Fréchet-Kolmogorov-Riesz-Weil’s theorem on locally compact groups via Arzelà-Ascoli’s theorem

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Abstract

In the paper, we present a functional analysis view on the Arzelà-Ascoli theorem for the Banach space $C_0(X)$, where $X$ is a locally compact Hausdorff space. The proof hinges upon the Banach-Alaoglu’s theorem. This approach is motivated by the work of Gabriel Nagy.

In the second part of the paper, we put forward the most natural proof (in author’s opinion) of Fréchet-Kolmogorov-Riesz-Weil theorem for locally compact Hausdorff groups $G$. The method basically amounts to the fact that boundedness, equicontinuity and equivanishing are preserved by convolution with continuous and compactly supported functions.

Keywords: Arzelà-Ascoli theorem, Fréchet-Kolmogorov-Riesz-Weil theorem, convolution, locally compact groups

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1 Introduction

The main objective of the paper is to present, in author’s opinion, the most natural proof of the Fréchet-Kolmogorov-Riesz-Weil theorem. In its basic form,
it characterizes the relatively compact families of the Banach space \( L^p(\mathbb{R}^N) \),
where \( 1 \leq p < \infty \). The proof is attributed to Maurice Fréchet (1878-1973),
Andrey Kolmogorov (1903-1987) and Frigyes Riesz (1880-1956). It can be found in [2], p. 111 (Theorem 4.26):

**Theorem 1.** *(Fréchet-Kolmogorov-Riesz theorem)*

Let \( F \) be a bounded set in \( L^p(\mathbb{R}^N) \) with \( 1 \leq p < \infty \). Assume that

\[
\lim_{|x| \to 0} \| R_x f - f \|_p = 0 \quad \text{uniformly in } f \in F
\]

where \( R_x f(y) = f(y + x) \). Then the closure of \( F|_\Omega \) in \( L^p(\Omega) \) is compact for
every measurable set \( \Omega \subset \mathbb{R}^N \) with finite measure.

Scrutinizing this theorem, two questions immediately spring to mind:

1. What is so special about \( \mathbb{R}^N \), could the theorem be generalized to more
   abstract spaces?

2. Could we get rid of the 'finite measure' assumption on the subset \( \Omega \)?

In 1940 André Weil (1906-1998) wrote a book ‘L’intégration dans les groupes
topologique’ (comp. [15]), in which he answered both questions:

**Theorem 2.** Let \( G \) be a locally compact Hausdorff group. A family \( F \subset L^p(G) \)
is relatively compact if and only if

1. \( F \) is bounded in \( L^p \)-norm,

2. for every \( \varepsilon > 0 \), there exists \( K \subset G \) (compact subset of \( G \)) such that

\[
\forall f \in F \quad \| f - f\mathbb{1}_K \| < \varepsilon,
\]

3. for every \( \varepsilon > 0 \), there exists an open identity neighbourhood \( V \) such that

\[
\forall f \in F \quad \| L_x f - f \|_p < \varepsilon,
\]

where \( L_x f(y) = f(xy) \).
The proof of this theorem is rather difficult to follow. The exposition is very terse and aptly avoids technical details. The fact that the book is written in French does not make matters easier.

As we have stated earlier, the main goal of this paper is to provide a simple and elegant proof of the Fréchet-Kolmogorov-Riesz-Weil theorem. In order to do that, we build upon the visionary work of Gabriel Nagy (comp. [12]). His beautiful paper characterized relatively compact families in $C(K)$, the space of continuous functions on compact space $K$. A crucial part of his approach was the use of Banach-Alaoglu’s theorem. We employ Nagy’s techniques to characterize relatively compact families in $C_0(X)$, the space of continuous functions, vanishing at infinity on locally compact Hausdorff space $X$. This is the content of Section 2.

A brief remark is in order. As far as Fréchet-Kolmogorov-Riesz-Weil theorem is concerned, proving the Arzelà-Ascoli theorem in an ‘extravagant’ way is not necessary. In fact, Theorem 5 could be concluded from the previous papers that the author has written on the subject (comp. [8, 9, 10]). Yet another way to derive Theorem 5 is to view $C_0(X)$, with $X$ locally compact Hausdorff space, as the ideal

$$\left\{ f \in C(\alpha(X)) : f(\infty) = 0 \right\},$$

where $\alpha(X)$ is the Alexandroff one-point compactification of $X$ ([11], p.185). From this perspective, it is enough to remark that the equivanishing of the family $F \subset C_0(X)$ is equivalent to equicontinuity at the point $\infty$. This approach is well-known in the folklore. However, having said all the above, it is not our aim to present the shortest possible proof of the Arzelà-Ascoli theorem. Instead, it is the author’s strong conviction that a new perspective on an old result (despite its length) is enlightening.

The crux of Section 3 lies in Lemmas 7, 8 and 9 which assert that for a suitable continuous function with compact support $\phi \in C_c(G)$, $G$ being a locally compact Hausdorff space, and $F \subset L^p(G)$ we have

$$L^p\text{-boundedness of } F \text{ implies pointwise boundedness of } F \ast \phi,$$
$$L^p\text{-equicontinuity of } F \text{ implies equicontinuity of } F \ast \phi,$$
$$L^p\text{-equivanishing of } F \text{ implies equivanishing of } F \ast \phi,$$

where $F \ast \phi = \left\{ f \ast \phi : f \in F \right\}$. Observe that the above implications shift the problem of characterizing the relatively compact families of $L^p(G)$ to an analogous problem in $C_0(G)$. However, we have already tackled this issue in
Finally, the Fréchet-Kolmogorov-Riesz-Weil theorem (Theorem 14) for locally compact Hausdorff groups is the climax of the paper.

