SUPPORTS OF REPRESENTATIONS OF THE RATIONAL CHEREDNIK ALGEBRA OF TYPE A

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Abstract. We first consider the rational Cherednik algebra corresponding to the action of a finite group on a complex variety, as defined by Etingof. We define a category of representations of this algebra which is analogous to “category $O$” for the rational Cherednik algebra of a vector space. We generalise to this setting Bezrukavnikov and Etingof’s results about the possible support sets of such representations. Then we focus on the case of $S_n$ acting on $\mathbb{C}^n$, determining which irreducible modules in this category have which support sets. We also show that the category of representations with a given support, modulo those with smaller support, is equivalent to the category of finite dimensional representations of a certain Hecke algebra.

1. Introduction

1.1. Linear actions. Let $W$ be a finite group acting faithfully on a finite dimensional $\mathbb{C}$-vector space $\mathfrak{h}$. The Weyl algebra $D(\mathfrak{h})$ of $\mathfrak{h}$ admits an action of $W$, so $\mathbb{C}[W] \otimes_{\mathbb{C}} D(\mathfrak{h})$ becomes an algebra in a natural way. We denote this algebra by $\mathbb{C}[W] \ltimes D(\mathfrak{h})$. The rational Cherednik algebra, defined by Etingof and Ginzburg [11], is a universal flat deformation of this algebra. It is named thus because it is a degeneration of the double affine Hecke algebra defined by Cherednik [6]. We recall the definition of the rational Cherednik algebra below:

Definition 1.1. We define the set of reflections in $W$ to be

$$S = \{ s \in W \mid \text{rk}(s - 1) = 1 \}.$$ 

For $s \in S$, let $\alpha_s^\vee \in \mathfrak{h}$ and $\alpha_s \in \mathfrak{h}^*$ be the nontrivial eigenvectors of $s$, with eigenvalues $\lambda_s^{-1}$ and $\lambda_s$, normalised so that $\langle \alpha_s^\vee, \alpha_s \rangle = 2$. Given a $W$-invariant function $c : S \to \mathbb{C}$, the rational Cherednik algebra $H_c(W, \mathfrak{h})$ is the unital associative $\mathbb{C}$-algebra generated by $\mathfrak{h}$, $\mathfrak{h}^*$ and $W$, with relations

$$wx = w_x w, \quad wy = w_y w,$$

$$[x, x'] = 0, \quad [y, y'] = 0,$$

$$[y, x] = \langle y, x \rangle - \sum_{s \in S} c(s) \langle y, \alpha_s \rangle \langle \alpha_s^\vee, x \rangle s,$$

for $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$ and $w \in W$.

If there is no risk of confusion, we denote this algebra simply by $H_c$.

Much progress has been made in the representation theory of $H_c$ by restricting attention to finitely generated modules on which $\mathfrak{h}$ acts locally nilpotently. The category of such modules, introduced by Opdam and Rouquier [15], is denoted by $\mathcal{O}(H_c)$ and displays many similarities with “category $O$” for semisimple complex Lie algebras; this point of view is explained in [19]. The natural homomorphism $\mathbb{C}[\mathfrak{h}] \to H_c$ allows us to think of such modules as coherent sheaves on the complex variety $\mathfrak{h}$. By completing at
various points of \( \mathfrak{h} \). Bezrukavnikov and Etingof [4] characterised the possible support sets of such a module, showing in particular that any irreducible component of this set is the set of fixed points of some subgroup of \( W \). Moreover they constructed the following flat connections from these modules (see Proposition 3.20 of [4]).

**Proposition 1.2.** Suppose \( M \in \mathcal{O}(H_c) \) and \( W' \) is a subgroup of \( W \). Let \( Y \) be the set of points in \( \mathfrak{h} \) whose stabiliser is \( W' \), and let \( i_Y : Y \hookrightarrow \mathfrak{h} \) be the inclusion. Denoting by \( \text{Sh}(M) \) the coherent sheaf on \( \mathfrak{h} \) corresponding to the \( \mathbb{C}[[\mathfrak{h}]] \)-module \( M \), there is a flat connection on the coherent sheaf pullback \( i_Y^* \text{Sh}(M) \) determined by

\[
\nabla_ym = ym - \sum_{s \in S \setminus W'} c(s)\langle y, \alpha_s \rangle \frac{2}{1 - \lambda_s} \frac{1}{\alpha_s} (s - 1)m
\]

for \( m \in M \) and \( y \in \mathfrak{h}^{W'} \).

This flat connection is a special case of Theorem 1.2(2) below. In fact this statement holds for any module \( M \) in the category \( H_c-\text{mod}_{coh} \) of modules finitely generated over \( \mathbb{C}[[\mathfrak{h}]] \subseteq H_c \). This allows us to give the following alternative characterisation of the category \( \mathcal{O}(H_c) \).

**Proposition 1.3.** The category \( \mathcal{O}(H_c) \) is a Serre subcategory of \( H_c-\text{mod}_{coh} \). Moreover given an irreducible \( M \in H_c-\text{mod}_{coh} \), let \( W' \subseteq W \) be a subgroup whose fixed point set is a component of \( \text{Supp} \ M \). Then \( M \) lies in \( \mathcal{O}(H_c) \) if and only if the flat connection of Proposition 1.2 has regular singularities.

1.2. **Actions on Varieties.** Now suppose \( W \) acts on a smooth complex algebraic variety \( X \), and \( \omega \) is a \( W \)-invariant closed 2-form on \( X \). We recall briefly the notion of twisted differential operators [1]. Let \( \mathcal{D}_\omega(X) \) denote the sheaf of algebras generated over \( \mathcal{O}_X \) by the tangent bundle \( TX \), with relations

\[
xy - yx = [x, y] + \omega(x, y), \quad xf - fx = x(f)
\]

for vector fields \( x \) and \( y \) and regular functions \( f \), where \([\cdot, \cdot]\) denotes the usual Lie bracket of vector fields (note that throughout this paper, scripted letters will generally denote sheaves of modules or algebras). To give an action of \( \mathcal{D}_\omega(X) \) on a quasi-coherent sheaf \( \mathcal{M} \) is equivalent to giving a connection on \( \mathcal{M} \) with curvature \( \omega \). Given an immersion of a smooth curve \( i : C \hookrightarrow X \), we obtain a connection on the pullback \( i^* \mathcal{M} \) which is trivially flat, so \( i^* \mathcal{M} \) may be thought of as an untwisted \( \mathcal{D} \)-module. We say \( \mathcal{M} \) has regular singularities if \( i^* \mathcal{M} \) has regular singularities in the usual sense for every such immersion. This definition was given by Finkelberg and Ginzburg [13] for 2-forms which are étale-locally exact. In fact we will only be interested in coherent sheaves \( \mathcal{M} \) over \( \mathcal{D}_\omega(X) \), and the existence of such a sheaf ensures that \( \omega \) is Zariski-locally exact.

Any 1-form \( \alpha \) gives rise to an isomorphism \( \mathcal{D}_\omega(X) \cong \mathcal{D}_{\omega + d\alpha}(X) \). Therefore by patching sheaves of algebras of the form \( \mathcal{D}_\omega(X) \), we obtain a sheaf of algebras \( \mathcal{D}_\psi(X) \) corresponding to any class \( \psi \in H^2(X, \Omega^1_X) \), where \( \Omega^1_X \) is the two step complex \( \Omega^1_X \rightarrow \Omega^2_X \) lying in degrees 1 and 2, \( \Omega^1_X \) is the sheaf of 1-forms and \( \Omega^2_X \) the sheaf of closed 2-forms. When \( X \) is affine, any such class is represented by a global 2-form. Note that our definition of regular singularities depends on a global 2-form chosen to represent the class.

Etingof [10] has defined a sheaf of algebras \( \mathcal{H}_{c,\psi}(W, X) \) on \( X/W \), generalizing Definition 1.1. We will recall this definition below (see Definition 2.2) after developing some preliminaries. There is a natural copy of the structure sheaf \( \mathcal{O}_X \) in \( \mathcal{H}_{c,\psi}(W, X) \), and we will consider the full subcategory \( \mathcal{H}_{c,\psi}-\text{mod}_{coh} \) of \( \mathcal{H}_{c,\psi}(W, X)-\text{mod} \), consisting of sheaves of modules which are coherent as \( \mathcal{O}_X \)-modules. Our first goal is to classify possible support sets of such modules, in analogy with the results of [4]. Explicitly, given a
subgroup $W' \subseteq W$, let

\[ X^{W'} = \{ x \in X \mid {}^w x = x \text{ for } w \in W' \}, \]

\[ X_{\text{reg}}^{W'} = \{ x \in X \mid \text{Stab}_W(x) = W' \}. \]

Also define

\[ P = \{ Y \mid Y \text{ is a component of } X^{W'} \text{ for some } W' \subseteq W \}. \]

These subsets are locally closed, and may be viewed as (non-affine) varieties. Let $P'$ denote the set of all $Y \in P$ such that $H_c(W', T_x X/T_x X^{W'})$ admits a nonzero finite dimensional module, where $x$ is any point of $Y$ and $W' = \text{Stab}_W(x)$. We will prove:

**Theorem 1.4.** Suppose $\mathcal{M} \in \mathcal{H}_{c,\omega}^\text{mod}_{\text{coh}}$.

1. Suppose $Z \subseteq X$ is a closed $W$-invariant subset of $X$, and consider the subsheaf of \textquotedblleft $Z$-torsion\textquotedblright elements in $\mathcal{M}$,

\[ \Gamma_Z(\mathcal{M})(U) = \{ m \in \mathcal{M}(U) \mid \text{Supp } m \subseteq Z \}. \]

That is, $\Gamma_Z(\mathcal{M})$ is the sum of all coherent subsheaves of $\mathcal{M}$ which are set-theoretically supported on $Z$. Then $\Gamma_Z(\mathcal{M})$ is an $\mathcal{H}_{c,\omega}$-submodule of $\mathcal{M}$.

2. Let $Y \in P$ and let $i_Y : Y \hookrightarrow X$ be the inclusion. The coherent sheaf pullback $i_Y^*(\mathcal{M})$ on $Y$ admits a natural action of $D_{\mathcal{Y}}^\omega(\psi)$. In particular, if $\psi = 0$, then $i_Y^*(\mathcal{M})$ admits a natural flat connection.

3. The set-theoretical support of $\mathcal{M}$ has the form

\[ \text{Supp } \mathcal{M} = \bigcup_{Y \in P_M} Y \]

for some $W$-invariant subset $P_M \subseteq P'$.

4. There is an integer $K > 0$, depending only on $c$, $W$ and $X$, such that any such $\mathcal{M}$ is scheme-theoretically supported on the $K^{\text{th}}$ neighbourhood of its set-theoretical support.

5. Every object of $\mathcal{H}_{c,\omega}^\text{mod}_{\text{coh}}$ has finite length.

6. If $\mathcal{M}$ is irreducible then we may take $P_M$ in part (3) to be a single $W$-orbit in $P'$.

We would like a sensible subcategory of $\mathcal{H}_{c,\omega}^\text{mod}_{\text{coh}}$ in which to study the representation theory of $\mathcal{H}_c$, analogous to the category $\mathcal{O}(H_c)$ in the linear case. Motivated by Proposition 1.3, we make the following definition. Again we need to choose a global 2-form $\omega$ representing the class $\psi$ for this definition.

**Definition 1.5.** Let $\mathcal{H}_{c,\omega}^\text{mod}_{RS}$ denote the Serre subcategory of $\mathcal{H}_{c,\omega}^\text{mod}_{\text{coh}}$, such that an irreducible $\mathcal{M} \in \mathcal{H}_{c,\omega}^\text{mod}_{\text{coh}}$ lies in $\mathcal{H}_{c,\omega}^\text{mod}_{RS}$ exactly when the connection on $i_Y^*(\mathcal{M})$ given in Theorem 1.4(2) has regular singularities, where $Y \in P_M$ as in Theorem 1.4(6).

For a linear action, Proposition 1.3 shows that this category coincides with $\mathcal{O}(H_c)$. Nevertheless we will use the notation $\mathcal{H}_{c,\omega}^\text{mod}_{RS}$ even in the linear case to avoid confusion with the structure sheaf of a variety.

1.3. **The Type A Case.** Taking $X$ to be an open subset of a vector space, the above will be of use in the sequel, in which we study representations of $H_c = H_c(S_n, \mathbb{C}^n)$, where $S_n$ is the symmetric group acting on $\mathbb{C}^n$ by permuting coordinates. The category $H_c-\text{mod}_{RS}$ is semisimple unless $c$ is rational with denominator between 2 and $n$ (see [2]), so we take $c = \frac{r}{m}$ where $m \geq 2$ is coprime with $r$. It is shown in [3] (and follows from Theorem 1.4) that the support of any module in $H_c-\text{mod}_{RS}$ is of the form

\[ X_q = \{ b \in \mathfrak{h} \mid \text{Stab}_{S_n}(b) \cong S_m^q \}. \]
for some integer $q$ with $0 \leq q \leq \frac{m}{n}$. It is known that the irreducible modules in $H_c$ are parameterised by the irreducible representations of $\mathbb{C}[S_n]$, which are in turn parameterised by partitions of $n$. Given a partition $\lambda \vdash n$, let $\tau_\lambda$ and $L(\tau_\lambda)$ denote the corresponding representation of $S_n$ and $H_c$ respectively. The support of the latter is determined by the following.

**Theorem 1.6.** If $c > 0$, then the support of the $H_c$-module $L(\tau_\lambda)$ is $X_{qm}(\lambda)$, where

$$q_m(\lambda) = \sum_{i \geq 1} \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{m} \right\rfloor.$$

If $c < 0$, the support of $L(\tau_\lambda)$ is $X_{qm}(\lambda')$, where $\lambda'$ is the transpose of $\lambda$.

Note that any $\lambda \vdash n$ can be uniquely written as $m\mu + \nu$ where $\mu \vdash q_m(\lambda)$ and $\nu'$ is $m$-regular. In particular, the above proves the following conjecture of Bezrukavnikov and Okounkov. While this paper was in preparation, this result was generalised to the cyclotomic case by Shan and Vasserot [21].

**Corollary 1.7.** Consider the universal enveloping algebra $A$ of the Heisenberg algebra, with generators $\{\alpha_i | i \in \mathbb{Z}, i \neq 0\}$ and relation $[\alpha_i, \alpha_j] = is_{i-j}$. Consider the grading on $A$ defined by $\deg(\alpha_i) = i$. Let $F$ denote Fock space, that is, the left $A$-module

$$F = A/\text{span}\{A\alpha_i | i > 0\}.$$

The number of irreducibles in $H_c$-mod$_RS$ whose support is $X_q$ is the dimension of the $qm$-eigenspace of the operator

$$\sum_{i>0} \alpha_{-im} \alpha_{im}$$

acting on the degree $n$ part of $F$.

Denoting by $H_c$-mod$_RS^q$ the Serre subcategory of $H_c$-mod$_RS$ consisting of all modules supported on $X_q$, we will determine the structure of the quotient category

$$H_c$-mod$_RS^q/H_c$-mod$_RS^{q+1}\mod$$

(where $H_c$-mod$_RS^{[n/m]+1}$ is the subcategory containing only the zero module). Explicitly, let $p = n - qm$ and $q = e^{2\pi i c}$, and consider the Hecke algebra $H_q(S_p)$ with generators $T_1, \ldots, T_{p-1}$ and relations

$$T_j T_j = T_j T_i \text{ if } |i - j| > 1,$$

$$T_j T_{i+1} T_i = T_{i+1} T_j T_{i+1},$$

$$(T_i - 1) (T_i + q) = 0.$$ The irreducible modules over $H_q(S_p)$ are indexed by $m$-regular partitions of $p$ [8]. Given $\nu \vdash p$ with $q_m(\nu') = 0$, let $D_\nu$ denote the corresponding irreducible. We will show:

**Theorem 1.8.** With $c = \frac{m}{n}$ and $q = e^{2\pi i c}$, the category $H_c$-mod$_RS^q/H_c$-mod$_RS^{q+1}$ is equivalent to the category of finite dimensional modules over $\mathbb{C}[S_q] \otimes \mathbb{C} H_q(S_p)$. If $\nu \vdash p$ is $m$-regular, and $\mu \vdash q$, the irreducible in $H_c$-mod$_RS^q$ corresponding to $\tau_\mu \otimes D_\nu$ under this equivalence is $L(\tau_{mq+\nu})$ if $c > 0$, and $L(\tau_{mq+\nu'})$ if $c < 0$.

1.4. Outline of the Paper. The paper is organised as follows. In Section 2, after some algebraic geometry preliminaries we recall the definition of the rational Cherednik algebra of a variety, and prove Theorem 1.4. In Section 3, we state some known results about the representation theory for linear actions, and in particular for $H_c(S_n, \mathbb{C}_n)$. From this we prove Proposition 1.3 and deduce one direction of Theorem 1.6. We restrict to the case $c = \frac{1}{m}$ in Section 4 and construct an explicit equivalence from the category of minimally supported representations. This enables us, in Section 5, to prove Theorem 1.8 for $c = \frac{1}{m}$. From this we deduce Theorems 1.8 and 1.6 in general.
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2. **Coherent Representations**

Suppose $X$ is a smooth algebraic variety over $\mathbb{C}$ and $W$ a finite group acting on $X$, such that the set of points with trivial stabiliser is dense in $X$. Let $\omega$ be a $W$-invariant closed 2-form on $X$. Suppose for the moment that $X$ is affine. In order to define the rational Cherednik algebra $H_{c,\omega}(W, X)$, we require the following lemma, which is shown in Section 2.4 of [10].

**Lemma 2.1.** Suppose $Z \subseteq X$ is a smooth closed subscheme of codimension 1. Let $O(X)$ denote the ring of regular functions on $X$, and $O(X)\langle Z \rangle$ the space of rational functions on $X$ whose only pole is along $Z$, with order at most 1. There is a natural $O(X)$-module homomorphism $\xi_Z : TX \to O(X)\langle Z \rangle/O(X)$ whose kernel consists of all vector fields preserving the ideal sheaf of $Z$.

Since $TX$ is a projective $O(X)$-module, we may lift $\xi_Z$ along the surjection

$$O(X)\langle Z \rangle \to O(X)\langle Z \rangle/O(X)$$

to an $O(X)$-module homomorphism $\zeta_Z : TX \to O(X)$.

**Definition 2.2** (Definitions 2.7 and 2.8 of [10]). Let $S$ denote the set of pairs $(Z, s)$, where $s \in W$ and $Z$ is an irreducible component of $X^s$ of codimension 1 in $X$. Let $c : S \to \mathbb{C}$ be a $W$-invariant function. Let $X_{\text{reg}}$ denote the set of points in $X$ with trivial stabiliser in $W$, and

$$D_\omega(X_{\text{reg}}) = \Gamma(X_{\text{reg}}, D_\omega(X))$$

the algebra of global algebraic twisted differential operators on the smooth scheme $X_{\text{reg}}$. For each vector field $v$ on $X$, we define the Dunkl-Opdam operator $D_v \in \mathbb{C}[W] \ltimes D_\omega(X_{\text{reg}})$ by

$$D_v = v + \sum_{(Z, s) \in S} \frac{2c(Z, s)}{1 - \lambda_{Z, s}} \zeta_Z(v)(s - 1),$$

where $\lambda_{Z, s}$ is the determinant of $s$ on $T_x X^s$ for any $x \in Z$. The rational Cherednik algebra $H_{c,\omega}(W, X)$ is the unital $\mathbb{C}$-subalgebra of $\mathbb{C}[W] \ltimes D_\omega(X_{\text{reg}})$ generated by $\mathbb{C}[W] \ltimes O(X)$ and the $D_v$.

**Remarks:**

1. Although $D_v$ depends on the choice of lift $\zeta_Z$, the algebra $H_{c,\omega}(W, X)$ does not.
2. Proposition 2.3 below shows that this algebra behaves well with respect to étale morphisms. Moreover if $\alpha$ is any $W$-invariant 1-form on $X$, the isomorphism

$${\mathbb{C}[W] \ltimes D_\omega(X_{\text{reg}})} \cong {\mathbb{C}[W] \ltimes D_{\omega + d\alpha}(X_{\text{reg}})}$$

identifies $H_{c,\omega}(W, X)$ with $H_{c,\omega+d\alpha}(W, X)$. Therefore if we do not assume $X$ is affine, and we take a $W$-invariant class $\psi \in H^2(X, \Omega_X^{\geq 1})^W$ rather than a global 2-form, we may patch algebras of the above form to construct a sheaf of algebras $H_{c,\psi}(W, X)$ on $X/W$. Nevertheless, for the moment we will continue to assume $X$ is affine and $\omega$ is a specified 2-form.

**Proposition 2.3.** Suppose $p : U \to X$ is a $W$-equivariant étale morphism, with $U$ affine. For each component $Z'$ of $U^s$ of codimension 1, the image of $Z'$ is a component $Z$ of $X^s$ of codimension 1, and we set $c'(Z', s) = c(Z, s)$. There is a natural $O(U)$-module isomorphism $O(U) \otimes O(X) \to H_{c,\omega}(W, X) \to H_{c',p^*\omega}(W, U)$, whose composition with $H_{c,\omega}(W, X) \to O(U) \otimes O(X)$ is a sheaf homomorphism. Moreover given any $H_{c,\omega}(W, X)$-module $M$, there is a natural action of $H_{c',p^*\omega}(W, U)$ on $O(U) \otimes O(X) M$. 

Proof. Since $\mathcal{O}(U)$ is flat over $\mathcal{O}(X)$, the inclusion $H_{c,\omega}(W, X) \subseteq \mathbb{C}[W] \ltimes D_\omega(X_{\text{reg}})$ induces an $\mathcal{O}(U)$-module monomorphism
\[
i: \mathcal{O}(U) \otimes \mathcal{O}(X) H_{c,\omega}(W, X) \rightarrow \mathcal{O}(U) \otimes \mathcal{O}(X) \mathbb{C}[W] \ltimes D_\omega(X_{\text{reg}})
= \mathbb{C}[W] \ltimes D_{p^*\omega}(U_{\text{reg}}).
\]
Moreover, for an appropriate choice of the lifts $\zeta$, for any $v \in TX$ the image of $D_v \in H_{c,\omega}(W, X)$ under $i$ is $D_{p^*v}$. Now $H_{c',p^*\omega}(W, U)$ is the subalgebra of $\mathbb{C}[W] \ltimes D_{p^*\omega}(U_{\text{reg}})$ generated by $\mathbb{C}[W] \ltimes \mathcal{O}(U)$ and $\{D_v \mid v \in TU\}$. But
\[
TU = \mathcal{O}(U) \otimes \mathcal{O}(X) TX,
\]
and
\[
[D_v, f] = v(f) + \sum_{(Z, s) \in S} \frac{2c(Z, s)}{1 - \lambda_{Z, s}} \zeta_Z(v)(s - f)s \in \mathbb{C}[W] \ltimes \mathcal{O}(U)
\]
for any $v \in TU$ and $f \in \mathcal{O}(U)$. It follows that $H_{c',p^*\omega}(W, U)$ is spanned by elements of the form
\[
fwD_{p^*v_1}D_{p^*v_2} \ldots D_{p^*v_k}
\]
for $f \in \mathcal{O}(U)$, $w \in W$ and $v_i \in TX$. Applying the same argument with $p$ equal to the identity on $X$, we see that $H_{c,\omega}(W, X)$ is spanned by
\[
fwD_{v_1}D_{v_2} \ldots D_{v_k}
\]
for $f \in \mathcal{O}(X)$, $w \in W$ and $v_i \in TX$. Thus the image of $i$ is exactly $H_{c',p^*\omega}(W, U)$, giving the required isomorphism $j: \mathcal{O}(U) \otimes \mathcal{O}(X) H_{c,\omega}(W, X) \rightarrow H_{c',p^*\omega}(W, U)$. Note that the composition of $i$ with $H_{c,\omega}(W, X) \rightarrow \mathcal{O}(U) \otimes \mathcal{O}(X) H_{c,\omega}(W, X)$ equals the composite
\[
H_{c,\omega}(W, X) \subseteq \mathbb{C}[W] \ltimes D_\omega(X_{\text{reg}}) \rightarrow \mathbb{C}[W] \ltimes D_{p^*\omega}(U_{\text{reg}}),
\]
which is an algebra homomorphism. In particular, $j$ also preserves the right $\mathcal{O}(X)$-module structure.

Now suppose $M$ is an $H_{c,\omega}(W, X)$-module. The multiplication map
\[
H_{c',p^*\omega}(W, U) \otimes_{\mathbb{C}} \mathcal{O}(U) \rightarrow H_{c',p^*\omega}(W, U)
\]
and the action map $H_{c,\omega}(W, X) \otimes_{\mathcal{O}(X)} M \rightarrow M$ give rise to a map
\[
H_{c',p^*\omega}(W, U) \otimes_{\mathbb{C}} \mathcal{O}(U) \otimes_{\mathcal{O}(X)} M
\rightarrow\nH_{c',p^*\omega}(W, U) \otimes_{\mathcal{O}(X)} M
\cong\n\mathcal{O}(U) \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X) \otimes_{\mathcal{O}(X)} M
\rightarrow\n\mathcal{O}(U) \otimes_{\mathcal{O}(X)} M.
\]
It is straightforward to check that this defines an action. \qed

As in [4], we will study representations of $H_{c,\omega}(W, X)$ by restricting to the formal neighbourhood of a point, or more generally a closed subset.

Proposition 2.4. Suppose $Z$ is a $W$-invariant closed subset of $X$, and let $\mathfrak{i} \subseteq \mathcal{O}(X)$ denote the ideal vanishing on $Z$. Consider the coordinate ring of the “formal neighbourhood” of $Z$,
\[
\hat{\mathcal{O}}_{X, Z} = \lim_{\longleftarrow} \mathcal{O}(X)/\mathfrak{i}^k.
\]
There is a natural algebra structure on $\hat{\mathcal{O}}_{X, Z} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X)$, and this algebra acts naturally on $\hat{\mathcal{O}}_{X, Z} \otimes_{\mathcal{O}(X)} M$ for any $M$ in $H_{c,\omega}(W, X)\text{-mod}_{\text{coh}}$. 

Proof. We define an algebra filtration $H_{c,\omega}^{\leq d}$ of $H_{c,\omega}(W, X)$ as follows. Let $H_{c,\omega}^{\leq 0} = \mathbb{C}[W] \ltimes \mathcal{O}(X)$ and

$$H_{c,\omega}^{\leq 1} = H_{c,\omega}^{\leq 0} + \mathbb{C}[W]\{D_v \mid v \in TX\}.$$  

This is independent of the choice of lift $\zeta$, since different choices of $D_v$ differ by elements of $\mathbb{C}[W] \ltimes \mathcal{O}(X)$. Finally let

$$H_{c,\omega}^{\leq d} = (H_{c,\omega}^{\leq 1})^d.$$  

As in the previous proof, if $v_1, \ldots, v_m$ generate $TX$ over $\mathcal{O}(X)$, then $H_{c,\omega}^{\leq d}$ is generated over $\mathcal{O}(X)$ by

$$wD_{v_1} \cdots D_{v_k}$$  

for $w \in W$ and $k \leq d$. In particular, $H_{c,\omega}^{\leq d}$ is a finitely generated left $\mathcal{O}(X)$-module.

