Phase Diagram of the Dissipative Hofstadter Model

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Abstract

A quantum particle moving in a uniform magnetic field and periodic potential (the Hofstadter model) has an energy band structure which varies in a discontinuous fashion as a function of the magnetic flux per lattice unit cell. In a real system, randomness of various kinds should "smooth out" this behavior in some way. To explore how this happens, we have studied the dissipative quantum mechanics of the Hofstadter model. We find, by virtue of a duality in a two-dimensional space parametrized by the dissipation constant and the magnetic field strength, that there are an infinite number of phase transition lines, whose density grows without limit as the dissipation goes to zero and the model reduces to the original Hofstadter model. The measurable quantity of greatest interest, the mobility, can be determined exactly in most of parameter space. The critical theory on the phase transition lines has yet to be characterized in any detail, but it has reparametrization invariance and defines a set of nontrivial backgrounds for open string theory.

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1. Hofstadter Model and Dissipation: Introduction and Issues

The quantum mechanics of an electron moving in two dimensions subject to a uniform magnetic field and a periodic potential (which we will call the Hofstadter problem for short) has a remarkable fractal energy band structure [1] which varies discontinuously as a function of the number of flux quanta per lattice unit cell. Such behavior must, however, be “smoothed out” by the inevitable randomness of a real physical system. An interesting model of dynamic randomness [2] (due ultimately to weak coupling of the electron to lattice vibrations) is provided by the dissipative quantum mechanics of Caldeira and Leggett (DQM) [3] and the topic of this paper is the application of DQM to the Hofstadter problem. A key point is that DQM converts quantum mechanics into one-dimensional statistical mechanics (the one dimension being Euclidean time on the electron’s world line) with long-range interactions strong enough to support phase transitions between, as it turns out, localized and delocalized long-time behavior. Thus, the discontinuities of the original quantum mechanics are replaced by phase transitions and the problem is to map out the phase diagram (in the magnetic field-dissipation constant plane). Presumably the density of phase transitions grows as dissipation decreases, recreating discontinuous dependence on the magnetic field in the zero-dissipation limit. We are able to show, using a duality argument, that this is exactly what happens and that the phase diagram is what was found years ago [4] in the study of theta-dependence of the confinement-deconfinement transition in Abelian lattice gauge theories! Away from the critical lines, the behavior of the system for most of the parameter space is very simple: the long-time mobility, for example, is given by the trivial theory for some value of the magnetic field and no potential. The critical theories, however, are definitely non-trivial and we have only been able to extract limited information about them. Apart from their condensed matter interest, they represent new solutions of open string theory [5]. Our results at least indicate that there is a rich vein of one-dimensional critical theories to explore, once the appropriate tools have been developed.

The outline of this paper is as follows. Section 2 contains a review of the dissipative Hofstadter model. In Section 3 we sketch the arguments showing that, at specific critical points, the theory is at a renormalization group fixed point and the mobility can be calculated exactly. In the fourth section, we derive the approximate duality transformations of the dissipative Hofstadter model, and, in the following section, we demonstrate that the duality transformations are exact for a discrete version of the model. In section 6, we use these transformations to map out the phase diagram of the dissipative Hofstadter model. In the final section, we present a brief discussion of our results.

2. Setup of the Dissipative Hofstadter Model

We begin with a brief outline of dissipative quantum mechanics. For details the reader is referred to [2] and [3]. The quantum mechanics of a particle subject to a scalar potential $V$ and vector potential $\vec{A}$ is described by the Euclidean path integral

$$Z_{QM} = \int [D X(t)] e^{-S_{QM}[X]} \quad (2.1)$$

with

$$S_{QM}[X] = \int dt \left\{ \frac{1}{2} M \dot{\vec{X}}^2 + V(X) \right\} + i \int dt A_i(X) \dot{X}^i. \quad (2.2)$$
If the quantum variable, $X^i$, is in some sense macroscopic (a good example is the trapped magnetic flux in a Josephson junction) there will typically exist an infinite set of degrees of freedom to which $X^i$ is at least weakly coupled and which give rise to dissipative effects on the motion of $X^i$. In the classical limit, dissipation can be described by adding a phenomenological friction term $-\eta \dot{X}^i$ to the equation of motion. It is then natural to ask whether there is a correspondingly simple and universal way to express the quantum effects of weak dissipation. A practical motivation is to assess whether coherent quantum effects, such as tunneling, can really be observed in macroscopic systems.

Caldeira and Leggett [3] addressed this problem via a simple model: In addition to the coordinates $X^i$, let there be a bath of harmonic oscillators, $q_\alpha$, with a distribution of frequencies, $\omega_\alpha$, coupled linearly to the $X^i$ with weak coupling strengths, $C_\alpha$. If the parameters satisfy the functional condition

$$\sum_\alpha \frac{C^2_\alpha}{2 \omega_\alpha} \delta(\omega - \omega_\alpha) = \frac{\eta \omega}{\pi}, \quad (2.3)$$

then, when the oscillators are `integrated out’ of the classical equations of motion for $X^i$, they supply the canonical $-\eta \dot{X}^i$ friction term. Since the dependence of the action on the $q_\alpha$ is only quadratic, they can be explicitly integrated out of the quantum path integral also. The result is a new path integral over the $X^i$ alone, where the quantum effect of friction is contained in a non-local term whose strength is set by the classical friction constant $\eta$:

$$Z^n_{QM} = \int [DX(t)] e^{-S_{QM}[X]-S_\eta[X]} \quad (2.4)$$

where

$$S_\eta[X] = \frac{\eta}{4\pi \hbar} \int_{-\infty}^{\infty} dt \, dt' \frac{(\dddot{X}(t) - \dddot{X}(t'))^2}{(t-t')^2}. \quad (2.5)$$

Because of the non-locality of the $\eta$-term, the path integral (2.4) is effectively that of a one-dimensional statistical system with long-range interactions. Such systems, unlike one-dimensional local systems, have phase transitions (the classic example being the Ising chain with $1/r^2$ interactions [3]). In the DQM context, the phase transitions are between different regimes of long-time behavior of Green’s functions (typically between localized and delocalized). The crucial qualitative information about DQM therefore concerns the phase structure of its long-time behavior and we will analyze the dissipative Hofstadter model from that point of view.

To study the Hofstadter model, we specialize (2.2) as follows: $\vec{X} = (x, y)$ is taken to be two-dimensional, the potential is taken to be periodic with period $a$ in both directions, and of strength $V_0$, so that

$$V(x, y) = V_0 \cos(2\pi x/a) + V_0 \cos(2\pi y/a); \quad (2.6)$$

and the magnetic field is taken to be uniform and in a linear gauge:

$$(A_x, A_y) = \frac{1}{2}(By, -Bx). \quad (2.7)$$
Leaving aside the potential term, the action (2.2) is Gaussian:

\[
S_g = \frac{\eta}{4\pi\hbar} \int_{-T/2}^{T/2} dt \int dt' \frac{(\dot{X}(t) - \dot{X}(t'))^2}{(t - t')^2} + \frac{M}{2\hbar} \int dt \dot{X}^2 + \frac{ieB}{2\hbar c} \int dt (\dot{x}y - \dot{y}x) .
\] (2.8)

The generating functional for the full theory is then given by

\[
Z[\eta, B, V, F] = \int [DX(t)] e^{-S_g[X] - \frac{i}{\hbar} \int V(\vec{X}) dt - S_F[X]},
\] (2.9)

where \(S_F[X]\) is a linear source term,

\[
S_F[X] = \int \vec{F}(t) \cdot \vec{X}(t) dt.
\] (2.10)

