EQUATIONS OF LONG WAVES WITH A FREE SURFACE IV. THE CASE OF CONSTANT SHEAR

Boris A. Kupershmidt †

† The University of Tennessee Space Institute
Tullahoma, TN 37388, USA
E-mail: bkupersh@utsi.edu

Abstract
A large class of two-dimensional free-surface hydrodynamical systems is determined that can be self-consistently reduced by the condition that the velocity profile has a constant shear. The reduced systems turn out to be Hamiltonian, and so does the reduction process itself. All reducible systems, Hamiltonian or not, are determined and shown to form a Lie algebra. All this is then generalized to the multilayer/multi-species representations.

1 Introduction
The classical one-dimensional long-wave system

\begin{align*}
h_t &= (hu)_x, \\
u_t &= uu_x + gh_x,
\end{align*}

has a venerable history. Here \( h = h(x,t) \) is the height of a free surface over the bottom \( \{y = 0\} \); \( u = u(x,t) \) is the horizontal component of velocity; \( t \) is the time coordinate (opposite in sign to the physical time); \(-\infty < x < \infty \); subscripts \( t \) and \( x \) denote partial derivatives; \( g \) is the gravitational acceleration.

The system (1.1) is Hamiltonian and integrable: it can be put into the form [5,6]

\begin{align*}
\begin{pmatrix} h \\ u \end{pmatrix}_t &= \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \begin{pmatrix} \delta H/\delta h \\ \delta H/\delta u \end{pmatrix},
\end{align*}

where

\begin{align*}
\partial &= \partial/\partial x, \\
H &= \frac{1}{2}(hu^2 + gh^2),
\end{align*}

and there exists an infinite number of conserved densities for that system.

In 1973 Benney [1] derived the following two-dimensional generalization of the system (1.1):

\begin{align*}
h_t &= \left( \int_0^h udy \right)_x, \\
u_t &= uu_x + gh_x - u_y \int_0^y u_x dy,
\end{align*}
where now \( u = u(x, y, t) \) depends also upon the second space coordinate, \( y : 0 \leq y \leq h \). Benney had found two remarkable properties of the two-dimensional system (1.5). First, if one introduces the moments of the velocity \( u(x, y, t) \):

\[
A_n = A_n(x, t) = \int_0^h u^n(x, y, t) dy, \quad n \in \mathbb{Z}_{\geq 0}.
\]

then the system (1.5) implies the autonomous evolution system

\[
A_{n,t} = A_{n+1,x} + gnA_{n-1}A_{0,x}, \quad n \in \mathbb{Z}_{\geq 0}.
\]

Second, the moments system (1.7) has an infinite number of polynomial conserved densities \( H_n \in A_n + Q[\{g; A_0, ..., A_{n-2}\}] : \)

\[
H_0 = A_0, \quad H_1 = A_1, \quad H_2 = A_2 + gA_0^2, ...
\]

Subsequently, Manin and myself showed [6,7] that:

(A) The moment system (1.7) is itself Hamiltonian: it can be written in the form

\[
A_{n,t} = \sum_{m \geq 0} B_{nm}(H_m), \quad H_m = \frac{\delta H}{\delta A_m},
\]

\[
B_{nm} = nA_{n+m-1} + \partial m A_{n+m-1},
\]

with \( H = \frac{1}{2}H_2 = \frac{1}{2}(A_2 + gA_0^2) \), and with the matrix (1.10) being Hamiltonian;

(B) The general system (1.9) is implied by the following two-dimensional free-surface system:

\[
h_t = \partial(mA_{m-1}H_m)
\]

\[
u_t = \partial(u^mH_m) - u_y \int_0^y dy(mu^mH_m)_x.
\]

We sum on repeated non-fixed indices unless directed otherwise;

(C) The Hamiltonians \( H_k \)’s (1.8) found by Benney are in involution with respect to the Hamiltonian structure (1.10). Therefore, the corresponding higher flows commute;

(D) When \( u \) is \( y \)-independent,

\[
u_y = 0,
\]

so that we are back to the classical one-dimensional case, the map (1.6) becomes

\[
A_n = hu^n, \quad n \in \mathbb{Z}_{\geq 0},
\]
and this map is Hamiltonian between the Hamiltonian structures (1.2) and (1.10).

The purpose of this note is to show that there exists an interesting reduced family of the full two-dimensional system (1.11) generalizing the purely one-dimensional reduction \( u_y = 0 \) (1.12), namely

\[
\begin{align*}
  u_y = s &= \text{const}, \quad (1.14) \\
  u(x, y, t) &= v(x, t) + sy, \quad s = \text{const}. \quad (1.15)
\end{align*}
\]

Thus, we consider the case when the shear is present but is constant.

We shall verify that: the constraint \( \{ u_y = s \} \) (1.14) is compatible with the flow (1.11) for any Hamiltonian \( H \); that on this constrained submanifold \( \{ u_y = s \} \) (1.15), the system (1.11) turns into a Hamiltonian system of the form

\[
\begin{pmatrix}
  h_t \\
  v_t
\end{pmatrix} = \begin{pmatrix}
  0 & \partial \\
  \partial & -s\partial
\end{pmatrix} \begin{pmatrix}
  \delta H / \delta h \\
  \delta H / \delta v
\end{pmatrix}; \quad (1.16)
\]

and that the corresponding reduction map

\[
A_n = \int_0^h (v + sy)^n dy, \quad n \in \mathbb{Z}_{\geq 0}, \quad (1.17)
\]

is Hamiltonian between the Hamiltonian structures (1.16) and (1.10).

We then determine when similar constant-shear reductions exist for other free-surface hydrodynamical systems.

