ANALYTIC SOLUTIONS OF NONLINEAR ELLIPTIC EQUATIONS ON RECTANGULAR TORI

YUNFENG SHI

Abstract. In this paper, we consider the nonlinear elliptic equations on rectangular tori. Using methods in the study of KAM theory and Anderson localization, we prove that these equations admit many analytic solutions.

1. Introduction and main results

In this paper, we investigate the following equation on rectangular tori

\[ -\Delta u - mu + \epsilon (f(x)u^p + g(x)) = 0, \quad x \in \prod_{i=1}^{d} (\mathbb{R}/2\pi\beta_i \mathbb{Z}), \quad d \geq 1, \]

where \( \beta = (\beta_1, \ldots, \beta_d) \in [\frac{1}{2}, 1]^d, \quad p > 1, \quad \epsilon \geq 0 \quad \text{and} \quad m > 0. \) We assume further \( f(x), g(x) \) are real trigonometric polynomials satisfying \( g \not\equiv 0. \) Let \( T^d := \mathbb{R}^d/(2\pi \mathbb{Z})^d. \) Performing a change of variables, it is convenient to consider instead the following equation

\[ -\Delta_\nu u - mu + \epsilon (f(x)u^p + g(x)) = 0, \quad x \in T^d, \]

where

\[ \Delta_\nu := -\sum_{i=1}^{d} \nu_i^2 \frac{\partial^2}{\partial x_i^2}, \quad \nu = (\beta_{-1}^1, \ldots, \beta_{-1}^d) \in [1, 2]^d. \]

Our aim of the present paper is to find periodic solutions of (1.2). In fact, it is easy to see \( u = 0 \) is not a solution of (1.2) if \( \epsilon \neq 0. \) Hence it is meaningful to look for periodic solutions of (1.2) with positive \( \epsilon. \)

We have the following main result.

Theorem 1.1. For any \( \delta > 0, \) there is some \( \epsilon_0 > 0 \) depending only on \( \delta, f, g, m, p, d \) such that, for \( 0 \leq \epsilon \leq \epsilon_0, \) there exists a set \( \Omega = \Omega(\epsilon) \subset [1, 2]^d \) of Lebesgue measure \( \text{mes}([1, 2]^d \setminus \Omega) \leq \delta \) such that, for \( \nu \in \Omega, (1.2) \) admits an analytic solution. More precisely, for any \( \nu \in \Omega, \) there exists some \( u_\nu \in C^\omega(T^d, \mathbb{R}) \) so that \( u_\nu(x) \) is a solution of (1.2).

To prove the existence of solutions for a nonlinear elliptic equation, methods of the calculus of variations, bifurcation theory and topological degree have been used. By contrast, we will make use of the methods developed in KAM (Kolmogorov-Arnold-Moser) and Anderson localization theory. To the best of our knowledge, there is no result about the existence of analytic solutions for nonlinear elliptic equations on rectangular tori.

Remark 1.2. Regarding the rectangular tori, there have been plenty of results on the study of Schrödinger equations on rectangular tori. Bourgain \[4\] firstly addressed the question of Strichartz estimates for Schrödinger equations on tori. In \[11\], Bourgain carried out the study

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of Strichartz estimates for Schrödinger equations on any 3-dimensional rectangular tori. Later, Guo-Oh-Wang [25] proved some new Strichartz estimates for linear Schrödinger equations on any d-dimensional irrational tori. Remarkably, Bourgain-Demeter [12] proved the sharp estimates for linear Schrödinger equations on any irrational tori. Recently, Deng-Germain-Guth [22] obtained the Strichartz estimates over large time scales for the Schrödinger equations on generic rectangular tori. This indeed inspires us to study nonlinear elliptic equations on most rectangular tori. So far, there are also many results about Sobolev norms growth for equations on rectangular tori, see e.g. [3, 17, 20, 21].

Remark 1.3. As we will see later, the small-divisors difficulty appears in the present work. In fact, the KAM techniques and CWB (Craig-Wayne-Bourgain) methods are powerful tools to overcome the small-divisors difficulty. The KAM results, such as the existence results of quasi-periodic (or almost-periodic) solutions for nonlinear Hamiltonian PDEs have been widely studied in the literature. In particular, the study for 1-dimensional PDEs has attracted a great deal of attention over years and is well understood [3, 6, 18, 19, 27, 31, 35, 37]. In high dimensional case, the first result was due to Bourgain [6]. Significantly, using the so called CWB methods, he proved the existence of quasi-periodic solutions for a class of nonlinear Schrödinger equations (NLS) on $\mathbb{T}^2$. Later in [8], by introducing techniques from the study of Anderson localization theory in [14], Bourgain established existence results of quasi-periodic solutions for NLS and nonlinear wave equations (NLW) in arbitrary dimension $d$. The standard KAM approach has been extended by Eliasson-Kuksin [23] to NLS on arbitrary dimensional torus $\mathbb{T}^d$. Recently, Wang [32] considered a class of completely resonant NLS on $\mathbb{T}^d$ with supercritical nonlinearities, and also showed the existence of quasi-periodic solutions. Berti-Bolle dealt with NLS [2] and NLW [1] on $\mathbb{T}^d$ with finitely differentiable nonlinearity and obtained the existence of Sobolev regular quasi-periodic solutions. Berti-Bolle made use of a modified Nash-Moser iterations together with the multi-scale analysis. In a latest work by Berti-Maspero [3], they proved the existence of Sobolev regular quasi-periodic solutions for the NLW and NLS on arbitrary rectangular tori. We should remark that all existence solutions mentioned above are at most Gevrey regular in time or space variables. In [38], Yuan developed a new KAM scheme so that he can deal with equations with normal frequencies having finite limit-points. In particular, Yuan proved the existence of quasi-periodic solutions for generalized Pochhammer-Chree equations on a.e. rectangular tori. Very recently, Wang [34] proved the existence of analytic quasi-periodic Floquet-Bloch solutions for NLS on $\mathbb{R}^d$. In fact, the present paper is also motivated by work of Berti-Maspero [3], Wang [34] and Yuan [38].

