Excitation of Orthogonal Radiation States

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Abstract—A technique for designing antenna excitation realizing orthogonal states is presented. It is shown that a symmetric antenna geometry is required in order to achieve orthogonality with respect to all physical quantities. A maximal number of achievable orthogonal states and a minimal number of ports required to excite them are rigorously determined from the knowledge of an antenna’s symmetries. The number of states and the number of ports are summarized for commonly used point groups (a rectangle, a square, and so on). The theory is applied to an example of a rectangular rim where the positions of ports providing the best total active reflection coefficient, an important metric in multiport systems, are determined. The described technique can easily be implemented in existing solvers based on integral equations.

Index Terms—Antenna theory, computer simulation, eigenvalues and eigenfunctions, electromagnetic modeling, method of moments, modal analysis.

I. INTRODUCTION

THE ever-growing requirements on data throughput capacity [1] and simultaneous full occupancy of the radio spectrum has led to many novel concepts in recent decades [2]. One of the most successful techniques is the multiple-input multiple-output (MIMO) method [3], [4] heavily utilized in modern communication devices [5], [6]. When considering MIMO spatial multiplexing, spatial correlation has a strong impact on ergodic channel capacity [7], therefore, low mutual coupling between the states generated by individual antennas is required [8], [9].

In this article, free-space channel capacity is increased by considering spatial multiplexing realized by orthogonal electromagnetic field states excited by a multiport radiator [10]–[12]. This assumes that orthogonal states are a good starting position for realistic channels where stochastic effects cannot be neglected. Instead of an array of transmitters [13], the orthogonality is provided by a general multiport antenna system. This approach addresses the question of how many orthogonal states can, in principle, be induced by a radiating system of a given geometry and how many localized ports are needed to excite them separately.

Previous research on this topic utilized characteristic modes (CMs) [14], [15], which provide orthogonal states in far-field. Unfortunately, as shown by the long history of attempts within the CM community [16]–[26], this task is nearly impossible to accomplish, as entire-domain functions defined over arbitrarily shaped bodies cannot be selectively excited by discrete ports [27].

Many other methods exist to characterize and approach the maximal capacity, for both a special case of spherical geometry [28], [29] and for arbitrarily shaped antennas [30]–[32]. The number of degrees of freedom represented by electromagnetic field states was studied on an information theory level as well [33]–[35]. As with the CMs approach, in all these cases, the optimal coefficients do not prescribe any particular excitation of a selected or optimized antenna designs. This issue was solved in [36] utilizing a singular value decomposition of excitation coefficients represented in spherical wave expansion and in [37] by employing a port-mode basis [38]. The orthogonal radiation patterns are excited; however, the schemes are not orthogonal with respect to other physical operators, leading to unpleasant effects, such as nonzero mutual reactances [39].

The situation changes dramatically for a structure invariant under certain symmetry operations, including rotation, reflection, or inversion. Certain symmetry operations were utilized in [10] and [40]; however, a general approach can be reached only by applying point group theory [41], which allows the modes computed by arbitrary modal decomposition to be classified into several irreducible representations (irreps) that are orthogonal to each other. Spherical harmonics [42] of a different order are a notable example of such an orthogonal set of states. A known property of physical states selected arbitrarily from two different irreps is that all mutual metrics are identically zero [43]. This useful property has already been utilized for the block diagonalization of the bodies of a revolution matrix [44] and further study reveals interesting properties regarding the simultaneous excitation of perfectly isolated states [45], [46]. An additional benefit is that selective excitation is possible since the antenna excitation vectors may follow the irreducible representations of the underlying structure [47].

The key instrument employed in this work is the group theory-based construction of a symmetry-adapted basis [41] and block diagonalization of the operators. This methodology leads to a fully automated design, without the necessity of a visual inspection or manual manipulation of the data [48]. The upper bound on the number of orthogonal states and the lower bound on the number of ports are rigorously derived only from the knowledge of symmetries. It is observed that the

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later number is significantly lower than the number of ports utilized in practice [16]. The placement of a given number of ports maximizing a selected antenna metric is investigated through combinatorial optimization [49] over vector adapted bases.

The entire design procedure can easily be incorporated into a simple algorithm, thus opening possibilities to analyze MIMO antennas automatically. All findings are demonstrated on a set of canonical geometries. The figure of merit is then solved in Section V via an exhaustive search. This article is concluded in Section VI.

This article is structured as follows. The theory is developed in Section II, primarily based on point group theory and eigenvalue decomposition. The basic consequences are demonstrated on an example in Section III. Section IV addresses the important questions of how many orthogonal states are available and how many ports are needed to excite them independently. The optimal placement of a given number of ports is then solved in Section V via an exhaustive search. This article is concluded in Section VI.

