SYSTEMS OF TWO SUBSPACES IN A HILBERT SPACE

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Abstract. We study two subspace systems in a separable infinite-dimensional Hilbert space up to (bounded) isomorphism. One of the main result of this paper is the following: Isomorphism classes of two subspace systems given by graphs of bounded operators are determined by unitarily equivalent classes of the operator ranges and the nullity of the original bounded operators giving graphs. We construct several non-isomorphic examples of two subspace systems in an infinite-dimensional Hilbert space.

KEYWORDS: Subspace, Hilbert space, Schatten class operator.

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1. Introduction

Let $E_1$ and $E_2$ be two closed subspaces in a Hilbert space $H$, then we say that $(H; E_1, E_2)$ is a two subspace system in $H$ or a system of two subspaces in $H$. Let $(L; F_1, F_2)$ be another two subspace system in $L$. We say that $(H; E_1, E_2)$ and $(L; F_1, F_2)$ are unitarily (resp. boundedly, algebraically) isomorphic if there exists a unitary operator (resp. bounded invertible operator, invertible operator) $V$ of $H$ to $L$ such that $V(E_1) = F_1$ and $V(E_2) = F_2$. Unitary isomorphism classes of two subspace systems are studied by many authors (cf. H. Araki, C. Davis, J. Dixmier, P. Halmos, M. Stone etc.).

It is easy to see that two subspace systems $(H; E_1, E_2)$ and $(L; F_1, F_2)$ are algebraically isomorphic if and only if $\text{Hdim}(E_1 \cap E_2) = \text{Hdim}(F_1 \cap F_2)$, $\text{Hdim}(E_1/(E_1 \cap E_2)) = \text{Hdim}(F_1/(F_1 \cap F_2))$, $\text{Hdim}(E_2/(E_1 \cap E_2)) = \text{Hdim}(F_2/(F_1 \cap F_2))$ and $\text{Hdim}(H/(E_1 + E_2)) = \text{Hdim}(L/(F_1 + F_2))$, where $\text{Hdim}(K)$ is a Hamel dimension of a vector space $K$. For a Hilbert space, we denote by $\dim H$ the Hilbert space dimension of $H$, that is, the cardinality of an orthonormal basis (or a complete orthonormal system) of $H$.

These three types of isomorphisms (unitary isomorphisms, bounded isomorphisms and algebraic isomorphisms) are different each other. Unitary isomorphisms and bounded isomorphisms of two subspace systems are distinguished by angles. Bounded isomorphisms and algebraic isomorphisms of two subspace systems are also distinguished. For example, put $a_n = 1/n$ and $b_n = 1/n^2$. Let $A$ be the diagonal
operator with diagonals \((a_n)_n\) and \(B\) be the diagonal operator with diagonals \((b_n)_n\) on \(K = \ell^2(\mathbb{N})\). Put \(H = K \oplus K\). Then two subspace systems \((H; K \oplus 0, graph(A))\) and \((H; K \oplus 0, graph(B))\) are algebraically isomorphic, but not boundedly isomorphic, since \(A\) and \(B\) belong to different Schatten classes. Bounded isomorphisms of systems of two subspaces have not been studied extensively compared with unitary isomorphisms.

In this paper we study two subspace systems up to bounded isomorphism. For this purpose, it is crucially important to investigate operator ranges. We recall an important paper \([FW]\) by Fillmore-Williams, which studies operator ranges.

One of the main result of this paper is the following: Isomorphism classes of two subspace systems given by graphs of bounded operators are determined by unitarily equivalent classes of the operator ranges and the nullity of the original bounded operators giving graphs. We describe a relation among derived three subspaces associated with two subspaces, \(A_2\)-Dynkin quiver and operator ranges. We give several examples of two subspace systems.

The classification problem of \(n\) subspaces in a Hilbert space up to unitary isomorphism arises naturally. But the problem for \(n \geq 3\) is *-wild in the sense of S.Kruglyak and Y.Samoilenko \([KS]\) and extremely difficult. See also S. Kruglyak, V. Rabanovich and Y. Samoilenko \([KRS]\), Y. Moskaleva and Y. Samoilenko \([MS]\) and Sunder \([Su]\) for the study of \(n\) subspaces. We study three subspaces \([EW3]\) and \(n\) subspaces \([EW1]\) up to bounded isomorphism which is weaker than unitary isomorphism. We should remark that in our former papers we just called “isomorphism” for “bounded isomorphism”.

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2. TWO SUBSPACE SYSTEMS GIVEN BY GRAPHS OF OPERATORS

In \([EW1]\), we studied several subspaces in an infinite dimensional Hilbert space in general. In this paper, we classify two subspaces up to bounded isomorphism in a separable Hilbert space.

