On a discrete analog of the Tzitzeica equation

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Abstract. A discrete analog of the Tzitzeica equation is found in the form of quad-equation. Its continuous symmetry is an inhomogeneous Narita–Bogoyavlensky type lattice equation which defines a discretization of the Sawada–Kotera equation. The integrability of these discretizations is proven by construction of the Lax representations.

Keywords: Tzitzeica equation, discrete differential geometry, Lax pair, higher symmetry

1 Introduction

The celebrated Tzitzeica equation [1, 2, 3, 4]

\[ H_{xy} = e^{H} - e^{-2H} \]  \hspace{1cm} (1)

is probably the most exotic integrable equation among those which can be written in such a compact form. It looks deceptively similar to the Liouville and the sine-Gordon equations, but the underlying spectral problem is of third order and this makes all properties more complicated. In particular, the problem of integrable discretization is rather difficult. The most natural pattern for a discrete version is given obviously by quad-equations, that is, difference equations on the square grid

\[ Q(h, h_1, h_2, h_{12}) = 0 \]

where \( h = h(n_1, n_2) \) and the subscript \( k \) denotes the shift \( n_k \to n_k + 1 \). However, up to the author’s knowledge, no such discretization was known till now. The discretization proposed by Schief and Bobenko [5, 6, 7] is
very natural from the geometric point of view, but it is given by a three-component system rather than one scalar equation:

\[ hh_{12}(h_1 h_2 - h_1 - h_2) + h_{12} + h - 1 - \frac{AB}{h} h_1 h_2 h_{12} = 0, \]

\[ \frac{A_2}{A} = \frac{h_1}{h}, \quad \frac{B_1}{B} = \frac{h_2}{h}. \]  

(2)

The discretization through the permutability properties of the Bäcklund transformations [8, 9, 10] also leads to equations with several components, because the Tzitzeica equation does not admit Bäcklund transformations of first order (in contrast to the sine-Gordon equation).

On the other hand, the discrete theory is usually in perfect parallel with the continuous one and therefore one should expect the existence of some quad-equation analogous to (1). The aim of this paper is to study the following candidate for this role:

\[ hh_{12}(c^{-1} h_1 h_2 - h_1 - h_2) + h_{12} + h - c = 0 \]  

(3)

or, in the exponential form,

\[ e^{H_1 + H_2 - C} - e^{H_1} - e^{H_2} = e^{C - H - H_{12}} - e^{-H} - e^{-H_{12}}. \]

The parameter \( c \neq 0, \infty \) is essential and cannot be scaled, but the values \( c, -c \) are equivalent under the change of sign \( h \to -h \) and \( c, c^{-1} \) are equivalent under the reflection \( h(n_1, n_2) \to 1/h(-n_1, n_2) \). The fixed points \( c = \pm 1 \) are special: at these points the equation degenerates into

\[ hh_{12}(h_1 - 1)(h_2 - 1) = (h - 1)(h_{12} - 1) \]  

(4)

which is one of the form of the well known discrete Liouville equation. This is a linearizable equation, namely, the substitution \( h = \frac{\tau_1 \tau_2}{\tau_{12}} \) maps solutions of the linear wave equation \( \tau_{12} - \tau_1 - \tau_2 + \tau = 0 \) into solutions of (4). The general solution of the latter is given therefore explicitly by the cross-ratio

\[ h = \frac{(a_1 - b)(a - b_2)}{(a - b)(a_1 - b_2)} \]

where \( a = a(n_1) \) and \( b = b(n_2) \) are arbitrary functions of one discrete variable. At \( c \neq \pm 1, \) no such explicit formula exists, however we will see in Section 3 that the above substitution still makes sense and brings the equation to the trilinear/bilinear forms.

