A Wick-rotatable metric is purely electric

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Abstract

We show that a metric of arbitrary dimension and signature which allows for a Wick rotation to a Riemannian metric necessarily has a purely electric Riemann and Weyl tensor.

1 Introduction

In quantum theories a Wick rotation is a mathematical trick to relate Minkowski space to Euclidean space by a complex analytic extension to imaginary time. This enables us to relate a quantum mechanical problem to a statistical mechanical one relating time to the inverse temperature. This trick is highly successful and is used in a wide area of physics, from statistical and quantum mechanics to Euclidean gravity and exact solutions.

In spite of its success, there is a question about its range of applicability. A question we can ask is: Given a spacetime, does there exist a Wick rotation to transform the metric to a Euclidean one?

Here we will give a partial answer to this question and will give a necessary condition for a Wick rotation (as defined below) to exist. However, before we prove our main theorem, we need to be a bit more precise with what we mean by a Wick rotation. Consider a pseudo-Riemannian metric (of arbitrary dimension and signature). We need to allow for more general coordinate transformations than the real diffeomorphisms preserving the metric signature – namely to complex analytic continuations of the real metric [1,2].

Consider a point $p$ and a neighbourhood, $U$, of $p$. Assume this neighbourhood is an analytic neighbourhood and that $x^\mu$ are coordinates on $U$ so that $x^\mu \in \mathbb{R}^n$. We will adapt the coordinates to the point $p$ so that $p$ is at the origin of this coordinate system. Consider now the complexification of $x^\mu \mapsto x^\mu + iy^\nu = z^\mu \in \mathbb{C}^n$. This complexification enables us to consider the complex analytic neighbourhood $U^C$ of $p$.

Furthermore, let $g^C_{\mu\nu}$ be a complex bilinear form induced by the analytic extension of the metric:

$$g^C_{\mu\nu}(x^\rho)dx^\mu dx^\nu \mapsto g^C_{\mu\nu}(z^\rho)dz^\mu dz^\nu.$$
Next, consider a real analytic submanifold containing $p$: $U \subset U^C$. The imbedding $\iota : \bar{U} \mapsto U^C$ enables us to pull back the complexified metric $g^C$ onto $\bar{U}$:

$$g \equiv \iota^* g^C.$$  \hfill (1)

In terms of the coordinates $\bar{x}^\mu$: $g = g_{\mu\nu}(\bar{x}^\rho) d\bar{x}^\mu d\bar{x}^\nu$. This bilinear form may or may not be real. However, if the bilinear form $\bar{g}_{\mu\nu}(\bar{x}^\rho) d\bar{x}^\mu d\bar{x}^\nu$ is real (and non-degenerate) then we will call it an analytic extension of $g_{\mu\nu}(x^\rho) d\rho^\mu d\rho^\nu$ with respect to $p$, or simply a Wick rotation of the real metric $g_{\mu\nu}(x^\rho) d\rho^\mu d\rho^\nu$. This clearly generalises the concept of Wick rotations from the standard Minkowskian setting to a more general setting $[10]$.

In the following, let us call the Wick rotation, in the sense above, for $\phi$; i.e., $\phi : U \mapsto \bar{U}$. We note that this transformation is complex, and we can assume, since $U$ is real analytic, that $\phi$ is analytic.

The Wick rotation in the sense above, leaves the point $p$ stationary. It therefore induces a linear transformation, $M$, between the tangent spaces $T_p U$ and $T_{\phi(p)} \bar{U}$. The transformation $M$ is complex and therefore may change the metric signature; consequently, even if the metric $\bar{g}_{\mu\nu}$ is real, it does not necessarily need to have the same signature of $g_{\mu\nu}$.

Consider now the curvature tensors, $R$ and $\nabla^{(k)} R$ for $g_{\mu\nu}$, and $\bar{R}$ and $\bar{\nabla}^{(k)} \bar{R}$ for $\bar{g}_{\mu\nu}$. Since both metrics are real, their curvature tensors also have to be real. The analytic continuation, in the sense above, induces a linear transformation of the tangent spaces; consequently, this would relate the Riemann tensors $R$ and $\bar{R}$ through a complex linear transformation. It is useful to introduce an orthonormal frame $e_\mu$. The orthonormal frames $e_\mu$ and $\bar{e}_\mu$ are related through their complexified frame $\bar{e}^C_\mu$. We can define a complex orthonormal frame requiring the inner product $\bar{g}^{\mu\nu}(\bar{e}^C_\mu, \bar{e}^C_\nu) = \delta_{\mu\nu}$. This inner product is invariant under the complex orthogonal transformations, $O(n, \mathbb{C})$. The real frames $e_\mu$ and $\bar{e}_\mu$ are obtained by restricting the complex frame to real frames; hence, we consider the real vector spaces $T_p U$ and $T_{\phi(p)} \bar{U}$ as embedded in the complexified vector space $(T_p U)^C \cong (T_{\phi(p)} \bar{U})^C$. The real frames are thus related through a restriction of a complex frame having an $O(n, \mathbb{C})$ structure group.

