SPECTRUM OF SYK MODEL II: CENTRAL LIMIT THEOREM

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Abstract. In our previous paper [6], we derived the almost sure convergence of the global density of eigenvalues of random matrices of the SYK model. In this paper, we will prove the central limit theorem for the linear statistic of eigenvalues and compute its variance.

1. Introduction

The SYK model is a random matrix model in the form of \[ \begin{align*}
H &= \frac{1}{\sqrt\left(q_n/n^2\right)} \sum_{1 \leq i_1 < i_2 < \ldots < i_{qn} \leq n} J_{i_1i_2\ldots i_{qn}} \psi_{i_1}\psi_{i_2}\ldots\psi_{i_{qn}},
\end{align*} \]

where \( J_{i_1i_2\ldots i_{qn}} \) are independent identically distributed (i.i.d.) random variables with mean 0 and variance 1; we further assume that the k-th moment of \( |J_{i_1i_2\ldots i_{qn}}| \) is uniformly bounded for any fixed k; \( \psi_j \) are Majorana fermions satisfying the algebra

\[ \{\psi_i, \psi_j\} := \psi_i\psi_j + \psi_j\psi_i = 2\delta_{ij}, \quad 1 \leq i, j \leq n. \]

Throughout the article, \( n \) is an even integer. As a remark, physicists care especially when \( q_n \) is an even integer, but the model is still a good one if \( q_n \) is odd from the mathematical point of view, and our main results apply to both cases.

By the representation of the Clifford algebra, \( \psi_i \) can be represented by \( L_n \times L_n \) Hermitian matrices with \( L_n = 2^{n/2}\) [14], and thus \( H \) is also Hermitian. Let \( \lambda_i, 1 \leq i \leq L_n \) be the eigenvalues of \( H \) which are real numbers. Let’s define the normalized empirical measure of eigenvalues of \( H \) as

\[ \rho_n(\lambda) := \frac{1}{L_n} \sum_{j=1}^{L_n} \delta_{\lambda_j}(\lambda). \]

In our first paper [6], we proved that \( \rho_n \) converges to a probability measure \( \rho_\infty \) with probability 1 (or almost surely). Such result can be viewed as a type of ‘law of large numbers’ in probability theory. Actually, let \( q_n \) be even, the limiting density \( \rho_\infty \) will depend on the limit of the quotient \( q_n^2/n \) if \( 1 \leq q_n \leq n/2 \) or \( (n - q_n)^2/n \) if \( n/2 \leq q_n < n \). The results for odd \( q_n \) are similar. We refer to Theorems 1 and 2 in [6] for the precise statements.

In this paper, other than the ‘law of large numbers’, we will prove the central limit theorem (CLT) for the linear statistic of eigenvalues of the SYK model and compute its variance as \( n \to \infty \). The CLT is one of the most important theorems in probability theory and random matrix theory. Our results indicate some useful...
information about the global 2-point correlation of eigenvalues, we also refer to the recent papers [8, 9] for the numerical results on the local behavior (or rescaling limit) of the 2-point correlation.

Given a test function \( f(x) \), the linear statistic of eigenvalues is

\[
L_n(f) := \frac{1}{L_n} \sum_{j=1}^{L_n} f(\lambda_j).
\]

In random matrix theory, the investigation of the CLT for the linear statistic of eigenvalues of random matrices dates back to Jonsson on Gaussian Wishart matrices [12]. Similar work for the Wigner matrices was derived by Sinai-Soshnikov [22] and more general results by Johansson [13]. There are many contexts on this, we refer to [15, 16], Chapter 9 in [2] and Chapter 3 in [18] for more details.

It’s also worth mentioning the CLT for the linear statistic of many other random point processes, where basically the variance of the linear statistic can be expressed as some energy functional of the test function. Actually it’s really hard to list all of these point processes, we refer to the following results and the references therein: Sodin-Tsirelson’s work on zeros of random polynomials and random analytic functions [20], Shiffman-Zelditch’s work on zeros of random holomorphic sections over the complex manifolds [19], Berman’s result regarding the Fekete points defined via the Bergman kernel on the complex manifolds [3] and Soshnikov’s result on the determinantal point processes [23].

In §3, we will prove the following CLT for the general SYK model,

**Theorem 1.** Let \( J_{i_1 i_2 \cdots i_q} \) be i.i.d. random variables with mean 0 and variance 1, and the \( k \)-th moment of \( |J_{i_1 i_2 \cdots i_q}| \) is uniformly bounded. Let’s denote \( \gamma := \mathbb{E} J_{i_1 i_2 \cdots i_q}^4 \) as the 4-th moment of the random variable. Let \( q_n \) be either even or odd integers. Let \( \rho_{\infty} \) be the limiting density of eigenvalues of the SYK model as in Theorem 1 (if \( q_n \) is even) or Theorem 2 (if \( q_n \) is odd) in [6], which also depends on the limit of the quotient \( q_n^2/n \) if \( 1 \leq q_n \leq n/2 \) or \((n - q_n)^2/n \) if \( n/2 \leq q_n < n \). Let \( f(x) \) be a real polynomial. Then we have the following convergence in distribution as \( n \to \infty \),

\[
\left( \frac{n}{q_n} \right)^{\frac{1}{2}} (L_n(f) - \mathbb{E} L_n(f)) \Rightarrow \langle x f'/2, \rho_{\infty} \rangle J,
\]

where \( J \) is the Gaussian distribution with mean 0 and variance \( \gamma - 1 \). In particular, the limit of its variance satisfies

\[
\lim_{n \to +\infty} \left( \frac{n}{q_n} \right) \mathrm{var}[L_n(f)] = \langle x f'/2, \rho_{\infty} \rangle^2 (\gamma - 1).
\]

**Remark 1.** Theorem 1 implies the CLT for the trace \( L_n^{-1} \mathrm{Tr} H^k \) for any fixed \( k \), for example, let’s take \( f(x) = x^2 \), then we will have the CLT for \( L_n^{-1} [\sum_{j=1}^{L_n} \lambda_j^2] \) which is \( L_n^{-1} \mathrm{Tr} H^2 \). In [22], Sinai-Soshnikov proved the CLT for \( \mathrm{Tr} W^k N \) where \( W \) are general \( N \times N \) Wigner matrices and \( k_N \) are some slowly growing functions of \( N \) (see [1, 2, 18] also). As a consequence, they proved the CLT for analytic test functions on the disk and the almost sure convergence of the largest eigenvalue. But for the SYK model, there are essential difficulties to prove such type of results.

Actually, the proof of Theorem 1 is based on careful estimates of the variance and covariance of the trace \( L_n^{-1} \mathrm{Tr} H^k \) (see [3]). The estimates will also imply that Theorem 1 holds for some class of analytic functions other than polynomials.
In the special case of the Gaussian SYK model, we can improve Theorem 1 to a larger class of functions.

**Theorem 2.** For the Gaussian SYK model where $J_{i_1, i_2, \cdots, i_{q_n}}$ are i.i.d. standard Gaussian random variables such that the fourth moment is $\gamma = 3$, Theorem 1 holds for a class of functions $f(x)$, where $f(x)$ are Lipschitz functions and $f'(x)$ are bounded uniformly continuous.

As a special case, Theorem 2 is true when test functions are smooth with compact support. But we do not know what happen in general, especially when the test functions are not smooth enough or singular. Such cases are intensively studied in random matrix theory, a good reference is [18].

Note that there is a symmetry between the systems with the interaction of $q_n$ fermions and $n - q_n$ fermions (see [6]), therefore, we only prove the main theorems for even $q_n$ with $2 \leq q_n \leq n/2$, the rest cases (even $q_n$ with $n/2 \leq q_n < n$ or odd $q_n$) follow immediately without any essential difference and we omit the proof.

2. Preliminary

2.1. Notations and basic properties. Let’s first review some notations and basic properties in [6] that we will make use of in this paper.

For a set $A = \{i_1, i_2, \cdots, i_m\} \subseteq \{1, 2, \cdots, n\}$, $1 \leq i_1 < i_2 < \cdots < i_m \leq n$, we denote

$$\Psi_A := \psi_{i_1} \cdots \psi_{i_m} \quad \text{and} \quad \Psi := I \quad \text{if} \quad A = \emptyset.$$  

We denote the set

$$I_n = \{(i_1, i_2, \cdots, i_{q_n}) \mid 1 \leq i_1 < i_2 < \cdots < i_{q_n} \leq n\}.$$  

Thus the cardinality of $I_n$ is

$$|I_n| = \binom{n}{q_n}.$$  

For any coordinate $R = (i_1, \cdots, i_{q_n}) \in I_n$, we denote

$$J_R := J_{i_1, \cdots, i_{q_n}} \quad \text{and} \quad \Psi_R := \psi_{i_1} \cdots \psi_{i_{q_n}}.$$  

Sometimes we identify $R$ with the set $\{i_1, \cdots, i_{q_n}\}$. Thus we can simply rewrite the SYK model as

$$H = \frac{i^{q_n/2}}{\sqrt{\binom{n}{q_n}}} \sum_{R \in I_n} J_R \Psi_R$$  

(4)

Given any set $X$ and any integer $k \geq 1$, we define $P_2(X^k)$ to be the tuples $(x_1, \cdots, x_k) \in X^k$ for which all entries $x_1, \cdots, x_k$ appear exactly twice. If $k$ is odd, then $P_2(X^k)$ is an empty set.
Throughout the article, we denote \( c_k \) as some constant depending only on \( k \) and independent of \( n \) and \( q_n \), but its value may differ from line to line, the same for \( c_{2k}, c'_k \) and so forth.

We will also need the following properties that one can find the proof in [6],

- Given a set \( A \subseteq \{1, 2, \cdots, n\} \),
  \[
  \text{Tr} \Psi_A = 0 \text{ and } \Psi_A \neq \pm I \text{ are always true for } A \neq \emptyset.
  \]
- For \( A, B \subseteq \{1, 2, \cdots, n\} \), then
  \[
  \Psi_A = \pm \Psi_B \text{ if and only if } A = B.
  \]
- And
  \[
  \Psi_A \Psi_B = \pm \Psi_{A \Delta B} \text{ where } A \Delta B := (A \setminus B) \cup (B \setminus A).
  \]
- For \( A_1, \cdots, A_k \subseteq \{1, 2, \cdots, n\} \), we have
  \[
  |\frac{1}{L_n} \text{Tr} \Psi_{A_1} \cdots \Psi_{A_k}| \leq 1.
  \]

### 2.2. Moments.

**Given any even integer \( k \), we define the set of 2 to 1 maps as**

\[
S_k = \left\{ \pi : \{1, 2, \cdots, k\} \rightarrow \{1, 2, \cdots, \frac{k}{2}\} \mid |\pi^{-1}(j)| = 2, 1 \leq j \leq \frac{k}{2} \right\}.
\]

The crossing number \( \kappa(\pi) \) for a pair-partition \( \pi \) is defined to be the number of subsets \( \{r, s\} \subset \{1, 2, \cdots, \frac{k}{2}\} \) such that there exists \( 1 \leq a < b < c < d \leq k, \pi(a) = \pi(c) = r, \pi(b) = \pi(d) = s \).

**Given \( a > 0 \), throughout the article, we denote**

\[
m^a_k := \begin{cases} 
\frac{1}{(k/2)!} \sum_{\pi \in S_k} e^{-2a\kappa(\pi)} & \text{if } k \text{ is even}, \\
0 & \text{if } k \text{ is odd}.
\end{cases}
\]

It’s further proved in [11] that

\[
m^0_k := \lim_{a \rightarrow 0} m^a_k = \begin{cases} 
(k-1)!! & \text{if } k \text{ is even}, \\
0 & \text{if } k \text{ is odd},
\end{cases}
\]

which is the \( k \)-th moment of the standard Gaussian measure; and

\[
m^\infty_k := \lim_{a \rightarrow \infty} m^a_k = \begin{cases} 
\frac{k!}{(k/2)!(k/2+1)!} & \text{if } k \text{ is even}, \\
0 & \text{if } k \text{ is odd},
\end{cases}
\]

which is the \( k \)-th moment of the semicircle law (or the Catalan numbers).

