ON TWO NATURAL EXTENSIONS OF VINNICOMBE’S METRIC: THEIR NONCOINCIDENCE YET EQUIVALENCE ON STABILIZABLE PLANTS OVER $A_+$

RUDOLF RUPP AND AMOL SASANE

Abstract. Let $A_+$ be the ring of Laplace transforms of complex Borel measures on $\mathbb{R}$ with support in $[0, +\infty)$ which do not have a singular nonatomic part. We compare the $\nu$-metric $d_{A_+}$ for stabilizable plants over $A_+$ given in [1] with yet another metric $d_{H^\infty|A_+}$, namely the one induced by the metric $d_{H^\infty}$ for the set of stabilizable plants over $H^\infty$ given in [7]. Both $d_{A_+}$ and $d_{H^\infty}$ coincide with the classical Vinnicombe metric defined for rational transfer functions, but we show here by means of an example that these two possible extensions of the classical $\nu$-metric for plants over $A_+$ do not coincide on the set of stabilizable plants over $A_+$. We also prove that they nevertheless give rise to the same topology on stabilizable plants over $A_+$, which in turn coincides with the gap metric topology.

1. Introduction

We recall the general stabilization problem in control theory. Suppose that $R$ is a commutative integral domain with identity (thought of as the class of stable transfer functions) and let $\mathbb{F}(R)$ denote the field of fractions of $R$. The stabilization problem is: Given $P \in \mathbb{F}(R)$ (an unstable plant transfer function), find $C \in \mathbb{F}(R)$ (a stabilizing controller transfer function), such that

$$H(P, C) := \begin{bmatrix} P & 1 \\ 1 & -CP \end{bmatrix}^{-1} \begin{bmatrix} 1 & -C \\ -C & 1 \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

(is stable).

In the robust stabilization problem, one goes a step further. One knows that the plant is just an approximation of reality, and so one would really like the controller $C$ to not only stabilize the nominal plant $P_0$, but also all sufficiently close plants $P$ to $P_0$. The question of what one means by “closeness” of plants thus arises naturally. So one needs a function $d$ defined on pairs of stabilizable plants such that $d$ is a metric on the set of all stabilizable plants, $d$ is amenable to computation, and stabilizability is a robust property of the plant with respect to this metric (that is, whenever a plant $P_0$ is stabilized by a controller $C$, then there is a small enough neighbourhood of
the plant $P_0$ consisting of plants which are stabilized by the same controller $C$). Such a desirable metric was introduced by Glenn Vinnicombe in \cite{8} and is called the $\nu$-metric. In that paper, essentially $R$ was taken to be the rational functions without poles in the closed unit disk. It was shown in \cite{8} that the $\nu$-metric is indeed a metric on the set of stabilizable plants, and that stabilizability is a robust property of the plant $P$.

The problem of what happens when $R$ is some other ring of stable transfer functions of infinite-dimensional systems was left open in \cite{8}. This problem of extending the $\nu$-metric from the rational case to nonrational transfer function classes of infinite-dimensional systems was addressed in \cite{1} where the approach taken was abstract. However when we focus on the set of stabilizable plants over $\mathbb{A}_+$, there are two possible natural extensions of the Vinnicombe metric for rational plants. We recall these two possibilities from \cite{1} and \cite{7} in the following section. The question of whether these two metrics coincide on the full set of stabilizable plants over $\mathbb{A}_+$ is a natural one. We prove that this is not the case by means of an example in Section 3. Notwithstanding this noncoincidence, we show that these two metrics do induce the same topology in Section 4.

2. Recap of the two $\nu$-metrics for unstable plants over $\mathbb{A}_+$

We recall the following standard definitions from the factorization approach to control theory.

2.1. The notation $\mathbb{F}(\mathbb{A}_+)$: $\mathbb{F}(\mathbb{A}_+)$ denotes the field of fractions of $\mathbb{A}_+$.

2.2. Normalized coprime factorization: For a $P \in \mathbb{F}(\mathbb{A}_+)$, a factorization $P = N/D$, where $N, D \in \mathbb{A}_+$, is called a coprime factorization of $P$ if there exist $X, Y \in \mathbb{A}_+$ such that $XN + YD = 1$. If moreover

$$\bar{N}(iy)N(iy) + \bar{D}(iy)D(iy) = 1 \quad (y \in \mathbb{R}),$$

then the coprime factorization is referred to as a normalized coprime factorization of $P$. Since we are dealing with functions rather than with matrices, it is not necessary to distinguish between left and right coprime factorizations.

2.3. The notation $G, \tilde{G}, K, \tilde{K}$: Given $P \in \mathbb{F}(\mathbb{A}_+)$ with normalized factorization $P = N/D$, we introduce the following matrices with entries from $\mathbb{A}_+$:

$$G = \begin{bmatrix} N \\ D \end{bmatrix} \quad \text{and} \quad \tilde{G} = \begin{bmatrix} -D & N \end{bmatrix}.$$  

Similarly, given an element $C \in \mathbb{F}(\mathbb{A}_+)$ with normalized coprime factorization $C = X/Y$, we introduce the following matrices with entries from $\mathbb{A}_+$:

$$K = \begin{bmatrix} Y \\ X \end{bmatrix} \quad \text{and} \quad \tilde{K} = \begin{bmatrix} -X & Y \end{bmatrix}.$$
2.4. **The notation**  $S(A_+)$: $S(A_+)$ denotes the set of all $P \in \mathbb{F}(A_+)$ that possess a normalized coprime factorization.

It follows from the proof of [4, Lemma 6.6.(e)] and [4, Theorem 5.2.8] that whenever $p \in \mathbb{F}(A_+)$ has a coprime factorization over $A_+$, it also has a normalized coprime factorization over $A_+$. However, it is known that not every element in $\mathbb{F}(A_+)$ possesses a coprime factorization; see for example [3].

