Singularity scattering laws for bouncing cosmologies: a brief overview

Philippe G. LeFloch

Abstract. For contracting/expanding bouncing cosmologies, the formulation of junction conditions at a bouncing was recently revisited by the author in collaboration with B. Le Floch and G. Veneziano. The regime of interest here is the so-called quiescent regime, in which a monotone behavior of the metric is observed and asymptotic expansions can be derived. Here, we overview our new methodology based on the notion of singularity scattering maps and cyclic spacetimes, and we present our main conclusions. In particular, we provide a classification of all allowed bouncing junction conditions, including three universal laws.

Keywords: Einstein equation; weak solution; singularity hypersurface; bouncing cosmology; singularity scattering map; cyclic spacetime.

1 Introduction

Self-gravitating matter fields. We overview recent developments [16,17,18] on weak solutions to Einstein’s field equations, established in collaboration with B. Le Floch (ENS, Paris) and G. Veneziano (CERN, Geneva), concerning the formulation of scattering laws allowing one to pass from a contracting phase to an expanding one, across a singularity hypersurface. The interest of the author for weak solutions to the Einstein equations began fifteen years ago [25,27,28,29], and continued until recent years in collaboration with B. Le Floch [12–15].

Since this is only a brief overview of the subject, only a few results and comments are discussed and the reader is referred to the cited papers for further background material and results on the subject.

We are interested in self-gravitating matter fields and physically realistic models, whose solutions may involve (possibly impulsive) gravitational waves (deformations of the spacetime geometry), shock waves (in the fluid), and phase transition interfaces (for complex fluids). Such waves are represented by singularity hypersurfaces across which the solutions to the Einstein equations exhibit a jump discontinuity or even a blow-up. We need various techniques of geometric analysis and mathematical physics, including Lorentzian geometry (in presence
of metrics with weak regularity) and arguments from the theory of partial differential equations (of nonlinear hyperbolic or elliptic type). We also pay attention to the underlying physical modeling, including (possibly modified) gravity modeling and continuum physics.

**New methodology.** In [16], we proposed a new (mathematical) perspective on bouncing cosmologies, that is, spacetimes containing contracting and expanding phases of big crunch and big bang type connected together by a bounce, which is regarded as a singularity hypersurface across which the small-scale physics has been “factored out” (see Section 5.4). On this subject, a very large literature exists, for instance by Ashtekar, Brandenberger, Ijjas, Gasperini, Lübbe, Pawłowski, Penrose, Peter, Steinhardt, Tod, Turok, Veneziano, Wilson-Ewing, and others. We do not attempt to review this literature and refer the reader to [2,3,6,7,34,37,38,40] and the references cited therein.

The framework proposed in [16,17,18] relies on a systematic study of bouncing junctions at geometric and fluid interfaces. For regular junction hypersurfaces, one can use Israel’s junction conditions [4], about which we refer the reader to Marc and Senovilla [31].

To deal with singularity hypersurfaces, we begin by analyzing the degrees of freedom and constraints. In the regime of “quiescent” cosmology (cf. Barrow [5] and Andersson and Rendall [1]), spacetimes have a monotone behavior (as opposed to BKL oscillations identified by Belinsky, Khalatnikov, Lifshitz) and asymptotic expansions of Fuchsian type can be established. The quiescent behavior on gravitational singularities is observed for large classes of matter models as well as for the vacuum Einstein equations in high dimensions, or for spacetimes admitting certain symmetries (for instance $T^2$ symmetry).

A classification of bouncing laws is established in [16], which is based on analyzing the scattering phenomena near a singularity hypersurface and formulating junction conditions via universal or model-dependent laws. This also naturally leads to the construction of cyclic spacetimes describing the collision of two gravitational waves beyond singularities, and to the resolution of the global plane-wave collision problem, as we call it.

We will not discuss the plane-symmetric problem here and we refer the reader to [18] as well as the earlier works [8,9].

