New Additive Results for Cauchy Dual and MP–Inverse of Weighted Composition Operators

Morteza Sohrabi

Abstract. In this paper, we prove some basic results for Cauchy dual of weighted composition operators. Also we introduce some new classes of operators, called $\dagger$-hyponormal, $\dagger$-quasi-hyponormal, and we provide necessary and sufficient conditions for Cauchy dual and MP–inverse of weighted composition operators on $L^2(\Sigma)$ to belong to these classes. In addition, we study the complex symmetry of these types of operators. Moreover, some examples are provided to illustrate the obtained results.

1. Introduction and Preliminaries

Let $(X, \Sigma, \mu)$ be a sigma finite measure space and let $\varphi : X \to X$ be a measurable transformation such that $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to $\mu$. It is assumed that the Radon-Nikodym derivative $h = d\mu \circ \varphi^{-1} / d\mu$ is finite-valued or equivalently $(X, \varphi^{-1}(\Sigma), \mu)$ is sigma finite. We use the notation $L^2(\varphi^{-1}(\Sigma))$ for $L^2(X, \varphi^{-1}(\Sigma), \mu |_{\varphi^{-1}(\Sigma)})$ and henceforth we write $\mu$ in place of $\mu |_{\varphi^{-1}(\Sigma)}$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. We denote that the linear space of all complex-valued $\Sigma$-measurable functions on $X$ by $L^0(\Sigma)$. The support of $f \in L^0(\Sigma)$ is defined by $\sigma(f) = \{ x \in X : f(x) \neq 0 \}$. For a finite valued function $u \in L^0(\Sigma)$, the weighted composition operator $T_{u,\varphi}$ on $L^2(\Sigma)$ induced by $\varphi$ and $u$ is given by $T_{u,\varphi}f = M_u \circ C_{\varphi}f$, where $M_u$ is a multiplication operator and $C_{\varphi}$ is a composition operator on $L^2(\Sigma)$ defined by $M_u f = uf$ and $C_{\varphi}f = f \circ \varphi$, respectively.

The associated $\varphi^{-1}(\Sigma) \subseteq \Sigma$, there exists an operator $E := E_{\varphi^{-1}(\Sigma)} : L^p(\Sigma) \to L^p(\mathcal{A})$, which is called conditional expectation operator. $D(E)$, the domain of $E$, contains the set of all non-negative measurable functions and each $f \in L^p(\Sigma)$ with $1 \leq p \leq \infty$, which satisfies

$$\int f d\mu = \int_A E(f) d\mu, \quad A \in \varphi^{-1}(\Sigma).$$

Recall that $E : L^2(\Sigma) \to L^2(\mathcal{A})$ is a surjective, positive and contractive orthogonal projection. For more details on the properties of $E$ see [19, 25, 27]. Since by the change of variable formula,

$$\int_X f \circ \varphi d\mu = \int_X h f d\mu, \quad f \in L^1(\Sigma),$$

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Email address: sohrabi.mo@lu.ac.ir (Morteza Sohrabi)
Definition 1.1. Let \( m \) \( T \) then space \( p \) \([22]\), the authors generalized the work done in \([4]\) and have obtained some characterizations of related classes. Hence, it is better that we consider the weighted case of composition operators. In \([10]\) and \([8]\), Lambert \([20, 25]\), Singh \([29]\), Campbell \([6–8]\) and Stochel \([13]\). In \([4]\) and \([5]\) some weak hyponormal weighted composition operators as separately. In \([22]\) some examples were presented to illustrate that weighted composition operators lie between those classes. This note is a continuation of the work done in \([22]\).

Composition operators as an extension of shift operators are a good tool for separating weak hyponormal classes. Classic seminormal (weighted) composition operators have been extensively studied by Harrington and Whitley \([18]\), Lambert \([20, 25]\), Singh \([29]\), Campbell \([6–8]\) and Stochel \([13]\). In \([4]\) and \([5]\) some weak hyponormal classes of composition operators are studied. In those work, examples were given which show that composition operators can be used to separate each partial normality class from quasinormal through \( w \)-hyponormal. But in some cases composition operators can not be separated some of these classes. Hence, it is better that we consider the weighted case of composition operators. In \([10]\) and \([22]\), the authors generalized the work done in \([4]\) and have obtained some characterizations of related \( p \)-hyponormal weighted composition operators as separately. In \([22]\) some examples were presented to illustrate that weighted composition operators lie between those classes. This note is a continuation of the work done in \([22]\).

Let \( B(H) \) be the algebra of all bounded linear operators on the infinite dimensional complex Hilbert space \( H \). Let \( T \) be the polar decomposition for \( T \in B(H) \), where \( U \) is a partial isometry and \(|T| = \sqrt{T^*T} \). Associated with \( T \in B(H) \) there is a useful related operator \( \tilde{T} = |T|^{1/2}U|T|^{1/2} \), called the Aluthge transform of \( T \).

**Definition 1.1.** Let \( m \in \mathbb{N}, n \in \mathbb{N} \cup \{0\} \) and \( p > 0 \). We denote by \( \mathcal{K}(p, m, n) \) the set of all operators such that on \( \mathcal{H} \) that \( T^n(T^mT^m)^pT^n \geq T^n(T^mT^m)^{p+1}T^n \).

Note that \( \mathcal{K}(p, m, n) \subseteq \mathcal{K}(p, m, n) \), for all \( n_1 \leq n_2 \). An operator \( T \) is \( p \)-paranormal if \( |||T|||\|U|||\|x\| \geq |||T|||\|x\|_p \) and \( T \) is absolute \( p \)-paranormal if \( |||T|||\|x\| \geq |||T|||\|x\|_p \) for all unit vectors \( x \in \mathcal{H} \).

Let \( B_C(H) \) be the set of all bounded linear operators on \( H \) with closed range. For \( T \in B_C(H) \), the Moore-Penrose inverse of \( T \), denoted by \( T^* \), is the unique operator \( T^* \) that satisfies following:

\[
T^* = T, \quad T^*T^* = T^*, \quad (T^*)^* = T^*, \quad (T^*)^* = T^*. 
\]

We recall that \( T^* \) exists if and only if \( T \in B_C(H) \). Note that if \( T \in B_C(H) \), then \( T^* \), \( |T| \) and \( T^* \) have closed range. The Moore-Penrose inverse is designed as a measure for the invertibility of an operator. If \( T = U|T| \) is invertible, then \( T^{-1} = T^\dagger \), \( U \) is unitary and so \(|T| = \sqrt{T^*T} \) is invertible. It is a classical fact that the polar decomposition of \( T^\dagger \) is \( U^\dagger|T| \). It is easy to check that \( U^\dagger|T| \) and \( |T|^\dagger U^\dagger|T| \) are the polar decomposition and Aluthge transform of \( T^\dagger \), respectively. For other important properties of \( T^\dagger \) see \([1, 12]\).

