C₀-HILBERT MODULES

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Abstract. We provide the definition and fundamental properties of algebraic elements with respect to an operator satisfying hypothesis (h). Furthermore, we analyze Hilbert modules using C₀-operators relative to a bounded finitely connected region Ω in the complex plane.

Introduction

The theory of contractions of class C₀ was developed by Sz.-Nagy-Foias [7], Moore-Nordgren [6], and Bercovici-Voiculescu [2,3], and J.A. Ball introduced the class of C₀-operators relative to a bounded finitely connected region Ω in the complex plane, whose boundary ∂Ω consists of a finite number of disjoint, analytic, simple closed curves. The theory of Hilbert modules over function algebras has been developed by Ronald G. Douglas and Vern I. Paulsen [4].

We analyze Hilbert modules using C₀-operators relative to Ω. Every operator T defined on a Hilbert space H satisfying hypothesis (h) is not a C₀-operator relative to Ω. Thus, we provide the definition of an algebraic element with respect to T.

If B is the set of algebraic elements with respect to T, and it is closed, then naturally we have a bounded operator T_B from the quotient space H/B to H/B. In section 2, we discuss the relationships between the algebraic elements with respect to T_B in H/B and the algebraic elements with respect to T in H.

In section 3, we define a module action on a Hilbert space H by using a C₀-operator T relative to Ω, and introduce a C₀-Hilbert module H_T. Naturally, this raises the following question:

If every element of H_T is algebraic with respect to T over A, then T is either a C₀-operator or not.

In this paper, we consider a case in which the rank of the C₀-Hilbert module H_T is finite, and we show that if a generating set

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\{h_1, \cdots, h_k\}(k < \infty)$ of a Hilbert module $H_T$ over $A$ is contained in $B$, then $T$ is a $C_0$-operator.

Furthermore, if $B$ is closed, then by using the Jordan model of a $C_0$-operator $T$ relative to $\Omega$, we show that there are locally maximal $C_0$-submodules $M_i(i = 0, 1, 2, \cdots)$ of $H_T$ such that $M_0 \subset M_1 \subset M_2 \subset \cdots$.

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1. Preliminaries and Notation

1.1. Hilbert Modules. Let $X$ be a compact, separable, metric space and let $C(X)$ denote the algebra of all continuous complex-valued functions on $X$. A function algebra on $X$ is a closed subalgebra of $C(X)$, which contains the constant functions and separates points of $X$.

**Definition 1.1.** Let $F$ be a function algebra, and let $H$ be a Hilbert space. We say that $H$ is a Hilbert module over $F$ if there is a separately continuous mapping $\phi : F \times H \to H$ in each variable satisfying:

(a) $\phi(1, h) = h$,
(b) $\phi(fg, h) = \phi(f, \phi(g, h))$,
(c) $\phi(f + g, h) = \phi(f, h) + \phi(g, h)$,
(d) $\phi(f, \alpha h + \beta k) = \alpha \phi(f, h) + \beta \phi(f, k)$,

for every $f, g$ in $F$, $h, k$ in $H$, and $\alpha, \beta$ in $\mathbb{C}$.

We will denote $\phi(f, h)$ by $f.h$. For $f$ in $F$, we let $T_f : H \to H$ denote the linear map $T_f(h) = f.h$. If $H$ is a Hilbert module over $F$, then by the continuity in the second variable we have that $T_f$ is bounded.

**Definition 1.2.** Let $H$ be a Hilbert module over $F$. Then the module bound of $H$, is

$$K_F(H) = \inf\{K : \|T_f\| \leq K\|f\| \text{ for all } f \text{ in } A\}.$$  
We call $H$ contractive if $K_F(H) \leq 1$.

If $H$ is a Hilbert module over $A$, then a set $\{h_\delta\}_{\delta \in \Gamma} \subset H$ is called a generating set for $H$ if finite linear sums of the form

$$\sum_i f_i h_\delta_i, f_i \in A, \delta_i \in \Gamma$$

are dense in $H$.

**Definition 1.3.** If $H$ is a Hilbert module over $A$, then rank$_A(H)$, the rank of $H$ over $A$, is the minimum cardinality of a generating set for $H$. 
In the last few decades, the theory of Hilbert modules over function algebras has been developed by Ronald G. Douglas and Vern I. Paulsen [4].

1.2. A Functional Calculus. Let $H$ be a Hilbert space. Recall that $H^\infty$ is the Banach space of all (complex-valued) bounded analytic functions on the open unit disk $D$ with supremum norm [7]. A contraction $T$ in $L(H)$ is said to be completely nonunitary if there is no invariant subspace $K$ for $T$ such that $T|K$ is a unitary operator.

