ON THE $L^2$-RESTRICTION NORM PROBLEM FOR CLOSED GEODESICS ON THE MODULAR SURFACE

DANA ABOU ALI

Abstract. Let $f$ be a Petersson normalized Hecke-Maass cusp form with spectral parameter $t \geq 2$ and let $C_D$ be the union of closed geodesics in $\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ associated to a fundamental discriminant $D > 0$. Following a suggestion by Sarnak in his letter to Reznikov, we express the restriction norm $\|f|_{C_D}\|_2^2$ as a weighted sum of central values of $L$-functions using Waldspurger’s formula. This allows us to get an unconditional improvement over the current bounds.

1. Introduction

This paper deals with a geodesic restriction problem: given a Riemann surface $(M, g)$, a normalized function $f: M \to \mathbb{C}$ and a closed geodesic $\ell \subseteq M$, we want to estimate $\|f|_{\ell}\|_2$ with respect to some parameter. The goal of this problem is, informally, to understand the extent to which $f$ can concentrate on small subsets of $M$ (\cite{14}, \cite{5}). The following standard result was proved by Reznikov \cite{18} and generalized by Burq, Gérard and Tzvetkov \cite{5}.

**Theorem.** If $M$ is a compact smooth Riemannian surface (without boundary), $\Delta$ is the Laplace operator associated to $g$ and $\ell \subseteq M$ a smooth curve then

$$\|f|_{\ell}\|_2 \ll (1 + r)^{\frac{1}{4}}$$

for any eigenvector $f$ of $\Delta$ with eigenvalue $-r^2$, $r \geq 0$.

It is possible to get better bounds by adding some restrictions on $M$, $\ell$ and $f$. For example, whenever we consider arithmetic objects, one can use techniques from number theory to get a smaller exponent (see \cite{1} or \cite{8} for example). In this paper, we take $M$ to be the modular surface $\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$, $f$ a Hecke-Maass cusp form and $\ell := C_D$ the union of all closed geodesics in $\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ associated to a fundamental discriminant $D > 0$.

Marshall \cite{14} proved the following

**Theorem.** Let $M$ be a compact congruence arithmetic hyperbolic surface arising from a quaternion division algebra over $\mathbb{Q}$, $\ell \subseteq M$ a geodesic segment of unit length and $f$ an $L^2$ normalized Hecke-Maass cusp form with spectral parameter $t \in \mathbb{R}$. Then

$$(1) \quad \|f|_{\ell}\|_2 \ll t^{\frac{3}{4} + \varepsilon}.$$  

Using \cite{3} and the bound $\|f\|_4 \ll t^{\varepsilon}$ announced by Sarnak and Watson in \cite{20}, the exponent in (1) can be improved so that

$$\|f|_{\ell}\|_2 \ll t^{\frac{1}{8} + \varepsilon}.$$  

Following a suggestion by Sarnak in a letter to Reznikov (see \cite{21}, p.6 example (c)), we prove
Theorem A. Let $f$ be a Hecke-Maass cusp form with $\|f\|_2 = 1$ and Laplace eigenvalue $\lambda = 1/4 + t^2$ for some $t \geq 2$. Let $C_D$ be the union of geodesics defined above. Then,

$$\|f|_{C_D}\|_2 \leq D, \epsilon \t^\theta + \epsilon$$

where $\theta \geq 0$ is any bound towards the Ramanujan conjecture for Maass forms.

In particular, assuming the Ramanujan conjecture we would get $\|f|_{C_D}\|_2 \leq \epsilon$. Using the Kim-Sarnak bound \[13\], we have the unconditional bound

$$\|f|_{C_D}\|_2 \leq D, \epsilon \t^{2\theta + \epsilon}$$

which is an improvement over the previously mentioned results.

Sketch of proof. We start the proof by finding an appropriate expression for $\|f|_{C_D}\|_2$ involving completed L-functions. To do so, we endow $C_D$ with a compact abelian group structure by constructing a homeomorphism to a disjoint union of circles. Then, using representation theory of compact abelian groups we find an orthonormal basis of $L^2(C_D)$ which allows us to write $\|f|_{C_D}\|_2$ as a sum of squares of Fourier coefficients. By Waldspurger’s formula, these Fourier coefficients are related to the values $\Lambda(1/2, f \times \theta_n)$ for some Hecke characters $\chi$. In particular, to prove Theorem A it will be enough to show that

$$\sum_{\psi \in H^+(K)} \sum_{n \in \mathbb{Z}} \frac{\gamma(1/2, f \times \theta_n^\psi)}{\gamma(1, \text{Ad} f)} L(1/2, f \times \theta_n^\psi) \leq \epsilon, \t^{2\theta + \epsilon}$$

where $K = \mathbb{Q}(\sqrt{D})$ and $H^+(K)$ is the dual of the ideal class group.

The quotient of gamma factors decays exponentially for $|n| \geq c_D \t$ where $c_D$ is a constant to be determined later. In particular, we are essentially dealing with a finite sum. Using the approximate functional equation, we can express $L(1/2, f \times \theta_n^\psi)$ as a sum in terms of the Fourier coefficients $\lambda_f(m)$ of $f$ and $\lambda_m^\psi(\theta_n)$ of $\theta_n^\psi$. By definition, $\lambda_m^\psi(\theta_n)$ consists of a sum over ideals in $K = \mathbb{Q}(\sqrt{D})$. However, after applying characters orthogonality to the $H^+(K)$ sum, we only need to consider sums over principal ideals with totally positive generators. In other words, the sum over ideals can be reduced to a sum over elements $\alpha$ in an appropriately chosen fundamental domain $\mathcal{F}_D$ for the action of the totally positive units on the totally positive integers of $K$.

Finally, we apply the Poisson summation formula to the $n$ sum to find some cancellations. As a result, we can further restrict our $\alpha$ sum to a small enough subset of generators whenever $|c_D \t - n|$ is large. Applying a dyadic partition of unity allows us to minimize the effect of the terms in the $n$ sum for which $|c_D \t - n|$ is small.

2. Background material and setup

2.1. Real quadratic number fields. Throughout this paper, we will let $D > 0$ be a fundamental discriminant and $K := \mathbb{Q}(\sqrt{D})$ the corresponding real quadratic number field. We denote the ring of integers, group of units and narrow class group of $K$ by $\mathcal{O}_K$, $U(K)$ and $H^+(K)$ respectively.

As a consequence of Dirichlet’s unit theorem, the group of totally positive units $U^+(K)$ is infinite cyclic. We denote by $\epsilon_K$ the unique generator of $U^+(K)$ such that $\epsilon_K > 1$ as an element of $\mathbb{R}$.

Finally, we consider the action of the group $U^+(K)$ on $\mathcal{O}_K^+$, the set of totally positive integers, via multiplication. The embedding $l_K : \alpha \mapsto (\alpha, \alpha^*)$ of $K$ into $\mathbb{R}^2$ allows us to obtain a fundamental domain $\mathcal{F}_D \subseteq \mathbb{R}^2$ for this action. More precisely,
Proposition 2.1. Let \( \beta_D := \frac{1 + \sqrt{D}}{2} \) for \( D \equiv 1 \mod 4 \) and \( \beta_D := \frac{\sqrt{D}}{2} \) otherwise. If we see \( K \) as a subset \( \mathbb{R}^2 \) using \( \iota_K \), a fundamental domain \( \mathcal{F}_D \) for the action of \( U^+(K) \) on \( \mathcal{O}_K^+ \) is equal to

\[
\mathcal{F}_D := C_D \cap L_D
\]

where \( C_D \) is the cone \( C_D := \{(x, y) \in (\mathbb{R}_{>0})^2 \mid y \leq x < \varepsilon_D^2y\} \) and \( L_D \) is the lattice \( L_D := \{(a + b\beta_D, a + b\beta_D^2) \mid a, b \in \mathbb{Z}\} \).

2.2. Closed geodesics of the modular surface. As mentioned previously, we will be working with \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \), the orbit space of the action of subgroup \( \text{SL}_2(\mathbb{Z}) \) on \( \mathbb{H} \). In this subsection, we give a summary of the description of closed geodesics in \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \) developed in Section 1 of [19].

Definition 1. (i) An element \( A \in \text{SL}_2(\mathbb{Z}) \) is said to be hyperbolic if the corresponding Möbius transformation has precisely two fixed points \( x, x' \in \mathbb{R} \).

(ii) A hyperbolic element \( A \in \text{SL}_2(\mathbb{Z}) \) is called primitive if we cannot find any \( B \in \text{SL}_2(\mathbb{Z}) \) and \( n \in \mathbb{Z} \setminus \{0, 1, -1\} \) with \( A = B^n \).

We have a right action of \( \text{SL}_2(\mathbb{Z}) \) on hyperbolic elements via \( A.B := B^{-1}AB \). This leads to an induced action of \( \text{PSL}_2(\mathbb{Z}) \) on primitive hyperbolic elements.

Definition 2. Let \( A \in \text{SL}_2(\mathbb{Z}) \) be hyperbolic and let \( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, |a| > 1 \) be the unique matrix of this form which is conjugate to \( A \). We define the norm of \( A \) to be \( N(A) = a^2 \).

Let \( A \in \text{SL}_2(\mathbb{Z}) \) be a primitive hyperbolic element and let \( x, x' \in \mathbb{R} \) be the fixed points of the corresponding Möbius transformation. Let \( \gamma_A \subseteq \mathbb{H} \) be the geodesic joining \( x \) and \( x' \) and let \( \gamma_A \) be its image in \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \). Then, \( \gamma_A \) is a closed geodesic in \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \) of length \( \log(N(A)) \). Moreover, this construction induces a bijection between closed geodesics in \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \) and conjugacy classes of primitive hyperbolic elements (up to identifying \( A \) with \( -A \)).

