A MODIFICATION OF GALERKIN’S METHOD FOR OPTION PRICING

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Abstract. We present a novel method for solving a complicated form of a partial differential equation called the Black-Scholes equation arising from pricing European options. The novelty of this method is that we consider two terms of the equation, namely the volatility and dividend, as variables dependent on the state price. We develop a Galerkin finite element method to solve the problem. More specifically, we discretize the system along the state variable and build new basis functions which we use to approximate the solution. We establish convergence of the proposed method and numerical results are reported to show the proposed method is accurate and efficient.

1. Introduction. The financial sector is usually one of the largest sections in an economy, and its effectiveness is supported by the presence of options and other derivatives. In addition, different types of derivatives are used in energy trading, agriculture trading, and insurance. Options are a type of financial derivatives that let the user speculate about a stock price in the future. The option buyer will profit if their prediction is fairly accurate. Options can also be used to minimize the risk. Buying an option means buying the right to buy stock at a specified price after a certain period of time (Call option) or to sell stock at a specified price after a certain period of time (Put option). The time when the option can be exercised is called the expiry date.

There are many different types of options. The most common options are European and American options. An American option lets the option holder exercise the option at any time until the expiry date, while a European option lets the option holder to exercise the option only at the expiry date. There are also other types of options such as exotic options. This paper considers the European Option valuation. Mathematically finding the fair option price eliminates arbitrage in option trading. By obtaining a theoretical price, the buyer and the seller can use it as a reference point in their dealings. Thus, it is important to develop numerical methods for pricing options accurately.

There is a vast volume of literature for mathematical methods of option pricing; see the review of these methods in [11]. To improve computational precision, Wang [8] considered numerical solution of this equation based on a so-called fitted finite
volume method. This was a discretization method based on a finite volume formulation of the problem coupled with a fitted local approximation to the solution and on an implicit time-stepping technique. The local approximation is determined by a set of two-point boundary value problems; this fitting technique is based on the idea proposed by [1] for convection-diffusion equations. Overall, this method represents a special case of Petrov-Galerkin’s method and increases the accuracy of calculation of the price comparing with the straightforward finite-difference method.

Angermann and Wang [2] extended the fitted finite volume method to American options. In [2] and related papers [9, 10, 12], some regularity problems were solved. It has to be clarified that there are several competing methods in the literature. In particular, it is common to use Monte-Carlo simulation that allows to find a price in a particular state point without solving partial differential equations; See [3, 7]. However, the methods based PDEs allow more precise estimation of the entire value function.

The novelty of this paper is to use Galerkin’s method for the Black-Scholes equation and construct basis functions using the state-dependent functions for volatility and dividends. Each basis function can be derived using the given parameters. Option value at any point along the state variable can be found by interpolating two basis functions.

The paper is organized as follows. In Section 2, we set up the problem by transforming the original Black-Scholes equation. In Section 3, we set up basis functions to apply Galerkin’s method. In Section 4, we introduce weak formulation of the problem as well as the main theorem that is important for solving the problem. In Section 5, we prove the theorem. In Section 6, we review some methods of solution: exact method and Crank-Nicolson’s method. In Section 7, we prove the convergence of the method. In Section 8, we test the method numerically.

2. Problem setting. We will consider a model of a stock price $S(t)$ described by the following stochastic Ito equation

$$dS(t) = S(t)(rdt + \sigma(S(t),t)dw(t)), \quad t > 0,$$

where $w(t)$ is a Wiener process, $r$ is a risk-free rate, $\sigma$ is the volatility of this stock. Assume that there are dividends $d(x)$ on the stock. The pricing problem can be formulated as follows: Let a random variable $X$ represent a payoff of a financial option. If $X = f(S(T))$ for some function $f : \mathbb{R} \rightarrow \mathbb{R}$ then the option is said to be of a European type. In this case, the “fair” option price at time $t$ is $P(t) = e^{-r(T-t)}E\{f(S(T)|S(t))\}$, i.e. it can be calculated as the conditional expectation.

We assume that one of the following conditions is satisfied:

1. $r = 0$
2. $f(x) = (x - K)^+ + (K - x)^+$, where $K > 0$ is given, for call and put options respectively. Here we use the notation $(x)^+ = \max(x, 0)$.

It can be shown, using Ito’s Lemma, that

$$P(t) = V(S(t), t),$$

where $V(x, t)$ is a solution of the boundary value problem for the following partial differential equation:
\[
\frac{\partial V}{\partial t}(x, t) + \frac{1}{2} \sigma(x)^2 \frac{\partial^2 V}{\partial x^2}(x, t) + x(r - d(x)) \frac{\partial V}{\partial x}(x, t) - rV(x, t) = 0,
\]
\[
V(x, T) = f(x), \quad x > 0,
\]
\[
\lim_{x \to \infty} \frac{V(x, t)}{f(x)} = R_0(t), \quad 0 < t < T,
\]
\[
V(0, t) = R_0(t)f(0), \quad 0 < t < T.
\]

(2)

Here, \(R_0(t) \equiv 1\) for the call option and \(R_0(t) \equiv e^{-r(T-t)}\) for the put option. Equation (2) is a so-called Black-Scholes equation which is a special case of a parabolic equation, see, e.g., [5, 6]. Here, \(x \in (0, \infty), t \in [0, T), \sigma(x)\) represents the volatility coefficient, \(d(x)\) represents the dividend rate, \(r\) represents the risk-free bank rate, \(T > 0\) is the terminal time. We assume that \(d(x)\) is a bounded function and that \(\sigma(x)\) is a bounded function with a bounded first derivative. In this paper we consider volatility \(\sigma(x)\) as a function of underlying stock price \(x\) rather than a constant.

We introduce a variable \(y\) such that
\[
x = e^y, \quad V(x, t) = \hat{V}(\ln x, t).
\]

(3)

Then \(\frac{\partial V}{\partial \sigma} = \frac{\partial \hat{V}}{\partial y} e^{-y}, \frac{\partial^2 V}{\partial x^2} = e^{-2y} (\frac{\partial^2 \hat{V}}{\partial y^2} - \frac{\partial \hat{V}}{\partial y}).\) Equation (2) takes the form
\[
\frac{\partial \hat{V}}{\partial t}(y, t) + \frac{1}{2} \sigma(y)^2 \frac{\partial^2 \hat{V}}{\partial y^2}(y, t) + (r - d(y)) \frac{\partial \hat{V}}{\partial y}(y, t) - r\hat{V}(y, t) = 0,
\]
\[
\hat{V}(y, T) = f(y), \quad x \in (-L, L),
\]
\[
\hat{V}(L, t) = R_0(t)f(L), \quad \hat{V}(-L, t) = R_0(t)f(-L), \quad t \in [0, T)
\]

(4)

Note that we write \(f(e^y)\) and \(\sigma(e^y)\) as \(f(y)\) and \(\sigma(y)\) respectively, to keep notations short. Next, we use the substitution:
\[
\hat{v}(y, t) = e^{-r(t-T)}\hat{V}(y, t).
\]

Let \(R(t) := R_0(t)e^{-r(t-T)}\). By the definitions, \(R(t) = 1\) for the call options.

