Degrees of maps between $S^3$-bundles over $S^5$

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Abstract
In this article, we compute all possible degrees of maps between $S^3$-bundles over $S^5$. It also provides a correction of an article by Lafont and Neofytidis [6].

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1 The main result
For two $n$-dimensional closed manifold $M$ and $N$, let $D(M, N)$ be the set of degrees of maps from $M$ to $N$, i.e.

$$D(M, N) = \{ \deg f : M \to N \}.$$  

Set $D(M) = D(M, M)$ for short.

The set of mapping degrees has been studied by many authors, for both lower dimensional and higher dimensional manifolds, cf. [1, 2, 9] etc. Unlike the complicated situation for lower dimensional manifolds, homotopy theory seems to be a powerful tool in the higher dimensional case.

In this article, the main objects concerned are $S^3$-bundles over $S^5$. Recall that isomorphism classes of 4-dimensional vector bundles over $S^5$ are one-to-one correspondence with elements in $\pi_4(SO(4)).$ Using the canonical diffeomorphism $SO(4) \cong S^3 \times SO(3)$, we have $\pi_4(SO(4)) \cong \pi_4(S^3) \oplus \pi_4(SO(3)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Under this correspondence, denote the total spaces of those associated sphere bundles as $M_{i,j}$ ($i = 0, 1$). Obviously, $M_{0,0} \cong S^3 \times S^5$, $M_{1,0} \cong SU(3)$. Moreover, we have:

Lemma 1.1 ([4, 5]).
1. $S^3 \times S^5$, $M_{0,1}$ and $SU(3)$ are not homotopy equivalent to each other.
2. $M_{1,1} \simeq SU(3)$.

Therefore, we only need to consider three objects: $S^3 \times S^5$, $M_{0,1}$ and $SU(3)$. The main result is the following:
\textbf{Theorem 1.2.} Let $M, N \in \{S^3 \times S^5, M_{0,1}, SU(3)\}$ and $M \neq N$, then

1. $D(M) = \begin{cases} \mathbb{Z}, & \text{if } M = S^3 \times S^5, M_{0,1} \\ 4\mathbb{Z} \cup \{2k + 1k \in \mathbb{Z}\}, & \text{if } M = SU(3) \end{cases}$.

2. $D(M, N) = \begin{cases} 4\mathbb{Z}, & \text{if } M = SU(3) \\ 2\mathbb{Z}, & \text{otherwise}. \end{cases}$

\textbf{Remark 1.} The motivation comes from the work of Lafont and Neofytidis [4]. They considered mapping degrees among principal $SU(2)$-bundles over $S^3$. Note that $SU(2) \cong S^3$. One aim of their article is to correct Pütmann’s result of $D(SU(3))$ [8]. However, their result about $D(SU(3), S^3 \times S^5)$ is wrong. The reason is that the image of their map $g$ does not lie in $SU(3)$. For example, choose $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \in SU(3)$, then $g(A) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \notin SU(3)$. It also leads to a wrong proof of their result on $D(SU(3))$, which relies on $D(SU(3), S^3 \times S^5)$ in their article, although the final result is miraculously correct.

\textbf{Remark 2.} In fact, any simply connected 8-manifold with homology isomorphic to that of $S^3 \times S^5$ is homotopy equivalent to either $S^3 \times S^5, M_{0,1}$ or $SU(3)$. It will be proved in a coming paper of the author. Therefore we actually obtained the set of mapping degrees among a larger class of manifolds.

\section{The proof}

We’ll analysis the homotopy sets between these manifolds. First recall some basic results for homotopy groups of spheres. Let $\iota_n \in \pi_n(S^n)$ be represented by the identity map of $S^n$, $\eta_2 \in \pi_3(S^2)$ and $\nu_4 \in \pi_7(S^4)$ be represented by Hopf maps, $\eta_n = \Sigma^{n-2}\eta_2 \in \pi_{n+1}(S^n)$.