II. ORTHOGONAL STATES

Let us assume antenna metric $p$ defined via quadratic form

$$ p(q_m, q_n) = \langle q_m, A(q_n) \rangle $$

(1)

where $q_m$ and $q_n$ are states of the system (e.g., modal current densities, modal far fields, or excitation states, see Table I), $A$ is a linear complex operator (see Appendix A for representative examples), and $\langle \cdot, \cdot \rangle$ denotes the inner product

$$ \langle a(r), b(r) \rangle = \int_{\Omega} a^*(r) \cdot b(r) \, dV, $$

(2)

where $a(r)$ and $b(r)$ are generic vector fields supported in region $\Omega$, $r \in \Omega$. For the purpose of this article, orthogonality of states is further defined as

$$ p(q_m \in S_i, q_n \in S_j) = \zeta_{ijmn} \delta_{ij}, $$

(3)

where $\{S_i\}$ are disjoint sets of states $q$, $\zeta$ are normalization constants, and $\delta_{ij}$ is a Kronecker delta.

In order to obtain a numerically tractable problem, procedures, such as the method of moments (MoM) [53] or finite element method (FEM) [54], are commonly employed, recasting states $q$, operators $A$, and sets $S$ into column vectors $Q$, matrices $A$ [55], and linear vector spaces $S$, respectively (see Table I and Appendix A). Within such a paradigm, the orthogonality (3) can be written as

$$ Q^H_i A Q_j = 0 : \, \, \, Q_i \in S_i, \, \, \, Q_j \in S_j $$

(4)

which means that matrix $A$ is block-diagonalized in the basis generated by these states.

Difficulties in finding orthogonal sets of vectors strongly depend on the number of operators $\{A_i\}$ with respect to which relation (4) must simultaneously be satisfied. In the case of a sole operator $\{A\}$ or two operators $\{A_1, A_2\}$, the solution to a standard $AQ = \lambda Q$ or a generalized $A_1 Q = \lambda A_2 Q$

| current densities | far fields | excitation states | port modes |
|-------------------|-----------|--------------------|------------|
| characteristic modes [14] | far-field patterns [51] | excitation modes [16] | port modes [52] |
| $q_m = J_m(r)$ | $q_n = F_m(t, \theta, \phi)$ | $q_m = B_m(r)$ | $A_1 = X_0, A_2 = R_0$ |
| $A_1 = \lambda_0, A_2 = R_0$ | $A = \lambda_0 = Re \{Z_0\}$ | $A = y = z^{-i}$ | $Q_m = I_m$ |

TABLE I

THREE EXAMPLES OF SYSTEM STATES $q_m$ AND ASSOCIATED OPERATORS $A$ PRESERVING ORTHOGONALITY IN THE SENSE OF (3). THE ALGEBRAIC REPRESENTATION OF STATES $V_m$ AND OPERATORS $A$ IS EXPRESSED IN A BASIS $\{\phi_m(r)\}$, SEE APPENDIX A FOR DETAILS. ALL QUANTITIES SHOWN IN THE TABLE ARE SUBSEQUENTLY INTRODUCED THROUGHOUT THIS ARTICLE

The entire design procedure can easily be incorporated into a simple algorithm, thus opening possibilities to analyze MIMO antennas automatically. All findings are demonstrated on a set of canonical geometries. The figure of merit is then solved in Section V via an exhaustive search. This article is concluded in Section VI.
which means that the CMs from different irreps (α ≠ β), or from the same irrep but different dimension (i ≠ j), are orthogonal with respect to stored energy as well. This statement can be generalized to all operators resulting from a MoM paradigm, see examples in Appendix A or in [59].

The relation (9) states that columns of matrices \( \Gamma^{(α,i)} \) form vector spaces \( S \) in (4); consequently, the columns of \( \Gamma^{(α,i)} \) can be desired vectors \( Q \). In such a case, the orthogonality (4) holds simultaneously for all operators \( \{A_i\} \) describing the physical behavior of the underlying structure whenever two vectors belong to different species.

Notice that the utilization of symmetries induces the pattern diversity, see the second column of Table I, and allows to create several orthogonal states, which are not affected by, and does not have the influence on, used multiplexing technique; thus, for example, time or code diversity can be applied at each orthogonal state to further increase the capacity of the system.

III. ILLUSTRATIVE EXAMPLE

This section demonstrates the usefulness of the point group-based block diagonalization (9) to obtain orthogonal states.

The design procedure is illustrated on the example of a rectangular plate of dimensions \( 2L \times L \) and of electrical size \( \varepsilon_{r}L \approx 10.19 \) (\( \varepsilon_{r} \) abbreviates a free-space wavenumber and \( a \) denotes the radius of the smallest sphere circumscribing the plate), which was used in [16] to construct orthogonal states via the selective excitation of CMs. The CMs in [16] were visually separated into four “groups” (using the nomenclature of [16]), and voltage sources (ports) were associated with each such group so as to provide maximum excitation of the dominant CM of each group. In order to independently control four sets of modes, eight voltage sources (delta gaps) were used. The structure and positions of voltage sources used in [16] are shown in Fig. 3. Unit voltages were considered with polarity determined by the second column of Table II.

The point group theoretical treatment introduced in Section II-A offers a different solution to the same problem. The underlying object has four point symmetries (identity, rotation of \( \pi \) around z-axis, and two reflections via the \( xy \) and \( yz \) planes) and belongs to the C_{nv} point group (see Fig. 2). The CMs of Table II. are visually separated into four “groups” (using the nomenclature of [16]), and voltage sources (ports) were associated with each such group so as to provide maximum excitation of the dominant CM of each group. In order to independently control four sets of modes, eight voltage sources (delta gaps) were used. The structure and positions of voltage sources used in [16] are shown in Fig. 3. Unit voltages were considered with polarity determined by the second column of Table II.

