Relax the Non-Collusion Assumption for Multi-Server PIR

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Abstract—For distributed protocols involving many servers, assuming that they do not collude with each other makes some secrecy problems solvable and reduces overheads and computational hardness assumptions in others. While the non-collusion assumption is pervasive among privacy-preserving systems, it remains highly susceptible to covert, undetectable collusion among computing parties. This work stems from an observation that if the number of available computing parties is much higher than the number of parties required to perform a secure computation, collusion attempts could be deterred.

We focus on the standard problem of multi-server private information retrieval (PIR) that inherently assumes that servers do not collude. For PIR application scenarios, such as those for blockchain light clients, where the available servers are plentiful, a single server’s deviating action is not tremendously beneficial to itself. We can make deviations undesired through small amounts of rewards and penalties, thus raising the bar for collusion significantly. For any given multi-server 1-private PIR (i.e. the base PIR scheme is constructed assuming no pairwise collusion), we provide a collusion mitigation mechanism. We first define a two-stage sequential game that captures how rational servers interact with each other during collusion, then determine the payment rules such that the game realizes the unique sequential equilibrium: a non-collusion outcome. We also offer privacy protection for an extended period beyond the time the query executions happen, and guarantee user compensation in case of a reported privacy breach. Overall, we conjecture that the incentive structure for collusion mitigation to be functional towards relaxing the strong non-collusion assumptions across a variety of multi-party computation tasks.

I. INTRODUCTION

Several privacy-preserving solutions require the non-collusion assumption among computing parties for their security. This assumption allows us to solve some important problems [13], [16], [37], with no known solutions without it. It also reduces overheads and allows us to bypass computational hardness results in other problems such as private information retrieval (PIR) [10]. In fact, the entire field of secure multi-party computation (MPC) [5], [54] relies on the non-collusion assumption.

While most of the privacy-preserving distributed protocols above do allow collusion among pre-defined subsets of parties, all bets are off once the adversary can lure a superset of parties. Indeed, the use of the non-collusion assumption remains highly susceptible to undetectable collusion among the computing parties: from a legal perspective, while collusion can be penalized using antitrust laws, collusion over Internet-based secure systems can be difficult to prove, if not impossible; parties may collude without even recognizing each other. As these distributed protocols are getting considered for applications such as blockchain privacy (e.g., [28], [43]), non-collusion is considered as an Achilles’ heel for large adoption.

In the context of database querying service, a PIR protocol allows a client to query a database without revealing which piece of information is of interest. Chor et al. [10] formalize the $k \geq 2$ server PIR problem and proposed a mechanism for fetching a single bit of an $N$-bit string, stored at $\ell \geq k$ servers (or parties/players), without disclosing the bit’s location to the $k$ queried servers. More generally, we can consider a setting where $\ell$ servers store an identical database $D$ comprising $N$ entries and offer data retrieval services for each entry.

PIR can be constructed from single server and multiple servers. Single server information-theoretic PIR or IT-PIR achieves information-theoretic privacy and is not possible beyond the trivial solution of sending the whole database to the user. Computational PIR or cPIR [36] approaches this problem by including computational hardness assumptions for problems like quadratic residuosity. The computational inefficiency of single-server computational PIR schemes is well known [3], [31], [46]. By involving more non-colluding servers ($k \geq 2$), the communication complexity can be substantially reduced to $O(\log N)$, which serves as a motivation for multi-server PIR schemes. But the key concern is that in reality, servers can easily collude through various covert and anonymous channels [47], and the collusion remains oblivious to PIR executions. And this is the motivation for this work.

The difficulty of privacy-related collusion detection lies in the fact that collusion can be orthogonal to protocol runs. Dong et al. [17] explores discouraging correctness-related collusion in cloud computing. The key element is to have colluding parties collude via a smart contract which can be verified by another contract quite easily. This is because correctness naturally implies verifiability and there, collusion is implemented with smart contracts completely observable by others. However, privacy-pertinent collusion can happen anywhere (e.g., outside the system under discussion), any time (e.g., long after the run of the protocol), and its existence is hard to verify. We follow the intuition from a bit guessing game and seek to derive a design with proper payment rules to achieve a non-collusion outcome.

Secret-Shared Bit Guessing Game. Consider a game where the game host commits to a secret unbiased bit $b$ and players
can make one open guess about \( b \). The correctness of guesses is announced after no more player wants to guess. A player receives $2 for a correct guess and loses $4 for a wrong guess. Rational players do not participate because the expected return is negative ($-1$). Suppose the host now secret-shares the bit among \( k \) parties and if the game rules stay the same, players can collaborate to recover the bit and make correct guesses. The collaboration process is oblivious to the game.

To stop collusion, the host can change the rules to reward the player making the first correct guess with $2 and take $4 from other \( k - 1 \) players, of which an additional $2 goes to this player. Assuming equal speed of submitting a guess when players are ready to guess, not colluding is strictly dominating when \( k > 2 \) and weakly dominating for \( k = 2 \). The implication is that it’s possible to make collusion visible and undesired via proper incentive structure.

This simple game is already expressive, but it also contains caveats that can cause the incentives to stop being effective in practice, e.g. a malicious host, a player with private knowledge about \( b \), etc. Proper designing of the structure and determination of the incentive-related parameters is crucial. We observe that in many settings such as blockchains, where the number of available computing parties can be much higher than the required set, an effective incentive structure can be constructed and made practical. While this observation has extensive applicability, this work focuses on the problem of mitigating collusion in PIR protocols.

A. Contributions

We study the collusion problem in multi-server PIR and design an incentive structure to mitigate such collusion. The high-level idea of our solution is to incentivize colluding servers to report collusion with verifiable evidence. Incentivizing colluding servers to make such a desired move is one sub-problem. Defining and verifying evidence is the other. On top of the two, we need to ensure the practicality of the design, e.g. payments involved should be reasonable. Incentive issue is resolved via a game-theoretic design, with intuitions from the bit-guessing game; evidence verification is settled cryptographically; economic feasibility is achieved by having many available servers. Overall, we make the following contributions.

First, we present a game-theoretic collusion deterrence mechanism for multi-server PIR assuming many servers being present, so that one can enjoy the efficiency of 1-privacy if\footnote{1-privacy PIR schemes are constructed assuming no pairwise collusion and provide effectively perfect efficiency in terms of downloading capacity, in the sense that to fetch a \( b \)-bit entry, one only needs to download \( b + 1 \) bits [45].}. Delegating collusion mitigation to our proposed mechanism external to PIR protocols imposes little additional efficiency cost.

The mechanism skeleton is essentially a sequential game, where non-colluding outcome is the sequential equilibrium. We consider general collusion protocols where parties can compute arbitrary functions on received queries or inputs derived from them via a secure MPC protocol, e.g. reconstructing the queried entry from responses. To achieve a practical incentive structure, we have many servers and let clients send companion queries. In this way, a single server does not bear too much information worth to leak before it is discouraged to collude (similar to when the players are reluctant to recover the target bit in the bit-guessing game). We also tackle manipulations of the mechanism from parties of information advantage, e.g., users or servers with private knowledge.

Second, we ensure longer-term privacy. Collusion can happen anytime after the queries, while the incentive structure is constructed for a single run of a PIR protocol. Thus, to ensure privacy for an extended time period, a privacy protection self-insurance is constructed. Due to the finite nature of the insurance pool, we set expiration dates for privacy protection.

Third, we extend the discussion to coalitional players. Servers inside a coalition/clique have complete trust for each member, thus immune to our incentive structure, and this is where the difficulty lies. If the group size is smaller than the number of queries needed for reconstruction, the problem is reduced to the previous one. For larger group size, we analyze the problem from cooperative game theory perspective and identify the conditions necessary to keep group size small.

Last, we implement and deploy the coordinator as a smart contract on Ethereum. The contract instantiates the game rules, maintains essential data, and resolves collusion accusations. We apply optimization to economize the gas costs, whose estimates reside in Table II.

Blockchain Application. While the proposed solution can utilize blockchains for implementation, blockchains can also find the design helpful [29], e.g., allowing light clients to query full nodes in a privacy-preserving way [43]. Existing solutions are either inefficient given the ever-changing nature of database [43], or require trusted hardware [38], [40], [52]. Blockchains are suitable application scenarios as they have many servers (full nodes) to achieve security and better performance, which admits more practical parameters for our design.

II. Preliminaries and Overview

A. Preliminaries

Multi-server PIR. We borrow the definition of multi-server PIR primitive described in [26]. Let \( I, Q, R, Y \) respectively denote the random variables for the entry index \( i \in [N] \) of interest, the query string \( q \in \{0, 1\}^\gamma \), the response string \( r \in \{0, 1\}^\gamma \) and the output \( y \in \mathbb{D} \) reconstructed by the user.

Definition 1. The interaction described by \( (I, Q, R, Y) \) provides correctness if \( \Pr[Y = \mathbb{D}_i | I = i] = 1 \) where \( \mathbb{D}_i \) is the \( i \)-th entry in database \( \mathbb{D} \).

The interaction described by \( (I, Q, R, Y) \) provides perfect I-privacy if \( \forall i, i^* \in [N], \Pr[Q = q | I = i] = \Pr[Q = q | I = i^*] \) where the probability is over all the random coin tosses made by the client.
The interaction described by \((I, Q, R, Y)\) provides computational 1-privacy if \(\forall i, i^* \in [N]\), the distribution ensembles \(\{Q[I = i]\}_{\lambda \in \mathbb{N}}\) and \(\{Q[I = i^*]\}_{\lambda \in \mathbb{N}}\) are computationally indistinguishable with security parameter \(\lambda\).

In this work we focus on \(k\)-out-of-\(k\) and 1-private PIR schemes. We denote the multi-server PIR scheme as \((\ell, k, 1)\)-PIR. A set of \(\ell\) servers \(S = \{S_i | i \in [\ell]\}\) maintain identical databases consisting of \(N\) entries, which we represent with \(D = \{D_i | i \in [N]\}\). In each round, to query the entry at index \(a\), user \(U\) generates \(k\) queries and sends them to \(k\) (distinct) servers, which we denote as \(S_M \subseteq S\) \((|S_M| = k)\). Each queried server \(S_j \in S_M\) locally computes results corresponding to the query and responds with answer string \(A_j\). \(U\) then collects all \(k\) responses and reconstructs the queried entry \(D_a\).

There is an abundance of available PIR constructions, including the ones for two servers [6], [10], [18], [44], [51], and three or more servers [2], [4], [9], [19], [26], [30], [35], [48], [55]. While all constructions suffice for the purpose of this work, we find Boyle-Gilboa-Ishai’s construction for 2-server cPIR [6] and Hafiz and Henry’s construction for more server case [26] to be suitable. (See Section E for further details.)

### B. Threat Model

We assume all servers to be rational and take actions that maximize their profits in analysis. (If servers turn out to be irrational, we aim to compensate the users for privacy loss.) Normally, servers provide PIR services and are compensated by the system in the form of service fees. They can potentially collude with each other to break user privacy and obtain what the recovered private information is worth. Servers can collude as individuals (Section III, IV), or form coalitions (Section VII), or leave the service (Section VI).

To make the results more general, we handicap ourselves by giving the adversary advantages in collusion. We allow servers to communicate with each other through two-way anonymous secret channels, which provide sender and receiver anonymity and keep the communication covert and undetectable. Further, we assume the communication is simulatable such that a server can generate the communication transcripts on its own and colluding servers can self-protect by denying having the conversation. This is because the correspondence scripts may not contain irrefutable proofs like secure digital signatures. The nature of the communication can be hard to automatically determine even if signatures are present. If a server decides to collude, we assume there exists a secure fair-information exchange protocol, or/and a secure fair MPC protocol allowing colluding parties to compute the result collectively.