Definition 3. A primitive indefinite quadratic form of discriminant \( D > 0 \) is a quadratic form \( Q(x, y) \) of the form \( Q(x, y) = ax^2 + bxy + cy^2 \) with \( a, b, c \in \mathbb{Z} \), \( \gcd(a, b, c) = 1 \) and \( b^2 - 4ac = D \).

We use the shorthand notation \([a, b, c]\) to denote the quadratic form \( Q(x, y) = ax^2 + bxy + cy^2 \).

From now on, unless specified otherwise, all quadratic forms we consider will be primitive indefinite.

We have an \( \text{SL}_2(\mathbb{Z}) \) action on the space of quadratic forms via change of variables. More precisely, given a quadratic form \( Q(x, y) \) and \( A \in \text{SL}_2(\mathbb{Z}) \), we define \((Q.A)(x, y) := Q(A(x, y))\) where \( A(x, y) \) is the image of \((x, y)\) by the linear transformation induced by \( A \). For any fundamental discriminant \( D > 0 \), this action induces an action of \( \text{PSL}_2(\mathbb{Z}) \) on the space of quadratic forms of discriminant \( D \).

Proposition 2.2. Let \([a, b, c]\) be a quadratic form. Then,

(i) the stabilizer of \([a, b, c]\) by the action of \( \text{PSL}_2(\mathbb{Z}) \) is the image of the infinite cyclic subgroup with generator \( M_{[a, b, c]} = \begin{pmatrix} x_D - by_D \\ ay_D \\ y_D \end{pmatrix} \) where \( D > 0 \) is the discriminant of \([a, b, c]\) and \((x_D, y_D) \in (\mathbb{Z}_{\geq 0})^2\) is the fundamental solution of the Pell equation \( x^2 - Dy^2 = 4 \),

(ii) the map \([a, b, c] \mapsto M_{[a, b, c]}\) gives a bijection of \( \text{PSL}_2(\mathbb{Z}) \)-sets between quadratic forms and primitive hyperbolic elements in \( \text{PSL}_2(\mathbb{Z}) \). In particular, this bijection descends to a bijection of the corresponding equivalence classes and

(iii) for any fundamental discriminant \( D > 0 \), the bijection \([a, b, c] \mapsto M_{[a, b, c]}\) restricts to a bijection between quadratic forms of discriminant \( D > 0 \) and primitive hyperbolic elements of norm \( \varepsilon_D^2 \).

As a consequence, we have an explicit description for the closed geodesics of \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \).
Theorem 1. Let $\gamma \subseteq \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ be a closed geodesic and let $\tilde{\gamma}$ be a lift of $\gamma$ to $\mathbb{H}$. Let $\theta_1, \theta_2 \in \mathbb{R}$ be the endpoints of $\tilde{\gamma}$. Then,

(i) there exists a fundamental discriminant $D > 0$ such that $\gamma$ has length $\log(\varepsilon_D^2)$,
(ii) there exists a quadratic form $[a,b,c]$ of discriminant $D$ such that $M_{[a,b,c]}$ fixes the endpoints of $\tilde{\gamma}$. In this case, $\theta_1$ and $\theta_2$ are roots of the polynomial $ax^2 + bx + c$ and in particular, $\theta_1, \theta_2 \in \mathbb{Q}(\sqrt{D})$ are Galois conjugates.

Finally, let $D > 0$ be a fundamental discriminant and $K = \mathbb{Q}(\sqrt{D})$. There is a natural bijection between the narrow class group $H^+(K)$ of $K$ and the equivalence classes of quadratic forms of discriminant $D$ modulo $\text{SL}_2(\mathbb{Z})$ (see the appendix in [9]). As a consequence, we have the following

Theorem 2. There exists a bijection between the narrow class group $H^+(K)$ and the set of closed geodesics of $\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ of length $\log(\varepsilon_D^2)$.

Remark 2.1. Alternatively, we could fix a set of primitive closed geodesics. In this case, whenever the quadratic form $[a,b,c]$ is in the same $\text{SL}_2(\mathbb{Z})$-orbit as $[-a,-b,-c]$, the length of the geodesic would be $\log(\varepsilon_D)$ instead of $2\log(\varepsilon_D)$. However, thanks to the normalization in (9), the restriction norm formula will not be affected.

We can now define the set

$$ C_D := \bigcup_{\gamma} \gamma $$

where the union is taken over all closed geodesics $\gamma$ in $\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}$ of length $\log(\varepsilon_D^2)$.

2.3. Hecke characters. In this paper, we will only be working with real quadratic number fields, and therefore, we define Hecke characters only in this particular context. For more details, we refer the reader to [16], Chapter 7.

Definition 4 (Hecke characters). Let $K = \mathbb{Q}(\sqrt{D})$ be a real quadratic field with discriminant $D > 0$ and let $\varepsilon_D > 1$ be a generator of $U^+(K)$. Consider a decomposition $H^+(K) \cong \mathbb{Z}/h_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/h_s\mathbb{Z}$ of $H^+(K)$ into cyclic groups and fix $J_1, \ldots, J_s$ ideals in $K$ such that $[J_i]$ generates the cyclic factor $\mathbb{Z}/h_i\mathbb{Z}$ in $H^+(K)$. Then, for any class group character $\psi$ and any $n \in \mathbb{Z}$ we define the Hecke character $\chi_{\psi,n}$ as

$$ \chi_{\psi,n}([a]) = \psi([a]) \left| \frac{x}{a^x} \right|^{\varepsilon_D n} $$

for $a = xJ_{t_1}^1 \cdots J_{t_s}^s$, $0 \leq t_i < h_i$.

Given a Hecke character $\chi_{\psi,m}$, we let $\theta_{\chi_{\psi,m}}$ be the corresponding theta function. This function is then a Maass form of level $D$, spectral parameter $\frac{m\pi}{\log(\varepsilon_D)}$ and nebentypus $\chi_D$, the Dirichlet character obtained using the Legendre symbol (for more details see [4], Chapter 1.9). The $n$-th Fourier coefficient of this Maass form are given by

$$ \lambda_{\theta_{\chi_{\psi,m}}}(n) = \begin{cases} \frac{1}{2} \sum_{N([a]) = |n|} \chi_{\psi,m}([a]) \\
|n| \sum_{N([a]) = \pm |n|} \operatorname{sgn}(n) \chi_{\psi,m}([a]) \end{cases} $$

Using the fact that Hecke characters are unitary, we can have the following estimate

Lemma 2.1. For a Hecke character $\chi$, we have $\lambda_{\theta_{\chi}}(n) \ll_{\varepsilon} |n|^\varepsilon$ for any $\varepsilon > 0$. 

24. L-functions. From now on, all Maass forms are assumed to be Hecke-Maass cusp forms. Given a Maass form \( f \) and \( n \in \mathbb{Z} \) we denote the \( n \)-th Fourier coefficient of \( f \) by \( \lambda_{f}(n) \).

In this paper we will be working with two types of L-functions: the Rankin-Selberg convolution \( L(s, f \times \theta_{\chi}) \) where \( f \) is a Maass form of level 1 and \( \chi \) is a Hecke character, and the adjoint square L-function \( L(1, \text{Ad } f) \) where \( f \) is once again a Maass form. For the definitions and properties of these L-functions, we refer the reader to [6] (Section 1.1) and [11] (p. 136–138). We briefly state a few results we will need in the proof of the main theorem.

Definition 5. For any \( s \in \mathbb{C} \) and \( x \in \mathbb{R} \) we define the function

\[
\gamma(s, f, x) := \pi^{-2s} \prod_{\pm} \Gamma \left( \frac{s + i(t \pm \frac{\pi x}{\log(\varepsilon D)})}{2} \right) \Gamma \left( \frac{s - i(t \pm \frac{\pi x}{\log(\varepsilon D)})}{2} \right).
\]

When the Maass forms \( f \) and \( \theta_{\chi_{\psi, n}} \) have the same parity we have \( \gamma(s, f, n) = \gamma(s, f \times \theta_{\chi_{\psi, n}}) \).

Lemma 2.2. Let \( f \) and \( g \) be two Maass forms with respective spectral parameters \( t \) and \( r \), both real and suppose that \( f \) and \( g \) have the same parity. Then, for \( s \in \mathbb{R} \) fixed,

\[
q_{\infty}(s, f \times g) \asymp_{s} (1 + (t + r)^2)(1 + (t - r)^2)
\]

where \( q_{\infty}(s, f \times g) \) is the analytic conductor of the L-function \( L(s, f \times g) \).

Using Stirling’s formula, we can estimate the gamma factors of the Rankin-Selberg and adjoint square L-function as follows

Lemma 2.3. Let \( f \) and \( g \) be Hecke-Maass cusp forms with real spectral parameters \( t \) and \( r \) respectively and suppose \( f \) and \( g \) have the same parity. For \( s = 1/2 \), we have

\[
\gamma \left( \frac{1}{2}, f \times g \right) \ll (1 + |t + r|^2)^{-\frac{1}{2}}(1 + |t - r|^2)^{-\frac{1}{2}}(e^{-\frac{\pi}{2}|t + r|})(e^{-\frac{\pi}{2}|t - r|})^2
\]

where the implied constant is absolute.

Lemma 2.4. Let \( f \) be a Hecke-Maass cusp form of level \( q \geq 1 \) with spectral parameter \( t \in \mathbb{R} \). Then,

\[
\gamma(1, \text{Ad } f) \gg e^{-\pi|t|}
\]

where the implied constant is absolute.

Finally, we conclude this section with a theorem by Hoffstein and Lockhart.

Theorem 3. Let \( f \) be a Hecke-Maass cusp form of level \( q = 1 \) and Laplace eigenvalue \( \lambda \). For any \( \varepsilon > 0 \), we have

\[
L(1, \text{Ad } f) \gg_{\varepsilon} \lambda^{-\varepsilon}.
\]

Proof. See [10].

