Representation theory of the $\alpha$-determinant and zonal spherical functions

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Abstract

We prove that the multiplicity of each irreducible component in the $\mathcal{U}(\mathfrak{gl}_n)$-cyclic module generated by the $l$-th power $\det(\alpha)(X)^l$ of the $\alpha$-determinant is given by the rank of a matrix whose entries are given by a variation of the spherical Fourier transformation for $(\mathfrak{S}_n, \mathfrak{S}_n^l)$. Further, we calculate the matrix explicitly when $n = 2$. This gives not only another proof of the result by Kimoto-Matsumoto-Wakayama (2007) but also a new aspect of the representation theory of the $\alpha$-determinants.

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1 Introduction

Let $n$ be a positive integer. We denote by $\mathfrak{S}_n$ the symmetric group of degree $n$. For a permutation $\sigma \in \mathfrak{S}_n$, we define

$$\nu(\sigma) \eqdef \sum_{i \geq 1} (i - 1)m_i(\sigma) \quad (\sigma \in \mathfrak{S}_n),$$

where $m_i(\sigma)$ is the number of $i$-cycles in the disjoint cycle decomposition of $\sigma$. We notice that $\nu(\cdot)$ is a class function on $\mathfrak{S}_n$. It is easy to see that $(-1)^{\nu(\sigma)} = \text{sgn} \sigma$ is the signature of a permutation $\sigma$.

Let $\alpha$ be a complex number and $A = (a_{ij})_{1 \leq i, j \leq n}$ an $n$ by $n$ matrix. The $\alpha$-determinant $\det^{(\alpha)}(A)$ of $A$ is defined by

$$\det^{(\alpha)}(A) \eqdef \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\nu(\sigma)} a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} n.$$  \hfill (1.1)

We readily see that the $\alpha$-determinant $\det^{(\alpha)}(A)$ coincides with the determinant $\det(A)$ (resp. permanent $\text{per}(A)$) of $A$ when $\alpha = -1$ (resp. $\alpha = 1$). Hence we regard the $\alpha$-determinant as a common generalization of the determinant and permanent.

The $\alpha$-determinant is first introduced by Vere-Jones [11]. He proved the identity

$$\det(I - \alpha A)^{-1/\alpha} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{1 \leq i_1, \ldots, i_k \leq n} \det^{(\alpha)} \begin{pmatrix} a_{i_1 i_1} & \cdots & a_{i_1 i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k i_1} & \cdots & a_{i_k i_k} \end{pmatrix}$$  \hfill (1.2)

for an $n$ by $n$ matrix $A = (a_{ij})_{1 \leq i, j \leq n}$ such that the absolute value of any eigenvalue of $A$ is less than 1. Here $I$ denotes the identity matrix of suitable size. His intention of the study of the $\alpha$-determinant is an application to probability theory. Actually, the identity (1.2) supplies a unified treatment of the multivariate binomial and negative binomial distributions. Further, Shirai and Takahashi [10] proved a Fredholm determinant version of (1.2) for a trace class integral operator and use it to define a certain one-parameter family of point processes. We note that a pfaffian analogue of the Vere-Jones identity (1.2) has been also established and is applied to
probability theory by Matsumoto [7]. It is also worth noting that (1.2) is obtained by specializing \( p_i(x) = \alpha^{i-1} \) and regarding \( y_1, \ldots, y_n \) as eigenvalues of \( A \) in the Cauchy identity
\[
\prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} = \sum_{\lambda} \frac{1}{z_{\lambda}} p_\lambda(x) p_\lambda(y),
\]
where \( \lambda \) in the right-hand side runs over the set of all partitions, \( z_\lambda \) denotes the cardinality of the centralizer of a permutation whose cycle type is \( \lambda \), and \( p_\lambda \) denotes the power-sum symmetric function corresponding to \( \lambda \) (see [8] for detailed information on symmetric functions). In fact, under the specialization, the left-hand side of (1.3) becomes \( \det(1 - \alpha t)^{-1/\alpha} \) and the right-hand side represents its expansion in terms of \( \alpha \)-determinants (see also [3]).