At the moment, let us record that the original two-dimensional Benney system (1.5) reduces on the submanifold \( \{ u_y = s, \ u = v + sy \} \), to the system

\[
\begin{pmatrix}
  h_t \\
  v_t
\end{pmatrix} = \begin{pmatrix}
  (vh + s\frac{h^2}{2})_x \\
  vv_x + gh_x
\end{pmatrix} = \begin{pmatrix}
  0 & \partial \\
  \partial & -s\partial
\end{pmatrix} \begin{pmatrix}
  \frac{(v + sh)^2}{2} + gh \\
  vh + s\frac{h^2}{2}
\end{pmatrix}, \quad (1.18)
\]

\[
H = \frac{1}{2}(A^*_2 + gA^*_0) = \frac{1}{2} \left( \frac{(v + sh)^3 - v^3}{3s} + gh^2 \right), \quad (1.19a)
\]

\[
\frac{\delta H}{\delta h} = \frac{(v + sh)^2}{2} + gh, \quad \frac{\delta H}{\delta v} = vh + s\frac{h^2}{2}. \quad (1.19b)
\]
2 Constant-Shear Flows

Denote by $(\cdot)^*$ the reduction of the object $(\cdot)$ on the submanifold

\[ u_y = s, \ u = v + sy. \]  \hspace{1cm} (2.1)

Thus,

\[ A_m^* = \int_0^h (v + sy)^{m+1} \frac{d}{ds} (v + sh)^{m+1} - v^{m+1}}{s(m + 1)}, \quad s \neq 0, \]  \hspace{1cm} (2.2)

\[ A_m^*,v = \frac{\partial A_m^*}{\partial v} = mA_{m-1}^* = \frac{(v + sh)^m - v^m}{s}, \]  \hspace{1cm} (2.3)

\[ A_m^*,h = \frac{\partial A_m^*}{\partial h} = (v + sh)^m, \]  \hspace{1cm} (2.4)

\[ H_h^* = \frac{\delta (H^* )}{\delta h} = \left( \frac{\delta H}{\delta A_m} \right)^* \frac{\partial A_m^*}{\partial h} = (v + sh)^m H_m^* \]  \hspace{1cm} (2.5)

\[ H_v^* = H_m^* \frac{\partial A_m^*}{\partial v} = (mA_{m-1} H_m)^* = \left( (v + sh)^m - v^m \right) \frac{H_m^*}{s}. \]  \hspace{1cm} (2.6a)

Now, denote temporarily

\[ F = u^m H_m, \]  \hspace{1cm} (2.7)

so that

\[ F_u = mu^{m-1} H_m. \]  \hspace{1cm} (2.8)

Differentiating equation (1.11b) with respect to $y$, we find:

\[ u_{yt} = \partial (F_u u_y) - u_{yy} \int_0^y dy (F_u)_x - u_y (F_u)_x = \]

\[ = F_u u_{yx} - u_{yy} \int_0^y dy (F_u)_x. \]  \hspace{1cm} (2.9)

When $u_y = s = \text{const}, u_{yt} = u_{yx} = u_{yy} = 0$. Thus, the flow (1.11) properly restricts on the constraint \{ $u_y = s, \ u = v + sy$ \}. Evaluating equation (1.11b) at $y = 0$, we obtain:

\[ v_t = \partial (v^m H_m^*). \]  \hspace{1cm} (2.10)
Equation (1.11a) becomes:

\[ h_t = \partial(mA_{m-1}H_m)^s \quad [\text{by (2.6a)}] = \partial \left( \frac{\delta H^s}{\delta v} \right), \]  

(2.11)

which proves the first half of formula (1.16). To prove the second half of that formula, we need to check, in view of the relation (2.10), that

\[ \partial(v^mH^*_m) = \partial(H^*_h) - s\partial(H^*_v) \]  

(2.12)

or

\[ v^mH^*_m = H^*_h - sH^*_v. \]  

(2.13)

By formulae (2.5,6b), we have to verify that

\[ v^mH^*_m = (v + sh)^mH^*_m - \left( (v + sh)^m - v^m \right)H^*_m, \]  

(2.14)

which is obviously true.

### 3 The Reduction Map Is Hamiltonian

We need to verify that the map (2.2),

\[ A^*_m = \frac{(v + sh)^{m+1} - v^{m+1}}{s(m + 1)}, \quad m \in \mathbb{Z}_{\geq 0}, \]  

(3.1)

is Hamiltonian between the Hamiltonian matrices

\[ b = \begin{pmatrix} 0 & \partial \\ \partial & -s\partial \end{pmatrix} \]  

(3.2)

and

\[ B_{nm} = nA_{n+m-1}\partial + \partial mA_{n+m-1}. \]  

(3.3)

This is equivalent to the equality

\[ B^* = JbJ^t, \]  

(3.4)

where \( J \) is the Fréchet Jacobian of the map (3.1):

\[ J_{n,h} = \frac{\partial A^*_n}{\partial h} = A^*_{n,h}, \quad J_{n,v} = \frac{\partial A^*_n}{\partial v} = A^*_{n,v}. \]  

(3.5)

In components, the equality (3.4) becomes:

\[ nA^*_{n+m-1}\partial + \partial mA^*_{n+m-1} = A^*_{n,v}A^*_{m,h} + (A^*_{n,h} - sA^*_{n,v})\partial A^*_{m,v}, \quad n, m \in \mathbb{Z}_{\geq 0}. \]  

(3.6)
This identity in turn, splits into the pair:

\[(n + m)A_{n+m-1}^* = A_{n,v}^* A_{m,h}^* + (A_{n,h}^* - sA_{n,v}^*)A_{m,v}^*, \quad (3.7)\]
\[mA_{n+m-1,x}^* = A_{n,v}^*(A_{m,h}^*)_x + (A_{n,h}^* - sA_{n,v}^*)(A_{m,v}^*)_x. \quad (3.8)\]

We start with the identify (3.7). From formulae (2.3,4) we have:

\[A_{n,h}^* - sA_{n,v}^* = v^n. \quad (3.9)\]

Denote

\[\sigma = v + sh. \quad (3.10)\]

Formula (3.7) becomes in view of formula (2.2):

\[(n + m)\frac{\sigma^{n+m} - v^{n+m}}{s(n + m)} = \frac{\sigma^n - v^n}{s}\sigma^m + v^n \frac{\sigma^m - v^m}{s}, \quad (3.11)\]

which is obviously true.