We now recall the definition of the two metrics $d_\nu$ on $S(A_+)$.  

2.5. **The metric** $d_{A+}$. Let $C_{\geq 0} := \{s \in \mathbb{C} \mid \text{Re}(s) \geq 0\}$ and let $A^+$ denote the Banach algebra

$$A^+ = \left\{ s(\in \mathbb{C}_{\geq 0}) \mapsto \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \mid f_a \in L^1(0, \infty), (f_k)_{k\geq 0} \in \ell^1, 0 = t_0 < t_1, t_2, t_3, \ldots \right\}$$

equipped with pointwise operations and the norm:

$$\|F\| = \|f_a\|_{L^1} + \|(f_k)_{k\geq 0}\|_{\ell^1}, \quad F(s) = \widehat{f_a}(s) + \sum_{k=0}^{\infty} f_k e^{-st_k} \quad (s \in C_{\geq 0}).$$

Here $\widehat{f_a}$ denotes the Laplace transform of $f_a$, given by

$$\widehat{f_a}(s) = \int_0^{\infty} e^{-st}f_a(t)dt, \quad s \in C_{\geq 0}.$$  

Similarly, define the Banach algebra $A$ as follows:

$$A = \left\{ iy(\in i\mathbb{R}) \mapsto \widehat{f_a}(iy) + \sum_{k=-\infty}^{\infty} f_k e^{-iyt_k} \mid f_a \in L^1(\mathbb{R}), (f_k)_{k\in \mathbb{Z}} \in \ell^1, \ldots, t_{-2}, t_{-1} < 0 = t_0 < t_1, t_2, \ldots \right\}$$

equipped with pointwise operations and the norm:

$$\|F\| = \|f_a\|_{L^1} + \|(f_k)_{k\in \mathbb{Z}}\|_{\ell^1}, \quad F(iy) := \widehat{f_a}(iy) + \sum_{k=-\infty}^{\infty} f_k e^{-iyt_k} \quad (y \in \mathbb{R}).$$

Here $\widehat{f_a}$ is the Fourier transform of $f_a$,

$$\widehat{f_a}(iy) = \int_{-\infty}^{\infty} e^{-iyt}f_a(t)dt \quad (y \in \mathbb{R}).$$

For $F(iy) = \widehat{f_a}(iy) + \sum_{k=-\infty}^{\infty} f_k e^{-iyt_k} \quad (y \in \mathbb{R})$ in $A$, we set

$$F_{AP}(iy) = \sum_{k=-\infty}^{\infty} f_k e^{-iyt_k} \quad (y \in \mathbb{R}),$$

and call it the almost periodic part of $F$.

Recall that the algebra $AP$ of complex valued (uniformly) almost periodic functions is the smallest closed subalgebra of $L^\infty(\mathbb{R})$ that contains all the functions $e_\lambda := e^{i\lambda y}$. Here the parameter $\lambda$ belongs to $\mathbb{R}$. For any $f \in AP$, its Bohr-Fourier series is defined by the formal sum  

$$\sum_{\lambda} f_\lambda e^{i\lambda y}, \quad y \in \mathbb{R}, \quad (2.1)$$
where
\[ f_\lambda := \lim_{N \to \infty} \frac{1}{2N} \int_{[-N,N]} e^{-i\lambda y} f(y) dy, \quad \lambda \in \mathbb{R}, \]
and the sum in (2.1) is taken over the set \( \sigma(f) := \{ \lambda \in \mathbb{R} \mid f_\lambda \neq 0 \} \), called the Bohr-Fourier spectrum of \( f \). The Bohr-Fourier spectrum of every \( f \in AP \) is at most a countable set.

We have \( \widehat{L^1(\mathbb{R})} \cap AP = \{0\} \). Indeed such an almost periodic function must have limit zero at \( \pm \infty \), and so it must be a constant equal to zero. This follows, for example, from the normality of the translates of almost periodic functions [2, Chapter I, Section 2, p.14], which says that if \( f \) is an almost periodic function, then any sequence of the form \( (f(x + h_n))_{n \in \mathbb{N}} \), where \( h_n \) are real numbers, one can extract a subsequence converging uniformly on the real line.

It can also be seen easily that \( \widehat{L^1(\mathbb{R})} \) is an ideal in \( A \), since if \( f_a \in L^1(\mathbb{R}) \) and \( F_{AP} := \sum_{k \in \mathbb{Z}} f_k e^{-iyt_k} \) then
\[
\sum_{k \in \mathbb{Z}} f_k f_a (\cdot - t_k)
\]
is an absolutely convergent series in \( L^1(\mathbb{R}) \), whose Fourier transform is precisely \( \widehat{f_a} \cdot F_{AP} \).

If \( R \) is a commutative unital ring, we denote by \( \text{inv} \) \( R \) the set of invertible elements of \( R \).

