More on counterterms in the gravitational action and anomalies

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Abstract

The addition of boundary counterterms to the gravitational action of asymptotically anti-de Sitter spacetimes permits us to define the partition function unambiguously, without background subtraction. We show that the inclusion of p-form fields in the gravitational action requires the addition of further counterterms which we explicitly identify. We also relate logarithmic divergences in the action dependent on the matter fields to anomalies in the dual conformal field theories. In particular we find that the anomaly predicted for the correlator of the stress energy tensor and two vector currents in four dimensions agrees with that of the $\mathcal{N} = 4$ superconformal $SU(N)$ gauge theory.

I. INTRODUCTION

The Maldacena conjecture [1], [2], [3], [4] asserts that there is an equivalence between a gravitational theory in a $(d+1)$-dimensional anti-de Sitter spacetime and a conformal field theory in a $d$-dimensional spacetime which can in some sense be viewed as the boundary of the higher dimensional spacetime. The formulation of this correspondence is made precise by equating the partition functions of the two theories

$$Z_{\text{AdS}}[\Phi_i] = Z_{\text{cft}}[\Phi^0_i].$$

(1.1)

In the supergravity theory, the fields $\Phi^0_i$ correspond to boundary data for bulk fields $\Phi_i$ which propagate in the $(d+1)$-dimensional spacetime. However, on the field theory side, these fields correspond to external source currents coupled to various operators.

An interesting consequence of the Maldacena conjecture is the natural definition of the gravitational action for asymptotically anti-de Sitter spacetimes without reference to a background [5], [6]. The consideration of the gravitational action has a long history, particularly in the context of black hole thermodynamics [7]. One difficulty that has always plagued this approach is that the gravity action diverges. The traditional approach to this problem
is to use a background subtraction whereby one compares the action of a spacetime with that of a reference background, whose asymptotic geometry matches that of the solution in some well-defined sense. However, this approach breaks down when there is no appropriate or obvious background.

The AdS/CFT correspondence tells us that if, as we expect, the dual conformal field theory has a finite partition function, then to make sense of (1.1) we must be able to remove the divergences of the gravitational action without background subtraction. The framework for achieving this is by defining local counterterms on the boundary [3], [8], [9], [10]. Consider the Einstein action in $(d+1)$ dimensions

$$I = -\frac{1}{16\pi G_{d+1}} \int_M d^{d+1}x \sqrt{g} (R + d(d-1)l^2) - \frac{1}{8\pi G_{d+1}} \int_N \sqrt{\gamma} K$$

(1.2)

where $G_{d+1}$ is the Newton constant and $R$ is the Ricci scalar. As usual a boundary term must be included for the equations of motion to be well-defined [7], with $K$ the trace of the extrinsic curvature of the $d$-dimensional boundary $N$ embedded into the $(d+1)$-dimensional manifold $M$. Then provided that the metric near the conformal boundary can be expanded in the asymptotically anti-de Sitter form

$$ds^2 = \frac{dx^2}{l^2 x^2} + \frac{1}{x^2} \gamma_{ij} dx^idx^j,$$

(1.3)

where in the limit $x \to 0$ the metric $\gamma$ is non-degenerate, we may remove divergent terms in the action by the addition of a counterterm action dependent only on $\gamma$ and its covariant derivatives of the form [8], [11]

$$I_{ct} = \frac{1}{8\pi G_{d+1}} \int_N d^d x \sqrt{\gamma} \{(d-1)l + \frac{1}{2(d-2)l} R(\gamma) + \frac{1}{2l^3(d-4)(d-2)^2} (R_{ij}(\gamma)R^{ij}(\gamma)$$

$$- \frac{d}{4(d-1)} R(\gamma)^2 + ....\}.$$

(1.4)

$R(\gamma)$ and $R_{ij}(\gamma)$ are the Ricci scalar and the Ricci tensor for the boundary metric respectively. Combined these counterterms are sufficient to cancel divergences for $d \leq 6$, with several exceptions. Firstly, in even dimensions $d = 2n$ one has logarithmic divergences in the partition function which can be related to the Weyl anomalies in the dual conformal field theory [9]. Secondly, if the boundary metric becomes degenerate one can no longer remove divergences by counterterm regularisation [12]; this is a manifestation of the fact that the dual conformal field theory does not have a finite partition function in the degenerate limit.

The purpose of this paper is to discuss a third case in which one needs to consider regularisation more carefully. Much of the discussion of the gravitational action to date has concerned the case where the only boundary data in (1.1) stems from the gravitational field. If there are matter fields on $M$ additional counterterms may be needed to regulate the action. The addition of scalar fields to the bulk action has been considered in several recent papers [13], [14], but the focus of our consideration here will be p-form fields. We will also briefly discuss scalar fields with potentials derived from maximal gauged supergravity theories in general dimensions.

The motivation for our work is to complete the construction of the theoretical tools required to investigate gravitational physics in anti-de Sitter backgrounds. With this in
mind, we are particularly interested in whether one needs counterterms to define the action for charged black brane solutions such as those recently constructed in [15], [16], [17], [18]. It turns out that for many of the gauged supergravity solutions constructed so far, such as charged black hole solutions in five dimensions [19], [20], one does not need any further counterterms to (1.4). However, one does need to include further counterterms both for charged black holes in four dimensions and more generally for magnetically charged branes in higher dimensions.

The related logarithmic terms that arise in the gravitational action when \(d\) is even have an interpretation in terms of anomalies arising from mixing conformal and other symmetries in the dual conformal field theories. Although the aim of this paper is not to reproduce in detail the anomalies in the dual conformal field theory we will discuss one particular case, namely the anomaly in the correlator of the stress energy tensor and two vector currents in four dimensions.

The plan of this paper is as follows. In \(\S\) II we summarise the matter dependent counterterms that are found to be required. In \(\S\) III we consider the analysis of counterterms for \(p\)-form fields. In \(\S\) IV we discuss scalar fields in the context of gauged supergravity theories. In \(\S\) V we consider in more detail gauged supergravity in seven dimensions. In \(\S\) VI we briefly consider the related anomalies in the dual conformal field theories.

