SOAP BUBBLES AND ISOPERIMETRIC REGIONS IN THE PRODUCT OF A CLOSED MANIFOLD WITH EUCLIDEAN SPACE

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Abstract. For any closed Riemannian manifold $X$ we prove that large isoperimetric regions in $X \times \mathbb{R}^n$ are of the form $X \times$ (Euclidean ball). We first show that isoperimetric boundaries in such ambient manifold are very regular, and then obtain apriori estimates for CMC hypersurfaces leading to the result. We prove that if $X$ has non-negative Ricci curvature then the only soap bubbles enclosing a large volume are the products $X \times$ (Euclidean sphere). We give an example of a surface $X$, with Gaussian curvature negative somewhere, such that the product $X \times \mathbb{R}$ contains stable soap bubbles of arbitrarily large enclosed volume which do not even project surjectively onto the $X$ factor.

1. Introduction

Definition. Given a Riemannian manifold $M$, possibly with boundary, and a positive number $v$, the isoperimetric problem asks for a region $\Omega \subset M$ whose volume is $v$ and whose perimeter is minimal among all regions of volume $v$ in $M$.

Solutions to the isoperimetric problem, if they exist, are called isoperimetric regions. Their boundaries are called isoperimetric boundaries; they are differentiable hypersurfaces except perhaps at some singular points that together make a closed subset which is either empty or of codimension at least 8 in the ambient manifold [13]. More precisely, these boundaries are smooth away from the singular points and from $\partial M$, and they are of class at least $C^1$ at the points where they touch $\partial M$, see e.g. [7]. Moreover, they have constant mean curvature in the smooth part away from $\partial M$.

Definition. A soap bubble in $M$ is any smooth embedded hypersurface $S \subset M$ which has constant mean curvature and is the boundary of some smooth domain.

While closely related, the two concepts are not equivalent. An isoperimetric boundary will not be a soap bubble if it has singular points or if it touches the boundary of the ambient manifold. A soap bubble may not minimize area among hypersurfaces enclosing the same volume.

The purpose of the present paper is to study the shape of soap bubbles and isoperimetric regions in Riemannian products $X \times \mathbb{R}^n$, where $X$ is a closed, connected Riemannian manifold of any dimension. Our first result gives symmetry and regularity for these objects, large or small. To make a precise statement we fix some conventions. We write $B(y,r)$ for the Euclidean ball with center $y$ and radius $r$ in $\mathbb{R}^n$, write $\overline{B}(y,r)$ for the closed ball, and $S(y,r)$ for the Euclidean sphere.

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Definition. A subset $E \subset X \times \mathbb{R}^n$ is normalized if there is an open set $A \subseteq X$, a function $u : A \to \mathbb{R}^+$, and a point $y \in \mathbb{R}^n$, such that $E$ is the union of the coaxial balls $\{x\} \times B(y, u(x))$, as $x$ ranges over $A$, or the closure of such union. The rotated graph of $u$ is the union of the coaxial spheres $\{x\} \times S(y, u(x))$ as $x$ ranges over $A$. The symmetry axis of these sets is $X \times \{y\}$.

We now state the symmetry and regularity result.

**Theorem 1.** Every isoperimetric region in $X \times \mathbb{R}^n$ is bounded and normalized. Every isoperimetric region in the compact manifold $X \times \overline{B}(y, r)$ is, up to a translation parallel to the $\mathbb{R}^n$ factor, normalized with symmetry axis $X \times \{y\}$. In both cases its boundary is, after deleting the symmetry axis, the rotated graph of a $C^1$ function $u : A \to \mathbb{R}^+$ which is smooth in all of $A$ for isoperimetric boundaries in $X \times \mathbb{R}^n$, and smooth in $\{u < r\} \subset A$ for isoperimetric boundaries in $X \times \overline{B}(y, r)$. Any soap bubble in $X \times \mathbb{R}^n$ is, after deleting the symmetry axis, the rotated graph of a smooth function.

A consequence is that singular points on an isoperimetric boundary, if any, can only exist where said boundary meets its symmetry axis. Isoperimetric regions always exist if the ambient manifold is compact. For the non-compact manifold $X \times \mathbb{R}^n$ one can use the argument in [19, page 129], which provides existence on ambient spaces that have an isometry action with compact quotient. In the present paper we give a direct existence proof for $X \times \mathbb{R}^n$ by showing that, for fixed volume $v$ and large radius $r$, there are isoperimetric regions of volume $v$ in $X \times \overline{B}(y, r)$ that do not reach $X \times S(y, r)$ at all. Details are given in Section 7.

Our two main results are the following.

**Theorem 2.** Large isoperimetric regions in $X \times \mathbb{R}^n$ are of the form $X \times (\text{ball})$.

**Theorem 3.** If $X$ has $\text{Ric} \geq 0$, then soap bubbles in $X \times \mathbb{R}^n$ with large enclosed volume are of the form $X \times (\text{sphere})$.

Several authors have studied isoperimetric regions in product spaces. Wu-Yi Hsiang and Wu-Teh Hsiang [12] determined them in the product of two hyperbolic spaces. Duzaar and Steffen [3] proved Theorem 2 for the cylinder spaces $X \times \mathbb{R}$. Pedrosa and Ritoré [21] proved Theorem 2 for $S^1 \times \mathbb{R}^n$ as well as the analogous result for the product of $S^1$ with a hyperbolic space. Ritoré and Vernadakis [22] have obtained a proof of Theorem 2 in the spirit of Geometric Measure Theory. Theorem 2 is also true [8] when the ambient anifold is the product $X \times \mathbb{H}^n$.

**Definition.** Let $M$ be a Riemannian manifold, and let $S \subseteq M$ be a soap bubble with unit normal $\nu$ and second fundamental form $II$. Let $\text{Ric}$ be the Ricci tensor of $M$. The index form of $S$ is the following quadratic form acting on functions $g : S \to \mathbb{R}$:

$$Q(g) = \int_S \left( |\nabla^S g|^2 + P g^2 \right) d\text{area} , \quad P := -\text{Ric}(\nu, \nu) - |II|^2 .$$

The Jacobi equation is $\Delta^S g - P g = 0$. We say that $S$ is stable if its index form is positive definite on the functions with zero average over $S$, except those of the form $\langle \nu, \xi \rangle$ where $\xi$ is a Killing vector field on $M$. 

While isoperimetric boundaries are global minima of area for fixed enclosed volume, stability is the necessary condition for a local minimum. It is natural to ask whether the condition \( \text{Ric} \geq 0 \) in Theorem 3 can be replaced by the hypothesis of large stable soap bubble. The answer is negative.

**Theorem 4.** There exist surfaces \( X^2 \), with Gauss curvature negative somewhere, such that in the manifold \( X \times \mathbb{R} \) we find a one-parameter family \( \{ S_v \}_{0 < v < \infty} \) of soap bubbles with the following properties:

1. The enclosed volume of \( S_v \) is \( v \).
2. There is a positive lower bound for the mean curvatures of the \( S_v \).
3. No \( S_v \) projects surjectively onto \( X \).
4. There is a \( v_0 > 0 \) such that all soap bubbles \( S_v \) with \( v \geq v_0 \) are stable.

The fact that the surfaces \( S_v \) with \( v \geq v_0 \) are stable implies that it is possible to get them by “inflating \( S_{v_0} \)”, so that they will never “burst” during the process. There are examples of this phenomenon where \( X \) is a surface in \( \mathbb{R}^3 \) with the induced metric; then the product \( X \times \mathbb{R} \) is a cylinder in \( \mathbb{R}^4 \). One such cylinder is sketched in Figure 1; the cylinder’s profile is a surface with a thin neck where Gaussian curvature is negative; three soap bubbles \( S_v \) are shown, getting larger but never exiting the domain \( (\text{neck}) \times \mathbb{R} \).

![Figure 1. Bubbles on a cylinder in \( \mathbb{R}^4 \)](image)

Since \( X \) is bounded and \( \mathbb{R}^n \) is infinitely large, we can think of \( X \times \mathbb{R}^n \) as a “slightly thickened Euclidean space”. Then Theorem 2 exhibits large isoperimetric regions in this space as “slightly thickened Euclidean isoperimetric regions”, and Theorem 3 gives a condition under which soap bubbles have the same behavior. The mean curvature of spheres in \( \mathbb{R}^n \) is proportional to \( 1/\text{radius} \), hence proportional to \( (\text{volume})^{-1/n} \). We expect a similar estimate in \( X \times \mathbb{R}^n \); yet the bubbles \( S_v \) of Theorem 4 become arbitrarily large while their mean curvature decreases to a positive constant. All these ideas are reflected in the following theorem, which is essential in the proof of Theorems 2 and 3.

**Theorem 5.** An isoperimetric region in \( X \times \mathbb{R}^n \) of volume \( v \) has the following bound for the mean curvature of its boundary:

\[
H \leq n \left( \omega_n |X| \right)^{1/n} v^{-1/n},
\]

where \( |X| \) denotes the Riemannian volume of \( X \).

Given \( v > 0 \), for \( r \) sufficiently large (depending on \( v \)), isoperimetric regions in \( X \times \overline{B}(y, r) \) of volume \( v \) satisfy the following bound:

\[
H \leq 2n \left( \omega_n |X| \right)^{1/n} v^{-1/n}.
\]
Suppose that $S \subset X \times \mathbb{R}^n$ is the closure of the rotated graph of $u : A \to \mathbb{R}^+$, that it is not too small, and that one of the following conditions is satisfied:

- either: $S$ is isoperimetric in $X \times \mathbb{R}^n$ or in some large $X \times \overline{B}(y,r)$,
- or: $S$ is a soap bubble and $X$ has $\text{Ric} \geq 0$,

then we have the estimates:

\[(3) \quad \max u - \min u \leq \text{const}, \quad H \leq \frac{\text{const}}{\max u}.
\]

For a large soap bubble in $X \times \mathbb{R}^n$ of volume $v$, and without the condition $\text{Ric} \geq 0$, we only have the weaker estimates:

\[(4) \quad \max u \leq \text{const} \cdot v^{1/n}, \quad H \leq \text{const}.
\]

The four constants depend only on $n$ and $X$.

The paper is organized as follows. The boundedness part of Theorem 1 is proved in Section 2, the symmetry part in Section 3, and the regularity part in Section 4. In Section 5 we prove the part of Theorem 5 about isoperimetric boundaries, and in Section 6 the part for soap bubbles. In Section 7 we prove existence of isoperimetric regions of every volume. The main theorems (2) and (3) are proved in Section 8 based on the estimate (3) in Theorem 5 and a gradient bound from the Appendix. In Section 9 we construct the families $\{S_v\}$ of Theorem 4. In Section 10 we show that in some of these families all large bubbles are stable.

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2. Monotonicity formula

In this section we shall obtain a lower estimate (6) for area and use it to prove that every isoperimetric region in $X \times \mathbb{R}^n$ is bounded. We shall also use (6) in later sections.

We use the monotonicity formula proved in [14, pages 483-484], which is true for all hypersurfaces of constant mean curvature. Their proof is for a Euclidean ambient space. We adapt it here and we indicate the small changes needed to make it work in all ambient Riemannian manifolds.

For each $m$ we use $\mathcal{H}^m$ to denote $m$-dimensional Hausdorff measure.

Definition. Let $M$ be a Riemannian manifold of dimension $d$ and let $S \subset M$ be a hypersurface. By mean curvature of $S$ we mean the scalar function $H : S \to \mathbb{R}$ such that for any deformation $\{S_t\}$ of any compact piece $S_0 \subset S$ we have:

\[(5) \quad \frac{d}{dt} \bigg|_{t=0} \mathcal{H}^{d-1}(S_t) = \int_{S_0} \xi \cdot (H \nu) \, d\mathcal{H}^{d-1} + \int_{\partial S_0} (\xi \cdot \eta) \, d\mathcal{H}^{d-2},
\]

where $\xi$ is the field of the velocities of motion for each point during the deformation, $\nu$ is a unit normal for $S$, and $\eta$ is the outer conormal to $\partial S_0$ in $S$.

For example, the unit sphere in $\mathbb{R}^d$ has $H = d - 1$ with respect to the outer unit normal.
In [14] they consider a smooth piece of hypersurface in \( S \subset \mathbb{R}^d \) that is of the form \( S = B(z_0, r_1) \cap \partial \Omega \) for some region \( \Omega \), and whose mean curvature is a positive constant \( H \). The point \( z_0 \) is assumed to lie on \( S \). Then define, for \( 0 < s < r_1 \), the following objects:

\[
S(s) = S \cap \overline{B}(z_0, s) ,
\]

\[
U(s) = \Omega \cap \overline{B}(z_0, s) \quad \text{(a solid)} ,
\]

\[
Q(s) = \Omega \cap \partial B(z_0, s) \quad \text{(a spherical piece)} ,
\]

\[
\nu_s = \text{the outer unit normal along } \partial U(s) ,
\]

and obtain a differential inequality satisfied by the area function \( a(s) \equiv \mathcal{H}^{d-1}(S(s)) \), and use it to estimate \( a(s) \) from below.

Choose orthonormal coordinates \( z_1, \ldots, z_d \) centered at \( z_0 \) and consider the vector field \( \mathbf{V} \equiv z_1 \partial_{z_1} + \cdots + z_d \partial_{z_d} \), whose flow is \( \varphi_t(z) = e^t z \) and whose divergence is \( d \). By formula (5), compute:

\[
(d - 1) a(s) = \frac{d}{dt} \bigg|_{t=0} e^{(d-1)t} a(s) = \frac{d}{dt} \bigg|_{t=0} \mathcal{H}^{d-1}(\varphi_t(S(s))) = \int_{S(s)} H \nu \cdot \mathbf{V} + \int_{\partial S(s)} \eta \cdot \mathbf{V} = H \int_{\partial U(s)} \mathbf{V} \cdot \nu_s - H \int_{Q(s)} \mathbf{V} \cdot \nu_s + \int_{\partial S(s)} \eta \cdot \mathbf{V} .
\]

Then estimate the three summands in the last expression:

\[
H \int_{\partial U(s)} \mathbf{V} \cdot \nu_s = d H \mathcal{H}^d(U(s)) \leq d \omega_d H s^d ,
\]

\[
-H \int_{Q(s)} \mathbf{V} \cdot \nu_s \leq 0 ,
\]

\[
\int_{\partial S(s)} \eta \cdot \mathbf{V} \leq s \frac{d}{ds} a(s) .
\]

The third inequality follows from the coarea formula for the function \( \sqrt{y_1^2 + \cdots + y_d^2} \).

We now have \( (d - 1) a(s) \leq d \omega_d H s^d + s \frac{d}{ds} a(s) \), or equivalently:

\[
\frac{d}{ds} \left( s^{1-d} a(s) \right) \geq -d \omega_d H .
\]

In the case \( H = 0 \) this differential inequality says that \( s^{1-d} a(s) \) is a monotone increasing function. This is why it is called monotonicity formula.

