CYCLIC ACTIONS ON RATIONAL RULED SYMPLECTIC
FOUR-MANIFOLDS

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Abstract. Let $(M, \omega)$ be a ruled symplectic four-manifold. If $(M, \omega)$ is rational, then every homologically trivial symplectic cyclic action on $(M, \omega)$ is the restriction of a Hamiltonian circle action.

1. Introduction

1.1. Theorem. Let $(M, \omega)$ be a rational ruled symplectic four-manifold. Then every homologically trivial symplectic cyclic action on $(M, \omega)$ is the restriction of a Hamiltonian circle action.

A ruled symplectic four-manifold is an $S^2$-bundle over a closed Riemann surface, with a symplectic ruling: a symplectic form on the total space that is nondegenerate on each fiber. It is rational if the Riemann surface is $S^2$, and irrational otherwise.

A cyclic action is an effective action of a cyclic group $\mathbb{Z}_r = \mathbb{Z}/r\mathbb{Z}$ of finite order $1 < r < \infty$. An action is called homologically trivial if it induces the identity map on homology.

An effective symplectic action of a torus $T = T^k = (S^1)^k$ on a symplectic manifold $(M, \omega)$ is Hamiltonian if there exists a moment map, that is, an invariant smooth map $\Phi: M \to \mathfrak{t}^* = \mathbb{R}^k$ such that $d\Phi_j = -\iota(\xi_j)\omega$ for all $j = 1, \ldots, k$, where $\xi_1, \ldots, \xi_k$ are the vector fields that generate the torus action. A Hamiltonian circle action is always homologically trivial because the circle group is connected.

A Hamiltonian action of a torus $T$ defines a one-to-one homomorphism from $T$ to the group of Hamiltonian diffeomorphisms $\text{Ham}(M, \omega)$. The image is a subgroup of $\text{Ham}(M, \omega)$ that is isomorphic to $T$. Every circle subgroup of $\text{Ham}(M, \omega)$ is obtained this way.

1.2. Corollary. Let $(M, \omega)$ be a rational ruled symplectic four-manifold. Then every finite cyclic subgroup of $\text{Ham}(M, \omega)$ embeds in a circle subgroup of $\text{Ham}(M, \omega)$.

Since every compact 4-dimensional Hamiltonian $S^1$-space is Kähler, i.e., admits a complex structure such that the action is holomorphic and the symplectic form is Kähler [11, Theorem 7.1], Theorem [11] implies the following corollary.

1.3. Corollary. Let $(M, \omega)$ be a ruled symplectic four-manifold equipped with a homologically trivial symplectic cyclic action. If $(M, \omega)$ is rational, there exists an integrable almost complex structure compatible with $\omega$ such that the cyclic action is holomorphic.
Related works. This study fits into the broader topological classification of group actions on four-manifolds. The question of whether a (pseudofree, homologically trivial, locally linear) cyclic action on a four-manifold \( M \) must be the restriction of a circle action is studied also in the holomorphic, differentiable, and topological categories; see \[7\]. When \( M = \mathbb{C}P^2 \), the answer is Yes in all three categories \[20\], (in fact, these categories coincide). When \( M \) is a rational ruled surface, the answer is Yes in the holomorphic category, however there exist “exotic” homologically trivial pseudofree cyclic actions on \( M \) that are not holomorphic but are smoothable with respect to some smooth structure \[21\]. By our result, Corollary \[1.3\], these actions cannot be symplectic.

In the symplectic category, Chen \[4\] showed that the answer is Yes when \((M, \omega)\) is \( \mathbb{C}P^2 \) with the standard Fubini–Study form. Furthermore, Chen \[4\] gave an answer in a non-closed case: the answer is Yes if \((M, \omega)\) is the four-ball \( B^4 \) with the standard symplectic form \( \sum_{i=1}^{2} dx_i \wedge dy_i \), and the cyclic action is linear near the boundary of \( B^4 \).

In a previous paper \[5\], we showed that the statement of Theorem \[1.1\] does not hold in general. On \((M, \omega)\), which is either a six-point blowup of certain sizes of \( \mathbb{C}P^2 \), or a three-point blowup of certain sizes of an irrational ruled symplectic four-manifold, we gave an example of symplectic action of \( \mathbb{Z}_2 \), acting trivially on homology, and showed that it cannot extend to a Hamiltonian circle action; we also showed that \((M, \omega)\) does admit a Hamiltonian circle action.

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2. Almost complex structures and pseudo-holomorphic spheres

In this section we recall definitions and results on almost complex structures and pseudo-holomorphic spheres, and deduce corollaries for pseudo-holomorphic spheres with respect to invariant almost complex structures.

Almost complex structures. An almost complex structure on a manifold \( M \) is an automorphism \( J: TM \to TM \) such that \( J^2 = -\text{Id} \). An almost complex structure \( J \) is integrable if it is induced from a complex manifold structure. It is compatible with \( \omega \) if \( \langle \cdot, \cdot \rangle := \omega(\cdot, J \cdot) \) is a Riemannian metric on \( M \). A Kähler manifold is a symplectic manifold with an integrable almost complex structure that is compatible with the symplectic form. The space of all compatible almost complex structures on a symplectic manifold \((M, \omega)\) is non-empty and contractible \[15, Proposition 4.1\]. We denote it by \( \mathcal{J} := \mathcal{J}(M, \omega) \). The first Chern class of the complex vector bundle \((TM, J)\) is independent of the choice of \( J \in \mathcal{J} \); we denote it by \( c_1(TM) \).
2.1. Let \((V, \omega)\) be a symplectic vector space and \(g\) an inner product on \(V\). Denote by \(g^\sharp : V \to V^*\) the isomorphism \(u \mapsto g(u, \cdot)\) and by \(\omega^\sharp : V \to V^*\) the isomorphism \(v \mapsto \omega(v, \cdot)\). Then \(A = (g^\sharp)^{-1} \circ \omega^\sharp\) is anti-symmetric with respect to \(g\). Moreover, \(AA^*\) is symmetric and positive definite. Let \(P\) be a positive square root of \(AA^*\). Then \(J = P^{-1}A\) is a compatible complex structure on \(V\). The factorization \(A = PJ\) is called the polar decomposition of \(A\).

