Approximation by Lipschitz, analytic maps on certain Banach spaces

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Abstract. We show that on separable Banach spaces admitting a separating polynomial, any uniformly continuous, bounded, real-valued function can be uniformly approximated by Lipschitz, analytic maps on bounded sets.

1. Introduction

1.1. History of the Problem. Investigating the uniform approximation of continuous functions by smooth functions has a long history (‘function’ in this paper shall mean real-valued function). For continuous functions $f$ defined on closed intervals in $\mathbb{R}$, Weierstrass’s well known classical theorem asserts that $f$ can be uniformly approximated by polynomials. For continuous functions $f$ on open subsets of $\mathbb{R}^n$, H. Whitney showed in the famous work [20] that $f$ can be uniformly approximated by (real) analytic maps.

For infinite dimensional (real) Banach spaces, the situation is far more difficult. In fact, it was proven in [18] that even on the closed unit ball of separable Hilbert space, $C^\infty$-smooth functions cannot generally be approximated by polynomials (see below for the relevant definitions). Nevertheless, for certain Banach spaces Whitney’s result can be extended to infinite dimensions. A classical and often cited theorem of Kurzweil [15] states in particular that if $X$ is a separable Banach space admitting a separating polynomial, then any continuous function on $X$ can be uniformly approximated by analytic maps on $X$. Examples of such $X$ include $l_p$ or $L_p$, with $p$ an even integer.

Most subsequent work concerning the uniform approximation of continuous functions on Banach spaces focused on approximation by (merely) $C^p$-smooth functions rather than analytic maps (see e.g., [5], [9]), the primary reason for this being the ability to employ $C^p$-smooth partitions of
unity in this context, which cannot be used in the analytic case. As a consequence, the number of results on analytic approximation in this area are few.

However, over the last ten years two important papers have appeared. In [6] it is shown that in $l_p$ or $L_p$, with $p$ an even integer, that any equivalent norm can be uniformly approximated on bounded sets by analytic norms. In [7] it is proven for example, that in $X = c_0$ or $X = C(K)$, with $K$ a countable compact, that any equivalent norm can be uniformly approximated by analytic norms on $X \setminus \{0\}$.

An independent line of investigation concerning approximation by Lipschitz, $C^p$-smooth functions was recently undertaken in a series of papers including [11], [12], [1], and [13]. For example, in [1] it is shown in particular that on separable Banach spaces admitting Lipschitz, $C^p$-smooth bump functions, that any bounded, uniformly continuous function can be uniformly approximated by Lipschitz, $C^p$-smooth functions. This result was then generalized to weakly compactly generated Banach spaces in [13]. In terms of smoothness class ($C^p$, $C^\omega$, etc.), the present paper in some sense completes this programme. These results have found applications in the area of deleting diffeomorphisms on Banach spaces [3], and in variational principles on Riemannian manifolds [2]. For superreflexive spaces, we note that the approximation of Lipschitz functions by Lipschitz, $C^1$-smooth maps can be accomplished via convolution techniques (see, e.g., [4], [16]).

The motivation for this article was to see if the recent results on Lipschitz, smooth approximation could be achieved in the context of analytic approximation in the spirit of Kurzweil’s work mentioned above. We give a positive solution to this problem for bounded and uniformly continuous functions on bounded sets in separable spaces admitting a separating polynomial. We remark that the uniform continuity is a necessary condition here. To our knowledge, this is the only result on approximation (of general functions as opposed to norms) by Lipschitz, analytic functions on infinite dimensional spaces.

Specifically, we establish,

**Theorem 1.** Let $X = (X, \| \cdot \|_X)$ be a separable, real Banach space that admits a separating polynomial, let $G \subset X$ be a bounded open set, and let $F : G \to \mathbb{R}$ be bounded and uniformly continuous. Then for each $\varepsilon > 0$ there is a real analytic function $K : G \to \mathbb{R}$ which is Lipschitz on $G$ and such that $|F(x) - K(x)| < \varepsilon$ for all $x \in G$.

**1.2. Basic Definitions.** Our notation is standard, with $X$ denoting a Banach space, and an open ball with centre $x$ and radius $r$ denoted $B_r(x)$, and the boundary of the unit ball in the space $X$ is denoted by $S_X$. Function shall always mean real-valued function. If $\{f_j\}_j$ is a sequence of Lipschitz functions on $X$, then we will at times say this family is uniform Lipschitz (UL) if there is a common Lipschitz constant for all $j$. A homogeneous polynomial of degree $n$ is a map, $P : X \to \mathbb{R}$, of the form $P(x) = A(x, x, ..., x)$,
where $A : X^n \to \mathbb{R}$ is $n$-multilinear and continuous. For $n = 0$ we take $P$ to be constant. A polynomial of degree $n$ is a sum $\sum_{i=0}^{n} P_i(x)$, where the $P_i$ are $i$-homogeneous polynomials.

Let $X$ be a Banach space, and $G \subset X$ an open subset. A function $f : G \to \mathbb{R}$ is called analytic if for every $x \in G$, there are a neighbourhood $N_x$, and homogeneous polynomials $P_n^x : X \to \mathbb{R}$ of degree $n$, such that

$$f(x + h) = \sum_{n \geq 0} P_n^x(h) \text{ provided } x + h \in N_x.$$ 

Further information on polynomials may be found, for example, in [SS].

For a Banach space $X$, we define its complexification $\tilde{X}$ in the standard way. That is, $\tilde{X} = X \oplus iX$ with norm

$$\|z\|_{\tilde{X}} = \|x + iy\|_X = \sup_{0 \leq \theta \leq 2\pi} \|\cos \theta x - \sin \theta y\|_X.$$ 

If $q(x)$ is a polynomial on $X$, there is a natural extension of $q(x)$ to a polynomial $\tilde{q}(z) = \tilde{q}(x + iy)$ on $\tilde{X}$ where for $y = 0$ we have $\tilde{q} = q$. For more information on complexification (and polynomials) we recommend [17].

We define $\tilde{c}_0 = \{(z_j) : z_j \in \mathbb{C}, \ |z_j| \to 0\}$, with norm $\|z\|_{\tilde{c}_0} = \|(z_j)\|_{\tilde{c}_0} = \max_j \{|z_j|\}$, and a similar definition for $\tilde{l}_\infty$. In the sequel, all extensions of functions from $X$ to $\tilde{X}$, as well as subsets of $\tilde{X}$, will be embellished with a tilde.

