NIELSEN REALIZATION FOR SPHERE TWISTS ON 3-MANIFOLDS

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ABSTRACT. For a 3-manifold \( M \), the twist group \( \text{Twist}(M) \) is the subgroup of the mapping class group \( \text{Mod}(M) \) generated by twists about embedded 2-spheres. We study the Nielsen realization problem for subgroups of \( \text{Twist}(M) \). We prove that a nontrivial subgroup \( G \subset \text{Twist}(M) \) is realized by diffeomorphisms if and only if \( G \) is cyclic and \( M \) is a connected sum of lens spaces, including \( S^1 \times S^2 \). We also apply our methods to the Burnside problem for 3-manifolds and show that \( \text{Diff}(M) \) does not contain an infinite torsion group when \( M \) is reducible and not a connected sum of lens spaces.

1. Introduction

The mapping class group \( \text{Mod}(M) \) of a smooth, closed oriented manifold \( M \), is the group of isotopy classes of orientation-preserving diffeomorphisms of \( M \). Denoting \( \text{Diff}^+(M) \) the group of orientation-preserving diffeomorphisms, there is a natural projection map

\[
\pi : \text{Diff}^+(M) \to \text{Mod}(M)
\]

sending a diffeomorphism to its isotopy class. We say that a subgroup \( i : G \hookrightarrow \text{Mod}(M) \) is realizable if there is a homomorphism of \( \rho : G \to \text{Diff}^+(M) \) such that \( \pi \circ \rho = i \).

The Nielsen realization problem asks which finite groups \( G < \text{Mod}(M) \) are realizable. When \( M \) is a surface, every finite subgroup of \( \text{Mod}(M) \) is realizable by work of Kerkchoff [Ker83]. For other manifolds, only sporadic results have been obtained; see [Par21] §2 for a summary of results in dimension 3, [FL21, BK19, Lee22, Kon22] for results in dimension 4, and [FJ90, RW08, BT22] for higher dimensions.

When \( M \) is a 3-manifold, the twist group \( \text{Twist}(M) < \text{Mod}(M) \) is the subgroup generated sphere twists (defined in §2). McCullough [McC90] proved that \( \text{Twist}(M) \cong \mathbb{Z}/2\mathbb{Z} \) for some \( d \geq 0 \). We address the Nielsen realization problem for subgroups of \( \text{Twist}(M) \). This problem was studied by Zimmermann [Zim21] for \( M = \#_k(S^1 \times S^2) \), but there is an error in his argument; see [Zim22]. Nevertheless, using some of the same ideas, we correct the argument, and we generalize from \( \#_k(S^1 \times S^2) \) to all 3-manifolds, giving a precise condition for which subgroups can be realized or not.

To state the main result, recall that for any pair of coprime integers \( p, q \), there is a lens space \( L(p, q) \). Every lens space is covered by \( S^3 \) with the exception of \( L(0, 1) \cong S^1 \times S^2 \).

**Main Theorem.** Fix a closed, oriented 3-manifold \( M \), and fix a nontrivial subgroup \( 1 \neq G < \text{Twist}(M) \). Then \( G \) is realizable if and only if \( G \) is cyclic and \( M \) is diffeomorphic to a connected sum of lens spaces.
Application to the Burnside problem for diffeomorphism groups. Recall that a torsion group is a group where every element has finite order. The existence of finitely-generated, infinite torsion groups, known as Burnside groups, was proved by Golod–Shafarevich [Gol64] [Gv64] and Adian-Novikov [NA68]. E. Ghys and B. Farb asked if the homeomorphism group of a compact manifold can contain a Burnside group; see [Fis11, Question 13.2] and [Fis17, §5]. As an application of the tools used to prove the Main Theorem, we prove the following result on Burnside problem for Diff(M).

**Theorem 1.1.** Let M be a compact oriented 3-manifold. Assume that M is reducible and not a connected sum of lens spaces. Then Diff(M) does not contain a Burnside group.

Smooth vs. topological Nielsen realization. The topological mapping class group \( \text{Mod}_H(M) \) is defined as the group of isotopy classes of orientation-preserving homeomorphisms of M. There is a natural projection map

\[
\text{Homeo}^+(M) \to \text{Mod}_H(M),
\]

and the Nielsen realization problem can also be asked for subgroups of \( \text{Mod}_H(M) \). For 3-manifolds, the smooth and topological mapping class groups coincide \( \text{Mod}(M) \cong \text{Mod}_H(M) \) by Cerf [Cer59], who proved that \( \text{Diff}^+(M) \) and \( \text{Homeo}^+(M) \) are homotopy equivalent. Surprisingly, the realization problem for finite groups is also the same in the topological and smooth categories in dimension 3. For dimension 4, even the connected components are different by work of Ruberman [Rub98]. The work of Baraglia–Konno [BK19] gives a mapping class that can be realized as homeomorphism but not as diffeomorphism and Konno [Kon22] generalizes the results to more manifolds.

**Theorem 1.2** (Pardon, Kirby–Edwards). Let M be a closed oriented 3-manifold. A finite subgroup \( G < \text{Mod}(M) \) is realizable by homeomorphisms if and only if it is realizable by diffeomorphisms.

In particular, this allows us to strengthen the conclusion of the Main Theorem. We emphasize that the group \( G \) in Theorem 1.2 is finite; however, we do not know an example of a 3-manifold \( M \) and an infinite group \( G < \text{Mod}(M) \) that is realizable by homeomorphisms but not diffeomorphisms.

**Proof of Theorem 1.2** Let \( \rho : G \to \text{Homeo}^+(M) \) be a realization of \( G < \text{Mod}(M) \). By Pardon [Par21], \( \rho \) can be approximated uniformly by a smooth action \( \rho' : G \to \text{Diff}^+(M) \). Since \( \text{Homeo}^+(M) \) is locally path-connected by Kirby–Edwards [EK71], we know that \( \rho'(g) \) and \( \rho(g) \) are isotopic in \( \text{Homeo}^+(M) \) for each \( g \in G \), and hence also isotopic in \( \text{Diff}^+(M) \) since \( \text{Mod}_H(M) = \text{Mod}(M) \).

Related work. The following remarks connect the Main Theorem to other previous work.

**Remark 1.3** (Twist group for \( S^1 \times S^2 \)). The group Twist(\( S^1 \times S^2 \)) is isomorphic to \( \mathbb{Z}/2\mathbb{Z} \). We construct a realization of this group in §5. This example seems to be overlooked in some of the literature on finite group actions on geometric 3-manifolds. It is a folklore conjecture of Thurston that any finite group action on a geometric 3-manifold is geometric (i.e. acts isometrically on some geometric structure). It is easy to see that our realization of the twist group, which also appears in work of Tollefson [Tol73], does not preserve any geometric structure on \( S^1 \times S^2 \), so it is a simple counterexample to Thurston’s conjecture.
According to Meeks–Scott [MS86], Thurston proved some cases of his conjecture, but these results were not published. Meeks–Scott [MS86] proved Thurston’s conjecture for manifolds modeled on $\mathbb{H}^2 \times \mathbb{R}$, $\text{SL}_2(\mathbb{R})$, $\text{Nil}$, $\mathbb{E}^3$, and $\text{Sol}$. In [MS86, Thm. 8.4] it is asserted (incorrectly) that Thurston’s conjecture also holds for 3-manifolds modeled on $S^2 \times \mathbb{R}$ (in particular $S^2 \times S^1$); they give an argument, but in the case when some $g \in G$ has positive-dimensional fixed set (as is the case for our realization of $\text{Twist}(S^1 \times S^2)$), they cite a preprint of Thurston that seems to have never appeared.

