Quasi-additivity of Tsallis entropies and correlated subsystems

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Abstract
We use Beck’s quasi-additivity of Tsallis entropies for \( n \) independent subsystems to show that like the case of \( n = 2 \), the entropic index \( q \) approaches 1 by increasing system size. Then, we will generalize that concept to correlated subsystems to find that in the case of correlated subsystems, when system size increases, \( q \) also approaches a value corresponding to the additive case.

Keywords: Non-extensive statistical mechanics, Tsallis entropies, Quasi-additivity relation, correlated subsystems.

1 Introduction
The formalism of non-extensive statistical mechanics has been developed over the past two decades as a beautiful generalization of ordinary statistical mechanics [1, 2, 3]. It is based on the extremization of the Tsallis entropies

\[ S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1}, \]

subject to suitable constraints. Here, the \( p_i \) are the probabilities of physical microstates, and \( q \) is the entropic index. The Tsallis entropies reduce to the Boltzmann-Gibbs (or Shannon) entropy \( S_{BG} = \sum_{i=1}^{W} p_i \ln p_i \) for \( q \to 1 \). Growing experimental evidence indicates that \( q \neq 1 \) yields a correct description of many complex physical phenomena [4]. Tsallis’ original suggestion was that this approach might be relevant for equilibrium systems with long-range interactions, but recently it was also pointed out that the formalism comes into play when systems are far from equilibrium. For example, for systems with fluctuating mean free path [5, 6] and temperature or energy dissipation rate [7, 8], Tsallis entropy may be relevant. The common feature in the examples of non-equilibrium systems mentioned is that the parameter \( q \) can be expressed by the relative

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variance of the fluctuations of a parameter $X$

$$q = 1 \pm \frac{\langle X^2 \rangle - \langle X \rangle^2}{\langle X \rangle^2},$$

(2)

provided that $X$ is $\chi^2$ (or gamma) distributed. Here the signs ‘+’ (‘−’) refer to $q > 1$ ($q < 1$) cases, respectively. From this point of view, the parameter $q$ can be treated in such systems as a measure of fluctuations of a parameter $X$ [5, 6].

Recently Beck [9] has suggested that for systems with fluctuations in temperature or energy dissipation rate, it is possible to make the Tsallis entropies quasi-additive by choosing different entropic indices at different spatial scales:

$$S^A_q + S^B_q = S^{A+B}_{q'},$$

(3)

where $q$ is used for statistically independent identical subsystems $A$ and $B$ and $q'$ for a composed system $A + B$. Eq. (3) for $q$ close to 1 results in

$$q' - 1 = (q - 1)\frac{\langle B^2 \rangle}{\langle B^2_i \rangle + \langle B_i \rangle^2},$$

(4)

where $B_i := \ln p_i$ is the so called "bit number" [10], its negative expectation $- < B_i > := - \sum_i p_i \ln p_i$ is the Shannon entropy, and its variance $- < B_i^2 > := - \sum_i p_i (\ln p_i)^2$ is related to fluctuations in entropy. From Eq. (4), it is clear that if $q > 1$, then $q > q' > 1$. This is what Beck interpreted as scale dependence. In other words, $q'$ related to the composed system is less than $q$ for each subsystem.

Moreover, there are experimental examples showing that $q$ is monotonously decreasing as a function of distance $r$ [11]. This idea has been used in [12] for independent non-identical subsystems with different entropic indices $q_1$ and $q_2$, where a power law for entropic index was extended as a function of distance $r$. The Tsallis entropies are non-extensive (there is little difference between extensivity and additivity and in this paper we will use extensivity and additivity interchangeably). Given two independent subsystems $A$ and $B$ with probabilities $p_i^A$ and $p_i^B$, respectively, the entropy of the composed system $A + B$ (with probabilities $p_{ij}^{A+B} = p_i^A p_i^B$) satisfies

$$S^{A+B}_{q} = S^A_q + S^B_q + (1 - q)S^A_q S^B_q.$$

(5)

Hence, for independent subsystems, there is additivity only for $q = 1$. More recently, another interesting concept has been reported [13, 14, 15]. Within this approach, equal and distinguishable subsystems can be strongly (globally) correlated such that for an adequate value of $q \neq 1$, $S_q$ becomes strictly additive. This means that additivity can exist even for correlated subsystems.

