ALMOST CONFORMAL VACUA AND CONFINEMENT

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Abstract:

Dynamics of confining vacua which appear as deformed superconformal theory with a non-Abelian gauge symmetry, is studied by taking a concrete example of the sextet vacua of $\mathcal{N} = 2$, $SU(3)$ gauge theory with $n_f = 4$, with equal quark masses. We show that the low-energy “matter” degrees of freedom of this theory consist of four magnetic monopole doublets of the low-energy effective $SU(2)$ gauge group, one dyon doublet, and one electric doublet. We find a mechanism of cancellation of the beta function, which naturally but nontrivially generalizes that of Argyres-Douglas. Study of our SCFT theory as a limit of six colliding $\mathcal{N} = 1$ vacua, suggests that the confinement in the present theory occurs in an essentially different manner from those vacua with dynamical Abelianization, and involves strongly interacting non-Abelian magnetic monopoles.

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1. Introduction

The true mechanism of confinement in QCD is still covered in a mystery, in spite of considerable amount of work dedicated to this problem. In a recent series of papers [1], the mechanism of confinement and of dynamical symmetry breaking has been studied in some detail in the context of $\mathcal{N} = 2$ supersymmetric quantum chromodynamics (SQCD), broken softly to $\mathcal{N} = 1$. An interesting fact emerged from these analyses: non-Abelian monopoles of the type studied by Goddard-Olive-Nuyts [2] make appearance as low-energy degrees of freedom, and play a central role in the infrared dynamics in some of the confining vacua [3].

The most intriguing type of vacua among those found in [1], however, are the ones based on deformation (perturbation) of superconformal field theories (SCFT) [6, 7]. The low-energy degrees of freedom involve relatively nonlocal dyons and there is no local effective Lagrangian describing them. Upon perturbation - small adjoint mass - confinement and dynamical symmetry breaking ensue, as can be demonstrated indirectly through various considerations, such as the study of the large $\mu$ effective action, supersymmetry and holomorphy, and the vacuum counting.

Though far-fetched it might sound, confinement is described precisely this way in many systems within the context of $\mathcal{N} = 2$ supersymmetric gauge theories with $n_f$ hypermultiplets. Examples are the $r = \frac{n_f}{2}$ vacua [1] in $SU(n_c)$ theories with equal mass flavors, and all of the confining vacua of $USp(2n_c)$ and $SO(n_c)$ theories with matter fields with zero bare masses [1]. In contrast, the Abelian dual superconductor mechanism [4, 5] is realized rather as exceptional cases: $r = 0$ or $r = 1$ vacua in $SU(n_c)$ theory, or in pure $\mathcal{N} = 2$ Yang-Mills theories, i.e., $n_f = 0$.

Whether a similar mechanism is at work in the standard Quantum Chromodynamics (QCD) is not known. The phenomenon is deep, though, and in our opinion deserves a closer look than has been given so far.

In this paper, we make a first step in that direction. We study the nature of the low-energy degrees of freedom in vacua in which confinement appears to be caused by a collaboration of relatively non-local light monopoles and dyons. As an example, we study the $r = 2$ vacua of $\mathcal{N} = 2$ supersymmetric $SU(3)$ gauge theory with $n_f = 4$ hypermultiplets. We show that the low-energy degrees of freedom of this theory consist of four monopole doublets of the effective $SU(2)$ gauge group, one dyon doublet, and one electric doublet. We show how they conspire to give a vanishing beta

\footnote{We recall that confining $\mathcal{N} = 1$ vacua arising by perturbing the $\mathcal{N} = 2$ $SU(n_c)$ gauge theories with the adjoint mass term are classified by different effective gauge groups, $SU(r) \times U(1)^{n_c-r-1}$.}
function, generalizing the Argyres-Douglas mechanism \[8, 9\] in a non-trivial manner.

Study of the superconformal theory as a limit of a number of colliding $\mathcal{N} = 1$ vacua, as realized by first considering the theory at unequal quark masses and then taking the limit of equal masses, indicates that confinement mechanism at the almost conformal vacua such as the one under consideration, is essentially distinct from the one at work in those vacua where confinement is due to the condensation of weakly coupled magnetic particles as first found in Ref. \[10, 11\].

2. The Sextet ($r = 2$) Vacua of $SU(3)$, $n_f = 4$ Gauge Theory

We study the sextet vacuum ($r = 2$ vacua) of $SU(3)$ gauge theory with four flavors of quarks. The Seiberg-Witten curve \[10, 11\] of this theory is (by setting $2\Lambda = 1$) \[12\]

$$y^2 = \prod_{i=1}^{3}(x - \phi_i)^2 - \prod_{a=1}^{4}(x + m_a) \equiv (x^3 - Ux - V)^2 - \prod_{a=1}^{4}(x + m_a). \quad (2.1)$$

For equal bare quark masses ($m_a = m$), it simplifies:

$$y^2 = \prod_{i=1}^{3}(x - \phi_i)^2 - (x + m)^4 \equiv (x^3 - Ux - V)^2 - (x + m)^4. \quad (2.2)$$

The sextet vacua correspond to the point, diag $\phi = (-m, -m, 2m)$, i.e.,

$$U = 3m^2; \quad V = 2m^3, \quad (2.3)$$

where the curve exhibits a singular behavior,

$$y^2 \propto (x + m)^4 \quad (2.4)$$

corresponding to the unbroken $SU(2)$ symmetry. This singularity splits to six separate singularities when the quark masses are taken to be slightly unequal and generic.

At a generic point $(U, V)$ near \[2.3\] the right hand side of the curve \[2.2\] has six square-root branch points, four of which are near $x = -m$, the other two are at separate points of $O(1)$. We draw the canonical cycles as in Fig.1 and define

$$a_{D1} = \oint_{\alpha_1} \lambda, \quad a_{D2} = \oint_{\alpha_2} \lambda, \quad a_1 = \oint_{\beta_1} \lambda, \quad a_2 = \oint_{\beta_2} \lambda, \quad (2.5)$$

where the (meromorphic) one-form $\lambda$ is given by \[12\]

$$\lambda = \frac{x}{2\pi d} \log \frac{\prod(x - \phi_i) - y}{\prod(x - \phi_i) + y}. \quad (2.6)$$
The masses of the particles having the magnetic and electric quantum numbers $(g_1, g_2; q_1, q_2)$ are then given by the formula

$$M_{(g_1, g_2; q_1, q_2)} = \sqrt{2} |g_1 a D_1 + g_2 a D_2 + q_1 a_1 + q_2 a_2|.$$  

(2.7)

Figure 1: The canonical homology cycles together with the cuts (thick lines), in the double-sheeted Riemann surface representation of the curve (2.1).