The continuous limit is given by the following substitutions:

\[ c \to 1 + \alpha \varepsilon^6, \quad h(n_1, n_2) \to 1 + \beta \varepsilon^2 h(x, y), \quad x = \varepsilon n_1, \quad y = \varepsilon n_2. \]
In the limit $\varepsilon \to 0$, the terms up to $\varepsilon^5$ in equation (3) vanish identically and the coefficients at $\varepsilon^6$ recover the equation

$$\beta^2(h h_{xy} - h_y h_x) = 2\beta^3 h^3 - 2\alpha$$

which is the Tzitzeica equation (1) up to the change $h = e^H$ and obvious scalings (certainly, if $c \equiv 1$ then $\alpha = 0$ and we obtain the Liouville equation, as it should be).

In the next section we discuss the zero curvature representation for equation (3). An associated differential-difference flow is derived in Section 4. It serves as a Volterra lattice type discretization of the Sawada–Kotera equation [11] and its modification given in Section 5 defines a continuous symmetry of the discrete Tzitzeica equation. The role of such symmetries in the theory of quad-equations is well known, see e.g. [12].

2 Linear problems

Integrability of equation (3) is based on the zero curvature representation which can be conveniently written as a system of second order difference equations.

**Statement 1.** The discrete Tzitzeica equation (3) is equivalent to the consistency conditions of the following equations:

\[
\begin{align*}
\psi_{11} - \mu \psi_1 &= \frac{h_1 - c}{h_1(h - c)}(\psi_1 - \mu \psi) + \frac{c - \mu}{h - c}(\psi_{12} - \nu \psi_1), \\
\psi_{12} + \psi &= h(\psi_1 + \psi_2), \\
\psi_{22} - \nu \psi_2 &= \frac{h_2 - c}{h_2(h - c)}(\psi_2 - \nu \psi) + \frac{c - \nu}{h - c}(\psi_{12} - \mu \psi_2)
\end{align*}
\]

(5)

where

\[
\mu = c - (c + 1)\lambda, \quad \nu = c - (c - 1)\lambda^{-1}
\]

and $\lambda$ is the spectral parameter.

The proof is obtained by straightforward computation. Clearly, one can substitute $\psi_{12}$ from the second equation into the other two and then the system can be rewritten in the matrix form

$$\Psi_1 = L\Psi, \quad \Psi_2 = M\Psi$$

where $\Psi = (\psi, \psi_1, \psi_2)^T$. The consistency condition then takes the standard form of the discrete zero curvature representation $L_2M = M_1L$ with $3 \times 3$ matrices. This equation is exactly equivalent to (3).

The spectral problem for the Tzitzeica equation (1)

\[
\begin{align*}
\psi_{xx} = v_x \psi_x + \lambda e^{-v} \psi_y, \quad \psi_{xy} = e^v \psi_x, \quad \psi_{yy} = \lambda^{-1} e^{-v} \psi_x + v_y \psi_y
\end{align*}
\]

(6)
can be easily recovered by the continuous limit. In the matrix form we obtain \( \Psi_x = U \Psi, \Psi_y = V \Psi \) where \( \Psi = (\psi, \psi_x, \psi_y)^T \), and the zero curvature representation \( U_y - V_x = [V, U] \) as the consistency condition.

Recall that equations (6) are geometrically nothing but the Gauss equations for the indefinite affine spheres \( \psi : R^2 \to R^3 \) (the definite ones correspond to the elliptic version of the Tzitzeica equation), see e.g. [13].

In order to clarify the geometric meaning of system (5), we compare it with the Schief–Bobenko discretization of Gauss equations [5, 6, 7]:

\[
\begin{align*}
\psi_{11} - \psi_1 &= \frac{h_1 - 1}{h_1(h-1)}(\psi_1 - \psi) + \frac{\lambda A}{h-1}(\psi_{12} - \psi_1), \\
\psi_{12} + \psi &= h(\psi_1 + \psi_2), \\
\psi_{22} - \psi_2 &= \frac{h_2 - 1}{h_2(h-1)}(\psi_2 - \psi) + \frac{\lambda^{-1} B}{h-1}(\psi_{12} - \psi_2).
\end{align*}
\]

(7)

In this case the consistency conditions give exactly system (2). Equations (7) describe the so-called discrete indefinite affine spheres, a class of discrete surfaces \( \psi : Z^2 \to R^3 \) characterized by two properties:

1) \( \psi \) is a discrete asymptotic net, that is, the points \( \psi, \psi_1, \psi_T, \psi_2, \psi_T \) are coplanar for any \((n_1, n_2) \in Z^2 \) (here, \( \bar{k} \) denotes the backward shift \( n_k \to n_k - 1 \));

2) \( \psi \) is a discrete affine Lorentz harmonic net, that is, all discrete affine normals (defined as vectors \( \psi_{12} - \psi_1 - \psi_2 + \psi \) attached to the centers of plaquettes \( \frac{1}{4}(\psi_{12} + \psi_1 + \psi_2 + \psi) \)) meet in one point, the origin.

Property 2) is expressed analytically by second equation (7) and we see that it is satisfied for system (5) as well.

Property 1) is expressed by first and last equations (7), with the coefficients specified by use of the consistency condition. We see that this property does not hold for the surface \( \psi(n_1, n_2) \) defined by system (5), but it holds for the surface obtained by the gauge transformation

\[
\tilde{\psi}(n_1, n_2) = \mu^{-n_1} \nu^{-n_2} \psi(n_1, n_2).
\]

Certainly, it is possible to rewrite the system in terms of \( \tilde{\psi} \), then property 1) will be satisfied, but property 2) will be distorted.

Therefore, one can say that system (5) defines a class of discrete surfaces which are, up to a simple coordinate-dependent scaling, both discrete asymptotic nets and discrete affine Lorentz harmonic nets, however in contrast to the Schief–Bobenko case these properties are not fulfilled simultaneously.

To finish this section, we notice that elimination of \( \psi_2 \) and \( \psi_{12} \) from two first equations (5) brings to the following ordinary difference equation of third order with respect to the shift \( T_1 \):

\[
u \psi_{111} + \psi_{11} = \mu (\psi_1 + u \psi), \quad u := \frac{h_{11}(c - h_1)}{h_{11}h_1(h-c)}
\]
(certainly, an analogous equation holds for the shift $T_2$). This linear problem is considered in more details in Section 4.

3 Bilinear equations

Consider the continuous case first. It is well known (see e.g. [6, 14]) that the substitution $h = -2(\log \tau)_{xy}$ maps the algebraic form of Tzitzeica equation

$$hh_{xy} - h_x h_y = h^3 - 1$$

into the trilinear form

$$4 \det \begin{pmatrix} \tau_{yy} & \tau_{xxy} & \tau_{xyy} \\ \tau_y & \tau_{xy} & \tau_{xy} \\ \tau_x & \tau_x & \tau_{xx} \end{pmatrix} = \tau^3. \quad (8)$$

The fact which apparently has not been paid attention before is that a couple of simpler bilinear equations can be added,

$$3(\tau_{xy} \tau_{xx} - \tau_x \tau_{xxy}) = \tau_y \tau_{xxx} - \tau \tau_{xxyy},$$

$$3(\tau_{xy} \tau_{yy} - \tau_y \tau_{xyy}) = \tau_x \tau_{yyy} - \tau \tau_{xyyy} \quad (9)$$

which are consistent with (8). These follow from the conservation laws

$$\left( \frac{h_{xx}}{h} \right)_y = 3D_x(h), \quad \left( \frac{h_{yy}}{h} \right)_x = 3D_y(h)$$

which admit integration after the substitution:

$$\frac{h_{xx}}{h} = -6(\log \tau)_{xx} + a(x), \quad \frac{h_{yy}}{h} = -6(\log \tau)_{yy} + b(y).$$

Notice that $\tau$-function is defined in (8) up to the multiplication by arbitrary functions on $x$ and on $y$, and this freedom can be fixed by setting $a = b = 0$ without loss of generality. Now substituting $\tau$ again yields relations

$$(\log \tau)_{xxx} = -6(\log \tau)_{xx}(\log \tau)_{xy}, \quad (\log \tau)_{xyy} = -6(\log \tau)_{yy}(\log \tau)_{xy}$$

which are (9).

