Black Hole Thermodynamics from String Theory

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ABSTRACT

In this note we consider a stringy description of black hole horizon. We start with a nonlinear sigma model defined on a two dimensional Euclidean surface with background Rindler metric. By solving the field equations, we show that to the leading order the Bekenstein-Hawking formula of black hole entropy can be produced. We also point out a relation between the present formalism and the 'tHooft formalism.
To construct a theory of quantum gravity in four dimensions is one of the most difficult and challenging subjects left in the modern theoretical physics since we have so far neither useful informations from experiments nor consistent quantum field theory. Under such a circumstance, it seems to be an orthodox attitude to attack concrete problems with logical conflicts and then learn the fundamental principle from which in order to construct a full-fledged theory. In the case of quantum gravity, as one of such unsolved problems, we have quantum black holes. In particular, it is widely known that there are at least three problems which remain to be clarified in quantum black hole, those are, the endpoint of Hawking radiation, the information loss paradox and the statistical origin of black hole entropy.

Recently, there have been some progresses on the last problem. Among them, the authors of Ref.[6] made an interesting observation that superstring theory might play an important role in deriving the Bekenstein-Hawking formula of the black hole entropy.

On the other hand, in previous works, 'tHooft has stressed that black holes are as fundamental as strings, so that the two pictures are really complementary. In fact, he has demonstrated that by properly taking account of a leading gravitational back-reaction of the black hole horizon, the gravitational shock wave, from hard particles, his S matrix which describes the dynamical properties of a black hole can be recast in the form of functional integral over the Nambu-Goto string action. Although his formalism has some weaknesses, it is extremely interesting from the physical viewpoint since quantum incoherence never be lost and all information of particles entering into a black hole is transmitted to outgoing particles owing to the Hawking radiation through the quantum fluctuations of the black hole horizon. As it is expected that superstring has many degrees of freedom and hairs associated with its many excited states, the 'tHooft formalism might also give us a clue to understanding of a huge entropy and quantum hairs of a black hole.

In this note, we shall simply assume that the dynamics of the event hori-
zon of a black hole can be described by the world sheet swept by a string in the Schwarzschild background, and then would like to discuss what physical consequences can be derived from this assumption. However, the Schwarzschild metric is rather complicated, so that we shall confine ourselves to the case of the Rindler spacetime. The case of the Schwarzschild metric will be reported in a separate paper. We will see that a nonlinear sigma action leads to the well-known Bekenstein-Hawking formula of black hole entropy, $S = \frac{1}{4G} A_H^{1/10}$, within the lowest order of approximation. Moreover, one obtains a covariant operator algebra on the horizon which is a natural generalization to the 'tHooft one\textsuperscript{11}. Thus our stringy approach to black hole physics might be fruitful in both black hole thermodynamics and 'tHooft formalism.

In relation to our stringy approach, some people might wonder that the black hole horizon never be composed of a string since a free falling observer crossing the horizon encounters nothing unusual. However, this apparent contradiction would be overcome by the principle of ”black hole complementarity” which has recently been advocated in Ref.[12]. This principle says that the reference frame of an asymptotic observer and that of a free falling observer approaching the horizon of a black hole are very different and the above-mentioned contradiction can be traced to unsubstantiated assumptions about physics at or beyond the Planck scale.

As an effective action describing the dynamical properties of the black hole horizon, we start with a classical action of a nonlinear sigma model which is given by

$$S_E = -\frac{T}{2} \int d^2 \sigma \sqrt{h} \alpha^\beta \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X),$$

where $T$ is a string tension having dimensions of mass squared. $h_{\alpha\beta}(\tau, \sigma)$ denotes the two dimensional world-sheet metric which has a Euclidean signature, and $h = det h_{\alpha\beta}$. $X^\mu(\tau, \sigma)$ maps the string into a four dimensional spacetime, and then
$g_{\mu\nu}(X)$ can be identified as the background spacetime metric in which the string is propagating. Note that $\alpha, \beta$ takes values 0, 1 and $\mu, \nu$ does values 0, 1, 2, 3.

