Non-commutative NLS hierarchies: dressing, solutions & time-like boundaries

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Abstract

We consider the generalized matrix non-linear Schrödinger hierarchy. By employing the universal Darboux-dressing scheme we derive solutions for the hierarchy of integrable PDEs via solutions of the matrix Gelfand-Levitan-Marchenko equation, and we also identify recursion relations that yield the Lax pairs for the whole matrix NLS hierarchy. These results are obtained considering either matrix-integral or general $n^{th}$ order matrix-differential operators as Darboux-dressing transformations. In this framework special links with the Airy and Burgers equations are also discussed. The matrix version of the Darboux transform is also examined leading to the non-commutative version of the Riccati equation. The non-commutative Riccati equation is solved and hence suitable conserved quantities are derived. In this context we also discuss the infinite dimensional case of the NLS matrix model as it provides a suitable candidate for a quantum version of the usual NLS model. We then focus on the time-like version of the familiar NLS model and we extend the notion of space-time dualities in the presence of integrable time-like boundary conditions. We derive the associated time-like “conserved” quantities and Lax pairs as well as the corresponding boundary conditions.

1 Introduction

The non-linear Schrödinger (NLS) model and its generalizations are among the most well studied prototypical integrable systems (see for instance [1]–[9] and references therein). Both continuum and discrete versions have been widely studied from the point of view of the inverse scattering method or the Darboux and Zakharov-Shabat (ZS) dressing methods [10]–[15]. Within the ZS scheme [12] solutions of integrable non-linear PDEs can be obtained by means of solutions of the associated linear problem, and also the hierarchy of Lax operators can be explicitly constructed. Similarly, a large amount of studies exists from the algebraic/Hamiltonian standpoint [13] in the case of periodic as well as generic integrable boundary conditions (see e.g. [16] [8]). In the algebraic scheme the hierarchy of Lax pairs can
also be constructed via the universal formula based on the existence of a classical $\tau$-matrix \cite{17}. This fundamental formula was extended to the case of integrable boundary conditions \cite{18}, as well as at the quantum level \cite{19}.

We consider in this study a generalized matrix version of the NLS model and we focus on three main objectives: 1) We investigate solutions of the generalized matrix NLS hierarchy based on the generic notion of Darboux-dressing transformation. We explicitly derive recursion relations that yield the Lax pairs for the whole hierarchy, and we also obtain generalized solutions by employing both discrete and continuous solutions of the associated linear problem and solving the matrix (non-commutative) Gelfand-Levitan-Marchenko (GLM) equation. Solutions of the generalized mKdV model in particular, a member of the hierarchy under study, are derived and expressed in terms of Airy functions, generalizing the results in \cite{20}. 2) We study the non-commutative/quantum Riccati flows for the matrix NLS hierarchy. In particular, we derive and solve the non-commutative Riccati equation and hence we obtain the auxiliary function and relevant non-commutative conserved quantities. In this context the matrix NLS model can be seen as a non-commutative version of the familiar NLS model, and be further upgraded to the quantum version of the usual NLS model provided that suitable commutation relations are imposed among the non-commutative fields. It is worth noting that relevant studies on the formulation of the quantum GLM equation for the NLS model were discussed in \cite{21}. 3) We introduce time-like boundary conditions for the NLS model extending the notion of the space-time dualities introduced in \cite{22, 23}. Our study is based on the Hamiltonian formulation, the existence of a classical $\tau$-matrix and the underlying classical reflection algebra.

Let us summarize below what is achieved in this investigation. In the first part of this study (sections 2 and 3) we implement the generalized Darboux-dressing method in order to derive solutions as well as recursion relations that produce the Lax pairs for the whole hierarchy. The Darboux transform in our frame is chosen to be either an integral or a differential operator. When choosing the Darboux transform to be an integral operator we basically apply the Zakharov-Shabat (ZS) dressing method as described in \cite{12}. One of the main advantages of the ZS dressing method is that no analyticity conditions are \textit{a priori} required for the solutions of the corresponding scattering problem, as is the case when applying the inverse scattering transform, thus this method provides essentially a linearization of non-linear ODEs and PDEs. Here, based on the universal Darboux-dressing scheme we produce solitonic solutions as well as formal expressions for generic solutions via the Fredholm theory (see also \cite{20}), and also derive sets of recursion relations that yield the Lax pairs for the integrable hierarchy at hand. Moreover, by exploiting the cubic dispersion relations for the third member of the hierarchy, i.e. the generalized matrix mKdV model, we are able to express the formal solutions in terms of Airy functions, and show that solutions of the corresponding linear problem satisfy the Airy equation, which is known to be a linearization of Painlevé II. Explicit expressions of the dressed Lax pairs for the first few members of the hierarchy are also reported. The viscous and inviscid matrix Burgers equations are also derived as special cases in our setting.

In section 4 we investigate non-commutative/quantum Riccati flows associated to the matrix NLS hierarchy; these are also directly related to the notion of Grassmannian flows studied in the context of non-local, non-linear PDEs in \cite{24, 25}. We solve the non-commutative
Riccati equation and subsequently derive suitable conserved quantities based on the solution of both time and space Riccati flows. A brief discussion on the relevance of the quantum version of NLS with our findings is also presented. The findings of this section are closely related to the results presented in [19] regarding the ideas on the quantum auxiliary linear problem and the quantum Darboux-Bäcklund transforms. Note that in [19] a certain version of the quantum discrete NLS model was studied, i.e. the quantum Ablowitz-Ladik model, so comparisons with the present findings can be made.

In section 5 we focus on the familiar NLS model and we generalize the idea of space-time dualities studied in [22, 23] in the case of generic time-like integrable boundary conditions. To achieve this we implement the Hamiltonian description based on the existence of the classical $r$-matrix, providing the underlying Poisson structure. As argued in [23] the time-like Poisson structure gives rise to an ultra-local Poisson algebra for the $t$ component of the Lax pair, with the same classical $r$-matrix as the ultra-local Poisson algebra satisfied by the $x$ component with respect to the usual Poisson bracket. We first briefly review the results found in [23] for integrable dual periodic systems by introducing time-like Poisson algebras. We then move on to the case of generic integrable boundary conditions, and based on the fundamental algebraic relations $(t$-Poisson) we extend the idea of Sklyanin’s modified monodromy [16] along the time axis. We produce novel results regarding time-like integrable boundary conditions via the derivation of the respective “conserved” quantities and the $x$-part of the Lax pairs in the presence of general open boundaries.

## 2 Dressing transformations as integral operators

The main aim in this section is the derivation of solutions as well as the construction of the Lax pairs of all the members of the matrix NLS hierarchy. The key idea is that solutions of the non-linear integrable PDEs are obtained via the matrix Gelfand-Levitan-Marchenko equation, which will be derived below, by employing solutions of the associated linear problem. We review now the main ideas of the generic Darboux-dressing scheme [12]. Let

$$D^{(n)} = \partial_{tn} + A^{(n)}, \quad L^{(n)} = \partial_{tn} + A^{(n)},$$

(2.1)

where $A^{(n)}$, $\hat{A}^{(n)}$ are some “bare” and “dressed” operators respectively related via the generic Darboux-dressing transformation

$$G \ D^{(n)} = L^{(n)} G \Rightarrow \partial_{tn} G = G A^{(n)} - \hat{A}^{(n)} G.$$ 

(2.2)

It is worth noting that this fundamental idea was generalized in [24, 25] and led to the construction of non-local non-linear PDEs as well as to their solution via solutions of the associated linear problem.

Given that $[D^{(n)}, D^{(m)}] = 0$ the latter transformation (2.2) leads to the generalized Zakharov-Shabat zero curvature relations, which define the integrable hierarchy

$$\partial_{tn} \hat{A}^{(m)} - \partial_{tm} \hat{A}^{(n)} + [A^{(n)}, \hat{A}^{(m)}] = 0.$$ 

(2.3)

In general, $A^{(n)}$, $\hat{A}^{(n)}$ can be matrix, differential or integral operators, but in our setting here we choose to consider them as matrix-differential operators, whereas the dressing transform $G$ is chosen to be either a matrix-integral operator or a matrix-differential one.
Let us introduce the necessary notation in the case of the Darboux transform $G$ as an integral operator. We now derive the matrix GLM equation from the fundamental factorization condition. Let $G = I + K^+$, and introduce the operators $K^\pm$, $F$ with integral representations ($f$ is the test function, an $N$-column vector in general):

\[
F(f)(x) = \int_R F(x,y)f(y) \, dy,
\]

\[
K^\pm(f)(x) = \int_R K^\pm(x,y)f(y) \, dy,
\]

(2.4)
such that: $K^+(x,y) = 0$, $x > y$, and $K^-(x,y) = 0$, $x < y$. The operators $K^\pm$, $F$ are required to satisfy the factorization condition

\[
(\mathbb{I} + K^+) (\mathbb{I} + F) = \mathbb{I} + K^-,
\]

(2.5)
which leads to the fact that the kernel $K^+(x,y)$ satisfies the Gelfand-Levitan-Marchenko equation and $K^-(x,y)$ obeys an analogous integral equation:

\[
K^+(x,z) + F(x,z) + \int_x^\infty dy K^+(x,y)F(y,z) = 0, \quad z > x,
\]

\[
K^-(x,z) = F(x,z) + \int_x^\infty dy K^+(x,y)F(y,z), \quad z < x.
\]

(2.6)

$F(x,y)$ is the solution of the linear problem, i.e. invariance of the differential operators $D^{(n)}$ under the action of the operator $F$ is required:

\[
F D^{(n)} = D^{(n)} F.
\]

(2.7)
Dependence on the universal time $t$ that includes all the time flows $t_n$ is implied but omitted for now for simplicity. Note that the factorization condition (2.5) is nothing but the analogue of the Darboux-dressing transformation acting on the linear solution $F$ and providing the transformed solution $K^-$. We shall address this issue again when dealing with the auxiliary matrix problem in the matrix language. The kernel $K^+$, as will be transparent in the following, is the quantity that produces solutions of the integrable PDEs emerging from the zero curvature condition. One may think of the GLM equation as a necessary intermediate step between the linear problem and the non-linear integrable PDE.