Put \( S_{u, \varphi} = M_{\varphi(u)}T_{u, \varphi} \). Then \( S_{u, \varphi} \in B_C(L_2(\Sigma)) \), because \( \frac{\varphi(u)}{T} \) bounded away from zero on \( X \). Since for each \( f \in L_2(\Sigma), T_{u, \varphi} f = hE(u f) \circ \varphi^{-1} \), then we have

\[
T_{u, \varphi} S_{u, \varphi} T_{u, \varphi} f = u(S_{u, \varphi} T_{u, \varphi} f) \circ \varphi = u(\frac{\varphi(u)}{T} hE(u^2) \circ \varphi^{-1} f) \circ \varphi = u(\frac{\varphi(u)}{T} hE(u^2) \circ \varphi^{-1} f) \circ \varphi = u(\frac{\varphi(u)}{T} \circ \varphi).
\]

Since \( u \geq 0 \) and \( \varphi(h \circ \varphi) = \varphi(h \circ \varphi E(u^2)) = \varphi(E(u^2)) \geq \varphi(u) \), and so

\[
T_{u, \varphi} S_{u, \varphi} T_{u, \varphi} f = (u(\frac{\varphi(u)}{T} \circ \varphi) \circ \varphi = (u(\frac{\varphi(u)}{T} \circ \varphi) = T_{u, \varphi} f.
\]
Theorem 1.2. Let $T$ and $S$ be left invertible operators. To be precise, if $T$ is left invertible, it easy to see that $T^*$ is a right inverse of $T$. The Cauchy dual of left invertible operators is introduced in [30] as a powerful tool in the model theory of left-invertible operators. We introduce now the notion of Cauchy dual for Moore–Penrose inverses.

Corollary 1.3. Let $C_p \in B_c(L^2(\Sigma))$. Then $C^+_p = M_{\omega(T)} C_p^{*}$.

The Cauchy dual of left invertible operators is introduced in [30] as a powerful tool in the model theory of left-invertible operators. To be precise, if $T$ is left invertible, it easy to see that $T^*T$ is invertible and the operator given by $L_T := (T^*T)^{-1}T^*$ is a canonical left inverse of $T$. The Cauchy dual of $T$ is then defined as

$$\omega(T) := T(T^*T)^{-1}L^*_T,$$

which is a right inverse of $T^*$. For more details on the properties of Cauchy dual see [9, 30, 32]. We introduce now the notion of Cauchy dual for Moore–Penrose inverse.

Definition 1.4. Let $T \in B_c(H)$. The Cauchy dual $T$ is is defined as

$$\omega(T) = T(T^*T)^{\dagger}.$$
Proposition 2.1. Let $T_{u,p} \in B_C(L^2(\Sigma))$, we have

(a) $\omega(T_{u,p}) = \frac{X_{\text{comp}}}{\text{comp}} T_{u,p} = (T_{u,p})^\dagger$.  
(b) $\omega(T_{u,p}) = \frac{X_{\text{comp}}}{\text{comp}} T_{u,p} = T^\dagger_{u,p} = (\omega(T_{u,p}))^\dagger$.  
(c) $\omega(T_{u,p})^* \omega(T_{u,p}) = \frac{X_{\text{comp}}}{\text{comp}}$, $\omega(T_{u,p}) \omega(T_{u,p})^* = \frac{X_{\text{comp}}}{\text{comp}} u E(u f)$.  
(d) $\omega(\omega(T_{u,p})) = T_{u,p}$.  
(e) $\omega(T^\dagger_{u,p}) = \omega(T_{u,p})^\dagger$.

Proof. (a) we know that $(TT^\dagger) = T^\dagger T^\dagger$. So, for each $f \in L^2(\Sigma)$, we have

$$\omega(T_{u,p}) f = T_{u,p} T^\dagger_{u,p} T^\dagger_{u,p} f = \frac{X_{\text{comp}}}{\text{comp}} f.$$ 

(b) We have 

$$\omega(T_{u,p})^* f = T^\dagger_{u,p} T^\dagger_{u,p} T^\dagger_{u,p} f = T^\dagger_{u,p} (\frac{X_{\text{comp}}}{\text{comp}} f).$$

(c) 

$$\omega(T_{u,p})^* \omega(T_{u,p}) f = \frac{X_{\text{comp}}}{\text{comp}} f$$

and 

$$\omega(T_{u,p}) \omega(T_{u,p})^* f = \frac{X_{\text{comp}}}{\text{comp}} f.$$ 

(d) $\omega(\omega(T_{u,p})) = (T_{u,p})^\dagger = T_{u,p}$.  
(e) By direct computations, we get that  

$$\omega(T^\dagger_{u,p}) f = T^\dagger_{u,p} T_{u,p} T^\dagger_{u,p} f = h E(u f) \circ q^{-1} = T^\dagger_{u,p} f = (\omega(T_{u,p}))^\dagger f.$$

□

Proposition 2.2. Let $T_{u,p} \in B_C(L^2(\Sigma))$. Then $\omega(T_{u,p}) = V[\omega(T_{u,p})]$ is the polar decomposition of $\omega(T_{u,p})$, such that

$$|\omega(T_{u,p})| f = \frac{1}{\sqrt{V}} f;$$  

$$V(f) = \frac{X_{\text{comp}}}{\text{comp}} T_{u,p} f.$$ 

We know that $T^\dagger_{u,p} f = \frac{1}{\sqrt{\frac{X_{\text{comp}}}{\text{comp}}}} u f \circ q$. Now turn to the computation of $\omega(T^\dagger_{u,p})$ and $\omega(T^\dagger_{u,p})$. By combining the previous results we obtain the following proposition.
Proposition 2.3. Let $T_{u, \varphi} \in B_C(L^2(\Sigma))$. Then

(a) $\omega(T_{u, \varphi}) = u \frac{X}{\sqrt{\int |\varphi|^2}} f \circ \varphi$.

(b) $\omega(T_{u, \varphi}) = u \frac{\overline{X} + \overline{\varphi}}{\overline{\int |\varphi|^2}} \sqrt{\frac{1}{\int |\varphi|^2}} f \circ \varphi$.

Corollary 2.4. Let $T_{u, \varphi} \in B_C(L^2(\Sigma))$. Then $\omega(T_{u, \varphi}) = \omega(T_{u, \varphi})$ if and only if $E(u^2 \sqrt{f}) = \sqrt{E(u^2)}$.

We recall that a conjugation on a Hilbert space $\mathcal{H}$ is an anti-linear operator $S : \mathcal{H} \to \mathcal{H}$ which is conjugate linear, involutive and isometric. By involutive and isometric we mean that $S^2 = I$ and $\langle Sx, Sy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$, respectively. An operator $T \in \mathcal{H}$ is said to be complex symmetric if there exists a conjugation $S$ on $\mathcal{H}$ such that $T = ST^2S$. The class of complex symmetric operators includes all normal and binormal operators, Hankel operators, truncated Toeplitz operators, and Volterra integration operators, see [14–16]. The problem of describing all complex symmetric weighted composition operators on various analytic function spaces is very active recently (see [2, 17, 33]). In the following, we show that the Cauchy dual of a weighted composition operator is complex symmetric.