Remark 1.4. Regarding the methods, we use mainly the Nash-Moser iterations in [8] as well as the Green’s function estimates developed by Bourgain [10]. As is well-known, the key to the Nash-Moser iterations is the appropriate estimates on the inverses (or Green’s functions) of the linearized operators. If we regard $(\nu_1, \cdots, \nu_d)$ as parameters, then the covariance property (see (3.3) in section 3) holds. More importantly, in this case the Green’s functions are quite similar to that in the study of Anderson localization for quasi-periodic operators on $\mathbb{Z}^d$. This observation may lead to good controls of the Green’s functions, i.e., off-diagonal exponential decays of the Green’s functions. Let us recall briefly some Anderson localization results. For $d = 1$, Bourgain-Goldstein [13] originally established the non-perturbative Anderson localization for general quasi-periodic Schrödinger operators with real analytic potentials. They introduced the powerful semi-algebraic sets methods to eliminate the resonances. Along this line and combining with the multi-scale analysis, Bourgain [7] even proved the Anderson localization for a class of ergodic operators with skew shifts and then got the localization and almost periodicity of the waves for some quantum kicked rotor model. When considering on $\mathbb{Z}^d$ ($d \geq 2$), the large deviation theorem (LDT) for Green’s functions can not be derived directly.
from the Diophantine properties of the frequencies and the semi-algebraic sets considerations. Actually, by posing an arithmetic condition on the frequency together the matrix-valued Car-itan estimate when proving the LDT, Bourgain-Goldstein-Schlag [14] showed the Anderson localization for quasi-periodic Schrödinger operators on $\mathbb{Z}^2$. Techniques of [14] were used by Bourgain and Wang to study KAM results for high dimensional PDEs as well as some spectral problems [8, 15, 16, 31–33]. For $d \geq 3$, it is difficult to impose a similar arithmetic condition on the frequencies. To overcome this problem, Bourgain [10] introduced new methods and successfully extended results of [14] to arbitrary dimension $d$. The basic techniques of [10] are also semi-algebraic sets and matrix-valued Cartan estimate, but involve more delicate analysis. Recently, methods of Bourgain [10] were used by Goldstein-Schlag-Voda [24] to study multi-frequency quasi-periodic Schrödinger operators on $\mathbb{Z}$, and by Jitomirskaya-Liu-Shi [26] to study a class of long-range quasi-periodic operators on $\mathbb{Z}^d$ with more general ergodic transformations. The results of [26] can be adapted to our problem here.

This paper is organized as follows. The basic notations are introduced in section 2. A reformulation of the problem on $\mathbb{Z}^d$ is presented in section 3. The key ingredient, i.e., LDT for Green’s functions is proved in section 4. In section 5, the main result is established by using Nash-Moser iterations. We want to point out that the present paper is not self-contained but relies heavily on [8, 10, 26].

2. Notations

We define $a \ll b$ if there is some small $\varepsilon > 0$ so that $a \leq \varepsilon b$. We write $a \sim b$ if $a \ll b$ and $b \ll a$. We write $a \pm$ to denote $a \pm \varepsilon$ for some small $\varepsilon$.

For any $x \in \mathbb{R}^{d_1}$ and $X \subset \mathbb{R}^{d_1+d_2}$, define the $x$-section of $X$ to be the set

$$X(x) = \{ y \in \mathbb{R}^{d_2} : (x, y) \in X \}.$$

We denote by $\lfloor x \rfloor$ the integer part of some $x \in \mathbb{R}$. For any $x \in \mathbb{R}^d$, let $|x| = \max_{1 \leq i \leq d} |x_i|$. For $U_1, U \subset \mathbb{R}^d$, we introduce

$$\text{diam}(U) = \sup_{n, n' \in U} |n - n'|, \quad \text{dist}(m, U) = \inf_{n \in U} |m - n|,$$

and $\text{dist}(U_1, U) = \inf_{n \in U_1} \text{dist}(n, U)$.

3. A Reformulation of the Equations on $\mathbb{Z}^d$

Notice that (1.2) can be transformed into nonlinear equations on lattice $\mathbb{Z}^d$ via the standard Fourier arguments. More precisely, one just needs consider the following equations

$$F(\hat{u})(n) = 0, \ n \in \mathbb{Z}^d,$$

where

$$F(\hat{u})(n) = \left( \sum_{i=1}^d \nu_i^2 n_i^2 - m \right) \hat{u}(n) + \epsilon(f \hat{u}^p + \gamma)(n).$$

We want to solve (3.1) and thus employ the Nash-Moser iterations. The linearized operator of $F$ at $u$ (we write $u = \hat{u}$ for simplicity) reads

$$F_u := D + \epsilon S_u, \ u \in \mathbb{C}^{\mathbb{Z}^d},$$
where
\[ D = \text{diag} \left( \sum_{i=1}^{d} \nu_i^2 n_i^2 - m \right) \quad n \in \mathbb{Z}^d \]
is a diagonal operator, and
\[ S_u(n, n') = \widehat{\text{fu}_n}^{-1}(n - n') \]
is a Töplitz operator. Define for \( \theta \in \mathbb{R}^d \) the following operators
\[ (3.2) \quad F_u(\theta) = \text{diag} \left( \sum_{i=1}^{d} (\nu_i n_i + \theta_i)^2 - m \right) \quad n \in \mathbb{Z}^d + \epsilon S_u. \]
Then the covariance property holds:
\[ (3.3) \quad F_u(\theta)(n + \pi, n' + \pi) = F_u(\theta + \pi\nu)(n, n'), \]
where
\[ \pi = (\pi_1, \cdots, \pi_d), \quad \pi\nu = (\pi_1\nu_1, \cdots, \pi_d\nu_d). \]