Suppose \((M, \omega)\) is a symplectic manifold and \(g\) a Riemannian metric on \(M\). Then we note that the polar decomposition is canonical and the above construction of a compatible almost complex structure is smooth. See [3, Sec 3.1].

2.2. **Claim.** For every symplectic action of a compact Lie group \(G\) on a compact symplectic manifold \((M, \omega)\) there exists a \(G\)-invariant \(\omega\)-compatible almost complex structure.

**Proof.** Take a \(G\)-invariant Riemannian metric \(g\) on \(M\). Such a metric is obtained from some Riemannian metric \(g'\) by averaging with respect to the action of \(G\) as follows:

\[
g(u, v) := \int_G g'(\sigma_a^* u, \sigma_a^* v) \, da
\]

for \(u, v \in TM\), where \(G \ni a \mapsto \sigma_a \in \text{Symp}(M, \omega)\) denotes the action and \(da\) denotes the Haar measure. The polar decomposition in §2.1 associated to the invariant metric \(g\) provides a \(G\)-invariant almost complex structure. \(\square\)

**J-holomorphic spheres in four-dimensional symplectic manifolds.** A (parametrized) \(J\)-holomorphic sphere, or \(J\)-sphere for short, is a map from \(\mathbb{CP}^1\) to an almost complex manifold, \(f : (\mathbb{CP}^1, j) \to (M, J)\) that satisfies the Cauchy–Riemann equations \(df \circ j = J \circ df\) at every point in \(\mathbb{CP}^1\). The image \(C = f(\mathbb{CP}^1)\) is an unparamterized \(J\)-sphere. A parametrized \(J\)-holomorphic sphere is called simple if it cannot be factored through a branched covering of the domain. An embedding is a one-to-one immersion which is a homeomorphism with its image. An embedded \(J\)-sphere \(C \subset M\) is the image of a \(J\)-holomorphic embedding \(f : (\mathbb{CP}^1, j) \to (M, J)\). If \(J\) is \(\omega\)-compatible, then such a \(C\) is an embedded \(\omega\)-symplectic sphere.

**Gromov’s compactness theorem** [8, 1.5.B] guarantees that, given a converging sequence of almost complex structures on a compact manifold, a corresponding sequence of holomorphic curves with bounded symplectic areas has a weakly converging subsequence; the limit under weak convergence might be a connected union of holomorphic curves.

In dimension four, \(J\)-holomorphic spheres admit nice properties, which we will use in this study, such as the adjunction formula [16, Corollary E.1.7], the Hofer–Lizan–Sikorav regularity criterion [10], and the positivity of intersections of \(J\)-holomorphic spheres [16, Appendix E and Proposition 2.4.4]. As a result of the positivity of intersections we have the following claim.
2.3. **Claim.** Consider a symplectic action of a compact Lie group $G$ on a symplectic four-manifold $(M, \omega)$. Assume that the action is trivial on $H_2(M; \mathbb{Z})$. Let $J_G$ be a $G$-invariant $\omega$-compatible almost complex structure on $M$. Let $C$ be a $J_G$-holomorphic sphere. Then $C$ is $\omega$-symplectic. Moreover, let $G \to \text{Symp}(M, \omega)$, $g \mapsto \sigma_g$, denote the action,  

- if $C$ is of negative self-intersection, then it is $G$-invariant, i.e., $\sigma_g(C) = C$ for all $g \in G$;  
- if $C$ is of self-intersection zero, then for $g \in G$, the image of $C$ under $\sigma_g$ either equals $C$ or is disjoint from $C$.

3. **Configurations of $J$-holomorphic spheres in rational ruled symplectic four-manifolds**

3.1. The underlying space of a rational ruled symplectic four-manifold is either $S^2 \times S^2$ (the trivial bundle) or $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ (the non-trivial bundle).

In $H_2(S^2 \times S^2; \mathbb{Z})$, denote the homology classes $B := [S^2 \times pt]$ and $F := [pt \times S^2]$.

In $H_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}; \mathbb{Z})$, denote by $L$ the homology class of a line $\mathbb{CP}^1$ in $\mathbb{CP}^2$ and by $E$ the class of the exceptional divisor; denote by $F$ the fiber class of the fibration $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2} \to S^2$. Note that $F = L - E$.

For $\lambda \geq 0$, denote by $\omega_0^\lambda$ the symplectic form $(1 + \lambda)\tau \oplus \tau$ on $S^2 \times S^2$ where $\tau$ is the standard area form on $S^2$ with $\int_{S^2} \tau = 1$.

For $\lambda > -1$, denote by $\omega_1^\lambda$ the symplectic form on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ for which there exist an embedded symplectic sphere of area $(2 + \lambda)$ in the class $L$ and an embedded symplectic sphere of area $(1 + \lambda)$ in the class $E$.

By the work of Gromov [8], Li–Liu [13], Lalonde–McDuff [12], McDuff [14] and Taubes [19], up to scaling, every symplectic form on $S^2 \times S^2$ is of the form $\omega^0_\lambda$ for $\lambda \geq 0$; every symplectic form on $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ is of the form $\omega^1_\lambda$ for $\lambda > -1$. See also [18] Examples 3.5, 3.7.

3.2. Consider $(M, \omega) = (S^2 \times S^2, \omega^0_\lambda := (1 + \lambda)\tau \oplus \tau)$ with $\lambda \geq 0$. Take $\ell = \lceil \lambda \rceil$, namely, the integer satisfying $\ell - 1 < \lambda \leq \ell$.

Let $J \in \mathcal{J}$. Note that if a class $A = a_B B + a_F F \in H_2(S^2 \times S^2; \mathbb{Z})$ can be represented by a simple $J$-holomorphic sphere, then, by the adjunction formula,

$$0 \leq A \cdot A - c_1(A) + 2 = 2a_Ba_F - 2a_B - 2a_F + 2 = 2(a_B - 1)(a_F - 1).$$

Also, since $J$ is $\omega$-compatible, $0 < \omega(A) = a_B(1 + \lambda) + a_F$. Hence either $a_B, a_F \geq 2$; or $a_B = 1$ and $a_F > -(1 + \lambda)$; or $a_B \geq 0$ and $a_F = 1$; see [1] Lemma 1.7].

