Superintegrable Systems, Multi-Hamiltonian Structures and Nambu Mechanics in an Arbitrary Dimension

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Abstract

A general algebraic condition for the functional independence of $2n - 1$ constants of motion of an $n$-dimensional maximal superintegrable Hamiltonian system has been proved for an arbitrary finite $n$. This makes it possible to construct, in a well-defined generic way, a normalized Nambu bracket which produces the correct Hamiltonian time evolution. Existence and explicit forms of pairwise compatible multi-Hamiltonian structures for any maximal superintegrable system have been established. The Calogero-Moser system, motion of a charged particle in a uniform perpendicular magnetic field and Smorodinsky-Winternitz potentials are considered as illustrative applications and their symmetry algebras as well as their Nambu formulations and alternative Poisson structures are presented.

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I. INTRODUCTION

Nambu mechanics is a generalization of the Hamiltonian formulation of classical mechanics in that it replaces the usual binary Poisson bracket (PB) to higher order $n$-ary bracket, generically called Nambu bracket (NB), and specifies the dynamics in terms of $n - 1$ "generalized Hamiltonian" functions [1,2]. The original motivation of Nambu was to show that the Hamiltonian mechanics is not the only formulation that makes a statistical mechanics possible. Relevance of Nambu mechanics to membrane theory has been put forward and a form of quantized Nambu mechanics has been purposed as a nonlinear generalization of geometric formulation of quantum mechanics [3,4]. Nambu formulation may also give some insights into the theory of higher order algebraic structures and their possible physical significance [5]. Unfortunately, up to now only few examples of dynamical systems which admit Nambu formulation have been given. The Euler equations for three dimensional (3D) rigid body were the only example given by Nambu. Then, the equations of Nahm system in the theory of static $SU(2)$ monopoles were realized in this formulation [2,6]. Connection between Nambu mechanics and so-called superintegrable systems has been the subject of some recent studies [7,8,9,10].

A Hamiltonian system of $n$ degrees of freedom is called to be completely integrable, in the Liouville-Arnold sense, if it admits $n$ functionally independent, globally defined constants of motion in involution (i.e., commuting with respect to PB) [12,13]. A completely integrable system is called superintegrable if it allows $k$ additional constants of motion. Not all constants of motion of a superintegrable system can be in involution but they must be functionally independent, otherwise the extra invariants are trivial. Superintegrability is said to be minimal if $k = 1$, and maximal if $k = n - 1$ [14,15,16]. An $nD$ maximally superintegrable Hamiltonian system can be specified by the following set

$$SI_H(2n - 1) = \{H, H_i, A_j : \{H, H_i\} = 0 = \{H_i, H_j\}, \{H, A_i\} = 0, \Gamma \neq 0\}$$ (1)

where $H$ is the Hamiltonian of the system, $H_i, A_j; i, j = 1, 2, \ldots, n - 1$ are the additional constants of motion and $\Gamma$ denotes the following $(2n - 1)$-form

$$\Gamma = dH \wedge dH_1 \wedge \cdots \wedge dH_{n-1} \wedge dA_1 \wedge \cdots \wedge dA_{n-1}.$$ (2)

Here $d$ and $\wedge$ denote the usual exterior derivative and exterior product of Cartan calculus. For functional independence $\Gamma$ must be different from zero on a dense subset of the underlying symplectic manifold endowed with the PB $\{\}.$

In this paper three main points concerning the fundamental structure of a $SI_H(2n - 1)$ system for any finite $n$ are established. We shall first prove that the constants of motion of a $SI_H(2n - 1)$ system are functionally independent where $(n - 1) \times (n - 1)$ matrix $B$ with elements $B_{ij} = \{H_i, A_j\}$ is nonsingular. Secondly we shall construct the normalized NB which produces the correct Hamiltonian time evolutions. This means that all $nD$ maximally superintegrable Hamiltonian systems admit Nambu formulation. We then show that every $SI_H(2n - 1)$ system admits $2n - 1$ multi-Hamiltonian structures. Statements that we have proved make these facts possible, in a well defined generic way and independently from the forms of Hamiltonians. Multi-Hamiltonian structures of maximally superintegrable systems were considered for the first time in Ref. [3] by using a different approach from ours. In this
context, our geometric proofs of Jacobi identity and of compatibility condition for alternative Poisson structures are different but validate and, in a sense, are complementary to that of Ref. [3].

In the next section main points of the Nambu mechanics, mainly those needed for the subsequent investigation are briefly reviewed (for more details we refer to [2]). In Section III a coordinate-free form of canonical NB and normalized NB are introduced, and the Jacobi identity for a normalized binary bracket induced from NB is established. The general algebraic condition for the functional independence is proved in section IV. Nambu formulation of a $SI_H(2n-1)$ system is established in section V where we also point out some general facts concerning the structure of symmetry algebras of maximally superintegrable systems. Multi-Hamiltonian structures are taken up in section VI. As applications the Calogero-Moser system, motion of a charged particle in a uniform perpendicular magnetic field (this will be referred to as (classical) Landau problem) and Smorodinsky-Winternitz potentials are considered in the final section where their symmetry algebras and explicit forms of their alternative Hamiltonian structures are established.

Our further notational conventions are as follows. We shall denote the linear spaces of all vector fields and (differential) $p$-forms on a smooth manifold $M$ of dimension $n$, respectively, by $X(M)$ and $\Lambda^p(M)$, $0 \leq p \leq n$. Together with their commutative (and associative) algebra structure with respect to usual point-wise product, the linear space of all smooth functions (0-forms) defined on $M$ will be represented by $\mathcal{A}$. We shall denote the vector fields by bold face letters, adopt the Einstein summation convention over repeated pair of contravariant and covariant indices and use the shorthand $\mathcal{A}^\otimes n = \mathcal{A} \otimes \cdots \otimes \mathcal{A}$ ($n$ times).

II. NAMBU MECHANICS

NB of order $n$ is the real multilinear map $\{ \ldots, \} : \mathcal{A}^\otimes n \rightarrow \mathcal{A}$ which has, for all $f_j, g_j \in \mathcal{A}$, the following properties.

i. Skew-symmetry

$$\{f_1, \ldots, f_n\} = (-1)^\varepsilon \{f_{\sigma(1)}, \ldots, f_{\sigma(n)}\},$$

where $\sigma$ is a member of the permutation group $S_n$ and $\varepsilon$ is the parity of permutation $\sigma$ ($\varepsilon = 1$ for odd permutations, and $\varepsilon = 0$ for even permutations).

ii. Derivation (the Leibniz rule)

$$\{f_1f_2, f_3, \ldots, f_{n+1}\} = f_1\{f_2, f_3, \ldots, f_{n+1}\} + f_2\{f_1, f_3, \ldots, f_{n+1}\}.$$  

iii. Fundamental identity (a kind of generalized Jacobi identity)

$$\{f_1, \ldots, f_{n-1}, \{g_1, \ldots, g_n\}\} = \sum_{k=1}^{n} \{g_1, \ldots, \{f_1, \ldots, f_{n-1}, g_k\}, \ldots, g_n\}.$$  

With respect to NB, $\mathcal{A}$ acquires another algebra structure henceforth denoted by $\mathcal{A}_N$.

Nambu dynamics is determined by $n-1$ Hamiltonian functions $h_1, \ldots, h_{n-1} \in \mathcal{A}_N$ and is described, for any $f \in \mathcal{A}_N$, by the Nambu-Hamilton (NH) equations of motion
\[
\frac{df}{dt} = X_{NH}(f) = \{f, h_1, \ldots, h_{n-1}\},
\]

(6)

where \(X_{NH}\) is called the NH vector field corresponding to \(h_1, \ldots, h_{n-1}\).

