Dynamical instability, Chaos, and Bloch oscillations of Bose-Einstein condensates in tilted optical lattices

Andrey R. Kolovsky
Max-Planck-Institut für Physik komplexer Systeme, D-01187 Dresden and
Kirensky Institute of Physics, Ru-660036 Krasnoyarsk.
(Dated: September 14, 2018)

We study the Bloch dynamics of a Bose-Einstein condensate of cold atoms by using the formalism of the discrete nonlinear Schrödinger equation. Depending on the static force magnitudes the system is shown to exhibit two qualitatively different regimes of Bloch oscillations – exponential decay for the static force magnitude less than some critical value (defined by the condensate density) and quasiperiodic oscillations in the opposite case. The relation of these regimes to the onset of chaos in the system is discussed.

PACS numbers: 03.75.Kk, 32.80.Pj, 03.75.Nt, 71.35.Lk

I. INTRODUCTION

Recently much attention has been payed to Bloch oscillations (BO) of Bose-Einstein condensate (BEC) of cold atoms in the optical lattices. It was found that, unlike the case of a dilute atomic gas, where the atom-atom interactions can be neglected, BO of BEC rapidly decay. A plausible explanation of this phenomenon is suggested by the dynamical instability of the Gross-Pitaevskii equation in the presence of periodic potential (see papers and references therein). In this paper we revisit this approach, assuming for simplicity the relatively deep optical lattices, where the Gross-Pitaevskii equation can be approximated by the discrete nonlinear Schrödinger equation (DNLSE). It is shown below that a strong static force can actually suppress the dynamical instability of the DNLSE and, as a consequence, BOs become not-decaying. We indicate a condition on the critical magnitude of the static force, and a condition on the critical density of BEC. This result is well supported by the numerical simulation of the system dynamics, performed for the realistic parameters of the present-days laboratory experiments with cold atoms.

II. DYNAMICAL INSTABILITY

According to DNLSE the (mean field) Bloch dynamics of a BEC is described by the system of coupled nonlinear equations

\[ i\hbar \dot{a}_l = -\frac{J}{2}(a_{l+1} + a_{l-1}) + g|a_l|^2a_l + dF a_l. \tag{1} \]

In Eq. (1) \( a_l(t) \) is the complex amplitude of a BEC of atoms localised in the \( l \)th well of the optical potential, \( J \) is the hopping or tunnelling matrix element (who’s value is uniquely defined by the lattice depth), \( d \) the lattice period, \( F \) magnitude of the static force, and \( g \) the nonlinear parameter given by the product of the microscopic interaction constant \( W \) and the filling factor \( \bar{n} \) (mean number of atoms per lattice site), \( g = \bar{n} W \). For a nonuniform density \( \bar{n} \) will mean the peak density.

Let us first consider the case of uniform initial conditions \( a_l(0) = 1 \). (Note that we do not normalise \( \sum_l |a_l|^2 \) to unity.) Subsequently using the gauge and Fourier transformation, \( b_\kappa = L^{-1/2}\sum_l \exp(ikl - i\omega_B t)a_l \), Eq. (1) can be written in the form,

\[ i\hbar \dot{b}_\kappa = -J \cos(d\kappa - \omega_B t)b_\kappa \]

\[ + \frac{g}{L} \sum_{\kappa_1,\kappa_2,\kappa_3} b_{\kappa_1}^\dagger b_{\kappa_2} b_{\kappa_3} \delta(\kappa - \kappa_1 + \kappa_2 - \kappa_3), \]

where \( \kappa \) is the quasimomentum \( -\pi/d \leq \kappa < \pi/d \), and \( \omega_B = dF/\hbar \) Bloch frequency. For the given initial condition Eq. (2) has trivial solution

\[ b_0(t) = \exp\left( i \frac{J}{dF} \sin(\omega_B t) - i \frac{g}{\hbar} t \right) \sqrt{L}, \quad b_{\kappa \neq 0}(t) = 0, \tag{3} \]

which is no other than the celebrated BO, where the kinetic and Stark energies of the system oscillate according to the cosine law with the Bloch period \( T_B = 2\pi/\omega_B = \hbar/dF \). However, for \( g \neq 0 \) the solution (3) can be unstable with respect to small perturbations. Following the standard approach we linearise Eq. (2) around the solution (3), which leads to the system of linear equations,