**Outline of this paper.** In Section 2, a brief presentation of recent results in mathematical general relativity is given. In Section 3, we discuss our methodology in order to deal with spacetimes with singularity hypersurfaces. Next, in Section 4, we introduce the notions of scattering maps and cyclic spacetimes and we state our local existence theory. Finally, Section 5 is devoted to the presentation of the classification of scattering maps.
2 Global nonlinear stability of Einstein spacetimes

2.1 Background

The initial value problem. The evolution problem for the Einstein equations is formulated as follows. The unknown is a Lorentzian four-manifold \((M, g)\) with signature \((-1,1,1,1)\) satisfying the field equations

\[ G = 8\pi T, \]  

supplemented with a prescribed data set representing the initial geometry and matter content. We are given a Riemannian 3-manifold \((M_0, g_0, k_0)\), representing a hypersurface embedded in the spacetime, together with a scalar matter density field \(\rho_0\) and a vector field \(J_0\), which should satisfy Einstein’s constraint equations, namely the Hamiltonian equation

\[ \text{Scal}_{g_0} + |k_0|^2 - \text{Tr}(k_0^2) = 16\pi \rho_0 \]  

and the momentum equation

\[ \text{div}_{g_0}(k_0 - \text{Tr}(k_0)g_0) = 8\pi J_0. \]  

Under a suitable gauge choice, the last two equations form a nonlinear elliptic equation while from the Einstein equations (2.1) one can also extract a nonlinear wave system satisfied by the metric.

The global nonlinear stability problem. Let us review a few results on the nonlinear stability of vacuum spacetimes under small perturbations. The theory was restricted to vacuum spacetimes having \(T = 0\) until recently (see next paragraph) and has a long history beginning in the 1990 with the pioneering contribution on Minkowski spacetime by Christodoulou and Klainerman, followed by important work by Bieri, Lindblad, Rodnianski, Hintz, Vasy, and others. More recently, further advances were made concerning the nonlinear stability of the Schwarzschild (stationary black hole) and Kerr (rotating) black hole spacetimes (Dafermos, Holzegel, Klainerman, Szeftel, Rodnianski, and others). In these works, the global dynamics of small perturbations of a given geometry are studied, and the analysis relies on numerous mathematical techniques for nonlinear wave equations and nonlinear elliptic equations: linearized stability, dispersive estimates, nonlinear structure, time decay, etc.

Matter fields, low decay, and singularities. Our objective in the present overview is to go beyond vacuum spacetimes. Several directions are of interest. In analyzing matter spacetimes as well as spacetimes beyond asymptotic symmetry. The author recently treated three types of interrelated problems: the nonlinear stability of Klein-Gordon fields \([21]–[24]\), the Einstein constraints beyond spherical symmetry \([26]\), and the evolution in presence of singularity hypersurfaces \([16,17,18]\), which is our main aim for the present overview.
2.2 Self-gravitating massive matter field

The global dynamics of massive fields. In presence of a massive scalar field, the Einstein equations exhibit a very complex dynamics, and analyzing the decay properties at timelike, null, and spacelike infinity is a very challenging problem. The first results on the nonlinear stability of self-gravitating massive matter fields were established in recent years by LeFloch and Ma [24] and by Ionescu and Pausader [10,11]. Two independent and very different proofs are thus now available. The method proposed by Ionescu and Pausader is based on the technique of spacetime resonances, originally developed (for simpler wave problems) by Shatah, Germain, and Masmoudi; see the references in [10,11]. The simpler class of solutions coinciding with the Schwarzschild spacetime outside a light cone was analyzed earlier in independent works by LeFloch and Ma [22,23] and by Wang [39].

In [21] and then in [24], a new vector field method is introduced, which we call the “Euclidian-hyperboloidal foliation method” and is relevant in order to solve the global existence problem for a broad class of coupled systems of nonlinear wave and Klein-Gordon equations. For a precise statement we refer the reader to [24] and we only sketch our main conclusion here.