Let \( 2 \leq q_n \leq n/2 \) be an even integer, then the main result proved in [6] is that, if \( q_n^2/n \rightarrow a \in [0, +\infty] \), then the expectation of the \( k \)-th moment of the normalized empirical measure \( \rho_n \) defined by (3) always satisfies

\[
m^a_{n,k} := \mathbb{E}(x^k, \rho_n) \rightarrow m^a_k, \; n \rightarrow \infty.
\]

### 3. Linear statistic and CLT

In this section, we will prove Theorem [11] by analyzing the limit of the covariance of \( L_n^{-1} \text{Tr} H^k \).
3.1. Limit of covariance.

**Lemma 1.** Let $q_n$ be even and $k, k' \geq 1$, $2 \leq q_n \leq n/2$. We assume $q_n^2/n \to a \in [0, +\infty]$ and denote the fourth moment $\gamma := \mathbb{E}J_n^2$, then we have

$$
\lim_{n \to +\infty} \left( \frac{n}{q_n} \right) \text{cov}(L_n^{-1} \text{Tr } H^k, L_n^{-1} \text{Tr } H^{k'}) = (m_k^2 k/2)(m_{k'}^2 k'/2)(\gamma - 1).
$$

**Proof.** We first consider the case when $k + k'$ is even. By (4), we have

$$
\frac{1}{L_n} \text{Tr } H^k = \frac{1}{L_n} \frac{i^{q_n k/2}}{q_n^{k/2}} \sum_{R_1, \ldots, R_k \in I_n} J_{R_1} \cdots J_{R_k} \text{Tr } \Psi_{R_1} \cdots \Psi_{R_k},
$$

and thus

$$
\text{cov}(L_n^{-1} \text{Tr } H^k, L_n^{-1} \text{Tr } H^{k'}) = \frac{1}{L_n^2} \frac{i^{q_n (k+k')/2}}{q_n^{(k+k')/2}} \sum_{R_1, \ldots, R_{k+k'} \in I_n} \text{Tr } J_{R_1} \cdots J_{R_{k+k'}} \Psi_{R_1} \cdots \Psi_{R_{k+k'}}.
$$

For every $R_1, \ldots, R_{k+k'} \in I_n$ and $A \in I_n$, let $|A| = |\{j: 1 \leq j \leq k+k', R_j = A\}|$. If some $R_i$ appears only once in $(R_1, \ldots, R_{k+k'})$, then

$$
\text{cov}(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{k+k'}}) = \mathbb{E}J_{R_1} \cdots J_{R_{k+k'}} - \mathbb{E}J_{R_1} \cdots \mathbb{E}J_{R_{k+1}} \cdots J_{R_{k+k'}} = 0
$$

by the independence of random variables. Hence, we can write

$$
\text{cov}(L_n^{-1} \text{Tr } H^k, L_n^{-1} \text{Tr } H^{k'}) = \text{cov}_1 + \text{cov}_2
$$

where

$$
\text{cov}_1 = \frac{1}{L_n^2} \frac{i^{q_n (k+k')/2}}{q_n^{(k+k')/2}} \sum_{(R_1, \ldots, R_{k+k'}) \in P_2(I_n^{k+k'})} \text{Tr } J_{R_1} \cdots J_{R_{k+k'}} \Psi_{R_1} \cdots \Psi_{R_{k+k'}},
$$

and

$$
\text{cov}_2 = \frac{1}{L_n^2} \frac{i^{q_n (k+k')/2}}{q_n^{(k+k')/2}} \sum_{(R_1, \ldots, R_{k+k'}) \in I_n^{k+k'} \setminus P_2(I_n^{k+k'}), \# R_i \geq 2} \text{cov}(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{k+k'}}) \cdot \text{Tr } \Psi_{R_1} \cdots \Psi_{R_k} \Psi_{R_{k+1}} \cdots \Psi_{R_{k+k'}}.
$$

Let’s first estimate $\text{cov}_1$. For $(R_1, \ldots, R_{k+k'}) \in P_2(I_n^{k+k'})$, we denote $A_1 := \{R_j | 1 \leq j \leq k\}$, $A_2 := \{R_j | k+1 \leq j \leq k+k'\}$ and $A_0 := A_1 \cap A_2$. Then we can decompose

$$
P_2(I_n^{k+k'}) = \bigcup_{j=0}^2 P_{2,j}^{(k,k')}(I_n^{k+k'})
$$

where

$$
P_{2,0}^{(k,k')}(I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in P_2(I_n^{k+k'}) | A_0 = \emptyset\},
$$

$$
P_{2,1}^{(k,k')}(I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in P_2(I_n^{k+k'}) | A_0 \neq \emptyset, \Psi_{R_1} \cdots \Psi_{R_k} = \pm I\},
$$

$$
P_{2,2}^{(k,k')}(I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in P_2(I_n^{k+k'}) | \Psi_{R_1} \cdots \Psi_{R_k} \neq \pm I\}.
$$

If $(R_1, \ldots, R_{k+k'}) \in P_{2,0}^{(k,k')}(I_n^{k+k'})$, then $J_{R_1} \cdots J_{R_k}$ and $J_{R_{k+1}} \cdots J_{R_{k+k'}}$ are independent, hence $\text{cov}(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{k+k'}}) = 0$. If $(R_1, \ldots, R_{k+k'}) \in P_{2,2}^{(k,k')}(I_n^{k+k'})$,
then $\text{Tr} \Psi_{R_1} \cdots \Psi_{R_k} = 0$ (see (21)). If $(R_1, \ldots, R_{k+k'}) \in P_{2,1}(I_n^{k+k'})$, then we easily have $\text{cov}(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{k+k'}}) = 1$. Thus, we have

$$\left| \text{cov}_1 \right| = \frac{1}{I_n^2} \frac{1}{q_n} \sum_{(R_1, \ldots, R_{k+k'}) \in P_{2,1}(I_n^{k+k'})} \text{cov}(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{k+k'}}),$$

$$\text{Tr} \Psi_{R_1} \cdots \Psi_{R_k} \text{Tr} \Psi_{R_{k+1}} \cdots \Psi_{R_{k+k'}} \left| \begin{array}{c}
\leq \frac{1}{I_n} \frac{1}{q_n} \sum_{(R_1, \ldots, R_{k+k'}) \in P_{2,1}(I_n^{k+k'})} 1 \\
= \left( \frac{n}{q_n} \right)^{-(k+k')/2} |P_{2,1}(I_n^{k+k'})|,
\end{array} \right.$$ 

where we used inequality 8.

Now we estimate $|P_{2,1}(I_n^{k+k'})|$. Let $m = |A_0| > 0$, then there exists $1 \leq i_1 < \cdots < i_m \leq k$ and $k + 1 \leq i'_1 < \cdots < i'_m \leq k + k'$ such that $A_0 = \{R_{i_1}, \ldots, R_{i_m}\} = \{R_{i'_1}, \ldots, R_{i'_m}\}$. Now we have $\Psi_{R_1} \cdots \Psi_{R_k} = \pm \Psi_{R_{i_1}} \cdots \Psi_{R_{i_m}}$ and

$$P_{2,1}(I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in P_{2}(I_n^{k+k'}) | m > 0, \Psi_{R_1} \cdots \Psi_{R_{i_m}} = \pm I\}.$$ 

Since $|A_1 \cup A_2| = (k+k')/2$, for every fixed $A_0$ there are $\binom{|I_n| - m}{(k+k')/2 - m}$ choices of $A_1 \cup A_2$; for every fixed $A_1 \cup A_2$, there are at most $((k+k')/2)^{k+k'}$ choices of $(R_1, \ldots, R_{k+k'})$. Let’s denote

\begin{equation}
B_m = \{(R_1, \ldots, R_m) \in I_n^m | \Psi_{R_1} \cdots \Psi_{R_m} = \pm I, R_i \neq R_j, \forall 1 \leq i < j \leq m\}.
\end{equation}

Then for fixed $m$, every $A_0$ corresponds to exactly $m!$ elements in $B_m$, thus the number of elements in $P_{2,1}(I_n^{k+k'})$ satisfying $|A_0| = m$ is at most $\binom{|I_n| - m}{(k+k')/2 - m} (\frac{k+k'}{2})^{k+k'} / (m!) \cdot |B_m|$. By the estimate of $|B_m|$ in Lemma 4 below, we will have

$$|P_{2,1}(I_n^{k+k'})| \leq \sum_{0 < m \leq (k+k')/2} \left( \frac{|I_n| - m}{(k+k')/2 - m} \right) \left( \frac{k+k'}{2} \right)^{k+k'} / (m!) \cdot |B_m|$$

$$\leq \sum_{0 < m \leq (k+k')/2} c_{k+k', m} |I_n|^{(k+k')/2 - m} |B_m|$$

$$\leq \sum_{0 < m \leq (k+k')/2} c_{k+k', m} |I_n|^{(k+k')/2 - m} c |I_n|^{m-1} n^{-\frac{1}{4}}$$

$$= c_{k+k'} |I_n|^{(k+k')/2 - 1} n^{-\frac{1}{4}}.$$ 

Thus we have

$$\left| \text{cov}_1 \right| \leq \left( \frac{n}{q_n} \right)^{-(k+k')/2} |P_{2,1}(I_n^{k+k'})|$$

$$\leq c_{k+k'} \left( \frac{n}{q_n} \right)^{-(k+k')/2} |I_n|^{(k+k')/2 - 1} n^{-\frac{1}{4}}$$