The proof of Theorem 1 is broken up into several sections and lemmas which we now present.

2. Preliminary Results

2.1. An extension of the Preiss norm. As developed in [10], there is a an analytic norm on $c_0$ (hereafter referred to as the Preiss norm) that is equivalent to the canonical supremum norm. Let us recall the construction. We may define this equivalent norm $\|\cdot\|$ as follows. Let $C : c_0 \to \mathbb{R}$ be given by $C(\{x_n\}) = \sum_{n=1}^{\infty} (x_n)^{2n}$. Let $W = \{x \in c_0 : C(x) \leq 1\}$. Then $\|\cdot\|$ is the Minkowski functional of $W$; that is, $\|x\|$ is the solution for $\lambda$ to $C(\lambda^{-1}x) = 1$. The Preiss norm is analytic at all non-zero points in $c_0$. To see this, let us define the function $\tilde{C} : V \to \mathbb{C}$ by $\tilde{C}(\{z_n\}) = \sum_{n=1}^{\infty} (z_n)^{2n}$ where $V$ is the subset of $\tilde{l}_\infty$, for which the series converges. Then $\tilde{C}$ is analytic at each $z \in \tilde{c}_0$. Indeed, the partial sums are analytic as a consequence of the analyticity of the projection functions $p_j(\{z_i\}) = z_j$, whose local differentiability is easily shown by a direct calculation. Since the series in the definition of $\tilde{C}$ converges locally uniformly at each $z \in \tilde{c}_0$ the analyticity of $\tilde{C}$ on $\tilde{c}_0$ follows. Also, for $z \in \tilde{c}_0$ sufficiently close to $c_0$ we have for $\lambda \in \mathbb{C} \setminus \{0\}$, $\frac{\partial \tilde{C}(\lambda^{-1}z)}{\partial \lambda} \neq 0$, hence one can apply the complex Implicit Function Theorem (see e.g., [8] page 265, where the real result for Banach spaces is easily extended to the analytic case) to $F(z, \lambda) = \tilde{C}(\lambda^{-1}z) - 1$ to obtain a unique
The function \( \lambda \) is defined on \( U \), \( 0 < c < U \) (real) analytic on \( U \), \( 0 \in \lambda (c) \) and \( j > j_0 \) such that \( y \in \lambda (c) \). Observe that \( 0 \in \lambda (c) \) and \( j > j_0 \) imply \( |y| < 3/4 \). Also, it is convenient to introduce a neighborhood \( N_1 \supset N_0 \) and a \( j_1 \) such that \( y \in \lambda (c) \). Observe that \( \overline{G}(\{y_j\}) \) converges uniformly on \( N_1 \). So \( \overline{G}(\{y_j\}) \) has a complex derivative on \( N_1 \), and it follows that \( C \) is (real) analytic on \( U \).

Now define \( F(\{z_j\}, \lambda) = F(z, \lambda) = \overline{G}(\lambda^{-1}z) - 1 \) as above, here on \( N_0 \times L \), where \( L \subset \mathbb{C} \setminus \{0\} \) is such that \( \lambda \in L \Rightarrow |\lambda - 1| \leq c \), for some fixed \( 0 < c < 1/8 \). Then \( F(z, \lambda) \) is (complex) analytic on \( N_0 \times L \) since \( \lambda^{-1}z \) is in \( N_1 \), and \( \overline{G}^{(z, \lambda)}(z, \lambda) = -\sum_{n=1}^{\infty} 2n (\lambda^{-1}z_n)^{2n} \lambda^{-1} \). Clearly if \( 0 \neq z \in N_0 \cap U \subset l_\infty \), then this last expression is not zero, and so by continuity of \( \overline{G}^{(z, \lambda)}(z, \lambda) \), choosing \( N_0 \) and \( c \) smaller if necessary, we have \( \overline{G}^{(z, \lambda)}(z, \lambda) \neq 0 \). Hence we may apply the complex Implicit Function Theorem to the equation \( F(z, \lambda) = \overline{G}(\lambda^{-1}z) - 1 \) to obtain a unique complex analytic solution \( \lambda(z) = \lambda(\{z_j\}) \) to \( \sum_{n=1}^{\infty} (\lambda^{-1}z_n)^{2n} = 1 \) on \( N_0 \times L \subset l_\infty \times \mathbb{C} \setminus \{0\} \). It follows that \( \lambda \) restricts to a (real) analytic function on \( U \). Easily \( \lambda|_{c_0} = ||\cdot|| \).

The function \( \lambda \) from the lemma, being the solution to \( \sum_{n=1}^{\infty} (\lambda^{-1}x_n)^{2n} = 1 \), is the Minkowski functional of \( S \), where

\[
S = \left\{ \{x_j\} \in U : C(\{x_j\}) = \sum_{n=1}^{\infty} (x_n)^{2n} \leq 1 \right\} \subset U.
\]

Observe that \( 0 \in \text{interior}(S) \). Now because the function \( C(\{x_j\}) \) is convex on \( U \), \( S \) is convex, and it follows that if \( x, y \in U \) with \( x + y \in U \), then \( \lambda(x + y) \leq \lambda(x) + \lambda(y) \). This shows that for such \( x, y \in U \), we have the 1-Lipschitz property, \( |\lambda(x) - \lambda(y)| \leq \lambda(x - y) \). Also, \( S \) is balanced, and hence for any \( x \in U \) and \( a \in \mathbb{R} \) with \( ax \in U \), we have in this case that
\[ \lambda(ax) = |a| \lambda(x). \] Next, if \( x = \{x_j\} \in U \), then from the argument establishing bounds for the Preiss norm, we also have here that \( \lambda(\{x_j\}) \geq \|x\|_\infty \). By the homogeneity property, this last inequality holds for any \( x \in U \) and \( a \in \mathbb{R} \) so that \( ax \in U \). Similarly, for \( x \in U \) we have as before that \( \frac{1}{2} \lambda(\{x_j\}) \leq \|x\|_\infty \), with this estimate holding for any \( x \in U \) and \( a \in \mathbb{R} \) so that \( ax \in U \). We use these estimates in the sequel.

2.2. Polynomials. Let \( X \) be a Banach space. A \textit{separating polynomial} on \( X \) is a polynomial \( q \) on \( X \) such that \( 0 = q(0) < \inf \{|q(x)| : x \in S_X\} \). It is known [FPWZ] that if \( X \) is superreflexive and admits a \( C^\infty \)-smooth bump function then \( X \) admits a separating polynomial. The following lemma makes precise, observations of Kurzweil in [K].

