On a Property of Nilpotent Matrices over an Algebraically Closed Field

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Suppose $F$ is an algebraically closed field. We prove that the ring $\prod_{n=1}^{\infty} M_n(F)$ has a special property which is, somewhat, in sharp parallel with (and slightly better than) a property established by Šter (LAA, 2018) for the rings $\prod_{n=1}^{\infty} M_n(\mathbb{Z}_2)$ and $\prod_{n=1}^{\infty} M_n(\mathbb{Z}_4)$, where $\mathbb{Z}_2$ is the finite simple field of two elements and $\mathbb{Z}_4$ is the finite indecomposable ring of four elements.

**Keywords:** nilpotent matrices, idempotent matrices, Jordan canonical form, algebraically closed fields.

**Bibliography:** 4 titles.
All rings $R$ are assumed here to be associative, containing the identity element 1 which differs from the zero element 0 of $R$. Recall that a ring $R$ is *nil-clean* provided that each its element is a sum of a nilpotent and an idempotent, is *$\pi$-regular* provided that for every element $r \in R$ there is $n \in \mathbb{N}$ such that $r^n = r^n R r^n$, and is *strongly $\pi$-regular* provided that $r^n \in r^{n+1} R$.

In his seminal paper [4], Ster showed that the ring $\prod_{n=1}^{\infty} M_n(\mathbb{Z}_2)$ is nil-clean but not strongly $\pi$-regular, whereas the ring $\prod_{n=1}^{\infty} M_n(\mathbb{Z}_4)$ is nil-clean but not $\pi$-regular. He utilizes an innovation of the method used in [1]. Specifically, for any $n \in \mathbb{N}$, it was proved there that, for every $n \times n$ matrix $A$ over the finite field $\mathbb{Z}_2$, there exists an idempotent matrix $E$ such that $(A - E)^4 = 0$, while the index of nilpotence over the finite ring $\mathbb{Z}_4$ is precisely 8. As usual, the symbol $I$ will stand in the sequel the standard matrix identity. Thereby, $A = N + E$ for some $N^4 = 0$ and hence $(I - E)A = (I - E)N$, but it is not clear at all whether $[(I - E)A]^4 = 0$ will hold eventually.

On the other side, in [2] we have examined rings $R$ having the property that, for each $a \in R$, there is an idempotent $e \in aR$ such that $(1 - e)a$ is nilpotent. We shall be here even rather more precise by considering an existing idempotent $e \in aRa$ with $[(1 - e)a]^2 = 0$.

It is well known that finite fields are, surely, *not* algebraically closed. So, the purpose of this very short note is to show that some (although little) improvement is possible by a strengthening of the technique utilized in [2] in the case of algebraically closed fields.

Before proceed by proving our chief result, we need the next two technical statements.

**Lemma 1.** Let $R$ be a unital ring, $n \geq 2$, and $A = \sum_{i=1}^{n-1} E_{i, i+1} \in M_n(R)$, where the $E_{i, j}$ denote matrix units. Then there exists an idempotent $B \in AM_n(R)A$ such that $((I - B)A)^2 = 0$.

**Proof.** First, suppose that $n = 2$. Then

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

and hence, taking $B = 0 \in AM_n(R)A$, we have $((I - B)A)^2 = A^2 = 0$. Let us therefore assume that $n \geq 3$, and let

$$B = A \left( \sum_{i=1}^{n-2} E_{i+2, i} \right) A = \left( \sum_{i=2}^{n-1} E_{i, i-1} \right) A = \sum_{i=2}^{n-1} E_{i, i}.$$

Then $B \in AM_n(R)A$, $B$ is clearly an idempotent, and

$$((I - B)A)^2 = ((E_{1, 1} + E_{n, n})A)^2 = E_{1, 2}^2 = 0,$$

as desired. $\square$

**Lemma 2.** Let $F$ be a field, $n \geq 1$, and $A \in M_n(R)$ a matrix in Jordan canonical form. Then there exists an idempotent $B \in AM_n(R)A$ such that $((I - B)A)^2 = 0$.

**Proof.** Write

$$A = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_m \end{pmatrix},$$

For citation:
P. V. Danchev, 2019, “On a Property of Nilpotent Matrices over an Algebraically Closed Field”, *Chebyshevskii sbornik*, vol. 20, no. 3, pp. 401–404.
where each $A_i$ is a Jordan block of size $n_i \times n_i$. For each $A_i$ we shall define a block $B_i$ of the same size, such that $B_i \in A_iM_{n_i}(F)A_i$ is idempotent.

