DUALITIES IN THE CLASSICAL SUPERGRAVITY LIMITS

Dualisations, dualities and a détour via 4k+2 dimensions

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Abstract. Duality symmetries of supergravity theories are powerful tools to restrict the number of possible actions, to link different dimensions and number of supersymmetries and might help to control quantisation. (Hodge-Dirac-)Dualisation of gauge potentials exchanges Noether and topological charges, equations of motion and Bianchi identities, internal rigid symmetries and gauge symmetries, local transformations with nonlocal ones and most exciting particles and waves. We compare the actions of maximally dualised supergravities (ie with gauge potential forms of lowest possible degree) to the non-dualised actions coming from 11 (or 10) dimensions by plain dimensional reduction as well as to other theories with partial dualisations. The effect on the rigid duality group is a kind of contraction resulting from the elimination of the unfaithful generators associated to the (inversely) dualised scalar fields. New gauge symmetries are introduced by these (un)dualisations and it is clear that a complete picture of duality (F(ull)-duality) should include all gauge symmetries at the same time as the rigid symmetries and the spacetime symmetries. We may read off some properties of F-duality on the internal rigid Dynkin diagram: field content, possible dualisations, increase of the rank according to the decrease of space dimension... Some recent results are included to suggest the way towards unification via a universal twisted self-duality (TS) structure. The analysis of this structure had revealed several profound differences according to the parity mod 4 of the dimension of spacetime (to be contrasted with the (Bott) period 8 of spinor properties). 1

1After the original lectures were delivered at this Cargèse school in May 1997, various developments have been presented at the Neuchâtel Workshop “Quantum aspects of Gauge theories, Supersymmetry and Unification” in September 1997 and at the Trieste Conference on “Superfivebranes and Physics in 5+1 dimensions” in April 1998. Work supported in part by EEC under TMR contracts ERBFMRX-CT96-0012 and -45
1. Introduction

The duality symmetries are invariances of equations of motion and even sometimes, for instance in odd dimensions, symmetries of a suitable action. The case of $4k + 2$ dimensions seems to be related to classical Lie-Poisson actions leading to Quantum groups, indeed the latter were discovered in 2 dimensional theories. The duality group gets its name from some of its elements that are actually Hodge dualities like in Dirac’s famous analysis of the exchange of electricity and magnetism in four dimensions which permutes strong and weak coupling expansions. This is also the nature of so-called discrete S-dualities in quantum four dimensional heterotic string theories as discussed by A. Sen. The $SL(2, \mathbb{R})$ symmetry in type IIB theory in 10 dimensions is also called S duality as it exchanges weak and strong string couplings. The string coupling is affected via the dilaton field. In four dimensions these symmetries involve Hodge duality in the target spacetime.

In string theory the so-called T-dualities do exchange strong and weak couplings of the sigma model on the worldsheet. They can be seen as generalisations of the Kramers-Wannier symmetry of the Ising model that permutes inverse temperature (or equivalently the euclidean period of time) and its inverse. T-dualities may also be realised as Hodge dualities but on the 2d worldsheet, as was shown by Buscher when the target space admits isometries. A generalisation called Lie-Poisson T-duality relaxes slightly the latter condition.

Three years ago Hull and Townsend unified the two main kinds of dualities (S and T) inside a much larger if conjectural discrete U-duality group of the quantum supergravity and string theories. These classical theories are now believed to be inequivalent limits of a quantum model called M-theory after Witten’s discovery of the correspondence between both formulations. The string coupling constant becomes geometrical, essentially equal to the length of the compactification circle along the eleventh dimension. An important point is that all solitonic excitations should be included in the quantum theory, as well as their duals which include fundamental strings in ten dimensions or fundamental membranes in 11.

Now the proposed U-duality groups are discrete subgroups of the duality symmetries of the equations of motion of maximally dualised supergravities which have been known since about 1980. We shall use the same letter for the discrete subgroup and the Lie group when it is clear from the context which one it is. The first classical supergravities under consideration were the fully dualised toroidal dimensional reductions of 11 dimensional (i.e. type II or maximal or $N_4 = 8$) supergravity. We shall only briefly mention their truncations to pure ($N_4 \leq 6$) supergravities. Type I supergravities possess also interesting duality symmetries of the classical equations of motion first
studied by Chamseddine [1]; considered together with those of type II [2] they strongly suggest that the two simply laced hyperbolic Kac-Moody algebras of maximal rank (equal to 10) should appear as symmetries of some huge space covering the set of unidimensional classical solutions of supergravity theories. Although there are only indications for that yet, let us give a name to these hyperbolic Kac-Moody groups: $E_{10}$ and $HD_{10}$ (called overextended $D_8$ in [2]), they correspond to type II resp. type I. It would be interesting to accommodate the heterotic theories in the hyperbolic game [3]. The situation there is still moving with hyperbolic Kac-Moody algebras and generalised Kac-Moody algebras (including Borcherds algebras) appearing in toroidal compactifications.

What has been established and extensively studied is the occurrence of infinite dimensional symmetry groups in the reduction to two dimensions, there the affine Kac-Moody extension $G^{(1)}$ enlarges $G$, the corresponding maximal U-duality group in three dimensions. In two dimensions scalar potentials (fields) are dual to scalar fields, as a result the exchange between gauge and internal (rigid) symmetries does not take place; the duality group acting on the appropriate set of fields covering the set of solutions becomes infinite dimensional. It is also an important problem to describe precisely and uniformly, namely for all dimensions, what the discrete (infinite) groups of U duality are. They are most probably groups over $\mathbb{Z}$, the rational integers, as defined by Chevalley. This conjecture is nicely compatible with the observation that any $\mathbb{R}$ factor group of the classical duality group has no infinite discrete duality analogue and disappears from the U-duality group at the quantum level.