By using $\phi$ we can relate the metrics $g = \phi^* \bar{g}$. Since the map is analytic (albeit complex), the curvature tensors are also related via $\phi$. If $R$ and $\bar{R}$ are the Riemann curvature tensors for $U$ and $\bar{U}$ respectively, then these are related, using an orthonormal frame, via an $O(n, \mathbb{C})$ transformation. Considering the components of the Riemann tensor as a vector in some $\mathbb{R}^N \subset \mathbb{C}^N$, then if there exists a Wick rotation of the metric at $p$, then the (real) Riemann curvature tensors of $U$ and $\bar{U}$ must lie in the same $O(n, \mathbb{C})$ orbit in $\mathbb{C}^N$.

**Note:** This definition of a Wick rotation does not include the more general analytic continuations defined by Lozanovski $[1]$. In particular, we consider one particular metric (thus not a family of them) and we require that the point $p$ is fixed and is therefore more of a complex rotation.

In the following we will utilise the study of real orbits of semi-simple groups, see e.g. $[5, 6]$. In particular, the considerations made in $[5]$ will be useful. For a

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1 This is a not really a proper inner product since it is not positive definite, but rather a $\mathbb{C}$-bilinear non-degenerate form.
more general introduction to the structure of Lie algebras including the Cartan involution, see, for example [7, 8].

2 The electric/magnetic parts of a tensor

Following [3], we can introduce the electric and magnetic parts of a tensor by considering the eigenvalue decomposition of the tensor under the Cartan involution $\theta$ of the real Lie algebras $\mathfrak{o}(p, q)$. This involution can be extended to all tensors, and to vectors $v \in T_p M$ in particular. Considering an orthonormal frame, so that:

$g(e_\mu, e_\mu) = \begin{cases} -1, & 1 \leq \mu \leq p \\ +1, & p + 1 \leq \mu \leq p + q = n, \end{cases}$

then the $\theta : T_p M \to T_p M$, can be defined as the linear operator:

$$\theta(e_\mu) = \begin{cases} -e_\mu, & 1 \leq \mu \leq p \\ +e_\mu, & p + 1 \leq \mu \leq p + q = n. \end{cases}$$

Clearly, this implies that the bilinear map:

$$(X, Y)_\theta := g(\theta(X), Y), \quad X, Y \in T_p M$$

defines a positive definite inner-product on $T_p M$. This Cartan involution can be extended to arbitrary tensor products.

Given a Cartan involution $\theta$, then since $\theta^2 = \text{Id}$, its eigenvalues are $\pm 1$ and any tensor $T$ has an eigenvalue decomposition:

$$T = T_+ + T_-, \quad \text{where } \theta(T_\pm) = \pm T_\pm.$$ 

A space is called purely electric (PE) if there exists a Cartan involution so that the Weyl tensor decomposes as $C = C_+$. Furthermore, a space is called purely magnetic (PM) if the Weyl tensor decomposes as $C = C_-$. If this property occurs also for the Riemann tensor, we call the space Riemann purely electric (RPE) or magnetic (RPM), respectively. Clearly, RPE implies PE.

3 The Riemann curvature operator

The Riemann curvature tensor can (pointwise) be seen as a bivector operator:

$$\text{Riem} : \bigwedge^2 \Omega_p(M) \to \bigwedge^2 \Omega_p(M).$$

In a pseudo-Riemannian space of signature $(p, q)$ the metric $g$ will provide an isomorphism between the space of bivectors, $\bigwedge^2 \Omega_p(M)$, and the Lie algebra $\mathfrak{g} = \mathfrak{o}(p, q)$. Consequently, the Riemann curvature operator can also be viewed as an endomorphism of $V := \mathfrak{g}$ as a vector space. Consider therefore any $R \in \text{End}(V)$:

$$R : V \to V.$$
This endomorphism can be split in a symmetric and anti-symmetric part, \( R = S + A \), with respect to the metric induced by \( g \) (proportional to the Killing form \( \kappa \) on \( V \)):

\[
g(S(x), y) = g(S(y), x), \quad g(A(x), y) = -g(A(y), x) \quad \forall x, y \in g.
\]

This metric is invariant under the Lie group action of \( G = O(p, q) \):

\[
g(h \cdot x, h \cdot y) = g(x, y),
\]

where \( h \cdot x \) is the natural Lie group action on the Lie algebra given by the adjoint: \( h \cdot x := Ad_h(x) = h^{-1}xh \).

