Prewavelet Solution to Poisson Equations

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Abstract

Finite element method is one of powerful numerical methods to solve PDE. Usually, if a finite element solution to a Poisson equation based on a triangulation of the underlying domain is not accurate enough, one will discard the solution and then refine the triangulation uniformly and compute a new finite element solution over the refined triangulation. It is wasteful to discard the original finite element solution. We propose a prewavelet method to save the original solution by adding a prewavelet subsolution to obtain the refined level finite element solution. To increase the accuracy of numerical solution to Poisson equations, we can keep adding prewavelet subsolutions.

Our prewavelets are orthogonal in the $H^1$ norm and they are compactly supported except for one globally supported basis function in a rectangular domain. We have implemented these prewavelet basis functions in MATLAB and used them for numerical solution of Poisson equation with Dirichlet boundary conditions. Numerical simulation demonstrates that our prewavelet solution is much more efficient than the standard finite element method.

1 Introduction

Finite element method is one of powerful numerical methods to solve PDE. Usually, if a finite element solution to a Poisson equation based on one level triangulation of the underlying domain is not accurate enough, one will discard the solution and then refine the triangulation and compute a new finite element solution at the refined level. It is wasteful to throw the original finite element solution away. In order to save the original solution and get the more
accurate new solution, we have to add $H^1$ orthogonal subsolution. That is, let $V_h$ be a finite element space over a triangulation $\Delta_h$ and $V_{h/2}$ be the finite element space over the refined triangulation. Since $V_h \subset V_{h/2}$, let $W_h = V_{h/2} \ominus V_h$ under $H^1$ norm, if $\Phi_h \in V_h$ is a finite element solution of Poisson equation with Dirichlet boundary condition, we can find $\Psi_h \in W_h$ so that $\Phi_h + \Psi_h$ is the finite element solution in $V_{h/2}$. In addition, suppose that $\phi_h$ is the most accurate solution that a computer can compute in the sense that it would be out of memory when computing a finite element solution $\Phi_{h/2}$ in $V_{h/2}$ directly. Since the size of the linear system associated with $\Psi_h$ is smaller than $\Phi_{h/2}$, if the computer can solve $\Psi_h$, we can add $\Psi_h$ to $\Phi_h$ to get $\Phi_{h/2}$ achieving the next level of accuracy. In this paper, we discuss how to compute $\Psi_h$. We shall construct compactly supported basis functions and a global supported basis function $\psi_{h,k}, k = 1, \ldots, N_h$ which span $W_h$. $\psi_{h,k}$’s are called prewavelets and $\Psi_h$ is a linear combination of these $\psi_{h,k}$’s and hence is called a prewavelet subsolution.

Prewavelets have been studied for more than 10 years (cf. [9], [5]). There are many methods available to construct compactly supported prewavelets over 2D domains under the $L_2$ norm. That is $W_h = V_{h/2} \ominus V_h$ under $L_2$ norm, e.g., in a series of papers [6], [7], [8], [11], and [4]. In 1997, Bastin and Laubin (2) explained how to construct compactly supported orthonormal wavelets in Sobolev space in the univariate setting. See also [1] for biorthogonal wavelets in Sobolev space. In [14], Lorentz and Oswald showed that there is no compactly supported prewavelets in Sobolev space or under $H^1$ norm based on integer translations of a box spline over $\mathbb{R}^2$. Since continuous piecewise linear finite element can be expressed by using box spline $B_{111}$, the result in [14] ruins a hope to find compactly supported prewavelets under $H^1$ norm. But this is not an end of story. It is possible to construct compactly supported prewavelets in a semi-norm in the univariate setting in [10]. It is also possible to construct compactly supported prewavelets in $H^r$ norm over each nested subspace, but the union of these prewavelets over all levels fails to be a stable basis for a Sobolev space (cf. [12]). Our new question is if we can find a prewavelet basis with as few as possible global supported prewavelet functions. Our answer is affirmative. That is, there is a prewavelet basis for $W_h$ with only one global supported basis function under the $H^1$ norm over rectangular domains. Also it is possible to find a compactly supported prewavelet basis for $W_h$ under the $H^1$ norm for Poisson equation over a triangular domain (cf. [13]).

The paper is organized as follows: We first explain that the Dirichlet boundary value problem of Poisson equation can be converted into a Poisson equation with zero boundary condition. An explicit conversion will be given. Thus the $H^1$ norm is now equivalent to the $H^1_0$ semi-norm. Then we introduce some notation to explain the weak solution of Poisson equation and its approximation to the exact solution. These explanations are well-known and given in the Preliminary section §2. In §3, we explain how to construct compactly supported prewavelets under $H^1_0$ semi-norm. In §4, we explain how to implement our prewavelet method for numerical solution of Poisson equation. Finally in §5 we present some numerical results. Our numerical experiment show that the time for computing a finite element solution by our prewavelet method is about half of the time by the standard finite element method using the direct method for inverting the linear systems. If using the conjugate gradient method for the linear systems for the finite element method, the prewavelet method is still faster than
for sufficiently accurate iterative solutions.

2 Preliminary

Let us start with a square domain \( \Omega = (0, 1) \times (0, 1) \in \mathbb{R}^2 \). Consider the Dirichlet boundary value problem for Poisson equation:

\[
\begin{align*}
-\Delta u(x, y) &= g(x, y), \quad (x, y) \in \Omega \\
u(x, y) &= f_1(x), \quad \text{for } y = 0 \text{ and } 0 \leq x \leq 1 \\
u(x, y) &= f_2(x), \quad \text{for } y = 1 \text{ and } 0 \leq x \leq 1 \\
u(x, y) &= f_3(y), \quad \text{for } x = 0 \text{ and } 0 \leq y \leq 1 \\
u(x, y) &= f_4(y), \quad \text{for } x = 1 \text{ and } 0 \leq y \leq 1 \\
\end{align*}
\]

Without lose of generality, we may assume that \( f_1(1) = f_2(1) = f_3(1) = f_4(1) = f_1(0) = f_2(0) = f_3(0) = f_4(0) = 0 \). Otherwise, letting \( f_1(0) = f_3(0) = a_1, f_3(1) = f_2(0) = a_2, f_2(1) = f_4(1) = a_3, f_4(0) = f_1(1) = a_4 \), we define \( h(x, y) = a_1 + (a_4 - a_1)x + (a_2 - a_1)y + (a_3 + a_1 - a_2 - a_4)xy \), and \( v(x, y) = u(x, y) - h(x, y) \). Then the above Dirichlet problem becomes to:

\[
\begin{align*}
-\Delta v(x, y) &= g(x, y), \quad (x, y) \in \Omega \\
v(x, y) &= f_1(x) - h(x, 0), \quad \text{for } y = 0 \text{ and } 0 \leq x \leq 1 \\
v(x, y) &= f_3(y) - h(0, y), \quad \text{for } x = 0 \text{ and } 0 \leq y \leq 1 \\
v(x, y) &= f_4(y) - h(1, y), \quad \text{for } x = 1 \text{ and } 0 \leq y \leq 1 \\
\end{align*}
\]

which satisfy the above assumption.

