TROPICAL COHOMOLOGY WITH INTEGRAL COEFFICIENTS FOR ANALYTIC SPACES

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ABSTRACT. We study tropical Dolbeault cohomology for Berkovich analytic spaces, as defined by Chambert-Loir and Ducros. We provide a construction that lets us pull back classes in tropical cohomology to classes in tropical Dolbeault cohomology as well as check whether these classes are non-trivial. We further define tropical cohomology with integral coefficients on the Berkovich space and provide some computations. Our main tool is extended tropicalization of toric varieties as introduced by Kajiwara and Payne.

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1. INTRODUCTION

Real valued differential forms and currents on Berkovich analytic spaces were introduced by Chambert–Loir and Ducros in their fundamental preprint [CLD12]. They provide a notion of bigraded differential forms and currents on these spaces that has striking similarities with the complex of smooth differential forms on complex analytic spaces. The definition works by formally pulling back Lagerberg’s superforms on \( \mathbb{R}^n \) along tropicalization maps. These tropicalization maps are induced by mapping open subsets of the analytic space to analytic tori and then composing with the tropicalization maps of the tori.

Payne, and independently Kajiwara [Pay09, Kaj08], generalized this tropicalization procedure from tori to general toric varieties and Payne showed that the Berkovich analytic space is the inverse limit over all these tropicalizations.

Shortly after the preprint by Chambert–Loir and Ducros, Gubler showed that one may, instead of considering arbitrary analytic maps to tori, restrict ones attention to algebraic closed embeddings if the analytic space is the Berkovich analytification of an algebraic variety [Gub16].

Let \( K \) be a field that is complete with respect to a non-archimedean absolute value and let \( X \) be a variety over \( K \). We write \( \Gamma = \log |K^*| \) for the value group of \( K \) and \( X^{an} \) for the Berkovich analytification of \( X \). Both the approach by Gubler and the one by Chambert–Loir and Ducros provide bigraded complexes of sheaves of differential forms \( (\mathcal{A}^{\bullet, \bullet}, d', d'') \) on \( X^{an} \). We denote by \( H^{\bullet, \bullet} \) (resp. \( H^{\bullet, \bullet}_c \)) the cohomology of the complex of global sections (resp. global sections with compact support) with respect to \( d'' \).

In this paper, we generalize Gubler’s approach, showing that one can define forms on Berkovich analytic spaces by using certain classes of embeddings of open subsets into toric varieties. Given a fine enough family of tropicalizations \( \mathcal{S} \) (see Section 3 for the definition of this notion) we obtain a bigraded complex of sheaves \( (\mathcal{A}^{\bullet, \bullet}_S, d', d'') \) on \( X^{an} \). We show that for many useful \( S \), our complex \( \mathcal{A}^{\bullet, \bullet}_S \) is canonically isomorphic to \( \mathcal{A}^{\bullet, \bullet} \).

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For the rest of the introduction, we make the very mild assumption the $X$ is normal and admits at least one closed embedding into a toric variety.

The general philosophy of this paper and also the definition of forms by Chambert-Loir and Ducros and Gubler is that we can transport constructions done for tropical varieties to Berkovich spaces by locally pulling back along tropicalizations. While Chambert-Loir, Ducros and Gubler used only tropicalization maps of tori, we will also allow tropicalization maps of general toric varieties. We will show advantages of this equivalent approach throughout the paper.

The definitions by both Chambert-Loir and Ducros as well as Gubler work with local embeddings. We show that we can also work with global embeddings. Our constructions provides us with the following: Let $\varphi : X \to Y_\Sigma$ be a closed embedding into a toric variety. Then we obtain pullback morphisms

\begin{align}
(1.1) & \quad \text{trop}^* : H^{p,q}(\text{Trop}(\varphi(X))) \to H^{p,q}(X^\text{an}) \quad \text{and} \\
(1.2) & \quad \text{trop}^* : H^{p,q}_c(\text{Trop}(\varphi(X))) \to H^{p,q}_c(X^\text{an})
\end{align}

in cohomology.

Note that (1.1) and (1.2) were not in general available in the approaches by Chambert–Loir and Ducros resp. Gubler. This construction allows us to explicitly construct classes in tropical Dolbeault cohomology.

For cohomology with compact support, we even obtain all classes this way:

**Theorem 1.1** (Theorem 7.2). We have

$$H^{p,q}_c(X^\text{an}) = \lim_{\varphi : X \to Y_\Sigma} H^{p,q}_c(\text{Trop}(\varphi(X))).$$

where the limit runs over all closed embeddings of $X$ into toric varieties.

We also show that the analogous result for $H^{p,q}$ is not true (Remark 7.3).

Further, in certain cases, we can check whether one of these classes is non-trivial on the tropical side.

**Theorem 1.2** (Theorem 8.2). Assume that $\text{Trop}(\varphi(X))$ is smooth. Then (1.1) and (1.2) are both injective.

We exhibit three examples in Section 8, namely Mumford curve, curves of good reduction and toric varieties.

Another construction that we transport over from the tropical to the analytic world is cohomology with coefficients other than the real numbers. For a subring $R$ of $\mathbb{R}$, we define a cohomology theory

$$H^{*,*}_\text{trop}(X^\text{an}, R) \quad \text{and} \quad H^{*,*}_\text{trop,c}(X^\text{an}, R)$$

with values in $R$-modules. Liu introduced in [Liu17] a canonical rational subspace $H^{p,q}(X^\text{an})_\mathbb{Q}$ of $H^{p,q}(X^\text{an})$. We show that this space agrees with $H^{p,q}(X^\text{an}, \mathbb{Q})$ as defined in this paper (Proposition 7.18).

We obtain the analogue of Theorem 1.1 where on the right hand side we have tropical cohomology with coefficients in $R$ (Proposition 7.7), and we provide an explicit isomorphism

$$d\mathbb{R} : H^{p,q}(X^\text{an}) \to H^{p,q}_\text{trop}(X^\text{an}, \mathbb{R}).$$

which is a version of de Rham’s theorem in this context (Theorem 7.15). Liu introduced in [Liu19] a monodromy operator

$$M : H^{p,q}_\text{trop,c}(X^\text{an}) \to H^{p-1,q+1}_\text{trop,c}(X^\text{an})$$
that respects $H^{*,*}(X^{an})_\mathbb{Q}$ if $\log |K^*| \subset \mathbb{Q}$ [Liu19, Theorem 5.5 (1)]. Mikhalkin and Zharkov introduce in [MZ14] a wave operator
\[ W: H^p_c(X,\mathbb{Q}) \to H^{p-1,q+1}(X^{an},\mathbb{R}). \]
Note that both these operators are also available without compact support.

We show that $W$ can be used to give an operator on $H^{*,*}(X^{an})$ and that this operator agrees with $M$ up to sign in Corollary 7.16.

We also obtain the following result regarding the interaction between the wave operator and the coefficients of the cohomology groups:

**Theorem 1.3.** Let $R$ be a subring of $\mathbb{R}$ and $R[\Gamma]$ the smallest subring of $\mathbb{R}$ that contains both $R$ and $\Gamma$. Then the wave operator $W$ restricts to a map
\[ W: H^p_c(X^{an},R) \to H^{p-1,q+1}(X^{an},R[\Gamma]). \]
As $W$ and $M$ agree up to sign and $H^{*,*}(X^{an},\mathbb{Q})$ agrees with $H^{*,*}(X^{an})_\mathbb{Q}$, this generalizes Liu's result for $\Gamma \subset R = \mathbb{Q}$.

We now sketch the organization of the paper. In Section 2 we recall background on toric varieties and their tropicalizations. In Section 3 we consider what we call families of tropicalizations which is what we will use to define forms on Berkovich spaces. We give the definitions and some examples of families that we will consider. In Section 4 we define for a fine enough family of tropicalizations $S$ a bigraded complex $A^{\bullet,\bullet}_S$ of sheaves of differential forms on Berkovich spaces. We also provide some conditions under which those complexes are isomorphic for different $S$. In Section 5, we prove that for so called admissible families $S$, the complexes $A^{\bullet,\bullet}_S$ are canonically isomorphic to $A^{\bullet,\bullet}$. In Section 6 we discuss integration of top-dimensional differential forms with compact support. In Section 7 we introduce tropical cohomology with coefficients for $X^{an}$ and compare it with $H^{*,*}$. Section 8 provides partial computations of $H^{p,q}(X^{an})$ and $H^{p,q}(X^{an},R)$ for curves and toric varieties, using our new approaches. In Section 9 we list open questions that one might ask as a consequence of our results. The Appendix A contains all constructions on tropical varieties that are needed for the paper. Most of these should be known to experts, however we still chose to list them for completeness. The main new result is the identification of the wave and monodromy operator, which is based on Lemma A.13.

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Parts of this work already appeared in a more a hoc and less conceptual way in the author’s PhD thesis [Jel16a].

**Notations and conventions**

Throughout $K$ is a field that is complete with respect to a (possibly trivial) non-archimedean absolute value. We denote its value group by $\Gamma := \log |K^*|$. If the absolute value is non-trivial, we normalize it in such a way that $\mathbb{Z} \subset \Gamma$. A variety $X$ is an geometrically integral separated $K$-scheme of finite type. For any variety $X$ over $K$, we will throughout the paper denote by $X^{an}$ the analytification in the sense of Berkovich [Ber90].
2. Toric varieties and tropicalization

2.1. Toric varieties. Let $N$ be a free abelian group of finite rank, $M$ its dual and denote by $N_\mathbb{R}$ resp. $M_\mathbb{R}$ the respective scalar extensions to $\mathbb{R}$.

Definition 2.1. A rational cone $\sigma \in N_\mathbb{R}$ is a polyhedron defined by equations of the form $\varphi(.) \geq 0$ with $\varphi \in M$, that does not contain a positive dimensional linear subspace. A rational fan $\Sigma$ in $N_\mathbb{R}$ is a polyhedral complex all of whose polyhedra are rational cones. For $\sigma \in \Sigma$ we define the monoid

$$S_\sigma := \{ \varphi \in M \mid \varphi(v) \geq 0 \text{ for all } v \in \sigma \}.$$  

We denote by $U_\sigma := \text{Spec}(K[S_\sigma])$. For $\tau \prec \sigma$ we obtain an open immersion $U_\tau \to U_\sigma$. We define the toric variety $Y_\Sigma$ to be the gluing of the $(U_\sigma)_{\sigma \in \Sigma}$ along these open immersions. For an introduction to toric varieties, see for example [Ful93].

Remark 2.2. The toric variety $Y_\Sigma$ comes with an open immersion $T \to Y_\Sigma$, where $T = \text{Spec}(K[M])$ and a $T$-action that extends the group action of $T$ on itself by translation. In fact any normal variety with such an immersion and action arises by the above described procedure ([CLS11, Corollary 3.1.8]). This was shown by Sumihiro.

Choosing a basis of $N$ gives an identification $N \cong \mathbb{Z}^r \cong M$ and $T \cong G_m^r$.

Definition 2.3. A map $\psi : Y_\Sigma \to Y_{\Sigma'}$ is called a morphism of toric varieties if it is equivariant with respect to the torus actions and restricts to a morphism of algebraic groups on dense tori. It is called an affine map of toric varieties if it is a morphism of toric varieties composed with a multiplicative torus translation.

Remark 2.4. A morphism of toric varieties $\psi : Y_\Sigma \to Y_{\Sigma'}$ is induced by a morphism of corresponding fans, meaning a linear map $N \to N'$ that maps cones in $\Sigma$ to cones in $\Sigma'$. 

2.2. Tropical toric varieties. Let $\Sigma$ be a rational fan in $N_\mathbb{R}$. We write $\mathbb{T} := \mathbb{R} \cup \{-\infty\}$. For $\sigma \in \Sigma$ we define $N(\sigma) := N_\mathbb{R}/(\sigma)_\mathbb{R}$. We write

$$N_\Sigma = \bigsqcup_{\sigma \in \Sigma} N(\sigma).$$

We call the $N(\sigma)$ the strata of $N_\Sigma$. Note that $N_\Sigma$ has a canonical action by $N$ and $N_\mathbb{R}$ and the strata are the strata of the action of $N_\mathbb{R}$. We endow $N_\Sigma$ with a topology in the following way:

For $\sigma \in \Sigma$ write $N_\sigma = \bigsqcup_{\tau \prec \sigma} N(\tau)$. This is naturally identified with $\text{Hom}_{\text{Monoids}}(S_\sigma, \mathbb{T})$.

We equip $\mathbb{T}^{S_\sigma}$ with the product topology and give $N_\sigma$ the subspace topology. For $\tau \prec \sigma$, the space $\text{Hom}(S_\tau, \mathbb{T})$ is naturally identified with the open subspace of $\text{Hom}_{\text{Monoids}}(S_\sigma, \mathbb{T})$ of maps that map $\tau \perp M$ to $\mathbb{R}$. We define the topology of $N_\Sigma$ to be the one obtained by gluing along these identifications.

Definition 2.5. We call the space $N_\Sigma$ a tropical toric variety.

Note that $N_\Sigma$ contains $N_\mathbb{R}$ as a dense open subset. For a subgroup $\Gamma$ of $\mathbb{R}$ and each stratum $N(\sigma)$ we call the set $N(\sigma)_{\Gamma} := (N \otimes \Gamma)/(\sigma)_{\Gamma}$ the set of $\Gamma$-points.

Let $\Sigma$ and $\Sigma'$ be fans in $N_\mathbb{R}$ and $N'_\mathbb{R}$ respectively. Let $L : N \to N'$ be a linear map such that $L_\mathbb{R}$ maps every cone in $\Sigma$ into a cone in $\Sigma'$. Such a map canonically induces a map $N_\Sigma \to N_{\Sigma'}$ that is continuous and linear on each stratum.

