The Differential equation method: 
calculation of vertex-type diagrams with one non-zero mass

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Abstract

The differential equation method is applied to evaluate analytically two-loop vertex Feynman diagrams. Three on-shell infrared divergent planar two-loop diagrams with zero thresholds contributing to the processes $Z \rightarrow b\bar{b}$ (for zero b mass) and/or $H \rightarrow gg$ are calculated in order to demonstrate a new application of this method.
1 Introduction.

In the two-loop order of the Standard Model large effects are due to the exchange of heavy particles, like $H$ and $t$. The present investigation is motivated by the need to perform accordingly high precision calculations of the decay rates $Z \to b\bar{b}$ and $H \to gg$. We demonstrate a new application of the differential equation method (DEM) to calculate analytically several scalar vertex diagrams for the above decays. Intense study of the relevant diagrams is in progress (see [1, 2, 3]) : semianalytically, using the technique of asymptotic expansion of Feynman diagrams [4, 5] and subsequent use of conformal mapping and summation by Padé approximation. We would like to note, however, that the technique of [1, 2] works best for diagrams with high thresholds. In the case of several massless virtual particles and in particular if the diagrams under consideration have zero thresholds, additional contributions (apart from the “naive” diagram, see e.g. [3]) are needed due to the underlying method of the large mass expansion. Moreover, in the above approach the convergence of the series w.r.t. the inverse masses is getting worse for large external momenta squared the fewer heavy masses are involved. Thus, to obtain analytic results is quite an important task in particular for those cases with many massless virtual particles. If only one non-zero mass is involved, in fact the DEM provides in a systematic manner these analytic results in a manageable amount of work. These analytic results in general are given in terms of a one-dimensional integral representation from which it is possible to obtain the Taylor series expansion in the external momentum squared. The latter is considered as the most efficient approach for the final numerical evaluation due to references [1, 2, 3].

In the next section we shall recall the basic formulae and the sense of the DEM [6]-[8] (and [9] for short review), which is used for the calculations. In Sections 3 and 4 we shall demonstrate the basic steps in the calculation of the massless two-loop diagram (Case 10, according to the notation of [2] and Fig. 1) and one diagram with one non-zero mass (Case 7). Section 5 contains only the results for another diagram (Case 8).

2 The Differential equation method

The DEM is a method which allows one to obtain results for massive diagrams, evaluating only diagrams having an essentially simpler structure than the initial ones. The method is based on the rule of integration by part [10]. Throughout the article we use the following notation. Assuming dimensional regularization, all the calculations are performed in Euclidean momentum space of dimension $D = 4 - 2\varepsilon$. The dotted and solid lines of any diagram correspond to massless and massive euclidean propagators

$$\frac{1}{(q^2)^\alpha} = \ldots \frac{1}{(q^2 + m^2)^\alpha} = \frac{\alpha}{m^2}$$

$\alpha$ and $m$ are called the index and mass of this line. Further (except when mentioned otherwise) all solid lines have the same mass $m$. Lines with index 1 and mass $m$ are not

1The transition to Minkowsky space for a diagram $I \to \tilde{I}$ is given by changing $q_0 \to i\tilde{q}_0$, $q_j \to \tilde{q}_j$ ($j = 1, 2, 3$) ($q$ and $\tilde{q}$ are the external momenta of the diagram in Euclidean and Minkowsky space, respectively). Hereafter all variables with “” denote Minkowsky space. For the diagrams under consideration, we will make these transitions at the final step of the calculation.
marked. Let us now consider the rules for our calculation.

**Rule 1.** Massive tadpoles \( T(\alpha_1, \alpha_2) \) are integrated due to the graphic identity (hereafter we use the Euclidean integral measure \( d^Dk/\pi^{D/2} \))

\[
T(\alpha_1, \alpha_2) \equiv \begin{array}{c}
\alpha_1 \ \alpha_2
\end{array} = \frac{R(\alpha_1, \alpha_2)}{(m^2)_{\alpha_1+\alpha_2-D/2}}
\]

where

\[
R(\alpha_1, \alpha_2) = \frac{\Gamma(D/2 - \alpha_1)\Gamma(\alpha_1 + \alpha_2 - D/2)}{\Gamma(\alpha_2)\Gamma(D/2)}
\]

and \( \Gamma \) is the Euler \( \Gamma \)-function.