## 2 Compact families in $C_0(X)$

Let us begin with a lemma, which is well-known in the mathematical folklore. However, due to the lack of reference, we provide a short proof.

**Lemma 3.** (comp. Proposition 1 in [12])
Let $X$ be a complete metric space. Set $A \subset X$ is relatively compact if and only if there does not exist an infinite set $B \subset \overline{A}$ such that

$$\inf_{x, y \in B \atop x \neq y} d(x, y) > 0. \tag{1}$$

*Proof.* For the first part, suppose that $\overline{A}$ is compact and that there exists a set $B \subset \overline{A}$ such that (1) is satisfied. Consequently, any sequence in $B$ cannot contain a convergent subsequence. This contradicts the compactness of $\overline{A}$ (comp. Theorem 3.28 in [1], p. 86).

For the second part, we investigate the situation when $A$ is not relatively compact. This means that there exists an $\varepsilon > 0$, for which there is no finite $\varepsilon$–net. In other words, for every finite family $(x_n)_{n=1}^N \subset \overline{A}$ there exists an element $x \in \overline{A}$ such that $d(x, x_n) \geq \varepsilon$ for every $n = 1, \ldots, N$. We may adjoin $x$ as the new element $x_{N+1}$, thus extending the finite list by one element. Continuing this procedure results in producing an infinite sequence which satisfies (1). This ends the proof. \qed

Our second result should be contrasted with Proposition 2 in [12]. Our exposition contains slight modifications of the notation but more importantly, the proof does not exploit the notion of Moore-Smith sequences (comp. [4], p. 49). It resorts solely to the familiar definition of continuity.

**Lemma 4.** Let $X$ be a normed vector space, $K$ be a compact subset of $X$ and let $\overline{B}_{X^*}(0, 1) \subset X^*$ be a closed unit ball. Then the restriction map

$$\Phi : (\overline{B}_{X^*}(0, 1), \tau^*) \ni \chi \mapsto \chi|_K \in C(K)$$

is continuous. In particular, $\Phi(\overline{B}_{X^*}(0, 1))$ is compact in $C(K)$. 

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Proof. Let $B_{C(K)}(ξ, ε)$ be an open ball in $C(K)$ of radius $ε$, centered at $ξ ∈ C(K)$. Choose

$$η ∈ Φ^{-1}(B_{C(K)}(ξ, ε)) = \{χ ∈ B_{X^*}(0, 1) : \|χ|_K - ξ\|_∞ < ε\}.$$ 

Since $K$ is compact, we find $(x_n)_{n=1}^N$ such that

$$∀ x ∈ K \exists n = 1, ..., N \|x - x_n\| < δ, \tag{2}$$

where $δ > 0$ is such that

$$3δ + \|η|_K - ξ\|_∞ < ε. \tag{3}$$

We consider the weak* open set

$$U(η, x_1, ..., x_N, δ) = \{χ ∈ B_{X^*}(0, 1) : ∀ n = 1, ..., N |χ(x_n) - η(x_n)| < ε\}$$

and prove that

$$U(η, x_1, ..., x_N, δ) ⊂ Φ^{-1}(B_{C(K)}(ξ, ε)).$$

For $x ∈ K$, we choose $n = 1, ..., N$ as in (2). If $χ ∈ U(η, x_1, ..., x_N, δ)$, then

$$|χ(x) - ξ(x)| ≤ |χ(x) - χ(x_n)| + |χ(x_n) - η(x_n)| + |η(x_n) - η(x)| + |η(x) - ξ(x)|$$

$$≤ \|χ\|_{X^*} \|x - x_n\| + δ + \|η\|_{X^*} \|x - x_n\| + \|η(x) - ξ(x)\| ≤ 3δ + \|η|_K - ξ\|_∞ < ε.$$ \[
\tag{8}
\]

Consequently, we have $\|χ|_K - ξ\|_∞ < ε$, i.e. $χ ∈ Φ^{-1}(B_{C(K)}(ξ, ε)).$

The second part of the theorem follows immediately from Banach-Alaoglu’s theorem (comp. [1], p. 235). \qed

At this point, we present the first main result of the paper.

**Theorem 5. (Arzelà-Ascoli theorem for $C_0(X)$)**

Let $X$ be a locally compact Hausdorff space. The family $F ⊂ C_0(X)$ is relatively compact if and only if

**(AA1)** $F$ is pointwise bounded, i.e. $\sup_{f ∈ F} |f(x)| < \infty$ for every $x ∈ X$,

**(AA2)** $F$ is equicontinuous in the sense

$$∀ ε > 0 \exists U_x ∈ τ_x \forall f ∈ F \forall y ∈ U_x \forall x \in X |f(y) - f(x)| < ε,$$
(AA3) \(\mathcal{F}\) is equi\-vanishing in the sense
\[
\forall \epsilon > 0 \exists K \in X \forall_{f \in \mathcal{F}} f(x) | < \epsilon.
\]

**Proof.** At first, suppose that \(\mathcal{F} \subset C_0(X)\) is relatively compact. It is obviously bounded and equicontinuity follows from a classical \(3\,\epsilon\)–argument (comp. Theorem 2 in [7]).