If \( F = \widehat{f_a} + F_{AP} \in \text{inv} \ A \), then we have for some \( G = \widehat{g_a} + G_{AP} \in A \) that
\[
(\widehat{f_a} + F_{AP})(\widehat{g_a} + G_{AP}) = \widehat{f_a}G + F_{AP}\widehat{g_a} + F_{AP}G_{AP} = 1.
\]
Using the fact that \( \widehat{L^1(\mathbb{R})} \) is an ideal in \( A \) and that \( L^1(\mathbb{R}) \cap AP = 0 \), we obtain \( F_{AP}G_{AP} = 1 \), and so \( F_{AP}(i) \in \text{inv} \ AP \) (see [5, Section 5.3] for a different proof). Also, again because \( \widehat{L^1(\mathbb{R})} \) is an ideal in \( A \), we have that \( F_{AP}^{-1} \widehat{f_a} \) is the Fourier transform of a function in \( L^1(\mathbb{R}) \), and so the map \( y \mapsto 1 + (F_{AP}(iy))^{-1}\widehat{f_a}(iy) = \frac{F(iy)}{F_{AP}(iy)} \) has a well-defined winding number \( w \) around 0; the definition is given below. Define \( W : \text{inv} \ A \to \mathbb{R} \times \mathbb{Z} \) by
\[
W(F) = (w_{av}(F_{AP}), w(1 + F_{AP}^{-1}\widehat{f_a})), \quad (2.2)
\]
where \( F = \widehat{f_a} + F_{AP} \in \text{inv} \ A \), and
\[
w_{av}(F_{AP}) := \lim_{R \to \infty} \frac{1}{2R} \left( \arg \left( F_{AP}(iR) \right) - \arg \left( F_{AP}(-iR) \right) \right),
\]
\[
w(1 + F_{AP}^{-1}\widehat{f_a}) := \frac{1}{2\pi} \left( \arg \left( 1 + (F_{AP}(iy))^{-1}\widehat{f_a}(iy) \right) \right)_{y=+\infty} - \left( \arg \left( 1 + (F_{AP}(iy))^{-1}\widehat{f_a}(iy) \right) \right)_{y=-\infty}.
\]
We also recall that \( F = \widehat{f_a} + F_{AP} \in A \) is invertible if and only if for all \( y \in \mathbb{R} \), \( F(iy) \neq 0 \) and \( \inf_{y \in \mathbb{R}} |F_{AP}(iy)| > 0 \).
**Definition 2.1.** For \( P_1, P_2 \in S(A_+) \), with the normalized coprime factorizations

\[
\begin{align*}
P_1 &= N_1/D_1, \\
P_2 &= N_2/D_2,
\end{align*}
\]

we define

\[
d_{A_+}(P_1, P_2) := \begin{cases} \\
\|\tilde{G}_2G_1\|_\infty & \text{if } G_1^*G_2 \in \text{inv } A \text{ and } W(G_1^*G_2) = (0, 0), \\
1 & \text{otherwise},
\end{cases}
\]

where the notation is as in Subsections 2.1-2.4.

It can be seen that this gives an extension of the classical Vinnicombe \( \nu \)-metric. Let \( RH^\infty \) denote the set of all rational functions that are holomorphic and bounded in the open right half plane \( \mathbb{C}_{>0} := \{ s \in \mathbb{C} : \text{Re}(s) > 0 \} \). We use the notation \( C(\mathbb{T}) \) for the algebra of complex-valued continuous functions defined on the unit circle \( \mathbb{T} := \{ z \in \mathbb{C} : |z| = 1 \} \), with all operations defined pointwise. If \( f \in \text{inv } C(\mathbb{T}) \), then \( f \) has a well-defined (integral) winding number \( w(f) \in \mathbb{Z} \) with respect to 0.

Let \( \varphi \) be the conformal map \( \varphi : D \to \mathbb{C}_{>0} \) from the open unit disk \( D \) to the open right half plane \( \mathbb{C}_{>0} \) given by

\[
\varphi(z) = \frac{1 + z}{1 - z} \quad (z \in D).
\]

Recall that the classical Vinnicombe \( \nu \)-metric is given as follows. For all \( P_1, P_2 \) in \( S(RH^\infty) \),

\[
d(P_1, P_2) = \begin{cases} \\
\|\tilde{G}_2G_1\|_\infty & \text{if } ((G_1^*G_2) \circ \varphi) \in \text{inv } C(\mathbb{T}) \text{ and } w((G_1^*G_2) \circ \varphi) = 0 \\
1 & \text{otherwise},
\end{cases}
\]

Clearly, if \( ((G_1^*G_2) \circ \varphi) \in \text{inv } C(\mathbb{T}) \), then the almost periodic part of \( G_1^*G_2 \) is a nonzero constant, and so the average winding number of \( G_1^*G_2 \) must be zero and that \( w((G_1^*G_2) \circ \varphi) = w((G_1^*G_2)) \). If \( ((G_1^*G_2) \circ \varphi) \not\in \text{inv } C(\mathbb{T}) \), then \( G_1^*G_2 \not\in \text{inv } A \), and so both \( d(P_1, P_2) \) and \( d_{A_+}(P_1, P_2) \) are equal to 1. Hence we have

\[
d(P_1, P_2) = d_{A_+}(P_1, P_2)
\]

whenever \( P_1, P_2 \in S(RH^\infty) \).

2.6. **The metric** \( d_{H^\infty}|_{A_+} \). Let \( H^\infty \) be the Hardy algebra, consisting of all bounded and holomorphic functions defined on the open unit disk

\[
D := \{ z \in \mathbb{C} : |z| < 1 \}.
\]

Given \( \rho \in (0, 1) \), let \( A_\rho \) be the open annulus

\[
A_\rho := \{ z \in \mathbb{C} : \rho < |z| < 1 \}.
\]

We set \( C_b(A_\rho) = \{ F : A_\rho \to \mathbb{C} : f \text{ is continuous and bounded on } A_\rho \} \).
Let $\rho \in (0, 1)$. With the norm defined by
\[
\|F\|_\infty := \sup_{z \in \mathbb{A}_\rho} |F(z)| \text{ for } F \in C_b(\mathbb{A}_\rho),
\]
$C_b(\mathbb{A}_\rho)$ is a unital semisimple commutative complex Banach algebra with the involution $\cdot^*$ defined by
\[
(F^*)(z) = \overline{F(z)} \quad (z \in \mathbb{A}_\rho, \ F \in C_b(\mathbb{A}_\rho)).
\]
Let $\rho \in (0, 1)$. For $f \in H^\infty$, define $I : H^\infty \to C_b(\mathbb{A}_\rho)$ by
\[
(I(f))(z) = f(z) \quad (z \in \mathbb{A}_\rho, \ f \in H^\infty).
\]
Then $I$ is an injective map. Henceforth we will identify $H^\infty$ as a subset of $C_b(\mathbb{A}_\rho)$ via this map $I$.

