Generalized solution concepts in games with possibly unaware players

Leandro C. Rêgo · Joseph Y. Halpern

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Abstract Most work in game theory assumes that players are perfect reasoners and have common knowledge of all significant aspects of the game. In earlier work (Halpern and Rêgo 2006, arxiv.org/abs/0704.2014), we proposed a framework for representing and analyzing games with possibly unaware players, and suggested a generalization of Nash equilibrium appropriate for games with unaware players that we called generalized Nash equilibrium. Here, we use this framework to analyze other solution concepts that have been considered in the game-theory literature, with a focus on sequential equilibrium.

Keywords Economic theory · Foundations of game theory · Awareness · Sequential equilibrium

JEL Classification C70 · C72

1 Introduction

Game theory has proved to be a useful tool in the modeling and analysis of many phenomena involving interaction between multiple agents. However, standard
models used in game theory implicitly assume that agents are perfect reasoners and have common knowledge of all significant aspects of the game. There are many situations where these assumptions are not reasonable. In large games, agents may not be aware of the other players in the game or all the moves a player can make. Recently, we proposed a way of modeling such games (Halpern and Rêgo 2006). Given an underlying game \( \Gamma \), a key feature of this approach is the use of a game based on \( \Gamma \), which represents the point of view of a player (or the modeler) about the game \( \Gamma \). Since the game is no longer assumed to be common knowledge, we need a set of such games, and a mapping linking every non-terminal history \( h \) in every game to an information set in a possibly different game that represents the perception of the game for the player who is moving at \( h \).

In games with possibly unaware players, standard solution concepts cannot be applied. For example, in a standard game a strategy profile is a Nash equilibrium if each agent’s strategy is a best response to the other agents’ strategies, so each agent \( i \) would continue playing his strategy even if \( i \) knew what strategies the other agents were using. In the presence of unawareness this no longer make sense, since the strategies used by other players may involve moves \( i \) is unaware of. We proposed a generalization of Nash equilibrium consisting of a collection of strategies, one for each pair \((i, \Gamma')\), where \( \Gamma' \) is a game that agent \( i \) considers to be the true game in some situation. Intuitively, the strategy for a player \( i \) at \( \Gamma' \) is the strategy \( i \) would play in situations where \( i \) believes that the true game is \( \Gamma' \). Roughly speaking, a generalized strategy profile \( \tilde{\sigma} \), which includes a strategy \( \sigma_{i, \Gamma'} \) for each pair \((i, \Gamma')\), is a generalized Nash equilibrium if \( \sigma_{i, \Gamma'} \) is a best response for player \( i \) if the true game is \( \Gamma' \), given the strategies being used by the other players in \( \Gamma' \). We showed that every game with awareness has a generalized Nash equilibrium by associating a game with awareness with a standard game (where agents are aware of all moves) and proving that there is a one-to-one correspondence between the generalized Nash equilibria of the game with awareness and the Nash equilibria of the standard game.

As is well known, Nash equilibrium may rely on the existence of incredible threats. For example, one Nash equilibrium of the game shown in Fig. 1 has \( A \) playing down _A_ and \( B \) playing across _B_. Intuitively, \( A \) plays down because of \( B \)’s “threat” to play across _B_. But this threat does not appear credible. If \( B \) is rational and ever gets to move, he will not choose to move across _B_ since it gives him a lower payoff than playing down _B_.

![Fig. 1 A simple game](image-url)
One way to justify the existence of incredible threats off the equilibrium path in a Nash equilibrium is to view the player as choosing a computer program that will play the game for him, and then leaving. Since the program is not changed once it is set in motion, threats about moves that will be made at information sets off the equilibrium path become more credible. However, in a game with awareness, a player cannot write a program to play the whole game at the beginning of the game because, when his level of awareness changes, he realizes that there are moves available to him that he was not aware of at the beginning of the game. He thus must write a new program that takes this into account. But this means we cannot sidestep the problem of incredible threats by appealing to the use of a pre-programmed computer to play a strategy. Once we allow a player to change his program, threats that were made credible because the program could not be rewritten become incredible again. Thus, the consideration of equilibrium refinements that block incredible threats becomes even more pertinent with awareness.\(^1\)

There have been a number of different solution concepts proposed in the literature to deal with this problem (and others), including \textit{perfect equilibrium} (Selten 1975), \textit{proper equilibrium} (Myerson 1978), \textit{sequential equilibrium} (Kreps and Wilson 1982), and \textit{rationalizability} (Bernheim 1984; Pearce 1984), to name just a few. Each of these solution concepts involves some notion of best response. Our framework allows for straightforward generalizations of all these solution concepts. As in our treatment of Nash equilibrium, if \(\Gamma_1 \neq \Gamma_2\), we treat player \(i\) who considers the true game to be \(\Gamma_1\) to be a different agent from the version of player \(i\) who considers \(\Gamma_2\) to be the true game. Each version of player \(i\) best responds (in the sense appropriate for that solution concept) given his view of the game. In standard games, it has been shown that, in each finite game, there is a strategy profile satisfying that solution concept. Showing that an analogous result holds in games with awareness can be nontrivial. Instead of going through the process of generalizing every solution concept, we focus here on \textit{sequential equilibrium} since (a) it is one of the best-known solution concepts for extensive games, (b) the proof that a generalized sequential equilibrium exists suggests an interesting generalization of sequential equilibrium for standard games, and (c) the techniques used to prove its existence in games with awareness may generalize to other solution concepts.

Sequential equilibrium refines Nash equilibrium (in the sense that every sequential equilibrium is a Nash equilibrium) and does not allow solutions such as (down\(_A\), across\(_B\)). Intuitively, in a sequential equilibrium, every player must make a best response at every information set (even if it is reached with probability 0). In the game shown in Fig. 1, the unique sequential equilibrium has \(A\) choosing across\(_A\) and \(B\) choosing down\(_B\). We propose a generalization of sequential equilibrium to games with possibly unaware players, and show that every game with awareness has a generalized sequential equilibrium. This turns out to be somewhat more subtle than the corresponding argument for generalized Nash equilibrium. Our proof requires us to define a generalization of sequential equilibrium in standard games. Roughly speaking, this generalization relaxes the implicit assumption in sequential equilibrium that

\[^1\text{We thank Aviad Heifetz and an anonymous referee for raising some of these issues.}\]
every history in an information set is actually considered possible by the player. We call this notion *conditional sequential equilibrium*.

Other issues arise when considering sequential equilibrium in games with awareness. For example, in a standard game, when a player reaches a history that is not on the equilibrium path, he must believe that his opponent made a mistake. However, in games with awareness, a player may become aware of her own unawareness and, as a result, switch strategies. In the definition of sequential equilibrium in standard games, play off the equilibrium path is dealt with by viewing it as the limit of “small mistakes” (i.e., small deviations from the equilibrium strategy). Given that there are alternative ways of dealing with mistakes in games with awareness, perhaps other approaches for dealing with off-equilibrium play might be more appropriate. While other ways of dealing with mistakes may well prove interesting, since for us a player’s awareness level is given exogenously, as part of the description of the game, we can motivate our generalization of sequential equilibrium in the presence of awareness in exactly the same way that sequential equilibrium is motivated in standard games. Specifically, once the modeler knows what players are aware of at each node, off-equilibrium play can be interpreted as the result of mistakes since equilibrium play already takes into consideration players’ differential awareness.

The rest of this paper is organized as follows. In Sect. 2, we give the reader the necessary background to understand this paper by reviewing our model for games with awareness. In Sect. 3, we review the definition of sequential equilibrium for standard games and define its generalization for games with awareness. In Sect. 4, we define the concept of conditional sequential equilibrium for standard games, and prove that there is a one-to-one correspondence between the generalized sequential equilibria of a game with awareness and the conditional sequential equilibria of the standard game associated with it. We conclude in Sect. 5. Proofs of the theorems can be found in the Appendix.

2 Games with awareness

In this section, we review the definition of games of awareness. The material is taken from Halpern and Rêgo (2006); we encourage the reader to consult our earlier paper for detailed definitions, motivation, and detailed comparison to related work in the literature.

Given a standard extensive-form game described by a game tree $\Gamma$, to model a game with awareness we need a set of games based on $\Gamma$. A *game $\Gamma^+$ based on $\Gamma$* is an extensive game where some possible histories of $\Gamma$ are removed, and the remaining histories may be augmented by moves of nature that change players’ awareness about the possible histories of $\Gamma$. The game $\Gamma^+$ is intended to represent the view of a player (or the modeler) of the game $\Gamma$ actually being played.

To make this precise, we first review the standard definition of finite extensive game, and the definition of a game based on an extensive game, taken from Halpern and Rêgo (2006). A *(finite) extensive game* is a tuple $\Gamma = (N, M, H, P, f_c, \{I_i : i \in N\}, \{u_i : i \in N\})$, where
• $N$ is a finite set consisting of the players of the game.
• $M$ is a finite set whose elements are the moves (or actions) available to players (and nature) during the game.
• $H$ is a finite set of finite sequences of moves (elements of $M$) that is closed under prefixes, so that if $h \in H$ and $h'$ is a prefix of $h$, then $h' \in H$.

2 Intuitively, each member of $H$ is a history. We can identify the nodes in a game tree with the histories in $H$. Each node $n$ is characterized by the sequence of moves needed to reach $n$. A terminal history or run in $H$ is a history that is not a strict prefix of any other history in $H$. Let $Z$ denote the set of terminal histories of $H$. Let $M_h = \{m \in M : h \cdot (m) \in H\}$ (where we use $\cdot$ to denote concatenation of sequences); $M_h$ is the set of moves that can be made after history $h$.