As for (AA3), let \((f_n)_{n=1}^{N}\) be an \(\frac{\epsilon}{2}\)–net for \(\mathcal{F}\). For each \(f_n\), let \(K_n\) be a compact set such that
\[
\forall x \in X \setminus K_n \ |f(x)| < \frac{\epsilon}{2}.
\]

Put \(K := \bigcup_{n=1}^{N} K_n\). Hence, for every \(f \in \mathcal{F}\) there exists \(n = 1, \ldots, N\) such that
\[
\forall x \in X \setminus K \ |f(x)| \leq |f(x) - f_n(x)| + |f_n(x)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

We proceed with proving the converse, i.e. we prove that (AA1), (AA2) and (AA3) together imply relative compactness of \(\mathcal{F}\). Any set \(A \subset \mathcal{F}\) induces a function
\[
\Psi_A : X \ni x \mapsto (f(x))_{f \in A} \in l^\infty(A).
\]
Such a function is well-defined due to (AA1). Moreover, it is continuous due to (AA2). Below, we prove that \(\Psi_A(X)\) is compact.

Fix \(\epsilon > 0\). By (AA3), we choose \(K \in X\) such that
\[
\forall_{f \in A \atop x \notin K} |f(x)| < \frac{\epsilon}{2}.
\] (4)

Since \(\Psi_A(K)\) is compact, then it has an \(\frac{\epsilon}{2}\)–net, \((\Psi_A(x_n))_{n=1}^{N}\). We adjoin to this \(\frac{\epsilon}{2}\)–net \(0 \in l^\infty(A)\) and prove that it is an \(\epsilon\)–net.

For every element \(\phi \in \Psi_A(X)\), there exists \(x \in X\) such that
\[
\|\phi - \Psi_A(x)\|_\infty < \frac{\epsilon}{2}.
\] (5)

If \(x \in K\), then there exists \(n = 1, \ldots, N\) such that
\[
\|\Psi_A(x) - \Psi_A(x_n)\|_\infty < \frac{\epsilon}{2}.
\] (6)

By the triangle inequality we have
\[
\|\phi - \Psi_A(x_n)\|_\infty \leq \|\phi - \Psi_A(x)\|_\infty + \|\Psi_A(x) - \Psi_A(x_n)\|_\infty < \epsilon.
\]
In the event that $x \not\in K$, we have
\[ \| \Psi_A(x) - 0 \|_\infty = \sup_{f \in A} |f(x)| < \frac{\varepsilon}{2}. \]

Again by the triangle inequality, we have $\| \phi - 0 \| < \varepsilon$. In conclusion, \((\Psi_A(x_n))_{n=1}^N \cup \{0\}\) is an $\varepsilon$–net for \(\overline{\Psi_A(X)}\). Hence \(\overline{\Psi_A(X)}\) is compact.

Suppose, for the sake of contradiction, that \(F\) is not relatively compact. By Lemma 3, there exists an infinite set \(A \subset F\) and \(\delta > 0\) such that
\[ \inf_{f,g \in A} \| f - g \|_\infty > \delta. \]

Let us consider a closed, unit ball \(B\) in \(l^\infty(A)^*\). By Lemma 4, the set
\[ \Phi(B) = \left\{ \chi_{|\Psi_A(x)|} : \chi \in B \right\} \subset C\left(\overline{\Psi_A(X)}\right) \]
is compact (in the norm topology). Furthermore, let us consider the projections \(\pi_f : l^\infty(A) \to \mathbb{C}\) given by \(\pi_f(\phi) = \phi(f)\). It is trivial to check that \(\pi_f \in B\) for every \(f \in A\). We have
\[ \pi_f \circ \Psi_A(x) = \pi_f \left( (g(x))_{g \in A} \right) = f(x), \]
which we depict in the commutative diagram below:

\[ \begin{array}{ccc}
X & \xrightarrow{\Psi_A} & l^\infty(A) \\
\downarrow f \in C_0(X) & & \downarrow \pi_f \in B \\
\mathbb{C} & \xleftarrow{\pi_f} & \end{array} \]

By the choice of \(A\), for every \(f, g \in A\) we can find \(x \in X\) such that \(|f(x) - g(x)| > \delta\), or equivalently
\[ |\pi_f \circ \Psi_A(x) - \pi_g \circ \Psi_A(x)| > \delta. \quad (7) \]

Since \(\left(\pi_f|_{\overline{\Psi_A(X)}}\right)_{f \in A} \subset \Phi(B)\), then (7) contradicts the compactness of \(\Phi(B)\). We conclude that \(F\) is relatively compact, which ends the proof.
3 Compact families in $L^p$–spaces

Throughout this section, we assume that

$G$ is a locally compact Hausdorff group with left-invariant Haar measure $\mu$.

Note that the inverse map $\iota : G \to G$ is a homeomorphism. At first, let us prove a technical lemma.

