SOME SURFACES WITH CANONICAL MAPS OF DEGREE 10, 11 AND 14

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Abstract. In this note we present examples of complex algebraic surfaces of general type with canonical maps of degree 10, 11 and 14. They are constructed as quotients of a product of two Fermat septics using certain free actions of the group $\mathbb{Z}_2^7$.

1. Introduction

Beauville has shown in [B79] that if the image of the canonical map $\Phi_K$ of a surface $S$ of general type is a surface, then the following inequality holds:

$$d := \deg(\Phi_K) \leq 9 + \frac{27 - 9q}{p_g - 2} \leq 36.$$

Here, $q$ is the irregularity and $p_g$ the geometric genus of $S$. In particular, $28 \leq d$ is only possible if $q = 0$ and $p_g = 3$. Motivated by this observation, the construction of surfaces with $p_g = 3$ and canonical map of degree $d$ for every value $2 \leq d \leq 36$ is an interesting, but still widely open problem [MLP21, Question 5.2]. In particular for most values $10 \leq d$, it is not known if a surface realising that degree exists. Indeed, for a long time, the only examples were the surfaces of Persson [Per78], with canonical map of degree 16 and Tan [Tan03], with degree 12. In recent years, this problem attracted the attention of many authors, putting an increased effort in the construction of new examples. As a result, previously unknown surfaces with degree $d = 12, 16, 20, 24, 32$ and 36 have been discovered, by Rito [Ri15, Ri17a, Ri17b, Ri19], Gleissner, Pignatelli and Rito [GPR18] and Nguyen [N19, N21]. In this work, we present surfaces with canonical maps of degree $d = 10, 11$ and 14. According to our knowledge they are the first surfaces for those degrees. They can be described using Pardini’s theory of branched abelian covers [Pa91], which is one of the standard techniques in this subject, cf. [MLP21]. However, we decided to present them in an elementary way using plane curves and basic algebraic geometry at graduate textbook level [S13]. Our construction is completely self-contained, basically reference free and easily accessible. It can be sketched as follows: the surfaces $S$, which all have $p_g = 3$, arise as quotients of a product of two Fermat septics

$$F = \{x_0^7 + x_1^7 + x_2^7 = 0\} \subset \mathbb{P}^2$$

modulo certain free and diagonal actions of the group $\mathbb{Z}_2^7$. Their explicit construction allows us to write the canonical system of each of them in terms of three $\mathbb{Z}_2^7$-invariant holomorphic two-forms on the product $F \times F$. It turns out that for none of them $|K_S|$ is base-point free, i.e. the canonical map $\Phi_K : S \dasharrow \mathbb{P}^2$ is just a rational map. To compute its degree, we resolve the indeterminacy by a sequence of blowups and compute the degree of the resulting morphism via elementary intersection theory.

We point out that our surfaces are particular examples of surfaces isogenous to a product, i.e. quotients of a product of two curves modulo a free action of a finite group. This construction goes back to an exercise in Beauville’s book on Complex Algebraic Surfaces [B96, Exercises X.13 4], where a free action of $\mathbb{Z}_2^3$ on a product

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of two Fermat quintics is used to construct a surface with \( p_g = q = 0 \). This example served as an important inspiration for our work. By now, many more surfaces isogenous to a product have been constructed. Apart from other works, that mainly deal with irregular surfaces, we want to mention the complete classification of surfaces isogenous to a product with \( p_g = q = 0 \) \([BCG08]\) and the classification for \( p_g = 1 \) and \( q = 0 \) under the assumption that the action is diagonal \([G15]\). However, for higher values of \( p_g \), a classification of regular surfaces isogenous to a product is much more involved and is not yet established. Recently similar constructions involving non-free actions on a product of Fermat curves have been used to provide other interesting projective manifolds that helped us to understand some important geometric phenomena. Most notably are the rigid but not infinitesimally rigid manifolds \([BP21]\) of Bauer and Pignatelli that gave a negative answer to a question of Kodaira and Morrow \([KM71, p.45]\) and, to a lesser degree, also the infinite series of \( n \)-dimensional infinitesimally rigid manifolds of general type with non-contractible universal cover for each \( n \geq 3 \), provided by Frapporti and the second author of this paper \([FG21]\).

