Periodical Solutions of Multi-Time Hamilton Equations

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Abstract

To our knowledge, there are two main references [9], [12] regarding the periodical solutions of multi-time Euler-Lagrange systems, even if the multi-time equations appeared in 1935, being introduced by de Donder. That is why, the central objective of this paper is to solve an open problem raised in [12]: what we can say about periodical solutions of multi-time Hamilton systems when the Hamiltonian is convex?

Section 1 recall well-known facts regarding the equivalence between Euler-Lagrange equations and Hamilton equations. Section 2 analyzes the action that produces multi-time Hamilton equations, and introduces the Legendre transform of a Hamiltonian together a new dual action. Section 3 proves the existence of periodical solutions of multi-time Hamilton equations via periodical extremals of the dual action, when the Hamiltonian is convex.

Key words: multi-time Hamilton action, periodical extremals, convex Hamiltonian.

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1 Classical Legendre transform of a Lagrangian

In this Section we recall the classical duality used in mechanics, based on the fact that the Legendre transform of a Lagrangian is a Hamiltonian (see [2], [7], [8], [11] for single-time theory, and [1], [3]-[6], [8]-[12] for multi-time theory).
1.1 Single-time Euler-Lagrange equations

The Lagrangian, $L : \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $(t, x, \dot{x}) \mapsto L(t, x, \dot{x})$, where $x = (x^1(t), ..., x^n(t))$ and $\dot{x} = (\dot{x}^1(t), ..., \dot{x}^n(t))$, produces the single-time Euler-Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^i} = \frac{\partial L}{\partial x^i}, \quad i = 1, ..., n. \quad (1)$$

1.2 Single-time Hamilton equations

We suppose that the Lagrangian $L$ defines the diffeomorphism $\dot{x}^i \rightarrow p_i = \frac{\partial L}{\partial \dot{x}^i}$ called the Legendre transformation.

The Legendre transformation of the Lagrangian $L$ is the Hamiltonian $H : \mathbb{R} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$, $(t, x, p) \mapsto H(t, x, p)$, $H(t, x, p) = \dot{x}^i p_i - L(t, x, \dot{x})$.

The equations (1) become

$$\frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i}, \quad i = 1, ..., n \quad (2)$$

because

$$\frac{\partial H}{\partial x^i} = p_j \frac{\partial \dot{x}^j}{\partial x^i} - \frac{\partial L}{\partial x^i} - \frac{\partial L}{\partial \dot{x}^j} \frac{\partial \dot{x}^j}{\partial x^i} = -\frac{\partial L}{\partial x^i} = -\frac{d}{dt} p_i$$

and

$$\frac{\partial H}{\partial p_i} = \dot{x}^i + p_j \frac{\partial \dot{x}^j}{\partial p_i} - \frac{\partial L}{\partial x^i} \frac{\partial \dot{x}^j}{\partial p_i} = \dot{x}^i. \quad (3)$$

So, the second order ODEs (1) in an $n$-dimensional space are equivalent with the first order ODEs (2) in a $2n$-dimensional space.

The Hamiltonian $H$ associated to $L$ was built in 1834 by Hamilton. With this function, the Euler-Lagrange equations (1) achieve a symmetrical structure in the form of the Hamilton equations (2).

More than that, in the case when the Lagrangian $L$ is an autonomous function (does not explicitly depend on $t$), the function $H$ is a first integral for the Hamilton equations, because

$$\frac{dH}{dt} = \frac{d}{dt} (p_i \dot{x}^i - L) = \frac{dp_i}{dt} \dot{x}^i + p_i \frac{d\dot{x}^i}{dt} - \frac{\partial L}{\partial t} - \frac{\partial L}{\partial x^i} \frac{dx^i}{dt} - \frac{\partial L}{\partial \dot{x}^i} \frac{d\dot{x}^i}{dt} = -\frac{\partial L}{\partial t} = 0.$$
Using the symplectic structure \( J = \begin{pmatrix} 0 & \delta^i_j \\ -\delta^i_j & 0 \end{pmatrix}, (\delta^i_j) \in M_n(R) \), we may write the Hamilton equations in the matrix form
\[
J \begin{pmatrix} \dot{x}^j \\ \dot{p}_i \end{pmatrix} + \begin{pmatrix} \frac{\partial H}{\partial x^i} \\ \frac{\partial H}{\partial p_i} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

