STONE PSEUDOVARIELTIES

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ABSTRACT. Profinite algebras are the residually finite compact algebras; their underlying topological spaces are Stone spaces. We extend the theory of profinite algebras to a more general setting of Stone topological algebras. We introduce Stone pseudovarieties, that is, classes of Stone topological algebras of a fixed topological signature that are closed under taking Stone quotients, closed subalgebras and finite direct products. Looking at Stone spaces as the dual spaces of Boolean algebras, we find a simple characterization of when the dual space admits a natural structure of topological algebra. This provides a new approach to duality theory which culminates in the proof that a Stone quotient of a Stone topological algebra that is residually in a given Stone pseudovariety is also residually in it, thereby extending the corresponding result of M. Gehrke for the Stone pseudovariety of all finite algebras over discrete signatures. The residual closure of a Stone pseudovariety is thus a Stone pseudovariety, and these are precisely the Stone analogues of varieties. A Birkhoff type theorem for Stone varieties is also established.

1. INTRODUCTION

In view of Stone duality, compact 0-dimensional spaces, also known as Stone spaces or Boolean spaces, deserve special attention. When the dual Boolean algebras have some additional structure, this is also carried over to the corresponding Stone spaces [14, 13, 12]. Although not initially formulated that way, this situation may be recognized in the seminal framework developed by Eilenberg [9] for the classification of classes of regular languages by pseudovarieties of finite semigroups and finite monoids, which provides a translation of combinatorial problems on languages to algebraic problems that sometimes enables the effective solution of the former. Eilenberg’s framework has been extended in various directions, including other algebraic structures than semigroups [24, 11, 29, 28].

The solution of the algebraic problems resulting from the translation mentioned in the preceding paragraph is often expressed in terms of verifiable pseudoidentities, which are just formal equalities between members of a suitable relatively free Stone topological algebra. In the classical setting, the

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fact that pseudovarieties are always defined by pseudoidentities is just Reit-

erman’s theorem [21, 4] and has served as motivation to understanding the
structure of relatively free profinite semigroups.

In the classical setting of Eilenberg’s theory, the Stone topological algebras
that arise are all profinite, which restricts the reach of the theory to the realm
of regular languages. Basically, because Stone topological algebras of certain
kinds, such as semigroups, are necessarily profinite, one cannot expect to
go beyond regular languages without looking at different kinds of algebraic
structures. Progress in that direction has been previously obtained (see [15])
but the theory remains confined to word languages.

Our aim with this paper is to develop a general study of Stone topological
algebras that may eventually lead to applications to more general classes of
languages. We extend the notion of pseudovariety of finite algebras to Stone
topological algebras, namely by considering classes $S$ of Stone topological
algebras that are closed under taking Stone quotients, closed subalgebras, and
finite direct products. Such classes have associated relatively free topological
algebras which play a central role in this paper.

Viewing Stone spaces as dual spaces of Boolean algebras provides an al-
ternative approach to Stone topological algebras. The possibility of defining
a natural topological algebraic structure on the dual space of a Boolean al-
gebra turns out to admit a rather simple characterization. This leads to an
alternative approach to duality, which may be compared with that developed
in [13, 14, 12]. The focus in those papers is somewhat different from ours:
first in that bounded distributive lattices, rather than Boolean algebras, are
considered there, so that Priestley duality plays the role of Stone duality; and
second because profinite algebras are viewed there as duals of lattices with
additional unary operations. Extra care needs to be taken in our approach
due to the fact that we are dealing with topological signatures. In the case of
a Stone signature, which is a significant generalization of a finite signature,
the duality turns out to be particularly simple. As our main application
we show that, for an arbitrary Stone pseudovariety $\mathcal{S}$, a Stone quotient of
a residually $\mathcal{S}$ Stone topological algebra is again residually $\mathcal{S}$. This further
generalizes the special case obtained in [12] beyond the profinite case and
discrete signatures. The residual closure of a Stone pseudovariety is, there-
fore, also a Stone pseudovariety. For such Stone pseudovarieties, which are
the Stone analogs of the classical Birkhoff varieties, we establish adequate
versions of Birkhoff’s theorem by showing that they are defined by Stone
pseudoidentities over Stonean spaces. The Reiterman theorem turns out to
be a special case.

Here is a short guide to the paper. Section 2 quickly introduces necessary
background. Section 3 defines Stone pseudovarieties, provides a construction
of their free Stone topological algebras, and discusses various properties of
these structures and how they relate to term algebras. In Section 4 we
define three conditions on a Boolean algebra of subsets of an algebra and,
which have both a topological and an algebraic/combinatorial character; two of
these conditions turn out to be equivalent in general while all three
conditions are equivalent for a Stone signature. The more general of those
conditions are shown in Section 5 to characterize such Boolean algebras
whose dual spaces have a natural structure of Stone topological algebra, thus leading to our approach to duality. Since free Stone topological algebras are compactifications of term algebras, a natural question is whether they are the most general ones, namely the Čech-Stone compactifications; we show that this is not the case for a discrete signature as long as there is at least one operation symbol of arity greater than one. The paper concludes with Section 6 where some first applications of duality are presented. Further applications are planned for the continuation of this work.

2. Preliminaries

We adopt the notion of topological algebra given in [22]. Briefly, we consider a signature to be a graded set $\Omega = \bigcup_{n \geq 0} \Omega_n$, where each $\Omega_n$ is a set, whose elements are called the $n$-ary operation symbols. An $\Omega$-algebra is a pair $(A, E)$, where $A$ is a nonempty set and $E = (E_n)_{n \geq 0}$ is a sequence of evaluation mappings $E_n : \Omega_n \times A^n \to A$. In case $A$ is endowed with a topology and so is each set $\Omega_n$, we say that the algebra $(A, E)$ is a topological $\Omega$-algebra if each mapping $E_n$ is continuous. Usually, the sequence $E$ is not explicitly mentioned and we refer to an algebra $(A, E)$ simply as the algebra $A$. The notions of homomorphism, subalgebra, direct product are the standard ones (see [6]); in the topological setting, we want homomorphisms to be continuous, subalgebras to have the induced topology and product algebras to be endowed with the product topology.

Given a topological property $\mathcal{P}$, we say that topological algebra $A$ is a $\mathcal{P}$ algebra if the topological space $A$ has property $\mathcal{P}$. However, we talk about a Stone topological algebra instead of Stone algebra when the underlying topological space is a Stone space because the latter designation already has a different meaning in the literature.

If each set $\Omega_n$ is a topological space with a certain property $\mathcal{P}$ (such as being compact, 0-dimensional, or discrete), then we say that the signature $\Omega = \bigcup_{n \geq 0} \Omega_n$ has property $\mathcal{P}$. The exception is finiteness for we say that a signature is finite if so it is as a set. Note that, unlike some literature, we assume that compact spaces are Hausdorff. Since we always want to consider only Hausdorff spaces, finite algebras are always viewed with the discrete topology.

For a class $\mathcal{C}$ of topological algebras, we say that a topological algebra $A$ is residually $\mathcal{C}$ if, for every pair $a, b$ of distinct elements of $A$, there is a continuous homomorphism $\varphi : A \to C$ into a member $C$ of $\mathcal{C}$ such that $\varphi(a) \neq \varphi(b)$.

A continuous function from a topological space $X$ to a topological algebra $A$ is said to be a generating mapping if its image generates (algebraically) a dense subalgebra of $A$. In case the mapping is inclusion, we then also say that $A$ is $X$-generated.

By a profinite algebra we mean a compact algebra which is residually finite. Equivalently, a profinite algebra is an inverse limit of finite algebras in the category of topological algebras. For our purposes, the following alternative characterization of profiniteness suffices.
Theorem 2.1 ([3]). A Stone topological algebra $S$ is profinite if and only if, for every clopen subset $L \subseteq S$, there exists a continuous homomorphism $\varphi : S \to F$ onto a finite algebra $F$ such that $L = \varphi^{-1}(\varphi(L))$.

In the terminology of formal language theory, Theorem 2.1 states that a Stone topological algebra is profinite if and only if its clopen subsets are the subsets that are recognized by continuous homomorphisms into finite algebras.

The following result is proved in [12] Theorem 4.3] for finite signatures using duality theory. An alternative approach using a characterization of profiniteness by syntactic congruences is presented in [3 Theorem 4.2], where an equivalent formulation is adopted that is not so convenient for our current purposes; it works in our setting of general topological algebras.

Theorem 2.2. Suppose that $\varphi : S \to T$ is an onto continuous homomorphism of topological algebras such that $T$ is $0$-dimensional and Hausdorff. If $S$ is profinite then so is $T$.

A pseudovariety is a class of finite $\Omega$-algebras that is closed under taking homomorphic images, subalgebras and finite direct products. For a pseudovariety $V$, a profinite algebra is said to be pro-$V$ if it is residually $V$.

The absolutely free $\Omega$-algebra over a set $X$ is denoted $T_\Omega(X)$. Its elements are the $\Omega$-terms on $X$, that is, the formal expressions that may be constructed from the elements of $X$ applying formally operation symbols according to their arity: the elements of $X$ are terms and, if $w \in \Omega_n$ and $t_1, \ldots, t_n$ are terms, then so is $w(t_1, \ldots, t_n)$. Terms may be represented by labeled trees recursively as follows: the labeled tree of $x \in X$ is a tree reduced to the root, of non-operation type which is labeled $x$; for $w \in \Omega_n$ and terms $t_1, \ldots, t_n$, the labeled tree of $w(t_1, \ldots, t_n)$ has root of operation type labeled $w$ with (ordered) children which are the roots of the labeled trees of $t_1, \ldots, t_n$, respectively. By the typed shape of a term we mean the corresponding labeled tree where labels are omitted but the type of node is retained.

In case $X$ is a topological space and $\Omega$ is a topological signature, we may view the term algebra as the topological sum of the spaces of terms of each typed shape; by the latter we mean the set of all terms of a given typed shape, which is viewed as the direct product of the spaces of possible labels of each individual node.

Proposition 2.3. The term algebra $T_\Omega(X)$ is the absolutely free topological $\Omega$-algebra on $X$ in the sense that the inclusion mapping $\iota : X \to T_\Omega(X)$ is such that, for every continuous mapping $\varphi : X \to A$ into a topological $\Omega$-algebra $A$, there is a unique continuous homomorphism $\hat{\varphi} : T_\Omega(X) \to A$ such that $\hat{\varphi} \circ \iota = \varphi$.

Proof. We first show that $T_\Omega(X)$ is a topological $\Omega$-algebra by showing that each evaluation mapping $E_n : \Omega_n \times (T_\Omega(X))^n \to T_\Omega(X)$ is continuous. Indeed, if $(\alpha_i, t_{1,i}, \ldots, t_{n,i})$ is a net converging in $\Omega_n \times (T_\Omega(X))^n$ to $(\alpha, t_1, \ldots, t_n)$, then we may assume that, for each $k \in \{1, \ldots, n\}$, all $t_{k,i}$ have the same typed shape as $t_k$, so that the label in each node of $t_{k,i}$ converges to the label of the corresponding node of $t_k$. Then the same property holds for
the terms of the net \( (o_i(t_{1,i}, \ldots, t_{n,i}))_i \) with respect to the term \( o(t_1, \ldots, t_n) \), which shows that this term is the limit of the net, thereby establishing that \( E_n \) is continuous.

Next, suppose that \( \varphi : X \to A \) is a continuous mapping into a topological algebra \( A \). By the universal property of \( T_\Omega(X) \) as a free algebra, we know that there is a unique homomorphism \( \hat{\varphi} : T_\Omega(X) \to A \) such that \( \hat{\varphi} \circ o = \varphi \).

Thus, it suffices to show that \( \hat{\varphi} \) is continuous. For this purpose, consider a net \( (t_i)_i \) converging in \( T_\Omega(X) \) to a term \( t \). We may again assume that all \( t_i \) have the same typed shape as \( t \). We need to show that

\[
\tag{2.1} \hat{\varphi}(t_i) \text{ converges to } \hat{\varphi}(t).
\]

We proceed by induction on the height of \( t \), assuming that the desired convergence \( (2.1) \) holds for smaller height than that of \( t \). As \( t_i \) converges to \( t \) in \( T_\Omega(X) \), we may assume that either \( t_i \) converges to \( t \) in \( X \), in which case the continuity of \( \varphi \) yields \( (2.1) \), or \( t_i = o_1(s_{1,i}, \ldots, s_{n,i}) \) and \( t = o(s_1, \ldots, s_n) \), where \( n \geq 0 \), \( o_i \) converges to \( o \) in \( \Omega_n \), and each \( s_{k,i} \) converges to \( s_k \) in \( T_\Omega(X) \).

Since each \( s_k \) has height smaller than that of \( t \), the induction hypothesis gives that \( \hat{\varphi}(s_{k,i}) \) converges to \( \hat{\varphi}(s_k) \) in \( A \). Since \( \hat{\varphi} \) is a homomorphism and the evaluation mapping \( E_n^A : \Omega_n \times A^n \to A \) is continuous, it follows that \( (2.1) \) holds, thereby achieving the induction step.

Let \( X \) be a topological space. We say that the space \( X \) is \( T_1 \) if, given any distinct points \( x, y \in X \), there is an open set \( U \) such that \( x \in U \) and \( y \notin U \).

The space \( X \) is completely regular if, whenever \( x \in X \) and \( C \subseteq X \) is a closed subset not containing \( x \), there is a continuous function \( \varphi : X \to \mathbb{R} \) into the reals that maps \( x \) to 0 and \( C \) to 1. A \( T_1 \) completely regular space is said to be a Tychonoff space. Compact spaces are Tychonoff [10, Theorems 1.5.11 and 3.1.9] and so are subspaces of Tychonoff spaces.

A compactification of the space \( X \) is a compact space \( C \) endowed with a homeomorphic embedding \( \varepsilon_C : X \to C \) whose image is a dense subspace of \( C \). Compactifications of \( X \) may be naturally ordered by letting \( C_1 \subseteq C_2 \) if there is a continuous mapping \( f : C_2 \to C_1 \) such that \( f \circ \varepsilon_{C_2} = \varepsilon_{C_1} \). The \v{C}ech-Stone compactification (also known as Stone-\v{C}ech compactification; since both papers [4, 27] were published in the same year, we prefer the alphabetical order) of \( X \) is a maximum compactification of \( X \) in the quasi-ordering \( \preceq \), which may or may not exist. If it exists for the space \( X \), it is clearly unique and it is denoted \( \beta X \). The \v{C}ech-Stone compactification exists exactly for Tychonoff spaces [10, Corollary 3.5.10]. Note that the existing literature also considers compactifications of more general spaces, but then the definition assumes that \( \varepsilon_C \) is only a continuous mapping.

Recall that, for a Tychonoff space \( T \) of cardinality \( \kappa \), any compactification of \( T \) has cardinality at most \( 2^{2\kappa} \) [10, Theorem 3.5.3]. Hence, for a topological space \( X \), the cardinality of an \( X \)-generated Stone topological \( \Omega \)-algebra \( S \) is bounded by \( 2^{2\kappa} \), where \( \kappa = \max\{|\Omega|, |\aleph_0|, |X|\} \) is a bound of the cardinality of the subalgebra of \( S \) (algebraically) generated by \( X \). By identifying homeomorphic-isomorphic Stone topological algebras, we conclude that the \( X \)-generated Stone topological algebras may be viewed as constituting a set.
By a Stone pseudovariety we mean a nonempty class of Stone topological algebras of a fixed topological signature that is closed under taking continuous homomorphic images that are again Stone spaces, closed subalgebras, and finite direct products. Clearly, every one-point algebra, also called a trivial (topological) algebra, is a Stone topological algebra which is a homomorphic image of every Stone topological algebra. Thus, the trivial algebras constitute the smallest Stone pseudovariety. We also mention that continuous bijections between compact spaces are homeomorphisms. Hence, a continuous mapping between Stone topological algebras that is an algebraic isomorphism is also a homeomorphism.

Note that pseudovarieties of finite algebras are Stone pseudovarieties. For a pseudovariety \( V \) of finite algebras, the class of all pro-\( V \) algebras is also a Stone pseudovariety because of Theorem 2.2.

The class of all Stone topological \( \Omega \)-algebras is a Stone pseudovariety, denoted \( \text{St} \Omega \). The pseudovariety of all finite Stone topological \( \Omega \)-algebras is denoted \( \text{Fin} \Omega \). Since the intersection of a nonempty family of Stone pseudovarieties is again a Stone pseudovariety, every class of Stone topological algebras generates a Stone pseudovariety, namely the smallest Stone pseudovariety that contains it.

The following are some further easy to describe examples of Stone pseudovarieties.

**Example 3.1.** On a Stone space, we may always define a structure of Stone topological algebra by choosing a point and declaring it to be the only value of all operations. Such a structure and the resulting Stone topological algebra are said to be null. The class \( \text{Null} \Omega \) of all null Stone topological \( \Omega \)-algebras is a Stone pseudovariety.

