UNIFORM AND NONSTANDARD EXISTENCE IN REVERSE MATHEMATICS

SAM SANDERS

ABSTRACT. Reverse Mathematics is a program in the foundations of mathematics which provides an elegant classification of theorems of ordinary mathematics based on computability. Our aim is to provide an alternative classification of theorems based on the central tenet of Feferman’s Explicit Mathematics, namely that a proof of existence of an object yields a procedure to compute said object. Our classification gives rise to the Explicit Mathematics theme (EMT) of Nonstandard Analysis. Intuitively speaking, the EMT states that a standard object with certain properties can be computed by a functional if and only if this object exists classically with these same standard and nonstandard properties. In this paper, we establish examples for the EMT ranging from the weakest to the strongest Big Five system of Reverse Mathematics. Our results are proved over the usual base theory of Reverse Mathematics, conservatively extended with higher types and Nelson’s internal approach to Nonstandard Analysis.

1. Introduction

1.1. Reverse Mathematics, explicitly. The subject of this paper is the development of Reverse Mathematics (RM for short; See Section 1.3 for an introduction) over a conservative extension of the usual base theory involving higher types and Nonstandard Analysis. This extended base theory, introduced in Section 2, is based on Nelson’s internal set theory ([43]) and Kohlenbach’s higher-order RM ([36]). The aforementioned development of RM, which takes place in Section 3–7, leads to the formulation of the Explicit Mathematics Theme (EMT for short), which we discuss now. We follow the notations from Nelson’s internal set theory.

Theme 1.1 (The theme from Explicit Mathematics). Consider a standard theorem of mathematics of the form:

\[ T^* \equiv (\forall^{st} x^{st})(A^{st}(x) \rightarrow (\exists^{st} y^{st})B^{st}(x, y)). \]

The nonstandard version of \( T^* \) is the statement:

\[ (\forall^{st} x^{st})(A^{st}(x) \rightarrow (\exists^{st} y^{st})B(x, y)), \]

\( (T^*) \)

where \( B^{st} \) is ‘transferred’ to \( B \), i.e. the standardness predicate ‘\( st \)’ is omitted. Furthermore, the uniform version of \( T \), is

\[ (\exists \Phi^{st}(\rightarrow))(\forall x^{st})(A(x) \rightarrow B(x, \Phi(x))). \]

\( (UT) \)

The Explicit Mathematics Theme (EMT) is the observation that for many theorems \( T \) as above, the base theory proves \( T^* \iff UT \).

Note that the EMT expresses that the mere existence of an object \( y \) as in \( T^* \), is equivalent to \( y \) being computable via a functional as in the uniform version \( UT \).

*Department of Mathematics, Ghent University, Belgium & Munich Center for Mathematical Philosophy, LMU Munich, Germany

E-mail address: ssander@cage.ugent.be.

1 For certain statements \( T \), there exists an additional interesting uniform version in which the existential quantifiers in \( A \) are also removed by a functional. When we encounter such statements, we shall make a distinction between \( UT_1 \) and \( UT_2 \).
In this paper, we provide evidence for the EMT by establishing the latter for various theorems $T$ studied in second-order RM, where $T$ and $UT$ range from provable in the base theory to provable only in the strongest Big Five system. This development takes place in Sections 3 to 7. Finally, for some motivation regarding this study, we refer to Section 1.2, while Sections 1.3 and 1.4 provide background information on Reverse and Explicit Mathematics.

1.2. Motivation. We discuss the foundational significance of the EMT.

(1) Central to the EMT is that statements involving higher-type objects like $UT$ are equivalent to statements $T^*$ involving only lower-type nonstandard objects. In this light, it seems incoherent to claim that higher-type objects are somehow ‘more real’ than nonstandard ones (or vice versa). The EMT thus suggests that higher-order RM is implicit in Friedman-Simpson RM, as Nonstandard Analysis is used in the latter. (See e.g. [1, 34, 47, 48, 57, 58, 62–64]). Moreover, the EMT gives rise to an example of a higher-order statement implicit in Friedman-Simpson RM, as discussed in Remark 5.15.

(2) In general, to prove $T^* \rightarrow UT$, one defines a functional $\Psi(\cdot, M)$ of (rather) elementary complexity, but involving an infinite number $M$. Assuming $T^*$, this functional is $\Omega$-invariant (See Definition 2.5) and the axiom $\Omega$-CA from the base theory provides the required standard functional for $UT$. As discussed in Section 3.5, these results can be viewed as a contribution to Hilbert’s program for finitistic mathematics, as infinitary objects (the functional from $UT$) are decomposed into elementary objects.

(3) Fujiwara and Kohlenbach have established the equivalence between (classical) uniform existence as in $UT$ and intuitionistic provability for rather rich formulas classes ([22, 23]). The EMT suggests that $T^*$ constitutes another way of capturing intuitionistic provability. Nonetheless, we establish the EMT for statements beyond the Fujiwara-Kohlenbach metatheorems.

(4) Our results reinforce the heuristic $WKL_0 \approx ACA_0 \approx \Pi^1_1$-CA$_0$ put forward in [54, I.11.7]. In particular, our treatment of the EMT for the fan theorem, i.e. the classical contraposition of WKL, and for ATR$_0$ are neigh identical. Furthermore, the EMT for $(S^2)$, the functional version of II$_1$-CA$_0$, is highly similar to the EMT for $(\exists^2)$, the functional version of ACA$_0$, thanks to the bounding result in Theorem 7.3. In conclusion, Nonstandard Analysis allows us to treat sets of numbers in the same way as one treats numbers.

Besides the previous arguments, a first general motivation for the study of higher-order RM is as follows: It was shown in [52] that higher-order statements are implicit in second-order RM. A second motivation, based on Feferman’s Explicit Mathematics (See Section 1.4) is discussed in Sections 1.3 and 1.4. Finally, we urge the reader to first consult Remark 2.13 so as to clear up a common misconception regarding Nelson’s framework.

1.3. Reverse Mathematics: a ‘computable’ classification. Reverse Mathematics (RM) is a program in the foundations of mathematics initiated around 1975 by Friedman ([19,20]) and developed extensively by Simpson ([53,54]) and others. The aim of RM is to find the axioms necessary to prove a statement of ordinary mathematics, i.e. dealing with countable or separable objects. The classical base

---

2In Constructive Reverse Mathematics ([52]), the base theory is based on intuitionistic logic.
theory RCA$_0$ of ‘computable’ mathematics’ is always assumed. Thus, the aim is to find the minimal axioms $A$ to derive a given statement $T$ in RCA$_0$; In symbols:

The aim of RM is to find the minimal axioms $A$ such that $\text{RCA}_0 \vdash [A \rightarrow T]$ for statements $T$ of ordinary mathematics.

Surprisingly, once the minimal axioms $A$ have been found, we almost always also have $\text{RCA}_0 \vdash [A \leftrightarrow T]$, i.e. not only can we derive the theorem $T$ from the axioms $A$ (the ‘usual’ way of doing mathematics), we can also derive the axiom $A$ from the theorem $T$ (the ‘reverse’ way of doing mathematics). In light of the latter, the discipline was baptised ‘Reverse Mathematics’.

In the majority of cases, for a statement $T$ of ordinary mathematics, either $T$ is provable in RCA$_0$, or the latter proves $T \leftrightarrow A_i$, where $A_i$ is one of WKL$_0$, ACA$_0$, ATR$_0$ or $\Pi^1_1$-CA$_0$. The latter together with RCA$_0$ form the ‘Big Five’ and the aforementioned observation that most mathematical theorems fall into one of the Big Five categories, is called the Big Five phenomenon ([11] p. 432)). Furthermore, each of the Big Five has a natural formulation in terms of (Turing) computability (See e.g. [54, I.3.4, I.5.4, I.7.5]). As noted by Simpson in [54, I.12], each of the Big Five also corresponds (loosely) to a foundational program in mathematics.

An alternative view of Reverse Mathematics is as follows (and expressed in part by [54, Remark I.8.9.5]): Reverse Mathematics studies theorems of mathematics ‘as they stand’, instead of the common practice in constructive mathematics of introducing extra (often perceived as unnatural) conditions to make these theorems provable constructively. In other words, rather than enforcing computability via extra conditions, RM takes a ‘relative’ stance: Assuming computable mathematics in the guise of RCA$_0$, how non-computable is a given theorem of mathematics, as measured by which of the other Big Five (or other principles) it is equivalent to?

In conclusion, Reverse Mathematics can be viewed as a classification of theorems of ordinary mathematics from the point of view of computability (See e.g. [54, I.3.4]). A natural question is if there are other interesting ways of classifying these theorems, which is part of the motivation of this paper, and discussed next.

1.4. Explicit Mathematics. Around 1967, Bishop introduced Constructive Analysis ([8]), an approach to mathematics with a strong focus on computational meaning, but compatible with classical, recursive, and intuitionistic mathematics. In order to provide a natural formalisation for Constructive Analysis, Feferman introduced Explicit Mathematics (EM) in [15-17]. To capture Bishop’s constructive notion of existence in a classical-logic setting, EM is built around the central tenet:

A proof of existence of an object yields a procedure to compute said object.

Soon after its inception, it was realised that the framework of EM is quite flexible and can be used for the study of much more general topics, such as reductive proof theory, generalised recursion theory, type theory and programming languages, etcetera. A monograph on the topic of EM is forthcoming as [18].

Similar to the second view of Reverse Mathematics (as classifying theorems based on computability, rather than ‘forcing computability onto theorems’), one can approach EM from the same relative point of view: Rather than enforcing the central tenet of EM, one can ask the following ‘relative’ question:

For a given theorem $T$, what extra axioms are needed to compute the objects claimed to exist by $T$?

---

3The system RCA$_0$ consists of induction $I\Sigma_1$, and the recursive comprehension axiom $\Delta^0_1$-CA.

4Exceptions are classified in the so-called Reverse Mathematics Zoo ([14]).
In other words, how strong is UT the uniform version of a theorem T? To this question, the EMT from Section [3.1] provides a surprising(ly) uniform answer.

2. About and around the base theory RCA_0^Ω

In this section, we introduce the base theory RCA_0^Ω in which we will prove our results. We discuss some basic results and introduce some notation.

2.1. The system RCA_0^Ω. In two words, RCA_0^Ω is a conservative extension of Kohlenbach’s base theory RCA_0 from [35] with certain axioms from Nelson’s Internal Set Theory (IIS) based on the approach from [15]. This conservation result is proved in [5], while certain partial results are implicit in [4]. In turn, RCA_0^Ω is a conservative extension of RCA_0 for the second-order language by [36, Prop. 3.1].

In Nelson’s syntactic approach to Nonstandard Analysis ([43]), as opposed to Robinson’s semantic one ([45]), a new predicate ‘st(x)’, read as ‘x is standard’ is added to the language of ZFC. The notations (∀stx) and (∃ṣty) are short for (∀x)(st(x) → ...) and (∃y)(st(y) ∧ ...). The three axioms Idealization, Standard Part, and Transfer govern the new predicate ‘st’ and give rise to a conservative extension of ZFC. Nelson’s approach has been studied in the context of higher-type arithmetic in e.g. [11,35].

Following Nelson’s approach in arithmetic, we define RCA_0^Ω as the system

\[ \text{E-PRA}_0^ω + \text{QF-AC}^{1,0} + \text{HAC}_{\text{int}} + I + \text{PF-TP}_γ \]

from [5, §3.2-3.3]. To guarantee that RCA_0^Ω is a conservative extension of RCA_0, Nelson’s axiom Standard part must be limited to Ω-CA defined below (which derives from HAC_{int}), while Nelson’s axiom Transfer has to be limited to universal formulas without parameters, as in PF-TP_γ. We have the following theorem

**Theorem 2.1.** The system E-PRA_0^ω + HAC_{int} + I + PF-TP_γ is a conservative extension of E-PRA_0. The system RCA_0^Ω is a Π^0_1-conservative extension of PRA.

**Proof.** See [5, Cor. 9].

The conservation result for E-PRA_0^ω + QF-AC^{1,0} is trivial. Furthermore, omitting PF-TP_γ, the theorem is implicit in [4, Cor. 7.6] as the proof of the latter goes through as long as EFA is available. We now discuss the new axioms in more detail.

2.2. The Transfer principle of RCA_0^Ω. We first discuss the Transfer principle included in RCA_0^Ω, which is as follows.

**Principle 2.2 (PF-TP_γ).** For any internal formula ϕ(x^γ) with all parameters shown, we have ( ∀^st x^γ )ϕ(x) → ( ∀x )ϕ(x).

A special case of the previous can be found in Avigad’s system NPRA_ω from [1]. The omission of parameters in PF-TP_γ is essential, as is clear from Theorem 2.3 for which we introduce:

\[ ( ∃^st f^1)(∀^st n)f(n) = 0 \rightarrow (∀n)f(n) = 0 ] \quad (Π^0_1{-}\text{TRANS}) \]

\[ (∃^2)(∀^γ g^1)(∃^0)g(x) = 0 \leftrightarrow ϕ(g) = 0 ] \quad (∃^2) \]

Note that standard parameters are allowed in f, and that (∃^2) is the functional version of ACA_0 ([54, III]), i.e. arithmetical comprehension.

**Theorem 2.3.** The system RCA_0^Ω proves Π^0_1{-}\text{TRANS} ⇔ (∃^2).

**Proof.** By [5, Cor 12]. We sketch part of the proof in Theorem 3.1 below.

Besides being essential for the proof of the previous theorem, PF-TP_γ implies that all functionals defined without parameters are standard, as discussed next.
Remark 2.4 (Standard functionals). We discuss an important advantage of the axiom PF-TP\(_\Sigma\). First of all, given the existence of a functional, like e.g. the existence of the fan functional (See e.g. [36, 42]) as follows:

\[
(\exists \Omega)(\forall \varphi)(\forall f^1, g^1 \leq 1)[\overline{\Omega}(\varphi) = 0 \Rightarrow \overline{\Omega}(\varphi) = 0],[\overline{\varphi}(g) = 0],
\]

(MUC)

we immediately obtain, via the contraposition of PF-TP\(_\Sigma\), that

\[
(\exists \Omega)(\forall \varphi)(\forall f^1, g^1 \leq 1)[\overline{\Omega}(\varphi) = 0 \Rightarrow \overline{\Omega}(\varphi) = 0],[\overline{\varphi}(g) = 0].
\]

(2.1)

In other words, we may assume that the fan functional is standard. The same holds for any functional of which the definition does not involve additional parameters.

Secondly, we may assume \(\Omega(\varphi)\) is the least number as in (MUC), which implies that \(\Theta(\varphi)\) from (2.1) can also be assumed to have this property. However, then \(\Theta(\varphi) = 0\) \(\Omega(\varphi)\) for any \(\varphi^2\), implying \(\Theta = \Omega\), i.e. if it exists, the fan functional is unique and standard. The same again holds for any uniquely-defined functional of which the definition does not involve additional parameters.

The previous observation prompted the addition to RCA\(_\Pi\) of axioms reflecting the uniqueness and standardness of certain functionals (See [5, 33.3]). It should be noted that Nelson makes a similar observation concerning IST in [43, p. 1166].

2.3. The Standard part principle of RCA\(_\Pi\). Next, we discuss the Standard Part principle, called \(\Omega\)-CA, included in RCA\(_\Pi\). Intuitively speaking, a Standard Part principle allows us to convert nonstandard into standard objects.

By way of example, the following type 1-version of the Standard part principle results in a conservative extension of WKL\(_0\) (See [6, 3.3]).

\[
(\forall X^1)(\exists Y^1)(\forall x^0)(x \in X \iff x \in Y).
\]

(2.3)

Here, we have used set notation to increase readability; We assume that sets \(X^1\) are given by their characteristic functions \(f^1_X\), i.e. \((\forall x^0)(x \in X \iff f_X(x) = 1)\). The set \(Y\) from (STP) is also called the standard part of \(X\).

We now discuss the Standard Part principle \(\Omega\)-CA, a very practical consequence of the axiom HAC\(_{int}\). Intuitively speaking, \(\Omega\)-CA expresses that we can obtain the standard part (in casu \(G\)) of \(\Omega\)-invariant nonstandard objects (in casu \(F(\cdot, M)\)). Note that we write ‘\(N \in \Omega\)’ as short for ‘\(\text{st}(N^0)\)’.

Definition 2.5. [\(\Omega\)-invariance] Let \(F^{(\sigma \times 0) \rightarrow 0}\) be standard and fix \(M^0 \in \Omega\). Then \(F(\cdot, M)\) is \(\Omega\)-invariant if

\[
(\forall x^0)(\forall N^0 \in \Omega)[F(x, M) = 0 F(x, N)].
\]

(2.2)

Principle 2.6 (\(\Omega\)-CA). Let \(F^{(\sigma \times 0) \rightarrow 0}\) be standard and fix \(M^0 \in \Omega\). For every \(\Omega\)-invariant \(F(\cdot, M)\), there is a standard \(G^{\sigma \rightarrow 0}\) such that

\[
(\forall x^0)(\forall N^0 \in \Omega)[G(x) = 0 F(x, N)].
\]

(2.3)

The axiom \(\Omega\)-CA provides the standard part of a nonstandard object, if the latter is independent of the choice of infinite number used in its definition.

Theorem 2.7. The system RCA\(_\Pi\) proves \(\Omega\)-CA.

Proof. See e.g. [19]; We also sketch the derivation of \(\Omega\)-CA from HAC\(_{int}\). The latter takes the form

\[
(\forall x)(\exists y)(x^0 y^0 \varphi(x, y) \rightarrow (\exists y)(\forall x^0 y^0 \varphi(x, y)),
\]

(HAC\(_{int}\))

where \(\varphi(x, y)\) is internal, i.e. not involving the standardness predicate ‘st’, and where \(\Phi(x)\) is a finite sequence of objects of the type of \(y\). Thus, HAC\(_{int}\) does not provide a witness to \(y\), but a sequence of possible witnesses.
Let \( F(\cdot, M) \) is \( \Omega \)-invariant, i.e. we have
\[
(\forall x)(\exists k^0)(\forall y) [F(x, M) =_{\Omega} F(x, N)].
\]
We immediately obtain (any infinite \( k^0 \) will do that
\[
(\forall x)(\exists k^0)(\forall y) [F(x, M) =_{\Omega} F(x, N)].
\]
By the induction axioms present in RCA\(_0^\omega\), there is a least such \( k \) for every standard \( x^\sigma \). By our assumption (2.4), such least number \( k^0 \) must be standard, yielding:
\[
(\forall x)(\exists k^0)(\forall y) [F(x, M) =_1 F(x, N)].
\]
Now apply HAC\(_{\text{int}}\) to obtain standard \( \Phi^\sigma \) such that
\[
(\forall x)(\exists k^0)(\forall y) [F(x, M) =_1 F(x, N)].
\]
Next, define \( \Psi(x) := \max_{\xi < \Phi(x)} \Phi(x)(i) \) and note that
\[
(\forall x)(\exists k^0)(\forall y) [F(x, M) =_1 F(x, N)].
\]
Finally, put \( G(x) := F(x, \Psi(x)) \) and note that \( \Omega\)-CA follows.

In light of the previous proof, one easily establishes the following corollaries.

**Corollary 2.8.** In RCA\(_0^\omega\), we have for all standard \( F^{(\sigma \times 0)^{-1}} \) that
\[
(\forall x)(\forall y) (\forall M, N \in \Omega) [F(x, M) =_{\Omega} F(x, N)]
\]
\[
= (\exists k^0)(\forall y) (\forall M^0, N^0 \geq k^0) [F(x, M^0) =_{\Omega} F(x, N^0)]
\]
where \( f^1 \approx g \) if \( (\forall n)(f(n) =_{\Omega} g(n)) \).

**Corollary 2.9.** In RCA\(_0^\omega\), for all standard \( F^{(\sigma \times 0)^{-1}} \) and internal formulas \( C \),
\[
(\forall x)(\forall y) (\forall M, N \in \Omega) [C(F, x) \rightarrow F(x, M) =_{\Omega} F(x, N)]
\]
\[
= (\exists k^0)(\forall y) (\forall M^0, N^0 \geq k^0) [C(F, x) \rightarrow F(x, M^0) =_{\Omega} F(x, N^0)]
\]

Applications of the previous corollaries are assumed to be captured under the umbrella-term \( \Omega\)-CA. Furthermore, by the above, if we drop the \( \Omega \)-invariance condition in \( \Omega\)-CA, the resulting system is a non-conservative extension of RCA\(_0^\omega\).

### 2.4. Notations and remarks.
We finish this section with some remarks. First of all, we shall use the same notations as in [5], some of which we repeat here.

**Remark 2.10** (Notations). We write \( (\forall x)(\exists y) \Phi(x, y) \) and \( (\exists x)(\forall y) \Psi(x, y) \) as short for \( (\forall x)(\exists y) [\exists \tau (\tau(x, y) \rightarrow \Phi(x, y))] \) and \( (\exists x)(\forall y) [\exists \tau (\exists \tau (\tau(x, y) \rightarrow \Psi(x, y))] \). We also write \( (\forall x)(\exists y) \Phi(x, y) \) and \( (\exists x)(\forall y) \Psi(x, y) \) as short for \( (\forall x)(\exists y) [\exists \tau (\tau(x, y) \rightarrow \Phi(x, y))] \) and \( (\exists x)(\forall y) [\exists \tau (\exists \tau (\tau(x, y) \rightarrow \Psi(x, y))] \). Furthermore, if \( \neg \Phi(x, y) \) (resp. \( \neg \Psi(x, y) \)), we also say that \( x^\sigma \) is ‘infinite’ (resp. finite) and write \( 'x^\sigma \in \Omega \). Finally, a formula \( A \) is ‘internal’ if it does not involve \( \tau \), and \( A^\tau \) is defined from \( A \) by appending ‘st’ to all quantifiers (except bounded number quantifiers).