II. COUNTERTERM REGULARISATION OF THE PARTITION FUNCTION

We will analyse here first a truncated action of a generic gauged supergravity theory, such that the Einstein action (1.2) is extended to include a scalar field \(\phi\) and a \(p\)-form \(F_p\) such that

\[
I_{\text{bulk}} = -\frac{1}{16\pi G_{d+1}} \int_M d^{d+1}x \sqrt{g} \left[ R - \frac{1}{2} (\partial \phi)^2 + V(\phi) - \frac{1}{2p} e^{\alpha \phi} F_p^2 \right].
\]  

(2.1)

We will assume the potential \(V(\phi)\) to be of the form arising in \((d + 1)\)-dimensional maximal (or for \(d=5\) the nearest to maximal that is known) supergravity. This restriction includes most interesting known solutions since the associated potentials fall into this category. Most of our analysis refers to \(p = 2\) since the main case of interest for \(p > 2\) fields is seven-dimensional gauged supergravity and the latter does not admit a truncation of this type. We will consider the analysis for seven-dimensional supergravity separately in \(\S\) IV.

It is perhaps useful to summarise here the results of the analysis of the next sections. In addition to counterterms depending only on the induced boundary metric (1.4) and logarithmic divergences relating to the Weyl anomalies of the dual theories in \(d = 2n\) we find the following. Restricting the scalar potential as above and as defined in more detail in \(\S\) IV, we find that there is a scalar field divergence only when \(d = 3\), which can be removed by a counterterm of the form

\[
I_{\text{ct}} = \frac{5l}{256\pi G_4} \int d^3x \sqrt{\gamma(\phi)^2}.
\]

(2.2)

For \(p\)-form fields, there will be no divergences for \(d < 2p\) but in \(d = 2p\) there will be a logarithmic divergence of the action.
\[ I_{\text{log}} = \frac{1}{32\pi p G_{d+1} l} \ln \epsilon \int d^{2p} x \sqrt{\gamma^0 (F_p^0)^2}, \]  

(2.3)

where \( F_p^0 \) is the induced field on the boundary, whilst for \( d > 2p \) we must include the counterterm

\[ I_{\text{div}} = \frac{1}{64\pi^2 p^2 G_{d+1} l (d-2p)} \int d^d x e^{\phi} (F_p)^2. \]  

(2.4)

There will be additional terms in the anomaly for \( d = 2p + 2n \) depending on derivatives of \( F_p \) and its coupling to the curvature, and correspondingly further counterterms for \( d > 2p + 2n \).

We derive these explicitly for \( p = 2 \) (3.2) in the context of Einstein-Maxwell theories and, in §V, discuss the absence of these terms in the context of seven-dimensional gauged supergravity.

### III. P-FORM FIELDS

Suppose that a \((d + 1)\)-dimensional manifold \( M \) of negative curvature has a regular \( d \)-dimensional conformal boundary \( N \) in these of [21]. Then in the neighbourhood of the boundary \( N \) we will assume that the metric can be expressed in the form (1.3) with the induced hypersurface metric \( \gamma \) admitting the expansion

\[ \gamma_{ij} = \gamma^0_{ij} + x^2 \gamma^2_{ij} + x^4 \gamma^4_{ij} + x^6 \gamma^6_{ij} \ldots \]  

(3.1)

If \( M \) is an Einstein manifold with negative cosmological constant, then according to [10], [22] such an expansion always exists. For solutions of gauged supergravity theories with matter fields, demanding that this expansion is well-defined as \( x \to 0 \) will impose conditions on the matter fields induced on \( N \). Note that when \( d \) is even there will in general also be a logarithmic term \( h^d \) appearing at order \( x^d \). However, it can be shown that \( \text{Tr}[(\gamma^0)^{-1} h^d] \) vanishes identically; it will not contribute to the action and can be neglected from here on.

In what follows we will use repeatedly the Ricci tensor for the metric (1.3) which has the following components

\[ \mathcal{R}_{xx} = -\frac{d}{x^2} - \frac{1}{2} \{ \text{Tr}(\gamma^{-1} \partial x^2 \gamma) - \frac{1}{x} \text{Tr}(\gamma^{-1} \partial x \gamma) - \frac{1}{2} \text{Tr}(\gamma^{-1} \partial x \gamma^{-1} \partial x \gamma) \} \]

\[ \mathcal{R}_{ij} = -\frac{dl^2 \gamma_{ij}}{x^2} - l^2 \left\{ \frac{1}{2} \partial x^2 \gamma - \frac{1}{2x} \partial x \gamma - \frac{1}{2} (\partial x \gamma)^{-1} (\partial x \gamma) + \frac{1}{4} (\partial x \gamma) \text{Tr}(\gamma^{-1} \partial x \gamma) \right\} + \mathcal{R}(\gamma) l^{-2} - \frac{(d-2)}{2x} \gamma (\partial x \gamma) \]

(3.2)

\[ \mathcal{R}_{xi} = \frac{1}{2} (\gamma^{-1})^{jk} [\nabla_i \gamma_{jk,x} - \nabla_k \gamma_{ij,x}] , \]

where \( \nabla \) is the covariant derivative associated with \( \gamma \). Let us include just a minimally coupled p-form field into the Einstein action (we will consider the scalar field case separately in the next section) so that

\[ I_{\text{bulk}} = -\frac{1}{16\pi G_{d+1}} \int_M d^{d+1} x \sqrt{g} \left[ \mathcal{R} + d(d-1)l^2 - \frac{1}{2p} F_p^2 \right] , \]  

(3.3)
where for the present we have ignored possible Chern-Simons terms. This action is a consistent truncation of gauged supergravity theories in $d < 6$ but not for $d = 6$ itself. However, it is interesting to consider cosmological Einstein-Maxwell theory in $d = 6$ in its own right and we shall do so here. Gauged supergravity in seven dimensions is discussed in §V. The equations of motion derived from the action (3.3) are

$$R_{mn} = -dl^2g_{mn} + \frac{1}{2}F_{(p)mq_{1}...q_{p-1}} F_{(p)m}^{q_{1}...q_{p-1}} + \frac{(1 - p)}{2p(d - 1)} F_{p}^{2}g_{mn};$$

$$\mathcal{R} = -d(d + 1)l^2 + \frac{(d + 1 - 2p)}{2p(d - 1)} F_{p}^{2};$$

$$d * F_{p} = 0; \quad dF_{p} = 0.$$

Let us assume that in the vicinity of the conformal boundary the $p$-form field can be expanded as a power series in $x$ as

$$F_{p} = F_{p}^{0} + xdx \wedge A_{p-1}^{1} + x^{2}F_{p}^{2} + x^{2}dx \wedge A_{p-1}^{2} + x^{3}F_{p}^{3}....$$

(3.5)

where $G_{k}^{i}$ is a $k$-form dependent only on $x^{i}$. One can justify this form for the expression retrospectively by demanding that one can satisfy all field equations. For example, having explicitly chosen the asymptotic form of the metric we can’t have terms in this expansion which diverge as $x \to 0$, as can be seen by inspection of the Einstein equations.

