Influence of Quantum Gravity on the Tunneling Radiation of Fermion
Bing-bing CHEN
Sichuan Minzu College, Kangding, Sichuan, 626001, China

Keywords: Quantum gravitational effect; Two-dimensional space-time; Fermion

Abstract. The tunneling behavior of two-dimensional space-time Fermion under the effect of quantum gravity is discussed. By modifying the relationship, Dirac equation is modified, and then the Fermion tunneling radiation is studied to obtain the corrected Hawking temperature. The results show that under the effect of quantum gravity, the Hawking temperature of the black hole does not only depend on the quality of the black hole itself, but also on the quality and energy of the radiation particles. At the same time, the modified temperature is lower than the original Hawking temperature, indicating that the modification of quantum gravity delays the increase of Hawking temperature.

Introduction

In 1974, Hawking used the viewpoint of quantum mechanics to prove the existence of thermal radiation in black holes\(^{[1]}\). Subsequently, people have carried out a lot of research on radiation phenomena and proposed several different research methods. One of the effective methods is semi-classical tunneling\(^{[2]}\). This method can truly describe the tunneling radiation of scalar particles and the modified Hawking temperature considering the varying background time and space and the self-gravity of the particles\(^{[3]}\). In Fermi field, the Hawking temperature of the black hole can also be obtained by discussing Fermion tunneling horizon\(^{[4]}\), but since the varying background time and space and reaction are ignored, the standard Hawking temperature of the black hole is obtained.

On the other hand, many quantum gravitational theories imply the existence of a minimum observable length\(^{[5-8]}\). This length can be implemented in Generalized Uncertainty Principle (GUP), which is also the result of modifying Communication Relationship of Heisenberg algebra. However, the modification of Communication Relationship is not unique, and different modifications can be obtained different effective models. On this basis, people have conducted extensive and in-depth discussion on the related properties of black holes. The Generalized Dirac Equation can also be obtained by modifying Communication Relationship.

This paper will discuss the two-dimensional space-time Fermion tunneling radiation obtained by Jackiw-Teitelboim Theory. Firstly, Generalized Dirac Equation is obtained by Communication Relationship, and the influence of quantum gravity on the tunneling behavior of Fermion is discussed, and the modified Hawking temperature is obtained.

Generalized Dirac Equation

The (GUP) model can be obtained by modifying the basic Communication Relationship of Heisenberg algebra:

\[
\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 + \beta \left(\Delta p\right)^2\right]
\]  

(1)

In which, \(\hbar\) is Reduced Planck Constant, \(\Delta x\) and \(\Delta p\) are respectively the uncertainty of particle position and momentum, \(\beta = \frac{\beta_0}{M_p^2}\) and \(\beta_0\) are the dimensionless parameters that characterize quantum gravity, \(M_p\) is Planck Quality\(^{[9]}\). Kempf and others firstly make modification to Communication Relationship\(^{[6]}\), and the modified basic Communication Relationship is:

\[
\Delta x \Delta p \geq \frac{\hbar}{2} \left[1 + \beta_0 \left(\Delta p\right)^2\right]
\]  

(2)
\[ [x_i, p_i] = i\hbar\delta_{ij}[1 + \beta p^2] \] (2)

In which \( x_i \) and \( p_i \) can be represented as\(^{[10]}\):

\[ x_i = x_{0i} \]
\[ p_i = p_{0i}(1 + \beta p^2) \]
\[ p^2 = p_0 p' = -\hbar^2\left[1 - \beta h^2 \left(\partial_j \partial'^j\right)\right]\partial_i \left[1 - \beta h^2 \left(\partial_j \partial'^j\right)\right]\partial^j \approx -\hbar^2 \left[\partial_i \partial^j - 2\beta h^2 \left(\partial_j \partial'^j\right)\right] \] (3)

In Equation (3), all high-order terms of \( \beta \) are ignored, and the regular Communication Relationship of \( x_{0i} \) and \( p_{0i} \) satisfies \([x_{0i}, p_{0j}] = i\hbar\delta_{ij}\).