### 1.3 Multi-time Euler-Lagrange equations

We consider the multi-time variable \( t = (t^1, ..., t^p) \in \mathbb{R}^p \), the functions \( x^i : \mathbb{R}^p \to \mathbb{R}, (t^1, ..., t^p) \to x^i(t^1, ..., t^p), i = 1, ..., n \), and we denote \( x^i_\alpha = \frac{\partial x^i}{\partial t^\alpha}, \alpha = 1, ..., p \). The Lagrange function

\[
L : \mathbb{R}^{p+n+np} \to \mathbb{R}, \quad (t^\alpha, x^i, x^i_\alpha) \to L(t^\alpha, x^i, x^i_\alpha)
\]

gives the Euler-Lagrange equations

\[
\frac{\partial}{\partial t^\alpha} \frac{\partial L}{\partial x^i_\alpha} = \frac{\partial L}{\partial x^i}, \quad i = 1, ..., n, \quad \alpha = 1, ..., p
\]

(second order PDEs system on the n-dimensional space).

### 1.4 Multi-time Hamilton equations

In 1935 de Donder [1] obtained the Hamilton equations in the multi-time case by using the partial derivatives

\[
p^\alpha_k = \frac{\partial L}{\partial x^k_\alpha}
\]

and the Hamiltonian \( H = p^\alpha_k x^k_\alpha - L \). If \( L \) satisfies some conditions, then the system (4) defines a \( C^1 \) bijective transformation \( x^i_\alpha \to p^\alpha_i \), called the Legendre transformation for the multi-time case. By this transformation we have:

\[
\frac{\partial H}{\partial p^\alpha_i} = x^i_\alpha + p^\beta_k \frac{\partial x^k_\alpha}{\partial p^\beta_i} - \frac{\partial L}{\partial x^k_\beta} \frac{\partial x^k_\beta}{\partial p^\alpha_i} = x^i_\alpha
\]
\[
\frac{\partial H}{\partial x^i} = p^\alpha_k \frac{\partial x^k}{\partial x^i} - \frac{\partial L}{\partial x^i} \frac{\partial x^k}{\partial x^\alpha} - \frac{\partial L}{\partial x^\alpha} \frac{\partial x^k}{\partial x^i} = -\frac{\partial L}{\partial x^i}.
\]

The \(np + n\) Hamilton equations \(\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p^\alpha_i}, \frac{\partial p^\alpha_i}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i}\) (summation after \(\alpha\), \(i = 1,...,n, \alpha = 1,...,p\) are first order PDEs on the space \(R^{n+p}\), equivalent to the Euler-Lagrange equations on \(R^n\).

### 1.5 The conservation of energy-moment tensor

The multi-time Hamiltonian is not conserved on the solutions of the multi-time Hamilton equations even if the Lagrangian is autonomous (does not depend on \(t^\alpha\)). But, we may observe that the Lagrangian \(L\) defines the energy-moment tensor

\[
T^\alpha_\beta = x^i_\beta \frac{\partial L}{\partial x^i_\alpha} - \delta^\alpha_\beta L,
\]

whose divergence is

\[
\frac{\partial}{\partial t^\alpha} T^\alpha_\beta = x^i_\beta \frac{\partial}{\partial x^i_\alpha} x^j_\alpha - \delta^\alpha_\beta \left( \frac{\partial}{\partial t^\alpha} x^i_\alpha + \frac{\partial}{\partial x^i_\alpha} x^j_\gamma \frac{\partial}{\partial x^\gamma_\beta} \right) =
\]

\[
= x^i_\beta \frac{\partial p^\alpha_i}{\partial t^\alpha} + x^i_\beta \frac{\partial p^\alpha_i}{\partial t^\alpha} - \frac{\partial L}{\partial x^i_\beta} \frac{\partial x^i_\alpha}{\partial t^\alpha} - \frac{\partial L}{\partial x^i_\beta} \frac{\partial x^i_\alpha}{\partial t^\alpha} =
\]

\[
= x^i_\beta \left( \frac{\partial p^\alpha_i}{\partial t^\alpha} - \frac{\partial L}{\partial x^i_\beta} \right) - \frac{\partial L}{\partial t^\beta} = -\frac{\partial L}{\partial t^\beta}.
\]

