Abstract

The main goal of this paper is to give a pedagogical introduction to Quantum Information Theory—to do this in a new way, using network diagrams called Quantum Bayesian Nets. A lesser goal of the paper is to propose a few new ideas, such as associating with each quantum Bayesian net a very useful density matrix that we call the meta density matrix.
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A Review of Classical and Quantum Bayesian Nets

References
1 Introduction

The main goal of this paper is to give a pedagogical introduction to Quantum Information Theory—to do this in a new way, using network diagrams called Quantum Bayesian (QB) Nets. The paper assumes no prior knowledge of Classical[1]-[2] or Quantum[3]-[9] Information Theory. It does assume a good understanding of the machinery of Quantum Mechanics, such as one would obtain by reading any reasonable textbook that explains Dirac bra-ket formalism. The paper reviews QB nets in an appendix. If you have difficulty understanding said appendix, you might want to read Ref.[10] before continuing this paper.

Most of the ideas discussed in this paper are not new. They are well-known, standard ideas invented by the pioneers (Bennett, Holevo, Peres, Schumacher, Wooters, etc.) of the field of Quantum Information Theory. What is new about this paper is that, whenever possible and advantageous, we rephrase those ideas in the visual language of QB nets. The paper does present a few new ideas, such as associating with each QB net a very useful density matrix that we call the meta density matrix of the net.

The topics covered in this paper are shown in the Table of Contents. The paper, in its present form, is far from being a complete account of the field of Quantum Information Theory. Some important topics that were left out (because the author didn’t have enough time to write them up) are: quantum compression, quantum error correction, channel capacities, quantum approximate cloning, entanglement quantification and manipulation. Future editions of this paper may include some of these topics. I welcome any suggestions or comments. To fill in gaps left by this paper, or to find alternative explanations of difficult topics, see Refs.[3]-[9] and references therein.
In this section, we will introduce certain notation which is used throughout the paper.

We define $Z_{a,b}$ to be the set \{a, a + 1, \ldots, b\} for any integers $a$ and $b$. Let $Bool = \{0, 1\}$. For any finite set $S$, let $|S|$ denote the number of elements in $S$.

The Kronecker delta function $\delta(x, y)$ equals one if $x = y$ and zero otherwise. We will often abbreviate $\delta(x, y)$ by $\delta^x_y$.

We will often use the symbol $\sum_{ri}$ to mean that one must sum whatever is on the right-hand side of this symbol over all repeated indices (a sort of Einstein summation convention). Likewise, $\sum_{al}$ will mean that one should sum over all indices. If we wish to exclude a particular index from the summation, we will indicate this by a slash followed by the name of the index. For example, in $\sum_{ri/f}$ or $\sum_{all/f}$ we wish to exclude summation over $f$.

The Pauli matrices $\sigma_x$, $\sigma_y$ and $\sigma_z$ are defined by

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$ (2.1)

For any real $p \in [0, 1]$, we define the binary entropy function $h(\cdot)$ by

$$h(p) = -p \log_2(p) - (1 - p) \log_2(1 - p).$$ (2.2)

When speaking of bits with states 0 and 1, we will often use an overbar to represent the opposite state: $\overline{0} = 1$, $\overline{1} = 0$.

We will underline random variables. For example, we might write $P(\underline{a} = a)$ for the probability that the random variable $\underline{a}$ assumes value $a$. $P(\underline{a} = a)$ will often be abbreviated by $P(a)$ when no confusion will arise. $S_a$ will denote the set of values which the random variable $\underline{a}$ may assume, and $N_a$ will denote the number of elements in $S_a$. With each random variable $\underline{a}$, we will associate an orthonormal basis $\{|a\rangle|a \in S_a\}$ which we will call the $\underline{a}$ basis. We will represent by $H_\underline{a}$ the Hilbert space spanned by the $\underline{a}$ basis. $|\underline{a} = a\rangle$ will mean the same thing as $|a\rangle$; $|\underline{a} = a\rangle$ is just a more explicit notation that indicates that $|a\rangle$ belongs to $H_\underline{a}$. If $\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_N$ are any $N$ random variables, we will use $H_{\underline{x}_1} \otimes H_{\underline{x}_2} \otimes \cdots \otimes H_{\underline{x}_N}$ to denote $H_{\underline{x}_1} \otimes H_{\underline{x}_2} \otimes \cdots \otimes H_{\underline{x}_N}$.

Whenever we use the word “ditto”, as in “$X$ (ditto, $Y$)”, we mean that the statement is true if $X$ is replaced by $Y$. For example, if we say “$A$ (ditto, $X$) is smaller than $B$ (ditto, $Y$)”, we mean “$A$ is smaller than $B$” and “$X$ is smaller than $Y$”.

This paper will also utilize certain notation associated with classical and quantum Bayesian nets. See Appendix A for a review of such notation.
3 Classical Entropy: Its Definition and Properties

In this section, we will define various classical entropies associated with a CB net.

Suppose $p_1, p_2, \ldots, p_N$ are $N$ non-negative numbers which add up to one. The classical entropy $H(\vec{p})$ of $\vec{p}$ is defined by

$$H(\vec{p}) = -\sum_{i=1}^{N} p_i \log_2 p_i .$$ (3.1)

$H(\vec{p})$ measures the spread of the probability distribution $\vec{p}$.

In Thermodynamics, entropy measures the disorder of a macroscopic system. See Ref. [5] for a discussion of the relationship between the entropy of Thermodynamics and Eq. (3.1).

In Communication Theory, one uses the words “information” and “entropy” interchangeably. In the context of communication theory, the word “information” means information content of an average message. Given any random variable $\vec{z}$, one may think of a sequence $x_1, x_2, \ldots, x_N$ of samples of $\vec{x}$ as a message. Then one makes the assumption that the more information an average message (of fixed length) carries, the higher the variance of $\vec{z}$ will be, and vice versa. Eq. (3.1) quantifies the variance of $\vec{z}$ if we replace $p_i$ and the sum over $1 \leq i \leq N$ by $P(\vec{z} = x)$ and a sum over $x \in S$, where $S$ is the set of values that $\vec{z}$ can assume.

When dealing with a CB net, it is convenient to rephrase Eq. (3.1) in terms of the node random variables of the net. Consider a CB net $\mathcal{N}^C$ with $N$ nodes labelled by the random variables $x_1, x_2, \ldots, x_N$. These $N$ random variables are related by a joint probability distribution $P(x.)$. Suppose $\Gamma_1$ and $\Gamma_2$ are non-empty subsets of $Z_{1,N}$. $\Gamma_1$ and $\Gamma_2$ need not be disjoint. The probability distributions $P[(x.)_{\Gamma_1}], P[(x.)_{\Gamma_2}]$ and $P[(x.)_{\Gamma_1 \cup \Gamma_2}]$ can be obtained by summing $P(x.)$ over the unwanted arguments, a process called marginalization. We define:

$$H[(\vec{z})_{\Gamma_1}] = -\sum_{(x.)_{\Gamma_1}} P[(x.)_{\Gamma_1}] \log_2 P[(x.)_{\Gamma_1}] ,$$ (3.2)

$$H[(\vec{z})_{\Gamma_1}|(\vec{z})_{\Gamma_2}] = -\sum_{(x.)_{\Gamma_1 \cup \Gamma_2}} P[(x.)_{\Gamma_1 \cup \Gamma_2}] \log_2 \left( \frac{P[(x.)_{\Gamma_1 \cup \Gamma_2}]}{P[(x.)_{\Gamma_2}]} \right) ,$$ (3.3)

$$H[(\vec{z})_{\Gamma_1} : (\vec{z})_{\Gamma_2}] = \sum_{(x.)_{\Gamma_1 \cup \Gamma_2}} P[(x.)_{\Gamma_1 \cup \Gamma_2}] \log_2 \left( \frac{P[(x.)_{\Gamma_1 \cup \Gamma_2}]}{P[(x.)_{\Gamma_1}]P[(x.)_{\Gamma_2}]} \right) .$$ (3.4)

For example, if $a$ and $b$ are nodes of a CB net, then

$$H(a) = -\sum_{a} P(a) \log_2 P(a) ,$$ (3.5)
$$H(a, b) = - \sum_{a,b} P(a, b) \log_2 P(a, b) ,$$  
(3.6)

$$H(a|b) = - \sum_{a,b} P(a, b) \log_2 P(a|b) ,$$  
(3.7)

$$H(a : b) = \sum_{a,b} P(a, b) \log_2 \left( \frac{P(a, b)}{P(a)P(b)} \right) ,$$  
(3.8)

where $P(a) = \sum_b P(a, b)$, $P(b) = \sum_a P(a, b)$, and the sums over $a$ (ditto, $b$) range over all $a \in S_a$ (ditto, $b \in S_b$).

Note that definitions Eqs. (3.2) to (3.4) are independent of the order of the node random variables within $(\mathbf{a})_{\Gamma_1}$ and $(\mathbf{a})_{\Gamma_2}$. For example, if $a, b, c$ are nodes of a CB net, then

$$H(a, b, c) = H(a, c, b), \quad H[a|(b, c)] = H[a|(c, b)] .$$  
(3.9)

It is convenient to extend definitions Eqs. (3.2) to (3.4) in the following two ways. First, we will allow $(\mathbf{a})_{\Gamma_1}$ (ditto, $(\mathbf{a})_{\Gamma_2}$) to contain repeated random variables. If it does, then we will throw out any extra copies of a random variable. For example, if $a, b, c$ are nodes of a CB net, then

$$H(a, a, b, c) = H(a, b, c), \quad H[a|(b, b, c)] = H[a|(b, c)] .$$  
(3.10)

Second, we will allow $(\mathbf{a})_{\Gamma_1}$ (ditto, $(\mathbf{a})_{\Gamma_2}$) to contain internal parentheses. If it does, then we will ignore the internal parentheses. For example, if $a, b, c, d$ are nodes of a CB net, then

$$H[(a, b), c] = H(a, b, c), \quad H[a|(b, c), d)] = H[a|(b, c, d)] .$$  
(3.11)

Let $\mathbf{X} = (\mathbf{a})_{\Gamma_1}$ and $\mathbf{Y} = (\mathbf{a})_{\Gamma_2}$. $H(\mathbf{X})$ measures the spread of the $P(\mathbf{X})$ distribution. $H(\mathbf{X}|\mathbf{Y})$ is called the conditional entropy of $\mathbf{X}$ given $\mathbf{Y}$. $H(\mathbf{X} : \mathbf{Y})$ is called the mutual entropy of $\mathbf{X}$ and $\mathbf{Y}$, and it measures the dependency of $\mathbf{X}$ and $\mathbf{Y}$; it is non-negative, and it equals zero iff $\mathbf{X}, \mathbf{Y}$ are independent random variables (i.e., $P(\mathbf{X}, \mathbf{Y}) = P(\mathbf{X})P(\mathbf{Y})$ for all $\mathbf{X} \in S_{\mathbf{X}}$ and $\mathbf{Y} \in S_{\mathbf{Y}}$).

Note that Eqs. (3.2) to (3.4) imply that

$$H(\mathbf{X}|\mathbf{Y}) = H(\mathbf{X}, \mathbf{Y}) - H(\mathbf{Y}) ,$$  
(3.12)

$$H(\mathbf{X} : \mathbf{Y}) = H(\mathbf{X}) + H(\mathbf{Y}) - H(\mathbf{X}, \mathbf{Y}) ,$$  
(3.13)

$$H(\mathbf{X} : \mathbf{Y}) = H(\mathbf{X}) - H(\mathbf{X}|\mathbf{Y}) ,$$  
(3.14)

$$H(\mathbf{X} : \mathbf{Y}) = H(\mathbf{Y}) - H(\mathbf{Y}|\mathbf{X}) .$$  
(3.15)
In Eq.(3.14), one may think of \( H(X) \) as the information about \( X \) prior to transmitting it, and \( H(X|Y) \) as the information about \( X \) once \( X \) is transmitted and \( Y \) is found out. Since \( H(X : Y) \) is the difference between the two, one may think of it as the information (or entropy) “transmitted” from \( X \) to \( Y \). This interpretation of \( H(X : Y) \) is an alternative to the dependency interpretation mentioned above.

Let \( X = (x)_{\Gamma_1}, Y = (x)_{\Gamma_2} \) and \( Z = (x)_{\Gamma_3} \), where the \( \Gamma_1, \Gamma_2, \Gamma_3 \) are non-empty, possibly overlapping, subsets of \( Z_{1,N} \). We can extend further the domain of the function \( H(\cdot) \) by introducing the following axioms

\[
H[(X,Y) : Z] = H[(X : Z), (Y : Z)], \quad (3.16)
\]

\[
H[(X : Y), Z] = H[(X, Z) : (Y, Z)]. \quad (3.17)
\]

Eq.(3.16) means that “:” distributes over “,”. According to Eq.(3.13), the LEFT hand side of Eq.(3.16) equals \( H(X,Y) + H(Z) - H(X,Y,Z) \). Eq.(3.17) means that “,” distributes over “:”. According to Eq.(3.13), the RIGHT hand side of Eq.(3.17) equals \( H(X, Z) + H(Y, Z) - H(X,Y, Z) \). With the help of the above distributive laws, the entropy of a compound expression with any number of “:” and “|” operators can be expressed as a sum of \((\pm 1)H(\cdot)\) functions containing “,” but not containing “:” and “|” in their arguments. For example, if \( a, b, c \) are nodes of a QB net, then

\[
H[(a : b) | c] = H[(a : b), c] - H(c)
= H[(a, c) : (b, c)] - H(c)
= H(a, c) + H(b, c) - H(a, b, c) - H(c). \quad (3.18)
\]

If some parentheses are omitted within the argument of \( H(\cdot) \), the argument may become ambiguous. For example, does \( H(a : b, c) \) mean \( H((a : b), c) \) or \( H(a : (b, c)) \)? Ambiguous arguments should be interpreted using the following operator precedence order, from highest to lowest precedence: comma(,), colon(:), vertical line(||). Thus, \( H(a : b, c) \) should be interpreted as \( H(a : (b, c)) \).

In the mathematical field called Set Theory, one defines the union \( A \cup B \), the intersection \( A \cap B \) and the difference \( A - B = A \cap \text{complement}(B) \) of two sets \( A \) and \( B \). One also defines functions \( \mu(\cdot) \) called measures. A measure \( \mu(\cdot) \) assigns a non-negative real number to any “measurable” set \( A \). \( \mu(\cdot) \) satisfies

\[
\mu(\emptyset) = 0, \quad (3.19)
\]

\[
\mu(\cup_{i=0}^{\infty} E_i) = \sum_{i=0}^{\infty} \mu(E_i), \quad (3.20)
\]

where \( \emptyset \) is the empty set, and the \( E_i \)'s are disjoint measurable sets. For example, for any set \( S = [a_1, b_1] \cup [a_2, b_2] \cup \ldots \cup [a_N, b_N] \), where the \([a_i, b_i]\)'s are disjoint closed intervals of real numbers, one can define \( \mu(S) = \sum_{i=0}^{N} (b_i - a_i) \).
There is a close analogy between the properties of entropy functions in Information Theory (IT) and those of measure functions in Set Theory (ST). If \( A, B \) are sets and \( a, b \) are node random variables, then it is fruitful to imagine the following correspondences[11]:

\[
\begin{align*}
\text{atoms} : & \quad A \leftrightarrow a \\
\text{binary operators} : & \quad A \cup B \leftrightarrow (a, b) \\
& \quad A \cap B \leftrightarrow (a : b) \\
& \quad A - B \leftrightarrow (a | b) \\
\text{real-valued function} : & \quad \mu(A) \leftrightarrow H(a)
\end{align*}
\]  
\hspace{1cm} (3.21)

In both ST and IT, one defines a real-valued function (i.e., \( \mu(\cdot) \) in ST versus \( H(\cdot) \) in IT). This real-valued function takes as arguments certain well-formed expressions. A well-formed expression consists of either a single atom (a set in ST versus a node random variable in IT) or a compound expression. A compound expression is formed by using binary operators (\( \cup \cap - \) in ST versus \( , : | \) in IT) to bind together either (1) 2 atoms or (2) an atom and another compound expression or (3) two compound expressions.

Table 3 gives a list of properties (identities and inequalities) satisfied by the classical entropy \( H(\cdot) \). Whenever possible, Table 3 matches each property of entropy functions with an analogous property of measure functions. See Refs.[1]-[9] to get proofs of those statements in Table 3 that are not proven in this paper.
| Property | Formula | Notes |
|----------|---------|-------|
| $\mu(A - B) = \mu(A \cup B) - \mu(B)$ | $H(\sum w_i \rho_i) = H(\sum w_i \rho_i) - H(\sum w_i \rho_i)$ | $(\cdot \text{ in terms of } \cup)$ |
| $\mu(A \cap B) = \mu(A) + \mu(B) - \mu(A \cup B)$ | $H(\sum w_i \rho_i) = H(\sum w_i \rho_i) - H(\sum w_i \rho_i)$ | $(\cap \text{ in terms of } \cup)$ |
| $\mu((A \cup B) \cap C) = \mu((A \cap C) \cup (B \cap C))$ | $H(\sum w_i \rho_i) = H(\sum w_i \rho_i)$ | $(\cap \text{ distributes over } \cup)$ |
| $\mu((A \cap B) \cup C) = \mu((A \cup C) \cap (B \cup C))$ | $H(\sum w_i \rho_i) = H(\sum w_i \rho_i)$ | $(\cap \text{ distributes over } \cup)$ |
| $0 \leq \mu(A)$ (non-negative) | $0 \leq H(\sum w_i \rho_i) \leq \log_2 N_\rho$ | |
| $\mu(B) \leq \mu(A \cup B)$ or $0 \leq \mu(A - B)$ | $H(\sum w_i \rho_i) \leq H(\sum w_i \rho_i)$ | Equality iff $\sum_w f(w_i)$ for some function $f()$. |
| $\mu(A \cup B) \leq \mu(A) + \mu(B)$ or $\mu(A - B) \leq \mu(A)$ or $0 \leq \mu(A \cap B)$ (sub-additivity) | $H(\sum w_i \rho_i) \leq H(\sum w_i \rho_i)$ or $H(\sum w_i \rho_i) \leq H(\sum w_i \rho_i)$ or $0 \leq H(\sum w_i \rho_i)$ | Equality iff $X$ and $Y$ are independent. |
| $\mu(A - (B \cup C)) \leq \mu(A - B)$ (strong sub-additivity) | $H(\sum w_i \rho_i) \leq H(\sum w_i \rho_i)$ | $H \rightarrow S_{\rho}$ |
| $S(U^{\rho U^\dagger}) = S(\rho)$ | | for any unitary matrix $U$. |
| $S(\rho) = H(\bar{\rho})$, where $p_j = \langle i | \rho | j \rangle > 0$. Equality iff $\langle i | \rho | j \rangle > 0$ for all $i \neq j$. | $S(\sum_j p_j | j > | j >) \leq H(\bar{\rho})$, where $p_j$ is a prob. distribution. Equality iff $\langle j | j' > \rangle = \delta(j, j')$. |
| $\sum_j p_j \log_2 \frac{p_j}{\bar{p}_j} \leq 0$ where $p_j$ and $\bar{q}_j$ are prob. distributions. Gibbs’ inequality. Equality iff $q_j = p_j$ for all $j$. | $-\text{tr}(\rho (\log_2 \rho - \log_2 \sigma)) \leq 0$ where $\rho, \sigma$ are density matrices. Equality iff $\rho = \sigma$. |
| $\sum w_i H(\tilde{\rho}_i) \leq H(\sum w_i \tilde{\rho}_i)$, where $w_i \geq 0$ and $\sum w_i = 1$. Convexity. Equality iff $\exists \tilde{\bar{\rho}}$ such that $\forall \alpha, \tilde{\bar{\rho}}_\alpha = \tilde{\bar{\rho}}$. | $\sum w_i S(\rho_{\alpha}) \leq S(\sum w_i \tilde{\rho}_i)$, where $w_i \geq 0$ and $\sum w_i = 1$. Convexity. Equality iff $\exists \bar{\rho}$ such that $\forall \alpha, \bar{\rho}_{\alpha} = \bar{\rho}$. |
| $H(\sum w_i \rho_i) \leq \sum w_i H(\bar{\rho}_i) - \sum w_i \log_2 w_i$, where $w_i \geq 0$ and $\sum w_i = 1$. Equality iff $\bar{\rho}_i \cdot \bar{\rho}_j = 0$ for $i \neq j$. Equality is Shannon grouping axiom for $H()$. | $S(\sum w_i \rho_{\alpha}) \leq \sum w_i S(\rho_{\alpha}) - \sum w_i \log_2 w_i$, where $w_i \geq 0$ and $\sum w_i = 1$. Equality iff $\rho_{\alpha} \bar{\rho}_j = 0$ for $\alpha \neq j$. Lanford-Robinson. |
4 CB Net Examples

In Section 3, we discussed entropic properties which are valid for all CB nets. In this section, we will discuss entropic properties that apply to particular CB nets.

