MINIMAL RATES OF ENTROPY CONVERGENCE
FOR RANK ONE SYSTEMS

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Abstract. If \((X, T)\) is a rank one system and \(g\) a positive concave function on \((0, \infty)\) such that \(g(x)/x^3\) is integrable, then \(\lim \sup_{n \to \infty} H(\alpha_n^{n-1})/g(\log_2 n) = \infty\), for all partitions \(\alpha\) of \(X\) into two sets with \(\lim_{n \to \infty} \max \{\mu(A) \mid A \in \alpha_n^{n-1}\} = 0\).

1. Introduction

In this paper we will continue our investigation of minimal entropy convergence rates that was begun in [2]. Let us review some standard notation: If \((X, \mathcal{B}, \mu)\) is a probability space, \(T: X \to X\) a measure-preserving transformation and \(\alpha\) a finite partition of \(X\), then the \(n\)-th refinement of \(\alpha\) under \(T\) is denoted by \(\alpha_n^{n-1}\). Furthermore, if \(f(x) := -x \log_2 x\) for \(x \in [0, 1)\), then the entropy of \(\alpha\) is defined as

\[
H(\alpha) := \sum_{A \in \alpha} f(\mu(A)).
\]

The following general result was shown in [2].

**Theorem 1.1.** Let \((X, T)\) be an aperiodic measure-preserving system and assume that \(g\) is a positive monotone increasing function defined on \([0, \infty)\) which satisfies the condition \(\int_1^\infty g(x)/x^2 \, dx < \infty\). If \(\alpha\) is a partition of \(X\) into two sets such that

\[
\lim_{n \to \infty} \max \{\mu(A) \mid A \in \alpha_n^{n-1}\} = 0
\]

then

\[
\lim \sup_{n \to \infty} \frac{H(\alpha_n^{n-1})}{g(\log_2 n)} = \infty.
\]

This theorem gives us a universal lower bound for entropy convergence rates of measure-preserving systems. It was shown in [3] that the statement of Theorem 1.1 cannot be improved in the given generality. In order to obtain stronger convergence rates we need to impose additional conditions on \((X, T)\). The following theorem gives us an example of this type. It was stated in [2] and proved in [3].

**Theorem 1.2.** If \((X, T)\) is rank one mixing, then for all non-trivial partitions \(\alpha\) of \(X\) into two sets we have

\[
\lim \sup_{n \to \infty} \frac{H(\alpha_n^{n-1})}{\log_2 n} > 0.
\]

The question that we intend to answer in the present paper is whether we can still obtain an improvement of Theorem 1.1 if we reduce the assumptions of Theorem

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1.2 from ‘rank one mixing’ to ‘rank one’. We will see that this is indeed the case (see Theorem 5.7), but the convergence rates for arbitrary rank one systems are in general weaker than the rate \( \log_2 n \) given in Theorem 1.2.

2. Definition of Rank One Transformations

We consider the probability space \((0, 1), \mathcal{B}, \mu\), where we denote by \(\mathcal{B}\) the \(\sigma\)-algebra of all Lebesgue measurable subsets of \([0, 1)\) and by \(\mu\) we denote Lebesgue measure on \([0, 1)\).

**Definition 2.1.** An \(n\)-tuple \(\tau = (B_0, \ldots, B_{n-1})\) of pairwise disjoint sets \(B_i \in \mathcal{B}\) is called a tower, if \(\mu(B_i) = \mu(B_j)\) for all \(i, j \in \{0, \ldots, n-1\}\). We will always refer to the sets \(B_i\) as the levels of \(\tau\). The union of all the levels will be denoted by \(|\tau|\), i.e.

\[
|\tau| := \bigcup_{i=0}^{n-1} B_i.
\]

In case that all the levels in a tower \(\tau\) are half open intervals, we assign to \(\tau\) a transformation \(T_\tau\) as follows:

**Definition 2.2.** Let \(\tau = (B_0, \ldots, B_{n-1})\) be a tower of intervals \(B_k = [a_k, b_k) \subset [0, 1)\). Then we define \(T_\tau : [0, 1) \to [0, 1)\) by

\[
T_\tau(x) := \begin{cases} 
    x + a_{k+1} - a_k, & \text{for } x \in [a_k, b_k) \text{ and } k \in \{0, \ldots, n-2\} \\
    x + a_0 - a_{n-1}, & \text{for } x \in [a_{n-1}, b_{n-1}) \\
    x, & \text{for } x \in [0, 1) \setminus |\tau|. 
\end{cases}
\]

This definition implies in particular that \(T_\tau(B_i) = B_{i+1}\) for all \(i \in \{0, \ldots, n-2\}\), \(T_\tau(B_{n-1}) = B_0\) and \(T_\tau|_{[0,1) \setminus |\tau|} \equiv \text{id}\).

The constructive definition of rank one transformations says that any such transformation can be obtained via ‘cutting and stacking’ of towers of subintervals of \([0, 1)\). So if \(\{\tau_n\}_{n=1}^\infty\) is a sequence of towers \(\tau_n = (B_0(n), \ldots, B_m(n))\), with \(\bigcup_{n=1}^\infty |\tau_n| = [0, 1)\) such that each \(B_i(n)\) is a subinterval of \([0, 1)\) and \(\tau_{n+1}\) is obtained from \(\tau_n\) by cutting \(\tau_n\) into vertical subtowers of equal measure and stacking them with possibly some new levels as ‘spacers’ added in between, then

\[
T := \lim_{n \to \infty} T_{\tau_n}
\]

is a rank one transformation on \([0, 1)\), where the limit is understood to be a pointwise limit. A more precise definition can be found in [4].

For the rest of our discussion we will assume \(T\) to be a rank one transformation with a defining sequence of towers \(\{\tau_n\}_{n=1}^\infty\) as explained above. It is clear that each \(\tau_n\) is a Rokhlin tower of \(T\), i.e. \(\tau_n = (B_0(n), TB_0(n), \ldots, T^{m_n-1}B_0(n))\). This allows us to abbreviate our notation by substituting \(B_n\) for \(B_0(n)\). So we have

\[
\tau_n = (B_n, TB_n, \ldots, T^{m_n-1}B_n).
\]

For the proof of the main lemma in Section 4 it will be useful to work only with towers of length \(2^k\) for some \(k \in \mathbb{N}\) and for this reason we need some additional notation.

**Definition 2.3.** For any \(n \in \mathbb{N}\) we define \(N_n := \lfloor \log_2 m_n \rfloor\), where \([x] := \max\{n \in \mathbb{N} \mid n \leq x\}\). Furthermore, for all \(k \in \mathbb{N}\) with \(k \leq N_n\) we define

\[
B_{n,k} := \bigcup_{i=0}^{2^{N_n-k-1}} T^{i2^k}B_n,
\]

\[
\sigma_{n,k} := (B_{n,k}, TB_{n,k}, \ldots, T^{2^k-1}B_{n,k}) \quad \text{and} \quad 
\mu_{n,k} := \frac{\mu}{\mu(B_{n,k})}.
\]
We notice that $|\sigma_{n,k}| = |\sigma_{n,j}|$ for all $k, j \leq N_n$.

3. Some Properties of 01-Names
First we will review some of the definitions and propositions that we already discussed in [2] and then we will discuss some additional properties of 01-names that will be essential for the proof of the main lemma in the next section. (The definitions in this paper are actually slightly modified compared to [2], because we will almost exclusively work with 01-names of length $2^k$ for some $k \in \mathbb{N}$.)

**Definition 3.1.** If $\alpha$ is a finite partition of $[0, 1)$ and $F \in \mathcal{B}$, then we define the entropy of $\alpha$ restricted to $F$ as $H_F(\alpha) := \sum_{A \in \alpha} f(\mu(A \cap F))$.

**Lemma 3.2.** If $\alpha$ is a finite partition of $[0, 1)$ and $F, G \in \mathcal{B}$ with $G \subset F$, then $H_F(\alpha) > H_G(\alpha) - 2$.

**Definition 3.3.** For any $E \in \mathcal{B}$, $x \in [0, 1)$, $n \in \mathbb{N}$ and $s \in \{0, 1\}^{2^n}$ we define
\[
\alpha(E) := \{E, [0, 1) \setminus E\}, \\
\tau_E^n(x) := (\chi_E(x), \chi_E(Tx), \ldots, \chi_E(Tn-1x)), \\
s_E^n(x) := \tau_{2^n}^n(x) \text{ and} \\
A_n^E(s) := (s_E^n)^{-1}(\{s\})
\]

With these definitions it is easy to see that $\alpha(E)_{2^n-1} = \{A_n^E(s) \mid s \in \{0, 1\}^{2^n}\}$. In order to understand how the measure of $A_n^E(s)$ depends on the properties of $s$, we need to define the period of a 01-name.

**Definition 3.4.** For $r \in \{0, 1\}^n$ we define the period of $r$ as
\[
q_n(r) := \min\{k \in \{1, \ldots, n - 1\} \mid r_i = r_{i+k} \text{ for all } i \in \{0, \ldots, n - k - 1\} \cup \{n\}\}
\]
Furthermore, for $s \in \{0, 1\}^{2^n}$ we define
\[
p_n(s) := q_{2^n}(s) \text{ and } S_n := \{s \in \{0, 1\}^{2^n} \mid p_n(s) \leq 2^{n-1}\}.
\]

The proofs of the following simple lemmas were given in [2].

**Lemma 3.5.** Let $\tau = (B, TB, \ldots, T^{2^n-1}B)$ be a Rokhlin tower, $E \in \mathcal{B}$ and $s \in \{0, 1\}^{2^n}$. Then
\[
\mu(A_n^E(s) \cap |\tau|) \leq \frac{2}{p_n(s)} \mu(|\tau|).
\]

**Lemma 3.6.** Let $\alpha$ be a finite partition of $[0, 1)$, $F \in \mathcal{B}$ and $c > 0$ such that
\[
\mu(A \cap F) \leq c \text{ for all } A \in \alpha.
\]
Then $H_F(\alpha) \geq \mu(F) \log_2 \frac{1}{c}$.