NB of order \(n\) induces infinite family of lower order NB, including the family of Poisson structures, all of which satisfy corresponding fundamental identities (FIs) that follow from (5). Below we shall concentrate only on the induced Poisson structures. For a fixed set of \(n - 2\) Hamiltonian functions \(f_i \in \mathcal{A}_N\) we define the Nambu induced PB as follows

\[
\{f, g\}_{NP} = \{f, g, f_1, \ldots, f_{n-2}\},
\]

(7)

where \(f, g \in \mathcal{A}_N\) are arbitrary. If in Eq. (5) we take

\[f_{n-1} = f, g_{n-1} = g, g_n = h, g_i = f_i; i = 1, \ldots, n - 2\]

then, by virtue of (3), the first \(n - 2\) terms at the right hand side (5) vanish and we get

\[
\{f, \{g, h\}_{NP}\}_{NP} + cp = 0,
\]

(8)

where \(cp\) stands for cyclic permutations. Eq. (8) reveals the fact that \(\{,\}_{NP}\) satisfies the Jacobi identity. Note that all the fixed functions in the definition (7) are Casimirs of \(\{,\}_{NP}\), that is, \(\{f_i, f\}_{NP} = 0\) for all \(f_i, i = 1, \ldots, n - 2\) and \(f \in \mathcal{A}_N\).

### III. CANONICAL NB AND NORMALIZED NB

The problem of constructing concrete realizations of NB is of great importance. In the case of \(M = \mathbb{R}^n\) the following form, called the canonical NB

\[
\{f_1, \ldots, f_n\} = \frac{\partial(f_1, \ldots, f_n)}{\partial(x^1, \ldots, x^n)},
\]

(9)

was provided by Y. Nambu. Here \((x^1, \ldots, x^n)\) denote the local coordinates of \(\mathbb{R}^n\) and the right hand side stands for the Jacobian of the mapping \(f = (f_1, \ldots, f_n): \mathbb{R}^n \to \mathbb{R}^n\).

We shall now introduce a coordinate-free expression of the canonical NB that provides a considerable ease in proving the technical points of the paper. For this purpose we first associate the \(n - 1\) form

\[
\gamma = dh_1 \wedge \cdots \wedge dh_{n-1}
\]

(10)

to \(n - 1\) Hamiltonian functions \(h_j \in \mathcal{A}_N\). We then recall the Hodge map \(* : \Lambda^p(\mathbb{R}^n) \to \Lambda^{n-p}(\mathbb{R}^n)\) defined for any p-form \(w = (1/p!)w_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\) as follows [17]

\[
* w = \frac{1}{p!(n-p)!} \epsilon^{i_1 \ldots i_p} w_{i_1 \ldots i_p} dx^{i_{p+1}} \wedge \cdots \wedge dx^{i_n},
\]

where \(w_{i_1 \ldots i_p}\) are antisymmetric covariant components of \(w\) and \(\epsilon^{i_1 \ldots i_n}\) with \(\epsilon^{1 \ldots n} = 1\) is the \(nD\) completely antisymmetric Levi-Civita symbol. Note that with respect to \(\mathcal{A}\) the Hodge map is linear and exterior product is bilinear. It is now obvious that, in local coordinates
\[ * (df \wedge \gamma) = \{ f, h_1, \ldots, h_{n-1} \} = \frac{\partial(f, h_1, \ldots, h_{n-1})}{\partial(x^1, \ldots, x^n)}. \] (11)

In that case, multilinearity, antisymmetry and derivation properties of the canonical NB are direct results of the linearity of the Hodge map with respect to \( \mathcal{A} \), and of the well-known properties of \( \wedge \)-product and \( d \). We can also associate the \( n-1 \)-form \( \beta = \beta_{i_1 \ldots i_{n-1}} dx^{i_1} \wedge \cdots \wedge dx^{i_{n-1}}/(n-1)! \) to the vector field \( \beta = (\beta^1, \ldots, \beta^n) \) with components \( \beta^k = \epsilon^{k i_1 \ldots i_{n-1}} \beta_{i_1 \ldots i_{n-1}}/(n-1)! \). Then, (11) can also be written as

\[ * (df \wedge \gamma) = \gamma \cdot \nabla f, \] (12)

where \( \nabla \) stands for \( n \)-dimensional gradient operator and \( \cdot \) denotes the usual inner product of \( \mathbb{R}^n \). The fundamental identity can be verified by taking \( f = \{ g_1, \ldots, g_n \} \) in Eq. (12).

We should note that Eq. (12) implies

\[ X_{NH} = (-1)^{n-1} (\gamma \wedge d) = \gamma \cdot \nabla, \] (13)

for the NH vector field corresponding to \( h_1, \ldots, h_{n-1} \). As illustrative examples, let us consider the cases \( n = 2, 3 \). The vector fields corresponding to \( \gamma = dh \) in the case of \( n = 2 \) and \( \gamma' = dh_1 \wedge dh_2 \) in the case of \( n = 3 \) are easily found to be

\[ \gamma = (\partial_2 h, -\partial_1 h), \quad \gamma' = \nabla h_1 \times \nabla h_2, \]

where \( \partial_j = \partial/\partial x_j \) and \( \times \) denotes the cross product of \( \mathbb{R}^3 \). The associated NH vector fields and canonical NBs can be written as follows

\[ X_{NH}(f) = \partial_2 h \partial_1 f - \partial_1 h \partial_2 f, \quad X'_{NH}(f) = (\nabla h_1 \times \nabla h_2) \cdot \nabla f. \]

The first is the usual PB of \( \mathbb{R}^2 \) and the second is the original NB first appeared in [1]. In the next two sections we shall generalize these expressions for a \( SI_H(2n-1) \) system in the case of arbitrary \( n \).

To be precise, from now on we shall adopt, in accordance with related literature, the following definition: If the time evolution equations of a dynamical system can be written in terms of (canonical) NB then the system will be called to admit equivalent Nambu formulation. As it will be apparent in the next two sections in order to get the correct dynamics the induced PBs must be properly normalized. For this purpose, in terms of \( n-2 \)-form

\[ \eta = df_1 \wedge \cdots \wedge df_{n-2}, \] (14)

we define

\[ \{ f, g \}_{NP} = C \{ f, g, f_1, \ldots, f_{n-2} \} = C^* (df \wedge dg \wedge \eta), \] (15)

where \( C \in \mathcal{A} \) is, for the time being, an arbitrary function. \( C \) will be referred to as the normalization coefficient and will be specified from the requirement that the NH equation produces the correct time evolution for any function. The generic form of \( C \) will be determined in the next section, but before that it must be emphasized that \( \{ , \} \_{NP} \) satisfies the Jacobi identity for any \( C \). To prove this let us consider
\begin{align}
\{h, \{f, g\}_{NP}\}_{NP}' &= C^*\{dh \wedge d[C^*(df \wedge dg) \wedge \eta] \wedge \eta\} 
\end{align} \label{eq:16}

and its cyclic permutations for three functions \(h, f\) and \(g\). Since in a \(nD\) space we have at most \(n\) functional independent functions and since in \((15)\) we have, apart from \(C\), \(n + 1\) functions, in the most general case one of them, say \(f\), must be functional dependent to others. Hence we can take

\begin{align}
df &= adh + bg + \sum_{i=1}^{n-2} c_if_i, 
\end{align} \label{eq:17}

where \(a, b\) and \(c_i\) are arbitrary constants. On substituting this in \((16)\) and in its \(cp\) we immediately see that their sum vanishes.

Obviously, in the case of \(n = 2\) we have \(C = 1\). For known 3D examples, namely, for free rigid body and the Nahm equations we also have \(C = 1\). However the requirement of nontrivial \(C\) is inevitable at least when \(n\) is an even integer greater than two. Although this normalization requirement have appeared in the literature, its generic form and important implications were not recognized.

