Network navigation using Page Rank random walks

Emilio Aced Fuentes
Escuela Politécnica Superior
Universidad Autónoma de Madrid
emilio.aced@estudiante.uam.es

Simone Santini*
Escuela Politécnica Superior
Universidad Autónoma de Madrid
simone.santini@uam.es

ABSTRACT
We introduce a formalism based on a continuous time approximation, to study the characteristics of Page Rank random walks. We find that the diffusion of the occupancy probability has a dynamics that exponentially "forgets" the initial conditions and settles to a steady state that depends only on the characteristics of the network. In the special case in which the walk begins from a single node, we find that the largest eigenvalue of the transition value ($\lambda_1 = 1$) does not contribute to the dynamic and that the probability is constant in the direction of the corresponding eigenvector. We study the process of visiting new node, which we find to have a dynamic similar to that of the occupancy probability. Finally, we determine the average transit time, which we find to exhibit certain connection with the corresponding time for Lévy walks. The relevance of these results reside in that Page Rank, which are a more reasonable model for the searching behavior of individuals, can be shown to exhibit features similar to Lévy walks, which in turns have been shown to be a reasonable model of a common large scale search strategy known as Area Restricted Search.

1. INTRODUCTION
The study of stochastic processes in graphs has a fairly long history in the analysis of transport and diffusion processes, from the study of epidemics [2, 7] and technical systems [19] to animal [15, 5] and human [8, 9] movements or the analysis of social interactions [10]. With the rise in popularity of the Internet around the turn of the century and of social networks a decade later, there has been a corresponding surge of interest in the study of random walks on graphs, this time seeing them as models of search strategies: a possible and common search strategy is, in fact, jumping from neighbor to neighbor until the information is found that one is after. In a standard random walks on a graph, one moves at each time step from a node to one of its neighbors picked at random [12]. These walks are a classic whose study precedes the applications to search and their characteristics are well understood. However, the connection between random walks and search has generated new interest in other kinds of walks.

In the continuum, for example, it has been observed that the so-called Lévy Walks (walks in which one makes jumps at a distance $d$ from the current location with probability $p(d) \sim d^{-\alpha}$ [1, 4]) exhibit diffusion characteristics similar to those of a common animal behavior known as Area Restricted Search (ARS) [17,18]. The clearest example of ARS is foraging: an animal will move around with small movements, staying essentially in the same patch as long as there is a lot of food to be found then, as the food becomes scarce, will do a long migration to find a new patch. This behavior, which in animals is controlled by dopamine, is at the base of many problem-solving behaviors in virtually all animals[1], from the nematode C.elegans hunting for food, to the saccadic eye movements of a person looking at an image. The reason for this universality (apart from the early appearance of dopamine control in evolution) is that ARS is optimal for "patchy" resources in which the location and characteristics of the patches are not known in advance [18].

In the continuum, Lévy walks have a dynamic behavior that mimics that of ARS walks, and this has led to their study in connection with search problems [18]. The analysis has been extended to Lévy walks on graphs, in which the walker jumps from a node to another at a distance $d$ with probability $p(d) \sim d^{-\alpha}$, as in the continuum case, but where now $d$ is the length of the shortest path between two nodes [16]. This study has led to useful insight into the behavior of these walks on graphs.

One issue that one might rise in connection with this work is whether a Lévy walk is a suitable model of search behavior. When one is searching the web, for example, one, with the exception of the immediate neighbors ($d = 1$), one has no idea what pages are at a shortest-path distance $d$ from the one they are looking at. In a social network, one might conceivably jump directly at the site of a friend of a friend ($d = 2$) but, beyond that, it is virtually impossible to decide to execute a jump at a distance $d$.

One, more reasonable, model seems to be the following.

*Simone Santini was supported in part by the Spanish Ministerio de Ciencia e Innovación under the grant N. PID2019-108965GB-I00 Más allá de la recomendación estática: equidad, interacción y transparencia

More precisely: the behavior is found in all eumetazoa that is, in all animals except porifera, a class that contains sponges and little more.
A person keeps looking for information moving from one page to another but at any given time, with probability \( u \), she invokes a second mechanism (e.g., a search engine) that leads to a new page that has no link-structural relation with the one she was just visiting. That is, from the point of view of a random walk, the behavior is the following: at any moment, with probability \( 1-u \), move to one of your neighbors chosen at random; with probability \( u \), jump to a random node in the graph. This type of walk is known as the Page Rank random walk\(^3\)\(^\[1\] \)\(^\[2\] \)\(^\[3\] \).

In this paper, we analyze the Page Rank random walk and derive some of its features, most notably, the dynamics that leads to the stationary solution, the number of new nodes visited as a function of time, and the average time of the walk between two random nodes of the graph.

2. BASIC EQUATIONS

Let \( G = (V,E) \) be an undirected graph with \( n \) nodes and \( m \) edges \((V = \{1,\ldots,n\}, E \subseteq V \times V, |E| = 2m)\), with adjacency matrix \( A \) \((a_{ij} = 1\text{ if } (i,j) \in E, a_{ij} = 0\text{ otherwise})\). Let \( d_i = \sum_j a_{ij} \) be the degree of node \( i \), that is, the number of its neighbors. Also, define the matrix \( D = \text{diag}(d_1,\ldots,d_n) \).

