Two Examples of Toric Arrangements

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Abstract

We show that the integral cohomology algebra of the complement of a toric arrangement is not determined by the poset of layers. Moreover, the rational cohomology algebra is not determined by the arithmetic matroid (however it is determined by the poset of layers). We also study the combinatorics of realizable arithmetic matroids over a modular matroid.

Introduction

A toric arrangement is a finite collection of hypertori lying in an algebraic torus. The complement of a toric arrangement is an open set: we are interested in studying its cohomology ring. The combinatorics of an arrangement is encoded in its poset of layers, i.e. the poset of connected components of the intersections of some hypertori. No description of this class of poset is known so far. In the case of hyperplane arrangements, the associated poset we consider is the poset of intersections, which turns out to be a geometric lattice. The combinatorial data of a hyperplane arrangement can be stored equivalently in another structure: a matroid. There are several equivalent definitions of matroid, we point to [Oxl11] for a general reference. Matroids can be generalized to the toric case in different ways: arithmetic matroids (see [DM13] and [BM14]), matroids over rings (see [FM16]) and G-semimatroids (see [DR18]). All these combinatorial data permit us to define the arithmetic Tutte polynomial (introduced first in [Moc12]), that is the analogous, in the case of toric arrangements, of the Tutte polynomial of a hyperplane arrangement.

The study of interplay between the cohomology of the complement of a toric arrangement and its combinatorics started in [Loo93], where the Betti numbers of the complement were computed; it follows that the Poincaré polynomial is a specialization of the arithmetic Tutte polynomial (see [Moc12]). The cohomology algebra of the complement of a hyperplane arrangement is the cohomology of the algebraic de Rham complex (see [Gro66]) that was combinatorially described in [OS80]. An analogous approach in the toric case started with [DP05] (see also [DP11]), where a description of the
graded\textsuperscript{1} cohomology algebra with complex coefficients $\text{Gr} H(M(A), \mathbb{C})$ was given. The graded algebra with rational coefficients $\text{Gr} H(M(A), \mathbb{Q})$ can be obtained from the Leray spectral sequence of the inclusion of the complement $M(A)$ into the algebraic torus, as shown in [Bib16] and [Dup15]. The graded algebra with integer coefficients $\text{Gr} H(M(A), \mathbb{Z})$ was studied in [CD17] and from the combinatorial point of view, in [Pag17]. Recently, presentations of the cohomology algebras $H(M(A), \mathbb{Q})$ and $H(M(A), \mathbb{Z})$ in the spirit of [OS80] was obtained in [CDD+18], generalizing [DP05]. The description of $H(M(A), \mathbb{Q})$ depends only on the poset of layers.

In Section \textbf{2} we show that the integral cohomology algebra $H(M(A), \mathbb{Z})$ of the complement of a toric arrangement is not combinatorial, i.e. it does not depend only on the poset of layers (Theorem \textbf{2.1}).

In Section \textbf{3} we study arithmetic matroids over a modular matroid and we show, in this case, that the posets of layers of all realizations are all isomorphic.

Moreover, in Section \textbf{4} we show that arithmetic matroids and matroids over $\mathbb{Z}$ contain less information than the poset of layers. Indeed, we build two central toric arrangements with the same arithmetic matroid, the same matroid over $\mathbb{Z}$, but with non-isomorphic posets of layers (Theorem \textbf{4.1}) and non-isomorphic cohomology algebra with rational coefficients. As consequence, there cannot exist a “cryptomorphism” between arithmetic matroids (respectively, matroids over $\mathbb{Z}$) and any class of posets such that – in the realizable cases – the poset associated with the matroid coincides with the poset of layers of any realization.

The following question about central toric arrangements remains open.

\textbf{Question 1.} Does the integral cohomology algebra of the complement of a central toric arrangement determine the toric arrangement?

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1 Definitions

Let $N = (n_{i,j})$ be a matrix with integer coefficients of size $r \times n$.\footnote{The filtration used in [DP05] coincides with the Leray filtration for the inclusion of the complement in the complex torus.}
Definition 1.1. The central toric arrangement $\mathcal{A}$ defined by $N$ is the collection of $n$ hypertori $H_1, \ldots, H_n$ in $T = (\mathbb{C}^*)^r$, where

$$H_j \overset{\text{def}}{=} \{ (x_1, \ldots, x_r) \in (\mathbb{C}^*)^r \mid x_1^{n_{1,j}} x_2^{n_{2,j}} \cdots x_r^{n_{r,j}} = 1 \}.$$  

We are interested in studying the topological invariants of the complement

$$M(\mathcal{A}) \overset{\text{def}}{=} T \setminus \bigcup_{H \in \mathcal{A}} H$$

of a toric arrangement $\mathcal{A}$.

Definition 1.2. Let $\mathcal{A} = \{H_1, \ldots H_n\}$ be a toric arrangement. A layer $W$ of $\mathcal{A}$ is a connected component of the intersection of some hypertori in $\mathcal{A}$. The poset of layers $S(\mathcal{A})$ of the toric arrangement $\mathcal{A}$ is the partially ordered set whose elements are all the layers of $\mathcal{A}$ ordered by reverse inclusions. The poset of layers is ranked by the codimension in $T$

$$\text{rk} W = \text{codim}_T W.$$  

Remark 1.3. The poset of layers does not coincide with the poset of torsions defined in [Mar18].

For each pair of layers $(W_1, W_2)$ there exists a unique meet $W_1 \wedge W_2$ of the two layers, i.e. the minimal layers containing both $W_1$ and $W_2$. However, the join $W_1 \vee W_2$ of two elements may not be unique (e.g. if the intersection of the two tori is not connected).

In the final part of this paper we will need the notions of arithmetic matroids and of matroids over $\mathbb{Z}$. A matroid is a pair $(E, \text{rk})$ where $E$ is a finite set and $\text{rk} : \mathcal{P}(E) \to \mathbb{N}$ is the rank function with some properties (see [Oxl11] for the exact definition). One of these properties is the submodularity: for each pair of subsets $S, T \subseteq E$ the following inequality holds:

$$\text{rk}(S) + \text{rk}(T) \geq \text{rk}(S \cap T) + \text{rk}(S \cup T).$$

An arithmetic matroid is a matroid $(E, \text{rk})$ with a multiplicity function $m : \mathcal{P}(E) \to \mathbb{N} \setminus \{0\}$ that satisfies five properties (listed for instance in [DM13] or [BML4] for the definition).

A matroid over $\mathbb{Z}$ is a ground set $E$ together with a $\mathbb{Z}$-module $M(S)$ for each subset $S$ of $E$ such that these modules satisfy some specific relations (see [FM16] for the definition).

