Null Dimensional Reduction of M2-Brane

J. Klusoň

Department of Theoretical Physics and Astrophysics
Faculty of Science, Masaryk University
Kotlářská 2, 611 37, Brno
Czech Republic
E-mail: klu@physics.muni.cz

ABSTRACT: We perform a general reduction of an M2-brane on a space-time that admits a null Killing vector leading to fundamental string and D2-brane action in Newton-Cartan background.
1. Introduction and Summary

Idea of non-relativistic string theory was firstly proposed twenty years ago in two papers [1, 2]. These works studied strings with non-relativistic spectrum in flat space-time and they were related to the analysis of strings in the background with critical electric fields [3] which is related to the famous relation between string theory and noncommutative geometry [3]. The generalization from flat to special curved background was performed in [3]. New interest in non-relativistic string theories begun recently in the context of AdS/CFT correspondence [6, 7] where torsional Newton-Cartan (NC) geometry was firstly observed as the geometry of boundary theory. More importantly, this torsional NC geometry is also consistent background for non-relativistic strings [8, 21, 26]. In fact, there are basically two ways how to construct non-relativistic string theory. The first one is based on specific non-relativistic limit [27, 28, 34] while the second one on null reduction of the world-sheet string in arbitrary general background with null isometry [8, 13, 21, 26]. It was then shown in [21] how these two non-relativistic string theories are related. In fact, it was also shown in [24] that the non-relativistic string in torsional Newton-Cartan background is defined by specific T-duality transformations along direction with null isometry. In this paper we would like to study the question whether it is possible to define non-relativistic string from M2-brane through double dimensional reduction along null isometry of the M-theory background. It is well known that dimensional reduction of M-theory along spatial circle leads to Type IIA theory where M2-brane maps either into fundamental string or to D2-brane [8, 11], for excelent review, see [30]. In this paper we

\footnote{For related works, see for example [1, 10, 11, 31, 32, 33, 34, 21, 22, 23, 24, 35, 27, 28, 29].}
proceed in similar way when we analyse M2-brane in the M-theory background with null isometry. This is very important problem which, as far as we know, has not been studied yet, with exception of recent paper [33] \(^2\). We firstly consider M2-brane extended along null isometry direction and perform its double dimensional reduction. In order to perform this procedure we write M2-brane using auxiliary three dimensional metric and presume that this metric has null isometry as well. As a result we obtain Polyakov like form of non-relativistic string with Newton-Cartan world-sheet metric so that it can be interpreted as a string with non-relativistic world-sheet. However when we eliminate components of the world-sheet Newton-Cartan gravity by solving their equations of motion we obtain standard square root form of the string action with the induced volume element corresponding to the target space-time Newton-Cartan gravity. We also find that for given eleven dimensional background with null isometry there is a family of Newton-Cartan backgrounds that are related by simple redefinitions which is similar result as in [28].

As the next step we consider M2-brane that is transverse to the null isometry direction. Following similar analysis as in case of spatial dimensional reduction [36] we find that this action corresponds to D2-brane action in Newton-Cartan background.

In summary, we consider null dimensional reduction of M-theory background and we find that M2-brane maps either to fundamental string or D2-brane in Newton-Cartan background. We find that these actions basically contain pull back of the volume element of target space Newton-Cartan background which is nice result that implies many new questions. The first one is how these non-relativistic strings and D2-brane actions are related to the non-relativistic string actions that were found in previous works. This question is closely related to the problem how null dimensional reduction of M-theory background is related to corresponding stringy Newton-Cartan background studied recently in [3]. And finally it would be certainly interesting to see how our work could be related to Matrix theory conjecture. We hope to return to these problems in future.

This paper is organized as follows. In the next section (2) we review double dimensional reduction of M2-brane along spatial dimensional direction of M-theory background. Then in section (3) we perform null dimensional reduction of M-theory and double dimensional reduction of M2-brane. In section (4) we discuss Hamiltonian formulation of this non-relativistic string. Finally in section (5) we perform transverse dimensional reduction of M2-brane that leads to D2-brane in non-relativistic background.

2. Review of Double Dimensional Reduction of M2-brane along Spatial Circle

In this section we perform double dimensional reduction of M2-brane that leads to fundamental string in Type IIA theory. This procedure is well known when M2-brane is defined using the square root form of the action (See for example excellent review [30]). However

\(^2\)Of course there is well known procedure of DLCQ of M-theory [38, 39, 40]. It would be certainly very interesting to see how our approach is related to the Matrix theory description of M-theory.
we would like to perform this analysis with the help of Polyakov form of the M2-brane action \(^3\).

