ANCIENT FINITE ENTROPY FLOWS BY POWERS OF CURVATURE IN $\mathbb{R}^2$

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ABSTRACT. We show the existence of non-homothetic ancient flows by powers of curvature embedded in $\mathbb{R}^2$ whose entropy is finite. We determine the Morse indices and kernels of the linearized operator of shrinkers to the flows, and construct ancient flows by using unstable eigenfunctions of the linearized operator.

1. INTRODUCTION

Given $\alpha > 0$, the $\alpha$-curve shortening flow ($\alpha$-CSF) is a family of complete convex curves $\Gamma_t$ embedded in $\mathbb{R}^2$ which evolves by the $\alpha$-power-of-curvature. Namely, the position vector $X(\cdot, t)$ of $\Gamma_t$ satisfies

$$\frac{\partial X}{\partial t}(p, t) = \kappa^\alpha(p, t)N(p, t),$$

where $\kappa$ is the curvature and $N$ is inward pointing unit normal vector of $\Gamma_t$.

We say that a flow $\Gamma_t$ is ancient if it exists for $t \in (-\infty, T)$ for some $T \in \mathbb{R} \cup \{+\infty\}$. Geometric flows satisfy parabolic equations so that there are in general only a few number of ancient flows. For example, Wang [23] showed that a closed convex embedded ancient curve shortening flow (CSF)$^1$ sweeping the entire plane is a shrinking circle, and Daskalopoulos-Hamilton-Sesum [20] showed that a closed convex embedded ancient CSF is a shrinking circle or an Angenent oval.$^2$ See also Bourni, Langford, and Tinaglia [7] for the classification of non-compact ones.

Ancient flows have been intensively studied in the mean curvature flow, a higher dimensional version of the CSF. In particular, ancient mean curvature flows are useful to investigate singularities. See [5, 6, 10, 11, 17, 18, 14] (c.f. Ricci flow [9, 4, 12]).

The $\alpha$-CSF is a fully nonlinear flow, which behaves like the $\alpha$-Gauss curvature flow in many aspects. In particular, if $\alpha = \frac{1}{3}$ ($\alpha = \frac{1}{n+2}$ in higher

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$^1$Curve shortening flow means the $\alpha$-CSF with $\alpha = 1$.

$^2$It looks like a shortening paper clip sweeping a slab.
dimensions) then a $\alpha$-CSF remains a $\alpha$-CSF under any affine transform (of determinant one) of the ambient space. The affine normal flow ($\frac{1}{3}$-CSF) have been widely studied due to its beauty from affine geometry. For example, Loftin and Tsui [21] showed that an ancient closed convex affine normal flow must be a shrinking circle or a shrinking ellipses. See also [15] for a rigidity result of ancient Gauss curvature flows.

Andrews, Guan, and Ni [3] introduced an important notion of entropy for $\alpha$-CSF. We recall that the support function $u_{z_0}$ with respect to $z_0 \in \mathbb{R}^2$ is

$$u_{z_0}(\theta) := \max_{z \in \Omega} \langle (\cos \theta, \sin \theta), z - z_0 \rangle,$$

and the entropy $\mathcal{E}_\alpha(\Omega)$ of a bounded convex region $\Omega \subset \mathbb{R}^2$ and its boundary $\partial \Omega$ is defined by

$$\mathcal{E}_\alpha(\partial \Omega) = \mathcal{E}_\alpha(\Omega) = \sup_{z_0 \in \Omega} \mathcal{E}_\alpha(\Omega, z_0),$$

where $\mathcal{E}_\alpha(\Omega, z_0)$ is

$$\mathcal{E}_\alpha(\Omega, z_0) = \begin{cases} \frac{\alpha}{\alpha - 1} \log \left( \int_{\mathbb{S}^1} u_{z_0}^{1-\frac{1}{\alpha}}(\theta)d\theta \right) - \frac{1}{2} \log \frac{|\Omega|}{\pi} & \text{if } \alpha \neq 1, \\ \int_{\mathbb{S}^1} \log u_{z_0}(\theta)d\theta - \frac{1}{2} \log \frac{|\Omega|}{\pi} & \text{if } \alpha = 1. \end{cases}$$

Here $|\Omega|$ denotes the area of it.

In [3], they showed that the entropy $\mathcal{E}_\alpha(\Gamma_t)$ of the $\alpha$-CSF decreases with respect to $t$. Hence, we say that an ancient $\alpha$-CSF has finite entropy if

$$\lim_{t \to -\infty} \mathcal{E}_\alpha(\Gamma_t) < +\infty.$$ 

Clearly, self-shrinking ancient solutions has finite entropy, since the entropy does not change under homothetic transformation. However, non-homothetic ancient $\alpha$-CSFs which have been discovered do not have finite entropy. See [20] and [8]. In this paper, we present families of non-homothetic closed ancient $\alpha$-CSFs which converge to a self-shrinker\(^3\) as $t \to -\infty$ after rescaling. Then, their entropy is less than that of the limiting shrinker, namely the ancient flows have the finite entropy. See Theorem 3.2.

To construct ancient flows asymptotic to a self-shrinking ancient flow, we first recall the classification result of self-shrinkers.

**Theorem 1.1 (Andrews [2]).** If $\alpha \in \left[ \frac{1}{8}, +\infty \right) \setminus \{1/3\}$, then the shrinker of (1.1) is a circle (denote it as $\Gamma_c^\alpha$). If $\alpha = \frac{1}{3}$, then a shrinker is an ellipse. If $\alpha \in (0, \frac{1}{8})$, then a shrinker is a circle or a curve $\Gamma_k^\alpha$ with $k$-fold symmetry, where $3 \leq k \in \mathbb{N}$ with $k < \sqrt{1 + 1/\alpha}$. The curves $\Gamma_k^\alpha$ depend smoothly

\(^3\)If $\Gamma_t = (-t)^{\alpha/1+\alpha} \Gamma_{-1}$ is the $\alpha$-CSF, then we call $\Gamma_{-1}$ a self-shrinker or a shrinker.
ANCIENT SOLUTION TO $\alpha$-CSF

on $\alpha < \frac{1}{k^2 - 1}$ and converge to regular $k$-sided polygons as $\alpha \searrow 0$ and to circles as $\alpha \nearrow \frac{1}{k^2 - 1}$. See Table 1 and Figure 1 for illustrations.

| $\alpha$       | $\Gamma_{\alpha}^c$ and $\Gamma_{\alpha}^k$       |
|----------------|-----------------------------------------------|
| $\left[ \frac{1}{8}, +\infty \right) \setminus \frac{3}{2}$ | $\bigcirc$                              |
| $\left[ \frac{1}{15}, \frac{1}{8} \right)$       | $\square$                              |
| $\left[ \frac{1}{24}, \frac{1}{15} \right)$       | $\triangle$                            |
| ...           | ...                                           |

**Table 1.** Enumeration of shrinkers for different $\alpha$.

**Figure 1.** The shape of $\Gamma_{\alpha}^k$ (normalized by (1.7)) when $k = 3$, $\alpha = \frac{1}{9}, \frac{1}{16}, \frac{1}{100}$ from left to right.

To fix the asymptotic self-shrinking ancient flow, we consider the normalized flow $\bar{\Gamma}_\tau$ defined by

$$\bar{X}(p, \tau) = (1 + \alpha)^{-\frac{1}{\alpha+1}} e^\tau X(p, e^{-(1+\alpha)\tau}), \quad (1.5)$$

By Proposition 2.1 the support function $\bar{u}(\theta, \tau)$ of $\bar{X}$ with respect to the origin satisfies

$$\bar{u}_\tau = -(\bar{u}_{\theta \theta} + \bar{u})^\alpha + \bar{u}. \quad (1.6)$$

Hence, the support function $h$ of a self-shrinker $\Gamma$ with respect to the origin satisfies

$$h_{\theta \theta} + h = h^{-1/\alpha}, \quad (1.7)$$
and thus the difference \( v = \bar{u} - h \) satisfies
\[
v_{\tau} = - (h_{\theta \theta} + h + v_{\theta \theta} + v)^{-\alpha} + (h + v) := L_\Gamma(v) + E_\Gamma(v). \tag{1.8}
\]
Here \( L_\Gamma \) is the linearization of the above equation at \( v = 0 \)
\[
L_\Gamma(v) := \alpha h^{1 + \frac{1}{\alpha}} (v_{\theta \theta} + v) + v \tag{1.9}
\]
and
\[
|E_\Gamma(v)| \leq C |v_{\theta \theta} + v|^2, \tag{1.10}
\]
for small enough \( v_{\theta \theta} + v \). See Proposition 3.3 for details.

It is easy to see that the Jacobi operator \( L_\Gamma \) is a self-adjoint operator on
the space \( L^2_h(S^1) = \{ f : \int_{S^1} f^2 h^{-1 - 1/\alpha} < \infty \} \), and thus it has a sequence
of eigenvalues and eigenfunctions which form the basis of \( L^2_h(S^1) \). We are
able to characterize its kernel and Morse index\(^4\) as follows.

**Theorem 1.2** (cf. Proposition 2.2 and Theorem 2.4). *Suppose \( 0 < \alpha \neq \frac{1}{3} \).

1. The Morse index of \( L_{\Gamma_{a}} \) is \( 2k - 1 \), and \( \ker L_{\Gamma_{a}} = \text{span} \{ h_{\theta} \} \), where

   \( h \) is the support function of \( \overline{\Gamma}_a \).

2. The Morse index of \( L_{\Gamma_{a}} \) is \( 2 \left[ \sqrt{1 + 1/\alpha} \right] - 1 \).\(^5\) If \( \alpha = \frac{1}{2^{k-1}} \), then

   \( \ker L_{\Gamma_{a}} = \text{span} \{ \cos k\theta, \sin k\theta \} \). Otherwise \( \ker L_{\Gamma_{a}} = \emptyset \).

The center manifold theory in functional analysis provides the existence
of an \( I \)-parameter family of ancient solutions to a class of fully nonlinear
parabolic equations, where \( I \) is the Morse index. See Lunardi [22, Chapter 9]. However, using the contraction mapping method, we can show the
existence of such ancient solutions and even including sharp asymptotic behaviors of the solutions with layer structures. See Choi and Mantoulidis [16] and Caffarelli-Hardt-Simon [13] for quasilinear parabolic and elliptic PDEs. Here comes the second main theorem of our paper.