**Rule 2.** For a triangle the following recurrence relation \([10], [6]-[9]\) is valid (here the line with index \( \alpha_i \) has mass \( m_i \))

\[
\begin{array}{c}
\alpha_2 \ 
\alpha_3
\end{array} \alpha_1 \ 
\begin{array}{c}
D - 2\alpha_1 - \alpha_2 - \alpha_3 =
-2m_1^2\alpha_1 \ 
\begin{array}{c}
\alpha_2 \ 
\alpha_3
\end{array} \alpha_1+1
\end{array} \ 
\begin{array}{c}
\alpha_2 \ 
\alpha_3
\end{array} \alpha_1 \ 
\begin{array}{c}
\alpha_2 \ 
\alpha_3
\end{array} \alpha_1+1
\]

\[
+\alpha_2 \left( \begin{array}{c}
\alpha_2+1 \ 
\alpha_3
\end{array} \alpha_1-1 - \begin{array}{c}
\alpha_2+1 \ 
\alpha_3
\end{array} \alpha_1 \ 
\begin{array}{c}
\alpha_2+1 \ 
\alpha_3
\end{array} \alpha_1 \right) + (\alpha_2 \leftrightarrow \alpha_3)
\]

We stress the fact that the basic line of the triangle always plays a special role (in the following we call it "distinctive").

This Rule, i.e. the rule of integration by part, is obtained by multiplying the integrand of the original diagram by \( D = dk^\mu/dk^\mu \) (\( k \) is the momentum along the distinctive line) and using \( \int d^Dk \ \text{div}() = 0 \) for the regularized Feynman integrals.

**Rule 3.** Massless loops \( L(q^2; \alpha_1, \alpha_2) \) with the external momentum \( q \) are integrated due to the graphic identity

\[
L(q^2; \alpha_1, \alpha_2) \equiv \begin{array}{c}
\alpha_1 \ 
\alpha_2
\end{array} q = \frac{a(\alpha_1) a(\alpha_2)}{a(\alpha_1 + \alpha_2 - D/2)} A(\alpha_1, \alpha_2)
\]

where

\[
A(\alpha_1, \alpha_2) = \frac{\Gamma(D/2 - \alpha)}{\Gamma(\alpha)}, \quad a(\alpha) = \frac{\Gamma(D/2 - \alpha)}{\Gamma(\alpha)}
\]

Stricly speaking, Rule 3 is not independent of Rules 1 and 2 because it may be obtained (at least for even values of \( \alpha_1 \) or \( \alpha_2 \)) in the limit \( m^2 \to 0 \) from equation (1) below. However, it is very convenient to consider it as third additional rule.

Consider a few specific examples.

1. A simple loop with one massive line

\[
\begin{array}{c}
q \ 
\alpha
\end{array} = I_1(q^2; \alpha)
\]
Applying Rule 2 \((\alpha_1 = 1, \alpha_2 = \alpha, \alpha_3 = 0)\) with the massless line as distinctive one, we get
\[
(D - 2 - \alpha)I_1(q^2; \alpha) = \alpha \left[ T(0, \alpha + 1) - (q^2 + m^2)I_1(q^2; \alpha + 1) \right]
\]
The last diagram \(-\alpha I_1(q^2; \alpha + 1)\) on the r.h.s. is the derivative with respect to \(m^2\) of the initial diagram \(I_1(q^2; \alpha)\). Hence, the r.h.s. has the form
\[
\alpha R(0, 1 + \alpha) \frac{1}{[m^2]^{\alpha+1-D/2}} + (q^2 + m^2) \frac{d}{dm^2} I_1(q^2; \alpha),
\]
Thus we get a differential equation with respect to \(m^2\) for the initial diagram. Its solution for \(I_1\) is
\[
q^{2(\alpha+1-D/2)} I_1(q^2; \alpha) = \frac{1}{t^{D-2-\alpha}} \int_0^t ds \frac{\alpha R(0, 1 + \alpha)}{s^{(2-D/2)(1-s)^{\alpha+1-D/2}}} (t = \frac{q^2}{q^2 + m^2})
\]
Notice that sometimes a more convenient representation of the diagram \(I_1\) is given in the form
\[
\begin{array}{c}
\alpha \\
q
\end{array} = \alpha R(0, 1 + \alpha) \int_0^1 ds \frac{d}{4m^2/s}{\int_0^1 \frac{ds}{(1-s)^{2-D/2}s^{\alpha+1-D/2}}} q^{(\alpha+1-D/2)/m^2/s} (1)
\]

