PROJECTIVELY DEFORMABLE LEGENDRIAN SURFACES

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ABSTRACT. Consider an immersed Legendrian surface in the five dimensional complex projective space equipped with the standard homogeneous contact structure. We introduce a class of fourth order projective Legendrian deformation called $\Psi$-deformation, and give a differential geometric characterization of surfaces admitting maximum three parameter family of such deformations. Two explicit examples of maximally $\Psi$-deformable surfaces are constructed; the first one is given by a Legendrian map from $\mathbb{P}^2$ blown up at three distinct collinear points, which is an embedding away from the $-2$-curve and degenerates to a point along the $-2$-curve. The second one is a Legendrian embedding of the degree 6 del Pezzo surface, $\mathbb{P}^2$ blown up at three non-collinear points. In both cases, the Legendrian map is given by a system of cubics through the three points, which is a subsystem of the anti-canonical system.

CONTENTS

1. Introduction
2. Legendrian surface
   2.1. Structure equation
   2.2. Flat asymptotic 3-web
   2.3. Vanishing cubic differential $\Psi$
   2.4. Isothermally asymptotic
   2.5. Asymptotically ruled
3. Second order deformation
   3.1. Structure equation
4. $\Psi$-deformation
   4.1. Structure equation
   4.2. Surfaces with maximum $\infty^3$ $\Psi$-deformations
   4.3. $(\Psi, \chi)$-deformations
5. Examples
   5.1. Flat surface
   5.2. Tri-ruled surface
References

1. Introduction

Let $Z^{2m+1}$ be a complex manifold of odd dimension $2m + 1 \geq 3$. A contact structure on $Z$ is by definition a hyperplane field $\mathcal{H} \subset TZ$ such that it is locally defined by $\mathcal{H} = (\alpha)\perp$ for a 1-form $\alpha$ that satisfies the nondegeneracy condition $\alpha \wedge (d\alpha)^m \neq 0$. A manifold with a contact structure is called a contact manifold. Typical examples of contact manifolds are the homogeneous adjoint varieties of simple Lie algebras including the odd dimensional projective spaces $\mathbb{P}^{2m+1}$, [LM].

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There exist a distinguished class of subvarieties in a contact manifold. A Legendrian subvariety \( M^m \subset \mathbb{P}^{2m+1} \) is a \( \mathcal{H} \)-horizontal subvariety of maximum dimension \( m \). Typical examples of Legendrian subvarieties are the homogeneous sub-adjoint varieties, \([LM]\). In contrast to the case of real and smooth category, where both contact structures and Legendrian subvarieties are flexible and admit unobstructed local deformation, there are relatively small set of known Legendrian subvarieties in complex contact manifolds, even in the simplest odd dimensional projective spaces.

Bryant showed that every compact Riemann surface can be embedded in \( \mathbb{P}^3 \) as a Legendrian curve in relation to the twistorial description of minimal surfaces in 4-sphere, \([Br1]\). Landsberg and Manivel adopted the idea from \([Br1]\) and showed that a K3 surface blown up at certain twelve points can be embedded in \( \mathbb{P}^5 \) as a Legendrian surface via an explicit birational contactomorphism from the projective cotangent bundle \( \mathbb{P}(T^*\mathbb{P}^5) \) to \( \mathbb{P}^5 \). Buczynski established a general principle of hyperplane sections for Legendrian subvarieties in projective spaces, \([Bu2]\). Successive hyperplane sections of the known examples then gave many new smooth Legendrian subvarieties. Buczynski also showed that the algebraic completion of the special linear group \( SL_3\mathbb{C} \subset \mathbb{P}^{17} \) is a smooth, Fano Legendrian subvariety with Picard number 1, \([Bu1]\).

The purpose of the present paper is to propose a differential geometric study of Legendrian surfaces in the five dimensional projective space \( \mathbb{P}^5 \) from the perspective of projective Legendrian deformation. Let us explain the motive. In his general investigation of the deformation of a submanifold in a homogeneous space during the period 1916 -1920, Cartan considered the following problem of third order projective deformation of a surface in the projective space \( \mathbb{P}^5 \), which built upon the earlier work of Fubini, \([Ca]\) and the reference therein; Let \( x: \Sigma \hookrightarrow \mathbb{P}^5 \) be a surface. Let \( x': \Sigma \hookrightarrow \mathbb{P}^5 \) be a deformation of \( x \). \( x' \) is a third order deformation of \( x \) when there exists an application map \( g: \Sigma \rightarrow SL_4\mathbb{C} \) such that for each \( p \in \Sigma \), \( x \) and \( g(p) \circ x' \) agree up to order three at \( p \). Which surfaces in \( \mathbb{P}^5 \) admit a nontrivial(application map \( g \) is nonconstant) third order deformation?

Cartan showed that a generic surface is rigid and does not admit such deformations, but that there exist two special sets of surfaces with nondegenerate second fundamental form that admit maximum three parameter family of third order deformations.

The present work was inspired especially by the observation that a quartic Kummer surface is one of the maximally deformable surfaces, \([Fe]\). This led us to consider the analogous problem for Legendrian surfaces in \( \mathbb{P}^5 \), which may allow one to obtain the detailed structure equations for a set of Legendrian surfaces with special properties. The integration of the structure equation so obtained may suggest a method of construction for new examples. In particular, one may hope to find the Legendrian analogue of Kummer surfaces.

**Main results.**

1. Let \( x: M \hookrightarrow \mathbb{P}^5 \) be a Legendrian surface in \( \mathbb{P}^5 \) viewed as a homogeneous space of the symplectic group \( Sp_3\mathbb{C} \). Assuming the second fundamental form of \( x \) is nondegenerate, the moving frame method is employed to determine the basic local invariants of a Legendrian surface as a set of three symmetric differentials \( (\Phi, \Psi, \chi) \) of degree \((3, 3, 2)\) and order \((2, 4, 5)\), Proposition 2.22 and \(2.23\).

2. A class of fourth order deformation called \( \Psi \)-deformation is introduced. Given a Legendrian surface \( x: M \hookrightarrow \mathbb{P}^5 \), a deformation \( x': M \hookrightarrow \mathbb{P}^5 \) is a \( \Psi \)-deformation when the pair \( (\Phi', \Psi') \) for \( x' \) is isomorphic to that of \( x \) at each point of \( M \) up to motion by \( Sp_3\mathbb{C} \). This choice of deformation is justified later by the analogy with the aforementioned Cartan’s result that there exist two special sets of Legendrian surfaces called \( D_0 \)-surfaces and \( D \)-surfaces that admit maximum three parameter family of \( \Psi \)-deformations, Proposition 4.9.

3. The structure equations for the maximally \( \Psi \)-deformable surfaces are determined. It turns out that the local moduli space of \( D_0 \)-surfaces depends on 1 arbitrary function of one variable, whereas the local moduli space of \( D \)-surfaces is finite dimensional, Theorem 4.21.
4. Two global examples of $D_0$-surfaces are constructed explicitly. The first one is given by a Legendrian map from $\mathbb{P}^2$ blown up at three distinct collinear points, which is an embedding away from the -2-curve (the proper transform of the line through the three points) and degenerates to a point along the -2-curve, Theorem 5.1. The second one is given by a Legendrian embedding of the degree 6 del Pezzo surface, $\mathbb{P}^2$ blown up at three non-collinear points, Theorem 5.7.

The paper is organized as follows. In Section 2, the basic local invariants of a Legendrian surface are defined as a set of three symmetric differentials ($\Phi$, $\Psi$, $\chi$). By imposing natural geometric conditions in terms of these invariants, we identify four distinguished classes of Legendrian surfaces, and determine their structure equations, Section 2.2 through 2.5. In Section 3 as a preparatory step for the analysis of deformation with geometric constraints in Section 4, we determine the structure equations for the second order Legendrian deformation, or equivalently the Legendrian deformation preserving the second order cubic differential $\Phi$. The analysis shows that there is no local obstruction for the second order deformation of a Legendrian surface with nondegenerate cubic $\Phi$. In Section 4, we introduce the $\Psi$-deformation, which is the main object of study in this paper. It is the second order deformation which also preserves the fourth order cubic differential $\Psi$. The analysis shows that a generic Legendrian surface does not admit any nontrivial $\Psi$-deformations, but that the two special sets of Legendrian surfaces called $D_0$-surfaces and $D_1$-surfaces admit maximum three parameter family of $\Psi$-deformations. Moreover, a subset of these maximally deformable surfaces admit $\Psi$-deformations that also preserve the fifth order quadratic differential $\chi$, Theorem 4.34. In Section 5, we choose and integrate two simple examples of structure equations for $D_0$-surfaces. In Section 5.1 the flat case is examined, where all the structure coefficients vanish. The structure equation is integrated, and one gets a rational Legendrian variety with a single, second order branch type isolated singularity. Its smooth resolution is $\mathbb{P}^2$ blown up at three distinct collinear points, with the exceptional divisor being the -2-curve. In Section 5.2, called tri-ruled (Section 2.5), the case where each leaf of the three $\Phi$-asymptotic foliations lies in a linear Legendrian $\mathbb{P}^2$. The structure equation is integrated, and one gets a smooth Legendrian embedding of $\mathbb{P}^2$ blown up at three non-collinear points. The embedding is given by a subsystem of the anti-canonical system and each of the six -1-curves is mapped to a line.

Throughout the paper, we freely apply the methods and results of exterior differential systems. We refer the reader to [BCG3] for the standard reference on the subject.

2. Legendrian surface

The method of moving frames is a process of equivariant frame adaptation for a submanifold in a homogeneous space. The algorithmic operation of successive normalizations reveals the basic local invariants of the submanifold as the coefficients of the structure equation and their derivatives. The method was developed by Élie Cartan, and Cartan himself applied it extensively to a variety of problems.

In this section, the method of moving frames is applied to immersed Legendrian surfaces in the five dimensional complex projective space $\mathbb{P}^5$. We establish the fundamental structure equation which depends on the sixth order jet of the Legendrian immersion, and identify the basic local invariants as the set of three symmetric differentials $\Phi$, $\Psi$, and $\chi$ of order 2, 4, and 5 respectively, Proposition 2.22. $\Phi$ and $\Psi$ are cubic, and $\chi$ is quadratic.

In order to understand the geometric implication of these invariants, we impose a set of conditions in terms of $\Phi$, $\Psi$, and $\chi$, and give an analysis for the Legendrian surfaces that satisfy these conditions. This in turn leads to four classes of Legendrian surfaces with interesting geometric properties, Section 2.2, 2.3, 2.4, and 2.5 respectively.

Let us give an outline of the analysis.

- $\Phi$, cubic differential of order 2: $\Phi$ represents the second fundamental form of the Legendrian surface. Assuming $\Phi$ is nondegenerate, the base locus of $\Phi$ defines a 3-web called asymptotic web. It is the lowest order local invariant of a Legendrian surface.
We give an analysis for surfaces with flat asymptotic web, Section 2.2. The condition for the asymptotic web to be flat is expressed by a single fourth order equation for the Legendrian immersion, (2.26). A differential analysis shows that this PDE becomes involutive after a partial prolongation with the general solution depending on five arbitrary functions of 1 variable, Proposition 2.28. It turns out that all of the surfaces that are of interest to us necessarily have flat asymptotic web, e.g., surfaces admitting maximum family of nontrivial Legendrian deformation, Section 4.

The moving frame computation associates to each asymptotic foliation a unique Legendrian $\mathbb{P}^2$-field that has second order contact with the given foliation, (2.40). An asymptotic foliation is called ruled when the associated Legendrian $\mathbb{P}^2$-field is leafwise constant. We give an analysis for surfaces with ruled asymptotic foliations, Section 2.5.

• $\Psi$, cubic differential of order 4: The moving frame computations show that there is no third order local invariants for a Legendrian surface. The pencil of cubics $(\Psi; \Phi)$ based at $\Phi$ accounts for roughly one-half of the fourth order invariants of a Legendrian surface.

We give an analysis for surfaces with vanishing $\Psi$, called $\Psi$-null surfaces, which can be considered as the Legendrian analogue of quadrics in $\mathbb{P}^3$, Section 2.3. The condition for $\Psi$ to vanish is expressed by a pair of fourth order equations for the Legendrian immersion, (2.30). A differential analysis shows that the structure equation for $\Psi$-null surfaces closes up with the general solution depending on one constant, Proposition 2.34.

More generally, we give an analysis for the class of surfaces called isothermally asymptotic surfaces, which is the case when $\Psi$ is proportional to $\Phi$ and the pencil $(\Psi; \Phi)$ degenerates, Section 2.4. It will be shown that this class of surfaces are examples of surfaces admitting maximum three parameter family of $\Psi$-deformations, Section 4.

• $\chi$, quadratic differential of order 5: The geometry of $\chi$ is examined in Section 4. It will be shown that there exist Legendrian surfaces which admit maximum one parameter family of deformations preserving the triple $(\Phi, \Psi, \chi)$.

For a modern exposition of Cartan’s equivalence method, we refer to [Ga] [IL].

2.1. Structure equation. Let $V = \mathbb{C}^6$ be the six dimensional complex vector space. Let $\varpi$ be the standard symplectic 2-form on $V$. Let $\mathbb{P}^5 = \mathbb{P}(V)$ be the projectivization equipped with the induced contact structure. The contact hyperplane field $\mathcal{H}$ on $\mathbb{P}^5$ is defined by

$$\mathcal{H}_x = [(\hat{x} \downarrow \varpi)^\perp], \text{ for } x \in \mathbb{P}^5,$$

where $\hat{x} \in V$ is any de-projectivization of $x$. $(\hat{x} \downarrow \varpi)^\perp \subset V$ is a codimension one subspace containing $\hat{x}$, and its projectivization $[(\hat{x} \downarrow \varpi)^\perp] \subset \mathbb{P}^5$ is a hyperplane at $x$. The symplectic group $\text{Sp}_3\mathbb{C}$ acts transitively on $\mathbb{P}^5$ as a group of contact transformation.

$\mathcal{H}$ inherits a conformal class of nondegenerate symplectic 2-form determined by the restriction of $\varpi$ on the quotient space $(\hat{x} \downarrow \varpi)^\perp/(\hat{x})$. A two dimensional Lagrangian subspace of $\mathcal{H}_x$ is called Legendrian. Let $\Lambda \to \mathbb{P}^5$ be the bundle of Legendrian 2-planes. Let $\text{Lag}(V)$ be the set of three dimensional Lagrangian subspaces of $V$. The symplectic group $\text{Sp}_3\mathbb{C}$ acts transitively on both $\Lambda$ and $\text{Lag}(V)$, and there exists the incidence double fibration;

$$\begin{array}{c}
\text{Sp}_3(\mathbb{C}) \\
\downarrow \pi \\
\Lambda = \text{Sp}_3(\mathbb{C})/P \\
\pi_0
\end{array}$$

$$\begin{array}{c}
\mathbb{P}^5 \\
\pi_1 \\
\text{Lag}(V)
\end{array}$$

Figure 2.1 Double fibration

The fiber of $\pi_0$ is isomorphic to $\text{Lag}(2, \mathbb{C}^4)$, and the fiber of $\pi_1$ is $\mathbb{P}^2$. 
To fix the notation once and for all, let us define the projection maps \( \pi, \pi_0, \) and \( \pi_1 \) explicitly. Let \( (e, f) = (e_0, e_1, e_2, f_0, f_1, f_2) \) denote the \( \text{Sp}_3 \mathbb{C} \subset \text{SL}_6 \mathbb{C} \) frame of \( V \) such that the 2-vector \( \omega_0 = e_0 \wedge f_0 \) is dual to the symplectic form \( \omega \). Define

\[
\begin{align*}
\pi(e, f) &= ([e_0], [e_0 \wedge e_1 \wedge e_2]), \\
\pi_0([e_0], [e_0 \wedge e_1 \wedge e_2]) &= [e_0], \\
\pi_1([e_0], [e_0 \wedge e_1 \wedge e_2]) &= [e_0 \wedge e_1 \wedge e_2].
\end{align*}
\]

In this formulation, the stabilizer subgroup \( P \) in Figure 2.1 is of the form

\[
P = \{ \begin{pmatrix} A & B \\ (A^t)^{-1} \end{pmatrix} \},
\]

where \( (A^{-1}B)^t = A^{-1}B \), and

\[
A = \{ \begin{pmatrix} t & * & * \\
* & * & * \\
* & * & *
\end{pmatrix} \}.
\]

Here \( ' \cdot ' \) denotes 0 and \( '* ' \) is arbitrary.