### IV. AN ALGEBRAIC EXPRESSION FOR FUNCTIONAL INDEPENDENCE

The phase-space of a Hamiltonian system is a \(2^n\) dimensional symplectic manifold \(M\) on which a symplectic structure is defined by a closed \((d\Omega = 0)\) and nondegenerate symplectic 2-form \(\Omega\). Two immediate implications of nondegeneracy are that \([11,12]\); (i) \(M\) is orientable with nowhere vanishing Liouville measure (volume form)

\begin{align}
V_L &= \frac{(-1)^{n(n-1)/2}}{n!} \Omega^n, 
\end{align} \label{eq:18}

where \(\Omega^n = \Omega \wedge \cdots \wedge \Omega\) (\(n\) times). (ii) There is a natural isomorphism between the vector fields and 1-forms defined by \(\xi \rightarrow \mu\xi = i\xi\Omega\), where \(i\xi : \Lambda^p(M) \rightarrow \Lambda^{p-1}(M)\) is called the interior product operator defined for any \(p\)-form \(\alpha\) by

\begin{align}
(i\xi\alpha)(\xi_1, \ldots, \xi_{p-1}) &= \alpha(\xi, \xi_1, \ldots, \xi_{p-1}).
\end{align} \label{eq:19}

If \(\mu\xi\) is exact, that is, if \(\mu\xi = df\), then \(\xi\) is called a Hamiltonian vector field corresponding to \(f \in \mathcal{A}\) and henceforth denoted by \(\xi_f : i\xi_f\Omega = df\). PB on \(M\) is defined by \(\Omega(\xi_f, \xi_g) = \{f, g\}\). According to Darboux theorem at each point of \(M\) there are local canonical coordinates \((q^1, \ldots, q^n, p_1, \ldots, p_n)\) in which \(\Omega\) takes the form \(\Omega = dq^1 \wedge dp_j\) and leads us to the following coordinate expressions

\begin{align}
\{f, g\} &= \Omega(\xi_f, \xi_g) = \partial_{q^i}f \partial_{p_j}g - \partial_{p_j}f \partial_{q^i}g, 
\xi_f &= \partial_{p_j}f \partial_{q^i} - \partial_{q^i}f \partial_{p_j},
V_L &= dq^1 \wedge \cdots \wedge dq^n \wedge dp_1 \wedge \cdots \wedge dp_n. 
\end{align} \label{eq:20}

By definition of exterior forms and their \(\wedge\)-products, the value of \(\alpha \wedge \beta; \alpha \in \Lambda^p, \beta \in \Lambda^q\) on \(p + q\) vectors \(\xi_k \in \mathcal{X}(M)\); \(k = 1, 2, \ldots, p + q \leq 2n\), is given, in the notation of \([12]\), by
where \( i_1 < \cdots < i_p \) and \( j_1 < \cdots < j_q \) such that \( (i_1, \ldots, i_p, j_1, \ldots, j_q) \) is a permutation of \((1, 2, \ldots, p+q)\). The summation in Eq. (21) is over all permutations of the permutation group \( S_{p+q} \), provided that the indices are partitioned into two ordered sets as given above.

By making use of Eq. (21) we now evaluate the Liouville form \( V_L \) given by (20c) on \( 2n \) Hamiltonian vector fields \( \xi_{f_1}; A = 1, \ldots, 2n \), as follows

\[
V_L(\xi_{f_1}, \ldots, \xi_{f_{2n}}) = \varepsilon_{i_1}^{\cdots} i_{2n} dq^1(\xi_{f_{i_1}}) \cdots dq^n(\xi_{f_{i_{n+1}}}) \cdots dp_n(\xi_{f_{i_{2n}}})
\]

\[
= (-1)^n \varepsilon_{i_1}^{\cdots} i_{2n} \partial_{p_1} f_{i_1} \cdots \partial_{p_n} f_{i_{n+1}} \partial_q f_{i_{n+2}} \cdots \partial_q f_{i_{2n}}
\]

\[
= (-1)^n \frac{\partial(f_{i_1+1}, \ldots, f_{i_2}, f_{i_1}, \ldots, f_{i_n})}{\partial(q^1, \ldots, q^n, p_1, \ldots, p_n)}.
\]

By the antisymmetry properties of determinant, or in view of \( V_L(\xi_1, \ldots, \xi_{2n}) = (-1)^n V_L(\xi_{n+1}, \ldots, \xi_{2n}, \xi_1, \ldots, \xi_n) \), the above relation can be written as

\[
V_L(\xi_{f_1}, \ldots, \xi_{f_{2n}}) = \frac{\partial(f_1, \ldots, f_{2n})}{\partial(q^1, \ldots, q^n, p_1, \ldots, p_n)}.
\]

We now evaluate \( \Omega^n \) on the same set of Hamiltonian vector fields as follows

\[
\Omega^n(\xi_{f_1}, \ldots, \xi_{f_{2n}}) = \sum_{S_{2n}} (-1)^\varepsilon \Omega(\xi_{f_{i_1}}, \xi_{f_{i_2}}) \cdots \Omega(\xi_{f_{k_1}}, \xi_{f_{k_2}})
\]

\[
= \sum_{S_{2n}} (-1)^\varepsilon \{ f_{i_1}, f_{i_2} \} \cdots \{ f_{k_1}, f_{k_2} \},
\]

where \( i_1 < i_2; \ldots; k_1 < k_2 \) such that \( (i_1, i_2, \ldots, k_1, k_2) \) is a permutation of \((1, 2, \ldots, 2n)\), and we have used Eq. (20a). Let us make the identifications

\[
f_1 = f, \ f_2 = H, \ f_{2+i} = H_i, \ f_{n+i+1} = A_i; \ i = 1, 2, \ldots, n - 1 ,
\]

where \( H, H_i, A_i \in SI_H(2n-1) \) and \( f \in \mathcal{A} \) is an arbitrary function. In that case, at the right hand side of Eq. (23) only the terms in which \( H \) is paired with \( f \) and each \( H_i \) is paired with one of \( A_j \) give non-zero contributions. All other possible pairing are zero by very definition of maximal superintegrability. With this in mind let us consider a fixed partition

\[
\{ f_{i_1}, f_{i_2} \} \{ f_{j_1}, f_{j_2} \} \cdots \{ f_{k_1}, f_{k_2} \}.
\]

There are \( n \) different places for \( \{ f_{i_1}, f_{i_2} \} \), and for a fixed place of \( \{ f_{i_1}, f_{i_2} \} \) there are \( n - 1 \) possible places for \( \{ f_{j_1}, f_{j_2} \} \), and \( n - 2 \) places for the next pair (provided that the places of

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1When we were about to submit this paper we came across to very recent study [10] which derive Eq. (23) (with Jacobian at the left hand side) without referring to the above symplectic techniques. They successfully use it in Nambu formulation of a class of systems whose symmetry algebras close into some simple Lie algebras (see the next section) and then they develop, in analogy to the classical case, interesting quantum versions of NB.
the first two pairs are fixed), and so on. In all this cases the signs of permutations are the same since each is obtained from the initial one by interchanging two fixed pairs of indices. Hence, in the right hand side of (23) there are \( n! \) identical copies of each non-zero term. Thus, (23) can be written, in view of identifications given by (24), as

\[
\Omega^n(\xi_{f_1}, \ldots, \xi_{f_{2n}}) = n!K(n)\{f, H\},
\]

where \( K(n) \) represents all different partitions which are non-zero

\[
K(n) = N\epsilon_1^{i_1} \cdots \epsilon_{n-1}^{i_{n-1}}\{H_1, A_1\} \cdots \{H_{n-1}, A_{i_{n-1}}\}.
\]