**Lemma 2.** Let \( X \) be a real Banach space with norm \( \|\cdot\|_X \) and suppose that there is a separating polynomial \( p \) of degree \( n \) on \( X \). Then there is a polynomial \( q \) on \( X \) such that \( \|y\|^{2n}_X \leq q(y) \) for all \( y \) in \( X \) with \( q(y) < 1 \). Furthermore, there is a constant \( K_1 > 0 \) such that \( q(y) \leq K_1 \max\{\|y\|_X,\|y\|^{2n}_X\} \) for all \( y \) in \( X \).

**Proof.** Let \( p = p_1 + p_2 + \cdots + p_n \), where \( p_i \) is \( i \)-homogeneous for \( 1 \leq i \leq n \). Define \( q = \sum_{i=1}^n q_i \), where \( q_i = p_i^2 \) is \( 2i \)-homogeneous for \( 1 \leq i \leq n \). Then there is some \( \eta > 0 \) such that \( q(x) \geq \eta \) for all \( x \in S_X \). By scaling, we may assume that \( \eta = 1 \). Suppose that \( y \in X \) satisfies \( \|y\|_X < 1 \). Then \( y = \alpha x \) where \( |\alpha| < 1 \) and \( x \in S_X \). We compute

\[
q(y) = q(\alpha x) = q_1(\alpha x) + q_2(\alpha x) + \cdots + q_n(\alpha x)
= \alpha^2 q_1(x) + \alpha^4 q_2(x) + \cdots + \alpha^{2n} q_n(x)
\geq \alpha^{2n} q(x) = \|y\|^{2n}_X q(x) \geq \|y\|^{2n}_X.
\]

Now, suppose that \( y = \alpha x \) and \( x \in S_X \) with no constraint on \( \alpha \), and that \( q(y) < 1 \). Then \( \alpha^{2i} q_i(x) < 1 \) for all \( i \). Were \( \alpha \geq 1 \) we would have \( 1 > q(y) \geq q(x) \) contradicting \( q(x) \geq 1 \). Thus \( \alpha < 1 \) and we obtain \( \|y\|^{2n}_X \leq q(y) \). For the second statement, let \( q_i(x) = M_i(x, x, \ldots, x) \) for \( 1 \leq i \leq n \), where \( M_i \) is a bounded and \( 2i \)-homogeneous multilinear functional for each \( i \). Then if \( y = \alpha x \) for \( x \in S_X \) we have

\[
q_i(y) = q_i(\alpha x) = \alpha^{2i} M_i(x, x, \ldots, x) \leq \alpha^{2i} A_i
\]
where \( \sup\{M_i(x, x, \ldots, x) : x \in B_X\} = A_i \). So

\[
q(y) = \sum_{i=1}^n q_i(y) \leq \sum_{i=1}^n A_i \|y\|^{2i}_X.
\]

Let \( K = \sum_{i=1}^n A_i \). Observing that \( \|y\|^{2i}_X \leq \max\{\|y\|_X,\|y\|^{2n}_X\} \) for each \( i \), the required inequality follows. \( \square \)
3. Main Results

Let us remark that if $F$ is uniformly approximated by Lipschitz functions, then a necessary condition on $F$ is that it be uniformly continuous. Also, if the norm on $X$ can be uniformly approximated by analytic functions, then it is easy to show that $X$ admits a $C^\infty$-smooth bump function. Finally, we observe that if $G$ is star-shaped as well as bounded, then $F$ is necessarily bounded on $G$ as it is uniformly continuous.

Without loss of generality, for the remainder of this paper we may and do assume that $0 \in G$. Let $q$ be the polynomial constructed in Lemma 2. We extend $q$ to a complex analytic map $\tilde{q}$ on $\tilde{X}$. Let $M \geq 1$ be such that $q < M$ on $G - G$. Such an $M$ exists because $G$ is bounded. Note that being the extension of a polynomial to $\tilde{X}$, $\tilde{q}$ is Lipschitz on bounded neighbourhoods of $G - G$ in $\tilde{X}$. For the remainder of the paper, let us fix such a bounded neighbourhood $\tilde{G} - G$ where $\tilde{G} \subset \tilde{X}$ is a bounded neighbourhood containing $G$, and choose $\tilde{M}$ so that $|\tilde{q}| \leq \tilde{M}$ on $\tilde{G} - G$. Also, fix $R > 1$ so that $G \subset B_R(0)$.

Since $F$ is bounded and real-valued, by composing with an appropriate first order polynomial, we may suppose without loss of generality that $1 \geq F \geq 1/3$ on $G$. Fix $\varepsilon \in (0, 1/4)$ for the rest of the proof, and fix $\delta > 0$ so that by uniform continuity, for $x, y \in G$ we have

$$\|x - y\|_X < \delta \text{ implies } |F(x) - F(y)| < \varepsilon.$$ 

Let $\gamma_1 < \gamma_2 < \gamma_3$ be positive with $\gamma_3 < 1$ so small that $q(y) < \gamma_3$ implies $\|y\|_X < \delta$ and such that $3\gamma_1 < \gamma_2/2 < 1$. Using separability, let $\{x_j\}_{j=1}^\infty$ be a dense subset of $G$, and define three open coverings of $G$ using the sets

$$C_j^i = \{x \in X : q(x - x_j) < \gamma_i\},$$

for $j \in \mathbb{N}$ and $i = 1, 2, 3$. That these form open covers follows from Lemma 2.

3.1. The functionals $\varphi_n$. The purpose of this section is to establish the following key lemma. We use the notation established above.

**Lemma 3.** For $\eta > 0$, there exists a sequence of complex analytic maps $\varphi_n : \tilde{X} \to \mathbb{C}$ with the following properties:

(i). The collection $\{\varphi_n|_X\}$ is Uniformly Lipschitz (UL) on $G$, with Lipschitz constant independent of $\eta$.

(ii). For each $x_0 \in G$, there exists $n$ with $\varphi_n(x_0) > 1/2$.