We will use the usual notations for rational and real numbers and functions as introduced in [39] p. 288–289 and [54] 1.8.1 for the former.

**Remark 2.11** (Real number). A (standard) real number \( x \) is a (standard) fast-converging Cauchy sequence \( q_{n,j}^1 \), i.e. \( (\forall n^0, i^1) ([q_n - q_{n+1}] \leq \frac{1}{2i}) \). We freely make use of Kohlenbach’s ‘hat function’ from [39] p. 289 to guarantee that every sequence \( f^1 \) can be viewed as a real. Two reals \( x, y \) represented by \( q_{i,j} \) and \( r_{i,j} \) are equal, denoted \( x \equiv y \), if \( (\forall n)([q_n - r_n] \leq \frac{1}{2^n}) \). Inequality \( < \) is defined similarly. We also write \( x \equiv y \) if \( (\forall n)([q_n - r_n] \leq \frac{1}{2^n}) \) and \( x \gg y \) if \( x \equiv y \wedge x \neq y \). Functions \( F \) mapping reals to reals are represented by functionals \( \Phi^{\sigma \rightarrow \tau} \) such that \( (\forall x, y)(x = y \rightarrow \Phi(x) = \Phi(y)) \), i.e. equal reals are mapped to equal reals.
As hinted at by Corollary 2.8 the notion of equality in RCA_0^0 is important.

**Remark 2.12 (Equality).** The system RCA_0^0 only includes equality between natural numbers ‘’=’’ as a primitive. Equality ‘’=’’ for type τ-objects x, y is then defined as follows:

\[ [x =_\tau y] \equiv (\forall z^1 \ldots z^k)[x z_1 \ldots z_k =_0 y z_1 \ldots z_k] \]  

(2.5)

if the type τ is composed as τ ≡ (τ_1 \rightarrow \ldots \rightarrow \tau_k \rightarrow 0). In the spirit of Nonstandard Analysis, we define ‘approximate equality ≈_\tau’ as follows:

\[ [x \approx_\tau y] \equiv (\forall z^1 \ldots z^k)[x z_1 \ldots z_k =_0 y z_1 \ldots z_k] \]  

with the type τ as above. The system RCA_0^w includes the axiom of extensionality for all ϕ^{τ→τ} as follows:

\[ (\forall x^\tau, y^\tau)[x =_\rho y \rightarrow \varphi(x) =_\tau \varphi(y)]. \]  

(E)

However, as noted in [4] p. 1973], the so-called axiom of standard extensionality is problematic and cannot be included in RCA_0^0.

Now, certain functionals (like (\mathbb{Z}^2)^st introduced above) are standard extensional, but others may not be. Hence, we shall sometimes prepend ‘E-’ to an axiom defining a functional to express that the latter is standard extensional. In particular, if AX^{st} ≡ (\mathbb{Z}^st \Phi^{ρ→τ})A^{st}(\Phi) for internal A, then E-AX^{st} is AX^{st} plus the statement that \Phi as in the latter satisfies standard extensionality as in (\mathbb{R}^{st}).

As an example illustrating the previous remark, the functional from UWKL^{st} is standard extensional if it outputs the left-most path in an infinite binary tree, and this property naturally emerges in the proof of Theorem 3.3. In Theorem 3.3 we show that standard extensionality as in E-UWKL^{st} follows naturally from UWKL.

In light of Corollary 2.8 it is obvious how Ω-CA can be further generalised to F((\sigma \times 0)→τ) using ‘≈_τ’ instead of ‘≡_1’; The same holds for ‘≃’ and real-valued F.

**Remark 2.13 (The computable nature of operations in RCA_0^0).** Tennenbaum’s theorem ([33] [11.3]) ‘literally’ states that any nonstandard model of PA is not computable. What is meant is that for a nonstandard model M of PA, the operations +_M and ×_M cannot be computably defined in terms of the operations + and × of the standard model \mathbb{N} of PA.

While Tennenbaum’s theorem is of interest to the semantic approach to Nonstandard Analysis involving nonstandard models, RCA_0^0 is based on Nelson’s syntactic framework, and therefore Tennenbaum’s theorem does not apply: Any attempt at defining the (external) function ‘+’ limited to the standard numbers’ is an instance of illegal set formation, forbidden in Nelson’s internal framework ([43] p. 1165]).

To be absolutely clear, lest we be misunderstood, Nelson’s internal set theory IST forbids the formation of external sets \{x ∈ A : st(x)\} and functions ‘f(x) limited to standard x’. Therefore, any appeal to Tennenbaum’s theorem to claim the ‘non-computable’ nature of + and × from RCA_0^0 is blocked, for the simple reason that the functions ‘+’ and × limited to the standard numbers’ simply do not exist. On a related note, we recall Nelson’s dictum from [43] p. 1166 as follows:

*Every specific object of conventional mathematics is a standard set.*

It remains unchanged in the new theory [IST].

In other words, the operations ‘+’ and ‘×’, but equally so primitive recursion, in (subsystems of) IST, are exactly the same familiar operations we know from (subsystems of) ZFC. Since the latter is a first-order system, we however cannot exclude the presence of nonstandard objects, and internal set theory just makes this explicit, i.e. IST turns a supposed bug into a feature.
3. The Explicit Mathematics theme around arithmetical comprehension

In this section, we establish the EMT from Section 3.1 for theorems \( T^* \leftrightarrow UT \leftrightarrow (\exists^2) \), i.e. at the level of arithmetical comprehension, the third Big Five system (See [54, III]) of Reverse Mathematics.

3.1. The EMT for weak König’s lemma. In this section, we establish the EMT for the weak König’s lemma, the defining axiom of the Big Five system WKL\(_0\) ([54, 1.10]). The uniform version of weak König’s lemma is defined as:

\[
(\exists \Phi \exists^1(\forall T^1 \leq 1) \left[ (\forall n^0)(\exists \beta^1)(\exists n \in T) \rightarrow (\forall x^0)(\Phi(T) \in T) \right],
\]

(WKL)

with set-theoretic notation rather than as in [54]. The nonuniform version is:

\[
(\exists^\ast T^1 \leq 1) \left[ (\forall n^0)(\exists \beta^1)(\exists n \in T) \rightarrow (\exists^\ast \alpha^1)(\forall m^0)(\exists n \in T) \right].
\]

(WKL*)

We are abusing notation by using ‘\( T \leq 1 \)’ to denote that \( T \) represents a binary tree. We will sometimes mention the notion of ‘infinite number’ and it will always be clear from context whether we mean the antecedent of UWKL or WKL*.

Furthermore, note the type mismatch between ‘infinite number’ and ‘infinite tree’.

Theorem 3.1. In \( \text{RCA}_0^\ast \), we have \((\exists^2)^{st} \leftrightarrow \Pi^0_1\)-TRANS \( \leftrightarrow \text{WKL}^* \leftrightarrow E\text{-UWKL}^{st} \).

Proof. By way of illustration of the use of \( \Omega\)-CA, we first prove \( \Pi^0_1\)-TRANS \( \rightarrow (\exists^2)^{st} \). We then prove

\[
\Pi^0_1\text{-TRANS} \rightarrow \text{WKL}^* \rightarrow E\text{-UWKL}^{st} \rightarrow (\exists^2)^{st},
\]

(3.1)

which establishes the theorem by Theorem 2.3 above.

First of all, assume \( \Pi^0_1\)-TRANS and let \( f^1 \) be standard. Clearly, \((\exists^\ast x^0)(f(x) = 0)\) is equivalent to \((\exists x^0 \leq M^0)(f(x) = 0)\), for any infinite \( M^0 \), even if \( f \) involves additional standard parameters. Define standard \( \psi^{(1 \times 0) \rightarrow 0} \) as follows.

\[
\psi(f, N^0) := \begin{cases} 
0 & (\exists x^0 \leq N^0)(f(x) = 0) \\
1 & \text{otherwise}
\end{cases}.
\]

By \( \Pi^0_1\)-TRANS, \( \psi(f, M^0) \) is \( \Omega \)-invariant, i.e.

\[
(\forall x^1)(\forall N, M \in \Omega)(\psi(f, M) =_0 \psi(f, N)).
\]

By \( \Omega\)-CA, there is standard \( \varphi^{1 \rightarrow 0} \) such that \((\forall x^1)(\forall N \in \Omega)(\psi(f, M^0) =_0 \varphi(f))\), and \((\exists^2)^{st}\) immediately follows by the definition of \( \psi \).

Secondly, the first implication in (3.1) is immediate: By Theorem 2.2, we have \((\exists^2)^{st} \) and hence WKL\(^\ast\), as arithmetical comprehension implies weak König’s lemma. Now, the consequent of WKL\(^{st}\) is \((\exists^\ast \alpha^1)(\forall m^0)(\exists n \in T)\), and applying \( \Pi^0_1\)-TRANS to the innermost \((\Pi^0_1)^{st}\)-formula in the latter yields WKL*.

Thirdly, the final implication follows from [38, Proposition 3.4]. Indeed, in the latter proof Kohlenbach establishes UWLK \( \rightarrow (\exists^2) \) by defining (primitive recursively) the functional \( \varphi \) from \( (\exists^2) \) to the functional \( \Phi \) from UWLK; The axiom of extensionality is only invoked for \( \Phi \). Furthermore, the axiom \( T\text{st} \) from \( \text{RCA}_0^\Omega \) states that the Kleene recursor constant \( R_0 \) is standard, implying that all functionals defined by primitive recursion in \( \text{RCA}_0^\Omega \) are standard in \( \text{RCA}_0^\ast \). Hence, Kohlenbach’s proof of UWLK \( \rightarrow (\exists^2) \) goes through relative to the standard world in \( \text{RCA}_0^\Omega \), i.e. we have the final implication in (3.1).

For the middle implication in (3.1), we first prove \((\exists^2)^{st} \rightarrow \text{UWLK}^{st}\) which will make what follows much clearer. To this end, for an infinite binary tree \( T \), consider:

\[
(\forall n^0)(\exists \alpha^0)((|\alpha| = n \wedge 0 \ast \alpha \in T) \vee (\exists^\ast n^0)(\exists \alpha^0)((|\alpha| = n \wedge 1 \ast \alpha \in T)),
\]

(3.3)
which expresses that one of the branches originating from the root node of $T$ is infinite. As is clear from (3.3), the notion of infinite branch is a $(\Pi^0_1)^\ast$-formula. To derive UWKL$^\ast$ from $(\exists^1)^\ast$, one starts at the root node of $T$ and determines if the 0-branch or the 1-branch is infinite using $(\exists^1)^\ast$, i.e. which disjunct of (3.3) holds. If the $\eta$-branch is chosen (for $\eta = 0,1$), we define $\Phi(T)(0)$, the first element of the path in $T$, as the number $n$. Next, we move to the node $n$ and repeat the previous construction relative to $n$ to define $\Phi(T)(1)$, and so on.

Intuitively speaking, to prove the implication WKL$^\ast \rightarrow E$-UWKL$^\ast$, we assume WKL$^\ast$ and repeat the construction of $\Phi$ from the previous paragraph, but with both quantifiers ‘$(\forall^\ast n_0^\eta)$’ in (3.2) replaced by $(\forall n_0 \leq M)$ for infinite $M$. Because of WKL$^\ast$, the resulting functional is $\Omega$-invariant and standard extensional. We now spell out the details of this heuristic sketch.

Thus, assume WKL$^\ast$ and define the functional $\Psi(T,M)$ as follows: We define $\Psi(f,M)(0)$ as 0 if $(\forall m_0 \leq M)(\exists \alpha_0)(|(\alpha_0)| = m \land 0 \star \alpha \in T)$, and 1 otherwise. For the general case, define

$$
\Psi(f,M)(n + 1) := \begin{cases} 
0 & (\forall m_0 \leq M)(\exists \alpha_0)(|(\alpha_0)| = m \land \Phi(T)(n) \star \alpha \in T) \\
1 & \text{otherwise}
\end{cases}.
$$

(3.4)

Next, we prove that $\Psi(T,M)$ is $\Omega$-invariant assuming WKL$^\ast$. First of all, define the subtree $T_\sigma$ for $\sigma \in T$ by $(\forall \tau_0 \leq 1)(\tau \in T_\sigma \iff \sigma \star \tau \in T)$. Next, fix infinite $M$ and let $T$ be some infinite binary tree. Now note that if $\Psi(T,M)(0) = 0$, then the 0-branch of $T$ is infinite, i.e. $(\forall^\ast m_0^0)(\exists \alpha_0^0)(|(\alpha_0^0)| = m \land 0 \star \alpha \in T)$. By definition, this implies that the tree $T_0$ is infinite, i.e. $(\forall^\ast m_0^0)(\exists \alpha_0^0)(|(\alpha_0^0)| = m \land \alpha \in T_0)$. Now apply WKL$^\ast$ to the infinite binary tree $T_0$ to obtain the existence of a standard binary sequence $\alpha$ such that $(\forall n_0^0)(\exists \alpha_0 \in T_0)$. However, this implies by definition that $(\forall m_0^0)(\exists \alpha_0^0)(|(\alpha_0^0)| = m \land 0 \star \alpha \in T)$. By the definition of $\Psi$, we must have $\Psi(f,M)(0) = \Psi(f,N)(0) = 0$ for any infinite $N$. Similarly, one proves that $\Psi(T,M)(n) = \Psi(T,N)(n)$ for any finite $n$ and infinite $M,N$, and we obtain $\Psi(T,M) \approx_1 \Psi(T,N)$ for infinite $N,M$ and $T$ any standard infinite binary tree.

As $\Omega$-CA requires quantification over all standard binary trees as in (3.5), we need to specify the behaviour on finite trees. Thus, we define $\Theta(T,M)$ as $\Psi(T,M)$ if $(\forall m_0 \leq M)(\exists \alpha_0)(|(\alpha_0)| = m \land \alpha \in T)$, and 0 otherwise. Using WKL$^\ast$ as in the previous paragraph, it is clear that $\Theta(T,M)$ is $\Omega$-invariant, i.e.

$$(\forall^\ast n_0^0, T_1 \leq 1)(\forall N, M \in \Omega)[\Theta(T,M) \approx_1 \Theta(T,N)].$$

(3.5)

Now let $\Phi$ be the ‘standard part’ of $\Theta$ provided by $\Omega$-CA, i.e. we have $(\forall^\ast T_1 \leq 1)(\forall M \in \Omega)[\Theta(T,M) \approx_1 \Phi(T)]$ and $\Phi$ is as required by UWKL$^\ast$. Now suppose $\alpha^1 \in T$ is a standard path to the left of $\Phi(T)$. By the definition of ‘to the left of’, there is some standard $n_0$ such that $\overline{\Phi(T)}n_0$ and $\alpha(n_0 + 1) < \Phi(T)(n_0 + 1)$. However, then the tree $T_{\overline{\Phi(T)}n_0}$ is infinite and applying WKL$^\ast$ to this tree, we have $\Phi(T)(n_0 + 1) = \Theta(T,M)(n_0 + 1) = \alpha(n_0 + 1)$, a contradiction. Thus, $\Phi(T,M)$ outputs the left-most path in $T$. This immediately implies the standard extensionality of $\Phi$, i.e. we have

$$(\forall^\ast T_1^1, S^1 \leq_1 1)(T \approx_1 S \rightarrow \Phi(T) \approx_1 \Phi(S)),$$

and E-UWKL$^\ast$ now follows from WKL$^\ast$, and we are done. \qed

The functional $\Psi(\cdot,M)$ from (3.4) is called the canonical approximation of the functional $\Phi$ from UWKL, as it mirrors the way the latter functional is defined using $(\exists^2)$. As discussed in Remark 3.14, the canonical approximation has rather
low complexity, with possible applications to Hilbert’s program for finitistic mathematics. Finally, in light of the above, the reader should be convinced that the equivalence involving WKL* is note merely a ‘coding trick’ (see also Theorem 3.3).

The series of implications (3.1) is useful as a template for establishing similar equivalences in a uniform way, as is clear from the proofs of the theorems in the next two sections. We first prove the following corollaries.

**Corollary 3.2.** In RCA0, we have \( (\exists^2) \leftrightarrow \Pi^1_1\text{-TRANS} \leftrightarrow \text{WKL}^* \leftrightarrow \text{UWKL}^* \).

**Proof.** Immediate from \[36\] Prop. 3.9 and \[5\] Cor. 14. Alternatively and similar to the proof of the latter, it is possible to modify E-UWKL so that PF-TP\(\gamma\) may be applied, yielding UWKL. This involves dropping the ‘st’ in the antecedent of UWKL and bringing the universal quantifier in the consequent to the front.

The principle E-UWKL may seem somewhat contrived, but actually follows directly from UWKL. In particular, the standard extensionality in the former can be proved quite elegantly using PF-TP\(\gamma\), as we establish now.

**Theorem 3.3.** In RCA0, the implication \( \text{UWKL} \rightarrow \text{E-UWKL}^* \) can be proved ‘directly’, i.e. without the use of the equivalences \( \Pi^1_1\text{-TRANS} \leftrightarrow (\exists^2) \leftrightarrow \text{UWKL}^* \). In particular, the functional from UWKL may be assumed to be standard extensional.

**Proof.** First of all, let \( \Psi \) be the functional from UWKL and note that the axiom of extensionality for \( \Psi \) can be brought into the form:

\[
(\forall S^1, R^1 \leq 1, k^0)(\exists N^0)\left( T N = S N \rightarrow \Psi(S)k = \Psi(R)k, \right)
\]

by resolving both occurrences of ‘\( \equiv \)’ and bringing all quantifiers to the front. Now bring all the quantifiers in UWKL to the front, yielding:

\[
(\exists \Psi^1, T^1 \leq 1, k^0)\left( \forall n^0, T^1 \leq 1 \right) \left( \exists m^0 \right) [\alpha = m \land \alpha \in T] \rightarrow \Psi(T)n \in T.
\]

Using QF-AC1.0, we obtain

\[
(\exists \Psi^1, T^1 \leq 1, k^0) \left( \forall n^0, T^1 \leq 1 \right) \left( \exists m^0 \right) [\alpha = m \land \alpha \in T] \rightarrow \Psi(T)n \in T,
\]

and the extensionality of \( \Psi \), we obtain

\[
(\exists \Psi^1, T^1 \leq 1, k^0) \left( \forall n^0, T^1 \leq 1 \right) \left( \exists m^0 \right) [\alpha = m \land \alpha \in T] \rightarrow \Psi(T)n \in T
\]

Applying PF-TP\(\gamma\) to the second conjunct, we obtain

\[
(\exists \Psi^1, T^1 \leq 1, k^0) \left( \forall n, T^1 \leq 1 \right) \left( \exists m^0 \right) [\alpha = m \land \alpha \in T] \rightarrow \Psi(T)n \in T
\]

Applying PF-TP\(\gamma\) to \[35\], we may assume the functionals \( \Psi \), \( \Phi \), and \( \Xi \) are standard. Now consider a standard binary tree \( T' \) such that \( (\forall^a k^0)(\exists a^0)[|\alpha| = k \land \alpha \in T'] \).

For standard \( n, \Xi(T', n) \) is standard and we have \( (\exists a^0)[|\alpha| = \Xi(T', n) \land \alpha \in T'] \), implying \( \Psi(T)n \in T' \) for any standard \( n \), i.e. we have UWKL\(^*\). Similarly, if \( S \approx_k R \) for standard binary trees \( S, R \), then \( \Phi(S, T, k) \) is standard for standard \( k \), implying \( \Psi(S)k = \Psi(R)(k) \) for standard \( k \). But the latter is just \( \Psi(T) \approx_k \Psi(S) \) and \( \Psi \) satisfies standard extensionality, implying E-UWKL\(^*\).

In light of the previous theorem, one can prove standard extensionality for any functional of type \( 1 \rightarrow 1 \) (or of similar typing) with a defining internal formula (without parameters). This is somewhat surprising in light of the discussion of \( [\text{B}] \) \(^*\) in \[4\] p. 1973. We now prove a result like Theorem 3.3 for \( (\exists^2) \).
In [5, Cor. 14], the equivalence $\Pi^0_2$-TRANS $\leftrightarrow$ (3) is proved indirectly using the operator $(\mu^2)$ from [25][36], defined as:

$$(\exists \mu^2)(\forall f^1)((\exists x^0)f(x) = 0 \rightarrow f(\mu(f)) = 0).$$

This axiom is equivalent to (3) over RCA$_{\omega}^+$ (See [38]) and $(\mu^2)$ yields $\Pi^0_2$-TRANS by applying PF-TP$\nu$ to the former, i.e. $\mu$ is standard, and so is the witness $\mu(f)$ for standard $f^1$. We now show that (3) implies $\Pi^0_2$-TRANS in the same way.