We will give the definition of arithmetic matroids and matroids over $\mathbb{Z}$ only in the realizable cases, since we deal only with the realizable ones.

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2The join of $W_1$ and $W_2$ is $W_1 \vee W_2$, an element greater of both $W_1$ and $W_2$ and minimal with this property.

3Also known as $\mathbb{Z}$-matroids.
Let \( N \in M(r, n; \mathbb{Z}) \) be an integer matrix. We denote with \( N[S] \), for \( S \subseteq [n] \), the sub-matrix of \( N \) made up with the columns indexed with \( S \). The \textit{realizable matroid over} \( \mathbb{Z} \) described by \( N \) is the function \( M \) that associates to every subset \( S \subseteq [n] \) the module \( \mathbb{Z}^n / \langle N[S] \rangle \), where \( \langle N[S] \rangle \) is the sub-module generated by the columns of \( N[S] \). The \textit{realizable arithmetic matroid} described by \( N \) is \((n, \text{rk}, m)\), where \( \text{rk}(S) = \text{rank}\langle N[S]\rangle \) and \( m(S) = |\text{tor}(\mathbb{Z}^n / \langle N[S]\rangle)| \) is the cardinality of the torsions of the quotient.

The following two definitions are standard in matroid theory.

**Definition 1.4.** A \( k \)-flat of a matroid \((E, \text{rk})\) is a maximal subset of \( E \) of rank \( k \). The \textit{poset of flats} is the partially ordered set whose elements are the flats of \((E, \text{rk})\) ordered by reverse inclusion.

The following class of modular matroids is quite small, though it contains free matroids and projective geometries. For a general reference on modular matroids, see [Oxl11, Section 6.9].

**Definition 1.5.** A pair of flats \((S, T)\) is modular if the following equality holds
\[
\text{rk}(S) + \text{rk}(T) = \text{rk}(S \cap T) + \text{rk}(S \cup T).
\]
A flat \( S \) is modular if for all flats \( T \) the pair \((S, T)\) is modular. A matroid is modular if all its flats are modular.

Let \( A = \bigoplus_{n \in \mathbb{N}} A^n \) be a graded-commutative algebra and consider for each \( \alpha \in A^1 \) the left multiplication \( \delta_\alpha : A^i \to A^{i+1} \). The pair \((A; \delta_\alpha)\) is a complex for each \( \alpha \in A^1 \).

**Definition 1.6.** The \( k \)-th resonance variety of \( A \) is
\[
\mathcal{R}^k(A) \overset{\text{def}}{=} \{ \alpha \in A^1 \mid H^k(A, \delta_\alpha) \neq 0 \}.
\]
The \( k \)-th resonance varieties (with coefficients in the ring \( R \)) for a toric arrangement \( \mathcal{A} \) is
\[
\mathcal{R}^k(\mathcal{A}; R) \overset{\text{def}}{=} \mathcal{R}^k(H^*(M(\mathcal{A}); R)).
\]

We will use only the first resonance variety \( \mathcal{R}^1(\mathcal{A}, R) \) of a toric arrangement \( \mathcal{A} \), where \( R \) is the ring \( \mathbb{Z} \) or \( \mathbb{Q} \).

## 2 First example

The example that we will expose in this section was already appeared in [Pag17, Example 7.1] as a generalization of [CD17, Example 7.3.2], but without the result [CDD+18, Theorem 7.4] it was not possible to complete the calculation.
Consider the arrangements $\mathcal{A}$ and $\mathcal{A}^n$ in $T = (\mathbb{C}^*)^2$ defined by the matrices:

$$
\begin{align*}
N &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \\
N^n &= \begin{pmatrix} 1 & a & a + 1 \\ 0 & n & n \end{pmatrix}
\end{align*}
$$

where $n$ is a positive integer and $a, a + 1$ are relatively prime to $n$.

As calculated in [Loo93, 2.4.3], the Poincaré polynomial of $M(\mathcal{A})$ is $1 + 5t + 6t^2$ and that of $M(\mathcal{A}^n)$ is $1 + 5t + (2n + 4)t^2$. The Tutte polynomial of the arithmetic matroid associated with $N$ (see [Moc12]) is $x^2 + x + y$, the one associated with $N^n$ is $x^2 + x + ny + 2n - 2$.

**Theorem 2.1.** Let $n > 5$ be a natural number relatively prime to 6, the arrangements $\mathcal{A}^1_n$ and $\mathcal{A}^2_n$ have isomorphic posets of layers but non isomorphic cohomology algebras with integer coefficients.

From [CDD+18, Theorem 6.13, Remark 6.15] the two arrangements $\mathcal{A}^1_n$ and $\mathcal{A}^2_n$ have isomorphic cohomology algebras with rational coefficients. We need a couple of lemmas to prove Theorem 2.1.

**Lemma 2.2.** Let $A$ be a graded-commutative algebra. The first resonance variety $\mathcal{R}^1(A)$ is the union (possibly infinite) of planes in $A^1$.

**Proof.** If $\alpha \in \mathcal{R}^1(A)$, then there exists $\beta \in A^1 \setminus a\mathbb{C}$ such that $\alpha \beta = 0$. Thus, the plane generated by $\alpha$ and $\beta$ is contained in $\mathcal{R}^1(A)$. We obtain the desired result from the arbitrariness of $\alpha \in \mathcal{R}^1(A)$. \hfill \square

We use coordinates $x, y$ on $T$. The cohomology ring of $M(\mathcal{A})$ is generated by the closed forms

$$
\begin{align*}
\omega_1 &= d \log(1 - x) \\
\omega_2 &= d \log(1 - y) \\
\omega_3 &= d \log(1 - xy),
\end{align*}
$$

associated to the hypertori $H_1, H_2, H_3$ respectively, together with the closed forms defined on the entire torus $(\mathbb{C}^*)^2$ (e.g., together with $\psi_1 = d \log(x)$ and $\psi_2 = d \log(y)$). The relations between those forms are described in general in [CDD+18, Lemma 3.2]. In our example the relations are:

$$
\begin{align*}
\omega_1 \omega_2 - \omega_1 \omega_3 + \omega_2 \omega_3 + \psi_1 \omega_3 &= 0 \\
\omega_1 \psi_1 &= 0 \\
\omega_2 \psi_2 &= 0 \\
\omega_3 \psi_1 + \omega_3 \psi_2 &= 0
\end{align*}
$$

(1)
Lemma 2.3. The first resonance variety $R^1(A;\mathbb{Q})$ of the complement of $A$ is the union of the following five planes of $H^1(M(A);\mathbb{Q})$;

$$P_1 = \langle \omega_1, \psi_1 \rangle$$
$$P_2 = \langle \omega_2, \psi_2 \rangle$$
$$P_3 = \langle \omega_3, \psi_1 + \psi_2 \rangle$$
$$P_4 = \langle \omega_1 - \omega_3, \omega_1 - \omega_2 - \psi_1 \rangle$$
$$P_5 = \langle \omega_2 - \omega_3, \omega_1 - \omega_2 + \psi_2 \rangle.$$ 

Proof. The multiplication map $f : H^1(M(A)) \otimes H^1(M(A)) \to H^2(M(A))$ is surjective and factors through $\bigwedge^2 H^1(M(A))$. The kernel of $\tilde{f} : \bigwedge^2 H^1(M(A)) \to H^2(M(A))$ has dimension $4 = \binom{5}{2} - 6$, hence $L \overset{\text{def}}{=} \ker \tilde{f} \simeq \mathbb{P}^3$ is a linear subspace of $\mathbb{P}(\bigwedge^2 H^1(M(A))) \simeq \mathbb{P}^9$.