**Theorem 1.3** (cf. Theorem 3.2). *Let \( \alpha \neq \frac{1}{3} \) and \( \lambda_1 \leq \cdots \leq \lambda_I < 0 \)
denote the negative eigenvalues of \( L_\Gamma \) where \( \Gamma = \overline{\Gamma}_k \) or \( \overline{\Gamma}_c \) and \( I \) is the
Morse index. There exists \( \beta \in (0, 1), \varepsilon_0 > 0 \) and an injective continuous
map \( S : B_{\varepsilon_0}(0)(\subset \mathbb{R}^{I-3}) \rightarrow C^{2,\beta}(\mathbb{S}^1 \times (-\infty, -1]) \) such that for each
\( \alpha = (a_1, \cdots, a_{I-3}) \in \mathbb{R}^{I-3} \) the image \( v = S(\alpha) \) is an ancient solution to
(1.8). Moreover, if \( 3 < k \leq I \) and \( a, b \in \mathbb{R}^{I-3} \) satisfy
\( a_{k-3} - b_{k-3} \neq 0 \) and \( a_j - b_j = 0 \) for all \( j > k - 3 \), then \( S \) satisfies
\[
S(\alpha)(\theta, \tau) - S(\beta)(\theta, \tau) = (a_{k-3} - b_{k-3}) e^{-\lambda_k \tau} \varphi_k(\theta) + o(e^{-\lambda_k \tau}) \tag{1.11}
\]
when $\lambda_{k-1} < \lambda_k$, and

$$S(a)(\theta, \tau) - S(b)(\theta, \tau) = e^{-\lambda_k \tau} \sum_{i=k-1}^{k} (a_{i-3} - b_{i-3}) \varphi_i(\theta) + o(e^{-\lambda_k \tau}) \quad (1.12)$$

when $\lambda_{k-1} = \lambda_k$, where $\varphi_i$ are eigenfunctions of $L_\Gamma$ with the eigenvalue $\lambda_i$ and $\langle \varphi_i, \varphi_j \rangle_{L^2} = \delta_{ij}$. In particular, $S(0)(\theta, \tau) = h(\theta)$ corresponds to the shrinker.

**Remark 1.4.** Notice that the first three eigenfunctions of $L_\Gamma$ are $h, \cos \theta, \sin \theta$ by Proposition 2.2, which accounts for dilations and transitions of the non-rescaled $\alpha$-CSF. See Proposition 3.4. Therefore, we consider $(I - 3)$-parameter family of ancient solutions rather than $I$-parameter.

Moreover, if $\Gamma = \Gamma^c$, then rotations accounts for 1-parameter. Namely, Theorem 1.3 provides a $(I - 4)$-parameter family ancient flows converging to a round shrinking circle up to rigid motions and dilations.

In short, given $\frac{1}{k^2 - 1} \leq \alpha < \frac{1}{(k-1)^2 - 1}$ with $3 \leq k \in \mathbb{N}$, by Theorem 1.3 there exist, up to rigid motions and dilations, a $(2k - 5)$-parameter family of closed convex ancient $\alpha$-CSFs converging to a round shrinking circle and a $(2m - 3)$-parameter family of closed convex ancient $\alpha$-CSFs converging to a shrinking $m$-fold symmetric curve for each integer $3 \leq m < k$.

In an following paper, the authors will classify ancient finite entropy $\alpha$-CSFs, and show that the solutions in Theorem 1.3 are the all solutions up to transitions and dilations with exhibiting the layer structure (1.11)-(1.12).

An outline of our paper is in order. In Section 2, we devote to studying the spectrum of the linear operator $L$. In Section 3, we construct ancient solutions converge with finite entropy by contraction mapping theorem.

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## 2. Spectra of Linearized Operators

We begin by deriving the evolution equation of the support function $\bar{u}$ of the normalized flow $\bar{\Gamma}_\tau$ given by (1.5).

**Proposition 2.1.** Let $\bar{\Gamma}_\tau$ be a normalized $\alpha$-CSF satisfying (1.5). The support function $\bar{u}$ of $\bar{\Gamma}_\tau$ satisfies

$$\bar{u}_\tau = -\tilde{\kappa}^\alpha + \bar{u} = -\left(\bar{u}_{\theta\theta} + \bar{u}\right)^{-\alpha} + \bar{u}. \quad (2.1)$$

**Proof.** By using (1.5), we have

$$\tilde{\kappa} = (1 + \alpha) \frac{1}{1+\alpha} e^{-\tau} \kappa, \quad (2.2)$$
and thus
\[ \partial \tau \ddot{X} = (1 + \alpha) \frac{\ddot{X}}{1 + \alpha} \exp(-\alpha \tau) \partial t \ddot{X} + \ddot{X} = \dot{\kappa} N + \ddot{X}. \] (2.3)

Therefore, \( \ddot{u} = \langle \ddot{X}, N \rangle \) and \( \ddot{u}_{\theta \theta} + \ddot{u} = \ddot{\kappa}^{-1} \) yield the desired evolution equation.

We are interested in normalized ancient flows \( \dot{\Gamma}_\tau \) converging to a shrinker \( \Gamma = \dot{\Gamma}_k \) or \( \dot{\Gamma}_c \) as \( \tau \rightarrow -\infty \). Namely, the difference \( v = \ddot{u} - h \) converges to zero, where \( h \) is the suppose function of \( \Gamma \) satisfying (1.7). Moreover, the evolution equation (1.8) has the linearized operator \( \mathcal{L} \) given by (1.9).

\[ \mathcal{L} = \alpha h \frac{1}{1 + \frac{1}{\alpha}} (\partial_\theta^2 + 1) + 1. \] (2.4)

We shall abbreviate \( \mathcal{L} \) as \( \mathcal{L}_\Gamma \) whenever there is no confusion.

We introduce the space \( L^2_h(S^1) = L^2(S^1, h^{-1/\alpha} d\theta) \) with norm \( ||f||^2_h = \int_{S^1} f^2 h^{-1/\alpha} \, d\theta \). It is equipped with the inner product
\[ (f, g)_h = \int_{S^1} f g h^{-1/\alpha} \, d\theta. \] (2.5)

Since \( h > 0 \) on \( S^1 \) and (1.7), this norm is equivalent to the standard \( L^2 \) norm.

It is easy to see that \( \mathcal{L} \) is a self-adjoint operator on \( L^2_h \), and thus \( \mathcal{L} \) has discrete eigenfunctions and eigenvalues. We remind that an eigenfunction \( \varphi \in L^2_h(S^1) \) and the corresponding eigenvalue \( \lambda \in \mathbb{R} \) satisfy
\[ \alpha h \frac{1}{1 + \frac{1}{\alpha}} (\varphi_{\theta \theta} + \varphi) + (\lambda + 1) \varphi = 0, \quad \text{on} \quad S^1. \] (2.6)

Moreover, there exists a sequence of the pairs \( (\lambda_i, \varphi_i) \) of eigenvalues and eigenfunctions such that \( \lambda_i \leq \lambda_{i+1}, \lim_{i \rightarrow \infty} \lambda_i = +\infty \), \( (\varphi_i, \varphi_j)_h = \delta_{ij} \), and \( \text{span}\{\varphi_1, \varphi_2, \cdots\} = L^2_h(S^1) \).

In this section, we will study eigenfunctions with negative or zero eigenvalues of \( -\mathcal{L} \).

**Proposition 2.2.** There are some known eigenvalues for \( -\mathcal{L} \).

1. \( \lambda = -1 - \alpha \) is an eigenvalue with the eigenfunction \( \varphi = h \). Since \( h \) is always positive, \( \lambda = -1 - \alpha \) is the lowest eigenvalue.
2. \( \lambda = -1 \) is an eigenvalue with the eigenfunctions \( \varphi = \sin \theta, \cos \theta \).
3. \( \mathcal{L}_{\Gamma_k} = \alpha (\partial_\theta^2 + 1) + 1 \) has eigenvalues
   \[ \lambda_1 = -\alpha - 1, \quad \lambda_{2l} = \lambda_{2l+1} = \alpha (l^2 - 1) - 1, \quad l \geq 1 \] (2.7)
   with the eigenfunctions \( \cos(l \theta) \) and \( \sin(l \theta) \). Notice that \( -\mathcal{L} \) has an eigenvalue \( \lambda = 0 \) only when \( \alpha = 1/(l^2 - 1) \) for some \( l \geq 2 \).
4. \( \mathcal{L}_{\Gamma_c} \) has zero eigenvalue \( \lambda = 0 \) with eigenfunction \( \varphi = h_\theta \). More importantly, \( \lambda = 0 \) is simple.
Proof. (1), (2), (3) are easy to verify. For (4), it is obtained by differentiating (1.7) with respect to \( \theta \), which gives \( h_\theta \) satisfies (2.6) when \( \lambda = 0 \). Indeed, \( h_\theta \) arises from rotations of \( \bar{\Gamma}^k \). Andrews [2, Lemma 7.3] shows that the eigenspace of \( \lambda = 0 \) has dimension ONE, which is span\{\( h_\theta \)\}. \( \square \)

In Proposition 2.2, we characterize all eigenfunctions of \( -L_{\bar{\Gamma}^\alpha} \) and neutral eigenfunctions of \( -L_{\bar{\Gamma}^\alpha} \). Thus, we will focus on \( \Gamma = \bar{\Gamma}^k \) and consider negative eigenvalues of \( -L_{\bar{\Gamma}^\alpha} \). We shall simply write \( L = L_{\bar{\Gamma}^\alpha} \) for the rest of this section.

The following lemma is equivalent to [1, Lemma 5] whose proof needs Brunn-Minkowski inequality there. We give a direct proof here.