**Corollary 3.4.** In RCA$_{\omega}^+$, the implication (3) $\rightarrow \Pi^0_2$-TRANS can be proved ‘directly’, i.e. without the use of the equivalence (3) $\leftrightarrow (\mu^2)$. In particular, the functional from (3) may be assumed to be standard extensional.

**Proof.** Similar to the proof of Theorem 5.3 obtain via PF-TP$\nu$ that (3) implies the standardness and standard extensionality of $\varphi$ as in (3). This amounts to applying the contraposition of PF-TP$\nu$ to

$$(\exists \varphi^2, \Xi^2)[(\forall f^1)(\exists x^0)f(x) = 0 \leftrightarrow \varphi(f) = 0]$$

$\land (\forall g^1, h^1)(\exists \Xi(g, h) = \exists \Xi(g, h) \rightarrow \varphi(g) = \varphi(h)),$

which follows from (3), extensionality [4], and QF-AC$^{1,0}$. Now assume $\Pi^0_2$-TRANS is false, i.e. there is standard $h^1$ such that $(\forall n^0)(h(n) = 0) \land (\exists m^0)(h(m) \neq 0$. Define the standard sequences $f^2 := 11\ldots$ and $g^2$ as:

$$g(n) := \begin{cases} 1 & (\nu k < n)h(k) = 0 \\ 0 & \text{otherwise} \end{cases}.$$  

Clearly $f \approx_1 g$, implying $1 = \varphi(f) = \varphi(g) = 0$, by the definition of $\varphi$ and standard extensionality. This contradiction implies that we must have $\Pi^0_2$-TRANS. □

A similar theorem can be proved for the Suslin functional and $\Pi^0_2$-TRANS from [5, §4.2]. Finally, the following corollary suggests that Theorem 5.3 provides a nice template for classifying uniform principles.

**Corollary 3.5.** In RCA$_{\omega}^+$, the implication UWKL $\rightarrow \Pi^0_2$-TRANS can be proved ‘directly’, i.e. without the use of the equivalence (3) $\leftrightarrow \Pi^0_2$-TRANS.

**Proof.** Proceed as in the proof of the theorem and derive (5.3), but without introducing $\Xi^2$, i.e. we have

$$(\exists \Psi^{-1}, \Phi^3)[(\forall T \leq_1 1)[(\forall m^0)(\exists \alpha^0)[|\alpha| = m \land \alpha \in T] \rightarrow (\forall n^0)\Psi(T)n \in T]\ (3.7)$$

$\land (\forall S^1, R^1 \leq_1 1, k^0)(T\Phi(S, R, k) = S\Phi(S, R, k) \rightarrow \Psi(S)k = \Psi(R)k].$

By PF-TP$\nu$, we may assume that the functionals $\Psi, \Phi$ are standard and that $\Psi$ is standard extensional as in the proof of the theorem. Now suppose $\Pi^0_2$-TRANS is false, i.e. there is standard $h^1$ such that $(\forall n^0)(h(n) = 0$ and $(\exists m^0)(h(m) \neq 0$. For $T'$ the (necessarily standard) full binary tree, define the standard binary tree $S'$ as (using $\Psi$ from (3.7)):

$$\sigma \in S' \leftrightarrow (\forall m < |\sigma|)h(m) = 0 \lor (\forall i = 0, 1)(\Psi(T')(1) = i \rightarrow (\forall j < |\sigma|)\sigma(j) = 1 - i].$$

However, then $T' \approx_1 S'$, implying $\Psi(T') \approx_1 \Psi(S')$, but this contradicts (3.7) as $S'$ is infinite (either 00... or 11... is completely in $S'$), but $\Psi(S')n$ is not in $S'$ for large enough $n$, by the assumption on $h$. This contradiction yields $\Pi^0_2$-TRANS. □

Finally, we point out that extensionality [4] is essential in obtaining the above results. In particular, Kohlenbach has proved that without full extensionality, the principle UWKL is not really stronger than WKL itself (38).
3.2. The EMT for $\Sigma^0_1$-separation. Next, we establish the EMT for the $\Sigma^0_1$-separation principle from [54, I.11.7], known to be equivalent to WKL; the non-standard version is:

**Principle 3.6 ($\Sigma^0_1$-SEP*).** For standard $f^1_1$ and $\varphi_i(n) \equiv (\exists n_i)(f_i(n_i, n) = 0)$ such that $(\forall^* n)\neg(\varphi^1_i(n) \land \varphi^2_i(n))$, there is standard $Z^1$ such that

\[
(\forall n^0)[\varphi_1(n) \rightarrow n \not\in Z \land \varphi_2(n) \rightarrow n \in Z].
\] (3.8)

The principle $\Sigma^0_1$-SEP* states the existence of a separating set for $(\Sigma^0_1)^{st}$-formulas, but for all numbers, not just the standard ones. Similarly, $\Sigma^0_1$-SEP is:

**Principle 3.7 ($U\Sigma^0_1$-SEP).** For $\varphi_i(n, f) \equiv (\exists n_i)(f(n_i, n) = 0)$, we have

\[
(\exists F((1 \times 1 \times 0) \rightarrow 0))[\forall f^1_1, g^1_1][\forall n] \neg(\varphi_1(n, f) \land \varphi_2(n, g)) \rightarrow
\]

\[
(\forall n^0)[\varphi_1(n, f) \rightarrow F(f, g, n) = 1] \land (\forall n)[\varphi_2(n, g) \rightarrow F(f, g, n) = 0].
\] (3.9)

**Theorem 3.8.** In $\text{RCA}_0^{\omega}$, $\Pi^0_1$-TRANS $\leftrightarrow \Sigma^0_1$-SEP* $\leftrightarrow E\Sigma^0_1$-SEP* $\leftrightarrow (\exists^0_1)^{st}$.

Proof. Clearly, we could exploit the equivalence between $\Sigma^0_1$-separation and weak König’s lemma ([54 IV.4.4]), but we instead prove the following implications:

\[
\Pi^0_1\text{-TRANS} \rightarrow \Sigma^0_1\text{-SEP} \rightarrow E\Sigma^0_1\text{-SEP} \rightarrow (\exists^0_1)^{st},
\] (3.10)

which establishes the theorem. The first implication in (3.10) is trivial as in (3.8) is a $\Pi^0_1$-formula, and the final implication follows from [46 Theorem 3.6]. One can also use the observation that the final part of the proof of [54 IV.4.4] implies that $U\Sigma^0_1$-SEP $\rightarrow$ UWKL, and the same for the standard extensional versions.

For the second implication, note that $\Sigma^0_1$-SEP* implies for standard $f^1_1$ that:

\[
(\forall^* n)[\neg\varphi^1_i(n, f_1) \lor \neg\varphi^2_i(n, f_2)] \rightarrow (\forall n)[\neg\varphi_1(n, f_1) \lor \neg\varphi_2(n, f_2)].
\] (3.11)

Now let $h^1_1$ be such that $h^1_1(k, M) = 0 \leftrightarrow (\forall n^0 \leq M) f^1_i(n_i, k) \neq 0$, and note that by (3.11), we have $h^1_1(k, M) = 0 \lor h^2_2(k, M) = 0$ for standard $k$ and $M \in \Omega$. Next, define the functional $\Psi$ as follows:

\[
\Psi(f_1, f_2, M)(n) :=
\begin{cases}
0 & h^1_1(n, M) = 0 \land h^2_2(n, M) \neq 0 \\
1 & h^1_1(n, M) \neq 0 \land h^2_2(n, M) = 0 \\
2 & h^1_1(n, M) \neq 0 \land h^2_2(n, M) \neq 0 \\
3 & h^1_1(n, M) = 0 \land h^2_2(n, M) = 0
\end{cases}
\] (3.12)

We now show that $\Psi$ is both as required for $U\Sigma^0_1$-SEP and $\Omega$-invariant, in case the standard $f_1$ satisfy $(\forall^* n)[\neg\varphi^1_i(n, f_1) \lor \neg\varphi^2_i(n, f_2)]$.

Indeed, for standard $n$, if $\varphi^1_i(n, f_1)$ then by assumption and (3.11), we have $\neg\varphi^2_2(n, f_2)$, and the second case in (3.12) holds (for any infinite $M$). If $\varphi^2_2(n, f_2)$ holds for standard $n$, then similarly $\neg\varphi^1_1(n, f_1)$ by the previous, and the first case in (3.12) holds (for any infinite $M$). Since $\varphi^2_2(n, f_2) \land \varphi^1_1(n, f_1)$ is impossible by assumption, the third case in (3.12) does not occur. If for some standard $n_0$, we have the fourth case (or even only $\neg\varphi^1_1(n_0, f_2) \land \neg\varphi^1_1(n_0, f_1)$), consider the functions $f_3, f_4$ defined for any $k$ as $f_3(k, n) = f_1(k, n)$ and $f_4(k, n) = f_2(k, n)$ for $n \neq n_0$ and $f_3(k, n_0) = f_4(k, n_0) = f_1(k, n_0) \times f_2(k, n_0)$. By assumption, we have $(\forall^* n)[\neg\varphi^1_1(n, f_3) \lor \neg\varphi^2_2(n, f_4)]$ and (3.11) applied to the latter yields $\neg\varphi^1_1(n_0, f_1) \land \neg\varphi^1_1(n_0, f_2)$. Hence, if the fourth case in (3.12) occurs, it does so for every $M \in \Omega$.

As $\Omega$-CA requires quantification over all standard sequences $f^1_1$ as in (3.13), we need to specify the behaviour when the separation assumption $(\forall^* n)[\neg\varphi^1_1(n, f_1) \lor \neg\varphi^2_2(n, f_2)]$ is not met. Thus, we define $\Theta(f, g, M)$ as $\Psi(f, g, M)$ if $(\forall n \leq M)[(\forall n \leq M)[(\forall n \leq M)$.
3. The EMT for the intermediate value theorem. Next, we establish the EMT for the intermediate value theorem (IVT). Although the latter is provable in RCA₀ (Theorem 3.6), the uniform version of UIVT is equivalent to $(\exists^2)$, as discussed by Kohlenbach in [36, p. 293] and in Remark 3.12.

With regard to notation, let ‘$f \in C[0, 1]$’ mean that $f$ is $\varepsilon$-$\delta$-continuous on $[0, 1]$ and ‘$f \in \overline{C}[0, 1]$’ that $f \in C[0, 1]$ and $f(0) \geq 0$ and $f(1) \leq 0$. Appending of ‘‘st’’ to $C$ and $\overline{C}$ means that all quantifiers are relative to ‘‘st’’. As explained in Remark 3.12, the exact choice of continuity (involving $C$ or $\overline{C}$) is immaterial. The uniform and nonstandard versions of IVT are:

\[
(\forall^st f^{1\rightarrow 1} \in \overline{C}[0, 1])(\exists^st x^1 \in [0, 1])(f(x) = 0). \tag{IVT^*}
\]

\[
(\exists^st (1\rightarrow 1))((\forall^st f^{1\rightarrow 1} \in \overline{C}[0, 1])(f(\Phi(f)) = 0). \tag{UIVT}
\]

Note that IVT^* is weaker than IVT^*, as the latter (and also UIVT^*) involves ‘‘$\approx$’’ rather than ‘‘$=$’’, which turns out to yield quite a difference in strength.

**Theorem 3.10.** In RCA₀, we have $(\exists^2)^{st} \leftrightarrow \Pi_1^0$-TRANS $\leftrightarrow$ IVT^* $\leftrightarrow$ E-UIVT^*.

**Proof.** To establish the equivalences in the theorem, we prove

\[
\Pi_1^0$-TRANS $\rightarrow$ IVT^* $\rightarrow$ E-UIVT^* $\rightarrow$ $(\exists^2)^{st}, \tag{3.14}
\]

which establishes the theorem by Theorem 2.3 above. The first implication in (3.14) is trivial as IVT is provable in RCA₀ (See e.g. [34, II.6.6]) and $z^1 = 0$ is a $\Pi_1^0$-formula for reals $z$. The final implication follows from the proofs of [35, Proposition 3.14] and [33, Prop. 3.9] in the same way the final implication in (3.1) is proved.

For the remaining implication in (3.14), it should first be noted that in the proof that [35] Proposition 3.14.3 implies $(\exists^2)$, the intermediate value functional $F$ is only applied to polynomials $f_{\bar{y}}(x) := yx - y$ and $f_{\bar{y}}(x) := yx$ to obtain a discontinuous function, and hence $(\exists^2)$. Hence, it suffices to obtain E-UIVT^* limited to standard polynomials (which can be coded by standard reals). This situation is reminiscent of the ‘usual’ Brouwerian counterexample involving IVT where only very special functions are used (See e.g. [11] p. 4 or [39] p. 11).

We now prove the remaining implication in (3.14). To this end, we prove that IVT^* implies E-UIVT^* limited to standard polynomials. To quantify over the latter, we will use the notation $(\forall^st f^{1\rightarrow 1} \in P)$, although polynomials may be coded by reals. We also write $(\forall^st f^{1\rightarrow 1} \in P)$ to mean that $f(0) > 0$ and $f(1) < 0$. Similar to the proof of Theorem 3.1, we will mimic the proof of $(\exists^2) \rightarrow$ UIVT involving the usual ‘interval-halving’ technique.
First of all, following [54, II.6.6-7], we have the ‘approximate’ IVT:
\[
(\forall^* f^{1\rightarrow 1} \in \mathcal{P})(\forall k^0)(\exists x^1 \in (0, 1))((f(x) < \frac{1}{2}).
\]
Now, by the continuity of \(f\), we can replace \((\exists x^1 \in (0, 1))((f(x) < \frac{1}{2})\) by a \(\Sigma^0_1\) formula, say \((\exists^* x^0)\varphi(f, x, n)\), with \(\varphi\) quantifier-free. Furthermore, it is easy to find a (primitive recursive) witnessing function for this existential quantifier (again using the continuity of \(f\)). Actually, this witnessing function is nothing more than a realizer for the constructive ‘approximate’ version of IVT (See e.g. [8, Theorem 4.8, p. 40]). Hence, we can treat \((\forall^* f^{1\rightarrow 1} \in \mathcal{P})(\exists x^1 \in (0, 1))((f(x) < \frac{1}{2})\) as a \(\Pi^0_1\)-formula, say \(\Psi(f, k, 0, 1)\), with \(\psi\) quantifier-free and the two last variable places for the interval end points in the former formula.

Now define \(\Psi(f, M)(0)\) as 0 if \((\forall k^0 \leq M)\psi(f, k, 0, \frac{1}{2})\), and \(\frac{1}{2}\) otherwise. In general, the functional \(\Psi\) is defined as:
\[
\Psi(f, M)(n+1) := \begin{cases} 
\Psi(f, M)(n) & \text{if } (\forall k^0 \leq M)\psi(f, k, \Psi(f, M)(n), \Psi(f, M)(n) + \frac{1}{2^{n+1}}) \\
\Psi(f, M)(n) + \frac{1}{2^{n+1}} & \text{otherwise}
\end{cases}
\]

We now prove that \(\Psi(f, M)\) is \(\Omega\)-invariant for standard \(f \in \mathcal{P}\) and \(M \in \Omega\). If \(\Psi(f, M)(0) = 0\), then \(f\) becomes arbitrarily small on \([0, \frac{1}{2}]\) relative to ‘\(\sim\)’, i.e. we have that \((\forall^* k^0)(\exists^* x^1 \in (0, \frac{1}{2}))((|f(x)| < \frac{1}{2})\). Since \(f\) is a standard polynomial, this implies \((\exists^* x^1 \in [0, \frac{1}{2}) )f(x_0) \approx 0\). Applying IVT* for the interval \([0, x_0]\), we obtain \((\exists^* x^1 \in (0, \frac{1}{2}))f(x) = 0\). But then \(f\) becomes arbitrarily small on \([0, \frac{1}{2}]\), i.e. \((\forall k^0)(\exists^* x^1 \in (0, \frac{1}{2}))((|f(x)| < \frac{1}{2})\), and we have \(\Psi(f, M)(0) = \Psi(f, N)(0) = 0\) for any \(N \in \Omega\) by definition.

Similarly, one proves that \(\Psi(f, M)(n) = \Psi(f, N)(n)\) for any finite \(n\) and infinite \(M, N\), and we obtain \(\Psi(f, M) \approx_1 \Psi(f, N)\) for infinite \(N, M\) and \(f\) any standard polynomial. Furthermore, it is easy to see that \(\Psi(f, M)\) provides the left-most intermediate value of \(f\). Applying \(\Omega\)-CA now yields E-UIVT\(^*\) limited to standard polynomials, and we are done.

\begin{corollary}
In R\(\text{CA}^0_1\), we have \((\exists^2) \Leftrightarrow \Pi^0_1\)-\(\text{TRANS} \Leftrightarrow \text{IVT}^* \leftrightarrow \text{UIVT}\).
\end{corollary}

\begin{proof}
Immediate from the theorem, [36 Prop. 3.14], and [3 Cor. 14]. We also sketch a ‘direct’ proof of UIVT \(\rightarrow \Pi^0_1\)-\(\text{TRANS} \) below in Remark 3.12
\end{proof}

We finish this section with a remark on continuity.

\begin{remark}[Continuity]
First of all, as is clear from [36 Prop. 3.14] and as noted in the previous proof, UIVT limited to various continuity classes is still equivalent to \((\exists^2\). Similarly, one can replace \(\mathcal{C}^*\) in IVT\(^*\) and E-UIVT\(^*\) by \(\mathcal{C}\) and the proof of IVT\(^*\) \(\rightarrow E\text{-UIVT}^*\) still goes through. Indeed, with \(\Psi\) defined as in the proof of the theorem, one can prove \((\forall^* f^{1\rightarrow 1})(\forall N, M \in \Omega)[f \in C \rightarrow \Psi(f, N) \approx_1 \Psi(f, M)\] and apply Corollary 2.39. To prove the former, it does seem WKL\(^*\) is needed (which follows from E-UIVT\(^*\) limited to standard polynomials). Since internal formulas are part of the original language (of R\(\text{CA}^0_0\) or R\(\text{CA}_0\), \(C\)-functions are arguably more interesting objects of study than \(C^*\)-functions (from the point of view of the EMT). One could also work with representations \(\Phi^{1\rightarrow 1} \in \mathcal{C}[0, 1]\), where the latter denotes the definition of continuity on Cantor space, i.e.
\[
(\forall \alpha \leq_1 1, k^0)(\exists \beta \leq_1 1)(\exists N \in \mathcal{P} N \rightarrow \Phi(\alpha)(k) = \Phi(\beta)(k)),
\]
and \(\Phi\) is extensional with regard to real equality as in Remark 2.11. Secondly, we can prove a version of Theorem 3.3 for the intermediate value theorem by replacing \(f^{1\rightarrow 1} \in \mathcal{C}[0, 1]\) by the definition of continuity from Reverse Mathematics (5.4 II.6.1)) involving a so-called associate of type 1. Indeed, by [37 Prop. 4.10 and
Prop. 4.4], every pointwise continuous function satisfies [54, II.6.1] given WKL, and the latter already follows from UIVT limited to polynomials.

For instance, to prove that the aforementioned limited version of UIVT implies \( \Pi^0_1\)-TRANS, assume \( h^1 \) does not satisfy the latter, let \( f \) be the function from the usual Brouwerian counterexample to IVT (See [59, p. 11]) and if \( \Phi(f) > \frac{1}{2} \) define \( g \) by \( g(x) := f(x) - \sum_{k=0}^{\infty} k(i) \frac{2^i}{i!} \), where \( k(i) = 1 \leftrightarrow (\exists n \leq i) h(n) \neq 0 \). Abusing notation somewhat, we have \( f \approx_1 g \), implying \( \Phi(f) \approx_1 \Phi(g) \), but this yields a contradiction as \( \Phi(g) \) must satisfy \( \Phi(g) \leq \frac{1}{2} \). For the case \( \Phi(f) \leq \frac{1}{2} \), define \( g(x) := f(x) + \ldots \), and we obtain UIVT \( \rightarrow \Pi^0_1\)-TRANS.

3.4. The EMT for the Weierstraß extremum theorem. Finally, we establish the EMT for the Weierstraß extremum theorem, equivalent to WKL by [54, IV.2.3].

\[
(\exists f^{(1 \rightarrow 1)})(\forall f \in C[0, 1])(\forall y \in [0, 1])(f(y) \leq f(\Phi(f))). \quad \text{(UWEMAX)}
\]

\[
(\exists f^{1\rightarrow 1})(\exists x \in [0, 1])(\forall y \in [0, 1])(f(y) \leq f(x)). \quad \text{(WEIMAX*)}
\]

By [36, Prop. 3.14], the exact choice of continuity in UWEMAX does not matter. Note that if \( f \in C[0, 1] \) does not involve ‘st’ in WEIMAX*.