The \( \text{Sp}_3 \mathbb{C} \)-frame \( (e, f) \) satisfies the structure equation

\[
d(e, f) = (e, f) \phi
\]

for the Maurer-Cartan form \( \phi \) of \( \text{Sp}_3 \mathbb{C} \). The components of \( \phi \) are denoted by

\[
\phi = \begin{pmatrix}
\omega & \eta \\
\theta & -\omega^t
\end{pmatrix},
\]

where \( \{ \omega, \theta, \eta \} \) are 3-by-3 matrix 1-forms such that \( \theta^t = \theta, \eta^t = \eta \). \( \phi \) satisfies the structure equation

\[
d\phi + \phi \wedge \phi = 0.
\]

Let \( x : M \hookrightarrow \mathbb{P}^5 \) be a \( \mathcal{H} \)-horizontal, immersed Legendrian surface. We employ the method of moving frames to normalize the \( \text{Sp}_3 \mathbb{C} \)-frame along \( x \). Our argument is local, and the action of certain finite permutation group that occurs in the course of normalization shall be ignored. This does not affect the analysis nor the result of moving frame computation for our purpose. The process of equivariant reduction terminates at the sixth order jet of the immersion \( x \).

1-adapted frame. By definition, there exists a unique lift \( \tilde{x} : M \hookrightarrow \Lambda \). Let \( \tilde{x}^* \text{Sp}_3 \mathbb{C} \to M \) be the pulled back \( P \)-bundle. We continue to use \( \phi \) to denote the pulled back Maurer-Cartan form on \( \tilde{x}^* \text{Sp}_3 \mathbb{C} \). From (2.1), (2.3), the initial state of \( \phi \) on \( \tilde{x}^* \text{Sp}_3 \mathbb{C} \) takes the form

\[
\phi = \begin{pmatrix}
\omega_{00} & \omega_{01} & \omega_{02} & \eta_{00} & \eta_{01} & \eta_{02} \\
\omega_{10} & \omega_{11} & \omega_{12} & \eta_{10} & \eta_{11} & \eta_{12} \\
\omega_{20} & \omega_{21} & \omega_{22} & \eta_{20} & \eta_{21} & \eta_{22} \\
\theta_{11} & \theta_{12} & -\omega_{00} & -\omega^1 & -\omega^2 \\
\theta_{21} & \theta_{22} & -\omega_{02} & -\omega_{12} & -\omega_{22}
\end{pmatrix},
\]

where \( \theta_{ij} = \theta_{ji}, \eta_{ij} = \eta_{ji}, \) and we denote \( \omega_{i0} = \omega^i, i = 1, 2 \). For any section \( s : M \to \tilde{x}^* \text{Sp}_3(\mathbb{C}), \{ s^*\omega^1, s^*\omega^2 \} \) is a local coframe of \( M \).

2-adapted frame. Differentiating \( \theta_{10} = 0, \theta_{20} = 0 \), one gets

\[
\begin{pmatrix} \theta_{11} & \theta_{12} \end{pmatrix} \wedge \begin{pmatrix} \omega^1 \\ \omega^2 \end{pmatrix} = 0.
\]

By Cartan’s lemma, there exist coefficients \( t_{ijk}; i, j, k = 1, 2, \) fully symmetric in indices such that

\[
\theta_{ij} = t_{ijk} \omega^k.
\]
The structure equation shows that the cubic differential
\[(2.5) \quad \Phi = \theta_{ij} \omega^i \omega^j = t_{ijk} \omega^i \omega^j \omega^k\]
is well defined on \(M\) up to scale. \(\Phi\) represents the second fundamental form of the Legendrian immersion.

**Definition 2.6.** Let \(x : M \hookrightarrow \mathbb{P}^5\) be an immersed Legendrian surface. Let \(\Phi\) be the cubic differential \[(2.5)\] which represents the second fundamental form of the immersion \(x\). The Legendrian surface is nondegenerate if the cubic differential \(\Phi\) is equivalent to an element in the unique open orbit of the general linear group \(GL_2 \mathbb{C}\) action on cubic polynomials in two variables, \([\text{Mc}]\).

**Remark 2.7.** The Segre embedding \(\mathbb{P}^1 \times Q^1 \subset \mathbb{P}^5\) is ruled by lines, and it has a degenerate second fundamental cubic.

We assume the Legendrian surface is nondegenerate from now on. By a frame adaptation, one may normalize \(t_{ijk}\) such that
\[(2.8) \quad \Phi = 3 \omega^1 \omega^2 (\omega^1 + \omega^2), \quad = -3 \omega^1 \omega^2 \omega^3,\]
where \(\omega^3 = -(\omega^1 + \omega^2)\). This is equivalent to
\[(2.9) \quad \begin{pmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{pmatrix} = \begin{pmatrix} \omega^2 & \omega^1 + \omega^2 \\ \omega^1 + \omega^2 & \omega^1 \end{pmatrix}.\]
The structure group \(P, (2.2)\), for the 2-adapted frame is reduced such that \(A = \{ \begin{pmatrix} a & \ast \\ \ast & a A' \end{pmatrix} \mid a \neq 0 \}\), where \(A'\) is the finite subgroup of \(GL_2 \mathbb{C}\) whose induced action leaves \(\Phi\) invariant.

Three asymptotic line fields are determined by \(\{ \omega^1, \theta^1, \theta^2 \}\). The set of respective foliations defines a 3-web called asymptotic web on the Legendrian surface. Since a planar 3-web has local invariants, e.g., web curvature, asymptotic web is the lowest order invariant of a nondegenerate Legendrian surface.

**3-adapted frame.** On the 2-adapted frame satisfying \((2.9)\), set
\[(2.10) \quad \omega_{ij} = \delta_{ij} \omega^0 + s_{ijk} \omega^k, \quad \text{for } i, j = 1, 2,\]
for coefficients \(s_{ijk}\). Differentiating \((2.9)\), one gets
\[(2.11) \quad -3 s_{212} + s_{221} + 2 s_{111} + 2 s_{211} = 0, \quad -\frac{1}{2} s_{121} + \frac{3}{2} s_{112} + \frac{1}{2} s_{212} - \frac{3}{2} s_{221} + s_{211} - s_{122} = 0, \quad -3 s_{121} + s_{112} + 2 s_{122} + 2 s_{222} = 0.\]
Exterior derivatives of \((2.10)\) show that
\[ds_{112} \equiv -\eta_{11} - \eta_{21} + \frac{1}{3} \omega^0, \quad ds_{211} \equiv -\eta_{22}, \quad ds_{212} \equiv -\eta_{21} - \eta_{22} + \omega^0, \quad ds_{121} \equiv -\eta_{11} - \eta_{21} + \omega^0, \quad ds_{122} \equiv -\eta_{11}, \quad ds_{221} \equiv -\eta_{21} - \eta_{22} + \frac{1}{3} \omega^0, \quad \mod \omega^1, \omega^2, \omega^0.\]
By a frame adaptation, one may translate the coefficients \( \{ s_{211} = 0, s_{122} = 0, 3s_{112} - s_{121} = 0 \} \), which forces \( 3s_{221} - s_{212} = 0 \) by (2.11). This set of normalizations is equivalent to adapting the \( \text{Sp}_3 \mathbb{C} \)-frame so that
\[
\begin{align*}
  de_1 &\equiv f_2 \omega^1 \mod e_0, e_1; \omega^2, \\
  de_2 &\equiv f_1 \omega^2 \mod e_0, e_2; \omega^1, \\
  d(e_1 - e_2) &\equiv (f_1 + f_2) \omega^2 \mod e_0, (e_1 - e_2); \omega^3.
\end{align*}
\]
By a further frame adaptation, one may translate \( s_{ijk} = 0 \) (we omit the details), and we have
\[
\begin{align*}
  \omega_{ij} &= \delta_{ij} 3 \omega_{00}.
\end{align*}
\]
For this 3-adapted frame, the structure equation shows that a triple of Legendrian \( \mathbb{P}^2 \)-fields is well defined along the Legendrian surface. Let \( (L_1, L_2, L_3) \) be the triple of Legendrian \( \mathbb{P}^2 \)-fields, or equivalently the triple of Lagrangian 3-plane fields, defined by
\[
\begin{align*}
  L_1 &= [e_0 \wedge e_1 \wedge f_2], \\
  L_2 &= [e_0 \wedge e_2 \wedge f_1], \\
  L_3 &= [e_0 \wedge (e_1 - e_2) \wedge (f_1 + f_2)].
\end{align*}
\]
Each \( L_i \) is the unique Legendrian \( \mathbb{P}^2 \)-field that has second order contact with the asymptotic foliation defined by \( \langle \omega^i \rangle \).

**4-adapted frame.** On the 3-adapted frame satisfying (2.12), set
\[
\begin{align*}
  \omega_{01} &= h_{01k} \omega^k, \\
  \omega_{02} &= h_{02k} \omega^k, \\
  \eta_{ij} &= \eta_{ji} = h_{ijk} \omega^k, \quad \text{for } i, j = 1, 2.
\end{align*}
\]
The structure equation shows that the cubic differential \( \Psi = \eta_{ij} \omega^i \omega^j \) is well defined up to scale, and up to translation by
\[
\Psi \rightarrow \Psi + (s_1 \omega^1 + s_2 \omega^2)((\omega^1)^2 + (\omega^2)^2), \quad \text{for arbitrary coefficients } s_1, s_2.
\]
By a frame adaptation, one may translate \( h_{111} = 0, h_{222} = 0 \) so that
\[
\Psi = \omega^1 \omega^2((h_{112} + 2h_{121}) \omega^1 + (2h_{122} + h_{221}) \omega^2).
\]
\( \Psi \) is now well defined up to scale. It is a fourth order invariant of the nondegenerate Legendrian surface. For the problem of projective deformation of Legendrian surfaces, \( \Psi \) will play the role of the third fundamental form for surfaces in \( \mathbb{P}^3 \).

Note that the derivative of (2.12) with the relation \( h_{111} = 0, h_{222} = 0 \) gives the compatibility equations
\[
\begin{align*}
  h_{011} &= h_{121} + h_{221}, \\
  h_{022} &= h_{112} + h_{122}, \\
  h_{012} &= \frac{3}{5} (h_{121} - h_{122}), \\
  h_{021} &= \frac{3}{5} (h_{121} + h_{122}).
\end{align*}
\]

**5-adapted frame.** On the 4-adapted frame satisfying (2.15), (2.16), set
\[
\begin{align*}
  \eta_{10} &= h_{10k} \omega^k, \\
  \eta_{20} &= h_{20k} \omega^k.
\end{align*}
\]
The structure equation shows that the quadratic differential \( \chi = \eta_{10} \omega^1 + \eta_{20} \omega^2 \) is well defined up to scale, and up to translation by
\[
\chi \rightarrow \chi + s_0 ( (\omega^1)^2 + (\omega^2)^2 ), \quad \text{for arbitrary coefficient } s_0.
\]
By a frame adaptation, one may translate \( h_{101} + h_{202} = 0 \) (we omit the details) so that
\[
(2.18) \quad \chi = h_{101} (\omega^1)^2 + (h_{102} + h_{201}) \omega^1 \omega^2 - h_{101} (\omega^2)^2.
\]
\( \chi \) is now well defined up to scale. It is a fifth order invariant of the nondegenerate Legendrian surface.

Restricting to the sub-bundle defined by the equation \( h_{101} + h_{202} = 0 \), we set
\[
(2.19) \quad \eta_{00} = h_{001} \omega^1 + h_{002} \omega^2.
\]
At this stage, no more frame adaptation is available, and the components of the induced Maurer-Cartan form \( \phi \) is uniquely determined, modulo at most a finite group action (this finite group does not enter into our analysis, and we shall not pursue the exact expression for the representation of this group). The reduction process of moving frame method stops here.

For a notational purpose, let us make a change of variables;
\[
(2.20) \quad h_{112} = a_1, \quad h_{101} = b_1, \quad h_{001} = c_1, \\
h_{121} = a_2, \quad h_{102} = b_2, \quad h_{002} = c_2, \\
h_{122} = a_3, \quad h_{201} = b_3, \\
h_{221} = a_4.
\]
The covariant derivatives are denoted by
\[
da_i = -\frac{4}{3} a_i \omega^0 + a_{ik} \omega^k, \\
db_i = -2 b_i \omega^0 + b_{ik} \omega^k, \\
dc_i = -\frac{8}{3} c_i \omega^0 + c_{ik} \omega^k.
\]
Differentiating (2.14), (2.17), (2.19), one gets a set of compatibility equations among the covariant derivatives.
\[
(2.21) \quad a_{11} = -2 b_2, \\
a_{22} = -\frac{3}{2} a_{21} + \frac{15}{2} b_3 + 8 b_1, \\
a_{31} = -\frac{3}{2} a_{21} + \frac{15}{2} b_3 + 10 b_1, \\
a_{32} = a_{21} + 5 b_2 - 12 b_1 - 5 b_3, \\
a_{42} = -2 b_3, \\
b_{12} = b_{21} - \frac{3}{5} a_3^2 + c_2 + a_4 a_1 - \frac{2}{5} a_2 a_3, \\
b_{32} = -b_{11} - c_1 + \frac{3}{5} a_2^2 + \frac{2}{5} a_2 a_3 - a_4 a_1, \\
c_{12} = c_{21} - 2 a_1 b_3 - 2 a_3 b_2 + 2 a_2 b_2 + 2 a_4 b_2.
\]
The structure equation (2.4) for \( \phi \) is now an identity with these relations. One may check that the structure equation with the coefficients \( \{ a_i, b_j, c_k \} \) becomes involutive after one prolongation with the general solution depending on one arbitrary function of 2 variables in the sense of Cartan, [BCG3].

**Proposition 2.22.** Let \( x : M \hookrightarrow \mathbb{P}^5 \) be a nondegenerate immersed Legendrian surface. Let \( \tilde{x} : M \hookrightarrow \Lambda \) be the associated lift to the bundle of Legendrian 2-planes. Let \( \tilde{x}^* \text{Sp}_3(\mathbb{C}) \rightarrow M \) be the pulled back bundle, Figure 2.1 \( \tilde{x}^* \text{Sp}_3(\mathbb{C}) \) admits a reduction to a sub-bundle with 1-dimensional fibers such that the induced
Maurer-Cartan form $\phi$ satisfies the structure equations \((2.9), (2.12), (2.14), (2.15), (2.16), (2.17), (2.19), (2.20),\) and \((2.21)\).

We shall work with the 5-adapted frame for the rest of the paper. Unless stated otherwise, ‘the structure equation’ would mean the structure equation for the 5-adapted frame.

Note that under the notations we chose, the invariant differentials $\Phi$, $\Psi$, and $\chi$, \((2.8), (2.15), (2.18),\) are expressed by
\[
\Phi = 3(\omega^1)^2\omega^2 + 3\omega^1(\omega^2)^2,
\]
\[
\Psi = \omega^1\omega^2((a_1 + 2a_2)\omega^1 + (2a_3 + a_4)\omega^2),
\]
\[
= (a_1 + 2a_2)(\omega^1)^2\omega^2 + (2a_3 + a_4)\omega^1(\omega^2)^2,
\]
\[
\chi = b_1(\omega^1)^2 + (b_2 + b_3)\omega^1\omega^2 - b_1(\omega^2)^2.
\]

In the next two sections, Section 3 and Section 4, we shall examine the deformability, or the rigidity, of Legendrian surfaces preserving these invariant differentials.

Before we proceed to the problem of deformation, let us examine four classes of Legendrian surfaces with special geometric properties. There exist a number of surfaces in $\mathbb{P}^3$ with notable characteristics, which have been the subject of extensive study, [Fe]. Some of the surfaces described below can be considered as the Legendrian analogues of these classical surfaces.