We assume the underlying PIR protocol is secure under its assumptions, which means that it ensures correctness and privacy (Definition 1). Each server receives at most one query in a round (i.e. a complete run of PIR for one entry between a user and selected servers). Servers hold a common prior probability distribution \(p = (p_1, ..., p_N)\) over all \(N\) entries in database \(D\). Servers can have arbitrary private information about certain users.

### C. Solution Overview

We now describe the solution and present pseudocodes in Figure 1. Servers are indexed and participants all know the indices. Due to the nature of our discussion, we also address servers as players or parties.

**System.** There are \(\ell\) servers maintaining the same database \(D\). At the beginning, servers make deposits to a coordinator contract (CC) which will be transferred back when servers exiting the PIR service. A user picks at random \(k\) servers and \(\omega \geq 1\) random entries \(D_{r_1}, \ldots, D_{r_\omega}\). The user then generates \(k\) queries for the intended entry \(D_a\) and for each of the \(\omega\) random entries (we call them companion queries). This forms \(k\) vectors of queries. The user then randomly permutes the order of queries in each vector and send query vectors to the \(k\) selected servers. Queries sent to queried servers are signed by the client. The client also post the queried server list onto a public server functioning as bulletin board (BB).

Queried servers post commitments of responses to BB and send de-commit information to the user through any secure channels. The user can reconstruct after accumulating the responses. This work is not specifically concerned with Byzantine behaviors of servers, including sending back wrong responses or not responding, but it’s straightforward to see that these two deviations can be mitigated with slight changes to the system, e.g., giving CC access to database \(D\).

**How collusion proceeds.** Queried servers participating in a collusion protocol can adopt (1) approach 1, where servers exchange information and recover the queried entry locally. Or they can adopt (2) approach 2, where servers compute some non-trivial (i.e., output depends on the inputs) function \(f(\cdot)\) on queries or responses with a secure 2-party computation (2PC) or multi-party computation (MPC) protocol, where they provide secret information received from clients as inputs and receive the output without learning about others’ inputs. For example, they can input the received queries to compute the queried index or its least significant bit. In approach 1, one colluding party holds evidence against one other party for each information exchange. In approach 2, one colluding server may hold evidence against \((k - 1)\) other parties.

**Intuition behind discouraging collusion.** We indirectly detect and deter collusion through a “reporting then verification” procedure. The intuition is to fine colluding servers and to reward collusion reporters. We construct an incentive structure in the form of a sequential game where non-collusion is the sequential equilibrium. The key is to determine practical payment rules properly to arrive at the desired non-collusion outcome. We state the following intuitions regarding payment amounts and a notation table can be found in Table I:

- Reporting reward always covers accusation fee.
- If no user compensation is required, service fee can be made arbitrarily low (by having arbitrarily high server deposit).
- When user compensation is desired, service fee increases according to privacy value to balance between user compensation (user being the victim) and defense against user manipulation (user being the attacker).
• Ceteris paribus, when fewer queries are sent in one round, higher server deposit is needed.

Collusion Mitigation Coordinator. The coordinator algo-

Normal Service (for query $D_a$)

1) $U$ samples $k$ distinct integers in $[l]$ at random and obtains $k$ corresponding server indices: $id_1, \ldots, id_k$.
2) $U$ runs one instance of PIR protocol to query $D_a$ and generates $k$ queries: $Q_1, \ldots, Q_k$.
3) $U$ samples another index $a_r$ uniformly at random, runs one instance of PIR protocol with $a_r$, and generates $k$ queries: $Q_1', \ldots, Q_k'$.
4) $U$ posts index list $(id_1, \ldots, id_k)$ onto BB along with the signature $\sigma_{id} = \text{Sign}_{sk_U}(id)$;
5) $U$ sends $\{(Q_i, \sigma_i), (Q_i', \sigma_i')\}_{\text{perm}}$ (the queries, signatures are permuted at random) to server $id_i$ through any secure channels where $\sigma_i = \text{Sign}_{sk_U}(Q_i)$ and $\sigma_i' = \text{Sign}_{sk_U}(Q_i')$ are the signatures. $U$ stores this permutation locally.
6) Server $id_i$ verifies the signatures and computes the answers $A_i, A_i'$ locally against database $D$.
7) Server $id_i$ posts a commitment $c_{d,i} = \text{Comm}((r_{d,i}, A_i || A_i') (r_{d,i}, r_{d,i}'))$ onto BB and sends decommit information to $U$.
8) $U$ retrieves the responses, verifies the commitments, and reconstructs queried entry $D_a$.

Collusion Resolution (Server $id_i$, accusing)

1) Server $id_i$ submits a collusion report to CC: $(\text{type}, ed)$, indicating evidence type and evidence $ed$.
   The evidence either includes information exchanged or input used in collusion, non-trivial circuits for computing function $f(\cdot)$ along with results computed from MPC.
2) Each accused server $id_j$ submits auxiliary info:
   • Type I-query: $(Q_{j,i} || Q_{j,i}', \sigma_{sk_f}(Q_{j,i}))$.
   • Type I-response: $(A_{j,i} || A_{j,i}'$, de-commit information)
   • Type II: input to $f(\cdot)$ (either the query or response).
3) If server $id_i$ fails to submit information in time, CC confirms the accusation. Otherwise:
   • Type I-query: if $ed$ matches $Q_{j,i}$ or $Q_{j,i}'$, confirm.
   • Type I-response: CC repeats the commitment on $A_{j,i} || A_{j,i}'$ with corresponding randomness. If the outputs match the commitment on board and $A_{j,i} = ed$, confirm. If the outputs do not match information on BB, confirm.
   • Type II: if the input does not pass verification or if the output of $f(\cdot)$ on provided inputs match $ed$, confirm.

Fig. 1: Overview of routines ($\omega = 1$ companion query). We omit token transfers and address servers by their IDs. $\text{Sign}(\cdot)$ is a signature scheme and $\text{Comm}(\cdot)$ is a commitment scheme.

| Meaning | Functions |
|---------|-----------|
| $v_S$ (â) | Service fee |
| $v_I$ (â) | Server Initial deposit |
| $v_A$ (â) | Accusation fee |
| $v_B$ (â) | Reporting Reward |
| $v_C$ (â) | Privacy worth |

Table I: Notation table. $\text{Sign} \downarrow, \uparrow, -$ indicates the parameter is desired to be low, high, or as it is.

One convenient implementation of coordinator is a smart contract on a public blockchain system. We give more details in Section V after exploring the sequential game. When one server accuses with evidence, the contract counts down an evidence collection window and the involved parties who failed to provide the information needed in time are marked as “culpable”. For different types of evidence, the involved servers provide different auxiliary information. Note that we do not distinguish between collusion that recovers the intended index $a$ in any way.

The accuser provides one of the following information as evidence to the evidence verification algorithm: (Type I) the original query received by the accused or the query response of the accused; (Type II) the circuit for computing function $f(\cdot)$ (not the MPC circuits), the inputs to $f(\cdot)$ and the output computed from MPC. They respectively correspond to collusion approach 1 and 2. To verify Type I-query evidence, we have the accused server share its received query and signature. If the server stays silent, the collusion report is marked as valid. If the signature scheme is secure, we can verify such evidence correctly. For Type I-response evidence, we have the accused server provide the de-commit information. If the commitment scheme is secure, we can verify this evidence correctly. For Type II evidence, we have the accused server(s) provide corresponding inputs to $f(\cdot)$. We verify inputs, confirm the non-triviality of $f(\cdot)$ (details in Section V-A), and compare the output of $f(\cdot)$ on all verified inputs to the evidence.

III. COPING WITH 2 COLLUDING PARTIES

We consider $(l, k, 1)$-PIR and aim to provide the same privacy guarantees the backbone PIR implies with our mechanism while relaxing the non-collusion assumption. Throughout this section, we let $k = 2$. We continue to denote the intended queried entry that can be reconstructed from the responses as $D_a$. As depicted in Section E, in Boyle-Gilboa-Ishai’s construction, $U$ sends the 2 function and key shares $(f_i, k_i)$, $(f_j, k_j)$ to two servers $S_i, S_j$. If the two collude together, they may exchange the received shares or exchange responses (Type I evidence), or compute the entry $D_a$ together without information exchange (generating Type II evidence).

Colluding servers verify the information input by the other party, e.g., by including circuits that examine the digital signatures from the user. We also consider occasions when servers implement blind collusion in Section B, where they
only infer the authenticity of information but never directly verify.

A. A sequential game for $\ell = 2$ parties

Order of events. We have two servers $S_1, S_2$. Consider the following sequential game $G_0$ in $(2, 2, 1)$-PIR:

(0) In the initialization round or round zero, server $S_1, S_2$ each decides whether to collude ($C$ or $\bar{C}$). If $< 2$ parties have colluding intention, the game immediately ends.

1) In round one, each colluding server selects one of two actions independently and simultaneously: amicable ($A$), i.e. exchange the correct information, or deceitful ($D$), i.e. send an incorrect or junk message.

2) In round two, the two servers have exchanged information or computed nontrivial functions. Crucially, a server knows whether the other server played $D$ in the previous round. In this round, each server selects one of two actions: report the collusion ($R$) or keep it secret ($\bar{R}$).

Then the game terminates.

Payoffs. If the game stops at initialization round, the servers receive service fees $v_{SI}$. Otherwise, the payoffs are determined by a payment rule $\phi(\cdot)$, which takes as input the actions of the agents throughout the game. A server receives $v_C$ if they successfully recover the user’s query. Since the colluding servers also retrieve $\omega$ randomly picked entries, the actual information gain is worth $v_C^2 \in \left[ \frac{v_C^2}{2^{\omega+1}}, v_C \right]$, depending on the private information parties hold. When they report, they pay an accusation fee $v_A$ and get a reporting bonus $v_{RI}$, upon a successful accusation but lose deposit $v_I$ if the report is incorrect. The accused server in a successful report loses its deposit $v_I$.

In more detail, (i) if they both play amicable ($A$) in round 1, they gain payoff $v_C^2$ from recovering the user’s queried entry $v_u$. In round 2, when one party reports ($R$), it pays an accusation fee $v_A$. It receives reporting bonus $v_{RI}$ and the other party loses their deposit $v_I$ upon successful accusation. This means that $\phi(A_1 R_1, A_2 \bar{R}_2) = (v_A + v_C^2 + v_{RI} - v_A, v_S + v_C^2 - v_I)$. We abbreviate this strategy notation as $(A, A, \bar{R})$ hereafter. (ii) If both parties play deceitful ($D$) in round 1, they do not receive $v_C^2$. Other rules stay the same. (iii) If only one of the two plays amicable ($A$), only the deceitful party receives $v_C^2$ for recovering the secret. Other payment rules stay the same.

The complete payoff tree is shown in Figure 2. We focus on the part of the tree where collusion can actually happen (the shaded area in Figure 2). This is because if at least one party does not participate, the collusion game ends in initialization and is not effectively started. There are some subtleties when implementing the payment rules. First, the service fee $v_S$ is not transferred to servers right after a round of PIR service for a user has been accomplished. Because collusion can happen long after the service, we set a privacy protection period and detain service fees until the protection expiration time (See Section VI for more detailed discussion). Second, collusion report bonus $v_{RI} \leq v_I$, because rewards for reporters come from fines from the accused.

Additionally, the accusation fee $v_A > \frac{1}{w+1} v_{RI}$, where $w$ is the number of companion queries sent along with a true intended query ($w \geq 1$). This indicates that the accusation fee is higher than the upper bound of what one can expect to gain from false accusations. We have this lower bound for accusation fee because with prior $\omega$ and arbitrary private knowledge about the true index $a$, one can trivially make successful false accusations with Type II evidence with probability $\leq \frac{1}{w+1}$. The success probability is not 1 because the accuser needs to specify which of the query in (or response to) the received queries correspond to the one used in collusion. Without actual collusion, one makes a guess. The user randomly permutes the queries, so the probability of guessing correctly is $\frac{1}{w+1}$.