2.1. Singularity Structure near the Conformal Point and Low-energy Degrees of Freedom

The conformally invariant vacua occur at the points where more than one singularity loci in the $u−v$ plane, corresponding to relatively nonlocal massless states, coalesce [6, 7]. The nature of the low-energy degrees of freedom at the SCFT vacua can be determined by studying the monodromy matrices around each of the singularity curves. In the case of the $r = 2$ vacua of $SU(3)$ theory with $n_f = 4$, it is necessary to study the behavior of the theory near the point Eq. (2.3). Set

$$U = 3m^2 + u, \quad V = 2m^3 + v, \quad x + m \to x.$$  

(2.8)

The discriminant of the curve factorizes [7] as

$$\Delta = \Delta_s \Delta_+ \Delta_-,$$  

(2.9)

where the squark singularity\textsuperscript{2} corresponds to the factor

$$\Delta_s = (m u - v)^4,$$  

(2.10)

\textsuperscript{2}At large $m$ hence at large $U$ and $V$ it represents massless quarks and squarks; as is well known, at small $m$ it becomes monopole singularity, due to the fact that the corresponding singularity goes under certain cuts produced by other singularities [13, 14].
where the fourth zero represents the flavor multiplicity \( n_f = 4 \);

\[
\Delta_+ = 4 m u + 36 m^2 u + 108 m^3 u + 108 m^4 u + u^2 + 24 m u^2 + 36 m^2 u^2 + \\
4 u^3 - 4 v - 36 m v - 108 m^2 v - 108 m^3 v - 18 u v - 27 v^2, \quad (2.11)
\]

\[
\Delta_- = -4 m u + 36 m^2 u - 108 m^3 u + 108 m^4 u + u^2 - 24 m u^2 + 36 m^2 u^2 + \\
4 u^3 + 4 v - 36 m v + 108 m^2 v - 108 m^3 v + 18 u v - 27 v^2, \quad (2.12)
\]

represent the loci where some other dyons become massless. The equations \( \Delta_\pm = 0 \) can be approximated by

\[
v = m u + \frac{1}{12 m + 4} u^2 + O(u^3), \quad (2.13)
\]

\[
v = m u + \frac{1}{12 m - 4} u^2 + O(u^3) \quad (2.14)
\]

at small \( m \). Thus at sufficiently small \( u, v, m \), the three equations \( \Delta_\pm = 0 \) can be replaced by

\[
v = m u, \quad v = m u + \frac{u^2}{4}, \quad v = m u - \frac{u^2}{4}. \quad (2.15)
\]

These three (complex) curves meet at \( u = v = 0 \) tangentially. For the purpose of studying the topological feature of the three curves one can further rescale \( u, v \) by \( u = m \tilde{u}, \; v = m^2 \tilde{v} \) so that the three curves are now

\[
\tilde{v} = \tilde{u}, \quad \tilde{v} = \tilde{u} + \frac{\tilde{u}^2}{4}, \quad \tilde{v} = \tilde{u} - \frac{\tilde{u}^2}{4}. \quad (2.16)
\]

In order to define uniquely the monodromies around the three curves near the SCFT point we consider the intersections of these curves with the \( S^3 \) sphere

\[
|\tilde{u}|^2 + |\tilde{v}|^2 = 1. \quad (2.17)
\]

A two-dimensional (Re \( \tilde{u} - \text{Re} \tilde{v} \)) projection of the curves (2.16), (2.17), is shown in Fig.2.

These intersections form one-dimensional closed curves (on the two dimensional projection in Fig.2 just two points are visible for each such intersection curve). We consider various closed curves lying on \( S^3 \) and encircling the curves (2.16) at various points. It is convenient to make first a stereographic projection from \( S^3 \to R^3 \) by

\[
X = \frac{w_1}{1 - w_4}; \quad Y = \frac{w_2}{1 - w_4}; \quad Z = \frac{w_3}{1 - w_4}, \quad (2.18)
\]
Figure 2: A two-dimensional (Re $\tilde{u}$ – Re $\tilde{v}$) projection of the curves (2.16) and the sphere $S^3$, (2.17).

where

$$\tilde{u} = w_1 + i w_2; \quad \tilde{v} = w_3 + i w_4,$$

after which the intersection curves take the form of the three linked rings. It can be shown that each pair of the rings are linked non-trivially with linking number two \[\text{[3]}\]. Furthermore, the linking between all three pairs of the rings can be shown to occur in the same direction. Topologically, the three intersection curves look like those in Fig. 3.

We consider now various closed paths in the space $(u, v)$, starting from a fixed reference point (for instance lying above the page), encircling various parts of the rings and coming back to the original point, as in the three paths shown in Fig. 4. These induce the movements of the four branch points near the origin (the slight movements of the furthest ones near $\pm 1$ are irrelevant). As the branch points move, the integration contours $\alpha_i$’s and $\beta_i$’s get entangled in a non-trivial way. For example, in the case of the central closed circuit of Fig. 4 in which the curve $\tilde{v} = \tilde{u}$ is encircled once near $\tilde{u} = 1$, $\tilde{v} = 1$, the two branch points closest to the origin (call $x_1, x_2$) rotate with respect to each other, Arg($x_1 - x_2$) going through a change of $4\pi$. The canonical

\[\text{[3]}\text{The linking number is defined by}\]

$$N = \frac{1}{4\pi} \oint \oint dx_i dy_j \epsilon_{ijk} \frac{(x - y)_k}{((x - y)^2)^{3/2}}$$

where the integrations are along the two rings.
cycles go through the change
\[ \alpha_1 \to \alpha_1, \quad \beta_1 \to \beta_1 - 4\alpha_1, \quad \alpha_2 \to \alpha_2, \quad \beta_2 \to \beta_2. \] (2.20)

The monodromy transformation is thus
\[
\begin{pmatrix}
  a_{D1} \\
  a_{D2} \\
  a_1 \\
  a_2
\end{pmatrix}
\to
M_1
\begin{pmatrix}
  a_{D1} \\
  a_{D2} \\
  a_1 \\
  a_2
\end{pmatrix},
\quad
M_1 = \tilde{M}_1^4,
\quad
\tilde{M}_1 =
\begin{pmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
 -1 & 0 & 1 & 0 \\
 0 & 0 & 0 & 1
\end{pmatrix}
\] (2.21)

From the well-known formula [12]
\[ M = \begin{pmatrix}
  1 + \vec{q} \otimes \vec{g} & \vec{q} \otimes \vec{q} \\
 -\vec{g} \otimes \vec{q} & 1 - \vec{g} \otimes \vec{q}
\end{pmatrix}, \quad (2.22)
\]
one concludes that the (four) massless particles at the singularity \( \tilde{v} = \tilde{u} \) have charges
\[ (g_1, g_2; q_1, q_2) = (1, 0; 0, 0). \] (2.23)