Remark 1.4 (Sphere twists in dimension 4). For a 4-manifold $W$, for each embedded 2-sphere $S \subset W$ with self-intersection $S \cdot S = -2$, there is a sphere twists $\tau_S \in \text{Mod}(W)$, which has order 2. There are several results known about realizing the subgroup generated by a sphere twist, both positive and negative; see Farb–Looijenga [FL21, Cor. 1.10], Konno [Kon22, Thm. 1.1], and Lee [Lee22, Rmk. 1.7]. It would be interesting to determine precisely when a sphere twist is realizable in dimension 4.

About the proof of the Main Theorem. The proof is divided into two parts: construction and obstruction (corresponding to the “if” and “only if” directions in the theorem statement). For the obstruction part of the argument, we prove the following constraint on group actions on reducible 3-manifolds.

**Theorem 1.5.** Let $M$ be a closed, oriented, reducible 3-manifold. Let $G < \text{Diff}^+(M)$ be a finite subgroup that acts trivially on $\pi_1(M)$. Then $G$ is cyclic. If $G$ is nontrivial, then $M$ is a connected sum of lens spaces.

Section outline. In §2 we recall results about sphere twists and the twist group. In §3 we explain results from minimal surface theory that allow us to decompose a given action into actions on irreducible 3-manifolds. In §4 and 5 we prove the “obstruction” and “construction” parts of the Main Theorem, respectively. In §6 we prove Theorem 1.1.

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2. **Sphere twists and the twist subgroup**

In this section we collect some facts about sphere twists and the group they generate. In §2.1 we recall the definition of sphere twists and recall that they act trivially on $\pi_1(M)$ and $\pi_2(M)$. In §2.2 we give a computation for the twist group of any closed, oriented 3-manifold (Theorem 2.3). The computation can be deduced by combining different results from the literature and gives a precise generating set.

### 2.1. Sphere twists and their action on homotopy groups.

Fix a closed oriented 3-manifold $M$. We recall the definition of a sphere twist. Fix an embedded 2-sphere $S \subset M$ with a tubular neighborhood $U \cong S \times [0, 1] \subset M$, and fix a closed path $\phi : [0, 1] \to \text{SO}(3)$ based at the identity that generates $\pi_1(\text{SO}(3))$. Define a diffeomorphism of $U$ by

$$T_S(x, t) = (\phi(t)(x), t)$$

and extend by the identity to obtain a diffeomorphism $T_S$ of $M$. The isotopy class $\tau_S \in \text{Mod}(M)$ of $T_S$ is called a sphere twist. The twist subgroup of $\text{Mod}(M)$, denoted $\text{Twist}(M)$, is the subgroup generated by all sphere twists.
Lemma 2.1 (Action of sphere twists on homotopy groups). Let $M$ be a closed, oriented 3-manifold with a 2-sided embedded sphere $S \subset M$. Then $\tau_S$ acts trivially on $\pi_1(M)$ and $\pi_2(M)$.

Remark 2.2 (Action on $\pi_1(M)$ vs. $\pi_1(M, \ast)$). When we refer to the action of a diffeomorphism $f \in \text{Diff}(M)$ on $\pi_1(M)$, we mean as an outer automorphism. Technically, this is not an action, but there is a well-defined homomorphism $\text{Diff}(M) \to \text{Out}(\pi_1(M))$. When $f$ has a fixed point $\ast \in M$, the action of $f$ on $\pi_1(M, \ast)$ refers to the induced automorphisms $f_\ast : \pi_1(M, \ast) \to \pi_1(M, \ast)$. This distinction will be important in later sections. If $f$ acts trivially on $\pi_1(M)$ it can be isotoped to $f' \in \text{Diff}(M, \ast)$ that acts on $\pi_1(M, \ast)$ by conjugation (generally nontrivial).

Lemma 2.1 is well-known. That sphere twists act trivially on $\pi_1(M)$ is implicit in [McC90]. This fact may be proved as follows. Let $\ast \in M$ be a fixed point of the diffeomorphism $T_S$ defined in (1). After choosing a prime decomposition of $M$, one can show that each element of $\pi_1(M, \ast)$ is represented by a loop contained entirely in the fixed set of $T_S$. That sphere twists act trivially on $\pi_2(M)$ follows from [McC90, Lem. 1.1]. It can also be deduced from a general result of Laudenbach that says that if $f \in \text{Diff}(M, \ast)$ acts trivially on $\pi_1(M, \ast)$, then it also acts trivially on $\pi_2(M, \ast)$; see [BBP22, Thm. 2.4].

2.2. Generators and relations in the twist group. In this section we compute $\text{Twist}(M)$ for every closed, oriented 3-manifold.

Theorem 2.3. Let $M$ be a closed, orientable 3-manifold with prime decomposition $M = \#_k(S^1 \times S^2)\#P_1\# \cdots \#P_\ell$, where the $P_i$ are irreducible. Let $\ell' \leq \ell$ be the number of the $P_i$ that are lens spaces. Then

\[
\text{Twist}(M) \cong \begin{cases} 
(Z/2Z)^k & \text{if } \ell = \ell' \\
(Z/2Z)^{k+\ell-\ell'}-1 & \text{if } \ell > \ell'.
\end{cases}
\]

We were not able to find Theorem 2.3 in the literature, although it can be deduced by combining various old results.

For the proof of Theorem 2.3 we will use the following explicit construction of $M$. Let $X$ be the complement of $\ell + 2k$ open disks in $S^3$. For each $P_i$ choose a closed embedded disk $D_i \subset P_i$, and let $Y$ be the compact manifold

\[
Y := \left[ \prod_{i=1}^\ell P_i \setminus \text{int}(D_i) \right] \sqcup \left[ \prod_k S^2 \times [-1,1] \right].
\]

We form $M$ by gluing $X$ and $Y$ along their boundary $\partial X \cong \partial Y$. See Figure 1.

![Figure 1](image_url)
Lemma 2.4 (Pants relation). Let $Z$ be the complement in $S^3$ of three disjoint 3-balls. Let $S_1, S_2, S_3$ be the three boundary components of $Z$. Then $\tau_{S_1}\tau_{S_2}\tau_{S_3} = 1$ in $\text{Mod}(Z)$.

Lemma 2.4 can be proved in an elementary fashion by constructing an explicit isotopy. We explain the idea briefly. Fix a properly embedded arc $\alpha \subset Z$ joining the spheres $S_1$ and $S_2$. Let $N$ be a regular neighborhood of $S_1 \cup \alpha \cup S_2$. Observe that $N$ is diffeomorphic to $Z$ and one of its boundary components $S'_3$ is parallel to $S_3$. Using $\alpha$ as a guide, one can isotope $T_{S_1} \circ T_{S_2}$ to a diffeomorphism supported on a collar neighborhood of $S'_3$ and that is a sphere twist on this collar neighborhood. Since $S'_3$ is parallel to $S_3$, this gives the desired relation.

