric-fine coreflection was used in both [19] and [20] to extend the results from $K$-scattered spaces to $\sigma - K$-scattered, where $K$ was the class of compact (resp. $\check{C}$ech-scattered paracompact) spaces. The metric-fine coreflection $m$ can be directly extended to pre-uniformities of topological spaces. In our situation, we define the metric-fine modification $m\mu$ of a pre-uniformity $\mu$ as the filter of all covers refined by covers $\mathcal{V}$ for which there is a countable closed cover $\{F_n : n \in \mathbb{N}\}$ of the underlying space such that for each $n \in \mathbb{N}$, there is $U_n \in \mu$ with $U_n \upharpoonright F_n \prec \mathcal{V} \upharpoonright F_n$.

The covers in $\lambda\nu \mu$ are obtained by means of Noetherian trees as follows. There is a Noetherian tree $T$ and mapping $\varphi : T \to 2^X$ with the property that for each $p \in T$, the immediate successors $S_p$ of $p$ in $T$ are divided into countably many parts $S_p^{(n)}$ such that for each $n$, the image $\varphi(S_p^{(n)})$ is a uniform cover of the closed subspace $\bigcup(\varphi(S_p^{(n)}))$ of $\varphi(p)$, and these subspaces form a countable cover of $\varphi(p)$. 

Locally Fine Coreflection in Product Spaces
6. Countable Products of $\sigma$-partition-complete supercomplete spaces.

In our proof of Theorem 7.5 that the locally fine coreflection of the product $\lambda\Pi F X_i$ contains all the normal covers, we will need the result that each finite product is supercomplete†. We establish here a stronger result for countable products. Indeed, we don’t have to stop at paracompactness. We will prove a product theorem for regular spaces $X$ having supercomplete pre-uniformities (monoids of covers), i.e., filters $\mu$ of coverings such that each open cover of $X$ is in $\mu$. We first state the result for partition-complete spaces.

**Theorem 6.1:** Let $(X_i : i \in \mathbb{N})$ be a countable family of partition-complete, regular spaces, and for each $i$, let $\mu_i$ be a supercomplete pre-uniformity for $X_i$. Then $\lambda\Pi \mu_i X_i$ is a supercomplete pre-uniformity for the topological product $\Pi X_i$.

**Proof.** The proof is very similar to the one given in [20] for the case of countably many Čech-scattered paracompacta. Therefore, we will only give the main steps of the proof and refer the reader to [20] for details. Let $\mathcal{G}$ be an open cover of $\Pi X_i$. We may assume that the factors $X_i$ are the same space $X$. (Replace the factors by their disjoint union.) For each $i$, let $T$ be the tree $T_\phi(X)$, where $\phi$ is the stationary winning strategy. Let $T'$ be the perverse power $\Pi_\mathcal{P} T$, where $\mathcal{P}$ is the standard set of perversities defined above (Section 2). Finally, let $T''$ be the subtree of $T'$ consisting of all $P \in T'$ which do not have a predecessor $Q$ with a uniform neighbourhood $N$ such that $G$ is a uniform cover of $N$. (The term ‘uniform’ refers here to the related pre-uniformities.)

We claim that $T''$ is well-founded. Indeed, suppose to the contrary that $T''$ has an infinite chain $(P_n : n \in \mathbb{N})$. By the definition of perverse products, the projections $(\pi_i[P_n] : n \in \mathbb{N})$ are again sequences with infinitely many distinct terms. Thus there are, for each $i$, relatively open sets $E^{(i)}_n \subset X$ such that

$$\ldots \supset E^{(i)}_n \supset \pi_i[P_n] \supset E^{(i)}_{n+1} \supset \ldots,$$

indeed, we may write $E^{(i)}_{n+1} = \phi(\pi_i[P_n])$. Therefore, the intersection

$$K_i = \bigcap_{n \in \mathbb{N}} \pi_i[P_n]$$

is compact, and it follows that

$$K = \bigcap_{n \in \mathbb{N}} P_n = \Pi_{i \in \mathbb{N}} K_i$$

is compact, too.

For each $x \in K$, choose a basic open set $B_x$ such that $x \in B_x \subset \bar{B}_x \subset G_x$ for some $G_x \in \mathcal{G}$. By the compactness of $K$, there is a finite set $\mathcal{B} = \{B_x : x \in F\}$ which covers $K$. It is easy to find (finite) families $\mathcal{B}_i$ of open subsets of the spaces $X_i$ which cover the sets $K_i$ such that for some $s \in \mathbb{N},$

$$\mathcal{B}' = \left( \bigcap_{i=1}^{s} \pi_i^{-1}[\mathcal{B}_i] \right) \times \mathcal{B}.$$

† Nevertheless, it is not sufficient that all the finite products are supercomplete. It was shown in [14] that there is a subspace $X$ of the real numbers such that each finite power $(FX)^n$ is supercomplete but $(FX)^\omega$ is not. The space $X$ is not partition-complete. On the other hand, it was shown in [15] that there are non-analytic subspaces $X$ of $\mathbb{R}$ such that $\lambda((FX)^\kappa) = (FX)^\kappa$ for all $\kappa$. 

