Q-CURVATURE FLOW WITH INDEFINITE NONLINEARITY

LI MA

Abstract. In this note, we study Q-curvature flow on $S^4$ with indefinite nonlinearity. Our result is that the prescribed Q-curvature problem on $S^4$ has a solution provided the prescribed Q-curvature $f$ has its positive part, which possesses non-degenerate critical points such that $\Delta_{S^4} f \neq 0$ at the saddle points and an extra condition such as a non-trivial degree counting condition.

Mathematics Subject Classification 2000: 53Cxx,35Jxx
Keywords: Q-curvature flow, indefinite nonlinearity, blow-up, conformal class

1. Introduction

Following the works of A.Chang-P.Yang [4], M.Brendle [3], Malchiodi and M.Struwe [6], we study a heat flow method to the prescribed Q-curvature problem on $S^4$. Given the Riemannian metric $g$ in the conformal class of standard metric $c$ on $S^4$ with Q-curvature $Q_g$. Then it is the well-known that

$$Q_g = -\frac{1}{12}(\Delta_g R_g - R_g^2 + 3|Rc(g)|^2) := Q,$$

where $R_g$, $Rc(g)$, $\Delta_g$ are the scalar curvature, Ricci curvature tensor, the Laplacian operator of the metric $g$ respectively.

Recall the Chern-Gaussian-Bonnet formula on $S^4$ is

$$\int_{S^4} Q_g dv_g = 8\pi^2,$$

Hence, we know that $Q_g$ has to be positive somewhere. This gives a necessary condition for the prescribed Q-curvature problem on $S^4$. Assuming the prescribed curvature function $f$ being positive on $S^4$, the heat flow for the Q-curvature problem is a family of metrics of the form $g = e^{2u(x,t)}c$ satisfying

$$u_t = \alpha f - Q, \quad x \in S^4, \quad t > 0,$$

where $u : S^4 \times (0,T) \rightarrow R$, and $\alpha = \alpha(t)$ is defined by

$$\alpha \int_{S^4} f dv_g = 8\pi^2.$$

Date: July 12th, 2008.

* The research is partially supported by the National Natural Science Foundation of China 10631020 and SRFDP 2006003002.
Here $dv_\mathcal{g}$ is the area element with respect to the metric $g$. It is easy to see that
\[
\alpha_t \int_{S^4} f dv_\mathcal{g} = 2\alpha \int_{S^4} (Q - \alpha f) f dv_\mathcal{g}.
\]
Malchiodi and M. Struwe [6] can show that the flow exists globally, furthermore, the flow converges at infinity provided $f$ possesses non-degenerate critical points such that $\Delta_{S^4} f \neq 0$ at the saddle points with the condition
\[
\sum_{\{p: \nabla f(p) = 0; \Delta_{S^4} f(p) < 0\}} (-1)^{\text{ind}(f,p)} \neq 0.
\]
Here $\Delta_{S^4} := \Delta$ is the Analyst’s Laplacian on the standard 4-sphere $(S^4, c)$. Recall that $\int_{S^4} dv_c = \frac{8}{3}\pi^2$. The purpose of this paper is to relax their assumption by allowing the function $f$ to have sign-changing or to have zeros.

Since we have
\[
Q = \frac{1}{2} e^{-4u}(\Delta^2 u - \text{div}(\frac{2}{3} R(c)c - 2Rc(c))du + 6),
\]
the equation (1) define a nonlinear parabolic equation for $u$, and the flow exists at least locally for any initial data $u|_{t=0} = u_0$. Clearly, we have
\[
\partial_t \int_{S^4} dv_\mathcal{g} = 2 \int S^4 u_t dv_\mathcal{g} = 0.
\]
We shall assume that the initial data $u_0$ satisfies the condition
\[
\int f e^{4u} dv_c > 0.
\]
We shall show that this property is preserved along the flow. It is easy to compute that
\[
Q_t = -4u_t Q - \frac{1}{2} Pu_t = 4Q(Q - \alpha f) + P(\alpha f - Q),
\]
where $P = P_g = e^{-4u}P_c$ and $P_c$ is the Paneitz operator in the metric $c$ on $S^4$ [4]. Using (4), we can compute the growth rate of the Calabi-type energy $\int_{S^4} |Q - \alpha f|^2 dv_\mathcal{g}$.

