SOLUTION OF CERTAIN PELL EQUATIONS

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Abstract. Let $a, b, c$ be any positive integers such that $c \mid ab$ and $d^\pm_i$ is a square free positive integer of the form $d^\pm_i = a^{2k}b^{2l} \pm ic^m$ where $k, l \geq m$ and $i = 1, 2$. The main focus of this paper is to find the fundamental solution of the equation $x^2 - d^\pm_i y^2 = 1$, with the help of the continued fraction of $\sqrt{d^\pm_i}$. We also obtain all the positive solutions of the equations $x^2 - d^\pm_i y^2 = \pm 1$ and $x^2 - d^\pm_i y^2 = \pm 4$ by means of the Fibonacci and Lucas sequences.

Furthermore, in this work, we derive some algebraic relations on the Pell form $F_{d^\pm_i}(x, y) = x^2 - d^\pm_i y^2$ including cycle, proper cycle, reduction and proper automorphism of it. We also determine the integer solutions of the Pell equation $F_{\Delta}(x, y) = 1$ in terms of $d^\pm_i$.

We generalized all the results of the papers [2], [9], [26] and [37].

1. Introduction

Let $d$ be a positive integer which is not a perfect square and $N$ be any nonzero fixed integer. Then the equation $x^2 - dy^2 = N$ is known as Pell equation after the name of English mathematician, John Pell. The equations $x^2 - dy^2 = 1$ and $x^2 - dy^2 = -1$ are known as the classical Pell equations. If $a^2 - db^2 = N$, we say that $(a, b)$ is a solution of the equation $x^2 - dy^2 = N$. We use the notation $(a, b)$ and $a + \sqrt{db}$ interchangeably to denote the solutions of the equation $x^2 - dy^2 = N$. Also, if $a$ and $b$ are positive, we say that $a + b\sqrt{d}$ is a positive solution to the equation $x^2 - dy^2 = N$. Among these there is a least solution $a_1 + b_1\sqrt{d}$, in which $a_1$ and $b_1$ have their least positive values. Then the number $a_1 + b_1\sqrt{d}$ is called fundamental solution of the equation $x^2 - dy^2 = N$. If $a + \sqrt{db}$ and $r + \sqrt{ds}$ are solutions of the equation $x^2 - dy^2 = N$, then $a = r$ iff $b = s$, and $a + \sqrt{db} < r + \sqrt{ds}$ iff $a < r$ and $b < s$.

An equation $x^2 - dy^2 = 1$ has infinite many solutions iff the equation $x^2 - dy^2 = -1$ has no solution. The continued fraction of $\sqrt{d}$ played a vital role to solve the Pell equation $x^2 - dy^2 = \pm 1$. Actually its period length is useful for knowing the solution of this equation. Let $d$ be a positive integer that is not a perfect square. Then there

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is a continued fraction expansion of $\sqrt{d}$ such that $\sqrt{d} = [a_0, a_1, a_2, \ldots, a_{n-1}, 2a_0]$ where $n$ is the period length and the $a_j$'s are given by the recursion formula; $a_0 = \sqrt{d}, a_k = \pm a_{k-1}$ and $a_{k+1} = \frac{1}{x_k-a_k}, k = 0, 1, 2, \ldots$

Recall that $a_n = 2a_0$ and $a_{n+k} = a_k$ for all $k \geq 1$. Then $n^{th}$ convergent of $\sqrt{d}$ is given by

$$\frac{p_n}{q_n} = [a_0, a_1, a_2, \ldots, a_{n-1}, a_n] = a_0 + \frac{1}{a_1 + \frac{1}{\ddots + \frac{1}{a_{n-1} + \frac{1}{a_n}}}}$$

In this paper, we give the fundamental solution of the equation $x^2 - d_i^\pm y^2 = \pm 1$ by means of the period length of the continued fraction expansion of $\sqrt{d_i^\pm}$, where $d_i^\pm = a^{2k}b^2 \pm ic^m$. After finding the fundamental solution of the Pell equation $x^2 - d_i^\pm y^2 = \pm 1$, we obtain the positive integer solutions of equation $x^2 - d_i^\pm y^2 = \pm 4$, for $d_i^\pm = a^{2k}b^2 \pm ic^m$ by the means of the generalized Fibonacci and Lucas sequences.

The main results of this paper also generalized the results presented in [2], [9] and [26].

Furthermore, in this work, we derive some algebraic relations on the Pell form $F_{d_i^\pm}(x, y) = x^2 - d_i^\pm y^2$ or of discriminant $\Delta d_i^\pm = 4d_i^\pm$ including cycle, proper cycle, reduction and proper automorphism of it. Also we determine the integer solutions of the Pell equation $F_{\Delta d_i^\pm}(x, y) = 1$ via $d_i^\pm$. The main results of this paper also generalized the results presented in [37].

2. Basic Setup

If $N$ is a quadratic non-residue modulo $d$, then the Pell Equation $x^2 - dy^2 = N$ has no integer solution. If $N$ is a perfect square, then the Pell Equation $x^2 - dy^2 = N$ is solvable in integers for all positive, non-square integers $d$. The equation $x^2 - dy^2 = 1$ has a solution in positive integers $x$ and $y$ for all positive, non-square integers $d$. If $(x, y) = (u, v)$ is a positive integer solution of $x^2 - dy^2 = 1$, then there exists a positive integer $m$ such that $u + v\sqrt{d} = x_1 + y_1m\sqrt{d}$, where $(x_1, y_1)$ is the fundamental solution of $x^2 - dy^2 = 1$.

If we know the fundamental solutions of the equations $x^2 - dy^2 = \pm 1$ and $x^2 - dy^2 = \pm 4$ then we can give all positive integer solutions to these equations. For more information about the Pell equation, one can consult [25] and [20].