**Definition 3.7.** For $n, i, j \in \mathbb{N}$ with $0 \leq i \leq j \leq n - 1$ and $s \in \{0, 1\}^n$ we denote a subname of $s$ by
\[
s_i^j := (s_i, \ldots, s_j).
\]
We will use the same notation also for finite subnames of doubly infinite sequences $s \in \{0, 1\}^\mathbb{Z}$. 

Definition 3.8. For \( n, m \in \mathbb{N} \) and \( s \in \{0, 1\}^n, r \in \{0, 1\}^m \) we define the concatenation of \( s \) and \( r \) as \((s, r) := (s_0, \ldots, s_{n-1}, r_0, \ldots, r_{m-1}) \in \{0, 1\}^{n+m}\).

Let us denote by \( \lambda \) the left shift on \( \{0, 1\}^\mathbb{Z} \), i.e. \( \lambda(s)_i = s_{i+1} \) for all \( s \in \{0, 1\}^\mathbb{Z} \) and \( i \in \mathbb{Z} \).

Definition 3.9. For \( s \in \{0, 1\}^\mathbb{Z} \) we say that \( s \) is periodic if there exists a \( k \in \mathbb{N} \) such that \( \lambda^k(s) = s \). In this case we define the period of \( s \) as

\[ p(s) := \min\{k \in \mathbb{N} | \lambda^k(s) = s\}. \]

If \( s, r \in \{0, 1\}^\mathbb{Z} \) are periodic then the distance between \( s \) and \( r \) is defined to be

\[ d(s, r) := \lim_{n \to \infty} \frac{1}{n} \|\lambda^i(s)_0^{n-1} - r_0^{n-1}\|_1. \]

The existence of the limit and the minimum is obvious (see Remark 3.10).

Remark 3.10. If \( s, r \in \{0, 1\}^\mathbb{Z} \) are periodic then we have

\[ d(r, s) = d(s,r) = \min_{0 \leq i \leq p(s)p(r)-1} \frac{1}{p(s)p(r)} \|\lambda^i(s)_0^{p(s)p(r)-1} - r_0^{p(s)p(r)-1}\|_1. \]

Definition 3.11. It is easy to see that for \( s \in \{0, 1\}^n \) there exists a unique \( \bar{s} \in \{0, 1\}^\mathbb{Z} \) such that \( \bar{s}_0^{n-1} = s \) and \( p(\bar{s}) = q_n(s) \). We call \( \bar{s} \) the extension of \( s \). Furthermore, we define \( s^+ := \bar{s}_0^{2n-1} \) and \( s^- := \bar{s}_0^{-1} \).

Remark 3.12. The \( 01 \)-names \( s^+ \) and \( s^- \) are the unique elements in \( \{0, 1\}^n \) with \( q_{2n}((s, s^+)) = q_{2n}((s^-, s)) = q_n(s) \). Furthermore, if \( r \neq s^+ \) and \( t \neq s^- \), then it is easy to see that \( q_{2n}((r, s)) > n/2 \) and \( q_{2n}((t, s)) > n/2 \).

Definition 3.13. For \( E \in \mathcal{B}, n, k, K \in \mathbb{N} \) with \( k \leq K \leq N_n \) and \( s, r, t \in \{0, 1\}^{2^k} \) we define

\[ p_k(s, r) := \min_{0 \leq i \leq 2^{k-1}} \frac{1}{2^k} \|\lambda^i(s)_0^{2^{k-1}} - r\|_1, \]

\[ f_{k, K}(s, r) := \begin{cases} \min\{p_k(s, r), 2^{-K}\}, & \text{if } p_k(s) > p_k(r) \leq 2^{k/4} \\
\min\{p_k(r, s), 2^{-K}\}, & \text{if } p_k(s) > p_k(r), p_k(r) \leq 2^{k/4} \\
\min\{p_k(s, r) + p_k(r, s)\}/2, 2^{-K}, & \text{if } p_k(s) = p_k(r) \leq 2^{k/4}, \end{cases} \]

\[ I_{k, K}^E(n) := \int_{B_{n,k} \times B_{n,k}} f_{k, K}(s_k^E(x), s_k^E(y)) d(\mu_{n,k} \times \mu_{n,k}). \]

Remark 3.14. From the definition above follows easily that the maps \( f_{k, K} \) are symmetric, i.e. \( f_{k, K}(s, r) = f_{k, K}(r, s) \) for all \( s, r \in \{0, 1\}^{2^k} \). Furthermore, if \( s \in S_k \) (i.e. \( p_k(s) \leq 2^{k-1} \)) and \( p_k(s, r) = 0 \), then

\[ \begin{align*}
a) & \quad \lambda^i(\bar{s}) = \bar{r}, \\
b) & \quad p_k(\bar{r}, s) = 0 \quad \text{and} \\
c) & \quad p_k(s) = p_k(r). \end{align*} \]

Notice: If we do not assume \( p_k(s) \leq 2^{k-1} \) then it is possible to have \( p_k(s, r) = 0 \) but \( p_k(r, s) \neq 0 \). This is for example the case for \( s := (0001) \) and \( r := (0010) \). Using the observations a), b) and c) it is not difficult to see that

\[ (s \sim r)_k \iff p_k(s, r) = 0 \]
defines an equivalence relation on $S_h$. The equivalence classes will be denoted by $[s]_k$. Given this definition it is easy to show that

$$f_{k,K}(s,r) = f_{k,K}(t,r) \text{ for all } t \in [s]_k,$$

whenever $p_k(s) < p_k(r)$ and $s \in S_h$. We also notice that for all $s \in S_h$ we have $s \sim s^+$, $s \sim s^-$ and the maps

$$h_+ : [s]_k \rightarrow [s]_k; r \mapsto r^+ \quad \text{and} \quad h_- : [s]_k \rightarrow [s]_k; r \mapsto r^-$$

are bijective and $(h_+)^{-1} = h_-.$

**Lemma 3.15.** If $k \in \mathbb{N}$ and $s, r \in \{0, 1\}^{2^k}$ with $p_k(s), p_k(r) \leq 2^{k/3}$, then $|\rho_k(s, r) - d(s, r)| \leq 2^{-k/3}d(s, \bar{r})$.

**Proof.** Let us define $c := \left[\frac{2^k}{p_k(s) p_k(r)}\right]$. It is easy to see that for any $i \in \{0, \ldots, 2^k - 1\}$ we have

$$c\|\lambda(\bar{s})^0 p_k(r)^{-1} - r 0 \| = \left\| \frac{\lambda(\bar{s})^{2^k - 1} - r 0 \|}{c + 1} \lambda(\bar{s})^{2^k - 1} - r 0 \| \right\|\leq \frac{2^{-k/3} \rho_k(s, r)}{c}.$$

Using Remark 3.10 it follows that

$$(1 - 2^{-k/3})d(s, \bar{r}) \leq \rho_k(s, r) \leq (1 + 2^{-k/3})d(s, \bar{r}).$$

This completes the proof.

**Corollary 3.16.** If $k \in \mathbb{N}$, $k \geq 3$ and $s, r \in \{0, 1\}^{2^k}$ with $p_k(s), p_k(r) \leq 2^{k/3}$, then

$$|\rho_k(s, r) - \rho_k(r, s)| \leq 2^{2-k/3} \rho_k(s, r).$$

**Proof.** Simple consequence of Lemma 3.15 and the fact that $d(s, \bar{r}) = d(\bar{r}, \bar{s})$ (see Remark 3.10).

**Corollary 3.17.** If $k, K \in \mathbb{N}$, $k \geq K \geq 3$ and $s, r \in \{0, 1\}^{2^k}$ with $p_k(s), p_k(r) \leq 2^{k/3}$, then

$$|f_{k,K}(s, r) - f_{k,K}(u, t)| \leq 2^{2-k/3} f_{k,K}(s, r) \text{ for all } t \in [r]_k \text{ and } u \in [s]_k.$$

**Proof.** According to Remark 3.14 we have $\bar{s} = \lambda^i(\bar{u})$ and $\bar{r} = \lambda^j(\bar{l})$ for some $i, j \in \{0, \ldots, 2^k - 1\}$. This implies clearly that $d(s, \bar{r}) = d(\bar{u}, \bar{l})$ and therefore we can apply Lemma 3.15 to obtain that

$$|\rho_k(s, r) - \rho_k(u, t)| \leq 2^{1-k/3} \rho_k(s, r).$$

The statement of the corollary follows now easily from the definition of $f_{k,K}$ and the proof is complete.