If $A_i$ is invertible as a matrix of $M_{n_i}(F)$, then the identity element $I_{n_i}$ of $M_{n_i}(F)$ is in $A_iM_{n_i}(F)A_i$, and we set $B_i = I_{n_i}$. If $A_i$ is not invertible, then either $n_i = 1$ and $A_i = (0)$, or $n_i \geq 2$ and $A_i = \sum_{j=1}^{n_i-1} E_{jj+1}$. In the first case, we let $B_i = (0)$, and in the second case, we take $B_i$ as in Lemma 1. Then, clearly, in each case, $B_i \in A_iM_{n_i}(F)A_i$ is idempotent, and it is easy to see that $((I_{n_i} - B_i)A_i)^2 = 0$ for each $i$.

It follows immediately that

$$
B = \begin{pmatrix}
B_1 & 0 & \ldots & 0 \\
0 & B_2 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & B_m
\end{pmatrix}
$$

has the desired properties. □

**Proposition 1.** Let $F$ be an algebraically closed field, and let $R = \prod_{n=1}^{\infty} M_n(F)$. Then for each $A \in R$ there is an idempotent $B \in ARA$ such that $((I - B)A)^2 = 0$.

**Proof.** For each $n$ let $A_n$ denote the projection of $A$ onto the component $M_n(F)$ in $R$. Since $F$ is algebraically closed, for each $n$ we can find an invertible matrix $C_n \in M_n(F)$ such that $D_n = C_nA_nC_n^{-1}$ is in Jordan canonical form. By Lemma 2, for each $n$ we can find an idempotent matrix $G_n \in D_nM_n(F)D_n$ such that $((I_n - G_n)D_n)^2 = 0$. Now, for each $n$ let $B_n = C_n^{-1}G_nC_n$, and let $B = (B_1, B_2, \ldots) \in R$. Since each $G_n$ is idempotent, the same holds for each $B_n$, and hence also for $B$. Also, since $G_n \in D_nM_n(F)D_n$ and $C_n$ is invertible, we have for each $n$ that

$$
B_n = C_n^{-1}G_nC_n \in C_n^{-1}D_nM_n(F)D_nC_n = A_nC_n^{-1}M_n(F)C_nA_n = A_nM_n(F)A_n,
$$

and hence $B \in ARA$. Finally, since $((I_n - G_n)D_n)^2 = 0$, for each $n$ we have

$$
((I_n - G_n)A_n)^2 = ((I_n - C_n^{-1}G_nC_n)A_n)^2 = (C_n^{-1}(I_n - G_n)C_nA_n)^2
$$

$$
= (C_n^{-1}(I_n - G_n)D_nC_n)^2 = C_n^{-1}((I_n - G_n)D_n)^2C_n = 0,
$$

from which it follows that $((I - B)A)^2 = 0$, as required. □

We end our work with the following challenging query:

**Problem 1.** Extend the considered above property for any field $F$ which is not necessarily algebraically closed.

An intuitive idea could be the following one: It is enough to establish the claim for a given $M_n(F)$ with the index of the nilpotent $(1 - e)a$ bounded independent of $n$. Since every matrix is the direct sum of a unit and a nilpotent (we do not need the field $F$ to be algebraically closed for this), it is enough to do the assertion for units and for nilpotents. For a unit $a$, we take $e = 1$. Now suppose $a$ is nilpotent. It is enough to do the statement for the Weyr canonical form of $a$—for more details the interested reader can see [3]. Thus assume $a$ has Weyr structure $(n_1, n_2, \ldots, n_r)$. The idea is to get an idempotent $e$ in $aRa$ that is diagonal, has $0$s in the first $n_1$ places and the last $n_r$, and such that $(1 - e)a$ has zero blocks (relative to the partition $n_1, \ldots, n_r$) except in the $(1, 2)$ block. Then index of the nilpotent $(1 - e)a$ is exactly 2.

We will illustrate in the case of a homogeneous structure $(3, 3, 3, 3)$ but the argument in the nonhomogeneous case is similar although a little trickier. Thus, in terms of $3 \times 3$ blocks and $I = I_3$, we will have that
\[
a = \begin{pmatrix}
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & I \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Let us now

\[
r = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
I & 0 & 0 & 0 \\
0 & I & 0 & 0
\end{pmatrix},
\]

and

\[
e = ara = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & I & 0 & 0 \\
0 & 0 & I & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then, one finds that

\[
(1 - e)a = \begin{pmatrix}
0 & I & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

is nilpotent of index 2, as expected.

**Acknowledgments.** The author owes his sincere thanks to Professor Zachary Mesyan from the University of Colorado, Colorado Springs, and to Professor Kevin O’Meara, for their valuable communication on the present object.

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