The classical field equations give us that

$$0 = T_{\alpha\beta} = -\frac{2}{T} \frac{1}{\sqrt{h}} \frac{\delta S_E}{\delta h_{\alpha\beta}},$$

$$= \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) - \frac{1}{2} h_{\alpha\beta} h^{\rho\sigma} \partial_\rho X^\mu \partial_\sigma X^\nu g_{\mu\nu}(X), \quad (2)$$

$$0 = \partial_\alpha (\sqrt{h} h^{\alpha\beta} g_{\mu\nu} \partial_\beta X^\nu) - \frac{1}{2} \sqrt{h} h^{\alpha\beta} \partial_\alpha X^\rho \partial_\beta X^\sigma \partial_\mu g_{\rho\sigma}. \quad (3)$$

In this note we consider the case that the background spacetime metric $g_{\mu\nu}(X)$ takes a form of the Euclidean Rindler metric

$$ds^2 = g_{\mu\nu} dX^\mu dX^\nu = +g^2 z^2 dt^2 + dx^2 + dy^2 + dz^2, \quad (4)$$

where $g$ is given by $g = \frac{1}{4M}$. This Rindler metric can be obtained in the large mass limit from the Schwarzschild black hole metric. Here it is important to notice that we have performed the Wick rotation with respect to the time component since now we would like to discuss the thermodynamic properties of the Rindler spacetime when we assume that the dynamics of the event horizon is controlled by a Euclidean string.

Now one can easily solve Eq.(2) as follows

$$h_{\alpha\beta} = G(\tau, \sigma) \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X), \quad (5)$$

where $G(\tau, \sigma)$ denotes the Liouville mode. Next we fix the gauge symmetries which
are the two dimensional diffeomorphisms and the Weyl rescaling by

\[ x(\tau, \sigma) = \tau, \quad y(\tau, \sigma) = \sigma, \quad G(\tau, \sigma) = 1. \]  

(6)

At this stage, let us impose an "axial" symmetry

\[ r(\tau, \sigma) = r(\tau), \quad t(\tau, \sigma) = t(\tau). \]

(7)

From Eq.s (5), (6) and (7), the world sheet metric \( h_{\alpha\beta} \) takes the form

\[ h_{\alpha\beta} = \begin{pmatrix} g^2 z^2 \dot{t}^2 + \dot{z}^2 + 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

(8)

where the dot denotes a derivative with respect to \( \tau \). And the remaining field equations (3) become

\[ \partial_\tau (\frac{z^2 \dot{t}}{\sqrt{h}}) = 0, \]

(9)

\[ \partial_\tau h = 0, \]

(10)

\[ \partial_\tau (\frac{\dot{z}}{\sqrt{h}}) - \frac{1}{\sqrt{h}} g^2 z \dot{t}^2 = 0, \]

(11)

where

\[ h = g^2 z^2 \dot{t}^2 + \dot{z}^2 + 1. \]

(12)

Now it is straightforward to solve the above field equations. We have two kinds of solutions. One solution is a trivial one given by

\[ z = \dot{z} = \ddot{z} = 0, \quad t(\tau) = \text{arbitrary}. \]

(13)

which corresponds to a world-sheet surface of the Euclidean string just lying on the black hole horizon. The next solution is the solution of "world sheet instanton"
described by

\[ z(\tau) = \sqrt{c_2(\tau - \tau_0)^2 + \frac{g^2 c_1^2}{c_2}}, \]

\[ t - t_0 = \frac{1}{g} \tan^{-1} \left( \frac{c_2}{g c_1} (\tau - \tau_0) \right), \]

(14)

where \( c_1, c_2, \tau_0, \) and \( t_0 \) are the integration constants, in other words, "the moduli parameters". To understand the physical meaning of this solution more vividly, it is convenient to rewrite \( z \) in terms of the time coordinate variable \( t \). From Eq.(14), we obtain

\[ z(t_E) = \frac{g c_1}{\sqrt{c_2}} \frac{1}{\cos g(t_E - t_{E0})}, \]

(15)

where we added the suffix \( E \) on \( t \) in order to indicate the Euclidean time clearly. Furthermore after Wick-rotating the time coordinate, we have in the real Lorentzian time \( t_L \)

\[ z(t_L) = \frac{g c_1}{\sqrt{c_2}} \frac{1}{\cosh g(t_L - t_{L0})}, \]

(16)

The explicit form of the solution (16) shows that this solution has a physical behavior of approaching the event horizon \( z = 0 \) asymptotically in the Lorentzian time coordinate \( t_L \). Incidentally, note that the solution (15) has a periodicity with respect to the Euclidean time component, \( \beta = \frac{2\pi}{g} \) whose inverse gives us nothing but the Hawking temperature \( T_H = \frac{1}{\beta} = \frac{g}{2\pi} = \frac{1}{8\pi M} \) of the Rindler spacetime \(^{13}\).

Next we would like to evaluate the black hole entropy to the leading order of approximation within the present formalism by a method developed by Gibbons and Hawking \(^{14}\). Before calculation, let us present a brief review of their method. The free energy \( F \) of a black hole in equilibrium with a radiation bath can be computed in terms of the Euclidean path integral

\[ e^{-\beta F} = \int_{\beta h} DX e^{-\frac{S}{\hbar}}, \]

(17)
where $S_E$ denotes the Euclidean action, and the path integral is performed under the boundary condition of being periodic in the Euclidean time with period $\beta \hbar$. Then the black hole thermodynamics can be recovered in the limit $\hbar \to 0$ by expanding $S_E$ around its saddle point. Thus evaluating the free energy $\beta$ to the leading term equals to substituting a classical solution into the Euclidean action.