We also have \( \lim_{s \to 0} s^{1-d} a(s) = \omega_{d-1} \), which combined with the monotonicity formula gives the following lower bound for area:

\[
a(s) \geq \left( \omega_{d-1} - d \omega_d H s \right) s^{d-1} .
\]

For a non–Euclidean ambient space \( M \) the above proof needs the following modifications. Choose \( (z_1, \ldots, z_d) \) to be canonical coordinates at \( z_0 \), i.e. coordinates for which the Christoffel symbols vanish at \( z_0 \). As long as we keep \( s \) small, the identities used above are all true in an approximate way. As examples: while in the Euclidean case we had \( \nabla \mathbf{V} = \text{id} \), now we have \( \nabla \mathbf{V} = \text{id} + O(s) \); while in the Euclidean case \( \mathcal{H}^d(B(z_0, s)) = \omega_d s^d \), now it is \( \mathcal{H}^d(B^M(z_0, s)) = (1 + O(s)) \omega_d s^d \).
Notice also that the calculation carries through for isoperimetric boundaries, provided $z_0$ is not a singular point.

Then for small $s$ we get
\[ \frac{d}{ds} \left( s^{1-d} a(s) \right) \geq -c_1 \ H \]
where $c_1$ is some positive constant close to $d\omega_d$ in value, and we obtain our desired lower estimate for area:

\[ a(s) \geq \left( \omega_d - c_1 H s \right) s^{d-1}. \]

In a complete manifold with bounded geometry (as is $X \times \mathbb{R}^n$) the constant $c_1$ and the radius of a ball where the above proof is valid may be chosen the same for all points $z_0$. Then formula (6) provides a range of radii for which we have a lower area bound near every regular point of $S$. The larger $H$ is, the shorter that range of radii is; formula (6) is useful only in combination with some upper bound for $H$.

**Proposition 6.** Every isoperimetric region in $X \times \mathbb{R}^n$ is bounded.

Let $d$ be the dimension of $X \times \mathbb{R}^n$. An isoperimetric boundary $\partial\Omega$ has constant mean curvature. No matter how large this constant is, it has a fixed value and thus provides a positive (if small) value $r_0$ and a constant $\epsilon > 0$ such that $H^{d-1}\left( \partial\Omega \cap B^{X \times \mathbb{R}^n}(z_0, r_0) \right) > \epsilon$ for every non-singular $z_0 \in \partial\Omega$. If $\Omega$ were unbounded then $\partial\Omega$ would be unbounded and it would contain an infinity of non-singular points $z_j$ with pairwise distances all greater than $2r_0$. But then the intersections $\partial\Omega \cap B^{X \times \mathbb{R}^n}(z_j, r_0)$ would be pairwise disjoint, the area of $\partial\Omega$ would be infinite and $\Omega$ could not be isoperimetric.

3. **Symmetry**

In this section we prove the symmetry part of Theorem 1.

If $S \subset X \times \mathbb{R}^n$ is a soap bubble, we can use A. D. Alexandrov’s reflection method, as described e.g. in [10], to prove that $S$ is a union of coaxial spheres $\{x\} \times S(y, u(x))$ as $x$ ranges over the image of $S$ under the projection $X \times \mathbb{R}^n \to X$.

This description forbids, in particular, that some parts of $S$ be surrounded by others. This is a rather obvious consequence of the maximum principle because the mean curvature is the same constant in all connected components. In the case $n = 1$, assuming that the axis is $X \times \{0\}$, the soap bubble is the union of the graphs of $u$ and $-u$.

Consider now the case of an isoperimetric region in $X \times \mathbb{R}^n$ or in $X \times \overline{B}(y, r)$. There are several symmetrization procedures associated with the names of Steiner and Schwarz, see for instance [1] page 78. All have the effect of preserving the volume of a (sufficiently smooth) set without increasing its boundary area.

We consider here the following symmetrization procedure in an arbitrary ambient manifold $M$. Fix a Killing vector field $V$ which admits an orthogonal hypersurface $M_1 \subset M$. If $\Omega$ is the region which is to be symmetrized, then for each orbit $\gamma$ of $V$ one replaces the intersection $\gamma \cap \Omega$ with a segment $\gamma \Omega$ centered at the point $\gamma \cap M_1$ and having the same one-dimensional measure as $\gamma \cap \Omega$. If $\gamma \cap \Omega$ is empty, then let $\gamma \Omega$ be also empty. The symmetrized set

\[ S\Omega \overset{\text{def}}{=} \bigcup_{\gamma} \gamma \Omega \]

has the same volume as $\Omega$ and is symmetric with respect to $M_1$. We claim that if $\Omega$ is sufficiently regular then the boundary area of $S\Omega$ is at most that of $\Omega$.

For regions with enough regularity one has three equivalent notions of boundary area:
– The standard area of the regular part of $\partial \Omega$.
– Perimeter, see e.g. [6].
– Minkowski content, defined as:
  \[ \lim_{h \to 0} \frac{1}{h} \left( \text{Vol}(\Omega^h) - \text{Vol}(\Omega) \right) \],
  with $\Omega^h = \{ z \mid \text{dist}(z,\Omega) \leq h \}$.

Isoperimetric regions have enough regularity so that these three notions coincide, see e.g. [15].
For such regions we can now explain why $\text{area} (\partial S \Omega) \leq \text{area} (\partial \Omega)$. Using Minkowski content to compute boundary area, the claimed inequality follows from $(S \Omega)^h \subseteq S(\Omega^h)$. The proof of this inclusion in [1] pages 78-79 only requires that the flow of $V$ preserve distance and one-dimensional measure, hence it applies to the general setting we have described. See also [23, page 203] for explicit pictures.

The existence of the pair $V, M_1$ provides local coordinates $v_1, ..., v_m$ in $M$ with respect to which the metric is expressed as $g \equiv g_0 + g_{mm} dv_m^2$, where $g_0$ is a metric on $(v_1, ..., v_{m-1})$-space and the function $g_{mm}$ is independent of $v_m$. Then the argument in [1, pages 108-111] applies to show that $S \Omega$ has strictly less boundary area than $\Omega$ unless $\Omega$ satisfies the following two conditions:

– $\Omega$ was already symmetric to start with (with respect to some image of $M_1$ under the flow of $V$).
– $\Omega$ is “convex in the direction of $V$”. This means that each orbit of $V$ intersects $\Omega$ in an orbit segment or the empty set.

We apply these conclusions to $M = X \times \mathbb{R}^n$ and choose $V$ to be any constant vector field along the $\mathbb{R}^n$ factor.
In the case of a region $\Omega$ which is isoperimetric in $X \times \mathbb{R}^n$, we conclude there is a point $y \in \mathbb{R}^n$ such that $\Omega$ is symmetric with respect to all hypersurfaces

\[ X \times (\text{Euclidean hyperplane through } y) \]

and convex in the direction of the $\mathbb{R}^n$ factor, hence a union of coaxial balls with $X \times \{ y \}$ as common axis.
In the case $\Omega$ is isoperimetric in $X \times \overline{B}(y, r)$, we let $P \subseteq \mathbb{R}^n$ be the Euclidean hyperplane through $y$ orthogonal to the direction of $V$ and choose $M_1 = X \times P$ as hypersurface orthogonal to $V$. We notice two properties of $X \times \overline{B}(y, r)$: it is symmetric with respect to $M_1$ and intersects any orbit of $V$ in a line segment. They imply that for every $\Omega \subseteq X \times \overline{B}(y, r)$ the symmetrized region $S \Omega$ is completely contained in $X \times \overline{B}(y, r)$. We conclude that there is a point $y' \in B(y, r)$ such that $\Omega$ is symmetric with respect to all hypersurfaces

\[ X \times (\text{Euclidean hyperplane through } y') \],

and convex in the direction of the $\mathbb{R}^n$ factor. Hence $\Omega$ is a union of coaxial balls \( \{ x \} \times \overline{B}(y', u(x)) \). Notice that if $u(x)$ achieves the value $r$ then necessarily $y' = y$. If $\max u < r$, then $y$ and $y'$ may be different.

We finally make a comment about the function $u$. Denote by $\Omega$ the region bounded by a soap bubble, or an isoperimetric region in $X \times \mathbb{R}^n$, or an isoperimetric region in $X \times \overline{B}(y, r)$. The interior $U$ of $\Omega$ is an open set in $X \times \mathbb{R}^n$, thus its image under the projection to the $X$ factor is an open set $A \subseteq X$. For each $x \in A$ the value $u(x)$ must be positive, because the intersection of $U$ with the slice $\{ x \} \times \mathbb{R}^n$ must be a non-empty open ball. Therefore $U$ is the union of the non-empty open balls
\{x\} \times B(y, u(x)) as x ranges over A. Such a union of balls is open in \(X \times \mathbb{R}^n\) if and only if \(u\) is lower semicontinuous. We are going to see in the next section that \(u\) is actually much more regular.

4. Regularity

Let \(u\) and \(A\) be as described in Section \(\S\). In this section we prove that \(u\) is \(C^1\) in \(A\) and that it is smooth in \(A \cap \{u < r\}\). We also show that \(u\) is continuous in its whole domain (the closure of \(\overline{A}\)) and vanishes on the frontier \(\overline{A} \setminus A\).

Let \(S\) be a soap bubble or an isoperimetric boundary, in all of \(X \times \mathbb{R}^n\) or in \(X \times \overline{B}(y_0, r)\). We may assume without loss of generality that \(y_0 = 0\) and that \(X \times \{0\}\) is the symmetry axis of \(S\).

Let \(\{\rho\} : X \times \mathbb{R}^n \to \mathbb{R}\) as follows:

\[
\rho(x, y) = \text{dist} \left( (x, y), X \times \{0\} \right) = \sqrt{y_1^2 + \cdots + y_n^2}.
\]

This function is smooth away from \(X \times \{0\}\).

The singular set of \(S\), if non-empty, is compact and projects to a compact set in \(X\) whose codimension in \(X\) is at least 7. The reason for this is that, due to the invariance of \(S\) under rotations of the \(\mathbb{R}^n\) factor, any image in \(X\) of a singular point comes from a whole \(S^{n-1}\)-worth of singular points on \(S\).

Denote by \(S_0\) the regular part of \(S \cap \{\rho > 0\}\). In \(S_0\) there is defined an outer unit normal \(\nu\). We denote by \(\rho_{\nu}\) the derivative of \(\rho\) along this normal.

Let \(\pi : S \to X\) be the restriction of the projection \(X \times \mathbb{R}^n \to X\). Let \(\Omega\) be the region bounded by \(S\). The interior of \(\Omega\) is the union of the coaxial balls \(\{x\} \times B(0, u(x))\) for some function \(u : A \to \mathbb{R}^+\) that may be described as \(u = \rho \circ \pi^{-1}\). At a point \(z \in S_0\) where \(\rho_{\nu} \neq 0\), the map \(\pi\) is a submersion; thus \(u\) is near \(\pi(z)\) as regular as \(S_0\) is near \(z\) (smooth or \(C^1\), depending on the case). On the other hand, if \(\rho_{\nu}(z) = 0\) then the gradient of \(u\) is infinite at \(\pi(z)\). In order to prove the regularity part of Theorem \(\|\|\) we study the vanishing of \(\rho_{\nu}\).

Lemma 7. Let \(M\) be a Riemannian manifold and \(S \subset M\) a hypersurface. Let \(\xi\) be a vector field on \(M\) and let \(\varphi_1\) be the flow of \(\xi\). Denote by \(H\) the mean curvature function of \(S\) and by \(H_i\) the same for \(\varphi_1(S)\). Write \(\nu\) for the unit normal of \(S\) and decompose \(\xi = \xi^\top + f \nu\). Then for each \(p \in S\) we have:

\[
\frac{\partial}{\partial t} \bigg|_{t=0} H_i(\varphi_1(p)) = \xi^\top_p H - \left( \text{Ric}(\nu, \nu) + |II|^2 \right)_p f(p) - (\Delta^S f)_p,
\]

where \(\text{Ric}\) is the Ricci tensor of \(M\) and \(II\) is the second fundamental form of \(S\).

If \(S\) is an isoperimetric boundary in \(X \times \overline{B}(0, r)\), let \(S_\infty\) denote \(S \cap \{\rho = r\}\). If \(S\) is a soap bubble or an isoperimetric boundary in \(X \times \mathbb{R}^n\), let \(S_\infty\) be just the empty set.

Proposition 8. In \(S_0 \setminus S_\infty\) the following identity holds:

\[
\Delta^S(\rho_{\nu}) = \left( \frac{n-1}{\rho^2} - \text{Ric}(\nu, \nu) - |II|^2 \right) \rho_{\nu},
\]

Proof. Both sides of equality \(\|\|\) are zero on the interior of the set \(S_0 \cap \{\rho_{\nu} = 0\}\). Also, both sides are smooth everywhere on \(S_0 \setminus S_\infty\). Therefore if we prove the identity on \((S_0 \setminus S_\infty) \cap \{\rho_{\nu} \neq 0\}\) it will also be true on the frontier points of \(S_0 \cap \{\rho_{\nu} = 0\}\) because these are limits of points where \(\rho_{\nu} \neq 0\).
Let \( z_0 = (x_0, y) \in S_0 \setminus S_\infty \) be a point with \( \rho_v(z_0) \neq 0 \). The function \( u \) is smooth in some neighborhood \( U_{x_0} \) of \( x_0 \) in \( X \), and a neighborhood \( S^{z_0} \) of \( z_0 \) in \( S \) is described as a rotated graph: \( \{ \rho = u(x), x \in U_{x_0} \} \). For any such rotated graph we have:

\[
H = \frac{n - 1}{u \sqrt{1 + |\nabla u|^2}} - \text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}},
\]

where gradient and divergence are taken in \( X \). In particular, the hypersurfaces \( S_t = \{ \rho = t + u(x), x \in U_{x_0} \} \) have the following mean curvatures:

\[
H_t = \frac{n - 1}{(t + u)^2} - \text{div} \frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} = H_0 - \frac{n - 1}{\rho^2} \rho_v t + O(t^2).
\]

But \( S_t \) is the image of \( S^{z_0} \) under the flow of \( \nabla \rho = (\nabla \rho)^T + \rho_v \nu \); thus Lemma 7 gives:

\[-\frac{n - 1}{\rho^2} \rho_v = (\nabla \rho)^T H_0 - \left( \text{Ric}(\nu, \nu) + |II|^2 \right) \rho_v - \Delta S(\rho_v) .
\]

In addition \( H_0 \) is constant, hence \( (\nabla \rho)^T H_0 \equiv 0 \). Thus (3) holds where \( \rho_v \neq 0 \). \( \square \)

Now (3) is a Schrödinger equation. By the results in [2], a non-negative solution to such equation that vanishes at a point must vanish everywhere. Notice that the object \( S \) is, by our hypotheses, the boundary of a region \( \Omega \) that is a union of coaxial balls \( \{ x \} \times B(0, u(x)) \). Since \( \nu \) points outward with respect to \( \Omega \), it is \( \rho_v \geq 0 \) everywhere on \( S_0 \). We conclude that on each connected component of \( S_0 \setminus S_\infty \) we have either \( \rho_v \equiv 0 \) or \( \rho_v > 0 \).

If \( S \) is a soap bubble or an isoperimetric boundary in \( X \times \mathbb{R}^n \), then we cannot have \( \rho_v \equiv 0 \) on a connected component of \( S_0 \), because then a whole orbit of \( \nabla \rho \) would be part of \( S \) and this contradicts boundedness. In fact, some orbits may be interrupted by the singular points of \( S \) but not all orbits, due to the large codimension of the singular set.