**Definition.** For Hilbert spaces \(H_1\) and \(H_2\), we denote by \(B(H_1, H_2)\) the set of bounded operators of \(H_1\) to \(H_2\). For \(H_1 = H_2 = H\), we denote by \(B(H)\) the algebra of bounded operators on a Hilbert space \(H\). An operator range in a Hilbert space \(H\) is a linear subspace of \(H\) that is the range of some bounded operator on \(H\). If a vector space \(\mathcal{R}\) is the range of a closed operator on \(H\), then \(\mathcal{R}\) is an operator range. Operator ranges \(\mathcal{R}\) and \(\mathcal{S}\) in \(H\) are called similar, if there is an invertible operator \(T \in B(H)\) such that \(\mathcal{S} = T\mathcal{R}\), and unitarily equivalent, if \(T\)
can be taken to be unitary. Operator ranges are similar if and only if they are unitarily equivalent (cf. [FW, Theorem 3.1]). Let \( T \) be a densely defined closed operator on \( H \). Then we denote by \( \text{Dom}(T) \) the domain of \( T \), by \( \text{ran}T \) the set \( T(H) \) and by \( \ker T \) the set \( T^{-1}\{\{0\}\} \). We denote by \( C(H) \) the set of compact operators on a Hilbert space \( H \). For vectors \( x \) and \( y \) in \( H \), the symbol \( x \otimes y \) represents an operator on \( H \) such that \((x \otimes y)z = \langle z, y \rangle x \) for \( z \in H \).

Any compact operator \( T \) on a Hilbert space \( H \) has a form \( T = \sum_{j=1}^{N} \mu_j(T) v_j \otimes u_j \) where \( N = \text{rank}(T), \mu_1(T) \geq \mu_2(T) \geq \cdots \geq \mu_N(T) \) (or \( \geq \cdots \) if \( N = \infty \)), \((u_j)\), \((v_j)\) are orthogonal families and \( \mu_j(T) \) is the j-th eigenvalues of \( |T| \). Using min-max principle we have the following known fact. For any \( T \in C(H) \) and \( X \in B(H) \), \( \mu_j(T) = \mu_j(T^*) \), \( \mu_j(XT) \leq \|X\|\mu_j(T) \) and \( \mu_j(TX) \leq ||X||\mu_j(T) \).

The (extended) Schatten class for \( \alpha > 0 \) is

\[
C^\alpha(H) := \{ T \in B(H) ; \text{Tr}(|T|^\alpha) < \infty \},
\]

see MacCarthy [M] for the case that \( 0 < \alpha < 1 \). For \( \alpha > 0 \), we say that a sequence \((a_n)_n \) of complex numbers is in \( \ell^\alpha(\mathbb{N}) \) if \( \sum |a_n|^\alpha < \infty \).

**Definition.** Let \( H \) be a Hilbert space and \( E_1, \ldots, E_n \) be \( n \) closed subspaces in \( H \). Then we say that \( S = (H; E_1, \ldots, E_n) \) is a system of \( n \)-subspaces in \( H \) or an \( n \)-subspace system in \( H \). Let \( T = (K; F_1, \ldots, F_n) \) be another system of \( n \)-subspaces in a Hilbert space \( K \). Then \( \varphi : S \to T \) is called a bounded homomorphism if \( \varphi : H \to K \) is a bounded linear operator satisfying that \( \varphi(E_i) \subset F_i \) for \( i = 1, \ldots, n \). And \( \varphi : S \to T \) is called a bounded isomorphism if \( \varphi : H \to K \) is a bounded invertible linear operator satisfying that \( \varphi(E_i) = F_i \) for \( i = 1, \ldots, n \). We say that systems \( S \) and \( T \) are bounded isomorphic if there is a bounded isomorphism \( \varphi : S \to T \).

**Definition.** Let \( A : K_1 \to K_2 \) be a closed operator of a Hilbert space \( K_1 \) to a Hilbert space \( K_2 \). Let \( H = K_1 \oplus K_2 \). Then the two subspace system \((H; K_1 \oplus 0, \text{graph}(A))\) is said to be given by a \( \text{graph}(A) \) of the operator \( A \).

**Definition.** Let \((H; E_1, E_2)\) be a two subspace system. Then we call the three subspace system \((H; E_1^0, E_1^+, E_2)\) the derived three subspace system (or the derived three subspaces) of \((H; E_1, E_2)\).

In this paper we mainly discuss isomorphisms by bounded invertible operators between \( n \) subspace systems.

The following known fact is useful to study bounded isomorphisms.

**Lemma 2.1.** ([FW Lemma2.1]) Let \( H \) be a Hilbert space and \( H_1 \) and \( H_2 \) be two closed subspaces of \( H \). Then the following are equivalent:
(1) \( H = H_1 + H_2 \) and \( H_1 \cap H_2 = 0 \).

(2) There exists a closed subspace \( M \subset H \) such that \( (H; H_1, H_2) \) is boundedly isomorphic to \( (H; M, M^\perp) \).

(3) There exists an idempotent \( P \in B(H) \) such that \( H_1 = \text{ran} P \) and \( H_2 = \text{ran}(1 - P) \).

The following result is well known as in P.Halmos [H]. Let \( M \) and \( N \) be closed subspaces of a Hilbert space \( H \). Then \( M \) and \( N \) are in generic position if

\[ M \cap N, M \cap N^\perp, M^\perp \cap N, M^\perp \cap N^\perp \]

are zero. For any such pair \( M, N \), there exist a Hilbert space \( K \) and a closed linear operator \( T \) having domain and range dense in \( K \) and zero kernel, such that a unitary operator of \( H \) onto \( K \oplus K \) carries \( M \) to \( K \oplus 0 \) and carries \( N \) to the graph of \( T \). The linear operator \( T \) can be chosen self-adjoint and positive, and if it is chosen so, then it is unique up to unitary equivalence.