**Notation:** Throughout the paper a surface \( S \) is a projective manifold of dimension two. We use standard terminology in surface theory, cf. \([B79]\).

## 2. The surfaces

In this section we construct a series of surfaces \( S \), as quotients of a product of two Fermat septics \( F \), modulo a suitable diagonal action of the group \( \mathbb{Z}_7^2 \). For any surface \( S \), we determine the canonical map \( \Phi_K \) and compute its degree.

On the first copy of \( F \) we define the action of \( \mathbb{Z}_7^2 \) as

\[
\phi: \mathbb{Z}_7^2 \to \text{Aut}(F), \quad (a, b) \mapsto [(x_0 : x_1 : x_2) \mapsto (x_0 : \zeta^a_7 x_1 : \zeta^b_7 x_2)].
\]

This action has 21 points with non trivial stabilizer. They form three orbits of length 7. A representative of each orbit and a generator of the stabilizer is given by:

| point   | (\(-1 : 0 : \zeta_7\)) | (\(-1 : \zeta_7 : 0\)) | (\(0 : -1 : \zeta_7\)) |
|---------|-------------------------|-------------------------|-------------------------|
| generator | \((1, 0)\)               | \((0, 1)\)               | \((6, 6)\)               |

Note that the automorphisms \( \phi(a, b) \) are precisely the deck transformations of the cover

\[
\pi: F \to \mathbb{P}^1, \quad (x_0 : x_1 : x_2) \mapsto (x_1^7 : x_2^7).
\]

The cover has degree 49 and is branched along \((0 : 1)\), \((1 : 0)\) and \((-1 : 1)\). In particular \( F/\mathbb{Z}_7^2 \simeq \mathbb{P}^1 \) and \( \pi \) is the quotient map.

On the second copy of \( F \), for which we use the homogenous variables \( y = (y_0 : y_1 : y_2) \), the group acts by \( \phi \circ A \), where \( A \in \text{Aut}(\mathbb{Z}_7^2) \) is an automorphism depending on the specific example. The explicit choices for \( A \) are stated in the tables below. To write the canonical systems of the corresponding unmixed quotients

\[
S := (F \times F)/\mathbb{Z}_7^2 \quad \text{modulo the diagonal actions} \quad \phi \times (\phi \circ A),
\]

we need to fix a suitable basis of the space \( H^0(F, \Omega^1_F) \) of global holomorphic 1-forms on \( F \). In affine coordinates such a basis is given by

\[
\{\omega_{jk} := u^j v^{k-6} du \mid j + k \leq 4\}, \quad \text{where} \quad u := \frac{x_1}{x_0} \quad \text{and} \quad v := \frac{x_2}{x_0}.
\]

Note that:

1) The action of \( \mathbb{Z}_7^2 \) on \( H^0(F, \Omega^1_F) \) under pullback with \( \phi \) is

\[
\phi(a, b)^*(\omega_{jk}) = \zeta_7^{a(j+1)+b(k-6)} \omega_{jk}.
\]
II) The space of canonical sections $H^0(K_S)$ is isomorphic to $(H^0(\Omega^1_F) \otimes H^0(\Omega^1_F))^\mathbb{Z}_2^7$, where the action on the tensor product is diagonal, i.e. $(a, b) \in \mathbb{Z}_2^7$ acts via

$$\phi(a, b)^* \otimes \phi(A(a, b))^*.$$  

The observations I) and II) imply:

**Lemma 2.1.** A basis of $H^0(K_S)$ is given by the $\mathbb{Z}_2^7$-invariant tensors $\omega_{jklm} := \omega_{jk} \otimes \omega_{lm}$. A tensor $\omega_{jklm}$ is invariant if and only if for all $(a, b) \in \mathbb{Z}_2^7$ it holds:

$$a(j + 1) + b(k - 6) + a'(l + 1) + b'(m - 6) \equiv 0 \mod 7,$$

where $$\begin{pmatrix} a' \\ b' \end{pmatrix} := A \begin{pmatrix} a \\ b \end{pmatrix}.$$ 

We can now state and prove our main result:

**Theorem 2.2.** For all $A \in \text{Aut}(\mathbb{Z}_2^7)$ in the table below, the diagonal action $\phi \times (\phi \circ A)$ of $\mathbb{Z}_2^7$ on the product of two Fermat septics is free. The quotient is a regular smooth projective surface $S$ of general type with $p_g = 3$. A basis of $H^0(K_S)$, the canonical map $\Phi_{K_S}$ in projective coordinates and its degree are stated in the table:

| No | $A$ | Basis of $H^0(K_S)$ | $\Phi_{K_S}(x, y)$ | deg($\Phi_{K_S}$) |
|----|-----|----------------------|---------------------|-------------------|
| 1. | (4 5) | $\{\omega_{1304}, \omega_{2210}, \omega_{3012}\}$ | $(x_1x_2x_3^2y_2^4 : x_1^2x_2^2y_0y_1 : x_0x_1y_0y_1y_2^2)$ | 10 |
| 2. | (2 6) | $\{\omega_{0011}, \omega_{1202}, \omega_{2040}\}$ | $(x_1^3y_0y_1y_2 : x_0x_1x_2^2y_0^2y_2^2 : x_0x_3^2y_1^2)$ | 11 |
| 3. | (3 3) | $\{\omega_{0103}, \omega_{1310}, \omega_{3031}\}$ | $(x_1^2x_2y_0y_2^2 : x_1x_2^2y_0^2y_1 : x_0x_1y_0y_1y_2^2)$ | 14 |

**Proof.** First we show that the three diagonal actions $\phi \times (\phi \circ A)$ on $F \times F$ are free. Indeed, as remarked above, the non-trivial stabilizers of the points on the first copy of $F$ are generated by $(1, 0), (0, 1)$ and $(6, 6)$. However, none of these elements have a fixed point on the second copy of $F$ under the twisted actions $\phi \circ A$. Thus, the actions are free and the quotient surfaces $S$ are smooth, projective and of general type. The latter holds because the genus of the Fermat septic is $g(F) = 15 \geq 2$. Moreover, they are regular surfaces, because they do not possess any non-zero holomorphic one/forms, since $F/\mathbb{Z}_2^7$ is biholomorphic to $\mathbb{P}^1$. The geometric genus of each $S$ is therefore equal to

$$p_g = \chi(O_S) - 1 = \frac{(g(F) - 1)^2}{|\mathbb{Z}_2^7|} - 1 = \frac{14^2}{49} - 1 = 3.$$ 

Using Lemma 2.1 we compute a basis of $H^0(K_S)$ for each surface $S$. Replacing the affine variables by $\frac{x_0}{y_0}$ and $\frac{y_0}{x_0}$ and multiplying by $x_0^4y_0^4$ we obtain the bi-quartics that define the canonical map.

It remains to determine the degree of $\Phi_{K_S}$ for each surface $S$. For this purpose we resolve the indeterminacy of these maps by a sequence of blowups, as explained in the textbook [B96] Theorem II.7:

$$\begin{array}{ccc}
\tilde{S} & \xrightarrow{\phi_{\tilde{S}}} & S \\
\xrightarrow{\Phi_{\tilde{S}}} & \Phi_{K_S} & \\
\xrightarrow{\Phi_{\tilde{S}}} & \mathbb{P}^2, & \\
\end{array}$$