Now the key point is that to preserve the asymptotic form of the metric we will have to restrict the leading order $p$-form contribution to the Ricci scalar to be of order $x^{2p}$ or smaller. If it were any larger, the leading order form of the metric would be changed. Since the bulk action includes a term of the form

$$I_{\text{bulk}} \sim \int_{M} d^{d+1}x \sqrt{g} F_{p}^{2};$$

(3.6)

there will be an induced infra-red divergence in the action only if $(d + 1) > 2p$. To justify why we can neglect Chern-Simons terms in (3.4) note that if we have terms in the action for $d = (3p - 2)$ of the form

$$I_{\text{CS}} = \int_{M} F_{p} \wedge F_{p} \wedge A_{p-1},$$

(3.7)

then their magnitude is constrained by (3.5) to be at least as small as $x^{2}$ as one takes the limit $x \to 0$. Hence Chern-Simons terms will affect only finite terms in the action and can be ignored here.

To satisfy the closure property for the $p$-form we will have to implement the conditions

$$dF_{p}^{0} = 0; \quad dA_{p-1}^{1} + 2F_{p}^{2} = 0, \quad dA_{p-1}^{2} + 2F_{p}^{3} = 0,$$

(3.8)

and so on. Expanding out the $p$-form equation of motion we find the leading order conditions that

$$D_{i_{0}}^{0}(F_{p}^{0})_{i_{1}...i_{p-1}} = (d - 2p)l^2(A_{p-1}^{1})^{i_{1}...i_{p-1}}; \quad D_{i_{0}}^{0}(A_{p-1}^{1})^{i_{1}...i_{p-1}} = 0,$$

(3.9)

where all indices are raised and lowered in the metric $\gamma^{0}$. The first equation tells us that $A^{1}$ acts as a source for $F^{0}$ whilst the second equation effectively picks out a gauge for $A^{1}$. Note that if $F^{0}$ vanishes - or in another words, the field induced on the boundary vanishes - then the field equations force the next order term in the $p$-form to be at least of order $x^{2}$. 


A. Vector fields

Minimally coupled scalar fields, corresponding to \( p = 1 \), have been considered in other work \[13\]. Let us now discuss the detailed analysis for \( p = 2 \) before considering the generalisation to \( p > 2 \). Expanding out the Einstein equations in powers of \( x \), the leading order terms determine \( \gamma^2 \) in terms of \( \gamma^0 \) as

\[
[\gamma^2_{ij}] = \frac{1}{(d - 2)l^2} \left[ R^0_{ij} - \frac{1}{2(d - 1)} R^0 R^0_{ij} \right],
\]

where \( R^0 \) and \( R^0_{ij} \) are the Ricci scalar and Ricci tensor respectively of the metric \( \gamma^0 \). For \( p \geq 2 \), the \( p \)-form does not affect the relationship of \( \gamma^2 \) to \( \gamma^0 \) (3.10) which is found in the absence of matter fields. The \( x^2 \) term in the \( \mathcal{R}_{xx} \) equation of motion gives us the relationship

\[
\text{Tr}[\gamma^{-1} \gamma^4] = \frac{1}{4} \text{Tr}[(\gamma^0)^{-1} \gamma^2 (\gamma^0)^{-1} \gamma^2] + \frac{1}{16(d - 1)^2} (F^0_2)^2,
\]

(3.11)

where we contract \( F^0_2 \) again using the leading order metric \( \gamma^0 \). Using the \( \mathcal{R}_{ij} \) equation of motion at the same order we find that

\[
[\gamma^4_{ij}] = \frac{1}{4(d - 4)l^2} (F^0_2)_{ik} (F^0_2)^k_j - \frac{3}{16(d - 1)(d - 4)l^2} (F^0_2)^2 \gamma^0_{ij},
\]

(3.12)

with the other component of \( \gamma^4 \) being given in terms of the curvature as the usual expression

\[
[\gamma^4_{ij}] = \frac{1}{(d - 2)^2(4 - d)l^2} \left\{ R^0_{ij} R^0_{kl} - \frac{1}{(d - 1)} R^0 R^0_{ij} + \frac{1}{4} (R^0)^2 \gamma^0_{ij} \right\} - \frac{(d + 4)}{4(d - 1)^2} (R^0)^2 \gamma^0_{ij} \ldots,
\]

(3.13)

where the ellipses denote terms involving covariant derivatives of the curvature. Note that although the expression (3.12) is ill-defined when \( d = 4 \) in this case we will only need to use the well defined trace (3.11) to determine divergent terms. We can expand out \( (F^0_2)^2 \) as

\[
(F^0_2)^2 = x^4 (F^0_2)^2 + 2l^2 x^6 (A^1_1)^2 - x^6 (dA^1_1)(F^0_2)^i_j + \frac{1}{(d - 2)(d - 1)l^2} x^6 R^0 (F^0_2)^2
\]

\[
- \frac{2}{(d - 2)l^2} x^6 (R^0)^i_j F^0_2(kj) F^0_2(kl),
\]

(3.14)

where all contractions use \( \gamma^0 \). Furthermore from the \( x^4 \) term in the \( \mathcal{R}_{xx} \) equation of motion we derive the relationship

\[
\text{Tr}[(\gamma_0)^{-1} \gamma^6] = -\frac{1}{3} \text{Tr}[(\gamma_0)^{-1} \gamma^2]^3 - \frac{1}{24} (A^1_1)^2 + \frac{1}{48l^2(d - 1)} (F^0_2)^2 (x^6),
\]

(3.15)

where the last subscript indicates that we use the coefficient of the \( x^6 \) term in (3.14). Now let us expand the metric; as well as terms depending only on \( \gamma^0 \) and its derivatives...
\[ \sqrt{g}^{(1)} = \frac{\sqrt{\gamma^0}}{l_{d+1}} \left( 1 + \frac{1}{2} x^2 \text{Tr}((\gamma^0)^{-1} \gamma^2) + \frac{1}{8} x^4 [\text{Tr}((\gamma^0)^{-1} \gamma^2)]^2 - \frac{1}{8} x^4 \text{Tr}((\gamma^0)^{-1} \gamma^2)^2 \right) \] (3.16)

\[ - \frac{3}{16} x^6 \text{Tr}((\gamma^0)^{-1} \gamma^2)^3 \text{Tr}((\gamma^0)^{-1} \gamma^2)]^2 + \frac{1}{4} x^6 \text{Tr}((\gamma^0)^{-1} \gamma^2)^3 \]

\[ + \frac{1}{16} x^6 \text{Tr}((\gamma^0)^{-1} \gamma^2)^3 - \frac{1}{2} x^6 \text{Tr}((\gamma^0)^{-1} \gamma^2)^3 \] 

\[ F_{2}^0 \]

\[ \frac{1}{16} x^6 \text{Tr}((\gamma^0)^{-1} \gamma^2)^3 \] 

\[ I_{\text{bulk}} = \frac{1}{8 \pi G_{d+1}} \int_{M} d^{d+1}x \sqrt{g}(d^2 + \frac{1}{4}(d-1)F^2_{2}). \] (3.18)