**Lemma 2.3.** There is NO eigenvalue of \(-L\) in \((-1 - \alpha, -1)\).

**Proof.** Suppose \( \varphi \) is an eigenfunction of \( -L \) satisfying \((\varphi, h)_h = 0\) and (2.6). Then there exists \( c \) such that \( \tilde{\varphi} = \varphi - ch \) satisfy \( \int_{S^1} \tilde{\varphi} = 0 \). Then \( \int_{S^1} \tilde{\varphi}^2 - |\tilde{\varphi}_\theta|^2 \leq 0 \). Multiplying (2.6) by \( \tilde{\varphi} h^{-1 - \frac{1}{\alpha}} \) and integrating over \( S^1 \) give

\[
\alpha \int_{S^1} (\varphi_{\theta\theta} + \varphi) \tilde{\varphi} + (\lambda + 1) \int_{S^1} \varphi \tilde{\varphi} h^{-1 - \frac{1}{\alpha}} = 0. \tag{2.8}
\]

Let us simplify the left-hand side. First, using the fact \( \int_{S^1} \varphi h^{-\frac{1}{2}} = 0 \), we have

\[
\int_{S^1} \varphi \tilde{\varphi} h^{-1 - \frac{1}{\alpha}} = \int_{S^1} \varphi^2 h^{-1 - \frac{1}{\alpha}} = (\varphi, \varphi)_h > 0.
\]

Second,

\[
\int_{S^1} (\varphi_{\theta\theta} + \varphi) \tilde{\varphi} = \int_{S^1} [\varphi_{\theta\theta} + \varphi] \tilde{\varphi} + c \int_{S^1} [h_{\theta\theta} + h] \tilde{\varphi}
= \int_{S^1} (\tilde{\varphi}^2 - \varphi^2) + c \int_{S^1} h^{-\frac{1}{2}} (\varphi - ch) \leq -c^2 \int_{S^1} h^{-1 - \frac{1}{\alpha}} \leq 0,
\]

where in the second equality we used (1.7).

If \( \lambda \in (-1 - \alpha, -1) \) is an eigenvalue of \( -L \), inserting the above two inequalities into the LHS of (2.8), one could find out the LHS < 0. Contradiction. \( \square \)

It follows from Proposition 2.2 and Lemma 2.3 that \( -L \) has eigenvalues

\[
\lambda_1 < \lambda_2 = \lambda_3 < \lambda_4 \leq \cdots \tag{2.9}
\]

where \( \lambda_1 = -1 - \alpha \), \( \lambda_2 = \lambda_3 = -1 \).

**Theorem 2.4.** Suppose \( k \geq 3 \), \( \alpha \in (0, 1/(k^2 - 1)) \). The negative eigenspace of \( -L_{\bar{\Gamma}^\alpha} \) has dimension \( 2k - 1 \). In particular,
If $k$ is odd, every negative eigenvalue except $\lambda_1$ has the eigenspace of dimension two. If $k$ is even, every negative eigenvalue except $\lambda_1, \lambda_k, \lambda_{k+1}$ has the eigenspace of dimension two. In both cases, any eigenfunction of $\lambda_{2l}$ and $\lambda_{2l+1}$, $1 \leq l \leq k-1$, have $2l$ zeros.

Furthermore, $\lambda_{2k} = 0$ and $\lambda_{2k+1} > 0$ are simple, namely

$$\lambda_{2k-1} < \lambda_{2k} (= 0) < \lambda_{2k+1} < \lambda_{2k+2} \leq \cdots .$$

In addition, both $\varphi_{2k}$ and $\varphi_{2k+1}$ have $2k$ nodal sets.

Easily one can see the dimension of eigenspace of each eigenvalue is at most 2. This is because (2.6) is a second order ODE and it has at most two linearly independent solutions.

For a function $\varphi$, the term zeros (or nodal sets) refers to the set $\{\theta : \varphi(\theta) = 0\}$. The term nodal domain refers to the connected components of the complement of the nodal sets.

**Lemma 2.5.** Each eigenfunction $\varphi$ satisfies $\varphi^2 + |\varphi'|^2 > 0$. Any eigenfunctions $\varphi$, except $\varphi \in \text{span}\{h\}$, has even number of zeros and even number of nodal domains.

**Proof.** If $\varphi(\theta_0) = \varphi'(\theta_0) = 0$ at some point $\theta_0 \in S^1$, then we have $\varphi = 0$ on $S^1$ by the uniqueness of solution to the second order ODE. Hence, $\varphi^2 + |\varphi'|^2 > 0$ everywhere. Therefore, if $\varphi(\theta_0) = 0$ at some $\theta_0 \in S^1$ then $\varphi(\theta_0 + \epsilon)\varphi(\theta_0 - \epsilon) < 0$ for small enough $\epsilon$. Namely, $\varphi$ change its signs at zeros. Hence, $\varphi$ has even number of zeros and thus it has even number of nodal domains.

**Lemma 2.6.** Suppose that $\lambda$ has a two dimensional eigenspace. Then, its eigenfunctions have the same number of nodal sets.

**Proof.** This follows from the Sturm separation theorem. For reader’s convenience, we give a proof. Suppose $\text{span}\{\varphi, \psi\}$ are the eigenspace of $\lambda$, where $\varphi, \psi$ are linearly independent. Then the Wronskian of $\varphi$ and $\psi$ is not zero for any $\theta$, that is

$$W[\varphi, \psi](\theta) = \begin{vmatrix} \varphi(\theta) & \psi(\theta) \\ \varphi'(\theta) & \psi'(\theta) \end{vmatrix} \neq 0. \quad (2.10)$$

If $\psi$ and $\varphi$ has different number of nodal sets, by Lemma 2.5, then without loss of generality one nodal set $\varphi$ is strictly contained in one nodal set of $\psi$. That is, there exists $\theta_1$ and $\theta_2$ such that

$$\varphi(\theta_1) = \varphi(\theta_2) = 0, \quad \varphi'(\theta_1) \varphi'(\theta_2) < 0, \quad \psi(\theta_1) \psi(\theta_2) > 0.$$

However, we will get

$$W(\theta_1)W(\theta_2) = \varphi'(\theta_1)\psi(\theta_1)\varphi'(\theta_2)\psi(\theta_2) < 0.$$

This is not possible, because $W$ does not change sign. \qed
Because the above lemma, we are eligible to say the number of nodal sets corresponding to an eigenvalue \( \lambda \).

**Lemma 2.7.** Suppose an eigenvalue \( \lambda_i \) has a two dimensional eigenspace, then for any \( \lambda_j > \lambda_i \), the number of nodal sets of the eigenfunctions corresponding to \( \lambda_j \) is greater than that of \( \lambda_i \). Similarly, if \( \lambda_j < \lambda_i \), then the number of nodal sets corresponding to \( \lambda_j \) is less than that of \( \lambda_i \).

**Proof.** Suppose the eigenspace of \( \lambda_i \) is \( \text{span}\{\psi_i, \tilde{\psi}_i\} \) and take any eigenfunction \( \varphi_j \) corresponding to \( \lambda_j \). Assume \( \varphi_j(\theta_0) = 0 \) for some \( \theta_0 \). One can find \( \omega \in [0, 2\pi] \) such that \( (\cos \omega) \psi_i(\theta_0) + (\sin \omega) \tilde{\psi}_i(\theta_0) = 0 \). Notice \( \varphi_i = (\cos \omega) \psi_i + (\sin \omega) \tilde{\psi}_i \) is an eigenfunction of \( \lambda_i \). Since \( \varphi_i(\theta_0) = \varphi_j(\theta_0) = 0 \), the conclusions follow from the Sturm-Picone comparison theorem. For reader’s convenience, we sketch the proof of the case \( \lambda_j > \lambda_i \). The other case is similar. It suffices to prove that there is at least a zero of \( \varphi_j \) which lies strictly between any two consecutive zeros of \( \varphi_i \). Assume that \( \varphi_i \) has two consecutive zeros \( a \) and \( b \), and \( \varphi_j \) has no zero in \((a, b)\). Without loss of generality, we assume \( \varphi_i > 0 \) and \( \varphi_j > 0 \) in \((a, b)\).

Denote the Wronskian \( W(\theta) = \varphi_i \varphi_j' - \varphi_i' \varphi_j \), then we can directly calculate \( W' = (\lambda_i - \lambda_j) \frac{1}{\alpha} h^{-1-1/\alpha} \varphi_j \varphi_i \). We have the following Picone’s identity

\[
\left( \frac{\varphi_i}{\varphi_j} W \right)' = \frac{1}{\alpha} h^{-1-1/\alpha} (\lambda_i - \lambda_j) \varphi_j^2 - \left( \frac{W}{\varphi_j} \right)^2
\]

(2.11)

wherever \( \varphi_j \neq 0 \).

If \( \varphi_j(a) > 0 \) and \( \varphi_j(b) > 0 \), then we integrate (2.11) from \( a \) to \( b \).

\[
\left. \frac{\varphi_i}{\varphi_j} W \right|_a^b < 0.
\]

This contradicts to \( \varphi_i(a) = \varphi_i(b) = 0 \).

If \( \varphi_j(a) = 0 \) and \( \varphi_j(b) > 0 \), then \( \varphi_j'(a) > 0 \). Integrating (2.11) from \( a - \epsilon \) to \( b \) for \( \epsilon > 0 \) small enough, we obtain

\[
\left. \frac{\varphi_i}{\varphi_j} W \right|_{a-\epsilon}^b < 0
\]

However, we know \( \varphi_i(a - \epsilon) > 0 \), \( \varphi_j(a - \epsilon) > 0 \) and \( W(a - \epsilon) < 0 \) for sufficiently small \( \epsilon > 0 \). Namely, the above quantity is positive. Hence it is an obvious contradiction.

The case of \( \varphi_j(a) > 0 \) with \( \varphi_j(b) = 0 \) and \( \varphi_j(a) = \varphi_j(b) = 0 \) can be ruled out in the same manner. \( \square \)

Recall that the support function \( h \) of \( \Gamma_k^\alpha \) is \( k \)-fold symmetric. It has \( 2k \) critical points. By rotating \( \Gamma_k^\alpha \), we may assume

\[
h'(n\pi/k) = 0,
\]

(2.12)
for all \( n \in \mathbb{Z} \). Then \( h \) has even reflection symmetry with respect to \( n\pi/k \) for any \( n \in \mathbb{Z} \), namely \( h(\theta) = h(2n\pi/k - \theta) \) for any \( n \in \mathbb{Z} \).