**Theorem 3.13.** In \( \text{RCA}_0^\Pi \), \( \Pi^0_1\)-TRANS \( \iff \) WEIMAX* \( \iff \) UWEMAX \( \iff \) (3^2)*.

**Proof.** To establish the theorem, we now prove:

\[
\Pi^0_1\text{-TRANS} \rightarrow \text{WEIMAX}^* \rightarrow \text{E-UWEMAX}^* \rightarrow (3^2)^*.
\]

First of all, the final implication again follows from [36, Proposition 3.14]. Furthermore, to obtain UIVT from UWEMAX, apply the latter to \( -|f| \) (for \( f \in C \)) and note that the maximum of \( -|f| \) must be an intermediate value of \( f \). However, as noted in the proof of Theorem [5, 10], UIVT is only applied to polynomials to obtain (3^2) in the proof of [36, Proposition 3.14]. Hence, UWEMAX and WEIMAX* may also be limited to certain elementary functions.

Secondly, we prove the first implication in (3.16). Using [36, Proposition 3.14] and [5, Cor. 15], \( \Pi^0_1\)-TRANS is equivalent to UWEMAX, and by PF-TP\(_\forall\), the functional in the latter is standard, immediately implying WEIMAX*. We also list more conceptual ‘direct’ proofs for Lipschitz continuous functions (with factor one) and for \( \Phi \in \mathfrak{C}[0, 1] \). In light of the results in [27], it should be straightforward to convert \( f \in C[0, 1] \) into \( \Phi \in \mathfrak{C}[0, 1] \) using (3^2). Note that the first implication in (3.16) is not trivial, as the innermost universal formula of the Weierstraß maximum theorem is \( \Pi^0_1 \). Nonetheless, this formula is equivalent to a \( \Pi^0_1\)-formula, namely if we restrict the real quantifier to the rationals. Thus, assume \( \Pi^0_1\)-TRANS and consider standard \( \Phi \in \mathfrak{C}[0, 1] \). We first prove that \( \Phi \in \mathfrak{C}^*[0, 1] \), i.e. that \( \Phi \) also is continuous relative to the standard world as in (3.16)*.

To this end, consider the proof of [37, Prop. 4.10] in which it is proved that a functional \( \Phi^{1 \rightarrow 1} \in \mathfrak{C}[0, 1] \) has a modulus of uniform continuity, assuming WKL. In the latter proof, Kohlenbach defines a sequence of infinite binary trees \( T_{k,n} \) using (only) \( \Phi \), and the sequence of paths through these trees (the sequence exists via WKL) is used to define the characteristic function of the formula in square brackets in the following formula:

\[
(\forall k, f^1 \leq_1 1)(\exists N)[(\forall h^1, g^1 \leq_1 1)(\exists N = \overline{N} = \overline{N} \rightarrow \Phi(h)(k) = \Phi(g)(k))] \quad (3.17)
\]

Now if \( \Phi \) is additionally standard, the tree \( T_{k,n} \) will also be standard, and UWKL (available via Theorem [5, 1] and PF-TP\(_\forall\)) yield a standard sequence of paths through (all of) \( T_{k,n} \). It is then easy to show that the formula in square brackets in (3.17)
now has a standard characteristic function. However, then Π_0^1\text{-TRANS} yields that for standard k and standard f^1 ≤_1 1 
(∃N∈N)(∀h^1,g^1 ≤_1 1)((fN=gN⇒f(k)=g(k))),

as the formula in square brackets may be treated as quantifier-free with standard parameters. With some coding, Φ ∈ C^{st}[0, 1] now follows, i.e. we have Π_{k+1}^1. Thirdly, since Π_0^1\text{-TRANS} → (∃N) → \text{WKL}^{st}, [54] IV.2.3] implies the Weierstraß maximum theorem relative to ‘st’. By [37] Prop. 4.4 and 4.10], we may apply the maximum theorem for standard Φ ∈ C[0, 1] (which satisfy Φ ∈ C^{st}[0, 1] by the previous paragraph). Hence, there is standard x_0 ∈ [0, 1] such that (∀q^0 ∈ [0, 1])(ϕ(q) ≤ ϕ(x_0)), where we used Π_0^1\text{-TRANS} to obtain the final Π_0^1-formula. However, for any Φ ∈ C[0, 1] and x ∈ [0, 1], we have 
(∀q^0 ∈ [0, 1])(ϕ(q) ≤ ϕ(x)) ↔ (∀q^1 ∈ [0, 1])(ϕ(q) ≤ ϕ(x)). \tag{3.18}

The reverse implication of \tag{3.18} is trivial, while the forward one is a simple application of Φ ∈ C[0, 1]. Hence, \tag{3.18} implies WEIMA \* for Φ ∈ C[0, 1]. For Lipschitz continuous functions with factor one, note that such functions are by definition also Lipschitz continuous relative to ‘st’. Therefore, the same proof involving \tag{3.18} yields WEIMA \* for such functions.

Finally, for the remaining implication in \tag{3.18}, we only provide a sketch, as the former is proved in much the same way as the middle implication in \tag{3.14}. Indeed, consider the following formula (∀k^0)(∃x^1 ∈ [0, 1])(\sup_{y∈[0, 1]} f(y) - f(x)| < \frac{1}{k}). Note that WEIMA \* implies the existence of the supremum by [54] IV.2.3]. As in the proof of Theorem \tag{3.10} there is a witnessing function for the existential quantifiers in the previous formula: The functional Ψ(f, M) is then built in the same way as in the proof of Theorem \tag{3.10} and we are done.

In conclusion, we note that a slick proof of this theorem proceeds by proving \tag{3.10} for standard Lipschitz continuous functions and then proving ‘full’ \tag{3.10} using the already established equivalences and (the proof of) [39] Prop. 3.14]. □

The above proofs reveal a template of the form which may be applied to obtain the EMT for RT(1) and [54] I.10.3.9], using [46] Theorems 4.2 and 4.3]. In Remark \tag{3.14} below, we elaborate on this template. Furthermore, as in Remark \tag{3.12}, a version of Theorem \tag{3.3} may be proved for the Weierstraß maximum theorem by restricting continuity to the usual definition in Reverse Mathematics.

### 3.5. Concluding remarks

We finish this section with some concluding remarks.

**Remark 3.14** (Hilbert’s program). We discuss the connection of the above results to Hilbert’s program for finitist mathematics. We are motivated by Tait’s analysis (56) that the formal system PRA captures Hilbert’s notion of finitist mathematics, and Burgess’ detailed study of how Reverse Mathematics contributes to Hilbert’s program (12). The following quote is essential:

[... ] whereas a finitist cannot know that everything provable in PRA is finitistically provable, a finitist can know that everything provable in a bounded fragment of PRA such as EFA is finitistically provable. This positive fact is the other side of the coin from the negative fact that bounded fragments do not exhaust finitistic provability as (according to the Tait analysis) PRA provability does. ([12] p. 139])

In short, to establish a partial realization of Hilbert’s program, it is essential according to Burgess that results are ultimately provable in a subsystem of PRA, like EFA (= IΔ₀ + EXP). Now, the functional Ψ(T, M) from \tag{3.3} is definable in the
$\Pi^0_1$-conservative extension of EFA from [51 Cor. 8], i.e. the latter nonstandard system has a PRA-consistency proof. Thus, after following the latter proof, a finitist can conclude that the latter functional is unproblematic finitistically.

Furthermore, one can prove in the same EFA-based system that every standard functional $\Xi^{\rightarrow 1}$ which outputs the left-most path $\Xi(T)$ in the standard binary tree $T$ satisfies $\Xi(T) \approx_{1} \Psi(T, M)$, and vice versa. In other words, a finitist can accept the correctness of the hypothetical statement ‘If a functional as in UWKL exists, then it equals a finitistically acceptable object’. It should be noted that a similar argument works for the fan functional (MUC) from Section 4 which happens to be inconsistent with UWKL.

As to intuitionistic mathematics, we remark the following.

**Remark 3.15.** For L.E.J. Brouwer, the real numbers $\mathbb{R}$ constituted a ‘unsplittable continuum’, exemplified by Brouwer’s rejection of $x > \mathbb{R} 0 \lor x \leq \mathbb{R} 0$, a special case of tertium non datur. A similar observation regarding the ‘syrupy’ continuum in intuitionistic mathematics is made by van Dalen in [10]. The results in the previous theorems and corollaries go the opposite way: In our system, the maximum or intermediate values of continuous functions are determined by discrete case distinctions as done in the canonical functional. In this way, a ‘very discrete’ picture of the continuum emerges. Furthermore, in our opinion, the canonical approximations endow the original functionals with plenty of ‘numerical meaning’, though not the kind envisaged by Brouwer and other constructivists. In Section 4 we establish the EMT for principles from intuitionistic mathematics.

Next, we discuss a connection to intuitionistic logic due to Kohlenbach.

**Remark 3.16.** As noted above, while the intermediate value theorem is provable in RCA$_0$, the uniform version is equivalent to $(\exists^2)$. Similarly, the statement SUP that every continuous function has a supremum is equivalent to the Weierstraß maximum theorem [54, IV.2.3], but the uniform version of SUP is much weaker than $(\exists^2)$ (See [36, §3] and Corollary 5.7). This behaviour can be explained as follows.

Following Kohlenbach ([36]), the cause of the difference in behaviour between the maximum theorem and SUP, is that the latter can be proved from the fan theorem in intuitionistic logic, while the former by contrast requires classical logic. This use of classical logic results in a discontinuity at the uniform level and hence $(\exists^2)$ due to so-called Grilliot’s trick (See [36, §3] and [25]). This leads us to the following conjecture, where ‘BISH’ is Errett Bishop’s *Constructive Analysis* ([8]).

**Conjecture 3.17.** For a theorem $T$ provable in ACA$_0$, there are two categories:

1. $\text{BISH} \vdash (T \rightarrow \text{WKL})$ We have $\Pi^0_1$-TRANS $\leftrightarrow (\exists^2) \leftrightarrow T^* \leftrightarrow UT$.
2. $\text{BISH} \vdash (\text{FAN} \rightarrow T)$ We have $T^* \leftrightarrow T^* \leftrightarrow UT^*$.

Examples of the second case of the conjecture are discussed in Section 5. In particular, we study the fan theorem itself, the Heine-Borel lemma, Riemann integration, and the supremum of continuous functions.

Following this conjecture, the following theorems should fall into the first category: Peano’s theorem for $y' = f(x, y)$, binary expansion of reals, Jordan matrix decomposition, Ramsey’s theorem RT(1), contraposition of Heine-Borel compactness, Gödel’s completeness theorems, Brouwer’s fixed point theorem, the Hahn-Banach theorem, Weierstraß approximation theorem, the Hilbert and Robson basis theorems ([53]), WWKL, etc.
Examples of theorems which should fall in the second category: contraposition of WWKL and \( \Sigma^0_1 \)-separation, Riemann integration of continuous functions, Heine-Borel compactness, theorems from the previous category involving unique existence (\([27]\)), existence of supremum for \( f \in C[0,1] \), etc.

Finally, we discuss our choice of framework.

**Remark 3.18.** As a consequence of the above results, we observe that the functional \( \Phi \) from UWKL (which may be assumed to output the left-most path) equals the functional \( \Theta(\cdot, M) \) from \([35]\) for infinite \( M \) and standard input. Similarly, the functional \( \varphi \) from \((3^2)\) equals \( \psi(\cdot, M) \) from \([32]\) for infinite \( M \) and standard input. The apparent restriction to standard input is only a limitation of our choice of framework: Indeed, in stratified Nonstandard Analysis, the unary predicate ‘\( \text{st}(x) \)’ is replaced by the binary predicate ‘\( x \equiv y \)’, to be read ‘\( x \) is standard relative to \( y \)’ \([28,31,44]\). In this framework, we could prove the following:

\[
(\forall \mathcal{F})(\forall \mathcal{M} \ni f)[\psi(f,M) =_0 \varphi(T)] \land (\forall T^1 \leq 1)(\forall \mathcal{M} \ni T)[\Theta(T,M) \approx_1 \Phi(T)],
\]

where \( x \equiv y \) is \( \neg x \equiv y \), i.e. \( x \) is nonstandard relative to \( y \). In other words, in stratified Nonstandard Analysis, the approximation of \( \Phi \) and \( \varphi \) from UWKL and \((3^2)\) works for any object, not just the standard ones. Of course, we have chosen Nelson’s framework for this paper, as this approach is more mainstream.

4. The Explicit Mathematics theme for the fan functional

In this section, we establish the EMT for the fan functional, defined as in \([14]\) below, a classically false principle (See Theorem \([43]\)). Hence, Corollary \([42]\) below implies that the EMT is not limited to statements of classical mathematics. For reasons of space, we only establish the EMT for one intuitionistic principle; In \([50]\), a large number of intuitionistic principles is studied from the point of view of the EMT, including Brouwer’s continuity theorem.

As to its history, the fan functional was introduced by Tait as the first example of a functional which is non-obtainable, i.e. not computable from lower-type objects (See \([12]\) p. 102). In intuitionistic mathematics, the fan functional emerges as follows: By \([59]\) 2.6.6, p. 141], if a universe of functions \( \mathcal{U} \) satisfies \( \text{EL} + \text{FAN} \), then the class \( \text{ECF}(\mathcal{U}) \) of extensional continuous functionals relative to \( \mathcal{U} \), contains a fan functional. Here, \( \text{EL} \) is a basic system of intuitionistic mathematics and \( \text{FAN} \) is the fan theorem, the classical contraposition of WWKL. Similar results are in \([24,60,61]\).

\[
(3^\mathcal{U})(\forall \varphi)(\forall \mathcal{F}^1, g^1 \leq_1 1)[\mathcal{F}(\varphi)] =_0 \mathcal{F}(\varphi) \rightarrow \varphi(f) =_0 \varphi(g)]. \quad (\text{MUC})
\]

Clearly, the existence of the fan functional implies that all type 2-functionals are continuous, which contradicts \((3^2)\) as the latter is equivalent to the existence of discontinuous functions by \([30]\) Prop. 3.12.

Now consider the following principle expressing that all standard type 2 objects are nonstandard continuous:

\[
(\forall^\mathcal{U} \mathcal{F}^2)(\forall \mathcal{F}^1, g^1 \leq_1 1)[f \approx_1 g \rightarrow \varphi(f) =_0 \varphi(g)]. \quad (\text{DR})
\]

We have the following theorem.

**Theorem 4.1.** In \( \text{RCA}_0 \), we have \( \text{MUC} \leftrightarrow (\text{DR}) \).

**Proof.** Let \( \{0,1\}^N \) be the set of binary sequences of length \( N \). For \( \varphi^2 \) and \( f^0 \in \{0,1\}^N \), we tacitly assume that \( \varphi(f) \) stands for \( \varphi(f \ast 00 \ldots) \).

First of all, assume \([29]\) and note that the latter immediately implies

\[
(\forall^\mathcal{U} \mathcal{F}^2)(\forall \mathcal{F}^1, g^1 \leq_1 1)(\exists^\mathcal{U} x^0)[(\mathcal{F}_x =_0 \mathcal{F}x) \rightarrow \varphi(f) =_0 \varphi(g)].
\]
Furthermore, we obtain for any fixed $M \in \Omega$,

$$\forall^a \varphi^2(\forall^a f^0, g^0 \in \{0, 1\}^M)(\exists^a x^0)[(\exists x =_0 \varphi x) \to \varphi(f) =_0 \varphi(g)].$$

(4.1)

The formula $\Phi(x, f, g, \varphi)$ in square brackets in (4.1) is decidable and we define $g(f, g, \varphi, M)$ as the least $x^0 \leq M$ such that $\Phi(x, f, g, \varphi)$. As the range of $f, g$ in (4.1) is discrete, we may compute $\max_{f, g}^M g(f, g, \varphi, M)$. However, this finite number does not depend on $f$ or $g$ anymore, and we obtain

$$\forall^a \varphi^2(\exists^a x^0)(\forall^a f^0, g^0 \in \{0, 1\}^M)(\exists x =_0 \varphi x) \to \varphi(f) =_0 \varphi(g).$$

(4.2)

Combining (23) and (4.2), we have that

$$\forall^a \varphi^2(\exists^a x^0)(\forall f^1, g^1 \leq 1)[(\exists x =_0 \varphi x) \to \varphi(f) =_0 \varphi(g)].$$

(47)

Note that if $y^0$ is as in (4.2), then in (47) we can take $x = y$. Now define $\Xi(\varphi^2, M^0)$ as the least $y^0 \leq M^0$ as in (47), i.e. the following ‘elementary in $\varphi$’ functional:

$$\Xi(\varphi^2, M^0) := (\mu y \leq M)(\forall f^0, g^0 \in \{0, 1\}^M)(\exists y =_0 \varphi y) \to \varphi(f) =_0 \varphi(g).$$

(7)

The functional $\Xi(\varphi^2, M^0)$ is clearly $\Omega$-invariant (because we assume (23), i.e.

$$\forall^a \varphi^2(\forall^a f^0, M^0)(\exists y =_0 \varphi y) = \Xi(\varphi, M),$$

(4.3)

and hence $\text{MUC}^\Omega$ follows from (23) and (4.3) by applying $\Omega$-CA to the latter.

Secondly, assume $\text{MUC}^\Omega$ and note that we may assume that $\Omega(\varphi)$ is minimal in that for $m < \Omega(\varphi)$, there are binary sequences $\alpha, \beta$ of length at most $m$ such that $\varphi(\alpha) \neq _0 \varphi(\beta)$. Indeed, we need only check a finite number of finite binary sequences to see if $\Omega(\varphi)$ is minimal in this sense. A simple bounded search can be used to redefine $\Omega(\varphi)$ if necessary. Now assume the following formula:

$$\forall^a \varphi^2(\forall f^1 \leq 1)(\varphi(f) =_0 \varphi(\overline{f} n)) \leftrightarrow n \geq_0 \Omega(\varphi).$$

(4.4)

As stated in [5, §3.3] and suggested in Remark 2.1, the language $\text{RCA}_0^\Omega$ contains a symbol $\Omega_0^\Omega$ with defining axiom

$$\forall^a \varphi^2(\forall f^1 \leq 1)(\varphi(f) =_0 \varphi(\overline{f} n)) \leftrightarrow n \geq_0 \Omega_0(\varphi).$$

(4.5)

where $M(\Omega)$ is the universal formula in $\text{MUC}$ with the additional requirement that $\Omega(\varphi)$ is minimal. The axiom (4.3) expresses that the fan functional, if it exists, is unique and standard. Thus, (4.4) yields

$$\forall^a \varphi^2(\forall f^1 \leq 1)(\varphi(f) =_0 \varphi(\overline{f} n)) \leftrightarrow n \geq_0 \Omega_0(\varphi),$$

(4.6)

which contains no parameters, i.e. (4.4) qualifies for PF-TP0 (after bringing the universal quantifier outside the square brackets). Hence, we obtain:

$$\forall^a \varphi^2(\forall f^1 \leq 1)(\varphi(f) =_0 \varphi(\overline{f} n)) \leftrightarrow n \geq_0 \Omega(\varphi),$$

(4.7)

Together with (4.4), (23) is now immediate.

Finally, we prove (4.4). The reverse direction of the latter is immediate by $\text{MUC}^\Omega$: For the forward direction, assume $\forall^a \varphi^2(\forall f^1 \leq 1)(\varphi(f) =_0 \varphi(\overline{f} m_0)) \land m_0 <_0 \Omega(\varphi)$ for some fixed standard $m_0^0$ and $\varphi^2$. Fix standard $f^1, g^1 \leq 1$ such that $\overline{f} m_0 = \overline{g} m_0$. We have $\varphi(f) =_0 \varphi(\overline{f} m_0)$ and $\varphi(g) =_0 \varphi(\overline{f} m_0)$ by assumption, and $\varphi(\overline{f} m_0) =_0 \varphi(\overline{g} m_0)$ by extensionality. However, we now have $\varphi(f) =_0 \varphi(g)$ for any $f, g$ such that $\overline{f} m_0 = \overline{g} m_0$ while $m_0 < \Omega(\varphi)$, by assumption. This contradicts the minimality of $\Omega(\varphi)$, and the forward direction of (4.4) follows.

Similarly to [36, Prop. 3.15], $\text{RCA}_0^\Omega + \text{MUC}^\Omega$ is a conservative extension of WKL0 by [31 Theorem 5]. Now consider the formula (23) from the proof and consider the following corollary, establishing the EMT for the fan functional.

**Corollary 4.2.** In $\text{RCA}_0^\Omega$, we have $\text{MUC} \leftrightarrow \text{MUC}^\Omega \leftrightarrow (23)$. 
Proof. Immediate from the previous proof, in particular (4.7), and Remark 2.4.

Unsurprisingly, the fan functional is inconsistent with classical mathematics.

**Theorem 4.3.** The principles \(\text{MUC}_0^{\text{st}}\) and \((\exists^2)^{\text{st}}\) are inconsistent with \(\text{RCA}_0^{\Omega}\).

**Proof.** Let \(\varphi_0\) be the functional defined by \((\exists^2)^{\text{st}}\) and let \(\Omega\) be as in \(\text{MUC}_0^{\text{st}}\).

Now define \(f^1\) as follows: \(f(n) = 1\) if \(n \geq \Omega(\varphi_0)\), and zero otherwise, and let \(1^1\) be the sequence which is 1 everywhere. Then we have \(\Omega(\varphi_0) = \Omega(\varphi_0)\), and hence \(\varphi_0(1) = 1\), by \(\text{MUC}_0^{\text{st}}\). However, by the definition of \(\varphi_0\), we have \(\varphi_0(f) = 0\), as clearly \(f(\Omega(\varphi_0) + 1) = 0\).