In a random walk, we start at time \( t = 0 \) from a node \( v_0 \in V \) and, at time \( t \) we are in node \( v_t \), we move to one of its neighbors with probability \( 1/d_i \). Define the matrix \( W = D^{-1}A \) with elements

\[
w_{ij} = \begin{cases} \frac{a_{ij}}{d_i} & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}
\]

Then \( w_{ij} \) is the probability of moving from node \( i \) to node \( j \) in a single step. Let \( p_i(t) = \mathbb{P}[v_t = i] \) be the probability of being in node \( i \) at time \( t \). The evolution of this probability is given by the Master Equation

\[
p_i(t + 1) = \sum_{j: (i,j) \in E} \frac{1}{d_j} p_j(t) = \frac{1}{d_i} \sum_{j=1}^n a_{ij} p_j(t) = \frac{1}{d_i} \sum_{j=1}^n w_{ij} p_j(t)
\]

If we collect the probabilities in a vector \( |p(t)\rangle = |p_1(t),\ldots,p_n(t)\rangle \) then the master equation can be rewritten as

\[
|p(t + 1)\rangle = W |p(t)\rangle
\]

That is, \( |p\rangle^0 \) is the initial probability distribution of the walker

\[
|p(t)\rangle = (W)^t |p^0\rangle
\]

It is easy to check that the probability distribution \( |\pi\rangle \), with

\[
\pi_i = \frac{d_i}{\sum_k d_k} = \frac{d_i}{2m}
\]

satisfies

\[
|\pi\rangle = W |\pi\rangle
\]

and is therefore a stationary distribution of the walk. It is also possible to show that this distribution is unique \[12]. From \[15] it follows that \( |\pi\rangle \) is an eigenvector of \( W \) corresponding to \( \lambda_1 = 1 \) (or, equivalently, a right eigenvector of \( W \): \( \langle \pi | = \langle \pi | W \) and the unicity of the stationary distribution implies that the eigenvalue \( \lambda_1 = 1 \) has multiplicity 1. Since the eigenvector \( |\pi\rangle \) has all positive components, it follows from the Frobenius-Perron theorem \[3\] that

\[
1 = \lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_n \geq -1
\]

We show in Appendix A that \( \lambda_n = -1 \) only for bipartite graphs, a case that we do not consider, so we shall always assume \( \lambda_n > -1 \). If we are at a node \( i \), the probability of moving to one of its neighbor is one. Consequently

\[
\sum_j w_{ij} = \sum_j \frac{a_{ij}}{d_i} = \frac{1}{d_i} \sum_{(i,j) \in E} 1 = 1
\]

If \( |1_n\rangle = |1,\ldots,1\rangle \), it is easy to see from \[3\] that \( W |1_n\rangle = |1_n\rangle \), or

\[
|1_n\rangle = \langle 1_n | W
\]

That is, the constant vector \( |1_n\rangle \) is the right eigenvector of \( W \) corresponding to \( \lambda_1 = 1 \).

The left eigenvectors other than the first correspond to eigenvalues with \( |\lambda| < 1 \). These have a property that will be quite relevant in our context:

**Lemma 2.1.** Let \( |v\rangle \) be a left eigenvector of \( W \) corresponding to an eigenvalue \( \lambda \neq 1 \). Then

\[
\sum_i v_i = 0
\]

**Proof.** The eigenvalue equation gives

\[
\sum_j w_{ij} v_j = \lambda v_i
\]

Summing the components we have

\[
\lambda \sum_i v_i = \sum_i \sum_j w_{ij} v_j = \sum_j \sum_i w_{ij} v_j = \sum_j v_j
\]

where the sum over \( j \) in the third expression is equal to 1 because of \[3\]. The equation

\[
\lambda \sum_i v_i = \sum_j v_j
\]

with \( \lambda \neq 1 \) has \[10\] as the only solution \( \square \).

The matrix \( W \) can be diagonalized as

\[
W' = T A T^{-1}
\]

Let \( T \) and \( T^{-1} \) be defined as

\[
T = [|\phi_1\rangle \cdots |\phi_n\rangle]
\]

and

\[
T^{-1} = \begin{bmatrix} \langle \psi_1 | \\ \vdots \\
\langle \psi_n | \end{bmatrix}
\]

Then \( |\phi_i\rangle \) are eigenvectors and \( \langle \psi_i | \) right eigenvectors of \( W' \), that is

\[
W' |\phi_i\rangle = \lambda_i |\phi_i\rangle
\]

\[
\langle \psi_i | W' = \lambda_i \langle \psi_i |
\]
Let $t_{ij}$ be the $i,j$ element of $T$ and $\tau_{ij}$ that of $T^{-1}$, then

$$t_{ij} = \langle \epsilon_i | \phi_j \rangle = \phi_{i,j} \quad \text{(the $i$th component of $|\phi_j\rangle$)},$$

and

$$\tau_{ij} = \langle \psi_i | e_j \rangle = \psi_{i,j} \quad \text{(the $j$th component of $|\psi_i\rangle$}).$$