In order to determine the phase structure of the dissipative Hofstadter model, we will be concentrating on the properties of the partition function and two-point function, which are obtained from \(Z[\eta, B, V, F]\) as follows:

\[
Z[\eta, B, V] = Z[\eta, B, V, 0]
\] (2.11)

and

\[
\langle X^\mu(t_1)X^\nu(t_2)\rangle(\eta, B, V) = \frac{1}{Z[\eta, B, V, 0]} \frac{\delta^2 Z[\eta, B, V, F]}{\delta F^\mu(t_1)\delta F^\nu(t_2)} \bigg|_{F=0} .
\] (2.12)

It is useful to define the dimensionless parameters \(\alpha = \eta a^2/2\pi\hbar\) and \(\beta = eBa^2/2\pi\hbar c\), to rescale \(\vec{X}\) by \(1/a\), and to reexpress the Gaussian action in Fourier space:

\[
S_g = \frac{i}{\hbar} \int \frac{d\omega}{2\pi} \left\{ \frac{\alpha}{2\pi} |\omega| + \frac{Ma^2}{2\hbar} \omega^2 \delta_{ij} + 2\pi\beta\epsilon_{ij}\omega \right\} \hat{X}^*_i(\omega)\hat{X}_j(\omega) .
\] (2.13)

One sees that the dissipation and magnetic field terms are dimension one operators (they grow like the first power of \(\omega\)) and therefore marginal from the renormalization group point of view. On the other hand, the ordinary kinetic energy term is dimension two and therefore irrelevant: it just acts as a regulator for short times and has no effect on universal critical properties. We will set it to zero from now on and use a more convenient short-distance cutoff wherever needed.

The action (2.13) with \(M = 0\) plays a very important role: it is the Gaussian fixed point theory which governs the long-time behavior of the system whenever the potential term (2.6) is irrelevant. We will shortly identify the region in the \(\alpha\)-\(\beta\) plane where this condition is met. Let us first summarize the essentials of the Gaussian theory: all information is contained in the coordinate two-point function \(\langle X^\mu(t_1)X^\nu(t_2)\rangle(\eta, B, V)\), the inverse of the quadratic form in (2.13). We will call it \(D_{ij}(t - t'; z)\), using the complex number \(z = \alpha + i\beta\) to summarize the parameters on which it depends. Explicitly,

\[
D_{ij}(t - t'; z) = -\frac{\alpha}{\alpha^2 + \beta^2} \log(t - t')^2 \delta_{ij} - i\frac{\pi\beta}{\alpha^2 + \beta^2} \text{sign}(t - t') \epsilon_{ij} .
\] (2.14)
The logarithmic growth of $D(t)$ clearly indicates that the Gaussian system is on the borderline between localization and delocalization. The coefficient of the $\delta_{ij}$ term is called, for obvious reasons, the mobility ($\mu$). For zero magnetic field, $\mu \to \infty$ as dissipation goes to zero, while for finite field, the mobility goes to zero in the same limit since the particles become stuck in Landau levels. The coefficient of the $\epsilon_{ij}$ term is essentially a Hall coefficient: it measures the response transverse to an applied field.

In Fourier space, the two-point function is

$$\tilde{D}_{ij}(\omega; z) = \frac{1}{|\omega|} M_{ij}(\omega; \frac{1}{z}), \quad (2.15)$$

where

$$M_{ij}(\omega, x + iy) = x \delta_{ij} + y \frac{|\omega|}{\omega} \epsilon_{ij}. \quad (2.16)$$

In Section 6, we will make use of the following simple rules for the addition and multiplication of $M_{ij}$:

$$M_{ij}(\omega; z_1) + M_{ij}(\omega; z_2) = M_{ij}(\omega; z_1 + z_2); \quad (2.17)$$

$$M_{ij}(\omega; z_1)M_{ij}(\omega; z_2) = M_{ij}(\omega; z_1 z_2); \quad (2.18)$$

and

$$c M_{ij}(\omega; z_1) = M_{ij}(\omega; cz_1) \quad (2.19)$$

for all $z_1, z_2 \in C$ and $c \in R$.

Given the explicit Gaussian two-point function, one can easily calculate the effective dimensionality of a perturbation such as (2.6) and identify the regions in the $z$-plane where it is irrelevant. This is a simple one-loop renormalization group calculation and it is easy to show that (2.6) is marginal when the mobility coefficient $\alpha/(\alpha^2 + \beta^2)$ is unity and irrelevant when $\alpha/(\alpha^2 + \beta^2)$ is greater than one. This condition defines a circle in the $z$-plane, of radius one-half and centered at $z = (\frac{1}{2}, 0)$, inside of which (2.6) is irrelevant and on which it is marginal. For zero field, the critical value is $\alpha = 1$, a fact which has been known for a long time [7]. Outside the circle, (2.6) is relevant and, if one knew that there were only two phases, one would have to identify that entire region with a localized phase. This is almost certainly correct on the zero magnetic field line, but, given the rich dual and self-similar structure of the pure quantum mechanics of the Hofstadter problem, certainly wrong for non-zero field.

Two loosely related problems arise at this point. The first is to map out the phase structure in the region outside the “first circle”, where the periodic potential is relevant. We believe we have a complete solution of this problem. As will be explained in subsequent sections, the key to the solution is an approximate duality symmetry (generated by two transformations $z \to 1/z$ and $z \to z + i$) which relates behavior in different regions of the $z$-plane. The second problem is to characterize the family of critical theories which live on the boundary of the first circle (and its images under duality). At present we know very little about them, although, as we shall explain in the next section, they become particularly simple at the points where $\beta/\alpha$ is integer. There is a further motivation for studying these critical theories, whose elaboration must await another paper. It is that, as explained in [8], these critical theories satisfy an infinite set of Ward identities deriving from reparametrization invariance, possess a manifest $SL(2, R)$ symmetry, and represent nontrivial solutions of open string theory!
3. Fermionization and Exact Fixed Points

In this section, we summarize some of the properties of the critical theories at the points on the “first circle” with integral $\beta/\alpha$. Using a convenient regulator, we show that at these points the theory has no logarithmic divergences and is therefore at a renormalization group fixed point. In addition, we show that the theory can be fermionized and that the free energy and mobility can be exactly calculated.

In order to calculate the partition function and correlation functions of the $\dot{X}^\mu$’s, it is most convenient to expand the path integral in powers of the cosine potential. We then have to evaluate correlators of products of $\dot{X}^\mu$’s and $e^{\pm iX^\nu}$’s (the latter coming from expanding the cosines as a sum of exponentials) and the result can be written in terms of integrals over products of factors of the type $\langle \dot{X}^\mu(t_i)X^\nu(t_j) \rangle_0$ and $e^{\pm iX(t_i)}X^\nu(t_j) \rangle_0$, where the expectation value is with respect to the gaussian propagator $D_{ij}(t-t';z)$ and the integrals are over the $t$ coordinates of the potential insertions. In order to get finite results at all stages of the calculation, we need both an infrared and an ultraviolet cutoff. We obtain the former by putting the parameter $t$ on a circle of circumference $T$, and the latter by multiplying the Fourier-transformed propagator $\tilde{D}_{ij}(\omega;z)$ by $e^{-|\omega|}$ (we do this instead of giving the particle an explicit mass). With these regulators, (2.14) is no longer correct, but it is not too hard to show that