So, the energy-moment tensor \(T^\alpha_\beta\) is conserved on the solutions of the multi-time Hamilton equations, if the Lagrangian \(L\) is autonomous.

### 2 Legendre transform of a multi-time Hamiltonian. Dual action

We may write the multi-time Hamilton equations in the form ([9],[12])

\[
\delta^\alpha_\beta \delta^i_j \frac{\partial p^\beta_i}{\partial t^\alpha} + \frac{\partial H}{\partial x^j} = 0, \quad -\delta^\alpha_\beta \delta^i_j \frac{\partial x^j_i}{\partial t^\alpha} + \frac{\partial H}{\partial p^\beta_i} = 0
\]
or

\[
(\delta \otimes J) \begin{pmatrix}
\frac{\partial x^j}{\partial t^a} \\
\frac{\partial p^i_{\alpha}}{\partial t^a} \\
\frac{\partial p^j_{\beta}}{\partial t^a}
\end{pmatrix} + \begin{pmatrix}
\frac{\partial H}{\partial x^j} \\
\frac{\partial H}{\partial p^i_{\alpha}} \\
\frac{\partial H}{\partial p^j_{\beta}}
\end{pmatrix} = \begin{pmatrix} 0 \\
0 \end{pmatrix},
\]

(5)

\(j = 1, ..., n; \; i = 1, ..., n; \; \alpha = 1, ..., p; \; \beta = 1, ..., p,\) where

\[
\delta \otimes J = \begin{pmatrix}
0 & \delta^\alpha_\beta \delta_i^j \\
-\delta^\alpha_\beta \delta_i^j & 0
\end{pmatrix}
\]

is a polysymplectic structure acting on \(R^{np+np^2}\) with values in \(R^{np}\).

Let \(T_0 = [0, T^1] \times [0, T^2] \times \cdots \times [0, T^p]\) and \(dt = dt^1 \cdots dt^p\) the volume element. The action \(\Psi\), whose Euler-Lagrange equations are the Hamilton equations, is

\[
\Psi (u) = \int_{T_0} \mathcal{L} \left( t, u(t), \frac{\partial u}{\partial t} \right) dt, \quad \text{where} \quad u = (x, p),
\]

\[
\mathcal{L} \left( t, u(t), \frac{\partial u}{\partial t} \right) = -\frac{1}{2} \left( \frac{\partial p^\alpha_{i}}{\partial t^\alpha} x^j - \frac{\partial x^i}{\partial t^\alpha} p^\alpha_{i} \right) - H (t, x, p) =
\]

\[
= -\frac{1}{2} \left( \frac{\partial p^\alpha_{i}}{\partial t^\alpha}, -\frac{\partial x^j}{\partial t^\beta} \right) \begin{pmatrix}
\delta_{ij} & 0 \\
0 & \delta^\alpha_\beta \delta_i^j
\end{pmatrix} \begin{pmatrix}
x^i \\
p^\alpha_{i}
\end{pmatrix} - H (t, x(t), p(t)) =
\]

\[
= -\frac{1}{2} G \left( (\delta \otimes J) \frac{\partial u}{\partial t}, u \right) - H (t, u),
\]

where the scalar product is represented by the matrix \(G = \begin{pmatrix}
\delta_{ij} & 0 \\
0 & \delta^\alpha_\beta \delta_i^j
\end{pmatrix}\) (standard Riemannian metric from \(R^{np}\)). Indeed, the Euler-Lagrange equations produced by \(\mathcal{L}\) are

