Self-dual cones, generalized lattice operations and isotone projections *

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Abstract

By using the metric projection onto a closed self-dual cone of the Euclidean space, M. S. Gowda, R. Sznajder and J. Tao have defined generalized lattice operations, which in the particular case of the nonnegative orthant of a Cartesian reference system reduce to the lattice operations of the coordinate-wise ordering. The aim of the present note is twofold: to give a geometric characterization of the closed convex sets which are invariant with respect to these operations, and to relate this invariance property to the isotonicity of the metric projection onto these sets. As concrete examples the Lorentz cone and the nonnegative orthant are considered. Old and recent results on closed convex Euclidean sublattices due to D. M. Topkis, A. F. Veinott and to M. Queyranne and F. Tardella, respectively are obtained as particular cases. The topic is related to variational inequalities where the isotonicity of the metric projection is an important technical tool. For Euclidean sublattices this approach was considered by G. Isac, H. Nishimura and E. A. Ok.

1. Introduction

A commonly used approach in establishing the solvability of variational inequalities and furnishing their solution is the usage of fixed point theorems and the iterative processes

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they engender, respectively (e.g., [1, 3, 9, 10, 11, 12, 14, 20, 21, 22, 23]).

A specific route to follow during this endeavour is to derive monotone and convergent iterative processes with respect to some order relations. For the particular case of nonlinear complementarity problems this approach was first initiated by G. Isac and A. B. Németh. Both the solvability and the approximation of solutions of nonlinear complementarity problems can be handled by using the metric projection onto the convex cone associated with the problem. The idea to relate the ordering induced by the convex cone and the metric projection onto the convex cone goes back to their paper [6], where a convex cone in the Euclidean space which admits an isotone projection onto it (called isotone projection cone) was characterized. The isotonicity is considered with respect to the order induced by the convex cone.

The isotone projection cones were used in the solution of some nonlinear complementarity problems [7], [8], [16]. Solving complementarity problems by successive approximation require repeated projection onto the underlying cone. It is particularly meaningful that this is an efficient procedure for isotone projection cones [15].

If the projection onto the closed convex set encountered in the definition of a variational inequality is monotone with respect to an appropriate order relation, then an iterative method can be worked out for its solution. An easily handleable order relation in the Euclidean space is the coordinate-wise ordering. G. Isac [5] showed that the projection onto a closed convex set is isotone with respect to this order relation if the set is a sublattice.

In a recent paper H. Nishimura and E. A. Ok [17] showed that latticiality is also a necessary condition for the isotonicity of the metric projection. In the last cited paper several applications were given for variational inequalities defined on closed convex sublattices and other related equilibrium problems. But how do the closed convex sublattices with nonempty interior of the coordinate-wise ordered Euclidean space look? The answer to this question seems to go back to the results of D. M. Topkis [24] and A. F. Veinott Jr. [25] and was settled recently by M. Queyranne and F. Tardella [18].

The positive cone of the coordinate-wise ordering is the nonnegative orthant of a Cartesian reference system in the Euclidean space. It is a self-dual latticial cone and defines well behaved lattice operations. Although largely investigated, they are very restrictive. Among the attempts to extend these lattice operations, one concerning self-dual cones and intrinsically related to metric projections is that proposed by M. S. Gowda, R. Sznajder and J. Tao [4]. Fortunately these extended lattice operations, apart from keeping several properties of lattice operations, seem to be good tools in handling the problem of the isotonicity of the metric projections.

In this note we characterize the closed convex sets which are invariant with respect to these operations showing that the metric projection onto these sets is isotone with respect to the order generated by the self-dual cone giving rise to the respective operations.

The structure of this paper is as follows. In Section 2, we will define the notion of self-dual cones and as particular examples the nonnegative orthant and the Lorentz cone. In Section 3, we will define the lattice operations for the nonnegative orthant and extend these operations to a self-dual cone. In the same section we state our main results, namely
Theorems 1, 2 and 3 which will be proved in Sections 7., 8. and 9. In Section 4. we will
investigate in Sections 11. and 10., respectively. Finally, we end our paper by making
some comments and raising some open questions in Section 12.

2. Self-dual cones

Denote by \( \mathbb{R}^m \) the \( m \)-dimensional Euclidean space endowed with the scalar product \( \langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R} \), and the Euclidean norm \( ||\cdot|| \) and topology this scalar product defines.

Throughout this note we shall use some standard terms and results from convex geometry (see e.g. [19]).

Let \( K \) be a convex cone in \( \mathbb{R}^m \), i. e., a nonempty set with (i) \( K + K \subset K \) and (ii) \( tK \subset K \), \( \forall t \in \mathbb{R}_+ = [0, +\infty) \). The convex cone \( K \) is called pointed, if \( (-K) \cap K = \{0\} \).

The cone \( K \) is generating if \( K - K = \mathbb{R}^m \).

For any \( x, y \in \mathbb{R}^m \), by the equivalence \( x \leq_K y \iff y - x \in K \), the convex cone \( K \) induces an order relation \( \leq_K \) in \( \mathbb{R}^m \), that is, a binary relation, which is reflexive and transitive. This order relation is translation invariant in the sense that \( x \leq_K y \) implies \( x + z \leq_K y + z \) for all \( z \in \mathbb{R}^m \), and scale invariant in the sense that \( x \leq_K y \) implies \( tx \leq_K ty \) for any \( t \in \mathbb{R}_+ \). If \( \leq \) is a translation invariant and scale invariant order relation on \( \mathbb{R}^m \), then \( x \leq_K y \iff x - y \in K \). If \( K \) is pointed, then \( \leq_K \) is antisymmetric too, that is \( x \leq_K y \) and \( y \leq_K x \) imply that \( x = y \). The elements \( x \) and \( y \) are called comparable if \( x \leq_K y \) or \( y \leq_K x \).

We say that \( \leq_K \) is a latticial order if for each pair of elements \( x, y \in \mathbb{R}^m \) there exist the lowest upper bound \( \sup\{x, y\} \) and the upperst lower bound \( \inf\{x, y\} \) of the set \( \{x, y\} \) with respect to the order relation \( \leq_K \). In this case \( K \) is said a latticial or simplicial cone, and \( \mathbb{R}^m \) equipped with a latticial order is called an Euclidean vector lattice.

The dual of the convex cone \( K \) is the set
\[
K^* := \{y \in \mathbb{R}^n : \langle x, y \rangle \geq 0, \forall x \in K\},
\]
with \( \langle \cdot, \cdot \rangle \) the standard scalar product in \( \mathbb{R}^n \).

The cone \( K \) is called self-dual, if \( K = K^* \). If \( K \) is self-dual, then it is a generating, pointed, closed cone.

In all that follows we shall suppose that \( \mathbb{R}^m \) is endowed with a Cartesian reference system with the standard unit vectors \( e_1, \ldots, e_m \). That is, \( e_1, \ldots, e_m \) is an orthonormal system of vectors in the sense that \( \langle e_i, e_j \rangle = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker symbol. Then, \( e_1, \ldots, e_m \) form a basis of the vector space \( \mathbb{R}^m \). If \( x \in \mathbb{R}^m \), then
\[
x = x^1 e_1 + \ldots + x^m e_m.
\]
can be characterized by the ordered \( m \)-tuple of real numbers \( x^1, \ldots, x^m \), called the coordinates of \( x \) with respect the given reference system, and we shall write \( x = (x^1, \ldots, x^m) \). With this notation we have \( e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \), with 1 in the \( i \)-th position and 0 elsewhere. Let \( x, y \in \mathbb{R}^m \), \( x = (x^1, \ldots, x^m) \), \( y = (y^1, \ldots, y^m) \), where \( x^i, y^i \) are the coordinates of \( x \) and \( y \), respectively with respect to the reference system. Then, the scalar product of \( x \) and \( y \) is the sum \( \langle x, y \rangle = \sum_{i=1}^{m} x^i y^i \).

The set \( \mathbb{R}^m_+ = \{ x = (x^1, \ldots, x^m) \in \mathbb{R}^m : x^i \geq 0, \ i = 1, \ldots, m \} \)

is called the nonnegative orthant of the above introduced Cartesian reference system. A direct verification shows that \( \mathbb{R}^m_+ \) is a self-dual cone.