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Furthermore, by the regularity of $X_i$ we can assume there are open neighbourhoods $N_i, M_i$ of $K_i$ covered by $\mathcal{B}_i$ such that $\bar{N}_i \subset M_i$. As every open cover of $X_i$ is in $\mu_i$, it follows that $\mathcal{B}'$ is a uniform cover of the uniform neighbourhood

$$N = \bigcap_{i=1}^{s} \pi_i^{-1}[N_i]$$

of $K$, and therefore so is $\mathcal{G}$. Now for each $i \in \{1, \ldots, s\}$ there is $n_i$ such that $\pi_i[P_{n_i}] \subset N_i$. Put $n = \max\{n_1, \ldots, n_s\}$. Then it is easy to see that $P_n \subset N$, which yields a contradiction, because $P_{n+1}$ is an element of $T''$.

We prove by using the $\lambda$-neighbourhood induction lemma that $\mathcal{G}$ is a $\lambda$-uniform cover of the root of $T''$. Let $P$ be an element of $T''$. If $P$ is a maximal element of $T''$, then by the definition of $T''$, there is a uniform neighbourhood $N_P$ of $P$ such that $\mathcal{G}$ is a uniform cover of $N_P$. (More precisely, there is a cover $\mathcal{U}$ in $\Pi \mu_1$ such that $\mathcal{U} \setminus N_P \prec \mathcal{G} \setminus N_P$.) We proceed by induction and assume that every successor $Q$ of $P$ in $T''$ has a $\lambda$-uniform neighbourhood $N_Q$ such that $\mathcal{G}$ is a $\lambda$-uniform cover of $N_Q$. Since $T'$ is a perverse product, and since $T''$ is a lower part (an ideal) of $T'$, there is $i \in \mathbb{N}$ such that the immediate successors $Q$ of $P$ differ only with respect to $i$. For $j \neq i$, $\pi_j[P] = \pi_j[Q]$, while $\pi_i[Q]$ is an immediate successor of $\pi_i[P]$ in $T_\phi(X_i)$. We recall that $T(X)$ was obtained through a countable iteration of the trees $T_\phi(S)$ for closed subsets $S$ of $X$, defining it as the union of the trees $T_n$. There are two possibilities. Either $\pi_i[P]$ is a maximal element of some $T_n$ and its immediate successors all belong to $T_{n+1}$, or else both $\pi_i[P]$ and its immediate successors belong to the same tree of the form $T_\phi(S)$. However, in each case, the immediate successors are divided into two parts: a closed set $F \subset \pi_i[P]$ and all the closures $\bar{V}$ of non-empty open sets $V \subset \pi_i[P] \setminus F$ such that $\bar{V} \cap F = \emptyset$.

Let $Q_F = \pi_i^{-1}[F] \cap P$ be the immediate successor of $P$ that corresponds to $F$. By the induction hypothesis, $Q_F$ has a $\lambda$-uniform neighbourhood $N_F$ such that $\mathcal{G}$ is a $\lambda$-uniform cover of $N_F$. Then $P - N_F$ has a closed $\lambda$-uniform neighbourhood $V_F$ such that $V_F \cap Q_F = \emptyset$, and such that the binary cover $\{N_F, V_F\}$ is a $\lambda$-uniform cover of $P$. Indeed, there is a $\lambda$-uniform cover $\mathcal{V}$ of $X'$ such that $\text{St}(P - N_F, \mathcal{V}) \cap Q_F = \emptyset$. (Simply take $\mathcal{V}$ as a $\lambda$-uniform cover such that $\mathcal{V} \subset N_F$.) We can assume that $\mathcal{V}$ consists of the closures of basic open sets. This follows from the assumption that the spaces $X_i$ are regular and that every open cover of $X_i$ belongs to $\mu_i$. (Recall that the locally fine coreflection of a product of pre-uniformities has a basis of covers consisting of rectangular elements.) Then each element $V$ of $\mathcal{V} \setminus P$ that meets $P - N_F$ satisfies $\pi_i[V] \cap \pi_i[Q_F] = \emptyset$, which implies that $V$ is an immediate successor of $P$. But then by the inductive hypothesis, each $V \subset \mathcal{V} \setminus P$ with $V \cap (P - N_F) \neq \emptyset$ has a $\lambda$-uniform neighbourhood $N_V$ such that $\mathcal{G}$ is $\lambda$-uniform cover of $N_V$. It then follows that $P$ has a $\lambda$-uniform neighbourhood $N_P$ satisfying the condition of our claim. In fact, by the $\lambda$-neighbourhood induction lemma the set $N_P$ has a $\lambda$-uniform neighbourhood $N'_P$, such that $\mathcal{G}$ is a $\lambda$-uniform cover of $N'_P$. The sets $\mathcal{V}$ such that $\mathcal{V} \cap (P \setminus N_F) \neq \emptyset$ together with $N_F$ form a $\lambda$-uniform cover of $P$. As $\mathcal{G}$ is $\lambda$-uniform over the sets $N_V$ and $N'_P$, the set $P$ has by a new application of the lemma a $\lambda$-uniform neighbourhood $N_P$ with the desired property. \n
For the following corollaries, notice that the pre-uniformities generated by all locally finite, locally countable, point-finite, point-countable, and countable open covers are locally fine (i.e., stable under the operation $\lambda$).

