Commutative hypergroups associated with a hyperfield
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Abstract

Let \( H \) be a commutative hypergroup and \( L \) a discrete commutative hypergroup. In the present paper we introduce a commutative hypergroup \( \mathcal{K}(H, \varphi, L) \) associated with a hyperfield \( \varphi \) of \( H \) based on \( L \). Moreover for the hyperfield \( \varphi \) of a compact commutative hypergroup \( H \) of strong type based on a discrete commutative hypergroup \( L \) of strong type, we introduce the dual hyperfield \( \hat{\varphi} \) of \( \hat{L} \) based on \( \hat{H} \) and show that \( \hat{\mathcal{K}}(H, \varphi, L) \cong \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}) \).

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1. Introduction

In a previous paper ([HKTY2]) we discussed the hypergroup structure of the space \( \mathcal{K}(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2)) \) under the assumptions that \( H \) is a compact hypergroup of strong type and \( H_0 \) is a closed subhypergroup of \( H \) with finite index \( [H : H_0] \), \( \hat{H} \) and \( \hat{H}_0 \) denoting the dual of \( H \) and \( H_0 \) respectively, and \( \mathbb{Z}_q(2) \) signifying the \( q \)-deformation of \( \mathbb{Z}_2 \) with \( 0 < q \leq 1 \) in the sense of [KTY]. It remained an open problem to investigate the hypergroup structure of the dual \( \mathcal{K}(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2)) \) of \( \mathcal{K}(H \cup H_0, \mathbb{Z}_q(2)) \) at least under additional assumptions on \( H \) and \( H_0 \). It required a different approach to master that problem. The present solution to the problem relies on a generalization of the notion of a hyperfield, which had been successfully applied in the case of finite hypergroups in [HKKK].

Let us emphasize that in the present discussion the dual hypergroup structure of \( \mathcal{K}(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2)) \) will be established for commutative hypergroups \( H \) without assuming compactness of \( H \) and finiteness of the index \( [H : H_0] \) of \( H_0 \) in \( H \).

The preliminary knowledge on hypergroups needed in the sequel can be taken from the traditional sources [BH] and [J]; some additional references on the structure of hypergroups to be consulted are [HK1], [HK2] and [HKY].

In section 3 the appropriate generalization of the hyperfield method is presented. We are considering hyperfields \( \varphi \) mapping elements \( \ell \) of a countable discrete commutative hypergroup \( L \) to compact subhypergroups \( H(\ell) \) of a commutative hypergroup \( H \). Then the space \( \mathcal{K}(H, \varphi, L) \) is introduced and shown to be a commutative hypergroup (Theorem 3.1). Next we define the hypergroup
\[ \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}) \] by the dual field of \( \hat{\varphi} \) of \( \varphi \) and obtain that for a compact commutative hypergroup \( H \) of strong type the desired dual \( \mathcal{K}(H, \varphi, L) \) is isomorphic to \( \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}) \) (Theorem 3.5). Natural conditions yield the identification of \( \mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}) \) with \( \hat{L} \lor \hat{H} \) and with the substitution \( S(Q \times L : Q \rightarrow H) \) introduced by Voit in [V1].

Section 4 contains various examples of the hyperfield method, in particular the extension of Voit’s result in [V2] to higher dimensional tori.

In section 5 we drop the assumption of compactness of \( H \) and identify \( \mathcal{K}(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2)) \) with \( \mathcal{K}(\hat{H}, \varphi, \mathbb{Z}_q(2)) \) (Theorem 5.2) using the character theory for induced representations developed in [HKY].

2. Preliminaries

For a locally compact space \( X \) we shall mainly consider the subspaces \( C_c(X) \) and \( C_0(X) \) of the space \( C(X) \) of continuous functions on \( X \) which have compact support or vanish at infinity respectively. By \( M(X) \), \( M^b(X) \) and \( M_c(X) \) we abbreviate the spaces of all (Radon) measures on \( X \), the bounded measures and the measures with compact support on \( X \) respectively. Let \( M^1(X) \) denote the set of probability measures on \( X \) and \( M^1_c(X) \) its subset \( M^1(X) \cap M_c(X) \). The symbol \( \delta_x \) stands for the Dirac measures in \( x \in X \).

A hypergroup \((K, \ast)\) is a locally compact space \( K \) together with a convolution \( \ast \) in \( M^b(K) \) such that \((M^b(K), \ast)\) becomes a Banach algebra and that the following properties are fulfilled.

(H1) The mapping

\[(\mu, \nu) \mapsto \mu \ast \nu \]

from \( M^b(K) \times M^b(K) \) into \( M^b(K) \) is continuous with respect to the weak topology in \( M^b(K) \).

(H2) For \( x, y \in K \) the convolution \( \delta_x \ast \delta_y \) belongs to \( M^1_c(K) \).

(H3) There exist a unit element \( e \in K \) with

\[ \delta_e \ast \delta_x = \delta_x \ast \delta_e = \delta_x \]

for all \( x \in K \), and an involution

\[ x \mapsto x^- \]

in \( K \) such that

\[ \delta_{x^-} \ast \delta_{y^-} = (\delta_y \ast \delta_x)^- \]

and

\[ e \in \text{supp}(\delta_x \ast \delta_y) \]

if and only if \( x = y^- \)

whenever \( x, y \in K \).
(H4) The mapping 

$$(x, y) \mapsto \text{supp}(\delta_x \ast \delta_y)$$

from $K \times K$ into the space $C(K)$ of all compact subsets of $K$ furnished with Michael topology is continuous.

A hypergroup $(K, \ast)$ is said to be commutative if the convolution $\ast$ is commutative. In this case $(M^b(K), \ast, -)$ is a commutative Banach *-algebra with identity $\delta_e$. There is an abundance of hypergroups and there are various constructions (polynomial, Sturm-Liouville) as the reader may learn from the pioneering papers on the subject.