Our main result is following

**Theorem 1.** Let $f$ be a positive somewhere, smooth function on $S^4$ with only non-degenerate critical points on the its positive part $f_+$ with its Morse index $\text{ind}(f_+, p)$. Suppose that at each critical point $p$ of $f_+$, we have $\Delta f \neq 0$. Let $m_i$ be the number of critical points with $f(p) > 0$, $\Delta_{S^4} f(p) < 0$ and $\text{ind}(f, p) = 4 - i$. Suppose that there is no solutions with coefficients $k_i \geq 0$ to the system of equations
\[
m_0 = 1 + k_0, m_i = k_{i-1} + k_i, 1 \leq i \leq 4, k_4 = 0.
\]
Then $f$ is the $Q$ curvature of the conformal metric $g = e^{2u}c$ on $S^4$. 
Note that this result is an extension of the famous result of Malchiodi-Struwe [6] where only positive $f$ has been considered. A similar result for Curvature flow to Nirenberg problem on $S^2$ has been obtained in [8]. See also J.Wei and X.Xu’s work [9].

For simplifying notations, we shall use the conventions that $dc = \frac{du}{3\pi^2}$ and $\bar{u} = \bar{u}(t)$ defined by

$$\int_{S^4} (u - \bar{u}) dv_c = 0.$$ 

2. Basic properties of the flow

Recall the following result of Beckner [2] that

$$\int_{S^4} (|\Delta u|^2 + 2|\nabla u|^2 + 12u) dc \geq \log(\int_{S^4} e^{4u} dc) = 0,$$

where $|\nabla u|^2$ is the norm of the gradient of the function $u$ with respect to the standard metric $c$. Here we have used the fact that $\int_{S^4} e^{4u} dc = 1$ along the flow (1).

We show that this condition is preserved along the flow (1). In fact, letting

$$E(u) = \int_{S^4} (Pu + 4Qc) u dc = \int_{S^4} (|\Delta u|^2 + 2|\nabla u|^2 + 12u) dc$$

be the Liouville energy of $u$ and letting

$$E_f(u) = E(u) - 3\log(\int_{S^4} e^{4u} dc)$$

be the energy function for the flow (1), we then compute that

$$\partial_t E_f(u) = -\frac{3}{2\pi^2} \int_{S^4} |\alpha f - Q|^2 dv_g \leq 0.$$

One may see Lemma 2.1 in [6] for a proof. Hence

$$E_f(u(t)) \leq E_f(u_0), \quad t > 0.$$ 

After using the inequality (5) we have

$$\log(1/\int_{S^4} e^{4u} dc) \leq E_f(u_0),$$

which implies that $\int_{S^4} e^{4u} dv_c > 0$ and

$$e^{E_f(u_0)} \int_{S^4} e^{4u} dc \leq \int_{S^4} e^{4u} dc.$$

Note also that $\int_{S^4} e^{4u} dc = 1/\alpha(t)$. Hence,

$$\alpha(t) \leq \frac{1}{e^{E_f(u_0)}}.$$
Using the definition of $\alpha(t)$ we have

$$\alpha(t) \geq \frac{1}{\max_{S^4} f}.$$  

We then conclude that $\alpha(t)$ is uniformly bounded along the flow, i.e.,

(8) \quad \frac{1}{\max_{S^4} f} \leq \alpha(t) \leq \frac{1}{e^{E_f(u_0)}}. \]

We shall use this inequality to replace (26) in [6] in the study of the normalized flow, which will be defined soon following the work of Machiodi and M.Struwe [6]. If we have a global flow, then using (6) we have

$$2 \int_0^\infty dt \int_{S^4} |\alpha f - Q|^2 dv_g \leq 4\pi (E_f(u_0) + \log \max_{S^4} f).$$

Hence we have a suitable sequence $t_l \to \infty$ with associated metrics $g_l = g(t_l)$ and $\alpha(t_l) \to \alpha > 0$, and letting $Q_l = Q(g_l)$ be the Q-curvature of the metric $g_l$, such that

$$\int_{S^4} |Q_l - \alpha f|^2 \to 0, \quad (t_l \to \infty).$$

Therefore, once we have a limiting metric $g_\infty$ of the sequence of the metrics $g_l$, it follows that $Q(g_\infty) = \alpha f$. After a re-scaling, we see that $f$ is the Gaussian curvature of the metric $\beta g_\infty$ for some $\beta > 0$, which implies our Theorem 1.