**Lemma 2.8.** Suppose that the eigenspace of \( \lambda_i \neq \lambda_1 \) is \( \text{span}\{\varphi_i\} \), namely \( \lambda_i \) is simple. Then, either \( \varphi_i \) has at least \( 2k \) zeros, or \( \varphi_i \) has exactly \( k \) zeros and \( k \) must be even. In the second case, zeros of \( \varphi_i \) are \( \{2n\pi/k : n \in \mathbb{Z}\} \) or \( \{(2n + 1)\pi/k : n \in \mathbb{Z}\} \) modulo \( 2\pi \), where \( h \) satisfies 2.12.

**Proof.** By the symmetry of \( h \), \( \varphi_i(2n\pi/k - \theta) \) is an eigenfunction of \( \lambda_i \). Since the eigenspace of \( \lambda_i \) has dimension one, we must have \( \varphi_i(\theta) = c\varphi_i(2n\pi/k - \theta) \) for some \( c(n) \neq 0 \) and any \( \theta \in S^1 \). Replacing \( \theta \) by \( 2n\pi/k - \theta \), one gets \( \varphi_i(2n\pi/k - \theta) = c(n)\varphi_i(\theta) \). Thus, \( c(n) \) must be 1 or \(-1\).

If \( c(n) = 1 \), then \( \varphi_i \) has even reflection symmetry with respect to \( n\pi/k \). Then, we have \( \varphi_i(n\pi/k) = 0 \), and because Lemma 2.5, we also \( \varphi_i(n\pi/k) \neq 0 \) for such \( n \). If \( c(n) = -1 \), then \( \varphi_i \) has an odd reflection symmetry with respect to \( n\pi/k \). Obviously, we have \( \varphi_i(n\pi/k) = 0 \) for such \( n \). Conversely, if \( \varphi_i(n\pi/k) \neq 0 \) then \( \varphi_i \) has even reflection symmetry with respect to \( n\pi/k \). If \( \varphi_i(n\pi/k) = 0 \), then \( \varphi_i \) has odd reflection symmetry with respect to \( n\pi/k \).

Now, we consider \( \varphi_i \) on \([0, n\pi/k]\). We divide it into three cases.

First, if \( \varphi_i \) has a zero in \((0, n\pi/k)\), then the reflect symmetries of \( \varphi \) guarantees at least \( 2k \) zeros in \( S^1 \).

Second, if \( \varphi_i(0) = \varphi_i(n\pi/k) = 0 \) and has no zero inside \((0, n\pi/k)\), then after the reflection symmetries of \( \varphi \) guarantees at least \( 2k \) zeros in \( S^1 \).

Last, \( \varphi_i \) has only one zero on the endpoint of \([0, n\pi/k]\), say \( \varphi_i(0) = 0 \) and \( \varphi_i \neq 0 \) for \((0, n\pi/k)\). In this case, the previous paragraph shows that \( \varphi_i \) has even reflection symmetry with respect to \((2n + 1)\pi/k\) and has odd reflection symmetry with respect to \(2n\pi/k\) for any \( n \). Moreover, \( k \) must be even since \( \varphi_i \) has even number of nodal sets. Counting the zeros of \( \varphi_i \), we find it is \( k \) in this case. \( \square \)

Now we can prove the first part of Theorem 2.4.

**Proof of Theorem 2.4 Part (1).** We will use the induction to prove that \( \lambda_{2l} = \lambda_{2l+1} \) for any \( 1 \leq l < k \), except that \( \lambda_k = \lambda_{k+1} \) when \( k \) is even. In any case, eigenfunctions corresponds to \( \lambda_{2l} \) and \( \lambda_{2l+1} \) have \( 2l \) nodal domains. If \( l = 1 \), then \( \lambda_2 = \lambda_3 \) by Proposition 2.2. Any eigenfunction in \( \text{span}\{\cos \theta, \sin \theta\} \) has 2 nodal domains. Suppose the induction is complete for any \( l \) such that \( 2l + 1 \leq k - 1 \), that is

\[
\lambda_1 < \lambda_2 = \lambda_3 < \cdots < \lambda_{2l} = \lambda_{2l+1} < 0
\]

and those eigenfunctions of \( \lambda_{2j} = \lambda_{2j+1} \) have \( 2j \) nodal domains for \( j \leq l \).
It follows from Courant nodal domain theorem [19, VI.6], $\varphi_{2(l+1)}$ has at most $2(l + 1)$ nodal domains. Because $2(l + 1) \leq 2(k - 1)$, Lemma 2.8 implies that $\lambda_{2(l+1)}$ will be repeated unless $k$ is even and $2(l + 1) = k$.

Let us first consider the case that $\lambda_{2(l+1)}$ is repeated. We can continue the induction. The dimension of the eigenspace of each eigenvalue is at most 2, thus $\lambda_{2l+3} = \lambda_{2(l+1)} > \lambda_{2l+1} = \lambda_{2l}$. Now we only need to prove $\varphi_{2l+2}$ and $\varphi_{2l+3}$ has $2l + 2$ nodal domains. First, they have the same number of nodal sets by Lemma 2.6, while Lemma 2.7 implies that they must have at least $2l + 2$ nodal sets. Combining the previous upper bound on the number of nodal sets, our induction for $l + 1$ is complete.

If $k$ is even and $\lambda_{2(l+1)} = \lambda_k$ is simple, then Lemma 2.8 says $\varphi_k$ has $k$ zeros. Now, we consider $\lambda_{k+1}$. By the Courant nodal domain theorem and Lemma 2.5, any eigenfunction associated to $\lambda_{k+1}$ also has $k$ zeros. Therefore, the second part of Lemma 2.7 implies that $\lambda_{k+1}$ is also simple. Then, we have $\lambda_{2l+2} < \lambda_{2l+3} < \lambda_{2l+4}$. By Lemma 2.8, $\lambda_{2l+4}$ is repeated, and its eigenfunctions has $2l + 4 = k + 2$ zeros. The induction on $l + 1$ and $l + 2$ is complete. Now, since $2(l + 2) > k$ in this case, Lemma 2.8 shows the rest negative eigenvalue are all repeated, therefore it can be continued as the previous case.

Putting everything together, we have the following relations

$$
\lambda_1 < \lambda_2 = \lambda_3 < \cdots < \lambda_{2l} = \lambda_{2l+1} < \cdots < \lambda_{2k-2} = \lambda_{2k-1}
$$

except that when $k$ is even, the relation $\lambda_k = \lambda_{k+1}$ will be replaced by $\lambda_k \leq \lambda_{k+1}$. All these eigenvalues are negative, because $\lambda = 0$ has the eigenfunction $h_0$ with $2k$ nodal sets and $\varphi_{2k-2}$ has $2k - 2$ nodal sets. So, Lemma 2.7 says $\lambda_{2k-2} < 0$. □

Next, in order to show Part (2) of Theorem 2.4, we consider the following eigenvalue problems in the Hilbert space $H^1([0, \pi/k])$ with various boundary conditions

$$
\begin{align*}
\psi'' + \psi &= -\frac{1}{\alpha} h^{-1-\frac{1}{\alpha}} (\mu + 1) \psi, \quad \psi(0) = \psi(\pi/k) = 0, \quad \text{(DD)} \\
\psi'' + \psi &= -\frac{1}{\alpha} h^{-1-\frac{1}{\alpha}} (\mu + 1) \psi, \quad \psi(0) = \psi'(\pi/k) = 0, \quad \text{(DN)} \\
\psi'' + \psi &= -\frac{1}{\alpha} h^{-1-\frac{1}{\alpha}} (\mu + 1) \psi, \quad \psi'(0) = \psi(\pi/k) = 0, \quad \text{(ND)} \\
\psi'' + \psi &= -\frac{1}{\alpha} h^{-1-\frac{1}{\alpha}} (\mu + 1) \psi, \quad \psi'(0) = \psi'(\pi/k) = 0. \quad \text{(NN)}
\end{align*}
$$
Here we also assume $h$ satisfies (2.12). According to the Sturm-Liouville theory, the eigenvalues $\mu_i^{AB}$ for the problems $(AB)$ where $A, B = D$ or $N$ satisfy,

$$\mu_1^{AB} < \mu_2^{AB} < \mu_3^{AB} < \cdots$$

and the eigenfunction $\psi_i^{AB}$ corresponding to $\mu_i^{AB}$ have $i - 1$ zeros.

For $(NN)$, it is easy to know $h \in H^1([0, \pi/k])$ and it is an eigenfunction to the first eigenvalue $\mu_1^{NN} = -1 - \alpha$.

**Proposition 2.9.** For $0 < \alpha < 1/(k^2 - 1)$, we have $\mu_2^{NN} > 0$ of $(NN)$.

**Proof.** We shall write $\mu_2 = \mu_2^{NN}$ for short within this proposition.

First, $\mu_2$ can not be equal to zero. Otherwise, we will have an eigenfunction $\psi_2$ on $[0, \pi/k]$ such that

$$\psi_2'' + \psi_2 = -\frac{1}{\alpha} h^{-1 - \frac{1}{\alpha}} \psi_2, \quad \psi_2'(0) = \psi_2'\left(\pi/k\right) = 0.$$ 

By reflecting $\psi_2$ about $n\pi/k$ evenly for any $n$, we can extend $\psi_2$ to a smooth function defined on $\mathbb{S}^1$. This contradicts to the fact that the eigenspace of $L$ for $\lambda = 0$ is one dimensional, because $\psi_2 \notin \text{span}\{h_{\theta}\}$.

Second, suppose $\mu_2 < 0$ and $\psi_2$ is an eigenfunction. In the following Lemma 2.10, we get a function $\eta(\theta)$ with

$$\eta'' + \eta = -\frac{1}{\alpha} h^{-1 - \frac{1}{\alpha}} \eta$$

with $\eta(0) > 0$ and $\eta'(0) = 0$, $\eta(\pi/k) < 0$, $\eta'(\pi/k) < 0$. See Figure 2 for illustration.

We claim that $\eta$ has only one zero in $(0, \pi/k)$.