• $P : (H - Z) \rightarrow N \cup \{c\}$ is a function that assigns to each nonterminal history $h$ a member of $N \cup \{c\}$. (We can think of $c$ as representing nature.) If $P(h) = i$, then player $i$ moves after history $h$; if $P(h) = c$, then nature moves after $h$. Let $H_i = \{h : P(h) = i\}$ be the set of all histories after which player $i$ moves.

• $f_c$ is a function that associates with every history for which $P(h) = c$ a probability measure $f_c(\cdot \mid h)$ on $M_h$. Intuitively, $f_c(\cdot \mid h)$ describes the probability of nature’s moves once history $h$ is reached.

• $I_i$ is a partition of $H_i$ with the property that if $h$ and $h'$ are in the same cell of the partition then $M_h = M_{h'}$, i.e., the same set of moves is available for every history in a cell of the partition. Intuitively, if $h$ and $h'$ are in the same cell of $I_i$, then $h$ and $h'$ are indistinguishable from $i$’s point of view; $i$ considers history $h'$ possible if the actual history is $h$, and vice versa. A cell $I \in I_i$ is called an $(i)$information set.

• $u_i : Z \rightarrow \mathbb{R}$ is a payoff function for player $i$, assigning a real number ($i$’s payoff) to each terminal history of the game.

In this paper, as in most work in game theory, we further assume that players have perfect recall: they remember all the actions that they have performed and the information sets they passed through; see Osborne and Rubinstein (1994) for the formal definition.

A game $\Gamma^+$ based on $\Gamma$ is an extensive game where all moves available to players in $\Gamma^+$ are also available to players in $\Gamma$, and the only moves in $\Gamma^+$ that are not in $\Gamma$ are moves by nature that, intuitively, are messages that change some players’ awareness of histories in $\Gamma$. Formally, $\Gamma^+ = (N^+, M^+, H^+, P^+, f_c^+, \{I_i^+ : i \in N^+\}, \{u_i^+ : i \in N^+\})$ is a game based on $\Gamma = (N, M, H, P, f_c, \{I_i : i \in N\}, \{u_i : i \in N\})$ if $\Gamma^+$ is an extensive game that satisfies the following conditions:

A1. $N^+ \subseteq N$.
A2. If $P^+(h) \in N^+$, then $\overline{h} \in H$, $P^+(h) = P(\overline{h})$ and $M_{\overline{h}}^+ \subseteq M_{\overline{h}^+}$, where $\overline{h}$ is the subsequence of $h$ consisting of all the moves in $h$ that are also in $M$. Moreover, if $h$ and $h'$ are in $H^+$, $\overline{h}$ and $\overline{h}'$ are in the same information set for player $i$ in $\Gamma$, player $i$ moves at $h$ and $h'$, and $h \cdot (m) \in H^+$, then $h' \cdot (m) \in H^+$. Intuitively, all the moves available to $i$ at $h$ must also be available to $i$ in the underlying game.

2 If $h = (m_1, m_2, \ldots, m_k)$, then the empty history and any history $h' = (m_1, m_2, \ldots, m_j)$, $j \leq k$, is a prefix of $h$. If $h'$ is a prefix of $h$ that is not equal to $h$, it is called a strict prefix.
\(\Gamma\) and, moreover, the set of moves available at \(h\) must be the same for any other history in \(H^+\) that corresponds to a history in the same information set as \(\bar{h}\) in the underlying game.

A3. If \(P^+(h) = c\), then either \(P(\bar{h}) = c\) and \(M^+_h \subseteq M^+_\bar{h}\), or \(P(\bar{h}) \neq c\) and \(M^+_h \cap M = \emptyset\). The moves in \(M^+_h\) in the case where \(M^+_h \cap M = \emptyset\) intuitively are messages sent by nature to various players that change their awareness of histories of \(\Gamma\). We assume for ease of exposition that each move in \(M^+ - M\) has the form \(m = (m_i)_{i \in N^+}\), where, intuitively, \(m_i\) is a message to \(i\) that changes \(i\)'s awareness of histories of \(\Gamma\).

We say that \(i\) has the same view in histories \(h\) and \(h'\) if he cannot distinguish the underlying sequence of moves in \(h\) and \(h'\), and nature sends exactly the same messages regarding awareness to \(i\) in \(h\) and \(h'\) at the same times.

A4. The histories \(h\) and \(h'\) are in the same information set for player \(i\) in \(\Gamma^+\) iff \(i\) has the same view in both \(h\) and \(h'\).

A5. \(\{z : z \in Z^+\} \subseteq Z\).

A6. For all \(i \in N^+\) and runs \(z \in Z^+\), if \(\bar{z} \in Z\), then \(u_i^+(z) = u_i(\bar{z})\). Thus, a player’s utility just depends on the moves made in the underlying game.

As we said, a game based on \(\Gamma\) describes either the modeler’s view of the game or the subjective view of the game of one of the players. A game with awareness collects all these different views, and describes, in each view, what view other players have. Formally, a game with awareness based on \(\Gamma = (N, M, H, P, f, \{I_i : i \in N\}, \{u_i : i \in N\})\) is a pair \((\Gamma^*, \mathcal{F})\), where

- \(\mathcal{G}\) is a countable set of games based on \(\Gamma\), of which one is \(\Gamma\);
- \(\mathcal{F}\) maps a game \(\Gamma^* \in \mathcal{G}\) and a history \(h\) in \(\Gamma^+\) such that \(P^+(h) = i\) to a pair \((\Gamma^h, I)\), where \(\Gamma^h \in \mathcal{G}\) and \(I\) is an \(i\)-information set in game \(\Gamma^h\).

Intuitively, \(\Gamma\) is the objective game or the game from the point of view of an omniscient modeler. The other games in \(\mathcal{G}\) describe a player’s subjective view of the situation. If player \(i\) moves at \(h\) in game \(\Gamma^* \in \mathcal{G}\) and \(\mathcal{F}(\Gamma^*, h) = (\Gamma^h, I)\), then \(\Gamma^h\) is the game that \(i\) believes to be the true game when the history is \(h\), and \(I\) consists of the set of histories in \(\Gamma^h\) that \(i\) currently considers possible.

The mapping \(\mathcal{F}\) must satisfy a number of consistency conditions that capture desirable properties of awareness. Consider the following constraints, where \(\Gamma^+ \in \mathcal{G}\), \(h \in H^+, P^+(h) = i\), and \(\mathcal{F}(\Gamma^+, h) = (\Gamma^h, I)\).

C1. \(h' \in I\) iff \(h'\) is a history in \(H^h\) where \(i\) has the same view as \(h\).

C2. If \(h' \in H^h\), then there exists \(h_1 \in H^+\) such that \(\bar{h}' = \bar{h}_1\). Moreover, if \(h' \cdot \langle m \rangle \in H^h\) and \(m \in M\), then for all \(h_1 \in H^+\) such that \(\bar{h}' = \bar{h}_1\) are in the same information set in \(\Gamma\) and \(P^h(h') = P^+(h_1)\), we have that \(h_1 \cdot \langle m \rangle \in H^+\). Intuitively, these conditions ensure that a player \(i\) cannot consider possible that another player \(j\) believes in a game with some history or move that he \((i)\) is unaware of.

C3. If \(h''\) is a history in \(\Gamma''\), and \(h''\) is a history in \(\Gamma''\) such that player \(i\) has the same view in \(h'\) and \(h''\), then \(\mathcal{F}(\Gamma', h') = \mathcal{F}(\Gamma'', h'')\).

C4. If \(h'\) is a prefix of \(h\), \(P^+(h') = i\), and \(\mathcal{F}(\Gamma^+, h') = (\Gamma^{h'}, I')\), then \(\bar{h}_1 : h'_1 \in H^{h'} \subseteq \{\bar{h}_1 : h_1 \in H^h\}\). Moreover, \(\Gamma^h \neq \Gamma^{h'}\) only if \(\bar{h}_1 : h'_1 \in H^{h'} \neq \{\bar{h}_1 : h_1 \in H^h\}\). Intuitively, the player never forgets some history he was previously...
aware of and he changes the game that he considers possible only if he becomes aware of more moves of the underlying game.

We define a *plausible history* to be history $h$ such that, for every player $i$, if $i$ makes a move $m$ in $h$, then $i$ is aware of $m$. More formally, a history $h$ in $\Gamma^+$ is *plausible* if for every prefix $h_1 \cdot \langle m \rangle$ of $h$, we have $F(\Gamma^+, h_1) = (\Gamma^1, I)$, where $m \in M^1_I$.

C5. There exists some plausible history in $I$.

It may seem that by making $F$ a function we cannot capture a player’s uncertainty about the game being played or uncertainty about opponents’ unawareness about histories. However, we can capture such uncertainty by folding it into nature’s initial move in the game the player consider possible while moving. It should be clear that this gives a general approach to capturing such uncertainties.

A standard extensive game $\Gamma$ can be identified with the game $((\Gamma), F)$, where for all histories $h$ in an $i$-information set $I$ in $\Gamma$, $F(\Gamma, h) = (\Gamma, I)$. (This definition of $F$ is actually forced by C1.) Thus, all players are aware of all the runs in $\Gamma$, and agree with each other and the modeler that the game is $\Gamma$. We call this the canonical representation of $\Gamma$ as a game with awareness.