In section 2 we shall recall the “silver rules” of supergravities and the building blocks of the U-groups. The latter use three ingredients: the dilatonic rescaling symmetry one obtains in 10 dimensions, the “Ehlers phenomenon” where a scaling symmetry becomes a whole $SL(2)$ and finally the fusing together of such $SL(2)$’s with the expected symmetry $GL(D-d)$ that comes from the dimensional reduction from $D$ to $d$ dimensions into a simple rigid internal U-duality group. Given $G$ the U-duality group, the three silver rules are that the scalar fields parameterise a symmetric space of the noncompact type $K\backslash G$ where $K$ is the maximal compact subgroup of $G$, that equations simplify dramatically upon restoring a local gauge invariance under $K$ and finally that in even dimensions $d = 2f$ the equations of motion of the field strengths of order $f$ and their Bianchi identities are unified in a twisted self-duality equation.

$$*S.V.F = V.F$$

where $S$ is an invariant operator acting on the appropriate representation of $K$ and where $V$ is the coset representative of the scalar field transforming
under $G$ on the right contragrediently to $\mathcal{F}$ on the left, and transforming also under the local gauge group $K$ on the left. The case of timelike compactification is somewhat different [4] but it is important for Euclidean signature, see [5]. The maximal compact subgroup is replaced in general by a noncompact subgroup and the quotient has a rather problematic topology.

In section 3 we proceed to study the effects of dualisations beginning with a comparison between pure gravity reduced from 4 to 3 dimensions and its dualised theory with Ehlers symmetry acting in a local way as a rigid $SL(2, \mathbb{R})$ invariance of the action. In the theory of integrable Hamiltonian systems, Bäcklund transformations (for instance Miura transformations) may exchange solutions of one system with those of another. Here dualisations are discrete duality transformations, essentially Legendre transformations, that modify the (perturbative) field contents of the action and the analogy is useful. Then we analyse in detail the case of maximal supergravity, also in three dimensions. It may seem at first that as soon as one leaves the simple case of maximally dualised theories one is in danger of losing oneself among all the possibilities. It is not so, the key choice is to pick a grading of the root space, for instance along a particular root, one can then show how undualisation of some scalar fields belonging to the corresponding highest level (or levels) do actually reduce the $U$-symmetry by changing the dimension of (and simultaneously contracting i.e. partially abelianising) the remaining subalgebra. In passing we note that large abelian subalgebras of dualities do occur and in fact those of maximal dimension as classified by Malcev [6] for the case of complex Lie algebras (or their normal real forms) do appear. In the case at hand there can be 36 commuting generators in the maximally noncompact (so-called split or normal) real form of $E_8$: $E_8(+8)$. This is to be contrasted with the compact form situation where the maximal abelian dimension is the rank.

In the fourth section we shall try to learn about the higher form fields from the Dynkin diagram of the $U$-group. It turns out that one can read off from the diagram the number of forms of various degrees, because they belong to very specific fundamental representations of $G$. In the case of maximal supergravities the Dynkin diagram reflects the possibility of dualisation between higher form fields by the existence of some outer automorphism. In general and most importantly it suggests that one should unify internal rigid symmetries with diffeomorphisms (or at least the $GL(D-d)$ subgroup) in a larger group of rank at least eleven (or twelve?). This can be applied to type IIA, IIB or I. In fact we find a purely group theoretical version of the Horava Witten orbifolding relating type II and type I string theories, it corresponds to using a Cartan involution of the $U$-duality in any dimension. The heterotic duality groups are interesting too, they are non-split real forms in which one still recognizes the expected linear diffeo-
morphism symmetries, but they will be treated separately. Note that the non-maximal supergravities also lead to non-split real forms of the duality groups [7].

Then we shall present the general analysis of duality symmetries in curved space with either Lorentzian or Euclidean signature. The matter will be taken in an N-plet of middle rank \((f = d/2)\)-forms plus sufficiently many scalar fields. Since our paper [8] appeared on the hep-th archive we realised that the general Lorentzian case seems to have been investigated by Tanii [9], some partial results on the 2 dimensional case were also obtained in [11]. The 4 dimensional case was systematically analysed in [10] but as one might expect the \(4k + 2\) dimensional case is quite different. The fact that self-duality becomes possible is not directly relevant here. We have also constructed constrained actions that preserve the U-symmetry and allow to simplify the computations by doubling the set of fields (third silver rule).

In the last section I shall summarize two research projects I have been working on for the past few years. On the one hand I want to stress the importance of the existence of “complementary” classical limits appropriate to different experimental situations. This may lead to some clarification of the formulation of Quantum Mechanics. There is no classical world only classical approximations: the limit of the Planck constant tending to zero is to be defined by a dimensionless criterion but more importantly by specifying what is being kept fixed. Particle limit and wave or classical field limit are in duality. On the other hand I would like to point out more publicly than before the intimate relation between intersection theory, fermionisation and charge quantization of dyons. The fundamental differences among even dimensions between \(d = 4k\) and \(d = 4k + 2\) will be discussed. The idea is that well defined statistics is what saves locality in theories of extended objects, it is associated to a charge quantization condition in two dimensions. But the latter has to be symmetric to allow for chiral (self-dual) particles [12] [13]. We show that higher dimensional fermionisation requires a détour through 6 (or 10) dimensions or twisting in 4d by some internal symmetry in the sense of section 2. In guise of conclusion I give a preview of a forthcoming paper where it is shown that indeed the twisted self-duality equation is universal excluding for the time being the graviton. There is a candidate for F-duality that brings us into the realm of graded superalgebras.

2. Silver rules

In the quest for supergravity actions the use of extra-dimensions of space made the deformation or so-called Noether method significantly simpler by reducing the number of scalar fields to zero in the best cases and thus
avoiding nonpolynomial expressions. The idea of higher dimensions is in fact quite natural in the context of extended Poincaré superalgebras, in particular the doubling of the number of fermionic charges can result from the addition of 2 dimensions. Interest in four-dimensional supergravity with the maximal possible number of supercharges (32) led us to consider ten dimensions. In fact we went directly to eleven dimensions with Majorana spinors as suggested by the spectrum analysis by Nahm at the linear level but also by practical considerations. No scalar fields are left in eleven dimensions and the spectrum is amazingly simple: a graviton, a gauge three-form and the Rarita-Schwinger field. Furthermore dimensional reduction is easier than its converse group disintegration [7]. For instance the theory with 24 supercharges in 6 dimensions predicted in [7] was only constructed 16 years later [14]. Returning to eleven dimensions, the requirement of local supersymmetry leads to a unique action for this set of classical fields. After toroidal compactification one obtains the expected $N = 8$ supergravity in 4 dimensions but also peculiar duality symmetries. Indeed in dimension $d$ one discovered (maximally) noncompact symmetry groups $E_{11-d}(11 - d)$ of the toroidally reduced theories; the number between parentheses is the real rank to be defined in the next paragraph.