We focus now on the sets of operators associated to the generalized matrix NLS model:

\[
D^{(0)} = W\partial_x, \quad D^{(1)} = \partial_t - W\partial_x, \quad D^{(n)} = (\partial_{t_n} - \partial_x^n)\mathbb{I}, \quad n > 1,
\]

(2.8)
\[
L^{(0)} = W\partial_x + U(x), \quad L^{(1)} = \partial_t \mathbb{I} - W\partial_x + a(x), \quad L^{(n)} = (\partial_{t_n} - \partial_x^n)\mathbb{I} + \sum_{k=0}^{n-1} a_k(x)\partial_x^k.
\]

(2.9)
$I$ is the $\mathcal{N} \times \mathcal{N}$ unit matrix ($\mathcal{N} + M = \mathcal{N}$),

\[
W = \begin{pmatrix} w_1 \mathbb{I}_{\mathcal{N} \times \mathcal{N}} & 0_{\mathcal{N} \times M} \\ 0_{M \times \mathcal{N}} & w_2 \mathbb{I}_{M \times M} \end{pmatrix}.
\]

(2.10)
and the quantities $U(x)$, $a(x)$, $a_x(x)$ are going to be identified via the dressing process.
In general, $W$ can be any $N \times N$ matrix, but we focus henceforth on the case where it is given in the block form above. Different choices of the $W$ matrix will naturally give rise to models with distinct underlying symmetries. The linear PDEs \[ W \partial_x F(x, y) + \partial_y F(x, y)W = 0, \]
\[ \partial_{t_n} F(x, y) - \partial_x^F(x, y) + (-1)^n \partial_y^F(x, y) = 0, \] (2.11)
the dependence of the functions on time $t_n$ is omitted for brevity. Let us now express the solutions of the linear problem for the matrix NLS as
\[ F(x, z) = \begin{pmatrix} 0_{N \times N} & f_{N \times M}(x, y) \\ \hat{f}_{M \times N}(x, y) & 0_{M \times M} \end{pmatrix} \] (2.12).
This will be used subsequently for obtaining solutions of the GLM equation and hence solutions for the tower of integrable non-linear PDEs of the matrix NLS hierarchy.

2.1 Solutions of the matrix GLM equation

The first step is the derivation of solutions of the GLM equation by means of the linear solutions above (2.12). Let the matrix kernel $K^+$ be expressed as
\[ K^+(x, y) = \begin{pmatrix} \mathbb{A}_{N \times N}(x, y) & \mathbb{B}_{N \times M}(x, y) \\ \mathbb{C}_{M \times N}(x, y) & \mathbb{D}_{M \times M}(x, y) \end{pmatrix}. \] (2.13)
Inserting the matrix expressions \[ \mathbb{A}, \mathbb{B} \] and \[ \mathbb{C}, \mathbb{D} \] into the GLM equation we obtain two independent sets of equations involving the matrix fields $\mathbb{A}$, $\mathbb{B}$
\[ \mathbb{B}(x, z) + f(x, z) + \int_x^\infty dy \ \mathbb{A}(x, y) \hat{f}(y, z) = 0, \] (2.14)
\[ \mathbb{A}(x, z) + \int_x^\infty dy \ \mathbb{B}(x, y) \hat{f}(y, z) = 0, \] (2.15)
and the fields $\mathbb{C}$, $\mathbb{D}$
\[ \mathbb{C}(x, z) + \hat{f}(x, z) + \int_x^\infty dy \ \mathbb{D}(x, y) \hat{f}(y, z) = 0, \] (2.16)
\[ \mathbb{D}(x, z) + \int_x^\infty dy \ \mathbb{C}(x, y) f(y, z) = 0. \] (2.17)
These two sets are independently solved and the matrix-fields $\mathbb{B}$ and $\mathbb{C}$ are then given by the expressions:
\[ \mathbb{B}(x, z) + f(x, z) - \int_x^\infty d\tilde{y} \int_x^\infty dy \ \mathbb{B}(x, \tilde{y}) \hat{f}(\tilde{y}, y) f(y, z) = 0, \]
\[ \mathbb{C}(x, z) + \hat{f}(x, z) - \int_x^\infty d\tilde{y} \int_x^\infty dy \ \mathbb{C}(x, \tilde{y}) f(\tilde{y}, y) \hat{f}(y, z) = 0. \] (2.18)
The solutions $\mathbb{B}, \mathbb{C}$ produce the fields of the matrix NLS hierarchy as will be shown in a subsequent section when constructing the associated Lax pairs via the dressing transform. We introduce below both discrete and continuum solutions of the linear equations and obtain the corresponding solutions of the matrix GLM equation.
A. Discrete solutions

We first consider discrete solutions of the linear problem, which can be expressed as

\[ f(x, z, t) = \sum_{\alpha=1}^{L} b_{\alpha} e^{\sum_{n} \Lambda_{\alpha}^{(n)} t_n - \kappa_{\alpha} x - \mu_{\alpha} z}, \quad \hat{f}(x, z, t) = \sum_{\alpha=1}^{L} \hat{b}_{\alpha} e^{\sum_{n} \hat{\Lambda}_{\alpha}^{(n)} t_n - \hat{\mu}_{\alpha} x - \hat{\kappa}_{\alpha} z}, \]

where we make the dependence on \( t = \{ t_n \} \) (the “universal” time containing all the time flows \( t_n \)) explicit in order to show the dispersion relations for each time flow. Indeed, the general dispersion relations immediately follow from (2.11), and are given as

\[ w_1 \kappa + w_2 \mu = 0, \quad w_1 \hat{\kappa} + w_2 \hat{\mu} = 0, \]

\[ \Lambda^{(n)} = (-1)^n \kappa^{(n)} + \mu^{(n)} = 0, \quad \hat{\Lambda}^{(n)} = (-1)^n \hat{\mu}^{(n)} + \hat{\kappa}^{(n)} = 0. \]

Taking into consideration the form of the solutions of the linear problem as well as equation (2.18) we consider the generic form for the matrix fields:

\[ B(x, z, t) = \sum_{\alpha=1}^{L} \ell_{\alpha}(x, t) e^{-\mu_{\alpha} z}, \quad C(x, z, t) = \sum_{\alpha=1}^{L} \ell_{\alpha}(x, t) e^{-\kappa_{\alpha} z}. \]

Let us also introduce some important objects: \( M \) and \( \hat{M} \) are operator valued matrices with elements \( M_{\alpha\beta}, \hat{M}_{\alpha\beta} \) being themselves \( M \times M \) and \( N \times N \) matrices respectively defined as:

\[ M(x, t) = I_{M \times M} \otimes I_{L \times L} - P(x, t), \quad \hat{M}(x, t) = I_{N \times N} \otimes I_{L \times L} - \hat{P}(x, t), \]

\[ P_{\beta\alpha}(x, t) = \sum_{\gamma=1}^{L} I_{\beta\gamma}(x, t) \hat{b}_{\gamma} b_{\alpha}, \quad \hat{P}_{\beta\alpha}(x, t) = \sum_{\gamma=1}^{L} \hat{I}_{\beta\gamma}(x, t) \hat{b}_{\gamma} \hat{b}_{\alpha}, \]

and we also define

\[ I_{\beta\gamma}(x, t) = e^{\sum_{n} (\Lambda_{\beta}^{(n)} + \Lambda_{\gamma}^{(n)}) t_n} e^{-(\mu_{\beta} + \mu_{\gamma}) x e^{-(\hat{\kappa}_{\gamma} + \kappa_{\alpha}) x}}, \]

\[ \hat{I}_{\beta\gamma}(x, t) = e^{\sum_{n} (\hat{\Lambda}_{\beta}^{(n)} + \hat{\Lambda}_{\gamma}^{(n)}) t_n} e^{-(\hat{\mu}_{\beta} + \hat{\mu}_{\gamma}) x e^{-(\hat{\kappa}_{\gamma} + \kappa_{\alpha}) x}}. \]

Then the matrices \( L, \hat{L} \) are identified as

\[ L(x, t) M(x, t) = -B(x, t), \quad \hat{L}(x, t) \hat{M}(x, t) = -\hat{B}(x, t). \]

\( B, \hat{B} \) are operator valued \( L \)-vectors with components: \( b_{\alpha} e^{\sum_{n} \Lambda_{\alpha}^{(n)} t_n - \kappa_{\alpha} x}, \hat{b}_{\alpha} e^{\sum_{n} \hat{\Lambda}_{\alpha}^{(n)} t_n - \mu_{\alpha} x} \) respectively. Provided that \( M, \hat{M} \) are invertible, i.e \( \det(1 - P) \neq 0, \det(1 - \hat{P}) \neq 0 \), then the general discrete solutions can be expressed in a formal series expansion as

\[ L(x, t) = -B(x, t) \left( I_{M \times M} \otimes I_{L \times L} + \sum_{m=1}^{\infty} P^{m} \right). \]

Similarly, for \( \hat{L} \): \( B \rightarrow \hat{B}, \hat{P} \rightarrow \hat{P} \) and \( I_{M \times M} \rightarrow I_{N \times N} \).
It will be instructive to provide the explicit expression for the “one-soliton” solution $L = 1$. Equations (2.24) in this case reduce to
\[
\begin{align*}
L(x, t) &= -\beta e^{\sum_n \Lambda^{(n)} t_n - \kappa x}, \\
\hat{L}(x, t) &= -\hat{\beta} e^{\sum_n \hat{\Lambda}^{(n)} t_n - \hat{\mu} x}, \\
f(x, t) &= e^{\sum_n (\Lambda^{(n)} + \hat{\Lambda}^{(n)}) t_n} e^{-\left(\mu + \hat{\mu}\right)x e^{-(\kappa + \hat{\kappa})x}} \\
&= (\mu + \hat{\mu})(\kappa + \hat{\kappa}),
\end{align*}
\] (2.26)
then the fields $B, C$ are given in a compact form as
\[
\begin{align*}
B(x, z, t) &= -\beta \left(1_{M \times M} + g(x, t)bb\right) e^{\sum_n \Lambda^{(n)} t_n - \kappa x - \mu z}, \\
C(x, z, t) &= -\hat{\beta} \left(1_{N \times N} + \hat{g}(x, t)b\hat{b}\right) e^{\sum_n \hat{\Lambda}^{(n)} t_n - \hat{\mu} x - \hat{\kappa} z},
\end{align*}
\] (2.27)
where $g = \frac{e^i}{\xi}$ and we have assumed $(bb)^2 = \xi \hat{b}\hat{b}$, (see also relevant results for the vector NLS in [2,9]). The kernels $\Lambda, \varnothing$ of the matrix kernel $K^+$ (3.2) can be identified by means of relations (2.14) and (2.17) respectively. If we further impose Temperley-Lieb type constraints for the quantities $b, \hat{b}$: $bbb = \xi b$ and $\hat{b}\hat{b}\hat{b} = \hat{\xi} \hat{b}$, we conclude,
\[
\begin{align*}
B(x, z, t) &= -\beta e^{\sum_n \Lambda^{(n)} t_n - \kappa x - \mu z} \\
&= \frac{1}{1 - \xi}, \\
C(x, z, t) &= -\hat{\beta} e^{\sum_n \hat{\Lambda}^{(n)} t_n - \hat{\mu} x - \hat{\kappa} z} \\
&= \frac{1}{1 - \hat{\xi}},
\end{align*}
\] (2.29)
As will become transparent later in the text $B(x, x), C(x, x)$ are proportional to the relevant NLS matrix fields. It is worth mentioning that for the one soliton case the constraint is actually justified as a consistency condition when computing also the quantities $\Lambda, \varnothing$ via equations (2.15) and (2.17).

