Q-chains in the U(1)-gauged Friedberg-Lee-Sirlin model

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Abstract – We construct static axially symmetric multi-Q-ball configurations in the U(1)-gauged two-component Fridberg-Lee-Sirlin model in a flat spacetime. The solutions represent electromagnetically bounded chains of stationary spinning charged Q-balls placed along the axis of symmetry. We discuss the properties of these configurations and exhibit their domain of existence.

Introduction. – Q-balls represent spatially localized finite energy field configurations in flat space [1–3], they may exist in a model with complex scalar field possessing an unbroken, continuous global symmetry (for a review, see, e.g., [4–6]). The Q-balls carry a Noether charge associated with this symmetry, they are time-dependent non-topological solitons with a stationary oscillating phase. Configurations of this type may exist both in the models with a single complex scalar field and a suitable non-renormalizable self-interaction potential [3], and in a two-component model with a symmetry breaking potential [2].

The Friedberg-Lee-Sirlin (FLS) model [2] is a simple two-component renormalizable scalar theory with fourth-order potential and a minimal interaction term. Notably, stable non-topological soliton solutions of that model also exist in the limiting case of vanishing potential [7,8] as the vacuum expectation value of the real component remains finite.

Evidently, one can generalize the model considering gauged Q-balls with local U(1) symmetry [9–16]. In the U(1)-gauged version of the FLS model [10], the Noether charge is associated with the harmonic time dependence of the complex field. Further, the U(1)-gauged Friedberg-Lee-Sirlin-Maxwell model can be considered as a truncated version of the Witten’s model of superconducting cosmic strings with U(1) × U(1) local gauge invariance [8,17,18].

It was pointed out that there are two families of the spinning Q-balls with positive and negative parity, the corresponding solutions are symmetric or antisymmetric with respect to reflections in the xy-plane [19]. Apart from the fundamental spherically symmetric Q-balls, there are both radially and angularly excited solutions [19–21]. The radially excited Q-balls are still spherically symmetric, however the scalar field possesses one or more nodes in the radial direction. Similar radially excited gauged Q-balls also exist in the U(1)-gauged model [16]. The angularly excited solutions can be parity even or parity odd.

The angularly excited axially symmetric Q-balls with non-zero angular momentum possess an additional azimuthal phase factor of the spinning field [6,19,20]. In the U(1)-gauged theory such spinning Q-balls may induce a toroidal magnetic field [22,23]. Solutions of that type can be viewed as vortons, the finite energy localized spinning loops stabilized by rotation [6,17,24,25].

It is known that gauged Q-balls may exist only for relatively small values of the gauge coupling [9,10,14]. Further, they are expected to become unstable for large values of the Noether charge Q because of the repulsive electric Coulomb force [9]. A possibility that, to our best knowledge, has not been considered before, is that in the U(1)-gauged model the electric repulsion can be balanced by the scalar and magnetic interactions of the axially symmetric spinning solitons. This may open a way to construct a new type of solutions, which correspond to chains of gauged spinning Q-balls. In these axially symmetric equilibrium configurations a number of constituents are located symmetrically with respect to the origin along the symmetry axis, the repulsive electric interaction is balanced by the scalar and magnetic forces between the spinning gauged Q-balls providing zero net effect. Similar chain solutions are known to exist in various systems, both for gravitating and flat space solitons, e.g., for non-Abelian monopoles and dyons [26–36], Skyrmions [37,38], and boson stars [39].

A main objective of this letter, which extends our previous consideration of the parity-even axially symmetric
spinning gauged Q-balls in the Friedberg-Lee-Sirlin-Maxwell theory [22], is to examine this possibility. We show that, indeed, there are new families of multi-component axially symmetric solutions of the model, which represent chains of spinning gauged Q-balls, they possess both a non-zero electric charge and a magnetic field. We found such configurations with even number of \( k \) constituents on the symmetry axis numerically and determine their domains of existence.

The model. – We consider the four-dimensional \( U(1) \)-gauged two-component Friedberg-Lee-Sirlin-Maxwell model, which describes a coupled system of the real self-interacting scalar field \( \psi \) and a complex scalar field \( \phi \), minimally interacting with the Abelian gauge field \( A_\mu \).