\[
\frac{1}{2} \frac{\partial p^\alpha_{i}}{\partial t^\alpha} = -\frac{1}{2} \frac{\partial p^\alpha_{i}}{\partial t^\alpha} - \frac{\partial H}{\partial x^i}, \quad \text{i.e.,} \quad \frac{\partial p^\alpha_{i}}{\partial t^\alpha} = -\frac{\partial H}{\partial x^i}
\]

and

\[
\frac{\partial x^i}{\partial t^\alpha} = \frac{\partial H}{\partial p^\alpha_{i}}.
\]
It is well-known that the Legendre transform of a Lagrangian $L$ is the Hamiltonian $H$. Besides this classical duality, we introduce a duality based on the Legendre transform of a Hamiltonian $H$.

The Legendre transformation $H^* (t, \cdot)$ of the Hamiltonian $H (t, \cdot)$ is defined by the implicit formula

$$H^* (t, v) = G (u, v) - H (t, u), \quad v = \nabla_u H (t, u),$$

when $\nabla H (t, u)$ is invertible. Automatically,

$$u = \nabla_v H^* (t, v),$$

so that $(\nabla H)^{-1} = \nabla H^*$. If $H (t, \cdot)$ is of class $C^1$, strictly convex and has the property $\frac{H (t, u)}{|u|} \to \infty$ for $|u| \to \infty$, then $H^* (t, \cdot) \in C^1 (R^{n+np}, R)$, according [2]. The norm $|u|$ is coming from the scalar product (Riemannian metric) $G$.

The Legendre transform $H^* (t, \cdot)$ of $H (t, \cdot)$ determines a new duality. Indeed, if we write $u = (x^i, p^\alpha_i)$, the Hamiltonian equations can be written in the compact form (5). Setting $v = (y^i, q^\alpha_i)$ defined by $y^i = -x^i, q^\alpha_i = -p^\alpha_i$, we obtain

$$(\delta \otimes J) \frac{\partial v}{\partial t} = \left( \begin{array}{c} \frac{\partial q^\alpha_j}{\partial t^\alpha} \\ \frac{\partial y^j}{\partial t^\beta} \end{array} \right) = \left( \begin{array}{c} -\frac{\partial p^\alpha_j}{\partial t^\alpha} \\ \frac{\partial x^j}{\partial t^\beta} \end{array} \right) = -(\delta \otimes J) \frac{\partial u}{\partial t} = \nabla H (t, u).$$

or equivalently

$$u = \nabla H^* (t, (\delta \otimes J) \frac{\partial v}{\partial t}).$$

if the Legendre transform $H^* (t, \cdot)$ of $H (t, \cdot)$ exists. On the other hand

$$G \left( (\delta \otimes J) \frac{\partial u}{\partial t}, u \right) = \left( \begin{array}{c} \frac{\partial p^\alpha_j}{\partial t^\alpha} - \frac{\partial x^j}{\partial t^\beta} \end{array} \right) \left( \begin{array}{cc} \delta_{ij} & 0 \\ 0 & \delta^{\alpha \beta} \delta_{ij} \end{array} \right) \left( \begin{array}{c} x^i \\ p^\alpha_i \end{array} \right) = \frac{\partial p^\alpha_i}{\partial t^\alpha} x^i - \frac{\partial x^i}{\partial t^\alpha} p^\alpha_i$$

and consequently,

$$G \left( (\delta \otimes J) \frac{\partial v}{\partial t}, v \right) = \frac{\partial q^\alpha_j}{\partial t^\alpha} y^i - \frac{\partial x^i}{\partial t^\alpha} q^\alpha_i = G \left( (\delta \otimes J) \frac{\partial u}{\partial t}, u \right).$$