**Example 3.2.** Let \( \kappa \) be an infinite cardinal. The class \( \text{St}^\kappa \Omega \) of all Stone topological \( \Omega \)-algebras of cardinal less than \( \kappa \) is a Stone pseudovariety. Note that \( \text{St}^{\aleph_0} \Omega = \text{Fin} \Omega \). Distinct infinite cardinals \( \kappa \) determine distinct Stone pseudovarieties \( \text{St}^\kappa \Omega \) and in fact even the Stone pseudovarieties \( \text{St}^\kappa \Omega \cap \text{Null} \Omega \) are distinct: first, for every infinite discrete space, its Alexandroff (one point) compactification is a Stone space, so that there are Stone spaces of every cardinality; second, such a Stone space admits a null structure.

**Example 3.3.** A Stone topological algebra \( S \) is said to be nilpotent if there exists an integer \( N \) such that, for every finite space \( X \) and every continuous homomorphism \( \varphi : T^X_\Omega \to S \), all terms with typed shape of height at least \( N \) have the same image. The class \( \text{St}_{\text{nil}} \) of all nilpotent Stone topological algebras is a Stone pseudovariety.

3.1. **Relatively free Stone topological algebras.** Given a topological space \( X \) and a Stone pseudovariety \( \delta \), an \( \delta \)-free Stone topological algebra over \( X \) is given by a continuous generating mapping \( \iota : X \to S \) into a Stone topological algebra \( S \) which is residually \( \delta \) and such that, for every continuous mapping \( \varphi : X \to T \) into a member \( T \) of \( \delta \), there is a unique continuous homomorphism \( \hat{\varphi} : S \to T \) such that \( \hat{\varphi} \circ \iota = \varphi \). Usually,
mapping \( \iota \) is understood from the context and we simply refer to \( S \) as an \( \delta \)-free Stone topological algebra.

The proof of the next result gives a construction for free Stone topological algebras relative to Stone pseudovarieties. The construction is nothing but the usual realization of an inverse limit in the topological or algebraic context. Since our considerations on Stone topological algebras go beyond the standard setting, the details of the proof are presented for the sake of completeness.

**Proposition 3.4.** Let \( \mathcal{S} \) be a Stone pseudovariety and let \( X \) be a topological space. Then there exists an \( \delta \)-free Stone topological algebra over \( X \) which is uniquely determined up to continuous isomorphism respecting generators.

**Proof.** By the cardinality considerations at the end of Section 2 there is a nonempty set \( \mathfrak{R} \) of generating mappings from \( X \) to members of \( \delta \) such that, for every generating mapping \( \varphi : X \to T \) into a member of \( \delta \), there is a unique member \( \psi : X \to R \) of \( \mathfrak{R} \) and a continuous isomorphism \( \alpha : T \to R \) such that \( \alpha \circ \psi = \varphi \). For a generating mapping \( \psi : X \to R \) in \( \mathfrak{R} \) we also write \( R\psi \) for \( R \). We order \( \mathfrak{R} \) as follows: for \( \varphi : X \to R_\varphi \) and \( \psi : X \to R_\psi \), we write \( \varphi \preceq \psi \) if there exists a continuous homomorphism \( \alpha_{\varphi,\psi} : R_\psi \to R_\varphi \) such that \( \alpha_{\varphi,\psi} \circ \psi = \varphi \). We observe that the mapping \( \alpha_{\varphi,\psi} \) is uniquely determined by the pair \( \varphi, \psi \) of elements of \( \mathfrak{R} \) such that \( \varphi \preceq \psi \) because \( R_\psi \) is \( X \)-generated. Also note that \( \preceq \) is a partial order on \( \mathfrak{R} \) which is upper directed: given \( \varphi_1 \) and \( \varphi_2 \) in \( \mathfrak{R} \), there is \( \varphi \in \mathfrak{R} \) such that \( \varphi_1 \preceq \varphi \) and \( \varphi_2 \preceq \varphi \).

The remainder of the proof consists in showing that the usual construction of the inverse limit of the inverse system \( \mathfrak{R} \) is a Stone topological algebra that is \( \delta \)-free over \( X \).

The product \( P = \prod_{\varphi \in \mathfrak{R}} R_\varphi \) of Stone topological algebras is itself a Stone topological algebra. Consider the subset \( F \) consisting of all \( (r_\varphi)_{\varphi \in \mathfrak{R}} \) such that, whenever \( \varphi, \psi \in \mathfrak{R} \) satisfy \( \varphi \preceq \psi \), the equality \( \alpha_{\varphi,\psi}(r_\psi) = r_\varphi \) holds. We claim that the (continuous) mapping \( \iota : X \to F \) given by \( \iota(x) = (\varphi(x))_{\varphi \in \mathfrak{R}} \) determines an \( \delta \)-free Stone topological algebra over \( X \).

It is routine to verify that \( F \) is a subalgebra of \( P \). We need to show that it is a closed subset of \( P \). For a pair \( \varphi, \psi \in \mathfrak{R} \) satisfying \( \varphi \preceq \psi \), we consider the subset \( F_{\varphi,\psi} = \{(r_\varphi)_{\varphi \in \mathfrak{R}} \in P \mid \alpha_{\varphi,\psi}(r_\psi) = r_\varphi \} \) of \( P \). Since the graph of the continuous mapping \( \alpha_{\varphi,\psi} \) is a closed subset of \( R_\psi \times R_\varphi \), the subset \( F_{\varphi,\psi} \) is closed in \( P \). Now \( F \) is closed because it is the intersection of all \( F_{\varphi,\psi} \) with \( \varphi, \psi \in \mathfrak{R} \) such that \( \varphi \preceq \psi \). As \( P \) is a Stone topological algebra that is residually \( \delta \), the same is true for its closed subalgebra \( F \).

The mapping \( \iota : X \to F \) is continuous. To show that it is a generating mapping, we recall that elements of the subalgebra algebraically generated by \( \iota(X) \) are given by terms in variables from the set \( X \). More formally and generally, since \( T_\Omega(X) \) is the free topological algebra over \( X \), each continuous mapping \( \varphi : X \to R \) can be extended to a unique continuous homomorphism \( \tilde{\varphi} : T_\Omega(X) \to R \), where the image of \( \tilde{\varphi} \) is exactly the subalgebra of \( R \) algebraically generated by \( \iota(X) \). We consider an arbitrary element \( r \in F \) and its neighborhood \( N \) and we want to show that \( N \) contains some point given by a term. We may assume that \( N = \prod_{\varphi \in \mathfrak{R}} N_\varphi \) is an open set from the basis of the product topology of \( P \). Thus, there is a finite subset \( Z \) of \( \mathfrak{R} \) such that \( N_\varphi = R_\varphi \) for \( \varphi \notin Z \) and \( N_\varphi \) is a proper open subset of \( R_\varphi \) for
\[ \varphi \in Z. \] Since the order of \( R \) is upper directed, there is \( \psi \in R \) such that \( \varphi \leq \psi \) for every \( \varphi \in Z \). We observe that the open subset \( V \) of \( R_\psi \) given by \( V = \bigcap_{\varphi \in Z} \alpha_{\varphi, \psi}(N_\varphi) \) contains the element \( r_\psi \). Since \( R_\psi \) is \( X \)-generated, there is a term \( t \) such that \( \psi(t) \in V \). It is easy to see that the condition \( \psi(t) \in N_\varphi \) holds for every \( \varphi \in Z \) and so, it holds for every \( \varphi \in R \).

To show that \( \iota : X \to F \) is an \( \delta \)-free Stone topological algebra over \( X \), it remains to prove the universal property. Let \( \varphi : X \to S \) be an arbitrary continuous mapping into a member \( S \) of \( \delta \). By the choice of \( R \), there exist \( \psi : X \to R_\psi \) in \( R \) and a continuous injective homomorphism \( \alpha : R_\psi \to S \).

Consider the restriction \( \pi_\psi : F \to R_\psi \) to \( F \) of the projection of \( P \) to the \( \psi \)-component. Then \( \alpha \circ \pi_\psi : F \to S \) is a continuous homomorphism such that \( \alpha \circ \pi_\psi \circ \iota = \varphi \). Uniqueness of a mapping \( F \to S \) with such properties follows from the fact that \( \iota \) is a generating mapping. This establishes the existence of an \( \delta \)-free topological algebra over \( X \).

Let \( \chi : X \to G \) be another \( \delta \)-free Stone topological algebra over \( X \). Let \( \varphi \) be an arbitrary element of \( R \). From the universal property of \( \chi \), we may deduce the existence of a continuous homomorphism \( \beta_\varphi : G \to R_\varphi \) such that \( \beta_\varphi \circ \chi = \varphi \). From the universal property of the product (both as an algebra and a topological space), we get a continuous homomorphism \( \beta : G \to P \). The image of \( \beta \) is equal to \( F \) because \( \chi \) is a generating mapping. Moreover, since \( G \) is residually \( \delta \), \( \beta \) is also injective. Altogether, \( \beta : G \to F \) is a continuous isomorphism and there is just one \( \delta \)-free Stone topological algebra over \( X \) up to continuous isomorphism.

For a Stone pseudovariety \( \delta \), we denote \( \overline{\Pi}_X \delta \) the \( \delta \)-free Stone topological algebra over \( X \). The mapping \( \iota : X \to \overline{\Pi}_X \delta \) is called the natural generating mapping.

**Proposition 3.5.** Let \( \delta \) be a nontrivial Stone pseudovariety and let \( X \) be a topological space. Then the continuous mapping \( \iota : X \to \overline{\Pi}_X \delta \) is a homeomorphic embedding if and only if \( X \) is 0-dimensional and Hausdorff.

**Proof.** Assume that \( \iota \) is a homeomorphic embedding. Since every Stone space is 0-dimensional and Hausdorff, and these properties are inherited by subspaces, we get that \( X \) is 0-dimensional and Hausdorff.

Now, assume that \( X \) is 0-dimensional and Hausdorff. Taking a pair of distinct elements \( x \) and \( y \) in \( X \), we know that there is a clopen subset \( U \) of \( X \) such that \( x \in U \) and \( y \notin U \). We consider a function \( \psi : X \to A \) into a nontrivial member \( A \) of \( \delta \) which maps \( U \) to a point and \( X \setminus U \) to a different point. Note that \( \psi \) is a continuous mapping. Since \( \psi \) factorizes through \( \iota \), there is a continuous (homomorphism) \( \alpha : \overline{\Pi}_X \delta \to A \) such that \( \alpha \circ \iota = \psi \).

Since \( A \) is a Stone space, there is is a clopen subset \( V \) containing \( \psi(U) \) but not the singleton set \( \psi(X \setminus U) \). Then \( \iota(U) \subseteq \alpha^{-1}(V) \) and \( \iota(X \setminus U) \subseteq \iota(X) \setminus \alpha^{-1}(V) \), so that \( \iota(U) = \alpha^{-1}(V) \cap \iota(X) \) is a clopen subset of \( \iota(X) \), thereby showing that \( \iota : X \to \iota(X) \) is an open mapping. In particular, we see that \( \iota(x) \neq \iota(y) \), which shows that \( \iota \) is injective. Hence, \( \iota : X \to \iota(X) \) is a homeomorphism. \( \square \)
Whenever the assumptions of Proposition 3.5 hold, we think of $X$ as a subset of $\overline{\Omega}_X \mathcal{S}$ and of the natural generating mapping as being the inclusion mapping.

Another natural question is whether $\overline{\Omega}_X \mathcal{S}$ contains as a subspace a homeomorphic-isomorphic copy of $T_{\Omega}(X)$, which is true in the case of pseudovarieties of finite algebras whenever certain nilpotent algebras are present in $\mathcal{S}$. We consider here their topological analogs which reflect the topologies on $X$ and $\Omega$.

**Lemma 3.6.** Let $K$ be a clopen subset of $T_{\Omega}(X)$ all of whose elements have the same typed shape such that $K$ is the product of clopen sets at each node in the typed shape. Then there is a continuous homomorphism $\varphi: T_{\Omega}(X) \to F$ into a nilpotent finite algebra $F$ such that $K = \varphi^{-1}(\varphi(K))$.

**Proof.** Let $\sigma$ be the typed shape of the elements of $K$. At each node $\tau$ of $\sigma$, $K$ is determined by a clopen subset $K_\tau$ of either $X$ or else of $\Omega_n$, for a node with $n_\tau \geq 0$ children; to uniformize the notation, we let $n_\tau = -1$ and $\Omega_{-1} = X$ if the space corresponding to the node $\tau$ is $X$. Since each $K_\tau$ is clopen and $\sigma$ is a finite tree, there is for each $n \geq -1$ a continuous retraction $\varphi_n: \Omega_n \to \Sigma_n$ onto a finite subset $\Sigma_n$ of $\Omega_n$ such that, for each node $\tau$ of $\sigma$, $\varphi_n^{-1}(\varphi_n(K_\tau)) = K_\tau$. Such a function may be obtained by collapsing to one of its points each atom in the Boolean subalgebra of $\mathcal{P}_\sigma(\Omega_n)$ generated by the clopen sets $K_\tau$ such that $n_\tau = n$.

Let $Y = \Sigma_{-1}$ and $\Sigma$ be the discrete signature whose $n$-ary symbols are those of $\Sigma_n$ for each $n \geq 0$. We define $F$ to consist of the $\Sigma$-terms over the set $Y$ of typed shape $\sigma$ together with all their subterms and an additional element $\bot$. When the term operations applied to the elements of $F$ produce a term in $F$, the operation is defined to give that value; otherwise, the value of the operation is $\bot$. Then, $F$ is a nilpotent $\Sigma$-algebra and, through the continuous functions $\varphi_n$, we may view $F$ as a nilpotent topological $\Omega$-algebra which is generated by the composite mapping $X \xrightarrow{\varphi_n} Y \xrightarrow{\varphi^{-1}} F$. This generating mapping extends uniquely to a continuous homomorphism $\varphi: T_{\Omega}(X) \to F$ which is readily seen to have the required property. □

**Proposition 3.7.** Let $X$ and each $\Omega_n$ be Hausdorff 0-dimensional spaces. Then the topological algebra $T_{\Omega}(X)$ is residually finite. Moreover, for every Stone pseudovariety $\mathcal{S}$ containing $\mathcal{S}_{\text{nil}} \cap \mathcal{F}_{\Omega}$, $T_{\Omega}(X)$ is homeomorphic-isomorphic to the subalgebra of $\overline{\Omega}_X \mathcal{S}$ generated by $X$.

**Proof.** Consider distinct terms $s$ and $t$ in $T_{\Omega}(X)$. Since $T_{\Omega}(X)$ is a Hausdorff 0-dimensional space, there is a clopen subset $K$ separating $s$ and $t$. By definition of the topology of $T_{\Omega}(X)$, $K$ is a union of clopen sets of constant typed shape, each of which is a product of clopen sets at the corresponding node sets. Hence, we may assume that $K$ is one of the clopen sets in the statement of Lemma 3.6. The lemma yields a continuous homomorphism $\varphi: T_{\Omega}(X) \to F$ into a finite algebra $F$ such that $K = \varphi^{-1}(\varphi(K))$, which, therefore, separates the terms $s$ and $t$. This establishes that $T_{\Omega}(X)$ is residually finite.

Let $\iota: T_{\Omega}(X) \to \overline{\Omega}_X \mathcal{S}$ be the natural continuous homomorphism given by Proposition 2.3 and let $\operatorname{Im} \iota$ be the image of $\iota$. To prove the last statement in the proposition it suffices to show that, for every clopen subset $K$ of $T_{\Omega}(X)$
with the property considered in Lemma 3.6, the set \( \iota(K) \) is the intersection of an open subset of \( \Omega X S \) with \( \text{Im } \iota \). Let \( \varphi : T_{\Omega}(X) \rightarrow F \) be the continuous homomorphism given by Lemma 3.6 and let \( \hat{\varphi} : \Omega X S \rightarrow F \) be the unique continuous homomorphism such that \( \hat{\varphi} \circ \iota = \varphi \). To conclude the proof, it remains to observe that \( \hat{\varphi}^{-1}(\varphi(K)) \cap \text{Im } \iota = \iota(K) \).

The universal property of \( S \)-free Stone topological algebras can be somewhat extended as follows to consider residually \( S \) Stone topological algebras.

**Proposition 3.8.** Let \( S \) be a Stone pseudovariety and let \( X \) be a topological space. Then, for every Stone topological algebra \( S \) that is residually \( S \) and every continuous mapping \( \varphi : X \rightarrow S \), there is a unique continuous homomorphism \( \hat{\varphi} : \Omega X S \rightarrow S \) such that \( \hat{\varphi} \circ \iota = \varphi \), where \( \iota : X \rightarrow \Omega X S \) is the natural generating mapping.

**Proof.** Without loss of generality, we may assume that \( \varphi \) is a generating mapping. Since \( S \) is residually \( S \), there is a continuous embedding \( \varepsilon : S \rightarrow \prod_{i \in I} S_i \) into a product of members of \( S \). Consider the following commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\iota} & \Omega X S \\
\varphi \downarrow & & \phi \downarrow \\
S & \xrightarrow{\varepsilon} & \prod_{i \in I} S_i \xrightarrow{\pi_i} S_i \\
\end{array}
\]

where the mapping \( \pi_i \) is the component projection, \( \hat{\varphi}_i \) is the unique continuous homomorphism such that the outer trapezoid commutes, given by the universal property defining \( \Omega X S \); \( \Phi \) is the mapping into the product induced by the \( \hat{\varphi}_i \), that is, such that the right triangle commutes for every \( i \in I \); and the existence of the mapping \( \hat{\varphi} \) follows from the fact that both the images of \( \Phi \) and \( \varepsilon \) in the product \( \prod_{i \in I} S_i \) are \( X \)-generated and, therefore they are equal. \( \square \)

In particular, we deduce that there is an onto continuous homomorphism \( \eta : \Omega X S \rightarrow \Omega X S' \) respecting generators whenever the pair of Stone pseudovarieties satisfies \( S' \subseteq S \). We call it the *natural continuous homomorphism* \( \Omega X S \rightarrow \Omega X S' \).

