Extracting fracture mechanics parameters (higher-order coefficients of Williams series expansion) from FEM analysis and digital photoelasticity method

A Mironov, D Petrova, Y Bakhareva and L Stepanova
Department of Mathematical Modeling in Mechanics, Samara National Research University, Samara, 443086, Russia
E-mail: Mironov.AV2020@yandex.ru, Daryamixpetrova@gmail.ru

Abstract. The paper describes new algorithms and procedures proposed for determining fracture mechanics parameters from finite element analysis using the over deterministic method. The multi-parameter crack tip stress field description is used. The algorithms and procedures based on multi-parameter stress field representations in series form are shown to be a powerful tool for reliable and accurate parameter determination. The technique is aimed at the determination of coefficients of the Williams series expansion from finite element analysis and is based on the over deterministic approach. The methodology is illustrated and applied to several cases of cracked specimens. Examples are presented for crack-tip fields recorded using digital photoelasticity. The results of finite element analysis are compared with the digital photoelasticity experiments. The results are in good agreement. The principal stresses obtained from finite element method are in good agreement with the isochromatic fringe patterns obtained by the photoelasticity method. Explanation has been made for giving guidance to a user on how best to approach implementation of the method from a practical standpoint.

1. Introduction
Defects and cracks play a decisive role in characterizing the strength and failure of these materials and structures [1-5]. Therefore, gaining insight on cracking processes is of crucial importance [1]. The first step in analyzing any fracturing process is to determine the crack tip asymptotic fields in order to characterize the stress, deformation and displacement near the crack tip, which requires the coefficients (unknowns) of the crack tip asymptotic field to be determined via reliable methods [1-10]. The first terms of the crack tip stress series expansion in isotropic linear elastic materials are singular, and, hence, dominant, in the proximate vicinity of the crack tip. Therefore, in the singular dominant zone the first terms are sufficient to characterize the crack tip fields. However, at further distances from the crack tip, the importance of the higher order terms become evident [6-11]. Thus, precise and simple algorithms are needed to reliably calculate coefficients in the multi-parameter crack-tip fields. Numerical methods and in particular finite element method (FEM) [6-11] allow us to extract the crack tip parameters. Moreover, it is worth noting that even determining the stress intensity factor is still the subject of investigations. For instance, in [12] direct extraction of stress intensity factors by a high-order numerical manifold method is realised. The proposed in [12] SIF extraction method is shown to
yield highly accurate results even without mesh refinement. Formulas extracting stress intensity factors (SIFs) of the biharmonic equations on cracked domains with clamped (or simply supported or free) boundary conditions along the crack faces are derived in [13]. In [13] it is shown the iteration methods quickly converge and the proposed enrichment method yields highly accurate stress intensity factors. It is also demonstrated that for a known true solution, the extraction formulas yield exact stress intensity factor. Thus, the determination of SIF still raises questions. The determination of higher order coefficients requires more accurate approaches [14-24].

This article applies the finite element over-deterministic method to determine the coefficients of the crack tip asymptotic fields. The present paper proposes the technique for extracting the higher order terms of the Williams series expansion (WE) for the crack tip stress field in a large variety cracked specimens.

2. Materials and methods

2.1. The multi-parameter crack tip stress field expansion

The main objective of this paper is the numerical determination of higher-order coefficients of WE. Polar coordinates $r, \theta$ are introduced and centered at the crack tip. In polar coordinates the Williams series solution for the near crack – tip stress field has the form [18,23,24]

$$\sigma_{ij}(r,\theta) = \sum_{m=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{k}^{mk} r^{2-1} f_{m,ij}^{(k)}(\theta),$$

where index $m$ is associated to the fracture mode; $a_{k}^{mk}$ are coefficients related to the geometric configuration, load and mode; $f_{m,ij}^{(k)}(\theta)$ are angular functions depending on stress components and mode.