In this article, we focus our attention on the representation-theoretic aspect of the \( \alpha \)-determinant. Let \( \mathcal{U}(\mathfrak{gl}_n) \) be the universal enveloping algebra of the general linear Lie algebra \( \mathfrak{gl}_n = \mathfrak{gl}_n(\mathbb{C}) \), and \( \mathcal{P}(\text{Mat}_n) \) be the polynomial algebra in the \( n^2 \) variable \( x_{ij} \) \( (1 \leq i, j \leq n) \). We put \( X = (x_{ij})_{1 \leq i, j \leq n} \) and write an element in \( \mathcal{P}(\text{Mat}_n) \) as \( f(X) \) in short. The algebra \( \mathcal{P}(\text{Mat}_n) \) becomes a left \( \mathcal{U}(\mathfrak{gl}_n) \)-module via
\[
E_{ij} \cdot f(X) = \sum_{s=1}^n x_{is} \frac{\partial f(X)}{\partial x_{js}},
\]
for \( f(X) \in \mathcal{P}(\text{Mat}_n) \) where \( \{E_{ij}\}_{1 \leq i, j \leq n} \) is the standard basis of \( \mathfrak{gl}_n \). Now we regard the \( \alpha \)-determinant \( \det^{(\alpha)}(X) \) of \( X \) as an element in \( \mathcal{P}(\text{Mat}_n) \) and consider the cyclic submodule
\[
V_{n,l}(\alpha) \overset{\text{def}}{=} \mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^l
\]
of \( \mathcal{P}(\text{Mat}_n) \). Since
\[
V_{n,1}(-1) = \mathcal{U}(\mathfrak{gl}_n) \cdot \det(X) \cong \mathcal{M}_n^{(1^n)}, \quad V_{n,1}(1) = \mathcal{U}(\mathfrak{gl}_n) \cdot \det(X) \cong \mathcal{M}_n^{(n)},
\]
the module \( V_{n,1}(\alpha) \) is regarded as an interpolation of these two irreducible representations. Here we denote by \( \mathcal{M}_n^{(\lambda)} \) the irreducible \( \mathcal{U}(\mathfrak{gl}_n) \)-module whose highest weight is \( \lambda \). We notice that we can identify the dominant integral weights with partitions as far as we consider the polynomial representations of \( \mathcal{U}(\mathfrak{gl}_n) \).

Our main concern is to solve the

**Problem 1.1.** Describe the irreducible decomposition of the \( \mathcal{U}(\mathfrak{gl}_n) \)-module \( V_{n,l}(\alpha) \) explicitly.

In [4], the following general result on \( V_{n,l}(\alpha) \) is proved.

**Theorem 1.2.** For each \( \lambda \vdash nl \) such that \( \ell(\lambda) \leq n \), there exists a certain square matrix \( F_{n,l}^{\lambda}(\alpha) \) of size \( K_{\lambda(l^n)} \) whose entries are polynomials in \( \alpha \) such that
\[
V_{n,l}(\alpha) \cong \bigoplus_{\lambda \vdash nl \atop \ell(\lambda) \leq n} (\mathcal{M}_n^{\lambda})^\oplus \otimes \mathcal{F}_{n,l}^{\lambda}(\alpha).
\]

Here \( K_{\lambda\mu} \) denotes the Kostka number and \( \ell(\lambda) \) is the length of \( \lambda \).

We call this matrix \( F_{n,l}^{\lambda}(\alpha) \) the transition matrix for \( \lambda \) in \( V_{n,l}(\alpha) \). We notice that the transition matrix is determined up to conjugacy. Thus, Problem 1.1 is reduced to the determination of the matrices \( F_{n,l}^{\lambda}(\alpha) \) relative to a certain (nicely chosen) basis. Up to the present, we have obtained an explicit form of \( F_{n,l}^{\lambda}(\alpha) \) in only several particular cases.

