ON THE FOURIER-JACOBI MODEL FOR SOME ENDOSCOPIC ARTHUR
PACKET OF $U(3) \times U(3):$ THE NON-TEMPERED CASE

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Abstract. For a tempered $L$-parameter of $U(n) \times U(n)$, it is known that there is a unique representation in their associated relevant Vogan $L$-packet which produces the unique Fourier-Jacobi model. We showed that this is true for some non-tempered Arthur parameter of $U(3) \times U(3)$. Furthermore, we specified such representation under the local Langlands correspondence for unitary group. The main tools we used are the see-saw identity, local theta correspondence and local Gross-Prasad conjecture for small rank cases.

1. Introduction

The Gross-Prasad conjecture deals with the restriction problem of the $p$-adic group. In this paper, we shall investigate it for some non-tempered case not yet treated before.

Let $E/F$ be a quadratic extension of number fields and $G = U(3)$ be the unitary group relative to $E/F$. Then $H = U(2) \times U(1)$ is the unique elliptic endoscopic group for $G$. In [19], Rogawski has defined a certain enlarged class of $L$-packet, or $A$-packet of $G$ using the endoscopic transfer of a one-dimensional character of $H$ to $G$. In more detail, let $\varrho = \otimes_v \varrho_v$ be an one-dimensional automorphic character of $H$. An $A$-packet $\Pi(\varrho) \simeq \otimes \Pi(\varrho_v)$ is the transfer of $\varrho$ with respect to functoriality for an embedding of $L$-groups $\xi: L_H \to L_G$. Then for all places $v$ of $F$, $\Pi(\varrho_v)$ contains a certain non-tempered representation $\pi^n(\varrho_v)$ and it contains an additional supercuspidal representation $\pi^s(\varrho_v)$ precisely when $v$ remains prime in $E$. Gelbart and Rogawski [11] showed that the representations in this $A$-packet arise in the Weil representation of $G$. Our goal is to study the branching rule of the representations in this $A$-packet.

For the branching problem, there is a fascinating conjecture, so called Gross-Prasad (GP) conjecture, which was first proposed by Gross and Prasad [6] for orthogonal group and later extended it to all classical group in [5]. Since our main theorem has to do with it, we shall give a brief review on the GP conjecture, especially for unitary group.

Let $E/F$ be a quadratic extension of local fields of characteristic zero. Let $V_{n+1}$ be a Hermitian space of dimension $n+1$ over $E$ and $W_n$ a skew-Hermitian space of dimension $n$ over $E$. Let $V_n \subset V_{n+1}$ be a nondegenerate subspace of codimension 1, so that if we set

$$G_n = U(V_n) \times U(V_{n+1}) \quad \text{or} \quad U(W_n) \times U(W_n)$$

and

$$H_n = U(V_n) \quad \text{or} \quad U(W_n),$$

then we have a diagonal embedding

$$\Delta : H_n \hookrightarrow G_n.$$
Let $\pi$ be an irreducible smooth representation of $G_n$. In the Hermitian case, one is interested in computing

$$\dim_{\mathbb{C}} \text{Hom}_{\Delta H_n}(\pi, \mathbb{C}).$$

We shall call this the Bessel case (B) of the GP conjecture. For the GP conjecture in the skew-Hermitian case, we need a notion of the Weil representation $\omega_{\psi, \chi, W_n}$, where $\psi$ is a nontrivial additive character of $F$ and $\chi$ is a character of $E^\times$ whose restriction to $F^\times$ is the quadratic character $\omega_{E/F}$ associated to $E/F$ by local class field theory. In this case, one is interested in computing

$$\dim_{\mathbb{C}} \text{Hom}_{\Delta H_n}(\pi, \omega_{\psi, \chi, W_n}).$$

We shall call this the Fourier–Jacobi case (FJ) of the GP conjecture. To treat both cases using one notation, we shall let $\nu = \mathbb{C}$ or $\omega_{\psi, \chi, W_n}$ in the respective cases.

By the result of [1], it is known

$$\dim_{\mathbb{C}} \text{Hom}_{\Delta H_n}(\pi, \nu) \leq 1$$

and so the next step is to specify irreducible smooth representations $\pi$ such that

$$\text{Hom}_{\Delta H_n}(\pi, \nu) = 1.$$ (A non-zero element of $\text{Hom}_{\Delta H_n}(\pi, \nu)$ is called the Bessel (Fourier-Jacobi) model of $\pi$ in the (skew) hermitian case.)

In [5], Gan, Gross, Prasad has brought this problem into a more general setting using the notion of relevant pure inner forms of $G_n$ and Vogan $L$-packets. A pure inner form of $G_n$ is a group of the form

$$G'_n = U(V'_n) \times U(V'_{n+1}) \text{ or } U(W'_n) \times U(W''_n)$$

where $V'_n \subseteq V'_{n+1}$ are $n$ and $n+1$ dimensional hermitian spaces over $E$ and $W'_n, W''_n$ are $n$-dimensional skew hermitian spaces over $E$.

Furthermore, if

$$V'_{n+1}/V'_n \cong V_{n+1}/V_n \text{ or } W'_n = W''_n,$$

we say that $G'_n$ is relevant pure inner form.

(Indeed, there are four pure inner forms of $G_n$ and among them, only two are relevant.)

If $G'_n$ is relevant, we set

$$H'_n = U(V'_n) \text{ or } U(W'_n),$$

so that we have a diagonal embedding

$$\Delta : H'_n \hookrightarrow G'_n.$$

Now suppose that $\phi$ is an $L$-parameter for the group $G_n$. Then the (relevant) Vogan $L$-packet $\Pi_{G_n}^R$ associated to $\phi$ consists of certain irreducible smooth representations of $G_n$ and its (relevant) pure inner forms $G'_n$ whose $L$-parameter is $\phi$. We denote the relevant Vogan $L$-packet of $\phi$ by $\Pi_{G_n}^R$.

With these notions, we can loosely state the result of Beuzart-Plessis ([2], [3], [4]) for Bessel case and Gan-Ichino ([5]) for Fourier-Jacobi case as follows:

**Theorem 1.1.** For a tempered $L$-parameter $\phi$ of $G_n$, the followings hold:

(i) $\sum_{\pi' \in \Pi_{G_n}^R} \dim_{\mathbb{C}} \text{Hom}_{\Delta H'_n}(\pi', \nu) = 1$.

(ii) Using the local Langlands correspondence for unitary group, we can pinpoint the unique $\pi' \in \Pi_{G_n}^R$ such that

$$\dim_{\mathbb{C}} \text{Hom}_{\Delta H'_n}(\pi', \nu) = 1.$$
To emphasize its dependence on the number \( n \), we denote the Bessel and Fourier-Jacobi case of Theorem 1.1 as (B)\(_n\) and (FJ)\(_n\) respectively and later we shall elaborate. Recall that an \( L \)-parameter for \( U(V) \) is called generic if \( L(s, \text{Ad} \circ \phi) = L(s, \text{As}^{(-1)^n} \circ \phi) \) is holomorphic at \( s = 1 \). The GP conjecture predicts that this theorem also hold for a generic \( L \)-parameter \( \phi \) of \( G_n \).

