Combinatorially refine a Zagier-Stanley result on products of permutations

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Abstract

In this paper, we enumerate the pairs of permutations that are long cycles and whose product has a given cycle-type. Our main result is a simple relation concerning the desired numbers for a few related cycle-types. The relation refines a formula of the number of pairs of long cycles whose product has $k$ cycles independently obtained by Zagier and Stanley relying on group characters, and was previously obtained by Féray and Vassilieva by counting some colored permutations first and then relying on some algebraic computations in the ring of symmetric functions. Our approach here is simpler and combinatorial.

Keywords: Product of long cycles; Plane permutation; Stirling number; Exceedance; Zagier-Stanley result; Féray-Vassilieva relation
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1 Introduction

Let $\mathcal{S}_n$ denote the symmetric group on $[n] = \{1, 2, \ldots, n\}$. We shall use the following two representations of a permutation $\pi \in \mathcal{S}_n$:

- **two-line form**: the top line lists all elements in $[n]$, following the natural order. The bottom line lists the corresponding images of elements on the top line, i.e.,

$$\pi = \begin{pmatrix}
1 & 2 & 3 & \cdots & n-2 & n-1 & n \\
\pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n-2) & \pi(n-1) & \pi(n)
\end{pmatrix}.$$

- **cycle form**: a permutation $\pi$ is decomposed into disjoint cycles. The set consisting of the lengths of these disjoint cycles is called the cycle-type of $\pi$. We can encode this set as an integer partition of $n$. An integer partition $\lambda$ of $n$, denoted by $\lambda \vdash n$, can be represented by a non-increasing integer sequence $\lambda = \lambda_1 \lambda_2 \cdots$, where $\sum_i \lambda_i = n$, or as $1^{m_1} 2^{m_2} \cdots n^{m_n}$, where we have $m_i(\lambda)$ of part $i$ and $\sum_i im_i = n$. A cycle of length $k$ is called a $k$-cycle. In addition, we denote the number of permutations of cycle-type $\lambda$ by $z_\lambda$. It is well known
that $z_\lambda = \frac{n!}{\prod_{i=1}^m i^{m_i} m_i!}$ if $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n}$. We also denote the length of $\lambda$, i.e., the number of positive parts in $\lambda$, by $\ell(\lambda)$.

Factorizations of permutations or products of permutations have been extensively studied in different contexts. Most of the related results in the field rely either partially or totally on a character theoretic approach (e.g., [9, 11, 14]). It is generally hard to obtain explicit and simple counting formulas. However, when one of the involved permutations is a long cycle, we may obtain some explicit formulas, see [3, 6, 8, 13] and references therein. In particular, Zagier [14] and Stanley [12] have independently obtained the following result: the number of $n$-cycles $s$ such that the product $(1 \, 2 \, \cdots \, n)^s$ has $k$ cycles is given by the surprisingly simple formula

$$2^{n-1} \frac{(n+1)}{(n+m)(n+1-m)} C_m(n+1, k),$$

where $C_m(n, k)$ is the number of permutations on $[n]$ with $k$ cycles and the elements in $[m]$ separated, i.e., an analogue of $C(n, k)$. This analogue was particularly used to answer a call of Stanley [10] for simple combinatorial proofs for the probability of separating $m$ elements due to Du and Stanley [10].

In this paper, we enumerate the pairs of long cycles whose product has a given cycle-type, refining the Zagier-Stanley result. Specifically, we obtain the theorem described below.

Let $\lambda = 1^{m_1} 2^{m_2} \cdots n^{m_n} \vdash n + 1$. For $i > 0$ (and $m_{i+1} \neq 0$), denote by $\lambda^{(i+1)}$ the partition $\mu = 1^{m_1} \cdots i^{m_i} (i+1)^{m_{i+1}-1} \cdots n^{m_n} \vdash n$, i.e., changing an $i+1$ part to an $i$ part. Let $p_\mu^{(n)}$ denote the number of pairs of $n$-cycles whose product has a cycle-type $\mu$.

