Numerical Prediction of cusps or kinks in the Nambu-Goto dynamics

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Abstract

It is known that Nambu-Goto world-sheet present some pathological structures, such as cusps and kinks, during its evolution. We propose a method through the Raychaudhuri equation for membranes to determine if there are cusps and kinks in the world-sheet. We generalize the Raychaudhuri equation for non-extremal membranes and use it as a tool for determining when a string in the Nambu-Goto action will form cusps or kinks in its evolution. Furthermore, we present two examples where we test graphically this method.

I. INTRODUCTION

It is well known that when we consider a string described by the Nambu-Goto action it has some pathological structures in its evolution, i.e., cusps or kinks can be formed. In some cases the pathology of a closed string is widely used, for example, in topological defects with cosmic string these linear topological defects are predicted in a wide class of elementary particle models and could be formed as a symmetry breaking phase transition in the early universe. Also, we have these pathologies in the emission of gravitational radiation, in gravitational waves produced by cosmic strings, or as a cusp anomaly of a light-like Wilson line (see for example).

Until now, we know that higher derivatives (more than fourth order) in the extrinsic curvature remove pathologies in the Nambu-Goto action, however does not exist a method to know whether such pathology arise in the Nambu-Goto dynamics. Here we present a method to determine if the string will form cusps or kinks in its evolution. In turn, this method will help to study the evolution of an extended object to know whether anomalies exist, for example in brane cosmology.

In a previous work, the Raychaudhuri equation (Rh) was generalized to extremal relativistic membranes. The Rh equation plays an important role in General Relativity (GR). One of its applications is to describe the incompleteness of the geodesic which is related to the singularities of space-time. For the case of membranes, the Rh equation is built on the world-sheet; thus, when we describe the Rh equation for membranes we are implicitly describing the evolution of an extended object.

The Rh equation in relativistic membranes is useful in two different cases. The first case is when we have some background with a given singularity, then the Rh equation tells us how the evolution of the extended object is affected due to the singularity of the background, and the second case is the one proposed here, where we use the Rh equation as a tool to determine the presence of such pathologies (cusps and kinks).

To this end, we generalize the Rh equation for non-extremal membranes. On this context, we use the Rh equation as a tool to determine if the world-sheet will exhibit cusps and kinks, whose Lagrangian depends on \( L = L(g_{ab}, K_{ab}^{i}, \bar{\nabla}_{a} K_{bc}^{i}, \ldots) \). The recipe consist in inserting the equation of motion obtained from the Lagrangian into the Rh equation, with this, we are able to analyze the behavior of Rh equation for non-extremal membranes and through a mathematical analysis, we relate the zeros of the solution with the cusps and kinks of the evolution of the string. Therefore, it is possible to predict whether the world-sheet will have cusps and kinks or not.

The paper is organized as follows. In Section II we briefly review some geometric aspects of the Rh model for extremal membranes, this section also introduces our notation. In Section III we extend the model to non extremal \( D \)-dimensional relativistic membranes. In Section IV we show some examples which illustrate the use of the Rh equation to compute the cusps and kinks, i.e., points where the world-sheet will collapse and form cusps and kinks. Finally, in Section V we present our conclusions and remarks, and we give a possible way to extend the model to relativistic \( D \)-dimensional membranes with \( D > 2 \).

II. THE GEOMETRY

Consider an extended object \( \Sigma \), of dimension \( D-1 \), evolving in a background space-time with \( N \)-dimension, with metric \( \eta_{\mu \nu} \), \( (\mu, \nu = 0, 1, \ldots, N - 1) \). The trajectory or world volume \( m \), of the extended object is \( D-
dimensional, and is described by the embedding:

$$x^\mu = X^\mu(\xi^I),$$  \hspace{1cm} (1)

where $x^\mu$ are local coordinates for the background space-time, $\xi^I$ are local coordinates for the world volume and $X^\mu$, are the embedding functions $(a, b = 0, 1, \ldots, D - 1)$. With the parameterization (1) we obtain a basis of tangent vectors to the world volume at each point of $\Sigma$. In this context, we introduce $N-D$, unitary normal vectors to the world volume denoted as $n^\mu_i$, $(i, j = 1, 2, \ldots, N - D)$. Normal vectors to $m$ are implicitly defined by $n^i \cdot e_a = 0$ and with normalization $n_i \cdot n_j = \delta_{ij}$. Thus, the vectors $\{e_a^i, n^i\}$ form an orthogonal basis in the world volume $m$, adapted to $\Sigma$.

