Jeans instability for an inert binary mixture: a kinetic theory approach in the Euler regime

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Abstract
The Jeans stability criterion for gravitational collapse is examined for the case of an inert binary mixture in local equilibrium, neglecting dissipative effects. The corresponding transport equations are established using kinetic theory within the Euler regime approximation. It is shown that the corresponding dispersion relation is modified; yielding corrections to the Jeans wave number that can be generalized for several interesting cases involving positive entropy production such as matter-antimatter systems and ordinary matter-dark matter dynamics.

1. Introduction
Structure formation through gravitational collapse is one of the most important processes present in astrophysics. First identified by James Jeans in 1902 [1], gravity acts through very long scales allowing density fluctuations to grow exponentially when the mass present at certain scales surpasses a given mass threshold. A very simple argument that leads to a fairly accurate approximation of this threshold consists in the consideration of the wave equation for the density fluctuation $\delta \rho$, valid for an isothermal one-component system with equilibrium density $\rho_0$ [2, 3]:

$$
\nabla^2 \delta \rho - \frac{1}{C_T^2} \frac{\partial^2 (\delta \rho)}{\partial t^2} + \frac{4\pi G \rho_0}{C_T^2} \delta \rho = 0,
$$

(1)

where $C_T^2$ is the isothermal speed of sound and $G$ is Newton’s gravitational constant. The wave-like solution $\delta \rho = A \exp[i(\mathbf{q} \cdot \mathbf{r} - \omega t)]$ leads to the dispersion relation:

$$
\frac{\omega^2}{C_T^2} - q^2 + \frac{4\pi G \rho_0}{C_T^2} = 0.
$$

(2)

Growing exponential modes associated to structure formation correspond to the existence of imaginary roots of equation (2).

The corresponding threshold is given by the Jeans wavenumber [4–6]:

$$
q_J = \sqrt[3]{\frac{4\pi G \rho_0}{C_T^2}}.
$$

(3)

Now, since the wavelength of the system is related to the wave number through the relation $\lambda = \frac{2\pi}{q}$, a rough estimation of the critical mass needed in order to form a structure through gravitational collapse in a volume scaled as $\lambda^3$ reads:

$$
M_J \approx \frac{\pi^{3/2} C_T^3}{G^{3/2} \rho_0^{1/2}}.
$$

(4)

For the case of a molecular cloud $\rho_0 \sim 1.6 \times 10^{-19} \text{kg m}^{-3}$ and $C_T \sim 0.435 \times 10^2 \text{m s}^{-1}$, yielding $M_J \sim 2.1 \times 10^{30} \text{kg}$, which is quite close to the value of a solar mass.
Heat fluxes associated with density gradients (Dufour effect) have been addressed for the case of a dilute two-component plasma \[7\], nevertheless, that work only established a dispersion relation for the whole system, not identifying the effect of the existence of different speeds of sound for the species. The case of a binary mixture in the Euler regime has been only partially addressed in phenomenological approaches neglecting temperature fluctuations \[8, 9\]. Another approach for these problems is the direct use of the Boltzmann equation \[10–12\] to establish the corresponding dispersion relation. In this type of formalism, it is very difficult to identify the different contributions of the species if dissipation is present in contrast with the averages approach used in the present work.

The purpose of the present paper is to fill this gap giving the kinetic foundations of the derivation of the dispersion relation for an inert dilute binary gas in the presence of Newtonian gravity. In order to accomplish this task, the paper has been divided as follows: section two corresponds to the establishment of the evolution equations of the averages of the conserved microscopic quantities (mass, linear momentum, and energy), special emphasis is made in the structure of the mass fluxes for each type of particle. In section three the transport equations are linearized. In section four a suitable dispersion relation is established identifying the effect of the second component of the mixture in the stability of the system. Finally, section five is devoted to a discussion of the results here obtained.

2. Balance equations

2.1. Evolution of the distribution function

The starting point of the formalism is the Boltzmann equation for two species in the presence of Newtonian gravity within the BGK approximation:

\[ \frac{\partial f_{i}^{a}}{\partial t} + v_{i}^{a} \frac{\partial f_{i}^{a}}{\partial x^{a}} + g_{i}^{a} \frac{\partial f_{i}^{a}}{\partial \tau_{i}^{a}} = - \frac{f_{i}^{a} - f_{i}^{(0)}{\tau}}, \quad i = 1, 2. \]  

(5)