The generalized Fibonacci and Lucas sequences $f_n(w, z)$ and $L_n(w, z)$ are given in the followings:

Let $w$ and $z$ be two nonzero positive integers with $w^2 + 4z \geq 0$. The generalized Fibonacci and Lucas sequences with initial conditions $f_0(w, z) = 0, f_1(w, z) = 1$ and $L_0(w, z) = 2, L_1(w, z) = 0$ are of the form $f_n(w, z) = w f_{n-1}(w, z) + z f_{n-2}(w, z)$, $L_n(w, z) = w L_{n-1}(w, z) + z L_{n-2}(w, z) \forall n \geq 2$, respectively. They can also be
Let $\frac{p_l}{q_l}$ be the convergent of the continued fraction expansion of $\sqrt{d}$, and let $l$ be the length of the expansion.

- If $l$ is even, then the fundamental solution of $x^2 - dy^2 = 1$ is given by
  \[ x = p_{l-1} \quad y = q_{l-1} \]
  and the equation $x^2 - dy^2 = -1$ has no solutions.

- If $l$ is odd, then the fundamental solution of $x^2 - dy^2 = 1$ is given by
  \[ x = p_{2l-1} \quad y = q_{2l-1} \]
  and $x = p_{l-1}, y = q_{l-1}$ is the fundamental solution of $x^2 - dy^2 = -1$.

**Theorem 2.2.** If $x_1, y_1$ is the fundamental solution of $x^2 - dy^2 = 1$, then every positive solution of the equation is given by $x_n, y_n$, where $x_n$ and $y_n$ are integers determined from $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$ \(n = 1, 2, 3, \ldots\)

**Theorem 2.3.** If $x_1, y_1$ is the fundamental solution of $x^2 - dy^2 = -1$, then every positive solution of the equation is given by $x_n, y_n$, where $x_n$ and $y_n$ are integers determined from $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^{2n-1}$ \(n = 1, 2, 3, \ldots\)

The following two theorems are given in [20].

**Theorem 2.4.** If $x_1, y_1$ is the fundamental solution of $x^2 - dy^2 = 4$, then every positive solution of the equation is given by $x_n, y_n$, where $x_n$ and $y_n$ are integers determined from $x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^n}{2^n}$ \(n = 1, 2, 3, \ldots\)

**Theorem 2.5.** If $x_1, y_1$ is the fundamental solution of $x^2 - dy^2 = -4$, then every positive solution of the equation is given by $x_n, y_n$, where $x_n$ and $y_n$ are integers determined from $x_n + y_n\sqrt{d} = \frac{(x_1 + y_1\sqrt{d})^n}{4^n}$ \(n = 1, 2, 3, \ldots\)

The following theorems are given in [32].

**Theorem 2.6.** Let $d \equiv 2 \pmod{4}$ or $d \equiv 3 \pmod{4}$. Then the equation $x^2 - dy^2 = -4$ has no solution if and only if the equation $x^2 - dy^2 = -1$ has positive solutions.

**Theorem 2.7.** Let $d \equiv 0 \pmod{4}$. If $x_1 + \frac{d}{4}y_1$ is the fundamental solution of the equation $x^2 - \frac{d}{4}y^2 = 1$, then the fundamental solution of the equation $x^2 - dy^2 = 4$ is given as $(2x_1, y_1)$. 
Theorem 2.8. Let $d \not\equiv 0 \pmod{4}$. If $x_1 + y_1 \sqrt{d}$ is the fundamental solution of the equation $x^2 - dy^2 = 1$ then the fundamental solution of the equation $x^2 - dy^2 = 4$ is $(2x_1, 2y_1)$.

3. Basic Setup 2

A real binary quadratic form (or just a form) $F$ is a polynomial in two variables $x$ and $y$ of the type

$$F = F(x, y) = ax^2 + bxy + cy^2$$

with real coefficients $a, b, c$. We denote $F$ briefly by $F = (a, b, c)$. The discriminant of $F$ is defined by the formula $b^2 - 4ac$ and is denoted by $\Delta$. A quadratic form $F$ of discriminant $\Delta$ is called indefinite if $\Delta > 0$, and is called integral if and only if $a, b, c \in \mathbb{Z}$. An indefinite quadratic form $F = (a, b, c)$ of discriminant $\Delta$ is said to be reduced if

$$|\sqrt{\Delta} - 2|a| < b < \sqrt{\Delta}$$

Most properties of quadratic forms can be giving by the aid of extended modular group $\Gamma$ (see [34]). Gauss defined the group action of $\Gamma$ on the set of forms as follows:

$$gF(x, y) = (ar^2 + brs + cs^2)x^2 + (2art + bru + bts + 2csu)xy + (at^2 + btu + cu^2)y^2$$

for $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \Gamma$. An element $g \in \Gamma$ is called an automorphism of $F$ if $gF = F$. If $\det g = 1$, then $g$ is called a proper automorphism of $F$ and if $\det g = -1$, then $g$ is called an improper automorphism of $F$. Let $\text{Aut}(F)^+$ denote the set of proper automorphisms of $F$ and let $\text{Aut}(F)^-$ denote the set of improper automorphisms of $F$ (for further details on binary quadratic forms see [13], [5], [7] and [24]). Let $\rho(F)$ denotes the normalization (it means that replacing $F$ by its normalization) of $(c, -b, a)$. To be more explicit, we set