**Lemma 6.** Let $K$ be a compact subset of $G$ and let $U$ be open and relatively compact identity neighbourhood. Then, there exists a compact set $D$ such that

$$ \forall_{x \notin D} \ xU \cap K = \emptyset. $$

**Proof.** We put $A = \overline{U} \cdot \iota(U)$, which is compact due to the continuity of multiplication. Observe that $(xU)_{x \in K}$ is an open cover of $K$, so we may choose a finite subcover $(x_n U)_{n=1}^N$. We put $D = \bigcup_{n=1}^N x_n A$, which is again a compact set.

Suppose that $x \notin D$ and, for the sake of contradiction, assume that there exists $y \in xU \cap K$. Hence, there exists $n = 1, \ldots, N$ such that

$$ x \in y \iota(U) \subset x_n \overline{U} \cdot \iota(U) \subset x_n A \subset D. $$

We reached a contradiction $x \in D$, which ends the proof. \hfill \Box

In the sequel, we will need the concepts of $L^p$–equicontinuity and $L^p$–equivanishing. A family $\mathcal{F} \subset L^p(G)$ is said to be $L^p$–**equicontinuous** if

$$ \forall_{\varepsilon > 0} \ \exists_{U_e \in \tau_G} \forall_{f \in \mathcal{F}} \sup_{x \in U_e} \| L_x f - f \|_p < \varepsilon \quad \text{and} \quad \sup_{x \in U_e} \| R_x f - f \|_p < \varepsilon, $$

where $L_x f(y) = f(xy)$ and $R_x f(y) = f(yx)$. A family $\mathcal{F} \subset L^p(G)$ is said to be $L^p$–**equivanishing** if

$$ \forall_{\varepsilon > 0} \ \exists_{K \in G} \forall_{f \in \mathcal{F}} \int_{G \setminus K} |f(y)|^p \, dy < \varepsilon^p. $$

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3.1 Inheritance of boundedness, equicontinuity and equivanning

In the three lemmas below, we prove that boundedness, equicontinuity and equivanning of $F$ are in a sense inherited when convoluted with a continuous function with compact support $\phi \in C_c(G)$.

**Lemma 7.** Let $F \subset L^p(G)$ be bounded in $L^p$–norm. If $\phi \in C_c(G)$ then $F \ast \phi \subset C^b(G)$ is bounded.

*Proof.* Let $M > 0$ be a $L^p$–bound on $F$. At first, we prove that $f \ast \phi$ is continuous for every $f \in F$. Fix $\varepsilon > 0$ and $x^* \in G$. By Proposition 2.41 in [5], p. 53 there exists a symmetric $U_\varepsilon \in \tau_G$ such that

$$ \forall x \in U_\varepsilon \left( \int_G |\phi \circ \iota(xy) - \phi \circ \iota(y)|^{p'} \, dy \right)^{\frac{1}{p'}} < \frac{\varepsilon}{M}. \quad (8) $$

For $x \in x^* U_\varepsilon$, we have

$$ |f \ast \phi(x) - f \ast \phi(x^*)| = \left| \int_G f(y) \phi \left( y^{-1}x \right) - f(y) \phi \left( y^{-1}x^* \right) \, dy \right| $$

Hölder \( \leq \norm{f}_p \left( \int_G |\phi \left( y^{-1}x \right) - \phi \left( y^{-1}x^* \right) |^{p'} \, dy \right)^{\frac{1}{p'}} \leq M \left( \int_G |\phi \left( y^{-1}x^{-1}y \right) - \phi \left( y^{-1}x^* \right) |^{p'} \, dy \right)^{\frac{1}{p'}} $$

$$ = M \left( \int_G |\phi \circ \iota \left( x^{-1}x^*y \right) - \phi \circ \iota \left( y \right) |^{p'} \, dy \right)^{\frac{1}{p'}} < \varepsilon. $$

This proves that $F \ast \phi$ is a family of continuous functions.

In order to prove that $F \ast \phi$ is a family of bounded functions and moreover, that it is bounded in the supremum norm, we have

$$ \forall f \in F \left( \int_G f(y) \phi \left( y^{-1}x \right) \, dy \right)^{\frac{1}{p'}} \leq \norm{f}_p \left( \int_G |\phi \left( y^{-1}x \right) |^{p'} \, dy \right)^{\frac{1}{p'}} \leq M \norm{\mu(x \cdot \iota(supp(\phi)))}^{\frac{1}{p'}} = M \norm{\phi}_\infty \left( \mu \left( \iota(\text{supp}(\phi)) \right) \right)^{\frac{1}{p'}}, \quad (9) $$

where the second inequality stems from the fact that if

$$ y^{-1}x \not\in \text{supp}(\phi) \iff y \not\in x \cdot \iota(\text{supp}(\phi)), $$

then $\phi(y^{-1}x) = 0$. We conclude that $F \ast \phi \subset C^b(G)$ is bounded. \( \Box \)
Lemma 8. Let $F \subset L^p(G)$ be $L^p$-equicontinuous. If $\phi \in C_c(G)$, then $F \star \phi$ is equicontinuous.