Clearly,
\[
\frac{\partial \hat{V}}{\partial t}(y, t) = re^{r(t-T)}\hat{v}(y, t) + \frac{\partial \hat{v}}{\partial t}(y, t)e^{r(t-T)},
\]
\[
\hat{V}(y, T) = e^{r(T-T)}f(y) = f(y),
\]
\[
\hat{v}(L, t) = R(t)f(L),
\]
\[
\hat{v}(-L, t) = R(t)f(-L).
\]

Equation (4) becomes
\[
re^{r(t-T)}\hat{v}(y, t) + \frac{\partial \hat{v}}{\partial t}(y, t)e^{r(t-T)} + \frac{1}{2}e^{r(t-T)}\sigma(y)^2 \frac{\partial^2 \hat{v}}{\partial y^2}(y, t) - \frac{\partial \hat{v}}{\partial y}(y, t))
\]
\[
+e^{r(t-T)}(r - d(y)) \frac{\partial \hat{v}}{\partial y}(y, t) - re^{r(t-T)}\hat{v}(y, t) = 0.
\]

Simplifying the above, we obtain that
\[
\frac{\partial \hat{v}}{\partial t}(y, t) + \frac{1}{2} \sigma(y)^2 \frac{\partial^2 \hat{v}}{\partial y^2}(y, t) + (r - d(y)) \frac{\partial \hat{v}}{\partial y}(y, t) - \frac{\partial \hat{v}}{\partial y}(y, t) = 0,
\]
\[
\hat{v}(y, T) = f(y), \quad \hat{v}(L, t) = R(t)f(L), \quad \hat{v}(-L, t) = R(t)f(-L).
\]

(5)
For brevity, we will rewrite this as
\[
\frac{\partial \hat{v}}{\partial t}(y, t) + \rho(y) \frac{\partial^2 \hat{v}}{\partial y^2}(y, t) + \eta(y) \frac{\partial \hat{v}}{\partial y}(y, t) = 0,
\]
\[
\hat{v}(y, T) = f(y), \quad \hat{v}(L, t) = R(t)f(L), \quad \hat{v}(-L, t) = R(t)f(-L)
\]
Here, \(\rho(y) = \frac{1}{2} \sigma(y)^2\), \(\eta(y) = r - d(y) - \frac{1}{2} \sigma^2(y)\) and \(R(t) = e^{\epsilon(t-T)}\). We assume that in (2), \(|f(x)| \leq c(1 + |x|)\), for some constant \(c > 0\) and we look for solution \(|\hat{v}(x, t)| \leq C_1(1 + |x|)\). Let us introduce a differential operator \(A\) such that
\[
A\hat{v} = \frac{\partial^2 \hat{v}}{\partial y^2} + \eta \frac{\partial \hat{v}}{\partial y}.
\]
We consider (5) for \(y \in (-L, L)\), where \(L > 0\) is a sufficiently large constant. We consider the boundary problem
\[
\frac{\partial \hat{v}}{\partial t} + A\hat{v} = 0, \quad \hat{v}(y, T) = f(y), \quad y \in \mathbb{R},
\]
\[
\hat{v}(-L, t) = R(t)f(-L),
\]
\[
\hat{v}(L, t) = R(t)f(L).
\]
We use the substitution
\[
v(y, t) = \hat{v}(y, t) - f(y)R(t).
\]
Then,
\[
\frac{\partial \hat{v}}{\partial t} = \frac{\partial v}{\partial t} + f(y) \frac{\partial R}{\partial t}.
\]
This gives the following problem:
\[
\frac{dv}{dt} = -Av - Af(t) - f(y) \frac{\partial R}{\partial t},
\]
\[
v(y, T) = 0,
\]
\[
v(-L, t) = v(L, t) = 0.
\]
Here \(y \in D, D = (-L, L)\).

3. Basis functions. Let \(\{y_k\}_{k=1}^{N+1} \subset D\) be selected such that \(-L = y_0 < y_1 < y_1 < \ldots < y_{N+1} = L\). Let us introduce the basis functions \(\phi(y) = \phi_k(y), k = 0, \ldots, N + 1\) that satisfy \(A\phi = 0\) on \((y_{k-1}, y_k) \cup (y_k, y_{k+1})\) and the following conditions:
1. \(\phi_k(y_k) = 1, \phi_k(y_{k-1}) = \phi_k(y_{k+1}) = 0\).
2. \(\phi_k(y) \geq 0\).
3. \(\phi_k(y) = 0\) for \(y \notin (y_{k-1}, y_{k+1})\).
4. \(\phi_k|_{[y_{k-1}, y_k]} \in C^2([y_{k-1}, y_k])\); \(\phi_k|_{[y_k, y_{k+1}]} \in C^2([y_k, y_{k+1}])\). Here, \(C^2\) is the space of twice differentiable functions.
5. \(\phi_k|_{[y_{k-1}, y_{k+1}]} \in W^1_\infty(\{y_{k-1}, y_k\})\). Here, \(W_\infty^1\) is a Sobolev space of functions with bounded first derivative.
6. \(A\phi_k(y) = 0\) for \(y \in (y_{k-1}, y_k) \cup (y_k, y_{k+1})\).
7. \(\phi_k(y) + \phi_{k+1}(y) = 1\) for \(y \in [y_k, y_{k+1}]\).

Diagrams of these functions form intersecting deformed triangles on the \(\phi - x\) plane. To find \(\phi's\), we need to solve the following equation:
\[
A\phi_k(y) = \rho(y) \frac{\partial^2 \phi_k}{\partial y^2}(y) + \eta(y) \frac{\partial \phi_k}{\partial y}(y) = 0.
\]
We consider two cases: (1) \( y \in [y_{k-1}, y_k] \) and (2) \( y \in [y_k, y_{k+1}] \). The boundary conditions are as follows:

\[
\phi_k(y_{k-1}) = 0, \quad \phi_k(y_k) = 1, \quad \phi_k(y_{k+1}) = 0. \quad (13)
\]

Let \( \gamma(y) = \frac{\partial \phi_k}{\partial y}(y) \). The equation (12) becomes

\[
\rho(y) \frac{\partial \gamma}{\partial y}(y) + \eta(y) \gamma(y) = 0.
\]

This equation can be solved exactly using the integrating factor method. It can be written as

\[
\frac{\partial \gamma}{\partial y}(y) + \frac{\eta(y)}{\rho(y)} \gamma(y) = 0.
\]

Let

\[
\omega_{k,-}(y) = \int_{y_{k-1}}^{y} \frac{\eta(x)}{\rho(x)} dx + c.
\]

Let integrating factor be \( \mu = C_1 e^{\omega_{k,-}(y)} \). Then

\[
\gamma_k(y) = C_1 e^{-\omega_{k,-}(y)}.
\]

To find \( \phi_k \), we integrate \( \gamma \):

\[
\phi_k(y) = \int_{y_{k-1}}^{y} \gamma(x) dx + C_2.
\]

Next, we check the initial conditions (13):

\[
\phi_k(y_{k-1}) = \int_{y_{k-1}}^{y_{k-1}} \gamma(x) dx + C_2 = 0,
\]

\[
C_2 = 0,
\]

\[
\phi_k(y_k) = \int_{y_{k-1}}^{y_k} \gamma(x) dx = C_1 \int_{y_{k-1}}^{y_k} e^{-\omega_{k,-}(x)} dx = 1.
\]

Hence,

\[
C_1 = \frac{1}{\int_{y_{k-1}}^{y_k} e^{-\omega_{k,-}(x)} dx}.
\]

Thus, we determined \( \phi_k(y) \) for \( y \in (y_{k-1}, y_k) \), and

\[
\phi_k(y) = \frac{1}{\int_{y_{k-1}}^{y_k} e^{-\omega_{k,-}(x)} dx} \int_{y_{k-1}}^{y} e^{-\omega_{k,-}(x)} dx.
\]

(14)

When the discretization step is small enough, this function will be increasing from \( y_{k-1} \) to \( y_k \).

Similarly, we find \( \phi_k(y) \) for \( y \in (y_k, y_{k+1}) \). Let

\[
\omega_{k,+}(y) = \int_{y_k}^{y} \frac{\eta(x)}{\rho(x)} dx + c.
\]

Let \( \mu = C_1 e^{\omega_{k,+}(y)} \) and \( \gamma_k(y) = C_1 e^{-\omega_{k,+}(y)} \). To find \( \phi_k(y) \), we integrate \( \gamma \):

\[
\phi_k(y) = \int_{y_k}^{y} \gamma(x) dx + C_2.
\]
Next, we match the initial conditions (13) to find $C_1$ and $C_2$ as follows:

$$\phi_k(y) = \int_{y_k}^{y_{k+1}} \gamma(x) dx + C_2 = 1,$$

$$C_2 = 1,$$

$$\phi_k(y+1) = \int_{y_k}^{y_{k+1}} \gamma(x) dx + 1 = C_1 \int_{y_k}^{y_{k+1}} e^{-\omega_k(x)} dx + 1 = 0,$$

$$C_1 = \frac{-1}{\int_{y_k}^{y_{k+1}} e^{-\omega_k(x)} dx}.$$

Thus, $\phi_k(y)$ for $y \in (y_k, y_{k+1})$ is determined as follows:

$$\phi_k(y) = \frac{-1}{\int_{y_k}^{y_{k+1}} e^{-\omega_k(x)} dx} \int_{y_k}^{y} e^{-\omega_k(x)} dx + 1.$$

We will use these basis functions to discretize the system. Any point $V(y)$ can be represented as a linear combination of two appropriate basis functions in the domain: $v_k(t)\phi_k(y) + v_{k+1}(t)\phi_{k+1}(y)$ if $y \in [y_k, y_{k+1}]$. Figure 1 shows an example of such a basis function.

