An Approach to Calculate Inverse of LR-Trapezoidal Fuzzy Matrix

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Abstract. In this paper, we devised a numerical algorithm for calculating the inverse of trapezoidal fuzzy numbers matrix. By analysing the existence of fuzzy inverse, the sufficient conditions are restricted. Furthermore, through calculating the approximate fuzzy inverse, the algorithm can be widely used in solving fully fuzzy linear system and it is validated by some given examples.

1. Introduction

Since the fuzzy linear system (FLS) was first introduced in 1998, it has been playing a crucial role in miscellaneous domains such as statistics, life science and astronomy. In dealing with any real applications, it is difficult to completely obtain the all coefficients of these systems. Therefore, one of the approaches that can be done to deal with the uncertainty problem is using fuzzy numbers in linear systems, aiming to change the linear systems to the fuzzy systems. The definition and arithmetic operations of fuzzy numbers were proposed and investigated firstly by Zadeh [1], Nahmias [2] and Dubois et al. [3]. Besides, Goetschel and Ma Ming proposed and improved diverse methods for calculating fuzzy numbers and fuzzy number spaces [4-7].

In 1998, to solve the complex FLS with fuzzy number, a common model was built by Friedman et al. [8]. Left-right FLS (LR-FLS) was used to extend FLS by using the triangular fuzzy numbers. Few studies about solving LR-FLS were investigated by Allahviranloo et al., R. Ghanbari [9]. To associate triangular fuzzy numbers and LR-FLS, a numeric approach using trapezoidal fuzzy numbers was developed by Nasseri [10] in 2011.

As is known to all that the matrix perform an important role in solving crisp linear system. Moreover, we associate that the fuzzy matrix could also be applied in addressing the fully fuzzy linear system (FFLS). Therefore, it will be significant to solve FFLS by investigating of fuzzy inverse matrix. In order to compute the inverse of a fuzzy matrix, Basaran [11] provided novel insights for solving a FFLS by approximating fuzzy inverse matrix. Later, the proposed method was partly corrected and supplied by M. Mosleh and M. Otadi [12]. In recent ten years, Guo et al. [13-18] have done a series of researches on linear fuzzy matrix equation.

In the study, to address calculating the inverse of LR-trapezoidal fuzzy matrix, we present a simple and powerful approach. Furthermore, we give an existence condition used in trapezoidal fuzzy inverse matrix and propose the computing approach.
2. Preliminaries
We provide several definitions as precondition for implying the notion of LR-trapezoidal fuzzy numbers (see [1, 2, 4]).

Definition 2.1 If the membership function of a fuzzy number \( \tilde{M} \) meet the following definition, it is called as a trapezoidal fuzzy number:

\[
\mu_{\tilde{M}} = \begin{cases} 
L \left( \frac{m-x}{\alpha} \right), & x < m, \alpha > 0, \\
1, & m \leq x \leq n, \\
R \left( \frac{x-n}{\beta} \right), & x > n, \beta > 0.
\end{cases}
\]  

(1)

If \( m, n \) are the left and right mean values and \( \alpha, \beta \) are left and right spreads of \( \tilde{M} \), then \( \tilde{M} = (m, n, \alpha, \beta)_{LR} \) is called the LR-trapezoidal fuzzy number. The function \( L(.) \) is called left shape function if the following hold:

1. \( L(x) = L(-x) \),
2. \( L(0) = 1 \) and \( L(1) = 0 \),
3. \( L(x) \) is a non increasing on \([0, +\infty)\).

The right shape function \( R(.) \) definition is similar to the \( L(.) \).

Definition 2.2 If \( \tilde{M} = (m, n, \alpha, \beta) \) satisfy \( m - \alpha > 0(n + \beta < 0) \), we called \( \tilde{M} \) is positive (negative) fuzzy number. If \( \tilde{M} \) satisfies \( m - \alpha < 0 < n + \beta \), then it is a near zero fuzzy number.

Definition 2.3 For arbitrary LR-trapezoidal fuzzy numbers \( \tilde{M}_1 = (m_1, n_1, \alpha_1, \beta_1)_{LR} \) and \( \tilde{M}_2 = (m_2, n_2, \alpha_2, \beta_2)_{LR} \), we have

1. Addition:
   \( \tilde{M}_1 + \tilde{M}_2 = (m_1 + m_2, n_1 + n_2, \alpha_1 + \alpha_2, \beta_1 + \beta_2)_{LR} \).
2. Subtraction:
   \( \tilde{M}_1 - \tilde{M}_2 = (m_1 - m_2, n_1 - n_2, \alpha_1 + \beta_2, \alpha_2 + \beta_1)_{LR} \).
3. Multiplication:
   If \( \tilde{M}_1 > 0 \) and \( \tilde{M}_2 > 0 \), then
   \( \tilde{M}_1 \times \tilde{M}_2 = (m_1 m_2, n_1 n_2, m_1 \alpha_2 + m_2 \alpha_1, n_1 \beta_2 + n_2 \beta_1) \).

3. Solving Fuzzy Inverse
In this section, we investigate the FLS \( \hat{A} \hat{X} = \hat{B} \) which based on LR-trapezoidal fuzzy numbers.

