Avoiding long Berge cycles

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Abstract

Let \( n \geq k \geq r + 3 \) and \( H \) be an \( n \)-vertex \( r \)-uniform hypergraph. We show that if 
\[
|H| > \frac{n-1}{k-2} \binom{k-1}{r}
\]
then \( H \) contains a Berge cycle of length at least \( k \). This bound is tight when \( k-2 \) divides \( n-1 \).

We also show that the bound is attained only for connected \( r \)-uniform hypergraphs in which every block is the complete hypergraph \( K_{k-1}^r \).

We conjecture that our bound also holds in the case \( k = r + 2 \), but the case of short cycles, \( k \leq r + 1 \), is different.

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1 Definitions, Berge \( F \) subhypergraphs

An \( r \)-uniform hypergraph, or simply \( r \)-graph, is a family of \( r \)-element subsets of a finite set. We associate an \( r \)-graph \( H \) with its edge set and call its vertex set \( V(H) \). Usually we take \( V(H) = [n] \), where \([n] \) is the set of first \( n \) integers, \([n] := \{1, 2, 3, \ldots, n\} \). We also use the notation \( H \subseteq \binom{[n]}{r} \).

Definition 1.1 (Anstee and Salazar [1], Gerbner and Palmer [5]). For a graph \( F \) with vertex set \( \{v_1, \ldots, v_p\} \) and edge set \( \{e_1, \ldots, e_q\} \), a hypergraph \( H \) contains a Berge \( F \) if there exist distinct vertices \( \{w_1, \ldots, w_p\} \subseteq V(H) \) and edges \( \{f_1, \ldots, f_q\} \subseteq E(H) \), such that if \( e_i = v_{i_1}v_{i_2} \), then \( \{w_{i_1}, w_{i_2}\} \subseteq f_i \). The vertices \( \{w_1, \ldots, w_p\} \) are called the base vertices of the Berge \( F \).

Of particular interest to us are Berge cycles.

Definition 1.2. A Berge cycle of length \( \ell \) in a hypergraph is a set of \( \ell \) distinct vertices \( \{v_1, \ldots, v_\ell\} \) and \( \ell \) distinct edges \( \{e_1, \ldots, e_\ell\} \) such that \( \{v_i, v_{i+1}\} \subseteq e_i \) with indices taken modulo \( \ell \).

A Berge path of length \( \ell \) in a hypergraph is a set of \( \ell + 1 \) vertices \( \{v_1, \ldots, v_{\ell+1}\} \) and \( \ell \) hyperedges \( \{e_1, \ldots, e_\ell\} \) such that \( \{v_i, v_{i+1}\} \subseteq e_i \) for all \( 1 \leq i \leq \ell \).

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Let $\mathcal{H}$ be a hypergraph and $p$ be an integer. The $p$-shadow, $\partial_p \mathcal{H}$, is the collection of the $p$-sets that lie in some edge of $\mathcal{H}$. In particular, we will often consider the 2-shadow $\partial_2 \mathcal{H}$ of a $r$-uniform hypergraph $\mathcal{H}$ in which each edge of $\mathcal{H}$ yields a clique on $r$ vertices.

2 Background

Erdős and Gallai [3] proved the following result on the Turán number of paths.

**Theorem 2.1** (Erdős and Gallai [3]). Let $k \geq 2$ and let $G$ be an $n$-vertex graph with no path on $k$ vertices. Then $e(G) \leq (k-2)n/2$.

This theorem is implied by a stronger result for graphs with no long cycles.

**Theorem 2.2** (Erdős and Gallai [3]). Let $k \geq 3$ and let $G$ be an $n$-vertex graph with no cycle of length $k$ or longer. Then $e(G) \leq (k-1)(n-1)/2$.

Győri, Katona, and Lemons [6] extended Theorem 2.1 to Berge paths in $r$-graphs. The bounds depend on the relationship of $r$ and $k$.

**Theorem 2.3** (Győri, Katona, and Lemons [6]). Suppose that $\mathcal{H}$ is an $n$-vertex $r$-graph with no Berge path of length $k$. If $k \geq r + 5$, then $e(\mathcal{H}) \leq \frac{n}{k-1} \binom{k-1}{r}$, and if $r \geq 3$, then $e(\mathcal{H}) \leq \frac{n(k-1)}{r+1}$.