We found ourselves in a situation similar to that of general relativists in front of the Ehlers $SL(2, \mathbb{R})$ symmetry or its extension for stationary electrovac $SU(2, 1)$. In the absence of conceptual understanding we began a systematic analysis of the symmetry Lie algebras for all dimensions and all number of supersymmetries. This led to a list of noncompact symmetric spaces, ie cosets of the form $K\backslash G$ of the U-symmetry groups $G$ by their maximal compact subgroups $K$. The real rank of the coset is the maximal dimension of the subspace of a Cartan subalgebra orthogonal to the compact directions (for short noncompact Cartan generators). The real rank $r$ is equal to the rank $l$ for a normal (also called split) real form, this is the case for the descendants of type I or II supergravities in 10 dimensions. It turns out that in all cases the real rank increases by one upon each step of dimensional reduction. One can also check that $d + l - N'_4 = 4$ is constant in the disintegration triangle [7] for type II(A) and other pure (in four dimensions) supergravities in various dimensions; $N'_4$ is the number of supersymmetries in four dimensions except for the maximal case where it is equal to seven. For Chamseddine's type I disintegration column, the constant is equal to 7 instead of 4. Now the scalar fields parameterize this coset and other fields transform as representations of $K$ or if one prefers of $G$ through the scalar group element which intertwines between $K$ and $G$ (we do not consider spinor fields here). It is quite suggestive to think of (the scalars as) a moving frame exchanging the ($K$) tangent space-Lorentz indices with ($G$) world indices.
The split or maximally noncompact real form of a complex simple Lie group is the generalisation to a general Lie group of $SL(2, \mathbb{R})$ inside $SL(2, \mathbb{C})$. One simply defines the split form as generated by a number of copies of $SL(2, \mathbb{R})$ equal to the rank $l$ and a finite set of relations, the Serre relations, which restrict their simple and multiple commutators. The same relations hold over the real numbers for the split form as over the complex for the complex form. Another characteristic of the split form is that it admits a Cartan subalgebra of $l = r$ commuting, noncompact but ad-diagonalisable generators. The rest of the generators come in real pairs associated to opposite roots and closing on Cartan generators to form again copies of $SL(2, \mathbb{R})$. In the non-split case, the rank $l$ is strictly larger than the number $r$ of linearly independent noncompact Cartan generators. However if we relax the diagonalisability assumption the number $a$ of linearly independent commuting generators might be strictly larger than $r$ and even than $l$. We shall see examples of this in the next section for real split forms. The split real form is also characterized by having the smallest compact subalgebra and consequently the largest dimension for the coset space $K\backslash G$.

This symmetric space structure $K\backslash G$ for the moduli space of scalar fields in the maximally dualised form of pure supergravity theories constitutes the first silver rule. Some of the gauge forms descending either from the gauge three form or from the metric, namely the Kaluza-Klein one forms, can be simultaneously dualised when their potentials can be covered with derivatives despite the presence of the Chern-Simons-like term in 11 dimensions. In fact all those potentials that are of degree $d-2$ can be dualised into scalars in the case of pure supergravities, this we shall call maximal dualisation in the scalar sector. That particular form of the action has the maximal rigid internal symmetry.

The second silver rule is the generalisation of the orthonormal moving frame technique to internal symmetries. In General Relativity the metric is given on a manifold, it does not depend on any frame choice but its components depend on coordinate choices in a tensorial way. Alternatively the same local information can be obtained from a frame of orthonormal 1-forms, but they are only defined up to a gauge Lorentz transformation. In our scalar manifold case, $K\backslash G$ can be parameterised either as a manifold by ignoring the coset structure or alternatively as the base of a principal bundle the group $G$ itself. Then the subgroup action (on the left here) is a gauge symmetry that compensates for the extra freedom that was added to make the $G$ symmetry manifest. Typically, our algebraic power being limited, the coset representatives we encounter first are very often in the triangular or Borel and more generally in the solvable $K$-gauge [15]. One
reason is that in that gauge the exponential parameterisation is bijective
but more important probably is the fact that unipotent elements and their
inverse are parameterised polynomially. The search for hidden symmetries
is consequently the restoration of the gauge freedom or at least, and this is
easier, the enlargement of the Borel symmetry to the full $G$ invariance. For
instance in two dimensions only the second process has been realised yet.

The third silver rule of supergravity is that the middle degree field
strengths are self-dual; more precisely their equations of motion and their
Bianchi identities can be unified in a covariant set of Bianchi identities for
a doubled set of fields restricted by a twisted self-duality condition so that
the original $n$ second order equations have been encoded into a set of $2n$
first order ones. This was first discovered in 4 dimensions with Minkowskian
signature such that the square of the Hodge dual is equal to minus one:
clearly the self-duality needs a twist in the form of a $G$ invariant operator
$\mathcal{S}$ acting on the representation $2n$ and of square equal to minus one in
four dimensions in order to compensate for the previous minus sign. As an
illustration let us recall that maximal supergravity in 4 dimensions has 28
vectors, with their dual potentials this makes 56. But this is precisely the
dimension of the fundamental representation of $E_7$, it is symplectic and can
be extended to a representation of $SP(56, \mathbb{R})$.

It is an easy exercise to check that the square of the Hodge operator is
equal to $(-1)^{(s-t)/2}$. Note that the difference $s - t$ also appears in the classifi-
cation of Majorana spinors. In fact instantons require Euclidean (or (2,2))
signature in four dimensions. But supersymmetry can provide some internal
degrees of freedom to compensate for the otherwise devastating minus sign,
in other words it recalls its higher (10 or 6) dimensional origin to allow for
generalised or we shall say twisted self-duality (TS). Twist can also be
invoked to allow for generalised spin structure or to define a generalised Ma-
jorana condition. This self-duality is a feature of toroidal compactifications
as the appearance of bare potentials rather than field strengths associated
to nonabelian gauge theories or simply (charged) matter couplings seems
to ruin the possibility of dualisation.