If $F = \sum_{i=1}^n A_i = \sum_{i=1}^n (a^i_B B + a^i_F F)$ and each $A_i$ is represented by a simple $J$-holomorphic sphere, then, since $\sum_{i=1}^n a^i_B = 0$, we must have $a^i_B = 0$ and $a^i_F = 1$ for all $i$, so $1 = \sum_{i=1}^n a^i_F = n$. We conclude that for every $J \in \mathcal{J}$ the class $F$ is $J$-indecomposable, i.e., it cannot be written as a sum of two or more classes that are represented by non-constant
$J$-holomorphic spheres. However, if $\lambda > 0$ then $B = (B - F) + F$, with $\int_{(B - F)} \omega_\lambda^0 = \lambda > 0$, $\int_F \omega_\lambda^0 = 1 > 0$, and $B - F$ is the homology class of the symplectically embedded antidiagonal $\{(s, -s) \in S^2 \times S^2\}$.

The following facts are derived from Gromov’s compactness theorem and the properties of $J$-holomorphic spheres in dimension four, see Gromov [3], Abreu [1], and Abreu–McDuff [2].

1. The space $\mathcal{J}_\lambda := \mathcal{J}(M, \omega) = \mathcal{J}(S^2 \times S^2, \omega_\lambda^0)$ is stratified as follows:

$$\mathcal{J}(S^2 \times S^2, \omega_\lambda^0) = U_0^0 \cup U_1^0 \cup \cdots U_\ell^0, \quad \ell = \lfloor \lambda \rfloor \in \mathbb{N},$$

where

$$U_k^0 := \{ J \in \mathcal{J}_\lambda^0 \mid B - kF \in H_2(S^2 \times S^2; \mathbb{Z}) \}$$

is represented by a simple $J$-holomorphic sphere.

In particular, $U_0^0$ is open and dense in $\mathcal{J}_\lambda^0$.

2. For every $J \in \mathcal{J}(S^2 \times S^2, \omega_\lambda^0)$, for any $p \in S^2 \times S^2$ there is an embedded $J$-holomorphic sphere in the class $F$ passing through the point $p$, unique up to reparametrizations. The family of these spheres forms the fibers of a fibration $S^2 \times S^2 \to S^2$.

3. For $J \in U_0^0$, the previous statement holds also in the class $B$ instead of $F$. Hence for such $J$ there are two foliations $\mathcal{F}_B^J$ and $\mathcal{F}_F^J$ whose leaves are embedded $J$-holomorphic spheres in $B$ and in $F$, respectively; the leaf in $\mathcal{F}_B^J$ (resp. $\mathcal{F}_F^J$) through a point $p$ is the unique $J$-holomorphic sphere in $B$ (resp. $F$) through $p$; any two spheres in the same foliation are disjoint; each sphere in $\mathcal{F}_B^J$ intersects each sphere in $\mathcal{F}_F^J$ at exactly one point and transversally.

4. Moreover, for $J \in U_0^0$, there is a diffeomorphism $\Psi_J : S^2 \times S^2 \to M$ such that $\Psi_J$ maps the $J_0$-foliations $\mathcal{F}_B^{J_0} = \{ S^2 \times \text{pt} \}$, $\mathcal{F}_F^{J_0} = \{ \text{pt} \times S^2 \}$ to the corresponding $J$-foliations, where pt $\in S^2$ and $J_0 = j \times j$ is the standard split complex structure on $S^2 \times S^2$. The symplectic form $\omega_J := \Psi_J^*(\omega)$ is cohomologous and linearly isotopic to $\omega_0 = \omega_\lambda^0$. See the construction in [16, Theorem 9.4.7]. Hence, by Moser’s lemma, there is a diffeomorphism $h : S^2 \times S^2 \to S^2 \times S^2$ such that $\Psi_J \circ h : (S^2 \times S^2, \omega_0) \to (M, \omega)$ is a symplectomorphism that induces the identity map on $H_2(S^2 \times S^2; \mathbb{Z})$.

3.4. Items (3) and (4) above generalize to any compact connected symplectic four-manifold $(M, \omega)$ and $\bar{J} \in \mathcal{J}(M, \omega)$ that satisfy the following conditions:

- There is no symplectically embedded 2-sphere with self-intersection number $-1$.
- There exist classes $B, F$ in $H_2(M; \mathbb{Z})$ such that $B \cdot B = F \cdot F = 0$, $B \cdot F = 1$, and both $B$ and $F$ are represented by embedded $\bar{J}$-holomorphic spheres.

In such a case, replace $J \in U_0^0$ in items (3) and (4) by $\bar{J}$, and replace the form $\omega_0$ in item (4) by $\bar{b} \tau \oplus \hat{f} \tau$ where $\bar{b} := \int_{\bar{B}} \omega$ and $\hat{f} := \int_{\bar{F}} \omega$. See the proof of [16, Theorem 9.4.7].
3.5. Consider \((M, \omega) = (\mathbb{CP}^2#\overline{\mathbb{CP}^2}, \omega^1)\) with \(\lambda > -1\). Take \(\ell = [\lambda]\). Abreu–McDuff [2] gave an analogous description of \(\mathcal{J}(\mathbb{CP}^2#\overline{\mathbb{CP}^2}, \omega^1)\).

(1) The space \(\mathcal{J}^1_\lambda := \mathcal{J}(\mathbb{CP}^2#\overline{\mathbb{CP}^2}, \omega^1)\) is stratified as follows:

\[
\mathcal{J}(\mathbb{CP}^2#\overline{\mathbb{CP}^2}, \omega^1) = U^1_0 \cup U^1_1 \cup \cdots \cup U^1_\ell, \quad \ell = [\lambda] \in \mathbb{N},
\]

where

\[
U^1_\ell := \{ J \in \mathcal{J}^1_\lambda \mid E - kF \in H_2(\mathbb{CP}^2#\overline{\mathbb{CP}^2}; \mathbb{Z}) \}
\]

is represented by a simple \(J\)-holomorphic sphere.

(2) For every \(J \in \mathcal{J}(\mathbb{CP}^2#\overline{\mathbb{CP}^2}, \omega^1)\), the fiber class \(F\) is represented by a 2-parameter family of embedded \(J\)-holomorphic spheres that fiber \(\mathbb{CP}^2#\overline{\mathbb{CP}^2}\).