If \( S \) is an isoperimetric boundary in \( X \times \overline{B}(0, r) \) and \( S_0 \setminus S_\infty \) has a connected component with \( \rho_v \equiv 0 \), then such component must reach the obstacle \( X \times S(0, r) \) and not be tangent to it, which contradicts the standard regularity for obstacle problems [2].

We conclude that \( \rho_v > 0 \) on all of \( S_0 \setminus S_\infty \). If \( S \) is a soap bubble, this already implies that \( u \) is smooth on all of \( A \).

If \( S \) is isoperimetric in \( X \times \overline{B}(0, r) \), we let \( A' = \{ x \in A \mid u(x) < r \} \); in this case \( S \cap \{ \rho = r \} \) is compact and \( A' \) is open because it equals \( A \setminus \pi(S \cap \{ \rho = r \}) \). If \( S \) is isoperimetric on \( X \times \mathbb{R}^n \), we let \( A' = A \). There is a subset \( K \subset A' \), of codimension at least 7 in \( A' \), such that \( u \) is smooth in \( A' \setminus K \). In addition \( K \) is locally closed, because it is the intersection of the open set \( A' \) with the (compact) image under \( \pi \) of the singular set.

We can switch from a rotated graph description to a Cartesian graph description. If \( (y_1, \ldots, y_n) \) are orthonormal coordinates in \( \mathbb{R}^n \), then \( S_0 \setminus (S_\infty \cup \{ y_n = 0 \}) \) is the union of two graphs \( \{ y_n = \pm f(x, y_1, \ldots, y_{n-1}) \} \), where \( f \) is the following function:

\[
f(x, y_1, \ldots, y_{n-1}) = \sqrt{u(x)^2 - y_1^2 - \cdots - y_{n-1}^2},
\]
A variation of area of pieces of $S$.

A preliminary estimate (15) comes from computing in two different ways the first variation of area. Let $u$ be a vector field parallel to the normal field of $S$. We first consider the obstacle case. It will be trivial to adapt the argument to the other two cases.

Given a tiny element of hypersurface with normal unit vector $v$, the first variation of area is

$$\frac{\partial}{\partial t} \varphi_t(0) = \int_S \left( \frac{\partial}{\partial t} \varphi_t(x) - \varphi_t(x) \frac{\nabla}{\nabla v} \varphi_t(x) \right) \, d\sigma,$$

where $\varphi_t(x)$ is the area of the hypersurface $S_t$ at time $t$. It is clear that $A' \times \{0\} \subset U$. What we have proved so far implies that $f$ is smooth except perhaps on a locally closed set whose codimension in $X \times \mathbb{R}^{n-1}$ is at least 7.

De Giorgi and Stampacchia have a theorem [5] which says that if $f$ is a $C^2$ function defined on $U \setminus K$, with $U \subset \mathbb{R}^d$ open and $K \subset U$ compact of codimension greater than 1, and if the graph of $f$ is minimal in $\mathbb{R}^{d+1}$, then $f$ extends to a $C^2$ function on all of $U$. L. Simon [25] has improved this theorem, so that we only need $K$ to be locally closed in $U$ and the graph of $f$ over $U \setminus K$ may satisfy a PDE of some general type which includes the case of constant mean curvature in $U \times \mathbb{R}$, with the factor $U$ having any smooth Riemann metric.

We can apply L. Simon’s theorem to the function $f|_{U \setminus K}$ defined by (10). It thus extends to a function $\tilde{f}$ which is smooth in all of $U$. We know from Section 3 that the function $u$ is lower semicontinuous; this alone does not force it to coincide with $\tilde{f}(x,0)$, but it did not coincide then the region $\Omega$ would not meet its boundary in the nice way that isoperimetric regions do, see e.g. [15]. Finally we have proved that $u$ is smooth on all of $A'$. In the case $A' \neq A$, at least we have $u$ of class $C^1$ on all of $A$ because $\rho_u = 1$ everywhere on $S \setminus \{\rho = r\}$.

We consider now what happens at the points $z_0 = (x_0,0) \in S \setminus (X \times \{0\})$, where the object $S$ meets its symmetry axis. Suppose there is a sequence $\{z_j\} \subset S_0$ that converges to $z_0$ and satisfies $\lim \rho(z_j) = \delta > 0$. Then the Euclidean ball $\{x_0\} \times B(0,\delta)$ is entirely contained in $S$ and we have $\rho_u = 0$ at points on $S$ near the axis but not on the axis. Such points belong to $S_0 \setminus S_\infty$ and we have a contradiction.

Therefore, along any sequence converging to $z_0$ we have $\liminf \rho(z_j) = 0$. Since $u$ is lower semicontinuous and non-negative, we conclude that $u$ extends continuously from $A$ to the closure $\overline{A}$ and its value on the frontier $\overline{A} \setminus A$ is identically zero.

The proof of Theorem 1 is now complete.

5. Estimates for isoperimetric boundaries

In this section we prove the isoperimetric boundary part of Theorem 5.

Recall from Section 3 that $\rho : X \times \mathbb{R}^n \to \mathbb{R}$ is given by $\rho(x,y) = |y|$. Again $S = \partial \Omega$ will be a soap bubble or an isoperimetric boundary, in all of $X \times \mathbb{R}^n$ or in $X \times \overline{B}(0,r)$. We refer to the last possibility as the obstacle case, because is such case $X \times S(0,r)$ acts as an obstacle that the isoperimetric region $\Omega$ may hit. In the three cases we assume $X \times \{0\}$ to be the symmetry axis of $\Omega$ and $S$.

We now know that $S \cap \{\rho > 0\}$ is the rotated graph of a differentiable function $u : A \to \mathbb{R}^+$. In the obstacle case $u$ is $C^1$ in all of $A$ and smooth in $\{u < r\}$. In the other two cases $u$ is smooth on all of $A$.

Let $k = \dim X$ and $d = k + n = \dim(X \times \mathbb{R}^n)$. We shall write “area” for $(d-1)$-dimensional Hausdorff measure.

A preliminary estimate [15] comes from computing in two different ways the first variation of area of pieces of $S$ under the flow of $\xi = y_1 \partial_{y_1} + \cdots + y_n \partial_{y_n}$, a vector field parallel to the $\mathbb{R}^n$ factor and vanishing along the axis $X \times \{0\}$.

Given a tiny element of hypersurface with normal unit vector $v$, the flow of a given vector field $\xi$ modifies the area of such piece at the rate $\text{div} \xi - v \cdot \nabla_v \xi$, and so the first variation of area is $\int_S (\text{div} \xi - v \cdot \nabla_v \xi)$ for any hypersurface $\Sigma$, compact or non-compact.

We first consider the obstacle case. It will be trivial to adapt the argument to the other two cases.
The hypersurface $S \cap \{\rho > 0\}$ is transverse to almost all level sets of $\rho$. Hence $S(\varepsilon) = S \cap \{\varepsilon \leq \rho \leq r - \varepsilon\}$ is a compact hypersurface with smooth boundary for almost every $\varepsilon \in (0, r/2)$. Denoting by $\eta_\varepsilon$ the outer conormal of $\partial S(\varepsilon)$ within $S$, and applying formula (5) of Section 2 to $S(\varepsilon)$, the result is:

$$\int_{S(\varepsilon)} (\nabla \cdot \nu \cdot \nu \cdot \xi) = \int_{S(\varepsilon)} H \nu \cdot \xi + \int_{S \cap \{\rho = \varepsilon\}} \xi \cdot \eta_\varepsilon + \int_{S \cap \{\rho = \rho - \varepsilon\}} \xi \cdot \eta_\varepsilon,$$

where the last term is non–negative because $S$ is a rotated graph $\{\rho = u(x)\}$ and $\xi \equiv \rho \nabla \rho$. We thus have:

$$\int_{S(\varepsilon)} (\nabla \cdot \nu \cdot \nu \cdot \xi) \geq \int_{S(\varepsilon)} H \nu \cdot \xi + \int_{S \cap \{\rho = \varepsilon\}} \xi \cdot \eta_\varepsilon.$$

The vector field $\xi$ satisfies $|\xi| = \varepsilon$ along $\{\rho = \varepsilon\}$ and the last term in (11) is bounded in absolute value by $\varepsilon \cdot \mathcal{H}^{d-2}(S \cap \{\rho = \varepsilon\})$. For each $t \in (0, r)$ define $\delta(t) = \inf_{0<\varepsilon<t} \varepsilon \cdot \mathcal{H}^{d-2}(S \cap \{\rho = \varepsilon\})$. The coarea formula gives:

$$\infty > \text{area}(S \cap \{0 < \rho < t\}) \geq \int_0^t \mathcal{H}^{d-2}(S \cap \{\rho = \varepsilon\}) \ d\varepsilon \geq \int_0^t \frac{\delta(t)}{\varepsilon} \ d\varepsilon,$$

which implies $\delta(t) = 0$ for all $t$. We deduce the existence of a sequence $\varepsilon_j \to 0$ such that $\int_{S \cap \{\rho = \varepsilon_j\}} \xi \cdot \eta_{\varepsilon_j} \to 0$. In the limit as $j \to \infty$, inequality (11) thus becomes:

$$\int_{S \cap \{0 < \rho < r\}} (\nabla \cdot \nu \cdot \nu \cdot \xi) \geq \int_{S \cap \{0 < \rho < r\}} H \nu \cdot \xi.$$

**Definition.** For each $t \in [0, r]$, define $X_t = \{x \in X \mid u(x) \geq t\}$, which is a region with smooth boundary for almost every $t$.

The **thick part** of an isoperimetric region $\Omega$ in $X \times \overline{B}(0, r)$ is $\Omega_{\text{thick}} = X_r \times B(0, r)$. See Figure 2.

![Figure 2. The thick part of $\Omega$.](image)

For any vector $\mathbf{v}$ tangent to $X \times \mathbb{R}^n$, the derivative $\nabla \nu \xi$ is the orthogonal component of $\mathbf{v}$ in the direction of the $\mathbb{R}^n$ factor. In particular $0 \leq \nu \cdot \nabla \nu \xi \leq 1$. It follows that:

$$n - 1 \leq \nabla \mathbf{v} \cdot \nabla \nu \xi \leq n.$$

Now (12) and the second inequality in (13) give:

$$n \cdot \text{area}(S) \geq \int_{S \cap \{0 < \rho < r\}} (\nabla \cdot \nu \cdot \nu \cdot \xi) \geq \int_{S \cap \{0 < \rho < r\}} H \nu \cdot \xi.$$

We assume that $X_r$ is a smooth region. If it is not, we can reach the same results by passing to a limit as $t \nearrow r$. The outer normal $\nu$ of $\Omega \setminus \Omega_{\text{thick}}$ coincides with $\nu$. 
along $S \cap \{0 < \rho < r\}$ and with $-\nu_r$ along $(\partial X_r) \times B(0, r)$, where $\nu_r$ is the obvious lift of the outer unit normal of $X_r$ in $X$. Since $\xi \cdot \nu_r = 0$, we have:

$$\int_{S \cap \{0 < \rho < r\}} \nu \cdot \xi = \int_{\partial (\Omega \setminus \Omega_{\text{thick}})} \theta \cdot \xi .$$

Using the divergence theorem, and writing $v$ for the volume of $\Omega$, we get:

$$\int_{S \cap \{0 < \rho < r\}} H \nu \cdot \xi = \int_{\partial (\Omega \setminus \Omega_{\text{thick}})} \theta \cdot \xi = n H \cdot \left( v - \mathcal{H}^d(\Omega_{\text{thick}}) \right) ,$$

and this together with inequality (14) yields our preliminary estimate for the mean curvature:

$$\left( v - \mathcal{H}^d(\Omega_{\text{thick}}) \right) H \leq \text{area} (S) .$$

This is for the obstacle case, but if $S$ is a soap bubble or is isoperimetric in all of $X \times \mathbb{R}^n$ then $\Omega_{\text{thick}}$ is empty and (15) simplifies to:

$$v H \leq \text{area} (S) .$$

We now prove the estimate (1) in Theorem 5. When $\Omega$ is an isoperimetric region in all of $X \times \mathbb{R}^n$, it has less boundary area than the cylinder $X \times (\text{ball})$ of the same volume, that is:

$$\text{area} (S) \leq n (\omega_n |X|)^{1/n} \cdot v^{n-1} ,$$

and this, combined with inequality (16), gives (1).

Inequality (2) in Theorem 5 follows in exactly the same way if $v - \mathcal{H}^d(\Omega_{\text{thick}})$ is greater than $v/2$ for $r$ sufficiently large. The next lemma thus finishes the proof of (2).

**Lemma 9.** For sufficiently large $r$, depending on $v$, the part $\Omega_{\text{thick}}$ contains less than half the volume of $\Omega$.

**Proof.** We assume that $v/2 \leq \mathcal{H}^d(\Omega_{\text{thick}})$ and derive an upper bound for $r$.

The hypothesis $v/2 \leq \mathcal{H}^d(\Omega_{\text{thick}})$ is equivalent to $\frac{1}{2^n} v r^{-n} \leq \mathcal{H}^k(X_r)$.

Notice that $\mathcal{H}^k(X_t)$ is a decreasing function of $t$. In particular, for any $s \in [0, \frac{r}{2}]$ and any $t \in [\frac{r}{2}, r]$ we have $\mathcal{H}^k(X_s) \geq \mathcal{H}^k(X_t)$, whence:

$$v = \mathcal{H}^d(\Omega) = \int_0^r n \omega_n s^{n-1} \mathcal{H}^k(X_s) ds \geq \omega_n \mathcal{H}^k(X_t) \int_0^{r/2} n s^{n-1} ds .$$

We then have, for all $t \in [\frac{r}{2}, r]$, the following inequalities:

$$\frac{1}{2^n} v r^{-n} \leq \mathcal{H}^k(X_r) \leq \mathcal{H}^k(X_t) \leq \frac{2^n}{\omega_n} v r^{-n} .$$

Make now the extra hypothesis $r^n \geq \frac{2^{n+1}}{\omega_n} \frac{v}{\mathcal{H}^k(X)}$, which by the last inequality in (18) ensures:

$$\mathcal{H}^k(X_t) \leq \frac{1}{2} \mathcal{H}^k(X) \quad \text{for} \quad t \in [\frac{r}{2}, r] .$$

Let $c_X$ be the **isoperimetric constant** of $X$, so that for every region $Y \subset X$ we have:

$$\mathcal{H}^{k-1}(\partial Y) \geq c_X \cdot \max \left( \mathcal{H}^k(Y), \mathcal{H}^k(X \setminus Y) \right)^{\frac{k-1}{k}} .$$
Again we assume the second inequality in (3) follows from (2).

The projection \( t \) almost every \( \varepsilon \).

Define:

\[
\H^{k-1}(\partial X_i) \geq c_X \cdot \left( \H^k(X_i) \right)^{k-1} \geq \text{const} \cdot (v \cdot r^{-n})^{k-1}.
\]

The coarea formula gives in turn the following estimate:

\[
\text{area}(S) > \int_{r/2}^{r} n \omega_n t^{n-1} \H^{k-1}(\partial X_i) \, dt \geq \text{const} \cdot r^n \cdot (v \cdot r^{-n})^{k-1},
\]

which implies:

\[
r^n \leq \text{const} \cdot v^{1-k} \cdot \left( \text{area}(S) \right)^k \leq \text{const} \cdot v^{1-(k/n)},
\]

the last inequality coming from (17).