Now let \( w(x) = v(x, y) - x(f_4(y) - h(1, y)) - (1 - x)(f_3(y) - h(0, y)) - y(f_2(x) - h(x, 1)) - (1 - y)(f_1(x) - h(x, 0)) \). Then \( w(x) \) satisfies the equation

\[
\begin{align*}
-\Delta w(x, y) &= g_1(x, y), \quad (x, y) \in \Omega \\
w(x, y) &= 0, \quad (x, y) \in \partial \Omega \\
\end{align*}
\]

with \( g_1(x, y) = g(x, y) + \frac{\partial}{\partial y}[-x(f_4(y) - h(1, y)) - (1 - x)(f_3(y) - h(0, y))] + \frac{\partial^2}{\partial x^2}[-y(f_2(x) - h(x, 1)) - (1 - y)(f_1(x) - h(x, 0))] \).

If we can find solution for \( w \), it is easy to get \( u(x, y) \). In the remaining paper, we only consider the Poisson equation with zero boundary condition:

\[
\begin{align*}
-\Delta u(x, y) &= g(x, y), \quad (x, y) \in \Omega \\
u(x, y) &= 0, \quad (x, y) \in \partial \Omega. \\
\end{align*}
\] 

Next we define

\[
H^1_0(\Omega) = \{v \in L^2(\Omega) : \langle v, v \rangle < \infty \text{ and } v(x, y) = 0, (x, y) \in \partial \Omega\},
\]
where the inner product $\langle u, v \rangle_s$ is defined by

$$\langle u, v \rangle_s = \int_0^1 \int_0^1 \frac{\partial u(x,y)}{\partial x} \frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} \frac{\partial v(x,y)}{\partial y} \, dxdy.$$ 

By using Poincare’s inequality, $\|u\|_s = \sqrt{\langle u, u \rangle_s}$ is a standard Sobolev norm for $H^1_0(\Omega)$. Suppose $u, v \in H^1_0(\Omega)$. Integration by parts yields

$$\langle g, v \rangle = \int_0^1 \int_0^1 g(x,y)v(x,y) \, dxdy$$

$$= \int_0^1 \int_0^1 -\Delta u(x,y)v(x,y) \, dxdy$$

$$= \int_0^1 \int_0^1 \frac{\partial u(x,y)}{\partial x} \frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} \frac{\partial v(x,y)}{\partial y} \, dxdy$$

$$= \langle u, v \rangle_s.$$ 

Thus, a weak solution $u$ to (1) is characterized by finding $u \in H^1_0(\Omega)$ such that

$$\langle u, v \rangle_s = \langle g, v \rangle, \quad \forall v \in H^1_0(\Omega). \quad (2)$$

The following result is well-known. For convenience, we present a short proof.

**Theorem 2.1.** Suppose $g \in C(\Omega)$ and $u \in C^2(\Omega)$ satisfy (2). Then $u$ is weak solution of (1).

**Proof.** Let $v \in H^1_0(\Omega)$. Then integration by parts gives

$$\langle g, v \rangle = \langle u, v \rangle_s$$

$$= \int_0^1 \int_0^1 \frac{\partial u(x,y)}{\partial x} \frac{\partial v(x,y)}{\partial x} + \frac{\partial u(x,y)}{\partial y} \frac{\partial v(x,y)}{\partial y} \, dxdy$$

$$= \int_0^1 \int_0^1 -\Delta u(x,y)v(x,y) \, dxdy$$

$$= \langle -\Delta u(x,y), v \rangle.$$ 

It follows that $\langle g - (-\Delta u(x,y)), v \rangle = 0$ for all $v \in H^1_0(\Omega)$. That is, $g \equiv -\Delta u$ and hence, $u$ satisfies (1). \qed

Next we introduce continuous linear spline space on $\Omega = [0,1] \times [0,1]$. For convenience, let $N_j = (2^j - 1)^2$ and $j \geq 1$. Denote $x_{ji} = \frac{i}{2^j}$ and $y_{jk} = \frac{j}{2^j}$ for $i = 1, \ldots, 2^j - 1$. Clearly, the lines segment of $x = x_{ji}$ and $y = y_{jk}$ divide the square $\Omega$ into $N_j$ sub-squares. The diagonal going from down-left to up-right of each sub-square divides the sub-square into two congruent triangle. We will refer to the set of all such triangles as a Type-1 triangulation of $\Omega$ (see Figure 1).
Define $\phi_{ik}^j$ to be linear spline with support on the hexagon with following vertices
\[(x_{j(i-1)},y_{j(k-1)}), (x_{ji},y_{j(k-1)}), (x_{j(i+1)},y_{j(k)}), (x_{j(i+1)},y_{j(k+1)}), (x_{j(i+1)},y_{j(k)}), (x_{j(i-1)},y_{j(k)}))\]
and $\phi_{ik}(x_{ji},y_{jk}) = \delta_{i,i'} \delta_{k,k'}$, where $\delta_{i,i'} = 0$ if $i' \neq i$ and 1 if $i' = i$.

Let $V_j = \text{span}\{\phi_{ik}^j, i = 1, \ldots, 2^j - 1, k = 1, \ldots, 2^j - 1\}$ be the subspace of $H_0^1(\Omega)$. By following lemma, there exists a unique $u_j \in V_j$ satisfying
\[
\langle u_j, v \rangle_s = \langle f, v \rangle \quad \forall v \in V_j.
\]
(3)

$u_j$ is the standard finite element solution in $V_j$. The following result is well-known. For completeness, we include a short proof.

**Lemma 2.1.** Given $g \in L^2(\Omega)$, (3) has a unique solution.

**Proof.** Reorder the basis functions $\phi_{ik}^j$ to $\phi_m$, $m = 1, \ldots, N_j$ and let $u_j = \sum a_m \phi_m$. Denote $k_{mn} = \langle \phi_m, \phi_n \rangle_s$ and $F_m = \langle f, \phi_m \rangle$ for $m = 1, \ldots, N_j$. Set $A = (a_m)$ to be the coefficient vector, $K = [k_{mn}]_{1 \leq m,n \leq N_j}$ to be the stiff matrix, and $F = (F_m)$ to be the right hand side vector. Then the solutions in (3) is written in the following matrix equation form
\[
KA = F.
\]
(4)

We claim that the solution for above equation always exists and is unique. Otherwise there is a nonzero vector $c$ such that $Kc = 0$. Write $c = (c_m, m = 1, \ldots, N_j)$ and let $v = \sum_{i=1}^{N_j} c_i \phi_i$ be the linear spline. Then $Kc = 0$ is equivalent to
\[
\langle v, \phi_m \rangle_s = 0 \quad \forall m = 1, \cdots, N_j.
\]

Multiplying $\langle v, \phi_m \rangle_s$ by $c_m$ and summing over $m$ yields $\langle v, v \rangle_s = 0$. Thus, $v = a + bx + cdxy$. Boundary condition implies $v \equiv 0$. Since $\{\phi_m\}$ are linear independent, $c \equiv 0$ and hence, the solution is unique. 

\[\square\]
Let us discuss the error between $u$ and $u_j$. It is standard in finite element analysis (cf. [3]). For completeness we present a simple derivation. Subtracting (3) from (2) implies

$$\langle u - u_j, w \rangle_s = 0 \quad \forall w \in V_j. \quad (5)$$

Then for any $v \in V_j$

$$\|u - u_j\|_s^2 = \langle u - u_j, u - u_j \rangle_s$$
$$= \langle u - u_j, u - v \rangle_s + \langle u - u_j, v - u_j \rangle_s$$
$$= \langle u - u_j, u - v \rangle_s$$
$$\leq \|u - u_j\|_s \|u - v\|_s$$

It follows that $\|u - u_j\|_s \leq \|u - v\|_s$ for any $v \in V_j$. Thus we have proved the following.