4. Proof of the main theorem

4.1. Proposition. Let \((M, \omega) = (S^2 \times S^2, \omega^0_\lambda := (1 + \lambda)\tau \oplus \tau)\) with \(\lambda \geq 0\) equipped with a homologically trivial symplectic action of a finite cyclic group \(G\). Let \(J = J_G\) be a \(G\)-invariant \(\omega\)-compatible almost complex structure on \(M\). Assume that \(J\) is in the stratum \(U^0_0\). Then the \(G\)-action is symplectically conjugate to the restriction of a standard circle action on \((M, \omega)\).

Note that if \(\lambda = 0\), the space \(\mathcal{J} = U^0_0\). Hence the proposition implies Theorem 1.1 for the case \((S^2 \times S^2, \tau \oplus \tau)\).

We call a symplectic circle action on \((M, \omega) = (S^2 \times S^2, \omega^0_\lambda)\) standard if the circle rotates the first sphere at speed \(a\) and the second at speed \(b\), where \(a\) and \(b\) are relatively prime integers.

Proof. Fix a generator \(g \in G\) and denote by \(\sigma_g\) the corresponding symplectomorphism defined by the action, i.e., \(\sigma_g\) is the image of \(g\) under the homomorphism \(G \to \text{Symp}(M, \omega)\). Since \(J\) is \(G\)-invariant and \(\sigma_g\) induces the identity map on \(H_2(S^2 \times S^2; \mathbb{Z})\), the map \(\sigma_g\) sends a \(J\)-holomorphic sphere representing a class in \(H_2(S^2 \times S^2; \mathbb{Z})\) to a \(J\)-holomorphic sphere representing the same class. Since \(J \in U^0_0\), items (3) and (4) in §3.2 apply; in particular there are foliations \(\mathcal{F}^B_J\) and \(\mathcal{F}^F_J\) of embedded \(J\)-holomorphic spheres; the leaves of the foliations through a given point are unique. Therefore, \(\sigma_g\) sends each leaf of \(\mathcal{F}^B_J\) (resp. \(\mathcal{F}^F_J\)) to a leaf of \(\mathcal{F}^B_J\) (resp. \(\mathcal{F}^F_J\)).

Let \(J_0 = j \times j\) be the standard split complex structure on \(S^2 \times S^2\). By item (4) in §3.2 there exists a symplectomorphism \(\Psi_J: (S^2 \times S^2, \omega) \to (S^2 \times S^2, \omega)\) that maps the \(J_0\)-foliations, \(\mathcal{F}^B_{J_0}, \mathcal{F}^B_{J_0}\), to the corresponding \(J\)-foliations, \(\mathcal{F}^B_J, \mathcal{F}^B_J\). Note that the leaves of the \(J_0\)-foliations coincide with the leaves of the \(\Psi_J\)-\(J\)-foliations.

Now, define a map \(\sigma_1: S^2 \to S^2\) by the following procedure. For \(x \in S^2\), \(v_x = \{ x \} \times S^2\) is a leaf in \(\mathcal{F}^F_{J_0}\); namely, a \(J_0\)-holomorphic sphere representing \(F\). Then \(\Psi_J^{-1}\sigma_1\Psi_J(v_x)\) is a
leaf in $F_{\lambda_0}^0$, and therefore of the form $\{x\} \times S^2$. Define $\sigma_1(x) = x'$. Similarly we define $\sigma_2 : S^2 \to S^2$ by comparing $u_y = S^2 \times \{y\}$ and $\Psi_j^{-1} \sigma_g \Psi_j(u_y)$. Note that each $\sigma_i$ generates an action of $G$ on $S^2$ because $(\Psi_j^{-1} \sigma_g \Psi_j)^r = \Psi_j^{-1} \sigma_g^r \Psi_j = \Psi_j^{-1} \sigma_g^r \Psi_j$. Observe that $\sigma_i$ is orientation preserving with respect to the orientation defined by $\omega_j$.

We check that $\Psi_j$ is $G$-equivariant with respect to the split action generated by $\sigma_1 \times \sigma_2$ and the given $G$-action, i.e.,

$$\Psi_j^{-1} \sigma_g \Psi_j(x, y) = (\sigma_1 \times \sigma_2)(x, y) \text{ for } (x, y) \in S^2 \times S^2.$$  

The image $p = \Psi_j(x, y)$ is the intersection of the $J$-holomorphic spheres $\Psi_j(v_x)$ and $\Psi_j(u_y)$. By item (3) in [3.2], they are the unique $J$-holomorphic spheres passing through $p$ in the classes $F$ and $B$ respectively. By the same token, $\sigma_g \Psi_j(v_x)$ and $\sigma_g \Psi_j(u_y)$ are the unique $J$-holomorphic spheres passing through $\sigma_g(p)$ in $F$ and $B$, and $\Psi_j^{-1} \sigma_g \Psi_j(v_x)$ and $\Psi_j^{-1} \sigma_g \Psi_j(u_y)$ the unique $J_0$-holomorphic spheres passing through $\Psi_j^{-1} \sigma_g \Psi_j(x, y)$. In other words, $\Psi_j^{-1} \sigma_g \Psi_j(x, y)$ is the intersection of $\Psi_j^{-1} \sigma_g \Psi_j(v_x)$ and $\Psi_j^{-1} \sigma_g \Psi_j(u_y)$, which is $(x', y') = (\sigma_1(x), \sigma_2(y)) =: (\sigma_1 \times \sigma_2)(x, y)$. We have the following commutative diagram of symplectomorphisms:

\[
\begin{array}{ccc}
(S^2 \times S^2, \omega) & \xrightarrow{\Psi_j} & (S^2 \times S^2, \omega) \\
\sigma_1 \times \sigma_2 \downarrow & & \downarrow \sigma_g \\
(S^2 \times S^2, \omega) & \xrightarrow{\Psi_j} & (S^2 \times S^2, \omega)
\end{array}
\]