Analogously, the monodromy transformations around the \( \tilde{v} = \tilde{u} + \frac{\vec{a}^2}{4}, \quad \tilde{v} = \tilde{u} - \frac{\vec{a}^2}{4} \)
are determined to be
\[ M_2 =
\begin{pmatrix}
  -1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
 -4 & 0 & 3 & 0 \\
 0 & 0 & 0 & 1
\end{pmatrix}, \quad M_6 =
\begin{pmatrix}
  1 & 1 & 1 & 0 \\
  0 & 1 & 0 & 0 \\
 0 & 0 & 1 & 0 \\
 0 & -1 & -1 & 1
\end{pmatrix}, \quad (2.24)
\]
In principle, this procedure could be carried out for all other parts of the rings, but there is a more systematic procedure, which takes the symmetry of the system into account. Indeed, various monodromy transformations are related by the conjugations,

\[
M_1 = M_6^{-1}A_5M_6, \quad A_2 = M_2^{-1}M_1M_2, \quad M_4 = M_5^{-1}A_2M_3, \quad A_5 = M_5^{-1}M_4M_5, \\
M_2 = M_1^{-1}A_6M_1, \quad A_3 = M_3^{-1}M_2M_3, \quad M_5 = M_4^{-1}A_3M_4, \quad A_6 = M_6^{-1}M_5M_6, \\
M_3 = M_2^{-1}A_1M_2, \quad A_4 = M_4^{-1}M_3M_4, \quad M_6 = M_5^{-1}A_4M_5, \quad A_1 = M_1^{-1}M_6M_1
\]

(2.25)

as can be easily verified by looking at Fig. 3. By knowing any three of them, for instance \(M_1, M_2, M_6\) above, these relations yield uniquely all the other monodromy matrices. The twelve monodromy matrices \(M_1 \sim M_6, A_1 \sim A_6\) determined this way are listed in Appendix A. The formula (2.22) then gives the charges

\[
M_1 : (1, 0; 0, 0)^4, \quad M_4 : (-1, 1; 0, 0)^4, \quad M_5 : (2, -2; -1, 0), \\
A_2 : (-1, 0; 1, 0)^4, \quad A_5 : (1, -1; -1, 0)^4, \quad A_3 : (-2, 2; -1, 0), \quad A_6 : (2, 0; 1, 0), \\
M_3 : (0, 1; -1, 0), \quad M_6 : (0, 1; 1, 0), \quad A_4 : (4, -3; -1, 0), \quad A_1 : (-4, 1; 1, 0),
\]

(2.26)
where the superscript 4 for $M_1, M_4, A_2, A_5$ indicates the fact that these charges appear four times (the monodromy matrix being the fourth power of an elementary monodromy matrix, corresponding to these charges). Note that all of them have a vanishing charge $q_2$; thus in the limit $u \to 0$, $v \to 0$ (where $\alpha_1, \beta_1, \alpha_2$ cycles shrink to zero cycles) all of them would become massless simultaneously.

![Figure 5: Exchanging the two equivalent necks of the bi-torus]

We wish to know how these particles behave under the $SU(2) \times U(1)$ group which is unbroken at $u = v = 0$. Although we know that the charges (2.26) refer to a (magnetic or electric) $U(1)^2$ subgroup of $SU(3)$, it is not a priori clear how it is related the unbroken $SU(2) \times U(1)$ group. A useful element is the transformation property under the canonical base change,

$$
\alpha_1 \to \alpha_2 - \alpha_1; \quad \beta_1 \to -\beta_1; \quad \alpha_2 \to \alpha_2; \quad \beta_2 \to \beta_1 + \beta_2.
$$

(2.27)

Note that the intersection numbers $\alpha_i \# \beta_j = \delta_{ij}$ are maintained. Geometrically, it corresponds to the exchange of the two equivalent “necks” of the bi-torus (Fig.4), and can be interpreted as an $SU(3)$ transformation in which the first and the second component of the fundamental multiplet are exchanged.

This transformation induces the change of charges

$$
g_1 \to -g_1; \quad g_2 \to g_1 + g_2; \quad q_1 \to -q_1 + q_2; \quad q_2 \to q_2.
$$

(2.28)

An inspection shows that the charges (2.26) are actually paired as doublets transforming into each other under it:

$$(1, 0; 0, 0) \leftrightarrow (-1, 1; 0, 0), \quad (-1, 0; 1, 0) \leftrightarrow (1, -1; -1, 0),$$
These pairs of charges can therefore be interpreted as belonging to various *doublets* of the unbroken $SU(2)$ gauge group. It is easy to introduce, accordingly, the magnetic and electric $U_1(1), U_2(1)$ charges such that the first is a subgroup of the $SU(2)$ while the second is orthogonal to it:

$$Q_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (2.30)$$

The corresponding magnetic and electric $U_1(1)$ charges are

$$\tilde{m}_1 = m_1; \quad \tilde{q}_1 = q_1 - \frac{1}{2} q_2; \quad (2.31)$$

and

$$\tilde{m}_2 = m_1 + 2 m_2; \quad \tilde{q}_2 = \frac{1}{2} q_2 \quad (2.32)$$

respectively. The normalization is chosen such that the transformation has determinant unity in the $\{ m_1, m_2; q_1, q_2 \}$ space. The charges of the doublets in the new basis are:

| Matrix | Charge |
|--------|--------|
| $M_1, M_4$ | $(\pm 1, 1, 0, 0)^4$ |
| $A_2, A_5$ | $(\pm 1, -1, \mp 1, 0)^4$ |
| $M_2, M_5$ | $(\pm 2, 2, \mp 1, 0)$ |
| $A_3, A_6$ | $(\pm 2, -2, \pm 1, 0)$ |
| $M_3, M_6$ | $(0, 2, \pm 1, 0)$ |
| $A_1, A_4$ | $(\pm 4, -2, \mp 1, 0)$ |

Table 1: The charges of the massless doublets in the new basis (2.31), (2.32).

Note that only the members of the same doublet are relatively local, i.e., have a vanishing relative Dirac unit \[4\]

$$\mathcal{N}_D = \sum_{i=1}^{2} (g_{Ai} q_{Bi} - q_{Ai} g_{Bi}). \quad (2.33)$$

The problem now is to find out which of these massless particles are actually present in the low-energy superconformal theory defined at $u = v = 0$, and see how the relatively non-local matter fields cooperate to give, together with the (dual) gauge fields and their superpartners, a vanishing beta function.
2.2. Low-Energy Coupling Constant

The low-energy effective coupling constants at the superconformal point can be determined by studying the behavior of the curve near that point. For \( m = 0 \) we have the following Riemann surface:

\[
y^2 = (x^3 - ux - v)^2 - x^4 = (x^3 + x^2 - ux - v)(x^3 - x^2 - ux - v).
\] (2.34)

At \( u = 0, v = 0 \) the branch points are at:

\[
x_1 = x_2 = x_3 = x_4 = 0, \quad x_5 = -1, \quad x_6 = 1.
\] (2.35)

The periods \( a_{D1}, a_{D2}, a_1 \) become small: this corresponds to the non-local charges of the previous section which become massless in the sextet vacuum.