Introduce generalized frequency by considering quantum gravitation\(^{[10]}\):

\[ \sigma = E\left(1 - \beta E^2\right) \] (4)

In the above equation, energy is defined as \( E = i\hbar \partial_0 \), use energy relation \( p^2 + m^2 = E^2 \), the general expression of particle energy\(^{[11,12]}\) can be obtained:

\[ \overline{E} = E\left[1 - \beta \left(p^2 + m^2\right)\right] \] (5)

In the curved space and time, the motion of the spin \( \frac{1}{2} \) Fermion follows Dirac Equation\(^{[11]}\):

\[ i\gamma^\mu \left(\partial^\mu + \Omega^\mu\right)\psi + \frac{m}{\hbar}\psi = 0 \] (6)

In which, \( m \) is the quality of tunneling particle, \( \Omega^\mu = \frac{i}{2} \Gamma^a_{\mu} \sum_{ab} \), \( \sum_{ab} = \frac{i}{4}[\gamma^\alpha, \gamma^\beta] \). \( \gamma^\mu \) matrix meets the reverse Communication Relationship \( \{\gamma^\alpha, \gamma^\beta\} = 2g^{\alpha\beta} \). Equation (6) can be shown as:

\[ -i\gamma^0 \partial_0 \psi = \left(i\gamma^j \partial_j + i\gamma^\mu \Omega^\mu + \frac{m}{\hbar}\right)\psi \] (7)

Put Equation (3) and (5) into Equation (7), ignore all high-order terms of \( \beta \) in the calculation to get:

\[ -i\gamma^0 \partial_0 \psi = \left(i\gamma^j \partial_j + i\gamma^\mu \partial_\mu + \frac{m}{\hbar}\right)(1 + \beta h^2 \partial_j \partial'^j - \beta m^2)\psi \] (8)

After the shifting term, Generalized Dirac Equation is obtained\(^{[13]}\):

\[ \left[i\gamma^0 \partial_0 + i\gamma^j \partial_j \left(1 - \beta m^2\right) + i\gamma^j \beta h^2 \left(\partial_j \partial'^j\right)\partial_i + \frac{m}{\hbar}(1 + \beta h^2 \partial_j \partial'^j - \beta m^2) + i\gamma^\mu \Omega^\mu \left(1 + \beta h^2 \partial_j \partial'^j - \beta m^2\right)\right]\psi = 0 \] (9)

The above equation is the modified Dirac Equation.

**Study on Tunneling Radiation of Fermion**

According to Jackiw-Teitelboim (JT) model, two-dimensional gravitational theory can be obtained\(^{[6]}\):
\[ A = \frac{1}{2} \int \sqrt{-g} d^2r \left[ \Phi R + \lambda^2 V(\Phi) \right] \]  \hspace{1cm} (10)

In which, \( V(\Phi) = (a+1)\Phi^a, a > -1 \); \( \lambda \) is two-dimensional cosmological constant; \( \Phi \) is a scalar field, the two-dimensional black hole solution of the action amount generating \( AdS \) space is as follows:

\[ ds = -f(r) dt^2 + \frac{1}{g(r)} dr^2 \]  \hspace{1cm} (11)

In which, \( f = g = \left( \Phi^{a+1} - \frac{2M}{\lambda} \right) \), \( \Phi = \lambda r \).

Below, we consider the effect of quantum gravity on the tunneling behavior of Fermions. For a spin \( \frac{1}{2} \) Fermion, it has two states which are spinning up and spinning down, corresponding to two different wave functions. Since the calculation results of the two states are the same, only the spinning up state is considered here. Assume that the wave function of the spinning up state is:

\[ \psi = \begin{pmatrix} A \\ 0 \\ B \\ 0 \end{pmatrix} \exp \left[ \frac{i}{\hbar} I(t, r) \right] \]  \hspace{1cm} (12)

In which, \( I \) is the amount of action of tunneling Fermion, \( A \) and \( B \) are the functions of \( t \) and \( r \). In order to obtain the solution of Equation (9), the following \( \gamma^\mu \) matrix is chosen:

\[ \gamma' = \frac{1}{\sqrt{f}} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma' = \sqrt{f} \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \]  \hspace{1cm} (13)

In the above gamma matrix, \( i = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, -i = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \sigma^3 = \begin{pmatrix} 0 & \sigma^3 \\ \sigma^3 & 0 \end{pmatrix} \) represents a Pauli matrix.