The set \( L_{m+1} = \{ (x, x^{m+1}) \in \mathbb{R}^m \otimes \mathbb{R} = \mathbb{R}^{m+1} : \|x\| \leq x^{m+1} \} \),

(or simply \( L \) if there is no confusion about the dimension) is a self-dual cone called \( m+1 \)-dimensional second order cone, or \( m+1 \)-dimensional Lorentz cone, or \( m+1 \)-dimensional ice-cream cone \([I]\).

The nonnegative orthant \( \mathbb{R}^m_+ \) and the Lorentz cone \( L \) defined above are the most important and largery used self-dual cones in the Euclidean space. But the family of self-dual cones is rather rich \([2]\).

### 3. Generalized lattice operations

A hypersubspace or a hyperplane through the origin, is a set of form

\[
H(u, 0) = \{ x \in \mathbb{R}^m : \langle u, x \rangle = 0 \}, \ u \neq 0.
\]

For simplicity the hypersubspaces will also be denoted by \( H \). The nonzero vector \( u \) in the above formula is called the normal of the hyperplane.

A hyperplane (through \( a \in \mathbb{R}^m \)) is a set of form

\[
H(u, a) = \{ x \in \mathbb{R}^m : \langle u, x \rangle = \langle u, a \rangle, \ u \neq 0 \}.
\]

A hyperplane \( H(u, a) \) determines two closed halfspaces \( H_-(u, a) \) and \( H_+(u, a) \) of \( \mathbb{R}^m \), defined by

\[
H_-(u, a) = \{ x \in \mathbb{R}^m : \langle u, x \rangle \leq \langle u, a \rangle \},
\]

and

\[
H_+(u, a) = \{ x \in \mathbb{R}^m : \langle u, x \rangle \geq \langle u, a \rangle \}.
\]

Taking a Cartesian reference system in \( \mathbb{R}^m \) and using the above introduced notations, the coordinate-wise order \( \leq \) in \( \mathbb{R}^m \) is defined by

\[
x = (x^1, \ldots, x^m) \leq y = (y^1, \ldots, y^m) \iff x^i \leq y^i, \ i = 1, \ldots, m.
\]
Using the notion of the order relation induced by a cone, defined in the preceding section, we see that \( \leq = \leq_{R^m} \).

With the above representation of \( x \) and \( y \), we define
\[
x \wedge y = (\min\{x_1, y_1\}, \ldots, \min\{x_m, y_m\}), \quad \text{and} \quad x \vee y = (\max\{x_1, y_1\}, \ldots, \max\{x_m, y_m\}).
\]

Then, \( x \wedge y \) is the upper lower bound and \( x \vee y \) is the lowest upper bound of the set \( \{x, y\} \) with respect to the coordinate-wise order. Thus, \( \leq \) is a lattice order in \( R^m \). The operations \( \wedge \) and \( \vee \) are called lattice operations.

The subset \( M \subset R^m \) is called a sublattice of the coordinate-wise ordered Euclidean space \( R^m \), if from \( x, y \in M \) it follows that \( x \wedge y, x \vee y \in M \).

Denote by \( P_D \) the projection mapping onto a nonempty closed convex set \( D \subset R^m \), that is the mapping which associate to \( x \in R^m \) the unique nearest point of \( x \) in \( D \):
\[
P_Dx \in D, \quad \text{and} \quad \|x - P_Dx\| = \inf\{\|x - y\| : y \in D\}.
\]

The nearest point \( P_Dx \) can be characterized by
\[
P_Dx \in D, \quad \text{and} \quad \langle P_Dx - x, P_Dx - y \rangle \leq 0, \forall y \in D. \quad (3)
\]

From the definition of the projection and the characterization (3) there follow immediately the relations:
\[
P_D(-x) = -P_-Dx, \quad (4)
\]
\[
P_{x+D}y = x + P_D(y - x) \quad (5)
\]
for any \( x, y \in R^m \).

\[
P_D(tx + (1 - t)P_Dx) = P_Dx, \quad \forall t \in [0, 1]. \quad (6)
\]

In all what follows next \( K \subset R^m \) will denote a self-dual cone.

Define the following operations in \( R^m \):
\[
x \sqcap y = P_{x-K}y, \quad \text{and} \quad x \sqcup y = P_{x+K}y, \quad (31).
\]

Assume the operations \( \sqcup \) and \( \sqcap \) have precedence over the addition of vectors and multiplication of vectors by scalars.

A direct checking yields that if \( K = R^m_+ \), then \( \sqcap = \wedge \) and \( \sqcup = \vee \). That is \( \sqcap \) and \( \sqcup \) are some generalized lattice operations. Moreover: \( \sqcap \) and \( \sqcup \) are lattice operations if and only if the self-dual cone used in their definitions is a nonnegative orthant of some Cartesian reference system.

The subset \( M \subset R^m \) is called invariant with respect to \( \sqcap \) and \( \sqcup \) if for any \( x, y \in M \) we have \( x \sqcap y, x \sqcup y \in M \). That is, such an invariant set is the analogous for generalized lattice operations of a sublattice for lattice operations.

We are now ready to state our main results in form of three theorems, namely Theorems \( \text{1, 2 and 3} \) which will be proved in Sections \( \text{7, 8 and 9} \), respectively.
Theorem 1 The closed convex set $C \subset \mathbb{R}^m$ with nonempty interior is invariant with respect to the operations $\sqcap$ and $\sqcup$ defined by some self-dual cone if and only if it is of form
\[ C = \bigcap_{i \in \mathbb{N}} H_-(u_i, a_i), \tag{7} \]
where each hyperplane $H(u_i, a_i)$ is tangent to $C$ and is invariant with respect to $\sqcap$ and $\sqcup$. The interest of this theorem resides in the reduction of the problem of invariance of a closed convex set $C \subset \mathbb{R}^m$ with nonempty interior with respect to the operations $\sqcap$ and $\sqcup$ to the characterization of the hyperplanes with this property in the representation (7) of $C$. In the important case of the Lorentz cone and respective the nonnegative orthant the invariant hyperplanes have rather simple geometric characterizations.

As we have remarked, in the case of $K = \mathbb{R}_+^m$ the invariant sets are the so called sublattices of the coordinate-wise ordered Euclidean space. As far as we know, the geometric characterization of closed convex sublattices of the coordinate-wise ordered Euclidean space goes back to D. M. Topkis [24] and A. F. Veinott [25] and it was revisited recently by M. Queyranne and F. Tardella [18]. The above theorem can be considered the generalization of the main result in the last cited paper with the remark that there the hyperplanes in (7) are geometrically characterized. (We shall give the characterization of these hyperplanes using an independent proof in the final section of our note, giving this way a different proof of the main result in [18].)

Let $\prec$ a given order relation in $\mathbb{R}^m$. A closed convex set $C$ is called isotone projection set and $P_C$ isotone projection with respect to $\prec$ if $P_C$ is order preserving with respect to $\prec$, i.e., if $x \prec y$ implies that $P_C x \prec P_C y$. In all what follows we take $\prec = \leq_K$ with a given fixed self-dual cone $K \subset \mathbb{R}^m$. Since there is no ambiguity, we shall use $\leq$ in place of $\leq_K$ and the term isotone projection in place of isotone projection with respect to $\leq$.

Theorem 2 Let $K \subset \mathbb{R}^m$ be a self-dual cone and $\sqcap$ and $\sqcup$ the above generalized lattice operations defined with the aid of $K$. Let $C \subset \mathbb{R}^m$ be a nonempty closed convex set. If $C$ is invariant with respect to the operations $\sqcap$ and $\sqcup$, then $C$ is an isotone projection set.

This result for $K = \mathbb{R}_+^m$ is due to G. Isac [5]. As have remarked recently H. Nishimura and E. A. Ok [17], for this case the converse of theorem is also true: from the isotonicity of $P_C$ it follows that $C$ is a sublattice.