**Corollary 6.2:** Let $(X_i : i \in \mathbb{N})$ be a countable family of regular paracompact (resp. para-Lindelöf, metacompact, meta-Lindelöf, Lindelöf) spaces. If the spaces $X_i$ are partition-complete, then the product space $\Pi X_i$ is paracompact (resp. para-Lindelöf, metacompact, meta-Lindelöf, Lindelöf).
Proof. Suppose that the spaces $X_i$ are paracompact and partition complete. Let $\mu_i$ be the pre-uniformity formed by all locally finite open covers of $X_i$. Then $\mu_i$ is supercomplete for $\Omega(X_i)$, because by paracompactness every open cover of $X_i$ can be refined by, and thus is, a member of $\mu_i$. By the theorem proved above, $\lambda\Pi\mu_i$ is supercomplete pre-uniformity for the topology of $\Pi X_i$. But $\Pi\mu_i$ has a basis consisting of products of finitely many locally finite open covers, and therefore so does $\lambda\Pi\mu_i$, because the pre-uniformity generated by the locally finite open covers is preserved by $\lambda$. It follows that every open cover of $\Pi X_i$ has an open locally finite refinement, and thus the product is paracompact. The other cases are proved in an analogous way.

This corollary can be stated also for other locally fine covering concepts, e.g., ultraparacompactness (refinement by clopen covers) etc. It was essentially proved by Plewe in [33]. He showed that the localic product of countably many partition-complete regular spaces is spatial. The result then follows from a theorem of Dowker and Strauss ([4]) that the localic product of regular paracompact (resp. metacompact, Lindelöf, ultraparacompact) spaces is again paracompact (resp. metacompact, Lindelöf, ultraparacompact). The case of paracompactness was already due to Isbell [24]. One could also obtain theorem 6.1 by combining Plewe’s result on spatiality and the relationship between $\lambda\Pi\mu_i$ and $\otimes\Omega(X_i)$, mentioned above. In addition to the topological corollaries, theorem 6.1 yields stronger combinatorial corollaries† for uniformities.

Corollary 6.3: Let $(\mu_i X_i : i \in \mathbb{N})$ be a countable family of supercomplete spaces. If the spaces $X_i$ are partition-complete, then the product space $\Pi\mu_i X_i$ is supercomplete.

Proof. As the spaces $\mu_i X_i$ are supercomplete, we have $\lambda\mu_i = \mathcal{F}(X_i)$ for each $i$. Therefore, by 6.1,

$$\lambda\Pi\mu_i = \lambda\Pi\lambda\mu_i = \lambda\Pi\mathcal{F}(X_i) = \mathcal{F}(\Pi X_i),$$

where we have used [22], Exercise VII 8a (p. 143).

As in [19], [20], we may extend the proof of 6.1 to the $\sigma$-partition-complete case. (Recall that a space is called $\sigma$-partition-complete if it is a countable union of partition-complete, closed subspaces.) Thus, we obtain the following result:

Theorem 6.4: Let $(X_i : i \in \mathbb{N})$ be a countable family of $\sigma$-partition-complete, regular spaces, and let $\mu_i$ be a supercomplete pre-uniformity for $X_i$. Then $\lambda m\Pi\mu_i$ is a supercomplete pre-uniformity for the topological product $\Pi X_i$.

Proof. Given a family $(X_i : i \in \mathbb{N})$ of regular spaces with supercomplete pre-uniformities $\mu_i$, each of which is a countable union of closed, partition-complete subspaces, we replace the trees $T_\phi(X_i)$ with a forest which contains a subtree $T_p(F_n)$ for each of the partition complete parts $F_n$. The perverse product construction ensures that only products of partition-complete parts are used. Since the partially ordered set $T''$ so constructed is Noetherian, $T''$ only has countably many ‘types’ of subproducts. We use the same induction as in 6.1, but we arrive, instead of the entire product of the $X_i$, at minimal elements $P$ which are subproducts and have $(\lambda$-uniform) neighbourhoods $N_P$ such that $\mathcal{G}^\omega$ is $\lambda$-uniform over $N_P$. As the successors of these minimal elements belong to the same type (i.e., their factors are contained in the same partition-complete parts), and there are only countably many types, we arrive at a countable, closed cover of the product such that $\mathcal{G}$ is $\lambda$-uniform over each member. Consequently, $\lambda(m\Pi\mu_i)$ is a supercomplete pre-uniformity.

† We also note that it gives a basis for effective product theorems, inasmuch as the factor spaces can be presented effectively.
7. Normal covers in uncountable products. In [30], the third author proved that every normal cover of an arbitrary product of complete metric spaces belongs to the locally fine coreflection of the corresponding product of the fine uniformities of the factor spaces. This result had been previously established for arbitrary products of Polish spaces (cf. [23], VII, Cor. 21.). In this section we will extend this to products of paracompact, partition-complete spaces. It turns out as in the preceding section that we may consider supercomplete pre-uniformities of regular partition-complete spaces.