Information sets. The game we have defined is a game of incomplete information: in the first and second round, an agent is not certain about which action the other agent is taking. An information set for a player $S_i$ contains all decision nodes, at which player $S_i$ takes an action, indistinguishable by $S_i$.

Many information sets in our game are trivial, i.e. containing a single node. For example, in round 1, both agents are aware of the other agent’s interests in collusion. However, there are non-trivial information sets, e.g., the ones prefixed “12” in Figure 2, where the two make simultaneous decisions.

Solution concept. The game we have defined is a sequential game. Nash equilibrium, the standard notion of equilibrium in game theory, are known to predict “unreasonable” behaviors in sequential games. In a Nash equilibrium of a sequential game a player can commit to empty/non-credible threats.\(^3\) We consider a refinement of Nash equilibrium for sequential games, namely Sequential Equilibrium [34]. In this type of equilibria, a player chooses the optimal action (the action that maximizes expected utility from now on) at each information set, no matter what has happened in the past. Moreover, a sequential equilibrium not only prescribes a strategy for each player, but also a belief about other players’ strategies.

To formalize the notion of sequential rationality, we first formalize the notion of a belief.

Definition 2. For each information set $I$, a belief assessment $b$ gives the conditional probability distribution $b(\cdot|I)$ over all decision nodes $v \in I$.

For example, $b(D_2|I^A_1)$ is the belief of player $S_1$ that player $S_2$ takes action $D$ in round one, given that she is in information set $I_1^A = \{(A,A), (A,D)\}$. A belief assessment expresses all such beliefs for all players at their information sets.

\(^3\)Consider the Ultimatum game with two players. One player proposes a way to divide a sum of money between the two players. If the responding player accepts the proposal, then the money is divided accordingly; otherwise, both receive nothing. The responder can threaten to accept only fair offers, but this threat is not credible: given any non-zero amount, the only rational action for the responder is to accept.
Definition 3. A strategy profile and belief assessment pair \((s,b)\) is a sequential equilibrium if \((s,b)\) is sequentially rational and \(b\) is consistent with \(s\).

Here, a strategy profile \(s\) specifies all actions of all players. Given \((s,b)\), \(s\) is called sequentially rational if at each information set \(I\), the player to take an action maximizes her expected payoff, given beliefs \(b(.|I)\) and that other players follow \(s\) in the continuation (remaining) game. Given a strategy profile \(s\), for any information set \(I\) on the path of play of \(s\), \(b(.|I)\) are consistent with \(s\) if and only if they are derived using Bayes’ rule. For example, let player 1 be at information set \(I\) containing two nodes, \(x, y\). Suppose according to \(s\), they are respectively reached with probability 0.2, 0.3 (product of probabilities of actions along the path). Then a consistent belief for player 1 is that \(b(x|I) = \frac{0.2}{0.2 + 0.3} = 0.4\) and \(b(y|I) = 0.6\).

Analysis. As mentioned before, we only consider the game not ended in the initialization round, since otherwise collusion does not take place. As noted, the general setting \(\Pi\) for 2 server collusion in \((2,2,1)\)-PIR include:

\[\Pi(a)\] Payment values are positive and \(v_I \geq v_R > v_A, v_A > \frac{v_R}{w+1} v_{R^C}\)

\[\Pi(b)\] Users do not manipulate the scheme.

\[\Pi(b)\] is relieved after Section IV-A.

Our main result for the two agent case is the following:

Proposition 1. In a 2-party collusion game in \((2,2,1)\)-PIR in setting \(\Pi\), the unique sequential equilibrium is the pair \((s,b)\), where \(s = (DR,DR)\) and \(b\) satisfies \(b(D_2|I_{s_1}^A) = b(D_2|I_{s_2}^D) = b(D_1|I_{s_2}^A) = b(D_1|I_{s_2}^D) = b(R_2|I_{s_2}^A) = b(R_2|I_{s_2}^D) = b(R_1|I_{s_2}^A) = b(R_1|I_{s_2}^D) = 1\), and \(b(.|I) = 0\) in all other cases.

Informal. In the game we have defined, both servers play deceitfully in collusion in a sequential equilibrium. Condition \(v_I > v_A\) encourages reporting collusion in round 2 when the other party plays amicable \((A)\). Working backwards, this encourages playing deceitful in round 1.

Proof. The strategy \(s\) or the move probabilities, \(\alpha_2, \alpha_4\) and \(\beta_2, \beta_4\), are shown in Figure 2. Belief probabilities at the information sets prefixed with “12” are implicit in the tree, and we denote as \(b(.|I)\). Beliefs need to be consistent with \(s\).

Several of the variables are straightforward to determine. At information set 3 (node “12.3”), \(\alpha_4 = 1\) because \((v_{R}^D + v_S + v_R - v_A) > (v_{R}^A + v_S)\). Likewise, \(\beta_4 = 1\). At information set 1 (node “12.1”), we also have \(\alpha_3 = 1\) because \(S_1\)’s expected return is

\[
\alpha_3 \beta_3(v_{R}^A + v_R - v_I) + \alpha_3 (1 - \beta_3)(v_{R}^A + v_S + v_R - v_A) + (1 - \alpha_3) \beta_3(v_{R}^A - v_I) + (1 - \alpha_3) (1 - \beta_3) (v_{R}^A + v_S) = \alpha_3 [\beta_3 \frac{v_I - v_R}{2} + (v_R - v_A)] - \beta_3 (v_I + v_S) + v_S + v_{R}^A
\]
and \( v_I > v_R, v_R > v_A \). Likewise, \( \beta_3 = 1 \). The expected return for both players is simply \( v_C^\Delta + \frac{\alpha - 2\alpha + v_A}{2} \).

Working backwards, we already know what players would do at information set 1-4. Now let’s consider information set 0 (node “12.0”). \( S_1 \)’s expected return can be calculated as follows:

\[
\alpha_2 \beta_2 (v_C^\Delta + \frac{v_R - v_I}{2} - v_A) + \alpha_2 (1 - \beta_2) (v_S - v_I) + (1 - \alpha_2) \beta_2 (v_C^\Delta + v_S + v_R - v_A) + (1 - \alpha_2) (1 - \beta_2) (v_S)
\]

\[
= \alpha_2 \beta_2 (v_I - v_R + v_I - v_S) - \alpha_2 \beta_2 (v_R + v_C + v_R - v_A) + v_S
\]

\( S_1 \)’s expected gain seems to depend on \( S_2 \)’s strategy. More specifically, if \( \beta_2 (v_I - v_R + v_I - v_S) > 0 \), \( \alpha_2 = 1 \) and \( \alpha_2 = 0 \) if the quantity < 0. But as we can see, the quantity is negative because \( v_I + 2v_S + v_R > 0 \). Then to maximize the expected return, \( \alpha_2 = 0 \). Similarly, \( \beta_2 = 0 \).

Due to the payment rules of the game, the strategy does not depend on beliefs. The current structure of the game allows players to know the path they have taken after they make a simultaneous move. This indicates that the beliefs consistent with \( s \) are those corresponding to the other party’s move probabilities. Therefore, we have \( b(D_2 | I^A_{S_1}) = b(D_2 | I^D_{S_1}) = b(D_1 | I^A_{S_1}) = b(D_1 | I^D_{S_1}) = 1 \) and \( b(R_2 | I^A_{S_1}) = b(R_2 | I^D_{S_1}) = b(R_1 | I^A_{S_1}) = b(R_1 | I^D_{S_1}) = 0 \).

We consider another structure for the sequential game in Section B-A where colluding parties do not verify others’ inputs. In this setting, players no longer know for sure the path being taken after making a simultaneous move (as shown in Figure 6). We define a deceitful player problem there and give Theorem B.1 in Section B-A.

Comments on simultaneous move in round 1. In round 1, the two parties select action \( A \) or \( D \) simultaneously. Alternatively, if simultaneity cannot be implemented, one party has to move first, e.g., to share the computed responses along with verification information first. It’s straightforward to see no matter what move this initiator player takes, the follower’s dominant strategy is \( D \). Therefore, to give the “initiator” less disadvantage, we assume the existence of fair collusion protocols and let the colluding parties move simultaneously.

To wrap up, as we see in the proof of Proposition 1, we used our assumptions on the payment values to argue about which actions players will pick. A natural question is whether feasible solutions exist for this simple case with 2 servers.

Hard to be feasible for \( \ell = 2 \). First, when the game is repeated between the same 2 players, cooperation can become equilibrium [39], especially when they expect to play the game infinitely; the Folk Theorem [21] states that if players are patient enough, then repeated interaction can result in virtually any average payoff in a Subgame Perfect Equilibrium. Second, there is only a single source of fines used for user compensation and reporting reward (as depicted in Table I). As the privacy protection upper bounds increases, the deposit amount increases and has to be greater than privacy worth. This also makes cooperation more attractive for multiple repetitions of this game. Third, if a user manipulates the game to “steal” deposits, it’s impossible to stop it in the current setting while keeping a feasible solution. This is because the client only pays \( v_S + v_A \) (both need to be low) and what it receives back is \( v_R \) and breach compensation (both need to be high) from “fake” accusations. We next study what happens when we increase \( \ell \).

B. A sequential game for \( \ell > 2 \) parties

Order of events. Let the set of agents be \( S = \{S_1, S_2, ..., S_I\} \). WLOG, we denote the two queried servers as \( S_1, S_2 \). Consider the following sequential game \( G_1 \) in \((\ell, 2, 1)\)-PIR:

(0) In round zero, the client queries 2 servers. Each server decides whether to collude.

1) In round one, one colluding server selects one of two actions: \( A \) or \( D \). Non-queried servers play deceitful \( (D) \) before learning any correct information.

2) In round two, colluding servers either report (\( R \)) the collusion or keep it secret (\( \bar{R} \)).

Analysis. Our result for the 2-party collusion in \( \ell > 2 \) server case is the following and the proof resides in Section A:

Proposition 2. In the 2-party collusion game in \((\ell, 2, 1)\)-PIR in setting II, the pair \((s, b)\) of strategy profile \( s = (D_R, ..., D_R, D_R) \) for all \( \ell \) servers and belief assessment \( b \) is the unique sequential equilibrium, where \( \forall S_i \in S, b \) satisfies \( b(D_{s_{\ref{2}}} | I^A_{S_i}) = b(D_{s_{\ref{2}}} | I^D_{S_i}) = b(R_{s_{\ref{2}}} | I^A_{S_i}) = b(R_{s_{\ref{2}}} | I^D_{S_i}) = 1 \), and \( b(\bar{R} | I) = 0 \) in all other cases.

Informal. In the defined colluding game for \( \ell > 2 \) servers, all servers play deceitfully in a sequential equilibrium. Similarly, condition \( v_R > v_A \) incentivizes reporting collusion.

Note that non-queried servers does not learn any queries or responses, because queried servers always play \( D \) when they are in a game with non-queried servers.