We should be slightly more careful though as we know examples of
dual pairs of theories one of which has bare potentials. For instance the
Freedman-Townsend-Thierry-Mieg dual sigma models are theories of two
forms in four dimensions with bare potentials. This is compatible with dual-
isation and even twisted self-duality [8], [16]. Indeed, in the most favorable
case, dualisation can use either one of two routes: one either introduces a
Lagrange multiplier for the field strength Bianchi identity which becomes
the dual potential, this is possible provided no potential appears in the
Lagrangian or in the Bianchi identity, or one may use another first order
formalism where now both the original potentials and field strengths are
considered as independent variables. In the sigma model case the first route is taken when going from the scalar field description towards the two-form version but the reverse route is of the second type. This reciprocity of the two routes occurs in general. In the dual sigma model case the alternative paths are not allowed. Furthermore doubling does not mean one can actually find a dual Lagrangian as the example of 11 dimensional supergravity shows [16] and [17]. There exists a doubled TS (twisted self-dual) formalism but the 3-form cannot be integrated out to give the dual 6-form theory.

Let us now return to the general discussion of the third silver rule of supergravity. The structure is a mixture of $K$ and $G$ group theory. The potentials form a $G$-multiplet but the field strength combinations that are self-dual are the $K$-multiplet. The key equation was already mentioned, it is $\ast S \nabla \mathcal{F} = \nabla \mathcal{F}$. This is to be supplemented by the $G$ tensor equation $\mathcal{F} = dA$. This coexistence of differentiation in curved space ($G$ representation) and self-duality in flat or tangent space ($K$ representation) seems to be quite general, it holds for instance also for various (super)brane actions and for Born-Infeld theories.

There are important remarks to make at this point. Firstly $S$ is an operator from the $K$ representation to itself. Secondly one should point out that there have been two ways to write this self duality equation, we just gave the second one, the first one was given in [18] and references therein: one constructs the analog of the metric (the monodromy in integrable systems). In the simple case of $SO/SL$ we may define using a $SO$-invariant positive definite metric

$$\mathcal{M}_{MN} := \nabla^t_A \eta_{AB} \nabla^B_N.$$  

Then the TS equation reads

$$\ast \Omega^P M \mathcal{F}_N = \mathcal{F}^P.$$  

The invariant tensor $\Omega$ is an invariant of the noncompact group $G$, for instance the symplectic form of $E_7$ in its 56 dimensional representation in the case of four spacetime dimensions.

How can we relate this formula to the previous one using $K$-tensors? It does not seem to have been written up in general in the litterature but we need to establish

$$\nabla^A \Omega^M N \nabla^t B = S^A C \eta^{-1} C B := \omega^{AB}$$

or alternatively

$$\nabla^t \omega^{-1} = \omega^{-1} \nabla \Omega$$

choosing the identity matrix for $\nabla$ we must identify $\omega$ and $\Omega$. Things becomes even simpler if we use the operator $S$ and rewrite the above formula
as

$\sigma(\mathcal{V}) = S^{-1} \mathcal{V} S$

where $\sigma$ is the Cartan involution\(^2\) The Cartan involution is “inner” in this representation, the condition for involutivity (that $S^2$ be central) is automatic here. This property of $\sigma$ has interesting consequences to which we hope to return in the future. For a general group, following [19], we must introduce $\sigma(\mathcal{V}^{-1}) \mathcal{V}$ to replace $\eta^{-1} \mathcal{M}$.

There is no one-to-one relation between the symmetry property of the twist-form $\omega$ and the parity of $(s-t)/2$. The latter is related to reality properties whereas the former is a linear question. The symmetry of the twist can be found by inspection, surprisingly it does not depend on the signature of spacetime. One instance of this “signature blindness”, namely the fact that duality transformations are canonical in $4k$ dimensions and not in $4k + 2$ dimensions in Euclidean signature as well as in the Minkowskian one, was actually discovered some time ago\(^3\). Let us recall also that the fact that dualities are not canonical in 2 dimensions was realised before Quantum groups were invented see for instance [20]. In $4k$ Lorentzian dimensions one has no real “self-dual” tensor but there is a Euclidean one and twist is antisymmetric, in $4k + 2$ dimensions on the other hand twist is symmetric. The existence of a symmetric resp. antisymmetric bilinear, invariant form for an irreducible complex representation of a group that is equivalent to its contragredient (here by conjugation by $\omega$) makes it a “real” resp. “quaternionic” representation [21].

The analysis is complicated here by the fact that as we shall see the subgroup $H$ that replaces $K$ in the Euclidean case is non-compact. The properties of the twist $\omega$, the invariant metric $\eta$ and the operator $S$ are intertwined in the relation $S := \omega \cdot \eta$. The origin of the two different symmetries of $\omega$ has not been completely clarified yet, but the detailed study of supergravities is quite suggestive. We shall review the systematic analysis of [8] in section 4. For comparison, let us recall that half-spinor representations in even Lorentzian dimensions are complex if $d = 4k$, real if $d = 8k + 2$ and quaternionic if $d = 8k + 6$. The distinction between quaternionic and real spinors is irrelevant for quadratic expressions like the Ramond-Ramond bosons under consideration here. More relevant is the fact that the only symplectic fundamental representation of $SL(D, \mathbb{C})$ is the self-dual tensor for $D = 4k + 2$ [21]. Let us also note that the fundamental representations of $SP(2N, \mathbb{C})$ and of $SO(2N, \mathbb{C})$ are obviously respectively symplectic and real.

\(^2\)Let us recall that the maximal compact subgroup of a real simple Lie group can be defined as the fixed point set of an involution $\sigma$, the Cartan involution of $G$. In the simple case under consideration $\sigma(\mathcal{V}) := \eta^{-1} \mathcal{V}^{-1} \eta$.

\(^3\)in a discussion with S. Deser in Sept. 1996.
Let us now review how U-dualities grow upon toroidal dimensional reduction. In eleven dimensions the classical equations of motion admit an engineering scaling symmetry because the dimensionful coupling constant appears as an overall factor in the action. In 10 dimensions this leads to an $\mathbb{R}$ internal and rigid symmetry. Technically one must specify an (Einstein) frame. This noncompact symmetry rescales the dilaton or the length of the compactification circle. Beyond this one generator there is an automatic $SL(11 - d)$ internal symmetry directly originating from the diffeomorphism symmetries of our starting point. The second fact to notice is that in 8 dimensions the scaling group becomes an $SL(2, \mathbb{R})$ factor. This is also the dimension in which the 4-form field strength can be self-dual. More surprising still is the fusion into a simple Lie group of both ingredients, $SL(2, \mathbb{R})$ and $SL(11 - d)$ below 8 dimensions.