In order to perform dimensional reduction we consider eleven dimensional background metric in the form [30]

\[
ds^2 = \hat{g}_{MN} dx^M dx^N = e^{-\frac{2}{3} \phi} g_{\mu\nu} dx^\mu dx^\nu + e^{\frac{4}{3} \phi} (dy - C_\mu dx^\mu)^2 ,
\]

where \(\mu, \nu = 0, 1, \ldots, 9\) and where \(y\) is compact direction. As is well known natural probe of M-theory is M2-brane which is three dimensional extended object with the action

\[
S = -T_{M2} \int d^3 \xi \sqrt{-\det \hat{g}_{\alpha\bar{\beta}}} = \hat{g}_{\alpha\beta} = \hat{g}_{MN} \partial_\alpha x^M \partial_\beta x^N ,
\]

where \(T_{M2}\) is M2-brane tension, \(\xi^\alpha, \bar{\alpha} = 0, 1, 2\) are coordinates that label three dimensional world-volume of M2-brane. Further, \(x^M, M, N = 0, 1, \ldots, 10\) label embedding of M2-brane into target space-time.

Let us now consider M2-brane that wraps compact direction labelled by \(y\). Standard procedure is to use double dimensional reduction in the action (2.2) [30]. However for letter purposes we proceed with slightly different way when we consider form of M2-brane action written using three dimensional auxiliary metric \(\gamma_{\bar{\alpha}\bar{\beta}}\) so that we can rewrite the action (2.2) into the form

\[
S = -\frac{T_{M2}}{2} \int d^3 \xi \sqrt{-\gamma} \gamma_{\bar{\alpha}\bar{\beta}} \hat{g}_{\bar{\alpha}\bar{\beta}} - 1 .
\]

In order to see an equivalence between this Polyakov-like M2-brane action (2.3) and (2.2) let us consider equation of motion for \(\gamma_{\bar{\alpha}\bar{\beta}}\) that follow from (2.3)

\[
\frac{1}{2} \sqrt{-\gamma} \gamma_{\bar{\alpha}\bar{\beta}} (\gamma^{\bar{\delta}} \hat{g}_{\bar{\delta}\bar{\gamma}} - 1) + \sqrt{-\gamma} \gamma_{\bar{\alpha}\bar{\beta}} = 0 .
\]

This equation can be solved for \(\gamma_{\bar{\alpha}\bar{\beta}}\) as \(\gamma_{\bar{\alpha}\bar{\beta}} = g_{\bar{\alpha}\bar{\beta}}\). Then inserting this result into (2.3) we obtain an action (2.2).

As we wrote above we presume that M2-brane wraps \(y\)-direction so that we can identify \(\xi^2\) with \(y\). As a result we get following components of induced metric

\[
\hat{g}_{22} = e^{\frac{4}{3} \phi} , \quad \hat{g}_{2\beta} = -e^{\frac{4}{3} \phi} C_\mu \partial_\beta x^\mu , \quad \hat{g}_{\alpha 2} = -e^{\frac{4}{3} \phi} \partial_\alpha x^\mu C_\mu , \quad \hat{g}_{\alpha\beta} = e^{-2\phi} g_{\alpha\beta} + e^{\frac{4}{3} \phi} C_\alpha C_\beta ,
\]

where \(\alpha, \beta = 0, 1\). Let us insert this ansatz into (2.2) and we obtain

\[
S = -T_{M2} \int dy d^2 \xi \sqrt{-\det \hat{g}_{\alpha\bar{\beta}}} = \hat{g}_{\alpha\beta} = T_{M2} \int d^2 \xi \sqrt{-\det g_{\alpha\beta}} = T_{FS} \int d^2 \xi \sqrt{-\det g_{\alpha\beta}}
\]

\(^3\)For simplicity we restrict ourselves to the case of vanishing three form field in M-theory keeping in mind that generalization of the analysis with inclusion of this field is straightforward.
which is the standard Nambu-Goto action for fundamental string in Type IIA theory when we performed identification between string tension \( T_{FS} \) and M2-brane tension in the form

\[
T_{FS} = T_{M2} \int dy .
\] (2.7)

Let us now perform the same double dimensional reduction in case of the Polyakov form of M2-brane action. Inserting the ansatz (2.5) into (2.3) we obtain

\[
S = -\frac{T_{M2}}{2} \int d^3\xi \sqrt{-\det \hat{\gamma}} (\hat{\gamma}^{\alpha\beta} \hat{g}_{\alpha\beta} + 2\hat{\gamma}^{\alpha\beta} \hat{g}_{\alpha} + \hat{\gamma}^{22} \hat{g}_{yy} - 1) .
\] (2.8)