An element $\alpha \in H^1(M(A))$ belongs to the first resonance varieties if and only if there exists $\beta \in H^1(M(A))$ such that $\alpha \beta = 0$ in $H^2(M(A))$ and $\beta \not\in \mathbb{C} \alpha$. This implies that $\alpha \land \beta$ is in $\ker \tilde{f}$ and so $[\alpha \land \beta]$ is in the linear subspace $L$. Viceversa if $[\gamma]$ belongs to $L$ and is a decomposable tensor (i.e. belongs to $\text{Gr}(2, H^1(M(A)))$) then $[\gamma] = [\alpha \land \beta]$ and the plane $\langle \alpha, \beta \rangle$ is contained in the first resonance variety.

Now we prove that the intersection $L \cap \text{Gr}(2, H^1(M(A)))$ is the disjoint union of five points. The relations in eq. (1) implies the following factorized equations

$$(\omega_1 - \omega_3)(\omega_1 - \omega_2 - \psi_1) = 0$$
$$(\omega_2 - \omega_3)(\omega_1 - \omega_2 + \psi_2) = 0.$$ 

These equations ensure that the five points $[P_i]$, $i = 1, \ldots, 5$ lie in this intersection. The dimension of the Grassmannian $\text{Gr}(k,V)$ is $k(\dim V - k)$, which in our case is equal to 6. Moreover, when $k = 2$ its degree coincides with the Catalan number $C_{\dim V - 2}$. Hence $\text{Gr}(2, H^1(M(A)))$ has degree 5 and every $\mathbb{P}^3 \subset \mathbb{P}^9$ intersects $\text{Gr}(2, 5)$ in five points (this is the general case) or in a sub-variety of positive dimension.

We exclude the latter case by explicit computation. Fix the Plücker coordinates $[x_{ij}]_{1 \leq i < j \leq 5}$ of $\mathbb{P}^9$, where $\{\omega_1, \omega_2, \omega_3, \psi_1, \psi_2\}$ is the chosen basis of $H^1(M(A))$. The coordinates of the five planes – written in lexicographical

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4 The formula for the degree of the Plücker embedding of the Grassmannian is due to Schubert in 1886, but we refer to [GW11].
order \([x_{1,2}, x_{1,3}, \ldots, x_{4,5}]\) are:

\[
P_1 = [0, 0, 1, 0, 0, 0, 0, 0, 0] \\
P_2 = [0, 0, 0, 0, 0, 1, 0, 0, 0] \\
P_3 = [0, 0, 0, 0, 0, 0, 1, 1, 0] \\
P_4 = [1, -1, 1, 0, 0, 0, -1, 0, 0] \\
P_5 = [1, -1, 0, 0, 1, 0, -1, 0, 1, 0].
\]

Thus the linear subspace \(L\) has equation given by the ideal

\[
I \overset{\text{def}}{=} (x_{15}, x_{24}, x_{45}, x_{12} + x_{13}, x_{13} + x_{23}, x_{13} - x_{34} + x_{35}).
\]

The equation of the Grassmannian are given by the Pfaffian of principal minors of size four of a skew-symmetric matrix. Thus the defining ideal is

\[
J \overset{\text{def}}{=} (x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23}, x_{12}x_{35} - x_{13}x_{25} + x_{15}x_{23}, \\
x_{12}x_{45} - x_{14}x_{25} + x_{15}x_{24}, x_{13}x_{45} - x_{14}x_{35} + x_{15}x_{34}, \\
x_{23}x_{45} - x_{24}x_{35} + x_{25}x_{34})
\]

and the union of the two ideals is

\[
I + J = (x_{15}, x_{24}, x_{45}, x_{14}x_{25}, x_{14}x_{35}, x_{25}x_{34}, x_{12} + x_{13}, x_{13} + x_{23}, \\
x_{13} - x_{34} + x_{35}, x_{12}x_{34} + x_{14}x_{23}, x_{12}x_{35} - x_{13}x_{25}).
\]

This last ideal is zero dimensional, so that the intersection of the subspace \(\mathbb{P}(\ker f)\) with the Grassmannian \(\text{Gr}(2, H^1(M(A)))\) is (scheme theoretically) the union of the five points \([P_i], i = 1, \ldots, 5\). Finally, we obtain that the first resonance variety \(R^1(A; \mathbb{Q})\) is the union of the five planes. \(\square\)

Notice that for every \(n\) and \(a\) the complement \(M(A_n^a)\) is a Galois covering of \(M(A)\) with Galois group \(\mathbb{Z}/n\mathbb{Z}\). We call \(\pi_a : M(A_n^a) \rightarrow M(A)\) the natural projection. The covering \(\pi_a\) is the restriction of the covering of tori defined by the inclusion of the lattice \(\langle e_1, ne_2 + ae_1 \rangle\) in \(\mathbb{Z}^2 = \langle e_1, e_2 \rangle\). The map \(\pi_a\) induces an inclusion

\[
\pi_a^* : H^\bullet(M(A) ; \mathbb{Z}) \rightarrow H^\bullet(M(A_n^a) ; \mathbb{Z})
\]

of cohomology rings with integer coefficients, this inclusion follows from the corresponding one with complex coefficients (see [DP05, Section 4.2]) together with the fact that the cohomology is torsion free (as shown in [DL15]).