By Proposition [3.1], the cyclic action on $S^2$ generated by $\sigma_i : S^2 \to S^2$ is conjugate to the restriction of a circle action on $S^2$ by an orientation-preserving diffeomorphism $h_i$. That is, we have the following commutative diagram

\[
\begin{array}{ccc}
(S^2 \times S^2, \omega) & \xrightarrow{h_i \times h_2} & (S^2 \times S^2, \omega) \\
\tilde{\sigma}_1 \times \tilde{\sigma}_2 \downarrow & & \downarrow \sigma_1 \times \sigma_2 \\
(S^2 \times S^2, \omega) & \xrightarrow{h_i \times h_2} & (S^2 \times S^2, \omega)
\end{array}
\]

where $\tilde{\sigma}_i : S^2 \to S^2$ denotes a generator of the restriction of a circle action on $S^2$ to the cyclic subgroup. More precisely, if $a_i$ is the rotation number of $\sigma_i$ on $S^2$, and $r$ stands for the order of $G$, then $\gcd(a_1, a_2, r) = 1$ because the given $G$-action is effective. Let $\gcd(a_1, a_2) = c$. Then $\gcd(c, r) = 1$, $\gcd(a_1/c, a_2/c) = 1$, and $\tilde{\sigma}_i$ is the restriction induced by the inclusion $G \hookrightarrow S^1$, $g \mapsto g^r$ of the circle action on $S^2$ by rotation at speed $a_i/c$. Note that $h_1 \times h_2$ preserves the $J_0$-foliations.

Since $\Psi_j$ and $h_1 \times h_2$ send the class $B$ to $B$ and the class $F$ to $F$, $\tilde{\omega}_j$ and $\omega_0 = \omega_\lambda^0$ are cohomologous. Since both $\omega_0$ and $\tilde{\omega}_j$ tame $(h_1 \times h_2)^* \Psi^* j, J$ and are $G$-invariant with respect to the action generated by $\tilde{\sigma}_1 \times \tilde{\sigma}_2$, the 2-form

$$\omega_t = (1 - t) \omega_0 + t \tilde{\omega}_j$$
is symplectic and $G$-invariant for all $t \in [0,1]$. By Moser’s method, one can find a $G$-equivariant vector field whose time-one flow gives a $G$-equivariant diffeomorphism $h : S^2 \times S^2 \to S^2 \times S^2$ such that $h^*\omega_J = \omega_0$. Namely,

$$
(S^2 \times S^2, \omega_0) \xrightarrow{h} (S^2 \times S^2, \tilde{\omega}_J)
$$

(4.4)

Moreover, by the proof of Proposition 4.1, there is a symplectomorphism $G$-invariant vector field whose time-one flow gives a $G$-equivariant diffeomorphism $h : S^2 \times S^2 \to S^2 \times S^2$ that conjugates the $G$-action and a standard action on $S^2 \times S^2$. If the $G$-action on $\{x\} \times S^2$ and $S^2 \times \{y\}$ is standard, then $\psi$ can be chosen to be the identity on these spheres.

In fact this holds for any symplectic form $\omega$ on $S^2 \times S^2$ instead of $\omega_0$, by (3.4) and the fact that every symplectic sphere in $S^2 \times S^2$ has even self-intersection due to homological reason.

Next we show an equivariant version of Gromov’s result [8] on $D^2 \times D^2$, as appeared in [1, Theorem 1.9].

Here $D^2$ denotes the unit disc in $\mathbb{R}^2$. Let $\omega_\lambda = (1 + \lambda)dx^1 \wedge dx^2 + dy^1 \wedge dy^2$ be a split symplectic form where $(x, y) = ((x^1, x^2), (y^1, y^2)) \in \mathbb{R}^2 \times \mathbb{R}^2$. We call an action of a finite cyclic group $G$ on $\mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{C} \times \mathbb{C}$ linear if $\sigma_g(x, y) = (\xi^a x, \xi^b y)$, where $g$ is a generator of $G$, $\sigma_g$ is the corresponding map defined by the action, and $\xi$ is the corresponding root of unity.

4.6. **Lemma.** Let a cyclic group $G$ of finite order act symplectically on the product of unit discs $D^2 \times D^2$ with a symplectic form $\omega$. Assume that $\omega$ equals $\omega_\lambda$ near the boundary, and the $G$-action is linear near the boundary.

Then the given $G$-action on $D^2 \times D^2$ is conjugate to the linear action by a symplectomorphism that is the identity on the boundary.

**Proof.** Consider the collapsing map

$$
\kappa : D^2 \times D^2 \to S^2 \times S^2
$$

that identifies the points of $\{x\} \times \partial D^2$, for $x \in D^2$, as $(x, \infty)$, and identifies the points of $\partial D^2 \times \{y\}$, for $y \in D^2$, as $(\infty, y)$. Let $J_G$ be a $G$-invariant $\omega$-compatible almost complex structure on $D^2 \times D^2$ that is equal to the standard split complex structure $J_0$ near the
boundary. Such $J_G$ exists by taking any $\omega$-compatible almost complex structure $J$ that is equal to $J_0$ near the boundary, averaging the metric $\omega(\cdot, J\cdot)$ with respect to the $G$-action, and then taking the structure provided by the polar decomposition, as in Claim 2.2.

Consider the image $S^2 \times S^2$ with the induced symplectic form and the induced (compatible) almost complex structure, again denoted by $\omega$ and $J_G$. Let $g \in G$ be a generator of $G$ and $\sigma_g$ be the corresponding symplectomorphism of $D^2 \times D^2$ defined by the action. Denote by $\bar{\sigma}_g$ the symplectomorphism of $S^2 \times S^2$ that pulls back to $\sigma_g$. On the image of the boundary, $\bar{\sigma}_g(x, \infty) = (\xi^a x, \infty)$ and $\bar{\sigma}_g(\infty, y) = (\infty, \xi^b y)$. On the complement of $S^2 \times \{\infty\} \cup \{\infty\} \times S^2$, we have $\bar{\sigma}_g(x, y) = \sigma_g(\kappa^{-1}(x, y))$.

The spheres $S^2 \times \{\infty\}$ and $\{\infty\} \times S^2$ in $S^2 \times S^2$ are $J_G$-holomorphic and invariant under $\bar{\sigma}_g$. Moreover, the $G$-action $\bar{\sigma}_g$ is standard on these spheres. As explained in Remark 4.5 by Proposition 4.1, there is a symplectomorphism $\psi: S^2 \times S^2 \to S^2 \times S^2$ conjugating $\bar{\sigma}_g$ and a standard action, and $\psi$ is the identity on $\{\infty\} \times S^2$ and $S^2 \times \{\infty\}$. Hence this standard action is the one induced from $(x, y) \mapsto (\xi^a x, \xi^b y)$.