B. Continuous Solutions

We come now to the more general scenario of continuum solutions of the linear problem. More precisely, the generic solution of the linear problem can be expressed as a continuum Fourier transform
\[
\begin{align*}
f(x, z, t) &= \int_{\mathbb{R}} dk \ b_k \exp \left[\sum_n \Lambda_k^{(n)} t_n + ikx + i\mu z\right], \\
\hat{f}(x, z, t) &= \int_{\mathbb{R}} \hat{k} \ \hat{b}_k \exp \left[\sum_n \hat{\Lambda}_k^{(n)} t_n + i\mu z + ikz\right],
\end{align*}
\] (2.31)
where the coefficients $b$ and $\hat{b}$ are $N \times M$ and $M \times N$ matrices respectively. As in the case of discrete solutions the dispersion relations then immediately follow from (2.11):
\[
w_1 + w_2 \mu_k = 0, \quad \Lambda_k^{(n)} - (i)^n k^n = 0, \quad \hat{\Lambda}_k^{(n)} - (i)^n \mu_k^n = 0.
\] (2.32)
Taking into account the solutions of the linear problem as well as equation (2.13) we consider for the matrix fields:
\[
\begin{align*}
B(x, z, t) &= \int_{\mathbb{R}} dk \ L(k, x, t) e^{i\mu z}, \\
C(x, z, t) &= \int_{\mathbb{R}} dk \ \hat{L}(k, x, t) e^{i\hat{\mu} z}.
\end{align*}
\] (2.33)
Let us also define the quantities:

\[ B(k, x, t) = b_k e^{\sum_n \lambda_{k+1}^{(n)} i_n + ikx}, \quad \hat{B}(k, x, t) = \hat{b}_k e^{\sum_n \lambda_{k+1}^{(n)} i_n + i\mu_k x}, \]

\[ \mathbb{P}(k_1, k, x, t) = \int_{\mathbb{R}} dk_2 f(k_1, k_2, k, x, t) b_{k_2} b_k, \]

\[ \hat{\mathbb{P}}(k_1, k, x, t) = \int_{\mathbb{R}} dk_2 \hat{f}(k_1, k_2, k, x, t) b_{k_2} \hat{b}_k \]

(2.34)

recall \( \mu_k = -\frac{\omega}{k} \), and

\[ f(k_1, k_2, k, x, t) = -e^{\sum_n (\lambda_{k+1}^{(n)} + \lambda_k^{(n)}) i_n} e^{i(\mu_1 + \mu_2) x} e^{i(k_2 + k) x}, \]

\[ \hat{f}(k_1, k_2, k, x, t) = -e^{\sum_n (\lambda_{k+1}^{(n)} + \lambda_k^{(n)}) i_n} e^{i(k_1 + k_2) x} e^{i(\mu_2 + \mu_3) x}. \]

(2.35)

Provided that the operators \( \mathbb{I} - \mathbb{P}, \mathbb{I} - \hat{\mathbb{P}} \) are invertible i.e. the Fredholm determinants are non-zero \( \det(\mathbb{I} - \mathbb{P}) \neq 0, \ det(\mathbb{I} - \hat{\mathbb{P}}) \neq 0 \), then the matrices \( \mathbb{L}, \hat{\mathbb{L}} \) are explicitly identified via the integral equations

\[ \int dk_1 \mathbb{L}(k_1, x, t) \left( \mathbb{I}_{M \times M} \delta(k_1, k) - \mathbb{P}(k_1, k, x, t) \right) = -B(k, x, t), \]

(2.36)

similarly, for \( \hat{\mathbb{L}}: \mathbb{B} \rightarrow \hat{\mathbb{B}}, \mathbb{P} \rightarrow \hat{\mathbb{P}} \) and \( \mathbb{I}_{M \times M} \rightarrow \mathbb{I}_{N \times N} \).

The latter relations provide the formal series expansion for \( \mathbb{L} \) and \( \hat{\mathbb{L}} \), i.e. the integral analogues of the matrix relations (2.25) presented in the discrete case previously:

\[ \mathbb{L}(k) = -B(k) - \sum_{m=1}^{\infty} \int_{\mathbb{R}} dk_1 \ldots \int_{\mathbb{R}} dk_m B(k_1) \mathbb{P}(k_1, k_2) \mathbb{P}(k_2, k_3) \ldots \mathbb{P}(k_m, k), \]

(2.37)

similarly for \( \hat{\mathbb{L}}: \mathbb{B} \rightarrow \hat{\mathbb{B}} \) and \( \mathbb{P} \rightarrow \hat{\mathbb{P}} \); dependence of \( x, t \) is implied in the expression above.

- Generalized mKdV solutions & Airy functions

It is interesting to focus now on solutions of the generalized mKdV equation \( (n = 3) \). The key observation is that due to the cubic dispersion relations in the case \( n = 3 \) (2.32) the solutions of the linear problem can be expressed in terms of Airy functions \( \text{Ai}(x) \). Indeed, after expressing the coefficients \( b_k, \hat{b}_k \) in terms of the initial values of the solutions of the linear problem \( f_0, \hat{f}_0 \) at \( t = 0 \), via an inverse Fourier transform

\[ b_k = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi f_0(\xi) e^{-ik\xi}, \quad \hat{b}_k = \frac{1}{2\pi} \int_{\mathbb{R}} d\xi \hat{f}_0(\xi) e^{-ik\xi}, \]

(2.38)

and recalling the definition of the Airy function

\[ \text{Ai}(x) = \frac{1}{\pi} \int_0^\infty dt \cos \left( \frac{t^3}{2} + xt \right), \]

(2.39)

we obtain

\[ f(x, z) = \frac{1}{\nu} \int_{\mathbb{R}} d\xi \text{Ai}(\frac{x + sz - \xi}{\nu}) f_0(\xi), \quad \hat{f}(x, z) = \frac{1}{\nu} \int_{\mathbb{R}} d\xi \text{Ai}(\frac{sx + z - \xi}{\nu}) \hat{f}_0(\xi), \]

(2.40)
where \( s = -\frac{w_1}{w_2} \), \( \nu = (3(1-s^2)t)^{\frac{1}{2}} \).

We can express the kernels appearing in the fundamental expressions (2.18) in terms of Airy functions, indeed let \( \hat{F}(x; \tilde{y}, z) = \int_x^\infty dy \hat{f}(\tilde{y}, y) \hat{f}(y, z) \), \( F(x; \tilde{y}, z) = \int_x^\infty dy \hat{f}(\tilde{y}, y) f(y, z) \), then

\[
\hat{F}(x; \tilde{y}, z) = \frac{1}{\nu^2} \int_x^\infty dy \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} d\tilde{\xi} \text{Ai}(\frac{\tilde{y} + sy - \xi}{\nu}) \text{Ai}(\frac{sy + z - \tilde{\xi}}{\nu}) f_0(\xi) \hat{f}_0(\tilde{\xi}),
\]

\[
F(x; \tilde{y}, z) = \frac{1}{\nu^2} \int_x^\infty dy \int_{\mathbb{R}} d\xi \int_{\mathbb{R}} d\tilde{\xi} \text{Ai}(\frac{\tilde{y} + y - \xi}{\nu}) \text{Ai}(\frac{y + sz - \tilde{\xi}}{\nu}) f_0(\xi) f_0(\tilde{\xi}).
\]

Distinct choices of the initial functions \( f_0, \hat{f}_0 \) give rise to different generic solutions, but an obvious choice of initial conditions is for instance, \( f_0(\xi) = \delta(\xi) m_{N \times M} \), \( \hat{f}_0(\xi) = \delta(\xi) \hat{m}_{M \times N} \). Given that \( f, \hat{f} \) satisfy the linear problem (2.11) for \( n = 3 \), it is straightforward to see that both \( \nu f(\xi + sz) \), \( \nu \hat{f}(\xi + sz) \) satisfy as expected the Airy equation, (linearization of Painlevé II (see also [20]))

\[
\frac{\partial^2 F(\zeta)}{\partial \zeta^2} - \zeta F(\zeta) = 0,
\]

with the parameter \( \zeta \) defined as \( \frac{s+1}{s+2} \) and \( \frac{s+3}{s+2} \) respectively, and having also assumed that \( F(\zeta) \to 0 \) when \( \zeta \to \infty \). The matrix fields \( B, C \) can be then expressed as a formal series expansion in terms of Airy functions via (2.18) and (2.41). These findings are in tune with the results derived in [20], where the connection between the mKdV and Painlevé II are discussed. Similar observations can be made for the matrix NLS model \( (n = 2) \) provided that the values of \( w_1, w_2 \) are suitably tuned. Indeed, in this case the solutions can be expressed in terms of the heat kernel. We shall comment in the next subsection on the matrix Burgers equation as a special case of our construction. As is well known in the viscous Burgers case, solutions can be obtained in terms of the heat kernel via the Cole-Hopf transformation.

### 2.2 Dressing of linear operators: the hierarchy

The next important task is to obtain the hierarchy of the Lax pairs via the dressing process (2.2), where \( G = 1 + K^+ \) is chosen to be an integral operator with kernel given in (2.4). Note that in the rest of this section dependence on the universal time \( t \) is implied, but omitted for simplicity. Let us first apply the dressing transformation for the \( L^{(0)} \) (the \( t \) independent part of the Lax pair):

\[
G \ W \partial_x = (W \partial_x + U) \ G,
\]

which directly leads to

\[
\int_x^\infty dy \left( W \partial_x K(x, y) + \partial_y K(x, y) W + U(x) K(x, y) \right) + \left( U(x) + K(x, x) W - W K(x, x) \right) f(x) = 0,
\]

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yielding the fundamental equations

\[ U(x) = WK(x, x) - K(x, x)W, \]

\[ W \partial_x K(x, y) + \partial_y K(x, y)W + U(x)K(x, y) = 0. \] (2.45)

The latter equations provide the form of \( U \) as well the \( K \)-matrix

\[ U(x) = \begin{pmatrix} 0 & \tilde{u}(x) \\ u(x) & 0 \end{pmatrix}, \] (2.46)

\[ \tilde{u}(x) = hB(x, x), \quad u(x) = -hC(x, x), \quad \text{and} \quad h = w_1 - w_2, \] (2.47)

and the set of constraints

\[ w_1 \left( \partial_x \tilde{u}(x, y) + \partial_y \tilde{u}(x, y) \right) = -hB(x, x)C(x, y), \]

\[ w_1 \partial_x B(x, y) + w_2 \partial_y B(x, y) = -hB(x, x)D(x, y), \]

\[ w_1 \partial_x C(x, y) + w_1 \partial_y C(x, y) = hC(x, x)\tilde{u}(x, y). \] (2.48)

Let us now implement the dressing transform for all the associated time flows \( t_n \), and derive the dressed operators \( L^{(n)} \), \( n > 0 \).