Our main theorem is to investigate (FJ)\(_3\) for some \( L \)-parameter of \( G_3 \) involving a non-tempered \( L \)-parameter of \( U(W_3) \). More precisely, we have

**Main Theorem.** For an irreducible smooth representation \( \pi_2 \) of \( U(W_3) \), let \( \pi = \pi^n(\theta) \otimes \pi_2 \) as a representation of \( G_3 \). Then

1. \( \text{Hom}_H(\pi, \omega_{\psi,\chi,W_3}) = 0 \) if \( \pi_2 \) does not come from the theta lift of \( U(V_2) \).
2. Assume that \( \pi_2 \) is the theta lift from \( U(V'_2) \) and let \( \phi = \phi^n \otimes \phi_2 \) be the \( L \)-parameter of \( \pi \). Then
   \[
   \sum_{\pi' \in \Pi^R} \text{dim}_C \text{Hom}_H(\pi', \omega_{\psi,\chi,W_3}) = 1.
   \]
3. Using the local Langlands correspondence for unitary groups, we can explicitly describe \( \pi' \in \Pi^R \) appearing in (ii) such that
   \[
   \text{dim}_C \text{Hom}_H(\pi', \omega_{\psi,\chi,W_3}) = 1.
   \]

**Remark 1.2.** As we shall see in Theorem 3.2, the \( L \)-parameter of \( \pi^n(\theta) \) is not only non-tempered but also non-generic. Thus if we choose the \( L \)-parameter of \( \phi_2 \) in \( \phi \) apart from those obtained by the theta lift from \( U(V_2) \) to \( U(V_3) \), then the first part of the Main Theorem tells us that the GP conjecture may not true for non-generic \( L \)-parameter of \( G_n \).

The proof of Main Theorem is based on the following see-saw diagram:

\[
\begin{array}{ccc}
U(W_3) \times U(W_3) & \rightarrow & U(V_2) \\
\downarrow & & \downarrow \\
U(W_3) & \rightarrow & U(V_1) \times U(V_1)
\end{array}
\]

Since all element in the \( A \)-packet \( \Pi(\theta) \) can be obtained by the theta lift from \( U(V_1) \), we can write \( \pi^n(\theta) = \Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega_{\psi,\chi,W_3}^\vee(\pi_2^\vee) \) where \( \sigma \) is an irreducible smooth character of \( U(V_1) \) and \( \psi, \chi \) are some characters, which are needed to fix a relevant Weil representation. Then by the see-saw identity, we have

\[
\text{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega_{\psi,\chi,W_3}^\vee, \pi_2^\vee) \simeq \text{Hom}_{U(V_1)}(\Theta_{\psi,\chi,V_2,W_3}(\pi_2^\vee), \sigma).
\]

From this, we see that for having \( \text{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega_{\psi,\chi,W_3}^\vee, \pi_2^\vee) \neq 0 \), it should be preceded \( \Theta_{\psi,\chi,V_2,W_3}(\pi_2^\vee) \neq 0 \). This accounts for (i) in the **Main Theorem** because

\[
\text{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega_{\psi,\chi,W_3}^\vee, \pi_2^\vee) \simeq \text{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \pi_2^\vee, \omega_{\psi,\chi,W_3}^\vee).
\]

If \( \Theta_{\psi,\chi,V_2,W_3}(\pi_2^\vee) \neq 0 \), then by the local theta correspondence, \( \pi_2^\vee \) should be \( \Theta_{\psi,\chi,V_2,V_3}(\pi_0) \), where \( \pi_0 \) is an irreducible representation of \( U(V_2) \). By applying (B)\(_1\), we can pinpoint \( \pi_0 \) and \( \sigma \) in the framework of local Langlands correspondence such that \( \text{Hom}_{U(V_1)}(\pi_0, \sigma) \neq 0 \). Next we shall use the precise local theta correspondences for \( (U(V_1), U(W_3)) \) and \( (U(V_1), U(W_1)) \) in order to transfer recipe for (B)\(_1\) to (FJ)\(_3\).

The rest of the paper is organized as follows; In Section 2, we shall give a brief sketch of the local Langlands correspondence for unitary group. In Section 3, we collect some results of the local theta correspondence for unitary group which we will use in the proof of our main results. In Section 4, we shall prove our Main Theorem as well as giving the recipe for the supercuspidal \( L \)-parameter.
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1.1. Notations. We fix some notations we shall use throughout this paper:

- $E/F$ be a quadratic extension of local fields of characteristic zero.
- $c$ is the non-trivial element of $\text{Gal}(E/F)$.
- $\text{Fr}_E$ is a Frobenius element of $\text{Gal}(\overline{E}/E)$.
- Denote by $\text{Tr}_{E/F}$ and $\text{N}_{E/F}$ the trace and norm maps from $E$ to $F$.
- $\delta$ is an element of $E$ such that $\text{Tr}_{E/F}(\delta) = 0$.
- Let $\psi$ be an additive character of $F$ and define
  \[
  \psi^E(x) := \psi(\frac{1}{2} \text{Tr}_{E/F}(\delta x)) \quad \text{and} \quad \psi^E_2(x) := \psi(\text{Tr}_{E/F}(\delta x)).
  \]
- Let $\chi$ be a character of $E^\times$ whose restriction to $F^\times$ is $\omega_{E/F}$, which is the quadratic character associated to $E/F$ by local class field theory.
- For an linear algebraic group $G$, denote its $F$-points by $G(F)$ or simply by $G$.

2. Local Langlands correspondence for unitary group

By the recent work of Mok [18] and Kaletha-Mínguez-Shin-White [17], the local Langlands correspondence is now known for unitary group conditional on the stabilization of the twisted trace formula. Since our main results are expressed using the local Langlands correspondence, we shall assume the local Langlands correspondence for unitary group. In this section, we list some of its properties which are used in this paper. Indeed, much of this section are excerpts from Section.2 in [8].

2.1. Hermitian and skew-Hermitian spaces. For $\varepsilon = \pm 1$, let $V$ be a finite $n$-dimensional vector space over $E$ equipped with a nondegenerate $\varepsilon$-Hermitian $c$-sesquilinear form $\langle \cdot, \cdot \rangle_V : V \times V \to E$. That means for $v, w \in V$ and $a, b \in E$,

\[
\langle av, bw \rangle_V = ab^c \langle v, w \rangle_V, \quad \langle w, v \rangle_V = \varepsilon \cdot \langle v, w \rangle_V^c.
\]

We define $\text{disc} V = (-1)^{(n-1)n/2} \cdot \det V$, so that

\[
\text{disc} V \in \begin{cases} 
F^\times/N_{E/F}(E^\times) & \text{if } \varepsilon = +1; \\
\delta^n \cdot F^\times/N_{E/F}(E^\times) & \text{if } \varepsilon = -1
\end{cases}
\]

and we could define $\epsilon(V) = \pm 1$ by

\[
\epsilon(V) = \begin{cases} 
\omega_{E/F}(\text{disc} V) & \text{if } \varepsilon = +1; \\
\omega_{E/F}(\delta^{-n} \cdot \text{disc} V) & \text{if } \varepsilon = -1.
\end{cases}
\]

By a theorem of Landherr, for a given positive integer $n$, there are exactly two isomorphism classes of hermitian spaces of dimension and they are distinguished form each other by $\epsilon(V)$. Let $U(V)$ be the unitary group of $V$ defined by

\[
U(V) = \{ g \in \text{GL}(V) \mid \langle gv, gw \rangle_V = \langle v, w \rangle_V \text{ for } v, w \in V \}.
\]

Then $U(V)$ turns out to be connected reductive algebraic group defined over $F$. 