Define

$$\Psi: D^2 \times D^2 \to D^2 \times D^2$$

as the identity on $D^2 \times \partial D^2 \cup \partial D^2 \times D^2$ and $\kappa^{-1} \circ \psi \circ \kappa$ on the complement of the boundary. Then $\Psi$ is a symplectomorphism conjugating the given action and the linear action, that is the identity on the boundary. \hfill \Box

We now prove Theorem 1.1 for the cases that are not covered by Proposition 4.1, i.e., if $(M, \omega) = (S^2 \times S^2, \omega^*_k)$ and $J = J_G$ is not in $U_0^*$ or if $(M, \omega) = (\mathbb{CP}^2 \# \mathbb{CP}^2, \omega^*_1)$. A $G$-invariant $\omega$-compatible almost complex structure $J_G$ on $(M, \omega)$ exists by Claim 2.2.

4.7. Proposition. Let a cyclic group $G$ of finite order act symplectically on $(M^*_k, \omega^*_k)$ for $\star = 0, 1$. Assume that the action is trivial on $H_2(M^*_k; \mathbb{Z})$. Let $J = J_G$ be a $G$-invariant $\omega^*_\star$-compatible almost complex structure. Set $A_k := B - kF$ if $\star = 0$ and $A_k := E - kF$ if $\star = 1$.

Assume that $A_k$ is represented by a simple $J$-holomorphic sphere $C_1$ for an integer $k > -\star$. Then the following hold.

1. The sphere $C_1$ is embedded and $G$-invariant and there is an embedded $G$-invariant sphere $C_2$ in the class $F$ such that $C_1$ and $C_2$ intersect at a point; the $G$-action on each sphere is, up to conjugation by a symplectomorphism of the sphere, a rotation, with rotation numbers $(a, -b)$ at $C_1 \cap C_2$.

2. The $G$-action is symplectically isomorphic to the standard $G$-action on the Hirzebruch surface $\text{Hirz}_n(a, -b)$ with the symplectic form $\eta_n$, as described in §A.1 and §A.3 where $n = 2k + \star$ and $\mu = 1 + \lambda - k$.

The proof of item (2) is an equivariant variation of the proof of [1, Lemma 3.5].

Proof.
(1) Fix a generator $g \in G$ and denote by $\sigma_g$ the corresponding symplectomorphism defined by the action. By Claim 2.3 since the self-intersection number of $A_k$ is negative, the $J$-holomorphic sphere $C_1$ in $A_k$ is $G$-invariant. By the adjunction formula [16, Corollary E.1.7] and since $A_k \cdot A_k - c_1(A_k) + 2 = 0$, the simple $J$-holomorphic sphere $C_1$ is embedded. By Proposition B.1 the restriction of the $G$-action to $C_1$ is conjugate to the restriction to a cyclic subgroup of a circle action, i.e., a rotation. In particular, it fixes two points on $C_1$. Let $p$ be one of these $G$-fixed points with rotation number $a$ at $p$. By item (2) in §3.2 and §3.5, there is an embedded $J$-holomorphic sphere $C_2$ in the class $F$ passing through the point $p$. By Claim 2.3 since $F \cdot F = 0$, the image $\sigma_g(C_2)$ either coincides with $C_2$ or is disjoint from it. However, since the $G$-fixed point $p$ is on $C_2$, $\sigma_g(C_2)$ must coincide with $C_2$. So $C_2$ is also invariant under the $G$-action. Hence, by Proposition B.1 up to conjugation by a symplectomorphism, the restriction of the $G$-action to $C_2$ is a rotation with rotation number $-b$ at $p$.

(2) By item (1), $C_1$ and $C_2$ are $G$-invariant symplectic spheres intersecting transversally at one point $p$ with rotation numbers $(a, -b)$ at $p$. Since $G$ is finite abelian, the tangent space at the fixed point $p$ decomposes into a direct sum of symplectically orthogonal eigenspaces of $G$; hence $C_1$ and $C_2$ in fact intersect $\omega^*_\lambda$-orthogonally at $p$. The sphere $C_1$ has self-intersection number $-n$ and size $\mu$; the sphere $C_2$ has self-intersection number 0 and size 1. Using the equivariant version of Weinstein’s symplectic neighbourhood theorem for such pair of submanifolds [9] (also see [17]), we can construct a diffeomorphism

$$\alpha : \Hirz_n \rightarrow M^*_\lambda$$

that sends the zero section $S_0$ to $C_1$ and the fiber at zero $F_0$ to $C_2$ and is an equivariant symplectomorphism from a neighbourhood $U$ of $S_0 \cup F_0$ in $(\Hirz_n(a, -b), \eta_\mu)$ onto a neighbourhood $U'$ of $C_1 \cup C_2$ in $(M^*_\lambda, \omega^*_\lambda)$ with the given $G$-action. (See §A.4 and §A.3 for the notations.) So it is enough to show that the $\alpha$-pull back of the $G$-action is the standard $G$-action on $\Hirz_n(a, -b)$, up to conjugation by a symplectomorphism.

Since $S_0$ and $F_0$ are invariant symplectic spheres, the restriction of the $\alpha$-pull back of the $G$-action to the complement of $S_0 \cup F_0$ in $\Hirz_n$ is a symplectic action with respect to $\alpha^* \omega^*_\lambda$. Since $\alpha$ is a symplectomorphism near $S_0 \cup F_0$, $\alpha^* \omega$ equals $\eta_\mu$ there.