Since \( \epsilon^1 \cdots \epsilon^{n-1} = 1 \), the factor \( N \) is found to be \((-1)^{(n-1)(n-2)/2}\) by computing the parity of permutation

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots & 2n-1 & 2n \\
1 & 2 & 3 & n+2 & 4 & n+3 & 5 & \ldots & n+1 & 2n
\end{pmatrix}
\]

As a result, by defining \( B_{ij} = \{H_i, A_j\} \) and the \((n-1) \times (n-1)\) matrix \( B = (B_{ij}) \) with determinant

\[
detB = \epsilon_1^{i_1} \cdots \epsilon_{n-1}^{i_{n-1}}B_{11} \cdots B_{n-1,n-1},
\]

Eq. (25) can be written as

\[
\Omega^n(\xi_{f_1}, \ldots, \xi_{f_{2n}}) = (-1)^{(n-1)(n-2)/2} n! detB\{f, H\}.
\]

It will be convenient to give the explicit calculation of (27) for \( n = 1, 2, 3 \). For \( n = 1 \) we immediately get \( \Omega(\xi_{f_1}, \xi_{f_2}) = \{f, H\} \), and \( detB = 1 \). In the case of \( n = 2 \) we obtain directly from (23)

\[
\Omega^2(\xi_{f_1}, \ldots, \xi_{f_4}) = (-1)^{\epsilon_1}\{f_1, f_2\}\{f_3, f_4\} + (-1)^{\epsilon_2}\{f_3, f_4\}\{f_1, f_2\},
\]

where \( \epsilon_1 \) is the parity of identity permutation and \( \epsilon_2 \) is the parity of

\[
\begin{pmatrix}
1 & 2 & 3 & 4 \\
3 & 4 & 1 & 2
\end{pmatrix} = (13)(24) \Rightarrow \epsilon_2 = 0 = \epsilon_1.
\]

Hence \( \Omega^2(\xi_{f_1}, \ldots, \xi_{f_4}) = 2\{f, H\}B_{11} \), and \( detB = B_{11} = \{H_1, A_1\} \). For \( n = 3 \) we have

\[
\Omega^3(\xi_{f_1}, \ldots, \xi_{f_6}) = \{f_1, f_2\} \sum_{i=1}^{3} [((-1)^{\epsilon_1^i} + (-1)^{\epsilon_2^i})\{f_3, f_5\}\{f_4, f_6\} +
\]

\[
((-1)^{\epsilon_1^i} + (-1)^{\epsilon_2^i})\{f_3, f_6\}\{f_4, f_5\}],
\]

where \( \epsilon_1^j ; j = 1, 2, 3, 4 \), are for the following permutations

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 5 & 4 & 6
\end{pmatrix} = (45) \Rightarrow \epsilon_1^1 = 1,
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 4 & 6 & 3 & 5
\end{pmatrix} = (3465) \Rightarrow \epsilon_1^2 = 1,
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 & 6 & 4 & 5
\end{pmatrix} = (465) \Rightarrow \epsilon_1^3 = 0,
\]

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 4 & 5 & 3 & 6
\end{pmatrix} = (345) \Rightarrow \epsilon_1^4 = 0.
\]
For $\epsilon_2$ and $\epsilon_3$, the first pair (12) must be interchanged, respectively, with the second pair and the last pair in the above permutations; in both cases we have the same signs as $\epsilon_1$. Hence Eq. (28) can be written as

$$\Omega^3(\xi_{f_1}, \ldots, \xi_{f_n}) = -3!\{f, H\}(B_{11}B_{22} - B_{12}B_{21}),$$

and we get $\det B = B_{11}B_{22} - B_{12}B_{21}$.

In view of Eq. (18) a comparison of Eqs. (22) and (27) yields

$$\frac{\partial(f, H, H_1, \ldots H_{n-1}, A_1, \ldots, A_{n-1})}{\partial(q^1, \ldots, q^n, p_1, \ldots, p_n)} = (-1)^{n+1}\det B\{f, H\}.$$

In terms of

$$d^n x = dx^1 \wedge \cdots \wedge dx^n,$$

$$d^n \dot{x}^i = dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n,$$

we obtain for $\Gamma$, by comparing (11) and (30)

$$\Gamma = \det B \sum_{i=1}^{n} (-1)^{i-1}[\partial_q H d^n q \wedge d^n \dot{p}_i - (-1)^n \partial_{p_i} H d^n q^i \wedge d^n p_i],$$

where the summation is explicitly written to avoid any confusion. Note that $^*(df \wedge \Gamma')$ is equal to the left hand side of Eq. (30). We conclude this section by the following statement.

If $H \neq \text{constant}$ then the constants of motion $H, H_i, A_i \in SI_H(2n - 1)$ are functionally dependent ($\Gamma = 0$) if and only if $\det B = 0$. These functions are functionally independent ($\Gamma \neq 0$) where $\det B \neq 0$.

V. NAMBU FORMULATION OF MAXIMALLY SUPERINTEGRABLE SYSTEMS AND THEIR SYMMETRY ALGEBRAS

Since the Jacobian determinant (30) is proportional to $\{f, H\}$, the correct time evolution of $f$ can be expressed by the properly normalized NB as in Eq. (15) with the normalization coefficient $C = (-1)^{n+1}/\det B$. Indeed, the bracket

$$\{f, H\}^{(0)}_{NP} = ^*(df \wedge \Gamma'),$$

$$= \frac{(-1)^{n+1}\partial(f, H, H_1, \ldots, H_{n-1}, A_1, \ldots, A_{n-1})}{\det B}\frac{\partial(q^1, \ldots, q^n, p_1, \ldots, p_n)}{\partial(q^1, \ldots, q^n, p_1, \ldots, p_n)},$$

written in terms of $(2n - 1)$-form

$$\Gamma' = \frac{(-1)^{n+1}}{\det B}\Gamma,$$

produces the correct Hamiltonian time evolution

$$\frac{df}{dt} = \{f, H\}^{(0)}_{NP} = \{f, H\}.$$
where \( f \) is an arbitrary function. Therefore, every \( nD \) maximal superintegrable time-independent Hamiltonian system defined by (1) admits equivalent Nambu formulation.

It must be emphasized that as we have proved in Sec. III the normalized NB defined, for two arbitrary functions \( h, f \in \mathcal{A} \), by

\[
\{ f, h \}^{(0)}_{NB} = \ast (df \wedge dh \wedge \Gamma_H),
\]

obeys the Jacobi identity. Here the \((2n - 2)\)-form \( \Gamma_H \) is defined as

\[
\Gamma_H = \frac{(-1)^{n+1}}{\det B} dH_1 \wedge \cdots \wedge dH_{n-1} \wedge dA_1 \wedge \cdots \wedge dA_{n-1}.
\]

The above discussion singles out an important special case in which the symmetry algebra of a \( SI_H(2n - 1) \) system is such that \( \det B \) is everywhere a non-zero constant. In such a case the constants of motion are globally functional independent and the Nambu formulation is possible without any (nontrivial) normalization coefficient. For this reason the rest of this section is devoted to a discussion of some general points of symmetry algebras of superintegrable systems.