Since we presume that M2-brane reduces in the same way as the background metric it is natural to consider the same ansatz for induced metric as in (2.1). In more details, let us presume that the world-volume metric has the form

\[
\hat{\gamma}^{y\beta} = e^{\frac{2}{3}\varphi} \gamma^{y\beta} , \quad \hat{\gamma}^{u\alpha} = -e^{\frac{4}{3}\varphi} \gamma^{u\alpha} + e^{\frac{4}{3}\varphi} c_{\alpha} c_{\beta} .
\] (2.9)

It is easy to see that

\[
\det \hat{\gamma}_{\alpha\beta} = \det \gamma_{\alpha\beta} .
\] (2.10)

Further, the metric inverse to (2.9) has the form

\[
\hat{\gamma}^{\alpha\beta} = e^{\frac{2}{3}\varphi} \gamma^{\alpha\beta} , \quad \hat{\gamma}^{y\beta} = e^{\frac{2}{3}\varphi} \gamma^{y\beta} , \quad \hat{\gamma}^{u\alpha} = -e^{\frac{4}{3}\varphi} \gamma^{u\alpha} + e^{\frac{4}{3}\varphi} c_{\alpha} c_{\beta} .
\] (2.11)

Then the action (2.8) is equal to

\[
S = -\frac{T_{M2}}{2} \int d^3\xi \sqrt{-\gamma} (e^{\frac{2}{3}\varphi} \gamma^{\alpha\beta} (e^{-\frac{4}{3}\phi} g_{\alpha\beta} + e^{\frac{4}{3}\phi} C_{\alpha} C_{\beta}) - 2e^{\frac{2}{3}\varphi} \gamma^{y\beta} C_{\alpha} e^{\frac{4}{3}\phi} + (e^{-\frac{4}{3}\varphi} + e^{\frac{4}{3}\varphi} c_{\alpha} \gamma^{\alpha\beta} c_{\beta}) e^{\frac{2}{3}\phi} - 1) .
\] (2.12)

In order to see an equivalence of the action (2.12) with (2.6) let us solve the equation of motion for \( c_{\alpha} \) that follow from (2.12)

\[
-e^{\frac{2}{3}(2\phi+\varphi)} \gamma^{\alpha\beta} C_{\beta} + e^{\frac{2}{3}(2\phi+\varphi)} \gamma^{\alpha\beta} c_{\beta} = 0
\] (2.13)

that has solution \( c_{\alpha} = C_{\alpha} \). Further, equation of motion for \( \varphi \) has the form

\[
e^{\frac{2}{3}\varphi} \gamma^{\alpha\beta} (e^{-\frac{4}{3}\phi} g_{\alpha\beta} + e^{\frac{4}{3}\phi} C_{\beta} C_{\alpha}) - 2e^{\frac{2}{3}\varphi} \gamma^{y\beta} C_{\alpha} e^{\frac{4}{3}\phi} + (e^{-\frac{4}{3}\varphi} + e^{\frac{4}{3}\varphi} c_{\alpha} \gamma^{\alpha\beta} c_{\beta}) e^{\frac{2}{3}\phi} = 0 .
\] (2.14)

Using \( c_{\alpha} = C_{\alpha} \) the previous equation reduces into

\[
e^{\frac{4}{3}(\varphi - \phi)} \gamma^{\alpha\beta} g_{\alpha\beta} - 2e^{\frac{2}{3}(\phi - \varphi)} = 0 .
\] (2.15)
Finally, the equation of motion for $\gamma^{\alpha\beta}$ has the form

$$
-\frac{1}{2} \gamma^{\beta\alpha}(e^{\frac{2}{3}\phi} \gamma^{\gamma\delta}(e^{-\frac{2}{3}\phi} g_{\gamma\delta} + e^{\frac{2}{3}\phi} C_\gamma C_\delta) -
-2e^{\frac{2}{3}\phi} c_\alpha \gamma^{\alpha\beta} \gamma^{\gamma\delta}(e^{-\frac{4}{3}\phi} C_\gamma C_\delta)
+ e^{\frac{2}{3}\phi} + e^{\frac{2}{3}\phi} c_\alpha \gamma^{\alpha\beta} C_\beta + e^{\frac{2}{3}\phi} + e^{\frac{2}{3}\phi} c_\alpha = 0
$$