Definition 2.6. A map $N_\Sigma \to N_{\Sigma'}$ that arises this way is called morphism of tropical toric varieties.

An affine map of tropical toric varieties is a map that is the composition of morphism of toric varieties with an $N_\mathbb{R}$-translation.
2.3. Tropicalization. Let $\Sigma$ be a rational fan in $N_{\mathbb{R}}$. Denote by $Y_\Sigma$ the associated toric variety and by $N_\Sigma$ the associated tropical toric variety.

**Definition 2.7.** Payne defined in [Pay09] a tropicalization map

$$\text{trop}_\Sigma : Y_\Sigma^{an} \rightarrow N_\Sigma$$

to the topological space $N_\Sigma$ as follows: For $\tau \prec \sigma$, the space $\text{Hom}(S_\tau, \mathbb{T})$ is naturally identified with the open subspace of $\text{Hom}_{\text{Monoids}}(S_\sigma, \mathbb{T})$ of maps which map $\tau^i \cap M$ to $\mathbb{R}$. The map $\text{trop} : U_\sigma^{an} \rightarrow \text{Trop}(U_\sigma)$ is then defined by mapping $|.| \in U_\sigma^{an}$ to the homomorphism $u \mapsto \log |u|_x \in \text{Trop}(U_\sigma) = \text{Hom}(S_\sigma, \mathbb{T})$. We will often write $\text{Trop}(Y_\Sigma) := N_\Sigma$.

For $Z$ a closed subvariety of $Y_\Sigma$ we define $\text{Trop}(Z)$ to be the image of $Z^{an}$ under $\text{trop} : Y_\Sigma^{an} \rightarrow \text{Trop}(Y_\Sigma)$.

**Definition 2.8.** The construction $\text{Trop}(Y_\Sigma)$ is functorial with respect to affine maps of toric varieties. In particular, for a morphism (resp. affine map) of toric varieties $\psi : Y_\Sigma \rightarrow Y_\Sigma'$, we obtain a morphism (resp. affine map) of tropical toric varieties $\text{Trop}(\psi) : \text{Trop}(Y_\Sigma) \rightarrow \text{Trop}(Y_\Sigma')$.

If $\psi$ is a closed immersion, then $\text{Trop}(\psi)$ is a homeomorphism onto its image.

**Example 2.9.** Affine space $\mathbb{A}^r = \text{Spec} K[T_1, \ldots, T_r]$ is the toric variety that arises from the cone $\{ x \in \mathbb{R}^r \mid x_i \geq 0 \text{ for all } i \in [r] \}$. By definition the tropicalization in then $\mathbb{T}^r$ and the map $\text{trop} : \mathbb{A}^{r, an} \rightarrow \mathbb{T}^r$; $|.| \mapsto (\log |T_i|)_{i \in [r]}$.

Let $\sigma$ be a cone in $N_{\mathbb{R}}$. We pick a finite generating set $b_1, \ldots, b_r$ of the monoid $S_\sigma$. Let $U_\sigma$ be the affine toric variety associated to a cone $\sigma$. Then we have a surjective map $K[T_1, \ldots, T_r] \rightarrow K[S_\sigma]$, which induces a toric closed embedding $\varphi_B : U_\sigma \rightarrow \mathbb{A}^r$. By functoriality of tropicalization we also get a morphism of tropical toric varieties $\text{Trop}(U_\sigma) \rightarrow \mathbb{T}^r$ that is a homeomorphism onto its image.

2.4. Tropical subvarieties of tropical toric varieties. In this section we fix a subgroup $\Gamma \subset \mathbb{R}$.

**Definition 2.10.** An integral $\Gamma$-affine polyhedron in $N_{\mathbb{R}}$ is a set defined by finitely many inequalities of the form $\varphi(\cdot) \geq \tau$ for $\varphi \in M$, $\tau \in \Gamma$. An integral $\Gamma$-affine polyhedron on $N_{\Sigma}$ is the topological closure of an integral $\Gamma$-affine polyhedron in $N(\sigma_r)$ for $\sigma_r \in \Sigma$.

Let $\tau$ be an integral $\Gamma$-affine polyhedron in $N_{\Sigma}$. For $\sigma \prec \sigma'$ we have that $N(\sigma') \cap \tau$ is a polyhedron in $N(\sigma')$ (that might be empty) and we consider this as a face of $\tau$. Further we denote by $L(\tau) = \{ \lambda(u_1 - u_2) \mid u_1, u_2 \in \tau, \lambda \in \mathbb{R} \} \subset N(\sigma_\tau)$ the *linear space* of $\tau$. If $\tau$ is integral $\Gamma$-affine, $L(\tau)$ contains a canonical lattice that we denote by $\mathbb{Z}(e)$.

**Definition 2.11.** A tropical subvariety of a tropical toric variety is given the support of a integral $\Gamma$-affine polyhedral complex with weights attached to its top dimensional faces, satisfying the balancing condition.

**Definition 2.12.** Let $Z$ be a closed subvariety of a toric variety $Y_\Sigma$. Then $\text{Trop}(Z) := \text{trop}(Z^{an}) \subset N_{\mathbb{R}}$ is a tropical subvariety of $\text{Trop}(Y_\Sigma)$. For a variety $X$ and a closed embedding $\varphi : X \rightarrow Y_\Sigma$ we write $\text{Trop}_\varphi(X) := \text{Trop}(\varphi(X))$ and $\text{trop}_\varphi := \text{trop} \circ \varphi^{an} : X^{an} \rightarrow \text{Trop}_\varphi(X)$.

We will never explicitly use the weights nor the balancing condition, so the reader may be happy with the fact that there are weights and that they satisfy the balancing condition. If they are not happy with this, let us refer them to the excellent introduction [Gub13].
3. Families of tropicalizations

In this section, $K$ is a complete non-archimedean field and $X$ is a $K$-variety.

3.1. Definitions. The philosophy throughout the paper will be that we can approximate non-archimedean analytic spaces through embedding them into toric varieties and tropicalizing. We will define families that approximate the analytic space well enough (fine enough families) as well as notions that tell us that two families basically contain the same amount of information (final and cofinal families).

**Definition 3.1.** A family of tropicalizations $\mathcal{S}$ of $X$ consists of the following data:

i) A class $\mathcal{S}_{\text{map}}$ containing closed embeddings $\varphi: U \to Y_\Sigma$ for open subsets $U$ of $X$ and toric varieties $Y_\Sigma$.

ii) For an element $\varphi: U \to Y_\Sigma$ of $\mathcal{S}_{\text{map}}$ a subclass $\mathcal{S}_\varphi$ of $\mathcal{S}_{\text{map}}$ that contains maps $\varphi': U' \to Y_{\Sigma'}$ for open subsets $U' \subset U$ and such that there exists an affine map of toric varieties $\psi_{\varphi,\varphi'}$, such that

$$
\begin{array}{ccc}
U' & \xrightarrow{\varphi'} & Y_{\Sigma'} \\
\downarrow & & \downarrow \\
U & \xrightarrow{\varphi} & Y_\Sigma \\
\end{array}
$$

commutes. Such a $\varphi'$ is called refinement of $\varphi$. The map $\psi_{\varphi,\varphi'}$ induces an affine map of toric varieties $\text{Trop}(\psi_{\varphi,\varphi'})$. The restriction of $\text{Trop}(\psi_{\varphi,\varphi'})$ to $\text{Trop}_{\varphi'}(U')$ depends only on $\varphi$ and $\varphi'$, so we denote this map by $\text{Trop}(\varphi,\varphi')$: $\text{Trop}_{\varphi'}(U') \to \text{Trop}_{\varphi}(U)$.

We further require that if $\varphi'$ is a refinement of $\varphi$ and $\varphi''$ is a refinement of $\varphi'$, then $\varphi''$ be a refinement of $\varphi$.

A subfamily of tropicalizations of $\mathcal{S}$ is a family of tropicalizations $\mathcal{S}'$ such that all embeddings and refinements in $\mathcal{S}$ are also in $\mathcal{S}'$. Further $\mathcal{S}'$ is said to be a full subfamily if whenever $\varphi, \varphi' \in \mathcal{S}'_{\text{map}}$ and $\varphi'$ is a refinement of $\varphi$ in $\mathcal{S}$, it is also a refinement of $\varphi$ in $\mathcal{S}'$.

Foster, Gross and Payne study in [FGP14] so-called “Systems of toric embeddings”. They only consider the case where all of $X$ is embedded into the toric variety, a case we later call global families of tropicalizations.

**Definition 3.2.** Let $\mathcal{S}$ be a family of tropicalizations on $X$. An $\mathcal{S}$-tropical chart is given by pair $(V, \varphi)$ where $\varphi: U \to Y_\Sigma \in \mathcal{S}_{\text{map}}$ and $V = \text{trop}^{-1}(\Omega)$ is an open subset of $X^\text{an}$ which is the preimage of an open subset $\Omega$ of $\text{Trop}_\varphi(U)$.

Another $\mathcal{S}$ tropical chart $(V', \varphi')$ is called an $\mathcal{S}$-tropical subchart of $(V, \varphi)$ if $\varphi'$ is a refinement of $\varphi$ and $V' \subset V$.

Note that we have $\Omega = \text{trop}_\varphi(V)$.

**Example 3.3.** The maximal family of tropicalizations is denoted $\mathcal{T}$. It contains embeddings of all open subsets of $X$ into toric varieties with all possible refinements.

**Definition 3.4.** Let $\mathcal{S}$ and $\mathcal{S}'$ be families of tropicalizations. We say $\mathcal{S}'$ is cofinal for $\mathcal{S}$ if for every embedding $\varphi: U \to Y_\Sigma$ in $\mathcal{S}_{\text{map}}$ and every $x \in U^{\text{an}}$ there exists $\varphi': U' \to Y_{\Sigma'}$ in $\mathcal{S}'_{\text{map}}$ with $x \in U'^{\text{an}}$ such that $\varphi'_{|U \cap U'}$ restricts to a closed embedding of $U \cap U'$ into an open torus invariant subvariety of $Y_{\Sigma'}$, and that embedding is a refinement of $\varphi$ in $\mathcal{T}$.

$\mathcal{S}'$ is said to be final for $\mathcal{S}$ for every $\varphi: U \to Y_\Sigma$ in $\mathcal{S}_{\text{map}}$ and $x \in U^{\text{an}}$, there exists a refinement $\varphi': U' \to Y_{\Sigma'}$ with $x \in U'^{\text{an}}$, a closed embedding $m: Y_{\Sigma'} \to Y_{\Sigma''}$, that is an
affine map of toric varieties, such that \( m \circ \varphi' \) is in \( S'_\text{map} \) and \( \varphi' \) is a refinement of \( m \circ \varphi \) via \( m \) in \( S \).

**Definition 3.5.** A family of tropicalizations is called *fine enough* if the sets \( V \) such that there exist \( S \)-tropical charts \((V, \varphi)\) form a basis of the topology of \( X^{\text{an}} \) and for each pair of \( S \)-tropical charts \((V_1, \varphi_1)\) and \((V_2, \varphi_2)\) there exists \( S \)-tropical charts \((V_i, \varphi_i)\), which are \( S \)-tropical subcharts of both \((V_1, \varphi_1)\) and \((V_2, \varphi_2)\), such that \( V_1 \cap V_2 = V_i \).

**Lemma 3.6.** Let \( S' \) be a full subfamily of \( S \) and assume that \( S \) is fine enough. If \( S' \) is final or cofinal in \( S \), then \( S' \) is also fine enough.

**Proof.** Let \( x \in X^{\text{an}} \). Since \( S \) is fine enough and \( S' \) is a full subfamily, it is sufficient to prove that given an \( S \)-tropical chart \((V, \varphi)\) with \( x \in V \), there exists an \( S \)-tropical chart \((V', \varphi')\) with \( x \in V' \) that is a \( S \)-tropical subchart of \((V, \varphi)\).

If \( S' \) is cofinal in \( S \), then we pick \( \varphi' \) as in Definition 3.4. Then \((V \cap U^{\text{an}}, \varphi')\) is a tropical chart.

If \( S' \) is final, then \((V \cap U^{\text{an}}, m \circ \varphi')\) is a tropical chart. \( \square \)

### 3.2. Examples.

In the following we will give examples of families of tropicalizations for a variety \( X \). We will always specify the class \( S_{\text{map}} \) and for \( \varphi \in S_{\text{map}} \) simply define \( S_{\varphi} \) to be those \( \varphi' \) where an affine map of toric varieties \( \psi_{\varphi, \varphi'} \) as required in Definition 3.1 (iii) exists. The exception to this rule are Example 3.14, where we require the map \( \psi_{\varphi, \varphi'} \) to be a coordinate projection in order for \( \varphi' \) to be a refinement of \( \varphi \) and Example 3.9.

**Example 3.7.** The family \( \mathcal{A} \) is the family whose embeddings are closed embeddings of affine open subsets of \( X \) into affine space. This family is fine enough by the definition of the topology of \( X^{\text{an}} \).

Suppose we are given an embedding \( \varphi : U \rightarrow Y_\Sigma \) of an open subset of \( X \) into a toric variety with \( x \in U^{\text{an}} \). Let \( Y_\sigma \) be an open affine toric subvariety of \( Y_\Sigma \) such that \( Y_\Sigma ^{\text{an}}(x) \) contains \( \varphi^{\text{an}}(x) \). Let \( \varphi' := \varphi|_{\varphi^{-1}(Y_\sigma)} \). Now we pick a toric embedding of \( m : Y_\sigma \rightarrow \mathbb{A}^n \) as in Example 2.9. This shows that \( \mathcal{A} \) is final in \( \mathcal{T} \).