What do we gain from increasing \( \ell \). Now each server has a smaller chance of being queried (2/\( \ell \)). In a round, two players play the game with each other with probability \( p^* = \frac{1}{\ell} = \frac{2}{(\ell-1)} \). This is not negligible and as mentioned, cooperation (not reporting in round 2) can become an equilibrium strategy in repeated prisoner’s dilemma. We first suppose the players have infinite horizon, and we apply discount rate \( \delta \in (0, 1) \) on future returns. From always cooperating, one earns the service fee and the privacy worth gain in each round the same two players play together:

\[
(v_S + v_C^\Delta) + p^* (v_S + v_C^\Delta) \sum_{i=1}^{\infty} (1 - \delta)^i
\]

This equals \((v_S + v_C^\Delta) + p^* (v_S + v_C^\Delta) \frac{1 - \delta}{\delta}\). From reporting collusion in the first game, one expect to earn bonuses in the first round and the equilibrium returns later:

\[
v_C^\Delta + v_S + v_R - v_A + p^* v_S \sum_{i=1}^{\infty} (1 - \delta)^i
\]
This equals $v_2^k + v_S + v_R - v_A + p^*v_S\frac{k-\delta}{\delta}$. The difference between the second quantity and the first is $-p^*v_S\frac{1-\delta}{\delta} + v_R - v_A$. Therefore, as long as $\frac{2(1-\delta)}{\delta} \frac{v_2^k}{v_R - v_A} < \ell(\ell-1)$ or $\ell \geq 1 + \sqrt{\frac{2(1-\delta)}{\delta} \frac{v_2^k}{v_R - v_A}}$, one reports collusion. When $\delta = 0.01$, we need $\ell \geq 1 + 14\sqrt{\frac{v_2^k}{v_R - v_A}}$, e.g., $\ell \geq 71$ if $v_2^k = 25(v_R - v_A)$. If players have limited horizon, the discovered equilibrium is preserved with looser requirements on $\ell$.

IV. MORE COLLUDING PARTIES

In this section we discuss user manipulation, and apply our sequential game design to more colluding parties.

A. Combating False Accusations

One issue arises with Type I-II evidence is the user $U$ taking advantage of the incentive design to steal deposits since she knows her messages. Assuming $\gamma$-reimbursement, to make manipulating the game not profitable for $U$, we need

$$(k-1)v_S > (k-1)(v_R - v_A) + \gamma v_2^k$$

where the left-hand side is the service fee paid to the $k-1$ “falsely accused” servers by $U$. The term on the right-hand side is the maximum bonus from accusing $k-1$ other servers with Type II evidence. The accusation proposer who is queried and controlled by $U$ pays the accusation fee $(k-1)v_A$ for accusing $k-1$ other servers and receives $(k-1)v_R$ after evidence being validated. The intuition is that service fee is of such amount that “deposit stealing” is not profitable for $U$.

False accusations from servers with arbitrary knowledge about the intended queried entry have been coped with in Section III. Now we have settled the false accusation problem.

B. $k$ server collusion

Suppose for a colluding server, finding one other queried server who is collusion-prone can be accomplished with probability $z \leq 1$. And finding $k-1$ such queried servers and have all of them join a collusion protocol can be fulfilled with probability $z^{k-1}$. This can diminish rather fast for $z < 1$, e.g. $0.8^5 = 0.33$. Therefore, as a next step, we increase $k$.

If $k \geq 2$ servers are needed to reconstruct the secret index, we mainly follow the same game as $G_1$ in $(\ell, k, 1)$-PIR case. One minor change in the new game $G_2$ is: in round 0, the user picks $k$ agents instead of exactly 2. We denote the set of selected servers as $S_M \subset S$. Now we only need setting $\Pi(a)$.

In the collusion game for $k > 2$ servers, the payoff tree has a slightly different structure, and we depict the one for $k = 3$ in Figure 3. The following is the major theorem for $k$-server collusion in $(\ell, k, 1)$-PIR:

Theorem IV.1. In $k$-party collusion game in $(\ell, k, 1)$-PIR in setting $\Pi(a)$, the pair $(s, b)$ of strategy profile $s = (\tilde{R}_1, \ldots, \tilde{R}_s, \tilde{R})$ for all $\ell$ servers and belief assessment $b$ is the unique sequential equilibrium if $v_I > (k-1)^2v_A$ and $v_S$ satisfies the following:

- $(k-1)v_S > (k-1)(v_R - v_A) + \gamma v_2^k$ where $\forall S_i \in S$, $b$ satisfies $b(D_{s_i}|I^S_i) = b(D_{s-i})|I^D_{s_i}) = b(R_{s_i}|I^S_i) = b(R_{s-i}|I^D_{s-i}) = 1$, and $b(|I) = 0$ in other cases.

Additionally, for discount rate $\delta \in (0, 1)$, if $\ell > \frac{\delta v_2^k}{v_R - v_A}$, the sequential equilibrium is preserved when the game is repeated with the same $k$ players.

Informal. Here, the additional inequality is to stop user manipulation. All players play deceitful in round 1 essentially because the queried servers are incentivized to take action $R$ in round 2 when $\leq 1$ party plays $D$, in setting $v_R > v_A$ and $v_I > (k-1)^2v_A$. The intuition is that with $v_R > v_A$, the only server playing $D$ in round 1 reports the $(k-1)$ amicable colluding parties. When they all play $A$, with probability $\frac{k}{k}$, one party reports first and receives bonuses. In this case, when $v_R > v_A$ and $v_I > (k-1)^2v_A$, playing $R$ produces the maximum expected returns, $\frac{v_2^k - k-1}{k}(v_I - v_R) - (k-1)v_A$. Working backwards, playing $D$ in round 1 maximizes one’s expected returns in the remaining game.

Proof. According to Section IV-A, inequality $(k-1)v_S > (k-1)(v_R - v_A) + \gamma v_2^k$ stops user manipulation. Therefore, we do not need to treat user as a player in the following discussion and can focus on reasoning about servers’ actions.

If the servers collude via approach 1, where they exchange information with one another, or approach 2 pairwise, we have the exact same payoff tree as Figure 2. Then Proposition 1 implies that they play $\tilde{D}R$ in equilibrium and have consistent beliefs. Otherwise, we have a new payoff tree with similar structure as Figure 2.

We first let $k = 3$. Let $S_1, S_2, S_3$ be 3 queried servers. As depicted in Figure 3, when there’s one deceitful player, it plays $\tilde{R}$ at information set 2 (e.g., $\zeta_1 = 1$) because $v_R > v_A$. At information set 1, $S_1$’s expected return can be calculated as follows:

$$\alpha_3\beta_3\zeta_3(v_2^k - \frac{2(v_I - v_R)}{3} - 2v_A) + \alpha_3[\beta_3(1 - \zeta_3) + (1 - \beta_3)\zeta_3] : (v_2^k - \frac{v_I - v_R}{2} - 2v_A) + \alpha_3(1 - \beta_3)(1 - \zeta_3)(v_2^k + v_R + 2v_R - v_A)) + (1 - \alpha_3)[\beta_3\zeta_3 + \beta_3(1 - \zeta_3) + (1 - \beta_3)\zeta_3](v_2^k - v_I) + (1 - \alpha_3)(1 - \beta_3)(1 - \zeta_3)(v_2^k + v_R)$$

$$= \alpha_3[\beta_3\zeta_3\frac{5v_R - 2v_I}{3} + \beta_3(1 - \beta_3)v_I - v_A + 2(2v_R - v_A)] + \tilde{R}$$

Here $\tilde{R}$ contains all the remaining terms independent of $\alpha_3$. Since $v_I > 4v_A$ and $v_R > v_A$, the coefficient term of $\alpha_3$ is always positive (because its minimum, when all others play $R$, is positive), thus $\alpha_3 = 1$. Likewise, we have $\beta_3, \zeta_3 = 1$. The expected return for all players are simply $(v_2^k - \frac{2(v_I - v_R)}{3} - 2v_A)$. Then it’s obvious to see that $\alpha_2 = \beta_2 = \zeta_2 = 0$.

Similarly, for a general $k \geq 3$, we know that when there’s more than one deceitful player, all parties play $\tilde{R}$. When there’s exactly one deceitful player, this player plays $R$ at the corresponding information set (e.g., information set 2 in Figure 3).
At information set 1 where all have played $A$ in round 1, all players play $R$ because $v_I > (k-1)^2v_A$ and $v_R > v_A$. More specifically, for $S_I$, when all other players report collusion, it earns $\alpha(v_I + \frac{k-1}{k}(v_I - v_R)) - (k-1)v_A$. The coefficient term is positive when $v_I > (k-1)v_A$. Thus all players play $R$ and obtain expected return $(v_I - \frac{k-1}{k}(v_I - v_R)) - (k-1)v_A)$. This is essentially a $k$-party prisoner’s dilemma. Working backwards, playing $D$ in round 1 maximizes expected returns at information set 0.

Similar to the arguments in Proposition 2, for a non-queried server, it can only play $D$ in the first round in all collusion games. Overall, the unique sequentially rational equilibrium $s$ dictates that all players select $D$ in round 1 and $R$ in round 2, with the unique corresponding belief assessment that is consistent with $s$.

**What do we gain from increasing $k$.** First, the game is started with probability $\leq z^k$, which decreases exponentially with $k$ when $z < 1$. But this effect becomes negligible after a certain point. For example, let $z = 0.5$. We know that $0.5^5 = 0.03$ and $0.5^{10} = 0.001$, at which point it becomes less necessary to continue to raise $k$. Second, higher $k$ means more sources of fines, and we can have a lower deposit $v_I$. Third, when players have infinite horizon, the requirement on $\ell$ for players to report collusion when the game is repeated is further loosened. Because now the same players are picked with probability $\frac{1}{\ell} \leq \frac{1}{k}$ for $2 \leq k \leq \ell - 2$ (equality if and only if $k = 2$ or $\ell - 2$).

**Compensating victim user.** When there’s confirmed privacy breach and privacy protection has not expired, the user receives reimbursement $\gamma v_C^\Delta$ ($\gamma < 1$). Now we examine whether we have enough funds to provide $\gamma$-reimbursement. Let $\Lambda$ denote the remaining funds after fining the accused and rewarding the accuser. In an $(\ell, k, 1)$-PIR service with $\omega$ companion queries, the most efficient case is when Type II evidence is presented and $k-1$ other servers are fined. We need

$$\gamma v_C^\Delta \leq \Lambda k := (k-1)(v_I - v_R + v_A)$$

The inequality is implied by Theorem IV.1 when $v_S < v_I$, which is convenient to enforce. Because in principle, service fee is desired to be small while the deposit can be close to privacy value. Thus, it’s not included as a core condition for collusion mitigation.

**V. A Game Design Flow**

Now we combine the previous analysis and give a demonstration for determining parameters in the sequential game for a specific system. Assume $(\ell, k, 1)$-PIR, $k \geq 2$ and $\ell$ is sufficiently large. As a system designer, we evaluate the participating servers to determine a recommended $k$. We then decide the privacy value upper bound $v_I$, we aim to protect. We compute the parameters for the sequential game with Algorithm 2. It outputs a list of feasible value assignments for parameters. There is no absolute standard to select one assignment over another but rather depends on the priorities.

```
ALGORITHM 1: Coordinator (Essential Functions)
Input: $S, k, l, v_I, v_R, v_A, v_S, pk, src, user, \gamma$
Function AccusationVal(id, e_type, evidence):
for pk \in Journal[id], pkList do
  if \text{alreadyFined}[id][pk] then
    PaymentExec(pk, id, e_type, evidence)
  else
    PaymentExec(witness, id, e_type)
Read from Journal[id] the user, pkList;
Take fines $v_I$ from each accused server pk;
For each pk, let alreadyFined[id][pk] = True;
Distribute reward $v_R$ to witness;
Distribute $\gamma$-compensation to user;
On new (id, e_type, evidence):
  Charge $v_A$ from accuser msg.sender;
  AccusationVal(id, e_type, evidence)
On new (pkList, requests):
  Lock $v_S$ from msg.sender for each pk \in pkList;
  Issue a unique identifier id for the request;
  Store requests, pkList in Journal[id];
  for pk \in pkList do
    alreadyFined[id][pk] = false;
```

Algorithm 1 presents essential functions for implementing the game and payment rules. In Line 15, a user submits queries to a list of servers. The coordinator initializes parameters for future possible accusation resolutions. In Line 12, an accusation is filed, and we call the accusation validation routine. Line 4 checks the validity of the accusation. Line 7 calls payment execution routine to realize payment rules. Note that the two subroutines shown here are only conceptual.