3. Normalized flow and the proof of Theorem 1

We now introduce a normalized flow. For the given flow $g(t) = e^{2u(t)}c$ on $S^4$, there exists a family of conformal diffeomorphisms $\phi = \phi(t) : S^4 \to S^4$, which depends smoothly on the time variable $t$, such that for the metrics $h = \phi^*g$, we have

$$\int_{S^4} x dv_h = 0, \quad for \ all \ t \geq 0.$$

Here $x = (x^1, x^2, x^3, x^4, x^5) \in S^4 \subset R^5$ is a position vector of the standard 4-sphere. Let

$$v = u \circ \phi + \frac{1}{4} \log(det(d\phi)).$$

Then we have $h = e^{2v}c$. Using the conformal invariance of the Liouville energy [4], we have

$$E(v) = E(u),$$

and furthermore,

$$Vol(S^4, h) = Vol(S^4, g) = \frac{8}{3}\pi^2, \quad for \ all \ t \geq 0.$$

Assume $u(t)$ satisfies (11) and (2). Then we have the uniform energy bounds

$$0 \leq E(v) \leq E(u) = E_f(u) + \log(\int_{S^4} f e^{4u} dc) \leq E_f(u_0) + \log(\max_{S^4} f).$$
Using Jensen’s inequality we have

\[ 2\bar{v} := \int_{S^4} 2v dc \leq \log(\int_{S^4} e^{4v} dc) = 0. \]

Using this we can obtain the uniform \( H^1 \) norm bounds of \( v \) for all \( t \geq 0 \) that

\[ \sup_t |v(t)|_{H^1(S^2)} \leq C. \]

See the proof of Lemma 3.2 in [6]. Using the Aubin-Moser-Trudinger inequality [1] we further have

\[ 4 \sup_{\{0 \leq t < T\}} \int_{S^4} |u(t)| dc \leq \sup_t \int_{S^4} e^{4|u(t)|} dc \leq C < \infty. \]

Note that

\[ v_t = u_t \circ \phi + \frac{1}{4} e^{-4v} \text{div}_{S^4} (\xi e^{4v}) \]

where \( \xi = (d\phi)^{-1} \phi_t \) is the vector field on \( S^2 \) generating the flow \( (\phi(t)), t \geq 0, \) as in [6], formula (17), with the uniform bound

\[ |\xi|_{L^\infty(S^4)} \leq C \int_{S^4} |\alpha f - K|^2 dv_g. \]

With the help of this bound, we can show (see Lemma 3.3 in [6]) that for any \( T > 0 \), it holds

\[ \sup_{0 \leq t < T} \int_{S^2} e^{4|u(t)|} dc < +\infty. \]

Following the method of Malchiodi and M.Struwe [6] (see also Lemma 3.4 in [5]) and using the bound (8) and the growth rate of \( \alpha \), we can show that

\[ \int_{S^4} |\alpha f - Q|^2 dv_g \to 0 \]

as \( t \to \infty \). Once getting this curvature decay estimate, we can come to consider the concentration behavior of the metrics \( g(t) \). Following [5], we show that

**Lemma 2.** Let \((u_l)\) be a sequence of smooth functions on \( S^4 \) with associated metrics \( g_l = e^{2u_l} c \) with \( \text{vol}(S^4, g_l) = \frac{8}{3} \pi^2, \ l = 1, 2, ... \) as constructed above. Suppose that there is a smooth function \( Q_\infty \), which is positive somewhere in \( S^4 \) such that

\[ |Q(g_l) - Q_\infty|_{L^2(S^4, g_l)} \to 0 \]

as \( l \to \infty \). Let \( h_l = \phi_l^* g_l = e^{2v_l} c \) be defined as before. Then we have either

1) for a subsequence \( l \to \infty \) we have \( u_l \to u_\infty \) in \( H^4(S^4, c) \), where \( g_\infty = e^{2u_\infty} c \) has \( Q \)-curvature \( Q_\infty \), or

2) there exists a subsequence, still denoted by \((u_l)\) and a point \( q \in S^4 \) with \( Q_\infty(q) > 0 \), such that the metrics \( g_l \) has a measure concentration that

\[ dv_{g_l} \to \frac{8}{3} \pi^2 \delta_q \]
weakly in the sense of measures, while \( h_1 \to c \) in \( H^4(S^4, c) \) and in particular, \( Q(h_1) \to 3 \) in \( L^2(S^4) \). Moreover, in the latter case the conformal diffeomorphisms \( \phi_l \) weakly converges in \( H^2(S^4) \) to the constant map \( \phi_\infty = q \).