**Example 3.8.** The family \( \mathcal{G} \) is the family whose embeddings are closed embeddings of very affine open subsets of \( X \) into \( \mathbb{G}_m^n \).

This family is also fine enough if the base field is non-trivially valued [Gub16, Proposition 4.16], but not when \( K \) is trivially valued [Jel16a, Example 3.3.1].

**Example 3.9.** Assume that \( K \) is algebraically closed. Let \( X \) be a variety and \( U \) a very affine open subset. Then \( M = \mathcal{O}^*(U)/K^* \) is a free abelian group of finite rank and the canonical map \( K[M] \rightarrow \mathcal{O}(U) \) induces a closed embedding \( \varphi_U : U \hookrightarrow T \) for a torus \( T \) with character lattice \( M \). The embedding \( \varphi_U \) is called the *canonical moment map* of \( U \). We denote by \( \mathcal{G}_{\text{can}} \) the family of tropicalizations where \( \mathcal{G}_{\text{can}, \text{map}} = \{ \varphi_U \mid U \subset X \text{ very affine} \} \) and refinements being the maps induced by inclusions.

It is easy to see that this family is cofinal in \( \mathcal{G} \).

**Definition 3.10.** A family of tropicalizations \( S \) for a variety \( X \) is called *global* if all \( \varphi \in S_{\text{map}} \) are defined on all of \( X \).

Global families of tropicalizations will play a special role, as they will allow us to construct classes in tropical Dolbeault cohomology.

**Example 3.11.** Let \( S \) be a global family of tropicalizations such that

\[
X^{\text{an}} = \lim_{\varphi \in S_{\text{map}}} \text{Trop}_\varphi(X)
\]  

(3.1)
and such that if \( \varphi_1 : X \to Y_{\Sigma_1} \) and \( \varphi_2 : X \to Y_{\Sigma_2} \) in \( S_{\text{map}} \) then also \( \varphi_1 \times \varphi_2 : X \to Y_{\Sigma_1} \times Y_{\Sigma_2} \in S_{\text{map}} \). Then \( S \) is fine enough. In fact it follows directly from (3.1) that \( S \)-tropical charts form a basis of the topology and from the product property that we can always locally find common subcharts.

This example settles the question for when a variety admits a fine enough global family of tropicalizations. The obviously necessary condition of admitting at least one closed embedding into a toric variety is sufficient.

Families with the properties from Example 3.11 were extensively studied in [FGP14]. When \( X \) is normal, this helps us say even say more:

**Example 3.12.** Let \( X \) be a normal variety that admits at least one embedding into a toric variety. Then we may consider the family of tropicalizations \( \mathcal{T}_{\text{global}} \), where \( \mathcal{T}_{\text{global, map}} \) is the class of all embeddings of \( X \) into toric varieties.

Let \( \varphi : U \to \mathbb{A}^n \) be a closed embedding of an open subset \( U \) of \( X \) into affine space given by regular functions \( f_1, \ldots, f_n \) on \( U \). Then by [FGP14, Theorem 4.2] there exists an embedding \( \overline{\varphi} : X \to Y_{\Sigma} \) such that \( U \) is the preimage of an open affine invariant subvariety \( U_{\sigma} \) and each \( f_i \) is the pullback of a character that is regular on \( U_{\sigma} \). This exactly shows that \( \overline{\varphi}|_U \) is a refinement of \( \varphi \) in \( \mathcal{T} \), which shows that \( \mathcal{T}_{\text{global}} \) is cofinal in \( \mathbb{A}_{\text{reg}} \).

**Example 3.13.** Let \( X \) be an affine variety and \( \mathbb{A}_{\text{global}} \) the family of tropicalizations whose class of embeddings are embeddings of all of \( X \) into affine space.

We show that this family is cofinal in \( \mathbb{A} \): Let \( U \) be an open subset of \( X \) and \( \varphi : U \to \mathbb{A}^n \) an closed embedding given by \( f_1/g_1, \ldots, f_n/g_n \), where \( f_i, g_i \) are regular functions on \( X \). In particular, \( U = D(g_1, \ldots, g_n) \). We pick regular functions \( h_1, \ldots, h_k \) on \( X \) such that the \( f_i, g_i, h_i \) generate \( \mathcal{O}(X) \). We consider the embedding \( \varphi : X \to \mathbb{A}^m \times \mathbb{A}^k \) given by \( f_1, \ldots, f_n, g_1, \ldots, g_n, h_1, \ldots, h_k \). Then \( \varphi'|_U \) gives a closed embedding of \( U \) into \( \mathbb{A}^m \times \mathcal{G}_m^m \times \mathbb{A}^k \), which is clearly a refinement of \( \varphi \) in \( \mathcal{T} \).

**Example 3.14.** Assume that \( K \) is algebraically closed. Wanner and the author considered in [JW18] the class of linear tropical charts for \( X = \mathbb{A}^1 \). In the language of the present paper, they use the following family of tropicalizations, which we denote by \( \mathbb{A}_{\text{lin}, \text{map}} \): \( \mathbb{A}_{\text{lin, map}} \) is the set of linear embeddings i.e. those embeddings \( \varphi : \mathbb{A}^1 \to \mathbb{A}^s \), where the corresponding algebra homomorphism \( K[T_1, \ldots, T_r] \to K[X] \) is given by mapping \( T_i \) to \( (X - a_i) \) for \( a_1, \ldots, a_r \in K \). A refinement of \( \varphi \) is then a map \( \varphi' : \mathbb{A}^1 \to \mathbb{A}^s \) given by \( (X - b_1, \ldots, X - b_s) \) where \( s > r \) and \( \{a_i\} \subset \{b_j\} \). This family of tropicalizations is strongly admissible and global, for details see [JW18, Section 3.2]. By factoring the defining polynomials of any map \( \mathbb{A}^1 \to \mathbb{A}^n \) into linear factors, it follows that \( \mathbb{A}_{\text{lin}} \) is cofinal in \( \mathbb{A}_{\text{global}} \).

**Example 3.15.** Let \( K \) be algebraically closed and \( X \) be a smooth projective Mumford curve. Let Smooth be the class of embeddings of \( X \) into toric varieties such that \( \text{Trop}_\varphi(X) \) is a smooth tropical curve. Then Smooth is cofinal in \( \mathcal{T}_{\text{global}} \) by [Jel18, Theorem A].

4. Differential forms on Berkovich spaces

In this section, \( K \) is a complete non-archimedean field and \( X \) is a variety over \( K \). Further \( S \) is a fine enough family of tropicalizations.

In this section, we define a sheaf of differential forms \( \mathcal{A}^{p,q}_S \) with respect to \( S \). We will also show that for final and cofinal families, these sheaves are canonically isomorphic. We
use the sheaves $\mathcal{A}^{p,q}$ of differential forms on tropical varieties which are recalled in the appendix.

**Definition 4.1.** Let $\mathcal{S}$ be a fine enough family of tropicalizations. For $V$ an open subset of $X^\text{an}$. An element $\alpha \in \mathcal{A}^{p,q}_S(V)$ is given by a family of triples $(V_i, \varphi_i, \alpha_i)_{i \in I}$, where

i) The $V_i$ cover $V$, i.e. $V = \bigcup_{i \in I} V_i$.

ii) For each $i \in I$ the pair $(V_i, \varphi_i)$ is an $\mathcal{S}$-tropical chart.

iii) For each $i \in I$ we have $\alpha_i \in \mathcal{A}^{p,q}(\text{trop}_S(V_i))$.

iv) For all $i, j \in I$ there exist $\mathcal{S}$-tropical subcharts $(V_{ijl}, \varphi_{ijl})_{l \in L}$ that cover $V_i \cap V_j$ such that

$$\text{Trop}(\varphi_i, \varphi_{ijl})^* \alpha_i = \text{Trop}(\varphi_j, \varphi_{ijl})^* \alpha_j \in \mathcal{A}^{p,q}(\text{trop}_{\mathcal{S}}(V_{ijl})).$$

Another such family $(V_j, \varphi_j, \alpha_j)_{j \in J}$ defines the same form $\alpha$ if their union $(V_i, \varphi_i, \alpha_i)_{i \in I \cup J}$ still satisfies iv).

For an open subset $W$ of $V$ we can cover $W$ by $\mathcal{S}$-tropical subcharts $(V_{ij}, \varphi_{ij})$ of the $(V_i, \varphi_i)$. Then we define $\alpha|_W \in \mathcal{A}^{p,q}_S(W)$ to be defined by $(V_{ij}, \varphi_{ij}, \text{Trop}(\varphi_i, \varphi_{ij})^*(\alpha_i))$.

The differentials $d'$ and $d''$ are well defined on $\mathcal{A}^{p,q}_S$ and thus we obtain a complex $(\mathcal{A}^{*,*}_S, d', d'')$ of differential forms on $X^\text{an}$.

**Lemma 4.2.** Let $\mathcal{S}$ be a fine enough family of tropicalization. Let $\alpha \in \mathcal{A}^{p,q}_S(V)$ be given by a single $\mathcal{S}$-tropical chart $(V, \varphi, \alpha')$. Then $\alpha = 0$ if and only if $\alpha' = 0$.

*Proof.* This works the same as the proof in [CLD12].

**Lemma 4.3.** Let $\mathcal{S}$ be a fine enough family of tropicalization. Let $\mathcal{S}'$ be a fine enough subfamily. Then there exists a canonical injective morphism of sheaves

$$\Psi_{\mathcal{S}', \mathcal{S}}: \mathcal{A}_{\mathcal{S}'} \to \mathcal{A}_{\mathcal{S}}.$$  

Furthermore if $\mathcal{S}'$ is final or cofinal this morphism is an isomorphism.

*Proof.* The map is constructed by defining a form $(V, \varphi, \beta)$ to be defined by the same tuple. Injectivity follows from Lemma 4.2.

Assume that $\mathcal{S}'$ is cofinal in $\mathcal{S}$ and that we are given a form $\alpha$ given by $(V, \varphi, \beta)$. Fixing $x \in V$ and picking $\varphi'$ as in Definition 3.4, we define a form $\alpha'$ to be given by $(V \cap U^\text{an}, \varphi', \text{Trop}(\psi_{\varphi'}, \varphi')^* \beta)$. Since $\varphi'$ was a refinement of $\varphi$ in $\mathcal{T}$, we have

$$\Psi_{\mathcal{S}, \mathcal{T}}(\alpha) = \Psi_{\mathcal{S}', \mathcal{T}}(\alpha') = \Psi_{\mathcal{S}, \mathcal{T}}(\Psi_{\mathcal{S}', \mathcal{S}}(\alpha'))$$

which proves $\Psi(\mathcal{S}', \mathcal{S})(\alpha') = \alpha$ locally at $x$ by injectivity of $\Psi_{\mathcal{S}, \mathcal{T}}$.

Assume that $\mathcal{S}'$ is final in $\mathcal{S}$ and we are given a form $\alpha$ given by a tuple $(V, \varphi, \beta)$. We pick $m$ as in Definition 3.4 and define $\alpha'$ to be given by $(\varphi \circ m_2, V, \beta)$. Note here that $m$ induces an isomorphism of the tropicalisations of $V$, thus “pushing $\beta$ forward along $\varphi'$” is possible. Then $\Psi_{\mathcal{S}', \mathcal{S}}(\alpha') = \alpha$ since $\varphi$ is a refinement of $\varphi \circ m$ via $m$.

**Corollary 4.4.** Let $\mathcal{S}$ be a fine enough family of tropicalization. Let $\mathcal{S}'$ be a full subfamily that is either final or cofinal. Then the canonical morphism

$$\Psi_{\mathcal{S}', \mathcal{S}}: \mathcal{A}_{\mathcal{S}'} \to \mathcal{A}_{\mathcal{S}}.$$  

is an isomorphism.

*Proof.* The family $\mathcal{S}'$ is fine enough by Lemma 3.6 and then the Corollary follows from 4.3.

**Definition 4.5.** Let $\mathcal{S}$ be a fine enough family of tropicalizations. We say that $\mathcal{S}$ is admissible if the canonical morphism $\Psi_{\mathcal{S}, \mathcal{T}}$ is an isomorphism.
Corollary 4.6. We have a canonical isomorphism
\[ A_k \cong A_T. \]

If \( X \) is affine, these are also canonically isomorphic to \( A_{\text{global}} \) and if \( X \) is normal and admits an embedding into a toric variety, then these are also canonically isomorphic to \( A_{T,\text{global}} \).

Theorem 4.7. Let \( S \) be a fine enough global family of tropicalizations. Let \( \alpha \in \mathcal{A}^{p,q}(V) \) be given by a finite family \((V_i, \varphi_i, \alpha_i)\), where \( \varphi_i : X \to Y_{\Sigma_i} \) in \( S_{\text{map}} \). Then \( \alpha \) can be defined by one \( S \)-tropical chart.

