Self-force of a point charge in the spacetime of a massive wormhole

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Abstract
We consider the self-potential and the self-force for an electrically charged particle at rest in a massive wormhole spacetime. We develop a general approach for the renormalization of the electromagnetic field of such a particle in the static spacetimes and apply it to the spacetime of a wormhole, the metric of which is a solution to the Einstein-scalar field equations in the case of a phantom massless scalar field. The self-force is found in manifest form; it is an attractive force. We discuss the peculiarities due to the parameter of the wormhole mass.

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1. Introduction
A wormhole is an example of the spacetime with nontrivial topology and it connects different universes or different parts of the same universe. A careful discussion of wormhole geometry and physics as well as a good introduction in the subject may be found in the Visser book [1] and in the review by Lobo [2]. The most recent growth of interest in wormholes was connected with the ‘time machine’, introduced by Morris, Thorne and Yurtsever in [3, 4].

In curved spacetimes there is an interesting interaction charged particle with its own electromagnetic field due to scattering on the curvature [5]. The origin of this self-force is associated with the nonlocal structure of the field, the source of which is the particle. For particles in the spacetime of a cosmic string [6], the self-force may be a unique local gravitational interaction on a particle. Therefore, in this case there is no local gravitational interaction between the particle and the cosmic string, but nevertheless there exists a repulsive self-interaction force which is of nonlocal origin. A discussion of the self-force in detail may be found in [7, 8].
In calculating the self-force acting on a particle, one typically encounters a divergent expression. The finite expression for the self-force should be obtained by using a certain renormalization prescription. In this paper, we develop a simple renormalization procedure for a vector potential which may be applied for the charged particle at rest in static spacetimes. The main idea of this renormalization method originates from the well-developed procedure of the renormalization of the Green’s functions in quantum field theory on the background of curved spacetimes. In the framework of this procedure, one subtracts some terms of the expansion of the corresponding Green’s function of the massive field from the divergent expression obtained. The quantities of terms to be subtracted are defined by a simple rule—they no longer vanish as the field’s mass goes to infinity. This approach has been developed in [9, 10] for the massless scalar field.

The self-force for the charged particle at rest in the static wormhole background was analyzed in detail in [11–14] for arbitrary and specific profiles of the throat. The spacetime was a spherical symmetric solution, the Einstein equations with the phantom scalar field as a source. Sometimes this background is called anti-Fisher due to the analogous solution obtained by Fisher [15] for the normal scalar field. The spacetime of the wormhole has had an ultrastatic form, $g_{tt} = 1$, and therefore the Newtonian potential equals zero. For this reason we call this background a massless wormhole. It was shown that for an electric charge the self-force is always attractive [11–13], whereas it may be attractive or repulsive for a scalar charge [14] depending on the non-minimal connection constant $\xi$. In this paper we analyze the self-force for an electric particle at rest in the spacetime of another kind of wormhole considered in [16, 17] with the phantom scalar field as a source. We call this spacetime a massive wormhole because the Newtonian potential is not zero (see section 2).

The organization of this paper is as follows. In section 2, we consider the spacetime properties of the background under consideration. In section 3 we obtain in a manifest form the solution of the Maxwell equations for the particle at rest in the background of the massive wormhole [16, 17]. Section 4 is devoted to the renormalization procedure for the electromagnetic field of such particle in the case of static spacetimes. The procedure suggested is applied for the well-known case, the Schwarzschild black hole. In section 5 we calculate in manifest form the self-energy and self-force for the electrically charged particle in the spacetime of the massive wormhole [16, 17]. We discuss the results in section 6. Our conventions are those of Misner, Thorne and Wheeler [18]. Throughout this paper, we use the units $c = G = 1$.

2. The spacetime

The line element of the massive wormhole has the following form [16, 17]:

$$ds^2 = -e^{-\alpha(\rho)} dt^2 + e^{\alpha(\rho)} d\rho^2 + r^2(\rho) d\Omega^2, \quad (1)$$

where

$$r^2(\rho) = p^2 e^{2\alpha(\rho)}, \quad (2)$$

$$\alpha(\rho) = \frac{m}{\sqrt{n^2 - m^2}} \left\{ \frac{\pi}{2} - \arctan \left( \frac{\rho}{\sqrt{n^2 - m^2}} \right) \right\} \quad (3)$$

and $p^2 = \rho^2 + n^2 - m^2$. The radial coordinate $\rho$ may be positive as well as negative too. The square of the sphere of the radial coordinate $\rho$, $S = 4\pi r^2(\rho)$, is minimized for $\rho = m$.

There are two parameters, $n$ and $m$, in metric (1) with the relation $m^2 < n^2$. We consider the case $m > 0$ without the loss of generality. For $m = 0$, we obtain the wormhole which
was called as massless ‘drainhole’ [17] with the throat radius \( n \). To reveal the role of the parameter \( m \), let us consider the gravitational acceleration, \( a \), acting for a particle. In the spherical symmetric case it has the well-known form [19, 20]

\[
a = \sqrt{g^{tt}} a_\rho = - \frac{1}{\sqrt{g_{\rho\rho}}} \frac{\mathrm{d} \ln \sqrt{-g_{tt}}}{\mathrm{d} \rho}.
\]

For the metric under consideration (1), we obtain

\[
a = - \frac{m}{\rho^2} e^{a\rho/2}.
\]

In the limit \( \rho \to +\infty \), we have the Newtonian gravitational attraction, \( a \approx -m/\rho^2 \), to the wormhole throat with the effective mass \( m \). For \( \rho \to -\infty \) we obtain the Newtonian gravitational repulsion, \( a \approx -m e^{\alpha m/2}/\rho^2 \), with the negative effective mass \( -m e^{\alpha m/2} \), where \( \alpha_{\text{max}} = \alpha_{\rho \to -\infty} \). Therefore, the parameter \( m \) characterizes the effective mass of the wormhole.