$$\leq c_{k+k'} \left( \frac{n}{q_n} \right)^{-1} n^{-\frac{1}{2}}.$$
Hence, we have
\[
\lim_{n \to \infty} \left( \frac{n}{q_n} \right) \text{cov}_1 = 0.
\]
Therefore, in order to prove Lemma \[1\] we only need to prove
\[
(12) \quad \lim_{n \to \infty} \left( \frac{n}{q_n} \right) \text{cov}_2 = (m_n^2 k/2)(m_n^2 k'/2)(\gamma - 1).
\]
If \((R_1, \ldots, R_{k+k'}) \in I_n^{k+k'} \setminus P_2(I_n^{k+k'})\) and \#\(R_i \geq 2\), then \(|\{R_j|1 \leq j \leq k + k'\}| \leq (k + k')/2 - 1\). If the equality holds, then there are two possibilities. Type 1: some \(R_i\) appear 4 times and all the rest appear exactly twice. Type 2: two distinct \(R_i\) appear 3 times and all the rest appear twice. We denote \(Q_j(I_n^{k+k'})\) as the set of \((R_1, \ldots, R_{k+k'})\) with Type \(j\) for \(j = 1, 2\) and
\[
Q_3(I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in I_n^{k+k'} : |\{R_j|1 \leq j \leq k + k'\}| \leq (k + k')/2 - 2\}.
\]
Then we have
\[
\{(R_1, \ldots, R_{k+k'}) \in I_n^{k+k'} \setminus P_2(I_n^{k+k'}) : \#\(R_i \geq 2\), \forall i\} = \bigcup_{j=1}^3 Q_j(I_n^{k+k'})
\]
and we can further decompose
\[
\text{cov}_2 = \text{cov}_{2, 1} + \text{cov}_{2, 2} + \text{cov}_{2, 3}
\]
where
\[
\text{cov}_{2, j} = \frac{1}{I_n^2} \left( \frac{n}{q_n} \right)^{k+k'}/2 \sum_{(R_1, \ldots, R_{k+k'}) \in Q_j(I_n^{k+k'})} \text{cov}(J_{R_1} \cdots J_{R_{k+k}}, J_{R_{k+1}} \cdots J_{R_{k+k'}})
\cdot \text{Tr} \Psi_{R_1} \cdots \Psi_{R_{k}} \text{Tr} \Psi_{R_{k+1}} \cdots \Psi_{R_{k+k'}} , \quad j = 1, 2, 3.
\]
If \((R_1, \ldots, R_{k+k'}) \in Q_2(I_n^{k+k'})\), we assume \(A\) and \(B\) appear 3 times, \(A \neq B\) and all the rest appear twice. Then by properties in \[2.1\] again, we have \(\Psi_{R_1} \cdots \Psi_{R_{k+k'}} = \pm \Psi_A \Psi_B = \pm \Psi_{A \Delta B} \neq \pm I\), thus \(\text{Tr} \Psi_{R_1} \cdots \Psi_{R_{k}} \neq \pm I\) or \(\Psi_{R_{k+1}} \cdots \Psi_{R_{k+k'}} \neq \pm I\), this implies \(\text{Tr} \Psi_{R_1} \cdots \Psi_{R_{k}} = 0\) or \(\text{Tr} \Psi_{R_{k+1}} \cdots \Psi_{R_{k+k'}} = 0\). Hence, we have identity
\[
\left( \frac{n}{q_n} \right) \text{cov}_{2, 2} = 0.
\]
Let’s denote \(k_1 := (k + k')/2\), then for \(n\) large enough, we have
\[
|Q_3(I_n^{k+k'})| \leq \sum_{B \subseteq I_n, |B| = k_1 - 2} |\{(R_1, \ldots, R_{k+k'}) \in I_n^{k+k'} : \forall 1 \leq i \leq k + k'\}|
= \sum_{B \subseteq I_n, |B| = k_1 - 2} (k_1 - 2)^{k+k'}
= (k_1 - 2)^k \left( \frac{|I_n|}{k_1 - 2} \right)^{k+k'}
\leq c_{k+k'} |I_n|^{(k+k')/2 - 2}.
\]
Thus we have
\[
|\text{cov}_{2, 3}| \leq \frac{1}{\left( \frac{n}{q_n} \right)^{k+k'}/2} \sum_{(R_1, \ldots, R_{k+k'}) \in Q_3(I_n^{k+k'})} c_{k+k'}
\]
Thus, we have
\[
\frac{c_{k+k'}|I_n|_{k+k'}^{k+k'-2}}{(n_{qn})^{k+k'-2}} = c_{k+k'}\left(\frac{n}{q_n}\right)^{-2}.
\]

Thus we have
\[
\left(\frac{n}{q_n}\right)_{\text{cov}2.3} \to 0
\]
which has no contribution to the left hand side of (12).

All of the rest effort is to estimate the last term \(\text{cov}_{2.1}\). For \((R_1, \ldots, R_{k+k'}) \in Q_1(I_n^{k+k'})\), we assume \(A\) appears 4 times and all the rest appear exactly twice.

Let’s denote \(A_1 := \{R_j|1 \leq j \leq k\}\), \(A_2 := \{R_j|k+1 \leq j \leq k+k'\}\), \(A_0 := A_1 \cap A_2\), \(k_0 := \{|j|1 \leq j \leq k, R_j = A\}\), \(A_0^* = A_0 \setminus \{A\}\) for \(k_0\) even and \(A_0^* = A_0\) for \(k_0\) odd, then we can further decompose
\[
Q_1(I_n^{k+k'}) = (\bigcup_{j=0}^{3} Q_{1,j}(I_n^{k+k'}))
\]
where
\[
Q_{1,0}^{(k,k')}(I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in Q_1(I_n^{k+k'})|A_0 = \emptyset\},
Q_{1,1}^{(k,k')}(I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in Q_1(I_n^{k+k'})|A_0 = \{A\}, k_0 = 2\},
Q_{1,2}^{(k,k')}(I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in Q_1(I_n^{k+k'})|A_0^* \neq \emptyset, \Psi_{R_1} \cdots \Psi_{R_k} = \pm 1\},
Q_{1,3}^{(k,k')}(I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in Q_1(I_n^{k+k'})|\Psi_{R_1} \cdots \Psi_{R_k} = \pm 1\}.
\]

As before, if \((R_1, \ldots, R_{k+k'}) \in Q_{1,0}^{(k,k')}(I_n^{k+k'})\), then \(J_{R_1} \cdots J_{R_k}\) and \(J_{R_{k+1}} \cdots J_{R_{k+k'}}\) are independent, hence \(\text{cov}(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{k+k'}}) = 0\). If \((R_1, \ldots, R_{k+k'}) \in Q_{1,3}^{(k,k')}(I_n^{k+k'})\) then \(\text{Tr} \Psi_{R_1} \cdots \Psi_{R_k} = 0\). Therefore, we have
\[
\text{cov}_{2.1} = \text{cov}_{2.1,1} + \text{cov}_{2.1,2}
\]
where
\[
\text{cov}_{2.1,j} = \frac{1}{I_n^2} \frac{q_{n,k+k'}/2}{(n_{qn})^{k+k'}/2} \sum_{(R_1, \ldots, R_{k+k'}) \in Q_{1,j}^{(k,k')}(I_n^{k+k'})} \text{cov}(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{k+k'}}) \cdot \text{Tr} \Psi_{R_1} \cdots \Psi_{R_k} \text{Tr} \Psi_{R_{k+1}} \cdots \Psi_{R_{k+k'}} , j = 1, 2.
\]

Let’s recall the following estimate proved in [10].

**Lemma 2.** Let \(q_n\) be even, for any \(k \geq 1\), we have
\[
(13) \quad \text{var}[L_n^{-1} \text{Tr} H_k] \leq c_k \left(\frac{n}{q_n}\right)^{-1}
\]
where \(c_k\) is some constant.

By Lemma 2 we easily have the upper bound
\[
|\text{cov}(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{k+k'}})| \leq c_{k+k'}.
\]

Thus, we have
\[
|\text{cov}_{2.1,2}| \leq \frac{1}{n_{qn}}^{k+k'}/2 \sum_{(R_1, \ldots, R_{k+k'}) \in Q_{1,2}^{(k,k')}(I_n^{k+k'})} c_{k+k'} = \left(\frac{n}{q_n}\right)^{-k_1} c_{k+k'} |Q_{1,2}^{(k,k')}(I_n^{k+k'})|.
\]
Let $m = |A_0^*| > 0$, then there exists $1 \leq i_1 < \cdots < i_m \leq k$ and $k + 1 \leq i_1' < \cdots < i_m' \leq k + k'$ such that $A_0^* = \{R_{i_1}, \ldots, R_{i_m}\} = \{R_{i_1'}, \ldots, R_{i_m'}\}$. Now we have
\[
\Psi_{R_1} \cdots \Psi_{R_k} = \pm \Psi_{R_{i_1}} \cdots \Psi_{R_{i_m}}
\]
and
\[
Q_{1,2}^{(k,k')} (I_n^{k+k'}) = \{(R_1, \ldots, R_{k+k'}) \in Q_1(I_n^{k+k'}) | m > 0, \; \Psi_{R_{i_1}} \cdots \Psi_{R_{i_m}} = \pm I\}.
\]
Following the same argument as $P_{2,1}(k,k')(I_n^{k+k'})$ above, we will have
\[
|Q_{1,2}^{(k,k')} (I_n^{k+k'})| \leq \sum_{0 < m \leq k_1} (k_1 - 1)^{k+k'} \left( \frac{|I_n| - m}{k_1 - m - 1} \right) / (m!) \cdot |B_m|
\]
\[
\leq \sum_{0 < m \leq k_1} (k_1 - 1)^{k+k'} \left( \frac{|I_n| - m}{k_1 - m - 1} \right) / (m!) \cdot |I_n|^{m-1}
\]
\[
\leq \sum_{0 < m \leq k_1} c_{k+k',m} |I_n|^{k_1-m-1} |I_n|^{m-1} = c_{k+k'} |I_n|^{k_1-2}.
\]
Hence, we have
\[
|\text{cov}_{2,1,2}| \leq \binom{n}{q_n}^{-k_1} c_{k+k'} |Q_{1,2}^{(k,k')} (I_n^{k+k'})|
\]
\[
\leq \binom{n}{q_n}^{-k_1} c_{k+k'} |I_n|^{k_1-2} \leq c_{k+k'} \left( \frac{n}{q_n} \right)^{-2},
\]
i.e.,
\[
|\binom{n}{q_n} \text{cov}_{2,1,2}| \rightarrow 0,
\]
which implies that
\[
(14) \quad \lim_{n \rightarrow \infty} \binom{n}{q_n} \text{cov}_2 = \lim_{n \rightarrow \infty} \binom{n}{q_n} \text{cov}_{2,1,1}.
\]
Now we estimate $\text{cov}_{2,1,1}$ which turns out to be an interesting term.

Recall the assumption that $k + k'$ is even in the beginning of the proof, for the case when $k$ and $k'$ are both odd, by definition, $Q_{1,1}^{(k,k')} (I_n^{k+k'})$ must be empty, and thus $\text{cov}_{2,1,1} = 0$. Now we discuss the case when $k$ and $k'$ are both even.

By definition, given $(R_1, \ldots, R_{k+k'}) \in Q_{1,1}^{(k,k')} (I_n^{k+k'})$ with $A_0 = A_1 \cap A_2 = \{A\}$ where $A$ appears twice in both $(R_1, \ldots, R_k)$ and $(R_{k+1}, \ldots, R_{k+k'})$, we must have $(R_1, \ldots, R_k) \in P_2(I_n^k)$ and $(R_{k+1}, \ldots, R_{k+k'}) \in P_2(I_n^{k'})$. Furthermore, we have
\[
E[J_{R_1} \cdots J_{R_{k+k'}}] = E[J_A^2] = \gamma
\]
and
\[
\text{cov}(J_{R_1} \cdots J_{R_{k+k}}, J_{R_{k+1}} \cdots J_{R_{k+k'}})
\]
\[
= E[J_{R_1} \cdots J_{R_{k+k}}] - E[J_{R_1} \cdots J_{R_k}] \cdot E[J_{R_{k+1}} \cdots J_{R_{k+k'}}]
\]
\[
= \gamma - 1.
\]
Hence, we have
\[
\text{cov}_{2,1,1} = \frac{\gamma - 1}{L_n^2} \frac{i^{q_n(k+k')/2}}{(q_n)^{(k+k')/2}} \sum_{(R_1, \ldots, R_{k+k'}) \in Q_{1,1}^{(k,k')} (I_n^{k+k'})} \text{Tr} \Psi_{R_1} \cdots \Psi_{R_k} \text{Tr} \Psi_{R_{k+1}} \cdots \Psi_{R_{k+k'}}.
\]
We first consider the case $a \in (0, +\infty)$. By definition (6), we can rewrite
\[
\text{cov}_{2,1} = \frac{\gamma - 1}{L_n} \frac{\pi^{n(k+k')/2}}{(n^k) (k+k')/2} \sum_{\pi \in S_k} \sum_{\pi' \in S_{k'}} \sum_{R_1, \ldots, R_{k+k'}/2-1 \in I_n, R_i \neq R_j \text{ if } i \neq j} \frac{\Tr \Psi_{R_{\pi(1)}(1)} \cdots \Psi_{R_{\pi(k)}(k)}}{(k/2 - 1)! (k'/2 - 1)!}.
\]
By the anticommutative relation (2), for any fixed $r$ and $q$, we easily have (see [6])
\[
(15) \quad \frac{\pi^{n(k+k')/2}}{L_n} \frac{\Tr \Psi_{R_{\pi(1)}(1)} \cdots \Psi_{R_{\pi(k)}(k)}}{(k/2 - 1)! (k'/2 - 1)!} = (-1)^{\sum_{k=1}^{n} |R_{\pi(k)} \cap R_{\pi(k')}|}.
\]
We also need the following lemma dealing with the cardinality of the intersection of the coordinates $|R_{\pi(k)} \cap R_{\pi(k')}|$. 