A similar degeneration of genus two Riemann surfaces was studied in [15]. The main result derived there is that in the limit in which three of the six branch points of a genus two curve coalesce, the period matrix \( \tau_{ij} \) splits as:

\[
\tau_{ij} = \begin{pmatrix} \tau_{11} & 0 \\ 0 & \tau_{22} \end{pmatrix}.
\] (2.36)

\( \tau_{22} \) is the modulus of the “large” torus. If the three colliding points coalesce in \( a \), and the other three branch points are respectively at \( 0, 1, \infty \), then \( ^4 \)

\[
a = \frac{\theta_{00}(0, \tau_{22})}{\theta_{10}(0, \tau_{22})}.
\] (2.37)

The modulus of the “small” torus \( \tau_{11} \) is well defined only if in our limit the angles formed by the three colliding points are kept constant. If their complex coordinates are given by \( b, c, d \), one has:

\[
\frac{c - b}{d - b} = \frac{\theta_{00}(0, \tau_{11})}{\theta_{10}(0, \tau_{11})} = \frac{\vartheta_3^2(0|\tau_{11})}{\vartheta_2^2(0|\tau_{11})}.
\] (2.38)

The right hand side is the inverse of the modular \( \lambda \) function \( \lambda(\tau) \equiv \frac{\vartheta_3^2(0|\tau)}{\vartheta_2^2(0|\tau)} \). The relation (2.38) coincides with the inversion formula given in [10]. See Appendix B.

In the sextet vacuum we have a special situation: the three colliding points coalesce with one of the other three branch points. So the “large” torus degenerates \( (\tau_{22} = 0) \), corresponding to the fact that the \( U(1) \) factor in the unbroken \( SU(2) \times U(1) \) gauge group is a trivial IR free theory, as at the singularities of the \( SU(2) \) Seiberg-Witten theory [10]. See Fig. 6.

\(^4\theta_{00} \) and \( \theta_{10} \) are defined [15] as: \( \theta_{00}(0, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi n^2} \) and \( \theta_{10}(0, \tau) = \sum_{n=-\infty}^{\infty} e^{i\pi(n+\frac{1}{2})^2} \). They coincide with the standard Jacobi Theta functions [10]: \( \theta_{00}(0, \tau) = \vartheta_3(0|\tau_11); \theta_{10}(0, \tau) = \vartheta_2(0|\tau) \).
Figure 6: The large torus degenerates ($\tau_{22} \to 0$). The Abelian factor is a trivial infrared free theory.

On the other hand, the shape of the small torus $\tau_{11}$ is a function of the relative angles formed by the four colliding branch points. Let $(x_1, x_2, x_3, x_4)$ be the coordinates of the four colliding points. By translating them as $(0, x_2 - x_1, x_3 - x_1, x_4 - x_1)$ and making the transformation $x \to \frac{1}{x}$: we obtain the positions of the three finite branch points (as appropriate for using the inversion formula (2.38)) at $(\frac{1}{x_2 - x_1}, \frac{1}{x_3 - x_1}, \frac{1}{x_4 - x_1})$ and one at infinity.

We must now identify the three finite branch points in (2.38) with $\frac{1}{x_2 - x_1}, \frac{1}{x_3 - x_1}$, to determine $\tau_{11}$. Notice that the results does not depend on the choice which of the four points is moved to infinity, but do depend on the way $(b, c, d)$ are identified with $(\frac{1}{x_2 - x_1}, \frac{1}{x_3 - x_1}, \frac{1}{x_4 - x_1})$. This must be done consistently with the charge assignment, i.e., in accordance with the choice of the homology cycles encircling the branch points, defining $a_{Di}, a_i$ (see Fig.1).

The main problem however is that of the uniqueness of the superconformal theory in the limit $u, v \to 0$. The behavior of the branch points (see Eq.(2.40), Eq.(2.43) below) as $u, v \to 0$ clearly shows that the ratios among $(\frac{1}{x_2 - x_1}, \frac{1}{x_3 - x_1}, \frac{1}{x_4 - x_1})$ depend on the way the limit is approached. How can one avoid the arbitrariness of the value
of $\tau_{11}$ in the limit, and hence of the superconformal theory defined in such a limit?

In the limit $u, v \to 0$ the positions of the four colliding branch points can be determined approximately from the following equation:

$$(x^2 - ux - v)(x^2 + ux + v) = 0,$$  \hspace{1cm} (2.39)

with the solution:

$$x_1 = \frac{1}{2}(-u + \sqrt{u^2 - 4v}); \quad x_2 = \frac{1}{2}(-u - \sqrt{u^2 - 4v});$$

$$x_3 = \frac{1}{2}(u + \sqrt{u^2 + 4v}); \quad x_4 = \frac{1}{2}(u - \sqrt{u^2 + 4v}).$$ \hspace{1cm} (2.40)

Upon introduction of the variables $\epsilon, \rho, z, w$ following [8], given by:

$$v = \epsilon^2; \quad u = \epsilon \rho; \quad x = \epsilon z; \quad y = \epsilon^2 w,$$ \hspace{1cm} (2.41)

these equations become:

$$(z^2 - \rho z - 1)(z^2 + \rho z + 1) = 0;$$ \hspace{1cm} (2.42)

$$z_1 = \frac{1}{2}(-\rho + \sqrt{\rho^2 - 4}); \quad z_2 = \frac{1}{2}(-\rho - \sqrt{\rho^2 - 4});$$

$$z_3 = \frac{1}{2}(\rho + \sqrt{\rho^2 + 4}); \quad z_4 = \frac{1}{2}(\rho - \sqrt{\rho^2 + 4}).$$ \hspace{1cm} (2.43)

The positions of the four branch points depend thus only on

$$\rho^2 = \frac{u^2}{v},$$ \hspace{1cm} (2.44)

and the same is true for $\tau_{11} = \tau_{11}(\rho)$. The function $\tau_{11}(\rho)$ has the following singularities:

$$\rho^2 = 4 \to v = \frac{1}{4}u^2; \quad \rho^2 = -4 \to v = -\frac{1}{4}u^2;$$ \hspace{1cm} (2.45)

$$\rho = \infty \to v = 0.$$ \hspace{1cm} (2.46)

Thus around the points $+2, -2, 2i, -2i, \infty$ the function $\tau_{11}(\rho)$ has non-trivial monodromies, which correspond to the $U_1(1) \subset SU(2)$ charges of the previous section.