Proof. Since \( S \) is fine enough and global, there exists a common refinement \( \varphi : X \to Y_{\Sigma} \) for all the \( \varphi_i \). Then \((V_i, \varphi_i)\) is a \( S \)-tropical subchart of \((V_i, \varphi_i)\) for all \( i \). Denote by \( \alpha'_i := \text{Trop}(\varphi_i, \varphi)^* \alpha_i \) and \( \Omega_i := \text{trop}_\varphi(V_i) \). Then \( \alpha|_{V_i} \) is given by both \((V_i \cap V_j, \varphi_i, \alpha'_i|_{\Omega_i \cap \Omega_j})\) and \((V'_i \cap V'_j, \varphi_i, \alpha'_i|_{\Omega_i \cap \Omega_j})\). By Lemma 4.2, the forms \( \alpha'_i \) and \( \alpha'_j \) agree on \( \Omega_i \cap \Omega_j \), thus glue to give a form \( \alpha' \in \mathcal{A}^{p,q}(\text{trop}_\varphi(V)) \). The form \( \alpha \in \mathcal{A}^{p,q}(V) \) is then defined by \((V, \varphi, \alpha')\). \( \square \)

Let \( S \) be a global admissible family of tropicalizations. Let \( \varphi : X \to Y_{\Sigma} \) be a closed embedding in \( S_{\text{map}} \). We define a map \( \text{trop}^* : \mathcal{A}^{p,q}(\text{Trop}_\varphi(X)) \to \mathcal{A}^{p,q}_{S,c}(X^\text{an}) \) by setting for a \( \beta \in \mathcal{A}^{p,q}(\text{Trop}_\varphi(X)) \) the image \( \text{trop}^* \beta \) to be the form given by the triple \((\varphi, X^\text{an}, \beta)\). One immediately checks that this is well defined. We define this similarly for forms with compact support.

Theorem 4.8. Let \( S \) be a fine enough global family of tropicalizations. Let \( V \) be an open subset of \( X^\text{an} \) such that there exists an \( S \)-tropical chart \((V, \varphi)\). Then we have
\[ \mathcal{A}^{p,q}_{S,c}(V) = \lim \mathcal{A}^{p,q}_{S,c}(\text{trop}_\varphi(V)) \]
where the limit runs over all \( S \)-tropical charts \((V, \varphi)\).

Proof. For any \( S \)-tropical chart \((V, \varphi)\), the pullback along the proper map \( \text{trop}_\varphi \) induces a well defined morphism \( \mathcal{A}^{p,q}(\text{trop}_\varphi(V)) \to \mathcal{A}^{p,q}_{S,c}(V) \). By definition this map is compatible with pullback between charts. Thus the universal property of the direct limit leads to a morphism \( \Psi : \lim \mathcal{A}^{p,q}_{S,c}(\text{trop}_\varphi(V)) \to \mathcal{A}^{p,q}_{c}(V) \), where the limit runs over all \( S \)-tropical charts of \( V \).

This map is injective by construction. For surjectivity, let \( \alpha \in \mathcal{A}^{p,q}_{c}(V) \) be given by \((V_i, \varphi_i, \alpha_i)_{i \in I} \). Let \( I' \) be a finite subset of \( I \) such that the \( V_i \) with \( i \in I' \) cover the support of \( \alpha \). Then \((V_i, \varphi_i, \alpha_i)_{i \in I'} \) defines \( \alpha|_{V'} \in \mathcal{A}^{p,q}_{S,c}(V') \), where \( V' = \bigcup_{i \in I'} V_i \). Since \( S \) is global and fine enough, the conditions of Theorem 4.7 are satisfied and \( \alpha|_{V'} \) can be defined by a triple \((V', \varphi', \alpha')\). By passing to a common refinement with \((V, \varphi)\) we may assume that \((V', \varphi')\) is a subchart of \((V, \varphi)\).

It follows from Lemma 4.2 that \( \text{supp}(\alpha') = \varphi'_{\text{trop}}(\text{supp}(\alpha)) \) (cf. [CLD12, Corollaire 3.2.3]). Thus \( \text{supp}(\alpha') \) is compact. We extend \( \alpha' \) by zero to \( \overline{\alpha}' \in \mathcal{A}^{p,q}_{\text{Trop}_{\varphi'(X)},c}(\text{trop}_{\varphi'}(V)) \).

Then \( \alpha \) is defined by \((V, \varphi', \overline{\alpha}')\), which is in the image of \( \Psi \). \( \square \)

5. Comparison theorems

5.1. Comparing with Gubler's definition. Since Gubler's definition only works when \( K \) is non-trivially valued and algebraically closed we assume for this subsection that this is the case.

Theorem 5.1. The family of tropicalizations \( \mathcal{G} \) from Example 3.8 is admissible.
This theorem is not a formal consequence of the definition, since the family \( \mathbb{G}_m \) does not see any boundary considerations. Since for algebraically closed \( K \), Gubler defined in [Gub16] the sheaf \( \mathcal{A} \) like we here defined \( \mathcal{A}_{\text{can}} \) this theorem is crucial for us as it proves that what we show in later sections actually applies to \( \mathcal{A} \).

**Proof.** We have to show that the canonical map \( \Psi := \Psi_{\mathbb{G}_m} : \mathcal{A}_{\mathbb{G}_m} \rightarrow \mathcal{A}_{\mathcal{T}} \) is surjective. We argue locally around a point \( x \in X^{\text{an}} \). Let \( \alpha \in \mathcal{A}_{\mathcal{T}} \) be locally given by \((V, \varphi, \beta)\), where \( \varphi : U \rightarrow Y_\Sigma \) is a closed embedding of an open subset \( U \) of \( X \) into a toric variety. We fix coordinates on the torus stratum \( T \) of \( Y_\Sigma \) that \( x \) is mapped to under \( \varphi^{\text{an}} \), identifying \( T \) with \( \mathbb{G}_m^n \) for some \( n \) and we denote by \( \overline{\varphi}^{-1}(T) \). Since very affine open subset form a basis of the topology of \( X \), there exists a closed embedding \( \varphi' : U' \rightarrow \mathbb{G}_m^n \) for \( x \in U'^{\text{an}} \subset U^{\text{an}} \) such that if \( \pi : \mathbb{G}_m^n \rightarrow \mathbb{G}_m^n \) is the projection to the first \( n \) factors, then \( \overline{\varphi}|_Z = \pi \circ \overline{\varphi} \circ \iota_Z \).

After shrinking \( V \) we may assume that \( \text{trop}_\varphi(V) \) is a neighborhood of \( \text{trop}_\varphi(x) \), on which \( \beta = \pi^*(\beta_0) \) holds, where \( \pi \) is the projection to the stratum \( \text{Trop}(T) \) of \( \text{Trop}(Y) \).

We define the form \( \alpha' \in \mathcal{A}_{\mathcal{T}} \) to be given by \((V', \varphi', \text{Trop}(\psi)^*\beta_0)\). Since \( \Psi(\alpha') \) is by definition given by the same triple, we have to show that \((V, \varphi, \beta)\) and \((V', \varphi', \text{Trop}(\psi)^*\beta_0)\) define the same form in a neighborhood of \( x \). We do that by pulling back to a common subcharts, namely \( \varphi \times \varphi' : U' \rightarrow Y_\Sigma \times \mathbb{G}_m^n \). When pulling back on the tropical side we find that both \( \text{Trop}(\pi_1)^*\beta \) as well as \( \text{Trop}(\pi_2)^*\beta_0 \) are simply the pullback of \( \beta_0 \), hence these forms agree indeed.

**Corollary 5.2.** The family of tropicalizations \( \mathbb{G}_{\text{can}} \) is admissible.

**Proof.** This follows from the Lemma 4.3, the fact that \( \mathbb{G}_{\text{can}} \) is cofinal in \( \mathbb{G} \) and Theorem 5.1.

### 5.2. Comparing with the analytic definition by Chambert–Loir and Ducros.

We denote by \( \mathcal{A}_{\text{an}}^{p,q} \) the analytically defined sheaf of differential forms by Chambert-Loir and Ducros [CLD12].

**Definition 5.3.** We define a map

\[
\Psi_{\text{an}} : \mathcal{A}_{\text{an}}^{p,q} \rightarrow \mathcal{A}_{\text{an}}^{p,q}.
\]

Let \((V, \varphi, \alpha)\) an \( \mathbb{A} \)-tropical chart, where \( \varphi \) is given by \( f_1, \ldots, f_r \). Let \( f_{i_1}, \ldots, f_{i_k} \) be those \( f_i \) that do not vanish at \( X \). Then those induce a map \( \varphi' : V \rightarrow \mathbb{G}_m^n \) and \( \text{trop}_{\varphi'}(V) = \text{trop}_{\varphi}(V) \cap \{ |z_{i_j}| = -\infty \text{ for all } j = 1, \ldots, k \} \), which is a stratum of \( \text{trop}_{\varphi}(V) \) and denote by \( \alpha_I \) the restriction of \( \alpha \) to that stratum.. Then we define \( \Psi(\alpha) \) to be given by \((V, \varphi_{\mathbb{G}_m}, \alpha_I)\).

**Theorem 5.4.** The map \( \Psi_{\text{an}} \) is an isomorphism.

**Proof.** We only have to prove the statement on stalks, so we fix \( x \in X^{\text{an}} \). By construction any form is around \( x \) given by one \( \mathbb{A} \)-tropical chart. Then \( \Psi_{\text{an}}(\alpha) \) is also given around \( x \) by one chart. If now \( \Psi_{\text{an}}(\alpha) = 0 \), then \( \alpha_I \) equals zero by [CLD12, Lemme 3.2.2] and thus \( \alpha \) equals zero. This shows injectivity.

To show surjectivity, we take a form \( \alpha \) that is given locally around \( x \) by a map \( V \rightarrow \mathbb{G}_m^{r+s} \) defined by \( f_1, \ldots, f_r \) and a form \( \alpha_1 \) on \( \text{Trop}_{f_1, \ldots, f_r}(V) \). The \( f_i \) are Laurent series and around \( x \) we may cut them of in sufficiently high degree to obtain Laurent polynomials \( f'_1, \ldots, f'_r \) such that \( |f_i| = |f'_i| \) for all \( i \) on a neighborhood \( V_1 \) of \( x \). Now choosing functions \( g_1, \ldots, g_s \in \mathcal{O}_X(U) \) for an open subset \( U \) of \( X \) such that \( x \in U^{\text{an}} \), the \( f'_i \) and \( g_i \) define a closed embedding \( \varphi : U \rightarrow \mathbb{A}^{r+s} \) with the property that there exists a neighborhood \( V_2 \) of \( x \) in \( V_1 \) and a tropical chart \( (V_2, \varphi) \). We may assume that \( g_1, \ldots, g_t \) are non vanishing at \( x \) and \( g_{t+1}, \ldots, g_s \) are. Now \( \alpha|_{V_2} \) is defined by
a form $\alpha_2$ on $\text{Trop}_{f_1,\ldots,f_t,g_1,\ldots,g_t}(V_2)$. By construction we have $\text{Trop}_{f_1,\ldots,f_t,g_1,\ldots,g_t}(V_2)$ is the stratum of $\text{Trop}_S(V_2)$, where the coordinates $t$-th are $-\infty$. Denote by $\pi: \text{Trop}_S(V_2) \to \text{Trop}_{f_1,\ldots,f_t,g_1,\ldots,g_t}(V_2)$ the projection. Let $\alpha_3 = \pi^* \alpha_2$. Then we define $\beta \in \mathcal{A}_{\mathbb{A}^n}^n(V_2)$ by $(V_2, \varphi, \alpha_3)$. Now $\Psi_{an}(\beta)$ is given by $V_2 \to \mathbb{C}^{r+t}$ given by $f_1,\ldots,f_t,g_1,\ldots,g_t$ and $\alpha_2$. Since $\text{trop} f_1,\ldots,f_t,g_1,\ldots,g_t = \text{trop} f_1,\ldots,f_t,g_1,\ldots,g_t: V_2 \to \mathbb{R}^{r+t}$, the result follows from [CLD12, Lemma 3.1.10].

**Lemma 5.5.** Let $K$ be non-trivial valued and $\alpha \in \mathcal{A}_{G}^{n,q}(V)$ given by $(V, \varphi, \alpha')$. Then $\Psi_{an} \circ \Psi_{\mathbb{A},T}^{-1} \circ \Psi_{G,T}(\alpha)$ is given by $(V, \varphi_{an}, \alpha')$.

**Proof.** Let $\varphi: U \to \mathbb{C}^r$ be given by invertible functions $f_1,\ldots,f_r$. Then the corresponding embedding into $\mathbb{A}^{2r}$, which we denote by $\varphi_\pm$ is given by $f_1, f_1^{-1},\ldots,f_r,f_r^{-1}$. Denote by $\pi: \mathbb{R}^{2r} \to \mathbb{R}^r$ the projection to the odd coordinates. Then by construction $\Psi_{an} \circ \Psi_{G,n}(\alpha)$ is given by $(V, \varphi_{an}, \pi^*(\alpha'))$. Since $\varphi_{an}^{\pm}$ is a refinement of $\varphi_{an}$ and the map induced on tropicalizations is precisely $\pi$, we find that $(V, \varphi_{an}, \pi^*(\alpha)) = (V, \varphi_{an}, \alpha') \in \mathcal{A}_{an}^{n,q}(V)$, which proves the claim. \hfill \square

6. Integration

In this section we denote by $n$ the dimension of $X$ and we let $S$ be a fine enough family of tropicalizations.

**Definition 6.1.** Let $\alpha \in \mathcal{A}_{S,c}^{n,n}(X_{an})$ be an $(n,n)$-form with compact support. A tropical chart of integration for $\alpha$ is a $S$-tropical chart $(U_{an}, \varphi)$ where $U$ is an open subset of $X$ and $\varphi: U \to Y_\Sigma$ is a closed embedding of $U$ into a toric variety, such that $\alpha|_{U_{an}}$ is given by $(U_{an}, \varphi, \beta)$ for $\beta_U \in \mathcal{A}_{c,n}(\text{Trop}_\varphi(U))$.

**Lemma 6.2.** There always exists an $\mathbb{A}$-tropical chart of integration. If $K$ is non-trivially valued and algebraically closed, there always exist $G$-tropical charts of integration. If $S$ is global, then there always exist $S$ tropical chart of integration.