The parameter \( n \) of the wormhole may be understood in terms of a charge of some massless scalar field going through the wormhole (for an electric charge, see [21]). Indeed, the equation for a scalar massless field in the background with the line element (1) admits the radial non-singular solution for the scalar field. The derivative of this field with respect to the radial coordinate reads \( \phi'(\rho) = -q/\rho^2 \) with some parameter \( q \) which corresponds to the scalar charge from the point of view of a distant observer. The stress–energy tensor of this field has the form \( 8\pi T_{\mu}^\nu = q^2 \phi''/\rho^4 \text{diag}(1, 1, -1, -1) \). On the other hand, metric (1) is a non-singular solution of the Einstein equations with the scalar massless field with the opposite sign of the stress–energy tensor which reads \( 8\pi T_{\mu}^\nu = -n^2 \phi''/\rho^4 \text{diag}(1, 1, -1, -1) \). Therefore, the parameter \( n \) plays the role of the scalar charge of the phantom scalar field which passes through the wormhole spacetime.

3. The Maxwell equations

The Maxwell equations under the covariant Lorentz gauge have the following form (\( e \) is an electric charge of the particle):

\[
g^{\alpha\beta} A_{\mu,\alpha\beta} - A_{\nu} R_{\mu}^{\nu} = -4\pi J_{\mu} = -4\pi e \int u_\mu(\tau) \delta^{(4)}(x - x(\tau)) \frac{\mathrm{d}r}{\sqrt{-g}}.
\]

In the manifest form, they read

\[
-4\pi e \phi J_t = A_{t,\rho\rho} + 2(\rho - m) \frac{A_{t,\rho}}{p^2} \frac{L^2 A_t}{p^2} - \frac{2m}{p^2} A_{\rho,tt},
-4\pi e \phi J_\rho = A_{\rho,\rho\rho} + 2(\rho + m) \frac{A_{\rho,\rho}}{p^2} \frac{L^2 A_\rho}{p^2} - \frac{2(\rho - m)}{p^4} \frac{L^2 A_\rho}{p^2} - \frac{2m}{p^2} A_{\rho,tt},
-4\pi e \phi J_\theta = A_{\theta,\rho\rho} + 2m \frac{A_{\theta,\rho}}{p^2} \frac{L^2 A_\theta}{p^2} - \frac{2(\rho - m)}{p^4} \frac{L^2 A_\theta}{p^2} - \frac{2m}{p^2} A_{\theta,tt},
-4\pi e \phi J_\phi = A_{\phi,\rho\rho} + 2m \frac{A_{\phi,\rho}}{p^2} \frac{L^2 A_\phi}{p^2} - \frac{2(\rho - m)}{p^4} \frac{L^2 A_\phi}{p^2} - \frac{2m}{p^2} A_{\phi,tt},
\]
and the Lorentz condition is
\[ A_{ρ,ρ} + \frac{2ρ}{p^2} A_ρ + \frac{1}{p^2} A_{θ,θ} + \frac{1}{p^2} A_ρ = e^{2α} A_{t,t} = 0, \]  
(7)

where \( p^2 = ρ^2 + n^2 - m^2 \) and \( \hat{L}^2 = \frac{p^2}{ρ} + \cot θ \partial_θ + \csc^2 θ \partial_θ^2 \).

For a particle at rest with the position \( x' \), the vector potential \( A_μ \) has no dependence on the time and we use the following ansatz: \( A_μ = (0, 0, 0, A_t) \). The equation for the Lorentz gauge is identically satisfied and the system of equations reduces for a single equation for \( A_t \):
\[ A_{t,ρρ} + \frac{2(ρ - m)}{p^2} A_{t,ρ} + \frac{1}{p^2} \hat{L}^2 A_t = \frac{4πe}{e^α p^2} \delta^{(3)}(x - x'). \]
(8)

Due to the spherical symmetry of the problem under consideration, we represent the potential in the following form:
\[ A_t = 4πe \sum_{l,m} Y_{lm}(Ω_1) Y_{lm}^*(Ω_1') g_l(ρ, ρ'), \]
(9)

where \( Y_{lm}(Ω) \) is the spherical functions of the argument \( Ω = (θ, ϕ) \). The radial part, \( g_l \), satisfies the equation
\[ g_{l,ρρ} + \frac{2(ρ - m)}{p^2} g_{l,ρ} + \frac{l(l+1)}{p^2} g_l = \frac{1}{e^α p^2} \delta(ρ - ρ'). \]
(10)