We have

$$\delta_{ij} = (T^{-1}T)_{ij} = \sum_k \tau_{ik} t_{kj} = \sum_k \psi_{i,k} \phi_{j,k} = \langle \psi_i | \phi_j \rangle \quad \text{(18)}$$

and

$$\delta_{ij} = (TT^{-1})_{ij} = \sum_k t_{ik} \tau_{kj} = \sum_k \phi_{i,k} \psi_{j,k} \quad \text{(19)}$$

The matrix $W$ is that of the standard random walk, and its only equilibrium point is the probability vector $|\pi\rangle$. This vector satisfies the equation $W|\pi\rangle = |\pi\rangle$, that is, $\pi$ is an eigenvector relative to $\lambda = 1$ and therefore $\phi_1 \sim \pi$, or

$$\phi_{1,i} = bd_i \quad \text{(20)}$$

with $b > 0$. From (10) we have

$$\langle 1_n | W' = \langle 1_n | \quad \text{(21)}$$

that is,

$$\langle \psi_1 | = a \langle 1_n | \quad \text{(22)}$$

with $a > 0$. One of the constants $a$ and $b$ can be determined by condition (18) (the other one is arbitrary):

$$1 = \langle \psi_1 | \phi \rangle = a \cdot \sum_i d_i \quad \text{(23)}$$

We can choose $b = 1 / \sum_i d_i$, leading to

$$\phi_{1,i} = \frac{d_i}{\sum_k d_k} = \pi_i \quad \text{(24)}$$

and $a = 1$, that is $\langle \psi_1 | = \langle 1_n |$. Note that

$$\sum_k \phi_{1,k} = 1 \quad \text{and} \quad \sum_k \psi_{1,k} = n \quad \text{(25)}$$

3. PAGE RANK WALK

In the Page Rank random walk, at each step we toss a coin with probability $u$ of giving heads. If the result is tail, we pick a random neighbor and move to it. If the result is head, we jump to a random node in the graph. That is, if $r_{ij}$ is the probability of jumping from a node $i$ to a node $j$, we have

$$r_{ij} = (1 - u) a_{ij} + u \frac{1}{n} = (1 - u) w_{ij} + u \frac{1}{n} \quad \text{(26)}$$

The master equation for the page rank walk follows the general schema of the standard walk, and is given by

$$p_i(t + 1) = \sum_j p_j(t) \omega_{ji}$$

$$= \sum_j (1 - u) w_{ji} p_j(t) + u \frac{1}{n} \sum_j p_j(t) \quad \text{(27)}$$

$$= \sum_j (1 - u) w_{ji} p_j(t) + u \frac{1}{n}$$

Defining the vector

$$|p(t)\rangle \triangleq [p_1(t), \ldots, p_n(t)]' \quad \text{(28)}$$

That is

$$|p(t + 1)\rangle = (1 - u) W' |p(t)\rangle + u \frac{1}{n} |1_n\rangle \quad \text{(29)}$$

The stationary point of this iteration (occupancy probability at steady state) is given by

$$|p^*\rangle = (1 - u) W' |p^*\rangle + u \frac{1}{n} |1_n\rangle \quad \text{(30)}$$

Since $1 \geq \lambda_i > -1$ and $u > 0$, the matrix $I - (1 - u) W$ has all the eigenvalues strictly positive and therefore it is non singular. Eq. (30) can then be solved:

$$|p^*\rangle = [I - (1 - u) W']^{-1} u |1_n\rangle \quad \text{(31)}$$

Equation (29) can be rewritten as

$$|p(t + 1)\rangle - |p(t)\rangle = - [I - (1 - u) W'] |p(t)\rangle + u \frac{1}{n} |1_n\rangle \quad \text{(32)}$$

with

$$Q = [I - (1 - u) W'] \quad \text{(33)}$$

and the equilibrium can be rewritten as

$$|p^*\rangle = Q^{-1} u \frac{1}{n} |1_n\rangle \quad \text{(34)}$$

4. DYNAMICS OF THE PAGE RANK WALK

The matrix $Q = [I - (1 - u) W']$ has the same eigenvectors as $W'$ and eigenvalues

$$\mu_i = 1 - (1 - u) \lambda_i \quad \text{(35)}$$

with

$$u = \mu_1 < \mu_2 \leq \cdots \leq \mu_n = 1 - (1 - u) \lambda_n < 2 - u \quad \text{(36)}$$

If $u = 0$ the matrix $Q$ is singular. We shall not consider this case, which reduces to the standard random walk. If $u > 0$, $Q$ can be decomposed as

$$Q = TL T' \quad \text{(37)}$$

Note that we place the eigenvalues in an usual order: normally the first eigenvalue is the largest while in out case $\mu_1$ is the smaller. We do this to maintain a simpler correspondence between $\mu_i$ and $\lambda_i$.