Both bounds in Theorem 2.3 are sharp for each $k$ and $r$ for infinitely many $n$. The remaining case of $k = r + 1$ was settled later by Davoodi, Győri, Methuku, and Tompkins [2]: if $\mathcal{H}$ is an $n$-vertex $r$-graph with $|E(\mathcal{H})| > n$, then it contains a Berge path of length at least $r + 1$. Furthermore, Győri, Methuku, Salia, Tompkins and Vizer [7] have found a better upper bound on the number of edges in $n$-vertex connected $r$-graphs with no Berge path of length $k$. Their bound is asymptotically exact when $r$ is fixed and $k$ and $n$ are sufficiently large.

The goal of this paper is to present a similar result for cycles.

3 Main result: Hypergraphs without long Berge cycles

Our main result is an analogue of the Erdős–Gallai theorem on cycles for $r$-graphs.

**Theorem 3.1.** Let $r \geq 3$ and $k \geq r + 3$, and suppose $\mathcal{H}$ is an $n$-vertex $r$-graph with no Berge cycle of length $k$ or longer. Then $e(\mathcal{H}) \leq \frac{n}{k-1} \binom{k-1}{r}$. Moreover, equality is achieved if and only if $\partial_2 \mathcal{H}$ is connected and for every block $D$ of $\partial_2 \mathcal{H}$, $D = K_{k-1}$ and $\mathcal{H}[D] = K_{k-1}^{(r)}$. 

Note that a Berge cycle can only be contained in the vertices of a single block of the 2-shadow. Hence the aforementioned sharpness examples cannot contain Berge cycles of length \( k \) or longer.

**Conjecture 3.2.** The statement of Theorem 3.1 holds for \( k = r + 2 \), too.

Similarly to the situation with paths, the case of short cycles, \( k \leq r + 1 \), is different. Exact bounds for \( k \leq r - 1 \) and asymptotic bounds for \( k = r \) were found in [9]. The answer for \( k = r + 1 \) is not known.

For convenience, below we will use notation

\[
C_r(k) := \frac{1}{k-2} \binom{k-1}{r}.
\]

(1)

(So \( C_2(k)(n-1) = (k-1)(n-1)/2 \).) Theorem 3.1 yields the following implication for paths.

**Corollary 3.3.** Let \( r \geq 3 \) and \( n \geq k + 1 \geq r + 4 \). If \( \mathcal{H} \) is a connected \( n \)-vertex \( r \)-graph with no Berge path of length \( k \), then \( e(\mathcal{H}) \leq C_r(k)(n-1) \).

This gives a \( \frac{k-2}{k-r} \) times stronger bound than Theorem 2.3 for connected \( r \)-graphs for all \( r \geq 3 \) and \( n \geq k + 1 \geq r + 4 \) and not only for sufficiently large \( k \) and \( n \). In particular, Corollary 3.3 implies the following slight sharpening of Theorem 2.3 for \( k \geq r + 3 \).

**Corollary 3.4.** Let \( r \geq 3 \) and \( n \geq k \geq r + 3 \). If \( \mathcal{H} \) is an \( n \)-vertex \( r \)-graph with no Berge path of length \( k \), then \( e(\mathcal{H}) \leq \frac{n}{r} \binom{k}{r} \) with equality only if every component of \( \mathcal{H} \) is the complete \( r \)-graph \( K_r^k \).

In the next section, we introduce the notion of representative pairs and use it to derive useful properties of Berge \( F \)-free hypergraphs for rather general \( F \). In Section 5 we cite Kopylov’s Theorem and prove two useful inequalities. In Section 6 we prove our main result, Theorem 3.1, and in the final Section 7 we derive Corollaries 3.3 and 3.4.

## 4 Representative pairs, the structure of Berge \( F \)-free hypergraphs

**Definition 4.1.** For a hypergraph \( \mathcal{H} \), a system of distinct representative pairs (SDRP) of \( \mathcal{H} \) is a set of distinct pairs \( A = \{\{x_1, y_1\}, \ldots, \{x_s, y_s\}\} \) and a set of distinct hyperedges \( A = \{f_1, \ldots, f_s\} \) of \( \mathcal{H} \) such that for all \( 1 \leq i \leq s \)

- \( \{x_i, y_i\} \subseteq f_i \), and
- \( \{x_i, y_i\} \) is not contained in any \( f \in \mathcal{H} - \{f_1, \ldots, f_s\} \).