One natural question to ask is under what condition solutions are guaranteed to exist and whether the solution is practical. It’s hard to have an absolute definition for a “practical” solution. Here we seek to have affordable economic $v_S$, higher
where \( \eta \in (0, 1) \) is a practicality parameter characterizing the affordability of \( v_S \). We state the following theorem.

**Theorem V.1 (Existence of Solution).** There exist practical (satisfying Equation (2)) parameter assignments for server collusion mitigation in \((\ell,k,1)\)-PIR satisfying inequality set \( \{1\text{-}4\} \) where

1. \( v_A < v_R \leq v_I \)
2. \( v_A \geq \frac{1}{w+1} v_R \)
3. \((k-1)v_S > (k-1)(v_R-v_A) + \gamma v_C \)
4. \( v_I > (k-1)^2 v_A \)

In the following proof, we by default let \( v_C = \frac{v_C}{\omega} \) for clearer representation (parameterize \( v_C \) with \( \omega \)). We note that the worst case of \( v_C \) approaching \( v_C \) can be seen by conceptually letting \( \omega = 0 \) in the analysis for Inequality 3. Because there, colluding servers recognizing the true index among all the recovered ones is mathematically equivalent to no companion query being sent.

**Proof.** For inequality 1 and 2 to have a feasible solution of \( v_R, v_A, v_A \) pair, we need \( \frac{1}{w+1} v_R \leq v_A < v_R \). In the best case for a colluding server, \( w = 1 \). We need \( \frac{v_A}{v_R} \leq v_A < v_R \). Then given a viable parameter set \( k, v_A, v_R \) derived from \( 1 \) and \( 2 \), and the maximum privacy worth \( v_C \) to protect, we can solve the remaining inequalities. From inequality 1 and circno4, we can set \( v_I \). Then we only need to make \( v_S \) satisfy inequality 3. We show that we can have practical \( v_S \) (satisfying Equation (2)) satisfying inequality 3.

For inequality 3, we need \((k-1)\frac{v_A}{v_R} \eta > (k-1)(v_R-v_A) + \gamma v_C \), which gives \( \eta \geq \frac{k(v_R-v_A) + \gamma v_C}{(k-1)v_R} \). When we do not provide reimbursement \((\gamma = 0)\), we can have a feasible \( \eta < 1 \) by mandating \( k(v_R-v_A) < v_C \). This is easy to satisfy because the reward amount is not tied with the privacy value but the accusation fees. In other words, we can have a small \((v_R-v_A)\) while still motivating servers to report collusion. More specifically, since for previous conditions to stand, \( v_R, v_A \) only need to satisfy \( \frac{v_A}{v_R} \leq v_A < v_R \), inequality 3 can be satisfied when \( v_A < \frac{1}{2} v_C \). This can be easily satisfied. For \( \gamma > 0 \), we need \( v_A < \frac{1}{2}(1 - \frac{\gamma}{k-1})v_C \).

The right-hand side grows with \( w \), so we can safely set \( w \) to 1 or conceptually 0 (we always send companion queries) for a tighter bound. Then because \( v_A \) does not have other constraints, it can be made small to satisfy this condition. \( v_R \) is then adjusted accordingly.

Overall, we first decide \( v_C, k, \gamma \) and \( z \), which only depends on the assumption on how hard it is to find colluding partners. Then we can decide accusation fee \( v_A \) according to condition \( v_A < \frac{1}{2}(1 - \frac{\gamma}{k-1})v_C \) (to satisfy inequality 3). In the next step, we determine \( v_R \) according to \( \frac{v_A}{v_R} \leq v_A < v_R \) (to satisfy inequality 1 and 2). We then determine \( v_I \) according to inequality 4 and finally decide \( v_S \) according to inequality 3. Because we incorporated the practicality constraint in Equation 2 into the derivation, the service fee is by construction practical. A sample feasible parameter region of \( v_S, v_I, v_A \) in setting \( z = 1, \ell = 100, v_C = 100, k = \{2, 3, 4, 5\}, w = 1, \gamma = 0, v_I \leq v_C \) is shown in Figure 4. An example solution is \( v_S = 5, v_I = 93, v_R = 15, v_A = 11 \). One observation is that to get low service fee and accusation fee, we need higher deposit.

### A. Implementation

**Smart contract implementation of CC.** We treat Ethereum as a public bulletin board and conveniently implement the coordinator contract as a smart contract on Ethereum. The source code in Solidity is available [25]. The contract maintains the complete life cycle of PIR service and resolves collusion accusations. For evidence verification, as shown in Figure 1, the accuser submits the evidence of the corresponding type, followed by the user or the involved server submitting auxiliary information. If no supplementary data is submitted in time, the accusation is automatically marked as successful. The accused server can be pinpointed through elimination in this case. After collecting information, the verification algorithm verifies digital signatures or commitments (Type I), or computes certain function circuits after verifying function non-triviality and inputs (Type II). Any secure commitment and signature schemes would suffice, e.g., cryptographic hash functions as commitment scheme and ECDSA signatures [32]. To facilitate Type II evidence verification, we provide a sample reconstruction function as a default \( f(\cdot) \) inside the contract. So the accuser can omit the step of providing a function circuit if \( f(\cdot) \) is the reconstruction algorithm. Otherwise, we ask the accuser to address the function simply as \( f(\text{bytes memory}[k]) \) when providing the circuit for simpler calling convention.

We summarize the gas costs of contract deployment and each function call in Table II.
TABLE II: Cost estimates in Gas. CheckCircuits(·) additionally needs to include Chainlink payments to the oracle.

| Function                  | Cost | Function           | Cost |
|---------------------------|------|--------------------|------|
| Deploy contract           | 4743771 | ClaimServiceFee(·) | 33147 |
| Deposit(·)                | 116792 | Accuse(·)          | 197399|
| UpdateKey(·)              | 67307  | VerifyType1(·)     | 112559|
| PostRequests(·)           | 347633 | CheckCircuits(·)   | 917494|
| SubmitResponse(·)         | 152494 | VerifyType2(·)     | 342634|

**Non-triviality of function circuits.** When parties collude via approach 2 and report with Type II evidence, we need to check the non-triviality of function $f(·)$ in addition to verifying function inputs and outputs. We view a function as non-trivial if its output depends on the inputs. Taint analysis [42] is employed for this purpose. The high level idea is to taint the inputs and all relevant data affected by the inputs along the execution of the program. When we implement the design on Ethereum with smart contracts, the functions do not contain pointers and are deterministic. According to a survey by Di Angelo and Salzer [15], one can perform control and data flow analysis on EVM (Ethereum Virtual Machine) bytecode of smart contracts off-chain via EthIR [1] and Securify [49], among other tools.

However, accomplishing this task on-chain can be impractically expensive since it essentially requires compiling smart contracts on-chain. Therefore, we treat the function circuit provided by the accuser as non-trivial by default and only verifies its non-triviality if the accused indicates its triviality before providing auxiliary information. To verify, we can create an oracle contract and a job concerning verifying non-triviality of smart contracts on an oracle service platform like Chainlink [8]. It collects responses in a decentralized manner. The coordinator contract create requests through API calls to the oracle and receive responses. The verification cost is afforded by the accused initially and charged from the accuser if the function is indeed trivial.

**Limitation of supported functions.** Overall we prefer $f(·)$ to be light computations, e.g., the entire or the most significant bit (MSB) of the reconstructed data entry. Because if the function is computation intensive, it might become uneconomical to verify when the privacy worth is not high enough. But we also consider this requirement to be undemanding. No matter what the colluding parties have computed with MPC, at least some party, with arbitrary private knowledge, learns some bit(s) or a fraction of a bit about the queried entry from the output of MPC. Otherwise, we consider the collusion to be not effective. If a party learns some bits, it should be able to generate a simple function for verification purpose.

**VI. ADVERSARIAL EXITING STRATEGIES**

We have been focusing on mitigating collusion at the time a user makes queries. In practice, it can happen long after the query has taken place. Although we put no restriction on the coordinator algorithm executions, there might be insufficient funds in servers’ deposit accounts to execute on if they carry out a massive breach. This is a concern because storing pertinent data is not impractical. One subtlety is that if there exists no external utility, a server is not motivated to do so if the future return is higher than what they can gain from collusion. Nevertheless, there can be scenarios where servers have intentions to stop providing the service. For example, the owners of some machines are on the edge of bankruptcy and are willing to take the chances then quit the service. We consider the following two exiting strategies: (i) The server who is leaving wants to collude and break user privacy before leaving. Deposits and service fees may be lost during the collusion. See Section VI-A. (ii) The server registers for withdrawing from the service and receives deposit, service fees back. After everything has been processed, the server sells its database. Nothing happens to this server because it’s outside the jurisdiction of the game rules. See Section VI-B.

Note that the latter dominates the former exiting strategy if the profits from selling raw data are greater than the secrets minus potential fines.

**A. A Server is Leaving the System**

The coordinator or the rules we have established can still exert some power over this server. The maximum fines we can take is the deposit $v_I$ and detained service fees. Our goal is to make our ends meet: to reward collusion reports and reimburse users who have their privacy breached.

We propose a Self-Insurance [20] design tailored for our needs. Suppose during $T$ time units, there have been a total of $k\Omega$ queries ($\Omega$ users, $k$ queries per user). The total accumulated service fees equal to $k\Omega v_S$. The expected number of queries this exiting server $S_e$ receives is $\frac{k\Omega}{T}$ thus the expected maximum number of victim users is $\frac{k\Omega}{T}$. We say “maximum” here because users can be recurrent. The worst case is that all collusion are successful (with probability $p_{succ} = z(\frac{1}{T} - \frac{1}{v_I})$). We expect $(k - 1)/k$ of the collusion to be reported by colluding servers other than $S_e$. The expected total amount of fines we cannot realize is $(p_{succ} \cdot k \cdot \frac{(k - 1)}{T} \cdot v_I) = (k - 1)\Omega p_{succ} l \cdot v_I$. This quantity increases with $\Omega$, indicating that more users result in more dead weight.

On the other hand, the service fee pool also grows with $\Omega$. There is $k \cdot \frac{k\Omega}{T}$ transaction fees that can be utilized to cover the potential loss from $S_e$’s malicious exiting strategy. We use

$$\sigma = \frac{(k - 1)\Omega p_{succ} l \cdot v_I}{k^2 \Omega v_S} = \frac{(k - 1) \Omega p_{succ} \Omega - l v_I}{k^2 \Omega v_S}$$

to denote the tension between the size of inexecutable fines and potential insurance pool. We want $\sigma \leq 1$ to be able to cover possible loss using the self-insurance. The intuition here is that in the worst case where $p_{succ} = 1$, we can have $l$ significantly larger than $k$ at the outset of system design to allow a small $v_S$. The intuition is that when we have many servers in the system, one server does not carry so much information that the insurance pool cannot afford to lose.

To be more realistic, we can assume an interest rate $r$ per time unit on these detained service fees and the accumulated service fees become $k\Omega v_S(1+r)^T$. We can discount the reimbursement for the users across time, which would necessitate
less and less effective fines as a query becomes ancient. We can simply apply a discount rate \( r' \) on deposits and obtain \( \sigma' \)

\[
\sigma' = \frac{(k-1)p_{\text{succ}}\Omega - l v_1 (1-r')^T}{k^2 w - v_S (1+r)^T}
\]

We also want \( \sigma' \leq 1 \) to relieve the tension. As an illustration example, let \( k = 2, l = 1000, T = 10000, \Omega = 5000 \) and assume the worst case that \( p_{\text{succ}} = 1 \). We have \( \sigma' = \frac{1}{v_S (1+r)^T} \cdot \frac{v_S (1+r)^T}{v_S (1+r)^T} \). Let \( r' = r = 0.0001 \), we can have \( \sigma' \leq 1 \) when \( v_S \geq 0.027v_1 \). Note that although the intuition goes through, the above is not the exact formulation as we are treating the probability of finding collusion-prone parties being the same, as \( \ell/t \) applies, with the probability of finding collusion-prone parties.