To solve this equation we use the approach developed in [11]. Namely, the solution of the above equation is represented in the following form:
\[ g_l(ρ, ρ') = \Psi_1(ρ - ρ') Ψ_2(ρ) + \Psi_1(ρ) Ψ_2(ρ'), \]
(11)

where \( Ψ_1 \) and \( Ψ_2 \) increase and decrease for the \( ρ \rightarrow ∞ \) solutions of the corresponding homogeneous equation with Wronskian \( W(Ψ_1, Ψ_2) = -e^{-α}/p^2 \). These solutions in turn are expressed in terms of the full set of solutions of the corresponding homogeneous equation for both domains of the space, namely for \( ρ > 0 \) and \( ρ < 0 \). The full set of solutions reads
\[ \phi_1^+ = c_1 e^{b \arctan x} P_1^0(ix), \quad ρ > 0, \]
(12a)
\[ \phi_2^+ = c_1 e^{b \arctan x} I_1^0(ix), \quad ρ > 0, \]
(12b)
\[ \phi_1^- = c_1 e^{b \arctan x} P_1^0(-ix), \quad ρ < 0, \]
(12c)
\[ \phi_2^- = c_1 e^{b \arctan x} Q_1^0(-ix), \quad ρ < 0, \]
(12d)

where
\[ x = \frac{ρ}{\sqrt{n^2 - m^2}}, \quad b = \frac{m}{\sqrt{n^2 - m^2}}, \]
(13)
and \( P_1^0 \) and \( Q_1^0 \) are the modified Legendre functions of the first and second kind, respectively [23]. Then we have to consider six different relations of \( ρ \) and \( ρ' \) [11] depending on the sign and the value of the radial coordinate. To calculate self-force, we need only \( ρ > ρ' > 0 \) and \( 0 > ρ' > ρ \). Taking into account the Wronskian (see [23])
\[ W(P_1^0(z), Q_1^0(z)) = \frac{e^{ixμ}}{1 - z^2} \frac{Γ(1 + v + μ)}{Γ(1 + v - μ)}, \]
(14)

3 We use the covariant component of the potential in contrast with [11], and for this reason there is no minus sign here on the right-hand side.
we obtain

\[ A^+_t = -\frac{\varepsilon b \arctan x + b \arctan x'}{\sqrt{x^2 - \rho^2}} \sum_{l=0}^{\infty} (2l + 1) \left\{ \text{ie}^{-2\pi b} P_l^{ib}(ix') Q_l^{-ib}(ix) + \frac{1}{\pi} Q_l^{ib}(ix') Q_l^{-ib}(ix) \right\}, \]

\[ A^-_t = -\frac{\varepsilon b \arctan x + b \arctan x'}{\sqrt{x^2 - \rho^2}} \sum_{l=0}^{\infty} (2l + 1) \left\{ \text{ie}^{-2\pi b} P_l^{ib}(-ix') Q_l^{-ib}(-ix) + \frac{1}{\pi} Q_l^{ib}(-ix') Q_l^{-ib}(-ix) \right\}, \]

where \( A^+_t \) is the vector potential for \( \rho > \rho' > 0 \) and \( A^-_t \) is for \( 0 > \rho' > \rho \).

To find expressions in close form for the series above, we take into account the Heine formula

\[ \sum_{l=0}^{\infty} (2l + 1) P_l(x') Q_l(x) = \frac{1}{x - x'}, \quad (15) \]

and integral representations for the Legendre functions [23]

\[ P_l^\mu(x') = \left( \frac{x'^2 - 1}{\mu} \right)^{\mu/2} \frac{1}{\Gamma(-\mu)} \int_{-1}^{1} (x' - \tau)^{-\mu-1}, \quad 9\mu > 0, \quad x' \notin [-1, 1], \quad (16) \]

\[ Q_l^{-\mu}(x) = e^{\pi i} \frac{(x^2 - 1)^{-\mu/2}}{\Gamma(\mu)} \int_{x}^{\infty} Q_l(t)(t - x)^{\mu-1}, \quad l + 1 > 9\mu > 0, \quad |x| > 1. \quad (17) \]

Then after some calculation, we obtain

\[ \sum_{l=0}^{\infty} (2l + 1) P_l^{ib}(ix') Q_l^{-ib}(ix) = -\frac{\text{ie}^{-\pi b}}{x - x'} e^{b + b \arctan x - b \arctan x'} \quad (18) \]

for \( x > x' > 0 \) or \( x < x' < 0 \). Taking into account the formula

\[ Q_l^{ib}(ix') = \frac{\pi}{2 \sinh \pi b} \left\{ (-1)^l P_l^{ib}(-ix') - P_l^{ib}(ix') \right\}, \quad (19) \]

we obtain another series in the close form:

\[ \sum_{l=0}^{\infty} (2l + 1) Q_l^{ib}(ix') Q_l^{-ib}(ix) = \frac{e^{b \arctan x + b \arctan x'} - e^{b \arctan x - b \arctan x'}}{2 \sinh \pi b}, \quad \forall x, x'. \quad (20) \]

Using these formulas, we obtain

\[ A^+_t = -\frac{\varepsilon}{\rho - \rho'} e^{-\alpha(\rho)} + \frac{\text{ie}^{\pi b}}{2 \sinh \pi b} \frac{e^{-\alpha(\rho)} - e^{-\alpha(\rho')}}{\rho - \rho'}, \quad \rho > \rho' > 0, \quad (21) \]

\[ A^-_t = -\frac{\varepsilon}{\rho' - \rho} e^{-\alpha(\rho)} + \frac{\text{ie}^{\pi b}}{2 \sinh \pi b} \frac{e^{-\alpha(\rho)} - e^{-\alpha(\rho')}}{\rho - \rho'}, \quad \rho < \rho' < 0. \quad (22) \]

As noted above, these expressions are divergent at \( \rho' \to \rho \), and should be renormalized.
4. Renormalization