$$\rho^{i+1}(F) = (c_j, -b_j + 2c_j r_j, c_j r_j^2 - b_j r_j + a_j),$$

where

$$r_j = \begin{cases} \text{sign}(c_j) \left\lfloor \frac{b}{2|c_j|} \right\rfloor, & \text{for } |c_j| \geq \sqrt{\Delta}; \\ \text{sign}(c_j) \left\lfloor \frac{b + \sqrt{\Delta}}{2|c_j|} \right\rfloor, & \text{for } |c_j| < \sqrt{\Delta}. \end{cases}$$

for $j \geq 0$. The number $r_j$’s called the reducing number and the form $\rho^{i+1}(F)$ is called the reduction of $F$. Further if $F$ is reduced, then so is $\rho^{i+1}(F)$. In fact, $\rho$ is a permutation of the set of all reduced indefinite forms. Let $\tau(F) = \tau(a, b, c) = (-a, b, -c)$. Then the cycle of $F$ is the sequence $((\tau \rho)^j(G))$ for $j \in \mathbb{Z}$, where $G = (A, B, C)$ is a reduced form with $A > 0$ which is equivalent to $F$. The cycle and proper cycle of $F$ is given by the following theorem [13]:
Theorem 3.1. Let $F = (a, b, c)$ be reduced indefinite quadratic form of discriminant $\Delta$. Then the cycle of $F$ is a sequence $F_0 \sim F_1 \sim F_2 \sim \ldots \sim F_{l-1}$ of length $l$, where $F_0 = F = (a_0, b_0, c_0)$,

\begin{equation}
 s_j = s(F_j) = \left\lfloor \frac{b_j + \sqrt{\Delta}}{2|c_j|} \right\rfloor \quad \text{and}
\end{equation}

\begin{equation}
 F_{j+1} = (a_{j+1}, b_{j+1}, c_{j+1}) = (|c_j|, -b_j + 2|c_j|s_j, -(c_j s_j^2 + b_j s_j + a_j))
\end{equation}

for $1 \leq i \leq l - 2$. If $l$ is odd, then the proper cycle of $F$ is $F_0 \sim \tau F_1 \sim F_2 \sim \tau F_3 \cdots \sim \tau F_{l-2} \sim F_{l-1} \sim F_0 \sim F_1 \sim F_2 \sim \cdots \sim F_{l-2} \tau F_{l-1}$ of length $2l$. In this case the equivalence class of $F$ is equal to the proper equivalence class of $F$, and if $l$ is even, then the proper cycle of $F$ is $F_0 \sim \tau F_1 \sim F_2 \sim \tau F_3 \cdots \sim F_{l-2} \sim \tau F_{l-1}$ of length $l$. In this case the equivalence class of $F$ is the disjoint union of the proper equivalence class of $F$ and the proper equivalence class of $\tau(F)$.

4. Main Results

In this section, we will give our main results about the positive solutions of the Pell equations $x^2 - dy^2 = \pm 1$, and $x^2 - dy^2 = \pm 4$ for particular values of $d$. More precisely, for $d = d^\pm = a^{2k}b^2 \pm ic^m$, all the positive solutions of the equation in terms of the generalized Fibonacci and Lucas sequences has been investigated. Throughout in this section $h$ will be a positive integer such that $h = \frac{a^kb^l}{c^m}$, because $c \mid ab$.

Theorem 4.1. Let $h = \frac{a^kb^l}{c^m}$ be a positive integer, then the continued fraction of

\begin{enumerate}
  \item[i:] $d^-_1$ has the form $[a^kb^l; 1, 2h-2, 1, 2a^kb^l - 2]$ \hspace{1cm} \text{ab} \geq 2$
  \item[ii:] $d^+_1$ has the form $[a^kb^l; 2h, 2a^kb^l]$
  \item[iii:] $d^-_2$ has the form $[a^kb^l - 1; 1, h - 2, 1, 2a^kb^l - 2]$ \hspace{1cm} \text{ab} \geq 3$
  \item[iv:] $d^+_2$ has the form $[a^kb^l; h, 2a^kb^l]$
\end{enumerate}
Proof. For $d_1^-$, the continued fraction

\[
\sqrt{a^{2k}b^{2l} - c^m} = a^k b^l - 1 + \frac{1}{\sqrt{a^{2k}b^{2l} - c^m + a^k b^l - 1}}
\]

\[
= a^k b^l - 1 + \frac{1}{\sqrt{2a^k b^l - c^m}}
\]

\[
= a^k b^l - 1 + \frac{1}{1 + \frac{1}{\sqrt{2a^k b^l - c^m}}}
\]

\[
= a^k b^l - 1 + \frac{1}{1 + \frac{1}{\sqrt{a^{2k}b^{2l} - c^m + a^k b^l - c^m}}}
\]

\[
= a^k b^l - 1 + \frac{1}{1 + \frac{1}{\sqrt{a^{2k}b^{2l} - c^m + a^k b^l - c^m}}}
\]

\[
= a^k b^l - 1 + \frac{1}{2h - 2 + \sqrt{a^{2k}b^{2l} - c^m + a^k b^l - c^m}}
\]

\[
= a^k b^l - 1 + \frac{1}{2h - 2 + \sqrt{a^{2k}b^{2l} - c^m + a^k b^l - c^m}}
\]

\[
= a^k b^l - 1 + \frac{1}{2h - 2 + \sqrt{a^{2k}b^{2l} - c^m + a^k b^l - c^m}}
\]

\[
= a^k b^l - 1 + \frac{1}{2h - 2 + \sqrt{a^{2k}b^{2l} - c^m + a^k b^l - c^m}}
\]

\[
= a^k b^l - 1 + \frac{1}{2h - 2 + \sqrt{a^{2k}b^{2l} - c^m + a^k b^l - c^m}}
\]