4. **The Main Lemma** The most important tool for the proof of Theorem 5.7 is Lemma 4.3. We will try to give an explanation for the significance of this lemma. The proof of Theorem 1.1 as given in [2] involved estimates of the form

$$H(\alpha(E)_{10}^{n-1}) \geq \sum_{k=1}^{n} k x_k$$
where $x_k$ stood for the measure of the set of points in the base level of a Rokhlin tower of length $2^n$ with periods ranging between $2^{k-1}$ and $2^k - 1$. The nature of the estimates that led to the statement of Theorem 1.1 can be summarized as follows: If $\{a_n\}_{n=1}^\infty$ is a positive monotone increasing sequence with

$$\sum_{n=1}^\infty \frac{a_n}{n^2} < \infty$$

and $\{x_n\}_{n=1}^\infty$ is a positive sequence with $\sum_{n=1}^\infty x_n = \infty$, then

$$\limsup_{n \to \infty} \frac{1}{a_n} \sum_{k=1}^n k x_k = \infty,$$

because otherwise there is a $c > 0$ such that

$$\infty > \sum_{n=1}^\infty \frac{a_n}{n^2} \geq c \sum_{n=1}^\infty \frac{1}{n^2} \sum_{k=1}^n k x_k \approx c \sum_{n=1}^\infty k x_k \int_k^\infty \frac{1}{x^2} \, dx = c \sum_{n=1}^\infty x_n = \infty,$$

and this is a contradiction. The purpose of Lemma 4.3 in the context of this type of estimate is to allow us to replace the assumption $\sum_{n=1}^\infty x_n = \infty$ by the stronger assumption

$$\sum_{n=1}^\infty x_n^2 = \infty.$$

This in turn will make it possible to improve the convergence rates $\{a_n\}_{n=1}^\infty$ by assuming only that

$$\sum_{n=1}^\infty \frac{a_n^2}{n^3} < \infty.$$

Under these assumptions we obtain $\limsup_{n \to \infty} \frac{1}{a_n} \sum_{k=1}^n k x_k = \infty$, because otherwise we can again find a $c > 0$ such that

$$\infty > \sum_{n=1}^\infty \frac{a_n^2}{n^3} \geq c \sum_{n=1}^\infty \frac{1}{n^3} \left( \sum_{k=1}^n k x_k \right)^2 \approx c \sum_{n=1}^\infty \frac{1}{n^3} \sum_{k=1}^n k^2 x_k^2$$

$$\approx c \sum_{k=1}^\infty k^2 x_k^2 \int_k^\infty \frac{1}{x^3} \, dx = c \sum_{k=1}^\infty x_k^2 = \infty,$$

and this is a contradiction. The estimate above will actually not appear in exactly this form in the proof of Theorem 5.7, but it illustrates nicely the motivation for Lemma 4.3, because there we obtain a lower estimate on the product of sums of the $x_k$, which is best interpreted as a lower estimate for each individual value $x_k^2$. This will eventually allow us to obtain estimates that are similar to the assumption $\sum_{k=1}^\infty x_k^2 = \infty$. Finally, we also wish to mention that the original idea for the proof of Lemma 4.3 is related to Kolmogorov’s inequality (see [1]). In fact, Kolmogorov’s inequality can be used directly to prove a statement analogous to Theorem 5.7 for periodic approximations of the von Neumann-Kakutani adding machine.

**Definition 4.1.** For $n, k \in \mathbb{N}$ with $k \leq N_n$, $s \in \{0, 1\}^2$ and $E \in \mathcal{B}$ we define

$$B_{n,k}^E(s) := \{x \in B_{n,k} \mid s_k^E(x) = s\}.$$
Furthermore, for all $i \in \{0, \ldots, 4k\}$ we define
\[
P_{n,k}^E(i) := \bigcup_{s \in \{0,1\}^{2k}} B_{n,k}^E(s) \quad \text{and} \quad x_{n,k}^E(i) := \mu_{n,k}(P_{n,k}^E(i)).
\]

**Remark 4.2.** The definition of $I_{E,k}(n)$ implies that
\[
I_{E,k}(n) = \sum_{s,r,u,t} f_{k,k}(s,r) \mu(B_{n,k}(s)) \mu(B_{n,k}(r)).
\]

**Lemma 4.3.** If $n, k, K \in \mathbb{N}$ with $3 \leq K \leq k < N_n$ and $E \in \mathcal{B}$ then
\[
I_{E,k+1,K}(n) - I_{E,k+1,K}(n) \leq \frac{2}{2^K} \left( \sum_{i=0}^{k} x_{n,k}^E(i) - x_{n,k+1}^E(i) \right) \left( \sum_{i=k+1}^{4k+4} x_{n,k+1}^E(i) \right) + \frac{64}{2^{K+3}}.
\]

**Proof.** Let us define
\[
U := \{s \in \{0,1\}^{2k} \mid p_k(s) \leq 2^{k/4}\},
\]
\[
V := \{s \in \{0,1\}^{2k} \mid 2^{k/4} < p_k(s) \leq 2^{(k+1)/4}\},
\]
\[
W := \{0,1\}^{2k} \setminus (U \cup V).
\]

In order to simplify our notation, we set
\[
I_k := I_{E,k,K}^E(n), \quad I_{k+1} := I_{E,k+1,K}^E(n),
\]
\[
\mu_k := \mu_{n,k}, \quad \mu_{k+1} := \mu_{n,k+1},
\]
\[
f_k := f_{k,k}, \quad f_{k+1} := f_{k+1,k},
\]
\[
\mu_k(s) := \mu_k(B_{n,k}^E(s)), \quad \mu_{k+1}(s,u) := \mu_{k+1}(B_{n,k+1}^E((s,u))).
\]

For all $s, u \in \{0,1\}^{2k}$. For $s \in S_k$, we set $[s] := [s]_k$. Now we choose one representative $s_i$ from each equivalence class $[s]$ where $s \in U \cup V \subset S_k$. So if $\{s_1, \ldots, s_q\}$ is the set of these representatives, then $U \cup V = \bigcup_{i=1}^{q} [s_i]$. Using Remark 4.2 we obtain:
\[
I_{k+1} = \sum_{s,r \in \{0,1\}^{2k+1}} f_{k+1}(s,r) \mu_{k+1}(B_{n,k+1}^E(s)) \mu_{k+1}(B_{n,k+1}^E(r))
\]
\[
= \sum_{s,r,u,t \in \{0,1\}^{2k}} f_{k+1}((s,u),(r,t)) \mu_{k+1}(s,u) \mu_{k+1}(r,t)
\]
\[
= \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} \sum_{t,u \in \{0,1\}^{2k}} f_{k+1}((s,u),(r,t)) \mu_{k+1}(s,u) \mu_{k+1}(r,t)
\]
\[
+ \sum_{i=1}^{q} \sum_{s \in [s_i]} \sum_{r \in W} \sum_{t,u \in \{0,1\}^{2k}} f_{k+1}((s,u),(r,t)) \mu_{k+1}(s,u) \mu_{k+1}(r,t)
\]
\[
+ \sum_{j=1}^{q} \sum_{r \in [s_j]} \sum_{s \in W} \sum_{t,u \in \{0,1\}^{2k}} f_{k+1}((s,u),(r,t)) \mu_{k+1}(s,u) \mu_{k+1}(r,t)
\]
\[
+ \sum_{r,s \in W} \sum_{t,u \in \{0,1\}^{2k}} f_{k+1}((s,u),(r,t)) \mu_{k+1}(s,u) \mu_{k+1}(r,t)
\]

(4.1)
Since clearly $p_{k+1}(s, u) \geq p_k(s)$ for all $s, u \in \{0, 1\}^{2k}$, the definition of $f_{k+1}$ and $W$ implies that term (4.3) is equal to zero. Furthermore, the symmetry of $f_{k+1}$ (see Remark 3.14) implies that the terms (4.1) and (4.2) are equal. Hence

$$I_{k+1} = \sum_{i, j = 1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} f_{k+1}((s, u), (r, t)) \mu_{k+1}(s, u) \mu_{k+1}(r, t)$$

$$+ 2 \sum_{i, j = 1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} f_{k+1}((s, u), (r, t)) \mu_{k+1}(s, u) \mu_{k+1}(r, t).$$

If $s \in \{0, 1\}^{2k}$ and $u \neq s^+$, then Remark 3.12 implies that $p_{k+1}(s, u) > 2^{k-1}$. Therefore, $f_{k+1}((s, u), (r, t)) = 0$ whenever $u \neq s^+$ and $t \neq r^+$. Hence

$$I_{k+1} = \sum_{i, j = 1}^{q} \sum_{s \in [s_i]} f_{k+1}((s, s^+), (r, r^+)) \mu_{k+1}(s, s^+) \mu_{k+1}(r, r^+)$$

$$+ 2 \sum_{i, j = 1}^{q} \sum_{s \in [s_i]} f_{k+1}((s, s^+), (r, t)) \mu_{k+1}(s, s^+) \mu_{k+1}(r, t) \quad (4.4)$$

$$+ \sum_{i, j = 1}^{q} \sum_{s \in [s_i]} f_{k+1}((s, u), (r, r^+)) \mu_{k+1}(s, u) \mu_{k+1}(r, r^+)$$

$$+ 2 \sum_{i, j = 1}^{q} \sum_{s \in [s_i]} \sum_{t \in \{0, 1\}^{2k}} f_{k+1}((s, u), (r, t)) \mu_{k+1}(s, u) \mu_{k+1}(r, t).$$

Again the symmetry of $f_{k+1}$ implies that the terms (4.4) and (4.5) are equal. If $r \in W$, $s \in [s_i]$ and $u \neq s^+$, then $f_{k+1}((s, u), (r, t)) = 0$ for all $t \in \{0, 1\}^{2k}$. Hence

$$I_{k+1} = \sum_{i, j = 1}^{q} \sum_{s \in [s_i]} f_{k+1}((s, s^+), (r, r^+)) \mu_{k+1}(s, s^+) \mu_{k+1}(r, r^+)$$

$$+ 2 \sum_{i, j = 1}^{q} \sum_{s \in [s_i]} f_{k+1}((s, s^+), (r, t)) \mu_{k+1}(s, s^+) \mu_{k+1}(r, t)$$

$$+ 2 \sum_{i, j = 1}^{q} \sum_{s \in [s_i]} \sum_{t \in \{0, 1\}^{2k}} f_{k+1}((s, s^+), (r, t)) \mu_{k+1}(s, s^+) \mu_{k+1}(r, t) \quad (4.6)$$

