Fiber-full modules and a local freeness criterion for local cohomology modules

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Abstract
Let $R$ be a finitely generated positively graded algebra over a Noetherian local ring $B$, and $m = \langle R \rangle_+$ be the graded irrelevant ideal of $R$. We provide a local criterion characterizing the $B$-freeness of all the local cohomology modules $H^i_m(M)$ of a finitely generated graded $R$-module $M$. We show that fiber-full modules are exactly the ones that satisfy this criterion. When we change $B$ by an arbitrary Noetherian ring $A$, we study the fiber-full locus of a module in $\text{Spec}(A)$: we show that the fiber-full locus is always an open subset of $\text{Spec}(A)$ and that it is dense when $A$ is generically reduced.

Keywords Fiber-full module · Local cohomology · Ext module · Freeness · Base change · Local duality · Gröbner degeneration · square-free

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1 Introduction

Let $(B, b)$ be a Noetherian local ring, $R$ be a finitely generated positively graded $B$-algebra, and $m = \langle R \rangle_+$ be the graded irrelevant ideal of $R$. This paper is motivated by the following two lines of research:

(a) Recently there has been a lot of interest in the closely related notions of algebras having liftable local cohomology introduced by Kollár and Kovács [13], and cohomologically full rings introduced by Dao, De Stefani and Ma [7]. The study of these concepts has produced a number of important results, see, e.g., [6, 7, 13]. Of great interest to us is the important work of Kollár and Kovács [13] on the flatness and base change of the cohomologies of relative dualizing complexes.

For the case of modules, the correct extension of these notions seems to be the notion of fiber-full modules that was recently coined by Varbaro in [16] (also, see [20]). Here we introduce and study a notion of fiber full-modules which is relative over the base ring $B$.

We say that a finitely generated graded $R$-module $M$ is fiber-full over $B$ if $M$ is a free
B-module and the natural map
\[ H^i_m(M/b^qM) \rightarrow H^i_m(M/bM) \]
is surjective for all \( i \geq 0 \) and \( q \geq 1 \) (see Definition 2.3; cf. [16, Definition 3.8]).

(b) An important question is to characterize when the local cohomology modules \( H^i_m(M) \) are free \( B \)-modules for a finitely generated graded \( R \)-module \( M \). This question has been addressed in various contexts by Hochster and Roberts [10, Theorem 3.4], by Kollár [12, Theorem 78], by Smith [18], and by Chardin and Simis in joint work [5] with the author of this paper.

Our main result is the following theorem that gives a local criterion characterizing the \( B \)-freeness of all the local cohomology modules \( H^i_m(M) \). This result shows that fiber-full modules are exactly the ones that enjoy the property that \( M \) is a free \( B \)-module and \( H^i_m(M) \) is a free \( B \)-module for all \( i \geq 0 \). So, in a sense, it paves a bridge between (a) and (b).

**Theorem A** (Theorem 2.15) Let \((B, b)\) be a Noetherian local ring, \( R \) be a finitely generated positively graded \( B \)-algebra, and \( m = [R]_+ \). Let \( T = B[x_1, \ldots, x_r] \) be a positively graded polynomial ring over \( B \) such that we have a homogeneous surjection \( T \rightarrow R \). Let \( M \) be a finitely generated graded \( R \)-module and suppose that \( M \) is a free \( B \)-module. Then, the following six conditions are equivalent:

1. \( H^i_m(M) \) is a free \( B \)-module for all \( 0 \leq i \leq r \).
2. \( \text{Ext}_T^i(M, T) \) is a free \( B \)-module for all \( 0 \leq i \leq r \).
3. \( H^i_m(M/b^qM) \) is a free \( B/b^qM \)-module for all \( 0 \leq i \leq r \) and \( q \geq 1 \).
4. \( \text{Ext}_T^i(M/b^qM, T/b^qT) \) is a free \( B/b^qM \)-module for all \( 0 \leq i \leq r \) and \( q \geq 1 \).
5. The natural map \( H^i_m(M/b^qM) \rightarrow H^i_m(M/bM) \) is surjective for all \( 0 \leq i \leq r \) and \( q \geq 1 \).
6. The natural map \( \text{Ext}_T^i(M/b^qM, \omega_{T/b^qT}) \rightarrow \text{Ext}_T^i(M/b^qM, \omega_{T/b^qT}) \) is injective for all \( 0 \leq i \leq r \) and \( q \geq 1 \), where \( \omega_{T/b^qT} \) denotes the graded canonical module of \( T/b^qT \).

Moreover, when any of the above equivalent conditions is satisfied, we have the following isomorphisms

\[
\begin{align*}
(i) & \quad H^i_m(M) \cong (\text{Ext}_T^{r-i}(M, T(-\delta)))^b \text{ where } \delta = \deg(x_1) + \cdots + \deg(x_r), \\
(ii) & \quad H^i_m(M) \otimes_B C \xrightarrow{\cong} H^i_m(M \otimes_B C), \text{ and} \\
(iii) & \quad \text{Ext}_T^i(M, T) \otimes_B C \xrightarrow{\cong} \text{Ext}_T^i(M \otimes_B C, T \otimes_B C)
\end{align*}
\]

for all integers \( i \) and any \( B \)-algebra \( C \).

A couple of words regarding Theorem A are in place. Because of the above theorem, fiber-full modules get six equivalent desirable definitions, and this makes them quite a versatile class of modules. For fiber-full modules, we obtain a graded local duality theorem relative to the base ring \( B \) (see Definition 2.7) and base change isomorphisms for all \( H^i_m(M) \) and \( \text{Ext}_T^i(M, T) \). Consider the morphism \( f : X = \text{Spec}(R) \rightarrow Y = \text{Spec}(B) \) and the relative dualizing complex \( \omega_{X/Y} \), and suppose that \( f \) is flat. Since \( f \) is embeddable into the smooth morphism \( \Lambda^*_B \rightarrow \text{Spec}(T) \rightarrow \text{Spec}(B) \), it follows that the \(-i\)-th cohomology \( h^{-i}(\omega_{X/Y} \otimes_{X/Y} \mathcal{F}) \) of \( \omega_{X/Y} \) coincides with the sheafification of \( \text{Ext}_T^{r-i}(R, T) \). Therefore, in our current setting when we consider the special case \( M = R \) and \( f \) of Theorem A recover the result of Kollár and Kovács [13] on the flatness and base change of \( h^{-i}(\omega_{X/Y} \otimes_{X/Y} \mathcal{F}) \).
Next, we concentrate on some applications of the theorem above.