25. Approximate functional equation. To prove Theorem A we will need to estimate the central values of the L-functions \( L(s, f \times \theta_{\chi}) \) i.e. the values \( L(1/2, f \times \theta_{\chi}) \) where \( \theta_{\chi} \) is the theta function of some Hecke character \( \chi \). Since \( s = 1/2 \) lies in the critical strip, we will use the so called approximate functional equation. For the proofs of the approximate functional equation and Lemma 2.5, we refer the reader to [11] pages 98 and 100 respectively.

Theorem 4 (Approximate functional equation). Let \( f, g \) be self-dual Maass cusp forms such that the completed L-function \( \Lambda(s, f \times g) \) is entire. Let \( G(u) = e^{u^2} \), keeping the notation of Section 2.4 we have

\[
L(1/2, f \times g) = (1 + \varepsilon(f \times g)) \sum_{m \geq 1} \chi_{f}(m)\chi_{g}(m) \sum_{n \geq 1} \frac{\lambda_{f}(n)\lambda_{g}(n)}{\sqrt{n}} V_{1/2} \left( \frac{m^2n}{\sqrt{q(f \times g)}} \right).
\]
where \( V_s(y) := \frac{1}{2\pi i} \int_{\mathfrak{R}(u)=3} \frac{\gamma(s+u, f \times g)}{\gamma(s, f \times g)} y^{-u} G(u) \frac{du}{u} \) and \( \varepsilon(f \times g) \) is the root number of \( L(s, f \times g) \).

Because of our choice of \( G(u) \), \( V_s(y) \) behaves essentially like a bump function with respect to \( y \). More precisely,

**Lemma 2.5.** Fix \( A > 0 \). Suppose \( \mathfrak{R}(s+\kappa_j) \geq 3\alpha > 0 \) for \( j = 1, \ldots, 4 \) where \( \kappa_j \) are the local parameters of \( L(s, f \times g) \) at infinity. Then, the function \( V_s(y) \) defined in Theorem 2 satisfies

\[
V_s(y) \ll_{\alpha,A} \left( 1 + \frac{y}{\sqrt{q_\infty(s,f \times g)}} \right)^{-A}.
\]

Using the fast decay of \( V_s(y) \), we can ignore all terms in the approximate functional equation which are large compared to the conductor \( q_\infty(1/2, f \times g) \).

**Corollary 2.1.** Let \( f, g \) be self-dual Maass cusp forms with respective Laplace eigenvalues \( t, r \in \mathbb{R} \). Assume that the completed \( L \)-function \( \Lambda(s, f \times g) \) is entire and the root number \( \varepsilon(f \times g) = 1 \). Then, for any \( B, c, \delta > 0 \) and \( M \geq c q_\infty(1/2, f \times g)^{1/2+\delta} \) we have

\[
L(1/2, f \times g) = 2 \sum_{m=1}^{M} \frac{\chi_f(m) \chi_g(m)}{m} \sum_{n=1}^{M} \frac{\lambda_f(n) \lambda_g(n)}{\sqrt{n}} V_1 \left( \frac{m^2 n}{\sqrt{q(f \times g)}} \right) + O_{c,B,\delta}(q_\infty(1/2, f \times g)^{-B})
\]

where \( V_s(y) \) is the function described in Lemma 2.5 and the implied constant depends only on \( c, B, \delta \) and \( q := q(f \times g) \).

**Remark 2.2.** In case \( f \) is a Maass form and \( \theta_{\chi, y} \) is the theta function associated to the Hecke character \( \chi \), \( V_s(y) \) depends only on \( n \) and not on \( \psi \) (since the gamma factor depends only on \( n \)). Using the function \( \gamma(s, f, x) \), we can define the function \( V_s(y, x) \) in the obvious way so that \( V_s(y, n) = V_s(y) \) whenever \( n \) is constant and \( f \) have the same parity.

Combining the approximate functional equation, Lemma 2.1 and Iwaniec’s bounds towards the Ramanujan conjecture from [12], we have a uniform convexity bound for \( L(1/2, f \times \theta_\chi) \). More precisely,

**Proposition 2.3 (Convexity bound).** Let \( f \) be a Hecke-Maass cusp form of level 1 and spectral parameter \( t \) and let \( \chi := \chi_{\psi, x} \) be a Hecke character for the quadratic number field \( \mathbb{Q}(\sqrt{D}) \). Then, for any \( \varepsilon > 0 \) we have

\[
L(1/2, f \times \theta_{\chi, y}) \ll_{\varepsilon,D} q_\infty(1/2, f \times \theta_{\chi, y})^{1/4+\varepsilon}.
\]

We conclude this subsection with Proposition 2.4 which roughly states that the derivatives of \( V_{1/2}(y, n) \) with respect to \( n \) are small. Intuitively, this implies that the approximate functional equation is more or less uniform in \( n \) over small intervals.

**Proposition 2.4.** Let \( f \) be a Maass form with spectral parameter \( t \geq 1 \) and let \( V_s(y, x) \) be defined as in Remark 2.2. Let \( 1 \leq T \ll D t \) and suppose that \( \frac{1}{2} \leq |c_D t - x| \leq 2T \) with \( c_D = \frac{\log(\varepsilon_D)}{\pi} \). Then, for any \( j \geq 1 \) we have

\[
\frac{\partial^j}{\partial x^j} V_{1/2}(y, x) \ll_{\varepsilon,j} y^{-\frac{j}{2}} \varepsilon D^{-j}.
\]

**Proof.** Let \( \delta := \varepsilon/4 \). Using Cauchy’s theorem we can move the integration in the definition of \( V_{1/2}(y, x) \) to the line \( \Re(s) = \delta \). Let \( F(u, f, x) := \frac{\gamma(1/2 + u, f \times g)}{\gamma(1/2, f \times g)} \) so that \( V_{1/2}(y, x) = I_1(y, x) + I_2(y, x) \) where we define

\[
I_1(y, x) := \frac{1}{2\pi i} \int_{\Re(u)=\delta} F(u, f, x) y^{-u} e^{u^2} du \quad \text{and} \quad I_2(y, x) := \frac{1}{2\pi i} \int_{\Re(u)=\delta} F(u, f, x) y^{-u} e^{u^2} du.
\]
Using Stirling’s formula and estimates of the polygamma functions, we can prove that for \( j \geq 0 \) and \( \Re(u) = \delta \) we have \( \frac{\partial^j}{\partial x^j} F(u, f, x) \ll_{j, \delta, D} |u| |F(u, f, x)| \ll_{j, \delta, D} |u| e^{2\pi|\Im(u)|} t^{4\delta} \) and, if in addition \( |\Im(u)| \leq \frac{4}{5} \), we then have

\[
\frac{\partial^j}{\partial x^j} F(u, f, x) \ll_{j, \delta, D} |u| |F(u, f, x)| T^{-j} \ll_{j, \delta, D} |u| e^{2\pi|\Im(u)|} t^{4\delta}.
\]

As a consequence, \( I_1(y, x) \) and \( I_2(y, x) \) both satisfy the conditions for differentiation under the integral sign which leads to the estimates

\[
\frac{\partial^j}{\partial x^j} I_1(y, x) \ll_{j, \delta, D} y^{-\delta} t^{4\delta} T^{-j} \int_{|\Re(u)| = \delta} e^{2\pi|\Im(u)|} e^{8\delta - |\Im(u)|^2} du \ll_{j, \delta, D} y^{-\delta} t^{4\delta} T^{-j}
\]

and

\[
\frac{\partial^j}{\partial x^j} I_2(y, x) \ll_{j, \delta, D} y^{-\delta} t^{4\delta} e^{-\frac{1}{2} (T/8 - \pi)} \int_0^\infty e^{2\pi w} e^{-w^2} dw \ll_{j, \delta, D} y^{-\delta} t^{4\delta} T^{-j}.
\]

\( \square \)

2.6. Dyadic partition of unity. Partitions of unity allow us to break down sums over \( \mathbb{Z} \) into sums over smaller intervals which are sometimes easier to estimate (see [2], Section 2.8).

**Proposition 2.5.** There exists a smooth function \( W(x) \) supported on \( \left[ \frac{1}{2}, 2 \right] \) and such that

\[
\sum_{k \geq 0} W\left( \frac{x}{2^k} \right) = 1
\]

for any \( x \geq 1 \).

2.7. Lipschitz principle. Finally, in the proof of Theorem A, we will need to estimate the number of lattice points contained inside a parallelogram in \( \mathbb{R}^2 \). The Lipschitz principle gives an upper bound with respect to the area of the parallelogram and the length of its boundary.

**Proposition 2.6.** Let \( R \) be a closed and bounded region in \( \mathbb{R}^2 \), \( L \) the length of the boundary of \( R \) and \( A \) the area of \( R \). Then,

\[
|R \cap \mathbb{Z}^2| \ll L + A + 1.
\]

**Proof.** This follows from the Lipschitz principle [7] or Jarnik’s inequality [22]. \( \square \)

**Corollary 2.2.** Let \( \Lambda \) be a lattice in \( \mathbb{R}^2 \) with (ordered) basis \( B = \{ \alpha_1, \alpha_2 \} \) and let \( v_1, v_2 \in \mathbb{R}^2 \) be two linearly independent vectors. If we let \( P \) be the parallelogram \( P := \{ av_1 + bv_2 \mid a, b \in [0, 1] \} \) then

\[
|P \cap \Lambda| \ll_B (||v_1|| + ||v_2||) + ||v_1||, ||v_2|| + 1
\]

where the implied constant depends only on the basis \( B \) of \( \Lambda \).

**Proof.** Let us first assume that \( \Lambda = \mathbb{Z}^2 \). Then, by Proposition 2.6 we know that \( |P \cap \mathbb{Z}^2| \ll L + A + 1 \) where \( L = 2(||v_1|| + ||v_2||) \) is the perimeter of \( P \) and \( A = ||v_1 \times v_2|| \leq ||v_1||, ||v_2|| \). The result follows immediately in this case.

