Solvability of a model problem sublimation of snow

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Abstract. The mathematical model water and air movement in snow is studied taking into account sublimation. Snow is simulated as a four-phase continuous porous medium. The pores are entirely occupied by a mixture of water, air and vapor. To describe the process, the mass conservation equations for each phase, the Musket–Leverett system of two-phase filtration equations for water and air, and the energy conservation equation for snow are used.

1. Introduction
Mathematical models of thawing snow cover are constructed using the general principles of dynamics of a multiphase medium [1]. These models necessarily include phase transitions and use the filtrational approximation; therefore, the basic equations of the model are mass and energy conservation laws and the Darcy law for moving phases [2, 3]. This approach has been used in studies of two-phase filtration with variable porosity [4–6]. Significant volumes of snow evaporate bypassing the liquid phase at negative temperatures. An experimental study of filtration in melting snow was done in [7]. A review of models was made in [8].

The foundations of the theory of the movement of water and air in melting snow are laid in the works of S.C. Colbeck [9] and his followers [10–17]. However, although snow was considered as a multiphase medium in these works, the variable ice porosity, its deformation, and phase transitions were not taken into account.

In the study [18] snow cover is considered as a three-phase medium (water, air and ice). Empirical dependences for the capillary jump (water–air) and empirical formulas for the snow conductivity coefficient are given. However, the authors neglect the movement of air and significantly simplify the equation for temperature. As a result, the three-phase model reduces to the equation for temperature and the equation for the volume concentration of the aqueous phase.

The first controlled measurements of sublimation of snow grains in air were made by Thorpe and Mason [19]. Single ice spheres were suspended on a fine fiber in a wind tunnel and the resulting mass loss was measured with a microbalance. The works [20, 21] are devoted to the assessment of the sublimation of drifting snow. This study confirmed the significant effect of drifting snow sublimation. In a study Bernhardt et al. [22] included gravitational snow transport in a model sublimation of snow.
2. Formulation of the problem

Snow is treated as a porous medium whose solid skeleton consists of motionless ice particles. Water and air move jointly in the porous medium. Snow is a four-phase medium consisting of water (i = 1), air (i = 2), ice (i = 3), and vapor (i = 4). The following system of equations is used to simulate the process of sublimation of ice in snow [23]

\[
\frac{\partial \rho_i}{\partial t} + \text{div}(\rho_i \vec{u}_i) = \sum_{j=1}^{4} I_{ji}, \quad i = 1, \ldots, 4, \quad I_{ji} = -I_{ij}, \quad \sum_{i,j=1}^{4} I_{ij} = 0; \tag{1}
\]

\[
\vec{v}_i = -K_0 \frac{k_{0i}}{\mu_i} (\nabla p_i - \rho_i^0 \vec{g}), \quad i = 1, 2, \quad p_2 - p_1 = p_c(s_1, \theta), \quad \sum_{i=1}^{3} s_i = 1; \tag{2}
\]

\[
(\sum_{i=1}^{4} \rho_i^0 c_i \alpha_i) \frac{\partial \theta}{\partial t} + (\sum_{i=1}^{4} \rho_i^0 c_i \vec{v}_i) \nabla \theta = \text{div}(\lambda_c \nabla \theta) - \frac{\partial \rho_c^0 \phi}{\partial t}. \tag{3}
\]

Here \( t \) is time, \( \phi \) is the snow porosity, \( \vec{v}_i = \phi s_i(\vec{u}_i - \vec{u}_0) \) is the filtration velocities of water and air \((i=1,2)\), \( \vec{u}_i \) is the velocity of the \( i \)th phase, \( \rho_i \) is the reduced density related to the true density \( \rho_i^0 \) and volumetric concentration \( \alpha_i \) by the formula \( \rho_i = \alpha_i \rho_i^0 \) \((\alpha_i = \phi s_i, \ i = 1, 2, 4, \alpha_3 = 1 - \phi)\), \( s_1, s_2, s_4 \) are the water, air and vapor saturations, \( I_{ij} \) is the rate of mass transfer from the \( j \)th to the \( i \)th component per unit volume in unit of time, \( K_0 \) is the porous medium permeability tensor, \( k_{0i} \) are the relative phase permeabilities \((k_{0i} = k_{0i}(s_i) \geq 0, k_{0i}|_{s_i=0} = 0)\), \( \mu_i \) is the dynamic viscosity, \( p_i \) are the phase pressures, \( p_c \) is the capillary pressure, \( \vec{g} \) is the acceleration vector due to gravity, \( \theta \) is the temperature of the medium \((\theta_i = \theta, \ i = 1, 2, 3, 4)\), \( c_i = \text{const} > 0 \) is the constant-volume heat capacity of the \( i \)th phase, \( i = \text{const} > 0 \) is the specific heat of ice sublimation, \( \lambda_c = a_c + b_c p_c^2 \) is the thermal conductivity of snow, \( \rho_c = \sum_{i=1}^{4} \rho_i^0 \alpha_i \), \( a_c = \text{const} > 0 \) and \( b_c = \text{const} > 0 \) [2].