6
Using the identity
\[
\frac{1}{2} G \left( - (\delta \otimes J) \frac{\partial u}{\partial t}, u \right) = \frac{1}{2} G \left( (\delta \otimes J) \frac{\partial v}{\partial t}, v \right) - G \left( (\delta \otimes J) \frac{\partial v}{\partial t}, u \right)
\]
\[
= \frac{1}{2} G \left( (\delta \otimes J) \frac{\partial v}{\partial t}, v \right) - G \left( (\delta \otimes J) \frac{\partial v}{\partial t}, u \right),
\]
the action
\[
\Psi (u) = \int_{T_0}^{T} \left[ \frac{1}{2} G \left( - (\delta \otimes J) \frac{\partial u}{\partial t}, u \right) - H \left( t, u \left( t \right) \right) \right] dt.
\]
can be written as
\[
\Psi (u) = \int_{T_0}^{T} \left[ \frac{1}{2} G \left( (\delta \otimes J) \frac{\partial v}{\partial t}, v \left( t \right) \right) + G \left( (\delta \otimes J) \frac{\partial v}{\partial t}, u \left( t \right) \right) - H \left( t, u \left( t \right) \right) \right] dt.
\]
In this way it appears the dual action
\[
\Phi (v) = \int_{T_0}^{T} \left[ \frac{1}{2} G \left( (\delta \otimes J) \frac{\partial v}{\partial t}, v \left( t \right) \right) + H^* \left( t, (\delta \otimes J) \frac{\partial v}{\partial t} \right) \right] dt. \tag{6}
\]

3 Extremals of dual action and multi-time Hamilton equations

In this section we describe some connections between the periodical extremals of the multi-time Hamilton dual action and the periodical solutions of the multi-time Hamilton equations.

The dual action, \( \Phi (v) \), is defined on a suitable space of \( T \)-periodic functions, where \( T = (T^1, ..., T^p) \). It uses the Lagrangian
\[
\mathcal{L}^* \left( t, v \left( t \right), \frac{\partial v}{\partial t} \right) = \frac{1}{2} \left( (\delta \otimes J) \frac{\partial v}{\partial t}, v \left( t \right) \right) + H^* \left( t, (\delta \otimes J) \frac{\partial v}{\partial t} \right).
\]
Another useful property of the dual action is \( \Phi (v + c) = \Phi (v), \forall c \in \mathbb{R}^{n+np} \). Indeed, for
\[
v^* = (\delta \otimes J) \frac{\partial v}{\partial t} \tag{7}
\]
we have

\[ \Phi (v + c) = \int_{T_0} \left[ (\delta \otimes J) \frac{\partial (v + c)}{\partial t}, v + c \right] dt = \int_{T_0} \left[ (\delta \otimes J) \frac{\partial v}{\partial t}, v \right] dt + H^* (t, (v + c)^*) dt = \int_{T_0} \left[ (\delta \otimes J) \frac{\partial v}{\partial t}, v \right] dt + H^* (t, v^*) dt = \Phi (v). \]

Then we may restrict the search of the critical points of \( \Phi (v) \) to the set

\[ W_{1,2}^{T} = \left\{ v \in W_{1,2}^{T} \mid \int_{T_0} v(t) dt = 0 \right\}. \]

There are situations when it is much easier to find the critical points of the dual action \( \Phi \) than those of the action \( \Psi \). This possibility was noticed by Clark in case of single-time theory.

Between the periodical critical points of the dual action \( \Phi \) and the periodical solutions of Hamilton multi-time equations, there is a close connection, as we will show in the following result.

**Theorem.** Let \( H : T_0 \times R^{n+np} \to R, (t, u) \to H(t, u) \) be a Hamiltonian, measurable in \( t \) for any \( u \in R^{n+np} \) and strictly convex and continuously differentiable in \( u \) for any \( t \in T_0 \). Suppose there are the constants \( \alpha > 0, \delta > 0 \), and the functions \( \beta, \gamma \in L^2(T_0, R^+) \) so that for any \( u \in R^{n+np} \) and any \( t \in T_0 \), to have

\[ \frac{\delta}{2} \left| u \right|^2 - \beta(t) \leq H(t, u) \leq \frac{\alpha}{2} \left| u \right|^2 + \gamma(t). \] \[(8)\]

Then, the dual action \( \Phi \) is continuously differentiable on \( W_{1,2}^{T} \).