The corresponding Lagrangian density is

\[
L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (\partial_\mu \psi)^2 + |D_\mu \phi|^2 - m^2 \psi^2 |\phi|^2 - U(\psi),
\]

where \( D_\mu = \partial_\mu + i g A_\mu \) denotes the covariant derivative. The electromagnetic field strength tensor is \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \), with electric components \( E_\mu = F_{0\mu} \) and magnetic components \( B_\mu = \varepsilon_{\rho\mu\nu} F^{\rho\nu} \), \( g \) denotes the gauge coupling constant and \( m \) is the scalar coupling constant, it defines the mass of the complex component \( \phi \). Without loss of generality we assume \( g \geq 0 \).

The symmetry-breaking potential of the real scalar field \( \psi \) is \( U(\psi) = \mu^2 (1 - \psi^2)^2 \), thus, \( \psi \to 1 \) in the vacuum and the local \( U(1) \) symmetry is broken in the interior of the Q-ball, where the gauge field \( A_\mu \) becomes massive. Evidently, system (1) represents a generalization of the Abelian Higgs model, the gauged Q-ball behaves like a superconductor [9] with the field component \( \psi \) playing the role of the order parameter.

On the other hand, model (1) can be considered as a reduced version of the \( U(1) \times U(1) \)-gauged model of superconducting strings [17]. Such a theory supports stationary vortex rings stabilized by charge, current and angular momentum, so-called vortons [24,25].

The model (1) is invariant under the local \( U(1) \) gauge transformations of the fields, the conserved Noether current is

\[
j_\mu = i (\phi D_\nu \phi^* - \phi^* D_\nu \phi),
\]

with the corresponding charge \( Q = \int d^3 x \ j^0 \). This current is a source in the Maxwell equation

\[
\partial^\nu F_{\mu\nu} = g j_\mu.
\]

Variation of the Lagrangian (1) with respect to the fields \( \psi \) and \( \phi \) yields the field equations

\[
\partial^\mu \partial_\mu \psi = -m^2 \psi |\phi|^2 + 2 \mu^2 \psi (1 - \psi^2),
\]

\[
D^\mu D_\mu \phi = -m^2 \psi^2 \phi.
\]

We are interested in stationary spinning axially-symmetric solutions of model (1). The corresponding parametrization of the scalar fields is

\[
\psi = X(r, \theta), \quad \phi = Y(r, \theta) e^{i(\omega t + n \varphi)},
\]

where \( \omega \) is the angular frequency of the spinning complex field \( \phi \), and \( n \in \mathbb{Z} \) is the azimuthal winding number. In the static gauge the electromagnetic potential can be written as

\[
A_\mu dx^\mu = A_0 (r, \theta) dt + A_\varphi (r, \theta) \sin \theta d\varphi.
\]

Substitution of the ansatz (5), (6) into the definition of the \( U(1) \) charge \( Q \) above gives \( Q = \int d^3 x \ (g A_0 + \omega) Y^2 \). The stationary spinning axially symmetric configurations possess angular momentum which is given by the \( T^0_\varphi \) component of the stress-energy tensor, \( J = \int d^3 x \ T^0_\varphi \). The angular momentum of the spinning gauged Q-ball is quantized in the units of the electric charge of the configuration, \( J = n Q \) [6].

The field equations resulting from the variation of the reduced action on the ansatz (5), (6) are

\[
\begin{align*}
\Delta r \phi + 2 \mu^2 (1 - X^2 - m^2 Y^2) X &= 0, \\
\Delta r \phi - \frac{1}{r^2} \left( g A_\theta (\cos \theta) \right)^2 + (g A_\theta + \omega)^2 - m^2 X^2 + 2 g \omega Y \phi &= 0, \\
\Delta r \phi - \frac{1}{r^2 \sin^2 \theta} - 2 g^2 Y^2 A_\theta &= \frac{2 n Q}{\sin^2 \theta} Y^2, \\
\Delta r \phi &= 2 g^2 Y^2 A_0 = 2 g \omega Y^2,
\end{align*}
\]

where \( \Delta r = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cos \theta}{r \sin \theta} \frac{\partial}{\partial \theta} + \frac{\cos^2 \theta}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \) is the angular part of the Laplace operator.