Finally, we briefly consider the classically correct \((\text{MUC})_0\) obtained by limiting \(\text{MUC}\) to \(\varphi^2 \in C(2^N)\), i.e.

**Remark 4.4.** As to positive results, \((\text{MUC})_0\) is equivalent to the following:

\[
(\exists^2\Sigma^0_1)(\forall^e\varphi^2 \in C(2^N))(\forall^e f^1, g^1 \leq 1)(\text{T}_{\Sigma^0_1}(\varphi) = \text{T}_{\Sigma^0_1}(\varphi) \rightarrow \varphi(f) = \varphi(g))
\]

\[
(\forall^e\varphi^2 \in C(2^N))(\exists^2 k^0)(\forall^e f^1, g^1 \leq 1)(\text{T}_{\Sigma^0_1} = 0 \rightarrow \varphi(f) = \varphi(g)),
\]

\[(\forall^e\varphi^2 \in C(2^N))\text{ is internal. These equivalences are proved as for Theorem 4.1. In particular, similar to (5.3), (the language of) \text{RCA}_0^{\Omega} contains a symbol \(\exists^2\) and}

\[
\text{st}(\Xi_0) \land (\text{st}\Gamma^2) \left[ N(\Gamma) \rightarrow (\forall^e\varphi^2 \in C(2^N))(\Xi_0(\varphi) = \Gamma(\varphi)) \right],
\]

\[(\text{4.10)}\]

\[
\text{where } N(\Xi) \text{ is the square-bracketed formula in (4.3) with the additional requirement that } \Xi(\varphi) \text{ is minimal. As to negative results, (4.9)} \text{ involves C}(2^N) \text{ and not } \text{C}(2^N)\text{ and it seems impossible to obtain } (\text{MUC})_0^{\text{st}} \leftrightarrow (\text{MUC})_0^{\text{st}}. \text{ Indeed, } \varphi \in C^{\text{st}}(2^N) \text{ in the latter makes it impossible to apply PF-TP}_v. \text{ For the forward implication, we do not have a way of proving that } \varphi^2 \in C(2^N) \text{ also yields } \varphi^2 \in C^N(2^N) \text{ for standard.}
\]

5. The Explicit Mathematics Theme Around Weak König’s Lemma

In this section, we establish the EMT for theorems \(T\) such that \(UT\) is at the level of weak König’s lemma, in line with Conjecture 5.1.

5.1. The EMT for the Fan Theorem. First of all, we study the fan theorem, the classical contraposition of weak König’s lemma, i.e.

\[
(\forall^a \alpha \leq 1)(\exists n^0)(\exists n^0 \notin T) \rightarrow (\exists^2 k^0)(\forall^a \alpha \leq 1)(\exists n^0 \leq 0 k_0)(\exists n^0 \notin T).
\]

(5.1)

While weak König’s lemma is universally rejected as ‘non-constructive’ in constructive mathematics, the fan theorem is accepted in intuitionistic mathematics (9).

Denote the principle obtained by the universal closure of (5.1) by \(\text{FAN}^\star\). For the nonstandard case, let \(\text{FAN}^\star\) be \(\text{FAN}^\star\) but with \((\forall^a \alpha \leq 1)\) in the consequent. Now, there are at least two possible candidates for the uniform version of the fan theorem, as follows.

**Principle 5.1 (UFAN1).** There is a functional \(\Phi^2\) such that for any binary tree \(T\):

\[
(\forall^a \alpha \leq 1)(\exists n)(\exists n \notin \Omega(T)) \rightarrow (\forall^a \alpha \leq 1)(\exists n^0 \leq 0 \Phi(T))(\exists n \notin T).
\]

(5.2)

**Principle 5.2 (UFAN2).** There is \(\Phi^{(1 \times 2) \rightarrow 0}\) such that for any \(T^1 \leq 1\) and \(g^2\):

\[
(\forall^a \alpha \leq 1)(\exists n)(\exists n \notin T) \rightarrow (\forall^a \alpha \leq 1)(\exists n^0 \leq 0 \Phi(T, g))(\exists n \notin T).
\]

(5.2)

\footnote{For completeness, define \(\varphi^2 \in C(2^N)\) as \((\forall^e f^1 \leq 1)(\exists n)(\forall^e f^1 \leq 1)(T N = \exists N \rightarrow \varphi(f) = \varphi(g))\). As usual, denote \(\varphi^2 \in C^\ast(2^N)\) as the previous formula relative to ‘st’.}
Note that UFAN$_2$ is essentially the BHK-interpretation of intuitionistic logic (See e.g. [11] p. 8): The functional $g^2$ witnesses how the tree $T$ has no path, and the functional $\Phi(T,g)$ has access to this information to determine the finite height of $T$. For this reason, we refer to UFAN$_2$ as the uniform version of FAN.

We have the following preliminary results for the fan theorem. Recall that RCA$^0$ + LEM is conservative over WKL$_0$ by [5] Theorem 5 and [6] Prop. 3.15.

**Theorem 5.3.** In RCA$^0$, FAN$^*$ \( \leftrightarrow \) FAN$^{st}$, FAN$^{st}$ \( \rightarrow \) FAN, UFAN$_1^{st}$ \( \leftrightarrow \) (\( \exists \beta \))$^{st}$, UFAN$_2^{st}$ \( \rightarrow \) UFAN$_2^{st}$.

**Proof.** By the very structure of FAN$^{st}$, it is clear that FAN$^*$ follows trivially from the latter: If a standard binary tree has finite height, then nonstandard paths are also cut off at this height, as the paths have to go through the standard binary sequences of any height. To prove that FAN$^{st}$ \( \rightarrow \) FAN, assume the former and note that for standard $T^1 \leq 1$ and standard $g^2$, we have

\[
(\forall \alpha^1 \leq 1)(\overline{\alpha}(\alpha) \not\in T) \rightarrow (\exists^2 \alpha^0_1)(\forall \alpha^1 \leq 1)(\exists \eta^0 \leq k_0)(\overline{\eta} \not\in T).
\]

The previous formula trivially implies:

\[
(\forall \alpha^1 \leq 1)(\overline{\alpha}(\alpha) \not\in T) \rightarrow (\exists^2 \alpha^0_1)(\forall \alpha^1 \leq 1)(\exists \eta^0 \leq k_0)(\overline{\eta} \not\in T),
\]

and by weakening the consequent we obtain:

\[
(\forall \alpha^1 \leq 1)(\overline{\alpha}(\alpha) \not\in T) \rightarrow (\exists \eta^0_1)(\forall \alpha^1 \leq 1)(\exists \eta^0 \leq k_0)(\overline{\eta} \not\in T),
\]

which holds for standard $g^2, T^1 \leq 1$. However, (5.3) is an internal formula, say \( \varphi(T,g) \), with all parameters shown, and (\( \forall \alpha^1, T^1 \leq 1 \) \( \varphi(T,g) \)) implies (\( \forall \alpha^1, T^1 \leq 1 \) \( \varphi(T,g) \)) via PT-TP$^\ast$. Using QF-AC$^{1,0}$, FAN is now immediate.

Now assume (\( \exists \beta \))$^{st}$, fix $M \in \Omega$, and define the functional $\Psi(T,M)$ as 0 if (\( \forall \alpha^1, T^1 \leq 1 \) \( \alpha^1 \not\in T \)), i.e. if $T$ is infinite, and as the least $k \leq M$ such that (\( \forall \alpha^0 \in \{0,1\}^*(\alpha) = M \rightarrow (\exists \eta^0 \leq k)(\overline{\eta} \not\in T) \)) otherwise. Clearly, $\Psi(T,M)$ is $\Omega$-invariant (distinguish between finite and infinite trees to see this), and UFAN$_1^{st}$ follows. For the remaining implication, we derive E-UWKL$^*$ from UFAN$_1^{st}$. Let $T$ be an infinite standard binary tree and let $\Phi$ be the functional from UFAN$_1^{st}$. Recall that $\beta \in T_0$ is defined as $\alpha \ast \beta \in T$. Now define $\Psi(T,g)$ as 0 if (\( \forall \alpha^0 \in \{0,1\}^*(\alpha) \not\in \Phi(T_1) \rightarrow (\exists \eta^0 \leq \Phi(T_1))(\overline{\eta} \not\in T_1) \)) and 1 otherwise. For the general case, define $\Psi(T,n+1)$ as 0 if

\[
(\forall \alpha^0 \in \{0,1\}^*(\alpha) \not\in \Phi(T_{\Psi(T(n)+1)})) \rightarrow (\exists \eta^0 \leq \Phi(T_{\Psi(T(n)+1)}))(\overline{\eta} \not\in T_{\Psi(T(n)+1)}),
\]

and 1 otherwise. Then $\Psi$ is as required for E-UWKL$^*$. For the final implication, define $\Phi(T,g)$ as $\max_{|\sigma| = \Omega(g) \wedge |\sigma| \leq \alpha_1} g(\sigma \ast 00 \ldots)$. Alternatively, define $\Psi(T,g,M)$ as follows:

\[
\Psi(T,g,M) := \begin{cases} 
0 & \text{otherwise}
\end{cases},
\]

where $h(T,M) := (\forall \alpha^0 \in \{0,1\}^*(\alpha) = M \rightarrow (\exists \eta^0 \leq k)(\overline{\eta} \not\in T)$. Now use Theorem 5.1 in particular the nonstandard continuity of $g$, to prove the $\Omega$-invariance of this functional. \( \square \)

**Theorem 5.4.** In RCA$^0$, we have FAN$^{st}$ \( \leftrightarrow \) UFAN$_2^{st}$.

**Proof.** The reverse direction is immediate as HAC$_{int}$ implies QF-AC$^{1,0}$ relative to ‘st’. To prove that FAN$^{st}$ \( \rightarrow \) UFAN$_2^{st}$, assume the former and note that for standard $T^1 \leq 1$ and standard $g^2$, we have

\[
(\forall \alpha^1 \leq 1)(\overline{\alpha}(\alpha) \not\in T) \rightarrow (\exists^2 \alpha^0_1)(\forall \alpha^1 \leq 1)(\exists \eta^0 \leq k_0)(\overline{\eta} \not\in T).
\]
The previous formula trivially implies (for any standard \( g^2, T^1 \leq 1 \)) that
\[
(\forall^* \alpha \leq 1)(\exists \bar{\alpha}(\alpha) \not\in T) \rightarrow (\exists^* k^0_0)(\forall \beta^1 \leq 1)(\exists n^0 \leq 0 \ k_0)(\bar{\beta}n \not\in T),
\]
where \( \bar{\alpha}(\alpha) \) is the least \( n \leq g(\alpha) \) such that \( \bar{\alpha}n \not\in T \).

We now bring both quantifiers relative to ‘st’ to the front, yielding
\[
(\exists^* \alpha \leq 1, k^0_0)(\exists \bar{\alpha}(\alpha) \not\in T) \rightarrow (\forall \beta^1 \leq 1)(\exists n^0 \leq 0 \ k_0)(\bar{\beta}n \not\in T),
\]
for any standard \( g^2, T^1 \leq 1 \). Note that we could replace the quantifier ‘(\exists^* \alpha \leq 1, k^0_0)’ by a type 0-quantifier ‘(\exists^* \sigma^0 \leq 1, k^0_0)’. In particular, for \( \sigma = \bar{\alpha}g(\alpha) \), we have \( \sigma = (\sigma * 00...0)\bar{g}(\sigma * 00...0) \not\in T \). This will only be relevant for the corollary.

Abbreviating the internal formula in square brackets in (5.5) by \( \psi(\alpha, T, g, k) \), the previous implies
\[
(\forall^* g^2, T^1 \leq 1)(\exists^* k^0_0, \alpha^1 \leq 1)\psi(\alpha, T, g, k),
\]
and let standard \( \Xi^{1 \times 2 \rightarrow (0 \times 1)^*} \) be the functional resulting from applying HAC\text{int} to (5.4). Defining \( \Phi(T, g) := \max_{i < |\Xi(T, g)|} \Xi(T, g)(1)(i) \), the previous yields
\[
(\forall^* g^2, T^1 \leq 1)(\exists^* \alpha^1 \leq 1)(\exists k \leq \Phi(T, g))\psi(\alpha, T, g, k).
\]
Note that we ignored the second component of \( \Xi(T, g) \). Bringing the existential quantifier ‘(\exists^* \alpha^1 \leq 1)’ back inside \( \psi \), we obtain for all standard \( g^2, T^1 \leq 1 \) that
\[
(\forall^* \alpha^1 \leq 1)(\exists \bar{\alpha}(\alpha) \not\in T) \rightarrow (\forall \beta^1 \leq 1)(\exists n^0 \leq 0 \ \Phi(\alpha, T, g))(\bar{\beta}n \not\in T),
\]
which yields UFAN\text{st}_2 and we are done. \( \square \)

The following corollary establishes the EMT for FAN as in the second part of Conjecture 3.17. Note that we obtain \( UT^{\text{st}} \leftrightarrow T^{\text{st}} \) without extra assumptions, but require the axiom of choice for the internal version of this equivalence. Nonetheless, Hunter notes in [26, §2.1.2] that any QF-AC\text{st}^\sigma 0 still results in a conservative extension of RCA_0.

**Corollary 5.5.** In RCA\text{st}_0, we have FAN\text{st} \leftrightarrow UFAN\text{st}_2 \leftrightarrow FAN^* . Adding QF-AC\text{st}^2 0, we have FAN\text{st} \leftrightarrow FAN \leftrightarrow UFAN_2.

**Proof.** We only need to prove the second line in the corollary. There, the first forward implication follows from the theorem and the final reverse implication is immediate using QF-AC\text{st}^1 0. For the implication FAN \rightarrow UFAN_2, repeat the first part of the proof of the theorem without ‘st’ to obtain (5.4) without ‘st’. We can make sure the formula \( \psi \) is quantifier-free by requiring \( |\beta| = k \) in the consequent of (5.5). Furthermore, as noted in the proof of the theorem, the type 1-quantifier in (5.5) can be replaced by a type 0-quantifier. Now apply QF-AC\text{st}^2 0 to the resulting formula to obtain:
\[
(\exists \Xi(2 \times 1 \rightarrow (0 \times 0)^*))(\forall g^2, T^1 \leq 1)(\forall \psi(\Phi(T, g)(2), T, g, \Xi(T, g)(1)) \wedge \Xi(T, g)(2) \leq 1),
\]
and note that UFAN\text{st}_2 follows by ignoring the first component of \( \Xi \). Furthermore, by PF-TP, \( \psi \), we may assume \( \Xi \) is standard; Hence if for standard \( g^2, T^1 \leq 1 \) we have \( (\forall^* \alpha^1 \leq 1)(\exists \bar{\alpha}(\alpha) \not\in T) \), then the tree \( T \) is bounded by \( \Xi(g, T)(1) \), which is a standard number, i.e. UFAN\text{st}_2 and FAN\text{st} also follow, and we are done. \( \square \)

As an exercise, the reader can prove the equivalence between the fan theorem and its alternative nonstandard version, defined as: For all standard \( T^1 \)
\[
(\forall \alpha^1 \leq 1)(\exists n^0)\bar{\alpha}n \not\in T \rightarrow (\exists^* k^0_0)(\forall^* \alpha^1 \leq 1)(\exists n^0 \leq 0 \ k_0)(\bar{\alpha}n \not\in T). \quad \text{(FAN\text{st}*)}
\]
As a further exercise, the reader can prove the equivalence between the standard part principle (STP) and (5.1)\text{st} for any binary tree.

Similar to the addition of QF-AC\text{st}^2 0 in the previous corollary, certain results in Friedman-Simpson Reverse Mathematics require extra induction (often \( \Pi^2_2 \)). We
will often not mention $\text{QF-AC}^2.0$ in the next section, but leave the associated results implicit. As shown in [50,51], $\text{QF-AC}^2.0$ plays a similar important role in the RM of Brouwer’s continuity theorem (and related principles) and in the study of uniform versions of principles from the RM zoo.

We finish this section with the following remark.

**Remark 5.6.** Simpson has the following to say with regard to the mathematical naturalness of logical systems in [54, 1.12].

From the above it is clear that the five basic systems $\text{RCA}_0$, $\text{WKL}_0$, $\text{ACA}_0$, $\text{ATR}_0$, $\Pi^1_1-\text{CA}_0$ arise naturally from investigations of the Main Question. The proof that these systems are mathematically natural is provided by Reverse Mathematics.

By Corollary 5.5 weak König’s lemma is equivalent to the uniform fan theorem UFAN$_2$ over a system conservative over $\text{RCA}_0$. Hence, said uniform principle should also count as mathematically natural. In the following sections, we shall prove a number of equivalences between weak König’s lemma and uniform principles (involving continuity, Riemann integration, et cetera), bestowing mathematical naturalness onto all these higher-order statements.

5.2. The EMT for theorems equivalent to weak König’s lemma. In this section, we establish the EMT for various principles equivalent to weak König’s lemma, including the Heine-Borel lemma, Riemann integration, and the existence of the supremum of continuous functions. As noted in Remark 5.19 these principles can be derived constructively using the fan theorem, in line with Conjecture 5.17.

5.2.1. The Heine-Borel lemma. We first establish the EMT for the Heine-Borel lemma HB from [54, IV.1]. Careful inspection of the proof in the latter of the equivalence between WKL and HB, reveals that this proof is uniform. Thus, let UHB be the ‘fully’ uniform version of HB, i.e. the statement that there is a functional $g_i(\omega \rightarrow 2) \rightarrow 0$ such that for all open covers $I_n^{\omega \rightarrow 1} = (c_n, d_n)$ and $g^2$, we have:

$$(\forall x \in [0,1]) (x \in (c_{g(x)}, d_{g(x)})) \rightarrow \left(\exists x \in [0,1] \right)(\exists n \leq \Phi(I_n, g))(x \in (c_n, d_n)).$$

The functional $g^2$ is essential as we otherwise would obtain a version of HB like UFAN$_1$, i.e. equivalent to $(\exists^2)$. Furthermore, let HB$^*$ be HB$^\mathbb{R}$, but with the statement that the finite cover covers all of $[0,1]$, including the nonstandard reals.

This corollary to Theorem 5.3 establishes the EMT for the Heine-Borel lemma.

**Corollary 5.7.** In $\text{RCA}^\mathbb{Q}_0$, we have $\text{FAN}^\mathbb{R} \leftrightarrow \text{HB}^\mathbb{R} \leftrightarrow \text{UHB}^\mathbb{R} \leftrightarrow \text{HB}^*$. Adding $\text{QF-AC}^2.0$, we have $\text{FAN}^\mathbb{R} \leftrightarrow \text{HB}^* \leftrightarrow \text{UHB}$.

**Proof.** For the first two equivalences, the uniformity of the proofs of [54] IV.1.1-2 implies UFAN$_2 \leftrightarrow \text{UHB}$, and the equivalence to $\text{FAN}^\mathbb{R}$ follows from the theorem.

For the second equivalence, to prove $\text{HB}^\mathbb{R} \rightarrow \text{HB}^*$ is straightforward: The upper bound $k_0$ of the finite cover from HB$^\mathbb{R}$ also satisfies $(\forall x^1 \in [0,1])(x \in \cup_{i \leq k_0} I_i)$.

Indeed, suppose $z \in [0,1]$ is such that $z \not\in \cup_{i \leq k_0} I_i$. Then for all $i \leq k_0$, we have either $d_i \leq z$ or $z \leq c_i$, but we cannot have $z \approx c_i$ or $z \approx d_i$, as $c_i, d_i \in [0,1]$ satisfy $c_i, d_i \in \cup_{i \leq k_0} I_i$, which would also cover $z$. Thus, fix infinite $M$ and let $i_0$ be that $i \leq k_0$ such that $d_i(M) = \sup_{i < j < k_0} [z]M$ and $d_i(M) = \sup_{i < j < k_0} [z]M$ is minimal ($i_0$ is the least if there are several). Similarly, let $j_0$ be that $j \leq k_0$ such that $c_j(M) > \sup_{i < j < k_0} [z]M$ and $[z]M - c_j(M)$ is minimal. By the previous, we have $d_{i_0} \not\approx z \not\approx c_{j_0}$, implying $d_{i_0} + \frac{1}{2} < z < c_{j_0} - \frac{1}{2}$ for some finite $N^0$. By the definitions of $d_{i_0}$ and $c_{j_0}$, there are standard reals in $[d_{i_0}, c_{j_0}]$ which are not covered by $\cup_{i \leq k_0} I_i$, a contradiction. Hence, we must have $(\forall x^1 \in [0,1]((x \in \cup_{i \leq k_0} I_i).$
The first theorem we consider is the statement ‘every continuous function on the unit interval is uniformly continuous’, which is equivalent to weak König’s lemma by [34, IV.2.3]. As noted in Remark [34], the ‘obvious’ approach involving \(\text{MUC}\), i.e. simply restricting the fan functional to continuous functionals, does not immediately yield an equivalence to the fan theorem. Therefore, we will study the following principle, called MUC(\(\xi\)), for various notions of continuity:

\[
(\exists \Theta)(\forall \varphi \in \mathcal{C}(2^N))(\forall \alpha, \beta \leq 1)(\pi_1 \Theta(\varphi) = \beta \Theta(\varphi) \rightarrow \varphi(\alpha) = \varphi(\beta)).
\] (5.7)

First of all, let \(\varphi \in \text{CC}(2^N)\) denote that \(\varphi \in \text{C}(2^N)\) with a modulus of continuity \(g_\varphi \in \text{C}(2^N)\) which in turn has a modulus of continuity \(h_\varphi\). Both moduli are implicitly given together with \(\varphi\), and ‘CC’ stands for ‘constructive continuity’.

