Continuous Limit of Multiple Lens Effect and the Optical Scalar Equation

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ABSTRACT

We study the continuous limit of the multiple gravitational lensing theory based on the thin lens approximation. Under the approximation, we define a new, light-path dependent angular diameter distance \( \tilde{d} \) and show that it satisfies the optical scalar equation. The distance provides relations between quantities used in the gravitational lensing theory (the convergence-, the shear- and the twist-term) and those used in the scalar optics theory (the rates of expansion, shear and rotation).

Subject headings: cosmology:theory — distance scale — gravitational lensing

1. Introduction

The distance-redshift relation is one of the most important and difficult subjects in observational cosmology (here, “distance” means the angular diameter distance). Recent
observations on Type Ia supernovae with high-redshifts suggest the large deacceleration parameter $q_0 = \Omega_0/2 - \lambda_0$ ($\Omega_0, \lambda_0$ are the present values of the density parameter and of the cosmological constant, respectively; Perlmutter et al. 1999). The reliability of the suggestion depends on that of the distance measure to supernova adopted in the observation.

Since our universe is assumed to be described by the Friedmann-Lemaître universe which is homogeneous and isotropic in a large scale, we have used the Mattig formula as the distance measure (e.g. Weinberg 1972, and see also Appendix A1). However we know that the universe is not homogeneous in a small scale and that light ray propagating in the locally inhomogeneous universe is gravitationally affected by clumps intervening between the source and the observer which is known as the gravitational lens effect. Then we have to estimate the distance to a source based on the light ray propagating in the homogeneous universe in a global scale but locally inhomogeneous universe. Unfortunately, however, we have not known the solution of the Einstein field equations describing such a universe yet.

A recipe to obtain a distance-redshift relation without the exact solution of the field equations was presented by Dyer & Roeder (1972, 1973). They gave a distance measure based on light bundle propagating away from all the clumps between the source and the observer, where the rate of shear in the scalar optics theory is negligible (the shear-free assumption). In general, their distance measure is larger than that given by the Mattig formula. Consequently, it yields the dimming effect on the observed flux compared with one based on the Mattig distance measure. They also showed that the distance measure depends on the dimming effect more seriously than the the gravitational magnification effect in the Swiss-cheese model (Bertotti 1966; Dyer & Roeder 1974).

However the shear-free assumption does not always hold for light bundles propagating in the general space-time. In fact Watanabe & Sasaki (1990) showed that when the light ray experiences multiple gravitational scattering due to intervening clumps, the contribution of shear-rate to the distance measure (the Weyl focusing) is comparable with one of matter inside the beam (the Ricci focusing). Taking into account the single gravitational lens effect, Weinberg (1976) showed that an average flux from source in a clumpy universe with a low deacceleration parameter $q_0$ is equal to the flux in the Friedmann-Lemaître universe with the same $q_0$. Peacock (1986) developed Weinberg’s argument to the case with an arbitrary $\Omega_0$ and the smoothness parameter $\bar{\alpha} \sim 1$. Their results mean that the Dyer-Roeder distance multiplied by $\langle \mu \rangle^{-1/2}$ ($\langle \mu \rangle$ is the averaged value of the gravitational magnification factor $\mu$) can be considered to be equivalent to the distance obtained by the Mattig formula. The averaged distance may be useful in analyses such as $N - m$ or $N - z$ relations, in which the gravitational lensing effect must be statistically taken into account (e.g. Omote & Yoshida 1990; Yoshida & Omote 1992).
However, in the analysis of the distance-redshift relation for individual objects the averaged distance is no longer available (Hammer & Nottale 1986). Then we are faced with the problem how we should estimate the distance to a specific object? We can assume that the distance may be given by $\tilde{d} = \mu^{1/2}d$ ($d$ denotes the Dyer-Roeder distance) because the observed flux from the source is brighter by factor $\mu$ than one with no lensing effects in the clumpy universe. Although the above definition of the distance $\tilde{d} = \mu^{1/2}d$ seems reasonable, it is not a trivial problem to show that the distance $\tilde{d}$ to the source is coincident with one obtained from the optical scalar equations, which is the subject that we will discuss in this paper. We will show that a light-path dependent distance $\mu^{-1/2}d$ satisfies the Raychaudhuri equation for a null geodesic and that the rates of expansion and shear in the scalar optics theory can be expressed in terms of quantities of the gravitational lens theory under the thin lens approximation.

In this paper we assume that the universe is described, on average, by the Friedmann-Lemaître model, but locally by the clumpy universe given by Dyer & Roeder (1973). The same relation between the cosmological time and the redshift$^1$ is used in both universe models (the homogeneous universe and the clumpy universe). On the other hand, the distances to an object from an observer and from another object when gravitational lensing effects are absent are assumed to be given, not by the Mattig formula, but by the Dyer-Roeder distance.

The organization of this paper is as follows. After a brief review of the multiple gravitational lens effect in §§2.1, we give a formulation of the continuous limit of the multiple gravitational lens effect (§§2.2). In §§3.1, a light-path dependent angular diameter distance $\tilde{d}$ will be defined, and we will obtain equations satisfied by the elements of the Jacobian matrix $A$ which gives a mapping from an observer plane to a source plane at redshift $z$. In §§3.2, we will investigate relations between quantities in the gravitational lens theory and those in the scalar optics theory. Finally, we shall give the summary and conclusion of this paper.

2. Basic Formulation

In this section we will present a brief review of multiple lensing effects and give a formulation of the continuous limit of the multiple lensing. Furthermore we will derive an equation satisfied by the Jacobian matrix of the multiple gravitational lensings.

$^1$Usually the redshift is given by $k^\mu u_\mu$, where $k^\mu$ and $u^\mu$ are the tangent to a null geodesic and an observer’s four velocity. Therefore the redshift of a object depends on the distribution of inhomogeneities. However, here, we ignore such dependence of the redshift (Dyer & Roeder 1974).
We assume that our universe is described by the Friedmann-Lemaître model in the very large scale (see Appendix A1), but by the clumpy model (Dyer & Roeder 1972, 1973) in the small scale. The fraction of the smoothly distributed matter to the average density \( \bar{\rho} \) of the universe is \( \bar{\alpha} \) (the smoothness parameter), and the rest part of the matter in the universe is assumed to be concentrated into the clumps which act as the gravitational lenses. In this model the angular diameter distance \( D(z_1; z_2) \) from a specific object at redshift \( z_1 \) to another at \( z_2 \) is given by the Dyer-Roeder angular diameter distance (see Appendix A2). We introduce an dimensionless angular diameter distance from \( z_1 \) to \( z_2 \) as \( d(z_1; z_2) = D(z_1; z_2)/(c/H_0) \), where \( H_0 \) is the Hubble constant.