**Theorem 1.1.** Suppose $m$ and $n$ have the same parity. Then for any partition $\lambda \vdash n + 1$ of length $m$, we have

$$\frac{n+1}{2} \sum_{\mu=\lambda^{(i+1)}, i>0} \imath m_i \mu p_\mu^{(n)} = (n-1)! z_\lambda.$$

(1)

Summing over all possible partitions of length $m$ in eq. (1) will give us the Zagier-Stanley result. An equivalent statement of Theorem 1.1 was previously obtained in Féray and Vassilieva [7], by counting some colored permutations first and then by some algebraic computations in the ring of symmetric functions. Our approach here is simpler and totally combinatorial, and is based on extending the plane permutation framework which was first introduced by the author and Reidys [3] and has proven to be effective in dealing with hypermaps, graph embeddings and even the genome rearrangement problems involving transpositions, block-interchanges and reversals [3-5].
2 Refining the Zagier-Stanley result

We begin with a review of some notation and results on plane permutations in [3].

Definition 2.1. A plane permutation on \([n]\) is a pair \(p = (s, \pi)\) where \(s = (s_i)_{i=0}^{n-1}\) is an \(n\)-cycle and \(\pi\) is an arbitrary permutation on \([n]\). Given \(s = (s_0 \ s_1 \cdots s_{n-1})\), a plane permutation \(p = (s, \pi)\) is represented by a two-row array:

\[
p = \begin{pmatrix}
s_0 & s_1 & \cdots & s_{n-2} & s_{n-1} \\
\pi(s_0) & \pi(s_1) & \cdots & \pi(s_{n-2}) & \pi(s_{n-1})
\end{pmatrix}.
\]

The permutation \(D_p\) induced by the diagonal-pairs (cyclically), i.e., \(D_p(\pi(s_{i-1})) = s_i\) for \(0 < i < n\), and \(D_p(\pi(s_{n-1})) = s_0\), is called the diagonal of \(p\).

We sometimes refer to \(s\), \(\pi\), \(D_p\) respectively as the upper horizontal, the vertical and the diagonal. Obviously, we have \(D_p = s \pi^{-1}\). It should be pointed out that, although as a cyclic permutation, there is no absolute left-right order for the elements in \(s\), in this paper, we generally assume there is a left-right order, with the leftmost element being \(s_0\).

In a permutation \(\pi\) on \([n]\), \(i\) is called an exceedance if \(i < \pi(i)\) following the natural order and an anti-exceedance otherwise. Note that \(s\) induces a linear order \(<_s\), where \(a <_s b\) if \(a\) appears before \(b\) in \(s\) from left to right (with the leftmost element \(s_0\)). Without loss of generality, we always assume \(s_0 = 1\) unless explicitly stated otherwise. These concepts then can be generalized for plane permutations as follows:

Definition 2.2. For a plane permutation \(p = (s, \pi)\), an element \(s_i\) is called an exceedance of \(p\) if \(s_i <_s \pi(s_i)\), and an anti-exceedance if \(s_i \geq_s \pi(s_i)\).

In the following, we mean by “the cycles of \(p = (s, \pi)\)” the cycles of \(\pi\) and any comparison of elements in \(s\), \(\pi\) and \(D_p\) references the linear order \(<_s\).

Obviously, each \(p\)-cycle contains at least one anti-exceedance as it contains a minimum, \(s_i\), for which \(\pi^{-1}(s_i)\) is an anti-exceedance. We call these trivial anti-exceedances and refer to a non-trivial anti-exceedance as an NTAE. Furthermore, in any cycle of length greater than one, its minimum is always an exceedance.