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As mentioned in Section II-A, any columns of matrices \( \Gamma^{(α,i)} \) [58] can be used as excitation vectors \( V^{(α,i)} \), see Appendix B, to enforce orthogonality. To minimize the number of voltage sources used, it is advantageous to select those columns that have nonzero elements at the same positions across all species. In the specific case of Fig. 3, matrices \( \Gamma^{(α,i)} \) also contain columns with only four nonzero entries (i.e., with

\[ \text{only one-dimensional irreps exist in this case, i.e., dimensionality of each irrep } \alpha \text{ is } g^{(α)} = 1. \]
vector \( \mathbf{V}^{(α, i)}(ξ) ∈ \mathbb{C}^{N_s × 1} \) that satisfies

\[
(\mathbf{V}^{(α, i)}(ξ))^H \mathbf{A} \mathbf{V}^{(β, j)}(ζ) = ζ_{αβ} δ_{αβ} δ_{ij}
\]  

for an arbitrary operator \( \mathbf{A} ∈ \mathbb{C}^{N_s × N_s} \) with \( N_s \) being the number of unknowns (a number of basis functions). The mapping (13) is characterized by the point group of structure \( G = \{ R \} \) consisting of symmetry operations \( R \), dimensionality \( g^{(α)} = \dim \mathbf{D}^{(α)} \) of irrep \( α \), the order of the point group \( g = \sum_α (g^{(α)})^2 \), mapping matrix \( \mathbf{C}(R) \), and irreducible matrix representation \( \mathbf{D}^{(α)} = [δ_{ij}^{(α)}] \) with \( \mathbf{D} = (\mathbf{D}^{-1})^T \) (see [41], [58, Sec. II-C] for more details). The application of (13) and the exact meaning of all variables used are illustrated in an example in Appendix C.

Throughout this article, excitation vector \( \mathbf{V}(ξ) \) represents an arbitrarily shaped port (e.g., delta-gap and coaxial probe) that lies entirely in the generator of the structure, see the highlighted areas in Fig. 2, and variable \( ξ \) is used to code the position of this port. As an example, assume that port No. 1 in Fig. 3 is a delta-gap port represented by excitation vector \( \mathbf{V}(1) \). Notice that it is placed in one of the quadrants, which are the generators of the structure. Each summand of (13) maps (changing orientation, position, and amplitude) this port on its symmetry positions 2, 3, and 4, creating symmetry-adapted excitation vector \( \mathbf{V}^{(α, i)}(1) \) for a particular species \( (α, i) \).

The first two questions from Section III can be answered by inspecting (13).

1) The maximum number of orthogonal states, \( N_s \), (orthogonal with respect to all physical operators), is equal to the number of species of the given point group, i.e., to the number of vectors \( \mathbf{V}^{(α, i)}(ξ) \) generated by (13) for a given set of ports in the generator of the structure, which is \( N_s = \sum_α g^{(α)} \).

2) The minimum number of ports, \( N_p \), needed to distinguish all orthogonal states mentioned above is equal to the number of symmetry operations in point group \( G \) since each summand of (13) maps initial excitation vector \( \mathbf{V}(ξ) \) onto a new position and there are as many summands as symmetry operations (see the detailed example in Appendix C). It is assumed that each mapping is unique, and otherwise, not all orthogonal states are reached—this possibility is discussed later in Section IV-A.

Table III summarizes the number of maximal reachable orthogonal states and minimal number of ports required to excite the states selectively.

When combined together, the answers to 1 and 2 show how orthogonal states can be efficiently established for a given point group. On the other hand, this procedure does not ensure that all states lead to the same optimal value of the selected antenna metric. This calls for a reply to question 3, which is addressed in Section V.

### Table II

| Set | Ports [16, Table IV] | irrep α | Four ports |
|-----|----------------------|---------|-----------|
| S₁  | 1⁺, 2⁻, 3⁺, 4⁻      | A₁      | 1⁺, 2⁻, 3⁺, 4⁻ |
| S₂  | 5⁺, 6⁺               | B₁      | 1⁺, 2⁻, 3⁺, 4⁺ |
| S₃  | 7⁺, 8⁺               | B₂      | 1⁺, 2⁺, 3⁺, 4⁺ |
| S₄  | 7⁺, 8⁻               | A₂      | 1⁺, 2⁺, 3⁻, 4⁻ |

Fig. 3. Port locations on a rectangular plate. Reprinted from [16, Fig. 5]. Arrows show the orientation of the voltage sources.