(2.16)

that for $c_\alpha = C_\alpha$ simplifies as

$$
-\frac{1}{2} \gamma^{\beta\alpha}(e^{\frac{2}{3}(\phi-\varphi)} g_{\beta\alpha} + e^{\frac{2}{3}(\phi-\varphi)} - 1) + e^{\frac{2}{3}(\varphi-\phi)} g_{\beta\alpha} = 0 .
$$

(2.17)

Then in order to solve (2.15) we presume that $\phi = \varphi$ and hence (2.15) gives $\gamma^{\alpha\beta} g_{\beta\alpha} = 2$. Inserting this result into (2.17) we finally obtain

$$
\gamma^{\alpha\beta} = g^{\alpha\beta} .
$$

(2.18)

Collecting all these results we find that the action (2.12) reduces into standard Nambu-Goto action for bosonic string

$$
S = -T_F S \int \sqrt{-g}
$$

(2.19)

which we wanted to show.

The main goal of this section was to demonstrate how to perform dimensional reduction with auxiliary world-sheet metric. In the next section we apply this procedure for null dimensional reduction of M-theory.

3. Non-Relativistic String

In this section we proceed to the analysis of null dimensional reduction of M-theory and corresponding extended probe which is M2-brane. To do this we presume that the background metric has light-like isometry [31, 32], for recent extended analysis, see [14]

$$
\begin{align*}
 ds^2 &= g_{MN} dx^M dx^N = 2T_\mu dx^\mu (du - M_\nu dx^\nu) + h_{\mu\nu} dx^\mu dx^\nu, \\
g_{\mu\nu} &= T_\mu , \quad g_{\mu\nu} \equiv \tilde{H}_{\mu\nu} = H_{\mu\nu} - M_\mu T_\nu - T_\mu M_\nu ,
\end{align*}
$$

(3.1)

where $\mu, \nu = 0, 1, \ldots, 9$ and where $u$—is light-like coordinate.

In this section we consider M2-brane extended along light-like direction $u$. To do this we use Polyakov form of M2-brane action [23] where we now presume that three dimensional world-volume metric has light-like isometry so that it can be written in the form

$$
\gamma^{\alpha\mu} = \tau_\alpha , \quad \gamma^{\alpha\beta} = h_{\alpha\beta} - m_\alpha \tau_\beta - \tau_\alpha m_\beta .
$$

(3.2)
Then we have

$$\det \gamma = \begin{vmatrix} 0 & \tau_\beta \\ \tau_\alpha & h_{\alpha\beta} - m_\alpha \tau_\beta - m_\beta \tau_\alpha \end{vmatrix} = \begin{vmatrix} 0 & \tau_\beta \\ \tau_\alpha & h_{\alpha\beta} \end{vmatrix} = \det(-\tau_\alpha \tau_\beta + h_{\alpha\beta}) .$$ (3.3)

We further have an inverse metric

$$\gamma^{uu} = 2\Phi, \quad \gamma^{ua} = -\hat{v}^a, \quad \gamma^{\alpha\beta} = h^{\alpha\beta},$$ (3.4)

where these fields obey the relation

$$h^{\alpha\beta} \hat{h}_{\beta\gamma} - \tau_\gamma v^\alpha = \delta^\alpha_\gamma, \quad \tau_\alpha v^\alpha = -1, \quad h_{\alpha\beta} v^\beta = h^{\alpha\beta} \tau_\beta = 0 ,$$ (3.5)

and where

$$\hat{v}^a = v^a - h^{\alpha\beta} m_\beta, \quad \Phi = -v^a m_\alpha + \frac{1}{2} h^{\alpha\beta} m_\alpha m_\beta .$$ (3.6)

Finally using the fact that the induced metric has the form

$$g_{uu} = 0, \quad g_{ua} = g_{\mu\nu} \partial_\alpha x^\mu = T_\alpha, \quad g_{\alpha\beta} = \hat{H}_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu$$ (3.7)

we obtain following action

$$S = -\frac{T_{M2}}{2} \int du \int d^2\xi \sqrt{-\det(-\tau_\alpha \tau_\beta + h_{\alpha\beta})} \times (-2\hat{v}^a T_\alpha + h^{\alpha\beta} \hat{H}_{\beta\alpha} - 1) .$$ (3.8)

Generally $\int du \to \infty$ since $u$ is non-compact coordinate. Then we can rescale M2-brane tension as $T_{M2} = \frac{T_{FS}}{\int du}$ so that

$$T_{FS} = T_{M2} \int du .$$ (3.9)

In summary, we have an action for non-relativistic string in the form

$$S = -\frac{T_{FS}}{2} \int d^2\xi \sqrt{-\det(-\tau_\alpha \tau_\beta + h_{\alpha\beta})(-2\hat{v}^a T_\alpha + h^{\alpha\beta} \hat{H}_{\beta\alpha} - 1) .}$$ (3.10)

This is final form of non-relativistic string that is defined by double dimensional reduction of M2-brane along null direction.