System (1)–(3) is supplemented with the hypotheses

\( I_{34} = I_{34}(\theta), \quad I_{12} = I_{34} = I_{34} = I_{24} = I_{14} = 0, \quad \vec{u}_3 = \vec{u}_4 = 0, \quad \phi = \phi(\theta), \quad s_4 = s_4(\theta). \)

3. Simple solution

For system (1)-(3), we consider the following problem [23]: snow occupies a region \((-\infty, ct)\), \( t > 0 \). At \( z = -\infty \), water and vapor is absent \((s_1 = s_4 = 0, v_1 = 0)\), air is motionless \((v_2 = 0)\), and the temperature \((\theta = \theta^-)\) is specified (lower than the ice melting point); at \( z = ct \), the velocities of water \((v_1 = v_1^+)\) and air \((v_2 = v_2^+)\) and the air pressure \((p_2 = p^+)\), saturation of water \((s_1 = s_1^+)\), saturation of vapor \((s_4 = s_4^+)\) are known and the temperature \( \theta = \theta^+ \) is specified (equal to the ice melting point). Assuming that all unknown functions depend only on the variable \( \xi = z - ct \) \((c \) is an unknown constant\), we obtain

\[-c \frac{d}{d\xi} (\phi s_1 \rho_1^0 + \phi s_4 \rho_4^0(1 - \phi) + \phi s_4 \rho_4^0) + \frac{d}{d\xi} (\rho_1^0 v_1) = 0, \quad -c \frac{d}{d\xi} (\phi s_2 \rho_2^0) + \frac{d}{d\xi} (\rho_2^0 v_2) = 0, \tag{4}\]

\[v_1 = -K_0 \frac{k_{01}}{\mu_1} (\frac{dp_1}{d\xi} - \rho_1^0 \vec{g}), \quad p_2 - p_1 = p_c(s_1, \theta), \quad s_1 + s_2 + s_4 = 1, \tag{5}\]

\[\sum_{i=1}^{4} \rho_i^0 c_i (v_i - c_0 \vec{v}_i) \frac{d\theta}{d\xi} - c_0 \frac{d\rho_c^0 \phi}{d\xi} = \frac{d}{d\xi} (\lambda_c \frac{d\theta}{d\xi}), \tag{6}\]

\[s_1|_{\xi \to -\infty} = 0, \quad s_4|_{\xi \to -\infty} = 0, \quad \theta|_{\xi \to -\infty} = \theta^- \frac{d\theta}{d\xi}|_{\xi \to -\infty} = 0, \quad v_1|_{\xi \to -\infty} = 0, \tag{7}\]

\[p_2(0) = p^+, \quad s_1(0) = s_1^+, \quad s_4(0) = s_4^+, \quad \theta(0) = \theta^+, \quad v_1(0) = v_1^+, \quad i = 1, 2. \tag{8}\]
3.1. Determination of filtration velocities
Following [23], we set views for filtering rates:

\[ v_1 = c\phi s_1 + c\rho_0^1 (\phi^- - \phi) + c\rho_0^2 \phi s_4, \quad v_2 = c\phi s_2 - c\phi^- . \]  (9)

3.2. Representation for temperature
We introduce finite values of the temperature \( \theta^- \), \( \theta_1 \) and \( \theta^+ \). Let \( 0 < \theta^- < \theta_1 < \theta^+ \), \( \phi_1 \equiv (1 - \phi^-)/(\theta^+ - \theta_1) \), \( \phi_2 \equiv \alpha_4/(\theta^+ - \theta_1) \). We assume that, for all \( \theta \in (0, \infty) \) the following relations hold:

\[ \phi(\theta) = \begin{cases} \frac{1}{\lambda}, & \theta \geq \theta^+, \\ \phi^- + \phi_1(\theta - \theta_1), & \theta_1 \leq \theta \leq \theta^+, \\ \phi^-, & \theta \leq \theta_1. \end{cases} \]

\[ \phi(\theta)_{s_4}(\theta) = \begin{cases} \alpha_4, & \theta \geq \theta^+, \\ \phi_2(\theta - \theta_1), & \theta_1 \leq \theta \leq \theta^+, \\ 0, & \theta \leq \theta_1. \end{cases} \]