Analytical expressions for angular eigenfunctions $f_{m,ij}^{(k)}(\theta)$ are available [16,17]:

$$f_{m,ij}^{(1)}(\theta) = (k/2) \left[(2+k/2+(-1)^{k}) \cos(k/2-1) \theta-(k/2-1) \cos(k/2-3) \theta\right],$$

$$f_{m,ij}^{(2)}(\theta) = (k/2) \left[(2-k/2-(-1)^{k}) \cos(k/2-1) \theta+(k/2-1) \cos(k/2-3) \theta\right],$$

$$f_{m,ij}^{(3)}(\theta) = (k/2) \left[-(k/2+(-1)^{k}) \sin(k/2-1) \theta+(k/2-1) \sin(k/2-3) \theta\right],$$

$$f_{m,ij}^{(4)}(\theta) = (k/2) \left[(2+k/2-(-1)^{k}) \sin(k/2-1) \theta-(k/2-1) \sin(k/2-3) \theta\right],$$

$$f_{m,ij}^{(5)}(\theta) = (k/2) \left[-(2-k/2+(-1)^{k}) \sin(k/2-1) \theta+(k/2-1) \sin(k/2-3) \theta\right].$$

The multi-parameter fracture mechanics concept consists in the idea that the crack-tip stress field is described by means of WE (1). In this work the central crack in an infinite plane medium is considered. Analytical determination of coefficients in crack-tip expansion for a finite crack in an infinite plane medium is given in [23,24]:

for Mode I crack:

$$a_{2n+1}^{2} = (-1)^{n+1} \frac{(2n)!}{2^{3n+1/2}(n)!^{2}(2n-1)a^{n-1/2}}, a_{1} = \frac{\sigma_{12}^{\infty}}{4}, a_{2k}^{1} = 0$$

(4)

$$a_{2n+1}^{2} = (-1)^{n} \frac{(2n)!}{2^{3n+1/2}(n)!^{2}(2n-1)a^{n-1/2}}, a_{2k}^{2} = 0$$

(5)

for Mode II crack.
The analytical solution (4), (5) allows us to validate the proposed method since one can compare the numerical results with the analytical ones. The crack length is less than the width and height of the plate. It is shown that the higher order terms in WE can play significant role in the description of the crack tip fields. Nowadays, various techniques are used to determine the parameters $a_k^n$ that characterize the crack-tip stress field. Now one can enumerate analytical [15,16,23,24], experimental [7-11, 25,26] and numerical [19] methods. One of the promising methods is FEM. One of the numerical examples discussed below is the large plate with the central crack. The finite element solution will be obtained and the results will be compared with the analytical formulae (4) and (5).

2.2. Finite element over-deterministic method

As it is noted in [1] the basic principle of the finite element over-deterministic method is the use of a large number of FE data points in order to calculate the crack tip parameters. This is done by forming an algebraic system of equations where the number of equations is more than the number of unknowns. In this case the over-deterministic system of equations is encountered. In the framework of using the over-deterministic method to determine the coefficients of (1) nodal stresses can provide the necessary set of equations. The over deterministic technique assumes more equations than unknowns in order to obtain more accurate values. This implies that one can form an over deterministic system. Taking data from different points at different distances from the crack tip is allowed as higher order terms are included in the stress equations. The algorithm is implemented using in the mathematical software Maple. One can use the approach described in [12] and present equation (1) in the matrix form as

$$\sigma = CA$$

The closed form solution of (6) for the unknown vector of fracture mechanics parameters $A$ can be written as

$$A = (C^T C)^{-1} C^T \sigma$$

where $A = (C^T C)^{-1} C^T$ is the pseudo-inverse of matrix $C$. The coefficients $A$ are estimated by minimizing the objective function which is of quadratic form for stress expression in terms of unknown parameters:

$$J(A) = (\sigma - CA)^T (\sigma - CA)/2$$

2.3. Numerical example

The first example is the plate with the small central crack. In this work, 2D finite element analysis (FEA) of cracked specimens is carried out using Abaqus software to estimate SIF, T-stresses and coefficients of higher-order terms of WE. The analysis is done with 8-noded plane strain elements. The quarter point element is used to capture square root singularity at the crack tip. The model of plate with center crack is of dimension 400 mm $\times$ 400 mm having a crack of 10 mm length. The mesh pattern around the crack tip is kept very fine to capture the high-stress gradient. The mesh convergence is achieved with 72 elements along circumferential and 60 along the radial direction. In total, there are 13 344 elements. The typical finite element mesh is shown in figure 1. To determine the higher order coefficients the stress tensor components from the nodes belonging to concentric circles are used. One can use different number of concentric circles. A class of numerical experiments with different numbers of concentric circles has been realized. The minimum number of stress tensor components was 219 since one can use the only circle with the following stress tensor components $\sigma_{11}, \sigma_{12}, \sigma_{22}$. Increasing the number of considered concentric circles surrounding the crack tip one can enhance the dimension of the system (6). The maximum number of equations in (6) in the numerical experiments performed was 3492 from which the first fifteen coefficients of WE have been obtained.
Figure 1. Typical mesh containing singular elements near the crack tips.

