AN EXPOSITION OF THE COMPACTNESS OF $L(Q^{cf})$

ENRIQUE CASANOVAS AND MARTIN ZIEGLER

Abstract. We give an exposition of the compactness of $L(Q^{cf}_C)$, for any set $C$ of regular cardinals.

§1. Introduction. We present here a new and short exposition of the proof of the compactness of the logic $L(Q^{cf}_C)$, first-order logic extended by the cofinality quantifier $Q^{cf}_C$, where $C$ is a class of regular cardinals. The logic and the proof of compactness are due to S. Shelah. The Compactness Theorem was stated and proved in [8], but this article is not self-contained and some fundamental steps of the proof must be found in the earlier article [7]. The interested reader consulting these two articles will soon realise that the structure of the proof is not completely transparent and that to fully understand the details requires a lot of work.

The most popular case of the cofinality quantifier is the logic $L(Q^{cf}_\omega)$ of the quantifier of cofinality $\omega$, that is, $C = \{\omega\}$. Our motivation comes from the application of $L(Q^{cf}_\omega)$ in [1] to an old problem on expandability of models. An anonymous referee of a preliminary version of [1] did not accept the validity (in ZFC) of the compactness proof presented in [8], apparently confused by the assumption of the existence of a weakly compact cardinal made at the beginning of the article. The assumption only applies to a previous result on a logic stronger than first-order logic even for countable models.

Our proof of compactness of $L(Q^{cf}_C)$ uses some ideas of [8], but it is more in the spirit of Keisler’s proof in [5] of countable compactness of the logic $L(Q_1)$ with the quantifier of uncountable cardinality. However, we use a simpler notion of weak model. J. Väänänen in the last chapter of [10] offers also a proof of compactness of $L(Q^{cf}_\omega)$ in Keisler’s style, but it is incomplete and only gives countable compactness (see I. Hodkinson’s review in [4]).

There are some other proofs in the literature, but also unsatisfactory. The proof by H. -D. Ebbinghaus in [3], based on a set-theoretical translation, is just an sketch and the proof of J. A. Makowsky and S. Shelah in [6] only replaces part of Shelah’s argument in [8] by a different reasoning and does not include all details.

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§2. Connections. For a linear ordering \((X, <)\), we use the expressions
\[
\exists^{\text{cf}} x \ A(x) \quad \text{and} \quad \forall^{\text{cf}} x \ A(x),
\]
for \(\forall x' \exists x \ (x' \leq x \land A(x))\) and \(\exists x' \forall x \ (x' \leq x \rightarrow A(x))\), respectively. The variables \(x, x'\) will range over the set \(X\), \(y, y'\) over \(Y\) and \(z, z'\) over \(Z\).

**Definition.** Let \(X\) and \(Y\) be two linear orderings. A connection between \(X\) and \(Y\) is a relation \(G \subset X \times Y\) which satisfies
\[
\exists^{\text{cf}} x \forall^{\text{cf}} y \ G(x, y) \quad \text{and} \quad \exists^{\text{cf}} y \forall^{\text{cf}} x \neg G(x, y).
\]
Note that \(X\) and \(Y\) cannot be connected if \(X\) or \(Y\) has a last element.

**Remark 2.1.**
1. If \(X\) has no last element, the relation \(x \leq y\) connects \(X\) with itself.
2. If \(G\) connects \(X\) and \(Y\), then \(\neg G^{-1} = \{(y, x) \mid \neg G(x, y)\}\) connects \(Y\) and \(X\).
3. If \(G\) connects \(X\) and \(Y\), and \(H\) connects \(Y\) and \(Z\), then
\[
K = \left\{(x, z) \mid \exists y' \left( \forall y \ (y' \leq y \rightarrow G(x, y)) \land H(y', z) \right) \right\}
\]
connects \(X\) and \(Z\).

**Proof.** 1. and 2. are easy to see. We will not use 3. and leave the proof to the reader.

**Remark 2.2.** If \(X\) and \(Y\) are connected by \(G\), then they are also connected by
\[
G' = \left\{(x, y) \mid \exists x' \left( x \leq x' \land \forall y' \ (y \leq y' \rightarrow G(x', y')) \right) \right\}.
\]
\(G'\) is antitone in \(x\) and monotone in \(y\).