4. ODE implied by the Galerkin method. Using $\phi_m$ from the previous section we can approximate $v$ in (12) as $V_N(y, t) = \sum_{k=1}^{N} v_k(t)\phi_k(y)$. The boundary conditions can be interpreted as $v_0(t) = 0$ and $v_{N+1}(t) = 0$. Let $S_N$ be the span of $\{\phi_k\}_{k=1,...,N}$. Let us consider a bilinear mapping $a : H^1_0(D) \times H^1_0(D) \to \mathbb{R}$ such that $(Au, w)_{L^2(D)} = a(u, w)$ for all $u, w \in H^1_0(D) \cap W^2_2(D)$. In a weak form the equation (12) is

$$\frac{du}{dt}, w)_{L^2(D)} = -a(u, w) - R(t)a(f, w) - R'(t)(f, w)_{L^2(D)}$$

(15)
for all $w \in H_0^1(D)$, where $H_0^1(D)$ is the space of functions belonging to $L_2(D)$ together with their first derivatives and such that they vanish at $\partial D$. Following the Galerkin Method, we look for $v_k$ such that

$$
(V_N', w)_{L_2(D)} = -a(V_N, w) - R(t)a(f_N, w) - R'(t)(f_N, w)_{L_2(D)},
$$

for all $w \in S_N$, $m = 1, \ldots, N$. Formally, equation (12) can be presented as

$$
\sum_{k=0}^{N+1} v_k'(t) \phi_k(y) = - \sum_{k=0}^{N+1} v_k(t)A\phi_k(y) - R(t) \sum_{k=0}^{N+1} \xi_k A\phi_k(y) - R'(t) \sum_{k=0}^{N+1} \xi_k \phi_k(y).
$$

Here, $\xi_k = f(y_k)$. Since $v_0 = v_{N+1} = 0$, we can rewrite the above as

$$
\sum_{k=1}^{N} v_k'(t) \phi_k(y) = - \sum_{k=1}^{N} v_k(t)A\phi_k(y) - R(t) \sum_{k=0}^{N+1} \xi_k A\phi_k(y) - R'(t) \sum_{k=0}^{N+1} \xi_k \phi_k(y).
$$

Multiplying by $\phi_m$, for $m = 1, \ldots, N$ and integration by $dy$ gives

$$
\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t)\phi_k(y)\phi_m(y)dy = - \int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t)A\phi_k(y)\phi_m(y)dy
$$

$$
- R(t) \int_{\mathbb{R}} \sum_{k=1}^{N} \xi_k A\phi_k(y)\phi_m(y)dy - R'(t) \sum_{k=0}^{N+1} \xi_k \phi_k(y)\phi_m(y)dy. \quad (17)
$$

Using the above we can construct the following system.

**Theorem 4.1.**

$$
My'(t) = Bv(t) + \varphi, \quad v(T) = 0, \quad (18)
$$

where

$$
\varphi = R(t)(B\xi + \zeta) - R'(t)(M\xi + \kappa). \quad (19)
$$

Here,

$$
v(t) = \begin{pmatrix} v_1(t) \\ v_2(t) \\ \vdots \\ v_{N-1}(t) \\ v_N(t) \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_{N-1} \\ \xi_N \end{pmatrix}, \quad \zeta = \begin{pmatrix} \zeta_1 \\ 0 \\ \vdots \\ 0 \\ \zeta_N \end{pmatrix}, \quad \kappa = \begin{pmatrix} \kappa_1 \\ 0 \\ \vdots \\ 0 \\ \kappa_N \end{pmatrix}, \quad (20)
$$

$$
M = \begin{pmatrix} 
\mu_{1,1} & \mu_{1,2} & 0 & 0 & \cdots & 0 \\
\mu_{2,1} & \mu_{2,2} & \mu_{2,3} & 0 & \cdots & 0 \\
0 & \mu_{3,2} & \mu_{3,3} & \mu_{3,4} & \cdots & 0 \\
0 & 0 & \cdots & \mu_{N-1,N-2} & \mu_{N-1,N-1} & \mu_{N-1,N} \\
0 & 0 & \cdots & 0 & \mu_{N-1,N} & \mu_{NN} 
\end{pmatrix}, \quad (21)
$$

$$
B = \begin{pmatrix} 
\beta_{1,1} & \beta_{1,2} & 0 & 0 & \cdots & 0 \\
\beta_{2,1} & \beta_{2,2} & \beta_{2,3} & 0 & \cdots & 0 \\
0 & \beta_{3,2} & \beta_{3,3} & \beta_{3,4} & \cdots & 0 \\
0 & 0 & \cdots & \beta_{N-1,N-2} & \beta_{N-1,N-1} & \beta_{N-1,N} \\
0 & 0 & \cdots & 0 & \beta_{N-1,N} & \beta_{NN} 
\end{pmatrix}, \quad (22)
$$
The coefficients of $M$ are
\[
\mu_{m,m-1} = \int_{y_{m-1}}^{y_m} \phi_{m-1}(y) \phi_m(y) dy,
\]
\[
\mu_{m,m} = \int_{y_{m-1}}^{y_{m+1}} \phi_m^2(y) dy,
\]
\[
\mu_{m,m+1} = \int_{y_m}^{y_{m+1}} \phi_{m+1}(y) \phi_m(y) dy.
\]

The coefficients of $B$ are
\[
\beta_{m,m-1} = \frac{\sigma^2(y_m)}{2} (\phi'_{m-1}(y_m - 0),
\]
\[
\beta_{m,m} = -\frac{\sigma^2(y_m)}{2} (\phi'_m(y_m + 0) - \phi'_m(y_m - 0)),
\]
\[
\beta_{m,m+1} = -\frac{\sigma^2(y_m)}{2} (\phi'_{m+1}(y_m + 0)).
\]

The coefficients $\xi$, $\zeta$ and $\kappa$ are
\[
\xi_k = f(y_k),
\]
\[
\zeta_1(t) = \beta_{1,0} \xi_0 = \frac{\sigma^2(y_1)}{2} \phi_0'(y_1 - 0)) \xi_0,
\]
\[
\zeta_N(t) = \beta_{N,N+1} \zeta_{N+1} = \frac{\sigma^2(y_N)}{2} \phi_{N+1}'(y_N + 0) \xi_{N+1}.
\]
\[
\kappa_1(t) = \xi_0(t) \int_{y_0}^{y_1} \phi_0(y) \phi_1(y) dy = \mu_{1,0} \xi_0.
\]
\[
\kappa_N(t) = \xi_{N+1}(t) \int_{y_N}^{y_{N+1}} \phi_{N+1}(y) \phi_N(y) dy = \mu_{1,0} \xi_{N+1}.
\]

5. Proof of the Theorem (4.1).

5.1. Finding the components for $M$ in (18). Let us look at the left hand side of (17). If $k < m - 1$ or $k > m + 1$ then $\phi_k \phi_m = 0$. Therefore, for $m = 2, ..., N - 1$, we have that
\[
\int_{\mathbb{R}} \sum_{k=1}^{N} v_k'(t) \phi_k(y) \phi_m(y) dy
\]
\[
= \frac{dv_{m-1}(t)}{dt} \int_{y_{m-1}}^{y_m} \phi_{m-1}(y) \phi_m(y) dy + \frac{dv_m(t)}{dt} \int_{y_{m-1}}^{y_{m+1}} \phi_m^2(y) dy
\]
\[
+ \frac{dv_{m+1}(t)}{dt} \int_{y_m}^{y_{m+1}} \phi_{m+1}(y) \phi_m(y) dy
\]
\[
= \mu_{m-1} \frac{dv_{m-1}(t)}{dt} + \mu_m \frac{dv_m(t)}{dt} + \mu_{m+1} \frac{dv_{m+1}(t)}{dt}. \tag{26}
\]