In bounded isomorphisms case, the situation is completely different. Let \( K = \mathbb{C}^2 \) and \( T_1 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \), \( T_2 = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \). Then \( T_1 \) and \( T_2 \) are not similar, but the two systems \( (H; K \oplus 0, \text{graph}(T_1)) \) and \( (H; K \oplus 0, \text{graph}(T_2)) \) are boundedly isomorphic.

Example. Consider a sequence \( (\theta_n) \) such that \( 0 < \theta_n < \pi/2 \) and \( \theta_n \to 0 \) \((n \to \infty)\). Let \( C(\text{resp.} S) \) be the diagonal operator with diagonals \( (\cos \theta_n) \) (resp. \( (\sin \theta_n) \)). Let \( K = \ell^2(\mathbb{N}) \) and \( H = K \oplus K \). Let \( E_1 = K \oplus 0 \) and \( E_2 = \text{ran} \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix} \). Then \( (H; E_1, E_2) \) is not boundedly isomorphic to \( (K \oplus K; K \oplus 0, 0 \oplus K) \). In fact, if \( (H; E_1, E_2) \) were isomorphic to \( (K \oplus K; K \oplus 0, 0 \oplus K) \), then \( E_1 + E_2 \) must be closed since \( K \oplus K = K \oplus 0 + 0 \oplus K \) is closed, but \( E_1 + E_2 = K \oplus \text{ran} S \) is not closed because the range of \( S \) is not closed. This is a contradiction.

We need to recall Hilbert representations of quivers studied in [EW2].

Definition. A quiver \( \Gamma = (V, E, s, r) \) is a quadruple consisting of the set \( V \) of vertices, the set \( E \) of arrows, and two maps \( s, r : E \to V \), which associate with each arrow \( \alpha \in E \) its support \( s(\alpha) \) and range \( r(\alpha) \). We sometimes denote by \( \alpha : x \to y \) an arrow with \( x = s(\alpha) \) and \( y = r(\alpha) \). Thus a quiver is just a directed graph. We denote by \( |\Gamma| \) the underlying undirected graph of a quiver \( \Gamma \). A quiver \( \Gamma \) is said to be finite if both \( V \) and \( E \) are finite sets.

Definition. Let \( \Gamma = (V, E, s, r) \) be a finite quiver. We say that \( (H, f) \) is a Hilbert representation of \( \Gamma \) if \( H = (H_v)_{v \in V} \) is a family of Hilbert
spaces and \( f = (f_\alpha)_{\alpha \in E} \) is a family of bounded linear operators \( f_\alpha : H_{s(\alpha)} \to H_{r(\alpha)} \).

**Definition.** Let \( \Gamma = (V, E, s, r) \) be a finite quiver. Let \( (H, f) \) and \( (K, g) \) be Hilbert representations of \( \Gamma \). A *bounded homomorphism* \( T : (H, f) \to (K, g) \) is a family \( T_v : H_v \to K_v \) satisfying, for any arrow \( \alpha \in E \)

\[
T_{r(\alpha)} f_\alpha = g_\alpha T_{s(\alpha)}.
\]

Let \( \Gamma = (V, E, s, r) \) be a finite quiver and \( (H, f), (K, g) \) be Hilbert representations of \( \Gamma \). We say that \( (H, f) \) and \( (K, g) \) are *boundedly isomorphic*, denoted by \( (H, f) \simeq (K, g) \), if there exists a bounded isomorphism \( \varphi : (H, f) \to (K, g) \), that is, there exists a family \( \varphi = (\varphi_v)_{v \in V} \) of bounded invertible operators \( \varphi_v \in B(H_v, K_v) \) such that, for any arrow \( \alpha \in E \),

\[
\varphi_{r(\alpha)} f_\alpha = g_\alpha \varphi_{s(\alpha)}.
\]

We say that \( \Gamma \) is the \( A_2 \)-Dynkin quiver if \( \Gamma = (V, E, s, t) \) is an oriented graph such that the vertex set of \( \Gamma \) is \( V = \{1, 2\} \), the arrow set of \( \Gamma \) is \( E = \{\alpha\} \) with

\[
\circ_1 \xrightarrow{\alpha} \circ_2
\]

A Hilbert representation \( (H, f) \) of the \( A_2 \)-Dynkin quiver \( \Gamma \) is called a Hilbert representation constructed by an operator \( T : H_1 \to H_2 \) if \( H_{s(\alpha)} \) is a Hilbert space \( H_1 \), \( H_{r(\alpha)} \) is a Hilbert space \( H_2 \) and \( f_\alpha = T \).

We mainly study two subspace systems which are given by graphs of bounded operators. The following is the main theorem of the paper.