We now work over an arbitrary Noetherian ring, and we are interested in studying the locus of fiber-fullness of a module. Let $A$ be a Noetherian ring, $R$ be a finitely generated positively graded $A$-algebra, and $m = [R]_+$. For any $p \in \text{Spec}(A)$, let $k(p) = A_p / pA_p$ be the residue field of $p$. For a finitely generated graded $R$-module $M$, we define the fiber-full locus of $M$ by

$$\text{Fib}(M) = \{ p \in \text{Spec}(A) \mid M \otimes_A A_p \text{ is fiber-full over } A_p \}.$$  

Interestingly, it turns out that the fiber-full locus is very well-behaved: $\text{Fib}(M)$ is always an open subset, and if $A$ is generically reduced then $\text{Fib}(M)$ is also dense. The following theorem gives some results regarding the fiber-full locus of a module.

**Theorem B** (Theorem 3.2) Let $A$ be a Noetherian ring, $R$ be a finitely generated positively graded $A$-algebra, and $m = [R]_+$. Let $M$ be a finitely generated graded $R$-module. Then, the following statements hold:

(i) $\text{Fib}(M)$ is an open subset of $\text{Spec}(A)$.

(ii) If $A$ is generically reduced, then there is an element $a \in A$ avoiding the minimal primes of $A$ such that $H^i_m(M) \otimes_A A_a$ is a projective $A_a$-module for all $i \geq 0$, and so $\text{Fib}(M)$ is a dense subset of $\text{Spec}(A)$.

(iii) For all $i \geq 0$ and $\nu \in \mathbb{Z}$ the function

$$\text{Spec}(A) \to \mathbb{N}, \quad p \mapsto \dim_{k(p)} \left( \left[ H^i_m(M) \otimes_A k(p) \right]_\nu \right)$$

is locally constant on $\text{Fib}(M)$.

Finally, as a consequence of our methods, in Theorem 4.1 we give an alternative proof of an important result of Conca and Varbaro [6], where they showed that for a given homogeneous ideal $I \subset S = k[x_1, \ldots, x_r]$ ($m = (x_1, \ldots, x_r)$) the Hilbert functions of $H^i_m(S/I)$ and $H^i_m(S/\text{in}_<(I))$ coincide provided the initial ideal in $\langle I \rangle$ is square-free. This result settled a previous conjecture of Herzog.

The basic outline of this paper is as follows. In Sect. 2, we prove our main result, that is, we give the proof of Theorem A. In Sect. 3, we obtain Theorem B. Finally, in Sect. 4, we study square-free Gröbner degenerations.

## 2 A local freeness criterion for local cohomology modules

In this section, we provide a local criterion for the freeness of local cohomology modules. Throughout this section, the following setup is set in place.

**Setup 2.1** Let $(B, b)$ be a Noetherian local ring. Let $R$ be a finitely generated positively graded $B$-algebra. Let $T = B[x_1, \ldots, x_r]$ be a positively graded polynomial ring over $B$ (that is, $[T]_0 = B$ and $\deg(x_1) > 0$), such that we have a homogeneous surjection $T \to R$. Let $m = (x_1, \ldots, x_r) \subset T$ be the graded irrelevant ideal of $T$. By abuse of notation, interchangeably, $m$ will also denote the graded irrelevant ideal $mR = [R]_+ = \bigoplus_{\nu \geq 1} [R]_\nu$ of $R$. For each $q \geq 1$, let $B_q := B/b^q$, $R_q := R \otimes_B B_q$ and $T_q := T \otimes_B B_q \cong B_q[x_1, \ldots, x_r]$. Let $\delta := \deg(x_1) + \cdots + \deg(x_r) \in \mathbb{N}$.

**Remark 2.2** We begin by recalling the following basic facts:
(i) For any R-module M, by the Independence Theorem (see [2, Theorem 4.2.1]) we can compute $H^1_m(M)$ by considering M as a $T$-module.

(ii) $H^*_m(T) \cong \frac{1}{x_1 \cdots x_r} B[x_1^{-1}, \ldots, x_r^{-1}]$ and $H^*_m(T) = 0$ for all $i \neq r$.

(iii) Since B is a Noetherian local ring, there is no distinction between the notions of B-flat, B-projective and B-free for a finitely generated B-module (see, e.g., [15, §7]). It then follows that the properties of B-flat, B-projective and B-free are equivalent for a finitely generated graded R-module.

(iv) $H^*_m(M) = 0$ for all $i \geq r + 1$ and any R-module M.

(v) If M is a finitely generated graded R-module, then for each $\nu \in \mathbb{Z}$ the graded component $[H^*_m(M)]_\nu$ is a finitely generated B-module (see, e.g., [9, Theorem III.5.2], [4, Theorem 2.1]). So the conditions B-flat, B-projective and B-free are equivalent for $H^*_m(M)$.

Our main object of study is the following interesting class of modules (cf. [16, Definition 3.8]). We introduce a notion which is relative over the base ring B.

**Definition 2.3** A finitely generated graded R-module M is fiber-full over B if M is a free B-module and the natural map $H^*_m(M/bqM) \to H^*_m(M/bM)$ is surjective for all $i \geq 0$ and $q \geq 1$.

An equivalent definition for the notion of fiber-full is given by the following lemma.

**Lemma 2.4** Let M be a finitely generated graded R-module. Then, the natural map $H^*_m(M/bqM) \to H^*_m(M/bM)$ is surjective if and only if the natural map $\text{Ext}^{r-1}_T(M/bM, \omega_T_q) \to \text{Ext}^{r-1}_T(M/bqM, \omega_T_q)$ is injective, where $\omega_T_q$ denotes the graded canonical module of $T_q$.