In fact, on the contrary assume $\eta(\theta_0) = \eta(\theta_1) = 0$ for $0 < \theta_0 < \theta_1 < \pi/k$. One can define a new function $\tilde{\eta}(\theta)$ such that it equals $\eta(\theta)$ if $\theta \in [\theta_0, \theta_1]$ and zero elsewhere. Then obviously $\tilde{\eta} \in H^1([0, \pi/k])$ and

$$\int_0^{\pi/k} \tilde{\eta}''^2 = \int_0^{\pi/k} \frac{1}{\alpha} h^{-1 - 1/\alpha} \tilde{\eta}^2.$$  \hfill (2.13)

Recall the following variational characterization of $\mu_1^{DD}$,

$$\mu_1^{DD} = \inf_{u \in H^1([0, \pi/k]), u \neq 0} \left\{ \frac{\int_0^{\pi/k} u''_\theta^2 - u'^2}{\int_0^{\pi/k} u'^2 \frac{1}{\alpha} h^{-1 - 1/\alpha}} - 1 \left| u(0) = u(\pi/k) = 0 \right. \right\}.$$ 

The infimum is achieved by the first eigenfunction of $(DD)$. Using (3) in Proposition 2.2, we have $\mu_1^{DD} = 0$ and eigenfunction $\psi_1^{DD} = h_{\theta}$, because $h_{\theta}$ does not change sign in $(0, \pi/k)$. However, (2.13) implies $\mu_1^{DD} < 0$. Contradiction. The claim is proved.

Let $a$ be the only zero of $\eta$ in $(0, \pi/k)$ and $b$ be that of $\psi_2$. Without loss of generality, we assume $\psi_2(\theta) > 0$ when $\theta \in (0, b)$. Otherwise one can
work on $-\psi_2$. Define $W[\psi_2, \eta] = \psi_2 \eta' - \psi_2' \eta$. Then, we have $W(0) = 0$, $W(\pi/k) > 0$ and

$$W'(\theta) = \frac{1}{\alpha} h^{-1-1/\alpha} \mu_2 \psi_2 \eta.$$

If $a \geq b$, then $W(b) \geq 0$ while $W' < 0$ in $(0, b)$. This is impossible because of $W(0) = 0$.

If $a < b$, then $W(b) < 0$, $W' < 0$ on $(b, \pi/k)$. This contradicts to $W(\pi/k) > 0$. Therefore $\mu_2$ can not be negative.

\[ \square \]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2}
\caption{The graph of $\eta$ with $\eta(0) = 1$ when $\alpha = 1/16$ and $k = 3$.}
\end{figure}

**Lemma 2.10.** Let $k \geq 3$ and $0 < \alpha < 1/(k^2 - 1)$. There exists a smooth function $\eta$ on $[0, 2\pi]$ satisfying

$$\eta'' + \eta + \frac{1}{\alpha} h^{-1-1/\alpha} \eta = 0$$

and $\eta(0) > 0$, $\eta'(0) = 0$, $\eta(\pi/k) < 0$ and $\eta'(\pi/k) < 0$.

**Proof.** We will use some notations in [2, Lemma 7.2]. Consider the function $U(\alpha, r, \theta)$ defined by

$$U_{\theta\theta} + U = -\frac{1}{\alpha}$$

$$U_\theta(\alpha, r, 0) = 0, \quad U_\theta(\alpha, r, \Theta(\alpha, r)) = 0$$

$$U_\theta(\alpha, r, \theta) < 0, \quad 0 < \theta < \Theta(\alpha, r)$$

$$U(\alpha, r, 0) = r U(\alpha, r, \Theta(\alpha, r))$$

where $\Theta = \Theta(\alpha, r)$ is the period function defined in [2, Definition 2.1]. Moreover, one can find

$$U(\alpha, r, \Theta(\alpha, r)) = \left( \frac{2\alpha}{1 - \alpha} \cdot \frac{1 - r^{-1-1/\alpha}}{r^2 - 1} \right)^{\frac{\alpha}{\alpha+1}}, \quad (2.15)$$

$$U(\alpha, r, 0) = r U(\alpha, r, \Theta(\alpha, r)). \quad (2.16)$$
We will omit dependence on $\alpha$ of $U$ and $\Theta$ in what follows. It follows from [2] that for each $\alpha \in (0, 1/(k^2 - 1))$, there exists a unique $r^* \geq 1$ such that $\Theta(r^*) = \pi/k$. The support function $h$ is given by $h(\theta) = U(r^*, \theta)$. Define $\eta(\theta) = \frac{d}{dr}U(r, \theta)|_{r=r^*}$. Then obviously $\eta$ satisfies (2.14). Since $U_\theta(r, 0) = 0$ for any $r$, we have $\eta_\theta(0) = 0$. Note that (2.16) implies

$$U(r, 0) = r \left( \frac{2\alpha}{1-\alpha} \cdot \frac{1 - r^{1-1/\alpha}}{r^2 - 1} \right)^{\alpha\pi/k}$$

Differentiating with respect to $r$ implies $\eta(0) > 0$. Since $U_\theta(r, \Theta(r)) = 0$, Differentiating with respect to $r$ gives

$${\eta_\theta}(\Theta(r)) + U_{\theta\theta}(r, \Theta(r)) \frac{d}{dr}\Theta(r) = 0. \tag{2.17}$$

Note that $d/dr \Theta(r) > 0$ if $\alpha \in (0, 1/3)$ by [2]. Also $U_{\theta\theta}(r_*, \pi/k) \geq 0$, because $U(r_*, \theta)$ attains the minimum at $r = r_*$. Here $U_{\theta\theta}(r_*, \pi/k)$ cannot be 0, otherwise combined with $U_\theta(r_*, \pi/k) = 0$, one gets $U$ is a constant. Inserting $r = r_*$ to the above equation, one can see $\eta_\theta(\pi/k) < 0$.

On the other hand, it follows from (2.16) that

$$U(r, \Theta(r)) = \left( \frac{2\alpha}{1-\alpha} \cdot \frac{1 - r^{1-1/\alpha}}{r^2 - 1} \right)^{\alpha\pi/k}$$

Taking the derivative with respect to $r$ of the above equation reveals $\frac{d}{dr}U(r, \Theta(r)) < 0$. Therefore $\eta(\pi/k) = \frac{d}{dr}U(r, \Theta(r))|_{r=r_*} < 0$. \hfill $\square$

**Proof of Theorem 2.4 Part (2).** Since $\lambda_{2k-1}$ has a two dimensional eigenspace with $2k-2$ nodal sets, Lemma 2.7 implies $\lambda_{2k-1} < \lambda_{2k}$. The Courant nodal domain and Lemma 2.5 imply that the eigenfunctions associated to $\lambda_{2k}$ must have $2k$ nodal sets. We need to show $\lambda_{2k} = 0$.

Towards a contradiction, suppose that $\lambda_{2k} < 0$. $h_\theta$ is an eigenfunction corresponding to the eigenvalue 0 and it also has $2k$ nodal sets. Thus, Lemma 2.7 implies that $\lambda_{2k}$ must be simple. Therefore, Lemma 2.8 says that the eigenfunction $\varphi_{2k}$ is even-reflection-symmetric with respect to $n\pi/k$ for all $n \in \mathbb{N}$ and it has exactly $2k$ nodal sets. Thus, the restriction of $\varphi_{2k}$ on $[0, \pi/k]$ is a Neumann eigenfunction to (NN). Since $\varphi_{2k}$ changes its sign exactly once in $[0, \pi/k]$, we have $\lambda_{2k} \geq \mu_2^{NN}$ which is the second Neumann eigenvalue. However, Proposition 2.9 says $\mu_2^{NN} > 0$.

Since $\lambda_{2k-1} < 0$ has $2k-2$ nodal domains and 0 is an eigenvalue having $2k$ nodal domains, we have $\lambda_{2k} = 0$. $\lambda_{2k}$ is simple by Proposition 2.2, and thus we will have the next eigenvalue $\lambda_{2k+1} > 0$. The nodal sets of eigenfunction associated to $\lambda_{2k+1}$ is $2k$ by Courant nodal domain theorem. Then the second part of Lemma 2.7 implies that $\lambda_{2k+1}$ also have to be simple. \hfill $\square$
We completed the proof the Theorem 2.4. From now on, we discuss about why we may not have $\lambda_k = \lambda_{k+1}$ when $k$ is even. These two eigenvalues are related to the (DN) and (ND).

**Lemma 2.11.** We have $\mu_1^{DN} < 0$ and $\mu_1^{ND} < 0$.

**Proof.** Suppose $\psi^{DN}$ is an eigenfunction corresponding to $\mu_1^{DN}$. Then make an even reflection of $\psi^{DN}$ with respect to $\pi/k$. We get $\psi^{DN}$ is an eigenfunction of

$$\psi'' + \psi = -\frac{1}{\alpha} h^{-1 - \frac{1}{2}} (\mu^{DN} + 1) \psi, \quad \psi(0) = \psi(2\pi/k) = 0$$

Since $\psi^{DN}$ does not change sign in $[0, 2\pi]$, it must be the first eigenfunction for the above problem. Note that the reflection of $h_\theta$ also makes an eigenfunction corresponds to $0$ for the above problem. We must have $\mu_1^{DN} < 0$.

The fact of (ND) can be proved through even reflection with respect to $\theta = 0$.

**Remark 2.12.** If $\lambda_k$ is simple, then Lemma 2.8 says $\varphi_k$ will have $k$ zeros. Moreover, Lemma 2.8 indicates that the restriction of $\varphi_k$ on $[0, \pi/k]$ will give us a first eigenfunction of (DN) or (ND) corresponds to $\mu_1^{DN}$ or $\mu_1^{ND}$. In fact $\lambda_k = \min\{\mu_1^{DN}, \mu_1^{ND}\}$. For the same reason, $\lambda_{k+1} = \max\{\mu_1^{DN}, \mu_1^{ND}\}$. A priori we do not know $\mu_1^{DN} = \mu_1^{ND}$.

### 3. Construction of Ancient Solutions

In this section, we construct ancient solutions converging to a shrinker $\Gamma$ after rescaling by using the Morse index $I$ we characterized in Section 2. Let us denote the Morse index of $L_{\Gamma^k}$ by $I(L_{\Gamma^k})$. In the section 2, we showed $I(L_{\Gamma^k}) = 2k - 1$ and $I(L_{\Gamma_c^1}) = 2\lceil 1 + 1/\alpha \rceil - 1$. Again, we shall simply suppress the notation to $I$ and $L$. One should interpret the following for each case $\Gamma = \Gamma^k$ or $\Gamma_c$ respectively.