Hence $\sqrt{a^{2k}b^{2l} - c^m}$ has the continued fraction of the form $[a^k b^l - 1; 1, 2h - 2, 1, 2a^k b^l - 2]$. Similarly for $d_2^-$, one can obtained the required form of the continued fraction.
For \(d_2^+\), the continued fraction
\[
\sqrt{a^{2k}b^{2l} + 2cm} = a^k b^l + (\sqrt{a^{2k}b^{2l} + 2cm} - a^k b^l)
\]
\[
= a^k b^l + \frac{1}{\sqrt{a^{2k}b^{2l} + 2cm - a^k b^l}} = a^k b^l + \frac{1}{h + \frac{1}{\sqrt{a^{2k}b^{2l} + 2cm - a^k b^l}}}
\]
Hence \(\sqrt{a^{2k}b^{2l} + 2cm}\) has the continued fraction of the form \([a^k b^l; h, 2a^k b^l]\).
Similarly for \(d_1^+\), one can obtained the required form of the continued fraction. 

**Corollary 4.2.** If \(c = 1\), then the continued fraction of
- i: \(d_1^-\) has of the form \([a^k b^l; 1, 2a^k b^l - 2]\)
- ii: \(d_1^+\) has the form \([a^k b^l; 2a^k b^l]\)
- iii: \(d_2^-\) has of the form \([a^k b^l - 1; 1, a^k b^l - 2, 1, 2a^k b^l - 2]\)
- iv: \(d_1^+\) has the form \([a^k b^l; a^k b^l, 2a^k b^l]\)

**Remark 4.3.** The continued fraction of \(d_3 = \sqrt{a^{2k} - a^k}\) is of the form \([a^k - 1; 2, 2a^k - 2]\)

**Theorem 4.4.** i: Let us consider the Pell equation \(x^2 - d_2^\pm y^2 = 1\), then the fundamental solution \((x_1^\pm, y_1^\pm)\) is of the form \((2ha^k b^l \pm 1, 2h)\) and the other solutions are \((x_n^\pm, y_n^\pm)\), where
\[
\frac{x_n^+}{y_n^+} = [a^k b^l; 2h, 2a^k b^l, 2h] \quad \text{(n-1)time}
\]
\[
\frac{x_n^-}{y_n^-} = [a^k b^l - 1; 1, 2h - 2, 1, 2a^k b^l - 2, 1] \quad \text{(n-1)time}
\]

ii: Let us consider the Pell equation \(x^2 - d_2^\pm y^2 = 1\), the fundamental solution \((x_1^\pm, y_1^\pm)\) is of the form \((ha^k b^l \pm 1, h)\) and the other solutions are \((x_n^\pm, y_n^\pm)\), where
\[
\frac{x_n^+}{y_n^+} = [a^k b^l; h, 2a^k b^l, h] \quad \text{(n-1)time}
\]
\[
\frac{x_n^-}{y_n^-} = [a^k b^l - 1; 1, h - 2, 1, 2a^k b^l - 2, 1] \quad \text{(n-1)time}
\]
Proof. The period length of the continued fraction of $\sqrt{d_i^+}$ is 2 by theorem 4.1. Since $p_{-2} = 0, p_{-1} = 1, q_{-2} = 1, q_{-1} = 0$, $p_k = a_kp_{k-1} + p_{k-2}, q_k = a_kq_{k-1} + q_{k-2}$, therefore the fundamental solution is of the form $p_1^+ = a_1p_0 + p_{-1} = 2ha^b_1+1, q_1^+ = a_1q_0 + q_{-1} = 2h$ for $d_1^+$ by using lemma 2.1. Similarly the equation $x^2 - d_1^+y^2 = 1$ has the required solution form due to the theorem 4.1 and lemma 2.1.

Now we assume that $(x_{n-1}, y_{n-1})$ is a solution, that is, $x_{n-1} - d_1^+y_{n-1}^2 = 1$. Then we have that

$$\frac{x_{n-1}^2}{y_{n-1}^2} = a_1b_1' + \frac{1}{a_1b_1'+1} = \frac{x_{n-1}^2}{y_{n-1}^2} = a_1b_1 + \frac{1}{a_1b_1'} = \frac{a_0}{y_n} = \frac{(ha^b_1+1)x_{n-1}+hdy_{n-1}}{hx_{n-1}+(ha^b_1+1)y_{n-1}}$$

(4.1)

Similarly for $d_1^+$ and $d_i^-$ we can get all positive solutions of the required form.

\[\square\]

Corollary 4.5. The equation $x^2 - d_i^+y^2 = -1$ has no positive integer solutions, except $x^2 - d_1^+y^2 = -1$ has solution $(a_1b_1', 1)$ only if $c = 1$.

Proof. The continued fraction of $\sqrt{d_i^-}$ have even length, therefore the equation $x^2 - d_i^+y^2 = -1$ has no solution by lemma 2.1 but if $c = 1$, then $\sqrt{d_i^-}$ have odd length, therefore the equation has solution by lemma 2.1 and get required solution.

\[\square\]

Theorem 4.6. The $n^{th}$ integer solution $(x_n^+; y_n^+)$ of $x^2 - d_i^+y^2 = -1$ can be given as a linear combination of $x_i^+; y_i^+$ and $d_i^+$ namely, for $n \geq 2$

$$x_n^+ = x_1^+x_{n-1} + y_1^+d_1^+\ y_{n-1}$$
$$y_n^+ = y_1^+x_{n-1} + x_1^+y_{n-1}$$

and also satisfy the recurrence relation for $n \geq 4$

$$x_n^+ = (2x_1^+ - 1)(x_{n-1} + x_{n-2}) - x_{n-3}$$
$$y_n^+ = (2x_1^+ - 1)(y_{n-1} + y_{n-2}) - y_{n-3}$$

Proof. The first assertion is easily seen from 4.1. The second assertion can be proved by induction on $n$.

\[\square\]

Theorem 4.7. All positive integer solutions of the equation $x^2 - d_i^+y^2 = 1$ are given by

$$(x_n^+, y_n^+) = \left(\frac{1}{2}L_n(x_{1}^+, -1), y_1^+f_n(x_{1}^+, -1)\right) \quad n = 1, 2, 3, \ldots$$
Proof. Consider the Pell equation \( x^2 - d_i^+ y^2 = 1 \), then by theorem 4.3 and theorem 2.2 all positive solution of the equation are given by \( x_n^+ + y_n^+ \sqrt{d_i^+} = (x_1^+ + y_1^+ \sqrt{d_1^+})^n \).