Proposition 2.5. Let $\sigma(f) = \sigma(h) = X, \varphi^2 = I$, the identify transformation. If $h(h \circ \varphi) = 1, u \in L^2(\mathcal{A})$, where $\mathcal{A} = \varphi^{-1}(\Sigma)$. Then $\omega(T_{u, \varphi})$ is complex symmetric.

Proof. Define $S(f) = \frac{\sigma}{\mu \varphi}$. Then $S$ is conjugate linear, $S^2 = I$ and for each $f \in L^2(\Sigma)$, we have

$$S\omega(T_{u, \varphi})S(f) = S(T_{u, \varphi})S(f) = S(\frac{X}{\mu}hE(uS(f)) \circ \varphi^{-1}) = \frac{u}{\mu} \frac{h \circ \varphi}{f \circ \varphi} uS(f) = \frac{uf \circ \varphi}{f \circ \varphi} = \omega(T_{u, \varphi})f.$$  

Also

$$\langle Sf, Sg \rangle = \int_X \frac{(f \circ \varphi)(g \circ \varphi)}{h \circ \varphi} d\mu = \int_X \frac{hf \circ g}{h} d\mu = \langle g, f \rangle.$$  

So, $\omega(T_{u, \varphi})$ is complex symmetric. □

Example 2.6. Suppose that $1 < a < \infty$. Let $X = [\frac{1}{a}, a], d\mu = dx$ and $\Sigma$ be the Lebesgue sets. Define the non-singular transformation $\varphi : X \to X$ by $\varphi(x) = \frac{1}{x}$. Put $u(x) = x^2$ and $\mathcal{A} = \varphi^{-1}(\Sigma)$. Then $h(x) = \frac{1}{x^2}$ and $E = I$. Simple computations show that $\sigma(f) = \sigma(h) = X, \varphi^2 = I$ and $h(h \circ \varphi) = 1$. Define $S(f) = \frac{\sigma}{\mu \varphi}$. It is clear that $S$ is conjugate linear. By direct computation we get that $\omega(T_{u, \varphi}) = \frac{1}{\pi} f(\frac{1}{x})$ and $S\omega(T_{u, \varphi})S(f) = \omega(T_{u, \varphi})f$.

Thus $\omega(T_{u, \varphi})$ is complex symmetric on $L^2(\Sigma)$.

In the following two theorems, to avoid tedious calculations, we investigate only $K(p, 1, n)$ and $K(p, m, 0)$ classes of weighted composition operators. Note that $T \in K(p, 1, 0)$ if and only if $T$ is $p$-hyponormal and $T \in K(p, 1, 1)$ if and only if $T$ is $p$-quasihyponormal. Recall that $T \in K(p, 1, n)$ if and only if $T^n(T^*)^p T^n \geq T^* T^p T^n$, and $T \in K(p, m, 0)$ if and only if $(T^m)^* T^p \geq (T^m)^* T^m$.

Theorem 2.7. Let $T_{u, \varphi} \in B_C(L^2(\Sigma))$, $n \in \mathbb{N}$. Then $\omega(T_{u, \varphi}) \in K(p, 1, n)$ if and only if

$$E_n(u^2 \frac{p}{P}) \geq E_n(u_n(h^p \circ \varphi)(E(u^2))^p - E(u_n^p)),$$

on $\sigma(f_n)$.
Proof. Let \( f \in L^2(\Sigma) \). Direct computations show that
\[
(\omega(T_{u,p})^*)^n(\omega(T_{u,p})^*)^p(\omega(T_{u,p})^*)^f(f)
\]
\[
= \frac{1}{n!} \int_{\frac{1}{n}} h_n E_n \left( \frac{\mu_n}{\mu} (\mu \sigma f \circ \varphi^n) \right) \circ \varphi^{-k} f
\]
\[
= \frac{1}{n!} \int_{\frac{1}{n}} h_n E_n \left( \frac{\mu_n^2}{\mu} \right) \circ \varphi^{-n} f
\]
and
\[
(\omega(T_{u,p})^*)^n(\omega(T_{u,p})^*)^p(\omega(T_{u,p})^*)^f(f)
\]
\[
= \frac{1}{n!} \int_{\frac{1}{n}} h_n E_n \left( \frac{\mu_n}{\mu} E(u)^p \sigma f \circ \varphi^n \right) \circ \varphi^{-n} f
\]
\[
= \frac{1}{n!} \int_{\frac{1}{n}} h_n E_n \left( \frac{\mu_n^2}{\mu} \sigma f \circ \varphi^n \right) \circ \varphi^{-n} f.
\]
Then \( \omega(T_{u,p}) \in \mathcal{K}(p, 1, n) \) if and only if
\[ E_n \left( \frac{\mu_n^2}{\mu} \right) \geq E_n \left( \frac{\mu_n}{\mu} E(u)^p \sigma f \circ \varphi^n \right) \circ \varphi^{-n} f, \quad \text{on } \sigma(J_n). \]

\[ \square \]

Corollary 2.8. Let \( C_\psi \in B_C(L^2(\Sigma)) \). Then \( \omega(C_\psi) \in \mathcal{K}(p, 1, n) \) if and only if \( E_n \left( \frac{1}{p} \right) \geq E_n \left( \frac{1}{p} \right) \) on \( \sigma(h_n) \). In particular, if \( n = 1 \), then \( \omega(C_\psi) \) is \( p \)-quasihyponormal if and only if \( E \left( \frac{1}{p} \right) \geq \frac{1}{p} \) on \( \sigma(h) \).

Lemma 2.9. [26] Let \( \alpha \) and \( \beta \) be nonnegative and measurable functions. Then for every \( f \in L^2(\Sigma) \),
\[
\int_X |f|^2 d\mu \geq \int_X |E_n(\beta f)|^2 d\mu
\]
if and only if \( \sigma(\beta) \subseteq \sigma(\alpha) \) and \( E_n \left( \frac{\mu_n}{\mu} X_{\sigma(\alpha)} \right) \leq 1 \).

Theorem 2.10. Let \( T_{u,p} \in B_C(L^2(\Sigma)) \). Then \( \omega(T_{u,p}) \in \mathcal{K}(p, m, 0) \) if and only if \( \sigma(u_m) \subseteq \sigma(J_m) \) and
\[
E_m \left( u_m^2 \right) \leq \left( \frac{1}{p} \right) \sigma(u_m^2).
\]

Proof. Let \( f \in L^2(\Sigma) \). Then we have
\[
\langle (\omega(T_{u,p})^*)^m \omega(T_{u,p})^m \rangle \sigma f, f \rangle = \int_X \frac{\mu_n(x)}{\mu} |f|^2 d\mu,
\]
and
\[
\langle (\omega(T_{u,p})^m \omega(T_{u,p})^m \rangle \sigma f, f \rangle
\]
conclusion follows from Lemma 2.9.

\[ \lambda > 0. \]

The following hold:

**Lemma 2.12.** \[31\] Let \( T \) be its polar decomposition. Suppose \( p \) be a positive real numbers. Then

\[ \alpha \in \mathbb{T}, \quad \beta \in \mathbb{T}, \quad \sigma(\alpha) \in \sigma(f), \quad \sigma(\beta) \in \sigma(E(u)). \]

Now, the desired conclusion follows from Lemma 2.9.