These nontrivial monodromies in the $\rho$ plane must be related to the massless states found earlier. We extract therefore the magnetic and electric charges related to $U_1(1) \subset SU(2)$ from Table [9] and determine the corresponding 2 by 2 monodromy matrices, by using the formula Eq.(2.22). This gives the result shown in Table [10]. Note that these coincide with the two by two submatrices, referring to the first and
Table 2: Simplified monodromy matrices and massless charges

| Section | Charge   | Matrix                      |
|---------|----------|-----------------------------|
| $M_1, M_4$ | $(\pm 1, 0)^4$ | $\begin{pmatrix} 1 & 0 \\ -4 & 1 \\ -3 & 4 \end{pmatrix}$ |
| $A_2, A_5$ | $(\pm 1, \mp 1)^4$ | $\begin{pmatrix} -4 & 5 \\ -1 & 1 \\ -4 & 3 \end{pmatrix}$ |
| $M_2, M_5$ | $(\pm 2, \mp 1)$ | $\begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$ |
| $A_3, A_6$ | $(\pm 2, \pm 1)$ | $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ |
| $M_3, M_6$ | $(0, \pm 1)$ | $\begin{pmatrix} -3 & 1 \\ -16 & 5 \end{pmatrix}$ |
| $A_1, A_4$ | $(\pm 4, \mp 1)$ | $\begin{pmatrix} 1 & 0 \\ -4 & 1 \\ -3 & 4 \end{pmatrix}$ |

third rows and columns, taken out of the complete monodromy matrices found earlier (and listed in Appendix A)\footnote{Although the matrices in Appendix A refer to the basis before the transformation Eq.(2.31), these submatrices remain invariant: $\tilde{m}_1 = m_1$; $\tilde{q}_1 = q_1$, since $q_2 = 0$.}

To assign the various matrices of Table 2 to the monodromies around the singularities in the $\rho$ plane, we have to take into account the consistency condition (see Fig. 8)

$$M(+2i)M(+2)M(-2i)M(-2) = M(\infty), \quad (2.47)$$

where $M(+2i)$ is the monodromy matrix around the point $+2i$, etc. It turns out that
the solutions is not unique. By making use of the relations

\[ M_6 A_1 M_4 = M_3 A_1 M_6 = -1, \quad A_1 M_2 M_1 = A_4 M_5 M_4 = -1, \]

\[ M_3 M_2 A_2 = M_6 M_5 A_5 = -1 \]

(2.48)

which hold among the matrices of Table 2, it is possible to construct different solutions, given in various columns of Table 3.

Figure 8: Monodromies in the \( \rho \) plane.

| \( M(+2i) \) | \( M_6 \) | \( A_1 \) | \( M_3 \) | \( M_3 \) | \( A_4 \) | \( M_6 \) | \( M_6 \) | \( \ldots \) |
| \( M(+2) \) | \( A_6 \) | \( M_2 \) | \( M_2 \) | \( A_3 \) | \( M_5 \) | \( M_5 \) | \( A_6 \) | \( \ldots \) |
| \( M(-2i) \) | \( M_3 \) | \( A_4 \) | \( M_6 \) | \( M_6 \) | \( A_1 \) | \( M_3 \) | \( M_3 \) | \( \ldots \) |
| \( M(-2) \) | \( A_3 \) | \( M_5 \) | \( M_5 \) | \( A_6 \) | \( M_2 \) | \( M_2 \) | \( A_3 \) | \( \ldots \) |
| \( M(\infty)^{-1} \) | \( M_1 M_4 \) | \( M_1 M_4 \) | \( A_2 A_5 \) | \( M_4 M_1 \) | \( M_4 M_1 \) | \( A_5 A_2 \) | \( M_1 M_4 \) | \( \ldots \) |

Table 3: Different sections of the singularities describing the same physics

A crucial observation is that in the \((u, v)\) space there are infinite number of copies of \( \rho \) plane, corresponding to different phases of \( \epsilon \). Namely, by varying the phase of \( \epsilon \) the linked rings in Fig. 3 are cut in different sections (different copies of \( \rho \) plane). See Fig. 4. It is therefore natural to identify the set of singularities in each section with the monodromy matrices of a given column in Table 3. Note that at constant \( \epsilon \), \( \rho \rightarrow -\rho \) corresponds to \( \{ v \rightarrow v, u \rightarrow -u \} \) (reflection with respect to the origin in Fig. 4): this explains the appearance of a pair of particles in each section with opposite charges with respect to \( U_1(1) \). They belong to a doublet of \( SU(2) \).

The main point of this discussion is that the three different columns of charges (note the three-column periodicity in Table 3 and Table 4) can be interpreted as
Figure 9: The linked rings cut in different sections.

Table 4: The same as Table 3 but with charges.

|      | (0, 1) | (4, -1) | (0, 1) | (0, -1) | (-4, 1) | (0, -1) | ... |
|------|--------|---------|--------|---------|---------|---------|-----|
| +2i  | (2, 1) | (2, -1) | (2, -1) | (-2, -1) | (-2, 1) | (-2, 1) | ... |
| 2    | (0, -1) | (-4, 1) | (0, -1) | (0, 1) | (4, -1) | (0, 1) | ... |
| -2i  | (-2, -1) | (-2, 1) | (-2, 1) | (2, 1) | (2, -1) | (2, -1) | ... |
| -2   | (±1, 0)⁴ | (±1, 0)⁴ | (±1, 0)⁴ | (±1, 0)⁴ | (±1, 0)⁴ | (±1, 0)⁴ | ... |
| ∞    | (±1, 0)⁴ | (±1, 0)⁴ | (±1, 0)⁴ | (±1, 0)⁴ | (±1, 0)⁴ | (±1, 0)⁴ | ... |

three different descriptions of the same physics, differing only by the redefinition of the charges. Indeed, one can pass from the first column to the second by a $SL(2, \mathbb{Z})$ transformation $p_1 = \begin{pmatrix} -1 & 4 \\ 0 & -1 \end{pmatrix}$; from the second to the third by $p_2 = \begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$; from the third to the first by $p_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

And this suggests how to solve the problem of non-uniqueness of the $u, v \to 0$ limit, hence of the superconformal theory at hand. We define, as was done in [8], the superconformal limit by

$$\epsilon \to 0, \quad \rho \to 0,$$

i.e., $\rho \to 0$ first. Note that $\rho = 0$ is the unique point where the symmetry of the
equation Eq.(2.42) for the singularities, \( \rho \to -\rho \), is maintained.