The next example is a plate with double edge notch. The model has a size of 100 mm×50 mm and having a notch of 8.5 mm length. The mesh convergence is achieved with 72 elements along circumferential and 20 along the radial direction. In total, there are 5,181 elements. A series of numerical experiments with different numbers of concentric circles has been realized. We selected 73 points on circles of different ranges to plot a path around the top of the left notch. Based on the results of a series of experiments, the optimal path was chosen. The radius of the circle on which the experimental points are located was 0.6 mm. Using the values of the stress components for each experimental point we obtained 15 values of the components of the Williams series.

2.4. Extraction of the coefficients of the Williams series expansion for the semi-circular bend specimen, the plate with two edge notches and the plate with two collinear cracks from the FEM analysis

In this part of the paper the semi-circular bend (SCB) specimen with an inclined crack shown in figure 2 is studied. The following notations are adopted. \( P \) is the applied load, \( S \) is loading span in the SCB specimen, \( a \) is crack length, \( \alpha \) is crack inclination angle. The semi-circular bend specimen subjected to three-point bending has received much attention in recent years for measuring the mixed mode I/II fracture resistance [16-19].

In this work, 2D FEA of semi-disks with vertical crack and inclined notches is carried out using Abaqus software. To estimate SIF, T-stress and higher-order terms and verify the experimental results obtained FEM calculations have been employed. The analysis is done with 8-noded strain elements. The subsequent crack configurations are the plate with two edge notches and the plate with two collinear cracks of different length. The last configuration is interesting since there is the well-known analytical solution of the problem based on the theory of complex variable of the plane elasticity. All the numerical results obtained by the FEM analysis were compared with the experimental results obtained by using the photoelasticity method.
2.5. Extraction the coefficients of the WE near the crack tip by digital photoelasticity

Photoelasticity is a whole field experimental technique to obtain stress fields in both 2-D and 3-D elasticity problems [18,20,25,26]. Digital photoelasticity method has rapidly progressed in the last few years and has matured into an industry-friendly technique. Recently there has been a lot of works devoted to various aspects of the method and its applications [18,20,27-30]. The experimental setup is shown in figure 3 (left). The experimental isochromatic fringe patterns in the plate with the central crack are shown in figure 3. The over deterministic method has been applied to the experimental data obtained from the photoelasticity observations. The stress optic law relates the fringe order N and the in-plane principal stresses $\sigma_1, \sigma_2$, as $Nf_\sigma/t=(\sigma_1-\sigma_2)$, where $f_\sigma$ is the material stress fringe and $t$ is the thickness of the specimen.

![Experimental setup of transmission photoelasticity and isochromatic images for 85 kg, 90 kg, 100 kg and 125 kg.](image)

3. Results and discussion

The results of extraction of the coefficients of WE in the vicinity of the crack tip are given in table 1 where the first column shows the coefficients of WE obtained from FEA whereas the second one shows the error in FEM comparatively with the analytical results given by formulae (4) and (5) for an infinite plate with the central crack. Errors in analytical decisions are less than 1%, which means that the results are well consistent.