Throughout this paper we use latin letters for world volume and greek letters for the background. The tensor in italics are also in the background. The metric induced (or first fundamental form) on the world volume is then given by: $g_{ab} = e_a \cdot e_b = \eta_{\mu\nu} e^\mu_a e^\nu_b$.

Notice that we define the world sheet projections of the space-time covariant derivatives as $D_a := e^\mu_a D_\mu$, where $D_\mu$ denotes the torsionless covariant derivative compatible with the space-time metric $\eta_{\mu\nu}$. Let us consider the world-sheet gradients of the basis vectors $\{e_a, n\}$. Since they are space-time vectors, they are always decomposed using the orthonormal basis $\{e^a_i, n^a\}$ \[\text{as:}\]

$$D_a e_b = \gamma^c_{ab} e_c - K^c_{ab} n_i,$$  \hspace{1cm} (2a)

$$D_a n^i = K^i_{ab} e^b + \omega^i_{ab} n_j,$$  \hspace{1cm} (2b)

where $\gamma^c_{ab}$ is the world-sheet Ricci rotation coefficients and $K^i_{ab}$ is the $ith$ extrinsic curvature of the world-sheet (or second fundamental form). Its symmetry in the tangential indices is a consequence of the integrability of the base $\{e_a\}$. Moreover, we have that $\omega^i_{ab}$ is the normal fundamental form (or twist potential) of the world-sheet.

The kinematical expressions in Eqs. (2), which describe the extrinsic geometry, are generalizations of the classical Gauss-Weingarten equations, which altogether with the integrability conditions completely describe the extrinsic geometry of the world-sheet.

We therefore introduce a world-sheet covariant derivative defined on fields. It transforms as a tensor under normal frame rotations as

$$\nabla_a \Phi^{i_1 \ldots i_n} = \nabla^a \Phi^{i_1 \ldots i_n} - \omega^a_{i_1 j} \Phi^{j i_2 \ldots i_n} - \ldots - \omega^a_{i_n j} \Phi^{i_1 \ldots i_{n-1}}$$  \hspace{1cm} (3)

where $\nabla_a$, is the intrinsic world-sheet covariant derivative. The gradients of the space-time basis $\{E_a, n^i\}$, along the directions orthogonal to the world-sheet, can be expressed as

$$D_i n_j = \gamma^k_{ij} n_k - J_{aij} e^a,$$  \hspace{1cm} (4a)

$$D_i e_a = S_{aij} n_j + S_{abi} e^b,$$  \hspace{1cm} (4b)

where $D_i = n^\mu_i D_\mu$. The above relations are analogous to the Gauss-Weingarten Eqs. (3). The quantities $J_{aij}$, and $S_{abi}$, are defined as:

$$J_{aij} \equiv D^i e_a \cdot n_j,$$  \hspace{1cm} (5)

$$S_{abi} \equiv D^i e_a \cdot e_b = -S^i_{ba},$$  \hspace{1cm} (6)

$$\gamma_{aij} \equiv D_i n_j - n_k,$$  \hspace{1cm} (7)

In general, Eq. (5) does not possess any symmetry under the interchange of the normal indices $i$, and $j$; this shows the fact that $\{n^i\}$ (unlike the $\{e_a\}$) do not generally form an integrable distribution. It is the analog of $K_{aij}$ in the Gauss-Weingarten equations. Moreover, Eq. (6) is the analog of $\omega_{abi}^j$ and the Ricci rotation coefficient $\gamma_{aij}$ is associated with the normal base.