The hypothesis \((a,n) = (a + 1, n) = 1 - \text{ for } a = 1, 2 -\) implies that \(H^1(M(A_n^a) ; \mathbb{Z})\) has rank five, equal to that of \(H^1(M(A) ; \mathbb{Z})\). Let \(\alpha\) and \(\beta\) be
Figure 1: The Hasse diagram of the poset of layers of $A^1_n$ which coincides with the one of $A^2_n$

the two canonical generators of $H^1(T;\mathbb{Z})$ as sub-lattice of $H^1(M(A^a_n);\mathbb{Z})$:

than the morphism $\pi^*_a$ is

$$
\pi^*_a(\psi_1) = \alpha \\
\pi^*_a(\psi_2) = n\beta + a\alpha \\
\pi^*_a(\omega_i) = \omega_i \quad \text{for } i = 1, 2, 3.
$$

Lemma 2.4. The first resonance variety $R^1(A^a_n;\mathbb{Z})$ is the union of the following five sub-lattices of $H^1(M(A^a_n);\mathbb{Z})$:

$$
Q_1 = \langle \omega_1, \alpha \rangle \\
Q_2 = \langle \omega_2, n\beta + a\alpha \rangle \\
Q_3 = \langle \omega_3, n\beta + (a+1)\alpha \rangle \\
Q_4 = \langle \omega_1 - \omega_3, \omega_2 - \omega_1 + \alpha \rangle \\
Q_5 = \langle \omega_2 - \omega_3, \omega_1 - \omega_2 + n\beta + a\alpha \rangle.
$$

Proof. The lattice $H^1(M(A^a_n);\mathbb{Z})$ is embedded in $H^1(M(A);\mathbb{Q})$ and the first resonance variety $R^1(A^a_n;\mathbb{Z})$ is the intersection

$$
R^1(A^a_n;\mathbb{Z}) = R^1(A;\mathbb{Q}) \bigcap H^1(M(A^a_n);\mathbb{Z}).
$$

Now we can complete the proof of Theorem 2.1.

Proof of Theorem 2.1. The posets of layers $S(A^1_n)$ and $S(A^2_n)$ are isomorphic because they have 3 connected hypertori that intersect in $n$ points $(1, \zeta_n^i)$ for $i = 0, \ldots, n-1$ (where $\zeta_n$ is a $n^{th}$ primitive root of unity). The Hasse diagram of the posets in the case $n = 7$ is represented in Figure 1. Suppose that there exists an isomorphism $\varphi : H^\bullet(M(A^1_n);\mathbb{Z}) \rightarrow H^\bullet(M(A^2_n);\mathbb{Z})$; then $\varphi$ must map $R^1(A^1_n;\mathbb{Z})$ isomorphically into $R^1(A^2_n;\mathbb{Z})$. Furthermore, $\varphi$ sends each component $Q^1_i$ into another component $Q^2_j$. For each $(i,j)$, consider the cardinality $c(i,j)$ of the torsion subgroup of $H^1(M(A^a_n);\mathbb{Z})/\langle Q^a_i, Q^a_j \rangle$ for $a = 1, 2$. The value of $c(i,j)$ is $n$ when $(i,j) = (1,2), (1,3), (2,3), (4,5)$ and 1 otherwise, both for $A^1_n$ and $A^2_n$. Thus, $\varphi$ maps
Let $Q_1^1, Q_2^1, Q_3^1$ be a finite set of elements of $\Lambda$. We define, for any sub-lattice $\Lambda$ of $H^1(M(A_n^a); \mathbb{Z})$, its radical $\text{Rad} \Lambda$:

$$\text{Rad} \Lambda = \{ x \in H^1(M(A_n^a); \mathbb{Z}) \mid \exists n \in \mathbb{N}_+ \text{ such that } nx \in \Lambda \}.$$ 

Notice that for $k = 1, 2$ the following equality holds

$$H^1((\mathbb{C}^*)^2; \mathbb{Z}) = \text{Rad} \left( \bigcap_{1 \leq i < j \leq 3} \langle Q_i^k, Q_j^k \rangle \right),$$

hence $\varphi$ preserves the sub-lattice $L \overset{\text{def}}{=} H^1((\mathbb{C}^*)^2; \mathbb{Z}) = \langle \alpha, \beta \rangle$. Now we claim that there is no linear map $\varphi_L : L \to L$ that sends the three sub-lattices

$$\{Q_1^1 \cap L, Q_2^1 \cap L, Q_3^1 \cap L\}$$

into $\{Q_1^2 \cap L, Q_2^2 \cap L, Q_3^2 \cap L\}$ in some order. The three one-dimensional lattices are $Q_1^1 \cap L = \langle \alpha \rangle$, $Q_2^1 \cap L = \langle n\beta + \alpha \rangle$, $Q_3^1 \cap L = \langle n\beta + 2\alpha \rangle$ for the arrangement $A_n^1$ and the lattices $Q_1^2 \cap L = \langle \alpha \rangle$, $Q_2^2 \cap L = \langle n\beta + 3\alpha \rangle$, $Q_3^2 \cap L = \langle n\beta + 3\alpha \rangle$ for the arrangement $A_n^2$. In the case $a = 1$ we can find generators for two of those lattices (e.g. $-\alpha$ and $n\beta + \alpha$) such that their sum belongs to the sub-lattice $nL$. This property does not hold for the arrangement $A_n^2$; indeed $\pm \alpha \pm (n\beta + 2\alpha), \pm \alpha \pm (n\beta + 3\alpha), \pm (n\beta + 2\alpha) \pm (n\beta + 3\alpha)$ are not in $nL$ (here we use $n \neq 5$). Thus, we conclude that the map $\varphi$ cannot exist. \hfill \Box

### 3 Modular matroid

In this section we introduce a family of groups $\{K_\Lambda(S)\}_S$ related to a toric arrangements. We study their properties and we will use it to describe the poset of layers of a toric arrangement. This is the key point in order to show that the two following arrangements $A$ and $A'$ have different posets of layers.

#### The groups $H_\Lambda(S)$

Let $\Lambda$ be a lattice and $\{v_1, \ldots, v_n\}$ be a finite set of elements of $\Lambda$. Let $[n]$ be the set $\{1, 2, \ldots n\}$. We define $\Gamma_S$, for $S \subseteq [n]$, to be the sub-lattice spanned by the vectors $v_i, i \in S$. The lattice $\Gamma_{[n]}$ has a main role in the following discussion, therefore we address to it as $\Gamma$. Consider the functions $m_\Lambda$ and $m_\Gamma$, from the subsets of $[n]$ to the positive integer, defined by

$$m_\Lambda(S) \overset{\text{def}}{=} |\text{Rad}_\Lambda \Gamma_S / \Gamma_S| = |\text{Ext}^1 \left( \Lambda / \Gamma_S, \mathbb{Z} \right)|$$

and by

$$m_\Gamma(S) \overset{\text{def}}{=} |\text{Rad}_\Gamma \Gamma_S / \Gamma_S| = |\text{Ext}^1 \left( \Gamma / \Gamma_S, \mathbb{Z} \right)|.$$
The multiplicity functions $m_\Lambda$ and $m_\Gamma$ are nothing else that the cardinality of the torsion subgroups of $\Gamma S$ and $\Gamma S$, respectively.