We begin by considering the inhomogeneous linear PDE

$$\partial_\tau v = Lv + E_\Gamma(v)$$

Fix $\beta \in (0, 1)$, and for any $f : S^1 \times \mathbb{R} \to \mathbb{R}$ we define the seminorm

$$|f(\tau)|_{C^\beta} = \sup_{(\theta, t) \in S^1 \times (\tau - 1, \tau)} \left\{ \frac{|f(\theta, t_1) - f(\theta, t_2)|}{|\theta_1 - \theta_2|^{\beta} + |t_1 - t_2|^{\beta/2}} : (\theta_1, t_1) \neq (\theta_2, t_2) \right\}.$$

We use the special symbol $C^\beta$ to denote the parabolic norm in what follows. Notice that we write $\tau$ explicitly in $|f(\tau)|_{C^\beta}$ to indicate that the parabolic norm is taken on $S^1 \times (\tau - 1, \tau)$. For $l \geq 0$, define the norm

$$||f(\tau)||_{C^l, \beta} := \sum_{i+2j \leq l} \sum_{S^1 \times (\tau - 1, \tau)} |\partial^i_\theta \partial^j_t f|_{C^\beta}.$$

(3.1)
For some $\delta > 0$, define the norm
\[
\| f \|_{C^{1,\beta,\delta}} := \sup_{\tau \leq 0} \{ e^{-\delta \tau} \| f \|_{C^2(\Delta \times (\tau-1, \tau))} \}.
\] (3.2)

Suppose $X^{\delta}$ is the Banach space equipped with the norm $\| f \|_{C^{1,\beta,\delta}} < \infty$.

We fix once and for all an $L^2_h$ orthonormal sequence of eigenfunctions $\varphi_j$ of $-\mathcal{L}$ such that $\mathcal{L}\varphi_j = \lambda_j \varphi_j$ and $(\varphi_j, \varphi_j)_h = 1$. Define $v_j = (v, \varphi_j)_h$ and $P_j v = (v, \varphi_j)_h \varphi_j$. We also define $P_{\leq j} = \sum_{i=1}^{j} P_i$ and
\[
P_- = \sum_{j=0}^{I} P_j, \quad P_+ = \sum_{\{j: \lambda_j > 0\}} P_j, \quad P_0 = \sum_{\{j: \lambda_j = 0\}} P_j.
\] (3.3)

In addition, we define
\[
\| f \|_{L^2,\delta} = \sup_{\tau \leq 0} \{ e^{-\delta \tau} \| f(\cdot, \tau) \|_h \}.
\]

For the rest of this section, we will always choose $\delta$ as some positive constant different from $-\lambda_j$ for any $j$. Denote $J = \{ j : \lambda_j < -\delta \} \subset \{1, \ldots, I\}$. For example, $J = \emptyset$ if $\delta > \lambda_1 = -1 - \alpha$.

**Lemma 3.1.** Fix any $0 < \delta \notin \{ -\lambda_j \}_{j=1}^{\infty}$ and recall the operator $\mathcal{L}$ in (1.9). If $\| f \|_{L^2,\delta} < \infty$, then the equation
\[
\partial_\tau u - \mathcal{L} u = f(\theta, \tau), \quad \text{on} \quad \mathbb{S}^1 \times \mathbb{R}_-
\]
has a unique solution $u$ satisfying $\| u \|_{L^2,\delta} < \infty$ and $P_j u(\cdot, 0) = 0$ for $j \in J$. Furthermore, there exists $C = C(\alpha, \beta, \delta)$ such that $\| u \|_{L^2,\delta} \leq C \| f \|_{L^2,\delta}$ and $\| u \|_{C^{1,\beta,\delta}} \leq C \| f \|_{C^{0,\beta,\delta}}$ hold.

**Proof.** Recall that $\{ \varphi_i \}$ is an orthonormal basis of $L^2_h(\mathbb{S}^1)$ with respect to $(\cdot, \cdot)_h$. It suffices to solve
\[
\partial_\tau u_i + \lambda_i u_i = f_i, \quad \mathbb{S}^1 \times \mathbb{R}_-,
\]
where $u_i = (u, \varphi_i)_h$ and $f_i = (f, \varphi_i)_h$. Denote
\[
u_j(\tau) := \int_\tau^0 e^{\lambda_j(\tau-s)} f_j(s) ds, \quad j \in J,
\]
\[
u_j(\tau) := \int_\tau^{\tau} e^{\lambda_j(\tau-s)} f_j(s) ds, \quad j \in J^c = \mathbb{Z}_+ \setminus J.
\]
Notice the integral on the RHS is well-defined because $|f_j(s)| \leq \| f(\cdot, \cdot) \|_h \leq \| f \|_{L^2,\delta}$. Define $u(\cdot, \tau) = \sum_{j=1}^{\infty} u_j(\tau) \varphi_j(\cdot)$. It is easy to see $P_j(u(\cdot, 0)) = 0$ for any $j \in J$. 
Choose \( \delta' \) and \( \delta'' \) satisfying \( \max_{j \in J} \{ \lambda_j \} < -\delta' < -\delta < -\delta'' < \min_{j \in J^c} \{ \lambda_j \} \). Note that for \( j \in J \)

\[
u^2_j(\tau) \leq \int_{\tau}^{0} e^{2(\lambda_j + \delta')(s-\tau)} ds \int_{\tau}^{0} e^{-2\delta'(s-\tau)} |f_j|^2 ds \leq C \int_{\tau}^{0} e^{-2\delta'(s-\tau)} |f_j|^2 ds
\]

and for \( j \in J^c \)

\[
u^2_j(\tau) \leq \int_{-\infty}^{\tau} e^{2(\lambda_j + \delta'')(s-\tau)} ds \int_{-\infty}^{\tau} e^{-2\delta''(s-\tau)} |f_j|^2 ds \leq C \int_{-\infty}^{\tau} e^{-2\delta''(s-\tau)} |f_j|^2 ds.
\]

Combining the above two inequalities and using \( |f_j(s)| \leq ||f||_{L^2,\delta} e^{\delta s} \), one obtains

\[
||u(\cdot, \tau)||^2_h = \sum_{j} \nu^2_j(\tau) \leq C \int_{\tau}^{0} e^{-2\delta'(s-\tau)} |f_j|^2 ds + C \int_{-\infty}^{\tau} e^{-2\delta''(s-\tau)} |f_j|^2 ds \\
\leq C||f||^2_{L^2,\delta} e^{2\delta \tau}.
\]

Therefore \( ||u||_{L^2,\delta} \leq C||f||_{L^2,\delta} \).

Let’s establish \( C^{2,\beta,\delta} \) bounds. By the interior parabolic Schauder estimates (for instance, see [16, (C.6)]), we have that for any \( \tau \leq 0 \),

\[
||u||_{C^{2,\beta}(S^1 \times (\tau-1,\tau))} \leq C \left( ||u||_{L^2(S^1 \times (\tau-1,\tau))} + ||f||_{C^{0,\beta}(S^1 \times (\tau-1,\tau))} \right).
\]

Multiplying by \( e^{\delta \tau} \) and taking the supremum over \( \tau \leq 0 \) yield

\[
||u||_{C^{2,\beta,\delta}} \leq C(||u||_{L^2,\delta} + ||f||_{C^{0,\beta,\delta}}) \leq C(||f||_{L^2,\delta} + ||f||_{C^{0,\beta,\delta}}) \leq C||f||_{C^{0,\beta,\delta}}.
\]

We shall use contraction mapping theorem and the above lemma repeatedly to construct ancient solutions. Let us introduce some necessary notations. For any \( a = (a_1, \cdots, a_I) \in \mathbb{R}^I \), denote \( |a| = (\sum_{i=1}^{I} a_i^2)^{\frac{1}{2}} \). We introduce auxiliary operators which maps any integer set \( J \subset \{1, 2, \cdots, I\} \) to functions,

\[
u^j: \mathbb{R}^I \to L^2_h \times \mathbb{R}_-,
\]

\[
u^j(a) := \sum_{j \in J} a_j e^{-\lambda_j \tau} \varphi_j.
\]

Denote \( L = [\lambda_1/\lambda_I] \). For each \( l = 1, \cdots, L \), define

\[
J^{(l)} = \{ m : (l+1)\lambda_I < \lambda_m \leq l\lambda_I \}.
\]

Then \( \bigcup_{l=1}^{L} J^{(l)} \) is a partition of \( \{1, \cdots, I\} \) according to the negative eigenvalues of \( L \). Choose \( \delta_l \) satisfying

\[
\max\{\lambda_j : j \in J^{(l+1)}\} \leq (l+1)\lambda_I < -\delta_l < \min\{\lambda_j : j \in J^{(l)}\}.
\]
Write \( X^{(l)} = X^{\delta_l} \), \( \iota^{(l)} = \iota^{j(l)} \), and \( P^{(l)} = \sum_{j, \lambda_j < -\delta_l} P_j \) for simplicity. In what follows, we will use the symbol \( \lesssim \) for inequalities that hold up to multiplicative constants that may depend on \( \alpha, h \).

Here is the main result of this section

**Theorem 3.2.** Let \( L = [\lambda_1/\lambda_l]^6 \). There exists some \( \varepsilon_0 > 0 \) satisfying the following significance. Given \( a = (a_1, \cdots, a_l) \in \mathbb{R}^l \) with \( |a| < \varepsilon_0 \), there exist a set of functions \( \{\iota^{(l)}\}_{l=1}^L \) uniquely determined and depending continuously on \( a \) such that for each \( l = 1, \cdots, L \), we have \( \iota^{(l)} - \iota^{(l)}(a) \in X^{(l)}, P^{(l)}(\iota^{(l)}(\alpha)) = 0 \), and \( \sum_{j=1}^l \iota^{(j)} \) is an ancient solution of (1.8) for \((-\infty, 0]\). More importantly

\[
\lim_{\tau \to -\infty} e^{\lambda_m \tau} (\iota^{(l)}(\cdot, \tau), \varphi_m)_h = a_m, \quad m \in J^{(l)}.
\]  

Let us first prove a proposition which will be needed in the proof of Theorem 3.2.