Proof of Theorem 2.3. We prove the theorem in 3 steps.

Step 1: a generating set for $\text{Twist}(M)$. For $i = 1, \ldots, \ell$, fix an embedded sphere $S_i \subset Y$ that is parallel to the boundary component of $P_i \setminus \text{int}(D_i)$. For $j = 1, \ldots, k$, let $S'_j \subset Y$ be the embedded sphere $S^2 \times \{0\}$ in the $j$-th copy of $S^2 \times [0,1]$. Let $\text{Twist}(Y)$ be the subgroup of $\text{Mod}(Y)$ generated by the sphere twists $\{\tau_{S_1}, \ldots, \tau_{S_\ell}\} \cup \{\tau_{S'_1}, \ldots, \tau_{S'_k}\}$. (Recall that the mapping class group $\text{Mod}(Y)$ of a manifold with boundary is the group $\text{Diff}_\partial(Y)$ of diffeomorphisms that restrict to the identity on $\partial Y$, modulo isotopies that are the identity on $\partial Y$.) Consider the composition

$$\mathbb{Z}/2\mathbb{Z})^{\ell+k} \xrightarrow{\rho} \text{Twist}(Y) \xrightarrow{\pi} \text{Twist}(M),$$

where

$$\rho(a_1, \ldots, a_\ell, b_1, \ldots, b_k) = \tau^{a_1}_{S_1} \cdots \tau^{a_\ell}_{S_\ell} \tau^{b_1}_{S'_1} \cdots \tau^{b_k}_{S'_k},$$

and $\pi$ is the restriction of the homomorphism $\text{Mod}(Y) \to \text{Mod}(M)$. The composition $\pi \circ \rho$ is surjective by [McC90, Prop. 1.2]. In the rest of the proof we compute the kernels of $\pi$ and $\rho$.

Step 2: global relation among sphere twists. In this step we compute the kernel of $\pi : \text{Twist}(Y) \to \text{Twist}(M)$.

Let $D \subset X \subset M$ be an embedded ball. Let $\text{Emb}(D, M)$ be the space of embeddings that respect the orientation. Let $\text{Emb}^+_D(X, M)$ be the space of embeddings $X \to M$ that (i) restrict to the inclusion on $D$ and (ii) that extend to a diffeomorphism of $M$. Consider the following diagram, which consists of two fiber sequences (c.f. [Pal60]).

$$\begin{align*}
\text{Diff}_\partial(Y) & \longrightarrow \text{Diff}(M, D) \longrightarrow \text{Emb}^+_D(X, M) \\
& \downarrow \quad \downarrow \\
& \text{Diff}^+(M) \quad \text{Emb}(D, M)
\end{align*}$$

The “horizontal” fiber bundle in the diagram splits as a product by [HM87, Thm. 1]. Consequently, $\text{Mod}(Y) \to \text{Mod}(M, D)$ is injective. Then from the proceeding diagram, we
obtain
\[
\pi_1(\text{Emb}(D, M)) \xrightarrow{\delta} \text{Mod}(Y) \xrightarrow{\delta} \text{Mod}(M, D) \xrightarrow{\delta} \text{Mod}(M)
\]

As is well-known, the space \(\text{Emb}(D, M)\) is homotopy equivalent to the (oriented) frame bundle of \(M\), which is diffeomorphic to \(M \times \text{SO}(3)\) since closed oriented 3-manifolds are parallelizable. Then \(\pi_1(\text{Emb}(D, M)) \cong \pi_1(M) \times \mathbb{Z}/2\mathbb{Z}\). A generator of the \(\mathbb{Z}/2\mathbb{Z}\) factor maps under \(\delta\) to a sphere twist about \(S := \partial D\). From Lemma \ref{lem:2.4} we deduce that \(\tau_S = \tau_{S_1} \cdots \tau_{S_k} \in \text{Mod}(M, D)\), so \(\tau_{S_1} \cdots \tau_{S_k}\) belongs to \(\ker(\pi)\).

We claim that \(\ker(\pi) = \langle \tau_{S_1} \cdots \tau_{S_i} \rangle\). To see that \(\ker(\pi)\) is not larger, it suffices to show that if \(\gamma \in \pi_1(M) < \pi_1(\text{Emb}(D, M))\) is nontrivial, then \(\delta(\gamma)\) is not in the image of \(\text{Twist}(Y) \to \text{Mod}(M, D)\). This is easy to see because each element of \(\text{Twist}(Y) < \text{Mod}(M, D)\) acts trivially on \(\pi_1(M, *)\) (where the basepoint \(*\) belongs to \(D\)), whereas \(\delta(\gamma)\) acts by a nontrivial conjugation on \(\pi_1(M, *)\) (note that the fundamental group of a reducible 3-manifold has trivial center).

**Step 3: local triviality of sphere twists.** Here we compute the kernel of the map \(\rho : (\mathbb{Z}/2\mathbb{Z})^{\ell+k} \to \text{Twist}(Y)\) defined in \([1]\). Since the spheres \(S_1, \ldots, S_k, S'_1, \ldots, S'_l\) belong to distinct components of \(Y\), it suffices to determine which of the given generators for \(\text{Twist}(Y)\) is trivial in \(\text{Mod}(M, D)\). Sphere twists in \(S^2 \times [-1, 1]\) components are nontrivial by \([\text{Lau73}]\). See also \([\text{BBP22}]\) who prove this by considering the action on framings. It remains then to consider when the twists \(\tau_{S_i}\) (about the boundary of \(P_i \setminus \text{int}(D_i)\)) are nontrivial. By Theorem \ref{thm:2.5} below, \(\tau_{S_i} \in \text{Mod}(P_i, D_i) < \text{Mod}(M)\) is trivial if and only if \(P_i\) is a lens space. Combining this with the proceeding steps finishes the proof. \(\square\)

**Theorem 2.5** (Hendriks, Friedman-Witt). Let \(P\) be an irreducible 3-manifold. Fix an embedded ball \(D \subset P\), and let \(\tau_S \in \text{Mod}(P, D)\) be the sphere twist about a sphere \(S\) parallel to \(\partial D\). Then \(\tau_S = 1\) if and only if \(P\) is a lens space.

**Proof.** By work of Hendriks \([\text{Hen77}]\) (see \([\text{FW86}, \text{Cor. 2.1}]\)), the twist \(T_S \in \text{Diff}_\rho(P \setminus \text{int}(D))\) is not homotopic to the identity (rel boundary), unless \(P\) is either a lens space or a prism manifold (the latter are manifolds covered by \(S^3\) whose fundamental group is an extension of a dihedral group).

First consider the case when \(P\) is a lens space. Since we assume \(P\) is irreducible, \(P \neq S^1 \times S^2\), so \(P\) is covered by \(S^3\). In this case \(T_S\) is isotopic to the identity (rel boundary) by \([\text{FW86}, \text{Lem. 3.5}]\). (Aside: \(T_S\) is also isotopic to the identity when \(P = S^1 \times S^2\), which can be seen using Lemma \ref{lem:2.4}.)