B. Sz.-Nagy and C. Foias introduced an important functional calculus for completely non-unitary contractions.

**Proposition 1.4.** Let $T \in L(H)$ be a completely non-unitary contraction. Then there is a unique algebra representation $\Phi_T$ from $H^\infty$ into $L(H)$ such that:

(i) $\Phi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator;
(ii) $\Phi_T(g) = T$, if $g(z) = z$ for all $z \in D$;
(iii) $\Phi_T$ is continuous when $H^\infty$ and $L(H)$ are given the weak$^*$-topology.
(iv) $\Phi_T$ is contractive, i.e. $\|\Phi_T(u)\| \leq \|u\|$ for all $u \in H^\infty$.

We simply denote by $u(T)$ the operator $\Phi_T(u)$.

B. Sz.-Nagy and C. Foias [7] defined the class $C_0$ relative to the open unit disk $D$ consisting of completely non-unitary contractions $T$ on $H$ such that the kernel of $\Phi_T$ is not trivial. If $T \in L(H)$ is an operator of class $C_0$, then

$$\ker \Phi_T = \{u \in H^\infty : u(T) = 0\}$$

is a weak$^*$-closed ideal of $H^\infty$, and hence there is an inner function generating $\ker \Phi_T$. The minimal function $m_T$ of an operator of class $C_0$ is the generator of $\ker \Phi_T$. Also, $m_T$ is uniquely determined up to a constant scalar factor of absolute value one [2]. The theory of class $C_0$ relative to the open unit disk has been developed by B.Sz.-Nagy, C. Foias ([7]) and H. Bercovici ([2]).

1.3. Hardy spaces. We refer to [9] for basic facts about Hardy space, and recall here the basic definitions.

**Definition 1.5.** The space $H^2(\Omega)$ is defined to be the space of analytic functions $f$ on $\Omega$ such that the subharmonic function $|f|^2$ has a harmonic majorant on $\Omega$. For a fixed $z_0 \in \Omega$, there is a norm on $H^2(\Omega)$ defined by

$$\|f\| = \inf\{u(z_0)^{1/2} : u \text{ is a harmonic majorant of } |f|^2\}.$$
Let $m$ be harmonic measure for the point $z_0$, let $L^2(\partial\Omega)$ be the $L^2$-space of complex valued functions on the boundary of $\Omega$ defined with respect to $m$, and let $H^2(\partial\Omega)$ be the set of functions $f$ in $L^2(\partial\Omega)$ such that $\int_{\partial\Omega} f(z)g(z)dz = 0$ for every $g$ that is analytic in a neighborhood of the closure of $\Omega$. If $f$ is in $H^2(\Omega)$, then there is a function $f^*$ in $H^2(\partial\Omega)$ such that $f(z)$ approaches $f^*(\lambda_0)$ as $z$ approaches $\lambda_0$ nontangentially, for almost every $\lambda_0$ relative to $m$. The map $f \rightarrow f^*$ is an isometry from $H^2(\Omega)$ onto $H^2(\partial\Omega)$. In this way, $H^2(\Omega)$ can be viewed as a closed subspace of $L^2(\partial\Omega)$.

A function $f$ defined on $\Omega$ is in $H^\infty(\Omega)$ if it is holomorphic and bounded. $H^\infty(\Omega)$ is a closed subspace of $L^\infty(\Omega)$ and it is a Banach algebra if endowed with the supremum norm. Finally, the mapping $f \rightarrow f^*$ is an isometry of $H^\infty(\Omega)$ onto a weak*-closed subalgebra of $L^\infty(\partial\Omega)$.

1.4. $C_0$-operators relative to $\Omega$. We will present in this section the definition of $C_0$-operators relative to $\Omega$. Reference to this material is found in Zucchi [10].

Let $H$ be a Hilbert space and $K_1$ be a compact subset of the complex plane. If $T \in L(H)$ and $\sigma(T) \subseteq K_1$, for $r = p/q$ a rational function with poles off $K_1$, we can define an operator $r(T)$ by $g(T)^{-1}p(T)$.

**Definition 1.6.** If $T \in L(H)$ and $\sigma(T) \subseteq K_1$, we say that $K_1$ is a spectral set for the operator $T$ if $\|r(T)\| \leq \max\{|r(z)| : z \in K_1\}$, whenever $r$ is a rational function with poles off $K_1$.

If $T \in L(H)$ is an operator with $\overline{\Omega}$ as a spectral set and with no normal summand with spectrum in $\partial\Omega$, i.e., $T$ has no reducing subspace $M \subseteq H$ such that $T|M$ is normal and $\sigma(T|M) \subseteq \partial\Omega$, then we say that $T$ satisfies hypothesis $(h)$.