In order to apply WKB approximation, put Equation (12) and (13) into Equation (9), and the high-order terms of \( \hbar \) are ignored in the calculation process, equations are sorted out:

\[ -\frac{iA}{\sqrt{f}} \partial_\tau I - B \left( 1 - \beta m^2 \right) \sqrt{f} \partial \tau I - Am\beta (\partial_\tau I)^2 + B\beta \sqrt{f} (\partial_\tau I)^3 + Am \left( 1 - \beta m^2 \right) = 0 \]

\[ \frac{iB}{\sqrt{f}} \partial_\tau I - A \left( 1 - \beta m^2 \right) \sqrt{f} \partial \tau I - Bm\beta (\partial_\tau I)^2 + A\beta \sqrt{f} (\partial_\tau I)^3 + Bm \left( 1 - \beta m^2 \right) = 0 \]  \hspace{1cm} (14)

Considering the spatiotemporal characteristics of black holes, the following variables are separated.

\[ I = -\omega t + W(r) \]  \hspace{1cm} (15)

In which, \( \omega \) is the energy of the tunneling particles measured by the infinity observer.

Put Equation (15) into Equation (14), eliminate A and B and get:

\[ D_6 (\partial_\tau W)^6 + D_4 (\partial_\tau W)^4 + D_2 (\partial_\tau W)^2 + D_0 = 0 \]  \hspace{1cm} (16)

In which
\[ D_b = \beta^2 f^4 \]
\[ D_t = \beta^2 f^4 (m_t^2 \beta - 2) \]
\[ D_s = f^2 \left( (1 - m_t^2 \beta)^2 + \beta \left( 2m_t^2 - 2m_i^2 \beta \right) \right) \]
\[ D_o = -m_t^2 (1 - m_t^2 \beta)^2 - \omega^2 \]

Solve Equation (16) and integrate at the event horizon while ignoring the high-order terms of \( \beta \):
\[
W_i(r) = \pm \sqrt{\frac{\omega^2 + m^2 f (1 - 2 \beta m^2)}{f}} \times \left( 1 + \beta m^2 + \frac{\beta \omega^2}{f} \right) dr
\]  
(18)

\( \pm \) is corresponding to exit solution (incident solution), discuss the event horizon, make:
\[ f = \left( \frac{\Phi^{\nu+1}}{\delta} - \frac{2M}{\lambda} \right) = 0, r_h = \frac{2M}{\sqrt{\lambda}} \]  
(19)

Deform Equation (19) and obtain:
\[ f = \frac{(r-r_h) \left( \frac{\Phi^{\nu+1}}{\delta} - \frac{2M}{\lambda} \right)}{(r-r_h)} = (r-r_h) F(r) \]  
(20)

Put Equation (20) into Equation (18) to obtain:
\[
W_i(r) = \pm \int \sqrt{\frac{\omega^2 + m^2 (1 - 2 \beta m^2) (r-r_h) F(r)}{(r-r_h) F(r)}} \times \left( 1 + \beta m^2 + \frac{\beta \omega^2}{(r-r_h) F(r)} \right) dr
\]
(21)

In which \( r_h = \frac{2M}{\sqrt{\lambda}} \), under the classical limit, the probability of particles entering a black hole is uniform, that is, the black hole can absorb all the matters near the horizon\(^{[14]}\), then the black hole horizon absorption rate \( P_{\text{absorption}} = 1 \). Therefore, the tunneling rate of the Fermion in the spinning up state at the horizon is:
\[
\Gamma = \frac{P(\text{emission})}{P(\text{absorption})} = \frac{\exp(-2 \text{Im} \mathcal{M})}{\exp(-2 \text{Im} W_i)} = \frac{\exp(-2 \text{Im} \mathcal{M})}{\exp(-2 \text{Im} W_i)} = \exp \left( \frac{-4 \pi \omega (1 + \frac{3}{2} m^2 \beta)}{(a+1) \lambda^{\nu+1} r_h^\nu} \right)
\]  
(22)