### 3.2. Free Stone topological algebras are not profinite

We may use the space \( \beta \mathbb{N} \) to show that free Stone topological algebras are almost never profinite.

**Theorem 3.9.** If \( X \) is a nonempty topological space and \( \Omega \neq \Omega_0 \) is an arbitrary topological signature, then the Stone topological algebra \( \Omega X S t_{\Omega} \) is not profinite.

**Proof.** By assumption, we may choose \( n \geq 1 \) such that \( \Omega_n \neq \emptyset \). Let \( u \in \Omega_n \) be fixed and consider the signature \( \Omega' = \Omega'_n = \{u\} \).

We define a structure of topological \( \Omega' \)-algebra on \( \beta \mathbb{N} \) by the evaluation mapping

\[ E_n : \Omega'_n \times (\beta \mathbb{N})^n \rightarrow \beta \mathbb{N} \]

which is given by the projection on the last component followed by addition by 1. Here, by addition by 1 we mean the unique extension to a continuous
mapping \( \beta \mathbb{N} \to \beta \mathbb{N} \) of addition by 1 viewed as a mapping \( \mathbb{N} \to \beta \mathbb{N} \). As \( \beta \mathbb{N} \) is a Stone space (cf. \[30\] Proposition 3.9), it is thus a Stone topological \( \Omega' \)-algebra, which is generated by the element 0. We claim that it is not profinite. Indeed, since \( \Omega' \) is finite, up to isomorphism there are only countably many finite \( \Omega' \)-algebras. As \( \beta \mathbb{N} \) is finitely generated, there are only finitely many continuous homomorphisms into a specific finite algebra. Hence, if \( \beta \mathbb{N} \) would be residually finite, then it would embed into an algebra of cardinality \( 2^{\aleph_0} \), which contradicts the fact that its cardinality is really \( 2^{\aleph_0} \) (cf. \[10\] Corollary 3.6.12)). Hence, the \( \Omega' \)-algebra \( \beta \mathbb{N} \) is not profinite.

We may also view \( \beta \mathbb{N} \) as a Stone topological \( \Omega' \)-algebra by interpreting every operation different from \( \omega \) as the constant operation with value 0. If this enriched structure would be profinite, so would be the original one.

By the universal property of \( \prod X \mathcal{S} \mathcal{T} \Omega \), the constant mapping \( \varphi : X \to \beta \mathbb{N} \) with value 0 determines a unique continuous homomorphism \( \hat{\varphi} : \prod X \mathcal{S} \mathcal{T} \Omega \to \beta \mathbb{N} \) such that \( \varphi \circ \iota = \varphi \). Since \( \beta \mathbb{N} \) is generated by the image of \( \varphi \), the mapping \( \hat{\varphi} \) is onto. Since \( \beta \mathbb{N} \) is not profinite as a topological \( \Omega' \)-algebra, by Theorem 2.2 we conclude that neither is \( \prod X \mathcal{S} \mathcal{T} \Omega \), as desired. \( \square \)

### 3.3. Finite quotients

One of the basic observations in the theory of profinite algebras is that all finite quotients of \( \Omega \mathcal{X} \mathcal{V} \) belong to \( \mathcal{V} \). We next show an analog of this statement in the realm of Stone pseudovarieties. We start with a technical observation which extends a well known result on pseudovarieties of finite algebras \[2\] Lemma 4.1).

**Lemma 3.10.** Let \( \mathcal{S} \) be a Stone pseudovariety and \( S \) be a Stone topological algebra which is residually \( \mathcal{S} \). If \( K \) is a clopen subset of \( S \), then there exist \( T \in \mathcal{S} \) and a continuous homomorphism \( \varphi : S \to T \) such that the equality \( \varphi^{-1}(\varphi(K)) = K \) holds.

**Proof.** Let \( s \in K \). Then, for each \( u \in S \setminus K \) there is a continuous homomorphism \( \alpha_u : S \to M_u \) with \( M_u \in \mathcal{S} \) such that \( \alpha_u(s) \neq \alpha_u(u) \). Since \( \{\alpha_u(s)\} \) is closed in \( M_u \), we obtain an open subset \( O_u = \alpha_u^{-1}(M_u \setminus \{\alpha_u(s)\}) \) of \( S \). We see that \( s \notin O_u \) and \( u \in O_u \). In this way we get an open cover \( \{O_u\}_{u \in S \setminus K} \) of the closed subset \( S \setminus K \) of the compact space \( S \). It follows that there is a finite set \( F \subseteq S \setminus K \) such that \( S \setminus K \subseteq \bigcup_{u \in F} O_u \). We consider the finite product \( T_s = \prod_{u \in F} M_u \in \mathcal{S} \) and the corresponding continuous homomorphism \( \gamma_s : S \to T_s \). We claim that \( \gamma_s(s) \notin \gamma_s(S \setminus K) \). Indeed, for \( x \in S \setminus K \) there is \( u \in F \) such that \( x \in O_u \) and so \( (\gamma_s(x))_u = \alpha_u(x) \in M_u \setminus \{\alpha_u(s)\} \) and \( \alpha_u(x) \neq \alpha_u(s) \) follows. This means that \( \gamma_s(x) \neq \gamma_s(s) \).

Since \( S \) and \( T_s \) are compact, each \( \gamma_s(S \setminus K) \) is closed. Hence there is an open subset \( U_s \subseteq T_s \) such that \( \gamma_s(s) \in U_s \) and \( U_s \cap \gamma_s(S \setminus K) = \emptyset \). We denote \( L_s = \gamma_s^{-1}(U_s) \) for which we have \( \gamma_s(L_s) \cap \gamma_s(S \setminus K) = \emptyset \) and \( s \in L_s \). Since there is such an open set \( L_s \) for every \( s \in K \), we may choose a finite subcover from the open cover \( \{L_s\}_{s \in K} \) of \( K \). Hence, there is a finite subset \( G \subseteq K \) such that \( K \) is covered by \( \{L_s\}_{s \in G} \). Again, we consider the finite product \( T = \prod_{s \in G} T_s \in \mathcal{S} \) and the corresponding continuous homomorphism \( \varphi : S \to T \) whose components are the \( \gamma_s \) with \( s \in G \). It remains to show that \( \varphi^{-1}(\varphi(K)) = K \). The inclusion \( K \subseteq \varphi^{-1}(\varphi(K)) \) being trivial, assume for a moment that \( \varphi^{-1}(\varphi(K)) \not\subseteq K \). Then there are \( y \notin K \) and \( z \in K \) such that \( \varphi(y) = \varphi(z) \). If we take some \( s \in G \) satisfying \( z \in L_s \), then \( (\varphi(y))_s = (\varphi(z))_s \).
Since \( \phi \) following commutative diagram, where \( \alpha \) homomorphism of algebras \( \beta \) logical algebra which is residually \( S \).

Now we are ready to prove a statement concerning finite quotients of a Stone topological algebra which is residually \( S \).

**Proposition 3.11.** Let \( S \) be a Stone pseudovariety and \( S \) be a Stone topological algebra which is residually \( S \). Then every finite quotient of \( S \) belongs to \( S \).

**Proof.** Let \( \alpha : S \to F \) be a continuous homomorphism onto a finite discrete algebra \( F \). For each element \( f \in F \) we may consider the clopen set \( \alpha^{-1}(f) \) and a continuous homomorphism \( \varphi_f : S \to T_f \) into a member of \( S \) such that \( \alpha^{-1}(f) = \varphi_f^{-1}(\varphi_f(\alpha^{-1}(f))) \), whose existence is ensured by Lemma 3.10.

Let \( T \) be the finite product \( \prod_{f \in F} T_f \), which lies in \( S \), and \( \varphi : S \to T \) the corresponding continuous homomorphism, whose components are the \( \varphi_f \) \((f \in F)\). We let \( T' \) be the image of \( \varphi \), which also belongs to \( S \).

We claim that if two elements of \( S \) are identified by \( \varphi \), then they are also identified by \( \alpha \). For this purpose, let \( s, t \in S \) be a pair of elements such that \( \varphi(s) = \varphi(t) \). We take \( g = \alpha(s) \) and recall that \( \varphi_f^{-1}(\varphi_f(\alpha^{-1}(g))) = \alpha^{-1}(g) \). Since \( \varphi_g(t) = \varphi_g(s) \in \varphi_g(\alpha^{-1}(g)) \), we see that \( t \in \alpha^{-1}(g) \). Hence, we have \( \alpha(t) = g = \alpha(s) \), which establishes the claim. It follows that there is a homomorphism of algebras \( \beta : T' \to F \) satisfying \( \beta \circ \varphi = \alpha \), as depicted in the following commutative diagram, where \( \pi_f \) denotes a component projection.

\[
\begin{array}{ccc}
F & \xrightarrow{\alpha} & S \\
\downarrow{\beta} & \Leftrightarrow & \downarrow{\phi} \\
T' & \xrightarrow{\pi_f} & T
\end{array}
\]

Moreover, for every \( f \in F \) we know that \( \beta^{-1}(f) = \varphi(\alpha^{-1}(f)) \) is a closed subset of \( T' \) because \( \alpha^{-1}(f) \) is clopen in \( S \). Hence the preimage of any (closed) subset of \( F \) is closed so that \( \beta \) is a continuous homomorphism. Since we already saw that \( \beta \) is onto and \( T' \in S \), the proof is complete. \( \Box \)

4. Preliminaries on Boolean algebra

Before proceeding in the next section with our approach to duality, we present here the required preliminaries on Boolean algebras. This includes three versions of continuity conditions involving the inverses of evaluation mappings on a topological algebra.

4.1. Boolean algebras of sets. Given Boolean subalgebras \( \mathcal{B}_i \) of \( \mathcal{P}(S_i) \) \((i = 1, \ldots, n)\), we may consider the Boolean subalgebra \( \bigoplus_{i=1}^{n} \mathcal{B}_i \) of \( \mathcal{P}(S_1 \times \cdots \times S_n) \) generated by all boxes of the form

\[ L_1 \times \cdots \times L_n \ (L_i \in \mathcal{B}_i). \]

Note that, if each \( \mathcal{B}_i \) is the Boolean algebra of all clopen subsets of a Stone space \( S_i \), then the members of \( \bigoplus_{i=1}^{n} \mathcal{B}_i \) are the clopen subsets of the product space \( \prod_{i=1}^{n} S_i \). The construction \( \bigoplus_{i=1}^{n} \mathcal{B}_i \) is a concrete realization of what is called in the Boolean algebra literature the sum or the tensor product of the
Boolean algebras $\mathcal{B}_i$. Since the intersection of two boxes is a box and the complement in $S_1 \times \cdots \times S_n$ of a box is a finite union of pairwise disjoint boxes, the members of $\bigoplus_{i=1}^n \mathcal{B}_i$ are finite unions of pairwise disjoint boxes. In fact, for each element $P$ of $\bigoplus_{i=1}^n \mathcal{B}_i$, we may find finite partitions

\begin{equation}
(4.1) \quad S_i = \bigcup_{j \in J_i} (L_{i,j} \in \mathcal{B}_i),
\end{equation}

called a $\mathcal{B}_i$-partition, $(i = 1, \ldots, n)$ such that $P$ is the union of a set of boxes of the form $L_{1,j_1} \times \cdots \times L_{n,j_n}$; we can take the $L_{i,j}$ to be the atoms in the Boolean algebra of subsets $L_i$ generated by the $i$th component projections of the given boxes defining $P$. Such a list $((1.1) : i = 1, \ldots, n)$ of partitions is said to be a $(\mathcal{B}_i)_i$-mesh for $P$. In case all $S_i$ are equal and all $\mathcal{B}_i$ are equal to $\mathcal{B}$, then we refer simply to a $\mathcal{B}$-mesh for $P$.

We proceed to present an alternative way of finding a $\mathcal{B}$-mesh for $P$ that in a sense gives a minimum choice, which is independent of any particular decomposition of $P$ as a union of boxes.

Given a subset $P$ of the Cartesian product $S_1 \times \cdots \times S_n$, we consider the binary relation on $S_i$ defined by $s_i \rho_i s_i'$ if, for all $s_j \in S_j (j \neq i)$, the following equivalence holds:

\begin{equation}
(4.1) \quad (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) \in P \iff (s_1, \ldots, s_{i-1}, s_i', s_{i+1}, \ldots, s_n) \in P.
\end{equation}

Note that $\rho_i$ is an equivalence relation on $S_i$. We denote by $s_i/\rho_i$ the $\rho_i$-class of $s_i$.

**Lemma 4.1.** Let $\mathcal{B}_i$ be a Boolean algebra of subsets of $S_i$ ($i = 1, \ldots, n$) and suppose that $P$ is an element of $\bigoplus_{i=1}^n \mathcal{B}_i$. Consider a list of partitions $((1.1) : i = 1, \ldots, n)$ defining a $(\mathcal{B}_i)_i$-mesh for $P$ and the equivalence relations $\rho_i$ defined above. Then, the following hold:

1. if $s, t \in L_{i,j}$ for some indices $i$ and $j$, then $s \rho_i t$;
2. each equivalence relation $\rho_i$ has finite index;
3. the classes of each $\rho_i$ belong to $\mathcal{B}_i$;
4. we have a finite disjoint decomposition

\begin{equation}
(4.2) \quad P = \bigcup_{(s_1, \ldots, s_n) \in P} (s_1/\rho_1) \times \cdots \times (s_n/\rho_n).
\end{equation}

**Proof.** Suppose that $s_k \in S_k (k \neq i)$ are such that

\begin{equation}
(4.3) \quad (s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n) \in P.
\end{equation}

By the definition of mesh for $P$, there exist $j_k \in J_k (k = 1, \ldots, n)$ such that

\begin{equation}
(4.1) \quad (s_1, \ldots, s_{i-1}, s, s_{i+1}, \ldots, s_n) \in L_{1,j_1} \times \cdots \times L_{n,j_n} \subseteq P.
\end{equation}

Because $[4.1]_i$ is a partition of $S_i$, it follows that $j_i = j$ and so

\begin{equation}
(4.4) \quad (s_1, \ldots, s_{i-1}, t, s_{i+1}, \ldots, s_n) \in P.
\end{equation}

By symmetry, $[4.4]$ also implies $[4.3]$, thereby establishing that $s \rho_i t$.

Properties (2) and (3) are an immediate consequence of (1) as the partition of $S_i$ determined by $\rho_i$ is refined by the finite partition $([4.1]_i)$.

(1) The inclusion from left to right in $[4.2]$ follows from the obvious fact that $(s_1, \ldots, s_n)$ belongs to $(s_1/\rho_1) \times \cdots \times (s_n/\rho_n)$. For the reverse inclusion,
suppose that \((s_1, \ldots, s_n) \in P\). We claim that, if \(s_i t_i \ (i = 1, \ldots, n)\), then \((t_1, \ldots, t_n) \in P\). To prove the claim, we establish by induction on \(i\) that
\[
(t_1, \ldots, t_i, s_{i+1}, \ldots, s_n) \in P.
\]
Indeed, (4.5) holds for \(i = 0\) by hypothesis. Assuming that (4.5) holds for a given \(i < n\) then, since \(s_{i+1} t_{i+1}\), we may replace \(s_{i+1}\) by \(t_{i+1}\) to conclude that (4.5) also holds for \(i + 1\).

The following application of Lemma 4.1 plays a key role in the sequel of this section.

**Corollary 4.2.** Suppose that \((\mathcal{B}_{ij})_{j \in J}\) is a nonempty family of Boolean algebras of subsets of the set \(S_i \ (i = 1, \ldots, n)\). Then, the following equality holds:
\[
\bigcap_{j \in J} \left( \bigoplus_{i=1}^n \mathcal{B}_{ij} \right) = \bigoplus_{i=1}^n \left( \bigcap_{j \in J} \mathcal{B}_{ij} \right).
\]

**Proof.** The inclusion from right to left in (4.6) is immediate. For the reverse inclusion, suppose that \(P\) is an element of the left hand side of (4.6).

Consider the equivalence relations \(\rho_i\) defined above for the set \(S_i\). Let \(\mathcal{M}\) be the mesh for \(P\) whose partition of \(S_i\) consists of the classes of \(\rho_i\). By Lemma 4.1 \(\mathcal{M}\) is a \((\mathcal{B}_{ij})_i\)-mesh for \(P\) for every \(j\) and, therefore, it is a \((\bigcap_{j \in J} \mathcal{B}_{ij})_i\)-mesh for \(P\), which entails that \(P\) belongs to the right hand side of (4.6).

We conclude this subsection with another technical result of a combinatorial nature that is used later.