Let \(\mu\), \(\lambda\) be two integer partitions of \(n\). We denote \(\mu \triangleright_k \lambda\) if \(\mu\) can be obtained from \(\lambda\) by splitting one part into \(k\) parts, or equivalently, \(\lambda\) from \(\mu\) by merging \(k\) parts into one part. Let \(\kappa_{\mu, \lambda}\) be the number of different ways of merging \(k\) parts of \(\mu\) in order to obtain \(\lambda\) provided that \(\mu \triangleright_k \lambda\). Note that we differentiate two parts of \(\mu\) even if the two parts are of the same value. For example, for \(\mu = 1^22^2\) and \(\lambda = 1^12^13^1\), we have \(\kappa_{\mu, \lambda} = 4\).

Let \(U^n_\eta\) denote the set of plane permutations on \([n]\) where the diagonal is of cycle-type \(\eta\) and the vertical is of cycle-type \(\lambda\). We always assume \(\ell(\lambda) + \ell(\eta)\) has the same parity as \(n + 1\). Otherwise we know \(U^n_\lambda = \emptyset\). We denote \(p^n_\lambda = |U^n_\lambda|\). In Chen and Reidys [3], while studying the transposition action on the diagonal of plane permutations (i.e., transposing two adjacent diagonal blocks where a diagonal block is a set of consecutive diagonal-pairs) and exceedances, motivated by the work [4], the following proposition has been proved.
Proposition 2.3. Let $\lambda, \eta \vdash n$ and $p_{\lambda,a}^\eta$ be the number of $p \in U_\lambda^n$ such that $p$ has a exceedances. Then we have
\[
\sum_{a \geq 0} (n - \ell(\lambda) - a) p_{\lambda,a}^\eta = \sum_{\mu > 2i+1, i > 0} \kappa_{\mu,\lambda} p_{\mu}^\eta ,
\]
(3)
\[
(n + 1 - \ell(\lambda)) p_{\lambda}^{(n)} = \sum_{\mu > 2i+1, i > 0} \kappa_{\mu,\lambda} p_{\mu}^{(n)} + (n - 1)!z_\lambda .
\]
(4)

Our new contribution starts from here. In eq. (3), if we sum over all $\eta \vdash n$, we realize
\[
\sum_{\eta \vdash n} \sum_{a \geq 0} ap_{\lambda,a}^\eta = (n - \ell(\lambda)) (n - 1)!z_\lambda - \sum_{\mu > 2i+1, i > 0} \kappa_{\mu,\lambda} (n - 1)!z_\mu .
\]
(5)

Note that the left-hand side of the above equation can be interpreted as the total number of exceedances of $p \in \bigcup_{\eta \vdash n} U_\lambda^n$. However, by directly counting the exceedances, we have

Lemma 2.4. The total number of exceedances
\[
\sum_{\eta \vdash n} \sum_{a \geq 0} ap_{\lambda,a}^\eta = \frac{n - m_1(\lambda)}{2} (n - 1)!z_\lambda .
\]
(6)

Proof. It should not be hard to observe that in $\bigcup_{\eta \vdash n} U_\lambda^n$, for any upper horizontal $s$, the total number of exceedances of the plane permutations with the upper horizontal $s$ is the same as the total number of exceedances of the plane permutations with the upper horizontal $(1 \; 2 \; \cdots \; n)$. The latter is really just counting the total number of exceedances of the (conventional) permutations of cycle-type $\lambda$. Note that if a permutation $\pi$ of cycle-type $\lambda$ has $a$ exceedances, then its inverse $\pi^{-1}$ is of cycle-type $\lambda$ with $n - m_1(\lambda) - a$ exceedances. Because if an element $x$ that is not a fixed point is an exceedance of $\pi$, $\pi(x)$ is an exceedance of $\pi^{-1}$ and not a fixed point. Thus, for each such pair, they have on average $\frac{n - m_1(\lambda)}{2}$ exceedances, completing the proof. \qed

Combining eq. (5) and eq. (6), we have

Proposition 2.5. For any $\lambda \vdash n + 1$, the following is true
\[
(n + 1 - \ell(\lambda)) z_\lambda = \sum_{\mu > 2i+1, i > 0} \kappa_{\mu,\lambda} z_\mu + \frac{z_\lambda}{2} \sum_{i > 0} (i + 1)m_{i+1}(\lambda) .
\]
(7)

Proof. Just note that $n + 1 - m_1(\lambda) = \sum_{i > 0} (i + 1)m_{i+1}(\lambda)$. \qed

As a consequence of eq. (4), we have the following corollary.