**Theorem 2.2.** Let \( K_1, K_2 \) be Hilbert spaces and \( T, T' \) be in \( B(K_1, K_2) \). We put \( H = K_1 \oplus K_2 \). Then the following are equivalent:

1. \((H; K_1 \oplus 0, \text{graph}(T)) \) is boundedly isomorphic to \((H; K_1 \oplus 0, \text{graph}(T'))\).
2. Derived three subspace systems \((H; K_1 \oplus 0, 0 \oplus K_2, \text{graph}(T))\) and \((H; K_1 \oplus 0, 0 \oplus K_2, \text{graph}(T'))\) are boundedly isomorphic.
3. Hilbert representations constructed by \( T : K_1 \to K_2 \) and \( T' : K_1 \to K_2 \) are boundedly isomorphic as Hilbert representations of the \( A_2 \)-Dynkin quiver.
4. Operator ranges \( \text{ran}T \) and \( \text{ran}T' \) are unitarily equivalent and \( \dim \ker T = \dim \ker T' \).
5. There exist invertible operators \( L \in B(K_2) \) and \( M \in B(K_1) \) such that \( T = LT'M \).

**Proof.** (1) \( \Rightarrow \) (4): Assume that (1) holds. Then there exists an invertible operator \( S \in B(K_1 \oplus K_2) \) such that \( S(K_1 \oplus 0) = K_1 \oplus 0 \) and \( S(\text{graph}(T)) = \text{graph}(T') \). Hence there exist operators \( A \in B(K_1), B \in B(K_2), C \in B(K_2, K_1) \) such that

\[
S = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}.
\]

Since
$S(K_1 \oplus 0) = K_1 \oplus 0$, $A$ is surjective. If $Ax = 0$ for $x \in K_1$, then

$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Since $S$ is invertible, $x = 0$. Thus $A$ is injective. Therefore $A$ is bounded invertible. Since $S$ is invertible, $B$ is surjective. Assume that $By = 0$ for $y \in K_2$. We put $x = -A^{-1}Cy$.

Then

$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (A + CT)x_1 \\ BTx_1 \end{pmatrix} = \begin{pmatrix} x_2 \\ T'x_2 \end{pmatrix}$ for some $x_2 \in K_1$.

Hence $y = 0$. Thus $B$ is injective and $B$ is bounded invertible. Since $S(K_1 \oplus 0) \cap (\text{graph}(T)) = \ker T \oplus 0$, the bounded isomorphism $S = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ gives $S(\ker T \oplus 0) = \ker T' \oplus 0$.

Hence $\dim ker T = \dim ker T'$. Thus (4) holds.

(4) $\iff$ (5): This follows from a result [FW, Theorem 3.4].

(5) $\implies$ (3): It is trivial.

(3) $\implies$ (2): Assume that (3) holds. There exist bounded invertible operators $G_1 \in B(K_1)$ and $G_2 \in B(K_2)$ such that $T'G_1 = G_2T$. We put $S = \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix}$. Then $S$ is an invertible map on $K_1 \oplus K_2$.

\[
S(\text{graph}(T)) = \left\{ \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} \begin{pmatrix} x_1 \\ Tx_1 \end{pmatrix} \mid x_1 \in K_1 \right\} = \left\{ \begin{pmatrix} G_1x \\ T'G_1x_1 \end{pmatrix} \mid x_1 \in K_1 \right\}.
\]

Since $\text{ran} G_1 = K_1$, we have $S(\text{graph}(T)) = \text{graph}(T')$. Thus (2) holds.

(2) $\implies$ (1): It is trivial. \qed

Remark. We see that $A + CT$ is also one to one as pointed out to us by R. Sato and Y. Ueda. Therefore we can directly show that (1) implies (5) without a result in [FW].

Remark. The theorem above does not hold if $T_1$ or $T_2$ is not bounded. Let $T$ be a densely defined closed operator with the domain $\text{Dom}(T)$ of $T$ in a Hilbert space $K$ and $H = K \oplus K$. Assume that $\text{Dom}(T) \neq K$. Let $U$ be a bounded operator on $K$. Then derived three subspace...
systems $\tilde{S}_1 = (H; K \oplus 0, 0 \oplus K, graph(T))$ and $\tilde{S}_2 = (H; K \oplus 0, 0 \oplus K, graph(U))$ are not boundedly isomorphic although $S_1 = (H; K \oplus 0, 0 \oplus K, graph(T))$ and $S_2 = (H; K \oplus 0, graph(U))$ are boundedly isomorphic to $(H; K \oplus 0, 0 \oplus K)$ by Lemma 2.1. On the contrary, suppose that $\tilde{S}_1$ were boundedly isomorphic to $\tilde{S}_2$, then there exists an invertible operator $W$ on $H$ such that $W(K \oplus 0) = K \oplus 0, W(0 \oplus K) = 0 \oplus K$. The operator $W$ has the form $W = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$, where $A$ and $B$ are invertible.

Since $W(graph(U)) = graph(T)$, $AK = Dom(T)$. Hence $Dom(T) = K$. This is a contradiction. Hence $\tilde{S}_1$ is not boundedly isomorphic to $\tilde{S}_2$.