If $s \in [s_i]$ and $r, t \in \{0, 1\}^{2k}$ then it is obvious that

$$\rho_{k+1}((s, s^+), (r, t)) \geq \frac{1}{2}(\rho_k(s, r) + \rho_k(s^+, t)).$$

Therefore, if $t \neq r^+$ or $r \in W$, the definition of $f_{k+1}$ implies that

$$f_{k+1}((s, s^+), (r, t)) \geq \frac{1}{2}(\min\{\rho_k(s, r), 2^{-K}\} + \min\{\rho_k(s^+, t), 2^{-K}\}).$$

$$\quad (4.7)$$
Using (4.6), (4.7), (4.8) and (4.9), we obtain:

\[ f_{k+1}(s,s^+,(r,r^+)) \geq \frac{1}{2}(f_k(s,r) + f_k(s^+,r^+)). \]  

(4.8)

Now let \( s \in [s_i] \) and \( r \in \{0,1\}^{2^t} \). If \( p_k(r) \leq 2^{k/3} \), then Corollary 3.16 implies that

\[
\min \{ \rho_k(s,r), 2^{-K} \} \geq (1 - 2^{2-k/3}) f_k(s,r).
\]

If \( p_k(r) > 2^{k/3} \), then

\[
f_k(s,r) = \min \{ \rho_k(s,r), 2^{-K} \} \text{ or } f_k(s,r) = 0,
\]

depending on whether \( p_k(s) \leq 2^{k/4} \) or \( p_k(s) > 2^{k/4} \). This shows that for all \( s \in [s_i] \) and all \( r \in \{0,1\}^{2^t} \) we have

\[
\min \{ \rho_k(s,r), 2^{-K} \} \geq (1 - 2^{2-k/3}) f_k(s,r).
\]

(4.9)

Using (4.6),(4.7),(4.8) and (4.9), we obtain:

\[
I_{k+1} \geq \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} 1/2 (f_k(s,r) + f_k(s^+,r^+)) \mu_{k+1}(s,s^+) \mu_{k+1}(r,r^+)
\]

\[
+ (1 - 2^{2-k/3}) \left( \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} \sum_{t \in \{0,1\}^{2^t}} (f_k(s,r) + f_k(s^+,t)) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t) \right)
\]

\[
+ \sum_{i=1}^{q} \sum_{s \in [s_i]} \sum_{r \in W} \sum_{t \in \{0,1\}^{2^t}} (f_k(s,r) + f_k(s^+,t)) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
= \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} 1/2 (f_k(s,r) + f_k(s^+,r^+)) \mu_{k+1}(s,s^+) \mu_{k+1}(r,r^+)
\]

\[
+ (1 - 2^{2-k/3}) \left( \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} \sum_{t \in \{0,1\}^{2^t}} f_k(s,r) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t) \right)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} \sum_{t \in \{0,1\}^{2^t}} f_k(s^+,t) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{t \in W} \sum_{r \in [s_j]} f_k(s,r) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{t \in \{0,1\}^{2^t}} f_k(s^+,t) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in W} \sum_{t \in \{0,1\}^{2^t}} f_k(s,r) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{t \in W} f_k(s^+,t) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in W} f_k(s^+,t) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in W} f_k(s^+,t) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in W} f_k(s^+,t) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in W} f_k(s^+,t) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in W} f_k(s^+,t) \mu_{k+1}(s,s^+) \mu_{k+1}(r,t)
\]
Using Remark 3.14 and Corollary 3.17 we conclude that

\[
I_{k+1} \geq \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} \frac{1}{2} \left( f_k(s, r) + f_k(s^+, r^+) \right) \mu_{k+1}(s, s^+) \mu_{k+1}(r, r^+)
\]

\[
+ \left( 1 - 2^{-k/3} \right) \left( \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} \sum_{t \in \{0,1\}^{2k}} f_k(s, r) \mu_{k+1}(s, s^+) \mu_{k+1}(r, t) \right)
\]

\[
+ \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} \sum_{t \in \{0,1\}^{2k}} f_k(s^+, t) \mu_{k+1}(s, s^+) \mu_{k+1}(r, t)
\]

\[
+ \sum_{i=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} \sum_{t \in W} f_k(s, r) \mu_{k+1}(s, s^+) \mu_{k+1}(r, t)
\]

\[
+ \sum_{i=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} \sum_{t \in \{0,1\}^{2k}} f_k(s^+, r) \mu_{k+1}(s, s^+) \mu_{k+1}(r, t)
\]
Using these definitions, we obtain

$$I_{k+1} \geq (1 - 2^{2^{-k/3}})^2 \left( \sum_{i,j=1}^{q} f_k(s_i, s_j) \mu_{k+1}(B(i)) \mu_{k+1}(B(j)) \right)$$

$$+ \sum_{i,j=1}^{q} f_k(s_i, s_j) \mu_{k+1}(B(i)) \mu_{k+1}(D^+(j))$$

$$+ \sum_{i,h=1}^{q} f_k(s_i, s_h) \mu_{k+1}(B(i)) \mu_{k+1}(D^-(h))$$

$$+ \sum_{i=1}^{q} \sum_{t \in W} f_k(s_i, t) \mu_{k+1}(B(i)) \mu_{k+1}(C^+(t))$$

$$+ \sum_{i=1}^{q} \sum_{r \in W} f_k(s_i, r) \mu_{k+1}(B(i)) \mu_{k+1}(C^-(r))$$

$$= (1 - 2^{2^{-k/3}})^2 \left( \sum_{i,j=1}^{q} f_k(s_i, s_j) \mu_{k+1}(B(i)) \mu_{k+1}(B(j)) \right)$$

$$+ \sum_{i,j=1}^{q} f_k(s_i, s_j) \mu_{k+1}(B(i)) (\mu_{k+1}(D^+(j)) + \mu_{k+1}(D^-(j)))$$

$$+ \sum_{i=1}^{q} \sum_{r \in W} f_k(s_i, r) \mu_{k+1}(B(i)) (\mu_{k+1}(C^+(r)) + \mu_{k+1}(C^-(r))) \right).$$ (4.10)

We will now find an upper estimate for $I_k$. Using the symmetry of $f_k$, Remark 3.14, Corollary 3.17 and the fact that $f_k(s, r) = 0$ for all $s, r \in W$, we obtain

$$I_k = \sum_{s, r \in \{0, 1\}^k} f_k(s, r) \mu_k(s) \mu_k(r)$$

$$= \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} f_k(s, r) \mu_k(s) \mu_k(r)$$

$$\leq \sum_{i,j=1}^{q} \sum_{s \in [s_i]} \sum_{r \in [s_j]} (1 + 2^{-k/3}) f_k(s_i, s_j) \mu_k(s) \mu_k(r) + 2 \sum_{i=1}^{q} \sum_{s \in [s_i]} \sum_{r \in W} f_k(s, r) \mu_k(s) \mu_k(r).$$

If $s \in [s_i]$, then

$$\mu_k(s) = \frac{1}{2} \left( \mu_{k+1}(s, s^+) + \mu_{k+1}(s^-, s) + \sum_{u \in \{0, 1\}^k \atop u \neq s^+} \mu_{k+1}(s, u) + \sum_{u \in \{0, 1\}^k \atop u \neq s^-} \mu_{k+1}(u, s) \right)$$

and for $r \in W$ we have

$$\mu_k(r) = \frac{1}{2} \sum_{t \in \{0, 1\}^k} (\mu_{k+1}(t, r) + \mu_{k+1}(r, t)).$$
Using these equalities and the symmetry of $f_k$ it is easy to see that

$$I_k \leq (1 + 2^{2-k/3}) \left( \frac{1}{4} \sum_{i,j=1}^{q} f_k(s_i, s_j)(2\mu_{k+1}(B(i)) + \mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-(i))) 
+ (2\mu_{k+1}(B(j)) + \mu_{k+1}(D^+(j)) + \mu_{k+1}(D^-(j))) 
+ \frac{1}{2} \sum_{i=1}^{q} \sum_{r \in W} f_k(s_i, r)(2\mu_{k+1}(B(i)) + \mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-(i))) 
+ \mu_{k+1}(C^+(r)) + \mu_{k+1}(C^-(r))) \right)$$

$$= (1 + 2^{2-k/3}) \left( \sum_{i,j=1}^{q} f_k(s_i, s_j)\mu_{k+1}(B(i))\mu_{k+1}(B(j)) 
+ \frac{1}{4} \sum_{i,j=1}^{q} f_k(s_i, s_j)\mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-)(D^-)(j)) 
+ \frac{1}{4} \sum_{i,j=1}^{q} f_k(s_i, s_j)\mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-)(j)) 
+ \mu_{k+1}(D^-)(j)) 
+ \frac{1}{2} \sum_{i=1}^{q} \sum_{r \in W} f_k(s_i, r)\mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-)(r)) + \mu_{k+1}(C^+(r)) + \mu_{k+1}(C^-(r))) \right)$$