• \( n = 1 \): equation (2.2) yields

\[
\int_x^\infty dy K(x, y) \left( \partial_{t_1} - W \partial_y \right) f(y) = a(x) f(x) + \left( \partial_{t_1} - W \partial_x \right) \int_x^\infty dy K(x, y) f(y) + \int_x^\infty dy a(x) K(x, y) f(y).
\] (2.49)

After integrating by parts and considering the boundary terms we conclude:

\[ a(x) = K(x, x)W - WK(x, x), \]

\[ \partial_{t_1} K(x, y) = W \partial_x K(x, y) + \partial_y K(x, y)W - a(x)K(x, y). \] (2.50)

• \( n > 1 \): from the basic dressing relation (2.2) we obtain

\[
\int_x^\infty dy K(x, y) \left( \partial_{t_n} - \partial_y^n \right) f(y) = X_x f(x) + \left( \partial_{t_n} - \partial_x^n \right) \int_x^\infty dy K(x, y) f(y) + \int_x^\infty dy X_x K(x, y) f(y),
\] (2.51)

where we define \( X_x = \sum_{k=0}^{n-1} a_k(x) \partial_x^k \). After repeated integrations by parts, and care-
fully taking into consideration the boundary terms by iteration, we conclude:

\[
\int_{x}^{\infty} dy \left( \partial_{x} K(x, y) - \partial_{y}^{n} K(x, y) + (-1)^{n} \partial_{y}^{n} K(x, y) + \chi_{x} K(x, y) \right) f(y)
\]

\[
+ \sum_{k=0}^{n-1} a_{k} \partial_{y}^{k} f(x) - \sum_{m=0}^{n-1} (-1)^{m} \partial_{y}^{m} K(x, y)|_{x=y} \partial_{x}^{n-m-1} f(x) + \partial_{x}^{n-1} \left( K(x, x) f(x) \right)
\]

\[
+ \sum_{m=0}^{n-2} \partial_{x}^{m} \left( \partial_{x}^{n-m-1} K(x, y)|_{x=y} f(x) \right) - \sum_{k=1}^{n-1} a_{k}(x) \partial_{x}^{k-1} \left( K(x, x) f(x) \right)
\]

\[
- \sum_{k=1}^{n-1} a_{k}(x) \sum_{m=0}^{k-2} \partial_{x}^{m} \left( \partial_{x}^{k-m-1} K(x, y)|_{x=y} f(x) \right) = 0.
\]

(2.52)

It follows from the expression above that \( K \) satisfies:

\[
\partial_{x} K(x, y) - \partial_{y}^{n} K(x, y) + (-1)^{n} \partial_{y}^{n} K(x, y) + \sum_{k=0}^{n-1} a_{k}(x) \partial_{y}^{k} K(x, y) = 0,
\]

(2.53)

whereas the use of the boundary terms for each order \( \partial_{x}^{m} f(x) \) provides the elements \( a_{k} \).

In fact, similar relations arise when considering a differential operator as the Darboux dressing transform, as is discussed in the next subsection.

• **Matrix Burgers equation**

Let us now focus on the linear operator \((\partial_{x} - \partial_{y}^{2}) I\) independently of the existence of a Lax pair. The dressed operator is required to be of the form \((\partial_{x} - \partial_{y}^{2}) I + V(x)\), then via the dressing process described in detail in this section (2.52), (2.53), we conclude \((n = 2):\)

\[
\partial_{x} K(x, y) - \partial_{y}^{2} K(x, y) + \partial_{y}^{2} K(x, y) + V(x) K(x, y) = 0,
\]

\[
V(x) = -2 \left( \partial_{x} K(x, y) + \partial_{y} K(x, y) \right)|_{x=y},
\]

(2.54)

where \( V \) and \( K \) are generic \( N \times N \) matrices. Let us also consider the following general condition for the kernel: \( K(x, y) = K(x + wy) \), \( w \) is a constant, then the equations above produce at \( x = y \) and after setting \( \chi = (1 + w)x \):

\[
\partial_{x} K(\chi) + (w^{2} - 1) \partial_{\chi}^{2} K(\chi) - 2(1 + w) \partial_{\chi} K(\chi) K(\chi) = 0.
\]

(2.55)

The latter yields both the viscous and inviscid matrix Burgers equations. Indeed, for \( w = 1 \), and after rescaling the time \( \tau = -4t \) we obtain the inviscid Burgers equation

\[
\partial_{\tau} K(\chi) + \partial_{\chi} K(\chi) K(\chi) = 0.
\]

(2.56)

For \( w \neq \pm 1 \), and after setting \( \tau = -2(1 + w)t \) and \( \nu = \frac{w-1}{w+1} \) we obtain the viscous Burgers equation

\[
\partial_{\tau} K(\chi) + \partial_{\chi} K(\chi) K(\chi) = \nu \partial_{\chi}^{2} K(\chi).
\]

(2.57)

Solutions of the latter can be obtained via the heat kernel given that the viscous Burgers equation can be mapped to the heat equation via the Cole-Hopf transformation. To generalize the transform in the matrix case let us consider solutions of
the form $K(\chi) = f(\chi)b$ where $b$ is an $N \times N$ matrix: $b^2 = \kappa b$ and $f(\chi)$ is a scalar, which then satisfies the scalar Burgers equation with rescaled time $\hat{\tau} = \frac{\kappa \tau}{\hat{\nu}}$. The scalar Cole-Hopf transformation can now be implemented: $f(\chi) = -2\hat{\nu} \partial_x (\log \varphi(\chi))$ where $\varphi$ is a solution of the heat equation: $\partial_t \varphi(\chi) = \hat{\nu} \partial_x^2 \varphi(\chi)$. Interestingly in [26] it was shown that the quantum discrete NLS model yields in the continuum limit the stochastic heat equation and hence the viscous Burgers equation. These are significant connections that are at the epicenter of ongoing investigations at both classical and quantum level (see also relevant findings in [19, 27] as well as section 4).

Some noteworthy comments are in order here. The issue of canonical quantization emerges naturally in this context. In general, the quantities $B(x,x), C(x,x)$ can be seen as the corresponding non-commutative fields, given the relevant definition of the fields (2.47). If one further considers the symmetric case $M = N \to \infty$, and requires that the fields satisfy canonical commutation relations, then one indeed recovers the quantum version of the NLS model with an underlying $sl_2$ algebra (see relevant results on the quantum GLM equation for the NLS model in [21]). The notion of the non-commutative/quantum Riccati equations associated to the system is thus a particularly pertinent issue, which will be discussed in section 4. In fact, the use of the Riccati equation is fundamental in solving the auxiliary problem and deriving the associated non-commutative conserved quantities (see also [19] for relevant findings).

## 3 Dressing transformations as differential operators

We have discussed in the preceding section the dressing formulation employing an integral operator as the Darboux-dressing transformation. We now turn our attention to the case where the Darboux transform is a differential operator. The fundamental transform is expressed as a first order differential operator:

$$ \mathcal{G} = \mathcal{L}_x + K(x), $$

where

$$ K(x) = \begin{pmatrix} A_{N \times N}(x,y) & B_{N \times M}(x,y) \\ C_{M \times N}(x,y) & D_{M \times M}(x,y) \end{pmatrix}. $$

(3.2)

Let us now solve the set of dressing relations:

$$ \left( \mathcal{L}_x + K(x) \right) W \partial_x = \left( W \partial_x + U(x) \right) \left( \mathcal{L}_x + K(x) \right), $$

(3.3)

$$ \left( \mathcal{L}_x + K(x) \right) \left( \mathcal{L}_{x_1} - W \partial_x \right) = \left( \mathcal{L}_{x_1} - W \partial_x + a(x) \right) \left( \mathcal{L}_x + K(x) \right), $$

(3.4)

$$ \left( \mathcal{L}_x + K(x) \right) \left( \partial_{x_n} - \partial_x \right) I = \left( \mathcal{L}_{x_n} - \partial_x^n + \sum_{k=0}^{n-1} a_k(x) \partial_x^k \right) \left( \mathcal{L}_x + K(x) \right), \quad n > 1 \quad (3.5) $$

Equation (3.3) provides:

$$ U(x)K(x) = -W \partial_x K(x), $$

$$ U(x) = K(x)W - WK(x), $$

(3.6)
and as in the integral case discussed above $U$ is defined in (2.46), where the fields are now identified as

$$\hat{u}(x) = -hB(x, x), \quad u(x) = hC(x, x), \quad \text{and} \quad h = w_1 - w_2.$$  

(3.7)

Also, the following relations emerge, similar to the ones appearing in the integral case (2.48)

$$w_1 \partial_x A(x) = hB(x)C(x), \quad w_2 \partial_x D(x) = -hC(x)B(x),$$

$$w_1 \partial_x B(x) = hB(x)D(x), \quad w_2 \partial_x C(x) = -hC(x)A(x).$$  

(3.8)

From the analysis above we conclude that the $L^{(0)}$ operator reads as

$$L^{(0)} = W \partial_x + \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix}. \quad (3.9)$$

The time dependent part of the dressing (3.4), (3.5) also gives sets of equations associated to each time flow $t_n$:

- **n = 1**: equation (3.4) yields: $a(x) = -U(x)$ and
  $$\partial_t K(x) = W \partial_x K(x) + \partial_x K(x)W + U(x)K(x). \quad (3.10)$$

- **n > 1**: equation (3.5) gives
  $$\partial_{t_n} K(x) - \partial_x^n K(x) + \sum_{k=1}^{n-1} a_k(x) \partial_x^k K(x) = 0, \quad (3.11)$$
  as well as the contributions proportional to $\partial_x^m$, which essentially determine each one of the factors $a_k$ of every $L^{(n)}$ operator

$$\sum_{k=1}^{n} a_{k-1}(x) \partial_x^k - \sum_{m=1}^{n-1} \binom{n}{m} \partial_x^{n-m} K(x) \partial_x^m + \sum_{k=0}^{n-1} \sum_{m=1}^{k-1} \binom{k}{m} a_k(x) \partial_x^{k-m} K(x) \partial_x^m = 0. \quad (3.12)$$

The latter relation gives rise to a tower of constraints among the various coefficients $a_k$, which can then be uniquely determined. Moreover, use of the extra constraint provided by (3.3) gives a solution (one soliton solution) of the underlying non-linear PDE.

We report below the Lax pairs $\mathcal{L}^{(n)}$, as well as the relevant equations of motion for $n = 1, 2, 3$, i.e. for the generalized transport, NLS and mKdV equations respectively.

- **n = 1**: **the matrix transport equation**
  The time component for the $t_1$ time flow reads as

$$\mathcal{L}^{(1)} = \partial_{t_1} \mathcal{L} - W \partial_x - \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix}. \quad (3.13)$$

The zero curvature condition (commutativity) for the pair $\mathcal{L}^{(0)}, \mathcal{L}^{(1)}$, leads to the matrix transport equation:

$$\partial_{t_1} u + w_2 \partial_x u = 0, \quad (3.14)$$

and similarly for $\hat{u}$, with $w_2 \to w_1$. 

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• \( n = 2 \): the matrix NLS equation

The time part of the Lax pair for the NLS model \( n = 2 \) is given by

\[ \mathbb{L}^{(2)} = (\partial_t - \partial^2_x) I + \frac{2}{h} \begin{pmatrix} -\frac{1}{w_1} \hat{u}u & -\partial_x \hat{u} \\ \partial_x u & \frac{1}{w_2} \hat{u}u \end{pmatrix}. \]  

(3.15)

Consequently the equations of motion for NLS read as

\[ \partial_t u - \frac{w_1 + w_2}{h} \partial^2_x u + \frac{2(w_1 + w_2)}{h w_1 w_2} \hat{u}u = 0, \]  

(3.16)

similarly for \( \hat{u} \) but with \( h \rightarrow -h \).