**Lemma 4.2.** Let \( \mathcal{H} \) be a hypergraph, let \( (A, A) \) be an SDRP of \( \mathcal{H} \) of maximum size. Let \( \mathcal{B} := \mathcal{H} \setminus A \) and let \( \partial_2 \mathcal{B} \) be the 2-shadow of \( \mathcal{B} \). For a subset \( S \subseteq B \), let \( \mathcal{B}_S \) denote the set of hyperedges that contain at least one edge of \( S \). Then for all nonempty \( S \subseteq B \), \( |S| < |\mathcal{B}_S| \).

**Proof.** Suppose for contradiction there exists a nonempty set \( S \subseteq B \) such that \( |S| \geq |\mathcal{B}_S| \). Choose a smallest such \( S \).
We claim that $|S| = |B_S|$. Indeed, if $|S| > |B_S|$ then $|S| \geq 2$ because $B_S \neq \emptyset$ by definition. Take any edge $e \in S$. The set $S \setminus e$ is nonempty and $|S \setminus e| = |S| - 1 \geq |B_S| \geq |B_S \setminus e|$, a contradiction to the minimality of $S$.

Consider the case $|S| = |B_S|$. By the minimality of $S$, each subset $S' \subset S$ satisfies $|S'| < |B_{S'}|$. Therefore by Hall’s theorem, one can find a bijective mapping of $S$ to $B_S$, where say the edge $e_i \in S$ gets mapped to hyperedge $f_i$ in $B_S$ for $1 \leq j \leq |S|$. Then $(A \cup \{e_1, \ldots, e_{|S|}\}, A \cup \{f_1, \ldots, f_{|S|}\})$ is a larger SDRP of $H$, a contradiction.

\[ \square \]

**Lemma 4.3.** Let $\mathcal{H}$ be a hypergraph and let $(A, A)$ be an SDRP of $\mathcal{H}$ of maximum size. Let $B := H \setminus A$, $B = \partial B$, and let $G$ be the graph on $V(H)$ with edge set $A \cup B$. If $G$ contains a copy of a graph $F$, then $H$ contains a Berge $F$ on the same base vertex set.

**Proof.** Let $\{v_1, \ldots, v_r\}$ and $\{e_1, \ldots, e_q\}$ be a set of vertices and a set of edges forming a copy of $F$ in $G$ such that the edges $e_1, \ldots, e_b$ belong to $B$. By Lemma 4.2, each subset $S$ of $\{e_1, \ldots, e_b\}$ satisfies $|S| < |B_S|$. So we may apply Hall’s Theorem to match each of these $e_i$’s to a hyperedge $f_i \in B$. The edges $e_i \in A$ can be matched to distinct edges of $A$ given by the SDRP. Since $A \cap B = \emptyset$ this yields a Berge $F$ in $H$ on the same base vertex set.

We have $|H| = |A| + |B|$. Note that the number of $r$-edges in $B$ is at most the number of copies of $K_r$ in its 2-shadow. Therefore Lemma 4.3 gives a new proof for the following result of Gerbner and Palmer (cited in [4]): for any graph $F$,

$$\text{ex}(n, K_r, F) \leq \text{ex}_r(n, \text{Berge} F) \leq \text{ex}(n, F) + \text{ex}(n, K_r, F).$$

Here $\text{ex}_r(n, \{F_1, F_2, \ldots\})$ denotes the Turán number of $\{F_1, F_2, \ldots\}$, the maximum number of edges in an $r$-uniform hypergraph on $n$ vertices that does not contain a copy of any $F_i$.

The generalized Turán function $\text{ex}(n, K_r, F)$ is the maximum number of copies of $K_r$ in an $F$-free graph on $n$ vertices.

## 5 Kopylov’s Theorem and two inequalities

**Definition:** For a natural number $\alpha$ and a graph $G$, the $\alpha$-disintegration of a graph $G$ is the process of iteratively removing from $G$ the vertices with degree at most $\alpha$ until the resulting graph has minimum degree at least $\alpha + 1$ or is empty. This resulting subgraph $H(G, \alpha)$ will be called the $(\alpha + 1)$-core of $G$. It is well known (and easy) that $H(G, \alpha)$ is unique and does not depend on the order of vertex deletion.