In this work, the notation \([\_]\) is used to identify the \(i\)th-species and \((\_\_\_\_)\) is used to denote the order of the distribution function with respect to the Knudsen number. In equation (5), \(\tau\) is the relaxation time to the equilibrium functions \(f_{i}^{(0)}\) and the Newtonian gravitational fields \(g_{i}^{a}\) are expressed in terms of the corresponding scalar potential \(\varphi_{i}\) given by

\[ g_{i}^{a} = - \frac{\partial \varphi_{i}}{\partial x^{a}}, \quad i = 1, 2. \]  

(6)

which in turn satisfies the Poisson equation:

\[ \nabla^{2} \varphi_{i} = 4\pi G \rho_{i}, \quad i = 1, 2. \]  

(7)

The total gravitational potential \(\varphi\) satisfies the equation

\[ \nabla^{2} \varphi = 4\pi G \rho = 4\pi G (\varphi_{1} + \varphi_{2}) \]  

(8)

Macroscopic variables are identified with statistical averages, these are given, for species with number density \(n_{i}\) by:

\[ \langle \psi_{i} \rangle = \frac{1}{n_{i}} \int \psi_{i} f_{i} d\vec{v}_{i}. \]  

(9)

The center of mass velocity for the binary system is defined as

\[ u^{a} = \frac{m_{1} n_{1} \langle v_{1}^{a} \rangle + m_{2} n_{2} \langle v_{2}^{a} \rangle}{m_{1} n_{1} + m_{2} n_{2}}, \]  

(10)

and the chaotic velocity for each species reads:

\[ e_{i}^{a} = v_{i}^{a} - u^{a}. \]  

(11)

Equations (10)–(11) lead to the important relation:

\[ m_{1} n_{1} \langle e_{1}^{a} \rangle + m_{2} n_{2} \langle e_{2}^{a} \rangle = 0. \]  

(12)

2.2. Mass balance

The mass balances for each species are obtained multiplying equations (5) by the collision invariants \(m_{1}\) and \(m_{2}\) and upon integration in the respective velocity spaces, yielding:
\[
\frac{\partial \rho_{[i]}}{\partial t} + \frac{\partial}{\partial x^a}(\rho_{[i]} u^a_{[i]}) = 0, \quad i = 1, 2. \tag{13}
\]

In equation (13), \( \rho_{[i]} = m_{[i]} n_{[i]} \) and use has been made of the definition of statistical average (equation (9)). It is interesting to notice that dissipative contributions may appear in the mass balance equations through equation (11):

\[
\frac{\partial \rho_{[i]}}{\partial t} + \frac{\partial}{\partial x^a}(m_{[i]} n_{[i]} u^a_{[i]}) = - \frac{\partial}{\partial x^a} J^a_{[i]}, \quad i = 1, 2, \tag{14}
\]

where

\[
J^a_{[i]} = m_{[i]} n_{[i]} \langle \epsilon^a_{[i]} \rangle, \tag{15}
\]

To complete the mass balance, we notice that the addition of equations (12) over both species lead to the mass balance for the mixture:

\[
\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^a}(\rho u^a) = 0, \tag{16}
\]

where the total density of the system is:

\[
\rho = \rho_{[1]} + \rho_{[2]} \tag{17}
\]

### 2.3. Momentum balance

Upon multiplication of equations (5) by the collision invariants \( m_{[1]} \eta_{[1]}^1 \) and \( m_{[2]} \eta_{[2]}^2 \), the addition of the equations over both species leads to the density momentum balance equation:

\[
\frac{\partial}{\partial t}(\rho u^a) + \frac{\partial}{\partial x^b}(\rho u^a u^b + p \delta^{ab} + \Pi^{ab}) = \rho \nabla \varphi \tag{18}
\]

where \( p \) is the total pressure \( p = p_{[1]} + p_{[2]} \) and \( \varphi = \varphi_{[1]} + \varphi_{[2]} \). The last equation has been obtained considering equation (10) and the corresponding averages through the distribution functions:

\[
f_{[i]} = f_{[i]}^{(0)} + f_{[i]}^{(1)}. \tag{19}
\]

The equilibrium distribution function \( f_{[i]}^{(0)} \) is given by:

\[
f_{[i]}^{(0)} = n_{[i]} \left( \frac{m_{[i]}}{2\pi kT} \right)^{\frac{3}{2}} e^{-\frac{m_{[i]} n_{[i]}^2}{2kT}}, \tag{20}
\]

where \( k \) is the Boltzmann constant and \( T \) is the temperature of the system. In equation (18)

\[
\int_{\Gamma} f_{[i]}^{(0)} \epsilon^a_{[i]} \epsilon^b_{[i]} d\bar{c}_{[i]} = n_{[i]} kT \delta^{ab}, \tag{21}
\]

and

\[
\Pi^{ab} = \int_{\Gamma} f_{[1]}^{(1)} \epsilon^a_{[1]} \epsilon^b_{[1]} d\bar{c}_{[1]} + \int_{\Gamma} f_{[2]}^{(1)} \epsilon^a_{[2]} \epsilon^b_{[2]} d\bar{c}_{[2]}, \tag{22}
\]

where \( \Gamma \) corresponds to all the velocity space; \( \Pi^{ab} \) vanishes in the Euler regime.