**Lemma 2.2. (Céa’s Lemma)** $\|u - u_j\|_s = \min \{\|u - v\|_s : v \in V_j\}$.

Given $u \in C^0(\Omega)$, let $u_j \in V_j$ be the interpolant of $v$:

$$u_j = \sum_{ik} u(x_{ji}, y_{jk})\phi_{ik}^{(j)}.$$

The following error estimate is well-known.

**Lemma 2.3.** Suppose $u \in C^2(\Omega)$. Then

$$\|u - u_j\|_s \leq \frac{\sqrt{12}}{2^j} \sqrt{\left|\frac{\partial^2 u}{\partial x^2}\right|_{L^\infty}^2 + \left|\frac{\partial u}{\partial x} \frac{\partial u}{\partial y}\right|_{L^\infty}^2 + \left|\frac{\partial^2 u}{\partial y^2}\right|_{L^\infty}^2}.$$

**Proof.** The proof is elementary and is left to the reader. See [13] for detail. \qed

## 3 Multiresolution and Prewavelets over Type-I triangulations

We start with the definition of multi-resolution approximation of $H^1_0(\Omega)$:

**Definition 3.1.** A multiresolution approximation of $H^1_0(\Omega)$ is a sequence of finite dimensions subspaces $V_j, j \in \mathbb{Z}^+$ of $H^1_0(\Omega)$ such that

1. $V_j \subset V_{j+1}, \quad j \in \mathbb{Z}^+$;
2. $\bigcup_{j=1}^\infty V_j$ is dense in $H^1_0(\Omega)$. 


Let $\Gamma_j$ be the type-1 triangulation with $2N_j$ triangles. Naturally, let $\Gamma_j^{j+1}$ be the uniform refinement of $\Gamma_j$. Let $V_j$ be the continuous piecewise linear spline space defined on the previous section. That is, $V_j = \text{span}\{\phi^j_{ik}, i = 1, \ldots, 2^j - 1, k = 1, \ldots, 2^j - 1\}$, where $\phi^j_{ik}$ are continuous piecewise linear functions which is 1 at $(x_{ji}, y_{jk})$ and zero at all other vertices.

Let $V_j^{j+1} = \text{span}\{\phi^{j+1}_{ik}, i = 1, \ldots, 2^{j+1} - 1, k = 1, \ldots, 2^{j+1} - 1\}$, and $(x_{j+1,i}, y_{j+1,k})$ are the vertices on the $j+1$ level Type-1 triangulation. Then the refinement equation is easily seen to be

$$
\phi^j_{ik} = \phi^{j+1}_{2i,2k} + \frac{1}{2} \phi^{j+1}_{2i-1,2k} + \frac{1}{2} \phi^{j+1}_{2i,2k-1} + \frac{1}{2} \phi^{j+1}_{2i+1,2k} + \frac{1}{2} \phi^{j+1}_{2i+1,2k-1} + \frac{1}{2} \phi^{j+1}_{2i+1,2k+1} + \frac{1}{2} \phi^{j+1}_{2i,2k+1}.
$$

See the Figure 2.

![Figure 2. Dilation relations](image)

The main purpose of this paper is to build a basis for the orthogonal complement $W_j$ of $V_j$ in $V_j^{j+1}$ under the inner product $\langle \cdot, \cdot \rangle_s$. Suppose we have the $W_j$. Then $V_j^{j+1} = V_j + W_j$ under the $H^0_1(\Omega)$ inner product. For a solution $u_j$ satisfying (3), we do not have to find out the solution for $u_{j+1} \in V_{j+1}$ such that $\langle u_{j+1}, v \rangle_s = \langle g, v \rangle \forall v \in V_{j+1}$.

Instead, we only need to find solutions for

$$
w_j \in W_j \text{ such that } \langle w_j, v \rangle_s = \langle g, v \rangle \forall v \in W_j.
$$

Then we have $w_j + u_j = u_{j+1}$. Ideally, we hope the supports of basis functions for $W_j$ are small, since small support can accelerate the calculations of $\langle g, v \rangle_s$. As explained in the Introduction, there is no compactly supported prewavelets for $W_j$. Nevertheless, we shall construct basis functions with only one globally supported basis function for $W_j$ in the following.

Clearly the $\Gamma_j$ can be continuously refined and hence we will have a nested sequence of subspaces

$$
V_1 \subset V_2 \subset V_3 \subset V_4 \subset V_5, \ldots
$$

to span $H^1_0(\Omega)$ by Lemma 2.3 since $C^2(\Omega)$ is dense in $H^1_0(\Omega)$. 

7
Let $W_j \subset V_{j+1}$ be the orthogonal complement of $V_j$ in $V_{j+1}$ for each refinement level $j$, i.e.,

$$V_{j+1} = V_j \bigoplus W_j.$$  

Then we get the decomposition

$$V_{j+1} = V_1 \bigoplus W_1 \bigoplus W_2 \bigoplus W_3 \bigoplus \ldots \bigoplus W_j$$

for any $j \geq 1$. The weak solution $u_{j+1}$ to the Poisson equation \((\text{I})\) at $V_{j+1}$ can be built by

$$u_{j+1} = u_1 + w_1 + w_2 + \cdots + w_j.$$

We now focus on building basis functions for the orthogonal complement $W_j$. By direct calculation, we obtain the following lemma immediately.

**Lemma 3.1.** We have $\langle \phi_{i,k}^j, \phi_{2i,2k}^{j+1} \rangle_s = 2,$

$$\langle \phi_{i,k}^j, \phi_{2i-1,2k}^{j+1} \rangle_s = 1/2, \quad \langle \phi_{i,k}^j, \phi_{2i-2,2k}^{j+1} \rangle_s = -1/2, \quad \langle \phi_{i,k}^j, \phi_{2i-3,2k}^{j+1} \rangle_s = -1/2,$$

$$\langle \phi_{i,k}^j, \phi_{2i-4,2k}^{j+1} \rangle_s = 0,$$

where $i, k \leq N_j$ and $i, k \in \mathbb{Z}$. The $N_j$ equations with $N_{j+1}$ coefficients, $b_{i,k}$. There are at least $N_{j+1} - N_j$ degrees of freedom. The solution space of these equation system should be the $W_j$. The linear independence of $\phi_{i',k'}^{j+1}$ implies that the coefficient matrix of the above linear system is of full rank. Hence, there are $N_{j+1} - N_j$ linear independent solutions which constitute a basis for $W_j$.

**Definition 3.2.** Let $V_{j+1}^m = \text{span}\{\phi_{i,k}^{j+1}, i = 1, \ldots, 2m - 1, k = 1, \ldots, 2m - 1\}$ be a subspace of $V_{j+1}$. Let $W_j^m$ be subspace of $W_j$ such that $W_j^m = W_j \cap V_{j+1}^m$.