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**IV. EXCITATION STATES BASED ON POINT GROUP THEORY**

Referring to [58, eq. (16)], symmetry-adapted excitation vectors can be constructed as

\[
\mathbf{V}^{(α, i)}(ξ) = g \sum_{R ∈ G} \mathbf{D}^{(α)}(R) \mathbf{C}(R) \mathbf{V}(ξ)
\]  

(13)

which is a linear map from excitation vector \( \mathbf{V}(ξ) ∈ \mathbb{C}^{N_s × 1} \) (see Appendix B) onto a symmetry-adapted excitation
TABLE III
MAXIMUM NUMBER OF SYMMETRY-BASED ORTHOGONAL STATES
N_s /MINIMAL NUMBER OF PORTS N_p NEEDED TO EXCITE ALL OF
THEM FOR A GIVEN POINT GROUP. SELECTED POINT GROUPS
ARE SHOWN IN FIG. 2. A SCHLENKFLIES NOTATION [41]
IS USED FOR POINT GROUPS NAMING

| n  | C_n | C_nv | C_nh | D_n | D_nh | D_n/4 | S_n |
|----|-----|------|------|-----|------|-------|-----|
| 2  | 2/2 | 4/4  | 4/4  | 4/4 | 8/8  | 6/8   | 2/2 |
| 3  | 3/3 | 6/6  | 6/6  | 6/6 | 10/10| 10/10 | 4/4 |
| 4  | 4/4 | 8/8  | 8/8  | 8/8 | 12/14| 12/14 | 6/6 |
| 5  | 5/5 | 10/16| 10/16| 10/16| 14/16| 14/16 | 8/8 |
| 6  | 6/6 | 12/16| 16/24| 16/28| 20/32| 16/28 | 10/24|
| 7  | 7/7 | 12/20| 16/24| 16/28| 20/32| 16/28 | 10/24|
| 8  | 8/8 | 12/20| 20/48| 16/60| 32/120| 16/60 | 10/24|

TABLE IV
SYMMETRY-ADAPTED DELTA-GAP NUMBER FIVE FROM Fig. 3

| R \ α | A_1 | A_2 | B_1 | B_2 |
|-------|-----|-----|-----|-----|
| E     | 5+  | 5+  | 5+  | 5+  |
| σ_x^z | -5  | -5  | -5  | -5  |
| C_5   | 6+  | 6+  | 6+  | 6+  |

A. Port Placed in the Reflection Plane

Formula (13) suggests that a problematic design appears when the port corresponding to excitation vector V(ζ) lies at the boundary of the generator of the structure [41], e.g., at the reflection plane. In this case, the port generally breaks the symmetry of the structure making the process of symmetry adaptation invalid. To give an example, imagine that a delta-gap port is placed at position ζ = 5 in Fig. 3. The reflection σ_x^z and identity operation E project this port onto itself but with different polarity. This collision is shown in Table IV. In this case, only states belonging to irreps A_2 and B_2 are realizable. More than four ports would be needed to establish four states.

V. PORTS’ POSITIONING

In order to answer the third question from Section III—where should ports be placed to maximize the performance of a device, with respect to a given physical metric, to maintain the orthogonality of states?—it is necessary to consider the particular requirements on the performance of the device. An example of investigating port positions to optimize the TARC of an antenna is used to demonstrate the sequence of steps to resolve this question. Instead of the rectangular plate shown in Fig. 3, a rectangular rim of dimensions 2L x L and width L/10 is considered (see the object in Fig. 4). The geometry of the rim belongs to the same point group as the plate but allows for the placement of discrete ports [60] at an arbitrary position without creating undesired short circuits.

A. Total Active Reflection Coefficient

The TARC [50], which is defined as

$$ t = \sqrt{1 - \frac{P_{\text{rad}}}{P_{\text{in}}}} $$

is used as a performance metric, where P_{rad} stands for radiated power and P_{in} stands for incident power. Within the MoM framework, (15) can be reformulated as

$$ t(v) = \sqrt{1 - \frac{4Z_0v^{H}P^{H}Y^{H}R_{0}YPv}{v^{H}k^{H}kv}} $$

with

$$ k = e + Z_0 y $$

where e is the identity matrix, Z_0 = 50 Ω is the characteristic impedance of all transmission lines connected to the ports, Y = Z^{-1} ∈ C^{N_p x N_p} is an admittance matrix, R_0 is the radiation part of the impedance matrix, and y ∈ C^{N_p x N} is the admittance matrix seen by N_p connected ports. Each port is represented by one column of matrix P and port voltages are all accumulated in vector v. Matrix P is therefore of size N_p x N_p and the excitation vector is given by V = Pv (see Appendices A, B, and D for detailed derivations).

B. Optimization Problem

The problem of TARC minimization with additional constraints on N_m orthogonal states is defined as to find port excitation vectors \{v_m\}, m ∈ \{1, ..., N_m\}, and port configuration P such as to fulfill

$$ \min_{\{P, v_m\}} t_{\text{rms}} $$

$$ \text{s.t. } v_m^{H}P^{H}A_n v_n = 0, \ m \neq n $$

$$ : = : $$

$$ v_m^{H}P^{H}A_{N_p} v_n = 0, \ m \neq n $$

(18)
where the root mean square (rms) metric based on (16) is adopted

\[
\text{rms} = \sqrt{\frac{1}{N_m N_f} \sum_{m=1}^{N_m} \sum_{f=1}^{N_f} t^2(v_m, o_f)}
\]  

(19)
to measure the overall performance over \(N_f\) frequency samples \(o_f\) and over multiple states. Matrices \(A_k\), \(k \in \{1, \ldots, N_k\}\), in the constraints above are placeholders for matrix operators from, e.g., Appendix A. These constraints enforce simultaneous orthogonality with respect to all operators describing the physical system at hand, e.g., with respect to far fields (\(A_k = Y^{HI}R_0Y\)), current densities (\(A_k = Y^{HI}\)), excitation vectors (\(A_k\) is the identity matrix), or energy stored by the states (\(A_k = Y^{HI}WY\)).