**Proof.** The case 1) can be proved as Lemma 3.6 in [6]. So we need only to prove the case 2). As in [6], we choose \( q_l \in S^4 \) and radii \( r_l > 0 \) such that

\[
\sup_{q \in S^4} \int_{B(q,r_l)} |K(g_l)| dv_{g_l} \leq \int_{B(q_l,r_l)} |K(g_l)| dv_{g_l} = 2\pi^2,
\]

where \( B(q,r_l) \) is the geodesic ball in \((S^4, g_l)\). Then we have \( r_l \to 0 \) and we may assume that \( q_l \to q \) as \( l \to \infty \). For each \( l \), we introduce \( \phi_l \) as in Lemma 3.6 in [6] so that the functions

\[
\hat{u}_l = u_l \circ \phi_l + \frac{1}{4} \log(det(d\phi_l))
\]

satisfy the conformal Q-curvature equation

\[
-P_{R^4} \hat{u}_l = 2\hat{Q}_l e^{4\hat{u}_l}, \text{ on } R^4,
\]

where \( \hat{Q}_l = Q(g_l) \circ \phi \) and \( P_{R^4} \) is the Paneitz operator of the standard Euclidean metric \( g_{R^4} \). Note that for \( \hat{g}_l = \phi^* g_l = e^{2\hat{u}_l} g_{R^4} \), we have

\[
Vol(R^4, \hat{g}_l) = Vol(S^4, g_l) = \frac{8}{3}\pi^2.
\]

Arguing as in [6], we can conclude a convergent subsequence \( \hat{u}_l \to \hat{u}_\infty \) in \( H^4_{loc}(R^4) \) where \( \hat{u}_\infty \) satisfies the Liouville type equation

\[
-\Delta^2_{R^4} \hat{u}_\infty = \hat{Q}_\infty(q) e^{4\hat{u}_\infty}, \text{ on } R^4,
\]

with

\[
\int_{R^4} K_\infty(q) e^{4\hat{u}_\infty} dz \leq \frac{8}{3}\pi^2.
\]

We only need to exclude the case when \( Q_\infty(q) \leq 0 \). Just note that by (7) we have

\[
\log(1/\int_{R^4} f \circ \phi_l e^{4\hat{u}_l}) \leq E_f(u_0).
\]

Hence, sending \( l \to \infty \), we always have \( f \circ \phi_l \to f \circ \phi(q) > 0 \) uniformly on any compact domains of \( R^4 \).

The remaining part is the same as in the proof of Lemma 3.6 in [6]. We confer to [6] for the full proof.

\(\square\)

We remark that some other argument can also exclude the case \( Q_\infty(q) < 0 \). It can not occur since there is no such a solution on the whole space \( R^4 \) (see also the argument in [7]). If \( Q_\infty(q) = 0 \), then \( \Delta_{R^4} \bar{u} := \Delta_{R^4} \bar{u}_\infty \) is a harmonic function in \( R^4 \). Let \( \bar{u}(r) \) be the average of \( u \) on the circle \( \partial B_r(0) \subset R^4 \). Then we have

\[
\Delta^2_{R^4} \bar{u} = 0.
\]
Hence $\Delta_{R^4} \bar{u} = A + Br^{-2}$ for some constants $A$ and $B$, where $r = |x|$. Since $\Delta_{R^4} \bar{u}$ is a continuous function on $[0, \infty)$, we have $\Delta_{R^4} \bar{u} = A$, which gives us that

$$\bar{u} = A + Br^2 + Cr^{-2}$$

for some constants $A$, $B$, and $C$. Again, using $\bar{u}$ is regular, we have $C = 0$ and $\bar{u} = A + Br^2$ with $B < 0$. However, it seems hard to exclude this case without the use of the fact (7).

With this understanding, we can do the same finite-dimensional dynamics analysis as in section 5 in [6]. Then arguing as in section 5 in [6] we can prove Theorem 1. By now the argument is well-known, so we omit the detail and refer to [6] for full discussion.

References

[1] Aubin, Thierry: Some nonlinear problems in Riemannian geometry (Springer Monographs in Mathematics). Springer, Berlin 1998 MR1636569 (99i:58001)
[2] W.Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, Ann. Math., 138(1993)213-242.
[3] S.Brendle, Global existence and convergence for a higher order flow in conformal geometry, Ann. math., 158(2003)323-343.
[4] S.Y.A.Chang, P.Yang, Extremal metrics of zeta function determinants on 4-manifolds, Ann. Math., 142(1995)171-212.
[5] M.Struwe, Curvature flows on surfaces, Annali Sc. Norm. Sup. Pisa, Ser. V, 1(2002)247-274.
[6] A.Malchiodi, M.Struwe, Q-curvature flow on $S^4$, J.Diff. Geom., 73(2006)1-44.
[7] L.Ma, Three remarks on mean field equations, preprint, 2008.
[8] M.C.Hong,L.Ma, Curvature flow to Nirenberg problem, preprint, 2008.
[9] J.Wei and X.Xu, On conformal deformation of metrics on $S^n$, J. Functional Analysis, 157(1998)292-325.

Li Ma, Department of mathematical sciences, Tsinghua University, Beijing 100084, China
E-mail address: lma@math.tsinghua.edu.cn