**Proof.** It is easy to see that \(G^{\text{anti}} = \{(x, y) \mid \exists x' \left( x \leq x' \land G(x', y) \right) \}\) connects \(X\) and \(Y\) and is antitone in \(x\). Now it can be seen that
\[
G' = \left(-((\neg G^{-1})^{\text{anti}})^{-1}\right)^{\text{anti}}.
\]

**Lemma 2.3.** Two linear orders without last element are connected if and only if they have the same cofinality.

**Proof.** If \(\text{cf}(X) = \text{cf}(Y) = \kappa\), choose two increasing cofinal sequences \((x_\alpha \mid \alpha < \kappa)\) and \((y_\alpha \mid \alpha < \kappa)\) in \(X\) and \(Y\). Then,
\[
G = \{(x, y) \mid \exists \alpha \ (x \leq x_\alpha \land y_\alpha \leq y)\}
\]
connects \(X\) and \(Y\).\(^1\)

\(^1\)It suffices to assume that the \(y_\alpha\) are increasing. Also, one can use \(G = \{(x_\alpha, y) \mid y_\alpha \leq y\}\).
For the converse, assume that \( \text{cf}(X) = \kappa \) and that \( G \) connects \( X \) and \( Y \). Choose a cofinal sequence \( (x_\alpha \mid \alpha < \kappa) \) in \( X \) and elements \( y_\alpha \) in \( Y \) such that \( y_\alpha \leq y \rightarrow G(x_\alpha, y) \) for all \( y \). Then the \( y_\alpha \) are cofinal in \( Y \). To see this we use that there are cofinally many \( y \) such that \( \neg G(x, y) \) for sufficiently large \( x \), which implies \( \neg G(x_\alpha, y) \) for some \( \alpha \). It follows that \( y < y_\alpha \).

**Lemma 2.4.** Assume that \( G \subset X \times Y \) satisfies
\[
\exists^\text{cf} x \exists y \ G(x, y) \quad \text{and} \\
\forall y' \exists x' \forall xy \ ((x' \leq x \land y \leq y') \rightarrow \neg G(x, y)).
\]
Then \( G' = \{(x, y) \mid \exists y' (y' \leq y \land G(x, y'))\} \) connects \( X \) and \( Y \).

Note that a connecting \( G \) which is monotone in \( y \) satisfies (3) and (4).

**Proof.** This is a straightforward verification. \( \dashv \)

§3. The main lemma. Consider a \( L \)-structure \( M \) with two (parametrically) definable linear orderings, \( <_\varphi \) and \( <_\psi \) of its universe, both without last element. We say that \( \varphi \) and \( \psi \) are **definably connected**, if there is a definable connection between \( (M, <_\varphi) \) and \( (M, <_\psi) \).

Recall that a formula \( \varphi(x) \) isolates a partial type \( \Sigma(x) \) in a theory \( T \) if it is consistent with \( T \) and implies \( \Sigma(x) \) in \( T \) (see Definition 4.1.1 in [9] or the definition of locally realizing a type in [2]). \( T \) isolates \( \Sigma(x) \) if some formula \( \varphi(x) \) does it in \( T \).

**Lemma 3.1.** If \( \varphi \) and \( \psi \) are not definably connected, and \( c \) is a new constant, the theory
\[
T' = \text{Th}(M, m)_{m \in M} \cup \{m <_\varphi c \mid m \in M\}
\]
does not isolate the partial type \( \Sigma(y) = \{n <_\psi y \mid n \in M\} \).

**Proof.** Assume that \( \gamma(c, y) \), for some \( L(M) \)-formula \( \gamma(x, y) \), isolates \( \Sigma(y) \) in \( T' \). This means that

1. \( T' \cup \{\gamma(c, y)\} \) is consistent.
2. \( T' \vdash \gamma(c, y) \rightarrow n <_\psi y \) for all \( n \in M \).

We show that the relation \( G \) defined by \( \gamma(x, y) \) has properties (3) and (4) of Lemma 2.4, where \( X = (M, <_\varphi) \) and \( Y = (M, <_\psi) \). This will contradict the hypothesis of our lemma.

That \( T' \cup \{\gamma(c, y)\} \) is consistent means that for all \( m \in M \), the theory \( \text{Th}(M, m)_{m \in M} \) does not prove \( m \leq_\varphi c \rightarrow \exists y \gamma(c, y) \), which means that \( M \models \exists x (m \leq_\varphi x \land \exists y \gamma(x, y)) \). This is exactly condition (3) of 2.4.