**Proof** Note that $T_q$ is a Cohen–Macaulay *complete *local ring with *maximal ideal (b + m)$T_q$ of *dimension r (under the notations of [3]). Thus, the graded local duality theorem [3, Theorem 3.6.19] implies that the natural map $H^*_m(M/bqM) \to H^*_m(M/bM)$ is surjective if and only if the natural map $\text{Ext}^{r-1}_T(M/bM, \omega_T_q) \to \text{Ext}^{r-1}_T(M/bqM, \omega_T_q)$ is injective. Since $M/bqM$ is b-torsion, the spectral sequence $E^{i,j}_2 = H^i_m(H^j_b(M/bqM)) \Rightarrow H^{i+j}_{b+m}(M/bqM)$ yields the isomorphism $H^*_m(M/bqM) \cong H^*_m(M/bM)$. So, the result is clear.

An explicit computation of the graded canonical module $\omega_T_q$ is given in the remark below.

**Remark 2.5** As in the proof of Lemma 2.4, we have $H^*_m(T_q) \cong H^*_m(T_q) \cong \frac{1}{x_1 \cdots x_r} B_q[x_1^{-1}, \ldots, x_r^{-1}]$. Consequently, [3, Theorem 3.6.19] gives the following isomorphism $\omega_T_q \cong \bigoplus_{\nu \in \mathbb{Z}} \text{Hom}_{B_q} \left( \left[ \frac{1}{x_1 \cdots x_r} B_q[x_1^{-1}, \ldots, x_r^{-1}] \right]_{-\nu}, \omega_{B_q} \right) \cong T_q(-\delta) \otimes_{B_q} \omega_{B_q}$ where $\omega_{B_q}$ denotes the canonical module of the Artinian local ring $B_q$.

For completeness, we show below that Definition 2.3 agrees with [16, Definition 3.8] under a general common setting.

**Remark 2.6** Let $C = \mathbb{k}[t]$ and $P = C[x_1, \ldots, x_r]$ be a positively graded polynomial ring over C. In [16, Definition 3.8] a finitely generated graded P-module M is called fiber-full...
if $M$ is $C$-flat and the natural map $\text{Ext}_P^i(M/tM, P) \rightarrow \text{Ext}_P^i(M/t^qM, P)$ is injective for all $i \geq 0, q \geq 1$. By a theorem of Rees (see, e.g., [3, Lemma 3.1.16]) this is equivalent to saying that $\text{Ext}_P^{i-1}/tqP(M/tM, P/t^qP) \rightarrow \text{Ext}_P^{i-1}/tqP(M/t^qM, P/t^qP)$ is injective for all $i \geq 0, q \geq 1$. Thus the sought equivalence between Definition 2.3 and [16, Definition 3.8] is given by Lemma 2.4 and Remark 2.5.

**Definition 2.7** For a graded $T$-module $M$, we denote the $B$-relative graded Matlis dual by

$$(M)^{\ast B} := \ast \text{Hom}_B(M, B) := \bigoplus_{\nu \in \mathbb{Z}} \text{Hom}_B([M]_{-\nu}, B).$$

Note that $(M)^{\ast B}$ has a natural structure of graded $T$-module. From the canonical perfect pairing of free $B$-modules in “top” local cohomology

$$[T]_\nu \otimes_B [H^T_m(T)]_{-\delta - \nu} \rightarrow [H^T_m(T)]_{-\delta} \cong B$$

we obtain a canonical graded $R$-isomorphism $H^T_m(T) \cong (T(-\delta))^{\ast B} = \ast \text{Hom}_B(T(-\delta), B).$

Then, for a graded complex $F_\bullet$ of finitely generated free $T$-modules, we obtain the isomorphism of complexes

$$H^m_m(F_\bullet) \cong \left(\text{Hom}_T(F_\bullet, T(-\delta))\right)^{\ast B}. \quad (1)$$

The lemma below is an important tool for us.

**Lemma 2.8** Let $M$ be a finitely generated graded $R$-module. Let $F_\bullet$ be a graded free $T$-resolution of $M$ by modules of finite rank. If $M$ is a free $B$-module, then $H^i_m(M \otimes_B C) \cong H_{r-i}(H^T_m(F_\bullet) \otimes_B C)$ for any $B$-algebra $C$ and all integers $i$.

**Proof** See [5, Lemma 3.4].

The next lemma provides a known base change isomorphism for local cohomology.

**Lemma 2.9** Let $M$ be a finitely generated graded $R$-module. Suppose that $M$ is a free $B$-module and $H^i_m(M)$ is a free $B$-module for all $i \geq 0$. Then

$$H^i_m(M) \otimes_B C \cong H^i_m(M \otimes_B C)$$

for all $i \geq 0$ and any $B$-algebra $C$.

**Proof** Let $F_\bullet$ be a graded free $T$-resolution of $M$ by modules of finite rank, and set $G_\bullet := H^m_m(F_\bullet)$. By Lemma 2.8, $H_i(G_\bullet)$ is a free $B$-module for all $i \geq 0$. Then, it follows that $H_i(G_\bullet) \otimes_B C \cong H_i(G_\bullet \otimes_B C)$ for all $i \geq 0$ (see, e.g., [5, Lemma 2.8]). By using Lemma 2.8 again we obtain

$$H^i_m(M) \otimes_B C \cong H_{r-i}(G_\bullet) \otimes_B C \cong H_{r-i}(G_\bullet \otimes_B C) \cong H^i_m(M \otimes_B C),$$

and so the result follows.

By using the property of exchange for local Ext’s (see [1, Theorem 1.9], [14, Theorem A.5]), we obtain the following lemma.
Lemma 2.10 Let $M$ be a finitely generated graded $R$-module and suppose that $M$ is a free $B$-module. Let $F_0 : \cdots \to F_i \to \cdots \to F_1 \to F_0$ be a graded free $T$-resolution of $M$ by modules of finite rank. Let

$$D^i_M := \text{Coker}(\text{Hom}_T(F_{i-1}, T) \to \text{Hom}_T(F_i, T))$$

for each $i \geq 0$. Then, the following statements hold:

(i) $\text{Ext}^i_T(M, T) = 0$ for all $i \geq r + 1$.

(ii) $D^i_M$ is a free $B$-module for all $i \geq r + 1$.

(iii) If $\text{Ext}^i_T(M, T)$ is a free $B$-module for all $0 \leq i \leq r$, then

$$\text{Ext}^i_T(M, T) \otimes_B C \xrightarrow{\sim} \text{Ext}^i_T(C \otimes_B C, T \otimes_B C)$$

for all $i \geq 0$ any $B$-algebra $C$.