Obviously $0 \subset V_{j+1}^1 \subset V_{j+1}^2 \subset \ldots \subset V_{j+1}^{2^{j+1}} = V_{j+1}$, and $0 \subset W_j^1 \subset W_j^2 \subset \cdots \subset W_j^{2^j} = W_j$. There is no nonzero solution of \((\text{III})\) in space of $V_{j+1}^1$. However, there are $\phi_{i,k}^{j+1}$ solutions of \((\text{IV})\) in space $V_{j+1}^2$. They are solutions of the following system of linear equations.
Using Lemma 3.1, we obtain the following equations.

\[
\sum_{1 \leq i, k \leq 3} b_{ik} \langle \phi_{ik}^{j+1}, \phi_{i1}^j \rangle_s = 0, \quad \sum_{1 \leq i, k \leq 3} b_{ik} \langle \phi_{ik}^{j+1}, \phi_{21}^j \rangle_s = 0,
\]

They are equivalent to the following equations.

\[
\begin{pmatrix}
\langle \phi_{11}^j, \phi_{11}^{j+1} \rangle_s & \langle \phi_{11}^j, \phi_{21}^{j+1} \rangle_s & \cdots & \langle \phi_{11}^j, \phi_{33}^{j+1} \rangle_s \\
\langle \phi_{11}^j, \phi_{11}^{j+1} \rangle_s & \langle \phi_{21}^j, \phi_{21}^{j+1} \rangle_s & \cdots & \langle \phi_{21}^j, \phi_{33}^{j+1} \rangle_s \\
\langle \phi_{12}^j, \phi_{11}^{j+1} \rangle_s & \langle \phi_{12}^j, \phi_{21}^{j+1} \rangle_s & \cdots & \langle \phi_{12}^j, \phi_{33}^{j+1} \rangle_s \\
\langle \phi_{12}^j, \phi_{11}^{j+1} \rangle_s & \langle \phi_{22}^j, \phi_{21}^{j+1} \rangle_s & \cdots & \langle \phi_{22}^j, \phi_{33}^{j+1} \rangle_s
\end{pmatrix}
\begin{pmatrix}
b_{11} \\
b_{21} \\
b_{31} \\
b_{12} \\
b_{22} \\
b_{32} \\
b_{13} \\
b_{23} \\
b_{33}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

Using Lemma 3.1, we obtain the following equations.

\[
\begin{pmatrix}
1 & 1/2 & -1 & 1/2 & 2 & 1/2 & -1 & 1/2 & 1 \\
0 & -1/2 & 1 & 0 & -1/2 & 1/2 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1/2 & 0 & -1/2 & 1 \\
0 & 0 & 0 & 1 & -1/2 & -1/2 & 0 & 1 & 1/2 & -1
\end{pmatrix}
\begin{pmatrix}
b_{11} \\
b_{21} \\
b_{31} \\
b_{12} \\
b_{22} \\
b_{32} \\
b_{13} \\
b_{23} \\
b_{33}
\end{pmatrix} = 
\begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}.
\]

The rank of the left matrix is four, because \(\phi_{11}^j, \phi_{21}^j, \phi_{12}^j, \phi_{22}^j\), are linear independent. So there are five solutions as shown below.

\[
\begin{pmatrix}
b_{11} \\
b_{21} \\
b_{31} \\
b_{12} \\
b_{22} \\
b_{32} \\
b_{13} \\
b_{23} \\
b_{33}
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix} \text{ or } 
\begin{pmatrix}
0 \\
0 \\
0 \\
-1 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix} \text{ or } 
\begin{pmatrix}
0 \\
0 \\
0 \\
-1 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix} \text{ or } 
\begin{pmatrix}
0 \\
0 \\
0 \\
1 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix} \text{ or } 
\begin{pmatrix}
0 \\
0 \\
0 \\
-1 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix}.
\]
More precisely,

\[
\psi_{0,1}^j = 2\phi_{1,1}^{j+1} + \phi_{1,3}^{j+1}
\]

as shown in Figure 3; \hspace{1cm} (7)

\[
\psi_{1,0}^j = 2\phi_{2,1}^{j+1} + \phi_{3,1}^{j+1}
\]

as shown in Figure 4; \hspace{1cm} (8)

\[
\psi_{1,1}^j = -\phi_{2,2}^{j+1} + \phi_{3,2}^{j+1} + \phi_{2,3}^{j+1} + \phi_{3,3}^{j+1}
\]

as shown in Figure 5; \hspace{1cm} (9)

\[
\psi_{1,1}^j = \phi_{1,1}^{j+1} + \phi_{2,1}^{j+1} + \phi_{1,2}^{j+1} - \phi_{2,2}^{j+1}
\]

as shown in Figure 6; \hspace{1cm} (10)

\[
\psi_{1,1}^j = \phi_{1,2}^{j+1} + \phi_{2,3}^{j+1} - \phi_{2,1}^{j+1} - \phi_{3,2}^{j+1}
\]

as shown in Figure 7. \hspace{1cm} (11)
Now we consider $V^3$. Similarly, there are 25 non-zero coefficient for linear system (6) and the coefficient matrix of rank 9. So the dimension of solution space of $W^3_j$ is $25 - 9 = 16$. The first five of them are the same to the wavelet functions in (7)–(11). The other 11 are given below.

\[
\begin{align*}
\psi_{1,2}^1 &= 2\phi_{1,4} + \phi_{1,5}^1, \\
\psi_{1,2}^1 &= 2\phi_{1,4}^1 + \phi_{5,1}^1, \\
\psi_{1,2}^3 &= \phi_{5,3}^1 + \phi_{3,4} + \phi_{4,3}^1 + \phi_{4,5} - \phi_{4,4}^1, \\
\psi_{1,2}^2 &= \phi_{3,5}^1 + \phi_{5,4} + \phi_{4,5} - \phi_{4,4}^1, \\
\psi_{2,1}^3 &= \phi_{5,3}^1 + \phi_{5,2} + \phi_{4,3}^1 + \phi_{4,2}^1, \\
\psi_{2,1}^2 &= \phi_{3,2}^1 + \phi_{4,1} + \phi_{3,1} - \phi_{4,2}^1, \\
\psi_{2,1}^4 &= \phi_{3,3}^1 + \phi_{4,3} + \phi_{3,4} - \phi_{4,4}^1, \\
\psi_{2,1}^3 &= \phi_{1,3}^1 + \phi_{2,3} + \phi_{1,4}^1 - \phi_{2,4}^1, \\
\psi_{2,1}^5 &= \phi_{1,4}^1 + \phi_{2,5}^1 - \phi_{2,3} - \phi_{3,4}^1, \\
\psi_{2,2}^3 &= \phi_{3,4}^1 + \phi_{4,5} - \phi_{4,3} - \phi_{5,4}^1, \\
\psi_{2,2}^5 &= \phi_{3,2}^1 + \phi_{4,3} - \phi_{4,1} - \phi_{5,2}^1, \\
\end{align*}
\]

as shown in Figure 8; as shown in Figure 9; as shown in Figure 10; as shown in Figure 11; as shown in Figure 12; as shown in Figure 13; as shown in Figure 14; as shown in Figure 15; as shown in Figure 16; as shown in Figure 17; as shown in Figure 18.
The above computation can be carried out on $V^n_j$ for $n = 3,..., 2^j - 1$. We have thus obtained five types of wavelet functions:

$$\psi^{j,1}_{0,k} = 2\phi^{j+1}_{1,k+1} + \phi^{j+1}_{1,k+2}$$

is supported next to the vertical boundary and is called vertical boundary wavelet.