Let \( \alpha_i^\pm = x_1^+ + y_1^+ \sqrt{d_1^+} \) and \( \beta_i^\pm = x_1^+ - y_1^+ \sqrt{d_1^+} \). Then \( \alpha_i^\pm + \beta_i^\pm = 2x_1^+ \), \( \alpha_i^\pm - \beta_i^\pm = 2y_1^+ \sqrt{d_1^+} \) and \( \alpha_i^\pm \beta_i^\pm = 1 \). Therefore \( x_n^+ + y_n^+ \sqrt{d_i^+} = (\alpha_i^\pm)^n, x_n^+ - y_n^+ \sqrt{d_i^+} = (\beta_i^\pm)^n \).

Thus it follows that \( x_n^+ = \frac{(\alpha_i^\pm)^n + (\beta_i^\pm)^n}{2y_1^+ \sqrt{d_i^+}} = \frac{y_1^+}{2}L_n(x_1^+, -1) \) and \( y_n^+ = \frac{(\alpha_i^\pm)^n - (\beta_i^\pm)^n}{2\sqrt{d_i^+}} = \frac{y_1^+}{2}f_n(x_1^+, -1) \).

\[ \square \]

**Theorem 4.8.** The fundamental solution of the Pell equation \( x^2 - d_i^+ y^2 = 4 \) is \( (x_1^+, y_1^+) = (2x_1^+, 2y_1^+) \).

**Proof.** We know that \( a^k b^l = c^m \) so \( (a^k b^l)^2 = (c^m)^2 \Rightarrow d_i^+ = c^{2m} h^2 \pm ic^m \). We will give proof for only \( d_i^+ = a^{2k} b^{2l} + 2c^m \). If \( c = 2s \) is even, then \( d_i^+ \equiv 0 \) (mod 4) and \( d_i^+ = 2^{m-2} s^2 h^2 + 2^{m-1} s^m \).

Hence by theorem 4.1 and 4.3 it follows that the equation \( x^2 - (2^{m-2} s^2 h^2 + 2^{m-1} s^m)y^2 = 1 \) has the fundamental solution \( (2^m h s^m + 1, 2h) \).

Then, by theorem 2.4, the fundamental solution to the equation \( x^2 - d_i^+ y^2 = 4 \) is of the form \( (2^{m+1} h^2 s^m + 2, 2h \sqrt{d_2^+}) \). Since \( c = 2s \) and \( 2^m h s^m = a^{2k} b^l \), so the the equation has the fundamental solution \( (2a^{2k} b^l + 2 + 2h \sqrt{d_2^+}) \).

Assume that \( c \) is odd. Then \( d_i^+ \equiv 2, 3 \) (mod 4) if \( a^{2k} b^l \) is even or odd respectively. Thus, by theorem 4.4 and 4.3 it follows that the fundamental solution of the equation \( x^2 - (2^{m-2} s^2 h^2 + 2^{m-1} s^m)y^2 = 1 \) is of the form \( (2^m h s^m + 2, 2h) \).

Then, by theorem 4.4, the fundamental solution to the equation \( x^2 - d_i^+ y^2 = 4 \) is of the form \( (2a^{2k} b^l + 2 + 2h \sqrt{d_2^+}) \). Similarly, we can proof for other values of \( d_i^+ \).

\[ \square \]

**Theorem 4.9.** The equation \( x^2 - d_i^+ y^2 = -4 \) has no positive integer solution, except \( x^2 - d_i^+ y^2 = -4 \) when \( c = 1 \).

**Proof.** Let \( d_i^+ = a^{2k} b^{2l} + c^m \) and assume that \( c \) is odd. Then \( d_i^+ \equiv 2, 3 \) (mod 4) if \( a^{2k} b^l \) is even or odd respectively. Thus, by theorem 2.6 and theorem 4.3 the equation \( x^2 - d_i^+ y^2 = -4 \) has no solution.

Now suppose that \( c \) is even and the positive integer \( f \) and \( g \) are the solution of the above equation, then \( f^2 - d_i^+ g^2 = -4 \). But \( d_i^+ \) is even and therefore \( f \) and \( g \) are even. Since \( (a^{k} b^{l})^2 = (c^m h^2) \) therefore, \( f^2 - (2^{m-2} s^2 h^2 + 2^{m-1} s^m)g^2 = -4 \) and implies that \( (\frac{f}{2})^2 - (2^{m-2} s^2 h^2 + 2^{m-1} s^m)g^2 = -1 \) this is impossible by theorem 4.4, similarly, for the other equations.

\[ \square \]

**Corollary 4.10.** The equation \( x^2 - d_i^+ y^2 = -4 \) has positive integer solutions \( (2a^k b^l, 2) \) if \( c = 1 \).