The proof is based on simple combinatorial properties of ‘basic subsets’ of products. Let \((X_i : i \in I)\) be a family of topological spaces. The basic sets \(B \subset \Pi X_i\) are of the form \(B = \cap \{ \pi_i^{-1}[B_i] : i \in F \}\), where \(B_i \subset X_i\) and \(F \subset I\) is finite. The basic open sets are those for which each \(B_i\) is open in \(X_i\). Given a subset \(S \subset \Pi X_i\), denote by \(\mathcal{B}(S)\) the collection of all basic subsets \(B\) of \(\Pi X_i\) such that \(B \subset S\). If \(B \in \mathcal{B}(\Pi X_i)\), then \(I(B)\) is the set of all \(i \in I\) such that \(\pi_i[B] \neq X_i\).

We first recall lemmas from [30]. It is useful to begin with the following:

Lemma 7.1: Let \(B_1, B_2\) be basic sets in a product \(\Pi X_i\), and let \(F = I(B_1) \cap I(B_2)\). Suppose that \(B_1, B_2 \neq \emptyset\).

1) If \(F = \emptyset\), then \(B_1 \cap B_2 \neq \emptyset\).
2) If \(F \neq \emptyset\), then \(B_1 \cap B_2 \neq \emptyset\) if and only if \(\pi_F[B_1] \cap \pi_F[B_2] \neq \emptyset\).

Lemma 7.2 ([30]): If \(\{ B_n : n \in \mathbb{N} \}\) is a family of non-empty basic sets in \(\Pi X_i\) such that the index sets \(I(B_n)\) are mutually disjoint, then \(\bigcup \{ B_n : n \in \mathbb{N} \} = X_i\).

Proof. Indeed, for a point \(x \in X\) and for a basic open neighbourhood \(B\) of \(x\) there is \(n\) with \(I(B) \cap I(B_n) = \emptyset\). By the preceding lemma, this implies \(B \cap B_n \neq \emptyset\), and thus \(x\) belongs to the closure of the union of the \(B_n\).

The following is the essential inclusion lemma for regularly open subsets of products:

Lemma 7.3 ([30]): Let \(G\) be an open subset and let \(R\) be a regular open subset of \(X = \Pi X_i\). Suppose that there are \(E \subset I\) and a sequence \(\{ B_n : n \in \mathbb{N} \}\) of basic subsets of \(R\) such that \(\pi_E[B_n] \supset \pi_E[G]\) for all \(n\), and that the index sets \(I(B_n) \setminus E\) are mutually disjoint. Then \(G \subset R\).

Lemma 7.4: Let \(R\) be a regular open proper subset of \(X = \Pi X_i\). Then there is a finite set \(F(R) \subset I\) which meets \(I(B)\) for each \(B \in \mathcal{B}(R)\).

Proof. Suppose that the condition of the lemma is not satisfied. Thus, for each finite subset \(B' \subset \mathcal{B}(R)\) there is \(B' \subset \mathcal{B}(R)\) such that \(I(B') \cap \bigcup \{ I(B) : B \in B' \} = \emptyset\). Therefore, \(\{ I(B) : B \in \mathcal{B}(R) \}\) contains an infinite disjoint subfamily, and by Lemma 7.2 we have \(R = X\), which contradicts the assumption.

In this paper, we will use two extended forms of these lemmas for finite unions of basic sets. They follow from the above versions.

Lemma 7.1': If \(\{ B_n : n \in \mathbb{N} \}\) is a family of finite families of basic sets in \(X = \Pi X_i\) such that the index sets \(I(B_n) = \bigcup \{ I(B) : B \in B_n \}\) are mutually disjoint, then the union of the sets \(\bigcup \{ B_n \}\) is dense in \(X\).

Proof. Indeed, this is weaker than the statement for one-element collections.

Lemma 7.2': Let \(G\) be an open subset and let \(R\) be a regular open subset of \(X = \Pi X_i\). Suppose that there is a sequence \(\{ B_n : n \in \mathbb{N} \}\) of finite families of basic subsets of \(R\) such that the index sets \(I(B_n)\) are mutually disjoint relative to a subset \(E \subset I\) for which \(\pi_E[\bigcup \{ B_n \}] \supset \pi_E[G]\) for all \(n\). Then \(G \subset R\).
**Proof.** By the assumption, and by the preceding lemma,

\[ G \subset \pi_E[G] \times \Pi_{I \in \mathcal{E}} X_i \subset \bigcup_{n \in \mathbb{N}} \left( \bigcup (B_n) \right), \]

because \( \pi_E[G] \subset \bigcup \{ \pi_E[B] : B \in B_n \} \) and the basic sets \( B \) satisfy the condition \( B = \pi_E[B] \times \pi_{I \in \mathcal{E}}[B] \). Thus, \( G \subset \text{int}(\bar{R}) \), and hence \( G \subset R \), because \( R \) is regular open. \( \Box \)

From this we obtain the following corollary:

**Lemma 7.3':** Let \( G, R \) be open subsets of \( X = \Pi I X_i \), where \( G \) is not contained in \( \text{int}(\bar{R}) \). Let \( E \subset I \) be finite. Then there is a finite set \( F(R) \subset I \) such that given any finite family \( B \) of basic open subsets \( B \subset R \) with \( \pi_E[\bigcup (B)] \supset \pi_E[G] \), we have

\[ (F(R) \cap (\bigcup \{ I(B) : B \in B \})) \setminus E \neq \emptyset. \]

**Proof.** If no such \( F(R) \) existed, we could inductively construct a sequence \( \{ B_n : n \in \mathbb{N} \} \) of finite families \( B_n \) of basic open sets such that \( \pi_E[\bigcup (B_n)] \supset \pi_E[G] \) and for which the index sets \( I(B_n) \) are pairwise disjoint relative to \( E \). But then by the preceding lemma, we would have \( G \subset R^* \), contradicting the assumption. \( \Box \)

For an open subset \( S \) of a topological space \( X \), the regular open extension \( S^* \) is the subset \( \text{int}(\bar{S}) \). Suppose that \( \mathcal{V} \) is a normal cover, i.e., \( \mathcal{V} \) is uniformizable. Let \( V_1 \) be a normal cover with \( \mathcal{V}_1 \prec^{**} \mathcal{V} \), and let \( \mathcal{R} \) be a locally finite, open refinement of \( \mathcal{V}_1 \) (which every normal cover has). Then the regular open extension \( \mathcal{R}^* = \{ R^* : R \in \mathcal{R} \} \) of \( \mathcal{R} \) refines \( \mathcal{V} \). Indeed, let \( R \in \mathcal{R} \), and let \( R \subset V \in \mathcal{V}_1 \), \( \text{St}(V, V_1) \subset V' \subset \mathcal{V} \). Then \( \text{int}(\bar{R}) \subset \text{St}(V, V_1) \subset V' \). Moreover, \( \mathcal{R}^* \) is locally finite. In the following proof, we will replace the given normal cover \( \mathcal{V} \) by the regular open extension of \( \mathcal{R}^{<\omega} \), where \( \mathcal{R}^{<\omega} \) is the directed cover consisting of the finite unions of elements of \( \mathcal{R} \), and \( \mathcal{R} \) is as above. Indeed, let \( \mathcal{V}, \mathcal{V}_1 \) and \( \mathcal{R} \) be as above. We claim that the regular open extension \( (\mathcal{R}^{<\omega})^* \) of \( \mathcal{R}^{<\omega} \) refines \( \mathcal{V}^{<\omega} \). To see this, let \( R_1, \ldots, R_n \in \mathcal{R} \), where we have sets \( V'_i \in \mathcal{V}_1 \) such that \( R_i \subset V_i \) and \( \text{St}(V'_i, V_1) \subset V'_i \subset V \). Then

\[ \text{int}(\bar{R}_1 \cup \cdots \cup R_n) \subset (V'_1 \cup \cdots \cup V'_n)^* \subset V_1 \cup \cdots \cup V_n. \]

We are now able to state the main result of this section.

**Theorem 7.5:** Let \( (X_i : i \in I) \) be a family of partition-complete, completely regular spaces, and for each \( i \in I \), let \( \mu_i \) be a supercomplete uniformity for \( X_i \). Then every normal cover \( \mathcal{V} \) of \( X = \Pi I X_i \) belongs to \( \lambda \Pi I \mu_i \).

**Proof.** Let \( \mathcal{R} \) be a locally finite cover of \( X \) by regular open sets such that \( (\mathcal{R}^{<\omega})^* \prec \mathcal{V}^{<\omega} \), and let \( \mathcal{G} \) be a cover consisting of basic open sets such that each member of \( \mathcal{G} \) meets only finitely many members from \( \mathcal{R} \). We will show that \( \mathcal{V}^{<\omega} \) belongs to \( \lambda \Pi I \mu_i \). We conclude that every normal cover belongs to this product uniformity.

We may assume that the cover \( \mathcal{R} \) is non-trivial, i.e., \( \mathcal{R} \neq X \) for all \( R \in \mathcal{R} \). Given any \( G \in \mathcal{G} \), consider the finite set \( \mathcal{R}_G \subset \mathcal{R} \) such that \( G \cap R = \emptyset \) for all \( R \in \mathcal{R} \setminus \mathcal{R}_G \). Then \( B \in \mathcal{B}(R) \) for such an \( R \) implies \( G \cap B = \emptyset \) and hence by Lemma 7.1, \( I(G) \cap I(B) \neq \emptyset \). On the other hand, as \( \mathcal{R} \) is non-trivial, Lemma 7.4 implies that for each \( R \in \mathcal{R}_G \) there is a finite set \( F(R) \subset I \) which meets each \( I(B), B \in \mathcal{B}(R) \). Hence,

\[ F = \bigcup \{ F(R) : R \in \mathcal{R}_G \} \cup I(G) \]
is a finite set which satisfies $F \cap I(B) \neq \emptyset$ for all $B \in \mathcal{B}(R)$ and all $R \in \mathcal{R}$. Recall from 6.1 that for the associated finite product, $\lambda \Pi_{F \mu_i}$ contains every open cover of $\Pi_{F}X_i$.