Substituting the explicit form for the metric (3.10) and (3.17) and integrating we find that as well as divergent terms depending only on \( \gamma^0 \) and its curvature invariants

\[ I^{(1)} = - \frac{(d-1)l}{8 \pi G_{d+1} \epsilon^d} \int d^d x \sqrt{g} \frac{(d-4)(d-1)}{16 \pi (d^2-4)G_{d+1} l \epsilon^{d-2}} \int d^d x \sqrt{g}  R^0_0 + ... \] (3.19)

which can be removed (at least for \( d \) odd) by the counterterms given in (3.4), there are additional possible divergences in the bulk action

\[ I_{\text{surf}} = - \frac{1}{8 \pi G_{d+1}} \int d^d x \sqrt{g} \frac{d-4}{32(d-1) l \epsilon^{d-4}} (F^0_2)^2 + ... \] (3.20)

as well as possible divergences in the surface action

\[ I = - \frac{\ln \epsilon}{64 \pi G_5 l^3} \int d^4 x \sqrt{\gamma^0} \left( R^0_0 \right)^2 - \frac{1}{3} (R^0)^2 \] (3.21)

and an additional logarithmic divergence in the action given by

\[ I_{\text{log}} = \frac{1}{64 \pi G_5 l \ln \epsilon} \int d^4 x \sqrt{\gamma^0} (F^0_2)^2. \] (3.22)

Note that the field equation (3.9) in this case implies that \( F^0_2 \) is both closed and co-closed in the metric \( \gamma^0 \). In \( d > 4 \) the same term will cause a power law divergence in the action.
\[ I_{\text{div}} = -\frac{1}{256\pi G_{d+1}l^{d-4}} (d-8) \int d^dx \gamma^0(F_2^0)^2, \]  

(3.24)

which can be removed by a counterterm of the form

\[ I_{\text{ct}} = \frac{1}{256\pi G_{d+1}l^{d-4}} \int d^dx \gamma^0 (d-8) (F_2)^2. \]  

(3.25)

as advertised in §4.

We should briefly mention another application of these results. We are concerned here with the definition of local counterterms which render the action finite as one takes the limit of \( \epsilon \to 0 \), that is, the boundary approaches the true boundary. In the context of the Randall-Sundrum scenario [23], however, one would keep the boundary at finite \( \epsilon \) as in [24]. In this case, (3.23) and (3.24) would correspond to part of the conformal field theory action on the brane.

In \( d = 6 \) as well as the logarithmic divergence associated with the Weyl anomaly of the dual theory, which is given by

\[ I_{\text{log}} = \frac{\ln \epsilon}{8^3\pi G_6 l^3} \int d^6x \sqrt{\gamma^0} \left\{ \frac{3}{50} (R^0)^3 + R^{(0)ij} R^{(0)kl} R^{(0)}_{ijkl} - \frac{1}{2} R^0 R^{(0)ij} R^{(0)}_{ij} \right\} \]

(3.26)

as was found in [9], we have an anomaly of the form

\[ I_{\text{log}} = \frac{1}{8\pi G_7 l^3} \ln \epsilon \int d^6x \sqrt{\gamma^0} \left\{ \frac{1}{16l^2} R^0(F_2^0)^2 - \frac{1}{8l} R^{(0)ij}(F_2^0)_i^j(F_2^0)_j^i + \frac{1}{16} (dA^1)^{ij}(F_2^0)_{ij} \right\}; \]

(3.27)

\[ = \frac{1}{8\pi G_7 l^3} \ln \epsilon \int d^6x \sqrt{\gamma^0} \left\{ \frac{1}{16} R^0(F_2^0)^2 - \frac{1}{8} R^{(0)ij}(F_2^0)_i^j(F_2^0)_j^i \right\} \]

where in the latter equality we have used the field equation (3.23). For \( d > 6 \), we will need to include an additional counterterm of the form

\[ I_{\text{ct}} = \frac{1}{8\pi G_{d+1}l^3} \int d^dx \sqrt{\gamma} \left\{ \frac{(5d-11)}{128(d-1)^2(d-2)(d-6)} R(F_2)^2 + \frac{(7d-66)}{48(d-6)(d-2)} R^0_i(F_2)_{ik}(F_2)^{ik} \right\} \]

(3.28)

\[ \frac{(d-8)(d-4)^2}{48(d-6)(d-2)} (D_iF_2^{ij})^2 + \frac{(d-12)(d-4)^2}{128(d-6)} (F_2)^{ij}(D_jD^k(F_2)_{ki} - D_kD^k(F_2)_{kj}) \}

The counterterms (3.27) and (3.28) will be adequate for \( d < 6 \) which includes all gauged supergravity theories of current interest. For completeness, let us mention that it is straightforward to extend the analysis to minimally coupled \( p \)-forms of higher order with the following results: the \( p \)-form term in the anomaly for \( d = 2p \) is given by

\[ I_{\text{log}} = \frac{1}{32\pi p G_{d+1}l} \ln \epsilon \int d^{2p}x \sqrt{\gamma^0}(F_p^0)^2, \]  

(3.29)

whilst for \( d > 2p \) we will find a divergence of the form
\[ I_{\text{div}} = -\frac{1}{64\pi p^2 G_{d+1} l^{d-2p}} \frac{(d-4p)}{(d-2p)} \int d^4x (F_p^0)^2, \] (3.30)

which can be removed by the counterterm

\[ I_{\text{div}} = \frac{1}{64\pi p^2 G_{d+1} l^{d-2p}} \frac{(d-4p)}{(d-2p)} \int d^4x (F_p^0)^2. \] (3.31)

There will of course be additional terms in the anomaly for \( d = 2p + 2n \) depending on derivatives of \( F_p \) and the curvature.