The following theorem is a consequence of Kopylov [8] about the structure of graphs without long cycles. We state it in the form that we need:

**Theorem 5.1 (Kopylov [8]).** Let $n \geq k \geq 5$ and let $t = \lfloor \frac{k - 1}{2} \rfloor$. Suppose that $G$ is a 2-connected $n$-vertex graph with no cycle of length at least $k$. Suppose that it is saturated, i.e., for every nonedge $xy$ the graph $G \cup \{xy\}$ has a cycle of length at least $k$. Then either

\[1\] A proof and a recent application can be found in [10].
Moreover, for $2 \leq w \leq k - 1$ one has $a_r(w) \leq (w - 1)\binom{k-1}{r} / (k - 2)$ with equality if and only if $w = k - 1$ and

— $w > r + 2$ and $\mathcal{H}$ is complete, or
— $w = r + 2$ and either $\mathcal{H}$ or $\overline{\mathcal{H}}$ is complete.

**Proof.** The case of $w \geq r + 2$ is a corollary of the classical Kruskal-Katona theorem, but one can give a direct proof by a double counting. If $\overline{\mathcal{H}}$ is empty, then $|\mathcal{H}| = \binom{w}{r}$ if and only if $\mathcal{H} = \binom{V(\mathcal{H})}{r}$. Otherwise, let $\overline{\mathcal{H}}$ denote the $r$-subsets of $V(\mathcal{H})$ that are not members of $\mathcal{H}$, $\overline{\mathcal{H}} = \binom{V(\mathcal{H})}{r} \setminus \mathcal{H}$. Each pair of $\overline{\mathcal{H}}$ is contained in $\binom{w - 2}{r - 2}$ members of $\mathcal{H}$ and each $e \in \overline{\mathcal{H}}$ contains at most $\binom{r}{2}$ edges of $\overline{\mathcal{H}}$. We obtain

$$|\overline{\mathcal{H}}| \binom{w - 2}{r - 2} \leq |\mathcal{H}| \binom{r}{2}.$$
Since $\binom{w-2}{r-2} \geq \binom{r}{r-2} = \binom{r}{2}$, $|\partial_2 \mathcal{H}| \leq |\mathcal{H}|$ with equality only when $w = r + 2$. Furthermore, if $\partial_2 \mathcal{H}$ and $\mathcal{H}$ are both nonempty, then for any $xy \in \partial_2 \mathcal{H}$ and $uv \in \partial_2 \mathcal{H}$ (with possibly $x = u$), any $r$-tuple $e$ containing $\{x, y\} \cup \{u, v\}$ is in $\mathcal{H}$ but contributes strictly less than $\binom{r}{2}$ edges to $\partial_2 \mathcal{H}$, implying $|\partial_2 \mathcal{H}| < |\mathcal{H}|$. This completes the proof of the case.

The case $w \leq r + 1$ is easy, and the calculation showing $a_r(w) \leq C_r(k)(w - 1)$ with equality only if $w = k - 1$ is standard. □

6 Proof of Theorem 3.1, the main upper bound

Proof. Let $\mathcal{H}$ be an $r$-uniform hypergraph on $n$ vertices with no Berge cycle of length $k$ or longer ($k \geq r + 3 \geq 6$). Let $(A, A)$ be an SDRP of $\mathcal{H}$ of maximum size. Let $B := \mathcal{H} \setminus A$, $B = \partial_2 B$. By Lemma 4.3, the graph $G$ with edge set $A \cup B$ does not contain a cycle of length $k$ or longer.

Let $V_1, V_2, \ldots, V_p$ be the vertex sets of the standard (and unique) decomposition of $G$ into 2-connected blocks of sizes $n_1, n_2, \ldots, n_p$. Then the graph $A \cup B$ restricted to $V_i$, denoted by $G_i$, is either a 2-connected graph or a single edge (in the latter case $n_i = 2$), each edge from $A \cup B$ is contained in a single $G_i$, and $\sum_{i=1}^{p} (n_i - 1) \leq (n - 1)$.

This decomposition yields a decomposition of $A = A_1 \cup A_2 \cup \cdots \cup A_p$ and $B = B_1 \cup B_2 \cup \cdots \cup B_p$, $A_i \cup B_i = E(G_i)$. If an edge $e \in E_i$ is contained in $f \in B$, then $f \subseteq V_i$ (because $f$ induces a 2-connected graph $K_r$ in $B$), so the block-decomposition of $G$ naturally extends to $B$, $B_i := \{f \in B : f \subseteq V_i\}$ and we have $B = B_1 \cup \cdots \cup B_p$, and $B_i = \partial_2 B_i$.