**Table 1. Coefficients of multi-parameter WE for a plate with a central crack of small length.**

| Coefficients of multi-parameter Williams series expansion for a plate with a central crack of small length | Error   |
|--------------------------------------------------|---------|
| $a_1^1=4.909 \text{ MPa} \cdot \text{m}^{1/2}$  | 0.01%   |
| $a_2^1=-2.449 \text{ MPa}$                       | 0.09%   |
| $a_3^1=2.484 \text{ MPa} \cdot \text{m}^{1/2}$  | 0.13%   |
| $a_4^1=-0.6263 \text{ MPa} \cdot \text{m}^{3/2}$| 0.22%   |
| $a_5^1=-0.3112 \text{ MPa} \cdot \text{m}^{5/2}$| 0.31%   |
| $a_6^1=-0.1951 \text{ MPa} \cdot \text{m}^{7/2}$| 0.35%   |
| $a_{11}^1=0.1361 \text{ MPa} \cdot \text{m}^{9/2}$| 0.44%   |
| $a_{13}^1=-0.1056 \text{ MPa} \cdot \text{m}^{11/2}$| 0.54%   |
| $a_{15}^1=0.0786 \text{ MPa} \cdot \text{m}^{13/2}$| 0.68%   |
Having obtained the solution for the plate with the central crack one can consider more complicated crack specimens. The plate with two edge notches was modelled in the multi-purpose software package Simulia Abaqus and explored. The numerical results are shown in figures 4-6. The distribution of the von Mises equivalent stress the plate with double edge notch is shown in figure 4. The distribution the stress component $\sigma_{11}$ stress the plate with double edge notch is shown in figure 5. The distribution of the stress component $\sigma_{22}$ for the plate with double edge notch is shown in figure 6. The crack tip neighborhood is encompassed by 72 singular finite elements. This allows us to obtain 219 equations used in the over-deterministic method. Thus, the total number of equations used in the over-deterministic method was 219. The first fifteen coefficients were determined and presented in table 2.

The results of extraction of the coefficients of WE in the vicinity of the notch tip given in table 2 indicate that the higher order coefficients of WE for the plate with two edge notches are determined stably and reliably. The next configuration considered in the analysis was the plate with two collinear cracks of different length. The distribution of the von Mises equivalent stress for plate with two collinear cracks of different length is shown in figure 7. The distribution the stress component $\sigma_{11}$ for plate with two collinear cracks of different length is shown in figure 8. The distribution of the stress component $\sigma_{22}$ for plate with two collinear cracks of different length is shown in figure 9.

The results of FEM analysis for the semi-circular bending specimen are shown in figures 10,11. Figure 10 (left) shows the distribution of the von Mises equivalent stress. Figure 10 (right) shows the
distribution of the stress component $\sigma_{11}$. Figure 11 shows the distributions of stress components $\sigma_{22}$ and $\sigma_{12}$.

**Table 2.** Coefficients of the Williams series expansion for the plate with double edge notch.

| Coefficients obtained by the finite element over-deterministic method |
|---|
| $a_1^1=110.2486 \text{ MPa} \cdot \text{m}^1$ |
| $a_2^2=-20.2416 \text{ MPa}$ |
| $a_3^3=20.5723 \text{ MPa} \cdot \text{m}^{1/2}$ |
| $a_4^4=-8.3117 \text{ MPa} \cdot \text{m}^1$ |
| $a_5^5=2.5757 \text{ MPa} \cdot \text{m}^{3/2}$ |
| $a_6^6=4.3562 \text{ MPa} \cdot \text{m}^2$ |
| $a_7^7=-3.6851 \text{ MPa} \cdot \text{m}^{5/2}$ |
| $a_8^8=-1.8220 \text{ MPa} \cdot \text{m}^3$ |
| $a_9^9=3.2209 \text{ MPa} \cdot \text{m}^{7/2}$ |
| $a_{10}^{10}=0.0611 \text{ MPa} \cdot \text{m}^4$ |
| $a_{11}^{11}=-2.0311 \text{ MPa} \cdot \text{m}^{9/2}$ |
| $a_{12}^{12}=1.0188 \text{ MPa} \cdot \text{m}^5$ |
| $a_{13}^{13}=0.2627 \text{ MPa} \cdot \text{m}^{11/2}$ |
| $a_{14}^{14}=-0.6395 \text{ MPa} \cdot \text{m}^{11/2}$ |
| $a_{15}^{15}=0.4357 \text{ MPa} \cdot \text{m}^{13/2}$ |

**Figure 7.** The distribution of the von Mises equivalent stress for plate with two collinear cracks of different length.
Figure 8. The distribution the stress component $\sigma_{11}$ for plate with two collinear cracks of different length.

Figure 9. The distribution of the stress component $\sigma_{22}$ for plate with two collinear cracks of different length.