2.2. \textit{L-parameters and component groups.} Let \( I_F \) be the inertia subgroup of \( \text{Gal}(\bar{F}/F) \). Let \( W_F = I_F \ltimes \langle \text{Fr}_F \rangle \) be the Weil group of \( F \) and \( WD_F = W_F \rtimes \text{SL}_2(\mathbb{C}) \) the Weil–Deligne group of \( F \). For a homomorphism \( \phi : WD_F \to \text{GL}_n(\mathbb{C}) \), we say that it is a representation of \( WD_F \) if

\begin{itemize}
  \item \( \phi \) is trivial on an open subgroup of \( I_F \),
  \item \( \phi \) is continuous and \( \phi(\text{Fr}_F) \) is semisimple,
  \item the restriction of \( \phi \) to \( \text{SL}_2(\mathbb{C}) \) is induced by a morphism of algebraic groups \( \text{SL}_2 \to \text{GL}_n \).
\end{itemize}

For a representation \( \phi \) of \( WD_F \), when the image of \( W_F \) is bounded, we say that \( \phi \) is tempered.

Define \( \phi^\vee \) by \( \phi^\vee(w) = t\phi(w)^{-1} \) and call this the contragredient representation of \( \phi \). If \( E/F \) is a quadratic extension of local fields and \( \phi \) is a representation of \( WD_E \), fix \( s \in W_F \setminus W_E \) and define a representation \( \phi^c \) of \( WD_E \) by \( \phi^c(w) = \phi(sw_1s^{-1}) \). It is known that the equivalence class of \( \phi^c \) is independent of the choice of \( s \). Then we say that \( \phi \) is conjugate self-dual if there is an isomorphism \( b : \phi \mapsto (\phi^\vee)^c \) and for \( \varepsilon = \pm 1 \), we say that \( \phi \) is conjugate self-dual with sign \( \varepsilon \) if \( (b^\vee)^c = \varepsilon \cdot b \).

Let \( V \) be an \( n \)-dimensional \( \varepsilon \)-hermitian space over \( E \) and an \( L \)-parameter for the unitary group \( U(V) \) is a homomorphism

\[
\phi : WD_F \longrightarrow L U(V) = \text{GL}_n(\mathbb{C}) \rtimes \text{Gal}(E/F)
\]

such that

- the composite of \( \phi \) with the projection onto \( \text{GL}_n(\mathbb{C}) \) is a representation of \( WD_F \)
- the composite of \( \phi \) with the projection onto \( \text{Gal}(E/F) \) is the natural projection of \( WD_F \) to \( \text{Gal}(E/F) \).

The following proposition in [5, §8] enable us to removes the cumbersome \( \text{Gal}(E/F) \)-factor in the definition of \( L \)-parameter of \( U(V) \).

\textbf{Proposition 2.1.} Restriction to \( W_E \) of \( WD_F \) in \( WD_F \) gives a bijection between the set of \( L \)-parameters for \( U(V) \) and the set of equivalence classes of conjugate self-dual representations

\[
\phi : WD_E \longrightarrow GL_n(\mathbb{C})
\]

of sign \((-1)^{n-1}\).

With this proposition, henceforth, we shall mean an \( L \)-parameter for \( U(V) \) by \( n \)-dimensional conjugate self-dual representation \( \phi \) of \( WD_E \) with sign \((-1)^{n-1}\).

Given an \( L \)-parameter \( \phi \) of \( U(V) \), we can write \( \phi \) as a direct sum

\[
\phi = \bigoplus_i m_i \phi_i
\]

with pairwise inequivalent irreducible representations \( \phi_i \) of \( WD_E \) with multiplicities \( m_i \). We say that \( \phi \) is square-integrable if it has no multiplicity (i.e. \( m_i = 1 \) for all \( i \)) and \( \phi_i \) is conjugate self-dual with sign \((-1)^{n-1}\) for all \( i \).

Given an \( L \)-parameter \( \phi \) for \( U(V) \), we can associate its component group \( S_\phi \) of \( \phi \). As explained in [5, §8], \( S_\phi \) is a finite 2-abelian group and so has a form

\[
S_\phi = \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j
\]

with a canonical basis \( \{a_j\} \), where the product ranges over all \( j \) such that \( \phi_j \) is conjugate self-dual with sign \((-1)^{n-1}\). If we denote the image of \(-1 \in \text{GL}_n(\mathbb{C})\) in \( S_\phi \) by \( z_\phi \), it is known that

\[
z_\phi = (m_j a_j) \in \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j.
\]
2.3. **Local Langlands correspondence for unitary group.** The aim of the local Langlands correspondence for unitary groups is to classify the irreducible smooth representations of unitary group. To state it, we first introduce some notations.

- Let $V^+$ and $V^-$ be the $n$-dimensional $\epsilon$-Hermitian spaces with $\epsilon(V^+) = +1$, $\epsilon(V^-) = -1$ respectively.
- For an $L$-parameter $\phi$ of $U(V^\pm)$, let $\Pi_\phi$ be the Vogan $L$-packet associated to $\phi$, which is a finite set of irreducible smooth representations of $U(V^\pm)$.
- Let $\text{Irr}(U(V^\pm))$ be the set of irreducible smooth representations of $U(V^\pm)$.

Then the local Langlands correspondence in an enhanced form by Vogan [21], say that there is one to one correspondence between

$$\text{Irr}(U(V^+)) \sqcup \text{Irr}(U(V^-)) \longleftrightarrow \bigcup_{\phi} \Pi_\phi,$$

where $\phi$ on the right-hand side runs over all equivalence classes of $L$-parameters for $U(V^\pm)$.

Then under the local Langlands correspondence, we may also decompose $\Pi_\phi$ as

$$\Pi_\phi = \Pi^+_\phi \sqcup \Pi^-_\phi,$$

where for $\epsilon = \pm 1$, $\Pi^\epsilon_\phi$ consists of the representations of $U(V^\epsilon)$ in $\Pi_\phi$.