**Proposition 3.3.** There exists some constants \( C = C(\alpha, h) \) and \( \epsilon = \epsilon(\alpha, h) > 0 \) such that if \( |v_{\theta \theta} + v| \leq \epsilon \) then

\[
|E_T(v)| \leq C|v_{\theta \theta} + v|^2,
\]  

\[
|E_T(v)(\tau)|_{C^\beta} \leq C|(v_{\theta \theta} + v)(\tau)|_{C^\beta} |(v_{\theta \theta} + v)(\tau)|_{C^0}.
\]

Moreover, if \( u, v \) satisfy \( |u_{\theta \theta} + u| + |v_{\theta \theta} + v| \leq \epsilon \) then

\[
|E_T(u)(\tau) - E_T(v)(\tau)|_{C^\beta} \leq C|((u - v)_{\theta \theta} + u - v)(\tau)|_{C^\beta} \left[ |(u_{\theta \theta} + u)(\tau)|_{C^0} + |(v_{\theta \theta} + v)(\tau)|_{C^0} \right]
\]

**Proof.** By the definition \( E_T \) in (1.8), we obtain

\[
E_T(v) = -h \left(1 + h^{\frac{\alpha}{2}}(v_{\theta \theta} + v)\right)^{-\alpha} + h - \alpha h^{1 + \frac{\alpha}{2}}(v_{\theta \theta} + v).
\]

Using the Taylor expansion of \((1 + x)^{-\alpha}\), it is easy to know \(|(1 + x)^{-\alpha} - 1 + \alpha x| \leq C(\alpha)x^2\) whenever \(|x| < 1/2\). Therefore we have

\[
|E_T(v)| \leq C(\alpha, h)|v_{\theta \theta} + v|^2
\]

whenever \(|v_{\theta \theta} + v| < \frac{1}{2} h^{-1/\alpha}\).

Our second conclusion follows from the following observations

\[
|(1 + x)^{-\alpha} + \alpha x - (1 + y)^{-\alpha} - \alpha y| \leq C(\alpha)(x - y)^2, \quad |x| + |y| < \frac{1}{2}
\]

\(^6[x]\) means the greatest integer less than or equal to \( x \).
and consequently for any \( t_1, t_2 \in [\tau - 1, \tau] \)
\[
|E_T(v)(\theta_1, t_1) - E_T(v)(\theta_2, t_2)|
\leq C(\alpha, h)|(v_{\theta\theta} + v)(\theta_1, t_1) - (v_{\theta\theta} + v)(\theta_2, t_2)|^2
\leq C(\alpha, h)|(|\theta_1 - \theta_2|^\beta + |t_1 - t_2|^\beta/2)|(v_{\theta\theta} + v)(\tau)|_{C^0}|(v_{\theta\theta} + v)(\tau)|_{C^0}.
\]
The estimates of \( |E_T(u)(\tau) - E_T(v)(\tau)|_{C^0} \) can be proved similarly. 

Now we can prove the main theorem of this section.

**Proof of Theorem 3.2.** We shall find all \( v^{(l)} \) by the induction. First, we notice that \( v^{(1)} \) is an ancient solution to the linear equation \( \partial_\tau v = L v \). Therefore, to find \( v^{(1)} \), we assume \( v^{(1)} = w + \iota^{(1)}(a) \) for some \( w \) to be determined. Then (1.8) is equivalent to
\[
\partial_\tau w = L w + (w + \iota^{(1)}(a))
\]
(3.7)
Here and in the following, we shall write \( E(v) = E_T(v) \) and \( r[v] = v_{\theta\theta} + v \) for short.

**Claim 1.** There exists small \( \varepsilon_0 \) such that if \( ||w||_{X(1)} + |a| < \varepsilon_0 \) then
\[
||E(w + \iota^{(1)}(a))||_{C^{0,\beta,\delta_1}} \lesssim ||w||_{X(1)}^2 + |a|^2,
\]
(3.8)
\[
||E(w_1 + \iota^{(1)}(a)) - E(w_2 + \iota^{(1)}(a))||_{C^{0,\beta,\delta_1}} \lesssim \varepsilon_0 ||w_1 - w_2||_{X(1)}.
\]
(3.9)
In fact, one can easily derive from (3.5) and (3.6) that there exists \( \varepsilon_0 > 0 \) and some constant \( C(\alpha, \beta, h) \) such that
\[
||E(v)(\tau)||_{C^{0,\beta}} \leq C||r[v](\tau)||_{C^{0,\beta}}^2
\]
provided \( ||r[v](\tau)||_{C^{0}} < \varepsilon_0 \). Furthermore
\[
||E(v_1)(\tau) - E(v_2)(\tau)||_{C^{0,\beta}}
\leq C(||r[v_1](\tau)||_{C^{0,\beta}} + ||r[v_2](\tau)||_{C^{0,\beta}})|r[v_1 - v_2](\tau)||_{C^{0,\beta}}
\]
(3.10)
provided \( ||r[v_1](\tau)||_{C^{0}} + ||r[v_2](\tau)||_{C^{0}} < \varepsilon_0 \). Therefore
\[
||E(w + \iota^{(1)}(a))(\tau)||_{C^{0,\beta}} \lesssim ||w(\tau)||_{C^{2,\beta}} + ||r\iota^{(1)}(a)(\tau)||_{C^{0,\beta}}^2
\]
\[
\lesssim ||w(\tau)||_{C^{2,\beta}} + |a|^2 e^{-2\lambda_T \tau}.
\]
Recall our definition of the norm (3.2), multiplying the above inequality by \( e^{\delta_1 \tau} \) and noticing \( -\delta_1 > 2\lambda_T \), one can get (3.8) holds. Moreover
\[
||E(w_1 + \iota^{(1)}(a)) - E(w_2 + \iota^{(1)}(a))||_{C^{0,\beta}(\mathbb{S}^1 \times (\tau - 1, \tau))}
\lesssim (||r[w_1](\tau)||_{C^{0,\beta}} + ||r[w_2](\tau)||_{C^{0,\beta}} + |a| e^{-\lambda_T \tau})||r[w_1 - r[w_2]](\tau)||_{C^{0,\beta}}
\]
which implies (3.9) holds. Thus, the claim is proved.
Define a map $S : \{ f \in X^{(1)} : \|f\|_{X^{(1)}} < \epsilon_0 \} \to X^{(1)}$ by $S(w) = u$ where $u$ is the solution of
\[ \partial_r u - \mathcal{L} u = E(w + \iota^{(1)}(a)) \quad \text{on} \quad S^1 \times \mathbb{R}_- \]
with $P^{(1)}(u(\cdot, 0)) = 0$. By Lemma 3.1, such $u \in X^{(1)}$ is unique, so $S(w)$ is well-defined. Moreover, Lemma 3.1 says,
\[ \|u\|_{X^{(1)}} \lesssim \|u\|_{X^{(1)}}^2 + |a|^2 \leq \epsilon_0^2 + |a|^2 \]
and
\[ \|S(w_1) - S(w_2)\|_{X^{(1)}} \lesssim (\epsilon_0 + |a|)\|w_1 - w_2\|_{X^{(1)}}. \]
Choosing $\epsilon_0$ small enough, $S$ will be a contraction mapping on $\{ f \in X^{(1)} : \|f\|_{X^{(1)}} < \epsilon_0 \}$. Therefore it has a fixed point $w$ which solves (3.7). Since $w \lesssim e^{\delta_1 \tau}$, we have
\[ \lim_{\tau \to -\infty} e^{\lambda \tau}(I^{(1)})_{\varphi_m}h = \lim_{\tau \to -\infty} e^{\lambda \tau}(\iota^{(1)}(a), \varphi_m)h = a_m, \quad m \in J^{(1)}. \]
Therefore, we have found $v^{(1)}$ and (3.4) is true for $l = 1$.

Suppose that we have found $v^l$ up to $v^{(l)}$, and (3.4) is established up to $l$ by the induction. If $J^{(l+1)} = \emptyset$, let $v^{(l+1)} = 0$. Obviously the theorem still holds for such $v^{(l+1)}$. If $J^{(l+1)} \neq \emptyset$, then we can find $v^{(l+1)}$ by the following process. Let $v^{(l+1)} = w + \iota^{(l+1)}(a)$. Since we require $\sum_{j=1}^{l+1} v^{(j)}$ is an ancient solution of (1.8), it suffices to find $w \in X^{(l+1)}$ such that
\[ \partial_r w = \mathcal{L} w + E^{(l+1)}(w) \quad (3.11) \]
where
\[ E^{(l+1)}(w) = E \left( w + \iota^{(l+1)}(a) + \sum_{j=1}^l v^{(j)} \right) - E \left( \sum_{j=1}^l v^{(j)} \right). \quad (3.12) \]

**Claim 2.** There exists $\epsilon_0$ small such that if $\|w\|_{X^{(l+1)}} + |a| < \epsilon_0$ then
\[ \|E^{(l+1)}(w)\|_{\mathcal{C}^{0, \beta}_{l+1}} \lesssim \|w\|_{X^{(l+1)}}^2 + |a|^2, \quad (3.13) \]
\[ \|E^{(l+1)}(w_1) - E^{(l+1)}(w_2)\|_{\mathcal{C}^{0, \beta}_{l+1}} \lesssim \epsilon_0 \|w_1 - w_2\|_{X^{(l+1)}}. \quad (3.14) \]

In fact, using (3.10), for $\tau[w_1]$ and $\tau[w_2]$ small
\[ \|E(w_1) - E(w_2)(\tau)\|_{\mathcal{C}^{0, \beta}} \lesssim (\|\tau[w_1](\tau)\|_{\mathcal{C}^{0, \beta}} + \|\tau[w_2](\tau)\|_{\mathcal{C}^{0, \beta}}) \|\tau[w_1] - \tau[w_2]\|_{\mathcal{C}^{0, \beta}}. \]
This implies
\[ \|E^{(l+1)}(w)(\tau)\|_{\mathcal{C}^{0, \beta}} \lesssim (\|w(\tau)\|_{\mathcal{C}^{2, \beta}} + |a|e^{-\lambda_I \tau})(\|w(\tau)\|_{\mathcal{C}^{2, \beta}} + |a|e^{-\lambda_I \tau}) \lesssim e^{-(l+2)\lambda_I \tau}(\|w\|_{\mathcal{C}^{2, \beta, -(l+1)\lambda_I}} + |a|)(\|w\|_{\mathcal{C}^{2, \beta, -(l+1)\lambda_I}} + |a|). \]
Recalling (3.2) and $(l + 2)\lambda_I < -\delta_{l+1} < (l + 1)\lambda_I$, one can see that (3.13) holds. The proof of (3.14) can be derived similarly.
Define a map $S : \{ f \in X^{(l+1)} : ||f||_{X^{(l+1)}} < \varepsilon_0 \} \to X^{(l+1)}$ by $S(w) = u$ where $u$ is the unique solution of
$$\partial_t u - Lu = E^{(l+1)}(w) \quad \text{on} \quad S^1 \times \mathbb{R}_-$$
with $P^{(l+1)}(u(\cdot, 0)) = 0$. Taking $\varepsilon_0$ small enough, $S$ is a contraction mapping. Note that (3.4) is satisfied for $m \in J^{(l+1)}$. The existence of $v^{(l+1)}$ is established.