From (9), we have

\[ \sum_{i=1}^{4} c_i \rho_i^0 (v_i - c) = c\rho_0^1 (1 - \phi)(c_1 - c_3) + c\rho_0^2 \phi s_4(c_1 - c_4) + A_1 c_1 + A_2 c_2. \]

Using conditions (7), we obtain

\[ \lambda_c \frac{d\theta}{d\xi} = f_1(\theta), \]  (10)

\[ f_1(\theta) \equiv c(c_1 - c_3)\rho_0^1 M(\theta) + c(c_1 - c_4)\rho_0^2 N(\theta) + (A_1 c_1 + A_2 c_2)(\theta - \theta^-) - \mu \rho_0^2(\phi - \phi^-), \]

where \( A_1 = -c\rho_0^2(1 - m^-), m^- = m(\theta^-), A_2 = -c\rho_0^2 m^- \),

\[ M(\theta) \equiv \int_{\theta^-}^{\theta} (1 - m(x)) dx = \begin{cases} \frac{(1 - m^-) (\theta^+ - \theta^- + \theta_1)}{2}, & \theta \geq \theta^+, \\ (1 - m^-)(\theta - \theta^-) - m_1 \frac{(\theta - \theta_1)^2}{2}, & \theta_1 \leq \theta \leq \theta^+, \\ (1 - m^-)(\theta - \theta^-), & \theta \leq \theta_1. \end{cases} \]

\[ N(\theta) \equiv \int_{\theta^-}^{\theta} m(x)s_4(x) dx = \begin{cases} \alpha_4 (\theta - \theta^+ - \theta_1), & \theta \geq \theta^+, \\ m_2 \frac{(\theta - \theta_1)^2}{2}, & \theta_1 \leq \theta \leq \theta^+, \\ 0, & \theta \leq \theta_1. \end{cases} \]

The solution of problem (7), (8), (10) can be written as (\( \lambda_c > 0 \)):

\[ I(\theta) \equiv \int_{\theta}^{\theta^+} \frac{dy}{f_1(y)} = \int_{\theta}^{0} \frac{dy}{\lambda_c(y)} = \psi(\lambda_c(\xi)), \]

\[ \theta(\xi) = I^{-1}(\psi(\lambda_c(\xi))). \]  (11)

For \( \theta \in [\theta_1, \theta^+] \), from (11) we obtain:

\[ I(\theta) = \int_{\theta}^{\theta^+} \frac{dy}{b_1(y - \theta_1)^2 + d_1(y - \theta_1) + a_1} = \psi(\lambda_c(\xi)). \]  (12)

Because \( \theta(\xi) \) is monotonic, a point \( \xi_1 \) exists such that \( \theta(\xi_1) = \theta_1 \).
For $\theta \in [\theta^-, \theta_1]$, from (11) we obtain:

$$\theta(\xi) = \theta^- + (\theta_1 - \theta^-) \exp \left( -b_2 \int_{\xi}^{\xi_1} \frac{dy}{\lambda_c(y)} \right).$$  \hspace{1cm} (13)

Thus, for the specified function $\lambda_c(s_1, \theta)$, representation (11) and its particular cases (12), (13) define the temperature for all $\xi \in (-\infty, 0)$.

3.3. Determination of saturations and pressure

Using the temperature equation (10) and following [23], the equation for the saturation can be written as:

$$a_0(s_1) \frac{ds_1}{d\xi} = f_2(s_1, \theta),$$  \hspace{1cm} (14)

$$f_2(s_1, \theta) \equiv \varphi_1 \varphi_2 \gamma(s_1) \frac{p'_0}{p_0} f_1 + \frac{1}{p_0} \gamma \varphi_1 \varphi_2 + \frac{1}{p_0} |c| \phi A s_1 - \frac{1}{p_0} |c| B (\phi - \phi^-) + \frac{1}{p_0} |c| D \phi s_4.$$  