From the hypothesis \( v/2 \leq \H^d(\Omega_{\text{thick}}) \) we have deduced that either \( r^n < \frac{2^n + 1}{\omega_n} \H^d(X) \) or \( r^n \leq \text{const} \cdot v^{1-(k/n)} \). These are two upper bounds for \( r \) which depend only on \( v \). Hence, for \( r \) larger than these bounds it must be \( \H^d(\Omega_{\text{thick}}) < v/2 \).

This proves Lemma 9 and, as we have explained, estimate (2) in Theorem 2. \( \square \)

We shall now prove estimate (3) for an isoperimetric boundary \( S \) in \( X \times \B(0, r) \).

Again we assume \( X \times \{0\} \) is the symmetry axis of \( S \).

We know that \( S \) has constant mean curvature in \( \{0 < r < r\} \). Assuming the volume \( v \) enclosed by \( S \) to be larger than some arbitrary value \( v_0 \), and \( r \) large enough for \( v \), the estimate (2) provides an upper bound \( H_0 \) for said constant mean curvature. Taking this mean curvature bound to the monotonicity inequality (16), we obtain constants \( \varepsilon \) and \( \delta \) such that:

\[
\text{if } z_0 \in S \cap \{ \varepsilon < \rho < r - \varepsilon \} \text{ then } \text{area} \left( S \cap B^{X \times \R^n}(z_0, \varepsilon) \right) \geq \delta.
\]

Define:

\[
\rho_0 = \min(\rho|_S), \quad \rho_1 = \max(\rho|_S).
\]

The projection \( X \times \R^n \rightarrow \R^n \) maps \( S \) onto a Euclidean ring with radii \( \rho_0 \) and \( \rho_1 \). Euclidean space \( \R^n \) has a packing constant \( C(n) \) such that a ring with those radii can pack \( \ell \) disjoint Euclidean balls of radius \( \varepsilon \), where \( \ell \geq C(n) \cdot (\rho_1^n - \rho_0^n)/\varepsilon^n \). These balls lift to disjoint distance balls in \( X \times \R^n \), centered at points of \( S \cap \{ \varepsilon < \rho < r - \varepsilon \} \).

It follows that:

\[
\text{area}(S) \geq \text{const} \cdot (\rho_1^n - \rho_0^n),
\]

the constant depending only on \( n \) and \( X \). Comparing \( S \) with a cylinder \( X \times (\text{sphere}) \) that encloses the same volume, we get \( \text{area}(S) \leq \text{const} \cdot \rho_1^{n-1} \), therefore:

\[
\rho_1^n - \rho_0^n \leq \text{const} \cdot \rho_1^{n-1},
\]

with the constant depending only on \( n \) and \( X \). Observing that:

\[
\rho_1^n - \rho_0^n = (\rho_1 - \rho_0) \cdot (\rho_1^{n-1} + \rho_1^{n-2} \rho_0 + \cdots + \rho_0^{n-1}) \geq (\rho_1 - \rho_0) \cdot \rho_1^{n-1},
\]

we deduce from (21) the inequality \( (\rho_1 - \rho_0) \cdot \rho_1^{n-1} \leq \text{const} \cdot \rho_1^{n-1} \), which is equivalent to the first inequality in (3). This estimate makes \( v \) comparable to \( \rho_1^n \), thus the second inequality in (3) follows from (2).
Remark. The constant in (21) really depends on \( n, X, \) and the chosen value \( v_0. \) This means that we have an estimate valid also for small enclosed volumes:

\[
\rho_1 - \rho_0 \leq \text{const}_v,
\]

where the constant depends on \( v \) but not on the radius \( r \) of the domain \( X \times B(0, r) \) where \( S \) is isoperimetric.

We also have proved now estimate (3) for regions isoperimetric in \( X \times \mathbb{R}^n \) of volume larger than \( v_0 \), because they are isoperimetric in any compact domain.

6. Estimates for soap bubbles

In this section we prove the soap bubble part of Theorem 5.

We start with the upper bounds for mean curvature: one under the hypothesis \( \text{Ric} \geq 0, \) the other under no special hypothesis. These bounds for \( H \) will in turn allow us to get the radius bounds. The idea for the mean curvature estimates is that large \( H \) would force \( S \) to “roll up” and bound a region of small volume. The first result along these lines was obtained by J. Serrin [24, pages 85-87] for surfaces in \( \mathbb{R}^3 \).

Serrin uses a formula of G. Darboux for parallel surfaces, and deals with the possible singularity of a parallel surface at a focal point. W. Meeks has a similar result in [16, page 544] for hypersurfaces in \( \mathbb{R}^n \). His calculation is equivalent to that of Serrin, but he works in the hypersurface instead of its parallel image and the singularities do not show up. He considers a height function \( x_n \) and the corresponding component \( \nu_n \) of the unit normal, then observes that (with our convention for \( H \)) the function \( x_n - \frac{n-1}{H} \nu_n \) is subharmonic in the hypersurface. If \( x_n|_{\partial S} \equiv 0, \) then \( x_n \leq \frac{n-1}{H} \) by the maximum principle. Intuitively, if the hypersurface is strongly curved then it cannot reach far out in a given direction.

We are going to imitate that argument here. We shall multiply a component of \( \nu \) by a constant parameter \( p \), then we choose suitable values for this parameter.

For any function \( \varphi \) on a domain of \( X \times \mathbb{R}^n \), define the tangential Laplacian as follows:

\[
\Delta^\top \varphi = \sum_{j=1}^{d-1} \text{Hess}(\varphi)(e_j, e_j),
\]

where \( e_1, \ldots, e_{d-1} \) is an orthonormal basis of \( TS \). With our convention for \( H \), the following holds:

\[
\Delta^S \rho = \Delta^\top \rho - H \rho_\nu \geq -H \rho_\nu,
\]

because the Hessian of \( \rho \) is positive semidefinite. This and formula (8) of Section 4 imply that for any constant \( p \) we have:

\[
\Delta^S (\rho - p \rho_\nu) \geq \left( p |II|^2 + p \text{Ric}(\nu, \nu) - p \frac{n-1}{\rho^2} - H \right) \rho_\nu.
\]

Restrict to \( p > 0, \) and recall that \( \rho_\nu > 0. \) Introduce now the hypothesis \( \text{Ric} \geq 0, \) then (24) simplifies to:

\[
\Delta^S (\rho - p \rho_\nu) \geq \left( p |II|^2 - H - p \frac{n-1}{\rho^2} \right) \rho_\nu,
\]

and Newton’s inequality \( |II|^2 \geq H^2/(d-1) \) leads to:

\[
\Delta^S (\rho - p \rho_\nu) \geq \left( p \frac{H^2}{d-1} - H - p \frac{n-1}{\rho^2} \right) \rho_\nu.
\]
Again let $\rho_1, \rho_0$ be the maximum and minimum, respectively, of $\rho$ over $S$. The useful choice here is $p = \rho_1/4$, then:

$$\Delta^S \left( \rho - \frac{\rho_1}{4} \rho_\nu \right) \geq \left( \frac{H^2}{4(d-1)} \rho_1 - H - \frac{n-1}{4 \rho^2} \rho_1 \right) \rho_\nu.$$  

Consider the hypersurface piece $\Sigma = S \cap \{ \rho \geq \rho_1/2 \}$. The boundary $\partial \Sigma$ may be empty, unless $\rho_0 < \rho_1/2$. At all points of $\partial \Sigma$ (if any) we have $\rho - (\rho_1/4) \rho_\nu \leq \rho_1/2$ while there are points on the interior of $\Sigma$ where $\rho - (\rho_1/4) \rho_\nu > \rho_1/2$. For example, a point $z \in S$ where $\rho(z) = \rho_1$ is interior to $\Sigma$ and gives $(\rho - (\rho_1/4) \rho_\nu) z = (3/4) \rho_1$. Therefore the maximum of $\rho - (\rho_1/4) \rho_\nu$ over $\Sigma$ is achieved at an interior point $z_0$. At $z_0$ we have $4 \rho^2 > \rho_1^2$ and $\Delta^S (\rho - (\rho_1/4) \rho_\nu) \leq 0$, hence:

$$(25) \quad 0 \geq \left( \frac{H^2}{4(d-1)} \rho_1 - H - \frac{n-1}{4 \rho^2} \rho_1 \right) (z_0) \geq \frac{H^2}{4(d-1)} \rho_1 - H - \frac{n-1}{\rho_1}.$$  

Suppose $n \geq 2$. The hypersurface $\{ \rho = \rho_1 \}$ touches $S$ tangentially from outside and has constant mean curvature $(n-1)/\rho_1$, thus $H \geq (n-1)/\rho_1 > 0$. Taking this lower bound for $H$ to (25), we obtain:

$$0 \geq \frac{H^2}{4(d-1)} \rho_1 - 2H,$$

and since $H > 0$ we deduce $H \leq 8 (d-1)/\rho_1$. If the ambient space is $X \times \mathbb{R}$, then $n-1 = 0$ and (25) reduces to:

$$0 \geq \frac{H^2}{4(d-1)} \rho_1 - H,$$

then $H$ is either 0 or a positive number not greater than $4 (d-1)/\rho_1$, in either case $H \leq 4 (d-1)/\rho_1 = (4/\rho_1) \cdot \dim(X)$.

We now prove the mean curvature bound when $S$ is a (non-isoperimetric) soap bubble enclosing volume $v$ and the Ricci curvature of $X$ is negative somewhere. Again we shall have to separate the case $n \geq 2$ from the case $n = 1$. We introduce the constant:

$$(26) \quad R_0 = \max_{|\nu|=1} \left( - \text{Ric} (\nu, \nu) \right)^+. $$

The number defined by (26) is the same whether we consider Ric as the Ricci tensor of $X$ and $\nu$ ranging over unit tangent vectors to $X$, or we consider Ric as the Ricci tensor of $X \times \mathbb{R}^n$ and $\nu$ ranging over unit tangent vectors to $X \times \mathbb{R}^n$. Notice that $\text{Ric} \geq 0$ is equivalent to $R_0 = 0$.

From (24) and Newton’s inequality, we now deduce for $p > 0$:

$$\Delta^S (\rho - p \rho_\nu) \geq \left( p \frac{H^2}{d-1} - H - R_0 p - p \frac{n-1}{\rho^2} \right) \rho_\nu.$$  

As pointed out above, if $n \geq 2$ then $H \geq (n-1)/\rho_1 > 0$. Let us see that the choice $p = d/H$ is useful in this situation, leaving the case $n = 1$ for later. First we obtain:

$$\Delta^S \left( \rho - \frac{d}{H} \rho_\nu \right) \geq \left( \frac{H}{d-1} - \frac{d R_0}{H} - \frac{(n-1) d}{H \rho^2} \right) \rho_\nu.$$  

Assume \( \rho_1 \geq 2 \). In particular, the hypersurface \( \Sigma = S \cap \{ \rho \geq 1 \} \) is non-empty. On \( \Sigma \) one has:

\[
\Delta^S \left( \rho - \frac{d}{H} \rho \nu \right) \geq \left( \frac{H}{d-1} - \frac{d R_0}{H} - \frac{(n-1)d}{H} \right) \rho \nu .
\]

The factor multiplying \( \rho \nu \) in the right-hand side is a strictly increasing function of \( H > 0 \), and it equals 0 for a unique positive value \( H_0 \) of \( H \). Let us see that for large enclosed volume we have \( H \leq H_0 \).

Suppose \( H > H_0 \), then \( \rho - (d/H) \rho \nu \) is strictly subharmonic on \( \Sigma \). This is impossible if \( \partial \Sigma \) is empty. If \( \partial \Sigma = S \cap \{ \rho = 1 \} \) is non-empty, then the maximum of \( \rho - (d/H) \rho \nu \) on \( \Sigma \) is reached somewhere on \( \partial \Sigma \). It follows that \( \rho \leq 1 + (d/H) < 1 + (d/H_0) \) on all of \( S \), which cannot be true if \( S \) encloses a large enough volume. Thus \( n \geq 2 \) plus large enclosed volume forces \( H \leq H_0 \).

Assume now \( n = 1 \). In this case, for each constant \( t \) the hypersurface \( \{ \rho = t \} \) is minimal. If \( \rho_0 > 0 \), then \( S \) is sandwiched between the two minimal hypersurfaces \( \{ \rho = \rho_0 \} \) and \( \{ \rho = \rho_1 \} \) that touch \( S \) tangentially; this implies \( 0 \leq H \leq 0 \), thereby forcing \( S \) to be minimal and, by the maximum principle, \( \rho_1 = \rho_0 \). Hence \( S \) must be of the form \( X \times \{-t,t\} \).

The remaining case is \( n = 1 \) and \( \rho_0 = 0 \). We know that \( S \) is a symmetric graph \( \{ \rho = \pm u(x) \} \), with \( x \) ranging over a proper subset \( X_0 \subset X \). This is, for instance, the situation for the surfaces \( S_v \) of Theorem 4. The minimal hypersurface \( \{ \rho = 0 \} \) is not tangent to \( S \) now, but \( \{ \rho = \rho_1 \} \) still is. Hence \( H \geq 0 \), and in fact it must be \( H > 0 \) by the maximum principle. Then formula (27) is valid again, adopting the following form:

\[
\Delta^S \left( \rho - \frac{d}{H} \rho \nu \right) \geq \left( \frac{H}{d-1} - \frac{d R_0}{H} \right) \rho \nu .
\]

We define \( H_0 \) by \( H_0 = \frac{d R_0}{H} \) and we deduce, as before, that \( H \leq H_0 \) for large enclosed volume.

Having proved the mean curvature bounds in (3) and (4), we shall now prove the radius bounds. Since \( S \) is supposed to be not too small, we have \( H \) less than some constant; then monotonicity plus a sphere packing argument, as we did in Section 5, yields again a lower area bound like (20) of Section 5. We need some upper bound for area in order to arrive at a radius estimate.

Once more we assume \( n \geq 2 \) and leave the \( n = 1 \) case for later. Since \( S \) is closed and everywhere smooth, we have:

\[
\int_S (\text{div} \xi - \nu \cdot \nabla_x \xi) = \int_S H \nu \cdot \xi .
\]

In Section 5 we used the second inequality in formula (13); now we use the first inequality in that formula, together with equality (28) and the divergence theorem, to deduce \( (n-1) \text{area}(S) \leq n H \nu \), and we can divide by \( n-1 \geq 1 \), to get:

\[
\text{area}(S) \leq \frac{n}{n-1} H \nu .
\]

If \( \text{Ric} \geq 0 \), then we have \( H \leq \text{const}/\rho_1 \) which transforms (29) into an inequality:

\[
\text{area}(S) \leq \text{const} \cdot \rho_1^{n-1},
\]

even though we do not assume \( S \) to be isoperimetric. We then obtain the radius oscillation estimate in (3) exactly as we did in Section 5.
If Ric is negative somewhere, then we only have $H \leq \text{const}$ and all we can get is the estimate:

$$\rho_1^n - \rho_0^n \leq \text{const} \cdot \text{area}(S) \leq \text{const} \cdot v,$$

and we consider the following dichotomy:

- if $\rho_1/\rho_0 > 2$, then $(1 - (1/2)^n) \rho_1^n \leq \text{const} \cdot v$,
- if $\rho_1/\rho_0 \leq 2$, then $v \geq \omega_n |X| \rho_0^n \geq \omega_n |X| (1/2)^n \rho_1^n$.