\textbf{Lemma 2.1 (11 12).} 1. $\pi_{n+1}(S^n) \cong \begin{cases} \mathbb{Z}\{\eta_2\}, & \text{if } n = 2 \\ \mathbb{Z}_2\{\eta_n\}, & \text{if } n \geq 3, \end{cases}$

2. $\pi_{n+2}(S^n) \cong \mathbb{Z}_2\{\eta_n\eta_{n+1}\}$ for $n \geq 2$;

3. $\pi_6(S^4) \cong \mathbb{Z}_{12}\{a_4\}$, $\pi_7(S^4) \cong \mathbb{Z}\{\nu_4\} \oplus \mathbb{Z}_{12}\{a_4\}, a_4 = \Sigma a_3, [\iota_4, \iota_4] = 2\nu_4 \pm a_4$;

4. $\pi_7(S^8) \cong \mathbb{Z}_2\{a_3\eta_6\}$, $\pi_8(S^8) \cong \mathbb{Z}_2\{\nu_4\eta_7\} \oplus \mathbb{Z}_2\{a_4\eta_7\}$.

Next write down the cell structures.

\textbf{Lemma 2.2 (cf. 3 7).} 1. $S^3 \times S^5 \simeq (S^3 \vee S^5) \cup_{[\iota_3, \iota_5]} D^8$;

2. $M_{0,1} \simeq (S^3 \vee S^5) \cup_{[\iota_3, \iota_5] + a_3\eta_6} D^8$;
3. \( SU(3) \simeq S^3 \cup_{\eta_3} D^5 \cup_{\xi} D^8 \), where \( \xi \) generates \( \pi_7(S^3 \cup_{\eta_3} D^5) \equiv \mathbb{Z} \).

Note that in this article, obvious inclusion maps are always omitted.

In the following, for \( M \in \{ S^3 \times S^5, M_{0,1}, SU(3) \} \), choose \( e_3^M \) and \( e_5^M \) to be generators of \( H^3(M) \equiv \mathbb{Z} \) and \( H^5(M) \equiv \mathbb{Z} \), such that \( e_3^M e_5^M \) equals to the orientation cohomology class \( \omega_M \) of \( M \).

**Proof of theorem 1.2.**

(1) \( D(S^3 \times S^5) = \mathbb{Z} \) is obvious.

(2) \( D(S^3 \times S^5, M_{0,1}) = 2\mathbb{Z} \).

Consider the Puppe sequence induced by the cofiber sequence \( S^7 \xrightarrow{[t_3, t_5]} S^3 \vee S^5 \rightarrow S^3 \times S^5 \):

\[
[S^3 \times S^5, M_{0,1}] \rightarrow [S^3 \vee S^5, M_{0,1}] \xrightarrow{[t_3, t_5]^*} [S^7, M_{0,1}]
\]

\([S^3 \vee S^5, M_{0,1}] \cong [S^3, M_{0,1}] \times [S^5, M_{0,1}] \cong \mathbb{Z} \{ t_3 \} \oplus \mathbb{Z} \{ \eta_3 \eta_4 \} \oplus \mathbb{Z} \{ \epsilon \} \). An element can be lifted to \([S^3 \times S^5, M_{0,1}]\) if and only if it maps to \(0\) under \([t_3, t_5]^*\). We have

\[
[t_3, t_5]^*(kt_3 + \epsilon_3 \eta_4 + \eta_5) = (kt_3 + \epsilon_3 \eta_4 + \eta_5)[t_3, t_5] = [kt_3, \eta_3 \eta_4 + \eta_5] = kl[t_3, t_5] + k\epsilon[t_3, \eta_3 \eta_4] = kl \alpha_3 \eta_6
\]

In the last step, \([t_3, t_5]\) = \(a_3 \eta_6\) in \(M_{0,1}\) and \([t_3, \eta_3 \eta_4]\) = \(0\) as \(S^3\) is a Lie group [13, Corollary X.7.8]. Since \(a_3 \eta_6\) is of order \(2\), \([t_3, t_5]^*(kt_3 + \epsilon_3 \eta_4 + \eta_5) = 0\) if and only if \(kl\) is even. Notice that \(kl\) is just the degree of the lifted map. Therefore the set of degrees coincides with all even numbers.