Let $(K, \ast)$ and $(L, \circ)$ be two hypergroups with units $e_K$ and $e_L$ respectively. A continuous mapping $\varphi : K \to L$ is called a hypergroup homomorphism if $\varphi(e_K) = e_L$ and $\varphi$ is the unique linear, weakly continuous extension from $M^b(K)$ to $M^b(L)$ such that

$$\varphi(\delta_x) = \delta_{\varphi(x)}, \quad \varphi(\delta_x^-) = \varphi(\delta_x)^- \quad \text{and} \quad \varphi(\delta_x \ast \delta_y) = \varphi(\delta_x) \circ \varphi(\delta_y)$$

whenever $x, y \in K$. If $\varphi : K \to L$ is also a homeomorphism, it will be called an isomorphism from $K$ onto $L$. An isomorphism from $K$ onto $L$ is called an automorphism of $K$. We denote by Aut$(K)$ the set of all automorphisms of $K$. Then Aut$(K)$ becomes a topological group equipped with the weak topology of $M^b(K)$. We call $\alpha$ an action of a locally compact group $G$ on a hypergroup $H$ if $\alpha$ is a continuous homomorphism from $G$ into Aut$(H)$. Associated with the action $\alpha$ of $G$ on $H$ one can define a semi-direct product hypergroup $K = H \rtimes_{\alpha} G$.

If the given hypergroup $K$ is commutative, its dual $\hat{K}$ can be introduced as the set of all bounded continuous functions $\chi \neq 0$ on $M^b(K)$ satisfying

$$\chi(\delta_x \ast \delta_y) = \chi(\delta_x)\chi(\delta_y) \quad \text{and} \quad \chi(\delta_x^-) = \overline{\chi(\delta_x)}$$

for all $x, y \in K$. This set of characters $\hat{K}$ of $K$ becomes a locally compact space with respect to the topology of uniform convergence on compact sets, but generally fails to be a hypergroup. If $\hat{K}$ is a hypergroup, then $K$ is called a strong hypergroup or a hypergroup of strong type. If the dual $\hat{K}$ of a strong hypergroup $K$ is also strong and $\hat{\hat{K}} \cong K$ holds, then $K$ is called a Pontryagin hypergroup or a hypergroup of Pontryagin type.

3. Hyperfields and hypergroups

Let $H = (H, M^b(H), \ast, -)$ be a commutative hypergroup with unit $h_0$ and $L = (L, M^b(L), \bullet, -) = \{\ell_0, \ell_1, \ldots, \ell_n, \ldots\}$ a countable discrete commutative hypergroup where $\ell_0$ is unit of $L$.

**Definition** For each $\ell \in L$, let $H(\ell)$ be a compact subhypergroup of $H$ satisfying the following conditions.
(1) $H(\ell_0) = \{h_0\}$ and $H(\ell^-) = H(\ell)$.

(2) $[H(\ell_i) \ast H(\ell_j)] \supset H(\ell_k)$ for $\ell_k \in \text{supp}(\varepsilon_{\ell_i} \ast \varepsilon_{\ell_j})$, where $[H(\ell_i) \ast H(\ell_j)]$ is the compact subhypergroup of $H$ generated by $H(\ell_i)$ and $H(\ell_j)$.

Then we call
$$\varphi : L \ni \ell \mapsto H(\ell) \subset H$$
a hyperfield of $H$ based on $L$.

We denote the normalized Haar measure $\omega_{H(\ell)}$ of $H(\ell)$ by $e(\ell)$ and note that condition (2) implies

(3) $e(\ell_i) \ast e(\ell_j) \ast e(\ell_k) = e(\ell_i) \ast e(\ell_j)$ for $\ell_k \in \text{supp}(\varepsilon_{\ell_i} \ast \varepsilon_{\ell_j})$.

Putting
$$K(H, \varphi, L) := \{(\delta_h \ast e(\ell)) \otimes \varepsilon_\ell \in M^b(H) \otimes M^b(L) : h \in H, \ell \in L\}$$
one sees that
$$K(H, \varphi, L) = Q(\ell_0) \cup Q(\ell_1) \cup \cdots \cup Q(\ell_n) \cup \cdots$$
is a locally compact space, where
$$Q(\ell) := H/H(\ell) \quad \text{for all } \ell \in L,$$
$$Q(\ell_0) := H.$$

Now we shall introduce a convolution $\circ$ and an involution $-$ on $K(H, \varphi, L)$ as elements of $M^b(H) \otimes M^b(L)$ in order to obtain the following theorem generalizing a result in [HKKK].

**Theorem 3.1** Let $\varphi$ be a hyperfield of a commutative hypergroup $H$ based on a countable discrete commutative hypergroup $L$. Then $K(H, \varphi, L)$ is a commutative hypergroup.

**Proof** The set $\{\delta_h \ast e(\ell) : h \in H\}$ is a commutative hypergroup isomorphic to the quotient hypergroup
$$Q(\ell) = H/H(\ell) \quad \text{for all } \ell \in L.$$

We see that
$$M^b(K(H, \varphi, L)) = \sum_{\ell \in L} M^b(Q(\ell)).$$

Next we examine the convolution on $K(H, \varphi, L)$ in detail. For $\ell_i, \ell_j \in L$ we denote the set $\{k : \ell_k \in \text{supp}(\varepsilon_{\ell_i} \ast \varepsilon_{\ell_j})\}$ by $s(\ell_i, \ell_j)$. Given $h_p, h_q \in H$ and $\ell_i, \ell_j \in L,$

$$(\delta_{h_p} \ast e(\ell_i)) \otimes \varepsilon_{\ell_i} \circ ((\delta_{h_q} \ast e(\ell_j)) \otimes \varepsilon_{\ell_j})$$

$$= (\delta_{h_p} \ast \delta_{h_q} \ast e(\ell_i) \ast e(\ell_j)) \otimes (\varepsilon_{\ell_i} \ast \varepsilon_{\ell_j})$$

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\[(\delta_{hp} \ast \delta_{hq} \ast e(\ell_i) \ast e(\ell_j) \ast e(\ell_k)) \otimes \left( \sum_{\ell \in s(\ell_i, \ell_j)} n_{ij}^k \varepsilon_{\ell_k} \right) \]

by condition (3) derived from the defining properties of the hyperfield \( \varphi \). In conclusion the convolution in \( K(H, \varphi, L) \) is well-defined, and its associativity holds.

Now we note that

\[\text{supp}(\((\delta_{hp} \ast e(\ell_i)) \otimes \varepsilon_{\ell_i}) \circ ((\delta_{hq} \ast e(\ell_j)) \otimes \varepsilon_{\ell_j})) = \bigcup_{k \in s(\ell_i, \ell_j)} ((h_p \ast h_q \ast [H(\ell_i) \ast H(\ell_j)]) / H(\ell_k))\]

is compact.