Next, formula (3.8) becomes:

\[m\left(\frac{\sigma^{n+m} - v^{n+m}}{s(n + m)}\right)_x = \frac{\sigma^n - v^n}{s}(\sigma^m)_x + v^n \left(\frac{\sigma^m - v^m}{s}\right)_x. \quad (3.12)\]

Since

\[\frac{\partial \sigma}{\partial v} = 1, \quad (3.13)\]

the identity (3.12) splits into the pair:

\[m\sigma^{n+m-1}\sigma h_x ? = (\sigma^n - v^n)m\sigma^{m-1}\sigma h_x + v^n m\sigma^{m-1}\sigma h_x, \quad (3.14)\]
\[m(\sigma^{n+m-1} - v^{n+m-1})v_x ? = (\sigma^n - v^n)m\sigma^{m-1}v_x + v^n m(\sigma^{m-1} - v^{m-1})v_x, \quad (3.15)\]

and each one of these identities is obviously true.

4 Other Two-Dimensional Systems

Free-surface systems, such as (1.11), are naturally attached to local Lie algebras, in particular to Poisson manifolds [4]. In the two-dimensional case, the general form of systems liftable into the space of moments has the form [4]

\[h_t = Q_mA_m + P_mA_{m,x} \quad (4.1a)\]
\[u_t = P_mu^m u_x + \bar{P}_m u^m - u_y \int_0^y dy(P_m(u^m)_x + Q_m u^m), \quad (4.1b)\]
where \( P_m, \bar{P}_m, Q_m \) are arbitrary functions of \( x \) and the \( A_n \)'s; the resulting evolution for the moments is:

\[
A_{n,t} = nA_{n+m-1}\bar{P}_m + A_{n+m}Q_m + A_{n+m,x}\bar{P}_m. \tag{4.2}
\]

The systems (1.11) we have looked at in the previous Sections are of the above form, with

\[
\bar{P}_m = H_{m,x}, \quad Q_m = (m + 1)H_{m+1,x}, \quad P_m = (m + 1)H_{m+1}. \tag{4.3}
\]

Let us determine when the system (4.1) can be self-consistently constrained onto the submanifold \( \{u_y = s\} \). Differentiating formula (4.1b) with respect to \( y \), we get:

\[
u_{y,t} = P_m(mu^{m-1}u_yu_x + uu_{m xy}) + \bar{P}_m mu^{m-1}u_y - \]

\[
u_{y y} \int_0^y dy (P_m(u^m)_x + Q_m u^m) - u_y(P_m mu^{m-1}u_x + Q_m u^m) =
\]

\[
= P_m u^m u_{xy} - u_{y y} \int_0^y dy (P_m (u^m)_x + Q_m u^m) +
\]

\[
+ u_y u^m((m + 1)\bar{P}_{m+1} - Q_m). \tag{4.4b}
\]

Hence, the system (4.1) is constrainable iff

\[
Q_m = (m + 1)\bar{P}_{m+1}, \quad m \in \mathbb{Z}_{\geq 0}. \tag{4.5}
\]

The resulting \( h, v \)-system can be read off formulae (4.1) for \( y = 0 \):

\[
h_t = (m + 1)\bar{P}_{m+1} A^x_m + P^x_m A^x_{m,x}, \tag{4.6a}
\]

\[
v_t = P^x_m v^m v_x + \bar{P}^x_m v^m. \tag{4.6b}
\]

Formula (4.3) shows that the relations (4.5) are satisfied for our original system (1.11).

The system (4.1) is of a general character. Among Hamiltonian systems of this type, there exists a two-parameter family \([2, \text{formula (2.99')}]\) given by the Hamiltonian matrix

\[
B_{\alpha,\beta}^{m,n} = (\alpha n + \beta)A_{n+m} \partial + \partial(\alpha m + \beta)A_{n+m}, \tag{4.7}
\]

where \( \alpha \) and \( \beta \) are arbitrary constants. For this case, we have:

\[
A_{n,t} = B_{\alpha,\beta}^{m,n}(H_m) = (\alpha n + \beta)A_{n+m} H_{m,x} + (\alpha m + \beta)(A_{n+m} H_m)_x =
\]

\[
= nA_{n+m} \alpha H_{m,x} + A_{n+m}(\alpha m + 2\beta)H_{m,x} + A_{n+m,x}(\alpha m + \beta)H_m. \tag{4.8}
\]

From formula (4.2) we see that

\[
\bar{P}_0 = 0; \quad \bar{P}_{m+1} = \alpha H_{m,x}; \quad Q_m = (\alpha m + 2\beta)H_{m,x}; \quad P_m = (\alpha m + \beta)H_m. \tag{4.9}
\]
Therefore, the constrainability criterion (4.5) is satisfied provided
\[(\alpha m + 2\beta)H_{m,x} = (m + 1)\alpha H_{m,x}, \quad m \in \mathbb{Z}_{\geq 0},\] (4.10a)
or
\[(\alpha m + 2\beta) = (m + 1)\alpha, \quad m \in \mathbb{Z}_{\geq 0},\] (4.10b)
and this happens iff
\[\alpha = 2\beta.\] (4.11)

This is a very puzzling result. To see why, notice that the Hamiltonian matrix \(B^{\alpha,\beta}\) (4.7) is linear in the field variables (the \(A_n\)'s). Hence, it corresponds to a Lie algebra. An easy calculation shows that this Lie algebra has the commutator
\[[X, Y]_k = \sum_{n+m=k} ((\alpha n + \beta)X_n Y_{m,x} - (\alpha m + \beta)Y_m X_{n,x}).\] (4.12)

Setting
\[f(x, p) = \sum_{n \geq 0} X_n p^n, \quad g(x, p) = \sum_{m \geq 0} Y_m p^m,\] (4.13)
we can convert the commutator (4.12) into the following Poisson bracket on \(\mathbb{R}^2\):
\[\{f, g\} = \beta(f_g p_x - f_x g_p) + \alpha(f_p g_x - f_x g_p).\] (4.14)

There is nothing in this Poisson bracket to indicate that the ratio
\[\alpha : \beta = 2 : 1\] (4.15)
is distinguished from all the other ratios.