(3) \( D(S^3 \times S^5, SU(3)) = 2\mathbb{Z} \).

Consider the Puppe sequence induced by the cofiber sequence \( S^7 \xrightarrow{[t_3, t_5]} S^3 \vee S^5 \rightarrow S^3 \times S^5 \):

\[
[S^3 \times S^5, SU(3)] \rightarrow [S^3 \vee S^5, SU(3)] \xrightarrow{[t_3, t_5]^*} [S^7, SU(3)]
\]

By the cell structure of \(SU(3)\), \(\pi_7(SU(3)) = 0\). Therefore, all elements in \([S^3 \vee S^5, SU(3)]\) can be lifted to \([S^3 \times S^5, SU(3)]\). \([S^3 \vee S^5, SU(3)] \cong [S^3, SU(3)] \times [S^5, SU(3)] \cong \mathbb{Z} \{ \alpha \} \oplus \mathbb{Z} \{ \beta \} \). Suppose \((k\alpha, l\beta)\) lifts to a map \(f_{k,l} : S^3 \times S^5 \rightarrow SU(3)\). Then we have

\[
f_{k,l}^*\omega_{SU(3)} = f_{k,l}^*(e_3^{SU(3)} e_5^{SU(3)}) = f_{k,l}^* e_3^{SU(3)} f_{k,l}^* e_5^{SU(3)}
\]

It is clear that the value of \(f_{k,l}^* e_3^{SU(3)}\) and \(f_{k,l}^* e_5^{SU(3)}\) correspond to the image of \(k\alpha\) and \(l\beta\) under the Hurwicz maps. By Hurwicz theorem, \(h_3 : \pi_3(SU(3)) \rightarrow \)}
$H_3(SU(3))$ is an isomorphism. To compute $h_5 : \pi_5(SU(3)) \to H_5(SU(3))$, use the fiber bundle $S^3 \to SU(3) \xrightarrow{\pi} S^5$:

$$
\pi_5(SU(3)) \xrightarrow{p^*} \pi_5(S^5) \xrightarrow{\pi_5} \pi_4(S^3) \xrightarrow{\pi_4} \pi_4(SU(3)) = 0
$$

The bottom isomorphism follows easily by the Gysin sequence. Therefore $h_5 : \pi_5(SU(3)) \to H_5(SU(3))$ is a multiplication by $\pm 2$ as $\pi_4(S^3) \cong \mathbb{Z}_2$. With those generators suitably chosen, we may assume the sign is $+$. Hence

$$
\int_{k,l}^* \omega_{SU(3)} = \int_{k,l}^* \epsilon_3^{SU(3)} \int_{k,l}^* \epsilon_5^{SU(3)} = 2kl \epsilon_3^{S^3 \times S^5} \epsilon_5^{S^5} = 2kl \omega_{S^3 \times S^5}
$$

which means that the set of degrees is just all even numbers.

(4) $D(M_{0,1}, S^3 \times S^5) = 2\mathbb{Z}$.

Consider the cofiber sequence $S^7 \xrightarrow{[\iota_3, \iota_5] + \alpha \gamma \eta} S^3 \vee S^5 \to M_{0,1}$. We have an exact sequence

$$
[M_{0,1}, S^3 \times S^5] \to [S^3 \vee S^5, S^3 \times S^5] \xrightarrow{([\iota_3, \iota_5] + \alpha \gamma \eta)^*} [S^7, S^3 \times S^5]
$$

$$
[S^3 \vee S^5, S^3 \times S^5] \cong [S^3, S^3 \times S^5] \times [S^5, S^3 \times S^5] \cong \mathbb{Z}_{\iota_3} \oplus \mathbb{Z}_2 \{\eta_3 \eta_4\} \oplus \mathbb{Z}_{\iota_5}.
$$