In order to verify the defining property of the involution \(-\) of \( K(H, \varphi, L) \) we compute

\[\((\delta_{hp} \ast e(\ell_i)) \otimes \varepsilon_{\ell_i}) \circ ((\delta_{hq} \ast e(\ell_j)) \otimes \varepsilon_{\ell_j})^- = \sum_{k \in s(\ell_i, \ell_j)} n_{ij}^k (\delta_{hp} \ast \delta_{hq}^{-1} \ast e(\ell_i) \ast e(\ell_j) \ast e(\ell_k)) \otimes \varepsilon_{\ell_k}.\]

These equalities imply that

\[((\delta_{hp} \ast e(\ell_i)) \otimes \varepsilon_{\ell_i}) = ((\delta_{hq} \ast e(\ell_j)) \otimes \varepsilon_{\ell_j})^-\]

holds if and only if

\[(h_0, \ell_0) \in \text{supp}(\((\delta_{hp} \ast e(\ell_i)) \otimes \varepsilon_{\ell_i}) \circ ((\delta_{hq} \ast e(\ell_j)) \otimes \varepsilon_{\ell_j})).\]

The remaining axioms of a hypergroup are easily verified. All together the desired conclusions are established. [Q.E.D.]

**Remark** If \( \varphi(\ell) = H(\ell) = \{h_0\} \) for all \( \ell \in L \), then \( K(H, \varphi, L) \) is the direct product hypergroup \( H \times L \).

Now let \( H \) be a compact commutative hypergroup of strong type such that \( \hat{H} \) is a discrete commutative hypergroup and \( L \) be a discrete commutative hypergroup of strong type such that \( \hat{L} \) is a compact hypergroup. For \( \chi \in \hat{H} \) we put

\[Y(\chi) := \{\ell \in L : \chi \in H(\ell)^\perp\}\]

where

\[H(\ell)^\perp := \{\chi \in \hat{H} : \chi(h) = 1 \text{ for all } h \in H(\ell)\}\]

denotes the annihilator of \( H(\ell) \) in \( \hat{H} \).
Lemma 3.2 Given $\chi \in \hat{H}$, $Y(\chi)$ is a subhypergroup of $L$ such that $Y(\chi^-) = Y(\chi)$, and $Y(\chi_i) \cap Y(\chi_j) \subset Y(\chi_k)$ for $\chi_k \in \text{supp}(\varepsilon_{\chi_i} \ast \varepsilon_{\chi_j})$.

Proof First of all we note that

$$H(\ell_i) \perp \cap H(\ell_j) \perp = [H(\ell_i) \ast H(\ell_j)] \perp \subset H(\ell_k) \perp$$

for $\ell_k \in \text{supp}(\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j})$ by the defining property (2) of the hyperfield $\varphi$ of $H$ based on $L$.

For $\ell_i, \ell_j \in Y(\chi), \chi \in H(\ell_i) \perp \cap H(\ell_j) \perp \subset H(\ell_k) \perp$ whenever $\ell_k \in \text{supp}(\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j})$, i.e., $\chi \in H(\ell_k) \perp$. Then $\text{supp}(\varepsilon_{\ell_i} \bullet \varepsilon_{\ell_j}) \subset Y(\chi)$. For $\ell \in Y(\chi)$ we have $\ell^- \in Y(\chi)$ by property (1) of the definition of the hyperfield $\varphi$ which states $H(\ell^-) = H(\ell)$ for all $\ell \in L$. Then $Y(\chi)$ appears to be a subhypergroup of $L$.

For $\ell \in Y(\chi_i) \cap Y(\chi_j)$ we see that $\chi_i \in H(\ell) \perp$ and $\chi_j \in H(\ell) \perp$. Since $H(\ell) \perp$ is a subhypergroup of $\hat{H}$ we get

$$\text{supp}(\varepsilon_{\chi_i} \ast \varepsilon_{\chi_j}) \subset H(\ell) \perp,$$

which implies that $\chi_k \in H(\ell) \perp$ for $\chi_k \in \text{supp}(\varepsilon_{\chi_i} \ast \varepsilon_{\chi_j})$. Altogether we arrive at the fact that $\ell \in Y(\chi_k)$, i.e., $\chi(\chi_i) \cap Y(\chi_j) \subset Y(\chi_k)$. [Q.E.D.]

Denoting $Y(\chi) \perp$ by $\hat{L}(\chi)$ for $\chi \in \hat{H}$ we easily see that $\hat{L}(\chi)$ is a closed subhypergroup of the compact hypergroup $\hat{L}$ satisfying properties (1) and (2) of the hyperfield $\varphi$ by Lemma 3.2. This leads to the following.

Lemma 3.3 The mapping

$$\hat{\varphi} : \hat{H} \ni \chi \longmapsto \hat{L}(\chi) \subset \hat{L}$$

is a hyperfield of $\hat{L}$ based on $\hat{H}$.

Definition The hyperfield $\hat{\varphi}$ is called the dual of the hyperfield

$$\varphi : L \ni \ell \longmapsto H(\ell) \subset H$$

of $H$ based on $L$. We note that the duality $\hat{\varphi} = \varphi$ holds if $H$ and $L$ are Pontryagin.

As a consequence of these preparations we obtain a commutative hypergroup

$$\mathcal{K}(\hat{L}, \hat{\varphi}, \hat{H}) = \{ (\delta_\rho \ast \varepsilon(\chi)) \otimes \varepsilon_\chi : \rho \in \hat{L}, \chi \in \hat{H} \},$$

where $\varepsilon(\chi)$ denotes the normalized Haar measure of $\hat{L}(\chi)$.

The following statements are easily verified.

Lemma 3.4

(i) For each $\chi \in \hat{H}$ and $\ell \in L$, $\ell \in Y(\chi)$ if and only if $\chi \in H(\ell) \perp$
(ii) For each $\chi \in \hat{H}$ and the Haar measure $e(\ell)$ of $H(\ell)$

$$\chi(e(\ell)) = \begin{cases} 
1 & \text{if } \chi \in H(\ell)^{\perp} \\
0 & \text{otherwise}
\end{cases}$$

(iii) For each $\ell \in L$ and the Haar measure $e(\chi)$ of $\hat{L}(\chi)$

$$e(\chi)(\ell) = \begin{cases} 
1 & \text{if } \ell \in Y(\chi) \\
0 & \text{otherwise}
\end{cases}$$

(iv) For each $\chi \in \hat{H}$ and $\ell \in L$, $\chi(e(\ell)) = e(\chi)(\ell)$.