Let us now consider what happens with system (4.8) for the case
\[\alpha = 2, \quad \beta = 1,\] (4.16)
when this system is restricted onto the submanifolds \(\{u_y = s\}\). By formula (4.9), the full system (4.1) has the form:
\[h_t = A_m H_{m,x} + (2m + 1)(A_m H_m)_x,\] (4.17a)
\[u_t = (2m + 1)u^m u_x H_m + 2u^{m+1} H_{m,x} - u_y \int_0^y dy((2m + 1)u^m)_x H_m + 2(m + 1)u^m H_{m,x}.\] (4.17b)
Hence, the restricted system becomes:

\[ h_t = A^*_m H^*_m, \]
\[ v_t = (2m + 1)v^m v_x H^*_m + 2v^{m+1} H^*_{m,x}. \]

**Proposition 4.19.** (i) The system (4.18) can be put into the following form:

\[
\begin{pmatrix}
  h \\
  v
\end{pmatrix}_t =
\begin{pmatrix}
  h \partial h + v \partial v \\
  v \partial v + s(v \partial + \partial v)
\end{pmatrix} \begin{pmatrix}
  \frac{\delta H^*}{\delta h} \\
  \frac{\delta H^*}{\delta v}
\end{pmatrix};
\]

(ii) The matrix

\[
\begin{pmatrix}
  h \partial h + v \partial v \\
  v \partial v + s(v \partial + \partial v)
\end{pmatrix}
\]

is Hamiltonian;

(iii) The reduction map \( A^*_m = \int_h v + sy)^n \, dy \) (2.2) is Hamiltonian between the Hamiltonian matrices \( b \) (4.20b) and \( B^{2,1} \) (4.6).

**Proof.** (i) We have to verify that

\[
(h \partial + \partial h)(H^*_h) + (v \partial + \partial v)(H^*_v) \approx (A^*_m \partial + (2m + 1) \partial A^*_m)(H^*_m),
\]
\[
(v \partial + \partial v)(H^*_h) - s(v \partial + \partial v)(H^*_v) \approx ((2m + 1)v^m v_x + 2v^{m+1} \partial)(H^*_m).
\]

Formulae (2.5, 6b, 13,) and (3.10) transform the identities (4.21) into the form:

\[
(h \partial + \partial h)\sigma^m + (v \partial + \partial v)\frac{\sigma^m - v^m}{s} \approx A^*_m \partial + (2m + 1) \partial A^*_m,
\]
\[
(v \partial + \partial v)v^m \approx (2m + 1)v^m v_x + 2v^{m+1} \partial.
\]

Formula (4.22b) is obvious.

Formula (4.22a) can be rewritten as

\[
(\sigma \partial + \partial \sigma)\sigma^m - (v \partial + \partial v)v^m \approx \frac{\sigma^{m+1} - v^{m+1}}{m + 1}(2m - 2) \partial + (2m + 1) \left( \frac{\sigma^{m+1} - v^{m+1}}{m + 1} \right)_x,
\]

and it follows from formula (4.22b);
(ii) Let $\mathcal{G}$ be a Lie algebra. It acts by derivations on itself. Hence, we can form the semidirect sum Lie algebra, with the commutator

$$
\begin{pmatrix}
  x_1 \\
y_1
\end{pmatrix},
\begin{pmatrix}
x_2 \\
y_2
\end{pmatrix} =
\begin{pmatrix}
  [x_1, x_2] \\
  [x_1, y_2] - [x_2, y_1] + \epsilon [y_1, y_2]
\end{pmatrix}
$$

(4.24)

where $\epsilon$ is an arbitrary constant that, when nonzero, can be scaled away to $\epsilon = 1$. The Hamiltonian matrix corresponding to the commutator (4.24) has the form

$$
\begin{pmatrix}
  B_h & B_v \\
  B_v & \epsilon B_v
\end{pmatrix},
$$

(4.25)

where $\epsilon = -s$ and $B = B_q = B_q(\mathcal{G})$ is the Hamiltonian matrix naturally attached to the Lie algebra $\mathcal{G}$ with the dual coordinates on $\mathcal{G}^*$ denoted by $q$. Our matrix $b_{(4.20)}$ is of the form (4.25), with $G = D(R^1)$ being the Lie algebra of vector fields on $R^1$;

(iii) We have to verify the identity

$$
JbJ^t = B_{2.1}^t,
$$

(4.26)

where the matrix $b$ is given by formula (4.20). In components, this identity becomes:

$$
(A^*_{n,h}(h\partial + \partial h) + A^*_{n,v}(v\partial + \partial v))A^*_{m,h} + v^n(v\partial + \partial v)A^*_{m,v} =
\ni = (2n + 1)A^*_{n+m}\partial + \partial(2m + 1)A^*_{n+m}.
$$

(4.27)