In light of the discussion in Section II, the simultaneous realization of all \(N_k \geq 2\) constraints in (18) is only possible on symmetric structures and only when excitation vectors \(V_m = P v_m\) are given by (13), i.e., \(V_m = V^{(a,i)} = P v^{(a,i)}\). This imposes specific requirements on port matrix \(P\) and port voltages \(v^{(a,i)}\).

First, ports represented by columns of port matrix \(P\) have to be symmetrically distributed on the structure. This is achieved by placing a port (a single column of matrix \(P\)) at arbitrary position \(\xi\) in the generator of the structure and then by replication of this port by the application of symmetry operations \(R \in G\) (column of matrix \(P\) is transformed by mapping matrices \(C(R)\)). Each replication results in a new port, i.e., new column\(^2\) of port matrix \(P\).

Second, port excitation vector \(v\) is constructed so that only ports placed in the region of the generator of the structure are excited (others are kept at zero voltage) and the symmetry adaptation (13) of vector \(V(\xi) = P v\) is processed. Here and further, \(\xi\) represents a particular position in the generator of the structure (see possible placements in Fig. 4).

Finally, port voltages \(v^{(a,i)}\) for species \((a,i)\) are acquired from excitation vector \(V^{(a,i)}\) as

\[
v^{(a,i)} = (P^HP)^{-1}P^HV^{(a,i)}
\]  

(20)
see (48) in Appendix D.

Being now equipped with symmetry-adapted excitation vectors \(V^{(a,i)} = P v^{(a,i)}\), constraints of (18) are automatically fulfilled irrespective of their number. The variables remaining for optimization (18) are therefore positions \(\xi\) of ports in the generator of the structure and their amplitudes. In a simplified case, when only one port exists in the generator of the structure, its amplitude is of no relevance and the only optimized variable is position \(\xi\), i.e., the optimization problem (18) reduces to

\[
\text{minimize } t_{\text{rms}}.
\]  

(21)

In order to give a simple set of instructions for the procedure above, the TARC minimization with fully orthogonal states iteratively performs the following.

1) Pick a position \(\xi\).
2) Create a port matrix \(P\), see (47) in Appendix D.
3) Construct vector \(v\) exciting only the ports in the generator of the structure.
4) Perform symmetry adaptation (13) of the vector \(V = Pv\) into all species \((a,i)\).
5) Get \(v^{(a,i)}\) via (20) for each species.
6) Calculate TARC \(t(v^{(a,i)})\) for all species (16).
7) Evaluate the fitness function \(t_{\text{rms}}\) via (19).

C. Single-Frequency Analysis

The optimization of the port’s placement in the generator of the structure (21) computed at the single-frequency sample \((N_f = 1)\) is analyzed in this section. The selected frequency corresponds to the antenna’s electrical size \(ka = 10.19\). The low number of tested positions and the possibility to precalculate all matrix operators \(A\) enable the use of an extensive search to evaluate (16) for each tested position \(\xi\) shown by the red in Fig. 4.

The results are presented in Fig. 5. As mentioned in Section IV-A, positions \(\xi = 1\) and \(\xi = 15\) are not able to excite all four orthogonal states since they are placed at the reflection plane. All other positions \(\xi\) result in a total of four symmetrically placed ports providing four orthogonal states.

Bars in Fig. 5 show TARC values (16) computed for each of the four species \((a,1)\), \(a \in \{A_1, A_2, B_1, B_2\}\). The values \(t_{\text{rms}}\) are represented by the black vertical lines. The optimal port position \(\xi\) in the generator of the structure is declared as the one with the lowest value of \(t_{\text{rms}}\), i.e., position \(\xi = 14\).

Radiated patterns were computed and plotted as 2-D cuts in Fig. 6 to confirm the orthogonality of the designed excitation vectors \(V^{(a,i)}(\xi)\). One can see that these patterns are similar to spherical harmonics that are orthogonal [42]. To reduce the complexity of radiation patterns in Fig. 6, radiation patterns were computed at \(ka = 1\), bearing in mind that the orthogonality between states is frequency independent.

D. Frequency Range Analysis

Multiport antenna systems typically operate in a wide frequency range. However, evaluating (21) at each frequency,
Fig. 6. Far-field cuts with polarization along direction $\hat{\phi}$ computed at $ka = 1$ for excitation vectors $V^{(\alpha,i)}$ for $\xi = 7$. (a) Cut at $\theta = \pi/2$. (b) Cut at $\phi = \pi/2$. Radiation patterns are orthogonal, which is confirmed by the envelope correlation coefficient [61] depicted in the table. The naming convention adapted for the states is the same as in Fig. 5 and Table IV.