**Remark.** If $T$ is a normal operator, then $kerT = kerT^* = (ranT)^\perp$. Therefore, if $T_1$ and $T_2$ are normal, then the condition (4) is equivalent to that $ranT_1$ and $ranT_2$ are unitarily equivalent.

### 3. Examples of non-isomorphic two subspace systems

At first we consider examples of two subspace systems given by graphs of compact operators.

**Proposition 3.1.** Let $A$ and $B$ be compact positive operators on a Hilbert space $K$. We may assume that there exist orthonormal systems $(x_n)_n$ and $(y_n)_n$ in $K$ such that $A = \sum_n \mu_n(A) x_n \otimes x_n$, $B = \sum_n \mu_n(B) y_n \otimes y_n$. Also assume that $\mu_n(A) \neq 0$ and $\mu_n(B) \neq 0$ for any $n \in \mathbb{N}$. Then the following (i) and (ii) are equivalent:

(i) $\dim kerA = \dim kerB$ and there exist positive numbers $\gamma_1, \gamma_2$ such that, for any $n \in \mathbb{N}$

$$\gamma_1 \mu_n(B) \leq \mu_n(A) \leq \gamma_2 \mu_n(B).$$

(ii) $(H; K \oplus 0, graph(A))$ is boundely isomorphic to $(H; K \oplus 0, graph(B))$.

**Proof.** (i)$\Rightarrow$ (ii): Assume (i). Since $\dim kerA = \dim kerB$, there exists a unitary operator $U$ such that $Uy_n = x_n$ for $n \in \mathbb{N}$. Then we have that

$$UBU^* = \sum_n \mu_n(B) Uy_n \otimes Uy_n = \sum_n \mu_n(B) x_n \otimes x_n.$$  

The positive sequence $(\nu_n)_n := (\mu_n(A)/\mu_n(B))_n$ is bounded and bounded below by the assumption (i). Take an orthonormal basis $\{z_n\}$ of $kerA$ and define a bounded invertible diagonal operator $Z := \sum \nu_n x_n \otimes x_n + \sum z_n \otimes z_n$. Then we have

$$UBU^* Z = A = \sum \mu_n(A) x_n \otimes x_n.$$  

By Theorem 2.2, we have (ii).
(ii)⇒(i): Assume (ii). Then there exist bounded invertible operators $C$ and $D$ such that $A = CBD$ by Theorem 2.2. Hence

$$\mu_n(A) = \mu_n(CBD) \leq ||C|| |D| \mu_n(B).$$

and

$$\mu_n(B) = \mu_n(C^{-1}AD^{-1}) \leq ||C^{-1}|| |D^{-1}| |\mu_n(A)|.$$  

Put $\gamma_1 = ||C^{-1}||^{-1} |D^{-1}|^{-1}$ and $\gamma_2 = ||C|| |D|$. Then we have

$$\gamma_3 \mu_n(B) \leq \mu_n(A) \leq \gamma_2 \mu_n(B).$$

Since $A = CBD$, $ker A = D^{-1}(ker B)$. Hence $dim ker A = dim ker B$.

Thus (i) holds.

We study two subspace systems given by graphs of Schatten class operators. We shall consider an invariant for such two subspace systems.

Let $T$ be a Schatten class operator on a Hilbert space $K$. We put

$$Sh(T) := \inf \{ \alpha > 0 : T \in C^\alpha(K) \}.$$

For example, let $T$ be the diagonal operator with diagonals $(1/n^s)_n$ for $s > 0$, then $Sh(T) = 1/s$.

**Proposition 3.2.** Let $T_1, T_2$ be Schatten class operators on a Hilbert space $K$. If $(H; K \oplus 0, \text{graph}(T_1))$ is boundedly isomorphic to $(H; K \oplus 0, \text{graph}(T_2))$, then $Sh(T_1) = Sh(T_2)$. But the converse does not hold.

**Proof.** For $1 \leq \alpha$, it is known that $C^\alpha(K)$ is a ideal in $B(K)$. For $0 < \alpha < 1$, it seems that this fact is not well known. So we shall give a proof for completeness. Heinz-inequality says that for positive operators $A, B \in B(K)$ and $0 \leq p \leq 1$, if $A \leq B$, then $A^p \leq B^p$. Using this inequality, we have

$$|BT|^{\alpha} = (|BT|^2)^{\alpha/2} = (T^*B^*BT)^{\alpha/2} \leq (T^*|B|^2T)^{\alpha/2} = |B|^\alpha (|T|^\alpha).$$

Therefore if $T \in C^\alpha(K)$, then $BT \in C^\alpha(K)$ for $\alpha > 0$. If $Tr(|T|^\alpha) < \infty$ for $0 < \alpha$, then

$$Tr(|T|^\alpha) = Tr(|T^*|^\alpha).$$

From this, if $T \in C^\alpha(K)$, then $TB \in C^\alpha(K)$ for $\alpha > 0$. Therefore $C^\alpha(K)$ is an ideal in $B(K)$, since it is known that for $0 < \alpha < 1$, $C^\alpha(K)$ is a linear space as in ([M],Theorem 2.8]), for example.