Now let \( \Lambda \) be any lattice and \( B \) be a fixed ordered basis. We reduce to the case above using an appropriate linear transformation. More precisely, let \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear isomorphism defined by \( T(\alpha_1) = (1, 0) \) and \( T(\alpha_2) = (0, 1) \). Note that \( T \) only depends on our (ordered) basis \( B \), and therefore, so does its operator norm. In other words, we have \( ||Tv_1|| \ll_B ||v_1|| \) and \( ||Tv_2|| \ll_B ||v_2|| \). In particular, since \( TP \) is the parallelogram \( TP = \{ a(Tv_1) + b(Tv_2) \mid a, b \in [0, 1] \} \), we obviously have the estimate

\[
|TP \cap \mathbb{Z}^2| \ll_B (||v_1|| + ||v_2||) + ||v_1||, ||v_2|| + 1.
\]

Finally, \( T \) is an isomorphism and \( TA = \mathbb{Z}^2 \), hence \( |P \cap \Lambda| = |TP \cap \mathbb{Z}^2| \) which concludes the proof. \( \square \)
3. Proof of Theorem A

We are finally ready to prove the main result. In what follows, we will let \( f \in L^2(\text{SL}_2(\mathbb{Z}) \setminus \mathbb{H}) \) be a Hecke-Maass cusp with spectral parameter \( t \geq 2 \).

3.1. Restriction norm formula. We start by using representation theory of compact abelian groups (or equivalently, Fourier theory of compact abelian groups) to derive an explicit formula for the restriction norm \( \|f|_{C_D}\|_2^2 \).

Keeping the notation defined in Section 2, we consider a closed geodesic \( \gamma \) in \( \text{SL}_2(\mathbb{Z}) \setminus \mathbb{H} \) of length \( \log(\varepsilon_D^2) \) and a lift \( \tilde{\gamma} \) of \( \gamma \) in \( \mathbb{H} \). Then \( \tilde{\gamma} \) is a semi-circle perpendicular to the real axis with endpoints \( \omega, \omega^* \in K \subseteq \mathbb{R} \) with \( \omega > \omega^* \) (as usual \( \omega^* \) denotes the Galois conjugate of \( \omega \)). Consider the matrix

\[
\kappa_\gamma := \frac{1}{\sqrt{\omega - \omega^*}} \begin{pmatrix} 1 & -\omega^* \\ 1 & -\omega \end{pmatrix} \in \text{SL}_2(\mathbb{R}).
\]

One easily checks that \( \kappa_\gamma, \omega = 0 \) and \( \kappa_\gamma, \omega^* = \infty \), and therefore, \( \kappa_\gamma \tilde{\gamma} = \gamma_0 \) where \( \gamma_0 = i\mathbb{R}^+ \) is the imaginary axis in the upper half plane. Since the hyperbolic measure \( ds \) is invariant under the action of \( \text{SL}_2(\mathbb{R}) \), for any \( g: \tilde{\gamma} \to \mathbb{C} \) we have \( \int_{\tilde{\gamma}} g(z) \, ds = \int_0^\infty g(\kappa_\gamma^{-1} iy) \frac{dy}{y} \). For any \( g: \gamma \to \mathbb{C} \), this leads to

\[
(5) \quad \int_{\gamma} g(z) \, ds = \int_1^{\varepsilon_D^2} g(\kappa_\gamma^{-1} iy) \frac{dy}{y}.
\]

We can therefore express integration over \( \gamma \) using the Lebesgue integral over \( \mathbb{R} \). Introducing the change of variables \( x := \log(\varepsilon_D^2)^{-1} \log(y) \) or equivalently, \( y = e^{\log(\varepsilon_D^2)x} \), (5) becomes

\[
(6) \quad \int_{\gamma} g(z) \, ds = \log(\varepsilon_D^2) \int_0^1 g(\kappa_\gamma^{-1} i e^{\log(\varepsilon_D^2)x}) \, dx.
\]

By abuse of notation, we will denote the set \( \{ iy \mid 1 \leq y < \varepsilon_D^2 \} \) by \( \kappa_\gamma \gamma \). Geometrically, if we define a parametrization of \( \kappa_\gamma \gamma \) as follows

\[
\Phi_0: [0, 1) \to \kappa_\gamma \gamma \\
\quad x \mapsto i e^{\log(\varepsilon_D^2)x}
\]

then, the right hand side of (6) is the integral of the pullback \( \Phi_0^* g(\kappa_\gamma^{-1} \cdot) \) along \( [0, 1) \). As a consequence, after canonically identifying \( [0, 1) \) to \( \mathbb{R}/\mathbb{Z} \) (which we endow with the Haar probability), we have a homeomorphism between \( \kappa_\gamma \gamma \) and a compact abelian group which is compatible with the measures on both spaces up to scaling. In particular, it preserves the corresponding \( L^2 \)-inner products in the sense that

\[
(7) \quad \langle g_1, g_2 \rangle = \log(\varepsilon_D^2) \langle \Phi_0^* g_1, \Phi_0^* g_2 \rangle
\]

for every \( g_1, g_2: \kappa_\gamma \gamma \to \mathbb{C} \). Of course, the inner product on the left hand side takes place in \( L^2(\kappa_\gamma \gamma) \) while the inner product on the right hand side takes place in \( L^2(\mathbb{R}/\mathbb{Z}) \).

Using Theorem 2 we can therefore parametrize the union of closed geodesics \( C_D \) using

\[
\Xi: \mathbb{R}/\mathbb{Z} \times H^+(K) \to C_D \\
\quad ([x], [a]) \mapsto \kappa_a^{-1} i e^{\log(\varepsilon_D^2)x}
\]

where \( \gamma_a \) is the closed geodesic corresponding to the class group element \([a]\). When we endow the finite group \( H^+(K) \) with the discrete topology, \( \Xi \) is a homeomorphism which is compatible with
integration on both spaces. Indeed, for any integrable \( g : \mathcal{C}_D \rightarrow \mathbb{C} \) we have
\[
\int_{\mathcal{C}_D} g(z) \, ds = \sum_{[a] \in H^+(K)} \int_{\gamma[a]} g(z) \, ds
\]
\[
= \log(\varepsilon_D^2) |H^+(K)| \int_{\mathbb{R}/\mathbb{Z} \times H^+(K)} \Xi^* g(z, [a]) \, d(z, [a])
\]
using the Fubini theorem and \([6]\). In terms of \( L^2 \) inner-products this translates to
\[
(g_1, g_2) = |C_D| \langle \Xi^* g_1, \Xi^* g_2 \rangle
\]
for any \( g_1, g_2 : \mathcal{C}_D \rightarrow \mathbb{C} \). We can therefore use Plancherel to express the \( L^2 \) norm of any \( g \in L^2(\mathcal{C}_D) \).

**Theorem 5.** For any \( g \in L^2(\mathcal{C}_D) \), we have
\[
\|g\|_2^2 = \frac{1}{|C_D|} \sum_{\psi \in H^+(K)} \sum_{n \in \mathbb{Z}} I(g, \chi_{\psi, n}).
\]

where \( \chi_{\psi, n} \) is the Hecke character obtained from \( \psi \in \hat{H}(K) \), \( n \in \mathbb{Z} \) and
\[
I(g, \chi_{\psi, n}) := \left| \sum_{[a] \in H^+(K)} \psi([a])^{-1} \int_1^{e_D^2} g(k^{-1} \gamma_i y)y^{-\frac{\pi n}{\log(\varepsilon_D)}} \, dy \right|^2.
\]

**Proof.** Using \([5]\) we have \( \|g\|_2^2 = |C_D| \langle \Xi^* g, \Xi^* g \rangle \) where the inner product on the right hand side takes place in \( L^2(\mathbb{R}/\mathbb{Z} \times H^+(K)) \). Since \( G := \mathbb{R}/\mathbb{Z} \times H^+(K) \) is a compact abelian Lie group equipped with the Haar probability, the characters of \( G \) form an orthonormal basis for \( L^2(G) \). The dual group of \( G \) is equal to
\[
\hat{G} = \{ \psi_{\psi, n} : (x, [a]) \mapsto e^{2\pi i nx} \psi([a]) \mid n \in \mathbb{Z}, \psi \in \hat{H}(K) \}.
\]

Therefore by Plancherel,
\[
\|g\|_2^2 = |C_D| \sum_{\psi \in H^+(K)} \sum_{n \in \mathbb{Z}} \left| \langle \Xi^* g, \psi_{\psi, n} \rangle \right|^2
\]
where the inner products on the right hand side take place in \( L^2(G) \). Finally, for any \( \psi \) and \( n \), we have
\[
\langle \Xi^* g, \psi_{\psi, n} \rangle = |H^+(K)|^{-1} \sum_{[a] \in H^+(K)} \int_0^{e_D^2} g(k^{-1} \gamma_i y) \frac{y^{-\frac{\pi n}{\log(\varepsilon_D)}}}{y} \, dy
\]
\[
= |C_D|^{-1} \sum_{[a] \in H^+(K)} \psi([a])^{-1} \int_0^{e_D^2} g(k^{-1} \gamma_i y) \frac{y^{-\frac{\pi n}{\log(\varepsilon_D)}}}{y} \, dy
\]
after applying the change of variables \( y := e^{\log(e_D^2) x} \). Combining everything, we recover \([9]\). \( \square \)

Whenever \( f \in L^2(\text{SL}_2(\mathbb{Z}) \backslash \mathbb{H}) \) is a Hecke-Maass cusp form, it follows from Waldspurger’s formula (Theorem 6.3.1 in \([17]\) or more generally Theorem 4.1 in \([15]\)) that up to some positive constant,
\[
I(f, \chi) \asymp \frac{1}{\Lambda(1, \text{Ad} f)} \Lambda(1/2, f \times \theta \chi)
\]
where \( \Lambda(1, \text{Ad} f) \) is the completed adjoint square L-function of \( f \) and \( \Lambda(1/2, f \times \theta \chi) \) is the completed Rankin-Selberg L-function. In this case, the restriction norm formula becomes
\[
\|f|_{\mathcal{C}_D}^2 \asymp \frac{1}{|C_D|} \sum_{\psi \in H^+(K)} \sum_{n \in \mathbb{Z}} \frac{1}{\Lambda(1, \text{Ad} f)} \Lambda(1/2, f \times \theta \psi, n).
\]
Remark 3.1. Whenever $f$ and $n$ have different parities (i.e. $f$ is an even Maass form and $n$ is an odd number or $f$ is an odd Maass form and $n$ is an even number), the integral $I(f, \chi_{\psi,n})$ vanishes by symmetry. Therefore, we are actually taking the sum over all $n$ with the same parity as $f$.