That \( T' \vdash \gamma(c, y) \rightarrow n <_\psi y \) means that there is an \( m \in M \) such that \( \text{Th}(M, m)_{m \in M} \) proves \( (m \leq_\varphi c \land \gamma(c, y)) \rightarrow n <_\psi y \), which means \( M \models \forall x y ((m \leq_\varphi x \land y \leq_\psi n) \rightarrow \neg \gamma(x, y)) \). The existence of such \( m \) for all \( n \) is exactly condition (4) of 2.4. \( \dashv \)
Corollary 3.2. Assume $\kappa$ is regular, $|M|, |L| \leq \kappa$, and $<_\varphi$ is a definable linear ordering of $M$ without last element. Then, there is an elementary extension $N$ of $M$ such that:

1. $M$ is not $<_\varphi$-cofinal in $N$.
2. If $<_\psi$ is a definable linear ordering of $M$ of cofinality $\kappa$, and $\psi$ and $\varphi$ are not definably connected, then $M$ is $<_\psi$-cofinal in $N$.

Proof. Let $c$ be a new constant and let $T' = Th(M, m)_{m \in M} \cup \{m <_\varphi c \mid m \in M\}$. By Lemma 3.1, $T'$ does not isolate any of the types $\Sigma_\psi(y) = \{n <_\psi y \mid n \in M\}$. Each $\Sigma_\psi(y)$ consists of a $<_\psi$-ordered chain of formulas increasing in strength. So by regularity of $\kappa$, for any $<_\psi$ of cofinality $\kappa$, the type $\Sigma_\psi(y)$ cannot be isolated neither by means of a set of $<_\kappa$ formulas. By the $\kappa$-Omitting Types Theorem (see Theorem 2.2.19 in [2]), there is a model of $T'$ omitting all types $\Sigma_\psi(y)$ for any $<_\psi$ of cofinality $\kappa$. This gives the elementary extension $N$.

This corollary applies in particular to the case $\kappa = \omega$. Here, the assumption on the cofinality of $<_\psi$ is not needed since it is the only possible cofinality in a countable model, and the Omitting Types Theorem used in the proof is the ordinary one for countable languages and countably many nonisolated types.

§4. Completeness. For a language $L$, let $L(Q^{cf})$ be the set of formulas which are built like first-order formulas but using an additional two-place quantifier $Q^{cf}xy \varphi$, for different variables $x$ and $y$. Let $C$ be a class of regular cardinals and $M$ an $L$-structure. For a binary relation $R$ on $M$, we write “$cf R \in C$” for “$R$ is a linear ordering of $M$, without last element and cofinality in $C$.”

The satisfaction relation $|=_C$ for $L$-structures $M$, $L(Q^{cf})$-formulas $\psi(\vec{z})$, and tuples $\vec{c}$ of elements of $M$ is defined inductively, where the $Q^{cf}$-step is $M |=_C Q^{cf}xy \varphi(x, y, \vec{c}) \iff cf \{(a, b) \mid M |=_C \varphi(a, b, \vec{c})\} \in C$.

We say that $M$ is a $C$-model of $T$, a set of $L(Q^{cf})$-sentences, if $M |=_C \psi$ for all $\psi \in T$.

A weak structure $M^* = (M, ...)$ is an $L^*$-structure, where $L^*$ is an extension of $L$ by an $n$-ary relation $R_\varphi$ for every $L(Q^{cf})$-formula $\varphi(x, y, z_1, ..., z_n)$. Satisfaction is defined using the rule $M^* |= Q^{cf}xy \varphi(x, y, \vec{c}) \iff M^* |= R_\varphi(\vec{c})$.

In weak structures, every $L(Q^{cf})$-formula is equivalent to a first-order $L^*$-formula, and conversely. So, the $L(Q^{cf})$-model theory of weak structures is the same as their first-order model theory.

Note that the $C$-semantics of $M$ is given by the semantics of the weak structure $M^*$ if one sets $M^* |= R_\varphi(\vec{c}) \iff M |=_C Q^{cf}xy \varphi(x, y, \vec{c})$.