Proof (i) Since $M$ is a free $B$-module, $F_0 \otimes_B B/b$ is a free $T_1$-resolution of $M/bM$. By using Hilbert’s syzygy theorem on the polynomial ring $T_1 = B/[x_1, \ldots, x_r]$, we get that $\text{Ext}^i_{T_1}(M/bM, T_1) = 0$ for all $i \geq r + 1$. In particular, the natural map

$$\text{Ext}^i_T(M, T) \otimes_B B/b \to \text{Ext}^i_{T_1}(M/bM, T_1) = 0$$

is surjective for all $i \geq r + 1$, however, [1, Theorem 1.9] or [14, Theorem A.5] imply that the map above is actually bijective. As $\text{Ext}^i_T(M, T)$ is a finitely generated graded $T$-module, each graded component $[\text{Ext}^i_T(M, T)]_\nu$ is a finitely generated $B$-module. Note that the condition $\text{Ext}^i_T(M, T) \otimes_B B/b = 0$ implies that $[\text{Ext}^i_T(M, T)]_\nu \otimes_B B/b = 0$ for all $\nu \in \mathbb{Z}$. Then, for all $i \geq r + 1$, Nakayama’s lemma yields that $[\text{Ext}^i_T(M, T)]_\nu = 0$ for all $\nu \in \mathbb{Z}$, and so $\text{Ext}^i_T(M, T) = 0$, as required.

(ii) Since $M$ is $B$-free, by considering the complex $\text{Hom}_T(F_0, T) \otimes_B C$ with $C$ any $B$-algebra, we obtain the four-term exact sequence

$$0 \to \text{Ext}^i_{T \otimes_B C}(M \otimes_B C, T \otimes_B C) \to D^i_M \otimes_B C \to \text{Hom}_T(F_{i+1}, T) \otimes_B C \to 0$$

for all $i \geq 0$. For all $i \geq r + 1$, due to the computations of part (i), when we substitute $C = B$ and $C = B/b$ in (2), we get the short exact sequences

$$0 \to D^i_M \to \text{Hom}_T(F_{i+1}, T) \to D^{i+1}_M \to 0$$

and

$$0 \to D^i_M \otimes_B B/b \to \text{Hom}_T(F_{i+1}, T) \otimes_B B/b \to D^{i+1}_M \otimes_B B/b \to 0,$$

respectively. For all $i \geq r + 1$, from (4) and the long exact sequence in Tor’s induced by tensoring (3) with $B/b$, we obtain that $\text{Tor}^1_B(D^{i+1}_M, B/b) = 0$, and so by the local flatness criterion (see [15, Theorem 22.3]) if follows that $D^{i+1}_M$ and $D^i_M$ are free $B$-modules.

(iii) Let $\gamma^i$ be the natural map $\gamma^i : \text{Ext}^i_T(M, T) \otimes_B B/b \to \text{Ext}^i_{T_1}(M/bM, T_1)$. By [1, Theorem 1.9] or [14, Theorem A.5], we have that under the assumption that $\gamma^i$ is surjective, then $\gamma^{i-1}$ is surjective if and only if $\text{Ext}^i_T(M, T)$ is a free $B$-module. Since $\gamma^{r+1}$ is surjective and $\text{Ext}^i_T(M, T)$ is a free $B$-module for all $0 \leq i \leq r$, by descending induction on $i$ we get that $\gamma^i$ is surjective for all $i \geq 0$. Finally, by using [1, Theorem 1.9] or [14, Theorem A.5] the result follows. □
The next proposition yields some version of graded local duality relative to the base ring $B$.

**Proposition 2.11** Let $M$ be a finitely generated graded $\mathbb{R}$-module and suppose that $M$ is a free $B$-module. Then, the following two conditions are equivalent:

1. $H^i_m(M)$ is a free $B$-module for all $0 \leq i \leq r$.
2. $\text{Ext}_T^i(M, T)$ is a free $B$-module for all $0 \leq i \leq r$.

Moreover, when any of the above equivalent conditions is satisfied, we have that

$$H^i_m(M) \cong \left(\text{Ext}_T^{i-1}(M, T(-\delta))\right)^*$$

for all integers $i$.

**Proof** Let $F^*_0 : \cdots \to F_i \to \cdots \to F_1 \to F_0$ be a graded free $T$-resolution of $M$ by modules of finite rank. Let $I^*_0 : I^0 \to I^1 \to \cdots \to I^i \to \cdots$ be an injective $B$-resolution of $B$.

(1) $\Rightarrow$ (2). Suppose that $H^i_m(M)$ is a free $B$-module for all $i \geq 0$ (see Remark 2.2(iv)). To simplify the notation, let $G_\bullet := H^i_m(F^*_\bullet)$. By Lemma 2.8, $H_t(G_\bullet)$ is a free $B$-module for all $i \geq 0$. From (1), we obtain the isomorphism of complexes $(G_\bullet)^*B \cong \text{Hom}_T(F^*_\bullet, T(\delta))$. Thus, to finish the proof it suffices to show that $H^i(\mathcal{(G_\bullet)^*B}) \cong (H_t(G_\bullet))^*B$ because $\text{Ext}_T^i(M, T(-\delta)) \cong H^i(\text{Hom}_T(F^*_\bullet, T(-\delta)))$.

Let $K^{p,q}_\bullet$ be the first quadrant double complex given by $K^{p,q}_\bullet := \bigoplus_{\nu \in \mathbb{Z}} \text{Hom}_B \left( [G_p]_{-\nu}, I^q \right)$. Note that the corresponding spectral sequences have respective second terms

$$I_{E^2}^{p,q} = \bigoplus_{\nu \in \mathbb{Z}} \text{Hom}_B(\text{Ext}_B^q ([G_p]_{-\mu}, B)) \quad \text{and} \quad \Pi_{E^2}^{p,q} = \bigoplus_{\nu \in \mathbb{Z}} \text{Ext}_B^q (\text{Hom}_B(\mathcal{G_q} \mathcal{G}_\bullet)_{-\nu}, B).$$

Finally, since $G_i$ and $H_t(G_\bullet)$ are free $B$-modules for all $i \geq 0$, we obtain the isomorphisms

$$\text{Ext}_T^i(M, T(\delta)) \cong H^i(\mathcal{(G_\bullet)^*B}) \cong (H_t(G_\bullet))^*B \cong (H^r_m(M))^*B$$

for all $i \geq 0$.