• \( n = 3 \): the generalized mKdV model

In this case the time component of the Lax pair is

\[ \mathbb{L}^{(3)} = (\partial_t - \partial^3_x) I + \frac{3}{h} \begin{pmatrix} -\frac{1}{w_1} \hat{u}u & -\partial_x \hat{u} \\ \partial_x u & \frac{1}{w_2} \hat{u}u \end{pmatrix} \partial_x 
+ \frac{3}{h^2} \begin{pmatrix} -\hat{u} \partial_x u + \frac{w_1}{w_2} \partial_x \hat{u} u & w_2 \partial^2_x \hat{u} - \frac{w_1}{w_1 w_2} \hat{u} \hat{u} \partial_x u \\ w_1 \partial^2_x u - \frac{w_1}{w_1 w_2} \hat{u} \hat{u} u & \partial_x u \hat{u} - \frac{w_2}{w_2} \partial_x \hat{u} \hat{u} \end{pmatrix}. \]

(3.17)

It is worth noting that for the derivation of the generalized mKdV L operator the use of the constraints (2.48), (3.8) in both the integral and differential cases has been essential. The equations of motion for the generalized mKdV then read as

\[ \partial_t u - \frac{E}{h^2} \partial^3_x u + \frac{3E}{h^2 w_1 w_2} (u \hat{u} \hat{u} + \partial_x u \hat{u}u) = 0, \]  

(3.18)

where \( E = w_1^2 + w_2^2 + w_1 w_2 \). A similar equation holds for \( \hat{u} \) provided that \( \hat{u} \rightarrow u, u \rightarrow \hat{u} \).

• Matrix Burgers equation

As discussed in the integral case to obtain the matrix Burgers equation we focus on the bare operator \((\partial_t - \partial^2_x) I\) and dressed operator \((\partial_t - \partial^3_x) I + V(x)\). We consider the first order differential operator \( \partial_x + K(x) \), \( V, K \) are \( N \times N \) matrices, as the fundamental Darboux transformation. Then via the dressing process as described previously in this subsection we conclude that \( V(x) = 2\partial_x K(x) \) and:

\[ \partial_t K(x) - \frac{1}{2} \partial^2_x K(x) + \partial_x K(x) K(x) = 0, \]  

(3.19)

where \( \tau = 2t \). The equation we obtain now is more restrictive compared to the one we got in the integral case. Now the diffusion constant is fixed, so we can only deal with the viscous Burgers equation as opposed to the integral case where both viscous and inviscid Burgers equations emerge.

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3.1 General Darboux transform & solutions

Consider now the most general scenario for the Darboux transform given as an $m^{th}$ order differential operator. We focus here on the time independent part of the transform and find the tower of differential equations obeyed by the coefficients. Let

$$G = \partial_x^m + \sum_{k=0}^{m-1} b_k(x) \partial_x^k. \quad (3.20)$$

Our aim is to identify the coefficients $b_k$ of the differential operator. Indeed, by means of the fundamental dressing relation

$$\left( \partial_x^m + \sum_{k=0}^{m-1} b_k(x) \partial_x^k \right) W \partial_x = \left( W \partial_x + U(x) \right) \left( \partial_x^m + \sum_{k=0}^{m-1} b_k(x) \partial_x^k \right), \quad (3.21)$$

we find recursion relations for the matrix coefficients $b_k$:

$$Wb_{k-1} - b_{k-1}W + W\partial_x b_k + U b_k = 0, \quad k \in \{1, \ldots, m-1\}$$

$$Ub_0 + W\partial_x b_0 = 0,$$

$$Wb_{m-1} - b_{m-1}W + U = 0, \quad (3.22)$$

which yield the form of $U$ and $b_k$ as well as solutions of the non-linear equations.

Note that in earlier works [5, 6, 9] the Lax pairs as well as the general Darboux transforms for the vector and matrix NLS models were expressed as $c$-number matrices. Here we are offering a unifying dressing scheme regardless of the particular form of the operators. We have already examined the case of integral and differential operators and we have seen their equivalence, while in section 4.1 we present the matrix dressing process, which is naturally in one to one correspondence with the dressing when considering the Darboux to be a differential operator.

It will also be instructive to comment on the distinct choices of Darboux-dressing transforms considered so far i.e. the integral Darboux versus the differential one. Indeed, the main advantage when considering the integral Darboux transform is that solutions of the GLM equation found via the linear data are provided in a straightforward way yielding in turn generic solutions of the associated integrable nonlinear PDEs. In the case of the differential Darboux transform on the other hand – as is the case of the matrix Darboux studied in section 4 – the dressing of the linear operators is a much simpler process given that one does not have to deal with the involved boundary terms emerging when performing the numerous integrations by parts in the integral case.

In the preceding subsection a detailed account on the dressing of the linear operators and the construction of the integrable NLS hierarchy was provided for the fundamental Darboux, whereas in this section we deal with the generic differential Darboux and obtain the fundamental dressing relations for the $t$-independent part of the Lax pairs $\textcircled{3.22}$. We restrict our attention to the construction of solutions via the set of the time independent constraints $\textcircled{3.22}$. Let us for simplicity focus on the one soliton case here to illustrate the process, a more exhaustive analysis on generic solutions via the recursion relations $\textcircled{3.22}$.
will be presented in a forthcoming publication. It will be important here as is the case in the usual Darboux transformation for the familiar NLS to consider the following ansatz for the one soliton solution, in accordance also with the results of the previous subsection:

\[ A(x) = A(x)\hat{b}, \quad D(x) = D(x)\hat{b}, \quad B(x) = B(x)b, \quad C(x) = C(x)\hat{b}, \quad (3.23) \]

where \( b, \hat{b} \) are \( N \times N \) and \( M \times M \) matrices. We also require: \( BC = k_1A - \xi_1A^2 \). A similar requirement is also considered for \( D \) with parameters \( k_2, \xi_2 \). Note that the ansatz (3.23) as well as the constraints on \( A, D \) are compatible with (3.8). After substituting the expressions (3.23) into the first of the equations (3.8) we obtain

\[ \partial_x A(x) = \frac{h\xi_1}{w_1} \left( \frac{k_1}{\xi_1} A(x) - A^2(x) \right) \Rightarrow A(x) = -\frac{k_1}{\xi_1} e^{-\frac{2hx}{w_1}(x-x_0)}. \quad (3.24) \]

Then from equation (3.8) we obtain a solution for \( C \)

\[ \partial_x C(x) = -\frac{h}{w_2} C(x)A(x) \Rightarrow C(x) = -\frac{C_0}{1 - e^{-\frac{2hx}{w_1}(x-x_0)}}, \quad (3.25) \]

where we have assumed \( \frac{w_1}{w_2} = 1 \). Similar solutions are obtained for the pair \( D, B \), but are omitted here for brevity. Obtaining solutions associated to more involved Darboux transformations becomes a highly involved algebraic problem given that one has to solve all the associated constraints (3.22). This is a very interesting direction to pursue and will be further discussed in future works.

4 Non-commutative Riccati flows

The main aim of the present section is the derivation and study of non-commutative/quantum Riccati flows associated to the matrix NLS hierarchy. As mentioned earlier in the text the notion of the non-commutative Riccati equation is fundamental in solving the auxiliary problem and deriving the related non-commutative conserved quantities. Before we focus on the non-commutative Riccati equation we review the dressing process for the matrix NLS hierarchy when the Lax pair are c-number matrices; which will then naturally lead to the non-commutative Riccati equation.

4.1 The Darboux matrix & the hierarchy

We briefly review the auxiliary linear problem and the dressing process for a Lax pair consisting of generic c-number \( d \times d \) matrices \((U, V)\). The Lax pair matrices depend in general on some fields and a spectral parameter, and obey the auxiliary linear problem:

\[ \partial_x \Psi(\lambda, x, t) = U(\lambda, x, t)\Psi(x, t), \quad \partial_t \Psi(\lambda, x, t) = V^{(n)}(\lambda, x, t)\Psi(x, t). \quad (4.1) \]

For the matrix NLS hierarchy the \( U \)-matrix is given as \( (d = M + N) \)

\[ U(\lambda, x, t_n) = \begin{pmatrix} \frac{1}{2}I_{N \times N} & \hat{u}(x, t) \\ u(x, t) & -\frac{1}{2}I_{M \times M} \end{pmatrix}, \quad (4.2) \]
where in general $\hat{u}$, $u$ are $N \times M$ and $M \times N$ matrices. The $U$-matrix (4.2) will be the starting point in our dressing process, whereas all the time components will be explicitly derived.

The first aim is to identify the “dressed” quantities $V^{(n)}$ of the hierarchy. Let $U_0$, $V_0^{(n)}$ be the “bare” Lax pairs:

$$U_0(\lambda) = \frac{\lambda}{2}\Sigma, \quad V_0^{(n)}(\lambda) = \frac{\lambda^n}{2}\Sigma; \quad (4.3)$$

where we define

$$\Sigma = \begin{pmatrix} \mathbb{I}_{N \times N} & 0 \\ 0 & -\mathbb{I}_{M \times M} \end{pmatrix}. \quad (4.4)$$

Also, the “dressed” time components of the Lax pairs can be expressed as formal series expansions

$$V^{(n)}(\lambda, x, t) = \frac{\lambda^n}{2}\Sigma + \sum_{k=0}^{n-1} \lambda^k w_k^{(n)}(x, t), \quad (4.5)$$

where the quantities $w_k^{(n)}$ will be identified via the dressing transform.

The Darboux transform $G$ is applied on the “bare” auxiliary function

$$\Psi(\lambda, x, t) = G(\lambda, x, t) \Psi_0(\lambda), \quad (4.6)$$

yielding (see also (2.2))

$$\partial_x G = U_G G - G U^{(0)}, \quad \partial_t G = V^{(n)} G - G V_0^{(n)}. \quad (4.7)$$

The fundamental Darboux matrix is chosen to be of the form

$$G(\lambda, x, t) = \begin{pmatrix} \lambda \mathbb{I}_{N \times N} + A_{N \times N}(x, t) & B_{N \times M}(x, t) \\ C_{M \times N}(x, t) & \lambda \mathbb{I}_{M \times M} + D_{M \times M}(x, t) \end{pmatrix} = \lambda \mathbb{I} + K. \quad (4.8)$$

This is the simplest case to consider, but nevertheless it fully describes the dressing process for the Lax pair. By solving the $x$-part of equations (4.7) we obtain

$$\hat{u} = -B, \quad u = C \quad \text{and} \quad \partial_x K = \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix} K. \quad (4.9)$$

Note that in all the expressions below $x$ and $t$ dependence is implied.