Nonlinear stability of self-gravitating massive fields: informal version

Consider the Einstein equations coupled to a Klein-Gordon field $\phi$, satisfying therefore the evolution equation

$$-\Box_g \phi + m^2 \phi = 0.$$  \hspace{1cm} (2.4)

Let $(M_0 \simeq \mathbb{R}^3, g_0, k_0, \phi_0, \phi_1)$ be an initial data set that is assumed to be sufficiently close to (vacuum) Minkowski data and to enjoy certain (possibly slow) decay conditions at spacelike infinity (for instance, possibly non-spherically symmetric at infinity). Suppose also that these data satisfy Einstein’s constraint equations. Then the corresponding initial value problem for the Einstein equations admits a globally hyperbolic Cauchy development $(M, g)$, which is also endowed with a global foliation by asymptotically flat hypersurfaces, is future causally geodesically complete, and is asymptotic to Minkowski spacetime in future causal directions, as well as in spacelike directions.

The Euclidian-hyperboloidal foliation method. Following the pioneering work by Lindblad and Rodnianski [30], we use the so-called wave gauge, and we thus introduce global coordinate functions satisfying the wave equation in the unknown metric. Our proof is based on a new methodology of proof which combines several novel ideas. We construct a foliation consisting of

- (1) asymptotically hyperboloidal slices in the interior of a light cone (which plays an important role in deriving decay in time),
- (2) asymptotically Euclidian slices in the exterior of this light cone (which plays an important role in deriving decay in space), and
- (3) we merge these two foliations in the vicinity of the light cone.
Fig. 1. The Euclidian-hyperboloidal foliation

See the illustration in Fig. 1. Moreover, the behavior in spacelike directions can be rather general; for instance the metric may be asymptotic with the Minkowski metric and the Schwarzschild metric in certain angular directions (with the exception of a cone with arbitrarily small angle where it may still enjoy $1/r$ decay [26]. See the illustration in Fig. 2. We also rely on (approximate) symmetries associated with the geometry of Minkowski spacetime but, importantly, we avoid the use of the scaling field, since it does not commute with the Klein-Gordon operator. We derive sharp energy estimates as well as sharp pointwise estimates for both the geometric and the matter variables. This requires us to establish new (weighted) Sobolev, Hardy, and Poincaré inequalities. Furthermore, we carefully analyze the nonlinear coupling taking place between the geometry and matter. More generally, our method applies to the global existence problem for a broad class of coupled nonlinear wave-Klein-Gordon equations.

Fig. 2. Asymptotic behavior at infinity.
3 Spacetimes with singularity hypersurfaces

3.1 Our standpoint

Asymptotics at singularities. We now present our framework to analyze spacetimes with singularities, and we seek a flexible framework for bouncing cosmologies involving contracting/expanding evolution phases. We are interested in covering physically meaningful junction conditions and, as mentioned earlier, we must go beyond Israel’s standard junction conditions since they only apply to regularity hypersurfaces. Here, we outline our new methodology and main results, while referring to the main papers for full statements and proofs [16,17,18].

In the regime of interest, we may encounter a rich and complex dynamics near singularities and we consider spacetimes in the quiescent regime which, by definition, enjoy certain monotone behavior and the absence of BKL oscillations (after Belinsky, Khalatnikov, and Lifshitz). Such spacetimes in the quiescent regime admit asymptotic expansions near a singularity hypersurface, and are found in a large variety of setups, including in the description of self-gravitating scalar field, stiff fluid, or compressible fluid, as well as all (matter or vacuum) spacetimes with symmetries (for instance, spatial $T^2$ symmetry).

Methodology. We work with suitably notions of rescaled metric, intrinsic curvature tensor, and matter fields. For the description of the geometry near a singularity hypersurface we require

- a set of past and future singularity data denoted by $(g^\pm, K^\pm, \phi_0^\pm, \phi_1^\pm)$, and
- a singularity scattering map denoted by

$$S : (g^-, K^-, \phi_0^-, \phi_1^-) \mapsto (g^+, K^+, \phi_0^+, \phi_1^+).$$

In turn, based on these notions and in a sense we introduce, we end up constructing spacetimes “beyond” singularities and, in the case of plane-symmetry, globally-defined $S$-cyclic spacetimes.

A first issue is to parametrize the degrees of freedom associated with the constraints at the singularity, while the second main issue is classifying the set of all possible bouncing laws. Interestingly, thanks to a systematic study of the set of singularity scattering maps we arrive at a general classification, by working with general spacetimes and relying on Einstein’s constraint equations. Observe that all earlier works in this question considered symmetric spacetimes or special junction conditions and, therefore, provided only a partial view on the problem. On the other hand, our work leads to a complete characterization of all physically-relevant maps, which clearly separates between universal and model-dependent features of the scattering on gravitational singularities.