Note that, in the static gauge both the electric and magnetic components of the gauge potential are set to vanish as \( r \to \infty \). Further, in the limiting case \( \omega^2 \approx m^2 \) the equations on the profile functions \( X(r, \theta), Y(r, \theta) \) (7) can be linearized. Then the asymptotic expansion of the system of eq. (7) suggests that, similar to the case of the spinning axially symmetric Q-balls [6,8,19–21], the angular part of the asymptotic solutions for the complex field function \( Y(r, \theta) \) is associated with the real spherical harmonics \( Y_{\ell n} (\theta, \varphi) \) which are proportional to the associated Legendre polynomials \( P^\ell_n (\cos \theta) \). Similarly, the angular part of the real-field function \( X(r, \theta) \) is proportional to the Legendre polynomials \( P_n (\cos \theta) \).

Thus, the field components of the axially symmetric spinning solutions of the \( U(1) \)-gauged FLS model (1) may be either symmetric with respect to reflections in the equatorial plane, \( \theta \to \pi - \theta \), or antisymmetric. However, since the real component is approaching a non-zero vacuum value \( X(r, \theta) = 1 \) as \( r \to \infty \), the field \( \psi \) has to be parity even. For example, the spherical harmonic \( Y^1_{\ell n} (\theta, \varphi) \) yields a parity-even component of the complex field, while the harmonic \( Y^3_{\ell n} (\theta, \varphi) \) corresponds to the parity-odd component. The simplest spherically symmetric solution corresponds to the spherical harmonic \( Y^0_{\ell n} \) for the complex component, and to the Legendre polynomial \( P_0 \), for the real component.

Recently, the existence of parity-even spinning gauged Q-balls was demonstrated [22]. In this paper, we present
strong numerical arguments that new mixed-parity angularly excited solutions of the nonlinear system of field equations (7) exist, for those the function \( Y(r, \theta) \) is parity odd while the real component \( X(r, \theta) \) is parity even.

The system (7) represents a set of four coupled elliptic partial differential equations with mixed derivatives. We solved this system numerically, by taking into account appropriate boundary conditions. As usual, they follow from the condition of regularity of the fields at the origin and the symmetry axis, the requirements of symmetry, and the finiteness of the energy of the system. In particular we have to take into account that in the static gauge the potential \( A_3 \), vanishing at the spatial boundary, as the real field \( X \) approaches its vacuum value. Accordingly, we impose \( X |_{r=\infty} = 1, Y |_{r=\infty} = 0, A_0 |_{r=\infty} = 0, A_\phi |_{r=\infty} = 0 \). Further, the restriction of regularity at the origin gives \( \partial_r X |_{r=0} = 0, Y |_{r=0} = 0, \partial_r A_0 |_{r=0} = 0, \partial_r A_\phi |_{r=0} = 0 \). Note that for spherically symmetric Q-ball the scalar field component \( Y(r, \theta) \) has a finite value at the origin, the corresponding boundary condition is \( \partial_r Y |_{r=0, \theta} = 0 \).

On the symmetry axis we impose \( \partial_\theta X |_{\theta=0} = 0, Y |_{\theta=0} = 0, \partial_\theta A_0 |_{\theta=0} = 0, A_\phi |_{\theta=0} = 0 \), and on the equatorial plane, for parity-odd solutions, we set \( X |_{\theta=\pi/2} = 1, Y |_{\theta=\pi/2} = 0, \partial_\theta A_0 |_{\theta=\pi/2} = 0, \partial_\theta A_\phi |_{\theta=\pi/2} = 0 \).

We have solved the boundary value problem for the coupled system of nonlinear partial differential equations (7) with boundary conditions above using a six-order finite difference scheme. The numerical calculations are mainly performed on an equidistant grid in spherical coordinates \( r \) and \( \theta \). The underlying linear system is solved with the Intel MKL PARDISO sparse direct solver [40] using the Newton-Raphson method. Calculations are performed with the packages FIDISOL/CADSOL [41] and the CESDSOL\(^1\) library. Estimated numerical errors are of order \( 10^{-3} \). For convenience, in our numerical calculations we fix the value of the mass parameter \( \mu^2 = 0.25 \), set the coupling constant \( m = 1 \) and restrict our consideration to angularly excited spinning gauged \( n = 1 \) Q-balls.