First, we will consider all possible CB nets with 2 and 3 nodes. Their nodes will be labelled by the random variables $a, b, c$.

![Figure 4.1: Two connected nodes.](image1)

Fig. (4.1) shows two connected nodes. By the definition of CB nets, the joint probability $P(a, b)$ of the two nodes of this net satisfies:

$$P(a, b) = P(b|a)P(a).$$  \hfill (4.1)

Taking the logarithms and then the expected values of both sides of the last equation yields

$$H(a, b) = H(b|a) + H(a).$$  \hfill (4.2)

![Figure 4.2: Diverging graph with 3 nodes.](image2)

Fig. (4.2) shows a “diverging” graph with 3 nodes. By the definition of CB nets, the joint probability $P(a, b, c)$ of all the nodes of this net satisfies:

$$P(a, b, c) = P(b)P(a|b)P(c|b).$$  \hfill (4.3)

The last equation implies the following entropic constraint:

$$H(a, b, c) = H(b) + H(a|b) + H(c|b)$$
$$= H(a, b) + H(c, b) - H(b),$$  \hfill (4.4)

which is equivalent to
\[ H[(a : c) | b] = 0. \] (4.5)

This means that at a fixed value of \( b \), \( a \) and \( c \) are independent random variables.

![Converging graph with 3 nodes.](image)

Figure 4.3: Converging graph with 3 nodes.

Fig. (4.3) shows a “converging” graph with 3 nodes. \( P(a, b, c) \) for this net must satisfy

\[ P(a, b, c) = P(a)P(c)P(b | a, c). \] (4.6)

Thus,

\[ H(a, b, c) = H(a) + H(c) + H(b | a, c) = H(a) + H(c) - H(a, c) + H(a, b, c), \] (4.7)

which is equivalent to

\[ H(a : c) = 0. \] (4.8)

This means that \( a \) and \( c \) are independent.

![Three node Markov chain.](image)

Figure 4.4: Three node Markov chain.

A Bayesian net consisting of a simple chain of \( N \) nodes connected by arrows all pointing in the same direction will be called an \( N \) node Markov chain. If the nodes are labelled by random variables \( q_1, q_2, \ldots, q_N \), we will denote the net by \( q_1 \rightarrow q_2 \rightarrow \cdots \rightarrow q_N \). Fig. (4.4) shows a 3 node Markov chain \( a \rightarrow b \rightarrow c \). \( P(a, b, c) \) for this net must satisfy:

\[ P(a, b, c) = P(c | b)P(b | a)P(a). \] (4.9)

Thus,
\begin{align*}
H(a, b, c) &= H(c|b) + H(b|a) + H(a) \\
&= H(c, b) - H(b) + H(b, a) ,
\end{align*}

which is equivalent to
\begin{equation}
H[(a : c)|b] = 0 .
\end{equation}

Note that Eq.(4.11) for the Markov chain Fig.(4.4) is the same as Eq.(4.5) for the diverging graph Fig.(4.2). This shows that two CB nets with different topologies can have the same entropic constraint.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure4_5}
\caption{Fully connected 3 node graph.}
\end{figure}

Fig.(4.5) shows a fully connected 3 node graph. \(P(a, b, c)\) for this net must satisfy:
\begin{equation}
P(a, b, c) = P(c|b, a)P(b|a)P(a) .
\end{equation}

Because the graph is fully connected, Eq.(4.12) is a tautology: it is satisfied by all probability distributions \(P(a, b, c)\). Eq.(4.12) implies
\begin{equation}
H(a, b, c) = H(c|b, a) + H(b|a) + H(a) .
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure4_6}
\caption{Fully connected 4 node graph.}
\end{figure}
Eq. (4.13) can be easily generalized to any number $N \geq 2$ of nodes. Consider a fully connected CB net with $N$ nodes labelled by the random variables $x_1, x_2, \ldots, x_N$. Fig. (4.6) shows the case $N = 4$. By the definition of CB nets, the joint probability of all the nodes must satisfy:

$$P(x_1, x_2, \ldots, x_N) = \prod_{j=1}^{N} P(x_j|x_{j-1}, \ldots, x_2, x_1). \quad (4.14)$$

Thus,

$$H(x_1, x_2, \ldots, x_N) = \sum_{j=1}^{N} H(x_j|x_{j-1}, \ldots, x_2, x_1). \quad (4.15)$$

Consider a 3 node Markov chain $q_1 \rightarrow q_2 \rightarrow q_3$. We shall demonstrate that:

$$0 = H(q_1|q_2) \leq H(q_1|q_3) \leq H(q_1|q_3), \quad (4.16)$$

and

$$H(q_1) = H(q_1 : q_1) \geq H(q_1 : q_2) \geq H(q_1 : q_3). \quad (4.17)$$

Eqs. (4.16) and (4.17) will be called fixed sender (or speaker) data processing (DP) inequalities. Eq. (4.16) tells us that the entropy of $q_1$ increases as “time” increases, because the “memory” $q_j$ of $q_1$ becomes a progressively less faithful representation of the original. Eq. (4.17) tells us that the dependency of $q_j$ on $q_1$ decreases as “time” $j$ increases. Alternatively, one might say that the amount of information transmitted from $q_1$ to $q_j$ decreases as the “distance” $j$ increases. The farther away the receiver is from the sender, the less information it gets. Eq. (4.17) follows trivially from Eq. (4.16). Just subtract $H(q_1)$ from each term of Eq. (4.16) and multiply the whole string of inequalities by $-1$. To prove Eq. (4.16), we begin by noticing that

$$P(q_1|q_2, q_3) = \frac{P(q_3|q_2)P(q_2|q_1)P(q_1)}{\sum_{q_1'} P(q_3|q_2)P(q_2|q_1')P(q_1')} = \frac{P(q_2|q_1)P(q_1)}{\sum_{q_1'} P(q_2|q_1')P(q_1')} = P(q_1|q_2). \quad (4.18)$$

This just means that once $q_2$ is known, finding out $q_3$ adds nothing new to our knowledge of $q_1$. Eq. (4.18) implies

$$H(q_1|q_2, q_3) = H(q_1|q_2). \quad (4.19)$$

Using the last equation and strong sub-additivity, we obtain

$$H(q_1|q_2) = H(q_1|q_2, q_3) \leq H(q_1|q_3). \quad (4.20)$$

QED.

The Markov chain $q_1 \rightarrow q_2 \rightarrow q_3$ also satisfies...
\[ H(q_3 | q_2) \leq H(q_3 | q_1), \quad (4.21) \]

and

\[ H(q_3 : q_2) \geq H(q_3 : q_1). \quad (4.22) \]

Eqs.\((4.21)\) and \((4.22)\) will be called fixed receiver (or listener) data processing (DP) inequalities. As in the fixed sender case, Eq.\((4.22)\) follows trivially from Eq.\((4.21)\). Just subtract \(H(q_2)\) from each term of the inequality and multiply by \(-1\). To prove Eq.\((4.21)\), we first realize that the method employed in Eq.\((4.18)\) can be used to show that

\[ P(q_3 | q_2, q_1) = P(q_3 | q_2). \quad (4.23) \]

Whereas in the fixed sender case, Eq.\((4.18)\) told us that we need only condition on the closest of the later times, Eq.\((4.23)\) instructs us to condition only on the closest of the earlier times. Eq.\((4.23)\) implies

\[ H(q_3 | q_2, q_1) = H(q_3 | q_2). \quad (4.24) \]

Using the last equation and strong sub-additivity, we obtain

\[ H(q_3 | q_2) = H(q_3 | q_2, q_1) \leq H(q_3 | q_1). \quad (4.25) \]

QED.

Eqs.\((4.17)\) and \((4.22)\) can be stated simultaneously as

\[ H(q_1 : q_3) \leq \min\{H(q_1 : q_2), H(q_2, q_3)\}. \quad (4.26) \]

Consider the 4 node Markov chain \(q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_4\). Then

\[ H(q_1 : q_3) \leq H(q_2 : q_3). \quad (4.27) \]

This follows from

\[ H(q_1 : q_4) \leq H(q_1 : q_3) \leq H(q_2 : q_3), \quad (4.28) \]

where we have used the fixed sender DP inequality first and the fixed receiver DP inequality second.

It is also interesting to note that the fixed receiver and fixed sender DP inequalities are related by time reversal. Indeed, suppose we are given a 3 node Markov chain \(q_1 \rightarrow q_2 \rightarrow q_3\). Then we can extend it to a 5 node Markov chain \(q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q'_2 \rightarrow q'_1\). We need to define the set of states and the transition matrices for nodes \(q'_2\) and \(q'_1\). Suppose we do this as follows:

\[ S_{q'_2} = S_{q_2}, \quad (4.29a) \]

15
\( S_{q'} = S_{q}, \)

(4.29b)

\[
P(q'_{2} = q_{2} | q_{3} = q_{3}) = P(q_{2} = q_{2} | q_{3} = q_{3}) = \frac{\sum_{q_{1}} P(q_{1}, q_{2}, q_{3})}{\sum_{q_{1}, q_{2}} P(q_{1}, q_{2}, q_{3})},
\]

(4.30a)

\[
P(q'_{1} = q_{1} | q'_{2} = q_{2}) = P(q_{1} = q_{1} | q_{2} = q_{2}) = \frac{\sum_{q_{3}} P(q_{1}, q_{2}, q_{3})}{\sum_{q_{1}, q_{3}} P(q_{1}, q_{2}, q_{3})},
\]

(4.30b)

where

\[
P(q_{1}, q_{2}, q_{3}) = P(q_{3} | q_{2}) P(q_{2} | q_{1}) P(q_{1}).
\]

(4.31)

Then, applying the fixed sender DP inequality leads to the fixed receiver one:

\[
H(q_{3} : q_{2}) = H(q_{3} : q'_{2}) \geq H(q_{3} : q'_{1}) = H(q_{3} : q_{1}).
\]

(4.32)

Can the DP inequalities, which are reminiscent of the Second Law of Thermodynamics, be generalized easily and naturally to Bayesian nets more complicated than merely Markov chains? Such a generalization could turn out to be very useful. After all, the Second Law of Thermodynamics is an extremely useful result. See [12] for a generalization.
In preparation for the next section, we will show in this section how to use a density matrix to generate a new, “reduced” density matrix. The Hilbert space acted upon by the reduced density matrix will have smaller dimension than the Hilbert space acted upon by the progenitor density matrix.

Recall that a density matrix is an operator $\rho$ acting on a Hilbert space $H$. In addition, $\rho$ must be a Hermitian operator with unit trace and non-negative eigenvalues. An operator with non-negative eigenvalues is called a non-negative (or positive indefinite) operator. Note that if $\sigma$ is a Hermitian operator that acts on a Hilbert space $H$, then $\sigma$ has non-negative eigenvalues iff $\langle \phi | \sigma | \phi \rangle \geq 0$ for all $| \phi \rangle \in H$. This is why. Let’s represent $\sigma$ by a matrix and the elements of $H$ by column vectors. Matrix $\sigma$ can be expressed as $\sigma = U \Lambda U^\dagger$, where $U$ is a unitary matrix and $\Lambda$ is a diagonal matrix whose diagonal entries $\lambda_i$ are the eigenvalues of $\sigma$. If $\phi$ is any vector in $H$, and $v_i$ are the components of vector $v = U^\dagger \phi$, then

$$\phi^\dagger \sigma \phi = \sum_i |v_i|^2 \lambda_i .$$

(5.1)

From the last equation, it is clear that $\phi^\dagger \sigma \phi \geq 0$ for all $| \phi \rangle \in H$ iff $\lambda_i \geq 0$ for all $i$.

For any operator $\sigma$ acting on $H_\alpha$ and for which $\text{tr}_\alpha \sigma \neq 0$, it is convenient to define the normalizing function $N(\sigma)$ by

$$N(\sigma) = \frac{\sigma}{\text{tr}_\alpha \sigma} .$$

(5.2)

Now suppose that $\rho$ is a density matrix acting on $H_\alpha \otimes H_\beta$, and $\pi_\alpha$ is a projection operator ($\pi_\alpha^2 = \pi_\alpha$) acting on $H_\alpha$. Let

$$K = \text{tr}_\alpha (\pi_\alpha \rho) .$$

(5.3)

If we define

$$| \phi_{ab} \rangle = (\pi_\alpha | a \rangle | b \rangle$$

(5.4)

for all $a \in S_\alpha$ and $b \in S_\beta$, then

$$K = \sum_{a,b} \langle \phi_{ab} | \rho | \phi_{ab} \rangle \geq 0 .$$

(5.5)

When $K \neq 0$, we can define the reduced density matrix $\text{red}_{\pi_\alpha} (\rho)$ by

$$\text{red}_{\pi_\alpha} (\rho) = N[\text{tr}_\alpha (\pi_\alpha \rho)] = K^{-1} \text{tr}_\alpha (\pi_\alpha \rho) .$$

(5.6)

Note that $\text{red}_{\pi_\alpha} (\rho)$ is indeed a density matrix. Clearly, it is Hermitian and it has unit trace. Furthermore, for any $| \beta \rangle \in H_\beta$, if we define
\[ |\chi_{a\beta} \rangle = (\pi_\alpha |a\rangle) |\beta\rangle \]  

(5.7)

for all \(a \in S_\alpha\), then

\[ \langle \beta | \text{red}_{\pi_\alpha} (\rho) |\beta\rangle = K^{-1} \sum_a \langle \chi_{a\beta} |\rho| \chi_{a\beta}\rangle \geq 0 . \]  

(5.8)

Some possibilities for \(\pi_\alpha\) are:

(a) \(\pi_\alpha = 1\). Then

\[ \text{red}_{\pi_\alpha} \rho = \text{tr}_\alpha \rho . \]  

(5.9)

Note that \(\text{tr}_\alpha (U \rho U^\dagger) = \text{tr}_\alpha (\rho)\) for any unitary matrix \(U\) acting on \(H_\alpha\). However, for other \(\pi_\alpha\)'s, it may happen that \(\text{red}_{\pi_\alpha} (U \rho U^\dagger) \neq \text{red}_{\pi_\alpha} (\rho)\). Thus, although not true for \(\text{tr}_\alpha (\cdot)\), \(\text{red}_{\pi_\alpha} (\cdot)\) may depend on the basis used to evaluate it.

(b) \(\pi_\alpha = |\alpha\rangle \langle \alpha|\), where \(|\alpha\rangle \in H_\alpha\). Then

\[ \text{red}_{\pi_\alpha} \rho = \frac{\langle \alpha|\rho|\alpha\rangle}{\langle \alpha| \text{tr}_\alpha (\rho) |\alpha\rangle} . \]  

(5.10)

If \(a, a' \in S_\alpha\), then some possibilities for \(|\alpha\rangle\) are \(|a\rangle\), \(\frac{1}{\sqrt{2}}(|a\rangle + |a'\rangle)\), and \(|\text{Av}(a)\rangle\), where

\[ |\text{Av}(a)\rangle = \frac{1}{\sqrt{N_\alpha}} \sum_{a \in S_\alpha} |a\rangle . \]  

(5.11)

We will call \(|\text{Av}(a)\rangle\) the average of the \(a\) basis.

Define

\[ E_\alpha = |\text{Av}(a)\rangle \langle \text{Av}(a)| , \]  

(5.12)

\[ K = \langle \text{Av}(a)| \text{tr}_\beta (\rho) |\text{Av}(a)\rangle . \]  

(5.13)

If \(K \neq 0\), we can define the entry sum \(E_{\Sigma_{\alpha}} (\rho)\) of \(\rho\) in the \(a\) basis by

\[ E_{\Sigma_{\alpha}} (\rho) = \text{red}_{E_\alpha} (\rho) . \]  

(5.14)

Thus,

\[ E_{\Sigma_{\alpha}} (\rho) = N [\text{tr}_\alpha (E_\alpha \rho)] = K^{-1} \langle \text{Av}(a)|\rho|\text{Av}(a)\rangle . \]  

(5.15)

\(E_{\Sigma_{\alpha}} (\rho)\) is called an entry sum because it can be expressed as

\[ E_{\Sigma_{\alpha}} (\rho) = N (\sum_{a_1, a_2} \langle a_1|\rho|a_2\rangle) , \]  

(5.16)

where the sum is over all \(a_1 \in S_\alpha\) and \(a_2 \in S_\alpha\).
6 Density Matrices Associated with a QB Net

In this section, we will describe a method for constructing many different density matrices associated with a single QB net.

Consider a QB net $N^Q$ with $N$ nodes labelled by the random variables $x_1, x_2, \ldots, x_N$. We will consider density matrices which act on $\mathcal{H}(x)$, where $\Gamma$ is a subset of $Z_{1,N}$. We will use $\Gamma(\rho)$ to represent the $\Gamma$ of density matrix $\rho$.

Let $A(x)$ be the amplitude assigned by $N^Q$ to story $x$. Assume that (see Appendix A)

$$\sum_x |A(x)|^2 = 1. \tag{6.1}$$

Then we can define the meta state-vector $|\psi_{\text{meta}}\rangle$ and the meta density matrix $\mu$ of $N^Q$ by

$$|\psi_{\text{meta}}\rangle = \sum_x A(x)|x\rangle, \tag{6.2}$$

$$\mu = |\psi_{\text{meta}}\rangle\langle\psi_{\text{meta}}|. \tag{6.3}$$

(Eq.(6.1) guarantees that $|\psi_{\text{meta}}\rangle$ has unit magnitude.) For example, if $N^Q$ has 3 nodes $a, b, c$, then

$$|\psi_{\text{meta}}\rangle = \sum_{a,b,c} A(a,b,c)|a,b,c\rangle, \tag{6.4}$$

$$\mu = \sum_{ij} A(a,b,c)A^*(a',b',c')|a,b,c\rangle\langle a',b',c'|. \tag{6.5}$$

Note that $|x\rangle$ in Eq.(6.2) represents a ket in the Hilbert space $\mathcal{H}_{x} = \mathcal{H}_{x_1} \otimes \mathcal{H}_{x_2} \otimes \cdots \otimes \mathcal{H}_{x_N}$. This is not the conventional use of a tensor product of Hilbert spaces. In Quantum Mechanics, such products are conventionally used to represent a “system” described by $\mathcal{H}_{x}$ which consists of $N$ “subsystems” such that the i’th subsystem is described by $\mathcal{H}_{x_i}$. ($x_1$ might correspond to the position and $x_2$ to the spin of the same particle, so the two subsystems may be associated with the same particle.) In our usage, the spaces $\mathcal{H}_{x_i}$ correspond to the nodes of a QB net. They need not correspond to separate subsystems. They might, for example, correspond to the same subsystem at two different times.

Because it acts on this unusual Hilbert space, the meta density matrix $\mu$ is unconventional. So why use it? Because it is uncontestably a density matrix in the formal sense (Hermitian, unit trace, non-negative.) Furthermore, as we shall see in what follows, $\mu$ proves to be a very useful tool for discussing QB nets. The reason why $\mu$ is so useful is not hard to see. $\mu$ is a vast storehouse of information about its QB net $N^Q$. In fact, it stores the amplitude of all the Feynman stories of $N^Q$. Applying to $\mu$ one or more red ($\Box$) operators of the type discussed in Section 5, we can generate
many different reduced density matrices, all pertaining to the same QB net $N^Q$. For example, for a QB net with 10 nodes, we might consider $E\Sigma_{x_1, x_2} \langle x_1 | \mu | x_1 \rangle$.