3.2 Formulation of the problem

The local ADM formulation. For simplicity in the present review we restrict attention to spacelike hypersurfaces. We introduce a local ADM formulation near
the singularity hypersurface under consideration, which consists of a Gaussian foliation covering a (small neighborhood in a) spacetime by spacelike hypersurfaces diffeomorphic to a given slice $\mathcal{H}_0$, say

$$\mathcal{M}^{(4)} = \bigcup_{\tau \in [\tau_-,\tau_+]} \mathcal{H}_\tau,$$

together with a spacetime metric

$$g^{(4)} = (g^{(4)}_{\alpha\beta}) = -d\tau^2 + g(\tau), \quad g(\tau) = g_{ij}(\tau)dx^i dx^j.$$

in which $\tau$ remains in a neighborhood of the origin, that is, $0 \in [\tau_-,\tau_+]$.

We require that Einstein’s evolution equations for the induced metric $g$ and the intrinsic curvature $K$ hold, that is,

$$\partial_\tau g_{ij} = -2K_{ij}, \quad \partial_\tau K^i_j = \text{Tr}(K)K^i_j + R^i_j - 8\pi M^i_j. \quad (3.1)$$

Here $M^i_j = \frac{1}{2} \rho g^i_j + T^i_j - \frac{1}{2} \text{Tr}(T)g^i_j$ denotes the matter contribution corresponding to a matter field $\phi$, typically satisfying the wave equation

$$\Box g^{(4)} \phi = 0. \quad (3.2)$$

The formulation is supplemented with Einstein’s constraint equations (Hamiltonian, momentum)

$$R + |K|^2 - \text{Tr}(K^2) = 16\pi \rho, \quad \nabla_i K^i_j - \nabla_j (\text{Tr}K) = 8\pi J_j. \quad (3.3)$$

When the time function is chosen to be such that the slices have constant mean curvature, we obtain the so-called CMC-Einstein flow which was studied extensively away from singularities by Andersson and Moncrief (local existence theory) and Anderson, Lott, Moncrief, Reiris, etc. (global dynamics theory). In the present discussion, we are interested in the local behavior near a singularity hypersurface.

**Example of asymptotic behavior: Kasner profiles.** As a first illustration of the asymptotic behavior that one should expect, let us consider the metric, extrinsic curvature, and matter field given by (with $\tau \in (-1,0)$)

$$g^{\text{Kasner}}(\tau,x) = (-\tau)^{2p_1}\{(dx^1)^2 + (-\tau)^{2p_2}(dx^2)^2 + (-\tau)^{2p_3}(dx^3)^2\},$$

$$K^{\text{Kasner}}(\tau,x) = -\frac{1}{\tau} \text{diag}(p_1,p_2,p_3)(x),$$

$$\phi^{\text{Kasner}}(\tau,x) = \phi^0_0(\tau) \log |\tau| + \phi^1_1(x). \quad (3.4)$$

Here, the Euclidean metric $g^-$ on $\mathcal{H} \simeq \mathbb{R}^3$ is chosen, while $K^-$ has constant eigenvectors and $K^- \equiv \text{diag}(p_1,p_2,p_3)$ in suitable coordinates. The functions $p_1,p_2,p_3$ are prescribed and defined on $\mathbb{R}^3$. In addition, we choose matter data $(\phi^0_0,\phi^1_1)$ which are also $x$-dependent.
Under suitable conditions on the data $p_1, p_2, p_3$ and $(\phi_0^- , \phi_1^- )$, this is an “asymptotic profile” in the limit $\tau \to 0$, in the sense we define next. We can also introduce the generalized Kasner spacetime metric

$$g_{\star (4)}^{\text{Kasner}} = -d\tau^2 + g_{\star \text{Kasner}}^*(\tau).$$

We distinguish between the particular cases.