### III. GENERALIZATION OF RAYCHAUDHURI EQUATION

For a geodesic curve, the Rh equation describes the evolution of $J^{ij} \equiv J_i^{0j}$, connecting neighboring geodesics along the curve, and giving specific values to $J^{ij}$, at some initial time. The complete set of equations governing the evolution of deformations can be obtained by taking the gradient of $J_{ij}$, (see appendix of \[\text{for an explicit calculation}\])

$$\nabla_b J^{ij}_a = -\nabla^i K^{ab}_{jk} - J^{jk}_{ab} K^{ij}_{ac} - K^{ab}_{ik} K^{ij}_{ac} + \mathcal{R}_{\alpha \beta \mu \nu} n^{\alpha a} e^\beta_b e^\mu_a n^{\nu j}.$$  \hspace{1cm} (8)

It should be noted that Eq. (8) does not depend on the equation of motion of the membrane. By taking the trace over the indices of the world-sheet we obtain

$$\nabla_a J^{aij} = -\nabla^i K^{aj}_{ak} - J^{aj}_{ik} K^{ik}_{ac} - K^{aj}_{ik} K^{ik}_{ac} + \mathcal{R}_{\alpha \beta \mu \nu} n^{\alpha a} e^\beta_b e^\mu_a n^{\nu i}.$$  \hspace{1cm} (9)

If we now anti-symmetrize Eq. (5) with respect to its world-sheet indices, we get

$$\nabla_a J^{ij}_b - \nabla_b J^{ij}_a = G^{ij}_{ab},$$  \hspace{1cm} (10)

where we have defined

$$G^{ij}_{ab} \equiv -J^{ik}_{ab} K^{kj}_{bc} - K^{ij}_{ak} K^{kj}_{bc} + \mathcal{R}_{\alpha \beta \mu \nu} n^{\alpha a} e^\beta_b e^\mu_a n^{\nu j} - (a \leftrightarrow b).$$

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1 Throughout this work, a midpoint denotes a contraction with the background metric $\eta_{\mu\nu}$. For example, $n^i \cdot e_a = \eta_{\mu\nu} n^\mu n^\nu a = 0$. 

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It is worth noticing that the source term which includes \( \nabla^i K^j \), has been canceled out independent of the background dynamics.

Notice that it is not straightforward to work with the quantity \( J^{ij}_a \); however, by analogy with mechanics of continuum, it is possible to decompose \( J^{ij}_a \), into its symmetric and antisymmetric parts with respect to the normal indices \( \Theta^{ij}_a \) and \( \Lambda^{ij}_a \) respectively as

\[
J^{ij}_a = \Theta^{ij}_a + \Lambda^{ij}_a. \tag{11}
\]

We further decompose \( \Theta^{ij}_a \), into its traceless and trace parts:

\[
\Theta^{ij}_a = \Sigma^{ij} + \frac{1}{N - D} \delta^{ij} \Theta_a. \tag{12}
\]

In one dimension \( \Theta, \Sigma, \), and \( \Lambda \), describe, respectively, the isotropic expansion, the shear and the vorticity of a trajectory with respect to neighboring trajectories. This is contained into the Rh equation in GR, but there is not a clear interpretation available at higher dimensions.

Now, we use Eq. (11) and Eq. (12) into Eq. (9), in order to obtain the equation of motion for \( \Theta_a, \Sigma^{ij}_a, \) and \( \Lambda^{ij}_a \). Then it is possible to separate the symmetric and antisymmetric parts and the trace in the form

\[
\tilde{\nabla}_a \Sigma^{ij} + (\Lambda^{a k} \Lambda^{j k} + \Sigma^{a k} \Sigma^{j k}) \text{str} + \frac{1}{N - D} \Sigma^{ij} \Theta_a = 0, \tag{13a}
\]