We adopt here a general approach to renormalization in curved spacetime [22], which means the subtraction of the first terms of the DeWitt–Schwinger asymptotic expansion of the Green’s function. The renormalization procedure for the spacetime with the line element (1) is not so simple as it was for the massless wormhole [11]. The point is that the spacetime under consideration has no ultrastatic form, that is \( g_{tt} \neq 1 \). For this reason the equation for \( A_t \) in the static case does not coincide with that for the scalar 3D Green’s function and we cannot use the standard formulas for the DeWitt–Schwinger expansion of the 3D Green’s function. Let us simplify the problem of renormalization by reducing them to the 3D case for the potential \( A_t \) of the charged particle at rest in a static spacetime,

\[
\text{d}s^2 = g_{tt}(x^i) \text{d}t^2 + \sum_{i=1}^{3} \text{d}x^i \text{d}x^j,
\]

where \( i, j, k = 1, 2, 3 \). This means that the equation for the Lorentz gauge is identically satisfied and one can write out the equation for the single nonzero component of \( A_\mu \) in the case of the massive field in the following way:

\[
g^{\mu\nu} A_{\mu;\nu} - A_t R^t_t - m^2_{\text{ef}} A_t = -4\pi e \int_{u_t(\tau)} \text{d}x^i \delta^{(3)}(x, x') \frac{\text{d}\tau}{\sqrt{-g}} = 4\pi e \sqrt{-g_{tt} / \sqrt{g^{(3)}}} \delta^{(3)}(x^i, x'^i),
\]

or

\[
g^{ij} \frac{\partial^2 A_i}{\partial x^i \partial x^j} - \left( g^{ik} \Gamma^k_{ij} + \frac{\partial^2 g_{tt}}{2g_{tt}} \frac{\partial A_t}{\partial x^i} + \frac{1}{4g_{tt}^2} \frac{\partial g_{tt}}{\partial x^i} \frac{\partial g_{tt}}{\partial x^j} + \Gamma^k_{ij} \frac{\partial g_{tt}}{2g_{tt} \partial x^k} \right) A_i - A_t R^t_t - m^2_{\text{ef}} A_t = 4\pi e \sqrt{-g_{tt} / \sqrt{g^{(3)}}} \delta^{(3)}(x^i, x'^i),
\]

where \( m_{\text{ef}} \) is the mass of the field, \( g^{(3)} = \det g_{ij} \), and we have taken into account that \( \frac{\text{d}\tau}{\text{d}t} = \sqrt{-g_{tt}} \). From this equation we obtain the following equation for the tetrad component of the vector field \( A_{(i)} = A_t / \sqrt{-g_{tt}} \):

\[
g^{ij} \left( \frac{\partial^2 A_{(i)}}{\partial x^i \partial x^j} - \Gamma^d_{ij} \frac{\partial A_{(d)}}{\partial x^d} \right) - m^2_{\text{ef}} A_{(i)} + \frac{g^{ik} \partial g_{tt} \partial A_{(i)}}{2g_{tt} \partial x^i \partial x^k} - \left( \frac{g^{ij} \partial g_{tt} \partial g_{tt}}{4g_{tt}^2 \partial x^i \partial x^j} + R^i_t \right) A_{(i)} = 4\pi e \sqrt{-g_{tt} / \sqrt{g^{(3)}}} \delta^{(3)}(x^i, x'^i).
\]

In the case of \( m_{\text{ef}} \gg 1/L \), we can obtain the solution of this equation near the point \( x' \) in terms of the distance along the geodesic between the separated points and purely geometrical quantities constructed out of the Riemann tensor [24, 25]. Let us consider the equation for the three-dimensional Euclidean Green’s function \( G_E(x^i, x'^i) \)

\[
\frac{1}{\sqrt{g^{(3)}}} \frac{\partial}{\partial x^i} \left( g^{(3)} g^{ik} \frac{\partial G_E(x^i, x'^i)}{\partial x^k} \right) - m^2_{\text{ef}} G_E(x^i, x'^i) + \frac{g^{ik} \partial g_{tt} \partial G_E(x^i, x'^i)}{2g_{tt} \partial x^j \partial x^k}
\]

\[
- \left( \frac{g^{ij} \partial g_{tt} \partial g_{tt}}{4g_{tt}^2 \partial x^i \partial x^j} + R^j_t \right) G_E(x^i, x'^i) = \frac{\delta^{(3)}(x^i, x'^i)}{\sqrt{g^{(3)}}},
\]

and introduce the Riemann normal coordinates \( y^i \) in the 3D space with the origin at the point \( x'^i \) [26]. In these coordinates, one has
\[ g_{ij}(y') = \delta_{ij} - \frac{1}{3} \tilde{R}_{ij} |_{y=0} y^i y^j + O \left( \frac{y^3}{L^3} \right), \quad (28) \]

\[ g^{ij}(y') = \delta^{ij} + \frac{1}{3} \tilde{R}^{ij} |_{y=0} y^i y^j + O \left( \frac{y^3}{L^3} \right), \quad (29) \]

\[ g^{(3)}(y') = 1 - \frac{1}{3} \tilde{R}^{ij} |_{y=0} y^i y^j + O \left( \frac{y^3}{L^3} \right), \quad (30) \]

where \( \delta_{ij} \) denotes the metric of a flat three-dimensional Euclidean spacetime and \( L \) is the characteristic curvature scale of the background geometry. \( \tilde{R}_{ij} \) and \( \tilde{R}^{ij} \) respectively stand for the components of Riemann and Ricci tensors of the three-dimensional spacetime with the metric \( g_{ij} \).