Furthermore, as explained in [5, §12], there is a bijection

$$J^\psi(\phi) : \Pi_\phi \to \text{Irr}(S_\phi)$$

which is canonical when $n$ is odd and depends on the choice of an additive character of $\psi : F^\times \to \mathbb{C}$ when $n$ is even. More precisely, such bijection is determined by the $N_{E/F}(E^\times)$-orbit of nontrivial additive characters

$$\begin{aligned}
\psi^E & : E/F \to \mathbb{C}^\times & \text{if } \epsilon = +1; \\
\psi & : F \to \mathbb{C}^\times & \text{if } \epsilon = -1.
\end{aligned}$$

According to this choice, when $n$ is even, we write

$$J^\psi = \begin{cases} 
J^\psi_E & \text{if } \epsilon = +1; \\
J^\psi & \text{if } \epsilon = -1,
\end{cases}$$

and even when $n$ is odd, we retain the same notation $J^\psi(\phi)$ for the canonical bijection.

Hereafter, if a nontrivial additive character $\psi : F \to \mathbb{C}^\times$ is fixed, we define $\psi^E : E/F \to \mathbb{C}^\times$ by

$$\psi^E(x) := \psi(\frac{1}{2} \text{Tr}_{E/F}(\delta x))$$

and using these two characters, we fix once and for all a bijection

$$J^\psi(\phi) : \Pi_\phi \to \text{Irr}(S_\phi)$$

as above.

With this fixed bijection, we could label all irreducible smooth representations of $U(V^\pm)$ with $\pi(\phi, \eta)$ for some unique $L$-parameter $\phi$ of $U(V^\pm)$ and $\eta \in \text{Irr}(S_\phi)$.
2.4. Properties of the local Langlands correspondence. We briefly list some properties of the local Langlands correspondence for unitary group, which we will use in this paper:

- \( \pi(\phi, \eta) \) is a representation of \( U(V^\epsilon) \) if and only if \( \eta(z_\phi) = \epsilon \).
- \( \pi(\phi, \eta) \) is tempered if and only if \( \phi \) is tempered.
- \( \pi(\phi, \eta) \) is square-integrable if and only if \( \phi \) is square-integrable.
- The component groups \( S_\phi \) and \( S_{\phi^V} \) are canonically identified. Under this canonical identification, if \( \pi = \pi(\phi, \eta) \), then its contragradient representation \( \pi^\vee \) is \( \pi(\phi^\vee, \eta \cdot \nu) \) where
  \[
  \nu(a_j) = \begin{cases} 
  \omega_{E/F}(-1)^{\dim \phi_i} & \text{if } \dim \phi \text{ is even}; \\
  1 & \text{if } \dim \phi \text{ is odd}.
  \end{cases}
  \]
- If \( \phi \) is a non-tempered \( L \)-parameter, we can decompose
  \[
  \phi = \phi_1 \oplus \cdots \oplus \phi_r \oplus \phi_0 \oplus (\phi_i^\vee)^\vee \oplus \cdots \oplus (\phi_i^\vee)^\vee,
  \]
  where
  - for \( 1 \leq i \leq r \), \( \phi_i \) is a \( k_i \)-dimensional representation of \( WD_E \) of the form \( \phi_i = \phi_i^0 \otimes | \cdot |^{e_i} \) for some tempered representation \( \phi_i^0 \) of \( WD_E \) and some real number \( e_i \) such that
    \[
    e_1 > \cdots > e_r > 0,
    \]
  - \( \phi_0 \) is a tempered \( L \)-parameter for \( U(V_0^\epsilon) \), where \( V_0^\epsilon \) are the \( \epsilon \)-Hermitian spaces of dimension \( n - 2(k_1 + \cdots + k_r) \) over \( E \).

We note that the natural map \( S_{\phi_0} \to S_\phi \) is an isomorphism.

3. Local theta correspondence

In this section, we state the local theta correspondence of unitary groups for two low rank cases. From now on, for \( \epsilon = \pm 1 \), we shall denote by \( V_n^\epsilon \) the \( n \)-dimensional Hermitian space with \( \epsilon (V_n^\epsilon) = \epsilon \) and by \( W_n^\epsilon \) the \( n \)-dimensional skew-Hermitian space with \( \epsilon (W_n^\epsilon) = \epsilon \), so that \( W_n^\epsilon = \delta \cdot V_n^\epsilon \).

3.1. The Weil representation for Unitary groups. In this subsection, we introduce the Weil representation.

Let \( E/F \) be a quadratic extension of local fields and \( (V_m, \langle \cdot, \cdot \rangle_{V_m}) \) be a \( m \)-dimensional Hermitian space and \( (W_n, \langle \cdot, \cdot \rangle_{W_n}) \) a \( n \)-dimensional skew-Hermitian space over \( E \). Define the symplectic space
\[
\mathbb{W}_{V_m, W_n} := \text{Res}_{E/F} V_m \otimes_E W_n
\]
with the symplectic form
\[
\langle v \otimes w, v' \otimes w' \rangle_{\mathbb{W}_{V_m, W_n}} := \frac{1}{2} \text{tr}_{E/F} \left( \langle v, v' \rangle_{V_m} \otimes \langle w, w' \rangle_{W_n} \right).
\]
We also consider the associated symplectic group \( Sp(\mathbb{W}_{V_m, W_n}) \) preserving \( \langle \cdot, \cdot \rangle_{\mathbb{W}_{V_m, W_n}} \) and the metaplectic group \( \tilde{Sp}(\mathbb{W}_{V_m, W_n}) \) satisfying the following short exact sequence:
\[
1 \to \mathbb{C}^\times \to \tilde{Sp}(\mathbb{W}_{V_m, W_n}) \to Sp(\mathbb{W}_{V_m, W_n}) \to 1.
\]

Let \( \mathcal{X}_{V_m, W_n} \) be a Lagrangian subspace of \( \mathbb{W}_{V_m, W_n} \) and we fix an additive character \( \psi : F \to \mathbb{C}^\times \). Then we have a Schrödinger model of the Weil Representation \( \omega_\psi \) of \( \tilde{Sp}(\mathbb{W}) \) on \( \mathcal{S}(\mathcal{X}_{V_m, W_n}) \), where \( \mathcal{S} \) is the Schwartz-Bruhat function space.
Throughout the rest of the paper, when it comes to a Weil representation of \( \chi \) where \( S \) is a character of \( E \) whose restriction to \( F \) is \( \omega_{E/F} \), which is the quadratic character associated to \( E/F \) by local class field theory, then \((\chi_{V_m}, \chi_{W_n})\) gives a splitting homomorphism

\[
\iota_{\chi_{V_m}, \chi_{W_n}} : U(V_m) \times U(W_n) \to \tilde{Sp}(W_{V_m}, W_n)
\]

and so by composing this to \( \omega_{\psi} \), we have a Weil representation \( \omega_{\psi} \circ \iota_{\chi_{V_m}, \chi_{W_n}} \) of \( U(V_m) \times U(W_n) \) on \( \mathbb{S}(X_{\chi_{V_m}, W_n}) \).

When the choice of \( \psi \) and \((\chi_{V_m}, \chi_{W_n})\) is fixed as above, we simply write

\[\omega_{\psi, W_n, V_m} := \omega_{\psi} \circ \iota_{\chi_{V_m}, \chi_{W_n}}.\]

Throughout the rest of the paper, when it comes to a Weil representations of \( U(V_m) \times U(W_n) \), we shall denote it by \( \omega_{\psi, W_n, V_m} \) with understanding the choices of characters \((\chi_{V_m}, \chi_{W_n})\) as above.