When \(P\) is a prism manifold, \(T_S\) is not isotopic to the identity (rel boundary). See \([\text{FW86}, \text{Thm. 2.2}]\). The statement there does not include one family of prism manifolds \(S^3/D_{4m}^*\). This is because the argument uses the (generalized) Smale conjecture, which was not proved for \(S^3/D_{4m}^*\) at the time the paper was written. See \([\text{FW86}, \text{Remark after Corollary 2.2}]\). Fortunately, the generalized Smale conjecture has now been confirmed for all prism manifolds (in fact for all elliptic 3-manifold, with the exception of \(\mathbb{R}P^3\)). See \([\text{HKMR12}]\) and \([\text{BK17}]\). \(\square\)
For later use, we record the following Corollary of Theorem 2.3. For the manifold $S^1 \times S^2$, we call a sphere of the form $* \times S^2$ a belt sphere (we use this terminology because this sphere can be viewed as the belt sphere of a handle attachment).

**Corollary 2.6.** Let $M = \#_k (S^1 \times S^2) \# P_1 \# \cdots \# P_l$, and assume each $P_i$ is a lens space. Then $\text{Twist}(M) \cong (\mathbb{Z}/2\mathbb{Z})^k$ is generated by twists about the belt spheres of the $S^1 \times S^2$ summands.

### 3. Decomposing finite group actions on 3-manifolds

In this section we explain some general structural results for certain finite group actions on 3-manifolds, which will allow us to decompose a $G$-manifold $M^3$ into simpler $G$-invariant pieces. For our application to the Main Theorem we are particularly interested in actions that are trivial on $\pi_i(M)$ for $i = 1, 2$.

#### 3.1. Equivariant sphere theorem.

The main result of this section is Theorem 3.1. In order to state it, we introduce some notation. Let $S$ be a collection of disjoint embedded spheres in a 3-manifold $M$. Define $M_S$ as the result of removing an open regular neighborhood of each $S \in S$ and capping each boundary component with a 3-ball. The 3-manifold $M_S$ is a closed, but usually not connected. This process is pictured in Figure 2.

![Figure 2. Cutting and capping along spheres $M \leadsto M_S$.](image)

**Theorem 3.1.** Let $M$ be a closed oriented 3-manifold and let $G$ be a finite subgroup of $\text{Diff}^+(M)$.

1. There exists a $G$-invariant collection $S$ of disjoint embedded spheres in $M$ such that the components of $M_S$ are irreducible.
2. If $M \neq S^1 \times S^2$ and $G$ acts trivially on $\pi_1(M)$ and $\pi_2(M)$, then $G$ preserves every element in $S$.

We call a collection of spheres as in the statement of Theorem 3.1 a sphere system for $G$.

**Remark 3.2.** Without loss of generality, one can assume that no $S \in S$ bounds a ball in $M$ by removing any sphere that bounds a ball from $S$. Similarly, if $G$ preserves every element of $S$, then we can also assume that no pair $S \neq S' \in S$ bound an embedded $S^2 \times [0, 1]$ in $M$.

Part (1) of Theorem 3.1 is due to Meeks–Yau; see [MY80, c.f. Thm. 7]. An alternate approach was given by Dunwoody [Dun85, Thm. 4.1]. The main tools used in these works are minimal surface theory (in the smooth and PL categories). For the proof of Theorem 3.1(2), we use the following lemmas.
Lemma 3.3. Let $S_0, S_1 \subset M$ be disjoint embedded spheres. If $S_0$ and $S_1$ are ambiently isotopic, then they bound an embedded $S^2 \times [0, 1]$ in $M$.

Lemma 3.3 follows from [Lau73, Lem. 1.2] and the Poincaré conjecture (Laudenbach proves that homotopic spheres bound an $h$-cobordism, and every $h$-cobordism is trivial by Perelman’s resolution of the Poincaré conjecture).

Lemma 3.4. Let $h$ be an orientation-preserving homeomorphism of $S^2 \times [0, 1]$. If $h$ that interchanges the two boundary components, then $h$ acts on $H_2(S^2 \times [0, 1]) \cong \mathbb{Z}$ by $-1$.

Proof of Lemma 3.4. Set $A = S^2 \times [0, 1]$. Consider the arc $\alpha = * \times [0, 1]$ and the sphere $\beta = S^2 \times 0$. After orienting $\alpha$ and $\beta$, we view them as homology classes $\alpha \in H_1(A, \partial A)$ and $\beta \in H_2(A)$, which generate these groups. Since $h$ interchanges the components of $\partial A$, $h(\alpha) = -\alpha$. Since $h$ is orientation-preserving,

$$\alpha \cdot \beta = h(\alpha) \cdot h(\beta) = -\alpha \cdot h(\beta).$$

This implies $h(\beta) = -\beta$ because the intersection pairing $H_1(A, \partial A) \times H_2(A) \to \mathbb{Z}$ is a perfect pairing by Poincaré–Lefschetz duality.

Proof of Theorem 3.1(2). Let $S$ be a $G$-invariant collection of embedded spheres as in Theorem 3.1(1). Fix $S \subset S$ and $g \in G$. We want to show that $g(S) = S$. Suppose for a contradiction that $g(S)$ is disjoint from $S$. Fix an embedding $f : S^2 \to M$ with $f(S^2) = S$. Since $g$ acts trivially on $\pi_2(M)$, the maps $f$ and $g \circ f$ are homotopic, hence isotopic by a result of Laudenbach and the Poincaré conjecture; c.f. [Lau73, Thm. 1].

By Lemma 3.3 the spheres $S$ and $g(S)$ bound a submanifold $A \cong S^2 \times [0, 1]$. Let $k \geq 2$ be the smallest power of $g$ so that $g^k(S) = S$.

First assume $k \geq 3$. Since we can find a $G$-invariant metric on $M$, the interiors of $A$ and $g(A)$ are either equal or disjoint. Since $k \geq 3$, $g^2(S) \neq S$, so $g(A) \neq A$. Consequently, $A \cup g(A)$ is diffeomorphic to $S^2 \times [0, 1]$. Similarly, we conclude that $A \cup g(A) \cup \cdots \cup g^{k-1}(A)$ is diffeomorphic to $S^2 \times S^1$ (this is the only $S^2$-bundle over $S^1$ with orientable total space). This contradicts the assumption that $M \neq S^2 \times S^1$.