Proof. Fix $\varepsilon > 0$ and $x_0 \in G$. Let $V = x_0 U_e$, where $U_e$ is symmetric and such that

$$\forall f \in F \sup_{x \in U_e} \| L_x f - f \|_p < \frac{\varepsilon}{\| \phi \|_\infty (\mu(\iota(supp(\phi))))^{\frac{1}{p'}}}. \quad (11)$$

For convenience, put $K := \iota(supp(\phi))$. Observe that for every $f \in F$, we have

$$f \star \phi(x) = \int_G f(y) \phi(y^{-1}x) \ dy \quad \text{by (12)}$$

Finally, for every $f \in F$ and $x \in V$, we obtain

$$| f \star \phi(x) - f \star \phi(x_0) | \leq \int_G | f(xy) - f(x_0y) | | \phi(y^{-1}) | \ dy \leq \left( \int_G | f(xy) - f(x_0y) |^p \ dy \right)^{\frac{1}{p}} \left( \int_G | \phi(y^{-1}) |^{p'} \ dy \right)^{\frac{1}{p'}} \leq \left( \int_G | f(xy) - f(x_0y) |^p \ dy \right)^{\frac{1}{p}} \| \phi \|_\infty (\mu(K))^{\frac{1}{p'}}$$

$$y \mapsto x^{-1} y \left( \int_G | f(x^{-1} y) - f(y) |^p \ dy \right)^{\frac{1}{p}} \| \phi \|_\infty (\mu(K))^{\frac{1}{p'}} \leq \varepsilon,$$

which ends the proof.

Lemma 9. Let $F \subset L^p(G)$ be $L^p$-equivanishing. If $\phi \in C_c(G)$ is such that $\phi(e) \neq 0$, then $F \star \phi$ is equivvanishing.

Proof. Fix $\varepsilon > 0$ and choose $K \in G$ such that

$$\forall f \in F \left( \int_{G \setminus K} | f |^p \ d\mu \right)^{\frac{1}{p}} < \frac{\varepsilon}{\| \phi \|_\infty (\mu(\iota(supp(\phi))))}. \quad (13)$$

Put $U := \iota(\{\phi \neq 0\})$, which is open and relatively compact identity neighbourhood with $U = \iota(supp(\phi))$. By Lemma 6 there exists $D \in G$ such that

$$\forall x \in D \ x U \cap K = \emptyset. \quad (14)$$

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Finally, for \( f \in \mathcal{F} \) and \( x \not\in D \) we have
\[
|f \ast \phi(x)| \leq \int_G |f(y)\phi(y^{-1}x)| \, dy \leq \|\phi\|_\infty \int_{xU} |f| \, d\mu
\]
\[
\leq \|\phi\|_\infty \left( \int_{xU} |f|^p \, d\mu \right)^{\frac{1}{p}} \left( \mu(xU) \right)^{\frac{1}{p'}} \leq \left( \int_{G \setminus K} |f|^p \, d\mu \right)^{\frac{1}{p}} \|\phi\|_\infty \left( \mu(xU) \right)^{\frac{1}{p'}} < \varepsilon,
\]
which ends the proof.

We need to show that \( \mathcal{F} \ast \phi \) is in a sense close to family \( \mathcal{F} \). In order to achieve this goal, we make use of the following result:

**Theorem 10.** (Minkowski’s integral inequality, comp. [6], p. 194 or [14], p. 271)
Let \( X, Y \) be \( \sigma \)-finite measure spaces, \( 1 \leq p < \infty \) and let \( F : X \times Y \to \mathbb{C} \) be measurable. Then
\[
\left( \int_X \left( \int_Y |F(x,y)| \, dy \right)^p \, dx \right)^{\frac{1}{p}} \leq \int_Y \left( \int_X |F(x,y)|^p \, dx \right)^{\frac{1}{p}} \, dy.
\] (15)

**Theorem 11.** (comp. Proposition 2.42 in [5], p. 53)
If \( \mathcal{F} \subset L^p(G) \) is \( L^p \)-equicontinuous, then for every \( \varepsilon > 0 \) there exists \( \phi \in C_c(G) \) such that
\[
\forall f \in \mathcal{F} \quad \|f \ast \phi - f\|_p < \varepsilon.
\]

**Proof.** Fix \( \varepsilon > 0 \). Pick \( \phi \in C_c(G) \) such that \( \phi(e) \neq 0 \), \( \int_G \phi \circ \iota \, d\mu = 1 \) and \( \text{supp}(\phi \circ \iota) \subset U \), where \( U \) is the open identity neighbourhood such that
\[
\forall f \in \mathcal{F} \quad \sup_{y \in U} \|R_y f - f\|_p < \varepsilon.
\] (16)