2. A simple loop with two massive lines
\[
\begin{array}{c}
q
\end{array} = I_2(q^2)
\]
In analogy with the previous consideration we get
\[
\begin{array}{c}
q
\end{array} = 2^{3-D} R(0, 2) \int_0^1 ds \frac{d}{4m^2/s}{\int_0^1 \frac{ds}{(1-s)^{1/2}s^{2-D/2}}} q^{(2-D/2)/4m^2/s} (2)
\]
It is clearly seen that Eqs.(1) and (2) allow us to reduce an \(l\)-loop diagram to an \((l - 1)\)-loop diagram with one propagator having “mass” \(m^2/s\) and \(4m^2/s\), respectively. Applications of these relations may be found in [3, 4].

Notice that if some lines have index 1, Rule 2 leads to a differential equation for the initial diagram with simpler diagrams in the inhomogeneous term. If, however, all indices are different from 1, Rule 2 does not lead to a simplification of the inhomogeneous term. In this case a more complicated technique, for example the method of Feynman parameters, is needed. Using the method of Feynman parameters the relation between \(l\)-loop and \((l - 1)\)-loop diagrams in general may be obtained in the form (as before \(\alpha_i\) corresponds to the mass \(m_i\)):
\[
\begin{array}{c}
\alpha_1, m_1 \\
q
\end{array} = \frac{\Gamma(\alpha_1 + \alpha_2 - D/2)}{\Gamma(\alpha_1) \Gamma(\alpha_2)} \int_0^1 ds \frac{d}{m_2^2/m_1^2 + m_2^2} q^{\alpha_1+\alpha_2-D/2} (3)
\]
For \(\alpha_1 = 1\) and \(\alpha_2 = \alpha\) and \(m_1 = 0\) (3) reduces to (1).
3. Vertex-type diagrams and $N$-point functions were studied in [7] and [8], respectively. Further simple examples are worked out in these Refs. Since the basic subject of our investigation is to calculate vertex diagrams in $Z \rightarrow b\bar{b}$ and $H \rightarrow gg$ decays, we consider here as another example the one-loop vertex diagram with the kinematics $q_1^2 = q_2^2 = 0$, $(q_1 - q_2)^2 = q^2$.

\[
\begin{align*}
q_1 - q_2 &= I_3(q^2, m^2) \\
&= I_3(q^2, m^2)
\end{align*}
\]

Applying Rule 2 with the massive and one of the massless as distinctive lines, we obtain, respectively

\[
(D - 4)I_3 = 2L(q^2, 1, 2) - 2m^2
\begin{align*}
L(q^2, 1, 2) - 2m^2
\end{align*}
\]

\[
(D - 4)I_3 = T(1, 2) + T(2, 1) - q^2
\begin{align*}
T(1, 2) + T(2, 1) - m^2
\end{align*}
\]

a) For the massless case, Eq.(4) leads to the result for $I_3(q^2, 0)$ in the form

\[
I_3(q^2, 0) = -\frac{1}{\varepsilon} A(1, 2) \frac{1}{q^{2(1+\varepsilon)}} = \frac{1}{\varepsilon^2} \frac{\Gamma(1+\varepsilon)}{q^{2(1+\varepsilon)}} \frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}
\]

b) For the massive case combining (4) and (5), we get the differential equation for $I_3(q^2, m^2)$ in the form

\[
\left(1 - \frac{2m^2}{q^2}\right)(D - 4)I_3 = 2f_3 + 2m^2 \left(1 - \frac{m^2}{q^2}\right) \frac{d}{dm^2} I_3,
\]

where

\[
f_3 = L(q^2, 1, 2) - \frac{m^2}{q^2} \left(T(1, 2) + T(2, 1)\right)
\]

is a combination of the massless loop and massive tadpoles.