The complement of $S_0 \cup F_0$ in $\Hirz_n$ is

$$\{([w, 1], [z, w^n, 1])\} \subset \mathbb{CP}^1 \times \mathbb{CP}^2$$

and the map

$$\beta : \mathbb{C}^2 \rightarrow \Hirz_n$$
given by \((w, z) \mapsto ([w, 1], [z, w^n, 1])\) pulls back \(\eta_\mu\) to
\[
\Omega_\mu = \frac{i}{2\pi} \left[ \mu \partial \bar{\partial} \log(1 + \|w\|^2) + \bar{\partial} \partial \log(1 + ||w||^{2n} + ||z||^2) \right] \\
= \frac{i}{2\pi} \left( \left[ \frac{\mu}{(1 + \|w\|^2)^2} + \frac{n^2 \|w\|^{2(n-1)}(1 + \|z\|^2)}{(1 + \|w\|^{2n} + \|z\|^2)^2} \right] dw \wedge d\bar{w} \\
+ \frac{(1 + \|w\|^{2n})}{(1 + \|w\|^{2n} + \|z\|^2)^2} dz \wedge d\bar{z} \right).
\]
A neighbourhood of \(\infty\) at \(\mathbb{C}^2\) is sent to a neighbourhood of \(S_0 \cup F_0\). The standard action on \(\text{Hirz}_n(a, -b)\) pulls back to a linear action, \((w, z) \mapsto (\xi^{-a}w, \xi^{b-na}z)\), and therefore the \(\alpha\)-pull back of the \(G\)-action on \(\text{Hirz}_n\) pulls back to an action on \(\mathbb{C}^2\) which is linear near infinity.

Denote \(f = \log((1 + \|w\|^2)^\mu(1 + \|w\|^{2n} + \|z\|^2))\).
Note that \(\Omega_\mu = \frac{i}{2\pi} \partial \bar{\partial} f\). Consider the gradient vector field \(X_f\) of \(f\) with respect to the Riemannian metric induced by \(\Omega_\mu\) and the standard almost complex structure \(J_0\) on \(\mathbb{C}^2\). Then
\[
L_{X_f} \Omega_\mu = \Omega_\mu.
\]
The function \(f\) strictly increases along rays from the origin. Moreover, the function \(f\) and the Riemannian metric are invariant under any linear \(G\)-action on \(\mathbb{C}^2\). Hence the backward flow of \(X_f\) commutes with linear \(G\)-actions, and takes the \((\beta \circ \alpha)\)-pull back of the \(G\)-action on \(\mathbb{C}^2\), which is linear near infinity, to an action on \(D^2 \times D^2\) which is linear near the boundary. Now the proposition follows from Lemma 1.6.

Note that the sphere \(S_0\) is the preimage of the top edge of \(H_{n, \mu}\) under a moment map of the Hamiltonian \(T^2\)-action (A.1) on \((\text{Hirz}_n, \eta_\mu)\); the sphere \(F_0\) is the preimage of the left edge; see Figure A.1. The complement of \(S_0 \cup F_0\) is the preimage of the complement of the top and left edges, so it corresponds to a standard star-shaped region in \(\mathbb{C}^2\), and the Liouville vector field \(X_f\) is a lift of the radial vector on the trapezoid in \(\mathbb{R}^2\) emitting from the bottom right corner.

\[\square\]

**Appendix A. Toric and cyclic actions on Hirzebruch surfaces**

A.1. **Hirzebruch surface.** Let \(O(n)\) denote the complex line bundle over \(\mathbb{CP}^1\) with first Chern class \(n\). The **Hirzebruch surface** \(\text{Hirz}_n\) is the complex manifold defined as a projective bundle, for any integer \(n\):
\[
\text{Hirz}_n \cong \mathbb{P}(O(0) \oplus O(-n)) \cong ((O(0) \oplus O(-n)) \setminus 0) / \mathbb{C}^*.
\]
It is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ obtained by gluing two copies $V_1, V_2$ of the form $\mathbb{C} \times \mathbb{CP}^1_{\text{fiber}}$ along

$$(\mathbb{C} \setminus \{0\}) \times \mathbb{CP}^1 \to (\mathbb{C} \setminus \{0\}) \times \mathbb{CP}^1 \quad (z, [x_1, x_2]) \mapsto \left( \frac{1}{z}, [x_1, z^n x_2] \right).$$

The zero section $S_0$ corresponds to $\{(z, [1, 0])\}$; the section at infinity $S_\infty$ corresponds to $\{(z, [0, 1])\}$; the fiber at zero $F_0$ refers to the fiber at $[1, 0] \in \mathbb{CP}^1_{\text{base}}$, namely, $\{(0, [x_1, x_2])\}$ in $V_1$; and the fiber at infinity $F_\infty$ refers to the fiber at $[0, 1] \in \mathbb{CP}^1_{\text{base}}$, namely, $\{(0, [x_1, x_2])\}$ in $V_2$. The self-intersection numbers of $S_0, S_\infty, F_0, F_\infty$ are $-n, n, 0, 0$, respectively.

One can equip the Hirzebruch surface $\text{Hirz}_n$ with a symplectic structure through symplectic cutting. Alternatively, note $\mathcal{O}(-1)$ is the tautological line bundle over $\mathbb{CP}^1$, whose fiber at $l \in \mathbb{CP}^1$ is the line $l$ in $\mathbb{C}^2$. Since $\mathcal{O}(-n)$ is the $n$-th tensor power of $\mathcal{O}(-1)$, the Hirzebruch surface can be identified with an algebraic submanifold of $\mathbb{CP}^1 \times \mathbb{CP}^2$ defined in homogeneous coordinates by

$$\{(w_1, w_2, [z_0, z_1, z_2]) \in \mathbb{CP}^1 \times \mathbb{CP}^2 \mid w_1^n z_2 = w_2^n z_1\}.$$ 

The isomorphism is given by

If $w_1 \neq 0$, 

$$(w_1, w_2, [z_0, z_1, z_2]) \mapsto \left( \frac{w_2}{w_1}, [z_0, z_1] \right) \in V_1$$

If $w_2 \neq 0$, 

$$(w_1, w_2, [z_0, z_1, z_2]) \mapsto \left( \frac{w_1}{w_2}, [z_0, z_2] \right) \in V_2$$

Any Kähler form on $\mathbb{CP}^1 \times \mathbb{CP}^2$ restricts to a Kähler form on $\text{Hirz}_n$. For $\mu \geq 0$, we denote by $\eta_\mu$ the sum of the Fubini–Study form on $\mathbb{CP}^1$ multiplied by $\mu$ and the Fubini–Study form on $\mathbb{CP}^2$. We shall use the same notation for its restriction to the Hirzebruch surface.