Consider the open cover

$$W_F = \pi_F[G] \wedge \pi_F[B(R)]$$

of the subproduct $\Pi_{F}X_i$ of $X$. (Here $\mathcal{B}(R)$ denotes the family of all basic sets contained in some member of $\mathcal{R}$.) By the result just mentioned, $W_F$ belongs to $\lambda(\Pi_{F \mu_i})$. Hence, there is a Noetherian tree $T_F$ consisting of subsets of $\Pi_{F}X_i$ satisfying the uniform refinement condition with respect to $\Pi_{F \mu_i}$ and for which $\text{End}(T_F) \prec W_F$. Moreover, we may assume that the elements of $T_F$ are closures of basic open sets.

We extend $T_F$ to a Noetherian tree $T'_F$ by adding, for each end element $P$, the perverse product of the trees $T_0(P)$ (with respect to the index set $F$) below $P$, where $P = \cap \{\pi_i^{-1}[P_i] : i \in F\}$. (We order the subset $F$ as $\{x_1, \ldots, x_n\}$ and use the the same perversities as in 6.1. The finite enlargements $F(P)$ defined below will extend this order. Notice that this limits the level of the elements from $T_0(P)$ to $n$.)

Finally, we extend $T'_F$ to a tree $T_0$ in $\mathcal{P}(\Pi_{F}X_i)$ by crossing each element with $\Pi_{F \setminus X_i}$. Then this new tree satisfies the uniform refinement condition with respect to $\Pi_{F \mu_i}$.

Let $\text{End}^*T_0$ be the set of all end elements $P$ of $T_0$ for which there is no $R' \in (\mathcal{R}^{<\omega})^*$, say $R' = (R_1 \cup \cdots \cup R_n)^*$, with open basic subsets $B \in \mathcal{B}(R)$ such that $B_i \subset R_i$ and $P \subset B_1 \cup \cdots \cup B_n$. If $\text{End}^*T_0 \neq \emptyset$, we continue the inductive definition of $T$. Let $P \in \text{End}^*T_0$. Then $\pi_F[P]$ refines, by the definition of $T_F$, the cover $\pi_F[G]$; let $G \in \mathcal{G}$ be such that $\pi_F[G] \supset \pi_F[P]$. On the other hand, $\pi_F[P]$ also refines $\pi_F[W]$; let $\pi_F[B] \supset \pi_F[P]$, where $B$ is a basic subset and $B \subset R$ for some $R \in \mathcal{R}$.

Denote by $\mathcal{R}(P)$ the set of all finite subsets $\mathcal{R}' \subset \mathcal{R}$, say $\mathcal{R}' = \{R_1, \ldots, R_n\}$, for which there are $B_i \in \mathcal{B}(R')$ such that

$$\pi_F[B_1 \cup \cdots \cup B_n] \supset \pi_F[P]$$

and additionally $B_i \cap P \neq \emptyset$ for all $i \in \{1, \ldots, n\}$. Thus, $\mathcal{R}(P) \neq \emptyset$, established by the one-element subsets. The set $\mathcal{R}_G$ of all $R \in \mathcal{R}$ which meet $G$ is finite. Consider an arbitrary element $\mathcal{R}' \in \mathcal{R}(P)$, $\mathcal{R}' = \{R_1, \ldots, R_n\}$, and a corresponding set $\{B_1, \ldots, B_n\}$ of basic sets $B_i \in \mathcal{B}(R_i)$ as defined above. In case $\mathcal{R}' \not\subset \mathcal{R}_G$, we have $R_i \cap G = \emptyset$ for some $R_i \in \mathcal{R}'$. But then $\pi_F[B_i] \cap \pi_F[G] \neq \emptyset$ and hence by Lemma 7.1’ we have $I(B_i) \cap I(G) \not\subset F$. Writing $\tilde{B} = B_1 \cup \cdots \cup B_n$ and $I(\tilde{B}) = I(B_1) \cup \cdots \cup I(B_n)$, we have $I(\tilde{B}) \cap I(G) \not\subset F$ for each such subset $\mathcal{R}' \subset \mathcal{R}_G$ and such basic sets $B_i \in \mathcal{B}(R_i)$.

On the other hand, given $\mathcal{R}' \subset \mathcal{R}_G$, Lemma 7.3’ implies that there is a finite set $F(\mathcal{R}') \subset I$ such that

$$(F(\mathcal{R}') \cap I(\tilde{B})) \cap (I \setminus F) \neq \emptyset,$$

whenever $\tilde{B}$ and $I(\tilde{B})$ are as above. (Because otherwise we would have $P \subset (\cup \mathcal{R}')^*$, which is ruled out by the assumption $P \in \text{End}^*(T_0)$.) Put

$$F(P) = \cup \{F(R) : \mathcal{R}' \subset \mathcal{R}_G \} \cup I(G) \cup F,$$

and note that $F(P)$ is finite. Let

$$W_F = \pi_{F(P)}[G] \wedge \pi_{F(P)}[\mathcal{B}(R)].$$

As before, there is a Noetherian tree $T_P$ consisting of subsets of $P$ satisfying the uniform refinement condition with respect to $(\Pi_{F(P)\mu_i}) \upharpoonright P$ and such that $\text{End}(T_P) \prec W_F$. For the following and subsequent perverse products, we order the set $F(P) \supset F$ extending the order of $F$. Extend $T_P$ to a Noetherian tree
T_p$ by adding the perverse product of the trees $T_{\phi_i}(\pi_i(Q))$ above the elements $Q$, where $Q \in \text{End}(T_p)$ and $i \in F(P)$. As above, we enlarge $T_p'$ to a Noetherian tree in $\mathcal{P}(\Pi_i X_i)$ by crossing the elements of $T_p'$, with the product of the $X_i$ for all the remaining factors. When this is done for each $P \in \text{End}^*(T_0)$, we obtain a Noetherian tree $T_1$.