2.2. Flat asymptotic 3-web. In this sub-section, we consider the class of Legendrian surfaces with flat asymptotic 3-web. For a comprehensive introduction to web geometry, we refer to [PP].

**Definition 2.24.** Let $M \hookrightarrow \mathbb{P}^5$ be a nondegenerate Legendrian surface. The asymptotic 3-web is the set of three foliations defined by \(\{ (\omega^1)\perp, (\omega^2)\perp, (\omega^3)\perp \}\) at 2-adapted frame, where $\omega^1 + \omega^2 + \omega^3 = 0$.

The following analysis shows that the differential equation describing the Legendrian surfaces with flat asymptotic web is in good form (involutive) and admits arbitrary function worth solutions locally.

The web curvature of the asymptotic 3-web can be expressed in terms of the structure coefficients of the Legendrian surface. From the structure equation (for 5-adapted frame),
\[
d\omega^i = \frac{2}{3} \omega_{00} \wedge \omega^i, \quad i = 1, 2, 3.
\]
The web curvature $K$ of the 3-web is given by
\[
\frac{2}{3} d\omega_{00} = K \omega^1 \wedge \omega^2,
\]
\[
= -\frac{4}{5} (a_2 - a_3)\omega^1 \wedge \omega^2.
\]
The asymptotic web is flat when
\[
a_2 - a_3 = 0.
\]

We wish to give an analysis of the compatibility equations derived from this vanishing condition.

Differentiating $a_2 - a_3 = 0$, one gets
\[
a_{21} = -2b_1 + 3b_2,
\]
\[
b_3 = -2b_1 + b_2.
\]

Differentiating the second equation for $b_3$, one gets
\[
b_{22} = 2b_2 - b_{11} - a_2^2 + a_4a_1 - c_1 + 2c_2,
\]
\[
b_{31} = -2b_{11} + b_{21}.
\]

Exterior derivative $d(d(a_2)) = 0$ with these relations then gives
\[
b_{21} = -b_{11} + a_2^2 - a_4a_1 + 3c_1 - 4c_2.
\]
The identities from \( d(d(b_1)) = 0, \ d(d(b_2)) = 0 \) determine the derivative of \( b_{11} \) by

\[
\left(2.27\right) \quad db_{11} + \frac{8}{3} b_{11} \omega_{00} = (-a_4 b_2 - \frac{1}{4} a_{41} a_1 - 2 b_1 a_2 - 2 b_1 a_1 + b_2 a_1 + \frac{1}{4} a_{12} a_4 + c_{22} + 2 c_{11} - 3 c_{21}) \omega^1
\]

\[
+ (a_4 b_2 + \frac{1}{4} a_{41} a_1 + 2 b_1 a_2 + 2 b_1 a_1 - b_2 a_1 - \frac{1}{4} a_{12} a_4 - c_{22} + c_{11}) \omega^2.
\]

At this step, we interrupt the differential analysis and invoke a version of Cartan-Kähler theorem, a general existence theorem for analytic differential systems, \([BCG3]\).

**Proposition 2.28.** The structure equation for the nondegenerate Legendrian surfaces with flat asymptotic web is in involution with the general solution depending on five arbitrary functions of \( 1 \) variable.

**Proof.** From the analysis above, the exterior derivative identities \( d(d(a_1)) = 0, \ d(d(a_4)) = 0, \ d(d(c_1)) = 0, \ d(d(c_2)) = 0, \ d(d(b_{11})) = 0 \) give 5 compatibility equations while the remaining independent derivative coefficients at this step are \( \{ a_{12}, a_{41}, c_{11}, c_{21}, c_{22}; b_{11} \} \). An inspection shows that the resulting structure equation is in involution with the last nonzero Cartan character \( s_1 = 5 \). \( \square \)

2.3. **Vanishing cubic differential \( \Psi \).** In this sub-section, we consider the class of Legendrian surfaces with vanishing cubic differential \( \Psi \), a fourth order invariant \( \left(2.15\right) \).

**Definition 2.29.** Let \( M \hookrightarrow \mathbb{P}^5 \) be a nondegenerate Legendrian surface. \( M \) is a \( \Psi \)-null surface if the fourth order cubic differential \( \Psi \) defined at \( 4 \)-adapted frame vanishes.

The following analysis shows that a \( \Psi \)-null surface necessarily has flat asymptotic web, and that the local moduli space of \( \Psi \)-null surfaces is finite dimensional.

From \( \left(2.28\right) \), \( \Psi \) vanishes when

\[
\left(2.30\right) \quad a_1 = -2 a_2, \quad a_4 = -2 a_3.
\]

Differentiating these equations, one gets

\[
 a_{21} = b_2, \quad a_{12} = -15 b_3 - 16 b_1 + 3 b_2,
\]

\[
 b_3 = -2 b_1 + b_2, \quad a_{41} = 10 b_1 - 12 b_2.
\]

Exterior derivatives \( d(d(a_2)) = 0, \ d(d(a_3)) = 0 \) with these relations give

\[
 b_{22} = 8 b_{11} + \frac{8}{5} a_2^2 - \frac{8}{5} a_2 a_3 + \frac{15}{2} b_{31} - \frac{3}{2} b_21,
\]

\[
 b_{31} = -\frac{6}{5} b_{11} + b_{21} - \frac{88}{125} a_3^2 + \frac{8}{5} c_2 + \frac{196}{125} a_2 a_3 - \frac{6}{5} c_1 + \frac{42}{125} a_2^2.
\]

Differentiating \( b_3 = -2 b_1 + b_2 \), one gets

\[
 b_{11} = \frac{22}{25} a_3^2 - 2 c_2 - \frac{49}{25} a_2 a_3 + \frac{3}{2} c_1 - \frac{21}{50} a_2^2,
\]

\[
 b_{21} = 2 c_1 - \frac{22}{25} a_2^2 - \frac{41}{25} a_2 a_3 + \frac{51}{50} a_3^2 - \frac{5}{2} c_2.
\]

Exterior derivatives \( d(d(b_1)) = 0, \ d(d(b_2)) = 0 \) with these relations give

\[
 c_{22} = c_{11} + \frac{68}{5} b_2 a_3 + \frac{58}{5} b_1 a_2 - \frac{68}{5} a_2 b_2 - \frac{78}{5} b_1 a_3,
\]

\[
 c_{21} = \frac{4}{3} c_{11} + \frac{266}{15} b_2 a_3 + 6 b_1 a_2 - 18 b_1 a_3 - \frac{32}{5} a_2 b_2.
\]
The identities from \( d(d(c_1)) = 0 \), \( d(d(c_2)) = 0 \) determine the derivative of \( c_{11} \) by

\[
dc_{11} + \frac{10}{3} c_{11} \omega_{00} = \left( \frac{6807}{875} a_2^3 - \frac{607}{35} c_2 a_3 - 6 b_2^2 + \frac{9993}{875} a_3^3 + 20 b_1 b_2 - \frac{6646}{875} a_2^2 a_3 - 12 b_1^2 \\
+ \frac{638}{35} c_1 a_3 - \frac{513}{35} c_1 a_2 - \frac{12779}{875} a_2 a_3^2 + \frac{447}{35} c_2 a_2) \omega^1 \\
+ \left( -\frac{10119}{875} a_2^3 - \frac{856}{35} c_2 a_3 + 62 b_2^2 + \frac{13016}{875} a_3^3 - 118 b_1 b_2 - \frac{8747}{875} a_2^2 a_3 + 54 b_1^2 \\
+ \frac{165}{7} c_1 a_3 - \frac{141}{7} c_1 a_2 - \frac{11763}{875} a_2 a_3^2 + \frac{771}{35} c_2 a_2) \omega^2. \right.
\]

Differentiating this equation again, one gets a compatibility equation of the form

\[
(a_2 - a_3) c_{11} = [a_i, b_j, c_k],
\]

where the right hand side is a polynomial in the variables \( a_i, b_j, c_k \). At this juncture, the analysis divides into two cases.

**Case** \( a_2 - a_3 \neq 0 \). It turns out that the condition \( a_2 - a_3 \neq 0 \) is not compatible with the vanishing of \( \Psi \), and there is no nondegenerate Legendrian surfaces with \( \Psi \equiv 0 \), and \( a_2 - a_3 \neq 0 \) . Some of the expressions for the analysis of this case are long. Let us explain the relevant steps of differential analysis, and omit the details of the long and non-essential terms.

From (2.31), solve for \( c_{11} \). Differentiating this, one gets a set of two equations, from which one solves for \( c_1, c_2 \). Differentiating these equations, one gets another set of two equations which imply \( b_1 = b_2 = 0 \). Differentiating these equations again, one finally gets two quadratic equations for \( a_2, a_3 \), which force \( a_2 = a_3 = 0 \), a contradiction.

**Case** \( a_2 - a_3 = 0 \). From Section (2.2), this is the case when the asymptotic 3-web is flat. Successive derivatives of the equation \( a_2 - a_3 = 0 \) imply the following.

\[
b_1 = b_2, \\
c_1 = c_2, \\
c_{11} = 2 a_2 b_2.
\]

Furthermore, these equations are compatible, i.e., \( d^2 = 0 \) is an identity.

The remaining independent coefficients at this step are \{ \( a_2, b_2, c_2 \). Let us remove the sub-script, and denote \{ \( a_2, b_2, c_2 \} = \{ a, b, c \} \). The structure equations for these coefficients are reduced to

\[
(2.32) \quad \begin{align*}
da &= -\frac{4}{3} a \omega_{00} + b(\omega^1 - \omega^2), \\
\omega^1 &= \frac{1}{3} \omega_{00}, \\
\omega^2 &= \frac{1}{3} \omega_{00}. \\
dc &= -\frac{8}{3} c \omega_{00} + 2 ab (\omega^1 - \omega^2).
\end{align*}
\]

The Maurer-Cartan form \( \phi \) takes the form

\[
(2.33) \quad \phi = \begin{pmatrix}
\omega_{00} & -a \omega^1 & -a \omega^2 & c(\omega^1 + \omega^2) & b(\omega^1 + \omega^2) & -b(\omega^1 + \omega^2) \\
\omega^1 & \frac{1}{3} \omega_{00} & . & b(\omega^1 + \omega^2) & -a \omega^2 & a(\omega^1 + \omega^2) \\
\omega^2 & . & \frac{1}{3} \omega_{00} & -b(\omega^1 + \omega^2) & a(\omega^1 + \omega^2) & -2 \omega^1 \\
. & . & . & -\omega_{00} & -\omega^1 & -\omega^2 \\
. & . & . & . & -\frac{1}{3} \omega_{00} & . \\
. & . & . & . & . & -\frac{1}{3} \omega_{00}
\end{pmatrix}.
\]
Proposition 2.34. Let $M \hookrightarrow \mathbb{P}^5$ be a nondegenerate, $\Psi$-null Legendrian surface. The asymptotic 3-web of $M$ is necessarily flat. The Maurer-Cartan form of the 5-adapted frame of $M$ is reduced to \((2.33)\), and the structure coefficients \{a, b, c\} satisfy the equation \((2.32)\). The local moduli space of $\Psi$-null Legendrian surfaces has general dimension 1.

**Proof.** Let $F \rightarrow M$ be the canonical bundle of 5-adapted frames from Proposition 2.22. \((2.32)\) shows that the invariant map \((a, b, c) : F \rightarrow \mathbb{C}^3\) generically has rank two. From the general theory of geometric structures with closed structure equation, [Br2], the local moduli space of this class of Legendrian surfaces has general dimension $\dim(\mathbb{C}^3) - \text{rank}(a, b, c) = 1$. A Legendrian surface in this class necessarily possesses a minimum 1-dimensional local group of symmetry. The line field $\langle \omega_{00}, \omega^1 - \omega^2 \rangle$ is tangent to the fibers of the invariant map \((a, b, c)\), and it generates a local symmetry. □

2.4. Isothermally asymptotic. In this sub-section, we consider the class of Legendrian surfaces which are the analogues of the classical isothermally asymptotic surfaces in $\mathbb{P}^3$, [Fe].

**Definition 2.35.** Let $M \hookrightarrow \mathbb{P}^5$ be a nondegenerate Legendrian surface. $M$ is isothermally asymptotic if the fourth order cubic differential $\Psi$, \((2.23)\), is a multiple of the second order cubic differential $\Phi$, \((2.8)\).

The following analysis shows that the differential equation describing the isothermally asymptotic Legendrian surfaces is in good form and admits arbitrary function worth solutions locally.

From \((2.23)\), $\Psi \equiv 0 \mod \Phi$ when

$$a_1 + 2a_2 = 2a_3 + a_4.$$ We wish to give an analysis of the compatibility equations derived from this condition.

Differentiating $a_1 + 2a_2 = 2a_3 + a_4$, one gets

$$a_{12} = 5a_{21} + 10b_2 - 27b_3 - 40b_1,$$

$$a_{41} = 5a_{21} - 2b_2 - 15b_3 - 20b_1.$$ The identities from $d(d(a_1)) = 0$, $d(d(a_2)) = 0$ determine the derivative of $a_{21}$ by

$$da_{21} + 2a_{21}\omega_{00} = (-2b_{21} + 8b_{11} - \frac{2}{5}b_{22} + \frac{27}{5}b_{31} - \frac{8}{25}a_{21} + \frac{8}{25}a_{13})\omega^1$$

$$+ (-4b_{11} + \frac{3}{5}b_{22} + 3b_{21} - \frac{3}{5}b_{31} + \frac{8}{5}a_{2}^2 - \frac{8}{5}a_{2}a_{3} + \frac{12}{25}a_{2}a_{1} - \frac{12}{25}a_{1}a_{3})\omega^2.$$ Exterior derivative $d(d(a_3)) = 0$ with these relations then gives

$$b_{22} = 5b_{11} + b_{31} + 5b_{21} + \frac{46}{5}a_{2}a_{1} - \frac{46}{5}a_{1}a_{3} - 15c_{1} - \frac{44}{5}a_{3}^2 + 20c_{2} + 5a_{1}^2 + \frac{21}{5}a_{2}^2 - \frac{2}{5}a_{2}a_{3}.$$ At this step, we interrupt the differential analysis and invoke a version of Cartan-Kähler theorem.

**Proposition 2.36.** The structure equation for the nondegenerate isothermally asymptotic Legendrian surfaces is in involutive with the general solution depending on five arbitrary functions of 1 variable.

**Proof.** From the analysis above, the exterior derivative identities $d(d(b_1)) = 0$, $d(d(b_2)) = 0$, $d(d(b_3)) = 0$, $d(d(c_1)) = 0$, $d(d(c_2)) = 0$, $d(d(a_{21})) = 0$ give 6 compatibility equations while the remaining independent derivative coefficients at this step are \{b_{11}, b_{21}, b_{31}, c_{11}, c_{21}, c_{22}; a_{21}\}. A short analysis shows that the resulting structure equation becomes involutive after one prolongation with the last nonzero Cartan character $s_1 = 5$. Since the prolonged structure equation does not enter into our analysis in later sections, the details shall be omitted. □
2.4.1. Isothermally asymptotic with flat asymptotic web. Consider the class of isothermally asymptotic Legendrian surfaces which have flat asymptotic 3-web. From (2.23) and (2.25), this is equivalent to the condition
\begin{equation}
(2.37) \quad a_1 = a_4, \quad a_2 = a_3.
\end{equation}

The following analysis shows that the differential equation describing such Legendrian surfaces is still in good form and admits arbitrary function worth solutions locally. Note that a $\Psi$-null surface is necessarily isothermally asymptotic with flat asymptotic web.

This is in contrast with the $\Psi$-null surface case, where the defining equation (2.30) is also a set of two linear equations among $a_i$’s and yet the resulting structure equations close up to admit solutions with finite dimensional moduli. This reflects the subtle well-posedness of the equation (2.37). The discovery of this class of Legendrian surfaces is perhaps most unexpected of the analysis in this section.