In Halpern and Rêgo (2006), we discussed generalizations of games with awareness to include situations where players may be aware of their own unawareness and, more generally, games where players may not have common knowledge of the underlying game; for example, players may disagree about what the payoffs or the information sets are. With these models, we can capture a situation where, for example, player $i$ may think that another player $j$ cannot make a certain move, when in fact $j$ can make such a move. For ease of exposition, we do not discuss these generalizations further here. However, it is not hard to show that the results of this paper can be extended to them in a straightforward way.

There has been a great deal of work recently in modeling (lack of) awareness. We briefly mention the most relevant papers here, and refer to the reader to Halpern and Rêgo (2006) for a more detailed comparison. In a sequence of papers, Feinberg (2004, 2005, 2009) considered unawareness in normal-form and extensive games; the definition in Feinberg (2009) is closest to that used here. (A detailed comparison can be found in Halpern and Rêgo (2006).) Feinberg (2004) deals with extensive-form games, and defines an extension of sequential equilibrium only indirectly, via a syntactic epistemic characterization. He does not provide a more direct semantic definition of sequential equilibrium, as we do. He gives an example showing that, much like the well-known result of Kreps et al. (1982) showing that, if there is a small probability of irrationality, there is a sequential equilibrium in finitely repeated prisoner’s dilemma with cooperation for almost all rounds of the game, if there is a small probability that one player is not aware of the possibility of defecting, then there is an extended sequential equilibrium in finitely repeated prisoner’s dilemma with cooperation for almost all rounds of the game. In later work, Feinberg (2009) gives a semantic definition of sequential equilibrium in games with awareness; we compare his approach to ours after we give our formal definition of generalized sequential equilibrium. Li (2006) has also provided a model of unawareness in extensive games, based on her earlier work on modeling unawareness (Li 2008, 2009). Heifetz et al. (2008) proposed a model of
dynamic awareness and defined a generalization of the concept of rationalizability in games with awareness.

3 Generalized sequential equilibrium

To explain generalized sequential equilibrium, we first review the notion of sequential equilibrium for standard games.

3.1 Sequential equilibrium for standard games

Sequential equilibrium is defined with respect to an assessment, a pair $(\tilde{\sigma}, \mu)$ where $\tilde{\sigma}$ is a strategy profile consisting of behavioral strategies and $\mu$ is a belief system, i.e., a function that determines for every information set $I$ a probability $\mu_I$ over the histories in $I$. Intuitively, if $I$ is an information set for player $i$, $\mu_I$ is $i$’s subjective assessment of the relative likelihood of the histories in $I$. Roughly speaking, an assessment is a sequential equilibrium if for all players $i$, at every $i$-information set, (a) $i$ chooses a best response given the beliefs he has about the histories in that information set and the strategies of other players, and (b) $i$’s beliefs are consistent with the strategy profile being played, in the sense that they are calculated by conditioning the probability distribution induced by the strategy profile over the histories on the information set.

Note that $\mu_I$ is defined even if $I$ is reached with probability 0. Defining consistency at an information set that is reached with probability 0 is somewhat subtle. In that case, intuitively, once information set $I$ is reached player $i$ moving at $I$ must believe the game has been played according to an alternative strategy profile. In a sequential equilibrium, that alternative strategy profile consists of a small perturbation of the original assessment where every move is chosen with positive probability.

Given a strategy profile $\tilde{\sigma}$, let $Pr_{\tilde{\sigma}}$ be the probability distribution induced by $\tilde{\sigma}$ over the possible histories of the game. Intuitively, $Pr_{\tilde{\sigma}}(h)$ is the product of the probability of each of the moves in $h$. For simplicity we assume $f_c > 0$, so that if $\tilde{\sigma}$ is such that every player chooses all of his moves with positive probability, then for every history $h$, $Pr_{\tilde{\sigma}}(h) > 0$. For any history $h$ of the game, define $Pr_{\tilde{\sigma}}(\cdot | h)$ to be the conditional probability distribution induced by $\tilde{\sigma}$ over the possible histories of the game given that the current history is $h$. Intuitively, $Pr_{\tilde{\sigma}}(h' | h)$ is 0 if $h$ is not a prefix of $h'$, is 1 if $h = h'$, and is the product of the probability of each of the moves in the path from $h$ to $h'$ if $h$ is a prefix of $h'$. Formally, an assessment $(\tilde{\sigma}, \mu)$ is a sequential equilibrium if it satisfies the following properties:

- **Sequential rationality.** For every information set $I$ and player $i$ and every behavioral strategy $\sigma$ for player $i$,

$$EU_i((\tilde{\sigma}, \mu) \mid I) \geq EU_i(((\tilde{\sigma}_{-i}, \sigma), \mu) \mid I),$$

where $EU_i((\tilde{\sigma}, \mu) \mid I) = \sum_{h \in I} \sum_{z \in Z} \mu_I(h) Pr_{\tilde{\sigma}}(z \mid h) u_i(z)$.

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3 See Myerson (1991) for a definition of sequential equilibrium in the case nature chooses some of its move with probability 0.
• Consistency between the belief system and strategy profile. If $\tilde{\sigma}$ consists of completely mixed (behavior) strategies, that is, ones that assign positive probability to every action at every information set, then for every information set $I$ and history $h$ in $I$,

$$\mu_I(h) = \frac{\Pr_{\tilde{\sigma}}(h)}{\sum_{h' \in I} \Pr_{\tilde{\sigma}}(h')}.$$ 

Otherwise, there exists a sequence $(\bar{\sigma}^n, \mu^n)$, $n = 1, 2, 3, \ldots$, of assessments such that $\bar{\sigma}^n$ consists of completely mixed strategies, $(\bar{\sigma}^n, \mu^n)$ is consistent in the above sense, and $\lim_{n \to \infty}(\bar{\sigma}^n, \mu^n) = (\bar{\sigma}, \mu)$.

Sequential equilibrium is not a reasonable solution concept for games with awareness; it requires that a player be aware of the set of possible strategies available to other players and to him. In order to define a generalized notion of sequential equilibrium for games with awareness, we first need to define a generalized notion of assessment for games with awareness. We first need a generalized notion of strategy, which we defined in our earlier paper.

Intuitively, a strategy describes what $i$ will do in every possible situation that can arise. This no longer makes sense in games with awareness, since a player no longer understands in advance all the possible situations that can arise. For example, player $i$ cannot plan in advance for what will happen if he becomes aware of something he cannot plan in advance for what will happen if he becomes aware of something he is initially unaware of. We solved this problem in our earlier paper as follows. Let $G_i = \{\Gamma' \in G : \text{for some } \Gamma^+ \in G \text{ and } h \in \Gamma^+, P^+(h) = i \text{ and } F(\Gamma^+, h) = (\Gamma', \cdot)\}$. Intuitively, $G_i$ consists of the games that $i$ views as the real game in some history. Rather than considering a single strategy in a game $\Gamma^* = (G, F)$ with awareness, we considered a collection $\{\sigma_i, \Gamma' : \Gamma' \in G_i\}$ of local strategies for every player $i \in N$. Intuitively, a local strategy $\sigma_i, \Gamma'$ for game $\Gamma'$ is the strategy that $i$ would use if $i$ were called upon to play and $i$ thought that the true game was $\Gamma'$. Thus, the domain of $\sigma_i, \Gamma'$ consists of pairs $(\Gamma^+, h)' \text{ such that } \Gamma^+ \in G, h \text{ is a history in } \Gamma^+, P^+(h)' = i, \text{ and } F(\Gamma^+, h)' = (\Gamma', I)$. Let $(\Gamma^h, I)^* = \{(\Gamma', h) : F(\Gamma', h) = (\Gamma^h, I)\}$; we call $(\Gamma^h, I)^*$ a generalized information set.

**Definition 3.1** Given a game $\Gamma^* = (G, F)$ with awareness, a local strategy $\sigma_i, \Gamma'$ for agent $i$ is a function mapping pairs $(\Gamma^+, h)' \text{ such that } h \text{ is a history where } i \text{ moves in } \Gamma^+ \text{ and } F(\Gamma^+, h)' = (\Gamma', I) \text{ to a probability distribution over } M'_{h_1}$. The moves available at a history $h' \in I$, such that $\sigma_i, \Gamma'(\Gamma_1, h_1) = \sigma_i, \Gamma'(\Gamma_2, h_2)$ if $(\Gamma_1, h_1)$ and $(\Gamma_2, h_2)$ are in the same generalized information set. Let $B_{i, \Gamma'}$ be the set of all local strategies $\sigma_i, \Gamma'$ for player $i$ at game $\Gamma'$. A generalized strategy profile of $\Gamma^* = (G, F)$ is an element of the Cartesian product $\times_{i \in N} \times_{\Gamma' \in G_i} B_{i, \Gamma'}$. That is, a generalized strategy profile $\tilde{\sigma}$ consists of a strategy for each agent $i$ and game $\Gamma' \in G_i$.

The belief system, the second component of the assessment, is a function from information sets $I$ to probability distributions over the histories in $I$. Intuitively, it captures how likely each of the histories in $I$ is for the player moving at $I$. For standard games this distribution can be arbitrary, since the player considers every history
in the information set possible. This is no longer true in games with awareness. It is possible that a player is playing game $\Gamma_1$ but believes he is playing a different game $\Gamma_2$. Furthermore, in a game, there may be some histories in an $i$-information set that include moves of which $i$ is not aware; player $i$ cannot consider these histories possible. To deal with these problems, we define $\mu$ to be a \textit{generalized belief system} if it is a function from generalized information sets to a probability distribution over the set of histories in the generalized information set that the player considers possible.