Remark 1: Before we continue, we shall do a "sanity check" to verify that indeed $p^*$ is a probability vector, that is, that $\sum_i \mu_i = 1 \times 1$. At equilibrium, we have

$$|p^*\rangle = Q^{-1} u \frac{1}{n} |1_n\rangle = u TL T^{-1} |1_n\rangle \quad \text{(38)}$$

From the shape of $T^{-1}$, we have

$$T^{-1} |1_n\rangle = \sum_k \psi_{i,k} \quad \text{(39)}$$

that is

$$T^{-1} |1_n\rangle = \sum_k \psi_{1,k} |e_k\rangle \quad \text{(40)}$$

From this it follows

$$L^{-1} T^{-1} |1_n\rangle = \sum_k \frac{1}{\mu_k} \sum_k \psi_{n,k} |e_k\rangle \quad \text{(41)}$$
That is

$$|p^*\rangle = \frac{u}{n} \sum_h \frac{1}{\mu_h} \sum_k \psi_{h,k} |\phi_h\rangle$$

and

$$\sum_i p_i^* = \frac{u}{n} \sum_h \frac{1}{\mu_h} \sum_k \psi_{h,k} \sum_i \phi_{h,i} \mu \geq \frac{u}{n} \mu = 1$$

Because of lemma 2.1 the only term remaining is that with $h = 1$ for which $\mu_1 = u$, therefore we have

$$\sum_i p_i^* = \frac{1}{n} \sum_k \psi_{1,k} \sum_i \phi_{1,i} = \frac{u}{n} \sum_k \psi_{1,k} \phi_{1,k}$$

where the equality † comes from 25.

(end of remark)

* * *

Let us now go back to the iteration (32). If we iterate it $\Delta$ times, assuming that the right-hand side is kept constant, we have

$$|p(t + \Delta)\rangle - |p(t)\rangle = \Delta \left[ -Q |p(t)\rangle + \frac{u}{n} |1_n\rangle \right]$$

If we now consider $t$ a continuous variable, divide by $\Delta$ and take the limit for $\Delta \to 0$ we obtain the continuous approximation of the walk, that is

$$\frac{d}{dt} |p\rangle = -Q |p\rangle + \frac{u}{n} |1_n\rangle$$

This equation can be interpreted as a diffusion equation under the spatial operator $Q$. We are interested in studying the time evolution of the probability as it approaches the steady state.

The solution of (47) is

$$|p(t)\rangle = \frac{u}{n} Q^{-1} |1_n\rangle + e^{-Qt} |C\rangle$$

where $|C\rangle$ is an arbitrary vector that depends on the initial conditions. Using the initial probabilities $|p^i\rangle$, we have

$$|p(t)\rangle = \frac{u}{n} \left( I - e^{-Qt} \right) Q^{-1} |1_n\rangle + e^{-Qt} |p^0\rangle$$

$$= \frac{u}{n} T \left( I - e^{-Lt} \right) L^{-1} |1_n\rangle + Te^{-Lt} T^{-1} |p^0\rangle$$

In order to analyze more in detail the dynamics of the walk, we move to the eigenvector basis. Define $|\zeta\rangle = T^{-1} |p\rangle$, $|\zeta^0\rangle = T^{-1} |p^0\rangle$, and

$$T^{-1} |1_n\rangle = |b\rangle = |b_1, \ldots, b_n\rangle' \ b_i = \sum_k \psi_{i,k}$$

With these we obtain

$$\frac{d}{dt} |\zeta\rangle = -L |\zeta\rangle + \frac{u}{n} |b\rangle$$

with solution

$$|\zeta\rangle(t) = \frac{u}{n} \left( I - e^{-Lt} \right) L^{-1} |b\rangle + e^{-Lt} |\zeta^0\rangle$$

The matrix that determines the dynamics are in this case diagonal, therefore each direction in the eigenvector space evolves independently. Consider the direction of the first eigenvector, corresponding to $\mu_1 = u$:

$$\zeta_1 = \frac{u}{n} \left( 1 - e^{-ut} \right) L^{-1} |b\rangle_1 + e^{-ut} |\zeta_1^0\rangle$$

$$= \frac{u}{n} \left( 1 - e^{-ut} \right) \frac{1}{u} \sum_k \psi_{1,k} + e^{-ut} T^{-1} |p^0\rangle_1$$

$$= 1 - e^{-ut} + e^{-ut} T^{-1} |p^0\rangle_1$$

If the walk begins at a specific node $m$, then $p^0_k = \delta_{k,m}$ that is

$$T^{-1} |p^0\rangle_1 = \sum_k \psi_{1,k} \delta_{k,m} = 1$$

leading to $\zeta_1 = 1$. The first eigenvalue $\mu_1$ does not contribute to the dynamics of the probability distribution, which is determined uniquely by $\mu_2, \ldots, \mu_n$, the only eigenvalues that actually depend on the structure of the graph.

5. NODE VISITATION

In the previous section we studied the evolution of the occupancy probability of the various nodes, on evolution that is controlled by the values $(\mu_i)_{i \geq 2}$.