This collection of vectors in $\Lambda$ defines a matroid $(\{n\}, \text{rk})$ that can be enriched by the multiplicity function $m_\Lambda$ and the triple $(\{n\}, \text{rk}, m_\Lambda)$ becomes a representable arithmetic matroid. Alternatively, we can enrich the matroid by the multiplicity function $m_\Gamma$ and obtain the representable arithmetic matroid $(\{n\}, \text{rk}, m_\Gamma)$.

Recall the definition of the radical of a sub-lattice $\Gamma' \subseteq \Lambda'$:

$$\text{Rad}_\Lambda \Gamma' \overset{\text{def}}{=} \{v \in \Lambda' \mid \exists n \in \mathbb{N}_+ \text{ such that } nv \in \Gamma\}.$$  

Consider for each subset $S$ of $\{n\}$ the short exact sequence:

$$0 \rightarrow \text{Rad}_\Gamma \Gamma S / \Gamma S \rightarrow \text{Rad}_\Lambda \Gamma S / \Gamma S \rightarrow H_\Lambda(S) \rightarrow 0.$$  

We call the rightmost group $H_\Lambda(S) \overset{\text{def}}{=} \text{Rad}_\Lambda \Gamma S / \text{Rad}_\Gamma \Gamma S$.

Since $\text{Rad}_\Gamma \Gamma T \cap \text{Rad}_\Lambda \Gamma S = \text{Rad}_\Gamma \Gamma S$, the map $i_{S,T}$ is injective. If $\text{rk}(S) = \text{rk}(T)$, then $i_{S,T}$ is the identity map because in this case we have the equalities $\text{Rad}_\Gamma \Gamma T = \text{Rad}_\Gamma \Gamma S$ and $\text{Rad}_\Lambda \Gamma T = \text{Rad}_\Lambda \Gamma S$. Let $G$ be the torsion of the group $\Lambda / \Gamma$ and notice that $H_\Lambda([n]) = G$, thus the groups $H_\Lambda(S), S \subseteq [n]$, are subgroups of $G$.

**Definition 3.1.** The **layer group** $\text{LG}_\Lambda(S)$ of a representation $\Lambda$ is the group

$$\text{LG}_\Lambda(S) \overset{\text{def}}{=} \text{Ext}^1 \left(\text{Rad}_\Lambda \Gamma S / \Gamma S, \mathbb{Z}\right) = \text{Ext}^1 \left(\Lambda / \Gamma S, \mathbb{Z}\right).$$

We also define the **relative layer group**

$$\text{LG}_\Gamma(S) \overset{\text{def}}{=} \text{Ext}^1 \left(\text{Rad}_\Gamma \Gamma S / \Gamma S, \mathbb{Z}\right) = \text{Ext}^1 \left(\Gamma / \Gamma S, \mathbb{Z}\right).$$

For any $S \subseteq T$ there is a natural map $\pi_{S,T} : \text{LG}_\Lambda(T) \rightarrow \text{LG}_\Lambda(S)$ which is injective if $\text{rk}(S) = \text{rk}(T)$ and surjective if $|T| - \text{rk}(T) = |S| - \text{rk}(S)$. The groups $\text{LG}_\Lambda(S)$, together with the natural maps between them, determine the poset of layers of the central toric arrangement described by $v_1 \in \Lambda$, as shown in [Len17]. Thus, we have a bijection between the connected components of $\bigcap_{s \in S} H_s$ and the elements of $\text{LG}_\Lambda(S)$.

We can think $H_\Lambda(\bullet)$ as a functor from the poset of flats (see Definition 1.5) of the underlying matroid $(\{n\}, \text{rk})$ to the category of finite abelian groups with inclusions. The group $H_\Lambda(S)$ has cardinality $m_\Lambda(S)$.

The following lemma holds for a pair of modular flats (see Definition 1.4).
Lemma 3.2. Let \( S, T \) be a pair of modular flats. Then the following equality holds

\[
H_\Lambda(S) \cap H_\Lambda(T) = H_\Lambda(S \cap T).
\]

Proof. The equality \( \text{Rad}_\Lambda \Gamma_S \cap \text{Rad}_\Lambda \Gamma_T = \text{Rad}_\Lambda \Gamma_{S \cap T} \) always holds. The modularity hypothesis implies that \( \Gamma_S \cap \Gamma_T = \Gamma_{S \cap T} \), thus

\[
H_\Lambda(S) \cap H_\Lambda(T) = \text{Rad}_\Lambda \Gamma_S \cap \text{Rad}_\Lambda \Gamma_T = \text{Rad}_\Lambda \Gamma_{S \cap T} \cap \Gamma = H_\Lambda(S \cap T)
\]

The groups \( K_\Lambda(S) \)

For any subset \( S \subseteq E \) we define \( K_\Lambda(S) = \text{Ext}^1(H_\Lambda(S), \mathbb{Z}) \). For \( S \subset T \) the dual of the diagram (2) is

\[
\begin{array}{cccccc}
0 & \longrightarrow & K_\Lambda(S) & \longrightarrow & L\Gamma_\Lambda(S) & \longrightarrow & L\Gamma_\Lambda(T) & \longrightarrow & 0 \\
\downarrow^{\pi_{S,T}} & & \downarrow^{r_{S,T}} & & \downarrow^{r_{S,T}} & & \downarrow^{r_{S,T}} & & \downarrow^{r_{S,T}} \\
0 & \longrightarrow & K_\Lambda(T) & \longrightarrow & L\Gamma_\Lambda(T) & \longrightarrow & L\Gamma_\Lambda(S) & \longrightarrow & 0
\end{array}
\]

whose rows are exact. The map \( p_{S,T} \) is always surjective and is an isomorphism if \( \text{rk}(S) = \text{rk}(T) \).

Lemma 3.3. Let \( T \) and \( S \) be two flats of a modular matroid \((E, \text{rk})\) and \( \Lambda \) be a representation of an arithmetic matroid \((E, \text{rk}, m)\). The following diagram is a pushout diagram.

\[
\begin{array}{ccc}
K_\Lambda(S \vee T) & \longrightarrow & K_\Lambda(S) \\
\downarrow & & \downarrow^{r} \\
K_\Lambda(T) & \longrightarrow & K_\Lambda(S \wedge T)
\end{array}
\]

Proof. By Lemma 3.2 the following diagram is a pullback diagram.

\[
\begin{array}{ccc}
H_\Lambda(S \wedge T) & \longrightarrow & H_\Lambda(S) \\
\downarrow & & \downarrow \\
H_\Lambda(T) & \longrightarrow & H_\Lambda(S \vee T)
\end{array}
\]

Applying the functor \( \text{Ext}^1(\ast, \mathbb{Z}) \) we obtain the claimed diagram. 