Suppose $\bar{\mathfrak{a}}$ is one of the nodes $\mathfrak{x}_j$ of the QB net, and consider red$_{\bar{\mathfrak{a}}} \mu$ for various $\pi_{\bar{\mathfrak{a}}}$.

(a) $\pi_{\bar{\mathfrak{a}}} = |a\rangle \langle a|$ for some $a \in S_{\bar{\mathfrak{a}}}$. Then red$_{\bar{\mathfrak{a}}} \mu = N(\langle a | \mu | a \rangle)$. This corresponds to an experiment in which node $\bar{\mathfrak{a}}$ is measured, and found to have a particular value $a$. The experiment is run repeatedly, and those runs for which $\bar{\mathfrak{a}} \neq a$ are rejected.

(b) $\pi_{\bar{\mathfrak{a}}} = 1$. Then red$_{\bar{\mathfrak{a}}} \mu = \text{tr}_{\bar{\mathfrak{a}}} \mu$. This corresponds to an experiment in which node $\bar{\mathfrak{a}}$ is measured without any expectations as to the value obtained. The experiment is run repeatedly. We sum over the various outcomes of the $\bar{\mathfrak{a}}$ measurement.

(c) $\pi_{\bar{\mathfrak{a}}} = |\mathcal{A}v(\bar{\mathfrak{a}})\rangle \langle \mathcal{A}v(\bar{\mathfrak{a}})|$. Then red$_{\bar{\mathfrak{a}}} \mu = E\Sigma_{\bar{\mathfrak{a}}} \mu$. This corresponds to an experiment in which node $\bar{\mathfrak{a}}$ is NOT measured.

Suppose $\rho$ is a density matrix obtained by reducing a meta density matrix $\mu$, and suppose $\rho$ acts on $\mathcal{H}_{(\mathfrak{x}) \Gamma(\mu)}$. Any node $\mathfrak{a}$ in $(\mathfrak{x}) \Gamma(\mu)$ will be said to be uncommitted, neither measured nor unmeasured. Any node $\mathfrak{a}$ in $(\mathfrak{x})_{Z_{1,N} \Gamma(\mu)}$ will be said to be either measured or unmeasured. It is unmeasured iff to go from $\mu$ to $\rho$, one of the reductions we performed was red$_{\bar{\mathfrak{a}}} = E\Sigma_{\bar{\mathfrak{a}}}$ as in case (c) above. If node $\mathfrak{a}$ is measured as in case (b) above (i.e., red$_{\bar{\mathfrak{a}}} = \text{tr}_{\bar{\mathfrak{a}}}$), we will say that it has been measured passively. We describe this measurement as passive because it does not involve data rejection by the observer like case (a) above.

Note that external nodes are always measured. If an observer does not measure them, they are still measured passively by the environment. Thus, if $\bar{\mathfrak{a}}$ is an external node, then $E\Sigma_{\bar{\mathfrak{a}}} (\mu)$ cannot be realized physically because $E\Sigma_{\bar{\mathfrak{a}}} (\mu)$ describes a situation in which $\bar{\mathfrak{a}}$ is not measured.

Suppose $\rho_{\text{out}}$ is obtained by e-summing $\mu$ over all internal nodes of the graph:

$$\rho_{\text{out}} = E\Sigma_{(\mathfrak{x})_{\mathcal{Z}_{\text{int}}}} (\mu) .$$

(6.6)

Then $\rho_{\text{out}}$ is a pure state. Here is why. Define

$$|\psi\rangle = \sum_{x} A(x.) |(x.)_{\mathcal{Z}_{\text{ext}}}\rangle .$$

(6.7)

Now note that

$$\langle \psi | \psi \rangle = \sum_{(x.)_{\mathcal{Z}_{\text{ext}}} \mathfrak{a}} \sum_{(x.)_{\mathcal{Z}_{\text{int}}} \mathfrak{a}} A(x.)^2 = 1 ,$$

(6.8)

and
\[ |\psi\rangle\langle \psi| = \sum_x \sum_{x'} A(x)x_A^* (x') |(x.)Z_{ext}\rangle\langle (x.)Z_{ext}| = \rho_{out}. \] (6.9)

QED.

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{figure6.1}
\caption{Fully connected 3 node graph.}
\end{figure}

To illustrate the definition of \( \rho_{out} \), consider Fig. (6.1), which shows a fully connected 3 node graph with nodes \( \text{a, b, c} \). Nodes \( \text{a, b} \) are internal and \( \text{c} \) is external. The \( \mu \) for this net is given by Eq. (6.5). Define \( \rho_{out} \) by

\[ \rho_{out} = E\Sigma_{a,b} \mu. \] (6.10)

If

\[ |\psi\rangle = \sum_{a,b,c} A(a,b,c) |c\rangle, \] (6.11)

then

\[ |\psi\rangle\langle \psi| = \sum_{a',b',c'} A(a,b)A^*(a',b',c') |c\rangle\langle c'| = \rho_{out}. \] (6.12)

\( \rho_{out} \) corresponds to a situation in which none of the internal nodes are measured and all the external ones are uncommitted. We will say that a density matrix has \textit{maximum internal coherence} if it corresponds to a situation in which none of the internal nodes are measured. \( \rho_{out} \) has maximum internal coherence. Reduced density matrices obtained by reducing \( \rho_{out} \) also have maximum internal coherence.
7 Probabilities Associated with a QB Net

In this section, we will define various probability distributions associated with a QB net.

Consider a QB net $\mathcal{N}^Q$ with $N$ nodes labelled by the random variables $x_1, x_2, \ldots, x_N$. Let $A(x)$ be the amplitude assigned by $\mathcal{N}^Q$ to story $x$. Suppose $\Gamma$ is a non-empty subset of $Z_{1,N}$. The probability of observing $(x)_{\Gamma}$ to have a value of $(x)_{\Gamma}$ is

$$ P[(x)_{\Gamma}] = \frac{\chi[(x)_{\Gamma}]}{\sum_{(y)_{\Gamma}} \chi[(y)_{\Gamma}]} , \quad (7.1) $$

where

$$ \chi[(x)_{\Gamma}] = \sum_{(x)_{z_{\text{ext}}-\Gamma}} \left( \sum_{(x)_{z_{\text{int}}-\Gamma}} A(x) \right)^2 . \quad (7.2) $$

In Eq. (7.2) we sum the amplitudes over all internal nodes except those in $\Gamma$, then we take the magnitude squared, then we sum that over all external nodes except those in $\Gamma$. We can express $P[(x)_{\Gamma}]$ in terms of the meta density matrix of the QB net:

$$ P[(x)_{\Gamma}] = \langle (x)_{\Gamma} | \left( \text{tr}_{(x)_{z_{\text{ext}}-\Gamma}} \mathcal{E}_{(x)_{z_{\text{int}}-\Gamma}} \mu \right) | (x)_{\Gamma} \rangle . \quad (7.3) $$

Thus, $P[(x)_{\Gamma}]$ corresponds to a situation in which the nodes in $\Gamma$ are projected to a single state, those in $Z_{\text{ext}} - \Gamma$ are passively measured, and those in $Z_{\text{int}} - \Gamma$ are not measured at all. Note that

$$ \sum_{(x)_{\Gamma}} P[(x)_{\Gamma}] = 1 , \quad (7.4) $$

as required for a probability distribution. However, if $\Gamma$ and $\Gamma'$ are non-empty disjoint subsets of $Z_{1,N}$, then it is possible that

$$ \sum_{(x)_{\Gamma \cup \Gamma'}} P[(x)_{\Gamma \cup \Gamma'}] \neq P[(x)_{\Gamma}] . \quad (7.5) $$

To illustrate the above definition of $P[(x)_{\Gamma}]$, consider the 3 node Markov chain $a \rightarrow b \rightarrow c$. Assume node $a$ has amplitudes $\psi_a$, where $\sum_a |\psi_a|^2 = 1$. Node $b$ (ditto, $c$) has amplitudes $U_{ba}$ (ditto, $V_{cb}$), where $U_{ba}$ (ditto, $V_{cb}$) are the entries of a unitary matrix. Then

$$ P(a) = \sum_{c} \left| \sum_{b} V_{cb} U_{ba} \psi_a \right|^2 = |\psi_a|^2 , \quad (7.6) $$

$$ P(b) = \sum_{c} \left| \sum_{a} V_{cb} U_{ba} \psi_a \right|^2 = \left| \sum_{a} U_{ba} \psi_a \right|^2 , \quad (7.7) $$

22
\[ P(c) = \left| \sum_{a,b} V_{cb} U_{ba} \psi_a \right|^2, \quad (7.8) \]

\[ P(b,c) = \frac{\left| \sum_a V_{cb} U_{ba} \psi_a \right|^2}{\sum_{b,c} \left| \sum_a V_{cb} U_{ba} \psi_a \right|^2} = \frac{\left| \sum_a V_{cb} U_{ba} \psi_a \right|^2}{\sum_{b,c} \left| \sum_a V_{cb} U_{ba} \psi_a \right|^2}, \quad (7.9) \]

\[ P(a, b, c) = \left| V_{cb} U_{ba} \psi_a \right|^2. \quad (7.10) \]

\[ \sum_{b,c} P(b, c) = 1, \quad (7.11) \]

but

\[ \sum_b P(b, c) \neq P(c). \quad (7.12) \]

We can define conditional probabilities using the unconditional ones \( P[(x.)_{\Gamma}] \) defined above. Suppose \( \Gamma_1 \) and \( \Gamma_2 \) are non-empty disjoint subsets of \( Z_{1,N} \). The conditional probability \( P[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] \) of observing \( (x.)_{\Gamma_1} \) to have a value of \( (x.)_{\Gamma_1} \), given or conditioned upon the fact that \( (x.)_{\Gamma_2} \) is known to have the value \( (x.)_{\Gamma_1} \), is

\[ P[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] = \frac{P[(x.)_{\Gamma_1\cup\Gamma_2}]}{P[(\mathcal{x})_{\Gamma_1}|(x.)_{\Gamma_2}]} , \quad (7.13) \]

where the denominator of this expression is defined by

\[ P[(\mathcal{x})_{\Gamma_1}|(x.)_{\Gamma_2}] = \sum_{(y.)_{\Gamma_1}} P[(y.)_{\Gamma_1}, (x.)_{\Gamma_2}]. \quad (7.14) \]

Note that

\[ \sum_{(x.)_{\Gamma_1}} P[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] = 1. \quad (7.15) \]

However, if \( \Gamma_1, \Gamma_1' \) and \( \Gamma_2 \) are non-empty disjoint subsets of \( Z_{1,N} \), then it is possible that

\[ \sum_{(x.)_{\Gamma_1\cup\Gamma_1'}} P[(x.)_{\Gamma_1\cup\Gamma_1'}|(x.)_{\Gamma_2}] \neq P[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}]. \quad (7.16) \]

To illustrate the definition of \( P[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] \), consider again the 3 node Markov chain \( \mathbf{a} \to \mathbf{b} \to \mathbf{c} \). One has

\[ P(a, b|c) = \frac{P(a, b, c)}{P(a, b)(c)}, \quad (7.17) \]

23
where

\[ P_{(a,b)}(c) = \sum_{a,b} P(a, b, c) . \]  

(7.18)

Note that

\[ \sum_{a,b} P(a, b|c) = 1 , \]  

(7.19)

but

\[ \sum_a P(a, b|c) \neq P(b|c) . \]  

(7.20)

We can easily extend the definition Eq. (7.13) of \( P[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] \) to the case that \( \Gamma_1 \) and \( \Gamma_2 \) overlap. We simply equate \( P[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] \) to \( P[(x.)_{\Gamma_1-\Gamma_2}|(x.)_{\Gamma_2}] \), and evaluate the latter with definition Eq. (7.13). For example, for a QB net with nodes \( a, b, c \), \( P[(a, b)|(b, c)] = P[a|(b, c)] \), and the right-hand side can be evaluated with Eq. (7.13).

Given any density matrix associated with the QB net \( \mathcal{N}^Q \), it is natural to define a probability distribution with its diagonal entries. Suppose \( \rho \) is a density matrix that acts on the Hilbert space \( H_{(x.)_{\Gamma^Q}} \), and suppose \( \Gamma \) is a non-empty subset of \( \Gamma(\rho) \). We define

\[ P_{\rho}[(x.)_{\Gamma}] = \langle (x.)_{\Gamma}|\text{tr}_{(x.)_{\Gamma}}(\rho)(x.)_{\Gamma} \rangle . \]  

(7.21)

In the last equation, we trace \( \rho \) over all nodes except those contained in \( \Gamma \), then we take the diagonal entries of the resulting operator. Note that

\[ \sum_{(x.)_{\Gamma}} P_{\rho}[(x.)_{\Gamma}] = 1 . \]  

(7.22)

Furthermore, if \( \Gamma \) and \( \Gamma' \) are non-empty disjoint subsets of \( \Gamma(\rho) \), then

\[ \sum_{(x.)_{\Gamma'}} P_{\rho}[(x.)_{\Gamma \cup \Gamma'}] = P_{\rho}[(x.)_{\Gamma}] . \]  

(7.23)

We can describe the last result by saying that the family of probability distributions \( \{ P_{\rho}[(x.)_{\Gamma}] | \Gamma \subset \Gamma(\rho) \} \) is closed under marginalization. We saw previously that the family \( \{ P[(x.)_{\Gamma}] | \Gamma \subset Z_{1,\mathcal{N}} \} \) does not possess this closure property.

To illustrate the definition of \( P_{\rho}[(x.)_{\Gamma}] \), consider a density matrix \( \rho \) which acts on \( H_{a,b,c} \). Then

\[ P_{\rho}(b, c) = \langle b, c|\text{tr}_{a}(\rho)|b, c \rangle , \]  

(7.24)

\[ \sum_{b,c} P_{\rho}(b, c) = 1 , \]  

(7.25)
\[
\sum_b P_\rho(b, c) = \langle c | \text{tr}_{\mathcal{A} \mathcal{B}} (\rho) | c \rangle = P_\rho(c) .
\] (7.26)

Note that for any probability distribution \( P[(x.)_\Gamma] \), we can find a density matrix \( \rho \) such that
\[
P[(x.)_\Gamma] = P_\rho[(x.)_\Gamma] .
\] (7.27)

Indeed, just set
\[
\rho = \text{tr}_{(\mathcal{Z})_{\text{ext-}\Gamma}} \left[ \mathcal{E}_\Sigma (\mathcal{Z})_{\text{int-}\Gamma} (\mu) \right] .
\] (7.28)

Suppose \( \mathcal{N}^C \) is the parent CB net of \( \mathcal{N}^Q \). Suppose \( \mu \) is the meta density matrix of \( \mathcal{N}^Q \). Then for any \( \Gamma \subset \mathcal{Z}_{1.N} \), \( P_\mu[(x.)_\Gamma] \) of \( \mathcal{N}^Q \) is identical to \( P[(x.)_\Gamma] \) of \( \mathcal{N}^C \). For example, if \( \mathcal{N}^Q \) had nodes \( a, b, c \) and amplitudes \( A(a, b, c) \), then \( P_\mu(a, b, c) \) for \( \mathcal{N}^Q \) and \( P(a, b, c) \) for \( \mathcal{N}^C \) both equal \( |A(a, b, c)|^2 \). Likewise, \( P_\mu(a, b) \) for \( \mathcal{N}^Q \) and \( P(a, b) \) for \( \mathcal{N}^C \) both equal \( \sum_c |A(a, b, c)|^2 \).

We can define conditional probability distributions using the unconditional ones \( P_\rho[(x.)_\Gamma] \) defined above. Suppose \( \Gamma_1 \) and \( \Gamma_2 \) are non-empty disjoint subsets of \( \Gamma(\rho) \). Then we define
\[
P_\rho[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] = \frac{P_\rho[(x.)_{\Gamma_1}, (x.)_{\Gamma_2}]}{\sum_{(y.)_{\Gamma_1}} P_\rho[(y.)_{\Gamma_1}, (x.)_{\Gamma_2}]} = \frac{P_\rho[(x.)_{\Gamma_1 \cup \Gamma_2}]}{P_\rho[(x.)_{\Gamma_2}]} .
\] (7.29)

Note that
\[
\sum_{(x.)_{\Gamma_2}} P_\rho[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] = 1 .
\] (7.30)

Furthermore, if \( \Gamma_1, \Gamma_1' \) and \( \Gamma_2 \) are non-empty disjoint subsets of \( \Gamma(\rho) \), then
\[
\sum_{(x.)_{\Gamma_1'}} P_\rho[(x.)_{\Gamma_1 \cup \Gamma_1'}|(x.)_{\Gamma_2}] = P_\rho[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] .
\] (7.31)

To illustrate the definition of \( P[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] \), consider a density matrix \( \rho \) which acts on \( \mathcal{H}_{\mathcal{A} \mathcal{B} \mathcal{C}} \). Then
\[
P_\rho(a, b|c) = \frac{P_\rho(a, b, c)}{\sum_{a', b'} P_\rho(a', b', c)} = \frac{P_\rho(a, b, c)}{P_\rho(c)} ,
\] (7.32)
\[
\sum_{a,b} P_\rho(a, b|c) = 1 ,
\] (7.33)
\[
\sum_a P_\rho(a, b|c) = P_\rho(b|c) .
\] (7.34)

We can easily extend the definition Eq.(7.29) of \( P_\rho[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] \) to the case that \( \Gamma_1 \) and \( \Gamma_2 \) overlap. We simply equate \( P_\rho[(x.)_{\Gamma_1}|(x.)_{\Gamma_2}] \) to \( P_\rho[(x.)_{\Gamma_1 \cap \Gamma_2}|(x.)_{\Gamma_2}] \), and evaluate the latter with definition Eq.(7.29).


Quantum Entropy: Its Definition and Properties

In this section, we will define various quantum entropies associated with a QB net. The von Neumann quantum entropy of a density matrix $\rho$ is defined by

$$S(\rho) = -\text{tr}(\rho \log_2 \rho) .$$

(8.1)

When $\rho$ is related to a QB net, it is convenient to rephrase Eq.(8.1) in terms of the node random variables of the net. Consider a QB net $N^Q$ with $N$ nodes labelled by the random variables $x_1, x_2, \ldots, x_N$. Suppose $\rho$ is a density matrix that acts on the Hilbert space $H_{(x)}\Gamma(\rho)$, and suppose $\Gamma, \Gamma_1$ and $\Gamma_2$ are non-empty subsets of $\Gamma(\rho)$. $\Gamma_1$ and $\Gamma_2$ need not be disjoint. We define:

$$S_\rho[(x)\Gamma] = S[\text{tr}(x)(\rho) \Gamma(\rho)],$$

(8.2)

$$S_\rho[(x)\Gamma_1|(x)\Gamma_2] = S_\rho[(x)\Gamma_1 \cup \Gamma_2] - S_\rho[(x)\Gamma_2],$$

(8.3)

$$S_\rho[(x)\Gamma_1 : (x)\Gamma_2] = S_\rho[(x)\Gamma_1] + S_\rho[(x)\Gamma_2] - S_\rho[(x)\Gamma_1 \cup \Gamma_2].$$

(8.4)

For example, suppose $a, b, c$ are nodes of a QB net. If $\rho$ is a density matrix which acts on $H_a$, then

$$S_\rho(a) = S(\rho) .$$

(8.5)

If instead, $\rho$ acts on $H_{a\downarrow b\downarrow c}$, then

$$S_\rho(a) = S[\text{tr}_{b\downarrow c} \rho],$$

(8.6)

$$S_\rho(a, b) = S[\text{tr}_{c} \rho],$$

(8.7)

$$S_\rho(a\mid b) = S_\rho(a, b) - S_\rho(b),$$

(8.8)

$$S_\rho(a : b) = S_\rho(a) + S_\rho(b) - S_\rho(a, b).$$

(8.9)

Eqs.(8.2) to (8.4) for the quantum entropy $S_\rho(\cdot)$ are very natural generalizations of Eqs.(3.2) to (3.4) for the classical entropy $H(\cdot)$.[13]

Note that definitions Eqs.(8.2) to (8.4) are independent of the order of the node random variables within $(x)\Gamma_1$ and $(x)\Gamma_2$. For example, if $\rho$ is a density matrix acting on $H_{a\downarrow b\downarrow c}$, then

$$S_\rho(a, b, c) = S_\rho(a, c, b), \quad S_\rho[a\mid(b, c)] = S_\rho[a\mid(b, c)].$$

(8.10)
It is convenient to extend definitions Eqs. (8.2) to (8.4) in the following two ways. First, we will allow \((x.)_{\Gamma_1}\) (ditto, \((x.)_{\Gamma_2}\)) to contain repeated random variables. If it does, then we will throw out any extra copies of a random variable. For example, if \(\rho\) is a density matrix acting on \(H_{a,b,c}\), then

\[
S_\rho(a,a,b,c) = S_\rho(a,b,c) , \quad S_\rho[a|(b,c)] = S_\rho[a|(b,c)] .
\]  

(8.11)

Second, we will allow \((x.)_{\Gamma_1}\) (ditto, \((x.)_{\Gamma_2}\)) to contain internal parentheses. If it does, then we will ignore the internal parentheses. For example, if \(\rho\) is a density matrix acting on \(H_{a,b,c}\), then

\[
S_\rho([a,b],c) = S_\rho(a,b,c) , \quad S_\rho[a|((b,c),d)] = S_\rho[a|(b,c,d)] .
\]  

(8.12)

Let \(X = (x.)_{\Gamma_1}\), \(Y = (x.)_{\Gamma_2}\) and \(Z = (x.)_{\Gamma_3}\), where the \(\Gamma_1, \Gamma_2, \Gamma_3\) are non-empty, possibly overlapping, subsets of \(Z_{1,N}\). As with the function \(H(\cdot)\), we will extend further the domain of the function \(S_\rho(\cdot)\) by introducing the following axioms

\[
S_\rho[(X,Y) : Z] = S_\rho[(X : Z), (Y : Z)] ,
\]  

(8.13)

\[
S_\rho((X : Y), Z] = S_\rho((X, Z) : (Y, Z)] .
\]  

(8.14)

Table 3 gives a list of properties (identities and inequalities) satisfied by the quantum entropy \(S_\rho(\cdot)\). Whenever possible, Table 3 matches each property of the quantum entropy \(S_\rho(\cdot)\) with an analogous property of the classical entropy \(H(\cdot)\). Analogous properties are indicated by \(H \rightarrow S_\rho\). See Refs. [1]-[9] to get proofs of those statements in Table 3 that are not proven in this paper.