In the discrete case the picture is very similar. The substitution $h = \frac{\tau_1 \tau_2}{\tau \tau_1 \tau_2}$ brings (3)

$$hh_{12}(c^{-1} h_1 h_2 - h_1 - h_2) + h_{12} + h = c = 0$$

to the trilinear form

$$c^{-1} \tau_{22} \tau_{12} \tau_{11} + \tau \tau_{122} \tau_{112} + \tau \tau_{22} \tau_{112} = \tau_{11} \tau_{22} \tau_{112} + \tau_1 \tau_2 \tau_{112} + c \tau \tau_{12} \tau_{112}$$
which can be rewritten as

\[
\begin{vmatrix}
\tau_{22} & \tau_{122} & \tau_{1122} \\
\tau_2 & c^{-1}\tau_{12} & \tau_{112} \\
\tau & \tau_1 & \tau_{11}
\end{vmatrix}
= (c - c^{-1})\tau_{12}\tau_{1122}.
\]

(10)

The additional bilinear equations

\[
\tau_{11}\tau_{12} - c\tau_{112} = c\tau_{1112} - \tau_{111}\tau_2,
\]

\[
\tau_{12}\tau_{22} - c\tau_{122} = c\tau_{1222} - \tau_{222}\tau_1
\]

(11)

can be extracted from the multiplicative conservation laws

\[
\frac{v_2}{u} = \frac{h}{h_{11}}, \quad u = \frac{h_{11}(c - h_1)}{h_{11}h_{1}h - c}; \quad \frac{v_1}{v} = \frac{h}{h_{22}}, \quad v = \frac{h_{22}(c - h_2)}{h_{22}h_{2}h - c}.
\]

(12)

These admit integration after the substitution:

\[
u = a(n_1)\frac{\tau_{1111}}{\tau_{11}}, \quad v = b(n_2)\frac{\tau_{2222}}{\tau_{22}},
\]

and again, since the \(\tau\)-function is defined up to multiplication by arbitrary functions on \(n_1\) and \(n_2\), hence one can chose integration constants \(a = b = 1\) without loss of generality. Replacing \(h\) in the expressions for \(u, v\) yields (11).

In the case of Bobenko–Schief discretization (2) the substitution for \(h\) is supplemented by \(A = a\frac{\tau_1^2}{\tau_{11}}, \quad B = b\frac{\tau_2^2}{\tau_{22}}\) with arbitrary \(a(n_1), b(n_2),\) and this leads to the trilinear equation [5, 6, 7]

\[
\begin{vmatrix}
\tau_{22} & \tau_{122} & \tau_{1122} \\
\tau_2 & \tau_{12} & \tau_{112} \\
\tau & \tau_1 & \tau_{11}
\end{vmatrix}
= ab\tau_{12}^3.
\]

(13)

The bilinear equations are not known for this discretization. Clearly, both equations (10) and (13) go to (8) under the corresponding continuous limits.

4 A difference analog of Sawada–Kotera equation

In the rest of the paper we change the notation: now we will consider only one discrete variable \(n\) (identified, say, with \(n_1\)) and it is more convenient to reserve subscripts for the order of shift along this variable rather than to distinguish between one-step shifts along different variables \(n_k\) as before. For instance, new \(h_4\) is the same as old \(h_{1111}\) and \(u_{-3}\) is the same as \(u_{\overline{111}}\).

The goal of this section is to derive the differential-difference equation

\[
u_t = u^2(\nu_2u_1 - u_{-1}u_{-2}) - u(u_1 - u_{-1}).
\]

(14)
Its close relation to the discrete Tzitzeica equation will be revealed in the next section, however, this lattice certainly deserves study on its own. It can be interpreted as a difference analog of the Sawada–Kotera equation [11]

\[ U_\tau = U_{xxxx} + 5U_{xxx} + 5U_x U_{xx} + 5U^2 U_x \]

which appears under the following continuous limit at \( \varepsilon \to 0 \):

\[ u(n, t) = \frac{1}{3} + \frac{\varepsilon^2}{9} U(x - \frac{4}{9} \varepsilon t, \tau + \frac{2\varepsilon^5}{135} t), \quad x = \varepsilon n. \]