$$
([\iota_3, \iota_5] + \alpha \gamma \eta_6)^* (k \iota_3 + \epsilon \eta_3 \eta_4 + l \iota_5) = (k \iota_3 + \epsilon \eta_3 \eta_4 + l \iota_5)([\iota_3, \iota_5] + \alpha \gamma \eta_6)
$$

$$
= [k \iota_3, \epsilon \eta_3 \eta_4 + l \iota_5] + ka_3 \gamma \eta_6
$$

$$
= kl[\iota_3, \iota_5] + k\epsilon[\iota_3, \eta_3 \eta_4] + ka_3 \gamma \eta_6
$$

$$
= ka_3 \gamma \eta_6
$$

In the last step, note that $[\iota_3, \iota_5] = 0$ in $S^3 \times S^5$. Since $ka_3 \gamma \eta_6 = 0$ if and only if $k$ is even, the possible degrees $kl$ are chosen from all even numbers.

(5) $D(M_{0,1}, M_{0,1}) = \mathbb{Z}$.

Consider the cofiber sequence $S^7 \xrightarrow{[\iota_3, \iota_5] + \alpha \gamma \eta} S^3 \vee S^5 \to M_{0,1}$. We have an exact sequence:

$$
[M_{0,1}, M_{0,1}] \to [S^3 \vee S^5, M_{0,1}] \xrightarrow{([\iota_3, \iota_5] + \alpha \gamma \eta)^*} [S^7, M_{0,1}]
$$

$$
([\iota_3, \iota_5] + \alpha \gamma \eta_6)^* (k \iota_3 + \epsilon \eta_3 \eta_4 + l \iota_5) = (k \iota_3 + \epsilon \eta_3 \eta_4 + l \iota_5)([\iota_3, \iota_5] + \alpha \gamma \eta_6)
$$

$$
= [k \iota_3, \epsilon \eta_3 \eta_4 + l \iota_5] + ka_3 \gamma \eta_6
$$

$$
= k(l + 1) a_3 \gamma \eta_6
$$

which equals to 0 if and only if $k(l + 1)$ is even. Thus we can choose $l = 1$ and $k$ arbitrary to make the mapping degree $kl$ realize all integers.
Lemma 2.3

Let \( S^3 \times S^5 \) be the bundle projection \( SU(3) \to S^5 \), then \( p : H_5(SU(3)) \to H_5(S^5) \) is an isomorphism. Therefore, the degree of \( f_k \times p : SU(3) \to S^3 \times S^5 \) is 4, and our result follows.

(8) \( D(SU(3), M_{0,1}) = 4\mathbb{Z} \).

It’s not as straight as the other cases. We’ll use the following observation.

Lemma 2.3 [10]. Let \( M, N \) be two closed \( n \)-manifolds and \( \overline{M}, \overline{N} \) be obtained by deleting an embedded \( D^n \) in \( M, N \) respectively. Then \( k \in D(M, N) \) if and only if there exists \( \overline{f} : \overline{M} \to \overline{N} \) such that the following diagram commutes up to homotopy:

\[
\begin{array}{ccc}
  S^{n-1} & \xrightarrow{i_M} & \overline{M} \\
  \downarrow{k_{n-1}} & & \downarrow{\overline{f}} \\
  S^{n-1} & \xrightarrow{i_N} & \overline{N}
\end{array}
\]

Here \( i_M, i_N \) denotes the inclusion of \( S^{n-1} \) to the boundary of \( \overline{M}, \overline{N} \) respectively.
Consider the Puppe sequence induced by the cofiber sequence $S^4 \xrightarrow{\eta} S^3 \xrightarrow{\epsilon} S^3 \cup_{\eta_3} D^5$:

$$[S^3 \cup_{\eta_3} D^5,M_{0,1}] \xrightarrow{\iota^*} [S^3,M_{0,1}] \xrightarrow{k} [S^4,M_{0,1}]$$

It is clear that $\eta_3 : \pi_3(M_{0,1}) \cong \mathbb{Z}\{t_3\} \to \pi_4(M_{0,1}) \cong \mathbb{Z}_2\{\eta_3\}$ is surjective. Therefore, for any map $f : S^3 \cup_{\eta_3} D^5 \to M_{0,1}$, if $f_* : H_3(S^3 \cup_{\eta_3} D^5) \cong \mathbb{Z} \to H_3(M_{0,1}) \cong \mathbb{Z}$ is a multiplication by $k$, then $k$ must be divisible by 2. It indicates that all possible mapping degrees from $SU(3)$ to $M_{0,1}$ must be divisible by 2.