(iii). For each $x_0 \in G$, there exist $\delta > 0$ and $n_0 > 1$ such that for $\|z\|_X < \delta$ and $n > n_0$ we have

$$|\varphi_n(x_0 + z)| < \eta.$$
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Let \( b \in C^\infty(\mathbb{R}, [0, 1]) \) such that \( b(t) = 1 \) iff \( t \notin (2\gamma_1, M + 2) \), and \( b(t) = 0 \) iff \( t \in [3\gamma_1, M + 1] \). Now define \( b_n : l^\infty_n \to [0, 1] \) by \( b_n(y_1, ..., y_n) = 1 - \| (b(y_1), ..., b(y_n)) \|_\infty \). Then \( \text{support}(b_n) = A_n \), and \( b_n = 1 \) on \( A'_n \). Moreover, \( b_n \) is Lipschitz with constant \( L_b \) independent of \( n \).

Define \( \nu_n : l^\infty_n \to \mathbb{R} \) by

\[
\nu_n(x) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j}(x_j - y_j)^2} \, dy,
\]

with

\[
T_n = \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j}y_j^2} \, dy
\]

\[
= \frac{1}{\kappa_n^{n/2}} \int_{\mathbb{R}^n} e^{-\sum_{j=1}^n 2^{-j}y_j^2} \, dy
\]

\[
= \frac{1}{\kappa_n^{n/2}} \hat{T}_n,
\]

and where the constants \( \kappa_n^{n/2} \geq (n!)^2 \left( \frac{\hat{T}_n}{\text{vol}(A_n)} \right) \) shall be further specified later.

Note that

\[
\nu_n(x) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j}(x_j - y_j)^2} \, dy
\]

\[
= \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(x - y) e^{-\kappa_n \sum_{j=1}^n 2^{-j}y_j^2} \, dy,
\]

and so
\[ |\nu_n(x) - \nu_n(x')| = \left| \frac{1}{T_n} \int_{\mathbb{R}^n} \left( b_n(x - y) - b_n(x' - y) \right) e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy \right| \]

\[ \leq \frac{1}{T_n} \int_{\mathbb{R}^n} |b_n(x - y) - b_n(x' - y)| e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy \]

\[ \leq L_b \|x - x'\|_\infty \frac{1}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy \]

\[ = L_b \|x - x'\|_\infty. \]

Hence, \( \nu_n \) is \( L_b \)-Lipschitz.

Next, consider the map \( \lambda_n : G \to l_\infty^n \) given by
\[ \lambda_n(x) = (q(x - x_1), \ldots, q(x - x_n)). \]

Then for \( n \geq 1 \) define (real) analytic maps \( \varphi_n : G \to \mathbb{R} \) by
\[ \varphi_n(x) = \nu_n(\lambda_n(x)) = \nu_n \left( \{q(x - x_j)\}_{j=1}^n \right) \]

\[ = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j} (q(x - x_j) - y_j)^2} dy. \]

Also, define \( \varphi_0(x) = 1 \) for all \( x \). Observe that
\[ \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} (q(x - x_j) - y_j)^2} dy = \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy = T_n, \]
and so in particular for any \( n \) (using \( b_n \in [0, 1] \))
\[ \varphi_n(0) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j} (q(-x_j) - y_j)^2} dy \]

\[ \leq \frac{1}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} (q(-x_j) - y_j)^2} dy \]

\[ = \frac{1}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^n 2^{-j} y_j^2} dy = 1. \]

Now, as to the Lipschitz properties of the \( \varphi_n \), recall that since \( G \) is bounded and \( q \) is a polynomial, \( q \) is Lipschitz on \( G - G \) with constant, say, \( L_q \). Now
\[ |\varphi_n(x) - \varphi_n(x')| = |\nu_n(\lambda_n(x)) - \nu_n(\lambda_n(x'))| \]
\[ \leq L_b \|\lambda_n(x) - \lambda_n(x')\|_\infty \]
\[ = L_b \|\{q(x - x_j) - q(x' - x_j)\}_{j=1}^n\|_\infty \]
\[ \leq L_b L_q \|x - x'\|_X, \]

and so the collection \{\varphi_n\} is UL on \(G\) with constant \(L_b L_q\). Using this fact, and \(\varphi_n(0) \leq 1\), we put \(W_1 = L_b L_q R + 1\), and note that \(|\varphi_n(x)| \leq W_1\) for all \(x \in G \subset B_R(0)\) and all \(n \geq 0\). In particular, \(W_1\) is independent of \(\kappa_n\).

We extend the maps \(\varphi_n\) to complex valued maps on \(\bar{X}\), calling them \(\tilde{\varphi}_n\). Namely (where \(x \in X\))

\[ \tilde{\varphi}_n(x + z) = \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^n 2^{-j} (\tilde{q}(x - x_j + z) - y_j)^2} dy \]

Note that the \(\tilde{\varphi}_n\) are complex analytic, and that the above calculation establishes (i) as \(\tilde{\varphi}_n|_X = \varphi_n\).

We next show (iii). For \(x \in X\) and \(z \in \bar{X}\), we have (see [15] which references [14])

\[ \tilde{q}(x - x_j + z) = q(x - x_j) + Z_j, \]

where \(Z_j \in \mathbb{C}\) with \(|Z_j| \leq C(1 + \|x - x_j\|_X)^m \|z\|_{\bar{X}}, C\) being a constant and \(m\) the degree of \(q\). As \(G\) is bounded, we can bound \(C(1 + \|x - x_j\|_X)^m\) above by some constant \(M_1\).

Now

\[ (\tilde{q}(x - x_j + z) - y_j)^2 = (q(x - x_j) - y_j + Z_j)^2 \]
\[ = (q(x - x_j) - y_j)^2 + 2(q(x - x_j) - y_j) Z_j + Z_j^2, \]
Hence

\[
\text{Re} (\bar{q} (x - x_j + z) - y_j)^2
\]

\[
= (q (x - x_j) - y_j)^2 + 2 (q (x - x_j) - y_j) \text{Re} Z_j + \text{Re} Z_j^2
\]

\[
\geq (q (x - x_j) - y_j)^2 - 2 |q (x - x_j) - y_j| |Z_j| - |Z_j^2|
\]

\[
= (|q (x - x_j) - y_j| - |Z_j|)^2 - 2 |Z_j|^2
\]

\[
\geq (|q (x - x_j) - y_j| - |Z_j|)^2 - 2 M_1^2 \|z\|^2_X.
\]