In the model just considered, it is easy to calculate the free energy. To do so we shall consider the solution (14) since this solution gives us the thermal temperature whose situation should be contrasted to the case of the other solution (13). The result is

$$F = -\frac{1}{\beta} \sqrt{c_2 + 1} TA_H,$$

where $A_H = \int dx dy$ which corresponds to the area of the black hole horizon if we consider the Schwarzschild black hole. By the formula which gives us the entropy

$$S = \beta^2 \frac{\partial F}{\partial \beta},$$

one can show that the entropy is given by

$$S = \sqrt{c_2 + 1} TA_H. \tag{20}$$

Note that the black hole entropy is proportional to the horizon area. Moreover, by selecting the string tension

$$T = \frac{1}{4\sqrt{c_2 + 1}G}, \tag{21}$$

we arrive at the famous Bekenstein-Hawking entropy formula $^{1,10}$

$$S = \frac{1}{4G} A_H. \tag{22}$$

Next we shall consider the relation between our model and 'tHooft one $^{11,15}$. According to 'tHooft, some quantum fluctuations of the event horizon can be in-
duced by hard particles having a large amount of momenta. Thus let us introduce "vertex operator" in the original action (1)

\[
S_E = -\frac{T}{2} \int d^2\sigma \sqrt{h} \alpha^\alpha \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}(X) + \int d^2\sigma \sqrt{h} P^\mu X^\nu g_{\mu\nu}(X),
\]

(23)

where it is now supposed that the target spacetime has a Lorentzian signature. Taking a variation with respect to \( X^\mu \) gives us the equation of \( X^\mu \) as follows

\[
0 = T \frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} \alpha^\alpha \partial_\beta X_\mu) + P_\mu,
\]

(24)

where we have replaced \( g_{\mu\nu} \) with the flat metric \( \eta_{\mu\nu} \). This approximation would become good when the black hole mass is large compared to the Planck mass. Moreover, we have fixed the world sheet metric \( h_{\alpha\beta}(\tau, \sigma) \) to be the metric on \( S^2 \). Therefore we obtain

\[
T \Delta_{tr} X^\mu + P_\mu = 0,
\]

(25)

where

\[
\Delta_{tr} = \frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} \alpha^\alpha \partial_\beta).
\]

(26)

Now we assume the following commutation relations which are motivated by the 'tHooft work

\[
\left[ X^\mu(\sigma), P^\nu(\sigma') \right] = i\eta^{\mu\nu} \delta^{(2)}(\sigma - \sigma'),
\]

(27)

in other words,

\[
P_\mu(\sigma) = -i \frac{\delta}{\delta X^\mu(\sigma)}.
\]

(28)
From (25) and (27), we have

\[
\left[ X^\mu(\sigma), X^\nu(\sigma') \right] = \frac{i}{T} \eta^{\mu\nu} f(\sigma, \sigma'),
\]  

(29)

where the Green function \( f(\sigma, \sigma') \) is defined as

\[-\Delta_{tr} f(\sigma, \sigma') = \delta^{(2)}(\sigma - \sigma').
\]  

(30)

The commutation relations (29) are just the covariant generalization to the relations which ’tHooft have obtained in the context of shock wave geometry\(^\text{11}\). Eq.(29) strongly suggests that distances between adjacent points on the black hole horizon in real world would be quantized with units of the order Planck scale. Thus we have succeeded in deriving the Lorentz covariant versions of the ’tHooft commutation relations by assuming that the dynamics of the horizon of a black hole is controlled by a Euclidean string.

In conclusion, the main point of this article has been to demonstrate that a black hole dynamics, in particular, the black hole thermodynamics can be understood in terms of string theory. We have seen that this is the case, at least in a specific field theoretical model. In this model, we have made an approximation that the Schwarzschild black hole can be described by the Rindler spacetime under the condition of the large black hole mass.

It is interesting to compare our derivation of the black hole entropy with that of Ref.[6] and [9]. There is a clear difference in the calculation method. The authors in Ref.[6] and [9] evaluated the black hole entropy by first inducing the Einstein-Hilbert action with the surface term and the possible higher derivative terms from string theory or pregeometry, and then considering the geometries containing conical singularities. On the other hand, we have directly evaluated the black hole entropy by using the string action without inducing the gravitational action from it.
Notes added

During the preparation of this article, we noticed that there is a recent work where the black hole is described by the membrane theory\textsuperscript{16}.

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