$$\langle \dot{X}^\mu(t_1)X^\nu(t_2) \rangle_0 = -\left(\frac{2\pi i}{T}\right)\frac{\alpha}{\alpha^2 + \beta^2} \left[ \frac{w_1^2 - w_2^2}{(w_1 - e^{-\epsilon} w_2)(w_1 - e^\epsilon w_2)} \right] \delta^{\mu\nu},$$

$$e^{\pm iX^\nu(t_1)X^\nu(t_2)} = \left[ \frac{e^\epsilon w_1 w_2}{(w_1 - e^\epsilon w_2)(w_1 - e^{-\epsilon} w_2)} \right] \left[ \frac{w_1 - w_2 e^\epsilon}{w_2 - w_1 e^\epsilon} \right]^{\pm i\epsilon \frac{\beta}{\alpha + \beta^2}}.$$

where we have defined a new position coordinate $w_j = e^{2\pi i t_j}/T$ which lies on the unit circle. Under this transformation, integrals over $t_j$ are converted into contour integrals of $w_j$ around the unit circle according to $\int dt \rightarrow \frac{T}{2\pi i} \oint \frac{dw_j}{w_j}$. We note that whenever $\alpha/(\alpha^2 + \beta^2) = 1$ and $\beta/\alpha$ is an integer, the expressions in equations (3.1) and (3.2) are rational functions of $w_j$ and $e^\epsilon$, so the integrands for the partition function and n-point functions will also be rational functions. Furthermore, because the integrand factors into a product of simple poles, we can perform the integrals by successively extracting the residues of the poles in the integration variables which lie within the unit circle. At each stage we are left with a rational function of the remaining $w_j$’s and $e^\epsilon$. This implies that, in the end, we can only get pole divergences in $\epsilon$ as $\epsilon \rightarrow 0$, and no logarithmic divergences. In particular, for correlators of marginal operators such as $\dot{X}(t)$, which can at worst have logarithmic divergences, there can be no divergences at all. As a result, when $\alpha/(\alpha^2 + \beta^2) = 1$ and $\beta/\alpha \in \mathbb{Z}$, the theory should be at a zero of the $\beta$-function. As we will show elsewhere, this also implies that the theory satisfies an infinite set of Ward Identities, which derive from reparametrization invariance. (As was explained in [3], these Ward...
identities are what make possible a string theory interpretation of systems of this kind.)
This is very much like the 2D case, where scale invariance implies conformal invariance.
We note that the absence of logarithmic divergences occurs for any value of the potential,
\(V_0\), which means that these fixed-point theories are really fixed lines.

We can use the regulator and contour integration procedure just described to calculate
the partition function and \(n\)-point functions to any order in \(V_0\), but it becomes extremely
tedious for higher orders. At the special point \(\alpha = 1, \beta = 0\), we can obtain a number of
results exact to all orders in \(V_0\) by fermionizing the unregulated theory and then regulating
the fermionic theory. We observe that if we express the \(X\)'s in terms of two free fermions
\(\psi_+(t)\) and \(\psi_-(t)\) according to
\[
e^{iX(t)} = \psi_+(t)\psi_-(t),
\]
and
\[
\dot{X}(t) = \frac{2\pi i}{T} \left[ \psi_+(t)\psi_+(t) - \psi_-(t)\psi_-(t) \right].
\]
and define the fermion propagators to be
\[
\langle \psi_+(t_1)\psi_+(t_2) \rangle = \langle \psi_-(t_1)\psi_-(t_2) \rangle = \frac{i\sqrt{w_1w_2}}{w_2 - w_1},
\]
and
\[
\langle \psi_+(t_1)\psi_-(t_2) \rangle = \langle \psi_-(t_1)\psi_+(t_2) \rangle = 0,
\]
we reproduce all the bosonic correlators of \(X\)'s and \(e^{\pm iX}\). Note that the set of operators
under discussion here generates an \(SU(2)\) algebra and is a particular example of the discrete
states which have been extensively discussed recently in two-dimensional quantum gravity [9].

Since the potential term, and all the operators of interest, are now quadratic in free
fermions, the unregulated problem reduces to free field theory. The ultraviolet-regulated
theory might not, however, and care must be taken here. We regulate by multiplying the
Fourier transform of the fermi propagators by \(e^{-\epsilon|\omega|}\), and again find that all the integrals for
the free energy and \(n\)-point functions are contour integrals of factored rational polynomials.
This time, all the integrands are just products of loops of the form \(\prod_{i=1}^{N} \frac{1}{(w_i - e^{-\epsilon}w_{i+1})}\),
with \(w_{N+1} = w_1\), so we can easily perform the integrals at any order in \(V_0\). Furthermore, at
least for the partition function and the two-point function, it is easy to resum the expansion
in powers of the potential. The result for the free energy density is
\[
F = \frac{1}{2\epsilon} \sum_{N=1}^{\infty} \frac{1}{N^2} (-1)^N \left( \frac{V_0T}{2} e^{-\langle X(0)X(0) \rangle} \right)^{(2N)} + O(\epsilon),
\]
while the result for the two point function is
\[
\langle \dot{X}(t_1)\dot{X}(t_2) \rangle = \left( \frac{2\pi}{T} \right)^2 \left( \frac{1}{1 + (\frac{V_0T}{2} e^{-\langle X(0)X(0) \rangle})^2} \right) \frac{2w_1w_2}{(w_1 - w_2)^2}.
\]
Note that both of these expressions depend on the peculiar dimensionless parameter
\(V_0T e^{-\langle X(0)X(0) \rangle}\). In the limit that the ratio of infrared to ultraviolet cutoff is large,
this parameter reduces to $V_0/\epsilon = \tilde{V}_0$. The only renormalization needed to render the two-point function (and, indeed, all n-point functions of the $\dot{X}$'s) finite is to choose $\tilde{V}_0$ to be finite. This doesn’t quite render the free energy density finite: it has a $1/\epsilon$ piece which we must subtract away by hand. We only attach physical significance to the finite piece left over after this subtraction and we note that in this particular case ($\alpha = 1$ and $\beta = 0$) this finite piece is zero. Note also that the functional form of the two-point function follows from $SU(1, 1)$ invariance if $\dot{X}$ transforms with weight unity. Both of these points and their string theory significance are explained at greater length in [4].

Apart from an overall $V_0$-dependent, finite, multiplicative renormalization, (3.8) is the same as the Gaussian two point function $D(t- t'; 1)$ transformed to the unit circle (consider (3.1) at $\alpha = 1, \beta = 0$). So, at the $\alpha = 1$ fixed point, the ratio of the renormalized mobility to its Gaussian value has a specific dependence on the potential strength:

$$\frac{\mu_{\text{eff}}}{\mu_{\text{Gauss}}} = (1 + (e^{-\langle X(0)X(0) \rangle}V_0T/2)^2)^{-1} = (1 + (\tilde{V}_0/2)^2)^{-1}.$$  (3.9)

This is in rough accord with the results of Guinea et. al. [8]. Since they worked in the tight-binding approximation, rather than in the weak-potential expansion used here, the renormalization of the Gaussian two-point function is different in the two cases. Rather surprisingly, even for non-vanishing potential strength, the higher n-point functions turn out to vanish for non-coincident points (i.e. when $t_i \neq t_j$ for $1 \leq i, j \leq n$). On the other hand, when all the $t_i$ are equal, the regulated 2n-point functions are proportional to $1/\epsilon^{2n}$. Thus, the n-point functions for $n > 2$ are zero except for contact terms. Because of the rescaling of the 2-point function and the contact terms for higher n-point functions, the fixed point theory when $\alpha = 1$ does not appear to be a simple gaussian theory. Precisely what it is is not yet known in any detail.