By formulae (2.2-4), this can be rewritten as

$$
\sigma^n(h\partial + \partial h) + \frac{\sigma^n - v^n}{s}(v\partial + \partial v)\sigma^m + v^n(v\partial + \partial v)\sigma^m - v^m
\i = (2n + 1)\frac{\sigma^{n+m+1} - v^{n+m+1}}{(n+m+1)s} \partial + \partial(2m + 1)\frac{\sigma^{n+m+1} - v^{n+m+1}}{(n+m+1)s}
\i = \frac{2}{s}(\sigma^{n+m+1} - v^{n+m+1})\partial + \frac{2m + 1}{s}(\sigma^{n+m}\sigma_x - v^{n+m}v_x),
$$

(4.28)

or as

$$
\sigma^n(\sigma\partial + \partial \sigma) - v^n(v\partial + \partial v)\sigma^m + v^n(v\partial + \partial v)\sigma^m - v^m =
\i = \sigma^n(\sigma\partial + \partial \sigma)\sigma^m - v^n(v\partial + \partial v)v^m
\i = 2(\sigma^{n+m+1} - v^{n+m+1})\partial + (2m + 1)(\sigma^{n+m}\sigma_x - v^{n+m}v_x),
$$

(4.29)

which follows from the single equality

$$
w^n(w\partial + \partial w)w^m = 2w^{n+m+1}\partial + (2m + 1)w^{n+m}w_x,
$$

(4.30)
which in turn follows from formula (4.22b) or is obvious in its own right ■

Remark 4.31. Formulae in [4,5] suggest - but not prove - that the constant-shear reductions of Hamiltonian systems do not exist in the $N + 1$ dimension for $N \neq 1$.

5 Lie Algebra Of Reducible Flows

A reducible dynamical system (4.2) in the momentum space has, by formula (4.5), the form

$$\dot{X}(A_n) = A_{n,t} = (n + m)A_{n+m-1}\dot{P}_m + A_{n+m,x}\dot{P}_m.$$

Suppose we have another reducible vector field, $\dot{Y}$:

$$\dot{Y}(A_n) = (n + m)A_{n+m-1}\dot{\Phi}_m + A_{n+m,x}\dot{\Phi}_m.$$

Theorem 5.3. The commutator of reducible vector fields is again reducible.

Proof. We shall show that

$$[\dot{X}, \dot{Y}](A_n) = (n + r)A_{n+r-1}\dot{\Omega}_r + A_{n+r,x}\dot{\Omega}_r,$$

where

$$\dot{\Omega}_r = \dot{X} (\Phi_r) - \dot{Y} (\Phi_r) + \sum_{k+m=r+1} \Phi_k \dot{P}_m +$$

$$+ \sum_{k+m=r} (\Phi_k \dot{P}_{m,x} - P_k \Phi_{m,x}),$$

$$\dot{\Omega}_r = \dot{X} (\Phi_r) - \dot{Y} (\Phi_r) + \sum_{k+m=r+1} (k\Phi_k \dot{P}_m - mP_m \dot{\Phi}_k) +$$

$$+ \sum_{k+m=r} (\Phi_k P_{m,x} - P_m \Phi_{k,x}).$$

We have:

$$[\dot{X}, \dot{Y}](A_n) - (n + r)A_{n+r-1}[\dot{X} (\Phi_r) - \dot{Y} (\Phi_r)] =$$

$$-A_{n+r,x} [\dot{X} (\Phi_r) - \dot{Y} (\Phi_r)] =$$

$$= \Phi_k (n + k)\dot{X} (A_{n+k-1}) + \Phi_k [\dot{X} (A_{n+k})]x -$$

$$= \dot{P}_m (n + m)\dot{Y} (A_{n+m-1}) + \dot{P}_m [\dot{Y} (A_{n+m})]x =$$

$$= \Phi_k (n + k)(n + k - 1 + m)A_{n+k-1+m-1}\dot{P}_m + A_{n+k-1+m,x}\dot{P}_m +$$

$$+ \Phi_k (n + k + m) (A_{n+k+m-1,x}\dot{P}_m + A_{n+k+m-1}\dot{P}_{m,x}) +$$

$$+ A_{n+k+m,x}\dot{P}_m + A_{n+k+m,x}\dot{P}_{m,x} -$$

$$ - \dot{P}_m (n + m) (n + m - 1 + k)A_{n+m-1+k-1}\dot{\Phi}_k + A_{n+m-1+k,x}\dot{\Phi}_k -$$

$$- \dot{P}_m (n + m + k) (A_{n+m+k-1,x}\dot{\Phi}_k + A_{n+m+k-1}\dot{\Phi}_{k,x}) +$$

$$+ A_{n+m+k,x}\dot{\Phi}_k + A_{n+m+k,x}\dot{\Phi}_{k,x}.$$


The first summands in (5.7c) and (5.8c) cancel out.

The second summands in (5.7a) and (5.8a), and the first summands in (5.7b) and (5.8b), combine into

$$A_{n+r, x} \sum_{k+m=r+1} (k \Phi_k \bar{P}_m - m P_m \Phi_k), \quad (5.9)$$

while the second summands in (5.7c) and (5.8c) yield

$$A_{n+r, x} \sum_{k+m=r} (\Phi_k P_{m,x} - P_m \Phi_{k,x}). \quad (5.10)$$

Formulae (5.6b, 9, 10) account for formula (5.5b).

What remains, the first summands in (5.7a) and (5.8a), and the second summands in (5.7b) and (5.8b), combine into

$$(n + r)A_{n+r-1}\{ \sum_{k+m=r+1} [(n+k) \Phi_k \bar{P}_m - (n+m) P_m \Phi_k] + \sum_{k+m=r} (\Phi_k \bar{P}_{m,x} - P_m \Phi_{k,x}) \} \quad (5.11)$$

which, together with the second summand in (5.6a), account for formula (5.5a).