Here

$$a_0 = -\varphi_1 \varphi_2 \gamma', \quad \gamma = g(p_1^0 - p_2^0), \quad p'_0 = \frac{dp_0}{d\theta},$$

$$\mu_i = \frac{\mu_i}{K_0 k_0}, \quad i = 1, 2,$$

$$A = \overline{\mu}_1 \varphi_2 + \overline{\mu}_2 \varphi_1, \quad B = \overline{\mu}_1 \varphi_2 \frac{\rho_3^0}{\rho_1^0} + \overline{\mu}_2 \varphi_1, \quad D = \overline{\mu}_1 \varphi_2 \frac{\rho_4^0}{\rho_1^0} + \overline{\mu}_2 \varphi_1,$$

$$\varphi_i = \begin{cases} 
0, & s_i \leq 0, \\
\frac{s_i}{s_i^0}, & 0 \leq s_i \leq 1, \\
1, & s_i \geq 1.
\end{cases}$$  

Equation (14) is examined for $\xi < 0$ and the condition $s_1(0) = s_1^+$, i.e., the Cauchy problem is studied for $\xi < 0$, and the condition $s_1|_{\xi=-\infty} = 0$ should be proved. The functions $f_1$, $\phi$, $s_4$ depend on temperature $\theta(\xi) \in [\theta^-, \theta^+]$, the condition $s_4 < \left( \frac{1}{p_1} \right)^{1/n_1}$. Regarding $p_0(\theta(\xi))$ the existence of a constant $\nu_0(\theta^-, \theta^+) > 0$ such that $\nu_0 \leq p_0 \leq \nu_0^{-1}$, $|p'_0| \leq \nu_0$ is postulated. The functions $\overline{\mu}_1$ and $\overline{\mu}_2$ can depend on $s_1(\xi)$ and $\theta(\xi)$, but must satisfy the inequalities $0 < \nu_i \leq \overline{\mu}_i \leq \nu_i^{-1} \leq \infty, \ i = 1, 2$.

To find the pressures $p_1$ and $p_2$, we consider the following equality implied by the Darcy law (5):

Using the Darcy law (5), the equation for the pressure can be written as:

$$\frac{dp}{d\xi} = -\frac{1}{(\varphi_1(s) + \varphi_2(s))} \left( \sum_{i=1}^{2} \left( \frac{\mu_i}{s_i^0} \varphi_i(\xi) - g p_i^0 \varphi_i - p'_0(\theta) f_1 \right) \right) \frac{1}{\lambda_c} (\varphi_1(s) g(s) + b(s) (\varphi_1(s) + \varphi_2(s))) \equiv f_3(s, \theta),$$

$$p(0) = p^+ - p_0(\theta^+) b(s^+), \quad p(\xi) \equiv p_2(\xi) - p_0(\theta) b(s), \quad b(s) = \int_0^s \frac{\varphi_1(y) \gamma'(y)}{\varphi_1(y) + \varphi_2(y)} dy.$$
Following [23], we obtain:

$$p(\xi) = p^+ - p_0(\theta^+)b(s_1^+) - \int_0^\theta f_3(s_1(x), \theta(x))dx,$$  \hspace{0.5cm} (15)

Here $s = s_1$ and $\varphi_1 + \varphi_2 > 0$ by virtue of Lemma 1 [23], and right part of the equality is a given function $s_1(\xi)$ and $\theta(\xi)$. Therefore, the function $p_1$ is determined up to an arbitrary constant, the value of which can be determined from the condition $p_2(0) = p^+$. The smoothness of $p_1$ and $p_2$ is determined smoothness of $s_1(\xi)$.

**Definition.** Weak solution to the problem (4)-(8) $R^- = (-\infty, 0)$ called functions $\theta(\xi)$, $s_1(\xi)$, $v_1(\xi)$, $p_i(\xi)$ and parameter $c_i$ if:

1) $\theta(\xi)$ has a continuous derivative and satisfies equation (10) and the conditions $\theta(0) = \theta^+$, $\theta|_{\xi \to -\infty} = \theta^-$, $\frac{\partial \theta}{\partial \xi}|_{\xi \to -\infty} = 0$;

2) $s(\xi)$ has a continuous derivative with weight $a(s)$ and satisfies equation (14) and the conditions $s_1(0) = s_1^+$, $s_1|_{\xi \to -\infty} = 0$;

3) $v_1(\xi)$ satisfy equalities (9) and the conditions $v_1(0) = v_1^+$, $v_1|_{\xi \to -\infty} = 0$;

4) $p_i(\xi)$ satisfy equalities (15) and the condition $p_2(0) = p^+$.