**Theorem 4.11.** All the positive solutions of the equation \( x^2 - d_i^+ y^2 = 4 \) are given as

\[ (x_i^+, y_i^+) = (L_n(2x_1^+, -1), y_1^+ f_n(2x_1^+, -1)) \quad n = 1, 2, 3, \ldots \]
Proof. We know by theorem 4.3 that $2x_1^\pm + 2y_1^\pm \sqrt{d_1^\pm}$ is the fundamental solution of the equation $x^2 - d_1^\pm y^2 = 4$. Therefore by theorem 2.4, all positive integer solution of the equation $x^2 - d_1^\pm y^2 = 4$ are given by $x_n^\pm + y_n^\pm \sqrt{d_1^\pm} = \frac{1}{2^{n-1}} (2x_1^\pm + 2y_1^\pm \sqrt{d_1^\pm})^n = 2^{\frac{n-1}{2}} (2x_1^\pm + 2y_1^\pm \sqrt{d_1^\pm})^n$. Now, let us consider $\alpha_i^\pm = \frac{2x_1^\pm + 2y_1^\pm \sqrt{d_1^\pm}}{2} \text{ and } \beta_i^\pm = \frac{2x_1^\pm - 2y_1^\pm \sqrt{d_1^\pm}}{2}$.

Then $\alpha_i^\pm + \beta_i^\pm = x_i^\pm \text{, } \alpha_i^\pm - \beta_i^\pm = 2y_i^\pm \sqrt{d_1^\pm} \text{ and } \alpha_i^\pm \beta_i^\pm = 1$. Thus it is easily seen that $x_n^\pm + y_n^\pm \sqrt{d_1^\pm} = 2(\alpha_i^\pm)^n \text{ and } x_n^\pm - y_n^\pm \sqrt{d_1^\pm} = 2(\beta_i^\pm)^n$. Therefore $x_n^\pm = (\alpha_i^\pm)^n + (\beta_i^\pm)^n = L_n(2x_1^\pm, -1) \text{ and } y_n^\pm = \frac{(\alpha_i^\pm)^n - (\beta_i^\pm)^n}{\sqrt{d_1^\pm}} = y_1^\pm i(n(2x_1^\pm, -1)$. □

Thus we can give the following corollaries.

Corollary 4.12. If $d_1^+ = a^{2k} + am$, then $\sqrt{d_1^+} = [a^k, 2a^{k-m}, 2a^k]$ and if $d_1^- = a^{2k} + am$, then $\sqrt{d_1^-} = [a^k, a^{k-m}, 2a^k]$.

Corollary 4.13. If $d = a^{2k} + 2am$, then the fundamental solution to the equation $x^2 - d^+ y^2 = 1$ is $x_1 + y_1 \sqrt{d_1^+} = a^{2k-m} + a^{k-m} \sqrt{d}$ and the equation $x^2 - d_1^- y^2 = -1$ has no solutions.

Corollary 4.14. Let $d_1^+ = a^{2k} + am$ . Then the fundamental solution to the equation $x^2 - d_1^+ y^2 = 1$ is $x_1 + y_1 \sqrt{d_1^+} = a^{2k-m} + 1 + a^{k-m} \sqrt{d}$ and the equation $x^2 - d_1^+ y^2 = -1$ has no solutions.

Remark 4.15. • If $c = b$ and $k = l = m = 1$, then the main results of [9] become the corollaries of our main results.

• If $c = a$ and $k = l = m = 1$ and $b = 1$, then the main results of [26] become the corollaries of our main results.

5. Main Results 2

Let us consider the matrix $M_{\pm i}$ associated with $d_i^\pm$ and corresponding fundamental solution $x_i^\pm, y_i^\pm$ as

$$M_{\pm i} = \begin{pmatrix} x_1^\pm & y_1^\pm d_1^\pm \\ y_1^\pm & x_1^\pm \end{pmatrix}$$

In the following theorem, we able to determine the $n^{th}$ power of $M_{\pm i}$ which we use it later. (Here, we note that $\binom{n}{2j} = \binom{n-2}{2j} + \binom{n-2}{2j-2} + 2\binom{n-2}{2j-1}$ for $j = 1, 2, \ldots, \frac{n-2}{2}$).

Theorem 5.1. If $n \geq 0$, then the $n^{th}$ power of $M_{\pm i}$ is given by

$$M_{\pm i}^n = \begin{pmatrix} M_{11}^n & M_{12}^n \\ M_{21}^n & M_{22}^n \end{pmatrix}$$

where
So it is true for \( M_{11}^n \), \( M_{12}^n \), and \( M_{21}^n \). We will prove it for 

(a): Here we will give the proof for 

\[
M_{11}^n = \sum_{j=0}^{n} \binom{n}{2j} (x_1^{\pm 1})^{n-2j} (y_1^{\pm 1})^{2j} (d_i^{\pm 1})^j = M_{22}^n
\]

\[
M_{12}^n = \sum_{j=0}^{n} \binom{n}{2j+1} (x_1^{\pm 1})^{n-2j} (y_1^{\pm 1})^{2j+1} (d_i^{\pm 1})^{j+1}
\]

\[
M_{21}^n = \sum_{j=0}^{n} \binom{n}{2j+1} (x_1^{\pm 1})^{n-2j} (y_1^{\pm 1})^{2j+1} (d_i^{\pm 1})^{j}
\]

(b): If \( n \) is odd

\[
M_{11}^n = \sum_{j=0}^{n-1} \binom{n}{2j} (x_1^{\pm 1})^{n-2j} (y_1^{\pm 1})^{2j} (d_i^{\pm 1})^j = M_{22}^n
\]

\[
M_{12}^n = \sum_{j=0}^{n-1} \binom{n}{2j+1} (x_1^{\pm 1})^{n-2j} (y_1^{\pm 1})^{2j+1} (d_i^{\pm 1})^{j+1}
\]

\[
M_{21}^n = \sum_{j=0}^{n-1} \binom{n}{2j+1} (x_1^{\pm 1})^{n-2j} (y_1^{\pm 1})^{2j+1} (d_i^{\pm 1})^{j}
\]

