Research Article

Algebro-Geometric Solutions of a (2+1)-Dimensional Integrable Equation Associated with the Ablowitz-Kaup-Newell-Segur Soliton Hierarchy

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The (2+1)-dimensional Lax integrable equation is decomposed into solvable ordinary differential equations with the help of known (1+1)-dimensional soliton equations associated with the Ablowitz-Kaup-Newell-Segur soliton hierarchy. Then, based on the finite-order expansion of the Lax matrix, a hyperelliptic Riemann surface and Abel-Jacobi coordinates are introduced to straighten out the associated flows, from which the algebro-geometric solutions of the (2+1)-dimensional integrable equation are proposed by means of the Riemann θ functions.

1. Introduction

Algebro-geometric solutions are an important class among exact solutions of soliton equations, which can be regarded as explicit solutions of the nonlinear integrable evolution equation and used to approximate more general solutions. Based on the nonlinearization technique of Lax pairs and direct method, many of algebro-geometric solutions of (1+1)-dimensional [1–3], (2+1)-dimensional [4, 5], and differential-difference [5, 6] soliton equations have been obtained, such as the Gerdjikov-Ivanov, modified Kadomtsev-Petviashvili, and Toda lattice equations [7–9]. The existence of infinitely many exact solutions is a reflection of this complete integrability.

Many other techniques for finding exact solutions have been also discovered: inverse scattering theory, Darboux transformation, Riemann-Hilbert method, etc. Recently, more exact solutions of soliton equations are found [10–13], and more dynamic behaviors are studied [14–16].

Ablowitz-Kaup-Newell-Segur (AKNS) soliton hierarchy is an important class of integrable equations, which can be reduced to Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV), sine-Gordon equation hierarchies, etc. The purpose of the paper is to further develop the direct method for constructing algebro-geometric solution of the following (2+1)-dimensional integrable equation [15] which concerns with the AKNS soliton hierarchy [17].

\[
\begin{align*}
    u_t &= -\frac{1}{2} u_{xy} + u \partial_x^{-1}(uv)_y, \\
    v_t &= \frac{1}{2} v_{xy} - v \partial_x^{-1}(uv)_y.
\end{align*}
\] (1)

In fact, system (1) is the Lax integrable equations from the AKNS soliton hierarchy, which has nonisospectral zero curvature representation. Bäcklund transformation for a splitting of \(sl(2)\) and a soliton exact solution for it was obtained [18].

The whole paper is organized as follows: in Section 2, we use Lenard operator pairs to briefly derive (1+1)-dimensional AKNS soliton hierarchy and give the (2+1)-dimensional integrable equation (1). Then, in Section 3, based on the solutions of the (1+1)-dimensional soliton equations and the elliptic coordinates, the solution of the (2+1)-dimensional integrable equation is reduced to solving ordinary differential equations. In Section 4, a hyperelliptic Riemann surface and Abel-Jacobi coordinates are introduced to straighten the associated flows. The Jacobi’s inversion problem is discussed, from which the
algebro-geometric solution of the \((2 + 1)\)-dimensional integrable equation is obtained in terms of the Riemann theta functions. A short summary is in Section 5.

2. The \((2 + 1)\)-Dimensional Soliton Equation

It is well known that the AKNS soliton hierarchy is isospectral evolution equation hierarchy associated with the spectral problem [17].

\[ \psi_x = U \psi = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix} \psi, \quad \lambda_i = 0, \quad \psi = (\psi_1, \psi_2)^T. \]

Consider the Lenard gradient sequence \(\{S_j\}_{j=0}^{\infty}\) by

\[ KS_{j+1} = JS_j, \]
\[ S_j|_{(q,r)=(0,0)} = 0, \]
\[ S_0 = (0, 0, 1)^T, \]

where \(S_j = (S^{(1)}_j, S^{(2)}_j, S^{(3)}_j)\) and

\[ K = \begin{pmatrix} \frac{1}{2} \partial & 0 & -r \\ 0 & -\frac{1}{2} \partial & -q \\ -q & r & \partial \end{pmatrix}, \]
\[ J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -q & r & \partial \end{pmatrix}. \]