We wish to give an analysis for the compatibility equations derived from (2.37). Differentiating the given equations $a_1 = a_4$, $a_2 = a_3$, one gets
\begin{align*}
a_{21} &= 4b_1 + 3b_3, \quad b_3 = -2b_1 + b_2, \\
a_{12} &= 4b_1 - 2b_2, \quad a_{41} = -2b_2.
\end{align*}

Exterior derivatives $d(d(a_1)) = 0$, $d(d(a_2)) = 0$, $d(d(a_3)) = 0$ with these relations give
\begin{align*}
b_{22} &= b_{31}, \\
b_{31} &= -\frac{5}{3}b_{11} + \frac{4}{3}b_{21} + \frac{1}{3}a_1^2 - \frac{1}{3}a_2^2 - c_1 + \frac{4}{3}c_2, \\
b_{21} &= -b_{11} + 3c_1 - 4c_2 + a_2^2 - a_1^2.
\end{align*}

Successively differentiating $b_3 = -2b_1 + b_2$, one gets
\begin{align*}
c_1 &= c_2, \\
c_{21} &= c_{11}, \quad c_{22} = c_{11} + 4b_1a_1 + 4b_1a_2.
\end{align*}
The identities from $d(d(b_1)) = 0$, $d(d(b_2)) = 0$ determine the derivative of $b_{11}$ by
\begin{align*}
&d b_{11} = -\frac{8}{3}b_{11}\omega_{00} + (2b_1a_2 + 3b_1a_1)(\omega^1 - \omega^2).
\end{align*}
Moreover, $d(d(b_{11})) = 0$ is an identity.

At this step, we invoke a version of Cartan-Kähler theorem.

**Proposition 2.38.** The structure equation for the nondegenerate isothermally asymptotic Legendrian surfaces with flat asymptotic web is in involution with the general solution depending on one arbitrary function of 1 variable.

**Proof.** From the analysis above, the exterior derivative identity $d(d(c_1)) = 0$ gives 1 compatibility equation while the remaining independent derivative coefficients at this step are $\{c_{11}; b_{11}\}$. By inspection, the resulting structure equation is in involution with the last nonzero Cartan character $s_1 = 1$. □

2.5. Asymptotically ruled. In this sub-section, we consider the class of Legendrian surfaces for which the asymptotic Legendrian $\mathbb{P}^2$-field defined at 3-adapted frame is constant along the corresponding asymptotic foliation.

**Definition 2.39.** Let $M \hookrightarrow \mathbb{P}^5$ be a nondegenerate Legendrian surface. Let $\mathcal{F}$ be an asymptotic foliation (one of the three) defined at 2-adapted frame. $\mathcal{F}$ is ruled if the corresponding Legendrian $\mathbb{P}^2$-field defined at 3-adapted frame is constant along $\mathcal{F}$.
With an abuse of terminology, we call the leaf-wise constant \( \mathbb{P}^2 \)-field *rulings* of the asymptotic foliation.

We wish to give a differential analysis for Legendrian surfaces with three, two, or one ruled asymptotic foliations in turn. Recall \( \omega^3 = -(\omega^1 + \omega^2) \). From the structure equation (2.3),

\[
\begin{align*}
    d\omega &= \frac{1}{3} \omega_0 \\
    \omega_0 &= a \omega^1 + a \omega^2 + a^2 (\omega^1 + \omega^2) \\
    &\vdots
\end{align*}
\]

(2.40)

\[
\begin{align*}
    de_0 &\equiv e_1 \omega^1, \mod e_0; \omega^2, \\
    de_1 &\equiv f_2 \omega^1, \mod e_0, e_1; \omega^2, \\
    df_2 &\equiv e_2 (a_4 \omega^1), \mod e_0, e_1, f_2; \omega^2, \\
    de_0 &\equiv e_2 \omega^2, \mod e_0; \omega^1, \\
    de_2 &\equiv f_1 \omega^2, \mod e_0, e_2; \omega^1, \\
    df_1 &\equiv e_1 (a_1 \omega^2), \mod e_0, e_2, f_1; \omega^1,
\end{align*}
\]

(2.41)

\[
\begin{align*}
    d(e_1 - e_2) &\equiv (f_1 + f_2) \omega^1, \mod e_0, (e_1 - e_2); \omega^3, \\
    d(f_1 + f_2) &\equiv e_1 (-a_1 + 2 a_3) + (2 a_2 + a_4) \mod e_0, (e_1 - e_2), (f_1 + f_2); \omega^3.
\end{align*}
\]

(2.42)

2.5.1. Tri-ruled. This is the class of surfaces for which all of the three asymptotic foliations are ruled. The following differential analysis shows that the local moduli space of tri-ruled Legendrian surfaces consists of two points.

Assume that each of the three Legendrian \( \mathbb{P}^2 \)-fields \( [e_0 e_1 f_2], [e_0 e_2 f_1] \), and \( [e_0 (e_1 - e_2) (f_1 + f_2)] \) is constant along the asymptotic foliations defined by \( \omega^2 = 0, \omega^1 = 0, \text{and} \omega^3 = 0 \) respectively. By (2.40), this implies

\[
a_4 = 0, \ a_1 = 0, \ a_2 = a_3 = a.
\]

A tri-ruled Legendrian surface has flat asymptotic 3-web, and it is also isothermally asymptotic. The cubic differential \( \Psi \) vanishes when \( a = 0 \), and a non-flat tri-ruled surface is distinct from \( \Psi \)-null surfaces discussed in Section 2.3.

A differential analysis shows that a tri-ruled surface necessarily has \( b_i = 0, \ c_1 = c_2 = a^2 \) (we omit the details). Maurer-Cartan form \( \phi \) is reduced to

\[
\phi = \begin{pmatrix}
    \omega_0 & \omega^1 & \omega^2 & \omega^1 + \omega^2 \\
    \omega^1 & \frac{1}{3} \omega_0 & \omega^2 & \omega^1 + \omega^2 \\
    -\omega_0 & -\omega^1 & -\omega^2 & a_1 \\
    -\omega_0 & -\omega^1 & -\omega^2 & a_1 \\
    \omega^1 + \omega^2 & \omega^1 & -\omega^2 & 0 \\
    \omega_0 & \omega^1 + \omega^2 & -\omega^2 & -\frac{1}{3} \omega_0 \\
    \omega_0 & \omega^1 & -\omega^2 & -\frac{1}{3} \omega_0 \\
    \omega^1 & \omega^2 & \omega^1 + \omega^2 & -\omega^2 \\
    \omega^1 & \omega^2 & \omega^1 + \omega^2 & -\omega^2 \\
    \omega^1 & \omega^2 & \omega^1 + \omega^2 & -\omega^2 \\
    \omega^1 & \omega^2 & \omega^1 + \omega^2 & -\omega^2
\end{pmatrix}
\]

with

\[
da = -\frac{4}{3} a \omega_0.
\]

**Proposition 2.43.** Let \( M \to \mathbb{P}^5 \) be a nondegenerate, tri-ruled Legendrian surface. \( M \) is necessarily isothermally asymptotic with flat asymptotic web. The Maurer-Cartan form of the 5-adapted frame of \( M \) is reduced to (2.41), and the single structure coefficient \( \{ a \} \) satisfies the equation (2.42). The local moduli space of tri-ruled Legendrian surfaces consists of two points.

**Proof.** From (2.42), the moduli space is divided into two cases; \( a \equiv 0 \), or \( a \neq 0 \). □

A differential geometric characterization of tri-ruled surfaces is presented in Section 5.
2.5.2. **Doubly-ruled.** Assume that each of the two Legendrian $\mathbb{P}^2$-fields $[e_0 \wedge e_1 \wedge f_2]$ and $[e_0 \wedge e_2 \wedge f_1]$ is constant along the asymptotic foliations defined by $\omega^2 = 0$ and $\omega^1 = 0$ respectively. By (2.40), this implies

\begin{equation}
(2.44) \quad a_4 = 0, \quad a_1 = 0.
\end{equation}

Note that such a doubly ruled surface with flat asymptotic web is necessarily tri-ruled.

Successively differentiating (2.44), one gets

\begin{align*}
b_2 &= 0, \\
b_3 &= 0, \\
a_{12} &= a_{41} = 0, \\
b_{21} &= b_{22} = b_{31} = 0, \\
b_{11} &= -c_1 + \frac{3}{5} a_2^2 + \frac{2}{5} a_2 a_3.
\end{align*}

Exterior derivative $d(d(b_1)) = 0$ with these relations gives

\begin{equation}
c_{21} = \frac{44}{5} b_1 a_3 - a_3 a_{21} + \frac{16}{5} b_1 a_2 - a_2 a_{21}.
\end{equation}

The identities from $d(d(a_2)) = 0, d(d(a_3)) = 0$ determine the derivative of $a_{21}$ by

\begin{equation*}
d a_{21} = -2 a_{21} \omega_{00} + (-8 c_2 + \frac{48}{25} a_2^2 + \frac{64}{25} a_2 a_3 + \frac{88}{25} a_3^2) \omega^1 + (8 c_1 + 12 c_2 + \frac{88}{25} a_2^2 - \frac{56}{25} a_2 a_3 - \frac{132}{25} a_3^2) \omega^2.
\end{equation*}

Differentiating this equation again, one gets

\begin{equation}
c_{22} = c_{11} - \frac{1}{2} a_2 a_{21} + \frac{1}{2} a_3 a_{21} - 8 b_1 a_3 - 2 b_1 a_2.
\end{equation}

Exterior derivatives $d(d(c_1)) = 0, d(d(c_2)) = 0$ with these relations determine the derivative of $c_{11}$ (the exact expression for $d c_{11}$ is long, and shall be omitted). Differentiating this equation, $d(d(c_{11})) = 0$ finally gives

\begin{equation}
c_{11} = \frac{1}{4(a_2 - a_3)} (-15 a_{21} c_1 - 7 a_3^2 a_{21} + 8 a_2 a_3 b_1 - 32 a_2^2 b_1 + 80 c_1 b_1 + 9 a_2^2 a_{21} - 100 c_2 b_1 \\
&\quad + 44 a_3^2 b_1 - 2 a_3 a_{21} a_2 + 15 c_2 a_{21}).
\end{equation}

Here we assumed that $a_2 \neq a_3$, or equivalently that the Legendrian surface is not tri-ruled.

Differentiating this equation again, and comparing with the formula for $d c_{11}$, one gets two polynomial compatibility equations for six coefficients $\{a_2, a_3, b_1, a_{21}, c_1, c_2\}$. Successive derivatives of these equations generate a sequence of compatibility equations for a nondegenerate Legendrian surface to admit exactly two asymptotic $\mathbb{P}^2$-ruled.

Partly due to the complexity of the polynomial compatibility equations, our analysis is incomplete. We suspect that if there do exist nondegenerate, doubly-ruled (and not tri-ruled) Legendrian surfaces, the moduli space of such surfaces is at most discrete.

2.5.3. **Singly-ruled.** Assume that the Legendrian $\mathbb{P}^2$-field $[e_0 \wedge (e_1 - e_2) \wedge (f_1 + f_2)]$ is constant along the asymptotic curves defined by $\omega^3 = 0$. From (2.40), this implies

\begin{equation}
a_1 + 2 a_3 = 2 a_2 + a_4.
\end{equation}

Note the equivalence relations.

\begin{align*}
\text{Singly ruled and flat asymptotic web} &\quad = \text{Singly ruled and isothermally asymptotic,} \\
&\quad = \text{Isothermally asymptotic and flat asymptotic web.}
\end{align*}

An analysis shows that the structure equation for a nondegenerate, singly-ruled Legendrian surface becomes involutive after one prolongation with the general solution depending on five arbitrary functions of 1 variable. We omit the details of differential analysis for this case.
3. SECOND ORDER DEFORMATION

Definition 3.1. Let \( x : M \hookrightarrow \mathbb{P}^5 \) be a nondegenerate Legendrian surface. Let \( x' : M \hookrightarrow \mathbb{P}^5 \) be a Legendrian deformation of \( x \). \( x' \) is a k-th order deformation if there exists a map \( g : M \to \text{Sp}_3 \mathbb{C} \) such that for each \( p \in M \), the k-adapted frame bundles of \( x' \) and \( g(p) \circ x \) are isomorphic at \( p \). When the application map \( g \) is constant, the deformation \( x' \) is trivial, and \( x' \) is congruent to \( x \) up to motion by \( \text{Sp}_3 \mathbb{C} \). Two deformations \( x_1' \) and \( x_2' \) are equivalent if there exists an element \( g_0 \in \text{Sp}_3 \mathbb{C} \) such that \( x_2' = g_0 \circ x_1' \). A 'deformation' would mean an 'equivalence class of deformations modulo \( \text{Sp}_3 \mathbb{C} \) action' for brevity.

It follows from the construction of adapted frames in Section 2 that a k-th order deformation is a \((k+1)\)-th order deformation when for each \( p \in M \), the application map \( g(p) \in \text{Sp}_3 \mathbb{C} \) not only preserves the k-adapted frame at \( p \), but also the first order derivatives of the k-adapted frame at \( p \) (this is a vague explanation, but the meaning is clear).

The definition of k-th order deformation indicates a way to uniformize the various geometric conditions that naturally occur in the theory of deformation and rigidity of submanifolds in a homogeneous space. Take for an example the familiar case of surfaces in three dimensional Euclidean space with the usual 1-adapted tangent frame of the group of Euclidean motions. One surface is a first order deformation of the other if they have the same induced metric, and it is a second order deformation if they also have the same second fundamental form. By Bonnet’s theorem, a second order deformation is a congruence, \([\text{Sp}]\).

Fubini, and Cartan studied the problem of third order deformation of projective hypersurfaces in \( \mathbb{P}^{n+1} \), \([\text{Ca}]\) and the reference therein. For \( n \geq 3 \), a third order deformation of a hypersurface with nondegenerate second fundamental form is necessarily a congruence, \([\text{JM}]\) for a modern proof. For \( n = 2 \), Cartan showed that a generic surface does not admit a nontrivial third order deformation, but that there exist two special classes of surfaces which admit maximum three parameter family of deformations.

The purpose of this section is to lay a foundation for generalizing Cartan’s work on projective deformation of surfaces in \( \mathbb{P}^3 \) to deformation of Legendrian surfaces in \( \mathbb{P}^5 \). As a preparation, we first consider the second order deformation. By applying a modified moving frame method, we determine the fundamental structure equation for the second order deformation of a nondegenerate Legendrian surface. The analysis shows that the resulting structure equation is in involution, and admits arbitrary function worth solutions locally. This implies that there is no local obstruction to second order deformation of a Legendrian surface.

The structure equation established in this section will be applied to the projective deformation of Legendrian surfaces with geometric constraints in Section 4.

3.1. Structure equation. Let \( x : M \hookrightarrow \mathbb{P}^5 \) be a nondegenerate Legendrian surface. Let \( F \to M \) be the associated canonical bundle of 5-adapted frames with the induced \( \text{Sp}_3 \mathbb{C} \)-valued Maurer-Cartan form \( \phi \). The pair \((F, \phi)\) satisfies the properties described in Proposition 2.22. Let \( x' : M \hookrightarrow \mathbb{P}^5 \) be a second order deformation of \( x \). Let \( F' \to M \) be the associated canonical bundle with the induced Maurer-Cartan form \( \pi \). From the definition of second order deformation, \( F' \) can be considered as a graph over \( F \) which agrees with \( F \) up to 2-adapted frame. By pulling back \( \pi \) on \( F \), we regard \( \pi \) as another \( \text{Sp}_3 \mathbb{C} \)-valued Maurer-Cartan form on \( F \).

Set

\[
\pi = \phi + \delta \phi.
\]

The components of \( \delta \phi \) are denoted by

\[
\delta \phi = \begin{pmatrix}
\delta \omega & \delta \eta \\
\delta \theta & -\delta \omega^t
\end{pmatrix},
\]
where \( \delta \theta^t = \delta \theta, \delta \eta^t = \delta \eta \). Maurer-Cartan equations for \( \pi \) and \( \phi \) imply the fundamental structure equation for the deformation \( \delta \phi \);

\[
(3.3)\quad d(\delta \phi) + \delta \phi \wedge \phi + \phi \wedge \delta \phi + \delta \phi \wedge \delta \phi = 0.
\]

Differentiating the components of \( \delta \phi \) from now on would mean applying this structure equation.