**Corollary 2.11.** \( \alpha(T_{u,p}) \) is \( p \)-hyponormal if and only if \( \sigma(u) \subseteq \sigma(f) \) and

\[ E \left( \frac{1}{f \circ q} \frac{u^2}{\sigma(E(u^2))} \right) \leq 1. \]

**Lemma 2.12.** [31] Let \( T \in B(H) \) and let \( U[T] \) be its polar decomposition. Suppose \( p \) be a positive real numbers. Then the following hold:

(a) \( T \) is \( p \)-paranormal if and only if for each \( \lambda > 0 \),

\[ |T^p| |T^p| - 2\lambda |T^p|^2 + \lambda^2 \geq 0. \]

(b) \( T \) is absolute-\( p \)-paranormal if and only if for each \( \lambda > 0 \),

\[ |T||T^p|^2 |T^p| - (p + 1)\lambda |T^p|^2 + p\lambda^{p+1} \geq 0. \]

**Theorem 2.13.** Let \( T_{u,p} \in B_C(L^2(\Sigma)) \). Then The following statements are hold.

(a) \( \alpha(T_{u,p}) \) is \( p \)-paranormal if and only if

\[ E \left( \frac{1}{f \circ q} \frac{u^2}{\sigma(E(u^2))} \right) \geq \frac{\chi_{\sigma(f)}}{h^p \circ q(E(u^2))^{p+1}} (E(u))^3, \quad \text{on } \sigma(I) \cap \sigma(E(u)). \]

(b) \( \alpha(T_{u,p}) \) is absolute-\( p \)-paranormal if and only if

\[ E \left( \frac{1}{f \circ q} \frac{u^2}{\sigma(E(u^2))} \right) \geq \frac{\chi_{\sigma(f)}}{h^p \circ q(E(u^2))^{p+1}} (E(u))^{2p+1}, \quad \text{on } \sigma(I) \cap \sigma(E(u)). \]

**Proof.** (a) Let \( f \in L^2(\Sigma) \). It is easy to check that

\[ |\omega(T_{u,p})|^{2p} f = \frac{\chi_{\sigma(f)}}{(f \circ q)E(u^2)} uE uf, \]

\[ |\omega(T_{u,p})|^{p} f = \frac{\chi_{\sigma(f)}}{(f \circ q)E(u^2)} uE uf, \]

\[ |\omega(T_{u,p})|^2 f = \frac{1}{f^p} f. \]

It follows that

\[ |T^p| |T^p| - 2\lambda |T^p|^2 + \lambda^2 \geq 0. \]

Now, by Lemma 2.12(a), \( \alpha(T_{u,p}) \) is \( p \)-paranormal if and only if

\[ E \left( \frac{1}{f \circ q} \frac{u^2}{\sigma(E(u^2))} \right) - 2\lambda \frac{\chi_{\sigma(f)}}{(f \circ q)E(u^2)} uE uf + \lambda^2, f \geq 0, \quad (1) \]
for each $\lambda \in (0, \infty)$. Put $f = \chi_{[\mu^{-1}B]}$ with $\mu(B) < \infty$. Hence, (2.1) holds if and only if
\[
\int_{\mu^{-1}B} \left( \frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \right) u E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right] d\mu \geq 0.
\]
Equivalently,
\[
\int_{\mu^{-1}B} \left( \frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \right) u E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right] d\mu \geq 0.
\]
But, this is equivalent to
\[
\frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \left( E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right) \geq 0.
\]
Set
\[
\frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \left( E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right) := a
\]
and
\[
b := \frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \left( E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right).
\]
Then $\omega(T_{u,q})$ is $p$-paranormal if and only if
\[
D(\lambda) := a - 2b\lambda + \lambda^2 \geq 0, \quad \lambda \in (0, \infty).
\]
Since
\[
\min_{\lambda \in (0, \infty)} D(\lambda) = D(b),
\]
it follows that
\[
D(b) \geq 0 \iff a \geq b^2
\]
\[
\iff \frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \left( E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right) \geq \frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \left( E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right)^2
\]
\[
\iff \frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \left( E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right)^2 \geq \frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \left( E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right)
\]
\[
\iff E\left( \frac{\mu}{\rho} \right) \geq \frac{X_{t}(q) u_F}{(J \circ q) \rho E(u^2)} \left( E\left( \frac{\mu}{\rho} \right) - \frac{2 \lambda}{(J \circ q) \rho E(u^2)} + \lambda^2 \right)^3, \quad \text{on} \, \sigma(J) \cap \sigma(E(u)).
\]
(b) The proof is similar to part(a). \qed

3. MP– inverse of weighted composition operators

In this section, we define the new classes of operators, called $\dagger$-hyponormal and $\dagger$-quasi-hyponormal. In addition, we discuss measure theoretic characterizations for Moore-Penrose inverse of weighted composition operators in these new classes. We then give some examples illustrating these classes. From now on, we assume that $T_{u,q}$ has closed range. Before our main results are presented, we state the following lemma.

Lemma 3.1. [21] Let $0 \leq v \in L^0(\mathcal{A})$, $0 \leq \omega \in L^0(\Sigma)$ and let $A := M_{u,v} E M_{\omega} \in B(L^2(\Sigma))$. Then for each $p \in (0, \infty)$, $A^p = M_{\sigma(u)vE(u^p)\rho^{-1}E\sigma(u)}$. 
\textbf{Theorem 3.2.} Let $T_{u,p} \in B_{C}(L^{2}(\Sigma))$, \(n \in \mathbb{N}\). Then $T_{u,p}^{*} \in \mathcal{K}(p, 1, n)$ if and only if
\[
\frac{E_n(u)E_n(E(\frac{u\chi}{1+n}))}{E_n(E(u^2))} \geq \frac{E_n(f^p \circ \varphi)}{E_n(f^p)} \cdot \frac{K_n}{J_n},
\]
on $\sigma(h_n)$. 

Where $K_n = h_nE_n(u_n) \circ \varphi^n$.