\[
\tilde{\nabla}_a \Lambda^{a j} + \Lambda^{a k} \Lambda^{k j} + \Sigma^{a k} \Sigma^{k j} - 2 \Lambda^{a k} \Sigma^{k j} + \frac{2}{N - D} \Lambda^{ij} \Theta_a = 0, \tag{13b}
\]

\[
\tilde{\nabla}_a \Theta_a - \Lambda^{a j} \Lambda_{a j} + \Sigma^{a j} \Sigma_{a j} + \frac{1}{N - D} \Theta_a \Theta_a - (M^2)_i = 0, \tag{13c}
\]

where the symbol \((\ldots)\text{str}\) denotes the symmetric traceless part of the matrix \((M^2)\) which is defined as:

\[
(M^2)^{ij} = -\tilde{\nabla}^i K^j - K^{ab} \tilde{\nabla}_a K^{bj} + \mathcal{R}_{a \beta \gamma \nu} \epsilon^a_{\beta \gamma} \epsilon^{\mu a} \epsilon^{\nu j}. \tag{14}
\]

Now using Eq. (11) and Eq. (12) into Eq. (10) and separating the symmetric and antisymmetric parts and the trace, we obtain:

\[
2 \tilde{\nabla}_{[a} \Sigma^{ij}_{b]} = -2 (\Lambda^{a k}_{[a} \Lambda^{j k}_{b]} + \Sigma^{a k}_{[a} \Sigma^{j k}_{b]} \text{str} + 4 \Lambda^{a k}_{[a} \Lambda^{k i}_{b]} + (\Lambda^{a k}_{[a} \Lambda^{i k}_{b]}), \tag{15a}
\]

\[
2 \tilde{\nabla}_{[a} \Lambda^{ij}_{b]} = -2 \Lambda^{k [j i}_{a} \Lambda^{k i}_{b]} - 2 \Sigma^{k [i}_{a} \Sigma^{j i}_{b]} - \Omega^{ij}_{a b}, \tag{15b}
\]

\[
2 \partial_{[a} \Theta_{b]} = 0. \tag{15c}
\]

Notice that \( \Omega^{ij}_{a b} \) is the curvature associated with the normal fundamental form \( \omega^{ij}_{a} \), defined by

\[
\Omega^{ij}_{a b} = \tilde{\nabla}_b \omega^{ij}_a - \tilde{\nabla}_a \omega^{ij}_b + \omega^{ik}_a \omega^{kj}_b - \omega^{ik}_b \omega^{kj}_a. \tag{16}
\]

From Eq. (15c), which describes the evolution of the generalized expansion \( \Theta_a \), it follows that: \( \Theta_a = \partial_a \Upsilon \). The above is implicit at least locally, for some potential function \( \Upsilon \). Inserting this expansion in Eq. (13) we obtain

\[
\Delta \Upsilon + \frac{1}{N - D} \partial_a \Upsilon \partial^a \Upsilon - \Lambda^2 + \Sigma^2 - M^2 = 0, \tag{17}
\]

where \( \Delta \) is the Laplace-Beltrami operator defined as: \( \Delta = (-g)^{-1/2} \partial_a (g^{ab} \sqrt{-g} \partial_b) \) and it is defined the world-sheet scalar quantities \( \Lambda^2 \equiv \Lambda^{a j} \Lambda_{a j}, \Sigma^2 \equiv \Sigma^{a j} \Sigma_{a j} \) and \( M^2 \equiv (M^2)_{i j} \), in Eq. (13) which describes the evolution of the expansion of the world-sheet; having a quasilinear hyperbolic partial differential equation of second order.