Now observe that the following two diagrams are equivalent, as $[(t_3,t_5) + a_3\eta_6)](2k\tau_7) = [t_3,t_5](2k\tau_7)$:

$$\begin{array}{c}
S^7 \xrightarrow{\xi} S^3 \cup_{\eta_3} D^5 \\
\downarrow \quad \downarrow 7 \\
S^7 \xrightarrow{2k\tau_7} S^3 \cup S^5
\end{array} \quad \begin{array}{c}
S^7 \xrightarrow{\xi} S^3 \cup_{\eta_3} D^5 \\
\downarrow \quad \downarrow 7 \\
S^7 \xrightarrow{2k\tau_7} S^3 \cup S^5
\end{array}$$

Therefore, combining lemma 2.3 and (7), the result follows.

(9) $D(SU(3)) = 4\mathbb{Z} \cup \{2k + 1|k \in \mathbb{Z}\}$.

As seen in [3] all maps $S^3 \cup_{\eta_3} D^5 \to SU(3)$ extends to $SU(3) \to SU(3)$ since $\pi_7(SU(3)) = 0$. Consider the Puppe sequence induced by the cofiber sequence $S^4 \xrightarrow{\eta} S^3 \xrightarrow{\epsilon} S^3 \cup_{\eta_3} D^5$:

$$0 = [S^4,SU(3)] \to [S^5,SU(3)] \xrightarrow{q} [S^3,M_{0,1}] \xrightarrow{\iota^*} [S^3,SU(3)] \to [S^4,SU(3)] = 0$$

Here $q : S^3 \cup_{\eta_3} D^5 \to S^5$ is the quotient map. Since $\pi_3(SU(3)) \cong \pi_5(SU(3)) \cong \mathbb{Z}$ and $[S^3 \cup_{\eta_3} D^5,SU(3)]$ is abelian, we have $[S^3 \cup_{\eta_3} D^5,SU(3)] \cong \mathbb{Z}\{\alpha\} \oplus \mathbb{Z}\{\beta\}$, where $\alpha$ is the inclusion of 5-skeleton of $SU(3)$ and $\beta$ is the generator of $\pi_5(SU(3))$ composed with $q$. Obviously,

$$\begin{align*}
\alpha^* : H^3(SU(3)) &\xrightarrow{\cong} H^3(S^3 \cup_{\eta_3} D^5), & H^5(SU(3)) &\xrightarrow{\cong} H^5(S^3 \cup_{\eta_3} D^5) \\
\beta^* : H^3(SU(3)) &\xrightarrow{0} H^3(S^3 \cup_{\eta_3} D^5), & H^5(SU(3)) &\xrightarrow{\times 2} H^5(S^3 \cup_{\eta_3} D^5)
\end{align*}$$

Therefore,

$$(k\alpha + l\beta)^* : H^3(SU(3)) \xrightarrow{\times k} H^3(S^3 \cup_{\eta_3} D^5)$$

$$(k\alpha + l\beta)^* : H^5(S^3 \cup_{\eta_3} D^5)$$

Extend $k\alpha + l\beta$ to $f_{k,l} : SU(3) \to SU(3)$, then $\deg f_{k,l} = k(k + 2l)$. We can realize all odd numbers by choosing $k = 1$ and $l$ arbitrary. Notice that if $\deg f_{k,l}$ is even, $k$ must be even. Let $k = 2k'$, then $\deg f_{k,l} = 4k'(k'+l)$, which is divisible by 4. Hence we can choose $k'$ arbitrary and $l = 1 - k'$ to realize $4\mathbb{Z}$. 

\qed
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