We consider the following two cases.

A. Bound the Loss Within the Current Model

Suppose the pre-established coalition is of size \( s \), and that the number of servers to be queried by the user is \( k (\leq \ell) \). We consider the following two cases.

First, if \( s < k \), there always exist some true queries that land outside the clique. Requesting collusion with outsiders is discouraged by our proposed design. The previous analysis applies, with the probability of finding collusion-prone parties changing from \( z^{k-1} \) to \( z^{k-s} \). Second, if \( s \geq k \), with some probability \( p_s \), the clique contains all the queries. We can calculate \( p_s \) as \( p_s = \binom{k}{s} \). One observation is that with \( k \) being the same, as \( \ell/t \) grows, the probability goes down. This implies that if the total number of servers is sufficiently large compared to the colluding parties, the probability that user privacy is compromised is small. Besides, servers inside the colluding ally may not recognize the true user secret among the \((w+1)\) total secrets retrieved. The expected utility from colluding together as \( p_s v_C^\Delta \). This implies that including more companion queries can decrease this loss. Overall, the introduced collusion deterrence mechanism ensures privacy for \( s \geq k \) case. The expected loss from privacy breach for the \( s \geq k \) case is \( p_s v_C^\Delta \).

B. Adding Spices

Cooperative game theory \cite{7} studies the formation of coalitions, and we consider strongly effective ones here. Our goal is to try to cap the coalition size. A cooperative game can be formalized with the set of \( \ell \) players \( N = \{S_1, \ldots, S_\ell\} \), each nonempty set of coalition \( S \) (of size \( s \)), and a characteristic/value function mapping a coalition to its collective payoff \( v : 2^N \rightarrow \mathbb{R} \). For example, the grand coalition has value \( v(N) = \ell v_C^\Delta + k v_S \). An allocation \( x \in \mathbb{R}^\ell \) specifies how to divide each coalition value among members.

Solution concept. We suppose the coalition structure (partition of \( N \)) is non-overlapping and solve for the allocation in the core of \( v(S) \). We parameterize the largest coalition and do not solve for \( v(N) \) directly because we want to cap the coalition size. Such a core allocation is a feasible allocation that no other coalition (\( \subseteq S \)) can improve on (i.e., give strictly higher payoff for all). Here, feasibility simply requires that one does not over-allocate the coalition value. For any \( S \) with size \( s \geq k \), its core is: \( \{S_j \in S, x(S_j) = p_s v_C^\Delta + k \ell \cdot v_S\} \). This clearly leads to the grand coalition being the most desired.

Alternatively, we can consider a slightly different value function where the coalition \( S \) only earns \( v(S) = p_s D(s, v_C^\Delta) + k \ell \cdot v_S \) instead of \( p_s v_C^\Delta + k \ell \cdot v_S \). Here \( D(\cdot) \) is a function non-increasing in \( s \). This is reasonable only if user privacy depreciates when more parties learn it. For example, the price of some stock is being frequently queried, and suppose this indicates upcoming valuation changes of the company. Individual profits decrease as more players learn this information.

Constant depreciation \( D(s, v_C^\Delta) = cv_C^\Delta \) where \( 0 < c < s \) is a constant promotes grand coalition. In this setting, the core is \( \{S_j \in S, x(S_j) = p_s v_C^\Delta + k \ell \cdot v_S\} \). By including one more member to \( S \), one existing member’s extra gain is \( (p_{s+1}/(s+1) - p_s/s) \cdot cv_C^\Delta > 0 \) because

\[
k \geq 2 \Rightarrow \frac{s}{s+1-k} > 1 \Rightarrow s \cdot \frac{k+1}{k} > (s+1) \cdot \frac{k}{k+1}
\]

This means that the coalition \( S \) has incentive to include new members until it becomes the grand coalition.

Similarly, linear depreciation \( D(s, v_C^\Delta) = cv_C^\Delta/s \) promotes grand coalition. This is essentially because the increasing rate of \( p_s \) is \( (s+1)/(s+1-k) \). Intuitively, when \( D(s, v_C^\Delta) \) depreciates faster than \( p_s \’s increasing rate, coalition size is kept at \( k \). Consider quadratic depreciation \( D(s, v_C^\Delta) = cv_C^\Delta/s^2 \). In this scenario, the core allocation is
\[ \forall S_j \in S, \mathcal{X}(S_j) = p_s/s^3 \cdot \ell \Delta + k/\ell \cdot v_S \]. By enforcing 
\[ s^3 p_{s+1} < (s+1)^2 p_s \], \( s \) is kept small. This indicates that 
\[ s^3 \left( \frac{s+1}{k} \right) < (s+1)^2 \left( \frac{k}{s} \right) \Rightarrow \frac{s^3}{s+1} < (s+1)^2 \]
\[ \Rightarrow k < \frac{3s^2 + 3s + 1}{(s+1)^2} \]

When \( k = 2 \), the inequality is satisfied. This means that when the privacy worth gain depreciates quadratically in \( s \), the coalition size is kept at 2 for \( k = 2 \) and the expected privacy loss is \( p_2 v_2^2 \). Note that as previously discussed, if the 2 members in the coalition respond to our provided incentives, it’s possible to make them report collusion inside by having a large \( \ell \). Overall, it’s hard to mitigate collusion when there exist strong coalitions.

VIII. Related Work

Deterring correctness-related collusion. Yakira et al. [53] presents a slashing mechanism for threshold cryptosystem collusion mitigation. All participants register in an escrow service. One can frame colluding players with proper evidence to the service, which slashes all other agents by burning their deposits and rewarding the reporting agent. There, privacy leakage and active false accusations are not a concern, unlike in PIR (finite number of entries). Besides, the nature of evidence and its verification are not discussed.

Dong et al. [17] focus on ensuring correctness in replication-based cloud computing and propose a solution involving inducing betrayal among colluding parties. The four major differences are that firstly, correct computation is not oblivious to the service under discussion, because unlike privacy-related collusion, correctness-related collusion violates security and affects the protocol. Second, collusion for privacy breach can happen any time after the PIR service. This means that even if correctness is ensured at the time of service, peers can still collude to break privacy later. Third, in privacy-related collusion, players exchange information and have arbitrary private knowledge, creating more complexities in analysis. Lastly, the computing parties in [17] collude via a smart contract, which can be input as evidence to a verification contract. It is limiting to expect collusion to happen on blockchains via explicit contracts.

Devet [14] discusses Byzantine server problem in Goldberg’s IT-PIR protocol, where Byzantine servers do not send correct responses back and such behavior is detected with certain probability. Devet designs a multi-round game for \( l \geq 2 \) servers, two of whose Nash equilibria are all servers playing Honest and all playing Byzantine. The insight for collusion mitigation is that if a server expects certain threshold proportion of the servers in the system to be Honest, then it also plays Honest. This conjecture necessitates collusion detection to be in place and unlike Byzantine responses where one can check correctness, it is not practical to detect malicious server communication directly and constantly. Wang et al. [50] consider collusion deterrence in MPC. One game-theoretic approach proposed there is to have undercover police disguised as corrupted clients to catch colluding parties. In the context of this work, sending fake collusion requests can reduce servers’ confidence in other servers compliantly collude. One challenge of this approach is that one needs to place certain level of trust on the entities initiating dummy collusion.

Ciampi et al. [11] present a collusion preserving secure (CP-secure) computation protocol with a collateral and compensation mechanism to disincentivize aborting. Parties deposit collateral at the initialization round, and they can withdraw only if all messages and executions are correct. Otherwise, the collateral is taken from dishonest players to reward others. In this design, correct messaging and executions are well-defined, self-contained and the aim is to stop subliminal communication inside the protocol. Yu et al. [56] considers collusion of rational peers in overlay multi-cast. They aim to provide safety-net (i.e. minimum profits) guarantees to non-deviating peers. Debt-links are introduced to allow links be used to send data only if a proper amount of debt coins are being paid. There, collusion also affects the protocol executions.

Clarke et al. [12] circumvent collusion problem in private data retrieval through anonymization. As a result, a privacy breach is not bound with an identified entity. It’s debatable whether this counts as privacy breach. Besides, a user needs to function as a node in the network.

Rational secret sharing (RSS). Halpern and Teague introduce the notion of RSS [27]. RSS studies the opposite problem of anti-collusion, which is how to have rational agents cooperate in secret reconstruction, with only assumptions on their preferences. They present the impossibility results assuming a server prefers as fewer servers learning the secret as possible, which is good news for collusion mitigation. But a randomized protocol can facilitate servers providing correct shares to others. Moreover, if we change the assumptions on server preferences, e.g., they only prefer to learn the secret than not and are indifferent to the number of fellow servers who also recover the secret, the impossibility results may no longer hold. This is bad news for collusion deterrence. Thus, due to the difficulty in assessing servers’ preferences accurately and arms race in encouraging cooperation versus deterring collusion, we prefer working with tokenized rewards and punishments.

IX. Conclusion and Future Work

In this work, we follow the intuition from a bit-guessing game and explore a sequential game design that mitigates collusion among individual players for \( (\ell, k, 1) \)-PIR systems, assuming a large \( \ell \). Longer-term privacy is ensured with a self-insurance pool. The discussion is later extended to collusion within coalitions. We also settle potential manipulation of the design by users and servers with private knowledge. Because we desire many servers to collectively provide the service, blockchains become a suited application scenario. Overall, privacy-related collusion is hard to tackle compared with
correctness-related ones and firm coalitions can cause even more difficulty.

**Future directions.** Our work opens up an interesting line of research. One intriguing direction for future work can be to study mechanisms for robust multi-server PIR [23] and other secret-sharing style applications or generic MPC. While we can directly apply the design to robust PIR, there’s fairness concern in punishing colluding servers after Type II evidence accusation. Another direction worth noting is accusations with information solely acquirable from collusion, e.g., identities in anonymous networks. When players do not have information advantages, false accusation problem becomes random accusation problem, which is much easier to tackle. More importantly, this approach can accommodate any function \( f(\cdot) \) computed by servers without requiring companion queries. It can also be the case that even before actual collusion, one party already learns others’ identities. This means that collusion attempt can be deterred, then user compensation may not be needed. This broadens the parameter feasibility region. Here the difficulty lies in defining this “identity”.

**X. Acknowledgement**

We thank Vassilis Zikas for valuable feedbacks on the system specifications. This work is supported partially by the National Science Foundation under grant CNS-1846316, NSERC through the Discovery Grants program and partly by a Google Research Scholar Award.

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For the two queried servers $S_1$ and $S_2$, Proposition 1 implies that they play $D\bar{R}$ in equilibrium and have beliefs consistent with the equilibrium strategy.

For a non-queried server $S_3$, it can only play $D$ in the first collusion game it plays. We are interested in whether they can learn anything through collusion. This is non-trivial because when they report collusion, they do not lose the deposit $\psi$ since they are not queried, thus not the owner of the information input into collusion. Suppose the counterparty is also a non-queried server $S_4$. Then they both play $D\bar{R}$ and does not learn any information about the queries or responses.

If the counterparty is a queried server, say $S_1$, then the game is deprived of information set 1 and 3 (or 2 and 4 if we put $S_3$ as the first player and $S_1$ as the second player) in Figure 2. Then playing $D$ is dominant strategy for queried server $S_1$. $S_3$ still does not learn about the queries or responses. Therefore, non-queried servers can only play $D$ in all collusion games. This means that the sequential equilibrium is preserved.  