In either case we deduce $\rho_1 \leq \text{const} \cdot v^{1/n}$. This completes the proof of (4) for $n \geq 2$.

Suppose now that $n = 1$. We still have $H$ bounded above by a constant and, by monotonicity, an inequality:

$$(30) \quad \text{area}(S) \geq \text{const} \cdot (\rho_1 - \rho_0).$$

We again need an upper bound for area. Formula (19) of Section 2 now reduces to $H = -\text{div} \left( \nabla u / \sqrt{1 + |\nabla u|^2} \right)$, involving only the divergence term. We take advantage of this by doing an integration by parts. Given a value $s > 0$, The function $u - s$ vanishes along the boundary of $X_s = \{ u \geq s \}$ and so:

$$\int_{X_s} (u H + 1) \geq \int_{X_s} (u - s) H + 1 = \int_{X_s} \left( \frac{|\nabla u|^2}{\sqrt{1 + |\nabla u|^2}} + 1 \right) \geq \int_{X_s} \sqrt{1 + |\nabla u|^2} = \frac{1}{2} \text{area}(S \cap \{ \rho \geq s \}),$$

for almost every $s > 0$. By letting $s \to 0$ we obtain:

$$(31) \quad \text{area}(S) \leq v H + 2 |X_0|.$$

If $\text{Ric} \geq 0$, then $H \leq \text{const}/\rho_1$ and (31) becomes $\text{area}(S) \leq \text{const}$. We take this to (30) and get the radius oscillation bound in (3).

If $\text{Ric}$ is negative somewhere, then $H \leq \text{const}$. Now (30) and (31) only give:

$$\rho_1 - \rho_0 \leq \text{const} \cdot v + 2 |X_0|,$$

which yields $\rho_1 - \rho_0 \leq \text{const} \cdot v$ for $v$ not too small. Then we consider the dichotomy $\rho_1/\rho_1 > 2$ or $\rho_1/\rho_0 \leq 2$, and in either case arrive at the radius bound in (3).

7. Existence

In this section we prove the following.

**Theorem 10.** In $X \times \mathbb{R}^n$ there are isoperimetric regions of every volume.

Fix a value $v > 0$ and for each $r$ let $\Omega(r)$ be any region isoperimetric of volume $v$ in $X \times B(0, r)$. These regions exist because the domains $X \times B(0, r)$ are compact. Up to a translation parallel to the $\mathbb{R}^n$ factor, we may assume that $X \times \{ 0 \}$ is the symmetry axis of all the $\Omega(r)$. Recall inequality (22) from Section 5 valid for $r$ large (depending on $v$) and where the constant depends on $v$ but not on $r$. The following calculation:

$$v \geq |X| \cdot \omega_n \cdot \rho_0^n \geq |X| \cdot \omega_n \cdot (\rho_1 - \text{const} v)^n,$$
provides a bound for $\rho_1 - \text{const}_v$ that depends on $v$ but not on $r$, thereby providing one such bound also for $\rho_1$. We have thus found a radius $r(v)$ such that for $r$ large enough the regions $\Omega(r)$ are contained in $X \times \overline{B}(0, r(v))$.

Choose a radius $r_0$ larger than $r(v)$ and large enough for $v$, and let $\Omega_0$ be any isoperimetric region in $X \times \overline{B}(0, r_0)$ of volume $v$, with symmetry axis $X \times \{0\}$. We claim that $\Omega_0$ is isoperimetric in all of $X \times \mathbb{R}^n$.

We first compare $\Omega_0$ with bounded regions. If $D$ is any bounded region in $X \times \mathbb{R}^n$ of volume $v$, there is an $r$ such that $D \subset X \times B(0, r)$ and $r$ is large enough for $v$. There is also a region $\Omega'$ isoperimetric of volume $v$ in $X \times \overline{B}(0, r)$, and so area $(\partial D) \geq \text{area} (\partial \Omega')$. By the above, a translate of the region $\Omega'$ is contained inside $X \times \overline{B}(0, r(v))$, where $\Omega_0$ is also isoperimetric of volume $v$. Therefore area $(\partial \Omega_0) = \text{area} (\partial \Omega')$, and so area $(\partial D) \geq \text{area} (\partial \Omega_0)$.

Let now $D'$ be an unbounded region of volume $v$ and finite boundary area. As $r$ goes to infinity the volume of $D'_r:= D' \cap (X \times B(0, r))$ approaches the volume of $D'$, and same for boundary area. For $r$ large choose a little ball $B$ in $X \times \mathbb{R}^n$, some distance apart from $D'_r$ and such that $D'_r \cup B$ has exactly volume $v$. Then

$$\text{area}(\partial \Omega_0) \leq \text{area}(\partial(D'_r \cup B)) = \text{area}(\partial D'_r) + \text{area} (\partial B),$$

and by letting $r \to \infty$ we get $\text{area}(\partial D) \geq \text{area}(\partial \Omega)$, due to $\text{area}(\partial B) \to 0$. In fact area $(\partial D') > \text{area}(\partial \Omega)$, because we proved in Section 2 that no unbounded region is isoperimetric in $X \times \mathbb{R}^n$.

8. Proof of Theorems 2 and 3

Let $S \subset X \times \mathbb{R}^n$. If $S$ is an isoperimetric boundary, or if $X$ has $\text{Ric} \geq 0$ and $S$ is a soap bubble, then for large enclosed volume $v$ we have estimate (3) from Theorem 3 this implies in particular that $\min u > 0$. Thus in these cases the function $u$, whose rotated graph is $S$, is defined and positive on all of $X$. Indeed, if it were $u : A \to \mathbb{R}^+$ with $A \neq X$ then the frontier of $A$ would be non-empty, and we saw at the end of Section 4 that $u$ would vanish there.

In view of this, Theorems 2 and 3 are corollaries of the following proposition.

**Proposition 11.** Fix a constant $C$. Let $u : X \to \mathbb{R}^+$ be a smooth function with the oscillation bound $\max u - \min u \leq C$ and such that the rotated graph $S = \{ \rho = u(x) \} \subset X \times \mathbb{R}^n$ has constant mean curvature. If $S$ encloses a sufficiently large volume (depending on $C$), then $u$ must be constant.

If $n = 1$ and $\min u > 0$, we have already explained in Section 6 that $S$ must be of the form $X \times \{-t, t\}$.

In the rest of this section we prove Proposition 11 for $n \geq 2$. Instead of the radius function $u$ we shall work with the slice volume function:

$$\sigma := \left( \frac{u}{n} \right)^n.$$

The choice of the factor $n^{-n}$ is not important, it just makes formulas a bit simpler. Consider the average $\overline{w^n} = (1/|X|) \int_X w^n$. The number $a = n^{-n} \cdot \overline{w^n}$ is the average of $\sigma$, thus:

$$\sigma \equiv a + \tau, \text{ for some function } \tau \text{ with } \int_X \tau = 0.$$
We fix the exponent $\alpha = \frac{n-1}{n} \in (0, 1)$. In terms of $\sigma$ we have:

\[
\text{area}\{ \{ \rho = u(x) \} \} = n^n \omega_n \int_X \sqrt{\sigma^{2\alpha} + |\nabla \sigma|^2}.
\]

Given a family \( \{ u_t \} \) of radius functions, and the corresponding family \( \{ \sigma_t \} \), we define \( \dot{\sigma} = \frac{d}{dt} |_{t=0} \sigma_t \). Direct differentiation in (32) gives:

\[
\frac{d}{dt} |_{t=0} \text{area}\{ \{ \rho = u_t(x) \} \} = n^n \omega_n \int_X \frac{\alpha \sigma^{2\alpha-1} \dot{\sigma} + \nabla \dot{\sigma} \cdot \nabla \sigma}{\sqrt{\sigma^{2\alpha} + |\nabla \sigma|^2}}.
\]

We consider the particular deformation \( S_t = \{ \rho = u_t(x) \} \) defined by:

\[
u_t = \left( \bar{u}^n + e^t (u^n - \bar{a}^n) \right)^{1/n},
\]

which satisfies \( S_0 = S \), and has \( \sigma_t = a + e^t \tau \), so that all \( S_t \) enclose the same volume. We must therefore have:

\[
0 = \int_X \frac{\alpha \sigma^{2\alpha-1} \tau + |\nabla \tau|^2}{\sqrt{\sigma^{2\alpha} + |\nabla \tau|^2}} \geq \int_X \alpha \sigma^{2\alpha-1} \tau + |\nabla \tau|^2.
\]

**Lemma 12.** There is a positive constant \( C'' \), depending only on \( n, X \), and the constant \( C \) from Proposition [7], such that for large enough enclosed volume we have:

\[
\int_X \frac{\sigma^{2\alpha-1} \tau + |\nabla \tau|^2}{\sqrt{\sigma^{2\alpha} + |\nabla \tau|^2}} \geq a^{-\alpha} \int_X \left( C'' |\nabla \tau|^2 - \frac{4}{n} \alpha^{2\alpha-2} \tau^2 \right) > 0.
\]

Using this lemma, we shall now finish the proof of Proposition [11]. Let \( \lambda_1(X) \) be the first eigenvalue of the Laplacian in \( X \) and observe that:

\[
\int_X \left( C'' |\nabla \tau|^2 - \frac{4}{n} \alpha^{2\alpha-2} \tau^2 \right) \geq \left( C'' - \frac{4}{n} \lambda_1(X) \alpha^{2\alpha-2} \right) \int_X |\nabla \tau|^2.
\]

The average \( a \) becomes arbitrarily large as the enclosed volume increases, and the coefficient \( C'' - \frac{4}{n} \lambda_1(X) \alpha^{2\alpha-2} \) becomes positive. Then the expression (36) is positive unless \( \nabla \tau \equiv 0 \), which forces \( \tau \) to be zero because it has zero average. Therefore, for large enclosed volume the integrals in (33) and (34) are positive unless \( \sigma \) and \( u \) are constants, which means that the equality in (34) only holds true if \( S = X \times \text{(sphere)} \). Proposition (11) is now proved, and also Theorems 2 and 3.

**Proof of Lemma 12.** Let us compare \( \sigma \) and \( \tau \) with \( a \), for large enclosed volume. Write \( u_1 \) for max \( u \) and \( u_0 \) for min \( u \), so we have \( u_1 - u_0 \leq C \). The inequalities:

\[
u_1 - C \leq u_0 \leq u_1 \leq u_0 + C
\]

give rise to the inequalities:

\[
\left(1 - \frac{C}{u_1}\right)^n \leq \left( \frac{u_0}{u_1} \right)^n \leq \frac{\sigma}{a} \leq \left( \frac{u_1}{u_0} \right)^n \leq \left(1 + \frac{C}{u_0}\right)^n.
\]

For large enclosed volume, \( u_0 \) and \( u_1 \) are arbitrarily large compared to \( C \), hence we may assume \( (1/2)a \leq \sigma(x) \leq 2a \) for all \( x \in X \).

The function \( u \to u^n \) has monotone increasing derivative, therefore:

\[
u_1^n - u_0^n = \frac{u_1^n - u_0^n}{n^n} \leq \frac{n}{n^n} (u_1 - u_0) u_1^{n-1} \leq C u_1^{n-1} u_0^{n-1} = C \left( \frac{u_1}{u_0} \right)^{n-1} \leq C (2a)^{\alpha} < 2 Ca^{\alpha},
\]
Theorem 5. Now Theorem 16 in the Appendix provides a constant which yields Lemma 12 by using Let $H$, where $C \sigma$ be an upper bound for mean curvature provided by the estimate (3 in $1$) it is $\sqrt{\tau} \geq \int_X (C'' a^{-\alpha} |\nabla \tau|^2 + a^\alpha - a \sigma - a \sigma^{-1})$. So we have $\tau$, we deduce:

$$|\tau| = |\sigma - a| \leq 2 C a^\alpha.$$

We estimate:

$$\frac{\sigma^{2\alpha-1} \tau + |\nabla \tau|^2}{\sqrt{\sigma^{2\alpha} + |\nabla \tau|^2}} = \sqrt{\frac{\sigma^{2\alpha}}{\sigma^{2\alpha} + |\nabla \tau|^2}} \geq \sqrt{\frac{\sigma^{2\alpha} |\nabla \tau|^2 - a \sigma^{2\alpha}}{\sigma^{2\alpha} + |\nabla \tau|^2}} \geq \sqrt{\frac{\sigma^{2\alpha} |\nabla \tau|^2 - a \sigma^{2\alpha}}{\sigma^{2\alpha} + |\nabla \tau|^2}} - a \sigma^{2\alpha}$$

and decompose the square root as $\sigma^\alpha$ plus a multiple of $|\nabla \tau|^2$:

$$\sqrt{\sigma^{2\alpha} + |\nabla \tau|^2} - \sigma^\alpha = \frac{|\nabla \tau|^2}{\sigma^{2\alpha} + |\nabla \tau|^2} + \sigma^\alpha.$$

Let $H_1$ be an upper bound for mean curvature provided by the estimate in Theorem 5. Now Theorem 16 in the Appendix provides a constant $C'$, depending only on $n, X, H_1, C$, such that $|\nabla u| \leq C'$. Thus:

$$\frac{|\nabla \tau|^2}{\sigma^{2\alpha} + |\nabla \tau|^2 + \sigma^\alpha} \geq \frac{|\nabla \tau|^2}{(1 + C'^2 + 1) \sigma^\alpha} \geq C'' a^{-\alpha} |\nabla \tau|^2,$$

where $C''$ is a positive constant that depends only on $n, X, C$. We now have:

$$\frac{\sigma^{2\alpha-1} \tau + |\nabla \tau|^2}{\sqrt{\sigma^{2\alpha} + |\nabla \tau|^2}} \geq C'' a^{-\alpha} |\nabla \tau|^2 + \sigma^\alpha - a \sigma^{-1} = C'' a^{-\alpha} |\nabla \tau|^2 + \sigma^{-1} \tau.$$

We further analyze:

$$\sigma^{\alpha-1} \tau = (a + \tau)^{\alpha-1} \tau = a^{\alpha-1} \left(1 + \frac{\tau}{a}\right)^{-1/n} \tau.$$

Since $|\tau| < 2 C a^\alpha$ and $\alpha < 1$, we may assume $-1/2 < \frac{\tau}{a} < 1/2$. But for $t \in (-1/2, 1/2)$ it is $\left|\frac{d}{dt} (1 + t)^{-1/n}\right| < \frac{4}{n}$. It follows that:

$$\left|1 - \frac{4}{n} \frac{\tau}{a}\right| \leq \left(1 + \frac{\tau}{a}\right)^{-1/n} \leq 1 + \frac{4}{n} \left|\frac{\tau}{a}\right|.$$

Where $\tau \geq 0$, use the first inequality in (37) and get:

$$\left(1 + \frac{\tau}{a}\right)^{-1/n} \tau \geq \tau - 4 \frac{\tau}{n} \frac{\tau}{a} \tau = \tau - 4 \frac{\tau^2}{n} \frac{\tau}{a}.$$

where $\tau < 0$, use the second inequality in (37) and get:

$$\left(1 + \frac{\tau}{a}\right)^{-1/n} \tau \geq \tau + 4 \frac{\tau}{n} \left|\frac{\tau}{a}\right| \tau = \tau + 4 \frac{\tau^2}{n} \frac{\tau}{a}.$$

So we have $\sigma^{\alpha-1} \tau \geq \sigma^{-1} \left(\tau - 4 \frac{\tau^2}{n} \sigma^{-1} \tau\right)$ everywhere, and we arrive at the inequality:

$$\int_X \frac{\sigma^{2\alpha-1} \tau + |\nabla \tau|^2}{\sqrt{\sigma^{2\alpha} + |\nabla \tau|^2}} \geq \int_X \left(C'' a^{-\alpha} |\nabla \tau|^2 + a^{\alpha-1} \tau - 4 \frac{\tau^2}{n} a^{-2} \tau^2\right),$$

which yields Lemma 12 by using $\int_X \tau = 0$ and taking the factor $a^{-\alpha}$ out of the integral. \qed
9. A special soap bubble family

In this section we prove the existence part of Theorem 4. The conditions that make the construction of the family \( \{S_v\} \) possible are stated in detail in Theorem 13 below. The ambient manifold is \( M = X \times \mathbb{R} \), with \( X \) a suitable 2–dimensional Riemannian manifold. In particular \( M \) is 3-dimensional and our family consists of surfaces. They lie in a region where the Ricci curvature is somewhere negative. It must be stressed that, in some of these families, the large soap bubbles are stable (proved in Section 10) but not isoperimetric, i.e. the same amount of volume can be enclosed using less area. We shall also see that, as the enclosed volume tends to infinity, their mean curvatures descend to a positive constant, not to zero.