Let $M \subset \mathbb{R}^m$ be a nonempty, closed convex set. The nonempty subset $M_0 \subset M$ is a face of $M$, if from $x, y \in M$ and $tx + (1 - t)y \in M_0$, for some $t \in [0, 1]$, it follows that $x, y \in M_0$. The face $M_0 \subset M$ is called proper face of $M$ if $M_0 \neq M$.

If $\operatorname{int} M \neq \emptyset$ and $M_0$ is a face of $M$ with $\operatorname{dim} M_0 = m - 1$, then $M_0$ is called a hyperface of $M$.

The subset
\[ C = \bigcap_{i=1}^q H_-(u_i, a_i), \tag{8} \]
is called a polyhedron.
Suppose that \( \text{int} C \neq \emptyset \) and that the representation (5) is *sharp* in the sense that no member in the intersection representing \( C \) is redundant. Then,

\[
C^i = C \cap H(u_i, a_i)
\]
is a hyperface of \( C \) \((i = 1, ..., q)\), and the normal \( u_i \) in the representation of \( H_- \) will be called a *normal of* \( C \) \((i = 1, ..., q)\). Obviously, \( \text{aff} C^i = H(u_i, a_i) \), where \( \text{aff} C^i \) denotes the affine hull of \( C^i \). In the particular case of a polyhedron \( C \) with nonempty interior, we can strengthen and join the results in Theorem 1 and Theorem 2 as follows:

**Theorem 3** Let \( C \) be a polyhedron with nonempty interior, represented by

\[
C = \bigcap_{i=1}^q H_-(u_i, a_i), \tag{9}
\]
where the representation (9) is sharp in the sense that each set \( H(u_i, a_i) \cap C \) is a hyperface of \( C \). Suppose further that \( K \) is a self-dual cone and \( \sqcap \) and \( \sqcup \) are the generalized lattice operations defined with the aid of it.

Then, the following assertions are equivalent:

(i) The polyhedron \( C \) is an invariant set with respect to the operations \( \sqcap \) and \( \sqcup \);
(ii) The projection \( P_C \) is isotone with respect to the order relation defined by \( K \);
(iii) Each hyperplane \( H(u_i, a_i), \ i = 1, ..., q \) is invariant with respect to the operations \( \sqcap \) and \( \sqcup \);
(iv) Each hyperplane \( H(u_i, a_i), \ i = 1, ..., q \) is an isotone projection set;
(v) Each proper face of \( C \) is invariant with respect to the operations \( \sqcap \) and \( \sqcup \).

### 4. Properties of \( \sqcap \) and \( \sqcup \)

In the particular case of the self-dual cone \( K \subset \mathbb{R}^m \), J. Moreau’s theorem ([13]) reduces to the following lemma:

**Lemma 1** For any \( x \) in \( K \) we have \( x = P_K x - P_K(-x) \) and \( \langle P_K x, P_K(-x) \rangle = 0 \). The relation \( P_K x = 0 \) holds if and only if \( x \in -K \).

**Lemma 2** The following relations hold for any \( x, y, z, w \in \mathbb{R}^m \) and any real scalar \( \lambda > 0 \).

(i) \( x \sqcap y = x - P_K(x-y) = y - P_K(y-x) \) and \( x \sqcup y = x + P_K(y-x) = y + P_K(x-y) \).
(ii) \( x \sqcap y = y \sqcap x \) and \( x \sqcup y = y \sqcup x \).
(iii) \( x \sqcap y \leq x \) and \( x \sqcap y \leq y \), and equalities hold if and only if \( x \leq y \) and \( y \leq x \), respectively.
(iv) $x \leq x \sqcup y$ and $y \leq x \sqcup y$, and equalities hold if and only if $y \leq x$ and $x \leq y$,
respectively.

(v) $x \sqcap y + x \sqcup y = x + y$

(vi) $(x + z) \sqcap (y + z) = x \sqcap y + z$ and $(x + z) \sqcap (y + z) = x \sqcap y + z$.

(vii) $(\lambda x) \sqcap (\lambda y) = \lambda x \sqcap y$ and $(\lambda x) \sqcup (\lambda y) = \lambda x \sqcup y$.

(viii) $\langle x - x \sqcap y, x \sqcup y - x \rangle = 0$,

(ix) $(-x) \sqcup (-y) = -x \sqcap y$.

(x) $\|x \sqcup y - z \sqcup w\| \leq \frac{2}{3}(\|x - z\| + \|y - w\|)$ and $\|x \sqcap y - z \sqcap w\| \leq \frac{2}{3}(\|x - z\| + \|y - w\|)$.

(xi) $x \sqcap y = z \sqcap w$, $\forall \ z = \lambda x + (1 - \lambda)x \sqcap y$, $w = \mu y + (1 - \mu)x \sqcap y$, $\lambda, \mu \in [0, 1]$,

$x \sqcup y = z \sqcup w$, $\forall \ z = \lambda x + (1 - \lambda)x \sqcup y$, $w = \mu y + (1 - \mu)x \sqcup y$, $\lambda, \mu \in [0, 1]$.

(xii) If $x \sqcap y = 0$, then $\langle x, y \rangle = 0$.

Proof.

(i) From equation (5) and Lemma 1 we have

$$x \sqcup y = P_{x+K}y = x + P_K(y - x) = x + (P_K(y - x) - P_K(x - y)) + P_K(x - y)$$

$$= x + (y - x) + P_K(x - y) = y + P_K(x - y).$$

A similar argument with $-K$ replacing $K$ shows that $x \sqcap y = x - P_K(x - y) = y - P_K(y - x)$.

(ii) It follows easily from item (i).

(iii) Since $x \sqcap y \in x - K$, it follows that $x \sqcap y \leq x$. By using item (ii) and the latter relation with $x$ and $y$ swapped, we get $x \sqcap y = y \sqcap x \leq y$. By item (i), the equality $x \sqcap y = x$ is equivalent to $P_K(x - y) = 0$. By Lemma 1 the latter relation is equivalent to $x \leq y$.

(iv) It can be shown similarly to item (iii).

(viii) By using item (i) and Lemma 1 we get

$$\langle x - x \sqcap y, x \sqcup y - x \rangle = \langle P_K(x - y), P_K(y - x) \rangle = 0.$$
Items (v) and (vi) follow immediately from item (i). Item (vii) follows easily from the positive homogeneity of $P_K$ and item (i). Item (ix) follows from (4) and item (i).

To verify item (x) we use item (i) and the Lipschitz property of the metric projection \((\ref{26})\), we obtain:

$$\|x \sqcup y - z \sqcup w\| = \|x - P_K(x - y) - z + P_K(z - w)\| \leq \|x - z\| + \|P_K(x - y) - P_K(z - w)\| \leq \|x - z\| + \|(x - y) - (z - w)\| \leq 2\|x - z\| + \|y - w\|,$$

and by symmetry

$$\|x \sqcap y - z \sqcup w\| \leq \|x - z\| + 2\|y - w\|.$$

By adding the obtained two relations we conclude the first relation in item (x). The second relation can be deduced similarly.

Using the definition of $x \sqcap y$ we have according to the formula (6) that

$$x \sqcap y = P_{x+K}y = P_{x+K}(\mu y + (1 - \mu)x \sqcap y) = P_{x+K}w = x \sqcap w = P_{w+K}x.$$

Using a similar argument we see that

$$x \sqcap w = z \sqcap w.$$

This is the first formula in item (xi). A similar argument yields the second relation in this item.

Item (xii) follows easily from items (v) and (viii). \qed

5. Subsets invariant with respect to $\sqcap$ and $\sqcup$

To shorten the writing the term invariant from now on will mean invariant with respect to the operations $\sqcap$ and $\sqcup$ defined with the aid of the given self-dual cone $K$.

Lemma 3

(i) The minimal invariant set containing the points $x, y \in \mathbb{R}^m$ is the set $\{x, y\}$ if $x$ and $y$ are comparable, and the set $\{x, y, x \sqcap y, x \sqcup y\}$ if $x$ and $y$ are not comparable;

(ii) The minimal invariant convex set containing the points $x, y \in \mathbb{R}^m$ is the closed line segment $[x, y]$ if $x$ and $y$ are comparable, and the planar rectangle with vertices $x$, $y$, $x \sqcap y$ and $x \sqcup y$ if $x$ and $y$ are not comparable.