**Example 1.3.** When \( l = 1 \), Problem 1.1 is completely solved in [8] as follows: For each positive integer \( n \), we have
\[
V_{n,1}(\alpha) = \mathcal{U}(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X) \cong \bigoplus_{\lambda \vdash n \atop f_\lambda(\alpha) \neq 0} \mathcal{M}_n^{\lambda} \otimes f^\lambda,
\]
(1.5)
where \( f_\lambda(\alpha) \) is a (modified) content polynomial

\[
f_\lambda(\alpha) \overset{\text{def}}{=} \prod_{i=1}^{\ell(\lambda)} \prod_{j=1}^{\lambda_i} (1 + (j - i)\alpha).
\]

In other words, for each \( \lambda \vdash n \), we have

\[
\text{multiplicity of } M^\lambda_n \text{ in } V_{n,1}(\alpha) = \begin{cases} 0 & \alpha \in \{1/k, 1 \leq k < \ell(\lambda)\} \cup \{-1/k, 1 \leq k < \lambda_1\}, \\ f^\lambda & \text{otherwise.} \end{cases}
\]

The transition matrix \( F^\lambda_{n,1}(\alpha) \) in this case is given by \( f_\lambda(\alpha)I \).

**Example 1.4.** When \( n = 2 \), the transition matrix \( F^\lambda_{2,l}(\alpha) \) is of size 1 (i.e. just a polynomial) and it is shown in [4, Theorem 4.1] that

\[
V_{2,l}(\alpha) = \mathcal{U}(\mathfrak{gl}_2) \cdot \det^{(\alpha)}(X)^l \cong \bigoplus_{k < \lambda, x} M^{(2l-s,s)}_2(\alpha) \neq 0,
\]

where we put

\[
F^{(2l-s,s)}_{2,l}(\alpha) = (1 + \alpha)^{l-s} G^l_s(\alpha),
\]

\[
G^l_s(\alpha) = \sum_{j=0}^l \frac{(-s)_j (l - s + 1)_j (-\alpha)^j}{(-l)_j j!}.
\]

Here \((a)_j = \Gamma(a + j)/\Gamma(a)\) is the Pochhammer symbol. We note that \( G^l_s(\alpha) \) is written by a Jacobi polynomial as

\[
G^l_s(\alpha) = \binom{s - l - 1}{s}^{-1} P^{(-l-1,2l-2s+1)}(1 + 2\alpha).
\]

In this paper, we show that the entries of the transition matrices \( F^\lambda_{n,l}(\alpha) \) are given by a variation of the spherical Fourier transformation of a certain class function on \( S_{nl} \) with respect to the subgroup \( \mathfrak{S}^\circ_n \) (Theorem 2.9). This result also provides another proof of Theorem 1.2. Further, we give a new calculation of the polynomial \( F^{(2l-s,s)}_{2,l}(\alpha) \) in Example 1.4 by using an explicit formula for the values of zonal spherical functions for the Gelfand pair \( (\mathfrak{S}_{2n}, \mathfrak{S}_n \times \mathfrak{S}_n) \) due to Bannai and Ito (Theorem 3.1).

## 2 Irreducible decomposition of \( V_{n,l}(\alpha) \)

Fix \( n, l \in \mathbb{N} \). Consider the standard tableau \( T \) with shape \((l^n)\) such that the \((i,j)\)-entry of \( T \) is \((i-1)l + j\). For instance, if \( n = 3 \) and \( l = 2 \), then

\[
T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}.
\]

We denote by \( K = R(T) \) and \( H = C(T) \) the row group and column group of the standard tableau \( T \) respectively. Namely,

\[
K = \{ g \in \mathfrak{S}_{nl} ; [g(x)] = [x], x \in [nl] \}, \quad H = \{ g \in \mathfrak{S}_{nl} ; g(x) \equiv x \pmod{l}, x \in [nl] \},
\]

where we denote by \([nl]\) the set \( \{1, 2, \ldots, nl\} \). We put

\[
e = \frac{1}{|K|} \sum_{k \in K} k \in \mathbb{C}[\mathfrak{S}_{nl}].
\]
This is clearly an idempotent element in $\mathbb{C}[S_{nl}]$. Let $\varphi$ be a class function on $H$. We put

$$\Phi \overset{\text{def}}{=} \sum_{h \in H} \varphi(h)h \in \mathbb{C}[S_{nl}].$$