Fig. 7. Best position of the port in the generator of the structure $\xi$ with respect to TARC value $t_{\text{rms}}$ evaluated at each frequency sample.

as was done in the previous section, does not provide a unique best position $\xi$ (see Fig. 7), where $N_{\xi} = 116$ frequency samples in the range corresponding to the antenna’s electrical size $ka \in (0.5, 12)$ was used.

The unique solution is accomplished by evaluating the rms value of TARC (19) over the frequency range in which the best position minimizing (21) is $\xi = 7$ (see Fig. 8). The realized TARC computed for this optimal position over the whole frequency band is shown in Fig. 9. It can be observed that there is no frequency where all four states radiate well, which results from their different current distributions. However, minimizing (21), by counting all frequencies of the selected band, provides a solution in which average TARC over all channels is the best.

The values in Fig. 8 are not so different and, in fact, are unsatisfactory. This is caused by the wide frequency range used and by employing the connected transmission lines of characteristic impedance $Z_0 = 50\,\Omega$, which is not an optimal value for the chosen structure. Optimization of the impedance matching would demand a topological change of the antenna structure (keeping the necessary symmetries), which is beyond the scope of this article.

E. More Ports Placed in the Generator of the Structure

The previous sections assumed the existence of a sole port placed in the generator of the structure that was symmetry-adapted. Nevertheless, a higher number of ports might give better radiation properties. In the case of $N_{\xi} > 1$ ports placed in the generator of the structure, in addition to all statements in Section V-B, the complex amplitudes connected to the ports also have significance.

As port excitation vector $v$ is constructed so that only ports placed in the generator of structure are excited (i.e., there is $N_{\xi}$ nonzero values) and because the symmetry-adaptation process (13) transforms these $N_{\xi}$ nonzero values to $N_p$ nonzero values in vector $v^{(\alpha,i)}$, the symmetry-adapted vector can also be expressed as

$$v^{(\alpha,i)} = p^{(\alpha,i)}k^{(\alpha,i)}$$

where $p \in \mathbb{R}^{N_p \times N_{\xi}}$ is a port-indexing matrix (each column in $p$ corresponds to one exclusively excited port in the generator of the structure) and vector $k$ of size $N_{\xi} \times 1$ contains only voltages of ports placed in the generator of the structure.

Substituting (22) into (16) leads to

$$t^{(\alpha,i)}(k^{(\alpha,i)}) = \sqrt{1 - \left(\frac{k^{(\alpha,i)}}{v^{(\alpha,i)}}\right)^H A^{(\alpha,i)} k^{(\alpha,i)}}$$

(23)
where

\[
A^{(α, i)} = 4 Z_0 (Y^p p^{(α, i)})^H R_0 Y^p p^{(α, i)} \in \mathbb{C}^{N_γ \times N_γ}
\]  

and

\[
B^{(α, i)} = (kp^{(α, i)})^H kp^{(α, i)} \in \mathbb{C}^{N_γ \times N_γ}.
\]

In order to minimize (23), a generalized eigenvalue problem

\[
A^{(α, i)} \lambda^{(α, i)} = \lambda^{(α, i)} B^{(α, i)}
\]

is solved and an eigenvector minimizing (23), i.e., one corresponding to the highest eigenvalue \( \lambda^{(α, i)} \), is chosen. This solution provides the best achievable TARC for a given species \((α, i)\), the value of which is

\[
\zeta^{(α, i)} = \sqrt{1 - \max(\lambda^{(α, i)})}.
\]

In the case of more ports placed in the generator of the structure, the process described in this section must be used in every step of optimization (21), i.e., vectors \( \zeta^{(α, i)} \) must be evaluated for each choice of \( \zeta \).

**F. Analysis With More Ports in the Generator of the Structure**

The optimal placement of two and three ports, \( N_γ = \{2, 3\} \), in the generator of the structure is studied in this section. The same metallic rim as in Section V-C operating at \( ka = 10.19 \) is used and the method from Section V-E is applied (see Table V for the results). It can be observed that the involvement of more ports significantly decreases the rms of TARC across the states. This is because the optimal current density reaching minimal TARC is better approximated with more excitation ports.

Table V shows the results for a port configuration adopted from [16], which was discussed in Section III. This configuration uses \( N_γ = 3 \) ports placed in the generator of the structure and \( N_p = 8 \) ports. Nevertheless, Table V reveals that better results may be obtained when the symmetry-adapted basis described in this article is utilized.

The frequency range analysis from Section V-D was repeated for the combination of \( N_γ = 2 \) ports placed in the generator of the structure. Ports at positions \( \zeta \in \{12, 14\} \) provide the lowest rms (19) \( t_{\text{rms}} = 0.605 \). However, the solution with a combination of more positions \( \{\zeta_2\} \) requires optimized port voltage amplitudes \( \zeta \) (22) that vary over frequency (see Fig. 10). Fig. 11 shows the realized TARC values reached by this configuration. The radiation efficiency is significantly improved compared with the previous solution shown in Fig. 9.

**VI. CONCLUSION**

The presence of symmetries was utilized via point group theory to describe a procedure that determines where to place ports on an antenna to achieve orthogonal states with respect to any radiation metric, such as radiation and total efficiency, antenna gain, or \( Q \)-factor.