Proof. The assertion (i) is the direct consequence of items (iii) and (iv) in Lemma 2.

If $x$ and $y$ are comparable, then any two points in the segment $[x, y]$ are comparable and their set is invariant by (i). Hence, $[x, y]$ is invariant, and being the minimal convex set containing $x$ and $y$, we arrive to the first assertion in item (ii).
If \( x \) and \( y \) are not comparable, by items (v) and (viii) of Lemma 2, \( x, y, x \sqcap y \) and \( x \sqcup y \) form a spatial quadruple with all the angles being rightangles. Hence, it must be a planar rectangle denoted by \( \Pi(x, y) \). The sides of this rectangle have comparable endpoints, hence the whole boundary of \( \Pi(x, y) \) must be contained in any invariant convex set containing \( x \) and \( y \). Let \( v \) be an arbitrary point in \( \Pi(x, y) \). The line through \( v \) parallel with the segment \([y, x \sqcap y]\) intersects the segment \([x, x \sqcap y]\) in \( z = \lambda x + (1 - \lambda)x \sqcap y \), the line through \( v \) parallel with \([x, x \sqcap y]\) meets \([y, x \sqcap y]\) at \( w = \mu y + (1 - \mu)x \sqcap y \), with some \( \lambda, \mu \in [0, 1] \).

See the below figure.

![Diagram](image.png)

Obviously, the rectangle with vertices \( z, x \sqcap y, w, v \) is contained in the rectangle \( \Pi(x, y) \), since they have the common points \( z, x \sqcap y, w \). The same is true for the rectangle with the vertices \( z, x \sqcap y, w \) and \( z \sqcup w \). Hence, the vertices \( v \) and \( z \sqcup w \) must coincide, that is, \( v = z \sqcup w \in \Pi(x, y) \). Hence, every point in the considered rectangle must be contained in any invariant convex set containing \( x \) and \( y \) and thus the whole rectangle \( \Pi(x, y) \) is contained in any invariant convex set containing the points \( x \) and \( y \).

We have to verify that \( \Pi(x, y) \) itself is invariant. Take \( u, v \in \Pi(x, y) \). If \( u \) and \( v \) are comparable, then they form an independent set. If not, we argue as follows. The lines through \( u \) and \( v \) parallel with the sides \([x, x \sqcup y]\) and \([y, x \sqcap y]\), respectively form a rectangle with opposite vertices \( u \) and \( v \). Denote by \( p \) and \( q \) its other opposite vertices. See the below figure.
A reasoning as above, combined with a case analysis shows that the rectangle with vertices $u$, $p$, $v$, $q$ must be contained in $\Pi(x, y)$ and hence it must coincide with $\Pi(u, v)$.

Indeed, assume e. g. that $u \in [p, z]$ with $z \in [x, x \sqcup y]$, and $v \in [p, w]$ with $w \in [y, x \sqcap y]$. Then, a reasoning as above shows that $p = z \sqcap w$, and using item (xi) in Lemma 2 we see that $p = u \sqcap v$. Thus, $u \sqcap v \in \Pi(x, y)$.

We can similarly see that $u \sqcup v \in \Pi(x, y)$.

\[ \blacksquare \]

**Lemma 4** If $M, M_i, \subset \mathbb{R}^m$, $i \in \mathcal{I}$ are invariant sets, then

(i) $\cap_{i \in \mathcal{I}} M_i$ is also invariant,

(ii) $\eta M + a$ is also invariant for any $a \in \mathbb{R}^m$ and $\eta \in \mathbb{R}$.

(iii) If the nonempty convex set $C$ is invariant, then its affine hull denoted by $\text{aff} C$ is invariant too.

(iv) The nonempty set $M \subset \mathbb{R}^m$ is an invariant convex set if and only if together with each pair $x, y$ of elements the convex hull $\text{co}\{x, y, x \sqcap y, x \sqcup y\}$ is contained in $M$.

**Proof.** The first assertion is trivial and the second follows easily from items (vi), (vii) and (ix) of Lemma 2.

To verify assertion (iii), we argue as follows: According to item (ii), we can suppose that $0 \in \text{icore} C$, where $\text{icore} C$ is the relative interior of $C$ with respect the topology of $\text{aff} C$ \textcolor{red}{[19]}.

Let $x, y \in \text{aff} C$ and take $t > 0$ such that $tx, ty \in C$. Then, $(tx) \sqcap (ty), (tx) \sqcup (ty) \in C$. Since $x \sqcap y = (1/t)((tx) \sqcap (ty))$ and $x \sqcup y = (1/t)((tx) \sqcup (ty))$, it follows that $x \sqcap y, x \sqcup y \in \text{aff} C$.

The proof of the assertion (iv) follows from item (ii) of Lemma 2. \[ \blacksquare \]
Corollary 1 Let \( x, y \in \mathbb{R}^n \) be incomparable elements. Then, the rectangle \( \Pi(x, y) \) with vertices \( x, y, x \cap y, \) and \( x \cup y \) is invariant according to item (ii) of Lemma 2. Assume that \( 0 \) is in the relative interior of \( \Pi(x, y) \). Then, the linear hull \( \Omega(x, y) := \text{aff} \, \Pi(x, y) \) is an invariant bidimensional subspace of \( \mathbb{R}^m \) by item (iii) of Lemma 4. In this subspace \( K_0 = K \cap \Omega(x, y) \) is a self-dual lattice cone and \( \cap \) and \( \cup \) restricted to \( \Omega(x, y) \) are the lattice operations with respect to the order relation that \( K_0 \) induces in this subspace. Hence, according to item (iv) of Lemma 4 every sublattice in \( \Omega(x, y) \) with respect to these lattice operations is an invariant set in \( \mathbb{R}^m \).

Proof. We shall use the notation cone \( M \) for the minimal closed convex cone in \( \mathbb{R}^m \) containing the nonempty set \( M \).

After a translation in \( \Omega(x, y) \), if necessary, we can suppose that \( x \cap y = 0 \). Hence, by item (iii) of Lemma 2 we get \( x, y \in K \) and by item (xii) of the same lemma it follows that \( \langle x, y \rangle = 0 \). We further have that \( x, y \in K_0 \) and hence cone \( \{ x, y \} \subseteq K_0 \). In fact we have that \( K_0 = \text{cone} \{ x, y \} \). Assuming the existence of some \( z \in K_0 \setminus \text{co} \{ x, y \} \), it would follow that \( \langle x, z \rangle < 0 \), or \( \langle y, z \rangle < 0 \). In any case we get a contradiction with the self-duality of \( K \). Thus, \( K_0 \) is a selfdual cone in \( \Omega(x, y) \).

In the bidimensional space every generating pointed cone is a latticial cone, hence so is \( K_0 \) in \( \Omega(x, y) \). The lattice operations with respect to the order relation \( \leq_{K_0} \) induced by \( K_0 \) in \( \Omega(x, y) \) can be characterized geometrically as follows: The infimum \( w \) of the set \( \{ u, v \} \subseteq \Omega(x, y) \) is given by the relation \( w = w - K_0 = (u - K_0) \cap (v - K_0) \). By using item (vii) of Lemma 2 we can suppose that \( u, v \in \Pi(x, y) \). Therefore, similar ideas to the proof of item (ii) of Lemma 3 yield that \( w = u \cap v \). Analogously, the supremum of the set \( \{ u, v \} \) with respect to \( \leq_{K_0} \) is exactly \( u \cup v \).

\[ \square \]

Lemma 5 The halfspace \( H_- \) is invariant if and only if the hyperplane \( H \) has this property.

Proof. According to item (ii) of Lemma 4 we can assume that \( 0 \in H \).