Consider the tensor product space $V = (\mathbb{C}^n)^{\otimes nl}$. We notice that $V$ has a $(U(\mathfrak{gl}_n), \mathbb{C}[S_{nl}])$-module structure given by

$$E_{ij} \cdot e_{i_1} \otimes \cdots \otimes e_{i_{nl}} \overset{\text{def}}{=} \sum_{s=1}^{nl} \delta_{i,j} e_{i_1} \otimes \cdots \otimes e_i \otimes \cdots \otimes e_{i_{nl}},$$

where $\{e_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{C}^n$. The main concern of this section is to solve the

**Problem 2.1.** Describe the irreducible decomposition of the left $U(\mathfrak{g}l_n)$-module $V \cdot e\Phi e$.

Here we show that Problem 2.1 includes Problem 1.1 as a special case. We consider the group isomorphism $\theta : H \rightarrow S_n$ defined by

$$\theta(h) = (\theta(h)_1, \ldots, \theta(h)_l): \quad \theta(h)_i(x) = y \iff h((x-1)i) = (y-1)i + i.$$  

We also define an element $D(X; \varphi) \in \mathcal{P}(\text{Mat}_n)$ by

$$D(X; \varphi) \overset{\text{def}}{=} \sum_{h \in H} \varphi(h) \prod_{p=1}^{l} \prod_{q=1}^{n} x_{\theta(h)_p(q)}.q = \sum_{h \in H} \varphi(h) \prod_{p=1}^{l} \prod_{q=1}^{n} x_{q,\theta(h)_p^{-1}(q)}.$$  

We note that $D(X; \alpha^{(\nu)}) = \det^{(\nu)}(X)^I$ since $\nu(\theta^{-1}(\sigma_1, \ldots, \sigma_l)) = \nu(\sigma_1) + \cdots + \nu(\sigma_l)$ for $(\sigma_1, \ldots, \sigma_l) \in S^n_l$.

Take a class function $\delta_H$ on $H$ defined by

$$\delta_H(h) = \begin{cases} 1 & h = 1 \\ 0 & h \neq 1. \end{cases}$$

We see that $D(X; \delta_H) = (x_1 x_2 \ldots x_n)^I$. We need the following lemma (see [3] Lemma 2.1) for the proof of (1). The assertion (2) is immediate.

**Lemma 2.2.** (1) It holds that

$$U(\mathfrak{g}l_n) \cdot e_{i_1}^{\otimes l} \otimes \cdots \otimes e_{i_n}^{\otimes l} = V \cdot e = \text{Sym}^l(\mathbb{C}^n)^{\otimes n},$$

$$U(\mathfrak{g}l_n) \cdot D(X; \delta_H) = \bigoplus_{i_{pq} \in \{1, \ldots, n\}} \mathbb{C} : \prod_{p=1}^{l} \prod_{q=1}^{n} x_{i_{pq}q} \cong \text{Sym}^l(\mathbb{C}^n)^{\otimes n}.$$  

(2) The map

$$\mathcal{T} : U(\mathfrak{g}l_n) \cdot D(X; \delta_H) \ni \prod_{p=1}^{l} \prod_{q=1}^{n} x_{i_{pq}q} \mapsto (e_{i_1} \otimes \cdots \otimes e_{i_1}) \otimes \cdots \otimes (e_{i_n} \otimes \cdots \otimes e_{i_n}) \cdot e \in V \cdot e$$

is a bijective $U(\mathfrak{g}l_n)$-intertwiner.
Lemma 2.3. It holds that
\[ \mathcal{U}(\mathfrak{gl}_n) \cdot D(X; \varphi) \cong V \cdot e\Phi e \]
as a left \( \mathcal{U}(\mathfrak{gl}_n) \)-module. In particular, \( V \cdot e\Phi e \cong V_{n,\ell}(\alpha) \) if \( \varphi(h) = \alpha^{\nu(h)} \).