An identity satisfied by \(S(\cdot)\) but with no classical counterpart is:

\[
S(U \rho U^\dagger) = S(\rho) ,
\]  

(8.15)

for any unitary matrix \(U\) acting on the same Hilbert space as the density matrix \(\rho\). We say that \(S(\cdot)\) is invariant under unitary transformations of its argument. Next we will rephrase Eq. (8.15) in terms of the node random variables of a QB net. Let \(\Gamma_1\) and \(\Gamma_2\) be disjoint sets whose union is \(\Gamma(\rho)\). Define \(X_1 = (x.)_{\Gamma_1}\), \(X_2 = (x.)_{\Gamma_2}\), and \(X = (x.)_{\Gamma(\rho)}\). Thus, \(X = (X_1, X_2)\). \(\rho\) acts on \(H_X\) so we can express it as:

\[
\rho = \sum_{ri} |X\rangle \rho_{X,X'} \langle X'| .
\]  

(8.16)

Suppose \(U\) acts on \(H_{X_1}\). Then

\[
U \rho U^\dagger = \sum_{ri} |Y\rangle |X_2\rangle U_{YX_1} \rho(x_1,x_2),(x_1',x_2') U_{X_1Y'}^\dagger \langle X'_2| \langle Y'| ,
\]  

(8.17)

\[
= \sum_{ri} |\psi_Y(X_1)\rangle |X_2\rangle \rho(x_1,x_2),(x_1',x_2') \langle X'_2| \langle \psi_Y(X_1)'| ,
\]  

where
\[ |\psi_Y(X_1)\rangle = \sum_Y |Y\rangle Y_{UX_1}. \]  

(8.18)

The Hilbert space \( \mathcal{H}_Y \) has the same dimension as \( \mathcal{H}_{X_1} \). The vectors \( |\psi_Y(X_1)\rangle \in \mathcal{H}_Y \) are orthonormal:

\[ \langle \psi_Y(X_1)|\psi_Y(X'_1)\rangle = \delta(X_1, X'_1). \]  

(8.19)

Thus,

\[ S_{U\rho U^\dagger}(Y, X_2) = \left[ S_{U\rho U^\dagger}(Y, X_2) \right]_{U=1} = S_{\rho}(X_1, X_2). \]  

(8.20)

Suppose \( X = (x)_r \) for some non-empty set \( \Gamma \subset \Gamma(\rho) \). The matrix \( \text{tr}_{(x)_r}(\rho) \) used in definition Eq. (8.2) of \( S_{\rho}(X) \) has diagonal entries which are the probabilities \( P_{\rho}(X) \) defined in Section 7. It is convenient to define a classical entropy for the \( P_{\rho}(X) \) distribution:

\[ H_{\rho}(X) = -\sum_X P_{\rho}(X) \log_2 P_{\rho}(X). \]  

(8.21)

Because the probability distributions \( P_{\rho}(X) \) are closed under marginalization, \( H_{\rho}(\cdot) \) satisfies all the identities and inequalities (see Table 3) satisfied by the classical entropy \( H(\cdot) \).

It follows from Table 3 that

\[ 0 \leq S_{\rho}(X) \leq H_{\rho}(X). \]  

(8.22)

Thus, \( H_{\rho}(X) \) is a useful upper bound on \( S_{\rho}(X) \).

The quantities \( H_{\rho}(X) \) and \( S_{\rho}(X) \) complement each other in what they tells us about \( \rho \) and \( X \). Indeed, note the following. Suppose \( X = (x)_r \) where \( \Gamma \subset \Gamma(\rho) \). Let \( \rho' = \text{tr}_{(x)_r} \rho(\rho) \) and \( M = \langle (x)_r|\rho'|(x)_r \rangle \) so that

\[ S_{\rho}(X) = -\text{tr}(M \log_2 M), \]  

(8.23)

\[ H_{\rho}(X) = -\sum_i M_{ii} \log_2 M_{ii}. \]  

(8.24)

\( M \) is a diagonal matrix iff \( S_{\rho}(X) = H_{\rho}(X) \). Knowing \( S_{\rho}(X) \) alone does not tell us if \( M \) is diagonal because \( \text{tr}(M \log_2 M) \) is invariant under unitary transformations of \( M \).

Henceforth, we will refer to the quantity

\[ Q_{\rho}(X) = H_{\rho}(X) - S_{\rho}(X) \]  

(8.25)

as the coherence of \( X \) in \( \rho \). Note that

\[ 0 \leq Q_{\rho}(X) \leq \log_2 N_X. \]  

(8.26)

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One has $Q_{\rho}(X) = 0$ (i.e., zero coherence) iff $H_{\rho}(X) = S_{\rho}(X)$, which is true iff $M$ is diagonal. One has $Q_{\rho}(X) = \log_2 N_X$ (i.e., max. coherence) iff $S_{\rho}(X) = 0$ and $H_{\rho}(X) = \log_2 N_X$. $S_{\rho}(X) = 0$ iff there exists some column vector $v$ such that $M = vv^\dagger$. $H_{\rho}(X) = \log_2 N_X$ iff the diagonal entries of $M$ are all equal. In fact, at max. coherence, all the entries of $M$ have the same absolute value $1/N_X$.

$Q_{\rho}(X) = 0$ iff $\rho' = \text{tr}_{(x)_{\Gamma(\rho)}}(\rho)$ is diagonal in the $X$-basis $\{|(x)_{\Gamma}\rangle\}$. Hence, $Q_{\rho}(X)$ can also be interpreted as the mismatch between $\rho'$ and the $X$ basis. At zero mismatch, the $X$ basis constitutes a set of eigenvectors of $\rho'$. 


9 Mixed States and Purification

In this section, we will show how any mixed state density matrix can be represented by a QB net.

Consider the QB net of Fig. (9.1), where

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| $j$   | $j = (j_1, j_2)$ | $\alpha_j$ | $\sum_j |\alpha_j|^2 = 1$ |
| $q$   | $q$    | $\delta(q, j_1)$ |          |
| $r$   | $r$    | $\delta(r, j_2)$ |          |

The meta density matrix $\mu$ for this net is

$$\mu = |\psi_{\text{meta}}\rangle\langle \psi_{\text{meta}}| ,$$

(9.1)

where

$$|\psi_{\text{meta}}\rangle = \sum_{ri} \alpha_{qr} |j = (q, r)\rangle |q = q\rangle |r = r\rangle.$$  

(9.2)

Define $\sigma$ and $\sigma_q$ by

$$\sigma = E\Sigma_j (\mu) = \sum_{ri} \alpha_{qr} \alpha^*_{q'r'} |q, r\rangle \langle q, r'|,$$

(9.3)

$$\sigma_q = \text{tr}_r (\sigma) = \sum_{ri} \alpha_{qr} \alpha^*_{q'r} |q\rangle \langle q'|.$$  

(9.4)

Clearly, $\sigma$ is a pure state and $\sigma_q$ is a mixed one. Since $\sigma$ is a pure state,

$$S_{\sigma}(q, r) = 0 .$$

(9.5)

By the Triangle Inequality (see Table 3),

$$S_{\sigma}(q) = S_{\sigma}(r) .$$

(9.6)
We’ve shown that some mixed state density matrices can be represented by a QB net. But can any mixed state density matrix be represented in this manner? Yes. This is why. Suppose \( \rho \) is

\[
\rho = \sum_{q} \beta_{qq'} |q\rangle \langle q' | .
\] (9.7)

Then the complex numbers \( \beta_{qq'} \) define a Hermitian matrix \( \beta \). One can always decompose \( \beta \) into \( \beta = U \Gamma U^\dagger \), where \( U \) is a unitary matrix and \( \Gamma \) is a diagonal matrix. If we let \( \alpha = U \sqrt{\Gamma} \), then

\[
\beta = \alpha \alpha^\dagger .
\] (9.8)

Thus,

\[
\rho = \sum_{q} \alpha_{qr} \alpha_{q'r}^* |q\rangle \langle q' | .
\] (9.9)

QED. The state

\[
|\psi\rangle = \sum_{q} \alpha_{q} |q, r\rangle
\] (9.10)

is called a *purification* of \( \rho \), because the mixed state \( \rho \) can be obtained from the pure state \( |\psi\rangle \) as follows:

\[
\text{tr}_z |\psi\rangle \langle \psi| = \rho .
\] (9.11)
10 Quantum System Interacting with Environment

In this section, we will consider QB nets that represents a quantum system interacting with its environment one or more times.

10.1 Single Interaction

![QB net diagram](image)

Figure 10.1: QB net for a system interacting once with its environment.

Consider the QB net of Fig. (10.1), where

| nodes | states | amplitudes | comments |
|-------|-------|-----------|----------|
| \( j \) | \( j = (j^1, j^2) \) | \( \alpha_j \) | \( \sum_j |\alpha_j|^2 = 1 \) |
| \( q \) | \( q \) | \( \delta(q, j^1) \) | |
| \( r \) | \( r \) | \( \delta(r, j^2) \) | |
| \( e \) | \( e \) | \( \beta_e \) | \( \sum_e |\beta_e|^2 = 1 \) |
| \( t \) | \( t = (t^1, t^2) \) | \( U(t|q, e) \) | \( \sum_t U(t|q, e)U^*(t|q', e') = \delta_{q q'} \delta_{e e'} \) |
| \( q_f \) | \( q_f \) | \( \delta(q_f, t^1) \) | |
| \( e_f \) | \( e_f \) | \( \delta(e_f, t^2) \) | |

Let \( \mathcal{N}_Q \) be the QB net which contains all the nodes shown in Fig. (10.1). Let \( \mathcal{N}_0^Q \) be the sub-net which contains only nodes \( j, q_1, r \).

The meta density matrix \( \mu_0 \) of \( \mathcal{N}_0^Q \) is

\[
\mu_0 = |\psi_{\text{meta}}^0\rangle\langle\psi_{\text{meta}}^0| ,
\]

(10.1)
where

\[ |\psi^0_{\text{meta}}\rangle = \sum_{ri} \alpha_{qr} |j = (q, r), q, r\rangle . \]  \hfill (10.2)

Define \( \rho_0 \) by

\[ \rho_0 = \mathbb{E} \Sigma |j \mu_0 = \sum_{ri} \alpha_{qr} \alpha^*_{qr'} |q, r\rangle \langle q', r'| . \]  \hfill (10.3)

\( \rho_0 \) is a pure state so

\[ S_{\rho_0}(r, q) = 0 . \]  \hfill (10.4)

The meta density matrix \( \mu \) of \( N^Q \) is

\[ \mu = |\psi_{\text{meta}}\rangle \langle \psi_{\text{meta}}| , \]  \hfill (10.5)

where

\[ |\psi_{\text{meta}}\rangle = \sum_{ri} U(q_f, e_f | q, e) \beta_{e} \alpha_{qr} |j = (q, r), q, r, e, t = (q_f, e_f), q_f, e_f\rangle . \]  \hfill (10.6)

Define \( \rho \) by

\[ \rho = \mathbb{E} \Sigma |_{q, e, t} \mu = \sum_{ri} U(q_f, e_f | q, e) \beta_{e} \alpha_{qr} U^*(q'_f, e'_f | q', e') \beta^*_{e'} \alpha^*_{qr'} |r, q_f, e_f\rangle \langle r', q'_f, e'_f| . \]  \hfill (10.7)

\( \rho \) is a pure state so

\[ S_{\rho}(r, q_f, e_f) = 0 . \]  \hfill (10.8)

By virtue of sub-additivity,

\[ S_{\rho}(r, e_f) = S_{\rho}(e_f) \leq S_{\rho}(r) . \]  \hfill (10.9)

By Eqs.(10.4) and (10.8) and the Triangle Inequality,

\[ S_{\rho}(r, e_f) = S_{\rho}(q_f) , \]  \hfill (10.10)

\[ S_{\rho}(r) = S_{\rho_0}(r) = S_{\rho_0}(q) . \]  \hfill (10.11)

Hence, Eq.(10.9) can be written as[14]

\[ S_{\rho}(q_f) - S_{\rho}(e_f) \leq S_{\rho_0}(q) . \]  \hfill (10.12)
### 10.2 Multiple Interactions

Consider the QB net of Fig. (10.2), where

| nodes   | states | amplitudes | comments |
|---------|--------|------------|----------|
| \( j \) | \( j = (j^1, j^2) \) | \( \alpha(j) \) | \( \sum_j |\alpha(j)|^2 = 1 \) |
| \( q_1 \) | \( q_1 \) | \( \delta(q_1, j^1) \) | |
| \( r \) | \( r \) | \( \delta(r, j^2) \) | |
| \( e_\lambda \) for \( \lambda \in Z_{1,2} \) | \( e_\lambda \) | \( \beta_\lambda(e_\lambda) \) | \( \sum_{e_\lambda} |\beta_\lambda(e_\lambda)|^2 = 1 \) |
| \( t_\lambda \) for \( \lambda \in Z_{1,2} \) | \( t_\lambda = (t^1_\lambda, t^2_\lambda) \) | \( U_\lambda(t_\lambda|q_\lambda, e_\lambda) \) | \( \sum_{t_\lambda} U_\lambda(t_\lambda|q_\lambda, e_\lambda)U^{*}_\lambda(t_\lambda'|q_\lambda', e_\lambda') = \delta_{q_\lambda q_\lambda'}^{q_\lambda q_\lambda'} \) |
| \( q_{\lambda f} \) for \( \lambda \in Z_{1,2} \) | \( q_{\lambda f} \) | \( \delta(q_{\lambda f}, t^1_\lambda) \) | Define \( q_2 = q_{1 f} \), \( q_3 = q_{2 f} \) |
| \( e_{\lambda f} \) for \( \lambda \in Z_{1,2} \) | \( e_{\lambda f} \) | \( \delta(e_{\lambda f}, t^2_\lambda) \) | |

Let \( \mathcal{N}^Q_0 \) be the net which contains only nodes \( j, q, r \). For \( \tau \in Z_{1,2} \), let \( \mathcal{N}^Q_\tau \) be the net which contains the previous net \( \mathcal{N}^Q_{\tau-1} \) plus nodes \( e_\tau, t_\tau, q_{\tau f}, e_{\tau f} \).

For \( \tau \in Z_{0,2} \), the meta density matrix \( \mu_\tau \) of net \( \mathcal{N}^Q_\tau \) is

\[
\mu_\tau = |\psi_{\tau \text{meta}}^\tau \rangle \langle \psi_{\tau \text{meta}}^\tau| ,
\]  

(10.13)

where

\[
|\psi_{\tau \text{meta}}^\tau \rangle = \sum_{\text{all}} \left( \prod_{\lambda=1}^{\tau} M_\lambda \right) \alpha(q_1, r)|\underline{j} = (q_1, r), q_1, r \rangle ,
\]  

(10.14)

where

\[
M_\lambda = U_\lambda(q_{\lambda f}, e_{\lambda f}|q_\lambda, e_\lambda)\beta_\lambda(e_\lambda)|e_\lambda, \underline{t}_\lambda = (q_{\lambda f}, e_{\lambda f}), q_\lambda, e_\lambda \rangle.
\]  

(10.15)
Define $\rho_\tau$ for $\tau \in Z_{0,2}$ by

$$\rho_\tau = \sum_{X(\tau)} (\mu_\tau) ,$$

(10.16)

where $X(\tau)$ represents all the internal nodes of $N^Q_\tau$. Thus, $\rho_0$ acts on $\mathcal{H}(\varrho_{q_1})$, $\rho_1$ acts on $\mathcal{H}(\varrho_{q_2})$, and $\rho_2$ acts on $\mathcal{H}(\varrho_{q_3})$. For any $\tau \in Z_{0,2}$, $\rho_\tau$ is a pure state so

$$S_{\rho_0}(r, q_1) = 0 ,$$

(10.17a)

$$S_{\rho_1}(r, q_{1f}, q_{1f}) = 0 ,$$

(10.17b)

$$S_{\rho_2}(r, q_{1f}, q_{2f}, q_{2f}) = 0 .$$

(10.17c)

Weak and strong sub-additivity imply

$$S_{\rho_2}(q_{1f}, q_{2f}) \leq S_{\rho_2}(q_{1f}, q_{2f}) \leq S_{\rho_2}(r) ,$$

(10.18)

which, by virtue of Eqs.(10.17), can be written as[14]

$$S_{\rho_2}(q_{2f}) - S_{\rho_2}(q_{1f}, q_{2f}) \leq S_{\rho_1}(q_{1f}) - S_{\rho_1}(q_{2f}) \leq S_{\rho_0}(q_1) .$$

(10.19)

Define $\sigma_\tau$ for all $\tau \in Z_{0,2}$ by

$$\sigma_\tau = \sum_{X(\tau)} (\mu_\tau) ,$$

(10.20)

where $X(\tau)$ now represents all the internal nodes of $N^Q_\tau$ except for $q_1, q_2, q_3$. Thus, $\sigma_0$ acts on $\mathcal{H}(\varrho_{q_1})$, $\sigma_1$ acts on $\mathcal{H}(\varrho_{q_2})$, and $\sigma_2$ acts on $\mathcal{H}(\varrho_{q_3})$. Next we will show that

$$0 = S_{\sigma_0}(q_1 | q_1) \leq S_{\sigma_1}(q_1 | q_2) \leq S_{\sigma_2}(q_1 | q_3) ,$$

(10.21)

which is a quantum counterpart of the classical fixed sender DP inequality Eq.(4.16). First note that

$$S_{\sigma_2}(q_1 | q_3) = S_{\sigma_2}(q_1 | q_{2f}) \geq S_{\sigma_2}(q_1 | q_{2f}, q_{2f}) ,$$

(10.22a)

where we’ve used $q_3 = q_{2f}$ and strong sub-additivity. Since $S(\cdot)$ is invariant under unitary transformations of its argument,

$$S_{\sigma_2}(q_1 | q_{2f}, q_{2f}) = \left[ S_{\sigma_2}(q_1 | q_{2f}, q_{2f}) \right]_{U_{2f}=1} = S_{\sigma_1}(q_1 | q_2) + \text{nil} ,$$

(10.22b)

where nil equals $S(\sum_{\beta, \beta'} \beta(e_2) \beta^*(e_2') | e_2)(e_2')$, which is zero. Combining Eqs.(10.22), we get

$$S_{\sigma_2}(q_1 | q_3) \geq S_{\sigma_1}(q_1 | q_2) .$$

(10.23)

QED. For an alternative proof of Eq.(10.21), see [12].
11 Two Mixtures Interacting

In this section, we will consider a QB net that represents two mixed states scattering once off each other.