**Proposition 1.7.** ([20], Theorem 3.1.4) Let $T \in L(H)$ be an operator satisfying hypothesis $(h)$. Then there is a unique algebra representation $\Psi_T$ from $H^\infty(\Omega)$ into $L(H)$ such that:

(i) $\Psi_T(1) = I_H$, where $I_H \in L(H)$ is the identity operator;

(ii) $\Psi_T(g) = T$, where $g(z) = z$ for all $z \in \Omega$;

(iii) $\Psi_T$ is continuous when $H^\infty(\Omega)$ and $L(H)$ are given the weak*-topology.

(iv) $\Psi_T$ is contractive, i.e., $\|\Psi_T(f)\| \leq \|f\|$ for all $f \in H^\infty(\Omega)$.

From now on we will indicate $\Psi_T(f)$ by $f(T)$ for all $f \in H^\infty(\Omega)$.

**Definition 1.8.** An operator $T$ satisfying hypothesis $(h)$ is said to be of class $C_0$ relative to $\Omega$ if there exists $u \in H^\infty(\Omega) \setminus \{0\}$ such that $u(T) = 0$. 

\[\begin{align*}
\text{YUN-SU KIM.}
\end{align*}\]
2. Algebraic Elements with respect to an Operator satisfying hypothesis \((h)\)

Every operator \(T\) satisfying hypothesis \((h)\) is not a \(C_0\)-operator relative to \(\Omega\), and so we provide the following definition.

**Definition 2.1.** Let \(T \in L(H)\) be an operator satisfying hypothesis \((h)\). An element \(h\) of \(H\) is said to be algebraic with respect to \(T\) provided that \(\theta(T)h = 0\) for some \(\theta \in H^\infty(\Omega) \setminus \{0\}\).

If not, \(h\) is said to be transcendental with respect to \(T\).

If \(A\) is a closed subspace of \(H\) generated by \(\{a_i \in H : i = 1, 2, 3, \cdots\}\), then \(A\) will be denoted by \(\bigvee_{n=1}^{\infty} a_i\).

**Proposition 2.2.** Let \(T \in L(H)\) be an operator satisfying hypothesis \((h)\).

(a) If \(h \in H\) is algebraic with respect to \(T\), then so is any element in \(\bigvee_{n=0}^{\infty} T^n h\).

(b) If \(h \in H\) is transcendental with respect to \(T\), then so is \(T^n h\) for any \(n = 0, 1, 2, \cdots\).

**Proof.** (a) Let \(\theta \in H^\infty(\Omega) \setminus \{0\}\) such that \(\theta(T)h = 0\).

Then for any \(n = 0, 1, 2, \cdots\),

\[
\theta(T)(T^n h) = T^n(\theta(T)h) = 0.
\]

It follows that \(\theta(T)h' = 0\) for any \(h' \in \bigvee_{n=0}^{\infty} T^n h\).

(b) Suppose that \(T^k h\) is algebraic with respect to \(T\) for some \(k > 0\). Thus there is \(f \in H^\infty(\Omega) \setminus \{0\}\) such that \(f(T)T^k h = 0\).

Let \(f_1(z) = z^k f(z)\) for \(z \in D\). Then \(f_1 \in H^\infty(\Omega) \setminus \{0\}\) and

\[
f_1(T)h = T^k f(T)h = f(T)T^k h = 0
\]

which contradicts to the fact that \(h\) is transcendental with respect to \(T\). \(\square\)

Note that \(T^0\) denote the identity operator on \(H\).

By Theorem 1 in [8], if \(h \in H\) is algebraic with respect to \(T\), then there is an inner function \(m_h \in H^\infty(\Omega)\) such that \(m_h(T)h = 0\) and \(m_h\) is said to be a minimal function of \(h\) with respect to \(T\).

**Theorem 2.3.** Let \(T \in L(H)\) be an operator satisfying hypothesis \((h)\), and \(B = \{h \in H : h\) is algebraic with respect to \(T\}\).

(a) If \(M = \{h_i : i = 1, 2, \cdots, k\}(k < \infty)\) is contained in \(B\), then so is \(\bigvee_{n=0}^{\infty} T^n M\).

(b) \(B\) is a subspace of \(H\).
Proof. (a) By Proposition 2.2 (a),
\begin{equation}
  (2.1) \quad m_{h_i}(T)(T^n h_i) = 0
\end{equation}
for any \( i = 1, \ldots, k \) and \( n = 0, 1, 2, \ldots \).

Let \( \theta = m_{h_1} \cdots m_{h_k} \). Then \( \theta \in H^\infty(\Omega) \setminus \{0\} \), and \( \theta = \theta_i m_{h_i} \) for some \( \theta_i \in H^\infty(\Omega) \setminus \{0\} \). Thus, by equation (2.1),
\begin{equation}
  (2.2) \quad \theta(T)(T^n h_i) = \theta_i(T) m_{h_i}(T)(T^n h_i) = 0
\end{equation}
for any \( i = 1, \ldots, k \) and \( n = 0, 1, 2, \ldots \).