$$\psi^{j,2}_{k,0} = 2\phi^{j+1}_{k+1,1} + \phi^{j+1}_{k+2,1}$$

called horizontal boundary wavelet, is supported next to the horizontal boundary. The next three types are supported inside the domain. The following

$$\psi^{j,3}_{i,k} = -\phi^{j+1}_{i+1,k+1} + \phi^{j+1}_{i+2,k+1} + \phi^{j+1}_{i+1,k+2} + \phi^{j+1}_{i+2,k+2}$$

is called interior wavelet of first kind. We call

$$\psi^{j,4}_{i,k} = -\phi^{j+1}_{2i,2k} + \phi^{j+1}_{2i-2,k} + \phi^{j+1}_{2i,2k-1} + \phi^{j+1}_{2i-2,k-1}$$

interior wavelet of second kind. The last one

$$\psi^{j,5}_{i,k} = \phi^{j+1}_{2i-1,2k} + \phi^{j+1}_{2i,2k+1} - \phi^{j+1}_{2i,2k-1} - \phi^{j+1}_{2i+1,2k}$$

is called interior wavelet of third kind.
Theorem 3.1. All the five types of wavelets in the $V_{nj}^{n+1}$ are linear independent for $1 \leq n \leq 2^j - 1$. That is, for each $1 \leq n \leq 2^j - 1$, the following functions

$$\psi_{j,1}, \quad k = 1, \ldots, n - 1,$$

$$\psi_{j,2}, \quad k = 1, \ldots, n - 1,$$

$$\psi_{j,3}, \quad 1 \leq i, k \leq n - 1,$$

$$\psi_{j,4}, \quad 1 \leq i, k \leq n - 1,$$

$$\psi_{j,5}, \quad 1 \leq i, k \leq n - 1$$

are linear independent.

Proof. Let us prove it by induction. It is true for $n = 2$ and for $n = 3$. Suppose it is true for $n = p$, that is,

$$\psi_{j,1}, \quad k = 1, \ldots, p - 1;$$

$$\psi_{j,2}, \quad k = 1, \ldots, p - 1;$$

$$\psi_{j,3}, \quad 1 \leq i, k \leq p - 1;$$

$$\psi_{j,4}, \quad 1 \leq i, k \leq p - 1;$$

$$\psi_{j,5}, \quad 1 \leq i, k \leq p - 1;$$

are linear independent. For $n = p + 1$, there are $6p - 1$ new functions which are

$$\psi_{j,1}, \quad k = p;$$

$$\psi_{j,2}, \quad k = p;$$

$$\psi_{j,3}, \quad i \text{ or } k = p;$$

$$\psi_{j,4}, \quad i \text{ or } k = p;$$

$$\psi_{j,5}, \quad i \text{ or } k = p.$$

Suppose they are not linear independent. That is, one can find

$$a_1^1,$$

$$a_2^2,$$

$$a_3^3, \quad i \text{ or } k = p;$$

$$a_4^4, \quad i \text{ or } k = p;$$

$$a_5^5, \quad i \text{ or } k = p$$

such that

$$a^1\psi_{0,p}^j + a^2\psi_{p,0}^j + \sum_{i \text{ or } k = p} a^i_{i,k} \psi_{i,k}^j + \sum_{i \text{ or } k = p} a^i_{i,k} \psi_{i,k}^j + \sum_{i \text{ or } k = p} a^i_{i,k} \psi_{i,k}^j + \psi' = 0, \quad (12)$$

where $\psi'$ is linear combination of the following functions:

$$\psi_{j,1}, \quad k = 1, \ldots, p - 1;$$

$$\psi_{j,2}, \quad k = 1, \ldots, p - 1;$$

$$\psi_{j,3}, \quad 1 \leq i, k \leq p - 1;$$

$$\psi_{j,4}, \quad 1 \leq i, k \leq p - 1;$$

$$\psi_{j,5}, \quad 1 \leq i, k \leq p - 1.$$
By the definition, $\phi_{2i+1,2k+1}^{j+1}, i = p$ or $k = p$ appear only once in $\psi_{i,k}^{j,3}, i = p$ or $k = p$, $\psi_{i,k}^{j,1}$ and $\psi_{p,0}^{j,2}$. Since $\phi^{j+1}$ are linear independent, that is, $a_{i,k}^{3} = 0, i \text{ or } k = p$, $a_{i,k}^{1} = 0$, and $a_{i,k}^{2} = 0$. Thus the equation (12) can be simplified to

$$\sum_{i \text{ or } k = p} a_{i,k}^{4} \psi_{i,k}^{j,4} + \sum_{i \text{ or } k = p} a_{i,k}^{5} \psi_{i,k}^{j,5} + \psi' = 0. \quad (13)$$

By the similar reason, $\phi_{2i,2k+1}^{j+1}, i = p$ or $k = p$ appear only once in $\psi_{i,k}^{j,4}, i = p$ or $k = p$. Since $\phi_{i,k}^{j+1}$ are linear independent, $a_{i,k}^{4} = 0, i \text{ or } k = p$. Thus the equation (13) can be further simplified to the following equation

$$\sum_{i \text{ or } k = p} a_{i,k}^{5} \psi_{i,k}^{j,5} + \psi' = 0.$$ 

Similarly, $a_{i,k}^{5} = 0, i \text{ or } k = p$ too. Thus the equation (12) is reduced to

$$\psi' = 0.$$

By induction hypothesis, all the coefficient of $\psi' = 0$ are zeros. Hence,

$$\psi_{0,k}^{j,1}, k = 1, \ldots, n - 1,$$
$$\psi_{k,0}^{j,2}, k = 1, \ldots, n - 1,$$
$$\psi_{i,k}^{j,3}, 1 \leq i, k \leq n - 1,$$
$$\psi_{i,k}^{j,4}, 1 \leq i, k \leq n - 1,$$
$$\psi_{i,k}^{j,5}, 1 \leq i, k \leq n - 1$$

are linear independent.

**Theorem 3.2.** All the five types of wavelets in the $W_{j}^{n}$ form a basis of $W_{j}^{n}$ for $1 \leq n \leq 2j - 1$. That is,

$$W_{j}^{n} = \text{span}\{\psi_{0,k}^{j,1}, \psi_{k,0}^{j,2}, \psi_{i,k}^{j,3}, \psi_{i,k}^{j,4}, \psi_{i,k}^{j,5}, 1 \leq i, k \leq n - 1\}$$

for $1 \leq n \leq 2j - 1$.

**Proof.** The dimension of $W_{j}^{n}$ is $(2n - 1)^{2} - (n)^{2} = 3n^{2} - 4n + 1$. It is easy to count that there are $(2n - 1)^{2} - (n)^{2} = 3n^{2} - 4n + 1$ functions in the following set

$$\psi_{0,k}^{j,1}, k = 1, \ldots, n;$$
$$\psi_{k,0}^{j,2}, k = 1, \ldots, n;$$
$$\psi_{i,k}^{j,3}, 1 \leq i, k \leq n;$$
$$\psi_{i,k}^{j,4}, 1 \leq i, k \leq n;$$
$$\psi_{i,k}^{j,5}, 1 \leq i, k \leq n$$

which all belong to the space $W_{j}^{n}$. Since they are linear independent, they form a basis for space $W_{j}^{n}$, where $1 \leq n \leq 2j - 1$. 

\[\square\]
Finally we need to find wavelets in $W^2_j \cap W^{2j-1}_j$. The computations are the same to the above except for that there is one globally supported basis function. In fact the following pictures show the basis functions located on the top boundary of the domain $\Omega$. (We omit the pictures for the basis functions on the right vertical boundary which are symmetric with respect to the line $y=x$ are those basic functions on the top horizontal boundary of $\Omega$.)