Now $I$ is $W$-invariant, so inside $H_{c,\omega}$ we have

$$(\mathbb{C}[W] \ltimes \mathcal{O}(X))I = I(\mathbb{C}[W] \ltimes \mathcal{O}(X))$$

and

$$[D_v, I] \subseteq [D_v, \mathcal{O}(X)] \subseteq \mathbb{C}[W] \ltimes \mathcal{O}(X).$$

It follows by induction on $k$ that $H_{c,\omega}^{\leq 1} I^{k+1} \subseteq I^k H_{c,\omega}^{\leq 1}$, so that $H_{c,\omega}^{\leq d} I^{k+d} \subseteq I^k H_{c,\omega}^{\leq d}$ for all $d, k \geq 0$. The multiplication map

$$H_{c,\omega}^{\leq d} \otimes H_{c,\omega}^{\leq e} \rightarrow H_{c,\omega}^{\leq d+e}$$

therefore naturally induces a map

$$H_{c,\omega}^{\leq d} \otimes (\mathcal{O}(X)/I^{d+k} \otimes \mathcal{O}(X)) H_{c,\omega}^{\leq e} \rightarrow \mathcal{O}(X)/I^k \otimes \mathcal{O}(X) H_{c,\omega}^{\leq d+e}.$$  

Taking inverse limits we obtain $\hat{H}_{c,\omega}^{\leq d} \otimes \hat{H}_{c,\omega}^{\leq e} \rightarrow \hat{H}_{c,\omega}^{\leq d+e}$, where

$$\hat{H}_{c,\omega}^{\leq d} = \lim_k H_{c,\omega}^{\leq d}.$$

In this way the space

$$\hat{H}_{c,\omega} = \bigcup_{d \geq 0} \hat{H}_{c,\omega}^{\leq d} = \hat{\mathcal{O}}_{X, Z} \otimes \mathcal{O}(X) H_{c,\omega}$$

becomes an associative algebra. Moreover for any $M \in H_{c,\omega}-\text{mod}_{\text{coh}}$, the action map induces

$$H_{c,\omega}^{\leq d} \otimes \left(\mathcal{O}(X)/I^{d+k} \otimes \mathcal{O}(X) M\right) \rightarrow \mathcal{O}(X)/I^k \otimes \mathcal{O}(X) M,$$

and taking inverse limit gives an action of $\hat{H}_{c,\omega}$ on $\hat{\mathcal{O}}_{X, Z} \otimes \mathcal{O}(X) M$. 

Next we show that the action of $W$ on $X$ looks, on the formal neighbourhood of the fixed point set, like a linear action.

Proposition 2.5. Suppose $Z \subseteq X$ is a smooth closed subset which is fixed pointwise by the action of $W$. Then every $x \in Z$ admits an affine open $W$-invariant neighbourhood $U \subseteq X$ such that there is a $W$-equivariant ring isomorphism $\phi$ making the following diagram commute:

$$\hat{\mathcal{O}}_{(U \cap Z) \times (T_x X/T_x Z), (U \cap Z) \times \{0\}} \xrightarrow{\phi} \hat{\mathcal{O}}_{U \cap Z} \quad \text{and} \quad \mathcal{O}(U \cap Z).$$

Here $W$ acts on the first ring according to its linear action on $T_x X/T_x Z$. 

Proof. Both rings are inverse limits, so it suffices to construct compatible $W$-equivariant ring isomorphisms
\[ \phi_k : \Gamma(X, \mathcal{O}_X / \mathcal{I}) \otimes \mathbb{C}[T_x X / T_x Z] / m^k \to \Gamma(X, \mathcal{O}_X / \mathcal{I}^k), \]
such that $\phi_1$ is the identity, where $m \subseteq \mathbb{C}[T_x X / T_x Z]$ is the ideal corresponding to the origin and $\mathcal{I} \subseteq \mathcal{O}_X$ is the ideal sheaf vanishing on $Z$.

Let $T_z Z$ denote the subspace of $T_x X$ vanishing on $T_z Z$. This is the image of $\Gamma(X, \mathcal{I})$ under the gradient map $\Gamma(X, \mathcal{I}) \to T_x X$. Let $a_1, \ldots, a_n$ be a basis for $T_x X$, and $b_1, \ldots, b_r$ a basis for $T_z Z$, such that $\langle a_i, b_j \rangle = \delta_{ij}$ for $1 \leq i \leq n$ and $1 \leq j \leq r$. Since $W$ is finite and acts linearly on $\Gamma(X, \mathcal{I})$ and $T_z Z$, and we are working over characteristic zero, Maschke's theorem implies the existence of a $W$-equivariant $\mathbb{C}$-linear map $\beta : T_z Z \to \Gamma(X, \mathcal{I})$ which is right inverse to the surjection $\Gamma(X, \mathcal{I}) \to T_z Z$. Let $f_j \in \Gamma(X, \mathcal{I})$ be the image of $b_j$. Also choose $v_i \in T_X$ mapping to $a_i \in T_x X$. Let $U$ denote the open neighbourhood of $x$ on which the matrix $(v_i(f_j))_{1 \leq i, j \leq r}$ is invertible. The functions $f_1, \ldots, f_r$ have linearly independent gradients on $U$, so their zero set $Z' \subseteq U$ has codimension $r$. However $Z' \supseteq Z \cap U$ since $f_j \in \Gamma(X, \mathcal{I})$, and the dimension $r$ of $T_z Z$ equals the codimension in $X$ of the component of $Z$ containing $x$. Therefore $Z$ coincides with $Z'$ on some neighbourhood of $x$. By shrinking $U$, we may suppose that $U$ is an affine, $W$-invariant open neighbourhood of $x$ such that $\Gamma(U, \mathcal{I})$ is generated by the $f_j$. We now inductively construct ring homomorphisms
\[ \gamma_k : \Gamma(U, \mathcal{O}_X / \mathcal{I}) \to \Gamma(U, \mathcal{O}_X / \mathcal{I}^k)^W \]
for $k \geq 1$, compatible with the projections $\Gamma(U, \mathcal{O}_X / \mathcal{I}^{k+1})^W \to \Gamma(U, \mathcal{O}_X / \mathcal{I}^k)^W$, and such that $\gamma_1$ is the identity (note that $W$ acts trivially on $\Gamma(U, \mathcal{O}_X / \mathcal{I}) = \Gamma(U \cap Z, \mathcal{O}_Z)$, since $Z$ is fixed pointwise by $W$). Suppose we have $\gamma_k$, where $k \geq 1$. Let $A = \mathbb{C}[x_1, \ldots, x_m]$ be a polynomial ring mapping surjectively to $\Gamma(U, \mathcal{O}_X)$, and let $p \subseteq A$ be the inverse image of the ideal $\Gamma(U, \mathcal{I})$. Note that $\Gamma(U, \mathcal{O}_X / \mathcal{I}) = A/p$. Now choose $y_i \in \Gamma(U, \mathcal{O}_X / \mathcal{I}^{k+1})^W$ mapping to $\gamma_k(x_i) \in \Gamma(U, \mathcal{O}_X / \mathcal{I}^k)^W$. We have a ring homomorphism $\gamma' : A \to \Gamma(U, \mathcal{O}_X / \mathcal{I}^{k+1})^W$ sending $x_i$ to $y_i$, and the composite with $\Gamma(U, \mathcal{O}_X / \mathcal{I}^{k+1})^W \to \Gamma(U, \mathcal{O}_X / \mathcal{I}^k)^W$ factors through $\gamma_k$. In particular, the composite kills $p$, so $\gamma'(p) \subseteq \Gamma(U, \mathcal{I}^k / \mathcal{I}^{k+1})^W$. Also the composite of $\gamma'$ with $\Gamma(U, \mathcal{O}_X / \mathcal{I}^{k+1})^W \to \Gamma(U, \mathcal{O}_X / \mathcal{I})$ is the natural projection $A \to \Gamma(U, \mathcal{O}_X / \mathcal{I})$. It follows that the restriction
\[ \delta = \gamma'|_p : p \to \Gamma(U, \mathcal{I}^k / \mathcal{I}^{k+1})^W \]
is an $A$-module homomorphism. Certainly then $\delta(p^2) = 0$. We have an exact sequence
\[ 0 \to p^2 \to p \to A/p \otimes_A T(\text{Spec } A)^* \to T(\text{Spec } A/p)^*, \]
where the map $p \to A/p \otimes_A T(\text{Spec } A)^*$ is the gradient map. Since
\[ \text{Spec } A/p = U \cap Z \]
is smooth, $T(\text{Spec } A/p)^*$ is a projective $A/p$-module. Therefore $\delta$ factors through the gradient map. Moreover $T(\text{Spec } A)^*$ is freely generated over $A$ by $dx_1, \ldots, dx_m$. Therefore we may find $z_1, \ldots, z_m \in \Gamma(U, \mathcal{I}^k / \mathcal{I}^{k+1})^W$ such that
\[ \delta(f) = \sum_{i=1}^m \frac{\partial f}{\partial x_i} z_i. \]
Let $\gamma'' : A \to \Gamma(U, \mathcal{O}_X / \mathcal{I}^{k+1})^W$ be the ring homomorphism sending $x_i$ to $y_i - z_i$. Since $k \geq 1$, we have
\[ \gamma''(f) = \gamma'(f) - \sum_{i=1}^m \frac{\partial f}{\partial x_i} z_i. \]
In particular, $\gamma''$ kills $p$, so it induces the required map $\gamma_{k+1} : A/p \to \Gamma(U, \mathcal{O}_X / \mathcal{I}^{k+1})^W$. 

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Now \( \mathbb{C}[T_xX/T_xZ] \) is freely generated by \( b_1, \ldots, b_r \), so we may extend \( \gamma_k \) to a homomorphism
\[
\phi_k^\prime : \Gamma(U, \mathcal{O}_X/I) \otimes \mathbb{C}[T_xX/T_xZ] \to \Gamma(U, \mathcal{O}_X/I^k)
\]
by sending \( b_j \) to \( f_j \). Note that the \( \phi_k^\prime \) are compatible as \( k \) varies, and are \( W \)-equivariant since \( \beta \) is. Since \( f_j \in \Gamma(X, I) \) for each \( j \), we have \( \phi_k^\prime(\Gamma(U, \mathcal{O}_X/I) \otimes m^l) \subseteq \Gamma(U, I^l/I^k) \) for \( 0 \leq l \leq k \). Thus \( \phi_k^\prime \) induces a map
\[
\phi_k : \Gamma(U, \mathcal{O}_X/I) \otimes \mathbb{C}[T_xX/T_xZ]/m^k \to \Gamma(U, \mathcal{O}_X/I^k),
\]
and to prove \( \phi_k \) is an isomorphism, it suffices to show that the induced maps
\[
\Gamma(U, \mathcal{O}_X/I) \otimes m^l/m^{l+1} \to \Gamma(U, I^l/I^{l+1})
\]
are isomorphisms, for \( 0 \leq l < k \). We took \( \gamma_1 \) to be the identity, so in fact this is a map of \( \Gamma(U, \mathcal{O}_X/I) \)-modules. That is, we are required to prove that \( \Gamma(U, I^l/I^{l+1}) \) is freely generated over \( \Gamma(U, \mathcal{O}_X/I) \) by degree \( l \) monomials in the \( f_j \). This is clear when \( l = 0 \). Moreover the monomials generate \( \Gamma(U, I^l) \) over \( \Gamma(U, \mathcal{O}_X) \), since the \( f_j \) generate \( \Gamma(U, \mathcal{O}_X) \) over \( \Gamma(U, \mathcal{O}_X) \). Finally suppose
\[
\sum_{\alpha} g_\alpha f_1^{\alpha_1} \cdots f_r^{\alpha_r} \in \Gamma(U, I^{l+1})
\]
for some \( g_\alpha \in \Gamma(U, \mathcal{O}_X) \), where each monomial has degree \( l \). The matrix \( (v_i(f_j))_{1 \leq i, j \leq r} \) is invertible on \( U \), so we may find vector fields \( v'_1, \ldots, v'_r \) on \( U \) satisfying \( v'_i(f_j) = \delta_{ij} \).

Since \( v'_i \Gamma(U, I^{l+1}) \subseteq \Gamma(U, I^l) \), we conclude that
\[
\sum_{\alpha} \alpha_\alpha g_\alpha f_1^{\alpha_1} \cdots f_i^{\alpha_i-1} \cdots f_r^{\alpha_r} \in \Gamma(U, I^l)
\]
for each \( 1 \leq i \leq r \). If \( l > 0 \) then for each \( \alpha \) we have \( \alpha_i \neq 0 \) for some \( i \), so by induction we conclude that \( g_\alpha \in \Gamma(U, I) \), as required.

The next proposition generalises Theorem 3.2 of [4], which applies to linear actions.

**Proposition 2.6.** Suppose \( Z \) is a smooth connected closed subset of \( X \), every point of which has the same stabiliser \( W' \) in \( W \). Suppose the \( W \)-translates of \( Z \) are all equal to or disjoint with \( Z \), and let \( WZ \) denote their union. Finally let \( W'' \) be the subgroup of \( W \) fixing \( Z \) setwise. Then
\[
\hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X) \cong \text{Mat}_{[W:W'\gamma]}(\mathbb{C}[W'\gamma] \otimes_{\mathbb{C}[W']} H_{c,\omega}(W', \text{Spf} \hat{\mathcal{O}}_{X,Z})),
\]
where \( H_{c,\omega}(W', \text{Spf} \hat{\mathcal{O}}_{X,Z}) \) is an algebra depending only on the following data:
- the ring \( \hat{\mathcal{O}}_{X,Z} \),
- the action of \( W' \) on \( \hat{\mathcal{O}}_{W,Z} \),
- the extension of \( \omega \) to a map \( \text{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,Z}) \land \text{Der}_{\mathbb{C}}(\hat{\mathcal{O}}_{X,Z}) \to \hat{\mathcal{O}}_{X,Z} \), and
- the parameters \( c(Z', s) \) for \( Z' \supseteq Z \).

The isomorphism is natural up to a choice of coset representatives for \( W'' \) in \( W \). There is a natural action of \( \mathbb{C}[W'\gamma] \otimes_{\mathbb{C}[W']} H_{c,\omega}(W', \text{Spf} \hat{\mathcal{O}}_{X,Z}) \) on \( \hat{\mathcal{O}}_{X,Z} \otimes_{\mathcal{O}(X)} M \) for any \( H_{c,\omega}(W, X) \)-module \( M \).

**Remarks:**

1. The construction of \( H_{c,\omega}(W, X) \) can be extended to allow \( X \) to be a formal scheme, and the algebra \( H_{c,\omega}(W', \text{Spf} \hat{\mathcal{O}}_{X,Z}) \) is an example of this construction. Nevertheless we give a self-contained definition of this algebra below without making reference to formal schemes.
2. When \( \omega = 0 \), the following proof can be simplified by embedding \( H_{c,\omega}(W, X) \) in \( \text{End}_{\mathbb{C}}(\mathcal{O}(X)) \) rather than \( \mathbb{C}[W] \rtimes D_\omega(X_{\text{reg}}) \).
Proof. We have a natural isomorphism
\[ \text{Der}_C(\hat{\mathcal{O}}_{X,WZ}) \cong \hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} TX, \]
so the closed 2-form
\[ \omega : TX \otimes_{\mathcal{O}(X)} TX \to \mathcal{O}(X) \]
extends naturally to a closed 2-form \( \text{Der}_C(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\hat{\mathcal{O}}_{X,WZ}} \text{Der}_C(\hat{\mathcal{O}}_{X,WZ}) \to \hat{\mathcal{O}}_{X,WZ} \).
We can now define an algebra \( D_\omega(\hat{\mathcal{O}}_{X,WZ}) \) in the same way that \( D_\omega(X) \) was defined, and
\[ D_\omega(\hat{\mathcal{O}}_{X,WZ}) \cong \hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} D_\omega(X) \]
as an \( \hat{\mathcal{O}}_{X,WZ} \)-module. Let \( K(\hat{\mathcal{O}}_{X,WZ}) \) be the localisation of \( \hat{\mathcal{O}}_{X,WZ} \) at all elements which are not zero divisors. There is a natural algebra structure on
\[ K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\hat{\mathcal{O}}_{X,WZ}} D_\omega(\hat{\mathcal{O}}_{X,WZ}) = K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X). \]
Since \( X \) is smooth and \( X_{\text{reg}} \) is dense in \( X \), the inclusion \( \mathcal{O}(X) \to \hat{\mathcal{O}}_{X,Z} \) extends to a monomorphism \( \Gamma(X_{\text{reg}}, \mathcal{O}_X) \to K(\hat{\mathcal{O}}_{X,WZ}) \). We therefore obtain an algebra monomorphism
\[ D_\omega(X_{\text{reg}}) = \Gamma(X_{\text{reg}}, \mathcal{O}_X) \otimes_{\mathcal{O}(X)} D_\omega(X) \hookrightarrow K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X). \]
From this we construct a monomorphism
\[ H_{c,\omega}(W, X) \hookrightarrow \mathbb{C}[W] \times D_\omega(X_{\text{reg}}) \to \mathbb{C}[W] \times (K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X)). \]
Therefore \( \hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X) \) may be naturally identified with a subalgebra of \( \mathbb{C}[W] \times (K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X)). \)
Let \( C \) be a set of left coset representatives for \( W'' \) in \( W \). We choose \( 1 \in C \) to be the representative for \( W'' \) itself. We have
\[ WZ = \coprod_{w \in C} wZ. \]
We have assumed that the closed sets on the right are pairwise disjoint, so
\[ K(\hat{\mathcal{O}}_{X,WZ}) = \bigoplus_{w \in C} K(\hat{\mathcal{O}}_{X,wZ}), \]
where \( K(\hat{\mathcal{O}}_{X,wZ}) \) is the field of fractions of \( \hat{\mathcal{O}}_{X,wZ} \). Let \( e \) denote the identity of \( \hat{\mathcal{O}}_{X,Z} \) in the above direct sum. Note that \( W'' \) fixes \( e \). Moreover for any \( w \in C \setminus \{1\} \) we have \( (\omega c)e = 0 \), since \( \omega c \) is the identity of \( \hat{\mathcal{O}}_{X,wZ} \) in this direct sum. It follows that there is an isomorphism
\[ \phi : \mathbb{C}[W] \times (K(\hat{\mathcal{O}}_{X,WZ}) \otimes_{\mathcal{O}(X)} D_\omega(X)) \to \text{Mat}_C(\mathbb{C}[W''] \times (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X))), \]
where \( \text{Mat}_C \) denotes the algebra of matrices with rows and columns indexed by \( C \). Explicitly, for \( w_1, w_2 \in C \) and \( a \in \mathbb{C}[W''] \times (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X)), \) \( \phi \) sends \( w_1 \omega a w_2^{-1} \) to the matrix with \( a \) in entry \( (w_1, w_2) \) and zeros elsewhere. Therefore to describe \( \hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X) \), it suffices to determine its image under \( \phi \).
Now \( \hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X) \) is generated as an algebra by \( \mathbb{C}[W] \otimes \hat{\mathcal{O}}_{X,WZ} \) and the Dunkl-Opdam operators. Under \( \phi \), the first subalgebra generates
\[ \text{Mat}_C(\mathbb{C}[W''] \times \hat{\mathcal{O}}_{X,Z}) \subseteq \text{Mat}_C(\mathbb{C}[W''] \times (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X))). \]
In particular, this contains \( \text{Mat}_C(\mathbb{C}) \), so the image of \( \hat{\mathcal{O}}_{X,WZ} \otimes_{\mathcal{O}(X)} H_{c,\omega}(W, X) \) is \( \text{Mat}_C(A) \), where \( A \) is the subalgebra of \( \mathbb{C}[W''] \times (K(\hat{\mathcal{O}}_{X,Z}) \otimes_{\mathcal{O}(X)} D_\omega(X)) \) generated
by $\mathbb{C}[W'] \ltimes \mathcal{O}_{X,Z}$ and the entries of the images of the Dunkl-Opdam operators under $\phi$. Recall that these operators are given by

$$D_v = v + \sum_{(Z', s) \in S} \frac{2c(Z', s)}{1 - \lambda_{Z', s}^Z} \zeta_{Z'}(v)(s - 1) \in \mathbb{C}[W] \ltimes D_\omega(X_{\text{reg}}),$$

for $v \in TX$. Given $(Z', s) \in S$, if $Z$ intersects $Z'$, then there is a point in $Z$ fixed by $s$, so $s \in W'$ and $Z \subseteq Z'$. On the other hand, if $Z'$ is disjoint with $Z$, then $\zeta_{Z'}(v)$ defines a regular function in $\mathcal{O}_{X,Z}$. Therefore $A$ is generated by $\mathbb{C}[W'] \ltimes \mathcal{O}_{X,Z}$ and the image of the map

$$D' : TX \to \mathbb{C}[W'] \ltimes (K(\mathcal{O}_{X,Z}) \otimes \mathcal{O}(X)) D_\omega(X),$$

$$D'(v) = v + \sum_{(Z', s) \in S} \frac{2c(Z', s)}{1 - \lambda_{Z', s}} \zeta_{Z'}(v)(s - 1).$$

Since $\text{Der}_C(\mathcal{O}_{X,Z}) \cong \mathcal{O}_{X,Z} \otimes \mathcal{O}(X) TX$, we may extend the $\mathcal{O}(X)$-linear maps $\zeta_{Z'} : TX \to \mathcal{O}(X)(Z')$ to $\mathcal{O}_{X,Z}$-linear maps

$$\tilde{\zeta}_{Z'} : \text{Der}_C(\mathcal{O}_{X,Z}) \to \mathcal{O}_{X,Z} \otimes \mathcal{O}(X)(Z') \subseteq K(\mathcal{O}_{X,Z}),$$

and the above formula then extends $D'$ to an $\mathcal{O}_{X,Z}$-linear map

$$\tilde{D}' : \text{Der}_C(\mathcal{O}_{X,Z}) \to \mathbb{C}[W'] \ltimes (K(\mathcal{O}_{X,Z}) \otimes \mathcal{O}(X)) D_\omega(X)).$$

Since $A$ contains $\mathcal{O}_{X,Z}$ and the image of $D'$, it also contains the image of $\tilde{D}'$. Given $s \in W'$, let $I_s \subseteq \mathcal{O}_{X,Z}$ denote the ideal generated by $f - f$ for $f \in \mathcal{O}_{X,Z}$. Let $Z'$ denote the component of $X^s$ containing $Z$, and let $I_{Z'} \subseteq \mathcal{O}(X)$ be the ideal vanishing on $Z'$. Then $I_s = \mathcal{O}_{X,Z} I_{Z'}$, so $(Z', s) \in S$ exactly when $I_s$ is locally principal. Moreover if $f \in \Gamma(U, \mathcal{O}_X)$ generates $I_{Z'}$ on some open subset $U \subseteq X$, then it generates $I_s$ on the corresponding open subset of $\text{Spec} \mathcal{O}_{X,Z}$, and

$$\tilde{\zeta}_{Z'}(v) = \frac{v(f)}{f} + \mathcal{O}_{X,Z} \otimes \mathcal{O}(X) \Gamma(U, \mathcal{O}_X)$$

for any $v \in \text{Der}_C(\mathcal{O}_{X,Z}, \mathcal{O}_{X,Z})$. This formula determines $\tilde{\zeta}_{Z'}(v)$ up to an element of $\mathcal{O}_{X,Z}$. Thus $\tilde{D}'$ is determined, up to a map $\text{Der}_C(\mathcal{O}_{X,Z}) \to \mathbb{C}[W'] \ltimes \mathcal{O}_{X,Z}$, by the action of $W'$ on $\mathcal{O}_{X,Z}$ and the parameters $c(Z', s)$ for $Z' \supseteq Z$. These data therefore determine the subalgebra

$$H_{e,\omega}(W', \text{Spf } \mathcal{O}_{X,Z}) \subseteq \mathbb{C}[W'] \ltimes (K(\mathcal{O}_{X,Z}) \otimes \mathcal{O}(X)) H_\omega(X)$$

generated by $\mathbb{C}[W'] \ltimes \mathcal{O}_{X,Z}$ and the image of $\tilde{D}'$. In particular, $H_{e,\omega}(W', \text{Spf } \mathcal{O}_{X,Z})$ is preserved by conjugation by $W'$, so $A = \mathbb{C}[W'] \otimes \mathbb{C}[W'] H_{e,\omega}(W', \text{Spf } \mathcal{O}_{X,Z})$, as required.

Finally consider any $H_{e,\omega}(W, X)$-module $M$. The previous proposition gives an action of $\mathcal{O}_{X,WZ} \otimes \mathcal{O}(X) H_{e,\omega}(W, X)$ on $\mathcal{O}_{X,WZ} \otimes \mathcal{O}(X) M$, so

$$\mathcal{O}_{X,Z} \otimes \mathcal{O}(X) M = e \mathcal{O}_{X,WZ} \otimes \mathcal{O}(X) M$$

admits an action of $e \mathcal{O}_{X,WZ} \otimes \mathcal{O}(X) H_{e,\omega}(W, X)e$. But $\phi(e)$ is the monomial matrix with a $1$ in the $(1, 1)$ entry, so restricting $\phi$ gives an isomorphism

$$e \mathcal{O}_{X,WZ} \otimes \mathcal{O}(X) H_{e,\omega}(W, X)e \cong \mathbb{C}[W'] \ltimes H_{e,\omega}(W', \text{Spf } \mathcal{O}_{X,Z}),$$

thus giving the required action. Note that this does not depend on the choice of coset representatives $C$, since we always take $C$ to contain $1$. 

We may now prove our first main result. In this proof we allow $X$ to not be affine, and our class $\psi \in H^2(X, \Omega^{\geq 1}_X)$ may not be represented by a global 2-form.
Proof of Theorem 1.4. (1) Since being a submodule is a local property, we may suppose $X$ is affine and that $\psi$ is represented by $\omega \in (\Omega^2_U)^W$. Consider the module of global sections $M = \Gamma(X, M) \in H_{c,\omega}(W, X)\text{-mod}$. Since $Z$ is $W$-invariant, it is clear that $\Gamma_Z(M)$ is preserved by $\mathbb{C}[W] \ltimes \mathcal{O}(X)$. It suffices to show that $D_v$ preserves $\Gamma_Z(M)$ for each $v \in TX$. Let $I \subseteq \mathcal{O}(X)$ denote the ideal vanishing on $Z$. Recall that $[D_v, \mathcal{O}(X)] \subseteq \mathbb{C}[W] \ltimes \mathcal{O}(X)$. Since $WI = IW$, it follows inductively that $I^{k+1}D_v \subseteq H_{c,\omega}I^k$ for each $k \geq 0$. Therefore if $m \in \Gamma_Z(M)$, then $I^km = 0$ for some $k$, whence $I^{k+1}D_vm = 0$, so $D_vm \in \Gamma_Z(M)$.