**Numerical results.** – Simplest spherically symmetric solutions of the gauged FLS model were constructed a long time ago [10]. These \( n = 0 \) gauged Q-balls may exist in some restricted domain of values of the parameters of the system. The repulsive electrostatic Coulomb interaction reduces the allowed range of values of the angular frequency of the spinning gauged Q-ball, the minimal value \( \omega_{\text{min}} \) is increasing as the gauge coupling \( g \) increases. As in the decoupled limit, the values of the scaled frequency \( \omega \) are also bounded from above by the mass of the complex scalar field, \( \omega \leq 1 \). However, the \( U(1) \)-gauged Q-balls may exist as a localized finite mass field configuration even in the limit \( \omega = 1 \) [11]. Further, radially excited \( U(1) \)-gauged Q-balls we studied recently in [15], the angularly excited parity-even solutions of the Friedberg-Lee-Sirlin-Maxwell model were considered in out previous work [22].

Here we study new type of axially symmetric parity-odd solutions of the model (1), which can be considered as composite configurations, the chains of spinning gauged Q-balls, located symmetrically with respect to the origin along the symmetry axis. These solutions can be classified by the winding number \( n \) and the number of constituents \( k \). A simple non-trivial configuration of that type represents a \( k = 2 \) pair of \( n = 1 \) Q-balls spinning in opposite phases, as shown in fig. 1, second row. The pair is stabilized by the magnetic field generated by the circular current. Our numerical calculations show that there is no similar axially symmetric \( n = 0 \) solution\(^2\).

A few chain solutions of that type on their fundamental branch for a given value of the mass parameter \( \mu^2 = 0.25 \) and angular frequency \( \omega = 0.90 \) are exhibited in figs. 1, 2. The plots in fig. 1 represent the scalar field functions \( X(r, \theta), Y(r, \theta) \) of the \( k = 1, 2, 4, 6 \) chains. For clarity, we

\[^{1}\text{Complex Equations – Simple Domain partial differential equations SOLver is a } C++ \text{ package being developed by one of us (IP).}

\[^{2}\text{Notably, the chains of solitons with zero angular momentum, stationary spinning on the symmetry axis, may exist in curved spacetime [39].} \]
have chosen polar coordinates $\rho = r \sin \theta$ and $z = r \cos \theta$ in the figures. These Q-chains possess $k$ constituents, as nicely seen by the number of peaks of the distributions of the charge density and the magnitude of the magnetic field, as seen in fig. 2.

Both for parity-even and for parity-odd axially symmetric solutions of the gauged model (1) the angular frequency $\omega$ is bounded from above. The gauged spinning solutions exist within a frequency interval $\omega_{\text{min}} \leq \omega \leq \omega_{\text{max}}$, the lower critical value of the frequency $\omega_{\text{min}}$ depends on the electromagnetic coupling.

The spinning gauged Q-balls arise as perturbative excitations as the angular frequency is decreasing slightly below the mass threshold $\omega_{\text{max}} = 1$. The $k$-chains of spinning Q-balls form the first branch of solutions, the angular frequency is decreasing along this branch. The constituents with non-zero angular momentum possess both the electric charge and circular magnetic field, which is generated by the Noether current $j_\mu$ (2). The corresponding solenoidal magnetic field has a set of $k$ pronounced maxima equidistantly located on the symmetry axis, while the electric charge of the constituents is pushed outwards, as seen in fig. 2.

The electromagnetic energy of the spinning solitons remains relatively small on the fundamental branch, where the electric Coulomb force is balanced mainly by the scalar interaction while the magnetic field is comparatively weak.

As for the parity-even gauged Q-balls, we refer to that branch as the “electric” one [22]. Since the local $U(1)$ symmetry becomes broken in the interior of the constituents of a chain, the electric branch corresponds to the “superconductive” phase.

Considering the frequency dependence of $n = 1$ $k$-chains of gauged Q-balls, we found that it is qualitatively the same as that for the parity-even spinning solutions, we discussed in [22]. Both the size and the electric charge of the constituents of the chain increases as the angular frequency is decreasing from the mass threshold $\omega_{\text{max}} = 1$. Hence, both the current $j_\mu$ and associated magnetic field $B$ become stronger. For some critical value of the frequency $\omega_{\text{cri}}$ the value of the real component of the configuration approaches zero, then the electromagnetic field becomes massless on some set of $k$ circular domains in space around the symmetry axis. As a result, the symmetry is restored in these regions and the energy of the magnetic field becomes higher than the electrostatic energy of the spinning Q-ball. In other words, the second, magnetic branch is formed [22]. The energy of the configuration, as well as the change $Q$ and the angular momentum $J$ are rapidly increasing with the angular frequency, the magnetic branch extends forward, as shown in fig. 3.