At \( \rho = 0 \), we find that the four colliding branch points are at \( x_1 = 2i, x_2 = -2i, x_3 = 2, x_4 = -2 \) (in the unit of \( \epsilon \)): they form a square. In order to apply (2.38), we must appropriately identify these four points, consistently with the charges given in the first column of Table 4.

We note that as \( \rho \) moves to \( 2i \) the branch points \( x_3 \) and \( x_4 \) coalesce. As the monodromy matrix around \( \rho = 2i \) implies the massless particle there to have the charge \( (0,1) \), we must assign the \( \beta_1 \) cycle to encircle the pair of branch points \( x_3 \) and \( x_4 \). An analogous consideration about the monodromy around \( \rho = +\infty \), where the branch points \( x_1 \) and \( x_4 \) coalesce, giving rise to massless monopoles, suggests that the \( \alpha_1 \) cycle defining the magnetic charge encircle \( x_1 \) and \( x_4 \) (see Table 4). We identify then

\[
e_3 = \frac{1}{x_4 - x_2} \to b, \quad e_2 = \frac{1}{x_1 - x_2} \to c, \quad e_1 = \frac{1}{x_3 - x_2} \to d,
\]

in Eq.(2.38). We obtain at \( \rho \to 0 \)

\[
\frac{1}{2} = \frac{\theta_{00}^1(0, \tau_{11})}{\theta_{10}^1(0, \tau_{11})}.
\]

which has the following solutions:

\[
\tau_{11} = \frac{\pm 1 + i}{2}, \quad \frac{\pm 3 + i}{10}, \ldots
\]

Other solutions of Eq.(2.51) can be found by acting repeatedly

\[
\tau \to \tau + 2; \quad \tau \to \frac{\tau}{1 - 2\tau}
\]

on the solutions (2.52).

3. Low-Energy Physics at a Renormalization-Group Fixed Point

The low-energy theory at the sextet vacua has a natural interpretation as a superconformal \( SU(2) \times U(1) \) gauge theory, with 4 magnetic monopole doublets \( (\tilde{m}_1, \tilde{q}_1) = (\pm 1, 0) \), a non-abelian electric doublet with charges \( (\tilde{m}_1, \tilde{q}_1) = (0, \pm 1) \) and a non-abelian dyon doublet with charges \( (\tilde{m}_1, \tilde{q}_1) = (\pm 2, \pm 1) \). See the first column of Table 4. The four magnetic doublets have the second \( (U_2(1)) \) abelian magnetic charge equal to \( \tilde{m}_2 = 1 \); the other two doublets have abelian magnetic charge equal to \( \tilde{m}_2 = 2 \).
The $SU(2)$ factor defines an interacting conformal theory. The $\beta$ function cancellation occurs in the following manner. The four magnetic monopole doublets cancel the contribution of the dual $SU(2)$ gauge bosons and of their supersymmetric partners as in a local $\mathcal{N} = 2$, $SU(2)$ gauge theory with $n_f = 4$. The non-trivial part of the cancellation occurs between the non-abelian electric and dyonic doublets. By considering the $U(1)$ subgroup of the $SU(2)$, their contribution to the first term of the beta function cancel if \[ \sum_i (q_i + m_i \tau)^2 = 0. \] (3.1)

With the dyon charges $(\tilde{m}_1, \tilde{q}_1) = (0, \pm 1), (\pm 2, \pm 1)$ at our disposal, this works if
\[
1 + (2 \tau + 1)^2 = 0, \quad \therefore \quad \tau^* = \frac{-1 + i}{2}.
\] (3.2)

Note that this value of critical coupling constant is precisely (one of) the value(s) one finds from the behavior of the small torus, (2.52).

As we change the basis of our theory by moving to another section, e.g., corresponding to the second column of Table 4, we redefine the homology cycles defining $a_{Di}, a_i$’s. The coupling constant $\tau$ is accordingly transformed:

\[
\tau^* \to \frac{-\tau^*}{-4\tau^* - 1} = \frac{3 + i}{10},
\] (3.3)

which is again the one following from the limiting behavior of the curve, (2.52). The condition for the beta function cancellation Eq.(3.1) with charges $(\tilde{m}_1, \tilde{q}_1) = (\pm 4, \mp 1), (\pm 2, \mp 1)$, precisely leads to $\tau^* = \frac{3 + i}{10}$, meaning that the infrared fixed-point condition is satisfied independently of the basis chosen to describing the theory.

Analogously, the transformation $\tau \to \frac{3\tau - 1}{4\tau - 1}$ allows to go from the second to the third basis, and $\tau \to \tau - 1$ to go from the third back to the first. The charges of massless fields and the critical coupling constant are transformed, but the condition for the infrared fixed point is always satisfied.

We find thus a natural but a highly non-trivial generalization of the Argyres-Douglas infrared fixed-point theory to one with a non-Abelian gauge symmetry.
4. Superconformal Vacuum as Limit of Six Colliding Vacua

The superconformal limit may be approached by first breaking it explicitly by unequal bare quark masses, by identifying the six nearby singularities $U_k, V_k$ ($k = 1, 2, \ldots, 6$) where the curve has the form,

$$y^2 = (x^3 - Ux - V)^2 - \prod_{a=1}^{4}(x + m_a) = (x - \alpha)^2(x - \beta)^2(x - \gamma)(x - \delta),$$  \hspace{1cm} (4.1)

and then by considering the limit of equal mass. In this limit, these singularities coalesce and becomes the conformal vacuum.

Each of the theories before the equal mass limit is taken is a local $U(1)^2$ gauge theory, with precisely two massless hypermultiplets, each of which carrying only one of the $U(1)$ charges. There are in all twelve massless hypermultiplets, and it is tempting to identify these degrees of freedom with the twelve massless particles found in the $m_i = 0$ theory ($4 + 1 + 1$ doublets of the effective $SU(2)$ gauge group).

A partial support comes from the observation that the massless states at one singularity and those at another singularity can be relatively non-local to each other. That this does occur can be explicitly verified by studying the movements of the four branch points near the origin ($|m_i| \ll \Lambda$), of the curve

$$y^2 = \prod_{i=1}^{3}(x - \phi_i)^2 - \prod_{a=1}^{4}(x + m_a) \equiv (x^3 - Ux - V)^2 - \prod_{a=1}^{4}(x + m_a),$$ \hspace{1cm} (4.2)

as one moves from a singularity $(U_1, V_1)$ to another $(U_2, V_2)$, by a numerical method. An example of such a non-trivial rearrangement of the branch points is illustrated in Fig. 10. There are also pairs of singularities which correspond to relatively local massless states (Fig. 11). This is consistent with the charges present in the theory (see the first column of Table 4).

