Quantum speedup for multi-qubit open systems

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Quantum speed limit (QSL) time captures the intrinsic minimal time interval for a quantum system evolving from an initial state to a target state. In single qubit open systems, it was found that the memory (non-Markovian) effect of environment plays an essential role in accelerating quantum evolution. In this work, we investigate the QSL time for multi-qubit open systems. We find that for certain class of states, the memory effect still acts as the indispensable requirement for speeding up quantum evolution, while for another class of states, speedup takes place even when the environment is of no memory. In particular, when the initial state is in product state |111⋯1⟩, there exists a sudden transition from no speedup to speedup in memoryless environment. On the other hand, we also display an evidence for the subtle connection between QSL time and entanglement that weak entanglement can accelerate quantum evolution even better.

I. INTRODUCTION

Quantum speed limit (QSL) time [1, 2], the intrinsic minimal time interval for a quantum system evolving from an initial state to a target state, is of crucial importance in the fields of quantum computation [3], quantum control [4–8], quantum metrology [9, 10], and non-equilibrium thermodynamics [11]. Recent decades have witnessed a great deal of research on QSL time both in closed [12–23] and open systems [24–29]. In particular, QSL time for an arbitrarily driven open system, governed by the time-dependent master equation $\dot{\rho}_t = L_\text{t} \rho_t$ ($L_\text{t}$ a superoperator), was presented by Deffner and Lutz [27]

$$\tau_{\text{QSL}} = \sin^2[B(\rho, \rho_t)] \max \left\{ \frac{1}{E_1^\text{r}}, \frac{1}{E_2^\text{r}}, \frac{1}{E_\infty^\text{r}} \right\}, \quad (1)$$

where $B(\rho, \rho_t) = \arccos(\sqrt{\langle \phi | \rho_t | \phi \rangle})$ denotes the Bures angle between the initial state $\rho = |\phi \rangle \langle \phi |$ and the target state $\rho_t$, and $E_\text{r}^\text{p}$ represents the average of $||L_t \rho_t||_p$ ($||\cdot||_p$ is the Schatten p norm) over actual driving time duration $\tau$, i.e., $E_\text{r}^\text{p} = (1/\tau) \int_0^\tau ||L_t \rho_t||_p dt$. It is found that memory effect of environment, characterized by non-Markovianity [30, 31], will accelerate the quantum evolution in the damped Jaynes-Cummings model of a single qubit [27, 28]. Furthermore, the transition from no speedup ($\tau_{\text{QSL}}/\tau = 1$) to speedup ($\tau_{\text{QSL}}/\tau < 1$) of quantum evolution is just the critical point when the memoryless environment becomes of memory [27, 28].

Although the memory effect of environment plays a decisive role in the acceleration of quantum evolution in single qubit case, question may arise that will it still be true even in multi-qubit cases? In this paper, we demonstrate that for a class of multi-qubit states, the answer to above question is correct, but for another class of multi-qubit states, memoryless environment can also speedup the quantum evolution.

The paper is organized as follows: In Sec. II, we present the QSL time for typical two-qubit, three-qubit, and n-qubit states, respectively. Discussion on the role of entanglement in QSL time is performed in Sec. III. Finally, conclusions are drawn in Sec. IV.

II. QUANTUM LIMITS TO MULTI-QUBIT DYNAMICAL EVOLUTION

We consider N independent two-level atoms (open system) each locally coupling to a leaky vacuum cavity (environment). The dynamics of multi-qubit open system is fully determined by each pair of atom-cavity [32] with the following Hamiltonian [33]

$$H = \omega_0 \sigma_+ \sigma_- + \sum_k \omega_k a_k a_k^\dagger + i \sum_k g_k (a_k \sigma_- - a_k \sigma_+), \quad (2)$$

where $\omega_0$ is the resonant transition frequency of the atom between the excited state $|1\rangle$ and the ground state $|0\rangle$, $\sigma_\pm$ are the Pauli raising and lowering operators. $\omega_k$ and $a_k (a_k^\dagger)$ denote the frequency and the annihilation(creation) operators of the $k$th mode of the cavity with $g_k$ the corresponding real coupling constant. The master equation for the reduced density matrix of the atom is given by $\dot{\rho}_t = L_\text{t} \rho_t$ with

$$L_\text{t} \rho_t = i\delta_t [\sigma_+ \sigma_- \rho_t] + \gamma_t (\sigma_+ \sigma_- \rho_t + \rho_t \sigma_+ \sigma_- - 2 \sigma_- \rho_t \sigma_+), \quad (3)$$

where $\delta_t = \text{Im}(\epsilon_t/c_t)$ and $\gamma_t = \text{Re}(\epsilon_t/c_t)$ are time-dependent Lamb shift and decay rate respectively, and $c_t$ is the decoherence function relying on the particular structure of cavity reservoirs [33]. The reduced density matrix of the atom with an initial state $\rho = (\rho_{mn})$ takes the form

