ON THE GROUP OF STRONG SYMPLECTIC HOMEOMORPHISMS

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Abstract. We generalize the "hamiltonian topology" on hamiltonian isotopies to an intrinsic "symplectic topology" on the space of symplectic isotopies. We use it to define the group $SSympeo(M, \omega)$ of strong symplectic homeomorphisms, which generalizes the group $Hameo(M, \omega)$ of hamiltonian homeomorphisms introduced by Oh and Muller. The group $SSympeo(M, \omega)$ is arcwise connected, is contained in the identity component of $Sympeo(M, \omega)$; it contains $Hameo(M, \omega)$ as a normal subgroup and coincides with it when $M$ is simply connected. Finally its commutator subgroup $[SSympeo(M, \omega), SSympeo(M, \omega)]$ is contained in $Hameo(M, \omega)$.

1. Introduction

The Eliashberg-Gromov symplectic rigidity theorem says that the group $Symp(M, \omega)$ of symplectomorphisms of a closed symplectic manifold $(M, \omega)$ is $C^0$ closed in the group $Diff^\infty(M)$ of $C^\infty$ diffeomorphisms of $M$ (see [8]). This means that the "symplectic" nature of a sequence of symplectomorphisms survives topological limits. Also Lalonde-McDuff-Polterovich have shown in [9] that for a symplectomorphism, being "hamiltonian" is topological in nature. These phenomena attest that there is a $C^0$ symplectic topology underlying the symplectic geometry of a symplectic manifold $(M, \omega)$.

According to Oh-Muller ([10]), the automorphism group of the $C^0$ symplectic topology is the closure of the group $Symp(M, \omega)$ in the group $Homeo(M)$ of homeomorphisms of $M$ endowed with the $C^0$ topology. That group, denoted $Sympeo(M, \omega)$ has been called the group of symplectic homeomorphisms:

$$Sympeo(M, \omega) =: \overline{Symp(M, \omega)}.$$

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The $C^0$ topology on $\text{Homeo}(M)$ coincides with the metric topology coming from the metric

$$d_0(g, h) = \max(\sup_{x \in M} d_0(g(x), h(x)), \sup_{x \in M} d_0(g^{-1}(x), h^{-1}(x)))$$

where $d_0$ is a distance on $M$ induced by some riemannian metric [11].

On the space $P\text{Homeo}(M)$ of continuous paths $\gamma : [0, 1] \to \text{Homeo}(M)$, one has the distance

$$\overline{d}(\gamma, \mu) = \sup_{t \in [0, 1]} \overline{d}^1(\gamma(t), \mu(t)).$$

Consider the space $P\text{Ham}(M)$ of all isotopies $\Phi_H = [t \mapsto \Phi^t_H]$ where $\Phi^t_H$ is the family of hamiltonian diffeomorphisms obtained by integration of the family of vector fields $X_H$ for a smooth family $H(x, t)$ of real functions on $M$, i.e.

$$\frac{d}{dt} \Phi^t_H(x) = X_H(\Phi^t_H(x))$$

and $\Phi^0_H = \text{id}$. Recall that $X_H$ is uniquely defined by the equation

$$i(X_H)\omega = dH$$

where $i(.)$ is the interior product.

The set of time one maps of all hamiltonian isotopies $\{\Phi^t_H\}$ form a group, denoted $\text{Ham}(M, \omega)$ and called the group of hamiltonian diffeomorphisms.

**Definition** The hamiltonian topology [11] on $P\text{Ham}(M)$ is the metric topology defined by the distance

$$d_{\text{ham}}(\Phi_H, \Phi_H') = \|H - H'\| + \overline{d}(\Phi_H, \Phi_H')$$

where

$$\|H - H'\| = \int_0^1 \text{osc}(H - H')dt.$$ 

and the oscillation of a function $u$ is

$$\text{osc}(u) = \max_{x \in M} u(x) - \min_{x \in M} u(x).$$
Let $Hameo(M,\omega)$ denote the space of all homeomorphisms $h$ such that there exists a continuous path $\lambda \in PHomeo(M)$ such that

$$\lambda(0) = id, \lambda(1) = h$$

and there exists a Cauchy sequence (for the $d_{ham}$ distance) of hamiltonian isotopies $\Phi_{H^n}$, which $C^0$ converges to $\lambda$ (in the $d$ metric).

The following is the first important theorem in the $C^0$ symplectic topology [11]:

**Theorem (Oh-Muller)**

The set $Hameo(M,\omega)$ is a topological group. It is a normal subgroup of the identity component $Sympeo_0(M,\omega)$ in $Sympeo(M,\omega)$. If $H^1(M,\mathbb{R}) \neq 0$, then $Hameo(M,\omega)$ is strictly contained in $Sympeo_0(M,\omega)$.

**Remark**

It is still unknown in general if the inclusion

$$Hameo(M,\omega) \subset Sympeo_0(M,\omega)$$

is strict.

The group $Hameo(M,\omega)$ is the topological analogue of the group $Ham(M,\omega)$ of hamiltonian diffeomorphisms.

The goal of this paper is to construct a subgroup of $Sympeo_0(M,\omega)$, denoted $SSympeo(M,\omega)$ and nicknamed the group of strong symplectic homeomorphisms, containing $Hameo(M,\omega)$, that is:

$$Hameo(M,\omega) \subset SSympeo(M,\omega) \subset Sympeo_0(M,\omega).$$

Like $Hameo(M,\omega)$, the group $SSympeo(M,\omega)$ is defined using a blend of the $C^0$ topology and the Hofer topology on the space $Iso(M,\omega)$ of symplectic isotopies of $(M,\omega)$.

We believe that $SSympeo(M,\omega)$ is "more right" than the group $Sympeo(M,\omega)$ for the $C^0$ symplectic topology. In particular the flux homomorphism seems to exist on $SSympeo(M,\omega)$. This will be the object of a future paper.
The results of this paper have been announced in [1].

The $C^0$ counterpart of the $C^\infty$ contact topology is been worked out in [5], [6].