Since the momentum map

$$A_n = \int_0^h u^n dy, \quad n \in \mathbb{Z}_{\geq 0}, \quad (5.12)$$

is injective, Theorem 5.3 implies that the 2+1-dimensional hydrodynamic systems (4.1) that are reducible in the physical space,

$$h_t = (m+1) \bar{P}_{m+1} A_m + P_m A_{m,x}, \quad (5.13a)$$

$$u_t = P_m u^m u_x + \bar{P}_m u^m - u^u \int_0^y dy (P_m (u^m)_x + (m+1) \bar{P}_{m+1} u^m), \quad (5.13b)$$

also form a Lie algebra. In particular, when there is no x-dependence, so that the $P_m$’s vanish and the $\bar{P}_m$’s depend on the $A_n$’s but not on the derivatives of the $A_n$’s, we get a free-surface analog of the Lie algebra ODE’s:

$$h_t = (m+1) \bar{P}_{m+1} A_m, \quad (5.14a)$$

$$u_t = \bar{P}_0 + \bar{P}_{m+1} (u^{m+1} - u_y \int_0^y dy (m+1) u^m). \quad (5.14b)$$

All these Lie algebras possess subalgebras induced by the constain

$$\{ h = A_0 = A_1 = ... = 0 \}. \quad (5.15)$$

**Remark 5.16.** The 2+1-dimensional free-surface hydrodynamics is infused with Lie algebras, such as (4.1), (4.2), (5.1), (5.13). This picture is very different from the theory of systems of hydrodynamical type in $1+1-d$, of the form

$$u_{i,t} = \sum_j \mathcal{M}_i^j(u) u_{j,x}, \quad (5.17)$$

where, in general, a commutator of two such systems is no longer of hydrodynamic type (see [8]).
6 Multilayer Representations

Imagine that the Benney system (1.5)

\[ h_t = \left( \int_0^h u \, dy \right)_x, \quad (6.1a) \]
\[ u_t = uu_x + gh_x - u_y \int_0^y u_x \, dy, \quad (6.1b) \]

is broken into \( N \) layers

\[ h_{k-1} \leq y \leq h_k, \quad k = 1, \ldots, N, \quad h_0 = 0, \quad (6.2) \]

such that in each layer the velocity profile \( u_k \) is \( y \)-independent. The Benney system then turns into the \( 2N \)-component system

\[ h_{k,t} = (u_k h_k)_x, \quad (6.3a) \]
\[ u_{k,t} = u_k u_{k,x} + gh_k, \quad k = 1, \ldots, N, \quad (6.3b) \]
\[ h = \sum_{k=1}^N h_k. \quad (6.3c) \]

This idea and the system (6.3) are due to Zakharov [10], who in addition showed that, rather mysteriously, the system (6.3) appears also as the zero-dispersion limit of a vector Nonlinear Schrödinger equation.

The system (6.3) can be considered as a multi-component version of the classical long-wave system (1.1), and it was analyzed in detail by Pavlov and Tsarev [9]. As Zakharov noted, the moment map (1.6)

\[ A_n = \int_0^h u^n dy, \quad n \in \mathbb{Z}_{\geq 0}, \quad (6.4) \]

which now becomes

\[ A_n = \sum_{k=1}^N h_k u_k^n, \quad n \in \mathbb{Z}_{\geq 0}, \quad (6.5) \]

maps the \( 2N \)-component system (6.3) into the infinite-component Benney momentum system (1.7)

\[ A_{n,t} = A_{n+1,x} + gn A_{n-1} A_{0,x}, \quad n \in \mathbb{Z}_{\geq 0}. \quad (6.6) \]

It’s easy to see that the same conclusion applies to all the higher Benney flows constructed in [6, 7], and indeed to any flow (1.9) in the Hamiltonian structures (1.10), (4.7) or any other linear Hamiltonian structure. The argument is as follows.
Let $\mathcal{G}$ be a Lie algebra, $C_A$ the (differential-difference) ring of functions on $\mathcal{G}^*$, and $B = B_A$ the natural Hamiltonian structure in $C_A$ attached to $\mathcal{G}$ (see [5].) Let $\mathcal{G}^{<N>}$ be the direct sum of $N$ copies of $\mathcal{G}$. The homomorphism of Lie algebras
\[ \varphi : \mathcal{G}^{<N>} \to \mathcal{G}, \]
(6.7a)
\[ \varphi \left( \bigoplus_{k=1}^{N} x_k \right) = \sum_{k=1}^{N} x_k, \]
(6.7b)
induces the corresponding linear Hamiltonian map
\[ \Phi : C_A \to C_A^{<N>}, \]
\[ \Phi(A_n) = \sum_{k=1}^{N} A_{n|k}. \]
(6.8)

If $C_V$ is another ring, with a Hamiltonian structure on it, and if
\[ w : C_A \to C_V \]
(6.9)
is a Hamiltonian map, then so is its $k^{th}$-copy version:
\[ w_k : C_{A|k} \to C_{V|k}. \]
(6.10)
The composition
\[ Z = \Phi \circ \Omega : C_A \to C_V^{<N>}, \]
\[ \Omega = \bigoplus_{k=1}^{N} w_k, \quad C_V^{<N>} = \bigoplus_{k=1}^{N} C_{V|k}, \]
(6.11)
is then a multilayer analog of the single-layer canonical map (6.9).