Figure 10: A schematic representation of a nontrivial rearrangement of the four nearby branch points in the $x$ space, as one moves from a singularity $(U_1, V_1)$ to another $(U_2, V_2)$. 
Nonetheless, there are reasons to believe that the mechanism of confinement in the superconformal theory, deformed by the adjoint mass term $\mu \text{Tr} \Phi^2$, cannot be understood this way. In fact, consider adding the term $\mu \text{Tr} \Phi^2$ in each of the six vacua $U_i, V_i$ with a $U(1)^2$ gauge symmetry. The superpotential of the theory has the form,

$$P = \sum_{i=1}^{2} \sqrt{2} A_D, M_i \tilde{M}_i + \mu U(A_{D1}, A_{D2}) + \text{mass terms} \quad (4.3)$$

The standard argument shows that the $k$-th vacuum is characterized by the equations

$$\mu = -\sum_{i=1}^{2} \frac{\partial a_{D_i}}{\partial U} M_i \tilde{M}_i; \quad 0 = \sum_{i=1}^{2} \frac{\partial a_{D_i}}{\partial V} M_i \tilde{M}_i, \quad (4.4)$$

where $\frac{\partial a_{D_i}}{\partial U}$ and $\frac{\partial a_{D_i}}{\partial V}$ are given by the period integrals of the holomorphic differentials, see Appendix C. Other equations following from (4.3) tell that the $M_i, \tilde{M}_i$ fields are massless. It can be seen easily that in the equal mass limit (superconformal limit), in which the four branch points come together,

$$\frac{\partial a_{D1}}{\partial U} \to \infty, \quad \frac{\partial a_{D2}}{\partial U} \to \infty, \quad (4.5)$$

hence

$$\langle M_1 \rangle, \langle \tilde{M}_1 \rangle, \langle M_2 \rangle, \langle \tilde{M}_2 \rangle \to 0. \quad (4.6)$$

The same holds for all massless hypermultiplets at the six $\mathcal{N} = 1$ vacua. This is analogous to the phenomenon discussed in [17] at the Argyres-Douglas point of $\mathcal{N} = 2, SU(2)$ gauge theory with $n_f = 1$. 

Figure 11: A trivial rearrangement of the branch points, as one moves from a singularity to another, having relatively local massless particles.
5. Non-Abelian vs Abelian Confinement Mechanism: QCD

This brings us to an apparently paradoxical situation. The consideration of the equal mass limit starting from the unequal mass cases allow us to find the candidate degrees of freedom and identify them with those which appear in the massless theory studied in the earlier sections. This also seems to allow to study the effect of the $\mathcal{N} = 1$ perturbation and the ensuing confinement and dynamical symmetry breaking, in terms of the local effective actions valid in each of the six vacua. However, in the equal-mass limit, all condensates are found to vanish (at finite $\mu$).

On the other hand, a detailed study of the $\mathcal{N} = 2$, $SU(3)$ gauge theory with $n_f = 4$ at a large adjoint mass $\mu$, made in [1], shows that at the sextet vacua confinement and dynamical symmetry breaking

$$SU(4) \times U(1) \rightarrow U(2) \times U(2)$$

(5.1)
do take place. By supersymmetry and holomorphy (physics depends on $\mu$, not on $|\mu|$), we know that the same result holds at small $\mu$. But then which is the order parameter of the confinement/dynamical symmetry breaking?

The fact that these (almost) superconformal theories are non-Abelian gauge theories with finite or infinite couplings, seems to show the way out. In the preceding discussion Eq.(4.3) - Eq.(4.6), the effects of the strong $SU(2)$ interactions, which is probably dominant, are entirely neglected. It is perfectly possible that the four magnetic monopole doublets of $SU(2)$ discussed in Sec 3., $M^i_\alpha$, ($\alpha = 1, 2, \ i = 1, \ldots, 4$), condense due to the $SU(2)$ interactions upon $\mu$ perturbation,

$$\langle M^i_\alpha M^j_\beta \rangle = \epsilon_{\alpha\beta} C^{ij} \neq 0,$$

(5.2)

where $C^{ij}$ is antisymmetric in the flavor indices. Such a structure is consistent with the known symmetry breaking pattern, Eq.(5.1).

We are thus led naturally to the conclusion that the microscopic mechanism of confinement and dynamical symmetry breaking in the almost superconformal vacua such as the one under consideration, should be essentially different from the Abelian confinement mechanism found in the original Seiberg-Witten work [10, 11], and involves strongly interacting non-Abelian magnetic degrees of freedom.

---

6 One of the $U(1)$ is a combination of the gauge $U(1)$ group and a subgroup of the flavor $SU(4)$. That Eq.(5.2) leaves the other $U(1)$ global symmetry is not so obvious. There is however a natural mechanism for the quantum quenching of the baryonic $U(1)$ charge for magnetic monopoles [18, 1].

7 Analogously, we believe that the vanishing of the monopole and charge condensates at the
The final aim of our efforts is to understand the microscopic mechanism of confinement in the standard QCD. As there are other reasons to believe that the model of confinement by weakly-coupled magnetic particles is not a good model for QCD [19], it is exceedingly interesting that in the class of confining vacua based on deformed superconformal theory we have a new mechanism of confinement, involving strongly interacting magnetic degrees of freedom. It has been noted also [1] that the pattern of dynamical symmetry breaking in these SCFT based vacua was most reminiscent of what happens in QCD.

Another line of thought along a recent work [3], appears to lead to a similar conclusion. There, the appearance of the non-Abelian monopoles as weakly coupled low-energy degrees of freedom in the $r$-vacua of $\mathcal{N} = 2$ supersymmetric QCD, $r < n_f^2$, was understood as a consequence of the flipping of the sign of the beta function coefficient:

$$b_0^{(dual)} \propto -2r + n_f > 0,$$

(5.3)

for dual $SU(r)$ theory, while in the fundamental $SU(n_c)$ theory

$$b_0 \propto -2n_c + n_f < 0.$$  

(5.4)

The dressing of the non-Abelian monopoles with the flavor quantum number is a necessary condition for this to happen. Indeed, in a pure $\mathcal{N} = 2$ Yang-Mills theory, or on a generic point of the moduli space in $\mathcal{N} = 2$ supersymmetric QCD, where this is not possible, only monopoles of Abelian variety appear as the low-energy degrees of freedom.

The appearance of Abelian monopoles as long-distance physical degrees of freedom is in itself consistent as $U(1)$ theories are infrared free. However, this phenomenon seems to be always accompanied by dynamical gauge symmetry breaking (dynamical Abelianization), with characteristic signals of enriched spectrum of Regge trajectories, etc.