Let us consider boundary cases $m = 1$ and $m = N$. For the case where $m = 1$,
\[
\frac{dv_0}{dt} \int_{y_0}^{y_1} \phi_0(y) \phi_1(y) dy + \frac{dv_1}{dt} \int_{y_0}^{y_2} \phi_1^2(y) dy + \frac{dv_2}{dt} \int_{y_1}^{y_2} \phi_2(y) \phi_1(y) dy
\]
\[
= 0 + \mu_1 \frac{dv_1}{dt} + \mu_2 \frac{dv_2}{dt}, \tag{27}
\]
5.2. Finding the components for $M\xi + \kappa$ in (18). We represent $f$ in (12) by $\xi_k$ for which $f_N(y) = \sum_{k=0}^{N+1} \xi_k \phi_k(y)$. We need to consider $\int_{-L}^{L} \sum_{k=0}^{N+1} \xi_k \phi_k(y) \phi_m(y) dy$. We found $\int_{-L}^{L} \sum_{k=1}^{N} \phi_k(y) \phi_m(y) dy$ in the previous section for $y_1, y_2, \ldots, y_N$. The result was $M$ which is applicable to $f_N(y)$. However, we consider boundary cases differently than in the previous section since $\xi_0$ and $\xi_{N+1}$ are not necessarily zeros. When $m = 1$,

$$\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k \phi_k(y) \phi_1(y) dy = \xi_0(t) \int_{y_0}^{y_1} \phi_0(y) \phi_1(y) dy + \xi_1(t) \int_{y_0}^{y_2} \phi_1^2(y) dy + \xi_2(t) \int_{y_1}^{y_2} \phi_2(y) \phi_1(y) dy.$$  

When $m = N$, we have

$$\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k \phi_k(y) \phi_N(y) dy = \xi_{N-1}(t) \int_{y_{N-1}}^{y_N} \phi_{N-1}(y) \phi_N(y) dy + \xi_N(t) \int_{y_{N-1}}^{y_{N+1}} \phi_N^2(y) dy + \xi_{N+1}(t) \int_{y_N}^{y_{N+1}} \phi_{N+1}(y) \phi_N(y) dy.$$  

To keep $M$ in (22) applicable to $\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k \phi_k(y) \phi_m(y) dy$ we introduce $\kappa = (\kappa_1, \ldots, \kappa_N)$, such that

$$\kappa_1 = (\phi_0, \phi_1)_{L^2(D)} \xi_0 = \xi_0(t) \int_{y_0}^{y_1} \phi_0(y) \phi_1(y) dy = \mu_{1,0} \xi_0.$$  

$$\kappa_N = (A \phi_{N+1}, \phi_N)_{L^2(D)} \xi_{N+1} = \xi_{N+1}(t) \int_{y_N}^{y_{N+1}} \phi_{N+1}(y) \phi_N(y) dy = \mu_{1,0} \xi_{N+1}.$$  

$$\kappa_k = 0$$

for $k = 2, \ldots, N - 1$.

5.3. Finding the coefficients for $B$ in (18). We consider the term in (17)

$$R'(t) \int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t) A \phi_k(y) \phi_m(y) dy = R'(t) v_k(t) \int_{\mathbb{R}} A \phi_k(y) \phi_m(y) dy.$$  

Let

$$(A \phi_k, \phi_m)_{L^2(D)} = \int_{-L}^{L} A \phi_k(y) \phi_m(y) dy.$$
and let
\[ \rho(y) = \frac{\sigma(y)^2}{2}, \quad \eta(y) = (r - d(y)) - \frac{1}{2}\sigma(y)^2. \]

We have that
\[ A\phi_k(y) = \frac{1}{2}\sigma(y)^2\phi''_k(y) + (r - d(y)) - \frac{1}{2}\sigma(y)^2\phi'_k(y) = \rho(y)\phi''_k(y) + \eta(y)\phi'_k(y) \]
\[ = (\rho(y)\phi'(y))' - \rho(y)'\phi'_k(y) + \eta(y)\phi'_k(y). \]

By the definitions,
\[ (A\phi_k, \phi_m)_{L^2(D)} = \int_{-L}^{L} (\rho(y)\phi'_k(y))\phi'_m(y)dy - \int_{-L}^{L} \rho(y)'\phi'_k(y)\phi'_m(y)dy \]
\[ + \int_{-L}^{L} \eta(y)\phi'_k(y)\phi'_m(y)dy \]
\[ = \rho(y)\phi'_k(y)\phi_m(y) |_{-L}^{L} - \int_{-L}^{L} \rho(y)\phi'_k(y)\phi'_m(y)dy \]
\[ - \int_{-L}^{L} \rho(y)'\phi'_k(y)\phi_m(y)dy + \int_{-L}^{L} \eta(y)\phi'_k(y)\phi_m(y)dy \]
\[ = 0 - \int_{-L}^{L} \rho(y)'\phi'_k(y)\phi'_m(y)dy - \int_{-L}^{L} \eta(y)\phi'_k(y)\phi_m(y)dy \]
\[ + \int_{-L}^{L} \eta(y)\phi'_k(y)\phi_m(y)dy. \]

For \( k \leq m - 2 \) or \( k \geq m + 2 \) we have \((A\phi_k, \phi_m)_{L^2(D)} = 0\), since for \( y \leq y_{m-1} \) or \( y \geq y_{m+1} \), \( \phi_m(y) = \phi'_m(y) = 0 \) and for \( y \leq y_{k-1} \) or \( y \geq y_{k+1} \), \( \phi_k(y) = \phi'_k(y) = 0 \) by definitions. It follows that we need to consider three cases: \( k = m - 1 \), \( k = m \), and \( k = m + 1 \). First, let us consider \( k = m - 1 \):

\[ (A\phi_{m-1}, \phi_m)_{L^2(D)} = -\int_{-L}^{L} \rho(y)\phi'_{m-1}(y)\phi'_m(y)dy - \int_{-L}^{L} \rho(y)'\phi'_{m-1}(y)\phi_m(y)dy \]
\[ + \int_{-L}^{L} \eta(y)\phi'_{m-1}(y)\phi_m(y)dy. \]

Since the \( \phi_{m-1}(y) \) and \( \phi_m(y) \) are multiplied inside the integrals, the boundaries for integrals will be \((y_{m-1}, y_m)\) since for \( y \leq y_{m-1} \), \( \phi_m(y) = \phi'_m(y) = 0 \) and for \( y \geq y_m \), \( \phi_{m-1}(y) = \phi'_{m-1}(y) = 0 \). Let
\[ J = \int_{y_{m-1}}^{y_m} \rho(y)\phi'_{m-1}(y)\phi'_m(y)dy. \]