If \( x \in \bigvee_{n=0}^\infty T^n M \), then there is a sequence \( \{x_n\}_{n=1}^\infty \) such that
\[
  \lim_{n \to \infty} x_n = x \quad \text{and} \quad x_n = \sum_{i=1}^k a_{n,i} P_{n,i}(T) h_i
\]
for some \( a_{n,i} \in \mathbb{C} \) and a polynomial \( P_{n,i} \). Then, equation (2.2) implies that
\[
  \theta(T)(x_n) = \theta(T) \left( \sum_{i=1}^k a_{n,i} P_{n,i}(T) h_i \right) = \sum_{i=1}^k a_{n,i} \theta(T) P_{n,i}(T) h_i = 0.
\]
It follows that \( \theta(T)(x) = 0 \) for any \( x \in \bigvee_{n=0}^\infty T^n M \). Thus \( x \in B \).

(b) Clearly, \( 0 \in B \). For \( h_1 \) and \( h_2 \) in \( B \), if \( m_1(T) h_1 = m_2(T) h_2 = 0 \), where \( m_i(i = 1, 2) \in H^\infty(\Omega) \setminus \{0\} \), then
\[
  (m_1 m_2)(T)(\alpha_1 h_1 + \alpha_2 h_2) = 0
\]
for any \( \alpha_i(i = 1, 2) \in \mathbb{C} \). Thus \( B \) is a subspace of \( H \).

Note that \( B \) does not need to be closed.

If \( T \) is a bounded operator on \( H \) and \( M \) is a (closed) invariant subspace for \( T \), then we can define a bounded operator \( T_M : H/M \to H/M \) defined by
\[
  T_M([h]) = [Th]
\]
where \( H/M \) is the quotient space. Since \( M \) is \( T \)-invariant, \( T_M \) is well-defined. Clearly, \( T_M \) is a bounded operator on \( H/M \).

Let \( R(\Omega) \) be the algebra of rational functions with poles off \( \overline{\Omega} \). We will say that a (closed) subspace \( N \) is \( R(\Omega) \)-invariant (or rationally invariant) for an operator \( T \) if it is invariant under \( u(T) \) for every \( u \in R(\Omega) \).

If \( N \) is a \( R(\Omega) \)-invariant subspace for an operator \( T \) satisfying hypothesis (h), then we can define \( \theta(T_N) : H/N \to H/N \) by
\[
  \theta(T_N)([h]) = [\theta(T)h]
\]
for \( \theta \in H^\infty(\Omega) \) and \([h] \in H/N \). Since \( N \) is \( R(\Omega) \)-invariant for the operator \( T \), \( T_N \) is well-defined. Clearly, \( T_N \) is a bounded operator on \( H/N \).
Definition 2.4. Let $T \in L(H)$ be an operator satisfying hypothesis $(h)$ and $M$ be an invariant subspace for $T$. An element $[h]$ of $H/M$ is said to be algebraic with respect to $T_M$ provided that $\theta(T_M)[h] = 0$ for some $\theta \in H^\infty(\Omega) \setminus \{0\}$.

If not, $h$ is said to be transcendental with respect to $T_M$.

Proposition 2.5. Let $T \in L(H)$ be an operator satisfying hypothesis $(h)$ and $B = \{h \in H : h$ is algebraic with respect to $T\}$. If $B$ is closed, then it is $R(\Omega)$-invariant.

Proof. Let $h \in B$ and $u \in R(\Omega)$. Then there is a nonzero function $\phi \in H^\infty(\Omega)$ such that $\phi(T)h = 0$.

It follows that $u(T)\phi(T)h = \phi(T)(u(T)h) = 0$, that is, $u(T)h \in B$. □

Theorem 2.6. Let $T \in L(H)$ be an operator satisfying hypothesis $(h)$ and $B = \{h \in H : h$ is algebraic with respect to $T\}$. If $B$ is a closed subspace of $H$, then the following statements are equivalent:

(i) $[a] \in H/B$ is algebraic with respect to $T_B$.
(ii) $a$ is algebraic with respect to $T$.

Proof. $(i) \rightarrow (ii)$ Since $[a] \in H/B$ is algebraic with respect to $T_B$, there is a nonzero function $\theta_1$ in $H^\infty(\Omega)$ such that $\theta_1(T)a \in B$.

It follows that

\[(2.3) \quad \theta_2(T)(\theta_1(T)a) = 0\]

for some $\theta_2 \in H^\infty(\Omega) \setminus \{0\}$.