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FIG. 1: (Color online) Quantum speed limit time ratio $\tau_{QSL}/\tau$ of two-qubit open system under memoryless environment as a function of $P_\tau$ with initial states $|\Psi_1\rangle = |01\rangle + \sqrt{1-\alpha^2}|10\rangle$ (black dotted line), $|\Psi_2(\alpha = 1)\rangle = |11\rangle$ (red solid curve), and $|\Psi_2(\alpha = 1/\sqrt{2})\rangle = (|11\rangle + |00\rangle)/\sqrt{2}$ (blue dashed curve), respectively. The red circle indicates the maximal speedup of quantum evolution for state $|\Psi_2(\alpha = 1)\rangle = |11\rangle$.

As the exact form of QSL time for a general pure initial state is cumbersome, we only consider two typical Bell-type initial states respectively, i.e., $|\Psi_1\rangle = |01\rangle + \sqrt{1-\alpha^2}|10\rangle$ and $|\Psi_2\rangle = |11\rangle + \sqrt{1-\alpha^2}|00\rangle$ with $\alpha \in [0, 1]$. According to the definition of QSL time in Eq. (1), we have

$$\rho_t = \begin{bmatrix} \rho_{11}|c_t|^2 & \rho_{10}|c_t| \\ \rho_{01}|c_t| & 1 - \rho_{11}|c_t|^2 \end{bmatrix},$$

where the excited state population $|c_t|^2$ is denoted by $P_t$ in the following.

### A. Two-qubit cases

In order to illustrate this phenomenon clearly, the QSL time ratio $\tau_{QSL}/\tau$ of initial state $|\Psi_1\rangle$ (black dotted line) under memoryless environment (the population $P_t$ monotonically decreases from 1 to the target $P_\tau$) was depicted in Fig. 1, where $\tau_{QSL}/\tau \equiv 1$.

However, a complex but interesting phenomenon appears for initial state $|\Psi_2\rangle$: quantum speedup can also take place even when the environment is of no memory. For instances, $\tau_{QSL}/\tau$ versus $P_\tau$ of initial states $|\Psi_2(\alpha = 1)\rangle = |11\rangle$ (red solid curve) and $|\Psi_2(\alpha = 1/\sqrt{2})\rangle = (|11\rangle + |00\rangle)/\sqrt{2}$ (blue dashed curve) are depicted respectively in Fig. 1, where $\tau_{QSL}/\tau \leq 1$ clearly illustrates the intrinsic acceleration of quantum evolution under memoryless environment. Especially, there exists a sudden change of no speedup to speedup with the critical point $P_\tau = 1/2$ for initial state $|\Psi_2(\alpha = 1)\rangle = |11\rangle$. To explain this phenomenon, we trace back to $||L_t\rho_t||_\infty$ defined in Eq. (1) with the following expression

$$||L_t\rho_t||_\infty = \begin{cases} -P_t(2\alpha^2P_t - \alpha^2 + \alpha), & P_t \geq \frac{1}{2}, \\ P_t(2\alpha^2P_t - \alpha^2 - \alpha), & P_t < \frac{1}{2}. \end{cases}$$

where we have employed the condition of memoryless environment $\dot{P}_t = -P_t$. Therefore, the QSL time ratio $\tau_{QSL}/\tau$ of Eq. (6) can be conveniently calculated as

$$\frac{\tau_{QSL}}{\tau} = \begin{cases} \frac{\alpha(1+P_\tau^2)}{1+P_\tau^2}, & P_\tau \geq \frac{1}{2}, \\ \frac{2\alpha(1-P_\tau^2)}{2(1-P_\tau^2)(1-\alpha P_\tau)}+1, & P_\tau < \frac{1}{2}. \end{cases}$$

One may check that $\tau_{QSL}/\tau \leq 1$ is always satisfied. In particular, when the initial state is in $|\Psi_2(\alpha = 1)\rangle = |11\rangle$, the QSL time ratio yields

$$\frac{\tau_{QSL}}{\tau} = \begin{cases} 1, & P_\tau \geq \frac{1}{2}, \\ \frac{2(1-P_\tau^2)}{2(1-P_\tau^2)+1}, & P_\tau < \frac{1}{2}. \end{cases}$$

The sudden change of quantum evolution from no speedup to speedup is therefore justified.