At first, we propose the definition and operation properties of LR-trapezoidal fuzzy number matrix.

Definition 3.1 A fuzzy number is regarded as one fuzzy number when it satisfies the following conditions:

1. \( 1 \) is the mean value;
2. \( \alpha, \beta \) are the left and right spreads \( (0 < \alpha, \beta < 1) \).

Then, the one fuzzy number is denoted by \( \tilde{1} = (1,1,\alpha,\beta) \).

When it satisfies the following conditions, a fuzzy number is regarded as zero fuzzy number:

1. \( 0 \) is the mean value;
2. \( \delta, \lambda \) are the left and right spreads \( (0 < \delta, \lambda < 1) \).

Then, the zero fuzzy number is denoted by \( \tilde{0} = (0,0,\delta,\lambda) \).

Fuzzy identity matrix is consisted of \( \tilde{1} \) as diagonal element and \( \tilde{0} \) as the off-diagonal elements.

Then, it is denoted by \( \tilde{I} \).

\[
\tilde{I} = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}
\]  

(5)
Definition 3.2 Suppose $\tilde{A} = (\tilde{a}_{ij})$ and $\tilde{B} = (\tilde{b}_{ij})$ be two $m \times n$ and $n \times p$ trapezoidal fuzzy matrices where $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$ are trapezoidal fuzzy numbers. We define $\tilde{A} \times \tilde{B} = \tilde{C} = (\tilde{c}_{ij})$ which is the $m \times p$ matrix, where

$$\tilde{c}_{ij} = \sum_{k=1}^{n} \tilde{a}_{ik} \times \tilde{b}_{kj} \quad (6)$$

Definition 3.3 Suppose $\tilde{A}$ and $\tilde{X}$ are trapezoidal fuzzy matrixes. If $\tilde{X}$ meet the given condition, $\tilde{A} \times \tilde{X} = \tilde{X} \times \tilde{A} = \tilde{I}$, it is regarded as the inverse of trapezoidal fuzzy matrix $\tilde{A}$ and we use $\tilde{A}^{-1} = \tilde{X}$ denotes it.

Theorem 3.1 Suppose $\tilde{A} = (A, B, M, N)$ is a nonnegative $n \times n$ trapezoidal fuzzy matrix and $\tilde{I} = (I, I, G, H)$ be a $n \times n$ fuzzy identity matrix. The nonnegative trapezoidal fuzzy inverse matrix of trapezoidal fuzzy matrix $\tilde{A}$ meets the following equation:

$$\tilde{A}^{-1} = \tilde{X} = (X, Y, Z, Q) = (A^{-1}, B^{-1}, A^{-1}(G - M|X|), B^{-1}(H - N|Y|)). \quad (7)$$

Proof. According to the Definition 3.2 and (2.3), we have

$$\tilde{A} \times \tilde{X} = (A, B, M, N) \times (X, Y, Z, Q) = (I, I, G, H).$$

i.e.

$$(A, B, M, N) \times (X, Y, Z, Q) = (AX, BY, AZ + MX, BQ + NY) = (I, I, G, H).$$

It is equivalent to

$$\begin{align*}
AX &= I \\
BY &= I \\
AZ + MX &= G \\
BQ + NY &= H
\end{align*}$$

Considering the conception of LR fuzzy numbers, the left and right spreads are not allowed to be negative. Thus, the both spreads X and Y should use absolute values. Then

$$AZ + M|X| = G, \quad BQ + N|Y| = H,$$

where the $|X|, |Y|$ are the absolute values matrix of X, Y.

By calculation, we obtain

$$\begin{align*}
X &= A^{-1}, \\
Y &= B^{-1}, \\
Z &= A^{-1}(G - M|X|), \\
Q &= B^{-1}(H - N|Y|).
\end{align*}$$

Definition 3.4 Suppose $\tilde{X} = (X, Y, Z, Q)$ denote a solution of $\tilde{A} \times \tilde{X} = \tilde{I}$, such that $Y - X \geq 0$ and $Z \geq 0, Q \geq 0$, $\tilde{X}$ is regarded as a strong LR-trapezoidal fuzzy inverse of fuzzy matrix $\tilde{A}$. Otherwise, $\tilde{X}$ is called a weak LR-trapezoidal fuzzy inverse. So $\tilde{A}^{-1} = \tilde{X} = \tilde{x}_{ij}$ where

$$\tilde{x}_{ij} = \begin{cases}
(x_{ij}, y_{ij}, z_{ij}, q_{ij}), & z_{ij} > 0, q_{ij} > 0, \\
(x_{ij}, y_{ij}, 0, \max\{-z_{ij}, q_{ij}\}), & z_{ij} < 0, q_{ij} > 0, \\
(x_{ij}, y_{ij}, \max\{z_{ij}, -q_{ij}\}, 0), & z_{ij} > 0, q_{ij} < 0, \\
(x_{ij}, y_{ij}, -q_{ij}, -z_{ij}), & z_{ij} < 0, q_{ij} < 0.
\end{cases}$$

If $\tilde{X} = \tilde{x}_{ij}$ is a weak LR-trapezoidal fuzzy inverse and $x_{ij} > y_{ij}$, the left mean value of $x_{ij}$ is $y_{ij}$ and the right mean value is $\frac{x_{ij} + y_{ij}}{2}$.