Now assume that $k = 2$. As in the preceding paragraph, if $A$ and $g(A)$ have disjoint interiors, then $M = S^2 \times S^1$. Therefore, $g(A) = A$. By Lemma 3.4, $[g(S)] = [S]$ in $H_2(M)$. By assumption that $G$ acts trivially on $\pi_2(M)$, we know $[g(S)] = [S]$. Therefore, $2[S] = 0$ in $H_2(M)$. If $S \subset M$ is non-separating, then there is a closed curve $\gamma \subset M$ so that $[S] : [\gamma] = 1$; this implies that $[S]$ has infinite order in $H_2(M)$, which is a contradiction. If $S \subset M$ is separating, then $M \setminus A$ is a union of two components $M_1 \cup M_2$ that are interchanged by $g$. This contradicts the fact that $G$ acts trivially on $\pi_1(M) \cong \pi_1(M_1) \star \pi_1(M_2)$. \hfill \qed

3.2. Decomposing an action along invariant spheres. Here we explain how we use Theorem 3.1 to decompose an action $G \cow M$ into smaller pieces. We also prove a result about the action on the fundamental group of the pieces under the assumption that $G$ acts trivially on $\pi_1(M)$.

Fix a finite subgroup $G < \text{Diff}^+(M)$ and assume $G$ acts trivially on $\pi_1(M)$. Let $S$ be a sphere system for $G$ (Theorem 3.1). We will always assume that $S$ has the additional properties discussed in Remark 3.2: no $S \in S$ bounds a ball and no pair $S, S' \in S$ bound an embedded $S^2 \times [0, 1]$.

Observe that there is an induced action of $G$ on $M_S$. To construct it, recall a classical result of Brouwer, Eilenberg, and de Kerékjártó [Bro19, dK19, Eh34] that every finite
subgroup of \( \text{Homeo}^+(S^2) \) is conjugate to a finite subgroup of \( \text{SO}(3) \), hence extends from the unit sphere \( S^2 \subset \mathbb{R}^3 \) to the unit ball \( D^3 \subset \mathbb{R}^3 \). In this way the action of \( G \) on \( M \setminus \bigcup_{S \in \mathcal{S}} S \) extends to an action on \( M_S \), which can be made smooth as well.

**Remark 3.5** (global fixed points). Since \( G \) acts trivially on \( S \), then the center of each of the added 3-balls in \( M_S \) contains a global fixed point for the \( G \)-action; we call these *canonical fixed points*. Each component of \( M_S \) contains at least one canonical fixed point.

The following proposition will be important for our proof of the Main Theorem.

**Proposition 3.6.** Fix \( G < \text{Diff}^+(M) \) acting trivially on \( \pi_1(M) \) and fix a \( G \)-invariant collection \( \mathcal{S} \) of disjoint, embedded spheres such that \( M_S \) has irreducible components. Let \( N \) be a component of \( M_S \), and let \( p \in N \) be a canonical fixed point, as defined in Remark 3.5. Then \( G \) acts trivially on \( \pi_1(N,p) \).

**Proof.** Fix \( g \in G \). We show that the action of \( g \) on \( \pi_1(N,p) \) is trivial. The statement is only interesting when \( N \) is not simply connected, so we assume this.

Let \( k \) be the number of elements of \( \mathcal{S} \) that meet \( N \). We separate the argument into the cases \( k = 1 \), \( k = 2 \), and \( k \geq 3 \).

**Case: \( k = 1 \).** Let \( S \in \mathcal{S} \) be the sphere that meets \( N \). The sphere \( S \) is separating and gives a description of \( M \) as a connected sum \( M \cong N \# N' \). Since \( g \in G \) has finite order and preserves \( S \), there is a fixed point \( q \in S^g \). By van Kampen’s theorem, \( \pi_1(M,q) \cong \pi_1(N,q) * \pi_1(N',q) \). By Lemma 2.1, the action of \( g \) on \( \pi_1(M,q) \) is by conjugation by some element \( \pi_1(M,q) \). Since \( g \) preserves the decomposition \( M = N \# N' \), it also preserves the factors in the splitting \( \pi_1(N,q) * \pi_1(N',q) \). Note that \( \pi_1(N',q) \) is nontrivial, since otherwise \( S \) bounds a ball in \( M \), contrary to our assumption. The only conjugation of \( A * A' \) that preserves both \( A \neq 1 \) and \( A' \neq 1 \) is the trivial conjugation, so \( g \) acts trivially on \( \pi_1(M,q) \). Consequently, \( g \) acts trivially on \( \pi_1(N,q) \), and also on \( \pi_1(N,p) \). (In general, changing the basepoint can change the automorphism to a nontrivial conjugation, but in this does not happen here since e.g. the points \( p,q \in N \) are connected by an arc contained in the fixed set \( N^g \).)

**Case: \( k = 2 \).** Let \( S, S' \in \mathcal{S} \) denote the spheres the meet \( N \). Observe that these sphere are either both separating or both nonseparating. Let \( N' \) be the closed 3-manifold such that \( M_S = N \sqcup N' \).

If \( S \) and \( S' \) are both separating, then the argument is similar to the case \( k = 1 \). Apply that argument to either sphere to see that the action is trivial at the corresponding canonical fixed point.

Assume then that both \( S \) and \( S' \) are nonseparating; in particular, this implies that \( N' \) is connected. Then \( M \) is obtained from \( N \sqcup N' \) by removing balls \( B_1, B_2 \subset N \) and \( B_1', B_2' \subset N' \) and gluing \( \partial B_i \) to \( \partial B_i' \). Choose a fixed points \( q \in S^g \) and \( q' \in (S')^g \). There is an isomorphism \( \pi_1(M) \cong \pi_1(N,q) * \pi_1(N',q) * \mathbb{Z} \). (The \( \mathbb{Z} \) factor is not important for this part of the argument, but will play a role when \( k \geq 3 \).)

By assumption \( g \) acts on \( \pi_1(M,q) \) by conjugation and preserves the free factors \( \pi_1(N,q) \) and \( \pi_1(N',q) \). Both of these groups is nontrivial, by our assumption that no two spheres in \( \mathcal{S} \) are parallel. Then as before, we conclude that \( g \) acts trivially on \( \pi_1(M,q) \), hence also on \( \pi_1(N,q) \) and \( \pi_1(N,p) \).

**Case \( k \geq 3 \).** Let \( S_0, \ldots, S_{k-1} \in \mathcal{S} \) denote the spheres that meet \( N \). If some \( S_i \) separates, then we can proceed similar to the case \( k = 1 \), so we can assume each \( S_i \) is nonseparating.
Then $M_S = N \sqcup N'$, where $N'$ is connected, and $M$ is obtained from $N \sqcup N'$ by removing balls $B_0, \ldots, B_{k-1} \subset N$ and $B_0', \ldots, B_{k-1}' \subset N'$ and gluing $B_i$ and $B_i'$ along their boundary (which is identified with $S_i \subset M$). Choose fixed points $q_i \in (S_i)^g$. There is an isomorphism

$$\pi_1(M, q_0) \cong \pi_1(N, q_0) \ast \pi_1(N', q_0) \ast F_{k-1}.$$ 

If $\pi_1(N', q_0) \neq 1$, then we can argue similar to the case $k = 2$. Therefore, we assume that $N'$ is simply connected, which means $\pi_1(M, q_0) \cong \pi_1(N, q_0) \ast F_{k-1}$.

The free group $F_{k-1}$ is generated by loops $\gamma_i = \eta_i \ast \eta_i'$, where $\eta_i$ is a path in $N$ from $q_0$ to $q_i$ (and disjoint from the interiors of the balls $B_0, \ldots, B_{k-1}'$), and $\eta_i'$ is a path in $N'$ from $q_i$ to $q_0$.