**Definition 3.2** A \textit{generalized belief system} $\mu$ is a function that associates each generalized information set $(\Gamma', I)^*$ with a probability distribution $\mu_{\Gamma', I}$ over the set $\{(\Gamma', h) : h \in I\}$. A \textit{generalized assessment} is a pair $(\vec{\sigma}, \mu)$, where $\vec{\sigma}$ is a generalized strategy profile and $\mu$ is a generalized belief system.

We can now define what it means for a generalized assessment $(\vec{\sigma}^*, \mu^*)$ to be a \textit{generalized sequential equilibrium} of a game with awareness. The definition is a straightforward modification of the definition. Rather than write out the whole definition here, we just explain the differences.

- the use of $EU_i$ in the definition of sequential rationality is replaced by $EU_i_{\Gamma', I}$, where $EU_i_{\Gamma', I}((\vec{\sigma}^*, \mu^*) \mid I)$ is the conditional expected payoff for $i$ in the game $\Gamma'$, given that strategy profile $\vec{\sigma}^*$ is used, information set $I$ has been reached, and player $i$’s beliefs about the histories in $I$ are described by $\mu^*_{\Gamma', I}$.
- We now require consistency between the \textit{generalized} belief system and \textit{generalized} strategy profile, which means we must consider generalized information sets rather than information sets.

It is easy to see that $(\vec{\sigma}, \mu)$ is a sequential equilibrium of a standard game $\Gamma$ iff $(\vec{\sigma}, \mu)$ is a (generalized) sequential equilibrium of the canonical representation of $\Gamma$ as a game with awareness. Thus, our definition of generalized sequential equilibrium generalizes the standard definition.

To better understand the concept of generalized sequential equilibrium concept, consider the following two examples.

**Example 3.3** Consider the game shown in Fig. 2. Suppose that both players 1 and 2 are aware of all terminal histories of the game, but player 1 (falsely) believes that player 2 is aware only of the terminal histories not involving $L$ and believes that player 1 is aware of these terminal histories as well. Also suppose that player 2 is aware of all of this; that is, player 2’s view of the game can be represented by the objective game $\Gamma$ shown in Fig. 2. While moving at node 1.1, player 1 considers the true game to be identical to $\Gamma$ except that from player 1’s point of view, while moving at 2.1, player 2 believes that the true game is $\Gamma_2$, shown in Fig. 3.

This game has a unique generalized sequential equilibrium where player 2 chooses $r$ and player 1 chooses $A$ in $\Gamma_2$. Believing that player 2 will move $r$, player 1 best responds by choosing $L$ at node 1.1. Since player 2 knows all this at node 2.1 in $\Gamma$, she chooses $l$ at this node. Thus, if players follow their equilibrium strategies, the payoff vector is $(-10, -1)$. In this situation, player 2 would be better off if she could let player 1 know that she is aware of move $L$, since then player 1 would play $A$ and both players would receive 1. On the other hand, if we slightly modify the game by making
Example 3.4 Consider the following variant of the entry game (Osborne 2004). There is a monopolist ($M$) that operates in a market and a single entrant firm ($E$) that is considering whether to be in ($I$) or stay out of ($O$) the market. If $E$ stays out, then $E$ receives a payoff of $3/2$ and $M$ receives a payoff of $5$. If $E$ decides to enter, then there is uncertainty about whether $E$ is strong enough to be competitive with $M$.\footnote{In the standard entry game, there is no uncertainty, and $E$ is assumed to be strong enough to compete with $M$.} There is a small probability $p$ that $E$ is strong. Suppose that

- $E$ is aware of all moves of the game;
- $M$ is not aware of the possibility that $E$ might not be strong if she enters the market.
- $E$ is uncertain about whether $M$ is aware that she ($E$) can enter the market without being strong. $E$ believes that $M$ is unaware with probability $q > 1/2$. Moreover, $E$ knows that if $M$ is aware of this move, then $M$ knows that $E$ is uncertain and knows the probability $q$.

To model this situation, we need three augmented games: the objective game $\Gamma$ described in Fig. 4, the game from the point of view of $M$, $\Gamma_M$, which does not include any history with the move $W$, and the game from the point of view of $E$, $\Gamma_E$, which has an initial move where nature chooses $M$ to be unaware of $W$ with probability $q$ and aware of $W$ with probability $1 - q$. 

$u_2((L, l)) = 3$, then player 2 would benefit from the fact that 1 believes that she is unaware of move $L$. □
If $p < 1/2$, then the standard game described in Fig. 4 has a unique sequential equilibrium: $E$ plays $O$ and $M$ plays $F$. However, in the game of awareness just described, where $M$ is unaware that $E$ could be weak, and $E$ places high probability on this possibility, there is a generalized Nash equilibrium where $E$ plays $I$, $M$ plays $A$ if he is unaware of $W$, and $F$ if he is aware of $W$. There is also a family of generalized Nash equilibria where $E$ plays $O$ and $M$ mixes between $A$ and $F$, but plays $F$ with sufficient probability that $E$ is deterred from entering. Only the equilibrium where $E$ plays $I$ is a generalized sequential equilibrium. The generalized Nash equilibria that are not sequential equilibria rely on the incredible threat that $M$ will fight even if he is not aware that $E$ may not be strong enough to compete in the market. In the unique generalized sequential equilibrium, $E$ enters the market because he considers it sufficiently likely that $M$ is not aware that $E$ might be weak.

There may seem to be a conceptual problem with generalized sequential equilibrium. Intuitively, an unexpected move made by player $B$ can lead player $A$ to believe that $B$ is aware of something that he ($A$) is not aware of. We cannot model such a belief in games of awareness as we have defined them; here, we are implicitly treating an unexpected move as a mistake, as is done in the standard notion of sequential equilibrium. However, we can also extend games of awareness to allow for awareness
of unawareness; see Halpern and Rêgo (2006). Once we do this, we can allow for the possibility that $A$ believes that $B$ is aware of something that $A$ is not aware of. Sequential equilibrium continues to make sense in this setting. We simply allow for the possibility that $A$’s move is justified by his belief that $B$ is aware of moves that he ($A$) is not aware of, and the potential consequences of these moves.

**Example 3.5** Consider the game shown in Fig. 5. If all players were aware of all moves, then there exist the following sequential equilibria in pure strategies:

- Player 1 chooses the strategy $ugi$, player 2 chooses the strategy $f$, and the belief system gives probability 1 to the history $\langle u \rangle$, and
- Player 1 chooses the strategy $dgi$, player 2 chooses the strategy $e$, and the belief system gives probability $p \leq 1/3$ to the history $\langle u \rangle$.

Suppose instead that player 2 is unaware of the move $u$ for player 1 at the beginning of the game. Now the choice that player 2 makes will depend on the game that she believes is being played when she moves. If she believes that the game is as shown in Fig. 6, and in addition believes that this is commonly believed to be the actual game, then if player 1 knows all this, then the game has a unique generalized sequential equilibrium in pure strategies, where player 1 chooses the strategy $dgi$ in the objective game $\Gamma$ and the strategy $di$ in game $\Gamma^2$, and player 2 chooses the strategy $e$ in game $\Gamma^2$. In this equilibrium, when player 2 moves, she believes that player 1 made a mistake by following his equilibrium strategy $d$. 
If instead player 2 believes that, at his second information set, player 1 has some move available that she is unaware of, that would lead them to a better payoff, then when she moves, player 2 should believe the game is as shown in Fig. 7. If \( a > 4 \) and \( b > 3 \), then the game again has a unique generalized sequential equilibrium, where player 1 chooses the strategy \( u \) in the objective game \( \Gamma \) and the strategy \( m \) in game \( \Gamma^2v \), and player 2 chooses the strategy \( f \) in game \( \Gamma^2v \). Note that in this equilibrium, player 2 does not think player 1 has made a mistake, as was the case in the previous situation. Now she believes that 1 has some available move that she is not aware of. It is worth mentioning that in this equilibrium player 2 falsely believes that player 1 chooses \( m \) at the beginning of the game; in fact, player 1 chooses \( u \).

Feinberg (2009) takes a different route to defining an extension of sequential equilibrium for games with awareness. Instead of considering local strategies and a generalized belief system, he considers a set of standard assessments, one for each game in \( G \), and defines this set of standard assessments to be an extended sequential equilibrium if each standard assessment is consistent and the strategies are sequentially rational. Feinberg imposes some consistency conditions that relate strategies in different games in \( G \); these turn out to make his set of strategies equivalent to our set of local strategies. However, Feinberg does not require that every information set contain some plausible history. This turns out to be relevant when defining consistency. Feinberg does not make clear how he defines consistency for information sets without plausible histories.

### 3.2 Existence of generalized equilibria

We now want to show that every game with awareness \( \Gamma^* \) has at least one generalized sequential equilibrium. To prove that a game with awareness \( \Gamma^* \) has a generalized Nash equilibrium, we constructed a standard game \( \Gamma^\nu \) with perfect recall and showed that there exists a one-to-one correspondence between the set of generalized Nash equilibrium of \( \Gamma^* \) and the set of Nash equilibrium of \( \Gamma^\nu \). Intuitively, \( \Gamma^\nu \) is constructed by essentially “gluing together” all the games \( \Gamma' \in G \), except that only plausible histories in \( \Gamma' \) are considered.