- Case $\phi_1^- \text{ constant}$: we then have $p_j(x) = 1/3 + f_j^1(x^2) - f_j^2(x^3)$, which is parametrized by three functions on $\mathbb{R}$, subject only to the inequality

$$\sum_j p_j(x)^2 \leq 1,$$  \hspace{1cm} (3.6)

easily satisfied for example by functions with all $|f_j(x^j)|$ sufficiently small. This is only an asymptotic solution (and generally not an exact solution) to the Einstein-scalar field system.

- Case $p_j \text{ constant}$: this is Kasner spacetime, not just an asymptotic profile, but the well-known solution to the (matter) Einstein equations. It is a vacuum solution only if, moreover, $\phi_0^-$ vanishes.

4 Fundamental notions and local existence theory

4.1 A construction scheme

Motivated by the work by Rendall \cite{rendall} and followers, we adopt the following strategy in order to parametrize a class of bouncing spacetimes.

- We make a gauge choice ensuring that the singularity hypersurface is located at $\tau = 0$.
- We solve from $\tau = 0$ toward the past ($\tau < 0$) and toward the future ($\tau > 0$).
- We derive an asymptotic ODE system referred to as the “velocity dominated” Ansatz which consists of (essentially) removing all spatial dependency in a choice of local coordinates.
- We apply argument from the theory of Fuchsian equations (Baouendi, Goulaouic, Rendall, Kichenassamy, . . .).
- We check that the asymptotic data and the asymptotic constraints hold on the singularity hypersurface.

In addition, we glue together past and future solutions, using a junction condition.

4.2 Singularity data and asymptotic profiles

We propose the following notions. See the illustration in Fig. 3.

**Definition 4.1.** A (past) singularity initial data set $(g^-, K^-, \phi_0^-, \phi_1^-)$ defined on a $3$-manifold $\mathcal{H}$ consists of two tensor fields $(g^-, K^-)$ and two scalar fields $(\phi_0^-, \phi_1^-)$ such that:
Fig. 3. Spacetime foliation by spacelike hypersurfaces $\mathcal{H}_\tau$.

(i) $g^-(g^2)$ is a Riemannian metric on $\mathcal{H}$.

(ii) $K^- = (K_i^-)$ is a CMC symmetric $(1,1)$-tensor, namely satisfying

$$g^{-ik}K^-_{kj} = g^{-jk}K^-_{ki}.$$  

(4.1)

(iii) The constant mean-curvature condition $\text{Tr}(K^-) = 1$ holds on $\mathcal{H}$.

(iv) The Hamiltonian constraint holds

$$1 - |K^-|^2 = 8\pi (\phi_0^-)^2.$$  

(4.2)

(v) The momentum constraints hold

$$\text{div}_{g^-}(K^-) = 8\pi \phi_0^- d\phi_1^-.$$  

(4.3)

Furthermore, the collection of all singularity data sets is denoted by $I(\mathcal{H})$.

**Definition 4.2.** A (past) **asymptotic profile** associated with a singularity initial data set $(g^-, K^-, \phi_0^-, \phi_1^-) \in I(\mathcal{H})$ is the following ancient geometric flow defined on $\mathcal{H}$

$$\tau \in (-\infty, 0) \mapsto (g^*, K^*, \phi^*)(\tau)$$  

(4.4)

as follows (with the exponential notation $|\tau|^{2K^-} = e^{2\log(|\tau|)K^-}$):

$$g^*(\tau) = |\tau|^{2K^-} g^-,$$

$$K^*(\tau) = -\frac{1}{\tau} K^-,$$

$$\phi^*(\tau) = \phi^-_0 \log |\tau| + \phi^-_1.$$  

(4.5)

The regime of interest in the present work corresponds to the so-called quiescent singularities having

$$K^- > 0.$$  

(4.6)

Below, we will also require the same sign condition after the bounce (“tame preserving”). In particular, this condition easily implies that the volume element decreases to zero as $\tau \to 0^-$, and then increases back to finite values for $\tau > 0$, as should be expected for a bounce.
4.3 Cyclic spacetimes

Our novel concepts are as follows.