Introducing the variable $x = q^2/m^2$ and using the boundary condition: $I_3 = 0$ for $m^2 \rightarrow \infty$, we get

\[
I_3(q^2, m^2) = \frac{1}{\varepsilon} \frac{\Gamma(1+\varepsilon)}{q^{2(1+\varepsilon)}} \int_0^x \frac{dy y^{-2\varepsilon}}{(1-y)^{1-\varepsilon}} \left[\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} - y^\varepsilon\right]
\]

\[
= \frac{1}{q^2} \left[\zeta(2) - \text{Li}_2(1-x)\right],
\]

where $\text{Li}_n$ is the polylogarithm and $\zeta(n)$ the Riemann $\zeta$-function.

In Minkowsky space (6) reads

\[
\tilde{I}_3(q^2, m^2) = \frac{i}{q^2} \left[\zeta(2) - \text{Li}_2(1+x)\right],
\]
where $\tilde{x} = \frac{\tilde{q}^2}{m^2}$. The appearance of an imaginary part for $iI_3$ from $\text{Li}_2(1 + \tilde{x})$ for $\tilde{x} > 0$ is seen from the analytic properties of $\text{Li}_2$.

The key idea of the DEM: using the rule of integration by part (Rule 2) with different distinctive lines, the results for massive Feynman diagrams are obtained as a differential equation with an inhomogeneous term containing simpler diagrams. Using this procedure repeatedly one can reduce the 2-loop diagrams to 1-loops. These are integrated either again with the help of Rule 2 (by analogy with example 3), decreasing step by step the number of lines, or using the method of Feynman parameters for more complicated cases.

The main difference from the massless case is the necessity to integrate the final result with respect to the mass several times. The basic advantage compared to the usual methods (for example, Feynman parameter method) to calculate massive Feynman diagrams is the appearance of complicated functions in the final step only. Hence, these functions do not interfere the process of calculation.

### 3 Two-loop vertex diagrams

At two-loop level we consider here three diagrams (see Fig. 1) out of ten characterized in [2]. They have the kinematics: $q_1^2 = q_2^2 = 0$, $(q_1 - q_2)^2 = q^2$. The evaluation of the massless diagram $J_{10}$ (*Case 10*) and one diagram with a non-zero mass $J_7$ (*Case 7*) will be fully presented. For another diagram (*$J_8$, i.e. Case 8*) we give only the final result.

Applying Rule 2 with line 6 (see Fig. 1) as distinctive one, we have for the $J_7$ diagram:

\[
(D - 4)J_7 = 2 - 2 - 2 - 2m^2 - m^2 \tag{7}
\]

Repeating the application of Rule 2 to the second diagram of Eq.(7) with the left vertical line as distinctive one, we get

\[
(D - 4) = 2 + 2 - 2 \tag{8}
\]

**Case 10.** For the massless case, Eqs.(7) and (8) lead to the result for $J_{10}$ in the following form

\[
(-2\epsilon)J_{10} = 2A(1, 2)\left(\frac{1}{q^2(1+\epsilon)} + \frac{1+\epsilon}{\epsilon} \right) - \frac{1}{\epsilon} A(2, 2) \tag{9}
\]

The first diagram on the r.h.s. of Eq.(9) was already calculated in the previous section, the third one may be done in an analog manner. The second diagram may be evaluated

\[
\text{Sometimes (see Eq.(8) below, for example) the differential equation can be brought to a simple arithmetic relation between the initial diagram and simpler diagrams in the inhomogeneous term.}
\]
by the method of Feynman parameters. The result for $J_{10}$ is known (see [11]) and has the form:

$$
J_{10} = \frac{1}{4 \varepsilon^4} \frac{\Gamma^2(1 + \varepsilon)}{(q^2)^{2(1 + \varepsilon)}} \left[ \frac{2 \Gamma^2(1 - \varepsilon) \Gamma(1 + \varepsilon)}{\Gamma(1 - 2 \varepsilon) \Gamma(1 + \varepsilon)} - 3 \frac{\Gamma(1 - 2 \varepsilon) \Gamma(1 + 2 \varepsilon) \Gamma(1 - \varepsilon)}{\Gamma(1 - 3 \varepsilon) \Gamma(1 + \varepsilon)} \right] + \frac{3}{4} \frac{\Gamma^2(1 - 2 \varepsilon) \Gamma(1 + 2 \varepsilon)}{\Gamma(1 - 3 \varepsilon) \Gamma(1 + \varepsilon)}
$$