(2) Consider $Y \in P$ and $x \in Y$, with stabiliser $W' \subseteq W$. Let $h = T_xX/T_xX^W$. Applying Proposition 2.5 to $Y$, there is a $W'$-invariant affine open neighbourhood $U$ of $x$, with $U \cap Y$ closed in $U$, and a $W'$-equivariant isomorphism

$$\phi : \hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y} \to \hat{\mathcal{O}}_{U, U \cap Y},$$

where we identify $U \cap Y$ with $(U \cap Y) \times \{0\} \subseteq (U \cap Y) \times h$. Moreover $\phi$ induces the identity on $\mathcal{O}(U \cap Y)$. Let $I$ be the kernel of the map $\hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y} \to \mathcal{O}(U \cap Y)$, let $C$ be a set of left coset representatives for $W'$ in $W$, and let

$$\bar{U} = \coprod_{w \in C} wU.$$

We have a natural étale morphism $\bar{U} \to X$, so by Propositions 2.8 and 2.10 for any $M \in H_{c,\psi}\text{-mod}_{\text{csh}}$ we have a natural action of $H_{c,\omega}(W', \text{Spf} \hat{\mathcal{O}}_{U, U \cap Y})$ on $M = \hat{\mathcal{O}}_{U, U \cap Y} \otimes_{\mathcal{O}(U)} \Gamma(U, M)$,

where $\omega \in (\Omega^2_U)^W$ represents $\psi$ on $U$. The isomorphism $\phi$ gives rise to an isomorphism

$$\phi_{c,\omega} : H_{c,\phi^*\omega}(W', \text{Spf} \hat{\mathcal{O}}_{U \cap Y} x_h, U \cap Y) \to H_{c,\omega}(W', \text{Spf} \hat{\mathcal{O}}_{U, U \cap Y}).$$

Let $\nu$ denote the pullback of $i_{U \cap Y}^*\omega$ to $(U \cap Y) \times h$ under the projection map $(U \cap Y) \times h \to U \cap Y$. By abuse of notation, we will also use $\nu$ to denote the completed map

$$\text{Der}_C(\hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y}) \otimes \mathbb{C} \text{Der}_C(\hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y}) \to \hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y}.$$

Then

$$\nu(v, v') - (\phi^*\omega)(v, v') \in I$$

for vector fields $v, v' \in \text{Der}_C(\hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y})$ which preserve $I$. It follows that $\nu - \phi^*\omega = d\alpha$ for some 1-form

$$\alpha : \text{Der}_C(\hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y}) \to \hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y}$$

which satisfies $\alpha(v) \in I$ whenever $v \in \text{Der}_C(\hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y})$ preserves $I$. Since we are working over characteristic 0, we may suppose $\alpha$ is $W'$-invariant. This gives an isomorphism

$$\alpha_{c,\omega} : H_{c,\nu}(W', \text{Spf} \hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y}) \to H_{c,\phi^*\omega}(W', \text{Spf} \hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y}).$$

Proposition 2.6 also gives a natural isomorphism

$$H_{c,\nu}(W', \text{Spf} \hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y}) \cong \hat{\mathcal{O}}_{(U \cap Y) x_h, U \cap Y} \otimes_{\mathcal{O}(U \cap Y)} \mathbb{C}[h] H_{c,\nu}(W', (U \cap Y) \times h),$$

and $H_{c,\nu}(W', (U \cap Y) \times h) = D_{i_{U \cap Y}^*\omega}(U \cap Y) \otimes H_c(W', h)$. Here for any $s \in W'$ acting by reflection on $h$, we take $c(s) = c(Z, s)$, where $Z$ is the component of $X^s$ containing $Y$. Composing these isomorphisms, we obtain an action of $D_{i_{U \cap Y}^*\omega}(U \cap Y)$ on $M$ commuting with the action of $\phi(\mathbb{C}[h]) \subseteq \hat{\mathcal{O}}_{U, U \cap Y}$. Therefore $D_{i_{U \cap Y}^*\omega}(U \cap Y)$ acts on

$$M/\phi(h^*)M = M/\phi(I)M = \Gamma(U \cap Y, i_Y^*\mathcal{M}).$$
We will show that the latter action is independent of the choices of $\phi$ and $\alpha$, thus proving that the action is natural and patches to give an action on all of $Y$.

Recall from the proof of Proposition 2.6 that $H_{c,\omega}(W', \text{Spf} \widehat{\mathcal{O}}_{U\cap Y})$ is a subalgebra of $\mathbb{C}[W'] \ltimes (K(\widehat{\mathcal{O}}_{U\cap Y}) \otimes \mathcal{O}_{U\cap Y}) D_{\omega}(U)$. The latter also contains a natural copy of $\text{Der}_C(\widehat{\mathcal{O}}_{U\cap Y})$. We constructed an $\widehat{\mathcal{O}}_{U\cap Y}$-linear map

$$\hat{D}': \text{Der}_C(\widehat{\mathcal{O}}_{U\cap Y}) \to H_{c,\omega}(W', \text{Spf} \widehat{\mathcal{O}}_{U\cap Y}).$$

If $v \in \text{Der}_C(\widehat{\mathcal{O}}_{U\cap Y})$ is $W'$-invariant then

$$\hat{D}'(v) \in v + \mathbb{C}[W'] \ltimes \widehat{\mathcal{O}}_{U\cap Y} \subseteq \mathbb{C}[W'] \ltimes (K(\widehat{\mathcal{O}}_{U\cap Y}) \otimes \mathcal{O}_{U\cap Y}) D_{\omega}(U)).$$

In particular $v \in H_{c,\omega}(W', \text{Spf} \widehat{\mathcal{O}}_{U\cap Y})$. Moreover by choosing $\hat{\zeta}_Z$ appropriately, we may ensure $\hat{D}'(v) \in v + \mathbb{C}[W'] \ltimes \phi(I)$ for any such $v$.

Consider a vector field $v$ on $U \cap Y$. We may extend $v$ naturally to a vector field on $(U \cap Y) \times \mathfrak{h}$, and therefore a derivation of $\widehat{\mathcal{O}}_{(U\cap Y)\times\mathfrak{h}Unity}$. Let $\bar{v}$ denote the pushforward of this derivation under $\phi$. This has the following properties:

(a) $\bar{v} \in \text{Der}_C(\widehat{\mathcal{O}}_{U\cap Y})$ is $W'$ invariant.

(b) The following diagram commutes:

$\begin{array}{ccc}
\widehat{\mathcal{O}}_{U\cap Y} & \xrightarrow{\bar{v}} & \widehat{\mathcal{O}}_{U\cap Y} \\
\downarrow & & \downarrow \\
\mathcal{O}(U \cap Y) & \xrightarrow{v} & \mathcal{O}(U \cap Y).
\end{array}$

As noted above, the first property ensures $\bar{v} \in H_{c,\omega}(W', \text{Spf} \widehat{\mathcal{O}}_{U\cap Y})$. The action of $v$ on $M$ constructed above is exactly the action of

$$\bar{v} + \phi \alpha(v) \in H_{c,\omega}(W', \text{Spf} \widehat{\mathcal{O}}_{U\cap Y}) \subseteq \mathbb{C}[W'] \ltimes (K(\widehat{\mathcal{O}}_{U\cap Y}) \otimes \mathcal{O}_{U\cap Y}) D_{\omega}(U)).$$

However, $\phi \alpha(v) \in \phi(I)$, so $v$ acts on $M/\phi(I)M$ as simply $\bar{v}$. Therefore it suffices to prove that the action of $\bar{v}$ on $M/\phi(I)M$ is determined by the above two properties. If $\bar{v}'$ also satisfies these properties, then

$$(\bar{v} - \bar{v}')(\widehat{\mathcal{O}}_{U\cap Y}) \subseteq \ker(\widehat{\mathcal{O}}_{U\cap Y} \to \mathcal{O}(U \cap Y)) = \phi(I).$$

That is, $\bar{v} - \bar{v}' \in \phi(I) \text{Der}_C(\widehat{\mathcal{O}}_{U\cap Y})$. Since $\bar{v}$ and $\bar{v}'$ are fixed by $W'$, this gives

$$\bar{v} - \bar{v}' \subseteq \phi(I)\hat{D}'(\text{Der}_C(\widehat{\mathcal{O}}_{U\cap Y})) + \phi(I)\mathbb{C}[W'] \subseteq \phi(I)H_{c,\omega}(W', \text{Spf} \widehat{\mathcal{O}}_{U\cap Y}).$$

Thus $\bar{v} - \bar{v}'$ acts as zero on $M/\phi(I)M$, as required.

(3) It is well known that a coherent sheaf with a connection is locally free, so the previous part shows that each $Y \in P$ is either contained in or disjoint with $\text{Supp} M$. Since

$$X = \coprod_{Y \in P} Y,$$

we conclude that $\text{Supp} M$ is a disjoint union of sets in $P$. Moreover $\text{Supp} M$ is closed and the closure $\overline{Y}$ of any $Y \in P$ is irreducible, so the irreducible components of $\text{Supp} M$ have the form $\overline{Y}$ for some $Y \in P$. Let $P_M$ denote the set of $Y \in P$ such that $\overline{Y}$ is an irreducible component of $\text{Supp} M$. The action of $W$ on $M$ ensures that $P_M$ is $W$-invariant, so it suffices to prove that $P_M \subseteq P'$.

Pick $Y \in P_M$ and let $x, W', U, \mathfrak{h}$, $\phi$ and $I$ be as above. Suppose $x \in \overline{Y}$ for some $Y' \in P_M$. Then $\overline{Y}$ is a connected component of $X^{W''}$ for some $W'' \subseteq W'$, and we must have $W'' \subseteq \text{Stab}_W(x) = W'$. But then $\overline{Y} \subseteq X^{W''}$, so $\overline{Y} \subseteq \overline{Y}'$ since
\( \mathcal{Y} \) is connected. Since \( \mathcal{Y} \) is an irreducible component of \( \text{Supp} \mathcal{M} \), we conclude that \( Y' = Y \). That is, \( \text{Supp} \mathcal{M} \) coincides with \( Y \) on some neighbourhood of \( x \).

By shrinking \( U \), we suppose that this holds on \( U \). Then some power of \( \phi(I) \) kills \( M \). It follows that

\[
N = k_x \otimes \mathcal{O}_{U \cap Y} M
\]

is finite dimensional, where \( k_x \) is the residue field of the point \( x \in U \cap Y \), and the map \( \mathcal{O}(U \cap Y) \to \mathcal{O}_{U \cap Y} \) is given by the map \( \phi \). Moreover the action of \( D_{U \cap Y} (U \cap Y) \otimes H_c(W', h) \) on \( M \) gives rise to an action of \( H_c(W', h) \) on \( N \).

Finally \( N \) is nonzero, since

\[
\mathbb{C}[h]/\mathcal{I}\mathbb{C}[h] \otimes_{\mathbb{C}[h]} N
\]

is the fibre of \( \mathcal{M} \) at \( x \in X \), which is nonzero by assumption.

(4) We keep the above notation. Since some power of \( I \) kills \( M \), and the ring \( \mathcal{O}(U \cap Y) \times_h \mathcal{O}(U \cap Y) / \mathcal{I}^l \) is finitely generated over \( \mathcal{O}(U \cap Y) \) for any \( l \), we conclude that \( M \) is a finitely generated module over \( \mathcal{O}(U \cap Y) \). Since it admits a connection, it is locally free. We will show in Lemma 3.1.2 that there is an integer \( K_Y \), depending only on \( h, W' \) and \( c \), such that \( N \) is killed by \( (h^*)^K_Y \). That is,

\[
I_Y^{K_Y} M \subseteq m_x M,
\]

where \( I_Y \subseteq \mathcal{O}(U) \) is the ideal vanishing on \( U \cap Y \), and \( m_x \subseteq \mathcal{O}(U \cap Y) \) is the maximal ideal corresponding to the point \( x \). Up to (non-canonical) isomorphism, the algebra \( H_c(W', h) \) is independent of the point \( x \in Y \). Therefore this equation holds for any \( x \in Y \cap U \). Together with local freeness, this ensures that \( I_Y^{K_Y} M = 0 \). Since \( \text{Supp} \mathcal{M} \) coincides with \( Y \) on \( U \), we conclude that \( \mathcal{I}^{K_Y} \mathcal{M} \) vanishes on \( U \), where \( I \subseteq \mathcal{O}_X \) is the ideal sheaf vanishing on \( \text{Supp} \mathcal{M} \). Now \( U \) was chosen to contain an arbitrary point on \( Y \), and \( \mathcal{I}^{K_Y} \mathcal{M} \) is \( \mathcal{I} \)-invariant, so \( \mathcal{I}^{K_Y} \mathcal{M} \) vanishes on the union \( WY' \) of all \( W \)-translates of \( Y \). That is, \( \mathcal{I}^{K_Y} \mathcal{M} \subseteq \Gamma_Z(M) \), where \( Z \) is the complement of \( WY \) in \( \mathcal{M} \). Note that \( Z \) is closed and \( \mathcal{I} \)-invariant.

It now follows by induction on \( \text{Supp} \mathcal{M} \) that \( M \) is killed by \( \mathcal{I} \) to the power of

\[
\sum_{Y \in \mathcal{P}', Y \subset \text{Supp} \mathcal{M}} K_Y.
\]

In particular, this proves the statement with \( K = \sum_{Y \in \mathcal{P}'} K_Y \).

(5) Let \( K \) be the integer constructed in the previous part. Again we prove the statement by induction on \( \text{Supp} \mathcal{M} \), and the case \( \text{Supp} \mathcal{M} = \emptyset \) is trivial. Suppose every module with smaller support has finite length. Choose \( Y \in \mathcal{P}_M \), and let \( I_Y \subseteq \mathcal{O}_X \) denote the ideal sheaf vanishing on \( \mathcal{Y} \). Let \( U \subseteq X \) be some open affine subset intersecting \( Y \), and let \( I_Y = \Gamma(U, I_Y) \). This is prime since \( Y \) is irreducible. Consider the ring

\[
R = \mathcal{O}(U)(I_Y) / I_Y^k \mathcal{O}(U)(I_Y);
\]

that is, the \( K \)-th formal neighbourhood of the (non-closed) generic point of \( Y \). Then \( I_Y^k R/I_Y^{k+1} R \) is finite dimensional over the field \( R/I_Y R \) for each \( k \), so \( R \) is Artinian. Since \( \mathcal{M} \) is coherent, it follows that

\[
R \otimes_{\mathcal{O}(U)} \Gamma(U, \mathcal{M})
\]

has finite length over \( R \). We may therefore assume by induction that the statement also holds for modules with the same support, for which the above \( R \)-module has smaller length.

Again let \( Z \) be the complement of \( WY \) in \( \text{Supp} \mathcal{M} \). Let \( \mathcal{I}_Z \) and \( \mathcal{I} \) be the ideal sheaves vanishing on \( Z \) and \( \text{Supp} \mathcal{M} \) respectively. Let \( \mathcal{M}' = \mathcal{M}/\Gamma_Z(\mathcal{M}) \) and \( \mathcal{M}'' = \mathcal{H}_{c, t} \mathcal{I}_Z^2 \mathcal{M}' \). Since \( \Gamma_Z(\mathcal{M}) \) and \( \mathcal{M}'/\mathcal{M}'' \) are supported on \( Z \), they have
finite length by induction. If $\mathcal{M}''$ is zero or irreducible, we are done. Suppose otherwise, and let $\mathcal{N}'$ be a proper nonzero submodule of $\mathcal{M}''$, and let $\mathcal{N}$ be the inverse image of $\mathcal{N}'$ in $\mathcal{M}$. We have $\mathcal{I}_Y\mathcal{I}_Z \subseteq \mathcal{I}$, so

$$\mathcal{I}_K^{\mathcal{I}} \mathcal{I}_Y \mathcal{I}_Z \subseteq \mathcal{I}_K \mathcal{M} = 0.$$ 

Therefore $\mathcal{I}_K^{\mathcal{I}} \mathcal{M} \subseteq \Gamma_Z(\mathcal{M})$, so $\mathcal{I}_K^{\mathcal{I}} \mathcal{M}' = 0$ and the map

$$\mathcal{O}(U)(h) \otimes \mathcal{O}(U) \Gamma(U, \mathcal{M}') \to R \otimes \mathcal{O}(U) \Gamma(U, \mathcal{M}')$$

is an isomorphism. Since localisation is exact, we conclude that

$$R \otimes \mathcal{O}(U) \Gamma(U, \mathcal{N}') \to R \otimes \mathcal{O}(U) \Gamma(U, \mathcal{M}')$$

is injective. Therefore we have a short exact sequence

$$R \otimes \mathcal{O}(U) \Gamma(U, \mathcal{N}') \to R \otimes \mathcal{O}(U) \Gamma(U, \mathcal{M}') \to R \otimes \mathcal{O}(U) \Gamma(U, \mathcal{M}'/\mathcal{N}')$$.

We claim that the first and last modules are nonzero. This is equivalent to $\text{Supp} \mathcal{N}'$ and $\text{Supp} \mathcal{M}'$ containing $Y$. Suppose the first fails. Then $\mathcal{N}'$ is supported on $Z$, so $\mathcal{I}_K^{\mathcal{I}} \mathcal{N}' = 0$. Thus

$$\mathcal{I}_K^{\mathcal{I}} \mathcal{N} \subseteq \mathcal{I}_K \Gamma_Z(\mathcal{M}) = 0.$$ 

Hence $\mathcal{N} \subseteq \Gamma_Z(\mathcal{M})$, so that $\mathcal{N}' = 0$, a contradiction. Now suppose $\mathcal{M}'/\mathcal{N}'$ is supported on $Z$. Then $\mathcal{I}_K^{\mathcal{I}} (\mathcal{M}'/\mathcal{N}') = 0$, so $\mathcal{N}' \supseteq \mathcal{I}_K^{\mathcal{I}} \mathcal{M}'$. Thus $\mathcal{N}' \supseteq H_{c, \psi} \mathcal{I}_K^{\mathcal{I}} \mathcal{M}' = \mathcal{M}''$, contradicting the assumption that $\mathcal{N}'$ is a proper submodule of $\mathcal{M}''$. This proves the claim, and we conclude that

$$\text{len}_R(R \otimes \mathcal{O}(U) \Gamma(U, \mathcal{N}')), \text{len}_R(R \otimes \mathcal{O}(U) \Gamma(U, \mathcal{M}'/\mathcal{N}')) < \text{len}_R(R \otimes \mathcal{O}(U) \Gamma(U, \mathcal{M})),$$

so $\mathcal{N}'$ and $\mathcal{M}'/\mathcal{N}'$ have finite length by induction. Again, since $\Gamma_Z(\mathcal{M})$ has finite length, we are done.

(6) Again let $Y$ be any element of $\mathcal{P}_\mathcal{M}$ and let $\overline{WY}$ and $Z$ be as above. Let $\mathcal{I}_{WY}$, $\mathcal{I}_Z$ and $\mathcal{I}$ be the ideal sheaves in $\mathcal{O}_X$ vanishing on $\overline{WY}$, $Z$ and $\text{Supp} \mathcal{M}$ respectively. Then

$$\mathcal{I}_K^{\mathcal{I}} \mathcal{I}_{WY} \mathcal{M} \subseteq \mathcal{I}_K \mathcal{M} = 0.$$ 

Thus $\mathcal{I}_K^{\mathcal{I}} \mathcal{M} \subseteq \Gamma_Z(\mathcal{M})$. Now $\Gamma_Z(\mathcal{M})$ is a submodule of $\mathcal{M}$ by (1), and it is proper since $Z \subseteq \text{Supp} \mathcal{M}$. Since $\mathcal{M}$ is irreducible, we conclude that $\Gamma_Z(\mathcal{M}) = 0$. Hence $\mathcal{I}_K^{\mathcal{I}} \mathcal{M} = 0$, so $\text{Supp} \mathcal{M} = \overline{WY}$ as required.

\[\square\]

3. Linear Actions

Now suppose $X = \mathfrak{h}$ is a finite dimensional vector space with a linear action. We briefly review some known results concerning representations of $H_c(\mathfrak{w}, \mathfrak{h})$; see [9]. We also prove Proposition [13] and one direction of Theorem [1,6].

3.1. Verma modules. Any $W$-module $\tau$ becomes a $C[W] \ltimes C[\mathfrak{h}^\ast]$-module by declaring that $\mathfrak{h}$ acts as 0. We may therefore construct an $H_c$-module

$$\mathcal{M}(\tau) = H_c \otimes_{C[W] \ltimes C[\mathfrak{h}^\ast]} \tau.$$ 

This is the Verma module corresponding to $\tau$. The multiplication map

$$C[\mathfrak{h}] \otimes C[W] \otimes C[\mathfrak{h}^\ast] \to H_c$$

is a vector space isomorphism, so $\mathcal{M}(\tau) \cong C[\mathfrak{h}] \otimes C[W] \otimes C[\mathfrak{h}^\ast]$ as a $C[W] \ltimes C[\mathfrak{h}]$-module. As a special case, when $\tau$ is the trivial representation, we obtain an action of $H_c$ on $C[\mathfrak{h}]$; this is called the polynomial representation. If $\tau$ is irreducible, then there is a unique
maximal proper submodule \( J(\tau) \) of \( M(\tau) \), and the quotient \( L(\tau) \) is irreducible. Let \( y_i \) be a basis of \( \frak{h} \), and \( x_i \) the dual basis of \( \frak{h}^* \). Define the Euler element by

\[
eu = \sum_i x_i y_i + \sum_{s \in S} c(s) \frac{2}{\lambda_s - 1} s \in H_c.
\]

This element has the useful property that \( \text{ad} \, \eu \) acts as 0 on \( W \), as 1 on \( \frak{h}^* \subseteq H_c \), and as \(-1\) on \( \frak{h} \subseteq H_c \). From this fact we deduce the following lemma, which was used in the proof of Theorem [14]

Lemma 3.1. There exists a positive integer \( K \), depending only on \( c \) and \( W \), such that \( \frak{h}^K \subseteq H_c \) and \( (\frak{h}^*)^K \subseteq H_c \) annihilate any finite dimensional \( H_c \)-module.

Proof. Let \( M \) be a finite dimensional \( H_c \)-module. Then \( M \) decomposes as

\[
M = \bigoplus_{\lambda \in \Lambda} M_\lambda,
\]

where \( M_\lambda \) is the generalised eigenspace of \( \eu \) with eigenvalue \( \lambda \), and \( \Lambda \subseteq \mathbb{C} \) is the finite set of eigenvalues. Since \( \text{ad} \, \eu \) acts as \(-1\) on \( \frak{h} \), it is clear that \( \frak{h} \) sends \( M_\lambda \) to \( M_{\lambda - 1} \). Thus \( \frak{h}^K \) kills \( M \), where \( K \) is any integer larger than \( d = \max(\text{Re}(\Lambda)) - \min(\text{Re}(\Lambda)) \).

The same is true of \( \frak{h}^* \), and it remains to bound \( d \) independently of \( M \). Pick \( \lambda \in \Lambda \) with \( \text{Re}(\lambda) \) minimal. Then \( \frak{h} \) acts as 0 on \( M_\lambda \). Since \( W \) commutes with \( \eu \), it preserves the eigenspace \( M_\lambda \). We may therefore find a subspace \( \tau \subseteq M_\lambda \) which is irreducible under the action of \( W \). Then

\[
\eu|_\tau = \left( \sum_i x_i y_i + \sum_{s \in S} c(s) \frac{2}{\lambda_s - 1} s \right)|_\tau = \sum_{s \in S} c(s) \frac{2}{\lambda_s - 1} s|_\tau.
\]

This depends only on the action of \( W \) on \( \tau \). Since \( W \) has only finitely many irreducible modules, there are only finitely many possible values for \( \lambda \), once \( c \) and \( W \) have been chosen. Similarly by writing

\[
\eu = \sum_i y_i x_i - \dim \frak{h} + \sum_{s \in S} c(s) \left( \frac{2}{\lambda_s - 1} + 2 \right) s,
\]

we see that there are only finitely many possibilities for \( \max(\text{Re}(\Lambda)) \). Thus \( d \) has only finitely many possible values, depending on \( W \) and \( c \), and may therefore be bounded independently of \( M \). \( \square \)

For a module \( M \in H_c - \text{mod}_{coh} \), the following conditions are equivalent:

1. The action of \( \eu \) on \( M \) is locally finite.
2. The action of \( \frak{h} \) on \( M \) is locally nilpotent.
3. Every composition factor of \( M \) is isomorphic to some \( L(\tau) \).

Proposition [13] states that the category of modules satisfying these conditions is exactly the category \( H_c - \text{mod}_{RS} \) of Definition [13]. We will require the following lemma to prove this.

Lemma 3.2. Let \( \frak{h} \) be a finite dimensional vector space, and suppose \( Z \subseteq \frak{h} \) is the zero set of a homogeneous ideal in \( \mathbb{C}[\frak{h}] \) (that is, \( Z \) is a cone). Let \( \xi \in D(\frak{h} \setminus Z) \) denote the Euler vector field. Then \( \xi \) acts locally finitely on the global sections of any \( O \)-coherent \( D \)-module on \( \frak{h} \setminus Z \) with regular singularities.

Proof. Let \( M \) be an \( O \)-coherent \( D \)-module on \( \frak{h} \setminus Z \) with regular singularities. Then \( M \) is locally free, so if \( U \) is an open subset of \( \frak{h} \setminus Z \), the restriction map

\[
\Gamma(\frak{h} \setminus Z, M) \to \Gamma(U, M)
\]
is injective. We may therefore replace \( \mathfrak{h} \setminus Z \) by any smaller \( \mathbb{C}^* \)-invariant open subset \( U \). Denoting by \( x_0, \ldots, x_n \) the coordinates on \( \mathfrak{h} \), we suppose that \( U \) is affine and disjoint with the zero set of \( x_0 \). We have an isomorphism

\[
\mathbb{C}[\mathfrak{h}][x_0^{-1}] \cong \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}[y_1, \ldots, y_n]
\]
geven by \( x_i \) to \( y_i \), for \( i > 0 \). Note the \( \mathbb{C}^* \) action on the left, which scales each \( x_i \), corresponds to the \( \mathbb{C}^* \) action on the right scaling only \( t \). Thus we have a \( \mathbb{C}^* \)-equivariant isomorphism

\[
U \cong \mathbb{C}^* \times Y
\]
where \( Y \) is some affine open subset of \( \text{Spec} \mathbb{C}[y_1, \ldots, y_n] \). In particular the vector field \( \xi \) on the left corresponds to \( t \partial_t \) on \( \mathbb{C}^* \).