Now, it is possible to consider \( \Upsilon \), as a generalized relative volume expansion potential. If \( l \), represents the characteristic length of the expansion, we can set \( \Upsilon = (N - D) \ln l \). With this change of variables, Eq. (17) becomes a linear equation

\[
\Delta l + \frac{1}{N - D} \left( \Sigma^2 - \Lambda^2 - M^2 \right) l = 0. \tag{17}
\]

Notice that, Eq. (17) is the Rh equation for non-extremal membranes, i.e., it is a wave equation on the world-sheet for a massive positive definite scalar field \( l \), with an effective mass term, \( \mu^2 = (\Sigma^2 - \Lambda^2 - M^2) / (N - D) \). Then, we have mapped the analysis of \( \Theta_a \), to the solution of a linear wave equation. However, we should have in mind that \( \mu^2 \) involves \( \Sigma^2_a \) and \( \Lambda_{a j} \), explicitly, as a result, it will depend implicitly also on \( \Theta_a \). We note that because of \( M^2 \) does not have a definite sign neither does \( \mu^2 \).

So far we have described the evolution of an extended object \( \Sigma \), in a generic space-time background and we obtained the Rh equation for non-extremal extended objects (see Eq. (17)). We no longer have a particle moving in a space-time, now will be an extended object moving in a space-time. Such that, we are interested in the geometry of that extended object.

### A. \( M^2 \) matrix analysis

Equation (17) is the generalized Rh equation for non-extremal membranes, the solution sometimes is not straightforward to find, because it is necessary to determine \( \Sigma^2, \Lambda^2, (M^2) \), which depend on the dimension, however if the symmetric traceless part of \((M^2)^{ij}\) is zero, then \( \Sigma^2 = \Lambda^2 = 0 \). In particular, we analyze the form
(\(M^2\)) in Eqs. (14) and (17) because we are interested on the evolution of the extended object.

We can solve the second order hyperbolic partial differential equation (17), with the use of variable separation in order to obtain \(D\)-ordinary differential equation. Using the theorem on the existence of zeros \([13]\), it is possible to obtain

\[
d^2z/dt^2 + H(t)z = 0,
\]

having at least one zero, if and only if \(H(t) > 0\). From this we can get important physical information, to compare Eq. (17) with Eq. (18) (after separation of variables), it is possible to see that both are second order differential equations and we can identify \(\Sigma^2 - \Lambda^2 - M^2 = H(t)\). Thus, only when \(\Sigma^2 = \Lambda^2 = 0\) (this is possible only if the symmetric traceless part of \((M^2)^i_j\) is zero), we can conclude that the isotropic expansion \(\Theta \to -\infty\), is possible if \(M^2 \leq 0\) in Eq. (17). Then, we have reduced the problem of finding the conjugate points to the problem of discovering the location of zeros in solutions of Eq. (17) or Eq. (20) and just in the zeros of this solution is when we will have the collapse of the world-sheet in a region and therefore we have the cusps or kinks formation.

Continuing with the analysis of the trace of Eq. (14) we have

\[
(M^2)^i_j = -\tilde{\nabla}^i K_i - \tilde{\nabla}^i K_i + 3\, \mathcal{R}_{\alpha\beta\mu\nu} \, e^{\alpha}_i e^{\beta}_j e^{\mu}_l e^{\nu}_i.
\]

Using the integrability conditions of the Gauss-Codazzi\([14]\)

\[
\mathcal{R}_{\alpha\beta\mu\nu} e^{\alpha}_i e^{\beta}_j e^{\mu}_l e^{\nu}_i = R_{abcd} - K_{\alpha\beta} K_{ijkl} + K_{\alpha\beta} K_{ijkl},
\]

and substituting the completeness relationship \(n^{\mu}_{\nu} n^{\nu}_{\alpha} = \eta^{\mu}_{\nu} - e^{\mu}_{\nu} e^{\nu}_{\alpha}\) of the basis vector \(\{e^{\mu}_{\alpha}, n^{\mu}_{\nu}\}\) of the world volume \(m\), in (19), and using (20) we find

\[
(M^2)^i_j = -\mathcal{R}_{\mu\nu} H^{\mu\nu} - \tilde{\nabla}^i K_i - 2 K_{ij} n^{ij}_K + K^i K_i + R.
\]

This terminology is borrowed from the perturbative analysis of\([15-18]\), where it appears as a variable mass in the world-sheet wave equation that describes the evolution of small perturbations.