On the one hand, $g(\gamma_i) \sim g(\eta_i) \ast \eta_i' \sim g(\eta_i) \ast \eta_i \ast \eta_i' = (g(\eta_i) \ast \eta_i) \ast \gamma_i$, so $g$ acts on $\gamma_i$ by left multiplication by the element $\beta_i = g(\eta_i) \ast \eta_i \in \pi_1(N, q_0)$.

On the other hand, $g$ acts on $\gamma_i$ by conjugation by an element $\alpha \in \pi_1(M, q_0)$. The only way these actions are equal is if both are trivial. If we use the word length on $\pi_1(M, q_0)$ given by the generating set $\{s : s \in \pi_1(N, q_0) \text{ or } s \in F_{k-1}\}$, then the word length of $\alpha \gamma_i \alpha^{-1}$ is odd, but the word length for $\beta_i \gamma_i$ is 2 unless $\beta_i = 1$. This implies that $\beta_i = 1$. Then we know that $\gamma_i = \alpha \gamma_i \alpha^{-1}$ for every $i$, which implies that $\alpha = 1$.

In particular, this implies that $g$ acts trivially on $\pi_1(M, q_0)$ and hence also on $\pi_1(N, q_0)$ and $\pi_1(N, p)$.

4. Obstructing realizations

In this section we prove the “only if” direction of the Main Theorem. This can be deduced quickly from the following more general statements.

**Theorem 4.1.** Let $N$ be a closed, oriented, irreducible 3-manifold with basepoint $p \in N$. Suppose there exists a nontrivial, finite-order element $f \in \text{Diff}^+(N, p)$ that acts trivially on $\pi_1(N, p)$. Then $N$ is a lens space.

**Theorem 4.2.** Let $M$ be a closed, oriented, reducible 3-manifold. Let $G < \text{Diff}^+(M)$ be a finite subgroup that acts trivially on $\pi_1(M)$. Then $G$ is cyclic.

**Proof of Main Theorem: obstruction.** Suppose $1 \neq G < \text{Twist}(M)$ is realizable. The fact that $\text{Twist}(M) \neq 1$ implies that either $M = S^2 \times S^1$ or $M$ is reducible. In the former case, there is nothing to prove, so we assume $M$ is reducible. This assumption together with Lemma 2.1 allow us to apply Theorem 4.2 and conclude that $G$ is cyclic.

It remains to show $M$ is a connected sum of lens spaces, or, equivalently, that each component of $M_S$ is a lens space. This is implied directly by Proposition 3.6 and Theorem 4.1.

Next we use Theorem 4.1 to deduce Theorem 4.2. Then we prove Theorem 4.1.

**Proof of Theorem 4.2.** Let $S$ be a sphere system for $G$ (Theorem 3.1). We also assume that no $S \in S$ bounds a ball and that no two spheres $S, S' \in S$ bound an embedded $S^2 \times [0, 1]$ (Remark 3.2).

Since $G$ acts trivially on $\pi_1(M)$, Proposition 3.6 and Theorem 4.1 combine to show that each component of $M_S$ is a lens space.

---

1Here the symbol $\sim$ indicates homotopic loops based at $q_0$. For the first homotopy, note that the paths $g(\eta_i')$ and $\eta_i'$ are homotopic rel endpoints because $N'$ is simply connected.
Suppose that there exists a component \( N \) of \( M_3 \) that is diffeomorphic to \( S^3 \). Let \( k \) be the number of elements of \( S \) that meet \( N \). If \( k = 1 \), then \( N \) is obtained from \( M \) by cutting along a sphere that bounds a ball. Similarly, if \( k = 2 \), then \( N \) is obtained by cutting \( M \) along two parallel spheres. Both of these are contrary to our assumption about \( S \). Therefore \( k \geq 3 \), which implies that \( N^G \) has at least 3 points. By the Smith conjecture [Mor83], the action of \( G \) on \( N \cong S^3 \) is conjugate into \( SO(4) \), and the fact that \( |N^G| \geq 3 \) implies that \( G \) is conjugate into \( SO(2) \). Therefore \( G \) is cyclic.

The remaining case is that every component of \( M_3 \) is a lens space different from \( S^3 \) and \( S^1 \times S^2 \). In this case \( \text{Twist}(M) \) is the trivial group by Corollary 2.6. □

We proceed to the proof of Theorem 4.1. Our argument is inspired by an argument of Borel [Bor83] that shows that a finite group \( G \) acting faithfully on a closed aspherical manifold \( N \) and \( \pi_1(N) \) has trivial center, then \( G \) also acts faithfully on \( \pi_1(N) \) (by outer automorphisms).

Before starting the proof, we recall some facts about lifting actions to universal covers. Let \( N \) be a closed manifold. Recall that \( \tilde{N} \) can be defined as the set of paths \( \alpha : [0,1] \to N \) with \( \alpha(0) = * \), up to homotopy rel endpoints. Using this description, there is a left action \( \pi_1(N,*) \times \tilde{N} \to \tilde{N} \) given by pre-concatenation of paths \([\gamma], [\alpha] = [\gamma * \alpha]\), and there is a left action

\[
\text{Diff}(N,*) \times \tilde{N} \to \tilde{N}
\]
given by post-composition \( f.[\alpha] = [f \circ \alpha] \). If \( f \in \text{Diff}(N,*) \) acts trivially on \( \pi_1(N,*) \), then \( F([\alpha]) = [f \circ \alpha] \) is a lift of \( f \) that commutes with the deck group action and fixes the homotopy class of the constant path (as well as every other homotopy class corresponding to an element of \( \pi_1(N,*) \)).

**Proof of Theorem 4.1.** As observed above, we can lift \( f \) to a finite-order diffeomorphism \( F \) that commutes with the deck group \( \pi_1(N,*) \) and has a global fixed point.

First we show that \( \pi_1(N) \) is finite. Suppose for a contradiction that \( \pi_1(N) \) is infinite. This implies \( \tilde{N} \) is contractible. By Smith theory [Smi34], the fixed set \((\tilde{N})^F \) is connected, and simply connected. Since \( F \) acts smoothly, \((\tilde{N})^F \) is a smooth 1-dimensional manifold, hence it is homeomorphic to \( \mathbb{R} \). Since \( \pi_1(N) \) commutes with \( F \), it acts on \((\tilde{N})^F \cong \mathbb{R} \), and this action is free and properly discontinuous since the action of \( \pi_1(N,*) \) on \( \tilde{N} \) has these properties. This implies that \( \pi_1(N,*) \cong \mathbb{Z} \), which contradicts the fact that \( N \) is a closed, aspherical 3-manifold (\( \mathbb{Z} \) is not a 3-dimensional Poincaré duality group).

Since \( \pi_1(N) \) is finite, its universal cover is diffeomorphic to \( S^3 \) by the Poincaré-conjecture. As in the preceding paragraph, consider the action of \( F \) on \( \tilde{N} \cong S^3 \). By Smith theory and smoothness of the action, the fixed set is a smooth, connected 1-dimensional manifold with nontrivial fundamental group. Hence \((\tilde{N})^F \cong S^1 \). Since \( \pi_1(\tilde{N}) \) acts freely on \((\tilde{N})^F \) this implies \( \pi_1(N) \) is cyclic, which implies that \( N \) is a lens space. □

5. **Constructing realizations**

In this section we prove the “if” direction of the Main Theorem. We state this as the following theorem.