Then
\[ (A\phi_{m-1}, \phi_m)_{L^2(D)} = -J - \int_{y_{m-1}}^{y_m} \rho(y)'\phi'_{m-1}(y)\phi_m(y)dy \]
\[ + \int_{y_{m-1}}^{y_m} \eta(y)\phi'_{m-1}(y)\phi_m(y)dy. \]
We have that
\[
J = \int_{y_{m-1}}^{y_m} \rho(y) \phi_{m-1}'(y) \phi_m'(y) \, dy
\]
\[
= \phi_m(y) \rho(y) \phi_{m-1}'(y) \bigg|_{y_{m-1}}^{y_m} - \int_{y_{m-1}}^{y_m} (\rho(y) \phi_{m-1}(y))' \phi_m(y) \, dy
\]
\[
= \rho(y_m) \phi_{m-1}'(y_m - 0) - \int_{y_{m-1}}^{y_m} (\rho(y) \phi_{m-1}'(y))' \phi_m(y) \, dy.
\]
Substituting \(J\) back in, we obtain that
\[
(A \phi_{m-1}, \phi_m)_{L_2(D)} = -\rho(y_m) \phi_{m-1}'(y_m - 0) + \int_{y_{m-1}}^{y_m} (\rho(y) \phi_{m-1}'(y))' \phi_m(y) \, dy
\]
\[-\int_{y_{m-1}}^{y_m} \rho(y) \phi_{m-1}'(y) \phi_m(y) \, dy
\]
\[+ \int_{y_{m-1}}^{y_m} \eta(y) \phi_{m-1}'(y) \phi_m(y) \, dy
\]
\[
= -\rho(y_m) \phi_{m-1}'(y_m - 0) + \int_{y_{m-1}}^{y_m} [(\rho(y) \phi_{m-1}'(y))' \phi_m(y) - \rho(y)' \phi_{m-1}(y) \phi_m(y)
\]
\[+ \eta(y) \phi_{m-1}'(y) \phi_m(y)] \, dy
\]
\[
= -\rho(y_m) \phi_{m-1}'(y_m - 0) + \int_{y_{m-1}}^{y_m} [(\rho(y) \phi_{m-1}'(y))' \phi_m(y)
\]
\[-\rho(y)' \phi_{m-1}(y) + \eta(y) \phi_{m-1}'(y) \phi_m(y)] \, dy.
\]
Since \(A \phi_{m-1} = (\rho(y) \phi_{m-1}'(y) - \rho(y)' \phi_{m-1}(y) + \eta(y) \phi_{m-1}'(y) - \rho(y)' \phi_{m-1}(y) = 0\) as defined in Section 3, we obtain that
\[
(A \phi_{m-1}, \phi_m)_{L_2(D)} = -\rho(y_m) \phi_{m-1}'(y_m - 0) = -\beta_{m,m-1}.
\]
\(\beta_{m,m-1}\) is the component to the left of the main diagonal in \(B\). Now, let us consider \(k = m:\)
\[
(A \phi_m, \phi_m)_{L_2(D)} = -\int_{-L}^{L} \rho(y) \phi_m'(y) \phi_m(y) \, dy - \int_{-L}^{L} \rho(y)' \phi_m'(y) \phi_m(y) \, dy
\]
\[+ \int_{-L}^{L} \eta(y) \phi_m'(y) \phi_m(y) \, dy.
\]
The boundaries for the integrals are \((y_{m-1}, y_{m+1})\) since for \(y \leq y_{m-1}, \phi_m(y) = \phi_m'(y) = 0\) and for \(y \geq y_{m+1}, \phi_m(y) = \phi_m'(y) = 0\). We will need to consider the following:
\[
(A \phi_m, \phi_m)_{L_2(D)} = -\int_{y_{m-1}}^{y_m} \rho(y) \phi_m'(y) \phi_m'(y) \, dy - \int_{y_{m-1}}^{y_m} \rho(y)' \phi_m'(y) \phi_m(y) \, dy
\]
\[+ \int_{y_{m-1}}^{y_m} \eta(y) \phi_m'(y) \phi_m(y) \, dy - \int_{y_{m-1}}^{y_m} \rho(y)' \phi_m'(y) \phi_m(y) \, dy
\]
\[+ \int_{y_{m-1}}^{y_{m+1}} \rho(y) \phi_m'(y) \phi_m(y) \, dy + \int_{y_{m}}^{y_{m+1}} \eta(y) \phi_m'(y) \phi_m(y) \, dy.
\]
Let
\[
(A \phi_m, \phi_m)_{L_2(D)} = Q_1 + Q_2,
\]
such that

\[ Q_1 = - \int_{y_{m-1}}^{y_m} \rho(y) \phi'_m(y) \phi''_m(y) dy - \int_{y_{m-1}}^{y_m} \rho(y)' \phi'_m(y) \phi_m(y) dy \\
+ \int_{y_{m-1}}^{y_m} \eta(y) \phi'_m(y) \phi_m(y) dy \]

and

\[ Q_2 = - \int_{y_{m-1}}^{y_{m+1}} \rho(y) \phi'_m(y) \phi''_m(y) dy - \int_{y_m}^{y_{m+1}} \rho(y)' \phi'_m(y) \phi_m(y) dy \\
+ \int_{y_m}^{y_{m+1}} \eta(y) \phi'_m(y) \phi_m(y) dy. \]

Let us consider \( Q_1 \) first. Let

\[ J_1 = \int_{y_{m-1}}^{y_m} \rho(y) \phi'_m(y) \phi''_m(y) dy. \]

Then

\[ Q_1 = -J_1 - \int_{y_{m-1}}^{y_m} \rho(y)' \phi'_m(y) \phi_m(y) dy + \int_{y_{m-1}}^{y_m} \eta(y) \phi'_m(y) \phi_m(y) dy, \]

We have that

\[ J_1 = \int_{y_{m-1}}^{y_m} \rho(y) \phi'_m(y) \phi''_m(y) dy \\
= \phi_m(y) \rho(y) \phi''_m(y) |_{y_{m-1}}^{y_m} - \int_{y_{m-1}}^{y_m} (\rho(y) \phi'_m(y))' \phi_m(y) dy \\
= \rho(y_m) \phi'_m(y_m - 0) - \int_{y_{m-1}}^{y_m} (\rho(y) \phi'_m(y))' \phi_m(y) dy. \]

Substituting \( J_1 \) back in, we obtain that

\[ Q_1 = -\rho(y_m) \phi'_m(y_m - 0) + \int_{y_{m-1}}^{y_m} (\rho(y) \phi'_m(y))' \phi_m(y) dy \\
- \int_{y_{m-1}}^{y_m} \rho(y)' \phi'_m(y) \phi_m(y) dy + \int_{y_{m-1}}^{y_m} \eta(y) \phi'_m(y) \phi_m(y) dy \\
= -\rho(y_m) \phi'_m(y_m - 0) + \int_{y_{m-1}}^{y_m} [ (\rho(y) \phi'_m(y))' \phi_m(y) \\
- \rho(y)' \phi'_m(y) \phi_m(y) + \eta(y) \phi'_m(y) \phi_m(y) ] dy \\
= -\rho(y_m) \phi'_m(y_m - 0) \\
+ \int_{y_{m-1}}^{y_m} [ (\rho(y) \phi'_m(y))' - \rho(y)' \phi'_m(y) + \eta(y) \phi'_m(y) ] \phi_m(y) dy. \]

Since \( A \phi_m(y) = (\rho(y) \phi'_m(y))' - \rho(y)' \phi'_m(y) + \eta(y) \phi'_m(y) = 0 \) as defined in Section 3, we obtain that

\[ Q_1 = -\rho(y_m) \phi'_m(y_m - 0). \]

Now, let us consider \( Q_2 \). Let

\[ J_2 = \int_{y_m}^{y_{m+1}} \rho(y) \phi'_m(y) \phi''_m(y) dy. \]
Then
\[ Q_2 = -J_2 - \int_{y_m}^{y_{m+1}} \rho(y) \phi_m'(y) \phi_m(y) dy + \int_{y_m}^{y_{m+1}} \eta(y) \phi_m'(y) \phi_m(y) dy. \]

We have that
\[ J_2 = \int_{y_m}^{y_{m+1}} \rho(y) \phi_m'(y) \phi_m(y) dy \]
\[ = \phi_m(y) \rho(y) \phi_m'(y) \bigg|_{y_m}^{y_{m+1}} - \int_{y_m}^{y_{m+1}} (\rho(y) \phi_m'(y))' \phi_m(y) dy \]
\[ = -\rho(y_m) \phi_m'(y_m + 0) - \int_{y_m}^{y_{m+1}} (\rho(y) \phi_m'(y))' \phi_m(y) dy. \]

Substituting \( J_2 \) back in, we obtain that
\[ Q_2 = \rho(y_m) \phi_m'(y_m + 0) + \int_{y_m}^{y_{m+1}} (\rho(y) \phi_m'(y))' \phi_m(y) dy \]
\[ - \int_{y_m}^{y_{m+1}} \rho(y) \phi_m'(y) \phi_m(y) dy + \int_{y_m}^{y_{m+1}} \eta(y) \phi_m'(y) \phi_m(y) dy \]
\[ = \rho(y_m) \phi_m'(y_m + 0) + \int_{y_m}^{y_{m+1}} [(\rho(y) \phi_m'(y))' \phi_m(y) - \rho(y) \phi_m'(y) \phi_m(y) \]
\[ + \eta(y) \phi_m'(y) \phi_m(y)] dy \]
\[ = \rho(y_m) \phi_m'(y_m + 0) + \int_{y_m}^{y_{m+1}} [(\rho(y) \phi_m'(y))' - \rho(y) \phi_m'(y) + \eta(y) \phi_m'(y)] \phi_m(y)] dy. \]

Since \( A \phi_m = (\rho(y) \phi_m')' - \rho(y) \phi_m' + \eta(y) \phi_m' = 0 \), we obtain that
\[ Q_2 = \rho(y_m) \phi_m'(y_m + 0). \]