There is an annulus \( Y \subset X \) such that all the surfaces in the family will be contained inside \( Y \times \mathbb{R} \) (this already prevents those surfaces from being of the form \( X \times S^0 \)), and so we need only worry about the geometry of the domain \( Y \times \mathbb{R} \). Then \( X \) can be any closed Riemannian surface containing an isometric copy of \( Y \).

We describe \( Y \) as \( I \times S^1 \), where \( I \) is an interval symmetric about 0. Denote by \( s \) the coordinate along \( I \) and by \( \theta \) the angle coordinate along \( S^1 = [0, 2\pi]/0 \sim 2\pi \).

Finally let \( y \) be the coordinate along the \( \mathbb{R} \) factor.

We endow \( Y \) with a rotationally symmetric metric:

\[
G_c = ds^2 + c^2 \cdot f(s)^2 d\theta^2,
\]

where \( f(s) \) is positive and even, that is \( f(-s) = f(s) \), and \( c \) is a positive constant.

The metric on \( Y \times \mathbb{R} \) is:

\[
G'_c = ds^2 + c^2 \cdot f(s)^2 d\theta^2 + dy^2.
\]

Notice that \( Y \times \mathbb{R} \) has an isometric circle action, defined by translating \( \theta \) by constants, and the following reflectional symmetries:

\[
(s, \theta, y) \mapsto (-s, \theta, y), \quad (s, \theta, y) \mapsto (s, \theta, \text{const} - y).
\]

The following auxiliary functions turn out to be very useful:

\[
F(s) = \int_0^s f(s) \, ds, \quad \varphi(s) = \frac{F(s)}{\bar{f}(s)}.
\]

Since \( f \) is even, both \( F \) and \( \varphi \) are odd. The basic identity relating \( f \) to \( \varphi \) is:

\[
\frac{1 - \varphi}{\varphi} = \frac{f_s}{\bar{f}}.
\]

A function \( \varphi(s) \) coming from this construction is not arbitrary: it has to be an odd function, vanish only at \( s = 0 \), and satisfy \( \varphi'(0) = 1 \). Conversely any \( \varphi(s) \) meeting these three criteria comes from a positive even function; in fact, the functions that \( \varphi \) comes from are the members of the following one-parameter family:

\[
f(s) = \frac{c}{\varphi(s)} \cdot \exp \int \frac{ds}{\varphi(s)}, \quad c \in \mathbb{R}^+,
\]

The value \( s = 0 \) is the only one where this formula may pose a problem. But if \( \varphi'(0) = 1 \) then \( \varphi(s) \equiv s + s^2 \hat{\varphi}(s) \) for some smooth odd function \( \hat{\varphi}(s) \), and thanks to \( \frac{1}{s + s^2 \hat{\varphi}} = \frac{1}{s} - \frac{\hat{\varphi}}{1 + s \hat{\varphi}} \) we rewrite (42) as:

\[
f(s) = \frac{c}{1 + s \hat{\varphi}(s)} \cdot \exp \int_0^s \frac{-\hat{\varphi}(s) \, ds}{1 + s \hat{\varphi}(s)},
\]

which is non-singular at \( s = 0 \) and defines a positive even function.
Theorem 13. Suppose \( \log F \) has a first inflection point at a value \( s_0 > 0 \). More concretely, suppose \((\log F)_s\) is negative in \( 0 < s < s_0 \) and positive in an interval starting at \( s_0 \). Then \( Y \times \mathbb{R} \) contains a family \( \{ S_{s_1} \}_{0 < s_1 < s_0} \) of soap bubbles, all contained in \((-s_0, s_0) \times S^1 \times \mathbb{R}\), and with enclosed volume going to infinity as \( s \to s_0 \).

Identity (41) gives \( (\log F)_s = -(f^2/F^2) \varphi_s \). The hypothesis in Theorem 13 is equivalent to \( \varphi_s \) being positive in \([0, s_0)\) and negative in some interval starting at \( s_0 \). In particular \( \varphi \) has a first local maximum at \( s = s_0 \), see Figure 3.

![Figure 3](image)

Here is an example with \( I = (-\pi + \varepsilon, \pi - \varepsilon) \) and \( s_0 = \pi/2 \):

\[
F = 2 \tan \frac{s}{2}, \quad f = \frac{1}{\cos^2(s/2)}, \quad \varphi = \sin s,
\]

and another one with \( I = (-1 + \varepsilon, 1 - \varepsilon) \) and \( s_0 = 1/\sqrt{3} \):

\[
F(s) = s(1 - s^2)^{-1/2}, \quad f(s) = (1 - s^2)^{-3/2}, \quad \varphi(s) = s - s^3.
\]

By Theorem 13, we expect the surface piece \( Y \) to have negative Gaussian curvature somewhere in \((-s_0, s_0) \times S^1\). Let us directly check this. Identity (41) implies:

\[
-\varphi f_{ss} = (f \varphi_s)_s - \varphi \frac{f^2_s}{f},
\]

and for \( s > 0 \) we get \(-\varphi f_{ss} \leq (f \varphi_s)_s\). Since the function \( f \varphi_s \) is positive at \( s = 0 \) and zero at \( s = s_0 \), its derivative has to be negative in some interval \( I_- \subset (0, s_0] \). Then \( f_{ss} \) is positive in that same interval. The Gaussian curvature of \( Y \) equals \(-f_{ss}/f\) and is thus negative in \( I_0 \times S^1 \) and also in the image of this set under the reflection \((s, \theta) \mapsto (-s, \theta)\).

Proof of Theorem 13. We consider constant mean curvature surfaces which are invariant under the circle action and the reflection \((s, \theta, y) \mapsto (s, \theta, -y)\), i.e. surfaces of the form:

\[
S_u = \{(s, \theta, \pm u(s)) : -s_1 \leq s \leq s_1, 0 \leq \theta \leq 2\pi\} \approx C_u \times S^1,
\]

which is the result of rotating a profile curve \( C_u = \{y = \pm u(s)\} \subset (sy \text{ plane}) \). We also allow translates of these in the \( y \)-direction. In our construction \( u(s) \) will be an even function defined in some symmetric interval \([-s_1, s_1] \subset I \) and satisfying:

\[
u(-s_1) = u(s_1) = 0, \quad u'(-s_1) = +\infty, \quad u'(s_1) = -\infty,
\]
so that the closed profile \( C_u \) is \( C^1 \). The condition for \( S_u \) to have constant mean curvature is an ODE on the profile curve \( C_u \), and elliptic regularity implies that \( C_u \) is actually smooth. Then \( S_u \) is a smooth surface diffeomorphic with \( \mathbb{T}^2 \). Both \( C_u \) and \( S_u \) are embedded if and only if \( u(s) \) satisfies:

\[
u(s) > 0 \quad \text{for all} \ s \in (-s_1, s_1),
\]
in which case the closed profile bounds a disk region \( D \) in the \( sy \) plane. Then \( S_u \) is the boundary of the solid torus \( \{ (s, \theta, y) : (s, y) \in D \} \subset Y \times \mathbb{R} \) and is thus a soap bubble.

We choose for \( S_u \) the unit normal \( \nu \) which points away from the solid torus. The mean curvature, as defined in Section 2, is given on \( S_u \cap \{ y > 0 \} \) by:

\[
H = \frac{-u''(s)}{(1 + u'(s)^2)^{3/2}} + \frac{-u'(s)}{(1 + u'(s)^2)^{1/2}} f'(s) f(s).
\]

Notice that the equation is the same if we replace \( f \) with any positive constant multiple \( c f \). Once \( \varphi \) is fixed, the family \( \{ c \cdot f \}_{c > 0} \) is fixed and a solution to (45) with \( H \) constant defines a surface that has constant mean curvature \( H \) with respect to all the metrics \( G_c \).

We can consider \( \nu \) as lying flat on the \( sy \)-plane and orthogonal to the profile. We define the angle \( \alpha(s) \), from the \( y \)-axis to \( \nu \), as follows:

\[
\frac{-\pi}{2} \leq \alpha(s) \leq \frac{\pi}{2}, \quad \nu|_{y \geq 0} = \sin \alpha \partial_s + \cos \alpha \partial_y,
\]

so that \( \tan \alpha(s) = -u'(s) \) and (45) becomes

\[
\frac{d}{ds} \left( f(s) \sin \alpha(s) \right) = H f(s).
\]

Since \( f \) is even, a solution \( u(s) \) to (45) is even if and only if \( u'(0) = 0 \). This condition is equivalent to \( \alpha(0) = 0 \). The conditions \( u'(\pm s_1) = \mp \infty \) are equivalent to \( \sin \alpha(\pm s_1) = \pm 1 \).

For a constant value \( H \), solutions to (45) with \( \alpha(0) = 0 \) are given by:

\[
\sin \alpha(s) = H \varphi(s).
\]

The conditions \( \sin \alpha(\pm s_1) = \pm 1 \) are now equivalent to \( H = 1/\varphi(s_1) \). Since we take \( s_1 \in (0, s_0) \), the constant \( H \) takes values in the interval \( [1/\varphi(s_0), +\infty) \); values smaller than \( 1/\varphi(s_0) \) will not appear in our construction.

Expressing \( u'(s) = -\tan \alpha \) in terms of \( \sin \alpha \), we arrive at:

\[
u(s) = \text{const} - \int_0^s \frac{\varphi(s) ds}{\sqrt{\varphi(s_1)^2 - \varphi(s)^2}} = \text{const} - \int_0^s \frac{\varphi(s) ds}{\sqrt{\varphi(s_1)^2 - \varphi(s)^2}},
\]

an explicit formula that involves only \( \varphi \). For each \( s_1 \in (0, s_0) \), the function:

\[
u_{s_1} : (-s_1, s_1) \to \mathbb{R}, \quad \nu_{s_1}(s) := -\int_0^s \frac{\varphi(s) ds}{\sqrt{\varphi(s_1)^2 - \varphi(s)^2}}
\]

is the solution to (45) with the following data:

\[
H = 1/\varphi(s_1), \quad u(0) = 0, \quad u'(0) = 0.
\]

We study its behavior in two cases: \( s_1 < s_0 \) or \( s_1 = s_0 \).
Case $s_1 < s_0$. For $s$ close to $s_1$, the integrand in (49) behaves like a positive multiple of $(s_1 - s)^{-1/2}$ because $\varphi(s)^2$ has positive derivative at $s = s_1$. Since $\int_{s_1 - \varepsilon}^{s_1} (s_1 - s)^{-1/2} ds$ is finite, $u_{s_1}(s)$ has a finite limit as $s \to s_1$. It has the same finite limit as $s \to -s_1$, and so it extends to the closed interval $[-s_1, s_1]$. The extended function is negative except at $s = 0$, and achieves its minimum at the endpoints $\pm s_1$. Then the function

$$\tilde{u}_{s_1} : [-s_1, s_1] \to \mathbb{R}, \quad \tilde{u}_{s_1}(s) := u_{s_1}(s) - u_{s_1}(\pm s_1)$$

is the solution to (45) with data:

$$H = 1/\varphi(s_1) \quad \tilde{u}(-s_1) = \tilde{u}(s_1) = 0,$$

and satisfies $\tilde{u}_{s_1}(s) > 0$ for $s \in (-s_1, s_1)$. The graph of $\tilde{u}_{s_1}$ meets the graph of $-\tilde{u}_{s_1}$ only at the endpoints $(\pm s_1, 0)$, where the derivative is infinite. By the previous discussion, the surface $S_{s_1} := S_{\tilde{u}_{s_1}}$ is a soap bubble in $Y \times \mathbb{R}$ with respect to all metrics $G_{\varepsilon}$. Doing this for all $s_1 \in (0, s_0)$, we get a soap bubble family $\{S_{s_1}\}_{0 < s_1 < s_0}$. Each member $S_{s_1}$ of this family is contained in the part $[-s_1, s_1] \times S^1 \times \mathbb{R}$, hence they all lie inside $(-s_0, s_0) \times S^1 \times \mathbb{R}$ which is a proper subset of $Y \times \mathbb{R}$.

Case $s_1 = s_0$. Since $\varphi(s)^2$ has a local maximum at $s = s_0$, for $s$ close to $s_0$ the integrand in (49) is at least as large as a positive multiple of $(s_0 - s)^{-1}$. From $\int_{s_0 - \varepsilon}^{s_0} (s_0 - s)^{-1} ds = +\infty$ we then deduce that the function $u_{s_0}$ tends to $-\infty$, at least at a logarithmic rate, as $s \to s_0$ and also as $s \to -s_0$. The undergraph:

$$E = \{ (s, y) : y \leq u_{s_0}(s) \} \subset (-s_0, s_0) \times \mathbb{R},$$

has infinite area both in the standard area measure and in the measure $cf(s) ds dy$. For $s_1 < s_0$, define a closed profile $\tilde{C}_{s_1}$ as the union of the graphs of $u_{s_1}$ and of the reflected function $2u_{s_1}(\pm s_1) - u_{s_1}(s)$. The point $(0, 0)$ is where $y$ is maximum on each $\tilde{C}_{s_1}$. Denote by $\Omega_{s_1}$ the region bounded by $\tilde{C}_{s_1}$ in the sy plane. For fixed $s$ the integral (49) is an increasing function of $s_1$. This implies that the D-shaped domains

$$D_{s_1} = \{ (s, y) : \min u_{s_1} \leq y \leq u_{s_1}(s), -s_1 \leq s \leq s_1 \}.$$

expand as $s_1 \nearrow s_0$, and they fill up $E$. A fortiori, the O-shaped regions $\Omega_{s_1}$ also fill up $E$, as shown in Figure 4. Therefore the area of $\Omega_{s_1}$ in the measure $cf(s) ds dy$ goes to infinity as $s_1 \to s_0$, and so does the volume enclosed by $S_{s_1}$ in $Y \times \mathbb{R}$.