**Remark 3.1.** When \( m = 1 \), the image of \( U(V_1) \) in \( \tilde{Sp}(W_{V_1}, W_n) \) coincides with the image of the center of \( U(W_n) \), and so we regard the Weil representation of \( U(V_1) \times U(W_n) \) as the representation of \( U(W_n) \). In this case, we denote the Weil representation of \( U(W_n) \) as \( \omega_{\psi, W_n} \). Furthermore, we can also use \( \chi_{V_1} = \chi^{-1} \) for the choice of splitting homomorphism \( \iota_{\chi_{V_1}, \chi_{W_n}} \) instead of \( \chi_{V_1} = \chi \). In this case, the Weil representation of \( U(W_n) \) is \( \omega_{\psi, W_n} \).

### 3.2. Local theta correspondence

Given a Weil representation \( \omega_{\psi, W_n, V_m} \) of \( U(V_m) \times U(W_n) \) and an irreducible smooth representation \( \pi \) of \( U(W_n) \), the maximal \( \pi \)-isotypic quotient of \( \omega_{\psi, W_n, V_m} \) is of the form

\[\Theta_{\psi, W_n, V_m}(\pi) \boxtimes \pi\]

for some smooth representation \( \Theta_{\psi, W_n, V_m}(\pi) \) of \( U(V_m) \) of finite length. By the Howe duality\(^1\), the maximal semisimple quotient \( \theta_{\psi, W_n, V_m}(\pi) \) of \( \Theta_{\psi, W_n, V_m}(\pi) \) is either zero or irreducible.

In this paper, we consider two kinds of theta correspondences for \((U(1) \times U(3))\) and \((U(2) \times U(3))\).

### 3.3. Case (i)

Now we shall consider the theta correspondence for \( U(V_1^\pm) \times U(W_3^\pm) \). The following is a compound of Theorem 3.4 and Theorem 3.6 in [13].

#### Theorem 3.2

Let \( \phi \) be a \( L \)-parameter of \( U(V_1^\pm) \). Then we have:

(i) For any \( \epsilon, \epsilon' = \pm 1 \) and any \( \pi \in \Pi_{\phi}^L \), \( \Theta_{\psi, W_3^\pm, V_1^\pm}^{\epsilon, \epsilon'}(\pi) \) is nonzero and irreducible.

(ii) \( \Theta_{\psi, W_3^\pm, V_1^\pm}(\pi) = \left\{ \begin{array}{ll} a \text{ non-tempered representation} & \text{if } \epsilon \left( \frac{1}{2}, \phi \otimes \chi^{-3}, \psi_E^F \right) \epsilon' = \epsilon \cdot \epsilon' \\
\text{a supercuspidal representation} & \text{if } \epsilon \left( \frac{1}{2}, \phi \otimes \chi^{-3}, \psi_E^F \right) = -\epsilon \cdot \epsilon', \end{array} \right.\]

where

\[\psi_E^F(x) = \psi(\text{Tr}_{E/F}(\delta x)).\]

(iii) The \( L \)-parameter \( \theta(\phi) \) of \( \Theta_{\psi, W_3^\pm, V_1^\pm}(\pi) \) has the following two forms:

\[\theta(\phi) = \left\{ \begin{array}{ll} \theta_m(\phi) = \chi| \cdot \left( \frac{1}{2}, \phi \cdot \chi^{-2} \right| \cdot | \chi |^{-\frac{1}{2}} & \text{if } \epsilon \left( \frac{1}{2}, \phi \otimes \chi^{-3}, \psi_E^F \right) = \epsilon \cdot \epsilon' \\
\theta_s(\phi) = \phi \cdot \chi^{-2} \otimes \chi \boxtimes S_2 & \text{if } \epsilon \left( \frac{1}{2}, \phi \otimes \chi^{-3}, \psi_E^F \right) = -\epsilon \cdot \epsilon', \end{array} \right.\]

where \( S_2 \) is the tautological 2-dimensional representation of \( SL_2(\mathbb{C}) \).

---

\(^1\)It was first proved by Waldspurger [22] for all residual characteristic except \( p = 2 \). Recently, Gan and Takeda [9], [10] have made it available for all residual characteristics.
(iv) For $\epsilon, \epsilon'$ such that $\epsilon(\frac{1}{2}, \phi \otimes \chi^{-3}, \psi_2^E) = \epsilon \cdot \epsilon'$, the theta correspondence $\pi \mapsto \theta_{\psi, W_1^3, V_1'}(\pi)$ gives a bijection

$$\Pi_\phi \longleftrightarrow \Pi_{\theta^* (\phi)}.$$

(v) For $\epsilon, \epsilon'$ such that $\epsilon(\frac{1}{2}, \phi \otimes \chi^{-3}, \psi_2^E) = -\epsilon \cdot \epsilon'$, the theta correspondence $\pi \mapsto \theta_{\psi, W_1^3, V_1'}(\pi)$ gives an injection

$$\Pi_\phi \hookrightarrow \Pi_{\theta^* (\phi)}.$$

Write

- $S_\phi = (\mathbb{Z}/2\mathbb{Z}) a_1$
- $S_{\theta^n (\phi)} = (\mathbb{Z}/2\mathbb{Z}) a_1$
- $S_{\theta^s (\phi)} = (\mathbb{Z}/2\mathbb{Z}) a_1 \times (\mathbb{Z}/2\mathbb{Z}) a_2$

where

$$\psi_2^E(x) = \psi(\text{Tr}_E/F(\delta x)).$$

(Note $\theta^s (\phi)$ is the square-integrable $L$-parameter of $U(W_3^3)$ and so $(\mathbb{Z}/2\mathbb{Z}) a_2$ of $S_{\theta^s (\phi)}$ arises from the summand $\chi \boxtimes S_2$ in $\theta^s (\phi)$.)

Since we are dealing only odd dimensions, there are three canonical bijections

- $J^v(\phi) : \Pi_\phi \longleftrightarrow \text{Irr}(S_\phi)$
- $J^v(\theta^n (\phi)) : \Pi_{\theta^n (\phi)} \longleftrightarrow \text{Irr}(S_{\theta^n (\phi)})$
- $J^v(\theta^s (\phi)) : \Pi_{\theta^s (\phi)} \longleftrightarrow \text{Irr}(S_{\theta^s (\phi)})$

Using these maps, the following bijection and inclusion

$$\text{Irr}(S_\phi) \longleftrightarrow \text{Irr}(S_{\theta^n (\phi)})$$

$$\eta \longleftrightarrow \theta^n (\eta),$$

$$\text{Irr}(S_\phi) \hookrightarrow \text{Irr}(S_{\theta^s (\phi)})$$

$$\eta \mapsto \theta^s (\eta)$$

induced by the theta correspondence can be explicated as follows:

$$\theta^n (\eta)(a_1) = \eta(a_1) \cdot \epsilon(\frac{1}{2}, \phi \otimes \chi^{-3}, \psi_2^E),$$

$$\theta^s (\eta)(a_1) = \eta(a_1) \cdot \epsilon(\frac{1}{2}, \phi \otimes \chi^{-3}, \psi_2^E), \quad \theta_2 (\eta)(a_2) = -1.$$

Remark 3.3. Note that $\theta^n (\phi)$ is the non-generic $L$-parameter. It is conjectured that an $L$-parameter is generic if and only if its associated $L$-packet $\Pi_\phi$ contains a generic representation (i.e. those possess Whittaker model). The Corollary 4.2.3 in [12], which asserts that all elements in $\Pi_{\theta^n (\phi)}$ have no Whittaker models, gives an evidence on this conjecture.

