Another point in homological algebra: Duality for discontinuous group actions

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Abstract
We consider discontinuous operations of a group $G$ on a contractible $n$-dimensional manifold $X$. Let $E$ be a finite dimensional representation of $G$ over a field $k$ of characteristics 0. Let $\mathcal{E}$ be the sheaf on the quotient space $Y = G \setminus X$ associated to $E$. Let $H^\bullet(Y; \mathcal{E})$ be the image in $H^\bullet(Y; \mathcal{E})$ of the cohomology with compact support. In the cases where both $H^\bullet(Y; \mathcal{E})$ and $H^\bullet(Y; \mathcal{E}^*)$ ($\mathcal{E}^*$ being the sheaf associated to the representation dual to $E$) are finite dimensional, we establish a non-degenerate duality between $H^m(Y; \mathcal{E})$ and $H^{n-m}(Y; \mathcal{E}^*)$. We also show that this duality is compatible with Hecke operators.

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1 Introduction
We consider discontinuous operations of a group $G$ on a contractible $n$-dimensional manifold $X$. Let $E$ be a finite dimensional representation of
$G$ over a field $k$ of characteristics 0, say. Associated to $E$ is a sheaf $\mathcal{E}$ on the quotient space $Y = G \setminus X$, and a theorem of Grothendieck [G] says that there is an isomorphism

$$H^\bullet(G; E) \cong H^\bullet(Y; \mathcal{E}).$$

This is a result which is basic for the cohomology theory of groups. In $H^\bullet(Y; \mathcal{E})$, we have the canonical subgroup $H^\bullet(Y; \mathcal{E})$ which is the image of the cohomology with compact supports. In Section 2, we show that there is a non-degenerate pairing

$$B : H^m_1(Y; \mathcal{E}) \otimes H^{n-m}_1(Y; \mathcal{E}^*) \to k.$$

This of course gives rise to isomorphisms

$$H^m_1(Y; \mathcal{E})^* \cong H^{n-m}_1(Y; \mathcal{E}^*)$$

where $\mathcal{E}^*$ is the sheaf associated to the dual representation $E^*$ and where $H^m_1(Y; \mathcal{E})^*$ is the dual of $H^m_1(Y; \mathcal{E})$.

Special instances of this form of Poincaré duality were frequently used. In full generality, we derive it by using Grothendieck’s arguments together with Poincaré-Verdier duality. In fact we feel that our paper is another (perhaps missing) chapter in the work [G] of Grothendieck.

In Section 3, we show that the above duality isomorphism (3) is compatible with Hecke operators.

## 2 Poincaré duality

Let $X$ be a connected, orientable, $n$-dimensional manifold and let $G$ be a discrete group operating effectively and smoothly on $X$ preserving the orientation. We assume that this operation is discontinuous, i.e. the following two conditions are satisfied:

- For each $x \in X$, the stabilizer $G_x$ is finite.
- Each $x$ admits a neighborhood $V$ such that

$$\gamma \cdot V \cap V = \emptyset \quad \text{for all} \quad \gamma \in G \setminus G_x.$$

We write $Y := G \setminus X$ and denote by $p : X \to Y$ the natural projection. Let $X_{\operatorname{reg}}$ be the open subset of $X$ consisting of all points with trivial stabilizer.

Since the smooth operation of a finite group looks, in the neighborhood of a fixed point of the group, like a linear operation [B1], we have the following two properties:
Fact 2.1 $X \setminus X_{\text{reg}}$ has at least codimension 2 in $X$.

Fact 2.2 Every point of $Y$ has an open neighborhood $U$ such that there exists an open subset $V$ of $X$ diffeomorphic to $\mathbb{R}^n$ and a finite subgroup $H$ of $G$ such that $H \cdot V = V$ and that $p$ induces a homeomorphism from $H \setminus V$ onto $U$.

Since paths in quotients by finite groups can be lifted ([B1], Th. 6.2, p. 91), we conclude from 2.2:

Fact 2.3 Let $A$ be a pathwise connected subset of $Y$ and let $A_0$ be a path component of $p^{-1}(A)$. Then:

a) $p^{-1}(A) = G \cdot A_0$.

b) For each $\gamma \in G$, either $\gamma \cdot A_0 = A_0$ or $\gamma \cdot A_0 \cap A_0 = \emptyset$.

Let $k$ be a field with the following property: For each $x \in X$, the characteristic of $k$ is prime to the order of $G_x$.

With the terminology of [B2], Def. V.9.1 and using [B2], Th. II.19.2 and [B2], Th. V.19.3, we then have:

Fact 2.4 $Y$ is an orientable $n$-dimensional homology manifold over $k$.