Next, for each \(x_0 \in G\) there exists \(j_0\) so that \(x_0 \in C_{j_0}\) and thus \(q (x_0 - x_{j_0}) < \gamma_1\), and also for \(y \in A_{j_0} \subset \text{support}(b_{j_0})\) we have \(y_{j_0} > 2 \gamma_1\). Hence, \(y_{j_0} - q (x_0 - x_{j_0}) > 2 \gamma_1 - \gamma_1 > \gamma_1\). Thus, for \(\|z\|_X < \frac{\gamma_1}{2 M_1}\) we have \(|q (x_0 - x_{j_0}) - y_{j_0}| - |Z_{j_0}| \geq |q (x_0 - x_{j_0}) - y_{j_0} - M_1 \|z\|_X| > \gamma_1/2\). It follows that for \(\|z\|_X < \frac{\gamma_1}{2 M_1}\) and \(n > j_0\), that

\[
\sum_{j=1}^{n} 2^{-j} \text{Re} (\bar{q} (x_0 - x_j + z) - y_j)^2
\]

\[
\geq \sum_{j=1}^{n} 2^{-j} \left( (|q (x_0 - x_j) - y_j| - |Z_j|)^2 - 2 M_1^2 \|z\|^2_X \right)
\]

\[
\geq 2^{-j_0} (|q (x_0 - x_{j_0}) - y_{j_0}| - |Z_{j_0}|)^2 - 2 M_1^2 \sum_{j=1}^{n} 2^{-j} \|z\|^2_X
\]

\[
> 2^{-j_0 - 1} \gamma_1 - 2 M_1^2 \|z\|^2_X.
\]

Hence, there exists \(\delta > 0\) and \(a > 0\) such that \(\|z\|_X < \delta\) and \(n > j_0\) implies \(\sum_{j=1}^{n} 2^{-j} \text{Re} (\bar{q} (x_0 - x_j + z) - y_j)^2 > a\).

Putting this together shows that for each \(x_0 \in X\) there exists \(\widehat{n}_0 > j_0\), \(\delta > 0\) and \(a > 0\) so that for all \(n > \widehat{n}_0\) and \(\|z\|_X < \delta\), we have
\[ |\tilde{\phi}_n(x_0 + z)| = \left| \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (\bar{q}(x_0 - x_j + z) - y_j)^2} \, dy \right| \]

\[ \leq \frac{1}{T_n} \int_{\mathbb{R}^n} b_n(y) e^{-\kappa_n \Re \sum_{j=1}^{n} 2^{-j} (\bar{q}(x_0 - x_j + z) - y_j)^2} \, dy \]

\[ \leq \frac{1}{T_n} \int_{A_n} e^{-\kappa_n a} \, dy = \frac{Vol(A_n)}{T_n} e^{-\kappa_n a}. \]

Now, by choice of \( \kappa_n \) (see above) and the definition of \( \hat{T}_n \) (which is independent of \( \kappa_n \))

\[ \frac{Vol(A_n)}{T_n} e^{-\kappa_n a} < \frac{Vol(A_n)}{T_n} \frac{n!}{\kappa_n^n a^n} = \frac{Vol(A_n)}{T_n} \frac{n!}{\kappa_n^{n/2} a^n} < \frac{1}{n! a^n}, \]

and so by choosing \( n_0 > \hat{n}_0 \) sufficiently large, we can guarantee that for \( n > n_0 \) and \( \|z\|_X < \delta \) we have \( |\tilde{\phi}_n(x_0 + z)| < \eta \), and so (iii) is proven.

Finally we show (ii). Now, since \( A'_n \subset A_n \), \( b_n = 1 \) on \( A'_n \), and the integrand is positive, we have

\[ \frac{1}{T_n} \int_{A'_n} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (q(x-x_j)-y_j)^2} \, dy \]

\[ = \frac{1}{T_n} \int_{A'_n} b_n(y) e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (q(x-x_j)-y_j)^2} \, dy \]

\[ < \frac{1}{T_n} \int_{A_n} b_n(y) e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (q(x-x_j)-y_j)^2} \, dy. \]

Finally, using \( b_n \leq 1 \), we observe that \( \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (q(x-x_j)-y_j)^2} \, dy - \int_{A_n} b_n(y) e^{-\kappa_n \sum_{j=1}^{n} 2^{-j} (q(x-x_j)-y_j)^2} \, dy \geq 0 \). Thus, putting this together and using (3.1) gives
\[ 1 - \varphi_n(x) \]
\[ = \frac{1}{T_n} \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j}(q(x-x_j)-y_j)^2} dy \]
\[ - \frac{1}{T_n} \int_{A_n} b_n(y) e^{-\kappa_n \sum_{j=1}^{n} 2^{-j}(q(x-x_j)-y_j)^2} dy \]
\[ \leq \frac{1}{T_n} \left( \int_{\mathbb{R}^n} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j}(q(x-x_j)-y_j)^2} dy \right) \]
\[ - \int_{A_n'} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j}(q(x-x_j)-y_j)^2} dy \]
\[ = \frac{1}{T_n} \int_{\mathbb{R}^n \setminus A_n} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j}(q(x-x_j)-y_j)^2} dy \]

(3.2)

Next, for each fixed \( x_0 \in G \), there exists \( j_0 \) with \( x_0 \in C_{j_0}^2 \) but with \( x_0 \notin G \setminus C_i^2 \) for \( 1 \leq i < j_0 \). Hence, \( q(x_0 - x_j) \geq \gamma_2 \) for all \( 1 \leq j < j_0 \). Now suppose that \( \left\| \{q(x_0 - x_j) - y_j\}_{j<j_0} \right\|_\infty < \gamma_2/2 \). Then \( y_j > q(x_0-x_j) - \gamma_2/2 > 3\gamma_1 \) for \( 1 \leq j < j_0 \). Also, \( y_j < q(x_0-x_j) + \gamma_2/2 \leq M + 1 \). Hence, putting \( p_0 = \{q(x_0-x_j)\}_{j<j_0} \in l^{\infty-1}_0 \), \( B_{x_0/2}(p_0) \subset A'_{j_0-1} \). Thus, writing \( q(x_0 - x_j) = (p_0)_j \), and for ease of notation \( n = j_0 - 1 \), we have, using (3.2)

\[ |1 - \varphi_n(x_0)| \leq \frac{1}{T_n} \int_{\mathbb{R}^n \setminus B_{\gamma_2/2}(p_0)} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j}(p_0)_j - y_j)^2} dy \]
\[ = \frac{1}{T_n} \int_{\mathbb{R}^n \setminus B_{\gamma_2/2}(0)} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j}y_j^2} dy. \]

