VELOCITY-DENSITY TWIN TRANSFORMS IN THIN DISK MODEL

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November 4, 2014

Abstract

Ring mass density and the corresponding circular velocity (squared) in thin disk model are known to be integral transforms of one another. It is less familiar that the transforms can be represented as one-fold integrals with identical kernels. It may be of interest that, unlike the equivalent Toomre’s formula, the resulting integral for the surface density does not involve the velocity derivative.

Disk integral transforms

In this paper we deal with axisymmetric and infinitesimally thin disk model. We use cylindrical coordinate system $\rho, \phi, z$.

Given a surface mass density $\sigma(\rho)$ in the disk plane $z = 0$, we can infer the rotation velocity $v(\rho)$ of test bodies moving on concentric circular orbits in that plane. Inversely, given a $v(\rho)$, we can find the corresponding $\sigma(\rho)$. Instead of $\sigma(\rho)$ it is more convenient to consider the ring density $\mu(\rho) = 2\pi G \rho \sigma(\rho)$. In the next section we show that

$\mu(\rho)$ and $v^2(\rho)$ are in one to one correspondence through the following pair of (virtually identical) disk integral transforms

\begin{align*}
v^2(\rho) &= \int_0^\infty w(x) \mu(x \rho) \, dx, \quad (1) \\
\mu(\rho) &= \int_0^\infty w(x) v^2(x^{-1} \rho) \, dx. \quad (2)
\end{align*}

The kernel (or weighting function) $w(x)$ in both integrals is given by the following combination of complete elliptic integrals $K$ and $E$ (defined in \cite{[1]})

\[ w(x) = \frac{1}{\pi} \left( \frac{K[k(x)]}{1 + x} + \frac{E[k(x)]}{1 - x} \right), \quad k(x) = \frac{2\sqrt{x}}{1 + x}. \]

Because $w(x)$ has an integrable singularity at $x = 1$, the integration in both integrals should be understood in the Cauchy principal value sense. The singularity is easily tractable numerically and presents itself no difficulty at all. The nature of the pole $x = 1$ is such that the integrals not extend to some extent on the radial gradients of the integrands $\mu$ and $v^2$, which is the characteristic feature of disk model. The strikingly symmetric integral forms cannot be simplified further. They are more suitable for numerical integration than the equivalent double integrals with highly oscillatory terms which we shall come to later.

The important point about Eq.\textsuperscript{[2]} is that it involves only $v^2$. A more familiar Toomre’s integrated formula for $\sigma$ found in 1963 \cite{[2]}, involves the derivative of $v^2$:

\[
\sigma(\rho) = \frac{G^{-1}}{\pi^2} \left[ \int_0^\rho \frac{dv^2(\tilde{\rho})}{d\tilde{\rho}} K\left(\frac{\tilde{\rho}}{\rho}\right) \, d\tilde{\rho} + \int_1^\infty \frac{dv^2(\tilde{\rho})}{d\tilde{\rho}} K\left(\frac{\rho}{\tilde{\rho}}\right) \, d\tilde{\rho} \right].
\]

Interestingly, Toomre did not give any expression in \cite{[2]} being a one-fold integral without the derivative of $v^2$. Also, no such expression can be found in Binney & Tremaine book on galactic dynamics \cite{[3]}. It seems therefore, that Eq.\textsuperscript{[2]} may not be widely known, if at all.

Toomre’s formula was pointed out to be of relatively little use in practice on account of the fact that the derivative of $v^2$ is usually subject to significant observational errors, resulting with a $\sigma$ varying in an erratic and unphysical way \cite{[3]}. We stress, that this drawback lies in the accuracy of measurements rather than in the disk model as such. The very feature of sensitivity to radial gradients we alluded to above, is a distinct phenomenon. Simply, disk model must be applied in due care.

We decided to focus on the integral Eq.\textsuperscript{[2]} in this separate paper, because of usefulness of the formula in modeling galactic disks. An elegant, maximally reduced one-fold integral form without the derivative of $v^2$, is needed also for practical reasons, especially for the accuracy and speeding up the numerical integration. This is the case, for example, in finding column mass densities of finite-width disks by means of recursions \cite{[7]}.
In the past we found a more complicated form of the Eq.2 formula [1]. There are also known various forms of equivalent double integral representations of Eq.2 (e.g. [3]), or numerical methods of finding \( \sigma \) from a given \( v^2 \) (e.g. [3]). An algebraic approach to inverting Eq.1 (given a sufficiently large fragment of \( v^2 \) sampled on a union of osculating rings) presented in [1], offers a very interesting representation of the direct formula Eq.2.