Recall that both flows

\[ u_\nu = u(u_1 - u_{-1}) \quad \text{and} \quad u_{\nu\nu} = u^2(u_2 u_1 - u_{-1} u_{-2}) \]

are very well known integrable equations: the first one is the celebrated Volterra lattice and the second one is the modified Narita–Bogoyavlensky lattice (see [15, 16, 17] for a detailed theory and the bibliography). In both cases, the continuous limit is the Korteweg–de Vries equation

\[ U_t = U_{xxx} + 6U U_x. \]

However, these lattices are not members of one and the same hierarchy, that is the flows \( \partial_\nu \) and \( \partial_{\nu\nu} \) do not commute. Therefore, one should not expect that their linear combination (14) is integrable. But it is, and we will prove it by constructing the Lax representation.

The starting point is the difference spectral problem

\[ w_{\psi_3} + \psi_2 = \mu(\psi_1 + w\psi). \quad (15) \]

This is a special reduction of a general third order problem, in parallel with the continuous case of the Sawada–Kotera equation. We write it in the operator form

\[ P\psi = \mu Q\psi, \quad P = (uT + 1)T^2, \quad Q = T + u \]

where \( T : n \to n + 1 \) is the shift operator. The isospectral deformations are defined by the Lax equations

\[ D_t(Q^{-1}P) = [A, Q^{-1}P] \iff P_t = BP - PA, \quad Q_t = BQ - QA. \]

These equations are equivalent, for the above \( P, Q \), to the system

\[ u_t = B(T + u) - (T + u)A, \quad B(T^2 - 1) = (T + u)AT - (uT + 1)A_2. \]

We assume that both \( A \) and \( B \) are difference operators, that is, Laurent polynomials in \( T \). The notation like \( A_2 \) is used for the operators with shifted coefficients, so that \( T^2 A = A_2 T^2 \). The second equation can be solved as follows:

\[ A = F(T - T^{-1}), \quad B = F_1 T + u(F - F_3) - F_2 T^{-1}, \]

\[ 7 \]
and then the substitution into the first equation yields

\[ u_t = TFu + uFT^{-1} - uF_3T - T^{-1}F_3u + F_1 - F_2 + u(F - F_3)u. \]  

(16)

Notice that this equation admits the reduction \( F^* = F \), with respect to the usual conjugation of difference operators: \( T^* = T^{-1}, \ (FG)^* = G^*F^* \) and \( f^* = f \) for functions \( f \). Let operator \( F \) be of the form

\[ F = f^{(k,k)}T^k + \cdots + f^{(k,1)}T + f^{(k,0)} + f^{(k,1)}_{-1}T^{-1} + \cdots + f^{(k,k)}_{-k}T^{-k}, \quad k > 0. \]

Then collecting coefficients at \( T^{k+1}, \ldots, T \) in equation (16) brings to the recurrent relations

\[
\begin{align*}
&u_{k+1}f_{1}^{(k,k)} - u_3f_{3}^{(k,k)} = 0, \\
u_kf_{1}^{(k,k-1)} - u_3f_{3}^{(k,k-1)} = f^{(k,k)}_2 - f^{(k,k)}_1 + uu_k(f^{(k,k)}_3 - f^{(k,k)}), \\
u_jf_{1}^{(k,j-1)} - u_3f_{3}^{(k,j-1)} = f^{(k,j)}_2 - f^{(k,j)}_1 + uu_j(f^{(k,j)}_3 - f^{(k,j)}) \\
&+ u_jf^{(k,j+1)}_2 - u_{j+1}f^{(k,j+1)}, \quad j = 1, \ldots, k - 1
\end{align*}
\]

and vanishing of the free term yields the desired equation for \( u \)

\[ u_t = 2u(f^{(k,1)} - f^{(k,1)}_2) + u^2(f^{(k,0)} - f^{(k,0)}_3) + f^{(k,0)}_1 - f^{(k,0)}_2. \]

We are interested only in a local evolution, that is, the recurrent relations must define all \( f^{(k,j)} \) as functions of a finite set of \( u \). It is easy to see that the first equation for \( f^{(k,k)} \) can be solved if and only if \( k \) is odd, namely

\[ f^{(k,k)} = u_{-1} \cdots u_{k-4}u_{k-2}, \quad k = 2m + 1 > 0. \]

In the simplest case \( k = 1 \) we find

\[ f^{(1,1)} = u_{-1}, \quad f^{(1,0)} = 1 - u_{-2}u_{-1} \]

and the corresponding flow is exactly (14). So, we arrive to the following statement.