Besides the vanishing PBs given by (1) the defining relations of the symmetry algebra of a \( SI_H(2n - 1) \) system contain the following

\[
\{ H_i, A_j \} = B_{ij}, \quad \{ A_i, A_j \} = C_{ij}.
\]

By the Jacobi identity \( B_{ij} \) and \( C_{ij} \) are constants of motion and each is functionally dependent to the constants of motion specified by the set \( SI_H(2n - 1) \). Hence, each of them can be expressed as \( X = X(H, H_i, A_j) \). By making use of the identity

\[
\{ R, X \} = \{ R, H \} \partial_H X + \sum_{i=1}^{n-1} (\{ R, H_i \} \partial_{H_i} X + \{ R, A_i \} \partial_{A_i} X)
\]

we obtain

\[
\{ H_i, X \} = \sum_{j=1}^{n-1} B_{ij} \partial_{A_j} X,
\]

\[
\{ A_i, X \} = \sum_{j=1}^{n-1} \left( -B_{ji} \partial_{H_j} X + C_{ij} \partial_{A_j} X \right),
\]

Obviously, the symmetry algebra of \( SI_H(2n - 1) \) system is a Lie algebra if and only if each of \( B_{ij} \) and \( C_{ij} \) is at most first order in \( H, H_i \) and \( A_j \). If each of \( B_{ij} \) and \( C_{ij} \) is a polynomial of degree at most \( k \), then the symmetry algebra is called to be a polynomial Poisson algebra of degree \( k \). There may also be cases in which the right hand sides of Eqs. (39) are polynomials but some of \( B_{ij} \) and \( C_{ij} \) are not. In such cases, by including the non-polynomial ones into the set of symmetry algebra we again obtain polynomial algebras. Although these possible cases are by no means exhaustive, especially for low values of \( n \) they are likely to occur as interesting structures \[14, 20, 21, 22\].
VI. MULTI-HAMILTONIAN STRUCTURES OF MAXIMALLY SUPERINTEGRABLE SYSTEMS

To ease the calculation of this section we shall use the notion \( H_k; k = 0, 1, \ldots, 2n - 2 \) such that \( H_0 = H \) and \( H_{n+i-1} = A_i, i = 1, \ldots, n - 1 \). Then, in terms of \((2n - 2)\)-forms

\[
\Gamma_H = \frac{(-1)^{n+k+1}}{\det B} dH_0 \wedge dH_1 \wedge \cdots \wedge dH_{k-1} \wedge d\hat{H}_k \wedge dH_{k+1} \wedge \cdots \wedge dH_{2n-2},
\]

where a hat over a quantity indicates that it should be omitted, we can define \( 2n - 1 \) different normalized NBs as follows

\[
\{ f, h \}^{(k)}_{NP} = \ast (df \wedge dh \wedge \Gamma_H), \quad k = 0, 1, \ldots, 2n - 2.
\]

Each of these brackets gives the original time evolution provided that we choose the new Hamiltonian function to be \( H_k \):

\[
\frac{df}{dt} = \{ f, H_k \}^{(k)}_{NP} = \{ f, H \}.
\]

In such a case, the system given by (1) has the so-called multi-Hamiltonian structures property: it can equally well be described by any one of the \( 2n - 1 \) pairs \((H_k, \{ \cdot \}^{(k)}_{NP})\).

We shall now prove that the above defined brackets are pairwise compatible, that is, \( \{ \cdot \}^{(k_1k_2)}_{NP} = a \{ \cdot \}^{(k_1)}_{NP} + b \{ \cdot \}^{(k_2)}_{NP} \) satisfies the Jacobi identity for all \( a, b \in \mathbb{R} \), independently from the form of \( detB \). Evidently, there is no loss of generality in taking \( k_1 = 0, k_2 = 1 \) and \( a = 1 = b \). Then let us consider

\[
\{ f, \{ g, h \}^{(01)}_{NP} \}^{(01)}_{NP} + cp = \{ \{ f, \{ g, h \}^{(0)}_{NP} \}^{(1)}_{NP} + cp \} + \{ \{ f, \{ g, h \}^{(1)}_{NP} \}^{(0)}_{NP} + cp \},
\]

\[
= \{ \ast [df \wedge d^* (dg \wedge dh \wedge \Gamma_{H_0} \wedge \Gamma_{H_1}) + cp] \} + \{ \ast [df \wedge d^* (dg \wedge dh \wedge \Gamma_{H_1} \wedge \Gamma_{H_0}) + cp] \},
\]

for three functions \( f, g \) and \( h \). In writing Eq. (43) we have made use of the fact that \( \{ \cdot \}^{(0)}_{NP} \) and \( \{ \cdot \}^{(1)}_{NP} \) satisfy the Jacobi identity separately. The definition of \( \{ \cdot \}^{(k)}_{NP} \) requires \( 2n - 1 \) independent functions but (43) involves \( 2n + 2 \) functions. Therefore, in the most general case any two of \((f, g, h)\), say \( f \) and \( g \), must depend on the other \( 2n \) independent functions. That is, we can take

\[
df = a_1 dh + \sum_{k=0}^{2n-2} b_k dH_k, \quad dg = a_2 dh + \sum_{k=0}^{2n-2} c_k dH_k,
\]

where \( a_1, a_2, b_k, c_k \) are arbitrary constants. When these are substituted in (43), with special care paid to the signs, the right hand side vanishes.

To point out another important property of superintegrable systems we shall use local canonical coordinates \( x^\gamma; \gamma = 1, \ldots, 2n \) such that \( x^j = q^j, x^{n+j} = p_j \) for \( j = 1, \ldots, n \). Then, in terms of the so-called Poisson tensor components

\[
\Lambda_{(k)}^{\alpha \beta} = \frac{(-1)^{n+k+1}}{\det B} \varepsilon^{\alpha \beta \gamma_0 \cdots \gamma_{2n-2}} \partial H_0 \partial H_1 \cdots \partial H_{k-1} \partial \hat{H}_k \partial H_{k+1} \cdots \partial H_{2n-2},
\]

were \( \Lambda_{(k)}^{\alpha \beta} \) is the \( \alpha \beta \)-component of the Poisson tensor. For example, \( \Lambda_{(1)}^{12} = \varepsilon_{123} \partial H_0 \partial H_1 \) and \( \Lambda_{(1)}^{12} = \varepsilon_{123} \partial H_0 \partial H_1 \). The \( \alpha \beta \)-component of the Poisson tensor is then given by

\[
\Lambda_{(k)}^{\alpha \beta} = \frac{(-1)^{n+k+1}}{\det B} \varepsilon^{\alpha \beta \gamma_0 \cdots \gamma_{2n-2}} \partial H_0 \partial H_1 \cdots \partial H_{k-1} \partial \hat{H}_k \partial H_{k+1} \cdots \partial H_{2n-2}.
\]
we rewrite Eq. (41) as

\[
\{ f, h \}_{NP}^{(k)} = \Lambda_{(k)}^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial h}{\partial x^\beta}, \quad k = 0, 1, \ldots, 2n - 2. \tag{45}
\]

In Eqs. (44-45) and below summations over repeated Greek letters range from 1 to 2\(n\). It is not hard to verify that in terms of Poisson tensor the Jacobi identity means that

\[
\Lambda_{(k)}^{\gamma\eta} \frac{\partial}{\partial x^\gamma} \Lambda_{(k)}^{\alpha\beta} + cp = 0, \tag{46}
\]

where \(cp\) indicates the cyclic sum with respect to superscripts \(\eta, \alpha\) and \(\beta\). We shall now prove that all of these Poisson tensors are singular, that is, \(\Lambda_{(k)}\) being the \(2n \times 2n\) matrix with elements \(\Lambda_{(k)}^{\alpha\beta}\) we have

\[
det(\Lambda_{(k)}) = \varepsilon_{\alpha_1 \cdots \alpha_{2n}} \Lambda_{(k)}^{1\alpha_1} \cdots \Lambda_{(k)}^{2n\alpha_{2n}} = 0. \tag{47}
\]