3.4. Case (ii). Now we shall consider the theta correspondence for $U(V_2') \times U(W_3^3)$. The following summarises some results of [7, 8], which are specialised to this case.

**Theorem 3.4.** Let $\phi$ be an $L$-parameter for $U(V_2^\pm)$. Then we have:

(i) Suppose that $\phi$ does not contain $\chi^3$.

(a) For any $\pi \in \Pi_{\phi'}$, $\theta_{\psi, W_3^3, V_2'}(\pi)$ is nonzero and $\theta_{\psi, W_3^3, V_2'}(\pi)$ has $L$-parameter

$$\theta(\phi) = (\phi \otimes \chi^{-1}) \oplus \chi^2.$$
(b) For each $\epsilon = \pm 1$, the theta correspondence $\pi \mapsto \theta_{\psi, W_3^\epsilon, V_2^\epsilon}(\pi)$ gives a bijection
$$\Pi_\phi \longleftrightarrow \Pi_{\theta(\phi)}.$$

(ii) Suppose that $\phi$ contains $\chi^3$.
(a) For any fixed $\pi \in \Pi_\phi$, exactly one of $\Theta_{\psi, W_3^+, V_2^+}(\pi)$ or $\Theta_{\psi, W_3^-, V_2^-}(\pi)$ is nonzero.
(b) If $\Theta_{\psi, W_3^+, V_2^+}(\pi)$ is nonzero, then $\theta_{\psi, W_3^+, V_2^+}(\pi)$ has $L$-parameter
$$\theta(\phi) = (\phi \otimes \chi^{-1}) \oplus \chi^2.$$
(c) The theta correspondence $\pi \mapsto \theta_{\psi, W_3^+, V_2^+}(\pi)$ gives a bijection
$$\Pi_\phi \longleftrightarrow \Pi_{\theta(\phi)}.$$

(iii) There is the bijection
$$J^\psi(\phi) : \Pi_\phi \longleftrightarrow \text{Irr}(S_\phi),$$
and we fix a bijection
$$J_{\psi, E}(\theta(\phi)) : \Pi_{\theta(\phi)} \longleftrightarrow \text{Irr}(S_{\theta(\phi)}),$$
where
$$\psi^E(x) = \psi^((\frac{1}{2}) \text{Tr}_{E/F}(\delta x)).$$

• If $\phi$ does not contain $\chi^3$, let
$$S_{\theta(\phi)} = S_\phi \times (\mathbb{Z}/2\mathbb{Z})b_1,$$
where the extra copy of $\mathbb{Z}/2\mathbb{Z}$ of $S_{\theta(\phi)}$ arises from the summand $\chi^2$ in $\theta(\phi)$.
Then for each $\epsilon$, using the above bijection $J$ and $J_{\psi, E}$, one has a canonical bijection
$$\text{Irr}(S_\phi) \longleftrightarrow \text{Irr}(S_{\theta(\phi)}),$$
$$\eta \longleftrightarrow \theta(\eta)$$
induced by the theta correspondence, where $\text{Irr}(S_{\theta(\phi)})$ is the set of irreducible characters $\eta'$ of $S_{\theta(\phi)}$ such that $\eta'(z_{\theta(\phi)}) = \epsilon$ and the bijection is determined by
$$\theta(\eta)|_{S_\phi} = \eta.$$

• If $\phi$ contains $\chi^3$, then $\phi \otimes \chi^{-1}$ contains $\chi^2$, and so
$$S_{\theta(\phi)} = S_\phi.$$
Thus, one has a canonical bijection
$$\text{Irr}(S_\phi) \longleftrightarrow \text{Irr}(S_{\theta(\phi)}),$$
$$\eta \longleftrightarrow \theta(\eta)$$
induced by the theta correspondence and it is given by
$$\theta(\eta) = \eta.$$

(iv) If $\pi$ is tempered and $\Theta_{\psi, W_3^+, V_2^+}(\pi)$ is nonzero, then $\Theta_{\psi, W_3^+, V_2^+}(\pi)$ is irreducible and tempered.
4. Main Theorem

In this section, we prove our main theorem. To prove it, we first state the precise result of Plessis which we shall use in the proof of Theorem 4.1.

**Theorem (B).** Let \( \phi = \phi^{(n+1)} \times \phi^{(n)} \) be a tempered \( L \)-parameter of \( U(V^\pm_{n+1}) \times U(V^\pm_n) \) and we write \( S_{\phi^{(n+1)}} = \prod_i (\mathbb{Z}/2\mathbb{Z})a_i \) and \( S_{\phi^{(n)}} = \prod_j (\mathbb{Z}/2\mathbb{Z})b_j \). Let \( \Delta : U(V^\pm_{n+1}) \times U(V^\pm_n) \hookrightarrow U(V^\pm_{n+1}) \times U(V^\pm_n) \) be the diagonal map. Then for \( \pi(\eta) \in \Pi^{R, \pm} = \Pi^{R, \pm}_{\phi^{(n+1)}} \times \Pi^{R, \pm}_{\phi^{(n)}} \) where \( \eta \in \text{Irr}(S_\phi) = \text{Irr}(S_{\phi^{(n+1)}}) \times \text{Irr}(S_{\phi^{(n)}}) \),

\[
\text{Hom}_{\Delta(U(V^\pm_n)) \times U(V^\pm_n)}(\pi(\eta), \mathbb{C}) = 1 \iff \eta = \eta_\bullet \quad \text{where}
\]

\[
\begin{align*}
\eta^{(1)}(a_i) &= \epsilon(\frac{1}{2}, \phi^{(n+1)}_i \otimes \phi^{(n)}, \psi^{-E}_2); \\
\eta^{(2)}(b_j) &= \epsilon(\frac{1}{2}, \phi^{(n+1)}_j \otimes \phi^{(n)}_j, \psi^{-E}_2).
\end{align*}
\]

**Theorem 4.1.** Let \( \phi^{(1)}, \phi^{(2)} \) be tempered \( L \)-parameters of \( U(V^\pm_1) \) and \( U(V^\pm_2) \) respectively and suppose that \( \phi^{(2)} \) does not contain \( \chi^3 \). Let

\[
\theta^\epsilon_n(\phi^{(1)}) = |\cdot|_\chi^{-1} \oplus \phi^{(1)} \cdot \chi^{-2} \oplus |\cdot|^{-\frac{1}{2}},
\]

\[
\theta^s(\phi^{(1)}) = \phi^{(1)} \cdot \chi^{-2} \oplus \chi \otimes S_2
\]

be the two \( L \)-parameters of \( U(W^\pm_3) \) appeared in (3.1) and let

\[
\theta(\phi^{(2)}) = \phi^{(2)} \otimes \chi \otimes \chi^{-2}
\]

be the \( L \)-parameters of \( U(W^\pm_3) \) appeared in Theorem 3.4 (ii).