The structure of the action (3.10) suggests that it has non-relativistic form on the world-sheet due to the presence of two dimensional Newton-Cartan metric. However in order to gain more insight into its structure let us solve equations of motion for $h^{\alpha\beta}, v^a$ and $m_\alpha$. To do this we use the fact that the matrix $A^{\alpha\beta} = -\tau_\alpha \tau_\beta + h_{\alpha\beta}$ is non-singular and hence it has an inverse matrix in the form $A^{\alpha\beta} = -v^\alpha v^\beta + h^{\alpha\beta}$ so that

$$\det A^{\alpha\beta} = \frac{1}{\det A^{\alpha\beta}} .$$ (3.11)

Then we can replace determinant $\det A^{\alpha\beta}$ with $\frac{1}{\det A^{\alpha\beta}}$ and hence (3.10) can be written in the form

$$S = -\frac{T_{FS}}{2} \int d^2\xi \frac{1}{\sqrt{-\det A^{\alpha\beta}}}(-2\hat{v}^a T_\alpha + h^{\alpha\beta} \hat{H}_{\beta\alpha} - 1) .$$ (3.12)
Now we are ready to perform variation of this action with respect to \( v^\alpha, h^{\alpha\beta} \) and \( m_\alpha \) treating them as independent variables. Firstly we get equation of motion for \( v^\alpha \)

\[
\tau_\alpha (-2 \dot{v}^\beta T_\beta + h^{\gamma\delta} \dot{H}_{\gamma\delta} - 1) - 2T_\alpha = 0 .
\]  

(3.13)

Further, equations of motion for \( h^{\alpha\beta} \) have the form

\[
(-\tau_\alpha \tau_\beta + h_{\alpha\beta})(-2 \dot{v}^\gamma T_\gamma + h^{\gamma\delta} \dot{H}_{\gamma\delta} - 1) - 2(H_{\alpha\beta} + (m_\alpha - M_\alpha)T_\beta + T_\alpha (m_\alpha - M_\beta)) = 0
\]

and finally equations of motion for \( m_\alpha \) has the form

\[
h^{\alpha\beta} T_\beta = 0 .
\]  

(3.15)

This is very important equation that has natural solution when we identify

\[
\tau_\alpha = T_\alpha .
\]  

(3.16)

Then the equation (3.13) reduces into simple equation

\[
h^{\alpha\beta} H_{\beta\alpha} - 1 = 0
\]

(3.17)

This equation has clearly solution in the form

\[
h_{\alpha\beta} = H_{\alpha\beta} - k_\alpha \tau_\beta - \tau_\alpha k_\beta ,
\]

(3.18)

where \( k_\alpha = K_\mu \partial_\alpha x^\mu \) is arbitrary two dimensional vector. To see this let us consider the defining equation

\[
h^{\alpha\beta} h_{\beta\gamma} - v^\alpha \tau_\gamma = \delta^\alpha_\gamma .
\]  

(3.19)

Taking its trace we get

\[
h^{\alpha\beta} h_{\beta\alpha} - v^\alpha \tau_\alpha = 2
\]

(3.20)

that implies

\[
h^{\alpha\beta} h_{\beta\alpha} = 1
\]

(3.21)

using the fact that \( v^\alpha \tau_\alpha = 1 \). Then we see that the equation (3.14) has the form

\[
-2\tau_\alpha \tau_\beta - 2(m_\alpha - M_\alpha)T_\beta - 2T_\alpha (m_\beta - M_\beta + k_\beta) = 0
\]

(3.22)

that has solution

\[
m_\alpha = M_\alpha + \frac{T_\alpha}{2} - k_\alpha .
\]  

(3.23)

Inserting (3.16) and (3.18) into the action (3.14) we find its square-root form

\[
S = -T_{FS} \int d^2 \xi \sqrt{-\det A_{\alpha\beta}} = T_{FS} \int d^2 \xi \sqrt{-\det(-T_\alpha T_\beta + H_{\alpha\beta} - k_\alpha T_\beta - T_\alpha k_\beta)} .
\]