This discussion can be adapted to the type I family. The starting point is in 10 dimensions [1] with one dilatonic symmetry. Now in 8 dimensions a second $SL(2, \mathbb{R})$ factor appears beyond the expected $SL(10 - 8, \mathbb{R})$, it incorporates one of the dilatonic symmetries of the 9 dimensional model. In lower dimensions the duality symmetries are the now familiar $SO(10 - d, 10 - d) \times \mathbb{R}$ enlarging $SL(10 - d, \mathbb{R})$. In 4 dimensions the last dilatonic subgroup becomes a full $SL(2, \mathbb{R})$ factor by dualisation of the two-form potential. Finally in three dimensions the group becomes simple namely $SO(8, 8)$.

Until now we focussed our attention on the maximally dualised theories. But if instead of deciding to lower as much as possible the degrees of the forms in the action by dualising, when possible, any $k$-form field strength to the lower degree $(k' = d - k)$-form we decide not to do it or to undualise some of the latter, the internal symmetry group action on the scalar fields that are being undualised is transmuted into a gauge group for the dual forms, at least above 2 dimensions. For the two-dimensional case see for instance [22] and references therein. The purpose of the next section is to discuss more systematically these choices (to dualise or not to dualise).

3. Dualisations

It may be advisable to begin with the simplest typical example rather than a list of definitions and formal properties. So let us follow J. Ehlers who recognized a $U(1)$ duality symmetry of Einstein’s vacuum solutions in 4 dimensions with one non-null Killing vector. One can view the situation as a fibration of spacetime over a three dimensional set of orbits. One question is whether the base inherits a geometrical structure namely a metric from the original Minkowskian manifold. Locally the answer is yes, it is the beautiful work of Kaluza (who actually developed the idea in 5 dimensions), it was
completed by O. Klein (after Quantum Mechanics), Jordan, Thiry ... In fact one may see that the Killing orbits being non-null the distribution of normal hyperplanes is transverse to the fibration, equivariant under the isometries by construction and thus defines a principal abelian connection (locally in all directions). Furthermore the length of the Killing vector defines a scalar field $C$ on the base. The construction of the geometry on the base is given in [23]. The case of null Killing vectors is much more subtle see [24] and references therein.

Now as we started from 4 dimensional spacetime the two degrees of freedom of the graviton have become a scalar field plus one polarisation of a vector gauge field (the connection we just presented). What are the symmetries? From 4d diffeomorphism invariance we expect a remaining scaling invariance of the cyclic (ie now internal) direction and of course the Maxwell-Weyl gauge invariance of the connection. However it turns out that the connection 1-form can be dualised, in other words it can be exchanged for a dual scalar field $B$ by a Legendre transform. Then the miracle in today’s state of affairs is that the two scalar fields do form a couple of coordinates for the Poincaré upper-half-plane. In other words they parametrise the symmetric Riemannian non-compact space $SO(2)\backslash SL(2, \mathbb{R})$.

Ehlers considered the $SO(2)$ subgroup of $SL(2, \mathbb{R})$ but the full group is now called Ehlers’ group. The action is pointwise as in a sigma model and rigid ie. independent on the position. But that means that it is actually non-local in terms of the original scalar $C$ and the original connection. Furthermore the other two generators are the expected scaling and a new shift of the second scalar $B$ which is defined up to an arbitrary additive constant by the Baecklund type formula:

$$dA \propto *dB \, C^4.$$  

See for instance [7] for complete formulas. It is also established there that the Ehlers rotations act at the linearised level as helicity rotations.

To summarise, the maximal dualisation leads to an unexpected symmetry that is nonlocal in terms of the original (non-dualised) fields. The undualisation of the new scalar field to the connection one-form hides the large symmetry but restores the gauge invariance of the connection field.

Our next example will be three dimensional maximal supergravity (type II but for our purpose the fermion fields may be set to zero). The maximally dualised theory is in a topological background gravity and its dynamics is that of a Riemannian symmetric space sigma model of the non-compact type. The 128 bosonic degrees of freedom span $SO(16)\backslash E_8$ or equivalently, but after gauge fixing the $SO(16)$ gauge invariance, they span a Borel (or upper triangular) subgroup. In a maximally noncompact real form the Cartan generators may be chosen noncompact and together with the positive
roots they generate a solvable group that is univocally parameterised by the exponential map. The compact generators are the differences of positive root generators and their opposite and disappear in the gauge fixed description. Let us briefly recall the origin of the various scalar fields. The three form in eleven dimension reduces to $8 \times 7 \times 6 / 3! = 56$ scalars plus 28 vectors (to be dualised to scalars) and some non-propagating components. The metric leads to $11 - 3 = 8$ vectors (also dualisable) as well as 36 scalars. It turns out that these fields fit into the graded decomposition of the Borel subalgebra along the simple positive root labeled 10 of the Dynkin diagram of $E_8$ in Figure 1.

$$
\begin{array}{ccccccc}
9 & 8 & 7 & 6 & 5 & 4 & 3 \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
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\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
$$

Figure 1: type IIA symmetry, Dynkin diagram of $E_8$ with affine extension.

The label indicates the dimension at which each vertex appears.

Let us recall that in the decomposition of a positive root into a linear combination of simple positive roots with non-negative coefficients there is a $\mathbb{Z}^5$ grading at our disposal. We claim that the simple root corresponding to the vertex 10 must be selected to find the symmetry of the bare reductions from 11 dimensional supergravity doing no dualisation at all. Firstly one must select the grading along that root, and then erase its vertex. Namely there are 28 positive roots of level 0 along that simple root (plus 8 Cartan generators) all inside the Borel subalgebra of $gl(7, \mathbb{R})$, as well as 56 level 1 roots, 28 at level 2 and 8 at level 3. Clearly the level 2 and 3 root generators form an abelian ideal because the level runs only up to 3. This means in practical terms that the corresponding 36 scalars can be undualised to vectors. They actually originated as such from dimensional reduction, and $GL(7, \mathbb{R})$ is the almost obvious symmetry after strict toroidal dimensional reduction.