More formally, given a game \( \Gamma^* = (G, F) \) with awareness, let \( \nu \) be a probability on \( G \) that assigns each game in \( G \) positive probability. (Here is where we use the fact that \( G \) is countable.) For each \( \Gamma' \in G \), let \( [H^{\Gamma'}] = \{ h \in H^{\Gamma'} : h \text{ is a plausible history} \} \).
Let $\Gamma^\nu$ be the standard game such that

- $N^\nu = \{(i, \Gamma') : \Gamma' \in \mathcal{G}_i\};$
- $M^\nu = \mathcal{G} \cup_{\Gamma' \in \mathcal{G}} [M^{\Gamma'}]$, where $[M^{\Gamma'}]$ is the set of moves that occur in $[H^{\Gamma'}];$
- $H^\nu = \langle \rangle \cup \{\langle \Gamma' \rangle \cdot h : \Gamma' \in \mathcal{G}, h \in [H^{\Gamma'}]\};$
- $P^\nu(\langle \rangle) = c$, and
  \[
P^\nu(\langle \Gamma^h \rangle \cdot h') = \begin{cases} (i, \Gamma^{h'}) & \text{if } P^h(h') = i \in N \text{ and } \mathcal{F}(\Gamma^h, h') = (\Gamma^{h'}, \cdot), \\ c & \text{if } P^h(h') = c; \end{cases}
\]
- $f^\nu(\langle \Gamma' \rangle | \langle \rangle) = v(\langle \Gamma' \rangle)$ and $f^\nu(|\langle \Gamma^h \rangle \cdot h') = f_c^h(|h')$ if $P^h(h') = c$;
- $T_{i,\Gamma^\nu}^\nu$ is a partition of $H_{i,\Gamma^\nu}^\nu$, where two histories $\langle \Gamma^1 \rangle \cdot h^1$ and $\langle \Gamma^2 \rangle \cdot h^2$ are in the same information set $\langle \Gamma', I \rangle^*$ iff $(\Gamma^1, h^1)$ and $(\Gamma^2, h^2)$ are in the same generalized information set $(\langle \Gamma', I \rangle)^*$;
- $u^\nu_{i,\Gamma^\nu}(\langle \Gamma^h \rangle \cdot z) = \begin{cases} u^h_i(z) & \text{if } \Gamma^h = \Gamma', \\ 0 & \text{if } \Gamma^h \neq \Gamma'. \end{cases}$

Unfortunately, while it is the case that there is a 1–1 correspondence between the Nash equilibria of $\Gamma^\nu$ and the generalized Nash equilibria of $\Gamma^*$, this correspondence breaks down for sequential equilibria. To see why consider the modified version of prisoner’s dilemma $\Gamma^p$, described in Fig. 8.

Besides being able to cooperate ($C_A$) or defect ($D_A$), player A who moves first has also the option of escaping ($E_A$). If player A escapes, then the game is over; if player A cooperates or defects, then player B may also cooperate ($C_B$) or defect ($D_B$). Suppose further that in the objective game,

- both $A$ and $B$ are aware of all histories of $\Gamma^p$;
- with probability $p$, $A$ believes that $B$ is unaware of the extra move $E_A$, and with probability $1 - p$, $A$ believes $B$ is aware of all histories;
- if $A$ believes $B$ is unaware of $E_A$, then $A$ believes that $B$ believes that it is common knowledge that the game being played contains all histories but $E_A$;
- if $A$ believes $B$ is aware of $E_A$, then $A$ believes that $B$ believes that there is common knowledge that the game being played is $\Gamma^p$;
- $B$ believes that it is common knowledge that the game being played is $\Gamma^p$. 


We need four games to model this situation:

- $\Gamma^p$ is the objective game;
- $\Gamma^A$ is the game from $A$’s point of view when she is called to move in the objective game;
- $\Gamma^{B.1}$ is the game from $B$’s point of view when he is called to move in game $\Gamma^A$ after nature chooses he is unaware of $E_A$ and is also the game from $A$’s point of view when she is called to move in $\Gamma^{B.1}$; and
- $\Gamma^{B.2}$ is the game from $B$’s point of view when he is called to move in game $\Gamma^A$ after nature chooses aware; $\Gamma^{B.2}$ is also the game from $B$’s point of view when he is called to move at $\Gamma^p$ and the game from $A$’s point of view when she is called to move in $\Gamma^{B.2}$.

Game $\Gamma^{B.2}$ is very similar to game $\Gamma^p$, except that it has an initial single move of nature $c^{B.2}$ that makes both players aware of all moves of $\Gamma^p$. For example, $\mathcal{F}(\Gamma^p, \langle \rangle) = (\Gamma^A, \{\text{unaware}_B, \text{aware}_B\}) \neq (\Gamma^{B.2}, \langle c^{B.2}\rangle) = \mathcal{F}(\Gamma^{B.2}, \langle c^{B.2}\rangle)$. For this reason, we use different labels for the nodes of these games. Let $A.2$ and $B.2$ be the labels of the nodes in game $\Gamma^{B.2}$ corresponding to $A$ and $B$ in $\Gamma^p$, respectively. $\Gamma^A$ and $\Gamma^{B.1}$ are shown in Figs. 9 and 10, respectively. 5

- In the objective game $\Gamma^p$, $A$ believes she is playing game $\Gamma^A$, and $B$ believes he is playing game $\Gamma^{B.2}$.
- In game $\Gamma^A$, nature chooses move unaware$_B$ with probability $p$ and aware$_B$ with probability $1 - p$. Then $A$ moves and believes she is playing $\Gamma^A$. At node $B.1$, $B$ believes he is playing $\Gamma^{B.1}$, and at node $B.2$, $B$ believes he is playing $\Gamma^{B.2}$.
- In game $\Gamma^{B.1}$, $A$ and $B$ both believe that the game is $\Gamma^{B.1}$.
- In game $\Gamma^{B.2}$, $A$ and $B$ both believe that the game is $\Gamma^{B.2}$.

The game $\Gamma^v$ is the result of pasting together $\Gamma^p$, $\Gamma^A$, $\Gamma^{B.1}$, and $\Gamma^{B.2}$. There are 5 players: $(A, \Gamma^A)$, $(A, \Gamma^{B.1})$, $(A, \Gamma^{B.2})$, $(B, \Gamma^{B.1})$, and $(B, \Gamma^{B.2})$. $(A, \Gamma^{B.1})$ and $(B, \Gamma^{B.1})$ are playing standard prisoner’s dilemma and therefore both should defect with probability 1; $(B, \Gamma^{B.1})$ must believe he is in the history where $(A, \Gamma^{B.1})$ defected with probability 1. $(A, \Gamma^A)$ and $(A, \Gamma^{B.2})$ choose the extra move $E_A$ with

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5 We abuse notation and use the same label for nodes in different games that are in the same generalized information set. For example, $B.2$ is a label at both $\Gamma^A$ and $\Gamma^{B.2}$.
probability 1, since it gives A a payoff of 5. The subtlety arises in the beliefs of \((B, \Gamma^{B.2})\) in the generalized information set \((\Gamma^{B.2}, \{\langle c^{B.2}, C_A \rangle, \langle c^{B.2}, D_A \rangle\})^*\), since this generalized information set is reached with probability zero. Note that \((\Gamma^{B.2}, \{\langle c^{B.2}, C_A \rangle, \langle c^{B.2}, D_A \rangle\})^* = \{(\Gamma^A, \langle \text{aware}_B, C_A \rangle), (\Gamma^A, \langle \text{aware}_B, D_A \rangle), (\Gamma^{B.2}, \langle c^{B.2}, C_A \rangle), (\Gamma^{B.2}, \langle c^{B.2}, D_A \rangle)\}\). By the definition of sequential equilibrium, player \((B, \Gamma^{B.2})\) will have to consider a sequence of strategies where all these histories are assigned positive probability. Although in general this is not a problem, note that \((B, \Gamma^{B.2})\) is meant to represent the type of player B that considers only histories in game \(\Gamma^{B.2}\) possible. Thus, intuitively, he should assign positive probability only to the histories \({\langle \Gamma^{B.2}, c^{B.2}, C_A \rangle, \langle \Gamma^{B.2}, c^{B.2}, D_A \rangle}\).

To see how this leads to a problem, first note that there is a sequential equilibrium of \(\Gamma^v\) where \((B, \Gamma^{B.2})\) believes with probability 1 that the true history is \(\langle \Gamma^A, \text{aware}_B, C_A \rangle\), \((A, \Gamma^{B.2})\) chooses \(E_A\) with probability 1, and \((B, \Gamma^{B.2})\) chooses \(C_B\) with probability 1. It is rational for \((B, \Gamma^{B.2})\) to choose \(C_B\) because \((B, \Gamma^{B.2})\) assigns probability 1 to the first move of nature in \(\Gamma^v\) be \(\Gamma^A\). Since his utility is 0 for every terminal history in \(\Gamma^v\) whose first move is \(\Gamma^A\), his expected utility is 0 no matter what move he makes at the generalized information set, given his beliefs.

There is no reasonable definition of generalized sequential equilibrium corresponding to this sequential equilibrium of \(\Gamma^v\). Player B while moving at node \(B.2\) would never cooperate, since this is a strictly dominated strategy for him in the game that he considers to be the actual game, namely \(\Gamma^{B.2}\).

The problem is that there is nothing in the definition of sequential equilibrium that guarantees that the belief system of a sequential equilibrium in \(\Gamma^v\) assigns probability zero to histories that players are unaware of in the game \(\Gamma^v\) with awareness. We want to define a modified notion of sequential equilibrium for standard games that guarantees that the belief system in \(\Gamma^v\) associates each information set with a probability distribution over a pre-specified subset of the histories in the information, which consists only of the histories in the information set player \(i\) actually considers possible. In this example, the pre-specified subset would be \({\langle \Gamma^{B.2}, c^{B.2}, C_A \rangle, \langle \Gamma^{B.2}, c^{B.2}, D_A \rangle}\).