With respect to the symplectic form $\eta_\mu$, we find that the symplectic spheres

$$S_0 = \{([w_1, w_2], [1, 0, 0])\} \text{ has size } \mu \quad \text{(zero section)},$$

$$S_\infty = \{([w_1, w_2], [0, w_1^n, w_2^n])\} \text{ has size } \mu + n \quad \text{(section at infinity)},$$

$$F_0 = \{([1, 0], [z_0, z_1, 0])\} \text{ has size } 1 \quad \text{(fiber at zero)},$$

$$F_\infty = \{([0, 1], [z_0, 0, z_2])\} \text{ has size } 1 \quad \text{(fiber at infinity)}.$$  

A.2. **Standard toric action on a Hirzebruch surface.** Let $S^1$ be identified with the unit circle in $\mathbb{C}$ and consider the Hamiltonian action of $T = (S^1)^2$ on $(\text{Hirz}_n, \eta_\mu)$ induced by

$$(A.1) \quad (s, t) \cdot ([w_1, w_2], [z_0, z_1, z_2]) = ([w_1, s w_2], [t z_0, z_1, s^n z_2]).$$

The moment map image of this torus action is a trapezoid, as in Figure [A.1], called a *standard Hirzebruch trapezoid* $H_{n, \mu}$. The zero section $S_0$, the section at infinity $S_\infty$, the fiber at zero $F_0$, and the fiber at infinity $F_\infty$ are $T$-invariant spheres, whose moment map images are the top, bottom, left, and right edges of the trapezoid, respectively. By
Delzant’s construction [6] there is an integrable \( T \)-invariant almost complex structure \( J_T \in \mathcal{J}(\text{Hirz}_n, \eta_\mu) \), for which the spheres \( S_0, S_\infty, F_0, F_\infty \) are \( J_T \)-holomorphic.

A.3. Standard cyclic action on a Hirzebruch surface. Let \( G \) be a cyclic group of finite order \( r \). Let a generator of \( G \) be identified with a primitive \( r \)th root of unity \( \xi \) in \( \mathbb{C} \). Consider the \( G \)-action on the Hirzebruch surface \( \text{Hirz}_n \) defined by

\[
\xi \cdot (z, [x_1, x_2]) = \begin{cases} 
(\xi^a z, [\xi^b x_1, x_2]) & \text{on } V_1, \\
(\xi^{-a} z, [\xi^b x_1, \xi^{na} x_2]) & \text{on } V_2,
\end{cases}
\]

where \( \gcd(a, b, r) = 1 \). Equivalently,

\[
(A.2) \quad \xi \cdot ([w_1, w_2], [z_0, z_1, z_2]) = ([w_1, \xi^a w_2], [\xi^b z_0, z_1, \xi^{na} z_2]).
\]

We shall call such an action a standard \( G \)-action on the Hirzebruch surface and refer to a Hirzebruch surface with such a standard \( G \)-action as \( \text{Hirz}_n(a, -b) \).

This \( G \)-action sends a fiber to a fiber while keeping two fibers invariant: \( F_0 \) and \( F_\infty \). The zero section \( S_0 \) and the section at infinity \( S_\infty \) are also \( G \)-invariant. By definition, the integers \((a, -b) \pmod{r}\) are the rotation numbers of the \( G \)-action at the fixed point \( S_0 \cap F_0 \), that is, the weights of the \( G \)-representation on the tangent space of the fixed point. The rotation numbers at the fixed point \( S_\infty \cap F_\infty \) are \((-a, b - na)\).

Moreover, let \( c = \gcd(a, b) \) and so \( \gcd(c, r) = 1 \); the standard \( G \)-action \((A.2)\) is the restriction induced by the inclusion \( G \hookrightarrow S^1 \), \( g \mapsto g^c \) of the circle action defined in the same way by replacing \( \xi \) by \( \theta \in S^1 \), \( a \) by \( a/c \), and \( b \) by \( b/c \), which can also be seen as the circle action obtained as the inclusion \( S^1 \hookrightarrow T^2 \), \( \theta \mapsto (\theta^{a/c}, \theta^{b/c}) \) followed by the toric action \((A.1)\).

Appendix B. Cyclic actions on the sphere

B.1. Proposition. A symplectic action of a cyclic group \( G \) of finite order on \((S^2, \tau)\) is conjugate to the restriction of a circle action by a symplectomorphism.

Proposition \((B.1)\) follows from well known results. Here is a sketch of the proof. By Claim \((2.2)\) there is a \( G \)-invariant \( \tau \)-compatible almost complex structure on \( S^2 \). It follows then from the Newlander–Nirenberg Theorem that it must be integrable. Hence, by the
Uniformization Theorem, it is biholomorphic to the standard almost complex structure on $S^2$. So $G$ is a subgroup of the automorphism group $\text{PSL}(2, \mathbb{C})$ of $\mathbb{CP}^1$. In $\text{PSL}(2, \mathbb{C})$ every element is conjugate to a rotation map, and in particular a generator of $G$ is conjugate to a rotation map. By further applying the equivariant version of Moser’s method, noting that the ingredients in the usual Moser’s method can be made $G$-equivariant, the symplectic action of the finite cyclic group $G$ on $(S^2, \tau)$ is conjugate by a symplectomorphism to the restriction of a circle action.

B.2. Remark. The circle action in the above proposition does not need to be assumed effective.

Suppose the order of the cyclic group $G$ is $r$, and suppose the weight of the cyclic action at a fixed point on $S^2$ is $a$. By Proposition [3.1] up to conjugation, the cyclic action is the restriction induced by the inclusion $G \hookrightarrow S^1$, $g \mapsto g^a$ of the effective circle action $\theta \cdot (z, x) \mapsto (\theta z, x)$, where $\theta \in S^1$ and $(z, x) \in S^2$ identified as the unit sphere in $\mathbb{R}^3 \cong \mathbb{C} \times \mathbb{R}$. On the other hand, this $G$-action can also be considered as a restriction of a non-effective circle action $\theta \cdot (z, x) \mapsto (\theta^a z, x)$ induced by the inclusion $g \mapsto g$. In either case, the integer $a \pmod{r}$ will be called the rotation number (at the chosen fixed point) of the $G$-action on the sphere $S^2$.

Moreover, the cyclic action in the above proposition does not need to be assumed effective either. If the cyclic action is effective, then $\gcd(a, r) = 1$. ♦

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