We should remark that the small-divisors here are
\[ \sum_{i=1}^{d} \nu_i^2 n_i^2 - m. \]

4. LDT for Green’s functions

The key of the Nash-Moser iterations is to get good estimates of
\[ G^u_{\Lambda}(\theta) = (R_{\Lambda} F_u(\theta) R_{\Lambda})^{-1}, \]
where \( R_{\Lambda} \) is the restriction operator on \( \Lambda \subset \mathbb{Z}^d \), and \( F_u(\theta) \) is defined by (3.2).
We call \( G^u_{\Lambda}(\theta) \) a Green’s function.

For some technical reasons, we need introduce elementary regions on \( \mathbb{Z}^d \). Given \( N > 0 \), \( \emptyset \neq I \subset \{1, \cdots, d\} \) and \( \varsigma = (\varsigma_i)_{i \in I} \in \{<, >\}^I \), define
\[ Q_N(I, \varsigma) := [-N, N]^d \setminus \{ n \in \mathbb{Z}^d : n_i \varsigma_i 0, i \in I \}. \]
Let
\[ \mathcal{E}^0_N := \bigcup_{\emptyset \neq I \subset \{1, \cdots, d\}, \varsigma \in \{<, >\}^I} \{Q_N(I, \varsigma)\} \]
and
\[ \mathcal{E}_N := \bigcup_{n \in \mathbb{Z}^d, Q \in \mathcal{E}^0_N} \{n + Q\}. \]

We write \( G^u_{\Lambda}(\theta) = G^u_{\Lambda}(\theta) \) if \( \Lambda \in \mathcal{E}^0_N \). From (3.3), the structure of \( G^u_{\Lambda}(\theta) \) is similar to that of Green’s functions in Anderson localization theory for lattice quasi-periodic Schrödinger operators in [10].

Recently, Jitomirskaya-Liu-Shi [26] extended Bourgain’s results [10] to more general ergodic transformations as well as long-range interactions case. Actually, the proofs of [26] can be used in the present work with some modifications, and then imply the LDT for \( G^u_{\Lambda}(\theta) \). The main techniques employed here are semi-algebraic sets analysis and the matrix-valued Cartan estimate.

Firstly, we introduce some useful facts about the semi-algebraic sets.
Definition 4.1 (Chapter 9, [8]). A set $S \subseteq \mathbb{R}^n$ is called a semi-algebraic set if it is a finite union of sets defined by a finite number of polynomial equalities and inequalities. More precisely, let \( \{ P_1, \cdots, P_s \} \subseteq \mathbb{R}[x_1, \cdots, x_n] \) be a family of real polynomials whose degrees are bounded by \( d \). A (closed) semi-algebraic set $S$ is given by an expression

$$S = \bigcup_j \bigcap_{\ell \in \mathcal{L}_j} \{ x \in \mathbb{R}^n : P_\ell(x) \varsigma_{j\ell} 0 \},$$

where $\mathcal{L}_j \subset \{1, \ldots, s\}$ and $\varsigma_{j\ell} \in \{\geq, \leq, =\}$. Then we say that $S$ has degree at most $sd$. In fact, the degree of $S$ which is denoted by $\text{deg}(S)$, means the smallest $sd$ over all representations as in (4.1).

Lemma 4.2 ([10]). Let $S \subseteq [0, 1]^{d_1+d_2}$ be a semi-algebraic set of degree $\text{deg}(S) = B > 0$ and $\text{mes}(S) \leq \eta > 0$, where

$$\log B \ll \log \frac{1}{\eta}.$$

Denote by $(x_1, x_2) \in [0, 1]^{d_1} \times [0, 1]^{d_2}$ the product variable. Fixing

$$\eta^{\frac{1}{d_2}} \leq \varepsilon \ll 1,$$

then there is a decomposition of $S$ as

$$S = S_1 \cup S_2$$

with the following properties. The projection of $S_1$ on $[0, 1]^{d_1}$ has small measure

$$\text{mes}_{d_1}(\text{Proj}_{x_1} S_1) \leq BC(d) \varepsilon,$$

and $S_2$ has the transversality property

$$\text{mes}_{d_2}(\mathcal{L} \cap S_2) \leq BC(d) \varepsilon^{-1} \eta^{\frac{1}{d_2}},$$

where $\mathcal{L}$ is a $d_2$-dimensional hyperplane in $[0, 1]^d$ s.t.,

$$\max_{1 \leq j \leq d_1} |\text{Proj}_{\mathcal{L}}(e_j)| < \varepsilon,$$

where we denote by $e_1, \cdots, e_{d_1}$ the $x_1$-coordinate vectors.