It is also possible to fermionize the theory at the other special points where $\beta/\alpha \in Z$ and $\alpha = \alpha^2 + \beta^2$. Although the $\dot{X}$'s require more care to fermionize because $\langle e^{iY(t_1)\dot{X}(t_2)} \rangle \propto \frac{d}{dt}\text{sign}(t_1 - t_2)$ is ill-defined without the regulator, the results for the free energy and 2-point functions are very similar to what we found at the $\alpha = 1$ fixed point.

### 4. Duality Symmetries from Coulomb Gas and Instanton Methods

We will now show that the dissipative Hofstadter model has at least an approximate duality symmetry under the modular group consisting of the transformations $z \rightarrow 1/z$, $z \rightarrow z + i$ and all their compositions, where $z = \alpha + i\beta$. In Section 6 we will use this duality to obtain an overall picture of the phase structure of the theory. To demonstrate the symmetry, we begin by deriving a generalized Coulomb gas representation of the DQM path integral. The first duality symmetry then follows from a manifest invariance of the Coulomb gas. Next, we derive a second Coulomb gas representation, which is related to the first by a transformation of $z$ and $V_0$. Insofar as both Coulomb gases are valid representations of the same model, we obtain the second duality transformation. These two transformations on $z$ are enough to generate the full modular group. This strategy was first applied to DQM without a magnetic field by Schmid [10] to obtain restricted duality under the transformation $\alpha \rightarrow 1/\alpha$ of the dissipation constant only. It was also used in reference [11] for the Hofstadter model with no dissipation.
For the first derivation of the Coulomb gas, we simply expand the path integral in powers of the potential and further expand the cosine potential as a sum of positive and negative frequency exponentials: 
\[ \cos X(\tau_j) = \frac{1}{2} \sum_{\epsilon_j = \pm 1} e^{i\epsilon_j X(\tau_j)}. \]
Because the potential \( (2.6) \) is a sum of terms periodic in \( X \) and \( Y \), we must actually define a two-dimensional vector of charges \( \vec{e}_i \). A collection of \( n \) of these charges, located at times \( \tau_i \), can be associated with a charge density
\[ \vec{\rho}^n(\tau) = \sum_{j=1}^{n} \delta(\tau - \tau_j). \] (4.1)

With these definitions, the generating function \( (2.9) \) becomes
\[ Z[z, V_0, F] = \int [D\vec{X}(t)] \sum_{n=0}^{\infty} \int d\tau_1 \ldots d\tau_n \left( \frac{V_0}{2} \right)^{n} \frac{1}{n!} \sum_{\vec{e}_j = \{\pm 1, 0\}} e^{-S_q}, \] (4.2)
where
\[ S_q = S_g + S_F - i \int \vec{X}(\tau) \cdot \vec{\rho}^n(\tau) d\tau. \] (4.3)

Because \( S_q \) is quadratic, the \( \vec{X} \) integration can be done exactly, with the result
\[ Z[z, V_0, F] = CZ_{\text{gen}} \left[ S_{\text{gen}}(z, F); \frac{V_0}{2} \right] = C \sum_{N} \sum_{\{\vec{e}_i\}} \int d\tau_1 \ldots d\tau_{2N} \frac{1}{(2N)!} \left( \frac{V_0}{2} \right)^{2N} \exp[-S_{\text{gen}}(z, F)], \] (4.4)
where the action, \( S_{\text{gen}}(z, F) \), is given by
\[ S_{\text{gen}}(z, F) = \frac{1}{2} \int d\tau d\tau' \left[ i\vec{F}(\tau) + \vec{\rho}^N(\tau) \right]^\dagger D(\tau - \tau'; z) \left[ i\vec{F}(\tau') + \vec{\rho}^N(\tau') \right]. \] (4.5)

The allowed values for the vector “charges” \( \vec{e}_i \) in the expression for \( Z_{\text{gen}} \) are \( (\pm 1, 0) \) and \( (0, \pm 1) \). Integrating over the zero mode of \( \vec{X} \) gives the additional requirement of charge neutrality: \( \sum_{i=1}^{2N} \vec{e}_i = 0 \).

To obtain the partition function, we set \( F = 0 \) in equation (4.4). It will be useful to note that the resulting object is identical to the grand canonical partition function for a certain generalized Coulomb gas. Let us define a Coulomb gas interaction “energy” between a collection of charges \( \vec{e}_i \) at locations \( \tau_i \) by
\[ S_{\text{CG}}(A + iB) = \frac{1}{2} \sum_{i,j} \vec{e}_i \cdot D(\tau_i - \tau_j; \frac{1}{A + iB}) \cdot \vec{e}_j, \] (4.6)
where \( D \) is the Gaussian two-point function of \( (2.14) \). Writing out the full expression for \( D \), we obtain the explicit expression
\[ S_{\text{CG}}(A + iB) = A \sum_{i<j} \vec{e}_i \cdot \vec{e}_j \ln(t_i - t_j)^2 + i\pi B \sum_{i<j} \text{sign}(\tau_i - \tau_j) \epsilon_{\mu\nu} e_i^\mu e_j^\nu. \] (4.7)
This is just a collection of two-body interaction terms, with real and imaginary parts whose strengths are set by $A$ and $B$ respectively. The real part looks like a standard Coulomb potential term. The imaginary part is of course unconventional and that is why we refer to this as a generalized Coulomb gas. Its partition function is

$$Z_{CG}[A + iB, \zeta] = \sum_N \sum \{\vec{e}_i\} \int d\tau_1 \ldots d\tau_{2N} \frac{1}{(2N)!} (\zeta)^{2N} \exp[-S_{CG}(A + iB)],$$

where $\zeta$ is the fugacity and the sum is over even numbers of charges only because we impose overall charge neutrality. The imaginary part of the interaction energy gives a phase to each term in the partition sum. In this regard it is similar to the $\theta$ term of QCD.

The partition function of interest to us, (4.4) evaluated at $F = 0$, can thus be written in terms of a generalized Coulomb gas as follows:

$$Z[z, V_0] = CZ_{CG}[1/z, V_0/2].$$

In short, the original system is equivalent to a generalized Coulomb gas in which each particle has charge $\vec{e}_j = (\pm 1, 0)$ or $(0, \pm 1)$ and fugacity $\zeta = V_0/2$, and the bonds between particles are assigned both an energy and a phase determined by the parameter $z = 1/(A + iB)$.

By varying the generating function (4.4) with respect to $\vec{F}$ and setting $\vec{F} = 0$ one gets expressions for n-point functions of the $X$’s in terms of Coulomb gas correlators of the charge density $\tilde{\rho}$. For the two-point function, in particular, we obtain

$$\langle \tilde{X}^\mu(\omega)\tilde{X}^\nu(-\omega) \rangle(z, V_0) = \tilde{D}^{\mu\nu}(\omega; z) - \tilde{D}^{\mu\sigma}(\omega; z) \langle \tilde{\rho}^\sigma(\omega)\tilde{\rho}^\lambda(-\omega) \rangle(1/z, V_0/2) \tilde{D}^{\lambda\nu}(\omega; z),$$

where by $\langle \tilde{\rho}^\sigma(\omega)\tilde{\rho}^\lambda(-\omega) \rangle(1/z, V_0/2)$ we mean the density-density correlation function in the Coulomb gas with coupling parameter $1/z$ and fugacity $V_0/2$. We should emphasize that although we have expanded in powers of $V_0$, no terms have been dropped, so our results to this point are exact.