Now, we arrive at the dual version of the statement of Theorem 3.1.

**Theorem 3.5** Let $\varphi$ be a hyperfield of a compact commutative hypergroup $H$ of strong type based on a discrete commutative hypergroup $L$ of strong type. Then

$$\hat{K}(H, \varphi, L) \cong K(\hat{L}, \hat{\varphi}, \hat{H}).$$

If $H$ and $L$ are Pontryagin hypergroups, then $K(H, \varphi, L)$ is also Pontryagin. Moreover the sequence

$$1 \rightarrow H \rightarrow K(H, \varphi, L) \rightarrow L \rightarrow 1$$

is exact and the dual sequence

$$1 \rightarrow \hat{L} \rightarrow K(\hat{L}, \hat{\varphi}, \hat{H}) \rightarrow \hat{H} \rightarrow 1$$

is exact as well. In particular $K(H, \varphi, L)$ and $K(\hat{L}, \hat{\varphi}, \hat{H})$ are extension hypergroups of $L$ by $H$ and $\hat{H}$ by $\hat{L}$ respectively.

**Proof** Clearly

$$\hat{K}(H, \varphi, L) \supset K(\hat{L}, \hat{\varphi}, \hat{H}).$$

It remains to be shown that

$$\hat{K}(H, \varphi, L) \subset K(\hat{L}, \hat{\varphi}, \hat{H}).$$

Let $\tau$ be a character of $K(H, \varphi, L)$. Then there exists $\chi \in \hat{H}$ such that

$$\tau((\delta_h * e(\ell)) \otimes \varepsilon_\ell) = \chi(h)\chi(e(\ell))\rho(\ell) = \rho(\ell)e(\chi)(\ell)\chi(h) = (\delta_\rho * e(\chi)) \otimes \varepsilon_\chi(\ell,h)$$

for some $\rho \in \hat{L}$ by Lemma 3.4. Consequently

$$\tau = (\delta_\rho * e(\chi)) \otimes \varepsilon_\chi \in K(\hat{L}, \hat{\varphi}, \hat{H}).$$

The assertion concerning the Pontryagin property follows from the fact that the dual $\hat{\varphi}$ of the dual hyperfield $\hat{\varphi}$ is $\varphi$. 

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Now let $e(H)$ denote the normalized Haar measure of the compact hypergroup $H$. Then

$$Q := \{(e(H) \otimes \varepsilon_{\ell_0}) \circ (\delta_h \otimes \varepsilon_{\ell_1}) : h \in H, \ell_1 \in L\}$$

$$= \{e(H) \otimes \varepsilon_{\ell_1} : \ell_1 \in L\}$$

is the quotient hypergroup $K(H, \varphi, L)/H$ isomorphic to $L$. This means that $K(H, \varphi, L)$ is an extension hypergroup of $L$ by $H$. [Q.E.D.]

Remark

(1) If $\varphi(\ell_0) = \{h_0\}$ and $\varphi(\ell) = H$ for all $\ell \in L$ such that $\ell \neq \ell_0$, then $K(H, \varphi, L)$ is the hypergroup join $H \lor L$. Moreover, if $H$ and $L$ are strong, then

$$K(\hat{L}, \hat{\varphi}, \hat{H}) = \hat{L} \lor \hat{H}.$$ 

(2) Let $Q := H/H_0$ for a closed subhypergroup $H_0$ of $H$. If $\varphi(\ell_0) = \{h_0\}$ and $\varphi(\ell) = H_0$ for all $\ell \in L$, $\ell \neq \ell_0$, then

$$K(H, \varphi, L) = S(Q \times L : Q \rightarrow H),$$

where the latter symbol denotes the substitution hypergroup obtained by substituting $Q$ in $Q \times L$ by $H$, in the sense of Voit [V1]. We note that $H_0$ is not assumed to be open which means that our definition is a generalization of Voit’s substitution.

4. Examples of hyperfields

Let $\mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ be the hypergroup of order two, where the convolution structure is given by

$$\varepsilon_{\ell_1} \cdot \varepsilon_{\ell_1} = q\varepsilon_{\ell_0} + (1 - q)\varepsilon_{\ell_1}$$

for $0 < q \leq 1$.

Example 4.1 If $H = \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{1\}$, $\varphi(\ell_1) = H(\ell_1) = C_n := \{z \in \mathbb{T} : z^n = 1\}$ ($n \in \mathbb{N}$), then

$$K(H, \varphi, L) = K(\mathbb{T}, \varphi, \mathbb{Z}_q(2)) = \mathbb{T} \cup \mathbb{T}$$

is a commutative hypergroup which coincides with Voit’s commutative hypergroup on the two tori $\mathbb{T} \cup \mathbb{T}$ ([V2]).

This means that $K(\mathbb{T}, \varphi, \mathbb{Z}_q(2))$ determines the commutative hypergroup structure on $\mathbb{T} \cup \mathbb{T}$ with parameter $(n, q)$, $n \in \mathbb{N}$, $0 < q \leq 1$.

Obviously $K(\mathbb{T}, \varphi, \mathbb{Z}_q(2))$ is Pontryagin and

$$\check{K}(\mathbb{T}, \varphi, \mathbb{Z}_q(2)) = K(\mathbb{Z}_q(2), \check{\varphi}, \mathbb{Z}).$$
where the dual field $\hat{\phi}$ of $\varphi$ is given as

$$\hat{\phi} : \mathbb{Z} \ni k \mapsto \hat{\phi}(k) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\phi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in n\mathbb{Z} \\ \mathbb{Z}_q(2) & \text{otherwise.} \end{cases}$$

**Example 4.2** If $H = \mathbb{T} \times \mathbb{T} = \mathbb{T}^2$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{(1, 1)\}$, $\varphi(\ell_1) = H(\ell_1) = C_n \times C_m$ ($n, m \in \mathbb{N}$), then

$$\mathcal{K}(H, \varphi, L) = \mathcal{K}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = \mathbb{T}_1 \cup \mathbb{T}_2$$

is a commutative hypergroup.