Lemma 4.3 (Lemma 1.18, [10]). Let $S \subseteq [0, 1]^{d_1+d_2}$ be a semi-algebraic set of degree $B$ and such that

$$\text{mes}_{d_1}(S(x)) < \eta \text{ for } \forall x \in [0, 1]^{d_2}.$$

Then the set

$$\left\{(y_1, \cdots, y_{2d_2}) \in [0, 1]^{d_1 \cdot 2^{d_2}} : \bigcap_{1 \leq i \leq 2^{d_2}} S(y_i) \neq \emptyset \right\}$$

is semi-algebraic of degree at most $BC$ and measure at most

$$BC \eta^{d_1-d_2} 2^{-d_2(d_2-1)/2},$$

where $C = C(d_1, d_2) > 0$. 
Eliminating multiple variables

Lemma 4.4 (Lemma 1.20, [10]). Let \( \emptyset \neq I \subset \{1, \cdots, d\} \) and let \( S \subset [0, L]^{\ell |I|} \) be a semi-
algebraic set of degree \( B \) such that
\[
\text{mes}_{|I|}(S) < \eta.
\]

For any
\[
\alpha = (\alpha_i)_{i \in I} \in \mathbb{R}^I, \ n = (n_i)_{i \in I} \in \mathbb{Z}^I,
\]
define
\[
n\alpha = (n_i \alpha_i)_{i \in I} \in \mathbb{R}^I.
\]

For any \( C > 1 \), let \( N_1, \cdots, N_{\ell - 1} \subset \mathbb{Z}^I \) be finite sets satisfying
\[
\min_{s \in I} |n_s| > (B \max_{s \in I} |n'_s|)^C,
\]
where \( n \in N_i, n' \in N_{i-1} \) (\( 2 \leq i \leq \ell - 1 \)).

Let \( L \sim \max_{n \in N_{\ell - 1}} |n| \). Then there is some \( C_0 = C_0(\ell, |I|) > 0 \) such that for any \( C \geq C_0 \) and
\[
\max_{n \in N_{\ell - 1}} |n|^C < \frac{1}{\eta}, \text{ one has}
\]
\[
\text{mes}\{\alpha \in [0, 1]^I : \ \exists n^{(1)} \in N_i \text{ s.t., } (\alpha, n^{(1)} \alpha, \cdots, n^{(\ell - 1)} \alpha) \in S\} \leq B^C \delta,
\]
where
\[
\delta^{-1} = \min_{n \in N_i} \min_{s \in I} |n_s|.
\]
The following result was indeed proved by Bourgain [10] in case \( d = 3 \) and it can be easily extended to any \( d \geq 1 \) (see [26] for details).

Lemma 4.5 (10). Let \( 0 < \tau_2 < \tau_1 < 1, C_* \gg 1 \) and \( N_2 \sim (\log N)^C \) for some \( C > 1 \). Suppose that for \( \emptyset \neq I \subset \{1, \cdots, d\}, \ \theta \in \mathbb{R}^d \) and \( \overline{N}^{\tau_2} \leq L \leq \overline{N}^{\tau_1} \), there is no sequence \( n^{(1)}, \cdots, n^{(2^d)} \in \mathbb{Z}^I \cap [-\overline{N}, \overline{N}]^I \) satisfying
\[
\min_{s \in I} |n^{(1)}_s| > L^{C_*},
\]
\[
\min_{s \in I} |n^{(i+1)}_s| \geq (L |n^{(i)}_s|)^{C_*} \ (1 \leq i \leq 2^d - 1),
\]
such that the following holds: for all $Q_L \in \mathcal{E}_L^0$ and $1 \leq i \leq 2^d$, $G_A^u(\theta)$ fails (4.2) + (4.3) for

\[ \Lambda = Q_L + \sum_{j \in J} n_j^{(i)} e_j \in \mathcal{E}_L, \]

where

\[ ||G_A^u(\theta)|| \leq e^{2N_2} \quad (0 < \kappa < 1), \]

\[ ||G_A^u(\theta)(n, n')|| \leq e^{-\kappa |n-n'|} \quad \text{for} \quad |n - n'| > N_2^{1+}, \]

and $|u(n)| \leq Ce^{-\rho |n|}, \rho > 0$.

Suppose moreover that

\[ \tau_2(C_* + 1)^{2(d-1)(d+2)} \leq \tau_1. \]

Then for $\theta \in \mathbb{R}^d$, the following statements hold:

(i) There is $N$ (depending on $\theta$) satisfying $\bar{N}^2 < N < \bar{N}^{\tau_1}$ such that if

\[ \Lambda = [-N, N]^d \setminus [-N^{\tau_1}, N^{\tau_1}]^d, \]

then

\[ ||G_A^u(\theta)|| \leq e^{3N_2}, \]

\[ ||G_A^u(\theta)(n, n')|| \leq e^{-(\kappa - \frac{N^{\tau_1}}{2}) |n-n'|} \quad \text{for} \quad |n - n'| > N_2^{1+}. \]

(ii) Let $Q_N \in \mathcal{E}_N^0$. Then for any $m \in Q_N$, there exist some $\bar{N}^2 < N < \bar{N}^{\tau_1}$ and $\Lambda_1, \Lambda \subset Q_N$ so that

\[ \text{diam}(\Lambda) \sim N, \text{diam}(\Lambda_1) \sim N^{\frac{\tau_1}{2}}, \quad m \in \Lambda_1 \subset \Lambda, \quad \text{dist}(m, Q_N \setminus \Lambda) \geq \frac{N}{2}, \]

and

\[ ||G_A^u|_{\Lambda \setminus \Lambda_1}(\theta)|| \leq e^{3N_2}, \]

\[ ||G_A^u|_{\Lambda_1}(\theta)(n, n')|| \leq e^{-(\kappa - \frac{N^{\tau_1}}{2}) |n-n'|} \quad \text{for} \quad |n - n'| > N_2^{1+}. \]

Finally, we introduce the powerful matrix-valued Cartan estimate with 1-dimensional parameters. For a generation to several variables case, we refer to [26].