### B. Magnetic strings in five dimensions

Solutions of gauged supergravity theories in which these anomalies and counterterms play a role have been constructed. The simplest example is the magnetic string solution of cosmological Einstein-Maxwell theory in five dimensions \[16\], \[17\] which is given by

\[
d s^2 = \left( l r \right)^{\frac{4}{3}} \left( \frac{1}{3 l r} \right)^{\frac{2}{3}} \left( d \tau^2 + d z^2 \right) + \frac{d r^2}{\left( l r + \frac{1}{3 l r} \right)^2} + r^2 \left( d \theta^2 + \sin^2 \theta d \phi^2 \right);
F_2 = -\frac{1}{\sqrt{3}l} \sin \theta d \theta \wedge d \phi.
\] (3.32)

Note that the magnetic charge is quantised in units depending on the cosmological constant \[26\]. Like the BPS electric black hole solutions in four and five dimensional gauged supergravity theories the magnetic string solution represents a naked singularity. Substituting the fields into the bulk action, we find that the potential logarithmic divergence is in fact absent. Note that in calculating the bulk action we have to introduce a UV cutoff around the singularity for the BPS solution but this won’t affect the determination of the IR divergences.

To check on this result, let us calculate the logarithmic terms in the action directly, using the boundary metric \( \gamma^0 \)

\[
\gamma_{ij}^0 d x^i d x^j = l^2 (d \tau^2 + d z^2) + (d \theta^2 + \sin^2 \theta d \phi^2).
\] (3.33)

The dual conformal theory hence has a background of \( R^2 \times S^2 \). Then the Weyl anomaly (3.22) is given by

\[
I_{\log} = -\frac{\ln \epsilon}{96\pi G_5 l^3} \int d^4 x \sqrt{\gamma^0}.
\] (3.34)

However, substituting into the expression (3.34), we find an equal and opposite contribution from the vector field, giving zero total divergence. We should mention here that the logarithmic term for the “dual” electric black holes in five dimensions vanishes \[11\]; this is because the Weyl anomaly for a field theory on \( R^1 \times S^3 \) vanishes. However, both anomaly cancellations appear to be accidental.
C. Magnetic three-branes in seven dimensions

Cosmological Einstein-Maxwell gravity in seven dimensions admits magnetic three-brane solutions of the form
\[
ds^2 = (lr)^2 \left(1 + \frac{1}{5l^2 r^2} \right) \left( d\tau^2 + ds^2(E^3) \right) + \frac{dr^2}{(lr + \frac{1}{5l^2 r^2})^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2); \]
\[
F_2 = -\frac{2}{\sqrt{5}l} \sin \theta d\theta \land d\phi. \tag{3.35}
\]
This solution is obviously very closely related to the magnetic string solution in five dimensions; however, it is not a solution of the gauged supergravity theories which arise in seven dimensions from compactification of eleven-dimensional supergravity on a sphere, since the latter does not admit an Einstein-Maxwell truncation. Calculating the bulk action, we find that again the logarithmic divergence does not appear. Calculation of the logarithmic divergence both directly and using the expression (3.27) allows us to check the coefficients in (3.27). Again there appears to be no profound reason why the total logarithmic anomaly should vanish.

IV. SCALAR FIELDS WITH GAUGED SUPERGRAVITY POTENTIALS

The action for a scalar field with potential is
\[
I_{\text{bulk}} = -\frac{1}{16\pi G_{d+1}} \int_M d^{d+1}x \sqrt{g} \left[ \mathcal{R} + l^2 V(\phi) - \frac{1}{2} (\partial \phi)^2 \right], \tag{4.1}
\]
where the equations of motion derived from the action are
\[
\mathcal{R}_{mn} = \frac{1}{2} (\partial_m \phi) (\partial_n \phi) - \frac{1}{(d-1)} l^2 V(\phi) g_{mn};
\]
\[
\mathcal{R} = \frac{1}{2} (\partial \phi)^2 - \frac{(d+1)}{(d-1)} l^2 V(\phi); \tag{4.2}
\]
\[
D_m \partial^m \phi = -l^2 \left( \frac{\partial V}{\partial \phi} \right). \]

The leading order terms in the Einstein equations imply that if the metric behaves as (1.3) near the conformal boundary the scalar field must tend to a value on $N$, $\phi \to \phi^0(x^i)$, such that the scalar potential takes the constant value $V(\phi^0(x^i)) = d(d-1)$. Furthermore, using the leading order term in the dilaton equation of motion, we find that
\[
\left( \frac{\partial V}{\partial \phi} \right)_{\phi=\phi^0(x^i)} = 0. \tag{4.3}
\]
In the vicinity of the conformal boundary the scalar field must behave as
\[
\phi(x, x^i) = \phi^0 + x^2 \phi^2(x^i) + x^4 \phi^4(x^i) \ldots, \tag{4.4}
\]
and the potential may be expanded about its boundary value as
\[ V(\phi) = d(d-1) + \frac{1}{2}V^2(x^i)(\phi - \phi^0)^2 + \frac{1}{6}V^3(x^i)(\phi - \phi^0)^2 + \ldots \quad (4.5) \]

\[ = d(d-1) + \frac{1}{2}x^2V^2(x^i)(\phi^1(x^i))^2 + x^3(V^2(x^i)\phi^1(x^i)\phi^2(x^i) + \frac{1}{2}V^3(x^i)(\phi^1(x^i))^2) + \ldots, \]

where \( V^i(x^i) \) is the \( i \)th derivative of \( V \) with respect to \( \phi \) evaluated at \( \phi^0 \). Then the derivative of the potential can be expanded as

\[
\left( \frac{\partial V}{\partial \phi} \right) = xV^2(x^i)\phi^1(x^i) + x^2(V^2(x^i)\phi^2(x^i) + \frac{1}{2}V^3(x^i)(\phi^1(x^i))^2) + \ldots. \quad (4.6)
\]

So far, we have taken the fields \( \phi^0 \) and \( V^i(x^i) \) to be arbitrary, but of course for the potentials which arise in maximal supergravities arbitrary values of the fields cannot be chosen; in particular, what is relevant here is that the second derivative of the potential at the boundary is fixed.