**Lemma 4.3.** Let \(P = \bigcup_{r \in R} B_r\) be a union of boxes \(B_r = L_{1,r} \times \cdots \times L_{n,r}\) such that, for every \(i \in \{1, \ldots, n\}\) and \(r, s \in R\), the sets \(L_{i,r}\) and \(L_{i,s}\) are either disjoint or equal. For each \(i \in \{1, \ldots, n\}\), let \(r_i \in R\) and let \(J_i\) be a nonempty subset of \(L_{i,r_i}\). If \(J_1 \times \cdots \times J_n\) is contained in \(P\), then it is contained in a box \(B_r\) for some \(r \in R\).

**Proof.** Note that, for \(r, s \in R\), the boxes \(B_r\) and \(B_s\) are either equal or disjoint. We proceed by induction on \(n\). The case \(n = 1\) being trivial, assume the result holds with \(n - 1\) in the place of \(n\). For each \(r \in R\), consider the box \(C_r = L_{1,r} \times \cdots \times L_{n-1,r}\). Then, projecting on the first \(n - 1\) components, we conclude that there is at least one index \(r \in R\) such that \(J_1 \times \cdots \times J_{n-1}\) is contained in the box \(C_r\). There may be more than one index \(r \in R\) with that property but the box \(C = C_r\) containing \(J_1 \times \cdots \times J_{n-1}\) is unique and in fact if a box \(C_r\) intersects \(J_1 \times \cdots \times J_{n-1}\) nontrivially, then it must be equal to \(C\). It follows that there must exist \(r \in R\) such that \(L_{n,r} = L_{n,r}\) and \(C_r = C\), so that \(J_1 \times \cdots \times J_n \subseteq B_r\), which achieves the induction step.

4.2. **An algebraized continuity condition.** Let \(S\) be an \(\Omega\)-algebra. Given a Boolean subalgebra \(\mathcal{B}\) of \(\mathcal{P}(S)\) and \(n \geq 0\), let \(\mathcal{B}_n = \mathcal{P}_{\text{ca}}(\Omega_n) \oplus \mathcal{B}^{(n)}\) where \(\mathcal{B}^{(n)} = \bigoplus_{r=1}^n \mathcal{B}\). Consider the following property of \(\mathcal{B}\), where \(E_n = E_n^S\) is the evaluation mapping of \(S\):

\[
\text{(C1)} \quad L \in \mathcal{B} \implies \forall n \geq 0, \ E_n^{-1}(L) \in \mathcal{B}_n.
\]
Lemma 4.4. Let \( \mathcal{B} \) be a Boolean subalgebra of \( \mathcal{P}(S) \). Then, there is a maximum Boolean subalgebra of \( \mathcal{B} \) satisfying \( (C) \).

Proof. Suppose \( \mathcal{B} \) is a chain of Boolean subalgebras of \( \mathcal{B} \) each of which satisfies \( (C) \) and let \( \mathcal{B} = \bigcup_{\mathcal{B} \in \mathcal{B}} \mathcal{D} \). It is routine to check that \( \mathcal{B} \) is a Boolean subalgebra of \( \mathcal{B} \). If \( L \in \mathcal{B} \) then there exists \( \mathcal{D} \in \mathcal{B} \) such that \( L \in \mathcal{D} \), whence \( E_n^{-1}(L) \in \mathcal{D}'_n \) because \( \mathcal{D} \) satisfies \( (C) \). Since \( \mathcal{D}'_n \subseteq \mathcal{B}'_n \), it follows that \( E_n^{-1}(L) \in \mathcal{B}'_n \). Hence, \( \mathcal{B} \) also satisfies \( (C) \). By Zorn’s Lemma, we conclude that there are maximal Boolean subalgebras of \( \mathcal{B} \) satisfying \( (C) \).

Next, suppose that \( \mathcal{C} \) and \( \mathcal{D} \) are Boolean subalgebras of \( \mathcal{A} \) satisfying \( (C) \) and let \( \mathcal{B} \) be the least Boolean subalgebra of \( \mathcal{A} \) containing \( \mathcal{C} \cup \mathcal{D} \). Each \( L \in \mathcal{B} \) is a finite union of sets of the form \( M \cap N \) such that \( M \in \mathcal{C} \) and \( N \in \mathcal{D} \). For such an intersection \( M \cap N \), we have

\[
E_n^{-1}(M \cap N) = E_n^{-1}(M) \cap E_n^{-1}(N) \in \mathcal{C}'_n \cup \mathcal{D}'_n \subseteq \mathcal{B}'_n,
\]

since \( \mathcal{C} \) and \( \mathcal{D} \) satisfy \( (C) \), where \( \mathcal{C}'_n \cup \mathcal{D}'_n \) denotes the least Boolean subalgebra of \( \mathcal{P}_c(\mathcal{O}_n) \oplus \mathcal{P}(S) \) containing \( \mathcal{C}'_n \cup \mathcal{D}'_n \). For a finite union of sets satisfying \( (C) \), we use a similar equality for union, and we conclude that \( \mathcal{B} \) satisfies \( (C) \). Hence, there is only one maximal Boolean subalgebra satisfying \( (C) \), which establishes the lemma.

Denote by \( \mathcal{B}_{\text{max}} \) the maximum Boolean subalgebra of \( \mathcal{P}_c(S) \) satisfying \( (C) \).

The following is an immediate application of Corollary 4.2.

Corollary 4.5. If the nonempty family \( (\mathcal{B}_i)_{i \in I} \) of Boolean subalgebras of \( \mathcal{P}(S) \) satisfies \( (C) \) then so does \( \bigcap_{i \in I} \mathcal{B}_i \).

Proof. Let \( L \) be an arbitrary element of \( \bigcap_{i \in I} \mathcal{B}_i \). By the assumption that each \( \mathcal{B}_i \) satisfies \( (C) \), we know that \( E_n^{-1}(L) \) belongs to \( (\mathcal{B}_i)'_n \). By Corollary 4.2 we have \( \bigcap_{i \in I} (\mathcal{B}_i)'_n = (\bigcap_{i \in I} \mathcal{B}_i)'_n \). Thus, \( E_n^{-1}(L) \) belongs to \( (\bigcap_{i \in I} \mathcal{B}_i)'_n \). Hence, the Boolean algebra \( \bigcap_{i \in I} \mathcal{B}_i \) satisfies \( (C) \).

The following result gives a special case in which it is easy to identify the Boolean algebra \( \mathcal{B}_{\text{max}} \).

Proposition 4.6. Let \( S \) be a Stone topological \( \Omega \)-algebra where \( \Omega \) is a Stone signature. Then the equality \( \mathcal{B}_{\text{max}} = \mathcal{P}_c(S) \) holds.

Proof. By continuity of \( E_n \), given \( K \in \mathcal{P}_c(S) \), the subset \( E_n^{-1}(K) \) of the Stone space \( \Omega_n \times S^n \) is clopen and, therefore, it is a union of boxes of the form \( K_0 \times K_1 \times \cdots \times K_n \) with clopen sides \( K_i \). Since \( E_n^{-1}(K) \) is compact, it is a finite union of such boxes, which shows that \( \mathcal{P}_c(S) \) satisfies condition \( (C) \).

The following example shows that the assumption that \( \Omega \) is a Stone signature cannot be dropped in Proposition 4.6.
Example 4.7. Let \( S = \mathbb{N} \cup \{\infty\} \) be the one-point compactification of the set of all natural numbers, which is viewed as a discrete space. Consider the unary signature \( \Omega = \Omega_1 = \{a_i : i \in \mathbb{N}\} \), also viewed as a discrete space. We define a structure of \( \Omega \)-algebra on \( S \) through the evaluation mapping \( E_1 : \Omega \times S \rightarrow S \) given by
\[
E_1(a_i, n) = \max\{0, n - i\} \quad (n \in \mathbb{N})
\]
\[
E_1(a_i, \infty) = \infty.
\]
For a finite subset \( K \) of \( \mathbb{N} \) (and these together with their complements in \( S \) are the clopen subsets of \( S \)), note that
\[
E_1^{-1}(K) = \bigcup_{i \in \mathbb{N}} \{a_i\} \times (K + i)
\]
where \( K + i = \{m + i : m \in K\} \). Since \( K + i \neq K + j \) for \( i \neq j \), \( E_1^{-1}(K) \) cannot be expressed as a finite union of boxes. Hence, the Boolean algebra \( \mathcal{P}_{co}(S) \) does not satisfy condition (C1).

The following result shows how condition (C1) is affected by continuous homomorphisms between topological algebras.

Proposition 4.8. Let \( \varphi : S \rightarrow T \) be a homomorphism of \( \Omega \)-algebras and let \( \mathcal{B} \) be a Boolean algebra of subsets of \( T \). If \( \mathcal{B} \) satisfies (C1) then so does \( \varphi^{-1}(\mathcal{B}) \) and the converse also holds if \( \varphi \) is onto.

Proof. We have the following commutative diagram for each \( n \geq 0 \):
\[
\begin{array}{ccc}
\Omega_n \times S^n & \xrightarrow{E_n^S} & S \\
\downarrow{id \times \varphi^n} & & \downarrow{\varphi} \\
\Omega_n \times T^n & \xrightarrow{E_n^T} & T
\end{array}
\]

Suppose first that \( \mathcal{B} \) satisfies (C1) and let \( L \in \mathcal{B} \). By assumption, there is an expression of the form
\[
(E_n^T)^{-1}(L) = \bigcup_{i=1}^r K_i \times L_{i,1} \times \cdots \times L_{i,n}
\]
where the \( K_i \) are clopen subsets of \( \Omega_n \) and the \( L_{i,j} \) belong to \( \mathcal{B} \). By the commutativity of Diagram (4.7), we get
\[
(E_n^S)^{-1}(\varphi^{-1}(L)) = (id \times \varphi^n)^{-1}((E_n^T)^{-1}(L)) = \bigcup_{i=1}^r K_i \times \varphi^{-1}(L_{i,1}) \times \cdots \times \varphi^{-1}(L_{i,n}),
\]

which shows that \( (E_n^S)^{-1}(\varphi^{-1}(L)) \) belongs to \( (\varphi^{-1}(\mathcal{B}))_n' \). Hence, \( \varphi^{-1}(\mathcal{B}) \) satisfies (C1).

Conversely, suppose that \( \varphi \) is onto and \( \varphi^{-1}(\mathcal{B}) \) satisfies (C1), and let \( L \in \mathcal{B} \). By assumption, there is an expression
\[
(E_n^S)^{-1}(\varphi^{-1}(L)) = \bigcup_{i=1}^r K_i \times \varphi^{-1}(L_{i,1}) \times \cdots \times \varphi^{-1}(L_{i,n}),
\]
where the $K_i$ are clopen subsets of $\Omega_n$ and the $L_{i,j}$ belong to $B$. Applying $\text{id} \times \varphi^n$ to both sides of (4.8), we get
\[
(E_T^n)^{-1}(L) = (\text{id} \times \varphi^n)(E_T^n)^{-1}(L) \\
= (\text{id} \times \varphi^n)((E_S^n)^{-1}(\varphi^{-1}(L))) \\
= \bigcup_{i=1}^r K_i \times L_{i,1} \times \cdots \times L_{i,n}
\]
where the first equality follows from the hypothesis that $\varphi$ is onto and the second one follows from the commutativity of Diagram (4.7). Hence, $B$ satisfies (C1).

4.3. A relaxed continuity condition. While it follows from the results in the previous subsection together with those in the next couple of subsections that condition (C1) adequately characterizes the Boolean algebras whose dual Stone spaces have a natural structure of Stone topological algebra in the case of a Stone signature, Example 4.7 shows that (C1) is no longer adequate for a general topological signature. We proceed to introduce a relaxed version of (C1) that provides the desired characterization.

Let $B$ be a Boolean subalgebra of $\mathcal{P}(S)$ and $\Omega$ a topological signature, where $S$ is an $\Omega$-algebra. Define $B''_n$ to be the set of all unions of the form
\[
\bigcup_{r \in R} K_r \times L_{1,r} \times \cdots \times L_{n,r}
\]
such that the following conditions are satisfied:
(C2.1) the sets $K_r$ form a clopen partition of $\Omega_n$;
(C2.2) each $L_{k,r}$ belongs to $B$;
(C2.3) for each $r_0 \in R$, the set \{ $r \in R : K_r = K_{r_0}$ \} is finite.

We emphasize that in (C2.1), the function $r \mapsto K_r$ may not be injective and (C2.3) expresses precisely the property that each preimage is finite. Note that $B'_n \subseteq B''_n$ while, by (C2.1), equality holds in case $\Omega$ is a Stone signature. We prove below that in general $B''_n$ is also a Boolean subalgebra of $\mathcal{P}(\Omega_n \times S^n)$.

There are two alternative and conflicting ways in which the decomposition (4.9) may be rewritten which we present in the next two lemmas.

**Lemma 4.9.** Every element of $B''_n$ admits a decomposition of the form (4.9) satisfying conditions (C2.1)–(C2.4), where
(C2.4) for each $r_0 \in R$ and each $i \in \{1, \ldots, n\}$, the sets of the form $L_{i,r}$ with $r$ such that $K_r = K_{r_0}$ are either disjoint or equal.

**Proof.** It suffices to note that, in view of (C2.3), there are only finitely many boxes of the form considered in (C2.4) and invoke Lemma 4.1.

**Lemma 4.10.** The elements of $B''_n$ are the subsets of $\Omega_n \times S^n$ that admit decompositions of the form
\[
\bigcup_{r \in R} K_r \times P_r
\]
where $r \mapsto K_r$ is an injective function satisfying (C2.7) and each $P_r$ belongs to $B^{(n)}$. □
Proof. Consider a decomposition (4.9) of an element $U$ of $B''_n$ satisfying (C2.1)–(C2.3). Let $\sim$ be the equivalence relation on $R$ given by $r_1 \sim r_2$ if $K_{r_1} = K_{r_2}$ and denote by $[r]$ the $\sim$-class containing $r$. Then the decomposition

$$U = \bigcup_{[r] \in R/\sim} K_r \times \left( \bigcup_{s \in [r]} L_{1,s} \times \cdots \times L_{n,s} \right)$$

satisfies the required conditions as the inner union is finite by (C2.3) and each of its terms belongs to $B(n)$ by (C2.2). For the converse, it suffices to express each $P_r$ in $B(n)$ as a finite union of boxes and distribute the Cartesian product of $K_r$ over that union. □

When $A$ is a subset of a set $X$ and it is clear from the context that it is considered as such, then we may write $A^C$ instead of $X \setminus A$.

**Lemma 4.11.** If $B$ is a Boolean subalgebra of $\mathcal{P}(S)$, then the set $B''_n$ is a Boolean subalgebra of $\mathcal{P}(\Omega_n \times S^n)$.

**Proof.** We need to show that $B''_n$ is closed under finite union and complement. Consider two sets $A_1$ and $A_2$ in $B''_n$, say with decompositions

$$A_i = \bigcup_{r \in R_i} K_{i,r} \times P_{i,r}$$

satisfying the conditions from Lemma 4.10. For the union $A_1 \cup A_2$ we may find a similar decomposition by first taking the partition $\{K_r : r \in R\}$ of $\Omega_n$ whose classes $K_r$ are the nonempty intersections of the form $K_{1,r_1} \cap K_{2,r_2}$ with $r_1 \in R_1$ and $r_2 \in R_2$ and then taking the decomposition

$$\bigcup_{r \in R} (K_r \times P_{1,r_1} \cup K_r \times P_{2,r_2}).$$

This shows that $A_1 \cup A_2$ belongs to $B''_n$.

For the complement $A_1^C$, we get the formula

$$A_1^C = \bigcup_{r \in R_1} K_{1,r} \times P_{1,r}^C$$

which ensures that $A_1^C \in B''_n$ by Lemma 4.10. □

Let $S$ be an $\Omega$-algebra. Consider the following condition on a Boolean subalgebra $B$ of $\mathcal{P}(S)$:

(C2) \quad $L \in B \implies \forall n \geq 0, \ (E_n^S)^{-1}(L) \in B''_n$.

Again, condition (C2) means that each mapping $(E_n^S)^{-1}$ defines a Boolean algebra homomorphism $B \to B''_n$.

Without any significant change, several results regarding condition (C1) can also be proved for condition (C2). We leave it to the reader to verify that the omitted proofs can be straightforwardly adapted.

**Lemma 4.12.** Let $\mathcal{A}$ be a Boolean subalgebra of $\mathcal{P}(S)$. Then, there is a maximum Boolean subalgebra of $\mathcal{A}$ satisfying (C2).
In particular, there is also a maximum Boolean subalgebra of \( \mathcal{P}_{co}(S) \) satisfying (C2), which we denote \( S^{\mathcal{B}}_{max2} \).

The next result is the analog of Proposition 4.6 but requires a much longer topological signature.

**Proposition 4.13.** Let \( S \) be a Stone topological \( \Omega \)-algebra where \( \Omega \) is an arbitrary topological signature. Then the equality \( S^{\mathcal{B}}_{max2} = \mathcal{P}_{co}(S) \) holds.