We claim that for each $i$,

$$|A_i| + |B_i| \leq C_r(k)(n_i - 1),$$

and hence

$$|\mathcal{H}| = |A| + |B| = \sum_{i=1}^{p} |A_i| + |B_i| \leq \sum_{i=1}^{p} C_r(k)(n_i - 1) \leq C_r(k)(n - 1),$$

completing the proof.

To prove (2) observe that the case $n_i \leq k - 1$ immediately follows from Lemma 5.3. From now on, suppose that $n_i \geq k$.

Consider the graph $G_i$ and, if necessary, add edges to it to make it a saturated graph with no cycle of length $k$ or longer. Let the resulting graph be $G'$. Kopylov’s Theorem (Theorem 5.1) can be applied to $G'$. If $G$ is $t$-disintegrable, then make $(n_i - k + 2)$ disintegration steps and let $W$ be the remaining vertices of $V_i$ ($|W| = k - 2$). For the edges of $A_i$ and $B_i$ contained in $W$ we use Lemma 5.3 to see that

$$|A_i| |W| + |B_i| |W| < C_r(k)(|W| - 1).$$

In the $t$-disintegration steps, we iteratively remove vertices with degree at most $t$ until we arrive to $W$. When we remove a vertex $v$ with degree $s \leq t$ from $G'$, $a$ of its incident edges are from $A$, and the remaining $s - a$ incident edges eliminate at most $\binom{s-a}{r-1}$ hyperedges from $B_i$ containing $v$. Therefore $v$ contributes at most $a + \binom{s-a}{r-1} \leq C_r(k)$ (by Lemma 5.2) to $|B_i| + |A_i|$.
It follows that

$$|A_i| + |B_i| < \left( \sum_{v \in G' \cap W} C_r(k) \right) + C_r(k)(|W| - 1) = C_r(k)(n_i - 1).$$

This completes this case.

Next consider the case (5.1.2), \(W := V(H(G, t)), |W| = s \leq k - 2\). We proceed as in the previous case, making \((n_i - s)\) disintegration steps. Apply Lemma 5.3 for \(|A_i[W]| + |B_i[W]|\) and Lemma 5.2 for the \((k - s)\)-disintegration steps (where \(k - s \leq t\)) to get the desired upper bound (with strict inequality).

Furthermore, if \(e(H) = |A| + |B| = C_r(k)(n - 1)\), then we have \(\sum_{i=1}^{p} (n_i - 1) = n - 1\) (so \(A \cup B\) is connected) and \(|A_i| + |B_i| = C_r(k)(n_i - 1)\) for each \(1 \leq i \leq p\). From the previous proof and Lemma 5.2, we see that this holds if and only if for each \(i\), \(n_i = k - 1\), and either \(B_i\) or \(A_i\) is complete. In particular, this implies that each block of \(A \cup B\) is a \(K_{k-1}\). We will show that each \(G_i\) corresponds to a block in \(\mathcal{H}\) that is \(K_{k-1}^{(r)}\) with vertex set \(V_i\).

In the case that \(B_i\) is complete for all \(1 \leq i \leq p\), we are done. Otherwise, if some \(A_i\) is complete (note \(r = k - 3\) by Lemma 3.2), then there are \(\binom{k-1}{3} = \binom{k-1}{r} = \binom{k-1}{1}\) hyperedges in \(A\) containing \(V_i\). If all such hyperedges are contained in \(V_i\), again we get \(H[V_i] = K_{k-1}^{(r)}\). So suppose there exists a \(f \in A\) which is paired with an edge \(xy \in E\) in the SDRP, but for some \(z \notin V_i\), \(\{x, y, z\} \subseteq f\). Then \(z\) belongs to another block \(G_j\) of \(A \cup B\). In \(A \cup B\), there exists a path from \(x\) to \(z\) covering \(V_i \cup V_j\) which avoids the edge \(xy\). Thus by Lemma 5.3, there is a Berge path from \(x\) to \(z\) with at least \(2(k-1) - 1\) base vertices which avoids this hyperedge \(f\) (since edge \(xy\) was avoided). Adding \(f\) to this path yields a Berge cycle of length \(2(k-1) - 1 > k\), a contradiction. \(\square\)

7 Corollaries for paths

In order to be self-contained, we present a short proof of a lemma by Győri, Katona, and Lemons [6].

**Lemma 7.1** (Győri, Katona, and Lemons [6]). Let \(\mathcal{H}\) be a connected hypergraph with no Berge path of length \(k\). If there is a Berge cycle of length \(k\) on the vertices \(v_1, \ldots, v_k\) then these vertices constitute a component of \(\mathcal{H}\).