Definition 4.3. A past-to-future singularity scattering map on a manifold $\mathcal{H}$ by definition is a map

$$S : I(\mathcal{H}) \rightarrow I(\mathcal{H}), \quad (g^-, K^-, \phi^-_0, \phi^-_1) \mapsto (g^+, K^+, \phi^+_0, \phi^+_1) \quad (4.7)$$

satisfying the two conditions:

- **Diffeomorphism-covariance**, that is, coordinate invariance.
- **Locality property**, that is, the restriction of $S(g^-, K^-, \phi^-_0, \phi^-_1)|_U$ depends only on the restriction of the data, for any open set $U \subset \mathcal{H}$.

Definition 4.4. Fix a singularity scattering map $S$. A $S$-cyclic spacetime $(\mathcal{M}^4, g)$ by definition satisfies the following conditions:

- $\mathcal{M}^4$ is a manifold endowed with a Lorentzian metric $g^{(4)}$ and a scalar field $\phi$.
- **Regularity domain**: $g^{(4)}$ and $\phi$ are defined outside a singularity locus $\mathcal{L} \subset \mathcal{M}^4$, and the Einstein equations hold in $\mathcal{M}^4 \setminus \mathcal{L}$, that is, the (evolution and constraint) Einstein equations $G^{(4)}_{\alpha\beta} = 8\pi T^{(4)}_{\alpha\beta}$ together with the matter evolution equation $\Box g^{(4)} \phi = 0$ (which actually follows from the former).
- **Local Gaussian foliations**: every point $p \in \mathcal{L}$ admits a neighborhood $U$ endowed with a foliation $\bigcup \tau \mathcal{H}_\tau$ containing $\tau = 0$ and such that $\mathcal{H}_0 = \mathcal{L} \cap U$.
  For $\tau \neq 0$, $\mathcal{H}_\tau$ are spacelike and diffeomorphic to $\mathcal{H}_0$ and the 4-metric reads $g^{(4)} = -d\tau^2 + g^{(3)}(\tau)$ for some 3-metrics $g(\tau)$ defined on $\mathcal{H}_\tau \simeq \mathcal{H}_0$.
- **Junction conditions on $\mathcal{L}$**: the future and past singularity data

$$ (g^\pm, K^\pm, \phi^\pm_0, \phi^\pm_1) = \lim_{\tau \rightarrow 0, \tau \neq 0} (|\tau|^2 \tau K g, -\tau K, \tau \partial_\tau \phi, \phi - \tau \log |\tau| \partial_\tau \phi)(\tau) \quad (4.8) $$

are related by the relations

$$ (g^+, K^+, \phi^+_0, \phi^+_1) = S(g^-, K^-, \phi^-_0, \phi^-_1). \quad (4.9) $$
4.4 Existence and asymptotic properties of cyclic spacetimes

We also say that a scattering map is quiescence-preserving provided $K^+ > 0$ whenever $K^- > 0$, that is, the map preserves the positivity of the intrinsic curvature. We then arrive at our existence result which we state somewhat informally. For more precise statements we refer the reader to [16].

\textbf{Theorem 4.1 (Existence of a class of cyclic spacetimes. The glueing technique)}. Consider a three-manifold $\mathcal{H}_0$ and quiescence-preserving scattering map $S : \mathbf{I}(\mathcal{H}_0) \to \mathbf{I}(\mathcal{H}_0)$ given over the space of singularity data. Consider a quiescent singularity data $(g^-, K^-, \phi_0^-, \phi_1^-)$ defined on $\mathcal{H}_0$, that is, satisfying the positivity condition $K^- > 0$. Then there exists a $S$-cyclic spacetime $(\mathcal{M}^{(4)}, g^{(4)})$ with singularity locus $\mathcal{H}_0$, together with a locally Gaussian foliation $\mathcal{M}^{(4)} = \bigcup_{\tau \in [\tau_-, \tau_+]} \mathcal{H}_{\tau}$ with time coordinate $\tau$, such that the flow $\tau \mapsto (g(\tau), K(\tau), \phi(\tau))$ satisfies the Einstein equations coupled to a scalar field $\phi$ away from $\tau = 0$, and $(g^+, K^+, \phi_0^+, \phi_1^+) = S(g^-, K^-, \phi_0^-, \phi_1^-)$ holds on $\mathcal{H}_0$.