Using this estimate, inequality (4.10), the definition of $U$ and $V$ and the fact that $f_k \leq 2^{-K}$, we obtain

$$I_k - I_{k+1} \leq \frac{1}{4} \sum_{i,j=1}^{q} f_k(s_i, s_j)\mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-)(j)) 
+ \mu_{k+1}(D^-)(j)) 
+ \frac{1}{2} \sum_{i=1}^{q} \sum_{r \in W} f_k(s_i, s_j)\mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-)(r)) + \mu_{k+1}(C^+(r)) + \mu_{k+1}(C^-(r))) 
+ 2^{-K}(2^{2-k/3} + 2^{3-k/3}) \left( \sum_{i,j=1}^{q} \mu_{k+1}(B(i))\mu_{k+1}(B(j)) 
+ \sum_{i,j=1}^{q} \mu_{k+1}(B(i))\mu_{k+1}(D^+(j)) + \mu_{k+1}(D^-)(j)) 
+ \mu_{k+1}(D^-)(j)) 
+ \frac{1}{4} \sum_{i,j=1}^{q} (\mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-)(i))(\mu_{k+1}(D^+(j)) + \mu_{k+1}(D^-)(j)) 
+ \sum_{i=1}^{q} \sum_{r \in W} \mu_{k+1}(B(i))\mu_{k+1}(D^+(r)) + \mu_{k+1}(C^+(r)) + \mu_{k+1}(C^-(r))) 
+ \sum_{i=1}^{q} \sum_{r \in W} (\mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-)(i))(\mu_{k+1}(C^+(r)) + \mu_{k+1}(C^-(r))) \right)$$
Using Remark 3.12 and the fact that 

\[ H \leq \frac{1}{2} \sum_{i,j=1}^{q} f_{k}(s_{i}, s_{j})(\mu_{k+1}(D^+)(i) + \mu_{k+1}(D^-)(i))(\mu_{k+1}(D^+)(j) + \mu_{k+1}(D^-)(j)) \]

\[ \frac{1}{2} \sum_{i,j=1}^{q} f_{k}(s_{i}, r)(\mu_{k+1}(D^+)(i) + \mu_{k+1}(D^-)(i))(\mu_{k+1}(C^+)(r) + \mu_{k+1}(C^-)(r)) \]

\[ + 2^{-K}2^{4-k/3}(2\mu_{k+1}(B_{n,k+1}))^{2} \]

Now we will examine the relation of the measure of the sets \( D^+(i), D^-(i), C^+(r) \) and \( C^-(r) \) to the values of \( x_{n,k}(l) \) and \( x_{n,k+1}(l) \) for \( l \in \{0, \ldots, 4k\} \). If \( l \leq k \), then it is easy to see that

\[ x_{n,k}(l) = \sum_{i=1}^{q} \mu_{k+1}(B(i)) + \frac{1}{2}(\mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-(i))) \] and

\[ x_{n,k+1}(l) = \sum_{i=1}^{q} \mu_{k+1}(B(i)). \]

Hence

\[ \sum_{l=0}^{k} (x_{n,k}(l) - x_{n,k+1}(l)) = \frac{1}{2} \sum_{i=1}^{q} (\mu_{k+1}(D^+(i)) + \mu_{k+1}(D^-(i))). \]

Using Remark 3.12 and the fact that \( p_{k+1}((s, r)) \geq p_k(r) \) for all \( s, r \in \{0, 1\}^{2^{k}} \) we conclude that for all

\[ x \in \left( \bigcup_{j=1}^{q} D^+(j) \cup D^-(j) \right) \cup \left( \bigcup_{r \in W} C^+(r) \cup C^-(r) \right) \]
Lemma 5.2. For all 

\[ p_{k+1}(s_{k+1}^E(x)) \geq 2^{(k+1)/4} \]

This implies

\[ \sum_{l=k+1}^{4k+4} x_{n,k+1}^E(l) \geq \frac{1}{2} \sum_{j=1}^{q} (\mu_{k+1}(D^+(j)) + \mu_{k+1}(D^-(j))) + \frac{1}{2} \sum_{r \in W} (\mu_{k+1}(C^+(r)) + \mu_{k+1}(C^-(r))). \]

(Note: The factor 1/2 appears, because the sets \( D^+(j), D^-(j), C^+(r) \) and \( C^-(r) \) are not all disjoint.) Finally, we obtain

\[ I_k - I_{k+1} \leq \frac{2}{2^k} \left( \sum_{l=0}^{k} x_{n,k}(l) - x_{n,k+1}(l) \right) \left( \sum_{l=k+1}^{4k+4} x_{n,k+1}^E(l) \right) + \frac{64}{2^{k+3}}, \]

and the proof is complete.

5. Conclusion of the Argument

The following two Lemmas will allow us to use estimates of the type mentioned in the introduction to Section 4.

Lemma 5.1. For \( n \geq 3 \) and \( s = (s_{-2n-2}, \ldots, s_{2n+2n-1}) \in \{0,1\}^{2n+1-2n-2} \) we define \( r_i := s_{i+2n-1}^{2n-1} \) for all \( i \in \{-2n-2, \ldots, 2n\} \). If \( p_n(s_{2n-1}^{2n-1}) > 2^{n-2} \), then there is an \( i \in \{-2n-2, \ldots, 2n\} \) such that for all \( k, j \in \{i, \ldots, i + 2^{n-2} - 1\} \) with \( k \neq j \).

Proof. If \( r_k \neq r_j \) for all \( k, j \in \{-2n-2, \ldots, 2n\} \) with \( k \neq j \), then we are done. So let us assume that there are \( k_0, j_0 \in \{-2n-2, \ldots, -1\} \) such that \( k_0 < j_0 \) and \( r_{k_0} = r_{j_0} \). Then we have

\[ q := q_{2n-1-j_0-k_0}(s_{k_0}^{2n-1+j_0-1}) \leq j_0 - k_0 < 2^{n-2}. \]

Since \( p_n(s_{2n-1}^{2n-1}) > 2^{n-2} \), there is an \( m \in \{2n-1+j_0-1, \ldots, 2n-1\} \) such that

\[ q_{m-k_0}(s_{k_0}^m) = q \quad \text{and} \quad q_{m-k_0+2}(s_{k_0}^m) > q. \]

It is now easy to see that for all \( k, j \in \{m-n-1+1, \ldots, m-n\} \) with \( k \neq j \) we have \( r_{k} \neq r_{j} \) and this completes the proof, because \( q < 2^{n-2} \).

Lemma 5.2. For all \( n, k \in \mathbb{N} \) and \( E \in \mathcal{B} \) with \( 3 \leq k \leq N_n \) we have

\[ H(\alpha(E)_0^{2k-1-1}) \geq \frac{\mu(|\tau_n|)}{32} \sum_{i=0}^{4k} i x_{n,k}^E(i) - 6 - \log_2 k. \]

Proof. For all \( x \in B_{n,k} \) with \( 2^{k-2} < p_k(s_{n,k}^E(x)) \leq 2^k \), we can find an \( i(x) \in \{-2^{k-2}, \ldots, 2^{k-1}\} \) such that for all \( j, k \in \{i(x), \ldots, i(x) + 2^{k-2} - 1\} \) with \( j \neq k \) we have

\[ s_{k-1}^E(T^j x) \neq s_{k-1}^E(T^k x) \quad \text{(by Lemma 5.1).} \]
Now we define

\[ \begin{align*}
N(i) &:= \{ x \in B_{n,k} \mid 2^{k-2} < p_k(s^E(x)) \leq 2^k \text{ and } i(x) = i \} \quad \text{and} \\
M(i) &:= T^i(N(i)) \quad \text{for all } i \in \{-2^{k-2}, \ldots, 2^{k-1}\}, \\
M &:= \bigcup_{i=-2^{k-2}}^{2^{k-1}} M(i), \\
\eta(i) &:= (P^E_{n,k}(i), T P^E_{n,k}(i), \ldots, T^{2^k-1} P^E_{n,k}(i)) \quad \text{for all } i \in \{0, \ldots, 4k-8\}, \\
S &:= \bigcup_{i=0}^{2^k-8} \{ \eta(i) \}, \\
\vartheta &:= (M, T M, \ldots, T^{2^k-2} M) \quad \text{and} \\
\beta &:= \{ \eta(0), \ldots, \eta(4k-8) \setminus [0,1) \setminus S \}.
\end{align*} \]

It is obvious that the sets \( N(i) \) are pairwise disjoint and since the length of \( \sigma_{n,k} \) is \( 2^k \) and \(-2^{k-2} \leq i \leq 2^{k-1}\), it is also clear that the sets \( M(i) \) are pairwise disjoint and that \( \vartheta \) is a Rokhlin tower. We also notice that \( S \cap \vartheta = \emptyset \) and the sets \( \{ \eta(i) \} \) are pairwise disjoint, i.e. \( \beta \) is a partition of \([0,1)\). Using the well known subadditivity for the entropy of finite partitions, Lemma 3.2 and the fact that \( H(\beta) < \log_2(4k) \), we obtain:

\[
H(\alpha(E)^{2^{k-1}-1}_0) \geq H(\alpha(E)^{2^{k-1}-1}_0 \cup \beta) - H(\beta) \\
= \sum_{i=0}^{4^k-8} H_{\eta(i)}(\alpha(E)^{2^{k-1}-1}_0) + H_{\{0,1\} \setminus S}(\alpha(E)^{2^{k-1}-1}_0) - H(\beta) \\
> \sum_{i=0}^{4^k-8} H_{\eta(i)}(\alpha(E)^{2^{k-1}-1}_0) + H_{\vartheta}(\alpha(E)^{2^{k-1}-1}_0) - 2 - \log_2(4k) \\
= \sum_{i=0}^{4^k-8} \sum_{s \in \{0,1\}^{2^k}} f(\mu(A^E_{k-1}(s) \cap \eta(i))) \\
+ \sum_{s \in \{0,1\}^{2^k}} f(\mu(A^E_{k-1}(s) \cap \vartheta)) - 4 - \log_2 k.
\]

Using (5.1), it is easy to see that

\[
\mu(A^E_{k-1}(s) \cap \vartheta) \leq \frac{\mu(\vartheta)}{2^k-2} \quad \text{for all } s \in \{0,1\}^{2^k-1}.
\]

For any \( s \in \{0,1\}^{2^k} \) with \( p_k(s) \leq 2^{k-2} \) it is not difficult to show that

\[
p_{k-1}(s^{2^k-1}_0) = p_k(s) \quad \text{(see also Lemma 1.11 in [2]).}
\]

