New large-$N_c$ relations among the nucleon and nucleon-to-$\Delta$ GPDs

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(Dated: March 26, 2022)

We establish relations which express the generalized parton distributions (GPDs) describing the $N \to \Delta$ transition in terms of the nucleon GPDs. These relations are based on the known large-$N_c$ relation between the $N \to \Delta$ electric quadrupole moment and the neutron charge radius, and a newly derived large-$N_c$ relation between the electric quadrupole ($E2$) and Coulomb quadrupole ($C2$) transitions. Namely, in the large-$N_c$ limit we find $C2 = E2$. The resulting relations among the nucleon and $N \to \Delta$ GPDs provide predictions for the $N \to \Delta$ electromagnetic form factors which are found to be in very good agreement with experiment for moderate momentum transfers.

PACS numbers: 12.38.Lg, 13.40.Gp, 25.30.Dh

Electromagnetic form factors (FFs) are the standard source of information on the structure of the nucleon and as such have been studied extensively. The phenomenon of asymptotic freedom in QCD, however, provides us with more sophisticated tools of describing the quark-gluon structure of hadrons. Generalized parton distributions (GPDs) provide the distribution of quarks as a function of both their momentum fraction and transverse position in the nucleon, see Refs. [1, 2, 3, 4] for reviews.

GPDs can be accessed by selecting a small size configuration of quarks and gluons, provided by hard exclusive reactions such as deeply virtual Compton scattering (DVCS). Figure 1 illustrates the leading (“handbag”) contribution to DVCS processes with the nucleon or the nucleon first excitation — $\Delta(1232)$ — in the final state. The crucial feature of such hard reactions is the possibility to separate the perturbative and nonperturbative stages of the interactions due to the factorization theorems [3, 6, 7]. The non-perturbative stage of these hard exclusive processes is described by GPDs. First DVCS experiments aimed to measure GPDs have recently been completed, many others are underway.

In this Letter we would like to establish general relations, based on the large number-of-colors ($N_c$) limit of QCD, between the $N \to N$ and $N \to \Delta$ FFs and GPDs. The nucleon GPDs are already well-constrained by the knowledge of the nucleon elastic FFs and forward parton distributions. Viable parametrizations of the nucleon GPDs exist in the literature [8, 9]. The GPDs describing the $N \to \Delta$ transition are, on the other hand, poorly known. While a comprehensive analysis of the $N \to \Delta$ FFs exists (see [10] for review), the corresponding forward parton distributions cannot be measured by means of deep inelastic scattering.

Recently, the possibility of accessing the $N \to \Delta$ GPDs in DVCS was proposed, see [11] for detailed estimates, and a first measurement of such a process has already been reported [12]. Furthermore, by generalizing the large-$N_c$ relations between the octet and decuplet baryons, it became possible to relate the dominant magnetic dipole ($M1$) $N \to \Delta$ transition GPD to the nucleon isovector GPD [13]. In this work we look for the large-$N_c$ relations for the electric ($E2$) and Coulomb ($C2$) quadrupole $N \to \Delta$ GPDs.

To introduce the electromagnetic $N \to \Delta$ transition it is useful to write down the effective Lagrangian:

$$\mathcal{L}_{\gamma N \Delta} = \frac{3ie}{2M_N(M_N + M_\Delta)} \bar{N} T^3 \times \left[ g_M \partial_\mu \Delta_\nu \tilde{F}^{\mu \nu} + ig_E \gamma_5 \partial_\mu \partial_\nu F^{\mu \nu} \right. \right. \left. \left. - \frac{g_C}{M_\Delta} \gamma_5 \gamma^\alpha (\partial_\alpha \Delta_\nu - \partial_\nu \Delta_\alpha) \partial_\mu F^{\mu \nu} \right] + \text{H.c.,} \right.$$

where $N$ denotes the nucleon (spinor) and $\Delta_\mu$ the $\Delta$-isobar (vector-spinor) fields, $M_N$ and $M_\Delta$ are respectively their masses, $F^{\mu \nu}$ and $\tilde{F}^{\mu \nu}$ are the electromagnetic field strength and its dual, $T^3$ is the isospin-1/2-to-3/2 transition operator. The important observation that we take from this expression is that the couplings $g_M$, $g_E$ and $g_C$ appear with the same structure of spin-isospin and field operators, and hence should scale with the same power of $N_c$, for large $N_c$.