We consider our derivation particularly simple. In Minkowsky space the diagram $\tilde{J}_{10}$ is represented in the form (10) with the replacement $q^2 \to -\tilde{q}^2$ and the additional factor $-1$.

**Case 7.** For the case with one non-zero mass, Eq.(7) leads to a differential equation for $J_7$ since the sum of the last two diagrams on the r.h.s. is the derivative of the initial diagram $J_7$ with respect to $m^2$. Solving it, we have

$$
J_7 = x^\varepsilon \int_0^x \frac{dy}{y^{1+\varepsilon}} f_7, \quad (11)
$$

where $f_7$ is the inhomogeneous term containing the two first diagrams of the r.h.s. of Eq.(7). These diagrams are effectively only one-loop vertex diagrams. Indeed, using for the second one Eq.(3) with $m_1^2 = 0$ to the first two diagrams of the r.h.s. of Eq.(8) and with $m_1^2 = m_2^2$ to the last one, we have

$$
\begin{align*}
\varepsilon \int_0^1 \frac{ds}{s(1-s)} & \left[ 1 + \frac{1+\varepsilon}{m^2/s} \ln(1+z) + \frac{1}{2} \ln^2(1+z) - \ln(x) \ln(1+z) \right] \\
& - \Phi_0^{(7)}(z) \left[ 1 + \frac{1}{2} \ln(x) \ln^2(1+z) + \frac{1}{2} \ln^2(x) \ln(1+z) \right],
\end{align*}
$$

where $z = xs(1-s)$,

$$
\Phi_0^{(7)}(z) = \ln[s(1-s)] \ln^2(1+z) - \frac{7}{6} \ln^3(1+z) - 12S_{1,2}(-z) - 5 \ln(1+z) \text{Li}_2(-z) - \zeta(2) \ln(1+z)
$$

and hereafter $\text{Li}_n(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$.
1. In Refs. [1, 2, 3] some diagrams contributing to the processes $Z \to b\bar{b}$ and $H \to gg$ were evaluated semianalytically, i.e. the first coefficients of the Taylor series in $q^2/m^2$ were calculated analytically (in terms of rational numbers and $\zeta(2), \zeta(3)$) and the diagram itself is reconstructed by means of conformal mapping and the Padé approximation technique. It turned out that this approach is particularly useful for the final numerical evaluation since it allows in general to calculate the diagram to higher precision than the numerical integration. Therefore we give below the Taylor series of (13). The importance of the obtained formula below is that arbitrary many Taylor coefficients can be obtained very easily while in Ref.[3] their calculation needed much computertime. This means that with such a representation the diagram can be calculated numerically to even higher precision with almost no effort.

\[ J_7 = -\frac{\Gamma^2(1+\varepsilon)}{(q^2)^2(m^2)^2 \varepsilon} \sum_{n=1}^{\infty} \frac{(-x)^n \Gamma^2(n)}{\Gamma(2n+1)} \left[ \frac{1}{\varepsilon^2} - \frac{1}{\varepsilon} \left( \ln(x) + S_1(n-1) \right) - \frac{3}{2} S_2(n-1) \right] - \frac{15}{2} S_1^2(n-1) + 4 S_1(n-1) S_1(2n) - \zeta(2) - \ln(x) S_1(n-1) + \frac{1}{2} \ln^2(x), \]

where

\[ S_l(n) = \sum_{k=1}^{n} \frac{1}{k^l} \]

2. In Minkowsky space the diagram $\tilde{J}_7$ is represented in the form (13) with the replacement $x \to -\tilde{x}$ and the additional factor $-1$. The imaginary part generates from the terms $\ln(-\tilde{x}) = \ln(\tilde{x}) - i\pi$ and $\ln^2(-\tilde{x})$ below the threshold at $\tilde{q}^2 = 4m^2$ and from the functions $S_{n,k}(\tilde{x}s(1-s))$ above.