The generalized Coulomb gas described by $Z_{CG}[A + iB, \zeta]$ possesses a symmetry under shifts in $A + iB$. Because all the charges are integral, the phase factor appearing in a generic term in the partition sum,

$$\exp[i\pi B \sum_{i<j} \text{sign}(\tau_i - \tau_j) e_i^\mu e_j^\nu],$$

is invariant under $B \rightarrow B + 1$. Consequently, both the partition function of the generalized Coulomb gas and the charge-density correlation functions remain invariant under the shifts $A + iB \rightarrow (A + iB) + in$, for integer $n$, so that

$$Z_{CG}[A + iB + in, \zeta] = Z_{CG}[A + iB, \zeta],$$

and

$$\langle \tilde{\rho}^\sigma(\omega)\tilde{\rho}^\lambda(-\omega) \rangle(A + iB + in, \zeta) = \langle \tilde{\rho}^\sigma(\omega)\tilde{\rho}^\lambda(-\omega) \rangle(A + iB, \zeta).$$
If we let \(1/z = A + iB\) and \(1/\tilde{z} = A + iB + in\), then we can use these equations and equations (4.9) and (4.10) to solve for the transformation of the partition function and two-point function of the Hofstadter model when \(z\) goes to \(\tilde{z} = z/(1 +inz)\). We find that the partition function remains the same and the coordinate two-point function transforms as

\[
\langle \tilde{X}(\omega) \tilde{X}(-\omega) \rangle(z, V_0) = \tilde{D}(\omega; z - i/n) + \omega^2 \tilde{D}(\omega; 1 + inz) \langle X(\omega) X(-\omega) \rangle\left(1 + \frac{z}{1 + inz}, V_0\right) \tilde{D}(\omega; 1 + inz).
\]

(4.14)

We believe this transformation to be exact because we have summed to all orders in perturbation theory. There are some subtleties having to do with regulation, but they should have no influence on the long-time behavior. In the next section, we will describe a discretized version of the dissipative Hofstadter model for which duality transformations like (4.14) are exact without any qualifications.

The second Coulomb gas is obtained by making an instanton expansion of the path integral. Here we assume that if the friction is small enough, the main contribution to the path integral comes from the “approximate” solutions to the Euclidean classical equations of motion in the absence of friction. In these solutions the particle moves between adjacent minima of the potential along instanton paths, which correspond to the particle tunneling from one minimum to another. These frictionless instantons for the Hofstadter model were studied in some detail in [11] and we will draw heavily on results presented there. When \(M = 0\), the classical action for the instanton is given by

\[
s = e^{-\frac{|\beta|}{\alpha h} + \omega^2} + \frac{\pi}{\alpha h} \approx 7.33,
\]

and when \(\beta \approx 0\), it is

\[
s = 4\sqrt{V_0/Ma^2}/\pi h.
\]

When \(M = 0\), the instanton solution starting at \((x, y) = (0, 0)\) and ending at \(2\pi \vec{e}_j\), with \(\vec{e}_j = (\pm 1, 0)\) or \((0, \pm 1)\), is given by [11]

\[
\vec{f}(\tau) = \vec{e}_j h(\tau) + (\hat{z} \times \vec{e}_j) g(\tau),
\]

where

\[
h(\tau) = 2 \tan^{-1} \left[ \sqrt{2} \sinh(4\pi^2 V_0 / \alpha h \tau) \right] + \pi,
\]

and

\[
g(\tau) = -i \cosh^{-1} \left[ 1 + 2 \sech(8\pi^2 V_0 / \alpha h \tau) \right].
\]

(4.17)

We can add together a series of these “kinks” to construct the approximate solution

\[
\tilde{X}_n(\tau) = \sum_{j=1}^{n} \vec{e}_j h(\tau - \tau_j) + (\hat{z} \times \vec{e}_j) g(\tau - \tau_j),
\]

(4.18)

whose Fourier transform can be written in terms of the charge distribution defined in equation (4.1):

\[
\tilde{X}_n(\omega) = h_{\omega} \tilde{\rho}_{\omega} + g_{\omega} (\hat{z} \times \tilde{\rho}_{\omega}).
\]

(4.19)

When we substitute this solution back into the full action and then sum over all such paths, we once again obtain a Coulomb gas expression for the partition function: the two signs...
of the charge correspond to instantons and anti-instantons; the locations of the charges correspond to the instanton center of mass collective coordinate; the fugacity is given by the exponential of minus the instanton action times a fluctuation determinant \(Ke^{-s}\), where \(K = V_0/\sqrt{\beta}\) when \(\alpha = 0\) and \(\beta >> 1\); and the instantons acquire long-range Coulomb interactions and phases from the dissipation and magnetic field terms in the action. The expression for the generating functional has the same structure as before. It is

\[
Z[z, V_0, F] = Z_{\text{gen}}[S_{\text{inst}}(z, F); Ke^{-s}],
\]

where now the energy function, \(S_{\text{gen}}\), in equation (4.4) is replaced by

\[
S_{\text{inst}}(z, F) = \frac{1}{2} \sum_{jk} \vec{e}_j^\dagger \Delta(\tau_j - \tau_k; z) \vec{e}_k + \int \frac{d\omega}{2\pi} \vec{F}_{-\omega} \cdot [h_\omega \vec{\rho}_\omega + g_\omega (\hat{z} \times \vec{\rho}_\omega)].
\]

Here, \(\Delta\) specifies the interaction energy and phases between two instantons and it is similar in structure to the Gaussian interaction function \(D(\tau_j - \tau_k; 1/z)\). In particular, its off-diagonal terms are exactly the same, but the diagonal terms depend in detail on the shape of the instanton solution. However, in the long-time limit \(|(\tau_j - \tau_k)4\pi^2V_0/(\alpha h)| >> 1\), \(\Delta\) turns out to have a universal form

\[
\Delta(\tau_j - \tau_k; z) \sim D(\tau_j - \tau_k; 1/z).
\]

Thus, in this limit the instanton interactions are identical to those of a Coulomb gas with interaction parameter \(z\) instead of \(1/z\). We will assume that properties of the critical points of the theory are sensitive only to this long-time limit. In that case, the partition function can again be written in terms of the Coulomb gas partition function given by equations (4.8) and (4.7). It is

\[
Z(z, V_0) = NZCG(z, K(V_0)e^{-s}),
\]

where \(N\) denotes the value of the partition function in the absence of tunneling.

We can again derive an expression for the two-point function in terms of the charge correlation function of the new Coulomb gas:

\[
\langle \vec{X}^\mu(\omega)\vec{X}^\nu(-\omega) \rangle(z, V_0) = H^{\mu\sigma}(\omega) \langle \hat{\rho}^\sigma(\omega)\hat{\rho}^\lambda(-\omega) \rangle(z, Ke^{-s}) H^{\lambda\nu}(\omega),
\]

where \(H(\omega)\) is

\[
H^{\mu\nu}(\omega) = h(\omega)\delta^{\mu\nu} + g(\omega)\epsilon^{\mu\nu}.
\]

The small \(\omega\) (large \(\tau\)) behavior of \(h\) and \(g\) is such that (4.24) reduces, as \(\omega \to 0\), to

\[
\langle \vec{X}^\mu(\omega)\vec{X}^\nu(-\omega) \rangle(z, V_0) = \tilde{D}^{\mu\sigma}(\omega; 1) \langle \hat{\rho}^\sigma(\omega)\hat{\rho}^\lambda(-\omega) \rangle(z, K e^{-s}) \tilde{D}^{\lambda\nu}(\omega; 1).
\]

The instanton method should be valid whenever the classical action of the Hofstadter model (without dissipation) is large and the coefficient of friction is much smaller than the classical action. We also assume that the large-time behavior of the system is not affected by the particular (regulator-dependent) small-distance form of the propagators.