It is easy to see that \(S_j\) is uniquely determined by the recursion relation. A direct calculation gives that

\[ S_1 = \begin{pmatrix} -r \\ -q \\ 0 \end{pmatrix}, \]
\[ S_2 = \begin{pmatrix} -\frac{1}{2} r_x \\ -\frac{1}{2} q_x \\ -\frac{1}{2} q_r \end{pmatrix}, \]

\[ S_3 = \begin{pmatrix} \frac{1}{4} r_{xx} + \frac{1}{2} q_x r \\ -\frac{1}{4} q_{xx} + \frac{1}{2} q_x^2 \\ \frac{1}{4} (rq_x - qr_x) \end{pmatrix}. \]

The auxiliary spectral of (2) is

\[ \psi_{tn} = V \psi = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \psi = \begin{pmatrix} \sum_{j=0}^{n} S^{(3)}_j \lambda^{-j} & \sum_{j=0}^{n} S^{(2)}_j \lambda^{-j} \\ \sum_{j=0}^{n} S^{(1)}_j \lambda^{-j} & -\sum_{j=0}^{n} S^{(3)}_j \lambda^{-j} \end{pmatrix} \psi. \]

The compatibility condition between (2) and (6) is the zero curvature equation:

\[ U_{tn} - V^{(n)}_{tn} + [U, V^{(n)}] = 0, \]

which is equivalent to the hierarchy of soliton equations

\[ X_n = \begin{pmatrix} q_{tn} \\ r_{tn} \end{pmatrix} = \begin{pmatrix} -2S^{(2)}_{m+1} \\ 2S^{(1)}_{m+1} \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} S^{(1)}_{n+1} \\ S^{(2)}_{n+1} \end{pmatrix}, \]

\(n = 1, 2, \ldots\)

The first two nontrivial members in the hierarchy are

\[ \begin{cases} q_{t_1} = \frac{1}{2} (q_{xx} - qr^2), \\ r_{t_1} = \frac{1}{2} (-r_{xx} + qr^2), \end{cases} \]

\[ \begin{cases} q_{t_2} = \frac{1}{4} (-q_{xxx} + 6qrq_x), \\ r_{t_2} = \frac{1}{4} (r_{xxx} + 6rqr_x). \end{cases} \]

Let \(t_2 = y, t_3 = t, u(x,y,t) = q(x,y,t), \) and \(v(x,y,t) = r(x,y,t)\) in (9) and (10); then, we can obtain the \((2 + 1)\)-dimensional equation (1) by the use of the following equation:

\[ (uv_x - u_x v)_x = -2(uv)_{yy}. \]

Therefore, if \(q\) and \(r\) are the compatible solutions of (9) and (10), then we can get that \(u = q\) and \(v = r\) are also the solutions of the \((2 + 1)\)-dimensional equation (1).

3. Variable Separation

In this section, we shall show how the \((1 + 1)\)-dimensional (9) and (10) are reduced to solvable ordinary differential
equations. Assume that (2) and (6) have two basic solutions\( \psi = (\psi_1, \psi_2)^T \) and \( \phi = (\phi_1, \phi_2)^T \). We define a matrix \( W \) of three functions \( f, g, h \) by

\[
 W = \frac{1}{2} (\phi \psi^T + \psi \phi^T) \sigma = \begin{pmatrix} f & g \\ h & -f \end{pmatrix}, \quad \sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

(12)

It is easy to verify by (2) and (6) that

\[
 W_x = [U, W], \quad W_{t_n} = \left[v^{(n)}, W\right],
\]

(13)

which imply that the functions \( \det W \) is a constant independent of \( x \) and \( t_m \). Equation (13) can be written as

\[
 g_x = -2g\lambda - 2qf, \quad h_x = 2h\lambda + 2rf,
\]

(14)

\[
 f_x = qh - rg, \quad g_t = -2gA - 2fB, \quad h_t = 2hA + 2fC,
\]