By a representation of $G$, we mean a finite dimensional vector space $E$ over $k$ with a linear operation of $G$ on it. A sheaf will always be a sheaf of $k$-vector spaces.

We consider the following functor $\mathcal{S}$ from the category of representations of $G$ to the category of sheaves on $Y$: For a representation $E$, the sheaf $\mathcal{E} := \mathcal{S}(E)$ is given by

$$
\Gamma(U; \mathcal{E}) := \{ \text{locally constant $G$-equivariant maps } p^{-1}(U) \to E \}
$$

for open subsets $U$ of $Y$. By 2.3, the stalks of $\mathcal{E}$ are given by

$$
\mathcal{E}_y = \{ G\text{-equivariant maps } p^{-1}(y) \to E \}.
$$

The constant sheaf on $X$ with stalks $E$, denoted by $E_X$, is a special instance of a $G$-sheaf on $X$. For any $G$-sheaf $\mathcal{F}$ on $X$, the direct image sheaf $p_*^G(\mathcal{F})$ is a $G$-sheaf on $Y$, where $G$ operates trivially on $Y$. Let $p_*^G(\mathcal{F})$ be the subsheaf of $p_*(\mathcal{F})$ consisting of those elements which are fixed under $G$. With this notation, we have

$$
\mathcal{S}(E) = p_*^G(E_X).
$$
More generally, following [G] (5.2.4), we introduce the sheaves
\[ \mathcal{H}^m(G; \mathcal{F}) := R^m p^*_s(\mathcal{F}) \]
on Y so that \( \mathcal{S}(E) = \mathcal{H}^0(G; E_X) \).

We denote by \( \Gamma^G_X(F) \) the \( G \)-invariant sections of \( F \) and put as in [G] (5.2.3):
\[ H^0_m(X; G, F) := R^m \Gamma^G_X(F) . \]
Moreover, let \( \Gamma^G_{X,c}(F) \) denote the elements in \( \text{Gamma}_X(F) \) with compact support and put as in [G] (5.7.2):
\[ H^0_c(X; G, F) := R^m \Gamma^G_{X,c}(F) . \]

By [G], Th. 5.2.1 there are two spectral sequences
\[ (H^p(Y; \mathcal{H}^q(G; \mathcal{F})))_{p,q} \Rightarrow H^\bullet(X; G, \mathcal{F}) , \]  
(4)  
\[ (H^p(G; H^q(X; \mathcal{F})))_{p,q} \Rightarrow H^\bullet(X; G, \mathcal{F}) . \]  
(5)

For a finite group \( G \), there are, by [G] (5.7.4) and (5.7.7), analogs for the cohomology with compact supports.
\[ (H^p_c(Y; \mathcal{H}^q(G; \mathcal{F})))_{p,q} \Rightarrow H^\bullet_c(X; G, \mathcal{F}) , \]  
(6)  
\[ (H^p(G; H^q_c(X; \mathcal{F})))_{p,q} \Rightarrow H^\bullet_c(X; G, \mathcal{F}) . \]  
(7)

From the spectral sequences (3) and (4), Grothendieck deduces the basic relation (1). Similarly, from (5) and (6) we get:

**Fact 2.5** If the group \( G \) is finite and if \( \text{char } k \) is prime to the order of \( G \), we have
\[ H^m_c(Y; \mathcal{E}) \cong H^m_c(X; E)^G . \]

For two sheaves \( \mathcal{F}, \mathcal{G} \) on \( Y \) and \( p \geq 0 \), we have the sheaf \( \mathcal{E}xt^p(\mathcal{F}, \mathcal{G}) \). It is the sheafification of the presheaf
\[ U \mapsto \text{Ext}^p(\mathcal{F}| U, \mathcal{G}| U) . \]

We write \( \mathcal{H}om \) instead of \( \mathcal{E}xt^0 \). Observe that the presheaf
\[ U \mapsto \text{Hom}(\mathcal{F}| U, \mathcal{G}| U) \]
is a sheaf, hence \( \mathcal{H}om(\mathcal{F}, \mathcal{G})(U) = \text{Hom}(\mathcal{F}| U, \mathcal{G}| U) . \)
Lemma 2.6 For two representations $E, F$ of $G$ and their associated sheaves $\mathcal{E}, \mathcal{F}$, there is a canonical isomorphism

$$ S(\text{Hom}_k(E, F)) \cong \text{Hom}(\mathcal{E}, \mathcal{F}). $$

Proof. Writing $G := S(\text{Hom}_k(E, F))$, we have a canonical map

$$ \alpha : G \to \text{Hom}(\mathcal{E}, \mathcal{F}). $$

The non-trivial point is to show that $\alpha_y : G_y \to \text{Hom}(\mathcal{E}, \mathcal{F})_y$ is surjective for all $y \in Y$. We have

$$ G_y = \{ \text{G-equivariant maps } \varphi : p^{-1}(y) \to \text{Hom}_k(E, F) \}. $$