### 4 Conditional sequential equilibrium

In the standard definition of sequential equilibrium for extensive games, it is implicitly assumed that every player considers all histories in his information set possible. This
is evident from the fact that if a strategy profile that is part of a sequential equilibrium assigns positive probability to every move, then by the consistency requirement the belief system also assigns positive probability to every history of every information set of the game. Therefore, this notion of equilibrium is not strong enough to capture situations where a player is certain that some histories in his information set will not occur. The notion of conditional sequential equilibrium, which we now define, is able to deal with such situations. It generalizes sequential equilibrium: in a game where every player considers every history in his information set possible, the set of conditional sequential equilibria and the set of sequential equilibria coincide.

Given a standard extensive game $\Gamma$, define a possibility system $K$ on $\Gamma$ to be a function that determines for every information set $I$ a nonempty subset of $I$ consisting of the histories in $I$ that the player moving at $I$ considers possible. We define a possibility system to be acceptable if, for every $i$-information set $I$, we have that $u_i(z) = u_i(z')$ for all terminal histories $z$ and $z'$ going through $I - K(I)$; that is, agent $i$ is indifferent between the runs that he does not consider possible. We assume that $K$ is common knowledge among players of the game, so that every player understands what histories are considered possible by everyone else in the game. If $I$ is an $i$-information set, intuitively, $i$ believes that only terminal histories going through $K(I)$ are possible, and is indifferent among the remaining histories. Note that we can obtain one possibility system by taking $K(I) = I$ for all information sets $I$; in our proofs it will be useful to consider other possibility systems.

Given a pair $(\Gamma, K)$, a $K$-assessment is a pair $(\sigma, \mu)$, where $\sigma$ is a strategy profile of $\Gamma$, and $\mu$ is a restricted belief system, i.e., a function that determines for every information set $I$ of $\Gamma$ a probability $\mu_I$ over the histories in $K(I)$. Intuitively, if $I$ is an information set for player $i$, $\mu_I$ is $i$’s subjective assessment of the relative likelihood of the histories player $i$ considers possible while moving at $I$, namely $K(I)$.

As in the definition of sequential equilibrium, a $K$-assessment $(\sigma, \mu)$ is a conditional sequential equilibrium with respect to $K$ if (a) at every information set where a player moves he chooses a best response given the beliefs he has about the histories that he considers possible in that information set and the strategies of other players, and (b) his restricted beliefs must be consistent with the strategy profile being played and the possibility system, in the sense that they are calculated by conditioning the probability distribution induced by the strategy profile over the histories considered possible on the information set. Formally, the definition of $(\sigma, \mu)$ being a conditional sequential equilibrium is identical to that of sequential equilibrium, except that the summation in the definition of $EU_i(\sigma, \mu) | I$ and $\mu_I(h)$ is taken over histories in $K(I)$ rather than histories in $I$. It is immediate that if $K(I) = I$ for every information set $I$ of the game, then the set of conditional sequential equilibria with respect to $K$ coincides with the set of sequential equilibria. The next lemma shows that set of conditional sequential equilibria for a large class of extensive games that includes $\Gamma^v$ is nonempty.

**Lemma 4.1** Let $\Gamma$ be an extensive game with perfect recall and countably many players such that (a) each player has only finitely many pure strategies and (b) each player’s payoff depends only on the strategy of finitely many other players. Let $K$ be an arbitrary acceptable possibility system. Then there exists at least one $K$-assessment that is a conditional sequential equilibrium of $\Gamma$ with respect to $K$.
We now prove that every game of awareness has a generalized sequential equilibrium by defining a possibility system $\mathcal{K}$ on $\Gamma^v$ and showing that there is a one-to-one correspondence between the set of conditional sequential equilibria of $\Gamma^v$ with respect to $\mathcal{K}$ and the set of generalized sequential equilibria of $\Gamma^*$.

**Theorem 4.2** For all probability measures $\nu$ on $\mathcal{G}$, if $\nu$ gives positive probability to all games in $\mathcal{G}$, and $\mathcal{K}(\langle \Gamma', I \rangle^*) = \{ \langle \Gamma', h \rangle : h \in I \}$ for every information set $\langle \Gamma', I \rangle^*$ of $\Gamma^v$, then $(\vec{\sigma}', \mu')$ is a generalized sequential equilibrium of $\Gamma^v$ with respect to $\mathcal{K}$, where $\sigma_i,\Gamma^v((\langle \Gamma^h \rangle, h')) = \sigma_i',\Gamma^v(\Gamma^h, h')$ and $\mu_i^\Gamma, I = \mu(\Gamma', I)^*$.

It can be easily shown that $\mathcal{K}$ as defined in Theorem 4.2 is an acceptable possibility system on $\Gamma^v$. Since $\Gamma^v$ satisfies all the conditions of Lemma 4.1, it easily follows from Lemma 4.1 and Theorem 4.2 that every game with awareness has at least one generalized sequential equilibrium.

Although it is not true that every conditional sequential equilibrium is also a sequential equilibrium of an arbitrary game, the next proposition shows there is a close connection between these notions of equilibrium. If $(\vec{\sigma}, \mu)$ is a conditional sequential equilibrium with respect to some possibility system $\mathcal{K}$, then there exists a belief system $\mu'$ such that $(\vec{\sigma}, \mu')$ is a sequential equilibrium.

**Proposition 4.3** For every extensive game $\Gamma$ with countably many players where each player has finitely many pure strategies and for every acceptable possibility system $\mathcal{K}$, if $(\vec{\sigma}, \mu)$ is a conditional sequential equilibrium of $\Gamma$ with respect to $\mathcal{K}$, then there exists a belief system $\mu'$ such that $(\vec{\sigma}, \mu')$ is a sequential equilibrium of $\Gamma$.

### 5 Conclusions

In this paper, we further developed the framework of games with awareness by analyzing how to generalize sequential equilibrium to such games. Other solution concepts can be generalized in a similar way. Although we have not checked all the details for all solution concepts, we believe that techniques like those used in our earlier paper to prove existence of generalized Nash equilibrium and ones similar to those used in this paper for generalized sequential equilibrium will be useful for proving the existence of other generalized solution concepts. For example, consider the notion of (trembling hand) perfect equilibrium (Selten 1975) in normal-form games. A strategy profile $\vec{\sigma}$ is a perfect equilibrium if there exists some sequence of strategies $(\vec{\sigma}^k)_{k=0}^{\infty}$, each assigning positive probability to every available move, that converges pointwise to $\vec{\sigma}$ such that for each player $i$, the strategy $\sigma_i$ is a best response to $\vec{\sigma}^k_{-i}$ for all $k$. The definition of generalized perfect equilibrium in games with awareness is the same as in standard games, except that we use generalized strategies rather than strategies, and require that for every local strategy $\sigma_i,\Gamma^v$ of every player $i$, $\sigma_i,\Gamma^v$ is a best response to $\vec{\sigma}_{-i,\Gamma^v}^k$ in game $\Gamma'$ for all $k$. To prove that every game with awareness in normal form has a generalized perfect equilibrium, we prove an analogue of Theorem 3.1(b) in Halpern and Rêgo (2006), giving a correspondence between the set of generalized perfect equilibria of $\Gamma^*$ and the set of perfect equilibria of $\Gamma^v$. The existence of a
generalized perfect equilibrium follows from the existence of a perfect equilibrium in \( \Gamma' \); the existence of a perfect equilibrium in \( \Gamma' \) follows from Lemma A.2.

In our earlier work, we showed that our definitions could be extended in a straightforward way to games with awareness of unawareness; that is, games where one player might be aware that there are moves that another player (or even she herself) might be able to make, although she is not aware of what they are. Such awareness of unawareness can be quite relevant in practice. We captured the fact that player \( i \) is aware that, at a node \( h \) in the game tree, there is a move that \( j \) can make she (\( i \)) is not aware of was by having \( i \)'s subjective representation of the game include a “virtual” move for \( j \) at node \( h \). Since \( i \) does not understand perfectly what can happen after this move, the payoffs associated with terminal histories that follow a virtual move represent what player \( i \) believes will happen if this terminal history is played and may bear no relationship to the actual payoffs in the underlying game. We showed that a generalized Nash equilibrium exists in games with awareness of unawareness. It is straightforward to define generalized sequential equilibrium for those games and to prove its existence using the techniques of this paper; we leave details to the reader.

We have focused here on generalizing solution concepts that have proved useful in standard games, where there is no lack of awareness. Introducing awareness allows us to consider other solution concepts. For example, Ozbay (2007) proposes an approach where a player’s beliefs about the probability of revealed moves of nature, that the player was initially unaware of, are formed as part of the equilibrium definition. We hope to explore the issue of which solution concepts are most appropriate in games with awareness in future work.

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Appendix A: Proof of theorems

A.1 Proof of Lemma 4.1, Theorem 4.2, and Proposition 4.3

**Lemma 4.1** Let \( \Gamma \) be an extensive game with perfect recall and countably many players such that (a) each player has only finitely many pure strategies and (b) each player’s payoff depends only on the strategy of finitely many other players. Let \( K \) be an arbitrary acceptable possibility system. Then there exists at least one \( K \)-assessment that is a conditional sequential equilibrium of \( \Gamma \) with respect to \( K \).