Let $\theta_3 = \theta_1 \cdot \theta_2 \in H^\infty(\Omega) \setminus \{0\}$. Then by equation (2.3), $\theta_3(T)a = 0$, and so $a \in B$.

$(ii) \rightarrow (i)$ If $a \in H$ is algebraic with respect to $T$, then there is a nonzero function $\theta$ in $H^\infty(\Omega)$ such that $\theta(T)a = 0$. Since $0 \in B$, $\theta(T_B)[a] = [\theta(T)a] = 0$. □

Corollary 2.7. Under the same assumption as Theorem 2.6, the following statements are equivalent:

(i) $[a] \in H/B$ is algebraic with respect to $T_B$.
(ii) $[a] = [0]$.

Proof. By Theorem 2.6 it is clear. □

Corollary 2.8. Let $T \in L(H)$ be an operator satisfying hypothesis $(h)$ and $M \subset B$ is a $R(\Omega)$-invariant subspace for $T$. Then the following statements are equivalent:
(i) \([a] \in H/M\) is algebraic with respect to \(T_M\).
(ii) \(a\) is algebraic with respect to \(T\).

Proof. It can be proven by the same way as the proof of Theorem 2.6.

Corollary 2.9. Let \(T \in L(H)\) be an operator satisfying hypothesis \((h)\) and \(M \subset B\) is a \(R(\Omega)\)-invariant subspace for \(T\). Then the following statements are equivalent:

(i) \([a] \in H/M\) is transcendental with respect to \(T_M\).
(ii) \(a\) is transcendental with respect to \(T\).

We recall that if \(K\) is a Hilbert space, \(H\) is a subspace of \(K\), \(V \in L(K)\), and \(T \in L(H)\), then \(V\) is said to be a dilation of \(T\) provided that

\[
T = P_H V|H.
\]

If \(T\) and \(V\) are operators satisfying hypothesis \((h)\) and \(V\) is a \(C_0\)-operator relative to \(\Omega\) satisfying equation (2.4), then \(V\) is said to be a \(C_0\)-dilation of \(T\). We will not discuss about \(C_0\)-dilation any more in this paper.

Lemma 2.10. Let \(T \in L(H)\) be an operator satisfying hypothesis \((h)\) and \(B' = \{h \in H : h\) is transcendental with respect to \(T\}\). It \(h \in B'\), then \(u(T)h \in B'\) for any \(u \in R(\Omega) \setminus \{0\}\).

Proof. Suppose that there is an element \(h\) in \(B'\) such that \(u(T)h\) is algebraic with respect to \(T\) for some \(u \in R(\Omega) \setminus \{0\}\).

Thus there is a nonzero function \(\phi \in H^\infty(\Omega)\) such that

\[
\phi(T)u(T)h = 0.
\]

Let \(\theta = \phi \cdot u\). Then \(\theta \in H^\infty(\Omega) \setminus \{0\}\) such that \(\theta(T)h = 0\) by equation (2.5). It contradicts to the fact that \(h \in B'\).

\(\square\)

3. \(C_0\)-Hilbert Modules

Let \(H\) be a Hilbert space and \(F\) be a function algebra on \(X\). Then \(H\) is a Hilbert module over \(F\) with the module action \(F \times H \to H\) given by

\[
f.h = f(x)h
\]

for a fixed \(x \in X\). Let \(H_x\) denote this Hilbert module over \(F\). Clearly, \(H_x\) is a contractive Hilbert module over \(F\) for any \(x \in X\).

Similarly, for an operator \(T\) on \(H\) satisfying hypothesis \((h)\), if \(A \subset H^\infty(\Omega)\) is a function algebra over \(\overline{\Omega}\) such that every polynomial is
contained in $A$, then $H$ is a Hilbert module over $A$ with the module action $A \times H \to H$ given by

\[(3.1) \quad f.h = f(T)h.\]

In this paper, $H_T$ denotes this Hilbert module over $A \subset H^\infty(\Omega)$. Clearly, $H_T$ is a contractive Hilbert module over $A$.

In this section, $A$ denotes a function algebra over $\overline{\Omega}$ such that every polynomial is contained in $A$ and $A \subset H^\infty(\Omega)$.

**Definition 3.1.** If $T \in L(H)$ is a $C_0$-operator relative to $\Omega$, then $H_T$ is called a $C_0$-Hilbert module.

**Definition 3.2.** Let $H$ and $K$ be Hilbert modules over $A$. Then a module map $X : H \to K$ is a bounded, linear map satisfying $X(f.h) = f.(Xh)$ for all $f$ in $A$, and $h$ in $H$. Two Hilbert modules are similar if there is an invertible module map from $H$ onto $K$, and are said to be isomorphic if there is a module map from $H$ onto $K$ which is a unitary.