Finally, the uniqueness and continuity of $v^{(l)}$ also follow from the contraction mapping theorem. □

According to Theorem 3.2, we can have $I$-parameter family of ancient solutions, but only $I - 3$ of them are important, because the first three of them depends on the time and space center we choose in the renormalization (1.5). That is, one can always just shift the non-rescaled ancient solution by time and space to edit the first three parameters.

**Proposition 3.4.** Let $\Gamma_t$ be an $\alpha$-CSF asymptotic to $\Gamma$ after rescaling where $\Gamma = \tilde{\Gamma}_\alpha$ or $\tilde{\Gamma}_c$, and let $\tilde{u}$ denote the support function of the rescaled flow $\tilde{\Gamma}_\tau$. Then, given $B = (b_1, b_2, b_3) \in \mathbb{R}^3$ the ancient flow
$$\Gamma^B_t = \Gamma_{t+b_1} + (b_2, b_3) \subset \mathbb{R}^2$$
(3.15)
satisfies
$$\tilde{u}^B(\theta, \tau) - \tilde{u}(\theta, \tau)
= (1 + \alpha)^{-\frac{1}{\alpha+1}} [b_2 \cos \theta + b_3 \sin \theta] e^{\tau} + \frac{b_1}{1 + \alpha} e^{(1+\alpha)\tau} h + o(e^{(1+\alpha)\tau})$$
where $\tilde{u}^B$ denotes the support function of the rescaled flow $\tilde{\Gamma}^B_\tau$. Consequently if $\tilde{\Gamma}_\tau$ is constructed from $\alpha$, then $\tilde{\Gamma}^B_\tau$ is constructed from using
$$\alpha + \left( \frac{b_1}{1 + \alpha}, (1 + \alpha)^{-\frac{1}{\alpha+1}} b_2, (1 + \alpha)^{-\frac{1}{\alpha+1}} b_3, 0, \cdots, 0 \right).$$

**Proof.** Our assumption implies
$$u^B(\theta, t) = u(\theta, t + b_1) + b_2 \cos \theta + b_3 \sin \theta$$
where $u^B$ and $u$ are the support functions of $\Gamma^B_t$ and $\Gamma_t$. It follows from (1.5) that
$$\tilde{u}^B(\theta, \tau) = (1 + \alpha)^{-\frac{1}{\alpha+1}} e^{\tau} u^B(\theta, -e^{-(1+\alpha)\tau})
= (1 + \alpha)^{-\frac{1}{\alpha+1}} e^{\tau} \left[ u(\theta, -e^{-(1+\alpha)\tau} + b_1) + b_2 \cos \theta + b_3 \sin \theta \right]
= e^{\tau - \tau_1} \tilde{u}(\theta, \tau_1) + (1 + \alpha)^{-\frac{1}{\alpha+1}} [b_2 \cos \theta + b_3 \sin \theta] e^{\tau}$$
(3.16)
where
$$\tau_1 = \frac{-1}{1 + \alpha} \log(e^{-(1+\alpha)\tau} - b_1).$$
As \( \tau \to -\infty \), we have \( \tau_1 = \tau + (1 + \alpha)^{-1}b_1e^{(1+\alpha)\tau} + o(e^{(1+\alpha)\tau}) \). Consequently \( e^{\tau_1} = 1 + (1 + \alpha)^{-1}b_1e^{(1+\alpha)\tau} + o(e^{(1+\alpha)\tau}) \).

Since \( \bar{\Gamma}_\tau \) converges to some self-shrinker \( \Gamma \) with support function \( h \), \( \bar{u}(\theta, \tau) \to h \) as \( \tau \to -\infty \). Plugging the above information of \( \tau_1 \) to (3.16), one gets the conclusion.

**Proof of Theorem 1.3.** It follows from Theorem 3.2 that there exists a map

\[
S : B_{\epsilon_0}(\subset \mathbb{R}^I) \to C^{2,\beta}(\mathbb{S}^1 \times (-\infty, 0])
\]

such that \( S(\mathbf{a}) = \sum_{j=1}^Lv^{(j)} \) is an ancient solution of (1.8).

Suppose that \( \mathbf{a}, \mathbf{b} \in \mathbb{R}^I \) satisfy \( a_k - b_k \neq 0 \) and \( a_i - b_i = 0 \) for all \( i > k \). Assume \( \lambda_k \in \mathcal{J}^{l+1} \) for some \( l \). Then for any \( j \in \{1, \ldots, l\} \), careful tracking the proof of Theorem 3.2 shows that \( v^{(j)}_a = v^{(j)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem. In the step to find \( v^{(l+1)}_a = v^{(l+1)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem. In the step to find \( v^{(l+1)}_a = v^{(l+1)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem. In the step to find \( v^{(l+1)}_a = v^{(l+1)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem. In the step to find \( v^{(l+1)}_a = v^{(l+1)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem. In the step to find \( v^{(l+1)}_a = v^{(l+1)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem. In the step to find \( v^{(l+1)}_a = v^{(l+1)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem. In the step to find \( v^{(l+1)}_a = v^{(l+1)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem. In the step to find \( v^{(l+1)}_a = v^{(l+1)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem. In the step to find \( v^{(l+1)}_a = v^{(l+1)}_b \), because of the uniqueness of them obtained through the contraction mapping theorem.

\[
S(\mathbf{a})(\theta, \tau) - S(\mathbf{b})(\theta, \tau) = v^{(l+1)}_a - v^{(l+1)}_b + O(e^{-\delta_{l+1}\tau})
= (a_k - b_k)e^{-\lambda_k\tau}\varphi_k(\theta) + O(e^{-\lambda_{k-1}\tau}) + O(e^{-\delta_{l+1}\tau})
= (a_k - b_k)e^{-\lambda_k\tau}\varphi_k(\theta) + o(e^{-\lambda_k\tau})
\]

when \( \lambda_{k-1} < \lambda_k \), and

\[
S(\mathbf{a})(\theta, \tau) - S(\mathbf{b})(\theta, \tau) = e^{-\lambda_k\tau} \sum_{i=k-1}^{k} (a_i - b_i)\varphi_i(\theta) + o(e^{-\lambda_k\tau})
\]

when \( \lambda_{k-1} = \lambda_k \), where \( \varphi_i \) are eigenfunctions of \( \mathcal{L}_\Gamma \) with the eigenvalue \( \lambda_i \) and \( \langle \varphi_i, \varphi_j \rangle_{L^2_h} = \delta_{ij} \).

Proposition 3.4 says it suffices to have the map for \( \mathbf{a} = (0, 0, 0, a_3, \ldots, a_I) \), because the other ancient solutions can be generated by these ones from a different choice of time and space center in the renormalization. Removing the first three zeros of \( \mathbf{a} \) and reindexing each components, we abuse the notation by still denoting \( \mathbf{a} = (a_1, \ldots, a_{I-3}) \). Therefore we have a map

\[
S : B_{\epsilon_0}(0) \subset \mathbb{R}^{I-3} \to C^{2,\beta}(\mathbb{S}^1 \times (-\infty, 0])
\]

with the desired property. \( \square \)

Notice that in Theorem 3.2 there is a restriction \( |\mathbf{a}| < \epsilon_0 \). It is possible to get around this by translating in the \( \tau \) (which is equivalent to parabolic scaling in the corresponding non-rescaled ancient solution). However, one has to pay the price that these ancient solutions may not live up to \( \tau = 0 \).
Theorem 3.5. There exists a continuous map $S$ (define in (3.18)) which maps any $a \in \mathbb{R}^I$ to $C^{2,\beta,-\lambda_I}(S^1 \times (-\infty, T(a))]$ such that $S(a)$ is an ancient solution of (1.8) on $(-\infty, T(a)]$, where $T(a)$ is defined in (3.17). Moreover there is a unique decomposition $S(a) = \sum_{l=1}^{L} v^{(l)}$ with
\[
\lim_{\tau \to -\infty} e^{\lambda m \tau} (v^{(l)}(\cdot, \tau), \varphi_m)_h = a_m, \quad \forall m \text{ satisfying } \lambda m \in ((l+1)\lambda_I, l\lambda_I]
\]
for any $l = 1, \cdots, L$.

Proof. For any $a \in \mathbb{R}^I$, let
\[
T(a) = \frac{1}{2(1+\alpha)} \max \{ \log \frac{\varepsilon_0}{|a|^2}, 0 \}
\]
then $\sum_{l=1}^{I} e^{-2\lambda m \tau} a^2_m < \varepsilon_0$. Then by the previous theorem, one can find \{\v^{(l)}\}_{l=1}^{L} such that $\sum_{l=1}^{L} v^{(l)}$ is an ancient solution of (1.8) on $\mathbb{S}^1 \times (-\infty, 0]$ and
\[
\lim_{\tau \to -\infty} e^{\lambda m \tau} (v^{(l)}(\cdot, \tau), \varphi_m)_h = e^{-\lambda m \tau} a_m, \quad m \in J^{(l)}.
\]
So we define a map $S$ by translating $v^{(l)}$
\[
S(a)(\cdot, \tau) = \sum_{l=1}^{L} v^{(l)}(\cdot, \tau - T).
\]
One can easily verify that $S(a)$ is an ancient solution of (1.8) on $\mathbb{S}^1 \times (-\infty, -T]$ and
\[
\lim_{\tau \to -\infty} e^{\lambda m \tau} (v^{(l)}(\cdot, \tau - T), \varphi_m)_h = a_m, \quad m \in J^{(l)}.
\]
\[\square\]

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