**Proof.** From the hypothesis, the Hamiltonian \( H(t, u) \) is strictly convex in \( u \) and has the property \( \frac{H(t, u)}{|u|} \to \infty \) when \( |u| \to \infty \). Then it follows (see [2]) that \( H^* (t, .) \in C^1 (R^{n+np}, R) \). From the relation (8) and the equality

\[ \max_{u \in R^{n+np}} \left( (v^*, u) - \frac{\alpha}{2} |u|^2 - \gamma \right) = \frac{\alpha^{-1}}{2} |v^*|^2 - \gamma, \] \[(9)\]
when $\alpha > 0$, we obtain the inequalities
\[ \frac{\alpha^{-1}}{2} |v^*|^2 - \gamma (t) \leq H^* (t, v^*) \leq \frac{\delta^{-1}}{2} |v^*|^2 + \beta (t) \] (10)

Taking into account the Proposition 2.2 from [2, page 34], with $N = n + np$ (and $n, p = q = 2$), we obtain
\[ |\nabla H^* (t, v^*)| \leq (|v^*| + \beta (t) + \gamma (t)) \delta^{-1} + 1 \leq C_1 |v^*| + C_2 (\beta (t) + \gamma (t) + 1), \] (11)
where $C_1, C_2$ are positive constants. As $\beta + \gamma + 1 \in L^2(T_0, R^+)$, from (10) and (11), the Lagrangian
\[ \mathcal{L}^* \left( t, v(t), \frac{\partial v}{\partial t} \right) = \frac{1}{2} \left( (\delta \otimes J) \frac{\partial v}{\partial t}, v \right) + H^* (t, v^*) \]

satisfies conditions similar to those of Theorem 1.4 from [2, page 10]. By consequence, the dual action $\Phi$ is continuously differentiable on $W^{1,2}_T$, and on $\tilde{W}^{1,2}_T$.

**Theorem.** The same hypothesis as in the previous theorem. If $v \in \tilde{W}^{1,2}_T$ is a critical point for the action $\Phi$, periodical, with the period $T = (T^1, ..., T^p)$, then the function $w (t) = \nabla H^* (t, v^* (t))$ verifies the problem
\[ (\delta \otimes J) \frac{\partial w}{\partial t} + \nabla H^* (t, w (t)) = 0, \quad w (0) = w (T) \]
and
\[ v(t) = w(t) - \frac{1}{T^1...T^p} \int_{T_0} w (t) \, dt. \]

**Proof.** We take $w = (z^i, r^\alpha_i), \quad v^* = (v^* i, v^* \alpha_i)$. From the definition of the dual action, we have
\[ v = (-x^i, -p^\alpha_i), \quad v^* = \left( -\frac{\partial p^\alpha_i}{\partial t^\alpha}, \frac{\partial x^i}{\partial t^i} \right). \]
Then
\[ \mathcal{L}^* \left( t, v(t), \frac{\partial v}{\partial t} \right) = \frac{1}{2} \left( -\frac{\partial x^i}{\partial t^\alpha} p^\alpha_i + \frac{\partial p^\alpha_i}{\partial t^\alpha} x^i \right) + H^* \left( t, -\frac{\partial p^\alpha_i}{\partial t^\alpha}, \frac{\partial x^i}{\partial t^i} \right). \]
If the function $v$ is a critical point for the action produced by $\mathcal{L}^*$, then the Euler-Lagrange equations
\[ \frac{\partial}{\partial t^\alpha} \frac{\partial \mathcal{L}^*}{\partial x^i} = \frac{\partial \mathcal{L}^*}{\partial v^* i}, \quad \text{where} \quad x^i = \frac{\partial x^i}{\partial t^i}, \]
are verified. So, we obtain
\[ \frac{-1}{2} \frac{\partial p^\alpha_i}{\partial t^\alpha} + \frac{\partial}{\partial t^\alpha} \frac{\partial H^*}{\partial v^* i} = \frac{1}{2} \frac{\partial p^\alpha_i}{\partial t^\alpha}. \]
or
\[ \frac{\partial}{\partial t^\alpha} \frac{\partial H^*}{\partial v_i^\alpha} = \frac{\partial p_i^\alpha}{\partial t^\alpha}. \]