Obviously $\mathcal{K}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2))$ is Pontryagin and

$$\hat{\mathcal{K}}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}^2),$$

where the dual field $\hat{\varphi}$ of $\varphi$ is given as

$$\hat{\varphi} : \mathbb{Z} \times \mathbb{Z} \ni k \mapsto \hat{\varphi}(k) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in n\mathbb{Z} \times m\mathbb{Z} \\ \mathbb{Z}_q(2) & \text{otherwise.} \end{cases}$$

**Example 4.3** If $H = \mathbb{T}$, $L = \mathbb{Z}_q(3) = \{\ell_0, \ell_1, \ell_2\}$ (see [KTY]) and $\varphi : \mathbb{Z}_q(3) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{1\}$, $\varphi(\ell_1) = H(\ell_1) = C_n$ and $\varphi(\ell_2) = H(\ell_2) = C_n$, then

$$\mathcal{K}(H, \varphi, L) = \mathcal{K}(\mathbb{T}, \varphi, \mathbb{Z}_q(3)) = \mathbb{T} \cup \mathbb{T} \cup \mathbb{T}$$

is a commutative hypergroup.

Obviously $\mathcal{K}(\mathbb{T}, \varphi, \mathbb{Z}_q(3))$ is Pontryagin and

$$\hat{\mathcal{K}}(\mathbb{T}, \varphi, \mathbb{Z}_q(3)) = \mathcal{K}(\mathbb{Z}_q(3), \hat{\varphi}, \mathbb{Z}),$$

where the dual field $\hat{\varphi}$ of $\varphi$ is given as

$$\hat{\varphi} : \mathbb{Z} \ni k \mapsto \hat{\varphi}(k) \subset \mathbb{Z}_q(3)$$

with

$$\hat{\varphi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in n\mathbb{Z} \\ \mathbb{Z}_q(3) & \text{otherwise.} \end{cases}$$
Example 4.4 If $H = \mathbb{T} \times \mathbb{T} = \mathbb{T}^2$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{(1, 1)\}$, $\varphi(\ell_1) = H(\ell_1) = C_n \times \mathbb{T}$ $(n \in \mathbb{N})$, then

$$K(H, \varphi, L) = K(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = \mathbb{T}^2 \cup \mathbb{T}$$

is a commutative hypergroup.

Obviously $K(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2))$ is Pontryagin and

$$\hat{K}(\mathbb{T}^2, \varphi, \mathbb{Z}_q(2)) = K(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}^2),$$

where the dual field $\hat{\varphi}$ of $\varphi$ is given as

$$\hat{\varphi} : \mathbb{Z} \times \mathbb{Z} \ni k \mapsto \hat{\varphi}(k) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in n\mathbb{Z} \times \{0\} \\ \mathbb{Z}_q(2) & \text{otherwise.} \end{cases}$$

Example 4.5 Let $H = K^\alpha(\mathbb{T}) = [-1, 1]$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{1\}$, $\varphi(\ell_1) = H(\ell_1) = K^\alpha(C_n)$ $(n \in \mathbb{N})$, where $K^\alpha(\mathbb{T}) = [-1, 1]$ is the orbital hypergroup defined by the action of $\mathbb{Z}_2 = \{e, g\}$ ($g^2 = e$) such that $\alpha_g(z) = \bar{z}$ on $\mathbb{T}$. Then

$$K(H, \varphi, L) = K(K^\alpha(\mathbb{T}), \varphi, \mathbb{Z}_q(2)) = [-1, 1] \cup [-1, 1]$$

is a commutative hypergroup.

Obviously $K(K^\alpha(\mathbb{T}), \varphi, \mathbb{Z}_q(2))$ is Pontryagin and

$$\hat{K}(K^\alpha(\mathbb{T}), \varphi, \mathbb{Z}_q(2)) = K(\mathbb{Z}_q(2), \hat{\varphi}, K^\alpha(\mathbb{Z})), $$

where $K^\alpha(\mathbb{Z}) = \{0, 1, 2, \cdots, n, \cdots\}$ is also the orbital hypergroup by the action of $\mathbb{Z}_2 = \{e, g\}$ ($g^2 = e$) such that $\alpha_g(n) = -n$ on $\mathbb{Z}$ and the dual field $\hat{\varphi}$ of $\varphi$ is given as

$$\hat{\varphi} : K^\alpha(\mathbb{Z}) \ni k \mapsto \hat{\varphi}(k) \subset \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(k) = \begin{cases} \{\ell_0\} & \text{for } k \in K^\alpha(n\mathbb{Z}) \\ \mathbb{Z}_q(2) & \text{otherwise.} \end{cases}$$

Example 4.6 Let $A$ be a commutative strong hypergroup and $C$ a compact strong hypergroup. If $H = A \times C$, $L = \mathbb{Z}_q(2) = \{\ell_0, \ell_1\}$ and $\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto H(\ell) \subset H$ with $\varphi(\ell_0) = H(\ell_0) = \{h_0\}$, $\varphi(\ell_1) = H(\ell_1) = C$, then

$$K(H, \varphi, L) = K(A \times C, \varphi, \mathbb{Z}_q(2)) = (A \times C) \cup A$$

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is the commutative hypergroup \( A \times (C \vee \mathbb{Z}_q(2)) \) of strong type and

\[
\tilde{\mathcal{K}}(A \times C, \varphi, \mathbb{Z}_q(2)) = \mathcal{K}(Z_q(2), \hat{\varphi}, \hat{A} \times \hat{C}) = \hat{A} \times (Z_q(2) \vee \hat{C}).
\]

In fact, the dual field \( \hat{\varphi} \) of \( \varphi \) is given as

\[
\hat{\varphi} : \hat{A} \times \hat{C} \ni (\chi, \rho) \longmapsto \hat{\varphi}(\chi, \rho) \subset Z_q(2)
\]

with

\[
\hat{\varphi}(\chi, \rho) = \begin{cases} 
\{\ell_0\} & \text{for } (\chi, \rho_0) \in \hat{A} \times \hat{C} \\
\mathbb{Z}_q(2) & \text{otherwise},
\end{cases}
\]

where \( \rho_0 \) is unit of \( \hat{C} \).