The easiest way to prove Eq. (47) may be as follows. Let \(\{e_\alpha : \alpha = 1, \ldots, 2n\}\) be a basis of vector space \(R^{2n}\) and let us define \(2n\) vectors \(v^{(\alpha)} = \Lambda_{(0)}^{\alpha\beta} e_\beta\), where, without any loss of generality, we have taken \(k = 0\). Then for \(k' = 1, \ldots, 2n - 2\) we have

\[
v^{(\alpha)} \cdot \nabla H_{k'} = \Lambda_{(0)}^{\alpha\beta} \frac{\partial H_{k'}}{\partial x^\beta} = \varepsilon^{\alpha\beta\gamma_1 \cdots \gamma_{2n-2}} \frac{\partial H_1}{\partial x^{\gamma_1}} \cdots \frac{\partial H_{2n-2}}{\partial x^{\gamma_{2n-2}}} \frac{\partial H_{k'}}{\partial x^\beta} = 0,
\]

because of the contraction of two symmetric and two antisymmetric indices. This proves that each of \(v^{(\alpha)}\) is perpendicular to the set of \(2n - 2\) linearly independent vectors \(S_H = \{\nabla H_{k'} : k' = 1, \ldots, 2n - 2\}\). Hence the \(2n\) vectors \(v^{(\alpha)}\) are linearly dependent and therefore the matrix of their components, which is the matrix \(\Lambda_{(0)}\), is singular. In fact the rank of this matrix is two since this is the dimension of the orthogonal complement of the set \(S_H\).

Evidently, all of the Poisson tensors defined above are pairwise compatible, but as they are singular they do not lead to any symplectic structure.

**VII. APPLICATIONS**

**A. Calogero-Moser system**

The Calogero-Moser system is one of the four \(nD\) systems which are known to be maximally superintegrable for any finite integer \(n\) [9,14,16]. The other three systems are the Kepler-Coulomb problem, harmonic oscillator with rational frequency ratios, and Winter- nitz system. Here we shall consider \(n = 2\) (two particles) case of the Calogero-Moser system described by

\[
H_{CM} = \frac{1}{2m} p^2 + \frac{g^2}{2(q^1 - q^2)^2}, \tag{48}
\]

where \(g\) is a constant. Constants of motion for \(H_{CM}\) can be written as
\[ H_1 = 2(q^1 + q^2)H_{CM} - q \cdot \mathbf{p} \frac{p_1 + p_2}{m}, \]
\[ A_1 = \frac{p_1 + p_2}{m}, \quad (49) \]
\[ A'_1 = \frac{1}{2m^2}(p_1 - p_2)^2 + \frac{g^2}{m(q^1 - q^2)^2}. \]

They obey the following relations
\[
\{ H_{CM}, H_1 \} = \{ H_{CM}, A_1 \} = \{ H_{CM}, A'_1 \} = 0, \]
\[
\{ A_1, A'_1 \} = 0, \quad \{ H_1, A_1 \} = 2A'_1, \quad \{ H_1, A'_1 \} = -2A_1A'_1. \quad (50)
\]

The relation \( \{ A_1, A'_1 \} = 0 \) implies (and is implied by) the fact that \( H_{CM}, A_1 \) and \( A'_1 \) are functionally dependent. From Eqs. (48) and (49) one can easily identify this dependence as 
\[ A'_1 = \frac{4H_{CM} - mA_1^2}{2m}. \]

In the former case the symmetry algebra is spanned by \( (H_{CM}, H_1, A_1) \) and since
\[ B_{11} = \{ H_1, A_1 \} = -A_1^2 + \frac{4}{m}H_{CM}, \quad (51) \]

it is a quadratic Poisson algebra. In the latter case the symmetry algebra is spanned by \( (H_{CM}, H_1, A'_1, B'_{11}) \), where \( B'_{11} = \{ H_1, A'_1 \} \) such that
\[ B'^2_{11} = -8A'_1^3 + \frac{16}{m}A_1^2H_{CM}. \]  

(52)

From Eqs. (39a) and (39b) it is found that
\[
\{ H_1, B'_{11} \} = \frac{1}{2} \frac{\partial B'^2_{11}}{\partial A'_1} = -12A'_1^2 + \frac{16}{m}A'_1H_{CM}, \quad (53a) 
\]
\[
\{ A'_1, B'_{11} \} = -\frac{1}{2} \frac{\partial B'^2_{11}}{\partial H_1} = 0. \quad (53b) 
\]

Hence we have again a quadratic Poisson algebra.

In both cases the time evolutions can be written in the Nambu and Hamiltonian mechanics equivalently as
\[
\frac{df}{dt} = -\frac{1}{B_{11}}\{ f, H_{CM}, H_1, A_1 \} = -\frac{1}{B'_{11}}\{ f, H_{CM}, H_1, A'_1 \} = \{ f, H_{CM} \}. \quad (54a) 
\]
\[
\frac{df}{dt} = -\frac{1}{B_{11}}\{ f, H_{CM}, H_1, A_1 \} = \{ f, H_{CM} \}. \quad (54b) 
\]

Both of the brackets
\[
\{ f, g \}^{(0)}_{NP} = -\frac{1}{B_{11}}\{ f, g, H_1, A_1 \}, 
\]
\[
\{ f, g \}^{(0)}_{NP} = -\frac{1}{B'_{11}}\{ f, g, H_1, A'_1 \}. 
\]
defined for two arbitrary functions $f, g$ satisfy the Jacobi identity. This is also the case for the normalized NBs corresponding to the Hamiltonian functions $H_1$

$$\{f, g\}^{(1)}_{NP} = \frac{1}{B_{11}} \{f, g, H_{CM}, A_1\},$$

$$\{f, g\}'^{(1)}_{NP} = \frac{1}{B_{11}'} \{f, g, H_{CM}, A_1'\},$$

and to $A_1$ and to $A_1'$

$$\{f, g\}^{(2)}_{NP} = \frac{1}{B_{11}} \{f, g, H_{CM}, H_1\},$$

$$\{f, g\}'^{(2)}_{NP} = \frac{1}{B_{11}'} \{f, g, H_{CM}, H_1'\}.$$ 

Finally in this subsection we should note that by redefinition of the constants of motion we can make the normalization coefficients trivial at the expense of restricting their domains of definition. As an example we consider the constant of motion

$$A'' = \frac{1}{2\hbar} \left( m \frac{A_1 - \hbar}{A_1 + \hbar} \right), \quad \hbar = 2 \sqrt{\frac{H_{CM}}{m}},$$

which satisfies $\{H_1, A''\} = \{H_1, A_1\} \partial A_1 A'' = -1$. Then, we can rewrite Eq. (54a) as

$$\frac{df}{dt} = \{f, H_{CM}, H_1, A''\} = \{f, H_{CM}\}. \quad (55)$$

Two more alternative brackets can be defined as

$$\{f, g\}''^{(1)}_{NP} = -\{f, g, H_{CM}, A''\},$$

$$\{f, g\}''^{(2)}_{NP} = \{f, g, H_{CM}, H_1\}. $$

Like others, all these brackets obey the Jacobi identity are pairwise compatible and each is degenerate. Explicit forms of corresponding Poisson tensors can also be found as will be done for the next application.

**B. Landau problem**

We now consider the well-known Landau Hamiltonian $H_L$ and first establish its super-integrability and Nambu formulation. Explicit expressions of three different Hamiltonian structures of this problem will be presented in the next subsection.

For a particle of charge $q > 0$ and mass $m$ moving on the $q^1 q^2$-plane under the influence of the perpendicular static and uniform magnetic field $B = \partial q^1 a_2 - \partial q^2 a_1$, $H_L$ is (in the Gaussian units)

$$H_L = \frac{1}{2m} (p - \frac{q}{c} a)^2 = \frac{1}{2} m (v_1^2 + v_2^2), \quad (56)$$
where \( c \) is the speed of light, \( \mathbf{a} = (a_1, a_2) \) is the vector potential and \( \mathbf{v} = (\mathbf{p} - \frac{2}{c} \mathbf{a})/m \) is the velocity vector \[23\]. Components of \( \mathbf{v} \) obey \( \{v_1, v_2\} = qB/m^2c \) for any \( B \).