Here, $|\tilde{M}|$ is a base-point free linear system. The self-intersection $\tilde{M}^2$ is positive if and only if $\Phi_{\tilde{S}}$ is not composed by a pencil. In this case $\Phi_{\tilde{S}}$ is onto and it holds:

$$\deg(\Phi_{K_S}) = \deg(\Phi_{\tilde{S}}) = \tilde{M}^2.$$
For the computation of the resolution, it is convenient to write the divisors of the bi-quartics defining \( \Phi_{K_S} \) as linear combinations of the reduced curves \( F_j := \{x_j = 0\} \) and \( G_k := \{y_k = 0\} \) on \( S \). Note that \( F_j \) and \( G_k \) intersect transversally in only one point and \((F_j, F_k) = (G_j, G_k) = 0\), for all \( j, k \). Thus, these curves can be illustrated as a grid of three vertical and three horizontal lines.

As an example, consider the first surface in the table. Here, the divisors of the three bi-quartics spanning the canonical system \(|K_S|\) are:

\[
F_1 + 3F_2 + 4G_2, \quad 2F_1 + 2F_2 + 3G_0 + G_1 \quad \text{and} \quad F_0 + 3F_1 + G_0 + G_1 + 2G_2.
\]

The fixed part of \(|K_S|\) is \( F_1 \) and the mobile part \(|M|\) has precisely four base-points:

\[
F_1 \cap G_2 = \{p_{12}\}, \quad F_2 \cap G_0 = \{p_{20}\}, \quad F_2 \cap G_1 = \{p_{21}\} \quad \text{and} \quad F_2 \cap G_2 = \{p_{22}\}.
\]

We blow up these points, take the pullback of the mobile part \(|M|\) of \(|K_S|\) and remove the fixed part of this new linear system. We repeat the procedure, until we obtain a base-point free linear system \(|\hat{M}|\). The degree of the canonical map can then be computed as \( \deg(\Phi_{K_S}) = \hat{M}^2 \). Alternatively, we can shortcut the computation by using Lemma 2.3 from below, which tells us the contribution of the difference \( M^2 - \hat{M}^2 \) coming from each base-point simply by looking at the coefficients of the divisors spanning the mobile part of \(|K_S|\). See Example 2.3 for an illustration.

Fortunately the conditions of Lemma 2.3 on the coefficients of the divisors are fulfilled also for any other surface of the table, providing us with an easier way to compute the degree of the canonical map.

\[\square\]

**Lemma 2.3.** Let \(|M|\) be a two-dimensional linear system on a surface \( S \), with only isolated base-points, which is spanned by \( D_1, D_2 \) and \( D_3 \). Assume that in a neighborhood of a basepoint \( p \), we can write the divisors \( D_i \) as

\[
D_1 = aH, \quad D_2 = bK \quad \text{and} \quad D_3 = cH + dK,
\]

where \( H \) and \( K \) are reduced, smooth and intersect transversally at \( p \) and \( a, b, c, d \) are non-negative integers, \( b \leq a \). Assume that

- \( d \geq b \) or
- \( b \neq 0 \) and \( c + md \geq a \), where \( a = mb + q \) with \( 0 \leq q < b \).

Then after blowing up at most \((ab)\)-times we obtain a new linear system \(|\hat{M}|\) such that no infinitely near point of \( p \) is a base-point of \(|\hat{M}|\). Moreover \( \hat{M}^2 = M^2 - ab \).

**Proof.** We prove the lemma by induction on \((a, b)\) with \( b \leq a \). Here we are considering the lexicographic order \( \leq \) defined on the lower diagonal \( \Delta^2 := \{(a, b) : a \geq b\} \subseteq \mathbb{N} \times \mathbb{N} \) as follows:

\[
(a', b') \leq (a, b) \text{ if and only if } a' < a \text{ or } a' = a \text{ and } b' \leq b.
\]

In this case \( \Delta^2 \) admits the well-ordering principle and so the principle of mathematical induction holds.