The methodology can play an essential role in the design of MIMO antennas when a few ports can orthogonalize several states (e.g., four ports on a rectangular structure generate four orthogonal states and eight ports on a square structure generate six orthogonal states). The maximal number of orthogonal states and the minimal number of ports needed to excite all of them is determined only from the knowledge of the point group to which the given geometry belongs. Due to the symmetries, the procedure of ports’ placement can be accelerated by the reduction of the section where the port located in the region of the generator of the structure can be placed and, subsequently, “symmetry-adapted” to the proper positions at the entire structure. It was also demonstrated that port positions intersecting reflection planes should not be used since they do not allow the excitation of all states.
A proper placement of ports was illustrated by an example—with a single frequency and frequency range analysis—featuring a simultaneous minimization of TARC across the realized orthogonal states. Leaving aside the final matching optimization, it has been clearly presented how symmetries can be utilized in the design of a multiport antenna.

APPENDIX A
MATRIX OPERATORS

Many antenna metrics are expressible as quadratic forms over time-harmonic current density $\mathbf{J}(\mathbf{r})$ [51], [59], which is represented in a suitable basis $\{ \mathbf{A}(\mathbf{r}) \}$ as

$$\mathbf{J}(\mathbf{r}) \approx \sum_{n=1}^{N_b} \mathbf{l}_n \mathbf{A}(\mathbf{r})$$

with $N_b$ being the number of basis functions. The metric $p$ is then given as

$$p = \langle \mathbf{J} \cdot \mathbf{A} \rangle \approx \mathbf{I}^{\text{H}} \left[ \langle \mathbf{A}(\mathbf{r}) \mathbf{A}^\dagger(\mathbf{r}) \rangle \right] \mathbf{I} = \mathbf{I}^{\text{H}} \mathbf{A} \mathbf{A}^\dagger \mathbf{I}$$

For example, the complex power balance [62] for radiator $\Omega$ made of a good conductor reads

$$P_{\text{rad}} + P_{L} + 2j\omega(W_{m} - W_{e}) \approx \frac{1}{2} \mathbf{I}^{\text{H}} \left[ \mathbf{Z}_{0} + \mathbf{R}_{p} \right] \mathbf{I}$$

where the vacuum impedance matrix $\mathbf{Z}_{0} = \mathbf{R}_{0} + j\mathbf{X}_{0}$ is defined elementwise as

$$Z_{0, mn} = -j\omega \mu_{0} \int_{\Omega} \int_{\Omega} \mathbf{A}(\mathbf{r}) \cdot \mathbf{G}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{A}^\dagger(\mathbf{r}') \, dS \, dS'$$

with $\omega$ being angular frequency, $\mu_{0}$ being vacuum permeability, and $\mathbf{G}$ being free-space dyadic Green’s function [63]. Ohmic losses $P_{L}$ are represented via matrix $\mathbf{R}_{p}$, which, under thin-sheet approximation [64], is defined elementwise as [59]

$$R_{p, mn} = \int_{\Omega} \rho(\mathbf{r}) \mathbf{A}(\mathbf{r}) \cdot \mathbf{A}^\dagger(\mathbf{r}) \, dS$$

where $\rho$ is the surface resistivity [64]. Another notable operator [65], [66]

$$\mathbf{W} = \frac{\varepsilon \mathbf{X}_{0}}{\varepsilon}$$

gives energy stored in the near field on a device, thus determining the bandwidth potential of a radiator [67].

APPENDIX B
EXCITATION VECTOR

The excitation of obstacle $\Omega$ is realized by an incident electric field intensity $\mathbf{E}^i(\mathbf{r})$ represented elementwise in a basis (28) as

$$V_{n} = \int_{\Omega} \mathbf{A}(\mathbf{r}) \cdot \mathbf{E}^i(\mathbf{r}) \, dS$$

with $\mathbf{V} = [V_{n}]$ called the excitation vector. Incident field $\mathbf{E}^i(\mathbf{r})$ can be nonzero everywhere (then, the vector $\mathbf{V}$ generally contains nonzero entries everywhere, e.g., a plane wave), or in a limited region only (then vector $\mathbf{V}$ is sparse, e.g., a delta-gap generator or a coaxial probe).

Fig. 12. (a) Five basis functions and their orientation on a star structure. (b) Excitation vector $\mathbf{V}(1) = [1, 0, 0, 0, 0]^T$ was symmetry-adapted by (13) to four irreps: (c) $\alpha = A_1$, (d) $\alpha = A_2$, (e) $\alpha = B_1$, and (f) $\alpha = B_2$.

Considering the electric field integral equation [53] in algebraic representation (28), the current solution $\mathbf{I}$ to a problem of given excitation $\mathbf{V}$ reads

$$\mathbf{Z} \mathbf{I} = \mathbf{V}$$

where $\mathbf{Z} = \mathbf{Z}_{0} + \mathbf{R}_{p}$ is the system (impedance) matrix.