Consider now a Cartan involution \( \theta : g \to g \). Then we define the inner-product on \( V = g \) as follows:

\[
\langle x, y \rangle_\theta = g(\theta(x), y),
\]

which is just proportional to \( \kappa_\theta (-, -) := -\kappa(-, \theta(-)) \). We can now, similarly, split any \( R \in \text{End}(V) \) in a symmetric and anti-symmetric part, \( R = R_+ + R_- \), with respect to the inner-product \( (-, -)_\theta \):

\[
\langle R_+(x), y \rangle_\theta = \langle R_+(y), x \rangle_\theta, \quad \langle R_-(x), y \rangle_\theta = -\langle R_-(y), x \rangle_\theta, \quad \forall x, y \in g.
\]

Let now \( \tilde{\theta} \) be a Cartan involution of another real form of \( \mathfrak{o}(n, \mathbb{C}) \). Assume the real form is \( \tilde{\mathfrak{g}} = \mathfrak{o}(\bar{\tilde{\theta}}, \bar{\tilde{\theta}}) \), with corresponding metric \( \tilde{g} \). We note that the Lie algebras \( g \) and \( \tilde{g} \) are isomorphic as vector spaces (let us call them \( V \) and \( \tilde{V} \), respectively) and both are subspaces of \( V^\mathbb{C} \); i.e., \( V, \tilde{V} \subset V^\mathbb{C} \).

As the space of endomorphisms, \( \text{End}(V) \), is also a vector space with the group action given by conjugation, we can thus define \( \mathcal{V} := \text{End}(V) \), and extend the Cartan involution, \( \theta \), as well as \( g \) tensorially to \( \mathcal{V} \). We define analogously an inner product on \( \mathcal{V} \):

\[
\langle \langle X, Y \rangle \rangle_\theta = g(\theta(X), Y), \quad X, Y \in \mathcal{V}.
\]

Defining \( \tilde{\mathcal{V}} \) similarly, we again have \( \mathcal{V}, \tilde{\mathcal{V}} \subset V^\mathbb{C} \).

Assume now that \( g = \mathfrak{o}(p, q) \) and \( \tilde{V} = \mathfrak{o}(n) \) (the compact real form of \( V^\mathbb{C} = \mathfrak{o}(n, \mathbb{C}) \)). Let \( \theta \) be a Cartan involution of \( \mathfrak{g} \) with Cartan decomposition \( \mathfrak{g} = T_0 \oplus P_0 \). Set \( C = T_0 \oplus iP_0 \) to be the compact real form of \( g^\mathbb{C} \). We will assume that \( \tilde{g} \) is embedded as a Lie algebra in \( \tilde{V}^\mathbb{C} \) by identifying the compact real forms \( C \) and \( \tilde{V} \) together. Define \( \tilde{V} \) to be the vector space copy of \( \tilde{g} \) in \( \tilde{V}^\mathbb{C} \). It follows that we can choose a Cartan involution:

\[
\theta : \tilde{V}^\mathbb{C} \to \tilde{V}^\mathbb{C}
\]

of \( V \) such that \( \theta_{|_V} = 1_V \). Let now \( \mathfrak{o}(p, q) \) be equipped with the metric \( \langle -, - \rangle_\theta \) as described earlier.

In what follows, we will consider the real orbits, \( \mathcal{O}(R) \) and its complexified orbit \( \mathcal{O}_C(R) \), defined by the action of the group on \( R \) as follows:

\[
\mathcal{O}(R) := \{ h \cdot R \mid h \in O(p, q) \} \subset \mathcal{V}
\]

\[
\mathcal{O}_C(R) := \{ h \cdot R \mid h \in O(n, \mathbb{C}) \} \subset \mathcal{V}^\mathbb{C}.
\]
There are many examples of purely electric spaces (see [3, 4] and references therein). In particular, a purely electric Lorentzian spacetime is of type G.

\[ \text{Corollary 3.2. A metric (of arbitrary dimension and signature) allowing for a Wick rotation at a point } p, \text{ has a purely electric Riemann tensor, and is consequently purely electric at } p. \]

4 Discussion

Using techniques from real invariant theory we have considered a class of metrics allowing for a complex Wick-rotation to a Riemannian space. We have showed that these necessarily are restricted, in particular, they are purely electric. The result is independent of dimension and signature and shows that if such a Wick rotation is allowable, then we necessarily restrict ourselves to classes of spaces where the "magnetic" degrees of freedom have to vanish (at the point \( p \)).

There are many examples of purely electric spaces (see [3, 4] and references therein). In particular, a purely electric Lorentzian spacetime is of type G,
Thus spacetimes not of these types provide with examples of spaces where such a Wick rotation is disallowed. Non-Wick-rotatable metrics include the classes of Kundt metrics [9] in Lorentzian geometry, and the Walker metrics [11] of more general signature. Also the metrics considered in [12] are in general non-Wick-rotatable metrics. Note that the plane-wave metrics are non-Wick-rotatable metrics.

It is clear that these results have profound consequences for quantum theories where such Wick-rotation is widely used. This result gives a clear restriction of the class of metrics that allows for such a Wick rotation. Clearly, also in the context of quantum gravity, the (real) gravitational degrees of freedom will be restricted by assuming the existence of such a Wick-rotation.

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