For every \( x \in G \), we have
\[
f \ast \phi(x) - f(x) = \int_G f(y)\phi(y^{-1}x) \, dy - f(x)\int_G \phi(y^{-1}) \, dy
\]
\[
= \int_G \left( f(xy) - f(x) \right) \phi(y^{-1}) \, dy.
\]
We put
\[ F(x, y) := (f(xy) - f(x))\phi(y^{-1}). \]
This function is measurable as a composition of the following measurable functions:
\[ F_1 : (x, y) \mapsto (x, y, y), \]
\[ F_2 : (x, y, z) \mapsto (x, y, z^{-1}), \]
\[ F_3 : (x, y, z) \mapsto (x, xy, \phi(z)), \]
\[ F_4 : (x, y, z) \mapsto (f(x), f(y), z), \]
\[ F_5 : (x, y, z) \mapsto (y - x)z. \]
Since \( f, \phi \) are integrable, then \( \text{supp}(f) \) and \( \text{supp}(\phi) \) are \( \sigma \)-compact. Hence, also the sets
\[ \text{supp}(f) \cdot \text{supp}(\phi) \times \iota(\text{supp}(\phi)) \quad \text{and} \quad \text{supp}(f) \times \iota(\text{supp}(\phi)) \]
are \( \sigma \)-compact. Next, we follow a series of logical implications:
\[ (x, y) \in \{ F \neq 0 \} \implies f(xy) - f(x) \neq 0 \quad \text{AND} \quad \phi(y^{-1}) \neq 0 \]
\[ \implies \left( xy \in \text{supp}(f) \quad \text{OR} \quad x \in \text{supp}(f) \right) \quad \text{AND} \quad y^{-1} \in \text{supp}(\phi) \]
\[ \implies \left( xy \in \text{supp}(f) \quad \text{AND} \quad y^{-1} \in \text{supp}(\phi) \right) \quad \text{OR} \quad \left( x \in \text{supp}(f) \quad \text{AND} \quad y^{-1} \in \text{supp}(\phi) \right) \]
\[ \implies (x, y) \in \text{supp}(f) \cdot \text{supp}(\phi) \times \iota(\text{supp}(\phi)) \quad \text{OR} \quad (x, y) \in \text{supp}(f) \times \iota(\text{supp}(\phi)). \]
We conclude that \( \{ F \neq 0 \} \) is \( \sigma \)-compact.

Finally, we are in position to apply Minkowski’s integral inequality:
\[
\forall f \in F \quad \| f \ast \phi - f \|_p = \left( \int_G \left( \int_G \left| f(xy) - f(x) \phi(y^{-1}) \right|^p \, dy \right)^{\frac{1}{p}} \, dx \right)^{\frac{1}{p}} \]
\[ \leq \int_G \left( \int_G \left| f(xy) - f(x) \right|^p \phi(y^{-1}) \, dy \right)^{\frac{1}{p}} \, dx \]
\[ = \int_G \| R_yf - f \|_p \phi(y^{-1}) \, dy \leq \sup_{y \in U} \| R_yf - f \|_p < \varepsilon, \]
which ends the proof. \( \Box \)
3.2 Young’s inequality and Fréchet-Kolmogorov-Riesz-Weil’s theorem

We begin with a version of Young’s inequality for locally compact groups. In [13], one can find a version for unimodular groups, but we do not impose such restriction. As far as the notation is concerned, for \( p \in [1, \infty) \) we understand \( p' \) to be the number satisfying

\[
\frac{1}{p} + \frac{1}{p'} = 1.
\]

**Theorem 12.** (Young’s inequality)

Let \( p, q, r \in [1, \infty) \) be such that

\[
\frac{1}{r} = \frac{1}{p} + \frac{1}{q} - 1. \tag{17}
\]

For \( f \in L^p(G) \) and \( g \in L^q(G) \), the convolution \( f \ast \Delta^{\frac{1}{p'}} g \) exists almost everywhere. Moreover, \( f \ast \Delta^{\frac{1}{p'}} g \in L^r(G) \) and we have

\[
\| f \ast \Delta^{\frac{1}{p'}} g \|_r \leq \| f \|_p \cdot \| g \|_q. \tag{18}
\]

**Proof.** Observe that it suffices to prove (18). This will immediately mean that \( f \ast \Delta^{\frac{1}{p'}} g \in L^r(G) \) and consequently that the convolution exists almost everywhere.

At first, we note a couple of equalities:

\[
\frac{1}{r} + \frac{1}{q'} + \frac{1}{p'} = \frac{1}{r} + \left( 1 - \frac{1}{q'} \right) + \left( 1 - \frac{1}{p'} \right) \equiv 1, \tag{17}
\]

\[
\left( 1 - \frac{p}{r} \right) q' \equiv p \left( 1 - \frac{1}{q} \right) q' = p, \tag{19}
\]

\[
\left( 1 - \frac{q}{r} \right) p' \equiv q \left( 1 - \frac{1}{p} \right) p' = q.
\]

13
With the use of Hölder’s inequality, we may perform the main calculation:

\[ |f \ast \Delta_{\vec{y}}^q g(x)| \leq \int_G \left( |f| (y)^{\frac{p}{n}} |g|(y^{-1}x)^{\frac{q}{n}} \right) \cdot |f|(y)^{(1-\frac{p}{n})} \cdot \left( |g|(y^{-1}x)^{(1-\frac{q}{n})} \right) \Delta_{\vec{y}}^q (y^{-1}x) \, dy \]

\[ \leq \left( \int_G |f| (y)^{p} |g|(y^{-1}x)^{q} \, dy \right)^{\frac{1}{p}} \left( \int_G |f| (y)^{(1-\frac{p}{n})} \, dy \right)^{\frac{1}{q}} \left( \int_G |g|(y^{-1}x)^{(1-\frac{q}{n})} \Delta_{\vec{y}}^q (y^{-1}x) \, dy \right)^{\frac{1}{q'}} \]