Using this observation, Lemma 3.5 and the definition of \( \eta(i) \) we obtain

\[
\mu(A^E_{k-1}(s) \cap \eta(i)) \leq \frac{2\mu(\eta(i))}{2^{k-1}/4} \quad \text{for all } s \in \{0,1\}^{2^k-1} \text{ and all } i \in \{0, \ldots, 4k-8\}.
\]

This allows us to apply Lemma 3.6 to conclude that

\[
H(\alpha(E)^{2^{k-1}-1}_0) \geq \sum_{i=0}^{4^k-8} \left( \frac{i-1}{4} - 1 \right) \mu(\eta(i)) + (k-2)\mu(\vartheta) - 4 - \log_2 k
\]
\[ \sum_{i=0}^{4k-8} \frac{i}{4} \mu(|\eta(i)|) + k \mu(|\vartheta|) - 6 - \log_2 k. \]

Furthermore,
\[ \mu(|\eta(i)|) = 2^{k-1} \mu(P_{n,k}^E(i)) = \frac{1}{2} x_{n,k}^E(i) \mu(|\sigma_{n,k}|) \geq \frac{1}{4} x_{n,k}^E(i) \mu(|\tau_n|) \]
and
\[ \mu(|\vartheta|) = 2^{k-2} \mu(\{ x \in B_{n,k} \mid 2^{k-2} \leq p_k(s_k^E(x)) \}) \]
\[ = 2^{k-2} \sum_{i=4k-7}^{4k} \mu(P_{n,k}^E(i)) \geq \frac{1}{8} \sum_{i=4k-7}^{4k} x_{n,k}^E(i) \mu(|\tau_n|). \]

Hence
\[ H(\alpha(E_{0})^{2^{k-1}-1}) \geq \frac{\mu(|\tau_n|)}{16} \sum_{i=0}^{4k-8} i x_{n,k}^E(i) + k \frac{\mu(|\tau_n|)}{8} \sum_{i=4k-7}^{4k} x_{n,k}^E(i) - 6 - \log_2 k. \]
\[ \geq \frac{\mu(|\tau_n|)}{32} \sum_{i=0}^{4k} i x_{n,k}^E(i) - 6 - \log_2 k. \]

This completes the proof.

The following Remark will be useful in the proof of Lemma 5.4 and Theorem 5.7.

**Remark 5.3.** Let \( E \in \mathcal{B} \), \( n, k \in \mathbb{N} \) with \( k < N_n \) and \( x \in B_{n,k} \). Then we have two possibilities:

1. \( x \in B_{n,k+1} \),
2. \( x \in T^2 B_{n,k+1} \).

If in the first case we have \( s_k^E(T^2 x) \neq s_k^E(x)^+ \) then Remark 3.12 implies that \( p_{k+1}(s_{k+1}^E(x)) > 2^{k-1} \). Similarly, if \( s_k^E(T^{-2} x) \neq s_k^E(x)^- \) in the second case, then \( p_{k+1}(s_{k+1}^E(T^{-2} x)) > 2^{k-1} \). Using this observation it is not difficult to show that
\[ x_{n,k}^E(i) \geq x_{n,k+1}^E(i) \text{ for all } i \in \{ 0, \ldots, 4k - 4 \}. \]

Lemma 5.4 shows how the assumption \( \lim_{n \to \infty} \max \{ \mu(A) \mid A \in \alpha(E_{0})^{n-1} \} = 0 \) is needed for the proof of Theorem 5.7. In a completely ergodic system any non-trivial partition will have this property (see Corollary 5.8).

**Lemma 5.4.** If \( 0 < \gamma \leq 1 \) and \( E \in \mathcal{B} \) such that
\[ \lim_{n \to \infty} \max \{ \mu(A) \mid A \in \alpha(E_{0})^{n-1} \} = 0, \]
then we have either
\[ \limsup_{n \to \infty} \frac{H(\alpha(E_{0})^{n-1})}{\log_2 n} > 0, \]

or for all \( \epsilon > 0 \) there exists an \( M \in \mathbb{N} \) such that for all \( n, k \in \mathbb{N} \) with \( M \leq k \leq N_n \) we have
\[ \sum_{i=\gamma n}^{i} x_{n,k}^E(i) < \epsilon \text{ for all } i \in \{ 0, \ldots, 4k \}. \]
**Proof.** Assume that there is an $\varepsilon > 0$ such that for all $M \in \mathbb{N}$ there exist $n, k \in \mathbb{N}$ with $M \leq k \leq N_n$ and
\[
\sum_{l \geq \gamma i} x_{n,k}^E(l) \geq \varepsilon \text{ for some } i \in \{0, \ldots, 4k\}.
\]

Then we need to show that
\[
\limsup_{n \to \infty} \frac{H(\alpha(E)_0^{n-1})}{\log_2 n} > 0.
\]
According to our assumption we can find sequences $\{n_j\}_{j=1}^\infty$, $\{k_j\}_{j=1}^\infty$ and $\{i_j\}_{j=1}^\infty$ such that $k_j \leq N_{n_j}$, $i_j \in \{0, \ldots, 4k_j\}$,
\[
\lim_{j \to \infty} k_j = \infty \quad \text{and} \quad (5.2)
\]
\[
\sum_{l \geq \gamma i_j} x_{n_j,k_j}^E(l) \geq \varepsilon \text{ for all } j \in \mathbb{N}. \quad (5.3)
\]

From the assumption $\lim_{n \to \infty} \max \{\mu(A) \mid A \in \alpha(E)_0^{n-1}\} = 0$ follows easily that for any fixed $i \in \mathbb{N}$ we have
\[
\lim_{j \to \infty} \sum_{l=0}^i x_{n_j,k_j}^E(l) = 0 \quad \text{(see Lemma 3.2 in [2])}.
\]
This shows that
\[
\lim_{j \to \infty} i_j = \infty. \quad (5.4)
\]

We consider now two cases:

**Case 1.** $i_j \geq 4k_j - 8$.

In this case we simply set $m_j := k_j$ and obtain
\[
H(\alpha(E)_0^{2m_j-1}) \geq \frac{\mu(|\tau_{n_j}|)}{32} \sum_{l \geq \gamma i_j} x_{n_j,m_j}^E(l) - 6 - \log_2 m_j \quad \text{(by Lemma 5.2)}
\]
\[
\geq \frac{\mu(|\tau_{n_j}|)}{32} \varepsilon \gamma i_j - 6 - \log_2 m_j \quad \text{(by (5.3))}
\]
\[
\geq \frac{\mu(|\tau_{n_j}|)}{8} \varepsilon \gamma (m_j - 2) - 6 - \log_2 m_j.
\]

**Case 2.** $i_j < 4k_j - 8$.

Here we define $m_j := \lfloor i_j/4 \rfloor + 2$. Then $4(m_j - 2) \leq i_j < 4(m_j - 1)$ and $m_j < k_j$.

Therefore, we can use Remark 5.3 and (5.3) to conclude that
\[
\sum_{l \geq \gamma i_j} x_{n_j,m_j}^E(l) \geq \varepsilon
\]
and with the same estimates as in case 1 we obtain
\[ H(\alpha(E)^{2^{m_j-1}}_0) \geq \frac{\mu(|\tau_{m_j}|)}{8} \varepsilon \gamma (m_j - 2) - 6 - \log_2 m_j. \]

It is clear that \( \lim_{j \to \infty} m_j = \infty \) because of (5.2), (5.4) and the definition of \( m_j \) in cases 1 and 2. Therefore, we finally conclude that
\[
\limsup_{n \to \infty} \frac{H(\alpha(E)_0^{n-1})}{\log_2 n} \geq \limsup_{j \to \infty} \frac{H(\alpha(E)^{2^{m_j-1}}_0)}{m_j - 1} \geq \limsup_{j \to \infty} \left( \frac{\mu(|\tau_{m_j}|)\varepsilon \gamma (m_j - 2) - 6 + \log_2 m_j}{m_j - 1} \right) = \frac{\gamma \varepsilon}{8} > 0.
\]
This completes the proof.

The following two results are important tools in order to make use of the main lemma in Section 4. Combining the statements of Lemma 5.5 and 5.6 we see that for large enough \( n \) we have
\[
I_{\alpha,n}^E(n) - I_{\alpha,K}^E(n) \approx \frac{3}{4} 2^{-K}.
\]
This will allow us to use the main lemma in order to obtain estimates which are closely related to the assumption \( \sum_{n=1}^{\infty} x_n^2 = \infty \) as explained in the introduction to Section 4. We also wish to point out that the proof of Lemma 5.6 is the only place where we use the rank one property of the transformation \( T \).