Suppose that \((a, b) = 0\). Then \(|M|\) is base-point free and so \( \hat{M} = M = M^2 - ab \). Now suppose that the statement is true for \((a', b') < (a, b)\). Our aim is to prove it for \((a, b)\). We blow up the base-point \( p \), take the pullback of the divisors \( D_i \) and remove the fixed part, which is the exceptional divisor \( bE \) of the blowup. In fact the pullback of \( D_3 \) contains \( c + d \) times \( E \) and \( c + d \geq b \), thanks to the assumptions \( c + md \geq a \) or \( d \geq b \):

If \( d \geq b \), then \( c + d \geq b \), otherwise if \( d < b \) and \( c + md \geq a \), then

\[
c + d - b \geq c + m(d - b) \geq c + md - a \geq 0.
\]

Restricted to the preimage of our neighborhood of \( p \), these divisors are:

\[
a\hat{H} + (a-b)E, \quad b\hat{K} \quad \text{and} \quad c\hat{H} + d\hat{K} + (c + d - b)E.
\]
Here, $\hat{H}$ and $\hat{K}$ are the strict transforms of $H$ and $K$. Let $|\hat{M}|$ be the linear system generated by these three divisors, then $\hat{M}^2 = M^2 - b^2$. If $a = b$ or $b = 0$, then $|\hat{M}|$ is base-point free and we are done. Otherwise, on the preimage, the linear system $|\hat{M}|$ has precisely one new base-point: the intersection point of $\hat{K}$ and $E$. Locally near this point the three divisors spanning $|\hat{M}|$ are:

$$(a-b)E, \quad b\hat{K} \quad \text{and} \quad d\hat{K} + (c+d-b)E.$$  

We need to distinguish two cases, when $m = 1$ or when $m > 1$. In the first case $a - b = q < b$, so that $(b, q) < (a, b)$. We can write $b = m'q + q'$, with $0 \leq q' < q$ and define new coefficients $a' := b, b' := q, c' := d$ and $d' := c + d - b$. Then they fulfill the inductive hypothesis, because:

If $c + d \geq a$, then

$$d' = c + d - b \geq a - b = q = b',$$

else if $d \geq b$, then

$$c' + m'd' \geq c' = d \geq b = a'.$$

By induction, the self-intersection of the new linear system $\hat{M}$ is equal to

$$\hat{M}^2 = (M^2 - b^2) - qb = M^2 - ab.$$  

In the case $m > 1$, then $b \leq a - b$ and $(a-b, b) < (a, b)$. We define new variables $a' := a - b$, $b' := b, c' := c + d - b$ and $d' := d$. Observe that $a' = a - b = (m-1)b' + q$ and we can define $m' := m - 1$. They satisfy the inductive hypothesis, because of the estimations:

If $c + md \geq a$, then

$$c' + m'd' = c + d - b + (m-1)d = c + md - b \geq a - b = a',$$

else if $d \geq b$, then $d' \geq b'$. Hence the self-intersection of the new linear system $\hat{M}$ is equal to

$$\hat{M}^2 = (M^2 - b^2) - (a-b)b = M^2 - ab.$$  

\hspace{1cm} \square

**Remark 2.4.** Lemma 2.3 fails if both of the conditions $c + md \geq a$ and $d \geq b$ are not satisfied. For example, taking $a = 3, b = 2$ and $c = d = 1$ we get that the correction term of $M^2 - \hat{M}^2$ is 5 instead of 6.

**Example 2.5.** We illustrate Lemma 2.3 by computing the degree of the canonical map of the first surface in our main Theorem 2.2 again. Recall that the mobile part of the canonical system is generated by:

$D_1 := 3F_2 + 4G_2, \quad D_2 := F_1 + 2F_2 + 3G_0 + G_1$ \quad and \quad $D_3 := F_0 + F_1 + G_0 + G_1 + 2G_2$.  