\[ \overset{(19)}{=} \left( \int_G |f|(y)^{p} |g|(y^{-1}x)^{q} \, dy \right)^{\frac{1}{p}} \left( \int_G |f|(y)^{q} \, dy \right)^{\frac{1}{q}} \left( \int_G |g|(y^{-1}x)^{q} \Delta_{\vec{y}}^q (y^{-1}x) \, dy \right)^{\frac{1}{q'}} \]

\[ = \left( \int_G |f|(y)^{p} |g|(y^{-1}x)^{q} \, dy \right)^{\frac{1}{p}} \left( \int_G |f|(y)^{q} \, dy \right)^{\frac{1}{q}} \left( \int_G |g|(y^{-1}x)^{q} \, dy \right)^{\frac{1}{q'}} \]

\[ = \left( \int_G |f|(y)^{p} |g|(y^{-1}x)^{q} \, dy \right)^{\frac{1}{p}} \|f\|_{p}^{\frac{p}{q}} \|g\|_{q}^{\frac{q}{q'}} = \left( |f|^p \ast |g|^q(x) \right)^{\frac{1}{p}} \|f\|_{p}^{\frac{p}{q}} \|g\|_{q}^{\frac{q}{q'}}. \]

(20)

The above estimates lead to

\[ \int_G \left| f \ast \Delta_{\vec{y}}^q g(x) \right|^r \, dx \leq \int_G |f|^p \ast |g|^q(x) \, dx \|f\|_{\frac{p}{r}}^{\frac{p}{r}} \|g\|_{\frac{q}{r}}^{\frac{q}{r'}} = \|f|^p \ast |g|^q \|_{1} \|f\|_{\frac{p}{r}}^{\frac{p}{r}} \|g\|_{\frac{q}{r}}^{\frac{q}{r'}} \]

\[ \leq \|f\|^p_1 \cdot \|g\|^q_1 \cdot \|f\|_{\frac{p}{r}}^{\frac{p}{r}} \|g\|_{\frac{q}{r}}^{\frac{q}{r'}} = \|f\|^p_1^{\frac{p}{r} + \frac{q}{r'}} \|g\|^q_1^{\frac{q}{r} + \frac{q}{r'}}, \]

where the second inequality follows from Theorem 1.6.2 in [3], p. 26. Taking the \( r \)-th root, we conclude that

\[ \|f \ast g\|_r \leq \|f\|_p^{\frac{p}{r} + \frac{q}{r'}} \|g\|_q^{\frac{q}{r} + \frac{q}{r'}} = \|f\|_p \|g\|_q, \]

which ends the proof.

We need one final theorem before the Fréchet-Kolmogorov-Riesz-Weil theorem.

**Theorem 13.** Let \( \mathcal{F} \) be \( L^p \)-bounded and let \( \phi \in C_c(G) \). If \( K \subset G \) is a compact set then \( \mathcal{F}|_K \ast \phi \) is relatively compact in \( C_0(G) \).
Proof. Suppose that $\|f\|_p \leq M$ for every $f \in \mathcal{F}$. At first, we show that the functions in $\mathcal{F}|_K \star \phi$ have common support, which is compact. For every $f \in \mathcal{F}$ observe that

$$\forall x \in G \left| f|_K \star \phi(x) \right| = \left| \int_K f(y) \phi(y^{-1}x) \, dy \right|,$$

and the integral on the right-hand side is 0 for $x \notin K \cdot \text{supp}(\phi)$. Hence

$$\text{supp}(f|_K \star \phi) \subset K \cdot \text{supp}(\phi)$$

for every $f \in \mathcal{F}$, and we conclude that the whole family $\mathcal{F}|_K \star \phi$ has a common compact support.

Next, we prove that $\mathcal{F}|_K \star \phi$ is equicontinuous. Pick $\varepsilon > 0$ and fix $x \in G$. Since $\phi$ is uniformly continuous (comp. \textit{[3]}, Lemma 1.3.6, p. 11), there exists a symmetric, open neighbourhood of the identity $U_\varepsilon \subset G$ such that

$$\forall u^{-1}v \in U_\varepsilon \left| \phi(u) - \phi(v) \right| < \frac{\varepsilon}{M \mu(K)^{\frac{1}{p'}}}. \tag{21}$$

Since

$$(z^{-1}y)^{-1}(z^{-1}x) \in U_\varepsilon \iff y^{-1}x \in U_\varepsilon \tag{22}$$

then

$$\forall f \in \mathcal{F} \left| f|_K \star \phi(y) - f|_K \star \phi(x) \right| = \left| \int_K f(z) \left( \phi(z^{-1}y) - \phi(z^{-1}x) \right) \, dz \right| \leq \|f\|_p \left( \int_K |\phi(z^{-1}y) - \phi(z^{-1}x)|^{p'} \, dz \right)^{\frac{1}{p'}} \leq \frac{\varepsilon}{M \mu(K)^{\frac{1}{p'}}}. \tag{21, 22}$$

The above estimate proves that $\mathcal{F}|_K \star \phi$ is equicontinuous. Since we already established that the whole family has a common compact support, $\mathcal{F}|_K \star \phi$ is obviously equivanishing.