The mean curvature $H = 1/\varphi(s_1)$ decreases as $s_1 \to s_0$, but the limit is the positive number $1/\varphi(s_0)$. Recall that values $H < 1/\varphi(s_0)$ never appear in this construction.

![Figure 4](image-url)

We have a diffeomorphism $(0, s_0) \to (0, +\infty)$ that maps $s_1 \in (0, s_0)$ to the volume enclosed by $S_{s_1}$. So we can use the enclosed volume as parameter in place of $s_1$. In this way the soap bubble family becomes $\{S_{s_0}\}_{v>0}$, as stated in Theorem 4.

□
10. Stability in the special family

In this section we prove the stability part of Theorem 14. The following theorem explicitly gives a condition under which the large surfaces in the family \( \{ S_u \} \) are stable. We shall also show that such condition is satisfied in some examples.

**Theorem 14.** Let \( f, F, \varphi, s_0 \) be as in Theorem 13. If \( (\log f)_{ss} > 0 \) in \((-s_0, s_0)\), then there is a number \( \beta > 0 \) such that if \( \varphi(s_1)/c > \beta \) then the soap bubble \( S_{s_1} \) is stable with respect to the metric \( G'_c \).

Fix a value \( c < \varphi(s_0)/\beta \) and let \( s_2 \in (0, s_0) \) be the solution to \( \varphi(s_2)/c = \beta \). Then the soap bubbles \( S_{s_1} \) with \( s_1 > s_2 \) are all stable with respect to \( G'_c \). The small \( S_{s_1} \), corresponding to \( s_1 \) close to zero, look like thin tubes around a circle and are not stable.

In example 13 we have \((\log f)_{ss} = f/2 > 0\). In example 44 we have \((\log f)_{ss} = (3 + 3s^2)/(1 - s^2)^2 > 0\). Both examples provide stable soap bubbles enclosing arbitrarily large volume in \( Y \times \mathbb{R} \) but whose projection to \( Y \) is not surjective.

We do not know whether for \( c \geq \varphi(s_0)/\beta \) the metric \( G'_c \) admits a family of non-isoperimetric stable soap bubbles with enclosed volume going to infinity.

Consider for a moment the problem of embedding \( (Y, G_c) = (I \times S^1, G_c) \) isometrically into \( \mathbb{R}^3 \) as a surface of revolution. Such embedding would be:

\[
(x_1, x_2, x_3) = \left( r(s) \cos \theta, r(s) \sin \theta, x_3(s) \right),
\]

for some functions \( r(s), x_3(s) : I \to \mathbb{R} \) satisfying the following equations:

\[
r'(s)^2 + x_3'(s)^2 = 1, \quad r(s) = c \cdot f(s) .
\]

The function \( r(s) \) is already given by the second equation, while \( x_3(s) \) is given by the formula \( x_3(s) = \int \sqrt{1 - c^2 f'(s)^2} \, ds \). Therefore \( x_3(s) \) exists if \( c |f'(s)| < 1 \), which is true for small enough \( c \). Once \( Y = I \times S^1 \) is thus embedded into \( \mathbb{R}^3 \), we can extend it to a closed surface \( X \subset \mathbb{R}^3 \). If \( c \) is also smaller than \( \varphi(s_0)/\beta \), then \( X \times \mathbb{R} \) is a cylinder in \( \mathbb{R}^4 \) admitting the stable family of soap bubbles.

**Proof of Theorem 14.** Let us first obtain a convenient formula for the index form of \( S_u \). We have the orthogonal bases \( \{ \partial_s + u'(s) \partial_y, \partial_y \} \) for the tangent spaces of \( S_u \), and we consider orthonormal bases \( \{ e_1, e_2 \} \) with \( e_1 \in \mathbb{R}(\partial_s + u'(s) \partial_y) \) and \( e_2 \in \mathbb{R} \partial_y \). In these orthonormal bases, the matrix of \( II \) is:

\[
\begin{bmatrix}
\frac{u_+}{u_+^2 + u_-^2} & 0 \\
0 & \frac{u_-}{u_+^2 + u_-^2}
\end{bmatrix}
\begin{bmatrix}
f \cdot 0 \\
0 & -f \cdot \sin \alpha
\end{bmatrix}
\]

independent of the constant \( c \). Using equalities (41) and (47), we rewrite that matrix as:

\[
\begin{bmatrix}
-H \varphi_+ & 0 \\
0 & H \cdot (\varphi_+ - 1)
\end{bmatrix},
\]

and so \( |II|^2 = H^2 \left( \varphi_+^2 + (\varphi_+-1)^2 \right) \).

For the Ricci term we have \( \text{Ric} (\nu, \nu) = -\frac{f_{ss}}{f} \sin^2 \alpha = -\frac{f_{ss}}{f} H^2 \varphi^2 \), also independent of \( c \). The relations \( \frac{f}{f} = \frac{1}{\varphi} \) and \( \frac{f}{f} = \frac{\varphi^2}{\varphi} + \frac{(1-\varphi)(1-2\varphi)}{\varphi^2} \) lead to:

\[-\text{Ric} (\nu, \nu) = H^2 \left( -\varphi \varphi_{ss} + (1-\varphi_+) (1-2\varphi_+) \right) .\]
Then, after a trivial simplification:

\[ P = -\text{Ric}(\nu, \nu) - |H|^2 = H^2 \cdot (-\varphi \varphi_{ss} - \varphi_s). \]

We can use \((s, \theta)\) as coordinates in \(S_u \cap \{y \geq 0\}\) and in \(S_u \cap \{y \leq 0\}\); accordingly we view \(ds\,d\theta\) as a measure on all of \(S_u\). In these coordinates:

- The metric induced on \(S_u\) is \((1 + u'^2)\,ds^2 + c^2\,f^2\,d\theta^2\),
- The area measure on \(S_u\) is \(\sqrt{1 + u'^2}\,c\,f\,ds\,d\theta\),

and for any \(g : S_u \to \mathbb{R}\) we find that:

\[
|\nabla^S g|^2\,d\text{area} = \left( \frac{c\,f}{\sqrt{1 + u'^2}} g_s^2 + \frac{\sqrt{1 + u'^2}}{c\,f} g_\theta^2 \right)\,ds\,d\theta.
\]

For \(S_{s_1}\) we can write \(1 + u'^2 = 1 + \frac{\varphi_s^2}{\varphi(s_1)^2 - \varphi^2} = \frac{\varphi(s_1)^2}{\varphi(s_1)^2 - \varphi^2}\), and finally obtain:

\[
Q(g) = \int_S \left( Ag_s^2 + \frac{1}{A} g_\theta^2 +Bg^2\right)\,ds\,d\theta,
\]

where:

\[
A = H\,cf(s)\,\sqrt{\varphi(s_1)^2 - \varphi(s)^2}, \quad B = H\,cf(s)\,\frac{-\varphi \varphi_{ss} - \varphi_s}{\sqrt{\varphi(s_1)^2 - \varphi(s)^2}}.
\]

In \(S_{s_1} \cap \{y \neq 0\}\), where \(s\) is a valid coordinate, the Jacobi operator \(\Delta^S g - Pg\) is the result of multiplying the following operator with a positive function:

\[
(A g_s)_s + \left( \frac{g_\theta}{A} \right)_\theta - B\,g.
\]

The surface \(S_{s_1}\) is invariant under the reflections:

\[(s, \theta, y) \longleftrightarrow (-s, \theta, y), \quad (s, \theta, y) \longleftrightarrow (s, \theta, -y),\]

and so it makes sense to define, for functions \(g : S_{s_1} \to \mathbb{R}\), the properties of being odd or even in the \(s\) variable, and the same for the \(y\) variable. We shall denote by \(g_s, g_y\) the odd and even parts of \(g\) with respect to \(s\), respectively, that is

\[g_s := \frac{1}{2} \left( g(s, \theta, y) - g(-s, \theta, y) \right), \quad g_y := \frac{1}{2} \left( g(s, \theta, y) + g(-s, \theta, y) \right),\]

and \(g_{s_1}, g_{s_1}\) shall have the analogous meaning in the \(y\) variable. In particular, we shall use the decomposition:

\[g = g_{s_1} + g_{s_1} + g_{s_1}.\]

The polar bilinear form of the index form admits the expression:

\[
Q(g, \tilde{g}) = \int_S \left( Ag_s \tilde{g}_s + \frac{1}{A} g_\theta \tilde{g}_\theta + Bg\tilde{g}\right)\,ds\,d\theta,
\]

and it is obvious, by the symmetries of \(A\) and \(B\), that functions with different parity in \(s\) or in \(y\) are \(Q\)-orthogonal. Therefore, for every \(g : S_{s_1} \to \mathbb{R}\) we have:

\[
Q(g) = Q(g_{s_1}) + Q(g_{s_1}) + Q(g_{s_1}),
\]

and we shall do a separate study of the positivity of each summand. The functions \(g_{s_1}\) and \(g_{s_1}\) always have zero average, hence \(g\) has zero average if and only if \(g_{s_1}\) has zero average.

The next result follows from the identity \(\gamma^2 |\nabla g_1|^2 = |\nabla(g_1 \gamma)|^2 - \nabla \gamma \cdot \nabla (g_1^2 \gamma)\).
Lemma 15. Let \( \Sigma \) be any compact surface with a Riemann metric. For any two functions \( g_1, g_2 : \Sigma \to \mathbb{R} \) the following holds:

\[
\int_\Sigma (|\nabla^\Sigma (g_1 g_2)|^2 + P (g_1 g_2)^2) = \\
= \int_\Sigma g_1^2 \gamma \cdot (P \gamma - \Delta \Sigma \gamma) + \int_\Sigma \gamma^2 |\nabla^\Sigma g_1|^2 + \int_{\partial \Sigma} (g_1^2 \gamma) \eta \cdot \nabla^\Sigma \gamma,
\]

where \( \eta \) is the outer conormal along \( \partial \Sigma \).

First case: \( g \) is odd in the variable \( y \). Since \( \partial_y \) is a Killing vector field, the function \( \psi = (1/H) \langle \partial_y, \nu \rangle \) is a solution to the Jacobi equation:

\[
\Delta^\Sigma \psi - P \psi = 0.
\]

The formula \( \psi = (\text{sgn } y) \cdot \sqrt{\varphi(s_1)^2 - \varphi^2} \) shows that \( \psi \) vanishes with non-zero derivative along the two circles defined as \( S_{s_1} \cap \{ y = 0 \} \). Any function \( g \) that is odd in the variable \( y \) vanishes along those circles too, hence \( g = \psi \cdot g_1 \) for some smooth function \( g_1 \) on \( S_{s_1} \). In this case Lemma 15 gives:

\[
Q(g) = Q(\psi \cdot g_1) = \int_\Sigma \psi^2 |\nabla^\Sigma g_1|^2.
\]

It follows that, for \( g \) odd in \( y \), the number \( Q(g) \) is positive unless \( g \) is a constant multiple of \( \langle \partial_y, \nu \rangle \).

Second case: \( g \) is even in \( y \) and odd in \( s \). Now we have \( g = \varphi g_1 \) for some smooth function \( g_1 \) on \( S_{s_1} \). Using formula 41 one can do a direct calculation that yields the following result:

\[
(A \varphi_s)_s = B \varphi - \frac{c \int \frac{\varphi(s_1)}{\sqrt{\varphi(s_1)^2 - \varphi^2}} \cdot \varphi \cdot (\log f)_{ss}}{\varphi^2}.
\]

The hypothesis of Theorem 14 then says that \( (A \varphi_s)_s - B \varphi \) and \( \Delta^\Sigma \varphi - P \varphi \) are negative multiples of \( \varphi \). In this case Lemma 15 gives us the following:

\[
Q(g) = Q(g_1 \varphi) = \int_{S_{s_1}} g_1^2 \varphi^2 \cdot \text{(positive)} + \int_{S_{s_1}} \varphi^2 |\nabla^\Sigma g_1|^2,
\]

and it is obvious that \( Q(g) > 0 \) unless \( g_1 \) and \( g \) are identically zero.

Third case: \( g \) is even in both \( s \) and \( y \). There is a closed profile \( C_u \subset (s, y) \) plane) such that \( S_{s_1} \) is like \( C_u \times S^1 \) with the coordinate \( \theta \) going along the \( S^1 \) factor. Moreover \( C_u \) is the union of two graphs \( \{ y = \pm u(s) \} \) with \( -s_1 \leq s \leq s_1 \). Consider the Fourier expansion:

\[
g = a_0 + \sum_{k \geq 1} \left( a_k \cos k\theta + b_k \sin k\theta \right),
\]

where the coefficients are functions \( a_k, b_k : C_u \to \mathbb{R} \) as symmetrical as \( g \) is:

\[
a_k(-s,y) = a_k(s,y) \quad \text{and} \quad a_k(s,-y) = a_k(s,y), \quad \text{same for} \quad b_k.
\]

Considering \( ds \) as a measure on all of \( C_u \), we can write:

\[
Q(g) = \int_{C_u} 2\pi \left[ A a_0^2 + B a_0^2 \right] ds + \\
+ \int_{C_u} \pi \left[ A \sum_{k \geq 1} (a_k^2 + b_k^2) + \sum_{k \geq 1} \left( \frac{k^2}{A} (a_k^2 + b_k^2) + B (a_k^2 + b_k^2) \right) \right] ds,
\]
We want the term \(\pi a_0\) in Theorem 14 and finishes the proof of Theorem 4.

Theorem 14 is that the following condition:

\[
\int_{C_s} (A a_0^2 + B a_0^2) \, ds + \pi \sum_{k \geq 1} \int_{C_s} \left( \frac{1}{A} + B \right) (a_k^2 + b_k^2) \, ds.
\]

The points \((s_1, 0), (0, u(0)), (-s_1, 0), (0, -u(0))\) separate \(C_s\) into four quadrants. That \(g\) has zero average in \(S_{s_1}\) is equivalent to \(a_0\) having zero average in \(C_s\) and, by the symmetries, it also has zero average on each quadrant. The first quadrant is the graph \(C'_s = \{(s, u(s)) : 0 \leq s \leq s_1\}\) and, since \(a_0\) has zero average on it, there is a value \(\overline{s} \in (0, s_1)\) such that \(a_0(\overline{s}, u(\overline{s})) = 0\).