\[ i\hbar \dot{b}_{\kappa \pm} = -J \cos(d\kappa - \omega_B t)b_{\kappa \pm} + 2\frac{g}{L} b_0^\dagger b_{\kappa \pm} + \frac{g}{L} b_{0\kappa \mp}^\dagger, \]

\[ i\hbar \dot{b}_{\kappa \mp} = -J \cos(d\kappa + \omega_B t)b_{\kappa \mp} + 2\frac{g}{L} b_0^\dagger b_{\kappa \mp} + \frac{g}{L} b_{0\kappa \pm}^\dagger, \tag{4} \]

where \( b_{\kappa \pm}(0) \) are arbitrary small. Substituting here \( b_0(t) \) from Eq. (3) and integrating Eq. (4) in time, we have

\[ b_{\kappa \pm}(t) \sim \exp(\nu t) b_{\kappa \pm}(0), \quad \nu \text{ given by the logarithm of the maximal eigenvalue of the Floquet matrix} \]

\[ U = \exp\left[ -i \frac{g}{\hbar} \int_0^{T_B} \left( -f(t) - 1 \right) dt \right], \tag{5} \]

\[ f(t) = \exp\left[ \frac{2J}{dF}(1 - \cos(d\kappa))\sin(\omega_B t) \right], \]
where the hat over the exponent denotes the time ordering. It is easy to see that the matrix $U$ is parametrised by the product $dx$, the ratio $g/dF$, and the ratio $J/dF$. Thus, without lost of generality, we can set two parameters ($J$ and $d$ in what follows) to unity. A typical dependence of the increment $\nu$ on the quasimomentum $\kappa$ and the static force magnitude $F$ is depicted in Fig. 1. It is seen in the figure that there is a critical value $F_{\text{cr}} \approx 1.2$ above which $\nu \equiv 0$ and, hence, the solution (3) is stable. Scanning over different $g$, we found $F_{\text{cr}} \approx 3g$ for $0.1 \leq g \leq 1$.\(^{12}\)

Suppression of the dynamical instability is easy to check numerically. These numerical simulations, performed for different lattice size $L$ and periodic boundary conditions, undoubtedly indicate existence of the critical $F$. Moreover, this is also valid for the non-uniform initial conditions $|a_i|^2 = \exp(-I^2/\sigma^2)$, realized in the experiment. Although in this case and for $F > F_{\text{cr}}$ BO slowly decay (with subsequent revivals, see Fig. 4 below), this decay is due to the trivial dephasing and fundamentally differs from the exponentially fast destruction of BO at $F < F_{\text{cr}}$, caused by the dynamical instability.

### III. TWO-SITE MODEL

One obtains a useful insight in physics of the discussed phenomenon by considering the limiting case of the lattice with only two sites,

$$H(t) = -J \cos(\omega_B t)(a_2^\dagger a_1 + a_1^\dagger a_2) + \frac{g}{2} \sum_{l=1}^{2} |a_l|^4,$$  

(6)
in the numerical simulations) and as an initial condition $|\Psi(0)\rangle$ we choose the ground state of (8) for $F = 0$, which is approximately given by the product of $N$ Bloch waves with zero quasimomentum. It is found that when the central stability island in Fig. 2 is large enough in comparison with the quantum of the phase space volume $2\pi\hbar$ decoherence is negligible. On the contrary, when the periodic point $(I, \theta) = (0, 0)$ is hyperbolic, we observe rapid decoherence of the system (see upper panel in Fig. 3). Note that the rate of decoherence can be well estimated by using the classical (i.e., mean field) dynamics and considering an ensemble of initial conditions scattered around the periodic point. Then the one-particle density matrix (9) is given by the correlation matrix, 

$$\rho_{l,m}(t) = L^{-1} \langle \langle a_l^* (t) a_m(t) \rangle \rangle ,$$

where the double-angle brackets denote the average over an ensemble of the initial conditions. The characteristic width of the distribution function for the action and angle variables is obviously given by the effective Planck’s constant or, equivalently, by the quantum fluctuations of the number of atoms in one well, $\Delta F \sim \hbar^2 \Delta n^2 \sim 1/\bar{n}$, and phase fluctuations, $\Delta \theta \sim 1/\Delta n^2 \sim 1/\bar{n}$ [12]. For $\bar{n} = 50$ the classical dynamics of $\rho^2(t)$ is depicted in the low panel of Fig. 3. By comparing both panels one concludes that decoherence of the quantum system (8) is actually due to the chaotic dynamics of its classical counterpart (9).