Fixing a point $x \in p^{-1}(y)$, the evaluation $\varphi \mapsto \varphi_x$ gives an isomorphism

$$ G_y \cong \text{Hom}_kG_x(E, F). $$

Given an element $\delta \in \text{Hom}(\mathcal{E}, \mathcal{F})_y$, we choose an open neighborhood $U$ of $y$ in $Y$ which is so small that each component of $p^{-1}(U)$ contains exactly one point of $p^{-1}(y)$ and that $\delta$ can be represented by an element $\psi \in \text{Hom}(\mathcal{E}|U, \mathcal{F}|U)$. Let $U_0$ be the component of $p^{-1}(U)$ containing $x$. Let $\xi \in U_0 \cap X_{\text{reg}}$ and $\eta := p(\xi)$. Evaluation in $\xi$ gives isomorphisms $\mathcal{E}_\eta \cong E$ and $\mathcal{F}_\eta \cong F$. Hence the homomorphism $\psi_\eta : \mathcal{E}_\eta \to \mathcal{F}_\eta$ gives a $k$-homomorphism $\varphi : E \to F$. Using 2.1, it is easy to see that $\varphi$ doesn’t depend on the choice of $\xi$. Moreover, this implies that $\varphi \in \text{Hom}_kG_x(E, F) \cong G_y$. It is clear that $\alpha_y(\varphi) = \delta$. \hfill $\square$

Given a representation $E$ of $G$, we denote by $E^* := \text{Hom}_k(E, k)$ the dual representation and write $\mathcal{E}^* := S(E^*)$.

Corollary 2.7 $\text{Hom}(\mathcal{E}, \mathcal{F}) \cong \text{Hom}(\mathcal{F}^*, \mathcal{E}^*)$.

Because of 2.4, we can apply Poincaré-Verdier duality (see e.g. [GM], (III. 70)) and obtain:

$$ \text{Hom}_k(H^m_c(Y; \mathcal{E}), k) \cong \text{Ext}^{n-m}(\mathcal{E}, k_Y). \quad (8) $$

Lemma 2.8 $\text{Ext}^p(\mathcal{E}, k_Y) = 0$ for $p > 0$.

Proof. This is a local property. By 2.2 it suffices to assume that $X$ is diffeomorphic to $\mathbb{R}^n$ and that $G$ is finite. We have

$$ \text{Ext}^p(\mathcal{E}, k_Y) \cong \text{Hom}_k(H^{n-p}_c(Y; \mathcal{E}), k) \quad \text{by (7)} $$

$$ \cong \text{Hom}_k(H^{n-p}_c(X; E^G), k) \quad \text{by 2.5}. $$

Since $X \cong \mathbb{R}^n$, we have $H^{n-p}_c(X; E) = 0$ for $p > 0$. \hfill $\square$
Lemma 2.9 For each $p \geq 0$, there is a canonical isomorphism

$$\text{Ext}^p(\mathcal{E}, k_Y) \cong H^p(Y; \mathcal{E}^*) .$$

Proof. We use the spectral sequence ([G], Th. 4.2.1)

$$(H^p(Y; \text{Ext}^q(\mathcal{E}, k_Y)))_{p,q} \Rightarrow \text{Ext}^\bullet(\mathcal{E}, k_Y) .$$

By 2.8, we obtain $\text{Ext}^p(\mathcal{E}, k_Y) \cong H^p(Y; \mathcal{H}om(\mathcal{E}, k_Y))$. Then apply 2.7. $\square$

There is a canonical homomorphism of sheaves

$$\mu : \mathcal{E} \otimes \mathcal{E}^* \rightarrow k_Y,$$

which induces

$$\mu_* : H^n_c(Y; \mathcal{E} \otimes \mathcal{E}^*) \rightarrow H^n_c(Y; k) .$$

Composing with the cup-product

$$\cup : H^n_c(Y; \mathcal{E}) \otimes H^{n-m}(Y; \mathcal{E}^*) \rightarrow H^n_c(Y; \mathcal{E} \otimes \mathcal{E}^*)$$

and with

$$\int : H^n_c(Y; k) \rightarrow k ,$$

we obtain for each $m \geq 0$ a homomorphism

$$B : H^n_c(Y; \mathcal{E}) \otimes H^{n-m}(Y; \mathcal{E}^*) \rightarrow k .$$

Combining Poincaré-Verdier duality (7) and in particular the remark in [GM] following the formula (III.70) with 2.9, we obtain