Figure 11.1: QB net for 2 mixtures interacting.

Consider the QB net of Fig.(11.1), where

| nodes   | states | amplitudes | comments |
|---------|--------|------------|----------|
| \( j_\lambda \) for \( \lambda \in \mathbb{Z}_{1,2} \) | \( j_\lambda = (j_\lambda^1, j_\lambda^2) \) | \( \alpha_\lambda(j_\lambda) \) | \( \sum_{j_\lambda} |\alpha_\lambda(j_\lambda)|^2 = 1 \) |
| \( q_\lambda \) for \( \lambda \in \mathbb{Z}_{1,2} \) | \( q_\lambda \) | \( \delta(q_\lambda, j_\lambda^1) \) | |
| \( r_\lambda \) for \( \lambda \in \mathbb{Z}_{1,2} \) | \( r_\lambda \) | \( \delta(r_\lambda, j_\lambda^2) \) | |
| \( t \) | \( t = (t^1, t^2) \) | \( U(t|q_1, q_2)U^*(t|q_1', q_2') = \delta_{q_1}^{q_1'}\delta_{q_2}^{q_2'} \) | |
| \( q_{\lambda f} \) for \( \lambda \in \mathbb{Z}_{1,2} \) | \( q_{\lambda f} \) | \( \delta(q_{\lambda f}, t^\lambda) \) | |

Let \( \mathcal{N}_\lambda^Q \) be the QB net which contains all the nodes shown in Fig.(11.1). For \( \lambda \in \mathbb{Z}_{1,2} \), let \( \mathcal{N}_\lambda^{Q_\lambda} \) be the sub-net which contains only nodes \( \tilde{j}_\lambda, q_\lambda, r_\lambda \).

For \( \lambda \in \mathbb{Z}_{1,2} \), the meta density matrix \( \mu_\lambda \) of \( \mathcal{N}_\lambda^Q \) is

\[
\mu_\lambda = |\psi^\lambda_{\text{meta}}\rangle\langle\psi^\lambda_{\text{meta}}|,
\]

where

\[
|\psi^\lambda_{\text{meta}}\rangle = \sum_{r_\lambda} \alpha_\lambda(q_\lambda, r_\lambda)|\tilde{j}_\lambda = (q_\lambda, r_\lambda)\rangle \quad (11.2)
\]
Define $\rho_\lambda$ by

$$\rho_\lambda = E\Sigma_{j\lambda} \mu_\lambda .$$

(11.3)

$\rho_\lambda$ acts on $\mathcal{H}_{(q_\lambda, r_\lambda)}$ and it is a pure state so

$$S_{\rho_\lambda}(q_\lambda, r_\lambda) = 0 .$$

(11.4)

The meta density matrix $\mu$ of $\mathcal{N}^Q$ is

$$\mu = |\psi_{\text{meta}}\rangle\langle \psi_{\text{meta}}| ,$$

(11.5)

where

$$|\psi_{\text{meta}}\rangle = \sum_{r_i} U(q_1, q_2|q_1, q_2) \left( \prod_{\lambda=1}^{2} \alpha_\lambda(q_\lambda, r_\lambda)|j_\lambda = (q_\lambda, r_\lambda)\rangle |t = (q_1, q_2), q_1, q_2\rangle .$$

(11.6)

Define $\rho$ by:

$$\rho = E\Sigma_{j\lambda} \mu_\lambda .$$

(11.7)

$\rho$ acts on $\mathcal{H}_{(q_1, q_2)}$ and it is a pure state so

$$S_{\rho}(q_1, q_2) = 0 .$$

(11.8)

According to Table 3,

$$|S_{\rho}(q_{\lambda f}) - S_{\rho}(q_{\lambda})| \leq S_{\rho}(q_{\lambda f}, q_{\lambda}) \leq S_{\rho}(q_{\lambda f}) + S_{\rho}(q_{\lambda}) ,$$

(11.9)

for $\lambda \in Z_{1,2}$. By Eq.(11.4) and the Triangle Inequality,

$$S_{\rho}(q_{\lambda}) = S_{\rho}(j_{\lambda}) = S_{\rho}(q_{\lambda}) .$$

(11.10)

By Eq.(11.8) and the Triangle Inequality,

$$S_{\rho}(q_{1f}, q_{1}) = S_{\rho}(q_{2f}, q_{2}) .$$

(11.11)

Hence, Eq.(11.9) can be rewritten as[15]

$$|S_{\rho}(q_{\lambda f}) - S_{\rho}(q_{\lambda})| \leq S_{\rho}(q_{\lambda f}, q_{1}) = S_{\rho}(q_{2f}, q_{2}) \leq S_{\rho}(q_{\lambda f}) + S_{\rho}(q_{\lambda}) .$$

(11.12)
12 POM

Given a Hilbert space $H$, a POM (Probability Operator Measure) is a set $\{ F_b | b \in S_b \}$ of non-negative Hermitian operators acting on $H$. In addition, the observables $F_b$ must form a “complete” set, meaning that

$$
\sum_b F_b = 1 .
$$

(12.1)

If $\rho$ is a density matrix acting on the same Hilbert space $H$ as the $F_b$’s, then we can define a probability distribution for the random variable $b$ by

$$
P(b) = \text{tr}(\rho F_b),
$$

(12.2)

for all $b \in S_b$. We call an experiment that yields the value $b$ for $b$ with a probability $P(b)$ a “generalized measurement”.

We say that the $F_b$’s are (pairwise) orthogonal if $F_b F_{b'} = 0$ for all $b, b' \in S_b$ such that $b \neq b'$. If the $F_b$’s are orthogonal, then we say that $\{ F_b | \forall b \}$ is an orthogonal POM.

An operator $F_b$ is said to have rank one if it can be represented in the form $|\psi\rangle\langle\psi|$, where $|\psi\rangle$ need not have unit magnitude. If $|\psi\rangle$ does have unit magnitude, then $F_b$ is a projector (i.e., $F_b^2 = F_b$). An $F_b$ which is projector is a pure state density matrix. For this reason, if the $F_b$’s are all projectors, then we say that $\{ F_b | \forall b \}$ is a pure POM.

A POM is both pure and orthogonal iff its $F_b$’s are (pairwise) orthogonal projectors (i.e., $F_b F_{b'} = F_b \delta(b, b')$ for all $b, b' \in S_b$). For such a POM, we can represent each $F_b$ by $|b\rangle\langle b|$, where the $|b\rangle$’s are an orthonormal basis of $H$. Eq.(12.1) then reduces to $\sum_b |b\rangle\langle b| = 1$. Such a POM is said to constitute a von Neumann or ideal measurement.

In this section, we will show how to represent a POM as a QB net. Part (a) will assume that the $F_b$’s are orthogonal projectors. Part (b) will not assume this.

12.1 Orthogonal Projector $F_b$’s

Consider the QB net of Fig.(12.1), where
Figure 12.1: QB net for orthogonal projector $F_b$'s.

| nodes | states  | amplitudes | comments                        |
|-------|---------|------------|---------------------------------|
| $j$   | $j = (j_1, j_2)$ | $\alpha_j$ | $\sum_j |\alpha_j|^2 = 1$           |
| $q$   | $q$     | $\delta(q, j_1)$ |                                |
| $r$   | $r$     | $\delta(r, j_2)$ |                                |
| $t$   | $t = (t_1, t_2)$ | $U(t|q, b)$ | $\sum_t U(t|q, b)U^*(t|q', b') = \delta_{q'}^q \delta_{b'}^b$ |
| $b$   | $b$     | $\delta(b, 0)$ |                                |
| $q_f$ | $q_f$   | $\delta(q_f, t_1)$ |                                |
| $b_f$ | $b_f$   | $\delta(b_f, t_2)$ |                                |

Suppose the unitary operator $U$ satisfies:

$$U|\phi\rangle_q \otimes |0\rangle_b = \sum_b \left( \sqrt{F_b} |\phi\rangle_q \right) \otimes |b\rangle_b ,$$

(12.3)

for any unit-magnitude vector $|\phi\rangle_q \in \mathcal{H}_q$. One can show that, for any POM $\{F_b|\forall b\}$ acting on $\mathcal{H}_q$, there exists a unitary operator $U$ that satisfies Eq.(12.3). Note that on the right-hand side of Eq.(12.3), the state $|b\rangle$ acts as a pointer that points towards a particular choice of $F_b$. Note that the completeness of the $F_b$’s and the unit-magnitude of $|\phi\rangle_q$ together imply that the right-hand side of Eq.(12.3) is a unit-magnitude vector. The vector $|\phi\rangle_q \otimes |0\rangle_b$, upon which $U$ acts is likewise a unit-magnitude vector. The fact that $U$ takes a unit-magnitude vector into another unit-magnitude vector (of the same dimension) is consistent with the unitarity of $U$.

Eq.(12.3) can be expressed in component form as follows:
\[ \sum_{r_i} U(q_f, b_f| q, b) \phi(q) \delta^b_0 = \sum_{r_i} \sqrt{F_b(q_f| q)} \phi(q) \delta^b_{b_f} \]  \hspace{1cm} (12.4)

for any function \( \phi(q) \). (\( \phi(q) \) need not be normalized since it appears on both sides of the equation.)

Let \( \mathcal{N}^Q \) be the QB net which contains all the nodes shown in Fig. (12.1). Let \( \mathcal{N}_0^Q \) be the sub-net which contains only nodes \( j, q, r \).

The meta density matrix \( \mu_0 \) of \( \mathcal{N}_0^Q \) is

\[ \mu_0 = \langle \psi^0_{meta} | \psi^0_{meta} \rangle, \]  \hspace{1cm} (12.5)

where

\[ | \psi^0_{meta} \rangle = \sum_{r_i} \alpha_{qr} \delta^b_0 | j = (q, r) \rangle . \]  \hspace{1cm} (12.6)

Define

\[ \rho_0 = \text{E} \sum_j \text{tr}_{\bar{L}} \mu_0 = \sum_{r_i} \alpha_{qr} \alpha_{q^* r} \langle q | q' \rangle . \]  \hspace{1cm} (12.7)

The meta density matrix \( \mu \) of \( \mathcal{N}^Q \) is

\[ \mu = \langle \psi_{meta} | \psi_{meta} \rangle , \]  \hspace{1cm} (12.8)

where

\[ | \psi_{meta} \rangle = \sum_{r_i} U(q_f, b_f| q, b) \alpha_{qr} \delta^b_{b_f} | j = (q, r) \rangle . \]  \hspace{1cm} (12.9)

By Eq. (12.4), \( | \psi_{meta} \rangle \) can also be expressed as

\[ | \psi_{meta} \rangle = \sum_{r_i} \sqrt{F_b(q_f|q)} \alpha_{qr} \delta^b_{b_f} | j = (q, r) \rangle . \]  \hspace{1cm} (12.10)

Define \( \rho \) by

\[ \rho = \text{E} \sum_{j, q, r} \text{tr}_{\bar{L}} \mu . \]  \hspace{1cm} (12.11)

In other words, we get \( \rho \) by tracing \( \mu \) over all the external nodes except \( b_f \), and e-summing it over all the internal nodes. \( \rho \) acts on \( \mathcal{H}_{b_f} \). Using the fact that the \( F_b \)'s are orthogonal projectors, it is easy to show that

\[ \rho = \sum_{r_i} F_b(q'|q) \alpha_{qr} \alpha^*_{q' r} | b_f = b \rangle \langle b_f = b | . \]  \hspace{1cm} (12.12)

Thus,

\[ \langle b | \rho | b \rangle = \text{tr}(F_b \rho_0) . \]  \hspace{1cm} (12.13)
### Figure 12.2: QB net for general $F_b$'s.

Consider the QB net of Fig. (12.2), where

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| $j$   | $j = (j_1, j_2)$ | $\alpha_j$ | $\sum_j |\alpha_j|^2 = 1$ |
| $q$   | $q$     | $\delta(q, j_1)$ |          |
| $r$   | $r$     | $\delta(r, j_2)$ |          |
| $t$   | $t = (t_1, t_2, t_3)$ | $U(t|q, b, x)$ | $\sum_t U(t|q, b, x)U^*(t|q', b', x') = \delta_q'(t)\delta_b'(t)\delta_x'(t)$ |
| $b$   | $b$     | $\delta(b, 0)$ |          |
| $x$   | $x$     | $\delta(x, 0)$ |          |
| $q_f$ | $q_f$   | $\delta(q_f, t_1)$ |          |
| $b_f$ | $b_f$   | $\delta(b_f, t_2)$ |          |
| $x_f$ | $x_f$   | $\delta(x_f, t_3)$ |          |

This is the same as the table in Section 12.1, except that there are two new nodes ($x, x_f$), and the states of node $t$ have 3 components instead of 2.

Instead of Eq. (12.3), we now suppose the unitary operator $U$ satisfies:

$$U|\phi\rangle_q \otimes |0\rangle_b \otimes |0\rangle_x = \sum_b \left( \sqrt{F_b} \right) |\phi\rangle_q \otimes |b\rangle_b \otimes |b\rangle_x,$$

for any unit-magnitude vector $|\phi\rangle_q \in \mathcal{H}_q$.

Eq. (12.14) can be expressed in component form as follows:

$$U|\phi\rangle_q \otimes |0\rangle_b \otimes |0\rangle_x = \sum_b \left( \sqrt{F_b} \right) |\phi\rangle_q \otimes |b\rangle_b \otimes |b\rangle_x$$.
\[
\sum_{i} U(q_f, b_f, x_f|q, b, x) \phi(q) \delta_{0}^{b} \delta_{0}^{x} = \sum_{i} \sqrt{F_b(q_f|q)} \phi(q) \delta_{b}^{b_f} \delta_{b}^{x_f} .
\] (12.15)

Let \( \mathcal{N}^Q \) be the QB net which contains all the nodes shown in Fig. (12.2). Let \( \mathcal{N}_0^Q \) be the sub-net which contains only nodes \( j, q, r, \).

\( \mu_0 \) and \( \rho_0 \) are defined as in Section 12.1 above.

The meta density matrix \( \mu \) of \( \mathcal{N}^Q \) is

\[
\mu = |\psi_{meta}\rangle \langle \psi_{meta}| ,
\] (12.16)

where

\[
|\psi_{meta}\rangle = \sum_{ri} U(q_f, b_f, x_f|q, b, x) \alpha_{qr}^{b} \delta_{0}^{b} \delta_{0}^{x} |j = (q, r), q, r, b, x, \bar{t} = (q_f, b_f, x_f), q_f, b_f, x_f\rangle .
\] (12.17)

By Eq. (12.15), \( |\psi_{meta}\rangle \) can also be expressed as

\[
|\psi_{meta}\rangle = \sum_{ri} \sqrt{F_b(q_f|q)} \alpha_{qr}^{b} \delta_{b}^{b_f} \delta_{b}^{x_f} |j = (q, r), q, r, b, x, \bar{t} = (q_f, b_f, x_f), q_f, b_f, x_f\rangle .
\] (12.18)

Define \( \rho \) by

\[
\rho = \mathbb{E}_{\bar{t},q,b,x} \text{tr}_{\bar{t},q,b,x} \mu .
\] (12.19)

In other words, we get \( \rho \) by tracing \( \mu \) over all the external nodes except \( \bar{b}_f \), and e-summing it over all the internal nodes. \( \rho \) acts on \( \mathcal{H}_{\bar{b}_f} \). It is easy to show that

\[
\rho = \sum_{ri} \text{tr}(F_b \rho_0) |\bar{b}_f = b\rangle \langle \bar{b}_f = b| .
\] (12.20)

Thus,

\[
\langle b| \rho |b \rangle = \text{tr}(F_b \rho_0) .
\] (12.21)

Whereas in Section 12.1, the orthogonal projector property of the \( F_b \)'s "forces" \( \rho \) to be diagonal, in this section, it is the tracing over node \( \bar{x}_f \), a passive measurement of that node, which forces \( \rho \) to be diagonal.
Suppose \( \{ w_a | a \in Z_{1,N} \} \) is a collection of non-negative numbers which add up to one. Suppose \( \{ \rho_a | a \in Z_{1,N} \} \) is a collection of density matrices all acting on the same Hilbert space \( \mathcal{H} \). Let
\[
\rho = \sum_a w_a \rho_a .
\]
We will say that \( \rho \) is a *weighted sum of density matrices*. We will call the collection \( \mathcal{E} = \{ (w_a, \rho_a) | \forall a \} \) a *signal ensemble*. We will call the \( w_a \)'s the weights of \( \mathcal{E} \) and the \( \rho_a \)'s the signal states or signals of \( \mathcal{E} \).

In Quantum Information Theory, one is often interested in density matrices like \( \rho \) and ensembles like \( \mathcal{E} \). One envisions sending a message encoded as a string (for example: \( \rho_1, \rho_5, \rho_3, \rho_1 \)) of signal states. (It is assumed that the states in the string are separated in some way, perhaps by intervening idle time periods.) To say something about the average behavior of such messages, one needs to consider \( \rho \) and \( \mathcal{E} \).

We’ll say two signals are *orthogonal* if \( \rho_a \rho_b = 0 \) for \( a \neq b \). A signal ensemble such that all its signals are mutually orthogonal will be called an *orthogonal ensemble*. Orthogonal ensembles play a special role in Quantum Information Theory, since their signals are perfectly distinguishable (by a generalized measurement with \( F_b = \rho_b / (\sum \rho_a) \)). Suppose we are given a non-orthogonal signal ensemble \( \{ (w_a, \rho_a) | \forall a \} \). Then we can always replace it by an orthogonal one. Indeed, if \( \{ |a\rangle | a \in Z_{1,N} \} \) is an orthonormal basis for some Hilbert space different from the one on which the \( \rho_a \)'s act, and we define
\[
\sigma_a = |a \rangle \langle a | \rho_a ,
\]
for all \( a \), then the ensemble \( \mathcal{E}' = \{ (w_a, \sigma_a) | \forall a \} \) is orthogonal. Let
\[
\sigma = \sum_a w_a \sigma_a = \sum_a w_a |a \rangle \langle a | \rho_a .
\]
Note how in \( \sigma \), each projector \( |a \rangle \langle a | \) acts as a pointer that points towards a particular choice of \( \rho_a \). We will say that \( \rho \) of Eq.(13.1) (ditto, \( \sigma \) of Eq.(13.3) ) is a weighted sum of density matrices with *scalar weights* (ditto, *orthogonal projector weights*). Next, we will show how both \( \rho \) and \( \sigma \) can be represented by a QB net.

### 13.1 Scalar Weights

Consider the QB net of Fig.(13.1), where
Figure 13.1: QB net for a weighted sum of density matrices with scalar weights.

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| \(a\) | \(a\)  | \(\sqrt{w_a}\) | \(\sum_a w_a = 1\) |
| \(j\) | \(j = (j_1, j_2)\) | \(\alpha(j|a)\) | \(\sum_j |\alpha(j|a)|^2 = 1\) |
| \(q\) | \(q\)  | \(\delta(q, j_1)\) |         |
| \(r\) | \(r\)  | \(\delta(r, j_2)\) |         |

The meta density matrix \(\mu\) for this net is

\[
\mu = |\psi_{\text{meta}}\rangle\langle \psi_{\text{meta}}| ,
\]

where

\[
|\psi_{\text{meta}}\rangle = \sum_{rj} \alpha(q,r|a) \sqrt{w_a} |a, j = (q,r), q,r\rangle .
\]

If we define \(\rho\) by

\[
\rho = \text{E}_\Sigma_{j} \text{tr}_{a,r} \mu ,
\]

then

\[
\rho = \sum_a w_a \rho_a ,
\]

where

\[
\rho_a = \sum_{rj/a} \alpha(q,r|a) \alpha^*(q',r'|a)|q\rangle\langle q'| .
\]

### 13.2 Orthogonal Projector Weights

Consider the QB net of Fig. (13.2), where
Figure 13.2: QB net for a weighted sum of density matrices with orthogonal projector weights.