Corollary 2.6. For any $\lambda \vdash n + 1$ and $i > 0$, we obtain
\[
(n + 1 - \ell(\lambda)) \frac{n + 1}{2} i m_i(\lambda^{(i+1)}) p_{\lambda^{(i+1)}}^{(n)} = \sum_{\mu > 2i+1, \lambda^{(i+1)}} \kappa_{\mu,\lambda} \frac{n + 1}{2} i m_i(\lambda^{(i+1)}) p_{\mu}^{(n)} + \frac{(i + 1)m_{i+1}(\lambda)}{2} (n - 1)!z_\lambda .
\]
(8)
Proof. Based on eq. (4), we first have
\[
(n + 1 - \ell(\lambda))P_{\lambda i(i+1)}^{(n)} = \sum_{\mu \triangleright 2j+1 \lambda^{i(i+1)}, j > 0} \kappa_{\mu, \lambda i(i+1)} P_{\mu}^{(n)} + (n - 1)!z_{\lambda i(i+1)}.
\]

Next, we observe \(z_{\lambda i(i+1)} = (\frac{i}{n+1} - 2m_{i+1}^{(\lambda)})z_{\lambda},\) and the proof follows.

In order to proceed, we need the following key lemma.

**Lemma 2.7.** For any \(\lambda \vdash n + 1,\) it holds that
\[
\sum_{i > 0} im_i(\lambda^{i(i+1)}) \sum_{\mu \triangleright 2j+1 \lambda^{i(i+1)}, \mu > j \lambda} \kappa_{\mu, \lambda i(i+1)} P_{\mu}^{(n)} = \sum_{j > 0} \kappa_{\mu, \lambda} \sum_{i > 0} im_i(\mu^{j(i+1)}) P_{\mu i(i+1)}^{(n)}. \tag{9}
\]

**Proof.** Note that both sides are eventually sums where each summand is indexed by a partition \(\mu\) of \(n.\) So it suffices to show the coefficients of the \(\mu\)-summands on both sides agree. Suppose \(\mu = \mu_1 \mu_2 \cdots.\) We identify \(\mu\) as the permutation \(\mu = (1 2 \cdots \mu_1)(\mu_1 + 1 \mu_2 + 1 \cdots \mu_1 + \mu_2)\). Note that the number \(\kappa_{\mu, \beta}\) for \(\mu \triangleright 2k+1 \beta\) can be interpreted as the number of permutations of cycle-type \(\beta\) that can be obtained by concatenating \(2k + 1\) cycles of the permutation \(\mu\) into one single cycle according to the cycle lengths (decreasingly) and the minimum elements of the cycles (increasingly). Next, we shall show that the considered coefficients counting the same subset of permutations of cycle-type \(\lambda\) on \([n + 1]\) associated with the permutation \(\mu.\)