**Theorem.** Let positive numbers $a_c, b_c, \mu, \phi^-, K_0, \theta^-, \theta_1, \theta^+, \rho_1^0, c_i, \alpha_i^2$, $(i = 1, 2, 3, 4)$, $s_1^+ \in (0, 1]$ and continuous in $s_1 \in [0, 1]$ and $\theta \in [\theta^-, \theta^+]$ functions $\phi(\theta), s_4(\theta)$, $k_0 = k_0(s_1, \theta)s_1^0$, $n_i > 1$, $\mu(s_1, \theta)$, $(i = 1, 2)$, $p_c(s_1, \theta) = p_0(\theta)\gamma(s_1)$, $\lambda_c = a_c + b_c\rho_2^c$, $\rho_2 = \rho_1^0s_1\phi + \rho_2^0s_2\phi + \rho_3^0(1 - \phi) + \rho_4^0s_4\phi$ satisfy the conditions:

(i) $\rho_2^0 < \rho_3^0 < \rho_4^0$, $c_4 < c_3 < c_1 < c_2$, $\alpha_4^2 < \frac{\rho_2^0\phi(c_1-c_3)}{\rho_2^0\phi(c_1-c_4)}(1 - \phi^-)$;

(ii) $s_4(\theta) < \left(\frac{\rho_4^0}{\rho_2^0}\right)^{1/n_2}$;

(iii) $0 < (a = -k_0s_1k_02\frac{d^2\phi}{d\theta^2}, k_01, k_02)$ for $s_1 \in (0, 1)$, $a|_{s_1=0,1} = k_0|_{s_1=0} = k_02|_{s_1=1} = 0$,

$$\frac{1}{s_1}k_02s_2\gamma|_{s_1=0} = 0, \quad \frac{d\phi}{d\theta} < 0,$$

$$a_0 \geq v_0(s_1(1 - s_1))^\kappa, \quad \kappa > 1, \quad \frac{a_0(s_1)}{s_1}|_{s_1=0} = 0, \quad \left(\|\frac{d\phi}{d\theta}\|\gamma|_{0,1}, \quad \|\frac{d\phi}{d\theta}\|\gamma|_{\theta^-, \theta^+}\right) \leq v_0, \quad 0 < v_0^{-1} < \left(\mu(s_1, \theta), k_02s_1, \theta, p_0(\theta)\right)\frac{d\phi(s_1)}{ds_1} \right| \leq v_0.

Then there is at least one weak solution to the problem (4)-(8), with the following properties:

$$0 \leq s_1(\xi) \leq 1, \quad \theta^- \leq \theta(\xi) \leq \theta^+, \quad c = \frac{(1 + \lambda)v_2^+}{(1 - \phi^-)(1 - \rho_1^0/\rho_2^0)} < 0.$$

In addition, there is a point $\xi_* \in (-\infty, \xi_1]$ such that $s_1(\xi) = 0$ for all $\xi \leq \xi_*$. Let $s_1(\xi)$ is a solution of problem (7), (8), (14) and $s_1^+ \in [0, 1]$, then $0 \leq s_1(\xi) \leq 1$.

**Lemma 1.** Let $s_1(\xi)$ be a solution of problem (7), (8), (14) and let the condition $\frac{s_1(s_1)}{s_1} - \varphi_2^2 \leq v_0, s_1 \in [0, 1]$ be satisfied. Then, a point $\xi_* < \xi_1$ exists such that $s_1(\xi) \equiv 0$ for all $\xi \leq \xi_*$. If $s_1 = s_1(\xi_1) = 0$ for all $\frac{s_1(s_1)}{s_1} - \varphi_2^2 |_{s_1=0} = 0$, we have $\xi_* = \xi_1$.