**Proof.** (a): Here we will give the proof for \( d_2^+ \) by mathematical induction on \( n \). If \( n = 2 \), then

\[
M_{21}^2 = \left( h^2 a^{2k} b^{2j} + 1 + 2h a^{k} b^j + h^2 d_2^+ \right), \quad \text{where}
\]

\[
M_{11}^2 = \sum_{j=0}^{2} \binom{2}{2j} (h a^{k} b^j + 1)^{2-2j} h^{2j} d_2^+ = (h a^{k} b^j + 1) + h^2 d_2^+ = M_{22}^2
\]

\[
M_{12}^2 = \sum_{j=0}^{2} \binom{2}{2j+1} (h a^{k} b^j + 1)^{1-2j} h^{2j+1} d_2^+ = 2h (h a^{k} b^j + 1)
\]

\[
M_{21}^2 = \sum_{j=0}^{2} \binom{2}{2j+1} (h a^{k} b^j + 1)^{1-2j} h^{2j+1} d_2^+ = 2h (h a^{k} b^j + 1)
\]

So it is true for \( n = 2 \). Let us assume that it is true for \( n - 2 \), that is, \( M_{11}^{n-2} = \left( M_{11}^{n-2} \ M_{12}^{n-2} \ M_{21}^{n-2} \right) \), where \( M_{11}^{n-2} = \sum_{j=0}^{n-3} \binom{n-3}{2j} (h a^{k} b^j + 1)^{n-3-2j} h^{2j} (d_2^+)^j = M_{22}^{n-2} \)

\[
M_{12}^{n-2} = \sum_{j=0}^{n-3} \binom{n-3}{2j+1} (h a^{k} b^j + 1)^{n-3-2j} h^{2j+1} (d_2^+)^j + M_{21}^{n-2} = \sum_{j=0}^{n-3} \binom{n-3}{2j+1} (h a^{k} b^j + 1)^{n-3-2j} h^{2j+1} (d_2^+)^j
\]

We will prove it for \( n \), since \( M_{11}^n = M_{11}^{n-2} M_{12}^{n-2} \), we get
From 4.6, we can write the solution for $x_i$ as

$$x_i = \sum_{j=0}^{\frac{n}{2}} \binom{n}{2j} (x_1^{\pm i})^{n-2j} (y_1^{\pm i})^{2j} (d_1^+)^{j}, \quad \text{if } n \text{ is even;}$$

$$x_i = \sum_{j=0}^{\frac{n-1}{2}} \binom{n}{2j+1} (x_1^{\pm i})^{n-2j} (y_1^{\pm i})^{2j} (d_1^+)^{j}, \quad \text{if } n \text{ is odd.}$$

Similarly, we can write the solution for $y_i$ as

$$y_i = \sum_{j=0}^{\frac{n-2}{2}} \binom{n}{2j+1} (x_1^{\pm i})^{n-2j} (y_1^{\pm i})^{2j+1} (d_1^+)^{j}, \quad \text{if } n \text{ is even;}$$

$$y_i = \sum_{j=0}^{\frac{n-3}{2}} \binom{n}{2j} (x_1^{\pm i})^{n-2j} (y_1^{\pm i})^{2j+1} (d_1^+)^{j}, \quad \text{if } n \text{ is odd.}$$

**Theorem 5.2.** The $n$th integer solution of $F_{\Delta d_i^+}(x, y) = 1$ is $(x_n^{\pm i}, y_n^{\pm i})$, where

$$x_n^{\pm i} = \sum_{j=0}^{\frac{n}{2}} \binom{n}{2j} (x_1^{\pm i})^{n-2j} (y_1^{\pm i})^{2j} (d_1^+)^{j}, \quad \text{if } n \text{ is even;}$$

$$x_n^{\pm i} = \sum_{j=0}^{\frac{n-1}{2}} \binom{n}{2j+1} (x_1^{\pm i})^{n-2j} (y_1^{\pm i})^{2j} (d_1^+)^{j}, \quad \text{if } n \text{ is odd.}$$

The proof follows from Theorem 4.6.
Now we can consider the Pell form $F_{\Delta_{d_1^+}}$ and note that this form is not reduced since $|\sqrt{4d} - 1| > 0$. So we can give the following theorem related to reduction of $F_{\Delta_{d_1^+}}$:

**Theorem 5.3.**

i: The reduction of $F_{\Delta_{d_1^+}}$ is

$$\rho^2(F_{\Delta_{d_1^+}}) = (1, 2hc^n, -c^m).$$

ii: The reduction of $F_{\Delta_{d_2^+}}$ is

$$\rho^2(F_{\Delta_{d_2^+}}) = (1, 2hc^n, -2c^m).$$

iii: The reduction of $F_{\Delta_{d_1^-}}$ is

$$\rho^2(F_{\Delta_{d_1^-}}) = (1, 2hc^n - 2, 1 - (2h - 1)c^m).$$

iv: The reduction of $F_{\Delta_{d_2^-}}$ is

$$\rho^2(F_{\Delta_{d_2^-}}) = (1, 2hc^n - 2, 1 - 2(h - 1)c^m).$$

**Proof.** Let $F_{\Delta_{d_1^+}} = d_{1,0}^+ = (1, 0, -d_1^+)$. Then from (3.5), we get $r_0 = 0$ and hence from (3.4), we have $\rho^1(F_{\Delta_{d_1^+}}) = (-d_1^+, 0, 1)$ which is not reduced. If we apply the reduction algorithm to $\rho^2(F_{\Delta_{d_1^+}})$ again, then we find that $r_1 = a^k b^l$ and so $\rho^2(F_{\Delta_{d_1^+}}) = (1, 2hc^n, -c^m)$ which is reduced. Similarly, we can get the reduction for others forms.