By definition, for any Hecke character $\chi_{\psi,n}$ of $K$ we have

$$\frac{\Lambda(1/2, f \times \theta_{\chi_{\psi,n}})}{\Lambda(1, \Ad f)} = \frac{D_{\psi}^z \gamma(1/2, f \times \theta_{\chi_{\psi,n}}) L(1/2, f \times \theta_{\chi_{\psi,n}})}{\gamma(1, \Ad f) L(1, \Ad f)}.$$

If we define

$$G(n) := \frac{\gamma(1/2, f \times \theta_{\chi_{\psi,n}})}{\gamma(1, \Ad f)}$$

to be the corresponding quotient of gamma factors, then, by Theorem 3 we have the estimate

$$\frac{1}{|C_D|} \sum_{\psi \in H^+(K)} \sum_{n \in \mathbb{Z}} \frac{1}{\Lambda(1, \Ad f)} \Lambda(1/2, f \times \theta_{\chi_{\psi,n}}) \ll \varepsilon \frac{D_{\psi}^z}{|C_D|} \sum_{\psi \in H^+(K)} \sum_{n \in \mathbb{Z}} G(n) L(1/2, f \times \theta_{\chi_{\psi,n}})$$

for any $\varepsilon > 0$.

Remark 3.2. For any $z \in \mathbb{C}$, $\Gamma(z)\Gamma(\overline{z}) = |\Gamma(z)|^2 \geq 0$. If follows from the definition of $G(n)$ that $G(n) \geq 0$ for all $n \in \mathbb{Z}$. As a consequence of Waldspurger’s formula we must also have $L(1/2, f \times \theta_{\chi_{\psi,n}}) \geq 0$ for any $\psi \in H^+(K)$ and $n \in \mathbb{Z}$.

3.2. Estimation of the gamma factors. To estimate the restriction norm, we start by applying Stirling’s formula to $G(n)$. As a consequence, we will deduce that the right hand side of (11) is essentially a finite sum.

Lemma 3.1. For any $n \in \mathbb{Z}$, let $r := \frac{\pi n}{\log(\varepsilon D)}$. Then,

$$G(n) \ll (1 + |t + r|^2)^{-\frac{1}{4}} (1 + |t - r|^2)^{-\frac{1}{4}} e^{-\pi \max\{|r|, |t|-|t|\}}$$

absolutely. Moreover, if $|n| \geq 2 \frac{\log(\varepsilon D)}{\pi} t$, then for any $\varepsilon > 0$ we have

$$G(n) \ll \varepsilon (1 + |t + r|^2)^{-\frac{1}{4} - \varepsilon} (1 + |t - r|^2)^{-\frac{1}{4} - \varepsilon} e^{-\frac{\pi}{4} |r|}.$$

Proof. By Remark 3.1 we may assume $n$ and $f$ have the same parity. Using Lemmas 2.3 and 2.4 we have

$$G(n) \ll \frac{(1 + |t + r|^2)^{-\frac{1}{4}} (1 + |t - r|^2)^{-\frac{1}{4}} (e^{-\frac{\pi}{4} |t+r|})^2 (e^{-\frac{\pi}{4} |t-r|})^2}{e^{-\pi |t|}}$$

absolutely. Using the well known equality $|t + r| + |t - r| = 2 \max\{|t|, |r|\}$, we get (12). To prove (13), we assume that $|r| \geq 2|t|$. In this case, $\max\{|t|, |r|\} = |r|$ and therefore

$$(e^{-\frac{\pi}{4} |t+r|})^2 (e^{-\frac{\pi}{4} |t-r|})^2 \leq e^{-\pi |t|} e^{-\frac{\pi}{4} |t+r|} e^{-\frac{\pi}{4} |t-r|} \ll \varepsilon e^{-\pi |t|} e^{-\frac{\pi}{4} |r|} (1 + |t + r|^2)^{-\varepsilon} (1 + |t - r|^2)^{-\varepsilon}.$$

Let $c_D := \frac{\log(\varepsilon D)}{\pi}$. We can split the sum we want to estimate as follows

$$\frac{t^z D_{\psi}^z}{|C_D|} \sum_{|n| \leq c_D} \sum_{\psi \in H^+(K)} G(n) L(1/2, f \times \theta_{\chi_{\psi,n}}) + \frac{t^z D_{\psi}^z}{|C_D|} \sum_{|n| > c_D} \sum_{\psi \in H^+(K)} G(n) L(1/2, f \times \theta_{\chi_{\psi,n}}).$$

Then, by Lemma 3.1 we expect the second part to be a negligible error term. We start by proving that it is indeed the case.
Proposition 3.1. For any \( \varepsilon > 0 \), we have
\[
\sum_{|n| > c_D t} \sum_{\psi \in H^+(K)} G(n) L(1/2, f \times \theta_{\chi, n}) \ll_{D, \varepsilon} t^\varepsilon.
\]

Proof. Using Proposition 2.3 and Lemma 2.2 we estimate \( L(1/2, f \times \theta_{\chi, n}) \) uniformly in \( \chi_{\psi, n} \). We then have
\[
L(1/2, f \times \theta_{\chi, n}) \ll_{\varepsilon} (1 + |t + n/c_D|^2)^{1/2 + \varepsilon}(1 + |t - n/c_D|^2)^{1/2 + \varepsilon}.
\]
Combining this estimate with (13) and taking the sum over all \( n \geq 2c_D t \) we see that
\[
\sum_{|n| \geq 2c_D t} G(n) L(1/2, f \times \theta_{\chi, n}) \ll_{D, \varepsilon} \sum_{n \geq 2c_D t} e^{-n/4c_D} \ll_{D, \varepsilon} t^{\varepsilon}.
\]
On the other hand, by (12) for \( c_D t < |n| < 2c_D t \) we have
\[
G(n) L(1/2, f \times \theta_{\chi, n}) \ll_{\varepsilon} (1 + |t + n/c_D|^2)^{1/2} (1 + |t - n/c_D|^2)^{1/2} e^{-\pi(n/2D - |t|)}.
\]
But in this case, \( |t + n/c_D| \leq 3|t| \) and \( |t - n/c_D| \leq |t| \) so that
\[
\sum_{c_D t < |n| < 2c_D t} G(n) L(1/2, f \times \theta_{\chi, n}) \ll_{\varepsilon} t^{4\varepsilon} \sum_{n = |c_D t|} \infty e^{-\pi(n/2D - |t|)} \ll_{\varepsilon, D} t^{4\varepsilon}.
\]
Finally, note that both estimates are independent of \( \psi \). Since \( |H^+(K)| \) is a finite constant depending only on \( D \), we have proved the proposition. \( \square \)

Remark 3.3. Applying Lemma 3.1 and the convexity bound to the main term leads to the bound
\[
\|f|c_D|\|_2 \ll_{\varepsilon, D} t^{1+\varepsilon}
\]
which is significantly worse than the bound from Theorem A. It is also interesting to note that the Lindelöf hypothesis automatically implies the bound \( \|f|c_D|\|_2 \ll_{\varepsilon, D} t^{\varepsilon} \).

3.3. First smooth partition of unity. We now move on to the main term. We start by introducing a smooth partition of unity centered about \( c_D t \). More precisely, let \( \tilde{U}(x) \) be a smooth bump function which is equal to 1 on \([0, \infty)\) and supported on \([-1/2, \infty)\). We can define the smooth bump function
\[
U(x) := \tilde{U}\left(\frac{x}{c_D t}\right)
\]
which is equal to 1 on \([0, \infty)\) and is supported on \([-c_D t/2, \infty)\). It is easy to see that for any \( x \in \mathbb{R} \), we have the inequality \( U(x) + U(-x) \geq 1 \). But for any \( n \in \mathbb{Z} \) and \( \psi \in H^+(K) \) we know that \( G(n) L(1/2, f \times \theta_{\chi, n}) \geq 0 \), \( G(n) = G(-n) \) and \( \theta_{\chi, n} = \theta_{\chi, -n} \), hence,
\[
\sum_{|n| \leq c_D t} G(n) L(1/2, f \times \theta_{\chi, n}) \leq 2 \sum_{|n| \leq c_D t} U(n) G(n) L(1/2, f \times \theta_{\chi, n}).
\]
Now let \( W(x) \) be the function obtained using Proposition 2.5 For each \( k \geq 0 \) we define the function
\[
W_k(x) := W\left(\frac{c_D t - x}{2^k}\right)
\]
By construction, for any \( k \geq 0 \), \( W_k \) is supported on \( 2^{k-1} \leq c_D t - x \leq 2^{k+1} \) and
\[
\sum_{k \geq 0} W_k(x) = \begin{cases} 
1 & \text{for } c_D t - x \geq 1 \\
0 & \text{for } x \geq c_D t.
\end{cases}
\]
Using Stirling’s formula and the convexity bound (Proposition 2.3), we can show that for any \( \varepsilon > 0 \) and \( \psi \in \mathcal{H}^+(K) \), we have
\[
(16) \quad \sum_{|n| \leq cDt} U(n)G(n)L(1/2, f \times \theta_{\psi,n}) = \sum_{k \geq 0} \sum_{n \in \mathbb{Z}} W_k(n)U(n)G(n)L(1/2, f \times \theta_{\psi,n}) + O_{D, \varepsilon}(t^\varepsilon).
\]
Let \( k_D := \log_2(\frac{3c_D t}{2}) + 1 \). Obviously, for \( t \geq 2 \) we have \( k_D \ll_D \log(t) \). Moreover, since \( U(n)W_k(n) = 0 \) for all \( n \in \mathbb{Z} \) whenever \( 2^{k-1} \geq \frac{3c_D t}{2} \), all terms in (16) with \( k \geq k_D \) may be ignored. Therefore
\[
||f|c_D||^2 \ll_{D, \varepsilon} t^\varepsilon \sup_{0 \leq k \leq k_D} \sum_{\psi \in \mathcal{H}^+(K)} \sum_{n \in \mathbb{Z}} W_k(n)U(n)G(n)L(1/2, f \times \theta_{\psi,n}) + t^\varepsilon
\]
for any \( \varepsilon > 0 \).