The following lemma is clear:
Lemma 4.1. The C-semantics of $M$ is given by the weak structure $M^*$ if and only if

$$M^* \models Q^\text{cf} xy \varphi(x,y,z) \iff \text{cf}\{(a,b) \mid M^* \models \varphi(a,b,z)\} \in C,$$

for all $\varphi$ and $z$.

The following property of weak structures $M^*$ can be expressed by a set $SA(L)$ of $L(Q^\text{cf})$ sentences (the Shelah Axioms):

- If the $L(Q^\text{cf})(M)$-formula $\varphi(x,y)$ satisfies $M^* \models Q^\text{cf} xy \varphi(x,y)$, then $\varphi$ defines a linear ordering $<_\varphi$ without last element. Furthermore, if $\psi(x,y)$ defines a linear ordering $<_\psi$ and $M^* \models \neg Q^\text{cf} xy \psi(x,y)$, there is no definable connection between $(M,<_\varphi)$ and $(M,<_\psi)$.

Lemma 4.2. L-structures with the C-semantics are models of $SA(L)$.

Proof. This follows from Lemma 2.3.

Theorem 4.3. Let $C$ be a nonempty class of regular cardinals, different from the class of all regular cardinals. An $L(Q^\text{cf})$-theory $T$ has a C-model if and only if $T \cup SA(L)$ has a weak model.

Proof. One direction follows from Lemma 4.2. For the other direction, assume that $T \cup SA(L)$ has a weak model.

Claim 1: If $L$ is countable, $T$ has an $\{\omega\}$-model of cardinality $\omega_1$.

Proof. Let $M_0^*$ be a countable weak model of $T \cup SA(L)$. Consider a linear ordering $<_\varphi$ without last element and $M_0^* \models Q^\text{cf} xy \varphi$. Then by Corollary 3.2 for $\kappa = \omega$ and the axioms $SA(L)$, there is an elementary extension $M_1^*$ such that $M_0$ is not $<_\varphi$-cofinal in $M_1$, but $<_\varphi$-cofinal in $M_1$ for every $\psi$ with $M_0^* \models Q^\text{cf} xy \psi$. We may assume that $M_1^*$ is countable. Continuing in this manner, taking unions at limit stages, one constructs an elementary chain of countable weak models $M_0^* < M_1^* \cdots$ of length $\omega_1$ with union $M^*$, such that

1. If $<_\varphi$ is a linear ordering of $M^*$ without last element and $M^* \models \neg Q^\text{cf} xy \varphi$, and if the parameters of $\varphi$ are in $M_\alpha$, then for uncountably many $\beta \geq \alpha$, $M_\beta$ is not $<_\varphi$-cofinal in $M_{\beta+1}$.
2. If $M^* \models Q^\text{cf} xy \psi$, and the parameters of $\psi$ are in $M_\alpha$, then $M_\alpha$ is $<_\psi$-cofinal in $M$.

It follows that, if $M^* \models \neg Q^\text{cf} xy \varphi$, then either $\varphi$ does not define a linear ordering without last element, or $<_\varphi$ has cofinality $\omega_1$. And, if $M^* \models Q^\text{cf} xy \psi$, then $<_\psi$ has cofinality $\omega$. By Lemma 4.1, $M$ is an $\{\omega\}$-model of the $L(Q^\text{cf})$-theory of $M^*$, and hence an $\{\omega\}$-model of $T$. This proves Claim 1.

Let $L'$ be the extension of $L$, which has for every $L(Q^\text{cf})$-formula $\varphi(x,y,z)$ a new relation symbol $V_\varphi$ of arity $2 + 2 \cdot |z|$. Let $SK$ be the set of axioms
which state that if \( \varphi(x,y,\tilde{c}_1) \) and \( \varphi(x,y,\tilde{c}_2) \) define linear orderings without last elements, and
\[
Q_{\text{cf}} xy \varphi(x,y,\tilde{c}_1) \leftrightarrow Q_{\text{cf}} xy \varphi(x,y,\tilde{c}_2),
\]
then \( V_\varphi(x,y,\tilde{c}_1,\tilde{c}_2) \) defines a connection between the two orderings.

Claim 2: \( T \cup SA(L') \cup SK \) has a weak model.