There are three demands to be considered: independency of subsystems, the same $q$ for the composed system and subsystems, and additivity of $S_q$. These demands can not coexist except when $q \to 1$ (Boltzmann-Gibbs entropy). Most early papers on non-extensive statistical mechanics deal with the first and the second demands and give up the third, Beck [9] keeps the first and the third, while Tsallis [13] considers the second and the third.
This paper is organized as follows. In section 2, Beck’s work is generalized by including \( n \) subsystems instead of the original two subsystems. In section 3, we will attempt to use quasi-additivity property (Eq. 3) for correlated subsystems where \( q \) is assumed to be close to the additive case \( q^* \). In section 4, a method of finding probabilities satisfying additivity relation is presented. In section 5, some numerical solutions of quasi-additivity relation for two correlated equal binary subsystems are given and, finally, we have a conclusion in section 6.

2 Quasi-additivity for \( n \) independent identical subsystems

Quasi-additivity relation (Eq. 3) for \( n \) identical subsystems can be written as:

\[
\sum_{r=1}^{n} S_{q}^{A_r} = S_{q}^{A_1+A_2+...+A_n},
\]

where \( S_{q}^{A_r} \) is the entropy of one of the subsystems and \( S_{q}^{A_1+...+A_n} \) is the entropy of the composed system. Our assumption is that \( q \) and \( q' \) are different and are close to 1. Parallel to what Beck did [9], one can obtain

\[
\sum_{i} p_i^q = \sum_{i} p_i \exp[(q - 1) \ln p_i] = 1 + (q - 1) \sum_{i} p_i \ln p_i + \frac{1}{2} (q - 1)^2 \sum_{i} p_i (\ln p_i)^2 + \ldots ,
\]

where the dots denote higher-order terms in \( q - 1 \). We can neglect the higher-order terms only if

\[
|(q - 1) \ln p_i| << 1, \quad \forall \ i
\]

or, if alternatively,

\[
p_i >> \exp\left(-\frac{1}{|q - 1|}\right).
\]

For example, for \( q = 1.1 \), it demands that \( p_i >> 4.54 \times 10^{-5} \) and for \( q = 0.7 \), that \( p_i >> 0.036 \). Most of the time, probabilities are not so small and the estimated value of \( q \) is very close to 1. So Eq. 9 is satisfied. The sum of entropies of \( n \) identical subsystems can be written as

\[
\sum_{r=1}^{n} S_{q}^{A_r} = nS_{q}^{A_1} = \frac{n}{q-1}(1 - \sum_{i} p_i^q) = -n \sum_{i} p_i \ln p_i - \frac{n}{2} (q - 1) \sum_{i} p_i (\ln p_i)^2 - \ldots ,
\]

and for the entropy of the composed system, we have

\[
S_{q'}^{A_1+A_2+...+A_n} = \frac{1}{q' - 1}(1 - \sum_{i_{1i_2...i_n}} p_{i_{1i_2...i_n}}^q) ,
\]

where \( p_{i_{1i_2...i_n}} \) are probabilities related to the composed system. Because \( q \) and \( q' \) are close to 1, the probabilities in different subsystems are nearly independent and, so

\[
\sum_{i_{1i_2...i_n}} p_{i_{1i_2...i_n}}^q = \sum_{i_{1i_2...i_n}} p_{i_1}^q p_{i_2}^q \cdots p_{i_n}^q = (\sum_{i} p_{i}^q)^n .
\]