**Lemma 4.6** (Matrix-valued Cartan estimate, [10, 14]). Let $T(x)$ be a self-adjoint $N \times N$ matrix function of a parameter $x \in [-\delta, \delta]$ satisfying the following conditions:

(i) $T(x)$ is real analytic in $x \in [-\delta, \delta]$ and has a holomorphic extension to

\[ D_{\delta, \delta_1} = \{ x \in \mathbb{C} : |\Re x| \leq \delta, \ |\Im x| \leq \delta_1 \} \]

satisfying

\[ \sup_{x \in D_{\delta, \delta_1}} ||T(x)|| \leq B_1. \]

(ii) For all $x \in [-\delta, \delta]$, there is subset $V \subset [1, N]$ satisfying the length

\[ |V| < M, \]

and

\[ \|(R|_{[1,N]}\setminus V)T(x)(R|_{[1,N]}\setminus V)^{-1}\| \leq B_2. \]

(iii) \[ \text{mes}\{x \in [-\delta, \delta] : \|T^{-1}(x)\| \geq B_3\} \leq 10^{-3}\delta_1(1 + B_1)^{-1}(1 + B_2)^{-1}. \]

Let

\[ 0 < \varepsilon \leq (1 + B_1 + B_2)^{-10M}. \]
Then
\[ \text{mes}\left\{ x \in \left[ \frac{\delta}{2}, \frac{\delta}{2} \right] : \| T^{-1}(x) \| \geq \varepsilon^{-1} \right\} \leq C\delta e^{-\frac{\varepsilon}{\delta^2}}. \]

We are ready to prove LDT for Green’s functions.

We first prove the LDT for the case when \( N > n \).

**Lemma 4.7.** Define for \( \theta \),
\[ \text{mes}(I) \leq C\delta \frac{\varepsilon}{\delta^2}, \] where \( C > 0 \) is an absolute constant.

**Proof.** The proof is trivial and we omit the details here. \( \square \)

**Lemma 4.8.** Let \( \xi \in (0, 1), \rho > 0 \). Then there exist \( N_0 > 1, \Sigma > 1, \kappa \in (0, 1) \) so that, for any \( u \) satisfying \( |u(n)| \leq Ce^{-\rho|n|} \), \( N_0 \leq N \leq |\log \varepsilon|^\Sigma \) and any \( \nu \in [1, 2]^d \),
\[ ||G_N^u(\theta)|| \leq e^{N\kappa}, \]
\[ |G_N^u(\theta)(n, n')| \leq e^{-\frac{\rho}{2}|n-n'|} \quad \text{for} \quad |n-n'| \geq \frac{N}{10}, \]
where \( \theta \) is outside a set \( X_N \subset \mathbb{R}^d \) satisfying for any \( 1 \leq j \leq d, \)
\[ \sup_{\theta_j \in \mathbb{R}^{d-1}} \text{mes}(X_N(\theta_j)) \leq e^{-N\gamma}, \]
with
\[ \theta_j = (\theta_1, \cdots, \theta_{j-1}, \theta_{j+1}, \cdots, \theta_d). \]

**Proof.** The proof is based on Lemma 4.7 and a standard Neumann series argument. \( \square \)

We define the following statements \((S)_N\) for \( N \in \mathbb{N} \).

**Definition 4.9 ((S)_N).** Let \( \kappa, \gamma, \zeta \in (0, 1) \). There is some semi-algebraic set \( \Omega_N \subset [1, 2]^d \) of degree at most \( N^d \) such that for \( \nu \in \Omega_N \), there exist some \( X_N \subset \mathbb{R}^d \) so that for any \( 1 \leq j \leq d, \)
\[ \sup_{\theta_j \in \mathbb{R}^{d-1}} \text{mes}(X_N(\theta_j)) \leq e^{-N\gamma}, \]
and for \( \theta \notin X_N, \)
\[ ||G_N^u(\theta)|| \leq e^{N\kappa}, \]
\[ |G_N^u(\theta)(n, n')| \leq e^{-(\rho_N-\kappa)|n-n'|} \quad \text{for} \quad |n-n'| \geq \frac{N}{10}, \]
where \( \rho_N > 0 \) and \( u_N \in \mathbb{C}^{2d} \) is rational in \( \nu \) so that
\[ \deg(u_N) \leq e^{(\log N)^4}, \quad |u_N(n)| \leq C e^{-\rho_N |n|.} \]
Moreover, \( \Omega_N \subset \Omega_{N_1} \cap \Omega_{N_2} \) with \( N_1 \sim (\log N)^{\frac{1}{\delta}}, N_2 \sim (\log N)^{\frac{1}{\delta}} < N \) and
\[ \text{mes}((\Omega_{N_1} \cap \Omega_{N_2}) \setminus \Omega_N) \leq N^{-\zeta}. \]

The main result of this section is the following LDT for Green’s functions.

**Theorem 4.10 (LDT).** There are \( \zeta, \kappa \in (0, 1) \) such that the following holds: Assume \( |\log \log \varepsilon| \geq N_0 \) and \((S)_N\) holds for \( N_0 \leq N < N \), \( \bar{N} \geq |\log \varepsilon|^\Sigma \). Then \((S)_{\bar{N}}\) holds with
\[ u_{\bar{N}} = u_{N_2}, \quad N_2 \sim (\log \bar{N})^{\frac{1}{\delta}}. \]
Moreover, the estimates (4.3) and (4.5) (for \( N = \bar{N} \)) remain valid if \( u_{\bar{N}} \) is replaced by some \( u \) satisfying: \( u \) is rational in \( \nu \) and
\[ \deg(u) \leq e^{(\log \bar{N})^4}, \| u - u_{\bar{N}} \|_{L^2} \leq e^{-\rho_{N_2}(\log \bar{N})^{\frac{1}{\delta}}} \].
Proof of Theorem 4.10