Now we can consider the cycle and proper cycle of $\rho^2(F_{\Delta_{d_1^+}})$.

**Theorem 5.4.** Let us consider the reduction $\rho^2(F_{\Delta_{d_1^+}})$ of $(F_{\Delta_{d_1^+}})$. Then

(1) The cycle of $\rho^2(F_{\Delta_{d_1^+}})$ is $(1, 2hc^n, -c^m) \sim (c^m, 2hc^n, -1).

(2) The cycle of $\rho^2(F_{\Delta_{d_2^+}})$ is $(1, 2hc^n - 2, 1 - (2h - 1)c^m) \sim (2hc^n - c^m - 1, 2hc^n - 2, -1).

(3) The cycle of $\rho^2(F_{\Delta_{d_1^-}})$ is $(1, 2hc^n - 2, 1 - (2h - 1)c^m) \sim (2hc^n, 2hc^n, -1).

(4) The cycle of $\rho^2(F_{\Delta_{d_2^-}})$ is $(1, 2hc^n - 4, 1 - 2(h - 1)c^m) \sim (2hc^n - 2c^m - 1, 2(h - 2)c^m, -2c^m) \sim (2hc^n, 2hc^n - 4c^m, -2hc^m + 2c^m + 1) \sim (2hc^n - 2c^m - 1, 2hc^n - 4, -1).

**Proof.** Let $\rho^2(F_{\Delta_{d_1^+}}) = \rho^2(F_{\Delta_{d_1^+}}) = (1, 2a^k b^l, -a^k b^l).$ Then from (3.6), we get $s_0 = 2h$ and thus $\rho^2(F_{\Delta_{d_1^+}}) = (a^k b^l, 2a^k b^l, -1).$ Again from (3.6), we get $s_1 = 2a^k b^l$ and hence $\rho^2(F_{\Delta_{d_1^+}}) = (1, 2a^k b^l, -a^k b^l) = \rho^2(F_{\Delta_{d_1^+}}).$ So the cycle of $\rho^2(F_{\Delta_{d_1^+}})$ is
\( \rho^2(F_{\Delta_{d_1^+}}) \sim \rho^2(F_{\Delta_{d_1^-}}) \). Note that \( l = 2 \), therefore from theorem 5.1 the proper cycle of \( \rho^2(F_{\Delta_{d_1^+}}) \) is \((1, 2a^kb', -a^kb')\) \( \sim (-2a^kb', 2a^kb', 1) \) of length 2. Similarly for \( \rho^2(F_{\Delta_{d_2^+}}) \) and \( \rho^2(F_{\Delta_{d_2^-}}) \), we can obtained the required cycle.

**Corollary 5.5.** The proper cycle of \( \rho^2(F_{\Delta_{d_1^+}}) \) is \((1, 2hc^m, -c^m) \sim (-c^m, 2hc^m, 1) \) of length 2.

The proper cycle of \( \rho^2(F_{\Delta_{d_2^+}}) \) is \((1, 2hc^m, -2c^m) \sim (2c^m, 2hc^m, -1) \) of length 2.

The proper cycle of \( \rho^2(F_{\Delta_{d_1^+}}) \) is \((1, 2hc^m - 2, 1 - (2h - 1)c^m) \sim (-2hc^m + c^m + 1, (2h - 1)c^m, c^m) \sim (-c^m, 2hc^m - 2c^m, 2hc^m - c^m - 1) \sim (-2hc^m + c^m + 1, 2hc^m - 2, 1) \) of length 4.

The proper cycle of \( \rho^2(F_{\Delta_{d_2^+}}) \) is \((1, 2hc^m - 2, 1 - 2(h - 1)c^m) \sim (-2hc^m + 2c^m + 1, 2(h - 2)c^m, 2c^m) \sim (-2c^m, 2hc^m - 4c^m, 2hc^m - 2c^m - 1) \sim (-2hc^m + 2c^m + 1, 2hc^m - 4, 1) \) of length 4.

Now we consider the proper automorphisms of \( F_{\Delta_{d_i^+}} \). To get this we first consider the following representations of the action of the group \( \Gamma \)

\[
g_{F_{\Delta_{d_i^+}}} = \left( \begin{array}{c} x_1^{\pm i} \\ y_1^{\pm i} \\ y_1^{\pm i} \\ x_1^{\pm i} \end{array} \right)
\]

Then we can give the following theorem which can be proved as in the same way that theorem 5.1 was proved.

**Theorem 5.6.** Let \( d_i^+ \) denote non-zero square free positive integer. Then

i: The set of proper automorphisms of \( F_{\Delta_{d_i^+}} \) is

\[
\text{Aut}^+(F_{\Delta_{d_i^+}}) = \{ \pm g_{F_{\Delta_{d_i^+}}}^n : n \in \mathbb{Z} \}.
\]

ii: The integer solutions of \( F_{\Delta_{d_i^+}}(x, y) = 1 \) are \((x_n^{\pm i}, y_n^{\pm i})\), where

\[
\left( \begin{array}{c} x_n^{\pm i} \\ y_n^{\pm i} \end{array} \right) = (g_{F_{\Delta_{d_i^+}}}^n) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \quad \text{for } n \geq 1.
\]

Remark 5.7. If \( c = a \) and \( k = l = m = 1 \) and \( b = 1 \), then the main results of [37] become the corollaries of our main results.

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