### 2.1. Multiple lens equations

Suppose that there are \( N \) lenses which are randomly distributed at redshift \( z_i (0 \leq z_1 < z_2 < \cdots < z_N) \) in the universe and that \( z_S = z_{N+1} > z_N \) is a redshift of a source. We set lens planes perpendicular to the line of sight at each redshift. The origin on each lens plane is located on the point intersected with the line of sight.

The multi-plane lens equation for the source is given by (see, e.g. Schneider et al. 1992; Yoshida, Nakamura & Omote 2004)

\[
\theta_S = \theta_1 - \sum_{j=1}^{N} \frac{d(z_j; z_S)}{d(0; z_S)} \alpha_j(D(0; z_j)\theta_j)
\]  

(1)

where \( \theta_j \) is the angular position on the \( i \)-th lens plane of the light from the source, and \( \alpha_j \) denotes the deflection angle due to the \( i \)-th lens. It is useful to introduce a renormalized deflection angle \( \tilde{\alpha}_j \) defined by \( \tilde{\alpha}_j = d(0; z_J)\alpha_j \). In equation (1) \( \theta_i \) are recursively expressed as follows:

\[
\theta_i = \theta_1 - \sum_{j=1}^{i} \frac{d(z_j; z_i)}{d(0; z_j)d(0; z_i)} \tilde{\alpha}_j(\theta_j), \quad \text{for } N \geq i > 2
\]  

(2)

\[
\theta_2 = \theta_1 - \frac{d(z_2; z_1)}{d(0; z_2)d(0; z_1)} \tilde{\alpha}_1(\theta_1). \quad \text{for } i = 2
\]  

(3)

From the above equations we see that \( \theta_i \) in equation (2) can be regarded as a source at redshift \( z_i \) for the foreground lenses. Furthermore, we can reduce a relation between \( \theta_{i-1}, \theta_i \) and \( \theta_{i+1} \) as follows:

\[
\frac{\theta_{i+1} - \theta_i}{\chi_i - \chi_{i+1}} - \frac{\theta_i - \theta_{i-1}}{\chi_{i-1} - \chi_i} = -(1 + z_i)\tilde{\alpha}_i(\theta_i),
\]  

(4)

where \( \chi_i \) is defined in the Appendix A3. The same equation is also derived from the time delay of the multiple lensing effects with the Fermat principle (Blandford & Narayan 1986).
The deflection angle $\tilde{\alpha}_i$ due to the $i$-th lens in the clumpy universe is given by
\begin{equation}
\tilde{\alpha}_i(\theta_i) = \frac{3}{2} \Omega_0 \frac{(1 + z_i)^2 d^2(0; z_i) \Delta z_i}{Y(z_i)} \frac{1}{\pi} \int_{\mathcal{D}} d^2 \theta' \Delta_{\delta}(\theta', Z(z_i)) \frac{\theta_i - \theta'}{|\theta_i - \theta'|^2},
\end{equation}
where $\Delta z_i = z_i - z_{i-1}$, $Z(z) = c[T_0 - T(z)]$ ($T_0$ and $T(z)$ are the present cosmological time and a cosmological time when the light emitted from the source at $z$, respectively) and $\Delta_{\delta}(\theta', Z(z_i))$ defined by
\begin{equation}
\delta_{\delta}(D(0; z)\theta, Z(z)) \equiv \delta_{\delta}(D(0; z)\theta, Z(z)) - \tilde{\alpha}(z) \equiv \rho(1 + z)^3 \Delta_{\delta}(\theta, z)
\end{equation}
is an inhomogeneity of the matter distribution on the $i$-th lens plane.

From equation (4) it can be shown that the Jacobian matrix $\mathbf{A}_i = \partial \theta_i / \partial \theta_1$ satisfies the differential equation
\begin{equation}
\frac{\mathbf{A}_{i+1} - \mathbf{A}_i}{\chi_i - \chi_{i+1}} - \frac{\mathbf{A}_i - \mathbf{A}_{i-1}}{\chi_{i-1} - \chi_i} = -(1 + z_i) \tilde{\mathbf{U}}(\theta_i) \mathbf{A}_i,
\end{equation}
where $\tilde{\mathbf{U}}$ is defined as
\begin{equation}
\tilde{\mathbf{U}}(\theta_i) = \frac{\partial \mathbf{A}_i(\theta_i)}{\partial \theta_i} = \frac{3}{2} \Omega_0 \frac{(1 + z_i)^2 d^2(0; z_i) \Delta z_i}{Y(z_i)} \frac{1}{\pi} \int_{\mathcal{D}} d^2 \theta' \Delta_{\delta}(\theta', Z(z_i)) \tilde{\mathbf{U}}'(\theta_i - \theta'),
\end{equation}
\begin{equation}
\tilde{\mathbf{U}}'(\eta) \equiv \left( \frac{\pi \delta^{(2)}(\eta) - \Gamma_x(\eta)}{\pi \delta^{(2)}(\eta) + \Gamma_x(\eta)}, \frac{-\Gamma_y(\eta)}{\pi \delta^{(2)}(\eta) + \Gamma_x(\eta)} \right).
\end{equation}
In equation (9) $\Gamma = (\Gamma_x, \Gamma_y)$ denotes the shear-term due to the $i$-th lens, and is given by
\begin{equation}
\Gamma_x(\eta) = \frac{\eta_x^2 - \eta_y^2}{|\eta|^4}, \quad \Gamma_y(\eta) = \frac{2\eta_x \eta_y}{|\eta|^4}.
\end{equation}
Seitz & Schneider (1992) derived a recursive formula of the Jacobian matrix $\mathbf{A}_i$ equivalent to equation (7). However, their formula is not useful to derive a differential equation of the Jacobian matrix by taking a continuous limit, as mentioned below.