**Lemma 5.5.** If \( E \in B \) such that
\[
\lim_{n \to \infty} \max \{ \mu(A) \mid A \in \alpha(E)_0^{n-1} \} = 0 \quad \text{and} \quad \limsup_{n \to \infty} \frac{H(\alpha(E)_0^{n-1})}{\log_2 n} = 0,
\]
then there exists an \( M \in \mathbb{N} \) such that for all \( n, K \in \mathbb{N} \) with \( M \leq K \leq N_n \) we have
\[
I_{\alpha,K}^E(n) > \frac{3}{4} 2^{-K}.
\]

**Proof.** It is obvious that for all \( s, r \in \{0,1\}^{2^K} \) we have
\[
\frac{1}{2^K} \|s - r\|_1 = 0 \quad \text{or} \quad \frac{1}{2^K} \|s - r\|_1 \geq 2^{-K}.
\]
Therefore, the definition of \( f_{K,K} \) (notice the equality of the two indeces) implies that
\[
f_{K,K}(s,r) = 0 \quad \text{or} \quad f_{K,K}(s,r) = 2^{-K} \quad \text{for all} \quad s, r \in \{0,1\}^{2^K}.
\]
Using Remark 3.14, it is also not difficult to see that
\[
f_{K,K}(s,r) = 0 \Rightarrow (p_K(s) = p_K(r) \quad \text{or} \quad p_K(s), p_K(r) > 2^{K/4}).
\]
Hence
\[
(\mu_{n,K} \times \mu_{n,K})(\{(x,y) \in B_{n,K} \times B_{n,K} \mid f_{K,K}(s^E_K(x), s^E_K(y)) = 0\}) \leq \sum_{l=0}^{K-1} x^E_{n,K}(l)^2 + \left( \frac{4K}{K} \sum_{l=K}^{x^E_{n,K}(l)} \right)^2.
\]
Since \( \limsup_{n \to \infty} H(\alpha(E)^{n-1}_0)/\log_2 n = 0 \), we can apply Lemma 5.4 with \( \varepsilon = \gamma = 1/4 \) to find an \( M \in \mathbb{N} \) such that for all \( K, n \in \mathbb{N} \) with \( M \leq K \leq N_n \) we have

\[
\sum_{l=K}^{4K} x_{n,K}^E(l) < \frac{1}{4}.
\] (5.6)

Using Lemma 5.4 with \( \varepsilon = 1/4 \) and \( \gamma = 1 \) we may assume that for all \( K, n \in \mathbb{N} \) with \( M \leq K \leq N_n \) we also have

\[
x_{n,K}^E(l) < \frac{1}{4} \text{ for all } l \in \{0, \ldots, K-1\}.
\] (5.7)

Since \( \sum_{l=0}^{4K} x_{n,K}^E(l) = 1 \), we can use (5.6) and (5.7) to conclude that

\[
\sum_{l=0}^{K-1} x_{n,K}^E(l) + \left( \sum_{l=K}^{4K} x_{n,K}^E(l) \right)^2 < \frac{1}{4}.
\]

Hence

\[
(\mu_{n,K} \times \mu_{n,K})(\{(x,y) \in B_{n,K} \times B_{n,K} | f_{K,K}(s_N^E(x), s_N^E(y)) > 0\}) > \frac{3}{4}.
\]

Using (5.5), we obtain

\[
I_{K,K}^E(n) > \frac{3}{4} 2^{-K},
\]

and the proof is complete.

**Lemma 5.6.** For all \( E \in \mathcal{B} \) and \( K \in \mathbb{N} \) we have

\[
\lim_{n \to \infty} I_{N_n,K}^E(n) = 0.
\]

**Proof.** We recall that by \( m_n \) we denoted the length of \( \tau_n \). For \( n \in \mathbb{N} \) and \( \varepsilon > 0 \) we define

\[
J_n := \int_{B_{n,N_n} \times B_{n,N_n}} \frac{1}{m_n} \| r_{m_n}^E(x) - r_{m_n}^E(y) \| \, d(\mu_{n,N_n} \times \mu_{n,N_n}),
\]

\[
H_n := \int_{B_{n,N_n} \times B_{n,N_n}} \frac{1}{2N_n} \| s_{N_n}^E(x) - s_{N_n}^E(y) \| \, d(\mu_{n,N_n} \times \mu_{n,N_n}) \text{ and}
\]

\[
\nu_n(\varepsilon) := \#\{i \in \{0, \ldots, m_n - 1\} | \mu_{n,N_n}(E \cap T^i B_{n,N_n}) > 1 - \varepsilon \text{ or } \mu_{n,N_n}(E \cap T^i B_{n,N_n}) < \varepsilon \}.
\]

The definition \( B_{n,N_n} \) implies that \( B_{n,N_n} = B_n \) and this means that \( \sigma_{n,N_n} \) is a subtower of \( \tau_n \) which consists of the first \( 2N_n \) levels of \( \tau_n \). This observation is very important, because it allows us to apply the Lebesgue density theorem (or the rank one property of \( T \)) to the sets \( T^i B_{n,N_n} \) which is not possible for \( B_{n,k} \) with \( k < N_n \). As a consequence we obtain

\[
\lim_{n \to \infty} \frac{\nu_n(\varepsilon)}{m_n} = 1 \text{ for all } \varepsilon > 0.
\]
It is not difficult to show that this implies (see Lemma 3.4 in [3])
\[
\lim_{n \to \infty} (\mu_n \times \mu_{n,N})(\{(x, y) \in B_n \times B_n \mid \|r_{m_n}(x) - r_{m_n}(y)\|_1 < \varepsilon m_n\}) = 1
\]
for all \( \varepsilon > 0 \) and therefore \( \lim_{n \to \infty} J_n = 0 \). Since \( H_n \leq m_n J_n / 2^{N_n} \leq 2 J_n \), it follows that \( \lim_{n \to \infty} H_n = 0 \). The definition of \( I_{N_n,K}^E(n) \) implies trivially that \( I_{N_n,K}^E(n) \leq H_n \) for all \( n, K \in \mathbb{N} \) with \( K \leq N_n \). So for a given \( K \in \mathbb{N} \) we have
\[
\lim_{n \to \infty} I_{N_n,K}^E(n) = 0,
\]
and the proof is complete.

**Theorem 5.7.** Let \( g : (0, \infty) \to \mathbb{R} \) be a positive concave function such that
\[
\int_1^\infty \frac{g(x)^2}{x^3} \, dx < \infty.
\]
If \( E \in B \) with \( \lim_{n \to \infty} \max\{\mu(A) \mid A \in \alpha(E)^{n-1}\} = 0 \), then
\[
\limsup_{n \to \infty} \frac{H(\alpha(E)_0^{n-1})}{g(\log_2 n)} = \infty.
\]

**Proof.** First we notice that it is sufficient to show that
\[
\limsup_{n \to \infty} \frac{H(\alpha(E)_0^{n-1})}{g(\log_2 n)} > 0, \tag{5.8}
\]
because for a given \( g \) it is not difficult to find a positive concave function \( h \) with \( \int_1^\infty h(x)^2/x^3 \, dx < \infty \) and
\[
\lim_{x \to \infty} \frac{h(x)}{g(x)} = \infty.
\]
Since \( h \) satisfies the same conditions as \( g \), statement (5.8) is also true for \( h \) and therefore
\[
\limsup_{n \to \infty} \frac{H(\alpha(E)_0^{n-1})}{g(\log_2 n)} = \limsup_{n \to \infty} \frac{H(\alpha(E)_0^{n-1})}{h(\log_2 n)} \lim_{n \to \infty} \frac{h(\log_2 n)}{g(\log_2 n)} = \infty.
\]
The concavity of \( g \) implies that \( g(x)/x \) is monotone decreasing and this shows that \( \lim_{x \to \infty} g(x)/x \) exists. If we had \( \lim_{x \to \infty} g(x)/x > 0 \), then
\[
\int_1^\infty \frac{g(x)^2}{x^3} \, dx \geq \left( \lim_{x \to \infty} \frac{g(x)}{x} \right)^2 \int_1^\infty \frac{1}{x} \, dx = \infty.
\]
Since this is a contradiction to our assumption, we conclude that
\[
\lim_{x \to \infty} \frac{g(x)}{x} = 0 \quad \text{or equivalently} \quad \lim_{x \to \infty} \frac{x}{g(x)} = \infty.
\]
We will use this observation to show that we may assume
\[
\limsup_{n \to \infty} \frac{H(\alpha(E)_0^{n-1})}{\log_2 n} = 0. \tag{5.9}
\]
If this were not the case, then
\[
\limsup_{n \to \infty} \frac{H(\alpha(E)^{n-1})}{g(\log_2 n)} = \limsup_{n \to \infty} \frac{H(\alpha(E)^{n-1})}{\log_2 n} \lim_{n \to \infty} \frac{\log_2 n}{g(\log_2 n)} = \infty.
\]
So the statement of the theorem would in this case be true and therefore we may indeed assume that \((5.9)\) is satisfied. According to Lemma 5.5 we can find an \(M \in \mathbb{N}\) such that for all \(n, K \in \mathbb{N}\) with \(M \leq K \leq N_n\) we have
\[
I^{E}_{K,K}(n) > \frac{3}{4} 2^{-K}.
\]
Furthermore, we may assume \(M\) to be so large that
\[
\sum_{k=M}^{\infty} \frac{32}{2^{k/3}} < \frac{1}{16}.
\]
Given a \(K > M\), we can use Lemma 5.6 to find an \(n \in \mathbb{N}\) (depending on \(K\)) with \(4K < N_n\) and
\[
I^{E}_{K,n,K}(n) < \frac{1}{8} 2^{-K}.
\]
Using Lemma 4.3, we obtain
\[
\frac{5}{8} 2^{-K} < I^{E}_{K,n,K}(n) - I^{E}_{K,n,K}(n) = \sum_{k = K}^{N_n - 1} I^{E}_{k,K}(n) - I^{E}_{k+1,K}(n)
\]
\[
< \frac{2}{2K} \sum_{k = K}^{N_n - 1} \left( \sum_{l=0}^{k} x^{E}_{n,k}(l) - x^{E}_{n,k+1}(l) \right) \left( \sum_{l=k+1}^{4k+4} x^{E}_{n,k}(l) \right) + \frac{2}{2K} \sum_{l=M}^{\infty} \frac{32}{2^{l/3}}.
\]
Hence
\[
\frac{1}{4} < \sum_{k = K}^{N_n - 1} \left( \sum_{l=0}^{k} x^{E}_{n,k}(l) - x^{E}_{n,k+1}(l) \right) \left( \sum_{l=k+1}^{4k+4} x^{E}_{n,k+1}(l) \right).
\]
This discussion shows that we can find sequences \(\{K_i\}_{i=1}^{\infty}\) and \(\{n_i\}_{i=1}^{\infty}\) such that for all \(i \in \mathbb{N}\) we have \(4K_i < N_{n_i} < K_{i+1}/4\) and
\[
\frac{1}{4} < \sum_{k = K_i}^{N_{n_i} - 1} \left( \sum_{l=0}^{k} x^{E}_{n_i,k}(l) - x^{E}_{n_i,k+1}(l) \right) \left( \sum_{l=k+1}^{4k+4} x^{E}_{n_i,k+1}(l) \right). \tag{5.10}
\]
For all \(i \in \mathbb{N}\) we define
\[
a_i := \log_2 K_i, \\
b_i := \log_2 N_{n_i}, \\
c_i := \lfloor b_i - a_i \rfloor, \\
d_i := \frac{b_i}{c_i} \\
and \\
\nu_i(j) := \left\lceil 2^{n_i + d_i j} \right\rceil \quad \text{for all } j \in \{0, \ldots, c_i\}.
\]
We notice that \(\nu_i(0) = K_i\) and \(\nu_i(c_i) = N_{n_i}\). Since \(4K_i < N_{n_i}\), we have \(d_i < 3/2\) and therefore
\[
\nu_i(j + 1) < 2^{d_i} \nu_i(j) + 2^{d_i} < 4 \nu_i(j) - 4 \quad \text{for all } j \in \{0, \ldots, c_i - 1\}.
\]
Now we can use Remark 5.3 to conclude that for all $k \in \{\nu_i(j), \ldots, \nu_i(j+1) - 1\}$ we have
\begin{equation}
0 \leq \sum_{l=0}^{k} x_{n_i,k}^E(l) - x_{n_i,k+1}^E(l) \leq \sum_{l=0}^{\nu_i(j+1) - 1} x_{n_i,k}^E(l) - x_{n_i,k+1}^E(l) \quad \text{and} \quad (5.11)
\end{equation}
\begin{equation}
\sum_{l=0}^{\nu_i(j)} x_{n_i,k}^E(l) \geq \sum_{l=0}^{\nu_i(j)} x_{n_i,k+1}^E(l). \quad (5.12)
\end{equation}
Since $\sum_{l=0}^{4k} x_{n_i,k}^E(l) = 1$ for all $k \leq N_{n_i}$, inequality (5.12) implies that for all $k \in \{\nu_i(j), \ldots, \nu_i(j+1)\}$ we have
\begin{equation}
\sum_{l=\nu_i(j)+1}^{4k} x_{n_i,k}^E(l) \leq \sum_{l=\nu_i(j)+1}^{4\nu_i(j+1)} x_{n_i,\nu_i(j+1)}^E(l). \quad (5.13)
\end{equation}
Using (5.10), (5.11) and (5.13) we obtain
\begin{equation}
\frac{1}{4} \leq \sum_{j=0}^{c_i - 1} \sum_{k=\nu_i(j)}^{\nu_i(j+1)-1} \left( \sum_{l=0}^{k} x_{n_i,k}^E(l) - x_{n_i,k+1}^E(l) \right) \left( \sum_{l=0}^{4\nu_i(j+1)} x_{n_i,k+1}^E(l) \right) \quad (5.15)
\end{equation}
So if for all $i \in \mathbb{N}$ and $j \in \{0, \ldots, c_i - 1\}$ we define
\begin{equation}
y_i(j) := \sum_{l=\nu_i(j)+1}^{4\nu_i(j+1)} x_{n_i,\nu_i(j+1)}^E(l),
\end{equation}
then
\begin{equation}
\frac{1}{4} \leq \sum_{j=0}^{c_i - 1} y_i(j)^2. \quad (5.14)
\end{equation}
According to Lemma 5.2 we have
\begin{equation}
H(\alpha(E)_{0}^{2\nu_i(j+1)-1}) \geq \frac{M(|\tau_{n_i}|)}{32} \sum_{l=0}^{4\nu_i(j+1)} l x_{n_i,\nu_i(j+1)}^E(l) - 6 - \log_2 \nu_i(j+1) \geq \frac{M(|\tau_{n_i}|)}{32} \nu_i(j) y_i(j) - 6 - \log_2 \nu_i(j+1). \quad (5.15)
\end{equation}
W.l.o.g. we may assume that \( \lim_{x \to \infty} \log_2 x/g(x) = 0 \), because