In other words, introducing a partition of unity allowed us to replace the sum in the restriction norm formula by sums over smaller intervals. On each such interval, the conductor \( q_\infty(1/2, f \times \theta_{\psi,n}) \) (and therefore the gamma factor \( G(n) \)) is almost constant with respect to \( n \). More precisely,

\textbf{Lemma 3.2.} Let \( 0 \leq k \leq k_D \) and write \( T := 2^k \). Then,
\[
(17) \quad W_k(n)G(n) \ll_D (Tt)^{-\frac{1}{2}}
\]
for any \( n \in \mathbb{Z} \) and
\[
(18) \quad q_\infty(s, f \times \theta_{\psi,n}) \asymp_D (Tt)^{2}
\]
for any \( n \geq -\frac{c_D t}{2} \) satisfying \( T/2 \leq c_D t - n \leq 2T \). In both cases, the implied constants depend only on \( D \).

\textbf{Proof.} For any \( n \geq -\frac{c_D t}{2} \) such that \( T/2 \leq c_D t - n \leq 2T \), we have \( t + \frac{n}{c_D t} \asymp t \). Since \( W_k \) is supported on \( T/2 \leq c_D t - n \leq 2T \), the results follow immediately from Lemmas 3.1 and 2.2 respectively. \( \square \)

Hence, as an immediate consequence of (17), for any \( T := 2^k \) satisfying the conditions of the lemma above
\[
\sum_{\psi \in \mathcal{H}^+(K)} \sum_{n \in \mathbb{Z}} W_k(n)G(n)U(n)L(1/2, f \times \theta_{\psi,n}) \ll_D (Tt)^{-\frac{1}{2}} \sum_{\psi \in \mathcal{H}^+(K)} \sum_{n \in \mathbb{Z}} W_k(n)U(n)L(1/2, f \times \theta_{\psi,n}).
\]

In what follows, we will prove that for any \( \varepsilon > 0 \), \( 0 \leq k \leq k_D \) and \( T := 2^k \) we have
\[
(19) \quad (Tt)^{-\frac{1}{2}} \sum_{\psi \in \mathcal{H}^+(K)} \sum_{n \in \mathbb{Z}} W_k(n)U(n)L(1/2, f \times \theta_{\psi,n}) \ll_{\varepsilon, D} t^{\varepsilon + 2\delta}
\]
with \( \theta \) any bound towards the Ramanujan conjecture.

\textbf{3.4. Character orthogonality.} We now fix a value of \( 0 \leq k \leq k_D \) and write \( T := 2^k \). The first step for proving (19) consists of writing out the approximate functional equation for \( L(1/2, f \times \theta_{\psi,n}) \) and applying character orthogonality for the finite group \( \mathcal{H}^+(K) \) to get some cancellations.

\textbf{Proposition 3.2.} For any \( k \) fixed, \( \delta > 0 \) and \( B \geq 1 \) we have
\[
(20) \quad (Tt)^{-\frac{1}{2}} \sum_{\psi \in \mathcal{H}^+(K)} \sum_{n \in \mathbb{Z}} W_k(n)U(n)L(1/2, f \times \theta_{\psi,n}) =
\]
\[
2|\mathcal{H}^+(K)| \sum_{m=1}^{(Tt)^{1+\delta}} \chi_D(m) \sum_{\alpha \in \mathcal{F}_D} \frac{\lambda_f(\alpha)}{\sqrt{|N(\alpha)|}} \Pi_\alpha(m) + O_{B, \delta, D}(t^{-B})
\]
where \( F_D \) is the fundamental domain of the action of \( U^+ (K) \) on \( O_K^+ \) from Proposition 2.1.

\[
\Pi_\alpha (m) := (T t)^{- \frac{1}{2}} \sum \sum \frac{\chi D(m)}{m} \frac{\lambda_f (\alpha)}{\sqrt{\lambda_f (\alpha)}} \Pi_\alpha (m) = (T t)^{- \frac{1}{2}} \sum \sum \frac{\chi D(m)}{m} \frac{\lambda_f (\alpha)}{\sqrt{\lambda_f (\alpha)}} \Pi_\alpha (m) W_\alpha (\alpha) U_T (\alpha).
\]

Proof. Fix \( n \in \mathbb{Z} \) such that \( W_k(n) U(n) \neq 0 \). Then, \( n \) lies in the support of \( W_k \) or equivalently, \( T/2 \leq c_D t - n \leq 2T \). By (18), \( q_{\infty} (1/2, f \times \theta_{\chi, n}) \ll_D (T t)^2 \) and therefore we can apply the approximate functional equation (Corollary 2.1) with \( M = (T t)^{1+\delta} \). By definition of \( \theta_{\chi, n} \) we then have

\[
\sum_{\psi \in H^+(K)} L(1/2, f \times \theta_{\chi, n}) = \sum_{\psi \in H^+(K)} \sum_{\alpha \in F} \frac{\chi_D(m)}{m} \frac{\lambda_f (\alpha)}{\sqrt{\lambda_f (\alpha)}} \Pi_\alpha (m) W_\alpha (\alpha) U_T (\alpha).
\]

where

\[
\Omega_\alpha := \sum_{\psi \in H^+(K)} \psi ([\alpha]) = \begin{cases} |H^+(K)| & \text{if } \alpha = (x) \text{ for some } x \in F_D \\ 0 & \text{else} \end{cases}
\]

by character orthogonality. We conclude the proof by taking the sum over \( n \in \mathbb{Z} \). Since the support of \( W_k(n) U(n) \) is contained in \([c_D t - 2T, c_D t - T/2]\), the sum consists of precisely \( 3T/2 \) terms which leads to the error term \( O_{\tilde{c}_D} (T^{-B+1} t^{-B}) = O_{\tilde{c}_D} (t^{-B}) \). On the other hand, we obtain the main term of (20) by permuting the order of summation and rearranging the terms. \( \square \)

To estimate the summation on the right hand side of (20) we introduce another smooth dyadic partition of unity. Let \( a_T \) be the smallest integer with \( a_T > (tT)^{1+\delta} \) and let \( U_T \) be a smooth bump function supported on \([0, a_T]\) and equal to 1 on \([1, (tT)^{1+\delta}]\). By abuse of notation, we write

\[
U_T (\alpha) := U_T (|N(\alpha)|)
\]

for all \( \alpha \in F_D \). Let \( W(x) \) be the function from Proposition 2.5 and define

\[
W_\alpha (\alpha) := W \left( \frac{|N(\alpha)|}{2^a} \right)
\]

for \( a \geq 0 \) and \( \alpha \in F_D \). Then, for all \( 1 \leq m \leq (T t)^{1+\delta} \) we have

\[
\chi D(m) \sum_{\alpha \in F_D} \frac{\lambda_f (\alpha)}{\sqrt{|N(\alpha)|}} \Pi_\alpha (m) = \sum_{\alpha \geq 0} \chi D(m) \sum_{\alpha \in F_D} \frac{\lambda_f (\alpha)}{\sqrt{|N(\alpha)|}} \Pi_\alpha (m) W_\alpha (\alpha) U_T (\alpha).
\]

Again, whenever \( 2^{a-1} \geq a_T \), \( W_\alpha (\alpha) U_T (\alpha) = 0 \) for all \( \alpha \in F_D \) so \( a \)-sum in the right hand consists of \( \ll_\delta \log(TT) \) pieces. Since \( T \ll_D t \), we therefore have the estimate

\[
(T t)^{- \frac{1}{2}} \sum \sum W_k(n) U(n) L(1/2, f \times \theta_{\chi, n}) \ll_D \epsilon \delta
\]

\[
t^\delta \sup_{0 \leq \epsilon \leq \sqrt{2} \log(TT)^{1+\delta}} \left| \sum m \chi D(m) \sum_{\alpha \in F_D} \frac{\lambda_f (\alpha)}{\sqrt{|N(\alpha)|}} \Pi_\alpha (m) W_\alpha (\alpha) U_T (\alpha) \right|.
\]
for any \( \varepsilon > 0, \delta > 0 \) and \( 1 \leq T \ll_d t \).