![Figure 19.](image1)

![Figure 20.](image2)

![Figure 21.](image3)

![Figure 22.](image4)

![Figure 23.](image5)
The last one (cf. Figure 24) is the only special basis function since it is not local supported. The numbers of all these wavelets in $W_{2j}^{2j} \setminus W_{2j-1}^{2j-1}$ amount to $2^{j+3} - 8$ which is equal to the number of dimension of $V_{j+1}^{2j+1} \setminus V_{j+1}^{2j-1}$.

**Theorem 3.3.** All the wavelets in the $W_{2j}^{2j} \setminus W_{2j-1}^{2j-1}$ are linear independent and form a basis for $V_{j+1}^{2j+1} \setminus V_{j+1}^{2j-1}$ which is spanned by the functions in $\{\phi_{i,k}^{2j+1}, 2^{j+1} - 2 \leq i, k \leq 2^{j+1} - 1\}$.

**Proof.** Let us just concentrate on the basis functions in $V_{j+1}^{2j+1} \setminus V_{j+1}^{2j-1}$ and in $W_{2j}^{2j} \setminus W_{2j-1}^{2j-1}$. Then the scaling matrix between two sets of basis functions is the following matrix up to a constant $A$:

$$A = \begin{pmatrix}
D & B1 & B2 \\
B1 & B2 \\
B1 & B2 & B1 & B2 \\
& & \ddots & \\
C3 & C3 & C3 & \cdots & C3 & C3 & C3 \\
& & & \ddots & \\
B2' & B1' & B2' & B1' & B2' \\
& & & & \ddots & \\
& & & & & B2' & B1'
\end{pmatrix},$$

where

$$D = \begin{pmatrix} 1 & 2 & 0 & 0 \end{pmatrix}, \quad B1 = \begin{pmatrix} 1 & 0 & 2 \\
1 & 0 & -1 \\
1 & 1 & 0 \\
1 & -1 \end{pmatrix}, \quad B2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
1 & 1 & 0 & 0 \end{pmatrix}.$$
\[
D' = \begin{pmatrix} 0 & 0 & 2 & 1 \end{pmatrix}, \quad B_1' = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ -1 & 0 & 1 & 0 \\ 2 & 0 & 1 & 1 \end{pmatrix}, \quad B_2' = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \]
\[
C_1 = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \\ 1 & -1 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}. \]

Let \( E = (m \ 0 \ 0) \). By the row operations we have
\[
\begin{pmatrix} E \\ B_1 & B_2 \\ B_1 & B_2 \end{pmatrix} = \begin{pmatrix} m & n & 0 \\ 1 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}
\[
\rightarrow \begin{pmatrix} m & n & -n & 2m & 2m & -n \\ 2m & -n & n & m & 2m + n & 2n & 0 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.
\]

Similar for \( B' \). Thus by row operations,
\[
A \rightarrow \begin{pmatrix} A_1 & G_1 & G_2 & \cdots \\ A_2 & G_2 & \cdots \\ A_3 & G_3 & \cdots \\ \vdots \end{pmatrix}
\[
\begin{pmatrix} A_{2j-2} & G_{2j-2} & C_{2j-2}' & C_{2j-2}' & A_{2j-2}' \\ \vdots \end{pmatrix},
\]
\[
A_{2j-2}'
\]
\[
G_{2j-2}'
\]
\[
C_{2j-2}'
\]
\[
A_{2j-2}'
\]
\[
G_{2j-2}'
\]
\[
C_{2j-2}'
\]
\[
A_{2j-2}'
\]
\[
G_{2j-2}'
\]
\[
C_{2j-2}'
\]
\[
A_{2j-2}'
\]
where $A_n$ is an upper triangular matrix of size $4 \times 4$ while $A'_n$ is a lower triangular matrix of size $4 \times 4$ which are given below.

\[
A_1 = \begin{pmatrix}
1 & 2 & 0 & 0 \\
-1 & 1 & 0 & 0 \\
1 & -1 & 2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
G_1 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
A_2 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
-1 & 2 & 0 & 0 \\
2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
G_2 = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
A'_n = \begin{pmatrix}
2 & -1 & n \\
0 & n & -1 \\
0 & 0 & 2 \\
0 & 0 & 0
\end{pmatrix},
G'_n = \begin{pmatrix}
0 & 0 & 0 & n \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

and the matrix $(C'_1 \ C'_2)$ is the following matrix

\[
(C'_1 \ C'_2) = \begin{pmatrix}
2^{j+1} - 5 & 2 & 1 & 0 & 2 \\
1 & 0 & -1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & -1 & -1 \\
1 & -1 & 1 & 1 & 1 & 1 \\
2^{j+1} - 5 & 0 & 0 & 0 & 1 & 0 \\
2 & 0 & 1 & 2 & 2^{j+1} - 5
\end{pmatrix}.
\]

It is easy to see the rank of $(C'_1 \ C'_2)$ is 8. Thus the rank of $A$ is $8(2^j) - 8$. Thus, all the prewavelet functions constructed above in the $W_{2^j}^{2j} \setminus W_{2^j-1}^{2j-1}$ are linear independent and hence form a basis of $V_{2^{j+1}}^{2j+1} \setminus V_{2^j}^{2j-1}$.

It is easy to see that the coefficients of the prewavelet functions in $W_{2^j-1}^{2j-1}$ in terms of the basis functions of $V_{2^{j+1}}^{2j+1} \setminus V_{2^j}^{2j-1}$ are all zeros. Thus the prewavelet functions in $W_{2^j-1}^{2j-1}$ together with the prewavelet functions in $V_{2^{j+1}}^{2j+1} \setminus V_{2^j}^{2j-1}$ are linear independent. It follows the main result in this paper.

**Theorem 3.4.** All the prewavelet functions in the $W_{2^j}^{2j} \setminus W_{2^j-1}^{2j-1}$ and the prewavelet functions in $W_{2^j-1}^{2j-1}$ form a basis for $W_j$. 

19
4 The Prewavelet Method for Poisson Equation

Let us use the basis functions of $V_j$ and $W_j$ to solve Poisson equation \([1]\). Mainly we explain how to compute $h_j \in W_j$. Let $g_j \in V_j$ and $g_{j+1} \in V_{j+1}$ be two FEM solutions. We aim to show that $h_j + g_j = g_{j+1}$.