### 2.2. Continuous limit of the multiple lens effect

In above equations (4) and (7), we take the limit of $z_{i+1} \rightarrow z_i (\Delta z_i \rightarrow 0)$ and obtain the following differential equations for $\theta(z)$ and $\mathbf{A}(z)$
\begin{equation}
\frac{d}{dz} \left[ \frac{dz}{d\chi} \frac{d}{dz} \theta(z) \right] = \frac{3}{2} \Omega_0 \frac{(1 + z)^3 d^2(0; z) 1}{Y(z)} \frac{1}{\pi} \int_{\mathcal{D}} d^2 \theta' \Delta_{\delta}(\theta', Z(z)) \frac{\theta(z) - \theta'}{|\theta(z) - \theta'|^2},
\end{equation}
\begin{equation}
\frac{d}{dz} \left[ \frac{dz}{d\chi} \frac{d}{dz} \mathbf{A}(\theta(z)) \right] = \left[ \frac{3}{2} \Omega_0 \frac{1}{Y(z)} \frac{(1 + z)^3 d^2(0; z) 1}{\pi} \int_{\mathcal{D}} d^2 \theta' \Delta_{\delta}(\theta', Z(z)) \tilde{\mathbf{U}}'(\theta(z) - \theta') \right] \mathbf{A}(\theta(z)).
\end{equation}
The equation (11) has a formal solution
\[
\theta(z) = \theta_0 - \frac{3}{2} \Omega_0 \int_0^z d\zeta \frac{(1 + \zeta^2) d(0; \zeta) d(\zeta; z)}{d(0; z) Y(\zeta)} \frac{1}{\pi} \int_D d^2 \theta' \Delta_\alpha(\theta', Z(\zeta)) \frac{\theta(\zeta) - \theta'}{|\theta(\zeta) - \theta'|^2},
\]
provided that \( \theta(z) \) satisfies initial conditions \( \theta(0) = \theta_0 \) and \( d\theta/dz|_{z=0} = 0 \) (see Appendix B1). If the intervening lenses distribution is known, \( \theta(z) \) can be obtained in the form of a power series with \( \Omega_0 \). A source at \( \theta(z) \) in the no lenses case \( (\Delta_\alpha(\theta', Z(z)) = 0) \) would be detected in the direction to the angular position \( \theta_0 \).

We can also obtain the following formal solution of equation (12) with the initial conditions \( A(\theta(0)) = I \) \((2 \times 2 \) unit matrix) and \( dA/dz|_{z=0} = 0 \) \((2 \times 2 \) zero matrix),
\[
A(\theta(z)) = I - \frac{3}{2} \Omega_0 \int_0^z d\zeta \frac{d(0; \zeta) d(\zeta; z)(1 + \zeta)^2}{d(0; z) Y(\zeta)} \hat{U}(\theta(\zeta), \zeta) A(\zeta),
\]
where the matrix \( \hat{U} \) is given in terms of its elements as follows:
\[
\hat{U}(\theta(z), z) = \frac{1}{\pi} \int_D d^2 \theta' \Delta_\alpha(\theta', z) \hat{U}'(\theta(z) - \theta')
\]
\[
= \begin{pmatrix}
\Delta_\alpha(\theta(z), z) - \gamma_x(\theta(z), z) & -\gamma_y(\theta(z), z) \\
-\gamma_y(\theta(z), z) & \Delta_\alpha(\theta(z), z) + \gamma_x(\theta(z), z)
\end{pmatrix}.
\]

In equation (15) \( \gamma(\theta(z), z) \) is defined by
\[
\gamma(\theta(z), z) = \frac{1}{\pi} \int_D d^2 \theta' \Delta_\alpha(\theta', z) \Gamma(\theta(z) - \theta').
\]

As described in Appendix B2, equation (14) can be rewritten in the form:
\[
A(\theta(z)) = I - \frac{3}{2} \Omega_0 \int_{\chi(0)}^{\chi(z)} d\chi_1 \int_{\chi(0)}^{\chi_1} d\chi_2 (1 + \zeta_2)^2 d^4(0; \zeta_2) \hat{U}(\theta(\zeta_2), \zeta_2) A(\theta(\zeta_2)).
\]

Using this expression, we can easily obtain the second derivative of \( A \) with respect to \( \chi(z) \):
\[
A''(\chi) = -W''(\chi) A(\chi),
\]
where the matrix \( W \) is given by
\[
W(\chi) = \frac{3}{2} \Omega_0 \int_{\chi(0)}^{\chi(z)} d\chi_1 \int_{\chi(0)}^{\chi_1} d\chi_2 (1 + \zeta_2)^2 d^4(0; \zeta_2) \hat{U}(\theta(\zeta_2))
\]
\[
= \begin{pmatrix}
\kappa(\chi) - S_x(\chi) & -S_y(\chi) \\
-S_y(\chi) & \kappa(\chi) + S_x(\chi)
\end{pmatrix}.
\]
In equation (19) the convergence $\kappa(\chi)$ and the shear $S(\chi)$ produced by a single lensing are defined as follows:

$$
\kappa(\chi) = \frac{3}{2} \Omega_0 \int_{\chi(0)}^{\chi(z)} d\chi_1 \int_{\chi(0)}^{\chi_1} d\chi_2 (1 + \zeta_2)^5 d^4(0; \zeta_2) \Delta_a(\theta(\zeta_2)),
$$

$$
S(\chi) = \frac{3}{2} \Omega_0 \int_{\chi(0)}^{\chi(z)} d\chi_1 \int_{\chi(0)}^{\chi_1} d\chi_2 (1 + \zeta_2)^5 d^4(0; \zeta_2) \gamma(\theta(\zeta_2)).
$$

In equation (18), $A(\chi), W(\chi)$ and the prime $'$ denote $A(\theta(z(\chi))), W(\theta(z(\chi)))$ and the derivative with respect to $\chi$, respectively. Appendix B2 gives a proof that equation (18) is the equivalent to equation (12). In evaluation of $A$, we must notice that $A$ depends on $\theta(z)$ which is the solution of equation (11).

3. Magnification Factor and the Optical Scalar Equations

3.1. Jacobian matrix

In the clumpy universe (Dyer & Roeder 1973) if the light ray is not affected by any clump which intervenes between a source and the observer, the flux $f_0$ from the source at redshift $z$ with absolute luminosity $L$ would be detected as

$$
f_0 = \frac{L}{4\pi (c/H_0)^2 (1 + z)^4 d^2(0; z)}. \quad (22)
$$

However, in the real case, observers detect a light ray which is gravitationally lensed by the clumps. Therefore the observed flux $f_{\text{obs}}$ is given by $f_{\text{obs}} = \mu f_0$, where $\mu$ is the gravitational magnification factor defined by $\mu = \det A^{-1}$. It is important to notice that $\mu$ depends on the light-path. Since observers recognize that the angular diameter distance to the source with flux $f_{\text{obs}}$ is given by $(H_0/c) \times L^{1/2}[4\pi (1 + z)^4 f_{\text{obs}}]^{-1/2}$, the new angular diameter distance $\tilde{d}$ is defined by

$$
\tilde{d}(\theta(z)|0; z) \equiv \mu^{-1/2}(\theta(z))d(0; z). \quad (23)
$$

We should notice that $\tilde{d}$ has a dependence on the light-path. Then, even if two sources have a same redshift and a same absolute luminosity, their distances are not always reduced to be same values. In the below we investigate the equation of the new distance $\tilde{d}$. Hereafter we assume $\mu \neq 0$, i.e. the light ray considered here does not pass through a conjugate point (Wald 1984).

