Geometrical clusterization of Polyakov loops in SU(2) lattice gluodynamics

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Abstract. The liquid droplet formula is applied to an analysis of the properties of geometrical (anti)clusters formed in SU(2) gluodynamics by the Polyakov loops of the same sign. Using this approach, we explain the phase transition in SU(2) gluodynamics as a transition between two liquids during which one of the liquid droplets (the largest cluster of a certain Polyakov loop sign) experiences a condensation, while the droplet of another liquid (the next to the largest cluster of the opposite sign of Polyakov loop) evaporates. The clusters of smaller sizes form two accompanying gases, which behave oppositely to their liquids. The liquid droplet formula is used to analyze the size distributions of the gas (anti)clusters. The fit of these distributions allows us to extract the temperature dependence of surface tension and the value of Fisher topological exponent \( \tau \) for both kinds of gaseous clusters. It is shown that the surface tension coefficient of gaseous (anti)clusters can serve as an order parameter of the deconfinement phase transition in SU(2) gluodynamics. The Fisher topological exponent \( \tau \) of (anti)clusters is found to have the same value \( 1.806 \pm 0.008 \). This value disagrees with the famous Fisher droplet model, but it agrees well with an exactly solvable model of the nuclear liquid-gas phase transition. This finding may evidence for the fact that the SU(2) gluodynamics and this exactly solvable model of nuclear liquid-gas phase transition are in the same universality class.

1. Introduction

The lattice simulations are presently considered as the only first principle tool to investigate the deconfinement phase transition (PT) in quantum chromodynamics (QCD). Such a PT is also expected in gluodynamics (GD) which is a pure non abelian gauge theory. The Svetitsky-Jaffe hypothesis [1, 2] relates the deconfinement PT in (d+1)-dimensional SU(N) GD to the magnetic PT in Z(N) symmetric spin model in d dimensions. The role of spin in GD is played by the local Polyakov loop. For lattice of size \( N_d^\sigma \times N_\tau \) it is defined by temporal gauge links \( U_4(\vec{x},t) \) as

\[
L(\vec{x}) = Tr \prod_{t=0}^{N_\tau-1} U_4(\vec{x},t). \tag{1}
\]

A high level of understanding of the spin systems along with the Svetitsky-Jaffe hypothesis led to a significant progress in studies of the SU(N) GD properties in the PT vicinity. Formation of geometrical clusters composed of the Polyakov loops is an important feature of GD [3, 4]. A similar phenomenon is well known in spin systems and it is responsible for percolation of...
clusters, which already was studied in GD with paying a special attention to the largest and the next to the largest clusters [5, 6]. However, in many respects the PT details are encoded in the properties of smaller clusters which is well-known from the famous Fisher Droplet Model (FDM) [7, 8]. An important finding of the FDM is that at the critical point the size distribution of physical clusters obeys a power law which is controlled by the Fisher topological exponent \( \tau \). Hence the value of \( \tau \) is rather important in order to develop a consistent theory of PT in QCD and to localize its critical point. Therefore, in this work we study the geometrical clusterization in SU(2) GD and analyze the properties of clusters of all possible sizes. This approach allows us to explain the deconfinement of color charges as a specific kind of the liquid-gas PT [9].

2. The Polyakov loop geometrical clusters
In case of SU(2) gauge group \( L(\vec{x}) \) has the real values from \(-1\) to \(1\). For a given lattice configuration the Polyakov loops being the nearest neighbors can be attributed to the same cluster, if they have the same sign. The boundaries of clusters with opposite signs of \( L(\vec{x}) \) are characterized by strong fluctuations. Therefore, similarly to Refs. [5, 6] we introduced the minimal absolute value of the Polyakov loop attributed to the clusters, i.e. a cut-off \( L_{\text{cut}} > 0 \). All space points \( \vec{x} \) with \( |L(\vec{x})| \leq L_{\text{cut}} \) are attributed to “auxiliary” or “confining” vacuum which volume fraction is independent of the inverse lattice coupling \( \beta = 1/g^2 \) [9], where \( g^2 \) is the lattice coupling constant. The above definition allows us to define the monomers, the dimers, etc. as the clusters made of a corresponding number of “gauge spins” of the same sign which are surrounded either by the clusters of opposite “spin” sign or by a vacuum [9]. Obviously, there are clusters of two types related to two signs of the local Polyakov loop. We introduce a formal definition of anticlusters, if their sign coincides with the sign of the largest \( n \)-mer existing at a given lattice configuration. The largest anticluster is the “anticluster droplet” and the other \( n \)-mers of the same sign correspond to the “gas of anticlusters”. The clusters are defined to have an opposite sign of the Polyakov loop and the largest of them is called the “cluster droplet”.