Substituting \( Q_1 \) and \( Q_2 \) back in we have that
\[ (A \phi_m, \phi_m)_{L^2(D)} = -\rho(y_m) \phi_m'(y_m - 0) + \rho(y_m) \phi_m'(y_m + 0) = -\beta_{m,m}. \]
(30)
\( \beta_{m,m} \) is the diagonal component in \( B \). Finally, let us consider \( k = m + 1 \).
\[ (A \phi_{m+1}, \phi_m)_{L^2(D)} = -\int_{-L}^{L} \rho(y) \phi_{m+1}'(y) \phi_m'(y) dy - \int_{-L}^{L} \rho(y) \phi_{m+1}'(y) \phi_m(y) dy \]
\[ + \int_{-L}^{L} \eta(y) \phi_{m+1}'(y) \phi_m(y) dy \]

Since the \( \phi_{m+1}(y) \) and \( \phi_m(y) \) are multiplied inside the integral, the boundaries for the integrals will be \((y_m, y_{m+1})\) since for \( y \leq y_m, \phi_{m+1}(y) = \phi_{m+1}'(y) = 0 \) and for \( y \geq y_{m+1}, \phi_m(y) = \phi_m'(y) = 0 \). Let
\[ J = \int_{y_m}^{y_{m+1}} \rho(y) \phi_{m+1}'(y) \phi_m'(y) dy. \]
\[ (A \phi_{m+1}, \phi_m)_{L^2(D)} = -J - \int_{y_m}^{y_{m+1}} \rho(y) \phi_{m+1}'(y) \phi_m(y) dy \]
\[ + \int_{y_m}^{y_{m+1}} \eta(y) \phi_{m+1}'(y) \phi_m(y) dy. \]
We have that

\[
J = \int_{y_m}^{y_{m+1}} \rho(y) \phi'_{m+1}(y) \phi_m(y) dy
\]

\[
= \phi_m(y) \rho(y) \phi'_{m+1}(y) |_{y_m}^{y_{m+1}} - \int_{y_m}^{y_{m+1}} (\rho(y) \phi'_{m+1}(y))' \phi_m(y) dy
\]

\[
= -\rho(y_m) \phi'_{m+1}(y_m + 0) - \int_{y_m}^{y_{m+1}} (\rho(y) \phi'_{m+1}(y))' \phi_m(y) dy.
\]

Substituting \( J \) back in, we obtain that

\[
(A \phi_{m+1}, \phi_m)_{L_2(D)} = \rho(y_m) \phi'_{m+1}(y_m + 0) + \int_{y_m}^{y_{m+1}} (\rho(y) \phi'_{m+1}(y))' \phi_m(y) dy
\]

\[
- \int_{y_m}^{y_{m+1}} \rho(y) \phi'_{m+1}(y) \phi_m(y) dy + \int_{y_m}^{y_{m+1}} \eta(y) \phi'_{m+1}(y) \phi_m(y) dy
\]

\[
= \rho(y_m) \phi'_{m+1}(y_m + 0) + \int_{y_m}^{y_{m+1}} [(\rho(y) \phi'_{m+1}(y))' \phi_m(y)
\]

\[
- \rho(y)' \phi'_{m+1}(y) \phi_m(y) + \eta(y) \phi'_{m+1}(y) \phi_m(y)] dy
\]

\[
= \rho(y_m) \phi'_{m+1}(y_m + 0) + \int_{y_m}^{y_{m+1}} [(\rho(y) \phi'_{m+1}(y))' - \rho(y)' \phi'_{m+1}(y)
\]

\[
+ \eta(y) \phi'_{m+1}(y)] \phi_m(y)] dy.
\]

Since \( A \phi_{m+1} = (\rho(y) \phi'_{m+1}(y))' - \rho(y)' \phi'_{m+1}(y) + \eta(y) \phi'_{m+1}(y) = 0 \), as defined in Section 3 we obtain that

\[
(A \phi_{m+1}, \phi_m)_{L_2(D)} = \rho(y_m) \phi'_{m+1}(y_m + 0) = -\beta_{m,m+1}.
\]

(31)

\( \beta_{m,m+1} \) is the component to the right of the main diagonal in \( B \).

Combining our findings and remembering that the term had a minus in the beginning, we obtain that

\[
-\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t) A \phi_k(y) \phi_m(y) dy = \rho(y_m) \phi'_{m-1}(y_m - 0) v_{m-1}(t)
\]

\[
+ \rho(y_m)(\phi'_{m}(y_m - 0) - \phi'_m(y_m + 0)) v_m(t)
\]

\[
- \rho(y_m) \phi'_{m+1}(y_m + 0) v_{m+1}(t)
\]

\[
= \beta_{m,m-1} v_{m-1}(t) + \beta_{m,m} v_m(t)
\]

\[
+ \beta_{m,m+1} v_{m+1}(t).
\]

(32)

Thus, we found the right side components for \( B \). Let us consider the boundary cases: \( m = 1 \) and \( m = N \). When \( m = 1 \), if we refer to (17), we will not consider the case when \( k = m - 1 = 0 \) since \( v_0 = 0 \). Thus,

\[
\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t) A \phi_k(y) \phi_1(y) dy = (-\rho(y_1) \phi'_1(y_1 - 0) + \rho(y_1) \phi'_1(y_1 + 0)) v_1(t)
\]

\[
+ \rho(y_1) \phi'_2(y_1 + 0) v_2(t).
\]

(33)
Similarly, when \( m = N \), we will not consider the case when \( k = m + 1 = N + 1 \) since \( v_{N+1} = 0 \). Thus
\[
\int_{\mathbb{R}} \sum_{k=1}^{N} v_k(t) \phi_k(y) \phi_N(y) dy = -\rho(y_N) \phi'_N(y_N - 0) v_{N-1}(t) + (-\rho(y_N) \phi'_N(y_N - 0) - \rho(y_N) \phi'_N(y_N + 0)) v_N(t).
\] (34)
Thus, we found the components for the matrix \( B \).

5.4. **Alternative way of finding the coefficients for \( B \) in (18).** Let us show an alternative shorter way of calculation of \( B \). This approach involves so called delta functions.

5.5. **Finding the components for \( B \xi + \zeta \) in (18).** We represent \( f \) in (12) by \( f_N(y) = \sum_{k=0}^{N+1} \xi_k \phi_k(y) \). We have found \((A \phi_k, \phi_m)_{L^2(D)} \) in the previous section for \( y_1, y_2, ..., y_N \). We have to consider boundary cases differently than in the previous section since \( \xi_0 \) and \( \xi_{N+1} \) are not necessarily zeros. For \( m = 1 \), we have
\[
\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A \phi_k(y) \phi_1(y) dy = -\rho(y_1) \phi'_0(y_1 - 0) \xi_0 + (\rho(y_1) \phi'_1(y_1 - 0) - \rho(y_1) \phi'_2(y_1 + 0) \xi_2.
\]
For \( m = N \), we have
\[
\int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A \phi_k(y) \phi_N(y) dy = -\rho(y_N) \phi'_N(y_N - 0) \xi_N - 1 + (-\rho(y_N) \phi'_N(y_N - 0) + \rho(y_N) \phi'_2(y_N + 0) \xi_{N+1}.
\]
To keep \( B \) in (22) applicable to \( \int_{\mathbb{R}} \sum_{k=0}^{N+1} \xi_k A \phi_k(y) \phi_m(y) dy \) we introduce \( \zeta = (\zeta_1, ..., \zeta_N) \), such that
\[
\zeta_1 = (A \phi_0, \phi_1)_{L^2(D)} \xi_0 = -\rho(y_1) \phi'_0(y_1 - 0) = -\beta_{1,0} \xi_0.
\]
\[
\zeta_{N} = (A \phi_{N+1}, \phi_N)_{L^2(D)} \xi_{N+1} = \rho(y_N) \phi'_N(y_N + 0) = -\beta_{N,N+1} \xi_N.
\]
\[
\zeta_k = 0, \text{ for } k = 2, ..., N - 1.
\]

5.6. **Proof of Theorem 4.1: Conclusion.** Combining the statements above we obtain the statement (18). This concludes the proof of Theorem 4.1.