By the Schur-Weyl duality, we have
\[ V \cong \bigoplus_{\lambda \vdash nl} \mathcal{M}_n^{\lambda} \boxtimes S^{\lambda}. \]

Here \( S^{\lambda} \) denotes the irreducible unitary right \( S_{nl} \)-module corresponding to \( \lambda \). We see that
\[ \dim (S^{\lambda} \cdot e) = \langle \text{ind}_K^{\mathfrak{gl}_n} 1_K, S^{\lambda} \rangle_{S_{nl}} = K_{\lambda(n)}^{(n)} \]
where \( 1_K \) is the trivial representation of \( K \) and \( \langle \pi, \rho \rangle_{S_{nl}} \) is the intertwining number of given representations \( \pi \) and \( \rho \) of \( S_{nl} \). Since \( K_{\lambda(n)}^{(n)} = 0 \) unless \( \ell(\lambda) \leq n \), it follows the

Theorem 2.4. It holds that
\[ V \cdot e\Phi e \cong \bigoplus_{\lambda \vdash nl, \ell(\lambda) \leq n} \mathcal{M}_n^{\lambda} \boxtimes (S^{\lambda} \cdot e\Phi e). \]

In particular, as a left \( \mathcal{U}(\mathfrak{gl}_n) \)-module, the multiplicity of \( \mathcal{M}_n^{\lambda} \) in \( V \cdot e\Phi e \) is given by
\[ \dim (S^{\lambda} \cdot e\Phi e) = \text{rk}_{\text{End}(S^{\lambda} \cdot e\Phi e)}(e\Phi e). \]

Let \( \lambda \vdash nl \) be a partition such that \( \ell(\lambda) \leq n \) and put \( d = K_{\lambda(n)}^{(n)} \). We fix an orthonormal basis \( \{e_1^{\lambda}, \ldots, e_f^{\lambda}\} \) of \( S^{\lambda} \) such that the first \( d \) vectors \( e_1^{\lambda}, \ldots, e_d^{\lambda} \) form a subspace \( (S^{\lambda})^K \) consisting of \( K \)-invariant vectors and left \( f^{\lambda} - d \) vectors form the orthocomplement of \( (S^{\lambda})^K \) with respect to the \( S_{nl} \)-invariant inner product. The matrix coefficient of \( S^{\lambda} \) relative to this basis is
\[ \psi_{ij}^{\lambda}(g) = \langle e_i^{\lambda}, g e_j^{\lambda} \rangle_{S^{\lambda}} \quad (g \in S_{nl}, 1 \leq i, j \leq f^{\lambda}). \tag{2.3} \]

We notice that this function is \( K \)-bi-invariant. We see that the multiplicity of \( \mathcal{M}_n^{\lambda} \) in \( V \cdot e\Phi e \) is given by the rank of the matrix
\[ \left( \sum_{h \in H} \varphi(h) \psi_{ij}^{\lambda}(h) \right)_{1 \leq i, j \leq d}. \]

As a particular case, we obtain the
Theorem 2.5. The multiplicity of the irreducible representation $\mathcal{M}_n^\lambda$ in the cyclic module $U(\mathfrak{gl}_n) \cdot \det^{(\alpha)}(X)^l$ is equal to the rank of

$$F_{n,l}^\lambda(\alpha) = \left( \sum_{h \in H} \alpha^{\nu(h)} \psi_{ij}^\lambda(h) \right)_{1 \leq i, j \leq d}, \quad (2.4)$$

where $\{\psi_{ij}^\lambda\}_{i,j}$ denotes a basis of the $\lambda$-component of the space $C(K\backslash\mathfrak{S}_{nl}/K)$ of $K$-biinvariant functions on $\mathfrak{S}_{nl}$ given by (2.3).