\textbf{Proof.} Let $f \in L^{2}(\Sigma)$. Direct computations show that
\[
(T_{u,p}^{*} - T_{u,p}')^n(T_{u,p}^{*} - T_{u,p}')^\prime(T_{u,p}^{*} - T_{u,p}')^n f
\]
\[= \frac{\chi_{\sigma(J_n \circ \varphi)}u_n(u \circ \varphi^n)E(u \frac{\chi_{\sigma(J_n \circ \varphi)}}{1+n}h_nE_n(u_n f) \circ \varphi^n \circ \varphi^n}{(J_n \circ \varphi^n)(f \circ \varphi^n)^pE(u^2) \circ \varphi^n} := a f,
\]
\[
(T_{u,p}^{*} - T_{u,p}')^n(T_{u,p}^{*} - T_{u,p}')^\prime(T_{u,p}^{*} - T_{u,p}')^n f
\]
\[= \frac{\chi_{\sigma(J_n \circ \varphi)}u_n(h_n \circ \varphi^n)E_n(u_n f) := b f}.
\]

Then, $W_{\lambda} \in \mathcal{K}(p, 1, n)$ if and only if
\[
\langle (a - b)f, f \rangle \geq 0,
\]
for each $\lambda \in (0, \infty)$. Put $f = \chi_{\varphi^n B}$ with $\mu(\varphi^{-n} B) < \infty$. Hence, (3.1) holds if and only if
\[
\int_{\varphi^{-n} B} \chi_{\sigma(J_n \circ \varphi)}u_n(u \circ \varphi^n)E(u \frac{\chi_{\sigma(J_n \circ \varphi)}}{1+n}h_nE_n(u_n f) \circ \varphi^n \circ \varphi^n
\]
\[= \frac{(J_n \circ \varphi^n)(f \circ \varphi^n)^pE(u^2) \circ \varphi^n - \chi_{\sigma(J_n \circ \varphi)}u_n(h_n \circ \varphi^n)E_n(u_n f)}{(J_n \circ \varphi^n)^2(f \circ \varphi^n)^p} \geq 0.
\]

Equivalently,
\[
\int_{B} \chi_{\sigma(J_n)}E_n(u) \circ \varphi^{-n}E_n(u)E_n(E(\frac{u\chi}{1+n} K_n))
\]
\[= \frac{J_n \circ \varphi^n)(f \circ \varphi^n)^pE(u^2) \circ \varphi^n - \chi_{\sigma(J_n \circ \varphi)}h_nE_n(u_n) \circ \varphi^{-n}2h_n d\mu \geq 0.
\]

But, this is equivalent to
\[
\frac{\chi_{\sigma(J_n)}}{E_n(f^p \circ \varphi)}E_n(E(u^2)) \left[ E_n(u) \frac{\chi_{\sigma(J_n)}}{J_n} \right] - \frac{\chi_{\sigma(J_n)}}{E_n(f^p)} \frac{K_n}{J_n} \geq 0,
\]
on $\sigma(J_n)$. Then $T_{u,p}^{*} \in \mathcal{K}(p, 1, n)$ if and only if
\[
\frac{E_n(u)E_n(E(\frac{u\chi}{1+n}))}{E_n(E(u^2))} \geq \frac{E_n(f^p \circ \varphi)}{E_n(f^p)} \cdot \frac{K_n}{J_n},
\]
on $\sigma(h_n)$. 

\[\square\]
Corollary 3.3. Let \( C_{t^p} \in \mathcal{B}(L^2(\Sigma)) \). Then \( C_{t^p}^\dagger \in \mathcal{K}(p, 1, n) \) if and only if \( E_n(h^p) \geq E_n(h^p \circ t^p) \) on \( \sigma(h_n) \). In particular, if \( n = 1 \), then \( C_{t^p}^\dagger \) is \( p \)-quasihyponormal if and only if \( E(h^p) \geq h^p \circ t^p \), on \( \sigma(h) \).

Theorem 3.4. Let \( T_{u^p} \in \mathcal{B}(L^2(\Sigma)) \). Then \( T_{u^p}^\dagger \in \mathcal{K}(p, m, 0) \) if and only if

\[
\frac{(E_m(t^p u^p))^2}{E_m(u^p)} \geq \frac{p^m \circ q^m}{E_m(p^m)} \text{ on } \sigma(J_m).
\]

Proof. Let \( f \in L^2(\Sigma) \). By direct computations and Lemma 3.1, we get that

\[
\int \frac{\chi_{\sigma(J_m)}}{J_m \circ q^m} (h_m \circ q^m) (E_m(u^2))^p - 1 u_m E_m(u_m f) - \frac{\chi_{\sigma(J_m)}}{J_m} f, f \geq 0,
\]

for each \( \lambda \in (0, \infty) \). Put \( \chi_{\sigma(J_m)} \) with \( \mu(q^{-m} B) < \infty \). Hence, (3.2) holds if and only if

\[
\int B \frac{\chi_{\sigma(J_m)}}{J_m \circ q^m} (h_m \circ q^m) (E_m(u^2))^p - 1 u_m E_m(u_m f) - \frac{\chi_{\sigma(J_m)}}{J_m} f, f \geq 0.
\]

Equivalently,

\[
\int B \frac{\chi_{\sigma(J_m)}}{J_m \circ q^m} (E_m(u_m) \circ q^{-m})^2 - \frac{1}{E_m(p^m) \circ q^{-m}} h_m d \mu \geq 0.
\]

But, this is equivalent to

\[
\frac{\chi_{\sigma(J_m)}}{J_m \circ q^m} (E_m(u_m) \circ q^{-m})^2 - \frac{1}{E_m(p^m) \circ q^{-m}} h_m d \mu \geq 0,
\]

on \( \sigma(J_m) \). Then, \( T_{u^p}^\dagger \in \mathcal{K}(p, m, 0) \) if and only if

\[
\frac{(E_m(t^p u^p))^2}{E_m(u^p)} \geq \frac{p^m \circ q^m}{E_m(p^m)} \text{ on } \sigma(J_m).
\]

\( \square \)

Corollary 3.5. Let \( T_{u^p} \in \mathcal{B}(L^2(\Sigma)) \). Then \( T_{u^p}^\dagger \) is \( p \)-hyponormal if and only if

\[
\frac{(E(u))^2}{E(u^2)} \geq \frac{p \circ q}{E(p)} \text{ on } \sigma(f).
\]

Definition 3.6. Let \( T \in \mathcal{B}(\mathcal{H}) \). We said that \( T \) is \( t \)-hyponormal if \( T^\dagger T^\dagger \geq T^\dagger T^\dagger \) and \( T \) is \( t \)-quasi hyponormal if \( T^\dagger (T^\dagger T^\dagger) \geq (T^\dagger T^\dagger) T^\dagger \).

Theorem 3.7. Let \( T_{u^p} \in \mathcal{B}(L^2(\Sigma)) \). Then The following statements are hold.

(a) \( T_{u^p}^\dagger \) is \( t \)-hyponormal if and only if \( E(u^2) \leq 1 \).

(b) \( T_{u^p}^\dagger \) is \( t \)-quasi hyponormal if and only if

\[
E(u) \geq I \circ q E(u), \text{ on } \sigma(E(u)).
\]
Proposition 3.8. Let $T$ be a partial isometry in a partial isometry. Consequently, we have the following proposition.

Proof. (a) Let $f \in L^2(\Sigma)$. Direct computations show that

\[
T_{u,p}^+T_{u,p}f = \int_X \frac{X_0(f)\mu}{\mu} \|f\|^2 d\mu
\]

\[
T_{u,p}T_{u,p}^+f = \int_X \frac{X_0(f)}{\mu} \left(\frac{h \circ \varphi}{\mu}\right) E(\mu^2) E(u f) d\mu
\]

\[
= \int_X \frac{|E(X_0(f) \left(\frac{h \circ \varphi}{\mu}\right) E(\mu^2) \mu| f^2 d\mu.}
\]

Hence by Lemma 2.9, $T_{u,p}^+T_{u,p}$ is $\dag$-hyponormal if and only if

\[
E(\frac{(f \circ \varphi)E(\mu^2)}{\mu}) X_0(f) \leq 1.
\]

Equivalently, $E(\mu^2) \leq 1$.