In general, if $n > 0$ and $T_n$ has been constructed, then the inductive step from $T_n$ to $T_{n+1}$ is entirely analogous to the above construction: We consider the set $\text{End}^*(T_n)$ of all end elements $P$ for which there is no finite subset $\mathcal{R}' \subset \mathcal{R}$ with $P \subset (\cup \mathcal{R}')^*$. Given such an element $Q$, there is a unique $P \in \text{End}^*(T_{n-1})$ below $Q$, and we define $\mathcal{R}(Q)$ as the set of all finite subsets $\mathcal{R}' = \{R_1, \ldots, R_n\} \subset \mathcal{R}$ for which there are basic open sets $B_i \subset R_i$ such that $\pi_{F(P)}[B_1 \cup \cdots \cup B_n] \supset \pi_{F(P)}[Q]$ and $B_i \cap Q \neq \emptyset$ for all $i$. The construction of $T_Q'$ then proceeds as above. We set

$$T_{n+1} = \cup\{T_Q' : Q \in \text{End}^*(T_n)\}.$$ 

Finally, we put

$$T = \cup\{T_n : n \in \mathbb{N}\}.$$ 

We claim that $T$ is Noetherian. Indeed, suppose to the contrary that $T$ has an infinite chain $(Q_n : n \in \mathbb{N})$. Now each $T_n$ is Noetherian, so there is in fact an infinite chain $(P_n : n \in \mathbb{N})$ of elements $P_n \in \text{End}^*(T_n)$ such that $P_k \supset Q_{n(k)}$. We have $P_{n+1} \subset P_n$ and $F(P_n) \subset F(P_{n+1})$. Let $J = \cup\{F(P_n) : n \in \mathbb{N}\}$. The ordering of $T$ along the chain $(P_n)$ is, because of the perverse orderings of the added trees $T'_p$, an ordering for which there is an infinite sequence $(\pi_i(P_{n(i)}))$ of subsets of $X_i$ for all $i \in J$ such that $\pi_i(P_{n+1}(i)) \subset \pi_i(P_{n(i)})$. By the construction there are relatively open sets $U_{n,i} = \phi_i(\pi_i(P_{n(i)}))$ such that

$$\pi_i(P_{n+1}(i)) \subset U_{n,i} \subset \pi_i(P_{n(i)}),$$

where $\phi_i$ denotes the stationary winning strategy associated with the space $X_i$.

It follows that each

$$K_i = \bigcap_{n \in \mathbb{N}} \pi_i(P_{n(i)})$$

is non-empty and compact. As $I(P_n) \subset J$ for all $n$, and each $P_n$ is basic, we obtain

$$\cap_{n \in \mathbb{N}} P_n = \pi_J[\cap_{n \in \mathbb{N}} P_n] \times \Pi_{I \setminus J} X_i \neq \emptyset$$

and the projection image (denote it by $K$) is compact.

Each $x \in \cap P_n$ has a basic open set $B_x$ such that $x \in B_x \subset R_x$ for some $R_x \in \mathcal{R}$. There is a minimal finite set $E$ such that

$$K = \pi_J[\cap P_n] \subset \cup\{\pi_J[B_x] : x \in E\}.$$ 

In fact, there is $n$ such that $k \geq n$ implies

$$\pi_J[P_k] \subset \cup\{\pi_J[B_x] : x \in E\}.$$ 

(To see this, assume that each $\pi_J[P_k]$ meets the complement of $\cup\{\pi_J[B_x] : x \in E\}$, say, in a closed set $A_k$. Then for each $i \in J$, the sets $\pi_i[A_k]$ form a filter base finer than $\cup(U_{n,1})$, and thus have cluster points $x_i \in K_i$. But $K$ is the product of the $K_i$ and the point $x = (x_i)_{i \in J}$ of $K$ has a neighbourhood which does not meet any member of the filter base $(A_k)$, contradicting that $x$ is a cluster point of the latter.)
Let \( R' = \{R_1, \ldots, R_n\} \), let \( \tilde{B} = \bigcup\{B_x : x \in E\} \) and write \( I(\tilde{B}) = \bigcup\{I(B_x) : x \in E\} \). We have \( R' \in R(P_k) \) for all \( k \geq n \).