2. The symplectic topology on $Iso(M,\omega)$

Let $Iso(M,\omega)$ denote the space of symplectic isotopies of a compact symplectic manifold $(M,\omega)$. Recall that a symplectic isotopy is a smooth map $H : M \times [0,1] \to M$ such that for all $t \in [0,1]$, $h_t : M \to M$, $x \mapsto H(x,t)$ is a symplectic diffeomorphism and $h_0 = id$.

The "Lie algebra" of $Symp(M,\omega)$ is the space $symp(M,\omega)$ of symplectic vector fields, i.e the set of vector fields $X$ such that $i_X\omega$ is a closed form.

Let $\phi_t$ be a symplectic isotopy, then

$$\dot{\phi}_t(x) = \frac{d\phi_t}{dt}(\phi_t^{-1}(x))$$

is a smooth family of symplectic vector fields.

By the theorem of existence and uniqueness of solutions of ODE's,

$$\Phi \in Iso(M,\omega) \mapsto \dot{\phi}_t$$

is a 1-1 correspondence between $Iso(M,\omega)$ and the space $C^\infty([0,1],symp(M,\omega))$ of smooth families of symplectic vector fields. Hence any distance on $C^\infty([0,1],symp(M,\omega))$ gives rise to a distance on $Iso(M,\omega)$.

An intrinsic topology on the space of symplectic vector fields.

We define a norm $||.||$ on $symp(M,\omega)$ as follows: first we fix a riemannian metric $g$ (which may be the one we used to define $d_0$ above, or any other riemannian metric), and a basis $B = \{h_1,..,h_k\}$ of harmonic 1-forms. For Hodge theory, we refer to [12].

Recall that the space $harm^1(M,g)$ of harmonic 1-forms is a finite dimensional vector space and its dimension is the first Betti number of $M$. 
On $\text{harm}^1(M, g)$, we put the following "Euclidean" norm:

for $H \in \text{harm}^1(M, g)$, $H = \sum \lambda_i h_i$, define:

$$|H|_B =: \sum |\lambda_i|.$$

This norm is equivalent to any other norm. Here we choose this one for convenience in the calculations and estimates to come later.

Given $X \in \text{sym}(M, \omega)$, we consider the Hodge decomposition of $i_X \omega$ [10]: there is a unique harmonic 1-form $H_X$ and a unique function $u_X$ such that

$$i_X \omega = H_X + du_X$$

Now we define a norm $||.||$ on the space $\text{symp}(M, \omega)$ by:

$$||X|| = |H_X|_B + \text{osc}(u_X).$$

(1)

It is easy to see that this is a norm. Let us just verify that $||X|| = 0$ implies that $X = 0$. Indeed $|H_X|_B = 0$ implies that $i_X \omega = du_X$, and $\text{osc}(u_X) = 0$ implies that $u_X$ is a constant, therefore $du_X = 0$.

**Remark**

This norm is not invariant by $\text{Symp}(M, \omega)$. Hence it does not define a Finsler metric on $\text{Symp}(M, \omega)$.

The norm $||.||$ defined above depends of course on the riemannian metric $g$ and the basis $B$ of harmonic 1-forms. However, we have the following:

**Theorem 1**

All the norms $||.||$ defined by equation (1) using different riemannian metrics and different basis of harmonic 1-forms are equivalent.

Hence the topology on the space $\text{symp}(M, \omega)$ of symplectic vector fields defined by the norm (1) is intrinsic : it is independent of the choice of the riemannian metric $g$ and of the basis $B$ of harmonic 1-forms.
For each symplectic isotopy $\Phi = (\phi_t)$, consider the Hodge decomposition of $i_{(\dot{\phi}_t)}\omega$

$$i_{(\dot{\phi}_t)}\omega = H_t^{\Phi} + du_t^{\Phi}$$

where $H_t^{\Phi}$ is a harmonic 1-form.

We define the length $l(\Phi)$ of the isotopy $\Phi = (\phi_t)$ by:

$$l(\Phi) = \int_0^1 (|H_t^{\Phi}| + osc(u_t^{\Phi}))dt = \int_0^1 ||\dot{\phi}_t||dt$$

One also writes

$$\int_0^1 ||\dot{\phi}_t||dt = ||\dot{\phi}_t||.$$

In the expressions above, we have written $|H_t^{\Phi}|$ for $|H_t^{\Phi}|_B$, where $B$ is a fixed basis of $harm^1(M,g)$, for a fixed riemannian metric $g$.

We define the distance $D_0(\Phi, \Psi)$ between two symplectic isotopies $\Phi = (\phi_t)$ and $\Psi = (\psi_t)$ by:

$$D_0(\Phi, \Psi) = |||\dot{\phi}_t - \dot{\psi}_t||| = \int_0^1 (|H_t^{\Phi} - H_t^{\Psi}| + osc(u_t^{\Phi} - u_t^{\Psi}))dt.$$ 

Denote by $\Phi^{-1} = (\phi_t^{-1})$ and by $\Psi^{-1} = (\psi_t^{-1})$ the inverse isotopies.

Remarks
1. The distance $D_0(\Phi, \Psi) \neq l(\Psi^{-1}\Phi)$ unless $\Psi$ and $\Phi$ are hamiltonian isotopies (see proposition 1).
2. $l(\Phi) \neq l(\Phi^{-1})$ unless $\Phi$ is hamiltonian.

In view of the remarks above, we define a more ”symmetrical” distance $D$ by:

$$D(\Phi, \Psi) = (D_0(\Phi, \Psi) + D_0(\Phi^{-1}, \Psi^{-1}))/2$$

Following [11], we define the symplectic distance on $Iso(M,\omega)$ by:

$$d_{symp}(\Phi, \Psi) = d(\Phi, \Psi) + D(\Phi, \Psi).$$

Definition. The symplectic topology on $Iso(M,\omega)$ is the metric topology defined by the distance $d_{symp}$.
Theorem 1’

The symplectic topology on \( \text{Iso}(M, \omega) \) natural: it is independent of all choices involved in its definition.