(b) By simple calculations we get that,

\[
T_{u,p}^+(T_{u,p}T_{u,p})f = \frac{X_0(f)hE(uf) \circ \varphi^{-1}}{f}
\]

\[
(T_{u,p}T_{u,p}^+)T_{u,p}f = hE(uf) \circ \varphi^{-1}.
\]

So, by the same of proof of Theorem 3.7, $T_{u,p}^+$ is $\dag$-quasi hyponormal if and only if

\[
\frac{X_0(f)E(u f) \circ \varphi^{-1}}{f} \geq E(u) \circ \varphi^{-1} \iff \frac{X_0(f)E(u f)}{f \circ \varphi} \geq E(u)
\]

\[
\iff E(u f) \geq f \circ \varphi E(u), \quad \text{on } \sigma(\mu(u)).
\]

Let $B(\mathcal{H})$ be the algebra of all bounded linear operators on the infinite dimensional complex Hilbert space $\mathcal{H}$. Let $T = U[T]$ be the polar decomposition for $T \in B(\mathcal{H})$, where $U$ is a partial isometry and $[T] = (T^* T)^{1/2}$. In the following we concentrate on the polar decomposition of $T_{u,p}^+$.

Let $f \in L^2(\Sigma)$. Then $(T_{u,p}^+T_{u,p}^+)(f) = \frac{X_0(f)hE(uf)}{\mu} (h \circ \varphi) E(uf)$ and so $|T_{u,p}^+|$ follows from Lemma 3.1. Moreover, by direct computations we have

\[
U_{u,p}^+ = M_{u,p} T_{u,p}.
\]

Moreover, it is easy to check that $U_{u,p}^+|T_{u,p}^+| = T_{u,p}^+U_{u,p}^+U_{u,p}U_{u,p}^+= U_{u,p}^+$ and $N(U_{u,p}) = N(T_{u,p}^+) = N(T_{u,p}^+)$.

Consequently, we have the following proposition.

Proposition 3.8. Let $T_{u,p} \in B_C(L^2(\Sigma))$. Then $T_{u,p}^+ = U_{u,p}^+|T_{u,p}^+|$ is the polar decomposition of $T_{u,p}^+$ such that

\[
|T_{u,p}^+|(f) = \frac{X_0(f)}{\sqrt{f \circ \varphi E(\mu^2)}} \mu E(uf);
\]

\[
U_{u,p}^+(f) = (\frac{X_0(f)}{f})^2 T_{u,p} f.
\]
Theorem 3.9. Let $T_{u,\varphi} \in B_C(L^2(\Sigma))$. Then the following statements are hold.

(a) $T^+_{u,\varphi}$ is $p$-paranormal if and only if

$$\frac{\langle E(u) \rangle}{\langle E(u^2) \rangle} \geq \frac{u}{\sqrt[p]{p}} \cdot \frac{1}{\sqrt[p]{p}}, \quad \text{on } \sigma(f).$$

(b) $T^+_{u,\varphi}$ is absolute-$p$-paranormal if and only if

$$\sqrt[p]{p} \cdot \frac{\langle E(u) \rangle}{\langle E(u^2) \rangle} \geq \frac{u}{\sqrt[p]{p}} \cdot \frac{1}{\sqrt[p]{p}}, \quad \text{on } \sigma(f).$$

Proof. Let $f \in L^2(\Sigma)$. It is easy to check that

$$|T^+_{u,\varphi}|^2 f = \frac{X_{\alpha(\varphi)}}{(J \circ \varphi)^p E(u^2)} u E(u f),$$

$$|T^+_{u,\varphi}|^p f = \frac{X_{\alpha(\varphi)}}{(J \circ \varphi)^p E(u^2)} u E(u f),$$

$$|T^+_{u,\varphi}|^2 f = \frac{1}{\sqrt[p]{p} f}.$$

It follows that

$$|T^+_{u,\varphi}|^2 |T^+_{u,\varphi}|^p |T^+_{u,\varphi}|^p f = \frac{X_{\alpha(\varphi)}}{(J \circ \varphi)^p E(u^2)} u E(u f).$$

Now, by Lemma 2.12(a), $T^+_{u,\varphi}$ is $p$-paranormal if and only if

$$\langle \frac{X_{\alpha(\varphi)}}{(J \circ \varphi)^p E(u^2)} u E(u f) - 2 \lambda \frac{X_{\alpha(\varphi)}}{\sqrt[p]{p} f} + \lambda^2, f \rangle \geq 0,$$

for each $\lambda \in (0, \infty)$. Put $f = X_{\varphi^{-1}B}$ with $\mu(\varphi^{-1}B) < \infty$. Hence, (3.3) holds if and only if

$$\int_B \frac{X_{\alpha(\varphi)}}{(J \circ \varphi)^p E(u^2)} u E(u f) - 2 \lambda \frac{X_{\alpha(\varphi)}}{\sqrt[p]{p} f} (\chi_n \circ \varphi) + \lambda^2) d\mu \geq 0.$$

Equivalently,

$$\int_B \frac{X_{\alpha(\varphi)}}{(J \circ \varphi)^p E(u^2)} u E(u f) - 2 \lambda \frac{X_{\alpha(\varphi)}}{\sqrt[p]{p} f} (\chi_n \circ \varphi) + \lambda^2) d\mu \geq 0.$$

But, this is equivalent to

$$\frac{X_{\alpha(\varphi)}}{(J \circ \varphi)^p E(u^2)} u E(u f) - 2 \lambda \frac{X_{\alpha(\varphi)}}{\sqrt[p]{p} f} (\chi_n \circ \varphi) + \lambda^2 \geq 0.$$

Set

$$\frac{E(u) \circ \varphi^{-1}}{\sqrt[p]{p} \circ \varphi^{-1} E(u^2) \circ \varphi^{-1}} E(u) \circ \varphi^{-1} := a$$

and

$$b := \frac{1}{\sqrt[p]{p} \circ \varphi^{-1}}.$$
Then, \( T_{u,\phi} \) is \( p \)-paranormal if and only if
\[
D(\lambda) := a - 2b\lambda + \lambda^2 \geq 0, \quad \lambda \in (0, \infty).
\]
Since
\[
\min_{\lambda \in (0, \infty)} D(\lambda) = D(b),
\]
it follows that
\[
D(b) \geq 0 \iff a \geq b^2
\]
\[
\iff \frac{E(u) \circ \varphi^{-1}}{\sqrt{\| f \|_p \circ \varphi^{-1}} \| E(u^2) \circ \varphi^{-1} \|^{1/2}} E(u) \circ \varphi^{-1} \geq \left( \frac{1}{\| f \|_p \circ \varphi^{-1}} \right)^2
\]
\[
\iff \frac{E(u)}{\sqrt{\| f \|_p (\| f \|_p \circ \varphi) E(u^2)}} \frac{E(u)}{\sqrt{\| f \|_p}} \geq 1\
\]
\[
\iff \frac{E(u)}{E(u^2)} \frac{E(u)}{\sqrt{\| f \|_p}} \geq \left( \frac{1}{\| f \|_p} \right)^2, \quad \text{on } \sigma(f)
\]