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| \( \tilde{j} \) | \( \tilde{j} = (\tilde{j}_1, \tilde{j}_2) \) | \( \sqrt{w_{\tilde{j}}} \delta(\tilde{j}_1, \tilde{j}_2) \) | \( \sum_{\tilde{j}_1} w_{\tilde{j}_1} = 1 \) |
| \( a \) | \( a \) | \( \delta(a, \tilde{j}_1) \) | |
| \( \tilde{r} \) | \( \tilde{r} \) | \( \delta(\tilde{r}, \tilde{j}_2) \) | |
| \( j \) | \( j = (j_1, j_2) \) | \( \alpha(j|a) \) | \( \sum_{j} |\alpha(j|a)|^2 = 1 \) |
| \( q \) | \( q \) | \( \delta(q, j_1) \) | |
| \( \tilde{r} \) | \( r \) | \( \delta(r, j_2) \) | |

The meta density matrix \( \mu \) for this net is

\[
\mu = |\psi_{\text{meta}}\rangle \langle \psi_{\text{meta}}| ,
\]

where

\[
|\psi_{\text{meta}}\rangle = \sum_{r_i} \alpha(q, r|a) \sqrt{w_a} |\underline{j} = (a, a), a, \underline{\tilde{r}} = a\rangle |\underline{j} = (q, r), q, r\rangle .
\]

If we define

\[
\sigma = \text{E}_{\underline{j}, \underline{\tilde{j}}} \text{tr}_{\underline{\tilde{r}}} \mu ,
\]

then

\[
\sigma = \sum_{a} (w_a|a\rangle \langle a|) \rho_a ,
\]

where
\[ \rho_a = \sum_{r_i/a} \alpha(q, r|a) \alpha^*(q', r|a) |q\rangle \langle q'|. \quad (13.13) \]

Note that

\[ S_\sigma(a, q) = -\sum_a \text{tr}_a \left[ w_a \rho_a \log_2(w_a \rho_a) \right] = H(\bar{w}) + \sum_a w_a S(\rho_a), \quad (13.14) \]

\[ S_\sigma(q) = S(\sum_a w_a \rho_a), \quad (13.15) \]

\[ S_\sigma(q) = H(\bar{w}). \quad (13.16) \]

Therefore,

\[ S_\sigma(q : q) = S(\sum_a w_a \rho_a) - \sum_a w_a S(\rho_a). \quad (13.17) \]
14 Signal Distinguishability

In this section, we will define two measures of signal distinguishability, the Holevo Information $\chi(\mathcal{E})$ and the Accessible Information $\chi_{\text{acc}}(\mathcal{E})$. Then we will use a QB net to prove that $\chi_{\text{acc}}(\mathcal{E}) \leq \chi(\mathcal{E})$, a result known as Holevo’s Inequality\[17\].

14.1 Holevo Information

Given a signal ensemble $\mathcal{E} = \{(w_a, \rho_a) | \forall a\}$, let

$$\rho = \sum_a w_a \rho_a . \quad (14.1)$$

The Holevo Information is defined by

$$\chi(\mathcal{E}) = S(\rho) - \sum_a w_a S(\rho_a) . \quad (14.2)$$

Some of the properties of $\chi(\mathcal{E})$ are:

(a) If the $\rho_a$’s are pure states, then $\chi(\mathcal{E}) = S(\rho)$.

(b) If the $\rho_a$’s are all the same, then $\chi(\mathcal{E}) = 0$. This result can be generalized as follows. The convexity of $S(\cdot)$ (see Table 3) implies $0 \leq \chi(\mathcal{E})$, with equality iff the $\rho_a$’s are all the same. Thus, $\chi(\mathcal{E})$ measures the indistinguishability of the signal states.

(c) If the $\rho_a$’s are orthogonal, then

$$S(\rho) = -\sum_a \text{tr}[w_a \rho_a \log_2(w_a \rho_a)] , \quad (14.3)$$

because orthogonal $\rho_a$’s “don’t mix” with each other so all sums over index $a$ collapse into a single outside sum. From Eq.(14.3), it follows that

$$S(\rho) = -\sum_a \text{tr}[w_a \rho_a (\log_2 w_a + \log_2 \rho_a)] = H(\vec{w}) + \sum_a w_a S(\rho_a) , \quad (14.4)$$

so

$$\chi(\mathcal{E}) = H(\vec{w}) . \quad (14.5)$$

We see that since orthogonal states are completely distinguishable, their quantum entropy is essentially classical. This result can be generalized as follows. According to Table 3,

$$\chi(\mathcal{E}) \leq H(\vec{w}) , \quad (14.6)$$

with equality iff the $\rho_a$’s are orthogonal.
If the $\rho_a$’s commute (i.e., $\rho_a \rho_b = \rho_b \rho_a$ for all $a, b$), then $\chi(\mathcal{E})$ reduces to a classical entropy. Indeed, because of the commutativity, the $\rho_a$’s can be simultaneously diagonalized in an orthonormal basis $\{|b\rangle\forall b\}$. In this basis, $S(\rho_a)$ for all $a$ and $S(\rho)$ reduce to classical entropies. To calculate $\chi(\mathcal{E})$ explicitly, define probabilities $P(a|b)$ and $P(a)$ by

$$\rho_a = \sum_b P(b|a) |b\rangle \langle b| , \quad (14.7)$$

$$P(a) = w_a . \quad (14.8)$$

Then

$$S(\rho) = S(\sum_{a,b} P(a,b) |b\rangle \langle b|) = S(\sum_b P(b) |b\rangle \langle b|) = H(b) , \quad (14.9)$$

$$\sum_a w_a S(\rho_a) = \sum_a P(a) \{- \sum_b P(b|a) \log_2 P(b|a)\} = H(b|a) , \quad (14.10)$$

so

$$\chi(\mathcal{E}) = H(a : b) . \quad (14.11)$$

### 14.2 Accessible Information

Suppose Alice sends Bob a signal $\rho_{a_0}$ using the signal ensemble $\mathcal{E} = \{ (w_a, \rho_a) \forall a \}$. Bob knows which ensemble Alice is using, but he doesn’t know $a_0$. To guess $a_0$, Bob devises and measures a POM $\{ F_b \forall b \}$. The value $b$ that he measures for $b$ will be characterized by:

$$P(b|a) = \text{tr}(F_b \rho_a) . \quad (14.12)$$

(This probability distribution specifies a so called *quantum channel.*) Since Bob knows $\mathcal{E}$, he can use

$$P(a) = w_a \quad (14.13)$$

as the a priori probability for signal $\rho_a$ for all $a \in S_2$. Bob would like to determine the posterior probabilities $P(a|b)$ in terms of what he knows ($P(b|a)$ and $P(a)$). He can do this with Bayes’ rule:

$$P(a|b) = \frac{P(b|a)P(a)}{\sum_{a'} P(b|a')P(a')} . \quad (14.14)$$

Bob will guess $a_0$ best if he uses the magical POM $\{ F_b \forall b \}$ that minimizes the $a$ spread of the probability distribution $P(a|b)$. This spread is measured by $H(a|b)$.  

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But $H(a : b)$ (called the “transmitted information”) equals $H(a) - H(a | b)$ and $H(a)$ is $F_b$ independent. So the magical POM also maximizes the transmitted information $H(a : b)$.

For any signal ensemble $\mathcal{E} = \{(w_a, \rho_a) | \forall a\}$, we define the Accessible Information by

$$\chi_{\text{acc}}(\mathcal{E}) = \max_{\{F_b | \forall b\}} H(a : b), \quad (14.15)$$

where $P(b | a)$ and $P(a)$ are defined by Eqs.(14.12) and (14.13). Since mutual entropies are always non-negative, $\chi_{\text{acc}}(\mathcal{E}) \geq 0$. One can show that equality is achieved iff the $\rho_a$’s are all the same. Hence, $\chi_{\text{acc}}(\mathcal{E})$ is a measure of indistinguishability of the signals $\rho_a$, just like $\chi(\mathcal{E})$ is. In fact, these two measures of indistinguishability are related by the so called Holevo’s Inequality[17]:

$$\chi_{\text{acc}}(\mathcal{E}) \leq \chi(\mathcal{E}), \quad (14.16)$$

which we will prove in the next section. It makes intuitive sense that $\chi_{\text{acc}}(\mathcal{E})$ is both a measure of indistinguishability and a measure of maximum information transmission. One expects that making more distinguishable the signals which compose a message will increase the information transmitted by the message.

### 14.3 Holevo’s Inequality

Next, we will use a QB net to prove Holevo’s Inequality.

![QB net for proving Holevo’s Inequality](image)
Consider the QB net of Fig.(14.1), where

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| $\tilde{j}$ | $\tilde{j} = (\tilde{j}_1, \tilde{j}_2)$ | $\sqrt{w_{\tilde{j}_1}} \delta(\tilde{j}_1, \tilde{j}_2)$ | $\sum_{\tilde{j}_1} w_{\tilde{j}_1} = 1$ |
| $\tilde{a}$ | $\tilde{a}$ | $\delta(a, \tilde{j}_1)$ | |
| $\tilde{r}$ | $\tilde{r}$ | $\delta(\tilde{r}, \tilde{j}_2)$ | |
| $\tilde{j}$ | $\tilde{j} = (\tilde{j}_1, \tilde{j}_2)$ | $\alpha(j|a)$ | $\sum_j |\alpha(j|a)|^2 = 1$ |
| $\tilde{q}$ | $\tilde{q}$ | $\delta(q, \tilde{j}_1)$ | |
| $\tilde{r}$ | $\tilde{r}$ | $\delta(r, \tilde{j}_2)$ | |
| $t$ | $t = (t_1, t_2, t_3)$ | $U(t|q, b, x)$ | $\sum_t U(t|q, b, x)U^*(t|q', b', x') = \delta_{q}^{q'} \delta_{b}^{b'} \delta_{x}^{x'}$ |
| $\tilde{b}$ | $\tilde{b}$ | $\delta(b, 0)$ | |
| $\tilde{x}$ | $\tilde{x}$ | $\delta(x, 0)$ | |
| $\tilde{q}_f$ | $\tilde{q}_f$ | $\delta(q_f, t_1)$ | |
| $\tilde{b}_f$ | $\tilde{b}_f$ | $\delta(b_f, t_2)$ | |
| $\tilde{x}_f$ | $\tilde{x}_f$ | $\delta(x_f, t_3)$ | |

The matrix $U$ must implement a general POM $\{F_b|\forall b\}$. Hence, it will be assumed to satisfy Eq.(12.15), which we restate:

$$
\sum_{ri} U(q_f, b_f, x_f|q, b, x)\phi(q)\delta_b^b \delta_x^x = \sum_{ri} \sqrt{F_b(q_f|q)} \phi(q)\delta_b^b \delta_x^x,
$$

(14.17)

for any function $\phi(q)$.

Let $\mathcal{N}_f^Q$ be the QB net which contains all the nodes shown in Fig.(14.1). Let $\mathcal{N}_0^Q$ be the sub-net which contains only nodes $\tilde{j}, \tilde{a}, \tilde{r}, \tilde{j}, q, r$.

The meta density matrix $\mu^0$ of $\mathcal{N}_0^Q$ was specified in Eq.(13.10). We also showed in Section 13 that if $\rho^0$ is defined by

$$
\rho^0 = \mathcal{E}_{\tilde{j}, \tilde{a}} \text{tr}_{\tilde{r}} \mu^0,
$$

(14.18)

then

$$
\rho^0 = \sum_a (w_a|a\rangle\langle a|)\rho_a,
$$

(14.19)

where
\[ \rho_a = \sum_{ri/a} \alpha(q, r|a) \alpha^*(q', r|a)|q\rangle\langle q'|. \]  

(14.20)

Furthermore, we showed that if \( \mathcal{E} = \{(w_a, \rho_a) | \forall a\} \), then

\[ S_{\rho}(a : q) = \chi(\mathcal{E}). \]  

(14.21)

The meta density matrix \( \mu^f \) of \( \mathcal{N}^Q_f \) is

\[ \mu^f = |\psi_{\text{meta}}^f\rangle\langle \psi_{\text{meta}}^f|, \]  

(14.22)

where

\[ |\psi_{\text{meta}}^f\rangle = \sum_{ri} U(q_f, b_f, x_f|q, b, x) \delta_0 \delta_0^t \alpha(q, r|a) \sqrt{w_a} \]  

\[ \tilde{j} = (a, a), a, \tilde{a} = a, \tilde{a}, \tilde{a} = (q, q'), q, r, b, x, \tilde{t} = (q, b, x), q, b, x_f \}. \]  

(14.23)

Define \( \rho^f \) by

\[ \rho^f = \text{tr}_{\mathcal{E}} E_{\prod j \tilde{j} q b x t} (\mu^f). \]  

(14.24)

In other words, we trace \( \mu^f \) over all the external nodes except \( q_f, b_f, x_f \), and we e-sum it over all internal ones except \( a \). Hence, \( \rho^f \) acts on \( H_{q_f} b_f x_f a \).

To prove Holevo’s Inequality, we begin by noticing that

\[ S_{\rho^f}(a : (b_f, q_f, x_f)) = S_{\rho^f}(a) + S_{\rho^f}(b_f, q_f, x_f) - S_{\rho^f}(a, b_f, q_f, x_f), \]  

(14.25a)

\[ S_{\rho^f}(a) = S_{\rho^f}(\tilde{a}), \]  

(14.25b)

\[ S_{\rho^f}(b_f, q_f, x_f) = [S_{\rho^f}(b_f, q_f, x_f)]_{U=1} = S_{\rho^f}(q), \]  

(14.25c)

\[ S_{\rho^f}(a, b_f, q_f, x_f) = [S_{\rho^f}(a, b_f, q_f, x_f)]_{U=1} = S_{\rho^f}(a, q). \]  

(14.25d)

Combining Eqs. (14.25) yields

\[ S_{\rho^f}[a : (b_f, q_f, x_f)] = S_{\rho^f}(a : q). \]  

(14.26a)

By virtue of strong sub-additivity,

\[ S_{\rho^f}(a : b_f) \leq S_{\rho^f}[a : (b_f, q_f, x_f)]. \]  

(14.26b)

Below, we will show that

\[ H_{\rho^f}(a : b_f) = S_{\rho^f}(a : b_f). \]  

(14.26c)
Combining Eqs.\((14.21)\) and \((14.26)\) finally yields Holevo’s Inequality:

\[
H_{\rho_f}(a : b_f) \leq S_{\rho_0}(a : q) = \chi(\mathcal{E}).
\]  

\[(14.27)\]

This can be understood as a special case of the Fixed Sender Data Processing Inequality \([12],[18]\). It says that when information is transmitted from \(a\), less reaches \(b_f\) than \(q\).

To show Eq.\((14.26c)\), we use Eq.\((14.17)\) to express \(\rho_f\) in terms of the POM \(\{F_b|\forall b\}\). It is then easy to show that

\[
\text{tr}_{q_f;x_f} \rho_f = \sum_{ri} \text{tr}(F_b \rho_a) w_a |a, b\rangle \langle a, b|.
\]  

\[(14.28)\]

Replacing \(\text{tr}(F_b \rho_a)\) and \(w_a\) by \(P(b|a)\) and \(P(a)\) (see Eqs.\((14.12)\) and \((14.13)\)) yields

\[
\text{tr}_{q_f;x_f} \rho_f = \sum_{a,b} P(a, b) |a, b\rangle \langle a, b|.
\]  

\[(14.29)\]

Eq.\((14.26c)\) now follows.

### 14.4 Example

![Figure 14.2: The vectors |\(\phi_1\rangle\), |\(\phi_2\rangle\), |\(\phi_3\rangle\).](image)

The following example (originally from Ref.\([19]\)) is often used to illustrate Holevo’s Bound.

Let

\[
|\phi_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |\phi_2\rangle = \frac{1}{2} \begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}, \quad |\phi_3\rangle = \frac{1}{2} \begin{bmatrix} -1 \\ -\sqrt{3} \end{bmatrix}.
\]

\[(14.30)\]
As shown in Fig. (14.2), these 3 vectors specify the corners of an equilateral triangle that lies on the real plane. Now consider the signal ensemble \( E = \{(w_a, \rho_a)\forall a\} \), with

\[
w_a = \frac{1}{3},
\]

\[
\rho_a = |\phi_a\rangle\langle\phi_a|,
\]

for \( a \in Z_{1,3} \). It is easy to show that

\[
\rho = \sum_{a=1}^{3} w_a \rho_a = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

so

\[
\chi(E) = S(\rho) = 1.
\]

Define a POM \( \{F_b|\forall b\} \) by

\[
F_b = \frac{2}{3}(1 - |\phi_b\rangle\langle\phi_b|),
\]

where \( b \in Z_{1,3} \). Then

\[
P(b|a) = \langle \phi_a|F_b|\phi_a \rangle = \begin{cases} 0 & \text{if } a = b \\ \frac{1}{2} & \text{if } a \neq b \end{cases}.
\]

According to Bayes’ rule, in this case the posterior probabilities \( P(a|b) \) are equal to \( P(b|a) \). Thus, if Bob measures this POM and obtains the value \( b \), he can safely conclude that Alice did not send signal \( b \), and he can assign equal posterior probabilities to the other two signals. One can show that this POM maximizes \( H(a:b) \). Therefore,

\[
\chi_{acc}(E) = H(a:b) = .5850.
\]

Holevo’s Inequality is satisfied, as expected.

Another interesting ensemble considered in Refs. [19] and [8] is

\[
w_a = \frac{1}{3},
\]

\[
\rho_a = |\Phi_a\rangle\langle\Phi_a|,
\]

\[
|\Phi_a\rangle = |\phi_a\rangle \otimes |\phi_a\rangle,
\]

where \( a \in Z_{1,3} \), and the vectors \( |\phi_a\rangle \) are those defined previously in Eq. (14.30). One finds \( \chi(E) = 1.5 \) and \( \chi_{acc}(E) = 1.3691 \).
In this section, we will consider a QB net that represents an EPR pair. An EPR pair consists of two spin half particles in a singlet state (i.e., a state of zero total spin).