### 3.5. Poisson summation

We now fix \( 0 \leq a \leq a_T \) and write \( A := 2^a \). We can define the smooth compactly supported function

\[
H_A(\alpha, m, x) := \frac{(Tt)^{-\frac{1}{2}}}{\sqrt{|N(\alpha)|}} W_a'(\alpha) U_T(\alpha) W_k(x) U(x) V_{\frac{t}{2}}\left(\frac{m^2|N(\alpha)|}{D}, x\right).
\]

Notice that \( \left| \frac{\alpha}{\alpha} \right|^{\frac{1}{2}\log|\epsilon_D|} = e^{\frac{x\log|\alpha|}{2\log|\epsilon_D|}} \), and we can therefore apply the Poisson summation formula to the \( n \)-sum. Our problem then reduces to estimating sums of the form

\[
(21) \quad \sum_{m=1}^{(Tt)^{1+\delta}} \frac{\chi_D(m)}{m} \sum_{\alpha \in \mathcal{F}_D} \lambda_f(\alpha) \sum_{\xi \in \mathbb{Z}} \mathcal{H}_A(\alpha, \epsilon, \xi) \leq \frac{1}{|\epsilon|^j} |\mathcal{H}_A^{(j)}(\alpha, \epsilon, \xi)|
\]

where the Fourier coefficients of \( \tilde{H}_A(\alpha, m, \xi) \) are taken with respect to the \( x \) variable. In particular, we need to estimate the Fourier coefficients \( \mathcal{H}_A(\alpha, m, \xi) \). For \( \xi \neq 0 \), a standard argument using integration by parts gives us the inequality

\[
(22) \quad \mathcal{H}_A(\alpha, m, \xi) \leq \frac{1}{|\epsilon|^j} |\mathcal{H}_A^{(j)}(\alpha, m, \xi)|
\]

where \( H_A^{(j)}(\alpha, m, \xi) \) is the \( j \)-th partial derivative of \( H_A(\alpha, m, x) \) with respect to \( x \). We estimate the Fourier coefficients of \( H_A^{(j)}(\alpha, m, x) \) using the following proposition:

**Proposition 3.3.** Let \( j \geq 0 \). Then,

\[
H_A^{(j)}(\alpha, m, x) = 0 \text{ unless } x \in \left[ c_D t - 2T, c_D t - \frac{T}{2} \right] \text{ and } |N(\alpha)| \in \left[ \frac{A}{2}, 2A \right].
\]

Moreover, for any \( \varepsilon > 0 \) we have

\[
H_A^{(j)}(\alpha, m, x) \ll_{\varepsilon,j,D} T^j \left( T^j A \right)^{-\frac{1}{2}}.
\]

**Proof.** We know that \( W_a'(\alpha) \) is zero whenever \( |N(\alpha)| \notin \left[ \frac{A}{2}, 2A \right] \) and \( W_k(x) \) is zero whenever \( x \notin \left[ c_D t - 2T, c_D t - \frac{T}{2} \right] \). As a consequence, the first statement must hold.

We now prove the upper bound on the partial derivatives with respect to \( x \). Of course, we may assume that \( |N(\alpha)| \in \left[ \frac{A}{2}, 2A \right] \). Using the product rule for derivatives, we see that

\[
H_A^{(j)}(\alpha, m, x) \ll_{j,T} (Tt)^{-\frac{j}{2}} \sum_{i=0}^{j} \left( W_k(x) U(x) \right)^{(j-i)} V_{\frac{t}{2}}\left(\frac{m^2|N(\alpha)|}{D}, x\right)
\]

where again all partial derivatives are taken with respect to \( x \). Recall that \( W_k(x) = W\left(\frac{c_D t - x}{T}\right) \) for some fixed smooth compactly supported function \( W(x) \). Therefore, by repeatedly applying the chain rule we easily see that \( W_k^{(j-i)}(x) \ll_{j,i} T^{-j-i} \). Similarly, since \( U(x) = \tilde{U}\left(\frac{x}{c_D t}\right) \) with \( \tilde{U}(x) \) a fixed bump function independent of \( t \) we have \( U^{(j-i)}(x) \ll_{D,i,j} T^{-j-i} \). Finally, by Proposition 2.4 we know that

\[
(23) \quad V_{\frac{t}{2}}\left(\frac{m^2|N(\alpha)|}{D}, x\right) \ll_{\varepsilon,i} D^\varepsilon T^{-i}
\]

for any \( i \geq 0 \). Combining everything, we get the desired estimate. \( \square \)

As an immediate consequence of Proposition 3.3 and (22) we get
Corollary 3.1. For any \( j \geq 0 \) and \( \xi \neq 0 \), we have
\[
\hat{H}_A(\alpha, m, \xi) \ll \varepsilon, D, T^{\frac{j}{2}} \left( tTA \right)^{-\frac{1}{2}}.
\]
Moreover for \( |N(\alpha)| \notin \left[ \frac{4}{A}, 2A \right] \), we have \( \hat{H}_A(\alpha, m, \xi) = 0 \).

Using this estimate, we show that the sums of the form (21) can essentially be restricted to finitely many \( \xi \in \mathbb{Z} \). Moreover, for each such \( \xi \), the only generators \( \alpha \in \mathcal{F}_D \) which survive lie in a small cone. More formally, we have

Proposition 3.4. For any \( m \geq 1 \) and \( 0 < \delta < 1 \), we have
\[
(24) \quad \sum_{\alpha \in \mathcal{F}_D} \lambda_f(\alpha) \sum_{\xi \in \mathbb{Z}} \hat{H}_A(\alpha, m, \xi - \frac{\log(\frac{\alpha}{\lambda})}{2\log(\varepsilon D)}) \ll \varepsilon, D, T^{\frac{j}{2}} \left( tTA \right)^{-\frac{1}{2}} \sum_{\xi \in \mathbb{Z}} |\lambda_f(\alpha)| \sum_{\frac{1}{2} \leq |N(\alpha)| \leq 2A} |\lambda_f(\alpha)|.
\]

where we define
\[
S_{\xi,R} := \left\{ (\alpha, \alpha^*) \in \mathcal{F}_D \mid \varepsilon_2^R \alpha^* \leq \alpha \leq \varepsilon_2^R \alpha^*, \frac{A}{2} \leq |N(\alpha)| \leq 2A \right\}.
\]
for any \( R > 0 \), \( \xi \in \mathbb{Z} \). By abuse of notation, we write \( \alpha \in S_{\xi,R} \) instead of \( (\alpha, \alpha^*) \in S_{\xi,R} \).

Proof. By definition, for any \( \alpha \in \mathcal{F}_D \) we have \( 0 \leq \frac{\log(\frac{\alpha}{\lambda})}{\log(\varepsilon D)} < 2 \). Therefore, by Corollary 3.1 we have
\[
\sum_{\alpha \in \mathcal{F}_D} \lambda_f(\alpha) \sum_{\xi \geq 3} \hat{H}_A(\alpha, m, \xi - \frac{\log(\frac{\alpha}{\lambda})}{2\log(\varepsilon D)}) \ll \varepsilon, D, T^{\frac{j}{2}} \left( tTA \right)^{-\frac{1}{2}} \sum_{\frac{1}{2} \leq |N(\alpha)| \leq 2A} |\lambda_f(\alpha)| \sum_{\xi \geq 3} \left| \frac{1}{\xi - \frac{\log(\frac{\alpha}{\lambda})}{2\log(\varepsilon D)}} \right|^2 \ll \varepsilon, D, T^{\frac{j}{2}} \left( tTA \right)^{-\frac{1}{2}} \sum_{\frac{1}{2} \leq |N(\alpha)| \leq 2A} |\lambda_f(\alpha)|.
\]

Similarly,
\[
\sum_{\alpha \in \mathcal{F}_D} \lambda_f(\alpha) \sum_{\xi \leq -2} \hat{H}_A(\alpha, m, \xi - \frac{\log(\frac{\alpha}{\lambda})}{2\log(\varepsilon D)}) \ll \varepsilon, D, T^{\frac{j}{2}} \left( tTA \right)^{-\frac{1}{2}} \sum_{\frac{1}{2} \leq |N(\alpha)| \leq 2A} |\lambda_f(\alpha)|.
\]

Let now \( \xi \in \{-1, 0, 1, 2\} \) and \( \alpha \in \mathcal{F}_D \). One easily sees that \( |\xi - \frac{\log(\frac{\alpha}{\lambda})}{2\log(\varepsilon D)}| \leq T^{-1+\delta} \) if and only if \( \alpha \in S_{2\xi,2T^{-1+\delta}} \) in which case we use the estimate \( \hat{H}_A(\alpha, m, \xi - \frac{\log(\frac{\alpha}{\lambda})}{2\log(\varepsilon D)}) \ll \varepsilon, D, T^{\frac{j}{2}} \left( tTA \right)^{-\frac{1}{2}} \) from Corollary 3.1. On the other hand, if \( \alpha \in \mathcal{F}_D \) satisfies \( |\xi - \frac{\log(\frac{\alpha}{\lambda})}{2\log(\varepsilon D)}| > T^{-1+\delta} \), then by Corollary 3.1 we have
\[
\hat{H}_A(\alpha, m, \xi - \frac{\log(\frac{\alpha}{\lambda})}{2\log(\varepsilon D)}) \ll \varepsilon, D, T^{\frac{j}{2}} \left( tTA \right)^{-\frac{j}{2}} \ll \varepsilon, D, T^{\frac{j}{2}} T^{-1} \left( tTA \right)^{-\frac{j}{2}}
\]

for \( j = \lceil \frac{2}{\delta} \rceil \). Combining everything and rearranging the order of summation, we get the expected estimate. \( \square \)
Notice that the right hand side of (24) does not depend on \( m \), so we can estimate the sum over \( m \) separately using the well known-bounds for the harmonic sum. As a result, if \( \lambda_f(n) \ll n^{\theta+\varepsilon} \) is any (uniform) bound towards the Ramanujan conjecture, we have

\[
T(TtA)^{-\frac{1}{2}} \sum_{\xi=-1}^{2} \sum_{\alpha \in S_{2\xi,2T-1+\delta}} |\lambda_f(\alpha)| \sum_{m=1}^{(Tt)^{1+\delta}} \frac{1}{m} \ll_{\varepsilon,\delta,D} t^{\varepsilon} A^{\theta+\varepsilon} T(TtA)^{-\frac{1}{2}} |S_{2\xi,2T-1+\delta}|
\]

and

\[
T^{-1}(TtA)^{-\frac{1}{2}} \sum_{\alpha \in F_D} \sum_{m=1}^{(Tt)^{1+\delta}} \frac{1}{m} \ll_{\varepsilon,\delta,D} t^{\varepsilon} A^{\theta+\varepsilon} T^{-1}(TtA)^{-\frac{1}{2}} \sum_{\alpha \in F_D} 1.
\]

But for any \( n \geq 1 \), one can easily show that \( \sum_{n(a)=1} 1 \ll n^{\varepsilon} \) where the sum is taken over all ideals \( a \) of \( \mathcal{O}_K \) with norm equal to \( n \). For this reason, (26) leads to

\[
T^{-1}(TtA)^{-\frac{1}{2}} \sum_{\alpha \in F_D} \sum_{m=1}^{(Tt)^{1+\delta}} \frac{1}{m} \ll_{\varepsilon,\delta,D} t^{2(\theta+\varepsilon)} T^{-1}.
\]

We conclude the proof by trivially estimating the cardinality of the sets \( S_{\xi,R} \) for any \( \xi \in \{-1,0,1,2\} \) and \( 0 < R \leq 2 \).