Using Eq. (7), we have

\[
(\sum_{i} p_{i}^q)^n = 1 + n(q' - 1) \sum_{i} p_i \ln p_i + \frac{n}{2} (q' - 1)^2 \sum_{i} p_i (\ln p_i)^2 + \frac{n(n - 1)}{2} (q' - 1)^2 (\sum_{i} p_i \ln p_i)^2 + \ldots ,
\]
where $|n(q' - 1) \ln p_i| << 1$, so that the higher-order terms are very small and negligible. Hence, the entropy of the composed system is obtained as follows:

$$S_{q'_{A_1+A_2+...+A_n}} = -n \sum p_i \ln p_i - \frac{n}{2} (q' - 1) \sum p_i (\ln p_i)^2 - \frac{n(n - 1)}{2} (q' - 1) (\sum p_i \ln p_i)^2 + \ldots.$$ (14)

Quasi-additivity, thus, implies a relation between $q$ and $q'$, namely,

$$\frac{q' - 1}{q - 1} = \frac{\sum p_i (\ln p_i)^2}{\sum p_i (\ln p_i)^2 + (n - 1)(\sum p_i \ln p_i)^2},$$ (15)

which can be rewritten in the following simplified form

$$\frac{q' - 1}{q - 1} = \frac{\langle B_i^2 \rangle}{\langle B_i^2 \rangle + (n - 1)\langle B_i \rangle^2}.$$ (16)

For $n = 2$, Eq. (4) is recovered. The R.H.S. of Eq. (16) is positive and less than 1, so $\frac{q' - 1}{q - 1}$ is also positive and we can write

$$0 < \left| \frac{q' - 1}{q - 1} \right| = \frac{\langle B_i^2 \rangle}{\langle B_i^2 \rangle + (n - 1)\langle B_i \rangle^2} < 1, \quad n > 1.$$ (17)

Eq. (17) shows the dependence of $q'$ on $n$. It also says that

$$|q' - 1| < |q - 1| \Rightarrow \begin{cases} 1 < q' < q & \text{if } q > 1 \\ q < q' < 1 & \text{if } q < 1 \end{cases}.$$ (18)

If we assume that $n$ increases by increasing scale $r$, it is clear from Eq. (17) that for $q > 1$ ($q < 1$), $q'(r)$ is a strictly monotonously decreasing (increasing) function of scale $r$, so that by increasing system size, $q'$ approaches 1. From Eq. (2), it is clear that the deviation of parameter $q$ from 1 can be interpreted as the fluctuations in temperature (or energy dissipation). Hence, approaching of $q'$ to 1 by increasing $n$, indicates that fluctuations become negligible when system size increases. Assuming that $n$ is proportional to volume ($V \propto r^3$), in the limit $n \rightarrow \infty$, it is clear from Eq. (16) that

$$|q' - 1| \propto \frac{1}{n} \propto \frac{1}{V}.$$ (19)

The dependence of $q$ on the size of the system has been studied in the hadronization process [16], where $q - 1$ is introduced as a measure of total heat capacity of the hadronizing system [17, 18]: $q - 1 = \frac{1}{\mathcal{C}}$, and then the assumption of $C \propto V$ results in: $q - 1 = \frac{1}{\mathcal{C}}$, where $q$ is $q'$ in our discussion. However, it should be noted that the higher-order terms in (13) are negligible, only if for each $n$ we have

$$|n(q' - 1) \ln p_i| << 1, \quad \forall p_i$$ (20)

and that this condition should also be confirmed also in the limit $n \rightarrow \infty$. By noting that $\frac{(B_i^2) - \langle B_i \rangle^2}{\langle B_i \rangle^2}$ is positive, one can change equality in Eq. (16) to the following inequalities:

$$|n(q' - 1)\ln p_i| \leq |q - 1|\frac{(B_i^2)}{(B_i)^2} \Rightarrow |n(q' - 1)\ln p_i| \leq |q - 1|\frac{(B_i^2)}{(B_i)^2} \ln p_i.$$ (21)
Hence, condition (20) is confirmed if

\[ A = |q - 1| \frac{\sum_i p_i (\ln p_i)^2}{(\sum_i p_i \ln p_i)^2} |\ln p_i| << 1 , \forall p_i . \]  

(22)

We use that condition to find permissible regions for binary subsystems, which have two possible states with the probabilities \( p \) and \( 1 - p \). The regions have been shown in Fig. (1) for \( A = 0.4 \) and \( A = 0.2 \). These values of \( A \) give a nearly good approximation for the expansion (13), because we hold the terms which contain \( A^2 \).