(2) $\Rightarrow$ (1). Suppose that $\text{Ext}_T^i(M, T)$ is a free $B$-module for all $0 \leq i \leq r$. Let $F^*_\bullet \leq r+1$ be the complex given as the truncation

$$F^*_\bullet \leq r+1 : 0 \to F_{r+1} \to F_r \to \cdots \to F_1 \to F_0,$$

and $P^*_\bullet := \text{Hom}_T(F^*_\bullet \leq r+1, T(\delta))$. By hypothesis $H^i(P^*_\bullet) \cong \text{Ext}_T^i(M, T)(\delta)$ is a free $B$-module for all $0 \leq i \leq r$. From Lemma 2.10, we also have that $H^{r+1}(P^*_\bullet) \cong D^r_M(-\delta)$ is a free $B$-module.

Let $L^{p,q}_\bullet$ be the second quadrant double complex given by $L^{p,q}_\bullet := \bigoplus_{\nu \in \mathbb{Z}} \text{Hom}_B \left( [P^p]_{-\nu}, I^q \right)$. Now the corresponding spectral sequences have respective second terms

$$I_{E^2}^{p,q} = \bigoplus_{\nu \in \mathbb{Z}} \text{Hom}_B(\text{Ext}_B^q ([P^p]_{-\mu}, B)) \quad \text{and} \quad \Pi_{E^2}^{p,q} = \bigoplus_{\nu \in \mathbb{Z}} \text{Ext}_B^q (\text{Hom}_B([P^q]_{-\nu}, B)).$$

Finally, since $P^1$ and $H^1(P^*_\bullet)$ are free $B$-modules for all $0 \leq i \leq r+1$, by using Lemma 2.8 and (1) we obtain the following isomorphisms

$$H^i_m(M) \cong H_t((P^*_\bullet)^*B) \cong (H^1(P^*_\bullet))^*B \cong (\text{Ext}_T^i(M, T(\delta)))^*B$$

for all $0 \leq i \leq r$. This concludes the proof of the proposition.
Lemma 2.13 Let the case of modules. Note that for every \( q \geq 1 \) and \( v \in \mathbb{Z} \), we have a natural map \([H^i_m(M)]_v \otimes_B B_q \to [H^i_m(M/b^qM)]_v\). Then, we obtain an induced natural map

\[ [H^i_m(M)]_v \to \lim_{\leftarrow} [H^i_m(M/b^qM)]_v, \]

where \([H^i_m(M)]_v \) denotes the completion of the finitely generated \( B \)-module \([H^i_m(M)]_v\) with respect to the maximal ideal \( b \subset B \).

Proposition 2.12 Let \( M \) be a finitely generated graded \( R \)-module. Then, the natural map

\[ [H^i_m(M)]_v \cong \lim_{\leftarrow} [H^i_m(M/b^qM)]_v, \]

is an isomorphism for all \( i \geq 0 \) and \( v \in \mathbb{Z} \).

Proof If we set \( T' = T[x_{r+1}] \) and \( M' = M \otimes_B B[x_{r+1}] \), then for all \( i \geq 0 \) we get the natural isomorphism \( H^i_{m+x_{r+1}T'}(M'/x_{r+1}M') \cong H^i_m(M) \). Thus, without any loss of generality, we assume that \( r \geq 2 \).

Let \( X = \text{Proj}(T) \) (here we may need \( r \geq 2 \)) and \( F = \tilde{M} \) be the corresponding coherent sheaf. For each \( q \geq 1 \), let \( M_q := M/b^qM \), \( X_q := X \times_{\text{Spec}(B)} \text{Spec}(B_q) \) and \( F_q := F \otimes_B B_q \). For \( i \geq 2 \), [9, Theorem III.11.1] or [19, Tag 02OC] yield the following natural isomorphism

\[ [H^i_m(M)]_v \cong H^{i-1}(X, F(v))^\sim \lim_{\leftarrow} H^{i-1}(X_q, F_q(v)) \cong \lim_{\leftarrow} [H^i_m(M_q)]_v. \]

(For the relations between sheaf and local cohomologies see, e.g., [8, Theorem A4.1], [11, Corollary 1.5]).

For \( i \leq 1 \), we have the following commutative diagram with natural maps

\[
\begin{align*}
0 \to [H^i_m(M)]_v \to [M]_v \to H^0(X, F(v))^\sim \to [H^1_m(M)]_v \to 0 \\
\alpha_1 \downarrow \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \\
0 \to \lim_{\leftarrow} [H^0_m(M_q)]_v \to \lim_{\leftarrow} [M_q]_v \to \lim_{\leftarrow} H^0(X_q, F_q(v)) \to \lim_{\leftarrow} [H^1_m(M_q)]_v \to 0.
\end{align*}
\]

The first row is exact due to the exactness of completion. Note that \([H^i_m(M_q)]_v\) and \( H^i(X_q, F_q(v)) \) are \( B_q \)-modules of finite length for all \( i \geq 0, v \in \mathbb{Z} \). Thus, by using [9, Proposition II.9.1] and [9, Example II.9.1.2] (i.e., the Mittag Leffler condition is satisfied), we obtain that the second row is also exact. The map \( \alpha_2 \) is an isomorphism by the definition of completion and the map \( \alpha_3 \) is an isomorphism by [9, Theorem III.11.1] or [19, Tag 02OC]. It then follows that \( \alpha_1 \) and \( \alpha_4 \) are also isomorphisms.

Therefore, the statement of the proposition holds for all \( i \geq 0 \).

Next, we have a basic lemma regarding the behavior of fiber-full modules (see Definition 2.3). The following result is an extension of [13, Proposition 4.5] and [6, Proposition 2.2] to the case of modules.

Lemma 2.13 Let \( C \) be an Artinian local ring which is a quotient of \( B \). Let \( M \) be a finitely generated graded \( R \)-module and suppose that \( M \) is fiber-full over \( B \).
(i) Let $N$ be a $C$-module and $\phi : N \to B/b$ be a surjective homomorphism. Then, the natural map

$$H^i_m(M \otimes_B \phi) : H^i_m(M \otimes_B N) \to H^i_m(M/bM)$$

is surjective for all $i \geq 0$.