By a reordering the indices $(i,k), 1 \leq i, k \leq 2^j$ in a linear fashion, let $V_j = \text{span}\{\phi^j_1, \ldots, \phi^j_{N_j}\}$. Also, we reorder all five type wavelet functions as well as the globally supported wavelet to denote $W_j = \text{span}\{\psi^j_1, \ldots, \psi^j_{N_{j+1}-N_j}\}$. Let $\Phi^j, \Psi^j$ be following vectors,

$$
\Phi^j = \begin{pmatrix}
\phi^j_1 \\
\phi^j_2 \\
\vdots \\
\phi^j_{N_j} 
\end{pmatrix}, \quad
\Psi^j = \begin{pmatrix}
\psi^j_1 \\
\psi^j_2 \\
\vdots \\
\psi^j_{N_{j+1}-N_j} 
\end{pmatrix}.
$$

Then we have the following equations

$$
\Phi^j = B_j \Phi^{j+1}, \quad \Psi^j = C_j \Phi^{j+1},
$$

where $B_j$ is $N_j \times N_{j+1}$ refinable matrix, and $C_j$ is a wavelet matrix of size $(N_{j+1}-N_j) \times N_{j+1}$. Let $D_j$ and $E_j$ be the following matrices:

$$
D_j = \begin{pmatrix}
\langle \phi^j_1, \phi^j_1 \rangle_s & \langle \phi^j_1, \phi^j_2 \rangle_s & \cdots & \langle \phi^j_1, \phi^j_{N_j} \rangle_s \\
\langle \phi^j_2, \phi^j_1 \rangle_s & \langle \phi^j_2, \phi^j_2 \rangle_s & \cdots & \langle \phi^j_2, \phi^j_{N_j} \rangle_s \\
\vdots & \vdots & \ddots & \vdots \\
\langle \phi^j_{N_j}, \phi^j_1 \rangle_s & \langle \phi^j_{N_j}, \phi^j_2 \rangle_s & \cdots & \langle \phi^j_{N_j}, \phi^j_{N_j} \rangle_s 
\end{pmatrix},
$$

$$
E_j = \begin{pmatrix}
\langle \psi^j_1, \psi^j_1 \rangle_s & \langle \psi^j_1, \psi^j_2 \rangle_s & \cdots & \langle \psi^j_1, \psi^j_{N_{j+1}-N_j} \rangle_s \\
\langle \psi^j_2, \psi^j_1 \rangle_s & \langle \psi^j_2, \psi^j_2 \rangle_s & \cdots & \langle \psi^j_2, \psi^j_{N_{j+1}-N_j} \rangle_s \\
\vdots & \vdots & \ddots & \vdots \\
\langle \psi^j_{N_{j+1}-N_j}, \psi^j_1 \rangle_s & \langle \psi^j_{N_{j+1}-N_j}, \psi^j_2 \rangle_s & \cdots & \langle \psi^j_{N_{j+1}-N_j}, \psi^j_{N_{j+1}-N_j} \rangle_s 
\end{pmatrix}.
$$

It is easy to see that $B_j D_{j+1} C_j^T = 0$ is equivalent to $V_j \perp W_j$. Clearly, we have $D_j = B_j D_{j+1} {B_j}^T$ and $E_j = C_j D_{j+1} {C_j}^T$.

Let $g_j$ be the projection of $g$ in $V_j$, and $h_j$ be the projection of $g$ in $W_j$. Since $V_j \bigoplus W_j = V_{j+1}$, $g + h$ will be equal to $g_{j+1}$. Let us write $g_j = \sum_{j=1}^{N_j} a_i \phi_i^j = (a_1, a_2, \ldots, a_{N_j}) \Phi^j$. Similarly, $h_j = (b_1, b_2, \ldots, b_{N_{j+1}-N_j}) \Psi^j$, and $g_{j+1} = (c_1, c_2, \ldots, c_{N_{j+1}}) \Phi^{j+1}$. By computing the weak solutions $h_j, g_j$, and $g_{j+1}$ in $W_j, V_j$, and $V_{j+1}$, respectively, we have

$$
D_j \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{N_j} \end{pmatrix} = \begin{pmatrix} \langle \phi^j_1, g \rangle \\ \langle \phi^j_2, g \rangle \\ \vdots \\ \langle \phi^j_{N_j}, g \rangle \end{pmatrix}.
$$
The above linear systems provide a computational method to find \( g_j, h_j \).

We now show \( h_j + g_j = g_{j+1} \). That is, \( g_{j+1} \) can be computed by using \( h_j \) and \( g_j \) only. Indeed, we have

\[
g_j = (a_1, a_2, \ldots, a_{N_j}) \Phi^j = (\Phi^j)^T (a_1, a_2, \ldots, a_{N_j})^T \\
= (\Phi^{j+1})^T B_j^T (a_1, a_2, \ldots, a_{N_j})^T \\
= (\Phi^{j+1})^T B_j^T D_j^{-1} ((\phi^j_1, g), (\phi^j_2, g), \ldots, (\phi^j_{N_j}, g))^T \\
= ((\Phi^{j+1})^T B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j ((\phi^j_1, g), (\phi^j_2, g), \ldots, (\phi^j_{N_j}, g))^T.
\]

Similarly,

\[
h_j = ((\Phi^{j+1})^T C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j ((\phi^j_1, g), (\phi^j_2, g), \ldots, (\phi^j_{N_j}, g))^T.
\]

and

\[
g_{j+1} = ((\Phi^{j+1})^T D_{j+1}^{-1} ((\phi^j_1, g), (\phi^j_2, g), \ldots, (\phi^j_{N_j}, g))^T.
\]
In order to show $h_j + g_j = g_{j+1}$, we only need to prove

$$B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j + C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j = D_{j+1}^{-1}. \quad (14)$$

Notice that $B_j$ and $C_j$ are not square matrices. That is we can not invert $B_j$ and $C_j$. Consider

$$(B_j D_{j+1} C_j D_{j+1}) \begin{pmatrix} B_j^T & C_j^T \end{pmatrix} = \begin{pmatrix} B_j D_{j+1} B_j^T & B_j D_{j+1} C_j^T \\ C_j D_{j+1} B_j^T & C_j D_{j+1} C_j^T \end{pmatrix} = \begin{pmatrix} B_j D_{j+1} B_j^T & \text{0} \\ \text{0} & C_j D_{j+1} C_j^T \end{pmatrix}$$

by using the orthogonal conditions of $V_j$ and $W_j$. Then we have the following equation

$$(B_j D_{j+1} C_j D_{j+1}) \begin{pmatrix} B_j^T & C_j^T \end{pmatrix} \begin{pmatrix} (B_j D_{j+1} B_j^T)^{-1} & \text{0} \\ \text{0} & (C_j D_{j+1} C_j^T)^{-1} \end{pmatrix} = I,$$

where $I$ stands for the identity matrix. In other words, we have

$$(B_j D_{j+1} C_j D_{j+1}) \begin{pmatrix} B_j^T & C_j^T \end{pmatrix} \begin{pmatrix} B_j D_{j+1} B_j^T & B_j \end{pmatrix} = I$$

which can be rewritten in the following form

$$(B_j^T (B_j D_{j+1} B_j^T)^{-1} & C_j^T (C_j D_{j+1} C_j^T)^{-1} \end{pmatrix} \begin{pmatrix} B_j D_{j+1} \\ C_j D_{j+1} \end{pmatrix} = I.$$

Hence we have

$$B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j D_{j+1} + C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j D_{j+1} = I$$

or

$$B_j^T (B_j D_{j+1} B_j^T)^{-1} B_j + C_j^T (C_j D_{j+1} C_j^T)^{-1} C_j = D_{j+1}^{-1}.$$

which is (14) and hence $h_j + g_j = g_{j+1}$.

## 5 Numerical Experiments

We have implemented the prewavelet method for numerical solution of Poisson equations over rectangular domains in MATLAB. We would like to demonstrate that our prewavelet method is more efficient than the standard finite element method.

In the following we provide three tables of CPU times for numerical solutions based on our prewavelet method and the standard finite element method for various levels of refinement of an initial triangulation ($\Gamma_0$ which consists of two triangles) of the standard domain $[0, 1] \times [0, 1]$. 