(15)

Now, suppose that the functions \( f, g, \) and \( h \) are finite-order polynomials in \( \lambda \):

\[
 f = \sum_{j=0}^{N+1} f_j \lambda^{N+1-j}, \quad g = \sum_{j=0}^{N+1} g_j \lambda^{N+1-j}, \quad h = \sum_{j=0}^{N+1} h_j \lambda^{N+1-j}.
\]

(16)

Substituting (16) into (14) yields

\[
 KG_{j-1} = f_j, \quad JG_j = 0, \quad KG_N = 0,
\]

(17)

\[
 G_j = \begin{pmatrix} g_j, h_j, f_j \end{pmatrix}^T.
\]

It is easy to see that \( JG_0 = 0 \) has the general solution:

\[
 G_0 = \alpha_0 S_0,
\]

(18)

where \( \alpha_0 \) is constant of integration. So, \( \text{Ker} f = \{ cS_0 | \forall c \} \). Acting with the operator \((J^{-1}K)^{k+1}\) upon (18), we can obtain from (3) and (17) that

\[
 G_k = \sum_{j=0}^{k} \alpha_j S_{k-j}, \quad k = 0, 1, \cdots
\]

(19)

where \( \alpha_0, \ldots, \alpha_k \) are integral constants. Substituting (19) into (17) obtains the following stationary evolution equation:

\[
 \alpha_0 KS_N + \cdots + \alpha_N KS_0 = 0.
\]

(20)

This means that expression (16) is existent.

In what follows, we decompose (9) and (10) into systems of integrable ordinary differential equations. Without loss of generality, let \( \alpha_0 = 1 \). From (3) and (19), we have

\[
 g_0 = 0, \quad \quad g_1 = -q,
\]

(21)

\[
 g_2 = \frac{1}{2} q_x - \alpha_1 q, \quad g_3 = -\frac{1}{4} q_{xx} + \frac{1}{2} q r + \frac{1}{2} q_x - \alpha_2 q,
\]

\[
 h_0 = 0, \quad \quad h_1 = -r, \quad h_2 = -\frac{1}{2} r_x - \alpha_1 r, \quad h_3 = -\frac{1}{4} r_{xx} + \frac{1}{2} q r^2 - \frac{1}{2} q_x - \alpha_2 r,
\]

(22)

\[
 f_0 = 1, \quad f_1 = \alpha_1, \quad f_2 = -\frac{1}{2} q r + \alpha_2,
\]

(23)

\[
 f_3 = \frac{1}{4} (r q_x - q r_x) - \alpha_1 q r + \alpha_3 q_4, \quad f_4 = \frac{1}{8} q_{xxx} - \frac{3}{4} q r q_x - \frac{1}{2} \left( \frac{1}{2} q_{xx} - q r^2 + \frac{1}{2} q_x - \alpha_3 q \right) + \frac{1}{2} q_x - \alpha_3 q,
\]

(24)

\[
 h_4 = -\frac{1}{8} r_{xxx} + \frac{3}{4} q r r_x + \frac{1}{2} \left( \frac{1}{2} r_{xx} + r^2 q \right) - \alpha_3 r - \alpha_3 r, \cdots
\]

(25)

We can write \( g \) and \( h \) as the following finite products:

\[
 g = -q \prod_{i=1}^{N} (\lambda - \mu_i), \quad h = -r \prod_{i=1}^{N} (\lambda - \nu_i).
\]

(26)

Comparing the coefficients of \( \lambda^{N-1}, \lambda^{N-2}, \) and \( \lambda^{N-3} \), we
get

\begin{align*}
g_2 &= q \sum_{j=1}^{N} \mu_j, \\
h_2 &= r \sum_{j=1}^{N} v_j, \\
g_3 &= -q \sum_{i<j} \mu_i \mu_j, \\
h_3 &= -r \sum_{i<j} v_i v_j, \\
g_4 &= q \sum_{i<j<k} \mu_i \mu_j \mu_k, \\
h_4 &= r \sum_{i<j<k} v_i v_j v_k,
\end{align*}