### B. Hypersurface

In the case of a hypersurface, there is only one normal vector \(i = 1\), where \(D = N - 1\). In this case \(\mathcal{G}_{\alpha\beta}\), and \(\omega^\alpha_i\), vanish identically due to the antisymmetry, similar to \(\Sigma^2\), and \(\Lambda^2\); thus, Eq. (14) becomes

\[
\partial_a J_b - \partial_b J_a = 0,
\]

so that \(J_{aij} := J_{a11} \equiv \Theta_a\). Moreover, it is also possible to align the tangent vectors along the normal direction, so that \(S^0_{\alpha\beta} = 0\), in an analogous way to a curve.

Therefore the Rh equation, in this special case, is reduced to

\[
\Delta \mathcal{Y} + \partial_\mu \mathcal{Y} \partial^\mu \mathcal{Y} - M^2 = 0,
\]

(23) again, setting \(\mathcal{Y} = (N - D) \ln l\), we have a hyperbolic second order partial differential equation

\[
\Delta l - (M^2) l = 0.
\]

(24) Then, \(M^2\) is now modified. From Eq. (21), the integrability conditions of Gauss-Codazzi\([14]\) and the relation of completeness, \((M^2)\) is now no longer a matrix

\[
(M^2) = -\mathcal{R}_{\mu\nu} H^{\mu\nu} - \nabla K - K^2 + 3R.
\]

(25) From Eq. (21) and Eq. (24), we notice that \((M^2)\) depends on the background only in the first term, so if there is a singularity in the space background, this will be manifested itself in the membrane through the Ricci scalar, \(\mathcal{R}\).

So far we have generalized the Rh equation for non-extremal membranes for any codimension (see Eq. (17)) and for the case of a hypersurface described by Eq. (24), where the modification is through the matrix \((M^2)\).

In order to describe this method, we address two examples for determining the presence of cusps or kinks in the world-sheet.

### IV. TESTING METHOD: EXAMPLES

The evolution of the string is through a two-dimensional extended object (world-sheet), in this context the Rh equation for non-extremal membranes tells us the singularities on the world-sheet (if exist) i.e. the string collapses, and with this new method we can predict if the extended object has cusps, kinks or both. In order to determine this, we propose the following recipe:

- First, the equation of motion is obtained from the Lagrangian and Eq. (21) is used (or Eq. (25) in the case of a hypersurface). After that, Eq. (17) it is used (or in Eq. (24) is used in the case of a hypersurface).

- Then, we solve the differential equation involved through the method of variables separation. The temporal part of the solutions is analyzed by checking if the solution cross or not the \(x\)-axis.

- Finally, we have two cases: a) crossing \(x\)-axis, in this case the world sheet collapses, implying the existences of cusps or kinks. b) not crossing \(x\)-axis, in this case the world sheet never collapses showing the stability of the structure.

In the following study we discuss the helical string in breathing mode and the model of circular loop to observe the stability or not of the world-sheet and identify the existence or not of the cusps or kinks.
A. Helical string in breathing mode

We will take the following simple case: consider extreme membranes, which satisfy the Nambu dynamics\cite{19}. The Nambu action is proportional to the area of the world-sheet,

\[ S[X^\mu] = -\alpha \int d^D \xi \sqrt{-g}, \]  

where \( \alpha \) is the tension of the membrane and its equation of motion is given by

\[ K^i = g^{ab} K^i_{ab} = 0. \]  

Substituting \( K^i = 0 \) in Eq. (21), \( (M^2) \) takes the form \( (M^2) = R \); therefore, the Rh equation \cite{17} for this case is

\[ \Delta l - \frac{R}{N - D} l = 0. \]  

Now we consider the helical string in breathing mode, \( \Sigma \), evolving into a Minkowski background space \( (3 + 1) \) dimensional parameterized as follows\cite{21},

\[ x^\mu X^\nu (x^a) = (\tau, Z(\tau) \cos \sigma, Z(\tau) \sin \sigma, q \sigma), \]  

where the tangent vectors will be given by \( e_a^a : (a = \tau, \sigma) \),

\[ e_\tau^a = (1, \dot{Z} \cos \sigma, \dot{Z} \sin \sigma, 0), \]  

\[ e_\sigma^a = (0, -Z \sin \sigma, Z \cos \sigma, q), \]  

the overdot denotes differentiation with respect to \( \tau \). This string is helical with breathing \( q \), and notice that in the limit \( q \to 1 \), corresponds to the flat world-sheet and the limit \( q \to 0 \), to the collapsing circular loop.