### 2.4. Energy balance

Multiplying equations (5) by the collision invariants \( \frac{1}{2} m_{[1]} \eta_{[1]}^1 \) and \( \frac{1}{2} m_{[2]} \eta_{[2]}^2 \), the addition of both equations leads to the energy density balance equation:

\[
\frac{\partial}{\partial t} \left( \frac{1}{2} \rho u^2 + \frac{3}{2} \rho k T \right) + \frac{\partial}{\partial x^b} \left( \frac{1}{2} \rho u^2 u^b + \frac{3}{2} \rho k T u^b + p u^b \delta^{ab} + f_{[i]}^{b} \right) = 0 \tag{23}
\]

where the heat flux \( J^a_{[i]} \) is given by:

\[
J^a_{[i]} = \frac{1}{2} m_{[1]} n_{[1]} \langle \epsilon^a_{[1]} \epsilon^2_{[1]} \rangle + \frac{1}{2} m_{[2]} n_{[2]} \langle \epsilon^a_{[2]} \epsilon^2_{[2]} \rangle. \tag{24}
\]

This heat flux vanishes in the Euler regime, but in the Navier–Stokes case \( f_{[i]}^{(1)} \) leads to useful expressions that contain the Fourier law and the Dufour effect [7].

### 3. Linearized transport equations

Equations (13), (18) and (23) constitute a set of non-linear partial differential equations which can be studied close to local equilibrium in terms of linear perturbations. In this type of formalism the vector containing the unknown variables \( x \) is expressed as the sum of its equilibrium value \( x_0 \) and its corresponding fluctuations \( \delta x \):
Neglecting second order terms, assuming a vanishing center of mass velocity and defining \( \delta \theta = \nabla \cdot \delta \vec{u} \), the linearized set of equations is easily obtained by direct substitution. The linearized mass balance reads:

\[
\frac{\partial}{\partial t} (\rho_{0|i}) + \rho_{0|i} \delta \theta = 0, \quad i = 1, 2.
\]

The longitudinal mode of the linear momentum balance equation becomes:

\[
\frac{\partial}{\partial t} (\delta \theta) + \left( \frac{\rho_{0[1]}}{\rho_0} C_{T[1]}^2 + \frac{\rho_{0[2]}}{\rho_0} C_{T[2]}^2 \right) \nabla^2 \left( \frac{\delta T}{T_0} \right) + C_{T[1]}^2 \nabla^2 \left( \frac{\delta \rho_{1[1]}}{\rho_0} \right) + C_{T[2]}^2 \nabla^2 \left( \frac{\delta \rho_{1[2]}}{\rho_0} \right) = -\nabla^2 \delta \varphi,
\]

and the set is closed using the linearized energy balance equation, together with the Poisson equation, respectively:

\[
\frac{\partial}{\partial t} (\delta T) + \frac{2}{3} T_0 \delta \theta = 0,
\]

\[
\nabla^2 (\delta \varphi) = 4\pi G (\delta \rho_{1[1]} + \delta \rho_{1[2]}).
\]

### 4. Dispersion equation and Jeans stability criterion

The linearized transport equations (27)–(30) can be treated algebraically taking successively in each equation the Fourier transform in space and the Laplace transform in time.

Defining:

\[
\delta \hat{X}(q^j, s) = \int_{0}^{\infty} \int_{-\infty}^{\infty} \delta x(\vec{r}, t) e^{iq^j \cdot \vec{r}} e^{-st} dt
dr
\]

the linearized transport equations can be immediately expressed as:

\[
\hat{A} \delta \hat{X}(q^j, s) = 0
\]

where

\[
\delta \hat{X} = (\delta \rho_{1[1]}, \delta \rho_{1[2]}, \delta \theta, \delta T, \delta \varphi)
\]

and

\[
\hat{A} = \begin{pmatrix}
\frac{s}{\rho_{0[1]}} & 0 & 1 & 0 & 0 \\
0 & \frac{s}{\rho_{0[2]}} & 1 & 0 & 0 \\
-C_{T[1]}^2 & 0 & \frac{s}{\rho_0} & \frac{s}{\rho_0} & -q^2 \\
0 & 0 & \frac{2}{3} & \frac{s}{\rho_0} & 0 \\
4\pi G & 4\pi G & 0 & 0 & q^2
\end{pmatrix}
\]