We may also define another distance \( D^\infty \) on \( \text{Iso}(M, \omega) \):

\[
D_0^\infty(\Phi, \Psi) = \sup_{t \in [0,1]}(|H_t^\Phi - H_t^\Psi| + \text{osc}(u_t^\Phi - u_t^\Psi))
\]

\[
D^\infty(\Phi, \Psi) = (D_0^\infty(\Phi, \Psi) + D_0^\infty(\Phi^{-1}, \Psi^{-1}))/2
\]

and

\[
d^\infty_{\text{symp}}(\Phi, \Psi) = \overline{d}(\Phi, \Psi) + D^\infty(\Phi, \Psi)
\]

Proposition 1.

Let \( \Phi = (\phi_t), \Psi = (\psi_t) \) be two hamiltonian isotopies and \( \sigma_t = (\psi_t)^{-1}\phi_t \) then

\[
|||\dot{\sigma}_t||| = |||\dot{\phi}_t - \dot{\psi}_t||| = \int_0^1 \text{osc}(u_t^\Phi - u_t^\Psi)dt
\]

Proof

This follows immediately from the equation

\[
\dot{\sigma}_t = (\psi_t^{-1})_*(\dot{\phi}_t - \dot{\psi}_t),
\]

which is a consequence of proposition 4. \( \square \)

Corollary.

The distance \( d_{\text{sym}} \) reduces to the hamiltonian distance \( d_{\text{ham}} \) when \( \Phi \) and \( \Psi \) are hamiltonian isotopies.

The symplectic topology reduces to the "hamiltonian topology" of [11] on paths in \( \text{Ham}(M, \omega) \).

A Hofer-like metric on \( \text{Symp}(M, \omega) \)
For any $\phi \in \text{Symp}(M, \omega)$, define:

$$e_0(\phi) = \inf(l(\Phi))$$

where the infimum is taken over all symplectic isotopies $\Phi$ from $\phi$ to the identity.

The following result was proved in [2].

**Theorem.**

The map $e : \text{Symp}(M, \omega) \to \mathbb{R} \cup \{\infty\}$:

$$e(\phi) =: (e_0(\phi) + e_0(\phi^{-1}))/2$$

is a metric on the identity component $\text{Symp}(M, \omega)_0$ in the group $\text{Symp}(M, \omega)$, i.e. it satisfies (i) $e(\phi) \geq 0$ and $e(\phi) = 0$ iff $\phi$ is the identity.

(ii) $e(\phi) = e((\phi)^{-1})$

(iii) $e(\phi, \psi) \leq e(\phi) + e(\psi)$.

The restriction to $\text{Ham}(M, \omega)$ is bounded from above by the Hofer norm.

Recall that the Hofer norm [8] of a hamiltonian diffeomorphism $\phi$ is

$$||\phi||_H = \inf(l(\Phi_H))$$

where the infimum is taken over all hamiltonian isotopies $\Phi_H$ from $\phi$ to the identity.

The Hofer-like metric above depends on the choice of a riemannian metric $g$ and a basis $\mathcal{B}$ of harmonic 1-forms. Hence it is not "natural". However, by theorem 1, all the metrics constructed that way are equivalent; so they define a natural topology on $\text{Symp}(M, \omega)$.

3. Strong symplectic homeomorphisms

**Definition:** A homeomorphism $h$ is said to be a strong symplectic homeomorphism if there exists a continuous path $\lambda : [0, 1] \to \text{Homeo}(M)$ such that $\lambda(0) = \text{id}; \lambda(1) = h$ and a sequence $\Phi^n = (\phi^n_t)$ of symplectic isotopies, which
converges to $\lambda$ in the $C^0$ topology (induced by the norm $\overline{d}$) and such that $\Phi^n$ is Cauchy for the metric $d_{\text{symp}}$.

We will denote by $SSympeo(M,\omega)$ the set of all strong symplectic homeomorphisms. This set is well defined independently of any riemannian metric or any basis of harmonic 1-forms.

Clearly, if $M$ is simply connected, the set $SSympeo(M,\omega)$ coincides with the group $Hameo(M,\omega)$.

We denote by $SSympeo(M,\omega)^\infty$ the set defined like in $SSympeo(M,\omega)$ but replacing the norm $d_{\text{symp}}$ by the norm $d_{\text{symp}}^\infty$.

Let $\mathcal{P}Homeo(M)$ be the set of continuous paths $\gamma : [0,1] \to Homeo(M)$ such that $\gamma(0) = id$, and let $\mathcal{P}^\infty(Harm^1(M))$ be the space of smooth paths of harmonic 1-forms.

We have the following maps:

$A_1 : Iso(M,\omega) \to \mathcal{P}Homeo(M), \Phi \mapsto \Phi(t)$

$A_2 : Iso(M,\omega) \to \mathcal{P}^\infty(Harm^1(M), \Phi \mapsto \mathcal{H}^\Phi$.

$A_3 : Iso(M,\omega) \to C^\infty(M \times [0,1], \mathbb{R}), \Phi \mapsto u^\Phi$.

Let $Q$ be the image of the mapping $A = A_1 \times A_2 \times A_3$ and $\overline{Q}$ the closure of $Q$ inside $I(M,\omega) =: \mathcal{P}Homeo(M) \times \mathcal{P}^\infty(Harm^1(M) \times C^\infty(M \times [0,1], \mathbb{R})$, with the symplectic topology, which is the $C^0$ topology on the first factor and the metric topology from $D$ on the second and third factor.

Then $SSympeo(M,\omega)$ is just the image of the evaluation map of the path at $t=1$ of the image of the projection of $Q$ on the first factor. This defines a surjective map:

$a : Q \to SSympeo(M,\omega)$

The symplectic topology on $SSympeo(M,\omega)$ is the quotient topology induced by $a$. 
Our main result is the following

**Theorem 2.**

Let \((M, \omega)\) be a closed symplectic manifold. Then \(SSympeo(M, \omega)\) is an arcwise connected topological group, containing \(Hameo(M, \omega)\) as a normal subgroup, and contained in the identity component \(Sympeo_0(M, \omega)\) of \(Sympeo(M, \omega)\).

If \(M\) is simply connected, \(SSympeo(M, \omega) = Hameo(M, \omega)\). Finally, the commutator subgroup \([SSympeo(M, \omega), SSympeo(M, \omega)]\) of \(SSympeo(M, \omega)\) is contained in \(Hameo(M, \omega)\).