**Proposition 3.3.** For operators $T_i (i = 1, 2)$ in $L(H)$ satisfying hypothesis (h), if $T_1$ and $T_2$ are similar operators, then $H_{T_1}$ and $H_{T_2}$ are similar Hilbert modules over $A$.

**Proof.** Let a module map $G : H \to H$ denote the similarity such that $GT_1 = T_2G$.

Define a linear map $Y : H_{T_1} \to H_{T_2}$ by

\[(3.2) \quad Y(f.h) = f.(Gh)\]

for $f \in A$ and $h \in H_{T_1}$.

Let $f_i.h_1 = f_2.h_2$ for $f_i \in A$ and $h_i \in H_{T_i}$. Then

\[(3.3) \quad f_1(T_1)h_1 - f_2(T_1)h_2 = 0.\]

Since $GT_1 = T_2G$, equation (3.3) implies that

$$f_1(T_2)Gh_1 = Gf_1(T_1)h_1 = Gf_2(T_1)h_2 = f_2(T_2)Gh_2.$$

It follows that $f_1(Gh_1) = f_2(Gh_2)$, that is, $Y$ is well-defined.

For $h \in H_{T_1}$,

\[(3.4) \quad Y(h) = Y(1.h) = G(h).\]

By equations (3.2) and (3.4), we can conclude that $Y$ is a module map. Since $G$ is bijective, so is $Y$.

**Corollary 3.4.** For operators $T_i (i = 1, 2)$ in $L(H)$ satisfying hypothesis (h), if $T_1$ and $T_2$ are unitarily equivalent, then $H_{T_1}$ and $H_{T_2}$ are isomorphic.
Proof. It is proven by the same way as the proof of Proposition 3.3.

If \( T \in L(H) \) is an operator satisfying hypothesis \((h)\) and \( M \) is a submodule of \( H_T \) over \( A \), then by the definition of module action given in equation (3.1), we have that \( M \) is \( T \)-invariant. Furthermore, \( M \) is an invariant subspace for each operator \( u(T) \) where \( u \in A \).

**Definition 3.5.** Let \( T \in L(H) \) be an operator satisfying hypothesis \((h)\). If \( M \) is a submodule of \( H_T \) (over \( A \)) such that \( T|M : M \to M \) is a \( C_0 \)-operator relative to \( \Omega \), then \( M \) is said to be a \( C_0 \)-submodule (over \( A \)) of \( H_T \).

**Definition 3.6.** Let \( T \in L(H) \). If there is an element \( h \in H \) which is not in the kernel of \( T \) such that \( \{T^n h : n = 0, 1, 2, \cdots \} \) is not linearly independent, then \( T \) is said to be dependent.

**Theorem 3.7.** If \( T \in L(H) \) is a dependent operator satisfying hypothesis \((h)\), then \( H_T \) always has a nonzero \( C_0 \)-submodule \( M \).

Proof. Since \( T \) is dependent, there is a nonzero element \( h \in H \) such that \( \{T^n h : n = 0, 1, 2, \cdots \} \) is linearly independent. It follows that

\[
\sum_{n=0}^{k} a_n T^n h = 0,
\]

for some nonzero polynomial \( p(z) = \sum_{n=0}^{k} a_n z^n (z \in D) \).

Let \( M \) be the closed subspace of \( H \) generated by \( \{\theta(T)h : \theta \in A\} \) and \( M' = \{f \in A : f(T)h = 0\} \). Since \( p \in M' \), \( M' \) is not empty.

Clearly, \( f.k \) is in \( M \) for every \( f \) in \( A \) and \( k \) in \( M \) and so \( M \) is a submodule of \( H_T \).

For any \( \theta \in A \) and \( f \in M' \),

\[
f(T)\theta(T)h = \theta(T)f(T)h = 0.
\]

It follows that \( f(T)h' = 0 \) for any \( f \in M' \) and \( h' \in M \).

Therefore, \( T_0 = T|M \) is a \( C_0 \)-operator relative to \( \Omega \), and so \( M \) is a \( C_0 \)-submodule of \( H_T \).

**Definition 3.8.** Let \( T \in L(H) \) be an operator satisfying hypothesis \((h)\). A \( C_0 \)-submodule \( M \) of \( H_T \) over \( A \) is said to be maximal provided that there is no submodule \( M' \) of \( H_T \) over \( A \) such that \( M \subset M' \) and \( T|M' \) is a \( C_0 \)-operator relative to \( \Omega \).

**Corollary 3.9.** Let \( T \in L(H) \) be an operator satisfying hypothesis \((h)\). If \( M \) is a maximal \( C_0 \)-submodule of \( H_T \) and \( h \in H_T \setminus M \), then \( \{T^n h : n = 0, 1, 2, \cdots \} \) is linearly independent.
Proof. Suppose that there is an element $h \in H_T \setminus M$ such that \( \{T^n h : n = 0, 1, 2, \cdots \} \) is linearly dependent.