**Proof.** As, by definition, \( S^{\mathcal{B}}_{max2} \) is contained in \( \mathcal{P}_{co}(S) \), it remains to show that the Boolean algebra \( \mathcal{P}_{co}(S) \) satisfies (C2). Let \( K \) be a clopen subset of \( S \) and consider the binary relation \( \rho \) on \( \Omega_n \) defined by

\[
(o, o') \in \rho \text{ if } \forall s_1, \ldots, s_n \in S \left( o_S(s_1, \ldots, s_n) \in K \iff o'_S(s_1, \ldots, s_n) \in K \right) .
\]

Note that \( \rho \) is an equivalence relation on \( \Omega_n \). We claim that its classes are open, whence they are clopen. To establish the claim, suppose that \((p_i)_{i \in I} \) is a net in the complement in \( \Omega_n \) of the class of \( o \) converging to \( p \); we need to show that \((o, p) \notin \rho \). Then, for each \( i \in I \), there exist \( s_{1,i}, \ldots, s_{n,i} \in S \) such that exactly one of \( o_S(s_{1,i}, \ldots, s_{n,i}) \) and \((p_i)_S(s_{1,i}, \ldots, s_{n,i}) \) belongs to \( K \).

Since we may replace \( K \) by \( K^c \) without changing \( \rho \), by taking a suitable subnet we may assume that it is \((p_i)_S(s_{1,i}, \ldots, s_{n,i}) \) that belongs to \( K \). Because \( S \) is compact, we may further assume that the net \((s_{1,i}, \ldots, s_{n,i})_{i \in I} \) converges to \((s_1, \ldots, s_n) \) in \( S^n \). As the evaluation mapping \( E_n \) is assumed to be continuous and \( K \) is clopen, it follows that \( p_S(s_1, \ldots, s_n) \in K \) and \( o_S(s_1, \ldots, s_n) \notin K \). Hence, the \( \rho \)-class of \( o \) is open.

Because \( E_{n}^{\Omega} \) is continuous, the set \((E_{n}^{\Omega})^{-1}(K) \) is open and, therefore it is a union

\[
(4.10) \quad \bigcup_{r \in R} K_r \times L_{1,r} \times \cdots \times L_{n,r}
\]

of boxes with the \( K_r \) open subsets of \( \Omega_n \) and the \( L_{k,s} \) clopen in \( S \). If \( o \in K_r \) and \((o, p) \notin \rho \), then \( \{p\} \times L_{1,r} \times \cdots \times L_{n,r} \subseteq (E_{n}^{\Omega})^{-1}(K) \) by the definition of \( \rho \). Hence, we may assume that each \( K_r \) is a union of \( \rho \)-classes. Moreover, we may break up each \( K_r \) into the \( \rho \)-classes contained in it, at the cost of splitting terms in the decomposition (4.10). The resulting decomposition clearly satisfies conditions (C2)–(C2), it only remains to modify it by showing that some terms may be dropped so as to satisfy (C2). For that purpose, we may first note that a similar decomposition

\[
\bigcup_{s \in R'} K'_s \times L'_{1,s} \times \cdots \times L'_{n,s}
\]

exists for \((E_{n}^{\Omega})^{-1}(K^c) \), where again each \( K'_s \) is a \( \rho \)-class and the sets \( L'_{k,s} \) are clopen in \( S \). For a fixed \( o \in \Omega_n \), the boxes \( L_{1,r} \times \cdots \times L_{n,r} \) such that \( o \in K_r \) together with the boxes \( L'_{1,s} \times \cdots \times L'_{n,s} \) with \( o \in K'_s \) constitute a clopen cover of the compact space \( S^n \) in which the former are disjoint from the latter. Hence, the union

\[
\bigcup_{o \in K_r} K_r \times L_{1,r} \times \cdots \times L_{n,r}
\]

coincides with a finite union involving only finitely many of the same terms. Applying this kind of reduction of (4.10) for a complete set of representatives
of the \( \rho \)-classes, we get the desired decomposition of \( (E_n^S)^{-1}(K) \) satisfying conditions (C2.1)–(C2.3), thereby completing the proof of the proposition. \( \square \)

The following result is again an application of Corollary 4.2.

**Corollary 4.14.** If the nonempty family \( (\mathcal{B}_i)_{i \in I} \) of Boolean subalgebras of \( \mathcal{P}(S) \) satisfies (C2) then so does \( \bigcap_{i \in I} \mathcal{B}_i \).

**Proof.** Let \( L \) be an arbitrary element of \( \mathcal{B} = \bigcap_{i \in I} \mathcal{B}_i \). By the assumption that each \( \mathcal{B}_i \) satisfies (C2), we know that \( (E_n^S)^{-1}(L) \) belongs to \( (\mathcal{B}_i)^{\prime \prime} \). For each \( i \in I \), there is a decomposition

\[
(E_n^S)^{-1}(L) = \bigcup_{r \in R_i} K_{i,r} \times P_{i,r}
\]

given by Lemma 4.10. Given \( o \in \Omega_n \), we conclude that there is a unique \( r_i \in R_i \) such that \( (E_n^S)^{-1}(L) \cap \{o\} \times S^n = \{o\} \times P_{i,r_i} \). By Corollary 4.2 it follows that there is \( P_o \in \mathcal{B}^{(n)} \) such that \( (E_n^S)^{-1}(L) \cap \{o\} \times S^n = \{o\} \times P_o \), that is, all the sets \( P_{i,r_i} \) are equal and belong to \( \mathcal{B}^{(n)} \). Thus, the decomposition \( (4.11) \) for every \( i \in I \) already shows that \( (E_n^S)^{-1}(L) \) belongs to \( \mathcal{B}^{\prime \prime} \). Hence, the Boolean algebra \( \mathcal{B} \) satisfies (C2). \( \square \)

We conclude this subsection with the analog of Proposition 4.8 whose proof can be easily adapted to handle condition (C2).

**Proposition 4.15.** Let \( \varphi : S \to T \) be a homomorphism of \( \Omega \)-algebras and let \( \mathcal{B} \) be a Boolean algebra of subsets of \( T \). If \( \mathcal{B} \) satisfies (C2) then so does \( \varphi^{-1}(\mathcal{B}) \) and the converse also holds if \( \varphi \) is onto.

### 4.4. A simplified version of the continuity condition

We show that condition (C2) may be simplified by omitting the partition assumption from condition (C2.1).

Let \( \mathcal{B} \) be a Boolean algebra and \( \Omega \) a topological signature. Define \( \mathcal{B}^{\prime \prime \prime}_n \) to be the set of all unions of the form

\[
\bigcup_{r \in R} K_r \times L_{1,r} \times \cdots \times L_{n,r}
\]

such that the following conditions are satisfied:

(C3.1) each set \( K_r \) is clopen;

(C3.2) each set \( L_{k,r} \) belongs to \( \mathcal{B} \);

(C3.3) for each \( o \in \Omega_n \), the set \( \{r \in R : o \in K_r\} \) is finite.

Although this definition is simpler than the definition of \( \mathcal{B}^{\prime \prime}_n \), it has some disadvantages. In particular, it is not clear how to prove properties like Corollary 4.14. The basic technical problem is that \( \mathcal{B}^{\prime \prime \prime}_n \) need not be a Boolean algebra in general. To see this, one may consider the Boolean algebra \( \mathcal{B} = \mathcal{P}(\mathbb{N}) \), \( n = 1 \), and \( \Omega_1 = \mathbb{N}^\ast \) the one-point compactification of \( \mathbb{N} \), where the limit of any unbounded sequence is the added point \( \infty \). Then the relation “less than” \( \mathcal{R}_\prec \) belongs to \( \mathcal{B}^{\prime \prime \prime}_n \) as \( \mathcal{R}_\prec = \bigcup_{n \in \mathbb{N}} \{n\} \times \{x \in \mathbb{N} : n < x\} \). However, one can check that \( \mathcal{R}_\prec^\mathcal{B} \) does not belong to \( \mathcal{B}^{\prime \prime \prime}_n \). Indeed, assuming \( \mathcal{R}_\prec^\mathcal{B} = \bigcup_{i \in I} K_i \times L_i \) satisfies (C3.1) and (C3.3), one can see that there is
some $i$ for which $L_i$ is infinite and $\infty \in K_i$, so that $K_i$ contains some element $n \in \mathbb{N}$ which is less than some member of $L_i$, which contradicts the assumption that $K_i \times L_i$ is contained in $R'_n$.

Although the set $\mathcal{B}_n^m$ does not coincide with the Boolean algebra $\mathcal{B}_n$, they are strongly related. In particular, the set $\mathcal{B}_n^m$ clearly contains $\mathcal{B}_n$ and we show that $\mathcal{B}_n^m$ is the maximum Boolean subalgebra of $\mathcal{B}_n$ in the following lemma.

**Lemma 4.16.** Let $S$ be an arbitrary set, $\mathcal{B}$ a Boolean subalgebra of $\mathcal{P}(S)$, $n$ an integer, and $\Omega$ a topological signature. If a set $K$ in $\mathcal{B}_n^m$ is such that $K^\mathcal{C}$ also belongs to $\mathcal{B}_n^m$, then $K$ belongs to $\mathcal{B}_n$.

**Proof.** Let $K \in \mathcal{B}_n^m$ be given by formula (1.12) and suppose that $K^\mathcal{C} \in \mathcal{B}_n^m$. We apply the idea from the proof of Proposition 4.13, namely, for this $L$ an integer, and $n$ any $\mathcal{B}_n^m$-tuple $\alpha$ of the form $\{\alpha_1, \ldots, \alpha_n\}$, indexed by a $\mathcal{B}_n$-integer, and a sequence of $S^\alpha$-sets $\{\alpha_1, \ldots, \alpha_n\}$ which contain a finite partition $\mathcal{B}_n^m$ of $\mathcal{B}_n$.

We see that $\rho$ is an equivalence relation and that we get the same relation, if we replace the set $K$ by $K^\mathcal{C}$ in the previous definition. We claim that $\rho$ is a clopen equivalence on the set $\Omega_n$. The proof of this claim occupies the following few paragraphs.

For an arbitrary $a \in \Omega_n$, we consider $[o]_{\rho} = \{p \in \Omega_n : (a, p) \in \rho\}$, the equivalence class containing $a$. To show that $[o]_{\rho}$ is open, let $(p_i)_{i \in I}$ be a net in $[o]_{\rho}$ converging to $p \in \Omega_n$. We improve the expression (1.12) for the considered set $K$; more precisely, we rewrite the part using indices from the subset $R' = \{r \in R : o \in K_r \ \vee \ p \in K_r\}$. Since the set $R'$ is finite, the subset $\{L_{i,r} : i \in \{1, \ldots, n\}, \ r \in R'\}$ of the Boolean algebra $\mathcal{B}$ is finite as well. Thus there exists a finite partition $S = \bigcup_{j \in J} M_j$, indexed by a finite set of integers $J = \{1, \ldots, k\}$ such that each $M_j$ belongs to $\mathcal{B}$ and, for every $r \in R'$ and $i = 1, \ldots, n$, there exist an integer $m(i, r)$ and a sequence of indices $j_{1(i,r)}, \ldots, j_{m(i,r)}$ from $J$ such that $L_{i,r} = M_{j_{1(i,r)}} \cup \cdots \cup M_{j_{m(i,r)}}$. We replace each such $L_{i,r}$ in the considered expression (1.12) for $K$ by this finite union $M_{j_{1(i,r)}} \cup \cdots \cup M_{j_{m(i,r)}}$ and distribute the product $K_r \times L_{1,r} \times \cdots \times L_{n,r}$ over each such union. In this way, the finite part $\bigcup_{r \in R'} K_r \times L_{1,r} \times \cdots \times L_{n,r}$ of the expression (1.12) is replaced by $\bigcup_{r \in R'} K_r \times P_r$, where $R'$ is a new finite index set, $K_r$ are the clopen sets used in the original expression but indexed by the new index set $R''$, and every $P_r$ is a basic box of the form $P_{1,r} \times \cdots \times P_{n,r}$ where all $P_{i,r} \in \mathcal{B}$ belong to the considered partition $\{M_j\}_{j \in J}$. In other words, if we define, for each $n$-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in J^n$, the basic box $P_\alpha$ by the formula $P_\alpha = M_{\alpha_1} \times \cdots \times M_{\alpha_n}$, then $P_r = P_\alpha$ for some $\alpha \in J^n$.

The last step in our improvement is that for a fixed basic box $P_\alpha$, we put together all $K_r$’s such that $P_r = P_\alpha$. Formally, we put $K_\alpha = \bigcup_{r \in R'' : P_r = P_\alpha} K_r$ whenever the index set $\{r \in R'' : P_r = P_\alpha\}$ is non-empty, and we put $K_\alpha = \emptyset$ otherwise. Then we replace the existing finite subparts of the considered expression of the form

$$\bigcup_{r \in R'' : P_r = P_\alpha} K_r \times P_r \text{ by } K_\alpha \times P_\alpha.$$
Additionally, we add the empty sets $K_{\alpha} \times P_\alpha$ for those $\alpha$ for which the index set $\{ r \in \mathbb{R}^n : P_r = P_\alpha \}$ is empty. Altogether, we obtain the expression

$$K = \bigcup_{r \in \mathbb{R}^n \setminus R'} K_r \times L_{1,r} \times \cdots \times L_{n,r} \cup \bigcup_{\alpha \in J^n} K_\alpha \times P_\alpha$$

with the following properties:

(i) $o, p \notin K_r$ for all $r \in \mathbb{R} \setminus R'$,
(ii) $P_\alpha \cap P_{\alpha'} \neq \emptyset$ if and only if $\alpha = \alpha'$, and
(iii) every $K_r$ (for $r \in \mathbb{R} \setminus R'$) and $K_\alpha$ (for $\alpha \in J^n$) is clopen in $\Omega_n$.

Notice that the same construction can be made for $K^\Omega$ independently.

Now, recall that we assume $(p_i, o) \notin \rho$ for every $i \in I$. For each $i \in I$, there is an $n$-tuple $(s_{1,i}, \ldots, s_{n,i}) \in S^n$ such that

$$| \{(o, s_{1,i}, \ldots, s_{n,i}), (p_i, s_{1,i}, \ldots, s_{n,i}) \} \cap K | = 1.$$ 

Whence, it is possible to choose a converging subnet of $(p_{i, \lambda})_{\lambda \in \Lambda}$ of the net $(p_i)_{i \in I}$ such that, choosing $K'$ equal to $K$ or $K^\Omega$, we have:

$$\forall \lambda \in \Lambda : \ ( (o, s_{1,i}, \ldots, s_{n,i}), (p_{i, \lambda}, s_{1,i}, \ldots, s_{n,i}) \notin K').$$

Without loss of generality, we may assume that $K' = K$. By the above property (ii), the fact $(o, s_{1,i}, \ldots, s_{n,i}) \in K$ means that $(o, s_{1,i}, \ldots, s_{n,i}) \in K_\alpha \times P_\alpha$ for some $\alpha \in J^n$. Since $J^n$ is finite, by taking an appropriate subnet of the net $(p_{i, \lambda})_{\lambda \in \Lambda}$ we may further assume that there is $\alpha \in J^n$ such that $(o, s_{1,i}, \ldots, s_{n,i}) \in K_\alpha \times P_\alpha$ for all $\lambda \in \Lambda$. We deduce that $o \in K_\alpha$ and $p_{i, \lambda} \notin K_\alpha$. Since the subset $K_\alpha$ is clopen by (iii), and since $p$ is a limit of the net $(p_i)_{i \in I}$ and consequently also the limit of the considered subnet $(p_{i, \lambda})_{\lambda \in \Lambda}$, we get that $p \notin K_\alpha$. We choose some element $(s_1, \ldots, s_n) \in P_\alpha$, which gives $(o, s_1, \ldots, s_n) \notin K$. Recall that $(s_1, \ldots, s_n) \notin P_{\alpha'}$ for $\alpha' \neq \alpha$ by (ii). Finally, as $p$ does not belong to $K_r$ for $r \in \mathbb{R} \setminus R'$, we deduce that $(p, s_1, \ldots, s_n) \notin K$. This leads to the conclusion $(o, p) \notin \rho$. This completes the proof of the claim that each $\rho$-class is open; therefore, it is also clopen.

Now we choose a set of representatives of all $\rho$-classes, that is, a set $O \subseteq \Omega_n$ such that for every $o \in \Omega_n$ there is a unique $o' \in O$ for which $(o', o) \in \rho$. For each $o \in O$, we consider the set $P_o \subseteq S^n$ determined by the property $K \cap \{ o \} \times S^n = \{ o \} \times P_o$. By the condition (C3), we know that $P_o$ is a finite union of boxes with sides in $\mathcal{B}$, including the case when $P_o$ is empty. Finally, we obtain the equality $K = \bigcup_{o \in O} [o]_\rho \times P_o$, which shows that $K \in \mathcal{B}''$.

We are prepared to give an alternative condition for (C2). Let $S$ be an $\Omega$-algebra. Consider the following condition on a Boolean subalgebra $\mathcal{B}$ of $\mathcal{P}(S)$:

\[ (C3) \quad L \in \mathcal{B} \implies \forall n \geq 0, \ (E_n^S)^{-1}(L) \in \mathcal{B}'' \]

The new condition is equivalent to condition (C2), as we show in the following statement.

**Proposition 4.17.** Let $S$ be an $\Omega$-algebra and $\mathcal{B}$ be a Boolean subalgebra of $\mathcal{P}(S)$. Then $\mathcal{B}$ satisfies (C2) if and only if it satisfies (C3).
Proof. If we consider the sets \( B''_n \) and \( B'''_n \) for a given Boolean subalgebra \( B \) and an integer \( n \), then we have \( B''_n \subseteq B'''_n \) which yields that property (C2) implies property (\( \text{C}_3 \)).