**Proof.** By compactness, we may assume that \( L \) is countable. Then \( T \) has an \( \{ \omega \} \)-model \( M \) of cardinality \( \omega \), by Claim 1. If \( \varphi(x,y,\tilde{c}_1) \) and \( \varphi(x,y,\tilde{c}_2) \) define linear orderings without last element, and \( M \models C \) \( Q_{\text{cf}} xy \varphi(x,y,\tilde{c}_1) \leftrightarrow Q_{\text{cf}} xy \varphi(x,y,\tilde{c}_2) \), then the two orderings have the same cofinality, namely \( \omega \) or \( \omega \), and there is a connection between them by Lemma 2.3. We interpret \( V_\varphi(x,y,\tilde{c}_1,\tilde{c}_2) \) by any such connection. This yields an expansion of \( M \), which is an \( \{ \omega \} \)-model of \( T \cup SA(L') \cup SK \). This proves Claim 2.

To prove the theorem, we choose two regular cardinals \( \lambda < \kappa \) such that \(|L| \leq \kappa \) and either \( \lambda \not\in C \) and \( \kappa \in C \) or conversely. Let \( M_0^* \) be a weak model of \( T \cup SA(L') \cup SK \). If \( M_0^* \) is finite, it is a \( C \)-model of \( T \) for trivial reasons.\(^2\)

Otherwise, we may assume that \( M_0^* \) has cardinality \( \kappa \) and all \( L(Q_{\text{cf}}) \)-definable linear orderings without last element have cofinality \( \kappa \). Let us first assume that \( \lambda \not\in C \) and \( \kappa \in C \).

Consider an \( L(Q_{\text{cf}}) \)-definable linear ordering \( <_\varphi \) without last element and \( M_0^* \models \neg Q_{\text{cf}} xy \varphi \). Then by Corollary 3.2 and the axioms \( SA(L') \), there is an elementary chain of weak models \( M_0^* < M_1^* < \cdots < M_\lambda^* \) of length \( \lambda \) with union \( M^* \), such that

1. If \( <_\varphi \) is an \( L(Q_{\text{cf}}) \)-definable linear ordering \( <_\varphi \) of \( M^* \) without last element and \( M^* \models \neg Q_{\text{cf}} xy \varphi \), and if the parameters of \( \varphi \) are in \( M_\alpha \), then for \( \lambda \)-many \( \beta \geq \alpha \), \( M_\beta \) is not \( <_\varphi \)-cofinal in \( M_{\beta+1} \).
2. If \( M^* \models Q_{\text{cf}} xy \psi \), and the parameters of \( \psi \) are in \( M_\alpha \), then \( M_\alpha \) is \( <_\psi \)-cofinal in \( M \).

It follows that, if \( M^* \models \neg Q_{\text{cf}} xy \varphi \), then either \( \varphi \) does not define a linear ordering without last element, or \( <_\varphi \) has cofinality \( \lambda \). And, if \( M^* \models Q_{\text{cf}} xy \psi \), then \( <_\psi \) has cofinality \( \kappa \). By Lemma 4.1, \( M \models L \) is an \( C \)-model of the \( L(Q_{\text{cf}}) \)-theory of \( M^* \), and whence a \( C \)-model of \( T \).

The proof in the case \( \lambda \in C \) and \( \kappa \not\in C \) is, mutatis mutandis, the same. \( \varnothing \)

**Corollary 4.4.** For every class \( C \) of regular cardinals, the logic \( L(Q_{\text{cf}}^C) \) is compact.

\(^2\)SA(L) is used here.
We have always assumed that whenever \(Q^{\text{cf}}_{\varphi}xy\varphi(x,y,\bar{c})\), the definable ordering \(<_\varphi\) linearly orders the universe. This is not exactly the assumption of Shelah in [8]: with his definition \(<_\varphi\) linearly orders \(\{x \mid \exists y \varphi(x,y,\bar{c})\}\), the domain of \(\varphi\). The results presented here, in particular completeness and compactness, also apply to this modification of the semantics, it suffices to add, for each such \(\varphi\), new relation symbols \(R_\varphi\) and \(H_\varphi\), and declare that for every \(\bar{c}\), \(R_\varphi(x,y,\bar{c})\) defines a linear ordering \(<'_\varphi\) on the universe and \(H_\varphi(x,y,\bar{c})\) connects \(<_\varphi\) and \(<'_\varphi\). This gives compactness. For the formulation of completeness (Theorem 4.3), one must adapt the axioms SA to the new situation.

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