**Statement 2.** Lattice (14) governs the isospectral deformation of the linear problem (15) defined by the equation

\[ \psi_t = A(\psi) = (u_{-1}T + 1 - u_{-1}u_{-2} + u_{-2}T^{-1})(T - T^{-1})(\psi). \]

(17)

The higher flows can be computed analogously. At \( k = 3 \) one finds

\[
\begin{align*}
f^{(3,3)} &= u_1u_{-1}, \quad f^{(3,2)} = w + w_{-1}, \quad f^{(3,1)} = 1 - u_{-1}(w + w_{-1} + w_{-2}), \\
f^{(3,0)} &= u_{-1}u_{-2}(w + w_{-1} + w_{-2} + w_{-3})
\end{align*}
\]
where \( w = u(1 - u_1u_{-1}) \) and

\[
\begin{align*}
  w_t & = u(w_1(w_3 + w_2 + w_1 + w) - w_{-1}(w + w_{-1} + w_{-2} + w_{-3}) \\
  & \quad - u_1(w_3 + w_{-1}) + u_{-1}(w_1 + w_{-3}).
\end{align*}
\]

One can check directly that this is a commuting flow for (14) indeed. It should be noted that its right hand side is a sum of three homogeneous polynomials:

\[
u_t^3 = P^{(3)} + P^{(5)} + P^{(7)}, \quad \deg P^{(m)} = m,
\]

where \( P^{(3)} \) and \( P^{(5)} \) define respectively the symmetries for the Volterra and the Narita–Bogoyavlensky lattices

\[
u_t' = P^{(3)} = u(u_1(u_2 + u_1 + u) - u_{-1}(u + u_{-1} + u_{-2})),
\]

\[
u_t'' = P^{(7)} = u^2u_1^2u_3u_4 + \ldots.
\]

5 Modified lattice

The role of discrete Miura type transformations for lattice (14) is played by the substitutions

\[
M^- : \quad u = \frac{h_2(c - h_1)}{h_2h_1h - c}, \quad M^+ : \quad \tilde{u} = \frac{(c - h_1)h}{h_2h_1h - c}
\]

(the first one already appeared in the end of Section 2).

**Statement 3.** The substitutions \( M^\pm \) map solutions of the lattice

\[
h_t = \frac{h(c - h)}{h_1hh_{-1} - c} \left( \frac{h(c - h_1)(c - h_{-1})(h_2h_1 - h_{-1}h_{-2})}{(h_2h_1h - c)(hh_{-1}h_{-2} - c)} - h_1 + h_{-1} \right)
\]

into solutions of lattice (14).

**Proof.** One can easily check that if \( \phi \) is a particular solution of linear problem (15) corresponding to the value of spectral parameter \( \mu = 1/c \) then the quotient \( h = \phi/\phi_1 \) is related to the potential \( u \) by the substitution \( M^- \). Therefore, the time evolution of \( h \) can be found by use of equation (17) for \( \phi_t \) which guarantees that \( h \) satisfies some modified lattice equation in the form of a conservation law

\[
(\log h)_t = (T - 1)S(h_1, h, h_{-1}, h_{-2})
\]

where

\[
S = \frac{\phi_t}{\phi} = \frac{1}{\phi} \left( u_{-1}(1 - u_{-1}u_{-2} + u_{-2}T^{-1}) - (\phi_{-1} - \phi_1) \right)
\]

\[
= u_{-1} \left( 1 - \frac{1}{h_1} \right) + u_{-2}(h_{-1}h_{-2} - 1) + (1 - u_{-1}u_{-2}) \left( h_{-1} - \frac{1}{h} \right).
\]
Replacing $u$ and some algebra bring to equation (18).