**Proof** We use the same ideas that are used to prove existence of standard sequential equilibrium, following closely the presentation in Myerson (1991). The proof goes as follows. Given \( \Gamma \), let \( \Gamma_M \) be the multiagent representation of \( \Gamma \) in normal form. We
prove (Lemma A.1) that for every perfect equilibrium $\sigma$ of $\Gamma_M$ and every possibility system $K$, there exists a restricted belief system $\mu$ such that $(\sigma, \mu)$ is a conditional sequential equilibrium of $\Gamma$ with respect to $K$. Then we show (Lemma A.2) that for $\Gamma$ satisfying the hypothesis of the theorem, $\Gamma_M$ has at least one perfect equilibrium.

We now review the relevant definitions. A normal-form game is a tuple $(N, x_{i\in N}C_i, \{u_i : i \in N\})$, where $N$ is the set of players of the game, $C_i$ is the collection of pure strategies available for player $i$ in the game, and $u_i$ is a payoff function that determines for each strategy profile in $\times_{i\in N}C_i$ the payoff for player $i$.

Given a standard extensive-form game $\Gamma = (N, M, H, P, f_c, \{I_i : i \in N\}, \{u_i : i \in N\})$, let $S^* = \cup_{i \in N} I_i$. Intuitively, we associate with each $i$-information set $I \in I_i$ a temporary player that has $M(I)$ as its set possible strategies; $S^*$ is just the set of all temporary players in $\Gamma$. For each temporary player $I$ we associate a payoff function $v_I : \times_{i\in S^*} M(I) \to \mathbb{R}$ such that if each temporary player $I$ chooses action $a_I$, and $\sigma$ is the pure strategy profile for $\Gamma$ such that for every $i \in N$ and $I \in I_i, \sigma_i(I) = a_I$, then $v_I(\times_{i\in S^*} a_I) = u_I(\sigma)$. The multiagent representation for $\Gamma$ in normal form is the tuple $(S^*, \times_{i\in S^*} M(I), \{v_I : I \in S^*\})$.

Given any countable set $B$, let $\Delta(B)$ be the set of all probability measures over $B$, and let $\Delta^0(B)$ be the set of all probability measures over $B$ whose support is all of $B$. Given a game in normal form $\Gamma = (N, x_{i\in N}C_i, \{u_i : i \in N\})$, a mixed strategy profile $\sigma \in x_{i\in N} \Delta(C_i)$ is a perfect equilibrium of $\Gamma$ if there exists a sequence $(\hat{\sigma}^k)_{k=1}^\infty$ such that (a) $\hat{\sigma}^k \in x_{i\in N} \Delta^0(C_i)$ for $k \geq 1$, (b) $\hat{\sigma}^k$ converges pointwise to $\sigma$, and (c) $\sigma_i \in \text{argmax}_{\tau_i \in \Delta(C_i)} \text{EU}_i(\hat{\sigma}^k_{-i}, \tau_i)$ for all $i \in N$.

The following lemmas are analogues of Theorems 5.1 and 5.2 in Myerson (1991).

**Lemma A.1** If $\Gamma_M$ is a multiagent representation of $\Gamma$ in normal form, then for every perfect equilibrium $\sigma$ of $\Gamma_M$ and every acceptable possibility system $K$, there exists a restricted belief system $\mu$ such that $(\sigma, \mu)$ is a conditional sequential equilibrium of $\Gamma$ with respect to $K$.

**Proof** The proof is almost identical to that of Theorem 5.1 in Myerson (1991). We focus on the necessary changes, leaving the task of verifying that the rest of the proof goes without change to the reader. Let $(\hat{\sigma}^k)_{k=1}^\infty \in x_{i\in S^*} \Delta(M(I))$ be a sequence of behavioral strategy profiles satisfying conditions (a), (b), and (c) of the definition of perfect equilibrium. For each $k$, define a belief system $\mu^k(I)$ such that, for each information set $I$, $\mu^k(I)$ is the probability over histories in $K(I)$ defined as

$$
\mu^k_I(h) = \frac{\text{Pr}_{\hat{\sigma}^k}(h)}{\sum_{h' \in K(I)} \text{Pr}_{\hat{\sigma}^k}(h')}.
$$

Thus, $\mu^k \in \prod_{I \in \mathcal{I}} \Delta(K(I))$, where $\mathcal{I}$ is the set of all information sets in $\Gamma$. If $\Omega = \sum_{I \in \mathcal{I}} ||\Delta(K(I))||$, then $\mu^k$ can be viewed as a point in $[0, 1]^\Omega$; by Tychonoff’s Theorem, $[0, 1]^\Omega$ is compact. Thus, there must be a convergent subsequence of $\mu^1, \mu^2, \ldots$. Suppose that this subsequence converges to $\mu$. It is easy to see that $\mu$ is consistent with $\sigma$ and $K$.

Let $Z(I)$ denote the set of terminal histories that do not contain any prefix in $I$. Let $I$ be an arbitrary $i$-information set in $\Gamma$. When agent $I \in S^*$ uses the randomized
strategy $\rho_I \in \Delta(M(I))$ against the strategies specified by $\tilde{\sigma}^k$ for all other agents, his expected payoff is

$$EU_I(\tilde{\sigma}^k_{-I}, \rho_I) = \sum_{h \in I} \Pr(\tilde{\sigma}^k_{-I}, \rho_I)(h)EU_I(\tilde{\sigma}^k_{-I}, \rho_I \mid h) + \sum_{z \in Z(I)} \Pr(\tilde{\sigma}^k_{-I}, \rho_I)(z)u_i(z).$$

Note that for $h \in I$ or $h \in Z(I)$, $\Pr(\tilde{\sigma}^k_{-I}, \rho_I)(h) = \Pr_{\tilde{\sigma}^k}(h)$, since this probability is independent of the strategy used by player $I$. Also note that for all $h \in I - K(I)$, since $K$ is acceptable, $EU_I(\tilde{\sigma}^k_{-I}, \rho_I \mid h)$ is independent of $\rho_I$. Thus,

$$EU_I(\tilde{\sigma}^k_{-I}, \rho_I) = \sum_{h \in K(I)} \Pr_{\tilde{\sigma}^k}(h)EU_I(\tilde{\sigma}^k_{-I}, \rho_I \mid h) + \sum_{z \in Z(I)} \Pr_{\tilde{\sigma}^k}(z)u_i(z) + C',$$

$$= (\sum_{h \in K(I)} h^k(h)EU_I(\tilde{\sigma}^k_{-I}, \rho_I \mid h))\left(\sum_{h \in K(I)} \Pr_{\tilde{\sigma}^k}(h)\right) + C'',$

where $C'$ and $C''$ are two constants independent of $\rho_I$.

The rest of the proof proceeds just as the proof of Theorem 5.1 in Myerson (1991); we omit details here. \hfill \Box

**Lemma A.2** If $\Gamma$ is an extensive-form game with perfect recall such that (a) there are at most countably many players, (b) each player has only finitely many pure strategies, and (c) the payoff of each player depends only on the strategy of finitely many other players, then $\Gamma_M$ has at least one perfect equilibrium.

**Proof** The proof is almost identical to that of Theorem 5.2 in Myerson (1991). Again, we focus on the necessary changes, leaving it to the reader to verify that the rest of the proof goes without change. We need to modify some of the arguments since $\Gamma_M$ is not a finite game; since it may contain countably many players. First, by the same argument used to prove that $\Gamma_v$ has at least one Nash equilibrium in our earlier work (Halpern and Rêgo 2006), we have that for any $\Gamma$ satisfying the hypothesis of the lemma, $\Gamma_M$ has at least one Nash equilibrium.

Let $C_i$ be the set of pure strategies available for player $i$ in $\Gamma_M$. By Tychonoff’s Theorem $\times_{i \in N}[0, 1]^{C_i}$ is compact. Since $\times_{i \in N} \Delta(C_i)$ is a closed subset of $\times_{i \in N}[0, 1]^{C_i}$, it is also compact. All the remaining steps of the proof of Theorem 5.2 in Myerson (1991) apply here without change; we omit the details. \hfill \Box

The proof of Lemma 4.1 follows immediately from Lemmas A.1 and A.2. \hfill \Box

**Theorem 4.2** For all probability measures $\nu$ on $\mathcal{G}$, if $\nu$ gives positive probability to all games in $\mathcal{G}$, and $K_i((\Gamma', I)^*) = \{(\Gamma', h) : h \in I\}$ for every information set $(\Gamma', I)^* \text{ of } \Gamma_v$, then $(\tilde{\sigma}', \mu')$ is a generalized sequential equilibrium of $\Gamma^*$ iff $(\tilde{\sigma}, \mu)$ is a conditional sequential equilibrium of $\Gamma_v$ with respect to $K$, where $\sigma_i, (\Gamma^h) \cdot h' = \sigma_i', (\Gamma^h) \cdot h'$ and $\mu_i, (\Gamma', I)^* = \mu_i, (\Gamma', I)^*$. 

**Proof** Let $Pr_{\tilde{\sigma}}^V(h)$ be the probability distribution over the histories in $\Gamma_v$ induced by the strategy profile $\tilde{\sigma}$ and $f_c^V$. For a history $h$ of the game, define $Pr_{\tilde{\sigma}}^V(\cdot \mid h)$ to be the conditional probability distribution induced by $\tilde{\sigma}$ and $f_c^V$ over the possible histories of the game given that the current history is $h$. Similarly, let $Pr_{\tilde{\sigma}}^H$ be the probability
distribution over the histories in $\Gamma^h \in \mathcal{G}$ induced by the generalized strategy profile $\tilde{\sigma}'$ and $f^h_c$. Note that if $Pr^+_\tilde{\sigma}(h') > 0$, then $h' \in [H^h]$. Thus, $(\Gamma^h) \cdot h' \in H^v$.

