ON SOME RECENT RESULTS ABOUT
INERTIAL MANIFOLDS AND KINEMATIC
DYNAMOS

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Abstract
The conditions imposed in the paper ['Inertial manifolds and completeness of eigenmodes for unsteady magnetic dynamos', Physica D 194 (2004) 297-319] on the fluid velocity to guarantee the existence of inertial manifolds for the kinematic dynamo problem are too demanding, in the sense that they imply that all the solutions tend exponentially to zero. The inertial manifolds are meaningful because they represent different decay rates, but the classical kinematic dynamos where the magnetic field is maintained or grows are not covered by this approach, at least until more refined estimates are found.

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1 Introduction
In [1], the existence of inertial manifolds for the kinematic dynamo problem under certain conditions is proved. The result is applied to the case of time-periodic flows in a spatially periodic box Ω, showing that any solution may be
represented as the sum of an exponentially decreasing function plus a finite sum of Floquet-like terms: time exponentials times functions periodic in time. This term corresponds to the solutions within a finite-dimensional inertial manifold \( \mathcal{M}(t) \). The hypotheses needed to prove the existence of \( \mathcal{M}(t) \) depend, as usual, on a spectral gap condition: the eigenvalues \( \mu_N \) of the Stokes operator must satisfy for some \( N \)

\[
\frac{2}{\sqrt{\mu_{N+1}} - \sqrt{\mu_N}} + \frac{1}{\sqrt{\mu_{N+1}}} < \frac{\eta}{3w_0},
\]

where \( \eta \) is the magnetic diffusivity and \( w_0 \) is the following norm on the velocity: assume that both the velocity \( v \) and its gradient \( \nabla v \) are uniformly bounded for all time in \( \Omega \), and let \( v_0 \) and \( u_0 \) be their respective maxima. Then \( w_0 = v_0 + u_0 \mu_1^{-1/2} \).

Inertial manifolds are often elusive objects in fluid dynamic problems, and this case is no exception. It is apparent that (1) is rather demanding, given the extremely small diffusivity occurring in realistic dynamo problems. We will show that in several cases, including the examples in [1], (1) implies that all the solutions tend exponentially to zero. In fact the conditions for this to occur are weaker than (1). Hence all the Floquet exponents have negative real part.

The inertial manifolds are still interesting because the decay rate within them is different from the decay rate transverse to them, but the original object of kinematic dynamo theory, which was to find velocity fields such that the magnetic field associated to them was maintained, or better grew exponentially, cannot be achieved with these examples: finer estimates are needed.

Although the authors restrict themselves to the space periodic case, their methods seem adaptable with minor modifications upon (1) to other boundary value problems, such as Dirichlet ones. We will also comment briefly upon this case in order to illustrate the general situation.

## 2 Energy inequalities

The induction equation satisfied by the magnetic field \( B \)

\[
\frac{\partial B}{\partial t}=\eta \nabla^2 B - \mathbf{v} \cdot \nabla B + B \cdot \nabla \mathbf{v},
\]
to which it must be added $\nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{v} = 0$ and adequate boundary conditions: $\mathbf{B}$ and $\mathbf{v}$ periodic, $\mathbf{B}$ of mean zero in $\Omega$ in the periodic case, $\mathbf{B} \mid_{\partial \Omega} = \mathbf{0}$ in the Dirichlet case. Energy inequalities are obtained by the standard method of multiplying (2) by $\mathbf{B}$ and integrating in $\Omega$. The diffusive term equals

$$\eta \int_\Omega \nabla^2 \mathbf{B} \cdot \mathbf{B} \, dV = \eta \int_{\partial \Omega} \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial n} \, d\sigma - \eta \int_\Omega |\nabla \mathbf{B}|^2 \, dV. \quad (3)$$

and the boundary term vanishes for these boundary conditions: obviously for the Dirichlet case, and in periodic problems because $\mathbf{B}$ is periodic and the normal vector antiperiodic at opposite sides of the box. Also the lagrangian term vanishes:

$$\frac{1}{2} \int_\Omega (\mathbf{v} \cdot \nabla \mathbf{B}) \cdot \mathbf{B} \, dV = \frac{1}{2} \int_\Omega \mathbf{v} \cdot \nabla |B|^2 \, dV = \frac{1}{2} \int_{\partial \Omega} \mathbf{B} \cdot \mathbf{n} \, d\sigma = 0. \quad (4)$$

As for the remaining term, it may be written in two ways. Directly

$$\int_\Omega \mathbf{B} \cdot \nabla \mathbf{v} \cdot \mathbf{B} \, dV, \quad (5)$$

or, after integration by parts,

$$\int_\Omega \mathbf{B} \cdot \nabla \mathbf{v} \cdot \mathbf{B} \, dV = \int_\Omega \mathbf{B} \cdot \nabla (\mathbf{B} \cdot \mathbf{v}) - \mathbf{B} \cdot \nabla \mathbf{B} \cdot \mathbf{v} \, dV$$

$$= \int_\Omega (\mathbf{B} \cdot \mathbf{v}) \mathbf{B} \cdot \mathbf{n} \, d\sigma - \int_\Omega \mathbf{B} \cdot \nabla \mathbf{B} \cdot \mathbf{v} \, dV, \quad (6)$$

and again the boundary integral vanishes. All this is classical (see e.g. [2]). Therefore, for any $\alpha \in [0, 1]$, we may write

$$\frac{1}{2} \frac{\partial}{\partial t} \int_\Omega B^2 \, dV = -\eta \int_\Omega |\nabla \mathbf{B}|^2 \, dV$$

$$+ \alpha \int_\Omega \mathbf{B} \cdot \nabla \mathbf{v} \cdot \mathbf{B} \, dV - (1 - \alpha) \int_\Omega \mathbf{B} \cdot \nabla \mathbf{B} \cdot \mathbf{v} \, dV. \quad (7)$$