### B. Three-qubit cases

In this subsection, we also consider two typical three-qubit states, i.e., W type state $|\Psi_3\rangle = |001\rangle + |010\rangle + \sqrt{1-\alpha^2-\beta^2}|100\rangle$ and GHZ type state $|\Psi_4\rangle = |111\rangle + \sqrt{1-\alpha^2}|000\rangle$. According to Eq. (1), the expressions of QSL time ratio are obtained

$$\frac{\tau_{QSL}}{\tau} = \begin{cases} \frac{1-P_\tau}{1-P_\tau^2}, & P_\tau \geq \frac{1}{2}, \\ \frac{2(1-P_\tau^2)}{2(1-P_\tau^2)+1}, & P_\tau < \frac{1}{2}. \end{cases}$$

for state $|\Psi_3\rangle$ and
\[ \frac{\tau_{QSL}}{\tau} = \frac{\alpha + \alpha(1 - \alpha^2)P_n(3 - 2P_n^2 - 3P_n) + \alpha(2\alpha^2 - 1)P_n^3}{\int_0^\tau \max \left\{ \left| \dot{P}_t(\pm \frac{1}{2}X + 3\alpha P_t - \frac{1}{2}\alpha) \right| \right\} dt}, \]  
\text{(11)}

for state \( |\Psi_4\rangle \), where \( X = \text{Sqrt}(4\alpha^2 P_n^4 - 8\alpha^2 P_n^3 + 8\alpha^2 P_n^2 - 5\alpha^2 P_n + P_n + \alpha^2) \).

Equation (10) bears a resemblance to the case of state \( |\Psi_1\rangle \), implying that quantum speedup of evolution only occurs in memory environment. As for Eq. (11), we consider a special case, i.e., \( |\Psi_4(\alpha = 1)\rangle = |111\rangle \) under the environment of no memory \( (\tau = -P) \). Therefore, the Eq. (11) can be simplified as:

\[ \frac{\tau_{QSL}}{\tau} = \begin{cases} 1, & P \geq \frac{1}{2}, \\ \frac{1 - P^2}{-3P + 3P - 2 - \frac{1}{2}}, & P < \frac{1}{2}, \end{cases} \]  
\text{(12)}

indicating that there also exists a sudden transition from no speedup to speedup of quantum evolution even the environment is of no memory.

C. N-qubit cases

In this subsection, we show that above phenomena are ubiquitous in n-qubit cases (n is an arbitrary positive integer). It is easy to check that if the N-qubit open system is initially prepared in state \( a_1 |100\cdots0\rangle + a_2 |01\cdots0\rangle + \cdots \), with \( \sum_{j=1}^N \alpha_j^2 = 1 \), the QSL ratio is exactly the same as Eqs. (5) and (10). Therefore, the memory effect of environment becomes the essential condition for speeding up the quantum evolution.

However, if the initial state is in \( |11\cdots1\rangle \), the QSL ratio is given by

\[ \frac{\tau_{QSL}}{\tau} = \begin{cases} 1, & P \geq \frac{1}{2}, \\ \frac{1 - P_{n+1}^2}{-3P + 3P_{n+1}^2 - 2 - \frac{1}{2}}, & P < \frac{1}{2}, \end{cases} \]  
\text{(13)}

In particular when the environment is memoryless, i.e., \( \dot{P}_{n+1} = -P \), Eq. (13) reduces to

\[ \frac{\tau_{QSL}}{\tau} = \begin{cases} 1, & P \geq \frac{1}{2}, \\ \frac{1 - P_{n+1}^2}{(1 - P_n)^2 + \frac{1}{2}}, & P < \frac{1}{2}. \end{cases} \]  
\text{(14)}

Clearly, \( P = 1/2 \) is the critical point when the open system experiences the sudden transition from no speedup to speedup of quantum evolution under memoryless environment.