**Example 4.7** Let \( A \) be a commutative hypergroup and \( C \) a compact hypergroup. If \( H = A \times C, L = \mathbb{Z}_q(3) = \{\ell_0, \ell_1, \ell_2\} \) and \( \varphi : \mathbb{Z}_q(3) \ni \ell \mapsto H(\ell) \subset H \) with \( \varphi(\ell_0) = H(\ell_0) = \{h_0\}, \varphi(\ell_1) = H(\ell_1) = C_0, \varphi(\ell_2) = H(\ell_2) = C_0 \), where \( C_0 \) is a closed subhypergroup of \( C \), then

\[
\mathcal{K}(H, \varphi, L) = \mathcal{K}(A \times C, \varphi, \mathbb{Z}_q(3)) = (A \times C) \cup (A \times Q) \cup (A \times Q)
\]

is the commutative hypergroup \( A \times (S(Q \times \mathbb{Z}_q(3) : Q \to C)) \), where \( Q = C/C_0 \).

### 5. Applications of the theorems

We assume \( H \) to be a (not necessarily compact) commutative hypergroup of strong type such that \( \hat{H} \) is a commutative hypergroup with unit character \( \chi_0 \). Let \( H_0 \) be a closed subhypergroup of strong type of \( H \), where the annihilator \( H_0^\perp \) in \( \hat{H} \) is a compact subhypergroup of \( \hat{H} \). For \( \tau \in \hat{H}_0 \) we consider the set

\[
A(\tau) = \{ \chi \in \hat{H} : \operatorname{res}^H_{H_0} \chi = \tau \}.
\]

As usual \( \omega_{H_0^\perp} \) denotes the normalized Haar measure of \( H_0^\perp \). Then there exists a unique \( H_0^\perp \)-invariant probability measure \( \mu_{A(\tau)} \) which is given by

\[
\mu_{A(\tau)} = ch(\tilde{\tau}) \cdot \omega_{H_0^\perp},
\]

for some \( \tilde{\tau} \in A(\tau) \).

**Definition** (see [HKY])

(i) For \( \tau \in \hat{H}_0 \) the character of \( \tau \) induced from \( H_0 \) to \( H \) is defined by

\[
\text{ind}_{H_0}^H ch(\tau) := \mu_{A(\tau)}.
\]

(ii) For \( \tau_i, \tau_j \in \hat{H}_0 \), \( ch(\tau_i) \cdot ch(\tau_j) \) is decomposed on \( \hat{H}_0 \) in the form

\[
ch(\tau_i) \cdot ch(\tau_j) = \int_C ch(\tau) \nu(d\tau),
\]

where \( \nu \) is the normalized Haar measure on \( C \).
where \( \nu \) is a probability measure on \( \hat{H}_0 \) and 

\[
C := \text{supp}(\nu) = \text{supp}(\text{ch}(\tau_i) \cdot \text{ch}(\tau_j))
\]
is compact.

Then we introduce 

\[
\text{ind}_{H_0}^H(\text{ch}(\tau_i) \cdot \text{ch}(\tau_j)) := \int_C \text{ind}_{H_0}^H \text{ch}(\tau) \nu(d\tau).
\]

The subsequent simple facts play an essential role in the upcoming discussion.

**Lemma 5.1** (see [HKY])

(i) For \( \tau \in \hat{H}_0 \)

\[
\text{res}_{H_0}^H(\text{ind}_{H_0}^H \text{ch}(\tau)) = \text{ch}(\tau).
\]

(ii) For \( \pi_i, \pi_j \in \hat{H} \)

\[
\text{res}_{H_0}^H(\text{ch}(\pi_i) \cdot \text{ch}(\pi_j)) = (\text{res}_{H_0}^H \text{ch}(\pi_i)) \cdot (\text{res}_{H_0}^H \text{ch}(\pi_j)).
\]

(iii) For \( \pi \in H \) and \( \tau_i, \tau_j \in \hat{H}_0 \)

\[
\text{ind}_{H_0}^H((\text{res}_{H_0}^H \text{ch}(\pi)) \cdot \text{ch}(\tau_i) \cdot \text{ch}(\tau_j)) = \text{ch}(\pi) \cdot \text{ind}_{H_0}^H(\text{ch}(\tau_i) \cdot \text{ch}(\tau_j))
\]

(iv) For \( \tau_i, \tau_j \in \hat{H}_0 \)

\[
\text{res}_{H_0}^H(\text{ind}_{H_0}^H(\text{ch}(\tau_i) \cdot \text{ch}(\tau_j))) = \text{ch}(\tau_i) \text{ch}(\tau_j).
\]

**Proof** (i) and (ii) are clear.

(iii) It is easy to check that 

\[
\text{ind}_{H_0}^H(\text{ch}(\tau_i) \cdot \text{ch}(\tau_j)) = \text{ch}(\tilde{\tau}_i) \cdot \text{ch}(\tilde{\tau}_j) \cdot \omega_{H_0^+}
\]
for \( \tilde{\tau}_i \in A(\tau_i) \) and \( \tilde{\tau}_j \in A(\tau_j) \). Then we see that 

\[
\text{ch}(\pi) \cdot \text{ind}_{H_0}^H(\text{ch}(\tau_i) \cdot \text{ch}(\tau_j)) = \text{ch}(\pi) \cdot \text{ch}(\tilde{\tau}_i) \cdot \text{ch}(\tilde{\tau}_j) \cdot \omega_{H_0^+}
\]
and 

\[
\text{ind}_{H_0}^H((\text{res}_{H_0}^H \text{ch}(\pi)) \cdot \text{ch}(\tau_i) \cdot \text{ch}(\tau_j)) = \text{ch}(\pi) \cdot \text{ch}(\tilde{\tau}_i) \cdot \text{ch}(\tilde{\tau}_j) \cdot \omega_{H_0^+}.
\]

(iv) For \( \tau_i, \tau_j \in \hat{H}_0 \)

\[
\text{res}_{H_0}^H(\text{ind}_{H_0}^H(\text{ch}(\tau_i) \cdot \text{ch}(\tau_j))) = \text{res}_{H_0}^H \left( \int_C \text{ind}_{H_0}^H \text{ch}(\tau) \nu(d\tau) \right)
\]
\begin{align*}
&= \int_C \text{res}^H_H_0 (\text{ind}^H_H_0 \text{ch}(\tau)) \nu(d\tau) \\
&= \int_C \text{ch}(\tau) \nu(d\tau) \\
&= \text{ch}(\tau_i) \cdot \text{ch}(\tau_j).
\end{align*}

**Remark** A pair \((H, H_0)\) for a commutative hypergroup of strong type is always an admissible hypergroup pair in the sense of [HKTY2] by Lemma 5.1.