If $M'$ is the closed subspace of $H_T$ generated by \( \{\theta(T) h : \theta \in A\} \), then by Theorem 3.4, $T|M'$ is a $C_0$-operator relative to $\Omega$. Since $T|M$ and $T|M'$ are $C_0$-operators relative to $\Omega$, there are nonzero functions $\theta_i \in H^\infty(\Omega) (i = 1, 2, \cdots)$ such that

\[
\theta_1(T|M) = 0 \quad \text{and} \quad \theta_2(T|M') = 0.
\]

It follows that $\theta_1 \theta_2(T|M \setminus M') = 0$, that is, $T|M \setminus M'$ is also a $C_0$-operator relative to $\Omega$. By maximality of $M$, $M \cap M' = M$ which contradicts to the fact that $h \in M' \setminus M$.

For an operator satisfying hypothesis $(h)$, $T \in L(H)$, $h \in H$ is said to be \textit{algebraic with respect to $T$ over $A$}, provided that

\[ \theta(T)h = 0 \quad \text{for some} \quad \theta \in A \setminus \{0\}. \]

If $B = \{h \in H : h \text{ is algebraic with respect to } T \text{ over } A\}$, then we could raise the question of whether the following sentence is true or not;

If every element of $H_T$ is algebraic with respect to $T$ over $A$, then $T$ is a $C_0$-operator.

In the next Theorem, we provide a condition in which that sentence is true.

**Theorem 3.10.** Let $T \in L(H)$ be an operator satisfying hypothesis $(h)$. If $H_T$ is a Hilbert module over $A$ with a generating set $\{h_1, \cdots, h_k\} (k < \infty)$ and $h_i \in B$ for $i = 1, 2, \cdots, k$, then $H_T = B$ and $T$ is a $C_0$-operator.

Proof. Since $h_i \in B$, there is a nonzero function $m_i$ in $A$ such that $m_i(T)h_i = 0$ for $i = 1, 2, \cdots, k$. Then for any $f \in A$, $m_i(T)f(T)h_i = m_i(T)f(T)m_i(T)h_i = 0$. It follows that $f.h_i \in B$ for any $f \in A$.

By Theorem 2.3 (b), $\{\sum_{i=1}^k f_i.h_i : f_i \in A\}$ is contained in $B$. Since $\{\sum_{i=1}^k f_i.h_i : f_i \in A\}$ is dense in $H_T$, it is enough to prove that $B$ is a closed subspace of $H_T$.

Let $b$ be an element in the closure of $B$ in the norm topology induced by the inner product defined in $H_T$. Then, there is a sequence $\{b_n\}_{n=1}^\infty$ in $\{\sum_{i=1}^k f_i.h_i : f_i \in A\}$ such that $\lim_{n \to \infty} b_n = b$.

Define a function $m = m_1 \cdots m_k$. Then, for any $f_i \in A$,

\[
\sum_{i=1}^k f_i(T)h_i = m(T) \sum_{i=1}^k f_i(T)h_i = \sum_{i=1}^k f_i(T)m(T)h_i = 0.
\]
Equation (3.6) implies that \( m(T)(b_n) = 0 \) for any \( n = 1, 2, \cdots \). Thus \( m(T)b = 0 \) so that \( b \in B \). Therefore, \( H_T = B \).

Since \( m(T)b = 0 \) for any \( b \in B(= H_T) \), \( m(T) = 0 \) which proves that \( T \) is a \( C_0 \)-operator.

\[
\square
\]

Recall that a nonzero function \( \theta \) in \( H(\Omega) \) is said to be inner if \(|\theta|\) is constant almost everywhere on each component of \( \partial \Omega \). Then the Jordan block \( S(\theta) \) is an operator acting on the space \( H(\theta) = H^2(\Omega) \ominus \theta H^2(\Omega) \) as follows:

\[
S(\theta) = P_{H(\theta)}S|H(\theta),
\]

where \( S \in L(H^2(\Omega)) \) is defined by \( (Sf)(z) = zf(z) \).

An operator \( T \in L(H) \) is called a quasi-affine transform of an operator \( T' \in L(H') \) if there exists an injective operator \( X \in L(H, H') \) with dense range such that \( T'X = XT \). \( T \) and \( T' \) are quasi-similar if \( T \prec T' \) and \( T' \prec T \).

**Proposition 3.11.**[10] Let \( H \) be a separable Hilbert space and \( T \in L(H) \) be an operator of class \( C_0 \) relative to \( \Omega \). Then there is a family \( \{ \theta_i \in H^\infty(\Omega) : i = 0, 1, 2, \cdots \} \) of inner functions such that

(i) For \( i = 1, 2, \cdots \), \( \theta_i \) divides \( \theta_{i-1} \), that is, \( \theta_{i-1} = \theta_i \varphi \) for some \( \varphi \in H^\infty(\Omega) \).