So let us consider here scalar fields in \( D \)-dimensional maximal supergravity: these parametrise the coset \( E_{11-D}/K \) where \( E_n \) is the exceptional group and \( K \) is its maximal compact subgroup. Focusing on the \( SL(N,R) \) subgroup of \( E_n \) and using the local \( SO(N) \) transformations to diagonalise the scalar potential we are led to the form

\[
V = \frac{d(d-1)}{N(N-2)} \left( \sum_{i=1}^{N} X_i^2 - 2(\sum_{i=1}^{N} X_i^2) \right), \quad (4.7)
\]

where \( d = D - 1 \) and the \( N \) scalars \( X_i \) can be parametrised in terms of \( (N-1) \) independent scalars \( \phi_\alpha \) as

\[
X_i = e^{-\frac{1}{2}b^\alpha_i \phi_\alpha}, \quad (4.8)
\]

where the \( b^\alpha_i \) are the weight vectors of the fundamental representation of \( SL(N;R) \) normalised so that

\[
b^\alpha_i b^\alpha_j = 8\delta_{ij} - \frac{8}{N}, \quad \sum b^\alpha_i = 0. \quad (4.9)
\]

Then the potential \((4.7)\) has a minimum at \( X_i = 1 \) for \( N \geq 5 \) (which includes all cases considered here), at which point \( \phi_\alpha = 0 \) and \( V = d(d-1) \). Explicitly differentiating we find that the second derivatives at this minimum are given by

\[
\frac{\partial^2 V}{\partial \phi_\alpha \partial \phi_\beta} = \frac{d(d-1)}{N(N-2)} b^\alpha_i b^\beta_i. \quad (4.10)
\]

Using the properties of the weight vectors we find that

\[
b^\alpha_i b^\beta_i = 4(N-4)\delta^{\alpha\beta} \quad \forall \alpha. \quad (4.11)
\]

We will be interested in maximal supergravities in \( D = 4, 5, 7 \) for which \( N = 8, 6, 5 \) respectively. Substituting in these values, we find that the second derivatives of the potentials are given by

\[
\frac{\partial^2 V}{\partial \phi_\alpha \partial \phi_\beta} = 2(d-2)\delta^{\alpha\beta}. \quad (4.12)
\]
The same expression is found for the non-maximal supergravity potential in six dimensions discussed in [25]; this potential arises naturally from the reduction of massive IIA supergravity on $S^4$. The significance of (4.12) is the following; since the form of the potential fixes the asymptotic values of the scalar fields to be zero, each scalar field can be expanded as

$$\phi(x, x^i) = x^k \phi^k(x^i) + x^{k+1} \phi^{k+1}(x^i) + ...$$

(4.13)

The leading order term in the scalar equation is given by

$$[k(d - k) - V^2] \phi^k(x^i) = 0,$$

(4.14)

and so using (4.12) we find that $k = d - 2$. Note that it is the specific form of the potential which forces the scalar to behave as $x^{d-2}$ at infinity; in a more general potential, we might have leading order terms with $k < d - 2$ which would give rise to further divergences in the action. The form of the potential effectively ensures that the scalar charge is finite; it can be expanded as

$$V(\phi) = d(d - 1) + (d - 2)x^{2(d-2)}(\phi^{d-2}(x^i))^2 + ...$$

(4.15)

Since the on-shell bulk term in the Einstein action is given by

$$I_{\text{bulk}} = \frac{l^2}{8\pi G_{d+1}(d - 1)} \int_M d^{d+1}x \sqrt{g}V(\phi),$$

(4.16)

using just the asymptotic form of the metric (1.3) the only possible scalar field divergences are in $d = 3, 4$. In the former case, the Einstein equations give us the relationship

$$\text{Tr}[(\gamma^0)^{-1}\gamma^2] = \frac{R^0}{4l^2} - \frac{3}{8}(\phi^1(x^i))^2,$$

(4.17)

which is enough to determine the dependence of the divergence on the scalar fields

$$I_{\text{div}} = -\frac{5l}{256\pi G_4\epsilon} \int d^3x \sqrt{\gamma^0} (\phi^1(x^i))^2.$$ 

(4.18)

This means that one needs an additional counterterm in the action of the form

$$I_{\text{ct}} = \frac{5l}{256\pi G_4} \int d^3x \sqrt{\gamma}(\phi)^2.$$ 

(4.19)

One will need this term to regularise the action of the four-dimensional charged black holes discussed in [27], [28], [29], [30]. For $d = 4$, $\gamma^2$ is unaffected by the scalar fields, but

$$\text{Tr}[(\gamma^0)^{-1}\gamma^4] = -\frac{1}{3}(\phi^2)^2,$$

(4.20)

which is enough to determine that

$$I_{\text{div}} = \frac{l}{48\pi G_5} \ln \epsilon \int d^4x \sqrt{\gamma^0}(V^2 - 4)(\phi^2(x^i))^2 = 0.$$ 

(4.21)

That is, the potential anomaly term in $d = 4$ is absent. We should be unsurprised by this for two reasons. The first is that the thermodynamics of charged black holes in five dimensions [13], [20] has been extensively discussed in the context of stability analysis and no logarithmic term in the action was found. Secondly, there is no candidate for an anomaly of this form in the dual conformal field theory.
V. GAUGED SUPERGRAVITY IN SEVEN DIMENSIONS

The analysis of the two previous sections assumes that one can consistently truncate a supergravity theory to an Einstein-Maxwell theory or to an Einstein-dilaton theory. However, this isn’t generally the case. Although solutions of Einstein-Maxwell theory are consistent solutions of four and five dimensional gauged supergravities, they are not solutions of gauged supergravity in seven dimensions.

Let us consider the following truncation of $D = 7$ gauged supergravity, containing gravity, two scalar fields $\phi_1, \phi_2$, two vector fields $F_1, F_2$ and a single three-form potential $C$ discussed in [15]

$$I_{\text{bulk}} = \frac{1}{16 \pi G_7} \int_M d^7 x \sqrt{g} \{ R + 2l^2 V(\phi_1, \phi_2) - 5(\partial(\phi_1 + \phi_2))^2 - (\partial(\phi_1 - \phi_2))^2$$

$$- e^{-4\phi_1}(F_1)^2 e^{-4\phi_2}(F_2)^2 + 4l^2 e^{-4\phi_1-4\phi_2}C^2$$

$$- \frac{l}{3} \epsilon^{mnpqrst} C_{mnp} \partial_q C_{rst} + \frac{1}{\sqrt{3}} \epsilon^{mnpqrst} C_{mnp} F_{(1)qr} F_{(2)st} + \ldots \},$$

where the ellipses denote terms that are not relevant here. We have simplified the field content by focusing on a $U(1)^2$ truncation of the gauge group and by diagonalising the potential. The resulting scalar potential is given by

$$V(\phi_1, \phi_2) = 8e^{2\phi_1+2\phi_2} + 4e^{-2\phi_1+4\phi_2} + 4e^{4\phi_1+2\phi_2} + e^{-8\phi_1-8\phi_2},$$

which is of the form discussed in the previous section and the field equations are

$$D_m D^m (3\phi_1 + 2\phi_2) = -e^{-4\phi_1}(F_1)^2 + 4l^2 e^{-4\phi_1-4\phi_2}C^2 - \frac{1}{2} \partial V \partial \phi_1;$$