It therefore suffices to consider a module \( M \) over \( \mathcal{D}(\mathbb{C}^* \times Y) = \mathcal{D}^*(\mathbb{C}^*) \otimes \mathcal{D}(Y) \) which is \( \mathcal{O} \)-coherent with regular singularities. Moreover we may suppose that \( M \) is irreducible. The Riemann-Hilbert correspondence [7] implies that \( M \) is of the form \( L \otimes \mathbb{C} N \) for some irreducible modules \( L \in \mathcal{D}(\mathbb{C}^*) \)-mod and \( N \in \mathcal{D}(Y) \)-mod, and that \( L = \mathcal{C}[t, t^{-1}]w \) with connection

\[
\nabla_{\partial_t} f(t, t^{-1}) v = (\partial_t f(t, t^{-1}) v + \lambda t^{-1} f(t, t^{-1}) v)
\]
for some \( \lambda \in \mathbb{C} \). Since \( t \partial_t \) acts as \( n + \lambda \) on \( t^n v \), and \( \{ t^n v \mid n \in \mathbb{Z} \} \) is a basis for \( L \), we are done.

**Proof of Proposition 1.3.** First we show each \( L(\tau) \) lies in \( \mathcal{H}_c \)-mod\( \mathcal{R} \). Choose \( Y \in \mathcal{P}_L(\tau) \), and let \( i_Y : Y \rightarrow X \) denote the inclusion. We are required to show that the connection on \( i_Y^* \mathcal{S}h(L(\tau)) \) has regular singularities. Certainly we have a surjection \( i_Y^* \mathcal{S}h(M(\tau)) \rightarrow i_Y^* \mathcal{S}h(L(\tau)) \) intertwining the connections, so it suffices to prove that the connection on \( i_Y^* \mathcal{S}h(M(\tau)) \) has regular singularities. However,

\[
\Gamma(Y, i_Y^* \mathcal{S}h(M(\tau))) \cong \mathcal{O}(Y) \otimes_{\mathbb{C}[\mathfrak{h}]} \mathbb{C}[\mathfrak{h}] \otimes \mathbb{C} \tau = \mathcal{O}(Y) \otimes \mathbb{C} \tau.
\]

Moreover \( \tau \) is killed by \( \mathfrak{h} \), so by Proposition 1.2 the connection on \( i_Y^* \mathcal{S}h(M) \) is described by

\[
\nabla_y m = -\sum_{s \in S} c(s)(y, \alpha_s) \frac{2}{1 - \lambda_s} \frac{1}{\alpha_s} (s - 1)m
\]
for \( m \in \tau \) and \( y \in \mathfrak{h}^\tau \), where \( \tau^* \) is the stabiliser of any point in \( Y \). Since this expression only contains poles of first order, the connection has regular singularities. Since \( \mathcal{H}_c \)-mod\( \mathcal{R} \) is a Serre subcategory of \( \mathcal{H}_c \)-mod\( \mathcal{C} \), by definition, this proves any module satisfying condition (3) above lies in \( \mathcal{H}_c \)-mod\( \mathcal{R} \).

Conversely, we will show that any \( M \in \mathcal{H}_c \)-mod\( \mathcal{R} \) satisfies condition (1) above. This condition is preserved by extensions, and Theorem 1.4(5) shows that \( M \) has finite length, so we may suppose \( M \) is irreducible. As usual let \( Y \) be in \( \mathcal{P}_M \) with stabiliser \( \tau^* \). We may find a homogeneous polynomial \( f \in \mathbb{C}[\mathfrak{h}] \) which vanishes on each \( Y' \subseteq \mathcal{P}_M \setminus \{ Y \} \), but not on \( Y \) itself. Note that the kernel of the map

\[
M \rightarrow \bigoplus_{w \in \tau^*} M[(w f)^{-1}]
\]
is \( \Gamma_Z(M) \), where \( Z \) is the common zero set of all the \( w f \). This is zero by Theorem 1.4(1) and the irreducibility of \( M \). Since \( \text{ad} \mathfrak{eu} \) acts as \text{deg} \( f \) on each \( w f \), the action of \( \mathfrak{eu} \) on \( M \) extends naturally to one on \( M[(w f)^{-1}] \). Therefore since \( \mathfrak{eu} \) is \( \tau^* \)-invariant, it suffices to show that the action of \( \mathfrak{eu} \) on \( M[(w f)^{-1}] \) is locally finite.

Let \( U \subseteq \mathfrak{h} \) denote the affine open subset on which \( f \) is nonzero, and let \( I_Y \subseteq \mathbb{C}[\mathfrak{h}][f^{-1}] \) denote the ideal vanishing on \( U \cap Y \). Note that \( I_Y \) is generated by some subspace \( V \subseteq \mathfrak{h}^* \), so \( I_Y^k M[f^{-1}] \) is invariant under \( \mathfrak{eu} \) for each \( k \). Moreover since \( M[f^{-1}] \) is supported on \( U \cap Y \), we have \( I_Y^k M[f^{-1}] = 0 \) for some \( K > 0 \). Therefore it suffices to show that
euler acts locally finitely on each $I_Y^k M[f^{-1}]/I_Y^{k+1} M[f^{-1}]$. Finally for each $k$ we have a surjective map

$$V^\otimes_k \otimes \mathbb{C} M[f^{-1}]/I_Y^k M[f^{-1}] \to I_Y^k M[f^{-1}]/I_Y^{k+1} M[f^{-1}]$$

intertwining $1 \otimes \text{euler}$ with $\text{euler} - k \deg f$. We may therefore consider just $k = 0$. But

$$M[f^{-1}]/I_Y^k M[f^{-1}] = \mathcal{O}(U \cap Y) \otimes \mathbb{C}[h] M$$

is an $\mathcal{O}$-coherent $\mathcal{D}$-module on $U \cap Y$ with connection given by Proposition 3.2. It has regular singularities by assumption. Note that $U \cap Y$ is the basic open subset of the vector space $\mathfrak{h}^W$ on which $f|_{\mathfrak{h}^W}$ is nonzero. Therefore the vector field $\xi$ of Lemma 3.2 acts locally finitely. To describe the action of $\xi$ explicitly, let $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_r\}$ be dual bases for $\mathfrak{h}^*$ and $\mathfrak{h}$, such that $\{y_1, \ldots, y_r\}$ span $\mathfrak{h}^W$. Then $x_{r+1}, \ldots, x_n$ are zero in $\mathcal{O}(U \cap Y)$, and a straightforward calculation shows that

$$\xi m = \text{euler} m + \sum_{s \in S \setminus W'} \frac{2c(s)}{1 - \lambda_s} m + \sum_{s \in S \setminus W'} \frac{2c(s)}{1 - \lambda_s} sm$$

for $m \in M$. Note that $\mathcal{O}(U \cap Y) \otimes \mathbb{C}[h] M$ admits an action of $W'$ commuting with $\mathcal{O}(U \cap Y)$. Since $\xi$ and $\text{euler}$ have the same commutator with any element of $\mathcal{O}(U \cap Y)$, this formula holds for any $m \in \mathcal{O}(U \cap Y) \otimes \mathbb{C}[h] M$. Therefore since $\xi$ and $\mathbb{C}[W']$ act locally finitely, so does $\text{euler}$, as required. \square

3.2. Characters of modules. Let $\mathbb{Z}[[t]]$ denote the space of formal $\mathbb{Z}$-linear combinations of powers of $t$, such that the exponents appearing belong to $A + \mathbb{Z}_{\geq 0}$ for some finite subset $A \subseteq \mathbb{C}$. Let $K(W - \text{mod}_{fd})$ denote the Grothendieck group of the category of finite dimensional representations of $W$. There is a homomorphism

$$\text{Ch} : K(H_c(W, \mathfrak{h}) - \text{mod}_{RS}) \to K(W - \text{mod}_{fd}) \otimes_{\mathbb{Z}} \mathbb{Z}[[t]]$$

sending $[M]$ to

$$\sum_{\lambda \in \mathbb{C}} [M_\lambda] t^\lambda,$$

where $M_\lambda \subseteq M$ is the generalised $\lambda$-eigenspace of $\text{euler}$, considered as a $W$-module. If $\tau$ is an irreducible $W$-module, then

$$\text{Ch}([M(\tau)]), \text{Ch}([L(\tau)]) \in [\tau] t^{h(\tau)} + K(W - \text{mod}_{fd}) \otimes \mathbb{Z}[[t]] t^{h(\tau) + 1},$$

where $h(\tau)$ is the scalar by which

$$\sum_{s \in S} c(s) \frac{2}{\lambda_s - 1}$$

acts on $\tau$. It follows that $\text{Ch}$ is injective, and both $\{[L(\tau)]\}$ and $\{[M(\tau)]\}$ form bases for $K(H_c(W, \mathfrak{h}) - \text{mod}_{RS})$. Moreover the matrix relating the $[M(\tau)]$ to $[L(\tau)]$ is upper triangular, when the irreducibles are ordered by $\text{Re}(h(\tau))$. It follows that the functor

$$\text{Verma}_W : W - \text{mod}_{fd} \to H_c(W, \mathfrak{h}) - \text{mod}_{RS},$$

$$\tau \mapsto M(\tau)$$

induces an isomorphism on Grothendieck groups.
3.3. Induction and restriction. Bezrukavnikov and Etingof \cite{4} also construct “parabolic induction and restriction” functors for rational Cherednik algebras. Statements (1-4) of the following theorem summarise Propositions 3.9, 3.10 and 3.14 of \cite{4}. Statement (5) follows from the construction of $\text{Res}_b$ and Proposition 2.21 of \cite{15}, and was communicated to the author by Etingof.

**Theorem 3.3.** Consider a point $b \in \mathfrak{h}$, with stabiliser $W' \subseteq W$. There exist exact functors $\text{Res}_b : H_c(W, \mathfrak{h}) - \text{mod}_{RS} \rightarrow H_c(W', \mathfrak{h}) - \text{mod}_{RS}$ and $\text{Ind}_b : H_c(W', \mathfrak{h}) - \text{mod}_{RS} \rightarrow H_c(W, \mathfrak{h}) - \text{mod}_{RS}$ with the following properties:

1. The functor $\text{Ind}_b$ is right adjoint to $\text{Res}_b$.
2. The support of $\text{Res}_b(M)$ is the union of the components of $\text{Supp} M$ passing through $b$.
3. The support of $\text{Ind}_b(N)$ is the union of $W$-translates of $\text{Supp} N$.
4. The induced maps $[\text{Res}_b]$, and $[\text{Ind}_b]$ on Grothendieck groups satisfy
   
   $[\text{Res}_b][\text{Verma}_W] = [\text{Verma}_W][\text{Res}]$,  
   $[\text{Ind}_b][\text{Verma}_W] = [\text{Verma}_W][\text{Ind}]$,  

   where $\text{Res} : W - \text{mod}_{fd} \rightarrow W' - \text{mod}_{fd}$ and $\text{Ind} : W' - \text{mod}_{fd} \rightarrow W - \text{mod}_{fd}$ are the usual restriction and induction functors.
5. If $M$ is a Verma module, then $\text{Res}_bM$ has a filtration whose successive quotients are Verma modules.

**Remark 3.4.**

We will use a definition of $\text{Res}_b$ that differs slightly from that in \cite{4} in two ways. Firstly, in the notation of \cite{4}, we do not include $\zeta$, so $\text{Res}_b$ produces modules in $H_c(W', \mathfrak{h}) - \text{mod}_{RS}$ rather than $H_c(W', \mathfrak{h}/\mathfrak{h}^{W'}) - \text{mod}_{RS}$.

Secondly, Theorem 3.2 of \cite{4} includes a shift from $b$ to the origin, which we omit. In both cases the functor we have omitted is an equivalence, so the properties of $\text{Res}_b$ are unchanged.

By its construction, $\text{Res}_b$ behaves well with respect to monodromy. We give two results to this effect. The first, concerning monodromy around a single hyperplane, follows easily from \cite{4}.

**Proposition 3.5.** Suppose $b, v \in \mathfrak{h}$ have stabilisers $W', W'' \subseteq W$ respectively. Suppose that $W'' \subseteq W'$, and that $C \subseteq W'$ is a subgroup acting faithfully on $\mathfrak{C} v$. Consider the map $\phi : C \rightarrow \mathfrak{h}$ given by $\phi(z) = b + z v$. There is a Zariski open subset $U \subseteq C^*$ such that $\phi$ maps $U$ into $Y = b \leftarrow \mathfrak{h}^{W''}$. Let $i_Y : Y \hookrightarrow \mathfrak{h}$ denote the inclusion. For any $M \in H_c(W, \mathfrak{h}) - \text{mod}_{RS}$, the $D$-modules $i_Y^* \text{Sh}(M)$ and $i_Y^* \text{Sh}(\text{Res}_bM)$ satisfy

$$C((z)) \otimes_{\mathfrak{O}_Y} \phi[^*]_Y i_Y^* \text{Sh}(M) \cong C((z)) \otimes_{\mathfrak{O}_Y} \phi[^*]_Y i_Y^* \text{Sh}(\text{Res}_bM)$$

as $\mathbb{C}[C] \ltimes C((z))[\partial_z]$-modules. In particular, the monodromies about the origin of the $D$-modules $\phi[^*]_U i_U^* \text{Sh}(M)$ and $\phi[^*]_U i_U^* \text{Sh}(\text{Res}_bM)$ are conjugate, and the same is true when these equivariant $D$-modules are pushed down to $U/C$.

To prove the next result, we required the following simple algebraic geometry lemma.

**Lemma 3.6.** Suppose $U$ is an open subset (not necessarily affine) of an affine reduced Noetherian scheme $\text{Spec} A$. Suppose $\mathcal{M}$ is a coherent sheaf on $U$ such that the support of any nonzero $m \in \Gamma(U, \mathcal{M})$ contains some irreducible component of $U$. Finally suppose $\Gamma(U \cap Z, \mathcal{O}_Z)$ is finitely generated over $A$ for each irreducible component $Z$ of $\text{Spec} A$. Then $\Gamma(U, \mathcal{M})$ is finitely generated over $A$.

**Proof.** By Exercise II.5.15 of \cite{16}, there is a finitely generated $A$-module $N$ such that $\mathcal{M} \cong \text{Sh}(N)[U]$. Let $P$ denote the set of irreducible components of $U$. For $Y \in P$, let $\overline{Y}$ denote the closure of $Y$ in $\text{Spec} A$. This is an irreducible component of $\text{Spec} A$,
and is therefore an integral affine Noetherian scheme with the reduced induced scheme structure. Let $K(Y)$ be the field of fractions of $\mathcal{O}(Y)$, and let

$$K = \bigoplus_{Y \in P} K(Y).$$

Let $N'$ be the kernel of $N \to K \otimes_A N$. If $n \in N'$, then the support of $n$ does not contain any $Y \in P$. Therefore the image of $n$ in $\Gamma(U, Sh(N)) = \Gamma(U, \mathcal{M})$ is zero, so $n$ is supported on $\text{Spec} A \setminus U$. Thus

$$Sh(N/N')|_U \cong Sh(N)|_U \cong \mathcal{M}.$$ 

Now $K \otimes_A N$ is finitely generated over $K$, so it is a direct sum of finite dimensional vector spaces over the $K(Y)$. We may write

$$K \otimes_A N = \bigoplus_{i=1}^m K(Y_i)x_i,$$

where $x_i \in K \otimes_A N$ and $Y_i \in P$ (where $Y_i$ may equal $Y_j$ for $i \neq j$). Since $N$ is finitely generated over $A$, we may choose the $x_i$ so that

$$N/N' \subseteq \bigoplus_{i=1}^m \mathcal{O}(Y_i)x_i.$$

Therefore

$$\Gamma(U, \mathcal{M}) = \Gamma(U, Sh(N/N')) \subseteq \bigoplus_{i=1}^m \Gamma(Y_i, \mathcal{O}(Y_i)).$$

By assumption, the right hand side is finitely generated over $A$. Since $A$ was assumed to be Noetherian, the result follows. \qed

We can now prove our second result relating $\text{Res}_b$ and monodromy.

**Proposition 3.7.** Suppose $b \in \mathfrak{h}$ with $\text{Stab}_b(W) = W'$. Suppose $M \in H_c(W, \mathfrak{h})$-mod$_{\text{RS}}$ is scheme theoretically supported on $\text{Supp} M$. Also suppose that for each $s \in S \setminus W'$, and for each subgroup $W'' \subseteq W'$ for which $W'' \cap \mathfrak{h}_{\text{reg}}$ has trivial monodromy around $Z(\alpha_s) \cap \mathfrak{h}_{\text{reg}}$. Then for each such $W''$, the $N_{W''}(W'')$-equivariant $D$-module $M|_{\mathfrak{h}_{\text{reg}}}$ is exactly the restriction of the $N_{W'}(W'')$-equivariant $D$-module $\text{Res}_b M|_{\mathfrak{h}_{\text{reg}}}$, where

$$\mathfrak{h}_{\text{reg}} = \{y \in \mathfrak{h} \mid \text{Stab}_{W''}(y) = W''\}.$$

**Proof.** Let

$$U = \{y \in \mathfrak{h} \mid \text{Stab}_W(y) \subseteq W'\}.$$

In particular, if $s \in S \setminus W'$ then $\alpha_s$ is nonzero on $U$. The action $\rho : H_c(W, \mathfrak{h}) \to \text{End}_C(M)$ naturally induces an action

$$\rho' : H_c(W', U) \to \text{End}_C(\mathcal{O}(U) \otimes \mathcal{O}(\mathfrak{h}) M)$$

such that

$$\rho'(y)(1 \otimes m) = 1 \otimes ym - \sum_{s \in S \setminus W'} \frac{2c(s)}{1 - \lambda_s} \langle y, \alpha_s \rangle \frac{1}{\alpha_s} \otimes (s - 1)m$$

for $y \in \mathfrak{h}$ and $m \in M$. According to Proposition 1 and the above construction, $M$ and $\mathcal{O}(U) \otimes \mathcal{O}(\mathfrak{h}) M$ induce the same $N_{W''}(W'')$-equivariant $D$-modules on $\mathfrak{h}_{\text{reg}}$. We will construct an $N \in H_c(W', \mathfrak{h})$-mod$_{\text{RS}}$ such that

$$\mathcal{O}(U) \otimes \mathcal{O}(\mathfrak{h}) N \cong \mathcal{O}(U) \otimes \mathcal{O}(\mathfrak{h}) M.$$
as \( H_c(W', U) \)-modules. This implies that
\[
\hat{O}_{h,b} \otimes O(h) N \cong \hat{O}_{h,b} \otimes O(U) (O(U) \otimes O(h) M)
\]
as \( H_c(W', \text{Spf } \hat{O}_{h,b}) \)-modules. By definition of \( \text{Res}_{h} \), this ensures that \( \text{Res}_{h} M \cong N \). Since \( O(U) \otimes O(h) N \cong O(U) \otimes O(h) M \), the \( N_{W'}(W'') \)-equivariant \( D \)-modules on \( h_{reg}'' \) induced by \( O(U) \otimes O(h) M \) is the restriction of that on \( h_{reg} \) induced by \( N \). The result follows.

It remains to construct \( N \). Let \( P \) be the set of subgroups \( W'' \) of \( W' \) for which \( h_{W''} \) is an irreducible component of \( \text{Supp} M \). Let
\[
Z = \bigcup_{W'' \in P} h_{W''}.
\]
Any component of \( \text{Supp} M \) not contained in \( Z \) is contained in \( h \setminus U \), so the support of \( O(U) \otimes O(h) M \) is \( Z \cap U \). By (1) of Theorem 3.8, we may replace \( M \) by a quotient of \( M \) and suppose that the support of any nonzero \( m \in \Gamma(U, \text{Sh}(M)) \) intersects \( h_{W''} \) for some \( W'' \in P \). Since \( M_{|h_{W''}} \) is locally free, any such support would then contain \( h_{W''} \).

By assumption, if \( W'' \in P \) then the coherent \( D \)-module \( M_{|h_{W''}} \) has trivial monodromy around each hyperplane in \( \hat{h}_{W''} \setminus h_{W''} \). Therefore there is a coherent \( D \)-module \( N(W'') \) on \( h_{W''} \) whose restriction to \( h_{W''} \) is \( M_{|h_{W''}} \). Patching these with \( O(U) \otimes O(h) M \), we obtain a coherent sheaf \( N \) on the open subset
\[
U' = (U \cap Z) \cup \bigcup_{W'' \in P} h_{W''}
\]
of \( Z \). The support of any nonzero \( n \in N = \Gamma(U', N) \) contains \( h_{W''} \) for some \( W'' \in P \), and therefore contains some component of \( U' \). Moreover for \( W'' \in P \),
\[
h_{W''} \setminus U' = \bigcup_{w_y \in h_{W''} \setminus W'} \{ y \in h_{W''} \mid w_y y = y = w_{2y} \}
\]
has codimension at least 2 in \( h_{W''} \). We may therefore apply Lemma 3.6 to conclude that \( N \) is finitely generated over \( O(Z) \). We may identify \( N \) with the submodule of \( O(U) \otimes O(h) M \) consisting of elements whose restriction to \( h_{W''} \) has no pole around \( Z(\alpha_s) \), for each \( W'' \in P \) and \( s \in S \setminus W'' \). This condition is preserved by the action of \( H_c(W', h) \), so \( N \in H_c(W', h)^{\mod RS} \). Finally
\[
O(U) \otimes O(h) N = \Gamma(U \cap Z, N) = O(U) \otimes O(h) M,
\]
as required.

3.4. Type A. Now let \( W = S_n \), the symmetric group \( S_n \) on \( n \) letters, acting on \( h = \mathbb{C}^n \) by permuting coordinates. The reflections in this case are transpositions. As they are all conjugate, the function \( c : S \to \mathbb{C} \) must be constant, and we identify it with its value in \( \mathbb{C} \). Let \( h/\mathbb{C} \) denote the quotient of \( h \) by the line fixed by \( W \). In this case we have the following simple criterion for when \( H_c(W, h/\mathbb{C}) \) admits a finite dimensional representation.

**Theorem 3.8** (3 Theorem 1.2). Suppose \( n > 1 \). The algebra \( H_c(W, h/\mathbb{C}) \) admits a nonzero finite dimensional representation if and only if \( c = \frac{r}{n} \) for some integer \( r \) coprime with \( n \). In this case the category of finite dimensional modules is semisimple with one irreducible. Moreover if \( c = \frac{1}{n} \), this irreducible is one dimensional.

Using this with Theorem 1.4(3) gives the following (see Example 3.25 of [3]).
**Theorem 3.9.** Suppose \( c = \frac{r}{m} \), where \( r \) and \( m \) are integers with \( m \) positive and coprime with \( r \). For each nonnegative integer \( q \leq n/m \), let

\[
X'_q = \left\{ b \in h \mid b_i = b_j \text{ whenever } \left\lfloor \frac{i}{m} \right\rfloor = \left\lfloor \frac{j}{m} \right\rfloor \leq q \right\} \quad \text{and} \quad X_q = \bigcup_{w \in W} wX'_q.
\]

Then any module in \( H_c(W, h) - \text{mod}_{coh} \) is supported on one of the \( X_q \). If \( c \) is irrational then every such module has full support.

We would like to determine more explicitly which irreducibles in \( H_c - \text{mod}_{RS} \) have which support sets. The irreducible representations of \( W = S_n \) are well known to be parameterised by partitions of \( n \). Given a partition \( \lambda \vdash n \), the corresponding irreducible is denoted \( \tau_\lambda \) and called the **Specht module** indexed by \( \lambda \). We will represent a partition \( \lambda \vdash n \) as a nonincreasing sequence of nonnegative integers, \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \), whose sum is \( n \), where two sequences are identified if their nonzero entries agree. If \( \lambda \vdash n \) and \( \mu \vdash m \), we may obtain a partition \( \lambda + \mu = (\lambda_1 + \mu_1, \lambda_2 + \mu_2, \ldots) \) of \( n + m \). We first prove a lemma about the induction functor introduced in Theorem 3.3.

**Lemma 3.10.** Suppose \( c \) is a positive real number, and suppose we have partitions \( \lambda \vdash n \) and \( \mu \vdash m \). Let \( b \) be a point in \( \mathbb{C}^{n+m} \) whose stabiliser in \( S_{n+m} \) is \( S_n \times S_m \). Then \( \text{Ind}_b(L(\tau_\lambda) \otimes L(\tau_\mu)) \) admits a nonzero map from \( M(\tau_{\lambda+\mu}) \).

**Proof.** We first prove

**Claim 1:** the lowest order term in \( \text{Ch}[\text{Ind}_b(M(\tau_\lambda) \otimes M(\tau_\mu))] \) is \( [\tau_{\lambda+\mu}]^{\text{h}(\tau_{\lambda+\mu})} \).

Indeed, the construction of Verma modules shows that \( M(\tau_\lambda) \otimes M(\tau_\mu) \cong M(\tau_\lambda \otimes \tau_\mu) \), so Theorem 3.3(4) implies that

\[
\text{Ch}[\text{Ind}_b(M(\tau_\lambda) \otimes M(\tau_\mu))] = \text{Ch}[M(\text{Ind}(\tau_\lambda \otimes \tau_\mu))].
\]

The Littlewood-Richardson rule describes how \( \text{Ind}(\tau_\lambda \otimes \tau_\mu) \) splits into Specht modules for \( S_{n+m} \). In particular, given partitions \( \alpha \) and \( \beta \) of \( n + m \), say \( \alpha \) dominates \( \beta \), denoted \( \alpha \geq \beta \), if

\[
\sum_{i=1}^p \alpha_i \geq \sum_{i=1}^p \beta_i
\]

for all \( p \geq 1 \). Then

\[
[\text{Ind}(\tau_\lambda \otimes \tau_\mu)] = [\tau_{\lambda+\mu}] + \sum_{\nu \leq \lambda+\mu} c^\nu_{\lambda\mu}[\tau_{\nu}]
\]

for some coefficients \( c^\nu_{\lambda\mu} \in \mathbb{Z} \). Certainly \( \text{Ch}[M(\tau_{\lambda+\mu})] \) has the required lowest order term, so it suffices to prove that \( h(\tau_\alpha) < h(\tau_\beta) \) whenever \( \alpha > \beta \). Recall that \( h(\tau_\alpha) \) is the action of

\[
-c \sum_{i \neq j} s_{ij}
\]

on \( S(\alpha) \). By the Frobenious character formula (see Exercise 4.17(c) of [14]), this is

\[
h(\tau_\alpha) = -c \sum_{i \geq 1} \frac{1}{2} \alpha_i^2 - \left( i - \frac{1}{2} \right) \alpha_i.
\]

Using the Abel summation formula,

\[
h(\tau_\alpha) - h(\tau_\beta) = -c \sum_{i \geq 1} (\alpha_i - \beta_i) \left( \frac{\alpha_i + \beta_i + 1}{2} - i \right)
\]

\[
= -c \sum_{i \geq 1} \left( \sum_{j=1}^{i} (\alpha_j - \beta_j) \right) \left( \frac{\alpha_i - \alpha_{i+1} + \beta_i - \beta_{i+1}}{2} + 1 \right)
\]

\(< 0,
\]
if $\alpha > \beta$. Here we have used that $\sum_{j=1}^{i} (\alpha_j - \beta_j) = 0$ for $i$ sufficiently large. This proves Claim 1. From this we will deduce

**Claim 2:** the lowest order term in $\text{Ch}[\text{Ind}_b(L(\tau_\lambda) \otimes M(\tau_\mu))]$ is $[\tau_{\lambda+\mu}]^{h(\tau_{\lambda+\mu})}$.