**Theorem 5.1.** Let \( M \) be a connected sum of lens spaces. Then every cyclic subgroup of \( \text{Twist}(M) \) is realizable.

\[\text{By Hurewicz, } \pi_3(\tilde{N}) \cong H_3(\tilde{N}). \text{ Since } \pi_1(N) \text{ is infinite, } \tilde{N} \text{ is noncompact, so } H_3(\tilde{N}) = 0. \text{ Similarly, all higher homotopy groups vanish by Hurewicz’s theorem.} \]
Fix $M$ as in Theorem 5.1 and write the prime decomposition

$$M = \#_k(S^1 \times S^2) \# P_1 \# \cdots \# P_t,$$

where each $P_i$ is a lens space different from $L(0,1) \equiv S^1 \times S^2$.

To prove Theorem 5.1 given a nontrivial element $g \in \text{Twist}(M)$ we define $\gamma \in \text{Diff}(M)$ such that $\gamma^2 = id$ and $[\gamma] = g$ in $\text{Mod}(M)$. The basic approach is to define an order-2 diffeomorphism of

$$\sqcup_k (S^1 \times S^2) \sqcup P_1 \sqcup \cdots \sqcup P_t$$

in such a way that the diffeomorphisms on the components can be glued to give an order-2 diffeomorphism of $M$. On each component of (4) we perform one of the following diffeomorphisms.

- (constant $\pi$ rotation) Define $R_0 : S^1 \times S^2 \to S^1 \times S^2$ by $id \times r$, where $r : S^2 \to S^2$ is any $\pi$ rotation (choose one – the particular axis is not important).
- (nonconstant $\pi$ rotation) Let $c : [0,1] \to \mathbb{R}P^2$ be a closed path that generates $\pi_1(\mathbb{R}P^2)$, and let $\alpha : \mathbb{R}P^2 \to \text{SO}(3)$ be the map that sends $\ell \in \mathbb{R}P^2$ to the $\pi$-rotation whose axis is $\ell$. Now define

$$R_1 : S^1 \times S^2 \to S^1 \times S^2$$

by $(t,x) \mapsto (t,\alpha(c(t))(x))$.

Since $\alpha \circ c : [0,1] \to \text{SO}(3)$ defines a nontrivial element of $\pi_1(\text{SO}(3))$, the diffeomorphism $R_1$ represents the generator of $\text{Twist}(S^1 \times S^2) \cong \mathbb{Z}/2\mathbb{Z}$. This shows that $\text{Twist}(S^1 \times S^2)$ is realized. This involution appears in [Tol73 §1] in a slightly different form.

- (lens space rotation) Fix $p,q$ relatively prime and with $p \geq 2$. View $L(p,q)$ as the quotient of $S^3 \subset \mathbb{C}^2$ by the $\mathbb{Z}/p\mathbb{Z}$ action generated by $(z,w) \mapsto (e^{2\pi i/p}z,e^{2\pi iq/p}w)$. Define

$$R_{p,q} : L(p,q) \to L(p,q)$$

as the involution induced by $(z,w) \mapsto (z,-w)$ on $S^3$ (which descends to $L(p,q)$ since it commutes with the $\mathbb{Z}/p\mathbb{Z}$ action).

Each of the diffeomorphisms $R_0$, $R_1$, and $R_{p,q}$ has 1-dimensional fixed set. The representation in the normal direction at a fixed point is the antipodal map on $\mathbb{R}^2$ (there is no other option since these diffeomorphisms are involutions). Lemma 5.2 below allows us to glue these actions along their fixed sets.

**Lemma 5.2.** Suppose $M,M'$ are oriented manifolds, each with a smooth action of a finite group $G$. Assume that $x \in M$ and $x' \in M'$ are fixed points of $G$, and that the representations $T_x M$ and $T_{x'} M'$ are isomorphic by an orientation reversing map. Then $M$ and $M'$ can be glued along regular neighborhoods $B$ and $B'$ of $x$ and $x'$ so that there is a smooth action of $G$ on $M \# M'$ that restricts to the given action on $M \setminus B$ and $M' \setminus B'$. □

**Remark 5.3.** The condition that the isomorphism $T_x M \cong T_{x'} M'$ be orientation-reversing appears because the connected sum of two oriented manifolds is defined by deleting an open ball from each and identifying the boundaries of these balls by an orientation-reversing diffeomorphism. This condition is always satisfied if each tangent space contains a copy of the trivial representation (choose an appropriate reflection).
Remark 5.4 (Useful isotopies). To prove that $\gamma \in \text{Diff}(M)$ is in the isotopy class of $g \in \text{Twist}(M)$, the following observation will be useful. The fixed set of $R_{p,q}$ acting on $L(p,q)$ contains the image $C$ of the circle $\{(z,0) : |z| = 1\} \subset S^3$. The isotopy $h_t(z,w) = (z,e^{\pi(1-t)}w)$, $0 \leq t \leq 1$, descends to $L(p,q)$ to give an isotopy between $R_{p,q}$ and the identity, and $h_t$ fixes $C$ for each $t$.

Similarly, it’s possible to isotope $R_1$ to $R'_1$, which is a constant $\pi$-rotation (say about the $z$-axis) on a neighborhood of $* \times S^2$, for some fixed basepoint $* \in S^1$ (observe that $R'_1$ is still an involution). Furthermore, we can isotope $R'_1$ to a diffeomorphism that is the identity near $* \times S^2$ and in such a way that the isotopy at time $t \in [0,1]$ is a rotation by angle $\pi(1-t)$ (about the $z$-axis) on each sphere in a regular neighborhood of $* \times S^2$. The fixed set restricted to a neighborhood of $* \times S^2$ remains constant during this isotopy.

Finally, we can isotope $R_0$ to the identity so that at time $t$ the diffeomorphism is a constant rotation by angle $\pi(1-t)$ (about the fixed axis).

On a neighborhood of a fixed point, the local picture of the isotopies of $R_{p,q}$, $R'_1$, and $R_0$ looks the same. This will allow us to perform these isotopies equivariantly on connected sums.

We proceed now to the proof of Theorem 5.1. First we warm up with the case $M = \#_k(S^1 \times S^2)$ and then we do the general case.

5.1. Realizations for connected sums of $S^1 \times S^2$. Fix $k \geq 1$ and consider

$$M_k := \#_k(S^1 \times S^2).$$

Let $S_i$ be a belt sphere in the $i$-th connect summand, and denote the sphere twist about $S_i$ by $\tau_i$. The twists $\tau_1, \ldots, \tau_k$ form a basis for $\text{Twist}(M_k) \cong (\mathbb{Z}/2\mathbb{Z})^k$, c.f. Theorem 2.3.