We are now interested in a different aspect of the walk, namely how fast we visit the nodes of the graph, and whether Page Rank allows us to visit the nodes faster than a regular random walk.

Let us consider a graph in which traveling a random edge leads us to a node with an average of $\bar{v} = \sum_v v / n$ nodes. With probability 1, the node we came from. At the node we went back to $v_1$ neighbors and, among these, there is exactly one that we visited.

There are two ways in which this can happen: either we moved to a neighbor $v'$ and from that we walked to one of the neighbors (this happened with probability $1 - \bar{v}$), or we decide to jump to a random node of the graph and with that jump we arrived at $v$ (this happened with probability $\bar{v}$).

Once we are at $v$, we also have two alternatives as to what to do: we can either move to a neighbor (with probability $1 - \bar{v}$) or jump to a random node (with probability $\bar{v}$).

Let us consider separately the two ways in which we may have arrived at $v$ and the possible ways in which we can leave $v$.

Arrived from a neighbor (probability $1 - \bar{v}$): the node $v$ has on average $\bar{q}$ neighbors and, among these, there is $v'$, the node we came from. At the node $v$ we toss a coin:

Move to a neighbor (probability $1 - \bar{v}$): The average number of neighbors is $\bar{q}$; with probability $1/\bar{q}$ we shall go back to $v'$, and not visit any new node.

Suppose now we have just arrived at a node $v$ in the graph. There are two ways in which this can happen: either we were at a neighbor $v'$ from that we walked to one of the neighbors (this happened with probability $1 - \bar{v}$), or we decide to jump to a random node of the graph and with that jump we arrived at $v$ (this happened with probability $\bar{v}$).

Once we are at $v$, we also have two alternatives as to what to do: we can either move to a neighbor (with probability $1 - \bar{v}$) or jump to a random node (with probability $\bar{v}$).

Let us consider separately the two ways in which we may have arrived at $v$ and the possible ways in which we can leave $v$.

Arrived from a neighbor (probability $1 - \bar{v}$): the node $v$ has on average $\bar{q}$ neighbors and, among these, there is $v'$, the node we came from. At the node $v$ we toss a coin:

Move to a neighbor (probability $1 - \bar{v}$): The average number of neighbors is $\bar{q}$; with probability $1/\bar{q}$ we shall go back to $v'$, and not visit any new node.
with probability
\[(1 - u)^2 \frac{q - 1}{q} \left(1 - \frac{mt}{n}\right) \quad (55)\]

Jump to a random node (probability \(u\)): In this case, it doesn’t really matter the fact that one of our neighbors has certainly been visited: we jump to a random node in the graph, so the probability of choosing this option and visiting a new node is
\[(1 - u)u \left(1 - \frac{mt}{n}\right) \quad (56)\]

Arrived from a jump (probability \(u\)): In this case, it really doesn’t matter how we decide to leave the node, either with another jump or through a neighbor: we are in a random area of the graph, our neighbors are random nodes. The probability of visiting a new node using this option is
\[u \left(1 - \frac{mt}{n}\right) \quad (57)\]

Putting it all together, the probability that with a step of the walk we visit a new neighbor is
\[\frac{\left(1 - u\right)^2 q - 1}{q} \left(1 - \frac{mt}{n}\right) + u \left(1 - \frac{mt}{n}\right) + u \left(1 - \frac{mt}{n}\right) \quad (58)\]

where \(\nu\) depends on \(u\) and can be written, after some manipulation, as
\[\nu \triangleq -\frac{1}{q} u^2 + \frac{2}{q} u + \frac{q - 1}{q} \quad (59)\]

From \(58\), we determine that the average increment of the number of nodes visited in the random walk is
\[m_{t+1} - m_t = \nu \left(1 - \frac{mt}{n}\right) \quad (60)\]

Taking infinitesimal time steps, we transform this into a differential equation
\[\frac{d}{dt} m(t) = \nu \left(1 - \frac{m(t)}{n}\right) \quad (61)\]

which, assuming that we begin the walk from a single node \((m(0) = 1)\) gives
\[m(t) = n \left(1 - e^{-\nu t/n}\right) + e^{-\nu t/n} \quad (62)\]

Note that this is a dynamic very similar to that of the probability spread (the two are, of course, related, so this should not come as a surprise). The visit is faster is \(\nu\) is large and, deriving \(62\) with respect to \(u\), we see that the maximum of \(\nu\) for \(u = 1\). That is, the completely random visit is the fastest. This is compatible with the results already obtained in \(16\).

Of course, in practical situations, this doesn’t mean that the random jumps on a social network are the best search strategy: one must consider that nearby site often have related information, so a strategy that stays around as long as there are resources and then jumps to a random node (that is, something akin to ARS, but with random jumps in lieu of Lévy flights) might be optimal. This, however, goes beyond the scope of this note.