In particular, for the Hamiltonian matrix (1.10)
\[ h_t = \partial (mA_{m-1}H_m) \]
(6.13a)
\[ u_t = \partial (u^mH_m) - u_y \int_0^y dy (mu^{m-1}H_m)_x. \]
(6.13b)
with the Hamiltonian map (1.13)
\[ w(A_n) = hu^n, \]
(6.14)
the general construction above gives:
\[ h_{k,t} = \partial_{u_k}^{\delta} Z(H), \]
(6.15a)
\[ u_{k,t} = \partial_{h_k}^{\delta} Z(H), \]
(6.15b)
\[ Z(A_n) = \sum_{k=1}^{N} h_k u_k^n. \]
(6.15c)
If $H$ is the $2^{nd}$ Hamiltonian (1.8),

$$H = \frac{1}{2}(A_2 + gA_0^2)$$  \hfill (6.16)

then

$$Z(H) = \frac{1}{2}\left(\sum_k h_k u_k^2 + g(\sum_k h_k)^2\right),$$  \hfill (6.17a)

$$\frac{\delta Z(H)}{\delta u_k} = h_k u_k, \quad \frac{\delta Z(H)}{\delta h_k} = \frac{u_k^2}{2} + gh,$$  \hfill (6.17b)

and we recover the Zakharov system (6.3). If $H$ is the $3^{rd}$ Hamiltonian,

$$H = \frac{1}{3}A_3 + gA_0A_1,$$  \hfill (6.18)

then

$$Z(H) = \frac{1}{3}\sum_k h_k u_k^3 + g(\sum_k h_k u_k)(\sum h_k),$$  \hfill (6.19a)

$$\frac{\delta Z(H)}{\delta u_k} = h_k u_k^2, \quad \frac{\delta Z(H)}{\delta h_k} = \frac{u_k^3}{3} + g u_k h + gA_1 \Rightarrow$$  \hfill (6.19b)

$$h_{k,t} = (h_k u_k^2)_x, \quad u_{k,t} = u_k^2 u_{k,x} + g(u_k h + A_1)_x, \quad k = 1, \ldots, N.$$  \hfill (6.20a)

All the odd-numbered higher Benney flows can be restricted onto the invariant submanifold [3]

$$\{A_1 = A_3 = A_5 = \ldots = 0\}.$$  \hfill (6.21)

The analog of this fact is this: Suppose $N$ is even:

$$N = 2M,$$  \hfill (6.22a)

and

$$h_{k+m} = h_k, \quad u_{k+m} = -u_k, \quad k = 1, \ldots, M.$$  \hfill (6.22b)

Then all the odd flows in the $(h, u)$-space can be properly reduced by the constrain (6.22). The $3^{rd}$ flow (6.20) becomes:

$$h_{k,t} = (h_k u_k^2)_x,$$  \hfill (6.23a)

$$u_{k,t} = u_k^2 u_x + 2g(u_x h)_x, \quad k = 1, \ldots, M,$$  \hfill (6.23b)

in the variables

$$h_k = h_k, \quad U_k = u_k^2, \quad k = 1, \ldots, M.$$  \hfill (6.24)
16

$$h_{k,t} = (h_k U_k)_x, \quad (6.25a)$$
$$U_{k,t} = U_k U_{k,x} + 2g(U_{k,x} h + 2U_k h_x), \quad k = 1, ..., M. \quad (6.25b)$$

Similar Hamiltonian construction applies to the case of constant shear. Equations (6.15) become

$$h_{k,t} = \partial \frac{\delta}{\delta v_k} Z(H), \quad (6.26a)$$
$$v_{k,t} = \partial \left( \frac{\delta}{\delta h_k} - s_k \frac{\delta}{\delta v_k} \right) Z(H), \quad k = 1, ..., N, \quad (6.26b)$$

where now

$$Z(A_n) = \sum_k \int_0^{h_k} (v_k + s_k y)^n dy = \sum_k \frac{(v_k + s_k h_k)^{n+1} - v_k^{n+1}}{s_k(n+1)}. \quad (6.27)$$

In particular, for the 2nd flow, we get:

$$h_{k,t} = (h_k v_k + \frac{s_k h_k^2}{2})_x, \quad (6.28a)$$
$$v_{k,t} = v_k v_{k,x} + gh_x, \quad k = 1, ..., N. \quad (6.28b)$$

This is a multicomponent version of the single component shear system (1.18), and a shear extension of the Zakharov system (6.3).

The shear analog of the constrain (6.22) now becomes:

$$h_{k+m} = h_k, \quad v_{k+m} = -v_k, \quad s_{k+M} = -s_k, \quad k = 1, ..., M. \quad (6.29)$$

The construction above works only for systems with linear Hamiltonian structures. It doesn’t apply to the hydrodynamic chain [3]

$$A_{n,t} = A_{n+1,x} + (an + b)A_n A_{0,x} + cA_0 A_{n,x}, \quad n \in \mathbb{Z}_{\geq 0}, \quad (6.30a)$$
$$a, b, c = const, \quad (6.30b)$$

with

$$b \neq 2c. \quad (6.30c)$$

Nevertheless, we now show that the Zakharov map (6.5) and its shear version (6.27) work in the most general circumstances.

**Theorem 6.31.** (i) The Zakharov map (6.5) applies to any flow (4.2):

$$A_{n,t} = n A_{n+m-1} \tilde{P}_m + A_{n+m} Q_m + A_{n+m,x} P_m. \quad (6.32)$$

(ii) The map (6.27) applies to any flow (5.1):

$$A_{n,t} = (n + m) A_{n+m-1} \tilde{P}_m + A_{n+m,x} P_m. \quad (6.33)$$
Proof. The idea is this. Either of the two maps is a local diffeomorphism between the variables \( h_1, u_1, \ldots, h_N, u_N \) and \( A_0, \ldots, A_{2N-1} \). Therefore, we can consider these maps as providing an invariant submanifold for each flow. This means that if we find -- by whatever means -- an evolution in the \( \{ u_k, h_k \} \)--space that is properly embedded into the evolution in the \( A \)--space, that’s it.