In the standard QCD, the coefficient 2 in front of the color multiplicity $n_c = 3$ or $r(\leq 3)$ in Eq.(5.4), Eq.(5.3), is replaced by 11. It is clearly very difficult to realize, having at our disposal only a few light flavors, the required sign flip. It is even more so, since the monopoles, being scalars, contribute less than the quarks in the original Lagrangian. On the other hand, there are no phenomenological indications that dynamical Abelianization takes place in the real world of strong interactions.

Argyres-Douglas point found in Ref. [17] means the inadequacy of these degrees of freedom as the order parameters, rather than signalling the deconfinement. In fact, in that model (an $\mathcal{N} = 2$, $SU(2)$ theory with $n_f = 1$), the standard squark condensate $\langle \tilde{Q}Q \rangle$ remains finite at the Argyres-Douglas point, which cannot be simply expressed in terms of the monopole/dyon degrees of freedom.
We are naturally led to a picture of confinement in QCD, still largely to be clarified, involving strongly-interacting non-Abelian magnetic degrees of freedom.

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Appendix A: Monodromy Matrices

The set of the twelve matrices:

\[ M_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

\[ M_3 = \begin{pmatrix} 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix}, \quad M_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & 1 & 0 \\ 4 & -4 & 0 & 1 \end{pmatrix}, \]

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M_5 = \begin{pmatrix} -1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & 3 & 0 \\ 4 & -4 & -2 & 1 \end{pmatrix}, \quad M_6 = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 1 \end{pmatrix},
\lambda_1 = \begin{pmatrix} -3 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -16 & 4 & 5 & 0 \\ 4 & -1 & -1 & 1 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} -3 & 0 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},
\lambda_3 = \begin{pmatrix} 3 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & -1 & 0 \\ 4 & -4 & 2 & 1 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} -3 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -16 & 12 & 5 & 0 \\ 12 & -9 & -3 & 1 \end{pmatrix},
\lambda_5 = \begin{pmatrix} -3 & 4 & 4 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 4 & 5 & 0 \\ 4 & -4 & -4 & 1 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},

(A.1)

satisfy all the conjugation relations, Eqs.(2.25). Note also that
\begin{align*}
M_1 &= (\tilde{M}_1)^4, \quad M_4 = (\tilde{M}_4)^4, \quad A_2 = (\tilde{A}_2)^4, \quad A_5 = (\tilde{A}_5)^4, \\
A_1 &= (\tilde{A}_1)^4, \quad A_4 = (\tilde{A}_4)^4, \quad A_6 = (\tilde{A}_6)^4,
\end{align*}

(A.2)

with
\begin{align*}
\tilde{M}_1 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{M}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix},
\tilde{A}_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{A}_4 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix},
\end{align*}

(A.3)

The use of the formula Eq.(2.22) then yields the charges Eq.(2.26).

**Appendix B: Inversion formula**

A genus one curve can be parametrized in terms of a Weierstrass function as
\[
y = \frac{dP(z; \omega_1, \omega_2)}{dz}, \quad x = P(z; \omega_1, \omega_2),
\]
which satisfies
\[ y^2 = 4 x^3 - g_2 x - g_3 = 4 (x - e_1) (x - e_2) (x - e_3), \]  
(B.2)

where
\[ e_1 = \mathcal{P}(\frac{\omega_1}{2}), \quad e_2 = \mathcal{P}(\frac{\omega_2}{2}), \quad e_3 = \mathcal{P}(\frac{\omega_1 + \omega_2}{2}). \]  
(B.3)

The shape parameter is given by \( \tau = \frac{\omega_1}{\omega_2} \),
\[ \omega_1 = \oint_{\alpha} \frac{dx}{y}, \quad \omega_2 = \oint_{\beta} \frac{dx}{y}, \]  
(B.4)

where \( \alpha \) and \( \beta \) cycles encircle the pairs of branch points \( (e_3, e_2) \) and \( (e_3, e_1) \), respectively. With these definitions, the inversion formula is:
\[ \lambda^{-1}(\tau) = \frac{\theta_3^4(0, e^{i\pi \tau})}{\theta_2^4(0, e^{i\pi \tau})} = \frac{e_3 - e_2}{e_3 - e_1}. \]  
(B.5)

Appendix C: Period Integrals at the Sextet Singularities

According to the by now standard result [12], the derivatives of \( A_{D_i} \) and \( A_i \)'s with respect to the vacuum parameters \( U = \langle \text{Tr} \Phi^2 \rangle \) and \( V = \langle \text{Tr} \Phi^3 \rangle \) are to be identified with the period matrices,
\[
\begin{align*}
\frac{dA_{D1}}{dU} &= \oint_{\alpha_1} \frac{dx}{y}, & \frac{dA_{D2}}{dU} &= \oint_{\alpha_2} \frac{dx}{y}, \\
\frac{dA_{D1}}{dV} &= \oint_{\alpha_1} \frac{x \, dx}{y}, & \frac{dA_{D2}}{dV} &= \oint_{\alpha_2} \frac{x \, dx}{y}, \\
\frac{dA_1}{dU} &= \oint_{\beta_1} \frac{dx}{y}, & \frac{dA_2}{dV} &= \oint_{\beta_2} \frac{dx}{y}, \\
\frac{dA_1}{dV} &= \oint_{\beta_1} \frac{x \, dx}{y}, & \frac{dA_2}{dV} &= \oint_{\beta_2} \frac{x \, dx}{y},
\end{align*}
\]  
(C.1)

where
\[ y^2 = (x^3 - U x - V)^2 - \prod_{a=1}^4 (x + m_a) = \prod_{i=1}^6 (x - e_i). \]  
(C.2)

At the superconformal vacua of our interest, four of the branch points \( e_i, i = 1, 2, 3, 4 \) coalesce whereas two others are at \( e_5, e_6 = \pm 1 \). By our choice, \( \alpha_1 \) cycle encircles the branch points \( e_1 \) and \( e_4 \); \( \beta_1 \) cycles encircle the branch points \( e_3 \) and \( e_4 \); \( \alpha_2 \) cycle
encircles all the branch points $e_1$, $e_2$, $e_3$, $e_4$, and the large $\beta_2$ cycle encircles the points $e_2$ and $e_5$. Near the SCFT point,

$$\frac{dA_{D_1}}{dU} \approx \frac{1}{\sqrt{e_5 e_6}} 2 \int_{e_1}^{e_4} \frac{dx}{\sqrt{(x - e_1)(x - e_4)(x - e_2)(x - e_3)}} \xrightarrow{SCFT \to \infty} \infty, \quad (C.3)$$

etc. It is easy to verify Eq.(4.3) similarly.