Defining scales
\[ N_1 = \left( (\log N)^{\frac{1}{2}} \right), \quad N_2 := \left\lfloor N_{1/2}^{1/2} \right\rfloor, \]
then
\[ N \sim e^{N_1^{1/2}}. \]

If we assume
\[ |\log \log \epsilon| \geq N_0, \]
and \((S)_N\) holds for \(N_0 \leq N < \overline{N}, \overline{N} \geq (\log \lambda)^{\Sigma},\) then \(N_1 > N_2 \geq N_0,\) which shows \((S)_N\) holds for \(N = N_1, N_2.\) Similar to \([10]\), the set in \((\nu, \theta)\) defined by \((4.4)\) and \((4.5)\) can be replaced by a semi-algebraic set of degree at most \(e^{C(\log N)^4}.)\]

Consider now any scale \(L\) with
\[ N_1 < L < \overline{N}. \]
Let \(S_L\) be the set of all \((\nu, \theta) \in \mathbb{R}^d \times \mathbb{R}^d\) such that for any \(n \in [-L, L]^d, \nu \in \Omega_{N_1} \cap \Omega_{N_2}, \theta + n\nu \in X_{N_2}.\) Direct computations gives
\[ \deg(S_L) \leq Ce^{C(\log N_1)^4}(2L + 1)^dN_1^{d} \leq L^{d}. \]

Fix \(1 \leq j \leq d\) and \(\theta_j^c \in \mathbb{R}^{d-1}.\) Thus we have
\[ \deg(S_L) \leq Ce^{C(\log N_1)^4}(2L + 1)^dN_1^{d} \leq L^{d}. \]

Moreover, if \((\nu, \theta) \in S_L,\) by using the resolvent identity (see \([26]\) for details), one has
\[ ||G^{N_2}_{N_2}(\theta)|| \leq e^{2N_2}, \]
\[ |G^{N_2}_{N_2}(\theta)(n, n')| \leq e^{-\rho_1|n-n'|} \text{ for } |n-n'| > N_1^{d+1}, \]
where
\[ \rho_1 = \rho_{N_2} - N_2^{0-}. \]

In the following, we will eliminate the variable \(\theta.\) This needs make further restrictions on \(\nu.\) By fixing \(I \subset \{1, \cdots, d\},\) define
\[ \mathcal{A} := \{ (\nu, \theta, y) \in [1, 2]^d \times \mathbb{R}^d \times \mathbb{R}^I : \nu \in \Omega_{N_2} \cap \Omega_{N_1}, (\nu, (\theta_j + y_j)_{j \in I}, (\theta_j)_{j \notin I}) \notin S_L \}. \]

Obviously, by \((4.6)\) and \((4.8),\)
\[ \deg(\mathcal{A}) \leq L^{d}. \]

Fix \(\nu\) and consider
\[ \mathcal{A}_1 := \mathcal{A}(\nu) \subset \mathbb{R}^d \times \mathbb{R}^I. \]

We note that \(G^{\nu_{N_2}}(\theta)\) satisfies off-diagonal exponential decay property for all \(\nu \in [1, 2]^d\) and \(|\theta| \geq \overline{N}.\) Without loss of generality, it suffices to assume \(\mathcal{A}_1 \subset [0, C\overline{N}]^d \times [0, C\overline{N}]^I.\) From \((4.7),\) for all \(\theta,\)
\[ \deg(\mathcal{A}_1(\theta)) \leq \eta := e^{-N_2^{d}}. \]

By \((4.9),\) \((4.10)\) and Lemma 4.3
\[ \mathcal{A}_2 = \left\{ (y_{i})_{1 \leq i \leq 2^d} : \bigcap_{1 \leq i \leq 2^d} A_1(y^{(i)}) \neq \emptyset \right\} \subset [0, C\overline{N}]^d \]
is a semi-algebraic set of degree
\[ \deg(\mathcal{A}_2) \leq L^{d}. \]
and measure

\begin{equation}
\mes(A_2) \leq \eta_1 := \frac{1}{N^{C \eta} |I|^{d-2d(d-1)/2}}.
\end{equation}

Notice that for $N_0 \gg 1$ and $C_* \gg 1$,

\[ \frac{1}{\eta_1} \gg N^{C_*}. \]

Then from (4.11), (4.12) and Lemma 4.4 the set $A_3 \subset [1, 2]^I$ containing $\nu_I := (\nu_j)_{j \in I}$, which is defined by the following: there is some sequence $n^{(1)}, \ldots, n^{(2d)} \in \mathbb{Z}^I \cap [-N, N]^I$ satisfying

\begin{align*}
\min_{s \in I} |n_s^{(1)}| &> L_{C_*}, \\
\min_{s \in I} |n_s^{(i+1)}| &\geq (L|n_s^{(i)}|)^{C_*} \quad (1 \leq i \leq 2d - 1),
\end{align*}

such that

\[(\nu_I, n^{(1)}, \nu_I, \ldots, n^{(2d)}, \nu_I) \in A_2, \]

satisfies

\begin{equation}
\mes(A_3) \leq L^{-C_*} L^C,
\end{equation}

where $C_* \gg C$. It is easy to see the total number of $n^{(1)}, \ldots, n^{(2d)} \in \mathbb{Z}^I \cap [-N, N]^I$ satisfying (4.13) and (4.14) can be bounded by $(2N + 1)^{2d}$. Recalling (4.11), $A_3$ is a semi-algebraic set of degree

\begin{equation}
\deg(A_3) \leq C(2N + 1)^{2d} L^C \leq N^{2d},
\end{equation}

if

\[ \log L \leq c_1 \log N, \]

where $0 < c_1 = c_1(d) \ll 1$. Define

\[ \Omega_N := \bigcap_{\emptyset \neq I \subset \{1, \ldots, d\}} \{ \nu \in \Omega_{N_2} \cap \Omega_{N_1} : \nu_I \notin A_3 \}. \]

If we assume

\[ \log L \geq c_2 \log N, \]

then by (4.15) and (4.16), for $C_* \gg C$,

\begin{align*}
\mes((\Omega_{N_2} \cap \Omega_{N_1}) \setminus \Omega_N) &\leq C(d) L^{-C_*} L^C \leq \frac{C}{N^\zeta}, \\
\deg(\Omega_N) &\leq C(d) L^C N_2^{d_2} N_1^{d_1} N^{2d} \leq N^{2d},
\end{align*}

where $c_2 > 0$ will be specified below and $\zeta$ depends on $c_2$.