We prove Claim 2 by descending induction on $h(\lambda)$; that is, suppose it holds for all pairs $(\nu, \mu)$ with $h(\nu) > h(\lambda)$. Of course this assumption is vacuous when $h(\lambda)$ is maximal, so we need not prove the base case. We have

$$[M(\tau_\lambda)] = [L(\tau_\lambda)] + \sum_{h(\nu) > h(\lambda)} d^\nu_\lambda [L(\tau_\nu)]$$

for some nonnegative integers $d^\nu_\lambda$. Tensoring by $M(\tau_\mu)$ and applying $\text{Ind}_b$ and $\text{Ch}$, we obtain

$$\text{Ch}[\text{Ind}_b(M(\tau_\lambda) \otimes M(\tau_\mu))] = \text{Ch}[\text{Ind}_b(L(\tau_\lambda) \otimes M(\tau_\mu))] + \sum_{h(\nu) > h(\lambda)} d^\nu_\lambda \text{Ch}[\text{Ind}_b(L(\tau_\nu) \otimes M(\tau_\mu))].$$

By Claim 1, the lowest order term in this expression is $[\tau_{\lambda+\mu}]^{h(\tau_{\lambda+\mu})}$. However, each summand on the right has nonnegative integer coefficients. Therefore this must be the lowest order term of one of the summands. For $h(\nu) > h(\lambda)$, the lowest order term of $\text{Ch}[\text{Ind}_b(L(\tau_\nu) \otimes M(\tau_\mu))]$ is $[\tau_{\nu+\mu}]^{h(\tau_{\nu+\mu})}$ by induction. Clearly if $\nu \neq \lambda$ then $\nu + \mu \neq \lambda + \mu$, so the term $[\tau_{\lambda+\mu}]^{h(\tau_{\lambda+\mu})}$ can only come from $\text{Ch}[\text{Ind}_b(L(\tau_\lambda) \otimes M(\tau_\mu))]$, as required. We now conclude

**Claim 3:** the lowest order term in $\text{Ch}[\text{Ind}_b(L(\tau_\lambda) \otimes L(\tau_\mu))]$ is $[\tau_{\lambda+\mu}]^{h(\tau_{\lambda+\mu})}$.

Indeed this follows from Claim 2 using the same argument by which Claim 2 followed from Claim 1. This proves that the $h(\tau_{\lambda+\mu})$-eigenspace of $\text{eu}$ in $\text{Ind}_b(L(\tau_\lambda) \otimes L(\tau_\mu))$ is isomorphic to $\tau_{\lambda+\mu}$ as an $S_{n+m}$-module, and is killed by $h \leq H_c(S_{n+m}, h)$. We therefore have a nonzero $\mathbb{C}[S_{n+m}] \ltimes \mathbb{C}[h^*]$-module homomorphism

$$\tau_{\lambda+\mu} \to \text{Ind}_b(L(\tau_\lambda) \otimes L(\tau_\mu)),$$

where $h$ is defined to act as 0 on $\tau_{\lambda+\mu}$. By definition of the Verma module, we obtain a nonzero map

$$M(\tau_{\lambda+\mu}) \to \text{Ind}_b(L(\tau_\lambda) \otimes L(\tau_\mu)),$$

as required. \qed

Now suppose $c = \frac{r}{m}$ where $r$ is coprime with $m$. We say $\lambda$ is $m$-regular if no part of $\lambda$ appears $m$ or more times. Any partition $\lambda \vdash n$ can be written uniquely as $\lambda = m\mu + \nu$ such that $\nu'$ is $m$-regular. Let $q_m(\lambda) = |\mu|$. More explicitly

$$q_m(\lambda) = \sum_{i \geq 1} \left\lfloor \frac{\lambda_i - \lambda_{i+1}}{m} \right\rfloor.$$

We will eventually show, in Theorem 1.6, that the support of $L(\tau_\lambda)$ is $X_{q_m(\lambda)}$ for $c > 0$ and $X_{q_m(\lambda')}$ for $c < 0$. For the moment we prove one direction:

**Theorem 3.11.** With $c = \frac{r}{m}$ and $h = \mathbb{C}^n$ as above, the support of the $H_c$-module $L(\tau_\lambda)$ is contained in $X_{q_m(\lambda)}$ if $c > 0$, and contained in $X_{q_m(\lambda')}$ if $c < 0$.

**Proof.** We denote $X_q$ by $X^n_q$ throughout this proof, as we will be considering support sets for other values of $n$. Suppose $c > 0$.

Let $q = q_m(\lambda)$, and write $\lambda = m\mu + \nu$ for some partitions $\mu \vdash q$ and $\nu \vdash n - qm$, such that $q_m(\nu) = 0$. By Proposition 9.7 and Theorem 9.8 of [3], the support of $L(\tau_{mq})$ is contained in $X^n_{qm}$. Thus the support of $L(\tau_{mq}) \otimes L(\tau_\nu)$ is contained in $X^n_{qm} \times \mathbb{C}^{n-qm} \subseteq X^n_q$. By Theorem 3.3, the same is true of $\text{Ind}_b(L(\tau_{mq}) \otimes L(\tau_\nu))$, where $b \in \mathbb{C}^n$ is a point whose stabiliser in $S_q$ is $S_{qm} \times S_{n-qm}$. By Lemma 3.10, we have a nonzero map

$$\phi : M(\tau_\lambda) \to \text{Ind}_b(L(\tau_{mq}) \otimes L(\tau_\nu)).$$
Now $L(\tau_\lambda)$ is the only irreducible quotient of $M(\tau_\lambda)$, so the image of $\phi$ admits $L(\tau_\lambda)$ as a quotient. Thus $L(\tau_\lambda)$ is a subquotient of $\text{Ind}_q(L(\tau_{\mu})) \otimes L(\tau_\nu)$, so its support must be contained in $X_q^n$. This proves the $c > 0$ case.

There is an automorphism of $\mathbb{C}[W]$ sending $s \in S$ to $-s$. Twisting by this automorphism sends $\tau_\lambda$ to $\tau_\lambda$. Moreover it extends to an isomorphism $H_c(W, \mathfrak{h}) \to H_{-c}(W, \mathfrak{h})$, which is the identity on $\mathfrak{h}$ and $\mathfrak{h}^\ast$. Therefore the statement for $c < 0$ follows from that for $c > 0$.

Finally we will require the following classification of two-sided ideals in $H_c(W, \mathfrak{h})$, due to Losev.

**Theorem 3.12** ([17] Theorem 4.3.1 and [12] Theorem 5.10). There are $[n/m] + 1$ proper two-sided ideals of $H_c(W, \mathfrak{h})$,

$$0 = J_0 \subseteq J_1 \subseteq \cdots \subseteq J_{[n/m]} \subseteq H_c(W, \mathfrak{h}).$$

Moreover if $c > 0$ then the polynomial representation admits a filtration

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_{[n/m]} \subseteq \mathbb{C}[\mathfrak{h}]$$

such that $Z(I_q) = X_q$ and $\text{Ann}_{H_c(W, \mathfrak{h})}(\mathbb{C}[\mathfrak{h}]/I_q) = J_q$. Each $I_q$ is radical if and only if $c = \frac{1}{m}$.

4. **Minimal Support for Type A**

In this section we consider the algebra $H_c = H_{-c}(S_n, \mathbb{C}^n)$, and study modules in $H_c$-mod$\mathfrak{RS}$ whose support is the smallest possible set, namely $X_q$ where $q = [n/m]$. In particular, we will show that the full subcategory of such modules is semisimple. We begin with a general lemma concerning the localisation functor for linear actions.

**Lemma 4.1.** Consider a finite group $W$ acting linearly on a finite dimensional vector space $\mathfrak{h}$. Suppose $\alpha \in \mathbb{C}[\mathfrak{h}]^W$ is a symmetric polynomial, and let $U \subseteq \mathfrak{h}$ denote the open set on which $\alpha$ is nonzero. Moreover suppose no nonzero module in $H_c(W, \mathfrak{h})$-mod$\mathfrak{RS}$ is supported on the zero set of $\alpha$. Then the localisation functor

$$L : H_c(W, \mathfrak{h})$`-mod$\mathfrak{coh} \to H_c(W, U)$`-mod$\mathfrak{coh}$

identifies $H_c(W, \mathfrak{h})$-mod$\mathfrak{RS}$ with a full subcategory of $H_c(W, U)$-mod$\mathfrak{coh}$ closed under subquotients.

**Proof.** Let $\mathcal{A}$ denote the full subcategory of $H_c(W, U)$-mod$\mathfrak{coh}$ consisting of modules $M$ such that every $m \in M$ is killed by $\mathfrak{h}^n\alpha^n$ for some $n$. This is clearly closed under subquotients.

Certainly $L$ is exact, since $\mathbb{C}[\mathfrak{h}]$ is flat over $\mathbb{C}[\mathfrak{h}]$, and its image lies in $\mathcal{A}$. Conversely suppose $M \in \mathcal{A}$, and let $V \subset M$ be a finite dimensional subspace generating $M$ over $\mathbb{C}[\mathfrak{h}]$. Multiplying $V$ by some power of $\alpha$, we may suppose that $\mathfrak{h}$ acts nilpotently on $V$. Since $H_c(W, \mathfrak{h}) = \mathbb{C}[\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[W] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{h}^\ast]$, the $H_c(W, \mathfrak{h})$-submodule $N$ of $M$ generated by $V$ is finitely generated over $\mathbb{C}[\mathfrak{h}]$, and $\mathfrak{h}$ acts locally nilpotently on $N$. Thus $N \in H_c(W, \mathfrak{h})$-mod$\mathfrak{RS}$. Moreover $M$ is generated by $N$ over $\mathbb{C}[\mathfrak{h}]$, so $M \cong L(N)$ is in the image of the localisation functor. Therefore $\mathcal{A}$ is exactly the image of $L$.

Define the functor $E : H_c(W, U)$-mod $\to H_c(W, \mathfrak{h})$-mod sending $M$ to the subspace $E(M) \subset M$ on which $\mathfrak{h}$ acts nilpotently. Note that $E(M)$ is stable under the action of $H_c(W, \mathfrak{h})$. We will show that if $M \in H_c(W, \mathfrak{h})$-mod$\mathfrak{RS}$, then $E(L(M))$ is naturally isomorphic to $M$. The kernel of the natural map $M \to L(M)$ is exactly the submodule $\Gamma_{Z(\alpha)}(M)$ defined in Theorem 3.4(1). This is zero since we have assumed no modules are supported on $Z(\alpha)$. We may therefore identify $M$ with a submodule of $E(L(M))$. Consider any $v \in E(L(M))$. As above, the $H_c(W, \mathfrak{h})$-submodule $N$ of $L(M)$ generated by $v$ is in $H_c(W, \mathfrak{h})$-mod$\mathfrak{RS}$. Thus $M' = M + N \subseteq L(M)$ is also in $H_c(W, \mathfrak{h})$-mod$\mathfrak{RS}$, and
Suppose a unital associative algebra $L$. Using the exactness of localisation, we conclude that $L(M'/M) = 0$, so that $M'/M$ is supported on $Z(\alpha)$. Again this implies that $M'/M = 0$, so $v \in M$. Thus $M$ is exactly $EL(M)$. We have shown above that the localisation functor $L : H_c(W, h) - \text{mod}_{RS} \to A$ is essentially surjective, so $E$ and $L$ are mutually inverse functors. 

We also require the following well-known result.

**Lemma 4.2.** Suppose a unital associative algebra $H$ contains an idempotent $e$ such that $HeH = H$. Then the functors

\[ \text{H} - \text{mod} \to eHe - \text{mod}, \quad M \mapsto eM \]

and

\[ eHe - \text{mod} \to H - \text{mod}, \quad N \mapsto He \otimes eHe N \]

are mutually inverse equivalences.

Now consider the algebra $H_c = H_{\frac{1}{m}}(S_n, \mathbb{C}^n)$. Let

\[ \mathfrak{h} = \mathbb{C}^n = \text{Spec} \mathbb{C}[x_1, \ldots, x_n], \]

where $S_n$ permutes the $x_i$. Let $q = \lceil n/m \rceil$ and write $n = qm + p$, so that $0 \leq p < m$. Let $\mathfrak{h}' = \mathbb{C}^{q+p}$, with coordinate ring $\mathbb{C}[\mathfrak{h}'] = \mathbb{C}[z_1, \ldots, z_q, t_1, \ldots, t_p]$, on which $S_q \times S_p$ acts naturally. Define

\[ \pi : \mathbb{C}[\mathfrak{h}] \to \mathbb{C}[\mathfrak{h}'] \]

by

\[ \pi(x_i) = \begin{cases} z_{\lceil i/m \rceil} & \text{if } i \leq qm \\ t_{i-qm} & \text{if } i > qm. \end{cases} \]

This identifies $\mathfrak{h}'$ with one of the components of $X_q \subseteq \mathfrak{h}$. The map $\pi$ restricts to a map $\pi : \mathbb{C}[\mathfrak{h}]^{S_n} \to \mathbb{C}[\mathfrak{h}']^{S_q \times S_p}$. Unfortunately this is not surjective if $p > 0$. We therefore localise as follows. Let $[n] = \{1, 2, \ldots, n\}$, and let

\[ \alpha_d = \sum_{j \in [n]} \left( \sum_{j \in J} x_j^d \right) \prod_{j \in J, i \notin J} (x_i - x_j). \]

Clearly $\alpha_d$ is symmetric. Moreover using that $p < m$, it can be shown that

\[ \pi(\alpha_d) = \left( \sum_{j=1}^{r} t_j^d \right) \prod_{1 \leq i \leq q} (z_i - t_j)^m. \]

Let $U \subseteq \mathfrak{h}$ and $U' \subseteq \mathfrak{h}'$ denote the affine open subsets on which $\alpha_0$ and $\pi(\alpha_0)$ are nonzero, respectively. Let $A = \mathcal{O}(U)^{S_n}$ and $B = \mathcal{O}(U')^{S_q \times S_p}$, and let $\phi : A \to B$ be the map induced by $\pi$. Now $B$ is generated as a $\mathbb{C}$-algebra by $\pi(\alpha_0)^{-1}$, $\sum_{j=1}^{p} t_j^d$ and $\sum_{i=1}^{q} z_i^d$. We have

\[ \sum_{j=1}^{p} t_j^d = \phi \left( \frac{\alpha_d}{\alpha_0} \right) \quad \text{and} \quad \sum_{i=1}^{q} z_i^d = \phi \left( \frac{1}{m} \sum_{i=1}^{n} x_i^d - \frac{p}{m} \frac{\alpha_d}{\alpha_0} \right), \]

so $\phi$ is surjective. Moreover since $X_q$ is the union of the $S_n$ translates of $\mathfrak{h}'$, the kernel of $\phi$ is $(\mathcal{O}(U)I_q)^{S_n}$, where $I_q$ is the ideal vanishing on $X_q$. 


Now consider the idempotents
\[
e = \frac{1}{n!} \sum_{w \in S_n} w \in \mathbb{C}[S_n] \subseteq H_c(S_n, \mathfrak{h}) \subseteq H_c(S_n, U),
\]
\[
e' = \frac{1}{d!p!} \sum_{w \in S_q \times S_p} w \in \mathbb{C}[S_q \times S_p] \subseteq H_{(m,c)}(S_q \times S_p, \mathfrak{h}') \subseteq H_{(m,c)}(S_q \times S_p, U),
\]
where \((m, c)\) indicates the function which takes the value \(m\) on transpositions in \(S_q\) and \(c\) on those in \(S_p\). For convenience, we omit the subscripts \(c\) and \((c, m)\) for the rest of this section. It is known (see Corollary 4.2 of [11]) that \(H(S_n, \mathfrak{h})eH(S_n, \mathfrak{h}) = H(S_n, \mathfrak{h})\), so that \(H(S_n, U)eH(S_n, U) = H(S_n, U)\), and \(M \mapsto eM\) is an equivalence of categories from \(H(S_n, U)\)-mod to \(eH(S_n, U)\)-mod by Lemma 1.2. Similar results hold for \(e'\).

In particular the faithful action of \(eH(S_q \times S_p, U')\)-mod to \(eH(S_q \times S_p, U)\)-mod by Lemma 1.2. Similar results hold for \(e'\).

**Proposition 4.3.** The image of the homomorphism \(\sigma : eH(S_n, U)e \to \text{End}_{\mathbb{C}}(B)\) describing the above action is exactly \(e'H(S_q \times S_p, U')e' \subseteq \text{End}_{\mathbb{C}}(B)\), and the kernel is generated by \(eI_q e\).

**Proof.** Let \(\{x_\alpha^i\}^n\) and \(\{z_i^\vee, t_j^\vee\}\) be the bases of \(\mathfrak{h}\) and \(\mathfrak{h}'\) dual to \(\{x_\alpha\}\) and \(\{z_i, t_j\}\). For \(d \geq 0\), let
\[
p_x(d) = \sum_{a=1}^n x_a^d,
\]
and similarly for \(p_t, p_z, p_{x^\vee}, p_{z^\vee}, p_{t^\vee}\). It is known that \(eH(S_n, \mathfrak{h})e\) is generated as an algebra by \(e\mathbb{C}[\mathfrak{h}]e\) and \(p_{x^\vee}(2)\) (see the proof of Proposition 4.9 of [11], and Corollary 4.9 of [2]). It follows that \(eH(S_n, U)e\) is generated by \(Ae\) and \(p_{x^\vee}(2)e\), and that
\[
e'H(S_q \times S_p, U')e'
\]
is generated by \(Be', \ p_{x^\vee}(2)e'\) and \(p_{t^\vee}(2)e'\). For \(f, g \in A\), we have
\[
\sigma(fe)\phi(g) = \phi(fg) = \phi(f)\phi(g),
\]
so the image under \(\sigma\) of \(Ae \subseteq eH(S_n, U)e\) is \(Be' \subseteq e'H(S_q \times S_p, U')e'\). Next, recalling that \(\phi : A \to B\) is the restriction of \(\pi : \mathcal{O}(U) \to \mathcal{O}(U')\), we show that
\[
\pi(\partial_{x_a^i} f) = \begin{cases} 
\frac{1}{m} \partial_{z_i^\vee} \phi(f) & \text{if } \pi(x_a) = z_i, \\
\partial_{t_j^\vee} \phi(f) & \text{if } \pi(x_a) = t_j
\end{cases}
\]
for \(f \in A\). (1)

Indeed, this holds when \(f = p_x(d)\), since
\[
\frac{1}{m} \partial_{z_i^\vee} \phi(p_x(d)) = \frac{1}{m} \partial_{z_i^\vee} (mp_x(d) + pt(d)) = dz_i^{d-1}
\]
and
\[
\partial_{t_j^\vee} \phi(p_x(d)) = \partial_{t_j^\vee} (mp_x(d) + pt(d)) = dt_j^{d-1}.
\]
Now as functions of \(f\), both sides of (1) are \(\mathbb{C}\)-derivations from \(A\) to \(\mathcal{O}(U')\). Therefore (1) holds on the subring generated by the \(p_x(d)\), namely \(\mathbb{C}[\mathfrak{h}]^{S_n}\), whence it holds on
\( \mathbb{C}[h]^{S_n}[\alpha_0^{-1}] = A \). Now for \( f \in A \) we have

\[
p_{x^\vee}(2)f = \sum_{a=1}^{n} x_a^\vee \partial_{x_a^\vee} f
\]

\[
= \sum_{a=1}^{n} \left( \partial_{x_a^\vee}^2 f - c \sum_{b \neq a} \frac{\partial_{x_a^\vee} f - s_{ab} \partial_{x_b^\vee} f}{x_a - x_b} \right)
\]

\[
= \Delta_x f - c \sum_{a \neq b} \frac{\partial_{x_a^\vee} f - \partial_{x_b^\vee} f}{x_a - x_b},
\]

where \( \Delta_x \) denotes the Laplacian in the variables \( x_a \). Similar formulae hold for the actions of \( p_{x^\vee}(2) \) and \( p_{t^\vee}(2) \) on \( B \). Define \( P \in e^i H(S_q \times S_p, U')e' \) by

\[
P = \frac{1}{m} p_{x^\vee}(2)e' + p_{t^\vee}(2)e' - 2 \sum_{1 \leq i < j \leq p} \frac{1}{z_i - t_j} \left( \frac{1}{m} z_i^\vee - t_j^\vee \right) e'.
\]

Note that

\[
\sum_{1 \leq i < j \leq p} \frac{1}{z_i - t_j} \left( \frac{1}{m} z_i^\vee - t_j^\vee \right)
\]

is symmetric under \( S_q \times S_p \), so final sum is an element of \( e^i H(S_q \times S_p, U')e' \), though the individual summands are not. We claim that

\[
\phi(p_{x^\vee}(2)f) - P\phi(f) = 0
\]

for \( f \in A \). We first show that the left hand side is a derivation. Indeed

\[
\phi(p_{x^\vee}(2)(fg)) = \phi \left( fp_{x^\vee}(2)g + gp_{x^\vee}(2)f + \sum_a (\partial_{x_a^\vee} f)(\partial_{x_a^\vee} g) \right)
\]

\[
= \phi(f)\phi(p_{x^\vee}(2)g) + \phi(g)\phi(p_{x^\vee}(2)f)
\]

\[
+ \sum_i \frac{1}{m} (\partial_{z_i^\vee} \phi(f))(\partial_{z_i^\vee} \phi(g)) + \sum_j (\partial_{t_j^\vee} \phi(f))(\partial_{t_j^\vee} \phi(g)),
\]

using (\text{II}). Also

\[
P(fg) = fP(g) + gP(f) + \sum_i \frac{1}{m} (\partial_{z_i^\vee} f)(\partial_{z_i^\vee} g) + \sum_j (\partial_{t_j^\vee} f)(\partial_{t_j^\vee} g).
\] (2)

Subtracting, we see that \( \phi(p_{x^\vee}(2)f) - P\phi(f) \) is indeed a derivation in \( f \). By the same argument as above, it suffices to check the equation when \( f = p_x(d) \). We calculate

\[
\phi(p_{x^\vee}(2)p_x(d)) = \phi \left( d(d-1)p_x(d-2) - cd \sum_{a \neq b} \frac{x_a^{d-1} - x_b^{d-1}}{x_a - x_b} \right).
\]

If \( \pi(x_a) = \pi(x_b) = z_i \), then

\[
\pi \left( \frac{x_a^{d-1} - x_b^{d-1}}{x_a - x_b} \right) = \pi \left( \sum_{i=0}^{d-2} x_a^i x_b^{d-2-i} \right) = (d-1)z_i^{d-2}.
\]
Therefore recalling that \( c = \frac{1}{m} \),
\[
\phi(p_{\nu'}(2)p_{\nu}(d)) = d(d-1)(mp_{\nu} (d - 2) + p_{t}(d - 2)) - cd(d-1)m(m-1)p_{\nu} (d - 2)
- cdm^2 \sum_{i \neq i'} \frac{z_{i}^{d-1} - z_{i'}^{d-1}}{z_{i} - z_{i'}} - 2cdm \sum_{i,j} \frac{z_{i}^{d-1} - t_{i}^{d-1}}{z_{i} - t_{j}},
\]
\[
= \left[ d(d-1)p_{\nu} (d - 2) - md \sum_{i \neq i'} \frac{z_{i}^{d-1} - z_{i'}^{d-1}}{z_{i} - z_{i'}} \right]
+ \left[ d(d-1)p_{t} (d - 2) - cd \sum_{j \neq j'} \frac{t_{j}^{d-1} - t_{j'}^{d-1}}{t_{j} - t_{j'}} \right]
- 2d \sum_{i,j} \frac{z_{i}^{d-1} - t_{j}^{d-1}}{z_{i} - t_{j}}
= p_{\nu'}(2)p_{\nu} (d) + p_{\nu'}(2)p_{t} (d) - 2 \sum_{i,j} \frac{d_{i}^{d-1} - d_{j}^{d-1}}{z_{i} - t_{j}}
= P(mp_{\nu} (d) + p_{t} (d))
= P(\phi(p_{\nu}(d)),
\]
as required. Therefore \( \text{Im} \, \sigma \) is the subalgebra of \( \text{End}_{C}(B) \) generated by \( Be' \) and \( P \). Now (2) shows that for \( fe' \in Be' \subseteq e'H(S_{q} \times S_{p}, U')e' \), we have
\[
[P, fe'] = P(f)e' + \sum_{i} \frac{1}{m} (\partial_{i} f) z_{i}^{\nu} e' + \sum_{j} (\partial_{j} f) t_{j}^{\nu} e'.
\]
In particular,
\[
[P, \pi(a_{0})e'] = P(\pi(a_{0}))e' + \sum_{1 \leq i \leq q, 1 \leq j \leq p} m\pi(a_{0}) \left( \frac{1}{m} z_{i}^{\nu} - t_{j}^{\nu} \right) e'.
\]
Therefore \( \text{Im} \, \sigma \) contains
\[
P + \frac{2}{m\pi(a_{0})} ([P, \pi(a_{0})e'] - P(\pi(a_{0}))e') = \frac{1}{m} p_{\nu'}(2)e' + p_{\nu'}(2)e'.
\]
We also have
\[
[P, p_{\nu} (2)e'] = P(p_{\nu} (2))e' + \frac{2}{m} \sum_{i} z_{i}^{\nu} e',
\]
so \( \text{Im} \, \sigma \) contains
\[
\left[ \frac{1}{m} p_{\nu'}(2)e' + p_{\nu'}(2)e', \sum_{i} z_{i}^{\nu} e' \right] = \frac{2}{m} p_{\nu'}(2)e'.
\]
Thus \( \text{Im} \, \sigma \) contains \( Be', p_{\nu'}(2)e' \) and \( p_{\nu'}(2)e' \), so it is exactly \( e'H(S_{q} \times S_{p}, U')e' \).