Because \( t(z^i, r_i^\alpha) = \nabla H^*(t, v^i, v_i^\alpha) \), we find \( r_i^\alpha = \frac{\partial H^*}{\partial v_i^\alpha} \) and then
\[ \frac{\partial r_i^\alpha}{\partial t^\alpha} = \frac{\partial p_i^\alpha}{\partial t^\alpha}. \] (12)

On the other side, the equality \( w = \nabla H^* \left( t, -\frac{\partial p_i^\alpha}{\partial t^\alpha}, \frac{\partial x^i}{\partial t^\alpha} \right) \) and the duality produce
\[ t \left( -\frac{\partial p_i^\alpha}{\partial t^\alpha}, \frac{\partial x^i}{\partial t^\alpha} \right) = \nabla H(t, w). \]

By using the relation (12) we have \( t \left( -\frac{\partial r_i^\alpha}{\partial t^\alpha}, \frac{\partial z^i}{\partial t^\alpha} \right) = \nabla H(t, w) \), and hence
\[ -\frac{\partial r_i^\alpha}{\partial t^\alpha} = \nabla z^i H(t, w). \] (13)

From the definition of the function \( H \) we find
\[ \frac{\partial z^i}{\partial t^\alpha} = \frac{\partial H}{\partial r_i^\alpha}(t, z^i, r_i^\alpha). \] (14)

From the relations (13) and (14) we obtain
\[ t \left( -\frac{\partial r_i^\alpha}{\partial t^\alpha}, \frac{\partial z^i}{\partial t^\alpha} \right) = \nabla H(t, w) \text{ or } (\delta \otimes J) \frac{\partial w}{\partial t} = \nabla H(t, w). \]

Consequently, \( w \) is the solution of the equation \( (\delta \otimes J) \frac{\partial w}{\partial t} + \nabla H^*(t, w) = 0 \).

We consider \( v \) a critical point for the dual action \( \Phi(v) \) having mean value zero.

Then, also \( v_c = v + c \) is a critical point for \( \Phi \) because \( \int_{T_0} \left( (\delta \otimes J) \frac{\partial v}{\partial t}, c \right) dt = 0 \) (see the periodicity of \( v \)) and
\[ \Phi(v + c) = \int_{T_0} \left( (\delta \otimes J) \frac{\partial (v + c)}{\partial t}, v + c \right) + H^*(t, (v + c)^*) dt = \]

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\[
= \int_{T_0} \left( (\delta \otimes J) \frac{\partial v}{\partial t}, v \right) + \left( (\delta \otimes J) \frac{\partial v}{\partial t}, c \right) + H^* (t, v^*) \, dt = \\
= \int_{T_0} \left( (\delta \otimes J) \frac{\partial v}{\partial t}, v \right) + H^* (t, v^*) \, dt = \Phi (v).
\]

Because \( v \) has the mean value zero, i.e., \( \int_{T_0} v (t) \, dt = 0 \), we find
\[
\int_{T_0} (v_c (t) - c) \, dt = 0,
\]
and hence
\[
c = \frac{1}{T^1 \ldots T^p} \int_{T_0} u_c (t) \, dt.
\]
If \( v_c \) is a critical point for \( \Phi \), then \( u_c = -v_c \) is a critical point for \( \Psi \), i.e., it verifies the equation \( (\delta \otimes J) \frac{\partial u_c}{\partial t} + \nabla H (t, u_c) = 0 \). By duality,
\[
u_c = \nabla H^* \left( t, (\delta \otimes J) \frac{\partial u_c}{\partial t} \right) = \nabla H^* \left( t, - (\delta \otimes J) \frac{\partial u_c}{\partial t} \right) = \nabla H^* (t, v^*) = w.
\]
Then \( v(t) = v_c (t) - c = w(t) - \frac{1}{T^1 \ldots T^p} \int_{T_0} w (t) \, dt \). Because \( v \) has the period \( T = (T^1, ..., T^p) \), we have \( v (0) = v (T) \), so \( w (0) = w (T) \).

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