Suppose that $(H; K \oplus 0, \text{graph}(T_1))$ is boundedly isomorphic to $(H; K \oplus 0, \text{graph}(T_2))$. By Theorem 2.2, there exist bounded invertible operators $L, M \in B(K)$ such that $T_1 = LT_2M$. Since $C^\alpha(K)$ is an ideal of
$B(K)$ for any $\alpha, T_1 \in C^\alpha(K)$ if and only if $T_2 \in C^\alpha(K)$. Therefore $Sh(T_1) = Sh(T_2)$.

But the converse does not hold. In fact, let $T_1$ be the diagonal operator with diagonals $(1/n)_n$ and $T_2$ the diagonal operator with diagonals $(1/\{(n + 1)\log(n + 1)\})_n$. Then $Sh(T_1) = Sh(T_2)$ and two subspace systems $(H; K \oplus 0, graph(T_1))$ and $(H; K \oplus 0, graph(T_2))$ are not boundedly isomorphic by Proposition 3.1. □

Example. Let $K$ be a Hilbert space with a basis $(e_n)_n$. Let $A = \Sigma (1/n)(e_{n+1} \otimes e_n)$ and $B = \Sigma (1/n)(e_n \otimes e_n)$. Then $A$ and $B$ are Schatten class operators. Clearly $Sh(A) = Sh(B) = 1$. But the two subspace systems given by the graphs of $A$ and $B$ are not isomorphic, since $\text{ran}A$ is not unitarily equivalent to $\text{ran}B$.

We note that Schatten class operators do not exhaust all compact operators. Consider a diagonal operator $T$ with diagonals $(1/\log(n + 1))_n$. Then $T$ is a compact operator but $T$ does not belong to any Schatten class operator. The next proposition can be applied for such an operator.

Proposition 3.3. Let $s, t$ be positive numbers and $s \neq t$. Let $c_n$ be a decreasing sequence of positive numbers with $\lim_{n \to \infty} c_n = 0$. Put $K = \ell^2(\mathbb{N})$ and $H = K \oplus K$. Let $A(\text{resp.} B)$ be a diagonal operator on $K$ with diagonals $(c_n^s)_n$ (resp. $(c_n^t)_n$). Then $(H; K \oplus 0, graph(A))$ is not boundedly isomorphic to $(H; K \oplus 0, graph(B))$.

Proof. On the contrary, suppose that $(H; K \oplus 0, graph(A))$ were boundedly isomorphic to $(H; K \oplus 0, graph(B))$. Then by Proposition 3.1 there exist positive numbers $\gamma_1, \gamma_2$ such that

$$\gamma_1 \mu_n(B) \leq \mu_n(A) \leq \gamma_2 \mu_n(B).$$

Hence

$$\gamma_1 c_n^t \leq c_n^s \leq \gamma_2 c_n^t.$$  

If $s > t$, by $\gamma_1 c_n^{s-t}$, this is a contradiction. If $s < t$, by $c_n^{s-t} \leq \gamma_2$, this is a contradiction. This proves the theorem. □

Example. Let $K = L^2[0, 1]$ and $s, t > 0$. Let $M_{x^s}$ be the multiplication operator on $K$ such that $M_{x^s}f(x) = x^sf(x)$ for $f \in K$ and $x \in [0, 1]$. Then $\text{ran}M_{x^s}$ is unitarily equivalent to $\text{ran}M_{x^t}$.

In fact we put $U: K \to K$ by

$$(Uf)(x) = \sqrt{s x^{s-1}} f(x^s)$$
for $f \in K$. Then $U$ is a unitary and $UM_x = M_xU$. Thus $\text{ran}M_x$ and $\text{ran}M_{x^*}$ are unitarily equivalent.

Next we shall consider when two subspaces are algebraically isomorphic. The Hamel dimension of any infinite dimensional separable Banach space is continuously infinite (cf.[L]). The Hamel dimension of an operator range in a separable Hilbert space $K$ is finite or continuous, since, for $A \in B(K)$, $K/\ker A$ is algebraically isomorphic to $\text{ran}A$. For any non-closed operator range $\mathcal{R}$ in a separable Hilbert space $K$, the Hamel dimension of $K/\mathcal{R}$ is continuous. See, for example, [FW, page.274, Cor1]. Let $c_0$ be the vector space of sequences which converges to 0 and let $c_{00}$ be the subspace of sequences with a finite support. Clearly the Hamel dimension of $c_{00}$ is countable. Thus $c_{00}$ cannot be an operator range in $\ell^2(\mathbb{N})$.

It is easy to see the following:

**Proposition 3.4.** Let $H$ and $L$ be Hilbert spaces. Then the following are equivalent.

1. Two subspace systems $(H; E_1, E_2)$ and $(L; F_1, F_2)$ are algebraically isomorphic.
2. 
   
   \[
   \text{Hdim}(E_1 \cap E_2) = \text{Hdim}(F_1 \cap F_2),
   \]
   
   \[
   \text{Hdim}(E_1/(E_1 \cap E_2)) = \text{Hdim}(F_1/(F_1 \cap F_2)),
   \]
   
   \[
   \text{Hdim}(E_2/(E_1 \cap E_2)) = \text{Hdim}(F_2/(F_1 \cap F_2))
   \]
   
   and
   
   \[
   \text{Hdim}(H/(E_1 + E_2)) = \text{Hdim}(L/(F_1 + F_2)).
   \]

The following proposition is a direct consequence of the proposition above.