Let $\varepsilon \in (0, 1)$ and $a_\varepsilon(s) \equiv a_0(s) + \varepsilon > 0$. For $\xi < 0$ instead of (7), (8), (10), (14) we consider the problem

$$a_\varepsilon(\varepsilon^s)\frac{d\varepsilon^s}{d\xi} = f_2(\varepsilon^s, \theta^s), \quad \frac{d\theta^s}{d\xi} = \frac{f_1(\theta^s)}{\lambda_c(\varepsilon^s, \theta^s)}, \quad s^s(0) = s^+, \quad \theta(0) = \theta^+.$$

$$s^s(0) = s^+, \quad \theta(0) = \theta^+. \hspace{0.5cm} (16)$$
A feature of the parent formulation of the problem is need to justify conditions for water saturation at infinity \((s_1(-\infty) = 0)\). To overcome these difficulties the \(\varepsilon\) – regularization method is used and the property of the finite velocity of perturbation propagation is established (Lemma 2). First we consider at the auxiliary problem (16). For every \(\varepsilon > 0\) the local solvability of Cauchy problem follows from Picard’s theorem [24]. The local solution can be continued to any finite interval since in Lemma 1 the physical principle of maximum for saturation is proved. We have uniform estimates for \(\varepsilon\). We can make the passage to the limit at \(\varepsilon \to 0\) based on the Arzela’s theorem. Then, the properties of the solution of the problem for saturation are refined for such an interval on which the structure of the right-hand side of the equation for saturation allows us to establish the property of the finite propagation velocity of perturbations (Lemma 2). The latter allows us to obtain the result of the theorem.

**Conclusion**

The problem in a self-similar formulation for the model of snow sublimation is studied. The final formulas for the filtration rates of water and air are obtained. Representations for temperature and pressures are found. The saturation of water is found from the solution of the Cauchy problem for an ordinary differential equation of the first order with degeneration on the solution.

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**References**

[1] Nigmatulin R I 1991 *Dynamics of Multiphase Media, Part 1* (New York: Hemisphere)
[2] Kuchment L S, Demidov V N and Motovilov Yu G 1983 *River Runoff Formation. Physical and Mathematical Models* (Moscow: Nauka) (in Russian)
[3] Korobkin A A, Papin A A and Khabikpasheva T I 2013 *Mathematical Models of Snow and Ice Cover* (Barnaul: ASU Publ.) (in Russian)
[4] Papin A A and Sibin A N 2019 *Fluid. Dyn. 54* (4) 520–34
[5] Sibin A 2017 *J. Phys.: Conf. Ser.* 894 012085
[6] Papin A A and Sibin A N 2016 *J. Phys.: Conf. Ser.* 722 012034
[7] Waldner P A, Schneebeli M, Schultz-Zimmermann U and Fluhler H 2008 *Hydrol. Process.* 18 (7) 1271–90
[8] Papin A A and Tokareva M A 2018 *IOP Conf. Ser.: Earth Environ. Sci.* 193 012055
[9] Colbeck S C 1972 *J. Glaciol.* 63 (11) 369–85
[10] Gray J M N T 1996 *Philos. T. R. Soc. A* 354 (1707) 465–500
[11] Sellers S 2000 *Cold. Reg. Sci. Technol.* 31 (1) 47–57
[12] Webb R W, Fassnacht S R, Gooseff M N and Webb S W 2018 *Transport Porous Med.* 123 457–76
[13] Liu H, Maghoul P and Hollander H M 2019 *Phys. Chem. Earth* 113 31–42
[14] Xu H, Wang D, Tan Y, Zhou J and Oeser M 2018 *J. Clean. Prod.* 170 1413–22
[15] Shagapov V Sh, Chiglintseva A S, Rusinov A A, Khasanov M K and Khusainov I G 2018 *J. Appl. Mech. Tech. Phys.* 59 (3) 422–33
[16] Xu H and Tan Y 2015 *Energy* 80 666–76
[17] Liu X, Rees S J and Spitler J D 2007 *Appl. Therm. Eng.* 27 1125–31
[18] Daanen R P and Nieber J D 2007 *J. Cold Reg. Eng.* 23 (2) 43–68
[19] Thorpe A D and Mason B J 1966 *Br. J. Appl. Phys.* 17 541–8
[20] Groot Zwaaftink C D, Mott R and Lehning M 2013 *Water Resour. Res.* 49 1581–90
[21] Groot Zwaaftink C D, Lowe H, Mott R, Bavay M and Lehning M 2011 *J. Geophys. Res.* 116 D16107
[22] Bernhardt M, Schulz K, Liston G E and Zangl G 2012 *J. Hydrol.* 424 196–206
[23] Papin A A 2008 *J. Appl. Mech. Tech. Phys.* 49 (4) 527–36
[24] Hartman P 1964 *Ordinary Differential Equations* (New York: Wiley)