We use the notation $C(\mathbb{T})$ for the Banach algebra of complex-valued continuous functions defined on the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, with all operations defined pointwise, with the supremum norm:
\[
\|f\|_\infty = \sup_{\zeta \in \mathbb{T}} |f(\zeta)| \text{ for } f \in C(\mathbb{T}),
\]
and the involution $\cdot^*$ defined pointwise:
\[
f^*(\zeta) = \overline{f(\zeta)} \quad (\zeta \in \mathbb{T}).
\]
If $F \in \text{inv} \ C_b(\mathbb{A}_\rho)$, then for each $r \in (\rho, 1)$, the map $F_r : \mathbb{T} \to \mathbb{C}$, given by
\[
F_r(\zeta) = F(r\zeta) \quad (\zeta \in \mathbb{T}),
\]
belongs to $C(\mathbb{T})$, and so each $F_r$ has a well-defined (integral) winding number $w(F_r) \in \mathbb{Z}$ with respect to $0$. By the local constancy of the winding number $w : \text{inv} \ C(\mathbb{T}) \to \mathbb{Z}$, $r \mapsto w(F_r)$ is constant on $(\rho, 1)$. That is, if $F \in \text{inv} \ C_b(\mathbb{A}_\rho)$, and $\rho < r < r' < 1$, then
\[
w(F_r) = w(F_{r'}). \quad (2.4)
\]
We now define the map $\mathcal{W} : \text{inv} \ C_b(\mathbb{A}_\rho) \to \mathbb{Z}$ by setting
\[
\mathcal{W}(F) = w(F_r) \quad (r \in (\rho, 1), \ F \in \text{inv} \ C_b(\mathbb{A}_\rho)).
\]
Then $\mathcal{W}$ is well-defined.

As before, let $\varphi$ be the conformal map $\varphi : \mathbb{D} \to \mathbb{C}_{>0}$ given by
\[
\varphi(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{D}).
\]
For $P_1, P_2 \in \mathcal{S}(\mathbb{A}_+)$, with the normalized coprime factorizations
\[
P_1 = N_1/D_1, \quad P_2 = N_2/D_2,
\]
we define
\[
d^p_{HK} \mid \mathbb{A}_+(P_1, P_2) = \begin{cases} 
\|(\mathcal{G}_2 \mathcal{G}_1) \circ \varphi\|_\infty & \text{if } (\mathcal{G}_1 \mathcal{G}_2) \circ \varphi \in \text{inv} \ C_b(\mathbb{A}_\rho) \text{ and } \\
1 & \text{otherwise},
\end{cases}
\]
(2.4)
where the notation is as in Subsections 2.1-2.4.

It follows from [7] that \( d_{H^\infty} \) defined by

\[
d_{H^\infty}(P_1, P_2) = \lim_{\rho \to 1} d_{H^\infty}^\rho(P_1, P_2)
\]

actually defines a metric and if \( P_1, P_2 \in \mathbb{S}(\mathbb{R}H^\infty) \), then \( d(P_1, P_2) = d_{H^\infty}(P_1, P_2) \).

Thus this is also an extension of the classical Vinnicombe metric.

3. An example of \( P_1, P_2 \) for which \( d_{A_+}(P_1, P_2) \neq d_{H^\infty}(P_1, P_2) \)

Let \( P \) be given by

\[
P(s) = \frac{\alpha}{\beta} e^{-s}.
\]

where \( \alpha, \beta \) are nonzero real numbers and \( \alpha^2 + \beta^2 = 1 \). Set

\[
N := \alpha e^{-s},
D := \beta.
\]

Then

\[
0 \cdot N + \frac{1}{\beta} \cdot D = 1,
\]

and so \( N, D \) are coprime in \( A_+ \). Also,

\[
N^* \cdot N + D^* \cdot D = \alpha e^{-s} \cdot \alpha e^{-s} + \beta^2 = \alpha^2 + \beta^2 = 1
\]

on \( i\mathbb{R} \). Thus

\[
P = N/D
\]

is a normalized coprime factorization of \( P \).

Now choose a real number \( r \) such that \( \frac{1}{\sqrt{2}} < r < 1 \), and set

\[
P_1 = \frac{r}{\sqrt{1-r^2}} e^{-s},
P_2 = \frac{r}{-\sqrt{1-r^2}} e^{-s}.
\]

Then we have

\[
G_1^* G_2 = \begin{bmatrix} N_1 & D_1 \end{bmatrix} \begin{bmatrix} N_2 \\ D_2 \end{bmatrix} = \begin{bmatrix} r e^{-s} & \sqrt{1-r^2} \\ \sqrt{1-r^2} & -r e^{-s} \end{bmatrix} = r^2 e^{-s-s} - (1-r^2) = r^2 e^{-2\text{Re}(s)} - (1-r^2).
\]

Thus \( (G_1^* G_2)|_{i\mathbb{R}} = r^2 - (1-r^2) = 2r^2 - 1 > 2 \cdot \frac{1}{2} - 1 = 0 \). Hence \( G_1^* G_2 \in \mathcal{A} \) and \( w_{av}(G_1^* G_2) = 0 \). Thus \( W(G_1^* G_2) = (0,0) \).