For all strategy profiles $\sigma$ and generalized strategy profiles $\sigma'$, if $\sigma', (\Gamma^h) \cdot h' = \sigma, (\Gamma^h) \cdot h')$, then it is easy to see that for all $h' \in H^h$ such that $Pr^h_\sigma(h') > 0$, we have that $Pr^h_{\tilde{\sigma}}((\Gamma^h) \cdot h') = \nu((\Gamma^h) Pr^h_{\tilde{\sigma}}(h'))$. And since $\nu$ is a probability measure such that $\nu((\Gamma^h) > 0$ for all $\Gamma \in \mathcal{G}$, we have that $Pr^h_{\tilde{\sigma}}(\langle \Gamma^h) \cdot h' > 0$ if $Pr^h_\sigma(h') > 0$. It is also easy to see that for all $h' \neq ()$ and all $h'' \in H^h$ such that $Pr^h_{\tilde{\sigma}}(h'' | h') > 0$, $Pr^h_{\tilde{\sigma}}((\Gamma^h) \cdot h'' | h') = Pr^h_{\tilde{\sigma}}(h'' | h')$.

Suppose that $(\tilde{\sigma}, \mu)$ is a conditional sequential equilibrium of $\Gamma^v$ with respect to $\mathcal{K}$. We first prove that $(\tilde{\sigma}', \mu')$ satisfies generalized sequential rationality. Suppose, by way of contradiction, that it does not. Thus, there exists a player $i$, a generalized $i$-information set $(\Gamma^+, I)_i^*$, and a local strategy $s'$ for player $i$ in $\Gamma^+$ such that

$$\sum_{h \in I, z \in Z^+} \mu'_{\Gamma^+, I}(h) Pr^+_{\tilde{\sigma}}(z | h) u^+_i(z) < \sum_{h \in I, z \in Z^+} \mu'_{\Gamma^+, I}(h) Pr^+_{\tilde{\sigma}'}((\Gamma^+) \cdot z | h) u^+_i(z).$$

Define $s$ to be a strategy for player $(i, \Gamma^+)$ in $\Gamma^v$ such that $s((\langle \Gamma^h) \cdot h' = s'(\langle \Gamma^h) \cdot h')$. Using the observation in the previous paragraph and the fact that $\mu'_{\Gamma^+, I} = \mu_{\Gamma^+, I}^*$ and $\mathcal{K}((\Gamma^+, I)_i^*) = \{ (\Gamma^+, h) : h \in I \}$, it follows that

$$\sum_{(\Gamma^+, h) \in \mathcal{K}((\Gamma^+, I)_i^*)} \sum_{z \in [Z^+]^+} \mu_{\Gamma^+, I}^*(h) Pr^v_{\tilde{\sigma}}((\Gamma^+) \cdot z | h) u^+_i(z)$$

$$< \sum_{(\Gamma^+, h) \in \mathcal{K}((\Gamma^+, I)_i^*)} \sum_{z \in [Z^+]^+} \mu_{\Gamma^+, I}^*(h) Pr^v_{\tilde{\sigma}'}((\Gamma^+) \cdot z | h) u^+_i(z). \quad (1)$$

By definition of $u^v_{\tilde{\sigma}'}(\langle \Gamma^, z) = u^v_{\tilde{\sigma}, I, \Gamma^+}((\Gamma^+), z)$. Replacing $u^+_i(z)$ by $u^v_{\tilde{\sigma}', I, \Gamma^+}$ in (1), it follows that $(\tilde{\sigma}', \mu')$ does not satisfy sequential rationality in $\Gamma^v$, a contradiction. So, $(\tilde{\sigma}', \mu')$ satisfies generalized sequential rationality. It remains to show that $\mu'$ is consistent with $\tilde{\sigma}'$.

Suppose that, for every generalized information set $(\Gamma^+, I)_i^*$, $\sum_{h \in I} Pr^v_{\tilde{\sigma}}(h) > 0$. By definition of $\mathcal{K}$ and the fact that for all $h' \in H^v$, $Pr^v_{\tilde{\sigma}}((\Gamma^+) \cdot h') > 0$ if $Pr^v_{\tilde{\sigma}}(h') > 0$, we have that for every information set $(\Gamma^+, I)_i^*$ of $\Gamma^v$,

$$\sum_{(\Gamma^+, h) \in \mathcal{K}((\Gamma^+, I)_i^*)} Pr^v_{\tilde{\sigma}}((\Gamma^+) \cdot h) > 0.$$
Since $\mu_{\Gamma^+,I}' = \mu_{(\Gamma^+,I)^*}, \mathcal{K}((\Gamma', I)^*) = \{(\Gamma', h) : h \in I\}$, and for all $h' \in H^h$ such that $Pr^h_{\sigma}(h') > 0$, we have that $Pr^h_{\sigma}((\Gamma^+ h) \cdot h') = v((\Gamma^+ h) Pr^h_{\sigma}(h'))$, it is easy to see that for every generalized information set $(\Gamma^+, I)^*$ and every $h \in I$,

$$\mu_{\Gamma^+,I}'(h) = \frac{Pr^h_{\sigma}(h)}{\sum_{h' \in I} Pr^h_{\sigma}(h')}.$$ 

Thus, $\mu'$ is consistent with $\sigma'$.

Finally, suppose that there exists a generalized information set $(\Gamma^+, I)^*$ such that $\sum_{h \in I} Pr^h_{\sigma}(h) = 0$. By definition of $\mathcal{K}$ and the fact that for all $h' \in H^h, Pr^h_{\sigma}((\Gamma^+ h') > 0$ if $Pr^h_{\sigma}(h') > 0$, we have that $\sum_{(\Gamma^+ h) \in \mathcal{K}((\Gamma^+,I)^*)} Pr^h_{\sigma}((\Gamma^+ h) h) = 0$. Thus, by the consistency of $\mu, \sigma$, and $\mathcal{K}$, there exists a sequence of $\mathcal{K}$-assessments $(\sigma^n, \mu^n)$ such that $\sigma^n$ consists of completely mixed strategies, $\mu^n$ is consistent with $\sigma^n$ and $\mathcal{K}$, and $(\sigma^n, \mu^n)$ converges pointwise to $(\sigma', \mu')$.

Define a sequence of $\mathcal{K}$-assessments $(\tau^n, \nu^n)$ such that $\nu^n_{\Gamma^+,I} = \mu^n_{(\Gamma^+,I)^*}$, and $\sigma^n_{\Gamma^+,I}(\Gamma^+ h') = \tau^n_{\Gamma^+,I}(\Gamma^+ h')$ for all $n$. Since $\sigma^n$ is completely mixed, so is $\tau^n$; it also follows from the earlier argument that $\nu^n$ is consistent with $\tau^n$ for all $n$. Since $(\sigma^n, \mu^n)$ converges pointwise to $(\sigma, \mu)$, it is easy to see that $(\tau^n, \nu^n)$ converges pointwise to $(\sigma', \mu')$. Thus, $\mu'$ is consistent with $\sigma'$, and $(\sigma', \mu')$ is a generalized sequential equilibrium of $\Gamma^*$, as desired. The proof of the converse is similar; we leave details to the reader.

\[\square\]

**Proposition 4.3** For every extensive game $\Gamma$ with countably many players where each player has finitely many pure strategies and for every acceptable possibility system $\mathcal{K}$, if $(\sigma, \mu)$ is a conditional sequential equilibrium of $\Gamma$ with respect to $\mathcal{K}$, then there exists a belief system $\mu'$ such that $(\sigma', \mu')$ is a sequential equilibrium of $\Gamma$.

**Proof** Since $(\sigma, \mu)$ is a conditional sequential equilibrium of $\Gamma$ with respect to $\mathcal{K}$, by the consistency of $\mu, \sigma$, and $\mathcal{K}$, there exists a sequence of $\mathcal{K}$-assessments $(\hat{\sigma}^k, \hat{\mu}^k)$ such that $\hat{\sigma}^k$ is completely mixed, $\hat{\mu}^k$ is consistent with $\hat{\sigma}^k$ and $\mathcal{K}$, and $(\hat{\sigma}^k, \hat{\mu}^k)$ converges pointwise to $(\sigma, \mu)$. Let $\tilde{\nu}^k$ be the belief system consistent with $\hat{\sigma}^k$. Using the same techniques as in the proof of Lemma A.1, we can construct a subsequence of $(\hat{\sigma}^k, \hat{\mu}^k)$ that converges pointwise to $(\sigma, \mu')$. Thus, $\mu'$ is consistent with $\sigma$. It remains to show that $(\sigma, \mu')$ satisfies sequential rationality.

Since $\mathcal{K}$ is acceptable, for every $i$-information set $I$ of $\Gamma$, player $i$ has the same utility for every terminal history extending a history in $I - \mathcal{K}(I)$, it is not hard to show that

$$EU_i((\sigma, \mu') \mid I) = C + \mu'((\mathcal{K}(I))EU_i((\sigma, \mu) \mid I),$$

where $C$ and $\mu'(\mathcal{K}(I))$ are independent of $\sigma_i(I)$. Since, by sequential rationality, $\sigma_i(I)$ is a best response given $\mu$, it is also a best response given $\mu'$. It follows that $(\sigma, \mu')$ is a sequential equilibrium of $\Gamma$, as desired. \[\square\]
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