The induced metric \( g_{ab} = \eta_{\mu \nu} e_a^\mu e_b^\nu \) will be now given by

\[ ds^2_S = g_{ab} d\xi^a d\xi^b = -(1 - \dot{Z}^2) d\tau^2 + (Z^2 + q^2) d\sigma^2. \]  

On the other hand, the normal vectors are

\[ n_\tau^i = \frac{1}{\sqrt{q^2 + Z^2}} (0, q \sin \sigma, -q \cos \sigma, Z), \]  

\[ n_\sigma^i = \frac{1}{\sqrt{1 - Z^2}} (\dot{Z}, \cos \sigma, \sin \sigma, 0). \]  

In addition, the extrinsic curvature will be

\[ K_{1ab} = \frac{q \dot{Z}}{\sqrt{q^2 + Z^2}} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right), \]  

\[ K_{2ab} = \frac{1}{\sqrt{1 - Z^2}} \left( \begin{array}{cc} -\dot{Z} & 0 \\ 0 & Z \end{array} \right), \]  

and the normal fundamental form of the world-sheet is given by

\[ (\omega_a^{ij}) = \frac{q}{\sqrt{(q^2 + Z^2)(1 - Z^2)}} \left( \begin{array}{cc} 0 \\ 1 \end{array} \right). \]  

Using the parameterization showed in Eq. (20) on the equation of motion described by Eq. (27), we find the following differential equation

\[ \frac{\ddot{Z}}{(1 - \dot{Z}^2)^{3/2}} + \frac{Z}{(q^2 + Z^2)(1 - Z^2)^{1/2}} = 0, \]  

whose general solution can be written as

\[ Z(\tau) = \sqrt{q^2 - \dot{q}^2} \cos \left( \frac{\tau - \tau_0}{\kappa} \right). \]

With the aim to illustrate the method we will reduce some constants, and therefore choosing the initial condition \( \tau_0 = 0 \) and \( \kappa = 1 \), then we substitute the solution \( Z(\tau) \) in Eq. (28) and takes the form,

\[ \frac{d^2}{d\sigma^2} (\tau, \sigma) - \frac{d^2}{d\tau^2} (\tau, \sigma) + \left[ \left( \frac{q^2 - 1}{(\cos^2 \tau + q^2 \sin^2 \tau)} \right) \right] l(\tau, \sigma) = 0. \]  

Notice that the derivative term with respect to \( \tau \) and \( \sigma \) does not appear due to the fact that the metric is diagonal. Now, we use separation of variables to obtain the equation for \( \tau \) since we are interested in the evolution of the system (notice that \( \tau \) is the proper time of the string):

\[ \frac{d^2}{d\tau^2} l(\tau) - \left[ \left( \frac{q^2 - 1}{(\cos^2 \tau + q^2 \sin^2 \tau)} \right) + m^2 \right] l(\tau) = 0, \]  

where \( m^2 \) is the constant of separation, for intermediate \( m \) the trajectory is never singular, and the extrinsic curvature peaks at approximately \( q^{-2} \sqrt{1 - q^2} \). Eq. (31) reminds us the equation of an oscillator, but with the difference that the term in square brackets explicitly depends on \( \tau \), into the oscillator problem this term is a constant and therefore, our solution is no longer oscillatory. However, Eq. (31) look like Eq. (13) with the term in brackets equal to \( H(\tau) \), then we can apply the theorem on the existence of zeros of ordinary differential equations.