which together with (23), we obtain

\begin{align*}
\frac{1}{2} \partial_t \ln q &= \sum_{i=1}^{N} \mu_i + \alpha_1, \\
\frac{1}{2} \partial_t \ln r &= -\sum_{i=1}^{N} v_i - \alpha_1, \\
\frac{1}{2} \partial_t \ln g &= \sum_{i<j} \mu_i \mu_j + \alpha_1 \sum_{i=1}^{N} \mu_i + \alpha_1^2 - \alpha_2, \\
\frac{1}{2} \partial_t \ln h &= -\sum_{i<j} v_i v_j - \alpha_1 \sum_{i=1}^{N} v_i + \alpha_1^2 + \alpha_2, \\
\frac{1}{2} \partial_t \ln g &= -\sum_{i<j<k} \mu_i \mu_j \mu_k - \alpha_1 \sum_{i<j} \mu_i \mu_j - (\alpha_1^2 - \alpha_2) \sum_{i=1}^{N} \mu_i - \alpha_1^3 + 2 \alpha_1 \alpha_2 - \alpha_3,
\end{align*}

\begin{align*}
\frac{1}{2} \partial_t \ln h &= \sum_{i<j<k} v_i v_j v_k + \alpha_1 \sum_{i<j} v_i v_j + (\alpha_1^2 - \alpha_2) \sum_{i=1}^{N} v_i + \alpha_1^3 - 2 \alpha_1 \alpha_2 + \alpha_3.
\end{align*}

Let us consider the function \(\det W\), which is a \((2N+2)\)-order polynomial in \(\lambda\) with constant coefficients of the flow and \(\tau_r\) flow:

\[-\det W = f^2 + gh = \prod_{j=1}^{2N+1} (\lambda - \lambda_j) \equiv R(\lambda)\]  \hspace{1cm} (32)

Substituting (16) into (32), comparing the coefficient of \(\lambda^{2N+1}\), \(\lambda^{2N}\), and \(\lambda^{2N-1}\), and considering (23), we can obtain

\begin{align*}
\alpha_1 &= -\frac{1}{2} \sum_{j=1}^{2N+2} \lambda_j, \\
\alpha_2 &= \frac{1}{2} \sum_{j \neq i} \lambda_i \lambda_j - \frac{1}{8} \left( \sum_{j=1}^{2N+2} \lambda_j \right)^2, \\
\alpha_3 &= \frac{1}{2} \sum_{j \neq i} \lambda_i \lambda_j + \frac{1}{4} \sum_{j=1}^{2N+2} \lambda_i \lambda_j - \frac{1}{16} \left( \sum_{j=1}^{2N+2} \lambda_j \right)^3.
\end{align*}

From (32) we see that

\begin{align*}
f_{|_{\lambda=\mu_k}} &= \sqrt{R(\mu_k)}, \\
f_{|_{\lambda=v_k}} &= \sqrt{R(\nu_k)}.
\end{align*}

Using (14) and (26), we get

\begin{align*}
g_{|_{\lambda=\mu_k}} &= -2qf_{|_{\lambda=\mu_k}} = -q \mu_k \prod_{i=1}^{N} (\mu_k - \mu_i), \\
h_{|_{\lambda=v_k}} &= 2rf_{|_{\lambda=v_k}} = -r \nu_k \prod_{i=1}^{N} (\nu_k - v_i).
\end{align*}

Together with (34), we get

\begin{align*}
\mu_k &= \frac{2 \sqrt{R(\mu_k)}}{\prod_{i=1}^{N} (\mu_k - \mu_i)}, \\
\nu_k &= -\frac{2 \sqrt{R(\nu_k)}}{\prod_{i=1}^{N} (\nu_k - v_i)}.
\end{align*}