For this purpose we need, first, to obtain an equation of the gravitational magnification factor $\mu$. In general, Jacobian matrix $A$ from the observe plane to a source plane at redshift
z is expressed as
\[ A = \begin{pmatrix} K_x + G_x & G_y + K_y \\ G_y - K_y & K_x - G_x \end{pmatrix} = K_x \sigma_0 + G_y \sigma_1 + iK_y \sigma_2 + G_x \sigma_3, \tag{24} \]
where \( \sigma_i \)'s are the Pauli matrices:
\[ \sigma_0 = I, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25} \]

In this expression, the magnification factor \( \mu \) is given by
\[ \mu = \det A^{-1} = (K \cdot K - G \cdot G)^{-1}, \]
where \( K, G \) denote the cumulative convergence-, twist- and shear-terms, respectively. In a single lensing case, \( A \) is symmetric, i.e., the twist-term \( K_y \) always vanishes. On the other hand, in the multiple lensing case, it does not vanish, in general.

According to Seitz et al. (1994), we can define an optical deformation matrix \( Q \) in terms of \( A \) and its derivative with respect to \( \chi \) as follows:
\[ Q \equiv A' A^{-1}, \tag{26} \]
which gives
\[ A' = QA. \tag{27} \]

Inserting equation (24) into equation (26), we obtain an explicit form of the optical deformation matrix:
\[ Q = (\ln \mu^{-1/2})' \sigma_0 + \varepsilon_{ab} \left( \hat{K}_a \hat{G}_b' - \hat{K}_a' \hat{G}_b \right) \sigma_1 + i\varepsilon_{ab} \left( \hat{K}_a \hat{K}_b' + \hat{G}_a' \hat{G}_b \right) \sigma_2 + \delta_{ab} \left( \hat{K}_a \hat{G}_b' - \hat{K}_a' \hat{G}_b \right) \sigma_3, \tag{28} \]
where \( \varepsilon_{xy} = \varepsilon_{yy} = 0, \varepsilon_{yx} = -\varepsilon_{yx} = 1, \hat{K}_a \equiv \mu^{1/2} K_a \) and \( \hat{G}_a \equiv \mu^{1/2} G_a \).

Using a relation (18), the definition of \( Q \) and its derivative with respect to \( \chi \), we can obtain a relation between \( Q \) and \( \mathbf{W}'' \) as follows
\[ Q' + Q^2 = -\mathbf{W}'', \tag{29} \]
where \( \mathbf{W}'' \) is written in terms of the Pauli matrices as \( \mathbf{W}'' = \kappa'' \sigma_0 - S''_y \sigma_1 - S''_x \sigma_3 \). Using equations (28) and (29), we find equations of the coefficients of the Pauli matrices in \( Q \) in the following
\[ (\ln \mu^{-1/2})'' + \left[ (\ln \mu^{-1/2})' \right]^2 + \left\{ \varepsilon_{ab} \left( \hat{K}_a \hat{G}_b' - \hat{K}_a' \hat{G}_b \right) \right\}^2 \]
Furthermore, inserting equations (37), (38) into equations (31)-(33), we find that
\[\delta_{ab} \left( \hat{K}_a \hat{G}_b' - \hat{K}'_a \hat{G}_b \right) - \varepsilon_{ab} \left( \hat{K}_a \hat{K}_b' + \hat{G}_a' \hat{G}_b' \right) = -\kappa'', \quad (30)\]
\[\varepsilon_{ab} \left( \hat{K}_a \hat{G}_b' - \hat{K}'_a \hat{G}_b \right) + 2 \left( \ln \mu^{-1/2} \right)' \left[ \delta_{ab} \left( \hat{K}_a \hat{G}_b - \hat{K}'_a \hat{G}_b' \right) \right] = S_x'', \quad (31)\]
\[\varepsilon_{ab} \left( \hat{K}_a \hat{G}_b' - \hat{K}'_a \hat{G}_b \right) + 2 \left( \ln \mu^{-1/2} \right)' \left[ \varepsilon_{ab} \left( \hat{K}_a \hat{G}_b' - \hat{K}'_a \hat{G}_b \right) \right] = S_y'', \quad (32)\]
\[\varepsilon_{ab} \left( \hat{K}_a \hat{K}_b' + \hat{G}_a' \hat{G}_b' \right) + 2 \left( \ln \mu^{-1/2} \right)' \left[ \varepsilon_{ab} \left( \hat{K}_a \hat{K}_b' + \hat{G}_a' \hat{G}_b' \right) \right] = 0. \quad (33)\]

Equation (30) is equivalent to one derived by Seitz et al. (1994). They gave the equation of the magnification factor in the clumpy universe by using a rate of shear derived from a Weyl term of the scalar optics theory in the linearized general relativity. On the other hand, equation (30) is derived by taking the continuous limit of the multiple gravitational lens theory. It should be noticed, therefore, that in our formulation we can obtain the right hand side of equation (30) from data of lens distributions itself, without referring to the Weyl term (the metric terms given by solving the Einstein equation).

The second derivative of \( \bar{d} \) with respect to the affine parameter \( v \) can be expressed as combination of the second derivative of \( d \) with respect to \( v \) and the second derivative of \( \mu^{-1/2} \) with respect to \( \chi \):
\[\frac{d^2}{dv^2} \bar{d}(\theta(z)|0; z) = \mu^{-1/2} \frac{d^2}{dv^2} d(0; z) + \frac{1}{d^2(0; z)} \frac{d^2}{d\chi^2} \mu^{-1/2}(\theta(z)). \quad (34)\]

Combining equation (30) with equation (34), the Dyer-Roeder equation (A13) yields
\[\frac{d}{dv} \tilde{\Theta} + \tilde{\Theta}^2 + \left( \delta_{ab} \tilde{\Sigma}_{ab} \right)^2 + \left( \varepsilon_{ab} \tilde{\Sigma}_{ab} \right)^2 - \left( \varepsilon_{ab} \tilde{\omega}_{ab} \right)^2 = -\frac{4\pi G}{H_0} \rho(D(0; z)\theta(z), Z(z))(1 + z)^2, \quad (35)\]
where
\[\tilde{\Theta} = \frac{d}{dv} \ln \bar{d}(\theta(z)|0; z), \quad (36)\]
\[\tilde{\Sigma}_{ab} = -\frac{\hat{K}_a \hat{G}_b' - \hat{K}'_a \hat{G}_b}{d^2(0; z)}, \quad (37)\]
\[\tilde{\omega}_{ab} = -\frac{\hat{K}_a \hat{K}_b' + \hat{G}_a' \hat{G}_b'}{d^2(0; z)}. \quad (38)\]