Lemma 2.10 The pairing $B$ is non-singular for each $m \geq 0$. $\square$

For a sheaf $\mathcal{F}$ on $Y$, let us write

$$H^\bullet_1(Y; \mathcal{F}) := \text{im}(H^\bullet_c(Y; \mathcal{F}) \rightarrow H^\bullet(Y; \mathcal{F})) .$$

The properties of the cup-product, in particular its anti-commutativity, show that 2.10 implies the following duality for $H^\bullet_1$:

Theorem 2.11 Let $X$ be an orientable, connected, $n$-dimensional manifold and let $G$ be a group operating smoothly and discontinuously on $X$ preserving the orientation. Let $k$ be a field such that for each $x \in X$, the characteristic of $k$ is prime to the order of the stabilizer $G_x$. Let $E$ be a representation of $G$ on a finite-dimensional $k$-vector space and let $E^*$ be the dual representation. We denote by $\mathcal{E}$ the sheaf on $Y = G \setminus X$ associated to $E$ and by $\mathcal{E}^*$ the sheaf associated to $E^*$ and assume that all cohomology groups $H^m(Y; \mathcal{E})$ and $H^m(Y; \mathcal{E}^*)$ are finite dimensional $k$-vector spaces. Then $B$ induces for each $m \geq 0$ a non-singular pairing of finite dimensional $k$-vector spaces

$$B : H^m_1(Y; \mathcal{E}) \otimes H^{n-m}_1(Y; \mathcal{E}^*) \rightarrow k .$$  (9)
3 Hecke operators

Let us now assume in addition that there is a group $\mathfrak{G}$ with

$$G \subseteq \mathfrak{G} \subseteq \text{Diff} X$$

and that $E$ is actually a representation of $\mathfrak{G}$. Let $g \in \mathfrak{G}$ be an element in the commensurator of $G$, that is, we assume that the two groups

$$\Delta' := gGg^{-1} \cap G,$$

$$\Delta'' := G \cap g^{-1}Gg = g^{-1}\Delta'g$$

are of finite index in $G$. Consider the following diagram:

$$\begin{array}{ccccccc}
X & \xrightarrow{p'} & \Delta' \setminus X & \xrightarrow{\varphi} & \Delta'' \setminus X & \leftarrow & X \\
p & & \pi' & & \pi'' & & p \\
G \setminus X & & G \setminus X & & & & \\
\end{array} \quad (10)$$

All the arrows except $\varphi$ are natural projections; $\varphi = \varphi_g$ is the diffeomorphism given by

$$\varphi(\Delta'x) := \Delta''g^{-1}x = g^{-1}\Delta'x .$$

As we shall explain, we obtain from (8) homomorphisms

$$\begin{array}{c}
H^\bullet(\Delta' \setminus X; \mathcal{E}') & \xrightarrow{\varphi^*} & H^\bullet(\Delta'' \setminus X; \mathcal{E}'') \\
\tau & & \pi''^* \\
H^\bullet(G \setminus X; \mathcal{E}) & & H^\bullet(G \setminus X; \mathcal{E}) \\
\end{array} \quad (11)$$

Here $\mathcal{E}$ is the sheaf introduced in section 2, and $\mathcal{E}'$ (resp. $\mathcal{E}''$) are the analogous sheaves on $\Delta' \setminus X$ (resp. $\Delta'' \setminus X$).

**Definition.** The Hecke operator $T(g)$ is the endomorphism of $H^\bullet(G \setminus X; \mathcal{E})$ given by

$$T(g) := \tau \circ \varphi^* \circ \pi''^* .$$

Let us now explain the three homomorphisms $\tau$, $\varphi^*$, and $\pi''^*$ in (9):

We have a canonical inclusion $i' : \mathcal{E} \hookrightarrow \pi'_*\mathcal{E}^*$. We obtain induced homomorphisms

$$\pi''^* : H^\bullet(G \setminus X; \pi'_*\mathcal{E}') \rightarrow H^\bullet(\Delta' \setminus X; \mathcal{E}').$$
which is an isomorphism by the Vietoris mapping theorem ([B2], Th. II. 11.1) and

\[ \pi'^* = \pi'^{n} \circ i'_* : H^\bullet(G \setminus X; \mathcal{E}) \to H^\bullet(\Delta' \setminus X; \mathcal{E}'). \]

Similarly for \( \mathcal{E}'' \) instead of \( \mathcal{E}' \). This explains \( \pi'^* \).