(ii) Let $C = I_0 \supset I_1 \supset \cdots \supset I_{\ell-1} \supset I_\ell = 0$ be a filtration with $I_j/I_{j+1} \cong B/b$ for all $0 \leq j \leq \ell - 1$. Then, we have the following short exact sequences

$$0 \to H^i_m(M \otimes_B I_{j+1}) \to H^i_m(M \otimes_B I_j) \to H^i_m(M \otimes_B I_j/I_{j+1}) \to 0$$

for all $i \geq 0$ and $0 \leq j \leq \ell - 1$.

**Proof** (i) Let $w \in N$ such that $\phi(w) = 1 \in B/b$. Take $q \geq 1$ such that we have a surjection $B_q \to C$ and so $N$ becomes a $B_q$-module. Let $\eta : B_q \to N$ be the map defined by $1 \in B_q \mapsto w \in N$. Therefore, the natural map $H^i_m(M/b^qM) \to H^i_m(M/bM)$ coincides with the composition of maps $H^i_m(M \otimes_B \phi) \circ H^i_m(M \otimes_B \eta)$, and so the result follows from the definition of fiber-full.

(ii) Fix $0 \leq j \leq \ell - 1$. Since $M$ is a free $B$-module, we have the short exact sequence

$$0 \to M \otimes_B I_{j+1} \to M \otimes_B I_j \to M \otimes_B I_j/I_{j+1} \to 0,$$

and so we get the long exact sequence in local cohomology

$$H^{i-1}_m(M \otimes_B I_{j+1}) \xrightarrow{B_{i-1}} H^i_m(M \otimes_B I_j/I_{j+1}) \to H^i_m(M \otimes_B I_j) \xrightarrow{\beta_i} H^i_m(M \otimes_B I_{j+1}).$$

By part (i) the map $\beta_i$ is surjective for all $i \geq 0$, and this implies the result.

The following proposition shows that fiber-fullness implies the freeness of certain local cohomology modules. This result is inspired by the techniques used in [13, §4].

**Proposition 2.14** Let $M$ be a finitely generated graded $R$-module and suppose that $M$ is fiber-full over $B$. Then $H^i_m(M/b^qM)$ is a free $B_q$-module for all $i \geq 0$ and $q \geq 1$.

**Proof** Let $C$ be an Artinian local ring which is a quotient of $B$. Let $C = I_0 \supset I_1 \supset \cdots \supset I_{\ell-1} \supset I_\ell = 0$ be a filtration with $I_j/I_{j+1} \cong B/b$ for all $0 \leq j \leq \ell - 1$. For ease of notation, let $\overline{M} := M \otimes_B C$. As $M$ is a free $B$-module, we have $M \otimes_B I_j \cong I_j \overline{M}$. Note that $I_1 = bC$, $I_{\ell-1} \cong B/b$ and $I_{\ell-1} = (t) \subset C$ is a principal ideal generated by some element $0 \neq t \in C$.

We shall prove that $H^i_m(M \otimes_B C)$ is a free $C$-module for all $i \geq 0$. We proceed by induction on the length $\ell = \text{length}(C)$ of $C$. If $\ell = 1$, then $C = B/b$ and the result is clear. Thus, we assume that $\ell \geq 2$ and that $H^i_m(M \otimes_B C')$ is a free $C'$-module for any quotient $C'$ of $B$ with $\text{length}(C') < \ell$.

From Lemma 2.13(iii), by successively composing the injections $H^i_m(M \otimes_B I_{j+1}) \hookrightarrow H^i_m(M \otimes_B I_j)$, we obtain the injection $H^i_m(M \otimes_B I_j) \hookrightarrow H^i_m(M \otimes_B C)$. Then, $0 \to I_j \overline{M} \to M \to M/I_j \overline{M} \to 0$ yields a long exact sequence in local cohomology that splits into the following short exact sequences

$$0 \to H^i_m(I_j \overline{M}) \to H^i_m(M) \to H^i_m(M/I_j \overline{M}) \to 0,$$  

(5)
because $H^i_m(I_j \overline{M}) \cong H^i_m(M \otimes_B I_j) \hookrightarrow H^i_m(M \otimes_B C) \cong H^i_m(\overline{M})$ is injective for all $i \geq 0$. Hence, multiplication by $t$ induces the following commutative diagram

$$
\begin{array}{c}
H^i_m(\overline{M}) \\
\downarrow t \\
H^i_m(t \overline{M})
\end{array}
\quad \phi
$$

(the surjectivity of $H^i_m(\overline{M}) \twoheadrightarrow H^i_m(t \overline{M})$ follows from Lemma 2.13(i), and the injectivity of $H^i_m(t \overline{M}) \hookrightarrow H^i_m(\overline{M})$ is given by (5) with $j = \ell - 1$). So we obtain that $t H^i_m(\overline{M})$ and $H^i_m(t \overline{M})$ coincide as submodules of $H^i_m(\overline{M})$. This latter fact together with (5) yields the following isomorphism

$$H^i_m(\overline{M}/t \overline{M}) \cong H^i_m(\overline{M})/t H^i_m(\overline{M}).$$

Since $\overline{M}/t \overline{M} \cong M \otimes_B C/(t)$, the induction hypothesis implies that every $H^i_m(\overline{M}/t \overline{M})$ is a free $C/(t)$-module, and so Lemma 2.9 gives the following isomorphisms

$$H^i_m(\overline{M}/I_j \overline{M}) \cong H^i_m(\overline{M}/t \overline{M} \otimes_{C/(t)} C/I_j)$$

$$\cong H^i_m(\overline{M}/t \overline{M}) \otimes_{C/(t)} C/I_j$$

$$\cong H^i_m(\overline{M})/I_j H^i_m(\overline{M})$$

for all $0 \leq j \leq \ell - 1$. Consequently, the Eqs. (5) and (6) give the equality $H^i_m(I_j \overline{M}) = I_j H^i_m(\overline{M})$ as submodules of $H^i_m(\overline{M})$. As for (5), now the short exact sequence $0 \rightarrow I_1 \overline{M} \rightarrow \overline{M} \rightarrow t \overline{M} \rightarrow 0$ gives the following short exact sequence

$$0 \rightarrow I_1 H^i_m(\overline{M}) \rightarrow H^i_m(\overline{M}) \rightarrow t H^i_m(\overline{M}) \rightarrow 0,$$

and so from the isomorphism $C/I_1 \cong (t)$ we obtain a natural isomorphism

$$H^i_m(\overline{M}) \otimes_{C/(t)} (t) \cong t H^i_m(\overline{M}).$$

Finally, the fact that $H^i_m(\overline{M})/t H^i_m(\overline{M})$ is a free $C/(t)$-module, the isomorphism of (7), and the local flatness criterion (see [15, Theorem 2.23]) imply that $H^i_m(\overline{M})$ is a free $C$-module. So, the proof of the proposition is complete.