If $\mathcal{H}_0$ is compact, then the volume $V(\tau) = \text{Vol}_{g(\tau)}(\mathcal{H}_{\tau})$ of the slices is shrinking toward the singularity

$$\lim_{\tau \to 0} V(\tau) = 0.$$ 

The solution exhibits a crushing singularity in the sense that the mean curvature of the slices blowup

$$\lim_{\tau \to 0} \tau H(\tau) = -1 \quad \text{on } \mathcal{H}_{\tau}$$

Moreover, the solution exhibits a curvature singularity at which the spacetime scalar (and Weyl) curvature $R^{(4)}$ blows up in a uniform way on the singularity hypersurface:

$$\lim_{\tau \to 0} \tau^2 R^{(4)}(\tau) = -8\pi(\phi_0^+)^2 \quad \text{on } \mathcal{H}_{\tau}$$

as well as the spacetime Weyl curvature except in degenerate cases.

As mentioned earlier, our technique of proof is of Fuchsian type and relies on a glueing argument. For additional developments on Fuchsian techniques and related issues, we refer to standard papers by Rendall and co-authors [35,36] and more recent contributions by Alexakis, Fournodavlos, Luk, Speck, and Rodnianski, referred to in [16].

5 Classification of scattering maps

5.1 Terminology

We continue with some further definitions.
Definition 5.1. A singularity scattering map $S$ is said to enjoy the locality property if, for all point $x \in \mathcal{H}$, $S(g^-, K^-, \phi^-_0, \phi^-_1)(x)$ depends upon $(g^-, K^-, \phi^-_0, \phi^-_1)(x)$ and possibly derivatives at $x$, only.

It is called a ultra-local map if it involves pointwise values only, that is, $S(g^-, K^-, \phi^-_0, \phi^-_1)(x)$ depends only on $(g^-, K^-, \phi^-_0, \phi^-_1)(x)$. By diffeomorphism invariance, the restrictions $S_x$ to every point $x$ then are the same.

A conformal map by definition is such that $g^*(\tau_-)$ and $g^*(\tau_+)$ differ by a conformal factor. A map is said to be rigidly conformal if $g^+$ and $g^-$ differ by a conformal factor.

Due to the ultralocality property, specifying a singularity scattering map $S$ on $\mathcal{H}$ is equivalent to specifying one on a unit ball of $\mathcal{H}$.

Definition 5.2. A singularity scattering map $S$ is said to be

- momentum-preserving if $K^+ = K^-$ and $\phi^+_0 = \pm \phi^-_0$;
- momentum-reversing if $K^+ = \frac{2}{3} \delta - K^-$ and $\phi^+_0 = \pm \phi^-_0$;
- idempotent if $S \circ S$ is the identity map on $\mathcal{I}(\mathcal{H})$;
- invertible if $S^{-1}$ is well-defined as a scattering map.

(Here, we denote by $\delta$ the Kronecker symbol $\delta^i_i$.)

5.2 Main classification results

Finally, we are in a position to state our main results.

Theorem 5.1 (Rigidly conformal maps). Only two classes of ultra-local spacelike rigidly conformal singularity scattering maps are available for bouncing of self-gravitating scalar fields. They are described as follows.

- Isotropic rigidly conformal bounce $S^{\text{iso}}_{\lambda, \varphi}$:

  \[ g^+ = \lambda^2 g^-, \quad K^+ = \delta/3, \quad \phi^+_0 = 1/\sqrt{12\pi}, \quad \phi^+_1 = \varphi, \]  

  parametrized by a conformal factor $\lambda = \lambda(\phi^-_0, \phi^-_1, \det K^-) > 0$ and a constant $\varphi$.