**Proposition 3.5.** Consider $(a_n)_n \in \ell^\infty(\mathbb{N})$ and $(b_n)_n \in \ell^\infty(\mathbb{N})$ such that $a_n \neq 0$ and $b_n \neq 0$ for any $n \in \mathbb{N}$. Let $A$ and $B$ be diagonal operators on $K = \ell^2(\mathbb{N})$ with diagonals $(a_n)_n$ and $(b_n)_n$ respectively. Put $H = K \oplus K$. Then the following hold.

1. If $\text{ran}A$ is closed and $\text{ran}B$ is non-closed, then two subspace systems $(H; K \oplus 0, \text{graph}(A))$ and $(H; K \oplus 0, \text{graph}(B))$ are not algebraically isomorphic.

2. If $\text{ran}A$ and $\text{ran}B$ are both closed or both non-closed, then two subspace systems $(H; K \oplus 0, \text{graph}(A))$ and $(H; K \oplus 0, \text{graph}(B))$ are algebraically isomorphic.

**Proof.** (i) We assume that $\text{ran}A$ is closed and $\text{ran}B$ is non-closed. Then we have that $\text{ran}A = K$ and

\[
(K \oplus K)/(K \oplus \text{ran}A) = (K \oplus K)/(K \oplus K) = 0,
\]
but \((K \oplus K)/(K \oplus \text{ran}B) \neq 0\). So these two subspace systems are not algebraically isomorphic.

(ii) Consider the case that the operator ranges \(\text{ran}A\) and \(\text{ran}B\) are non-closed. Then \((K \oplus 0) \cap (\text{graph}A) = \ker A \oplus 0 = 0\) and \((K \oplus 0) \cap (\text{graph}B) = \ker B \oplus 0 = 0\). And \(K \oplus 0, \text{graph}A\) and \(\text{graph}B\) are all algebraically isomorphic to \(K\). Moreover the Hamel dimensions of \((K \oplus K)/(K \oplus \text{ran}A) = K/\text{ran}A\) and \((K \oplus K)/(K \oplus \text{ran}B) = K/\text{ran}B\) are both continuous, because \(\text{ran}A\) and \(\text{ran}B\) are non-closed. Hence the two subspace systems \((H; K \oplus 0, \text{graph}(A))\) and \((H; K \oplus 0, \text{graph}(B))\) are boundedly isomorphic. Next consider the case that the operator ranges \(\text{ran}A\) and \(\text{ran}B\) are closed. Then \(\text{ran}A = K = \text{ran}B\). Similar consideration implies the conclusion. \(\square\)

For example, let \(a_n = 1/n\) and \(b_n = 1/n^2\). Then two subspace systems \((H; K \oplus 0, \text{graph}(A))\) and \((H; K \oplus 0, \text{graph}(B))\) are algebraically isomorphic, but not boundedly isomorphic.

**Example.** Let \(A\) be the diagonal operator with diagonals \((n^2)_n\) and \(A'\) be the diagonal operator with diagonals \((2)_n\) on \(K = \ell^2(\mathbb{N})\). We put \(H = K \oplus K\). Then \((H; K \oplus 0, \text{graph}(A))\) and \((H; K \oplus 0, \text{graph}(A'))\) are boundedly isomorphic. In fact this follows from lemma 2.1, since \(K \oplus 0 + \text{graph}(A) = K \oplus K\), \(K \oplus 0 + \text{graph}(A') = K \oplus K\), \((K \oplus 0) \cap \text{graph}(A) = 0\) and \((K \oplus 0) \cap \text{graph}(A') = 0\).

**Example.** Let \(A\) (resp. \(A', C\)) be the diagonal operator with diagonals \((n^2)_n\), (resp. \((2)_n\), \((1/n^2)_n\) on \(K = \ell^2(\mathbb{N})\). We put \(H = K \oplus K\). Then \((H \oplus H; H \oplus 0, \text{graph}(A \oplus C))\) and \((H \oplus H; H \oplus 0, \text{graph}(A' \oplus C))\) are boundedly isomorphic. In fact \((H; K \oplus 0, \text{graph}(A))\) and \((H; K \oplus 0, \text{graph}(A'))\) are boundedly isomorphic, and \((H \oplus H; H \oplus 0, \text{graph}(A \oplus C))\) is boundedly isomorphic to \((H; K \oplus 0, \text{graph}(A)) \oplus (H; K \oplus 0, \text{graph}(C))\).

We give a condition when two subspace systems given by graphs of unbounded operators are boundedly isomorphic.

**Proposition 3.6.** Let \(T_1, T_2\) be densely defined closed operators on a Hilbert space \(K\) such that \(T_1^{-1}, T_2^{-1}\) are boundedly operators. If \(||T_1^{-1} - T_2^{-1}|| < 1\), then two subspace systems \((H; K \oplus 0, \text{graph}(T_1))\) and \((H; K \oplus 0, \text{graph}(T_2))\) are boundedly isomorphic.