6. TRANSIT TIME

We are now interested in determining the “speed” at which the walk moves, that is, the average time that it takes, if we start from a node \(i\), to reach a node \(j\). We begin by rewriting the master equation for continuous time. Note that the values \(p_{ij}(t)\) are discrete probabilities on the nodes but probability densities if we consider time. That is, if we start from node \(i\), the probability of arriving at node \(j\) in the time interval \([t, t+dt]\) is \(p_{ij}(t)dt\). With this in mind, we can write the master equation as
\[p_{ij}(t) = \delta_{ij} \delta(0) + \int_0^t p_{ij}(t - \tau) F_{ij}(\tau) d\tau \quad (63)\]

where \(\delta_{ij}\) is the Kronecker delta, \(\delta(0)\) the Dirac’s, and \(F_{ij}(t)\) is the probability density of first passage, that is, \(F_{ij}(t)dt\) is the probability that, starting from node \(i\), we go for the first time through node \(j\) in the interval \([t, t+dt]\). Taking the Laplace transform of this relation, we have
\[\tilde{p}_{ij}(s) = \delta_{ij} + \tilde{p}_{jj}(s) \tilde{F}_{ij}(s) \quad (64)\]

that is
\[\tilde{F}_{ij}(s) = \frac{\tilde{p}_{ij}(s) - \delta_{ij}}{\tilde{p}_{jj}(s)} \quad (65)\]

This quantity is related to the expected time that we are looking for. The expected time to go from \(i\) to \(j\) is the average of the first passage, that is
\[\langle T_{ij} \rangle = \int_0^\infty t F_{ij}(t) \ dt = -\tilde{F}_{ij}(0) \quad (66)\]

In order to compute the derivative, we follow a procedure similar to that in \(16\), translating it to continuous time. We have
\[\tilde{p}_{ij}(s) = \int e^{-st} p_{ij}(t) dt = p_j^* \int e^{-st} + \int e^{-st} [p_{ij}(t) - p_j^*] dt \quad (67)\]

\[\triangleq p_j^* e^{-st} + \int e^{-st} \tilde{p}_{ij}(t) dt \]

where we have defined \(\tilde{p}_{ij}(t) \triangleq p_{ij}(t) - p_j^*\). Computing the first integral and expressing the exponential in the second as a series, we have
\[\tilde{p}_{ij}(s) = \frac{p_j^*}{s} + \sum_{n=0}^\infty \frac{(-1)^n}{n!} s^n \int t^n \tilde{p}_{ij}(t) dt \]

\[\triangleq \frac{p_j^*}{s} + \sum_{n=0}^\infty \frac{(-1)^n}{n!} s^n R_{ij}^n \quad (68)\]

where we have defined the moments
\[R_{ij} = \int_0^\infty t^n \tilde{p}_{ij}(t) dt \quad (69)\]

and the shortcut
\[Q_{ij} = \sum_{n=0}^\infty \frac{(-1)^n}{n!} s^n R_{ij}^n \]
Note that $Q_{ij}(0) = R_{ij}^{(0)}$. We can now rewrite (65) as

$$
\tilde{F}_{ij}(s) = \frac{1}{p_j^* + Q_{ij}(s) - \delta_{ij}} \left[ p_j^* + Q_{ij}(s) - \delta_{ij} \right] - \frac{(Q_{ij}(s) + sQ_{ij}')(s)}{p_j^* + sQ_{ij}(s)}
$$

(71)

It derivative is

$$
\tilde{F}_{ij}'(s) = \frac{Q_{ij}(s) - \delta_{ij} + sQ_{ij}'}{p_j^* + sQ_{ij}(s)} - \frac{(Q_{ij}(s) + sQ_{ij}')(s)}{(p_j^* + sQ_{ij}(s))^2}
$$

(72)

Computing it in $s = 0$ we obtain

$$
\langle T_{ij} \rangle = \frac{1}{p_j^*} \left[ R_{ij}^{(0)} - R_{ij}^{(0)} + \delta_{ij} \right]
$$

(73)

Taking the average over all pair of nodes, we have

$$
\langle T \rangle = \sum_{i \neq j} \langle T_{ij} \rangle p_j^* = \sum_i R_{ii}^{(0)}
$$

(74)

In order to compute the values $R_{ii}^{(0)}$ we need to express $p_{ij}(t)$. From (48) we have

$$
p_{ij}(t) = \frac{u}{N} (e_j | I - \alpha Q | e_i) + \langle e_j | e^{-\alpha t} Q^{-1} | e_i \rangle
$$

(75)

that is

$$
\dot{p}_{ij}(t) = \langle e_j | e^{-\alpha t} Q^{-1} | e_i \rangle - \frac{u}{N} \langle e_j | e^{-\alpha t} | e_i \rangle - \frac{u}{N} \langle e_j | e^{-\alpha t} Q^{-1} | e_i \rangle
$$

(76)

Using the definitions (15), (16), and (50), we can expand this equation as

$$
\dot{p}_{ij}(t) = \sum_k e^{-\mu_k t} \phi_{k,j} \psi_{k,i} - \frac{u}{N \mu_k} \phi_{k,j} b_k
$$

(77)