**Theorem 5.8.** In \(\text{RCA}_0^+ + \text{QF-AC}^{2,0}\), we have FAN \(\Leftrightarrow\) MUC(CC). This equivalence holds relative to ‘st’ in \(\text{RCA}_0^\ast\).

**Proof.** For the first reverse implication, let \(T\) be a binary tree such that \((\forall \alpha^1 \leq 1)(\exists \alpha)(\exists \Theta(\varphi) \notin T)\) and use QF-AC\(^{1,0}\) to obtain \(g^2\) such that \((\forall \alpha^1 \leq 1)(\exists \Theta(\varphi) \notin T)\). Define \(\tilde{g}(\alpha, T)\) as \(\Theta(\varphi))\) if \(\varphi(\alpha) \notin T\) and zero otherwise. By assumption, \(\tilde{g}(\cdot, T)\) is continuous as in \(\text{CC}(2^N)\): In particular, this function is its own modulus of continuity. Applying MUC(CC) yields an uniform upper bound for \(\tilde{g}(\cdot, T)\), implying that \(T\) is finite, and FAN follows.

For the first forward implication, following the proof of [37, Prop. 4.4], an associate \(\alpha^1\) for \(\Phi^2\) can be defined (uniformly) from \(\Phi\) and a continuous modulus of pointwise continuity \(g_\Phi\). By definition, the associate satisfies:

\[
(\forall \beta^1 \leq 1 \forall k^0)(\exists k^0)\alpha(\beta k) > 0 \land (\forall \beta^1 \leq 1, k^0)(\alpha(\beta k) > 0 \rightarrow \Phi(\beta) + 1 = \alpha(\beta k)).
\] (5.8)

Furthermore, if \(g_\Phi\) has a modulus of continuity, say \(h_\Phi\), one easily defines (uniformly in \(h_\Phi\)) a witnessing function \(i_\Phi\) for \((\forall \beta^1 \leq 1)(\exists k^0)(\alpha(\beta k) > 0, i.e., we have \((\forall \beta^1 \leq 1, k^0)(\alpha(\beta k) > 0\). Finally, define a tree \(T\) by \(\sigma \in T \leftrightarrow \alpha(\sigma) > 0\) and apply UFAN\(_2\) to obtain the functional from MUC(CC). The previous clearly relativizes to the standard world, and HAC\(_\text{int}\) implies QF-AC\(^{2,0}\) relative to ‘st’.

This result is not satisfying as the CC-notion of continuity is very restrictive. We therefore study the notion of continuity used in RM in more detail. Recall that continuity in the sense of [34, II.6.1] amounts to the existence of a modulus of pointwise continuity, i.e. the treatment of continuous functions as in RM entails a slight constructive enrichment, which is not problematic for the RM of WKL\(_0\) by [37, Prop. 4.10]. We now obtain a ‘nonstandard’ enrichment due to the RM-definition of continuity. This result was first obtained in [52].

**Remark 5.9 (Continuity).** In two words, the ‘nonstandard’ enrichment implicit in working with associates is as follows: A standard function which is given by an associate and is continuous relative to standard Cantor space, is automatically uniformly continuous everywhere there, given weak König’s lemma. For type 2-functionals, we can only conclude this continuity relative to ‘st’.

To establish the previous claim, consider a standard function \(\alpha^1\) such that \((\forall k^0)(\exists k^0)(\alpha(\beta k) > 0)\), which represents some function \(\phi\) on Cantor space. In other words, ‘\(\alpha\) is a code for \(\phi\)’ in the sense of [34, II.6.1] and one writes symbolically \(\phi(\beta) = \alpha(\beta k)(\alpha(\beta m) > 0)\). Now clearly \((\forall \beta^1 \leq 1)(\exists k^0 \leq N)(\alpha(\beta k) > 0)\) for some standard \(N^0\) by FAN\(_\ast\) (See also Corollary [55,3] and this...
implies $\forall \gamma, \beta \leq 1 (\forall \gamma N = 3N \to \phi(\gamma) = \phi(\beta))$, i.e. $\phi$ is uniformly continuous on all of Cantor space. We also obtain nonstandard continuity as follows:

$$\forall \beta^1, \gamma^1 \leq 1 (\beta \approx 1 \gamma \to \phi(\beta) = \phi(\gamma)) \quad (5.9)$$

By contrast, repeating the proof of [37, Prop. 4.10] for standard $\Phi^2 \in C(\mathbb{N})$ relative to ‘st’, we only obtain $\exists\alpha$ modulus for every continuous $1 \to \alpha$ weak principle by [37, Prop. 4.8], as the latter shows that the axiom guaranteeing the RM-definition of continuity by [37, Prop. 4.4]. Furthermore, MOD seems to be nonstandard versions (like (5.19)) in the following theorem is left as an exercise.

Theorem 5.11. In RCA$_0^\Omega$, we have $\text{FAN}^{\text{st}} \leftrightarrow \text{MUC}(C_{\text{rm}})^{\text{st}}$.
In RCA$_0^\Omega$ + ASC, we have $\text{FAN}^{\text{st}} \leftrightarrow \text{MUC}(M)^{\text{st}} \leftrightarrow \text{MUC}(M) \leftrightarrow \text{MUC}(C_{\text{rm}})$.
In RCA$_0^\Omega$ + ASC, we have $[\text{FAN}^{\text{st}} + \text{MOD}] \leftrightarrow [\text{MUC}(C)^{\text{st}} + \text{MOD}] \leftrightarrow \text{MUC}(C)$.

Proof. First of all, we prove the second line in the theorem except for the final forward implication. The first reverse implication follows as in the proof of the previous theorem. The second reverse implication follows from applying PF-TP$_\gamma$ to MUC($M$) and observing that by ASC, a standard functional $\phi^2 \in M^{\text{st}}(\mathbb{N})$ satisfies $\phi \in M(\mathbb{N})$. For the third reverse implication, a continuous modulus uniformly yields an associate by the proof of [37, Prop. 44].

For the remaining forward implications, assume $\text{FAN}^{\text{st}}$ and note that for standard $\phi^2 \in M(\mathbb{N})$, the latter’s standard modulus yields a standard associate $\alpha^1$ as in the proof of [37, Prop. 4.4], i.e. we have $\phi^2$ and $\phi^2$ has a standard modulus of continuity. Applying $\text{FAN}^\star$ to the latter yields $\forall \beta^1 \leq 1 (\exists \beta^0 \leq N)(\exists \beta^1 \leq 1 (\exists \beta^0 \leq N)\alpha(\beta^1) > 0)$ for some standard $N^0$. Define $\Psi(\varphi, K)$ as $(\mu K)(\forall \alpha^0, \beta^0 \leq 1 (|\alpha| = |\beta| = K \land \mu K = \mu K \to \varphi(\alpha + 0 \ldots) = \varphi(\beta + 0 \ldots)))$, and note that $\forall \varphi^2 \in M(\mathbb{N}))\forall L, K \in \Omega)\Psi(\varphi, K) = \Psi(\varphi, L)$. Since the formula $\forall \varphi \in M(\mathbb{N})$ is internal, there is (by Corollary 2.9) a standard $\Theta^3$ such that

$$\forall \varphi^2 \in M(\mathbb{N}))\forall K \in \Omega)\Psi(\varphi, K) = \Theta(\varphi), \quad (5.10)$$

and we have proved [37] from Remark 4.4 for $M$ instead of $C$. To obtain MUC($M$) from this weaker version of [37], proceed as in Remark 4.4 and the proof of Theorem 11.1. By ASC, $\varphi^2 \in M^{\text{st}}(\mathbb{N})$ implies $\varphi \in M(\mathbb{N})$, and MUC($M$) also follows from the weaker version of [37].
Secondly, we prove the first line and the remaining implication in the second line. For the reverse implication in the first line, define (for a standard binary tree \( T \)) the function \( \alpha^1 \) as \( \alpha(\sigma) = 0 \) if \( \sigma \in T \) and 2 otherwise. Applying MUC\((C_{rm})_{st}\) implies that \( T \) is bounded if it has no path (all relative to ‘st’). For the forward implication in the first line, obtain a version of \( (5.8) \) for \( C_{rm}(2^N) \) instead of \( M(2^N) \) in the same way as the first part of the proof. Since FAN\(^{st}\) implies \( \alpha^3 \in C_{rm}(2^N) \rightarrow \alpha^1 \in C_{rm}(2^N) \) for standard \( \alpha^1 \), MUC\((C_{rm})_{st}\) follows from this weak version of \( (5.8) \). As above, this weak version also implies MUC\((C_{rm}) \) by PF-TP\(_N\).

Thirdly, we prove the third line. Assume FAN\(^{st}\) and consider MOD, i.e.

\[
(\forall^s \Phi^2 \in C(2^N))(\exists^s g^2)(\forall \alpha^1, \beta^1 \leq 1)(\Xi(\Phi)(\alpha^1) = \bar{\Xi} g(\alpha) \rightarrow \varphi(\alpha) = \varphi(\beta)).
\]  

(5.11)

As ‘\( \Phi^2 \in C(2^N) \)’ is internal, we may apply HAC\(_{st}\) to (5.11), yielding standard \( \Theta^{2 \rightarrow 2} \) such that \((\exists g^2 \in \Theta(\Phi)) \) in (5.11). Now define standard \( \Xi^{2 \rightarrow 2} \) as follows: \( \Xi(\Phi)(\alpha^1) := \max_{i \leq j(\Phi)} \Theta(\Phi)(i)(\alpha) \). Clearly \( \Xi \) outputs a standard modulus of continuity for standard \( \Phi \) as input. Now proceed as above to obtain a version of (5.10) and use ASC to obtain MUC\((C)_{st}\). Furthermore, the latter implies FAN\(^{st}\) as in the first part of this proof. Next, apply PF-TP\(_N\) to MUC\((C) \) to obtain MOD. The remaining equivalences follow from the previous parts of the proof.

\[ \square \]

The results in the theorem suggest that we can either directly work with type 1-associates without additional assumptions, or work with ‘representation-free’ type 2-functions and adopt additional axioms. Since the first route is the one taken in RM, we shall also adopt this approach.

The previous proof reveals a general technique for treating uniform theorems relating to continuity: One works with the internal notion of continuity, e.g. \( \varphi^2 \in M(2^N) \) rather than \( \varphi \in M^{st}(2^N) \), to obtain a version of (5.10) by Corollary 2.3. Since the standard notion of continuity is included in the internal one (by definition or by ASC), the theorem follows. In this light, we shall discuss two more examples of the EMT, namely Riemann integration and supremum for continuous functions.

**Definition 5.12.**

1. We write \( y = \sup_{x \in [0,1]} f(x) \) as an abbreviation for:
\[
(\forall z^1 \in [0,1]) [f(x) \leq y \land (\forall k^0)(\exists z^0 \in [0,1])(y - \frac{1}{k} < f(z))].
\]  

(5.12)

2. We write ‘\( \phi \in C_{rm}[0,1] \)’ for \( \phi \) given by (\( \Phi^1, g^2 \)) such that \( \Phi \) is a code for \( \phi : [0,1] \rightarrow \mathbb{R} \) as in [51] II.6.1, and \( g \) is a modulus of continuity of \( \phi \).

Note that the extra modulus in the second part of the definition does not really constitute an enrichment of the RM-definition of continuity by [37] Prop. 4.4. We consider the following principles.

\[
(\forall f \in C_{rm}[0,1] ) (\exists y^1 ) (y = \sup_{x \in [0,1]} f(x)).
\]  

(SUP)

\[
(\exists y^1 ) (\forall f \in C_{rm}[0,1] ) (\Phi(f) = \sup_{x \in [0,1]} f(x)).
\]  

(USUP)

\[
(\exists y^1 ) (\forall f \in C_{rm}[0,1] ) (\forall N \in \omega_1 ) [\Psi(f, N) \approx \Theta(f)].
\]  

(SUP\(^*\))

**Corollary 5.13.** In RCA\(_0\), FAN\(^{st}\) \( \leftrightarrow \) SUP\(^{st}\) \( \leftrightarrow \) USUP\(^{st}\) \( \leftrightarrow \) SUP\(^*\) \( \leftrightarrow \) USUP\(^*\).

**Proof.** The first equivalence follows from [51] IV.2.3. Now assume FAN\(^{st}\), consider standard \( f \in C_{rm}[0,1] \) and define \( \Psi(f, M) := \max_{z \leq M} |f| \) for a real \( z \) represented by the sequence \( w_k \). Since by definition also \( f \in C_{rm}[0,1] \), \( f \) has a standard supremum \( y \) and it is easy to prove that \( \Psi(f, M) \approx y \approx \Psi(f, N) \) for \( M, N \in \Omega \). Applying Corollary 2.3, there is a standard \( \Theta^{3 \rightarrow 1} \) such that

\[
(\forall f \in C_{rm}[0,1] ) (\forall N \in \Omega ) [\Psi(f, N) \approx \Theta(f)].
\]  

(5.13)
Corollary 5.14. In we now provide an example of a 'nonstandard' enrichment in the form of nonstandard continuity (5.9). Similarly, the statement that every continuous function \( f \in C_{rm}[0,1] \) implies \( f \in C_{rm}[0,1] \), and USUP* is now immediate from (5.14). The remaining forward implications are proved as for Theorem 4.4 and Remark 4.3 as the supremum of \( f \in C_{rm}[0,1] \) is unique. In particular, to represent a standard binary codes \( \Theta_0 \) function \( \phi \) on Cantor space via a pair of codes \( (\gamma,\alpha) \) and (5.13) together with the properties of \( \Psi(f,M) \) now yields:

\[
\langle \forall f \in C_{rm}[0,1] \rangle \sup_{x \in [0,1]} f(x) \rangle^\ast.
\]

As in the previous proof, \( f \in C_{rm}[0,1] \) implies \( f \in C_{rm}[0,1] \), and USUP* is now immediate from (5.14). The remaining forward implications are proved as for Theorem 4.4 and Remark 4.3 as the supremum of \( f \in C_{rm}[0,1] \) is unique. In particular, to represent a standard binary codes \( \Theta_0 \) function \( \phi \) on Cantor space, we should observe was first made in [52]. Secondly, to represent a standard continuous function \( \phi \) on Cantor space, we have

\[
O(\Theta_0) \wedge (\forall^\ast \Xi \rightarrow O(\Xi) \rightarrow (\forall^\ast f \in C_{rm}[0,1])(\Theta_0(f) \approx \Xi(f)))
\]

where \( O(\Theta) \) is (5.14). Now consider \( O(\Theta_0) \) and drop the 'st' on the existential quantifier in the second conjunct of (5.12)* to obtain a formula of the form \( (\forall^\ast \phi)(\forall \rho(\forall \Xi)) \) where \( \phi \) internal and without parameters as \( \Theta_0 \) is part of the language of RCA\(^0\). Applying PF-TP\(\phi \) now yields USUP and SUP*. Finally, USUP and SUP* imply SUP*, as can be seen by using the intermediate value theorem.

With minor adaptation, the proof of the previous corollary also applies to Riemann integration. Indeed, let INT, UINT, and INT* be SUP, USUP, and SUP*, but with (5.12) replaced by \( y = \int_0^1 f(x) \, dx \), which has an obvious definition (54 IV.2.6). The following corollary establishes the EMT for Riemann integration.

Corollary 5.14. In RCA\(^0\), \( \text{FAN}^\ast \leftrightarrow \text{INT}^\ast \leftrightarrow \text{UINT}^\ast \leftrightarrow \text{INT}^\ast \).

In Remark 5.9, we showed that the definition of continuity used in RM constitutes a ‘nonstandard’ enrichment in the form of nonstandard continuity 5.9. Similarly, we now provide an example of a uniform principle implicit in 53 IV.2.3, i.e. the statement that every continuous function \([0,1]\) is uniformly continuous. This observation was first made in 52.

Remark 5.15. First of all, by 37 Prop. 4.4], the RM definition of continuity implicitly involves a modulus, and we shall make the latter explicit. In other words, we represent a continuous function \( \phi \) on Cantor space via a pair of codes \( (\alpha,\beta) \), where \( \alpha \) codes \( \phi \) and \( \beta \) codes its continuous modulus of pointwise continuity \( \omega_\phi \). Thus, \( \alpha \) and \( \beta \) satisfy \( (\forall \gamma^\ast \leq 1) (\exists N^0)\alpha(\gamma^\ast N) > 0 \) and \( (\forall \gamma^\ast \leq 1) (\exists N^0)\beta(\gamma^\ast N) > 0 \); the values of \( \omega_\phi \) and \( \phi \) at \( \gamma^\ast \leq 1 \), denoted \( \omega_\phi(\gamma^\ast) \) and \( \phi(\gamma^\ast) \), are \( \beta(\gamma^\ast N) - 1 \) and \( \alpha(\gamma^\ast N) - 1 \) for any \( k^\ast \), such that the latter are non-negative. By the previous:

\[
(\forall \gamma^\ast \leq 1) (\exists N^0) (\exists K^\ast)[K \geq \alpha(\gamma^\ast N) > 0]
\]

and \( (\forall \gamma^\ast \leq 1) (\exists N^0) (\exists K^\ast)[K \geq \beta(\gamma^\ast N) > 0] \).

Obviously, there are other ways of guaranteeing that \( \phi \) and \( \omega_\phi \) map standard binary sequences to standard numbers, but whichever way we guarantee that \( \omega_\phi \) and \( \phi \) are standard for standard input, (5.15) yields that

\[
(\forall \gamma^\ast \leq 1) (\exists N^0) (\forall \gamma^\ast \leq 1) (\exists K^\ast)[K \geq \alpha(\gamma^\ast N) > 0]
\]

since \( \omega_\phi(\gamma^\ast) \) is assumed to be standard for standard binary \( \gamma^\ast \). Combining (5.17) and (5.10), we obtain \( (\forall \gamma^\ast \leq 1) (\exists N^0)(\forall \gamma^\ast \leq 1) (\exists K^\ast)[K \geq \beta(\gamma^\ast N) > 0] \). Applying FAN*, which follows from weak König’s lemma by Corollary 5.3, we obtain \( (\forall \gamma^\ast \leq 1) (\exists N \leq k)(\exists \gamma^\ast N) > 0 \), for some standard \( k^\ast \). Hence, for every standard and continuous (in the sense of RM) function \( \phi \) on Cantor space, we have

\[
(\exists N^0)(\forall \gamma^\ast \leq 1) (\exists \gamma^\ast N)[K \geq \alpha(\gamma^\ast N) > 0](\exists \gamma^\ast N)(\forall \gamma^\ast N)(\exists \gamma^\ast N)
\]

(5.18)
given weak König’s lemma (or equivalently [54 IV.2.3]) by Corollary 5.5. In other words, implicit in weak König’s lemma (or again [54 IV.2.3]) is the fact that all standard continuous functions are uniformly continuous on all of Cantor space. The associated statement in the higher type framework is as follows:

\[
(\forall^\mathbb{N}_1 \varphi^2 \in M(\mathbb{N}^N)(\exists^\mathbb{N} N)(\forall^\mathbb{N} \zeta, \gamma \leq 1) (\mathbb{N} \subseteq \mathbb{N} \rightarrow \varphi(\zeta) = \varphi(\gamma))).
\]  

(5.19)

Applying HAC\textsubscript{int} to the previous formula, we obtain MUC\textsubscript{st} \(M^\text{st}\). In conclusion, we have established that the latter uniform statement is implicit in the non-uniform statement [54 IV.2.3]. Similar results hold for other theorems related to continuity, like those concerned with Riemann integration.

6. The Explicit Mathematics theme around arithmetical transfinite recursion

In this section, we establish the EMT for the fourth Big Five system, called ATR\textsubscript{0}, which formalises arithmetical transfinite recursion ([54 V]). Our theorems and proofs associated with ATR\textsubscript{0} show a striking resemblance to those obtained for the fan theorem in Section 5.1. Simpson has previously pointed out a connection between WKL\textsubscript{0} and ATR\textsubscript{0} in [54 I.11.7], and this connection apparently manifests itself quite strongly at the uniform level.

For reasons of space, we only consider some examples of the EMT around ATR\textsubscript{0}. We will work with the functional version of the latter, which is a mere cosmetic difference. Indeed, let WO\textsubscript{st}(X) and \(H_f(X,Y)\) be the formula \(H_\delta(X,Y)\) from [54 V.1.1 and V.2.2] for \(\theta(n^0,Y^1) \equiv (\forall k^0)\{f(k,n,Uk) = 0\}. Then ATR\textsubscript{0} in our framework is:

\[(\forall f^1, X^1)[WO(X) \rightarrow (\exists Y^1)H_f(X,Y)]. \tag{ATR\textsubscript{0}}\]

Recall that WO\((X)\) means that the countable linear order \(\leq_X\) is well-founded. Then define the following uniform version of ATR\textsubscript{e} as:

\[(\exists \Phi^1 \rightarrow 1)(\forall f^1, X^1)[WO(X) \rightarrow H_f(X, \Phi(f,X))], \tag{UATR\textsubscript{e}}\]

and the (non-trivial) nonstandard version of ATR\textsubscript{e} as:

\[(\forall^\mathbb{N}_1 f^1, X^1)[WO(X) \rightarrow (\exists^\mathbb{N} Y^1)H^\mathbb{N}_f(X,Y)]. \tag{ATR^\mathbb{N}_e}\]

The proof of the following theorem should be compared to that of Theorem 5.4 and Corollary 5.5. By [18 Theorem 2.2], the base theory is not stronger than ACA\textsubscript{0}.