When \( B \) is constant the most general form of the vector potential is \( \mathbf{a} = (B/2)(-q^2, q^1) + \nabla q \chi \), where \( \chi \equiv \chi(q) \) is an arbitrary gauge function. In such a case we have, in any gauge \( \chi \), two constants of motion

\[
H_1 = m(v_2 + \omega q^1), \quad A_1 = -m(v_1 - \omega q^2),
\]

(57)

where \( \omega = qB/mc \) is the cyclotron frequency. \( (H_1, A_1)/m\omega \) correspond to coordinates of the cyclotron centre and they satisfy the gauge-independent relations \( \{v_j, H_1\} = 0 = \{v_j, A_1\}, j = 1, 2 \), and

\[
\{H_L, H_1\} = 0 = \{H_L, A_1\}, \quad \{H_1, A_1\} = -m\omega.
\]

(58)

These relations explicitly show that \( H_L, H_1 \) and \( A_1 \) are functional independent constants of motion and they close into a Lie algebra structure which can be identified as the centrally extended Heisenberg-Weyl algebra. This completes the superintegrability of the Landau problem.

Let us now consider the 3-form \( \Gamma = (m\omega)^{-1}dH_L \wedge dH_1 \wedge dA_1 \), which can be written as

\[
\Gamma = (v_1 dq^2 - v_2 dq^1) \wedge [dp_1 \wedge dp_2 - \frac{q}{c}(dp_1 \wedge da_2 - dp_2 \wedge da_1)] + m\omega \mathbf{v} \cdot d\mathbf{p} \wedge dq^1 \wedge dq^2.
\]

(59)

Making use of this we immediately have

\[
*(dq \wedge \Gamma) = \mathbf{v},
\]

\[
*(dp \wedge \Gamma) = \frac{q}{c}(v_1 \nabla a_1 + v_2 \nabla a_2).
\]

It is straightforward to check that the right hand sides of these equations are the right hand sides of the canonical Hamiltonian equations for \( H_L \). Therefore, we can write them collectively as

\[
\frac{d}{dt} \mathbf{u} = *(d\mathbf{u} \wedge \Gamma),
\]

(60)

where \( \mathbf{u} = (q^1, q^2, p_1, p_2) \). One may also write Eq. (60) for any function \( f \).

C. Three Hamiltonian Structures of Landau Problem

Since \( B_{11} = \{H_1, A_1\} = -m\omega \) we can define the following three different Poisson tensors

\[
\Lambda_{(0)}^{\alpha\beta} = \frac{1}{m\omega} \epsilon^{\alpha\beta\gamma_1\gamma_2} \frac{\partial H_1}{\partial x^{\gamma_1}} \frac{\partial A_1}{\partial x^{\gamma_2}},
\]

\[
\Lambda_{(1)}^{\alpha\beta} = -\frac{1}{m\omega} \epsilon^{\alpha\beta\gamma_1\gamma_2} \frac{\partial H_L}{\partial x^{\gamma_1}} \frac{\partial A_1}{\partial x^{\gamma_2}},
\]

\[
\Lambda_{(2)}^{\alpha\beta} = \frac{1}{m\omega} \epsilon^{\alpha\beta\gamma_1\gamma_2} \frac{\partial H_L}{\partial x^{\gamma_1}} \frac{\partial H_1}{\partial x^{\gamma_2}},
\]

(61)

for the Landau problem. These are all computed and in terms of \( 2 \times 2 \) matrices.
\[ Y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} \partial_2 a_1 & \partial_2 a_2 \\ -\partial_1 a_1 & -\partial_1 a_2 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & v_1 \\ 0 & v_2 \end{pmatrix}, \quad S_2 = \begin{pmatrix} -v_1 & 0 \\ -v_2 & 0 \end{pmatrix}, \]

and

\[ l = \det Z, \quad l_1 = v_1 \partial_1 a_1 + v_2 \partial_1 a_2, \quad l_2 = v_1 \partial_2 a_1 + v_2 \partial_2 a_2, \]

they can be expressed by the following \(4 \times 4\) matrices

\[
\begin{align*}
\Lambda_{(0)} &= J_0 + \frac{1}{B} \begin{pmatrix} \xi Y & Z \\ -\tilde{Z} & \xi l_Y \end{pmatrix}, \\
\Lambda_{(1)} &= -v_2 \Lambda_{(0)} + \begin{pmatrix} 0 & S_1 \\ -\tilde{S}_1 & \xi l_1 Y \end{pmatrix}, \\
\Lambda_{(2)} &= v_1 \Lambda_{(0)} + \begin{pmatrix} 0 & S_2 \\ -\tilde{S}_2 & \xi l_2 Y \end{pmatrix}.
\end{align*}
\]

Here \(\tilde{Z}\) denotes the transposition of the matrix \(Z\) and in terms of \(2 \times 2\) zero matrix \(0\) and unit matrix \(1\)

\[ J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

stands for the standard \(4 \times 4\) symplectic matrix.

We first should note that these Hamiltonian structures are valid in any gauge \(\chi\). Then, as applications of general statements proved in the main text one can explicitly verify the following.

(i) Each of \(\Lambda_{(k)}\) provides the correct Hamiltonian equations for the Landau problem which can be cast in the following matrix forms

\[ \frac{d}{dt} \tilde{u} = \Lambda_{(k)} \nabla H_k; \quad k = 0, 1, 2, \]

where \(H_0 = H_L,\ H_2 = A_1\) and \(\nabla H_k\) is the column matrix of gradient \(H_k\).

(ii) \(\det \Lambda_{(k)} = 0\) and in fact each of \(\Lambda_{(k)}\) has rank two.

(iii) All \(\Lambda_{(k)}\) satisfy the Jacobi identity (46) and they are pair-wise compatible.

**D. Smorodinsky-Winternitz potentials**

We now consider a set of four potentials first found by Winternitz and co-workers [24]. Relaying on the assumptions; (i) Hamiltonians are of potential form, (ii) integrals of motion are at most quadratic in momenta, they found the following four potentials

\[
\begin{align*}
V^{(1)} &= \frac{1}{2} k r^2 + \frac{1}{2} \left( \frac{\alpha_1}{q_1^2} + \frac{\beta_1}{q_2^2} \right), \\
V^{(2)} &= \omega (4q_1^2 + q_2^2) + \alpha_2 q_1 + \frac{\beta_2}{q_2^2}, \\
V^{(3)} &= \frac{1}{2r} \left( \kappa + \frac{\alpha_3}{r + q_1} + \frac{\beta_3}{r - q_1} \right), \\
V^{(4)} &= \frac{1}{2r} (\sigma + \alpha_4 \sqrt{r + q_1} + \beta_4 \sqrt{r - q_1}).
\end{align*}
\]
where \( k, \omega, \kappa, \sigma \) and \( \alpha_j, \beta_j \) are some real constants and \( r^2 = q_1^2 + q_2^2 \). All these potentials accept separation of variables in at least two coordinate systems and each has, in addition to Hamiltonian function \( H^{(j)} = (p_i^2/2m) + V^{(j)} \), two constants of motion \( H_1^{(j)}, A_1^{(j)}; j = 1, 2, 3, 4 \). That is, all the Smorodinsky-Winternitz systems are superintegrable and they contain two-dimensional harmonic oscillator and Kepler-Coulomb problem as special cases. 

D-dimensional version of (67a) for \( D \geq 2 \) is known as the Winternitz system \([13]\).