It is customary to characterize the three different types

![Diagram](image-url)
of the $\gamma N\Delta$ transition in terms of the Jones–Scadron FFs $G_M^\gamma$, $G_E^\gamma$, $G_C^\gamma$. The contribution of the effective couplings Eq. (1) to these FFs can be straightforwardly computed with the following result:

\[
\begin{align*}
G_M^\gamma(Q^2) &= g_M + [-M_\Delta \omega g_E + Q^2 g_C] / Q^2 _+ , \\
G_E^\gamma(Q^2) &= [-M_\Delta \omega g_E + Q^2 g_C] / Q^2 _+ , \\
G_C^\gamma(Q^2) &= -2M_\Delta [\omega g_E + M_\Delta g_E] / Q^2 _+ , \\
\end{align*}
\]

where we introduced $Q_+ = \sqrt{(M_\Delta + M_N)^2 + Q^2}$ and the photon energy in the $\Delta$ rest frame: $\omega = (M_\Delta^2 - M_N^2 - Q^2)/(2M_\Delta)$. We immediately note that at $Q^2 = 0$,

\[
\begin{align*}
G_E^\gamma(0) &= -\frac{\Delta}{2(M_N + M_\Delta)} g_E , \\
G_C^\gamma(0) &= -\frac{2M_\Delta}{M_N + M_\Delta} \left[ \frac{M_N + M_\Delta}{2M_\Delta} \frac{\Delta}{M_N} g_C + g_E \right] ,
\end{align*}
\]

where $\Delta = M_\Delta - M_N$ is the $\Delta$-nucleon mass difference. In the large-$N_c$ limit this mass difference goes as $1/N_c$, whereas the baryon masses increase proportionally to $N_c$:

\[
M_N(\Delta) = O(N_c) , \quad \Delta = O(N_c^{-1}) .
\]

Given the fact that $g_E$ and $g_C$ scale with the same power of $N_c$, we observe that the first term in Eq. (3b) is suppressed by $1/N_c^2$ and therefore obtain the following large-$N_c$ relation:

\[
G_C^\gamma(0) \approx \frac{2M_\Delta}{M_N + M_\Delta} \frac{2M_\Delta}{\Delta} G_E^\gamma(0) .
\]

Of special interest are the multipole ratios: $R_{EM} = E2/M1$ and $R_{SM} = C2/M1$, which can be expressed in terms of the Jones-Scadron FFs:

\[
R_{EM} = -\frac{G_E^\gamma}{G_M^\gamma} , \quad R_{SM} = -\frac{Q_+ - Q}{4M_\Delta^2} \frac{G_E^\gamma}{G_M^\gamma} .
\]

It is easy to see that our relation (4) translates into $R_{SM} = R_{EM}$, at $Q^2 = 0$. Using Eqs. (5a) and (6) one readily verifies the result of Ref. [14]: $R_{EM} = O(1/N_c^2)$.

We now turn to GPDs and recall that in general they depend on three variables: $x$, $\xi$, and $Q^2$. The light-cone momentum fraction $x$ is defined by $k^+ = xP^+$, where $k$ is the quark loop momentum and $P$ is the average nucleon momentum $P = (p + p^{'})/2$, where $p(p^{'})$ are the initial (final) baryon four-momenta, see Fig. [1]. The skewedness variable $\xi$ is defined by $q^+ = -2\xi P^+$, where $q = p^-' - p$ is the overall momentum transfer in the process, and where $2\xi \rightarrow x_B/(1-x_B)/2$ in the Bjorken limit; $x_B = Q^2_f/(2p \cdot q_h)$ is the usual Bjorken scaling variable, with $Q^2_f = -q_h^2 > 0$ the virtuality of the hard photon. Finally, the third variable is the total momentum transfer squared: $Q^2 = -q^2$.