Suppose that \( H \) is invariant, but \( H_- \) is not. Then, there exist some \( x, y \in H_- \) such that \( x \cup y \not\in H_- \) or \( x \cap y \not\in H_- \). Assume that \( x \cap y \not\in H_- \). Then, \( x \cap y \in \text{int} \, H_+ \). The line segment \( [x, x \cap y] \) meets \( H \) in \( z = \lambda x + (1 - \lambda)x \cap y, \lambda \in [0, 1] \), the line segment \( [y, x \cap y] \) meets \( H \) at \( w = \mu y + (1 - \mu)x \cap y, \mu \in [0, 1] \). According to item (xi) in Lemma 2 we have then

\[ z \cap w = x \cap y \not\in H, \]

which contradicts the invariance of \( H \).

Suppose now that \( H_- \) is invariant, but \( H \) is not. Then, there exist some \( x, y \in H \) such that \( x \cap y \not\in H \) or \( x \cap y \not\in H \). Since \( H_- \) is invariant, we can assume that \( x \cup y \in \text{int} \, H_- \). Let \( u \) be the normal of \( H \). Then, \( \langle u, x \cap y \rangle < 0 \). By using the relation in item (v), we have then

\[ 0 = \langle u, x + y \rangle = \langle u, x \cap y \rangle + \langle u, x \cap y \rangle. \]
Whereby, by using the relation \( \langle u, x \sqcup y \rangle < 0 \), we conclude that
\[
\langle u, x \sqcap y \rangle > 0,
\]
that is, \( x \sqcap y \in \text{int} H_+ \), contradicting the invariance of \( H_- \).

\[ \square \]

**Lemma 6** If the nonempty closed convex set \( C \subset \mathbb{R}^m \) is invariant, then so is every face \( C_0 \) of \( C \).

**Proof.** Take \( x, y \in C_0 \). Then, \( (1/2)(x + y) \in C_0 \), because \( C_0 \) is convex. Using the standard relation
\[
\frac{1}{2}(x + y) = \frac{1}{2} x \sqcap y + \frac{1}{2} x \sqcup y,
\]
the inclusions \( x \sqcap y, x \sqcup y \in C \), and the definition of the face, we have that
\[
x \sqcap y, x \sqcup y \in C_0.
\]

\[ \square \]

**Lemma 7** A linear subspace \( S \) of \( \mathbb{R}^m \) is invariant if and only if \( S \) is invariant with respect to \( P_K \), i.e., \( P_K(S) \subset S \).

**Proof.** Suppose that \( S \) is invariant and let any \( x \in S \). Then, \( 0 \in S \) and item (i) of Lemma 2 it follows that \( P_K(x) = 0 + P_K(x - 0) = 0 \sqcup x \in S \).

Conversely, suppose that \( P_K(S) \subset S \). Hence, by using again item (i) of Lemma 2 and the invariance of a linear subspace under linear combinations, for any \( x, y \in S \) we have
\[
x \sqcap y = x - P_K(x - y) \in S \text{ and } x \sqcup y = x + P_K(y - x) \in S.
\]

\[ \square \]

Denote by \( \text{bdr} C \) the boundary of a set \( C \).

**Lemma 8** Suppose that \( C \) is an invariant closed convex set with nonempty interior, and \( H \) is a hyperplane tangent to \( C \) in some point of \( \text{bdr} C \). Then, \( H \) is invariant.

**Proof.** According to item (ii) of Lemma 4 we can assume that \( 0 \in \text{bdr} C \), that \( H \) is tangent to \( C \) at 0, and that \( C \subset H_- \).

We shall prove our claim by contradiction: we assume that \( H \) is not invariant.

Since \( H \) is not invariant, there exist some \( z, w \in H \) such that \( z \sqcup w \) or \( z \sqcap w \) is not in \( H \). Suppose that \( u \) is the normal of \( H \). From the relation in item (v) of Lemma 2 we have then
\[
0 = \langle u, z + w \rangle = \langle u, z \sqcap w \rangle + \langle u, z \sqcup w \rangle,
\]
whereby it follows that \( z \sqcup w \) and \( z \sqcap w \) are in opposite open half-spaces determined by \( H \).
Suppose that \( z \cap w \in \text{int} \ H_+ \). Taking \( x = z - (z + w)/2 \), we have \( -x = w - (z + w)/2 \). Then, by our working hypothesis that \( 0 \in H \), it follows that the line segment \([-x, x] \subset H\). We can easily check that \((-x) \cap x \in \text{int} \ H_+ \). Denoting by \( B \) the unit ball in \( \mathbb{R}^m \), then there exists some \( \delta > 0 \) such that

\[
(-x) \cap x + \delta B \subset \text{int} \ H_+. \tag{10}
\]

We have the relation

\[
[-x, x] = \{tx : t \in [-1, 1]\}.
\]

Next we project \([-x, x]\) in the direction of \( u \) onto \( \text{bdr} \ C \). All the above reasonings are valid when we change \( x \) with its positive multiple, hence we can chose \( x \) small enough, so that the above projection to make a sense.

Denote by \( \gamma(t) \) the image of \( tx \) in \( \text{bdr} \ C \) by this projection. Since \( H \) is a tangent hyperplane, the segment \([-x, x]\) will be tangent to \( \gamma \) at \( t = 0 \), \( \gamma(0) = 0 \), \( \gamma'(0) \) exists, and \( \gamma'(0) = x \).

Since \( \gamma \) is differentiable in \( t = 0 \), we have the following representations around 0:

\[
\gamma(t) = tx + \eta(t), \ t > 0, \tag{11}
\]

and

\[
\gamma(-t) = -tx + \zeta(-t), \ t > 0, \tag{12}
\]

where

\[
\frac{\eta(t)}{t} \to 0 \quad \text{and} \quad \frac{\zeta(-t)}{t} \to 0, \quad \text{as} \ t \to 0, \ t > 0. \tag{13}
\]

Using item (x) of Lemma 2 as well as the relations (11) and (12), we have then

\[
\|(-tx) \cap (tx) - \gamma(-t) \cap \gamma(t)\| \leq \frac{3}{2}(\| -tx - \gamma(-t)\| + \|tx - \gamma(t)\|) = \frac{3}{2}(\|\zeta(-t)\| + \|\eta(t)\|).
\]

Dividing the last relation by \( t > 0 \), and using the relation in item (vii) of Lemma 2, we obtain that

\[
\|(-x) \cap x - \frac{1}{t}\gamma(-t) \cap \gamma(t)\| \leq \frac{3}{2} \left( \left\| \frac{\zeta(-t)}{t} \right\| + \left\| \frac{\eta(t)}{t} \right\| \right). \tag{14}
\]

Take now \( t > 0 \) small enough in order to have by (13)

\[
\frac{3}{2} \left( \left\| \frac{\zeta(-t)}{t} \right\| + \left\| \frac{\eta(t)}{t} \right\| \right) < \delta.
\]

For such a \( t > 0 \) we have, by using (14), that

\[
\frac{1}{t}(\gamma(-t) \cap \gamma(t)) \in \text{int} \ H_+,
\]

and thus

\[
\gamma(-t) \cap \gamma(t) \in \text{int} \ H_+,
\]

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that is, \(\gamma(-t), \gamma(t) \in C\), but
\[\gamma(-t) \sqcap \gamma(t) \notin C,\]
contradicting the invariance of \(C\).

The obtained contradiction shows that \(H\) must be invariant with respect to the operations \(\sqcup\) and \(\sqcap\).

\(\Box\)

6. Isotonicity of the projection onto a hyperplane

**Lemma 9** Let \(H \subset \mathbb{R}^m\) be a hyperplane through the origin with unit normal vector \(u \in \mathbb{R}^m\). Then, \(P_H\) is isotone if and only if
\[\langle x, y \rangle \geq \langle u, x \rangle \langle u, y \rangle,\]
for any \(x, y \in K\).

**Proof.** Since \(P_H\) is linear, it follows that \(P_H\) is isotone if and only if
\[P_Hx = x - \langle u, x \rangle u \in K,\] (15)
for any \(x \in K\). By the self-duality of \(K\), it follows that relation (15) is equivalent to
\[\langle x, y \rangle = \langle u, x \rangle \langle u, y \rangle + \langle x - \langle u, x \rangle u, y \rangle \geq \langle u, x \rangle \langle u, y \rangle,\]
for any \(x, y \in K\). \(\Box\)

**Lemma 10** Let \(H \subset \mathbb{R}^m\) be a hyperplane through the origin with unit normal vector \(u \in \mathbb{R}^m\). If \(P_H\) is isotone, then \(H\) is invariant.