(3.24)

This is the final form of the action for fundamental string in theory that arises through null reduction of M-theory. We see that it has Nambu-Goto form despite of the fact that its Polyakov like form contains world-sheet metric that is non-relativistic. Further, we see that there is family of Newton-Cartan metrics corresponding to given null background due to the presence of the arbitrary vector \( k_\alpha \). This is characteristic property of the Newton-Cartan background that is defined with the help of null reduction of higher dimensional background.
4. Hamiltonian Formalism

In this section we find Hamiltonian formulation of the non-relativistic string with the action given in (3.24). From (3.24) we obtain following conjugate momenta

\[ p_\mu = -T_{FS} A_{\mu \nu} \partial_\beta x^\nu \, \mathcal{A}^{\beta 0} \sqrt{- \det \mathcal{A}} \, , \tag{4.1} \]

where \( A_{\mu \nu} = -T_\mu T_\nu + H_{\mu \nu} + k_\mu T_\nu + T_\mu k_\nu \). It is important to stress that the matrix \( A_{\mu \nu} \) is non-singular and hence it has an inverse matrix equal to

\[ A^{\mu \nu} = H^{\mu \nu} - \frac{1}{1 + 2\Phi} \hat{V}^\mu \hat{V}^\nu \, , \tag{4.2} \]

where

\[ \hat{V}^\mu = V^\mu - H^{\mu \nu} k^\nu \, , \quad \Phi = -k_\mu V^\mu + \frac{1}{2} k_\mu H^{\mu \nu} k^\nu \, . \tag{4.3} \]

Then the bare Hamiltonian is equal to

\[ H_B = p_\mu \partial_0 x^\mu - \mathcal{L} = -T_{FS} A_{0 \beta} \mathcal{A}^{\beta 0} + \sqrt{- \det \mathcal{A}} = 0 \tag{4.4} \]

as we should expect for theory that is invariant under two dimensional diffeomorphism. On the other hand we have two primary constraints that follow from definition of momenta \( p_\mu \)

\[ \mathcal{H}_S = p_\mu \partial_1 x^\mu \approx 0 \tag{4.5} \]

and also

\[ \mathcal{H}_T = p_\mu A^{\mu \nu} p_\nu + T_{FS} A_{\mu \nu} \partial_1 x^\mu \partial_1 x^\nu \approx 0 \, . \tag{4.6} \]

Then the total Hamiltonian is the sum of two primary constraints

\[ H = \int d\xi (N^S \mathcal{H}_S + N^T \mathcal{H}_T) \, . \tag{4.7} \]

It would be simple exercise to show that \( \mathcal{H}_S, \mathcal{H}_T \) are first class constraints which again reflect invariance of the action under two dimensional diffeomorphism.

5. Transverse dimensional reduction

Let us consider M2-brane that is transverse to the light-like direction. In this case the induced metric has the form

\[ \tilde{g}_{\dot{\alpha} \dot{\beta}} = \dot{H}_{\dot{\alpha} \dot{\beta}} + T_\dot{\alpha} \partial_{\dot{\beta}} \dot{u} + \partial_{\dot{\alpha}} T_{\dot{\beta}} \]

so that the action (2.3) has the form

\[ S = -\frac{T_{M2}}{2} \int d^3 \xi \sqrt{-\gamma} (\dot{H}_{\dot{\alpha} \dot{\beta}} + \partial_{\dot{\alpha}} T_{\dot{\beta}} + T_\dot{\alpha} \partial_{\dot{\beta}} \dot{u} - 1) \, . \tag{5.2} \]
We would like to introduce vector field dual to \( \partial_\alpha u \) which is not possible directly in the action (5.2) since there are no terms quadratic in \( \partial_\alpha u \). In order to have terms quadratic in \( \partial_\alpha u \) let us introduce two auxiliary fields \( A_\alpha, B_\alpha \) and consider following expression

\[
\sqrt{-\gamma}(\gamma^{\alpha\beta}\partial_\alpha u \partial_\beta u + A_\alpha \gamma^{\alpha\beta}\partial_\beta u + B_\alpha \gamma^{\alpha\beta}\partial_\beta u + A_\alpha \gamma^{\alpha\beta}B_\beta)
\]

(5.3)

that we add to the action (5.2). Let us now consider equations of motion for \( A_\alpha, B_\alpha \) that follow from (5.3)

\[
\partial_\alpha u + B_\alpha = 0, \quad \partial_\alpha u + A_\alpha = 0.
\]

(5.4)