In a frame where the virtual photon momentum $q_h$ and the average nucleon momentum $P$ are along the $z$-axis and opposite to each other, one can parameterize the non-perturbative piece of the $N \rightarrow \Delta$ DVCS amplitude as [2, 13]:

\[
\frac{1}{2\pi} \int dy e^{ixP^+} y^{-}(\Delta|\bar{\psi}(\frac{y}{2}) \gamma \cdot n \gamma_\psi(\frac{y}{2})|N) \left| y^+ = y_+ = 0 \right.
\]

\[
= \sqrt{2/3} u^a(p') \left\{ H_M(x, \xi, Q^2) (-K_{\alpha\mu}^M) + H_E(x, \xi, Q^2) (-K_{\alpha\mu}^E) + H_C(x, \xi, Q^2) (-K_{\alpha\mu}^C) + H_A(x, \xi, Q^2) (g_{\alpha\mu}\gamma_5) \right\} n^\mu u(p) ,
\]

where $\psi$ is the quark field of flavor $q$, $u$ the nucleon spinor, $u^a(p')$ is the Rapita-Schweringer spinor for the $\Delta$-field, $n^\mu$ is a light-cone vector along the negative $z$-direction, $\tau_3/2$ is the third isospin operator for quarks, and $\sqrt{2/3}$ is the isospin factor for the $p \rightarrow \Delta^+$ transition. Furthermore, the covariants $k_{\alpha\mu}^{M,E,C}$ are [13]:

\[
\begin{align*}
K_{\alpha\mu}^M &= -\frac{3(M_\Delta + M_N)}{2MNQ^2_+} \varepsilon_{\alpha\mu\lambda\sigma} P^\lambda q^\sigma , \\
K_{\alpha\mu}^E &= -K_{\alpha\mu}^M - \frac{6(M_\Delta + M_N)}{MNQ^2_+ Q^2_\perp} \varepsilon_{\alpha\sigma\lambda\rho} P^\lambda q^\sigma \varepsilon_{\mu\delta\rho} P^\delta q^\gamma_5 , \\
K_{\alpha\mu}^C &= -\frac{3(M_\Delta + M_N)}{MNQ^2_+ Q^2_\perp} q_\alpha (q_\mu P_\nu - q_\nu P_\mu) \gamma_5 .
\end{align*}
\]

The thus introduced GPDs $H_M$, $H_E$, and $H_C$ correspond with the three $N \rightarrow \Delta$ Jones–Scadron FFs and relate to them via the sum rules:

\[
\int dx H_{M,E,C}(x, \xi, Q^2) = 2 G_{M,E,C}^\gamma(Q^2) .
\]

The fourth $N \rightarrow \Delta$ vector GPD $H_4$ in Eq. (7) vanishes first moment.

For the magnetic dipole $N \rightarrow \Delta$ transition, it was shown that, in the large-$N_c$ limit, the relevant $N \rightarrow \Delta$ GPD $H_M$ can be expressed in terms of the nucleon isovector GPD $E^u - E^d$ as [13]:

\[
H_M = 2 \frac{G_M^\gamma(0)}{\kappa_V} \left\{ E^u - E^d \right\} ,
\]

where $\kappa_V = \kappa_p - \kappa_n = 3.70$ is the nucleon isovector anomalous magnetic moment. The large-$N_c$ limit value $G_M^\gamma(0) = \kappa_V/\sqrt{2}$ is about 20% smaller than the experimental number, and therefore we will use the phenomenological value $G_M^\gamma(0) \approx 3.02$ [10] in the calculations below.