We employ the method of moving frames to normalize the frame bundle \( F' \) based at \( F \). In effect, one may adopt the following analysis as the constructive definition of \( (F', \pi) \). The equivariant reduction process for \( F' \) in this section can be considered as the derivative of the one applied for \( F \) in Section 2. To avoid repetition, some of the details of non-essential terms in the analysis below shall be omitted.

1, and 2-adapted frame. Let \( (e', f') \) and \( (e, f) \) denote the 5-adapted \( Sp_3 \mathbb{C} \)-frames of \( F' \) and \( F \) respectively, \( (2.3) \). The condition of second order deformation and the definition of 2-adapted frame imply that there exist frames such that

\[
(3.4)\quad e'_0 = e_0,
\]

\[
(e'_1, e'_2) \equiv (e_1, e_2) \mod e_0.
\]

We take this identification as the initial circuit for the algorithmic process of moving frame computation.

From the general theory of moving frames, \( (3.4) \) shows that one may adapt \( F' \) to normalize \( \delta \omega^0 = 0 \), \( \delta \omega^1 = 0 \), \( \delta \theta^0 = 0 \), \( \delta \theta^1 = 0 \), \( \delta \theta^2 = 0 \).

Differentiating these equations, one gets

\[
(\delta \omega^1 - \delta \omega^0 \delta \omega^2 \delta \omega^1 - \delta \omega^0 \delta \omega^0) \wedge (\omega^1 \omega^2) = 0,
\]

\[
(\delta \theta^1 \delta \theta^2 \delta \theta^1 \delta \theta^2) \wedge (\omega^1 \omega^2) = 0.
\]

By Cartan’s lemma, there exist coefficients \( \delta s_{ijk}, \delta t_{ijk}; i, j, k = 1, 2 \), such that

\[
\delta \omega^i_j - i^j \delta \omega^0_k = \delta s_{ijk} \omega^k, \quad \text{where} \quad \delta s_{ijk} = \delta s_{ikj},
\]

\[
\delta \theta^i_j = \delta t_{ijk} \omega^k, \quad \text{where} \quad \delta t_{ijk} \text{ fully symmetric in indices.}
\]

The coefficients \( \{ \delta t_{ijk} \} \) depend on the second order jet of the immersion \( x' \). By the assumption of second order deformation, the cubic differential

\[
\Phi' = (\theta^i_j + \delta \theta^i_j) (\omega^i_j + \delta \omega^i_j)(\omega^j_i + \delta \omega^j_i),
\]

\[
= \Phi + \delta t_{ijk} \omega^j_i \omega^k, \quad \text{by (3.3)},
\]

must be a nonzero multiple of \( \Phi \). One may thus use the group action that corresponds to \( \delta \omega^0 \) to scale so that

\[
(3.6)\quad \delta t_{ijk} = 0.
\]

3-adapted frame. On the 2-adapted frame satisfying \( (3.6) \), set \( \delta \omega^0 = \delta s_{00k} \omega^k \). There are \( 6 + 2 = 8 \) independent coefficients in \( \{ \delta s_{ijk} = \delta s_{ikj}, \delta s_{00k} \} \). Differentiating \( \delta \theta^i_j = 0, i, j = 1, 2 \), one gets 3 linear relations among them. By the group action that corresponds to \( \{ \delta \omega^0_1, \delta \omega^0_2, \delta \eta^1_1, \delta \eta^1_2, \delta \eta^2_2 \} \), one may translate the remaining 5 coefficients so that

\[
(3.7)\quad \delta \omega^0_0 = 0,
\]

\[
\delta \omega^i_j = 0, \quad \text{for} \ i, j = 1, 2.
\]
At this step, the deformation $\delta \phi$ is reduced to

$$(3.8) \quad \delta \phi = \begin{pmatrix} \delta \omega_{01} & \delta \omega_{02} & \delta \eta_{10} & \delta \eta_{11} & \delta \eta_{20} & \delta \eta_{21} & \delta \eta_{22} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$  

Note that since all of the third order terms $\{ \delta s_{ijk}, \delta s_{00k} \}$ are absorbed by frame adaptations, a second order deformation of a nondegenerate Legendrian surface is automatically a third order deformation, see remark below Definition 3.1.

**4-adapted frame.** On the 3-adapted frame satisfying (3.7), set

$$\delta \omega_{0i} = \delta h_{0ik} \omega^k, \quad \text{for } i = 1, 2,$$

$$\delta \eta_{ij} = \delta h_{ijk} \omega^k, \quad \text{for } i, j = 1, 2.$$  

There are 10 independent coefficients in $\{ \delta h_{0ij}, \delta h_{ijk} \}$. Differentiating (3.7), one gets 5 linear relations among them. By the group action that corresponds to $\{ \delta \eta_{10}, \delta \eta_{20} \}$, one may translate $\delta h_{111} = 0, \delta h_{222} = 0$. The structure coefficients can be normalized accordingly so that

$$(3.9) \quad \delta \omega_{01} = (u_0 + u_1) \omega^1, \quad \delta \omega_{02} = (u_0 + u_2) \omega^2,$$

$$(\delta \eta_{ij}) = \begin{pmatrix} u_2 \omega^2 & u_0 (\omega^1 + \omega^2) \\ u_0 (\omega^1 + \omega^2) & u_1 \omega^1 \end{pmatrix},$$

for 3 coefficients $\{ u_0, u_1, u_2 \}$.

Note that the first equation of (3.9) implies that the web curvature of the asymptotic 3-web is invariant under the second order deformation. Let us denote the covariant derivatives of $\{ u_0, u_i \}$ by

$$du_0 = -\frac{4}{3} u_0 \omega_0 + u_{0k} \omega^k,$$

$$du_i = -\frac{4}{3} u_i \omega_0 + u_{ik} \omega^k, \quad \text{for } i = 1, 2.$$  

**5-adapted frame.** On the 4-adapted frame satisfying (3.9), set

$$\delta \eta_{i0} = \delta h_{i0k} \omega^k, \quad \text{for } i = 1, 2.$$  

By the group action that corresponds to $\{ \delta \eta_{00} \}$, one may translate $\delta h_{101} + \delta h_{202} = 0$. Introduce variables $\{ v_0, v_1, v_2; w_1, w_2 \}$, and put

$$(3.10) \quad \hat{\delta \eta}_{10} = v_0 \omega^1 + v_2 \omega^2,$$

$$\hat{\delta \eta}_{20} = v_1 \omega^1 - v_0 \omega^2,$$

$$\hat{\delta \eta}_{00} = w_1 \omega^1 + w_2 \omega^2.$$  

At this step, no more frame adaptation is available. The reduction process of moving frame method stops here. Let us denote the covariant derivatives of $\{ v_i, w_i \}$ by

$$dv_i = -2 v_i \omega_0 + v_{ik} \omega^k,$$

$$dw_i = -\frac{8}{3} w_i \omega_0 + w_{ik} \omega^k, \quad \text{for } i = 1, 2.$$
The identity from the exterior derivative
among the deformation coefficients \( \{u_0, u_1; v_0, v_1; w_1\} \). Differentiating \( \{\delta \omega_{01}, \delta \omega_{02}\} \) from (3.9), one gets
(3.11)
\[
\begin{align*}
    u_{01} &= -u_{21} - 2v_0 + v_2, \\
    u_{02} &= -u_{12} + 2v_0 + v_1.
\end{align*}
\]

Differentiating \( \{\delta \eta_{11}, \delta \eta_{22}, \delta \eta_{12}\} \), one gets
(3.12)
\[
\begin{align*}
    v_0 &= \frac{1}{2}(-v_1 + v_2), \\
    u_{12} &= -2v_1, \\
    u_{21} &= -2v_2.
\end{align*}
\]

The identity from the exterior derivative \( d(d(u_0)) = 0 \) implies
(3.13)
\[
v_{12} = 2v_{11} + v_{21} - 2v_{22} + \frac{8}{5}(a_2 - a_3)u_0.
\]

Differentiating \( \{\delta \eta_{10}, \delta \eta_{20}\} \) from (3.10) with these relations, one gets
(3.14)
\[
\begin{align*}
    v_{11} &= 3v_{21} - 6w_1 + 8w_2 - 2u_0^2 + \frac{16}{5}a_2u_0 - \frac{36}{5}a_3u_0 + 2a_4u_2 + 2a_1u_1 + 2u_1w_2, \\
    v_{22} &= 3v_{21} + 6w_2 - 4w_1 - 2u_0^2 + \frac{12}{5}a_2u_0 - \frac{32}{5}a_3u_0 + 2a_4u_2 + 2a_1u_1 + 2u_1w_2.
\end{align*}
\]

Differentiating \( \{\delta \eta_{00}\} \) from (3.10), one finally gets
(3.15)
\[
\begin{align*}
    w_{12} &= w_{21} + (2v_2 - 2v_1 + 2b_2 - 2b_3)u_0 + (2v_2 + 2b_2)u_1 + (-2v_1 - 2b_3)u_2 \\
    &\quad + (-2a_1 - 2a_3)v_1 + (2a_4 + 2a_2)v_2.
\end{align*}
\]

Proof. a) The structure equation for second order deformation becomes involutive after one prolongation with the general solution depending on five arbitrary functions of 1 variable.

b) The second order deformation \( x' \) is necessarily a third order deformation. If \( x' \) is a fourth order deformation of \( x \), \( x' \) is congruent to \( x \).

**Proposition 3.16.** Let \( x : M \hookrightarrow \mathbb{P}^5 \) be a nondegenerate Legendrian surface. Let \( x' : M \hookrightarrow \mathbb{P}^5 \) be a second order Legendrian deformation of \( x \). Let \( \pi = \phi + \delta \phi \) be the induced Maurer-Cartan form of \( x' \), (3.2), where \( \phi \) is the induced Maurer-Cartan form of \( x \). There exists a 5-adapted frame for \( x' \) such that the coefficients of \( \delta \phi \) satisfy the structure equations (3.8) through (3.15). These equations furthermore imply that:

a) The structure equation for second order deformation becomes involutive after one prolongation with the general solution depending on five arbitrary functions of 1 variable.

b) The second order deformation \( x' \) is necessarily a third order deformation. If \( x' \) is a fourth order deformation of \( x \), \( x' \) is congruent to \( x \).

**Proof.** a) We show that the structure equation for deformation becomes involutive after a partial prolongation. The identities from exterior derivatives \( d(d(v_1)) = 0, d(d(v_2)) = 0 \) determine the derivative of \( v_{21} \) by
(3.17)
\[
v_{21} \equiv (-w_{11} - 3w_{22} + 3w_{21})\omega_1 + (w_{11} - w_{22} - w_{21})\omega_2, \quad \text{mod } \omega_0; u_{11}, u_{22}; u_0, u_i, v_i, w_i.
\]

Note the relation
(3.18)
\[
\begin{pmatrix}
    du_1 \\
    du_2 \\
    dv_1 \\
    dv_2
\end{pmatrix} =
\begin{pmatrix}
    u_{11} & \cdot & u_{22} \\
    \cdot & u_{22} \\
    w_{11} & u_{21} \\
    w_{21} & w_{22}
\end{pmatrix}
\begin{pmatrix}
    \omega_1 \\
    \omega_2
\end{pmatrix}, \quad \text{mod } \omega_0; u_0, u_i, v_i, w_i.
\]

By inspection, the structure equations (3.17) and (3.18) are in involution with the last nonzero Cartan character \( s_1 = 5 \).

b) For the first part, see the remark at the end of 3-adapted frame. For the second part, the condition for the fourth order deformation implies \( u_0 = u_1 = u_2 = 0 \). The compatibility equations (3.12) and (3.14)
then force the remaining deformation coefficients to vanish so that \( \delta \phi = 0 \). The rest follows from the uniqueness theorem of ODE, \( \Box \).

We shall examine the second order Legendrian deformation with the additional condition that it preserves the fourth order differential \( \Psi \), or that it preserves both \( \Psi \) and the fifth order differential \( \chi \). The primary object of our analysis will be to give characterization of such surfaces that support maximum parameter family of nontrivial deformations.

4. \( \Psi \)-deformation

In this section, we apply the fundamental structure equation for second order deformation to the geometric situation where the deformation is required to preserve a part of fourth order invariants of a Legendrian surface.

**Definition 4.1.** Let \( x : M \hookrightarrow \mathbb{P}^5 \) be a nondegenerate Legendrian surface. Let \( x' : M \hookrightarrow \mathbb{P}^5 \) be a second order deformation of \( x \). \( x' \) is a \( \Psi \)-deformation if the application map \( g : M \to \text{Sp}_3 \mathbb{C} \) for the second order deformation \( x' \) is such that for each \( p \in M \), the fourth order cubic differential \( \Psi' \) of \( x' \) and \( \Psi \) of \( g(p) \circ x \) are isomorphic at \( p \).

As noted in Proposition 3.16, there is no local obstruction for the second order deformation of a Legendrian surface, whereas if one requires the second order deformation to preserve all of the fourth order invariants, the deformation is necessarily a congruence. The idea is to impose a condition that balances between these two extremes.

Let us give a summary of results in this section. The condition for a second order deformation to be a \( \Psi \)-deformation is expressed as a pair of linear equations on the deformation coefficients, (4.3). A more or less basic over-determined PDE analysis of these equations shows that the resulting structure equation for \( \Psi \)-deformation closes up admitting at most three parameter family of solutions, (4.7). The class of isothermally asymptotic surfaces with flat asymptotic web discussed in Section 2.4.1 are examples of such surfaces admitting maximum parameter family of \( \Psi \)-deformations, which we call \( D_0 \)-surfaces, (4.15). Analysis of the structure equation shows that there exist another class of surfaces with finite local moduli that admit maximum parameter family of \( \Psi \)-deformations, which we call \( D \)-surfaces, (4.18). \( D_0 \)-surfaces and \( D \)-surfaces account for the set of maximally \( \Psi \)-deformable Legendrian surfaces, Theorem 4.21. Further analysis shows that there exist subsets called \( S_0 \)-surfaces and \( S \)-surfaces which admit \( \Psi \)-deformations that also preserve the fifth order differential \( \chi \).

We continue the analysis of Section 3.