**Proof.** Let \(J(x, y) := (y, x)\) for \(x, y \in K\). Then \(J\) gives an bounded isomorphism of \((H; K \oplus 0, \text{graph}(T))\) to \((H; 0 \oplus K, \text{graph}(T^{-1}))\).

We put \(\Phi = \begin{pmatrix} I & 0 \\ T_2^{-1} - T_1^{-1} & I \end{pmatrix}\). Since \(||T_2^{-1} - T_1^{-1}|| < 1\), \(\Phi\) is invertible and

\[\Phi(\text{graph}(T_1^{-1})) = \text{graph}(T_2^{-1})\] and \(\Phi(0 \oplus K) = 0 \oplus K\).
Hence \((H; 0 \oplus K, \text{graph}(T_1^{-1}))\) and \((H; 0 \oplus K, \text{graph}(T_2^{-1}))\) are boundedly isomorphic. Therefore two systems \((H; K \oplus 0, \text{graph}(T_1))\) and \((H; K \oplus 0, \text{graph}(T_2))\) are boundedly isomorphic. \(\square\)

We recall a useful lemma in [FW].

**Lemma 3.7.** [FW] Lemma 3.2] Let \(A = \int_0^M \lambda dE_{\lambda}\) and \(B = \int_0^N \lambda dF_{\lambda}\) be positive operators on a Hilbert space \(H\). Assume that \(\text{ran}A = \text{ran}B\). Then there is a positive constant \(K \geq 1\) such that

\[
\dim E[\alpha, \beta]H \leq \dim F[\alpha/K, K\beta]H.
\]

and

\[
\dim F[\alpha, \beta]H \leq \dim E[\alpha/K, K\beta]H.
\]

whenever \(0 < \alpha \leq \beta\).

The above lemma 3.7 enables us to consider the following examples.

**Example.** Let \(A\) (resp. \(B, C, D\)) be the diagonal operator with diagonals \((n^2)_n\) (resp. \((n^3)_n, (1/n^2)_n, (1/n^3)_n)\) on \(K = \ell^2(\mathbb{N})\). Then \((H \oplus H; H \oplus 0, \text{graph}(A \oplus C))\) and \((H \oplus H; H \oplus 0, \text{graph}(B \oplus D))\) are not boundedly isomorphic. In fact, let \(A'\) be the diagonal operator with diagonals \((2)_n\) on \(K = \ell^2(\mathbb{N})\). Then \((H \oplus H; H \oplus 0, \text{graph}(A \oplus C))\) and \((H \oplus H; H \oplus 0, \text{graph}(B \oplus D))\) are boundedly isomorphic, and \((H; K \oplus 0, \text{graph}(B \oplus D))\) and \((H; K \oplus 0, \text{graph}(A' \oplus D))\) are also boundedly isomorphic. Put \(c_n = 1/n^2\) and \(d_n = 1/n^3\). Let \(E\) be a spectral measure for \(A' \oplus C\) and \(F\) a spectral measure for \(A' \oplus D\).

By lemma 3.7, there exists a positive constant \(K \geq 1\) such that

\[
\dim (E[\alpha, 1/K]H) = \# \{n \in \mathbb{N}; \alpha \leq c_n \leq (1/K)\} \\
\leq \dim (F[\alpha/K, 1]H) = \# \{n \in \mathbb{N}; \alpha/K \leq d_n \leq 1\} \\
= \# \{n \in \mathbb{N}; \alpha \leq Kd_n \leq K\}.
\]

Put

\[
m_0 := \min \{m \in \mathbb{N}; c_m \leq 1/K\}
\]

For any \(n \in \mathbb{N}\), put \(\alpha = c_{n+m_0}\). Then

\[
n + 1 = \# \{\ell \in \mathbb{N}; \alpha \leq c_\ell \leq 1/K\} \leq \# \{\ell \in \mathbb{N}; \alpha \leq Kd_\ell \leq K\}.
\]

Hence \(\alpha \leq Kd_{n+1} \leq Kd_n\). Therefore \(c_{n+m_0} \leq Kd_n\). Thus for any \(n \in \mathbb{N}\), we have that

\[
n^3/(n + m_0)^2 \leq K
\]

This implies a contradiction. Therefore these two subspaces are not boundedly isomorphic.

Next we give another example.

**Example.** Let \(T \in B(L^2[2, 3])\) be a multiplication operator defined by \((Tf)(t) = tf(t)\) for \(f \in L^2[2, 3]\). Let \(C\) be the diagonal operator
with diagonals \((1/n^2)\_n\) and \(D\) be the diagonal operator with diagonals \((1/n^3)\_n\) on \(K = \ell^2(\mathbb{N})\). We put \(H = K \oplus K\). Then operators \(T \oplus C\) and \(T \oplus D\) have continuous spectrum and \((H \oplus H; H \oplus 0, graph(T \oplus C))\) and \((H \oplus H; H \oplus 0, graph(T \oplus D))\) are not boundedly isomorphic.

Use lemma [3.7] similarly for the intervals not containing the interval \([2,3]\).

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