Using the large-$N_c$ estimate of Eq. (10), the sum rule Eq. (9) for $G_M^\gamma$ can be written as:

\[
G_M^\gamma(Q^2) = \frac{G_M^\gamma(0)}{\kappa_V} \int_{-1}^{+1} dx \left\{ E^u - E^d \right\}(x, \xi, Q^2) ,
\]

\[
= \frac{G_M^\gamma(0)}{\kappa_V} \left\{ \frac{F_2^u(Q^2) - F_2^d(Q^2)}{\kappa_V^2} \right\} ,
\]

where $F_2^u - F_2^d$ is the (isovector) combination of the proton ($p$) - neutron ($n$) Pauli FFs. Because the sum
rule (11) is independent of \( \xi \), we only need to constrain the GPD \( E^\pi \) for \( \xi = 0 \) in order to evaluate \( G_M^* \). In Ref. [6], a 3-parameter modified Regge parametrization for the nucleon GPDs was found to provide a good quantitative description for all four nucleon elastic form factors over the whole \( Q^2 \) range. This is illustrated in Fig. 2 for the ratio of the proton Pauli over Dirac FFs. Using the sum rule prediction based on the large-\( N_c \) estimate of Eq. (11), the \( N \rightarrow \Delta \) magnetic FF \( G_M^* \) follows as a prediction. It is seen from Fig. 2 that the large-\( N_c \) relation reproduces well the experimentally observed, faster than dipole, fall-off of \( G_M^* \). One sees that this is also in agreement with the corresponding fall-off of the nucleon isovector Pauli form factor, confirming the finding of Ref. [17] using a Gaussian model for the GPDs.

We shall now relate the \( N \rightarrow \Delta \) GPDs for the electric quadrupole (\( H_E \)) and Coulomb quadrupole (\( H_C \)) transitions with the nucleon GPDs. We start from a large-\( N_c \) relation between the \( N \rightarrow \Delta \) quadrupole moment \( Q_{p \rightarrow \Delta^+} \) and the neutron charge radius \( r_n^2 \) [18]:

\[
Q_{p \rightarrow \Delta^+} = \frac{1}{\sqrt{2}} r_n^2. \tag{12}
\]

Using the Jones-Scadron \( N \rightarrow \Delta \) form factors, we can express Eq. (12) as a relation for \( G_E^* \), which reads (to leading order in the \( 1/N_c \)-expansion) as:

\[
G_E^*(0) = -\frac{M_\Delta^2 - M_N^2}{12\sqrt{2}} r_n^2. \tag{13}
\]

Note that the experimental value of the \( R_{EM} \) ratio at the real photon point (\( R_{EM} = -2.5 \pm 0.5\% \) [19]) yields:

\[
Q_{p \rightarrow \Delta^+} = -(0.085 \pm 0.003) \text{ fm}^2. \tag{14}
\]

Using the experimental value of the neutron charge radius \( r_n^2 \), the large-\( N_c \) relation of Eq. (12) yields:

\[
Q_{p \rightarrow \Delta^+} = -0.08 \text{ fm}^2, \tag{15}
\]

in close agreement with experiment.

For small values of \( Q^2 \), the neutron electric FF is expressed as \( G_E^*(Q^2) \approx -r_n^2 Q^2/6 \). Therefore, an extension of the large-\( N_c \) relation (13) to finite \( Q^2 \) is given by:

\[
G_E^*(Q^2) \approx \frac{1}{\sqrt{2}} \frac{M_\Delta^2 - M_N^2}{2Q^2} G_E^*(Q^2). \tag{16}
\]

An analogous prediction for the Coulomb quadrupole \( N \rightarrow \Delta \) GPD, \( H_C \), can be made by using the newly derived relation Eq. (5). Extending this relation to finite \( Q^2 \), we have

\[
G_C^*(Q^2) \approx \frac{4M_\Delta^2}{M_\Delta^2 - M_N^2} G_C^*(Q^2). \tag{17}
\]

Using Eq. (9), we can turn this into a relation between the \( R_{EM} \) and \( R_{SM} \) ratios as:

\[
R_{SM}(Q^2) \approx \frac{Q^4 - Q^2 - R_{EM}(Q^2)}{M_\Delta^2 - M_N^2}. \tag{18}
\]

At \( Q^2 = 0 \), one recovers the relation \( R_{SM} = R_{EM} \).