Finally, we are now ready for the main result of this paper.

**Theorem 2.15** Assume Setup 2.1. Let $M$ be a finitely generated graded $R$-module and suppose that $M$ is a free $B$-module. Then, the following six conditions are equivalent:

1. $H^i_m(M)$ is a free $B$-module for all $0 \leq i \leq r$.
2. $\text{Ext}^i_T(M, T)$ is a free $B$-module for all $0 \leq i \leq r$.
3. $H^i_m(M/b^q M)$ is a free $B_q$-module for all $0 \leq i \leq r$ and $q \geq 1$.
4. $\text{Ext}^i_{T_q}(M/b^q M, T_q)$ is a free $B_q$-module for all $0 \leq i \leq r$ and $q \geq 1$.
5. The natural map $H^i_m(M/b^q M) \rightarrow H^i_m(M/b M)$ is surjective for all $0 \leq i \leq r$ and $q \geq 1$.
6. The natural map $\text{Ext}^i_{T_q}(M/b M, \omega_{T_q}) \rightarrow \text{Ext}^i_{T_q}(M/b^q M, \omega_{T_q})$ is injective for all $0 \leq i \leq r$ and $q \geq 1$.

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Moreover, when any of the above equivalent conditions is satisfied, we have the following isomorphisms

\[(i) \ H^i_m(M) \cong (\text{Ext}_T^{r-i}(M, T(-\delta)))^B,\]
\[(ii) \ H^i_m(M) \otimes_B C \xrightarrow{\sim} H^i_m(M \otimes_B C), \text{ and} \]
\[(iii) \ Ext_T^{r-i}(M, T) \otimes_B C \xrightarrow{\sim} Ext_T^{r-i}(M \otimes_B C, T \otimes_B C)\]

for all integers \(i\) and any \(B\)-algebra \(C\).

**Proof** The equivalences \((1) \iff (2)\) and \((3) \iff (4)\) are obtained from Proposition 2.11. The equivalence \((5) \iff (6)\) is a consequence of Lemma 2.4. The implication \((1) \implies (3)\) follows from Lemma 2.9. The implication \((3) \implies (5)\) is obtained from Lemma 2.9, and Proposition 2.14 yields the implication \((5) \implies (3)\). The additional statements regarding the isomorphisms \((i)\), \((ii)\) and \((iii)\) would follow from Proposition 2.11, Lemma 2.9 and Lemma 2.10, respectively.

Therefore, to conclude the proof of the theorem it suffices to show the implication \((3) \implies (1)\). Suppose that \(H^i_m(M/b^qM)\) is a free \(B_q\)-module for all \(i \geq 0\) and \(q \geq 1\). By Lemma 2.9 we have the natural isomorphism

\[H^i_m(M/b^{q+1}M) \otimes_{B_q} B_q \xrightarrow{\sim} H^i_m(M/b^qM).\]

This implies that the natural map \(H^i_m(M/b^{q+1}M) \rightarrow H^i_m(M/b^qM)\) is surjective. Then, the conditions of [19, Tag 0912] are satisfied for the inverse system \(\{H^i_m(M/b^qM)\}_{q \geq 1}\), and we get that the inverse limit \(\lim_{\leftarrow} H^i_m(M/b^qM)\) is a flat \(B\)-module. As a consequence of Proposition 2.12, we have that \(H^i_m(M)\) is a free \(B\)-module for each \(\nu \in \mathbb{Z}\), and so the proof of the theorem is complete.

### 3 Fiber-full locus of a module

In this section, we provide some applications that follow from Theorem 2.15.

Let \(A\) be a Noetherian ring, \(R\) be a finitely generated positively graded \(A\)-algebra, and \(m = [R]_+\) be the graded irrelevant ideal of \(R\). For any \(p \in \text{Spec}(A)\), let \(k(p) := A_p/pA_p\) be the residue field of \(p\).

**Definition 3.1** Let \(M\) be a finitely generated graded \(R\)-module. The **fiber-full locus** of \(M\) is defined as

\[\mathfrak{Fib}(M) := \{p \in \text{Spec}(A) \mid M \otimes_A A_p \text{ is fiber-full over } A_p\}\]

(see Definition 2.3).

The following theorem gives some properties of the fiber-full locus of a module.

**Theorem 3.2** Let \(M\) be a finitely generated graded \(R\)-module. Then, the following statements hold:

\[(i) \ \mathfrak{Fib}(M) \text{ is an open subset of } \text{Spec}(A).\]
\[(ii) \text{ If } A \text{ is generically reduced, then there is an element } a \in A \text{ avoiding the minimal primes of } A \text{ such that } H^i_m(M) \otimes_A A_a \text{ is a projective } A_a \text{-module for all } i \geq 0, \text{ and so } \mathfrak{Fib}(M) \text{ is a dense subset of } \text{Spec}(A).\]
For all \( i \geq 0 \) and \( \nu \in \mathbb{Z} \) the function

\[
\text{Spec}(A) \to \mathbb{N}, \quad p \mapsto \dim_{k(p)} \left( [H^i_m(M \otimes_A k(p))]_\nu \right)
\]

is locally constant on \( \text{fib}(M) \).

**Proof**

(i) It is a known result that for a finitely generated graded \( R \)-module \( L \), the following locus

\[
U_L := \{ p \in \text{Spec}(A) \mid L \otimes_A A_p \text{ is a free } A_p\text{-module} \}
\]

is an open subset of \( \text{Spec}(A) \) (see, e.g., [5, Lemma 2.5, Notation 2.6]). Similarly to Setup 2.1, we choose a positively graded polynomial ring \( T = A[x_1, \ldots, x_r] \) over \( A \) such that we have a homogeneous surjection \( T \to R \). Then by Theorem 2.15 we obtain the equality

\[
\text{fib}(M) = U_M \cap U_{\text{Hom}_T(M, T)} \cap U_{\text{Ext}_T^1(M, T)} \cap \cdots \cap U_{\text{Ext}_T^n(M, T)},
\]

and so the result follows.