Write

- \( S_{\phi^{(1)}} = S_{\theta^\epsilon_n(\phi^{(1)})} = (\mathbb{Z}/2\mathbb{Z})a_1 \);
- \( S_{\theta^s(\phi^{(1)})} = S_{\phi^{(1)}} \times (\mathbb{Z}/2\mathbb{Z})a_2 \);
- \( S_{\phi^{(2)}} = ((\mathbb{Z}/2\mathbb{Z})b_1 \quad \text{if} \ \phi^{(2)} \ \text{is irreducible};
\)
- \( (\mathbb{Z}/2\mathbb{Z})b_1 \times (\mathbb{Z}/2\mathbb{Z})b_2 \quad \text{if} \ \phi^{(2)} = \phi^{(2)}_1 \oplus \phi^{(2)}_2 \ \text{is reducible}
\)
- \( S_{\theta(\phi^{(2)})} = S_{\phi^{(2)}} \times (\mathbb{Z}/2\mathbb{Z})c_1
\)

where \( c_1 \) comes from the component \( \chi^{-2} \) of \( \theta(\phi^{(2)}) \).

We use the fixed character \( \psi \) to fix the local Langlands correspondence for \( \Pi_{\theta(\phi^{(2)})} \leftrightarrow \text{Irr}(S_{\theta(\phi^{(2)})}) \).

For \( x = n, s \), let

\[
\theta^x(\phi^{(1)}, \phi^{(2)}) = \theta^x(\phi^{(1)}) \times \theta(\phi^{(2)})
\]

be a \( L \)-parameter of \( G^\pm_3 = U(W^\pm_3) \times U(W^\pm_3) \) and \( \pi^x(\eta) \in \Pi^R_{\theta^x(\phi^{(1)}, \phi^{(2)})} \) a representation of a relevant pure inner form of \( G^\epsilon_3 \). Then,

\[
\text{Hom}_{\Delta(U(W^\pm_3))}(\pi^x(\eta), \omega_\psi, W^\pm_3) \neq 0 \iff \eta = \eta_\bullet
\]

where \( \eta_\bullet \in \text{Irr}(S_{\theta^x(\phi^{(1)}, \phi^{(2)})}) \) is specified as follows;

(i) When \( \phi^{(2)} \) is irreducible,

\[
\begin{align*}
\eta^{(1)}_n(a_i) &= \epsilon(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi^{-E}_2) \cdot \epsilon(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}_j, \psi^{-E}_2) \\
\eta^{(2)}_n(b_j) &= \epsilon(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}_j, \psi^{-E}_2) \\
\eta^{(1)}_n(c_1) &= \epsilon(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi^{-E}_2)
\end{align*}
\]
Furthermore, for each $\phi^3 \in \Phi_3^2$ such that $\chi(\phi^3) = 1$ and $\phi^3$ is reducible,

\[
\begin{align*}
\eta^*_{n_1}(a_1) &= \epsilon(\frac{1}{2}, \phi^3(1) \otimes \chi^3, \psi^E_2) \cdot \epsilon(\frac{1}{2}, \phi^3(2) \otimes \psi_2^E) \\
\eta^*_{n_2}(a_2) &= -1 \\
\eta^*_{n_3}(b_1) &= \epsilon(\frac{1}{2}, \phi^3(1) \otimes \phi^3(2), \psi^E_2) \\
\eta^*_{n_3}(c_1) &= -\epsilon(\frac{1}{2}, \phi^3(1) \otimes \chi^3, \psi^E_2)
\end{align*}
\]

Furthermore,

$$\dim \text{Hom}_{\Delta U(W^+_3)}(\pi^\times(\eta^*_n), \omega_{\psi, W^+_3}) = 1$$

Proof. For each $x = n, s$, we first prove the existence of some $\epsilon_x \in \{\pm 1\}$ and $\pi^\times(\eta) \in \Pi_{\theta_s(\phi^3)} R_x \epsilon_x$ such that

$$\text{Hom}_{\Delta U(W^+_3)}(\pi^\times(\eta), \omega_{\psi, W^+_3}) \neq 0.$$ 

For $a \in F^\times$, let $L_a$ be the 1-dimensional Hermitian space with form $a \cdot N_{E/F}$. Then

$$V_2^+/V_1^+ \cong V_2^-/V_1^- \cong L_{-1}.$$ 

We consider the following see-saw diagram : $(\epsilon, \epsilon')$ will be determined soon

\[
\begin{array}{ccc}
U(W^+_3) \times U(W^+_{-1}) & U(V^+_2) \\
U(W^+_3) & U(V^+_2) \times U(L_{-1})
\end{array}
\]

(4.1)

In this diagram, we shall use three theta correspondences:

(i) $U(V^+_2) \times U(W^+_3)$ relative to the pair of characters $(\chi^2, \chi^3)$;
(ii) $U(V^{-}_1) \times U(W^-_{3})$ relative to the pair of characters $(\chi, \chi^3)$;
(iii) $U(L_{-1}) \times U(W^+_3)$ relative to the pair of characters $(\chi^3, \chi^3)$.

By $(B)_1$, there is a unique $\epsilon' \in \{\pm 1\}$ and a unique pair of components characters

$$\eta_2, \eta_1 \in \text{Irr}^{\epsilon'}(S_{(\phi^3)^{-1}}) \times \text{Irr}^{\epsilon'}(S_{(\phi^3)^{-1}})$$

such that

$$\pi^\times(\eta_2), \pi^\times(\eta_1) < \Pi_{\theta_s(\phi^3)^{-1}} \times \Pi_{\theta_s(\phi^3)^{-1}} \iff \text{Hom}_{\Delta U(V^+_1)}(\pi^\times(\eta_2) \otimes \pi^\times(\eta_1), \mathbb{C}) \neq 0.$$ 

In particular, $\epsilon' = \eta_1(a_1) = \epsilon(\frac{1}{2}, (\phi^3)^{-1} \otimes (\phi^3)^{-1}, \psi^E_{-2}) = \epsilon(\frac{1}{2}, \phi^3(1) \otimes \phi^3(2), \psi^E_2)$. 