4.1. *Structure equation.* Let \( x : M \hookrightarrow \mathbb{P}^5 \) be a nondegenerate Legendrian surface. Let \( x' : M \hookrightarrow \mathbb{P}^5 \) be a \( \Psi \)-deformation of \( x \). Let \( \pi = \phi + \delta \phi \) be the induced Maurer-Cartan form of \( x' \), where \( \phi \) is the induced Maurer-Cartan form of \( x \). From (3.9), the deformation of the invariant differentials \( \Psi \) and \( \chi \) are given by

\[
\delta \Psi = \delta (\eta_{ij} \omega^i \omega^j),
\]

\[
= (u_2 + 2u_0) (\omega^1)^2 \omega^2 + (u_1 + 2u_0) \omega^1 (\omega^2)^2,
\]

\[
\delta \chi = \delta (\eta_{10} \omega^1 + \eta_{20} \omega^2),
\]

\[
= v_0 (\omega^1)^2 + (v_1 + v_2) \omega^1 \omega^2 - v_0 (\omega^2)^2.
\]

The condition for the deformation to preserve \( \Psi \) is expressed by the pair of linear equations

\[
(4.3) \quad u_1 + 2u_0 = 0, \quad u_2 + 2u_0 = 0.
\]

We wish to give an analysis of the compatibility equations for the deformation \( \delta \phi \) derived from (4.3).
Differentiating (4.3), one gets

\[ v_2 = -v_1, \]
\[ u_{11} = 2v_1, \]
\[ u_{22} = -2v_1. \]

Since \( v_0 = \frac{1}{2}(-v_1 + v_2), \) we observe that a \( \Psi \)-deformation leaves \( \chi \) invariant when \( v_1 = v_2 = 0. \)

Differentiating \( v_1 + v_2 = 0, \) one gets

\[ w_2 = w_1 + \frac{6}{5}(-a_2 + a_3)u_0, \]
\[ v_{21} = -\frac{1}{2}w_1 - \frac{3}{2}u_0^2 + \left( a_4 + a_1 + \frac{8}{5}a_2 - \frac{3}{5}a_3 \right)u_0. \]

Differentiating the first equation of (4.5) for \( w_2, \) one gets

\[ w_{21} = w_{11} + \frac{6}{5}(a_2 - a_3)v_1 + (9b_3 + 12b_1 - 3a_{21})u_0, \]
\[ w_{22} = w_{11} + 4v_1u_0 + (-2a_4 - 2a_1 - 2a_3 - 2a_2)v_1 + (-12b_1 - 4b_3 + 4b_2)u_0. \]

The identity from the exterior derivative \( d(d(v_1)) = 0 \) with these relations finally gives

\[ w_{11} = -2u_0v_1 + \left( -\frac{12}{5}a_3 + \frac{22}{5}a_2 + a_4 + a_1 \right)v_1 + \left( 24b_1 - 4b_2 + 15b_3 + a_{41} + a_{12} - 3a_{21} \right)u_0. \]

At this step, the remaining independent deformation coefficients are \( \{ u_0, v_1, w_1 \}. \) Moreover, they satisfy a closed structure equation, i.e., their derivatives are expressed as functions of themselves and do not involve any new variables. Let us record the structure equations for \( \{ u_0, v_1, w_1 \}. \)

\[ du_0 + \frac{4}{3}u_0\omega_{00} = v_1(-\omega^1 + \omega^2), \]
\[ dv_1 + 2v_1\omega_{00} = \left( \frac{3}{2}u_0^2 + \left( \frac{3}{5}a_3 - a_4 - a_1 - \frac{8}{5}a_2 \right)u_0 + \frac{1}{2}w_1 \right)\omega^1 \]
\[ + \left( -\frac{3}{2}u_0^2 + \left( a_3 + a_4 + a_1 \right)u_0 - \frac{1}{2}w_1 \right)\omega^2, \]
\[ dw_1 + \frac{8}{3}w_1\omega_{00} = w_{11}\omega^1 + w_{12}\omega^2, \]

where \( w_{11} \) is in (4.6) and

\[ w_{12} = 2u_0v_1 + \left( -\frac{28}{5}a_3 + \frac{18}{5}a_2 - a_4 - a_1 \right)v_1 + \left( 36b_1 - 6b_2 + 26b_3 + a_{41} + a_{12} - 6a_{21} \right)u_0. \]

Exterior derivative \( d(d(w_1)) = 0 \) gives a universal integrability condition for \( \Psi \)-deformation;

\[ (-60b_1 + 8b_2 - 42b_3 - a_{41} - a_{12} + 10a_{21})v_1 + \frac{12}{5}(a_2 - a_3)w_1 \equiv 0 \mod u_0. \]

The full expression for the right hand side of (4.8) is given by

\[
\text{RHS of (4.8)} = -\frac{44}{15}(a_2 - a_3)u_0^2 + (-5c_1 + 8c_2 - 17b_{11} + 10b_{21} - \frac{4}{3}b_{22} - \frac{26}{3}b_{31})v_1
\]
\[ + \left( \frac{32}{15}a_4a_2 - \frac{32}{15}a_4a_3 - \frac{112}{25}a_3^2 + \frac{123}{25}a_2^2 - \frac{86}{25}a_2a_3 - \frac{32}{15}a_1a_3 \right)u_0
\]
\[ + \left( \frac{32}{15}a_1a_2 + 3a_4a_1 - \frac{1}{3}a_{121} + 2a_{211} - \frac{1}{3}a_{411} - a_{212} + \frac{1}{3}a_{122} + \frac{1}{3}a_{412} \right)u_0, \]

where \( a_{ijk} \) denote the covariant derivative of \( a_{ij} \) as before.
Proposition 4.9. Let $M \hookrightarrow \mathbb{P}^5$ be a nondegenerate Legendrian surface. A $\Psi$-deformation of $M$ is determined by three parameters $\{u_0, v_1, w_1\}$ by (4.3), (4.4), and (4.5). These three deformation parameters satisfy a closed structure equation (4.7). The structure equation, and the universal integrability condition (4.8) imply that:

a) A nondegenerate Legendrian surface admits at most three parameter family of $\Psi$-deformations.

b) If the asymptotic web of $M$ is not flat, $M$ admits at most two parameter family of $\Psi$-deformations.

c) Assume the asymptotic web of $M$ is flat. If the structure coefficients of $M$ do not satisfy the differential relation $(a_{41} + a_{12} + 4b_2 - 4b_1) = 0$, $M$ admits at most one parameter family of $\Psi$-deformations.

d) Assume the asymptotic web of $M$ is flat and the structure coefficients satisfy the relation $(a_{41} + a_{12} + 4b_2 - 4b_1) = 0$. If the structure coefficients of $M$ do not satisfy the additional relation $c_1 = c_2$, $M$ does not admit nontrivial $\Psi$-deformations.

e) A nondegenerate Legendrian surface $M$ admits maximum three parameter family of $\Psi$-deformations if, and only if $M$ has flat asymptotic web, and the structure coefficients of $M$ satisfy the following relations.

\[
\begin{align*}
a_2 &= a_3, \\
b_1 - b_2 &= \frac{1}{4}(a_{41} + a_{12}), \\
c_1 &= c_2.
\end{align*}
\]

Proof. a) It follows from the uniqueness theorem of ODE, [Gr].

b) If the web curvature (2.25) of the asymptotic web does not vanish identically, one can solve (4.8) for $w_1$ on a dense open subset of $M$.

c) The asymptotic web is flat when $a_2 - a_3 = 0$. By the structure equations from Section 2.2, (4.8) is reduced to

\[
v_1(a_{41} + a_{12} + 4b_2 - 4b_1) \equiv 0 \mod u_0.
\]

Under the assumption of c), one can solve for $v_1$ as a function of $u_0$. Differentiating this, (4.7) implies that $w_1$ is also determined as a function of $u_0$.

d) and e) When $a_2 - a_3 = 0$ and $a_{41} + a_{12} + 4b_2 - 4b_1 = 0$, (4.8) is reduced to

\[
u_0(c_1 - c_2) = 0.
\]

If $c_1 - c_2 = 0$, the structure equation (4.7) is compatible and admits solutions with maximum three dimensional moduli. If $c_1 - c_2$ does not vanish identically, $u_0 = 0$. The structure equation (4.7) then implies $v_1 = w_1 = 0$. □

Example 4.10. The analysis of Section 2 shows that the following classes of Legendrian surfaces admit maximum three parameter family of nontrivial $\Psi$-deformations.

a) $\Psi$-null surfaces, Section 2.3

b) Isothermally asymptotic surfaces with flat asymptotic web, Section 2.4.1

c) Tri-ruled surfaces, Section 2.5.1

Note that a) and c) are subsets of b).

It is evident that a generic nondegenerate Legendrian surface does not admit any nontrivial $\Psi$-deformations. In consideration of the main theme of the paper, to understand Legendrian surfaces with special characteristics, we do not pursue to formulate the explicit criteria for $\Psi$-rigidity.

4.2. Surfaces with maximum $\infty^3$ $\Psi$-deformations. The structure of the moduli space of solutions to the deformation equation (4.7) depends on the geometry of the base Legendrian surface. Among the variety of cases, we consider in this subsection the class of Legendrian surfaces that admit maximum three parameter family of $\Psi$-deformations. The rationale for this choice comes from the fact that Kummer’s quartic surface constitutes an example of Cartan’s maximally third order deformable surfaces in $\mathbb{P}^3$, [Fe].
Let $M \hookrightarrow \mathbb{P}^5$ be a nondegenerate Legendrian surface with maximum three parameter family of $\Psi$-deformations. From e) of Proposition 4.9, such surfaces are characterized by the following three relations on the structure coefficients:

\begin{align}
& a_2 - a_3 = 0, \\
& b_1 - b_2 = \frac{1}{4} (a_{41} + a_{12}), \\
& c_1 - c_2 = 0.
\end{align}

We wish to give an analysis of the compatibility conditions derived from these relations, and determine the structure equation for the maximally $\Psi$-deformable surfaces.

Since $M$ has flat asymptotic web, let us assume the results of Section 2.2 and continue the analysis from that point on. Differentiating the third equation of (4.11), one gets

\begin{align}
& c_{21} = c_{11}, \\
& c_{22} = c_{11} + (4a_1 + 4a_2)b_1 + (-2a_1 + 2a_4)b_2.
\end{align}

The remaining undetermined derivative coefficients at this step are $\{a_{41}, c_{11}; b_{11}\}$. The identities from exterior derivatives $d(d(a_1)) = 0$, $d(d(a_4)) = 0$ determine the derivative of $a_{41}$ by

\begin{align}
& da_{41} + 2a_{41}\omega_{00} = 2(b_{11} - a_2^2 + c_1 + a_4a_1)(\omega^1 + \omega^2) + 4b_{11}\omega^2.
\end{align}

Moreover, $d(d(a_{41})) = 0$ is an identity.

Exterior derivative $d(d(b_{11})) = 0$, (2.27), with these relations gives the universal integrability condition to admit maximum $\infty^3$ family of $\Psi$-deformations.

\begin{align}
& (a_1 - a_4)c_1 + 2(a_1 - a_4)b_{11} + 3(-b_1 + b_2)(a_{41} + 2b_2) + (a_1 - a_4)(a_4a_1 - a_2^2) = 0.
\end{align}

At this juncture, the analysis divides into two cases.

Case $a_1 - a_4 = 0$. This is the case of isothermally asymptotic surfaces with flat asymptotic web. As noted in Example 4.10, this class of surfaces satisfy the defining relations (4.11) and admit maximum three parameter family of $\Psi$-deformations.

**Definition 4.15.** A Legendrian $D_0$-surface is an immersed, nondegenerate Legendrian surface in $\mathbb{P}^5$ which is isothermally asymptotic with flat asymptotic 3-web.

Let us record the full structure equation for $D_0$-surfaces.

\begin{align}
& da_1 + \frac{4}{3} a_1\omega_{00} = -2b_2\omega^1 + (4b_1 - 2b_2)\omega^2, \\
& da_2 + \frac{4}{3} a_2\omega_{00} = (-2b_1 + 3b_2)\omega^1 + (-4b_1 + 3b_2)\omega^2, \\
& db_1 + 2b_1\omega_{00} = b_{11}(\omega^1 - \omega^2), \\
& db_2 + 2b_2\omega_{00} = (-b_{11} + a_2^2 - c_1 - a_1^2)(\omega^1 + \omega^2) - 2b_{11}\omega^2, \\
& dc_1 + \frac{8}{3} c_1\omega_{00} = c_{11}(\omega^1 + \omega^2) + 4b_1(a_1 + a_2)\omega^2, \\
& db_{11} + \frac{8}{3} b_{11}\omega_{00} = b_1(2a_2 + 3a_1)(\omega^1 - \omega^2).
\end{align}
The induced Maurer-Cartan form $\phi$ takes the following form. 

\begin{equation}
\phi = \begin{pmatrix}
\omega_{00} & (a_1 + a_2)\omega^1 & (a_1 + a_2)\omega^2 & c_1(\omega^1 + \omega^2) & b_1\omega^1 + b_2\omega^2 & (-2b_1 + 2b_2)\omega^1 - b_1\omega^2 \\
\omega^1 & \frac{1}{3}\omega_{00} & \cdot & b_1\omega^1 + b_2\omega^2 & a_1\omega^2 & a_2(\omega^1 + \omega^2) \\
\omega^2 & \cdot & \frac{1}{3}\omega_{00} & (2b_1 + 2b_2)\omega^1 - b_1\omega^2 & a_2(\omega^1 + \omega^2) & a_1\omega^1 \\
\cdot & \cdot & \cdot & -\omega_{00} & -\omega^1 & -\omega^2 \\
\cdot & \omega^2 & \omega^1 + \omega^2 & (a_1 + a_2)\omega^1 & -\frac{1}{3}\omega_{00} & \cdot \\
\cdot & \omega^1 + \omega^2 & \omega^1 & (a_1 + a_2)\omega^2 & \cdot & -\frac{1}{3}\omega_{00}
\end{pmatrix}
\end{equation}

Case $a_1 - a_4 \neq 0$. The structure equation closes up in this case. First, solve (4.14) for $c_1$. Differentiating this, one can solve for $c_{11}$. At this step, the structure equation for this class of surfaces closes up with 7 independent structure coefficients $\{a_1, a_2, a_4, b_1, b_2; a_{41}, b_{11}\}$. Moreover, an analysis shows that the resulting structure equation is compatible, i.e., $d^2 = 0$ is an identity and does not impose any new compatibility conditions.

**Definition 4.18.** A Legendrian $D$-surface is an immersed, nondegenerate Legendrian surface in $\mathbb{P}^5$ which satisfies the following conditions.

a) it is not isothermally asymptotic,

b) it has flat asymptotic 3-web, and satisfies the differential relations (4.11), (4.12), (4.13), and (4.14).

Let us record the full structure equation for $D$-surfaces.

\begin{equation}
da_1 + \frac{4}{3}a_1\omega_{00} = -2b_2\omega^1 + (-a_{41} - 4b_1)\omega^2, \\
da_2 + \frac{4}{3}a_2\omega_{00} = (2b_1 + 3b_2)\omega^1 + (-4b_1 + 3b_2)\omega^2, \\
da_4 + \frac{4}{3}a_4\omega_{00} = a_{41}\omega^1 + (4b_1 - 2b_2)\omega^2, \\
db_1 + 2b_1\omega_{00} = b_{11}(\omega^1 - \omega^2), \\
db_2 + 2b_2\omega_{00} = (b_{11} + c_1 + a_{41})\omega^1 + (\omega^1 + \omega^2), \\
db_4 + 2a_{41}\omega_{00} = (b_{11} - a_1 - a_2^2 + c_1 + a_{41})\omega^1 + (\omega^1 + \omega^2), \\
db_{11} + \frac{8}{3}b_{11}\omega_{00} = -\frac{1}{4}(8b_1a_1 + 4b_2a_4 + a_{41}a_1 - 8b_1a_2 - 4b_1a_4 + a_{41}a_4)(\omega^1 - \omega^2).
\end{equation}

The induced Maurer-Cartan form $\phi$ takes the following form.

\begin{equation}
\phi = \begin{pmatrix}
\omega_{00} & (a_4 + a_2)\omega^1 & (a_1 + a_2)\omega^2 & c_1(\omega^1 + \omega^2) & b_1\omega^1 + b_2\omega^2 & (-2b_1 + 2b_2)\omega^1 - b_1\omega^2 \\
\omega^1 & \frac{1}{3}\omega_{00} & \cdot & b_1\omega^1 + b_2\omega^2 & a_1\omega^2 & a_2(\omega^1 + \omega^2) \\
\omega^2 & \cdot & \frac{1}{3}\omega_{00} & (2b_1 + 2b_2)\omega^1 - b_1\omega^2 & a_2(\omega^1 + \omega^2) & a_1\omega^1 \\
\cdot & \cdot & \cdot & -\omega_{00} & -\omega^1 & -\omega^2 \\
\cdot & \omega^2 & \omega^1 + \omega^2 & (a_4 + a_2)\omega^1 & -\frac{1}{3}\omega_{00} & \cdot \\
\cdot & \omega^1 + \omega^2 & \omega^1 & (a_1 + a_2)\omega^2 & \cdot & -\frac{1}{3}\omega_{00}
\end{pmatrix}
\end{equation}

where $c_1$ is given by (4.14).

**Theorem 4.21.** The set of nondegenerate Legendrian surfaces in $\mathbb{P}^5$ which admit maximum three parameter family of $\Psi$-deformations fall into two categories: Legendrian $D_0$-surfaces, or Legendrian $D$-surfaces. A general Legendrian $D_0$-surface depends on one arbitrary function of 1 variable, whereas a general Legendrian $D$-surface depends on four constants.
**Proof.** The generality of solutions for the structure equation for $D_0$-surfaces is treated in Proposition 2.38. For the generality of $D$-surfaces, consider the invariant map $I = (a_1, a_2, a_4, b_1, b_2; a_{11}, b_{11}) : F \to \mathbb{C}^7$, where $F$ is the canonical bundle of 5-adapted frames. Since $I$ generically has rank 3, the local moduli space of $D$-surfaces has general dimension dim $(\mathbb{C}^7) - \text{rank}(I) = 4$. □

The analogy of Theorem 4.21 with Cartan’s classification of maximally third order deformable surfaces in $\mathbb{P}^3$ is obvious, [Ca]. Cartan’s classification is also divided into two cases; one case with infinite dimensional local moduli, and the other case with finite dimensional local moduli. This analogy in a way conversely justifies our choice of $\Psi$-deformations.