To summarize, the preceding calculations demonstrate that the original partition function is equivalent to the partition function for two different Coulomb gases of “\(X\)” and “\(Y\)”
particles which correspond to the $X$ and $Y$ components of $\vec{c}_j$, respectively. The particles have charges $\pm 1$ and a fugacity which depends on which Coulomb gas is under consideration. The structure of the interaction energy is the same for both gases: like particles (both $X$ or both $Y$) interact via $-2AE_{ij}^Xe_0^X\log |\tau_i-\tau_j|$ for $|\tau_i-\tau_j| \gg 1$, and unlike particles via $i\pi B \text{sign}(\tau_i-\tau_j)e_{ij}^Xe_0^X$. When we treat the potential as a perturbation, the fugacity is given by $V_0/2$ and the interaction strengths by $A+iB=1/z$ (where $z$ is the parameter introduced in (2.14)). When we use the instanton method, the system is described as a gas with fugacity $/zeta = K(V_0,z)e^{-s}$ and interaction strength $A+iB = z$. Therefore, we have established that the system with dissipation and flux per unit cell given by $z$ and $V_0$ is dual to the system with $z \to 1/z$ and a new potential $\tilde{V}_0/2 = K(V_0,z)e^{-s}$.

The two-point function of the original system can be expressed in terms of the charge-density correlation function for either of the two Coulomb gases described above. Upon eliminating the charge density correlators between the two expressions, we get the following duality relation between two-point functions of the original theory in dual regions of parameter space (we expect it to be valid for large times, or, equivalently, small $\omega$):

$$
\langle \tilde{X}(\omega)\tilde{X}(-\omega)\rangle(z,V_0) = \tilde{D}(\omega;z) - \omega^2 \tilde{D}(\omega;z) \langle \tilde{X}(\omega)\tilde{X}(-\omega)\rangle(1/z,\tilde{V}_0) \tilde{D}(\omega;z) .
$$

(4.27)

Note that the fugacity transforms along with $z$ ($V_0 \to \tilde{V}_0$). This sort of relation has the strongest consequences in circumstances where there is in fact no dependence of the two-point function on fugacity. As we shall see, this covers a lot of territory.

Lastly, we may combine the “exact” duality transformation $z \to z/(1+inz)$ with the approximate duality transformation $z \to 1/z$. A complication here is that under the second duality, $V_0$ transforms in a $z$-dependent way. Consequently we learn that the dissipative Hofstadter system at $z$ and $V_0$ is dual to the system at $z+ni$ and some other potential strength. The two systems should have the same partition function up to an analytic prefactor. Additionally, if for some reason the two-point function is not explicitly dependent on $V$, then, by composing (4.27) and (4.14), we obtain

$$
\langle \tilde{X}(\omega)\tilde{X}(-\omega)\rangle(z+in) = \langle \tilde{X}(\omega)\tilde{X}(-\omega)\rangle(z) .
$$

(4.28)

We conclude that the system has an exact symmetry under $z \to z/(1+inz)$ and an approximate symmetry under all compositions of $z \to 1/z$ and $z \to z+i$. We will draw the consequences of this for the phase structure of the theory in the final section.

5. The Discrete Gaussian Model: an Instructive Example

Because our model has an approximate duality under $z \to z+i$ and $z \to 1/z$, it is natural to suspect that there is a similar model which has an exact duality under these transformations. The discrete gaussian chain with the interaction $V(r) = \alpha/(r^2 + 1/4)$ is known to be exactly dual to one with $\alpha$ replaced by $(1/\alpha)$ [14]. We find that if we couple two such chains via a magnetic interaction, the system is then exactly dual under $z \to z+i$ and $z \to 1/z$. We define such a system as follows. There is a one-dimensional lattice located at the sites $i = 1, 2, ..., N$ and a second dual lattice at $i' = 1/2, 3/2, ..., N-1/2$. At each site
there are height variables $h_x^i$ and $h_y^{i'}$ with $h_{x_i+N}^i = h_x^i$ and $h_{y_i+N}^{i'} = h_y^{i'}$. The height variables can take on any integer value except for $h_{x_i}^N$ and $h_{y_i}^{N-1/2}$, which we set to zero in order to keep the partition function finite. We will define our discrete Gaussian Hamiltonian with magnetic interaction to be

$$H_{DG}^N = \frac{1}{2} \sum_{i \neq j} V_N(i - j)(h_x^i - h_x^j)^2 + \frac{1}{2} \sum_{i' \neq j'} V_N(i' - j')(h_y^{i'} - h_y^{j'})^2$$

$$- \frac{1}{2} 2\pi i \beta \sum_i h_x^i (h_y^{i+\frac{1}{2}} - h_y^{i-\frac{1}{2}}) + \frac{1}{2} 2\pi i \beta \sum_{i'} h_y^{i'} (h_x^{i'+\frac{1}{2}} - h_x^{i'-\frac{1}{2}}),$$

(5.1)

where $V$ is chosen so that its Fourier transform $\tilde{V}$ satisfies

$$W_N(k) = \tilde{V}_N(0) - \hat{V}_N(k) = 2\pi \sin \frac{k}{2}.$$  

(5.2)

The partition function

$$Z_{DG}^N[z] = \sum_{h_x^i} \sum_{h_y^{i'}} e^{-H_{DG}^N},$$

(5.3)

depends on two parameters $\alpha$ and $\beta$, which we have combined into a complex parameter $z = \alpha + i\beta$. With these definitions, the momentum space version of (5.1) reads

$$H_{DG}^N = \frac{1}{2} \sum_k \vec{h}_{-k} \cdot \vec{D}^{-1}(k; z) \cdot \vec{h}_k$$

(5.4)

where $h_x^k$ and $h_y^k$ are the Fourier transforms of $h_x^i$ and $h_y^{i'}$,

$$h_x^j = \sum_k e^{ikj} h_x^k, \quad h_y^{j'} = \sum_k e^{ikj'} h_y^k,$$

(5.5)

(\text{where $k = 2\pi r/N$ and $r$ runs over the integers satisfying $-N/2 < j \leq N/2$), and $D^{-1}(k; z) = 2W_N(k)M(k; z)$, with $M$ defined by}

$$M(k; z) = \begin{pmatrix} \alpha & \frac{\beta k}{|k|} \\ -\beta & \alpha \end{pmatrix}.$$  

(5.6)

$D(k; z)$ is the discrete-time version of the propagator $D(\omega; z)$ originally introduced in (2.14).