- Non-isotropic rigidly conformal bounce $S^{\text{aniso}}_{\lambda, \varphi}$:

  \[ g^+ = c^2 \mu^2 g^-, \quad K^+ = \mu^{-3}(K^- - \delta/3) + \delta/3, \]  

  \[ \phi^+_0 = \mu^{-3} \phi^-_0 / F(\phi^-_1), \quad \phi^+_1 = F(\phi^-_1), \]  

  parametrized by a constant $c > 0$ and a function $f: \mathbb{R} \to [0, +\infty)$

  \[ \mu(\phi_0, \phi_1) = (1 + 12\pi (\phi_0)^2 f(\phi_1))^{1/6}, \quad F(\phi_1) = \int_0^{\phi_1} (1 + f(\varphi))^{-1/2} d\varphi. \]  

An analogous result holds for general maps; see [10] for the full statement.
Theorem 5.2 (General classification). Only two classes of ultra-local space-like singularity scattering maps for self-gravitating scalar fields, which represent either an isotropic bounce denoted by $S_{\lambda, \phi}^{iso}$ or a non-isotropic bounce denoted by $S_{\phi, c}^{ani}$. Now, $\lambda$ is a two-tensor, $\Phi$ is a canonical transformation, and $c$ is a constant.

5.3 The three universal laws of quiescent bouncing cosmology

We complete our presentation by stating three universal laws obeyed by any ultra-local bounce.

– **First law: scaling of Kasner exponents.** There exists a (dissipation) constant $\gamma \in \mathbb{R}$ such that

$$|g^+|^{1/2} \dot{K}^+ = -\gamma |g^-|^{1/2} \dot{K}^-$$

in which we recall that $g$ denotes the (rescaled) spatial metric in synchronous gauge, $|g|^{1/2}$ is the corresponding volume factor, and $\dot{K}$ denotes the traceless part of the extrinsic curvature (as a $(1,1)$ tensor).

– **Second law: canonical transformation of matter.** There exists a nonlinear map $\Phi: (\pi_\phi, \phi)^- \mapsto (\pi_\phi, \phi)^+$, preserving the volume form in the phase space $d\pi_\phi \wedge d\phi$ and depending solely on the scalar invariant $\det(\dot{K}_-)$. Here, $\pi_\phi \sim \phi_0$ denotes the conjugate momentum.

– **Third law: directional metric scaling.** One has

$$g^+ = e^{(\sigma_0 + \sigma_1 K + \sigma_2 K^2)} g^-,$$

which involves a nonlinear scaling in each proper direction of $K$. When $\gamma = 0$, we have an isotropic scattering and no restriction on $\sigma_0, \sigma_1, \sigma_2$. When $\gamma \neq 0$, one has a non-isotropic scattering, explicit formulas for $\sigma_0, \sigma_1, \sigma_2$ are available in terms of $\Phi, \gamma$.

5.4 Role of the small-scale physics

Let us conclude this overview with some further references. In ongoing work, we are interested in deriving scattering maps associated with specific theories. Our methodology encompasses junction conditions that were proposed in a variety of contexts: pre-Big Bang scenario (Gasperini, Veneziano, etc.), modified gravity-matter models (Brandenberger, Peter, Steinhardt, Turok, etc.), and loop quantum cosmology (Ashtekar, Pawlowski, Wilson-Ewing, etc.) among other theories. A different proposal was also made by Penrose [34] and studied by Tod [38] and followers.

Interestingly, as pointed out in [13] the notion of singularity scattering map naturally connects with the notion of “kinetic relation” that was proposed for sharp interface models of phase transition dynamics, for instance for two-phase flows of fluids or elastic materials. In this context [20], the small-scale parameters
of interest, which can be neglected at the macro-scale level of description, account for the viscosity, surface tension, heat conduction, etc. The essential macro-scale features are captured by junction conditions of Rankine-Hugoniot type, as well as kinetic relations or DLM families of paths (after Dal Maso, LeFloch, and Murat); see [19].

In turn, we have proposed a flexible framework for dealing with junction conditions in general relativity. We have uncovered all possible classes of junctions that are both geometrically and physically meaningful, and our classification distinguish between junctions of conformal or non-conformal type, spacelike or null or timelike type, etc. The methodology applies to scalar fields as well as stiff fluids or compressible fluids, and provides us with a guide in order to uncover relevant structures for each model of interest. In particular, our three universal laws constrain the macroscopic aspects of spacetime bounces, regardless of their origin from different microscopic corrections

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