(b) The proof is similar to part(a). \( \Box \)

**Corollary 3.10.** Let \( C_{\psi} \in B_C(L^2(\Sigma)) \). Then The following statements are hold.

(a) \( C_{\psi}^+ \) is \( p \)-paranormal if and only if
\[
E\left( \frac{1}{\sqrt{\| f \|_p}} \right) \geq \left( \frac{h \circ \varphi}{h \sqrt{h}} \right)^p, \quad \text{on } \sigma(h).
\]

(b) \( C_{\psi}^+ \) is absolute-\( p \)-paranormal if and only if
\[
\sqrt{h} E\left( \frac{1}{\sqrt{\| f \|_p}} \right) \geq \left( \frac{h \circ \varphi}{h \sqrt{h}} \right)^p, \quad \text{on } \sigma(h).
\]

In the following, we prove that the weighted composition operators are complex symmetric

**Corollary 3.11.** Let \( \sigma(h) = X, q^2 = 1 \), the identity transformation. If \( h(h \circ \varphi) = 1, u \in L^0(\mathcal{A}), \) where \( \mathcal{A} = \varphi^{-1}(\Sigma) \). Then \( T_{u,\varphi} \) is complex symmetric.

**Proof.** By the same argument in the proof of Proposition 2.5, we define
\[
S : L^2(\Sigma) \to L^2(\Sigma)
\]
\[
S(f) = \frac{u}{|\tilde{f} \circ \varphi|} \sqrt{h \circ \varphi}.
\]
According to assumptions it is clear that \( S \) is a conjugation and \( S(T_{u,\varphi}^*)S = T_{u,\varphi} \). \( \Box \)

**Example 3.12.** Let \( X = (0, 1) \) equipped with the Lebesgue measure \( d\mu = dx \) on the Lebesgue measurable subsets of \( X \) and let \( \varphi : X \to X \) a non-singular measurable transformation defined by
\[
\varphi(x) = \begin{cases} 
2x & 0 < x \leq \frac{1}{2}, \\
2 - 2x & \frac{1}{2} \leq x < 1.
\end{cases}
\]
Then for each $f \in L^2(\Sigma)$ and $x \in X$ we have

$$h(x) = \left| \frac{d}{dx} \left( \frac{x}{2} \right) \right| + \left| \frac{d}{dx} \left( \frac{2 - x}{2} \right) \right| = 1;$$

$$(E f)(x) = \frac{f(x) + f(1 - x)}{2};$$

$$(E f \circ \varphi^{-1})(x) = \frac{1}{2} (f(\frac{x}{2}) + f(1 - \frac{x}{2})).$$

Put $u(x) = \sqrt{x}$. Direct computation shows that

$$E(u) = \frac{\sqrt{x} + \sqrt{1-x}}{2},$$

$$E(u^2) = \frac{1}{2}, \quad E(u^2 \sqrt{f}) = \frac{1}{2} \sqrt{2}.$$}

$$J = \frac{1}{2}, \quad J \circ \varphi = \frac{1}{2}.$$

Then we have

$$\omega(T_{u,\varphi}) = \omega(T_{u,\varphi}) = \omega(T_{u,\varphi}) = \begin{cases} 2 \sqrt{x} f(2x) & 0 < x \leq \frac{1}{2}, \\ 2 \sqrt{2} f(2 - 2x) & \frac{1}{2} \leq x < 1. \end{cases}$$

In this case by Theorem 2.7 and Theorem 2.10, $\omega(T_{u,\varphi}) \in \mathcal{K}(p, 1, 1) \setminus \mathcal{K}(p, 1, 0)$. Also by Theorem 3.2, $T_{u,\varphi}^\dagger \in \mathcal{K}(p, 1, 0) \iff x = \frac{1}{2}$.

**Example 3.13.** Let $X = (1, \infty)$ equipped with the Lebesgue measure $d\mu$ on the Lebesgue measurable subsets. The transformation $\varphi$ and the weighted function $u(x)$ are given by $\varphi(x) = \sqrt{x}$ and $u(x) = \frac{1}{\sqrt{1-x}}$. Then $h(x) = 2x$, $E = 1$,

$$(E f)(x) = \frac{2}{1-x}, h \circ \varphi(x) = 2 \sqrt{x}, J \circ \varphi(x) = \frac{2}{1-x} \sqrt{x}. \quad \text{In this case by Theorems 2.7 and 2.10, } \omega(C_\varphi) \text{ is in } \mathcal{K}(p, 1, 0) \setminus \mathcal{K}(p, 1, 1).$$

Also by Theorem 3.7, $T_{u,\varphi}^\dagger$ is quasi-hyponormal but it is not quasi-hyponormal.

Note that the following example was used in [11] to show that the $p$-hyponormal classes are distinct for $p$ with $0 < p < 1$. Now we will show that block matrix operators can separate weak $p$-hyponormal classes.

**Example 3.14.** Let $M := \{A_{ij}\}_{1 \leq i, j \leq 6}$ be a block matrix operator whose blocks are $6 \times 3$ matrices such that $A_{ij} = 0, i \neq j$ and $A = A_0 = A_1 = A_2 = ..., \text{ where } A_n = A_{nn}$ for each $n \in \mathbb{N}_0$ and

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \sqrt{x_1} & 0 \\ 0 & 0 & \sqrt{x_2} \end{pmatrix}.$$
Then by [11, Proposition 2.2], \( C_\varphi \) is unitarily equivalent to the block matrix \( M \) such that for every \( k \in \mathbb{N}_0 \)

\[
\sqrt{\frac{m_{k+4}}{m_{3k+1}}} = \sqrt{x_1} \quad \text{and} \quad \sqrt{\frac{m_{k+5}}{m_{3k+2}}} = \sqrt{x_2}, \quad k \in \mathbb{N}_0.
\]

Moreover, by the same argument in the proof of [11, Proposition 3.1] \( M \) is in \( \mathcal{K}(p, 1, 0) \cap \mathcal{K}(p, 1, 1) \) if and only if \( C_\varphi \in \mathcal{K}(p, 1, 0) \cap \mathcal{K}(p, 1, 1) \). But this equivalent to \( C_\varphi \in \mathcal{K}(p, 1, 0) \cap \mathcal{K}(p, 1, 1) \). By Theorem 3.2 and Theorem 3.4, \( C_\varphi \) is par normal if and only if \( E(h^\varphi) \geq h^\varphi \circ \varphi \), this condition is equivalent to

\[
\left( \frac{m(q^{-1}(\varphi(n))))}{m(\varphi(n))} \right)^p \leq \frac{1}{m(q^{-1}(\varphi(n)))} \sum_{j \in q^{-1}(\varphi(n))} m_j \left( \frac{m(q^{-1}(j))}{m_j} \right)^p.
\]