---

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Let $V_j$ be the continuous linear finite element space over triangulation $\Gamma_j$ which is the $j$th refinement of $\Gamma_0$. For a test function $u$ which is the exact solution of Poisson equation (1), the finite element method is to compute $u_j \in V_j$ directly while our prewavelet method computes $u_j$ by computing $w_k, k = 1, \ldots, j$, i.e., $u_j = u_1 + w_1 + \cdots + w_{j-1}$.

In the following we present three tables of CPU times for computing numerical solutions $u_j, j = 4, 5, 6$ for three test solutions by using these two methods. Note that we use the direct method coded in MATLAB to solve the associated linear equations. We shall present tables of CPU times based on Conjugate Gradient Method for the systems of equations next.

For an exact solution $u(x, y) = \sin(2\pi x)\sin(2\pi y)$ which clearly satisfies the zero boundary conditions, we list CPU times for computing numerical solutions $u_j, j = 4, 5, 6$ by using these two methods in Table 1.

Table 1. CPU times to compute $u_j$ by the two methods

|     | FEM method     | Prewavelet Method |
|-----|----------------|-------------------|
| $j=4$ | 0.164531 seconds | 0.204067 seconds  |
| $j=5$ | 0.593587 seconds | 0.519293 seconds  |
| $j=6$ | 13.960323 seconds | 6.222679 seconds  |

For an exact solution $u(x, y) = xy(1-x)(1-y)$, the CPU times for numerical solutions by these two methods are given in Table 2.

Table 2. CPU times for computing $u_j$ by the two methods

|     | FEM method     | Prewavelet Method |
|-----|----------------|-------------------|
| $j=4$ | 0.150836 seconds | 0.218282 seconds  |
| $j=5$ | 0.574085 seconds | 0.558071 seconds  |
| $j=6$ | 13.896825 seconds | 6.202557 seconds  |

We list the CPU times for computing numerical solutions $u_j, j = 4, 5, 6$ of $u(x, y) = xy(1-x)(1-y)e^{8xy}$ by using these two methods in Table 3.

Table 3. CPU times for computing $u_j$ by the two methods

|     | FEM method     | Prewavelet Method |
|-----|----------------|-------------------|
| $j=4$ | 0.144159 seconds | 0.186389 seconds  |
| $j=5$ | 0.584828 seconds | 0.459181 seconds  |
| $j=6$ | 13.877403 seconds | 6.139101 seconds  |

It is clear from these three tables that the prewavelet method is much more efficient.
Next we use the Conjugate Gradient Method to solve the linear systems associated with FEM. Let us consider iterative solution to \( u_j \) for \( j = 6 \) with various accuracy. First let us consider the exact solution \( u(x, y) = \sin(2\pi x) \sin(2\pi y) \).

Table 4. CPU times for approximating the FEM solution \( u_6 \) by Conjugate Gradient Method

| \( \epsilon \)   | CPU times         |
|------------------|-------------------|
| \( 10^{-8} \)    | 5.411852 seconds  |
| \( 10^{-9} \)    | 5.783497 seconds  |
| \( 10^{-10} \)   | 6.221683 seconds  |
| \( 10^{-11} \)   | 6.616816 seconds  |
| \( 10^{-12} \)   | 6.917468 seconds  |
| \( 10^{-13} \)   | 7.836775 seconds  |

To approximate the FEM solution \( u_6 \) of the exact solution \( u(x, y) = xy(1-x)(1-y) \) by the Conjugate Gradient Method, we list the CPU times in Table 5.

Table 5. CPU times for approximating the FEM solution \( u_j \) by Conjugate Gradient Method

| \( \epsilon \)   | CPU times         |
|------------------|-------------------|
| \( 10^{-8} \)    | 4.476794 seconds  |
| \( 10^{-9} \)    | 4.878259 seconds  |
| \( 10^{-10} \)   | 5.306747 seconds  |
| \( 10^{-11} \)   | 5.887849 seconds  |
| \( 10^{-12} \)   | 6.811317 seconds  |
| \( 10^{-13} \)   | 6.754465 seconds  |

Finally let us consider the CPU times to approximate the FEM solution \( u_6 \) of \( u(x, y) = xy(1-x)(1-y)e^{x+y} \) by the Conjugate Gradient Method.

Table 6. CPU times for approximating the FEM solution by Conjugate Gradient Method

| \( \epsilon \)   | CPU times         |
|------------------|-------------------|
| \( 10^{-8} \)    | 10.110517 seconds |
| \( 10^{-9} \)    | 10.740035 seconds |
| \( 10^{-10} \)   | 11.319618 seconds |
| \( 10^{-11} \)   | 11.810142 seconds |
| \( 10^{-12} \)   | 12.320903 seconds |
| \( 10^{-13} \)   | 13.103407 seconds |

It is clear from all six tables, if we want an accurate iterative solution of \( u_6 \) within \( 10^{-12} \), the prewavelet method appears better.
References

[1] F. Bastin and C. Boigelot, Biorthogonal wavelets in $H^m(R)$, J. Fourier Anal. Appl. 4(1998), 749–768.

[2] F. Bastin and P. Laubin, Regular compactly supported wavelets in Sobolev spaces, Duke Math. J., 87(1997), 481–508.

[3] S. C. Brenner and L. R. Scott, The Mathematical Theory of Finite Element Methods. Springer-Verlag (1994)

[4] M. D. Buhmann, O. Davydov, and T. N. T. Goodman, Box spline prewavelets of small support, J. Approx. Theory 112 (2001), 16–27.

[5] C. K. Chui, J. Stöckler, and J. D. Ward, On compactly supported box-spline wavelets, Approx. Theory Appl. 8(1992), 77–100.

[6] M. S. Floater and E. G. Quak, Piecewise linear prewavelets on arbitrary triangulations. Numer. Math. 82 (1999), 221–252.

[7] M. S. Floater and E. G. Quak, Piecewise Linear Wavelets over Type-2 Triangulations. Computing Supplement 14 (2001), 89-103

[8] D. Hong and Y. Mu, Construction of prewavelets with minimum support over triangulations, Wavelet analysis and multiresolution methods (Urbana-Champaign, IL, 1999), 145–165, Dekker, New York, 2000.

[9] R. Q. Jia and C. A. Micchelli, Using the refinement equations for the construction of pre-wavelets. II. Powers of two, in Curves and surfaces (Chamonix-Mont-Blanc, 1990), pp. 209–246, Academic Press, Boston, MA, 1991.

[10] R. Q. Jia, J. Z. Wang, and D. X. Zhou, Compactly supported wavelet bases for Sobolev spaces, Applied and Computational Harmonic Analysis, 15(2003), 224–241.

[11] U. Kotyczka and P. Oswald, Piecewise linear prewavelets of small support, in Approximation Theory VIII, vol. 2, C. K. Chui and L. L. Schumaker, eds., World Scientific, Singapore, 1995, 235–242.

[12] M. J. Lai, Construction of multivariate compactly supported prewavelets in $L_2$ spaces and pre-Riesz basis in Sobolev spaces, J. Appr. Theory 142(2006), 83–115.

[13] H. P. Liu, Prewavelets for Numerical Solution of Poisson Equations, Ph.D. Dissertation, University of Georgia, Athens, GA. under preparation, 2007.

[14] R. A. Lorentz and P. Oswald, Nonexistence of compactly supported box spline prewavelets in Sobolev spaces. Surface fitting and multiresolution methods (Chamonix-Mont-Blanc, 1996), 235–244, Vanderbilt Univ. Press, Nashville, TN, 1997.