3. Size distributions of clusters
The described scheme of the (anti)cluster identification was realized numerically. The Polyakov loops were obtained at each spatial point of 3+1 dimensional lattice with the spatial and temporal extents \( N_s = 24 \) and \( N_t = 8 \), respectively. The simulations were performed for 13 values of the inverse lattice coupling \( \beta \) inside the interval \( \beta \in [2.31, 3] \). The physical temperature \( T \) is defined via two-loop \( \beta \) dependence of the lattice spacing \( a(\beta) \) as \( 1/T = N_t a(\beta) \) [5]. The \( \beta \) points where distributed not uniformly. They where concentrated in the PT region which is of principal interest for this study. The identification of (anti)clusters was performed for two values of the Polyakov loop cut-off \( L_{\text{cut}} = 0.1 \) and \( L_{\text{cut}} = 0.2 \). For most of \( \beta \) values the number
of (anti)clusters of each size was averaged over the ensemble of 800 and 1600 gauge field lattice configurations. The distributions obtained in this way were the same within the statistical errors. In case of gaseous anticlusters at three largest values of \( \beta \) statistics was increased to 2400 configurations. The right hand side vicinity of PT was also analyzed with such a statistics. In case of gaseous anticlusters at three largest values of \( \beta \), the size of (anti)clusters of each size was averaged over the ensemble of 800 and 1600 gauge field lattice configurations. The 4-parametric fit of the LDM formula. All these results refer to the cut-off \( L_{\text{cut}} = 0.2 \).

The Fisher exponent \( \tau \) is the same due to existing global Z(2) symmetry. If \( k = 1 \) we got \( \mu_{\text{min}} = 2 \) and \( \tau = 1.806(8) \) both for clusters and for anticlusters. We also found that \( \tau = 2 \) for clusters and for anticlusters, which agrees with the exactly solvable model of the nuclear liquid-gas PT [11] and contradicts to the FDM [7, 8]. Thus, from a four parametric fit of the LDF we found that \( k_{\text{min}} = 2 \) and \( \tau = 1.806(8) \) both for clusters and for anticlusters.

**Figure 2.** The Fisher exponent \( \tau \) for several values of \( k_{\text{min}} \) and for a few values of \( \beta \) found by the 4-parametric fit of the LDM formula. All these results refer to the cut-off \( L_{\text{cut}} = 0.2 \).
Figure 3. Reduced chemical potential (left panel) and reduced surface tension (right panel) as function of $\beta$ obtained for $L_{\text{cut}} = 0.2$. The curves shown in the right panel represent equation (4).

For fixed values of $k_{\min}$ and $\tau$ we performed a three parametric fit of the (anti)cluster size distributions to define $C_A$, $\mu_A$ and $\sigma_A$ with high precision. The typical value of $\chi^2/\text{dof} \approx 1$ was obtained for any $\beta$, which signals about high quality of the data description. The $\beta$-dependences of $\mu_A$ and $\sigma_A$ are shown on figure 3 for $L_{\text{cut}} = 0.2$. For $L_{\text{cut}} = 0.1$ the results are similar.