Also,

\[
\tilde{G}_2 G_1 = \begin{bmatrix} -D_2 & N_2 \end{bmatrix} \begin{bmatrix} N_1 \\ D_1 \end{bmatrix} = \begin{bmatrix} \sqrt{1-r^2} & r e^{-s} \\ \sqrt{1-r^2} & r e^{-s} \end{bmatrix} = 2r \sqrt{1-r^2} e^{-s}.
\]
Hence by the Arithmetic Mean-Geometric Mean inequality, we have

$$\| \tilde{G}_2 G_1 \|_\infty = 2r \sqrt{1 - r^2} < r^2 + (1 - r^2) = 1,$$

where we do have strict inequality since $r^2 \neq 1 - r^2$ (because $r \neq \frac{1}{\sqrt{2}}$).

Consequently,

$$d_{A_+}(P_1, P_2) = \| \tilde{G}_2 G_1 \|_\infty < 1.$$

Next we will show that $d_{H_\infty}(P_1, P_2) = 1$. Note that if $\rho$ is in $(0, 1)$, then the circle $\rho \mathbb{T}$ is mapped under the conformal map $\varphi : \mathbb{D} \to \mathbb{C}_{>0}$, given by

$$\varphi(z) = \frac{1+z}{1-z} \quad (z \in \mathbb{D}),$$

onto the circle

$$\frac{1+\rho^2}{1-\rho^2} + \frac{2\rho}{1-\rho^2} \mathbb{T}$$

in the open right half plane. This circle intersects the real axis at the points $z_1 < z_2$, where

$$z_1 = \frac{1+\rho^2}{1-\rho^2} - \frac{2\rho}{1-\rho^2} = \frac{1-\rho}{1+\rho},$$

$$z_2 = \frac{1+\rho^2}{1-\rho^2} + \frac{2\rho}{1-\rho^2} = \frac{1+\rho}{1-\rho}.$$

It is clear that for $\rho$ close enough to 1,

$$(G_1^* G_2)(z_1) \approx r^2 \cdot 1 - (1 - r^2) = 2r^2 - 1 > 2 \cdot \frac{1}{2} - 1 = 0,$$

$$(G_1^* G_2)(z_2) \approx r^2 \cdot 0 - (1 - r^2) = -(1 - r^2) < 0.$$

But $G_1^* G_2 = r^2 e^{-2\text{Re}(\alpha)} - (1 - r^2)$ is always real-valued. By the Intermediate Value Theorem, it follows that it must be a zero somewhere in $A_\rho$ and $G_1^* G_2$ can’t belong to $\text{inv} \ C_b(\mathbb{D}_\rho)$. In fact, all zeros belong to an arc of a circle with center on the real axis, tangent to $z = 1$, as can be seen easily in the right half plane. Hence $d_{H_\infty}(P_1, P_2) = 1$.

4. Equivalence of $d_{A_+}$ and $d_{H_\infty}|_{A_+}$

4.1. An alternative expression for $d_{H_\infty}|_{A_+}$. We begin by giving an alternative expression for $d_{H_\infty}$.

If $M \in \mathbb{C}^{p \times m}$, then the set of nonzero eigenvalues of $MM^*$ and $M^* M$ coincide. We denote by $\sigma(M)$ the square root of the largest eigenvalue of $M^* M$ (or equivalently $MM^*$). For a matrix $M \in \mathbb{A}^{p \times m}$, we set

$$\|M\|_\infty = \sup_{y \in \mathbb{R}} \sigma(M(iy)). \quad (4.1)$$

Proposition 4.1. If $P_1, P_2 \in \mathbb{S}(A_+)$, then for each $\rho \in (0, 1)$,

$$d_{H_\infty}^\rho(P_1, P_2) = \inf_{Q \in \text{inv} \ C_b(\mathbb{D}_\rho), \ W(Q) = 0} \|G_1 - G_2 Q\|_\infty.$$
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Proof. Let $Q \in \text{inv } C_b(\mathbb{A}_\rho)$ and $\mathcal{W}(Q) = 0$. We have

$$\|G_1 - G_2 Q\|_{\infty} = \left\| \left[ \begin{array}{c} G_2^* \\ G_2 \\
 \end{array} \right] (G_1 - G_2 Q) \right\|_{\infty}$$

(as $\left[ \begin{array}{c} G_2 \\ \tilde{G}_2^* \\
 \end{array} \right] \left[ \begin{array}{c} G_2 \\ G_2 \\
 \end{array} \right] = I$)

$$= \left\| \left[ \begin{array}{c} G_2^* G_1 - Q \\ \tilde{G}_2 G_1 \\
 \end{array} \right] \right\|_{\infty}$$

(since $\tilde{G}_2 G_2 = 0$ and $G_2^* G_2 = I$)

$$\geq \|\tilde{G}_2 G_1\|_{\infty}.$$ 

So if $G_2^* G_1 \in \text{inv } C_b(\mathbb{A}_\rho)$ and $\mathcal{W}(G_2^* G_1) = 0$, then from the above it follows that $\|G_1 - G_2 Q\|_{\infty} \geq \|\tilde{G}_2 G_1\|_{\infty} = d_{H,\infty}^p(P_1, P_2)$. As the choice of $Q$ above was arbitrary, we obtain

$$\inf_{Q \in \text{inv } C_b(\mathbb{A}_\rho), \mathcal{W}(Q) = 0} \|G_1 - G_2 Q\|_{\infty} \geq d_{H,\infty}^p(P_1, P_2).$$

(4.2)