Clearly \( \pi_{F(P_k)}[P_k] \subset \pi_{F(P_k)}[\tilde{B}] \) for \( k \geq n \). On the other hand, \( P_{k+1} \) does not refine \( \cup(R') \) (because it belongs to \( \text{End}^*(T_{k+1}) \)), and we have

\[
(F(R') \cap I(\tilde{B})) \setminus F(P_k) \neq \emptyset.
\]

But the definition of \( F(P_{k+1}) \) then implies

\[
(F(P_{k+1}) \cap I(\tilde{B})) \setminus F(P_k) \neq \emptyset.
\]

This is valid for all \( k \geq n \), and hence gives the contradiction that \( I(\tilde{B}) \) is infinite. Thus, \( T \) is Noetherian, as claimed.

The end elements \( P \in \text{End}(T) \) form a cover of the product space, and refine the cover \( (R^{<\omega})^* \). The construction of \( T \) implies that the cover \( \text{End}(T) \), and hence \( (R^{<\omega})^* \), belongs to \( \lambda(\Pi_{i \in I} \mu_i) \), and therefore so does \( \mathcal{V}^{<\omega} \). As this is valid for all normal covers \( \mathcal{V} \) of the product space, we conclude that \( \mathcal{V} \in \mu/p\mu \), where \( \mu = \lambda(\Pi_{i \in I} \mu_i) \) and \( p\mu \) denotes the uniformity consisting of all finite covers \( \mathcal{U} \in \mu \) (see [35]). But \( \mu/p\mu \subset \lambda\mu = \mu \) and hence \( \mathcal{V} \in \mu \), as desired. \( \blacksquare \)

**Remark 7.6:** Note that while the counterpart of Theorem 7.5 for countably many factors – 6.1 – is stated for regular spaces and pre-uniformities, Theorem 7.5 is 'restricted' to uniform spaces. This is needed in the last paragraph of the proof, in order to apply [35]. In case we considered directed covers, we could state: Every directed normal cover of a product \( \Pi X_i \) of regular spaces belongs to \( \lambda \Pi \mu_i \), where the \( \mu_i \) are complete pre-uniformities on the spaces \( X_i \).

8. Additional remarks: The locally fine coreflection and ‘formal topology’.

One of the motivations for studying ‘supercompleteness’ of products \( \Pi \mathcal{F} X_i \) instead of the mere paracompactness of the corresponding topological products is the recursive or inductive construction of the refinements of open covers from the factor covers. This was the motivation behind the series [11], [13], [14], [15], [19], [20] independently of the questions of ‘formal spaces’ ([5], [37]). A similar motivation can be found in the inductive derivation of covers for a constructive proof of Tychonoff’s Theorem in the context of frames in [29]. Effective presentations of formal spaces have been studied in [38].

On the other hand, the ‘spatiality’ (see [24]) (where it was called ‘primality’),[25], [32]) of the localic products \( \otimes \Omega(X_i) \) is equivalent to the condition

\[
(\ast) \quad \lambda(\Pi \mathcal{F} X_i) = \mathcal{F}(\Pi X_i)
\]

whenever the product is paracompact. (Here \( \Omega(X) \) denotes the topology of \( X \) as a locale. The spatiality of the product \( \otimes \Omega(X_i) \) means that it is isomorphic to the locale \( \Omega(\Pi X_i) \) over the topological product.) Indeed, these three fields have been brought together in a recent result of the first-named author [16] in the following sense: For a space \( X \), let \( \mathcal{O}(X)^* \) denote the (fine) monoid of all covers refinable by an open cover. Given a family \( (X_i) \) of regular spaces, the localic product \( \otimes \Omega(X_i) \) is spatial if, and only if, the equation

\[
(\ast\ast) \quad \lambda \Pi \mathcal{O}(X_i)^* = \mathcal{O}(\Pi X_i)^*
\]

holds. This is an analogue of (\(\ast\)), but without any reference to paracompactness.
Formal spaces essentially are counterparts of locally fine pre-uniformities in pre-orders \((P, \preceq)\) (partial order without antisymmetry), defined by ‘covering relations’ \(\mu \subset P \times 2^P\) which satisfy the following axioms:

1) If \(a \in U\), then \(\mu(a, U)\).
2) If \(a \preceq b\), then \(\mu(a, \{b\})\).
3) If \(\mu(a, U)\) and \(\mu(a, V)\), then \(\mu(a, U \land V)\), where \(U \land V\) denotes the set of all \(b \in P\) majorized by both \(U\) and \(V\).
4) If \(\mu(a, U)\) and for all \(u \in U\), \(\mu(u, V)\), then \(\mu(a, V)\).

A set \(U \in 2^P\) such that \((a, U) \in \mu\) is considered a ‘cover’ of the element \(a \in P\). By the above axioms, the covers of \(a\) form a ‘locally fine’ filter for each \(a \in P\). (For a pre-uniformity \(\mu\), one takes a filter of covers of the maximal element (the underlying set)).

Their ‘effective constructions’ are connected – via the above theorem – with the constructions for locally fine coreflections of products. However, we leave the details of obtaining effective refinements of open covers of products to future papers. Let it be mentioned, however, that the definition of the product of covering relations is reflected in the perverse product of trees. In the application of perverse products in Section 6, the immediate successors of a given element vary with respect to one coordinate only. Furthermore, the set of perversities is given effectively, and so is the associated product, relative to the factors.

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