**Definition** (see [HKTY2]) On the space 
\[\mathcal{K}(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2)) := \{(\text{ch}(\pi), \circ), (\text{ch}(\tau), \bullet) : \pi \in \hat{H}, \tau \in \hat{H}_0\}\]
we define the convolution \(\ast\) by the following properties:

1. \((\text{ch}(\pi_i), \circ) \ast (\text{ch}(\pi_j), \circ) := (\text{ch}(\pi_i) \cdot \text{ch}(\pi_j), \circ),\)
2. \((\text{ch}(\pi), \circ) \ast (\text{ch}(\tau), \bullet) := ((\text{res}^H_H_0 \text{ch}(\pi)) \cdot \text{ch}(\tau), \bullet),\)
3. \((\text{ch}(\tau), \bullet) \ast (\text{ch}(\pi), \circ) := (\text{ch}(\tau) \cdot (\text{res}^H_H_0 \text{ch}(\pi)), \bullet),\)
4. \((\text{ch}(\tau_i), \bullet) \ast (\text{ch}(\tau_j), \bullet) := q(\text{ind}^H_H_0 (\text{ch}(\tau_i) \cdot \text{ch}(\tau_j)), \circ) + (1 - q)(\text{ch}(\tau_i) \cdot \text{ch}(\tau_j), \bullet).\)

**Definition** Given \(\mathbb{Z}_q(2) = \{\ell_0, \ell_1\} (0 < q \leq 1)\) we introduce the set \(\mathcal{K}(\hat{H}, \varphi, \mathbb{Z}_q(2))\) via the hyperfield \(\varphi : \mathbb{Z}_q(2) \ni \ell \mapsto \hat{\varphi}(\ell) \subset \hat{H}\) given by
\[\varphi(\ell) = \begin{cases} 
\{\chi_0\} & \text{if } \ell = \ell_0 \\
H_0^\perp & \text{if } \ell = \ell_1
\end{cases}\]
as in section 3.

Then we have

**Theorem 5.2** Let \(H\) be a commutative hypergroup of strong type and \(H_0\) a closed subhypergroup of \(H_0\) such that \(H_0^\perp\) is compact in \(\hat{H}\). Then \(\mathcal{K}(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2))\) is a commutative hypergroup and
\[\mathcal{K}(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2)) \cong \mathcal{K}(\hat{H}, \varphi, \mathbb{Z}_q(2)).\]

**Proof** In order to show that \(\mathcal{K}(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2))\) is a hypergroup we should check the following associativity relations. For \(\pi_i, \pi_j, \pi_k, \pi \in \hat{H}\) and \(\tau_i, \tau_j, \tau_k, \tau \in \hat{H}_0\)

\[((\text{ch}(\pi_i), \circ) \ast (\text{ch}(\pi_j), \circ)) \ast (\text{ch}(\pi_k), \circ) = (\text{ch}(\pi_i), \circ) \ast ((\text{ch}(\pi_j), \circ) \ast (\text{ch}(\pi_k), \circ)).\]
(A2) \((ch(\pi_i), o) \ast (ch(\pi_j), o) \ast (ch(\tau), \bullet) = (ch(\pi_i), o) \ast (ch(\pi_j), o) \ast (ch(\tau), \bullet)\).

(A3) \((ch(\pi), o) \ast (ch(\tau), \bullet) = (ch(\pi), o) \ast (ch(\tau), \bullet)\).

(A4) \((ch(\tau_i), \bullet) \ast (ch(\tau_j), \bullet) = (ch(\tau_i), \bullet) \ast (ch(\tau_j), \bullet)\).

However these relations are shown in a similar way to the proof of Proposition 3.6 in our paper [HKTY2] combined with the above Lemma 5.1 so that we omit the details. It is easy to check the remaining axioms of a hypergroup for \(K(H \cup \hat{H}_0, \mathbb{Z}_q(2))\). The desired conclusion is obtained.

Next we introduce an isomorphism \(\psi : K(H \cup \hat{H}_0, \mathbb{Z}_q(2)) \rightarrow K(\hat{H}, \varphi, \mathbb{Z}_q(2))\) by

\[
\psi((ch(\pi), o)) = ch(\pi) \otimes \varepsilon_{\ell_0}, \quad \psi((ch(\tau), \bullet)) = (ch(\tilde{\tau}) \cdot \omega_{\hat{H}_0}) \otimes \varepsilon_{\ell_1}.
\]

It is easy to see that \(\psi\) is bijective. We only show that \(\psi\) is homomorphic.

\[
1. \quad \psi((ch(\pi_i), o) \ast (ch(\pi_j), o)) = \psi((ch(\pi_i) \cdot ch(\pi_j), o)) = (ch(\pi_i) \cdot ch(\pi_j)) \otimes \varepsilon_{\ell_0} = (ch(\pi_i) \otimes \varepsilon_{\ell_0}) \circ (ch(\pi_j) \otimes \varepsilon_{\ell_0}) = \psi((ch(\pi_i), o)) \circ \psi((ch(\pi_j), o)),
\]

\[
2. \quad \psi((ch(\pi), o) \ast (ch(\tau), \bullet)) = \psi(((\text{res}_H^\hat{H}_0 ch(\pi)) \cdot ch(\tau), \bullet)) = (ch(\pi) \cdot ch(\tilde{\tau}) \cdot \omega_{\hat{H}_0}^1) \otimes \varepsilon_{\ell_1} = (ch(\pi) \otimes \varepsilon_{\ell_0}) \circ (ch(\tilde{\tau}) \cdot \omega_{\hat{H}_0}^1 \otimes \varepsilon_{\ell_1}) = \psi((ch(\pi), o)) \circ \psi((ch(\tau), \bullet)),
\]

\[
3. \quad \psi((ch(\tau), \bullet) \ast (ch(\pi), o)) = \psi((ch(\tau), \bullet)) \circ \psi((ch(\pi), o))
\]

is obtained similarly.