Let

\[ c_2(C_* + 1)^{2(d-1)(d+2)} = c_1. \]

Then for $\nu \in \Omega_N$, the assumptions of Lemma 4.5 are satisfied.

We then construct $X_N$ and finish the proof. Again, it suffices to restrict the considerations on $B(0, 3N) = \{ \theta \in \mathbb{R}^d : |\theta| \leq 3N \}$. As in [10], on each unit cube in $B(0, 3N)$, using Lemma 4.6 Lemma 4.5 and the resolvent identity, one can find such $X_N$ for $\nu \in \Omega_N$. We refer to [26] (see also [10]) for details. \qed
Remark 4.11. From the definition of $\Omega_{\mathcal{N}}$, the set $\Omega_{\mathcal{N}}$ is basically stable under perturbations of order $e^{-(\log \mathcal{N})^{\gamma}}$, $0 < \gamma \ll 1$. More precisely, one can replace $\Omega_{\mathcal{N}}$ by the set

$$\bigcup_{i: I_i \cap \Omega_{\mathcal{N}} \neq \emptyset} I_i,$$

where the union runs over a partition of $[1, 2]^d$ of cubes $I_i$ of side length $e^{-(\log \mathcal{N})^{\gamma}}$. This leads to a reformulation of Theorem 4.10, i.e., Theorem 4.12 below. This point will be useful in Section 5.

We fix a large integer $A$ satisfying

$$1 \ll N_0 \leq A \leq |\log \epsilon|^{\Sigma}.$$

Theorem 4.12. Let $0 < \epsilon \ll 1$ and $r \geq 1$. Then there exists a collection $\Gamma_r$ of cubes $I$ (in $[1, 2]^d$) of side length $e^{-(r \log A)^{\gamma}}$ satisfying the following:

(i) If $\nu \in I \in \Gamma_r$, then there exists some set $X_{\mathcal{A}^r}$ such that, for $1 \leq j \leq d$,

$$\text{mes}(X_{\mathcal{A}^r}(\theta_j)) \leq e^{-A^{\gamma r}},$$

and for $\theta \notin X_{\mathcal{A}^r}$,

$$||G_{\mathcal{A}^r}^\nu(\theta)|| \leq e^{A^{\gamma r}},$$

$$|G_{\mathcal{A}^r}^\nu(\theta)(n, n')| \leq e^{-\frac{c}{2}|n-n'|} \text{ if } |n-n'| \geq \frac{A^r}{10}.$$

(ii) Each $I \in \Gamma_r$ is contained in a cube $I' \in \Gamma_{r-1}$ and

$$\Gamma_r = \{[1, 2]^d\} (1 \leq r \leq r_*)$$

$$\text{mes} \left( \bigcup_{I' \in \Gamma_{r-1}} I' \setminus \bigcup_{I \in \Gamma_r} I \right) \leq A^{-\frac{c}{2}} (r > r_*),$$

where $r_* = \left\lfloor \frac{\log |\log \epsilon|^{\Sigma}}{\log A} \right\rfloor$.

Proof. We refer to [26] for a detailed proof.

5. The proof of Main theorem: Nash-Moser algorithm

From Nash-Moser algorithm, the approximate solution $q_{r+1}$ at $(r+1)^{th}$ step can be derived from

$$q_{r+1} = q_r + \Delta_{r+1} q,$$

where the correction $\Delta_{r+1} q$ satisfies for $N = A^{r+1}$,

$$\Delta_{r+1} q = -G_N^q(0)F(q_r),$$

whenever $G_N^q(0)$ is good, i.e., $G_N^q(0)$ satisfies the following estimates

$$||G_N^q(0)|| \leq A^{(r+1)c},$$

$$|G_N^q(0)(n, n')| \leq e^{-c|n-n'|} \text{ for } |n-n'| \geq (r+1)c,$$

where for some $C, c > 0$.

As in [8], to prove Theorem 1.1 it suffices to establish the following iteration arguments.
Theorem 5.1. For small $\epsilon$, there exists some $A(\epsilon) \gg 1$ such that for any $r \geq 1$ there is some $q_r(\nu) \in \mathbb{C}^{2d}$ satisfying the following:

(i) $\text{supp}(q_r) \subset B(0, A^r)$.

(ii) $\sup_{\nu \in [1,2]^d} ||\Delta_r q||_2 \leq \sigma_r$, $\sup_{\nu \in [1,2]^d} ||\partial_\nu \Delta_r q||_2 \leq \overline{\sigma}_r$,

where $\log \log \frac{1}{\sigma_r + \overline{\sigma}_r} \sim r$.

(iii) $|q_r(n)| \leq Ce^{-\rho|n|}, \rho > 0$.

(iv) There is a collection $\Gamma_r$ of intervals $I$ in $\mathbb{R}^d$ of side length $A^{-r^C}$ so that the following holds:

(a) On $I \in \Gamma_r$, $q_r(\nu, n)$ is given by a rational function in $\nu$ of degree at most $A^r^3$.