Fix a nonzero element

$$g = a_1\tau_1 + \cdots + a_k\tau_k$$

in $\text{Twist}(M_k)$. We start by defining an involution $\hat{\gamma}$ of $\sqcup_k(S^1 \times S^2)$. For ease of exposition, let $W_i = S^1 \times S^2$ denote the $i$-th component of $\sqcup_k(S^1 \times S^2)$. Define $\hat{\gamma}$ on $W_i$ to be $R_0$ or $R_1$, depending on whether the coefficient $a_i$ is 0 or 1, respectively. Next we glue using Lemma 5.2 to obtain an involution $\gamma$ of $M_k = W_1 \# \cdots \# W_k \cong \#_k(S^1 \times S^2)$. There are multiple ways to describe the gluing; for example, choose $k-1$ distinct fixed points $x_1, \ldots, x_{k-1} \in W_k$, and for $1 \leq i \leq k-1$, glue $W_i$ to $W_k$ along regular (equivariant) neighborhoods of $x_i$ and an arbitrary fixed point $y_i \in W_i$ (the neighborhoods of $x_1, \ldots, x_k$ should be chosen to be small enough so that they are disjoint).

To see that $\gamma \in \text{Diff}(M_k)$ is in the isotopy class $g$, recall the short exact sequence of Laudenbach

$$1 \to \text{Twist}(M_k) \to \text{Mod}(M_k) \to \text{Out}(\pi_1(M_k)) \to 1.$$ 

It’s easy to check that $\gamma$ acts trivially on $\pi_1(M_k)$, so $\gamma$ represents a mapping class in $\text{Twist}(M_k)$. The particular isotopy class is determined by the action on trivializations of the tangent bundle of $M_k$, and in this way one can check that $[\gamma] = g$ in $\text{Twist}(M_k)$. We do not spell out the details of this because we give an alternate argument in the next section in the general case.

Remark 5.5. We cannot realize a non-cyclic subgroup of $\text{Twist}(M_n)$ using this construction because it is not possible to choose the axis for $R_0$ so that (1) $R_0$ and $R_1$ have a common fixed point and (2) $R_0$ and $R_1$ commute. Indeed, §4 proves no non-cyclic subgroup of $\text{Twist}(M_n)$ is realized.

\footnote{It’s possible that the fixed set is larger (this is true for $L(2,1) \cong \mathbb{RP}^3$), but this is not important.}
5.2. Realizations for connected sum of lens spaces. Now we treat the general case 
\[ M = \#_k(S^1 \times S^2)\#L(p_1, q_1)\# \cdots \#L(p_\ell, q_\ell), \]
where each \( L(p_j, q_j) \) is a lens space different from \( L(0, 1) \cong S^1 \times S^2 \). Our approach is similar to the preceding section.

Recall from Corollary 2.6 that \( \text{Twist}(M) \cong (\mathbb{Z}/2\mathbb{Z})^k \) is generated by twists \( \tau_1, \ldots, \tau_k \) in the belt spheres of the \( S^1 \times S^2 \) summands.

Fix a nonzero element 
\[ g = a_1\tau_1 + \cdots + a_k\tau_k \]
in \( \text{Twist}(M) \). We start by defining an involution \( \hat{\gamma} \) of 
\[ \sqcup_k(S^1 \times S^2) \sqcup L(p_1, q_1) \sqcup \cdots \sqcup L(p_\ell, q_\ell). \]
Let \( W_i \) denote the \( i \)-th component diffeomorphic to \( S^1 \times S^2 \). Define \( \hat{\gamma} \) on \( L(p_j, q_j) \) to be \( R_{p_j, q_j} \), and on \( W_i \) to be \( R_0 \) or \( R'_1 \), depending on whether the coefficient \( a_i \) is 0 or 1, respectively. (Recall that \( R'_1 \) is similar to \( R_1 \), but it has a product region.)

Next we glue using Lemma 5.2 to obtain an involution \( \gamma \) of \( M \). We glue by the following pattern. First we glue \( W_1, \ldots, W_k \). Choose \( k - 1 \) distinct fixed points \( x_1, \ldots, x_{k-1} \in W_k \), and for \( 1 \leq i \leq k - 1 \), glue \( W_i \) to \( W_k \) along regular (equivariant) neighborhoods of \( x_i \) and an arbitrary fixed point \( y_i \in W_i \) (as was done in the preceding section). Next glue \( L(p_j, q_j) \) to \( W_k \) in a region where \( \hat{\gamma} \) acts as a product (we can choose \( x_1, \ldots, x_{k-1} \) and the regular neighborhoods of these points to ensure that there is room to do this). In this way we obtain an involution \( \gamma \in \text{Diff}(M) \).

We need to check that \( \gamma \) is in the isotopy class of \( g \). Using the isotopies defined in Remark 5.4, we can isotope \( \hat{\gamma} \) to a map that is the identity on each \( L(p_j, q_j) \) component and each component \( W_i \) such that \( a_i = 0 \), and is a sphere twist on each component \( W_i \) such that \( a_i = 1 \). By construction these isotopies glue to give an isotopy of \( \gamma \) to a product of sphere twists representing \( g \).

This completes the proof of Theorem 5.1. \( \square \)

Question 5.6. Are any two realizations of \( g \in \text{Twist}(M) \) conjugate in \( \text{Diff}(M) \)?

6. Burnside Problem for \( \text{Diff}(M) \) for reducible 3-manifold \( M \)

In this section, we prove Theorem 1.1 which follows quickly from Lemma 6.1.

Lemma 6.1. Fix a closed, oriented 3-manifold \( M \), and consider the group 
\[ K := \ker \left[ \text{Diff}(M) \to \text{Out}(\pi_1(M)) \right]. \]
If \( M \) is reducible and that \( M \) is not a connected sum of lens spaces, then \( K \) is torsion free.

Remark 6.2 (A strong converse to Lemma 6.1). If \( M \) is a connected sum of lens spaces, then \( M \) has a faithful \( S^1 \)-action, so \( K \) contains \( S^1 \) as a subgroup. To see this, observe that each lens space has an \( S^1 \)-action with global fixed points, so by performing the connected sum equivariantly along fixed points (similar to the construction in Section 5) we obtain an \( S^1 \)-action on \( M \).

Proof of Theorem 1.1. If \( \text{Diff}(M) \) has an infinite torsion group \( G \), then either the image of \( \Phi \) is an infinite torsion group or the kernel \( K \) of \( \Phi \) has infinite torsion group. By [HM13, Thm. 5.2], \( \text{Out}(\pi_1(M)) \) contains a torsion-free finite-index subgroup. This implies that \( \Phi(G) \) is finite, since every element of \( \Phi(G) \) has finite order. On the other hand, by Lemma 6.1 that \( K \) is torsion free, so \( G \cong \Phi(G) \) is finite. \( \square \)
Proof of Lemma 6.1. Fix a nontrivial subgroup $G = \mathbb{Z}/p\mathbb{Z} \leq K$, and fix a sphere system $S$ for $G$ (Theorem 3.1). By Proposition 3.6, the action of $G$ on each component $N$ of $M_S$ is trivial on $\pi_1(N,p)$ as an automorphism. This implies that each component $N$ of $M_S$ is a lens space by Theorem 4.1. □

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