**Theorem 6.1.** In RCA\textsubscript{0} + (\(\exists^2\)), we have ATR\textsubscript{e} \(\equiv\) UATR\textsubscript{e} \(\equiv\). In RCA\textsubscript{0} + QF-AC\textsubscript{1,1} + (\(\exists^2\)), ATR\textsubscript{e} \(\leftrightarrow\) UATR\textsubscript{e} \(\leftrightarrow\) ATR\textsubscript{e} \(\leftrightarrow\) UATR\textsubscript{e} \(\leftrightarrow\) ATR\textsubscript{e}.

**Proof.** The respective uniform principles clearly imply their non-uniform counterparts. Furthermore, ATR\textsubscript{e} implies

\loss{(\forall^\mathbb{N}_1 f^1, X^1)(\exists^\mathbb{N} Y^1, h^1)[WO(X, h)^{st} \rightarrow H_f(X,Y)^{st}],}

(6.1)

where \((\forall h^1)WO(X, h) \equiv WO(X), i.e. the former is the latter with the only type 1-quantifier brought to the front. By [54 V.2.3], the formula in square brackets in (6.1) is arithmetical (relative to ‘st’) and we may drop all ‘st’ inside the square brackets due to \(\Pi^0_1\)-TRANS, obtained via (\(\exists^2\)). Since we now have:

\loss{(\forall^\mathbb{N}_1 f^1, X^1)(\exists^\mathbb{N} Y^1, h^1)[WO(X, h) \rightarrow H_f(X,Y)],}

(6.2)

we apply HAC\textsubscript{int} and obtain a standard \(\Psi\) such that

\loss{(\forall^\mathbb{N}_1 f^1, X^1)(\exists Y^1, h^1 \in \Psi(f, X))[WO(X, h) \rightarrow H_f(X,Y)],}

Since \(\Psi\) is standard, we also obtain, ignoring the second component of \(\Psi\), that

\loss{(\forall^\mathbb{N}_1 f^1, X^1)(\exists Y^1 \in \Psi(f, X)(1))(\exists^\mathbb{N} h^1)[WO(X, h) \rightarrow H_f(X,Y)].}
Since the formula in square brackets is arithmetical, we may again introduce ‘st’ everywhere using $\Pi^0_1$-TRANS. We obtain:

$$(\forall^st f, X^1)(\exists Y^1 \in \Psi(f, X)(1))[\text{WO}(X, h)^{st} \to H_f(X, Y)^{st}],$$

which yields by definition that

$$(\forall^st f, X^1)(\exists Y^1 \in \Psi(f, X)(1))[\text{WO}(X)^{st} \to H_f(X, Y)^{st}].$$

Since $H_f(X, Y)^{st}$ is arithmetical (relative to ‘st’), we can use ($\exists^2$) to test which entries of $\Psi(f, X)(1)$ satisfy the former. Thus, define $\Phi(f, X)$ as $\Psi(f, X)(1)(i_0)$ where $i_0 < |\Psi(f, X)(1)|$ is the least number $i^0$ such that $\Psi(f, X)(1)(i)$ satisfies $H_f(X, \cdot)^{st}$, if such there is, and the empty set otherwise. By definition, we have

$$(\forall^st f, X^1)[\text{WO}(X)^{st} \to H_f(X, \Phi(f, X))^{st}].$$

Indeed, if $\text{WO}(X)^{st}$ then by [54, Lemma V.2.3] relative to ‘st’, if standard $Z^1, W^1$ both satisfy $H_f(X, \cdot)^{st}$, then $Z \approx_1 W$, and $Z \approx_1 W$ by $\Pi^0_1$-TRANS. In other words, there is a unique standard $Y^1$ satisfying $H_f(X, \cdot)^{st}$ and this $Y^1$ is exactly the one computed by $\Phi(f, X)$ in case WO$(X)^{st}$.

Clearly, $\text{ATR}_e$ implies [6.2] without ‘st’. In the resulting formula, use ($\exists^2$) to make the formula in square brackets quantifier-free, and $\text{QF-AC}^{1,1}$ yields:

$$(\exists^1\Phi^{1\rightarrow 1})(\forall^st f, X^1)[\text{WO}(X, \Phi(f, X))(2) \to H_f(X, \Phi(f, X)(1))]. \quad (6.3)$$

Now $\text{UATR}_e$ follows by ignoring the second component of $\Phi$ in [6.3]. Since the latter does not involve parameters, $\Phi$ is standard by $\text{PF-TP}_\forall$. Thus, (6.3) implies

$$(\exists^1\Phi^{1\rightarrow 1})(\forall^st f, X^1)[\text{WO}(X, h) \to H_f(X, \Phi(f, X)(1))]. \quad (6.4)$$

Using $\Pi^0_1$-TRANS, $\text{ATR}_e^{st}$ is now immediate.

Next, note that in [6.1], we can drop all ‘st’ using $\Pi^0_1$-TRANS, except for in $(\forall^st f, X^1)$. Since the resulting formula has no parameters, we may apply $\text{PF-TP}_\forall$ to obtain $\text{ATR}_e$ (from $\text{ATR}_e^{st}$).

Finally, $\text{ATR}_e^{st}$ clearly implies $\text{ATR}_e$ given $\Pi^0_1$-TRANS. To obtain $\text{UATR}_e$ from $\text{ATR}_e^{st}$, drop the ‘st’ in ($\exists^2$) and apply $\text{PF-TP}_\forall$. \hfill $\square$

**Corollary 6.2.** In $\text{RCA}_0 + \text{QF-AC}^{1,1}$, we have $[\text{ATR}_e + (\exists^2)] \leftrightarrow \text{UATR}_e$.

**Proof.** Apply $\text{UATR}_e$ for the well-order $\{0\}$ to obtain ($\exists^2$). \hfill $\square$

The principle CWO from [54, V.6.8] has the same syntactical structure as $\text{ATR}_0$ by [54, V.2.7 and V.2.8]. Hence, it is straightforward to obtain an equivalence between $\text{ATR}_e$ and the (obvious) uniform version of CWO.

By Theorems [6.3] and [6.4], it is clear that there is a big difference between the two versions of the uniform fan theorem from Section 5.1. In particular, the inclusion of a realiser for the antecedent of the fan theorem makes a big difference (in logical strength). We now obtain a similar result for the statement PST that:

**A tree with uncountably many paths has a nonempty perfect subtree.**

The principle PST is equivalent to $\text{ATR}_0$ by [54, V.5.5].

**Principle 6.3 (UPST$_1$).** There is a functional $\Phi^{1\rightarrow 1}$ such that $\Phi(T)$ is a nonempty perfect subtree of any tree $T$ with uncountably many paths.

A tree $T$ is said to have uncountably many paths if

$$(\forall f^0_n)(\exists f^1 \in T)(\forall n^0)(\exists m^0)(f(m) \neq f_n(m)). \quad (6.5)$$

**Principle 6.4 (UPST$_2$).** There is a functional $\Phi^{1\rightarrow 1}$ such that $\Phi(T, g)$ is a nonempty perfect subtree for any tree $T$ with uncountably many paths, and any $g^{1\rightarrow 1}$ witnessing this, i.e. $(\forall f^0_n)(\forall n^0)(\exists m^0)[g(f^1(\cdot))(m) \neq f_n(m) \land g(f^1(\cdot)) \in T]$. 
The following theorem shows that the differences between UFAN$_1$ and UFAN$_2$ from Section 6.2 align perfectly with those between UPST$_1$ and UPST$_2$. The Suslin functional ($S^2$) is the functional version of $\Pi^1_1$-CA$_0$, and discussed in Section 4.

**Theorem 6.5.** In RCA$_0$, we have UPST$_1$ ↔ ($S^2$).
In RCA$_0^\Omega + (\exists^2) +$ QF-AC$^{1,1}$, we have UPST$_2$ ↔ ATR$_{eq}$.

**Proof.** The first equivalence follows from [40] Theorem 4.4]. For the equivalence on the second line, the forward implication is immediate by [54, V.5.5]. For the remaining implication, assume ATR$_{eq}$ and use [54, V.5.5] to obtain PST:

$$\forall T^1 \exists f \in T \exists m (f(m) \neq f_n(m)) \rightarrow (\exists S^1)P(S,T).$$

where $T$, $S$ are variables ranging over trees, and $P(S,T)$ is the arithmetical formula denoting that $S$ is a non-empty perfect subtree of $T$ (See [54, V.4.1]). Since ($\exists^2$) is available, we may treat arithmetical formulas as quantifier-free. As is common in RM, we also treat type $0 \rightarrow 1$-objects as type 1-objects. By QF-AC$^{1,1}$, we obtain

$$(\forall T^1, g^{1 \rightarrow 1} \exists f_n^{0 \rightarrow 1}) (\forall n^0)(\exists m^0)[g(f_{\cdot}(\cdot))(m) \neq f_n(m) \land g(f_{\cdot}(\cdot))) \in T \rightarrow (\exists S^1)P(S,T).$$

Bringing the set quantifiers to the front:

$$(\forall T^1, g^{1 \rightarrow 1} \exists f_n^{0 \rightarrow 1}, S^1)(\forall n^0)(\exists m^0)[g(f_{\cdot}(\cdot))(m) \neq f_n(m) \land g(f_{\cdot}(\cdot))) \in T \rightarrow P(S,T).$$

The formula in square brackets is arithmetical, and applying QF-AC$^{1,1}$ yields $\Phi^{1 \rightarrow 1}$ witnessing the existential quantifiers. Ignoring the first component of $\Phi$ (involving the witness to ($\exists f_n^{0 \rightarrow 1}$)), we obtain for all $T^1, g^{1 \rightarrow 1}$ that

$$(\forall f_n^{0 \rightarrow 1}, g^{0 \rightarrow 0})(\exists m^0)[g(f_{\cdot}(\cdot))(m) \neq f_n(m) \land g(f_{\cdot}(\cdot))) \in T \rightarrow P(\Phi(T, g), T).$$

which is exactly as required, and we are done.

In the same way as for Theorem 6.1 we can establish the following, where PST$_1^*$ is PST$^{st}$ with the ‘st’ dropped from the antecedent.

**Corollary 6.6.** In RCA$_0$, we have UPST$_1^*$ ↔ PST$_1^*$ ↔ UPST$_1$ ↔ ($S^2$).
In RCA$_0^\Omega + (\exists^2) +$ QF-AC$^{1,1}$, PST$_{eq}^*$ ↔ UPST$_2^*$ ↔ PST ↔ UPST$_2$ ↔ ATR$_{eq}$.

Recall Kohlenbach’s heuristic from [36, p. 293] on the connection between increased logical strength at the uniform level and the essential use of the law of excluded middle in proofs. In this light, the behaviour of UPST$_1$ is not that surprising as the proof of the perfect set theorem in ATR$_0$ makes use of the law of excluded middle for $\Pi^1_1$-formulas (See [54, p. 187]).

On the other hand, the contraposition of $\Sigma^1$-separation ([54, V.5.1]) has the same syntactical form as PST, and it is possible to obtain an equivalence between ATR$_{eq}$ and the uniform version of this contraposition (as in UPST$_2$). Further principles with the same syntactical structure as PST are: The contrapositions of [54, V.5.2-3] and [54, V.6.9.2], Ulm’s theorem ([54, V.7.3]), Fraïssé’s conjecture$^\dagger$ and certain equivalent principles from [40], and the principle TC from [21]. None of these seem to have nice nonstandard versions as in $T^*$ of the EMT. As to principles with a syntactical structure different from PST, we list *Jullien’s theorem* as in [40] (equivalent to Fraïssé’s conjecture) and the extendibility of $\zeta$, the linear order of the integers, from [13] (equivalent to ATR$_0$).

**Remark 6.7** (Mathematical naturalness). Recall Remark 5.6 concerning the naturalness of logical systems. Given the above results, UATR$_0$ and related principles also seem to deserve the label ‘mathematically natural’. As to exceptional principles, as well as a potential RM ‘zzo’ ([14]) between ACA$_0$ and ATR$_0$, one can limit ATR$_0$ to specific well-orders, like the natural numbers.

---

$^\dagger$To the best of our knowledge, the exact RM-classification of this theorem is not known.
Finally, we suggest further similarities between WKL_0 and ATR_0.

**Remark 6.8 (WKL_0 versus ATR_0).** As noted in Section 2.3, (STP) is the nonstandard version of weak König’s lemma. After Corollary 5.5, it is noted that (STP) is equivalent to WKL^n generalized to all finite trees. In light of the similarities between WKL_0 and ATR_0 pointed out in [54, I.11.7], it is natural question is whether there is a version of the Standard Part principle which corresponds to ATR_0. Intuitively speaking, ATR_0^n generalized to any f is equivalent to the statement expressing that one can take the standard part at each step in a (quantifier-free) transfinite recursion, and hand over this standard set to the next step, i.e. one can take standard parts along any countable well-order. This will be explored in future research, as it is beyond the scope of this paper.

7. The Explicit Mathematics theme around \( \Pi_1^1 \)-comprehension

In this section, we establish the EMT for theorems T such that UT is equivalent to \( \Pi_1 \)-comprehension. For reasons of space, we only consider some examples. Similar to the similarities between the fan theorem and arithmetical transfinite comprehension from the previous section, we establish in Section 7.1 the existence of strong similarities between arithmetical comprehension and \( \Pi_1 \)-comprehension.

We will work with the functional version of \( \Pi_1 \)-CA_0, the so-called Suslin functional ([3],[33],[45]), defined as follows:

\[
(\exists S)(\forall f^1)[S(f) = 0 \leftrightarrow (\exists g^1)(\forall x^0)(f(gx) \neq 0)] \quad (S^2)
\]

As shown in [5] Cor. 14, the Suslin functional \( (S^2) \) is equivalent to:

\[
(\forall^e f^1)[(\forall^e g^1)(\exists^e x^0)f(gx) = 0 \leftrightarrow (\forall^e g^1)(\exists^e x^0)f(gx) = 0]. \quad (\Pi_1^1\text{-TRANS})
\]

We now sketch our approach to the EMT around \( \Pi_1 \)-comprehension. In particular, we discuss an interesting analogy between \( (7.1) \) and \( (S^2) \).

**Remark 7.1 (Bounded formulas).** As discussed in the proof of Theorem 3.1, central to the development of the EMT in Section 7.1 is that \( (\Pi_1)\text{-TRANS} \) can be replaced by equivalent bounded formulas by simply replacing \( (\forall^e k^0) \) by \( (\forall k \leq M) \) for any \( M \in \Omega \) (assuming of course \( T^* \)). As will become clear in Section 7.4 Nonstandard Analysis also allows us to treat \( \Pi_1 \)-formulas as bounded formulas, as will become clear in the following two sections. In this way, the EMT for \( (S^2) \) can be established (in Section 7.2) in much the same way as for \( (3^2) \).

7.1. Bounding \( \Pi_1 \)-formulas. In this section, we show that \( \Pi_1 \)-formulas are equivalent to natural bounded formulas as in (7.1), assuming \( \Pi_1 \text{-TRANS} \).

To this end, consider the following principle. For finite sequences \( \tau^0, \sigma^0 \), the notation ‘\( \tau \leq_0 \sigma \)’ is defined as \( |\tau| = |\sigma| \land (\forall i < |\sigma|)(\tau(i) \leq_0 \sigma(i)) \).

**Principle 7.2 (RB).** There is a standard functional \( \Phi^{1\rightarrow 1} \) such that for all standard \( f^1 \) and \( M \in \Omega \), we have:

\[
(\forall g^0, M)(\exists x^0 \leq M)(f(gx) = 0) \leftrightarrow (\forall g^1)(\exists x^0)(f(gx) = 0). \quad (7.1)
\]

Intuitively speaking (RB) expresses that it suffices to look for witnesses to \( \Sigma_1 \)-formulas \( (\exists g^1)(\forall x^0)f(gx) \neq 0 \) below \( \Phi(f) \) for standard \( f \).

We have the following theorem, where the base theory is a conservative extension of WKL_0 (See [34]).

**Theorem 7.3.** In \( \text{RCA}_0 + \text{(STP)} \), we have \( (S^2)\text{-TRANS} \leftrightarrow \text{(RB)} \).
Proof. For the reverse implication, it is easy to derive $\Pi^1_1$-TRANS from (RB) using some coding. Then, the right-hand side of (7.1) may be replaced, using [STP] and $\Pi^1_1$-TRANS, by $(\forall^a g^1)(\exists^a x^0)(f(\overline{g}x) = 0)$. By $\Omega$-CA, there is a standard functional $\Xi(f)$ deciding the truth of the left-hand side of (7.1), yielding $(S^2)^a$.

For the forward implication, we recall that $(S^2)^a \iff \Pi^1_1$-TRANS by [5] Theorem 13]. Assuming $\Pi^1_1$-TRANS, the following formula is trivially true:

$$
(\forall^a f^1)(\exists^a h^1)[(\forall g^1 \leq_1 h)(\exists x)f(\overline{g}x) = 0 \to (\forall g^1)(\exists x)f(\overline{g}x) = 0].
$$

(7.2)

Since the formula in square brackets in (7.2) is internal, we may apply HAC$_\mathbb{int}$ to obtain standard $\Psi^{1\to 1}$ such that

$$
(\forall^a f^1)(\exists^a h^1 \in \Psi(f))[(\forall g^1 \leq_1 h)(\exists x)f(\overline{g}x) = 0 \to (\forall g^1)(\exists x)f(\overline{g}x) = 0].
$$

Note that $\Psi(f)$ does not provide a witness to $h$ in (7.2), but only a finite sequence of possible witnesses. However, we can simply define the standard functional $\Phi^{1\to 1}$ by $\Phi(f)(n) := \max_{i \leq n}[\Psi(f)](i)(n)$. Hence, we obtain

$$
(\forall^a f^1)[(\forall g^1 \leq_1 \Phi(f))(\exists x)f(\overline{g}x) = 0 \to (\forall g^1)(\exists x)f(\overline{g}x) = 0],
$$

and trivially also the reverse implication:

$$
(\forall^a f^1)[(\forall g^1 \leq_1 \Phi(f))(\exists x)f(\overline{g}x) = 0 \leftrightarrow (\forall g^1)(\exists x)f(\overline{g}x) = 0].
$$

Using [STP] and $\Pi^1_1$-TRANS and the fact that $\Phi$ is standard, we easily obtain:

$$
(\forall g^0 \leq_0 f^0)[\Phi(f)(M)(\exists x \leq M)f(\overline{g}x) = 0 \leftrightarrow (\forall g^1 \leq_1 \Phi(f))(\exists x)f(\overline{g}x) = 0],
$$

(7.3)

for any standard $f^1$ and $M \in \Omega$. We now immediately obtain (RB).

By the proof of the theorem, the functional $\Phi$ from (RB) is already present in $\text{RCA}_0^\Omega$, but we only obtain (RB) if $\Pi^1_1$-TRANS is present. Note that we can repeat the above proof for any special case of $\Pi^1_1$-TRANS.

Corollary 7.4. In $\text{RCA}_0^\Omega + \text{[STP]} + \text{QF-AC}^{1,1}$, $(S^2) \leftrightarrow \Pi^1_1$-TRANS $\leftrightarrow$ (RB).

Proof. Immediate from [5] Cor. 14] and the theorem.

Example 7.5 (Searching through the reals). If a $\Sigma^0_1$-sentence $(\exists n)\varphi(n)$ with $\varphi$ quantifier-free, is known to be true, one need only test $\varphi(0), \varphi(1), \ldots$ to eventually find a witness to $(\exists n)\varphi(n)$. Hence, once can 'search through the natural numbers' for a witness to a true $\Sigma^0_1$-sentence, i.e. this infinite search terminates. The previous is well-known and it is usually added that 'one cannot search through the real numbers (in a similarly basic way)'. Nonetheless, (RB) allows us to 'search through the reals' for a witness to a $\Sigma^0_1$-formula as in (7.1) by testing all sequences $\sigma$ such that $|\sigma| = M \land (\forall i < M)(\sigma(i) \leq \Phi(f)(i))$ for $(\forall x \leq M)f(\overline{\sigma}x) \neq 0$. Now $\Phi$ is already present in $\text{RCA}_0^\Omega$ and if $\Pi^1_1$-TRANS is given, this search will find a witness.

In light of the previous remark, (RB) provides us with a suitable bounding result for $\Pi^1_1$-formulas, as suggested in Example 7.3. We could prove a similar result for $\Delta^1_1$-comprehension (See [54] I.11.8) and the associated Transfer principle, but this is beyond the scope of this paper.