6. **Solution of the ODE (18).**

6.1. **Exact solution of Equation (18).** To solve the resulting system, we use the exponential matrix method: From the properties of linear ODEs, we obtain that
\[
V(t) = -\int_{t}^{T} e^{E(t-s)} M^{-1} \phi ds.
\]
In particular,

\[ V(0) = - \int_0^T e^{E(s)} M^{-1} \varphi ds \]

will give us the option value at the present time.

\[ V(t_k) = e^{Et_k} \int_{t_k}^T -e^{E(s)} M^{-1} \varphi ds. \]

Let \( E = M^{-1}B \). Note that since matrix \( M \) is tridiagonal, finding its inverse is numerically feasible.

6.2. Crank-Nicolson method. We can also solve system (18) using Crank-Nicolson method backwards as

\[
\frac{u^{i+\Delta t} - u^i}{\Delta t} = M^{-1} \frac{1}{2} [ (Bu^{i+\Delta t} + \varphi(t + \Delta t)) + (Bu^i + \varphi(t))].
\]

(35)

This gives

\[
u^i = (M + \Delta t B)^{-1} [Mu^{i+\Delta t} - \Delta t \frac{1}{2} Bu^{i+\Delta t} - \Delta t (\varphi(t) + \varphi(t + \Delta t))].
\]

(36)

6.3. Backwards substitution. After solving the system we obtain the vector \( v_k, k = 1..N \). To solve the original equation, we have to reverse the substitutions (3), (5), and (9). The last substitution made was (9). We transform the answer:

\[
\hat{v}_k(t) := v_k(t) + \xi_k, \quad k = 1..N.
\]

Consider (5). To reverse it:

\[
\hat{V}_k(t) := \hat{v}_k(t)e^{r(t-T)}.
\]

Now, consider (3). Clearly, \( y = \ln x \). Therefore, finding \( V(x) \) from the original Black-Scholes equation is equivalent to finding \( V_N(y) = V_N(\ln x) \). To find \( V(x,t) \) for particular \( x \),

\[
V(x,t) = \hat{V}_k(t)\phi_k(\log x) + \hat{V}_{k+1}(t)\phi_{k+1}(\log x), \quad y_k \leq \log x \leq y_{k+1}.
\]

7. Convergence. To show convergence we will apply Theorem 7.1 from Douglas and Dupont’s paper [4]. In what follows, we will show all the conditions for convergence in Douglas and Dupont’s result are satisfied by our method.

**Theorem 7.1.** (Theorem 7.1 [4]) There exist constants \( C \) and \( \delta \) which depend on \( T, n, D, K_0, C_0 \) and \( C_1 \), such that for \( v \) and \( V_N \), solutions to (15) and (16), respectively, and any function \( \tilde{u} \) of the form \( \sum_{i=1}^N \alpha_i \tilde{u}_i \)

\[
\sup_{0 \leq t \leq T} \| u(\cdot, t) - V_N(\cdot, t) \|_{L^2(D)} + \delta \int_0^T \| u(\cdot, t) - V_N(\cdot, t) \|_{H_0^1(D)}^2 dt \\
\leq C \sup_{0 \leq t \leq T} \| u(\cdot, t) - \tilde{u}(\cdot, t) \|_{L^2(D)} + \int_0^T \| u(\cdot, t) - \tilde{u} \|_{H_0^1(D)}^2 dt \\
+ \left\| \frac{\partial}{\partial t}(u - \tilde{u}) \right\|_{L^2(D \times (0,T))}^2.
\]
First, we want to define our problem in the terms used in [4]. Note that we will refer to the following notations from [4] in a special way to avoid confusion: \( a(u, w) \) as \( \tilde{a}(u, w) \), \( A \) as \( \tilde{A} \), \( (f, w) \) as \( (\tilde{f}, w) \) and \( \eta \) as \( \tilde{\eta} \). Also, we will use \( y \) instead of \( x \) from [4]. We need to rewrite (4.1) in our paper as (7.2) in [4]:

\[
\langle \frac{\partial u}{\partial t}, w \rangle + \tilde{a}(u, w) = \langle f(u), w \rangle.
\]

Here,

\[
\tilde{a}(u, w) = \int_D \tilde{A}(y, t, u(y, t), \frac{\partial w}{\partial y}(y, t))dy
\]

and

\[
\tilde{f}(u) = \tilde{f}(y, t, u(y, t), \frac{\partial u}{\partial y}(y, t))
\]

for some measurable functions \( \tilde{A}(y, t, u, p) : D \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) and \( \tilde{f}(y, t, u, p) : D \times [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \). Note that in this paper \( p \) is not a vector. We will need to find the functions \( \tilde{A} \) and \( \tilde{f} \) to obtain (7.4) in [4]: Let us consider \( a(u, w) \) and \( R(t)a(f, w) \) in (15).

\[
a(u, w) = (Au, w)_{L^2(D)} = (\rho \phi_{yy}^\prime, w)_{L^2(D)} + (\eta \phi_y^\prime, w)_{L^2(D)}
\]

\[
= ((\rho u_y^\prime)_y, w)_{L^2(D)} - (\rho u_y^\prime u_y^\prime, w)_{L^2(D)} + (\eta u_y^\prime, w)_{L^2(D)}
\]

\[
= \int_{-L}^{L} (\rho u_y^\prime)_y w dy - \int_{-L}^{L} \rho u_y^\prime u_y^\prime w dy + \int_{-L}^{L} \eta u_y^\prime w dy
\]

\[
= 0 - \int_{-L}^{L} \rho u_y^\prime u_y^\prime w dy + \int_{-L}^{L} \eta u_y^\prime w dy.
\]

(37)

Similarly,

\[
R(t)a(f, w) = R(t)(Af, w)_{L^2(D)}
\]

\[
= R(t)((\rho f_y^\prime)_y, w)_{L^2(D)} - R(t)(\rho f_y^\prime f_y^\prime, w)_{L^2(D)} + R(t)(\eta f_y^\prime, w)_{L^2(D)}
\]

\[
= 0 - R(t) \int_{-L}^{L} \rho f_y^\prime f_y^\prime w dy + R(t) \int_{-L}^{L} \rho f_y^\prime f_y^\prime w dy
\]

\[
+ R(t) \int_{-L}^{L} \eta f_y^\prime w dy.
\]

(38)

Now, let us consider \( R'(t)(f, w)_{L^2(D)} \) in (15).

\[
R'(t)(f, w)_{L^2(D)} = R'(t) \int_{-L}^{L} f w dy.
\]

(39)

We combine the terms with \( w_y^\prime \) of the equations (37) and (38) to obtain \( \tilde{a}(u, w) \) from 7.2 in [4].

\[
\tilde{a}(u, w) = \int_{-L}^{L} (-\rho u_y^\prime w_y^\prime + R(t) \rho f_y^\prime w_y^\prime) dy.
\]

So,

\[
\tilde{A}(y, t, u, u_y^\prime) = -\rho u_y^\prime - R(t) \rho f_y^\prime.
\]
This gives
\[ \tilde{A}(y, t, u, p) = -\rho p - R(t)\rho f'_y. \]

Next, we combine the terms with \( w \) of the equations (37), (38) and (39) to obtain \((\tilde{f}, w)\) from [4].
\[
(\tilde{f}, w) = \int_{-L}^{L} \rho_y' u'_y w \, dy - \int_{-L}^{L} \eta u'_y \, dy + R(t) \int_{-L}^{L} \rho_y f'_y w \, dy - R(t) \int_{-L}^{L} \eta f'_y \, dy + R'(t) \int_{-L}^{L} f \, w \, dy.
\]
Next, we check the conditions from [4] on \( A \) and \( \tilde{f} \) in order for the theorem to hold. First, it is clear that both functions are measurable. Second, \( \frac{\partial a}{\partial u} = -\rho \) is such that \( C_0 \leq C_1, 0 \leq C_0 \leq C_1 \) for some positive \( C_0 \) and \( C_1 \). Next, we check the condition (7.11a) in [4]:
\[
\| \tilde{A}(y, t, w(y), p(y)) \|^2_{L^2(D)} + \| \tilde{f}(y, t, w(y), p(y)) \|^2_{L^2(D)} \leq C[\| w(y) \|^2_{L^2(D)} + \| p(y) \|^2_{L^2(D)} + 1].
\]
(40)
Observe that,
\[
\| \tilde{A}(y, t, w(y), p(y)) \|^2_{L^2(D)} = \| -\rho p - \rho f'_y \|^2_{L^2(D)} \leq C(\| p \|^2_{L^2(D)} + 1)
\]
and
\[
\| \tilde{f}(y, t, w(y), p(y)) \|^2_{L^2(D)} = \| \rho_y' p - \eta p + \rho_y f'_y - R(t)\eta f'_y + R'(t)f \|^2_{L^2(D)} \leq C(\| p \|^2_{L^2(D)} + 1).
\]
Since \( \rho, \rho_y' \) and \( \eta \) are bounded and \( f'_y \) is bounded, and \( R(t) \) and \( R'(t) \) are bounded, the condition (40) holds.