Explicit expressions of constants of motion and their nonvanishing PBs are given altogether in the Table I. In addition to that given in third column of the table one must add the following relations

\[
\{H_1^{(j)}, H^{(j)}\} = \{A_1^{(j)}, H^{(j)}\} = \{B_1^{(j)}, H^{(j)}\} = 0, \quad (68)
\]

to the defining relations of symmetry algebras. The constants of motion \( B_1^{(j)} = \{H_1^{(j)}, A_1^{(j)}\} \) are found to be

\[
B_1^{(1)} = -\frac{4}{m} \left[ L \left( \frac{p_1 p_2}{m} + k q_1 q_2 \right) - \alpha_1 q_2 p_2 q_1^2 + \beta_1 q_1 p_1 q_2 \right],
\]

\[
B_1^{(2)} = -\frac{2}{m} \frac{q_2 p_2 (8 \omega q_1 + \alpha_2)}{m - 2 \omega q_2^2 + \frac{2 \beta_2 q_2}{q_2}},
\]

\[
B_1^{(3)} = -\frac{2 p_1 L^2}{m^2} + \frac{2 q_2 L}{mr} \left( \frac{\alpha_3}{2} + \frac{\beta_3}{r + q_1} + \frac{\beta_3}{r - q_1} \right) + \frac{q_2^2}{mr} q \cdot p \left[ \frac{\alpha_3}{(r + q_1)^2} - \frac{\beta_3}{(r - q_1)^2} \right],
\]

where \( B_1^{(4)} \) is not written due to its length and we have defined

\[
\gamma_j^\pm = \alpha_j \pm \beta_j, \quad L = p_2 q_1 - p_1 q_2.
\]

Since \( B_1^{(j)} \)'s are cubic polynomials in the momenta they can not be written as polynomials in the constants of motion \([22]\). But their squares can be expressed as follows

\[
B_1^{(1)} = -\frac{16}{m} \left[ A_1^{(1)} (H_1^{(1)})^2 - 2 H_1^{(1)} H^{(1)} + k A_1^{(1)} - 2 \gamma_1^+ k \right] + 4 \alpha_1 H^{(1)} \]

\[
\frac{16}{m} \gamma_1^- (2 H_1^{(1)} H^{(1)} - \gamma_1^- k),
\]

\[
B_1^{(2)} = \frac{8}{m} \left[ 4 H_1^{(2)} (H_1^{(2)})^2 - H^{(2)} - \omega A_1^{(2)} H_1^{(2)} - H^{(2)} - \beta_2 (16 \omega H_1^{(2)} + \alpha_2) \right],
\]

\[
B_1^{(3)} = -\frac{1}{m} \left[ H_1^{(3)} (4 A_1^{(3)} - 8 H_1^{(3)} H^{(3)} + 8 \gamma_3^+ H^{(3)} - \kappa^2) - 2 \gamma_5^- (\kappa A_1^{(3)} + \gamma_3^- H^{(3)}) + \kappa^2 \gamma_3^+ \right],
\]

\[
B_1^{(4)} = \frac{1}{m} \left[ H_1^{(4)} (2 H_1^{(4)} H_1^{(4)} - \frac{1}{2} \gamma_4^+ \gamma_4^-) + A_1^{(4)} (2 H_1^{(4)} A_1^{(4)} - A_4 \beta_4) - \frac{\sigma}{2} \left( H^{(4)} + \alpha_2 + \frac{\beta_2^2}{2} \right) \right].
\]

As a result, the symmetry algebras for all the Smorodinsky-Winternitz potentials are five dimensional quadratic Poisson algebras generated by \( H^{(j)}, H_{1}^{(j)}, A_{1}^{(j)}, B_{11}^{(j)} \) and 1. Finally in this section we will be content with writing the normalized NB of this system as

\[
\frac{df}{dt} = -\frac{1}{B_1^{(j)} (*) (df \wedge dH^{(j)} \wedge dH_{1}^{(j)} \wedge dA_{1}^{(j)}) = \{f, H^{(j)}\}, \quad (71)
\]

It is also straightforward to write out the alternative Poisson structures for each member of this class of potentials.
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| H(1) | Constants of Motion | Nonvanishing PBs |
|------|---------------------|------------------|
| H(1) | \( H^{(1)}_1 = \frac{p^2}{m} + kq_1^2 + \frac{\alpha^+}{q_1} \) | \{H^{(1)}_1, B^{(1)}_{11}\} = -\frac{8}{m}[H^{(1)}_1(H^{(1)}_1 - 2H^{(1)}) + 2k(A^{(1)}_1 - \gamma^+_1)] |
|      | \( A^{(1)}_1 = \frac{L^2}{m} + r^2 \left( \frac{\alpha^+}{q_1} + \frac{\partial}{q_1^2} \right) \) | \{A^{(1)}_1, B^{(1)}_{11}\} = \frac{16}{m}(H^{(1)}_1 A^{(1)}_1 - H^{(1)} A^{(1)}_1 - \gamma_1 H^{(1)}) |
| H(2) | \( H^{(2)}_1 = \frac{L^2}{2m} p_1^2 + 4\omega q_1^2 + \alpha q_1 \) | \{H^{(2)}_1, B^{(2)}_{11}\} = \frac{4}{m}(\alpha^2 H^{(2)}_1 - 2 \omega A^{(2)}_1 - \alpha H^{(2)}) |
|      | \( A^{(2)}_1 = \frac{2}{m} L p_2 - q_2^2 (4\omega q_1 + \alpha^2) + \frac{4\beta q_1}{q_2} \) | \{A^{(2)}_1, B^{(2)}_{11}\} = -\frac{16}{m}(3H^{(2)}_1)^2 - 4H^{(2)} H^{(2)}_1 + H^{(2)}_1^2 + \frac{\alpha^2}{4} A^{(2)}_1^2 - 4\omega \beta_2 \) |
| H(3) | \( H^{(3)}_1 = \frac{L^2}{m} + r \left( \frac{\alpha^+}{r+q_1} + \frac{\beta^+}{r-q_1} \right) \) | \{H^{(3)}_1, B^{(3)}_{11}\} = -\frac{1}{m}(4H^{(3)}_1 A^{(3)} - \kappa \gamma^+_3) |
|      | \( A^{(3)}_1 = \frac{L}{m} p_2 - \frac{1}{2r} \left( \alpha^+_3 \frac{r-q_1}{r+q_1} + \beta^+_3 \frac{r+q_1}{r-q_1} + \kappa q_1 \right) \) | \{A^{(3)}_1, B^{(3)}_{11}\} = \frac{2}{m}[A^{(3)}_1^2 - 2H^{(3)}_1(2H^{(3)}_1 - \gamma^+_3 \gamma^-_3)] \) |
| H(4) | \( H^{(4)}_1 = \frac{1}{2r}[\sigma q_2 - \alpha^+_4 (r - q_1) \sqrt{r + q_1}] + \frac{L}{m} p_2 + \frac{\beta^+_4}{2r}(r + q_1) \sqrt{r - q_1} \) | \{H^{(4)}_1, B^{(4)}_{11}\} = \frac{2}{m}(H^{(4)}_1 A^{(4)}_1 - \frac{1}{4} \alpha \beta_4) |
|      | \( A^{(4)}_1 = -\frac{2}{2r} (\alpha^+_4 \sqrt{r - q_1} - \beta_4 \sqrt{r + q_1}) + \frac{L}{m} p_1 - \frac{\sigma q_2}{2r} \) | \{A^{(4)}_1, B^{(4)}_{11}\} = -\frac{2}{m}(H^{(4)}_1 H^{(4)}_1 - \frac{1}{8} \gamma^+_4 \gamma^-_4) |