In this section, we studied quasi-additivity relation for \( n \) independent identical subsystems. Our expansion makes sense only when \( q \) is close to 1. It should be noted that 1 is the value of \( q \) for which the Tsallis entropy becomes additive in independent subsystems. At this point, the question arise as to what we can say about correlated subsystems if we want to use quasi-additivity relation for them. The answer is provided in the following section.

3 Quasi-additivity property for correlated identical subsystems

As implied, if subsystems are independent, then the Tsallis entropy of a composite system becomes additive for \( q \to 1 \) (Boltzmann-Gibbs entropy) and for other values of \( q \), the Tsallis entropy is not additive (5). However, when there are some correlations between subsystems, the Tsallis entropy may be additive for an appropriate \( q \neq 1 \). In this section, we consider \( n \) equal subsystems, which are specially correlated. So additivity condition is established for \( q^* \neq 1 \):

\[ \sum_{r=1}^{n} S_{q^*}^{A_r} = S_{q^*}^{A_1 + A_2 + \ldots + A_n} . \]  

(23)

It may be interesting to use the quasi-additivity property (Eq. (6)) for correlated subsystems. Similar to the case of independent subsystems where \( q \) and \( q' \) are assumed to be close to 1, namely, additive case, here also we assume that \( q \) and \( q' \) are close to the additive case \( q^* \). Supposing that

Figure 1: Permissible region for \(|q - 1|\) plotted and colored as a function of \( p \), for two-state subsystems with the probabilities \( p \) and \( 1 - p \).
\[ q = q^* + \delta, \] 

One obtains

\[
\sum_i p_i^{q^*+\delta} = \sum_i p_i^q \exp^{\delta \ln p_i} = \sum_i p_i^q + \delta \sum_i p_i^q \ln p_i + \frac{1}{2} \delta^2 \sum_i p_i^q (\ln p_i)^2 + \ldots ,
\]

where only if \(|\delta \ln p_i| << 1\), it is possible to neglect the higher-order terms in the expansion. We have a coefficient \(\frac{1}{q-1}\) in the entropy and are interested in expanding it to powers of \(\delta\)

\[
\frac{1}{q-1} = \frac{1}{q^*-1+\delta} = \frac{q^*-1-\delta}{(q^*-1)^2-\delta^2} = \frac{q^*-1-\delta}{(q^*-1)^2} (1 + \frac{\delta^2}{(q^*-1)^3} + \ldots) 
\]

So the entropy \(S_q^{A_1}\) can be written as

\[
S_q^{A_1} = \frac{1 - \sum_i p_i^q}{q-1} = (1 - \sum_i p_i^q - \delta \sum_i p_i^q \ln p_i - \frac{1}{2} \delta^2 \sum_i p_i^q (\ln p_i)^2 + \ldots) (\frac{1}{q^*-1} - \frac{\delta}{(q^*-1)^2} + \frac{\delta^2}{(q^*-1)^3} + \ldots),
\]

and to second-order of \(\delta\) we obtain

\[
S_q^{A_2} = \frac{1 - \sum_i p_i^q}{q^*-1} - \delta \left( \frac{1 - \sum_i p_i^q - \sum_i p_i^q \ln p_i}{(q^*-1)^2} + \frac{1}{2} \frac{1}{q^*-1} \right) + \delta^2 \left( \frac{1 - \sum_i p_i^q - \sum_i p_i^q \ln p_i - \frac{1}{2} \sum_i p_i^q (\ln p_i)^2}{(q^*-1)^3} \right) + \ldots .
\]