Now by the same argument in the proof of [11, Proposition 3.3], we deduce that (5) is equivalent to

\[
\left( \frac{x_1}{4} \right)^p + \left( \frac{x_2}{4} \right)^p \geq 2
\]

Let \( 0 < q < p \) and \( M \) be in \( \mathcal{K}(p, 1, 0) \cap \mathcal{K}(p, 1, 1) \). Then by using (6), we can find \( x_1 \) and \( x_2 \) such that \( M \) is not in \( \mathcal{K}(q, 1, 0) \cap \mathcal{K}(q, 1, 1) \). Namely for \( x_1 = 3.5 \) and \( x_2 = 4.25 \), by using (6) it is easy to see that \( M \) is in \( \mathcal{K}(9, 1, 0) \cap \mathcal{K}(9, 1, 1) \), but it is not in \( \mathcal{K}(8, 1, 0) \cap \mathcal{K}(8, 1, 1) \). Also by the same argument \( M \) is is p-paranormal if and only if \( \omega(C_\varphi) \) is p-paranormal and by Theorem (2.13)(a), \( \omega(C_\varphi) \) is p-paranormal if and only if \( E(h^\varphi) \geq \frac{1}{\varphi(p)} \). This is equivalent to

\[
\left( \frac{4}{x_1} \right)^p + \left( \frac{4}{x_2} \right)^p \geq 2,
\]

particular in the above relations if we take \( x_1 = 3.5 \) and \( x_2 = 4.25 \), then by using (7) it is easy to see that for \( p \geq 6 \), \( M \) is p-paranormal but for \( p \leq 5 \), it is not p-paranormal.

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References

[1] A. Ben-Israel and T. N. E. Greville, Generalized inverses: theory and applications, Second Ed. Springer, 2003.
[2] P. S. Bourdon and S. Waleed Noor, Complex symmetry of invertible composition operators, J. Math. Anal. Appl. 429 (2015), 105-110.
[3] N. L. Braha and I. Hoxha, \((p, r, q)\)-Paranormal and absolute-\((p, r)\)-paranormal operators, J. Math. Anal. Appl. 4 (2013), 14-22.
[4] C. Burnap and I. Jung, Composition operators with weak hyponormality, J. Math. Anal. Appl. 337 (2008), 686-694.
[5] C. Burnap, I. Jung and A. Lambert, Separating partial normality classes with composition operators, J. Operator Theory 53 (2005), 381-397.
[6] J. Campbell and J. Jamison, On some classes of weighted composition operators, Glasgow Math. J. 32 (1990), 87-94.
[7] J. Campbell, M. Embry-Wardrop and R. Fleming, Normal and quasinormal weighted composition operators, Glasgow Math. J. 33 (1991), 275-279.
[8] J. Campbell and W. Hormor, Localising and seminormal composition operators on \( L^2 \), Proc. Roy. Soc. Edinburgh Sect. A 124 (1994), 301-316.
[9] S. Chavan, On operators Cauchy dual to 2-hyperexpansive operators, Proc. Edinburgh Math. Soc., 50 (2007), 637-552.
[10] M. Cho and T. Yamazaki, Characterizations of p-hyponormal and weak hyponormal weighted composition operators. Acta Sci. Math. (Szeged) 76 (2010), 173-181.
[11] G. Exner, I. Jung and M. Lee, Block matrix operators and weak hyponormalities, Integral Equations Operator Theory 65 (2009), no. 3, 345-362.
[12] D. S. Djordjević and N. C. Dinčić, Reverse order law for the Moore-Penrose inverse, J. Math. Anal. Appl. 361 (2010) 252-261.
[13] A. Daniluk, and J. Stochel, Seminormal composition operators induced by affine transformations, Hokkaido Math. J. 26 (1997), 377-404.
[14] S. R. Garcia and M. Putinar, Complex symmetric operators and applications I, Trans. Amer. Math. Soc. 358 (2006), 1285-1315.
[15] S. R. Garcia and M. Putinar, Complex symmetric operators and applications II, Trans. Amer. Math. Soc. 359 (2007), 3913-3931.
[16] S.R. Garcia and W. Wogen, Some new classes of complex symmetric operators, Trans. Amer. Math. Soc. 362 (2010), 6065-6077.
[17] P. V. Hai and L. H. Khoi, Complex symmetry of weighted composition operators on the Fock space, J. Math. Anal. Appl. 433 (2016), 1757-1771.
[18] D. Harrington and R. Whitley, Seminormal composition operators, J. Operator theory 11 (1984), 125-135.
[19] J. Herron, Weighted conditional expectation operators, Oper. Matrices 5 (2011), 107-118.
[20] T. Hoover, A. Lambert and J. Queen, The Markov process determined by a weighted composition operator, Studia Math. (Poland) LXII (1982), 225-235.
[21] M. R. Jabbarzadeh and M. Sohrabi Chegeni, Moore-Penrose inverse of conditional type operators, Oper. Matrices 11 (2017), 289-299.
[22] M. R. Jabbarzadeh and M. R. Azimi, Some weak hyponormal classes of weighted composition operators, Bull. Korean. Math. Soc. 47 (2010), 793-803.
[23] S. Jung, Y. Kim, E. Ko, J. E. Lee, Complex symmetric weighted composition operators on $H^2(D)$, J. Funct. Anal. 267 (2014), 323-351.
[24] D. Jung, M. Y. Lee and S. H. Lee, On classes of operators related to paranormal operators, Sci. Math. Jpn. 53 (2001), 33-43.
[25] Alan Lambert, Localising sets for sigma-algebras and related point transformations, Proc. Roy. Soc. Edinburgh Sect. A 118 (1991), 111-118.
[26] A. Lambert, Hyponormal composition operators, Bull. London Math. Soc. 18 (1986), 395-400.
[27] M. M. Rao, Conditional measure and applications, Marcel Dekker, New York, 1993.
[28] D. Senthilkumar and P. Maheswari Naik, Weyl’s theorem for algebraically absolute-$(p, r)$-paranormal operators, Banach J. Math. Anal. 5 (2011), 29-37.
[29] R. K. Singh and J. S. Manhas, Composition operators on function spaces, North Holland Math. Studies 179, Amsterdam 1993.
[30] A. Olofsson, Wandering subspace theorems, Integral Equ. Operator Theory, 51 (2005), 395-409.
[31] T. Yamazaki and M. Yanagida, A further generalization of paranormal operators, Sci. Math 31 (2000), 23-31.
[32] S. Shimorin, Wold-type decompositions and wandering subspaces for operators close to isometries, J. Reine Angew. Math. 531 (2001), 147-189.
[33] M. Wang and X. Yao, Complex symmetry of weighted composition operators in several variables, Internat. J. Math. 27 (2016) 1650017.