We just gave the simplest example of the phenomenon: the big Borel duality group of the maximally dualised theory is quotiented by the abelian ideal of the undualised scalar generators to become typically a semidirect product, here (after restoring the full linear group) $GL(7, \mathbb{R}) \ltimes \mathbb{R}^{56}$. What has happened is that the 36 arbitrary constants in the definition of the new scalar fields obtained by dualisations are not available before dualisations and their shifts by corresponding group elements drop out because the realisation of the Borel group becomes non faithful; furthermore non
commuting generators of $E_8$, like those at level 1, become commuting after undualisation because their commutators vanish by the previous mechanism. This phenomenon has been uncovered in [8] but abelian ideals have many other uses see [15]. If one wants to restore gauge invariance under the compact subgroup (second silver rule) one may alternatively describe the division by the ideal of the Borel group as a double coset construction for the full U-duality group [8].

Such undualisation along the root 10 of all the $E_{11-d}$ Borel subgroups leads to symmetries of a similar type: semidirect products of the corresponding linear subgroup by some abelian group which are the symmetries obtained from bare dimensional reduction from 11 dimensions. We did not give the details here but the $SL(11-d)$ is visible as a subdiagram of the $E_8$ Dynkin diagram that grows together with $E_{11-d}$ from the vertex labeled 9 towards the vertex labeled 3... For all dimensions one uses abelian ideals, those of the maximal possible dimension occur in various places as one can check following [6] who found them by inspection. A more conceptual discussion will be attempted in a further paper. But let us illustrate their usefulness on more examples.

Let us now turn briefly to three other natural choices of undualisations. We shall consider undualisation of all NS-NS fields (the even forms), it corresponds to exchanging the role of the simple roots 10 and 9 in Figure 1, namely to undualising generators whose root has the highest coefficient along the simple root 9. We could also consider type IIB supergravity in 10 dimensions, reduce to 9 dimensions and below without dualising the 5 form field-strength. This series is obtained by using root 8 (and simultaneously root 10) in a similar way as root 10 or 9 in the previous examples. The manifest and expected symmetry is now only $GL(9-d, \mathbb{R})$. Finally the truncation of $E_{11-d}$ to $E_{10-d}$ can also be obtained by deleting the vertex labelled $d$ at the end of the $SL(11-d)$ line inside ($E_{11-d}$) that shortens when $d$ increases. The origin of this property is that any vector in dimension $d+1$ gives rise to a new scalar in dimension $d$. Malcev has shown that the maximal dimension of a maximal abelian subalgebra of $E_7(7)$ is equal to 27, but $27=63-36$ which corresponds precisely to the contraction of $E_7(7)$ towards $E_6$. The rest of the $E$ family gives similar results.

The last series we would like to discuss is the type I family between 10 and 3 dimensions. The diagrams are subdiagrams of Figure 2:
Again the labels denote the dimension at which the Cartan generators appear, sometimes with a full $SL(2, \mathbb{R})$ but not immediately. We may conclude this section with the remark that there is also a couple of undualisations that are easy to describe in the type I case, namely those of the 2-form. They are done by the same type of contraction as for type II by using the simple root labelled 10 in Figure 2.

4. Higher order potentials

In the previous section we have seen how specific locations on the Dynkin diagram of internal dualities correspond to specific sets of dualisations and in the process of studying these we have seen that root labelled $d$ is associated to vectors. We shall now discover, in type II supergravities, that root 10 is associated to the 3-form, root 8 to the 4-form and root 9 to the 2-form potentials. Let us consider for instance dimension 7, there are 4 vectors from the metric plus 6 from the 3-form, together they build the representation 10 of $SL(5, \mathbb{R})$ which corresponds to the fundamental dominant weight on vertex $d = 7$. Similarly in dimension 6 the 3-form can (and should) be dualised to a 16th vector to implement the symmetry $SO(5, 5)$ and again this is the fundamental half spinorial representation associated to vertex $d = 6$. If the 3-form is associated to vertex 10 we might guess that the symmetry of the Dynkin diagram in 6 dimensions is responsible for the possibility of dualising it away.

In dimension 2 one knows that the U-symmetry of type I supergravity is the affine $D_8^{(1)}$. Note that there is a Weyl reflection that exchanges the highest root (it is the opposite of the root labelled (2) corresponding to the affine extension) with the simple positive root labelled 3. Roots 3 and (2) are connected by a simple line and so belong to a $SL(3, \mathbb{R})$ subgroup. This is related to the fact that 1-forms can be dualised to scalars in 3 dimensions. In fact for any dimension, and also in type I theories, 1-form potentials in the maximally dualised form belong to the representation of
highest weight equal to the fundamental dominant weight along the last root of the $SL(10 - d)$ line (root labelled $d$, $d \leq 9$ for type I) at least before they can be dualised away.

Scalars correspond to the adjoint representation of highest weight equal to the highest root. Their dualisability, which is a spacetime property, somehow can be seen in the internal symmetry Dynkin diagram as the contiguity of the highest weights. Any weight is Weyl conjugate to a dominant weight but not necessarily to a fundamental one. Conjugacy seems to be a necessary condition for the dualisability. For higher forms, the critical dimensions for which the degree of forms can be lowered (3 for vectors to scalars, 5 for 2-forms to 1-forms etc...) are most visible on the Dynkin diagram when dualisability corresponds to the existence of an outer automorphism.

Type II theories provide more examples of this. I have checked them one by one! The last vertex of the Dynkin diagram to appear (label $d$) is always associated to the highest weight of the vector representation (it is the fundamental weight for that vertex), the vertex labeled 9 in Figure 1 gives the highest weight of the 2-forms and similarly the vertex labeled 10 there corresponds to the 3-form. Note that $E_{11-d}$ has outer automorphisms visible as symmetries of its Dynkin diagram whenever $k$-forms and $(k' = d - 2 - k)$-forms are in duality, this symmetry exchanges the locations of the corresponding (fundamental) weights in all 9 cases of duality for degrees of potential-forms between 0 and 4!