(ii) Suppose that \( A \) is generically reduced. Let \( \{ p_1, \ldots, p_k \} \) be the minimal primes of \( A \). By part (i), we choose an ideal \( I \subset A \) such that \( \text{fib}(M) = \text{Spec}(A) \setminus V(I) \). As \( A \) is generically reduced, \( A_{p_1} \) is a field for any minimal prime \( p_1 \), and so it is clear that \( M \otimes_A A_{p_1} \) is fiber-full over \( A_{p_1} \). This implies that \( p_1 \nsubseteq I \). Finally, the prime avoidance lemma gives an element \( \alpha \in A \) such that \( \alpha \in 1 \setminus (p_1 \cup \cdots \cup p_k) \).

(iii) Fix any \( i \geq 0 \) and \( \nu \in \mathbb{Z} \). Let \( p \in \text{fib}(M) \). Since \( \text{fib}(M) \subset \text{Spec}(A) \) is an open subset by part (i), we can choose \( 0 \neq \alpha \in A \) such that \( p \in \text{Spec}(A_{\alpha}) \subset \text{fib}(M) \). Then, by using Theorem 2.15, we obtain that \( [H^i_m(M)]_\nu \otimes_A A\alpha \) is a finitely generated locally free \( A\alpha \)-module, and the base change isomorphism

\[
H^i_m(M \otimes_A k(\alpha)) \cong H^i_m(M) \otimes_A k(\alpha)
\]

for all \( q \in \text{Spec}(A_{\alpha}) \). By invoking [19, Tag 00NX] there exists an open subset \( V \subset \text{Spec}(A_{\alpha}) \) containing \( p \) such that the function

\[
\text{Spec}(A) \to \mathbb{N}, \quad q \mapsto \dim_{k(q)} \left( [H^i_m(M)]_\nu \otimes_A k(q) \right)
\]

is constant on \( V \). Therefore, the proof is complete.

### 4 Square-free Gröbner degenerations à la Conca–Varbaro

In this section, we give our alternative proof of the main result of [6]. Another alternative proof of this result was given in [16].

Let \( k \) be a field, \( S = k[x_1, \ldots, x_r] \) be a positively graded polynomial ring and \( m = (x_1, \ldots, x_r) \subset S \). Let \( A = k[t] \) be a polynomial ring. Let \( R = A \otimes_k S = S[t] \) be a polynomial with grading induced from \( S \); i.e., \( x_i \in R \) maintains the degree as an element of \( S \) and \( \deg(t) = 0 \). Let \( < \) be a term order on \( S \).

Our proof of the following result is a simple consequence of our previous developments.

**Theorem 4.1** (Conca–Varbaro) Let \( I \subset S \) be a homogeneous ideal. If the initial ideal \( \text{in}_{<}(I) \) is square-free, then

\[
\dim_k \left( [H^i_m(S/I)]_\nu \right) = \dim_k \left( [H^i_m(S/\text{in}_{<}(I))]_\nu \right)
\]

for all \( i \geq 0 \) and \( \nu \in \mathbb{Z} \).
Proof We can choose a weight vector $\omega \in \mathbb{N}^r$ such that the $\omega$-homogenization $J = \text{hom}_\omega(I) \subset \mathcal{R}$ satisfies the following conditions:

(i) $J$ is an $\mathcal{R}$-homogeneous ideal,
(ii) $\mathcal{R}/J$ is a free $\mathcal{A}$-module,
(iii) $\mathcal{R}/J \otimes _\mathcal{A} \mathcal{A}/(t) \cong \mathcal{S}/(\text{in}_<(I))$ and,
(iv) $\mathcal{R}/J \otimes _\mathcal{A} k(t) \cong \mathcal{S}/I$ (see, e.g., [8, §15.8]). The residue fields of the prime ideals $(0) \subset \mathcal{A}$ and $(t) \subset \mathcal{A}$ are given by $k((0)) = k(t)$ and $k((t)) = k$, respectively. We have that $\mathcal{R}/J \otimes _\mathcal{A} \mathcal{A}/(t)$ is fiber-full over $\mathcal{A}/(t)$ by [6, Proposition 2.3] or [16, Corollary 3.4]. Finally, Theorem 3.2 implies that

$$\dim _k \left( \left[ H_i^m(\mathcal{S}/I) \right]_\nu \right) = \dim _{k(t)} \left( \left[ H_i^m(\mathcal{R}/J \otimes _\mathcal{A} k(t)) \right]_\nu \right) = \dim _k \left( \left[ H_i^m(\mathcal{R}/J \otimes _\mathcal{A} \mathcal{A}/(t)) \right]_\nu \right) = \dim _k \left( \left[ H_i^m(\mathcal{S}/\text{in}_<(I)) \right]_\nu \right)$$

for all $i \geq 0$ and $\nu \in \mathbb{Z}$. □

For a finitely generated graded $\mathcal{S}$-module $M$, we denote the Castelnuovo-Mumford regularity and the depth of $M$ by $\text{reg}(M)$ and $\text{depth}(M)$, respectively, and a non-zero Betti number $\beta_{i,j}(M) := \dim _k \left( \left[ \text{Tor}_{i}^\mathcal{S}(k,M) \right]_{j} \right)$ is called extremal if $\beta_{h,k}(M) = 0$ for every $h \geq i, k \geq j$ with $(h,k) \neq (i,j)$.

Corollary 4.2 Let $I \subset \mathcal{S}$ be a homogeneous ideal. If $\text{in}_<(I)$ is square-free, then $\mathcal{S}/I$ and $\mathcal{S}/\text{in}_<(I)$ have the same extremal Betti numbers, and, in particular,

$$\text{depth}(\mathcal{S}/I) = \text{depth}(\mathcal{S}/\text{in}_<(I)) \quad \text{and} \quad \text{reg}(\mathcal{S}/I) = \text{reg}(\mathcal{S}/\text{in}_<(I)).$$

Proof It follows from Theorem 4.1 and [17, Theorem 3.11]. □

Data Availability Not applicable.

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