Inserting this result into (5.3) we see that this expression is zero and hence original and extended action for M2-brane are the same when \( A_\alpha, B_\alpha \) are on-shell. However thanks to the presence of the term quadratic in derivatives of \( u \) we can dualize scalar \( u \). To do this let us now introduce \( Y_\tilde{\alpha} = \partial_\alpha u \). Clearly this vector obeys

\[
e^{\tilde{\alpha}\tilde{\beta}\gamma} \partial_\beta Y_\gamma = 0,
\]

(5.5)

where \( e^{\tilde{\alpha}\tilde{\beta}\gamma} \) is totally antisymmetric symbol with \( e^{012} = 1 \). Let us now interpret \( Y_\tilde{\alpha} \) as an independent field while condition (5.3) is replaced by following term in the action

\[
T_{M2} \int d^3 \xi e^{\tilde{\alpha}\tilde{\beta}\gamma} \partial_\alpha A_\beta Y_\gamma = \frac{T_{M2}}{2} \int d^3 \xi e^{\tilde{\alpha}\tilde{\beta}\gamma}(\partial_\alpha A_\beta - \partial_\beta A_\alpha)Y_\gamma,
\]

(5.6)

where now variation with respect to \( A_\beta \) gives (5.5). In summary, the extended form of M2-brane action has the form

\[
S = -\frac{T_{M2}}{2} \int d^3 \xi \sqrt{-\gamma}(\gamma^{\alpha\beta}\tilde{H}_{\alpha\beta} + Y_\alpha T_\beta + T_\alpha Y_\beta) + \\
+\gamma^{\alpha\beta}Y_\alpha Y_\beta + A_\alpha \gamma^{\alpha\beta}Y_\beta + B_\alpha \gamma^{\alpha\beta}Y_\beta + A_\alpha \gamma^{\alpha\beta}B_\beta - 1) + \frac{T_{M2}}{2} \int d^3 \xi e^{\tilde{\alpha}\tilde{\beta}\gamma}F_{\alpha\beta}Y_\gamma,
\]

(5.7)

where \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \). Finally we eliminate \( Y_\tilde{\alpha} \) by solving their equations of motion that follow from the action above and we get

\[
-\sqrt{-\gamma} \gamma^{\alpha\beta}(Y_\beta + T_\beta + \frac{1}{2}A_\beta + \frac{1}{2}B_\beta) + \frac{1}{2}e^{\tilde{\alpha}\tilde{\beta}\gamma}F_{\beta\gamma} = 0.
\]

(5.8)

From this equation we can express \( Y_\alpha \) as

\[
Y_\alpha = \frac{1}{2\sqrt{-\gamma}}\gamma^{\alpha\tilde{\alpha}\tilde{\beta}\gamma}F_{\beta\gamma} - \frac{1}{2}(A_\alpha + B_\alpha + 2T_\alpha).
\]

(5.9)

Then inserting this result into the action (5.7) we obtain

\[
S = -\frac{T_{M2}}{2} \int d^3 \xi \sqrt{-\gamma}(\gamma^{\alpha\beta}\tilde{H}_{\alpha\beta} - \frac{1}{2}F_{\alpha\beta} \gamma^{\beta\gamma}F_{\gamma\gamma}^\alpha) - \\
-\frac{1}{4}(A - B)\alpha \gamma^{\alpha\beta}(A - B)\beta - (A + B)\alpha \gamma^{\alpha\beta}T_\beta - T_\alpha \gamma^{\alpha\beta}T_\beta - 1) - \frac{T_{M2}}{4} \int d^3 \xi e^{\tilde{\alpha}\tilde{\beta}\gamma}F_{\alpha\beta}(A_\gamma + B_\gamma + 2T_\gamma)
\]

(5.10)

\footnote{Similar procedure can be found in \cite{34, 35}.}
using the fact that
\[ \frac{1}{(\sqrt{-g})^2} F_{\gamma\beta} e^{\gamma\delta\alpha} e^{\alpha\delta\beta} F_{\gamma\delta} = -2 F_{\alpha\beta} \gamma^{\alpha\delta} \gamma^{\beta\delta} F_{\alpha\beta} . \] (5.11)