**APPENDIX B**

**Deceitful Player Problem**

Forging information in collusion. When colluding parties do not verify the information shared by colluding buddies, one interesting scenario is where a server $S_j$ prefers to retrieve the index alone or simply desires to hide its information to avoid being reported. We are interested in the probability of colluding servers fabricating information and fooling each other. In Boyle-Gilboa-Ishai’s construction, a query (I) is computationally hard to forge except with negligible probability. Responses (I) can be correctly forged with non-negligible probability. The forgeability of inputs into an MPC protocol (II) depends on the input type.

We denote the probability of one party fooling the other with respect to a prior $p$ as $p^1$ hereafter. As a simple illustration, consider a simple xor-based naive PIR: let server $S_j$ receive index vector $(j_1, j_2, j_m)$ and $S_j$ receive $(j_1, j_2, j_m, 0)$ so that the xor of the two vectors recovers index $a$. In collusion, to fool the other party, a server can guess an index and constructs a fake vector to redirect the xor-ing to some other index. But they both can cheat. If they have the same redirecting strategy $x \rightarrow y$ (guessing $j_x$, redirecting to $j_y$), then apparently they recover $a$. Other scenarios where they recover a single index are shown in Figure 5. Overall we prefer working with constructions that make $p^1$ arbitrarily close to 0.

A deceitful player guesses an index and tries to redirect the collusion protocol to output a targeted index. If the underlying PIR protocol ensures that faking information (a query or response) in collusion is computationally hard, then $p^1_{faking}$, the probability of creating a valid fake input, is 0. This would mean $p^1 = 0$, and we are back to the discussion we have in the main body. Therefore, the following analysis applies to
cases where $p_{faking} > 0$. For the worst case, we consider
$p_{faking} = 1$. This means that as long as a server can picture
a redirecting strategy, it can always fabricate such an input.

Problem statement. In a $(l, k, 1)$-PIR scheme, $l$ servers
maintain the same database $D$ containing $N$ entries. Servers
have a common prior $p$. A user $U$ wants to query entry $D_a$
and sends queries to a set of $k \geq 2$ servers $S_{M_i}$. Queried
servers participate in a collusion protocol. They do not verify others’
inputs directly and infer their validity through the output of
collusion protocol. They can choose to send correct or fake
information. What is the probability of servers being “fooled”
to believing the correctness of others’ actually fake inputs?

Analysis. We first discuss $k = 2$. Let server $S_1$ receive query
$q_1$ and $S_2$ ($i \neq j$) receive query $q_2$ such that $D_a$ is the queried
entry. To form a redirecting strategy, a server first guesses the
intended query. One can guess $D_a$ correctly with probability
$p_a$ ($p_a \leq \max_x p$). Then the server creates a fake input to
redirect the output to another index. Take the naive xor-based
construction as an example. We let $q_1 = (j_1, j_2, \ldots, j_m)$, $q_2 =$
$j_1, j_2, \ldots, j_m, a$, and summarize the strategies as follows:

$$
\begin{array}{c|c|c}
S_1 & S_2 & \text{Rationale} \\
\hline
\{j_1, j_2, \ldots, j_m\} & \{j_1, j_2, \ldots, j_m, a\} & \text{Amicable} \\
\{j_1, j_2, \ldots, j_m, a\} \cup \{j_1, j_2, \ldots, j_m+1\} & \{j_1, j_2, \ldots, j_m-1, a\} & j_m \to a \\
\{j_1, j_2, \ldots, j_m, a\} \cup \{j_1, j_2, \ldots, j_m+1\} & \{j_1, j_2, \ldots, j_m-1, a\} & j_m \to j_1 \\
\{j_1, j_2, \ldots, j_m, a, j_m+1\} & \{j_1, j_2, \ldots, j_m-1, a\} & a \to j_m \\
\{j_1, j_2, \ldots, j_m, a, j_m+1\} & \{j_1, j_2, \ldots, j_m+1, a\} & a \to j_m+1 \\
\{j_1, j_2, \ldots, j_m+1, a\} \cup \{j_1, j_2, \ldots, j_m+2\} & \{j_1, j_2, \ldots, j_m, a+1\} & j_m+1 \to j_1 \\
\{j_1, j_2, \ldots, j_m+1, a\} \cup \{j_1, j_2, \ldots, j_m+2\} & \{j_1, j_2, \ldots, j_m, a+1\} & j_m+1 \to j_2 \\
\end{array}
$$

TABLE III: Redirecting strategy summary. Each time an index
gets “guessed” or “redirected to”, its presence in the output is
flipped. Originally only index $a$ is flipped.

We are interested in the probability of a server being fooled.
Intuitively, if the outputs after collusion protocol contain more
than one index or gibberish when a server itself provide correct
inputs, this server knows that the colluding party is cheating.
Collusion does not effectually happen, so our proposed design
can have a rest. Otherwise, if the collusion output contains
only one index, the server may believe collusion is successful.
To find out this probability, we summarize four scenarios
where a single index is output in Table IV.

Row 1 is straightforward. Index $a$ is retrieved w.p. 1. Row
2 and 3 is because there is a single output in one cheating
player case when the deceitful party either guesses the index
correctly (with probability $p_a$) or guesses the index incorrectly
(w.p. $(1 - p_a)$) and redirects the result to $D_a$ (w.p. $p_a$). And
$p_a + (1 - p_a)p_a = p_a(2 - p_a)$, which we will denote as $p^1$.
Index $a$ is never returned because the only deceitful player
either flips $a$ or produces more than one index in output. The
deceitful player can always recover $a$ through deduction.

In $(D, D)$ case, by observing Table III, we notice that when
both parties have the same strategy or opposite strategies, they
recover $a$. The intuition is that the two servers cancel out each
other’s actions by flipping the flipped back. This happens w.p.
$p_{2,1} = \sum_i \sum_{j \neq i} 2p_i^2 p_j^2$. For other cases of single output, as
presented in Figure 5, in the left four subgraphs, the node
$j_x$ with edge from itself to $a$ or from $a$ to itself becomes the
potential output because $a$ is the original true index, and it gets
cancelled out. When the other deceitful player plays a strategy
that goes from $j_y$ to $j_x$ or from node $j_x$ to $j_y$, $j_y$ becomes the
output because $j_x$ is now cancelled out. These occur with
probability $p_{2,2} = \sum_i \sum_{j \neq i} \frac{1}{4} p_i p_j^2$. Note that $i = a$
or $j = a$ case has been included in output $a$ category. Now index $a$ is returned w.p. $p_{2,1}$ and because deceitful
servers cannot differentiate between different subcases, they
cannot make correct accusations deterministically but rather
probabilistically, w.p. $p_{2,1}$. This is to say, a deceitful party
report w.p. $p_{2}^1 = p_{2,1} + p_{2,2}$ and succeed w.p. $p_{2,1}^1$.

Now we can extend our discussion to a more general $k > 2$
case. When all servers play amicable, $a$ is output w.p. 1. When
1 server plays deceitful, the collusion protocol outputs a single
index w.p. $p_a(2 - p_a)$. Similarly, index $a$ is never returned,
and the only deceitful player can always make successful
accusations through deductions. When 2 servers play deceitful,
they have a single output w.p. $\sum_i \sum_{j \neq i} 2p_i^2 p_j^2$ where the 2
deceitful agents either have the same strategy or opposite
strategies.

When $k' < k$ servers play deceitful, they have a single output w.p. $\sum_{i_1} \sum_{i_2 \neq i_1} \cdots \sum_{i_{k'} \neq i_{k'+1}} 2p_i^2 p_j^2 \cdots p_{k'}^2$, where
the $k'$ deceitful agents either have the same strategy or opposite
strategies. When $k$ servers play deceitful, they output a single index with probability
$p_k^1 = \sum_{i_1} \sum_{i_2 \neq i_1} \cdots \sum_{i_{k'} \neq i_{k'+1}} 2p_i p_j^2 \cdots p_{k'}^2 + \sum_{i_1 \neq a} \sum_{i_2 \neq i_1} \cdots \sum_{i_{k'} \neq i_{k'+1}} 2^k p_a p_i^2 \cdots p_{k'}^2$. In these
scenarios with 2 or more deceitful players, deceitful servers
cannot distinguish between different subcases with single
output, and they do not report collusion due to condition
specified in setting $\Pi(a)$, $v_A = \frac{1}{w+1} v_R$.

A. Implications for the sequential game

As shown in Figure 6, information set 1 now consists of
four nodes. This means that $\alpha_3, \beta_3$ are no longer 1. From

\begin{table}
\centering
\begin{tabular}{|c|c|c|}
\hline
$(S_i, S_j)$ & Possible Single Output & Probability $p^1$ \\
\hline
$(A, A)$ & $a$ & 1 \\
$(A, D)$ & $\{j_m, j_{m+1}\}$ & $p_a(2 - p_a)$ \\
$(D, A)$ & $\{j_m, j_{m+1}\}$ & $p_a(2 - p_a)$ \\
$(D, D)$ & $\{j_m, j_{m+1}, j_{m+2}, a, j_1\}$ & $P^*$ \\
\hline
\end{tabular}
\caption{2-server collusion with single index output. Here $P^* = \sum_i \sum_{j \neq i} 2p_i^2 p_j^2 + \sum_{i \neq a} \sum_{j \neq i, a} 4p_a p_i^2 p_j^2$.}
\end{table}
Proof. According to Section IV-A, inequality $(k-1)v_S > (k-1)(v_R - v_A) + \gamma v_C^A$ stops user manipulation. Therefore, we can focus on reasoning about servers’ actions.

We first look at information set 1. WLOG, we examine from one queried party $S_1$’s perspective. Players hold consistent beliefs. By Bayes’ formula, when $S_1$ plays $A$ at information set 0, it believes that it arrives at information set 1 along move $AA$ with probability:

$$b(A_2|I_1^{AA}) = \frac{\beta_2}{\beta_2 + (1 - \beta_2)p_1^A}$$

Similarly, it beliefs it arrives at information set 1 along move $AD$ with probability $b(D_2|I_1^{AD}) = \frac{(1 - \beta_2)p_1^A}{\beta_2 + (1 - \beta_2)p_1^A}$. When $S_1$ plays $D$ at information set 0, we can calculate $b(A_2|I_1^{DD}) = \frac{\beta_2 p_1^{A}}{(1 - \beta_2)p_1^A + (1 - \beta_2)p_1^D}$ and $b(D_2|I_1^{DD}) = \frac{(1 - \beta_2)p_1^D}{(1 - \beta_2)p_1^A + (1 - \beta_2)p_1^D}$. If $S_1$ chooses $R$ ($\alpha_3 = 1$), then sequential rationality requires that

By laying out the expected returns, it’s easy to verify that when one plays $D$ and arrive at the corresponding information set 1, $R$ is the dominant strategy. This means that $\alpha_4 = 1$. Similarly, $\beta_4 = 1$. This is not to say that when one plays $D$, they always report (counter example is information set 4). But rather, if one plays $D$ and happen to arrive at information set 1 instead of 4, it always reports.

Now we solve for $\alpha_3$ and $\beta_3$. For a clearer representation, we temporarily denote $b(A_2|I_1^{AA})$ as $x$. At information set 1, there are three possible supports to consider for $S_1$: $\{R\}, \{R, R\}, \{R, R, R\}$. First we consider support $\{R\}$. To make $R$ sequentially rational, then we need the expected returns from playing $R$ to be higher than playing $R$:

$$R : \left[\beta_3(v_C^A - v_I - v_R - v_A) + (1 - \beta_3)\left(v_C^A + v_S + v_R - v_A\right)\right]x + (-v_I - v_A)(1 - x)$$

Because we have $v_R > v_A$ and $v_I > v_A$, the difference between the first and second quantity are positive. This means that playing $R$ always yield higher returns regardless of the other parties actions. Then because $S_1$ is capable of selfishness, we have $\alpha_3 = 1$. Similarly, $\beta_3 = 1$.