If we define $Q_0 := G_2^* G_1 \in C_b(\mathbb{A}_\rho)$, then $Q_0 \in \text{inv } C_b(\mathbb{A}_\rho)$ and $\mathcal{W}(Q_0) = 0$, and so

$$\inf_{Q \in \text{inv } C_b(\mathbb{A}_\rho), \mathcal{W}(Q) = 0} \|G_1 - G_2 Q\|_{\infty} \leq \|G_1 - G_2 Q_0\|_{\infty} = \left\| \left[ \begin{array}{c} G_2^* G_1 - Q_0 \\ \tilde{G}_2 G_1 \\
 \end{array} \right] \right\|_{\infty}$$

$$= \left\| \left[ \begin{array}{c} 0 \\ \tilde{G}_2 G_1 \\
 \end{array} \right] \right\|_{\infty} = \|\tilde{G}_2 G_1\|_{\infty} = d_{H,\infty}^p(P_1, P_2).$$

From this and (4.2), the claim in the proposition follows for the case when $G_2^* G_1 \in \text{inv } C_b(\mathbb{A}_\rho)$ and $\mathcal{W}(G_2^* G_1) = 0$.

Now let $Q \in \text{inv } C_b(\mathbb{A}_\rho)$ be such that $\mathcal{W}(Q) = 0$ and $\|G_1 - G_2 Q\|_{\infty} < 1$. Using $G_1^* G_1 = 1$, we see that $\|G_1^*\|_{\infty} = 1$ and

$$\|1 - G_2^* G_2 Q\|_{\infty} = \|G_1^* (G_1 - G_2 Q)\|_{\infty} \leq \|G_1^*\|_{\infty} \|G_1 - G_2 Q\|_{\infty} < 1 \cdot 1 = 1.$$ 

So $G_1^* G_2 Q = 1 - (1 - G_2^* G_2 Q)$ is invertible as an element of $C_b(\mathbb{A}_\rho)$. Consider the map $H : [0, 1] \rightarrow \text{inv } C_b(\mathbb{A}_\rho)$ given by $H(t) = 1 - t (1 - G_1^* G_2 Q)$, $t \in [0, 1]$.

By the homotopic invariance of the index $\mathcal{W}$ [11 Proposition 2.1],

$$0 = \mathcal{W}(1) = \mathcal{W}(H(0)) = \mathcal{W}(H(1)) = \mathcal{W}(G_1^* G_2 Q).$$

As $\mathcal{W}(Q) = 0$, we obtain that $\mathcal{W}(G_1^* G_2) = 0$. So we have shown that if there is a $Q \in C_b(\mathbb{A}_\rho)$ such that $Q \in \text{inv } C_b(\mathbb{A}_\rho)$, $\mathcal{W}(Q) = 0$ and $\|G_1 - G_2 Q\|_{\infty} < 1$, then $G_1^* G_2 \in \text{inv } C_b(\mathbb{A}_\rho)$ and $\mathcal{W}(G_1^* G_2) = (0, 0)$. Thus we have that if either $G_1^* G_2 \not\in \text{inv } C_b(\mathbb{A}_\rho)$ or $G_1^* G_2 \in \text{inv } C_b(\mathbb{A}_\rho)$ but $\mathcal{W}(G_1^* G_2) \neq 0$, then for all elements $Q \in C_b(\mathbb{A}_\rho)$ such that $Q \in \text{inv } C_b(\mathbb{A}_\rho)$, $\mathcal{W}(Q) = 0$, we have that $\|G_1 - G_2 Q\|_{\infty} \geq 1$, and so

$$\inf_{Q \in \text{inv } C_b(\mathbb{A}_\rho), \mathcal{W}(Q) = 0} \|G_1 - G_2 Q\|_{\infty} \geq 1 = d_{H,\infty}^p(P_1, P_2).$$
Also, with \( Q_n := \frac{1}{n} I \) \((n \in \mathbb{N})\), \( Q_n \in \text{inv } C_b(A_\rho) \) and \( W(Q_n) = 0 \). We have
\[
\| G_1 - G_2 Q_n \|_\infty \leq \| G_1 \|_\infty + \| G_2 \|_\infty \| Q_n \|_\infty \leq 1 + \frac{1}{n}.
\]
Hence
\[
\inf_{Q \in \text{inv } C_b(A_\rho)} \| G_1 - G_2 Q \|_\infty \leq \inf_{n \in \mathbb{N}} \| G_1 - G_2 Q_n \|_\infty \leq \left(1 + \frac{1}{n}\right) = 1 = d_{H_\infty}^\rho(P_1, P_2).
\]
Consequently, \( \inf_{Q \in \text{inv } C_b(A_\rho)} \| G_1 - G_2 Q \|_\infty = 1 = d_{H_\infty}^\rho(P_1, P_2) \). □

For \( P \in S(A_+) \), set \( \mu_{\text{opt}, A_+}(P) := \sup_{C \in S(A_+)} \mu_{P,C} \), where
\[
\mu_{P,C} = \begin{cases} \| H(P, C) \|_\infty^{-1} & \text{if } P \text{ is stabilized by } C, \\ 0 & \text{otherwise,} \end{cases}
\]
and
\[
H(P, C) := \left[ \begin{array}{c} P \\ 1 \end{array} \right] (1 - CP)^{-1} \left[ \begin{array}{c} -C \\ 1 \end{array} \right].
\]
We remark that first of all \( \mu_{\text{opt}, A_+}(P) > 0 \) because every \( P \in S(A_+) \) has a coprime factorisation, and we know that the coprime factorization gives a stabilizing controller. Secondly, as \( \mu_{P,C} \) is always bounded above by 1 (see [1, Remark 4.3]), we have that \( \mu_{\text{opt}, A_+}(P) \leq 1 \).

4.2. The gap-metric. In this subsection we will recall the gap-metric topology for unstable plants over the ring \( A_+ \). We will also recall a few known results from [6] lemmas which will be used in the next subsection in order to prove our claimed equivalence.