For the reverse implication, let \( L \in B \) be arbitrary. If we denote \( K = (E^S_n)^{-1}(L) \in B''_n \), then the set \( K^C = (E^S_n)^{-1}(L^C) \) also belongs to \( B'''_n \) because \( L^C \in B \). Hence we get \( K \in B''_n \) by Lemma 4.16. \( \square \)

5. Duality

For a Boolean algebra \( B \), denote \( B^* \) the Stone dual space of \( B \). Recall that \( B^* \) may be viewed as the set of all ultrafilters of \( B \); a basis of the topology is given by the sets \( U_L \) \((L \in B)\), where \( U_L \) consists of all ultrafilters containing \( L \).

For a set \( S \), let \( \mathcal{P}(S) \) be the Boolean algebra of all subsets of \( S \). If \( B \) is a Boolean subalgebra of \( \mathcal{P}(S) \), then we let \( \iota : S \to B^* \) be defined by

\[
\iota(s) = s^\uparrow = \{ L \in B : s \in L \}.
\]

In case \( S \) is a topological space, we let \( \mathcal{P}_{co}(S) \) denote the Boolean algebra of all clopen subsets of \( S \).

Proposition 5.1. Let \( B \) be a boolean algebra of subsets of a set \( S \). Then the following properties hold for an arbitrary element \( L \) of \( B \):

1. \( \overline{\iota(L)} = U_L \);
2. \( \iota^{-1}(\overline{\iota(L)}) = L \).

Moreover, the mapping \( \varphi : L \mapsto \overline{\iota(L)} \) is an isomorphism of \( B \) with \( \mathcal{P}_{co}(B^*) \).

Proof. (1) The closure \( \overline{\iota(L)} \) consists of all ultrafilters \( u \in B^* \) such that, for all \( K \in B \), \( u \in U_K \) implies \( U_K \cap \iota(L) \neq \emptyset \). The former condition means that \( K \in u \) while the latter means that there is \( s \in L \) such that \( s^\uparrow \in U_K \), that is, \( s \in K \), and thus it holds if and only if \( K \cap L \neq \emptyset \). Since \( u \) is an ultrafilter, \( L \) having nonempty intersection with all \( K \in u \) is equivalent to \( L \in u \), that is \( u \in U_L \).

(2) In view of (1), we need to show that \( \iota^{-1}(U_L) = L \). Indeed, \( s \in \iota^{-1}(U_L) \) holds if and only if \( s^\uparrow \in U_L \), that is, \( s \in L \).

By (2), the mapping \( \varphi \) is injective. Since, for \( K, L \in B \), we have \( U_K \cup U_L = U_{K \cup L} \) and \( U_K^C = U_{K^C} \), it follows from (1) that \( \varphi \) is a homomorphism of Boolean algebras. It remains to show that every clopen subset \( C \) of \( B^* \) belongs to the image of \( \varphi \). Now, as \( C \) is open, it is a union of basic open sets \( U_L \) with \( L \in B \); as \( C \) is compact, it is a finite union of such sets. Since \( \bigcup_{i=1}^m U_{L_i} = U_{\bigcup_{i=1}^m L_i} \), we conclude that \( C = U_L \) for some \( L \in B \), as desired. \( \square \)

5.1. From Boolean algebras to Stone topological algebras.

The next result shows how to obtain Stone topological algebras from the Stone dual of a Boolean algebra of subsets of \( S \). The algebraic structure, for which suitable properties are stated and proved below, is given by the following definition.
Let $S$ be a $\Omega$-algebra and let $\mathcal{B}$ be a Boolean subalgebra of $\mathcal{P}(S)$. For $o \in \Omega_n$ and $u_1, \ldots, u_n \in \mathcal{B}$, let
\begin{equation}
oindent (5.1) \quad o_{\mathcal{B}}(u_1, \ldots, u_n) = \{L \in \mathcal{B} : \forall i \in \{1, \ldots, n\} \exists L_i \in u_i : o_S(L, \ldots, L_n) \subseteq L\}.
\end{equation}

**Theorem 5.2.** Let $\Omega$ be a topological signature, $S$ an $\Omega$-algebra, and $\mathcal{B}$ a Boolean subalgebra of $\mathcal{P}(S)$ satisfying (C2). Then, (5.1) defines a structure of $\Omega$-algebra on $\mathcal{B}^*$ which makes it a Stone topological algebra. The mapping $\nu_{\mathcal{B}} : S \to \mathcal{B}^*$ defined by sending each $s \in S$ to $s^\uparrow = \{L \in \mathcal{B} : s \in L\}$ is a homomorphism with dense image such that $\mathcal{B} = \nu_{\mathcal{B}}(\mathcal{P}_o(\mathcal{B}^*))$. Moreover, in case $S$ is a topological $\Omega$-algebra, the mapping $\nu_{\mathcal{B}}$ is continuous if and only if $\mathcal{B}$ is contained in $\mathcal{P}_o(S)$.

**Proof.** Given $o \in \Omega_n$ and $u_1, \ldots, u_n \in \mathcal{B}$, let $u = o_{\mathcal{B}}(u_1, \ldots, u_n)$ be the set defined by (5.1). We claim that $u$ is an ultrafilter of $\mathcal{B}$. It is immediate to check that it is a proper filter of $\mathcal{B}$. To show that it is an ultrafilter, we must show that, given $L \in \mathcal{B}$, either $L$ or its complement $L^C$ belongs to $u$. By the assumption that $\mathcal{B}$ satisfies (C2), we know that both $o_S^{-1}(L)$ and $o_S^{-1}(L^C)$ belong to $\mathcal{B}^*$: for instance, $o_S^{-1}(L)$ is the projection on the last $n$ components of $(E_n^n)^{-1}(L) \cap \{o\} \times S^n$. It follows that there are $n$ finite partitions $S = \bigsqcup_{j=1}^n L_{i,j}$ into elements of $\mathcal{B}$ ($i = 1, \ldots, n$) such that each box $L_{1,j_1} \times \cdots \times L_{n,j_n}$ is entirely contained in either $o_S^{-1}(L)$ or $o_S^{-1}(L^C)$.

Since each $u_i$ is an ultrafilter of $\mathcal{B}$, there is a unique $\ell_i \in \{1, \ldots, k_i\}$ such that $L_{i,\ell_i} \in u_i$. For this choice of $\ell_i$, since the product $L_{1,\ell_1} \times \cdots \times L_{n,\ell_n}$ is contained in either $o_S^{-1}(L)$ or $o_S^{-1}(L^C)$, we conclude respectively that $L \in u$ or $L^C \in u$.

Next, we claim that each evaluation mapping $E_n^\mathcal{B}$ is continuous. Let $L \in \mathcal{B}$. By Lemma [L.3] there is a decomposition
\begin{equation}
oindent (5.2) \quad (E_n^n)^{-1}(L) = \bigcup_{r \in R} K_r \times L_{1,r} \times \cdots \times L_{n,r}
\end{equation}
satisfying conditions (C2)[1]: (C2)[4]. The claim follows from the formula
\begin{equation}
oindent (5.3) \quad (E_n^\mathcal{B})^{-1}(\mu_L) = \bigcup_{r \in R} K_r \times \mu_{L_{1,r}} \times \cdots \times \mu_{L_{n,r}}
\end{equation}
which we proceed to establish. For the inclusion from right to left, suppose that $(o, u_1, \ldots, u_n)$ is such that there is $r \in R$ with $o \in K_r$ and $L_{i,r} \in u_i$ ($i = 1, \ldots, n$). From (5.2), it follows that $o_S(L_{1,r}, \ldots, L_{n,r}) \subseteq L$, so that $L \in o_{\mathcal{B}}(u_1, \ldots, u_n)$, that is, $(o, u_1, \ldots, u_n)$ belongs to the left side of (5.3).

For the inclusion from left to right in (5.3), suppose that $(o, u_1, \ldots, u_n)$ belongs to $(E_n^\mathcal{B})^{-1}(\mu_L)$. Then, for each $i \in \{1, \ldots, n\}$, there exists $J_i \in u_i$ such that $o_S(J_1, \ldots, J_n) \subseteq L$, which means that $\{o\} \times J_1 \times \cdots \times J_n \subseteq (E_n^n)^{-1}(L)$. Let $R_o = \{r \in R : o \in K_r\}$ and note that conditions (C2)[1] and (C2)[3] together imply that $R_o$ is a finite set. By (5.2), we deduce that $J_i \subseteq \bigcup_{r \in R_o} L_{i,r}$, so that this finite union also belongs to $u_i$; hence, there is $r_i$ such that $L_{i,r_i} \in u_i$ so that $J_i \cap L_{i,r_i} \in u$ and we may as well assume that $J_i$ is contained in some $L_{i,r_i}$. By Lemma [L.3] it follows that there exists $r \in R$ such that $\{o\} \times J_1 \times \cdots \times J_n \subseteq K_r \times L_{1,r} \times \cdots \times L_{n,r}$, thereby showing that $(o, u_1, \ldots, u_n)$ belongs to the right side of (5.3).
Next, we show that the correspondence \( \iota_{\mathcal{B}} : s \mapsto s^\uparrow \) is a homomorphism \( S \to \mathcal{B}^* \). Indeed, given \( s_1, \ldots, s_n \in S \) and \( o \in \Omega_n \), the ultrafilter \( o_{\mathcal{B}^*}(s_1^\uparrow, \ldots, s_n^\uparrow) \) consists of all \( L \in \mathcal{B} \) such that, for each \( i \), there exists \( L_i \in s_i^\uparrow \) such that \( o_\mathcal{S}(L_1, \ldots, L_n) \subseteq L \), so that, in particular, we have \( o_\mathcal{S}(s_1, \ldots, s_n) \in L \), that is, \( L \) belongs to \( o_\mathcal{S}(s_1, \ldots, s_n)^\uparrow \). Conversely, if \( o_\mathcal{S}(s_1, \ldots, s_n) \in L \), that is, \( (o, s_1, \ldots, s_n) \in (E_{\mathcal{S}}^\mathcal{S})^{-1}(L) \) then, since \( \mathcal{B} \) is assumed to satisfy condition (C2), there are \( L_i \in \mathcal{B} \) such that \( s_i \in L_i \) \((i = 1, \ldots, n)\) and \( \{o\} \times L_1 \times \cdots \times L_n \subseteq (E_{\mathcal{S}}^\mathcal{S})^{-1}(L) \), which implies that \( L \) belongs to \( o_{\mathcal{B}^*}(s_1^\uparrow, \ldots, s_n^\uparrow) \). We conclude that \( o_{\mathcal{B}^*}(s_1^\uparrow, \ldots, s_n^\uparrow) = o_\mathcal{S}(s_1, \ldots, s_n)^\uparrow \), thereby showing that \( \iota_{\mathcal{B}} \) is a homomorphism. Given a nonempty basic open set \( \mathcal{U}_L \), with \( L \in \mathcal{B} \setminus \{\emptyset\} \), we have \( s^\uparrow \in \mathcal{U}_L \) for every \( s \in L \), which shows that the image of \( \iota_{\mathcal{B}} \) is dense in \( \mathcal{B}^* \).

The equality \( \iota_{\mathcal{B}}^{-1}(\mathcal{P}_{\mathcal{C}^*}(\mathcal{B}^*)) = \mathcal{B} \) follows from Proposition 5.1.

It remains to deal with the continuity of \( \iota_{\mathcal{B}} \). By Proposition 5.1, for every \( L \in \mathcal{B} \), we have \( \iota_{\mathcal{B}}^{-1}(\mathcal{U}_L) = L \). Hence, \( \iota_{\mathcal{B}} \) is continuous if and only if \( \mathcal{B} \) consists of open subsets of \( S \). Since \( \mathcal{B} \) is closed under complementation, we deduce that \( \iota_{\mathcal{B}} \) is continuous if and only if \( \mathcal{B} \) is contained in \( \mathcal{P}_{\mathcal{C}^*}(S) \). \( \square \)

The next result shows how inclusion of Boolean subalgebras of \( \mathcal{P}(S) \) reflects on the corresponding Stone topological algebras.

**Theorem 5.3.** Let \( \Omega \) be a topological signature, \( S \) an \( \Omega \)-algebra, and \( \mathcal{B} \) and \( \mathcal{C} \) be Boolean subalgebras of \( \mathcal{P}(S) \) satisfying condition (C2) such that \( \mathcal{B} \subseteq \mathcal{C} \). Consider on the dual spaces \( \mathcal{B}^* \) and \( \mathcal{C}^* \) the structure of Stone topological algebras given by Theorem 5.2. Then the dual (surjective continuous) mapping \( \xi^* : \mathcal{C}^* \to \mathcal{B}^* \) of the inclusion \( \xi : \mathcal{B} \to \mathcal{C} \) is a homomorphism of \( \Omega \)-algebras such that \( \xi^* \circ \iota_{\mathcal{C}} = \iota_{\mathcal{B}} \).

**Proof.** We start by noting that, for an ultrafilter \( u \in \mathcal{C}^* \), we have the equality \( \xi^*(u) = u \cap \mathcal{B} \). To prove that \( \xi^* \) is a homomorphism of \( \Omega \)-algebras, consider an operation symbol \( o \in \Omega_n \) and ultrafilters \( u_1, \ldots, u_n \in \mathcal{C}^* \). From the definition of the interpretation of \( o \) in the dual spaces \( \mathcal{B}^* \) and \( \mathcal{C}^* \), we see that

\[
o_{\mathcal{B}^*}(u_1 \cap \mathcal{B}, \ldots, u_n \cap \mathcal{B}) = \{ L \in \mathcal{B} : \exists L_i \in u_i \cap \mathcal{B}, \ o_\mathcal{S}(L_1, \ldots, L_n) \subseteq L \} \subseteq \{ L \in \mathcal{B} : \exists L_i \in u_i, \ o_\mathcal{S}(L_1, \ldots, L_n) \subseteq L \} = o_{\mathcal{C}^*}(u_1, \ldots, u_n) \cap \mathcal{B}.
\]

Since both \( o_{\mathcal{B}^*}(u_1 \cap \mathcal{B}, \ldots, u_n \cap \mathcal{B}) \) and \( o_{\mathcal{C}^*}(u_1, \ldots, u_n) \cap \mathcal{B} \) are ultrafilters of \( \mathcal{B} \), it follows that they are equal. Finally, for \( s \in S \), both ultrafilters \( \xi^*(\iota_{\mathcal{C}}(s)) \) and \( \iota_{\mathcal{B}}(s) \) are the set of all \( L \in \mathcal{B} \) such \( s \in L \), that is, they are equal. \( \square \)

**5.2. From Stone topological algebras to Boolean algebras.** We next show how from a Stone topological algebra we may obtain a Boolean subalgebra of \( \mathcal{P}_{\mathcal{C}^*}(S) \). This requires some preparation.

**Lemma 5.4.** Let \( \varphi : S \to T \) be a continuous mapping between two topological algebras and suppose that the restriction of \( \varphi \) to a dense subalgebra \( A \) of \( S \) is a homomorphism. Then \( \varphi \) is a homomorphism.
Proof. Let $s_1,\ldots,s_n$ be elements of $S$ and let $o \in \Omega_n$. Since $A$ is dense in $S$, for each $k \in \{1,\ldots,n\}$ there is a net $(a_{k,i})_i$ in $A$ converging to $s_k$, where we may assume that the same index set is used for all $k$. By continuity of the mappings $o_T, \varphi$, and $o_S$ together with the assumption that $\varphi|_A$ is a homomorphism, we obtain the following equalities:

$$o_T(\varphi(s_1),\ldots,\varphi(s_n)) = \lim o_T(\varphi(a_{1,i}),\ldots,\varphi(a_{n,i}))$$
$$= \lim \varphi(o_S(a_{1,i},\ldots,a_{n,i}))$$
$$= \varphi(o_S(s_1,\ldots,s_n)).$$

Hence, $\varphi$ is a homomorphism. \hfill \Box 

**Theorem 5.5.** Suppose that $\varphi : S \rightarrow T$ is a continuous homomorphism between topological algebras whose image is dense in $T$. Let $\mathcal{C}$ be a Boolean subalgebra of $\mathcal{B}_{\text{max}2}^T$ satisfying condition (C2) and let $\mathcal{B} = \varphi^{-1}(\mathcal{C})$. Consider the mapping $\tilde{\varphi} : T \rightarrow \mathcal{B}^*$ defined by $\tilde{\varphi}(t) = \varphi^{-1}(t^\Delta)$, where $t^\Delta = \{ K \in \mathcal{C} : t \in K \}$. Then $\tilde{\varphi}$ is a continuous homomorphism with dense image such that $\tilde{\varphi} \circ \varphi = \iota_\mathcal{B}$. Moreover, $\tilde{\varphi}$ distinguishes two elements of $T$ if and only if there is a member of $\mathcal{C}$ that separates them.