Furthermore, inserting equations (37), (38) into equations (31)-(33), we find that
\[\frac{d}{dv} \left( \delta_{ab} \tilde{\Sigma}_{ab} \right) + 2 \tilde{\Theta} \delta_{ab} \tilde{\Sigma}_{ab} = \frac{4\pi G}{H_0^2} \tilde{\rho}(z)(1 + z)^2 \gamma_x(\theta(z)), \quad (39)\]
\[\frac{d}{dv} \left( \varepsilon_{ab} \tilde{\Sigma}_{ab} \right) + 2 \tilde{\Theta} \varepsilon_{ab} \tilde{\Sigma}_{ab} = \frac{4\pi G}{H_0^2} \tilde{\rho}(z)(1 + z)^2 \gamma_y(\theta(z)), \quad (40)\]
and
Here we define \( \bar{\Sigma} = (\delta_{ab} + i\varepsilon_{ab})\Sigma_{ab} \) and \( \bar{\omega} = \varepsilon_{ab}\bar{\omega}_{ab} \) and then rewrite equations (35), (39)-(41) as follows:

\[
\frac{d}{dv} \bar{\Theta} + \bar{\Theta}^2 + |\bar{\Sigma}|^2 - \bar{\omega}^2 = \bar{\mathcal{R}},
\]

(42)

\[
\frac{d}{dv} \bar{\Sigma} + 2\bar{\Theta}\bar{\Sigma} = \bar{\mathcal{F}},
\]

(43)

\[
\frac{d}{dv} \bar{\omega} + 2\bar{\Theta}\bar{\omega} = 0,
\]

(44)

where

\[
\bar{\mathcal{R}} = -\frac{4\pi G}{H_0^2} \rho(D(0; z)\Theta(z)|0; z)(1 + z)^2,
\]

(45)

\[
\bar{\mathcal{F}} = \frac{4\pi G}{H_0^2} \sigma(z) \left[ \gamma_x(\Theta(z)) + i\gamma_y(\Theta(z)) \right] (1 + z)^2
\]

\[
= \frac{4G}{H_0^2} \int d^2\theta' \delta_\alpha \rho(D(0; z)\Theta', Z(z)) \Gamma(\Theta(z) - \Theta')(1 + z)^2,
\]

(46)

and \( \Gamma(z) = \Gamma_x(z) + i\Gamma_y(z) \) denotes the complex shear term due to a lens at redshift \( z \).

From equations (36) and (42), we find that our new distance (23) satisfies the following equation:

\[
\frac{d^2}{dv^2} \tilde{d}(\Theta(z)|0; z) + \left[ |\tilde{\Sigma}|^2 + \frac{4\pi G}{H_0^2} \rho(D(0; z)\Theta(z), Z(z))(1 + z)^2 \right] \tilde{d}(\Theta(z)|0; z) = 0.
\]

(47)

We should notice that we cannot obtain solution of equation (47) without solving equation (11).

### 3.2. Relations between the Jacobian matrix and the optical scalars

Since the cross-sectional area of the light bundle, \( A \), is proportional to square of the angular diameter distance (Dyer & Roeder 1972), we can rewrite \( \bar{\Theta} \) as

\[
\bar{\Theta} = \frac{1}{2} \frac{d}{dv} \ln A,
\]

(48)

which is the rate of expansion in the scalar optics theory. Furthermore, equation (45) can be regarded as the Ricci term, which is given by \( \bar{\mathcal{R}} = -(1/2)R_{\mu\nu}k^\mu k^\nu \). Therefore, we can show that equation (35) or (42) can be regarded as the optical scalar equation of the rate
of expansion, if $\dot{\Sigma}$ and $\dot{\omega}$ correspond to the rates of shear and rotation in the scalar optics theory.

First, let us investigate $\dot{\omega}$. Combining initial conditions of $A$ and definitions of $\dot{\Theta}$ and $\dot{\omega}$, we obtain a solution of (44), $\dot{\omega} = 0$. In the scalar optics, the rotation rate $\omega \equiv \sqrt{\omega_{\alpha\beta}\omega^{\alpha\beta}/2}$ always vanishes, because the covariant derivative of the tangent vector of the null geodesic is symmetric, i.e., $\omega_{\alpha\beta} = \nabla_{[\alpha}k_{\beta]} = 0$. From this fact and equation (44), we can regard $\dot{\omega}$ as the rotation rate in the scalar optics theory. Furthermore, we find that the optical deformation matrix $Q$ is symmetric. In a discrete multiple gravitational lensing theory, the symmetry of a matrix corresponding to the optical deformation matrix was shown by Seitz & Schneider (1992) by means of their recursive formula of the Jacobian matrix. However, in their paper, the reasons of the symmetry was not clarified.

Next, let us notice the right hand side of equation (43), $\tilde{F}$. This is similar to the Weyl term in the scalar optics theory. The difference between $\tilde{F}$ and the Weyl term in the linear perturbation theory of the general relativity (Sasaki 1987; Futamase & Sasaki 1989) exists in the following point: while the Weyl term at arbitrary point $(D(0; z)\theta(z), Z(z))$ on the light-path in the linear perturbation theory has some contribution from the matter in the whole universe outside the light bundle, $\tilde{F}$ in our formalism has a contribution from only the matter on the lens plane with the same redshift $z$. This comes from the fact that our formalism is based on the thin lens approximation. Therefore, we can regard $\tilde{F}$ as the Weyl term in scalar optics theory as long as the thin lens approximation is valid.

Finally we find that equations (42)–(44) are equivalent to the optical scalar equations (A11) in an inhomogeneous universe under the thin lens approximation. Consequently the optical scalars are expressed in terms of the elements of the Jacobian matrix and the Dyer–Roeder angular diameter distance as follows:

$$\Theta = \frac{d}{d\mu} \ln \left[ \mu^{-1/2}d(0; z) \right], \quad \Sigma = -\left( \delta_{ab} + i\epsilon_{ab} \right) \frac{\dot{K}_aG'_b - \dot{K}'_aG_b}{d^2(0; z)}, \quad \omega = -\epsilon_{ab} \frac{\dot{K}_aK'_b + \dot{G}'_aG_a}{d^2(0; z)}$$

(49)

where the last equation is exactly zero.