\[ R_{ij} |_{y=0} = \tilde{R}_{ij} |_{y=0} - \frac{g_{ij,t}}{2g_{tt}} + \frac{g_{ij,k} g_{tt,k}}{4g_{tt}^2} \]

\[ R_{ij} |_{y=0} = \tilde{R}_{ij} |_{y=0} - \frac{\delta^{ij} g_{tt,j}}{2g_{tt}} - \frac{\delta^{ij} g_{tt,i} g_{tt,j}}{4g_{tt}^2}. \quad (31) \]

Defining \( \overline{G}(y') \) by the relation

\[ \overline{G}(y') = \sqrt{g} G_{E}(y'), \quad (32) \]

and retaining in (27) only terms with coefficients involving two derivatives of the metric or fewer, one finds that \( \overline{G}(y') \) satisfies the equation

\[ \delta^{ij} \frac{\partial^2 \overline{G}}{\partial y^i \partial y^j} - m^2 \overline{G} + \delta^{ij} \frac{g_{ij,t}}{2g_{tt}} \frac{\partial \overline{G}}{\partial y^j} + \delta^{ij} \left( \frac{g_{ij,k} g_{tt,k}}{2g_{tt}^2} - \frac{g_{ij,t} g_{tt,j}}{2g_{tt}^2} \right) y^j \frac{\partial \overline{G}}{\partial y^j} \]

\[ + \overline{R}_{ij} y^i y^j \frac{\partial^2 \overline{G}}{\partial y^i \partial y^j} + \delta^{ij} \left( \frac{g_{ij,t}}{2g_{tt}} - \frac{g_{ij,t} g_{tt,i}}{2g_{tt}^2} \right) \overline{G} + \frac{\tilde{R}}{3} \overline{G} = -\delta^{(3)}(y), \quad (33) \]

where the coefficients here and below are evaluated at \( y^i = 0 \) (i.e. at the point \( x^i \)). Let us represent the Green’s function \( \overline{G}(y') \) as a series

\[ \overline{G}(y') = \overline{G}_0(y') + \overline{G}_1(y') + \overline{G}_2(y') + \cdots, \quad (34) \]

where \( \overline{G}_a(y') \) incorporates the geometrical coefficients involving the \( a \) derivatives of the metric at point \( y^i = 0 \). These functions satisfy the set of equations

\[ \delta^{ij} \frac{\partial^2 \overline{G}_0}{\partial y^i \partial y^j} - m^2 \overline{G}_0 = -\delta^{(3)}(y), \quad (35) \]

\[ \delta^{ij} \frac{\partial^2 \overline{G}_1}{\partial y^i \partial y^j} + \frac{g_{ij,t}}{2g_{tt}} \frac{\partial \overline{G}_0}{\partial y^j} = 0, \quad (36) \]

\[ \delta^{ij} \frac{\partial^2 \overline{G}_2}{\partial y^i \partial y^j} - m^2 \overline{G}_2 + \frac{g_{ij,t}}{2g_{tt}} \frac{\partial \overline{G}_1}{\partial y^j} + \delta^{ij} \left( \frac{g_{ij,k} g_{tt,k}}{2g_{tt}^2} - \frac{g_{ij,t} g_{tt,j}}{2g_{tt}^2} \right) \overline{G}_0 \]

\[ + \overline{R}_{ij} y^i y^j \frac{\partial^2 \overline{G}_0}{\partial y^i \partial y^j} + \delta^{ij} \left( \frac{g_{ij,t}}{2g_{tt}} - \frac{g_{ij,t} g_{tt,i}}{2g_{tt}^2} \right) \overline{G}_0 + \frac{\tilde{R}}{3} \overline{G}_0 = 0. \quad (37) \]

The function \( \overline{G}_0(y') \) obeys the condition

\[ \overline{R}_{ij} y^i y^j \frac{\partial^2 \overline{G}_0}{\partial y^i \partial y^j} - \overline{R}_j y^j \frac{\partial \overline{G}_0}{\partial y^j} = 0, \quad (38) \]
since $\overline{G}_0(y^i)$ should be the function of $\delta_{ij} y^i y^j$ only. Therefore, equation (37) may be rewritten as

$$
\delta^{ij} \frac{\partial^2 \overline{G}_2(y^i)}{\partial y^i \partial y^j} - m^2 \overline{G}_2(y^i) + \delta^{ij} \frac{g_{tt,ij}}{2g_{tt}} \frac{\partial \overline{G}_1}{\partial y^i} + \left[ \frac{1}{3} \tilde{R}_k^k + \frac{\delta^{ij}}{2} \left( \frac{g_{tt,kj}}{2g_{tt}} - \frac{g_{tt,k} g_{tt,j}}{2g_{tt}^2} \right) \right] y^i \frac{\partial \overline{G}_0}{\partial y^j} + \frac{\delta^{ij}}{3} \left( \frac{g_{tt,kj}}{2g_{tt}} - \frac{g_{tt,k} g_{tt,j}}{2g_{tt}^2} \right) \overline{G}_0 + \frac{\tilde{R}}{3} \overline{G}_0 = 0.
$$

(39)

Let us introduce the local momentum space associated with the point $y^i = 0$ by making the three-dimensional Fourier transformation

$$
\overline{G}_a(y^i) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dk_1 dk_2 dk_3}{(2\pi)^3} e^{ik_ay^i} \overline{G}_a(k^i).
$$