Next, we check the conditions (7.11a-7.11e) in [4] for our \( \tilde{A}(x, t, u, u'_y) \) and \( \tilde{f}(x, t, u, u'_y) \) Suppose that \( \tilde{f} = \tilde{f}_1 + \tilde{f}_2 \) and that there exists \( K_0 \) such that for \((y, t) \in D \times (0, T), r \) and \( s \) in \( \mathbb{R} \), and \( p \) and \( q \) in \( \mathbb{R} \).
Let us check the condition (7.11a) in [4]:
\[
| \tilde{A}(y, t, r, u'_y(y, t)) - \tilde{A}(y, t, s, u'_y(y, t)) | \leq K_0|r - s|.
\]
This holds since
\[
| \tilde{A}(y, t, r, u'_y) - \tilde{A}(y, t, s, u'_y) | = -\rho u'_y - \rho f'_y - (-\rho u'_y - \rho(y) f'_y) = 0.
\]
Let us check the condition (7.11b) in [4]:
\[
| \tilde{f}_1(y, t, u(y, t), p) - \tilde{f}_1(y, t, u(y, t), q) | \leq K_0|p - q|.
\]
This holds since
\[ |\bar{f}_1(y, t, u(y, t), p) - \tilde{f}_1(x, t, u(y, t), q)| \]
\[ = |\rho' \rho - \eta p + \rho' f' y - R(t)\eta f' y + R'(t)f - (\rho' \eta q - \eta q + \rho' f' y - R(t)\eta f' y + R'(t)f)| \]
\[ \leq K_0|p - q|. \]

and \( \rho(y), \rho'(y) \) and \( \eta(y) \) are bounded. Let us check the condition (7.11c)[4]:
\[ |\bar{f}_1(y, t, r, p) - \tilde{f}_1(y, t, s, p)| \leq K_0|r - s|. \]

This holds since
\[ |\bar{f}_1(y, t, r, p) - \tilde{f}_1(y, t, s, p)| = 0. \]

Let us check the condition (7.11d)[4]:
\[ |\bar{f}_2(y, t, r, u'(y, t)) - \tilde{f}_2(y, t, s, u'(y, t))| \leq K_0|r - s|. \]

This holds since
\[ |\bar{f}_2(y, t, r, u'(y, t)) - \tilde{f}_2(y, t, s, u'(y, t))| = 0. \]

Let us check the condition (7.11e)[4]:
\[ |\bar{f}_2(y, t, r, p) - \tilde{f}_2(y, t, r, q)| \leq K_0|p - q|. \]

This holds since
\[ |\bar{f}_2(y, t, r, p) - \tilde{f}_2(y, t, r, q)| \]
\[ = \rho' \rho p - \eta p + \rho' f' y - R(t)\eta f' y + R'(t)f 
- (\rho' \eta q - \eta q + \rho' f' y - \eta f' y - R(t)\eta f' y + R'(t)f) \]
\[ = (\rho' \eta - \eta):(p - q) \leq K_0|p - q| \]

and \( \rho'(y) \) and \( \eta(y) \) are bounded. Theorem 7.1 follows from Theorem 7.1 in [4] which shows our method is strongly convergent.

8. Numerical experiments.

8.1. Matching the Black-Scholes price. In these experiments we calculated the price for a put option in the classical Black-Scholes model using our method. We considered the payoff \((K - S(T))^+\). We found the solution of the equation (4) with \( f(x) = (K - x)^+ \) using our method and compared it with the solution given by the Black-Scholes formula. Table (1) shows the error
\[ E = \sup_x |V(x, 0) - V_{BS}(x, 0)|. \]

Here, \( V \) is the solution of (4) obtained by our method and \( V_{BS} \) is the exact solution of (4) given by the Black-Scholes formula. In addition, the table shows where the maximum (41) is achieved. In this table, \( N \) is the parameter introduced in Section 3 representing the rate of discretization in \( x \). \( N_t \) is the number of points along the time axis. We used \( \sigma = 0.3, T = 0.1, L = 10, r = 0, d = 0 \) and \( K = 1 \). For \( r = 0.025 \) It is shown that the error is decreasing as we use more points along the state price and time axes. Note that the largest error mostly occurs near the point where the final condition becomes zero. Figure 2 shows the comparison between the Black-Scholes and numerical solutions.
Table 1. Error of calculation of the put option for r=0.

| $N_i, N_t$ | E            |
|-----------|--------------|
| 20, 20    | 0.003902114  |
| 40, 40    | 0.006529201  |
| 80, 80    | 0.007018533  |
| 160, 160  | 0.003593509  |
| 320, 320  | 0.0003050627 |
| 640, 640  | 7.043937e-05 |

Table 2. Error of calculation of the put option for r=0.025

| $N_i, N_t$ | E            |
|-----------|--------------|
| 20, 20    | 0.003319792  |
| 40, 40    | 0.005971542  |
| 80, 80    | 0.006657621  |
| 160, 160  | 0.003484265  |
| 320, 320  | 0.001151376  |
| 640, 640  | 0.0003031325 |

Table 3. Error of calculation of the put option for r=0.05

| $N_i, N_t$ | E            |
|-----------|--------------|
| 20, 20    | 0.002830843  |
| 40, 40    | 0.00547276   |
| 80, 80    | 0.006323301  |
| 160, 160  | 0.003382576  |
| 320, 320  | 0.001133573  |
| 640, 640  | 0.0003015444 |

8.2. Experiments for state-dependent volatility. In these experiments we approximate a certain function using our approach. We will select the function $U(y, t)$, $d(y)$, and $\sigma(y)$, to satisfy the equation (5) exactly, and will estimate the error for our numerical method. Let us select

\[
U(y, t) = e^{t-T}(L^2 - y^2),
\]
\[
f(y) = L^2 - y^2,
\]
\[
R(t) = R_0(t) = 1,
\]
\[
\sigma(y) = \sqrt{(L^2 + 0.9(sin(y))^2)y},
\]
\[
d(y) = -(0.9(sin(y))^2 + y)/2 + (\sigma(y))^2/2,
\]
\[
r = 0, \quad L = 1, \quad T = 1.
\]

It can be verified directly that (5) is satisfied. Table (4) shows the error

\[
E = \sup_y |V(y, 0) - U(y, 0)|.
\]

Here, $V$ is the solution of (4) obtained by our method and $U(y, 0)$ is obtained by putting in the appropriate values into the chosen $U(y, t)$. In this table, $N$ is the
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Figure 2. Comparison of exact solution $U$ and numerical solution $V$

parameter introduced in Section 3 representing the rate of discretization in $y$ and $N_t$ is the parameter introduced similarly to the previous section. Again, it is shown that the maximum error is decreasing as we use more points along the state price and time axes. Note that the biggest error mostly occurs near the boundary point. Figure 3 shows the comparison between the exact and numerical solutions.

Table 4. Error of calculation of the case of state-dependent volatility.

| $N$, $N_t$ | $E$         |
|-----------|------------|
| 20, 20    | 8.60202    |
| 40, 40    | 0.1838133  |
| 80, 80    | 0.09596427 |
| 160, 160  | 0.04898591 |
| 320, 320  | 0.02474154 |
| 640, 640  | 0.01243255 |

9. Conclusion. In this paper we presented a method for solving the Black-Scholes equation considering the volatility is a function rather than a constant. We used a Galerkin’s method to discretize the system. This resulted in an ODE. We suggested two methods of solving the ODE: the exact method and the Crank-Nicolson method. Various numerical experiments were conducted which suggest the method’s convergence with the solution.

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Figure 3. Comparison of exact function $U$ and numerical solution $V$ for the case of non-constant $\sigma$.

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