**Proof.** Let \(V = \{v_1, \ldots, v_k\}, E = \{e_1, \ldots, e_k\}\) form the Berge cycle in \(\mathcal{H}\). If some edge, say \(e_1\) contains a vertex \(v_0\) outside of \(V\), then we have a path with vertex set \(\{v_0, v_1, \ldots, v_k\}\) and edge set \(E\). Therefore each \(e_i\) is contained in \(V\). Suppose \(V \neq V(\mathcal{H})\). Since \(\mathcal{H}\) is connected, there exists an edge \(e_0 \in \mathcal{H}\) and a vertex \(v_{k+1} \notin V\) such that for some \(v_i \in V\), say \(i = k\), \(\{v_k, v_{k+1}\} \subseteq e_0\). Then \(\{v_1, v_2, \ldots, v_k, v_{k+1}\}, \{e_1, \ldots, e_{k-1}, e_0\}\) is a Berge path of length \(k\). \(\square\)

**Proof of Corollary 3.3** Suppose \(n \geq k + 1\) and \(\mathcal{H}\) is a connected \(n\)-vertex \(r\)-graph with \(e(\mathcal{H}) > C_r(k)(n-1)\). Then by Theorem 3.1, \(\mathcal{H}\) has a Berge cycle of length \(\ell \geq k\). If \(\ell \geq k + 1\), then removing any edge from the cycle yields a Berge path of length at least \(k\). If \(\ell = k\), then by Lemma 7.1, \(\mathcal{H}\) again has a Berge path of length \(k\). \(\square\)

Now Theorem 3.1 together with Corollary 3.3 directly imply Corollary 3.4.
Proof of Corollary 3.4: Suppose $k \geq r + 3 \geq 6$ and $\mathcal{H}$ is an $r$-graph. Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_s$ be the connected components of $\mathcal{H}$ and $|V(\mathcal{H}_i)| = n_i$ for $i = 1, \ldots, s$.

If $n_i \leq k - 1$, then $|\mathcal{H}_i| \leq \binom{n_i}{r} < \frac{n_i}{r^k} \binom{k}{r}$. If $n_i \geq k + 1$, then by Corollary 3.3, $|\mathcal{H}_i| \leq C_r(k)(n_i - 1) < \frac{n_i}{r^k} \binom{k}{r}$. Finally, if $n_i = k$, then $|\mathcal{H}_i| \leq \binom{k}{r} = \frac{n_i}{r^k} \binom{k}{r}$, with equality only if $\mathcal{H}_i = K^r_k$. This proves the corollary.

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References

[1] R. Anstee and S. Salazar, Forbidden Berge hypergraphs, Electron. J. Combin. 24 (2017), Paper 1.59, 21 pp.

[2] A. Davoodi, E. Győri, A. Methuku, and C. Tompkins, An Erdős-Gallai type theorem for hypergraphs, European J. Combin. 69 (2018), 159–162.

[3] P. Erdős and T. Gallai, On maximal paths and circuits of graphs, Acta Math. Acad. Sci. Hungar. 10 (1959), 337–356.

[4] D. Gerbner, A. Methuku, and M. Vizer, Asymptotics for the Turán number of Berge-$K_{2,t}$, arxiv:1705.04134, (2017), 24 pp.

[5] D. Gerbner and C. Palmer, Extremal results for Berge-hypergraphs, SIAM Journal on Discrete Mathematics, 31 (2017), 2314–2337.

[6] E. Győri, Gy. Y. Katona, and N. Lemons, Hypergraph extensions of the Erdős-Gallai theorem, European Journal of Combinatorics, 58 (2016), 238–246.

[7] E. Győri, A. Methuku, N. Salia, C. Tompkins and M. Vizer, On the maximum size of connected hypergraphs without a path of given length, arxiv:1710.08364, (2017), 6 pp.

[8] G. N. Kopylov, Maximal paths and cycles in a graph, Dokl. Akad. Nauk SSSR 234 (1977), 19–21. (English translation: Soviet Math. Dokl. 18 (1977), no. 3, 593–596.)

[9] A. Kostochka, and R. Luo, On $r$-uniform hypergraphs with circumference less than $r$, in preparation.

[10] R. Luo, The maximum number of cliques in graphs without long cycles, Journal of Combinatorial Theory, Series B, 128 (2018), 219–226.