Let \( p_i : G \to K_i \), for \( i = 1, 2 \), be two quotients of \( G \) by the subgroups \( L_i \). We denote the pushout of \( p_i \) and \( p_j \) with

\[
K_{i,j} \overset{\text{def}}{=} K_i \sqcup K_j \sqcup_{p_i(z) \sim p_j(z)} = G/L_i + L_j.
\]
together with the two natural surjections \( s_i : K_i \to K_{i,j} \) and \( s_j : K_j \to K_{i,j} \).
The pullback of \( s_i \) and \( s_j \) is

\[
K_i \times_{K_{i,j}} K_j \overset{\text{def}}{=} \{(x, y) \in K_i \times K_j \mid s_i(x) = s_j(y)\} = G / L_i \cap L_j.
\]

**Lemma 3.4.** Let \( G \) and \( G' \) be two finite abelian groups of the same cardinality, and for \( i \leq n \) let \( p_i : G \to K_i \) and \( p_i' : G' \to K_i' \) be quotients of \( G \) and \( G' \). We denote with \( K_{i,j} \) (and with \( K'_{i,j} \)) the pullback of \( p_i \) and \( p_j \) (respectively, of \( p_i' \) and \( p_j' \)). Suppose that there exist bijections \( f_i : K_i \overset{1:1}{\to} K_i' \), for \( i \leq n \), such that for every \( i, j \) the diagram

\[
\begin{array}{ccc}
G & \xrightarrow{p_i} & K_i \\
\downarrow & & \downarrow \quad f_i \\
K_j & \xrightarrow{p_j} & K_j'
\end{array}
\]

commutes and that the induced map \( K_{i,j} \to K_{i,j}' \) is an isomorphism. Then there exist a bijection \( f : G \to G' \) such that \( p_i' \circ f = f_i \circ p_i \) for all \( i \).

**Proof.** Suppose first \( n = 1 \), for all \( x \in K_1 \) the sets \( p_1^{-1}(x) \) and \( p_1'^{-1}(f_1(x)) \) has the same cardinality. We can choose a bijection \( f_x : p_1^{-1}(x) \to p_1'^{-1}(f_1(x)) \) for all \( x \in K_1 \) and define \( f(y) = f_{p_1}(y) \). This function \( f \) is the sought function.

For \( n > 1 \), we want to reduce to the case \( n = 1 \). Let us fix \( i \neq j \) and consider the pullback \( K_i \times_{K_{i,j}} K_j \) of the two maps \( K_i \to K_{i,j} \) and \( K_j \to K_{i,j} \). We want to find a bijection

\[
f_{i,j} : K_i \times_{K_{i,j}} K_j \overset{1:1}{\to} K_i' \times_{K'_{i,j}} K_j'.
\]

Consider the pullback diagram

\[
\begin{array}{ccc}
G & \xrightarrow{(p_i, p_j)} & K_i \times_{K_{i,j}} K_j \\
\downarrow & & \downarrow \\
K_j & \xrightarrow{f_j} & K_j'
\end{array}
\]

and observe that by diagram (4) the two map from \( K_i \times_{K_{i,j}} K_j \) to \( K_{i,j}' \) induced by \( f_i \) and \( f_j \) coincides. Notice that \(|K_i \times_{K_{i,j}} K_j| = \frac{|K_i||K_j|}{|K_{i,j}|}\) and so \(|K_i \times_{K_{i,j}} K_j| = |K_i' \times_{K'_{i,j}} K_j'|\). Since the pullbacks in the category of \( \mathbb{Z} \)-modules and in the category of sets coincide, we have a well defined bijection \( f_{i,j} : K_i \times_{K_{i,j}} K_j \overset{1:1}{\to} K_i' \times_{K'_{i,j}} K_j' \). The bunch of \( n-1 \) maps \( \{f_k\}_{k \neq i,j} \cup \{f_{i,j}\} \) satisfy the hypothesis, so by induction we construct the map \( f : G \to G' \).
Lemma 3.5. Let Λ and Λ′ be two representations of the same torsion-free arithmetic matroid whose underlying matroid is modular. Then the functor $K_{\Lambda}$ and $K_{\Lambda'}$ are equivalent, i.e. there exist a bijection $f : K_{\Lambda}(E) \to K_{\Lambda'}(E)$ such that for every flat $S \subset E$ the map $f$ induces a bijection $f_S : K_{\Lambda}(S) \to K_{\Lambda'}(S)$.

Proof. We define $f$ by induction on the poset of flats of the underlying matroid. Since $K_{\Lambda}(\emptyset) = 0 = K_{\Lambda'}(\emptyset)$, the base case is done. Suppose that we have defined $f_S : K_{\Lambda}(S) \to K_{\Lambda'}(S)$ for every flat $S$ of rank less than $k$, compatibly with the restrictions. For each flat $T$ of rank $k$ consider the set $\{S_1, \ldots, S_m\}$ of flats of rank $k - 1$ contained in $T$. We apply Lemma 3.4 to $G = K_{\Lambda}(T)$, $G' = K_{\Lambda'}(T)$, $K_i = K_{\Lambda}(S_i)$ and $K'_i = K_{\Lambda'}(S_i)$. Lemma 3.3 implies $K_{i,j} = K_{\Lambda}(S_i \land S_j)$ and $K'_{i,j} = K_{\Lambda'}(S_i \land S_j)$, so the compatibility between $f_{S_i}$, $f_{S_j}$ and $f_{S_i \land S_j}$ ensure the condition (1). Thus we have a bijection $f_T : K_{\Lambda}(T) \to K_{\Lambda'}(T)$ compatible with the restrictions. We repeat this procedure for every flat $T$ of rank $k$ and inductively for every flats.

Theorem 3.6. Let $(E, \text{rk}, m)$ be an arithmetic matroid. Suppose $m(\emptyset) = 1$ and that at least one of the following holds:

1. the matroid $(E, \text{rk})$ is modular,
2. or $m(E) = 1$.

Then the posets of layers of all realizations of the arithmetic matroid are isomorphic.