That is

$$
R_{ij}^{(0)} = \int_0^\infty \dot{p}_{ij}(t) = \sum_k \frac{1}{\mu_k} \left[ \phi_{k,j} \psi_{k,i} - \frac{u}{N \mu_k} \phi_{k,j} b_k \right]
$$

(78)

$$
= \sum_k \frac{1}{\mu_k} \psi_{k,i} \psi_{k,i} - \frac{u}{N \mu_k^2} \sum_k \phi_{k,j} b_k
$$

(79)

The first summation over $i$ is equal to 1 because of (17), while of the second only the first term remains, the others being zero because of lemma 2.1. Considering that $\mu_1 = u$ and applying (20), we have

$$
\langle T \rangle = \sum_k \frac{1}{\mu_k} - \frac{u}{N \mu_k^2} \sum_i \phi_{k,i} \psi_{k,i}
$$

(80)

This result is directly comparable with that of (16). If $u = 0$, the result is the same as that of (16) when $\alpha \to \infty$. Note however that in (16) the dependence on the parameter $\alpha$ is hidden in the dependence of $\lambda_k$ while, in our case, $\lambda_k$ is a constant depending only on the structure of the graph, and the dependence on $u$ is explicit.

Note that, just as in the case of the dynamics, the result does not depend on $\mu_1$, but only on the eigenvalues $\mu_2, \ldots, \mu_n$ which describe the structure of the graph.
and the error is more pronounced for small $u$. The model slightly overestimates the number of nodes, was very small, and it is not shown to avoid clutter in the figure. As a function of $u$ closer to the standard random walk.

Figure 2: The average time to transit between nodes as a function of $u$: simulation data vs. model predictions.

7. SIMULATIONS

We consider a Random graph created using the algorithm of Leskovec et al. [11] with 1,000 nodes. On the graph, we execute 200 random walks starting at a randomly selected node and terminating when all the nodes have been visited. Using these walks we estimate the number of new nodes visited as a function of the number of steps, as well as the average time to move from one node to another (the first node of the estimation is, for each walk always the origin of the walk).

Figure 4 shows the number of nodes visited and the predicted value for $u = 0.1, 0.5, 0.8$. The variance of the data was very small, and it is not shown to avoid clutter in the figure. The model slightly overestimates the number of nodes, and the error is more pronounced for small $u$ (viz. for walks closer to the standard random walk.

Figure 2 shows the average transit time between nodes as a function of $u$. In this case, and coherently with the overestimation of Figure 1 the model underestimates the actual time, although it always remains within the standard deviation of the data, and the error is especially pronounced for small $u$. The eigenvalues of the 1000 x 1000 matrix used for the model were estimated using an iterative method [14], and their accuracy was not verified.

8. CONCLUSIONS

We have used a continuous approximation and spectral analysis to determine the diffusion characteristics of Page Rank random walks. In this, we have found some parallel with Lévy Random walks, which also give raise to superdiffusion [17]. The usefulness of this result is apparent especially in modeling searches: we now have a plausible model of people’s individual search behavior that can provably generate walks whose statistical characteristics match those of macroscopic search behaviors, such as ARS.

9. REFERENCES

[1] Intizar Ali, Ajay Gopinathan, et al. Levy walks in non-euclidean spaces. Bulletin of the American Physical Society, 63, 2018.
[2] Andrew Barbour and Denis Mollison. Epidemics and random graphs. In Stochastic processes in epidemic theory, pages 86–9. Springer, 1990.
[3] Türker Biyikoglu, Josef Leydold, and Peter F Stadler. Laplacian eigenvectors of graphs: Perron-Frobenius and Faber-Krahn type theorems. Springer, 2007.
[4] Christoph Boërgers and Claude Greengard. On the mean square displacement in levy walks. SIAM Journal on Applied Mathematics, 80(3):1175–96, 2020.
[5] Denis Boyer, Gabriel Ramos-Fernández, Octavio Miramontes, José L Mateos, Germinal Cocho, Hernán Larralde, Humberto Ramos, and Fernando Rojas. Scale-free foraging by primates emerges from their interaction with a complex environment. Proceedings of the Royal Society B: Biological Sciences, 273(1595):1743–50, 2006.
[6] Sergey Brin and Lawrence Page. The anatomy of a large-scale hypertextual web search engine. In Proceedings of the Seventh International World Wide Web Conference, 1998.
[7] Tom Britton, Svante Janson, and Anders Martin-Löf. Graphs with specified degree distributions, simple epidemics, and local vaccination strategies. Advances in Applied Probability, 39(4):922–48, 2007.
[8] Dirk Brockmann, Lars Hufnagel, and Theo Geisel. The scaling laws of human travel. Nature, 439(7075):462–5, 2006.
[9] Marta C Gonzalez, Cesar A Hidalgo, and Albert-Laszlo Barabasi. Understanding individual human mobility patterns. nature, 453(7196):779–782, 2008.
[10] José Luis Iribarren and Esteban Moro. Impact of human activity patterns on the dynamics of information diffusion. Physical review letters, 103(3):038702, 2009.
[11] Jure Leskovec, Lars Backstrom, Ravi Kumar, and Andrew Tomkins. Microscopic evolution of social networks. In Proceedings of the 14th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 462–470. ACM, 2008.
[12] László Lovász. Random walks on graphs: a survey. In D. Milos, V. T. Sos, and T. Szony, editors, Combinatorics, Paul Erdős is Eighty, pages 353–98. Budapest, János Bolyai Mathematical Society, 1996.
[13] Lawrence Page, Sergey Brin, Rajeev Motwani, and Terry Winograd. The PageRank citation ranking: Bringing order to the web. Technical report, Stanford InfoLab, 1999.
[14] William H. Press, Brian P. Flannery, Saul A. Teukolsky, and William T. Vetterling. Numerical Recipes, The Art of Scientific Computing. Cambridge University Press, 1986.
[15] Gabriel Ramos-Fernández, José L Mateos, Octavio Miramontes, Germinal Cocho, Hernán Larralde, and Barbara Ayala-Orozco. Lévy walk patterns in the foraging movements of spider monkeys (ateles geoffroyi). Behavioral ecology and Sociobiology, 55(3):223–30, 2004.
[16] A. P. Riascos and José L. Mateos. Long-range navigation on complex networks using Lévy random
Therefore, for the structure of networks, we consider now the vector $W$. From this and the structure of the adjacency matrix $A$, we have