\[
\int_1^\infty \frac{(\log_2 x)^2}{x^3} \, dx < \infty.
\]

In fact the function \( g \) typically converges to \( \infty \) like for example \( x/\ln x \) which is much faster than \( \log_2 x \) (to be more precise, we could use a similar argument as for proving the sufficiency of (5.8)). Hence

\[
\limsup_{n \to \infty} \frac{H(\alpha(E)_0^{n-1})}{g(\log_2 n)} \geq \limsup_{i \to \infty} \max_{0 \leq j < \nu_i} \frac{H(\alpha(E)_0^{2^{\nu_i(j+1)-1}})}{g(\nu_i(j+1)-1)} \\
\geq \limsup_{i \to \infty} \max_{0 \leq j < \nu_i} \frac{\nu_i(j) y_i(j)}{32 g(\nu_i(j+1)-1)} \quad \text{(by (5.15)).}
\]

So in order to prove (5.8), we only need to show that

\[
\limsup_{i \to \infty} \max_{0 \leq j < \nu_i} \frac{\nu_i(j) y_i(j)}{g(\nu_i(j+1)-1)} > 0. \tag{5.16}
\]

Let us assume that this \( \limsup \) is equal to zero. Then we can find an \( L \in \mathbb{N} \) such that for all \( i \geq L \) and all \( j \in \{0, \ldots, c_i-1\} \) we have

\[
y_i(j) < \frac{g(\nu_i(j+1)-1)}{\nu_i(j)}.
\]

Using the fact that \( g(x)/x \) is monotone decreasing and the estimate \( d_i < 3/2 \) (see above), we obtain

\[
y_i(j)^2 < 16 \frac{g(\nu_i(j+1)-1)^2}{(\nu_i(j)+1)^2} \leq 16 \frac{(\nu_i(j)+1)^2}{(\nu_i(j)+1)^2} \leq 16 \frac{g(2^{\nu_i(j)+d_i})}{2^{2(\nu_i(j)+d_i)}} \leq 16 \int_{j-1}^{j} \frac{g(2^{\nu_i(j)+d_i})}{2^{2(\nu_i(j)+d_i)}} \, dx = \frac{16}{d_i \ln 2} \int_{2^{\nu_i(j)+d_i}}^{2^{\nu_i(j)+d_i+1}} \frac{g(u)^2}{u^3} \, du.
\]

Now we apply (5.14) and the assumption \( N_{n_i} < K_{i+1}/4 \) to conclude that

\[
\frac{1}{4} < \sum_{j=0}^{\nu_i(j)-1} y_i(j)^2 < 32 \int_{2^{\nu_i(j)-d_i}}^{2^{\nu_i(j+1)-d_i}} \frac{g(x)^2}{x^3} \, dx \\
\leq 32 \int_{K_i/4}^{N_{n_i}} \frac{g(x)^2}{x^3} \, dx \leq 32 \int_{N_{n_i-1}}^{N_{n_i}} \frac{g(x)^2}{x^3} \, dx.
\]

It follows that

\[
\int_1^\infty \frac{g(x)^2}{x^3} \, dx \geq \sum_{i=2}^\infty \int_{N_{n_i-1}}^{N_{n_i}} \frac{g(x)^2}{x^3} \, dx = \infty.
\]

Since this is a contradiction, we have shown (5.16) and the proof is complete.

**Corollary 5.8.** Assume that \( (0,1), T \) is completely ergodic and \( g: (0,\infty) \to \mathbb{R} \) is a positive concave function such that

\[
\int_1^\infty \frac{g(x)^2}{x^3} \, dx < \infty.
\]
If $E \in \mathcal{B}$ with $0 < \mu(E) < 1$, then
\[
\limsup_{n \to \infty} \frac{H(\alpha(E)^n_0^{-1})}{g(\log_2 n)} = \infty.
\]

6. Some Related Questions It is an open question whether the statement of Theorem 5.7 is the best possible one, but we believe that this is indeed the case. It should be possible to prove this by using the von Neumann-Kakutani adding machine. More precisely, given a $g$ which violates the assumption of the integrability of $g(x)^2/x^3$, there should be an $E \in \mathcal{B}$ with
\[
\lim_{n \to \infty} \max \{ \mu(A) \mid A \in \alpha(E)^n_0^{-1} \} = 0 \quad \text{and} \quad 
\limsup_{n \to \infty} \frac{H(\alpha(E)^n_0^{-1})}{g(\log_2 n)} < \infty,
\]
where the refinements are generated by using the adding machine.

Another question is whether Theorem 5.7 can be generalized to infinite rank transformations. We believe that this is also possible in the sense that the rate $g$ would depend on the growth rate of the number of towers in the construction of $T$. For infinite rank transformations the rates $g$ would typically be slower than the rates in Theorem 5.7.

Finally we wish to remark that as a consequence of Theorem 5.7 we see that some of the transformations constructed in Chapter 5 of [3] are not rank one transformations. This is so, because these transformations were designed to demonstrate that the statement of Theorem 1.1 is in general the best possible one, i.e. we generated convergence rates that were slower than the statement of Theorem 5.7 would allow.

REFERENCES

[1] Robert B. Ash, "Real Analysis and Probability", Academic Press, San Diego, 1972.
[2] F. Blume, Possible Rates of Entropy Convergence, Ergodic Theory and Dynamical Systems, 17 (1997), 45–70.
[3] F. Blume, The Rate of Entropy Convergence, Doctoral Dissertation, University of North Carolina at Chapel Hill, 1995.
[4] S. Ferenczi, Systeme de Rang Fini, These de Doctorat, Universite d’Aix-Marseille 2, 1990.

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