(40)

It is easy to check that

$$
\overline{G}_0(k^i) = \frac{1}{k^2 + m^2_e},
$$

(41)

$$
\overline{G}_1(k^i) = i \frac{\delta^{ij} g_{tt,kj}}{2g_{tt}} k^j (k^2 + m^2_e)^2,
$$

(42)

$$
\overline{G}_2(k^i) = \frac{k^i k^j \delta^{ik} g^{jj}}{(k^2 + m^2_e)^3} \left( \frac{2}{3} \tilde{R}_{kk} + \frac{g_{tt,kl}}{g_{tt}} - \frac{5g_{tt,k} g_{tt,l}}{4g_{tt}^2} \right),
$$

(43)

where $k^2 = \delta^{ij} k^i k^j$. Taking into account equations (40)–(43) in (34) and integrating over $k_1, k_2, k_3$ we obtain

$$
\overline{G}_0(y^i) + \overline{G}_1(y^i) + \overline{G}_2(y^i) = \frac{\exp(-m_e y)}{8\pi} \left\{ \frac{2}{y} \frac{g_{tt,ij} y^i}{2g_{tt}} + \frac{1}{m_e} \left[ \delta^{ij} \left( \frac{g_{tt,ij}}{4g_{tt}} - \frac{5\delta^{ij} g_{tt,kl}}{16g_{tt}^2} \right) \right] + \tilde{R} \right\}
$$

$$
+ \left( -\frac{g_{tt,ij}}{4g_{tt}} + \frac{g_{tt,kl}}{4g_{tt}^2} - \frac{\tilde{R}_{ij}}{6} \right) y^i y^j \frac{1}{y},
$$

(44)

where

$$
y = \sqrt{\delta_{ij} y^i y^j}.
$$

(45)

Using the definition of $\overline{G}(y^i)$ (equation (32)), expansions (30) and (31), one finds

$$
G_E(y^i) = \frac{1}{8\pi} \left\{ \frac{2}{y} \frac{g_{tt,ij} y^i}{2g_{tt}} - \frac{m_e}{m_e} \left[ \frac{R}{6} + \delta^{ij} \left( \frac{5g_{tt,ij} g_{tt,kl}}{12g_{tt}^2} - \frac{19g_{tt,kl} g_{tt,ij}}{48g_{tt}^2} \right) \right] \right\}
$$

$$
+ \frac{m_e}{m_e} g_{tt,ij} y^i \frac{1}{2g_{tt}} + \frac{m_e^2}{m_e} y \left[ \delta^{ij} \left( -\frac{5g_{tt,ij}}{12g_{tt}} + \frac{19g_{tt,kl} g_{tt,ij}}{48g_{tt}^2} \right) - \frac{R}{6} \right] y
$$

$$
+ \left( \frac{g_{tt,ij}}{6g_{tt}} + \frac{5g_{tt,kl} g_{tt,ij}}{24g_{tt}^2} + \frac{R_{ij}}{6} \right) y^i y^j \frac{1}{y} + O \left( \frac{1}{m_e^3} L^3 \right) + O \left( \frac{m_e^3 y^2}{L^3} \right) + O \left( \frac{y^3}{L^3} \right) \right\}.
$$

(46)
In the arbitrary coordinates, this expression reads

\[
G_E(x^i; x'^i) = \frac{1}{8\pi} \left( \frac{2}{\sqrt{2\alpha}} - 2m_{\text{ef}} + \frac{g_E^{r' r'} \sigma^{r'} i'}{2g_E^{r r'}} + \frac{1}{m_{\text{ef}}} \left( \frac{5g_E^{r' r'} \sigma^{r'} i'}{12g_E^{r r'}} - \frac{19g_E^{r' r'} e_{r'} e_{i'}}{48g_E^{r r'}} \right) \right) 
+ m_{\text{ef}}^2 \frac{1}{\sqrt{2\alpha}} - m_{\text{ef}} \frac{g_E^{r' r'} \sigma^{r'} i'}{2g_E^{r r'}} + \left( -\frac{5g_E^{r' r'} \sigma^{r'} i' \sigma^{r'} i'}{12g_E^{r r'}} + \frac{19g_E^{r' r'} e_{r'} e_{i'}}{48g_E^{r r'}} - \frac{R}{6} \right) \sqrt{2\alpha} 
+ \left( -\frac{g_E^{r' r'} \sigma^{r'} i'}{6g_E^{r r'}} \frac{5g_E^{r' r'} \sigma^{r'} i' \sigma^{r'} i'}{24g_E^{r r'}} + \frac{R}{6} \right) \frac{\sigma^{r'} i' \sigma^{r'} i'}{\sqrt{2\alpha}} + O \left( \frac{1}{m_{\text{ef}}^2 L^3} \right) 
+ O \left( \frac{\sqrt{\alpha}}{m_{\text{ef}} L^3} \right) + O \left( \frac{\sigma}{L^3} \right) + O \left( \frac{m_{\text{ef}} \sigma^{3/2}}{L^3} \right) + O \left( m_{\text{ef}}^3 \sigma \right). \tag{47}
\]

where \(\sigma\) is one-half of the square of the distance between the points \(x^i\) and \(x'^i\) along the shortest geodesic connecting them, \(\sigma^{i'} = g^{i' j} \partial \sigma / \partial x^{i'}\), and we have taken into account that \(g_M\) is the scalar field in the 3D space with the metric \(g_M\). The renormalization procedure lies in the fact that we subtract the terms of this expansion which are survived in the limits \(m_{\text{ef}} \to \infty\) and \(\sigma \to 0\):

\[
A_{(i)}^{\text{sing}} (x^i; x'^i) = -4\pi e G_E^{\text{sing}} (x^i; x'^i) = -e \left( \frac{1}{\sqrt{2\alpha}} - m_{\text{ef}} + \frac{g_E^{r' r'} \sigma^{r'} i'}{4g_E^{r r'} \sqrt{2\alpha}} \right). \tag{48}
\]

The remaining terms in (47) give us the possibility of computing the self-potential and self-force for the particle at rest which is the source of the massive field.