**Definition 4.2** (Graph of a system). For \( P \in S(A_+) \), with the normalized coprime factorization \( P = N/D \), we define the graph of \( P \), denoted by \( \mathcal{G} \), to be the following subspace of the Hardy space \( H^2 \times H^2 \):
\[
\mathcal{G} = GH^2 = \left\{ \left[ \begin{array}{c} N \varphi \\ D \varphi \end{array} \right] : \varphi \in H^2 \right\}.
\]
Here \( H^2 \) denotes the Hardy space of all holomorphic functions defined in the open right half plane \( \mathbb{C}_{>0} := \{ s \in \mathbb{C} : \text{Re}(s) > 0 \} \) such that
\[
\sup_{\zeta > 0} \| f(\zeta + i \cdot) \|_{L^2(\mathbb{R})} < +\infty.
\]
One can see that \( \mathcal{G} \) is a closed subspace of \( H^2 \times H^2 \). Suppose that
\[
\left[ \begin{array}{c} N \\ D \end{array} \right] \varphi_n \overset{n \to \infty}{\longrightarrow} \left[ \begin{array}{c} f \\ g \end{array} \right]
\]
in $H^2 \times H^2$. If $X, Y \in \mathcal{A}_+$ are such that $XN + YD = 1$, then using the fact that elements from $\mathcal{A}_+$ are bounded and holomorphic in the right half plane, we obtain that

$$\varphi_n \xrightarrow{n \to \infty} Xf + Yg =: \varphi$$

in $H^2$. Consequently, using the fact that $N, D$ are bounded and holomorphic in the open right half plane, we obtain

$$\left[ \begin{array}{c} N \\ D \end{array} \right] \varphi_n \xrightarrow{n \to \infty} \left[ \begin{array}{c} N \\ D \end{array} \right] \varphi \in \mathcal{G}.$$

We denote the orthogonal projection from $H^2 \times H^2$ onto $\mathcal{G}$ by $P_G$.

**Definition 4.3 (Gap-metric $d_g$).** For $P_1, P_2 \in S(\mathcal{A}_+)$, with the normalized coprime factorizations $P_1 = N_1/D_1$ and $P_2 = N_2/D_2$, we define

$$d_g(P_1, P_2) := \|PG_1 - PG_2\|_{L(H^2 \times H^2)}. \tag{4.3}$$

We recall [6, Proposition 4.9 and Theorem 1.1]:

**Proposition 4.4.** If $P_1, P_2 \in S(\mathcal{A}_+)$, then

$$d_g(P_1, P_2) = \inf_{Q \in \text{inv \, \mathcal{A}_+}} \|G_1 - G_2Q\|_{\infty}. \tag{4.4}$$

**Proposition 4.5.** For $P_1, P_2 \in S(\mathcal{A}_+)$:

$$d_g(P_1, P_2) \mu_{\text{opt, } \mathcal{A}_+}(P_1) \leq d_{A_+}(P_1, P_2) \leq d_g(P_1, P_2). \tag{4.4}$$

4.3. **Equivalence.** Let $P_1, P_2 \in S(\mathcal{A}_+)$. Then

$$d_{H^\infty}^\rho(P_1, P_2) = \inf_{Q \in \text{inv \, } C_h (\mathcal{A}_+), \ W(Q) = 0} \|G_1 - G_2Q\|_{\infty}$$

$$\leq \inf_{Q \in \text{inv \, } C_h (\mathcal{A}_+), \ W(Q) = 0} \|G_1 - G_2Q\|_{H^\infty}$$

$$= \inf_{Q \in \text{inv \, } H^\infty} \|G_1 - G_2Q\|_{\infty}$$

$$\leq \inf_{Q \in \text{inv \, } \mathcal{A}_+} \|G_1 - G_2Q\|_{\infty}$$

$$= d_g(P_1, P_2)$$

$$\leq d_{A_+}(P_1, P_2) \mu_{\text{opt, } \mathcal{A}_+}(P_1).$$

Consequently,

$$d_{H^\infty}(P_1, P_2) = \lim_{\rho \to 1} d_{H^\infty}^\rho(P_1, P_2) \leq \frac{d_{A_+}(P_1, P_2)}{\mu_{\text{opt, } \mathcal{A}_+}(P_1)}. \tag{4.5}$$

Next we will show that

$$d_{A_+}(P_1, P_2) \mu_{\text{opt, } \mathcal{A}_+}(P_1) \leq d_{H^\infty}(P_1, P_2).$$

This inequality is trivially satisfied if $d_{H^\infty}(P_1, P_2) \geq \mu_{\text{opt, } \mathcal{A}_+}(P_1)$, since we know that $d_{A_+}(P_1, P_2) \leq 1$. 
So we will only consider the case when \( d_{H^\infty}(P_1, P_2) < \mu_{\text{opt}, A_+}(P_1) \). In particular, \( \mu_{\text{opt}, A_+}(P_1) > 0 \). This inequality implies that there is an element \( C_0 \in S(A_+) \) that stabilizes \( P_1 \). Moreover, \( d_{H^\infty}(P_1, P_2) < \mu_{P_1, C_0} \). Using the fact \( [7, \text{Theorem 3.15}] \) that 
\[
\mu_{P_2, C_0} \geq \mu_{P_1, C_0} - d_{H^\infty}(P_1, P_2),
\]
it follows that \( C_0 \) stabilizes (in \( H^\infty \)) \( P_2 \) as well. But by the corona theorems for \( H^\infty \) and for \( A_+ \) it follows that \( C_0 \) stabilizes \( P_2 \) in \( A_+ \) too.