$$D_m D^m (3\phi_2 + 2\phi_1) = -e^{-4\phi_2}(F_2)^2 + 4l^2 e^{-4\phi_1-4\phi_2}C^2 - \frac{1}{2} \partial V \partial \phi_2;$$

$$D_m (e^{-4\phi_1} F_1^{mn}) = \frac{1}{2\sqrt{3}} \epsilon^{mnpqrst} D_m (F_{(2)pq} C_{rst});$$

$$D_m (e^{-4\phi_2} F_2^{mn}) = \frac{1}{2\sqrt{3}} \epsilon^{mnpqrst} D_m (F_{(1)pq} C_{rst});$$

$$e^{-4\phi_1-4\phi_2} C_{mnp} = \frac{1}{12l} \epsilon^{qrst} \partial_q C_{rst} - \frac{1}{8\sqrt{3}l^2} \epsilon^{qrst} F_{(1)qr} F_{(2)st}. $$

The key point is that one cannot consistently set the scalars and the three-form potential to zero when the vector fields are excited. If the wedge product of the two vector field strengths vanishes, then one can set the three-form to zero, as for the known electric black hole solutions [15], but even this is not possible in general.

A consistent further truncation is to set $F_2 = 0, C = 0$ and $2\phi_1 + 3\phi_2 = 0$, in which case the field equations become

$$D_m D^m \phi = -\frac{3}{5} e^{-4\phi} F^2 - \frac{12}{5} l^2 (e^{2\phi} - e^{-8\phi}); \quad D_m (e^{-4\phi} F^{mn}) = 0;$$

$$R_{mn} = -\frac{2l^2}{5} \left( 12e^{\frac{2\phi}{3}} + 3e^{\frac{8\phi}{3}} \right) + \frac{10}{3} \partial_m \phi \partial_n \phi + 2e^{-4\phi} \left( F_{mp} F_n - \frac{1}{10} F^2 g_{mn} \right),$$
where we have dropped the subscripts on the fields for notational simplicity. Following the arguments of the previous section, using the leading order form of the asymptotic metric and the Einstein equations fixes the expansion of the fields about the boundary to be

\[ \phi(x, x^i) = x^k \phi_k(x^i) + x^{k+2} \phi^{k+2} + \ldots; \]

\[ F = F_2^0 + x dx \wedge A_1^1 + x^2 F_1^2 + x^2 dx \wedge A_2^1 + \ldots, \]

where \( k > 0 \) and we use the same notation for the vector field as in §II. Now substituting into the scalar field equation (5.4) we get the following expression

\[ l^2 (k^2 - 6k + 8)x^k \phi_k(x^i) + \frac{3}{5} x^4 (F_2^0)^2 = \mathcal{O}(x^{k+2}, x^6), \]

where as usual we raise and lower indices in the leading order induced metric \( \gamma^0 \). The only way to balance leading order terms is to take \( k = 4 \) and \( \phi^4 \neq 0 \) (as found in the previous section) but this forces \( F_2^0 = 0 \).

Hence a finite vector field is not induced on the boundary; one cannot have a non-zero magnetic charge in the bulk with this ansatz. Furthermore, since \( F_2^0 \) vanishes, as previously mentioned in §II the Maxwell equations fix the leading order behaviour of \( F^2 \) to be \( x^8 \) or smaller near the boundary, and thus the vector field will not contribute to any divergences of the action.

This result should be independent of the particular truncation we have taken. Analysing (5.3) one sees that the scalar field equations will always force the magnetic charge in the Abelian truncation of the theory to vanish, given the form of the scalar potential. One would also expect this obstruction to persist in the full non-Abelian theory.

Another consistent truncation is to set \( F_1 = F_2 = 0 \) and \( \phi_1 = \phi_2 \); this is relevant if we are trying to induce finite magnetic charge with respect to the three form potential. The relevant field equations are then

\[ D_m D^m \phi = \frac{4}{5} e^{-8\phi} C^2 - \frac{8}{5} \left( 2e^{4\phi} + 3e^{-6\phi} - e^{-16\phi} \right); \]

\[ e^{-8\phi} C_{mnp} = \frac{1}{12l^2} \varepsilon_{mnp} \partial_4 C_{rst}. \]

In this case it is the field equation for the three form which is important; the scalar field equation would not prohibit finite magnetic three-form charge. Suppose we look for a leading order expansion of \( C \) of the form

\[ C = x^a C^a_3 + x^b dx \wedge A_2^b + \ldots \]

where \( C^a_3 \) and \( A_2^b \) are three and two forms respectively, defined on the induced six-dimensional hypersurface. Then the self-duality equation (5.7) fixes \( a = 2 \) and \( C^2_3 \) to be self-dual in the six-dimensional induced metric \( \gamma^0 \), in agreement with the discussions of §III. Furthermore, the next order terms in this equation fix \( b = 3 \) with

\[ A_2^3 = \frac{2}{l^2} \ast dC^2_3, \]
where the dual is again taken in the metric $\gamma^0$. This means that the leading order contribution to the action from terms in $C$ is $x^{10}$, and hence there are no IR divergences resulting from the inclusion of $C$ terms. Physically, this means that one cannot find solutions of the gauged supergravity with an asymptotic metric of the form (1.3) which have finite magnetic three-form charge.

For completeness, let us also consider the truncation with $\phi_1 = \phi_2 = 0$ and $F_1 = F_2$. Consistency of the truncation requires that

$$F^2 = 4l^2 C^2; \quad D_m F^{mn} = \frac{1}{2\sqrt{3}} e^{mnprst} D_m (F_{pq} C_{rst});$$

(5.10)

and we also have to satisfy the last field equation in (5.3). Together these conditions prove to be very restrictive, and the leading order behaviour of $C$ and $F$ is fixed to be

$$C = x^2 C_3^2 + x^3 dx \wedge A_3^2 + ...; \quad F = x^3 dx \wedge A_1^3 + ....,$$

(5.11)

where $C_3^2$ is self-dual in $\gamma^0$ and $A_3^2$ satisfies the condition (5.9) as before. $A_3^3$ satisfies the gauge condition $D_m (A_3^1)^m = 0$ and satisfies the relation $(A_3^1)^2 = 2(C_3^2)^2$. As in the above truncations, no finite magnetic fluxes - and hence no anomalies - are induced in the boundary field theory.