From the time part of (4.7) we obtain a set of recursion relations, in exact analogy to the case of dressing via a differential operator, i.e.

$$w_{n-1}^{(n)} = \frac{1}{2} [K, \Sigma],$$

$$w_k^{(n)} = -w_k^{(n)} K, \quad k \in \{1, 2, \ldots, n-1\}$$

$$\partial_{t_n} K = w_0^{(n)} K. \quad (4.10)$$

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We solve the latter recursion relations, and identify the first few time components of the Lax pairs:

\[ V^{(0)} = \frac{1}{2} \Sigma, \]
\[ V^{(1)} = \lambda V^{(0)} + \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix}, \]
\[ V^{(2)} = \lambda V^{(1)} + \begin{pmatrix} -\hat{u}u & \partial_x \hat{u} \\ -\partial_x u & \hat{u}u \end{pmatrix}, \]
\[ V^{(3)} = \lambda V^{(2)} + \begin{pmatrix} \hat{u} \partial_x u - \partial_x \hat{u} u & -2\hat{u}u\hat{u} - \partial_x^2 \hat{u} \\ -2u\hat{u}u - \partial_x^2 u & u \partial_x \hat{u} - \partial_x u \hat{u} \end{pmatrix}. \quad (4.11) \]

In general, the \( V^{(n)} \) operator is identified as
\[ V^{(n)} = \lambda V^{(n-1)} + w^{(n)}_0. \]
Moreover, the recursion relations (4.10) lead to
\[ w^{(n)}_k = w^{(n-1)}_{k-1}, \quad w^{(n)}_0 = (-1)^{n-1} w^{(1)}_0 K^{n-1}, \quad (4.12) \]
where the latter relations together with the constraints (4.9) suffice to provide \( w^{(n)}_0 \) at each order.

Note that (4.7), (4.8) also produce the fundamental Darboux-Bäcklund transformation (BT) that connects two different solutions of the underlying integrable PDEs, i.e the pair \( U, V^{(n)} \) are associated to the fields \( u, \hat{u} \), whereas \( U_0, V^{(n)}_0 \) are associated to the fields \( u_0, \hat{u}_0 \). Then the \( x \)-part of (4.7) leads to the matrix Darboux-BT relations:
\[ B = -(\hat{u} - \hat{u}_0), \quad C = u - u_0, \]
\[ \partial_x A = \hat{u}C - B u_0, \quad \partial_x D = uB - C \hat{u}_0, \]
\[ \partial_x B = \hat{u}D - A \hat{u}_0, \quad \partial_x C = uA - D u_0. \quad (4.13) \]

For the dressing process we have clearly used the trivial solution \( u_0 = \hat{u}_0 = 0 \).

Let us also briefly discuss the general Darboux expressed as a formal \( \lambda \)-series expansion
\[ G = \lambda^m + \sum_{k=0}^{m-1} \lambda^k g_k, \quad (4.14) \]
where \( g_k \) are \( \mathcal{N} \times \mathcal{N} \) matrices to be identified. We focus on the fundamental recursion relations arising from the \( x \)-part of the Darboux transform (4.7):\n\[ \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix} = \frac{1}{2} [g_{m-1}, \Sigma], \quad (4.15) \]
\[ \partial_x g_0 = \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix} g_0, \quad \partial_x g_k = \frac{1}{2} [\Sigma, g_{k-1}] + \begin{pmatrix} 0 & \hat{u} \\ u & 0 \end{pmatrix} g_k. \quad (4.16) \]

The infinite series expansion \( m \to \infty \) is particularly interesting, in this case the “boundary” conditions (4.15) do not hold and one has to solve the infinite set of recursion relations (4.10), in order to identify solutions in analogy to the differential Darboux case discussed in subsection 3.1.
4.2 The non-commutative Riccati equation

We come now to our main task, which is the study of the non-commutative Riccati equation for the matrix NLS hierarchy. The primary aim is to identify solutions of the auxiliary linear problem and hence non-commutative conserved quantities. The auxiliary function is expressed as \( \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \), where we choose \( \Psi_1 \) to be an \( N \times N \) matrix and consequently \( \Psi_2 \) an \( M \times N \) matrix. We could have chosen equivalently \( \Psi_1, \Psi_2 \) to be \( N \times M \) and \( M \times M \) matrices respectively. We proceed essentially as in the scalar case (see e.g. [13]), noting that the non-commutativity between the components \( \Psi_1, \Psi_2 \) leads to highly non-trivial behavior when explicitly solving the associated Riccati equation. Consider now the \( x \)-part of the auxiliary problem (4.1):

\[
\partial_x \Psi_1 = \frac{\lambda}{2} \Psi_1 + \hat{u} \Psi_2,
\]

\[
\partial_x \Psi_2 = -\frac{\lambda}{2} \Psi_2 + u \Psi_1.
\] (4.17)

Define the \( M \times N \) matrix \( \Gamma = \Psi_2 \Psi_1^{-1} \), then from the latter expressions one arrives at the non-commutative Riccati equation obeyed by \( \Gamma \):

\[
\partial_x \Gamma = u - \lambda \Gamma - \hat{u} \Gamma.
\] (4.18)

The next important task is to identify the element \( \Gamma \). This can be achieved as in the usual NLS case [13] by expressing \( \Gamma \) in a formal power series expansion \( \Gamma = \sum_k \Gamma^{(k)} \lambda^k \) and solve the Riccati equation at each order. Then (4.18) reduces to:

\[
\partial_x \Gamma^{(k)} = -\Gamma^{(k+1)} - \sum_{l=1}^{k-1} \Gamma^{(l)} \hat{u} \Gamma^{(k-l)}, \quad k > 0.
\] (4.19)

Let us report below the first few terms of the expansion

\[
\Gamma^{(1)} = u, \quad \Gamma^{(2)} = -\partial_x u, \quad \Gamma^{(3)} = \partial_x^2 u - uu, \ldots
\] (4.20)

Some particularly relevant findings on Grassmannian/Riccati flows are presented in [24, 25], where the infinite Grassmannian was exploited in order to produce non-local PDEs. Here on the other hand we deal with the finite Grassmannian \( Gr(N|N+M) \), although at the quantum level we are interested in the infinite limit of \( Gr(N|2N) \) as \( N \to \infty \). A more exhaustive investigation on the significant notion of non-commutative Riccati flows as well as comparison with the relevant findings at the discrete quantum level [19] will be presented in forthcoming works.

The next natural step is to identify the associated conserved quantities by means of the auxiliary linear problem relations. Let us express the time components of the Lax pairs as \( V^{(n)} = \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix} \), and also recall for both the \( x \) and \( t \)-parts of the linear problem:

\[
\partial_x \Psi_1 \Psi_1^{-1} = \frac{\lambda}{2} + \hat{u} \Gamma,
\]

\[
\partial_t \Psi_1 \Psi_1^{-1} = \alpha_n + \beta_n \Gamma.
\] (4.21)
After cross differentiation of the equations above we conclude

\[ \partial_t n(\hat{u} \Gamma) = \partial_x (\alpha_n + \beta_n \Gamma) + [\alpha_n + \beta_n \Gamma, \hat{u} \Gamma]. \]  

(4.22)

In the usual (commutative) NLS case the commutator in the latter equation is zero, and thus the quantities

\[ I^{(k)} = \int dx \hat{u}(x) \Gamma^{(k)}(x), \]  

(4.23)

are automatically conserved (see also [13]); we have assumed vanishing boundary conditions at \( \pm \infty \). In the non-commutative case however the commutator in (4.22) is in principle non-zero, except when certain constraints are imposed. Let us focus on the first two simple members of the hierarchy i.e. \( V^{(0)}, V^{(1)} \) by just replacing the corresponding \( \alpha_n, \beta_n, n = 0, 1 \) we can immediately see that the commutator is zero. Of course we are interested in the case \( M = N \) and more precisely in the thermodynamic limit \( N \to \infty \).

Indeed, let us focus on the symmetric case \( M = N \) and in the limit \( N \to \infty \). It is important to note that to systematically address the issue of conservation laws in the non-commutative and in particular the quantum cases we need to assume some kind of ultra-locality, especially if we also wish to make a direct connection with the discrete quantum case. The obvious ultra-locality condition is of the form

\[ [\partial_x u(x), u(x)] = 0, \]  

(4.24)

which naturally reflects the fact, in the discrete quantum set up, that any two matrices \( A, B \) acting on different “quantum” spaces commute, i.e. \( A_1 B_2 = B_2 A_1 \) where in principle \( A_\alpha B_\alpha \neq B_\alpha A_\alpha \). The consistent quantum continuum limit should lead to (4.22) (see also a very relevant description at the classical level in [28]). These ultra-locality conditions should also be compatible with the quantum involution of the charges \( I^{(k)} \), i.e. compatible with a quantum Hamiltonian formulation.

Note also that in the continuum limit \( N \to \infty \), that matrix operators \( \Gamma, \hat{u}, \alpha, \beta \) with elements \( \Gamma_{ij}, u_{ij}, \ldots \) (we have suppressed the subscript \( n \) for simplicity) turn to integral operators with kernels \( \Gamma(\xi, \eta), u(\xi, \eta), \ldots \). It is clear that by taking the trace in both cases (discrete and continuous) we eliminate the unwanted commutator in (4.22), and we obtain the following conserved quantities:

\[ \mathcal{T}^{(k)} = \int dx \sum_{i,j=1}^N \tilde{u}_{ij}(x) \Gamma_{ji}^{(k)}(x) \quad \text{Discrete case}, \]  

(4.26)

\[ \mathcal{T}^{(k)} = \int dx \int d\xi \int d\eta \tilde{u}(x; \xi, \eta) \Gamma^{(k)}(x; \eta, \xi) \quad \text{Continuous case}, \]  

(4.27)

where the interval of integration \( I \) can be in general \( \mathbb{R} \). The conserved quantities above have been derived independently of the existence of ultra-locality conditions, however these are scalar objects as opposed to the block (potentially quantum) objects \( I^{(k)} \). We shall

\[ A_1 = A \otimes \mathbb{I}, \quad A_2 = \mathbb{I} \otimes A, \]  

(4.25)

where \( \mathbb{I} \) is the \( d \times d \) identity matrix.
report on these important subjects, and in particular on the issue of how the ultra-locality conditions modify the conserved quantities emerging from (4.22) in detail in a separate publication.

5 Time-like integrable boundary conditions

While in the preceding sections we have been exclusively focused on the generalized dressing scheme and the construction of solutions of the NLS integrable hierarchy, in this section we focus on the study of space-time dualities and more precisely on the implementation of time-like integrable boundary conditions, extending the results of [23]. A considerable amount of work has been devoted to the issue of integrable boundary conditions for the NLS model and its generalizations, but in the majority of studies space-like boundary conditions are considered. Here we are going to reverse the picture and consider time-like boundary conditions exploiting recent results on the time-space duality in the NLS case. In [22, 23] the concept of “dual” integrable 1+1 dimensional models was introduced, specifically in reference to the NLS model. We focus our attention here on the time-like version of the NLS hierarchy and extend the description of [23] in the presence of integrable time-like boundary conditions.

Let us first recall the space-like description. The starting point is the $U$-operator of the Lax pair $(U, V)$ introduced in the previous section. Assuming that the $U$-operator satisfies the linear Poisson structure

$$\{U_1(x, \lambda), U_2(y, \mu)\}_S = \left[ r_{12}(\lambda - \mu), U_1(x, \lambda) + U_2(y, \mu) \right] \delta(x - y), \quad (5.1)$$

where the $r$-matrix is a solution of the classical Yang-Baxter equation, we conclude that

$$\{u(x), \hat{u}(y)\}_S = \delta(x - y). \quad (5.2)$$

The subscript $S$ denotes space-like Poisson structure. Equation (5.1) acts on $V \otimes V$, where $V$ is in general an $d$ dimensional space, and the indices $1, 2$ in (5.1) denote the first and second space respectively. In general, for any $d \times d$ matrix $A$ the quantities $A_1, A_2$ are defined as in (4.25), i.e. $A_1$ acts non-trivially on the first space, whereas $A_2$ acts on the second one. The $r$-matrix acts on both spaces, and for the particular example we are going to examine here, $r$ is the Yangian solution [29],

$$r_{12}(\lambda) = \frac{1}{\lambda} \sum_{i,j=1}^d e_{ij} \otimes e_{ji}, \quad (5.3)$$

where $e_{ij}$ are $d \times d$ matrices with elements $(e_{ij})_{kl} = \delta_{ik} \delta_{jl}$; the quantity $\sum_{i,j} e_{ij} \otimes e_{ji}$ is the so called permutation operator.