By choosing \( \kappa_n \) larger if necessary, we can guarantee that

\[ \frac{1}{T_n} \int_{\mathbb{R}^n \setminus B_{\gamma_2/2}(0)} e^{-\kappa_n \sum_{j=1}^{n} 2^{-j}y_j^2} dy < 1/2, \]

and so \( \varphi_n(x_0) > 1/2 \). If however \( j_0 = 1 \) then \( \varphi_{j_0-1}(x_0) = \varphi_0(x_0) = 1 > 1/2 \). \( \square \)
3.2. The functionals $\psi_j$. Let $\zeta^1 : [0, M] \to \mathbb{R}$ satisfy $\zeta^1 (t) \geq 1/8$ for $t \in [0, M]$, be real analytic and Lipschitz, and be such that $\zeta^1 (t) < 1/4$ if $t \leq \gamma_2$, and $\zeta^1 (t) \geq 1$ if $t \geq \gamma_3$. We can choose $\zeta^1$ to be a polynomial, and hence it can be extended as a complex analytic map on $\mathbb{C}$, Lipschitz on $B_{\zeta^1} (0) \subset \mathbb{C}$, and if so we keep the same notation. Using this, define on $G$, for each $j \geq 1$

$$f_j (x) = \zeta^1 (q (x - x_j)),$$

Each $f_j$ is real analytic on $G$, and the collection $\{f_j\}$ is UL on $G$ since as pointed out above, $q$ is bounded (and Lipschitz) on $G - G$, and $\zeta^1$ is Lipschitz. Let us write

$$\tilde{f}_j (z) = \zeta^1 (\tilde{q} (z - x_j)),$$

noting that similarly $\{\tilde{f}_j\}$ is UL on $\tilde{G}$.

Next, for $j \geq 1$ define $\zeta^2 : [0, W_1] \to \mathbb{R}$ to be strictly positive with $\zeta^2 (t) \geq 1/8$ for $t \in [0, W_1]$, real analytic, Lipschitz, and such that $\zeta^2 (t) \geq 2$ if $t \leq 1/4$, $\zeta^2 (t) < 1/4$ if $t \geq 1/2$. We can choose $\zeta^2$ to be a polynomial, and hence it can be extended as a complex analytic map on $\mathbb{C}$, which is Lipschitz on $B_{W_1+1} (0) \subset \mathbb{C}$ with constant $L_2 \geq 1$, and if so we keep the same notation. For each $j \geq 1$ define a real analytic mapping on $G$ by

$$g_j (x) = \zeta^2 (\varphi_{j-1} (x)).$$

And for each $j \geq 1$ define a complex analytic mapping on $\tilde{X}$ by

$$\tilde{g}_j (x) = \zeta^2 (\tilde{\varphi}_{j-1} (x)).$$

The collection $\{g_j\}$ is UL on $G$ as the same is true of the $\varphi_j$.

It will be convenient to define the constant $T$ by

$$T = \sup \{ |z| : z \in \zeta^1 (B_{\zeta^1} (0)) \} + \sup \{ |z| : z \in \zeta^2 (B_{W_1+1} (0)) \}. $$

Let $h : [0, T] \to [0, 1]$ be strictly greater than 0, real analytic, Lipschitz, and such that $h (t) < \epsilon/4 < 1/10$ if $t \geq 3/4$, and $h (t) \geq 4/5$ if $t \leq 1/2$. We can choose $h$ to be a polynomial, and hence it can be extended as a complex analytic map on $\mathbb{C}$, Lipschitz on $B_T (0) \subset \mathbb{C}$ with constant $L_h \geq 1$, and if so we keep the same notation.

Now define maps

$$\psi_j (x) = f_j (x) + g_j (x),$$

and the maps

$$\tilde{\psi}_j (x) = \tilde{f}_j (x) + \tilde{g}_j (x).$$

Note that on $G$, each $\psi_j$ is analytic and the collection $\{\psi_j\}$ is UL on $G$. We pause to summarize the important properties of the $\psi_j$ functions in a lemma.
Lemma 4. For the functions $\psi_j$ defined above, we have

(i) $\psi_j(x) \geq 1$ for all $x \in G \setminus C_j^3$.

(ii) For each $x \in G$, there is a $j_0$ such that $\psi_{j_0}(x) < 1/2$.

(iii) For $x \in G$, $z \in \bar{X}$, there is a $j_x$ and a $\delta = \delta_x > 0$ such that

$$|\tilde{\psi}_j(x + z) - \tilde{\psi}_j(x)| < 1/(10L_h)$$

for $\|z\|_{\bar{X}} < \delta$ and $j > j_x$. Moreover, for $j > j_x$ we have $\psi_j(x) > 1$.

Proof. (i) For any $x \in G$

$$\psi_j(x) = f_j(x) + g_j(x) \geq f_j(x) = \zeta^1_j(q(x - x_j)).$$

But if $x \in G \setminus C^3_j$, $q(x - x_j) \geq \gamma_3$, implying $\zeta^1_j(q(x - x_j)) \geq 1$.

(ii) Fix $x \in G$. There is a $j_0$ with $x \in C_{j_0}^2$ but with $x \in G \setminus C_i^2$ for $1 \leq i < j_0$. From the construction of the $f_j$ and $g_j$ and using Lemma 3 (ii), $f_{j_0}(x) < 1/4$ and $g_{j_0}(x) < 1/4$.

(iii) Fix $x \in G$, and choose $\delta_1$ satisfying $1 \geq \delta_1 > 0$ so that $\|z\|_{\bar{X}} < \delta_1$ implies $x + z \in \bar{G}$. Since $\{x_j\}$ is dense, there is a $j'_x > 1$ such that $x \in C_j^1_{j_x}$. Now from Lemma 3 (iii) with $\eta = 1/40L_2L_h$, we have that for $x \in C_j^1_{j_x}$ there exists $\delta$ satisfying $0 < \delta < \delta_1$ so that $\|z\|_{\bar{X}} < \delta$ and $j > j'_x$ imply $|\tilde{\psi}_j(x + z)| < \eta$. Thus we have, for all $\|z\|_{\bar{X}} < \delta$ and all $j > j'_x$

$$|\tilde{\psi}_j(x + z) - \tilde{\psi}_j(x)| \leq |\tilde{\psi}_j(x + z)| + |\tilde{\psi}_j(x)|$$

$$< 2\eta \leq 1/(20L_2L_h) < 1,$$

and hence if $j > j_x \equiv j'_x + 1$

$$|\tilde{g}_j(x + z) - \tilde{g}_j(x)| = |\zeta^2(\tilde{\psi}_{j-1}(x + z)) - \zeta^2(\tilde{\psi}_{j-1}(x))|$$

$$\leq L_2|\tilde{\psi}_{j-1}(x + z) - \tilde{\psi}_{j-1}(x)|$$

$$\leq L_2 \cdot 1/(20L_2L_h) \leq 1/(20L_h).$$

Now $f_j(x) \geq 0$ and $\tilde{f}_j$ is Lipschitz on $G$ with constant independent of $j$, and so choosing $\delta > 0$ again smaller if necessary, we have $|\tilde{f}_j(x + z) - \tilde{f}_j(x)| < 1/(20L_h)$ for all $j$ and $\|z\|_{\bar{X}} < \delta$. We therefore have, for all $\|z\|_{\bar{X}} < \delta$ and
all \( j > j_x \)

\[
\left| \tilde{\psi}_j(x + z) - \tilde{\psi}_j(x) \right|
\]