Table 3. Coefficients of the Williams series expansion for the plate with two collinear cracks of different lengths.

| Coefficients obtained by the finite element over-deterministic method |
|-------------------------------------------------------------|
| $a_1^1$=96.3622 MPa·m$^{1/2}$                               |
| $a_2^2$=-43.5650 MPa                                       |
| $a_3^3$=44.4752 MPa·m$^{-1/2}$                            |
| $a_4^4$=0.7748 MPa·m$^{-1}$                                |
| $a_5^5$=-12.0939 MPa·m$^{3/2}$                            |
| $a_6^6$=1.5523 MPa·m$^{-2}$                                |
| $a_7^7$=6.4957 MPa·m$^{-5/2}$                              |
| $a_8^8$=-2.7905 MPa·m$^{-3}$                               |
| $a_9^9$=-0.7255 MPa·m$^{-7/2}$                             |
| $a_{10}^{10}$=3.3891 MPa·m$^{-4}$                          |
| $a_{11}^{11}$=-3.4391 MPa·m$^{-9/2}$                       |
| $a_{12}^{12}$=-1.0432 MPa·m$^{-5}$                         |
| $a_{13}^{13}$=8.5231 MPa·m$^{-11/2}$                       |
| $a_{14}^{14}$=7.1988 MPa·m$^{-11/2}$                       |
| $a_{15}^{15}$=4.4892 MPa·m$^{-13/2}$                       |
Figure 10. Distributions of the von Mises equivalent stress (left) and the stress component $\sigma_{11}$ (right) for the semi-circular bend specimen.

Figure 11. Distributions of the stress component $\sigma_{22}$ (left) and $\sigma_{12}$ (right) for the semi-circular bending specimen.

The results of calculations are tabulated in Table 4. Thus, one can see that the overdeterministic method based on FEM analysis allows us to estimate the higher-order coefficients with high accuracy. Isochromatic fringe pattern in the semi-circular bend (SCB) specimen with an inclined crack is shown in Figure 12.

Table 4. Crack tip fracture parameters for the SCB specimen

| Fracture parameter | $n = 2$ | $n = 4$ | $n = 8$ |
|--------------------|---------|---------|---------|
| $K_1$ (MPa$\sqrt{m}$) | 23.90   | 23.99   | 24.00   |
| $K_{II}$ (MPa$\sqrt{m}$) | 0.45    | 0.41    | 0.40    |
| $a_2^1$ (MPa) | -0.44   | -0.456  | -0.457  |
| $a_3^1$ (MPa$\cdot$m$^{-1/2}$) | 0.145   | 0.146   |
| $a_4^1$ (MPa$\cdot$m$^{-1}$) | 0.001   | 0.000   |
| $a_5^1$ (MPa$\cdot$m$^{-3/2}$) |         |         | 0.021   |
| $a_6^1$ (MPa$\cdot$m$^{-2}$) |         |         | 0.0006  |
| $a_7^1$ (MPa$\cdot$m$^{-5/2}$) |         |         | 0.0004  |
| $a_8^1$ (MPa$\cdot$m$^{-3}$) |         |         | 0.0002  |
Figure 12. Isochromatic fringe pattern in the semi-circular bend (SCB) specimen with an inclined crack.

Table 5. Coefficients of the Williams series expansion for the plate with the central crack with the geometric parameters as in the experimental photoelasticity method.