Remark 2.6. (1) By the definition of the basis $\{\psi_{ij}^\lambda\}_{i,j}$ in (2.3), we have $F_{n,l}^\lambda(0) = I$.

(2) Since $\alpha^{\nu(g^{-1})} = \alpha^{\nu(g)}$ and $\psi_{ij}^\lambda(g^{-1}) = \psi_{ji}^\lambda(g)$ for any $g \in \mathfrak{S}_{nl}$, the transition matrices satisfy $F_{n,l}^\lambda(\alpha)^* = F_{n,l}^\lambda(\overline{\alpha})$.

(3) In Examples 1.3 and 1.4, the transition matrices are given by diagonal matrices. We expect that any transition matrix $F_{n,l}^\lambda(\alpha)$ is diagonalizable in $\text{Mat}_{K^l(n)}(\mathbb{C}[\alpha])$.

Example 2.7 (Example 1.3). If $l = 1$, then $H = G = \mathfrak{S}_n$ and $K = \{1\}$. Therefore, for any $\lambda \vdash n$, we have

$$F_{n,1}^\lambda(\varphi) = \frac{n!}{f_{\lambda}} \langle \varphi, \chi^\lambda \rangle_{\mathfrak{S}_n} I \quad (2.5)$$

by the orthogonality of the matrix coefficients. Here $\chi^\lambda$ denotes the irreducible character of $\mathfrak{S}_n$ corresponding to $\lambda$. In particular, if $\varphi = \alpha^{\nu(\cdot)}$, then

$$F_{n,1}^\lambda(\alpha) = f_{\lambda}(\alpha)I \quad (2.6)$$

since the Fourier expansion of $\alpha^{\nu(\cdot)}$ (as a class function on $\mathfrak{S}_n$) is

$$\alpha^{\nu(\cdot)} = \sum_{\lambda \vdash n} \frac{f_{\lambda}}{n!} f_{\lambda}(\alpha) \chi^\lambda, \quad (2.7)$$

which is obtained by specializing the Frobenius character formula for $\mathfrak{S}_n$ (see, e.g. [3]).

The trace of the transition matrix $F_{n,l}^\lambda(\alpha)$ is

$$F_{n,l}^\lambda(\alpha) \overset{\text{def}}{=} \text{tr} \ F_{n,l}^\lambda(\alpha) = \sum_{h \in H} \alpha^{\nu(h)} \omega^\lambda(h), \quad (2.8)$$

where $\omega^\lambda$ is the zonal spherical function for $\lambda$ with respect to $K$ defined by

$$\omega^\lambda(g) \overset{\text{def}}{=} \frac{1}{|K|} \sum_{k \in K} \chi^\lambda(kg) \quad (g \in \mathfrak{S}_{nl}).$$

This is regarded as a generalization of the modified content polynomial since $F_{n,1}^\lambda(\alpha) = f^\lambda f_{\lambda}(\alpha)$ as we see above. It is much easier to handle these polynomials than the transition matrices. If we could prove that a transition matrix $F_{n,l}^\lambda$ is a scalar matrix, then we would have $F_{n,l}^\lambda = d^{-1} F_{n,l}^\lambda(\alpha)I$ and hence we see that the multiplicity of $\mathcal{M}_n^\lambda$ in $V_{n,l}(\alpha)$ is completely controlled by the single polynomial $F_{n,l}^\lambda(\alpha)$. In this sense, it is desirable to obtain a characterization of the irreducible representations whose corresponding transition matrices are scalar as well as to get an explicit expression for the polynomials $F_{n,l}^\lambda(\alpha)$. We will investigate these polynomials $F_{n,l}^\lambda(\alpha)$ and their generalizations in [3].