Since $π(η_1)$ is unitary,

$$\text{Hom}_{ΔU(V_1)}(π(η_2), π^\vee(η_1)) \neq 0$$

and by Theorem 3.4 (i) and (iv), $Θ_{ψ,\mathcal{V}_1',\mathcal{V}_3'}(π(η_2)) = \pi(η_2)$ for any $ε \in \{±1\}$. Thus by the see-saw identity and Remark 3.1 we have

$$\text{Hom}_{ΔU(W_3)}(Θ_{ψ,\mathcal{V}_3'},π^\vee(η_1)) \neq 0$$

and since $Θ_{ψ,\mathcal{V}_3',\mathcal{V}_2'}(π(η_2))$ is tempered and so unitary, we have

$$\text{Hom}_{ΔU(W_3)}(Θ_{ψ,\mathcal{V}_3'},π^\vee(η_1)) \neq 0.$$

By Theorem 3.2 (iii) and Theorem 3.4 (i), the $L$-parameter of $Θ_{ψ,\mathcal{V}_3',\mathcal{V}_2'}(π^\vee(η_1)) \otimes Θ_{ψ,\mathcal{V}_3',\mathcal{V}_2'}(π(η_2))$ is

$$\begin{cases}
ε(φ^{(1)}, φ^{(2)}) & \text{if } ε = \epsilon(\frac{1}{2}, φ^{(1)} \otimes χ^{-3}, ψ_E) \cdot \epsilon(\frac{1}{2}, φ^{(2)} \otimes ψ_E) \\
ε(φ^{(1)}, φ^{(2)}) & \text{if } ε = -\epsilon(\frac{1}{2}, φ^{(1)} \otimes χ^{-3}, ψ_E) \cdot \epsilon(\frac{1}{2}, φ^{(2)} \otimes ψ_E)
\end{cases}$$

and by Theorem 3.2 (v), Theorem 3.4 (iii), we see that their associated component characters are $η^{\bullet}_{n_1}, η^{\bullet}_{n_2}$ in each case.

Next we shall prove that these are the unique representations which yield Fourier-Jacobi model in each $L$-packets $Π_{θ^ε(φ^{(1)}, φ^{(2)})}$ and $Π_{θ^ε(φ^{(1)}, φ^{(2)})}$.

Since $θ^ε(φ^{(1)}, φ^{(2)})$ is tempered $L$-parameter, the uniqueness easily follows from (FJ) in this case. Therefore, we shall only consider the non-tempered $L$-parameter $θ^ε(φ^{(1)}, φ^{(2)})$. For some $ε$, there is a representation $π_2 \otimes π_1 \in Π_{θ^ε(φ^{(1)}, φ^{(2)})} = ε(π_2, π_1) \otimes Π_{θ^ε(φ^{(2)})}$ satisfying

$$\text{Hom}_{ΔU(W_3)}(π_2 \otimes π_1, ω_ψ, W_3) \neq 0$$

and in turn

$$\text{Hom}_{ΔU(W_3)}(π_2 \otimes ω_ψ, W_3, π^\vee) \neq 0.$$

(The existence of such $π_2 \times π_1$ was ensured by the previous step.) By Theorem 3.2 (iv), we can write $π_2 = Θ_{ψ,\mathcal{V}_3',\mathcal{V}_2'}(π^{(1)})$ for some $π^{(1)} \in Π'$ where

$$ε' = ε \cdot \epsilon(\frac{1}{2}, φ^{(1)} \otimes χ^{-3}, ψ_E).$$

Then by applying the see-saw duality in the see-saw diagram in (3.1), one has

$$\text{Hom}_{ΔU(W_3)}(π_2 \otimes ω_ψ, W_3, π^\vee) \cong \text{Hom}_{V_1'}(π^{(2)}, π^{(1)}) \neq 0$$

where $π^{(2)} = Θ_{ψ,\mathcal{V}_3',\mathcal{V}_2'}(π^{(1)})$.

Note that $π^{(2)} \neq 0$ and so it has the tempered $L$-parameter $θ^{(2)}$. Then by the $(B)_1$, $π^{(2)}, π^{(1)}$ is the unique pair in the $L$-packet $Π_{θ^{(2)}} \times Π_{θ^{(1)}}$ such that

$$\text{Hom}_{V_1'}(π^{(2)}, π^{(1)}) \neq 0$$

and so $(π_2, π_1)$ should be $(Θ_{ψ,\mathcal{V}_3',\mathcal{V}_2'}(π^{(1)}), Θ_{ψ,\mathcal{V}_3',\mathcal{V}_2'}(π^{(2)}))$.

This settles down the uniqueness issue.
Remark 4.2. When the $L$-parameter $\phi^{(2)}$ of $U(V_2^\pm)$ contains $\chi^{-3}$, write $\phi^{(2)} = \phi_0 \oplus \chi^{-3}$ for an $L$-parameter of $U(V_1^\pm)$. Then

$$\theta(\phi^{(2)}) = \begin{cases} 3 \cdot \chi^{-2} & \text{if } \phi_0 = \chi^{-3}, \\ \phi_0 \cdot \chi \oplus 2 \cdot \chi^{-2} & \text{if } \phi_0 \neq \chi^{-3} \end{cases}$$

and

$$S_{\theta(\phi^{(2)})} = \begin{cases} (\mathbb{Z}/2\mathbb{Z})b_1 & \text{if } \phi_0 = \chi^{-3}, \\ (\mathbb{Z}/2\mathbb{Z})b_1 \times (\mathbb{Z}/2\mathbb{Z})c_1 & \text{if } \phi_0 \neq \chi^{-3}. \end{cases}$$

If one develops a similar argument in this case, one could have a recipe for the non-tempered case. Since the proof is essentially same with the previous Theorem 4.1, we only state the result and skip the proof for brevity.

**Theorem 4.3.** We retain the same notation as in Theorem 4.1 except for replacing $\theta(\phi^{(2)})$ as above. Then,

$$\text{Hom}_{U(W)^{(2)}}(\pi^{\eta}(\eta), \omega_{\phi, W}) \neq 0 \iff \eta = \eta_{\phi}^{\star}$$

where $\eta_{\phi}^{\star} \in \text{Irr}(S_{\phi^{(2)}}) = \text{Irr}(S_{\phi^{(1)}}) \times \text{Irr}(S_{\phi^{(2)}})$ is specified as follows:

- When $\theta(\phi^{(2)}) = 3 \cdot \chi^{-2}$,

$$\begin{cases} \eta_{\phi}^{\star}(a_1) = \epsilon\left(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi^{E}_2\right) \\ \eta_{\phi}^{\star}(b_1) = \epsilon\left(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi^{E}_2\right). \end{cases}$$

- When $\theta(\phi^{(2)}) = \phi_0 \cdot \chi \oplus 2 \cdot \chi^{-2}$,

$$\begin{cases} \eta_{\phi}^{\star}(a_1) = \epsilon\left(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi^{E}_2\right) \\ \eta_{\phi}^{\star}(b_1) = \epsilon\left(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi^{E}_2\right) \\ \eta_{\phi}^{\star}(c_1) = \epsilon\left(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi^{E}_2\right). \end{cases}$$

**Remark 4.4.** It is remarkable that for the supercuspidal $L$-parameter $\theta_{\phi}^{(1)}(\phi^{(2)})$ with $\theta(\phi^{(2)})$ as above, the recipe, which is suggested in (FJ)3, does not occur from the theta lift from $U(V_1^\pm)$ and $U(V_2^\pm)$. This is quite similar with the Proposition 4.4 in [13], which concerns the non-tempered aspect of Bessel case of the GP conjecture.

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