$$W = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

with $W_1 \in \mathbb{R}^{n \times m}$ and $W_2 \in \mathbb{R}^{m \times n}$. From the general properties of transition matrix, we have

$$W|_{1_{n+m}} = |1_{n+m}|$$

From this and the structure of $W$, it follows

$$W_1|_{1_n} = |1_n|$$

$$W_2|_{1_m} = |1_m|$$

Consider now the vector

$$b = \begin{bmatrix} |1_n| \\ -|1_m| \end{bmatrix}$$

For the structure of $W$ and the equations that we use in the paper use $W$. The two, however, have the same eigenvalues—but not the same eigenvectors, of course.

Suppose now that $W$ has an eigenvalue $-1$. Let $b$ its eigenvector, and write

$$|c| = W|b|$$

Clearly $c_i = -b_i$.

We first show that all the components of $b$ have the same absolute value. Suppose, by contradiction

$$b_k = \min\{b_i\} < \max\{b_i\} = b_k$$

Then

$$|c_k| = \sum_j w_{kj} b_j \leq \sum_j w_{kj} |b_j| \leq |b_k| \sum_j w_{kj} = |b_k|$$

where the inequality (‡) is strict because of (8) and the last equality is due to (8). This contradicts $c_i = -b_i$.

Since all the elements have the same absolute value, we can consider a vector $b$ composed only of unitary elements, that is $b \in \{-1, 1\}^{n+m}$, and we can shift rows and columns of $W$ so that all the $1$s are in the first positions, that is, $b$ is as in (85). Consider now the element $c_i$ with $i \leq n$. Since $b$ is an eigenvector for the eigenvalue $-1$, we must have:

$$c_i = \sum_j w_{ij} b_j = \sum_{j=1}^n w_{ij} - \sum_{j=n+1}^{n+m} w_{ij} = -b_i = -1$$

Since $w_{ij} \geq 0$ and (25) holds, the only way to obtain -1 is to have $w_{ij} = 0$ for all $j = 1, \ldots, n$, so that

$$\sum_{j=n+1}^{n+m} w_{ij} = 1$$

The same argument applies to all $i = 1, \ldots, n$. A similar argument for $i = n+1, \ldots, n+m$ shows that in that case we must have $w_{ij} = 0$ for $j = n+1, \ldots, n+m$.

Putting the two together we have the condition

$$w_{ij} = 0 \quad \text{if } (i \leq n \text{ and } j \leq m) \text{ or } (i > n \text{ and } j > n)$$

This condition gives us the structure (85) for $W$, which implies that the graph is bipartite. \qed

**APPENDIX**

**A. EIGENVALUE $\lambda = -1$**

Let $W$ be the random walk transition matrix of a graph, which entails

$$w_{ij} \geq 0$$

as well as (8). We want to prove the following:

**Theorem A.1.** $W$ has an eigenvalue $\lambda = -1$ iff the graph is bipartite.

**Proof.** Suppose first the graph is bipartite, with the two sets of nodes containing $n$ and $m$ nodes. We can label the nodes so that the nodes in the first set are $1, \ldots, n$ and those in the second are $n+1, \ldots, m$. This entails that the adjacency matrix $A$ has structure

$$A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$$

and the walk matrix is

$$W = \begin{bmatrix} 0 & W_1 \\ W_2 & 0 \end{bmatrix}$$

Consider now the vector

$$b = \begin{bmatrix} |1_n| \\ -|1_m| \end{bmatrix}$$

Therefore $b$ is an eigenvector with eigenvalue $-1$. \[Note that we determine the eigenvalue of $W$ while the equations that we use in the paper use $W$. The two, however, have the same eigenvalues—but not the same eigenvectors, of course.\]