| Coefficients obtained by the photoelasticity method | Coefficients obtained by the FEM analysis |
|----------------------------------------------------|------------------------------------------|
| \( a_1 = 7.2528 \text{ MPa} \cdot \text{m}^2 \) | \( a_1 = 7.2527 \text{ MPa} \cdot \text{m}^2 \) |
| \( a_2 = 2.7516 \text{ MPa} \) | \( a_2 = 2.7516 \text{ MPa} \) |
| \( a_3 = 2.1406 \text{ MPa} \cdot \text{m}^{-1/2} \) | \( a_3 = 2.0163 \text{ MPa} \cdot \text{m}^{-1/2} \) |
| \( a_4 = -0.337 \text{ MPa} \cdot \text{m}^{-1} \) | \( a_4 = -0.302 \text{ MPa} \cdot \text{m}^{-1} \) |
| \( a_5 = -0.2844 \text{ MPa} \cdot \text{m}^{3/2} \) | \( a_5 = -0.2757 \text{ MPa} \cdot \text{m}^{3/2} \) |
| \( a_6 = -0.0919 \text{ MPa} \cdot \text{m}^{-2} \) | \( a_6 = -0.0985 \text{ MPa} \cdot \text{m}^{-2} \) |
| \( a_7 = 0.0765 \text{ MPa} \cdot \text{m}^{-5/2} \) | \( a_7 = 0.0712 \text{ MPa} \cdot \text{m}^{-5/2} \) |
| \( a_8 = 0.0025 \text{ MPa} \cdot \text{m}^{-3} \) | \( a_8 = 0.0019 \text{ MPa} \cdot \text{m}^{-3} \) |
| \( a_9 = 0.0340 \text{ MPa} \cdot \text{m}^{-7/2} \) | \( a_9 = 0.0315 \text{ MPa} \cdot \text{m}^{-7/2} \) |
| \( a_{10} = 0.0017 \text{ MPa} \cdot \text{m}^{-4} \) | \( a_{10} = 0.0016 \text{ MPa} \cdot \text{m}^{-4} \) |
| \( a_{11} = 0.0098 \text{ MPa} \cdot \text{m}^{-9/2} \) | \( a_{11} = 0.0076 \text{ MPa} \cdot \text{m}^{-9/2} \) |
| \( a_{12} = 0.0019 \text{ MPa} \cdot \text{m}^{-5} \) | \( a_{12} = 0.0012 \text{ MPa} \cdot \text{m}^{-5} \) |
| \( a_{13} = 0.0056 \text{ MPa} \cdot \text{m}^{-11/2} \) | \( a_{13} = 0.0050 \text{ MPa} \cdot \text{m}^{-11/2} \) |
| \( a_{14} = 0.0008 \text{ MPa} \cdot \text{m}^{-11/2} \) | \( a_{14} = 0.0007 \text{ MPa} \cdot \text{m}^{-11/2} \) |
| \( a_{15} = 0.0018 \text{ MPa} \cdot \text{m}^{-13/2} \) | \( a_{15} = 0.00147 \text{ MPa} \cdot \text{m}^{-13/2} \) |

The results obtained by the photoelasticity method are almost equal to the results obtained by the finite element method.

4. Conclusion
In this paper, we propose and describe an algorithm for constructing the stress field expansion coefficients at the crack tip and notch tip from finite element analysis data. The algorithms are tested
on several examples and the results are compared with the results of the photoelastic experiments. The comparison showed good agreement between the values of the coefficients of the multi-parametric asymptotic expansion. It is shown that higher approximations in the asymptotic expansion are especially significant when processing the entire set of experimental information.

Acknowledgement
Stepanova L and Bakhareva Y are grateful for financial support of the Russian Foundation for Basic Research (project No. 19-01-00631).