### 4.1. The Schwarzschild spacetime

Let us verify the above scheme for the well-known case of black hole [27–29]. The electromagnetic potential of a charged particle at rest in the Schwarzschild spacetime with metric

\[
\text{d}s^2 = - \left( 1 - \frac{2M}{r} \right) \text{d}t^2 + \left( 1 - \frac{2M}{r} \right)^{-1} \text{d}r^2 + r^2 (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2)
\]

was obtained by Copson [30] and corrected by Linet in [31]

\[
A_i = - \frac{e}{rr'} \left( \frac{r - M}{r} - M \cos \chi - \frac{eM}{rr'} \right), \tag{49}
\]

where \(R^2 = (r - M)^2 + (r' - M)^2 - 2(r - M)(r' - M) \cos \chi - M^2 \sin^2 \chi\) and \(\cos \chi = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi')\). In the limit of coincided angle variables, we obtain

\[
A_i = - \frac{e}{|r - r'|} \left( 1 - \frac{2M}{r} \right). \tag{50}
\]

and the tetrad component has the following form:

\[
A_{(i)} = - e \frac{1}{|r - r'|} \sqrt{1 - \frac{2M}{r}.} \tag{51}
\]

The singular part of this quantity as was shown above reads

\[
A_{(i)}^{\text{sing}} = - e \left( \frac{1}{\sqrt{2\alpha}} + \frac{g_E^{r' r'} \sigma^{r'} i'}{4g_E^{r r'} \sqrt{2\alpha}} \right), \tag{52}
\]

where the half-square of the radial geodesic distance in this case is

\[
\sigma (r, r') = \frac{1}{2} s(r, r') = \frac{1}{2} \left( \int_{r'}^{\infty} \frac{\text{d}r}{\sqrt{1 - 2M/r}} \right)^2. \tag{53}
\]
Straightforward calculations give the following expression for the divergent part:

$$A^{\text{sing}}_{(t)} = -\frac{e}{|r - r'|} \sqrt{1 - \frac{2M}{r'}}. \quad (54)$$

To renormalize the potential we subtract the divergent part and make the coincidence limit

$$A^{\text{ren}}_{(t)} = \lim_{r' \to r} (A_{(t)} - A^{\text{sing}}_{(t)}) = -\frac{eM}{r' \sqrt{1 - \frac{2M}{r'}}} \quad (55)$$

Then we define the self-potential in a usual manner (the minus sign is due to the covariant index of the 4-potential)

$$U^{\text{self}} = -\frac{e}{2} A^{\text{ren}}_{(t)} = \frac{e^2 M}{2r'^2}. \quad (56)$$

With this definition of self-energy, we have a classical definition of force (tetrad component) as a minus gradient of the potential,

$$F^{(r)} = eF_{r}^{\text{self}} u' \sqrt{-g_{rr}} = -\partial_r U^{\text{self}}, \quad (57)$$

and the sign of the potential gives the information about an attractive or a repelling character of the force. By using this expression, we recover the well-known expression for self-force (tetrad component) in the Schwarzschild spacetime [27–29]

$$F^{(r)} = \frac{Me^2}{r^3}. \quad (58)$$

5. The self-force

Let us proceed here to calculate the self-force in metric (1). According to the scheme developed above, we have to subtract from $A_{(t)}$ the following expression:

$$A^{\text{sing}}_{(t)} = -e \left( \frac{1}{\sqrt{2}\sigma} + \frac{g_{rr}u'\sigma^k}{4g_{tr}'\sqrt{2}\sigma} \right). \quad (59)$$

The curve with constants $t, \theta, \phi$ is the geodesic line and we have

$$\sigma(\rho, \rho') = \frac{1}{2} s^2(\rho, \rho') = \frac{1}{2} \left( \int_{\rho}^{\rho'} e^{a(x)/2} dx \right)^2. \quad (60)$$

By using this expression, we obtain the term to be subtracted ($\rho > \rho' > 0$)

$$A^{\text{sing}}_{(t)} = -\frac{e}{\rho - \rho'} e^{-a(\rho')/2}. \quad (61)$$

Therefore, we obtain the expression for the renormalized potential

$$A^{\text{ren}}_{(t)} = \lim_{\rho' \to \rho} (A^{+}_{(t)} - A^{\text{sing}}_{(t)}) = \frac{e}{\rho^2 + n^2 - m^2} \frac{me^{-a/2}}{\tanh \pi b}. \quad (62)$$

the self-potential

$$U^{\text{self}} = -\frac{e}{2} A^{\text{ren}}_{(t)} = -\frac{e^2}{\rho^2 + n^2 - m^2} \frac{me^{-a}}{2 \tanh \pi b}. \quad (63)$$

and the tetrad component of the self-force

$$F^{(r)} = -\partial_r U^{\text{self}} = \frac{e^2}{\rho^2 + n^2 - m^2} \frac{m(m - \rho)e^{-a}}{\tanh \pi b}. \quad (64)$$
Figure 1. The self-energy (a) and the self-force (b) for the massless case (thin curves) and for the massive wormhole case (thick curves) for $m/n = 0.7$. The zero value of $l$ corresponds to the sphere of the minimal square in both cases.