The key object in this setting is the space monodromy, a solution of the first of the equations (4.1),

$$T_S(a, b, \lambda) = P \exp \left( \int_b^a U(x, \lambda) dx \right), \quad a > b,$$
which satisfies a quadratic algebra, and guarantees space Poisson commutativity and thus integrability:
\[ \{ \text{tr} T_S(\lambda), \text{tr} T_S(\mu) \}_S = 0. \quad (5.4) \]

In [22, 23] the picture was reversed, that is, it was assumed that \( V \), as well as \( U \), satisfies a linear algebra (see also [30] on further emphasis on the algebraic/r-matrix description). Indeed, it was noticed in [23] that the time-like Poisson bracket could be constructed from an equivalent linear algebraic expression regarding the time component of the Lax pair:
\[ \{ V_1(t_1, \lambda), V_2(t_2, \mu) \}_{\mathcal{T}} = \left[ r_{12}(\lambda - \mu), V_1(t_1, \lambda) + V_2(t_2, \mu) \right] \delta(t_1 - t_2), \quad (5.5) \]

where \( r \) is the same classical \( r \)-matrix as in (5.1), and the subscript \( \mathcal{T} \) denotes the time-like Poisson structure. Then the time monodromy \( T \), a solution to the time part of (4.1):
\[ T_T(a, b, \lambda) = P \exp \left( \int_a^b V(t, \lambda) dt \right), \quad a > b \]
satisfies the quadratic algebra
\[ \{ T_{T1}(\lambda), T_{T2}(\mu) \}_T = \left[ r_{12}(\lambda - \mu), T_{T1}(\lambda)T_{T2}(\mu) \right], \quad (5.6) \]
and consequently one obtains commuting operators, with respect to the time-like Poisson structure.
\[ \{ \text{tr} T_T(\lambda), \text{tr} T_T(\mu) \}_T = 0. \quad (5.7) \]

Inspired by the form of the \( V \)-operator for the NLS model we express our starting operator \( V \) in the following form (\( d = 2 \)):
\[ V(\lambda) = \begin{pmatrix} \frac{\lambda^2}{2} - u & \lambda \hat{u} + \pi \\ \lambda u - \hat{\pi} & -\frac{\lambda^2}{2} + u \hat{u} \end{pmatrix}. \quad (5.8) \]

We require \( V \) to satisfy the time-like Poisson structure (5.5) and we then produce the time-like algebra for the fields, which reads as (we only write below the non zero commutators, see also [23]):
\[ \{ u(t), \pi(t') \}_T = \{ \hat{u}(t), \hat{\pi}(t') \}_T = \delta(t - t'). \quad (5.9) \]

Henceforth, we focus only on time-like Poisson structures thus we drop the subscript \( T \) whenever this applies.

### 5.1 Periodic Boundary Conditions

We start by briefly deriving the results found in [23] for dual systems with periodic boundary conditions, in the language of Lax pairs. The starting point for this construction is the auxiliary linear problem, (4.1), and the algebraic relation (5.5). Here we exclusively discuss time-like boundary conditions, however note that space-like boundaries have been discussed from the Hamiltonian point of view in [16, 8].
The key object in our analysis in the time monodromy matrix (5.6), which can be decomposed as

\[ T(t, t'; \lambda) = (1 + W(t; \lambda)) e^{Z(t,t'; \lambda)} (1 + W(t'; \lambda))^{-1}, \quad t > t', \]  

(5.10)

where W is anti-diagonal and Z is purely diagonal. Then one obtains the typical Riccati equation for W

\[ \partial_t W + \left[ W, V_D \right] + W V_A W - V_A = 0, \]  

(5.11)

\[ \partial_t Z = V_D + V_A W, \]  

(5.12)

where \( V_D, V_A \) are the diagonal and anti-diagonal parts of the operator \( V \). Solutions of the pair of equations above are given in the Appendix for the first several members of the \( \lambda \) expansion, i.e. after considering: \( W = \sum_n W(n), Z = \sum_n Z(n) \).

Taking the trace and logarithm of the time-like monodromy, we find the generator for an infinite tower of conserved quantities associated to the system:

\[ G(\lambda) = \ln \left( \text{tr} (T(\tau, -\tau, \lambda)) \right). \]

Taking into consideration the decomposition of \( T \) in (5.10) as well as the fact that the leading contribution in \( e^Z \) comes from the \( e^{Z_{11}} \) term as \( \lambda \to \infty \) (see the expression for \( Z_{(-2)} \) in the Appendix), then we conclude that \( G(\lambda) = Z_{11}(\lambda) \), having also assumed vanishing or periodic boundary conditions at \( \pm \tau \).

As in the space-like description we may derive the generating function that provides the hierarchy of \( U \)-operators associated to each one of the time-like Hamiltonians. Indeed, taking into consideration the zero curvature condition as well the time-like Poisson structure satisfied by \( V \) one can show that the generating function of the \( U \)-components of the Lax pairs is given by (see also [23])

\[ U_2(t, \lambda, \mu) = t^{-1}(\lambda) t r_1 \left( T_1(\tau, t, \lambda) r_1(\mu) T_1(t, -\tau, \lambda) \right), \]  

(5.13)

where \( t(\lambda) = tr(T(\lambda)) \). In the case where the \( r \)-matrix is the Yangian (5.3), and after taking into consideration the decomposition (5.10), the latter expression becomes

\[ U(t, \lambda, \mu) = \frac{t^{-1}(\lambda)}{\lambda - \mu} T(t, -\tau, \lambda) T(\tau, t, \lambda) \]

\[ = \frac{1}{\lambda - \mu} (1 + W(t, \lambda)) D (1 + W(t, \lambda))^{-1}, \]  

(5.14)

where \( D \) is a \( 2 \times 2 \) matrix: \( D = \text{diag}(1, 0) \).

Now using the expression for the generating function \( G(\lambda) = Z_{11}(\lambda) \) and the findings presented in the Appendix we identify the first couple of integrals of motion for the time-like hierarchy, in analogy to the space-like case (see e.g. [13]). Consequently, we make note

\[ ^2\text{Conserved with respect to time variations for the space-monodromy matrix, and \textquotedblleft conserved\textquotedblright\ with respect to spatial variations for the monodromy matrix built using \( V \).} \]
here of the first few integrals of motion (see also [23]):

\[ H^{(1)} = \int_{\tau}^{\tau} (u\pi - \hat{\pi}u) dt, \]
\[ H^{(2)} = \int_{\tau}^{\tau} ((u\hat{u})^2 - u_t\hat{u} - \hat{\pi}u) dt, \]
\[ H^{(3)} = \int_{\tau}^{\tau} (\hat{\pi}_t \hat{u} - u_t\pi) dt, \]
\[ H^{(4)} = \int_{\tau}^{\tau} (u_{tt} \hat{u} + \hat{\pi}_t \hat{\pi} - u\hat{u})(2u_t\hat{u} + u\hat{u}_t - \pi^2 u^2 - u^2 \pi^2 + 2\hat{\pi}\pi u\hat{u})) dt. \]

(5.15)

where we use the shorthand notation: \( F_t = \partial_t F, F_{tt} = \partial_{tt} F \), and so on. It is also worth noting that this description is in analogy to the relativistic case e.g. sine-Gordon model [13], where the Hamiltonian is expressed in terms of the sine-Gordon field \( \phi \) and its conjugate \( \pi \), where \( \pi \) is in turn expressed as a time derivative of the field via the corresponding equations of motion.

In addition to the derivation of the time-like charges in involution above we can also compute the corresponding \( U \)-operators of the time-like hierarchy via the expansion in powers of \( \frac{1}{k} \) of (5.14). The pair \((U^{(k)}, V)\) gives rise to the same equations of motion as Hamilton’s equations with the Hamiltonian \( H^{(k)} \) associated to the \( x_k \) flow. We provide below the first few members of the series expansion of \( U \) corresponding to the charges (5.15):

\[ U^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]
\[ U^{(2)} = \begin{pmatrix} \lambda & \hat{u} \\ u & 0 \end{pmatrix}, \]
\[ U^{(3)} = \begin{pmatrix} \lambda^2 - uu & \lambda\hat{u} + \pi \\ \lambda u - \hat{\pi} & u\hat{u} \end{pmatrix}, \]
\[ U^{(4)} = \begin{pmatrix} \lambda^3 - \lambda uu + \hat{\pi}\hat{u} - u\pi & \lambda^2 \hat{u} + \lambda\pi + \hat{u}_t \\ \lambda^2 u - \lambda\pi - u_t & \lambda uu - \hat{\pi}\hat{u} + u\pi \end{pmatrix}. \]

(5.16)

Having identified both the charges in involution as well as the various \( U \)-operators, let us focus on the second member of the hierarchy. In particular, let us obtain via the Hamiltonian \( H^{(2)} \) (and the time-like Poisson relations) and/or the Lax pair \((V, U^{(2)})\) the corresponding equations of motion. First we obtain (see also [23])

\[ \pi(x,t) = \partial_x \hat{u}(x,t), \quad \hat{\pi}(x,t) = \partial_x u(x,t). \]

(5.17)

and the equations of motion read as

\[ \partial_t u - \partial_x^2 u - 2\hat{u}u^2 = 0. \]

(5.18)

Similarly, for \( \hat{u} \) (\( u \to -\hat{u}, \hat{u} \to u \)). This concludes our brief review of the results for dual integrable systems with periodic boundary conditions.
5.2 Open Boundary Conditions

We come now to the more interesting scenario where integrable boundary conditions are implemented along the time axis. Space-like boundary conditions for NLS and its generalizations have been investigated (see e.g. [16, 8]), so we only concern ourselves here with time-like boundaries. This is indeed the first time that such conditions are systematically implemented and studied in the context of integrable models. Based on the fundamental relations (V-Poisson) we extend the idea of Sklyanin’s modified monodromy along the time axis. Then via the time-like reflection algebra we are able to construct the generating function of time-like Hamiltonians as well as the generating function of the $U^\pm$-operators in the presence of boundaries.