According to the Boltzmann relationship \( \Gamma \propto \exp (-\beta E), \beta \) is the inversion temperature at the horizon of the universe. The relationship between \( \beta \) and the temperature is \( \beta = T^{-1} \). \( E \) is the energy of the particle. The Hawking temperature at the black hole horizon can be obtained:
\[
T = \frac{(a+1) \lambda^{\nu+1} r_h^\nu}{4 \pi (1 + \frac{3}{2} m^2 \beta)} = T_0 \left( \frac{1}{1 + \frac{3}{2} m^2 \beta} \right)
\]  
(23)
In which \( T_0 = \frac{(a+1)^{a+1} \hbar^a}{4\pi} \) is the standard Hawking temperature.

Obviously, considering the influence of quantum gravity, Hawking temperature has a small modification. The modified temperature does not only depend on the number of quantum (quality and energy) of emitted particles, but also on the quality of the black hole. The modified temperature is lower than the original Hawking temperature, indicating that the modification of quantum gravity delays the increase of Hawking temperature.

**Conclusion**

The tunneling behavior of two-dimensional space time Fermion under the effect of quantum gravity is studied. Firstly, by modifying the position operator and selecting the appropriate gamma matrix, the Generalized Dirac Equation is obtained. Then, the tunneling radiation of Fermion is discussed to obtain the modified Hawking temperature. The results show that the Hawking temperature of the black hole does not only depend on the quality of the black hole itself, but also on the quality and energy of the radiation particles. The modified temperature is lower than the original Hawking temperature, indicating that the modification of quantum gravity delays the increase of Hawking temperature.

**Acknowledgments**

This work is supported in part by the Major Project of Education Department in Sichuan (Grant No. 17ZA0294).

**References**

[1] Hawking S W. Particle creation by black holes [J]. Communications in Mathematical Physics, 1975, 43(3): 199-220.

[2] Parikh M P, Wilczek F. Hawking Radiation as Tunneling [J]. Physical Review Letters, 2000, 85(24) : 242-245.

[3] Chen Deyou, Wu Houwen, Yang Haitang. Effects of quantum gravity on black holes [J]. International Journal of Modern Physics. A, 2014, 29(26): 154-196.

[4] Shokrollahi A. Free motion of a Dirac particle with a minimum uncertainty in position [J]. Reports on Mathematical Physics, 2012, 71(1): 245-249.

[5] Isi M, Mureika J, Nicolini P. Self-Completeness and the Generalized Uncertainty Principle [J]. JHEP, 2013, 139(12): 1311-1341.

[6] Banerjee R, Ghosh S. Generalised uncertainty principle, remnant mass and singularity problem in black hole thermodynamics [J]. Modern Physics Letters B, 2010, 224(24): 688-700.

[7] Oakes T L A, Francisco R O, Fabris J C. Ground State of the Hydrogen Atom via Dirac Equation in a Minimal Length Scenario [J]. European Physical Journal C, 2013, 73(12): 2495-2502.

[8] Kober M. Gauge theories under incorporation of a generalized uncertainty principle [J]. Physical Review D-Particles, 2010, 82(25): 145-150.

[9] Nozari K, Saghafi S. Natural cutoffs and quantum tunneling from black hole horizon [J]. JHEP, 2012, 11(12): 223-234.

[10] Zeynali K, Darabi F, Motavalli H. Black hole thermodynamics and modified GUP consistent with doubly special relativity [J]. Modern Physics Letters A, 2012, 22(45): 345-351.
[11] Nozari K, Karami M. Minimal, length and generalized Dirac equation [J]. Modern Physics Letters A, 2005, 8(20): 3095-3099.

[12] Hossenfelder S, Bleicher M, Hofmann S, et al. Stocker. Signatures in the Planck regime [J]. Modern physics letters B, 2003, 575(85): 132-139.

[13] Chen Deyou, Jiang Qingquan, Yang Haitang, et al. Remnants, fermions’ tunnelling and effects of quantum gravity [J]. Advances in High Energy Physics, 2013, 12(24): 412-421.

[14] Mitra P. Hawking temperature from tunnelling formalism [J]. Modern physics letters B, 2007, 648(2): 240-242.