**Conjectures**

1. Let \((M, \omega)\) be a closed symplectic manifold, then
\[
[SSympeo(M, \omega), SSympeo(M, \omega)] = Hameo(M, \omega).
\]

2. The inclusion \(SSympeo(M, \omega) \subset Sympeo_0(M, \omega)\) is strict.

3. The results in theorem 2 hold for \(SSympeo(M, \omega)\).

Conjecture 3 is supported by a result of Muller asserting that \(Hameo(M, \omega)\) coincides with \(Hameo(M, \omega)\) which is defined by replacing the \(L^{1, \infty}\) Hofer norm by the \(L^\infty\) norm [8].

**Measure preserving homeomorphisms**

On a symplectic \(2n\) dimensional manifold \((M, \omega)\), we consider the measure \(\mu_\omega\) defined by the Liouville volume \(\omega^n\). Let \(Homeo_0^{\omega\infty}(M)\) be the identity component in the group of homeomorphisms preserving \(\mu_\omega\). We have:

\[
Sympeo_0(M, \omega) \subset Homeo_0^{\omega\infty}(M).
\]

Oh and Muller [11] have observed that \(Hameo(M, \omega)\) is a sub-group of the kernel of Fathi’s mass-flow homomorphism [7]. This is a homomorphism \(\theta : Homeo_0^{\omega\infty}(M) \to H_1(M, \mathbb{R})/\Gamma\), where \(\Gamma\) is some sub-group of \(H_1(M, \mathbb{R})\). Fathi proved that if the dimension of \(M\) is bigger than 2, then \(\text{Ker}\theta\) is a simple group. This leaves open the following question [11]:

Is $\text{Homeo}_0^\omega(S^2) = \text{Symp}_0(S^2, \omega)$ a simple group?

But $\text{Symp}_0(S^2, \omega)$ contains $\text{Hameo}(S^2, \omega)$ as a normal subgroup. The question is to decide if the inclusion

$$\text{Hameo}(S^2, \omega) \subset \text{Symp}_0(S^2, \omega)$$

is strict. Since $SS\text{ympeo}(S^2, \omega) = \text{Hameo}(S^2, \omega)$, our conjecture 2 implies that $\text{Homeo}_0^\omega(S^2) = \text{Symp}_0(S^2, \omega)$ is not a simple group, a conjecture of [9].

Questions

1. Is $SS\text{ympeo}(M, \omega)$ a normal subgroup of $\text{Symp}_0(M, \omega)$?
2. Is $[\text{Symp}_0(M, \omega), \text{Symp}_0(M, \omega)]$ contained in $\text{Hameo}(M, \omega)$?

4. Proofs of the results

4.1. Proof of theorem 1

If $B$ and $B'$ are two basis of $\text{harm}^1(M, g)$, then elementary linear algebra shows that $|.|_B$ and $|.|_{B'}$ are equivalent. This implies that the corresponding norms on $\text{symp}(M, \omega)$ are also equivalent.

Let us now start our construction with a riemannian metric $g$ and a basis $B = (h_1, h_k)$ of $\text{harm}^1(M, g)$. We saw that for any $X \in \text{symp}(M, \omega)$,

$$i_{X} \omega = H_{X} + du_{X}$$

and we wrote $H_{X} = \sum \lambda_{i} h_{i}$.

Let $g'$ be another riemannian metric. The $g'$-Hodge decomposition of $i_{X} \omega$ is:

$$i_{X} \omega = H'_{X} + du'_{X}$$

where $H'_{X}$ is $g'$-harmonic.

Consider the $g'$-Hodge decompositions of the members $h_{i}$ of the basis $B$ i.e.

$$h_{i} = h'_{i} + dv_{i}$$
where $h'_i$ is $g'$ harmonic.

$B' = (h'_1, \ldots, h'_k)$ is a basis of $\text{harm}^1(M, g')$. Indeed if $\sum r_i h'_i = 0$, then $\sum r_i h_i = d(\sum r_i v_i)$. Hence $\sum r_i h_i$ is identically zero because it is an exact harmonic form. Therefore all $r_i$ are zero since $\{h_i\}$ form a basis.

The 1-form

$$H'_X =: \sum \lambda_i h'_i$$

is a $g'$-harmonic form representing the cohomology class of $i_X \omega$. By uniqueness, $H'_X = H''_X$.

Hence

$$|H'_X|_{g'} = \sum |\lambda_i| = |H_X|_g$$

Furthermore $H'_X = \sum \lambda_i (h_i - dv_i) = H_X + dv$ where $v = - \sum \lambda_i v_i$. Hence

$$i_X \omega = H'_X + du'_X = H_X + d(v + u'_X)$$

By uniqueness in the $g$-Hodge decomposition of $i_X \omega$,

$$u_X = v + u'_X.$$

Denote by $||X||_{g'}$, resp. $||X||_g$, the norm of $X$ using the riemannian metric $g'$ and the basis $B'$, resp. using the riemannian metric $g$ and the basis $B$. Then:

$$||X||_{g'} = |H'_X|_{g'} + \text{osc}(u'_X) = |H'_X|_{g'} + \text{osc}(u_X - v)$$

$$\leq |H'_X|_{g'} + \text{osc}(u_X) + \text{osc}(-v)$$

$$= |H_X|_{g} + \text{osc}(u_X) + \text{osc}(v) = ||X||_g + \text{osc}(v).$$

Similarly,

$$||X||_g = |H_X|_{g} + \text{osc}(u_X) = |H_X|_{g} + \text{osc}(v + u'_X)$$

$$\leq (|H_X|_{g} + \text{osc}(u'_X)) + \text{osc}(v) = ||X||_{g'} + \text{osc}(v).$$
Setting \( a = \|X\|_g, b = \|X\|_{g'}, c = \text{osc}(v) \), we proved \( a \leq b + c \) and \( b \leq a + c \). Subtracting these inequalities, we get \( a - b \leq b - a \) and \( b - a \leq a - b \). This gives \( a \leq b \) and \( b \leq a \), i.e. \( a = b \).