Further increase of the frequency leads to expansion of the domains of normal phase in a chain, where the real component becomes trivial, $\psi = 0$, and both the electromagnetic and complex scalar fields are massless. The critical value of the frequency $\omega_{\text{cri}}$ depends on the gauge coupling $g$, see fig. 3, it increases with $g$, as the electric and magnetic branches become shorter. It also increases for larger chains with higher number of constituents $k$ as the gauge coupling $g$ remains fixed, see fig. 3, where we exhibit the total energy $E$ of the gauged $n = 1$ $k$-chains as functions of the angular frequency $\omega$ at $g = 0.07$ and $\mu^2 = 0.25$. We observe that the separation between the constituents of the chain on the magnetic branch decreases as $\omega$ increases, further, the chain becomes energetically unstable.

Note that, both the for parity-even and parity-odd solutions, the magnetic branch may exist only for a non-zero values of the mass parameter $\mu$, it extends forward almost linearly with $\omega$. When $\mu$ becomes higher, the critical value of the frequency $\omega_{\text{cri}}$ is increasing [22].

Finally, we note that, similar to the case of the usual Q-balls, both the energy and the charge of the gauged $k$-chains are minimal at some critical value of the frequency $\omega_{\text{cri}} > \omega_{\text{min}}$. This is an indication of a cusp in the curve of $E(Q)$ dependence, i.e., the existence of different solutions with the same value of charge $Q$. Hence, one can expect the more energetic gauged $k$-chains on the upper magnetic branch are unstable. More general, the multi-tori structure of the $k$-chain solutions, which represent saddle

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3Note that for the gauged spinning Q-balls a multi-cusp pattern is generally observed [16,22].
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Fig. 3: Axially symmetric $n = 1$ chains of gauged Q-balls: the total energy $E$ is shown as a function of the angular frequency $\omega$ for the parity-even ($k = 1$) solution, the parity-odd pair of Q-balls ($k = 2$) and for the $k = 4$ and $k = 6$ chains at $g = 0.07$ (upper plot) and for some set of values of the gauge coupling $g$ (bottom plot). The mass parameter is fixed as $\mu^2 = 0.25$.

points of the energy functional, is energetically unfavorable as compared to the single torus-like structure of the parity-even solutions.

Conclusions. – The main purpose of this work was to present a new type of axially symmetric solutions of the $U(1)$ gauged FLS model, which represent electromagnetically bounded chains of even number of stationary spinning charged Q-balls with non-zero angular momentum coupled with a solenoidal magnetic field.

The chains emerge from the perturbative fluctuations at a maximal value of the angular frequency, given by the mass of the real scalar field. They then ascend along their fundamental electric branch, until a minimal value of the frequency is reached, which, for a fixed value of the mass parameter, is determined by the gauge coupling strength $g$. Then the second, magnetic branch is formed, it extends forward as the angular frequency is increasing. Along this branch, strong magnetic field of the vortex destroys the superconductive phase in some circular domains of space around the symmetry axis. On both branches the energy density of the $k$-chain solutions is a set of $k$ tori located symmetrically with respect to the equatorial plane.

We constructed $k$-chains of the gauged spinning Q-balls with even number of constituents. The building block of these solutions is the $\phi$-odd $n = 1$ configuration whose angular dependence is given by the spherical harmonic $Y^1_1(\theta, \varphi)$. This is a pair of spinning charged loops coupled to a toroidal magnetic field, which forms a vortex encircling the configuration. By analogy with similar chains of boson stars in the Einstein-Klein-Gordon theory [39], one can look for solutions with odd number of constituents, they would represent a deformation of the parity-even gauged spinning Q-balls considered recently in [22]. However, we were not able to find such solutions, it is possible that the odd chains may exist for higher values of the winding number $n$. We hope to address this problem in our future work.

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