Next, there is a sheaf homomorphism

\[ \sigma : \pi'_* \mathcal{E}' \to \mathcal{E} \]

defined as follows: Choose elements \( \gamma_1, \ldots, \gamma_m \in G \) with

\[ G = \coprod_{i=1}^{m} \gamma_i \Delta'. \]

For \( U \) open in \( G \setminus X \), define

\[ \sigma_U : \Gamma(U; \pi'_* \mathcal{E}') \to \Gamma(U; \mathcal{E}) \]

\[ \sigma_U(f)(x) := \sum_{i=1}^{m} \gamma_i \cdot f(\gamma_i^{-1} \cdot x) \]

for \( x \in p^{-1}(U) \). The definition of \( \sigma \) doesn’t depend on the choice of the \( \gamma_i \). We then have a transfer

\[ \tau := \sigma_* \circ (\pi'^{n})^{-1} : H^\bullet(\Delta' \setminus X; \mathcal{E}') \to H^\bullet(G \setminus X; \mathcal{E}) . \quad (12) \]

Observe that \( \tau \) is a special case of the transfer introduced in [B2], p.140-141. (Bredon’s assumption that \( G \) is a finite group is not at all used for the definition of the transfer called \( \mu^*_G/H \) by him.)

Finally, to explain \( \varphi^* \), we have to specify a \( \varphi \)-cohomomorphism

\[ \Phi : \mathcal{E}'' \sim \to \mathcal{E}' \]

(using the terminology of [B2], I. 4): For each \( z \in \Delta' \setminus X \), we need a homomorphism

\[ \Phi_z : \mathcal{E}''_{\varphi(z)} \to \mathcal{E}'_z . \]

Choosing an element \( x \in p'^{-1}(z) \), we have

\[ \mathcal{E}''_{\varphi(z)} = \{ \Delta'' \text{-equivariant maps } f : g^{-1}\Delta'x \to E \} , \]

\[ \mathcal{E}'_z = \{ \Delta' \text{-equivariant maps } \Delta'x \to E \} . \]

We put

\[ \Phi_z(f) : \delta'x \mapsto g \cdot f(g^{-1}\delta'x) . \]
Observe that this is the place where we actually use that $E$ is a representation of the group $\mathcal{G}$. In any case, together with the cohomomorphism $\Phi$, we obtain the induced homomorphism $\varphi^*$. This completes the definition of the Hecke operator $T(g)$.

The same construction works for cohomology with compact supports. Hence we get also Hecke operators

$$T(g) : H^*_c(G \setminus X; E) \to H^*_c(G \setminus X; E),$$

$$T(g) : H^*_1(G \setminus X; E) \to H^*_1(G \setminus X; E),$$

The following properties are easy to verify:

**Fact 3.1** On $H^0(G \setminus X; E) = E^G$, the operator $T(g)$ is given by

$$T(g) \cdot e = \sum_{i=1}^{m} \gamma_i g \cdot e$$

where the $\gamma_i$ are as above.

**Fact 3.2** If $\gamma, \gamma'$ are in $G$, we have

$$T(\gamma g \gamma') = T(g).$$

The transfer $\tau$ has the following well-known properties:

- $\tau \circ \pi^*$ is multiplication by the number $[G : \Delta]$.
- $\tau : H^0(\Delta' \setminus X; k) \to H^0(G \setminus X; k)$ is the homomorphism $[G : \Delta'] : k \to k$.
- The following diagram is commutative (use 2.1):

\[
\begin{array}{ccc}
H^*_c(\Delta' \setminus X; k) & \xrightarrow{\tau} & H^*_c(G \setminus X; k) \\
\downarrow{f} & & \downarrow{f} \\
k & & k
\end{array}
\]

- $\tau(b \cup \pi^*(c)) = \tau(b) \cup c$ (cf. [B2], p.405)

From these formulas, we conclude that Hecke operators behave well with respect to the pairing

$$B : H^m_c(Y; E) \otimes H^{n-m}(Y; E^*) \to k.$$ 

from section 2: We have

$$B(T(g) \cdot u \otimes v) = B(u \otimes T(g^{-1}) \cdot v).$$

(13)

So far in this section, we haven’t used any restriction on the characteristics of $k$; it is not even necessary that $k$ is a field. Now we see:
Theorem 3.3 Under the assumptions of the duality theorem 2.11, the pair $(H^m_\ell(Y; \mathcal{E}), T(g))$ consisting of a vector space and an endomorphism is dual to the pair $(H^{n-m}_\ell(Y; \mathcal{E}^*), T(g^{-1}))$.

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