Now for the details. Writing \( (\cdot) \) instead of \( (\cdot)_t \), we convert, first for the Zakharov case, formula (4.2) into

\[
\sum_k (\dot{h}_k u_k^n + n h_k u_k^{n-1} \dot{u}_k) = \sum_k \{ n h_k u_k^{n+m-1} \bar{P}_m + h_k u_k^{n+m} Q_m^* + [h_k, x] u_k^{n+m} + (n + m) h_k u_k^{n+m-1} u_{k,x} \} P_m^*,
\]

where

\[
(\cdot)^* = Z(\cdot), \quad Z(A_n) = \sum_k h_k u_k^n, \quad n \in \mathbb{Z}_{\geq 0}.
\]

We drop the \( \sum_k \) operator from each side, multiply the resulting equation by \( u_k^{-n} \), and find:

\[
\dot{h}_k + n h_k u_k^{-1} \dot{u}_k = h_k u_k^n Q_m^* + [h_k, x] u_k^m + m h_k u_k^{m-1} u_{k,x} \bar{P}_m^* + n h_k u_k^{-1} [u_k^m \bar{P}_m + u_k^m u_{k,x} P_m^*],
\]

whence

\[
\dot{h}_k = h_k u_k^m Q_m^* + [h_k, x] u_k^m + m h_k u_k^{m-1} u_{k,x} \bar{P}_m^*,
\]

\[
\dot{u}_k = u_k^m \bar{P}_m + u_k^m u_{k,x} P_m^*.
\]

Next, denoting

\[
\sigma_k = v_k + s_k h_k,
\]

so that

\[
Z(A_n) = \sum_k \frac{\sigma_k^{n+1} - v_k^{n+1}}{(n + 1)s_k},
\]

we convert formula (5.1) into

\[
\dot{A}_n = \sum_k \frac{(\sigma_k^{n+1} - v_k^{n+1})}{(n + 1)s_k} = \sum_k \{ \sigma_k^n \dot{h}_k + \frac{\sigma_k^n - v_k^n}{s_k} \dot{v}_k \} = \sum_k \{ \sigma_k^{n+m} - v_k^{n+m} \bar{P}_m^* + \left( \frac{\sigma_k^{n+m+1} - v_k^{n+m+1}}{(n + m + 1)s_k} \right) x P_m^* \},
\]
where again

\[(\cdot)^* = Z(\cdot),\]  \hspace{1cm} (6.41)

but with the map \(Z\) given by formula (6.39).

We now drop the operator \(\sum_k\) from each side of formula (6.40) and get, for each \(k\), the equation

\[
\sigma_k^n h_k + \frac{\sigma_k^n - v_k^n}{s_k} \dot{v}_k = \frac{\sigma_k^{n+m} - v_k^{n+m}}{s_k} P_m^* + \left(\frac{\sigma_k^{n+m+1} - v_k^{n+m+1}}{(n+m+1)s_k}\right) P_m^*.
\]  \hspace{1cm} (6.42)

For \(n = 0\), equation (6.42) yields:

\[
\dot{h}_k = \frac{\sigma_k^m - v_k^m}{s_k} P_m^* + \left(\frac{\sigma_k^{m+1} - v_k^{m+1}}{(m+1)s_k}\right) P_m^*.
\]  \hspace{1cm} (6.43)

Substituting this back into equation (6.42), we find:

\[
\frac{\sigma_k^n - v_k^n}{s_k} \dot{v}_k = [\sigma_k^{n+m} - v_k^{n+m} - \sigma_k^n (\sigma_k^m - v_k^m)] s_k^{-1} P_m^* + \\
+[(\sigma_k^{n+m} - \sigma_k^n \sigma_k^m) \sigma_k x - (v_k^{n+m} - \sigma_k^n v_k^m) v_k x] s_k^{-1} P_m^* \Leftrightarrow
\]  \hspace{1cm} (6.44)

\[
\dot{v}_k = v_k^m P_m^* + v_k^m v_k x P_m^* \]  \hspace{1cm} (6.45)

References

[1] Benney, D. J. *Some Properties of Long Nonlinear Waves*, Stud. Appl. Math. **L11** (1973) 45-50.

[2] Kupershmidt, B. A., *Deformations of Integrable Systems*, Proc. Roy. Irish Acad. **83** A. No. 1 (1983) 45-74.

[3] Kupershmidt, B. A., *Normal and Universal Forms in Integrable Hydrodynamical Systems in Proc. of NASA Ames-Berkley Conf. on Nonlinear Problems in Optimal Control and Hydrodynamics*, L. R. Hunt and C. F. Martin Ed-s, Math. Sci. Press (1984) 357-378.

[4] Kupershmidt, B. A., *Hydrodynamical Poisson Brackets and Local Lie Algebras*, Phys. Lett. **121A** (1987) 167-174.

[5] Kupershmidt, B. A., *The Variational Principles of Dynamics*, World Scientific (Singapore, 1992).
[6] Kupershmidt, B. A. and Manin, Yu I., *Long-Wave Equation with Free Boundaries. I. Conservation Laws*, Funct. Anal. App. 11:3 (1977) 31 - 42 (Russian); 188-197 (English).

[7] Kupershmidt, B. A. and Manin, Yu I., *Equations of Long Waves with a Free Surface. II Hamiltonian Structure and Higher Equations*, Funct. Anal. Appl. 12:1 (1978) 25-37 (Russian); 20-29 (English).

[8] Pavlov, M. V., Svinolupov, S. I., and Sharipov, R. A., *An Invariant Criterion for Hydrodynamic Integrability*, Funktsional. Anal. i Prilozhen. 30 no. 1 (1996) 18–29, 96 (Russian); Funct. Anal. Appl. 30 no. 1 (1996) 15–22 (English); arXiv:solv-int/9407003.

[9] Pavlov, M. V., and Tsarev, S. P., *Conservation Laws for the Benney Equations*, Uspekhi Mat. Nauk 46 no. 4 (1991) 169–170 (Russian); Russian Math. Surveys 46 no. 4 (1991) 196–197 (English).

[10] Zakharov, V. E., *Benney Equations and Quasiclassical Approximation in the Inverse Problem Method*, Funktsional. Anal. i Prilozhen. 14 no. 2 (1980) 15–24 (Russian); Functional Anal. Appl. 14 no. 2 (1980) 89–98 (English).