Finally since \( H(S_{n}, U)e'H(S_{n}, U) = H(S_{n}, U) \), the two-sided ideals of the algebra \( e'H(S_{n}, U)e \) are in one to one correspondence with those in \( H(S_{n}, U) \). The latter are determined by their inverse image in \( H(S_{n}, \mathfrak{h}) \). Clearly the kernel of \( \sigma \) is proper and contains \( eI_{q}e \), so by Theorem 3.12 it is generated by \( eI_{q}e \). \( \square \)

We have identified \( \mathfrak{h}' \) with a subspace of \( \mathfrak{h} \), and the stabiliser of a generic point of \( \mathfrak{h}' \) in \( S_{n} \) is a natural copy of \( S_{n}' \subseteq S_{n} \). Let \( \mathfrak{h}'_{\text{reg}} \) denote the elements of \( \mathfrak{h}' \) with exactly this stabiliser. Note that \( \mathfrak{h}'_{\text{reg}} \) is contained in \( U \), since the zero set of
\[
\pi(a_{0}) = p \prod_{1 \leq i \leq q} (z_{i} - t_{j})^{m}
\]
consists of elements whose stabiliser in $S_n$ is at least as large as $S_{m+1} \times S_m^{q-1}$. Also note that the normaliser subgroup $N_{S_n}(S_m^n) \subseteq S_n$ acts on $h_{\text{reg}}'$. Each coset of $S_m^n$ in $N_{S_n}(S_m^n)$ has a unique representative of minimal length, and these representatives form a subgroup isomorphic to $S_q \times S_p$. The induced action of $S_q \times S_p$ on $h' = \text{Spec } \mathbb{C}[z_1, \ldots, z_q, t_1, \ldots, t_p]$ is the natural one. We now use the homomorphism $\sigma$ to analyse the subcategory of $H(S_n, h) - \text{mod}_R^q$ of minimally supported modules. Some of this information is generalised by Theorems 1.11 and 1.15.

**Theorem 4.4.** Let $H(S_n, h) - \text{mod}_R^q$ be the full subcategory of $H(S_n, h) - \text{mod}_R$, consisting of modules supported on $X_q$. There is an equivalence of categories

$$F : H(S_n, h) - \text{mod}_R^q \rightarrow H(S_q \times S_p, h') - \text{mod}_R,$$

such that $O(h_{\text{reg}}') \otimes_{\mathbb{C}[q]} M \cong O(h_{\text{reg}}') \otimes_{\mathbb{C}[q]} F(M)$ as $\mathbb{C}[S_q \times S_p] \times D(h_{\text{reg}}')$-modules. In particular, $H(S_n, h) - \text{mod}_R^q$ is semisimple, and its irreducibles are exactly the $L(\tau)$ for which $q_m(\lambda) = q$.

**Proof.** Let $eH(S_n, U)e - \text{mod}_{\text{coh}}$ denote the full subcategory of $eH(S_n, U)e - \text{mod}$ consisting of modules finitely generated over $e\mathbb{C}[U]e$, and similarly for $H(S_q \times S_p, U')$. Recall the equivalences $H(S_n, U) - \text{mod}_{\text{coh}} \cong eH(S_n, U)e - \text{mod}_{\text{coh}}$ and $H(S_q \times S_p, U') - \text{mod}_{\text{coh}} \cong e'H(S_q \times S_p, U')e' - \text{mod}_{\text{coh}}$; we denote the functors by $e$ and $e'$. By Lemma 1.1 we also have localisation functors $L : H(S_n, h) - \text{mod}_R \rightarrow H(S_n, h) - \text{mod}_{\text{coh}}$ and $L' : H(S_q \times S_p, h') - \text{mod}_R \rightarrow H(S_q \times S_p, h') - \text{mod}_{\text{coh}}$ which identify their domains with full subcategories of their codomains closed under subquotients. Finally the previous proposition gives a functor $\sigma^*$ identifying $e'H(S_q \times S_p, U')e' - \text{mod}_{\text{coh}}$ with a full subcategory of $eH(S_n, U)e - \text{mod}_{\text{coh}}$, again closed under subquotients.

$$\begin{align*}
H(S_n, h) - \text{mod}_R & \xrightarrow{\text{L}} H(S_n, U) - \text{mod}_{\text{coh}} \xrightarrow{\text{e}} e'H(S_q \times S_p, U')e' - \text{mod}_{\text{coh}} \\
H(S_q \times S_p, h') - \text{mod}_R & \xrightarrow{\text{L'}} H(S_q \times S_p, U') - \text{mod}_{\text{coh}} \xrightarrow{\text{e'}} e'H(S_q \times S_p, U')e' - \text{mod}_{\text{coh}}.
\end{align*}$$

Note that these functors are all exact. We make the following claims about them:

**Claim 1:** If $M \in H(S_n, h) - \text{mod}_R^q$, then the $O$-coherent $D$-module $O(h_{\text{reg}}') \otimes_{\mathbb{C}[q]} M$ on $h_{\text{reg}}'$ has regular singularities and trivial monodromy around each irreducible component of $Z(\alpha_0) \cap h'$.

**Claim 2:** If $N \in H(S_q \times S_p, U') - \text{mod}_{\text{coh}}$ is such that the $O$-coherent $D$-module $O(h_{\text{reg}}') \otimes_{O(U')} N$ on $h_{\text{reg}}'$ has regular singularities and trivial monodromy around each component of $Z(\alpha_0) \cap h'$, then $N \cong L'(N')$ for some $N' \in H(S_q \times S_p, h') - \text{mod}_R$. 

**Claim 3:** If $M \in H(S_n, U) - \text{mod}_{\text{coh}}$ and $N \in H(S_q \times S_p, U') - \text{mod}_{\text{coh}}$ are such that $eM \cong \sigma^*e'N$, then the $S_q \times S_p$-equivariant $D$-modules $O(h_{\text{reg}}') \otimes_{O(U)} M$ and $O(h_{\text{reg}}') \otimes_{O(U')} N$ on $h_{\text{reg}}'$ are isomorphic.

Let us first see how these results imply the statements of the theorem. The composites $\text{eL}$ and $\sigma^* e'L'$ identify $H(S_n, h) - \text{mod}_R^q$ and $H(S_q \times S_p, h') - \text{mod}_R$ with full subcategories of $eH(S_n, U)e - \text{mod}_{\text{coh}}$ closed under subquotients, which we call the images of $\text{eL}$ and $\sigma^* e'L'$. Now consider any $M \in H(S_n, h) - \text{mod}_R^q$. Theorem 4.12 shows that $I_q$ kills $M$, so $\text{eL}(M)$ is killed by $eI_q e'$, and therefore by the kernel of $\sigma$. Thus $\text{eL}(M) \cong \sigma^* e'N$ for some $N \in H(S_q \times S_p, U') - \text{mod}_{\text{coh}}$. By claim 3,

$$O(h_{\text{reg}}') \otimes_{O(U')} N \cong O(h_{\text{reg}}') \otimes_{O(U)} L(M) \cong O(h_{\text{reg}}') \otimes_{\mathbb{C}[q]} M$$

as $D(h_{\text{reg}}')$-modules. By claim 1, this $D$-module has regular singularities and trivial monodromy around each component of $Z(\alpha_0) \cap h'$. Therefore by claim 2, $N \cong L'(N')$ for some $N' \in H(S_q \times S_p, h') - \text{mod}_R$, so that $\text{eL}(M) \cong \sigma^* e'L'(N')$. This proves that the image of $\text{eL}$ is contained in that of $\sigma^* e'L'$, so there is an exact fully faithful embedding $F : H(S_n, h) - \text{mod}_R^q \rightarrow H(S_q \times S_p, h') - \text{mod}_R$ such that $\text{eL} = \sigma^* e'L'F$, and whose
image is closed under subquotients. Now $H(S_q \times S_p, h')$—mod$_{RS}^q$ is semisimple and has $\mathfrak{p}_n\mathfrak{p}_n$ (isomorphism classes of) irreducibles, where $\mathfrak{p}_n$ is the number of partitions of $n$. On the other hand, by Theorem 3.11 the $\mathfrak{p}_q\mathfrak{p}_q$ distinct irreducibles

$$\{L(\tau_{\lambda, \nu}) | \lambda \vdash q, \nu \vdash p\}$$

lie in $H(S_n, h')$—mod$_{RS}^q$. It follows that $F$ is an equivalence of categories, and that the above are all of the irreducibles in $H(S_n, h')$—mod$_{RS}^q$. Finally for $M \in H(S_n, h')$—mod$_{RS}^q$, we have $eL(M) \cong \sigma^*eL'F(M)$, so claim 3 implies

$$O(h'_{\text{reg}}) \otimes \mathbb{C}[h] M = O(h'_{\text{reg}}) \otimes O(U') L(M) \cong O(h'_{\text{reg}}) \otimes O(U') L'F(M) = O(h'_{\text{reg}}) \otimes \mathbb{C}[h'] F(M)$$

as $S_q \times S_p$-equivariant $D$-modules on $h'_{\text{reg}}$. It remains to prove the three claims.

**Proof of claim 1:** Fix $M \in H(S_n, h')$—mod$_{RS}^q$ RS. By assumption, $O(h'_{\text{reg}}) \otimes \mathbb{C}[h]$ $M$ has regular singularities. Let $b \in h'$ be a “generic” point of $\{\alpha_0\}$, that is, a point whose stabilizer is $W' \cong S_{m+1} \times S_m^{-1}$. By Proposition 3.3 it suffices to show that the $H(W', h')$-module $N = \text{Res}_b M$ is such that $O(h'_{\text{reg}}) \otimes \mathbb{C}[h]$ $N$ has trivial monodromy around $b$. This depends only on the action of $H(S_{m+1}, C^m) \subset H(W', h')$, so we may suppose $n = m + 1$. But $N$ has minimal support by Theorem 3.3.2. The proof of Corollary 4.7 of [4] shows that the only irreducible in $H_e(S_{m+1}, \mathbb{C}^m)$—mod$_{RS}$ with minimal support is $L(\mathbb{C})$, where $\mathbb{C}$ denotes the trivial representation of $S_{m+1}$. Moreover Lemma 2.9 of [15] shows that $\text{Ext}^1(L(\mathbb{C}), L(\mathbb{C})) = 0$, so $N$ is a direct sum of copies of $L(\mathbb{C})$. Thus $O(h'_{\text{reg}}) \otimes \mathbb{C}[h] N$ is a free module with trivial connection, and in particular has trivial monodromy.

**Proof of claim 2:** Suppose $N \in H(S_q \times S_p, U')$—mod$_{coh}$ has the given properties. Let $U'' \subset h'$ denote the subset of points with trivial stabiliser in $S_q \times S_p$, so that $h'_{\text{reg}} = U' \cap U''$. Then $Sh(N)|_{h'_{\text{reg}}}$ is an $O$-coherent $D$-module on $h'_{\text{reg}}$ with regular singularities and trivial monodromy around $Z(\pi(\alpha_0))$, so it extends to an $O$-coherent $D$-module $N''$ on $U''$ with regular singularities. We may glue $Sh(N)$ and $N''$ to obtain a coherent sheaf $N_1$ on $U' \cup U''$. Note that the complement of $U' \cup U''$ in $h'$ has codimension 2, so $\Gamma(U' \cup U'', O_{U'}) = \mathbb{C}[h']$. Also $N''$ is torsion free since it is an $O$-coherent $D$-module. Therefore $N_1$ is torsion free, so Lemma 3.6 shows that

$$N' = \Gamma(U' \cup U'', N_1)$$

is finitely generated over $\mathbb{C}[h']$. Also $H(S_q \times S_p, U'') \cong D(U'')$ acts on $N''$ consistently with the action of $H(S_q \times S_p, U')$ on $N$, so we obtain an action of $H(S_q \times S_p, h')$ on $N'$. Finally since $N''|_{U''} = N''$ has regular singularities, we have $N' \in H(S_q \times S_p, h')$—mod$_{RS}$ and $L'(N') = O(U'') \otimes \mathbb{C}[h'] N' = N$.

**Proof of claim 3:** We have a natural $\mathbb{C}[S_q \times S_p] \ltimes D(h'_{\text{reg}})$ action on $O(h'_{\text{reg}}) \otimes O(U') H(S_q \times S_p, U')$ given by

$$w(f \otimes a) = \omega f \otimes wa$$

and

$$\nabla_y (f \otimes a) = y(f) \otimes a + f \otimes ya - \sum_{s \in S'} e_s \langle y, \alpha_s \rangle \frac{f}{\alpha_s} \otimes (s-1)a,$$

where $S' \subset S_q \times S_p$ is the set of reflections. For $N \in H(S_q \times S_p, U')$—mod$_{coh}$, the natural isomorphism

$$O(h'_{\text{reg}}) \otimes O(U') H(S_q \times S_p, U') \otimes_{H(S_q \times S_p, U')} N \cong O(h'_{\text{reg}}) \otimes O(U') N$$

preserves the equivariant $D(h'_{\text{reg}})$-module structures by Proposition 1.2. By Lemma 4.2 we may write the above as

$$O(h'_{\text{reg}}) \otimes O(U') N \cong L' \otimes e'H(S_q \times S_p, U')e' e'N,$$

where $L'$ is the $(\mathbb{C}[S_q \times S_p] \ltimes D(h'_{\text{reg}}), e'H(S_q \times S_p, U')e')$-bimodule

$$L' = O(h'_{\text{reg}}) \otimes O(U') H(S_q \times S_p, U')e'.$$
Since $H(S_q \times S_p, U') = O(U') \mathbb{C}[[h']] \mathbb{C}[S_q \times S_p]$, we have
\[
L' = O(h'_{\text{reg}}) \otimes O(U') H(S_q \times S_p, U') e' H(S_q \times S_p, U') e' = O(h'_{\text{reg}}) \otimes O(U') \mathbb{C}[[h']] e' H(S_q \times S_p, U') e' = \bigcup_{d \geq 0} O(h'_{\text{reg}}) \otimes O(U') \mathbb{C}[[h']]^\leq_0 e' H(S_q \times S_p, U') e'.
\]
But
\[
f \circ y a = \nabla_y (f \circ a) - y(f) \circ a + \sum_{s \in S} c_s \langle y, e_s \rangle \frac{f}{e_s} \otimes (s - 1)a,
\]
so
\[
O(h'_{\text{reg}}) \otimes O(U') \mathbb{C}[[h']]^\leq_0 e' H(S_q \times S_p, U') e' \subseteq D(h'_{\text{reg}}) \otimes O(U') \mathbb{C}[[h']]^\leq_0 e' H(S_q \times S_p, U') e'.
\]
Since $\mathbb{C}[[h']]^\leq_0 = \mathbb{C}$, it follows that $L'$ is generated as a bimodule by $1 \otimes e'$. Now $H(S_q \times S_p, U')$ acts faithfully on $O(U')$, so we have an injection
\[
H(S_q \times S_p, U') e' \hookrightarrow \text{Hom}_C(e' O(U'), O(U')).
\]
Note that $e' O(U') = O(U')^{S_q \times S_p} = B$. Also $O(h'_{\text{reg}})$ is flat over $O(U')$, so we obtain an inclusion
\[
i : L' \rightarrow \text{Hom}_C(B, O(h'_{\text{reg}})).
\]
Now $\mathbb{C}[S_q \times S_p] \times D(h'_{\text{reg}})$ and $e' H(S_q \times S_p, U') e'$ act on $\text{Hom}_C(B, O(h'_{\text{reg}}))$ from the left and right, respectively, via their inclusions in $\text{End}_C(O(h'_{\text{reg}}))$ and $\text{End}_C(B)$, and $i$ is a homomorphism of bimodules. Finally $i(1 \otimes e')$ is the natural map $\phi : B \rightarrow O(h'_{\text{reg}})$, so $L'$ is the sub-bimodule of $\text{Hom}_C(B, O(h'_{\text{reg}}))$ generated by $\phi$.

Similarly $O(h'_{\text{reg}}) \otimes O(U) H(S_n, U)$ admits an action of $\mathbb{C}[S_q \times S_p] \times D(h'_{\text{reg}})$ given by
\[
w(f \circ a) = \circ f \circ wa
\]
and
\[
\nabla_y (f \circ a) = y(f) \circ a + f \circ ya - \frac{1}{m} \sum_{s \in S \setminus S_m} \langle y, e_s \rangle \frac{f}{e_s} \otimes (s - 1)a,
\]
where $S \subseteq S_n$ is the set of reflections. We are now identifying $S_q \times S_p$ with a subgroup of $S_n$ as discussed before the theorem. Again for $M \in H(S_n, U) - \text{mod}_c$, we have a $\mathbb{C}[S_q \times S_p] \times D(h'_{\text{reg}})$-module isomorphism
\[
O(h'_{\text{reg}}) \otimes O(U) M \cong O(h'_{\text{reg}}) \otimes O(U) H(S_n, U) e \otimes e H(S_n, U) e M.
\]
Therefore if $e M \cong \sigma e' N$, then
\[
O(h'_{\text{reg}}) \otimes O(U) M \cong L \otimes e' H(S_q \times S_p, U') e' \otimes e' N
\]
where $L$ is the $(\mathbb{C}[S_q \times S_p] \times D(h'_{\text{reg}}), e' H(S_q \times S_p, U') e')$-bimodule
\[
L = O(h'_{\text{reg}}) \otimes O(U) H(S_n, U) e \otimes e H(S_n, U) e e' H(S_q \times S_p, U') e'.
\]
The exact sequence
\[
e H(S_n, U) \cong e H(S_n, U) e \rightarrow e' H(S_q \times S_p, U') e'
\]
gives rise to a right exact sequence of right $e H(S_n, U) e$-modules
\[
O(h'_{\text{reg}}) \otimes O(U) H(S_n, U) \cong e H(S_n, U) e \rightarrow O(h'_{\text{reg}}) \otimes O(U) H(S_n, U) e \rightarrow L,
\]
since $H(S_n, U) e H(S_n, U) = H(S_n, U)$. As above, we have
\[
O(h'_{\text{reg}}) \otimes O(U) H(S_n, U) e = \bigcup_{d \geq 0} O(h'_{\text{reg}}) \otimes O(U) \mathbb{C}[[h']]^\leq_0 e H(S_n, U) e.
\]
Now $\mathfrak{h}$ is spanned by $\mathfrak{h}'$ and $x_i^\vee - x_j^\vee$, for all $i \neq j$ such that $x_i - x_j$ vanishes on $\mathfrak{h}'$. Fix such $i$ and $j$, and let $g \in \mathbb{C}[\mathfrak{h}]$ vanish on all components of $X_q$ except $\mathfrak{h}'$, such that $g$ is nonzero on $\mathfrak{h}'$. Then $\mathfrak{h}'f - f$ is divisible by $x_i - x_j$ for any $f \in \mathbb{C}[\mathfrak{h}]$, so $g(s_{ij} - 1) \in H(S_n, \mathfrak{h})$ sends $\mathbb{C}[\mathfrak{h}]$ into $I_q$. But the annihilator of $\mathbb{C}[\mathfrak{h}]/I_q$ in $H(S_n, \mathfrak{h})$ is $H(S_n, \mathfrak{h})I_qH(S_n, \mathfrak{h})$, so

$$H(S_n, U)I_qH(S_n, U) \ni [x_i^\vee, g(s_{ij} - 1)] = [x_i^\vee, g](s_{ij} - 1) + g(x_i^\vee - x_j^\vee)s_{ij}.$$  

Note that $[x_i^\vee, g](s_{ij} - 1) \in \mathbb{C}[\mathfrak{h}][C_n]$. But $g$ is invertible in $\mathcal{O}(\mathfrak{h}^\prime_{\text{reg}})$, so

$$x_i^\vee - x_j^\vee \in \mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}) \otimes \mathbb{C}[\mathfrak{h}] H(S_n, U)I_qH(S_n, U) + \mathcal{O}(\mathfrak{h}^\prime_{\text{reg}})C_n,$$  

whence

$$\mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}) \otimes \mathcal{O}(U) \mathbb{C}[\mathfrak{h}]^{\leq d+1}eH(S_n, U)e \subseteq \mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}) \otimes \mathcal{O}(U) \mathbb{C}[\mathfrak{h}]^{\leq d}eH(S_n, U)e + \mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}) \otimes \mathcal{O}(U) H(S_n, U)I_qH(S_n, U)e.$$  

Applying $\rho$, and using the same argument as above, we obtain

$$\rho\left(\mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}) \otimes \mathcal{O}(U) \mathbb{C}[\mathfrak{h}]^{\leq d+1}eH(S_n, U)e\right) \subseteq D(\mathfrak{h}^\prime_{\text{reg}})\phi\left(\mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}) \otimes \mathcal{O}(U) \mathbb{C}[\mathfrak{h}]^{\leq d}eH(S_n, U)e\right).$$  

Thus $L$ is generated as a bimodule by $\rho(1 \otimes e)$. Finally $H(S_n, U)$ acts on $\mathcal{O}(U)/I_q\mathcal{O}(U)$ with annihilator $H(S_n, U)I_qH(S_n, U)$, so we have an inclusion

$$H(S_n, U)e/H(S_n, U)I_qH(S_n, U)e \hookrightarrow \text{Hom}_C(e(\mathcal{O}(U)/I_q\mathcal{O}(U)), \mathcal{O}(U)/I_q\mathcal{O}(U))$$  

with $(\mathcal{O}(U)/I_q\mathcal{O}(U), eH(S_n, U)e)$-bimodules. Note that $e(\mathcal{O}(U)/I_q\mathcal{O}(U)) = B$. Since $\mathcal{O}(\mathfrak{h}^\prime_{\text{reg}})$ is flat over $\mathcal{O}(U)/I_q\mathcal{O}(U)$, we obtain an inclusion

$$j : L \hookrightarrow \text{Hom}_C(B, \mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}))$$  

of $(\mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}), e'H(S_q \times S_p, U')e')$-bimodules. Again $j$ preserves the left action of

$$\mathbb{C}[S_q \times S_p] \cong D(\mathfrak{h}^\prime_{\text{reg}}),$$  

and sends $\rho(1 \otimes e)$ to $\phi$, so $L$ is the sub-bimodule of $\text{Hom}_C(B, \mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}))$ generated by $\phi$. In particular, $L \cong L'$ as bimodules, so

$$\mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}) \otimes \mathcal{O}(U) M \cong L \otimes e'H(S_q \times S_p, U')e' \cong L' \otimes e'H(S_q \times S_p, U')e' \cong \mathcal{O}(\mathfrak{h}^\prime_{\text{reg}}) \otimes \mathcal{O}(U') N$$  

as $\mathbb{C}[S_q \times S_p] \cong D(\mathfrak{h}^\prime_{\text{reg}})$-modules, as required.  

5. The Monodromy Functors for Type A

We continue to study the case $W = S_n$, $\mathfrak{h} = \mathbb{C}^n$ and $c = \frac{1}{m}$. Previously we studied the modules in $H_c - \text{mod}_{RS}$ supported on $X_q$, where $q = [n/m]$. Now let $q$ be any integer satisfying $0 \leq qm \leq n$, and let $H_c - \text{mod}_{RS}$ denote the Serre subcategory of $H_c - \text{mod}_{RS}$ consisting of all modules supported on $X_q$. Our goal in this section is to construct an equivalence of categories from $H_c - \text{mod}_{RS}^q/H_c - \text{mod}_{RS}^{q+1}$ to the subcategory of finite dimensional representations of a certain Hecke algebra (where $H_c - \text{mod}_{RS}^{[n/m]+1}$ is the subcategory containing only the zero module). From this we will deduce Theorem 1.6.

Let us fix an integer $q$ as above, and put $p = n - qm$. Consider the natural copy $S_m^q \subseteq S_n$, and put $\mathfrak{h}' = \mathfrak{h}^\prime_{S_n^q}$ and $\mathfrak{h}^\prime_{\text{reg}} = \mathfrak{h}^\prime_{S_n^q}$. As in the previous section, we may identify $\mathfrak{h}'$ with Spec $\mathbb{C}[z_1, \ldots, z_q, t_1, \ldots, t_p]$ via the map

$$\pi : \mathbb{C}[\mathfrak{h}] \rightarrow \mathbb{C}[z_1, \ldots, z_q, t_1, \ldots, t_p], \quad \pi(x_i) = \begin{cases} z_{[i/m]} & \text{if } i \leq qm \\ t_{i-qm} & \text{if } i > qm. \end{cases}$$
Note that \( h' \) is one of the components of \( X_q \subseteq h \). By Proposition 1.2 we have a functor
\[
\text{Loc}^q : H_c - \mod_{RS}^q \rightarrow \mathbb{C}[S_q \times S_p] \times D(h'_\text{reg}) - \mod_{RS},
\]
\[
M \mapsto O(h'_\text{reg}) \otimes \mathbb{C}[h],
\]
where \( \mathbb{C}[S_q \times S_p] \times D(h'_\text{reg}) - \mod_{RS} \) denotes the category of coherent \( S_q \times S_p \)-equivariant \( D(h'_\text{reg}) \)-modules with regular singularities.

**Lemma 5.1.** The functor \( \text{Loc}^q \) is exact, and induces a faithful functor
\[
H_c - \mod_{RS}^q/H_c - \mod_{RS}^{q+1} \rightarrow \mathbb{C}[S_q \times S_p] \times D(h'_\text{reg}) - \mod_{RS},
\]
which we also denote by \( \text{Loc}^q \).

**Proof.** Any \( M \in H_c - \mod_{RS}^q \) is annihilated by some power of \( I_q \). By Theorem 3.12 \( I_q M = 0 \). Let \( b \) be any point in \( h'_\text{reg} \), and let \( U \subseteq h \) denote the affine Zariski open set
\[
U = \{ x \in h \mid x_i \neq x_j \text{ whenever } b_i \neq b_j \}. \]
Then \( U \cap h' = U \cap X_q = h'_\text{reg} \), so \( I_q O(U) \rightarrow O(U) \rightarrow O(h'_\text{reg}) \) is right exact. Thus \( O(U) \otimes_{\mathbb{C}[h]} M \rightarrow O(h'_\text{reg}) \otimes_{\mathbb{C}[h]} M \) is an isomorphism. Since \( M \mapsto O(U) \otimes_{\mathbb{C}[h]} M \) is exact, so is \( \text{Loc}^q \). Clearly the objects killed by \( \text{Loc}^q \) are exactly those in \( H_c - \mod_{RS}^{q+1} \). The result now follows from general categorical considerations.