We next solve for $\alpha_2$ and $\beta_2$. At information set 0, there are three possible supports to consider for $S_1$: $\{A\}, \{D\}, \{A, D\}$. First we consider support $\{A\}$. To make $A$ sequentially rational, then we need the expected returns from playing $A$ to be higher than playing $D$. The simplified expected returns from playing the two moves are as follows:

$$A : \beta_2(v_C^A - \frac{v_I - v_R}{2} - v_A)$$

$$+ (1 - \beta_2)[p_1^A(-v_A - v_S) + v_S - v_I]$$

$$D : \beta_2(v_C^A + v_S + v_R - v_A)$$

$$+ (1 - \beta_2)[p_1^D(v_C^A - \frac{v_I - v_R}{2} - v_S) - p_2^A v_A + v_S]$$

The difference of the first and second quantity is negative since $v_R > v_A$ and $(2 - p_1)p_1^D v_S < \left(p_1^A + p_2^A + \frac{p_2^D p_1^A}{2}\right)v_A + (1 - \frac{p_1^A p_2^D}{2})v_I + p_2^A v_2 v_C^A$. This means that $\alpha_2 = 0$. Likewise, $\beta_2 = 0$. Then the unique sequential equilibrium is all players playing $DDR$ and players have beliefs consistent the equilibrium strategy.
ALGORITHM 2: Designer

Input: \( l, k, t, v_C^2, v, p, \gamma, \omega \)
Output: \( v_R, v_I, v_A, v_S, \text{expLoss} \)

1. assign \( \left\lceil \frac{v_C^2}{v_R} \right\rceil \leftarrow v_C^2 / (\omega + 1) \), \( \text{expLoss} \leftarrow 0 \);
2. \( v_S^2 = \text{floor}(vC/k); \)
3. for \( v_S \) in range \((1, v_S^2)\) do
4. \( v_I^2 = v_C/k, v_I^2 = v_C^2; \)
5. for \( v_I \) in range \((v_I^2, v_I^2)\) do
6. for \( v_R \) in range \((1, v_R - 1)\) do
7. for \( v_R \) in range \((1, v_R - 1)\) do
8. bonus = \( \text{floor}(k - 1)(v_R - v_A); \)
9. if \( v_A \geq v_R/(\omega + 1) \wedge (k - 1)v_S \geq \) bonus + \( \gamma v_C^2 \) \wedge v_I \geq (k - 1)^2v_A \) then
10. assign.append \((v_S, v_I, v_A); \)
11. if \( t \geq k \) then
12. \( \text{expLoss} \leftarrow \sum_{q=k}^{t} \left( \frac{v_S^2}{v_C^2} \right); \)
13. return assign, \( \text{expLoss} \)

APPENDIX C

ALGORITHM

APPENDIX D

BEYOND MULTI-SERVER PIR

We note that while it is extremely meaningful to mitigate collusion in general MPC protocols, we start with PIR as a first step, due to its focus on privacy protection and its well-defined and clear semantics. Evidence can be carefully, explicitly defined and efficiently verified. More importantly, PIR protocols are one-round, starting with the user sending requests and ending with servers submitting responses without further interactions. Nevertheless, we take some steps towards dealing with collusion the problem beyond PIR.

The mitigation scheme we described so far employs a structure that can be generalized to collusion in settings with a secret sharing flavor other than PIR, such as t-privacy in MPC, in out-sourced secure computation utilizing MPC [33], in an anonymous broadcast system for whistleblowers [41] necessitating non-colluding servers. We denote the protocol facing collusion problem as \( \Gamma \). The backbone collusion mitigation solution for \( \Gamma \) comprises the following three components.

A collusion detection mechanism. (1) If the collusion inevitably triggers signals visible to parties other than the colluding servers, the mechanism can capture the signals and take measures. If the collusion can be carried out in a covert way similar to the two-way anonymous channel we allow servers to communicate with, then there are two scenarios we need to consider. (2) When there exists direct verifiable evidence for collusion such as unforgeable communication scripts, then the scheme only needs to verify the evidence and execute payment rules. (3) Otherwise the incentive structure we have described can be employed. The burden of motivating collusion reporting is on the game and payment rules. The verifiability of evidence necessitates specific functionality from \( \Gamma \). Generically, this requires the ability to recognize the secret of interest. Depending on the scenario, the mechanism can involve the party with the knowledge of this secret in a verification scheme, or it can enable offline verification via cryptographic arrangements.

Game and payment rules \((G, \phi)\) that motivate honest behavior and discourage collusion. Note that the payment rules here can incorporate banning a player from the system, which means to take away all potential future gains for this player. (1) When there exists unforgeable evidence, the game and payment rules only need to motivate reporting of collusion. (2) Otherwise, the rules also need to deal with the problem of false accusations. If false accusations cannot succeed with probability 1, one can add accusation fees and accusation substantiation threshold to increase the cost of false accusations to a level where it has negative expected returns. If false accusations can always succeed, changing the “evidence” to the ones that are not trivially forgeable is necessary.

Coping with adversarial exiting strategies. If the system can still exert power on the existing adversary, one can utilize a self-insurance design that extends privacy protection or other goals \( \Gamma \) sets out to achieve to a period of time into the future. The intuition is to construct and maintain an insurance pool, which can consist of temporarily detained fees like the service fees and realized extra fines. If the exiting adversary can bring about breaches from outside the system, a more sophisticated design needs to be in place to hinder the holding of undesired information by an ex-member, e.g. proof of secure erasure.

APPENDIX E

MULTI-SERVER PIR CONSTRUCTIONS

We find Boyle-Gilboa-Ishai’s construction for 2-server cPIR [6] and Hafiz and Henry’s construction for more server case [26] to be suitable for our applications to blockchain in mind. The two constructions are both based on distributed point functions (DPF) and provide neat designs, computational privacy guarantees plus good download, upload, computation costs, and round complexity.

A. 2-server PIR

Before we introduce the 2-server PIR construction, we describe 2 crucial related primitives, Function Secret Sharing (FSS) and its special case Distributed Point Function (DPF). An \( m \)-party FSS scheme for a function family \( F \) from \( \{0, 1\}^n \) to an Abelian group \( G \) comprises key generation function \( \text{Gen}(\cdot) \) and an evaluation function \( \text{Eval}(\cdot) \). For a function \( f \in F \), algorithm \( \text{Gen}(\cdot) \) returns \( m \)-tuple keys \( k_1, k_2, \ldots, k_m \) where each key \( k_i \) defines function \( f_i(x) = \text{Eval}(i, k_i, x) \) and \( f(x) = \sum_{i=1}^{m} f_i(x) \). With less than \( m \) keys, it’s computationally hard to derive \( f \). DPF [22] is FSS for point functions. A point function \( f_{a,b} : \{0, 1\}^n \rightarrow G \) for \( a \in \{0, 1\}^n, b \in G \) evaluates to \( b \) on input \( a \) and \( 0 \) on other inputs.

Boyle-Gilboa-Ishai’s Construction for 2-server cPIR. This construction utilizes a PRG-based 2-party DPF scheme. The client wants to retrieve \( \text{D}_a \) from \( \mathbb{D} = (\mathbb{D}_1, \ldots, \mathbb{D}_N) \) and needs to distribute the point function \( f_{a,1} : [N] \rightarrow \mathbb{Z}_2 \) to two servers. She generates a pair of keys \((k_1, k_2)\) with a secure PRG, obtains function shares \((f_1, f_2)\) and sends \((f_i, k_i)\) to server \( i \).
Server \( i \) responds with \( \sum_{j=1}^{N} D_j f_i(j) \). The client then obtains \( D_a \) by xor-ing two responses.

**B. \( k \)-server PIR**

We have been representing the database \( D \) as a vector of entries \( D = (D_1, \ldots, D_N) \). Alternatively, we can work in an arbitrary finite field \( \mathbb{F} \) and represent each entry in \( \mathbb{F} \). Suppose it takes \( s \) elements to describe each entry. Then \( D \) can be represented in \( \mathbb{F} \) as an \( s \times N \) matrix. We denote each entry as a vector \( D_i \) in this subsection instead of \( D_i \) to signal this \( s \)-element representation of an entry. \( D_{ij} \) locates the \( j \)-th element of entry \( D_i \).

**Hafiz-Henry’s construction for \( k \)-server cPIR.** We present the protocol for querying \( k = 2^K, K \in \mathbb{Z}^+ \) servers. The client intends to fetch \( D_a \) and needs to distribute the point function \( f_{a,1} : [N] \rightarrow \mathbb{Z}_2 \) to \( k \) servers. She generates \( K \) independent (2,2)-DPF key pairs, \( (k_0^{(0)}, k_1^{(0)}), \ldots, (k_0^{(K-1)}, k_1^{(K-1)}) \), for the point function \( f_{a,1} \). Here \( k_{i}^{(t)} \) (\( i \in \{0, \ldots, K - 1\} \)) represents the key and is different from the parameter \( k \). The client now sends to each server \( j \) the key share \( (k_{i,j}^{(K-1)}, \ldots, k_{i,j}^{0}) \). Each server \( j \) expands each of these keys into a length-\( N \) vector of bits and then concatenates the \( K \) vectors component-wise to obtain a length-\( N \) vector \( v \) of \( k \)-bit strings. Server \( j \) then goes through \( v \) component by component xor-ing the \( v[i] \)-th word of \( D_i \) into a running total when \( v[i] = 1 \). By construction, the \( a \)-th bit produced by each DPF key pair differs and all others are equal; thus, the XOR that each server produces is identical up to but not including which word of \( D_a \) it includes; moreover, one server includes no word of \( D_a \) at all. Taking this latter server’s response and xor-ing it with the responses from each of the other servers yields each of the words comprising \( D_a \) (in some random, but known to the querier, order).

**APPENDIX F**

**SELF-insurance**

Suppose users arrive at rate \( \frac{\Omega}{T} \). At each time unit, there are \( \frac{\Omega}{T} \) queries. The amount of accumulated service fees is \( \sum_{t=1}^{T} \frac{\Omega}{k} v_I (1 + r)^{-t} \) and the amount of fines needed is \( \sum_{t=1}^{T} \frac{(k-1)p_{\text{success}} \Omega}{k} v_I (1 - r')^{-t} \). We continue to take the ratio of the two quantities

\[
\sigma = \frac{\sum_{t=0}^{T} (k-1)p_{\text{success}} \Omega v_I (1 - r')^{-t} - v_I}{\sum_{t=1}^{T} k^2 \frac{\Omega}{k} v_I (1 + r)^{-t}}
\]

\[
= \frac{\sum_{t=0}^{T} (k-1)p_{\text{success}} v_I (1 - r')^{-t} - \frac{\Omega}{k^2} v_I}{\sum_{t=1}^{T} k^2 v_I (1 + r)^{-t}}
\]

We need \( \sigma \leq 1 \). After setting a privacy protection period \( T \), we determine the privacy devaluation rate \( r' \) and service fee interest rat \( r \) to balance the tension between inexecutable fines and insurance caused by a server’s exiting strategy.

If there are \( s > 1 \) servers adopting this exiting strategy during \( T \) time units, the amount of fines needed becomes \( \sum_{t=1}^{T} \frac{(k-s)p_{\text{success}} \Omega}{k} v_I (1 - r')^{-t} - sv_I \). The accumulated service fees stay the same. We can derive \( \sigma \) accordingly.