Lemma 3. When $q_n^2/n \rightarrow a \in (0, \infty)$, if we choose $\{R_1, \cdots, R_{k'}\}$ uniformly from $I_n^+$ with $R_i \neq R_j$ if $i \neq j$, then the intersection numbers $|R_{\pi(k)} \cap R_{\pi(k')}|, k = 1, \cdots, \kappa(\pi)$ are approximately independently Poisson($a$) distributed. Here, $\{r_k, s_k\}_{k=1}^{\kappa}$ are crossings of $\pi$.

With indentity (15) and Lemma 3 for any fixed map $\pi \in S_k, \pi' \in S_{k'}$, let $\{r_1, s_1\}, \{r_2, s_2\}, \cdots, \{r_{\kappa(\pi)}, s_{\kappa(\pi)}\}$ be the crossings of $\pi$ and $\{r'_1, s'_1\}, \{r'_2, s'_2\}, \cdots, \{r'_{\kappa(\pi')}, s'_{\kappa(\pi')}\}$ be the crossings of $\pi'$, then we have
\[
\lim_{n \to \infty} \frac{1}{L_n} \frac{\pi^{n(k+k')/2}}{(n^k) (k+k')/2} \sum_{R_1, \cdots, R_{k+k'/2-1} \in I_n, R_i \neq R_j \text{ if } i \neq j} \frac{\Tr \Psi_{R_{\pi(1)}(1)} \cdots \Psi_{R_{\pi(1+k')/2-1}} \cdots \Psi_{R_{\pi(k'+1)}(k'+1)}}{(k/2 - 1)! (k'/2 - 1)!} = \lim_{n \to \infty} \frac{1}{L_n} \frac{\pi^{n(k+k')/2}}{(n^k) (k+k')/2} \sum_{R_1, \cdots, R_{k+k'/2-1} \in I_n, R_i \neq R_j \text{ if } i \neq j} (-1)^{\sum_{k=1}^{\kappa(\pi)} |R_{\pi(k)} \cap R_{\pi(k')}|} (-1)^{\sum_{k=1}^{\kappa(\pi')} |R_{\pi'(k+k')} \cap R_{\pi'(k+k')}|}
\]
\[
= \sum_{m_1 \geq 0, 1 \leq i \leq \kappa(\pi)} (-1)^{m_1 + \cdots + m_{\kappa(\pi)}} \frac{a^{m_1 + \cdots + m_{\kappa(\pi)}}}{m_1! \cdots m_{\kappa(\pi)}!} e^{-a\kappa(\pi)},
\]
\[
= e^{-2a\kappa(\pi)} e^{-2a\kappa(\pi')}.
\]
Therefore, by definition (7), we will get
\[
(16) \quad \lim_{n \to \infty} \frac{n}{q_n} \text{cov}_{2,1} = \gamma - 1 \frac{\pi^{n(k+k')/2}}{(k/2 - 1)! (k'/2 - 1)!} \sum_{\pi \in S_k} e^{-2a\kappa(\pi)} \sum_{\pi' \in S_{k'}} e^{-2a\kappa(\pi')} = (m_k^a k/2)(m_{k'}^a k'/2)(\gamma - 1).
\]
Actually, (16) is also true if $k$ and $k'$ are both odd, since $\text{cov}_{2,1}$ and $m_k^a$ are both 0 for such case. The above arguments making use of the crossing numbers still work for the case $a = 0$. Therefore, we prove (12) when $a \in [0, \infty)$ and $k + k'$ is even.
The above arguments do not work for the case for \( a = +\infty \), but we can use Lemma 5 in [3] to conclude that if \( k + k' \) is even, we still have
\[
\lim_{n \to \infty} \left( \frac{n}{q_n} \right) \text{cov}_2 = \lim_{n \to \infty} \left( \frac{n}{q_n} \right) \text{cov}_2,1,1 = (m_k^2 k/2) (m_k^2 k'/2) (\gamma - 1).
\]

To summarize, combining the estimate of \( \text{cov}_1 \) and \( \text{cov}_2 \), for \( k + k' \) even and \( a \in [0, \infty] \), we finally prove
\[
\lim_{n \to \infty} \left( \frac{n}{q_n} \right) \text{cov}(L_n^{-1} \text{Tr} H^k, L_n^{-1} \text{Tr} H^{k'}) = (m_k^2 k/2) (m_k^2 k'/2) (\gamma - 1).
\]

In particular, for any \( k \), we have
\[
\lim_{n \to \infty} \left( \frac{n}{q_n} \right) \text{var}[L_n^{-1} \text{Tr} H^k] = (m_k^2 k/2)^2 (\gamma - 1).
\]

In the end, if \( k \) is odd and \( k' \) is even, we have
\[
\left| \left( \frac{n}{q_n} \right) \text{cov}(L_n^{-1} \text{Tr} H^k, L_n^{-1} \text{Tr} H^{k'}) \right|^2 \leq \left( \frac{n}{q_n} \right) \text{var}[L_n^{-1} \text{Tr} H^k] \left( \frac{n}{q_n} \right) \text{var}[L_n^{-1} \text{Tr} H^{k'}]
\rightarrow (m_k^2 k/2)^2 (m_k^2 k'/2)^2 (\gamma - 1)^2 = 0,
\]
since \( m_k^0 = 0 \) when \( k \) is odd by definition.

Therefore, for \( k + k' \) odd, we have
\[
\lim_{n \to \infty} \left( \frac{n}{q_n} \right) \text{cov}(L_n^{-1} \text{Tr} H^k, L_n^{-1} \text{Tr} H^{k'}) = 0 = (m_k^2 k/2) (m_k^2 k'/2) (\gamma - 1).
\]

This completes the proof except the estimate of \( |B_m| \).

Now we prove the following technical lemma on the estimate of \( |B_m| \) to finish the proof of Lemma 1.

**Lemma 4.** Let
\[
B_m = \{(R_1, \ldots, R_m) \in I_n^m | \Psi_{R_1} \cdots \Psi_{R_m} = \pm I, R_i \neq R_j, \forall 1 \leq i < j \leq m \}.
\]

Then we have the estimate
\[
|B_m| \leq c |I_n|^{m-1} n^{-\frac{m}{2}},
\]
where \( c \) is an absolute constant independent of \( m, n, q_n \).

**Proof.** By definition we have \( B_1 = B_2 = \emptyset \), and we only need to consider the case \( m \geq 3 \). Let \( B_m^* = \{(R_1, \ldots, R_m) \in I_n^m | \Psi_{R_1} \cdots \Psi_{R_m} = \pm I \} \), then we have \( B_m \subseteq B_m^* \) and \( |B_m| \leq |B_m^*| \). And we need to estimate \( B(m, n, q_n) = |B_m^*| / |I_n|^{m-1} \).

**Case 1:** \( m = 3 \). We have \( B_m^* = \{(R_1, R_2, R_3) \in I_n^m | \Psi_{R_1} \Psi_{R_2} = \pm \Psi_{R_3} \} = \{(R_1, R_2, R_3) \in I_n^m | \Psi_{R_1} \Delta R_2 = \pm \Psi_{R_3} \} = \{(R_1, R_2, R_3) \in I_n^m | R_1 \Delta R_2 = R_3 \} \) if \( (R_1, R_2, R_3) \in B_m^* \), then \( |B_m| = |R_1 \Delta R_2| = |R_3| + |R_2| - 2 |R_1 \cap R_2| \) and \( |R_3| = |R_2| = q_n/2 \). There are \( \binom{n}{q_n/2} \) choices of \( R_1 \), for every fixed \( R_1 \), there are \( \binom{n-q_n/2}{q_n/2} \) choices of \( R_2 \) satisfying \( |R_1 \cap R_2| = q_n/2 \). Therefore, \( |B_m| = (\binom{n}{q_n/2}) (\binom{n-q_n/2}{q_n/2}) \) and
\[
B(3, n, q_n) = |B_m^*/|I_n|^2 = (\binom{n}{q_n/2}) (\binom{n-q_n/2}{q_n/2}) / (\binom{n-q_n/2}{q_n/2}) = (\binom{n-q_n/2}{q_n/2} / (n-q_n/2) / q_n).
\]

If \( q_n \) is odd or \( n < 3 q_n/2 \), then \( B(3, n, q_n) = |B_m| = 0 \). If \( q_n \) is even and \( n > 3 q_n/2 \), then \( B(3, n, q_n) / B(3, n-1, q_n) = (\binom{n-q_n/2}{q_n/2} / (n-q_n/2) / q_n) = (n-q_n/2) / (n-q_n/2) / q_n) = \frac{n-q_n}{n-3q_n/2} \frac{n-q_n}{n} = 1 - \frac{q_n(n-2q_n)}{(2n-3q_n)n} \) (Notice that the expression of \( B(3, n, q_n) \) is well
defined for every positive integer \( n \). Thus for fixed \( q_n \), \( B(3, n, q_n) \) is increasing for \( 3q_n/2 \leq n \leq 2q_n \) and decreasing for \( n \geq 2q_n \), which implies \( B(3, n, q_n) \leq B(3, 2q_n, q_n) \). We notice that

\[
B(3, 2q_n, q_n) = \left( \frac{q_n}{q_n/2} \right)^2 / \left( \frac{2q_n}{2q_n} \right) = \frac{(q_n)^2}{((q_n/2)^2)^{(2q_n)}} = \prod_{j=1}^{q_n/2} \left( \frac{2j}{4j(4j-1)(4j-2)(4j-3)} \right)^{2j-1}
\]

Thus, if \( R_n \), then

\[
\prod_{j=1}^{q_n/2} \left( \frac{2j-1}{2j} \right) \left( 1 - \frac{1}{4(2j-1)^2} \right) = \prod_{j=1}^{q_n/2} \left( \frac{j+1}{2j} \right)^{1/2}
\]

Therefore, if \( 3q_n/2 \leq n \leq 3q_n \), then

\[
B(3, n, q_n) \leq B(3, 2q_n, q_n) \leq (q_n/2 + 1)^{1/2} \leq cn^{1/2}.
\]

If \( n > 3q_n \), then \( \frac{n-2q_n}{(2n-3q_n)} > \frac{1}{3} \) and \( B(3, n, q_n) / B(3, n-1, q_n) = 1 - \frac{q_n(n-2q_n)}{(2n-3q_n)n} \leq 1 - \frac{q_n}{3n} \). Thus, if \( n > 3q_n \) and \( q_n > 2 \), then

\[
B(3, n, q_n) = B(3, 3q_n, q_n) \prod_{j=3q_n+1}^{n} B(3, j, q_n) / B(3, j-1, q_n)
\]

\[
\leq B(3, 2q_n, q_n) \prod_{j=3q_n+1}^{n} \left( 1 - \frac{q_n}{3j} \right) \leq (q_n/2 + 1)^{1/2} \prod_{j=3q_n+1}^{n} \left( 1 - \frac{1}{3} \right)
\]

\[
= (q_n/2 + 1)^{1/2} \frac{3q_n}{n} \leq cn^{1/2}.
\]

If \( n > 3q_n \) and \( q = 2 \), then

\[
B(3, n, q_n) = \begin{pmatrix} 2 \\ n \end{pmatrix} / \begin{pmatrix} 2 \\ 1 \end{pmatrix} / \begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{4(n-2)}{n(n-1)} \leq \frac{4}{n} \leq cn^{1/2}.
\]

Therefore, \( B(3, n, q_n) \leq cn^{1/2} \) is always true and \( |B_m| \leq |B^*_m| = B(3, n, q_n)|I_n|^{m-1} \leq cn^{-1/2} \).