VI. ANOMALIES IN CONFORMAL FIELD THEORIES

The general structure of the gravitational action for $d$ even is

$$I = \frac{1}{16\pi G_{d+1}} \int d^d x \left[ \sqrt{\gamma^0} \left( a^0 \epsilon^{-d} + ... + a^{d-2}\epsilon^{-2} + a^d \ln \epsilon \right) \right] + I_{\text{fin}},$$

(6.1)

where $I_{\text{fin}}$ is finite in the $\epsilon \to 0$ limit. After subtraction of the divergent counterterms, including the logarithmic term, we are left with a renormalised effective action with a finite limit as $\epsilon \to 0$. Its variation under a conformal transformation $\delta \gamma^0 = 2\delta \lambda \gamma^0$ is of the form

$$\delta I_{\text{fin}} = \int d^d x \sqrt{\gamma^0} A \delta \lambda,$$

(6.2)

where the anomaly $A$ is given by

$$A = \frac{a^d}{8\pi G_{d+1}}.$$  (6.3)

Hence as first discussed in [9] one can relate the logarithmic divergence of the gravitational action to Weyl anomalies of the conformal field theory. On general grounds [31], [32], the gravitational part of the coefficient that appears in the anomaly must be of the form

$$a^d = d l^{1-d} (E^d + I^d + D^{(0)i} J^{d-1}_i),$$

(6.4)

where $E^d$ is proportional to the d-dimensional Euler density (the type A anomaly) and $I^d$ is a conformal invariant (type B anomaly). The current term depending on $J^{d-1}$ is trivial in the sense that it can be obtained by variation of covariant counterterms.
Anomalies which combine Weyl invariance with other symmetries can be analysed in much the same way as pure Weyl anomalies. Consider vector field sources in \( d = 4 \); using the duality relations

\[
G_5 = 8\pi^3 l^3 g_s^2 \quad \text{and} \quad l = (4\pi g_s N)^{-\frac{1}{4}}
\]

we find that the anomaly is

\[
A_{F_2}^0 = \frac{1}{256\pi^4 g_s^2 l^6} (F_2^0)^2 = \frac{N^2}{16\pi^2} f^2,
\]

where in the latter equality we have used the fact that (magnetic) fields in the bulk behave as \( 1/l \) to define \( f = LF_2^0 \) as the more natural field in the dual theory. One expects the anomaly in the correlator of the stress tensor and two vector currents for any conformal theory in \( d = 4 \) to be proportional to the Maxwell Lagrangian density \([33], [34]\); in \( d = 4 \) the only Weyl invariant term of the right dimension involving \( f \) is the Maxwell Lagrangian density. For the \( N = 4 \) \( SU(N) \) Yang-Mills theory, following \([9]\), the anomaly should be

\[
(N^2 - 1) [6\alpha_s + 2\alpha_f + \alpha_v] f^2,
\]

where the coefficients \( \alpha_s, \alpha_f \) and \( \alpha_v \) are for non-interacting scalars, fermions and vectors respectively and we have included the multiplet of six scalars, two fermions and one vector. The factor of \( (N^2 - 1) \) derives as usual from the fields taking values in the adjoint of \( SU(N) \). We assume that, as for the Weyl anomaly, there is a renormalisation theorem which protects the coefficients so that we can ignore interactions. Now the coefficients \( \alpha_i \) are well-known; indeed they were effectively calculated in the original papers \([33] \) and \([34] \) (see also \([35] \) and \([36] \)). Using the explicit values

\[
\alpha_s = \frac{1}{256\pi^2}; \alpha_f = \frac{1}{64\pi^2}; \alpha_v = \frac{1}{128\pi^2},
\]

we find that the Yang-Mills anomaly (6.6) in the large \( N \) limit agrees with what we found in the supergravity theory (6.5) even though we have ignored interactions! Such an agreement is perhaps less surprising given that it is already known that the pure Weyl anomaly is not renormalised \([9]\); the latter should be related to the vector anomaly by supersymmetry.

Now let us consider vector fields in \( d = 6 \). Since there is a non-vanishing logarithmic divergence in the action when one includes magnetic vector fields in cosmological Einstein-Maxwell theory, we expect that in the (unknown) dual conformal field theory there should be an anomaly in the correlator of the stress-energy tensor and two vector currents. In analogy to the discussions of the Weyl anomaly, we need to construct a basis of Weyl invariant polynomials involving the curvature and the vector field. Appropriate polynomials will be constructed from one curvature tensor and two Maxwell fields, or from two derivatives and two Maxwell fields. A basis of possible contractions is given by

\[
[V_6] = \{ R f_{ij} f^{ij}, R^{ij} f_{ij} f^i, (D^i f_{ij})^2, f_{ij} D_{[i} D_{j]} f^i, \Box (f_{ij} f_{ij}), D_i D_j f_{ij},\}
\]

where we have rescaled the vector field as above. Now there is a conformal invariant given by

\[
I = 2V_1 - 4V_2 + V_3 + 2V_4 - 2V_5 + 2V_6;
\]
one should be able to construct other conformal invariants from \( \{V_a\} \) but they do not appear in the anomaly. Note also that we can write as derivatives of currents

\[
D_i J^i_1 = V_3 + V_4; \quad D_i J^i_2 = V_5; \quad D_i J^i_3 = V_6.
\] (6.10)

We then find that the anomaly is given by

\[
\mathcal{A}_f = \frac{1}{256\pi^5 G_7} \left( I - D_i J^i_1 + 2D_i J^i_2 - 2D_i J^i_3 \right),
\] (6.11)
in accordance with the general anomaly form (6.4).

Finally, let us interpret the results of §V in terms of the dual conformal field theory. Gauged supergravity in a seven dimensional anti-de Sitter background should be dual to a boundary field theory consisting of \((0,2)\) tensor multiplets coupled to \((0,2)\) conformal supergravity in six dimensions. The boundary values of the bulk fields are in one-to-one correspondence with the fields of (off-shell) conformal supergravity defined at the boundary.

The coupling of a single \((0,2)\) tensor multiplet to conformal supergravity in six dimensions was discussed in [38]. Given the known result for the Weyl anomaly, one could determine the vector and three-form anomalies using supersymmetry. One could also try to determine these results directly by analysing a single \((0,2)\) tensor multiplet as was done for the Weyl anomaly in [39] (although this won’t give the right answer since it doesn’t for the Weyl anomaly). One should find that the predicted anomaly involves the invariants given above; there is no reason a priori why it should vanish.

However, what we have argued in §V is that the bulk equations of motion prevent us from inducing finite vector or three form fields on the boundary. That means that the bulk theory in this case can only tell us about tensor multiplets in backgrounds without vector or three form currents switched on.

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