The FFs \( G_E^* \) and \( G_C^* \) are obtained from the first moment of the \( N \rightarrow \Delta \) GPDs \( H_E \) and \( H_C \) through the sum rules of Eq. (5). We can therefore use Eqs. (11) and (15) to obtain relations between the \( N \rightarrow \Delta \) GPDs \( H_E \) and \( H_C \) and the neutron electric GPD combination as:

\[
H_E = \frac{1}{\sqrt{2}} \frac{M_\Delta^2 - M_N^2}{Q^2} \left\{ H^{(n)} - \frac{Q^2}{4M_N^2} E^{(n)} \right\},
\]

\[
H_C = \frac{4M_\Delta^2}{M_\Delta^2 - M_N^2} H_E, \tag{17}
\]

where the (neutron) GPDs \( H^{(n)} \) and \( E^{(n)} \) are defined in terms of the u- and d-quark flavor GPDs as:

\[
H^{(n)} = -1/3H^u + 2/3H^d, \quad E^{(n)} \equiv -1/3E^u + 2/3E^d.
\]

The prediction which follows from the large-\( N_c \) motivated expression of Eq. (14) is tested in Fig. 3 by comparing the \( Q^2 \) dependence of the neutron electric FF \( G_E^* \) and the \( N \rightarrow \Delta \) \( R_{EM} \) and \( R_{SM} \) ratios. Although the above relations are derived assuming small \( Q^2 \), we explore their empirical validity at moderate \( Q^2 \) as well.
a surprisingly (for a large-$N_c$) width with the experimental data. The prediction of the relatively flat predictions for Cleon GPDs, we thus have obtained parameter-free pre-

tation for the nucleon GPDs, these relations were found to yield a surprisingly good agreement with experiment, providing an explanation why $R_{EM}$ remains nearly constant for $Q^2$ values up to several GeV$^2$.

A worthwhile topic for future work is to perform model calculations for the $N \rightarrow \Delta$ GPDs, as well as provide lattice QCD estimates for its moments, in order to cross-check the above relations.

This work is supported in part by DOE grant no. DE-FG02-04ER41302 and contract DE-AC05-06OR23177 under which Jefferson Science Associates operates the Jefferson Laboratory.

![Neutron electric FF $G_E^p$](image)

**FIG. 3:** (Color online) Neutron electric FF $G_E^p$ (upper panel) in comparison with the $N \rightarrow \Delta R_{EM}$ (middle panel) and $R_{SM}$ (lower panel) ratios. The curves result from the "modified Regge" GPD parametrization [9], where in computing $R_{EM}$ and $R_{SM}$, the large-$N_c$ relations [14] for $G_E^p$, [15] for $G_C$, and [11] for $G_M$ are used. Data points for $G_E^p$ are from MAMI (red circles), Nikhef (blue square), and JLab (black triangles), see [1] for references. Data points for $R_{EM}$ and $R_{SM}$ are from Bates [25] (blue squares), MAMI [22, 23, 24] (red circles), JLab/CLAS [20] (black triangles), and JLab/HallA [27] (blue stars).

description of the available double polarization data. The $R_{EM}$ and $R_{SM}$ ratios are then computed using the large-$N_c$ relations [14] for $G_E^p$, [15] for $G_C$, and [11] for $G_M$.

By using the three parameter Regge form for the nuclear GPDs, we thus have obtained parameter-free predictions for $R_{EM}$ and $R_{SM}$. One sees that this yields a $R_{EM}$ ratio which has both the right size and displays a relatively flat $Q^2$ behavior, up to $Q^2$ of about 2 GeV$^2$. The prediction of the $Q^2$ dependence for both ratios is in a surprisingly (for a large-$N_c$ estimate) good agreement with the experimental data.

Summarizing, we derived new large-$N_c$ relations expressing the $N \rightarrow \Delta$ GPDs in terms of the nucleon GPDs. In particular, the $R_{EM}$ and $R_{SM}$ ratios were related to the neutron electric form factor. Using a parameterization for the nucleon GPDs, these relations were found to provide an explanation why $R_{EM}$ remains nearly constant for $Q^2$ values up to several GeV$^2$.