**Proof.** This follows from [Jel16a, Lemma 3.2.57], [Gub16, Proposition 5.13] and Theorem 4.7. Note that Gubler assume that $K$ is algebraically closed, but the proof goes through here. \hfill \square

**Theorem 6.3.** Let $\alpha \in \mathcal{A}_{S,c}^{n,n}(X_{an})$. and $(U_{an}, \varphi, \alpha_U)$ a $S$-tropical chart of integration. Then the value

$$\int_X^{\alpha} := \int_{\text{Trop}_\varphi(U)} \alpha_U$$

depends only on $\alpha$, in the sense that it is independent of $U$, $\varphi$ and $\alpha_U$. Further it is also independent of $S$, in the sense that

$$\int_X^{\alpha} = \int_X^{\Psi_{S,T}(\alpha)}.$$ 

**Proof.** Assume first that $K$ is non-trivial valued and algebraically closed. Then Gubler showed in [Gub16, Lemma 5.15] that as a consequence of the Sturmfels-Tevelev-formula [ST08, BPR16] the first part of the statement holds for $S = \mathbb{G}$.

Let $\varphi: U \to Y_\Sigma$. We may assume that $\varphi(U)$ meets the dense torus $T$. Denote by $\tilde{U} = \varphi^{-1}(T)$ and $\tilde{\varphi} := \varphi|_{\tilde{U}}$. We have supp$(\alpha) \subset \tilde{U}$ and supp$(\alpha_U) \subset \text{Trop}_{\tilde{\varphi}|_{\tilde{U}}}(\tilde{U})$.
by [Jel16a, Proposition 3.2.56], which implies that $(\mathring{U}^{an}, \varphi, \alpha_U|_{\text{Trop}_\varphi(U)})$ is a $G$-tropical chart of integration for $\alpha$. Thus we get

$$\int_{X^{an}}^S \alpha = \int_{\text{Trop}_\varphi(U)} \alpha_U = \int_{\text{Trop}_\varphi(U)} \alpha_U|_{\text{Trop}_\varphi(U)} = \int_{X^{an}} G^\Psi S, T \circ \Psi^{-1}_G T \alpha$$

The right hand side of equation (6.1) does not depend on $U, \varphi$ or $\alpha_U$ by Gubler's result, hence neither does the left hand side. So we proved the first part of the theorem when $K$ is non-trivially valued. Then second part follows also from (6.1) applied once to $S$ and once to $T$.

We reduce to the non-trivially valued case by picking a non-archimedean, complete, algebraically closed non-trivially valued extension $L$ of $K$. Let $p: X_L \rightarrow X$ be the canonical map. Since tropicalization is invariant under base field extension (cf. [Pay09, Section 6 Appendix]), we can define $\alpha \in A^{n,n}_{an}(X^{an})$ to be given by $(U^{an}_L, \varphi_L, \alpha_U)$. Now we have

$$\int_{\text{Trop}_\varphi(U)} \alpha_U = \int_{\text{Trop}_\varphi(U_L)} \alpha_U$$

and the right hand side depends only on $\alpha_L$, which depends only on $\alpha$. The last part follows because the maps $\Psi$ are also compatible with base change.

\begin{definition}
We define

$$\int_{X^{an}} \alpha = \int_{X^{an}} \alpha_U.$$

\end{definition}

\begin{lemma}
$\int_{X^{an}} X^{an}$ does not change when extending the base field.
\end{lemma}

\begin{proof}
This follows from the last part of the proof of Theorem 6.3.
\end{proof}

Chambert-Loir and Ducros also define an integration for $(n,n)$-forms with compact support, which we denote by $\int^{\text{CLD}}$.

\begin{lemma}
$\int^{\text{CLD}}$ does not change when extending the base field.
\end{lemma}

\begin{proof}
The base changes of any atlas of integration in the sense of [CLD12] is still an atlas of integration for the base changed form. Then since tropicalizations and the multiplicity $d_D$ from their definition does not changes, we obtain the result.
\end{proof}

\begin{theorem}
Let $\alpha \in A^{n,n}_{an}(X^{an})$. Then

$$\int^{\text{CLD}}_{X^{an}} \Psi_{an}(\alpha) = \int_{X^{an}} \alpha.$$

\end{theorem}

\begin{proof}
Let $L$ be a non-trivially valued non-archimedean extension of $K$. After replacing $K$ by $L$, $X$ by $X_L$ and $\alpha$ be $\alpha_L$, we may, since neither integral changes by Lemmas 6.5 and 6.6, assume that $K$ is non-trivially valued.

Then, by Corollary 5.2, $G_{\text{can}}$ is an admissible family of tropicalizations and thus by Theorem 6.3, we may take $\alpha \in A^{n,n}_{G_{\text{can}}, c}$, say given by $(V_i, \varphi_i, \alpha_i)_{i \in I}$. Then by Lemma 5.5 we have to show

$$\int^{\text{CLD}}_{X^{an}} (V_i, \varphi_i^{an}, \alpha_i)_{i \in I} = \int_{X^{an}} (V_i, \varphi_i, \alpha_i)_{i \in I}.$$

This was precisely shown in [Gub16, Section 7].
\end{proof}
From now on we will always assume that $S$ is an admissible family of tropicalizations and often just write $\mathcal{A}^{p,q}$ for $\mathcal{A}^{p,q}_S$.

7. Cohomology

7.1. Tropical Dolbeault cohomology.

**Definition 7.1.** We define tropical Dolbeault cohomology to be the cohomology of the complex $(\mathcal{A}^{p,\bullet}(X^{\text{an}}), d'')$, i.e.

$$H^{p,q}(X^{\text{an}}) := \frac{\ker(d'': \mathcal{A}^{p,q}(X^{\text{an}}) \to \mathcal{A}^{p,q+1}(X^{\text{an}}))}{\text{im}(d'': \mathcal{A}^{p,q-1}(X^{\text{an}}) \to \mathcal{A}^{p,q}(X^{\text{an}}))}.$$ 

Similarly we define cohomology with compact support $H^{p,q}_c(X^{\text{an}})$ as the cohomology of forms with compact support.

For a tropical variety $\text{Trop}_\phi(X)$ we defined similarly in the appendix $H^{p,q}(\text{Trop}_\phi(X))$ and $H^{p,q}_c(\text{Trop}_\phi(X))$

**Theorem 7.2.** Let $S$ be an admissible global family of tropicalizations. Then we have

$$H^{p,q}_c(X^{\text{an}}) = \lim_{\varphi \in \mathcal{S}} H^{p,q}_c(\text{Trop}_\varphi(X)).$$

**Proof.** The map is induced by $\text{trop}^*$. The theorem follows from the fact that taking cohomology commutes with forming direct limits and Theorem 4.8. 

**Remark 7.3.** The corresponding statement for $H^{p,q}(X^{\text{an}})$ fails. We sketch the argument. If $X$ is an affine variety, then we may pick $S = \mathbb{A}^n_{\text{global}}$, the class of closed embeddings into affine varieties.

One can show that $H^{n,n}(Y) = 0$ for all tropical subvarieties of $\mathbb{T}^n = \text{Trop}(\mathbb{A}^n)$. Then if the analogue of Theorem 7.2 would be true when removing compact support, this would imply $H^{n,n}(X^{\text{an}}) = 0$ for all affine varieties $X$.

Let $K$ be algebraically closed, $E$ be an elliptic curve of good reduction and let $e$ be a rational point of $E$. Let $V$ be an open neighborhood of $e$ that is isomorphic to an open annulus and let $X = E \setminus e$.

Using the Mayer-Vietoris sequence for the cover $(X^{\text{an}}, V)$ for $E^{\text{an}}$ gives the following exact sequence

$$H^{1,0}(V \setminus e) \to H^{1,1}(E^{\text{an}}) \to H^{1,1}(X^{\text{an}}) \oplus H^{1,1}(V)$$

By our assumption, $H^{1,1}(X^{\text{an}}) = 0$ and since $H^{1,0}(V \setminus e)$ and $H^{1,1}(V)$ are finite dimensional by [JW18, Theorem 5.7] this shows that $H^{1,1}(E^{\text{an}})$ is finite dimensional. However, it was shown in [Jel19, Theorem B] that this is not the case when the residue field of $K$ is $\mathbb{C}$.

7.2. Tropical cohomology with coefficients. In this section we assume that $S$ is a fine family of tropicalizations for $X$ that is cofinal in $\mathcal{T}$. We let $R$ be a subring of $\mathbb{R}$. We will use the sheaf of tropical cochains $\mathcal{C}^{p,q}(R)$ and the constructions from the Appendix.

**Definition 7.4.** Let $V$ be an open subset of $X^{\text{an}}$. An element of $\mathcal{C}^{p,q}(V, R)$ is given by a family $(V_i, \phi_i, \eta_i)_{i \in I}$ such that:

i) The $V_i$ cover $V$, i.e. $V = \bigcup_{i \in I} V_i$.

ii) For each $i \in I$ the pair $(V_i, \phi_i)$ is an $S$-tropical chart.

iii) For each $i \in I$ we have $\eta_i \in \mathcal{C}^{p,q}(\text{trop}_{\phi_i}(V_i), R)$.
iv) For all $i, j \in I$ there exist $S$-tropical subcharts $(V_{ijl}, \varphi_{ijl})_{l \in L}$ that cover $V_i \cap V_j$ such that
\[ \text{Trop}(\varphi_i, \varphi_{ijl})^* \eta_l = \text{Trop}(\varphi_j, \varphi_{ijl})^* \eta_j \in \mathcal{C}_S^{p,q}(\text{trop}_{\varphi_{ijl}}(V_{ijl}), R). \]

Another such family $(V_j, \varphi_j, \alpha_j)_{j \in J}$ defines the same form $\alpha$ if their union $(V_i, \varphi_i, \alpha_i)_{i \in I \cup J}$ still satisfies iv).

For each $p$ we obtain a complex of sheaves $(\mathcal{C}_S^p, R, \partial)$ on $X_{\text{an}}$. We will write $\mathcal{C}_S^{p,q}(R) := \mathcal{C}_S^{p,q}$.\]

**Remark 7.5.** It follows the same way as in Section 4 for $A^{p,q}$ that $\mathcal{C}_S^{p,q}$ is canonically isomorphic to $\mathcal{C}_{S'}^{p,q}$ when $S'$ is final or cofinal for $S$.

The author does not know whether one gets canonically isomorphic sheaves when one applies Definition 7.4 with for example $S = G$.

**Lemma 7.6.** Let $\eta \in \mathcal{C}_S^{p,q}(V, R)$ be given by $(V, \varphi, \eta')$. Then $\eta = 0$ if and only if $\eta' = 0$.

**Proof.** The proof for forms [Jel16a, Lemma 3.2.12] works word for word. \[ \square \]

**Proposition 7.7.** Comparing with tropical cohomology, we obtain
\[ \mathcal{C}_S^{p,q}(X_{\text{an}}, R) = \lim_{\varphi \in S} \mathcal{C}_{\text{trop},c}^{p,q}(\text{Trop}_\varphi(X), R) \]
and\[ H_{\text{trop},c}^{p,q}(X_{\text{an}}, R) = \lim_{\varphi \in S} H_{\text{trop},c}^{p,q}(\text{Trop}_\varphi(X), R). \]

**Proof.** This follows from Lemma 7.6 in the same way as for forms in Theorems 4.8 and 7.2 follow from Lemma 4.2. \[ \square \]

**Definition 7.8.** The maps $dR$ defined in the appendix define maps $dR : A^{p,q} \to \mathcal{C}_S^{p,q}(R)$

the induces a morphism of complexes of sheaves.

**Definition 7.9.** The cohomology $H_{\text{trop}}^{p,q}(X_{\text{an}}, R) := H^q(\mathcal{C}_S^{p,*}(X_{\text{an}}, R), \partial)$ is called tropical cohomology with coefficients in $R$ of $X_{\text{an}}$. Similarly $H_{\text{trop},c}^{p,q}(X_{\text{an}}, R) := H^q(\mathcal{C}_c^{p,*}(X_{\text{an}}, R), \partial)$ is called tropical cohomology with coefficients in $R$ with compact support of $X_{\text{an}}$.

**Definition 7.10.** We denote by $\mathcal{F}^p_R := \ker(\partial : \mathcal{C}^{p,0}(R) \to \mathcal{C}^{p,1}(R))$.

**Lemma 7.11.** The complex
\[ 0 \to \mathcal{F}^p_R \to \mathcal{C}^{p,0}(R) \to \mathcal{C}^{p,1}(R) \to \cdots \to \mathcal{C}^{p,n}(R) \to 0 \]
is exact.

**Proof.** Exactness on the tropical side is true by [JSS19, Proposition 3.11 & and Lemma 3.14] (with real coefficients, but the proof goes through here). It is then automatically true on the analytic side using the definitions (cf. the proof for forms [Jel16b, Theorem 4.5]). \[ \square \]
Remark 7.12. The sheaves $A^{p,q}_S$ admit partitions of unity, which can be shown the same way as it was shown by Gubler for $S = \mathbb{G}$ in [Gub16, Proposition 5.10]. This proof however uses the $\mathbb{R}$-structure of those sheaves.

The sheaves $\mathcal{C}^{p,q}(R)$ on a tropical variety (as defined in the appendix) are flasque sheaves [JSS19, Lemma 3.14], hence in particular acyclic.

However, it is not clear whether this property also holds for $\mathcal{C}^{p,q}(R)$ on the analytic space $X^{an}$ in general. We will prove some partial results in the next Lemma.

Lemma 7.13. If $R = \mathbb{R}$ or if $X$ is normal and admits at least one embedding into a toric variety the sheaves $\mathcal{C}^{p,q}(R)$ are acyclic with respect to the functor of global sections as well as global sections with compact support.