Let us choose dimension 6 for instance, the group $G$ is $SO(5,5)$ and the 2-forms can be selfdual under the symmetry of the diagram that exchanges the two half-spinor ends, i.e., the vertices 10 and 6 and leaves invariant the vertex 9. These vertices correspond to the fundamental weights of the 3-form, dual 1-form and self-dual 2-forms respectively. The scalars in the adjoint correspond to the lowest root whose vertex is adjacent to the vertex 8 related to the potentially dual 4-form as we saw at the end of the previous section by a Weyl reflection. Let us note in passing the strange rule that when the affine diagram is cyclic ($SL(p, \mathbb{R})^{(1)} p \geq 3$) the sum of the degrees of the potential forms whose vertex is attached to the (affine) extension vertex is its dual degree; for example in dimension 8 scalars are dual to 6 forms and 6=4+2, and similarly in dimension 7.

Let us now remark that the outer automorphisms of the internal Lie algebra of dualities $g$ should be defined as simultaneously dualising the world indices. This is quite easy to implement by similar symmetries of the Dynkin diagram of $SL(d)$. It is well known that the fundamental representations of $SL(d)$ are from one end to the other the antisymmetric powers of the vector representation in increasing order: the vector, then the antisymmetric twice contravariant tensor etc... The symmetry under the exchange of the two ends of $SL(d)$ is nothing but duality provided we consider field
strengths. By this we mean that the tensor character of the field strength should define the fundamental weight and consequently the vertex to be dualised. Under dimensional reduction and its inverse group disintegration [7] the linear group is shared in a $d$-dependent way between internal and external spaces. Let us denote by a star the scaling factor $\mathbb{R}$ of symmetry, the general picture for type IIA is given in Figure 3.

![Figure 3: type II $F''_{11}$ subgroups of F-symmetry](image)

The vertex labeled $w$ represents scaling symmetry.

The diagram obtained by adding two bonds to Figure 3 to connect a regular $SL(2, \mathbb{R})$ replacing the $\mathbb{R}$ factor to the rest of the diagram contains all the above diagrams in any dimension. Its rank is eleven let us call it $F''_{11}$, it has an analogue $F'_{11}$ for type I. Note that $SL(12, \mathbb{R})$ cannot be found in them by naive inspection. However it could still be included in there, like $SL(9, \mathbb{R})$ inside $E_8$. But $GL(12, \mathbb{R})$ is unlikely to be included and this could be related to the same property of F-theory. Note that supersymmetry implies that the twelfth dimension would be timelike and hence the subgroup of $SL(12, \mathbb{R})$ would be $SO(10, 2)$ [25] so it may be not so surprising that the $SL(12)$ is hidden.

$F'_{11}$ contains of course $E_{10}$. $HD_{10}$ lies analogously in $F'_{11}$ and M-ology suggests that types I and II should be unified to make the full F-group. In fact it is encouraging to discover that orbifolding by an involution (a Cartan involution) leads in all dimensions from the type II to the type I U-duality. The invariant set of the involution is given by erasing the vertex 9 of Figure 1 and adding vertex 10 of Figure 2. This group theoretical construction should be compared to the geometrical orbifolding of [26]. It is tantalising now to propose that the one-loop diagram obtained either by attaching through a new single vertex (as the simplest choice) vertices 8 and 10 of Figure 2 or vertices 9 and 3 of Figure 1 or 3 should be studied more carefully. This diagram has rank 12 and a $\mathbb{Z}_2$ symmetry.

In summary dualisability seems to require the existence of (outer) involutions of the F-duality group of a very special type. The F-group should be universal but then split into a product of spacetime symmetry by an internal symmetry factor in a type- and $d$-dependent way.
Let us now review the possible duality symmetries of a Lagrangian field theory in curved space with scalar fields and a N-plet of middle rank \((f = d/2)\) forms. In [10] it was established that in Minkowskian signature the set of equations of motion for the N vector fields and sufficiently many scalar fields are invariant at most under the symplectic group \(SP(2N, \mathbb{R})\). Then the energy-momentum tensor is automatically invariant. In fact if there is no scalar compensator like in the free Maxwell theory the available linear invariance is under \(GL(N, \mathbb{C})\). So there remains some work to be done to clarify the situation with a few scalars. For the free Maxwell case we should use the invariance of the energy-momentum tensor as an input instead of deriving it as in [10] but even then the discussion may be more involved. In [8] the analysis was extended to all dimensions higher than two and also to Euclidean signature. It is assumed that the half dimension forms occur at most quadratically in the action and again that there are enough scalar fields to be able to reduce the symmetry as in [10].

The Minkowskian analysis seems to have been done a long time ago as we mentioned but the Euclidean case is interesting because it illustrates the fact that a change of (non-degenerate) signature does not change the duality group; it changes, in general, the coset however. The non-compactness of the subgroup \(H\) that replaces \(K\) implies some chaos. What we have shown is that the case of dimension \(d = 4k\) resembles that of 4 dimensions but that of \(4k + 2\) dimensions however admits maximal duality group of the type \(O(N, N)\) with compact subgroup (Lorentzian case) \(O(N)^2\) resp. subgroup \(H = O(N, \mathbb{C})\). For completeness let us mention that in \(4k\) dimensions \(K = U(N)\) is similarly replaced in the Euclidean signature by \(H = GL(N, \mathbb{R})\).

This analysis was motivated in part by my desire to clarify the difference between two-dimensional quantisation of single charges by requiring locality (symmetric quadratic form) and the four-dimensional Dirac-Schwinger-Zwanziger antisymmetric quantisation condition. As we have seen this alternation between symmetry and antisymmetry of the quadratic invariant \(\omega\) for every other dimension can be checked by inspection and is rather robust as it does not depend on the signature of spacetime. The supersymmetric origin of the supergravity bosonic actions is not relevant here but let us recall that dualities are often coupled to chiral rotations in such models. Now it has been noticed repeatedly that the first nontrivial example of what we would like to call a quantisation dimension namely 6 is closely related to quaternions and hyperKaehler structures [27]. But \(k\) was arbitrary in the previous discussion so we should not restrict ourselves to the supersymmetry domain.

One conclusion at this stage is that, as one had maybe anticipated, the symmetry must be defined by a very basic counting, namely parity of the half-dimension \(f := d/2\); this number precisely determines the symmetry
or antisymmetry of the intersection form of the corresponding cohomology.