We now want to determine the “image” of \( \text{Loc}^q \). The Riemann-Hilbert correspondence gives an equivalence of categories
\[
\mathbb{C}[S_q \times S_p] \times D(h'_\text{reg}) - \mod_{RS} \rightarrow D(h'_\text{reg}/S_q \times S_p) - \mod_{RS},
\]
\[
\Rightarrow \pi_1(h'_\text{reg}/S_q \times S_p) - \mod_1,
\]
where the latter denotes the category of finite dimensional representations of the group \( \pi_1(h'_\text{reg}/S_q \times S_p) \) over \( \mathbb{C} \). Let \( \phi \) denote the composite functor. Under the identification \( h' = \text{Spec} \mathbb{C}[z_1, \ldots, z_q, t_1, \ldots, t_p] \cong \mathbb{C}^{q+p} \), the open subset \( h'_\text{reg} \) consists of points with all coordinates distinct. By the correspondence between covering spaces of \( h'_\text{reg}/S_{q+p} \) and subgroups of \( \pi_1(h'_\text{reg}/S_{q+p}) \), we have a homomorphism
\[
\mu : \pi_1(h'_\text{reg}/S_{q+p}) \rightarrow S_{q+p},
\]
and \( \pi_1(h'_\text{reg}/S_q \times S_p) = \mu^{-1}(S_q \times S_p) \). The fundamental group \( \pi_1(h'_\text{reg}/S_{q+p}) \) is known as the braid group \( B_{q+p} \). We need to describe some explicit elements of \( B_{q+p} \).

Suppose \( x_1, x_2, \ldots, x_{q+p} \in \mathbb{C} \) are the vertices of a convex \((q+p)\)-gon in the complex plane, listed counterclockwise. We take the point
\[
x = (x_1, x_2, \ldots, x_{q+p}) \in \mathbb{C}^{q+p} \cong h'
\]
as the basepoint of \( \pi_1(h'_\text{reg}) \). Suppose \( 1 \leq i, j \leq q+p \) and \( i \neq j \). Pick \( \epsilon > 0 \) such that \( 2\epsilon |x_i - x_j| < |x_i + x_j - 2x_k| \) for all \( k \). Let \( \gamma_{ij} : [0, 1] \rightarrow h'_\text{reg} \) be the path
\[
\gamma_{ij}(t) = \begin{cases} ts_{ij} x + (1 - t) x & \text{if } |t - \frac{1}{2}| > \epsilon \\ \frac{1}{2} (s_{ij} x + x) + i \epsilon e^{i \pi (t - \frac{1}{2})} (x - s_{ij} x) & \text{otherwise.} \end{cases}
\]
from \( x \) to \( s_{ij} x \). Geometrically, as \( \gamma_{ij} \) is traversed, \( x_i \) and \( x_j \) switch positions linearly, traversing small semicircles counterclockwise around \( \frac{1}{2}(x_i + x_j) \) to avoid intersecting. Pushing this path down to \( h'_\text{reg}/S_{q+p} \) gives an element \( S_{ij} \in B_{q+p} \) such that \( \mu(S_{ij}) = s_{ij} \).

Note that there is a unique component \( Z \) of \( h' \setminus h'_\text{reg} \) passing through \( \frac{1}{2}(s_{ij} x + x) \), and for \( M \in \mathbb{C}[S_q \times S_p] \times D(h'_\text{reg}) - \mod_{RS} \), the action of \( S_{ij} \) on \( \phi(M) \) is conjugate to the monodromy around \( Z \) of the induced local system on \( h'_\text{reg}/S_q \times S_p \).

**Lemma 5.2.** We have a surjection \( \pi_1(h'_\text{reg}/S_q \times S_p) \rightarrow B_q \times B_p \) whose kernel is generated by \( \{ S_{ij}^2 \mid i \leq q, j > q \} \).
Proof. Let \( U_q \subseteq \mathbb{C}^q \) and \( U_p \subseteq \mathbb{C}^p \) denote the open subsets on which all coordinates are distinct. We have a natural continuous map \( h_{\text{reg}}' \to U_q \times U_p \) and the natural action of \( S_q \times S_p \) on the latter space makes this map equivariant. We may therefore identify \( h_{\text{reg}}'/S_q \times S_p \) with an open subset of \((U_q/S_q) \times (U_p/S_p)\), and the complement \( Z \) is the image of

\[
\{ x \in U_q \times U_p \subseteq \mathbb{C}^{q+p} \mid x_i = x_j \text{ for some } i \leq q, j > q \}.
\]

Note that \((U_q/S_q) \times (U_p/S_p)\) is a smooth complex variety, and \( Z \) is an irreducible divisor. Moreover \( Z \) is invariant under the action of \( \mathbb{R}^n \), and is therefore contractible. Van Kampen’s Theorem now gives a surjection

\[
\pi_1(h_{\text{reg}}'/S_q \times S_p) \to \pi_1((U_q/S_q) \times (U_p/S_p)) = B_q \times B_p,
\]

whose kernel is generated by a loop around \( Z \). That is, the kernel is generated by \( S^2_{ij} \) for any \( i \leq q \) and \( j > q \).

The group \( B_q \) has a standard presentation; namely it is generated by \( \{ S_{i,i+1} \mid 1 \leq i < q \} \) subject to the braid relations

\[
S_{i,i+1}S_{i+1,i+2}S_{i,i+1} = S_{i+1,i+2}S_{i,i+1}S_{i+1,i+2},
\]

\[
S_{i,i+1}S_{j,j+1} = S_{j,j+1}S_{i,i+1} \text{ if } |i - j| > 1.
\]

The kernel of \( B_q \to S_q \) is generated by \( \{ S^2_{ij} \} \). For any \( q \in \mathbb{C}^* \), the Hecke algebra of type \( A_{p-1} \) is the algebra

\[
H_q(S_p) = \mathbb{C}[B_p]/\langle (S_{ij} - 1)(S_{ij} + q) \rangle.
\]

We will take \( q = e^{2\pi i/m} \). Combining the above, we have a surjection

\[
\psi : \mathbb{C}[\pi_1(h_{\text{reg}}'/S_q \times S_p)] \to \mathbb{C}[S_q] \otimes \mathbb{C} H_q(S_p)
\]

whose kernel is generated by

\[
\{ S^2_{ij} - 1 \mid i \leq q \text{ or } j \leq q \} \cup \{ (S_{ij} - 1)(S_{ij} + q) \mid i,j > q \}.
\]

This gives rise to a fully faithful embedding

\[
\psi^* : \mathbb{C}[S_q] \otimes \mathbb{C} H_q(S_p) - \text{mod}_{fd} \to \pi_1(h_{\text{reg}}'/S_q \times S_p) - \text{mod}_{fd},
\]

whose image is the full subcategory \( \mathcal{A} \) of \( \pi_1(h_{\text{reg}}'/S_q \times S_p) - \text{mod}_{fd} \) consisting of modules on which the relations (3) vanish.

**Lemma 5.3.** The composite functor \( \phi \text{Loc}^q \) sends each module into \( \mathcal{A} \).

**Proof.** Consider any \( M \in H_c - \text{mod}_{RS}^q \). We must show that each of the relations (3) vanish on \( \phi \text{Loc}^q(M) \). That is, for each component \( Z \) of \( h' \setminus h_{\text{reg}}' \), we must show that the monodromy of the induced local system on \( h_{\text{reg}}'/S_q \times S_p \) around \( Z \) satisfies the appropriate equation. Let \( b \in Z \) be a “generic” point, that is, chosen from \( Z \) so that its stabiliser \( W^b \subseteq S_n \) is minimal. By Proposition 5.5 it suffices to prove the appropriate equation for the monodromy of the local system corresponding to \( N = \text{Res}_b M \). There are three possibilities, depending on which two coordinates we have set equal.

**Case 1:** \( W^b \cong S_m^{n-2} \times S_{2m} \). We are required to show that the monodromy of the local system on \( h_{\text{reg}}' \) (ignoring the equivariance structure) around \( Z \) is trivial. The monodromy depends only on the action of \( H_c(S_{2m}, \mathbb{C}^{2m}) \subseteq H_c(W', h) \), so we may suppose \( n = 2m \). But \( N \) has minimal support by Theorem 4.2. By Theorem 4.4 it suffices to consider the module \( F(N) \in H_m(S_2, \mathbb{C}^2) - \text{mod}_{RS} \). However, the latter category is semisimple with irreducibles \( L(\tau_{(2)}) \) and \( L(\tau_{(1,1)}) \), so it suffices to check that these modules give rise to local systems with trivial monodromy around the diagonal. This is clear from the description of the connection given in Proposition 1.2.

**Case 2:** \( W^b \cong S_m^{n-1} \times S_{m+1} \). Again we must show that the monodromy of the local system on \( h_{\text{reg}}' \) around \( Z \) is trivial. Now the monodromy depends only on the action of
such that as an $U$

Proposition 5.4. The functor $\text{Loc}^g$ induces an equivalence

$$H_c - \text{mod}^g_{RS}/H_c - \text{mod}^{g+1}_{RS} \cong \phi^{-1}A.$$

Proof. By the above lemmas, $\text{Loc}^g$ induces a faithful functor

$$H_c - \text{mod}^g_{RS}/H_c - \text{mod}^{g+1}_{RS} \to \phi^{-1}A,$$

and it remains to show that this functor is full and essentially surjective. Both statements will follow from the existence of a functor $G : \phi^{-1}A \to H_c - \text{mod}^g_{RS}$ such that $\text{Loc}^g G$ is naturally equivalent to the identity functor. This will take some work, so we proceed in a sequence of lemmas.

Lemma 5.5. Let $N \subseteq S_n$ denote the normaliser of the subgroup $S^n \subseteq S_n$, and choose a set $C$ of left coset representatives for $N$ in $S_n$. Let $\mathfrak{h}^\perp$ denote the orthogonal complement to $\mathfrak{h}$ with respect to the natural $S_n$-invariant inner product on $\mathfrak{h} = \mathbb{C}^n$. There is a functor

$$\bar{G} : \mathbb{C}[S_q \times S_p] \ltimes D(\mathfrak{h}_\text{reg}) - \text{mod}_{RS} \to H_c(S_n, \mathfrak{h}) - \text{mod}$$

such that

$$\bar{G}M \cong \mathbb{C}[S_n] \otimes \mathbb{C}[N] M = \bigoplus_{w \in C} wM$$

as an $S_n$-module, and the actions of $x \in \mathfrak{h}^*$ and $y \in \mathfrak{h}$ are given by

$$x(w \otimes v) = w \otimes \pi(x^w)v,$$

$$y(w \otimes v) = w \otimes \nabla_y(wv) - \frac{1}{\pi} \sum_{x \neq \pi(x)} \langle y^w, x_i - x_j \rangle (w + ws_{ij}) \otimes \frac{1}{\pi(x_i - x_j)}v,$$

where $\pi : \mathbb{C}[\mathfrak{h}] \to \mathbb{C}[\mathfrak{h}']$ is as above, and $\rho : \mathfrak{h} \to \mathfrak{h}'$ is the projection with kernel $\mathfrak{h}^\perp$. Note that if $\pi(x_i) \neq \pi(x_j)$, then $\pi(x_i - x_j)$ is invertible in $O(\mathfrak{h}_\text{reg})$.

Proof. Under the natural identification $\mathfrak{h}' \times \mathfrak{h}^\perp \cong \mathfrak{h}$, the set $\mathfrak{h}_\text{reg} \times \mathfrak{h}^\perp$ is identified with an open subset $U \subseteq \mathfrak{h}$. Let

$$\bar{U} = S_n \times N U = \coprod_{w \in C} wU.$$
We have a natural étale morphism $\bar{U} \to \mathfrak{h}$, so by Propositions 2.3, 2.4, and 2.6 we have algebra homomorphisms

$$H_c(S_n, \mathfrak{h}) \to H_c(S_n, \bar{U}) \to \text{Mat}_{[S_n:N]}(\mathbb{C}[N] \ltimes_{\mathbb{C}[S_m]} H_c(S_m^q, \text{Spf} \bar{U}')).$$

But

$$H_c(S_m^q, \text{Spf} \bar{U}') = \bar{U}' \otimes \mathcal{O}(U) H_c(S_m^q, U) = \left( \lim_{\to} \mathcal{O}(U)/I^k \right) \otimes \mathcal{O}(U) H_c(S_m^q, U),$$

where $I \subseteq \mathcal{O}(U)$ is the kernel of $\mathcal{O}(U) \to \mathcal{O}(\bar{U}')$. Since $U \cong \bar{U}' \times \mathfrak{h}^+$, we have

$$H_c(S_m^q, U) = H_c(S_m^q, \mathfrak{h}^+) \otimes \mathbb{C} D(\bar{U}').$$

Moreover $\mathfrak{h}^+ \cong (\mathbb{C}^n/\mathbb{C})^q$, so by Theorem 3.8 and Lemma 3.1 we have an algebra homomorphism $H_c(S_m^q, \mathfrak{h}^+) \to \mathbb{C}$ whose kernel contains $(\mathfrak{h}^+)^\ast$. This induces a homomorphism

$$H_c(S_m^q, \mathfrak{h}^+) \otimes \mathbb{C} D(\bar{U}') \to D(\bar{U}').$$

which kills $I H_c(S_m^q, \mathfrak{h}^+) \otimes \mathbb{C} D(\bar{U}')$. This therefore factors through the completion, and we obtain

$$\mathbb{C}[N] \ltimes_{\mathbb{C}[S_m]} H_c(S_m^q, \bar{U}') \to \mathbb{C}[N] \ltimes_{\mathbb{C}[S_m]} D(\bar{U}') = \mathbb{C}[N/S_m] \ltimes D(\bar{U}').$$

Since $N/S_m \cong S_q \times S_p$, we obtain functors

$$\mathbb{C}[S_q \times S_p] \otimes D(\bar{U}') \mod_{RS} \to \mathbb{C}[N] \ltimes_{\mathbb{C}[S_m]} H_c(S_m^q, \bar{U}') \mod \cong \text{Mat}_{[S_n:N]}(\mathbb{C}[N] \ltimes_{\mathbb{C}[S_m]} H_c(S_m^q, \bar{U}')) \mod \to H_c(S_n, \mathfrak{h}) \mod.$$

Let $\check{G}$ be the composite functor. Following through the construction in Proposition 2.6 we see that $\check{G}M$ is exactly as described in the statement.

Now let

$$z_i^\vee = m \rho(x_{mi}^\vee) = \sum_{j=0}^{m-1} x_{mi-j}^\vee, \quad t_a^\vee = \rho(x_{mq+a}^\vee) = x_{mq+a}^\vee.$$

These form the basis of $\mathfrak{h}'$ dual to the basis $\{z_1, t_a\}$ of $(\mathfrak{h}')^\ast$. As in the previous section, denote

$$p_{x^\vee}(2) = \sum_{i=1}^n (x_i^\vee)^2 \in H_c(S_n, \mathfrak{h}),$$

and similarly for $p_{x^\vee}(2)$ and $p_{t}^\vee(2)$.

**Lemma 5.6.** The element $e_u \in H_c(S_n, \mathfrak{h})$ acts locally finitely on $\check{G}M$. Also $p_{x^\vee}(2)$ acts on symmetric elements of $\check{G}M$ via the formula

$$p_{x^\vee}(2) \sum_{w \in S_n} w \otimes u = \sum_{w \in S_n} w \otimes \zeta u,$$

where $\zeta \in D(\bar{U}')$ is the differential operator

$$\zeta = \frac{1}{m} p_{x^\vee}(2) + p_{t^\vee}(2) - 2 \sum_{i \neq j} z_i^\vee z_j^\vee - 2 \sum_{i,a} \frac{z_i^\vee - m t_a^\vee}{z_i - t_a} - \frac{2}{m} \sum_{a \neq b} \frac{t_a^\vee - t_b^\vee}{t_a - t_b}.$$

**Proof.** Consider the Euler vector field

$$\xi = \sum_{i=1}^q z_i z_i^\vee + \sum_{i=1}^r t_i t_i^\vee \in D(\mathfrak{h}').$$
This acts locally finitely on $M$ by Lemma 5.2. Calculating the action of the Euler element $\mathbf{eu} \in H^*_c$ on $v \in M \subseteq \tilde{G}M$ gives

$$
\mathbf{eu} \cdot v = \nabla_x v - \frac{n(n-1)}{2m} v.
$$

Since $\mathbf{eu}$ centralises $S_n$, we conclude that $\mathbf{eu}$ acts locally finitely on $\tilde{G}M$. The second statement also follows by a straightforward calculation.

Clearly the support of $\tilde{G}M$ lies in $X_R$, and $\mathcal{O}(\mathfrak{h}'_{\text{reg}}) \otimes \mathbb{C}[\mathfrak{b}] \tilde{G}M \cong M$. If $\tilde{G}M$ were finitely generated over $\mathcal{O}([\mathfrak{h}])$, this would be the required module in $H_c-\text{mod}_{\text{RS}}^d$. Unfortunately it is too large. We will construct the required module $GM$ as a submodule of $\tilde{G}M$. For each irreducible component $Z$ of $\mathfrak{h}' \setminus \mathfrak{h}'_{\text{reg}}$, let $x_Z \in (\mathfrak{h}')^*$ be an element with kernel $Z$. Let

$$
\alpha = \prod_{Z} x_Z^2 \in \mathbb{C}[\mathfrak{h}]^{S_n \times S_p}.
$$

Note that the zero set of $\alpha$ is exactly $\mathfrak{h}' \setminus \mathfrak{h}'_{\text{reg}}$.

**Lemma 5.7.** For each component $Z$ of $\mathfrak{h}' \setminus \mathfrak{h}'_{\text{reg}}$, there is a subspace $D_Z M \subseteq M$ satisfying:

1. $D_Z M$ is functorial in $M$,
2. $D_Z M$ is preserved by the actions of $\zeta$ and $\pi(\mathbb{C}[\mathfrak{h}]^{S_n}) \subseteq \mathbb{C}[\mathfrak{h}']$
3. If $U \subseteq \mathfrak{h}'$ is a Zariski open subset such that $V \subseteq \mathcal{O}(U \cap \mathfrak{h}'_{\text{reg}}) \otimes \mathcal{O}(\mathfrak{h}_{\text{reg}}) M$ freely generates $\mathcal{O}(U \cap \mathfrak{h}'_{\text{reg}}) \otimes \mathcal{O}(\mathfrak{h}_{\text{reg}}) M$ over $\mathcal{O}(U \cap \mathfrak{h}'_{\text{reg}})$, then setting $U' = U \cap \text{int}(\mathfrak{h}_{\text{reg}} \cup Z)$,

we have $D_Z M \subseteq x_Z^{-K} \mathcal{O}(U') V$ for some integer $K$.
4. If $M \in \phi^{-1} A$ then $M^{S_n \times S_p} = \mathbb{C}[\alpha^{-1}] (D_Z M)^{S_n \times S_p}$.

**Proof.** Choose a generic point $b \in Z$ with stabiliser $W' \subseteq S_n$. Let $B \subseteq \mathfrak{h}'$ be an open ball around $b$ which doesn’t intersect the other components of $\mathfrak{h}' \setminus \mathfrak{h}'_{\text{reg}}$. Since $Z \subseteq \mathfrak{h}'$ is a codimension 1 complex subspace, $\pi_1(B \cap \mathfrak{h}'_{\text{reg}}) \cong \mathbb{Z}$. Let $\tilde{B} \to B \cap \mathfrak{h}'_{\text{reg}}$ denote the universal cover, and let $\mathcal{O}_B^{an}$ and $\mathcal{O}_B^{an, s}$ denote the rings of analytic functions on $B$ and $\tilde{B}$ respectively. Now $\mathcal{O}_B^{an, s} \otimes \mathcal{O}(\mathfrak{h}_{\text{reg}})$ is naturally an analytic $\mathcal{O}$-coherent $D$-module on $\tilde{B}$. Since $\tilde{B}$ is simply connected, we have

$$
\mathcal{O}_B^{an, s} \otimes \mathcal{O}(\mathfrak{h}_{\text{reg}}) M = \mathcal{O}_B^{an, s} \otimes \mathcal{O}(\mathfrak{h}_{\text{reg}}) \mathcal{O}_B \otimes \mathcal{O}(\mathfrak{h}_{\text{reg}}) M.
$$

where $M_{\text{flat}} \subseteq \mathcal{O}_B^{an, s} \otimes \mathcal{O}(\mathfrak{h}_{\text{reg}}) M$ is the space of flat sections. Let $i: M \to \mathcal{O}_B^{an, s} \otimes \mathcal{O}(\mathfrak{h}_{\text{reg}}) M$ denote the inclusion. Again we consider three cases, depending on which two coordinates are equal on $Z$.

**Case 1:** $W' \cong S_m^l \times S_2$, so $Z$ is the kernel of $x_Z = t_a - t_b \in (\mathfrak{h}')^*$. Let $s \in S_q \times S_p$ be the transposition switching $t_a$ with $t_b$, and let $\mathcal{O}_{B, s}^{an, s}$ denote the subspace of $\mathcal{O}_B^{an}$ fixed by $s$. We may think of $\mathcal{O}_{B, s}^{an, s}$ as consisting of functions involving only even powers of $x_Z$. Let $\lambda = \frac{1}{m} + 1$, and let

$$
D_Z M = i^{-1} \left( \mathcal{O}_B^{an, s} M_{\text{flat}} + x_Z^\lambda \mathcal{O}_B^{an, s} M_{\text{flat}} \right),
$$

noting that $x_Z^\lambda$ is a well-defined function in $\mathcal{O}_B^{an}$. We first check that this is independent of the choice of $b$ and $B$. Suppose $b'$ and $B'$ are chosen to satisfy the same conditions. Since $\mathfrak{h}_{\text{reg}}^{W'}$ is connected, we may find a path joining $b$ to $b'$, and some tubular neighbourhood $T$ of this path in $\mathfrak{h}'$ will contain $B$ and $B'$, such that the inclusions $B \cap \mathfrak{h}_{\text{reg}} \to T \cap \mathfrak{h}_{\text{reg}}$ and $B' \cap \mathfrak{h}_{\text{reg}} \to T \cap \mathfrak{h}_{\text{reg}}$ are homotopy equivalences. This allows us to identify $\tilde{B}$ and
with open subsets of the universal cover \( \tilde{T} \) of \( T \cap b'_{\text{reg}} \). Thus \( i \) may be expressed as a composite

\[
M \xrightarrow{\lambda} O_T^{an} \otimes O(\bar{b}'_{\text{reg}}), \quad M \xrightarrow{\bullet} O_B^{an} \otimes O(b'_{\text{reg}}). \]

Now \( O_T^{an,s} + x_T^\lambda O_T^{an,s} \) is the inverse image of \( O_B^{an,s} + x_B^\lambda O_B^{an,s} \) under \( O_T^{an} \to O_B^{an} \), so

\[
O_T^{an,s} M_{\text{flat}} + x_T^\lambda O_T^{an,s} M_{\text{flat}} = k^{-1}(O_B^{an,s} M_{\text{flat}} + x_B^\lambda O_B^{an,s} M_{\text{flat}}).
\]

It follows that \( D_Z M = j^{-1}(O_T^{an,s} M_{\text{flat}} + x_T^\lambda O_T^{an,s} M_{\text{flat}}) \), so using \( b' \) and \( B' \) instead of \( b \) and \( B \) produces the same subspace. Property (1) is clear.

Now \( \pi : \mathbb{C}[b] \to \mathbb{C}[b'] \) is equivariant with respect to the action of \( S_q \times S_p \), so

\[
\pi(C[b]^{S_q}) \subseteq \mathbb{C}[b']^{S_q \times S_p} \subseteq O_B^{an,s}.
\]

Therefore \( D_Z M \) is preserved by the action of \( \pi(C[b]^{S_q}) \). To show it is preserved by \( \zeta \), note that

\[
\zeta x_Z^\lambda = 2\lambda(\lambda - 1)x_Z^{\lambda - 2} - \frac{4}{m} \lambda x_Z^{\lambda - 2} - 2 \sum_i \left( \frac{1}{t_a - z_i} - \frac{1}{t_b - z_i} \right) \lambda x_Z^{\lambda - 1}
\]

\[
= 2 \sum_i \left( \frac{1}{t_a - z_i}(t_a - z_i) - \frac{1}{t_b - z_i}(t_b - z_i) \right) \lambda x_Z^{\lambda - 1}
\]

Also setting \( x'_Z = t'_a - t'_b \), we may write

\[
\zeta \in \frac{1}{2}(x_Z^{\lambda})^2 - \frac{2}{m} x_Z^{\lambda \mu} + \sum_{y \in Z} C y^2 + O_B^{an,y},
\]

so for \( f \in O_B^{an,s} \), we have

\[
[\zeta, f] \in x_Z^{\lambda}(f)x_Z - \frac{2}{m} x_Z^{\lambda \mu}(f) + \sum_{y \in Z} C y(f)y + O_B^{an,y}(f) + O_B^{an,s}.
\]

But \( x_Z^{\lambda}(f)/x_Z \in O_B^{an,s} \) and \( s \) fixes \( \zeta \), so we conclude that

\[
[\zeta, O_B^{an,s}] \subseteq O_B^{an,s} x_Z x_Z^{\lambda} + O_B^{an,s} + O_B^{an,s}.
\]

Hence

\[
\zeta(O_B^{an,s} M_{\text{flat}} + x^\lambda O_B^{an,s} M_{\text{flat}})
\]

\[
\subseteq [\zeta, O_B^{an,s}] M_{\text{flat}} + \zeta O_B^{an,s} x_Z^{\lambda} M_{\text{flat}} + O_B^{an,s} \zeta(x_Z^{\lambda}) M_{\text{flat}}
\]

proving property (2).

Now suppose that \( U \) and \( V \) are as in property (3). If \( U \cap Z = \emptyset \) then \( O(U \cap b'_{\text{reg}}) = O(U') \), and the property is clear. Otherwise we may suppose that \( B \subseteq U \). Then

\[
O_B^{an} \otimes \mathbb{C} M_{\text{flat}} = O_B^{an} \otimes O(\bar{b}'_{\text{reg}}) = O_B^{an} \otimes \mathbb{C} V,
\]

so \( M_{\text{flat}} = XV \), where \( X \in GL_{O_B^{an}}(O_B^{an} \otimes \mathbb{C} V) \) satisfies

\[
X \in \sum_{\mu \in \Lambda, \ i \in L} \text{End}_\mathbb{C}(V) \otimes O_B^{an} x_Z^\mu(\log x_Z)^I,
\]

\[
X^{-1} \in \sum_{\mu \in \Lambda, \ i \in L} \text{End}_\mathbb{C}(V) \otimes O_B^{an} x_Z^{-\mu}(\log x_Z)^I.
\]
for some finite subsets $\Lambda \subseteq \mathbb{C}$ and $L \subseteq \mathbb{Z}_{\geq 0}$. Thus

$$M_{\text{flat}} \subseteq \sum_{\mu \in \Lambda, \ i \in L} \mathcal{O}^n_B x_Z^\mu (\log x_Z)^i V,$$

$$V \subseteq \sum_{\mu \in \Lambda, \ i \in L} \mathcal{O}^n_B x_Z^{-\mu} (\log x_Z)^i M_{\text{flat}}.$$

Thus

$$D_Z M \subseteq \left(\mathcal{O}(U \cap h_{\text{reg}}') \otimes_C V\right) \cap \left(\sum_{\mu \in \Lambda, \ i \in L} \left(\mathcal{O}^n_B x_Z^\mu (\log x_Z)^i + \mathcal{O}^n_B x_Z^{-\mu} (\log x_Z)^i\right) V\right)$$

$$\subseteq x_Z^K \mathcal{O}(U') V$$

for some integer $K$, proving (3).