Real values for \( s \) in the dispersion relation are identified with exponentially growing/decaying modes. In the Euler regime, and using the center of mass velocity in the balance equations (equation (10)), a very simple dispersion relation is obtained. The resulting expression reads:

\[
s^2 \left[ s^2 - 4\pi G (\rho_{1[1]} + \rho_{1[2]}) + q^2 \left( \frac{5}{3} C_{T[1]}^2 \rho_{0[1]} + \frac{5}{3} C_{T[2]}^2 \rho_{0[2]} \right) \right] = 0
\]

The critical wavenumbers correspond to non-trivial real values of \( s \) in equation (34), yielding:

\[
q^2 = \frac{4\pi G (\rho_{0[1]} + \rho_{0[2]})}{\frac{5}{7} \left( C_{T[1]}^2 \rho_{0[1]} + C_{T[2]}^2 \rho_{0[2]} \right)}.
\]

The Jeans Mass for the binary system is established following the same arguments included in the introduction. The result obtained is:
References

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5. Final remarks

The use of a representation of the local variables involving the center of mass velocity allows establishing simple expressions for the dispersion relation (equation (34)). Indeed, for a three-component system, the corresponding matrix in the Fourier–Laplace space reads:

\[
B = \begin{pmatrix}
\frac{s}{\rho_{0[1]}} & 0 & 0 & 0 & 0 \\
0 & \frac{s}{\rho_{0[2]}} & 0 & 1 & 0 \\
0 & 0 & \frac{s}{\rho_{0[3]}} & 1 & 0 \\
- \frac{C_{T[1]}^2 q^2}{\rho_0} & - \frac{C_{T[2]}^2 q^2}{\rho_0} & - \frac{C_{T[3]}^2 q^2}{\rho_0} & \frac{s}{\rho_0} & - q^2 \left( \frac{C_{T[1]}^2 \rho_{0[1]}}{\rho_0} + \frac{C_{T[2]}^2 \rho_{0[2]}}{\rho_0} + \frac{C_{T[3]}^2 \rho_{0[3]}}{\rho_0} \right) - q^2 \right] & 0 \\
4G\pi & 4G\pi & 4G\pi & 0 & \frac{s}{\rho_0} \\
0 & 0 & 0 & \frac{s}{\rho_0} & 0
\end{pmatrix}
\]

This matrix leads to the dispersion relation:

\[
s^5 \left[ s^2 + q^2 \left( C_{T[1]}^2 \frac{\rho_{0[1]}}{\rho_0} + \frac{5}{3} C_{T[2]}^2 \frac{\rho_{0[2]}}{\rho_0} + \frac{5}{3} C_{T[3]}^2 \frac{\rho_{0[3]}}{\rho_0} \right) - 4\pi G \rho_0 \right] = 0
\]  

(37)

where \( \rho_0 = \rho_{0[1]} + \rho_{0[2]} + \rho_{0[3]} \). The corresponding Jeans wave number reads:

\[
q_c^2 = \frac{4\pi G \rho_0}{s^2 \left( C_{T[1]}^2 \frac{\rho_{0[1]}}{\rho_0} + \frac{5}{3} C_{T[2]}^2 \frac{\rho_{0[2]}}{\rho_0} + \frac{5}{3} C_{T[3]}^2 \frac{\rho_{0[3]}}{\rho_0} \right)}
\]

(38)

In general, for a multicomponent dilute inert mixture of \( n \) components, the critical wavenumber \( q_c \) reads:

\[
q_c^2 = \frac{4\pi G \rho_0}{s^2 \sum_{i=1}^{n} \left( \frac{\rho_{0[i]}}{\rho_0} C_{T[i]}^2 \right)}
\]

(39)

Although similar results have been obtained since the pioneering work of Grishchuk and Zel’dovich [9], there are quite a few works in which kinetic theory has been directly applied to the analysis of the gravitational instability of dilute mixtures of gases [10]. The approach here presented leads to remarkably simple expressions for the eigenfrequencies of the dynamical system and the critical wave number. This type of representation for the transport equations motivates further analysis of the Jeans instability involving diffusion and other dissipative effects in different temperature scales. This will be the subject of future work.
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