**Proof.** By Lemma 7 it is enough to show that if \(\langle u, z \rangle = 0\), then \(\langle u, P_Kz \rangle = 0\). Suppose that \(\langle u, z \rangle = 0\). Then, \(P_Kz \in K\), \(P_Kz - z = P_K(-z) \in K\) and \(\langle P_Kz - z, P_Kz \rangle = 0\) by Lemma 1. By using Lemma 9 with \(x = P_Kz\) and \(y = P_Kz - z\), we get
\[\langle u, P_Kz \rangle^2 = \langle u, P_Kz \rangle \langle u, P_Kz - z \rangle \leq \langle P_Kz, P_Kz - z \rangle = 0.\]
Hence, it follows that \(\langle u, P_Kz \rangle = 0\). \(\Box\)
7. The proof of Theorem 1

It is known (see e.g. [19], Theorem 25.5) that if \( C \subset \mathbb{R}^m \) is a closed convex set with nonempty interior, then \( \text{bdr} \, C \) contains a dense subset of points where this surface is differentiable. Since the topology of \( \text{bdr} \, C \) possesses a countable basis, we can select from this dense set a countable dense set \( \{ a_i : \, i \in \mathbb{N} \} \subset \text{bdr} \, C \) such that there exist the tangent hyperplanes \( H(u_i, a_i) \) to \( C \) and \( C \subset H_-(u_i, a_i) \), \( i \in \mathbb{N} \). Since the set \( \{ a_i, \, i \in \mathbb{N} \} \) is dense in \( \text{bdr} \, C \), a standard convex geometric reasoning shows that in fact

\[
C = \bigcap_{i \in \mathbb{N}} H_-(u_i, a_i). \tag{16}
\]

Now, if \( C \) is invariant, then so is \( H(u_i, a_i) \), \( i \in \mathbb{N} \) by Lemma 8. Hence, the necessity of the condition in Theorem 1 is proved.

Conversely, if we have the representation \( (16) \) with the hyperplanes \( H(u_i, a_i) \), \( i \in \mathbb{N} \) invariant, then, by Lemma 5, the halfspaces \( H_-(u_i, a_i) \), \( i \in \mathbb{N} \) are also invariant. Then, by using item (i) of Lemma 4 and the representation \( (16) \), we see that \( C \) is invariant with respect to the operations \( \cap \) and \( \cup \) and the sufficiency of Theorem 1 is proved.

8. The proof of Theorem 2

Assume that the closed convex set \( C \) is invariant. Let \( x, y \in \mathbb{R}^m \) with \( x \leq y \) and denote \( u = P_C x, \, v = P_C y \).

Assume that \( u \leq v \) is false. Then, from \( u \cup v \in C \), the definition of the projection and item (iii) of Lemma 2 we have \( \| y - v \| < \| y - u \cup v \| \). Hence, from

\[
\| y - v \|^2 = \| y - u \cup v \|^2 + \| u \cup v - v \|^2 + 2 \langle y - u \cup v, u \cup v - v \rangle,
\]

it follows that

\[
\| u \cup v - v \|^2 < 2 \langle u \cup v - y, u \cup v - v \rangle.
\]

On the other hand, since \( u \cap v \in C \), we have \( \| x - u \| \leq \| x - u \cap v \| \), and thus we have similarly that

\[
\| u \cap v - u \|^2 \leq 2 \langle u \cap v - x, u \cap v - u \rangle.
\]

Summing up the latter two inequalities and using item (v) of Lemma 2, it follows that

\[
\langle u \cup v - v, u \cup v - v \rangle = \| u \cup v - v \|^2 < \langle u \cup v - y, u \cup v - v \rangle + \langle x - u \cap v, u \cup v - v \rangle.
\]

Thus,

\[
\langle y - x - (v - u \cap v), u \cup v - v \rangle < 0.
\]

Combining the latter inequality with item (viii) of Lemma 2 we obtain that

\[
\langle y - x, u \cup v - v \rangle < 0.
\]

But this is a contradiction, because \( y - x \in K = K^* \) and \( u \cup v - v \in K \) (by item (iii) of Lemma 2).

The obtained contradiction shows that \( P_C \) must be isotone.
**Corollary 2** Let $H$ be a hyperplane in $\mathbb{R}^m$. Then, $H$ is invariant if and only if it is an isotone projection set.

**Proof.** The proof follows from the joint application of Theorem 2 and Lemma 10. \hfill $\Box$

9. The proof of Theorem 3

Let us verify first the following equivalences

(i) $\iff$ (iii) $\iff$ (v). \hfill (17)

From Theorem 1 we have the equivalence

(i) $\iff$ (iii).

From Lemma 6 it follows

(i) $\Rightarrow$ (v).

If (v) holds then every hyperface $C \cap H(u_i, a_i)$ must be invariant. But then, as $H(u_i, a_i)$ is the affine hull of this hyperface, it must be invariant too, by item (iii) of Lemma 4. Hence

(v) $\Rightarrow$ (iii)

and (17) has been verified.

From Theorem 2 we have

(i) $\Rightarrow$ (ii).

We shall show next, that

(ii) $\Rightarrow$ (iv).

Assume the contrary: $C$ is an isotone projection set, but some hyperplane $H = H(u_i, a_i)$ in its sharp representation is not.

Bearing in mind item (ii) of Lemma 4, we can assume that 0 is in the relative interior of the hyperface $F = C \cap H$. If $B$ denotes the unit ball in $\mathbb{R}^m$, then for an appropriate positive $\delta > 0$ we can realize that

$H \cap \delta B \subset F$.

Since $C \subset H_-$, for each element $z \in \delta B \cap H_+$ we have

$P_C z = P_H z \in F$. \hfill (18)

Indeed, from $P_H z = P_{H_-} z$ and $C \subset H_-$ we have on the one hand

$\|z - P_H z\| = \|z - P_{H_-} z\| \leq \|z - P_C z\|$, \hfill (19)
and on the other hand $P_H z \in \delta B \cap H \subset F \subset C$ (as $P_H$ is nonexpansive) and then

$$\|z - P_H z\| \leq \|z - P_C z\|.$$  \hfill (20)

The relations (19) and (20) yield

$$\|z - P_C z\| = \|z - P_H z\|,$$

which together with $P_H z \in C$ and the unicity of the best approximation conclude that $P_C z = P_H z$.

From our working hypothesis that $P_H$ is not isotone and the linearity of this mapping (from the condition $0 \in H$), this is equivalent with the existence of some $z \in K$ with $P_H z \notin K$. The same is true for any positive multiple of $z$. Hence, we can assume at once that $z \in \delta B$.

Suppose that $z \in H_+$. From the isotonicity of $P_C$, we have as $0 \leq z$ and $P_C(0) = 0$, that

$$P_C z = P_C z - P_C(0) \in K,$$

which is impossible since by (18)

$$P_C z = P_H z \notin K.$$

Suppose that $z \in H_-$. Then, $-z \in \delta B \cap H_+$ and then

$$P_C(-z) = P_H(-z) = -P_H z \notin -K.$$  \hfill (21)

Since $-z \leq 0$, the isotonicity of $P_C$ yields

$$P_C(-z) \leq P_C(0) = 0$$

and hence $P_C(-z) \in -K$, contradicting (21).

The obtained contradictions conclude that $P_H$ must be isotone.