We proved that given the couple \((g, B)\) of a riemannian metric \(g\) and a basis of \(g\)-harmonic 1-forms, and any other riemannian metric \(g'\), there is a basis \(B'\) of \(g'\)-harmonic 1-forms so that \(\|X\|_g = \|X\|_{g'}\), hence the norm \(\|\cdot\|\) is independent of the riemannian metric up to the equivalence relation due to change of basis. In conclusion, all the norms on \(\text{symp}(M, \omega)\) given by formula (1) are equivalent. \(\Box\)

For the purpose of the proof of the main theorem, we fix a riemannian metric \(g\) and a basis \(B = (h_1, \ldots, h_k)\) of \(\text{harm}^1(M, g)\). The norm of a harmonic 1-form \(H\) will be simply denoted \(\|H\|\) and the norm of a symplectic vector field \(X\) will be simply denoted \(\|X\|\).

4.2. Proof of theorem 2

Let \(h_i \in S\text{Sympeo}(M, \omega)\) \(i = 1, 2\) and let \(\lambda_i\) be continuous paths in \(\text{Homeo}(M)\) with \(\lambda_i(0) = \text{id}, \lambda_i(1) = h_i\) and let \(\Phi^n_i\) be \(d_{\text{symp}}\)-Cauchy sequences of symplectic isotopies converging \(C^0\) to \(\lambda_i\).

Then \(\Phi^n_1(\Phi^n_2)^{-1}\) converges \(C^0\) to the path \(\lambda_1(t)(\lambda_2(t))^{-1}\). Here \(\Phi^n_1(\Phi^n_2)^{-1}(t) = \phi^n_1(t)(\phi^n_2(t))^{-1}\).

By definition of the distance \(d_{\text{symp}}\), \(\Phi^n\) is a \(d_{\text{symp}}\)-Cauchy sequence if and only if both \(\Phi^n\) and \((\Phi^n)^{-1}\) are \(D_0\) - Cauchy and \(\overline{D}\) - Cauchy sequences.

Main lemma.

If \(\Phi^n = (\phi^n_i)\) and \(\Psi^n_i = (\psi^n_i)\) are \(d_{\text{symp}}\)-Cauchy sequences in \(\text{Iso}(M)\), so is \(\rho^n_i = \phi^n_i \psi^n_i\).

It will be enough to prove that \(\rho^n_i\) is a \(D_0\) - Cauchy sequence. Indeed since \((\Phi^n)^{-1}\) and \((\Psi^n)^{-1}\) are \(D_0\) - Cauchy by assumption, the main lemma applied to their product implies that their product is also \(D_0\) Cauchy. Hence \((\Psi^n)^{-1}(\Phi^n)^{-1} = ...)
\((\Phi^n\Psi^n)^{-1} = (\rho^n)^{-1}\) is a \(D_0\) - Cauchy sequence. This will conclude the proof that \(SSympeo(M,\omega)\) is a group.

We will use the following estimate:

**Proposition 2.** There exists a constant \(E\) such that for any \(X \in \text{symp}(M,\omega)\), and \(H \in \text{harm}^1(M,g)\)

\[
|H(X)| = \sup_{x \in M}|H(x)(X(x))| \leq E||X|||H|
\]

*Proof.* Let \((h_1, \ldots, h_r)\) be the chosen basis for harmonic 1-forms and let \(E = \max_i E_i\) and \(E_i = \sup_V (\sup_{x \in M}|h_i(x)(V(x))|\) where \(V\) runs over all symplectic vector fields \(V\) such that \(||V|| = 1\).

Without loss of generality, we may suppose \(X \neq 0\) and set \(V = X/||X||\). Let 
\[H = \sum \lambda_i h_i.\] Then \(H(X) = ||X|| \sum \lambda_i h_i(V)\). Hence

\[
|H(X)| \leq ||X|| \sum |\lambda_i| \sup_x (|h_i(x)(V(x))|) \leq ||X|| \sum |\lambda_i|E = E||X|||H|.
\]

\(\square\)

We will also need the following standard facts:

**Proposition 3.**

Let \(\varphi\) be a diffeomorphism, \(X\) a vector field and \(\theta\) a differential form on a smooth manifold \(M\), Then

\[
(\varphi^{-1})^*[i_X \phi^*\theta] = i_{\varphi_*X}\theta
\]

**Proposition 4.**

If \(\phi_t, \psi_t\) are any isotopies, and if we denote by \(\rho_t = \phi_t\psi_t\), and by \(\dot{\varphi}_t = (\phi)^{-1}_t\)
then

\[
\dot{\rho}_t = \dot{\phi}_t + (\dot{\phi}_t)_* \psi_t
\]

and

\[
\dot{\phi}_t = -((\phi)_t^{-1})_* (\dot{\phi}_t)
\]
Proposition 5.

Let $\theta_t$ be a smooth family of closed 1-forms and $\phi_t$ an isotopy, then

$$\phi^*_t \theta_t - \theta_t = dv_t$$

where

$$v_t = \int_0^t (\theta_t(\dot{\phi}_s) \circ \phi_s) ds$$

Proof of the main lemma

If $\phi_t, \psi_t$ are symplectic isotopies, and if $\rho_t = \phi_t \psi_t$, propositions 3, 4 and 5 give:

$$i(\dot{\rho}_t) \omega = \mathcal{H}^\phi_t + \mathcal{H}^\psi_t + dK(\Phi, \Psi)$$

where $K = K(\Phi, \Psi) = u_t^\phi + (u_t^\psi) \circ (\phi_t)^{-1} + v_t(\Phi, \Psi)$, and

$$v_t(\Phi, \Psi) = \int_0^t (\mathcal{H}_t^\psi(\dot{\phi}_s) \circ \phi_s^{-1}) ds.$$

Let now $\phi^n_t, \psi^n_t$ be Cauchy sequences of symplectic isotopies, and consider the sequence $\rho^n_t = \phi^n_t \psi^n_t$.