4. Summary and Conclusions

In this paper we have formulated the continuous limit of the multiple gravitational lens effect by taking the limit in which the number of lens planes is infinite and then the redshift-intervals of lenses are infinitesimal with keeping the present average density $\bar{\rho}_0$ fixed. It, however, does not mean that the number of lenses must be infinite. In this case, we can
interpret that the finite number of lenses have some finite masses in their lens planes and other infinite number of lenses have zero masses in the planes, and that in the zero-mass lens planes, the deflection angles, $\tilde{\alpha}$, should be 0 and the matrices, $\tilde{U}$, should be $I$. The formalism is able to be adopted in the case where there are some finite number of lenses between sources and us.

Using the formalism we found that the new angular diameter distance, $\bar{d} = \mu^{-1/2}d$ satisfies the optical scalar equation of the expansion rate $\tilde{\Theta}$ in an inhomogeneous universe. It is interesting that the influence of inhomogeneities on the angular diameter distance is completely recovered by multiplication of $\mu^{-1/2}$ to the Dyer–Roeder distance, which is obtained by ignoring the influence of all clumps on the light beams.

We also found that quantities defined in the scalar optics theory (the rates of expansion $\Theta$, shear $\Sigma$ and rotation $\omega$) are expressed in terms of quantities defined in the gravitational lens theory (the terms of convergence $K_x$, twist $K_y$ and shear $G$). Seitz & Schneider (1994) showed that the absolute square of the shear rate, $|\Sigma|^2$, can be expressed by some combination of the determinants and traces of the Jacobian matrices in the discrete multiple gravitational lensing theory. It is equivalent to one in our formalism, which is also written as

$$|\tilde{\Sigma}|^2 = \frac{\tilde{G}' \cdot \tilde{G}' - \tilde{K}' \cdot \tilde{K}'}{d^4(0;z)}.$$ 

This expression gives an immediate estimation of the shear rate by tracing the lens equation (11) from the raw data of local distribution of matters.

The formalism includes the framework of the weak gravitational lensing theory (e.g. Bartelmann & Schneider 2001), in which the smoothness parameter $\bar{\alpha}$ is assumed to be unity (the distance measure is given by the Mattig formula and only single gravitational lensing is taken in account). In their framework, when the light ray propagates through some underdense region, $\rho < \bar{\rho}$, the light is assumed to be deflected by a lens with some negative mass (deflecting mass $\delta \rho = \rho - \bar{\rho} < 0$). Therefore such lenses with negative masses give some brightening effect similar to the usual clumps on the flux when the light beam passes away from such lenses, because the shear rate due to the negative mass lenses contributes to the Raychaudhuri equation in the form of $|\delta \rho|^2$. Then there is some possibility that the weak lensing theory does overestimate the distance measure due to the negative lenses unless the convergence due to the negative mass lens in the light beam balances with the shear outside the light beam due to the other ones. In our formalism, unlike the weak lensing theory, we do not need to introduce such lenses with the negative masses in order to take into account the underdense regions in which the matter density is only considered to be $\rho = \bar{\alpha} \bar{\rho}$. Therefore there is not the above possibility because the shear in such underdense region does not contribute to the distance measure.
Finally, we comment on equation (47) which is the generalized Dyer-Roeder equation. Schneider et al. (1992) also mentioned a similar equation, which is slightly different from ours. In their equation, the density of clumps \( \delta \bar{\alpha} \rho = \rho - \bar{\alpha} \bar{\rho} \) is not considered, because they assumed that the light beam does not pass through bound clumps. However, among real situations, there are cases in which the contributions of both the matter density of the bound clump in the light bundle and the shear rate of the light bundle are compatible (Watanabe \& Sasaki 1990). Even in such intermediate case, equation (47) gives the angular diameter distance to a target source, by solving the lens equations (11) based on observed distribution of foreground clumps.

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A. Relations between redshift \( z \), affine parameter \( v \) and \( \chi \)-function

In this appendix, we briefly describe the relation between \( z \) (redshift), \( v \) (affine parameter) and \( \chi \) (\( \chi \)-function).

A.1. Background universe

We assume that the universe is described by the Friedmann-Lemaître universe on average. Then the geometry on average is expressed as

\[
ds^2 = c^2dT^2 - a^2(T) \left[ \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right], \tag{A1}\n\]

where \( T \) is the cosmological time, \( a(T) \) denotes the scale factor of the universe which has the dimension of distance and \( k \) is related with the curvature of the universe. In this geometry, the Einstein field equations can be written as

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}, \tag{A2}\n\]

\[
\Lambda c^2 = 2 \frac{\ddot{a}}{a} + \left( \frac{\dot{a}}{a} \right)^2 + \frac{kc^2}{a^2}, \tag{A3}\n\]
where $\bar{\rho}$ and $\Lambda$ are the mean density of the universe and the cosmological constant, respectively, and the dot denotes the derivative with $T$. Since $a(T)$ and $\bar{\rho}$ are given in terms of their present values $a_0$ and $\bar{\rho}_0$ by
\[
\frac{\dot{a}}{a} = \Omega_0 (1 + z)^3 + \lambda_0 - K (1 + z)^2 \equiv H_0^2 Y^2(z),
\]
(A4)
\[
-\frac{\ddot{a}}{a^2} = q(z) = \frac{\Omega_0 (1 + z)^3}{Y^2(z)},
\]
(A5)
where $H_0$ is the present Hubble constant, $\Omega_0 = 8\pi G \bar{\rho}_0 / 3 H_0^2$, $\lambda_0 = \Lambda c^2 / 3 H_0^2$ and $K = kc^2 / a_0^2 H_0^2$. The above equations gives relations between $\Omega_0, \lambda_0$ and $K$:
\[
K = \Omega_0 + \lambda_0 - 1,
\]
(A6)
\[
q_0 = \frac{1}{2} \Omega_0 - \lambda_0,
\]
(A7)
where $q_0$ is the present deacceleration parameter defined as $-(\ddot{a}a/\dot{a}^2)_0$. It follows from equation (A4) that the relation between $T$ and $z$ is given by:
\[
-cdT = \frac{c}{H_0 (1 + z) Y(z)}.
\]
(A8)
Furthermore, in the Friedmann-Lemaître universe, an affine parameter $v$ defined along a null geodesic is given in terms of redshift $z$ by
\[
dv = \frac{dz}{(1 + z)^2 Y(z)}.
\]
(A9)
In this universe, the angular diameter distance $D_{\text{FL}}(z_2; z_1)$ from $z_1$ to $z_2$ is given as
\[
D_{\text{FL}}(z_2; z_1) \equiv a(T_1) r(z_2; z_1) = \frac{c}{H_0 \sqrt{K (1 + z_1)}} \sin \left[ \sqrt{K} \int_{z_2}^{z_1} \frac{dz}{Y(z)} \right].
\]
(A10)
This is the Mattig formula.