Similarly for the composed system, one can write

\[
S_q^{A_1+A_2+\ldots+A_n} = \frac{1 - \sum_i p_i^q}{q^*-1} - \varepsilon \left( \frac{1 - \sum_i p_i^q - \sum_i p_i^q \ln p_i}{(q^*-1)^2} + \frac{\sum_i p_i^q \ln p_i}{q^*-1} \right) 
\]

\[ + \varepsilon^2 \left( \frac{1 - \sum_i p_i^q - \sum_i p_i^q \ln p_i}{(q^*-1)^3} + \frac{\sum_i p_i^q \ln p_i}{(q^*-1)^2} - \frac{1}{2} \frac{\sum_i p_i^q \ln p_i}{q^*-1} \right) + \ldots .
\]

where \(\varepsilon = q' - q^*\), the sum is over all \(i_r\) and \(p_{112\ldots in}\)s satisfy extensivity condition (Eq. (23)).

From the above relations, it is clear that the entropies \(S_q^{A_1}\) and \(S_q^{A_1+\ldots+A_n}\) in the first approximation are equal to \(S_q^{A_1}\) and \(S_q^{A_1+\ldots+A_n}\), respectively, where \(q^*\) corresponds to the extensive case. Using quasi-additivity property, Eq. (5) in conjunction with Eq. (23) yields

\[
\begin{align*}
&n\delta \left( \frac{1 - \sum_i p_i^q}{(q^*-1)^2} + \frac{\sum_i p_i^q \ln p_i}{q^*-1} \right) - n\delta^2 \left( \frac{1 - \sum_i p_i^q}{(q^*-1)^3} + \frac{\sum_i p_i^q \ln p_i}{(q^*-1)^2} - \frac{1}{2} \frac{\sum_i p_i^q \ln p_i}{q^*-1} \right) \\
&= \varepsilon \left( \frac{1 - \sum_i p_i^q}{(q^*-1)^2} + \frac{\sum_i p_i^q \ln p_i}{q^*-1} \right) \\
&- \varepsilon^2 \left( \frac{1 - \sum_i p_i^q}{(q^*-1)^3} + \frac{\sum_i p_i^q \ln p_i}{(q^*-1)^2} - \frac{1}{2} \frac{\sum_i p_i^q \ln p_i}{q^*-1} \right).
\end{align*}
\]

The above relation can be rewritten as

\[
n\delta A - n\delta^2 B = \varepsilon C - \varepsilon^2 D ,
\]

(30)
where

\[ A = 1 - \sum_i p_i^q \ln p_i \left( q^* - 1 \right)^2 + \sum_i p_i^q \ln p_i, \quad (31) \]

\[ B = \frac{A}{q^* - 1} - \frac{1}{2} \sum_i p_i^q (\ln p_i)^2, \quad (32) \]

\[ C = 1 - \sum_{i_1 i_2 \ldots i_n} p_{i_1 i_2 \ldots i_n}^q \ln p_{i_1 i_2 \ldots i_n}, \quad (33) \]

\[ D = \frac{C}{q^* - 1} - \frac{1}{2} \sum_{i_1 i_2 \ldots i_n} p_{i_1 i_2 \ldots i_n}^q (\ln p_{i_1 i_2 \ldots i_n})^2. \quad (34) \]

By defining

\[ \gamma \equiv \frac{\varepsilon}{\delta} = \frac{q' - q^*}{q - q^*}, \quad (35) \]

Eq. (30) can be rewritten as

\[ D\gamma^2 - \frac{C}{\delta} \gamma + \frac{nA}{\delta} - nB = 0, \quad (36) \]

and its solution is obtained as follows

\[ \gamma = \frac{1}{2D} \left( \frac{C}{\delta} \pm \sqrt{\frac{C^2}{\delta^2} - 4Dn(A - B)} \right) = \frac{1}{2D} \left( \frac{C}{\delta} \pm \sqrt{1 - \frac{4Dn}{C^2} (A\delta - B\delta^2)} \right). \quad (37) \]