**Case 2:** \( m = 4 \). We have \( B^*_m = \{ (R_1, R_2, R_3, R_4) \in I_n^m : \Psi_R \Psi_R = \pm \Psi_R \} = \{ (R_1, R_2, R_3, R_4) \in I_n^m : \Psi_R \Psi_R = \pm \Psi_R \} = \{ (R_1, R_2, R_3, R_4) \in I_n^m : R_1 \sim \alpha R_2 = R_3 \sim \alpha R_4 \}. \) If \( (R_1, R_2, R_3, R_4) \in B^*_m \), let \( A = R_1 \Delta R_2 = R_3 \Delta R_4 \), then \( |A| = |R_1 \Delta R_2| = |R_1| + |R_2| - 2|R_1 \cap R_2| = 2q_n - 2|R_1 \cap R_2| \) is even and \( |A| \leq 2q_n \), \( |A \cap R_1| = |R_1| - |R_1 \cap R_2| = q_n - (2q_n - |A|)/2 = |A|/2 \), and the other we also have \( |A| = |R_1 \Delta R_2| = 2|R_1 \cup R_2| - |R_1| - |R_2| \leq 2n - 2q_n \). For fixed \( A \), assume \( |A| = 2k \), then there are \( \binom{2k}{k} \binom{n-2k}{q_n-k} \) choices of \( R_1, R_2 \) satisfying \( |R_1| = q_n, |A \cap R_1| = k \) and \( R_2, R_3 \) is uniquely determined by \( R_1, A \). Similarly there are \( \binom{2}{k} \binom{n-2k}{q_n-k} \) choices of \( (R_3, R_4) \). Moreover for every fixed integer \( k, 0 \leq k \leq \min(q_n, n - q_n) \), there are \( \binom{n}{2k} \binom{2k}{k} \binom{n-2k}{q_n-k} \) choices of \( A \) satisfying \( |A| = 2k \). Therefore, we have

\[
|B^*_m| = \sum_{k=0}^{\min(q_n, n-q_n)} \binom{n}{k} \binom{2k}{k} \binom{n-2k}{q_n-k}^2.
\]
Notice that
\[
\binom{n}{2k} \binom{n-2k}{k} = \frac{n!}{(2k)!(n-2k)!} \frac{(2k)!}{(2k)!} \frac{(n-2k)!}{(n-qn-k)!} = \frac{n!}{k!(q_n-k)!k!(n-q_n-k)!} = \binom{n}{q_n} \binom{n}{k} \binom{n-q_n}{k},
\]
then we have
\[
|B_n^*| = \sum_{k=0}^{\min(q_n, n-q_n)} \binom{n}{q_n} \binom{2}{k} \binom{n-2k}{q_n-k} \binom{n-q_n}{k} \binom{n}{2k}
\]
and
\[
B(4, n, q_n) = |B_n^*|/|I_n|^3 = \binom{n}{q_n}^{-1} \sum_{k=0}^{\min(q_n, n-q_n)} \binom{q_n}{k} \binom{n-2k}{k} \binom{n-q_n}{k} \binom{n}{2k}
\]
Therefore, \(B(4, n, q_n) = B(4, n, n-q_n)\), thus we only need to consider the case \(2 \leq q_n \leq n/2\). If \(2 \leq q_n \leq n/10\), then \(\binom{n}{2k} \binom{n-2k}{k} \geq \binom{n-q_n}{2k}\) for \(0 \leq k \leq q_n\) we have
\[
B(4, n, q_n) \leq \binom{n}{q_n}^{-1} \sum_{k=0}^{q_n} \binom{2q_n}{2k} \leq \binom{n}{q_n}^{-1} \sum_{j=0}^{2q_n} \sum_{j=0}^{2q_n} \binom{2q_n}{j} = \binom{n}{q_n}^{-1} 2^{3q_n}
\]
If \(n/10 \leq q_n \leq n/2\), then for \(0 \leq k \leq q_n\) and \(n\) even, we have
\[
\binom{n-q_n}{k} \binom{n}{k}^{-1} = \binom{2k}{k} \prod_{j=0}^{k-1} \frac{(n-q_n-j)(n-q_n-j)}{(n-2j)(n-2j-1)} \leq \binom{2k}{k} \prod_{j=0}^{k-1} \frac{(n/2-j)^2}{(n-2j)(n-2j-1)} = \prod_{j=1}^{k'} \frac{(2j-1-n+2-2j}{(2j+1)} \leq \prod_{j=1}^{k'} \frac{2j-1}{2j+1} \frac{1}{n} \frac{n}{q_n} \leq \frac{9}{n} \leq cn^{-\frac{1}{2}}.
\]
where \(k' = \min(k, n/2-k) \leq n/4\). For \(j = 0, 1, 2\), we have
\[
a_j := \sum_{k=0}^{q_n} \binom{k}{j} \binom{q_n}{k} \binom{n-q_n}{k} = \sum_{k=j}^{q_n} \binom{q_n-j}{k-j} \binom{n-q_n}{k} = \binom{q_n}{j} \binom{n-j}{k-j} \binom{n-q_n}{k-j} \binom{n-q_n-j}{k-j}
\]
and \((k-\mu)^2 = 2\binom{k}{2} - (2\mu-1)\binom{k}{1} + \mu^2\binom{k}{0}\). Therefore, for \(\mu = \frac{2(n-q_n)}{n}\), we have
\[
\sum_{k=0}^{q_n} (k-\mu)^2 \binom{q_n}{k} (n-q_n-k) = 2a_2 - (2\mu-1)a_1 + \mu^2a_0
\]
\[
\begin{align*}
&= 2 \binom{n}{k} \binom{n-2}{q_n-2} - (2\mu - 1)q_n \binom{n-1}{q_n-1} + \mu^2 \binom{n}{q_n} \\
&= \binom{n}{q_n} \left(\frac{q_n(q_n-1)(n-q_n)(n-q_n-1)}{n(n-1)}\right) - (2\mu - 1)q_n \binom{n}{q_n} + \mu^2 \\
&= \binom{n}{q_n} \mu \left(\frac{(q_n-1)(n-q_n-1)}{n(n-1)}\right) - \mu + 1 = \binom{n}{q_n} \mu \binom{n}{q_n} \left(\frac{n}{n(n-1)}\right) 
\end{align*}
\]

as \(9n/100 \leq \mu \leq n/4\) (using \(n/10 \leq q_n \leq n/2\)) and \((k-\mu)^2/\mu^2 + \mu^{-\frac{1}{2}} \geq \epsilon^{-1} \left(\frac{2}{k'(k'+1)}\right)^{\frac{1}{2}}\) for \(k' = \min(k, n/2 - k)\) (one can check this by discussing the cases \(|k - \mu| \leq \mu/2\) and \(|k - \mu| \geq \mu/2\) separately), which implies

\[
\left(\binom{n}{q_n} \binom{n-2}{q_n-2} \binom{n}{2k}\right)^{-1} \leq c \left(\frac{(k-\mu)^2}{\mu^2} + \mu^{-\frac{1}{2}}\right)
\]

and that

\[
B(4, n, q_n) = \binom{n}{q_n}^{-1} \sum_{k=0}^{q_n} \binom{n}{k} \binom{n-2}{q_n}^2 \binom{n}{2k}^{-1} \\
\leq c \left(\binom{n}{q_n}^{-1} \sum_{k=0}^{q_n} \binom{n}{k} \binom{n-2}{q_n} \left(\frac{(k-\mu)^2}{\mu^2} + \mu^{-\frac{1}{2}}\right)\right) \\
= c \binom{n}{q_n}^{-1} \binom{n}{q_n} \left(\frac{\mu}{\mu^2} \binom{n-2}{q_n} + \mu^{-\frac{1}{2}}\right) \\
= c \left(\frac{1}{n-1} + \mu^{-\frac{1}{2}}\right) \leq cn^{-\frac{1}{2}}.
\]

If \(n/2 \leq q_n \leq n - 2\), then \(B(4, n, q_n) = B(4, n, n - q_n) \leq cn^{-\frac{1}{2}}\). Therefore, \(B(4, n, q_n) \leq cn^{-\frac{1}{2}}\) is always true and \(|B_m| \leq |B_m^*| = B(4, n, q_n)|I_n|^{m-1} \leq c|I_n|^{m-1}n^{-\frac{1}{2}}\).

**Case 3:** \(m \geq 4\). We need to prove that \(|B_m^*| \leq |B_m^*||I_n|^{-4}\). Since \(\text{Tr} \Psi_{R_1} \cdots \Psi_{R_m} = 0\) for \(\Psi_{R_1} \cdots \Psi_{R_m} \neq \pm I\), we have

\[
|B_m^*| = L_n^{-2} \sum_{(R_1, \ldots, R_m) \in I_m^m} (\text{Tr} \Psi_{R_1} \cdots \Psi_{R_m})^2 \\
= L_n^{-2} \sum_{(R_1, \ldots, R_m) \in I_m^m} \text{Tr}(\Psi_{R_1} \cdots \Psi_{R_m} \otimes \Psi_{R_1} \cdots \Psi_{R_m}) \\
= L_n^{-2} \sum_{(R_1, \ldots, R_m) \in I_m^m} \text{Tr}(\Psi_{R_1} \otimes \Psi_{R_1}) \cdots (\Psi_{R_m} \otimes \Psi_{R_m}) \\
= L_n^{-2} \text{Tr} \left(\sum_{R_1 \in I_n} \Psi_{R_1} \otimes \Psi_{R_1}\right) \cdots \left(\sum_{R_m \in I_n} \Psi_{R_m} \otimes \Psi_{R_m}\right) \\
= L_n^{-2} \text{Tr} \left(\sum_{R \in I_n} \Psi_R \otimes \Psi_R\right)^m.
\]

Since \(\Psi_R\) is a \(L_n \times L_n\) Hermitian or anti-Hermitian matrix, \(\Psi_R \otimes \Psi_R\) is a \(L_n^2 \times L_n^2\) Hermitian matrix and \(\tilde{H} = \sum_{R \in I_n} \Psi_R \otimes \Psi_R\) is a \(L_n^2 \times L_n^2\) Hermitian matrix. Let’s
assume that the eigenvalues of \( \tilde{H} \) are \( \mu_j \in \mathbb{R} \) (1 \( \leq j \leq L_n^2 \)), then we have

\[
|B_m^*| = L_n^{-2} \text{Tr} \tilde{H}^m = L_n^{-2} \sum_{j=1}^{L_n^2} \mu_j^m.
\]

As \( \Psi_R^2 = \pm I \), we have \( (\Psi_R \otimes \Psi_R)| = |x| \) for \( x \in \mathbb{C}^{L_n^2} \). Therefore, \( |H| \leq \sum_{R \in I_n} |\Psi_R \otimes \Psi_R| = |I_n||x| \) which implies \( |\mu_j| \leq |I_n| \).