APPENDIX C
SYMMETRY ADAPTATION OF A VECTOR

The process of symmetry adaptation of a vector (13) is illustrated and explained in the example of a simple structure consisting of five RWG [68] basis functions [see Fig. 12(a)]. The delta-gap ports are connected directly to the basis functions, i.e., ports’ positions $\xi$ are identical to the numbering of basis functions. This structure belongs to the same point group $C_{3v}$ as the rectangular plate introduced in Section III and is thus invariant to the same four symmetry operations: identity (E), rotation by $\pi$ around $z$-axis ($C_{3}$), and two reflections by the $xz$ and $yz$ planes ($\sigma_{xz}, \sigma_{yz}$). The point group $C_{3v}$ consists of four irreps $\alpha \in \{A_1, A_2, B_1, B_2\}$, with dimensionality $g^{(\alpha)} = 1$ for each irrep $\alpha$.

Mapping matrix $\mathbf{C}(\mathbf{R})$, for each symmetry operation $\mathbf{R}$, is constructed so as to interlink pairs of basis functions that are mapped onto each other (respecting their orientation) via...
A given symmetry operation. Mapping matrices for the star structure from Fig. 12(a) read

\[
C(E) = \text{diag}([+1, +1, +1, +1, +1])
\]

\[
C(C_2) = \begin{bmatrix}
0 & 0 & 0 & 0 & -1 \\
0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & +1 \\
0 & 0 & 0 & 0 & +1 \\
0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & +1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1
\end{bmatrix}
\]

A general framework of how to obtain irreducible matrix representations \( D^{(\alpha)}(R) \) is described in [58, Sec. II.B]. However, for one-dimensional irreps, the matrices \( D^{(\alpha)}(R) \) can be obtained directly from the character table, see the character table for the \( C_{2v} \) point group in Table VI. These character tables are known [41] and unique for all point groups. For each irrep \( \alpha \) (row) and each symmetry operation \( R \) (column), the entry in the character table, called “a character”, is \( \chi^{(\alpha)}(R) = \text{trace}(D^{(\alpha)}(R)) \). Since the dimensionality of all irreps of the point group \( C_{2v} \) is one (\( g^{(\alpha)} = 1 \) for each irrep \( \alpha \)), values in the character table are equal to the irreducible matrix representations \( D^{(\alpha)}(R) \) (matrices of size 1 × 1).

The position of the initial port \( \xi \) can be freely chosen within the generator of the structure. Let us pick the position at \( \xi = 1 \) and construct an excitation vector \( V(1) = [1, 0, 0, 0, 0]^T \) [see Fig. 12(b)].

Once matrices \( C(R) \) and \( D^{(\alpha)}(R) \) are known, a symmetry adaptation of the excitation vector \( V(1) \) into a given species \( (\alpha, i) \) can be processed. Equation (13) can be read as: an initial port recorded in \( V(\xi) \) is mapped onto its “doublet” under symmetry operation \( R \) via mapping matrix \( C(R) \) while multiplying by a proper value from matrix \( D^{(\alpha)}(R) \) (in this case only values ±1) adds and provides a orthogonality property to the final symmetry-adapted vector \( V_{\text{adapt}}^{(\alpha,i)} \).

\[
V_{\text{adapt}}^{(A_1,1)} = [+1, -1, 0, +1, -1]^T
\]

\[
V_{\text{adapt}}^{(A_2,1)} = [+1, +1, 0, -1, -1]^T
\]

\[
V_{\text{adapt}}^{(B_1,1)} = [+1, +1, 0, +1, +1]^T
\]

\[
V_{\text{adapt}}^{(B_2,1)} = [+1, -1, 0, -1, +1]^T.
\]

These solutions are shown in Fig. 12(c)–(f). The normalization \( g^{(\alpha)} / g = 1 / 4 \) is intentionally omitted for each of solutions.

**APPENDIX D**

**TOTAL ACTIVE REFLECTION COEFFICIENT**

In order to derive (16), incident power \( P_\text{in} \) is written using incident power waves \( a \in C_{N_r} \times 1 \) at antenna ports [69] as

\[
P_\text{in} = \frac{1}{2} a^H a
\]

and the radiated power is written as [53]

\[
P_\text{rad} = \frac{1}{2} R_\text{rad}^H R_\text{rad}
\]

where \( R_\text{rad} \in \mathbb{R}_{N_r \times N_r} \) is a radiation part of impedance matrix \( Z \in \mathbb{C}_{N_r \times N_r} \) and \( I \in \mathbb{C}_{N_r \times 1} \) is a vector of expansion coefficients within the MoM solution to the electric field integral equation (EFIE) [53] (see Appendix A). Using (35), it holds that

\[
P_\text{rad} = \frac{1}{2} V^H Y^H R_\text{rad} Y V.
\]

Assume an antenna fed by ports connected to transmission lines of real characteristic impedance \( Z_0 \). Within the MoM paradigm [53], the excitation vector is

\[
V = P v
\]

where \( v \) are port voltages and matrix \( P \in \mathbb{R}_{N_r \times N_r} \) is a matrix the columns of which are the representations of separate ports. Notice that

\[
v = (P^H P)^{-1} P^H v.
\]

The incident power waves can be expressed as [69]

\[
a = \frac{1}{2V_0} (e + Z_0 y) v
\]

where \( e \) is an identity matrix and \( y \) is the admittance matrix [69] for port-like quantities.

Substituting (46), (47), and (49) into (15) results in (16). More details about TARC for multiport lossy antennas can be found in [37].

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