**A.2. The Dyer-Roeder distance**

Sachs (1961) gave geometrical equations of light bundle propagating in the general space-time from the geodesic deviation for the light ray as follows:
\[
\frac{d}{dv} \Theta + \Theta^2 + |\Sigma|^2 = \mathcal{R}, \quad \frac{d}{dv} \Sigma + 2 \Sigma \Theta = \mathcal{F},
\]
(A11)
where $\Theta$ and $\Sigma$ denote the rates of expansion and shear factor of the light bundle, $R$ and $F$ are the Ricci and Weyl terms. The rate of expansion $\Theta$ is defined as $\Theta = d \ln \sqrt{A}/dv$, where $A$ denotes the cross-sectional area of the light bundle. Since the angular diameter distance $D$ to a source at redshift $z$ is proportional to $\sqrt{A}$, equations (A11) are rewritten as

$$\frac{d^2}{dv^2} D = (R - |\Sigma|^2) D, \quad \frac{1}{D^2} \frac{d}{dv} (D^2 \Sigma) = F. \quad (A12)$$

In general it follows that the angular diameter distance depends not only on the Ricci term but also on the Weyl term, where the Weyl term is due to the inhomogeneities outside the light bundle while the Ricci term originates from the matter density in the light bundle.

Dyer & Roeder (1972, 1973) assumed that for the light beams far away from all the inhomogeneities, the contribution of $\Sigma$ to the distance $D$ from the source is negligible and that the matter-density in the light bundle is given by $\bar{\alpha}\bar{\rho}$, where $\bar{\alpha}$ (smoothness parameter) is the fraction of the density smoothly distributed in the universe to the mean density of the universe, and derived the angular diameter distance in the clumpy universe, which satisfies the following equation:

$$\frac{d^2}{dv^2} D + 4\pi G \frac{\bar{\rho}}{H_0^2}(1 + z)^2 D = 0. \quad (A13)$$

The relation (A9) between the affine parameter along the light ray and redshift allows us to express equation (A13) as follows:

$$\frac{d}{dz} \left[ (1 + z)^2 Y(z) \frac{d}{dz} d(z_0; z) \right] + \frac{3}{2} \Omega_0 \bar{\alpha} (1 + z)^3 d(z_0; z) = 0, \quad (A14)$$

where $d(z_0; z)$ denotes the dimensionless angular diameter distance $D(z_0; z)/(c/H_0)$ from an object at redshift $z_0$ to another at $z (z \leq z_0)$. The initial conditions at $z = z_0$ are given by

$$d(z_0; z) \bigg|_{z = z_0} = 0,$$

$$\frac{d}{dz} d(z_0; z_b) \bigg|_{z = z_0} = \frac{1}{(1 + z_0) Y(z_0)}, \quad (A15)$$

(Dyer & Roeder 1973), where the last condition is the Hubble law at redshift $z_0$. In the case with $\bar{\alpha} = 1$, equation (A10) is a solution of the Dyer-Roeder equation.

A.3. The $\chi$-function

The $\chi$-function is introduced by Schneider et al. (1992) as follows

$$\chi(z) \equiv \int_0^\infty \frac{dz}{(1 + z)^2 Y(z) d^2(0; z)}, \quad (A16)$$
where $d(0; z)$ is the dimensionless Dyer-Roeder angular diameter distance. The relation between $\chi$, $v$ and $z$ is then expressed as

$$d\chi = -\frac{dv}{d^2(0; z)} = -\frac{dz}{(1 + z)^2Y(z)d^2(0; z)}.$$  \hspace{1cm} (A17)

Accordingly we can use the $\chi$-function as a variable instead of redshift $z$ or affine parameter $v$.

The deference between values of $\chi$ at redshifts $z_i$ and at $z_j$ ($z_i < z_j$) has an interesting relation to the Dyer-Roeder angular diameter distance:

$$\chi(z_i; z_j) \equiv \chi(z_i) - \chi(z_j) = \frac{d(z_i; z_j)}{(1 + z_i)d(0; z_i)d(0; z_j)}. \hspace{1cm} (A18)$$

In the rest part of this subsection we will prove this relation. First of all, we investigate the differential equation which $d(0; z)\chi(z_i; z)$ satisfies. Let $f(z)$ denote $d(0; z)\chi(z_i; z)$ and differentiate it with respect to affine parameter $v$:

$$\frac{d}{dv}f(z) = \left[\frac{d}{dv}d(0; z)\right]\chi(z_i; z) + \frac{dz/dv}{(1 + z)^2Y(z)d(0; z)} = \left[\frac{d}{dv}d(0; z)\right]\chi(z_i; z) + \frac{1}{d(0; z)}. \hspace{1cm} (A19)$$

The second derivative of $f(z)$ with respect to $v$ is also given by

$$\frac{d^2}{dv^2}f(z) = \left[\frac{d^2}{dv^2}d(0; z)\right]\chi(z_i; z) = -\frac{4\pi G}{H_0^2}\bar{\alpha}\bar{\rho}(z)(1 + z)^2f(z). \hspace{1cm} (A20)$$

Therefore we find that $f(z)$ is a solution of the Dyer-Roeder equation (A13).

Next we investigate what initial conditions $f(z)$ satisfies. It is evident from equations (A18) and (A19) that $f(z_i) = 0$ and

$$\frac{d}{dv}f(z)\bigg|_{z=z_i} = \frac{dz/dv|_{z=z_i}}{(1 + z_i)^2Y(z_i)d(0; z_i)} = \frac{1}{d(0; z_i)}.$$  \hspace{1cm} (A21)

From equations (A20) and (A21), we find that $(1 + z_i)d(0; z_i)f(z)$ is equivalent to $d(z_i; z)$.

The proof of equation (A18) is thus completed. Functions similar to the $\chi$-function are defined in different forms by several authors (Nottale & Hammer 1984; Padmanabhan & Subramanian 1988) for their own purposes. Nottale & Hammer defined a distance as an optical distance, which is equivalent to the inverse of the $\chi$-function. Padmanabhan & Subramanian defined the function as an independent solution of the usual Mattig distance which satisfies the Raychaudhuri equation of the null geodesic in the homogeneous universe model.
B. Some Proofs

B.1. Equation (13) is a solution of equation (11)

In this appendix, we prove that in the continuous limit, the angular position $\theta(z)$ of images in the lensing plane at redshift $z$ satisfies equation (11). First of all, we differentiate equation (13) with respect to redshift $z$ and then obtain

$$\frac{d}{dz}\theta(z) = -\frac{3}{2} \Omega_0 \int_0^z d\zeta \frac{(1 + \zeta)^2 d(0; \zeta)}{Y(\zeta)} \left[ \frac{d}{dz} \left\{ \frac{d(\zeta; z)}{d(0; z)} \right\} \right] \frac{1}{\pi} \int_D d^2\theta' \Delta_\alpha(\theta', Z(\zeta)) \frac{\theta(\zeta) - \theta'}{|\theta(\zeta) - \theta'|^2}. \tag{B1}$$