IV. BLOCH OSCILLATIONS

Although it looks problematic to extend the above phase-space analysis of Sec. III for $L \gg 1$, some conjectures are still possible. Indeed, using the polar presentation for complex amplitudes $a_l$, one maps the $L$-site system into a system of $L$ driven coupled pendula and there is no doubt that this system would be generally chaotic [7]. On the other hand, absence of the dynamical instability for $F > F_{cr}$ indicates the presence of the stable periodic trajectory in the multi-dimensional phase space of the system corresponding to the stable BO. The crucial point is, however, whether the stability island surrounding this periodic trajectory is large enough to support the quantum state. To answer this question we numerically solve the DNLSE for randomly varied initial conditions,

$$|a_l|^2 = |a_l|^2 + \xi \left( \frac{|a_l|^2}{\bar{n}} \right)^{1/2} , \quad |a_l|^2 = \exp \left( -\frac{t^2}{\sigma^2} \right) ,$$

$$\theta_l = \xi \left( 4\bar{n}|a_l|^2 \right)^{-1/2} , \quad a_l = |a_l| \exp(i\theta_l)$$

where $\xi$ is a random Gaussian variable with $\overline{\xi^2} = 1$, which accounts for the quantum fluctuations. The results are presented in Fig. 4 which shows dynamics of the mean atomic momentum for $g = 0.4, \bar{n} = 200, \sigma = 25.6, F = 1.3$ (upper panel) and $F = 0.7$ (lower panel). It is seen that the stable regime of BO is also stable against the quantum fluctuations. This validates our conclusion about two qualitatively different regimes of BO. It interesting to estimate critical values of the parameters for a typical laboratory experiment. Taking as an example the recent experiment [4] with $^{87}$Rb atoms in a vertically oriented quasi one-dimensional lattice of the depth 4 recoil energies, one has $dF = 0.28, J = 0.17$, and
$W = 0.71 \cdot 10^{-5}$ recoil energies, respectively. Then the condition on the critical magnitude of the static force is formulated as a condition on the critical density of the condensate and corresponds to $\bar{n}_{cr} = 120$ atoms per lattice site. Since the peak density in Ref. [1] is larger than $\bar{n}_{cr}$, BOs should decay, which agrees with the experimental finding.

V. CONCLUSION

We have studied the dynamics induced by a static force of a BEC of cold atoms in an optical lattice. Depending on the static force magnitude $F$ (or, equivalently, on the density of the condensate $\bar{n}$) the system is shown to have two qualitatively different regimes of BO – exponential decay of oscillations for $F < F_{cr}(\bar{n})$ and ‘quasiperiodic’ oscillations for $F > F_{cr}(\bar{n})$. It is argued in the paper that the former regime reflects the chaotic dynamics of the system, where the transition to chaos coincides with onset of the dynamical instability in DNLSE.

It is worth of noting that the above results are obtained within the mean field approach. Because this approach assumes the limit $\bar{n} \to \infty$, $W \bar{n} = g/\bar{n} \to 0$, for any finite $\bar{n}$ the classical (macroscopic) dynamics of the system, governed by DNLSE, deviates from the quantum (microscopic) dynamics, governed by the Bose-Hubbard model. It is a challenge both for theory and experiment to establish the relation between the classical and quantum results, depending on the parameter $\tilde{\hbar} = 1/\bar{n}$. We would also like to stress that for considering the problem of quantum-to-classical correspondence one has to average the solution of DNLSE over an appropriate ensemble of the initial conditions. Without this additional procedure, the solutions of the DNLSE may have nothing to do with the solutions of the Bose-Hubbard model.