Example 2.8. Let us calculate $F_{n,l}^{(n-1,1)}(\alpha)$. We notice that $\chi^{(n-1,1)}(g) = \text{fix}_{nl}(g) - 1$ where $\text{fix}_{nl}$ denotes the number of fixed points in the natural action $\mathfrak{S}_{nl} \curvearrowright [nl]$. Hence we see that

$$F_{n,l}^{(n-1,1)}(\alpha) = \sum_{h \in H} \alpha^{\nu(h)} \frac{1}{|K|} \sum_{k \in K} (\text{fix}_{nl}(kh) - 1) = \sum_{h \in H} \alpha^{\nu(h)} \frac{1}{|K|} \sum_{k \in K} \sum_{x \in [nl]} \delta_{khx,x} - \sum_{h \in H} \alpha^{\nu(h)}.$$
It is easily seen that \(khx \neq x\) for any \(k \in K\) if \(hx \neq x\) \((x \in [n]l)\). Thus it follows that
\[
\frac{1}{|K|} \sum_{k \in K} \sum_{x \in [n]l} \delta_{khx,x} \sum_{x \in [n]l} \delta_{hx,x} \frac{1}{|K|} \sum_{k \in K} \delta_{kh,x} = \frac{1}{l} \tilde{h}x_{nm}(h) \quad (h \in H).
\]
Therefore we have
\[
F_{n,l}^{(n-1,1)}(\alpha) = \frac{1}{\sqrt{\gamma}} \sum_{h \in H} \alpha^{\nu(h)} \tilde{h}x_{nm}(h) - \sum_{h \in H} \alpha^{\nu(h)} = F_{n,1}^{(n)}(\alpha)^{l-1} F_{n,1}^{(n-1,1)}(\alpha)
\]
\[
= (n-1)(1-\alpha)(1-(n-1)\alpha)^{l-1} \prod_{i=1}^{n-2} (1+i\alpha)^{l}.
\]
We note that the transition matrix \(F_{n,l}^{(n-1,1)}\) is a scalar one (see [3]), so that the multiplicity of \(\mathcal{M}_{\alpha}^{(n-1,1)}\) in \(V_{n,l}(\alpha)\) is zero if \(\alpha \in \{1, -1, -1/2, \ldots, -1/(n-1)\}\) and \(n-1\) otherwise.

### 3 Irreducible decomposition of \(V_{2,l}(\alpha)\) and Jacobi polynomials

In this section, as a particular example, we consider the case where \(n = 2\) and calculate the transition matrix \(F_{2,l}^{(\alpha)}\) explicitly. Since the pair \((\mathcal{G}_2, K)\) is a Gelfand pair (see, e.g., [6]), it follows that
\[
K_{\lambda(l)} = \left[ \text{ind}_{\mathcal{G}_2}^{\mathcal{G}_2} 1_{K}, S^{\lambda} \right]_{\mathcal{G}_2} = 1
\]
for each \(\lambda \vdash 2n\) with \(\ell(\lambda) \leq 2\). Thus, in this case, the transition matrix is just a polynomial and is given by
\[
F_{2,l}^{(\alpha)} = \text{tr} F_{2,l}^{(\alpha)} = \sum_{h \in H} \alpha^{\nu(h)} \omega^{\lambda}(h) = \sum_{s=0}^{l} \binom{l}{s} \omega^{\lambda}(g_s) \alpha^s.
\]  \(\text{(3.1)}\)