In Fig. 1 it is possible to observe the Rh equation of the helical string in breathing mode, in Minkowski space-time background. In the Top panel we plot the solution with different values of \( m \), while in the Bottom panel we plot different values of \( q \), where \( q \) is the winding number per unit of length. The oscillations of the graphic are due to breather mode (and this is precisely what prevents the collapse). Notice that the graph never crosses the \( x \)-axis, which implies that the world-sheet will never have a cusps or kinks in the evolution, which is in agreement with Sakellariadou\cite{21}.

B. Circular loop

In this example, we start writing down the Nambu-Goto action (see Eq. (20)), whose equation of motion is
FIG. 1. Numerical solution of Eq. (31), with initial condition
\[ q = 0.55, \quad q = 0.57, \quad q = 0.64, \quad q = 0.99 \]
where the tangent vectors are now given by
\[ \frac{\dot{q}}{1 - Z^2} \cos \sigma, \frac{\dot{q}}{1 - Z^2} \sin \sigma \]

and the normal vector takes the form:
\[ n^\mu = \frac{1}{\sqrt{1 - Z^2}} \left( \frac{\dot{Z}}{Z}, \cos \sigma, \sin \sigma \right) \]
then, the extrinsic curvature will be
\[ K_{\tau\tau} = \frac{\dot{Z}}{\sqrt{1 - Z^2}}, \quad K_{\sigma\sigma} = \frac{Z}{\sqrt{1 - Z^2}} \]
Replacing the embedding relation written in Eq. (32) into
Eq. (27), it takes the specific form
\[ \ddot{Z}Z - \dot{Z}^2 + 1 = 0 \]
whose general solution is:
\[ Z(\tau) = \kappa \cos \left( \frac{\tau - \tau_0}{\kappa} \right) \]
In order to solve the differential equation we have chosen
\[ \tau_0 = 0, \quad \kappa = 1 \] as initial condition, thus one obtains the
canonical form of the loop trajectory (32). Now, we use
the fact that \( K = 0 \), in the equation of the hypersurface
(Eq. (25)) and we find that \( (M^2) \) takes the form \( (M^2) = 3R \). Substituting the solution \( Z(\tau) \) into Eq. (24) and
using separation of variables, the equation becomes
\[ \frac{d^2}{d\tau^2} l(\tau) = (6 \sec^2 \tau - m^2) l(\tau) = 0 \]
where \( m^2 \) is the new constant of separation. Then, we
solve Eq. (36) numerically and plot the solution in Fig. 2.
We see that the circular loop form cusps and kinks in the
world-sheet, because of the graph cuts the x-axis and, in
this way, the circular loop collapses at a point. The
extrinsic curvature invariants, become singular at this
point, hence rigidity would be indicated by a retardation
of the collapse or a positive correction to the amplitude
of the loop. The solution that we find is also in a good
agreement with Barbashov[1].

V. CONCLUSIONS AND REMARKS

We have provided a new method to determine if a
string will present or not cusps and kinks to examine
the evolution of deformations of relativistic world-sheet,
propagating in a background space-time of arbitrary
codimension. The construction of the method was
motivated by the Raychaudhuri equation for relativistic
membranes, although the method is algebraic, so the
numerical analysis is necessary to determine the solution
of the final differential equation.

In GR, the most general way to determine if a space-
time is singular is by the incompleteness geodesic condition
which is determined by using the Rh equation, and
by analyzing the global structure of the space-time. In
relativistic membranes, this condition is given by Eq. (17)
or Eq. (24) which have a general structure. It is important to remark that they contain information about the background, across the Ricci tensor and also possess, in intrinsic curvature and the scalar curvature. With these equations it is possible to determine if the world sheet collapse i.e., there are formation of cusps or kinks.

In the example presented in section IV B, we could appreciate the importance of the Rh equation for membrane via the extrinsic curvature model will have cusps or kinks in its evolution. This is ongoing research that will be presented elsewhere.

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