### 4.3. $(\Psi, \chi)$-deformations

In this subsection, we examine which of the maximally $\Psi$-deformable surfaces admit deformations that leave invariant both $\Psi$ and the fifth order quadratic differential $\chi$, (2.15).

**Definition 4.22.** Let $x : M \hookrightarrow \mathbb{P}^5$ be a nondegenerate Legendrian surface. Let $x' : M \hookrightarrow \mathbb{P}^5$ be a $\Psi$-deformation of $x$. $x'$ is a $(\Psi, \chi)$-deformation if the application map $g : M \to \text{Sp}_3\mathbb{C}$ for the $\Psi$-deformation $x'$ is such that for each $p \in M$, the fifth order quadratic differentials $\chi'$ of $x'$ and $\chi$ of $g(p) \circ x$ are isomorphic at $p$.

Let us give a summary of results in this subsection. The condition for a $\Psi$-deformation to be a $(\Psi, \chi)$-deformation is expressed by a single linear equation on the deformation coefficients, (4.23). An over-determined PDE analysis of this equation shows that the resulting structure equation for $(\Psi, \chi)$-deformations closes up admitting at most one parameter family of solutions, (4.24). The structure equation for the subset of maximally $\Psi$-deformable surfaces which admit one parameter family of $(\Psi, \chi)$-deformations is then determined, Theorem 4.34.

We continue the analysis of Section 4.2, specifically from (4.14).

Let $x : M \hookrightarrow \mathbb{P}^5$ be a nondegenerate, maximally $\Psi$-deformable Legendrian surface. Let $x' : M \hookrightarrow \mathbb{P}^5$ be a $\Psi$-deformation of $x$. Let $\pi = \phi + \delta \phi$ be the induced Maurer-Cartan form of $x'$, where $\phi$ is the induced Maurer-Cartan form of $x$. From (4.2), the condition for the deformation to preserve $\chi$ is expressed by the single linear equation

$$v_1 = 0.$$  

We wish to give an analysis of the compatibility equations for the $\Psi$-deformation $\delta \phi$ derived from (4.23).

Differentiating (4.23), one gets

$$w_1 = u_0(-3 u_0 + 2 a_1 + 2 a_2 + 2 a_4).$$

Since $v_1 = 0$ and $w_1$ is a function of $u_0$, there exists at most one parameter family of $(\Psi, \chi)$-deformations. Differentiating the equation for $w_1$ again, one gets the integrability equation

$$u_0(a_{41} - 4 b_1 + 2 b_2) = 0.$$  

If $a_{41} - 4 b_1 + 2 b_2 \neq 0$, this forces $u_0 = 0$ and the deformation is trivial. Hence we must have

$$a_{41} - 4 b_1 + 2 b_2 = 0.$$  

**Remark 4.26.** A similar analysis shows that for a general nondegenerate Legendrian surface, either it admits maximum one parameter family of $(\Psi, \chi)$-deformations, or it does not admit any such deformations. The Legendrian surfaces which admit $(\Psi, \chi)$-deformations are characterized by the following two relations on the structure coefficients;

$$a_2 - a_3 = 0,$$

$$a_{41} - a_{12} = 4 b_1.$$
Successively differentiating (4.25), one gets a set of three compatibility equations.

\[
\begin{align*}
(4.27) \quad & b_1 = 0, \quad b_{11} = 0, \\
& (a_1 - a_4)b_2 = 0, \\
& (a_1 - a_4)(c_1 + a_1a_4 - a_2^2) = 0.
\end{align*}
\]

At this juncture, the analysis divides into two cases.

**Case** \(a_1 - a_4 = 0\). This is a subset of \(D_0\)-surfaces. It is easily checked that the structure equation (4.16) remains in involution with the additional condition \(b_1 = b_{11} = 0\).

**Definition 4.28.** A Legendrian \(S_0\)-surface is a Legendrian \(D_0\)-surface for which the structure coefficients satisfy the additional relation \(b_1 = b_{11} = 0\).

Let us record the full structure equation for \(S_0\)-surfaces.

\[
(4.29) \quad da_1 + \frac{4}{3}a_1\omega_{00} = -2b_2(\omega^1 + \omega^2),
\]
\[
da_2 + \frac{4}{3}a_2\omega_{00} = 3b_2(\omega^1 + \omega^2),
\]
\[
db_2 + 2b_2\omega_{00} = -(c_1 + a_1^2 - a_2^2)(\omega^1 + \omega^2),
\]
\[
dc_1 + \frac{8}{3}c_1\omega_{00} = c_{11}(\omega^1 + \omega^2).
\]

The induced Maurer-Cartan form \(\phi\) takes the following form.

\[
(4.30) \quad \phi = \begin{pmatrix}
\omega_{00} & (a_1 + a_2)\omega^1 & (a_1 + a_2)\omega^2 & c_1(\omega^1 + \omega^2) & b_2\omega^2 & b_2\omega^1 \\
\omega^1 & \frac{1}{3}\omega_{00} & . & b_2\omega^2 & a_1\omega^2 & a_2(\omega^1 + \omega^2) \\
\omega^2 & . & \frac{1}{3}\omega_{00} & b_2\omega^1 & a_2(\omega^1 + \omega^2) & a_1\omega^1 \\
. & . & . & -\omega_{00} & -\omega^1 & -\omega^2 \\
. & . & \omega^1 + \omega^2 & \omega^1 + \omega^2 & -(a_1 + a_2)\omega^1 & -\frac{1}{3}\omega_{00} \\
. & . & \omega^1 + \omega^2 & \omega^1 & -(a_1 + a_2)\omega^2 & -\frac{1}{3}\omega_{00}
\end{pmatrix}
\]

Note that the subset of \(\Psi\)-null surfaces with the structure coefficient \(b = 0\), and tri-ruled surfaces are examples of Legendrian \(S_0\)-surfaces.

**Case** \(a_1 - a_4 \neq 0\). This is a subset of \(D\)-surfaces. It is easily checked that the structure equation (4.16) remains compatible with the additional condition \(b_1 = b_2 = a_{41} = b_{11} = 0, \quad c_1 = a_2^2 - a_1a_4\).

**Definition 4.31.** A Legendrian \(S\)-surface is a Legendrian \(D\)-surface for which the structure coefficients satisfy the additional relation \(b_1 = b_2 = a_{41} = b_{11} = 0, \quad c_1 = a_2^2 - a_1a_4\).

Let us record the full structure equation for \(D\)-surfaces.

\[
(4.32) \quad da_1 + \frac{4}{3}a_1\omega_{00} = 0,
\]
\[
da_2 + \frac{4}{3}a_2\omega_{00} = 0,
\]
\[
da_4 + \frac{4}{3}a_4\omega_{00} = 0.
\]
The induced Maurer-Cartan form $\phi$ takes the following form.

$$
\phi = \begin{pmatrix}
\omega_{00} & (a_4 + a_2)\omega^1 & (a_1 + a_2)\omega^2 & (a_2^2 - a_1 a_4)(\omega^1 + \omega^2) & \cdot & \cdot \\
\omega^1 & \frac{1}{3}\omega_{00} & \cdot & \cdot & a_1\omega^2 & a_2(\omega^1 + \omega^2) \\
\omega^2 & \cdot & \frac{1}{3}\omega_{00} & \cdot & a_2(\omega^1 + \omega^2) & a_4\omega^1 \\
\cdot & \cdot & -\omega_{00} & -\omega^1 & -\omega^2 & \cdot \\
\cdot & \omega^1 + \omega^2 & -a_1 a_2\omega^1 & -\frac{1}{3}\omega_{00} & \cdot & \cdot \\
\cdot & \omega^1 + \omega^2 & \omega & -a_1 a_2\omega^2 & -\frac{1}{3}\omega_{00} & \cdot \\
\end{pmatrix}
$$

(4.33)

Notice that when $a_4 = a_1$, the structure equation for $S$-surfaces degenerates to the structure equation for $S_0$-surfaces with the additional condition $b_2 = 0$, $c_1 = -a_1^2 + a_2^2$.

**Theorem 4.34.** The set of maximally $\Psi$-deformable Legendrian surfaces in $\mathbb{P}^5$ which admit one parameter family of $(\Psi, \chi)$-deformations fall into two categories: Legendrian $S_0$-surfaces, or Legendrian $S$-surfaces. A general Legendrian $S_0$-surface depends on one arbitrary function of 1 variable, whereas a general Legendrian $S$-surface depends on two constants.

**Proof.** The structure equation for $S_0$-surfaces is in involution with the last nonzero Cartan character $s_1 = 1$ (we omit the details). For $S$-surfaces, consider the invariant map $I = (a_1, a_2, a_4) : F \to \mathbb{C}^3$, where $F$ is the canonical bundle of $S$-adapted frames. Since $I$ generically has rank 1, the local moduli space of $S$-surfaces has general dimension $\dim (\mathbb{C}^3) - \text{rank}(I) = 2$. □

5. Examples

In this final section, we give a differential geometric characterization of tri-ruled surfaces, Section 2.5.1, which are examples of Legendrian $S_0$-surfaces. In Section 5.1 the flat case is characterized as a part of a Legendrian map from $\mathbb{P}^2$ blown up at three distinct collinear points. In Section 5.2 the non-flat case is characterized as a part of a Legendrian embedding from $\mathbb{P}^2$ blown up at three non-collinear points. In both cases, the Legendrian map is given by a system of cubics through the three points.

Let $(X_0, X_1, X_2, Y_0, Y_1, Y_2)$ be the standard adapted coordinate of $\mathbb{C}^6$ such that the symplectic 2-form $\omega = dX_0 \wedge dY_0 + dX_1 \wedge dY_1 + dX_2 \wedge dY_2$.

5.1. Flat surface. This is the class of surface for which all the structure coefficients vanish; $a_i = b_j = c_k = 0$.

Since $d\omega_{00} = 0$, take a section of the frame for which $\omega_{00} = 0$, and $d\omega^1 = d\omega^2 = 0$ consequently. Introduce a local coordinate $(x, y)$ such that $\omega^1 = dx$, $\omega^2 = dy$, and express the Maurer-Cartan form $\phi = A dx + B dy$ for constant coefficient matrices $A$, $B$. Since $d\phi = -\phi \wedge \phi = 0$, $A$ and $B$ commute and $g = \exp(Ax + By)$ is a solution of the defining equation

$$g^{-1} dg = \phi.$$

The exponential can be computed, and by definition of $\phi$ in Section 2, the first column of $g$ gives the following local parametrization of the flat Legendrian surface.

$$x_1(x, y) = \begin{pmatrix}X_0 \\ X_1 \\ X_2 \\ Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix}1 \\ x \\ y \\ -\frac{x^2}{2}(x + y) \\ xy + \frac{1}{2} \\ xy + \frac{x^2}{2} \end{pmatrix}.$$
Theorem 5.1. Let \( \pi : M_3 \rightarrow \mathbb{P}^2 \) be the rational surface obtained by blowing up \( \mathbb{P}^2 \) at three distinct collinear points \( \{ p_1, p_2, p_3 \} \). Let \( L \) be the line through \( p_i \)'s, \( E_i = \pi^{-1}(p_i) \) be the exceptional divisor, and let \( H \) be the linear divisor of \( \mathbb{P}^2 \). Let \( \tilde{L} \) be the -2-curve, the proper transform of \( L \). There exists a six dimensional proper subspace \( W \) of the linear system \( [\pi^*(3H) - E_1 - E_2 - E_3] \) which gives a Legendrian map \( \tilde{x} : M_3 \rightarrow \mathbb{P}^5. \) \( \tilde{x} \) is an embedding on \( M_3 - \tilde{L} \), and it degenerates to a point on \( \tilde{L} \).

a) A flat Legendrian surface is locally equivalent to a part of \( \tilde{x}(M_3 - \bigcup E_i) \).

b) The system \( W \) for \( \tilde{x} \) is a six dimensional subspace of the proper transform of the set of cubics through \( p_i \)'s. Each -1-curve \( E_i \) is mapped to a line under \( \tilde{x} \).

c) The asymptotic web is given by the proper transform of the three pencils of lines through \( p_i \)'s.

Proof of theorem is presented below in four steps.

Step 1. Consider the birational map \( x : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5 \) associated to \( x_1(x, y) \) defined by

\[
(5.2) \quad x([x, y, z]) = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} z^3 \\ xz^2 \\ yz^2 \\ -\frac{xy}{2}(x + y) \\ (xy + \frac{x^2}{2}z) \\ (xy + \frac{x^2}{2}z) \end{pmatrix},
\]

where \([x, y, z]\) is the standard projective coordinate of \( \mathbb{P}^2 \). It is undefined at three points

\[
p_1 = [1, 0, 0], \quad p_2 = [0, 1, 0], \quad p_3 = [1, -1, 0].
\]

At \( p_1 \), introduce the parametrization of the blow up by \([x, y, z] = [1, \lambda_1 z, \lambda_1 z]\) for the blow up parameter \( \lambda_1 \). The birational map becomes

\[
(5.3) \quad x([x, y, z]) = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{z^3}{\lambda_1 z^3} \\ \frac{z^2}{\lambda_1 z^2} \\ \frac{\lambda_1 z^2}{\lambda_1 z + \frac{1}{2}} \\ 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \\ -\frac{\lambda_1}{2} \\ 0 \\ 0 \end{pmatrix}, \quad \text{as } z \rightarrow 0.
\]

Similar formulae for \( p_2, p_3 \) show that the exceptional divisors \( E_2, E_3 \) are mapped to

\[
(5.4) \quad E_2 \rightarrow \begin{pmatrix} 0 \\ 0 \\ -\frac{\lambda_1}{2} \\ 0 \end{pmatrix}, \quad E_3 \rightarrow \begin{pmatrix} 0 \\ 0 \\ \frac{\lambda_1}{2} \\ -\frac{1}{2} \end{pmatrix}.
\]

Let \( L = \{z = 0\} \subset \mathbb{P}^2 \) be the line through \( p_i \)'s. By definition, \( x(L) = [0, 0, 0, 1, 0, 0] = x_0 \in \mathbb{P}^5 \). One may check that \( x : \mathbb{P}^2 - L \rightarrow \mathbb{P}^5 \) is an embedding, and that the image \( x(\mathbb{P}^2 - L) \) is disjoint from the exceptional loci \([5.3], [5.4]\).

A computation with \([5.3]\) at \( p_1 \), and similar computations at \( p_2, p_3 \) show that the associated lift \( \hat{x} : M_3 \rightarrow \mathbb{P}^5 \) is well defined and holomorphic, and that \( \hat{x} : M_3 - \tilde{L} \rightarrow \mathbb{P}^5 \) is a smooth embedding.
Step 2. Consider alternatively the following polynomial equations satisfied by $x$.

$$3X_0Y_0 + X_1Y_2 + X_2Y_2 = 0,$$

$$X_0^2Y_0 + \frac{1}{2}X_1X_2(X_1 + X_2) = 0,$$

$$X_0Y_1 - X_1X_2 - \frac{1}{2}X_2^2 = 0,$$

$$X_0Y_2 - X_1X_2 - \frac{1}{2}X_2^2 = 0,$$

$$\left( X_1(Y_2 - \frac{1}{2}Y_1) - X_2(Y_1 - \frac{1}{2}Y_2) \right) Y_0 = Y_1Y_2(Y_1 - Y_2).$$

By a direct computation, one can verify that this set of equations have rank 3 on $\hat{x}(M_3) = x([p^2 - L])$, except at $x_0$. One can also check that at $x_0$, $\hat{x}(M_3)$ is not smooth and has a second order branch type singularity (we omit the details. Note that $\hat{L}$ is a -2-curve and it cannot be blown down).