7.2. Two examples of the EMT around $\Pi^1_1$-CA$_0$. In this section, we provide two examples of the EMT around $\Pi^1_1$-CA$_0$. We make essential use of the fact that $\Pi^1_1$-formulas can be replaced by bounded ones, as shown in the previous section.
7.2.1. The EMT for $\Sigma^0_1$-determinacy. We establish the EMT for the $\Sigma^0_1$-determinacy principle, which is equivalent to ATR$_0$ by [54] V.8.7. We refer to [54] V.8 for definitions and notations; The principle $\Sigma^0_1$-DET is as follows.

**Principle 7.6 ($\Sigma^0_1$-DET).** For $\varphi(h^1, f^1) \equiv (\exists k^0) f(\overline{k}) = 0$, we have

$$\forall f^1 \exists S^1_0 \varphi(S_0 \otimes S_1, f) \lor \exists S^1_1 \varphi(S_0 \otimes S_1, f).$$

(7.4)

Clearly, if $f$ in $\varphi(S_0 \otimes S_1, f)$ ignores $S_0$ in $S_0 \otimes S_1$, then (7.4) just expresses the $\Pi^1_1$-law of excluded middle. The uniform version of $\Sigma^0_1$-DET is as follows.

**Principle 7.7 ($\Sigma^0_1$-DET).** For $\varphi(h^1, f^1) \equiv (\exists k^0) f(\overline{k}) = 0$, we have

$$\exists S^0 \varphi(S_0 \otimes S_1, f) \lor \exists S^1 \varphi(S_0 \otimes S_1, f).$$

(7.5)

Finally, let $\Sigma^0_1$-DET$^*$ be $\Sigma^0_1$-DET$^*$ without ‘st’ in the innermost $\Pi^1_1$-formulas.

**Theorem 7.8.** In RCA$_0^+$ $\oplus$ [STP], we have $\Sigma^0_1$-DET$^* \iff U\Sigma^0_1$-DET$^* \iff (S^2)^{st}$.  

**Proof.** We prove the following implications:

$$\Pi^1_1$-TRANS $\rightarrow \Sigma^0_1$-DET$^* \rightarrow U\Sigma^0_1$-DET$^* \rightarrow (S^2)^{st}. \tag{7.6}$$

For the first implication in (7.6), $\Pi^1_1$-TRANS implies $(S^2)^{st}$ and the latter implies ATR$_0$ and hence $\Sigma^0_1$-SEP$^*$. The first implication in (7.6) is now trivial as (7.4) is a $\Pi^1_1$-formula. The final implication in (7.6) is also immediate: For any $f^1$, let $f$ be such that in $\varphi(S_0 \otimes S_1, f)$, $\hat{f}$ ignores $S_0$ in $S_0 \otimes S_1$, i.e. $\hat{f}(S_0 \otimes S_1 k) = f(S_0^1 k)$. Then $\Phi(\hat{f})$ supplies a witness to $(\exists^1 g^1)(\forall^1 k^0)(\overline{f(k)}) = 0$, if such there is. Such a functional is known as $(\mu_1)$ (See [2] §8.4.1 and [2]) and implies $(S^2)^{st}$.

For the remaining implication in (7.6), we repeat the proof of Theorem 7.3 for a particular instance of $\Pi^1_1$-TRANS provided by $\Sigma^0_1$-DET$^*$. The latter can easily be seen to imply $\Pi^1_1$-TRANS and we will treat arithmetical formulas as decidable. Furthermore, let $A(f, S_0, S_1)$ be the innermost $\Pi^1_1$-formula in (7.4). Then:

$$\forall^1 f^1 \exists^1 S^1_0, S^1_1 [A(f, S_0, S_1) \land A(f, S_0, S_1)^{st} \rightarrow A(f, S_0, S_1)].$$

Now let $(\forall S^2_0, S^2_1) B(S_2, S_3, f, S_0, S_1)$ be $A(f, S_0, S_1)$, with $B$ arithmetical. Similar to (7.2) in the proof of Theorem 7.3 we obtain

$$\forall^1 f^1 \exists^1 S^1_0, S^1_1 [A(f, S_0, S_1) \land (\forall S^2_0, S^1_2 \leq_1 h) B(S_2, S_3, f, S_0, S_1) \rightarrow A(f, S_0, S_1)].$$

As in the aforementioned proof, apply HAC$^*$ to obtain $\Psi$ such that $(\exists S_0, S_1, h^1 \in \Psi(f))$ for standard $f^1$. Define $\Phi^1 \equiv \Phi(f)^{st}(n) := \max \{< k < \Psi(f)^{st}(n) \mid \Phi(f)(S_2, S_3, f, S_0, S_1) \}

\Phi$ ignores $S_0, S_1$ and computes the maximum of all possible witnesses to $h$ provided by $\Psi$. The previous considerations, together with the standardness of $\Phi$, yield that

$$\forall^1 f^1 \exists^1 S^1_0, S^1_1 [A(f, S_0, S_1) \land (\forall S^2_0, S^1_2 \leq_1 \Phi(f)) B(S_2, S_3, f, S_0, S_1) \rightarrow A(f, S_0, S_1)].$$

By definition, the inverse implication is again trivial:

$$\forall^1 f^1 \exists^1 S^1_0, S^1_1 [A(f, S_0, S_1) \land (\forall S^2_0, S^1_2 \leq_1 \Phi(f)) B(S_2, S_3, f, S_0, S_1) \rightarrow A(f, S_0, S_1)].$$

Similar to (7.3), $$(\forall S^2_0, S^1_2 \leq_1 \Phi(f)) B(S_2, S_3, f, S_0, S_1)$$ is equivalent to

$$\forall S^2_0, S^1_2 \leq_{\Phi(f)} M \forall k \leq M \forall S_0 \otimes S_2 k = 0 \lor (\forall k \leq M) f(S_0 \otimes S_2 k) \neq 0,$$

for standard $f$, $S_0, S_1$ and $M \in \Omega$. Hence, we may treat the innermost $\Pi^1_1$-formula in $\Sigma^0_1$-DET$^*$ as quantifier-free. One easily verifies that HAC$^*$ implies QF-AC$^{1,1}$ relative to ‘st’, and applying the latter to $\Sigma^0_1$-DET$^*$ yields $U\Sigma^0_1$-DET$^*$.
Corollary 7.9. In \( \text{RCA}_0^\omega \) + \((\text{STP}) + \text{QF-AC}_{1,1} \), we have

\[
\Pi^1_1\text{-TRANS} \iff \Sigma^0_3\text{-DET}^* \iff \text{U}^\omega_3\text{-DET} \iff (S^2)'.
\] (7.7)

Proof. In [5, Cor. 15], the equivalence \( \Pi^1_1\text{-TRANS} \iff (S^2) \) is proved. The final equivalence in (7.7) follows from the implication \( \text{U}^\omega_3\text{-DET} \to (S^2) \) from the proof of the theorem, as the reverse implication of the equivalence is immediate. \( \square \)

7.2.2. The EMT for \( \Sigma^1_1\text{-separation} \). We establish the EMT for \( \Sigma^1_1\text{-separation} \), which is equivalent to \( \text{ATR}_0 \) by [54, V.5.1]. The nonstandard version is:

**Principle 7.10 (\( \Sigma^1_1\text{-SEP}^* \)).** For standard \( f_1 \) and \( \varphi_i(n) \equiv (\exists g_1^n)(\forall n_i)(f_i(g_1^n, n) \neq 0) \), there is standard \( Z^1 \) such that

\[
(\forall n)^0[\varphi_1(n) \to n \not\in Z \land \varphi_2(n) \to n \in Z].
\] (7.8)

The principle \( \Sigma^1_1\text{-SEP}^* \) states the existence of a separating set for \( (\Sigma^1_1)^{st} \)-formulas, but for all numbers, not just the standard ones. Similarly, \( \text{U}^\omega_3\text{-SEP} \) is:

**Principle 7.11 (\( \text{U}^\omega_3\text{-SEP} \)).** For \( \varphi_i(n, f) \equiv (\exists g_1^n)(\forall n_i)(f(g_1^n, n, i) \neq 0) \), we have

\[
(\exists f^{(1 \times 1 \times 0)^0}(\forall f^{1, g^{1}})[(\forall n)^0] [\varphi_1(n, f) \land \varphi_2(n, g) \to (\forall n)^0[\varphi_1(n, f) \to F(f, g, n, i) = 0]]
\] (7.9)

**Theorem 7.12.** In \( \text{RCA}_0^\omega + \text{STP} \), we have \( \Sigma^1_1\text{-SEP}^* \iff \text{E-U}^\omega_3\text{-SEP} \iff (S^2)^{st} \).

Proof. We prove the following implications:

\( \Pi^1_1\text{-TRANS} \to \Sigma^1_1\text{-SEP}^* \to \text{E-U}^\omega_3\text{-SEP} \iff (S^2)^{st} \). (7.10)

The first implication in (7.10) is trivial as \( (7.8) \) is a \( \Pi^1_1 \)-formula. For the second implication in (7.10), clearly \( \text{E-U}^\omega_3\text{-SEP} \to \Sigma^0_3\text{-SEP}^* \), i.e., we may use \( \Pi^1_1\text{-TRANS} \) by Theorem 3.8. Next, note that \( \Sigma^1_1\text{-SEP}^* \) implies for all standard \( f_1, f_2 \) that

\[
(\forall n)^0[\varphi_1^1(n, f_1) \lor \varphi_2^1(n, f_2) \to (\forall n)[\neg \varphi_1(n, f_1) \lor \neg \varphi_2(n, f_2)]
\] (7.11)

Similar to the proof of Theorem 6.8 (involving the functions \( f_3, f_4 \)), this yields:

\[
(\forall n)^0[\varphi_1^2(n, f_1) \lor \varphi_2^2(n, f_2) \to (\forall n)[\neg \varphi_1(n, f_1) \lor \neg \varphi_2(n, f_2)]
\] (7.12)

\[
(\forall n)^0[\varphi_1^3(n, f_1) \land \varphi_2^3(n, f_2) \to (\forall n)[\neg \varphi_1(n, f_1) \land \neg \varphi_2(n, f_2)]
\] (7.13)

Now, the consequent (resp. antecedent) of both (7.12) and (7.13) is a \( \Pi^1_1 \)-formula (resp. relative to 'st'). In other words, the previous centered formulas are instances of \( \Pi^1_1\text{-TRANS} \) and we can repeat the proof of Theorem 6.8 for the conjunction of (7.12) and (7.13). This yields the existence of a standard functional \( \Phi \) such that for all standard \( f_1, f_2, n \) and \( M \in \Omega \), we have

\[
[\neg \varphi_1^M^1(n, f_1) \lor \neg \varphi_2^M^1(n, f_2)] \iff [\neg \varphi_1(n, f_1) \lor \neg \varphi_2(n, f_2)]
\] (7.14)

where we abbreviate \( \neg \varphi_i^M(n, f, n) \equiv [(\forall g_0^n) (\exists x_i \leq M) \Phi(f)(g_0^n, x, n) = 0] \). Now define the functional \( \Psi \) as follows:

\[
\Psi(f_1, f_2, M)(n) := \begin{cases} 
0 & \varphi_1^M(n, f_1) \land \neg \varphi_2^M(n, f_2) \\
1 & \neg \varphi_1^M(n, f_1) \land \varphi_2^M(n, f_2) \\
2 & \varphi_1^M(n, f_1) \land \neg \varphi_2^M(n, f_2) \end{cases}
\] (7.15)
Using (7.11), it is now straightforward (by following the proof of Theorem 3.8) that \( \Psi \) is both as required for \( U\Sigma^2_1 \)-SEP\(^\ast \) and \( \Omega \)-invariant, in case the standard \( f_i \) satisfy (\( \forall^* n \)[\( \neg \varphi^i_2(n, f_1) \lor \neg \varphi^i_2(n, f_2) \)]. We also show this explicitly now.

Indeed, for standard \( n \), if \( \varphi^i_2(n, f_i) \) then \( \varphi^i_2(n, f_1) \) by \( \Pi^0_1\)-TRANS; By assumption and (7.11), we have \( \neg \varphi^i_2(n, f_2) \), and the first case in (7.15) holds by (7.11) (for any infinite \( M \)). If \( \varphi^i_2(n, f_2) \) holds for standard \( n \), then similarly \( \neg \varphi^i_2(n, f_1) \) by the previous, and the second case in (7.15) holds (for any infinite \( M \)). Since \( \varphi^i_2(n, f_2) \land \varphi^i_2(n, f_1) \) is impossible by assumption, the final case in (7.15) does not occur. If for some standard \( n_0 \) the third case holds, we have it for all \( M \in \Omega \) by the second conjunct of (7.14). Hence, if the third case in (7.15) occurs, it does so for all \( M \in \Omega \).

As \( \Omega\)-CA requires quantification over all standard sequences \( f^i_1 \) as in (7.10), we need to specify the behaviour when the separation assumption (\( \forall^* n \)[\( \neg \varphi^i_1(n, f_1) \lor \neg \varphi^i_2(n, f_2) \)]) is not met. Thus, let \( \Xi(f_1, f_2, n, M) \) be the \( \Omega \)-invariant characteristic function of the left-hand side of the first conjunct in (7.11), and let \( \Lambda(f_1, f_2, n) \) be its standard part obtained via \( \Omega\)-CA. Now define \( \Theta(f, g, M) \) as \( \Psi(f, g, M) \) if (\( \forall n \leq M \))\( \Lambda(f, g, n) = 1 \), and 0 otherwise. Using \( \Pi^0_1\)-TRANS and \( \Sigma^0_1\)-SEP\(^* \) as in the previous paragraph, it is clear that \( \Theta(T, M) \) is \( \Omega \)-invariant, i.e. we have

\[
(\forall n, f^1, g^1)(\forall N, M \in \Omega) [\Theta(f, g, M) \approx_1 \Theta(f, g, N)].
\]

(7.16)

The axiom \( \Omega\)-CA provides a standard functional \( \Phi(\cdot) \approx_1 \Theta(\cdot, M) \) which satisfies \( U\Sigma^1_1\)-SEP\(^\ast \). As to the standard extensionality of \( \Phi \), note that if \( \varphi^i_1(n, f_i) \), i.e. in one of the two first cases of (7.15), this extensionality property is immediate due to (3.11). By the latter, the third case of (3.12) also does not occur for standard \( h_1, h_2 \) such that \( h_1 \approx_1 f_i \). For the final case in (3.12), a similar argument involving the functions \( f_3, f_4 \) from Theorem 3.3 guarantees standard extensionality.

For reasons of space, the proof of \( U\Sigma^0_1\)-SEP \( \rightarrow (S^2) \) is left to the reader. \( \square \)

By [37 Cor. 4.9] and [46 Theorem 2.2], QF-AC\(^{1,1} \), and hence the base theory of the following theorem, is quite weak.

**Corollary 7.13.** In \( RCA_0^\Omega + \text{STP} + \text{QF-AC}^{1,1} \), we have

\[
\Pi^0_1\text{-TRANS} \leftrightarrow \Sigma^0_1\text{-SEP}^* \leftrightarrow U\Sigma^1_1\text{-SEP} \leftrightarrow (S^2).
\]

(7.17)

**Proof.** In [3 Cor. 15], the equivalence \( \Pi^0_1\text{-TRANS} \leftrightarrow (S^2) \) is proved. The final equivalence in (7.17) follows from the implication \( U\Sigma^1_1\text{-SEP} \rightarrow (S^2) \) from the proof of the theorem, as the reverse implication of the latter is immediate. \( \square \)

In light of the uniformity of the proof in [33 §4], the uniform version of the extendibility of \( \zeta \), the linear order of the integers, seems equivalent to uniform \( \Sigma^0_1\)-separation, and the same for the nonstandard versions.

Finally, it should be possible to formulate a version of Conjecture 3.17 for ATR\(_0 \) and \( \Pi^0_1\)-CA\(_0 \) after studying more examples of the EMT around \( \Pi^0_1\)-CA\(_0 \).

**Acknowledgement 7.14.** This research was supported by the following funding bodies: FWO Flanders, the John Templeton Foundation, the Alexander von Humboldt Foundation, and the Japan Society for the Promotion of Science. The author expresses his gratitude towards these institutions. The author would like to thank Ulrich Kohlenbach, Karel Hrbacek, Benno van den Berg, Steffen Lempp, Paul Shafer, Mariya Soskova, and Denis Hirschfeldt for their valuable advice.
References

[1] Jeremy Avigad, *Weak theories of nonstandard arithmetic and analysis*. See [53].
[2] Jeremy Avigad and Solomon Feferman, *Gödel’s functional (“Dialectica”) interpretation*, Handbook of proof theory, Stud. Logic Found. Math., vol. 137, 1998, pp. 337–405.
[3] Jeremy Avigad and Jeremy Helzner, *Transfer principles in nonstandard intuitionistic arithmetic*, Archive for Mathematical Logic 41 (2002), 581–602.
[4] Benno van den Berg, Eyvind Briseid, and Pavol Safarik, *A functional interpretation for nonstandard arithmetic*, Ann. Pure Appl. Logic 163 (2012), no. 12, 1962–1994.
[5] Benno van den Berg and Sam Sanders, *Transfer equals Comprehension*, Submitted (2014). Available on arXiv: [http://arxiv.org/abs/1409.6881](http://arxiv.org/abs/1409.6881).
[6] Benno van den Berg and Eyvind Briseid, *Weak systems for nonstandard arithmetic*, In preparation.
[7] Josef Berger and Hajime Ishihara, *Brouwer’s fan theorem and unique existence in constructive analysis*, MLQ Math. Log. Q. 51 (2005), no. 4, 360–364.
[8] Errett Bishop, *Foundations of constructive analysis*, McGraw-Hill Book Co., New York, 1967.
[9] Douglas Bridges and Fred Richman, *Varieties of constructive mathematics*, Springer, 1975, pp. 87–139. LNM 450.
[10] Dirk van Dalen, *How connected is the intuitionistic continuum?*, J. Symbolic Logic 62 (1997), no. 4, 1147–1150.
[11] Douglas S. Bridges and Luminiţa Simona Vîţă, *Techniques of constructive analysis*, Universitext, Springer, New York, 2006.
[12] John P. Burgess, *On the outside looking in: a caution about conservativeness*, Kurt Gödel: essays for his centennial, Lect. Notes Log., vol. 33, Assoc. Symbolic Logic, 2010, pp. 128–141.
[13] Rodney G. Downey, Denis R. Hirschfeldt, Steffen Lempp, and Reed Solomon, *Computability-theoretic and proof-theoretic aspects of partial and linear orderings*, Israel J. Math. 138 (2003), 271–289.
[14] Damir D. Dzhafarov, *Reverse Mathematics Zoo*. [http://rmzoo.uconn.edu/](http://rmzoo.uconn.edu/)
[15] Solomon Feferman, *A language and axioms for explicit mathematics*, Recursion theory and set theory: a marriage of convenience, Generalized recursion theory II, Stud. Logic Foundations Math., vol. 94, North-Holland, 1978, pp. 55–98.
[16] moreover, *Constructive theories of functions and classes*, Logic Colloquium ’78 (Mons, 1978), Stud. Logic Foundations Math., vol. 97, North-Holland, 1979, pp. 159–224.
[17] Solomon Feferman, Gerhard Jäger, and Thomas Strahm, *Foundations of Explicit Mathematics*, In progress.
[18] Harvey Friedman, *Some systems of second order arithmetic and their use*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, 1975, pp. 235–242.
[19] moreover, *Systems of second order arithmetic with restricted induction*, I & II (Abstracts), Journal of Symbolic Logic 41 (1976), 557–559.
[20] Harvey M. Friedman and Jeffry L. Hirst, *Reverse mathematics and homeomorphic embeddings*, Ann. Pure Appl. Logic 54 (1991), no. 3, 229–253.
[21] Makoto Fujitake, *Techniques of constructive analysis*, Mathematical Logic Quarterly 41 (1995), 581–602.
[22] Makoto Fujitake, *Intuitionistic and uniform provability in reverse mathematics*, PhD thesis, Mathematical Institute, Tohoku University, Sendai (2015). To appear.
[23] Robin Gandy and Martin Hyland, *Computable and recursively countable functions of higher type*, North-Holland, 1977, pp. 407–438. Studies in Logic and Found. Math. 87.
[24] Thomas J. Grilliot, *On effectively discontinuous type-2 objects*, J. Symbolic Logic 36 (1971), 245–248.
[25] James Hunter, *Higher-order reverse topology*, ProQuest LLC, Ann Arbor, MI, 2008. Thesis (Ph.D.)–The University of Wisconsin - Madison.
[26] Jeffry L. Hirst, *Representations of reals in reverse mathematics*, Bull. Pol. Acad. Sci. Math. 55 (2007), no. 4, 303–316.
[27] Karel Hrbacek, *Stratified set theory: Internal View*, J. Log. Anal. 1 (2009), Paper 8, pp. 108.
[28] moreover, *Relative Set Theory: Internal View*, J. Log. Anal. 1 (2009), Paper 8, pp. 108.
[29] Karel Hrbacek, Olivier Lessmann, and Richard O’Donovan, *Analysis with ultrasmall numbers*, Amer. Math. Monthly 117 (2010), no. 9, 801–816.
[30] Hajime Ishihara, *Reverse mathematics in Bishop’s constructive mathematics*, Philosophy Scientiae (Cahier Spécial) 6 (2006), 43–59.