4. Order parameters of the deconfinement phase transition

From figure 3 it is seen that the behavior of the reduced surface tension coefficient drastically changes when $\beta$ exceeds the critical value $\beta_c = 2.52$. Indeed, for $\beta \leq \beta_c$ this quantity is constant and it is identical for clusters and anticlusters, while $\sigma_{\text{acl}}$ monotonically decreases and $\sigma_{\text{cl}}$ monotonically increases with $\beta$ for $\beta > \beta_c$. The qualitatively different behavior of $\sigma_{\text{cl}}$ and $\sigma_{\text{acl}}$ allows us to treat them as an order parameter of the deconfinement PT in SU(2) GD. On the other hand, the average value of Polyakov loop $\langle L(\vec{x}) \rangle$ is traditionally considered as an order parameter of the deconfinement PT in SU(2) GD. In [9] we demonstrated that $\langle L(\vec{x}) \rangle$ is mainly defined by the largest (anti)cluster droplets whose average size (see figure 4) is given by

$$\text{max} K_A = \sum_{k=1} k^{1+\tau} n_A(k) / \sum_{k=1} k^{\tau} n_A(k). \quad (3)$$

The $\beta$ dependence of $\sigma_A$ and $\text{max} K_A$ in the right hand side vicinity of $\beta_c$ is parametrized as

$$\sigma_A(\beta) = \sigma_A(\beta_c) \pm d_A (\beta - \beta_c)^{B_A}, \quad (4)$$

$$\text{max} K_A(\beta) = \text{max} K_A(\beta_c) \pm a_A (\beta - \beta_c)^{b_A} \quad (5)$$

where the signs “+” and “-” correspond to $A=\text{acl}$ and $A=\text{cl}$, respectively, and $b_A$ and $B_A$ are the critical exponents, whereas $d_A$ and $a_A$ are the normalization factors. Values of these parameters found from the fit are shown in table 1. It is remarkable that the found exponents $b_A$ are close to the critical exponent $\beta_{\text{Ising}} = 0.3265 \pm 0.0001$ of the 3-dimensional Ising model [12] and to the critical exponent $\beta_{\text{liquids}} = 0.335 \pm 0.015$ of simple liquids [13].

5. Conclusions

In this contribution we present a novel approach to study the deconfinement PT in the SU(2) GD in terms of the geometrical clusters composed of the Polyakov loops of the same sign. We justify the separation of (anti)clusters into “liquid” droplet and “gas” of smaller fragments and investigate their physical properties. We also explain the deconfinement PT as a special kind of the liquid-gas transition between two types of liquid whose behavior is drastically different in the region of broken global $Z(2)$ symmetry. Above PT the cluster liquid droplet evaporates, whereas
Table 1. The fit parameters according to equation (4) and equation (5)

| $L_{cut}$ | Type | $d_A$ | $B_A$ | $\chi^2$/dof | $a_A$ | $b_A$ | $\chi^2$/dof |
|-----------|------|-------|-------|--------------|-------|-------|--------------|
| 0.1       | Cl   | 0.485(14) | 0.2920(12) | 1.43 /4 | 3056(246) | 0.2964(284) | 16.32 /4 |
| 0.1       | aCl  | 2.059(28) | 0.4129(77) | 1.68 /4 | 2129(160) | 0.3315(269) | 8.94 /4 |
| 0.2       | Cl   | 0.2796(118) | 0.2891(16) | 1.11/4 | 4953(443) | 0.3359(289) | 12.3/3 |
| 0.2       | aCl  | 1.344(33) | 0.4483(21) | 0.66/2 | 2462(88)  | 0.3750(129) | 2.068/4 |

Figure 4. Dependence of the mean size of the maximal (anti)cluster found for $L_{cut} = 0.1$ (left panel) and for $L_{cut} = 0.2$ (right panel). The curves represent equation (5).

The anticluster liquid droplet condensates the accompanying gas of anticlusters. A successful application of the LDF to the description of the size distributions of all gaseous (anti)clusters excluding the monomers is the main result of this study. This approach allowed us to determine the $\beta$-dependences of the reduced chemical potential and the reduced surface tension coefficient. While in the symmetric phase these quantities are identical for fragments of both kinds, their behavior is drastically different in the deconfined phase. Another important finding of this study is a high precision determination of the Fisher topological constant $\tau = 1.806 \pm 0.008$ which is the same both for clusters and for anticlusters. This result is in line with the exactly solvable model of the nuclear liquid-gas PT [11] and disproves the FDM prediction that $\tau > 2$ [7, 8]. We showed that the reduced surface tension coefficient and the mean size of the largest (anti)cluster can be used as the new order parameters of deconfinement PT in SU(2) GD.

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