Suppose \( |+\rangle \) and \(-\rangle \) are the states of spin up and down in the +Z direction. We define \( |\psi_{EPR}\rangle \) by

\[
|\psi_{EPR}\rangle = \frac{1}{\sqrt{2}}(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle).
\] (15.1)

Let

\[
|+\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\] (15.2)

\[
|-\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\] (15.3)

If \( e = (e_1, e_2) \in \text{Bool}^2 \), then \( \psi_{EPR}(e) = \langle e |\psi_{EPR}\rangle \) is

\[
\psi_{EPR}(e) = \frac{1}{\sqrt{2}} [\delta_{e_1,e_2} - \delta_{e_1,0}].
\] (15.4)

Consider the QB net of Fig. (15.1), where

| nodes | states | amplitudes | comments |
|-------|-------|------------|----------|
| \( e \) | \( e = (e_1, e_2) \in \text{Bool}^2 \) | \( \psi_{EPR}(e) = \frac{1}{\sqrt{2}} [\delta_{e_1,e_2} - \delta_{e_1,0}] \) |          |
| \( x \) | \( x \in \text{Bool} \) | \( \delta(x, e_1) \) |          |
| \( y \) | \( y \in \text{Bool} \) | \( \delta(y, e_2) \) |          |

The meta density matrix \( \mu \) of this net is

\[
\mu = |\psi_{meta}\rangle \langle \psi_{meta}|,
\] (15.5)

where
\[ |\psi_{\text{meta}}\rangle = \sum_{ri} \psi_{\text{EPR}}(x, y)|\xi = (x, y), x, y \rangle. \quad (15.6) \]

Define \( \rho \) by:

\[
\rho = \mathbb{E} \sum_{\mu} \mu. \quad (15.7)
\]

Then

\[
\rho = \sum_{ri} \frac{1}{2} (\delta_{0,0}^{x,y} - \delta_{0,1}^{x,y})(\delta_{0,1}^{x',y'} - \delta_{1,0}^{x',y'}) \langle x, y | x', y' \rangle, \quad (15.8)
\]

\[
[\langle x, y | \rho | x', y' \rangle] = \frac{1}{2} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (15.9)
\]

\( \rho \) is a pure state so \( S_{\rho}(x, y) = 0 \) and \( S_{\rho}(x) = S_{\rho}(y) \). It is easy to show that

\[
\text{tr}_{x} \rho = \frac{1}{2} \sum_{y} \langle y | y \rangle, \quad (15.10)
\]

\[
\text{tr}_{y} \rho = \frac{1}{2} \sum_{x} \langle x | x \rangle, \quad (15.11)
\]

Thus,

\[
S_{\rho}(x) = 1, \quad H_{\rho}(x) = 1 \quad \text{(zero coherence)}
\]

\[
S_{\rho}(y) = 1, \quad H_{\rho}(y) = 1 \quad \text{(zero coherence)}
\]

\[
S_{\rho}(x, y) = 0, \quad H_{\rho}(x, y) = 1 \quad \text{(not max. coherence since } H_{\rho}(x, y) \neq 2)\]

\[
S_{\rho}(x | y) = -1, \quad H_{\rho}(x | y) = 0
\]

\[
S_{\rho}(y | x) = -1, \quad H_{\rho}(y | x) = 0
\]

\[
S_{\rho}(x : y) = 2, \quad H_{\rho}(x : y) = 1 \quad (15.12)
\]

Define \( \rho(y) \) by

\[
\rho(y) = \mathbb{E} \sum_{\mu} \langle y | \mu | y \rangle = \langle y | \rho | y \rangle. \quad (15.13)
\]

\( \rho(y) \) acts on \( \mathcal{H}_{x} \). It is easy to show that

\[
\rho(y) = |0 \rangle \langle 0 | \delta_{y}^{y} + |1 \rangle \langle 1 | \delta_{0}^{y}. \quad (15.14)
\]

Thus,
These results can be interpreted as follows. We start with an EPR pair of particles. One particle goes to Alice ($x$). The other goes to Bob ($y$). The density matrix called $\rho$ above corresponds to a situation in which Bob ignores his particle. The particle is still measured passively by the environment. Alice gets no information from the environment, so her particle has a 50/50 chance of being either up or down along any direction. The density matrix called $\rho(y)$ above corresponds to a situation in which instead of ignoring his particle, Bob measures it along the $+Z$ direction and communicates the result to Alice. The experiment is repeated many times. When Bob reports result $+z$, Alice sticks her particle into bin Bob+, and when he reports $-z$, she sticks it into bin Bob−. Alice’s particles in bin Bob+ (ditto, bin Bob−) behave as if they were in pure state $|−z\rangle$ (ditto, $|+z\rangle$). (Note that Alice’s particle points opposite to Bob’s. This is expected since the initial state $\psi_{EPR}$ of the two particles has zero total spin, and this quantity is conserved during the experiment.)
16 Quantum Eraser

In this section, we will consider a QB net that represents a situation in which one member of an EPR pair is measured in a special way so as to exhibit a phenomenon loosely called “quantum erasing”.

Suppose $|+_n\rangle$ and $|-_n\rangle$ are the states of spin up and down in the $+_n$ direction, where $n$ is either X or Z. Let

$$|+_n\rangle = |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$|-_n\rangle = |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$|+x\rangle = |0_x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$|-x\rangle = |1_x\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$  \hspace{1cm} (16.1, 16.2, 16.3, 16.4)

Define $U$ by

$$U = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$  \hspace{1cm} (16.5)

Note that

$$U|+_z\rangle = |+x\rangle,$$

$$U|-_z\rangle = |-x\rangle.$$  \hspace{1cm} (16.6, 16.7)

Also note that for $y, r \in \text{Bool}$,

$$\langle r | U | y \rangle = \frac{1}{\sqrt{2}} (-1)^{yr}.$$  \hspace{1cm} (16.8)

Consider the QB net of Fig.(16.1), where

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| $e$   | $e = (e_1, e_2) \in \text{Bool}^2$ | $\psi_{EPR}(e) = \frac{1}{\sqrt{2}} [\delta_{0,1} e_1, e_2 - \delta_{1,0} e_1, e_2]$ | |
| $x$   | $x \in \text{Bool}$ | $\delta(x, e_1)$ | |
| $y$   | $y \in \text{Bool}$ | $\delta(y, e_2)$ | |
| $r$   | $r \in \text{Bool}$ | $U(r | y) = \frac{1}{\sqrt{2}} (-1)^{yr}$ | |
Let $\mathcal{N}^Q$ be the QB net which contains all the nodes shown in Fig. (16.1). Let $\mathcal{N}^Q_0$ be the sub-net which contains only nodes $x, e, y$.

The meta density matrix $\mu_0$ of $\mathcal{N}^Q_0$ was given in Section 15. Let $\rho_0 = E \Sigma_{x,y} \mu_0$. Thus, $\rho_0$ corresponds to what we called simply $\rho$ in Section 15.

The meta density matrix $\mu$ of $\mathcal{N}^Q$ is

$$
\mu = |\psi_{\text{meta}}\rangle \langle \psi_{\text{meta}}|,
$$

where

$$
|\psi_{\text{meta}}\rangle = \sum_{r} U(r|y) \psi_{\text{EPR}}(x, y) |e = (x, y), x, y, r\rangle.
$$

Define $\rho$ by:

$$
\rho = E \Sigma_{x,y} \mu.
$$

Then

$$
\rho = \sum_{r} \frac{1}{4} \left[ (-1)^r \delta_0^x - \delta_1^y \right] |(-1)^r \delta_0^y - \delta_1^x\rangle |x, r\rangle \langle x', r'|,
$$

$$
[|x, r\rangle \langle x', r'|] = \frac{1}{4} \left[\begin{array}{cccc}
1 & -1 & -1 & -1 \\
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1 \\
-1 & 1 & 1 & 1
\end{array}\right].
$$

$\rho$ is a pure state so $S_\rho(x, r) = 0$ and $S_\rho(x) = S_\rho(r)$. It is easy to show that

$$
\text{tr}_x \rho = \frac{1}{2} \sum_r |r\rangle \langle r|,
$$

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\[
\text{tr}_r \rho = \frac{1}{2} \sum_x |x\rangle \langle x|,
\]
(16.15)

Thus,

\[
\begin{align*}
S_{\rho}(x) &= 1, & H_{\rho}(x) &= 1 \quad \text{(zero coherence)} \\
S_{\rho}(r) &= 1, & H_{\rho}(r) &= 1 \quad \text{(zero coherence)} \\
S_{\rho}(x, r) &= 0, & H_{\rho}(x, r) &= 2 \quad \text{(max. coherence)} \\
S_{\rho}(x|r) &= -1, & H_{\rho}(x|r) &= 1 \\
S_{\rho}(r|x) &= -1, & H_{\rho}(r|x) &= 1 \\
S_{\rho}(x : r) &= 2, & H_{\rho}(x : r) &= 0
\end{align*}
\]
(16.16)

Define \( \rho(r) \) by

\[
\rho(r) = 2E\Sigma_{\mu\nu} \langle r|\mu|r \rangle = 2\langle r|\rho|r \rangle.
\]
(16.17)

\( \rho(r) \) acts on \( \mathcal{H}_x \). It is easy to show that

\[
\rho(r) = |0_X\rangle \langle 0_X|\delta_{01}^r + |1_X\rangle \langle 1_X|\delta_{00}^r.
\]
(16.18)

Thus,

\[
S_{\rho(r)}(x) = 0, & H_{\rho(r)}(x) &= 1 \quad \text{(max. coherence)} \quad .
\]
(16.19)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure16.2.png}
\caption{Comparison of Feynman stories for QB net Fig.(15.1) representing an EPR pair and QB net Fig.(16.1) representing a quantum eraser.}
\end{figure}
These results can be interpreted as follows. We start with an EPR pair of particles. One particle goes to Alice (\(x\)). The other goes to Bob (\(y, r\)). Bob passes his particle through a Stern-Gerlach magnet that separates it into its \(\pm_x\) parts. The density matrix called \(\rho\) above corresponds to a situation in which Bob ignores his particle after it leaves the Stern-Gerlach magnet. The particle is still measured passively by the environment. Alice gets no information from the environment, so here particle has a 50/50 chance of being either up or down along any direction. The density matrix called \(\rho(r)\) above corresponds to a situation in which instead of ignoring his particle, Bob measures it along the +X direction and communicates the result to Alice. The experiment is repeated many times. When Bob reports result +\(x\), Alice sticks her particle into bin Bob+, and when he reports −\(x\), she sticks it into bin Bob−. Alice’s particles in bin Bob+ (ditto, bin Bob−) behave as if they were in pure state \(|−_x\rangle\) (ditto, \(|+_x\rangle\)). (Note that Alice’s particle points opposite to Bob’s. This is expected since the initial state \(\psi_{EPR}\) of the two particles has zero total spin, and this quantity is conserved during the experiment.)

This is all very similar to Section 15. But note that in Section 15, Alice’s particle ends in state +\(z\) (or −\(z\), depending on the result of Bob’s measurement), whereas now it ends in state +\(x\) (or −\(x\)). As shown in Fig. (16.2), if the value of \(y\) is fixed, then there is only one possible Feynman story. On the other hand, if the value of \(r\) is fixed, there are two possible Feynman stories. A related fact: In Section 15, Alice’s particle ends in a state characterized by the density matrix \(|+_z\rangle\langle+_z|\) which is diagonal in the \(|±_z\rangle\) basis, whereas now it ends in a state characterized by a density matrix \(|+_x\rangle\langle+_x|\) which isn’t diagonal in the \(|±_z\rangle\) basis.

We often say that an experiment of this sort is a "quantum eraser". By this, we mean the following. According to Eqs. (15.12) and (16.19)

\[
S_{\rho_0}(x) = 1, \quad H_{\rho_0}(x) = 1 \quad \text{(zero coherence)} ,
\]

\[
S_{\rho(r)}(x) = 0, \quad H_{\rho(r)}(x) = 1 \quad \text{(max. coherence)} .
\]

In Eq. (16.20), Bob ignores his particle. In Eq. (16.21), he passes it through a Stern-Gerlach magnet and reports the result of his measurement to Alice. We can go from minimum coherence (Eq. (16.20)) to the maximum coherence (Eq. (16.21)) for node \(x\) simply by asking Bob to do some extra processing. This extra processing seems to erase the coherence destroying mechanism.

Note that the density matrix \(\rho\) defined above acts on \(H_{x, r}\) and that

\[
\langle x \mid \langle r \mid \rho \mid r \rangle \mid x \rangle = \langle r \mid \langle x \mid \rho \mid x \rangle \mid r \rangle .
\]

That is, the order in which we apply \(\text{red}_{\langle x \mid} \langle r \mid \rho \mid r \rangle\) and \(\text{red}_{\langle r \mid} \langle x \mid \rho \mid x \rangle\) does not matter. This is often called the “delayed choice” phenomenon.

Note that we found \(H_{\rho(x, y)} = 0\) in this section, whereas we found \(H_{\rho_0(x, y)} = 1\) in Section 15. That is, \(x\) and \(y\) are independent whereas \(x\) and \(y\) aren’t. That’s because \(x\) and \(y\) must have opposite values whereas \(x\) and \(r\) don’t have to.

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17 Teleportation

In this section, we will consider a QB net that represents the phenomenon known as Teleportation[20].

Consider the QB net of Fig.(17.1), where

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| e     | \(e = (e_1, e_2) \in \text{Bool}^2\) | \(\psi_{EPR}(e) = \frac{1}{\sqrt{2}}[\delta_{e_1,0,e_2} - \delta_{e_1,1,e_2}]\) |          |
| x     | \(x \in \text{Bool}\) | \(\delta(x, e_1)\) |          |
| y     | \(y \in \text{Bool}\) | \(\delta(y, e_2)\) |          |
| a     | \(a \in \text{Bool}\) | \(\alpha_a\) | \(\sum_a |\alpha_a|^2 = 1\) |
| f     | \(f = (f_1, f_2) \in \text{Bool}^2\) | \(U(f|a, x)\) | \(U\) specified below |
| b     | \(b \in \text{Bool}\) | \(R(b|f, y)\) | \(R\) specified below |

Consider the so called “Bell basis” vectors \(|\Psi(f)\rangle\):  
\[
|\Psi(f)\rangle = \frac{1}{\sqrt{2}}(|0, f_1\rangle + (-1)^{f_2}|1, \bar{f}_1\rangle),
\]

where \(f \in \text{Bool}^2\), and \(\bar{0} = 1, \bar{1} = 0\). \(f_1\) tells us whether the two particles are in the same or different states (different state iff \(f_1 = 1\)). \(f_2\) tells us the sign between the two kets being summed (minus sign iff \(f_2 = 1\)). For example,

\[
|\Psi(1, 1)\rangle = \frac{1}{\sqrt{2}}(|0, 1\rangle - |1, 0\rangle).
\]

The state \(\psi_{EPR}(e)\) given above equals \(\langle e|\Psi(1, 1)\rangle\).

We define the matrix \(U\) mentioned above by
\[ U(f|a,x) = \langle \Psi(f)|a,x \rangle = \frac{1}{\sqrt{2}} \left[ \delta_{0,f_1}^{a,x} + (-1)^{f_2} \delta_{1,f_1}^{a,x} \right], \quad (17.3) \]

\[
[U(f|a,x)] = \frac{1}{\sqrt{2}}
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 \\
1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0
\end{bmatrix}.
\quad (17.4)
\]

The columns of \( U \) are clearly orthonormal so \( U \) is a unitary matrix.

The matrix \( R \) mentioned above can be defined in terms of \( U \) by

\[
R(b|f,y) = U(f|b,\bar{y})(-1)^{\bar{y}}(-1)^{f_1,f_2} \sqrt{2}.
\quad (17.5)
\]

Our reasons for defining \( R \) in this way will become clear as we go on. Note that

\[
\sum_b |R(b|f,y)|^2 = 1,
\quad (17.6)
\]

as required by the definition of QB nets.

It is convenient to define a function \( K(\cdot) \) by

\[
K(x,y,a,f,b) = R(b|f,y)U(f|a,x)\psi_{EPR}(x,y) .
\quad (17.7)
\]

Substituting explicit expressions for \( R, U \) and \( \psi_{EPR} \) into the last equation yields

\[
K(x,y,a,f,b) = \frac{(-1)^{f_1,f_2}}{2} \delta_{b,\bar{y}}^{a,x} (\delta_{0,f_1}^{a,x} + \delta_{1,f_1}^{a,x}) .
\quad (17.8)
\]

From this expression for \( K(\cdot) \), it follows that

\[
\sum_{x,y} K = \frac{(-1)^{f_1,f_2}}{2} \delta_{b}^{a}, \quad \sum_{x,y,f} K = \delta_{b}^{a} ,
\quad (17.9a)
\]

\[
\sum_{x,y} |K|^2 = \frac{1}{4} \delta_{b}^{a}, \quad \sum_{x,y,f} |K|^2 = \delta_{b}^{a} .
\quad (17.9b)
\]

Define the following kets:

\[
|\psi_{in}\rangle = \sum_a \alpha_a |a = a\rangle,
\quad (17.10a)
\]

\[
|\psi_{in}'\rangle = \sum_a \alpha_a |b = a\rangle,
\quad (17.10b)
\]

\[
|\psi_{out}\rangle = \sum_x A(x)|x_zext\rangle = \sum_{all} K(x,y,a,f,b)\alpha_a |b\rangle,
\quad (17.10c)
\]

\[
|\psi_{out}(f)\rangle = 2 \sum_{all/f} K(x,y,a,f,b)\alpha_a |b\rangle .
\quad (17.10d)
\]

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Note that we don’t sum over $f$ in the equation for $|\psi_{\text{out}}(f)|$. It follows by Eqs.(17.9) that the kets of Eqs.(17.10) have unit magnitude and that

$$|\psi_{\text{out}}(f)| = (-1)^{f_1 f_2} |\psi'_{\text{in}}|, \quad (17.11)$$

$$|\psi_{\text{out}}| = |\psi'_{\text{in}}|. \quad (17.12)$$

Because of Eq.(17.11), one says that the QB net of Fig.(17.1) “teleports” a quantum state from node $a$ to node $b$. Without knowing the state $|\psi_{\text{in}}\rangle$, Alice at $f$ measures the joint state delivered to her by $a$ and $x$. She obtains result $f$ which she sends by classical means to Bob at $b$. Bob can choose to allow any value of $f$, or he can ignore those repetitions of the experiment in which $f$ does not equal a particular value, say $(0, 1)$. In either case, the state $|\psi_{\text{out}}(f)|$ emerging from Bob’s lab $b$ is equal to $\pm |\psi'_{\text{in}}\rangle$.

Note that according to Eq.(17.12), even if Alice does not measure $f$, and instead she sends a quantum message to Bob, $|\psi_{\text{out}}\rangle$ equals $|\psi'_{\text{in}}\rangle$. However, this is not “true” teleportation. In “true” teleportation, we allow Alice to receive quantum messages but not to send them.

The meta density matrix $\mu$ for the net of Fig.(17.1) is

$$\mu = |\psi_{\text{meta}}\rangle\langle\psi_{\text{meta}}|, \quad (17.13)$$

where

$$|\psi_{\text{meta}}\rangle = \sum_{all} K(x, y, a, f, b) \alpha_a |\underline{\underline{x}}, x, y, a, f, b\rangle. \quad (17.14)$$

Note that by Eqs.(17.9), $|\psi_{\text{meta}}\rangle$ has unit magnitude.

Define the reduced matrix $\sigma$ by

$$\sigma = E\Sigma_{x, y} (\mu). \quad (17.15)$$

It is easy to show that

$$\sigma = |\phi_{a\overline{b}}\rangle\langle\phi_{a\overline{b}}| - |\phi_{\overline{f}}\rangle\langle\phi_{\overline{f}}|, \quad (17.16)$$

where

$$|\phi_{a\overline{b}}\rangle = \sum_a \alpha_a |a = a, b = a\rangle, \quad (17.17)$$

$$|\phi_{\overline{f}}\rangle = \sum_f \frac{(-1)^{f_1 f_2}}{2} |f\rangle. \quad (17.18)$$

Define

$$H_{\text{in}} = - \sum_{a \in \text{Bool}} |\alpha_a|^2 \log_2(|\alpha_a|^2). \quad (17.19)$$
Next we will calculate classical and quantum entropies for various possible density matrices $\rho$:

(a) $\rho = \text{tr}_b \sigma$

Then

$$\rho = \left( \sum_a |\alpha_a|^2 |a\rangle \right) |\phi_f\rangle \langle \phi_f| . \quad (17.20)$$

It is easy to show from Eq. (17.20) that

$$S_\rho(a) = H^{in}, \quad H_\rho(a) = H^{in} \quad \text{(zero coherence)}$$
$$S_\rho(b) = 0, \quad H_\rho(b) = 2 \quad \text{(max. coherence)}$$
$$S_\rho(a, f) = H^{in}, \quad H_\rho(a, f) = H^{in} + 2$$
$$S_\rho(a | f) = H^{in}, \quad H_\rho(a | f) = H^{in}$$
$$S_\rho(f | a) = 0, \quad H_\rho(f | a) = 2$$
$$S_\rho(a : f) = 0, \quad H_\rho(a : f) = 0 . \quad (17.21)$$

(b) $\rho = \mathcal{N} \langle f | \sigma | f \rangle$

Then

$$\rho = |\phi_{a,b}\rangle \langle \phi_{a,b}| . \quad (17.22)$$

Note that we get the same density matrix if we reduce $\sigma$ by projecting, tracing or e-summing over node $f$

$$\mathcal{N} \langle f | \sigma | f \rangle = \text{tr}_f \sigma = \Sigma_f \sigma . \quad (17.23)$$

It is easy to show from Eq. (17.22) that

$$S_\rho(a) = H^{in}, \quad H_\rho(a) = H^{in} \quad \text{(zero coherence)}$$
$$S_\rho(b) = H^{in}, \quad H_\rho(b) = H^{in} \quad \text{(zero coherence)}$$
$$S_\rho(a, b) = 0, \quad H_\rho(a, b) = H^{in}$$
$$S_\rho(a | b) = -H^{in}, \quad H_\rho(a | b) = 0$$
$$S_\rho(b | a) = -H^{in}, \quad H_\rho(b | a) = 0$$
$$S_\rho(a : b) = 2H^{in}, \quad H_\rho(a : b) = H^{in} \quad \text{transmitted info: quantum = 2 classical} . \quad (17.24)$$
Qubit Bouncing (a.k.a. Dense Coding)

Ref.\[21\] was the first to discuss a phenomenon that we will call qubit bouncing. Qubit bouncing is often called “quantum super dense coding”. In this section, we will consider a QB net that represents qubit bouncing.