Finally, we depict sufficient condition used in case when the strong fuzzy solution exists.

Theorem 3.2 Suppose $\tilde{A} = (A, B, M, N)$ and $\tilde{I} = (I, I, G, H)$ are non-negative LR-trapezoidal fuzzy matrix.

The trapezoidal fuzzy matrix $\tilde{A}$ possesses strong LR trapezoidal fuzzy inverse when $A^{-1}, B^{-1}$ are non-negative matrix and $GA \geq 0, HB \geq 0$, respectively.

Proof. When $A^{-1}$ and $B^{-1}$ are non-negative matrix, the facts $X = A^{-1} \geq 0$ and $Y = B^{-1} \geq 0$ are apparent.
Due to $GA \geq 0$ and $HB \geq 0$, we have $Z = A^{-1}(G - MA^{-1}) \geq 0$ and $Q = B^{-1}(H - NB^{-1}) \geq 0$. Thus, the fuzzy matrix $\tilde{A}$ possesses strong LR-trapezoidal fuzzy inverse $\tilde{X} = (X, Y, Z, Q)$.

4. Numerical Applications

Example 4.1 Let the given non-negative fuzzy matrix

$$\tilde{A} = \begin{pmatrix}
(4,6,2,1) & (5,8,1,2) & (3,10,1,1) \\
(5,6,1,1) & (6,10,1,1) & (4,6,2,2) \\
(3,4,1,1) & (4,6,2,1) & (3,6,1,1)
\end{pmatrix}$$

= \begin{pmatrix}
4 & 5 & 3 \\
5 & 6 & 4 \\
3 & 4 & 3
\end{pmatrix}
\begin{pmatrix}
6 & 8 & 10 \\
6 & 10 & 6 \\
4 & 6 & 6
\end{pmatrix}
\begin{pmatrix}
2 & 1 & 1 \\
1 & 1 & 2 \\
1 & 2 & 1
\end{pmatrix}.

Let

$$G = \begin{pmatrix}
0.2 & 0.3 & 0.4 \\
0.1 & 0.3 & 0.2 \\
0.3 & 0.2 & 0.3
\end{pmatrix}, \quad H = \begin{pmatrix}
0.4 & 0.1 & 0.3 \\
0.1 & 0.2 & 0.5 \\
0.3 & 0.2 & 0.3
\end{pmatrix}.$$.

Applying Theorem 3.1, we get

$$X = A^{-1} = \begin{pmatrix}
-2 & 3 & -2 \\
3 & -3 & 1 \\
-2 & 1 & 1
\end{pmatrix},$$

$$Y = B^{-1} = \begin{pmatrix}
3 & 1.5 & -6.5 \\
-1.5 & -0.5 & 3 \\
-0.5 & -0.5 & 1.5
\end{pmatrix}.$$.

And

$$Z = A^{-1}(G - M|X|) = \begin{pmatrix}
10.3 & 15.9 & 6.2 \\
-9.4 & -15.8 & -7.1 \\
-1.0 & 1.9 & 1.7
\end{pmatrix},$$

$$Q = B^{-1}(H - N|Y|) = \begin{pmatrix}
4.15 & 2.05 & 10.45 \\
-2.25 & -1.15 & -5.55 \\
-1.30 & -0.60 & -3.20
\end{pmatrix}.$$.

i.e.

$$(X, Y, Z, Q) = \begin{pmatrix}
(-2 & 3 & -2) \\
(3 & -3 & 1) \\
(-2 & 1 & 1)
\end{pmatrix}
\begin{pmatrix}
(10.3 & 15.9 & 6.2) \\
(-9.4 & -15.8 & -7.1) \\
(-1.0 & 1.9 & 1.7)
\end{pmatrix},$$

$$\tilde{A}^{-1} = \begin{pmatrix}
(-2.3, 10.3, 4.15) & (1.5, 2.25, 15.9, 2.05) & (-6.5, -4.25, 6.2, 10.45) \\
(-1.5, 2.25, 2.25, 9.4) & (-3, -0.5, 1.15, 15.8) & (1.3, 5.55, 7.1) \\
(-2, -0.5, 1.3, 1) & (-0.5, -0.25, 1.9, 0) & (1.15, 1.7, 0)
\end{pmatrix}.$$.

5. Conclusion

In the study, we developed calculating approach which can be used in computing LR-trapezoidal fuzzy inverse. The inverse is obtained by four crisp linear matrix equations. Through given equations, we obtained the left and right mean value and spreads of the trapezoidal fuzzy inverse. We devised a numerical algorithm for calculating the inverse of trapezoidal fuzzy numbers matrix. At the meantime, we give the definition of the sufficient condition when the strong trapezoidal fuzzy inverse exists.
Numerical example exhibits that the method has bright prospects in calculating the LR-trapezoidal fuzzy inverse matrix.

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