Now we have

\[
|B_m| \leq |B_m^*| = L_n^{-2} \sum_{j=1}^{L_n^2} \mu_j^m \leq L_n^{-2} \sum_{j=1}^{L_n^2} \mu_j^4 |I_n|^{m-4} = |B_1^*||I_n|^{m-4}
\]

\[
= B(4, n, q_n)|I_n|^{m-1} \leq c|I_n|^{m-1} n^{-\frac{1}{2}}.
\]

This completes the proof. \( \Box \)

### 3.2. Proof of Theorem 1

Let \( f(x) = \sum_{k=0}^{m} a_k x^k \) be a real polynomial, then

\[
\mathcal{L}_n(f) = \sum_{k=0}^{m} a_k \langle x^k, \rho_n \rangle = \sum_{k=0}^{m} a_k L_n^{-1} \text{Tr} H^k.
\]

We first have

**Lemma 5.** With the same assumptions as in Theorem 1, if \( 2 \leq q_n \leq n/2 \) is even, then

\[
\lim_{n \to +\infty} \left( \frac{n}{q_n} \right) \var[\mathcal{L}_n(f)] = \langle x^f/2, \rho_\infty \rangle^2 (\gamma - 1).
\]

**Proof.** Since \( \mathcal{L}_n(f) - a_0 = \sum_{k=1}^{m} a_k L_n^{-1} \text{Tr} H^k \), by Lemma 1 we have

\[
\lim_{n \to +\infty} \left( \frac{n}{q_n} \right) \var[\mathcal{L}_n(f)] = \lim_{n \to +\infty} \left( \frac{n}{q_n} \right) \sum_{k=1}^{m} \sum_{k'=1}^{m} a_k a_{k'} \text{cov}(L_n^{-1} \text{Tr} H^k, L_n^{-1} \text{Tr} H^{k'})
\]

\[
= \sum_{k=1}^{m} \sum_{k'=1}^{m} a_k a_{k'} (m_k^2 k/2)(m_{k'}^2 k'/2) (\gamma - 1)
\]

\[
= \left( \sum_{k=1}^{m} a_k m_k^2 k/2 \right)^2 (\gamma - 1) = \left( \sum_{k=1}^{m} a_k \langle x^k, \rho_\infty \rangle k/2 \right)^2 (\gamma - 1)
\]

\[
= \left( \sum_{k=1}^{m} a_k x^k k/2, \rho_\infty \right)^2 (\gamma - 1).
\]

Since \( xf'(x)/2 = \sum_{k=1}^{m} a_k x^k k/2 \), this completes the proof. \( \square \)

Now we can finish the proof of Theorem 1.
Proof. Let $2 \leq q_n \leq n/2$ be even. We first consider the case $f(x) = x^2$ where
\[
\langle f, \rho_n(\lambda) \rangle = L_n^{-1} \text{Tr} H^2 = \frac{1}{L_n} \left( \frac{q_n}{n^2} \right) \sum_{R_1, R_2 \in I_n} J_{R_1} J_{R_2} \text{Tr} \Psi_{R_1} \Psi_{R_2}.
\]

As discussed in [2], if $R_1 \neq R_2$, then $\Psi_{R_1} \Psi_{R_2} \neq \pm I$, $\text{Tr} \Psi_{R_1} \Psi_{R_2} = 0$; if $R_1 = R_2$, then $i^n \Psi_{R_1} \Psi_{R_2} = I$. Therefore
\[
\langle f, \rho_n(\lambda) \rangle = L_n^{-1} \text{Tr} H^2 = \frac{1}{(q_n/n)} \sum_{R \in I_n} J_R^2.
\]
Since $\mathbb{E}[J_R^2] = 1$, we have
\[
\langle f, \rho_n(\lambda) \rangle = \mathbb{E}[\langle f, \rho_n(\lambda) \rangle] = \frac{1}{(q_n/n)} \sum_{R \in I_n} (J_R^2 - 1).
\]

The random variables $J_R^2 - 1$ are independent with $\mathbb{E}[J_R^2 - 1] = 0$ and $\text{var}[J_R^2 - 1] = \mathbb{E}[J_R^4] - 1 = \gamma - 1$. By assumptions, $\mathbb{E}[J_R^2 - 1]$ is uniformly bounded, therefore, the random variables $J_R^2 - 1$, $R \in I_n$ satisfy the Lyapunov condition. Thus, by Lindeberg-Feller central limit law, we have
\[
\left( \frac{n}{q_n} \right)^{1/2} (\langle f, \rho_n \rangle - \mathbb{E}[\langle f, \rho_n \rangle]) \to J,
\]
where $J$ is Gaussian random variable with mean 0 and variance $\gamma - 1$. Since $m_a^q = 1$ for $a \in [0, +\infty]$, thus $\langle x f' / 2, \rho_\infty \rangle = \langle x^2, \rho_\infty \rangle = m_a^2 = 1$, this will imply that Theorem 4 is true for $f(x) = x^2$.

Now we consider the case for general polynomials $f(x)$. Let $\mu = \langle x f' / 2, \rho_\infty \rangle$ and define $f_1 = f - \mu x^2$, then we have
\[
\langle x f_1' / 2, \rho_\infty \rangle = \langle x f' / 2, \rho_\infty \rangle - \mu \langle x^2, \rho_\infty \rangle = 0.
\]
Thus, by Lemma 3 we have
\[
\lim_{n \to +\infty} \left( \frac{n}{q_n} \right) \text{var}[\langle f_1, \rho_n \rangle] = \langle x f_1' / 2, \rho_\infty \rangle^2 = (\gamma - 1) = 0.
\]
Therefore, we have
\[
\left( \frac{n}{q_n} \right)^{1/2} (\langle f_1, \rho_n \rangle - \mathbb{E}[\langle f_1, \rho_n \rangle]) \to 0
\]
in probability. Now we have
\[
\left( \frac{n}{q_n} \right)^{1/2} (\langle f, \rho_n \rangle - \mathbb{E}[\langle f, \rho_n \rangle]) = \left( \frac{n}{q_n} \right)^{1/2} (\langle f_1, \rho_n \rangle - \mathbb{E}[\langle f_1, \rho_n \rangle]) + \mu \left( \frac{n}{q_n} \right)^{1/2} (\langle x^2, \rho_n \rangle - \mathbb{E}[\langle x^2, \rho_n \rangle]).
\]

The first term tends to 0 in probability and the second term tends to $\mu J$ in distribution, therefore, we prove
\[
\left( \frac{n}{q_n} \right)^{1/2} (\langle f, \rho_n \rangle - \mathbb{E}[\langle f, \rho_n \rangle]) \Rightarrow \mu J.
\]
By definition of $\mu$, Theorem 4 is proved when $2 \leq q_n \leq n/2$ is even. For even $n/2 \leq q_n < n$, the results follow immediately since there is a natural symmetry between the cases of $n - q_n$ and $q_n$. For odd $q_n$, the results follow with almost the same proof. Thus we complete the proof of Theorem 4.
Remark 2. As a remark, let $c_k$ be the constants in \((13)\), then Theorem 7 holds for a class of analytic functions $f(x) = \sum_{k=0}^{\infty} a_k x^k$ with $\sum_{k=0}^{\infty} |a_k| c_k^k < +\infty$.

To see this, we first have

\[
\left( \frac{n}{q_n} \right) \var[(f, \rho_n)] = \left( \frac{n}{q_n} \right)^{+\infty} a_k a_{k'} \cov(L_n^{-1} \Tr H^k, L_n^{-1} \Tr H^{k'}). 
\]

By \((13)\), we have the upper bound

\[
\left( \frac{n}{q_n} \right) |\cov(L_n^{-1} \Tr H^k, L_n^{-1} \Tr H^{k'})| \leq (c_k c_{k'})^{1/2}.
\]

Then by assumption, we have

\[
\sum_{k=1}^{+\infty} \sum_{k'=1}^{+\infty} |a_k a_{k'}|(c_k c_{k'})^{1/2} = \left( \sum_{k=0}^{\infty} |a_k| c_k^k \right)^2 < +\infty.
\]

Therefore, by the dominated convergence theorem, Lemma 4 holds for $f(x)$ if we take the limit on both sides of \((17)\), and hence Theorem 1.

4. Improved CLT for the Gaussian SYK

In this section, we will prove that the CLT for the linear statistic of the Gaussian SYK model holds for a more general class of functions. We will prove Theorem 2 by approximations making use of the Féjér kernel and Theorem 4.

4.1. Estimate of variance. We first need the following estimate in the Gaussian case which is more precise compared with Lemma 3.

**Lemma 6.** Let $q_n$ be even, for the Gaussian SYK model, we have

\[
\var[L_n^{-1} \Tr H^k] \leq c_k \left( \frac{n}{q_n} \right)^{-1}
\]

for any $k \geq 1$ with $c_k = 2^{k+1} k! k^2$.

**Proof.** As in the beginning of the proof of Lemma 4, we first have

\[
\var[L_n^{-1} \Tr H^k] = \frac{1}{L_n^2} \frac{(-1)^{q_n, k/2}}{q_n^k} \left( \sum_{(R_1, \ldots, R_{2k}) \in P_2(I^2_k)} + \sum_{(R_1, \ldots, R_{2k}) \in I^2_k \setminus P_2(I^2_k)} \right) \cov(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{2k}}) \cdot \Tr \Psi_{R_1} \cdots \Tr \Psi_{R_k} \Tr \Psi_{R_{k+1}} \cdots \Tr \Psi_{R_{2k}} \equiv V_1 + V_2.
\]

We can write $J_{R_1} \cdots J_{R_k} = J_{R_1}^{a_1} \cdots J_{R_k}^{a_k}$, $J_{R_{k+1}} \cdots J_{R_{2k}} = J_{R_1}^{b_1} \cdots J_{R_k}^{b_k}$ such that $J_{R_1}^{a_1}, \ldots, J_{R_k}^{a_k}$ are distinct and $a_j, b_j \geq 0$ are integers. Then we have $J_{R_1} \cdots J_{R_{2k}} = J_{R_1}^{a_1+b_1} \cdots J_{R_k}^{a_k+b_k}$. Let $\gamma_j := \Ec_{J_{R_1}}$, $R \in I_n$, $j \geq 0$, $j \in \Z$, then $\gamma_j$ satisfies $\gamma_j = (2j-1)!!$ for $j$ even and $\gamma_j = 0$ for $j$ odd, $\gamma_j \geq 0$ and $\gamma_{j+m} \geq \gamma_j \gamma_m$. Thus we have

\[
\cov(J_{R_1} \cdots J_{R_k}, J_{R_{k+1}} \cdots J_{R_{2k}}) = \Ec_{J_{R_1}} \cdots J_{R_{2k}} - \Ec_{J_{R_1}} \cdots J_{R_{k}} \Ec_{J_{R_{k+1}}} \cdots J_{R_{2k}} = \Ec_{J_{R_1}^{a_1+b_1}} \cdots J_{R_k}^{a_k+b_k} - \Ec_{J_{R_1}^{a_1}} \cdots J_{R_k}^{a_k} \Ec_{J_{R_1}^{b_1}} \cdots J_{R_k}^{b_k}.
\]
By (15) where \(|L_n^{-1} \text{Tr} \Psi_{R_1} \cdots \Psi_{R_k}| \leq 1 \) and \(|L_n^{-1} \text{Tr} \Psi_{R_{k+1}} \cdots \Psi_{R_{2k}}| \leq 1\), we have

\[
|V_2| \leq \binom{n}{q_n}^{-k} \sum_{(R_1, \ldots, R_{2k}) \in I_{2k}^n \setminus P_2(I_{2k}^n)} |\text{cov}(J_{R_1} \cdots J_{R_{2k}}, J_{R_{k+1}} \cdots J_{R_{2k}})|
\]

\[
\leq \binom{n}{q_n}^{-k} \sum_{(R_1, \ldots, R_{2k}) \in I_{2k}^n \setminus P_2(I_{2k}^n)} \mathbb{E}J_{R_1} \cdots J_{R_{2k}}
\]

\[
= \binom{n}{q_n}^{-k} \left( \sum_{(R_1, \ldots, R_{2k}) \in I_{2k}^n \setminus P_2(I_{2k}^n)} \mathbb{E}J_{R_1} \cdots J_{R_{2k}} - \sum_{(R_1, \ldots, R_{2k}) \in P_2(I_{2k}^n)} \mathbb{E}J_{R_1} \cdots J_{R_{2k}} \right)
\]

\[
= |I_n|^{-k} \left( \mathbb{E} \left( \sum_{R \in I_n} J_R \right) - |P_2(I_{2k}^n)| \right) = \gamma_{2k} - |I_n|^{-k} |P_2(I_{2k}^n)|
\]

\[
= (2k - 1)!! - |I_n|^{-k} \binom{|I_n|}{k} k!(2k - 1)!! = (2k - 1)!! \left( 1 - \prod_{j=0}^{k-1} \frac{|I_n| - j}{|I_n|} \right)
\]

\[
\leq (2k - 1)!! \sum_{j=0}^{k-1} \frac{j}{|I_n|} = (2k - 1)!! \frac{k(k-1)}{2} |I_n|.
\]

Here, we used the fact that \(\sum_{R \in I_n} J_R\) has the standard Gaussian distribution.