Proof. Using the map $dR : A^{0,0} \to \mathcal{C}^{0,0}(\mathbb{R})$, we see that $\mathcal{C}^{0,0}(\mathbb{R})$ admits partitions of unity. Hence $\mathcal{C}^{0,0}(\mathbb{R})$ is a fine sheaf and since $\mathcal{C}^{p,q}(\mathbb{R})$ is a $\mathcal{C}^{0,0}(\mathbb{R})$-module (via the cap product) we see that $\mathcal{C}^{p,q}(\mathbb{R})$ is also a fine sheaf.

In general, if $X$ is normal and admits at least one embedding into a toric variety, then $T_{global}$ is a global family of tropicalizations that is cofinal in $A$, which is final in $T$. Hence $\mathcal{C}^{p,q}_{global}(R) \cong \mathcal{C}^{p,q}(R)$. Any section of $\mathcal{C}^{p,q}_{global}(R)$ that is defined by finitely many charts $(V_i, \varphi_i, \eta_i)$ can be defined by a single chart. This can be shown the same way as for forms in Theorem 4.7.

Since any section over a compact subset of $X^{an}$ is defined by finitely many charts, each such section can be defined by one chart $(X^{an}, \varphi, \eta)$. Now since the sheaf $\mathcal{C}^{p,q}(R)$ on $\text{Trop}_\varphi(X)$ is flasque, this section can be extended to a global section. This shows that the sheaf $\mathcal{C}^{p,q}(R)$ on $X^{an}$ is c-soft in the sense of [KS94, Definition 2.5.5], which implies that it is acyclic for the functor of global sections with compact support [KS94, Proposition 2.58 & Corollary 2.5.9].

Since $\mathcal{C}^{p,q}(R)$ is c-soft, to show that it is acyclic for the functor of global sections, we have to show that it admits a countable cover by compact sets [KS94, Proposition 2.5.10]. Since $\mathbb{A}^{n,an}$ is covered by countably many discs, this holds if $X$ is affine. Since general $X$ is covered by finitely many affine varieties, the claim follows. $\square$

Corollary 7.14. If $R = \mathbb{R}$ or $X$ is normal and admits at least one embedding into a toric variety we have

$$H^{p,q}_{\text{trop}}(X^{an}, R) = H^{q}(X^{an}, F^p_R) \text{ and } H^{p,q}_{\text{trop},c}(X^{an}, R) = H^{q}_{c}(X^{an}, F^p_R).$$

Further, we have

$$H^{p,q}_{\text{trop}}(X^{an}, R) = H^{p,q}_{\text{trop}}(X^{an}, \mathbb{Z}) \otimes R \text{ and } H^{p,q}_{\text{trop},c}(X^{an}, R) = H^{p,q}_{\text{trop},c}(X^{an}, \mathbb{Z}) \otimes R.$$

Proof. This follows from Lemmas 7.11 and 7.13, the fact that since $R$ is torsion free, thus a flat $\mathbb{Z}$-module and $F^p_R = F^p_\mathbb{Z} \otimes \mathbb{Z}$. $\square$

Theorem 7.15 (Tropical analytic de Rham theorem). There exist canonical isomorphisms

$$H^{p,q}(X^{an}) \cong H^{p,q}_{\text{trop}}(X^{an}, \mathbb{R}) \text{ and } H^{p,q}_{c}(X^{an}) \cong H^{p,q}_{\text{trop},c}(X^{an}, \mathbb{R}).$$

Proof. We have a map

$$dR : A^{p,q} \to \mathcal{C}^{p,q}(\mathbb{R})$$
that is locally given by using the de Rham map on the tropical side constructed in A.11. This makes the following diagram commutative:

This is now a commutative diagram of acyclic resolutions of $\mathcal{F}^p$, which proves the theorem. □

Since both the monodromy operator $M$ on superforms and the wave operator defined in the Appendix A on tropical cochains commute with pullbacks along affine maps on tropical toric varieties, we obtain maps

\[ M : H^{p,q}(X^{an}) \to H^{p-1,q+1}(X^{an}) \text{ and } W : H^{p,q}_{trop}(X^{an},\mathbb{R}) \to H^{p-1,q+1}_{trop}(X^{an},\mathbb{R}). \]

**Theorem 7.16.** The wave and the monodromy operator agree on cohomology up to sign by virtue of the isomorphism $dR$, meaning that the diagram

\[
\begin{array}{ccc}
  H^{p,q}(X^{an}) & \xrightarrow{(-1)^{p-1}M} & H^{p-1,q+1}(X^{an}) \\
  \downarrow{dR} & & \downarrow{dR} \\
  H^{p,q}_{trop}(X^{an},\mathbb{R}) & \xrightarrow{W} & H^{p-1,q+1}_{trop}(X^{an},\mathbb{R})
\end{array}
\]

commutes. The same is true for cohomology with compact support.

**Proof.** The wave and monodromy operator give morphisms of complexes is a morphism of complexes

\[ W : \overline{C}^{p,*}(\mathbb{R}) \to \overline{C}^{p-1,*}(\mathbb{R})[1] \text{ and } M : \mathcal{A}^{p,*} \to \mathcal{A}^{p-1,*}[1], \]

hence it is sufficient to show that

\[
\begin{array}{ccc}
  \mathcal{A}^{p,*} & \xrightarrow{(-1)^{p-1}M} & \mathcal{A}^{p-1,*}[1] \\
  \downarrow{dR} & & \downarrow{dR} \\
  \overline{C}^{p,*}(\mathbb{R}) & \xrightarrow{W} & \overline{C}^{p-1,*}(\mathbb{R})[1]
\end{array}
\]

commutes in the derived category. Replacing both $\mathcal{A}^{p,*}$ and $\overline{C}^{p,*}$ with the quasi-isomorphic $\mathcal{F}^p$, we have to show that

\[
\begin{array}{ccc}
  \mathcal{F}^p & \xrightarrow{(-1)^{p-1}M} & \mathcal{F}^{p-1,1} \\
  \downarrow{dR} & & \downarrow{dR} \\
  \overline{C}^{p-1,1}(\mathbb{R})
\end{array}
\]
commutes. This follows directly from Theorem A.13.

In [Liu19], Liu defined a $\mathbb{Q}$-subsheaf of $\mathcal{F}^p_\mathbb{R}$ and defined rational classes in tropical Dolbeault cohomology.

**Definition 7.17** (Liu). Denote by $\mathcal{J}^p$ the $\mathbb{Q}$-subsheaf of $\mathcal{F}^p$ generated by sections of the form $(V, \varphi, \alpha)$, where $\varphi: U \to T$ and $\alpha \in M_\mathbb{Q}$. Also define $H^{p,q}(X)_\mathbb{Q} := H^q(X^\text{an}, \mathcal{J})$.

**Proposition 7.18.** Assume that $X$ is normal and admits at least one embedding into a toric variety. We have canonical isomorphisms $\mathcal{F}^p = \mathcal{J}^p$ and $H^{p,q}(X^\text{an}, \mathbb{Q}) = H^{p,q}(X^\text{an})_\mathbb{Q}$.

**Proof.** The explicit computation [JSS19, Proposition 3.11] of $\mathcal{F}^p$ works also for rational coefficients. Then this follows directly from the definitions and Corollary 7.14.

It is a priori not clear that $H^{p,q}(X^\text{an}, \mathbb{Z})$ as we defined it does not agree with $H^{p,q}(X^\text{an}, \mathbb{Q})$. The following statement in particular shows that this is not the case.

**Theorem 7.19.** Assume that $X$ is normal and admits at least one embedding into a toric variety. Then is a non-trivial $\mathbb{R}$-linear map $\cap[X^\text{an}]_R: H^{n,n}(X^\text{an}, R) \to R$.

If $R = \mathbb{R}$, then this agrees with the map induced by integration via $dR$.

**Proof.** The maps $[\text{Trop}_\varphi(X)]_R$ as defined in Definition A.16 are compatible with pullback along refinements, so by Proposition 7.7 we get a well defined map on $H^{p,n}(X^\text{an}, R)$.

The last part of the statement follows from Proposition A.17.

Liu showed that if the value group of $K$ is equal to $\mathbb{Q}$, then his monodromy map $M$ maps rational classes to rational classes [Liu19, Theorem 5.5 (1)]. We generalize to the following statement:

**Theorem 7.20.** Assume that $X$ is normal and admits at least one embedding into a toric variety. The wave operator $W$ (and by virtue of Corollary 7.16 also the monodromy map $M$) restricts to a map $W: H^{p,q}_{\text{trop},c}(X^\text{an}, R) \to H^{p-1,q+1}_{\text{trop},c}(X^\text{an}, R[\Gamma])$.

**Proof.** By Proposition 7.7, it is sufficient to prove this theorem for $\text{Trop}_\varphi(X)$. Since $\text{Trop}_\varphi(X)$ is an integral $\Gamma$-affine tropical variety, this follows from Proposition A.15.

Mikhalkin and Zharkov conjectured that for a smooth tropical variety $X$, the iterated wave operator

$W^{p-q}: \mathcal{H}^{p,q}_{\text{trop}}(X, \mathbb{R}) \to \mathcal{H}^{q,p}_{\text{trop}}(X, \mathbb{R})$

is an isomorphism for all $p \geq q$ [MZ14, Conjecture 5.3].

Liu conjectured that if $K$ is such that the residue field $\tilde{K}$ is the algebraic closure of a finite field and $X$ is smooth and proper, then the iterated monodromy operator

$M^{p-q}: H^{p,q}(X) \to H^{q,p}(X)$

is an isomorphism for all $p \geq q$ [Liu19, Conjecture 5.2].

As a consequence of Corollary 7.16 we can tie together both of these conjectures.

**Proposition 7.21.** Let $X$ be a proper variety. Assume there exists a global admissible family of tropicalizations $\mathcal{S}$ for $X$ such that for all $\varphi \in \mathcal{S}_\text{map}$ the tropical variety $\text{Trop}_\varphi(X)$ satisfies Mikhalkin’s and Zharkov’s conjecture. Then $X$ satisfies Liu’s conjecture.

**Proof.** This follows directly from Corollary 7.16 and Theorem 7.2.
8. Non-trivial classes

In this section we (partially) compute tropical cohomology with coefficients in three examples: Curves of good reduction, toric varieties and Mumford curves.

We now show that the map $H^{p,q}(X^{an}, \mathbb{Z}) \to H^{p,q}(X^{an}, \mathbb{R}) \otimes H^{p,q}(X^{an}, \mathbb{Z}) \otimes \mathbb{R}$ need not be injective. In other words, there may be torsion classes in $H^{p,q}(X^{an}, \mathbb{Z})$.

For the first theorem assume that the value group $\Gamma$ of $K$ is a subring of $\mathbb{R}$. We denote by $\log |\mathcal{O}_X^\times|$ the sheaf of real valued functions on $X^{an}$ that are locally of the form $\log |f|$ for an invertible function $f$ on $X$.

**Theorem 8.1.** Let $K$ be algebraically closed and $X$ be a smooth projective curve of good reduction such that the Picard group of the reduction $\text{Pic}^0(\tilde{X})$ contains torsion classes. Then $H_{\text{trop}}^{1,1}(X^{an}, \mathbb{Z})$ contains torsion classes.

The proof is a variant of the proof in [Jel19] for real coefficients.

**Proof.** We have the following exact sequence

$$0 \to \Gamma \to \log |\mathcal{O}_X^\times| \to \mathcal{F}_Z^1 \to 0,$$

which is a non-archimedean version of a well-known exponential sequence from tropical geometry [MZ08, Definition 4.1]. This induces the following exact sequence in cohomology groups

$$0 \to H_{\text{trop}}^{1,0}(X^{an}, \mathbb{Z}) \to H_{\text{trop}}^{0,1}(X^{an}, \Gamma) \to H_{\text{trop}}^{1}(X^{an}, \log |\mathcal{O}_X^\times|) \to H_{\text{trop}}^{1,1}(X^{an}, \mathbb{Z}) \to 0.$$

In particular, since $X$ is a curve of good reduction, $X^{an}$ is contractible and $H_{\text{trop}}^{0,1}(X^{an}, \Gamma) = H^{1}(X^{an}, \Gamma) = 0$. Hence we have that

$$H^{1}(X^{an}, \log |\mathcal{O}_X^\times|) \to H^{1,1}(X^{an}, \mathbb{Z}),$$

is an isomorphism. Therefore it is sufficient to prove that $H^{1}(X^{an}, \log |\mathcal{O}_X^\times|)$ contains torsion classes. We have an exact sequence

$$0 \to \log |\mathcal{O}_X^\times| \to H_Z \to \iota_* \text{Pic}^0(\tilde{X}) \to 0,$$

where $H_Z$ is the sheaf of real valued functions on $X^{an}$ that locally factor as the retraction to a skeleton composed with a function with integer slopes and values in $\Gamma$ on the edges of said skeleton. This is sequence is the integral version of [Thu05, Lemme 2.3.22]. The associated long exact sequence

$$0 \to \Gamma \to \Gamma \to \text{Pic}^0(\tilde{X}) \to H^{1}(X^{an}, \log |\mathcal{O}_X^\times|)$$

shows that $H^{1}(X^{an}, \log |\mathcal{O}_X^\times|)$ contains torsion classes. \hfill $\square$

**Theorem 8.2.** Let $\varphi: X \to Y$ be a closed embedding of $X$ into a toric variety $Y$. Assume that $\text{Trop}_\varphi(X)$ is a smooth tropical variety. Then $\text{trop}^* : H^{p,q}(\text{Trop}_\varphi(X)) \to H^{p,q}(X^{an})$ and $\text{trop}^* : H^{p,q}_c(\text{Trop}_\varphi(X)) \to H^{p,q}_c(X^{an})$ are injective.