As the next step we introduce two fields \( X_{\bar{\alpha}} = A_{\bar{\alpha}} - B_{\bar{\alpha}} \) and \( 2Z_{\bar{\alpha}} = A_{\bar{\alpha}} + B_{\bar{\alpha}} \). We see that the equations of motion for \( X_{\bar{\alpha}} \) implies that \( X_{\bar{\alpha}} = 0 \). Then the action has the form
\[
S = -\frac{T_{M2}}{2} \int d^3 \xi \sqrt{-\gamma} (\gamma^{\alpha\bar{\beta}} (T_{\alpha\beta} + H_{\alpha\bar{\beta}} - (M_{\bar{\alpha}} + Z_{\bar{\alpha}}) T_{\beta} + T_{\alpha} (M_{\beta} + Z_{\beta})) - \frac{1}{2} F_{\alpha\beta} \gamma^{\beta\delta} F_{\beta\gamma\delta} \gamma^{\alpha\bar{\delta}} - 1) - \frac{T_{M2}}{2} \int d^3 \xi \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} (Z_{\gamma} + T_{\gamma}) .
\] (5.12)

Note that this is similar form of the action as was derived in \(^{[36]}\) in case of standard dimensional reduction of M-theory.

Finally we should solve equations of motion for \( \gamma^{\alpha\bar{\beta}} \) that follow from the action above
\[
-\frac{1}{2} \gamma^{\alpha\bar{\beta}} \gamma^{\beta\delta} \gamma^{\alpha\bar{\delta}} H_{\beta\gamma\delta} - \frac{1}{2} F_{\alpha\beta} \gamma^{\beta\delta} F_{\beta\gamma\delta} \gamma^{\alpha\bar{\delta}} - 1 + H_{\alpha\bar{\beta}} - F_{\gamma\alpha} F_{\beta\bar{\alpha}} \gamma^{\bar{\delta} \gamma} = 0 ,
\] (5.13)
where we defined
\[
H_{\alpha\bar{\beta}} = T_{\alpha\beta} + H_{\alpha\bar{\beta}} - (M_{\bar{\alpha}} + Z_{\bar{\alpha}}) T_{\beta} - (M_{\beta} + Z_{\beta}) T_{\alpha} .
\] (5.14)

In the leading order approximation it has solution \(^{[36]}\) \( \gamma^{\alpha\bar{\beta}} = H_{\alpha\bar{\beta}} \) so that in the leading order approximation the action has the form
\[
S = -T_{M2} \int d^3 \xi \sqrt{-\det H_{\alpha\bar{\beta}} (1 - \frac{1}{4} F_{\alpha\bar{\alpha}} H^{\alpha\bar{\delta}} F_{\delta\gamma} H^{\gamma\bar{\alpha}}) = -T_{M2} \int d^3 \xi \sqrt{-\det H_{\alpha\bar{\beta}} (1 - \frac{1}{2} F_{\alpha\bar{\alpha}} H^{\alpha\bar{\delta}} F_{\delta\gamma} H^{\gamma\bar{\alpha}}) = -T_{M2} \int d^3 \xi \sqrt{-\det (H_{\alpha\bar{\beta}} + F_{\alpha\bar{\beta}}) ,
\] (5.15)
where in the last step we used relations that hold in three dimensions
\[
\det (H_{\alpha\bar{\beta}} + F_{\alpha\bar{\beta}}) = \det H_{\alpha\bar{\beta}} \det (\delta^{\bar{\beta}}_{\alpha} + F_{\alpha\gamma} H^{\gamma \bar{\beta}}) = \det H_{\alpha\bar{\beta}} \left( 1 - \frac{1}{2} F_{\alpha\bar{\alpha}} H^{\alpha\bar{\delta}} F_{\delta\gamma} H^{\gamma\bar{\alpha}} \right) .
\] (5.16)

The action (5.15) is final form of D2-brane action in non-relativistic string theory that is defined by null dimensional reduction of M-theory. We again observe an interesting fact that given M-theory background with null isometry is mapped into family of Newton-Cartan backgrounds that differ by redefinition of \( M_\mu \). In particular, choosing \( Z_\mu = -M_\mu \) we get the action
\[
S = -T_{M2} \int d^3 \xi \sqrt{-\det (-T_{\alpha\beta} + H_{\alpha\bar{\beta}} + F_{\alpha\bar{\beta}}) - \frac{T_{M2}}{2} \int d^3 \xi \epsilon^{\alpha\beta\gamma} F_{\alpha\beta} (T_{\gamma} - M_{\gamma}) .
\] (5.17)
In summary, we found D2-brane action in the theory that is defined by null dimensional reduction of M-theory. It would be certainly interesting to extend this analysis to the case of M5-brane and its null dimensional reduction and study how the procedure suggested in this paper could be related to the recent discussion of null dimensional reduction of M5-branes presented in [33].

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