We shall now use the one-dimensional version of Lemma 15, see [4, page 107]. For \(0 \leq s \leq \overline{s}\) use \(\gamma = \psi\) and get:

\[
\int_{C'_s \cap \{0 \leq s \leq \overline{s}\}} (A a_0^2 + B a_0^2) \, ds = \int_{C'_s \cap \{0 \leq s \leq \overline{s}\}} A \psi^2 \left[ \frac{d}{ds} \frac{a_0}{\psi} \right]^2 \, ds + \left[ A \frac{\psi}{\psi} a_0^2 \right]_{0}^{\overline{s}} \geq 0,
\]

and we see that \(\int_{C'_s \cap \{0 \leq s \leq \overline{s}\}} (A a_0^2 + B a_0^2) \, ds \geq 0\), with strict inequality unless \(a_0|_{0 \leq s \leq \overline{s}}\) is a constant multiple of \(\psi\). For \(\overline{s} \leq s \leq s_1\) use \(\gamma = \varphi\) and obtain:

\[
\int_{C'_s \cap \{\overline{s} \leq s \leq s_1\}} (A a_0^2 + B a_0^2) \, ds \geq -A \bigg| \frac{\varphi}{\varphi} a_0^2 \bigg|_{s=\overline{s}} = 0,
\]

with strict inequality unless \(a_0 \equiv 0\) on \(C'_s \cap \{\overline{s} \leq s \leq s_1\}\).

But if \(a_0|_{C'_s}\) has to be a constant multiple of \(\psi\) on \(0 \leq s \leq \overline{s}\), and zero on \(\overline{s} \leq s \leq s_1\), then it must be identically zero. We conclude that \(\int_{C'_s} (A a_0^2 + B a_0^2) \, ds > 0\) unless \(a_0 = 0\) everywhere zero.

We want the term \(\pi \int_{C_s} \left( \frac{1}{A} + B \right) (a_k^2 + b_k^2) \, ds\) to be positive unless \(a_k = b_k = 0\) for \(k \geq 1\). Thus we want to ensure that \(\frac{1}{A} + B\) is a positive function. The formulas:

\[
\frac{1}{A} = \frac{1}{Hc} \int \frac{1}{\sqrt{\varphi(s_1)^2 - \varphi^2}} = \frac{\varphi(s_1)}{c} \int \frac{1}{f \varphi(s_1)^2 - \varphi^2},
\]

\[
B = \frac{c}{\varphi(s_1)} \int \frac{-\varphi \varphi_{ss} - \varphi_s}{\sqrt{\varphi(s_1)^2 - \varphi^2}} = \frac{1}{f \cdot \sqrt{\varphi(s_1)^2 - \varphi^2}} \cdot f^2 \cdot (-\varphi \varphi_{ss} - \varphi_s),
\]

show that the following condition:

\[
\left( \frac{\varphi(s_1)}{c} \right)^2 > \beta^2 := \max_{[0, s_0]} \left( f^2 \cdot | -\varphi \varphi_{ss} - \varphi_{s1} | \right),
\]

implies that \(\frac{1}{A} + B\) is a positive function. This takes care of the third case.

Putting the three cases together we see that, for \(\varphi(s_1)/c > \beta\) and \(g : S_{s_1} \rightarrow \mathbb{R}\) with zero average, it is \(Q(g) > 0\) unless \(g\) is a constant multiple of \(\langle \partial_y, \nu \rangle\). This proves Theorem 14 and finishes the proof of Theorem 4.

\(\square\)
APPENDIX: slope estimate

We prove here a gradient estimate for a rotated graph $S = \{ \rho = u(x) \} \subset X \times \mathbb{R}^n$ that has constant mean curvature $H$ and is some distance apart from its symmetry axis $X \times \{0\}$. The gradient bound depends on the radius oscillation of $S$.

More concretely, the radius function $u : X \to \mathbb{R}$ is defined on all of $X$ and we assume $\min u > 1$. The radius oscillation is the number:

$$C = \max u - \min u .$$

**Theorem 16.** There is a constant $C'$, depending only on $n, X, C$, such that:

$$|\nabla u(x)| \leq C' , \quad \text{for all } x \in X .$$

First we prove the theorem for $n \geq 2$. Points on $X \times \mathbb{R}^n$ will be described as $p = (x, y, y_n)$, where $x \in X$, $y = (y_1, \ldots, y_{n-1}) \in \mathbb{R}^{n-1}$, and $y_n \in \mathbb{R}$. The function:

$$f(x, y) = \sqrt{u(x)^2 - |y|^2} ,$$

already considered in formula (10) of Section 4, is defined in an open subset of $X \times \mathbb{R}^{n-1}$ that contains the closure of the following domain:

$$U_1 = \{ (x, y) : |y| < 1 \} = X \times B^{n-1}(0, 1) ,$$

and $S_+ := S \cap \{|y| < 1 , \ y_n > 0\}$ is described as a Cartesian graph:

$$S_+ = \{ y_n = f(x, y) , (x, y) \in U_1 \} .$$

The function $f$ has the following bounds:

$$\min u - 1 < f \leq \max u .$$

A gradient bound for $f$ will provide the same for $u$, because $u(x) = f(x, 0)$.

Denote by $\nu$ the unit normal of $S$ which along $S_+$ points to the positive $y_n$-direction. Consider also the function:

$$w = \sqrt{1 + |\nabla f|^2} .$$

We use the method of N. Korevaar in [13]. Let $\lambda : X \times \mathbb{R}^n \to \mathbb{R}$ be a non-negative continuous function, smooth at the points of $S$ where it is positive, and zero on $S \cap \{|y| \geq 1\}$. For small $\varepsilon > 0$ push $S$ off itself by taking each point $z \in S$ to the endpoint $z_\varepsilon$ of the geodesic segment with length $\varepsilon \lambda(p)$ and initial velocity $\nu_p$. This deforms $S$ to a new hypersurface $S_\varepsilon$ whose mean curvature we denote $H_\varepsilon$.

The new hypersurface $S_\varepsilon$ lies above $S$ in $\{y_n > 0\}$ and coincides with $S$ outside $\{|y| \geq 1 + O(\varepsilon)\}$. The points of $S \cap \{|y| \geq 1\}$ are not moved at all. Therefore the height of $S_\varepsilon$ over $S$ is maximized at some point $z_{0,\varepsilon} \in S_\varepsilon$ which is the end of a geodesic segment issuing from some $z(\varepsilon) \in S_+$. We can apply to $S_\varepsilon$ a downward translation in the $y_n$-direction until it touches $S$ from underneath at the translated point of $z_{0,\varepsilon}$. Hence $H_\varepsilon(z_{0,\varepsilon}) \geq H$. But formula (7) of Section 4 gives the following expression for $H_\varepsilon$:

$$H_\varepsilon(z_{\varepsilon}) = H - (|II|^2 \lambda + \text{Ric}(\nu, \nu) \lambda + \Delta^S \lambda)_z \cdot \varepsilon + \mathcal{E}_1 ,$$

for all $z \in S \cap \{\lambda > 0\}$.

The error term $\mathcal{E}_1$ is an $O(\varepsilon^2)$ depending at most on third derivatives of $(f, \lambda, G)$; here $G$ denotes the metric on $X$. The inequality $H_\varepsilon(z_{0,\varepsilon}) \geq H$ then implies:

$$\left( \Delta^S \lambda - R_0 \lambda \right)_{z(\varepsilon)} \leq \frac{1}{\varepsilon} \mathcal{E}_1 ,$$

where $R_0$ is the number defined in formula (26) of Section 4 and $\frac{1}{\varepsilon} \mathcal{E}_1$ is an $O(\varepsilon)$ depending at most on third derivatives of $(f, \lambda, G)$; it may be positive or negative.
We shall see that (54) leads to an apriori bound \( w(z(\varepsilon)) \leq w_0 \) at the special point \( z(\varepsilon) \). The idea is the following: Korevaar constructs \( \lambda \) with large positive second derivative \( \Delta\mu_{y_n} \); if \( S \) were close to vertical at \( z(\varepsilon) \), then \( \Delta^S \lambda \) \( (z(\varepsilon)) \) would be a large positive number and (54) would be contradicted for small \( \varepsilon \).

The number \( w_0 \) is independent of \( \varepsilon \). Let us see that, in the limit as \( \varepsilon \to 0 \), the quantity \( \lambda w \) is maximized at \( z(\varepsilon) \).

For \( \varepsilon \) sufficiently small, depending on first derivatives of \( (f, \lambda, G) \), the hypersurface \( S_\varepsilon \cap \{|\tilde{y}| \leq 1, y_n > 0\} \) is the graph of a function \( f_\varepsilon(x, \tilde{y}) \). Define \( f(z) = f(x, \tilde{y}) \) if \( z = (x, \tilde{y}, y_n) \), and similarly for \( f_\varepsilon \) and \( w \). We have the following formula for the height of \( S_\varepsilon \) above \( S \):

\[
f_\varepsilon(z) - f(z) = \varepsilon \lambda(z) w(z) + \mathcal{E}_2,
\]

where the error term \( \mathcal{E}_2 \) is an \( O(\varepsilon^2) \) depending at most on second derivatives of \( (f, \lambda, G) \). See the picture in [13, page 85] for a convincing proof. Since the quantity \( (f_\varepsilon - f)/\varepsilon \) is maximized at \( z(\varepsilon) \), we have:

\[
\lambda(z) w(z) \leq \lambda(z(\varepsilon)) w(z(\varepsilon)) + \mathcal{E}_3
\]

for all \( z \in S_+ \),

where \( \mathcal{E}_3 \) is an \( O(\varepsilon) \) depending at most on second derivatives of \( (f, \lambda, G) \). As we make \( \varepsilon \to 0 \) the point \( z(\varepsilon) \) keeps moving inside \( S_+ \) and the term \( \mathcal{E}_3 \) tends to zero. Given the bound \( w(z(\varepsilon)) \leq w_0 \), obtained from (54) at the special points \( z(\varepsilon) \), in the limit we have:

\[
\lambda(z) w(z) \leq \left( \max_{S_+} \lambda \right) \cdot w_0,
\]

which gives a slope bound at those \( z \in S_+ \) with \( \lambda(z) \) not too small. Consequently, to the conditions already imposed on \( \lambda \) we add the following one: \( \lambda > 0 \) on \( S_+ \cap \{|\tilde{y}| = 0\} \).

The function \( \lambda \) is first given by the ansatz \( \lambda(z) \equiv e^{C_1 \mu(z) - 1} \), so that \( \lambda \) is positive where \( \mu \) is positive and is zero where \( \mu \) is zero. We compute:

\[
\Delta^S \lambda = e^{C_1 \mu} \cdot C_1 \cdot (\Delta^S \mu + C_1 |\nabla^S \mu|^2).
\]

The factor \( e^{C_1 \mu} C_1 \geq C_1 \) is going to be large. We want the expression in parenthesis to be large at steep points of \( S_+ \).

The function \( \mu \) is given by a second ansatz:

\[
\mu(x, y) = \left( 1 - |\tilde{y}|^2 - \frac{(y_n - \min f)^+}{2 + 2C} \right)^+,
\]

notice that it does not depend on the point \( x \in X \). Both \( \mu \) and \( \lambda \) vanish on \( S \cap \{|\tilde{y}| \geq 1\} \). The denominator \( 2 + 2C \) is necessary to ensure that \( \mu \) and \( \lambda \) are positive on \( S_+ \cap \{|\tilde{y}| = 0\} \).

Consider now the formula \( \Delta^S \mu = \Delta^T \mu - H \mu \nu \). Obviously \( |\mu \nu| \leq \frac{1}{2 + 2C} + 2 \leq 3 \) and \( \Delta^T \mu \geq -2(n - 1) = 2 - 2n \), and so:

\[
\Delta^S \mu \geq 2 - 2n - 3H \quad \text{at any point of } S_+.
\]

This only prevents \( \Delta^S \mu \) from being a large negative number. We need to choose \( C_1 \) so that \( C_1 |\nabla^S \mu|^2 \) is large where \( \lambda > 0 \). To estimate \( |\nabla^S \mu| \), we use the unit length vector \( \nu \) which defines the steepest direction in \( S \). At points where \( \mu \) is positive:

\[
\nu = \frac{1}{w} \left( \frac{\nabla f}{|\nabla f|}, |\nabla f| \right) \implies |\nabla^S \mu| \geq |\nu \mu| \geq \frac{1}{w} \left( \frac{|\nabla^2 f|}{2C} - 2 \right).
\]
The last expression goes to $1/(2C)$ as $|∇f| → ∞$. Thus $C_1$ must be a multiple of $C^2$ to make $C_1 |∇Sμ|^2$ large. An easy calculation shows:

$$|∇f| > 20C \implies \frac{1}{w} \left( \frac{|∇f|}{2C} - 2 \right) > \frac{1}{3C} \implies |∇Sμ| > \frac{1}{3C}. $$

Accordingly we are going to choose $C_1$ satisfying $C_1 \cdot \left( \frac{1}{ε^2} \right)^2 ≥ 2n + 3H$, i.e. $C_1 ≥ (18n + 27H)C^2$. With this choice, at any point where $μ$ and $λ$ are positive (which certainly include the special points $z(ε)$), we have:

$$|∇f| > 20C \implies Δ^Sμ + C_1 \cdot |∇Sμ|^2 > 2 - 2n - 3H + 2n + 3H = 2 \implies Δ^Sλ > 2C_1 e^{C_1μ} > C_1 + C_1 λ \implies Δ^Sλ - R_0 λ > C_1 + (C_1 - R_0) λ.$$

Fix $μ$ and $λ$ by choosing $C_1 = \max \left( (18n + 27H)C^2, R_0 \right)$. For such a choice, and for $ε$ such that $\frac{1}{ε}E_1 < C_1$, the special points $z(ε)$ satisfy $|∇f(z(ε))| ≤ 20C$ and $w(z(ε)) ≤ 1 + 20C$. Making now $ε → 0$, we conclude that for $z \in S_+$ with $λ(z) ≠ 0$ it is:

$$w(z) ≤ \frac{1}{λ(z)} \cdot \left( \max_{S_+} λ \right) \cdot (1 + 20C) ≤ \frac{e^{C_1}}{λ(z)} \cdot (1 + 20C).$$

If $z = (x, 0, y_n) \in S_+$ is any point with $7 = 0$, then $μ(z) ≥ 1/2$ and $λ(z) ≥ e^{C_1/2} - 1$. The desired estimate is then:

$$|∇f(x)| = |∇f(x, 0)| < \frac{e^{C_1}}{e^{C_1} - 1} (1 + 20C), \text{ for all } x \in X.$$

The proof for $n = 1$ is almost the same, with some tiny simplifications that we next explain. Now we do not need to define $f$, because $S$ is already the disjoint union of two Cartesian graphs $\{y_1 = ±u(x)\}$. We give $λ$ by the same ansatz as before, and $μ$ by this one:

$$μ = 1 - \frac{(y_1 - \min u)^+}{2 + 2C}. $$

This time we have $|μ| < 1$ and $Δ^Tμ = 0$, hence $Δ^Sμ = 0 - Hμ ≥ -H$. We choose $C_1$ satisfying $C_1 \cdot \left( \frac{1}{ε^2} \right)^2 ≥ 2 + H$ and $C_1 ≥ R_0$. Under these conditions, at points where $|∇u| < 20C$ we have:

$$Δ^Sμ + C_1 |∇Sμ|^2 ≥ -H + 2 + H = 2 \quad \text{and} \quad Δ^Sλ - R_0 λ > C_1,$$

and we recover the estimate (56) with this new choice for $C_1$.

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