Define \( Q_0 := (\widetilde{K_0}G_1)^{-1}\widetilde{K_0}G_2 \). By \([1, \text{Proposition 4.4}] \), we know that \( \widetilde{K_0}G_2 \) is invertible in \( A_+ \). We have 
\[
G_2 - G_1Q_0 = G_2 - G_1(\widetilde{K_0}G_1)^{-1}\widetilde{K_0}G_2 = (I - G_1(\widetilde{K_0}G_1)^{-1}\widetilde{K_0})G_2.
\]
Also 
\[
I - \left[ \begin{array}{c} P_1 \\ 1 \end{array} \right] (1 - C_0 P_1)^{-1} \left[ \begin{array}{cc} -C_0 & 1 \\ 0 & -P_1 \end{array} \right] = \left[ \begin{array}{cc} 1 & 0 \\ C_0 & (1 - P_1 C_0)^{-1} \end{array} \right] \left[ \begin{array}{cc} 1 & -P_1 \\ 0 & 1 \end{array} \right].
\]
that is, 
\[
I - G_1(\widetilde{K_0}G_1)^{-1}\widetilde{K_0} = K_0(\widetilde{G_1}K_0)^{-1}\widetilde{G_1}.
\]
Thus 
\[
G_2 - G_1Q_0 = K_0(\widetilde{G_1}K_0)^{-1}\widetilde{G_1}G_2.
\]
Then we use \( \|K_0\| \leq 1 \) (since \( K_0^*K_0 = 1 \)) to obtain
\[
\|G_2 - G_1Q_0\|_{\infty} = \|K_0(\widetilde{G_1}K_0)^{-1}\widetilde{G_1}G_2\|_{\infty} \\
\leq \|K_0\|_{\infty}\|\widetilde{G_1K_0}^{-1}\widetilde{G_1G_2}\|_{\infty} \\
\leq 1 \cdot \|\widetilde{G_1K_0}^{-1}\widetilde{G_1G_2}\|_{\infty} \\
\leq \|\widetilde{G_1K_0}^{-1}\|_{\infty}\|\widetilde{G_1G_2}\|_{\infty}.
\]
As for each \( C \), \( \mu_{P_1, C} \leq 1 \), we have \( \mu_{\text{opt}, A_+}(P_1) \leq 1 \). So 
\[
d_{H^\infty}(P_1, P_2) < \mu_{\text{opt}, A_+}(P_1) \leq 1,
\]
and we obtain \( d_{H^\infty}(P_1, P_2) = \|\widetilde{G_1G_2}\|_{\infty} \).

From \([1, \text{Propositions 4.2, 4.5}] \), \( \|\widetilde{G_1K_0}^{-1}\|_{\infty} = 1/\mu_{C_0, P_1} = 1/\mu_{P_1, C_0} \). So
\[
\|G_2 - G_1Q_0\|_{\infty} \leq \|\widetilde{G_1K_0}^{-1}\|_{\infty}\|\widetilde{G_1G_2}\|_{\infty} \leq \frac{d_{H^\infty}(P_1, P_2)}{\mu_{P_1, C_0}}.
\]
Thus 
\[
d_g(P_1, P_2) = \inf_{Q \in \text{inv } A_+} \|G_1 - G_2Q\|_{\infty} \leq \|G_1 - G_2Q_0\|_{\infty} \leq d_{H^\infty}(P_1, P_2)/\mu_{P_1, C_0}.
\]
But 
\[
d_g(P_1, P_2) \geq d_{A_+}(P_1, P_2).
\]
Hence 
\[
\mu_{P_1, C_0} \cdot d_{A_+}(P_1, P_2) \leq d_{H^\infty}(P_1, P_2).
\]
As this inequality holds for any \( C_0 \) that stabilizes \( P_1 \) (in \( A_+ \)) for which there holds \( d_{H^\infty}(P_1, P_2) < \mu_{P_1, C_0} \), we can choose a sequence \( (C_{0,n})_{n \in \mathbb{N}} \) such \( \mu_{P_1, C_{0,n}} \to \mu_{\text{opt}, A_+}(P_1) \) as \( n \to \infty \). Thus
\[
\mu_{\text{opt}, A_+}(P_1) \cdot d_{A_+}(P_1, P_2) \leq d_{H^\infty}(P_1, P_2).
\]
(4.6)
Finally, from (4.5) and (4.6), we have

$$\mu_{\text{opt}, \mathcal{A}_+}(P_1) \cdot d_{\mathcal{A}_+}(P_1, P_2) \leq d_{H^\infty}(P_1, P_2) \leq \frac{d_{\mathcal{A}_+}(P_1, P_2)}{\mu_{\text{opt}, \mathcal{A}_+}(P_1)}.$$ 

Remark 4.6. We also mention that in this article we have only considered single input and single output control systems. However, the metrics $d_{\mathcal{A}_+}$, $d_{H^\infty}$ can also be defined on plants with multiple inputs and/or outputs as well; see [1] and [7]. One can ask if the induced topologies (on such matricial stabilizable plants over $\mathcal{A}_+$) are still equivalent. We leave this as an open problem.

Our route of proving the equivalence in the case of single input single output systems in this article is by appealing to the results from [6], which unfortunately are also available in only the scalar case. Whether the matricial analogue of the result from [6] holds is also open. If that result were available, then the same proof in this article, mutatis mutandis, would also yield the extension of the result in this article to the matricial case.

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Fakultät Allgemeinwissenschaften, Georg-Simon-Ohm Hochschule Nürnberg, Kesslerplatz 12, D-90489 Nürnberg, Germany.

E-mail address: rudolf.rupp@ohm-hochschule.de

Department of Mathematics, London School of Economics, Houghton Street, London WC2A 2AE, United Kingdom.

E-mail address: a.j.sasane@lse.ac.uk