The key object in our analysis is Sklyanin’s modified monodromy matrix along the time axis, which is given as

$$ T(\lambda) = T(\lambda)K^-(\lambda)\hat{T}(\lambda), \quad \text{(5.19)} $$

where $T$ is the time-like monodromy, $\hat{T}(\lambda) = V T^t(-\lambda)V$ with $V = \text{antidiag}(i, -i)$. Let $K^\pm$ be $c$-number solutions of the classical reflection equation [16]:

$$ \left\{ K^+_1(\lambda), K^+_2(\mu) \right\} = \left[ r_{12}(\lambda - \mu), K^+_1(\lambda)K^+_2(\mu) \right] + K^+_1(\lambda)r_{12}(\lambda + \mu)K^+_2(\mu) - K^+_2(\mu)r_{12}(\lambda + \mu)K^+_1(\lambda), \quad \text{(5.20)} $$

and consequently $T$ is also a solution of the reflection equation. Note that a $c$-number solution of the reflection equation is a “non-dynamical” solution: $\left\{ K^+_1(\lambda), K^+_2(\mu) \right\} = 0$. For $r$ being the Yangian (5.3), the most general $K^\pm$-matrices (up to some overall multiplicative factor) are given by [31]

$$ K^\pm(\lambda) = \begin{pmatrix} \lambda + i\xi^\pm & i\kappa^\pm \lambda \\ i\kappa^\pm \lambda & -\lambda + i\xi^\pm \end{pmatrix}, \quad \text{(5.21)} $$

where $\xi^\pm, \kappa^\pm$ are some arbitrary constants.

As in the periodic case we define the generating function of the time-like Hamiltonians for the model with open boundary conditions:

$$ G(\lambda) = \ln \left( t(\lambda) \right), \quad t(\lambda) = \text{tr} \left( K^+(\lambda)T(\lambda)K^-(\lambda)\hat{T}(\lambda) \right), \quad \text{(5.22)} $$

Taking into account the standard decomposition of the monodromy (5.10), as well the behavior of the $Z$ as $\lambda \to \infty$ we conclude

$$ G(\lambda) = Z_{11} + \hat{Z}_{11} + \ln \left( \mathbb{W}^+ \right) + \ln \left( \mathbb{W}^- \right), \quad \text{(5.23)} $$

where we define in general $\hat{f}(\lambda) = f(-\lambda)$, and after taking into account the standard decomposition of the monodromy matrix (5.10):

$$ \mathbb{W}^+ = \left( (1 + \hat{W}^t(t, \lambda)) V K^+(\lambda)(1 + W(t, \lambda)) \right)_{11}, \quad \text{(5.24)} $$

$$ \mathbb{W}^- = \left( (1 + W(-t, \lambda))^{-1} K^-(\lambda) V (1 + \hat{W}(-t, \lambda))^{-1} \right)^t_{11}. \quad \text{(5.25)} $$

---

3We could allow these to actually be functions of the evolution variable, i.e. “dynamical” boundary conditions. Doing so would have no effect on our derivations (except to make the expressions bulkier by writing in the parameter dependence), so we choose to ignore this case for now.
We focus here on the “dual point”, that is when both time and space-like descriptions lead to the same integrable PDEs; this precisely corresponds to the NLS model [23]. We first derive the boundary Hamiltonian, expressed in three distinct parts: the bulk Hamiltonian generated by \((Z_{11} + \bar{Z}_{11})\), and the two boundary Hamiltonians \(H^{(2)}_{\pm}\), generated by \(\ln(W_{\pm})\) (we have multiplied (5.23) by \(\frac{1}{2}\))

\[
H^{(2)} = \int_{-\tau}^{\tau} ((u\dot{u})^2 - u_t\dot{u} - \pi\dot{u}) dt + H_{\pm}^{(2)} + H^{(2)},
\]

where the boundary contributions evaluated at \(t = \pm \tau\) are given by

\[
H_{\pm}^{(2)} = \left(\frac{\xi^\pm u}{\kappa^\pm} - \frac{i\pi}{\kappa^\pm} + \frac{u^2}{2}\right)\bigg|_{t=\pm \tau}, \quad H^{(2)} = \left(\frac{\xi^-\dot{u}}{\kappa^-} - \frac{i\pi}{\kappa^-} + \frac{\dot{u}^2}{2}\right)\bigg|_{t=-\tau}.
\]

As in the periodic case using the fact that \(T\) and \(\hat{T}\) satisfy a quadratic algebra as well as the zero curvature condition we can derive in analogy to [18] the explicit form of the generating function of the boundary \(U\)-operators:

\[
U_2(t, \lambda) = t^{-1}(\lambda) \text{tr}_1 \left( K^+_1(\lambda) T_1(\tau, t, \lambda) r_{12}(\lambda - \mu) T_1(\tau, -\tau, \lambda) K^-_1(\lambda) \hat{T}_1(\tau, -\tau, \lambda) \right) + t^{-1}(\lambda) \text{tr}_1 \left( K^+_1(\lambda) T_1(\tau, -\tau, \lambda) K^-_1(\lambda) \hat{T}_1(\tau, -\tau, \lambda) r_{12}(\lambda + \mu) \hat{T}_1(\tau, t, \lambda) \right), \quad t \neq \pm \tau,
\]

and at the boundary points \(\pm \tau\):

\[
U_2(\tau, \lambda, \mu) =\]

\[
U_2(-\tau, \lambda, \mu) = t^{-1}(\lambda) \text{tr}_1 \left( K^+_1(\lambda) r_{12}(\lambda - \mu) T_1(\tau, -\tau, \lambda) K^-_1(\lambda) \hat{T}_1(\tau, -\tau, \lambda) \right) + t^{-1}(\lambda) \text{tr}_1 \left( K^+_1(\lambda) T_1(\tau, -\tau, \lambda) r_{12}(\lambda + \mu) K^-_1(\lambda) \hat{T}_1(\tau, -\tau, \lambda) \right).
\]

Next, we supply the \(U\)-matrices associated to the boundary Hamiltonian \(H^{(2)}\). From expression (5.28) we obtain the familiar bulk NLS \(U\)-operator (we have multiplied expressions (5.28)-(5.30) by \(\frac{1}{\frac{1}{2}}\))

\[
U^{(2)} = \left(\frac{\lambda^\pm}{2} - \frac{i\pi}{\kappa^\pm} + \frac{\lambda^\pm + u + \xi^\pm}{\kappa^\pm}\right).
\]

We now turn to the boundary \(U\)-matrices, evaluated at \(\pm \tau\) from expressions (5.29), (5.30):

\[
U^{(2)}_+ = \left(\frac{\lambda^\pm}{2} - \frac{i\pi}{\kappa^\pm} + \frac{\lambda^\pm + u + \xi^\pm}{\kappa^\pm}\right), \quad U^{(2)}_- = \left(\frac{\lambda^\pm}{2} + \tilde{\lambda} + \frac{\tilde{\lambda}}{\kappa^-} + \frac{\tilde{\lambda}}{\kappa^+} - \frac{\dot{u}^2}{2} + \frac{i\pi}{\kappa^-}\right).
\]
$U_\pm = U + \delta U_\pm$, by requiring $\delta U_\pm = 0$ we directly identify the boundary conditions, which in this case read as

\begin{equation}
\hat{u}(\tau) = \frac{\xi^+}{\kappa^+}, \quad u(\tau) = 0,
\end{equation}

\begin{equation}
\hat{u}(-\tau) = 0, \quad u(-\tau) = \frac{\xi^-}{\kappa^-},
\end{equation}

subject to the extra constraint $\kappa^\pm, \xi^\pm \gg 1$, so that the $\lambda$-dependence in the anti-diagonal terms in (5.32) becomes negligible.

In our discussion here we have used the strong integrability argument based on the existence of the classical $r$-matrix and the underlying Poisson structure. In general, derivation of space and time-like boundary conditions based exclusively on the existence of a Lax pair and the dressing process (together with a suitable boundary GLM equation) is one of the next key issues to address. Within this spirit the issue of time-like dressing in the absence of Poisson structure naturally emerges. The time-like dressing should involve time-like differential and integral operators as dressing transformations, which should in turn provide the solutions of the associated integrable PDEs. We hope to report on these and relevant important open issues mentioned throughout the text in forthcoming publications.

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A Time Riccati equation: $W, Z$ matrices

In this Appendix we compute the first few members of the expansion $\sum \frac{W^{(n)}}{\lambda^n}, \sum \frac{Z^{(n)}}{\lambda^n}$ by solving the time Riccati equation (5.11). Indeed, solving the time Riccati equation at each
order of the $\frac{1}{\lambda}$ expansion we obtain:

$$W^{(1)} = \begin{pmatrix} 0 & -\hat{u} \\ u & 0 \end{pmatrix},$$

$$W^{(2)} = \begin{pmatrix} 0 & -\pi \\ -\hat{\pi} & 0 \end{pmatrix},$$

$$W^{(3)} = \begin{pmatrix} 0 & -u_t + u\hat{u} \\ -u_t + u\hat{u} & 0 \end{pmatrix},$$

$$W^{(4)} = \begin{pmatrix} 0 & -\pi_t - \pi\hat{u}^2 \\ -\pi_t - \pi\hat{u}^2 & 0 \end{pmatrix},$$

$$W^{(5)} = \begin{pmatrix} 0 & 0 \\ u_t - \pi^2 \hat{\psi} - u^2 \hat{u}_t - 2u(u_t \hat{u} + \pi \hat{\pi}) & 0 \end{pmatrix}.$$  

We can use these above expressions to calculate the first few elements in the expansion of $Z$, via the equation for the diagonal part $Z^{(-2)}$ [5,12]

$$Z^{(-2)} = \begin{pmatrix} \tau & 0 \\ 0 & -\tau \end{pmatrix},$$

$$Z^{(1)} = \begin{pmatrix} \int_{-\tau}^{\tau} (u\pi - \hat{u}\hat{\pi}) dt \\ 0 \end{pmatrix} - \begin{pmatrix} \int_{-\tau}^{\tau} (u\pi - \hat{u}\hat{\pi}) dt \\ \int_{-\tau}^{\tau} ((u\hat{\pi})^2 + u\hat{u}_t - \pi\hat{\pi}) dt \end{pmatrix},$$

$$Z^{(2)} = \begin{pmatrix} \int_{-\tau}^{\tau} ((u\hat{\pi})^2 - u_t\hat{\pi} - \pi\hat{\pi}) dt \\ 0 \end{pmatrix} - \begin{pmatrix} \int_{-\tau}^{\tau} ((u\hat{\pi})^2 + u\hat{u}_t - \pi\hat{\pi}) dt \\ \int_{-\tau}^{\tau} ((u\hat{\pi})^2 + u\hat{u}_t - \pi\hat{\pi}) dt \end{pmatrix},$$

$$Z^{(3)} = \begin{pmatrix} \int_{-\tau}^{\tau} (\hat{\pi}_t \hat{u} - u_t \pi) dt \\ 0 \end{pmatrix} - \begin{pmatrix} \int_{-\tau}^{\tau} (\hat{\pi}_t \hat{u} - u_t \pi) dt \\ \int_{-\tau}^{\tau} ((u\hat{\pi})^2 + u\hat{u}_t - \pi\hat{\pi}) dt \end{pmatrix},$$

$$Z^{(4)} = \int_{-\tau}^{\tau} \left( u_{tt}\hat{u} + \hat{\pi}_t \pi - u\hat{u}(2u_t \hat{u} + u\hat{u}_t) \right) dt - \int_{-\tau}^{\tau} (\hat{\pi}^2 \hat{u}^2 + u^2 \pi^2 - 2\pi\hat{\pi}u\hat{u}) dt \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

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