We have:

$$|||\dot{\rho}^n_t - \dot{\rho}^m_t||| = \int_0^1 |\mathcal{H}^\phi_t - \mathcal{H}^\phi_t^n + \mathcal{H}^\psi_t - \mathcal{H}^\psi_t^n + \text{osc}(K(\Phi^n, \Psi^n) - K(\Phi^m, \Psi^m))| dt$$

$$\leq \int_0^1 |\mathcal{H}^\phi_t - \mathcal{H}^\phi_t^n| dt + \int_0^1 |\mathcal{H}^\psi_t - \mathcal{H}^\psi_t^n| dt$$

$$+ \int_0^1 \text{osc}(u_t^\phi - u_t^{\phi^n}) dt + \int_0^1 \text{osc}(u_t^\psi) \circ (\phi^n_t)^{-1} - u_t^{\psi^n} \circ (\phi^m_t)^{-1}) dt$$

$$+ \int_0^1 \text{osc}(v_t(\Phi^n, \Psi^n) - v_t(\Phi^m, \Psi^m)) dt$$

$$= |||\dot{\phi}^n_t - \dot{\phi}^m_t||| + \int_0^1 |\mathcal{H}^\phi_t^n - \mathcal{H}^\phi_t^m| dt + A + B$$
where
\[ A = \int_0^1 \text{osc}(u^\Psi_n t) \circ (\phi^n_t)^{-1} - u^\Psi_m t \circ (\phi^m_t)^{-1}) dt \]
and
\[ B = \int_0^1 \text{osc}(v_t(\Phi^n, \Psi^n) - v_t(\Phi^m, \Psi^m) dt \]

We have:
\[ A \leq \int_0^1 \text{osc}(u^\Psi_n t) \circ (\phi^n_t)^{-1} - u^\Psi_m t \circ (\phi^m_t)^{-1}) dt + \int_0^1 \text{osc}(u^\Psi_m t) \circ (\phi^m_t)^{-1} - (u^\Psi_m t) \circ (\phi^m_t)^{-1}) dt \]
\[ = \int_0^1 \text{osc}(u^\Psi_n t - u^\Psi_m t) dt + C \]

where
\[ C = \int_0^1 \text{osc}(u^\Psi_m t \circ (\phi^n_t)^{-1} - u^\Psi_m t \circ (\phi^m_t)^{-1}) dt. \]

Hence
\[ |||\dot{\rho}^n_t - \dot{\rho}^m_t||| \leq |||\dot{\rho}^n_t - \dot{\rho}^m_t||| \]
\[ + \int_0^1 |H^\Psi_n t - H^\Psi_m t)|dt + \int_0^t \text{osc}(u^\Psi_n t - u^\Psi_m t) dt + B + C \]
\[ = |||\dot{\rho}^n_t - \dot{\rho}^m_t||| + |||\dot{\psi}^n_t - \dot{\psi}^m_t||| + B + C \]

We now show that \( C \to 0 \) when \( m, n \to \infty \).

**Sub-lemma 1 (reparametrization lemma [11])**

\( \forall \epsilon \geq 0, \exists m_0 \) such that
\[ C = \int_0^1 \text{osc}(u^\Psi_m t \circ (\phi^n_t)^{-1} - u^\Psi_m t \circ (\phi^m_t)^{-1}) dt =: |||u^\Psi_m t \circ (\phi^n_t)^{-1} - u^\Psi_m t \circ (\phi^m_t)^{-1})|| \leq \epsilon \]

if \( m \geq m_0 \) and \( n \) large enough

**Remark**

This is the "reparametrization lemma" of Oh-Muller [11] (lemma 3.21. (2)). For the convenience of the reader and further references, we include their proof.
**Proof**

For short, we write $u_m$ for $u_t^m$ and $\mu^n_t$ for $(\phi^n_t)^{-1}$.

First, there exists $m_0$ large such that $||u_m - u_{m_0}|| \leq \epsilon/3$ for $m \geq m_0$, since $(u_m)$ is a Cauchy sequence for the distance $d(u_n, u_m) = \int_0^1 osc(u_n - u_m)dt$.

Therefore

$$||u_{m} \circ \mu^n_t - u_{m_0} \circ \mu^n_t)|| \leq ||u_m - u_{m_0}|| + ||u_{m_0} \circ \mu^n_t - u_{m_0} \circ \mu^m_t)|| + ||u_{m_0} \circ \mu^m_t - u_{m_0} \circ \mu^n_t)||$$

$$\leq (2/3)\epsilon + ||u_{m_0} \circ \mu^n_t - u_{m_0} \circ \mu^m_t)||$$

By uniform continuity of $u_{m_0}$, there exists a positive $\delta$ such that if $d(\mu^n_t, \mu^m_t) \leq \delta$, then $\max osc((u_{m_0} \circ \mu^n_t - u_{m_0} \circ \mu^m_t)) \leq \epsilon/6$. Hence $||u_{m_0} \circ \mu^n_t - u_{m_0} \circ \mu^m_t)|| \leq \epsilon/3$ for $n, m$ large. Recall that $\mu^n_t$ is a $d$- Cauchy sequence. \qed

To show that $\dot{\rho}^n_t$ is a Cauchy sequence, the only thing which is left is to show that $B \to 0$ when $n, m \to \infty$.

Let us denote $v_t(\Phi^n, \Psi^n)$ by $v^n_t$ , $\mathcal{H}^t$ by $\mathcal{H}^t_n$ or $\mathcal{H}_n$ and $(\phi^n_t)^{-1}$ by $\mu^n_t$.

For a function on $M$, we consider the norm

$$|f| = \sup_{x \in M} |f(x)|$$

We have:

$$|v^n_t - v^n_m| = \int_0^t (H_n(\dot{\mu}^n_s) \circ \mu^n_s - H_m(\dot{\mu}^m_s) \circ \phi^n_s)ds$$

$$\leq \int_0^1 |((H_n - H_m)(\dot{\mu}^n_s)) \circ \mu^n_s|ds$$

$$+ \int_0^1 |H_m(\dot{\mu}^n_s - \dot{\mu}^m_s) \circ \mu^m_s|ds$$

$$+ \int_0^1 |H_m(\dot{\mu}^n_s - \mathcal{H}^m_n(\dot{\mu}^n_s)) \circ \mu^m_s|ds$$
The last integral can be estimated as follows:

\[ \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^m| ds \]

\[ \leq \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^n| ds \] (1)

\[ + \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^n - \mathcal{H}_m(\dot{\mu}_s^m) \circ \mu_s^n| ds \] (2)

\[ + \int_0^1 |\mathcal{H}_m(\dot{\mu}_s^n) \circ \mu_s^m - \mathcal{H}_m(\dot{\mu}_s^m) \circ \mu_s^n| ds \] (3)

for some integer \( n_0 \).