Proof. By Proposition 4.13 the Boolean subalgebra $\mathcal{B}$ of $\mathcal{P}_{\text{co}}(S)$ satisfies (C2). Note that the mapping $\psi : K \mapsto \varphi^{-1}(K)$ is an onto homomorphism of Boolean algebras $\mathcal{C} \rightarrow \mathcal{B}$. We claim that it is an isomorphism. Indeed, if $K$ and $L$ are distinct members of $\mathcal{C}$ then the symmetric difference $K \triangle L$ is a nonempty open subset of $T$, whence it contains some element of the form $\varphi(s)$ for some $s \in S$, which implies that $\varphi^{-1}(K) \triangle \varphi^{-1}(L)$ is also nonempty, thereby showing that $\psi$ is injective. We thus get the external commuting square in the following diagram of continuous mappings, where the vertical arrows are homomorphisms with dense images given by Theorem 5.2:

$$\begin{array}{ccc}
S & \xrightarrow{\varphi} & T \\
\iota_\mathcal{C} \downarrow & & \downarrow \iota_\mathcal{B} \\
\mathcal{B}^* & \xrightarrow{\psi^*} & \mathcal{C}^* \\
\tilde{\varphi} & & \\
\end{array}$$

By Lemma 5.3, we conclude that $\psi^*$ is an isomorphism of Stone topological algebras. Further noting that $(\psi^*)^{-1} \circ \iota_\mathcal{B} = \tilde{\varphi}$, we deduce that $\tilde{\varphi}$ is indeed a continuous homomorphism with dense image and the equality $\tilde{\varphi} \circ \varphi = \iota_\mathcal{B}$ follows from the commutativity of the outer square in the above diagram. The mapping $\tilde{\varphi}$ distinguishes two points of $T$ if and only if so does $\iota_\mathcal{B}$, which is equivalent to the condition that the points in question are separated by the members of $\mathcal{C}$. \hfill \Box 

**Corollary 5.6.** Let $\varphi : S \rightarrow T$ be a continuous homomorphism with dense image, where $S$ is a topological algebra and $T$ is a Stone topological algebra. Then the Boolean algebra $\mathcal{B}_\varphi = \{ \varphi^{-1}(K) : K \in \mathcal{P}_{\text{co}}(T) \}$ satisfies (C2) and $T$ is isomorphic with the Stone topological algebra $\mathcal{B}_\varphi^*$ of Theorem 5.2.

Proof. By Propositions 4.13 and 4.15, $\mathcal{B}_\varphi$ satisfies (C2). In the notation of the proof of Theorem 5.3, all that remains to show is that $\iota_\mathcal{B}$ is an isomorphism of Stone topological algebras, where $\mathcal{C} = \mathcal{B}_{\text{max}2}^T$. In fact, we know that $\mathcal{C} = \mathcal{P}_{\text{co}}(T)$ by Proposition 4.13 hence $\iota_\mathcal{B}$ is a homeomorphism. Since
it is a homomorphism by Theorem 5.2, it is indeed an isomorphism of Stone topological algebras. □

In Corollary 5.6 one may take \( S \) to be the term algebra \( T_\Omega(X) \) for a generating subset \( X \) of \( T \). Then, Corollary 5.6 shows that all Stone topological algebras are duals of Boolean algebras of clopen subsets of a term algebra satisfying (C2). This may be viewed as an alternative approach to duality compared with that adopted in [12].

**Corollary 5.7.** Let \( \varphi : T_\Omega(X) \rightarrow \beta X \) be the natural homomorphism. Then the Boolean algebra \( B_\varphi \) defined in Corollary 5.6 is the largest Boolean subalgebra of \( \mathcal{P}_{co}(T_\Omega(X)) \) satisfying (C2).

**Proof.** By Propositions 4.13 and 4.15, \( B_\varphi \) is indeed a Boolean subalgebra of \( \mathcal{P}_{co}(T_\Omega(X)) \) satisfying (C2). By Theorem 5.2, \( (B_{\max^2}^{\text{T}_{\Omega}(X)})^* \) is an \( X \)-generated Stone topological algebra. The dual of the inclusion mapping \( \eta : B_\varphi \rightarrow B_{\max^2}^{\text{T}_{\Omega}(X)} \) is an onto continuous mapping \( \eta^* : (B_{\max^2}^{\text{T}_{\Omega}(X)})^* \rightarrow B_\varphi^* \) which is a homomorphism respecting the generating mappings from the space \( X \) by Theorem 5.3. But, by Corollary 5.6, \( B_\varphi^* \) is freely generated by \( X \). Hence, \( \eta^* \) must be injective and, dually, \( \eta \) must be surjective. This shows that \( B_\varphi = B_{\max^2}^{\text{T}_{\Omega}(X)} \).

Note that, in view of Theorem 5.2 and Corollary 5.6, Corollary 5.7 may be thought as providing a construction of the absolutely free Stone topological algebra \( \overline{\Omega}_X \text{St}_{\Omega} \), modulo the identification of the Boolean algebra \( B_{\max^2}^{\text{T}_{\Omega}(X)} \), for which we have no constructive description.

**Theorem 5.8.** Let \( \Omega \) be a topological signature for which there is \( n > 1 \) such that \( \Omega_n \) and \( X \) are both nonempty discrete spaces. Then \( B_{\max^2}^{\text{T}_{\Omega}(X)} \) is a proper Boolean subalgebra of \( \mathcal{P}_{co}(T_\Omega(X)) \).

**Proof.** Fix \( n > 1 \) such that \( \Omega_n \) is nonempty and discrete and consider the signature \( \Omega' = \Omega_n' = \Omega_n \). Notice, that \( T_{\Omega'}(X) \) is a closed subset of \( T_\Omega(X) \), and for each \( t \in T_{\Omega'}(X) \), the set \( \{t\} \) is an open subset of \( T_\Omega(X) \). Thus, every subset of \( T_{\Omega'}(X) \) is clopen in \( T_\Omega(X) \). Let \( L \) consist of all elements of \( T_{\Omega'}(X) \) of the form \( o(t, \ldots, t) \) with \( o \in \Omega_n \) and \( t \in T_{\Omega'}(X) \). For any \( o \in \Omega_n \), the set \( E_{n-1}^{-1}(L) \cap \{o\} \times (T_{\Omega}(X))^n \) is not a finite union of boxes, while \( L \) is a clopen subset of \( T_\Omega(X) \). Hence, the Boolean algebra \( \mathcal{P}_{co}(T_{\Omega}(X)) \) does not satisfy (C2), so that \( B_{\max^2}^{T_\Omega(X)} \) must be a proper Boolean subalgebra of \( \mathcal{P}_{co}(T_\Omega(X)) \).

We may now derive a result showing that, at least beyond the unary case, the Čech-Stone compactification does not provide a construction for the free Stone topological algebra \( \overline{\Omega}_X \text{St}_{\Omega} \).

**Corollary 5.9.** Let \( \Omega \) be a discrete signature with at least one operation symbol of arity greater than 1 and let \( X \) be a nonempty discrete space. Then it is not possible to define on the Stone space \( \beta(T_\Omega(X)) \) a structure of topological algebra in which \( T_\Omega(X) \) is a subalgebra.

**Proof.** Suppose on the contrary that the Stone space \( S = \beta(T_\Omega(X)) \) admits a structure of topological algebra extending the structure of \( T_\Omega(X) \) and let
\( \varphi : T_{1\Omega}(X) \rightarrow S \) be the inclusion mapping. By Theorem 5.5, we have an associated Boolean subalgebra \( \mathcal{B}_\varphi = \{ \varphi^{-1}(K) : K \in \mathcal{P}_{co}(S) \} \) of \( \mathcal{P}(T_{1\Omega}(X)) \) satisfying (C2). Since \( T_{1\Omega}(X) \) is a discrete space, \( \mathcal{B}_\varphi \) is equal to \( \mathcal{P}(T_{1\Omega}(X)) \) (see, for instance, [17, Theorem 3.27]). But, since \( \mathcal{B}_\varphi \) satisfies (C2), it must be contained in \( \mathcal{B}_{\text{max}2}(X) \). It follows that \( \mathcal{B}_{\text{max}2}(X) = \mathcal{P}(T_{1\Omega}(X)) \), which contradicts Theorem 5.8. \( \square \)

The special case of Theorem 5.3 for a profinite algebra \( T \) is particularly interesting. Suppose that \( T \) is generated by a continuous mapping \( \varphi : X \rightarrow T \) and consider the unique continuous homomorphic extension \( \hat{\varphi} : T_{1\Omega}(X) \rightarrow T \). Recall that \( \mathcal{B}_{\hat{\varphi}} \) consists of all subsets of \( T_{1\Omega}(X) \) of the form \( \hat{\varphi}^{-1}(K) \) where \( K \) is a clopen subset of \( T \). Now, by Theorem 2.1, a subset \( K \) of \( T \) is clopen if and only if there is a continuous homomorphism \( \psi : T \rightarrow F \) onto a finite algebra \( F \) such that \( K = \psi^{-1}(\psi(K)) \). Hence, \( \mathcal{B}_{\hat{\varphi}} \) consists of the sets of the form \( L = \hat{\varphi}^{-1}(\psi^{-1}(P)) \) with \( P \) an arbitrary subset of \( F \). In case \( T = \Pi_X V \) is a relatively free profinite algebra, over a pseudovariety \( V \) of finite algebras, the composite homomorphisms \( \psi \circ \hat{\varphi} \) may be characterized simply as the onto continuous homomorphisms \( T_{1\Omega}(X) \rightarrow F \) with \( F \in V \). This leads to the notion of \( V \)-recognizable tree language (better known as regular tree language [11, 26] in case \( V = \text{Fin}_0 \) of computer science and so \( \mathcal{B}_{\hat{\varphi}} \) consists precisely of all such languages over the “alphabet” \( X \). Characterizations of the Boolean algebras \( \mathcal{B}_{\hat{\varphi}} \) with \( X \) finite and discrete can be found as part of the analog of Eilenberg’s Correspondence Theorem (or “Variety Theorem”) [11, 28, 26].

6. Stone varieties

In this section, we present one of our main results and applications. It is a generalization for topological signatures of Theorem 2.2 that result being the special case of the Stone pseudovariety \( \text{Fin}_{1\Omega} \).

**Theorem 6.1.** Let \( \Omega \) be an arbitrary topological signature and let \( S \) be an arbitrary Stone pseudovariety. Let \( \varphi : S \rightarrow T \) be an onto continuous homomorphism of Stone topological algebras where \( S \) is residually \( S \). Then \( T \) is also residually \( S \).

**Proof.** Let \( t_1, t_2 \) be a pair of distinct points of \( T \). Since \( T \) is a Stone space, there is a clopen subset \( K \) of \( T \) such that \( t_1 \in K \) and \( t_2 \notin K \). Applying Lemma 5.10 to the clopen subset \( L = \varphi^{-1}(K) \) of \( S \), we obtain a continuous homomorphism \( \psi : S \rightarrow U \) onto a member of \( S \) such that \( L = \psi^{-1}(\psi(L)) \). Then, in the notation of Corollary 4.10 by Theorem 5.3 the Boolean subalgebras \( \mathcal{B}_\varphi \) and \( \mathcal{B}_\psi \) of \( \mathcal{P}_{co}(S) \) both satisfy (C2). By Corollary 4.15 the Boolean algebra \( \mathcal{B} = \mathcal{B}_\varphi \cap \mathcal{B}_\psi \) also satisfies (C2). Consider the various inclusions between the four Boolean algebras \( \mathcal{B}, \mathcal{B}_\varphi, \mathcal{B}_\psi, \) and \( \mathcal{P}_{co}(S) \), where we denote \( \mathcal{B} \hookrightarrow \mathcal{B}_\varphi \) by \( \xi \). Dualizing, in view of Theorem 5.3 we obtain the following
commutative diagram of continuous homomorphisms between Stone topological algebras, where \( \tilde{\phi} \) is given by Theorem 5.5 and all mappings are onto:

In particular, since \( U \) belongs to the Stone pseudovariety \( \mathcal{S} \), so does the Stone topological algebra \( \mathcal{B}^* \). To finish the proof, it suffices to show that \( \delta(t_1) \neq \delta(t_2) \).

Note that \( L \in \mathcal{B}_\varphi \cap \mathcal{B}_\psi = \mathcal{B} \). Let \( s_i \in S \) be such that \( \varphi(s_i) = t_i \) (\( i = 1, 2 \)). Then we have \( s_1 \in L \) while \( s_2 \notin L \). As \( \delta \circ \varphi = \psi \) and, for each \( s \in S \), we have \( \psi(s) = \{ J \in \mathcal{B} : s \in J \} \), we get \( \delta(t_1) = \psi(s_1) \neq \psi(s_2) = \delta(t_2) \), thereby completing the proof of the theorem.

For a class \( \mathcal{C} \) of Stone topological algebras, we denote by \( \hat{\mathcal{C}} \) the class of all residually \( \mathcal{C} \) Stone topological algebras. Note that \( \hat{\hat{\mathcal{C}}} = \hat{\mathcal{C}} \). Thus, the correspondence \( \mathcal{C} \mapsto \hat{\mathcal{C}} \) is a closure operator on the class of all classes of Stone topological algebras. We call \( \hat{\mathcal{C}} \) the residual closure of \( \mathcal{C} \). We say that a class \( \mathcal{C} \) of Stone topological algebras \( \mathcal{S} \) is residually closed if \( \mathcal{C} = \hat{\mathcal{C}} \).

**Corollary 6.2.** Let \( \Omega \) be a topological signature and \( \mathcal{S} \) a Stone pseudovariety. Then \( \hat{\mathcal{S}} \) is also a Stone pseudovariety and, for every topological space \( X \), we have \( \Omega_X \mathcal{S} \simeq \Omega_X \hat{\mathcal{S}} \).

**Proof.** To check that \( \hat{\mathcal{S}} \) is a Stone pseudovariety, the only nontrivial requirement is that it be closed under taking Stone continuous homomorphic images, which follows from Theorem 6.1. The existence of a continuous homomorphism \( \varphi : \Omega_X \mathcal{S} \to \Omega_X \hat{\mathcal{S}} \) respecting generating mappings is an immediate consequence of Proposition 6.3. The existence of a continuous homomorphism \( \Omega_X \hat{\mathcal{S}} \to \Omega_X \mathcal{S} \) also respecting generating mappings follows from the obvious fact that \( \psi : \mathcal{S} \subseteq \hat{\mathcal{S}} \). To conclude the proof, it suffices to observe that \( \varphi \) and \( \psi \) are mutually inverse mappings.

By a *Stone variety* we mean a nonempty class of Stone topological algebras that is closed under taking continuous homomorphic images that are again Stone spaces, closed subalgebras, and arbitrary direct products.

**Proposition 6.3.** A class of Stone topological algebras is a Stone variety if and only if it is a residually closed Stone pseudovariety.

**Proof.** A Stone variety \( \mathcal{V} \) is obviously a Stone pseudovariety. It is residually closed because, if \( S \) is residually \( \mathcal{V} \), then there is an embedding of \( S \), as a closed subalgebra, into a product of members of \( \mathcal{V} \), and so \( S \) belongs to \( \mathcal{V} \).

Conversely, we claim that, if \( \mathcal{S} \) is a residually closed Stone pseudovariety, then \( \mathcal{S} \) is a Stone variety. Let \( (S_i)_{i \in I} \) be a nonempty family of members of \( \mathcal{S} \). Given two distinct elements of the product \( S = \prod_{i \in I} S_i \), they are distinguished by the projection on some component \( S_i \), and so \( S \) is residually \( \mathcal{S} \). Since \( \mathcal{S} \) is residually closed, we deduce that \( S \) belongs to \( \mathcal{S} \), which proves the claim.

\( \square \)
Note that, for a Stone variety $\mathcal{V}$ and an arbitrary topological space $X$, the relatively free Stone topological algebra $\Omega^X \mathcal{V}$ belongs to $\mathcal{V}$, a fact that follows from the construction of $\Omega^X \mathcal{V}$ in the proof of Proposition 3.4. We deduce that a Stone topological algebra belongs to $\mathcal{V}$ if and only if it is a continuous homomorphic image of some $\Omega^X \mathcal{V}$.

An immediate consequence of Proposition 6.3 is that the residual closure of a Stone pseudovariety is the Stone variety it generates.

The proof of Theorem 6.1 may be adapted to establish the following result.

**Theorem 6.4.** If $(S_i)_{i \in I}$ is a nonempty family of Stone pseudovarieties then
\[ \bigcap_{i \in I} \hat{S_i} = \bigcap_{i \in I} \hat{\hat{S_i}}. \]

**Proof.** Since the inclusion $\bigcap_{i \in I} S_i \subseteq \bigcap_{i \in I} \hat{S_i}$ is clear, we need to show that, if $T$ is a Stone topological algebra that is residually $S_i$ for each $i \in I$, then $T$ is also residually $\bigcap_{i \in I} S_i$. Let $t_1$ and $t_2$ be distinct elements of $T$ and choose a clopen subset $K$ of $T$ such that $t_1 \in K$ and $t_2 \notin K$. By Lemma 3.10, there are onto continuous homomorphisms $\varphi_i : T \to S_i$ such that $S_i \in S_i$ and $K = \varphi_i^{-1}(\varphi_i(K)) (i \in I)$. Since every mapping $\varphi_i$ is closed, the image $\varphi_i(K)$ is a clopen subset of $S_i$.

Consider the Boolean subalgebras $B_{\varphi_i} = \{ \varphi_i^{-1}(L) : L \in P_{co}(S_i) \}$ of $P_{co}(T)$, which satisfies condition (C2) by Corollary 5.6. By Corollary 4.14, their intersection $B$ is also a Boolean subalgebra of $P_{co}(T)$ satisfying (C2).