(3.3)

\[
= \left| \tilde{f}_j(x + z) - \tilde{f}_j(x) + \tilde{g}_j(x + z) - \tilde{g}_j(x) \right|
\]

\[
\leq \left| \tilde{f}_j(x + z) - \tilde{f}_j(x) \right| + \left| \tilde{g}_j(x + z) - \tilde{g}_j(x) \right|
\]

\[
\leq 1/(20L_h) + 1/(20L_h) = 1/(10L_h).
\]

Finally, for \( j > j_x \), \( g_j(x) \geq 2 \) since \( \varphi_{j-1}(x) = |\varphi_{j-1}(x)| < \eta \leq 1/4 \). And since \( f_j(x) \geq 0 \), \( \psi_j(x) > 1 \).

3.3. Main Theorem. We finally present the proof of Theorem 1, using the previous constructions, results, and notation.

Proof of Theorem 1. We continue to use the notation introduced in the previous lemmas. Define the real analytic map

\[
u_j(x) = h(\psi_j(x)),
\]

noting that on \( G \), the collection \( \{u_j\} \) is UL. As usual we write \( \tilde{u}_j(z) = h(\tilde{\psi}_j(z)) \). Applying Lemma 4(iii), we obtain the following. For each \( x \in G \) there exists \( \delta = \delta(x) > 0 \) and \( j_x > 1 \) so that for \( z \in \tilde{X} \) with \( \|z\| \tilde{X} < \delta \) we have for all \( j \geq j_x \) that \( |\tilde{u}_j(x + z)| < 1/5 \). Indeed, using (3.3)

\[
|\tilde{u}_j(x + z) - \tilde{u}_j(x)| = \left| h(\tilde{\psi}_j(x + z)) - h(\tilde{\psi}_j(x)) \right|
\]

\[
\leq L_h \left| \tilde{\psi}_j(x + z) - \tilde{\psi}_j(x) \right|
\]

\[
\leq L_h \cdot 1/(10L_h) = 1/10
\]

But \( \psi_j(x) = \tilde{\psi}_j(x) > 1 \) implying that \( |\tilde{u}_j(x)| = h(\psi_j(x)) < 1/10 \). We conclude that

\[
|\tilde{u}_j(x + z)| \leq |\tilde{u}_j(x)| + 1/10 < 1/10 + 1/10 = 1/5.
\]

A similar calculation with Lemma 4(i) gives that \( x \in G \setminus C^3_j \) implies \( u_j(x) < \varepsilon/4 \). Also, by Lemma 4(ii) we note that for each \( x \) there is a \( j_0 \) with \( \psi_{j_0}(x) < 1/2 \), and hence \( u_{j_0}(x) \geq 4/5 \). Using the notation from section
2.1, the above shows that for each \( x \in G \), \{\( u_j(x) \)\} \( \in U \), and moreover, for any \( x,y \in G \) and \( a,b \in \mathbb{R} \) with \(|a| \leq 15/8\) and \(|b| \leq 15/8\) we have

\[
(3.4) \quad \{au_j(x)\} \in U
\]

\[
\{au_j(x)\} + \{bu_j(x)\} \in U
\]

It follows from this and the subadditivity of \( \lambda \) that for such \( x,y \) and \( a,b \) we have the 1-Lipschitz property

\[
(3.5) \quad |\lambda(\{au_j(x)\}) - \lambda(\{bu_j(y)\})| \leq \lambda(\{au_j(x)\}) - (\{bu_j(y)\})).
\]

Now put

\[
A(x) = \sum_{j=1}^{\infty} u_j(x)2^j.
\]

We next show that \( A \) is analytic on \( G \). From the calculation above, we have that \( \widetilde{A}(x+z) = \sum_{j=1}^{\infty} \tilde{u}_j(x+z)2^j \) converges locally uniformly at \( x \).

Since the \( \tilde{u}_j \) are complex analytic, we conclude that \( \tilde{A} \) is complex analytic in a neighbourhood of \( x \) in \( X \). Hence, since \( x \in G \) was arbitrary, in some neighbourhood \( U \) with \( G \subset U \subset X \) we have that \( A \) is complex analytic and so \( A(x) \) is real analytic on \( G \). Similarly from the calculations above, it follows that for each \( x \in G \) the map \( H(z,\mu) = \sum_{j=1}^{\infty} \left(\mu^{-1}\tilde{u}_j(z)\right)2^j \) is complex analytic on some neighbourhood \( \tilde{N}_x = \tilde{U}_x \times V \), where \( \tilde{U}_x \subset \tilde{X} \) contains \( x \) and \( V = \{w \in \mathbb{C} : |w| > 7/30\} \) (the lower bound 7/30 will be necessary later when we consider the sequence \( \{F(x,y)u_j(x)\} \)).

Next, for fixed \( x \in G \), consider the map \( Q : (8/15,\infty) \to U \) defined by \( Q(\mu) = \sum_{j=1}^{\infty} \left(\mu^{-1}u_j(x)\right)2^j \). This map is continuous (see the argument above establishing the analyticity of \( H \)). Also, for \( \mu > \sqrt{2} \), \( u_j(x)/\mu \leq 1/\mu < 1/\sqrt{2} \), and so \( \sum_j (u_j(x)/\mu)2^j \leq \sum_j \left(\frac{1}{2}\right)^j < 1 \). On the other hand, as noted, there exists \( j_0 \) with \( u_{j_0}(x) \geq 4/5 \), and so if \( \mu < 5/4 \) then \( \sum_j (u_j(x)/\mu)2^j \geq (u_{j_0}(x)/\mu)2^{j_0} > 1 \). Hence, by the Intermediate Value Theorem, for each \( x \in G \) the equation \( 0 = F(x,\mu) : H(x,\mu) - 1 \) has a (in this case unique) solution \( \mu = \mu(x) \).