Proof. If $m(E) = 1$, then by [Pag17, Theorem 3.12] there exist a unique representation of the arithmetic matroid. An explicit description of the poset of layers of a central toric arrangement is given in [Len17] in terms of the sets $(\text{LG}(S))_{S \subset E}$ and the maps $\pi_{S,T}$ between them. Any two realization $\Lambda$ and $\Lambda'$ are covering of the central toric arrangement $\Gamma$, as shown in [Pag17]. Observe that $\text{LG}_{\Lambda}(S) = \text{LG}_{\Gamma}(S) \times K_{\Lambda}(S)$ as a set and $\pi_{S,T} = (\gamma_{S,T}, p_{S,T})$ as map between sets. Analogously, $\text{LG}_{\Lambda'}(S) = \text{LG}_{\Gamma}(S) \times K_{\Lambda'}(S)$ and $\pi'_{S,T} = (\gamma_{S,T}, p'_{S,T})$. We observe that $K_{\Lambda}(S) = K_{\Lambda}(S)$, where $S$ is the minimal flat containing $S$. Consider the bijections $f_{S}$ of Lemma 3.3 the maps $\text{Id} \times f_S : \text{LG}_{\Gamma}(S) \times K_{\Lambda}(S) \to \text{LG}_{\Gamma}(S) \times K_{\Lambda}(S)$ are compatible with $(\gamma_{S,T}, p_{S,T})$ since $f_S \circ p_{S,T} = p'_{S,T} f_T$ for all flats $S$ and $T$. Therefore, the two poset of layers are isomorphic.

4 Second example

Consider the three matrices

$$N = \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 5 & 0 & 5 \\ 0 & 5 & 5 & 5 \end{pmatrix}, \quad N' = \begin{pmatrix} 1 & 4 & 1 & 6 \\ 0 & 5 & 0 & 5 \\ 0 & 0 & 5 & 5 \end{pmatrix}, \quad N'' = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$
These integer matrices describe three central toric arrangements $A$, $A'$ and $A''$ in $T = (\mathbb{C}^*)^3$. Both $A$ and $A'$ are Galois coverings of $A''$ with Galois group $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$.

Let $([4], \text{rk}, m)$ be the arithmetic matroid defined by $\text{rk}(S) = \min(|S|, 3)$ and by

$$ m(S) = \begin{cases} 
1 & \text{if } |S| \leq 1 \\
5 & \text{if } |S| = 2 \\
25 & \text{if } |S| \geq 3 
\end{cases} $$

Let $M$ be the matroid over $\mathbb{Z}$ defined by

$$ M(S) = \begin{cases} 
\mathbb{Z}^3 & \text{if } |S| = 0 \\
\mathbb{Z}^2 & \text{if } |S| = 1 \\
\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} & \text{if } |S| = 2 \\
\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z} & \text{if } |S| \geq 3 
\end{cases} $$

**Theorem 4.1.** The matrices $N$ and $N'$ are representations of the arithmetic matroid $([4], \text{rk}, m)$ and of the matroid $M$ over $\mathbb{Z}$. Moreover, the posets $S(A)$ and $S(A')$ are not isomorphic.

**Proof.** The first assertion follows from the Smith normal form of $N[S]$ and of $N'[S]$, the matrices obtained from $N$ and $N'$ by taking only the columns indexed by $S$. The second one follows from Lemma 4.2 below. \qed

The Poincaré polynomial of the complements $M(A)$ and $M(A')$ coincides with

$$ P(t) = P'(t) = 160t^3 + 41t^2 + 7t + 1. $$

The one of $M(A'')$ is $P''(t) = 14t^3 + 17t^2 + 7t + 1$. The Tutte polynomial of the arithmetic matroid $([4], \text{rk}, m)$ is $x^3 + x^2 + 25x + 25y + 48$ and the one associated with $N''$ is $x^3 + x^2 + x + y$.

Consider the following property

$$ \exists \{i, j, k, l\} = [4] \forall a = i \lor j, b = k \lor l \exists a \lor b. \quad (P) $$

**Lemma 4.2.** The property $(P)$ holds for $S(A)$ but not for $S(A')$.

**Proof.** We first discuss the poset $S(A')$. Consider $(i, j, k, l) = (1, 2, 3, 4)$, there are five possible joins $1 \lor 2$ that correspond to the five layers

$$ a_\mu : \begin{cases} 
x = 1 \\
y = \mu 
\end{cases} $$

where $\mu$ runs over all the fifth roots of unity. Analogously, the joins of 3 and 4 are the five layers

$$ b_\zeta : \begin{cases} 
x = z^{-5} \\
y = \zeta z^{-5} 
\end{cases} $$

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where ζ runs over all the fifth roots of unity. The join $a_\mu \lor b_\zeta$ exists if and only if the system

$$
\begin{align*}
  x &= 1 \\
  y &= \mu \\
  y &= \zeta z^5 \\
  z^5 &= 1
\end{align*}
$$

(5)

admits a solution. If $\zeta = \mu$, then the system has five solutions, otherwise there are no solutions. In particular, the property $[\mathbb{P}]$ does not holds for the poset $\mathcal{S}(\mathcal{A}')$.

A case by case analysis shows that the six systems analogous to (5) which we can write for the arrangement $\mathcal{A}_1$, have always a unique solution. Since $\text{LG}_\Gamma(S) = 0$ for all $S \subseteq E$, we have $\text{Ext}^1(\mathcal{H}_\Lambda(S), \mathbb{Z}) = \text{LG}_\Lambda(S)$. Recall that $\mathcal{H}_\Lambda([4]) = G$: in our case this group is generated by $\{e_2, e_3\}$ and isomorphic to $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$. The following table contains the subgroups $\mathcal{H}_\Lambda(S)$ and $\mathcal{H}_\Lambda'(S)$ for each 2-flat $S \subset [4]$.

| $S$   | $\mathcal{H}_\Lambda(S)$ | $\mathcal{H}_\Lambda'(S)$ |
|-------|----------------|----------------|
| $\{1, 2\}$ | $\langle e_2 \rangle$ | $\langle e_2 \rangle$ |
| $\{1, 3\}$ | $\langle e_3 \rangle$ | $\langle e_3 \rangle$ |
| $\{1, 4\}$ | $\langle e_2 + e_3 \rangle$ | $\langle e_2 + e_3 \rangle$ |
| $\{2, 3\}$ | $\langle e_2 + 4e_3 \rangle$ | $\langle e_2 + e_3 \rangle$ |
| $\{2, 4\}$ | $\langle e_2 + 2e_3 \rangle$ | $\langle e_2 + 3e_3 \rangle$ |
| $\{3, 4\}$ | $\langle e_2 + 3e_3 \rangle$ | $\langle e_2 \rangle$ |

We call $v_2$ and $v_3 \in \text{LG}_\Lambda(E)$ the dual elements of $e_2$ and $e_3$, respectively. The natural map between the layer groups is surjective $\pi_{i,j} : \text{LG}_\Lambda(E) \rightarrow \text{LG}_\Lambda(\{i,j\})$, call $I(i,j)$ its kernel. We report in the following table the ideals $I(S)$ and $I'(S)$ for the representations $\Lambda$ and $\Lambda'$, respectively.