Suppose \(\ell(\lambda) = \ell(\mu) - 2j.\) Let us start with the left-hand side and describe the subset \(A\) of permutations associated with \(\mu.\) A permutation \(\gamma \in A\) if \(\gamma\) gives us the permutation \(\mu\) following the procedure: (i) erasing \(n + 1\) from the cycle of \(\gamma\) containing it to obtain \(\gamma';\) (ii) splitting one cycle of \(\gamma'\) into \(2j + 1\) cycles. Now suppose \(\mu \triangleright 2j + 1 \lambda^{i(i+1)}\). We claim that there are \(im_i(\lambda^{i(i+1)})\kappa_{\mu, \lambda i(i+1)}\) associated permutations contained in \(A.\) This can be seen from the other way around: merging \(2j + 1\) cycles of \(\mu\) to obtain a permutation \(\gamma'\) of cycle-type \(\lambda^{i(i+1)}\) in \(\kappa_{\mu, \lambda i(i+1)}\) different ways; and next inserting \(n + 1\) into a cycle of length \(i\) in \(\gamma'\) to obtain a permutation \(\gamma\) of cycle-type \(\lambda\) in \(im_i(\lambda^{i+1})\) different ways. Denote the subset of permutations by \(A_i.\) Next it is obvious that for \(i \neq j, A_i \cap A_j = \emptyset,\) because \(n + 1\) is in cycles of different lengths. Therefore, \(A = \bigcup_{i > 0} A_i\) and the coefficient of the \(\mu\)-term on the left-hand side is \(|A|\).

Denote \(B\) the subset of associated permutations on the right-hand side, where a permutation \(\gamma \in B\) if \(\gamma\) gives us the permutation \(\mu\) following the procedure: (1) splitting one cycle of \(\gamma\) into \(2j + 1\) cycles to obtain \(\gamma';\) and (2) erasing \(n + 1\) from the cycle of \(\gamma'\) containing it. Now suppose \(\mu = \mu'^{j(i+1)}\) and \(\mu' \triangleright 2j + 1 \lambda.\) Analogously, we can conclude there are \(im_i(\mu'^{j(i+1)})\kappa_{\mu', \lambda}\) different associated permutations. Denote the subset of permutations by \(B_i.\) We also have \(B_i \cap B_j = \emptyset.\) Hence, the coefficient of the \(\mu\)-term on the right-hand side is \(|B|\). Finally, we can easily check that actually \(A = B.\) Therefore, the lemma follows.

For any \(\lambda \vdash n + 1,\) we denote \(T_{\lambda} = \sum_{i > 0} \frac{n+1}{i} im_i(\lambda^{i(i+1)})P_{\lambda i(i+1)}^{(n)}.\) Based on eq. (8) and
eq. (9), we obtain:

\[
(n + 1 - \ell(\lambda)) T_\lambda = \sum_{\mu > \lambda/2} \kappa_{\mu, \lambda} T_\mu + \frac{(n - 1)! z_\lambda}{2} \sum_{i > 0} (i + 1) m_{i+1}(\lambda). \tag{10}
\]

Now we are ready to prove our main theorem.

**Proof of Theorem 1.1.** Based on eq. (7) and eq. (10), we observe that both sides of the equality in the theorem satisfy the same recurrence. Then, it suffices to compare the respective initial conditions. Note that the initial cases correspond to the cases that \(\lambda = 1^{a} 2^{b}\) and \(a + 2b = n + 1\) for some \(a \geq 0\) and \(b \geq 0\). In [3], it is proved that the number of factorizations of a fixed permutation of cycle-type \(\lambda\) into two long cycles is \(\frac{n!}{n + 2 - a - b}\).

Then we can compute

\[
\frac{n + 1}{2} \sum_{\mu = \lambda^i(i+1), i > 0} im_i(\mu)p_{\mu}^{(n)} = \frac{n + 1}{2} (a + 1)p_{1^{a+1}2^{b-1}}^{(n)}
\]

\[
= \frac{n + 1}{2} (a + 1) \frac{z_{1^{a+1}2^{b-1}}}{(n - 1)!} \frac{(n - 1)! (n - 1)!}{n + 1 - (a + 1) - (b - 1)}
\]

\[
= \frac{(n - 1)! (n + 1)!}{1^{a+1}2^{b}a!b!} = (n - 1)! z_\lambda.
\]

This completes the proof. \(\square\)

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