Similarly, using (6) \((n = 2, n = 3)\), (16), (26), and (34), we get

\begin{align*}
\mu_{k_2} &= \frac{2 \sqrt{R(\mu_k)}}{\prod_{i=1}^{N} (\mu_k - \mu_i)} \left( \frac{\mu_k^2}{\prod_{i=1}^{N} (\mu_k - \mu_i)} + \sum_{i=1}^{N} \mu_i + \alpha_1 \right) \mu_k - \sum_{i<j} \mu_i \mu_j - \alpha_1 \sum_{i=1}^{N} \mu_i - \alpha_1^2 + \alpha_2, \\
\nu_{k_2} &= -\frac{2 \sqrt{R(\nu_k)}}{\prod_{i=1}^{N} (\nu_k - v_i)} \left( \frac{\nu_k^2}{\prod_{i=1}^{N} (\nu_k - v_i)} + \sum_{i=1}^{N} v_i + \alpha_1 \right) v_k - \sum_{i<j} v_i v_j - \alpha_1 \sum_{i=1}^{N} v_i - \alpha_1^2 + \alpha_2.
\end{align*}
\[ \mu_{\alpha} = \frac{2\sqrt{B(\mu_k)}}{\prod_{i=1}^{N}(\mu_k - \mu_i)} \left( \mu_k^2 + \left( \sum_{j=1}^{N} \mu_j + a_i \right)^2 \right) \mu_k^{N-1} + \sum_{j=1}^{N} \mu_j \mu_k^2 + a_i \sum_{j=1}^{N} \mu_j \mu_k + (\alpha_i - a_i) \sum_{j=1}^{N} \mu_j + a_i^2 - 2a_i a_j + a_j. \]

(39)

Therefore, if \( \lambda_1, \cdots, \lambda_{2N+2} \) are \( 2N + 2 \) distinct parameters and \( \mu_k, \nu_k, (k = 1, \cdots, N) \) are compatible solutions of differential equations (36), (37), and (39), then \( q \) and \( r \) determined by (28) are the compatible solution of (9) and (10), so we can get that \( u \) and \( v \) are also the solution of the \((2 + 1)\)-dimensional equation (1).

### 4. Algebro-Geometric Solution

We first introduce the hyperelliptic Riemann surface

\[ \Gamma : \xi^2 = R(\lambda), \]

\[ R(\lambda) = \prod_{j=1}^{2N+2} (\lambda - \lambda_j), \]

(40)

with genus \( g = N \). On \( \Gamma \), there are two infinite points \( \infty_1 \) and \( \infty_2 \), which are not points of \( \Gamma \). Equip \( \Gamma \) with the canonical basis of cycles \( a_1, \cdots, a_N, b_1, \cdots, b_N \), and the holomorphic differentials

\[ \bar{\omega}_i = \frac{\lambda^{-1} d\lambda}{\sqrt{R(\lambda)}}, I = 1, 2, \cdots, N. \]

(41)

Then, the period matrices \( A \) and \( B \) are defined by

\[ A_{ij} = \int_{a_i} \omega_j, \]

\[ B_{ij} = \int_{b_j} \omega_i. \]

(42)

Using \( A \) and \( B \), we can define the matrices \( C \) and \( \tau \), where

\[ C = (C_{ij}) = A^{-1}, \]

\[ \tau = (\tau_{ij}) = CB = A^{-1}B. \]

(43)

Then, matrix \( \tau \) can be shown to be symmetric, and it has positive define imaginary part. We normalize \( \bar{\omega}_j \) into the new basis \( \omega_j \):

\[ \omega_j = \sum_{i=1}^{N} C_{ij} \bar{\omega}_i, I = 1, 2, \cdots, N. \]

(44)

which satisfy

\[ \int_{a_k} \omega_i = \sum_{i=1}^{N} C_{ij} \int_{a_k} \omega_j = \sum_{i=1}^{N} C_{ij} A_{ik} = \delta_{jk}, \]

\[ \int_{b_k} \omega_i = \sum_{i=1}^{N} C_{ij} \int_{b_k} \omega_j = \sum_{i=1}^{N} C_{ij} B_{ik} = \tau_{jk}. \]

(45)