Consider the QB net of Fig.\((18.1)\), where

| nodes | states | amplitudes | comments |
|-------|--------|------------|----------|
| \(e\) | \(e = (e_1, e_2) ∈ \text{Bool}^2\) | \(ψ_{EPR}(e) = \frac{1}{\sqrt{2}}[δ_{0,1}^{e_1,e_2} − δ_{1,0}^{e_1,e_2}]\) | \(\sum_a |α_a|^2 = 1\) |
| \(x\) | \(x ∈ \text{Bool}\) | \(δ(x, e_1)\) | \(R\) specified below |
| \(y\) | \(y ∈ \text{Bool}\) | \(δ(y, e_2)\) | \(U\) specified below |
| \(a\) | \(a = (a_1, a_2) ∈ \text{Bool}^2\) | \(α_a\) | \(\sum_a |α_a|^2 = 1\) |
| \(t\) | \(t ∈ \text{Bool}\) | \(R(t|a, x)\) | \(R\) specified below |
| \(b\) | \(b = (b_1, b_2) ∈ \text{Bool}^2\) | \(U(b|t, y)\) | \(U\) specified below |

The matrix \(U\) in this section is identical to its namesake in the Teleportation section:

\[
U(b|t, y) = \frac{1}{\sqrt{2}}(δ_{0,b_1}^{t,y} + (-1)^{b_2}δ_{1,b_1}^{t,y}).
\]

The matrix \(R\) can be defined in terms of \(U\) by

\[
R(t|a, x) = U(a|t, \bar{x})(−1)^x \sqrt{2}.
\]

Our reasons for defining \(R\) in this way will become clear as we go on. Note that

\[
\sum_t |R(t|a, x)|^2 = 1,
\]
as required by the definition of QB nets.

It is convenient to define a function \( K(\cdot) \) by
\[
K(x, y, a, t, b) = U(b|t, y)R(t|a, x)\psi_{EPR}(x, y) .
\]  
(18.4)

Substituting explicit expressions for \( R, U \) and \( \psi_{EPR} \) into the last equation yields
\[
K(x, y, a, t, b) = \frac{1}{2} \delta_{b_1}^{a_1} [\delta^t_{y_0} + (-1)^{a_2+b_2} \delta^t_{y_1}] .
\]  
(18.5)

From this expression for \( K(\cdot) \), it follows that
\[
\sum_{x,y} K = \frac{1}{2} \delta_{b_1}^{a_1} \delta^t_{y_0} + (-1)^{a_2+b_2} \delta^t_{y_1} ,
\]  
(18.6a)
\[
\sum_{x,y,t} |K|^2 = \frac{1}{4} \delta_{b_1}^{a_1} \delta^t_{y_0} + (-1)^{a_2+b_2} \delta^t_{y_1} = 1 .
\]  
(18.6b)

Define the following kets:
\[
|\psi_{in}\rangle = \sum_a \alpha_a |a = a\rangle ,
\]  
(18.7a)
\[
|\psi_{in}'\rangle = \sum_a \alpha_a |b = a\rangle ,
\]  
(18.7b)
\[
|\psi_{out}\rangle = \sum_x A(x.)|(x.)Z_{ext}\rangle = \sum_{all} K(x, y, a, t, b)\alpha_a |b\rangle ,
\]  
(18.7c)

It follows by Eqs.(18.6) that the kets of Eqs.(18.7) have unit magnitude and that
\[
|\psi_{out}\rangle = |\psi_{in}'\rangle .
\]  
(18.8)

The meta density matrix \( \mu \) for the net of Fig.(18.1) is
\[
\mu = |\psi_{meta}\rangle \langle \psi_{meta}| ,
\]  
(18.9)

where
\[
|\psi_{meta}\rangle = \sum_{all} K(x, y, a, t, b)\alpha_a |\xi = (x, y), x, y, a, t, b\rangle .
\]  
(18.10)

Note that by Eqs.(18.6), \(|\psi_{meta}\rangle \) has unit magnitude.

Define the reduced matrix \( \sigma \) by
\[
\sigma = E\Sigma_{x,y} (\mu) .
\]  
(18.11)

It is easy to show that
\[
\sigma = |\phi_{meta}\rangle \langle \phi_{meta}| ,
\]  
(18.12)

where
\[ |\phi_{a,t,b}\rangle = \sum_{a} \frac{1}{2} \delta_{a_1}^a [\delta_0^t + (-1)^{a_2+b_2} \delta_1^t] |a,t,b\rangle , \quad (18.13) \]

Define

\[ H^{in} = - \sum_{a \in \text{Bool}} |\alpha_a|^2 \log_2(|\alpha_a|^2) , \quad (18.14) \]

\[ w_{a_1} = \sum_{a_2} |\alpha_{a_1 a_2}|^2 , \quad (18.15) \]

\[ H^{in}_1 = - \sum_{a_1 \in \text{Bool}} w_{a_1} \log_2(w_{a_1}) . \quad (18.16) \]

Next we will calculate classical and quantum entropies for various possible density matrices \( \rho \):

(a) \( \rho = \text{tr}_b \sigma \)

Then

\[ \rho = \sum_{a_1,t} (w_{a_1,t} |a_1,t\rangle \langle a_1,t|) \rho_{a_1,t} , \quad (18.17a) \]

where

\[ w_{a_1,t} = \frac{1}{2} w_{a_1} , \quad (18.17b) \]

\[ \rho_{a_1,t} = |\phi_{a_1,t}\rangle \langle \phi_{a_1,t}| , \quad (18.17c) \]

where

\[ |\phi_{a_1,t}\rangle = \frac{1}{\sqrt{w_{a_1}}} \sum_{a_2} [\delta_0^t + (-1)^{a_2} \delta_1^t] |a_1 a_2\rangle . \quad (18.17d) \]

It is easy to show from Eqs.(18.17) that

\[
\begin{align*}
S_\rho(a) &= H^{in} , & H_\rho(a) &= H^{in} & \text{(zero coherence)} \\
S_\rho(1) &= 1 , & H_\rho(1) &= 1 & \text{(zero coherence)} \\
S_\rho(a:t) &= 1 + H^{in}_1 , & H_\rho(a:t) &= 1 + H^{in} \\
S_\rho(a|t) &= H^{in}_1 , & H_\rho(a|t) &= H^{in} \\
S_\rho(t|a) &= 1 + H^{in}_1 - H^{in} , & H_\rho(t|a) &= 1 \\
S_\rho(a : t) &= H^{in} - H^{in}_1 , & H_\rho(a : t) &= 0
\end{align*}
\]

(b) \( \rho = E \Sigma_{\frac{1}{2}} \sigma \)
Then
\[ \rho = |\phi_{a,b}\rangle\langle \phi_{a,b}| \]  
(18.19a)

where
\[ |\phi_{a,b}\rangle = \sum_a \alpha_a |a = a, b = a\rangle. \]
(18.19b)

It is easy to show from Eqs.
\[ S_{\rho}(a) = H^{in}, \quad H_{\rho}(a) = H^{in} \quad \text{(zero coherence)} \]
\[ S_{\rho}(b) = H^{in}, \quad H_{\rho}(b) = H^{in} \quad \text{(zero coherence)} \]
\[ S_{\rho}(a, b) = 0, \quad H_{\rho}(a, b) = H^{in} \]
\[ S_{\rho}(a|b) = -H^{in}, \quad H_{\rho}(a|b) = 0 \]
\[ S_{\rho}(b|a) = -H^{in}, \quad H_{\rho}(b|a) = 0 \]
\[ S_{\rho}(a : b) = 2H^{in}, \quad H_{\rho}(a : b) = H^{in} \quad \text{transmitted info: quantum = 2 classical} \]
(18.20)
A Review of Classical and Quantum Bayesian Nets

In this Appendix, we give a brief review of Classical Bayesian (CB) and Quantum Bayesian (QB) nets. For more information, see Ref.[10].

First, we will state those properties which CB and QB nets have in common. We call a graph (or a diagram) a collection of nodes with arrows connecting some pairs of these nodes. The arrows of the graph must satisfy certain constraints that will be specified below. We call a labelled graph a graph whose nodes are labelled. A CB net (ditto, a QB net) consists of two parts: a labelled graph with each node labelled by a random variable, and a collection of node matrices, one matrix for each node. These two parts must satisfy certain constraints that will be specified below.

An internal arrow is an arrow that has a starting (source) node and a different ending (destination) one. We will use only internal arrows. We define two types of nodes: an internal node is a node that has one or more internal arrows leaving it, and an external node is a node that has no internal arrows leaving it. It is also common to use the terms root node or prior probability node for a node which has no incoming arrows (if any arrows touch it, they are outgoing ones).

We restrict our attention to acyclic graphs; that is, graphs that do not contain cycles. (A cycle is a closed path of arrows with the arrows all pointing in the same sense.)

We assign a random variable to each node of a CB net. Suppose the random variables assigned to the $N$ nodes are $x_1, x_2, \cdots, x_N$. For each $j \in Z_{1,N}$, the random variable $x_j$ will be assumed to take on values within a finite set $S_{x_j}$ called the set of possible states of $x_j$. If $\Gamma = \{k_1, k_2, \cdots, k_{|\Gamma|}\} \subset Z_{1,N}$, and $k_1 < k_2 < \cdots < k_{|\Gamma|}$, define $(x.)_\Gamma = (x_{k_1}, x_{k_2}, \cdots, x_{k_{|\Gamma|}})$ and $(x_\Gamma) = (x_{k_1}, x_{k_2}, \cdots, x_{k_{|\Gamma|}})$. Sometimes, we also abbreviate $(x.)_{Z_{1,N}}$ (i.e., the vector that includes all the possible $x_j$ components) by just $x.$, and $(x)_{Z_{1,N}}$ by just $x$. We often refer to $X = (x.)_\Gamma$ as a node collection. We say $X$ is empty if $|\Gamma| = 0$. If $|\Gamma| = 1$, we say it is a single-node node collection, and if $|\Gamma| > 1$, we say it is a compound node collection. Given two node collections $X_1 = (x)_{\Gamma_1}$ and $X_2 = (x)_{\Gamma_2}$, we say that $X_1$ and $X_2$ are disjoint (ditto, $X_1$ is a subset of $X_2$), if $\Gamma_1$ and $\Gamma_2$ are disjoint (ditto, $\Gamma_1 \subset \Gamma_2$).

Let $Z_{ext}$ be the set of all $j \in Z_{1,N}$ such that $x_j$ is an external node, and let $Z_{int}$ be the set of all $j \in Z_{1,N}$ such that $x_j$ is an internal node. Clearly, $Z_{ext}$ and $Z_{int}$ are disjoint and their union is $Z_{1,N}$.

Each possible value $x.$ of $x$ defines a different net story. For any net story $x.$, we call $(x.)_{Z_{int}}$ the internal state of the story and $(x.)_{Z_{ext}}$ its external state.

Define $\Gamma_j$ to be the set of all $k$ such that an arrow labelled $x_k$ (i.e., an arrow whose source node is $x_k$) enters node $x_j$.

Next, we will state those properties which are different in CB and QB nets.
For each net story $x.$ of a CB net, we assign a non-negative number $P_j[x_j|(x.)_{\Gamma_j}]$ to each node $x_j$. We call $P_j[x_j|(x.)_{\Gamma_j}]$ the probability of node $x_j$ within net story $x.$.

The function $P_j$ with values $P_j[x_j|(x.)_{\Gamma_j}]$ determines a matrix that we call the node matrix of node $x_j$. $x_j$ is the matrix’s row index and $(x.)_{\Gamma_j}$ is its column index. We require that the values $P_j[x_j|(x.)_{\Gamma_j}]$ be conditional probabilities; i.e., that they satisfy:

$$P_j[x_j|(x.)_{\Gamma_j}] \geq 0 , \tag{A.1}$$

$$\sum_{x_j} P_j[x_j|(x.)_{\Gamma_j}] = 1 , \tag{A.2}$$

where the sum in Eq.(A.2) is over all the states that the random variable $x_j$ can assume, and where Eqs.(A.1) and (A.2) must be satisfied for all $j \in Z_{1,N}$ and for all possible values of the vector $(x.)_{\Gamma_j}$ of random variables. The left-hand side of Eq.(A.2) is just the sum over the entries of a column of the node matrix.

The probability of net story $x.$, call it $P(x.)$, is defined to be the product of all the node probabilities $P_j[x_j|(x.)_{\Gamma_j}]$ for $j \in Z_{1,N}$. Thus,

$$P(x.) = \prod_{j \in Z_{1,N}} P_j[x_j|(x.)_{\Gamma_j}] . \tag{A.3}$$

We require $P(x.)$ to satisfy:

$$\sum_x P(x.) = 1 . \tag{A.4}$$

Call a CB pre-net a labelled graph and an accompanying set of node matrices that satisfy Eqs.(A.1), (A.2) and (A.3), but don’t necessarily satisfy the overall normalization condition Eq.(A.4). It can be shown that all acyclic CB pre-nets satisfy Eq.(A.4). If one considers only acyclic graphs as we do in this paper, then there is no difference between CB nets and CB pre-nets.

(b) Quantum Bayesian Net

For each net story $x.$ of a QB net, we may assign a a complex number $A_j[x_j|(x.)_{\Gamma_j}]$ to each node $x_j$. We call $A_j[x_j|(x.)_{\Gamma_j}]$ the amplitude of node $x_j$ within net story $x.$. The function $A_j$ with values $A_j[x_j|(x.)_{\Gamma_j}]$ determines a matrix that we call the node matrix of node $x_j$. $x_j$ is the matrix’s row index and $(x.)_{\Gamma_j}$ is its column index. We require that the quantities $A_j[x_j|(x.)_{\Gamma_j}]$ be probability amplitudes that satisfy:

$$\sum_{x_j} |A_j[x_j|(x.)_{\Gamma_j}]|^2 = 1 , \tag{A.5}$$
where the sum in Eq.(A.5) is over all the states that the random variable $x_j$ can assume, and where Eq. (A.5) must be satisfied for all $j \in Z_{1,N}$ and for all possible values of the vector $(x_j)_{\Gamma_j}$ of random variables.

The amplitude of net story $x$, call it $A(x)$, is defined to be the product of all the node amplitudes $A_j[x_j|(x.)_{\Gamma_j}]$ for $j \in Z_{1,N}$. Thus,

$$A(x) = \prod_{j \in Z_{1,N}} A_j[x_j|(x.)_{\Gamma_j}]. \quad (A.6)$$

We require $A(x)$ to satisfy:

$$\sum_{(x)_Z^{ext}} \left| \sum_{(x)_Z^{int}} A(x) \right|^2 = 1 \quad (A.7)$$

and

$$\sum_x |A(x)|^2 = 1. \quad (A.8)$$

Note that as a consequence of Eqs.(A.5) and (A.8), given any QB net, one can construct a special CB net by replacing at each node the value $A[x_j|(x.)_{\Gamma_j}]$ by its magnitude squared. We call this special CB net the parent CB net of the QB net from which it was constructed. We call it so because, given a parent CB net, one can replace the value of each node by its square root times a phase factor. For a different choice of phase factors, one generates a different QB net. Thus, a parent CB net may be used to generate a whole family of QB nets.

A QB pre-net is a labelled graph and an accompanying set of node matrices that satisfy Eqs.(A.5), (A.6) and (A.7), but don’t necessarily satisfy Eq.(A.8). A QB pre-net that is acyclic satisfies Eq.(A.8), because its parent CB pre-net is acyclic and this implies that Eq.(A.8) is satisfied. If one considers only acyclic graphs as we do in this paper, then there is no difference between QB nets and QB pre-nets. One can check that all the examples of QB nets considered in this paper satisfy Eq.(A.8). Eq.(A.8) is true iff the meta state $|\psi_{meta}\rangle$ defined by Eq.(6.2) has unit magnitude.

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[9] B. Schumacher, Lectures given at University of Innsbruck, from 28 May to 12 Jun 1998. Available at: http://www2.kenyon.edu/people/schumacb/lectures.htm

[10] R. R. Tucci, Int. Jour. of Mod. Physics B9, 295 (1995). Available as Los Alamos eprint quant-ph/9706039. The theory of this paper is implemented by a computer program called “Quantum Fog”, available at www.ar-tiste.com.

[11] This analogy between Information Theory and Set Theory, and its pictorial representation in terms of Venn diagrams, has been known since time immemorial. I’m not sure who was the first to point it out, but it seems to have been common knowledge less than five years after Shannon’s 1948 paper that started it all. I suspect that the analogy can be phrased more generally and rigorously within the mathematical field of Lattice Algebras, but I know of no references to support this claim.

[12] R.R. Tucci, “Data Processing Inequalities for Bayesian Nets”, Los Alamos eprint quant-ph/?

[13] This is very much in the spirit of N. J. Cerf, C. Adami, “Negative entropy and information in quantum mechanics”, Phys.Rev.Lett. 79 (1997) 5194 (available as Los Alamos eprint quant-ph/951202. Note other Los Alamos eprints by same authors on similar topics.) Like us, Cerf and Adami advocate defining quantum conditional and mutual entropies so as to preserve the Venn diagrams which have been used in classical information theory for decades. However, there are some big differences between our work and theirs (apart from the obvious fact that they don’t use Bayesian nets). For them the A and B in \( S(A|B) \) refer to separate “sub-systems” at the same instant of time. For us they are node random variables which need not represent separate subsystems. They might, for example, represent the same sub-system at different instants.

[14] B. Schumacher, M. A. Nielsen, “Quantum data processing and error correction”, Los Alamos eprint quant-ph/9604022.
B. Schumacher, “Sending quantum entanglement through noisy channels”, Los Alamos eprint quant-ph/9604023.

It is also called a POVM, which stands for ((Positive Operator) Valued) Measure. The reason for the long name is as follows.

In classical probability, one speaks of an event space $\Omega$ and a function $\mu : \Omega \to (\text{Non-negativeReals})$ called a real-valued measure. A random variable $b$ on $\Omega$ is a function $b : \Omega \to S_b$, where $S_b$ is the set of values that $b$ may assume. $P(b = b)$ is defined by

$$P(b = b) = \mu(\{\omega \in \Omega \mid b(\omega) = b\}).$$

In quantum mechanics, one speaks of an event space $\Omega$, a Hilbert space $H$, a density matrix $\rho$ acting on $H$, and a function $\mu_{op} : \Omega \to (\text{Non-negativeOperatorsActingOn } H)$ called an operator-valued measure. A random variable $b$ is still a function $b : \Omega \to S_b$. For each $b \in S_b$, one defines an operator $F_b$ acting on $H$ by

$$F_b = \mu_{op}(\{\omega \in \Omega \mid b(\omega) = b\}).$$

Then $P(b = b)$ is defined by

$$P(b = b) = \text{tr}(F_b \rho).$$

It’s really $\mu_{op}$ that is a POM, but since the set $\{F_b \mid \forall b\}$ partly specifies $\mu_{op}$, we call this set a POM too. For more information about POMs, see [6] and references therein.

A. S. Holevo, “Information Theoretical Aspects of Quantum Measurement”, (Engl. Transl.) Problems of Information Transmission, 9, 177-183 (1973).

Andreas Winter, quant-ph/9907077; R. Ahlswede, P. Loeber, quant-ph/9907081. These workers from the Uni. of Bielefeld have also shown (working independently from me, and using a $C^*$ Algebra approach) that Holevo’s Inequality follows from a Data Processing Inequality.

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