It is observed that there are two solutions which contain the term \( \frac{C}{2D\delta} \). Because \( \delta \) is small, \( \frac{C}{2D\delta} \) becomes very large. But we are interested in finding finite solutions. If the coefficients of \( \delta \) and \( \delta^2 \) in the radical are finite, then Eq. (37) can be rewritten as

\[ \gamma = \frac{C}{2D\delta} \pm \frac{|C|}{2D\delta} \left( 1 - \frac{2DA n}{C^2} \delta + \frac{2DB n^2}{C^4} \delta^2 \right). \quad (38) \]

We can choose ‘+’ or ‘−’ signs from Eq. (38) (depending on the sign of \( |C| \)) so that the terms proportional to \( \delta^{-1} \) are eliminated and the solution is finite. It should be noted that when \( q^* \to 1 \) or in cases where probabilities are very small so that their logarithms become infinite, the coefficients \( A, B, C \) and \( D \) can be infinite and, hence, Eq. (38) will not be valid.

### 4 Finding probabilities which satisfy additivity relation

We assumed that \( p_{i_1 i_2 \ldots i_n} \) s are the probabilities which make the Tsallis entropy additive, but there is a problem of finding those probabilities for each \( n \). We restrict ourselves to \( n \) equal binary subsystems which have been investigated in [14] by Tsallis, Gell-Mann and Sato. Because our subsystems are binary, we have two possible microstates for each subsystem. Namely, for \( n = 1 \), the probabilities of microstates are \( p_1 = p \) and \( p_2 = 1 - p \). For \( n = 2 \), the probabilities are \( p_{11}, p_{12} = p_{21} \) and \( p_{22} \) and they satisfy the following relations

\[ p_{11} + p_{12} = p_{1} = p, \quad (39) \]

\[ p_{21} + p_{22} = p_{2} = 1 - p. \quad (40) \]
Eqs. (39) and (40) imply that for two subsystems $A$ and $B$, the sum of probabilities of the composed system $A + B$ over microstates of $B$ with a specified microstate of $A$ results in the probability of that microstate of $A$, which is reasonable. Adding Eq. (39) to (40) yields

$$p_{11} + 2p_{12} + p_{22} = 1.$$  (41)

In the case of independent subsystems, $p_{11} = p^2, p_{12} = p(1 - p)$ and $p_{22} = (1 - p)^2$, so Eq. (11) becomes trivial. Similarly, for three equal binary subsystems, we can write

$$p_{111} + p_{112} = p_{11},$$  (42)

$$p_{121} + p_{122} = p_{12},$$  (43)

$$p_{221} + p_{222} = p_{22},$$  (44)

and with the help of Eqs. (41) to (44), we obtain

$$p_{111} + 3p_{112} + 3p_{122} + p_{222} = 1,$$  (45)

where $p_{112} = p_{121} = p_{211}$ and $p_{122} = p_{212} = p_{221}$. Eqs. (39), (40) and (42) to (44) are referred to as “scale-invariance” (or scale-freedom) relations which can be generalized to $n$ subsystems. They are called ‘Leibnitz rule’ in another notation [14] which relates the probabilities of $n$ systems to the probabilities of $n - 1$ systems. Eq. (45) can be generalized to $n$ subsystems using ‘Pascal triangle’. However, in every stage, the number of equations are 1 less than the number of unknown probabilities. So it is not possible to exactly find the probabilities. For example, for $n = 2$, we have two Eqs. (39) and (40) and three unknown probabilities $p_{11}, p_{12}$ and $p_{22}$. Another equation that can be added to find our probabilities is additivity condition for entropy. In this way, for two equal binary subsystems, it can be written as

$$S_{q^*}^{A+B} = 2S_{q^*}^A \Rightarrow \frac{1 - p_{11}^q - 2p_{12}^q - p_{22}^q}{q^* - 1} = 2 \frac{1 - p^q - (1 - p)^q}{q^* - 1},$$  (46)

and with the help of scale-invariance conditions, we have

$$\frac{1 - p_{11}^q - 2(p - p_{11})^q - (1 - 2p + p_{11})^q}{q^* - 1} = 2 \frac{1 - p^q - (1 - p)^q}{q^* - 1}. \quad (47)$$