For a fixed point \( \rho_0 \), then we introduce Abel-Jacobi coordinate as follows:

\[ \rho_m = (\rho_m^{(1)}, \rho_m^{(2)}, \cdots, \rho_m^{(N)})^T, m = 1, 2, \]

(46)

whose components are

\[ \rho^{(j)}_1(x, y, t) = \sum_{k=1}^{N} \int_{x_0}^{x} \rho^{(j)}_1(y, t) \omega_j = \sum_{k=1}^{N} \sum_{l=1}^{N} C_{jl} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \]

\[ \rho^{(j)}_2(x, y, t) = \sum_{k=1}^{N} \int_{y_0}^{y} \rho^{(j)}_2(x, t) \omega_j = \sum_{k=1}^{N} \sum_{l=1}^{N} C_{jl} \frac{\lambda^{l-1} d\lambda}{\sqrt{R(\lambda)}}, \]

(47)

From (47) and first expression of (36), we get

\[ \partial_x \rho^{(j)}_1 = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\mu^{l-1} \mu_{k-l}}{\sqrt{R(\mu_k)}} = \sum_{k=1}^{N} \sum_{l=1}^{N} \frac{\tau_{lk}^{l-1}}{\prod_{k=1}^{N} (\mu_k - \mu_1)}, \]

\[ = 2C_{jN} \Omega^{(1)}_0, j = 1, \cdots, N, \]

(48)

with the help of the following equality

\[ \sum_{k=1}^{N} \frac{\mu^{l-1}}{\prod_{i=1}^{N} (\mu_k - \mu_i)} = \delta_{NI}, I = 1, \cdots, N. \]

(49)

In a similar way, we obtain from (36)–(39), (47), and (48) that

\[ \partial_y \rho^{(j)}_1 = 2(-C_{jN} \mu_1 + a_i C_{jN} - (\alpha_i^2 - a_i) C_{jN}) = \Omega^{(1)}_0, \]

\[ \partial_y \rho^{(j)}_2 = 2(-C_{jN} \mu_1 + a_i C_{jN} - (\alpha_i^2 - a_i) C_{jN}) = \Omega^{(2)}_0, \]

\[ \partial_x \rho^{(j)}_2 = -\Omega^{(2)}_0, \]

\[ \partial_x \rho^{(j)}_2 = -\Omega^{(2)}_0. \]

(50)

On the basis of these results, we get the following:

\[ \rho^{(j)}_1(x, y, t) = \Omega^{(1)}_0 x + \Omega^{(1)}_1 y + \Omega^{(2)}_0 t + \gamma^{(1)}_0, \]

\[ \rho^{(j)}_2(x, y, t) = -\Omega^{(2)}_0 x - \Omega^{(1)}_0 y - \Omega^{(2)}_1 t + \gamma^{(1)}_0, \]

(52)
where
\[
Y_0^{(j)} = \sum_{k=1}^{N} \int_{p_k} \omega_j, \\
Y_1^{(j)} = \sum_{k=1}^{N} \int_{p_k} \omega_j.
\] (53)

An Abel map on \( \Gamma \) is defined as
\[
A(p) = \int_{p_0}^{p} \omega, \quad \omega = (\omega_1, \ldots, \omega_N)^T, \\
A\left( \sum n_k p_k \right) = \sum n_k A(p_k).
\] (54)

Consider two special divisors \( \sum_{k=1}^{N} p_1^{(k)} (m = 1, 2) \), and we have
\[
A\left( \sum_{k=1}^{N} p_1^{(k)} \right) = \sum_{k=1}^{N} A\left( p_1^{(k)} \right) = \sum_{k=1}^{N} \int_{p_k} \omega = \rho_1, \\
A\left( \sum_{k=1}^{N} p_2^{(k)} \right) = \sum_{k=1}^{N} A\left( p_2^{(k)} \right) = \sum_{k=1}^{N} \int_{p_k} \omega = \rho_2,
\] (55)