References
[1] Aytollahi M R, Nejati M and Ghouli S 2020 The finite element over-deterministic method to calculate the coefficients of crack tip asymptotic fields in anisotropic planes Engineering Fracture Mechanics 231 106982
[2] Vivekanandan A and Ramesh K 2019 Study of interaction effects of asymmetric cracks under biaxial loading using digital photoelasticity Theoretical and Applied Fracture Mechanics 99 pp 104-117
[3] Jobin T M, Khaderi S N and Ramji M 2020 Experimental evaluation of the strain intensity factor at the inclusion tip using digital photoelasticity Optics and Lasers in Engineering 126 105855
[4] Ramesh K and Pandey A 2018 An improved normalization technique for white light photoelasticity Optics and Lasers in Engineering 109 pp 7-16
[5] Sasikumar S and Ramesh K Applicability of colour transfer techniques in Twelve fringe photoelasticity (TFP) Optics and Lasers in Engineering 127 105963
[6] Guo E, Liu Y, Han Y, Arola D and Zhang D 2018 Full-field stress determination in photoelasticity with phase shifting technique Measurement Science and Technology 29(4) 045208
[7] Stepanova L V, Dolgikh V S and Turkova V A 2017 Digital photoelasticity for calculating coefficients of the Williams series expansion in plate with two collinear cracks under mixed mode loading Ceur Workshop Proceedings 1904 pp 200-208
[8] Stepanova L V and Dolgikh V S 2018 Interference-optical methods in mechanics for the multi-parameter description of the stress fields in the vicinity of the crack tip Journal of Physics: Conference series 1096(1) 012117
[9] Stepanova L V 2020 The algorithm for the determination of the Williams asymptotic expansion coefficients for notched semidisics using the photoelasticity method and finite element method AIP Conference Proceedings 2216 020013
[10] Dolgikh V S and Stepanova L V 2020 A photoelastic and numeric study of the stress field in the vicinity of two interacting cracks: Stress intensity factors, T-stresses and higher order terms AIP Conference Proceedings 2216 020014
[11] Stepanova Larisa, Roslyakov Pavel and Lomakov Pavel 2016 A photoelastic study for multiparametric analysis of the near crack tip stress field under mixed mode loading Procedia Structural Integrity 2 pp 1797-1804
[12] Wu J, Wang Y, Cai Y and Ma G 2020 Direct extraction of stress intensity factors for geometrically elaborate cracks using a high-order Numerical Manifold Method Engineering Fracture Mechanics 230 106963
[13] Kim S, Palta B and Oh H-S 2020 Extraction formulas of stress intensity factors for biharmonic equations containing crack singularities Computers and Mathematics with Applications 80 pp 1142-1163
[14] Stepanova L 2017 Influence of higher-order terms of the Williams expansion on the crack-tip stress field for mixed-mode loadings: Asymptotic solutions and interference-optical methods of solid mechanics ICF 2017 -14th International Conference on Fracture 2 pp 110-111
[15] Stepanova L V and Roslyakov P S 2015 Complete asymptotic expansion m.williams near the crack tips of collinear cracks of equal lengths in an infinite plane medium PNRPU Mechanics Bulletin 4 pp 188-225
[16] Stepanova L V and Igonin S A 2014 Perturbation method for solving the nonlinear eigenvalue problem arising from fatigue crack growth problem in a damaged medium Applied Mathematical Modelling 38(14) pp 3436-3455
[17] Stepanova L V and Yakovleva E M 2015 Asymptotic stress field in the vicinity of a mixed-mode crack under plane stress conditions for a power-law hardening material Journal of Mechanics of Materials and Structures 10(3) pp 367-393
[18] Stepanova L V and Roslyakov P S 2016 Complete Williams asymptotic expansion of stress field near the crack tip: Analytical solutions, interference-optic methods and numerical experiments AIP Conference Proceedings 1785 030029
[19] Patil P , Vysasarayani C.P and Ramji M 2017 Linear least squares approach for evaluating crack tip fracture parameters using isochromatic and isoclinic data from digital photoelasticity Optics and Lasers in Engineering 93 pp 182-194
[20] Tabanyukhova M V 2020 Photoelastic analysis of the stressed state of a flat element with geometrical stress concentrators (cutout and cuts) Key Engineering Material 827 pp 330-335
[21] Leon J C D, and Restrepo-Martinez A and Branch-Bedoya J W 2019 Computational analysis of Bayer colour filter arrays and demosaicking algorithms in digital photoelasticity Optics and Lasers in Engineering 122 pp 195-208
[22] Stepanova L V and Dolgich V S 2017 The investigation of cracks’ parameters of the Williams series asymptotic expansion using photoelasticity method ICF 2017 – 14th International Conference on Fracture 2 530-531
[23] Hello G, Tahar M B and Roelandt J M 2012 Analytical determination of coefficients in crack-tip stress expansions for a finite crack in an infinite plane medium International Journal of Solids and Structures 49 556-566
[24] Hello G 2018 Derivation of complete crack-tip stress expansions from Westergaard-Sanford solutions International Journal of Solids and Structures 144-145 265-275
[25] Ramesh K and Promod B R 1992 Digital image processing of fringe patterns in photomechanic Opt. Eng. 31(7) 148
[26] Ramesh K, Gupta S and Kelkar A A 1997 Evaluation of stress field parameters in fracture mechanics by photoelasticity – revisited Eng. Fracture Mechanics 56(1) 25-45
[27] Ramesh K and Sasikumar S 2020 Digital photoelasticity: Recent developments and diverse applications Optics and Lasers in Engineering 135 106186
[28] Stepanova L V 2020 Experimental determination and finite element analysis of coefficients of the multi-parameter Williams series expansion in the vicinity of the crack tip in linear elastic materials. Part I PNRPU Mechanics Bulletin 4 237-249
[29] Stepanova L V 2020 Experimental determination and finite element analysis of coefficients of the multi-parameter Williams series expansion in the vicinity of the crack tip in linear elastic materials. Part II PNRPU Mechanics Bulletin 4 250-
[30] Hamza A A, Sokkar T Z N, Azzam M and Azzam A 2020 A quantitative study on using for characterizing the effect of the stretching speed on the necking phenomenon J Polm Eng 40(9) 753-762