At threshold $\tilde{q}^2 = 4m^2$ we can get a closed form for diagram $\tilde{J}_7$. Splitting $\tilde{J}_7$ into real and imaginary part: $\tilde{J}_7 = \text{Re} \tilde{J}_7 + i \text{Im} \tilde{J}_7$, we have

\[ \text{Re} \tilde{J}_7 = \frac{\Gamma^2(1+\varepsilon)}{16(m^2)^2 \varepsilon^2} \left[ \frac{3}{\varepsilon^2} \zeta(2) - \frac{1}{\varepsilon} \left( 7\zeta(3) + 6\zeta(2) \ln(2) \right) \right] + 6 \zeta(2) \ln^2(2) - 14 \zeta(3) \ln(2) - \frac{511}{4} \zeta(4) + 16 U_{3,1}, \]

\[ \frac{1}{\pi} \text{Im} \tilde{J}_7 = \frac{\Gamma^2(1+\varepsilon)}{16(m^2)^2 \varepsilon^2} \left[ \frac{3}{\varepsilon^2} \zeta(2) - 6 \zeta(2) \ln(2) + 7 \zeta(3) \right], \]

where the function

\[ U_{3,1} = \sum_{n>m>0} \frac{(-1)^{n+m}}{n^3 m} = \frac{1}{2} \zeta(2) \ln^2(2) - \frac{1}{12} \ln^4(2) + \frac{1}{2} \zeta(4) - 2 \text{Li}_4 \left( \frac{1}{2} \right) \]

was defined by Broadhurst in [13].
Case 8. The scheme for calculating diagram $J_8$ is similar to the one of the previous subsection. Here we will present only the final result for this diagram.

1. The result of the diagram $J_8$ may be given in the following form

$$J_8 = \frac{\Gamma^2(1 + \varepsilon)}{(q^2)^2(m^2)^2} \left[ \frac{1}{\varepsilon^2} \Phi_2^{(8)}(x) + \frac{1}{\varepsilon} \Phi_1^{(8)}(x) + \Phi_0^{(8)}(x) \right], \quad (14)$$

where $\Phi_1^{(8)}(x) = \Phi_1^{(8)}(x) + \Phi_2^{(8)}(x)$ and $\Phi_0^{(8)}(x) = \Phi_0^{(8)}(x) + \Phi_2^{(8)}(x)$ and

$$\begin{align*}
\Phi_2^{(8)}(x) &= \zeta(2) - \text{Li}_2(1 - x), \\
\Phi_1^{(8)}(x) &= 3\text{Li}_3(1 - x) - 3S_{1,2}(1 - x) - 2\zeta(2)\ln(1 - x), \\
\Phi_0^{(8)}(x) &= 4\int_0^x \frac{dy}{1 - y}\text{Li}_2(-y) \\
&= \left[ \ln(1 - x)\text{Li}_2(-x) + \frac{1}{2}S_{1,2}(x^2) - S_{1,2}(x) - S_{1,2}(-x) \right], \\
\Phi_0^{(8)}(x) &= 2\zeta(3) - 7S_{1,3}(1 - x) - 9\text{Li}_4(1 - x) + 11S_{2,2}(1 - x) - \frac{3}{2}\text{Li}_2^2(1 - x) \\
&+ \zeta(2) \left[ \ln^2(1 - x) + 6\text{Li}_2(1 - x) \right] \\
\Phi_0^{(8)}(x) &= 4\int_0^x \frac{dy}{1 - y}\left[ \frac{3}{2}\text{Li}_3(-y) + 4S_{1,2}(-y) + \ln(1 - y)\text{Li}_2(-y) \right] \\
&- \Phi_1^{(8)}(x)\ln(1 - x)
\end{align*}$$

or the expansion w.r.t. $q^2/m^2$:

$$J_8 = \frac{\Gamma^2(1 + \varepsilon)}{(q^2)^2(m^2)^2} \sum_{n=1}^{\infty} \frac{x^n}{n} \left[ \frac{1}{\varepsilon^2} \left( \frac{1}{n} - \ln(n) \right) + \frac{1}{\varepsilon} \left( \Phi_1^{(8)}(n) - 3\ln(n)\left[ S_1 + \frac{1}{n} \right] + \frac{3}{2}\ln^2(n) \right) \right]$$

$$+ \Phi_0^{(8)}(n) - \ln(n)\Phi_1^{(8)}(n) + \frac{1}{2}\ln^2(n)\left[ 5S_1 + \frac{7}{n} - \frac{7}{6}\ln^3(n) \right],$$

where

$$\begin{align*}
\Phi_1^{(8)}(n) &= 3S_2 - 4K_2 + \frac{3S_1}{n} + \frac{3}{n^2} - \zeta(2), \\
\Phi_0^{(8)}(n) &= \frac{7}{2}S_2 + \frac{9}{2}S_1^2 + \frac{5S_1}{n} + \frac{7}{n^2} - 3\zeta(2) \\
\Phi_0^{(8)}(n) &= 9S_1S_2 + 2S_3 - 2K_3 - 12K_{2,1} - 4S_1K_2 + \frac{9S_1^2 + 7S_2}{2n} + \frac{5S_1}{n^2} + \frac{7}{n^3} \\
&- \zeta(2) \left[ 7S_1 + \frac{3}{n} \right] - 2\zeta(3)
\end{align*}$$

and

$$S_l \equiv S_l(n - 1), \quad K_l = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k^l} \quad \text{and} \quad K_{2,1} = \sum_{k=1}^{n-1} \frac{(-1)^{k+1}}{k^2} S_1(k - 1).$$
2. In Minkowsky space the diagram $\tilde{J}_8$ is represented in the form (14) with the replacement $x \rightarrow -\tilde{x}$ and the additional factor $-1$. The imaginary part generates from the functions $S_{n,k}(1 + \tilde{x})$, when $\tilde{x} > 0$.

The calculation of the diagram $\tilde{J}_8 = \operatorname{Re} \tilde{J}_8 + i \operatorname{Im} \tilde{J}_8$ at threshold $\tilde{q}^2 = m^2$ leads to the following result:

$$\operatorname{Re} \tilde{J}_8 = \frac{\Gamma^2(1 + \varepsilon)}{(m^2)^{2+2\varepsilon}} \left[ \frac{1}{2\varepsilon^2} \zeta(2) ight. + \frac{1}{2\varepsilon} \left( \frac{1}{4} \zeta(3) - 15\zeta(2) \ln(2) \right) + \frac{45}{4} \zeta(2) \ln^2(2) - \frac{401}{16} \zeta(4) + \frac{15}{2} U_{3,1} \right]$$

$$\frac{1}{\pi} \operatorname{Im} \tilde{J}_8 = \frac{\Gamma^2(1 + \varepsilon)}{(m^2)^{2+2\varepsilon}} \left[ \frac{1}{\varepsilon^2} \ln(2) ight. + \frac{3}{2\varepsilon} \left( \zeta(2) - \ln^2(2) \right) + \frac{3}{2} \ln^3(2) - 14\zeta(2) \ln(2) + \frac{53}{8} \zeta(3) \right]$$

4 Conclusion

The differential equation method has been applied to demonstrate a new approach to calculate two-loop vertex-type diagrams with one non-zero mass. Analytic results have been obtained for several scalar two-loop diagrams of the processes $Z \rightarrow b\bar{b}$ and $H \rightarrow gg$, which will be relevant for tests of the Standard Model. As main result we obtained a closed formula for the Taylor coefficients for the diagrams under consideration, which allows their easy numerical evaluation with extremely high precision. The obtained coefficients were checked against those of Ref. [3], which were, however, only known explicitly up to a certain highest index in terms of rational numbers and $\zeta(2), \zeta(3)$.

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Figure 1. Planar diagrams with zero threshold occurring in $Z \to b\bar{b}$ and $H \to gg$ according to notation of Ref. [3]. Dashed lines are massless and solid lines massive.