Now we can do a Poisson resummation trick, exactly as in [12], to reexpress the discrete Gaussian system as a Coulomb gas. The procedure is to replace the $h_i$ and $h_{i'}$ by continuous variables $\nu_i$ and $\nu_{i'}$ (over which we will integrate) but to include delta functions of the form $\sum_{q_j = -\infty}^{\infty} \delta_{q_j} e^{2\pi i q_j \nu_i}$ to pick out the integer values of $\nu$. On integrating out the $\nu_i$’s we obtain

$$Z_{DG}^N[z] = NC_N(\alpha^2 + \beta^2)^{-(N-1)/2} Z_{CG}^N[1/z],$$

(5.7)

13
where \( C_N = \prod_{k \neq 0} \pi/W(k) \) and \( Z^{CG}_N \) is given by

\[
Z^{CG}_N[1/z] = \sum_{q_i^x} \sum_{q_i^y} \exp \left[ - \sum_{k \neq 0} \frac{\pi^2}{W_N(k)} \vec{q}_{-k} \cdot M(k; 1/z) \cdot \vec{q}_k \right].
\]  

(5.8)

Here, \( \vec{q}_k = (qq^x_k, qq^y_k) \), where \( qq^x_k \) and \( qq^y_k \) are the Fourier transforms of \( q_i^x \) and \( q_i^y \), respectively. The \( q_i \) are constrained to satisfy \( \sum q_i^x = \sum q_i^y = 0 \). It is not too hard to recognize (5.8) as a discretized generalized Coulomb gas. Indeed, in the limit \( N \to \infty \), the interactions between the charges are exactly the same as in the model of the previous section. The key differences are that: the charges can now take on any integer value (they were previously restricted to \( \pm 1 \)); there is only one charge per lattice site; and the fugacity is 1 rather than being freely variable.

Next, we perform Cardy’s transformation \([13]\) on the expression for \( Z^{CG}_N \). We define the new height variables, \( l_i^x \) and \( l_i^y \), by

\[
q_i^x = l_i^{x+1/2} - l_i^{x-1/2}, \quad q_i^y = l_i^{y+1/2} - l_i^{y-1/2}, \quad \text{and} \quad l_i^0 = l_i^{1/2} = 0.
\]  

(5.9)

On making this variable substitution in (5.8) we find, on comparing with (5.4), that we have reproduced \( Z^{DG}_N \) with \( z \to 1/z \). Specifically,

\[
Z^{DG}_N[z] = NC_N(\alpha^2 + \beta^2)^{-\frac{(N-1)}{2}} Z^{DG}_N[1/z].
\]  

(5.10)

This exact duality between the system at \( z \) and \( 1/z \) depends on the precise form (5.2) and (5.4) of the interaction. The partition function in equation (5.3) is also manifestly invariant under \( z \to z + i \). Thus, the system is exactly dual to itself under all compositions of the two transformations \( z \to 1/z \) and \( z \to z + i \). We will shortly explore the consequences of this.

By a slight variant of the above arguments, one can derive, following \([12]\), an exact duality relation for the two-point function:

\[
\langle \tilde{h}(k)^{\mu} \tilde{h}(-k)^{\nu}(z) \rangle = \tilde{D}^{\mu\nu}(k; z) - M^{\mu\rho}(k; 1/z) \langle \tilde{h}(k)^{\rho} \tilde{h}(-k)^{\lambda}(1/z) \rangle M^{\lambda\nu}(k; 1/z).
\]  

(5.11)

The duality relation (4.27) for the continuum problem is equivalent to this if, for some reason, the two-point functions don’t depend on the potential amplitude or there is some self-dual value of the fugacity for which \( V = \bar{V} \).

At the self-dual point, \( z = 1 \), equation (5.11) determines the two point function exactly. It is \( \langle \tilde{h}(k)^{\mu} \tilde{h}(-k)^{\nu}(1) \rangle = \frac{1}{2} \tilde{D}(k; 1) \), which is half the value for the continuous gaussian model. (Note that there is a certain discontinuity here: anywhere inside the first circle, no matter how close to \( z = 1 \), the two-point function is given by \( \tilde{D}(k, z) \).) We can apply the transformations \( z \to z + i \) and \( z \to 1/z \) to this expression to find the value of the mobility for all \( z \) given by

\[
z = \frac{d - ib}{a + ic} = \frac{1}{a^2 + c^2} \cdot \frac{\frac{1}{2} ab + cd}{a^2 + c^2}, \quad \text{for} \quad ad - bc = 1, \quad a, b, c, d \in Z.
\]  

(5.12)

It is

\[
\langle h_k^{\mu} h_{-k}^{\nu}(z) \rangle = \frac{1}{2} (a^2 + c^2) \tilde{D}(k, 1).
\]  

(5.13)

We observe that at these special points (which include the special points on the first circle mentioned in the previous section) the off-diagonal propagator (or the Hall coefficient) vanishes even though the magnetic field at these points is not in general zero.
6. Duality and the Phase Diagram

Now we want to use these results to map out the phase structure of the dissipative Hofstadter model in the \((\alpha, \beta)\) plane. Along the zero-field line, there is no doubt that the system is localized for \(\alpha > 1\) and delocalized for \(\alpha < 1\). The Coulomb gas treatment of Section 4 allows us to expand on this somewhat. There are two ‘dual’ Coulomb gases: in the first, the charges correspond to insertions of the cosine potential and the fugacity is one-half of the potential strength \(V_0\); in the second, the charges correspond to instantons describing tunnelings between potential minima and the fugacity is essentially the hopping probability of the particle. The one-loop renormalization group argument applied to a Coulomb gas says that fugacity scales to zero (and the associated charges become irrelevant) when the coefficient in front of the logarithmic self-interaction energy is greater than a critical value. Applied to the first Coulomb gas, this argument says that the periodic potential insertions are irrelevant everywhere within the circle \(\alpha^2 + \beta^2 = \alpha\). Applied to the second Coulomb gas, it tells us that the fugacity for instantons scales to zero (i.e. the hopping operator is irrelevant and the particle is localized) everywhere to the right of \(\alpha = 1\). Thus, it seems very plausible that the system is delocalized within the ‘first circle’ \(\alpha^2 + \beta^2 = \alpha\) and localized in the half-plane to the right of \(\alpha = 1\). As partial confirmation of this, recall that, in Section 3, we were able to exactly calculate the two-point function at those points on the ‘first circle’ where \(\beta/\alpha \in \mathbb{Z}\) and indeed found delocalized behavior.

With the help of duality arguments, we can parlay these two results into a global picture of the phase structure. There are two levels of duality which we could imagine using: The discrete Gaussian model has an exact invariance under the two-element modular group \(z \mapsto z + i\) and \(z \mapsto 1/z\), as described in Section 5. The Hofstadter model, however, is exactly invariant only under the subgroup \(z \mapsto z/(1 + inz)\) (for \(n\) integer), as described in Section 4. What’s missing for the Hofstadter model are the transformations \(z \mapsto z + ni\) or \(z \mapsto 1/z\): under those parameter transformations, the strength of the periodic potential transforms as well. However, in the regions where the potential is effectively irrelevant (not just inside the first circle, as we shall see), the shifts and inversions should still be valid symmetries of the asymptotic behavior of the theory.

As a first step, let us see how far we can get with the exact symmetry of the Hofstadter model. Under \(z \mapsto z/(1 + inz)\), the half-plane \(\alpha > 1\) is mapped into the circle of radius \(1/(2n^2)\) and center \(1/(2n^2) - i/n\). This is tangent to the first circle at a point where \(\beta/\alpha = -n\) and also to the \(\beta\)-axis at \(\beta = -1/n\). These circles are displayed in Fig. 1. Now we use the exact relation (4.14) to compute the mobility (i.e. the long-time or low-frequency limit of the two-point function) in the circles in terms of the mobility in the region \(\alpha > 1\). Since this latter is a region of localization, its mobility is zero, and (4.14) expresses the mobility in the circles in terms of the mobility of the Gaussian theory, encoded in (2.14). The result is

\[
\langle X^\mu(t)X^\nu(0)\rangle(z, V) = D^{\mu\nu}(t; z + \frac{i}{n}) \quad \text{for } |t| >> 0,
\]

(6.1)

which is to say that, as far as mobility is concerned, the \(n\)-th circle is just a translation parallel to the \(\beta\) axis of the Gaussian mobility evaluated within an appropriate sub-circle of the first circle. The simple renormalization group argument implies that the mobility is indeed Gaussian within the first circle, so this is consistent with the \(z \mapsto z + i\) duality. These results are completely different from what one would find if the periodic potential (2.4) were zero: in that case the mobility would be given exactly by (2.14) everywhere in
the z-plane. So, although there is no explicit dependence on \( V_0 \), the potential is clearly relevant outside the first circle!