The relation

(iv) ⇒ (iii)

is a direct consequence of Lemma [10]

10. Particular case: the Lorentz cone

Lemma 11 For any $x, y, a \in \mathbb{R}^m$ the following inequality holds:

$$\langle x, y \rangle + \|x\| \|y\| \|a\|^2 \geq \langle a, x \rangle \langle a, y \rangle$$  \hfill (22)
**Proof.** Denote by $\varphi, \theta, \rho \in [0, \pi]$ the angles of the vectors $\{a, x\}, \{a, y\}$ and $\{x, y\}$, respectively in radians. Then, it is known that $\rho \leq \varphi + \theta$. Since the cosine function is decreasing in the interval $[0, \pi]$, the latter inequality gives

$$\cos \rho \geq \cos(\varphi + \theta) = \cos \varphi \cos \theta - \sin \varphi \sin \theta \geq \cos \varphi \cos \theta - 1.$$ 

Thus, $\cos \rho + 1 \geq \cos \varphi \cos \theta$, from where it follows

$$\langle x, y \rangle \left\| \frac{x}{\|x\|} \right\| \left\| \frac{y}{\|y\|} \right\| + 1 \geq \langle a, x \rangle \left\| \frac{a}{\|a\|} \right\| \left\| \frac{x}{\|x\|} \right\| \langle a, y \rangle \left\| \frac{a}{\|a\|} \right\| \left\| \frac{y}{\|y\|} \right\|,$$

or equivalently inequality (22).

\[
\text{Lemma 12} \quad \text{Let } m > 1 \text{ and } K \subset \mathbb{R}^{m+1} \text{ be the Lorentz cone }
\]

\[K = \{(x, x^{m+1}) \in \mathbb{R}^m \otimes \mathbb{R} : \|x\| \leq x^{m+1}\}, \]

and $H \subset \mathbb{R}^{m+1}$ a hyperplane through the origin with unit normal vector $(a, a^{m+1})$, where $a \in \mathbb{R}^m$ and $a^{m+1} \in \mathbb{R}$. Then, $P_H$ is isotone if and only if $a^{m+1} = 0$.

**Proof.** Let $b = (a, a^{m+1})$. By Lemma 9 we have to show that for any $u = (x, x^{m+1}) \in K$ and $v = (y, y^{m+1}) \in K$ we have $\langle u, v \rangle - \langle b, u \rangle \langle b, v \rangle \geq 0$ if and only if $a^{m+1} = 0$. Suppose that $a^{m+1} = 0$. Then, by using Lemma 11 we have

$$\langle u, v \rangle - \langle b, u \rangle \langle b, v \rangle = \langle x, y \rangle + x^{m+1}y^{m+1} - \langle a, x \rangle \langle a, y \rangle \geq (\langle x, y \rangle + \|x\| \|y\|)\|a\|^2 - \langle a, x \rangle \langle a, y \rangle \geq 0$$

Conversely, suppose that for any $u, v \in K$ we have $\langle u, v \rangle - \langle b, u \rangle \langle b, v \rangle \geq 0$. Since $m > 1$, there exists $z \in \mathbb{R}^m$ such that $\langle a, z \rangle = 0$ and $\|z\| = 1$. Let $u = (z, 1)$ and $v = (-z, 1)$. Then, $u, v \in K$ and thus

$$0 \leq \langle u, v \rangle - \langle b, u \rangle \langle b, v \rangle = -\|z\|^2 + 1 - (\langle a, z \rangle + a^{m+1})(-\langle a, z \rangle + a^{m+1}) = -\langle a^{m+1} \rangle^2.$$ 

Therefore, $a^{m+1} = 0$. \hfill \Box

Bearing in mind, item (ii) of Lemma 11 the working hypotheses $0 \in H$ and $\| (a, a^{m+1}) \| = 1$ can be ignored in the applications of the above lemma.

**Corollary 3** Let $M$ be a closed convex subset with nonempty interior in $\mathbb{R}^{m+1} = \mathbb{R}^m \otimes \mathbb{R}$ with $m > 1$. Consider the following assertions:

(i) $M$ is invariant with respect to the operations $\sqcap$ and $\sqcup$ defined by the Lorentz cone $K$,

(ii) $M$ is an isotone projection set,
\[ M = C \times \mathbb{R}, \quad (23) \]
where \( C \) is a closed convex set with nonempty interior in \( \mathbb{R}^m \).

Then
\[ (iii) \iff (i) \Rightarrow (ii). \]

**Proof.** From the convex geometry it follows that if \( M \) is of the form \(23\), then it can be represented as
\[ M = \cap_{i \in \mathbb{N}} H_-((a_i,0),(b_i,b_i^{m+1})). \quad (24) \]
Since every hyperplane \( H_-((a_i,0),(b_i,b_i^{m+1})) \) is isotone by Lemma 12, it follows from Corollary 2 that each \( H_-((a_i,0),(b_i,b_i^{m+1})) \) is invariant too. But then according to Theorem 1, \( M \) is an invariant set. The usage of Theorem 2 then shows that \( M \) is an isotone projection set.

If \( M \) is invariant, by Theorem 1 and Lemma 12 it must be of form \(24\). Putting \( C = \mathbb{R}^m \cap (\cap_{i \in \mathbb{N}} H_-((a_i,0),(b_i,b_i^{m+1}))) \), we arrive to the required representation \(23\) of \( M \).

\[ \square \]

**Remark 1**

1. The implication \((iii) \Rightarrow (ii)\) of the above corollary can be shown directly as well. Indeed, by the definition of the projection it easily follows that
\[ P_M(x,x^{m+1}) = P_{C \times \mathbb{R}}(x,x^{m+1}) = (P_Cx,x^{m+1}), \]
because for any \((y,y^{m+1}) \in M = C \times \mathbb{R}\) we have
\[ \| (y,y^{m+1}) - (x,x^{m+1}) \|^2 = \| y-x \|^2 + \| y^{m+1} - x^{m+1} \|^2 \geq \| P_Cx - x \|^2 \]
and \((P_Cx,x^{m+1}) \in C \times \mathbb{R}\). Now let \((x,x^{m+1}) \leq (y,y^{m+1})\). Then,
\[ \| y-x \| \leq y^{m+1} - x^{m+1}. \]
On the other hand, by the nonexpansivity of the projection \( P_C \), we have
\[ \| P_Cy - P_Cx \| \leq \| y - x \|. \]
Thus, the latter two inequalities imply \( \| P_Cy - P_Cx \| \leq y^{m+1} - x^{m+1} \), or equivalently
\[ P_M(x,x^{m+1}) = (P_Cx,x^{m+1}) \leq (P_Cy,y^{m+1}) = P_M(y,y^{m+1}). \]
Hence, \( P_M \) is isotone.

2. In the case \( m = 1 \) the Lorentz cone \( K \) is nothing else as the rotated \( \mathbb{R}_+^2 \) and hence in this case the investigations of the next section take effect.

3. The conditions \( m > 1 \) and that the interior of the convex set is nonempty is essential in the assertions of the corollary above. By Corollary 2 and the next section, it can be seen that the invariant sets of dimension 2 can have a different shape.
11. Particular case: the cone \( \mathbb{R}^m_+ \)

In this case the invariant sets are the sublattices of the coordinate-wise ordered Euclidean space. The following lemma is the sufficiency part of Lemma 2.1 in [17]. We include here its proof for the sake of completeness.

**Lemma 13** If the closed convex set \( C \subset \mathbb{R}^m \) admits an isotone projection \( P_C \) with respect to the coordinate-wise order in \( \mathbb{R}^m \), then \( C \) is a sublattice.

**Proof.** Suppose that \( P_C \) is isotone and take \( x, y \in C \). Let us see that \( x \lor y \in C \).

From the characterization (3) of the projection we have

\[
\langle P_C(x \lor y) - x \lor y, P_C(x \lor y) - y \rangle \leq 0. \tag{25}
\]

Since \( x \leq x \lor y \) and \( P_C \) is isotone, it follows that \( x = P_C x \leq P_C(x \lor y) \). Similarly, \( y \leq P_C(x \lor y) \) and hence \( x \lor y \leq P_C(x \lor y) \). We have also

\[
0 \leq P_C(x \lor y) - x \lor y \leq P_C(x \lor y) - y. \tag{26}
\]

The two terms in the scalar product (25) are in \( K = \mathbb{R}^m_+ \), and since \( K \) is self-dual, we must have the equality:

\[
\langle P_C(x \lor y) - x \lor y, P_C(x \lor y) - y \rangle = 0. \tag{27}
\]

By using again the self-duality of \( K \), the relation (26), as well as (27), it follows that

\[
0 \leq \langle P_C(x \lor y) - x \lor y, (P_C(x \lor y) - y) - (P_C(x \lor y) - x \lor y) \rangle = -\|P_C(x \lor y) - x \lor y\|^2,
\]

thus we must have

\[
P_C(x \lor y) = x \lor y,
\]

and since \( C \) is closed, \( x \lor y \in C \).