We get the following diagram of inclusions between Boolean algebras:

\[
\begin{array}{ccc}
P_{co}(T) & \xmapsto{\varphi_i} & B_{\varphi_i} \\
\downarrow & & \downarrow \\
B & & B^*
\end{array}
\]

The dual diagram of continuous homomorphisms yields the lower triangle of the following commutative diagram for every $i \in I$:

\[
\begin{array}{ccc}
T & \overset{\varphi_i}{\longrightarrow} & S_i \\
\downarrow^{\iota_B \varphi_i} & & \downarrow^{\hat{\varphi_i}} \\
B_{\varphi_i}^* & \xmapsto{\psi_i} & B^*
\end{array}
\]

By the assumptions on the $S_i$, we see that $K \in B_{\varphi_i}$ for every $i \in I$, whence $K \in B$. By Theorem 5.5 we get $\iota_B(t_1) \neq \iota_B(t_2)$. On the other hand, the continuous homomorphisms $\psi_i \circ \hat{\varphi_i}$ are onto. Since each $\hat{S_i}$ is a Stone pseudovariety, it follows that $B^*$ belongs to $\bigcap_{i \in I} \hat{S_i}$. Hence, $T$ is residually $\bigcap_{i \in I} \hat{S_i}$.

Thus, the residual closure operator on Stone pseudovarieties is a complete meet endomorphism of the lattice $\mathcal{L}_\Omega$ of all Stone pseudovarieties.

If $\mathcal{S}$ is a residually closed Stone pseudovariety, then we may consider the family of all Stone pseudovarieties $\mathcal{S}_i$ such that $\hat{\hat{S_i}} = \mathcal{S}$. By Theorem 6.4 we conclude that $\bigcap_{i \in I} \hat{S_i} = \mathcal{S}$. This proves the following result.
Corollary 6.5. Given a Stone variety \( \mathcal{V} \), the set of all Stone pseudovarieties \( \hat{\mathcal{V}} \) with \( \hat{\mathcal{V}} = \mathcal{V} \) is an interval \([\mathcal{V}_{\min}, \mathcal{V}]\) of the lattice \( \mathcal{L}_\Omega \).

In the profinite case, the minimum of the interval of Corollary 6.5 admits a particularly simple description.

Theorem 6.6. If \( \mathcal{V} \) is a Stone variety of profinite algebras then \( \mathcal{V}_{\min} = \mathcal{V} \cap \text{Fin}_\Omega \).

Proof. From \( \mathcal{V} \cap \text{Fin}_\Omega \subseteq \mathcal{V} \), we deduce that \( \mathcal{V} \cap \text{Fin}_\Omega \subseteq \hat{\mathcal{V}} = \mathcal{V} \). On the other hand, if \( S \) belongs to \( \mathcal{V} \), then \( S \) is profinite and so \( S \) embeds in a product of its finite quotients, thus in a product of members of \( \mathcal{V} \cap \text{Fin}_\Omega \), which proves that \( \mathcal{V} \subseteq \mathcal{V} \cap \text{Fin}_\Omega \). Hence, \( \mathcal{V} \cap \text{Fin}_\Omega \) is a pseudovariety of finite algebras whose residual closure is \( \mathcal{V} \). To complete the proof, we claim that, if \( V \) and \( W \) are pseudovarieties of finite algebras such that \( V \subseteq W \subseteq \hat{V} = \mathcal{V} \), then \( V = W \). Indeed, if \( S \) is an arbitrary element of \( W \), then \( S \) is residually \( V \), that is, it embeds in a direct product of algebras from \( V \). Since \( S \) is finite, it suffices to consider only finitely many factors in such a product to achieve the embedding. Hence, \( S \) belongs to \( V \).

Example 6.7. Let \( \Omega \) be a 0-dimensional signature. By Corollary 6.3 and Proposition 6.3, the class \( \text{Fin}_\Omega \) of all profinite algebras is a Stone variety. Suppose also that the space \( X \) is 0-dimensional. By Proposition 3.7, the topological algebra \( \text{T}_\Omega(X) \) is residually finite. Hence, the variety of all profinite \( \Omega \)-algebras satisfies no nontrivial identities. Of course, neither does the variety \( \text{St}_\Omega \) of all Stone topological \( \Omega \)-algebras, which contains non-profinite algebras by Theorem 3.9. This shows that one cannot expect Stone varieties to be defined by identities as in the classical Birkhoff theorem for varieties of discrete algebras over discrete signatures [5]. Nevertheless, of course every set \( \Sigma \) of identities still defines a Stone variety, namely the class of all Stone topological algebras that satisfy \( \Sigma \).

A topological space \( X \) is said to be extremally disconnected if the closure of every open subset of \( X \) is open. It is easy to see that every Hausdorff extremally disconnected space is totally disconnected. The compact extremally disconnected spaces are sometimes called Stonean spaces and turn out to be, up to homeomorphism, the Stone duals of complete Boolean algebras. Complete Boolean algebras are known to be injective in the category of Boolean algebras [23] (that is, given a Boolean subalgebra \( A \) of a Boolean algebra \( B \), a homomorphism from \( A \) to a complete Boolean algebra \( C \) extends to a homomorphism \( B \rightarrow C \)). Dually, and more generally, Stonean spaces are known to be precisely the projective spaces in the category of compact spaces [16]. For more details, see [18]. Note that, since a Boolean algebra may always be embedded in a complete Boolean algebra, dually, every Stone space is a continuous image of some Stonean space, a fact that is used below.

In spite of Example 6.7, it is still natural to expect Stone varieties to be defined by some sort of identities and it turns out that Stone pseudoidentities, which we proceed to introduce, play that role.

Let \( X \) be a topological space and let \( \mathcal{S} \) be a Stone pseudovariety. Let \( \iota : X \rightarrow \overline{\mathcal{O}_X}\mathcal{S} \) be the natural generating mapping. We associate with each continuous mapping \( \varphi : X \rightarrow S \) into a Stone topological algebra \( S \) which
is residually $\mathcal{S}$ the unique continuous homomorphism $\hat{\varphi} : \overline{\Omega}X\mathcal{S} \to S$ such that $\hat{\varphi} \circ \iota = \varphi$. By a Stone $\mathcal{S}$-pseudoidentity over $X$ we mean a pair $(u, v)$ of elements of $\overline{\Omega}X\mathcal{S}$, usually written as a formal equality $u = v$. We say that a residually $\mathcal{S}$ Stone topological algebra $S$ satisfies $u = v$ if, for every continuous homomorphism $\varphi : X \to S$, the equality $\hat{\varphi}(u) = \hat{\varphi}(v)$ holds. In case $\mathcal{S} = \mathcal{S}t_{\Omega}$ consists of all Stone topological $\Omega$-algebras, we refer simply to Stone pseudoidentities over $X$.

The following result is the suitable analog of Birkhoff’s theorem for Stone varieties. It comes at the cost of allowing proper classes of Stone pseudoidentities for describing varieties for, unlike the classical discrete setting, one cannot reduce to the case where (pseudo)identities are written over finite sets of variables. More precisely, we consider classes consisting of Stone $\mathcal{S}$-pseudoidentities over arbitrary Stonean spaces $X$. For such a class $\Sigma$, we denote $[\Sigma]_\mathcal{S}$ the class of all members of $\mathcal{S}$ that satisfy all members of $\Sigma$ and we call $\Sigma$ a basis of the class $[\Sigma]_\mathcal{S}$. In case $\mathcal{S} = \mathcal{S}t_{\Omega}$, we drop the index $\mathcal{S}$, writing simply $[\Sigma]$.

**Theorem 6.8.** A class $\mathcal{V}$ of Stone topological algebras is a Stone variety if and only if there is a class $\Sigma$ of Stone pseudoidentities over Stonean spaces such that $\mathcal{V} = [\Sigma]$.

**Proof.** Suppose first that $\mathcal{V}$ is a Stone variety: let $\Sigma$ be the class consisting of all Stone pseudoidentities over Stonean spaces that are valid in $\mathcal{V}$. We claim that $[\Sigma] \subseteq \mathcal{V}$, the reverse inclusion being obvious from the choice of $\Sigma$. Let $S$ be an arbitrary element of $[\Sigma]$ and consider a generating mapping $\varphi : X \to S$ with $X$ a Stone space, where we could simply take the identity mapping on $S$. Since $X$ is a continuous image of some Stonean space, we may as well assume that $X$ is a Stonean space. Then, with the above choice of $\iota$ and $\hat{\varphi}$, we may also consider the natural continuous homomorphism $\eta$ in Diagram (6.1) (cf. end of Subsection 3.1).

\[
\begin{array}{ccc}
X & \xrightarrow{\iota} & \overline{\Omega}X\mathcal{S}t_{\Omega} \\
\downarrow{\varphi} & & \downarrow{\eta} \\
S & \xrightarrow{\psi} & \overline{\Omega}X\mathcal{V}
\end{array}
\] (6.1)

Note that, if the pair of elements $u, v \in \overline{\Omega}X\mathcal{S}t_{\Omega}$ is such that $\eta(u) = \eta(v)$, then from the universal property of $\overline{\Omega}X\mathcal{V}$ we deduce that the Stone pseudoidentity $u = v$ belongs to $\Sigma$. By the assumption that $S$ belongs to $[\Sigma]$, it follows that $\hat{\varphi}(u) = \hat{\varphi}(v)$ for all such pairs $u, v$. Hence, there is a unique homomorphism $\psi$ such that Diagram (6.1) commutes and continuity of $\psi$ follows from the continuity of $\hat{\varphi}$ and $\eta$ and compactness of $\overline{\Omega}X\mathcal{S}t_{\Omega}$. Since $\varphi$ is a generating mapping, $\psi$ is onto. Since $\mathcal{V}$ is a Stone variety and $\overline{\Omega}X\mathcal{V}$ belongs to $\mathcal{V}$, we deduce that $S \in \mathcal{V}$. This establishes the equality $\mathcal{V} = [\Sigma]$ and proves half of the theorem.

The proof of the converse consists in showing that $\mathcal{V} = [\Sigma]$ is a Stone variety for every class $\Sigma$ of Stone pseudoidentities over Stonean spaces. Thus, we should show that $\mathcal{V}$ is closed under taking quotients that are Stone spaces, closed subalgebras, and arbitrary direct products. Except for the case of quotients, the verification is standard and amounts to a straightforward
argument that is omitted. The exception is what leads us to consider Stone pseudoidentities over Stonean spaces rather than over arbitrary topological spaces.

So, let $\alpha : S \to T$ be an arbitrary onto continuous homomorphism between Stone topological algebras and assume that $S \in \mathcal{V}$. Let $u = v$ be a member of $\Sigma$, say $u, v \in \Omega_X \delta t_{\Omega}$ for a Stonean space $X$. Let $\varphi : X \to T$ be a continuous mapping. Since $X$ is projective in the category of Stone spaces (cf. above discussion), there is a continuous mapping $\psi : X \to S$ such that $\alpha \circ \psi = \varphi$. Consider the induced continuous homomorphisms $\hat{\varphi}$ and $\hat{\psi}$ such that the following diagram commutes:

$$
\begin{array}{ccc}
X & \xrightarrow{\psi} & S \\
\downarrow{\varphi} & & \downarrow{\alpha} \\
\Omega_X \delta t_{\Omega} & \xrightarrow{\eta} & T.
\end{array}
$$

Since $S$ belongs to $\mathcal{V}$, $S$ satisfies $u = v$ and so the equality $\hat{\psi}(u) = \hat{\psi}(v)$ holds. From the commutativity of the diagram, it follows that $\hat{\varphi}(u) = \hat{\varphi}(v)$, which shows that $T$ also satisfies $u = v$. This establishes that $T \in \mathcal{V}$ and completes the proof of the theorem. $\square$

In particular, the Stone variety $\hat{\text{Fin}}_{\Omega}$ of all profinite $\Omega$-algebras is defined by some class of Stone pseudoidentities over Stonean spaces. Other than the basis provided by the proof of Theorem 6.8 we know of no simple basis for $\hat{\text{Fin}}_{\Omega}$. Yet, for some pseudovarieties $\mathcal{V}$ of finite algebras, usual identities (between terms) are sufficient to define the Stone variety $\hat{\mathcal{V}}$ of all pro-$\mathcal{V}$ algebras. Semigroups, groups, rings distributive lattices and lattices [20, 8] are examples where this phenomenon occurs.

We may derive Reiterman’s theorem from our previous results. Although the proof is not shorter than a direct proof, it shows how Reiterman’s theorem can be viewed as a special case of Theorem 6.8.

**Theorem 6.9.** A class of finite topological algebras is a Stone pseudovariety if and only if it is of the form $[\Sigma]_{\text{Fin}_\Omega}$ for some set $\Sigma$ of Stone $\text{Fin}_\Omega$-pseudoidentities over finite discrete spaces.

**Proof.** Let $\mathcal{S}$ be a Stone pseudovariety and $X$ a Stonean space. For $u, v \in \Omega_X \delta$, we may choose $u', v' \in \Omega_X \delta t_{\Omega}$ such that $\eta(u') = u$ and $\eta(v') = v$, where $\eta : \Omega_X \delta t_{\Omega} \to \Omega_X \delta$ is the natural continuous homomorphism. We claim that, for an arbitrary Stone topological algebra $S$ which is residually $\mathcal{S}$, $S$ satisfies $u = v$ if and only if $S$ satisfies $u' = v'$. Indeed, if $S$ satisfies $u = v$ and $\varphi : \Omega_X \delta t_{\Omega} \to S$ is a continuous homomorphism, then there is a continuous homomorphism $\psi : \Omega_X \delta \to S$ such that $\psi \circ \eta = \varphi$. By assumption, $\varphi(u) = \varphi(v)$ and so we get that $\psi(u') = \psi(v')$. The converse is even simpler and is left to the reader. Applying the claim to the Stone pseudovariety $\mathcal{S} = \text{Fin}_\Omega$, it follows that, for $\Sigma$ as in the statement of the theorem,

$$
[\Sigma]_{\text{Fin}_\Omega} = [u' = v' : (u = v) \in \Sigma] \cap \text{Fin}_\Omega
$$
is the intersection of two Stone pseudovarieties by Theorem 6.8 whence it is also a Stone pseudovariety.

For the converse, let \( V \) be a pseudovariety of finite topological algebras. By Theorem 6.8, there is a set \( \Sigma \) of \( \delta\Omega \)-pseudoidentities over Stonean spaces such that \( \hat{V} = [\Sigma] \). For each member \( u = v \) of \( \Sigma \), say with \( u, v \in \overline{\Omega}_X \delta\Omega \), we may consider the \( \hat{V} \)-pseudoidentity \( \eta(u) = \eta(v) \), where \( \eta : \overline{\Omega}_X \delta\Omega \to \overline{\Omega}_X \hat{V} \) is the natural continuous homomorphism. Let \( \Sigma' \) be the set of all such \( \hat{V} \)-pseudoidentities. By the considerations at the beginning of the proof and Theorem 6.6, we obtain the equalities

\[
\llbracket \Sigma' \rrbracket_{\text{Fin}\Omega} = \llbracket \Sigma \rrbracket_{\text{Fin}\Omega} \cap \text{Fin}\Omega = \hat{V} \cap \text{Fin}\Omega = V.
\]

It remains to show that we only need to take \( \text{Fin}\Omega \)-pseudoidentities over finite discrete spaces. Given a \( \text{Fin}\Omega \)-pseudoidentity \( u = v \), we consider all pseudoidentities of the form \( \gamma(u) = \gamma(v) \) where \( \gamma : \overline{\Omega}_X \text{Fin}\Omega \to \overline{\Omega}_Y \text{Fin}\Omega \) is an arbitrary continuous homomorphism and \( Y \) is an arbitrary finite subspace of a fixed countable discrete space \( Z \). A routine argument shows that a profinite algebra satisfies \( u = v \) if and only if it satisfies all such pseudoidentities over finite discrete spaces. Thus, if we consider all pseudoidentities over finite subspaces of \( Z \) so associated with pseudoidentities from \( \Sigma' \), we obtain a set of pseudoidentities \( \Sigma \) such that \( \llbracket \Sigma \rrbracket_{\text{Fin}\Omega} = V \). □

We finish with an example showing that Stone varieties are not characterized by their profinite members.

**Example 6.10.** Let \( \Omega = \Omega_1 \cup \Omega_2 \) be the signature given by \( \Omega_1 = \{ \alpha, \beta \} \) and \( \Omega_2 = \{ \gamma \} \). Consider the Stone variety \( V \) defined by the Jónsson-Tarski identities [19]:

\[
(6.2) \quad \alpha(\gamma(x, y)) = x, \quad \beta(\gamma(x, y)) = y, \quad \gamma(\alpha(x), \beta(x)) = x.
\]

Note that an \( \Omega \)-algebra \( S \) satisfies the identities (6.2) if and only if the mappings \( \gamma_S : S \times S \to S \) and \( (\alpha_S, \beta_S) : S \to S \times S \) are mutual inverses. In particular, the only finite algebra in \( V \) is the trivial one. Since \( V \) is a Stone variety, it follows that it does not contain any nontrivial profinite algebra. Note also that \( V \) is nontrivial as the Cantor set \( C \) admits a homeomorphism \( C \times C \to C \) and so it has a structure that makes it a member of \( V \).

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