Proposition 2 gives \( E|\mathcal{H}_m|D_0((\Phi^n)^{-1}, (\Phi^{n_0})^{-1}) \leq 2E|\mathcal{H}_m|D((\Phi^n), (\Phi^{n_0})^{-1}) \) as an upper bound for (1) and (3).

It also gives the following estimates:

\[ \int_0^1 |(\mathcal{H}_n - \mathcal{H}_m)(\dot{\mu}_s^n) \circ \mu_s^n| ds \leq E|\mathcal{H}_n - \mathcal{H}_m| \int_0^1 ||\dot{\mu}_s^n|||ds \]

\[ = E. |\mathcal{H}_n - \mathcal{H}_m|. I((\Phi^n)^{-1}) \]

and

\[ \int_0^1 |(\mathcal{H}_m(\dot{\mu}_s^n - \dot{\mu}_s^m)) \circ \mu_s^m| ds \leq E.|\mathcal{H}_m| \int_0^1 |(\dot{\mu}_s^n - \dot{\mu}_s^m)||ds \]

\[ = E.|\mathcal{H}_m|D_0((\Phi^n)^{-1}, (\Phi^m)^{-1}) \leq 2E|\mathcal{H}_m|D(\Phi^n, \Phi^m). \]

Therefore, we get the following estimate:

\[ |v_t^n - v_t^m| \leq E.|\mathcal{H}_n - \mathcal{H}_m|. I((\Phi^n)^{-1}) + E.|\mathcal{H}_m|2(D(\Phi^n, \Phi^m) + 4D(\Phi^n, \Phi^{n_0})) + G \]
where
\[ G = \int_0^1 |\mathcal{H}_m(\tilde{\mu}_s^n) \circ \mu_s^n - \mathcal{H}_m(\tilde{\mu}_s^n) \circ \mu_s^m|ds \]

Since \( osc(v^n_t - v^m_t) \leq 2|v^n_t - v^m_t| \), we see that
\[
\int_0^1 osc(v^n_t - v^m_t)dt \leq 2(l(\Phi^n)^{-1}) \int_0^1 |\mathcal{H}_n^t - \mathcal{H}_m^t|dt \\
+ E(2D(\Phi^m, \Phi^n) + 4D(\Phi^n, \Phi^{n_0})\int_0^1 |\mathcal{H}_m^t|dt + \int_0^1 Gdt
\]

We need the following facts:

**Sub-lemma 2 (Reparametrisation lemma)**

\( \forall \epsilon > 0, \exists n_0 \) such that
\[
L = \int_0^1 Gdt = \int_0^1 (\int_0^1 |\mathcal{H}_m(\tilde{\mu}_s^n) \circ \mu_s^n - \mathcal{H}_m(\tilde{\mu}_s^n) \circ \mu_s^m|ds)dt \leq \epsilon
\]

for \( n \geq n_0 \) and \( m \) sufficiently large.

**Proposition 6**

\( l((\Phi^n)^{-1}) \) and \( \int_0^1 |\mathcal{H}_m^t|dt \) are bounded for every \( n, m \).

We finish first the estimate for \( \int_0^1 osc(v^n_t - v^m_t)dt \) using sub-lemma 2 and proposition 6.

Putting together all the information we gathered, we see that:
\[
\int_0^1 osc(v^n_t - v^m_t)dt \leq 2(l(\Phi^n)^{-1}) \int_0^1 |\mathcal{H}_n^t - \mathcal{H}_m^t|dt \\
+ E(2D(\Phi^m, \Phi^n) + 4D(\Phi^n, \Phi^{n_0})\int_0^1 |\mathcal{H}_m^t|dt + L
\]

\( \leq 2l((\Phi^n)^{-1})D(\Phi^n, \Phi^m) + E(2D(\Phi^m, \Phi^n) + 4D(\Phi^n, \Phi^{n_0})\int_0^1 |\mathcal{H}_m^t|dt + L
\)
Therefore:

\[
\int_0^1 \text{osc}(v^n_t - v^m_t) dt \to 0
\]

when \( n, m \to \infty \), and \( n_0 \) is chosen sufficiently large. This finishes the proof of the main lemma.

\[\square\]

**Proof of proposition 6**

This follows from the estimates:

\[
l((\Phi^n)^{-1}) \leq D((\Phi^n)^{-1}, \Phi^{n_0}) + l(\Phi^{n_0})
\]

and

\[
\int_0^1 |H^t_m| dt \leq \int_0^1 |H^t_m - H^t_{n_0}| dt + \int_0^1 |H^t_{n_0}| dt
\]

\[
\leq D(\Phi^m, \Phi^{n_0}) + \int_0^1 |H^t_{n_0}| dt
\]

for any \( n_0 \). Hence if \( n_0 \) is sufficiently large, \( l((\Phi^n)^{-1}) \) and \( \int_0^1 |H^t_m| dt \) are bounded.

\[\square\]