**Proof.** By [JSS19, Theorem 4.33], since $\text{Trop}_\varphi(X)$ is smooth there is a perfect pairing

$$H^{p,q}(\text{Trop}_\varphi(X)) \times H^{p,n-q}_c(\text{Trop}_\varphi(X)) \to \mathbb{R}$$

induced by the wedge product and integration of superforms. Thus given a $d'$-closed $\alpha$ in $\mathcal{A}^{p,q}(\text{Trop}_\varphi(X))$ whose class $[\alpha] \in H^{p,q}(\text{Trop}_\varphi(X))$ is non-trivial, there exists $[\beta] \in$
H_{\mathbb{C}}^{n-p,n-q}(\text{Trop}_\varphi(X)) such that \( \int_{\text{Trop}_\varphi(X)} \alpha \wedge \beta \neq 0 \). Thus we have

\[
\int_{X^{\text{an}}} \text{trop}_\varphi^* \alpha \wedge \text{trop}_\varphi^* \beta \neq 0.
\]

Since integration and the wedge product are well defined on cohomology this means that \( [\text{trop}_\varphi^* \alpha] \wedge [\text{trop}_\varphi^* \beta] \) and consequently \([\text{trop}_\varphi^* \alpha] \) is not trivial. The argument for \( \alpha \in H_{\mathbb{C}}^{p,q}(\text{Trop}_\varphi(X)) \) works the same except \([\beta] \in H^{n-p,n-q}(\text{Trop}_\varphi(X)) \). \( \square \)

**Example 8.3.** Let \( Y_\Sigma \) be a smooth toric variety. Then \( Y_\Sigma \) is locally isomorphic to \( \mathbb{A}^n \) and hence \( \text{Trop}(Y) \) is locally isomorphic to \( \text{Trop}(\mathbb{A}^n) \) and hence is a smooth tropical variety. Thus

\[
\text{trop}^* : H^{p,q}(\text{Trop}(Y)) \to H^{p,q}(Y^{\text{an}})
\]

is injective by Theorem 8.2. Let \( Y_\Sigma(\mathbb{C}) \) be the complex toric variety associated with \( \Sigma \). Then \( H_{\text{Hodge}}^{p,q}(Y_\Sigma(\mathbb{C})) \cong H^{p,q}(\text{Trop}(Y_\Sigma), \mathbb{C}) \) [IKMZ19, Corollary 2]. In particular we have \( \dim_{\mathbb{C}} H_{\text{Hodge}}^{p,q}(Y_\Sigma) \geq \dim_{\mathbb{C}} H^{p,q}(\text{Trop}(Y_\Sigma(\mathbb{C}))). \) One may figure out the latter in terms of \( \Sigma \) using [Ful93, Section 5.2] or with the help of a computer and in terms of the polytope of \( Y \) using the package cellularSheaves for polymake [KSW17]. Note that \( H_{\text{Hodge}}^{p,q}(Y(\mathbb{C})) = 0 \) unless \( p = q \) by [Dan78, Corollary 12.7].

**Example 8.4.** Let \( K \) be algebraically closed, \( X \) be a smooth projective curve and \( \varphi : X \to Y \) be a closed embedding of \( X \) into a toric variety such that \( \text{Trop}_\varphi(X) \) is a smooth tropical variety (this exists if and only if \( X \) is a Mumford curve by [Jel18]). Then

\[
H_{\text{trop}}^{p,q}(\text{Trop}_\varphi(X), R) \to H_{\text{trop}}^{p,q}(X^{\text{an}}, R)
\]

is an isomorphism. In particular we have \( H_{\text{trop}}^{0,0}(X^{\text{an}}, R) \cong H_{\text{trop}}^{1,1}(X^{\text{an}}, R) \cong R \) and \( H_{\text{trop}}^{1,0}(X^{\text{an}}, R) \cong H_{\text{trop}}^{0,1}(X^{\text{an}}, R) \cong R^g. \)

**Proof.** Assume that \( \text{Trop}_\varphi(X) \) is smooth. Then \( \text{trop}_\varphi \) is a homeomorphism from a skeleton of \( X^{\text{an}} \) onto \( \text{Trop}_\varphi(X) \) [Jel16b, Theorem 5.7]. Using comparison with singular cohomology we obtain \( H_{\text{trop}}^{0,0}(\text{Trop}_\varphi(X), R) = R \) and \( H_{\text{trop}}^{0,1}(\text{Trop}_\varphi(X), R) = R^g. \) Using duality with coefficients in \( R \) as proven in [JRS18, Theorem 5.3] and comparison with singular homology, we also obtain \( H_{\text{trop}}^{1,1}(X^{\text{an}}, R) = R \) and \( H_{\text{trop}}^{1,0}(\text{Trop}_\varphi(X), R) = R^g. \) One immediately verifies all transition maps induced by refinements in the family Smooth defined in Example 3.15 are isomorphisms and hence the claim follows from Theorem 7.2. \( \square \)

9. Open Questions

In this section, we let \( X \) be a variety over \( K \).

When \( X \) is smooth, Liu constructed cycles class maps [Liu17], meaning canonical maps \( \text{cyc}_k : \text{CH}(X)^k \to H^{k,k}(X^{\text{an}}) \).

**Question 9.1.** What is the image of \( \text{cyc}_k \)?

In light of the tropical Hodge conjecture and Corollary 7.16, one might conjecture that the image of \( \text{CH}(X)_Q \) is \( H^{k,k}(X^{\text{an}}, Q) \cap \ker(M). \) One might start with the case \( k = \dim X - 1 \). Here one knows the answer tropically [JRS18], but the non-archimedean analogue is not a direct consequence.
**Question 9.2.** Does there exists an embedding $\varphi: X \to Y_\Sigma$ such that 
\[
\text{trop}^*: H^{p,q}(\text{Trop}_\varphi(X)) \to H^{p,q}(X^{\text{an}}) \quad \text{and} \quad \text{trop}^*: H^c_{\cdot,q}(\text{Trop}_\varphi(X)) \to H^c_{\cdot,q}(X^{\text{an}})
\]
are isomorphisms?

The statement for $H^c_{\cdot,q}(X^{\text{an}})$ is implied by the finite dimensionality of $H^{p,q}(X^{\text{an}})$ via Theorem 7.2. It is in fact equivalent to the finite dimensionality of $H^{p,q}(X^{\text{an}})$ if one knew that $H^c_{\cdot,q}(\text{Trop}_\varphi(X))$ is always finite dimensional, though the author is not aware of such a result (without regularity assumptions on $\text{Trop}_\varphi(X)$).

Other questions related to this concern smoothness of the tropical variety.

**Question 9.3.** Let $\varphi: X \to Y$ be a closed embedding of $X$ into a toric variety $Y$ such that $\text{Trop}_\varphi(X)$ is smooth. Are then 
\[
\text{trop}^*: H^{p,q}(\text{Trop}_\varphi(X)) \to H^{p,q}(X^{\text{an}}) \quad \text{and} \quad \text{trop}^*: H^c_{\cdot,q}(\text{Trop}_\varphi(X)) \to H^c_{\cdot,q}(X^{\text{an}})
\]
isomorphisms?

This is certainly a natural questions and “optimistically expected” to be true by Shaw [Sha17, p.3]. We now know it holds for curves 8.4, but even the case $X = Y$ is open in dimension $\geq 2$.

**Question 9.4.** Let $\varphi: X \to Y$ be a closed embedding of $X$ into a toric variety $Y$ such that $\text{Trop}_\varphi(X)$ is smooth. Does the diagram
\[
\begin{array}{ccc}
Z^k(X) & \xrightarrow{\text{trop}} & \text{CH}(X) \\
\downarrow \text{trop} & & \downarrow \text{cyc}_k \\
Z^k(\text{Trop}(\varphi(X))) & \xrightarrow{Z \mapsto \int_Z} & H^{n-k,n-k}(\text{Trop}_\varphi(X))^* \\
& & \xrightarrow{\text{PD}^{-1}} H^{k,k}(\text{Trop}_\varphi(X))
\end{array}
\]
commute? Here PD denote the Poincaré duality duality isomorphism on tropical varieties [JSS19] and cyc$_k$ denotes Liu’s cycles class map [Liu17].

Let us finish with the remark that the author does not know of any variety $X$ with $\dim(X) \geq 2$ and any $0 < p \leq \dim(X)$ and $0 < q \leq \dim(X)$ with $(p, q) \neq (1, 1)$ where we know $\dim_{\mathbb{R}} H^q(X^{\text{an}})$. (No, not even $H^2(\mathbb{P}^2,\mathbb{R})$ or $H^2(\mathbb{A}^2,\mathbb{R})$.)

**APPENDIX A. CONSTRUCTIONS IN COHOMOLOGY OF TROPICAL VARIETIES**

In this section, $R$ is a ring such that $\mathbb{Z} \subset R \subset \mathbb{R}$ and $\Gamma$ is a subgroup of $\mathbb{R}$ that contains $\mathbb{Z}$. Further $N$ is a free abelian group of finite rank, $M$ is its dual, and $\Sigma$ is a rational fan in $N_R$. Additionally, $X$ is an integral $\Gamma$-affine tropical subvariety of $N_\Sigma$.

**Definition A.1.** Let $U \subset N_R$ an open subset. A superform of bidegree $(p, q)$ is an element of 
\[
\mathcal{A}^{p,q}(U) = C^\infty(U) \otimes \Lambda^p M \otimes \Lambda^q M
\]

**Remark A.2.** There are differential operators $d'$ and $d''$ and a wedge product which are induced by the usual differential operator and wedge product on differential forms.

**Definition A.3.** For an open subset $U$ of $N_\Sigma$ we write $U_\tau := U \cap N_\tau$. A superform of bidegree $(p, q)$ on $U$ is given by a collection $\alpha = (\alpha_\sigma)_{\sigma \in \Sigma}$ such that $\alpha_\sigma \in \mathcal{A}^{p,q}(U_\sigma)$ and for each $\sigma$ and each $x \in U_\sigma$ there exists an open neighborhood $U_x$ of $x$ in $U$ such that for each $\tau \prec \sigma$ we have $\pi^{\star}_{\sigma,\tau} \alpha_\sigma = \alpha_\tau$. We call this the condition of compatibility at the boundary.
For a polyhedron $\sigma$ in $N_{\Sigma}$ we can define the restriction of a superform $\alpha$ to $\sigma$. Let $\Omega$ be an open subset of $|C|$ for a polyhedral complex $C$ in $N_{\Sigma}$. The space of superforms of bidegree $(p,q)$ on $\Omega$ is defined as the set of pairs $(U,\alpha)$ where $U$ is an open subset of $N_{\Sigma}$ such that $U \cap |C| = \Omega$ and $\alpha \in A^{p,q}(U)$. Two such pairs are identified if their restrictions to $\sigma \cap \Omega$ agree for every $\sigma \in C$.

**Definition A.4.** For a tropical subvariety $X$ of a tropical toric variety we obtain a double complex of sheaves $A^{\bullet,\bullet}(d',d'')$ on $X$. We define $H^{p,q}(X)$ (resp. $H^{p,\tau}_{c}(X)$) as the cohomology of the complex of global sections (resp. global sections with compact support) with respect to $d''$.

**Definition A.5.** There exists an iteration map $\int_{X} : A^{n,n}_{c}(X) \to \mathbb{R}$ that satisfies Stokes’ theorem and thus descends to cohomology.

**Remark A.6.** Superforms on tropical subvarieties of tropical toric varieties are functorial with respect to affine maps of tropical toric varieties [JSS19].

**Definition A.7.** Let $X$ be a tropical variety and $x \in X$ and denote by $\sigma$ the cone of $\Sigma$ such that $x \in N(\sigma)$. Then the tropical multitangent space at $x$ is defined to be

$$F^{R}_{p}(\tau) = \left( \sum_{\sigma \in X \cap N(\sigma), x \in \sigma} \Lambda^{p} \Lambda(\sigma) \right) \cap \Lambda^{p} R^{n} \subset \Lambda^{p} N(\sigma).$$

If $\nu$ is a face of $\tau$, then there are transition maps $\iota_{\tau,\nu} : F^{R}_{p}(\tau) \to F^{R}_{p}(\nu)$ that are just inclusions if $\tau$ and $\nu$ live in the same stratum and compositions with projections to strata otherwise.

We denote by $\Delta_{q}$ the standard $q$-simplex.

**Definition A.8.** A smooth stratified $q$-simplex is a map $\delta : \Delta_{q} \to X$ such that:

- If $\sigma$ is a face of $\Delta_{q}$ then there exists a polyhedron $\tau$ in $X$ such that $\delta$ is mapped into $\hat{\tau}_{i}$.
- Let $\Delta_{q} = [0,\ldots,q]$. Then if $\delta(i)$ is in contained in a the closure of a stratum of $N_{\Sigma}$, then so is $\delta(j)$ for $j \leq i$.
- For each stratum $X_{i}$ of $X$ the map $\delta : \delta^{-1}(X_{i}) \to X_{i}$ is $C^{\infty}$.

We denote the free abelian group of smooth stratified $q$-simplicies $\delta$ satisfying $\delta(\Delta_{q}) \subset \hat{\tau}$ by $C_{q}(\tau)$.

There is a boundary operator $\partial_{p,q} : C_{p,q}(X,R) \to C_{p,q-1}(X,R)$ that is on the simplex side given by the usual boundary operator and on the coefficient side by the maps $\iota_{\tau,\nu}$, when necessary. Dually we have $\partial_{p,q}^{c} : C_{p,q}(X,R) \to C_{p,q+1}(X,R)$.