Next note that \( h \) is strictly bounded above 0, and so for each \( x \in G \) it follows from continuity that \( \frac{\partial H(z,\mu)}{\partial \mu} \neq 0 \) in some neighbourhood \( \tilde{N}'_x = \tilde{U}'_x \times V' \), where \( \tilde{U}'_x \subset \tilde{U}_x \) contains \( x \) and \( V' \subset V \) contains \( \mu(x) \). From the above considerations, and an argument analogous to the one in section 2.1, it now follows by the complex Implicit Function Theorem that on some neighbourhood \( \tilde{W} \) of \( (x,\mu(x)) \) in \( \tilde{X} \times \mathbb{C} \) there is a unique solution \( \mu(z) \) to the equation \( F(z,\mu) = 0 \) which is complex analytic on \( \tilde{W} \). In particular, its restriction to \( W = \tilde{W} \cap (G \times \mathbb{R}) \) is real analytic. Noting that \( \{u_j(x)\} \in U \), an examination of the construction in section 2.1 of the extension of the
Preiss norm to an analytic function \( \lambda \) on \( U \subset I_\infty \), shows from the uniqueness conclusion of the Implicit Function Theorem that \( \mu(x) = \lambda(\{u_j(x)\}_j) \).

Thus, the map \( x \to \lambda(\{u_j(x)\}) \) is analytic on \( G \). As well, from the previous estimates on \( \lambda \) it follows that

\[
(3.6) \quad \frac{1}{2} \lambda(\{au_j(x)\}) \leq \|\{au_j(x)\}\|_\infty \leq \lambda(\{au_j(x)\}),
\]

and these inequalities hold for any \( a \in \mathbb{R} \) with \( \{au_i(x)\} \in U \). Finally define \( K: G \to \mathbb{R} \) by

\[
K(x) = \frac{\lambda(\{F(x)u_j(x)\})}{\lambda(\{u_j(x)\})},
\]

for all \( x \in G \). That the numerator of \( K \) is (real) analytic follows the same argument used to establish that the map \( x \to \lambda(\{u_j(x)\}) \) is analytic, where now one also uses the bounds \( 1 \geq F(x_j) \geq 1/3 \). Recall again that for each \( x \) there is a \( j_0 \) such that \( u_{j_0}(x) \geq 4/5 \). So, using (3.6) we obtain

\[
\lambda(\{u_j(x)\}) \geq \|\{u_j(x)\}\|_\infty \geq u_{j_0}(x) \geq \frac{4}{5},
\]

Now the functions \( F(x)u_j(x) \) are Lipschitz with constant independent of \( j \), since the collection \( \{u_j\} \) is UL and \( F \) is bounded. Also, by (3.5), using \( 0 < F \leq 1 \), \( \lambda \) is 1-Lipschitz on points of the form \( \{F(x)u_j(x)\}_j \), and so the composition \( \lambda(\{F(x)u_j(x)\}) \) is Lipschitz. These observations, combined with \( \lambda(\{u_j(x)\}) \geq 4/5 \) show that \( K \) is analytic and Lipschitz. Then using (3.4) and (3.5) we obtain the estimate

\[
|K(x) - F(x)| = \left| \frac{\lambda(\{F(x)u_j(x)\})}{\lambda(\{u_j(x)\})} - F(x) \right|
\]

\[
= \left| \frac{\lambda(\{F(x)u_j(x)\})}{\lambda(\{u_j(x)\})} - \frac{F(x)\lambda(\{u_j(x)\})}{\lambda(\{u_j(x)\})} \right|
\]

\[
= \left| \frac{1}{\lambda(\{u_j(x)\})} |\lambda(\{F(x)u_j(x)\}) - F(x)\lambda(\{u_j(x)\})| \right|
\]

\[
= \left| \frac{1}{\lambda(\{u_j(x)\})} |\lambda(\{F(x)u_j(x)\}) - \lambda(\{F(x)u_j(x)\})| \right|
\]

\[
\leq \left| \frac{1}{\lambda(\{u_j(x)\})} |\lambda(\{F(x)u_j(x)\}) - \lambda(\{F(x)u_j(x)\})| \right|
\]

\[
= \left| \frac{1}{\lambda(\{u_j(x)\})} |\{F(x)u_j(x)\} - \{F(x)u_j(x)\}| \right|
\]

\[
= \left| \frac{1}{\lambda(\{u_j(x)\})} |\{F(x)u_j(x)\} - \{F(x)u_j(x)\}| \right|
\]

\[
= \left| \frac{1}{\lambda(\{u_j(x)\})} |\{F(x)u_j(x)\} - \{F(x)u_j(x)\}| \right|
\]
Now by (3.6) and the comment following
\[ \lambda (\{ u_j (x) (F (x_j) - F (x)) \}) \leq 2 \| \{ u_j (x) (F (x_j) - F (x)) \} \|_\infty \]
\[ = 2 \max_j \{ u_j (x) |F (x_j) - F (x)| \}. \]
Set \( J = \{ j : x \in C^3_j \} \). For \( j \in J \) we have \( \| x - x_j \|_X < \delta \) and so
\[ u_j (x) |F (x_j) - F (x)| < u_j (x) \varepsilon. \]
It follows that for \( j \in J \)
\[ \frac{u_j (x) |F (x_j) - F (x)|}{\lambda (\{ u_j (x) \})} < \frac{u_j (x) \varepsilon}{\| \{ u_j (x) \} \|_\infty} \leq \varepsilon. \]
On the other hand, for \( j \notin J \) we have by part (i) of Lemma 3, as noted above, that
\[ u_j (x) |F (x_j) - F (x)| \leq 2u_j (x) < 2 \cdot \frac{\varepsilon}{4} = \varepsilon/2. \]
Hence, given that \( \lambda (\{ u_j (x) \}) \geq 4/5 \), we have for \( j \notin J \)
\[ \frac{u_j (x) |F (x_j) - F (x)|}{\lambda (\{ u_j (x) \})} \leq \left( \frac{5}{4} \right) \left( \frac{\varepsilon}{2} \right) < \varepsilon. \]
It follows that
\[ |K(x) - F(x)| < \varepsilon. \]
\[ \square \]

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