After multiplying $dz/d\chi = -(1 + z)^2 d^2(0; z) Y(z)$ to both sides of this equation, we differentiate it with respect to $z$, again.

$$\frac{d}{dz} \left[ \frac{dz}{d\chi} \frac{d\theta(z)}{dz} \right] = \frac{3}{2} \Omega_0 (1 + z)^4 d^3(0; z) \left[ \frac{d}{dz} \left\{ \frac{d(\zeta; z)}{d(0; z)} \right\} \right] \bigg|_{\zeta = z}$$

$$+ \frac{3}{2} \Omega_0 \int_0^z d\zeta \frac{(1 + \zeta)^2 d(0; \zeta)}{Y(\zeta)} \frac{d}{dz} \left[ (1 + z)^2 d^2(0; z) Y(z) \frac{d}{dz} \left\{ \frac{d(\zeta; z)}{d(0; z)} \right\} \right]$$

$$\times \frac{1}{\pi} \int_D d^2\theta' \Delta_\alpha(\theta', Z(\zeta)) \frac{\theta(\zeta) - \theta'}{|\theta(\zeta) - \theta'|^2}. \tag{B2}$$

Here, we apply the initial conditions of the Dyer-Roeder angular diameter distance, (A15), to the first term of the right hand side of equation (B2) to obtain

$$\left[ \frac{d}{dz} \left\{ \frac{d(\zeta; z)}{d(0; z)} \right\} \right] \bigg|_{\zeta = z} = \frac{1}{(1 + z) Y(z) d(0; z)}. \tag{B3}$$

In the second term of the right hand side of equation (B2), equation (A9) provides that

$$\frac{d}{dz} \left\{ (1 + z)^2 d^2(0; z) Y(z) \frac{d}{dz} \left\{ \frac{d(\zeta; z)}{d(0; z)} \right\} \right\}$$

$$= \frac{1}{(1 + z)^2 Y(z)} \frac{d}{dv} \left[ d^2(0; z) \frac{d}{dv} \left\{ \frac{d(\zeta; z)}{d(0; z)} \right\} \right]$$

$$= \frac{1}{(1 + z)^2 Y(z)} \left[ d(0; z) \frac{d^2}{dv^2} \frac{d(\zeta; z)}{d(0; z)} - d(\zeta; z) \frac{d^2}{dv^2} d(0; z) \right]. \tag{B4}$$

vanishes because both $d(0; z)$ and $d(\zeta; z)$ are solutions of the same equation (A13). Finally we find that equation (B2) is rewritten as

$$\frac{d}{dz} \left[ \frac{dz}{d\chi} \frac{d\theta(z)}{dz} \right] = \frac{3}{2} \Omega_0 (1 + z)^3 d^2(0; z) \frac{1}{Y(z)} \frac{1}{\pi} \int_D d^2\theta' \Delta_\alpha(\theta', Z(\zeta)) \frac{\theta(\zeta) - \theta'}{|\theta(\zeta) - \theta'|^2}. \tag{B5}$$

Similarly we can prove that equation (14) satisfies equation (12).
B.2. About equation (17)

In this appendix, we prove that if arbitrary function \( g(z) \) has a finite value at \( z = 0 \),

\[
\int_0^z d\zeta \frac{(1 + \zeta)^2 d(0; \zeta) d(\zeta; z)}{d(0; z) Y(z)} g(\zeta) = \int_{\chi(0)}^{\chi(z)} d\chi_1 \int_{\chi(0)}^{\chi(z_1)} d\chi_2 (1 + \zeta_2)^5 d^4(0; \zeta_2) g(\zeta_2),
\]

(B6)

where \( \chi_{1,2} = \chi(\zeta_{1,2}) \) defined in the Appendix A3. This relation is used in the text when equation (17) is derived from equation (14).

To begin with, we replace \( d(\zeta; z) \) in the left hand side of (B6) by the \( \chi \)-function by using relation (A18):

\[
\int_0^z d\zeta \frac{(1 + \zeta)^2 d(0; \zeta) d(\zeta; z)}{d(0; z) Y(z)} g(\zeta) = \int_0^z d\zeta (1 + \zeta)^3 d^2(0; \zeta) \frac{\chi(\zeta) - \chi(z)}{Y(\zeta)} g(\zeta)
\]

\[
= \int_0^z d\zeta \frac{(1 + \zeta)^3 d^2(0; \zeta)}{Y(\zeta)} g(\zeta) - \chi(z) \int_0^z d\zeta \frac{(1 + \zeta)^3 d^2(0; \zeta)}{Y(\zeta)} g(\zeta)
\]

\[
= - \int_{\chi(0)}^{\chi(z)} d\chi_1 \chi_1 (1 + \zeta_1)^5 d^4(0; \zeta_1) g(\zeta_1) + \chi(z) \int_{\chi(0)}^{\chi(z)} d\chi_1 (1 + \zeta_1)^5 d^4(0; \zeta_1) g(\zeta_1).
\]

(B7)

In equation (B7), we introduce a function

\[
h(\chi) \equiv \int_{\chi(0)}^{\chi(z)} d\chi_1 (1 + \zeta_1)^5 d^4(0; \zeta_1) g(\zeta_1),
\]

and rewrite equation (B7) as

\[
\int_0^z d\zeta \frac{(1 + \zeta)^2 d(0; \zeta) d(\zeta; z)}{d(0; z) Y(z)} g(\zeta) = \chi h(\chi) - \int_{\chi(0)}^{\chi(z)} d\chi_1 \frac{d}{d\chi_1} h(\chi_1)
\]

\[
= \chi(0) h(\chi(0)) + \int_{\chi(0)}^{\chi(z)} d\chi_1 h(\chi_1),
\]

(B8)

where we define \( \chi(0) h(\chi(0)) \) as a value of \( \chi(\epsilon) h(\chi(\epsilon)) \) in the limit of \( \epsilon \to 0 \). The behaviors of \( \chi(\epsilon) \) and \( h(\chi(\epsilon)) \) for small \( \epsilon \) are \( \chi(\epsilon) \sim O(1/\epsilon) \) and \( h(\chi(\epsilon)) \sim \epsilon^3 g(\epsilon) \). We thus find that \( \chi(\epsilon) h(\chi(\epsilon)) \sim \epsilon^2 g(\epsilon) \to 0 \) if \( g(0) \) is finite. Finally we have completed the proof of equation (B6).

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