In this present paper we analyse the problem of quantum-to-classical correspondence for BO by using a two-site model. Both the similarities and discrepancies were found. In particular, we show that the classically chaotic dynamics of the system for $F < F_{cr}$ is responsible for decoherence of the system, which is the deep reason for the decay of BOs. At the same time, the rate of decoherence, obtained by the means of the DNLSE, appears to be correct only for the short-time dynamics.

To conclude, we briefly discuss the relation of the above results with those of our recent papers [8, 9, 10], devoted to the Bloch dynamics of the system in the deep quantum regime $\hbar \sim 1/\bar{n} \sim 1$. Clearly, in this case one cannot appeal to the results of the classical (DNLSE) analysis and the system should be treated microscopically. Remarkably, this microscopical analysis also reveals two regimes of BO – regular (not-decaying) oscillations for a strong forcing $\hat{E}$ and rapid decay (decoherence) of BO in the case of a weak static force $\hat{E}$.

The author acknowledges discussions with J. Brand, A. Buchleitner, S. Flach, and S. Sinha.

[1] O. Morsch, J. H. M"uller, M. Cristani, D. Ciampini, and E. Arimondo, Phys. Rev. Lett. 87, 140402 (2001).
[2] M. Cristani, O. Morsch, N. Malossi, M. Jona-Lasinio, M. Anderlini, E. Courtade, and E. Arimondo, e-print cond-mat/0311160 (2003).
[3] Biao Wu and Qian Niu, New J. Phys. 5, 104 (2003).
[4] G. Roati, E. de Mirandaes, F. Ferlaino, H. Ott, G. Modugno, M. Inguscio, Phys. Rev. Lett. 92, 230402 (2004).
[5] This result coincides with an estimate $F_{cr} \approx 3.03g$ ($g/J < 1$), reported in the recent paper by Yi Zheng, M. Kostrun, and J. Javanainen, Phys. Rev. Lett. 93, 230401 (2004).
[6] Indeed, for $F > F_{cr}$ the complex amplitudes evolve as $a_t(t) \approx \exp(-i\omega_B t - ig|a|^2) a_t(0)$. Then, substituting here $|a|^2 = \exp(\hat{P}/\hbar)\hat{N}$ one obtains the characteristic dynamics of BO shown in Fig. 2. Note that this regime was considered earlier by Trombettoni and Smerzi [A. Trombettoni and A. Smerzi, Phys. Rev. Lett. 86, 2353 (2001)] by using a variational ansatz.
[7] Q. Thommen, J. C. Garreau, and V. Zehnle, Phys. Rev. Lett. 91, 210405 (2003).
[8] A. R. Kolovsky, Phys. Rev. Lett. 90, 213002 (2003).
[9] A. R. Kolovsky and A. Buchleitner, Phys. Rev. E68, 056213 (2003).
[10] A. Buchleitner and A. R. Kolovsky, Phys. Rev. Lett. 91, 253002 (2003).
[11] A similar Hamiltonian appears when one studies a BEC of cold atoms in a driven double-well potential, see the recent paper [12] and references therein. We also note that with a change $\sqrt{1 - F^2} \to 1$ the Hamiltonian 6 corresponds to the so-called Double Resonance Model, which was intensively studied during last four decades first in the connection to classical and then to quantum chaos [see, for example, a review G. P. Berman and A. R. Kolovsky, Sov. Phys. Usp. 35, 303 (1992)].
[12] K. W. Mahmud, H. Perry, and W. P. Reinhardt, e-print cond-mat/0312016 (2004).
[13] In a more general context a relation between the onset of dynamical instability and transition to chaos is discussed in G. P. Berman and A. R. Kolovskii, Sov. Phys. JETP 50, 1116 (1984).
[14] The effective Planck’s constant $\hbar$ is given by $\hbar = 2N$, where $N = 2\bar{n} + 1 \approx 4\bar{n}$ is dimension of the Hilbert space.
[15] More regorously, the distribution function for the classical ensamble is given by Husimi image of the initial quantum state.
[16] This regime is typically realised in 3D lattices. Note that the microscopic interaction constant $W$ in 3D lattices is typically three order of magnitude larger than in the quasi 1D lattices which, together with $\bar{n} \sim 1$, gives approximately the same value for the nonlinear parameter $g$. 