\[
4. \quad \psi((ch(\tau_i), \bullet) \ast (ch(\tau_j), \bullet)) = \psi(q(\text{ind}_{\hat{H}_0}^H (ch(\tau_i) \cdot ch(\tau_j)), o) + (1-q)(ch(\tau_i) \cdot ch(\tau_j), \bullet)) = q\psi((\text{ind}_{\hat{H}_0}^H (ch(\tau_i) \cdot ch(\tau_j)), o) + (1-q)\psi((ch(\tau_i) \cdot ch(\tau_j), \bullet)) = q((ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{\hat{H}_0}^1) \otimes \varepsilon_{\ell_0}) + (1-q)((ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{\hat{H}_0}^1) \otimes \varepsilon_{\ell_1}) = (ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{\hat{H}_0}^1) \otimes (q\varepsilon_{\ell_0} + (1-q)\varepsilon_{\ell_1}) = (ch(\tilde{\tau}_i) \cdot ch(\tilde{\tau}_j) \cdot \omega_{\hat{H}_0}^1) \otimes (\varepsilon_{\ell_1} \bullet \varepsilon_{\ell_1}) = ((ch(\tilde{\tau}_i) \cdot \omega_{\hat{H}_0}^1) \otimes \varepsilon_{\ell_1}) \circ ((ch(\tilde{\tau}_j) \cdot \omega_{\hat{H}_0}^1) \otimes \varepsilon_{\ell_1}) = \psi((ch(\tau_i), \bullet)) \circ \psi((ch(\tau_j), \bullet)). \quad [Q.E.D.]
\]
Now let $H$ be a discrete commutative hypergroup of Pontryagin type and $H_0$ a closed subhypergroup of $H$. Since $\hat{H}$ is a compact hypergroup, $H_0^\perp$ is a compact subhypergroup. Then $K(\hat{H}, \varphi, \mathbb{Z}_q(2))$ is defined via the hyperfield $\varphi$ given by

$$\varphi(\ell) = \begin{cases} \{\chi_0\} & \text{if } \ell = \ell_0 \\ H_0^\perp & \text{if } \ell = \ell_1. \end{cases}$$

The dual field $\hat{\varphi}$ of $\varphi$ is the field

$$\hat{\varphi} : H \ni h \mapsto \hat{\varphi}(h) \in \mathbb{Z}_q(2)$$

with

$$\hat{\varphi}(h) = \begin{cases} \{\ell_0\} & \text{if } h \in H_0 \\ \mathbb{Z}_q(2) & \text{otherwise}. \end{cases}$$

Applying Theorem 3.5 together with Theorem 5.2 we obtain

**Theorem 5.3** Let $H$ be a discrete commutative hypergroup of Pontryagin type and $H_0$ a closed subhypergroup of $H$ such that $H_0^\perp$ is compact in $\hat{H}$. Then

$$\hat{K}(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2)) \cong K(\mathbb{Z}_q(2), \hat{\varphi}, H).$$

6. Examples of hypergroup duals of $K(\hat{H} \cup \hat{H}_0, \mathbb{Z}_q(2))$

**Example 6.1** If $H = \mathbb{Z}$ and $H_0 = n\mathbb{Z}$ $(n \in \mathbb{N})$, then

$$K(\hat{\mathbb{Z}} \cup \hat{n\mathbb{Z}}, \mathbb{Z}_q(2)) = K(\hat{T}, \varphi, \mathbb{Z}_q(2)) = \hat{T} \cup T$$

(in Example 4.1)

and

$$\hat{K}(\hat{\mathbb{Z}} \cup \hat{n\mathbb{Z}}, \mathbb{Z}_q(2)) = K(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z})$$

(in Example 4.1).

**Example 6.2** If $H = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $H_0 = n\mathbb{Z} \times m\mathbb{Z}$ $(n, m \in \mathbb{N})$, then

$$K(\hat{\mathbb{Z}^2} \cup \hat{n\mathbb{Z} \times m\mathbb{Z}}, \mathbb{Z}_q(2)) = K(\hat{T}^2, \varphi, \mathbb{Z}_q(2)) = \hat{T}^2 \cup T^2$$

(in Example 4.2)

and

$$\hat{K}(\hat{\mathbb{Z}^2} \cup \hat{n\mathbb{Z} \times m\mathbb{Z}}, \mathbb{Z}_q(2)) = K(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}^2)$$

(in Example 4.2).

**Example 6.3** If $H = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $H_0 = n\mathbb{Z} \times \{0\} \cong n\mathbb{Z}$ $(n \in \mathbb{N})$, then

$$K(\hat{\mathbb{Z}^2} \cup \hat{n\mathbb{Z}}, \mathbb{Z}_q(2)) = K(\hat{T}^2, \varphi, \mathbb{Z}_q(2)) = \hat{T}^2 \cup T$$

(in Example 4.4)

and

$$\hat{K}(\hat{\mathbb{Z}^2} \cup \hat{n\mathbb{Z}}, \mathbb{Z}_q(2)) = K(\mathbb{Z}_q(2), \hat{\varphi}, \mathbb{Z}^2)$$

(in Example 4.4).
Example 6.4 If $H = \mathcal{K}^{\alpha}(\mathbb{Z})$ and $H_0 = \mathcal{K}^{\alpha}(n\mathbb{Z})$ ($n \in \mathbb{N}$), then

$$\mathcal{K}(\hat{\mathcal{K}}^{\alpha}(\mathbb{Z}) \cup (\hat{\mathcal{K}}^{\alpha}(n\mathbb{Z})), \mathbb{Z}_q(2)) = \mathcal{K}(\mathcal{K}^{\alpha}(T), \varphi, \mathbb{Z}_q(2))$$

(in Example 4.5)

and

$$\hat{\mathcal{K}}(\hat{\mathcal{K}}^{\alpha}(\mathbb{Z}) \cup (\hat{\mathcal{K}}^{\alpha}(n\mathbb{Z})), \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, \mathcal{K}^{\alpha}(\mathbb{Z}))$$

(in Example 4.5).

Example 6.5 Let $B$ be a commutative Pontryagin hypergroup and $D$ a discrete commutative Pontryagin hypergroup. If $H = B \times D$ and $H_0 = B$, then

$$\mathcal{K}(\hat{B} \times D \cup \hat{B}, \mathbb{Z}_q(2)) = \mathcal{K}(\hat{B} \times \hat{D}, \varphi, \mathbb{Z}_q(2)) = \hat{B} \times (\hat{D} \vee \mathbb{Z}_q(2))$$

(in Example 4.6)

and

$$\hat{\mathcal{K}}(\hat{B} \times D \cup \hat{B}, \mathbb{Z}_q(2)) = \mathcal{K}(\mathbb{Z}_q(2), \hat{\varphi}, B \times D) = B \times (\mathbb{Z}_q(2) \vee D))$$

(in Example 4.6).

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