Here we put \(g_s = (1, l+1)(2, l+2)\ldots(s, l+s) \in \mathcal{G}_2\). Now we write \(\lambda = (2l-p, p)\) for some \(p\) \((0 \leq p \leq l)\). The value \(\omega^{(2l-p, p)}(g_s)\) of the zonal spherical function is calculated by Bannai and Ito [2, p.218] as
\[
\omega^{(2l-p, p)}(g_s) = Q_p(s; -l-1, -l-1, l) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \binom{2l-p+1}{j} \binom{l}{j}^2 \binom{s}{j},
\]
where
\[
Q_n(x; \alpha, \beta, N) \overset{\text{def}}{=} \tilde{\frac{3}{2}} F_2 \left( -n, n+\alpha+\beta+1, -x \atop \alpha+1, -N \right) \overset{\text{def}}{=} \sum_{j=0}^{N} (-1)^j \binom{n}{j} \left( -n - \alpha - \beta - 1 \atop j \right) \left( -\alpha - 1 \atop j \right)^{-1} \left( N \atop j \right)^{-1} \left( x \atop j \right)
\]
is the Hahn polynomial (see also [6, p.399]). We also denote by \(n+1 \tilde{F}_p \left( a_1, \ldots, a_p \atop b_1, \ldots, b_q \right)_{(-N, x)}\) \((b_1, \ldots, b_q, -N) \in \mathbb{N}\) the hypergeometric polynomial
\[
\tilde{F}_p \left( a_1, \ldots, a_p \atop b_1, \ldots, b_q \right)_{(-N, x)} = \sum_{j=0}^{N} \frac{(a_1)_j \ldots (a_p)_j}{(b_1)_j \ldots (b_q)_j (-N)_j} \frac{x^j}{j!}
\]
for \(p, q, N \in \mathbb{N}\) in general (see [11]). Further, if we put
\[
G_p^{(l)}(x) \overset{\text{def}}{=} \frac{3}{2} \tilde{F}_1 \left( -p, l-p+1 \atop -l \right) = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \binom{l-p+j}{j} \binom{l}{j}^{-1} x^j,
\]
then we have the
Theorem 3.1. Let \( l \) be a positive integer. It holds that
\[
F_{2l}^{(2l-p,p)}(\alpha) = \sum_{s=0}^{l} \left( \begin{array}{c} s \\ l \end{array} \right) Q_p(s; l-1, l-1, l) \alpha^s = (1 + \alpha)^{l-p} G_p^l(\alpha)
\]
for \( p = 0, 1, \ldots, l \).

Proof. Let us put \( x = -1/\alpha \). Then we have
\[
\sum_{s=0}^{l} \left( \begin{array}{c} s \\ l \end{array} \right) Q_p(s; l-1, l-1, l) \alpha^s = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \binom{2l-p+1}{j} \binom{l}{j}^{-1} \alpha^j (1 + \alpha)^{l-j} = x^{-l} (x-1)^{l-p} \sum_{j=0}^{p} \binom{p}{j} \binom{2l-p+1}{j} \binom{l}{j}^{-1} (x-1)^{p-j}
\]
and
\[
(1 + \alpha)^{l-p} G_p^l(\alpha) = x^{-l} (x-1)^{l-p} \sum_{j=0}^{p} (-1)^j \binom{p}{j} \binom{l-p+j}{j} \binom{l}{j}^{-1} (-x)^{p-j}.
\]
Here we use the elementary identity
\[
\sum_{s=0}^{l} \left( \begin{array}{c} s \\ l \end{array} \right) \left( \begin{array}{c} l \\ j \end{array} \right) \alpha^s = \left( \begin{array}{c} l \\ j \end{array} \right) \alpha^j (1 + \alpha)^{l-j}.
\]
Hence, to prove the theorem, it is enough to verify
\[
\sum_{i=0}^{p} \binom{p}{i} \binom{l-p+i}{i} \binom{l}{i}^{-1} x^{p-i} = \sum_{j=0}^{p} \binom{p}{j} \binom{2l-p+1}{j} \binom{l}{j}^{-1} (x-1)^{p-j}.
\] (3.2)
Comparing the coefficients of Taylor expansion of these polynomials at \( x = 1 \), we notice that the proof is reduced to the equality
\[
\sum_{i=0}^{r} \binom{l-i}{l-r} \binom{l-p+i}{l-p} = \binom{2l-p+1}{r}
\] (3.3)
for \( 0 \leq r \leq p \), which is well known (see, e.g. (5.26) in [9]). Thus we have the conclusion. \( \square \)

Thus we give another proof of the irreducible decomposition (1.7).

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