3.6. Trivial estimates. To estimate \( |S_{\xi,R}| \), we will first reduce our problem to counting lattice points in a parallelogram then apply the Lipschitz principle. We start by introducing the parallelogram

\[
P_{\xi,R} := \left\{ a(\varepsilon_D \sqrt{2A}, \varepsilon_D^{1-\xi-R} \sqrt{2A}) + b(0, 2\varepsilon_D R \sqrt{2A}) \mid a, b \in [0,1] \right\}.
\]

To simplify the notation, for \( \xi \) and \( R \) fixed we define

\[
v_1 = (\varepsilon_D \sqrt{2A}, \varepsilon_D^{1-\xi-R} \sqrt{2A}) \text{ and } v_2 = (0, 2\varepsilon_D R \sqrt{2A}).
\]

**Lemma 3.3.** For \( \xi \geq -1 \) and \( 0 < R \leq 2 \) we have

\[
S_{\xi,R} \subseteq P_{\xi,R} \cap L_D
\]

where \( L_D \) is the lattice defined in Proposition [2.1]

**Proof.** Since \( S_{\xi,R} \) is a subset of the fundamental domain \( F_D \) which itself is a subset of \( L_D \), it is enough to prove that \( S_{\xi,R} \subseteq P_{\xi,R} \).

Given any vector \( (x,y) \in \mathbb{R}^2 \), we can write

\[
(x,y) = \frac{x}{\varepsilon_D \sqrt{2A}}v_1 + \frac{y-\varepsilon_D^{-\xi-R} x}{2\varepsilon_D^5 R \sqrt{2A}}v_2
\]

so that

\[
(27) \quad (x,y) \in P_{\xi,R} \iff \begin{cases} 0 \leq x \leq \varepsilon_D \sqrt{2A} \\ \varepsilon_D^{-\xi-R} x \leq y \leq \varepsilon_D^{-\xi-R} x + 2\varepsilon_D^5 R \sqrt{2A}. \end{cases}
\]

Hence, for any \( (x,y) \in S_{\xi,R} \), it is enough to prove that \( x \) and \( y \) satisfy the inequalities above. By definition of \( F_D \), we know that \( \varepsilon_D^{-2} x \leq y \leq x \). The inequality \( 0 \leq x \leq \varepsilon_D \sqrt{2A} \) now follows easily from \( N(x) = xy \).
On the other hand, the condition \( \varepsilon_D^{\xi-R} y \leq x \leq \varepsilon_D^{\xi+R} y \) from the definition of \( S_{\xi,R} \) is equivalent to \( \varepsilon_D^{\xi-R} x \leq y \leq \varepsilon_D^{\xi+R} x \). By the mean value theorem applied to the function \( a \mapsto \varepsilon_D^a \), we have

\[
\varepsilon_D^{\xi-R} x = \varepsilon_D^{\xi-R} x + (\varepsilon_D^{\xi+R} - \varepsilon_D^{\xi-R}) x \leq \varepsilon_D^{\xi-R} x + 2\varepsilon_D^{2-\xi+R} R \sqrt{2A}.
\]

Since we assumed \(-1 \leq \xi \) and \( R \leq 2 \), we must have \( \varepsilon_D^{\xi-R} \leq \varepsilon_D^{3} \), which concludes the proof. \( \square \)

As a result, we can apply Corollary 2.2 using the basis \( \mathcal{B} = \{(1,1), (\beta_D, \beta_D^* )\} \) of \( L_D \) to show that for any \( \xi \geq -1 \) and \( 0 < R \leq 2 \) we have

\[
|S_{\xi,R}| \ll_D RA + \sqrt{A}.
\]

Hence, for any \( \varepsilon > 0, T \ll_D t \) and \( A \ll_\varepsilon (Tt)^{1+\varepsilon} \), \([25]\) becomes

\[
T(TtA)^{-\frac{1}{2}} \sum_{\xi=-1}^{2} \sum_{\alpha \in S_{2\xi,2T-1+\varepsilon}} \lambda_f(\alpha) \sum_{m=1}^{(Tt)^{1+\varepsilon}} \frac{1}{m} \ll_D, t^{3\varepsilon} A^{\theta T(TtA)^{-\frac{1}{2}}(T^{-1+\varepsilon} A + A^\frac{1}{2})} \ll_D, t^{5\varepsilon} t^{20}.
\]

This shows that, as expected, for any \( \varepsilon > 0 \), we have

\[
\|f|c_D|^2 \|_{2} \ll_D, t^{2(\theta + \varepsilon)}
\]

where \( \lambda_f(n) \ll_\varepsilon n^{\theta + \varepsilon} \) is any bound towards the Ramanujan conjecture.

References

[1] Valentin Blomer, Rizwanur Khan, and Matthew Young. Distribution of mass of holomorphic cusp forms. *Duke Mathematical Journal*, 162(14), November 2013.

[2] Valentin Blomer, Étienne Fouvry, Emmanuel Kowalski, Philippe Michel, Djordje Milicević, and Will Sawin. The second moment theory of families of \( l \)-functions, 2018. URL https://arxiv.org/abs/1804.01450.

[3] J Bourgain. Geodesic restrictions and \( lp \)-estimates for eigenfunctions of riemannian surfaces. *American Mathematical Society Translations - Series 2*, 226:27–35, 2009.

[4] Daniel Bump. *Automorphic Forms and Representations*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997. doi: 10.1017/CBO9780511609572.

[5] N. Burq, P. Gérard, and N. Tzvetkov. Restrictions of the Laplace-Beltrami eigenfunctions to submanifolds. *Duke Mathematical Journal*, 138(3):445 – 486, June 2007.

[6] Christophe Cornut and Vinayak Vatsal. Nontriviality of rankin-selberg \( l \)-functions and cm points. In David Burns, Kevin Buzzard, and JanEditors Nekovár, editors, *L-Functions and Galois Representations*, London Mathematical Society Lecture Note Series, pages 121—186. Cambridge University Press, 2007.

[7] Harold Davenport. On a Principle of Lipschitz. *Journal of the London Mathematical Society*, s1-26(3):179–183, 07 1951. ISSN 0024-6107.

[8] Amit Ghosh, Andre Reznikov, and Peter Sarnak. Nodal domains of maass forms i. *Geometric and Functional Analysis*, 23(5):1515–1568, jul 2013.

[9] Gergely Harcos. Equidistribution on the modular surface and \( l \)-functions. *Clay Mathematics Proceedings*, 10:377–387, 2010.

[10] Jeffrey Hoffstein and Paul Lockhart. Coefficients of maass forms and the siegel zero. *Annals of Mathematics*, 140(1):161–176, 1994. ISSN 0003486X.

[11] H. Iwaniec and E. Kowalski. *Analytic Number Theory*. American Mathematical Society Colloquium Publications. American Mathematical Society, 2004. ISBN 9780821836330.

[12] Henryk Iwaniec. The spectral growth of automorphic \( l \)-functions. *Journal für die reine und angewandte Mathematik*, 428:139–160, 1992.

[13] Henry Kim and Peter Sarnak. Refined estimates towards the ramanujan and selberg conjectures. *Journal of the American Mathematical Society*, 16(1):175–181, 2003.
[14] Simon Marshall. Geodesic restrictions of arithmetic eigenfunctions. *Duke Mathematical Journal*, 165(3), Feb 2016.

[15] Kimball Martin and David Whitehouse. Central L-Values and Toric Periods for GL(2). *International Mathematics Research Notices*, 2009(1):141–191, 2009. ISSN 1073-7928.

[16] W. Narkiewicz. *Elementary and Analytic Theory of Algebraic Numbers*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2004. ISBN 9783540219026.

[17] Alexandru A. Popa. Central values of rankin l-series over real quadratic fields. *Compositio Mathematica*, 142(4):811–866, 2006.

[18] Andre Reznikov. Norms of geodesic restrictions for eigenfunctions on hyperbolic surfaces and representation theory, 2004. URL https://arxiv.org/abs/math/0403437.

[19] Peter Sarnak. Class numbers of indefinite binary quadratic forms. *Journal of Number Theory*, 15(2):229–247, 1982. ISSN 0022-314X.

[20] Peter Sarnak. Spectra of hyperbolic surfaces. *Bulletin of the American Mathematical Society*, 40:441–478, 2003.

[21] Peter Sarnak. 2008 - letter to a. reznikov on restrictions of eigenfunctions, 2008. URL https://publications.ias.edu/node/498.

[22] Hugo Steinhaus. Sur un théorème de m. v. jarník. *Colloquium Mathematicum*, 1(1):1–5, 1947.