(ii) \( T \) is quasi-similar to \( \bigoplus_{i=0}^{\infty} S(\theta_i) \).

If \( T \in L(H) \) is a \( C_0 \)-operator relative to \( \Omega \), then by Definition[1.8] \( \ker \Psi_T \neq \{0\} \) and there is an inner function \( \theta \), called a minimal function of \( T \), in \( H^\infty(\Omega) \) such that \( \ker \Psi_T = \theta H^\infty(\Omega) \)[10]. We denote by \( m_T \) the minimal function of \( T \).

**Definition 3.12.** Let \( M \) be a \( C_0 \)-submodule of \( H_T \) with the following property:

If \( M_1 \) is a \( C_0 \)-submodule of \( H_T \) such that \( M \subset M_1 \) and \( m_{T|M} = m_{T|M_1} \), then \( M = M_1 \).

Then \( M \) is said to be a locally maximal \( C_0 \)-submodule of \( H_T \).

**Theorem 3.13.** Let \( H \) be a separable Hilbert space and \( T \in L(H) \) be an operator satisfying hypothesis (h). If \( B = \{ h \in H : h \) is algebraic with respect to \( T \) over \( A \} \) is a closed subspace of \( H \) and \( \text{rank}_A H_T < \infty \), then there are locally maximal \( C_0 \)-submodules \( M_i(i = 0, 1, 2, \cdots) \) of \( H_T \) such that

\[
M_0 \subset M_1 \subset M_2 \subset \cdots
\]
Proof. Let $T' = T|B$. For given element $h \in B$, we have a function $m_h \in A \setminus \{0\}$ such that
\[ m_h(T)h = 0. \]
Then $m_h(T)(\varphi,h) = m_h(T)\varphi(T)h = \varphi(T)m_h(T)h = 0$ for any $\varphi$ in $A$. Thus, $B$ is a submodule of $H_T$ so that $B = H_{T'}$.

Since
\[ \text{rank}_AH_{T'} = \text{rank}_AB \leq \text{rank}_AH_T < \infty, \]
and every elements $h$ in $B$ is algebraic with respect to $T'$ over $A$, Theorem 3.10 implies that $T' = T|B$ is a $C_0$-operator.

Thus by Proposition 3.11 there are inner functions $\theta_i (i = 0, 1, 2, \ldots)$ such that $\theta_{i+1}$ divides $\theta_i$ and $T|B$ is quasisimilar to $\bigoplus_{i=0}^{\infty} S(\theta_i)$.

For each $\theta_i (i = 0, 1, 2, \ldots)$, we have a bounded linear operator $\theta_i(T) : H \to H$ such that
\[ \theta_i(T)(f,h) = \theta_i(T)f(T)h = f(T)\theta_i(T)h = f.(\theta_i(T)h) \]
for any $f \in A$ and $h \in H$. Thus $\theta_i(T)(i = 0, 1, 2, \ldots)$ is a module map.

It follows that $M_i = \ker(\theta_i(T))$ is a submodule of $H_T$ and clearly, $T_i = T|M_i$ is a $C_0$-operator such that $\theta_i(T_i) = 0$. Thus $M_i$ is a $C_0$-submodule of $H_T$.

Let $i \in \{0, 1, 2, \ldots\}$ be given and $M$ be a $C_0$-submodule of $H_T$ such that
\[ (3.7) \quad M_i \subset M \quad \text{and} \quad m_{T|M} = m_{T|M_i}. \]
Since $m_{T|M_i} = \theta_i$, by equation (3.7), $m_{T|M} = \theta_i$. Thus, $\theta_i(T|M) = 0$ so that
\[ (3.8) \quad M \subset \ker(\theta_i(T)) = M_i. \]
From equations (3.7) and (3.8), $M = M_i$. Thus, $M_i$ is a locally maximal $C_0$-submodule of $H_T$ for each $i = 1, 2, 3, \ldots$.

Since $\theta_{i+1}$ divides $\theta_i$ for $i = 0, 1, 2, \ldots$, $M_i \subset M_{i+1}$. □

In fact, in the proof of Theorem 3.13, $T|B$ is quasisimilar to $\bigoplus_{i=0}^{k} S(\theta_i)$ where $k \leq \text{rank}_AH_T < \infty$. Thus, we have a finite number of locally maximal $C_0$-submodules $M_i(i = 0, 1, 2, \ldots, k)$.

Naturally, the following question remains: When is $B$ closed? However, we will not discuss this question in this paper.
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