So $p_{11}, p_{12}$ and $p_{22}$ are obtained to show that additivity can exist for correlated subsystems with $q^* \neq 1$. After finding the probabilities of two composed systems, it is possible to obtain the probabilities of three composed systems using scale-invariance and additivity conditions. These processes may be continued until all the probabilities which make the entropy additive for each $n$ are found.
5 Numerical solution of quasi-additivity relation for two correlated equal binary subsystems

5.1 Exact solution:

Similar to the additive case, quasi-additivity relation for two correlated identical binary subsystems can be written as

\[ S_{q^+} = 2S_q \Rightarrow 1 - p_{11}^q - 2p_{12}^q - p_{22}^q = 2 \frac{1 - p^q - (1 - p)^q}{q - 1}, \]  

(48)

where \( p_{11}, p_{12} = p - p_{11} \) and \( p_{22} = 1 - 2p + p_{11} \) are the probabilities related to the additive case and can be obtained from Eq. (47). For a specified \( p \) and \( q \) (\( q \) is assumed to be close to \( q^* \) as in the case of independent subsystems where \( q \) is close to 1), it is possible to find \( q' \) from Eq. (48) and compare it with \( q \). Also \( \gamma \) is obtained from Eq. (35), which we designate as \( \gamma_e \) to represent the word ‘exact’.

5.2 Approximate solution:

It is not always possible to find the solution of quasi-additivity relation for \( n \) correlated subsystems exactly. One may, therefore, use the approximate solution given in (37). If we are now interested in finding \( q' \) or \( \gamma \) for two equal binary correlated subsystems with specified \( p, q \) and \( q^* \) using our approximate relation, then for \( A, B, C \) and \( D \) from Eqs. (31) to (34) we obtain

\[
A = \frac{1 - p_{11}^{q^*} - (1 - p)^q - p_{11}^{q^*} \ln p + (1 - p)^q \ln (1 - p)}{(q^* - 1)^2} + \frac{p_{11}^{q^*} \ln p + (1 - p)^q \ln (1 - p)}{q^* - 1},
\]  

(49)

\[
B = \frac{A - 2p_{11}^{q^*} - p_{22}^{q^*} - p_{12}^{q^*} \ln p_{11} + 2p_{12}^{q^*} \ln p_{12} + p_{22}^{q^*} \ln p_{22}}{(q^* - 1)^2} + \frac{p_{12}^{q^*} \ln p_{12} + p_{22}^{q^*} \ln p_{22}}{q^* - 1},
\]  

(50)

\[
C = \frac{1 - p_{11}^{q^*} - 2p_{12}^{q^*} - p_{22}^{q^*} - p_{11}^{q^*}}{(q^* - 1)^2} + \frac{p_{11}^{q^*} \ln p_{11} + 2p_{12}^{q^*} \ln p_{12} + p_{22}^{q^*} \ln p_{22}}{q^* - 1},
\]  

(51)

\[
D = \frac{C - 2p_{11}^{q^*} \ln p_{11} + 2p_{12}^{q^*} \ln p_{12} + p_{22}^{q^*} \ln p_{22}}{(q^* - 1)^2} + \frac{p_{12}^{q^*} \ln p_{12} + p_{22}^{q^*} \ln p_{22}}{q^* - 1}.
\]  

(52)

So from Eq. (37), \( \gamma \) is obtained and designated as \( \gamma_a \), where \( a \) represents ‘approximate’. We can, then, obtain \( q' \) from Eq. (35).