An important shift of one occurs though as $f = 1$ in the symmetric case here, we shall elaborate on it later, it is related to the time extension.

Let us finish this section by giving some technical advice for the handling of non-manifestly duality invariant actions. It was inspired by [28] and consists in doubling as usual the set of forms of degree $f$ as we explained above and in keeping along the TS constraint which is also invariant. This leads to nicer couplings to scalar fields [8] for instance and to a derivation of the Noether current of duality which had to be guessed in [10].

5. Complementarity, dyons, TS and F-duality and conclusion

Let us now elaborate on the relationship between charge quantisation conditions for dyons in 4 dimensions and for chiral fermions in 2d. I noticed in March 1996 that the famous Skyrme minus sign, namely the fermionic character of his vertex operator had to be understood in units where $\hbar = 2\pi$

$$e^{2\pi(eg' + e'g)} = -1$$

This should have been (and maybe has been) analysed long ago. The analogy between chirality of spinors and helicity of vectors is a Lorentz fact. As we mentioned above supersymmetric dualities act simultaneously on the fermionic fields by chiral transformations. As it appears from [13] chiral fermions are obtained by a vertex construction if their charge obeys the quantisation condition: $e = g = \sqrt{s}/2$, where $s$ is an integer (a squarefree integer for the algebra of observables to be maximal). The model is the chiral $U(1)$ current algebra. The idea is that a nonlocal expression can only satisfy causality, or fermionic causality, if it obeys a quantisation condition à la Dirac. In the monopole case one wants the Dirac string to be invisible, in the fermionisation problem one wants the Mandelstam-Skyrme string to be almost invisible, giving only a minus sign but actually an unavoidable one. In the Sine-Gordon case the quantisation is the celebrated $\beta^2 = 4\pi$ relation. It could be rewritten, compare [12], as the relation $eg' = \frac{1}{4}$ but the symmetry of the quadratic form was overlooked there and generalisations attempted in 4 dimensions. We now know much more about affine Kac-Moody algebras (then 2 years old in Mathematics and to be born in Physics). The finiteness of the Schwinger term in 1 (null) dimension or in 1+1 dimensions is the possibility of central extension of Loop groups. A lot of effort has been devoted to find higher dimensional generalisations but divergences have spoiled the game. Yet non-relativistic versions of 2+1 or 3+1 dimensional fermionisation have been proposed since 1975 in the presence of gauge fields. The role of the Dirac quantisation condition ap-
peared immediately through the formula for angular momentum, which gives straightforwardly the minus sign of the Schwinger-Zwanziger condition. It is very natural to compare this to the Bohr-Rosenfeld uncertainty relation for electromagnetic fields. This is in 4 dimensions but free field theory and hence (bad)-divergence free.

Interestingly enough the quantisation condition for extended objects (p-branes) with \( p \geq 2 \) were given in 1985-86 without addressing the question of the sign. It was thus very exciting to check that in 6d supergravities \( (N_4 = 8 \text{ and } N_4 = 6 \ [7]) \) the duality symmetry was not symplectic but pseudo-Riemannian. Since then a few papers have appeared to fill the gap mentioned above by direct arguments and to verify the sign difference between \( 4k \) and \( 4k + 2 \) dimensions for the generalised Dirac-Schwinger-Zwanziger condition [29]. One may remark that these papers exclude the fermionic case. Now dimensional reduction techniques are available to exploit a desired fermionisation that could be done in six dimensions without any internal symmetry, so as to do it in lower dimensions with an automatic twist. The simplest candidate is a two-form (self-dual if one prefers) in 6d and in fact such a theory has become quite fashionable in the meantime starting with [30]. We can check that the fibration of the 6d Minkowskian space on a torus over either Euclidean or Minkowskian base gives in both cases the \( SL(2, \mathbb{Z}) \) modular group as duality group on the base for Maxwell theory. It was clear already from the disintegration magic of [7] that the rather fundamental nature of the so-called ADE structure seemed to play a role in supergravity theory. Since then it appeared in CFT, in the applications of singularity theory (where it was already a dominant character) to string theory, etc... Its home, intersection theory seems to be the most important tool for fermionisation and charge quantisation problems [31].

The second project I have been interested in that relates to dualities is the full analysis of the 1975 observation that classical field theory is obtained by letting Planck’s constant tend to zero but hiding some of them in rebaptised parameters like \( E=e/h \) and \( M=m/h \). The upper case parameters are relevant to the wave world in which one does not count quanta. Note that \( e/m=E/M \), so the anniversary of the electron as a particle should have been next year instead of last year. Now this relates to dualities because in this (electric) classical field theory limit one can study magnetic particles and their solitonic “quantised” charge. Clearly the dual theory, ignoring its strong coupling for the moment, has magnetic waves and electric particles at the new classical level. One message is that the words “classical limit” are so ambiguous that they confuse everybody. Let us mention in passing that non-relativistic limits are equally ambiguous as anybody at ease with Unit Systems can testify. How do classical people measure Planck’s
constant? They in fact combine two complementary limits to measure for instance two vertices of the \([e, E \text{ or } g, \alpha]\) triangle. The validity of a particular classical limit is to be decided case by case and one may be better than another. Is it true that the collection of all classical limits contains all the information about the quantum theory? This is postulated in the usual interpretation of Quantum Mechanics, but can we make this precise and restrict the set of limits that is necessary? For instance non-relativistically invariant classical limits are sometimes necessary for instance to obtain particles rather than quanta in order to avoid the Klein paradox. Similarly non-relativistically invariant limits of supergravity theories might be useful for some perturbative computations.

To conclude with less ambitious goals let me sketch some results of [16] in order to motivate the reader to dig deeper into the present lines. It turns out that indeed the TS idea can be implemented for all differential forms of the type II supergravity theories in all dimensions between 3 and 11. The structure is particularly simple of course in 11 dimensions but the equations can be rewritten in a dimension independent way. It applies to sigma models both gauge fixed and not gauge fixed. The restriction to toroidal compactifications remains and should be relaxed in the future. Our present reduction of type II to type I suggests that this might be doable. Bosonic strings and heterotic strings should be reconciled with the present group theoretical approach that smells of integrable systems, in a loose sense as chaos generally follows non-compact groups.

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