For the massless wormhole, $m \rightarrow 0$, we recover results obtained in [11]. Far from the wormhole’s throat, we obtain

$$\frac{U_{\rho \rightarrow +\infty}^{\text{self}}}{U_{\rho \rightarrow -\infty}^{\text{self}}} = \left| \frac{F_{\rho \rightarrow +\infty}^{(\rho)}}{F_{\rho \rightarrow -\infty}^{(\rho)}} \right| = e^{2\pi b}. \quad (65)$$

The self-force equals zero at the sphere of minimal square and the extreme of the force is at the points $\rho = m \pm n/\sqrt{3}$.

To compare our results with the massless case, we should plot pictures in the same coordinates. We plot the self-energy and the self-force as a function of the proper radial coordinate $l$, which is

$$l = \int_m^\rho e^{\alpha(x)/2} \, dx. \quad (66)$$

It is merely $\rho$ for the massless case. The modulo of this coordinate is the proper distance from the sphere of the minimal square. The self-energy and the tetrad component of the self-force are shown in figure 1. We observe that the parameter of the mass, $m$, brakes the symmetry of the figure. The extrema of the self-force are at the points $\rho = m \pm n/\sqrt{3}$.

Let us analyze our formulas for $n \approx m$. For $m \rightarrow n$, we have

$$\alpha = \begin{cases} \frac{2n}{\rho} + O\left(\frac{n-m}{\rho}\right) + O\left(\frac{n^2(n-m)}{\rho^2}\right), & \rho > 0, \\ \pi \sqrt{\frac{2n}{n-m}} + O\left(\frac{1}{\rho}\right), & \rho < 0, \end{cases} \quad (67)$$

and therefore the self-energy and self-force tend to zero for $\rho < 0$. But they have the form shown in figure 1 as the functions of the coordinate $l$ because $l$ tends to minus infinity for $\rho = 0$. The self-energy at its extreme in this case tends to zero as

$$U(\rho = 0) \approx -\frac{e^3}{4(n-m)} e^{-\pi \sqrt{\frac{n-m}{n+m}}}. \quad (68)$$

But the self-energy at extremes tends to finite values,

$$\lim_{m \rightarrow n} F_{\rho \rightarrow \rho_0}^{(\rho)} \left( \rho = m + \frac{n}{\sqrt{3}} \right) = \frac{e^2}{n^2} \left( 9 - 21 \frac{\sqrt{3}}{4} \right) e^{-3\sqrt{3}} \approx \frac{e^2}{n^2} \times 0.026 \ldots \quad (69)$$
lim_{m \to n} F^{(\rho)} (\rho = m - \frac{n}{\sqrt{3}}) = \frac{e^2}{n^2} \left( 9 + 21 \frac{\sqrt{3}}{4} \right) e^{-3 - \sqrt{3}} \approx + \frac{e^2}{n^2} 0.159 \ldots \quad (70)

and the positions of the extremes are \( l/n = -2.76 \ldots \) and \( l/n = 1.27 \ldots \).

6. Conclusion

In this paper, we considered in detail the self-interaction on an electrically charged particle at rest in a spacetime of a wormhole, the metric of which is a solution to the Einstein-scalar field equations in the case of a phantom massless scalar field. The main aim of the calculations is to reveal the role of the massive parameter, \( m \), of the wormhole in the self-force. The spacetime of the massive wormhole is taken in the form (1) which was obtained in [16, 17]. The parameter of mass reveals itself in the gravitational force acting on the test particle and it is positive in one part (\( \rho > 0 \)) of the wormhole spacetime and negative in the part with \( \rho < 0 \).

The spacetime under consideration has no ultrastatic form, the component \( g_{tt} \neq -1 \).

In this case we developed the renormalization procedure for the tetrad component of the 4-potential and found the terms which have to be subtracted (48) from the potential for renormalization. The application of this scheme gives the well-known result (58) for the self-force of the particle at rest in the Schwarzschild spacetime. The definition of the self-potential in the form (56) gives us the standard expression for the tetrad component of the self-force (57) as minus gradient of the self-potential

\[ F_{\text{self}} = -\nabla U_{\text{self}}, \quad (71) \]

and the sign of the potential marks the repelling or attractive character of the self-force as in classical mechanics.

For the spacetime under consideration, we found simple and close expressions for self-potential (63) and self-force (64). The spacetime has no symmetry for changing \( \rho \to -\rho \). The self-force and the self-energy in turn have no symmetry, too. More precisely, the ratio of the self-forces or self-potentials at infinity and minus infinity equals \( \exp(2\pi m/\sqrt{n^2 - m^2}) \) (65).

The self-force attracts particles to the throat and it falls down as \( \rho^{-3} \) far from the throat. The extrema of the self-force are at the points \( \rho = m \pm n/\sqrt{3} \). The self-force and the self-energy have the form plotted in figure 1. We observe that the self-force as a function of an invariant proper radial coordinate \( l \) has the form similar to that in the massless wormhole case. The parameter of mass changes asymmetrically the form of the curve and it saves the attractive character of the force.

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