For the estimate of \(V_1\), we first easily have

\[
V_1 \leq \binom{n}{q_n}^{-k} |P_{2,1}(I_{2k}^n)|.
\]

In the Gaussian case, following the proof of Lemma 2 (see [6] for the detailed proof), we further have the uniform estimate

\[
|P_{2,1}(I_{2k}^n)| = \sum_{0 < m \leq k, 2 | k - m} \binom{|I_n| - m}{k - m} \frac{(k - m)!}{2^{k - m} m!} \cdot |B_m|
\]

\[
\leq \sum_{0 < m \leq k, 2 | k - m} \frac{|I_n|^{k - m}}{(k - m)!} 2^{k - m} (k!)^2 / (2k - m)! \cdot |I_n|^{m-1}
\]

\[
= |I_n|^{k-1} \sum_{0 < m \leq k, 2 | k - m} \binom{k}{m} \frac{(k)!}{m!} \leq (2k^2 k! |I_n|^{k-1}.
\]
Using $|I_n| = \binom{n}{q_n}$, we have
\[ \text{var}[I_n^{-1} \text{Tr } H^k] = V_1 + V_2 \leq \binom{n}{q_n}^{-k} |P_{2,1}(I_n^{2k})| + V_2 \]
\[ \leq (2^k k!) \binom{n}{q_n}^{-k} |I_n|^{k-1} + (2k-1)! k(k-1)/2 \leq (2k)!! (1 + k(k-1)/2) \leq (2k)!! k^2 = 2^k k^2 =: c_k. \]
This completes the proof. \qed

4.2. Flichter kernel and approximations. Let $K_\lambda(x)$ be the Fichter kernel
\[ K_\lambda(x) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \left|\frac{\xi}{\lambda}\right|\right) e^{i\xi x} d\xi = \frac{\lambda}{2\pi} \left(\frac{\sin(\lambda x/2)}{\lambda x/2}\right)^2, \quad \lambda > 0. \]

Lemma 7. $K_\lambda \in C^\infty(\mathbb{R})$ and the derivatives $|K_\lambda^{(k)}(x)| \leq \frac{\lambda^{k+1}}{2\pi} \max(1, \lambda x)^{-2}$.

Proof. By definition of $K_\lambda(x)$, we have $K_\lambda \in C^\infty(\mathbb{R})$ and
\[ K_\lambda^{(k)}(x) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) (i\xi)^k e^{i\xi x} d\xi \]
for $k \geq 0, k \in \mathbb{Z}^+$. Therefore,
\[ |K_\lambda^{(k)}(x)| \leq \frac{1}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) |\xi|^k d\xi \leq \frac{\lambda^k}{2\pi} \int_{-\lambda}^{\lambda} \left(1 - \frac{|\xi|}{\lambda}\right) d\xi = \frac{\lambda^{k+1}}{2\pi}. \]
On the other hand, if $x \in \mathbb{R} \setminus \{0\}$, we integrate by parts to get
\[ K_\lambda^{(k)}(x) = \frac{i^k}{2\pi (ix)^2} \left( \int_{-\lambda}^{\lambda} (k(k-1)\xi^{k-2} - k(k+1)\frac{|\xi|}{\lambda}\xi^{k-2}) e^{i\xi x} d\xi + \xi^{-1} e^{i\xi x}|_{-\lambda}^{\lambda} \right), \]
therefore,
\[ |K_\lambda^{(k)}(x)| \leq \frac{1}{2\pi x^2} \left( \int_{-\lambda}^{\lambda} (k(k-1)\xi^{k-2} - k(k+1)\frac{|\xi|}{\lambda}\xi^{k-2}) d\xi + 2\lambda^{k-1} \right) \]
\[ = \frac{1}{2\pi x^2} \left( \int_{-\lambda}^{\lambda} k(k-1) \left(1 - \frac{|\xi|}{\lambda}\right) \xi^{k-2} - 2k \frac{|\xi|}{\lambda}\xi^{k-2} d\xi + 2\lambda^{k-1} \right) \]
\[ \leq \frac{1}{2\pi x^2} \left( \int_{-\lambda}^{\lambda} k(k-1) \left(1 - \frac{|\xi|}{\lambda}\right) |\xi|^{k-2} + 2k \frac{|\xi|^{k-1}}{\lambda} d\xi + 2\lambda^{k-1} \right) \]
\[ = \frac{1}{2\pi x^2} \left( \int_{-\lambda}^{\lambda} k(k-1) |\xi|^{k-2} - k(k-3) \frac{|\xi|^{k-1}}{\lambda} d\xi + 2\lambda^{k-1} \right) \]
\[ = \frac{1}{2\pi x^2} \left( 2k \lambda^{k-1} \chi_{\{k>1\}} - 2(k-3) \frac{\lambda^k}{\lambda} + 2\lambda^{k-1} \right) \]
\[ \leq \frac{1}{2\pi x^2} \left( 2k \lambda^{k-1} - 2(k-3) \lambda^{k-1} + 2\lambda^{k-1} \right) = \frac{8\lambda^{k-1}}{2\pi x^2} \leq \frac{\lambda^{k+1}}{2\pi} (\lambda x)^{-2}. \]
Thus we have $|K_x^h(x)| \leq \frac{\lambda^{k+1}}{2\pi}$ for $x \in \mathbb{R}$ and $|K_\lambda^h(x)| \leq \frac{\lambda^{k+1}}{2\pi} (\lambda x/3)^{-2}$ for $x \in \mathbb{R} \setminus \{0\}$. This completes the proof. \hfill \Box

**Lemma 8.** For $f \in L^\infty(\mathbb{R})$, we have $f_\lambda := f \ast K_\lambda \in C^\infty(\mathbb{R})$ and $|f_\lambda^h(x)| \leq 2\lambda^k \|f\|_{L^\infty(\mathbb{R})}$, thus $f_\lambda$ is real analytic.

**Proof.** By Lemma 7, we have $K_\lambda^h \in L^1(\mathbb{R})$, actually

$$
\|K_\lambda^h\|_{L^1(\mathbb{R})} \leq \frac{\lambda^{k+1}}{2\pi} \max(1, \lambda x/3)^{-2} \|K_\lambda^h\|_{L^1(\mathbb{R})} = \frac{\lambda^{k+1}}{2\pi} \max(1, x)^{-2} \|K_\lambda^h\|_{L^1(\mathbb{R})}
$$

$$
= \frac{\lambda^{k+1}}{2\pi} \cdot 4 = \frac{12\lambda^k}{2\pi} \leq 2\lambda^k.
$$

Since $f_\lambda = f \ast K_\lambda$, $f \in L^\infty(\mathbb{R})$ and $K_\lambda \in C^\infty(\mathbb{R})$, thus $f_\lambda \in C^\infty(\mathbb{R})$ and $f_\lambda^h = f \ast K_\lambda^h$. Furthermore,

$$
|f_\lambda^h(x)| \leq \|f\|_{L^\infty(\mathbb{R})} \|K_\lambda^h\|_{L^1(\mathbb{R})} \leq 2\lambda^k \|f\|_{L^\infty(\mathbb{R})}.
$$

We can assume $f$ is real valued. By Taylor expansion, for $n \in \mathbb{Z}^+$, we have

$$
f_\lambda(x) = \sum_{k=0}^{n-1} \frac{f_\lambda^{(k)}(0)}{k!} x^k + \frac{f_\lambda^{(n)}(\theta x)}{n!} x^n, \quad x \in \mathbb{R},
$$

here $\theta = \theta(n, x) \in (0, 1)$. Now we have

$$
\frac{|f_\lambda^{(n)}(\theta x)|}{n!} x^n \leq \frac{2\lambda^n \|f\|_{L^\infty(\mathbb{R})}}{n!} x^n = \frac{2(\lambda|x|)^n \|f\|_{L^\infty(\mathbb{R})}}{n!}.
$$

Since $\lim_{n \to +\infty} \frac{(\lambda|x|)^n}{n!} = 0$, this implies $\lim_{n \to +\infty} \frac{f_\lambda^{(n)}(\theta x)}{n!} x^n = 0$ and

$$
f_\lambda(x) = \sum_{k=0}^{+\infty} \frac{f_\lambda^{(k)}(0)}{k!} x^k, \quad x \in \mathbb{R},
$$

thus $f_\lambda$ is real analytic. This completes the proof. \hfill \Box

**Lemma 9.** If $f \in L^\infty(\mathbb{R})$ is uniformly continuous, then $\lim_{\lambda \to +\infty} \|f - f_\lambda\|_{L^\infty(\mathbb{R})} = 0$.

**Proof.** Let $\omega(a) = \sup_{x \in \mathbb{R}} |f(x) - f(x-a)|$ for $a \in \mathbb{R}$, then $0 \leq \omega(a) \leq 2\|f\|_{L^\infty(\mathbb{R})}$, $\omega(a + b) \leq \omega(a) + \omega(b)$ and $\lim_{a \to 0} \omega(a) = 0$ since $f$ is uniformly continuous, which also implies the continuity of $\omega$. By definition of $f_\lambda$ and the fact that $\int_R K_\lambda(y)dy = 1$, we have

$$
|f(x) - f_\lambda(x)| = \frac{1}{\pi} \left| \int_R \left( f(x) - \frac{2y}{\lambda} \right) \left( \frac{\sin y}{y} \right)^2 dy \right|
$$

$$
\leq \frac{1}{\pi} \int_R \left| f(x) - \frac{2y}{\lambda} \right| \left( \frac{\sin y}{y} \right)^2 dy \leq \frac{1}{\pi} \int_R \omega \left( \frac{2y}{\lambda} \right) \left( \frac{\sin y}{y} \right)^2 dy,
$$

thus

$$
\|f - f_\lambda\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\pi} \int_R \omega \left( \frac{2y}{\lambda} \right) \left( \frac{\sin y}{y} \right)^2 dy.
$$
By dominated convergence theorem, we have

$$\limsup_{\lambda \to +\infty} |f - g_\lambda|_{L^\infty(R)} \leq \frac{1}{\pi} \lim_{\lambda \to +\infty} \int_R \omega \left( \frac{2y}{\lambda} \right) \left( \frac{\sin y}{y} \right)^2 dy$$

which completes the proof.

4.3. Proof of Theorem

To prove Theorem we further need two lemmas.

**Lemma 10.** For the Gaussian SYK model, let the test function $f : \mathbb{R} \to \mathbb{R}$ be Lipschitz. Then

$$\left( \frac{n}{q_n} \right) \text{var} \left[ \mathcal{L}_n(f) \right] \leq C \| f' \|_{L^\infty(R)}^2$$

for some universal constant $C$.