Because the mobility inside the first circle and all the circles shown in Fig. 1 has no explicit \( V \)-dependence, we should be able to use the duality relation under \( z \to z + in \) (equation (4.28)) to find the mobility for all points that are translates of these regions. As indicated above, this symmetry is probably correct for the asymptotic behavior of the Hofstadter model. This claim is supported by the observation that for the two circles immediately above and below the first circle, we have just shown that the mobility is \( D(t; z + i) \), which is exactly what we would expect from the duality transformations given by equation (4.28). When we apply the translation by \( im \) to the first circle, we obtain a chain of circles tangent to each other and to the lines \( \alpha = 0 \) and \( \alpha = 1 \). Next, we can repeatedly apply \( z \to z/(1 + inz) \) and \( z \to z + in \) to these circles, which starts filling in the regions between the line \( \alpha = 0 \) and the first chain of circles with new circles that are tangent to the \( \beta \)-axis and already existing circles. The result is shown in fig. 2. It is the same phase diagram found by Cardy for abelian lattice gauge theories with theta terms [4]. In the diagram, between any two circles and the \( \beta \)-axis, there is another, smaller circle. So, the boundaries of these circles, which, as explained below, should identify lines of phase transitions, get denser and denser as one gets closer and closer to the \( \alpha = 0 \) line (the zero dissipation line). On this line, every point with rational \( \beta \) is tangent to a circle, and every point with irrational \( \beta \) is not. This connects nicely to the Hofstadter model (with no dissipation) where the behavior is quite different depending on whether \( \beta \) is rational or irrational.

According to the equations (4.14) and (4.28) for the transformation of the two-point function, we find that inside all the circles the mobility is Gaussian: in a circle tangent to the \( \beta \)-axis at \( z = p/q \), with \( p \) and \( q \) relatively prime, the mobility is given by \( D(t; z - ip/q) \). The triangular regions between the circles belong to some other phase which this argument doesn’t help us identify.

Along the phase transition circles, one expects to find a non-trivial critical theory. Each of these circles is a mapping of the critical line at \( \alpha = 1 \) (or the critical circle \( \alpha^2 + \beta^2 = \alpha \)) by compositions of the transformations \( z \to z + i \) and \( z \to 1/z \). Under these transformations, the partition function is multiplied by an analytic function (which depends on \( C \) and \( N \) in equations (4.9) and (4.23)) and the strength of the periodic potential changes. Because the existence of the phase transition on the first circle is independent of \( V_0 \) (at least for small \( V_0 \)), neither of these two changes to the partition function should affect the fact that it is critical. Consequently, we expect all the circles to be critical circles, at least for some range of values of the potential strength. However, because we do not yet know exactly how \( V_0 \) changes under these transformations, we know little about the critical theories, except at the points where \( \beta/\alpha = n \) for \( n \) an integer, which is where the other circles are tangent to the first circle. For these points, we can apply the exact duality transformation (4.14) to the two-point function at \( z = 1 \) (given in equation (3.8)) and then simplify the resulting expression by using equations (2.17), (2.18), and (2.19) to obtain

\[
\langle \vec{X}(t)\vec{X}(0)\rangle(z_n, V_0) = D(t; z_n) + (f(V_0) - 1) D(t; z_n^2)
\]

where \( z_n = 1/(1 + in) \) and \( f(V_0) \) is the factor by which the two-point function at \( z = 1 \) differs from the Gaussian two-point function (2.14). With the specific regulator used in
Sect. 3, we found that
\[ f(V_0) = \left(1 + \frac{V_0 T}{2} e^{-\langle X(0) X(0) \rangle} \right)^2 = 1 + (\tilde{V}_0/2)^2, \tag{6.3} \]
but, in general, its exact form depends on how the theory is regulated.

We note that this two-point function depends on the strength of the potential. For the discrete Gaussian model, there is no parameter corresponding to \(V_0\), but the two-point function is consistent with equation (6.2) with \(f(V_0) = 1/2\) or \(\tilde{V}_0 = 2\). For this special value of \(f(V_0)\), the two-point function at \(z_n = 1/(1 + in)\) becomes
\[ \langle \vec{X}(t) \vec{X}(0) \rangle(z_n, V_0) = \frac{1}{2}(1 + n^2)D(t;1). \tag{6.4} \]

A point to emphasize here is that the off-diagonal part of the mobility (the Hall coefficient) vanishes at these points for this particular value of \(V_0\), even though the magnetic field is not zero. These considerations make it quite plausible that the discrete Gaussian model of Sect. 5 has the same critical behavior as the continuum dissipative Hofstadter model for a special value of the potential strength.

In sum, the potential strength is an exactly marginal variable. The higher-point functions for \(\alpha = 1\) reduce to products of two-point functions plus contact terms (which have not been explicitly computed) and for the other points on the first circle we have evidence that some of the higher \(n\)-point functions are non-zero, SU(1,1) invariant functions. Therefore, even if we don’t know the details, we are guaranteed that the critical theory is not identical to the Gaussian theory.

### 7. Loose Ends and Problems for the Future

The main result is that we have discovered a phase structure for the Hofstadter model (and a discrete version of it) which shows how the fractal energy level structure of the pure quantum mechanics is matched by a fractal structure of phase transitions in the dissipative theory. There are several issues to pursue. We have identified critical surfaces with critical behavior depending on at least two parameters: potential strength and arc position in the \(\alpha\)-\(\beta\) plane. We know little about the detailed properties of these critical theories and it would be very interesting to study them in more detail, especially in the discrete Gaussian model, where things should be simple. We have seen in Section 3 that at the special points on the first circle something remarkable happens. We have hints that at these values of \(\alpha\) and \(\beta\), most of the \(n\)-pt functions (for \(n > 2\)) reduce to contact terms, which is reminiscent of topological field theory, but we do not yet know how to pursue this connection. These critical theories have an interpretation as solutions of open string theory with nontrivial backgrounds. It will obviously be interesting to explore the spacetime interpretation of these states. We also want to understand the nature of the phases in Fig. 2 which fill in the interstitial regions between the circles. They will presumably be some sort of Coulomb phase which is neither localized nor delocalized. We have yet to determine what this means quantitatively for the behavior of the mobility and Hall coefficient.

Another line of development takes us into N-body physics. We have seen that the transport coefficients (especially the Hall coefficient) of a single electron have remarkable
structure in the presence of dissipation. We can actually turn the Hall effect off by choosing the right potential amplitude at special values of the field. Filling Landau levels with such electrons would doubtless lead to new quantized Hall effect phenomenologies. There is a speculative connection with nonperturbative open string theory, because inserting many boundaries into the string possibly corresponds to having many electrons in the dissipative quantum system.

The final big issue is the possibility of constructing experimental realizations of the physics we have uncovered. Work is already under way to observe, separately, the Hofstadter spectrum and the macroscopic tunneling effects for a two level system. It may be possible that specially constructed junction arrays could be used to experimentally investigate the combined effects of the magnetic field, periodic potential and dissipation.

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