We determine the correction term to the self intersection number for each of the four base-points:

$$p_{12}, \quad p_{20}, \quad p_{21} \quad \text{and} \quad p_{22}, \quad \text{where} \quad \{p_{ij}\} = F_i \cap G_j.$$

- Around $p_{12}$, the divisors $D_i$ are given by $4G_2, F_1$ and $2F_1 + 2G_2$. In the notation of the Lemma $a = 4, b = 1$ and $c = d = 2$. This implies $c + md = 10 \geq 4$ and $d \geq q - 1 = -1$. The correction term is $ab = 4$.
- Around $p_{20}$ the divisors are $3F_2, 2F_2 + 3G_0$ and $G_0$. In this case $a = 3, b = 1, c = 2, d = 3$ and the correction term is $ab = 3$.
- Around $p_{21}$, we have $3F_2, 2F_1 + G_1$ and $G_1$, which yields 3 as correction term.
- Around $p_{22}$ we have $3F_2 + 4G_2, 2F_2$ and $2G_2$, thus the correction term is 4.

The degree of the canonical map is therefore given by

$$\deg(\Phi_{K_2}) = (3F_2 + 4G_2)^2 - 4 - 3 - 3 - 4 = 10.$$  

For completeness, we point out:
Remark 2.6. Our surfaces in Theorem 2.2 are examples of Beauville surfaces of unmixed type, because $F/\mathbb{Z}^2 \simeq \mathbb{P}^1$ and the quotient cover $\pi: F \to \mathbb{P}^1$ is branched in three points, as we explained above. Beauville surfaces are precisely the rigid surfaces isogenous to a product, i.e. those that allow no non-trivial deformations [BGV15].

While we cannot classify all regular surfaces isogenous to a product with $p_g = 3$, it is possible to classify those that arise under an unmixed diagonal action of an abelian group, thanks to the MAGMA algorithm [BCP97] from the paper [GPR18]. In particular, we know all unmixed Beauville surfaces $S$ with abelian group and $p_g = 3$. They form seven biholomorphism classes, which can all be realized as quotients of a product of two Fermat septic modulo $\mathbb{Z}^2$. Three of the four examples which are not in the table of Theorem 2.2 have generically finite canonical maps of degree 5, 7 and 14, whilst the canonical map of the fourth surface is composed with a pencil.

Extending the table of Theorem 2.2, we have:

| No | $A$ | Basis of $H^0(K_S)$ | $\Phi_{K_S}(x, y)$ | $\deg(\Phi_{K_S})$ |
|----|-----|---------------------|-------------------|------------------|
| 4  | $\begin{pmatrix} 3 & 3 \\ 6 & 2 \end{pmatrix}$ | $\{\omega_{10201}, \omega_{10004}, \omega_{3112}\}$ | $(x_0^2x_2^3y_1y_2 : x_0^3x_1y_1^2 : x_1^2x_2y_0y_1y_2^2)$ | 5 |
| 5  | $\begin{pmatrix} 5 & 4 \\ 6 & 5 \end{pmatrix}$ | $\{\omega_{1022}, \omega_{2131}, \omega_{4010}\}$ | $(x_0^3x_1y_1^2y_2^2 : x_0x_1^2x_2y_1y_2^2 : x_1^4y_0y_1)$ | 7 |
| 6  | $\begin{pmatrix} 1 & 1 \\ 6 & 2 \end{pmatrix}$ | $\{\omega_{10101}, \omega_{1313}, \omega_{3030}\}$ | $(x_0^3x_2y_0y_2^2 : x_1x_2^3y_1y_2^2 : x_0x_1^2y_0y_1^4)$ | 14 |
| 7  | $\begin{pmatrix} 2 & 2 \\ 6 & 3 \end{pmatrix}$ | $\{\omega_{2002}, \omega_{2121}, \omega_{4040}\}$ | $(x_0^2x_2^2y_0y_2^2 : x_0x_1^3y_0y_1^2y_2 : x_1^4y_1^4)$ | $\text{im}(\Phi_{K_S}) = \{y^2 = xz\} \subset \mathbb{P}^2$ |

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