**Definition A.9.** The groups of smooth tropical $(p,q)$-cell and cocells are respectively

$$C_{p,q}(X,R) := \bigoplus_{\tau \subset X} F^{R}_{p}(\tau) \otimes C_{q}(\tau)$$

$$C^{p,q}(X,R) := \text{Hom}_{\mathbb{R}}(C_{p,q}(X,R),R)$$

We denote by $\underline{C}_{p,q}(R)$ the sheafification of $C_{p,q}(X,R)$ as defined in [JSS19, Definition 3.13] and by $F^{R}_{p} = \ker(\partial : \underline{C}^{p,0}(R) \to \underline{C}^{p,1}(R))$. Note that we have $F^{R}_{p} = F^{p}_{\mathbb{R}} \otimes R$.

**Definition A.10.** We denote by $H^{p,q}_{\text{trop}}(X) := H^{q}(C_{p,\bullet}(X,R),\partial) = H^{q}(C^{p,\bullet}(X,R),\partial)$ and call this the tropical cohomology of $X$ with coefficients in $R$. Similarly we define tropical cohomology of $X$ with coefficients in $R$ with compact support.
It was shown in [JSS19] that the morphisms of complexes
\[ F^p \to A^p \] and \[ F^p \to C^p, \]
that are given by inclusion in degree 0 are in fact quasi isomorphisms. We now want to construct a de Rham morphism, meaning a quasi isomorphism \( dR : A^p \to C^p \) that is compatible with the respective inclusions of \( F^p \).

**Remark A.11.** Let \( v \otimes \delta \) be a smooth tropical \((p,q)\)-cell on an open subset \( \Omega \) of \( X \). Then we define for a \((p,q)\) form \( \alpha \in A^{p,q}(\Omega) \)
\[ \int_{v \otimes \delta} \alpha = \int_{\Delta_q} \delta^{-1}(\alpha; v) \]
We have to argue that this integral is well defined, since \( \delta \) might map parts of \( \Delta_q \) to infinity: Let \( X_0 \) be the stratum of \( X \) to which the baricenter of \( \Delta_q \) is mapped. Let \( \Delta_{q,0} := \delta^{-1}(X_0) \). Then we have \( \text{supp}(\delta^* \alpha) \subset \Delta_{q,0} \) by the condition of compatibility at the boundary for \( \alpha \), hence the integral is finite.

This defines a morphism
\[ dR : A^{p,q}(\Omega) \to C^{p,q}(\Omega) \]
\[ \alpha \mapsto \left( v \otimes \delta \mapsto \int_{v \otimes \delta} \alpha \right) \]
and one directly verifies using the classical Stokes' theorem that this indeed induces a morphism of complexes
\[ dR : A^p \to C^p. \]
that respects the respective inclusions of \( F^p \). Since both \( C^p \) and \( A^p \) for acyclic resolutions of \( F^p \), the map \( dR \) is a quasi-isomorphism.

**Definition A.12.** The **monodromy operator** is the map
\[ M : A^{p,q} \to A^{p-1,q+1}; \]
\[ dx_I \wedge d'x_J \mapsto \sum_{k=1}^p (-1)^{p-k} dx_{I \setminus i_k} \wedge d'x_{i_k} \wedge d''x_I \wedge d''x_J. \]
The wave operator \( W : C^{p,q}(\mathbb{R}) \to C^{p-1,q+1}(\mathbb{R}) \) is the sheafified version of the map dual to
\[ C_{p-1,q+1}(X, \mathbb{R}) \to C_{p,q}(X, \mathbb{R}); \]
\[ v \otimes \delta \mapsto v \wedge (\iota(\delta(1)) - \delta(0)) \otimes \delta|_{[1,\ldots,q+1]}. \]

**Lemma A.13.** The diagram
\[ \begin{array}{ccc}
F^p & \xrightarrow{W} & A^{p-1,1} \\
\downarrow & & \downarrow dR \\
\downarrow & & \downarrow \\
C^{p-1,1}(\mathbb{R})
\end{array} \]
commutes.
where $\delta: [0, 1] \to X$ is a smooth stratified 1-simplex and $v \in \mathbb{R}^{p-1}(\tau)$, where $\delta((0, 1)) \subset \tilde{\tau}$ and $\alpha \in \mathcal{F}^p$. After picking bases, using multilinearity for both $v$ and $\alpha$ we may assume that $\alpha = d'x_1 \wedge \cdots \wedge d'x_p$ and $v = x_1 \wedge \cdots \wedge x_{p-1}$. Then we have

$$\int_{[0,1]} \delta^* \langle M(\alpha), v \rangle = \sum_{i=1}^{p} (-1)^{p-i} \int_{[0,1]} \delta^* \langle d'x_1 \wedge \cdots \wedge d'x_i \wedge d''x_p, x_1 \wedge \cdots \wedge x_{p-1} \rangle \wedge d''x_i$$

$$a = \int_{[0,1]} \delta^* \langle d'x_1 \wedge \cdots \wedge d''x_p, x_1 \wedge \cdots \wedge x_{p-1} \rangle \wedge d''x_p$$

and

$$\langle d'x_1 \wedge \cdots \wedge d'x_p, \delta(1) - \delta(0) \rangle = (-1)^{p-1} dx_p(\delta(1)) - dx_p(\delta(0)),$$

where $dx_p$ denotes the $p$-th coordinate functions with respect to $x_1, \ldots, x_n$. This calculation holds true as long as the $p$-th coordinate functions is bounded on $\delta([0, 1])$. If this is not the case however, then $d''x_p$ vanishes in a neighborhood of $\delta(0)$ (resp. $\delta(1)$) by the compatibility condition and we may replace $[0, 1]$ by $[\varepsilon, 1]$ resp. $[\varepsilon, 1-\varepsilon]$ resp. $[0, 1-\varepsilon].$ □

**Theorem A.14.** The wave and the monodromy operator agree on cohomology up to sign by virtue of the isomorphism $d\mathbb{R}$, meaning that the diagram

$$\begin{array}{ccc}
\mathbb{H}^{p,q}(X) & \xrightarrow{(-1)^{p-1}M} & \mathbb{H}^{p-1,q+1}(X) \\
d\mathbb{R} & & \downarrow d\mathbb{R} \\
\mathbb{H}^{p,q}_{\text{trop}}(X, \mathbb{R}) & \xrightarrow{W} & \mathbb{H}^{p-1,q+1}_{\text{trop}}(X, \mathbb{R})
\end{array}$$

commutes. The same is true for cohomology with compact support.

**Proof.** The proof of Theorem 7.16 works word for word. □

**Proposition A.15.** The wave operator descends to an operator on cohomology

$$W: \mathbb{H}^{p,q}_{\text{trop}}(X, R) \to \mathbb{H}^{p-1,q+1}_{\text{trop}}(X, R[\Gamma])$$

and

$$W: \mathbb{H}^{p,q}_{\text{trop, c}}(X, R) \to \mathbb{H}^{p-1,q+1}_{\text{trop, c}}(X, R[\Gamma]),$$

where $R[\Gamma]$ is the smallest subring of $\mathbb{R}$ that contains both $R$ and $\Gamma$.

**Proof.** We pick a triangulation of $X$ with smooth stratified simplices such that all vertices are $\Gamma$-points. We can now compute $\mathbb{H}_{c}^{p,q}(X, R)$ and the wave homomorphism using this triangulation by [MZ14, Section 2.2]. For a $(p, q)$-chain $\delta$ with respect to this triangulation and with coefficients in $R$, we now have that $W(\delta)$ has coefficients in $R[\Gamma]$. Hence $W$ restricts to a map $\mathbb{H}^{p,q}(X, R) \to \mathbb{H}^{p-1,q+1}(X, R[\Gamma])$ resp. $\mathbb{H}_{c}^{p,q}(X, R) \to \mathbb{H}_{c}^{p-1,q+1}(X, R[\Gamma]).$ □

**Definition A.16.** We denote by

$$\cap[X]_R: C_{c}^{n,n}(X, R) \to R$$

the evaluation against the fundamental class, as defined in [JRS18, Definition 4.8] and also the induced map on cohomology $\mathbb{H}^{n,n}_{\text{trop, c}}(X, R) \to R.$
**Proposition A.17.** The following diagram commutes

\[
\begin{array}{ccc}
\mathcal{A}^{n,n}_{c}(X) & \xrightarrow{f_X} & \mathbb{R} \\
\downarrow \scriptstyle{dR} & & \downarrow \scriptstyle{n[X]} \\
\mathcal{C}^{n,n}_{c}(X, \mathbb{R}) & & \\
\end{array}
\]

Proof. This is a straightforward calculation using the definitions. 

**REFERENCES**

[Ber90] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff. Nonarchimedean geometry, tropicalization, and metrics on curves. *Algebr. Geom.*, 3(1):63–105, 2016.

[CLD12] Antoine Chambert-Loir and Antoine Ducros. Formes différentielles réelles et courants sur les espaces de Berkovich. 2012. [http://arxiv.org/abs/1204.6277](http://arxiv.org/abs/1204.6277).

[CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[Dan78] V. I. Danilov. The geometry of toric varieties. *Uspekhi Mat. Nauk*, 33(2(200)):85–134, 247, 1978.

[FGP14] Tyler Foster, Philipp Gross, and Sam Payne. Limits of tropicalizations. *Israel J. Math.*, 201(2):835–846, 2014.

[Ful93] William Fulton. *Introduction to toric varieties*, volume 131 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1993. The William H. Roever Lectures in Geometry.

[Gub13] Walter Gubler. A guide to tropicalizations. In *Algebraic and combinatorial aspects of tropical geometry*, volume 589 of *Contemp. Math.*. pages 125–189. Amer. Math. Soc., Providence, RI, 2013.

[Gub16] Walter Gubler. Forms and currents on the analytification of an algebraic variety (after Chambert-Loir and Ducros). In Matthew Baker and Sam Payne, editors, *Nonarchimedean and Tropical Geometry*, Simons Symposia, pages 1–30. Switzerland, 2016. Springer.

[IKMZ19] Ilia Itenberg, Ludmil Katzarkov, Grigory Mikhalkin, and Ilia Zharkov. Tropical homology. *Math. Ann.*, 374(1-2):963–1006, 2019.

[Jel16a] Philipp Jell. Differential forms on Berkovich analytic spaces and their cohomology. 2016. PhD Thesis, available at [http://epub.uni-regensburg.de/34788/1/ThesisJell.pdf](http://epub.uni-regensburg.de/34788/1/ThesisJell.pdf).

[Jel16b] Philipp Jell. A Poincaré lemma for real-valued differential forms on Berkovich spaces. *Math. Z.*, 282(3-4):1149–1167, 2016.

[Jel18] Philipp Jell. Constructing smooth and fully faithful tropicalizations for Mumford curves. 2018. [https://arxiv.org/abs/1805.11594](https://arxiv.org/abs/1805.11594).

[Jel19] Philipp Jell. Tropical Hodge numbers of non-archimedean curves. *Israel J. Math.*, 229(1):287–305, 2019.

[JRS18] Philipp Jell, Johannes Rau, and Kristin Shaw. Lefschetz (1,1)-theorem in tropical geometry. *Épijournal Geom. Algébrique*, 2:Art. 11, 2018.

[JSS19] Philipp Jell, Kristin Shaw, and Jascha Smacka. Superforms, tropical cohomology, and Poincaré duality. *Adv. Geom.*, 19(1):101–130, 2019.

[JW18] Philipp Jell and Veronika Wanner. Poincaré duality for the tropical Dolbeault cohomology of non-archimedean Mumford curves. *J. Number Theory*, 187:344–371, 2018.

[Kaj08] Takeshi Kajiwara. Tropical toric geometry. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 197–207. Amer. Math. Soc., Providence, RI, 2008.

[KS94] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1994. With a chapter in French by Christian Houzel, Corrected reprint of the 1990 original.
Lars Kastner, Kristin Shaw, and Anna-Lena Winz. Cellular sheaf cohomology of \textit{polymake}. In \textit{Combinatorial algebraic geometry}, volume 80 of \textit{Fields Inst. Commun.}, pages 369–385. Fields Inst. Res. Math. Sci., Toronto, ON, 2017.

Yifeng Liu. Tropical cycle classes for non-archimedean spaces and weight decomposition of de Rham cohomology sheaves. 2017. \url{https://users.math.yale.edu/~yl2269/deRham.pdf}, to appear in \textit{Ann. Sci. Éc. Norm. Supér.}

Yifeng Liu. Monodromy map for tropical Dolbeault cohomology. \textit{Algebr. Geom.}, 6(4):384–409, 2019.

Grigory Mikhalkin and Ilia Zharkov. Tropical curves, their Jacobians and theta functions. In \textit{Curves and abelian varieties}, volume 465 of \textit{Contemp. Math.}, pages 203–230. Amer. Math. Soc., Providence, RI, 2008.

Grigory Mikhalkin and Ilia Zharkov. Tropical eigenwave and intermediate Jacobians. In \textit{Homological mirror symmetry and tropical geometry}, volume 15 of \textit{Lect. Notes Unione Mat. Ital.}, pages 309–349. Springer, Cham, 2014.

Sam Payne. Analytification is the limit of all tropicalizations. \textit{Math. Res. Lett.}, 16(3):543–556, 2009.

Kristin Shaw. Superforms and tropical cohomology. 2017.

Bernd Sturmfels and Jenia Tevelev. Elimination theory for tropical varieties. \textit{Math. Res. Lett.}, 15(3):543–562, 2008.

Amaury Thuillier. Théorie du potentiel sur les courbes en géométrie analytique non archimédiennne. Applications à la théorie d’Arakelov. 2005.

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