**Proof.** The linear statistic $\mathcal{L}_n(f)$ is $(\frac{n}{q_n})^{-1/2} \| f' \|_{L^\infty(R)}$-Lipschitz if we view $\mathcal{L}_n(f)$ as a function of Gaussian vectors $x := (J_R)_{R \in I^n} \in \mathbb{R}^{n \choose n}$ (see part (a) of Lemma 1 in [7]). By the standard concentration of measure theorem for the Gaussian vectors (see [11]), we have

$$\mathbb{P}[|\mathcal{L}_n(f) - \mathbb{E}\mathcal{L}_n(f)| > t] \leq C e^{-c t^2 / t^2}, \quad t > 0,$$

where $L = (\frac{n}{q_n})^{-1/2} \| f' \|_{L^\infty(R)}$. Therefore, we have

$$\text{var} \left[ \mathcal{L}_n(f) \right] = \mathbb{E} |\mathcal{L}_n(f) - \mathbb{E}\mathcal{L}_n(f)|^2 = \int_0^{+\infty} 2t \mathbb{P}[|\mathcal{L}_n(f) - \mathbb{E}\mathcal{L}_n(f)| > t] dt$$

$$\leq \int_0^{+\infty} 2t C e^{-c t^2 / t^2} dt \leq C L^2 = C \| f' \|_{L^\infty(R)}^2 \left( \frac{n}{q_n} \right)^{-1}.$$

This completes the proof.

**Lemma 11.** For the Gaussian SYK model, let $f = f_1 + f_2$ such that $f_1$ is Lipschitz and $f_1'$ is bounded uniformly continuous, $f_2$ is a polynomial, $f_1, f_2$ are real valued, then

$$\lim_{\lambda \to +\infty} \left( \frac{n}{q_n} \right) \text{var} \left[ \mathcal{L}_n(f) \right] = 2 \langle x f'/2, \rho_\infty \rangle.$$

**Proof.** Let $g = f_1'$, $g_\lambda = g \ast K_\lambda$ for $\lambda > 0$, by Lemma 9 we have

$$\lim_{\lambda \to +\infty} \| g - g_\lambda \|_{L^\infty(R)} = 0.$$

By Lemma 8 $g_\lambda$ is real analytic and

$$|g_\lambda^{(k)}(x)| \leq 2\lambda^k \| g \|_{L^\infty(R)} = 2\lambda^k \| f_1' \|_{L^\infty(R)}.$$

As $f_2$ is a polynomial, we can write $f_2(x) = \sum_{k=0}^m a_k x^k$. Let $F_\lambda(x) = \int_0^x g_\lambda(y) dy + f_2(x)$, by Taylor expansion we have

$$F_\lambda(x) = \sum_{k=0}^{+\infty} \frac{g_\lambda^{(k)}(0)}{(k+1)!} x^{k+1} + \sum_{k=0}^m a_k x^k := \sum_{k=0}^{+\infty} b_k x^k,$$
with \( b_0 = a_0, b_k = \frac{g_{\lambda k}^{(k-1)}(0)}{k!} + a_k \) for \( 0 < k \leq m \) and \( b_k = \frac{g_{\lambda k}^{(k-1)}(0)}{k!} \) for \( k > m \).

For \( c_k = 2^k k!^2 \) as in Lemma 6 we have
\[
\sum_{k=0}^{\infty} |b_k| c_k^2 \leq \sum_{k=0}^{m} |a_k| c_k^2 + \sum_{k=1}^{\infty} \frac{|g_{\lambda k}^{(k-1)}(0)}{k!} (2^k k!) \frac{k}{k} \leq \sum_{k=0}^{m} |a_k| c_k^2 + \sum_{k=1}^{\infty} 2\lambda k^{-1} ||f_1'||_{L^\infty(\mathbb{R})} (2^k k!) \frac{k}{k} k < +\infty.
\]

Since \( L_n(F_\lambda) - b_0 = \sum_{k=1}^{\infty} b_k L_n \text{Tr} H_k \), we have
\[
\left( \frac{n}{q_n} \right) \text{var}[L_n(F_\lambda)] = \left( \frac{n}{q_n} \right) \sum_{k=1}^{+\infty} \sum_{k'=1}^{+\infty} b_k b_{k'} \text{cov}(L_n^{-1} \text{Tr} H_k, L_n^{-1} \text{Tr} H_{k'}).
\]

By Lemma 6 for the Gaussian case, we will have
\[
\left( \frac{n}{q_n} \right) \text{cov}(L_n^{-1} \text{Tr} H_k, L_n^{-1} \text{Tr} H_{k'}) \leq (c_k c_{k'})^\frac{1}{2},
\]
where \( c_k = 2^k k!^2 \). We notice that
\[
\sum_{k=1}^{+\infty} \sum_{k'=1}^{+\infty} |b_k b_{k'}| (c_k c_{k'})^\frac{1}{2} = \left( \sum_{k=0}^{\infty} |b_k| c_k^2 \right)^2 < +\infty.
\]

Therefore, by the dominated convergence theorem and Lemma 11 we have
\[
\lim_{n \to +\infty} \left( \frac{n}{q_n} \right) \text{var}[L_n(F_\lambda)] = \lim_{n \to +\infty} \left( \frac{n}{q_n} \right) \sum_{k=1}^{+\infty} \sum_{k'=1}^{+\infty} b_k b_{k'} (m_{k}^n k'/2)(m_{k'}^n k'/2) = \left( \sum_{k=1}^{+\infty} b_k m_{k}^n k/2 \right)^2
\]
\[
= \left( \sum_{k=1}^{+\infty} b_k (x^k, \rho \infty) k/2 \right)^2 = \left( \sum_{k=1}^{+\infty} b_k x^k k/2, \rho \infty \right)^2.
\]

Since \( x F'_n(x)/2 = \sum_{k=1}^{+\infty} b_k x^k k/2 \), we finally have
\[
\text{(18)} \quad \lim_{n \to +\infty} \left( \frac{n}{q_n} \right) \text{var}[L_n(F_\lambda)] = 2 \langle x F'_n(x)/2, \rho \infty \rangle^2.
\]

Since \( (f - F_\lambda)' = f' - F'_\lambda = f_1' + f_2' - (g_\lambda + f_2') = g - g_\lambda \), by Lemma 10 we have
\[
\left( \frac{n}{q_n} \right) \text{var}[L_n(f - F_\lambda)] \leq C ||(f - F_\lambda)'||_{L^\infty(\mathbb{R})}^2 = C ||g - g_\lambda||_{L^\infty(\mathbb{R})}^2.
\]

If we combine this with
\[
|\langle \text{var}[L_n(f)] \rangle^\frac{1}{2} - \langle \text{var}[L_n(F_\lambda)] \rangle^\frac{1}{2} | \leq \langle \text{var}[L_n(f - F_\lambda)] \rangle^\frac{1}{2},
\]
we have,
\[
\left( \frac{n}{q_n} \right) \text{var}[L_n(F_\lambda)]^\frac{1}{2} - C ||g - g_\lambda||_{L^\infty(\mathbb{R})} \leq \left( \frac{n}{q_n} \right) \langle \text{var}[L_n(f)] \rangle^\frac{1}{2}.
\]
\[
\left( n\right) ^{\frac{1}{2}} (\text{var}[L_n(F_\lambda)])^{\frac{1}{2}} + C \|g - g_\lambda\|_{L^\infty(\mathbb{R})}.
\]

Taking limit, we have
\[
\lim_{n \to +\infty} \left( n\right) ^{\frac{1}{2}} (\text{var}[L_n(F_\lambda)])^{\frac{1}{2}} - C \|g - g_\lambda\|_{L^\infty(\mathbb{R})} \leq \liminf_{n \to +\infty} \left( n\right) ^{\frac{1}{2}} (\text{var}[L_n(f)])^{\frac{1}{2}}
\]
\[
\leq \limsup_{n \to +\infty} \left( n\right) ^{\frac{1}{2}} (\text{var}[L_n(f)])^{\frac{1}{2}} \leq \lim_{n \to +\infty} \left( n\right) ^{\frac{1}{2}} (\text{var}[L_n(F_\lambda)])^{\frac{1}{2}} + C \|g - g_\lambda\|_{L^\infty(\mathbb{R})}.
\]

Notice that (using (18))
\[
\lim_{n \to +\infty} \left( n\right) ^{\frac{1}{2}} (\text{var}[L_n(F_\lambda)])^{\frac{1}{2}} = 2\frac{1}{2} \|xf'_{\lambda}/2, \rho_\infty\|,
\]
and that
\[
\|\langle xf'/2, \rho_\infty\rangle - |\langle xf'_{\lambda}/2, \rho_\infty\rangle|\| \leq \|xf - F_\lambda\|_{L^\infty(\mathbb{R})} \leq \|g - g_\lambda\|_{L^\infty(\mathbb{R})}/2,
\]

(here, we use the fact that \( \langle 1, \rho_\infty\rangle = \langle x^2, \rho_\infty\rangle = 1, \langle x, \rho_\infty\rangle \leq \langle (1 + x^2)/2, \rho_\infty\rangle = 1 \) for the limiting densities in all cases) we will have
\[
2\frac{1}{2} \|xf'/2, \rho_\infty\| - C \|g - g_\lambda\|_{L^\infty(\mathbb{R})} \leq \liminf_{n \to +\infty} \left( n\right) ^{\frac{1}{2}} (\text{var}[L_n(f)])^{\frac{1}{2}}
\]
\[
\leq \limsup_{n \to +\infty} \left( n\right) ^{\frac{1}{2}} (\text{var}[L_n(f)])^{\frac{1}{2}} \leq 2\frac{1}{2} \|xf'/2, \rho_\infty\| + C \|g - g_\lambda\|_{L^\infty(\mathbb{R})}.
\]

Since \( \lim_{\lambda \to +\infty} \|g - g_\lambda\|_{L^\infty(\mathbb{R})} = 0 \), letting \( \lambda \to +\infty \), we finally have
\[
\lim_{n \to +\infty} \left( n\right) ^{\frac{1}{2}} (\text{var}[L_n(f)])^{\frac{1}{2}} = 2\frac{1}{2} \|xf'/2, \rho_\infty\|, \]
which completes the proof. \( \square \)

Now we can finish the proof of Theorem 2. Let \( f(x) \) be Lipschitz and \( f'(x) \) is bounded uniformly continuous. Let’s denote \( f_1 := f \) and \( f_2 := -Ax^2 \) with \( A := \langle xf'/2, \rho_\infty\rangle \) where it’s easy to see \( |A| < \infty \). Then we have \( \langle xf_1 + f_2', \rho_\infty\rangle = \langle xf'/2, \rho_\infty\rangle - A x^2, \rho_\infty\rangle = \langle xf'/2, \rho_\infty\rangle - A = 0 \). Thus, by Lemma 11 we have
\[
\lim_{n \to +\infty} \left( n\right) \text{var}[\langle [f_1 + f_2], \rho_n\rangle] = 2 \langle x[f_1 + f_2'], \rho_\infty\rangle^2 = 0.
\]
Therefore, we have
\[
\left( n\right) ^{\frac{1}{2}} (\langle [f_1 + f_2], \rho_n\rangle - \mathbb{E}[\langle [f_1 + f_2], \rho_n\rangle]) \to 0
\]
in probability. By definitions of \( f, f_1, f_2 \) and \( \mu \) above, we have
\[
\left( n\right) ^{\frac{1}{2}} (\langle f, \rho_n\rangle - \mathbb{E}[\langle f, \rho_n\rangle])
\]
\[
= \left( n\right) ^{\frac{1}{2}} (\langle [f_1 + f_2], \rho_n\rangle - \mathbb{E}[\langle [f_1 + f_2], \rho_n\rangle]) + A \left( n\right) ^{\frac{1}{2}} (\langle x^2, \rho_n\rangle - \mathbb{E}[\langle x^2, \rho_n\rangle]).
\]
The first term tends to 0 in probability and Theorem 1 implies that the second term tends to $AJ$ in distribution where $J$ is the Gaussian distribution with mean 0 and variance 2, therefore, we have

$$\left(\frac{n}{q_n}\right)^{\frac{1}{2}} \left(\langle f, \rho_n \rangle - \mathbb{E}[\langle f, \rho_n \rangle]\right) \Rightarrow AJ,$$

which completes the proof of Theorem 2.

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