Proceeding with the above notation, suppose now that $M \in \phi^{-1} A$. Let $S$ be the endomorphism of $\mathcal{O}^n_B \otimes_{\mathcal{O}(h_{\text{reg}}')} M$ given by

$$S(f \otimes v) = (\tilde{s}^* f) \otimes sv,$$

where $\tilde{s}^* \in \text{End}_{\mathcal{O}}(\mathcal{O}^n_B)$ is induced by the automorphism $\tilde{s} : \tilde{B} \to \tilde{B}$ obtained by lifting $s : B \to B$. That is, $\tilde{s}^*$ sends $x_Z$ to $e^{-\pi i} x_Z$, and fixes any function killed by $x_Z^\alpha$. The restriction of $S$ to $M_{\text{flat}}$ is exactly the monodromy of the corresponding local system on $h_{\text{reg}}'/S_q \times S_p$ around $Z$, which is assumed to satisfy

$$(S|M_{\text{flat}} - 1)(S|M_{\text{flat}} + q) = 0.$$ 

Thus $M_{\text{flat}}$ decomposes into eigenspaces $M^1_{\text{flat}}$ and $M^{-q}_{\text{flat}}$ for $S$. The above shows that

$$M \subseteq \sum_{\mu \in \Lambda, \ i \in L} \mathcal{O}^n_B x_Z^{-\mu} (\log x_Z)^i M_{\text{flat}}$$

$$= \sum_{\mu \in \Lambda, \ i \in L} \mathcal{O}^n_B x_Z^{-\mu} (\log x_Z)^i M^1_{\text{flat}} + \mathcal{O}^n_B x_Z^{-\mu} (\log x_Z)^i M^{-q}_{\text{flat}}.$$

But $-q = e^{\pi i}$, so the elements fixed by $s$ must be contained in

$$M^s \subseteq \mathcal{O}^n_B x_Z^{-2} [x_Z^{-2}] M^1_{\text{flat}} + \mathcal{O}^n_B x_Z^{-2} [x_Z^{-2}] M^{-q}_{\text{flat}}.$$ 

If $v \in M^S_q \times S_p \subseteq M^s$, then for some $K > 0$ we have

$$\alpha^K v \in M \cap \left(\mathcal{O}^n_B x_Z^{-2} M^1_{\text{flat}} + \mathcal{O}^n_B x_Z^{-2} M^{-q}_{\text{flat}}\right) \subseteq D_Z M.$$

Since $\alpha$ is also $S_q \times S_p$-invariant, we conclude that $M^S_q \times S_p \subseteq \mathbb{C}[\alpha^{-1}](D_Z M)S_q \times S_p$, proving (4).

**Case 2:** $W' \cong S_{3m}^n \times S_2$, so $Z$ is the kernel of $x_Z = z_i - z_j \in (h')^*$. Let $s \in S_q \times S_p$ be the transposition switching $z_i$ with $z_j$, and again let $\mathcal{O}_B^{an,s}$ denote the subspace of $\mathcal{O}_B^{an}$ fixed by $s$. We now set $\lambda = 2m + 1$, and again define

$$D_Z M = i^{-1} \left(\mathcal{O}_B^{an,s} M_{\text{flat}} + x_2^{\lambda} \mathcal{O}_B^{an,s} M_{\text{flat}}\right).$$
The arguments proceed as in the previous case, the only difference being the following two calculations. We have
\[
\zeta x^\lambda_Z = \frac{2\lambda(\lambda - 1)}{m} x^\lambda_Z - 4\lambda x^{\lambda-2} - 2 \sum_{k \neq i,j} \left( \frac{1}{z_i - z_k} - \frac{1}{z_j - z_k} \right) \lambda x^{\lambda-1}
\]
\[
- \frac{2}{m} \sum_a \left( \frac{1}{z_i - t_a} - \frac{1}{z_j - t_a} \right) \lambda x^{\lambda-1}
\]
\[
= 2 \sum_{k \neq i,j} \frac{1}{(z_i - z_k)(z_j - z_k)} \lambda x^\lambda_Z + 2 \sum_a \frac{1}{(z_i - t_a)(z_j - t_a)} \lambda x^\lambda_Z
\]
\[
\subseteq x^\lambda_Z \mathcal{O}^{\text{an},s}_B.
\]
Also, setting \(x^\zeta_Z = z_i^\zeta - z_j^\zeta\), we have
\[
\zeta \in \frac{1}{2m} (x^\zeta_Z)^2 - 2 \frac{x^{\zeta}_Z}{x^\lambda_Z} + \sum_{y \in Z} C_y^2 + \mathcal{O}^{\text{an},y}_B y,
\]
so for \(f \in \mathcal{O}^{\text{an},s}_B\), we have
\[
[\zeta, f] \in \frac{1}{m} x^{\zeta}_Z(f) x^\lambda_Z - 2 \frac{x^{\zeta}_Z(f)}{x^\lambda_Z} + \sum_{y \in Z} C_y(f)y + \mathcal{O}^{\text{an},y}_B y(f) + \mathcal{O}^{\text{an},s}_B.
\]
The properties follow as above.

Case 3: \(W' \cong S_m^n \times S_{m+1}\), so \(Z\) is the kernel of \(x^\lambda_Z = z_i - t_a\). Let \(x^{\zeta}_Z = z_i^\zeta - mt_a^\zeta\), and define
\[
\mathcal{O}^{\text{an},s}_B = \{ f \in \mathcal{O}_B \mid x^{\zeta}_Z(f) \in x_Z \mathcal{O}^{\text{an},s}_B \},
\]
\[
D_Z M = i^{-1}(\mathcal{O}^{\text{an},s}_B \circ \text{flat}^*).
\]
Properties (1), (3) and (4) follow as in the previous two cases. Now suppose \(f \in \mathcal{C}[\mathfrak{h}]^{S_n}\), and pick \(j, k\) such that \(\pi(x_j) = z_i\) and \(\pi(x_k) = t_a\). Then \((x^\zeta_j - x^\zeta_k)f\) is antisymmetric under \(s_{jk}\), so \((x^\zeta_j - x^\zeta_k)f \in (x_j - x_k)\mathcal{C}[\mathfrak{h}]\). Using equation 11 from the proof of Proposition 4.3, applying \(\pi\) gives
\[
\frac{1}{m} x^\lambda_Z \pi(f) \in x_Z \mathcal{C}[\mathfrak{h}'].
\]
Therefore \(\pi(\mathcal{C}[\mathfrak{h}]^{S_n}) \subseteq \mathcal{O}^{\text{an},s}_B\), so \(D_Z M\) is preserved by the action of \(\pi(\mathcal{C}[\mathfrak{h}]^{S_n})\). Finally note that, for \(j \neq i\) we have
\[
\frac{1}{z_j - z_i} - \frac{1}{z_j - t_a} = \frac{x_Z}{(z_j - z_i)(z_j - t_a)} \in x_Z \mathcal{O}^{\text{an}}_B \text{ and }
\]
\[
x^\zeta_Z \left( \frac{m}{z_j - z_i} + \frac{1}{z_j - t_a} \right) = m \frac{x_Z}{(z_j - z_i)^2} \frac{m}{(z_j - t_a)^2}
\]
\[
= \frac{mx_Z(2z_j - z_i - t_a)}{(z_j - z_i)^2(z_j - t_a)^2} \in x_Z \mathcal{O}^{\text{an}}_B, \text{ whence }
\]
\[
\frac{m}{z_j - z_i} + \frac{1}{z_j - t_a} \in \mathcal{O}^{\text{an},s}_B.
\]
Thus
\[-2\frac{z_i^\vee - z_j^\vee}{z_i - z_j} \cdot \frac{2}{m} \frac{z_j^\vee - mt_a^\vee}{z_j - t_a} = \frac{2}{m+1} \left( \frac{1}{z_j - z_i} - \frac{1}{z_j - t_a} \right) (z_i^\vee - mt_a^\vee) + \frac{2}{m+1} \left( \frac{m}{z_j - z_i} + \frac{1}{z_j - t_a} \right) (z_i^\vee + t_a^\vee - (1 + \frac{1}{m})z_j^\vee) \leq x_Z \mathcal{O}_B^{an} x_Z^\vee + \mathcal{O}_B^{an+} Z.\]

Similarly for \( b \neq a \), we have
\[-2 \frac{z_i^\vee - mt_b^\vee}{z_i - t_b} - \frac{2}{m} \frac{t_a^\vee - t_b^\vee}{t_a - t_b} = \frac{2}{m(m+1)} \left( \frac{1}{t_b - z_i} + \frac{1}{t_a - t_b} \right) (z_i^\vee - mt_a^\vee) + \frac{2}{m(m+1)} \left( \frac{m}{t_b - z_i} - \frac{1}{t_a - t_b} \right) (z_i^\vee + t_a^\vee - (m+1)t_b^\vee) \leq x_Z \mathcal{O}_B^{an} x_Z^\vee + \mathcal{O}_B^{an+} Z.\]

Finally
\[
\frac{1}{m} (z_i^\vee)^2 + (t_a^\vee)^2 = \frac{1}{m(m+1)} (x_Z^\vee)^2 + \frac{1}{m+1} (z_i^\vee + t_a^\vee)^2,
\]
so
\[
\zeta \in \frac{1}{m(m+1)} (x_Z^\vee)^2 - \frac{2}{m} \frac{x_Z^\vee}{x_Z} + x_Z \mathcal{O}_B^{an} x_Z^\vee + \sum_{y \in Z} C y^2 + \mathcal{O}_B^{an+} y.
\]

Now suppose \( f \in \mathcal{O}_B^{an+} \), so \( x_Z^\vee(f)/x_Z \in \mathcal{O}_B^{an} \). We have \([x_Z^\vee, x_Z] = (m+1)\), so
\[
x_Z^\vee(\zeta(f)) \leq \frac{1}{m(m+1)} x_Z (x_Z^\vee)^2 - \frac{2}{m} \frac{x_Z^\vee}{x_Z} + x_Z \mathcal{O}_B^{an} x_Z^\vee + \sum_{y \in Z} C y^2 x_Z^\vee(f) + x_Z \mathcal{O}_B^{an+} y(f) \leq x_Z \mathcal{O}_B^{an} + \sum_{y \in Z} C y^2 (x_Z \mathcal{O}_B^{an}) + \mathcal{O}_B^{an+} y(x_Z \mathcal{O}_B^{an}) \leq x_Z \mathcal{O}_B^{an}.\]

Thus \( \zeta(f) \in \mathcal{O}_B^{an+} \), proving property (2).

**Lemma 5.8.** Consider the intersection
\[
DM = \bigcap_{Z} D_Z M
\]
over all components of \( \mathfrak{h}' \setminus \mathfrak{h}'_{reg} \). This subspace has the following properties:

1. \( DM \) is functorial in \( M \).
2. \( DM \) is preserved by the actions of \( \zeta \) and \( \pi(\mathbb{C}[\mathfrak{h}]^{S_n}) \subseteq \mathbb{C}[\mathfrak{h}'] \).
3. \( DM \) is finitely generated over \( \pi(\mathbb{C}[\mathfrak{h}]^{S_n}) \).
4. If \( M \in \phi^{-1}A \) then \( M^{S_\pi \times S_\pi} = \mathbb{C}[\pi^{-1}(DM)]^{S_\pi \times S_\pi} \).
Theorem 2.3 of [18] shows that $\text{EM}$ is a homomorphism of $\text{C}_{\text{GM}}$ where, as usual, $\text{eu}$ we conclude that $\text{EM}$ $\text{EM}$ $\text{EM}$ is Noetherian, so (3) follows.

Theorem 5.9.

Proof. Certainly (1), (2) and (4) follow immediately from the corresponding properties of $D_2M$. To prove (3), let $\{U_i\}$ denote a Zariski open cover of $\text{h}_1'$ where $g_i \neq 0$, for some $g_i \in \text{C}[h']$. Moreover we may suppose $g_i$ does not vanish on any component of $\text{b}'_{\text{reg}}$. Now $\text{C}[h']$ is a UFD, so we have the notion of the order of pole of any element of $\text{O}(U_i) = \text{C}[h'][a^{-1}, g_i^{-1}]$ along some component $Z$. Property (3) of $D_2M$ states that the coefficients of any $v \in D_2M$ relative to $V_i$ have poles along $Z$ of order at most $K$, for some integer $K$. Thus

$$DM \subseteq \text{O}(U'_i)\alpha^{-K}V_i,$$

where $U'_i = \text{Spec C}[h'][g_i^{-1}] \subseteq h'$. In particular, $\text{O}(U'_i)DM$ is finitely generated over $\text{O}(U'_i)$, so $\text{O}(U')DM$ is a coherent sheaf on $U'$, where $U'$ is the union of the $U'_i$. We have $\text{O}(U')DM \subseteq M$, and the latter is locally free, so $\text{O}(U')DM$ is torsion free. Also $h' \setminus U'$ is contained in $\text{h}' \setminus \text{b}'_{\text{reg}}$, but doesn’t contain any component of the latter, so it has codimension at least 2. Therefore $\text{O}(U') = \text{C}[h']$, so Lemma 5.9 implies that $\text{C}[h']DM$ is finitely generated over $\text{C}[h']$. Finally $\text{C}[h']$ is finite over $\pi(\text{C}[h']^\Sigma)$, and the latter ring is Noetherian, so (3) follows.

We may now complete the proof of Proposition 5.3. Consider the subspaces

$$EM = \left\{ \sum_{w \in S_n} w \otimes v \left| v \in (DM)^{S_q \times S_p} \right\} \subseteq e\text{GM},
GM = H_cEM,$$

where, as usual, $e = \frac{1}{q!} \sum_{w \in S_n} w$. Property (2) of $DM$ implies that $EM$ is preserved by the actions of $p_{q,2}(v)e \in H_c$ and $\text{C}[h]^{S_q e} \subseteq H_c$. These generate the subalgebra $eH_c e \subseteq H_c$, so $EM$ is an $eH_c e$-submodule of $eGM$. Also $EM \cong (DM)^{S_q \times S_p}$ as $\text{C}[h]^{S_n}$-modules, so $EM$ is finitely generated over $\text{C}[h]^{S_n}$. Since $\text{eu}$ acts locally finitely on $\text{GM}$, we conclude that $EM$ decomposes into finite dimensional generalised eigenspaces for $\text{eu}$, with eigenvalues in $\Lambda + \text{Z}_{\geq 0}$ for some finite subset $\Lambda \subset \text{C}$. The same is true of $GM$, since $H_c$ is finite over $eH_c e$ and $ad \text{eu}$ is locally finite on $H_c$. This ensures that $GM \subseteq H_c - \text{mod}_{\text{RS}}$.

Moreover the composite

$$\eta : \text{Loc}^gGM = \text{O}(\text{h}'_{\text{reg}}) \otimes_{\text{C}[h]} GM \hookrightarrow \text{O}(\text{b}'_{\text{reg}}) \otimes_{\text{C}[h]} \text{GM} \twoheadrightarrow M$$

is a homomorphism of $\text{C}[S_q \times S_p] \otimes D(\text{h}'_{\text{reg}})$-modules, by Proposition 5.2. But $GM \supsetneq EM$, so if $M \in \phi^{-1}A$, then property (4) of $DM$ ensures that $\text{Im} (\eta)$ contains $M^{S_q \times S_p}$.

Theorem 2.3 of [18] shows that

$$e' = \frac{1}{q!} \sum_{w \in S_q \times S_p} w$$

generates $\text{C}[S_q \times S_p] \otimes D(\text{b}'_{\text{reg}})$ as a two-sided ideal, so $M^{S_q \times S_p} = e'M$ generates $M$ over $\text{C}[S_q \times S_p] \otimes D(\text{b}'_{\text{reg}})$. Therefore $\eta$ is an isomorphism, proving the result.

Although we have considered $c = \frac{1}{m}$ so far in this section, the following result will allow us to generalise to $c = \frac{2}{m}$. The first statement follows from Corollary 4.3 of [4] in the case $m = 2$, and from Theorem 5.12 and Proposition 5.14 of [20] when $m > 2$. The second is Theorem 5.10 of [20].

Theorem 5.9. Suppose $r > 0$ is coprime with $m$. There is an equivalence of categories $H_{\frac{1}{m}} - \text{mod}_{\text{RS}} \cong H_{\frac{1}{m}, \frac{2}{m}} - \text{mod}_{\text{RS}}$ which identifies $H_{\frac{1}{m}, \frac{2}{m}} - \text{mod}_{\text{RS}}$ with $H_{\frac{1}{m}, \frac{2}{m}} - \text{mod}_{\text{RS}}$ and sends $L(\tau_\lambda) \in H_{\frac{1}{m}, \frac{2}{m}} - \text{mod}_{\text{RS}}$ to $L(\tau_\lambda) \in H_{\frac{1}{m}, \frac{2}{m}} - \text{mod}_{\text{RS}}$. There is a $\text{C}$-algebra isomorphism $H_{\frac{1}{m}, \frac{2}{m}}(S_p) \cong H_{\frac{1}{m}, \frac{2}{m}}(S_p)$ which sends $D_v \in H_{\frac{1}{m}, \frac{2}{m}}(S_p) - \text{mod}_{fd}$ to $D_v \in H_{\frac{1}{m}, \frac{2}{m}}(S_p) - \text{mod}_{fd}$.
We may now combine the above to prove our remaining main results.

Proof of Theorems 3.11 and 1.10 As in the proof of Theorem 3.11 it suffices to assume $c > 0$. By the previous theorem, we may suppose $c = \frac{1}{m}$. We have seen that $\phi \text{Loc}^\alpha$ and $\psi^*$ identify $H_c - \mod_{RS}^d/H_c - \mod_{RS}^{q+1}$ and $C[S_q] \otimes_C H_q(S_p) - \mod_{id}$, respectively, with the full subcategory $A \subseteq \pi_1(b_{\text{reg}}^\gamma/S_q \times S_p) - \modid$. This proves the first statement of Theorem 3.11.

Let $F_{n,q}: H_\frac{1}{m}(S_n, C^n) - \mod_{RS}^d/H_\frac{1}{m}(S_n, C^n) - \mod_{RS}^{q+1} \to C[S_q] \otimes_C H_q(S_p) - \mod_{id}$ denote the equivalence.

Next we prove by induction on $q$ that if $q_m(\lambda) = q$, then $\text{Supp}(L(\tau_\lambda)) = X_q$. Suppose it holds for $q' < q$. If the support of $L(\tau_\lambda)$ is $X_q$, then $q_m(\lambda) \leq q$ by Theorem 3.11. However we cannot have $q_m(\lambda) < q$ by the inductive hypothesis. Therefore the irreducibles in $H_c - \mod_{RS}^d/H_c - \mod_{RS}^{q+1}$ are a subset of $\Omega = \{ L(\tau_\lambda) \mid q_m(\lambda) = q \}$.

We have a bijection

$$\{ \mu \vdash q \} \times \{ \nu \vdash p \mid q_m(\nu) = 0 \} \to \{ \lambda \vdash n \mid q_m(\lambda) = q \}$$

given by $(\mu, \nu) \mapsto m\mu + \nu$. The irreducibles in $C[S_q] - \mod_{id}$ are indexed by $\{ \mu \vdash q \}$, and those in $H_q(S_p) - \mod_{id}$ by $\{ \nu \vdash p \mid q_m(\nu) = 0 \}$. Therefore $|\Omega|$ is exactly the number of irreducibles in $C[S_q] \otimes_C H_q(S_p) - \mod_{id}$. Since the latter category is equivalent to $H_c - \mod_{RS}^d/H_c - \mod_{RS}^{q+1}$, it follows that each module in $\Omega$ must be an irreducible in $H_c - \mod_{RS}^d/H_c - \mod_{RS}^{q+1}$; that is, they must all be supported on $X_q$. This completes the induction.

Finally we must show that $F_{n,q}(L(\tau_{m\mu + \nu})) \cong \tau_\mu \otimes D_{\nu'}$, where $\mu \vdash q$ and $\nu \vdash p$ with $q_m(\nu) = 0$. This is known when $q = 0$ (see Corollary 4.7 of [3]).

Next suppose that $p = 0$, so $n = qm$. We will prove the statement in this case by induction on $q$. Now $M(\tau_{n\mu})$ is the polynomial representation, and $L(\tau_{n\mu}) = \mathcal{O}(X_q)$. This induces the trivial local system on $b_{\text{reg}}^{n\mu}$, so $F_{n,q}(L(\tau_{n\mu})) = \tau_{(q)}$. This proves the statement for $q \leq 2$. Suppose $q > 2$ and that the statement holds for $q - 1$. Let $b \in \mathfrak{H}$ be a point whose stabiliser in $S_n$ is $W' = S_{n-m} \times S_m$. Note that any minimally supported module $M$ in $H_c(W', C^n) - \mod_{RS}$ is of the form $M \cong M' \otimes L(\tau_{m\mu})$ for some $M' \in H_c(S_{n-m}, C^{n-m}) - \mod_{RS}^{-1}$; by abuse of notation, we’ll write $F_{n,q}(M) = M'$ to mean $F_{n,q}(M)$. By (4) and (5) of Theorem 3.3, there is a filtration of $\text{Res}_b M(\tau_{m\mu})$ whose successive quotients are the Verma modules corresponding to the composition factors of $\text{Res}_{W'}^{S_{n-m}}(\tau_{m\mu})$. We have a surjection

$$\text{Res}_b M(\tau_{m\mu}) \twoheadrightarrow \text{Res}_b L(\tau_{m\mu}).$$

By (2) of Theorem 3.3, the latter module has minimal support, so it is semisimple by Theorem 4.4. For $\alpha + n - m$ and $\beta + m$, the only irreducible quotient of $M(\tau_\alpha \otimes \tau_\beta)$ is $L(\tau_\alpha \otimes \tau_\beta)$, and this has minimal support only if $\beta = (m)$ and $\alpha = m\gamma$ for some $\gamma + q = 1$. Moreover by the Littlewood-Richardson rule, $\tau_{m\gamma} \otimes \tau_{(m)}$ is a composition factor of $\text{Res}_{W'}^{S_{n-m}}(\tau_{m\mu})$ only if $\tau_\gamma$ is a composition factor of $\text{Res}_{S_{n-m}}^{S_{n-m-1}}(\tau_\mu)$, and in this case it has multiplicity one. Therefore $\text{Res}_b L(\tau_{m\mu})$ is a submodule of

$$\bigoplus_{\text{Hom}_{S_{n-m}}^{\gamma+q-1}(\tau_\gamma, \tau_\mu) \neq 0} L(\tau_{m\gamma} \otimes \tau_{(m)}).$$

By the proofs of Lemmas 5.1 and 5.3, $L(\tau_{m\mu})$ is scheme-theoretically supported on $X_q$, and the induced local system on $b_{\text{reg}}^{n\mu}$ has trivial monodromy around $Z(\alpha_s)$ for each
shown that the number of irreducibles in $H$.

Proof of Corollary 1.7. Let us again denote by $□$ and we are done.

By the inductive hypothesis, we cannot have $q > 0$ and $p = 0$ cases shown above. In fact the left hand side is irreducible since $F_{n,q}$ is an equivalence, so

$$F_{n,q}(L(\tau_\lambda)) \cong \tau_\mu \otimes D_{\nu'}$$

By the inductive hypothesis, we cannot have $h(\tau_\lambda) > h(\tau_{m\mu_+\nu})$. Therefore $\lambda = m\mu + \nu$ and we are done.

Proof of Corollary 1.7. Let us again denote by $p_n$ the number of partitions of $n$, and let $p_{n,m} = |\{\lambda \vdash n \mid q_m(\lambda) = 0\}|$ denote the number of $m$-regular partitions of $n$. We have shown that the number of irreducibles in $H^{\geq m}_{\geq q_m}(S_n, \mathbb{C}^n)$ whose support is $X_q$ is $p_q p_{n-qm,m}$. This is the coefficient of $s^m t^{qm}$ in the formal power series

$$N(s,t) = \sum_{p,q \geq 0} p_q p_{p,m} s^{qm} t^{qm}.$$ 

It is well known that

$$\sum_{n \geq 0} p_n t^n = \prod_{n > 0} \frac{1}{1 - t^n}.$$ 

Every partition $\lambda$ of $n$ can be written uniquely as $\lambda = m\mu + \nu$ where $\nu$ is $m$-regular. Therefore

$$\sum_{n > 0} s^n = \sum_{n \geq 0} p_{n,m} t^n = \sum_{p,q \geq 0} p_q p_{p,m} t^{qm+p} = \left( \prod_{q > 0} \frac{1}{1 - t^{qm}} \right) \left( \sum_{p \geq 0} p_{p,m} t^p \right),$$

giving

$$\sum_{p \geq 0} p_{p,m} t^p = \prod_{n > 0} \frac{1}{1 - t^n}.$$
Thus

\[
N(s, t) = \left( \sum_{p \geq 0} P_{p, m} s^p \right) \left( \sum_{q \geq 0} P_q (st)^q \right) = \left( \prod_{p \geq 0} \frac{1}{1 - s^p} \right) \left( \prod_{q \geq 0} \frac{1}{1 - (st)^q} \right).
\]

Now consider the operator

\[
A_m = \sum_{i > 0} \alpha_{-im} \alpha_{im}
\]

acting on Fock space \( F \). The elements

\[
\prod_{i > 0} \alpha_{\nu_i} + \text{span}\{A\alpha_i \mid i > 0\} \in F
\]

form a basis for \( F \), where the \( \nu_i \) are nonnegative integers with only finitely many nonzero. This element is an eigenvector for \( A_m \) with eigenvalue

\[
\sum_{i > 0} \nu_i.
\]

Therefore

\[
\text{tr}_F(sA_1 tA_m) = \sum_{\nu} \left( \prod_{i > 0} s^{\nu_i} \right) \left( \prod_{i > 0} t^{\nu_i} \right)
\]

\[
= \left( \prod_{i > 0} \sum_{\nu_i \geq 0} s^{\nu_i} \right) \left( \prod_{i > 0} \sum_{\nu_i \geq 0} (st)^{\nu_i} \right)
\]

\[
= \left( \prod_{i > 0} \frac{1}{1 - s^i} \right) \left( \prod_{i > 0} \frac{1}{1 - (st)^i} \right)
\]

\[
= N(s, t),
\]

as required. □

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