**Proof of sub-lemma 2**

\[
G = \int_0^1 |\mathcal{H}_m(\hat{\mu}^{n_0}_s) \circ \mu^n_s - \mathcal{H}_m(\hat{\mu}^{n_0}_s) \circ \mu^m_s| ds
\]

\[
\leq \int_0^1 |\mathcal{H}_m(\hat{\mu}^{n_0}_s) \circ \mu^n_s - \mathcal{H}_m(\hat{\mu}^{n_0}_s) \circ \mu^m_s| ds
\]

\[
+ \int_0^1 |\mathcal{H}_{n_0}(\hat{\mu}^{n_0}_s) \circ \mu^n_s - \mathcal{H}_{n_0}(\hat{\mu}^{n_0}_s) \circ \mu^m_s| ds
\]

\[
+ \int_0^1 |\mathcal{H}_{n_0}(\hat{\mu}^{n_0}_s) \circ \mu^n_s - \mathcal{H}_m(\hat{\mu}^{n_0}_s) \circ \mu^m_s| ds
\]

for some \( m_0 \).
Exactly like in the proof of sub-lemma 1
\[ G(t, n, m) \leq 2|\mathcal{H}_m^t - \mathcal{H}_m^{t_0}|L(\Psi^{n_0})^{-1}) + F \]
where
\[ F = \int_0^1 |\mathcal{H}_{m_0}(\dot{\mu}_s^{n_0}) \circ \mu_s^n - \mathcal{H}_{m_0}(\dot{\mu}_s^{n_0}) \circ \mu_s^m|ds \]
By uniform continuity of \( \mathcal{H}_{m_0}(\dot{\mu}_s^{n_0}) \), \( F \to 0 \) when \( n, m \to \infty \) since \( \mu_s^n \) is Cauchy.

By similar arguments as in the sub-lemma 1, \( G \to 0 \) and hence \( L \to 0 \) when \( m, n \to \infty \).

This concludes the proof of that \( SSympeo(M, \omega) \) is a group.

The fact that it is arcwise connected in the ambient topology of \( Homeo(M) \) is obvious from the definition.

\( Hameo(M, \omega) \) is a normal subgroup of \( SSympeo(M, \omega) \) since it is normal in \( Sympeo(M, \omega) \) [11].

Let \( h, g \in SSympeo(M, \omega) \) and let \( \Phi^n, \Psi^n \) be symplectic isotopies which form Cauchy sequences and \( C^0 \) converge to \( h, g \). By the main lemma the sequence \( \Phi^n \Psi^n(\Phi^n)^{-1}(\Psi^n)^{-1} \) is a Cauchy sequence. It obviously converges \( C^0 \) to the commutator \( hgh^{-1}g^{-1} \in SSympeo(M, \omega) \).

It is a standard fact that \( \Phi^n \Psi^n(\Phi^n)^{-1}(\Psi^n)^{-1} \) is a hamiltonian isotopy.

Indeed let \( \phi_t \) and \( \psi_t \) be symplectic isotopies, and let \( \sigma_t = \phi_t \psi_t \phi_t^{-1} \psi_t^{-1} \), then
\[ \dot{\sigma}_t = X_t + Y_t + Z_t + U_t \]
with \( X_t = \dot{\phi}_t, Y_t = (\phi_t)_* \dot{\psi}_t, Z_t = -((\phi_t)_\# \dot{\psi}_t^{-1})_* \phi_t, \) and \( U_t = -((\sigma_t)_\# \dot{\psi}_t) \).

By proposition 5, \( i(X_t + Z_t)\omega \) and \( i(Y_t + U_t)\omega \) are exact 1-forms. Hence \( \sigma_t \) is a hamiltonian isotopy.

By proposition 1, the metric \( D \) coincides with the one for hamiltonian isotopies. Hence \( \Phi^n \Psi^n(\Phi^n)^{-1}(\Psi^n)^{-1} \) is a Cauchy sequence for \( d_{ham} \). Therefore:
\[ [SSympeo(M, \omega), SSympeo(M, \omega)] \subset Hameo(M, \omega) \].
This finishes the proof of the main result. \hfill\Box

Appendix

For the convenience of the reader, we give here the proofs of propositions 3, 4, and 5.

**Proof of proposition 3**

Let $\theta$ be a $p$-form, $X$ a vector field and $\phi$ a diffeomorphism. For any $x \in M$ and any vector fields $Y_1, \ldots, Y_{p-1}$, we have:

\[
(\phi^{-1})^* [i_X \phi^* \theta](x)(Y_1, \ldots, Y_{p-1}) = (i_X \phi^* \theta)(\phi^{-1}(x))(D_x \phi^{-1}(Y_1(x)), \ldots, D_x \phi^{-1}(Y_{p-1}(x)))
\]

\[
= (\phi^* \theta)(\phi^{-1}(x))(X_{\phi^{-1}(x)}, D_x \phi^{-1}(Y_1(x)), \ldots, D_x \phi^{-1}(Y_{p-1}(x)))
\]

\[
= \theta(\phi^{-1}(x))(D_{\phi^{-1}(x)} \phi(X_{\phi^{-1}(x)}), D_{\phi^{-1}(x)} \phi D_x \phi^{-1}(Y_1(x)), \ldots, D_{\phi^{-1}(x)} \phi D_x \phi^{-1}(Y_{p-1}(x))
\]

\[
= \theta(x)((\phi_* X)_x, Y_1(x), \ldots, Y_{p-1}(x))
\]

\[
= (i(\phi_* X) \theta)(x)(Y_1, \ldots, Y_{p-1})
\]

since $D_{\phi^{-1}(x)} \phi D_x \phi^{-1} = D_x (\phi \phi^{-1}) = id$.

Therefore $(\phi^{-1})^* [i_X \phi^* \theta] = i(\phi_* X) \theta$ \hfill\Box

**Proof of proposition 4**

This is just the chain rule. See [6] page 145. \hfill\Box

**Proof of proposition 5**

For a fixed $t$, we have

\[
\frac{d}{ds} \phi_*^s \theta_t = \phi_*^s (L_{\phi_*^s \theta_t}),
\]

where $L_X$ is the Lie derivative in the direction $X$. Since $\theta$ is closed, we have:

\[
\frac{d}{ds} \phi_*^s \theta_t = \phi_*^s (di_{\phi_*^s \theta_t}) = d(\phi_*^s (\theta_t(\phi_*^s))) = d(\theta_t(\phi_*^s) \circ \phi_*).
\]

Hence for every $u$

\[
\phi_*^u \theta_t - \theta_t = \int_0^u \frac{d}{ds} \phi_*^s \theta_t ds = d(\int_0^u (\theta_t(\phi_*^s) \circ \phi_* ds)
Now set $u = t$. □

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