Finally, we prove that $\mathcal{F}|_K \star \phi$ is bounded in $C_0(G)$. We have

$$\forall f \in \mathcal{F} \left\| f|_K \star \phi \right\|_\infty = \sup_{x \in G} \left| \int_G f|_K(y) \phi(y^{-1}x) \, dy \right| \leq M \|\phi \circ \iota\|_{p'},$$

which proves that $\mathcal{F}|_K \star \phi$ is bounded in $C_0(G)$. We conclude the proof by applying Theorem \textit{[5]}.
Below, we present the crowning result of the paper.

**Theorem 14.** (Fréchet-Kolmogorov-Riesz-Weil’s theorem on a locally compact group)
A family $\mathcal{F} \subset L^p(G)$ is relatively compact if and only if

1. **(FKRW1)** $\mathcal{F}$ is bounded in $L^p(G)$—norm,
2. **(FKRW2)** $\mathcal{F}$ is $L^p$—equicontinuous,
3. **(FKRW3)** $\mathcal{F}$ is $L^p$—equivanishing.

**Proof.** At first, suppose that $\mathcal{F}$ is relatively compact. Then **(FKRW1)** follows immediately. For the rest of the proof, fix $\varepsilon > 0$.

As far as **(FKRW2)** is concerned, let $\{f_n\}_{n=1}^N \subset \mathcal{F}$ be an $\frac{\varepsilon}{3}$—net. By Proposition 2.41 in [5], p. 53 for every $n = 1, \ldots, N$ there exists an open identity neighbourhood $U_n$ such that

$$
\sup_{x \in U_n} \|L_x f_n - f_n\|_p < \frac{\varepsilon}{3} \quad \text{and} \quad \sup_{x \in U_n} \|R_x f_n - f_n\|_p < \frac{\varepsilon}{3}.
$$

Put $U = \bigcap_{n=1}^N U_n$, which is obviously open. Then for every $f \in \mathcal{F}$, there exists $n = 1, \ldots, N$ such that

$$
\sup_{x \in U} \|L_x f - f\|_p \leq \sup_{x \in U} \|L_x f - L_x f_n\|_p + \sup_{x \in U} \|L_x f_n - f_n\|_p + \|f_n - f\|_p
$$

$$
= 2\|f_n - f\|_p + \sup_{x \in U} \|L_x f_n - f_n\|_p \lesssim \varepsilon.
$$

An analogous reasoning works for $\|R_x f - f\|_p$, which proves that $\mathcal{F}$ is $L^p$—equicontinuous.

As far as **(FKRW3)** is concerned, let $\{g_n\}_{n=1}^N$ be an $\frac{\varepsilon}{2}$—net for $\mathcal{F}$. For every $n = 1, \ldots, N$ there exists $K_n \subset G$ such that

$$
\int_{G \setminus K_n} |g_n|^p \, d\mu < \frac{\varepsilon}{2}.
$$

Put $K = \bigcup_{n=1}^N K_n$, which is also compact. Then, for every $f \in \mathcal{F}$ there exists $n = 1, \ldots, N$ such that

$$
\int_{G \setminus K} |f|^p \, d\mu \leq \int_{G \setminus K} |f - g_n|^p \, d\mu + \int_{G \setminus K} |g_n|^p \, d\mu \lesssim \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$
This proves that $\mathcal{F}$ is $L^p$--equiavanzing.

At this point we prove the converse, namely that (FKRW1), (FKRW2) and (FKRW3) imply relative compactness of $\mathcal{F}$ in $L^p(G)$. By Theorem 11 pick $\phi \in C_c(G)$ such that

$$\forall f \in \mathcal{F} \|f \ast \phi - f\|_p < \frac{\varepsilon}{3}. \quad (25)$$

Now let $K \subset G$ be a compact set such that

$$\forall f \in \mathcal{F} \int_{G \setminus K} |f|^p \, d\mu < \left(\frac{\varepsilon}{3\|\Delta^{-\frac{1}{p}}\phi\|_1}\right)^p. \quad (26)$$

The family $\mathcal{F}\ast\phi$ is relatively compact in $C_0(G)$ by Theorem 5 and Lemmas 7, 8 and 9. By Theorem 13, we obtain the relative compactness of $\mathcal{F}|_K \ast \phi$ in $C_0(G)$. Hence, there exists a finite sequence $(h_n)_{n=1}^N \subset C_c(G)$ with $\text{supp}(h_n) \subset K \cdot \text{supp}(\phi)$ such that

$$\forall f \in \mathcal{F} \exists n = 1, \ldots, N \|f|_K \ast \phi - h_n\|_\infty < \frac{\varepsilon}{3\mu(K \cdot \text{supp}(\phi))}. \quad (27)$$

Finally, for every $f \in \mathcal{F}$ there exists $n = 1, \ldots, N$ such that

$$\|f - h_n\|_p \leq \|f - f \ast \phi\|_p + \|f \ast \phi - f|_K \ast \phi\|_p + \|f|_K \ast \phi - h_n\|_p \leq \frac{\varepsilon}{3} + \|f - f|_K\|_p \cdot \|\Delta^{-\frac{1}{p}}\phi\|_1 + \left(\int_{K \cdot \text{supp}(\phi)} \left|\left(\int_{K \cdot \text{supp}(\phi)} \left|f|_K \ast \phi - h_n\right|\right)(x)\right|\right)^\frac{1}{p} < \varepsilon,$$

which ends the proof. \hfill \square

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