| $S$   | $I(S)$ | $I'(S)$ |
|-------|--------|--------|
| $\{1, 2\}$ | $\langle v_3 \rangle$ | $\langle v_3 \rangle$ |
| $\{1, 3\}$ | $\langle v_2 \rangle$ | $\langle v_2 \rangle$ |
| $\{1, 4\}$ | $\langle v_2 - v_3 \rangle$ | $\langle v_2 - v_3 \rangle$ |
| $\{2, 3\}$ | $\langle 4v_2 - v_3 \rangle$ | $\langle v_2 - v_3 \rangle$ |
| $\{2, 4\}$ | $\langle 2v_2 - v_3 \rangle$ | $\langle 3v_2 - v_3 \rangle$ |
| $\{3, 4\}$ | $\langle 3v_2 - v_3 \rangle$ | $\langle v_3 \rangle$ |

Hence, we consider $a$ and $b$ as elements of $\text{LG}_\Lambda(\{i,j\})$ and $\text{LG}_\Lambda(\{k,l\})$, respectively. The condition that $a \lor b$ exists is equivalent to the fact that there exists an element $v \in \text{LG}_\Lambda(E)$ that maps to $a$ and $b$. Since all pairs of ideals $I(\{i,j\})$ and $I(\{k,l\})$ are relatively primes in $\mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, the element $v$ always exists (and is unique) by Chinese remainder theorem. Therefore, the poset $\mathcal{S}(\mathcal{A})$ satisfies the condition $[\mathbb{P}]$. □
Proposition 4.3. The complement $M(\mathcal{A})$ and $M(\mathcal{A'})$ of the two toric arrangement defined above have non-isomorphic cohomology algebras with rational coefficients, i.e.

$$H^*(M(\mathcal{A}); \mathbb{Q}) \ncong H^*(M(\mathcal{A'}); \mathbb{Q}).$$

Proof. Suppose that an isomorphism $\varphi : H^*(M(\mathcal{A}); \mathbb{Q}) \to H^*(M(\mathcal{A'}); \mathbb{Q})$ exists. We claim that $\varphi(H^*(T; \mathbb{Q})) = H^*(T; \mathbb{Q})$ where $T$ is the ambient torus. The proof of the claim is analogous to the one of Lemma 2.3. The first resonance variety of $M(\mathcal{A})$ and $M(\mathcal{A'})$ are the union of the four planes

$$Q_1 = \langle \omega_1, \alpha \rangle \quad Q'_1 = \langle \omega_1, \alpha \rangle$$
$$Q_2 = \langle \omega_2, 4\alpha + 5\beta \rangle \quad Q'_2 = \langle \omega_2, \alpha + 5\beta \rangle$$
$$Q_3 = \langle \omega_3, \alpha + 5\gamma \rangle \quad Q'_3 = \langle \omega_3, \alpha + 5\gamma \rangle$$
$$Q_4 = \langle \omega_4, 3\alpha + 5\beta + 5\gamma \rangle \quad Q'_4 = \langle \omega_4, 6\alpha + 5\beta + 5\gamma \rangle,$$

since the unique relations in cohomology of degree two are $\omega_i \psi_i = 0$. Thus there exists a bijection $f : [4] \to [4]$ such that $\varphi$ sends $Q_i$ into $Q'_{f(i)}$, for $i = 1, \ldots, 4$. Since $H^1(T; \mathbb{Q}) = \bigcap_{i=1}^4 (Q_j)_{j \neq i}$ in $H^1(M(\mathcal{A}); \mathbb{Q})$ and $H^1(T; \mathbb{Q}) = \bigcap_{i=1}^4 (Q'_{j})_{j \neq i}$ in $H^1(M(\mathcal{A'}); \mathbb{Q})$, the map $\varphi$ preserve the subspace $H^*(T; \mathbb{Q})$. Consider now the quotients $S^* = H^*(M(\mathcal{A}); \mathbb{Q})/(H^1(T; \mathbb{Q}))$ and $S^* = H^*(M(\mathcal{A'}); \mathbb{Q})/(H^1(T; \mathbb{Q}))$. The multiplication map $S^1 \times S^2 \to S^3$ has rank 51, instead the map $S^{41} \times S^{42} \to S^{43}$ has rank 43. The rank of the two multiplication maps can be calculated with a computer. Therefore the map $\varphi$ cannot be an isomorphism. 

The difference between the rank of $S^1 \times S^2 \to S^3$ and $S^{41} \times S^{42} \to S^{43}$ can be explained intuitively. For all hypertori $H_k \in \mathcal{A}$, $k = 1, \ldots, 4$, consider the five connected components $W^a_{i,j}$ of the intersection $H_i \cap H_j$ of two of them. On each connected component $W^a_{i,j}$ is supported a closed form $\omega^a_{i,j}$ introduced in [DP05] and in [CDD+18]. Analogously, on each of 25 points of the intersection $\bigcap_{i=1}^4 H_i$ are supported four closed forms $\omega^b_{i,j,k}$. For each $b \in \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$, the Orlik–Solomon relations in $S^*$ are

$$\sum_{i=1}^4 \omega^b_{i,j,k} = 0.$$ 

The product in $S^*$ is defined by

$$\omega^a_{i,j,k} \omega^b_{i',j,k} = \sum_{b \in \mathbb{Z}/5\mathbb{Z}} \omega^b_{i,j,k}.$$ 

The analogous definitions and formulas hold for the arrangement $\mathcal{A'}$. In the algebra $S^{4*}$ the following relations holds for $a \in \mathbb{Z}/5\mathbb{Z}$

$$(\omega^1_1 - \omega^2_2 + \omega^3_3 - \omega^4_4)(\omega^0_{1,2} + \omega^0_{3,4}) = 0$$
$$(\omega^1_1 + \omega^2_2 - \omega^3_3 - \omega^4_4)(\omega^0_{1,4} + \omega^0_{2,3}) = 0$$

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since $\pi'_{1,2} = \pi'_{3,4}$ and $\pi'_{1,4} = \pi'_{2,3}$. These relations give ten independent relations, the corresponding relations in the algebra $S^*$ are only two:

$$\left(\omega_1 - \omega_2 + \omega_3 - \omega_4\right)\left(\sum_{a=1}^{5} \omega_{1,2}^{a} + \omega_{3,4}^{a}\right) = 0$$

$$\left(\omega_1 + \omega_2 - \omega_3 - \omega_4\right)\left(\sum_{a=1}^{5} \omega_{1,4}^{a} + \omega_{2,3}^{a}\right) = 0$$

since $\pi_{1,2} \neq \pi_{3,4}$ and $\pi_{1,4} \neq \pi_{2,3}$.

Notice that the $G$-semimatroids described by $N$ and $N'$ are different, by \cite[Theorem E]{DR18}.

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