5.3 Results

The results are given in Tables 1 and 2 for \( q^* = 0.3 \) and \( q^* = 0.5 \) and for a given \( q \) close to \( q^* \) as \( p \) changes from 0.1 to 0.9 in columns. \( p_{11}, p_{12} \) and \( p_{22} \) (the probabilities related to the additive case) are given in the three first columns. The results of the exact solution of quasi-additivity equation (section 5.1) and also approximate solution (section 5.2) are shown in the following columns. It is seen that in both approximate and exact solutions, \( \gamma \) is positive and less than 1, namely, \( |q' - q^*| < |q - q^*| \) which means \( q \) approaches \( q^* \) by increasing system size, or alternatively, spatial scales which are similar to independent subsystems where \( q' \) approaches 1 by increasing system size. It is an interesting result which means that by increasing spatial scales, fluctuations become negligible.


6 Conclusion

In this paper, Beck’s concept of quasi-additivity was used for \( n \) independent subsystems and the relation between \( q \) and \( q' \) was found for \( q \) and \( q' \) to be close to 1 (Eq. (16)). It strictly shows that by increasing \( n \), or system size, \( q' \) approaches 1 so that \( q' \to 1 \) when \( n \to \infty \). The deviation of the parameter \( q \) from 1 in non-equilibrium systems describes the fluctuations in temperature (or energy dissipation)(Eq. (2)). Approaching of \( q' \) to 1 by increasing \( n \) shows how fluctuations become negligible when system size increases. We have also used the quasi-additivity relation for correlated subsystems where it is assumed that \( q \) and \( q' \) are close to \( q^* \) (specific value of \( q \), which makes Tsallis entropy additive). It has been found that for the case of correlated subsystems, \( q \) also approaches \( q^* \) which depicts a behaviour similar to the case of independent subsystems. The results for the two correlated equal binary subsystems have been given in the Tables.

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Table 1: $\gamma_e$ and $\gamma_a$ are given for $q^* = 0.3$. The upper lines in the Tables correspond to $q = 0.35$ and the lower ones to $q = 0.25$. Results are rounded to two digits after decimal point. As shown, $\gamma_e$ and $\gamma_a$ are less than 1.

| $q = 0.35$ | $q^* = 0.3$ |
|------------|-------------|
| $q = 0.25$ | $p_{11}$ $p_{12}$ $p_{22}$ $\gamma_e$ $\gamma_a$ |
| $p = 0.1$ | 0.09 0.01 0.89 | 0.52 |
|           | 0.09 0.009 0.89 | 0.52 |
| $p = 0.3$ | 0.28 0.02 0.68 | 0.32 |
|           | 0.28 0.02 0.68 | 0.32 |
| $p = 0.5$ | 0.02 0.48 0.02 | 0.26 |
|           | 0.02 0.48 0.02 | 0.26 |
| $p = 0.7$ | 0.68 0.02 0.28 | 0.32 |
|           | 0.68 0.02 0.28 | 0.32 |
| $p = 0.9$ | 0.89 0.01 0.09 | 0.52 |
|           | 0.89 0.01 0.09 | 0.52 |

Table 2: $\gamma_e$ and $\gamma_a$ are given for $q^* = 0.5$. The upper lines in the Tables correspond to $q = 0.55$ and the lower ones to $q = 0.45$. Results are rounded to two digits after decimal point. As shown, $\gamma_e$ and $\gamma_a$ are less than 1.

| $q = 0.55$ | $q^* = 0.5$ |
|------------|-------------|
| $q = 0.45$ | $p_{11}$ $p_{12}$ $p_{22}$ $\gamma_e$ $\gamma_a$ |
| $p = 0.1$ | 0.07 0.03 0.87 | 0.62 |
|           | 0.07 0.03 0.87 | 0.63 |
| $p = 0.3$ | 0.24 0.06 0.64 | 0.43 |
|           | 0.24 0.06 0.64 | 0.43 |
| $p = 0.5$ | 0.44 0.06 0.44 | 0.37 |
|           | 0.44 0.06 0.44 | 0.36 |
| $p = 0.7$ | 0.64 0.06 0.24 | 0.43 |
|           | 0.64 0.06 0.24 | 0.43 |
| $p = 0.9$ | 0.87 0.03 0.07 | 0.62 |
|           | 0.87 0.03 0.07 | 0.63 |