The second substitution follows either from the involution $h \to h^{-1}$, $c \to c^{-1}$ which preserves equation (18) or from the form invariance of linear problem (15) with respect to the change $n \to -n$.

In order to relate lattice (18) with the discrete Tzitzeica equation (3) we notice that $\hat{u}/u = h/h_2$ and compare this with the first of conservation laws (12). It is clear that if the discrete variable $n$ in (18) is identified with $n_1$ in (3) then the map $u \to \hat{u}$ can be identified with the shift $T_2$ along the second discrete variable $n_2$. This argument is not quite rigorous, since the conservation law is not exactly equivalent to equation (3). Nevertheless, the following statement holds true.

**Statement 4.** The discrete Tzitzeica equation (3) admits the evolution symmetry (18) along any of the coordinate directions $n = n_1$ or $n = n_2$, that is, the differentiation $D_t$ in virtue of lattice (18) is consistent with the discrete equation:

$$D_t(Q)|_{Q=0} = 0.$$ 

The proof of this identity is straightforward, although rather involved: $D_t(Q)$ contains variables $h$ in 12 points $(n_1 + k, n_2)$, $(n_1 + k, n_2 + 1)$, $k = -2, \ldots, 3$, so that 5 copies of the quad-equation $T^{-2}(Q) = \cdots = T^2(Q) = 0$ are used.

It is worth noticing that an infinite sequence of conservation laws for lattice (14) can be extracted from the single conservation law (19) for the modified lattice, by means of the classical trick with the inversion of Miura transformation as a power series in the spectral parameter $[18]$. Indeed, the equation $u = M^{-}(h)$ can be solved with respect to $h$ as the formal power series

$$h = -\frac{1}{u}(1 + ch^{(1)} + c^2h^{(2)} + \ldots)$$

with the coefficients recursively found as polynomials in $u_j$:

$$h^{(1)} = u_1(1 - uu_2),$$

$$h^{(2)} = -u_1u_2((1 - u_2u)(1 - u_3u_1) - u_3u(1 - u_4u_2)), \ldots$$

Then the coefficients of the expansion

$$\log h = -\log(-u) + ch^{(1)} + c^2\left(h^{(2)} - \frac{1}{2}(h^{(1)})^2\right) + \ldots$$

provide densities of the conservation laws $D_t(\rho^{(k)}) = (T - 1)\sigma^{(k)}$:

$$\rho^{(0)} = \log u, \quad \rho^{(1)} = u(1 - u_1u_{-1}), \ldots$$
6 Concluding remarks

It should be noted that the discrete Tzitzeica equation (3) falls out of the classification scheme based on the notion of 3D-consistency. Indeed, this notion is essentially equivalent to the existence of a zero curvature representation with $2 \times 2$ matrices [19], while the construction in Section 2 brings to the $3 \times 3$ matrices and it is difficult to expect that the size can be reduced. Up to the author’s knowledge, this is the first example of a quad-equation associated with a third-order spectral problem. According to [20], any ‘generic’ quad-equation can be included into a consistent triple, but the situation remains unclear for equations with degenerate biquadratics. In this terminology, equation (3) is degenerate, and this example demonstrates that this class of equations can be more complicated than it seems at a first glance. The alternative approaches to the integrability of quad-equations are developed, e.g. in [21, 22, 23] and one may hope that more examples of such kind will be discovered by these methods.

An open problem is to construct the Bäcklund transformations for equation (3). Probably, the commutativity of these transformations can be formulated as 3D-consistency of equation (3) with some discrete equations of second order in shifts rather than quad-equations.

The exhaustive classification of integrable Volterra-type lattices $u_t = f(u_1, u, u_{-1})$ was obtained by Yamilov [15, 16]. However, a little is known about the higher order lattices, even in the polynomial case, so that the study of lattices like (14) is of interest as well.

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