Simple derivation of disk transforms

\( \Box \). Surface density in axial symmetry is naturally expressed in terms of Hankel transforms. For our purposes we intentionally rewrite the result of Toomre’s method [2] into the chained form corresponding to Eq.2

\[
2\pi G \rho \sigma(\rho) = \int \left( \int \lambda \rho J_0(\lambda \rho) J_1(\lambda \tilde{\rho}) \, d\lambda \right) v^2(\tilde{\rho}) \, d\tilde{\rho}.
\]

Toomre called it as too formal to be of any direct use and, having integrated by parts, gave his integrated formula as the final result. Nevertheless, the above form with Bessel functions is useful in finding analytical expressions for \( \sigma \), given a \( v^2 \) (or vice versa), e.g. [3]. Kalnajs [3] considered a more general case of homoeoidal column density and related it to the rotation curve \( v^2 \) through an analogous intermediate transform with additional flattening parameter. In the limit of flat homoeoids, Kalnajs’ formulae should reduce to Toomre’s chained form.

In what follows, to prove Eq.2 we shall calculate the integral in the round brackets of the above chained form.

The inverse chained form corresponding to Eq.1 can be easily deduced, e.g. [10], and we arrange the result into a form resembling the previous integral

\[
v^2(\rho) = \int \left( \int \lambda \rho J_0(\lambda \rho) J_1(\lambda \rho) \, d\lambda \right) 2\pi G \rho \tilde{\sigma}(\tilde{\rho}) \, d\tilde{\rho}.
\]

If we rewrite it as \( v^2(\rho) = \int K(\rho, \tilde{\rho}) \mu(\tilde{\rho}) \, d\tilde{\rho} \), then the inverse formula reduces to \( \mu(\rho) = \int K(\rho, \tilde{\rho}) v^2(\tilde{\rho}) \, d\tilde{\rho} \).

Next, by substituting in these integrals \( \tilde{\rho} = x \rho \) and \( \rho = \rho/x \), respectively, we obtain \( v^2(\rho) = \int \rho K(\rho, x \rho) \mu(x \rho) \, dx \)

\( \mu(\rho) = \int (\rho/x) K(\rho/x, x) v^2(\rho/x) \, dx \). Furthermore, it is easily seen that \( K(\rho_1, \rho_2) = \frac{1}{\rho_1} u \left( \frac{\rho_2}{\rho_1} \right) \), where

\[
u(x) = \int \omega J_0(\omega x) J_1(\omega) \, d\omega.
\]

As so, \( v^2(\rho) = \int u(x) \mu(x \rho) \, dx \) and \( \mu(\rho) = \int u(x) v^2(\rho/x) \, dx \), which explains why the kernels in Eq.1 and Eq.2 are identical.

To complete our derivation, it remains to determine \( u(x) \). Instead of using tables of integrals, it will be more instructive to deduce \( u(x) \) from comparing the previous expression for \( v^2 \) with another one obtained by carrying out the well known textbook calculation concerning the axisymmetric gravitational potential \( \Phi(\rho, z) \) of a thin disk. To this end, we first arrange the expression for \( \Phi \), so as to isolate the elliptic function \( K \) (for a fixed \( \phi \) we make use of a new integration variable \( \phi \to \alpha : 2\alpha = \phi - \phi + \pi \))

\[
\Phi(\rho, z) = -4G \int_0^\infty \frac{\tilde{\rho} \sigma(\tilde{\rho}) \, d\tilde{\rho}}{\sqrt{(\rho + \tilde{\rho})^2 + z^2}} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - \frac{4\rho \tilde{\rho}}{(\rho + \tilde{\rho})^2 + z^2} \sin^2(\alpha)}).
\]

By differentiating \( \Phi \) with respect to \( \rho \), using the property

\[
K'(k) = \frac{E(k)}{k(1-k)} - \frac{K(k)}{k}
\]

and taking the limit \( z \to 0 \), we can obtain the desired result from the force equilibrium condition \( \rho^{-1}v^2(\rho) = \partial_x \Phi(\rho, 0) \) for circular orbits in the disk plane. The result simplifies to

\[
\frac{v^2(\rho)}{\rho} = 2G \int_0^\infty \frac{K(k(x))}{1 + x} \sigma(x) \, dx.
\]

From this result we immediately see that \( u(x) \equiv w(x) \), which in turn proves the relation Eq.2 \( \Box \) (a similar expression to Eq.2 we obtained in a not so straightforward way already in [11] and it is connected with the present form by the inversion \( x \to x^{-1} \)).

References

[1] We use for \( K \) and \( E \) the notational convention from Ryzhik, I. M., Gradshtein, I. S. Tables of Integrals, sums, series and products, Moscow, Leningrad, 1951:

\[
K(\varepsilon) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \varepsilon^2 \sin^2 \phi}}, \quad E(\varepsilon) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - \varepsilon^2 \sin^2 \phi}}
\]

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