Step 3. Let $D$ the hyperplane section $D = \hat{x}^{-1}\{ Y_0 = 0 \}$. From (5.2), the divisor consists of the proper transform of three lines $\{ L_1, L_2, L_3 \} = \{ y = 0, \text{ or } x = 0, \text{ or } x + y = 0 \} \subset \mathbb{P}^2$. Hence the linear system

$$[D] = \left[ \sum_i (\pi^*(H) - E_i) \right] = [\pi^*(3H) - E_1 - E_2 - E_3].$$

By definition, $W$ is a subspace of the linear system $[\pi^*(3H) - E_1 - E_2 - E_3]$, the proper transform of cubics through $p_i$’s. Finally, $\langle D, E_i \rangle = 1$, and each $E_i$ is mapped to a line.

Step 4. One may check by direct computation that the asymptotic web is given by the foliations $dy = 0, dx = 0, dx + dy = 0$, which represent three pencils of lines through $p_1, p_2, p_3$ respectively.

5.1.1. Generalization. The construction of flat surface admits a straightforward generalization.

Let $f_k(x, y)$ be a homogeneous polynomial of degree $k$ for $k = 3, 4, \ldots, m$, such that the top degree $f_m(x, y)$ has no multiple factors (product of $m$ mutually non-proportional linear functions in $x, y$). Consider the associated birational map $x : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ defined by

$$x([x, y, z]) = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \frac{z^m}{x^{m-1}} \\ \frac{x^{m-1}}{y^{m-1}} \\ -\sum_{k=3}^m f_k z^{m-k} \\ \sum_{k=3}^m \sum_{k=3}^{m-1} \left( \frac{\partial f_k}{\partial y} \right) z^{m-k+1} \\ \sum_{k=3}^m \sum_{k=3}^{m-1} \left( \frac{\partial f_k}{\partial y} \right) z^{m-k+1} \end{pmatrix}.$$

A direct computation shows that $x$ is Legendrian.

$x$ is undefined at $m$ points $\{ z = 0, f_m(x, y) = 0 \} \subset \mathbb{P}^2$. Let $\pi : M_m \rightarrow \mathbb{P}^2$ be the rational surface obtained by blowing up $\mathbb{P}^2$ at these points. Let $E_i = \pi^{-1}(p_i)$, $i = 1, 2, \ldots, m$, be the exceptional divisor, and let $\hat{L}$ be the proper transform of the line $\{ z = 0 \}$. An analysis similar as above shows that $x$ admits a well-defined smooth lift $\hat{x} : M_m \rightarrow \mathbb{P}^5$. But when $m \geq 4$, the singular locus of $\hat{x}$ consists of $\hat{L}$, and one point from each $E_i - \hat{L}$.

This class of Legendrian surfaces were first introduced in [Bu1].

5.2. Tri-ruled surface. This is the class of surface with the induced Maurer-Cartan form (2.41). Since the flat case is already treated, we examine the case $a \neq 0$.

From (2.42), one may scale $a = 1$. Then $\omega_{00} = 0$, and $d\omega^1 = d\omega^2 = 0$ consequently. Introduce a local coordinate $(s, t)$ such that $\omega^1 = \frac{i}{\sqrt{2}} ds$, $\omega^2 = \frac{i}{\sqrt{2}} dt$, where $i^2 = -1$, and express the Maurer-Cartan form $\phi = A ds + B dt$ for constant coefficient matrices $A, B$. As in Section 5.1, the equation $g^{-1} dg = \phi$
can be integrated and one gets the following local parametrization of a tri-ruled Legendrian surface up to conformal symplectic transformation.

\[
x_1(s, t) = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} \sin(s + t) \\ \sin(s) \\ \sin(t) \\ -\cos(s + t) \\ \cos(s) \\ \cos(t) \end{pmatrix}.
\]

By a conformal symplectic transformation, we mean a linear transformation of \(P\) through \(l_p\) with a nonzero scale, e.g., a linear transformation \((1, 1, 1, 1)\) of the birational map becomes \((1, 1, 1, 1)\) for nonzero \(l_1, l_2\).

An analysis shows that this local parametrization gives rise to a Legendrian embedding of \(\mathbb{P}^1 \times \mathbb{P}^1 = Q^1 \times Q^1 \subset \mathbb{P}^2 \times \mathbb{P}^2\), the product of two conics, blown up at two points. Since this surface is isomorphic to \(\mathbb{P}^2\) blown up at three non-collinear points, consider the associated Legendrian birational map \(x : \mathbb{P}^2 \dashrightarrow \mathbb{P}^5\) defined by

\[
x([x, y, z]) = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} (x^2 - y^2z) \\ (y^2 - z^2)x \\ (z^2 - x^2)y \\ (x^2 + y^2z) \\ (y^2 + z^2)x \\ (z^2 + x^2)y \end{pmatrix},
\]

where \([x, y, z]\) is the standard projective coordinate of \(\mathbb{P}^2\). Lemma \([5.12]\) is undefined at three points \(p_1 = [1, 0, 0], p_2 = [0, 1, 0], p_3 = [0, 0, 1]\).

**Theorem 5.7.** Let \(\pi : N_3 \rightarrow \mathbb{P}^2\) be the rational surface obtained by blowing up \(\mathbb{P}^2\) at three non-collinear points \(\{p_1, p_2, p_3\}\). Let \(L_k\) be the line through \((p_i, p_j), (ijk) = (123), E_i = \pi^{-1}(p_i)\) be the exceptional divisor, and let \(H\) be the linear divisor of \(\mathbb{P}^2\). Let \(\tilde{L}_k\) be the proper transform of \(L_k\). There exists a six dimensional proper subspace \(W\) of the linear system \(\pi^*(3H) - E_1 - E_2 - E_3\) which gives a Legendrian embedding \(\tilde{x} : N_3 \rightarrow \mathbb{P}^5\).

1. A non-flat tri-ruled Legendrian surface is locally equivalent to a part of \(\tilde{x}(N_3 - \cup E_i \cup \tilde{L}_i)\).

2. The system \(W\) for \(\tilde{x}\) is a six dimensional subspace of the proper transform of the set of cubics through \(p_i\)'s. Each of the six \(-1\)-curves \(E_i\) and \(\tilde{L}_k\) is mapped to a line under the embedding.

3. The asymptotic web is given by the proper transform of the three pencils of lines through \(p_i\)'s.

Proof of theorem is presented below in four steps.

**Step 1.** \(\tilde{x}\) is an immersion:

By a direct computation, it is verified that \(x\) is an immersion on \(\mathbb{P}^2 - \cup p_i\).

At \(p_1\), introduce the parametrization of the blow up by \([x, y, z] = [1, \lambda_1 z, z]\) for the blow up parameter \(\lambda_1\). The birational map becomes

\[
x([x, y, z]) = \begin{pmatrix} X_0 \\ X_1 \\ X_2 \\ Y_0 \\ Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} (x^2 - y^2z) \\ (y^2 - z^2)x \\ (z^2 - x^2)y \\ (x^2 + y^2z) \\ (y^2 + z^2)x \\ (z^2 + x^2)y \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 0 \\ (\lambda_1 - 1)z \\ (\lambda_1^2 - 1)z \\ (1 + \lambda_1^2 z)^2 \\ (\lambda_1^3 + 1)z \\ (\lambda_1^2 + 1)z \\ (\lambda_1^3 + 1) \end{pmatrix}, \text{ as } z \rightarrow 0.
\]
Similar computations for $p_2, p_3$ show that $x$ admits a lift $\tilde{x} : N_3 \to \mathbb{P}^5$ such that $\tilde{x}(N_3) = x(\mathbb{P}^2 - \cup p_i)$. The exceptional divisors are respectively mapped to

$$E_2 \to \begin{pmatrix} -\lambda_2 \\ 1 \\ \lambda_2 \\ 1 \\ 0 \end{pmatrix}, \quad E_3 \to \begin{pmatrix} 0 \\ -\lambda_3 \\ 1 \\ 0 \\ \lambda_3 \end{pmatrix},$$

for blow up parameters $\lambda_2, \lambda_3$.

From (5.8), one may check that the three vectors $x$, $\frac{\partial x}{\partial z}$, $\frac{\partial x}{\partial \lambda_1}$ are independent at $z = 0$. Similar computations for $p_2, p_3$ show that $\tilde{x}$ is an immersion on the exceptional divisors $E_i$. Hence $\tilde{x}$ is an immersion on $N_3$.

**Step 2.** $\tilde{x}$ is injective:

It is clear that $\tilde{x}$ is injective on $\cup E_i$, and that $\tilde{x}(\cup E_i)$ is disjoint from $x(\mathbb{P}^2 - \cup p_i)$. It suffices to show that $x$ is injective on $\mathbb{P}^2 - \cup p_i$.

Suppose $x([x, y, z]) = x([x', y', z'])$. Then

$$x^2 y = \mu x'^2 y', \quad x^2 z = \mu x'^2 z',$$
$$y^2 z = \mu y'^2 z', \quad y^2 x = \mu y'^2 x',$$
$$z^2 x = \mu z'^2 x', \quad z^2 y = \mu z'^2 y',$$

for a nonzero scaling parameter $\mu$.

Case $x = 0; y, z \neq 0$. Then $x'y' = x'z' = 0$. If $x' \neq 0$, then $y'z' = 0 = yz$, a contradiction. Hence $x' = 0$. The remaining equations then show that $\frac{y}{z} = \frac{y'}{z'}$.

Case $x, y, z \neq 0$. One has $\frac{x}{y} = \frac{x'}{y'}, \frac{y}{z} = \frac{y'}{z'}, \frac{x}{z} = \frac{x'}{z'}$. Hence $[x, y, z] = [x', y', z']$.

**Step 3.** Let $D$ the hyperplane section $D = \tilde{x}^{-1}(\{X_0 + X_1 + X_2 = 0\})$. From (5.3), $X_0 + X_1 + X_2 = (x - y)(y - z)(z - x)$. Hence the linear system

$$[D] = \left[ \sum_i (\pi^*(H) - E_i) \right] = [\pi^*(3H) - E_1 - E_2 - E_3].$$

$W$ is a subspace of the linear system of the proper transform of cubics through $p_i$’s, and $\tilde{x}$ is not normal.

Since $[\tilde{L}_k] = [\pi^*(H) - E_i - E_j]$, $(ijk) = (123)$, one has $\langle D, E_i \rangle = \langle D, \tilde{L}_k \rangle = 1$, and each $E_i$ and $\tilde{L}_k$ is mapped to a line.

**Step 4.** The equations for the asymptotic web can be checked on the affine chart $[x, y, 1]$ by a direct computation. Let $C$ be a line on $\mathbb{P}^2$ that passes through exactly one of $p_i$’s. The proper transform $\tilde{C}$ of $C$ has the divisor class $\pi^*(H) - E_i$. Hence $\langle D, \tilde{C} \rangle = 2$, and each leaf of the asymptotic foliations is mapped to a linear $\mathbb{P}^2 \subset \mathbb{P}^5$ which is necessarily Legendrian from the defining properties of the tri-ruled surfaces. \(\square\)

**Remark 5.9.** The linear system of conics through three non-collinear points gives the classical quadratic transformation of $\mathbb{P}^2$.

As the three points degenerate to become collinear, $\tilde{x}(N_3)$ degenerates to the flat surface in Section 5.1. The isolated singularity of the flat Legendrian surface thus admits a smoothing.

It is not known if every del Pezzo surface admits a Legendrian embedding. Legendrian embeddings of a set of degree 4 del Pezzo surfaces were constructed in [Bu2].

Note the algebraic equations satisfied by $\tilde{x}$(this is not a complete intersection).

$$X_0^2 - Y_0^2 = X_1^2 - Y_1^2 = X_2^2 - Y_2^2,$$
$$X_0(X_1X_2 + Y_1Y_2) + Y_0(X_1Y_2 + Y_1X_2) = 0.$$
Legendrian surface $\tilde{x}(N_3)$ can thus be considered as a complexification of the homogeneous special Legendrian torus with parallel second fundamental form, [HIL].

5.2.1. Generalization. The construction of tri-ruled surface admits a straightforward generalization.

Let $m, n$ be a pair of positive integers. Let $f_m, g_m$ be the homogeneous polynomials of degree $m$ of two variables which represent $\sin(ms), \cos(ms)$;

$$
\sin(ms) = f_m(\sin(s), \cos(s)) = \sum_{0 \leq k \leq \frac{m-1}{2}} (-1)^k \binom{m}{2k+1} \sin^{2k+1}(s) \cos^{m-(2k+1)}(s),
$$

$$
\cos(ms) = g_m(\sin(s), \cos(s)) = \sum_{0 \leq k \leq \frac{m}{2}} (-1)^k \binom{m}{2k} \sin^{2k}(s) \cos^{m-2k}(s).
$$

Consider the following local parametrization of a Legendrian surface.

$$
(5.11) \quad x_1(s, t) = \begin{pmatrix}
X_0 \\
X_1 \\
X_2 \\
Y_0 \\
Y_1 \\
Y_2
\end{pmatrix} = \begin{pmatrix}
\sin(ms + nt) \\
\sqrt{m} \sin(s) \\
\sqrt{n} \sin(t) \\
-\cos(ms + nt) \\
\sqrt{m} \cos(s) \\
\sqrt{n} \cos(t)
\end{pmatrix} = \begin{pmatrix}
f_m(u_1, u_2)g_n(v_1, v_2) + g_m(u_1, u_2)f_n(v_1, v_2) \\
(-1)^{m+n-1} \sqrt{m} u_1 u_0^{m-1} v_0^n \\
(-1)^{m+n-1} \sqrt{n} v_1 u_0^{m-1} v_0^n \\
f_m(u_1, u_2)f_n(v_1, v_2) - g_m(u_1, u_2)g_n(v_1, v_2) \\
(-1)^{m+n-1} \sqrt{m} u_2 u_0^{m-1} v_0^n \\
(-1)^{m+n-1} \sqrt{n} v_2 u_0^{m-1} v_0^n
\end{pmatrix},
$$

where $u_0^2 + u_1^2 + u_2^2 = 0, v_0^2 + v_1^2 + v_2^2 = 0$.

**Lemma 5.12.** The local parametrization $(5.11)$ is equivalent to the following Legendrian birational map $x : \mathbb{P}^2 \rightarrow \mathbb{P}^5$ up to conformal symplectic transformation.

$$
(5.13) \quad x([x, y, z]) = \begin{pmatrix}
X_0 \\
X_1 \\
X_2 \\
Y_0 \\
Y_1 \\
Y_2
\end{pmatrix} = \begin{pmatrix}
x^{2m}z - y^{2n}z^2 - 2m - 2n - 1 \\
\sqrt{m}(z^2 - x^2)x^m - 1 y^n z^{m-n} \\
\sqrt{n}(y^2 - z^2)x^m y^{n-1} z^{m-n} \\
x^{2m}z + y^{2n}z^2 - 2m - 2n + 1 \\
x^{2m}z + y^{2n}z^2 - 2m - 2n + 1 \\
\sqrt{n}(y^2 - z^2)x^m y^{n-1} z^{m-n}
\end{pmatrix}.
$$

**Proof.** Let $[x, y, z]$ be the homogeneous coordinate of $\mathbb{P}^2$. Take the following birational map $\varphi : \mathbb{P}^2 \rightarrow Q^1 \times Q^1 \subset \mathbb{P}^2 \times \mathbb{P}^2$;

$$
\varphi([x, y, z]) = ([u_0, u_1, u_2], [v_0, v_1, v_2]),
$$

$$
= ([2xz, z^2 - x^2, i(z^2 + x^2)], [2yz, -z^2 + y^2, i(z^2 + y^2)]).
$$

Lemma follows from de Moivre’s formula,

$$(u_2 \pm iu_1)^m(v_2 \pm iv_1)^n = (g_m(u_1, u_2) \pm if_m(u_1, u_2))(g_n(v_1, v_2) \pm if_n(v_1, v_2)).$$

Consider the case $m = n$. $(5.13)$ is undefined at the three points of $(5.6)$. Let $\tilde{x} : N_3 \rightarrow \mathbb{P}^5$ be the induced lift. It is never an immersion except when $(m, n) = (1, 1)$. For example when $m \geq 2$, the line $\{y = 0\}$ degenerates to a point.

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