Equation (14) also implies that there exists a maximal acceleration condition for state \( |11\cdots1\rangle \) when \( P \rightarrow 0 \), and the corresponding minimal QSL time ratio is given by:

\[ \frac{\tau_{QSL}}{\tau} \big|_{\text{min}} = \frac{2^{n-1}}{2n-1}. \]  
\text{(15)}

Especially when \( n = 2 \), Eq. (15) reduces to 2/3, which is marked in Fig. 1 as the red circle.

D. Memory effect on QSL time

In this subsection, we intend to show that memory effect of environment is still an important element for quantum acceleration for multi-qubit open systems. The memory environment we consider here is characterized by the Lorentzian spectral distribution \( J(\omega) = \frac{1}{2\pi} \frac{\gamma}{(\gamma^2 - \omega^2)^{1/2}} \), where \( \gamma_0 \) is the Markovian decay rate and \( \gamma \) is the spectral width [33]. \( P_t \) is now written as [33]

\[ P_t = e^{-\gamma t} \left| \cosh\left(\frac{dt}{2}\right) + \frac{\lambda}{d} \sinh\left(\frac{dt}{2}\right) \right|^2, \]  
\text{(16)}

where \( d = \sqrt{2\gamma_0 \lambda - \lambda^2} \). In Fig. 2, we take two-qubit as an example and the QSL time \( \tau_{QSL} \) versus coupling strength \( \gamma_0/\omega_0 \) is plotted with three typical two-qubit states \( |\Psi_1\rangle \) (black dotted line), \( |\Psi_2(\alpha = 1)\rangle \) (red solid curve), and \( |\Psi_2(\alpha = 1/\sqrt{2})\rangle \) (blue dashed curve), respectively, with parameters \( \lambda = 50 \), \( \omega_0 = 1 \), and \( \tau = 1 \). According to Ref. [33], we know that \( \gamma_0 = \lambda/2 \) is the transient point from memoryless environment \( (\gamma_0 < \lambda/2) \) to one of memory \( (\gamma_0 > \lambda/2) \). As is clearly shown in Fig. 2, further acceleration will take place when the environment enters the memory region.
In closed composite systems, entanglement has been taken as an essential role in the speed up of quantum evolution \[34\text{–}37\]. In this subsection, we go a step further to the connection between entanglement and QSL time in bipartite open systems. The initial state we consider here is an arbitrary pure state

\[|\phi\rangle = \alpha_1|11\rangle + \alpha_2|10\rangle + \alpha_3|01\rangle + \alpha_4|00\rangle, \]  

(17)

with \(\sum_{j=1}^{4} \alpha_j^2 = 1\), which is generated by Monte Carlo method, and the related entanglement is characterized by concurrence \(C\) in Ref. [38], with \(C = 0\) for a disentangled state and \(C = 1\) for a maximally entangled state.

In Fig. 3, 20000 random pure states are generated and their QSL ratio \(\tau_{QSL}/\tau\) versus concurrence are marked by tiny blue dots. As is clearly displayed in Fig. 3, there exists a maximal value of \(\tau_{QSL}/\tau\) equal to 1 (no speed up), no matter which value concurrence adopts. In addition, we have found that the upper bound of \(\tau_{QSL}/\tau\) can always be reached by states \(|\Psi_1\rangle = \alpha|01\rangle + \sqrt{1-\alpha^2}|10\rangle\) (dark red dots in Fig. 3), which means that the concurrence has nothing to do with \(\tau_{QSL}/\tau\) in its upper bound.

However, the minimal values of \(\tau_{QSL}/\tau\) are lower bounded by a subset of states \(|\Psi_2\rangle = \alpha|11\rangle + \sqrt{1-\alpha^2}|00\rangle\) (light green dots in Fig. 3), which do strongly depend on concurrence: \(\tau_{QSL}/\tau\) is proportional to concurrence, implying that weak entanglement will cause faster acceleration of quantum evolution.

IV. CONCLUSION

In summary, we have explored the quantum speed limit time for multi-qubit open systems. For a certain class of initial states, we have demonstrated that the quantum evolution can also be accelerated in memoryless (Markovian) environment. Moreover, we have found that entanglement plays a subtle role in the speedup of quantum evolution: weak entanglement is better for speeding up quantum evolution under certain circumstances.

We have only treated non-correlated environments in this paper. It will also be of importance and interest to study the QSL time of multi-qubit systems with the presence of initial correlations among the subsystems of composite environments [39].

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