(b) For $\nu \in \bigcup_{I \in \Gamma_r} I$,

$$||F(q_r)||_2 \leq \mu_r, ||\partial_\nu F(q_r)||_2 \leq \overline{\mu}_r,$$

where $\log \log \frac{1}{\mu_r + \overline{\mu}_r} \sim r$.

(c) For $\nu \in \bigcup_{I \in \Gamma_r} I$ and $N = A^r$,

$$||G^N_\nu(0)|| \leq A^{r^C},$$

$$|G^N_\nu(0)(n, n')| \leq e^{-\frac{C}{2}|n-n'|} \text{ for } |n-n'| \geq \nu^C.$$

(d) Each $I \in \Gamma_r$ is contained in an interval $I' \in \Gamma_{r-1}$ and

$$\text{mes} \left( [1,2]^d \setminus \bigcup_{I \in \Gamma_1} I \right) \leq A^{-\frac{C}{2}}.$$

$$\text{mes} \left( \bigcup_{I' \in \Gamma_{r-1}} I' \setminus \bigcup_{I \in \Gamma_r} I \right) \leq A^{-\frac{C}{2}}, \ r \geq 2.$$

(v)

$$\sigma_r < \sqrt{C}A^{-(\frac{1}{2})r}, \mu_r < \sqrt{C}A^{-(\frac{1}{2})r + 2},$$

$$\overline{\sigma}_r < \sqrt{C}A^{-\frac{1}{2}(\frac{1}{2})r}, \overline{\mu}_r < \sqrt{C}A^{-\frac{1}{2}(\frac{1}{2})r + 2}.$$

Proof. We start from $q_0 = 0$ and note that $F(q_0) = e\hat{g}$. We will construct $q_1$ firstly. It is easy to see

$$F_{q_0}(n, n) = \text{diag} \left( \sum_{i=1}^d n_i^2 n_i^2 - m \right)$$

and $F_{q_0}(n, n') = 0$ for $n \neq n'$. Let $A \gg 1$. Due to $m > 0$, then for any $0 < \delta < m$, one has

$$|F_{q_0}(n, n)| > \delta \text{ for } |n| \leq A$$

where $n$ is outside a set $R_1$ of measure

$$\text{mes}(R_1) \leq CA^d \delta^{\frac{1}{2}}.$$
Thus if $\nu \notin R_1$,

$$||\Delta_1 q||_{L^2} \leq \epsilon \delta^{-1} ||G||_{L^2}.$$  

Using standard Neumann series arguments, one can prove the theorem for any $1 \leq r \leq K \sim |\log \epsilon|^{\frac{1}{2}}$.

We will prove this theorem is true for any $r > K$. This can be completed by using LDT and make further restrictions on $\nu$. Assume $q_r$ ($\nu' \leq r, r > K$) have been constructed and fulfill all the properties in (i)–(v). We want to construct $q_{r+1}$ and this needs to study the inverse of $R_N F_{q_r}(0) R_N$, where $N = A^{r+1}$.

Firstly, in view of $||q_r - q_{r-1}||_{L^2} \leq \sigma_r \ll e^{-A^r}$, a standard perturbation argument implies for $\nu \in I \in \Gamma_r$

\begin{align}
||G_{A^r}^q(0)|| &\leq A^{rC}, \\
|G_{A^r}^q(n, n')| &\leq e^{-\frac{2}{3}|n-n'|} \text{ for } |n-n'| \geq rC.
\end{align}

Then we consider in $U = \{n \in \mathbb{Z}^d : \frac{1}{2^{r+1}} \leq |n| \leq N\}$. To prove $G_{A^r}^q(0)$ has estimates \((5.1), \ (5.2)\) with $Q_{A^r}$ being replaced by $U$, it needs make further restrictions on $\nu$ by using LDT and some semi-algebraic sets analysis arguments. Let $M_0 = A^{r_0}$ satisfy

$$M_0 \sim (\log N)^{\frac{1}{2}}.$$ 

Fix $I \in \Gamma_{r_0}$ and consider the following set

$$S_N = \{(\nu, \theta) \in I \times [-N, N]^d : \nu \in I, G_{A^0}^{q_0}(\theta) \text{ is not good}\}.$$ 

Obviously, $S_N$ is a semi-algebraic set of degree at most $e^{C(\log M_0)^3}$. By Theorem 4.12 we have

$$\mes(S_N) \leq CN^d e^{-M_0} \leq e^{-\frac{1}{2}M_0}.$$ 

Using Lemma 4.2 as in [10] (see also [20]), the set

$$S_* = \{\nu \in I : \exists n \in U \text{ s.t. }, (\nu, n\nu) \in S_N\}$$

has measure at most $A^{-\frac{1}{2}}$.

Sum over $I \in \Gamma_{r_0}$ and define $\Gamma_{r+1}$ to be the collection of cubes of side length $A^{-(r+1)C}$ satisfying the following: elements of $\Gamma_{r+1}$ are derived from dividing $I' \in \Gamma_r$ into cubes $I$ of side length $A^{-(r+1)C}$ so that $I \cap (\mathbb{R}^d \setminus S_*) \neq \emptyset$.

Finally, on $I \in \Gamma_{r+1}$, using $||q_r - q_0||_{L^2} \ll e^{-(\log M_0)}$ and the resolvent identity implies the good Green’s function $G_{A^r}^q(0)$. The remainder then becomes clear and we refer to Chapter 18 of [8], or [10] for details. \qed

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(Y. Shi) School of Mathematical Sciences, Peking University, Beijing, China

E-mail address: yunfengshi18@gmail.com