Denoting by $| \cdot |$ the $L^2(\Omega)$-norm, and using elementary bounds,

$$\frac{1}{2} \frac{\partial}{\partial t} |\mathbf{B}|^2 \leq -\eta |\nabla \mathbf{B}|^2 + \alpha u_0 |\mathbf{B}|^2 + (1 - \alpha) v_0 |\mathbf{B}||\nabla \mathbf{B}|. \quad (8)$$

As asserted, denote by $0 < \mu_1 < \mu_2 < \ldots$ the eigenvalues of the Stokes operator (which coincide with those of minus the laplacian) in the space under consideration: $H^2_{per}(\Omega)$ for the periodic case, $H^2(\Omega) \cap H_0^1(\Omega)$ for Dirichlet conditions. Then

$$|\nabla \mathbf{B}|^2 = \sum_{j=1}^\infty \mu_j |\mathbf{B}_j|^2 \geq \mu_1 \sum_{j=1}^\infty |\mathbf{B}_j|^2 = \mu_1 |\mathbf{B}|^2, \quad (9)$$
where $B_j$ is the $j$-th component of the field in the orthogonal base of eigenvectors of the Stokes operator. Hence

$$\frac{1}{2} \frac{\partial}{\partial t} |B|^2 \leq -\eta \nabla B^2 + \alpha u_0 \mu_1^{-1} |B|^2 + (1 - \alpha) v_0 \mu_1^{-1/2} |\nabla B|^2. \quad (10)$$

If $-k = -\eta + \alpha u_0 \mu_1^{-1} + (1 - \alpha) v_0 \mu_1^{-1/2} < 0$, we have an inequality

$$\frac{1}{2} \frac{\partial}{\partial t} |B|^2 \leq -k |\nabla B|^2 \leq -k \mu_1 |B|^2, \quad (11)$$

which implies that any solution decays exponentially in the $L^2(\Omega)$-norm.

### 3 Analysis of the estimates

Obviously the condition

$$\alpha u_0 \mu_1^{-1} + (1 - \alpha) v_0 \mu_1^{-1/2} < \eta, \quad (12)$$

holds for some $\alpha \in [0, 1]$ if and only if it holds at some of the extremes of the interval, i.e. $u_0 \mu_1^{-1} < \eta$ or $v_0 \mu_1^{-1/2} < \eta$. Notice that both of these quantities are smaller than $w_0 \mu_1^{-1/2}$.

Let us make a small insert to comment that neither the estimates in (1) nor the previous ones are modified by scale changes. This is because if $B(t, x)$ is a solution of the kinematic dynamo problem in $\Omega$ with velocity $v(t, x)$ and the boundary conditions, the solution in $R\Omega$ is $B(R^{-2}t, R^{-1}x)$, associated to the velocity $R^{-1}v(R^{-2}t, R^{-1}x)$. The eigenvalues of the Stokes operator become now $R^{-2} \mu_N$. Hence the new value of $v_0$ is $R^{-1}v_0$, the one of $u_0 \mu_1^{-1/2}$ is $R^{-1}u_0 \mu_1^{-1/2}$, and therefore (1) holds equally. Also the new values of $v_0 \mu_1^{-1/2}$ and $u_0 \mu_1^{-1}$ coincide with the previous ones, so that any of the bounds on them holds. Hence we may restrict ourselves to domains of fixed size when studying these problems.

Let us compare (12) with (1). For Dirichlet problems in general domains, the classical theorem of Rayleigh-Faber-Krahn states that the domain with minimal $\mu_1$ among those of given measure is given by the ball. Thus we can restrict ourselves to balls of radius 1, whose first eigenvalues are given by the squares of the smallest zero of the Bessel functions in dimension two, or the
spherical Bessel functions in dimension three. Since those are well known, we can assert
\[ \mu_1^{-1} u_0, \mu_1^{-1/2} v_0 < \frac{1}{2.4048} w_0, \]  
(13)
in dimension two, and
\[ \mu_1^{-1} u_0, \mu_1^{-1/2} v_0 < \frac{1}{\pi} w_0, \]  
(14)
in dimension three. Hence any estimate of the form \( w_0 < r\eta \) is improved by \( v_0 \mu_1^{-1/2} \) and \( u_0 \mu_1^{-1} \).

For periodic problems, all the eigenvalues are well known. In particular, for the case studied in [1] of square two and three-dimensional boxes, \( \mu_1 = 1 \) and \( \sqrt{\mu_{N+1}} - \sqrt{\mu_N} \leq 1 \). Using this, it is proved in the paper that an inertial manifold exists in dimension two if \( w_0 < \eta/6 \), in dimension three if \( w_0 < \eta/12 \). This obviously implies that (12) holds even for all \( \alpha \), and by a large margin. Hence all solutions decay exponentially. Thus the examples do not cover kinematic dynamos with nondecaying magnetic fields, but this should not detract from the fact that the argument is correct. The task is to refine the estimates in (1) so that they are weaker than the conditions for general decay. Let us mention that a solution bounded in \( L^2 \)-norm is also uniformly bounded: see [5].

4 Conclusions

The conditions put forward in [1] for the existence of finite-dimensional inertial manifolds for the kinematic dynamo problem in the space periodic case turn out to be so strong that all the solutions tend exponentially to zero. A similar situation is likely to occur for other boundary conditions. Therefore the results cannot be directly applied to classical kinematic dynamos where the magnetic field is at least maintained, at least until refined estimates are found.

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