Similar reasonings show that \( x \land y \in C \). \( \square \)

**Lemma 14** The hyperplane \( H \) through \( 0 \) with the normal \( u = (u^1, \ldots, u^m) \) is a sublattice if and only if

\[
u^i u^j \leq 0, \quad \text{whenever } i \neq j.
\]

**Proof.** By Corollary 2 it is enough to prove that \( P_H \) is isotone if and only if the conditions of the lemma hold.

In the following reasoning, for sake of simplicity, suppose that \( \|u\| = 1 \). Since \( P_H \) is linear, in order to characterize the hyperplane \( H \) with the property that \( x \leq y \) implies \( P_H x \leq P_H y \), it is sufficient to give necessary and sufficient conditions on the unit vector \( u \) such that

\[
P_H e_i \geq 0, \quad i = 1, \ldots, m, \tag{28}
\]
where \( e_i = (0, ..., 0, 1, 0...0) \), \( i = 1, ..., m \) are the standard unit vectors of the Cartesian reference system.

Since \( u \) is a unit vector, the conditions (28) can be written in the form:

\[
P_{He_i} = e_i - \langle u, e_i \rangle u = (0, ..., 0, 1, 0, ..., 0) - u^i(u^1, ..., u^m) \geq 0, \quad i = 1, ..., m.
\]

These conditions yield

\[
u^i u^j \leq 0, \quad \text{whenever} \quad i \neq j,
\]

and

\[
1 - (u^i)^2 \geq 0, \quad i = 1, ..., m.
\]

But the conditions (31) are trivially satisfied by the condition \( \|u\| = 1 \).

If \( \|u\| \neq 1 \), we can carry out the proof with \( u/\|u\| \) in place of \( u \) and we get the same conditions (30) on the coordinates of \( u \).

\[\Box\]

By putting together Theorem 1, Lemma 13 and Lemma 14, we obtain the following corollary:

**Corollary 4** Let \( C \) be a closed convex set with nonempty interior of the coordinate-wise ordered Euclidean space \( \mathbb{R}^m \). Then, the following assertions are equivalent

(i) The set \( C \) is a sublattice;

(ii) The projection \( P_C \) is isotone;

(iii)

\[C = \cap_{i=N} H_{-}(u_i, a_i),\]

where each hyperplane \( H(u_i, a_i) \) is tangent to \( C \) and the normals \( u_i \) are nonzero vectors \( u_i = (u_i^1, ..., u_i^m) \) with the properties \( u_i^k u_i^l \leq 0 \) whenever \( k \neq l, \quad i \in \mathbb{N} \).

The equivalence of items (i) and (iii) says slightly more than the main result in [18].

### 12. Comments and open questions

Motivated by isotone iterative methods for variational inequalities, the second author put the following very general and still open question: Which are the closed convex sets which possess a projection onto them which is isotone with respect to an order relation defined by a given cone? A related at least as interesting question is: Which are the closed convex sets for which there exist a cone such that the projection onto them are isotone with respect to order relation defined by the cone? Although these very general questions seem extremely difficult to handle, the present paper partially answered the first question for self-dual cones. The investigation led to interesting connections with the invariant sets with respect to the extended lattice operations defined by a self-dual cone. Another
question is: Can this invariance approach be extended for more general cones, e.g., by introducing extended lattice operations with respect to both the cone and its dual? We expect this paper to open a new area, providing a general tool for studying variational inequalities and related equilibrium problems by using isotonicity with respect to orders defined by cones, and greatly widening the field of similar previous investigations.

References

[1] A. Auslander. *Optimization Méthodes Numériques*. Masson, Paris, 1976.

[2] G. P. Barker and J. Foran. Self-dual cones in Euclidean spaces. *Linear Algebra Appl.*, 13:147–155, 1976.

[3] D. P. Bertsekas and J. N. Tsitsiklis. *Parallel and Distributed Computation: Numerical Methods*. Prentice-Hall, Inc, Englewood Cliffs, New Jersey, 1989.

[4] M. S. Gowda, R. Sznajder, and J. Tao. Some p-properties for linear transformations on Euclidean Jordan algebras. *Linear Algebra Appl.*, 393:203–232, 2004.

[5] G. Isac. On the order monotonicity of the metric projection operator. *Approximation Theory, Wavelets and Applications, ed. S. P. Singh*, 1995.

[6] G. Isac and A. B. Németh. Monotonicity of metric projections onto positive cones of ordered Euclidean spaces. *Arch. Math.*, 46(6):568–576, 1986.

[7] G. Isac and A. B. Németh. Isotone projection cones in Hilbert spaces and the complementarity problem. *Boll. Un. Mat. Ital. B.*, 7(4):773–802, 1990.

[8] G. Isac and S. Z. Németh. Regular exceptional family of elements with respect to isotone projection cones in Hilbert spaces and complementarity problems. *Optim Lett.*, 2(3):567–576, 2008.

[9] A. N. Iusem and B. F. Svaiter. A variant of Korpelevich’s method for variational inequalities with a new search strategy. *Optimization*, 42(4):309–321, 1997.

[10] E. N. Khobotov. A modification of the extragradient method for solving variational inequalities and some optimization problems. *Zhurnal Vychislitel’noi Matematiki i Matematicheskoi Fiziki*, 27(10):1462–1473, 1987.

[11] G. M. Korpelevich. The extragradient method for finding saddle points and other problems. *Matecon*, 12:747–756, 1976.

[12] P. Marcotte. Application of Khobotov’s algorithm to variational inequalities and network equilibrium problems. *Information Systems and Operational Research*, 29:258–270, 1991.
[13] J. J. Moreau. Décomposition orthogonale d’un espace hilbertien selon deux cônes mutuellement polaires. *C. R. Acad. Sci.*, 255:238–240, 1962.

[14] A. Nagurney. *Network Economics - A Variational Inequality Approach*. Kluwer Academic Publishers, Dordrecht, The Netherlands, 1993.

[15] A. B. Németh and S. Z. Németh. How to project onto an isotone projection cone. *Linear Algebra Appl.*, 433(1):41–51, 2010.

[16] S. Z. Németh. Iterative methods for nonlinear complementarity problems on isotone projection cones. *J. Math. Anal. Appl.*, 350(1):340–347, 2009.

[17] H. Nishimura and E. A. Ok. Solvability of variational inequalities on Hilbert lattices. *Preprint*, pages 1–28, 2012.

[18] M. Queyranne and F. Tardella. Bimonotone linear inequalities and sublattices of $\mathbb{R}^n$. *Linear Algebra Appl.*, 413:100–120, 2006.

[19] R. T. Rockafellar. *Convex Analysis*. Princeton: Princeton Univ. Press, 1970.

[20] M. Sibony. Méthodes itératives pour les équations et inéquations aux dérives partielles non linéaires de type monotone. *Calcolo*, 7:65–183, 1970.

[21] M. V. Solodov and B. F. Svaiter. A new projection method for variational inequality problems. *SIAM J. Control Optim.*, 37(3):765–776, 1999.

[22] M. V. Solodov and P. Tseng. Modified projection-type methods for monotone variational inequalities. *SIAM J. Control Optim.*, 34(5):1814–1830, 1996.

[23] D. Sun. A class of iterative methods for nonlinear projection equations. *J. Optim. Theory Appl.*, 91(1):123–140, 1996.

[24] D. M. Topkis. The structure of sublattices of the product of n lattices. *Pacific J. Math.*, 65:525–532, 1976.

[25] A. F. Veinott. Repräsentation of general and polyhedral sublattices and sublattices of product spaces. *Linear Algebra Appl.*, 114/115:172–178, 1981.

[26] E. Zarantonello. Projections on convex sets in Hilbert space and spectral theory, I: Projections on convex sets, II: Spectral theory. *Contrib. Nonlin. Functional Analysis, Proc. Sympos. Univ. Wisconsin, Madison*, pages 237–424, 1971.