where \( p_1^{(k)} = (\mu_k, \xi(\mu_k)), p_2^{(k)} = (\bar{\mu}_k, \xi(\bar{\mu}_k)) \). The Riemann theta function of \( \Gamma \) is defined as
\[
\theta(\zeta) = \sum_{z \in \mathbb{Z}^N} \exp\left( \pi i (\tau z + 2 \pi i \zeta, z) \right), \quad \zeta \in \mathbb{C}^N,
\] (56)

where \( \zeta = (\zeta_1, \ldots, \zeta_N)^T, (\zeta, z) = \sum_{j=1}^{N} \zeta_j z_j \). According to the Riemann theorem, there exist two constant vector \( M_1, M_2 \in \mathbb{C}^N \) such that
\[
F_m = \theta(A(p) - \rho_m - M_m), \quad m = 1, 2,
\] (57)

has exactly zeros at \( \mu_1, \ldots, \mu_m \) for \( m = 1 \) or \( \nu_1, \ldots, \nu_N \) for \( m = 2 \) and \( m = 3 \). To make the function single valued, the surface \( \Gamma \) is cut along all \( a_1, b_1 \) to form a simple connected region, whose boundary is denoted by \( \gamma \). Notice the fact that the integrals
\[
\frac{1}{2\pi i} \int_{\gamma} \lambda^k \ln F_m(\lambda) = I_k(\Gamma), \quad k \geq 1,
\] (58)

are constants independent of \( \rho_1, \rho_2 \) with \( I = I(\Gamma) = \sum_{j=1}^{N} \int_{a_j} \lambda^k \omega_j \). By the residue theorem, we have
\[
I_k(\Gamma) = \sum_{j=1}^{N} \mu_j^k + \sum_{j=1}^{N} \Re s_{\lambda \rightarrow 0} \lambda^k \ln F_1(\lambda),
\] (59)

\[
I_k(\Gamma) = \sum_{j=1}^{N} \nu_j^k + \sum_{j=1}^{N} \Re s_{\lambda \rightarrow 0} \lambda^k \ln F_2(\lambda).
\] (60)

Here, we only need to compute the residues in (59) for \( k = 1, 2, 3 \). In the way similar to calculations in [1, 2, 4], we obtain
\[
\Re s_{\lambda \rightarrow 0} \lambda \ln F_m(\lambda) = \Re s_{\lambda \rightarrow 0} \lambda^{-1} \ln F_m(z^{-1})
= (-1)^s \partial_s \ln \theta_1^{(m)}, \quad s = 1, 2; \quad m = 1, 2.
\] (61)

where \( \theta_1^{(1)} = \theta(\Omega_2 x + \Omega_1 y + \Omega_2 t + \pi_1), \theta_2^{(2)} = \theta(-\Omega_2 x - \Omega_1 y - \Omega_2 t + \pi_1), \pi_1, \pi_2 \) are constants. Thus from, we arrive at
\[
\sum_{j=1}^{N} \mu_j = I_1 - \partial_s \ln \theta_1^{(1)}, \\
\sum_{j=1}^{N} \nu_j = I_1 - \partial_s \ln \theta_2^{(2)}.
\] (62)

Similarly, we obtain
\[
\sum_{j=1}^{N} \mu_j^2 = I_2 + \frac{1}{2} \partial_{s^2} \ln \theta_1^{(1)} \theta_2^{(2)} - \frac{1}{2} \partial_{s^3} \ln \theta_1^{(1)} \theta_2^{(2)}, \\
\sum_{j=1}^{N} \nu_j^2 = I_2 + \frac{1}{2} \partial_{s^2} \ln \theta_1^{(2)} \theta_2^{(2)} - \frac{1}{2} \partial_{s^3} \ln \theta_1^{(2)} \theta_2^{(2)},
\] (63)

Then, we can get
\[
\partial_s \ln q = 2 \left( I_1 - \partial_s \ln \theta_1^{(1)} \right) + 2 \alpha_1 = \Theta_1, \\
\partial_s \ln r = -2 \left( I_1 - \partial_s \ln \theta_1^{(2)} \right) - 2 \alpha_1 = \Lambda_1.
\]
Data Availability

All data and models generated or used during this study appear in the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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